Point interactions in two- and three-dimensional Riemannian manifolds

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Abstract

We present a non-perturbative renormalization of the bound state problem of $n$ bosons interacting with finitely many Dirac-delta interactions on two- and three-dimensional Riemannian manifolds using the heat kernel. We formulate the problem in terms of a new operator called the principal or characteristic operator $\Phi(E)$. In order to investigate the problem in more detail, we then restrict the problem to one particle sector. The lower bound of the ground state energy is found for a general class of manifolds, e.g. for compact and Cartan–Hadamard manifolds. The estimate of the bound state energies in the tunneling regime is calculated by perturbation theory. Non-degeneracy and uniqueness of the ground state is proven by the Perron–Frobenius theorem. Moreover, the pointwise bounds on the wave function is given and all these results are consistent with the one given in standard quantum mechanics. Renormalization procedure does not lead to any radical change in these cases. Finally, renormalization group equations are derived and the $\beta$ function is exactly calculated. This work is a natural continuation of our previous work based on a novel approach to the renormalization of point interactions, developed by Rajeev.

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1. Introduction

The studies of Dirac-delta interactions in quantum mechanics (which are also called zero range, contact or point interactions, or Fermi pseudopotentials in the literature) date back to the work of Kronig and Penney [1] who introduced the periodic delta interactions describing the non-relativistic electrons moving in a one-dimensional fixed crystal lattice. The historical development of point interactions has been given extensively in the monograph [2]. Although it was Thomas [3] who pointed out that the problem of point interactions in three dimensions
could not be physically acceptable due to the ultraviolet divergences, Thorn [4] realized that we did not have to abandon these interactions and physical results could be obtained after regularization and renormalization, well-known procedures in quantum field theory.

The motivation of studying point interactions in two and three dimensions is based on trying to understand the concept of renormalization in a simple context rather than field theory. There are large amount of works on the renormalization of point interactions in the literature from several point of views [3–11]. Regularization schemes can be performed either in coordinate space [5–8] or in momentum space [4, 9–15]. A single-point interaction in two-dimensional flat space is also an instructive example of dimensional transmutation [4, 9, 16, 17], that is, the original Hamiltonian does not contain any intrinsic energy scale due to the dimensionless coupling constant in natural units, but a new parameter specifying the bound state energy is introduced after the renormalization procedure, which then fixes the energy scale of the system and this is called dimensional transmutation. This implies a violation of \( SO(2, 1) \) symmetry of the scale-invariant potential, so it is one of the simplest examples of anomaly or quantum mechanical symmetry breaking [10]. Furthermore, renormalization group equations of point interactions have been discussed in [5, 15, 18] and the \( \beta \) function has been calculated exactly so that the theory has been found as asymptotically free in two dimensions.

Mathematically rigorous treatments of point interactions are given in the context of self-adjoint extension theory [19] and a detailed exposition of this subject has been discussed in the monograph [2]. The Dirac-delta interaction is considered as a self-adjoint extension of a formally Hermitian-free Hamiltonian on a space with a removed point. The result of this method is identical to that of the renormalization method if a certain relation between the parameter in the extension and the renormalized (or bare) coupling constant is satisfied [10]. Self-adjoint extension method can also be analyzed within the Green’s function method [20, 21].

The many-body version of this problem on \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) is known as the formal non-relativistic limit of the \( \lambda \phi^4 \) scalar field theory in (2+1) and (3+1) dimensions. All these are extensively discussed first in the unpublished thesis of Hoppe [22] and later from a new perspective in [23, 24]. Rajeev [23] introduced a new non-perturbative renormalization method for point interactions which can be applied to several many-body theories: quantum mechanics with point interactions, fermionic and bosonic quantum fields interacting with a point source, many-body problems with point interactions and non-relativistic field theory with polynomial interactions. One of the main advantages of this approach is that all the information about the spectrum of the model is described by an explicit formula instead of imposing the boundary conditions on the operator as in the case of self-adjoint extension theory. This method is also particularly useful for dealing with the bound state problems because of its non-perturbative nature. We are not going to review the ideas developed in there. Instead, we suggest the reader to familiarize themselves with the paper [23] to make the understanding of this paper easier.

Following the original ideas developed in [23], we previously considered the bound state problem for \( N \) point interactions in two- and three-dimensional Riemannian manifolds [25]. The lower bounds on the ground state energy are found for few special manifolds \( S^2, \mathbb{H}^2 \) and \( \mathbb{H}^3 \) [25]. The construction of the relativistic extension of this model in two dimensions is also possible [26]. We also applied the same method based on the heat kernel to the non-relativistic Lee model [27] and its relativistic version has been constructed later on [28]. Our primary motivation comes from the question, how the renormalization method for the point interactions in quantum mechanics should be performed non-perturbatively on Riemannian manifolds, hoping that we can extend our understanding to the realm of quantum field theory. It is worth pointing out that this problem on two-dimensional Riemannian manifolds also
displays a kind of dimensional transmutation, where new energy scales different from the intrinsic energy scales of the system appear after the renormalization [25].

We organize this paper as follows. In section 2, we construct the bound state problem for \( n \) bosons living in two- and three-dimensional Riemannian manifolds interacting with \( N \) external Dirac-delta interactions. This construction is motivated by the work [23] in which the many-body version of this problem is renormalized non-perturbatively in flat spaces. We then restrict the problem to \( n = 1 \), and find the finite formulation of the problem in terms of a new operator, which is called the principal operator (or the characteristic operator) [23] and the result is consistent with the one that we have obtained in [25]. In section 3, we find the wave function for the bound states and show that ground state energy of the \( N + 1 \) center case is smaller than the \( N \) center case using Cauchy interlacing theorem. Then, we calculate the bound state energies in the tunneling regime using a version of perturbation theory. The results for our own purposes on the upper and the lower bounds of the heat kernel in the mathematics literature is given shortly for compact and Cartan–Hadamard manifolds in subsection 4.1 and in 4.2, respectively. Section 5 presents pointwise bounds on the wave function using the upper bounds of the heat kernel given in the previous section and it is shown that the result is consistent with the classical result found in the standard quantum mechanics. Section 6 establishes the lower bound of the ground state energy for a more general class of Riemannian manifolds. The proof of the lower bound for the ground state energy has the same spirit with the one given in our previous work [25] but the proof given here is generalized to a rich class of manifolds, such as compact and Cartan–Hadamard manifolds. Then, in section 7, non-degeneracy and positivity of the ground state is proven with the help of Perron–Frobenius theorem. Finally, we proceed with the study of the renormalization group equations and the \( \beta \) function is calculated exactly. Some important properties and asymptotic expansion of the heat kernel is summarized in appendix A and the proof of the existence of the Hamiltonian in two dimensions is explicitly given in the appendix B.

2. Construction of the model

We consider \( n \) non-relativistic bosons living in a two- or three-dimensional \((D = 2, 3)\) Riemannian manifold and they interact with \( N \) external attractive Dirac-delta potentials. In the second quantized language, the Hamiltonian of the system is

\[
H = \int_{\mathcal{M}} d^D g x \left[ \phi^\dagger_g(x) \left( -\frac{\hbar^2}{2m} \nabla^2_g \right) \phi_g(x) - \sum_{i=1}^{N} \lambda_i \phi^\dagger_g(x) \delta_g(x, a_i) \phi_g(x) \right],
\]

where \( d^D g x = \sqrt{\det g} d^D x \) is the \( D \)-dimensional volume element, \( \nabla^2_g \) is Laplace–Beltrami operator in the local coordinates \( x = (x^1, x^2, \ldots, x^D) \)

\[
\nabla^2_g = \frac{1}{\sqrt{\det g}} \sum_{\alpha,\beta = 1}^{D} \frac{\partial}{\partial x^\alpha} \left( g^{\alpha\beta} \sqrt{\det g} \frac{\partial}{\partial x^\beta} \right),
\]

and \( \phi^\dagger_g(x), \phi_g(x) \) is defined as the bosonic creation–annihilation operators on the Riemannian manifold with the metric structure \( g \). Here, \( a_i \) denotes the location of the \( i \)th external Dirac-delta potential on the manifold and \( \lambda_i \in \mathbb{R}^+ \) is the strength of the delta interaction at \( a_i \), so-called coupling constant.

It is easy to show that the number of bosons \( \int_{\mathcal{M}} d^D x \phi^\dagger_g(x) \phi_g(x) \) is conserved. The lowest eigenvalue of the Hamiltonian \( H \) of our problem in a sector with fixed number of bosons is either zero or negative infinite, so the energy of the ground state is not bounded from
below [25]: $E \to -\infty$. The first step we must do is to regularize the model. The natural regularization of Hamiltonian can be chosen as
\[
H_\epsilon = H_0 - \sum_{i=1}^{N} \lambda_i(\epsilon) \int_{\mathcal{M}} d^D x \, d^D y \, K_{\epsilon/2}(x, a_i; g) K_{\epsilon/2}(y, a_i; g) \phi_\epsilon^\dagger(x) \phi_\epsilon(y),
\] 
where $H_0$ is the free Hamiltonian. $K_\epsilon(x, y; g)$ is the heat kernel defined on the Riemannian manifold $(\mathcal{M}, g)$ and it converges to the Dirac-delta function $\delta_g(x, y)$ as $\epsilon \to 0^+$. In this limit, one can see that we recover the original Hamiltonian we are interested in. We write the heat kernel as $K_\epsilon(x, y; g)$ throughout the paper in order to specify which metric structure it is associated with. Some essential properties of the heat kernel on Riemannian manifolds that we have used in this paper are given in appendix A.

Now we will consider the resolvent of the regularized Hamiltonian in a Fock space formalism with arbitrary number of bosons. Following the same methodology developed for the model in the plane [23], we shall extend the bosonic Fock space $\mathcal{B}$ that we have started with to $\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathcal{B} \otimes \mathbb{C}^N$ by defining new creation and annihilation operators at the locations of the Dirac-delta interactions. These are called *angels*, which are first introduced in [23]. The angel states allow us to rewrite the model in such a way that the coupling constant appears additively rather than multiplicatively. As a result, we can renormalize the model non-perturbatively by simply normal ordering. We assume that the angel operators obey the orthofermionic algebra [29] defined by the following product relations (not with commutators):
\[
\chi_i \chi_j^\dagger + \delta_{ij} \sum_{k=1}^{N} \chi_k \chi_k = \mathbf{1} \delta_{ij}, \quad \chi_i \chi_j = 0 = \chi_i^\dagger \chi_j^\dagger,
\]
where $\mathbf{1}$ is the identity operator and $i, j, k = 1, 2, \ldots, N$. It is more convenient for our purposes to write the angel algebra in terms of projection operators:
\[
\chi_i \chi_j^\dagger = \delta_{ij} \Pi_0, \quad \chi_i \chi_j = 0 = \chi_i^\dagger \chi_j^\dagger,
\]
where
\[
\Pi_1 = \sum_{k=1}^{N} \chi_k \chi_k, \quad \Pi_0 = \mathbf{1} - \Pi_1
\]
are the projection operators onto the 1-angel and no-angel states, respectively. Now we define the augmented regularized Hamiltonian $\tilde{H}_\epsilon$ on $\tilde{\mathcal{B}}$:
\[
\tilde{H}_\epsilon = H_0 \Pi_0 + \left[ \sum_{i=1}^{N} \int_{\mathcal{M}} d^D x \, K_{\epsilon/2}(x, a_i; g) \phi_\epsilon(x) \chi_i^\dagger + \text{h.c.} \right] + \sum_{i=1}^{N} \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \chi_i.
\]
If we split the Hilbert space according to the angel number, the corresponding operator $\tilde{H}_\epsilon - E \Pi_0$ can be written in the following matrix form:
\[
\tilde{H}_\epsilon - E \Pi_0 = \begin{pmatrix}
\begin{array}{c}
a_i \\
\bar{b}_i^\dagger \\
d_e
\end{array}
\end{pmatrix},
\]
with $a : \mathcal{B} \to \mathcal{B}, b_i^\dagger : \mathcal{B} \otimes \mathbb{C}^N \to \mathcal{B}, d_e : \mathcal{B} \otimes \mathbb{C}^N \to \mathcal{B} \otimes \mathbb{C}^N$. Here,
\[
a = H_0 - E, \quad d_e = \sum_{i=1}^{N} \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \chi_i
\]
and
\[
b_i^\dagger = \sum_{i=1}^{N} \int_{\mathcal{M}} d^D x \, K_{\epsilon/2}(x, a_i; g) \phi_\epsilon(x) \chi_i.
\]
Then, one can construct the augmented regularized resolvent
\[
\tilde{R}_\epsilon(E) = \frac{1}{\tilde{H}_\epsilon - E - \Pi_0} = \left( \begin{array}{cc} \alpha_\epsilon & \beta_\epsilon \\ \beta_\epsilon & \delta_\epsilon \end{array} \right).
\] (10)

One can find \(\alpha_\epsilon, \beta_\epsilon, \delta_\epsilon\) in terms of \(a, b, \epsilon, d_\epsilon\) by direct computation. This could be done by apparently different but equivalent ways and the formulas were obtained in the appendix of [23]:
\[
\alpha_\epsilon = \left[ a - b_\epsilon^* d_\epsilon^{-1} b_\epsilon \right]^{-1} = \frac{1}{H_\epsilon - E} = R_\epsilon(E).
\] (11)

This means that \(\tilde{R}_\epsilon(E)\) projected to \(\mathcal{B}\) is just the resolvent of the operator \(H_\epsilon\). We also have another formula for \(\alpha_\epsilon\) [23]:
\[
\alpha_\epsilon = a^{-1} + a^{-1} \left[ d_\epsilon - b_\epsilon a^{-1} b_\epsilon^* \right]^{-1} b_\epsilon a^{-1},
\] (12)
or
\[
R_\epsilon(E) = \frac{1}{H_0 - E} + \frac{1}{H_0 - E} \phi_\epsilon(E)^{-1} b_\epsilon \frac{1}{H_0 - E},
\] (13)

where
\[
\Phi_\epsilon(E) = \sum_{i=1}^N \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \chi_i - \sum_{i,j=1}^N \int_{\mathcal{M}}^d \int_{\mathcal{M}}^d K_\epsilon(x, a_i; g) K_\epsilon(y, a_j; g) \chi_i \chi_j \phi_\epsilon(y) \left( \frac{1}{H_0 - E} \right) \phi_\epsilon(x) \chi_i \chi_j.
\] (14)

The operator \(\Phi_\epsilon(E)\) is called the regularized principal operator (or the regularized characteristic operator [23]). Note that writing the resolvent of \(H_\epsilon\) in this way allows us to write the coupling constant additively. The renormalization procedure can then be done if we can separate the singular part of the operator \(\Phi_\epsilon\). We will see that this is possible by normal ordering of the operators in the principal operator. By using eigenfunction expansions (A.11) and (A.13) of the operators \(\phi_\epsilon(x)\), \(\phi_\epsilon^*(x)\) and that of the heat kernel (or their analogs for non-compact manifolds), one can shift the operator \(\phi_\epsilon^*(x)\) in (14) to the left
\[
\frac{1}{H_0 - E} \phi_\epsilon^*(x) = \int d^d x' \phi_\epsilon^*(x') \int_0^\infty \frac{dt}{\bar{h}} e^{-\bar{h}(H_0 - E)t} K_t(x, x'; g),
\] (15)
and one can also shift the operator \(\phi_\epsilon(x)\) to the right using a similar equation. Then, the normally ordered principal operator can be written by using the properties of the heat kernel and separating the \(i = j\) term from the sum
\[
\Phi_\epsilon(E) = \sum_{i=1}^N \frac{1}{\lambda_i(\epsilon)} \chi_i^\dagger \chi_i - \sum_{i,j=1}^N \int_{\mathcal{M}}^d \int_{\mathcal{M}}^d K_\epsilon(x, a_i; g) K_\epsilon(y, a_j; g) \chi_i \chi_j \phi_\epsilon(y) \left( \frac{1}{H_0 - E} \right) \phi_\epsilon(x) \chi_i \chi_j
\]
\[\quad \times \chi_i^\dagger \chi_j - \sum_{i=1}^N \int_0^\infty \frac{dt}{\bar{h}} K_t(a_i, a_i; g) e^{-\bar{h}(H_0 - E)t} \chi_i \chi_i
\]
\[\quad - \sum_{i=1}^N \int_0^\infty \frac{dt}{\bar{h}} K_t(a_i, a_j; g) e^{-\bar{h}(H_0 - E)t} \chi_i \chi_j.
\] (16)

