Perfect Pseudo-Matchings in cubic graphs

HERBERT FLEISCHNER\textsuperscript{a}, BEHROOZ BAGHERI GH.\textsuperscript{a,b}, and BENEDIKT KLOCKER\textsuperscript{a}

\textit{a Algorithms and Complexity Group}\newline Vienna University of Technology\newline Favoritenstrasse 9-11, 1040 Vienna, Austria\newline

\textit{b Department of Mathematics}\newline West Virginia University\newline WV 26506-6310, Morgantown, USA

Abstract

A perfect pseudo-matching $M$ in a cubic graph $G$ is a spanning subgraph of $G$ such that every component of $M$ is isomorphic to $K_2$ or to $K_{1,3}$. In view of snarks $G$ with dominating cycle $C$, this is a natural generalization of perfect matchings since $G \setminus E(C)$ is a perfect pseudo-matching. Of special interest are such $M$ where $G/M$ is planar because such $G$ have a cycle double cover. We show that various well known classes of snarks contain planarizing perfect pseudo-matchings, and that there are at least as many snarks with planarizing perfect pseudo-matchings as there are cyclically $5-$edge-connected snarks.

Keywords: Snark; Perfect Pseudo-Matching; Eulerian graph; Transition system; Compatible cycle decomposition; Cycle double cover.

1 Introduction and preliminaries

All concepts not defined in this paper can be found in \cite{1, 5, 9}, giving preference to a definition as stated in \cite{5} if it differs from the corresponding definition in \cite{1}. A cycle

\textsuperscript{*}Research supported by FWF Project P27615-N25.
C in a graph $G$ is dominating if $E(G \setminus V(C)) = \emptyset$. A cycle $C$ in a graph $G$ is called stable if there exists no other cycle $D$ in $G$ such that $V(C) \subseteq V(D)$.

Our point of departure is the following.

**Sabidussi’s Compatibility Conjecture (SC Conjecture)** Given a connected eulerian graph $G$ with $\delta(G) > 2$ and an eulerian trail $T_\epsilon$ of $G$, there is a cycle decomposition $S$ of $G$ such that edges consecutive in $T_\epsilon$ belong to different elements in $S$. Whence one calls $S$ compatible with $T_\epsilon$.

The converse of the SC Conjecture is easily proved (see [15] but also [5, Theorem VI.1]): Given a cycle decomposition $S$ in the above $G$, then $G$ has an eulerian trail $T_\epsilon$ such that $S$ and $T_\epsilon$ are compatible. As has been noted before it suffices to consider such eulerian graphs $G$ for which $4 \leq \delta(G) \leq \Delta(G) \leq 6$, [4, Lemma 1].

In turn, such $G$ can be transformed into a cubic graph by splitting away from each vertex the transitions of a given eulerian trail $T_\epsilon$ of $G$, thus transforming $T_\epsilon$ into a cycle $C_\epsilon$. This transformation of $T_\epsilon$ into $C_\epsilon$ can also be viewed as a detachment of $G$ (see, e.g., [5, Corollaries V.10 and V.13]). Now, for a vertex $v$ of degree 4 in $G$ and the corresponding vertices $v'$ and $v''$ in $C_\epsilon$ add an edge $v'v''$, whereas for a vertex $w$ of degree 6 in $G$ and the corresponding $w', w'', w'''$ in $C_\epsilon$ introduce a new vertex $w^*$ and the edges $w^*w', w^*w'', w^*w'''$. The resulting cubic graph $G_3$ contains $C_\epsilon$ as a dominating cycle. Call $(G_3, C_\epsilon)$ associated with $(G, T_\epsilon)$. It is intuitively clear how one obtains $(G, T_\epsilon)$ from $(G_3, C_\epsilon)$. The following is well known [4, pp. 236–237].

**Proposition 1** Let $(G, T_\epsilon)$ and $(G_3, C_\epsilon)$ be as above. $G$ has a cycle decomposition compatible with $T_\epsilon$ if and only if $G_3$ has a cycle double cover $S$ with $C_\epsilon \in S$.

**Cycle Double Cover Conjecture (CDC Conjecture)** In every bridgeless graph $G$ with $E(G) \neq \emptyset$ there is a collection $S$ of cycles such that every edge of $G$ belongs to exactly two elements of $S$.

In dealing with the CDC Conjecture, it suffices to consider snarks, i.e., cyclically 4–edge-connected cubic graphs which are not 3−edge-colorable. In our understanding of snarks we omit the usual requirement that the girth must exceed 4. For snarks, the following conjecture has been formulated which is equivalent to various other conjectures, [2, 10, 16].
Dominating Cycle Conjecture (DC Conjecture) Every snark has a dominating cycle.

Proposition 1 and the DC Conjecture demonstrate the close relationship between SC Conjecture and CDC Conjecture.

Definition 2 Let $G$ be a $2$-connected eulerian graph. For each vertex $v \in V(G)$, let $\mathcal{T}(v)$ be a set of disjoint edge-pairs of $E(v)$, and $\mathcal{T} = \bigcup_{v \in V(G)} \mathcal{T}(v)$. Then, $\mathcal{T}$ is called a transition system, and a cycle decomposition $\mathcal{C}$ of $G$ is compatible with $\mathcal{T}$ if $|E(C) \cap P| \leq 1$ for every member $C \in \mathcal{C}$ and every $P \in \mathcal{T}$.

For planar eulerian graphs $G$ one can prove a more general result than the SC Conjecture; namely it suffices to assume that the transition system is non-separating; one even has a compatible cycle decomposition (CCD) in an arbitrary eulerian graph $G$ if the given transition system in $G$ is non-separating and does not yield a SUD-$K_5$-minor\(^1\). This has been shown recently in [9].

We note explicitly that the compatible cycle decomposition problem has been verified for planar graphs by Fleischner [6], for $K_5$-minor-free graphs by Fan and Zhang [3], and for SUD-$K_5$-minor-free graphs by Fleischner et al. [9].

On the other hand, when focusing on the SC Conjecture with all vertices of degree 4 and 6 only in a given graph $G$ with eulerian trail $T_\iota$, one has a direct connection to cubic graphs with a dominating cycle $C_\iota$ where $C_\iota$ corresponds to $T_\iota$ and viceversa, as noted above (see Proposition 1). This leads us to the following definition.

Definition 3 A perfect pseudo-matching $M$ in a graph $G$ is a collection of vertex-disjoint subgraphs of $G$ such that

- every member of $M$ is isomorphic to either $K_2$ or $K_{1,3}$, and
- $\bigcup_{H \in M} V(H) = V(G)$.

Definition 4 Let $M$ be a perfect matching (perfect pseudo-matching) in a graph $G$. $M$ is called

- planarizing if $G/M$ is a planar graph;
- $K_5$-minor-free if $G/M$ has no $K_5$-minor;

\(^1\)See the definition of a SUD-$K_5$-minor in a transitioned graph $(G, \mathcal{T})$ in [9].
• **SUD-\(K_5\)-minor-free** if \((G/M, \mathcal{T}_M)\) has no SUD-\(K_5\)-minor, where \(\mathcal{T}_M\) is the transition system induced by contracting \(M\).

Note that every perfect matching is a perfect pseudo-matching; and for every dominating cycle \(C\) in a cubic graph \(G\), \(M := G \setminus E(C)\) is also a perfect pseudo-matching.

Note that \(G \setminus E(M)\) is a set \(C_0\) of disjoint cycles in \(G\) for any perfect pseudo-matching \(M\) of a cubic graph \(G\). We conclude that if \(G/M\) has a compatible cycle decomposition (with the transitions defined by the pairs of adjacent edges in \(C_0\) and thus yielding a transition system \(\mathcal{T}_M\)), then \(G\) has a cycle double cover \(C\) with \(C_0 \subseteq C\).

**Example 1** Consider the perfect pseudo-matching \(M\), visualized as bold-face edges in the Petersen graph \(P\) in Figure 1. \(P \setminus E(M)\) yields the dominating cycle \(C_0 = v_1v_2v_3v_4v_9v_7v_5v_8v_6v_1\). Then

\[
\mathcal{C} = \{C_0, v_0v_1v_2v_7v_5v_0, v_0v_1v_6v_9v_1v_0, v_0v_4v_3v_8v_5v_0, v_2v_3v_6v_9v_7v_2\}
\]

is a cycle double cover of the Petersen graph containing \(C_0\) obtained from a compatible cycle decomposition of \((P/M, \mathcal{T}_M)\). This CDC \(C\) exists because \(P/M\) is planar; i.e., \(M\) is a planarizing perfect pseudo-matching.

![Figure 1](image-url)  
*Figure 1: The Petersen graph and its planarizing perfect pseudo-matching visualized as bold-face edges.*

## 2 Planarizing perfect pseudo-matchings in snarks

In view of Fleischner’s result on compatible cycle decomposition in planar eulerian \(G\) with \(\delta(G) \geq 4\) and with given non-separating transition system \(\mathcal{T}\), [6], it is of particular interest to search for planarizing perfect pseudo-matchings \(M\) in snarks. We study various well-known types of graphs in this direction.
By a computer search we found that all snarks with up to 26 vertices contain a planarizing perfect pseudo-matching except two snarks of order 26; one of them is given in Figure 6.

(a) Blanuša snarks

Consider the three subgraphs which we call Blanuša blocks $B_0$, $B_1$, and $B_2$ in Figure 2. For every $n$ we can construct a Blanuša snark $B_j^n$, $j = 1, 2$, as follows, [17, 11]. Let $H_i$, $1 \leq i \leq n - 1$, be $n - 1$ copies of the block $B_0$ and let $H_n$ be a copy of $B_j$, $j = 1, 2$. Then connect the half-edges $a'$ and $b'$ of each $H_i$ to the half-edges $a$ and $b$ of the block $H_{i+1}$, respectively, for $i = 1, \ldots, n - 1$, and likewise connect $H_n$ to $H_1$.

Note that $B_1^1$ is the Petersen graph, and $B_2^1$ is the first Blanuša snark. The set of all members of all $B_1^n$'s and all $B_2^n$'s are called generalized Blanuša snarks.