Due to the singular behavior of the diagonal heat kernel (A.17) near \(t = 0\), we expect that the third term is divergent as \(\epsilon \to 0^+\). Therefore, if we choose the coupling constant
\[
\frac{1}{\lambda_\epsilon(\epsilon)} = \int_0^\infty \frac{dt}{\bar{h}} K_\epsilon(a_i, a_i; g) e^{-\bar{h}t},
\] (17)
where $-\mu_i^2$ corresponds to experimentally measured bound state energy of the individual $i$th Dirac-delta center, we find the principal operator after taking the limit $\epsilon \to 0^+$

$$\Phi(E) = \lim_{\epsilon \to 0^+} \Phi_\epsilon(E) = \sum_{i=1}^{N} \int_0^\infty \frac{dt}{\hbar} K_i(a_i, a_i; g)(e^{-\frac{\bar{\hbar}}{\hbar} t \mu_i^2} - e^{-\frac{\bar{\hbar}}{\hbar} t (H_0 - E)}) \chi_i^\dagger \chi_i$$

$$- \sum_{i,j=1}^{N} \int_0^\infty \frac{dt}{\hbar} \int_{M(x)} d^D x d^D y K_i(a_i, x; g) K_i(a_j, y; g) \phi_j^\dagger(x) e^{-\frac{\bar{\hbar}}{\hbar} t (H_0 - E)} \phi_j(y) \chi_j$$

$$- \sum_{i \neq j}^{N} \int_0^\infty \frac{dt}{\hbar} K_i(a_i, a_j; g) e^{-\frac{\bar{\hbar}}{\hbar} t (H_0 - E)} \chi_i^\dagger \chi_j.$$

This can be written in a more compact way

$$\Phi(E) = \sum_{i,j=1}^{N} \Phi_{ij}(E) \chi_i^\dagger \chi_j,$$

where $\Phi_{ij}(E)$ can be read from (18). Once we have a proper definition of the principal operator, the divergence is completely removed since the spectrum of the problem can be found from the resolvent. We are now in a position to get the full resolvent of our problem in terms of the principal operator

$$R(E) = \lim_{\epsilon \to 0^+} R_\epsilon(E) = \frac{1}{H_0 - E} + \sum_{k=1}^{N} \phi_k^\dagger(a_k) \chi_k \Phi^{-1}(E) \sum_{l=1}^{N} \phi_l(a_l) \chi_l = \frac{1}{H_0 - E} + \sum_{i,j=1}^{N} \phi_j^\dagger(a_i) \Phi_{ij}^{-1}(E) \phi_j(a_j) \frac{1}{H_0 - E}.$$

where we have used $\Phi^{-1}(E) = \sum_{i,j=1}^{N} \Phi_{ij}(E) \chi_i^\dagger \chi_j$ and the algebra of angel operators (5) with the fact that $R(E) : \mathcal{B} \to \mathcal{B}$. We note that the principal operator can be extended to its largest domain of definition in the complex energy plane by analytic continuation.

Since $R(E) : \mathcal{B} \to \mathcal{B}$, we can consider the resolvent kernel between $n$ bosons with no angel states. Up to here, we have generalized the construction of the problem given in [25] to the many-boson cases. From now on, we shall study one-boson problem for simplicity, that is,

$$\int_{M(x)} d^D x \psi(x) \phi_j^\dagger(x) |0\rangle \otimes |\Omega\rangle,$$

where $|\Omega\rangle$ is the vacuum for the angel state and $\psi(x)$ is the wave function for the boson. Then, the resolvent kernel corresponding to this state satisfies the following equation after a straightforward calculation:

$$R(x, y|E) = R_0(x, y|E) + \sum_{i,j=1}^{N} R_0(x, a_i|E) \Phi_{ij}^{-1}(E) R_0(a_j, y|E),$$

where $R_0(x, y|E)$ is the free resolvent kernel which can be written in terms of the heat kernel (A.16). Here $\Phi_{ij}(E)$ can be analytically continued to its largest set in the entire complex plane so $E$ should be considered as a complex variable. Equation (21) gives the relation between the resolvent defined on an infinite-dimensional space and the principal matrix defined on a finite-dimensional space. Such formulas were extensively discussed in problems associated with self-adjoint extensions of operators, notably by Krein and his school, and also for such singular interactions in flat spaces [2, 10, 21]. Therefore, our problem can also be considered as a kind of self-adjoint extension of the free Hamiltonian. We will come back to this point at the end of section 5. The resolvent essentially includes all the information about the spectrum. We will restrict ourselves only to the bound state problem since the scattering problem requires a
deeper analysis. The discrete spectrum of the Hamiltonian is the set of numbers $E$ at which the resolvent does not exist and the continuous spectrum corresponds to the unbounded resolvent. The poles corresponding to bound states must be due to the $\Phi^{-1}(E)$ which can be seen from (21). In other words, the roots of

$$
\Phi(E)|\Psi\rangle = 0
$$

(22)
determine the bound state spectrum of the model. Since $\Phi(E) : B \otimes \mathbb{C}^N \rightarrow B \otimes \mathbb{C}^N$, let us try to consider $|\Psi\rangle$ as a direct product of no boson with one angel state:

$$
|\Psi\rangle = |0\rangle \otimes \sum_{k=1}^{N} A_k |e_k\rangle,
$$

(23)

where $|e_k\rangle \equiv \chi^{|\Omega}\rangle$ is a set of complete orthonormal basis for $\mathbb{C}^N$. Then, equation (22) yields

$$
\sum_{i=1}^{N} \int_{0}^{\infty} \frac{dt}{\hbar} K_t(a_i, a_i; g)\left(e^{-i\mu_i t} - e^{+E_i t}\right) A_i |e_i\rangle - \sum_{i \neq j}^{N} \int_{0}^{\infty} \frac{dt}{\hbar} K_t(a_i, a_j; g) e^{+E_j} A_j |e_j\rangle = 0,
$$

(24)

and the result can be written as a matrix equation

$$
\sum_{j=1}^{N} \Phi_{ij}(E) A_j = 0,
$$

(25)

where

$$
\Phi_{ij}(E) = \begin{cases} 
\int_{0}^{\infty} \frac{dt}{\hbar} K_t(a_i, a_i; g)\left(e^{-i\mu_i t} - e^{+E_i t}\right) & \text{if } i = j \\
- \int_{0}^{\infty} \frac{dt}{\hbar} K_t(a_i, a_j; g) e^{+E_j} & \text{if } i \neq j.
\end{cases}
$$

(26)

Thanks to the symmetry property of the heat kernel (A.9), the principal matrix is Hermitian for real values of energy and the explicit form of it in terms of the heat kernel has been first obtained by a different method in our previous work [25]. Many important aspects of the model can be understood by working out the principal matrix as we will see in the following sections.

It is well known that the same problem with a single delta potential in flat spaces is a good example of a dimensional transmutation in quantum mechanics. Our problem in $D = 2$ realizes a generalized dimensional transmutation [17]. In our case, the coupling constants $\lambda_i$ have the same dimension as $\hbar^2$ by the dimensional analysis. In contrast to the flat case, we also have intrinsic scales coming from the geometry of the space, such as curvature and the geodesic distance between centers $d_{ij}$. However, after the renormalization procedure, we obtain a set of new-dimensional parameters $\mu_i^2$ from relation (17). Hence, the energy is not determined by naive-dimensional analysis. However, in the case of a single delta attractor for the flat case there is no combination of dimensional parameters to come up with an energy scale, whereas in the case of a manifold we have geometric length scales which already may define an energy scale. The dimensional transmutation is most striking in cases where there is no intrinsic energy scale.

3. Interlacing theorem and perturbation theory

A mathematically satisfactory calculation of the wave function should proceed from the resolvent equation. Since the eigenvalues are isolated we can find the projection operator to
the subspace corresponding to this eigenvalue by a contour integral [30]:
\[ \langle x | P_k | y \rangle = \psi_k(x)\psi^*_k(y) = -\frac{1}{2\pi i} \oint_{\Gamma_k} dz \, R(x, y | z), \] (27)
where \( \Gamma_k \) is a small contour enclosing the isolated eigenvalue \(-\nu_k^2\). We note that the free Green’s functions \( R_0(x, y | z) \) will not contain any poles on the negative real axis, so all the poles on the negative real axis will come from the poles of the inverse principal matrix \( \Phi^{-1}_1(z) \). For simplicity we can assume that the eigenvalues are non-degenerate and let us denote the \( k \)th eigenvalue of the principal matrix as \( \omega_k \) so the eigenvalue problem is
\[ \sum_{j=1}^N \Phi_{ij}(-\nu_k^2) A^k_j(-\nu_k^2) = \omega_k(-\nu_k^2) A^k_i(-\nu_k^2). \] (28)
Since the principal matrix is Hermitian on the real line and
\[ \Phi^*_1(z) = \Phi_1(z^*), \] (29)
on the complex plane, there exists a holomorphic family of projection operators on the complex plane [31], so that we can apply the spectral theorem for the principal matrix \( \Phi_1(z) \):
\[ \Phi_{ij}(z) = \sum_k \omega_k(z) P_k(z)_{ij}, \] (30)
where \( P_k(z)_{ij} = A^k_i(z)A^k_j(z) \) and \( A^k_i(z) \) is the normalized eigenvector corresponding to the eigenvalue \( \omega_k(z) \). Similarly, we can write the spectral resolution of the inverse principal matrix:
\[ \Phi^{-1}_{ij}(z) = \sum_k \frac{1}{\omega_k(z)} P_k(z)_{ij}. \] (31)
The residue can then be found:
\[ \lim_{z \to -\nu_k^2} R_0(x, a_i | z + \nu_k^2) \Phi^{-1}_{ij}(z) R_0(a_j, y | z) \]
\[ = R_0(x, a_i | -\nu_k^2) \left[ \frac{\partial \omega_k(z)}{\partial z} \right]_{z = -\nu_k^2}^{-1} P_k(-\nu_k^2)_{ij} R_0(a_j, y | -\nu_k^2). \] (32)
Now we will look at the variations of the eigenvalues of \( \Phi \) as we change the parameters \( \nu \).

Using
\[ \omega_k(-\nu^2) = (A^k(-\nu^2), \Phi(-\nu^2)A^k(-\nu^2)), \] (33)
and as a consequence of the Feynman–Hellman theorem [32] in the non-degenerate case, we have
\[ \frac{\partial \omega_k(-\nu^2)}{\partial \nu} = \left( A^k(-\nu^2), \frac{\partial \Phi(-\nu^2)}{\partial \nu} A^k(-\nu^2) \right) \]
\[ = \sum_{i,j=1}^N A^k_{ij}(-\nu^2) \frac{\partial \Phi_{ij}(-\nu^2)}{\partial \nu} A^k_{ji}(-\nu^2). \] (34)
Taking the derivative of the principal matrix with respect to \( \nu \) from (26)
\[ \frac{\partial \Phi_{ij}(-\nu^2)}{\partial \nu} \bigg|_{\nu = \nu_k} = \int_0^\infty \frac{dt}{h} \left( \frac{2 v_k t}{h} \right) K_t(a_i, a_j; g) e^{-\frac{2 \nu_k t}{h}}, \] (35)
and inserting equation (35) into equation (34), we obtain
\[
\frac{\partial \omega^k(z)}{\partial z} = -\frac{1}{2v} \frac{\partial \omega^k(-v^2)}{\partial v} = -\frac{1}{2v} \sum_{i,j=1}^{N} A^*_i(-v^2) \int_{0}^{\infty} \frac{dt}{h} \left( \frac{2vt}{h} \right) K_i(a_i, a_j; g) e^{-\frac{v^2}{2t}} A_j^*(-v^2).
\] (36)

If we evaluate (36) at \( z = -v_k^2 \), or at \( v = v_k \), it yields
\[
\frac{\partial \omega^k(z)}{\partial z} \bigg|_{z=-v_k^2} = -\sum_{i,j=1}^{N} A^*_i(-v_k^2) \int_{0}^{\infty} \frac{dt}{h} \left( \frac{t}{h} \right) K_i(a_i, a_j; g) e^{-\frac{v_k^2}{2t}} A_j^*(-v_k^2).
\] (37)

Note that the integral is finite in two and three dimensions due to upper bounds on the heat kernel. If we combine all these results, we get
\[
\psi_k(x)\psi^*_k(y) = -\frac{1}{2\pi} (2\pi i) R_0(x, a_i; -v_k^2) \left[ -\sum_{i,j=1}^{N} A^*_i(-v_k^2) \int_{0}^{\infty} \frac{dt}{h} \left( \frac{t}{h} \right) K_i(a_i, a_j; g) e^{-\frac{v_k^2}{2t}} A_j^*(-v_k^2) \right]^{1/2} A_i(-v_k^2)^* A_j(-v_k^2) R_0(a_j, y; -v_k^2).
\] (38)

Then, we can directly read off the bound state wave function from the equation above:
\[
\psi_k(x) = \left[ \sum_{i,j=1}^{N} A^*_i(-v_k^2) \int_{0}^{\infty} \frac{dt}{h} \left( \frac{t}{h} \right) K_i(a_i, a_j; g) e^{-\frac{v_k^2}{2t}} A_j^*(-v_k^2) \right]^{-1/2} \times \int_{0}^{\infty} \frac{dt}{h} e^{-\frac{v_k^2}{2t}} \sum_{i=1}^{N} A_i(-v_k^2) K_i(a_i, x; g),
\] (39)

from which one can easily determine that \( \psi_k(x) \) is finite except at \( x = a_i \). This is exactly the same result obtained by a different method in [25]. Note that we have written \( A^*_i(-v_k^2) \) as \( A_i(-v_k^2) \) for simplicity. Incidentally equation (37) implies an interesting result for the variation of eigenvalues
\[
\frac{\partial \omega^k(-v^2)}{\partial v} \bigg|_{v=v_k} = \sum_{i,j=1}^{N} A^*_i(-v_k^2) \int_{0}^{\infty} \frac{dt}{h} \left( \frac{2vt}{h} \right) K_i(a_i, a_j; g) e^{-\frac{v_k^2}{2t}} A_j^*(-v_k^2) = \int_{0}^{\infty} \frac{dt}{h} \left( \frac{2vt}{h} \right) e^{-\frac{v_k^2}{2t}} \int_{M} \sum_{i=1}^{N} K_i(a_i; g) A_i(-v_k^2)^2.
\] (40)

where we have used the fact that \( v_k \in \mathbb{R}^+ \) for all \( k \) and properties of the heat kernel (A.8) and (A.9) and the order of the integration and the finite sum can be interchanged. We can easily determine that the above equation (40) is strictly positive due to the positivity of heat kernel, so
\[
\frac{\partial \omega^k(-v^2)}{\partial v} \bigg|_{v=v_k} > 0.
\] (41)

Energy eigenvalues \( E = -v^2 \) are obtained from the zeros of the eigenvalues of the principal matrix, that is, \( \omega^k(-v_k^2) = 0 \), and there is a unique solution for each \( \omega^k(-v^2) \). We also know
that for sufficiently small values of \( v \), the matrix \( \Phi(-v^2) \) becomes negative; hence, no zeros exist beyond some critical point.

With this result in mind, let us see what can be said about the comparison of the energy eigenvalues for the different number of delta centers. In order to see this, we need the Cauchy interlacing theorem in mathematics literature [33], which states that if we delete the last row and column of a Hermitian \((N + 1) \times (N + 1)\) matrix, the eigenvalues of the original matrix is interlaced by the eigenvalues of the new matrix, i.e. if \( \omega^2((-v^2)) \leq \omega^2((-v^2)) \leq \cdots \leq \omega^2((-v^2)) \) lists the eigenvalues of the original \((N + 1) \times (N + 1)\) matrix and if \( \tilde{\omega}^2((-v^2)) \leq \tilde{\omega}^2((-v^2)) \leq \cdots \leq \tilde{\omega}^2((-v^2)) \) lists the eigenvalues of the reduced matrix (any \( N \times N \) principal submatrix of the \((N + 1) \times (N + 1)\) matrix), then we have

\[
\omega^2((-v^2)) \leq \tilde{\omega}^2((-v^2)) \leq \omega^2((-v^2)) \leq \cdots \leq \tilde{\omega}^2((-v^2)) \leq \omega^2((-v^2)) \tag{42}
\]

We assume that the \((N + 1) \times (N + 1)\) matrix in the above-mentioned theorem is the principal matrix \( \Phi^{N+1}(-v^2) \) corresponding to a certain arrangement of \( N + 1 \) delta potentials. If we now delete the last row and the column it means that we remove the \((N + 1)\)st delta center from the system. The bound state problem of \( N \) centers corresponds to zero eigenvalue of the principal matrix \( \Phi^N \), and let us denote that bound state energy as \( \tilde{E}_k \):

\[
\tilde{\omega}^k(-\tilde{v}_k^2) = 0, \quad \tilde{E}_k = -\tilde{v}_k^2, \tag{43}
\]

if it exists. By the Cauchy interlacing result, we then expect the following inequality:

\[
\cdots < \omega^2(-\tilde{v}_k^2) < \tilde{\omega}^2(-\tilde{v}_k^2) = 0 < \omega^2(-\tilde{v}_k^2) < \cdots \tag{44}
\]

From the positivity of the derivative of the eigenvalues with respect to argument \((41)\), the eigenvalues \( \omega \) are monotonically increasing functions. Hence, in order to get a zero root of \( \omega^2(-\tilde{v}_k^2) \), we should increase \( \tilde{v}_k^2 \) to a higher value \( \tilde{v}_k^* \). As a result, \( \omega^2(-\tilde{v}_k^2) = 0 \) if \( \tilde{v}_k^2 < \tilde{v}_k^* \), i.e. \( E_k = -\tilde{v}_k^2 < \tilde{E}_k = -\tilde{v}_k^* \). Thus, the energies also interlace in the same manner—this is a nonlinear analog of Sturm’s comparison theorem of the eigenvalues. Moreover, \( E^{N+1}_{gr} < E^N_{gr} < E^N_{gr} = -\mu^2_k \), that is, the ground state is always negative and approaches the bound state energy for the 1-delta center as \( N \) gets smaller. We can also generalize these results to the degenerate cases but the proof is more cumbersome.

It is worth noting that we do not have to solve the energy eigenvalues for the bound state while performing our non-perturbative renormalization method. In other words, although our renormalization method makes the problem well defined and finite, the energy eigenvalues must be found after this finite formulation has been constructed. If we cannot solve the problem exactly after the renormalization, we must apply the standard approximation methods, such as perturbation theory and variational techniques. An interesting estimate for our problem can be given by perturbation theory. For simplicity, we assume that all binding energies \( \mu_k^2 \)'s are different and the magnitude of the minimum binding energy of the \( k \)th singular potential is much larger than the correlation energy between the \( k \)th fixed center and the \( l \)th center, that is to say, we assume \( \frac{\hbar^2}{2m(\cos \alpha_{k,l})} \ll \mu_{k,min}^2 \) (on a non-compact manifold we may assume that the geodesic distance between the centers is large). This assumption makes the off-diagonal elements of the principal matrix much smaller than its diagonal elements. For this reason, let us separate the principal matrix for \( E_k = -\tilde{v}_k^2 \) as the sum of a diagonal matrix and an off-diagonal matrix, which is very small compared to the diagonal part:

\[
\Phi(-\tilde{v}_k^2) = \Phi_D(-\tilde{v}_k^2) + \delta \Phi(-\tilde{v}_k^2). \tag{45}
\]

Since \( \Phi(-\tilde{v}_k^2) \) is Hermitian, we can apply standard perturbation techniques to our problem. The eigenvalue problem for the principal matrix we wish to solve is given in \((28)\). We
again suppose that there is no degeneracy for simplicity. The energy eigenvalue changes to
\[ E_k = E_k^{(0)} + \delta E_k \] or
\[ \nu_k = \nu_k^{(0)} + \delta \nu_k. \] (46)
From the fundamental idea of the perturbation theory in finite-dimensional spaces, we have the following expansions for the eigenvalues and eigenvectors:
\[ \omega_k = \omega_k^{(0)} + \omega_k^{(1)} + \omega_k^{(2)} + \cdots, \]
\[ A_k = A_k^{(0)} + A_k^{(1)} + A_k^{(2)} + \cdots, \] (47)
and the solution to the related unperturbed eigenvalue problem
\[ \sum_{j=1}^{N} \left[ \Phi_D(-v_k^2) \right]_{ij} A_j^{(0)}(-v_k^2) = \omega^{(0)}(-v_k^2) A_j^{(0)}(-v_k^2) \] (48)
is given by
\[ \omega^{(0)}(-v_k^2) = \int_0^\infty \frac{dt}{\hbar} K_t(a_k, a_k; g) \left[ e^{-t \nu_k^2/\hbar} - e^{-t \nu_k^2/\hbar} \right]. \] (49)
Then, the energy eigenvalues can easily be found from the condition \( \omega^{(0)}(-v_k^2) = 0 \):
\[ E_k^{(0)} = -\mu_k^2 \quad \text{or} \quad \nu_k^{(0)} = \mu_k, \] (50)
and the eigenvectors are
\[ A_k^{(0)}(\mu_k) \equiv A_k^{(0)} \equiv e_k \equiv \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \] (51)
where 1 is located in the kth position of the column and other elements of it are zero or we can write \( A_k^{(0)} = e_k = \delta_{kj} \). Here \( e_k \)'s form a complete orthonormal set of basis:
\[ \sum_{i=1}^{N} e_i^k e_j^k = \delta_{ij}. \] (52)
We must emphasize that there is no distinction between upper and lower indices for our purposes. Once we have found the solution of the diagonal part of the principal matrix or unperturbed eigenvalue problem, we can perturbatively solve the whole problem. The standard perturbation theory gives us the first- and second-order eigenvalues:
\[ \omega^{(1)}(-v_k^2) = \sum_{i,j=1}^{N} e_i^k \left[ \delta \Phi(-v_k^2) \right]_{ij} e_j^k = \left[ \delta \Phi(-v_k^2) \right]_{kk} = 0, \]
\[ \omega^{(2)}(-v_k^2) = \sum_{i,j=1}^{N} \left[ \sum_{l=1, l \neq k}^{N} e_i^l \left[ \delta \Phi(-v_k^2) \right]_{lj} e_j^l \right]^2 = \sum_{i,j=1}^{N} \frac{\Phi_i(-v_k^2) \Phi_j(-v_k^2)}{\omega^{(0)}(-v_k^2) - \omega^{(0)}(-v_k^2)} \] (53)
respectively. Hence the energy eigenvalues of the whole problem can be determined from
\[ \omega^k(-\mu_k^2 + \delta E_k) = \omega^{(0)}(-\mu_k^2 + \delta E_k) + \omega^{(1)}(-\mu_k^2 + \delta E_k) + \cdots = 0. \] (54)
and \( \omega^{k(0)}(-v_k^2) \) and \( \Phi_{iL}(-v_k^2) \) for \( k \neq l \) can be expanded around \( v_k = \mu_k \):

\[
\omega^{k(0)}(-\mu_k^2 + \delta E_k) = \left. \frac{\partial \omega^{k(0)}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \delta v_k + O(\delta^2 v_k),
\]

\[
\Phi_{iL}(-\mu_k^2 + \delta E_k) = \Phi_{iL}(-\mu_k^2) + \left. \frac{\partial \Phi_{iL}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \delta v_k + O(\delta^2 v_k),
\]

\[(55)\]

where we have used the fact \( \omega^{k(0)}(-\mu_k^2) = 0 \). If we substitute (55) into (54) and (53), and use the Feynman–Hellman theorem (34), we obtain

\[
0 = \left. \frac{\partial \Phi_{kk}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \delta v_k - \sum_{l = 1}^{N} \frac{1}{\Phi_{ll}(-\mu_k^2)} \Phi_{lL}(-\mu_k^2) \Phi_{lk}(-\mu_k^2) + \left( \Phi_{iL}(-\mu_k^2) \frac{\partial \Phi_{il}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} - \Phi_{il}(-\mu_k^2) \frac{\partial \Phi_{il}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \delta v_k \right) - \sum_{l = 1}^{N} \Phi_{il}(-\mu_k^2) \Phi_{lk}(-\mu_k^2) \Phi_{kl}(-\mu_k^2)
\]

\[
\times \left[ 1 + \frac{1}{\Phi_{il}(-\mu_k^2)} \right] - \frac{\partial \Phi_{il}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \delta v_k + O(\delta^2 v_k). \tag{56}\]

If we also expand the last factor in the powers of \( \delta v_k \) and ignore the second-order terms and combine the terms using the symmetry property of the principal matrix, we find

\[
\left[ \left. \frac{\partial \Phi_{iL}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \right] + \sum_{l = 1}^{N} \Phi_{iL}(-\mu_k^2) \Phi_{lk}(-\mu_k^2) \left. \frac{\partial \Phi_{iL}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \delta v_k
\]

\[
- 2 \sum_{l = 1}^{N} \Phi_{iL}(-\mu_k^2) \Phi_{lL}(-\mu_k^2) \left. \frac{\partial \Phi_{lL}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \delta v_k
\]

\[
= \sum_{l = 1}^{N} \Phi_{iL}(-\mu_k^2) \Phi_{lk}(-\mu_k^2) + O(\delta^2 v_k). \tag{57}\]

Ignoring the second and third terms on the left-hand side of the inequality due to the fact that \( \Phi_{kk}(-v_k^2) > |\Phi_{iL}(-v_k^2)| \), we get the change in \( v_k \):

\[
\delta v_k \approx \left( \left. \frac{\partial \Phi_{kk}(-v_k^2)}{\partial v_k} \right|_{v_k = \mu_k} \right)^{-1} \sum_{l = 1}^{N} \frac{\Phi_{iL}(-\mu_k^2) \Phi_{lk}(-\mu_k^2)}{\Phi_{iL}(-\mu_k^2)} + O(\delta^2 v_k), \tag{58}\]

so the change in the energy is \( \delta E_k \approx -2\mu_k \delta v_k + O(\delta^2 v_k) \). Let us now consider how the bound state energy changes in the tunneling regime for our problem in which \( \frac{\hbar^2}{2md^2} \ll \mu_k \). We must first calculate the asymptotic behavior of the off-diagonal element of the principal matrix in this regime. In order to see this, we make the scaling transformation \( t = \frac{u}{B} \), where \( B = \frac{h}{2md^2} \) and use the scaling property of the heat kernel (A.15). In two dimensions, we have

\[
\Phi_{iL}(-\mu_k^2) = -\int_0^\infty \frac{dt}{h} K_\mu(a_i, a_j; Bg) e^{-\frac{\mu_k^2}{g}}. \tag{59}\]

In the tunneling regime, the most significant contribution to the integral comes from the region \( u = 0 \) due to the fact that the integrand is suppressed by the exponential term for large values.
of \( u \). Hence we can use the short-time asymptotic of the heat kernel given in (A.20). The result is an integral representation of the modified Bessel function of the third kind [34]:

\[
\Phi_{ij}(-\mu_k^2) \sim -\frac{2d_{ij}^{1/2}}{4\pi\hbar^2/2m} K_0\left(\frac{\sqrt{2md_{ij}^2\mu_k^2}}{\hbar^2}\right) \sum_l \psi_l^{-1/2}(x, y),
\]

where we have used \( d_{ij} \to B_{1/2}d_{ij} \) and \( \psi_l \to \psi_l/B_{1/4} \) for two dimensions under the scaling transformation \( g \to Bg \). The asymptotic expansion of \( K_0(x) \) for large values of \( x \)

\[
K_0(x) \sim \sqrt{\pi} \frac{e^{-x}}{2x}
\]

leads to

\[
\Phi_{ij}(-\mu_k^2) \sim -\sqrt{\pi} \sum_l \frac{\psi_l^{-1/2}(x, y)}{\hbar^2 / 2m \mu_k^2} \frac{1}{2m \mu_k^2} e^{-\frac{\sqrt{2md_{ij}^2\mu_k^2}}{\hbar}}.
\]

Here, large values of \( x \) corresponds to the tunneling regime in our problem. For the three-dimensional case, the idea is the same and the result would be

\[
\Phi_{ij}(-\mu_k^2) \sim -\sqrt{\pi} \frac{\psi_l^{-1/2}(x, y)}{m \hbar} \frac{1}{(4\pi \hbar/2m)^{3/2}} e^{-\frac{\sqrt{2md_{ij}^2\mu_k^2}}{\hbar}}.
\]

Therefore, in the tunneling regime, we can find the change in the bound state energy \( \delta E_k \) in the presence of other delta interactions by substituting (62) and (63) into (58). In agreement with the naive expectation in the standard quantum mechanics, we show that the bound state energy in the tunneling regime gets exponentially smaller with increasing distance between the centers.

Before we prove the lower bound for the ground state energy and find pointwise bounds on the wavefunctions, the results in mathematics literature for the upper and lower bounds of the heat kernel for compact and Cartan–Hadamard manifolds will be briefly given in the next section.

4. Upper- and lower-bound estimates of the heat kernel

4.1. Compact manifolds

(1) Upper bound of the diagonal heat kernel. The following result is a simplified version of the corollary 3.6 given in [35], which assumes that some geometrical conditions must hold on the boundary. The global upper-bound estimate of the diagonal heat kernel in [35] includes whole-boundary information via an explicitly calculable strictly positive constant \( A \equiv A(d, H, R, K, V(M)) \), where \( d \) is the diameter of the manifold, and \( K \) is the lower bound on the Ricci curvature, and also \( H \) and \( R \) are parameters related to boundary conditions. We then state the following corollary by safely removing the boundary effects in corollary 3.6 in [35] since \( A \) is strictly positive.

Let \( M \) be a compact manifold. Suppose that the Ricci curvature of \( M \) satisfies \( \text{Ric}_M \geq -K, K \geq 0 \). Then \( \forall t > 0 \) and \( x \in M \):

\[
K_t(x, x; g) \leq \frac{1}{V(M)} + A'(ht/2m)^{-D/2},
\]

where \( A' \equiv A'(d, K, V(M)) \).
(2) Upper bound of the off-diagonal heat kernel. The following corollary given in [36] constrains the off-diagonal elements of the heat kernel from above.