For every $B_1^n$ let $M_1$ be a set of all edges of $< u_0, u_1, u_2, u_6 >$, $u_3u_5$, and $u_4u_7$ in all copies of $B_0$ and the edges of $< v_0, v_1, v_2, v_6 >$, $v_3v_5, v_4v_7$, and $v_8v_9$ in the copy of $B_1$.

For every $B_2^n$ let $M_2$ be a set of all edges of $< u_0, u_3, u_4, u_5 >$, $u_1u_2$, and $u_6u_7$ in all copies of $B_0$ and the edges $w_0w_1, w_2w_4, w_3w_5, w_6w_7$, and $w_8w_9$ in the copy of $B_2$.

By a drawing of Blanuša snarks as exemplified in Figure 3 and by Theorem 8 below, one can see that $M_j$ is a planarizing perfect pseudo-matching in $B_j^n$, for $j = 1, 2$.

(b) Flower snarks

For an odd integer $k \geq 3$, the flower snark $J_k$ is constructed as follows. $V(J_k) = \{v_1, v_2, \ldots, v_k\} \cup \{u_1, u_2, u_3, u_4, u_5, \ldots, u_k, u_1^2, u_2^2\}$. $J_k$ is comprised of a cycle
Figure 3: The Blanuša snarks $B^1_2$ and $B^2_2$ and their planarizing perfect pseudo-matchings exhibited by the bold-face edges.

$C_1 = u_1^1u_2^1 \ldots u_k^1u_1^1$ of length $k$ and a cycle $C_2 = u_1^2u_2^2 \ldots u_k^3u_1^3 \ldots u_k^1u_1^1$ of length $2k$, and in addition, each vertex $v_i$, $1 \leq i \leq k$, is adjacent to $u_i^1$, $u_i^2$, and $u_i^3$.

Note that for even $k \geq 4$, $C_1$ is an even cycle. Color the edges of $C_1$ with color set $\{2, 3\}$; color $u_i^iv_i$, $i = 1, \ldots, k$, $j = 2, 3$, with color $j$; color $u_{2i-1}^3u_{2i}^3$, $i = 1, \ldots, k/2$, with color 2; color $u_{2i-1}^2u_{2i}^2$, $i = 1, \ldots, k/2$, with color 3; and color the remaining edges with color 1. Thus $J_k$ is 3–edge-colorable for even $k$.

Now consider the perfect pseudo-matching $M = \{v_iu_i^1, v_iu_i^2, v_iu_i^3 : 1 \leq i \leq k\}$ in the flower snark $J_k$, $k \geq 3$. In fact, contract every claw induced by $\{u_i^1, u_i^2, u_i^3, v_i\}$ to a new vertex $v'_i$, $1 \leq i \leq k$. Then $J_k/M$ consists of three cycles $v'_1v'_2 \ldots v'_kv'_1$, so $J_k/M$ is planar.

(c) Goldberg snarks

Goldberg [12] constructed an infinite family of snarks, $G_5, G_7, \ldots$. For every odd $k \geq 5$, the vertex set of the Goldberg snark $G_k$ satisfies

$$V(G_k) = \{v_j^t : 1 \leq t \leq k, 1 \leq j \leq 8\}.$$

The subgraph $B_t$ induced by

$$\{v_1^t, v_2^t, \ldots, v_8^t\} \text{ and } \{v_1^tv_2^t, v_1^tv_7^t, v_2^tv_8^t, v_3^tv_4^t, v_3^tv_5^t, v_4^tv_7^t, v_5^tv_6^t, v_6^tv_8^t, v_7^tv_8^t\}$$

is called a basic block. The Goldberg snark is constructed by joining each basic block $B_t$ with $B_{t+1}$ by the edges $v_1^tv_1^{t+1}, v_4^tv_3^{t+1}$, and $v_3^tv_5^{t+1}$ where the subscripts of basic blocks and the superscripts of vertices are read modulo $k$.

Now consider the perfect matching $M = \{v_1^tv_7^t, v_2^tv_8^t, v_3^tv_4^t, v_5^tv_6^t : 1 \leq t \leq k\}$ in the Goldberg snark $G_k$, $k \geq 5$. 
Figure 4: The Goldberg snark $G_5$ and a planarizing perfect matching $M$ visualized by bold-face edges.

By such a drawing of $G_k$ as exemplified by Figure 4 and by Theorem 8 below, $G_k/M$ is planar.

(d) Celmins-Swart snarks

Planarizing perfect pseudo-matchings for the two Celmins-Swart snarks are shown in Figure 5.

Figure 5: The Celmins-Swart snarks and their planarizing perfect pseudo-matchings exhibited by the bold-face edges (cf. Theorem 8 below).

We also examined other snarks and determined planarizing perfect pseudo-matchings. However, we cannot conclude that every snark has a planarizing perfect pseudo-matching (otherwise we had an easy proof of the CDC Conjecture).
Example 2 By using computer programming we found a snark of order 26 shown in Figure 6 which has no planarizing perfect pseudo-matching but it has a SUD-$K_5$—minor-free perfect pseudo-matching.