Assume that for some points \( x, y \in \mathcal{M} \) (\( \mathcal{M} \) is any Riemannian manifold) and \( \forall t > 0 \):

\[
K_t(x, x; g) \leq \frac{C}{f(t)} \quad \text{and} \quad K_t(y, y; g) \leq \frac{C}{g(t)},
\]  

where \( f \) and \( g \) are increasing positive functions on \( (0, \infty) \) satisfying the regularity condition given below. Then, for any \( C_2 > 2 \) and for all \( t > 0 \),

\[
K_t(x, y; g) \leq 4A\sqrt{f(\varepsilon t)g(\varepsilon t)} \exp\left(-\frac{md^2(x, y)}{\bar{h}C_2 t}\right),
\]  

where \( \varepsilon = \varepsilon(C_2, a) \), and \( A \) and \( a \) are the constants from the regularity condition below.

Regularity condition. There are numbers \( A \geq 1 \) and \( a > 1 \) such that

\[
\frac{f(as)}{f(s)} \leq Af(at) \quad \text{for all } 0 < s < t.
\]  

By comparing equations (64) and (65), we realized that the right-hand side of (64) can be an explicit candidate for the functions \( f \) or \( g \) in the theorem above. Hence, we could have

\[
f(t) = g(t) = \left[\frac{1}{V(\mathcal{M})} + A'(\bar{h}t/2m)^{-D/2}\right]^{-1}
\]  

by choosing \( C = 1 \). It is easy to check that these functions are positive and increasing. We can also verify that they satisfy the regularity condition (67) with \( A = a^{D/2} \). Therefore, we have obtained the upper bound for the off-diagonal elements of the heat kernel

\[
K_t(x, y; g) \leq 4A \left[\frac{1}{V(\mathcal{M})} + B(\varepsilon)(\bar{h}t/2m)^{-D/2}\right] \exp\left(-\frac{md^2(x, y)}{\bar{h}C_2 t}\right),
\]  

where \( B(\varepsilon) = A'e^{-D/2} \).

(3) The lower bound of the heat kernel.

We have a direct theorem about the lower bound on the heat kernel (cf theorem 5.6.1 in [37]). We will just give the statement of the theorem. Let \( \mathcal{M} \) be a complete Riemannian manifold with \( \text{Ric}_\mathcal{M} \geq 0 \). Then, we have

\[
K_t(x, y; g) \geq (4\pi \bar{h}t/2m)^{-D/2} \exp\left(-\frac{md^2(x, y)}{2\bar{h}t}\right)
\]  

for all \( x, y \in \mathcal{M} \) and \( t > 0 \). In particular, we find the lower bound to be

\[
K_t(x, x; g) \geq (4\pi \bar{h}t/2m)^{-D/2}.
\]

4.2. Cartan–Hadamard manifolds

A manifold \( \mathcal{M} \) is called a Cartan–Hadamard manifold [38] if \( \mathcal{M} \) is a geodesically complete, simply connected, non-compact Riemannian manifold with a non-positive sectional curvature everywhere. The \( D \)-dimensional flat \( \mathbb{R}^D \) and hyperbolic spaces \( \mathbb{H}^D \) are the best-known examples of Cartan–Hadamard manifolds.
(1) The upper bound of the heat kernel. In order to give an upper bound for Cartan–Hadamard manifolds, we need to give some definitions and related theorems in the literature.

Isoperimetric inequalities. Isoperimetric inequalities are the relations between the boundary area of regions and their volume. We say that manifold $M$ admits the isoperimetric function $I$ if for any precompact open set $\Omega \subset M$ with smooth boundary
\[ A(\partial \Omega) \geq I(v), \quad (72) \]
where $A(\partial \Omega)$ is the area of boundary of the region $\Omega$, and $v = V(\Omega)$ is the volume of the region. Any Cartan–Hadamard manifold $M$ of dimension $D$ admits the isoperimetric function $I(v) = \kappa v^{\frac{D-1}{D}}$, $\kappa > 0$ [39]. We have an important theorem [40] given below.

Assume that manifold $M$ admits a non-negative continuous isoperimetric function $I(v)$ such that $I(v)/v$ is non-increasing. Let us define the function $f(t)$ by
\[ t = 4 \int_0^{f(t)} \frac{dv}{v I^2(v)}, \quad (73) \]
assuming that the integral does not diverge at $t = 0$. Then for all $x \in M$, $t > 0$ and $\varepsilon > 0$,
\[ K_t(x, x; g) \leq 2\varepsilon^{-1} f((1 - \varepsilon)t). \quad (74) \]
Moreover, if the function $f$ satisfies in addition the regularity condition (67), then for all $x, y \in M$, $t > 0$, $C_2 > 2$ and some $\varepsilon > 0$,
\[ K_t(x, y; g) \leq \frac{4A}{f(\varepsilon t)} \exp \left( \frac{-md^2(x, y)}{\hbar C_2 t} \right). \quad (75) \]
The isoperimetric function for Cartan–Hadamard manifolds given above satisfies all the requirements above, that is, it is a non-negative continuous function and $I(v)/v$ is non-increasing. Substituting the above isoperimetric function for Cartan–Hadamard manifolds into (73), we obtain the function $f(t)$ by a simple integration
\[ f(t) = \left( \frac{\kappa^2}{2D} \right)^{D/2}, \quad (76) \]
and it meets all the requirements of this theorem including the regularity condition. Hence, the upper bound on the off-diagonal elements of the heat kernel on Cartan–Hadamard manifolds is given by
\[ K_t(x, y; g) \leq \frac{C(\varepsilon, \kappa)}{(4\pi \hbar t / 2m)^{D/2}} \exp \left( \frac{-md^2(x, y)}{\hbar C_2 t} \right), \quad (77) \]
where $C(\varepsilon, \kappa) = \frac{4A}{(\varepsilon^2/2D)^{D/2}}$, and the physical parameters $\hbar, m$ are introduced for dimensional reasons. This upper-bound estimate of the heat kernel is also valid for minimal submanifolds, which are submanifolds of $\mathbb{R}^D$ whose normal mean curvature vector $H(x) = (H_1(x), H_2(x), \ldots, H_D(x))$ vanishes for all $x \in M$, because minimal submanifolds admit the same form of the isoperimetric function [41] with the Cartan–Hadamard manifolds.

(2) The lower bound of the diagonal heat kernel. On Cartan–Hadamard manifolds, the lower bounds of the diagonal elements of the heat kernel are obtained in [38]. Assume that the sectional curvature inside some ball $B(x, r)$ is bounded below by $-K_{\text{max}}^2(r)$, where $x \in M$. Then, for all $t > 0$ and $\delta > 0$,
\[ K_t(x, x; g) \geq \frac{c}{(4\pi \hbar t / 2m)^{D/2}} \exp \left[ \frac{-(\sigma_1(M) + \delta) \hbar t}{2m} \right], \quad (78) \]
where \( c \equiv c(x, \delta) > 0 \). The minimum eigenvalue of the Laplacian \( \sigma_1(M) \) is restricted to the following range \([42, 43]\):

\[
\frac{1}{2} (D - 1)^2 K_{\text{max}}^2 \geq \sigma_1(M) \geq \frac{1}{2} (D - 1)^2 K_{\text{min}}^2
\]

for a Cartan–Hadamard manifold whose sectional curvature is bounded from above by \(-K^2_{\text{min}}\).

5. Pointwise bounds on the wavefunction

There is extensive amount of literature on the exponential decays of the wave functions of the Schrödinger operators, which states that \( L^2 \) solutions of \((-\nabla^2 + V) \psi = E \psi\) obey pointwise bounds of the form

\[
|\psi(r)| \leq C_a e^{-ar},
\]

if the potential energy \( V \) is continuous and bounded below and \( E \) is in the discrete spectrum of \(-\nabla^2 + V\) (see [30] for the review of the subject). The proofs given in the literature do not include the potentials which require renormalization and they are valid only for \( \mathbb{R}^D \). We shall prove that it is still possible to get exponential pointwise bounds for our problem.

It is easy to see the upper bound of wave function (39) by applying the Cauchy–Schwartz inequality

\[
|\psi_k(x)| \leq \alpha \left| \sum_{i=1}^N \right| A_i \left( -v_i^2 \right) \int_0^\infty \frac{dt}{\hbar} e^{-\frac{v_i^2}{\hbar} t} K_i(a_i, x; g) \right|
\]

\[
\leq \alpha \left[ \sum_{i=1}^N \left| \int_0^\infty \frac{dt}{\hbar} e^{-\frac{v_i^2}{\hbar} t} K_i(a_i, x; g) \right|^2 \right]^{1/2}
\]

\[
\leq \alpha \sum_{i=1}^N \int_0^\infty \frac{dt}{\hbar} e^{-\frac{v_i^2}{\hbar} t} K_i(a_i, x; g),
\]

where we call the coefficient (which is independent of the coordinates \( x \)) in front of the integral in (39) as \( \alpha \) and we have used the fact that \( \sum_{i=1}^N |A_i( -v_i^2)|^2 = 1 \). Thanks to the upper bound on the heat kernel given in (69) and (77), we show that the wave function is pointwise bounded on \( M \). For compact manifolds, the upper bound (69) of the heat kernel gives

\[
|\psi_k(x)| \leq 8\alpha A \sum_{i=1}^N \left[ \frac{1}{V(M)} \sqrt{md^2(a_i, x)} K_1 \left( 2 \sqrt{md^2(a_i, x)v_i^2} \right) \right]
\]

\[
+ \frac{B(\epsilon)}{\hbar (\hbar/2m)^{D/2}} \left( \frac{md^2(a_i, x)}{v_i^2 C_2} \right)^{1/2} K_{\frac{1}{2}-1} \left( 2 \sqrt{md^2(a_i, x)v_i^2} \right)
\]

where we use the following integral representation of the modified Bessel functions of the third kind (or sometimes called Macdonald’s functions) [34]:

\[
K_v(z) = \frac{1}{2} \left( \frac{z}{2} \right)^v \int_0^\infty ds e^{-s-(z/4s)} s^{-v-1} |\arg z| < \frac{\pi}{4}, \quad \text{Re}(v) > -\frac{1}{2}.
\]

For \( D = 2 \) and \( D = 3 \), we can also find the upper bounds on Bessel functions \( K_0 \) and \( K_1 \) with another useful integral representation [34]:

\[
K_v(z) = \frac{\sqrt{\pi} e^v}{2v \Gamma(v + 1/2)} \int_0^\infty ds e^{-z \cos s} \sinh s^v, \quad \text{Re}(z) > 0, \quad \text{Re}(v) > -\frac{1}{2}.
\]
Using the inequality \( \cosh s = \frac{e^s + e^{-s}}{2} > \frac{e^s}{2} \) for all \( s \geq 0 \) in (84), we have an upper bound \( K_0(x) \) for \( x \in \mathbb{R}^+ \):

\[
K_0(x) < \int_0^\infty ds \, e^{-x^2}. \tag{85}
\]

By subsequent change of variables \( \xi = e^s \) and \( \eta = \xi - 1 \), we get

\[
K_0(x) < \int_0^\infty d\eta \, \frac{e^{-\eta(1+\eta)}}{\eta + 1}. \tag{86}
\]

If we also define a new variable \( z = x\eta \), we have

\[
K_0(x) < \int_0^\infty dz \, \frac{e^{-\frac{1}{z+x}}}{z+x} \leq \frac{2}{x} e^{-\frac{1}{x}}. \tag{87}
\]

Alternatively, we can find a sharper bound for \( K_0 \) if we divide the above integral into two parts as follows:

\[
K_0(x) < e^{-\frac{1}{x}} \left[ \int_0^1 dz \, \frac{e^{-\frac{1}{z+x}}}{z+x} + \int_1^\infty dz \, \frac{e^{-\frac{1}{z+x}}}{z+x} \right]
\leq e^{-\frac{1}{x}} \left[ \int_0^1 dz \, \frac{1}{z+x} + \frac{1}{1+x} \int_1^\infty dz \, e^{-\frac{1}{z+x}} \right]. \tag{88}
\]

Hence, we have the following bound for \( K_0 \) which shows the logarithmic singularity near \( x = 0 \):

\[
K_0(x) \leq e^{-\frac{1}{x}} \left[ \ln \left( \frac{x+1}{x} \right) + \frac{2(1-e^{-1/2})}{1+x} \right]
\leq \frac{2}{1+x} e^{-\frac{1}{x}} + e^{-\frac{1}{x}} \ln \left( \frac{x+1}{x} \right). \tag{89}
\]

Using \( \sinh^2 s = \left( \frac{e^s - e^{-s}}{2} \right)^2 < \frac{e^{2s}}{4} \) for all \( s \geq 0 \) and following the same steps as above for \( K_1 \), we find

\[
K_1(x) \leq e^{-\frac{1}{x}} \left( \frac{1}{x} + \frac{1}{2} \right). \tag{90}
\]

Substituting (89) and (90) into (82) for \( D = 2 \), we get

\[
|\psi_k(x)| \leq 8\alpha A \sum_{i=1}^N \left[ \frac{1}{2V(M)} \sqrt{\frac{md^2(a_i, x)}{v_i^2 h^2 C_2}} e^{-\sqrt{md^2(a_i, x)v_i^2 h^2 C_2}} \left( \frac{1}{\sqrt{md^2(a_i, x)v_i^2 h^2 C_2}} + 1 \right) + \frac{B(e)}{h (h/2m)} e^{-\sqrt{md^2(a_i, x)v_i^2 h^2 C_2}} \ln \left( \frac{2\sqrt{md^2(a_i, x)v_i^2 h^2 C_2} + 1}{2\sqrt{md^2(a_i, x)v_i^2 h^2 C_2}} \right) \right]
\leq \frac{2}{1 + 2\sqrt{md^2(a_i, x)v_i^2 h^2 C_2}} \right], \tag{91}
\]
and for $D = 3$, we have

$$|\psi_k(x)| \leq 8\alpha A \sum_{i=1}^{N} \left[ \frac{1}{2V(M)} \sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}} e^{-\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}} \right]$$

$$\times \left[ \frac{1}{\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}} + 1 \right] + \frac{B(\varepsilon) \sqrt{\pi}}{\hbar (\hbar/2m)^{3/2}} \frac{e^{-2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}}}{2\sqrt{\frac{md^2(a_i, x)}{\hbar C_2}}} \right], \quad (92)$$

where we have used the explicit exact expression for $K_1$:

$$K_1(u) = \frac{\sqrt{\pi}}{2u} e^{-u}. \quad (93)$$

We can repeat the same steps for Cartan–Hadamard manifolds by using the upper bounds of the heat kernel given in (77) and the result is

$$|\psi_k(x)| \leq 2\alpha A \sum_{i=1}^{N} \left[ \frac{C(\varepsilon, \kappa) \sqrt{\pi}}{h(4\pi h/2m)^{3/2}} \frac{e^{-2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}}}{2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}} \right]$$

$$\times \left[ \ln \left( \frac{2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}} + 1}{2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}} \right) \right] \quad (94)$$

for $D = 2$ and

$$|\psi_k(x)| \leq \alpha \sum_{i=1}^{N} \left[ \frac{C(\varepsilon, \kappa) \sqrt{\pi}}{h(4\pi h/2m)^{3/2}} \frac{e^{-2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}}}{2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}} \right]$$

$$\times \left[ \ln \left( \frac{2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}} + 1}{2\sqrt{\frac{md^2(a_i, x)}{v_i^2 \hbar^2 C_2}}} \right) \right] \quad (95)$$

for $D = 3$. In standard quantum mechanics, the pointwise exponential bounds do not take into account singular interactions. Nevertheless, we prove that they are still valid for our problem in two and three dimensions.

In order to understand heuristically why our problem can be considered as a self-adjoint extension, which is also suggested by Krein’s formula obtained in section 2, it is interesting to calculate the expectation value of the free energy for the bound state. Result (39) permits us to write the expectation value as

$$\langle \psi_k | H_0 | \psi_k \rangle = \left[ \sum_{i,j=1}^{N} A_i^*(-v_i^2) \int_0^\infty \frac{dt}{\hbar} \left( \frac{t^2}{\hbar^2} \right) K_i(a_i, a_j; \gamma) e^{-v_i^2 t^2} A_j(-v_j^2) \right]^{-1}$$

$$\times \int_{M} d^D g \times \left[ \int_0^\infty \frac{dt}{\hbar} e^{-v_i^2 t^2} \sum_{i=1}^{N} A_i^*(-v_i^2) K_i(a_i, x; \gamma) \right]$$

$$\times \left[ \int_0^\infty \frac{dt}{\hbar} e^{-v_j^2 t^2} \sum_{j=1}^{N} A_j(-v_j^2) \left( -\frac{\hbar^2}{2m} \nabla_g^2 K_{ij}(a_j, x; \gamma) \right) \right]. \quad (96)$$

Using (A.6) and making an integration by parts to the $t_2$ integral, we have
\[ \langle \psi_k | H_0 | \psi_k \rangle = - \left[ \sum_{i,j=1}^{N} A_i^*(-v_i^2) \int_0^\infty \frac{dt}{h} \left( \frac{t}{h} \right)^N K_i(a_i, a_j; g) e^{-\frac{v_i^2}{2}} A_j(-v_j^2) \right]^{-1} \]

\[ \times \int d^D x \left[ \int_0^\infty \frac{dt}{h} e^{-\frac{v_i^2}{2}} \sum_{i=1}^{N} A_i^*(-v_i^2) K_i(a_i, x; g) \right] \left[ -\sum_{j=1}^{N} A_j(-v_j^2) \delta_g(x, a_j) \right. \]

\[ \left. + \sum_{j=1}^{N} A_j(-v_j^2) \right] \int_0^\infty \frac{dt}{h} v_j^2 e^{-\frac{v_j^2}{2}} K_j(a_j, x; g) \right] \] (97)

where we have used the initial condition of the heat kernel (A.7). Integrating with respect to \( x \) and using the semigroup property of the heat kernel (A.8), we obtain

\[ \langle \psi_k | H_0 | \psi_k \rangle = \left[ \sum_{i,j=1}^{N} A_i^*(-v_i^2) \int_0^\infty \frac{dt}{h} \left( \frac{t}{h} \right)^N K_i(a_i, a_j; g) e^{-\frac{v_i^2}{2}} A_j(-v_j^2) \right]^{-1} \]

\[ \times \left[ \int_0^\infty \frac{dt}{h} e^{-\frac{v_i^2}{2}} \sum_{i,j=1}^{N} A_i^*(-v_i^2) A_j(-v_j^2) K_i(a_i, a_j; g) \right. \]

\[ \left. - \int_0^\infty \frac{dt}{h} \int_0^\infty \frac{dt}{h} \sum_{i,j=1}^{N} A_i^*(-v_i^2) A_j(-v_j^2) K_i(a_i, a_j; g) \right] \] (98)

By change of variables \( u = t_1 + t_2 \) and \( v = t_1 - t_2 \), we find

\[ \langle \psi_k | H_0 | \psi_k \rangle = \left[ \sum_{i,j=1}^{N} A_i^*(-v_i^2) \int_0^\infty \frac{dt}{h} \left( \frac{t}{h} \right)^N K_i(a_i, a_j; g) e^{-\frac{v_i^2}{2}} A_j(-v_j^2) \right]^{-1} \]

\[ \times \left[ \int_0^\infty \frac{dt}{h} e^{-\frac{v_i^2}{2}} \sum_{i,j=1}^{N} A_i^*(-v_i^2) A_j(-v_j^2) K_i(a_i, a_j; g) \right. \]

\[ \left. - \frac{1}{2} \int_0^\infty \frac{du}{h} \left( \int_u^\infty dv \right) \sum_{i,j=1}^{N} A_i^*(-v_i^2) A_j(-v_j^2) K_i(a_i, a_j; g) e^{-\frac{v_i^2}{2}} v_i^2 \right] \] (99)

One can easily see that the \( i = j \) term of the sum for the first term in parenthesis

\[ \int_0^\infty \frac{dt}{h} e^{-\frac{v_i^2}{2}} A_i(-v_i^2)^2 K_i(a_i, a_i; g) \] (100)

is divergent due to diagonal short-time asymptotics of the heat kernel (A.17) for \( D \geq 2 \) and one can also show that the off-diagonal terms of this sum in parenthesis and all terms in the second sum is convergent by the upper bound on the heat kernel given in (69) and (77). Hence we find that the expectation value of the free Hamiltonian is divergent:

\[ \langle \psi_k | H_0 | \psi_k \rangle \to \infty. \] (101)

It is a well-known fact that point interactions on \( \mathbb{R}^D \) can be considered as a self-adjoint extension of the free Hamiltonian [2, 10]. We may heuristically think of our problem as a kind of self-adjoint extension, since the wavefunction \( \psi_k(x) \) that we have found does not belong to the domain of the free Hamiltonian, so the self-adjoint extension of the free Hamiltonian extends the domain of it such that the states corresponding to the eigenfunctions \( \psi_k(x) \) are included.
6. Lower bound of the ground state energy $E_{gr}$

Although we renormalize the model, we have not completely proven that the energy is bounded from below. A well-known theorem in matrix analysis, called the Geršgorin theorem [44], states that all the eigenvalues $\omega$ of a matrix $\Phi \in M_N$ are located in the union of $N$ disks:

$$
\bigcup_{i=1}^{N} \{ |\omega - \Phi_{ii}| \leq R'_i(\Phi) \} \equiv G(\Phi),
$$

where $R'_i(\Phi) \equiv \sum_{j \neq i}^{N} |\Phi_{ij}|$ is the deleted absolute row sums. Since the matrix $\Phi_{ij}(E)$ is Hermitian due to the symmetry property of the heat kernel, we have $\omega \in \mathbb{R}$. Indeed, all eigenvalues $\omega$ are zero in our problem. If there is a lower bound on energy, that is, ground state energy, then we must expect that there would be no solution at all beyond this lower bound, say $E^* = -\nu_2^2$. Then, we want $\omega = 0$ not to be an eigenvalue, thereby none of the disks defined above should contain the zero eigenvalue when $E \leq E^*$. This means that we should impose

$$
|\Phi_{ii}(E)| = \Phi_{ii}(E) > \sum_{i \neq j}^{N} |\Phi_{ij}(E)|
$$

for all $i$, that is, the principal matrix must be strictly diagonally dominant in order not to have a zero eigenvalue. However, before imposing this condition we can simplify the problem. We note that

$$
|\Phi_{ii}(E)| \geq |\Phi_{ii}(E)|_{\min} \sum_{i \neq j}^{N} |\Phi_{ij}(E)|_{\max},
$$

so the above condition (103) is implied by the stronger requirement

$$
|\Phi_{ii}(E)|_{\min} > (N - 1)|\Phi_{ij}(E)|_{\max}.
$$

Once we obtain a solution to this inequality, it is satisfied for all $E \leq E^*$ since the diagonal part of the principal matrix (26) is a decreasing function of $E$ and the off-diagonal part of it is an increasing function for given $a_i$, $a_j$ and $N$. This means that there is no solution beyond this critical value $E^*$. Hence, the ground state energy must be larger than the critical value $E^*$:

$$
E_{gr} \geq E^*.
$$

The basic idea of the proof was given for special manifolds $S^2$, $H^2$ and $H^3$ in our previous work [25]. In the next subsections, we will find the lower bound of the ground state energy for a more general class of manifolds.

6.1. Compact manifolds

For compact manifolds, the upper bound for the off-diagonal elements of heat kernel and the lower bound for the on-diagonal part of the heat kernel has been given in (69) and (71), respectively. Using these bound estimates in (26), we find a lower bound for the principal matrix:

$$
\Phi_{ij}(E) \geq \begin{cases} 
\frac{1}{(4\pi \hbar^2/2m)} \ln \left( \frac{v^2}{\mu^2} \right) & \text{if } D = 2 \\
\frac{2\sqrt{\pi}}{\hbar(4\pi \hbar/2m)^{3/2}} \left( \sqrt{\frac{v^2}{\hbar}} - \sqrt{\frac{\mu^2}{\hbar}} \right) & \text{if } D = 3,
\end{cases}
$$

(107)
and an upper bound for it:

$$|\Phi_{ij}(E)| \leq \begin{cases} 
4A \left[ \frac{\pi}{C_2} K_1 \left( \frac{\sqrt{2}}{C_2} \mu_d \right) \sqrt{\frac{2B(\varepsilon)K_0 \left( \sqrt{2} \mu_d \right)}{4\pi \hbar^2/2m}} \right] & \text{if } D = 2 \\
4A \left[ \frac{\pi}{C_2} K_1 \left( \frac{\sqrt{2}}{C_2} \mu_d \right) \sqrt{2\pi C_2 B(\varepsilon) \mu_d \exp \left( -\sqrt{\frac{2}{C_2}} \mu_d \right)} \right] & \text{if } D = 3, 
\end{cases}$$

(108)

where \( i \neq j \) and we have used the monotonic behavior of the functions in \( \Phi_{ij}(E) \) so that we could maximize the principal matrix in which we defined \( d \equiv \min_{i \neq j} d_{ij} \) and \( \mu \equiv \max_i \mu_i \).

We also introduced a natural energy scale \( \mu_d^2 \equiv \frac{\hbar^2}{2md^2} \) for simplicity. In order to solve the inequality analytically, we must estimate the bounds on the Bessel and logarithm functions. We can estimate the lower bound of the logarithmic function [45]

$$\ln u > \frac{u - 1}{u} \quad \text{for } u > 0, u \neq 1.$$

(109)

Let us first consider the two-dimensional case. As a result of bounds (87), (90) given for the Bessel functions and the one for the logarithmic function given above, we find

$$\Phi_{ij}(E) \geq \frac{2m}{\pi \hbar^2} \left( 1 - \frac{1}{v/\mu} \right)^2$$

(110)

and

$$|\Phi_{ij}(E)| \leq 4A \exp \left( -\frac{1}{2C_2 \mu_d} \right) \left[ \frac{1}{V(M) \mu_d^2} \left( 1 + \frac{1}{\sqrt{2C_2}} \right) + \frac{2\sqrt{2C_2} B(\varepsilon)}{(4\pi \hbar^2/2m)v/\mu_d} \right].$$

(111)

Since \( v > \mu_d \), we may have

$$|\Phi_{ij}(E)| \leq 4A \exp \left( -\frac{1}{2C_2 \mu_d} \right) \left[ \frac{1}{V(M) \mu_d^2} \left( 1 + \frac{1}{\sqrt{2C_2}} \right) + \frac{2\sqrt{2C_2} B(\varepsilon)}{(4\pi \hbar^2/2m)} \right].$$

(112)

Therefore, there will be no solution to the eigenvalue equation for values of the ground state energy below a critical value \( v > v_* \) if the following inequality is satisfied:

$$\frac{2m}{\pi \hbar^2} \left( 1 - \frac{1}{v/\mu} \right) > 4A(N - 1) \exp \left( -\frac{1}{2C_2 \mu_d} \right) \left[ \frac{1}{V(M) \mu_d^2} \left( 1 + \frac{1}{\sqrt{2C_2}} \right) + \frac{2\sqrt{2C_2} B(\varepsilon)}{(4\pi \hbar^2/2m)} \right].$$

(113)

Hence, we can solve this and get the lower bound for the ground state energy

$$E_{gr} \geq -v_*^2 = -\mu_d^2 \left[ \frac{\mu}{\mu_d} + \sqrt{2C_2} W((N - 1)A_1) \right]^2,$$

(114)

where

$$A_1 = \exp \left( -\frac{1}{\sqrt{2C_2}} \right) \left[ \frac{1}{V(M) \mu_d^2} \left( 1 + \frac{1}{\sqrt{2C_2}} \right) + \frac{2\sqrt{2C_2} B(\varepsilon)}{(4\pi \hbar^2/2m)} \right] 4A(\pi \hbar^2/2m),$$

(115)

and \( W \) is called the Lambert-W function, also called the Omega or the ProductLog function [46], and it is defined as the inverse function of \( x e^x \). In other words,

$$y = x e^x \iff x = W(y).$$

(116)
As for the three-dimensional case, the off-diagonal part of the principal matrix have the following upper bound:

\[
|\Phi_{ij}(E)| \leq 4A \exp \left( -\frac{1}{2} v \frac{\mu}{C^2} \right) \left[ \frac{1}{V(M)\mu_d^2} \left( 1 + \frac{1}{\sqrt{2C^2}} \right) + \sqrt{2\pi C^2 B(\varepsilon)\mu_d} \frac{h}{(4\pi h^2/2m)} \right].
\]  

(117)

where we use a less sharp upper bound by using \( \nu > \mu_d \) in order to solve the inequality. Therefore, we conclude that there exists a critical value of bound state energy \( \nu > \nu^* \) for a given \( N \) and \( d \) such that \( \omega \neq 0 \) so the ground state energy cannot be less than \(-\nu^*^2\):

\[
E_{gr} \geq -\nu^2 = -\mu_d^2 \left( \frac{\sqrt{2}}{h} W((N-1)A_2) \right)^2,
\]  

(118)

where

\[
A_2 \equiv \exp \left( -\sqrt{\frac{C^2}{\pi}} \ln \left( \frac{\nu^2}{h} + \xi \right) \right)
\]  

\[
\times \left( \frac{V(M)\mu_d^2}{\sqrt{2C^2}} \left( 1 + \sqrt{\frac{2\pi C^2 B(\varepsilon)\mu_d}{h}} \right) \frac{h^{3/2} 4A (4\pi h^2/2m)^{3/2}}{\mu_d} \right).
\]  

(119)

6.2. Cartan–Hadamard manifolds

Similarly, using the upper and lower bounds of the heat kernel for Cartan–Hadamard manifolds, we have obtained the upper and lower bound on the off- and on-diagonal parts of the principal matrix, respectively:

\[
\Phi_{ij}(E) \geq \begin{cases} 
\frac{e}{(4\pi h^2/2m)} \ln \left( \frac{\nu^2}{\mu} + \xi \right) & \text{if } D = 2 \\
\frac{2\sqrt{\pi} c}{h (4\pi h/2m)^{3/2}} \left( \frac{\nu^2}{h} + \xi - \sqrt{\frac{\mu^2}{h} + \xi} \right) & \text{if } D = 3,
\end{cases}
\]  

(120)

with \( \xi \equiv \frac{h(\sigma_1(M)+\delta)}{2m} \geq 0 \) and

\[
\Phi_{ij}(E) \leq \begin{cases} 
\frac{2C(\varepsilon, \kappa)}{(4\pi h^2/2m)} K_0 \left( \frac{2}{C^2} \frac{v}{\mu_d} \right) & \text{if } D = 2 \\
\frac{\sqrt{2\pi C^2 C(\varepsilon, \kappa)\mu_d}}{h^{3/2} (4\pi h/2m)^{3/2}} \exp \left( -\frac{2}{C^2} \frac{v}{\mu_d} \right) & \text{if } D = 3
\end{cases}
\]  