![Figure 6: An snark of order 26 without any planarizing perfect pseudo-matchings and its SUD-$K_5$—minor-free perfect pseudo-matching visualized by bold-face edges.](image)

In fact, for all snarks with up to 32 vertices which contain a stable dominating cycle it was checked whether they contain a planarizing / $K_5$—minor-free / SUD-$K_5$—minor-free perfect pseudo-matching (the perfect pseudo-matching does not have to be the complement of the stable dominating cycle). It was determined that

there are 4615 snarks $G_3$ with up to 32 vertices which have a stable dominating cycle. 4612 of them contain a planarizing perfect pseudo-matching, 2 have no planarizing perfect pseudo-matching, but a $K_5$—minor-free perfect pseudo-matching, and one has no $K_5$—minor-free perfect pseudo-matching but a SUD-$K_5$—minor-free perfect pseudo-matching. So all 4615 snarks have a SUD-$K_5$—minor-free perfect pseudo-matching $M$ (which in turn guarantees that $G_3/M$ has a compatible cycle decomposition).

Moreover, we checked also directly the complements of the stable dominating cycles and found the following. Surprisingly, for all the stable dominating cycles in all snarks with up to 32 vertices the complements where never (!) planarizing. Furthermore, there is also no stable dominating cycle whose complement is a $K_5$—minor-free perfect pseudo-matching. Out of the 4615 snarks with stable dominating cycles there
are 3045 snarks which contain a stable dominating cycle whose complement is a SUD-$K_5$-minor-free perfect pseudo-matching. For the other 1570 snarks the complements of all stable dominating cycles are a SUD-$K_5$-minor perfect pseudo-matching, but nevertheless the contraction of these perfect pseudo-matchings have a compatible cycle decomposition. So in this case we see that the complement of all stable dominating cycles is at least a perfect pseudo-matching after whose contraction the eulerian graph has a cycle decomposition compatible with the eulerian trail corresponding to the given dominating cycle in the snark.

Moreover, 3-edge-colorability and compatible cycle decompositions can be related; see our next result.

**Theorem 5** Let $G$ be a cubic graph and let $M$ be a perfect pseudo-matching in $G$. Let $T$ be the transition system in $G/M$ defined by pairs of adjacent edges in $G_0 = G\setminus E(M)$. For a compatible cycle decomposition $\mathcal{S}$ of $(G/M, T)$ define the intersection graph $I(\mathcal{S})$ whose vertices correspond to the elements of $\mathcal{S}$ and $xy$ is in $E(I(\mathcal{S}))$ if and only if the corresponding cycles $C_x$ and $C_y$ in $\mathcal{S}$ have at least one vertex in common. The following (i) and (ii) are equivalent.

(i) $G$ is 3-edge-colorable.

(ii) $\chi(I(\mathcal{S})) \leq 3$ for at least one compatible cycle decomposition $\mathcal{S}$ of $(G/M, T)$.

Furthermore,

(iii) $\chi(I(\mathcal{S})) = 2$ if and only if $M$ is a perfect matching and represents a color class in a 3-edge-coloring of $G$.

**Proof.** Let $G$ be a cubic graph with a proper edge coloring $c$ with color set $\{1, 2, 3\}$. The coloring $c$ induces a color on every edge of $G/M$. Since $G$ is 3-regular and $c$ is proper, every vertex of $G/M$ is adjacent to 0 or 2 edges of color $i$, $i = 1, 2, 3$. Therefore, the subgraph induced by $i$-colored edges in $G/M$ is a disjoint union of $i$-colored cycles, $i = 1, 2, 3$; moreover, each such $i$-colored cycle is a compatible cycle. Let $\mathcal{S}$ be the set of all such compatible $i$-colored cycles for $i = 1, 2, 3$. Now, give color $i$ to every vertex of $I(\mathcal{S})$ which corresponds to a compatible $i$-colored cycle in $\mathcal{S}$, $i = 1, 2, 3$. Thus $I(\mathcal{S})$ has a proper vertex coloring with color set $\{1, 2, 3\}$.

By reversing the argument the implication (ii)$\rightarrow$(i) follows easily. The remainder of the proof is easily established. \[\square\]
A planarizing perfect pseudo-matching $M$ of the Petersen graph is shown in Figure 1, which in turn served as the basis for constructing cubic graphs $G$ with a stable dominating cycle $C$ with a planarizing perfect pseudo-matching $M$ having only 2 components which are $K_{1,3}$. This in turn led to a simple uniquely hamiltonian graph of minimum degree 4, [8]. We note in passing that any stable dominating cycle in a cubic graph can be used as the basis for constructing a simple uniquely hamiltonian graph of minimum degree 4.

In view of our remarks preceding Theorem 5, we are led to the following question.

**Question 1** Given a cubic graph $G_3$ with a stable dominating cycle $C$ and corresponding perfect pseudo-matching $M = E(G_3) \setminus E(C)$. Is it true that $(G_3/M, T_M)$ has a compatible cycle decomposition?