(121)

for \( i \neq j \). For \( D = 2 \), we have

\[
\ln \left( \frac{\nu^2}{\mu} + \xi \right) \geq \ln \left( \frac{\xi}{\mu} + \xi \right)
\]  

(122)

and if we use the same bounds for the Bessel function and the fact that \( v > \mu_d \), we obtain the lower bound for the ground state energy

\[
E_{gr} \geq -2C^2 \mu_d^2 W^2 \left( \frac{2(N-1)C(\varepsilon, \kappa)}{\ln \left( \frac{\xi}{\mu} + \xi \right)} \right).
\]  

(123)
For the three-dimensional case, we must do some additional assumption in order to get an analytic solution. We will assume that
\[
\nu \geq \mu^2 + \bar{h}\xi \quad \text{and this assumption should be checked whether it is consistent or not after we have found the solution. Hence,}
\]
\[
\sqrt{\frac{\nu^2}{\bar{h}} + \xi} - \sqrt{\frac{\mu^2}{\bar{h}} + \xi} \geq \left( \frac{\nu}{\bar{h}^{1/2}} - \sqrt{\frac{\mu^2}{\bar{h}} + \xi} \right),
\]
and finally
\[
E_{gr} \geq -\mu^2 d \left[ \frac{1}{\mu_d} \sqrt{\mu^2 + \bar{h}\xi} + \sqrt{\frac{C_2}{2} W(A_3(N - 1))} \right]^2,
\]
where
\[
A_3 \equiv \frac{C(\varepsilon, \kappa)}{e} \exp \left(-\frac{1}{\mu_d} \sqrt{\frac{2}{C_2} (\mu^2 + \bar{h}\xi)} \right).
\]

7. Non-degeneracy and positivity of the ground state

The rigorous proof of non-degeneracy and positivity of the ground state in standard quantum mechanics is given in [30, 47], which does not include the singular potentials. Therefore, it is necessary to check whether this is still valid for our problem. The proof in our case is based on the Perron–Frobenius theorem [44]. It states that if \( A \in M_N \) and if we suppose that \( A > 0 \) (that is, all \( A_{ij} > 0 \)), then
(a) \( \rho(A) > 0 \);
(b) \( \rho(A) \) is an eigenvalue of \( A \);
(c) there is an \( x \in \mathbb{C}^N \) with \( x > 0 \) and \( Ax = \rho(A)x \);
(d) \( \rho(A) \) is an algebraically (and hence geometrically) simple eigenvalue of \( A \);
(e) \( |\omega| < \rho(A) \) for every eigenvalue \( \omega \neq \rho(A) \), that is, \( \rho(A) \) is the unique eigenvalue of maximum modulus. Here \( \rho(A) = \max\{|\omega| : \omega \text{ is an eigenvalue of } A\} \) and called the spectral radius. A simple proof of this theorem for the positive symmetric matrices is given in [48].

Since the principal matrix (26) is not a positive matrix, we cannot directly apply the Perron–Frobenius theorem. Nevertheless, we can make the principal matrix strictly positive by subtracting the maximum of the diagonal part of it corresponding to the lower bound of energy \( E = E_* \), which is found in section 6, and reversing the overall sign
\[
\Phi'(E) = -\left[ \Phi(E) - (1 + \varepsilon)1\Phi^{\max}_m(E_*) \right] > 0, \quad \varepsilon > 0,
\]
where 1 is an \( N \times N \) identity matrix. Hence, considering the transformed principal matrix \( \Phi' \) in the light of this theorem, we conclude that there exists a strictly positive eigenvector which corresponds to the unique eigenvalue of maximum modulus:
\[
\sum_{j=1}^N \Phi'_{ij}(E)A_j(E) = \rho(\Phi')A_i(E).
\]
Here it must be noted that \( \Phi' \) has the same eigenvector with \( \Phi \) so it guarantees that there exists a strictly positive eigenvector for the principal matrix \( \Phi \). Using the eigenvalue problem
\[
\sum_{j=1}^N \Phi_{ij}(E)A_j(E) = \omega(E)A_i(E),
\]
where \( \rho(\Phi') = -\omega^{\min}(E) + (1 + \epsilon)\Phi_{ij}(E_\nu) \). For a given \( E = E_{k} \) or \( \nu = \nu_{k} \), there is a unique corresponding \( \omega^{\min}(E) \) and since we are looking for the zeros of the eigenvalues \( \omega(E) = 0 \), this minimum flows to zero at \( \nu = \nu^{\max} = \nu_{*} \). This means that the positive eigenvector \( A_{i} \) corresponds to the ground state energy so we prove that the ground state energy is unique and the associated eigenvector \( A_{i} \) is strictly positive. Due to the positivity property of the heat kernel, it is easy to see that the ground state wave function is strictly positive from equation (39):

\[
\psi_{k}(x) = \left[ \sum_{i,j=1}^{N} A_{i}(\nu_{k}^{2}) \int_{0}^{\infty} \frac{dt}{h} \left( \frac{t}{h} \right) K_{ij}(a_{i}, a_{j}; g) e^{-\frac{t^{2}}{4\pi t}} A_{j}(\nu_{k}^{2}) \right]^{-\frac{1}{2}} \\
\times \int_{0}^{\infty} \frac{dt}{h} e^{-\frac{t^{2}}{4\pi t}} \sum_{i=1}^{N} A_{i}(\nu_{k}^{2}) K_{i}(a_{i}, g) > 0. \tag{130}
\]

Hence, we prove that despite the singular character of the interaction, the ground state is still non-degenerate and unique.

8. Renormalization group equations

Renormalization group equations of our problem (for the \( N = 1 \) case) in flat spaces have already been given in the literature [6, 15, 18]. Here, we will show that this can also be derived explicitly for our problem.

8.1. Two-dimensional case

In this section, we shall choose the natural units \( \hbar = 2m = 1 \) for simplicity. It is useful to work with the dimensionless coupling constant. One possible way for the renormalization scheme in order to determine how the coupling constant changes with the energy scale is to define the following renormalized coupling constant \( \lambda^{R}_{i}(M_{i}) \) in terms of the bare coupling constant \( \lambda_{i}(\epsilon) \):

\[
\frac{1}{\lambda^{R}_{i}(M_{i})} = \frac{1}{\lambda_{i}(\epsilon)} - \int_{\epsilon}^{\infty} \frac{dt}{4\pi t} e^{-M_{i}^{2}/4t}, \tag{131}
\]

where \( M_{i} \) is the renormalization scale (it is of dimension \( [E]^{1/2} \)). Then, the renormalized principal matrix in terms of the renormalized coupling constant in natural units is given as

\[
\Phi_{ij}^{R}(E) = \begin{cases} 
\frac{1}{\lambda^{R}_{i}(M_{i})} - \int_{0}^{\infty} dt \left( K_{i}(a_{i}, a_{j}; g) e^{E} - \frac{e^{-M_{i}^{2}/4t}}{4\pi t} \right) & \text{if } i = j \\
- \int_{0}^{\infty} dt K_{i}(a_{i}, a_{j}; g) e^{E} & \text{if } i \neq j,
\end{cases} \tag{132}
\]

and the bound state energy is determined from the condition \( \det \Phi_{ij}^{R}(E) = 0 \) and it determines the relation between \( \lambda^{R}_{i}(M_{i}) \) and \( M_{i} \). Explicit dependence on \( M_{i} \) cancels the implicit dependence on \( M_{i} \) through \( \lambda^{R}_{i}(M_{i}) \). Physics is determined by the value of \( \lambda^{R}_{i}(M_{i}) \) at an arbitrary value of the renormalization point \( M_{i} \). However, this is not a proper way to look at our problem since we have to deal with several renormalized coupling constants with the same kind of interaction, which essentially differs from one another with arbitrary constants. These
constants can be determined by deciding the excited energy levels. We shall instead prefer one renormalized coupling constant by redefining the meaning of the renormalized coupling constant without altering physics. This could be done in the following way. As an external input, we decide about the relative strengths of individual delta interactions and do not use the ground state energy to fix the flow. We know that $-\mu_i^2$ is the bound state energy of the $i$th Dirac-delta center so it corresponds to the solution $\Phi^R_{1i}(-\mu_i^2) = 0$. Without loss of generality, let us assume that $\Phi^R_{11}(-\mu_1^2) = 0$ and this allows us to choose the renormalized coupling constant

$$\lambda^R(M) = \lambda_i(\epsilon) - \int_\epsilon^\infty dt \frac{e^{-M^2t}}{4\pi t}, \quad (133)$$

at some scale $M$. Once the renormalized coupling constant is fixed under this condition, we must also satisfy $\Phi^R_{ii}(-\mu_i^2) = 0$ for $i \neq 1$ with this choice at the same scale $M$. This is always possible if we add a constant term to the definition of the renormalized coupling constant. Let us consider the $i = 2$ case:

$$\Phi^R_{22}(-\mu_2^2) = \frac{1}{\lambda^R(M)} + \lim_{\epsilon \to 0+} \left[ \int_\epsilon^\infty dt \frac{e^{-M^2t}}{4\pi t} - \int_\epsilon^\infty dt K_t(a_2, a_2; g) e^{-\mu_2^2t} \right] - \Sigma_2,$$

$$= \int_0^\infty dt \left[ K_t(a_1, a_1; g) e^{-\mu_1^2t} - K_t(a_2, a_2; g) e^{-\mu_2^2t} \right] - \Sigma_2 = 0, \quad (134)$$

where we have used (133) and $\Phi^R_{11}(-\mu_1^2) = 0$. This means that there always exists a constant $\Sigma_i$ depending only on $\mu_i$ with $\Sigma_1 = 0$ and $\Sigma_i \neq 0$ for $i \neq 1$ such that the condition $\Phi^R_{ii}(-\mu_i^2) = 0$ can be satisfied. Hence, the renormalized coupling constant becomes

$$\frac{1}{\lambda^R(M)} = \frac{1}{\lambda_i(\epsilon)} - \int_\epsilon^\infty dt \frac{e^{-M^2t}}{4\pi t} + \Sigma_i, \quad (135)$$

and the choice of $\Sigma_i$’s refers to the relative strengths of delta interactions in this renormalization scheme. As a result, the renormalized principal matrix is

$$\Phi^R_{ij}(E) = \begin{cases} \frac{1}{\lambda^R(M)} - \int_0^\infty dt \left( K_t(a_i, a_i; g) e^{iE} - \frac{e^{-M^2t}}{4\pi t} \right) - \Sigma_i & \text{if } i = j \\ -\int_0^\infty dt K_t(a_i, a_j; g) e^{iE} & \text{if } i \neq j. \end{cases} \quad (136)$$

The renormalization condition is given by

$$M \frac{d\Phi^R_{ij}(M, \lambda^R(M), E; g)}{dM} = 0 \quad (137)$$

or

$$\left( M \frac{\partial}{\partial M} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} \right) \Phi^R_{ij}(M, \lambda^R(M), E; g) = 0, \quad (138)$$

where

$$\beta(\lambda_R) = M \frac{\partial \lambda_R}{\partial M} \quad (139)$$

is called the $\beta$ function and equation (138) is the renormalization group (RG) equation. In [15], the renormalization condition (137) corresponding to the problem in flat space has been written in terms of the $T$-matrix. Using (136) in (138), we can find the exact $\beta$ function:

$$\beta(\lambda_R) = -\frac{\lambda^2_R}{2\pi} < 0. \quad (140)$$
This result is the same as the one in flat spaces given in the literature [6, 14, 15, 18], so our problem is asymptotically free. From the explicit expression of the renormalized principal matrix, one can easily see the scaling property of it under a change of energy and metric scale \( \gamma \) using the scaling property of the heat kernel (A.15):

\[
\Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = \Phi_{ij}^R(\gamma^{-1} M, \lambda_R(M), E; g).
\]  

(141)

It is important to note that we also need to scale the metric and the idea of the metric scaling in deriving the renormalization group equation was motivated by [49] in the context of the renormalization group in quantum field theory on curved spaces. Hence, we have

\[
\gamma \frac{d}{d\gamma} \left[ \Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = \Phi_{ij}^R(\gamma^{-1} M, \lambda_R(M), E; g) \right].
\]  

(142)

This leads to the renormalization group equation for \( \Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) \) :

\[
\gamma \frac{d}{d\gamma} \Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) + M \frac{d}{dM} \Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = 0
\]  

(143)

or

\[
\left[ \gamma \frac{d}{d\gamma} - \beta(\lambda_R) \frac{d}{d\lambda_R} \right] \Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = 0.
\]  

(144)

If we postulate the following functional form for the principal matrix:

\[
\Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = f(\gamma) \Phi_{ij}^R(M, \lambda_R(\gamma M), E; g),
\]  

(145)

and substitute it into (144), we obtain an ordinary differential equation for the function \( f \):

\[
\gamma \frac{df(\gamma)}{d\gamma} = 0.
\]  

(146)

This gives the solution \( f(\gamma) = 1 \) using the initial condition at \( \gamma = 1 \). Therefore, we get

\[
\Phi_{ij}^R(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = \Phi_{ij}^R(M, \lambda_R(\gamma M), E; g),
\]  

(147)

which means that there is no anomalous scaling. After integrating

\[
\beta(\lambda_R) = \tilde{M} \frac{\partial \lambda_R(\tilde{M})}{\partial \tilde{M}} = -\frac{\lambda_R^2(\tilde{M})}{2\pi}
\]  

(148)

between \( \tilde{M} = M \) to \( \tilde{M} = \gamma M \) we can find the flow equation for the coupling constant:

\[
\lambda_R(\gamma M) = \frac{\lambda_R(M)}{1 + \frac{1}{\pi} \lambda_R(M) \ln \gamma}.
\]  

(149)

One can explicitly check relation (147) if the coupling constant evolves according to (149). First, we add and subtract a term in the time integral:

\[
\Phi_{ij}^R(M, \lambda_R(M), E; g) = \frac{1}{\lambda_R(M)} + \frac{1}{2\pi} \ln \gamma - \int_0^\infty dr \left( \Pi_i, a_i, g \right) e^{rE} - \frac{e^{-M^2 r_i}}{4\pi t_i} = \Sigma_i
\]  

(150)
and then using the scaling property of heat kernel (A.15), we get

\[
\frac{1}{\lambda_R(M)} = \int_0^\infty dt \left( \gamma^{-2} K_{\gamma^{-2} t}(a_i, a_i; \gamma^{-2} g) e^{\frac{-e^{-M^2 t}}{4\pi t}} \right) - \Sigma_i
\]

\[
= \frac{1}{\lambda_R(M)} - \int_0^\infty ds \left( K_s(a_i, a_i; \gamma^{-2} g) e^{\frac{-e^{-M^2 s}}{4\pi s}} \right) - \Sigma_i
\]

\[
= \Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g).
\] (151)

The off-diagonal term can be directly checked using just the scaling property of the heat kernel (A.15). In another way of thinking, one can find how the coupling constant evolves, given (149) from the scaling relation (147).