**Example 3** By using computer programming we found a snark $G_3$ of order 28 shown in Figure 7 which has exactly one stable dominating cycle $C$. The perfect pseudo-matching $M = E(G_3) \setminus E(C)$ is visualized by bold-face edges. $G_3/M$ contains a SUD-$K_5$--minor.

![Figure 7](image-url)

Figure 7: An snark of order 28 with precisely one stable dominating cycle $C$.

**Proposition 6** Suppose Question 1 has a positive answer. Then

(a) the SC Conjecture is true;

(b) the CDC Conjecture can be reduced to the DC Conjecture.
Proof. Suppose Sabidussis Compatibility Conjecture (SC Conjecture) is false. Then it is false even for an eulerian graph $G$ with eulerian trail $T_e$ and $4 \leq \delta(G) \leq \Delta(G) \leq 6$. Let $G_3$ be the cubic graph and $C_e$ the dominating cycle in $G_3$ corresponding to $G$ and $T_e$, respectively, as mentioned in the introduction following the statement of the SC Conjecture. We distinguish between two cases.

1) $C_e$ is a stable cycle. Then $E(G_3) \setminus E(C_e)$ is a perfect pseudo-matching $M$, and by the supposition of this proposition $G = G_3/M$ has a cycle decomposition compatible with $T_e$ (corresponding to $C_e$), contrary to our supposition that SC Conjecture is false for $G$ with eulerian trail $T_e$.

2) $C_e$ is not a stable cycle. Then there is a dominating cycle $C_1$ in $G_3$ with $V(C_e) \subseteq V(C_1)$. Considering $G = G_3/M$ and $T_e$ as in case 1), it follows that $C_1$ corresponds to a spanning trail $T_1$ in $G$. Moreover, $G \setminus E(T_1)$ is a set of totally disjoint compatible cycles $S^{(1)}$ in $(G, T)$.

Let $G^{(1)}$ be the subgraph of $G$ induced by $E(T_1)$. In $G^{(1)}$, $T_1$ is an eulerian trail. Suppressing in $G^{(1)}$ the vertices of degree 2 we transform $(G^{(1)}, T_1)$ into $(G', T')$ with $G'$ having vertices of degree 4 and 6 only, and $T'$ being an eulerian trail of $G'$. Let $G'_3$ be the cubic graph corresponding to $G'$, and let $C'$ be the dominating cycle of $G'_3$ corresponding to $T'$ and thus to $C_1$.

Now we consider the two cases 1) and 2) with $(G'_3, C')$ in place of $(G_3, C)$; and so on. Ultimately, for some $j > 0$, $G^{(j)}$ is nothing but a cycle $C^*$, or it has a cycle decomposition $S^*$ compatible with the eulerian trail $T^{(j)}$ of $G^{(j)}$ because the corresponding dominating cycle $C^{(j)}$ of $G^{(j)}_3$ is stable. Thus, $S^{(1)} \cup \ldots \cup S^{(j-1)} \cup \{C^*\}$ or $S^{(1)} \cup \ldots \cup S^{(j-1)} \cup S^*$, respectively, corresponds to a cycle decomposition $S$ of $G$ compatible with $T_e$ after step-by-step inserting anew the suppressed vertices of degree 2. The validity of SC Conjecture for $G$ follows, contrary to the original supposition.

Moreover, since it suffices to consider snarks when dealing with the CDC Conjecture we may first consider the DC Conjecture. Thus for a given snark $G_3$, if we can find a dominating cycle $C$, then we can construct $(G, T)$ by contracting the perfect pseudo-matching $M = G_3 \setminus E(C)$ with $G = G_3/M$ and $T$ corresponding to $C$. Now we argue algorithmically as above to obtain a cycle decomposition $S$ of $G$ compatible with $T$. Clearly, $S$ corresponds to a set $S_3$ of cycles in $G_3$ covering the edges of $M$.
Snarks of order $\leq 32$

| $n$ | $s(n)$ | $sppm(n)$ | $spppm(n)$ | $spmK_5(n)$ | $sppmK_5(n)$ | $spmSK_5(n)$ | $sppmSK_5(n)$ |
|-----|--------|-----------|------------|-------------|--------------|--------------|--------------|
| 10  | 1      | 1         | 0          | 1           | 0            | 1            | 0            |
| 18  | 2      | 1         | 0          | 1           | 0            | 1            | 0            |
| 20  | 6      | 5         | 0          | 5           | 0            | 4            | 0            |
| 22  | 31     | 29        | 0          | 29          | 0            | 14           | 0            |
| 24  | 155    | 146       | 0          | 146         | 0            | 97           | 0            |
| 26  | 1297   | 1239      | 2          | 1239        | 0            | 822          | 0            |
| 28  | 12517  | 12102     | 45         | 12102       | 15           | 8374         | 0            |
| 30  | 139854 | 136850    | 933        | 136850      | 578          | 105321       | 33           |
| 32  | 1764950| 1740342   | 24268      | 1740342     | 18537        | 1430228      | 1062         |