8.2. Three-dimensional case

Since it is convenient to work with the dimensionless coupling constant, we define a dimensionless coupling constant in three dimensions:

\[
\hat{\lambda}_R(M) = M \lambda_R(M).
\] (152)

Then, by similar arguments developed for two dimensions, the renormalized principal matrix in the natural units is

\[
\Phi^R_{ij}(E) = \begin{cases} 
\frac{M}{\hat{\lambda}_R(M)} - \int_0^\infty dt \left( K_t(a_i, a_i; g) e^{\frac{-e^{-M^2 t}}{(4\pi t)^{3/2}}} \right) - \Sigma_i & \text{if } i = j \\
- \int_0^\infty dt K_t(a_i, a_j; g) e^{\frac{-e^{-M^2 t}}{(4\pi t)^{3/2}}} & \text{if } i \neq j.
\end{cases}
\] (153)

The renormalization condition (138) in this case leads to the following \(\beta\) function:

\[
\beta(\hat{\lambda}_R) = M \frac{\partial \hat{\lambda}_R(M)}{\partial M} = \hat{\lambda}_R(M) - \frac{1}{4\pi} \hat{\lambda}_R^2(M),
\] (154)

which is in agreement with the result for flat space [15]. From the explicit expression of the renormalized principal matrix for three dimensions, one can easily see the scaling property of it under a change of scale \(\gamma\) using the scaling property of the heat kernel (A.15):

\[
\Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = \gamma \Phi^R_{ij}(\gamma^{-1} M, \hat{\lambda}_R(M), E; g),
\] (155)

so we have

\[
\gamma \frac{d}{dy} \left[ \Phi^R_{ij}(\gamma^{-1} M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = \gamma \Phi^R_{ij}(\gamma^{-1} M, \hat{\lambda}_R(M), E; g) \right].
\] (156)

This leads to the renormalization group equation for \(\Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g)\):

\[
\left[ \gamma \frac{d}{dy} - 1 + \beta(\hat{\lambda}_R) \frac{\partial}{\partial \hat{\lambda}_R} \right] \Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = 0
\] (157)

or

\[
\left[ \gamma \frac{d}{dy} - 1 - \beta(\hat{\lambda}_R) \frac{\partial}{\partial \hat{\lambda}_R} \right] \Phi^R_{ij}(M, \hat{\lambda}_R(M), \gamma^2 E; \gamma^{-2} g) = 0.
\] (158)

If we again postulate (145) and substitute it into (158), we obtain an ordinary differential equation for the function \(f\):

\[
\gamma \frac{df(\gamma)}{dy} = f.
\] (159)
The solution is \( f(\gamma) = \gamma \) by using the initial condition at \( \gamma = 1 \). Therefore, we have
\[
\Phi_{\beta}^{R}(M, \hat{\lambda}_{R}(M), \gamma^2 E; \gamma^{-2} g) = \gamma \Phi_{\beta}^{R}(\gamma M, E; g).
\] (160)
This means that there is also no anomalous scaling in three dimensions. After integrating
\[
\beta(\hat{\lambda}_{R}) = \bar{M} \frac{\partial \hat{\lambda}_{R}(\bar{M})}{\partial \bar{M}} = \hat{\lambda}_{R}(\bar{M}) - \frac{1}{4\pi} \hat{\lambda}_{R}(\bar{M}),
\] (161)
between \( \bar{M} = M \) and \( \bar{M} = \gamma M \) we can similarly find the flow equation for the coupling constant:
\[
\hat{\lambda}_{R}(\gamma M) = \frac{\gamma \hat{\lambda}_{R}(M)}{1 - \frac{1}{4\pi} \hat{\lambda}_{R}(M)(1 - \gamma)^{-1}}.
\] (162)
One can similarly check relation (160) if the coupling constant evolves according to (162). In this case, we have
\[
\gamma \Phi_{\beta}^{R}(M, \hat{\lambda}_{R}(\gamma M), E; g)
= \frac{M}{\hat{\lambda}_{R}(M)} + \frac{1}{4\pi} M(1 - \gamma) - \gamma \int_{0}^{\infty} dt \left( K_{1}(a_{i}, a_{i}; g) e^{t E} - \frac{e^{-M^{2} t}}{(4\pi t)^{3/2}} \right) - \Sigma_{i}
= \frac{M}{\hat{\lambda}_{R}(M)} + \frac{1}{4\pi} M(1 - \gamma) - \gamma \int_{0}^{\infty} dt \left( K_{1}(a_{i}, a_{i}; g) e^{t E} - \frac{e^{-M^{2} t} e^{-\gamma^{-2} t}}{(4\pi t)^{3/2}} \right) - \Sigma_{i}
+ \frac{e^{-M^{2} t} e^{-\gamma^{-2} t}}{(4\pi t)^{3/2}} - \Sigma_{i}
= \frac{M}{\hat{\lambda}_{R}(M)} - \gamma \int_{0}^{\infty} dt \left( K_{1}(a_{i}, a_{i}; g) e^{t E} - \frac{e^{-M^{2} t} e^{-\gamma^{-2} t}}{(4\pi t)^{3/2}} \right) - \Sigma_{i},
\] (163)
and then using the scaling property of the heat kernel (A.15), we get
\[
\frac{M}{\hat{\lambda}_{R}(M)} - \gamma \int_{0}^{\infty} dt \left( \gamma^{-3} K_{1} a_{i} a_{i}; \gamma^{-2} g) e^{t E} - \frac{e^{-M^{2} t} e^{-\gamma^{-2} t}}{(4\pi t)^{3/2}} \right) - \Sigma_{i}
= \frac{M}{\hat{\lambda}_{R}(M)} - \gamma \int_{0}^{\infty} ds \left( K_{s}(a_{i}, a_{i}; \gamma^{-2} g) e^{s \gamma^{-2} E} - \frac{e^{-M^{2} s}}{(4\pi s)^{3/2}} \right) - \Sigma_{i}
= \Phi_{\beta}^{R}(M, \hat{\lambda}_{R}(M), \gamma^2 E; \gamma^{-2} g).
\] (164)
One can similarly find how the coupling constant evolves, given in (162) from the scaling relation (160).

9. Conclusion

In this paper, we studied the bound state problem for several Dirac-delta interactions in various two- and three-dimensional Riemannian manifolds. Our renormalization method is basically inspired from [23] developed for the many-body version of this problem on flat spaces. This method allows us to renormalize the model non-perturbatively. It has also been shown that the heat kernel plays a key role in the renormalization procedure and helps us to find lower bounds on the ground state energy due to the sharp upper-bound estimates on it for several classes of manifolds. We proved that many well-known theorems given in standard quantum mechanics are still valid, such as pointwise bounds on the wave functions, non-degeneracy and uniqueness of the ground state although we have singular interactions. The renormalization procedure does not change these well-known results in standard quantum mechanics. Finally, we studied the renormalization group equations and the \( \beta \) function is exactly calculated.
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Appendix A. Heat kernel on Riemannian manifolds

Let \((M, g)\) be a compact connected Riemannian manifold; then, there exists a complete orthonormal system of \(C^\infty\) eigenfunctions \(\{f_l\}_{l=0}^\infty\) in \(L^2(M, d^D_g x)\) and the spectrum \(\sigma(M, g) = \{\sigma_l\} = \{0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots\}\), with \(\sigma_l\) tending to infinity as \(l \to \infty\) and each eigenvalue has finite multiplicity: \(-\nabla_g^2 f_l(x; g) = \sigma_l f_l(x; g)\). The multiplicity of the first eigenvalue \(\sigma_0 = 0\) is 1 and the corresponding eigenfunction is constant and given by \(f_0(x; g) = 1/\sqrt{V(M)}\), where \(V(M)\) is the manifold \((M, g)\). This theorem is also called the Hodge theorem for functions or the spectral theorem \([43, 50]\). This theorem is valid for Neumann, Dirichlet and mixed eigenvalue problems except for \(\sigma_0 > 0\), provided that the appropriate boundary condition is imposed.

The operator \(-\nabla_g^2\) is formally self-adjoint with respect to the \(L^2(M, d^D_g x)\), that is, the inner product is defined as

\[
(\psi_1, \psi_2) = \int_M d^D_g x \psi_1^*(x) \psi_2(x). \tag{A.1}
\]

Since it will be necessary for our purposes to know how the eigenfunctions change under the scaling transformations in the metric, we will use a notation which specifies the metric structure of the functions, such as \(f_l(x; g)\). The spectral theorem provides us with all the tools of the Fourier analysis, so that we can expand any function \(\psi(x) \in L^2(M, d^D_g x)\) in terms of the complete orthonormal eigenfunctions \(f_l(x; g)\):

\[
\psi(x) = \sum_{l \geq 0} (\psi(x), f_l(x; g)) f_l(x; g) = \sum_{l \geq 0} C_l f_l(x; g). \tag{A.2}
\]

Orthogonality and completeness of the eigenfunctions on compact manifolds are

\[
\delta_{kl} = \int_M d^D_g x f_k^*(x; g) f_l(x; g),
\]

\[
\delta^D_g (x, a_i) = \sum_{l \geq 0} f_l(x; g) f_l^*(a_i; g), \tag{A.3}
\]

where \(C_l\)'s are expansion coefficients. \(\delta^D_g(x, a_i)\) is the \(D\)-dimensional normalized delta function at point \(a_i \in M:\)

\[
\int_M d^D_g x \delta^D_g (x, a_i) = 1. \tag{A.4}
\]

Note that extra labels in the eigenfunction expansion must be taken into account if the problem admits degeneracy.

Moreover, one may extend heuristically the problem on to some non-compact manifolds in such a way that the spectral theorem is still applied. The relations such as completeness and orthogonality relations are defined by an appropriate generalization of the measures to the continuous distributions in the sense of \([51]\). Since the spectrum does not have to be discrete
for non-compact manifolds, we may have in general
\[
\psi(x) = \int d\mu(l) \psi(l) f_l(x; g),
\]
\[
\delta_{kl} = \int_M d^D x f^*_k(x; g) f_l(x; g),
\]
\[
\delta^D_{g}(x, a_i) = \int d\mu(l) f_l(x; g) f^*_l(a_i; g),
\]

where \(d\mu(l)\) is the measure and it includes the continuous spectrum as well as the point spectrum. This should be taken with a grain of salt and one must not forget that we may not have a rigorous proof of the spectral theorem for some special non-compact manifolds. From now on, we assume that we are dealing with manifolds which do not have such pathologies, that is, the spectral theorem is applicable.

Although the notion of the heat kernel can be defined on any Riemannian manifold, the explicit formulas only exist for some special class of manifolds, for example, Euclidean spaces \(\mathbb{R}^D\) [38] and hyperbolic spaces \(H^D\) [52]. We will list some of the well-known properties of the heat kernel on any \((M, g)\), and give the short-time asymptotic expansion of the heat kernel. Also we derive the scaling property of the heat kernel. They are all needed in our calculations.

1. **Heat equation.** It satisfies the heat equation since it is a fundamental solution to it by definition:
\[
\hbar \frac{\partial K_t(x, y; g)}{\partial t} - \frac{\hbar^2}{2m} \nabla^2_g K_t(x, y; g) = 0,
\]
where \(\nabla^2_g\) acts on the \(x\) variable.

2. **Initial condition.** It solves the Cauchy problem
\[
\lim_{t \to 0^+} K_t(., y; g) = \delta_g(., y).
\]

3. **Semi-group property:**
\[
\int_M d^D z K_t(x, z; g) K_{t_2}(z, y; g) = K_{t+t_2}(x, y; g).
\]

4. **Symmetry property:**
\[
K_t(x, y; g) = K_t(y, x; g).
\]

5. **Positivity property:**
\[
K_t(x, y; g) > 0 \quad \text{for all} \quad t > 0.
\]

If \(M\) is compact, then we have
\[
K_t(x, y; g) = \sum_{l \geq 0} e^{-\frac{\hbar}{2m} l^2} f_l^*(x; g) f_l(y; g),
\]
which converges uniformly on \(M \times M\). The analog of this sum on non-compact manifolds can be given as
\[
K_t(x, y; g) = \int d\mu(l) e^{-\frac{\hbar}{2m} l^2} f_l^*(x; g) f_l(y; g).
\]

We also have the eigenfunction expansion of the creation and annihilation operators
\[
\phi_k^+(x) = \sum_{l \geq 0} \phi^+_k(l) f_l^*(x; g),
\]
or in non-compact manifolds

\[ \phi_k^g(x) = \int d\mu(\ell) \phi_k^\ell(f^\ast(x; g)) \tag{A.14} \]

When the manifold \( M \) is a complete Riemannian manifold with the Ricci curvature bounded from below then the heat kernel satisfies the stochastic completeness property \([38, 53]\):

\[ \int_M d^D_x K_t(x, y; g) = 1. \]

On a compact manifold, stochastic completeness is always satisfied \([54]\). Using the properties of the heat kernel and stochastic completeness, one can derive the scaling property of the heat kernel

\[ K_t(x, y; g) = a^D K_{\alpha^2 t}(x, y; \alpha^2 g). \tag{A.15} \]

The free resolvent can be written in terms of the heat kernel:

\[ R_0(x, y|z) = \langle x | (H_0 - z)^{-1} | y \rangle = \frac{1}{h} \int_0^\infty dt e^{\frac{t}{h}} K_t(x, y; g), \tag{A.16} \]

where \( \text{Re}(z) < 0 \). It can be defined for complex values \( z \) by analytic continuation. We have short-time asymptotics of the diagonal heat kernel for any manifold \([55]\)

\[ \lim_{t \to 0^+} K_t(x, x; g) \sim \frac{1}{(4\pi \hbar t/2m)^{D/2}} \sum_{k=0}^{\infty} u_k(x, x)(\hbar t/2m)^{k/2}, \tag{A.17} \]

for every \( x \in M \). Here the functions \( u_k(x, x) \) are scalar polynomials in the curvature tensor of the manifold and its covariant derivatives at the point \( x \). When there is no boundary, the odd terms in the expansion, i.e. \( k = 1, 3, 5, \ldots \) vanish \([56]\). In this paper, we always assume that the manifolds have no boundary. Indeed, we also have short-time asymptotic of the heat kernel for any \( x \) and \( y \) with the several assumptions about the structure of the set of geodesics which join the points \( x \) and \( y \) \([57]\). It is shown that (see theorem 2.1 and 2.2 in \([57]\)) for all \( y \) sufficiently close to \( x \) (so that \( x \) and \( y \) can be joined by a unique shortest geodesic \( \gamma_{x,y} \) along which \( x \) and \( y \) are non-conjugate)

\[ K_t(x, y; g) \sim \frac{e^{\frac{-m^2 t^2}{4\hbar \Omega_{x,y}}}}{(4\pi \hbar t/2m)^{D/2}} d(D-1)/2 \Psi^{-1/2}_{x,y}(x, y), \tag{A.18} \]

where \( d(x, y) \) is the geodesic distance between \( x \) and \( y \) and \( \Psi_{x}(x, y) \) characterizes the divergence of the geodesic flow near \( y \), that is, if we emit a beam of geodesics from \( x \) along \( y \) in the solid angle \( d\Omega \) illuminating a hypersurface of area \( dS \) at \( y \) orthogonal to \( \gamma \), then \( \Psi(x, y) = dS/d\Omega \). The function \( \Psi(x, y) \) can also be written in terms of the Jacobi fields orthogonal to the geodesic \( \gamma_{x,y}(s) \), where \( 0 \leq s \leq d(x, y) \). If the number of shortest geodesics joining \( x \) and \( y \) is greater than 1 or if \( x \) and \( y \) are conjugate along some of them, then the result takes the following form (up to a bounded factor):

\[ K_t(x, y; g) = O((\hbar t / 2m)^{-(D+1)/2} e^{D(D-1)/2} \Psi^{-1/2}_{x,y}(x, y)), \tag{A.19} \]

where the index \( k = k(x, y) \) depends on the character of the degeneracy of the geodesic flow between \( x \) and \( y \). Since all our calculations essentially give the same physical result for all cases and subcases \([57]\), we will just consider a generic case (case 3.1 in \([57]\)). The set \( \Omega_{x,y} \) consists of finite number of geodesics \( \gamma_1, \ldots, \gamma_n \) and \( x \) and \( y \) are non-conjugate along each of them. In this case, for each \( \gamma_i \) we can define \( \Psi_i(x, y) \) by considering the Jacobi fields along \( \gamma_i \). Then, we have (theorem 3.1 in \([57]\))

\[ K_t(x, y; g) \sim \frac{e^{D(D-1)/2} \Psi^{-1/2}_{x,y}(x, y)}{(4\pi \hbar t/2m)^{D/2}} d(D-1)/2 \sum_i \Psi_i^{-1/2}(x, y), \tag{A.20} \]
Appendix B. Existence of the Hamiltonian