Table 1: \[s(n) = \#\text{Snarks of order } n; \quad sppm(n) = \#\text{Snarks of order } n \text{ with no planarizing perfect matching}; \quad spppm(n) = \#\text{Snarks of order } n \text{ with no planarizing perfect pseudo-matching}; \quad sppmK_5(n) = \#\text{Snarks of order } n \text{ with no } K_5\text{-minor-free perfect matching}; \quad sppmK_5(n) = \#\text{Snarks of order } n \text{ with no } K_5\text{-minor-free perfect pseudo-matching}; \quad sppmSK_5(n) = \#\text{Snarks of order } n \text{ with no SUD-}K_5\text{-minor-free perfect matching}; \quad sppmSK_5(n) = \#\text{Snarks of order } n \text{ with no SUD-}K_5\text{-minor-free perfect pseudo-matching.}

However, we also made a computer search establishing the usefulness of perfect pseudo-matchings - we include Table 1 produced by the third author and discuss the advantage of perfect pseudo-matchings vis-a-vis perfect matchings.

In Table 1, compare column $2i$ with column $2i - 1$, $i = 2, 3, 4$, in which a small portion of snarks of order $n$ have some perfect matching with a property $X$ whereas almost all snarks of order $n$ have some perfect pseudo-matching with the property $X$. For example, out of the 1297 snarks of order 26, 58 have some planarizing perfect matching while all snarks of order 26 except two of them (one is shown in Figure 6) have some planarizing perfect pseudo-matching; and all snarks of order 26 have some SUD-$K_5$-minor-free perfect pseudo-matching while 475 of them have some SUD-$K_5$-minor-free perfect matching.

Nonetheless, snarks with planarizing perfect pseudo-matchings exist in abundance. To show this we need to define a certain drawing of a cubic graph $G$ with given perfect
Lemma 7 Let a simple cubic graph $G$ with perfect pseudo-matching $M$ be given. Then we can draw $G$ in the plane in such a way that the crossings of the drawing involve only edges of $G \setminus E(M)$. Moreover, no three edges of $G$ cross each other in the same point of the plane.

Proof. The validity of the lemma rests on the following facts.

(i) One can draw $M$ in the Euclidean plane $\mathbb{R}^2$ such that any two points of different components of $M$ are of distance $> 1$ apart, say, and such that the drawing of $M$ does not contain any closed curve;

(ii) viewing $M$ as a point set, the set $\mathbb{R}^2 \setminus M$ is connected;

(iii) every edge $e = xy \in E(G) \setminus E(M)$ can be drawn as a smooth curve in $\mathbb{R}^2$ such that all of $e$ except $x, y$ lie in $\mathbb{R}^2 \setminus M$;

(iv) a desired drawing of $G$ in $\mathbb{R}^2 \setminus M$ can be achieved in a step-by-step manner. Namely, if we have already drawn a subgraph $G^\circ$ of $G$ with $G^\circ$ containing $M$ such that no three edges cross each other in the same point, and if $X^\circ$ is the set of crossing points of pairs of edges of $G^\circ$, then $\mathbb{R}^2 \setminus (E(M) \cup X^\circ)$ is still a connected point set (and so the next edge can be drawn without passing through an element of $X^\circ$).

In addition we may achieve the property that

(v) no pair of edges in the drawing of $E(G)$ has more than one common point. 

We call a drawing having the property stated in Lemma 7, a drawing with $M$-avoiding intersections. Note that if such a drawing exists, then there exists even one where adjacent edges do not intersect in $\mathbb{R}^2 \setminus M$ (see (v) above).

Thus we may relate drawings with $M$-avoiding intersections to planarizing perfect pseudo-matchings $M$ as follows.

Theorem 8 Let $G$ be a simple cubic graph, and let $M$ be a perfect pseudo-matching. $M$ is planarizing if and only if there exists a drawing of $G$ with $M$-avoiding intersections such that if $e, f \in E(G) \setminus E(M)$ intersect, then $e$ and $f$ are incident to different vertices of a component of $M$. 

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Proof. Let $M$ be a planarizing perfect pseudo-matching of a cubic graph $G$. So there is a drawing of $G$ with $M$-avoiding intersections by Lemma 7 such that $G/M$ is planar. Such drawing of $G$ can be obtained by starting from a plane embedding of $G/M$ and by replacing the vertices of $G/M$ by the components of $M$. Thus $e$ and $f$ are adjacent to different vertices of a component of $M$, for every edge-crossing involving $\{e, f\} \subseteq E(G) \setminus E(M)$ in this drawing of $G$; otherwise, the edges corresponding to $e$ and $f$ in $G/M$ intersect in said drawing.

Suppose conversely that $G$ has a drawing with $M$-avoiding intersections such that if $e, f \in E(G) \setminus E(M)$ intersect, then $e$ and $f$ are adjacent to different vertices of a component of $M$, but suppose that $G/M$ is not planar. So there exist some edge-crossing involving $\{e', f'\}$ in any embedding of $G/M$. Thus, $e'$ and $f'$ can be assumed to be adjacent to four different vertices in $V(G/M)$. Since every vertex of $G/M$ corresponds to a component of $M$, two edges $e, f \in E(G) \setminus E(M)$ corresponding to $e'$ and $f'$ intersect but they are not adjacent to different vertices of a component of $M$, which is a contradiction.

Lemma 9 Let $G$ be a 2–connected simple cubic graph having an embedding in the plane with precisely one edge-crossing $\{xx''', x' x''\} \subseteq E(G)$. Define $G^*_x := (G \setminus \{xx''', x' x''\}) \cup B_0$ obtained by connecting $x$ with $a$ and $x'''$ with $a'$, $x'$ with $b$ and $x''$ with $b'$, and where $B_0$ is the first Blanuša block and $a, a', b, b'$ are half-edges of $B_0$ (see Figures 2 and 8). Also let $M$ be a perfect pseudo-matching of $G \setminus \{xx''', x' x''\}$. Then the following is true.

(i) $G^*_x$ is 3–edge-colorable if and only if $G$ is 3–edge-colorable.

(ii) $G^*_x$ has a spanning subgraph homeomorphic to $G$.

(iii) $G^*_x$ has a planarizing perfect pseudo-matching $M^*_x$ with $M \subset M^*_x$.

(iv) Every CDC of $G$ can be extended to a CDC of $G^*_x$.

Proof. Suppose $xx''', x' x'' \in E(G)$ are involved in the only edge-crossing in the given embedding of $G$, and suppose $M$ is a perfect pseudo-matching of $G \setminus \{xx''', x' x''\}$. Let $G^*_x$ be the graph, obtained from $G$ as described in the statement of the lemma and Figure 8.
Let $G$ be easily extended to a 3-edge-colorable graph. Then $G$ is 3-edge-colorable, and a 3-edge-coloring of $G$ can be easily extended to a 3-edge-coloring of $G^*_x$. Therefore, we can conclude that (i) is true.

(ii) Clearly, $(G \setminus \{xx''', x'x''\}) \cup \{xx_0x_3x_4x_7x'', x'x_2x_1x_6x_5x''\}$ is a spanning subgraph of $G^*_x$ homeomorphic to $G$.

(iii) Let $M^*_x = M \cup E(H) \cup \{x_3x_5, x_4x_7\}$ where $H$ is a copy of $K_{1,3}$ induced by $\{x_0, x_1, x_2, x_6\}$. Obviously, $M^*_x$ is a planarizing perfect pseudo-matching of $G^*_x$ containing $M$ (see Figure 8).

(iv) Let $C$ be a CDC of $G$ and let $C_1, C_2, C_3,$ and $C_4$ be some cycles in $C$ such that $xx''' \in E(C_1) \cap E(C_2)$, and $x'x'' \in E(C_3) \cap E(C_4)$.

Put $P_1 = xx_0x_3x_4x_7x''$, $P_2 = xx_0x_1x_6x_7x''$, $P_3 = x'x_2x_1x_6x_5x''$, and $P_4 = x'x_2x_4x_3x_5x''$. Now set $C'_1 = C_1 \setminus \{xx''\} \cup P_1$, $C'_2 = C_2 \setminus \{xx'''\} \cup P_2$, $C'_3 = C_3 \setminus \{x'x''\} \cup P_3$, $C'_4 = C_4 \setminus \{x'x'''\} \cup P_4$, and $C' = x_0x_1x_2x_4x_7x_6x_5x_3x_0$.

Note that if some cycle in $C$ covers both edges $xx'''$ and $x'x''$, then suppose that $C_1 = C_3$ and that likewise $C_2 = C_4$ if $C$ contains two cycles traversing $xx'''$ and $x'x''$ each. In this case, put $C'_1 = C'_3 = C_1 \setminus \{xx''', x'x''\} \cup P_1 \cup P_3$ if $C_1 = C_3$ and set $C'_2 = C'_4 = C_2 \setminus \{xx''', x'x''\} \cup P_2 \cup P_4$ if $C_2 = C_4$.

It follows that

$$C^*_x = (C \setminus \{C_1, C_2, C_3, C_4\}) \cup \{C'_1, C'_2, C'_3, C'_4, C'\}$$
is a CDC of $G^*_x$.

With the help of Theorem 8 and Lemma 9, we can prove the following.

**Theorem 10** Given a cyclically $4-$edge-connected cubic graph $G$ with a perfect pseudo-matching $M$. Then there exists a cubic graph $G^*$ with the following properties.

(i) $G^*$ is a snark if and only if $G$ is a snark.

(ii) $G^*$ has a spanning subgraph homeomorphic to $G$.

(iii) $G^*$ has a planarizing perfect pseudo-matching $M^*$ with $M \subset M^*$; and moreover, $G^*$ admits a CDC containing the cycles of $G^* \setminus E(M^*)$.

(iv) Every CDC of $G$ can be extended to a CDC of $G^*$, but the converse is, unfortunately, not true.