Let $\Delta$ be a subset of the complex plane. A family $J(E)$, $E \in \Delta$ of bounded linear operators on the Hilbert space $\mathcal{H}$ under consideration, which satisfies the resolvent identity

$$J(E_1) - J(E_2) = (E_1 - E_2)J(E_1)J(E_2)$$

for $E_1, E_2 \in \Delta$ is called a pseudo-resolvent on $\Delta$ [58]. The following corollary (corollary 9.5 in [58]) gives the condition for which there exists a densely defined closed linear operator $A$ such that $J(E)$ is the resolvent family of $A$. Let $\Delta$ be an unbounded subset of $\mathbb{C}$ and $J(E)$ be a pseudo-resolvent on $\Delta$. If there is a sequence $E_n \in \Delta$ such that $|E_n| \to \infty$ as $n \to \infty$ and

$$\lim_{n \to \infty} -E_n J(E_n)x = x,$$

for all $x \in \mathcal{H}$, then $J(E)$ is the resolvent of a unique densely defined closed operator $A$. In order to show that the resolvent kernel that we have found in (21) corresponds to a unique densely defined closed operator $H$, we need to prove that it satisfies the resolvent identity, i.e.

$$R(x, y|E_1) - R(x, y|E_2) = (E_1 - E_2) \int_{\mathcal{M}} d^Dz R(x, z|E_1)R(z, y|E_2).$$

Substituting (21) into (B.3), we obtain

$$R_0(x, y|E_1) - R_0(x, y|E_2) + \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1)R_0(a_j, y|E_1)$$

$$= (E_1 - E_2) \int_{\mathcal{M}} d^Dz \left[ R_0(x, z|E_1)R_0(z, y|E_2) + \sum_{i,j=1}^{N} R_0(x, z|E_1)R_0(z, a_i|E_2)\Phi^{-1}_{ij}(E_2)R_0(a_j, y|E_2) + \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1)R_0(a_j, z|E_1)R_0(z, y|E_2) + \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} R_0(x, a_i|E_1)\Phi^{-1}_{ij}(E_1)R_0(a_j, z|E_1)R_0(z, y|E_2) \right].$$

Using formula (A.16), it is easy to see that the free resolvent satisfies the resolvent identity

$$(E_1 - E_2) \int_{\mathcal{M}} d^Dz R_0(x, z|E_1)R_0(z, y|E_2)$$

$$= (E_1 - E_2) \int_{\mathcal{M}} d^Dz \left[ \int_{0}^{\infty} \frac{dt_1}{h} K_{t_1}(x, z; g) \left( e^{iE_1t_1h/\hbar} \right) \int_{0}^{\infty} \frac{dt_2}{h} K_{t_2}(z, y; g) \left( e^{iE_2t_2h/\hbar} \right) \right]$$

$$= (E_1 - E_2) \int_{0}^{\infty} \frac{dt_1}{h} \int_{0}^{\infty} \frac{dt_2}{h^2} K_{t_1+t_2}(x, y; g) \left( e^{i(E_1+t_1h)E_2/\hbar} \right)$$

$$= \frac{(E_1 - E_2)}{2} \int_{0}^{\infty} \frac{du}{h} \left[ \int_{-\infty}^{\infty} \frac{dv}{h} K_{u}(x, y; g) \left( e^{iuE_1/2h} e^{ivE_2/2h} - e^{iuE_2/2h} e^{ivE_1/2h} \right) \right]$$

$$= \int_{0}^{\infty} \frac{du}{h} K_{u}(x, y; g) \left( e^{iuE_1/2h} - e^{iuE_2/2h} \right) = R_0(x, y|E_1) - R_0(x, y|E_2),$$

(B.5)
where we have used the semigroup property of the heat kernel (A.8) and made the change of variables \( u = t_1 + t_2, v = t_1 - t_2 \). Then, equation (B.4) becomes

\[
\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, y|E_1) = \sum_{i,j=1}^{N} R_0(x, a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2)
\]

\[
= (E_1 - E_2) \int d^{D}z \left[ \sum_{i,j=1}^{N} R_0(x, z|E_1)R_0(z, a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2) \right.
\]

\[
+ \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, z|E_1)R_0(z, y|E_2)
\]

\[
+ \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, z|E_1)
\]

\[
\times \sum_{i,j=1}^{N} \int d^{D}z R_0(x, z|E_1)R_0(z, a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2) \right].
\]

(B.6)

If we add and subtract the terms \( \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)R_0(a_j, y|E_1) = \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2) \) on the left-hand side of (B.6), and rearrange all the terms, we obtain

\[
\sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1)[R_0(a_j, y|E_1) - R_0(a_j, y|E_2)]
\]

\[
+ \sum_{i,j=1}^{N} R_0(x, a_i|E_1)[\Phi_{ij}^{-1}(E_1) - \Phi_{ij}^{-1}(E_2)]R_0(a_j, y|E_2)
\]

\[
+ \sum_{i,j=1}^{N} [R_0(x, a_i|E_1) - R_0(x, a_i|E_2)]\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2)
\]

\[
= (E_1 - E_2) \sum_{i,j=1}^{N} R_0(x, a_i|E_1)\Phi_{ij}^{-1}(E_1) \int_{M} d^{D}z R_0(a_j, z|E_1)R_0(z, y|E_2)
\]

\[
+ \sum_{i,j=1}^{N} R_0(x, a_i|E_1)[\Phi_{ij}^{-1}(E_1) - \Phi_{ij}^{-1}(E_2)]R_0(a_j, y|E_2) + (E_1 - E_2)
\]

\[
\times \sum_{i,j=1}^{N} \int_{M} d^{D}z R_0(x, z|E_1)R_0(z, a_i|E_2)\Phi_{ij}^{-1}(E_2)R_0(a_j, y|E_2),
\]

(B.7)

where we have used result (B.5) in the first and third terms. The second term can be written as

\[
\sum_{i,j=1}^{N} \sum_{k,l=1}^{N} R_0(x, a_i|E_1)\Phi_{ik}^{-1}(E_1) \left[ \Phi_{kj}(E_2) - \Phi_{kj}(E_1) \right] \Phi_{lj}^{-1}(E_2)R_0(a_j, y|E_2).
\]

(B.8)

It is important to note that the difference of the principal matrix equals the difference in the free resolvent kernel, that is,

\[
\Phi_{ij}(E_2) - \Phi_{ij}(E_1) = R_0(a_i, a_j|E_1) - R_0(a_i, a_j|E_2)
\]

\[
= (E_1 - E_2) \int_{M} d^{D}z R_0(a_i, z|E_1)R_0(z, a_j|E_2)
\]

(B.9)
for any $i$ and $j$. After substituting (B.9) into (B.8), equation (B.7) becomes

$$
(E_1 - E_2) \sum_{i,j=1}^{N} R_0(x, a_i | E_1) \Phi^{-1}_{ij}(E_1) \int_{M} d^2 y \ R_0(a_j, y | E_1) R_0(z, y | E_2)
$$

$$
+ (E_1 - E_2) \sum_{i,j=1}^{N} \sum_{l,k=1}^{N} R_0(x, a_i | E_1) \Phi^{-1}_{ij}(E_1)
$$

$$
\times \int_{M} d^2 y \ R_0(a_j, y | E_1) R_0(z, a_k | E_2) \Phi^{-1}_{kl}(E_2) R_0(a_l, y | E_2) + (E_1 - E_2)
$$

$$
\times \int_{M} d^2 y \ R_0(x, z | E_1) R_0(z, a_i | E_2) \Phi^{-1}_{ij}(E_2) R_0(a_j, y | E_2).
$$

(B.10)

This is exactly equal to (B.6) so the resolvent identity is satisfied. We must now imply the following condition in the $L^2$ norm:

$$
\|E_n R(E_n) f + f\| \to 0,
$$

(B.11)
as $n \to \infty$ and $f$ belongs to the Hilbert space $\mathcal{H}$ under consideration and the norm is taken with respect to $\mathcal{H}$. Let us choose the sequence $E_n = -nE_0$ since the resolvent is well defined in this resolvent set, in which we have no spectrum below the absolute value of the bound $E_0$ that we have found for the ground state energy. Without loss of generality, we can set $E_0 = c |E_i|$, where $c > 2$. Then, we have

$$
\|n E_0 R(-nE_0) f - f\| \to 0,
$$

(B.12)
as $n \to \infty$. Using (21) and separating the free part, we get

$$
\|n E_0 R(-nE_0) f - f\| \leq \|n E_0 R_0(-nE_0) f - f\|
$$

$$
+ n E_0 \|R_0(-nE_0) \Phi^{-1}(-nE_0) R_0(-nE_0) f\|.
$$

(B.13)

Since it is well known that the first part of the sum converges to zero as $n \to \infty$, that is, the free resolvent defines a densely defined closed operator (Laplacian), we investigate only the second term

$$
n E_0 \|R_0(-nE_0) \Phi^{-1}(-nE_0) R_0(-nE_0) f\|
$$

$$
\leq n E_0 \left[ \sum_{i,j=k,l=1}^{N} \int_{M} d^2 y \ R_0(a_j, y | -nE_0) R_0(x, a_i | -nE_0)
$$

$$
\times \int_{M} d^2 y \ R_0(a_j, y | -nE_0) R_0(y, a_k | -nE_0) \Phi^{-1}_{ij}(-nE_0) \| \Phi^{-1}_{kl}(-nE_0) \right]^{1/2},
$$

(B.14)

where we have used the fact that the Hilbert space norm of an operator is smaller than its Hilbert–Schmidt norm: $\|Af\| \leq Tr^{1/2}(A^* A)$ with $A = R_0(-nE_0) \Phi^{-1}(-nE_0) R_0(-nE_0)$. By using (A.16) and the change of variables $u = t_1 + t_2$, $v = t_1 - t_2$, we get

$$
\int_{M} d^2 y \ R_0(a_j, y | -nE_0) R_0(x, a_i | -nE_0) = \int_{0}^{\infty} \int_{0}^{\infty} \frac{d_1 d_2}{\hbar^2} K_{t_1+t_2}(a_j, a_i; g) e^{-\frac{|t_1-t_2| \hbar g}{\hbar^2}}
$$

$$
= \int_{0}^{\infty} \frac{dt}{\hbar^2} K_t(a_j, a_i; g) e^{-\frac{t \hbar g}{\hbar^2}}.
$$

(B.15)
Let us first consider the diagonal case $i = l$ and $k = j$ in the above. Then, equation (B.14) becomes

$$
n E_0 \left[ \sum_{i,j=1}^{N} \int_{M} d^D g \, R_0(a_i, x) - n E_0) R_0(x, a_i) - n E_0) \right]
\times \int_{M} d^D g \, R_0(a_j, y) - n E_0) R_0(y, a_j) - n E_0) \right|^{1/2}.
$$

(B.16)

In this case, the upper bound of equation (B.15) can be found from the upper bound of the heat kernel for compact manifolds

$$
\int_{M} d^D g \, R_0(a_j, y) - n E_0) R_0(x, a_j) - n E_0)
\leq \frac{4A \frac{1}{V(M) E_{1/2}^2 h^D}}{h^2 (\hbar / 2m)^D/2} \left( \frac{n E_0}{h} \right)^{2 - 2} \Gamma \left( 2 - \frac{D}{2} \right),
$$

(B.17)

and for Cartan–Hadamard manifolds

$$
\int_{M} d^D g \, R_0(a_j, y) - n E_0) R_0(x, a_j) - n E_0)
\leq \frac{C(\epsilon, \kappa)}{h^2 (4\pi h / 2m)^{D/2}} \left( \frac{n E_0}{h} \right)^{2 - 2} \Gamma \left( 2 - \frac{D}{2} \right).
$$

(B.18)

In order to give the upper bound for the inverse principal matrix, we decompose the principal matrix into two positive matrices:

$$
\Phi = D - K,
$$

(B.19)

where $D$ and $K$ stand for the diagonal and off-diagonal parts of the principal matrix. Then, it is easy to determine $\Phi = D(1 - D^{-1} K)$. The principal matrix is invertible if and only if $1 = D^{-1} K$, and $(1 - D^{-1} K)$ has an inverse if the norm $\| D^{-1} K \| < 1$. Then, we write the inverse of $\Phi$ as a geometric series:

$$
\Phi^{-1} = (1 - D^{-1} K)^{-1} D^{-1}
= (1 + (D^{-1} K) + (D^{-1} K)^2 + \cdots) D^{-1},
$$

(B.20)

where we must have $\| D^{-1} K \| < 1$. Since we are not concerned with the sharp bounds on $\Phi^{-1}$ for this problem, we can choose $\| D^{-1} K \| < 1/2$ by adjusting the $n E_0$ sufficiently large without loss of generality and get

$$
|\Phi^{-1}| \leq 2|D^{-1}|.
$$

(B.21)

The lower bound of the diagonal principal matrix for compact (107) and Cartan–Hadamard manifolds (120) gives the upper bound of the inverse principal matrix. Hence, we find

$$
|\Phi^{-1}_{ji}(-n E_0)| \leq \begin{cases} 
\frac{(4\pi h^2 / 2m) \ln^{-1}(n E_0 / \mu^2)}{2\sqrt{\pi}} \left( \frac{n E_0}{h} \right)^{-1} & \text{if } D = 2 \\
\frac{h(4\pi h / 2m)^{3/2}}{2\sqrt{\pi}} \left( \frac{n E_0}{h} \right)^{-1} \left( \frac{\mu^2}{h} \right)^{-1} & \text{if } D = 3
\end{cases}
$$

(B.22)

for compact manifolds and

$$
|\Phi^{-1}_{ji}(-n E_0)| \leq \begin{cases} 
\frac{(4\pi h^2 / 2m)}{c} \ln^{-1} \left( \frac{n E_0}{h^2} \right) \left( \frac{n E_0}{h^2} + \xi \right)^{-1} & \text{if } D = 2 \\
\frac{h(4\pi h / 2m)^{3/2}}{2\sqrt{\pi} c} \left( \frac{n E_0}{h} + \xi \right)^{-1} \left( \frac{\mu^2}{h} + \xi \right)^{-1} & \text{if } D = 3
\end{cases}
$$

(B.23)
for Cartan–Hadamard manifolds. If we substitute the results (B.17) and (B.18) into (B.16) for $D = 2$, and take the limit $n \to \infty$, the result goes to zero. Since the norm is always positive, we prove

$$\|nE_0 R(-nE_0) f - f\| \to 0$$

as $n \to \infty$. It is almost evident that the off-diagonal terms in the sum also vanishes in this limit primarily because these terms are exponentially damped $e^{-\sqrt{n}}$ due to the upper bound of the heat kernel. Unfortunately, the proof for $D = 3$ is more subtle and the volume growth conditions of the manifolds are very delicate in the analysis so we will not be able to prove it for the three-dimensional case by the same approach. The flat case can be done by the Fourier transform.

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