**Proof.** Suppose $M$ is a perfect pseudo-matching of $G$. By Lemma 7, there exists a drawing with $M$-avoiding intersections of $G$. Let $G^*$ be the graph, obtained from $G$ by repeatedly applying the same conversion we did in the proof of Lemma 9, for all edge-crossings in $E(G) \setminus E(M)$ in this drawing with $M$-avoiding intersections of $G$. Therefore, by repeatedly using Lemma 9, we can check that all statements in Theorem 10 are true: in particular, $G^*$ is cyclically $4-$edge-connected since $G$ is. Thus, (i) in Lemma 9 translates into (i) in Theorem 10. Furthermore, since $M^*$ is planarizing, $(G^*/M^*, T_{M^*})$ has a compatible cycle decomposition which can be readily translated into a CDC of $G^*$ containing the cycles of $G^* \setminus E(M^*)$. Finally, it is straightforward to see that $G^*$ may have a CDC which cannot be transformed into a CDC of $G$.

One is tempted to improve Theorem 10 by using a perfect matching in $G$ and using the Blažuša block $B_2$ instead of $B_0$ for the crossings in a corresponding drawing of $G$. The larger graph $G^*$ would, in fact, contain a planarizing perfect matching (see the perfect matching of $B_2$ in $B_2^2$ in Figure 3). However, due to the $3-$edge-coloring of $B_2$ we cannot draw the same conclusions as for $G^*$ in Theorem 10.

Finally we observe that the construction of a cyclically $4-$edge-connected cubic graph with planarizing perfect pseudo-matching as expressed by Theorem 10, tells us that there are at least as many snarks with planarizing perfect pseudo-matching as there are cyclically $5-$edge-connected snarks. ‘As many’ is to be understood in terms of infinite cardinalities of sets. This is expressed in our final theorem.
Theorem 11 Given the family $\mathcal{F}_5$ of all cyclically 5–edge-connected snarks together with a drawing in the plane in accordance with Lemma 7, for each element $G \in \mathcal{F}_5$. Call this drawing also $G$ and construct $G^*$ from $G$ in accordance with Theorem 10. Define a mapping

$$f : \mathcal{F}_5 \rightarrow \mathcal{F}_4$$

(the latter denoting the family of cyclically 4–edge-connected snarks having a planarizing perfect pseudo-matching) by setting

$$f(G) = G^*.$$

It follows that $f$ is injective.

Proof. Let $G$ and $H$ be cyclically 5–edge-connected snarks together with respective drawings in the plane in accordance with Lemma 7, and let $G^*$ and $H^*$ be constructed from $G$ and $H$, respectively, in accordance with Theorem 10. Suppose $G^*$ and $H^*$ are isomorphic; let $h^*(G^*) = H^*$ be such an isomorphism. It follows that the only cyclic 4–edge-cuts in $G^*$, $H^*$, respectively, are of the form

$$F = \{x_0x, x_5x''', x_2x', x_7x''\}$$

(see Figure 8), separating a Blanusa block $B_0$ from the rest of $G^*$, $H^*$, respectively. We denote these edge-cuts in $G^*$, $H^*$, respectively, by

$$F_{G^*} = \{x_{0G^*}, x_{5G^*}, x_{2G^*}, x_{7G^*}, x''_{G^*}\}$$

and

$$F_{H^*} = \{x_{0H^*}, x_{5H^*}, x_{2H^*}, x_{7H^*}, x''_{H^*}\}.$$

Now, assume without loss of generality that $h^*(F_{G^*}) = F_{H^*}$. It follows that

$$h^*(\{x_{0G^*}, x_{7G^*}, x''_{G^*}\}) = \{x_{0H^*}, x_{7H^*}, x''_{H^*}\}$$

and

$$h^*(\{x_{5G^*}, x_{2G^*}, x'_{G^*}\}) = \{x_{5H^*}, x_{2H^*}, x'_{H^*}\}$$

(otherwise we could rotate the notation). This leads to an isomorphism $h$ between $G$ and $H$ by setting $h(e) = h^*(e)$ for every edge $e$ which is not an element of a pair of crossing edges;
and set $h(x_G x''_G) = x_H x''_H$ and $h(x'_G x''_G) = x'_H x''_H$ for every pair of crossing edges in the drawing of $G$ and $H$, respectively. In other words, non-isomorphic cyclically 5-edge-connected snarks correspond to non-isomorphic cyclically 4-edge-connected snarks with planarizing perfect pseudo-matching. That is, $f$ is injective.

3 Final remarks

We have demonstrated the usefulness of the concept of perfect pseudo-matchings $M$ in categorizing snarks $G$, be it in connection with compatible cycle decompositions in $G/M$, be it more generally in connection with cycle double covers. In particular, we showed that when aiming at solving the CDC Conjecture we may restrict ourselves to snarks with stable dominating cycle (Question 1 and Proposition 6); or alternatively, we only need to deal with snarks having no SUD-$K_5$-minor-free perfect pseudo-matching.

As for several well-known classes of snarks we have shown that they have planarizing perfect pseudo-matchings, and that planarizing perfect pseudo-matchings appear in many snarks (Theorems 10 and 11). As a matter of fact, an old conjecture of the first author of this paper claims implicitly that if a snark $G$ has a planarizing perfect pseudo-matching which corresponds to a dominating cycle $C$ in $G$, then $G$ has a 5-cycle double cover containing $C$ (see [7, Conjecture 10]). This would lend support to Hoffmann-Ostenhof’s Strong 5-Cycle Double Cover Conjecture (see [13]).

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