Signatures of Noise Enhanced Stability in Metastable States

A. Fiasconaro$^1$, B. Spagnolo$^1$ and S. Boccaletti$^2$

$^1$ Dipartimento di Fisica e Tecnologie Relative and INFM-CNR, Group of Interdisciplinary Physics$^1$, Università di Palermo, Viale delle Scienze pad. 18, I-90128 Palermo, Italy

$^2$ Istituto dei Sistemi Complessi del CNR, Sezione di Firenze, Via Madonna del Piano, Sesto Fiorentino, Florence, Italy

The lifetime of a metastable state in the transient dynamics of an overdamped Brownian particle is analyzed, both in terms of the mean first passage time and by means of the mean growth rate coefficient. Both quantities feature non monotonic behaviors as a function of the noise intensity, and are independent signatures of the noise enhanced stability effect. They can therefore be alternatively used to evaluate and estimate the presence of this phenomenon, which characterizes metastability in nonlinear physical systems.

PACS numbers: 05.40-a,87.23Cc,89.75-k

Metastability is a generic feature of many nonlinear systems, and the problem of the lifetime of a metastable state involves fundamental aspects of non-equilibrium statistical mechanics. Metastable states, indeed, have been proved to play a crucial role e.g. in protein folding dynamics, Ising spin glasses, complex dynamics of large molecules at surfaces, enhancement of cellular memory, and in dynamics of cellular reactive oxygen species [1]. The problem of the lifetime of a metastable state has been addressed in a variety of areas, including first-order phase transitions, Josephson junctions, field theory and chemical kinetics [2].

Recently, the investigation of nonlinear dynamics in the presence of external noisy sources led to the discovery of some resonance-like phenomena, among which we recall stochastic resonance [3], resonant activation [4], and noise enhanced stability (NES) [5, 6, 7]. All these phenomena are characterized by a nonmonotonic behavior of some quantity as a function of the forcing noise intensity or the driving frequency, that reflects a constructive and counterintuitive effect of the noise acting on the nonlinear system. In particular, several theoretical studies have shown that the average escape time from a metastable state in fluctuating and static potentials has a non monotonic behavior as a function of the noise intensity [8, 9, 10, 11]. This resonance-like behavior, which contradicts the monotonic behavior predicted by the Kramers theory [10, 11], is called the NES phenomenon: the stability of metastable or unstable states can be enhanced by the noise and the average life time of the metastable state is larger than the deterministic decay time. Furthermore, if more realistic noise sources (such as colored noise with a finite correlation time) are considered, the value of the noise intensity at which the maximum of the average escape time occurs is even larger than that corresponding to the white noise case, meaning that the NES effect could be easy to measure experimentally, because of the finite time correlations involved in any realistic noisy source [8].

When considering a Brownian particle in the presence of a metastable potential, the NES effect is always obtained [3, 5], regardless on the unstable initial position of the particle. More precisely, two different dynamical regimes occur: one is characterized by a non monotonic behavior of the average escape time, as a function of noise intensity, and the other features a divergence of the mean escape time when the noise intensity tends to zero, implying that the Brownian particle remains trapped within the metastable state in the limit of small noise intensities. The description of the transition from one dynamical regime to the other is yet an open question.

In this Letter we analyze in more detail the divergent dynamical regime, and suggest an approach for detecting the stability of metastable states, that is alternative to the mean-first-passage-time (MFPT) technique. In particular, we find a non monotonic behavior of the MFPT with a minimum and a maximum as a function of the noise intensity for initial positions of the Brownian particle close to the point $x_c$, where the potential shape intersects the $x-$axis (see the inset of Fig. 1). In this regime the standard deviation of the escape time has a divergent behavior for small noise intensity. This means that the average escape time, that is a quantitative measure of the average life time of the metastable state, has in fact some statistical limitations to fully describe the stability of metastable states.

To complement the analysis of the transient dynamics of metastable states, we then introduce a different approach, based on the evaluation of the mean growth rate coefficient $\Lambda$ as a function of the noise intensity. The $\Lambda$ coefficient is evaluated by the use of a procedure similar to that for the calculation of the Lyapunov exponent in stochastic systems, e.g. we consider the evolution of the separation $\delta x(t)$ between two neighboring trajectories of the Brownian particle [12, 13, 14]. We will show that

--

$^*$E-mail address: afiasconaro@gip.dft.unipa.it

$^1$http://gip.dft.unipa.it
also the $\Lambda$ coefficient displays a non monotonic behavior (with a clear minimum), as a function of the noise intensity, thus representing an independent way for detecting and estimating the NES effect.

The starting point of our analysis is a Brownian particle in one spatial dimension, obeying the following Langevin equation

$$\dot{x} = -\frac{dU(x)}{dx} + \sqrt{D}\xi(t),$$  

where $\xi(t)$ is a white Gaussian noise source with zero mean and $\delta$-correlated in time ($\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t+\tau) \rangle = \delta(\tau)$), and $U(x) = 0.3x^2 - 0.2x^3$ is a cubic potential, whose shape is shown in the inset of Fig. 1. The potential profile has a local stable state at $x_s = 0$, an unstable state at $x_u = 1$, and intersects the $x$-axis at $x_c = 1.5$.

After fixing a given target position $x_F > x_c$, the MFPT $\tau(x_0, x_F)$ (the average time for a particle starting from an initial position $x_0$ to reach $x_F$) is given by the closed analytical expression

$$\tau(x_0, x_F) = \frac{2}{D} \int_{x_0}^{x_F} e^{2u(z)} \int_{-\infty}^{z} e^{-2u(y)} \, dy \, dz,$$  

where $u(x)$ is the dimensionless potential profile, obtained by normalizing $U(x)$ to the noise intensity $D$ ($u(x) = \frac{U(x)}{D}$). The double integral can be evaluated (in part analytically, in part numerically), giving rise to

$$\tau(x_0, x_F) = \frac{2}{D} \int_{x_0}^{x_F} e^{2u(z)} G(z) \, dz,$$

where $G(z) = 0.6046 \, e^{-z[I_{-1/3}(z) + I_{1/3}(z)]} - \frac{1}{2} F_2(\frac{1}{3}, \frac{1}{3}; 1; -2z) + \int_{-\infty}^{z} e^{-2u(y)} \, dy$, $z = 1/(10D)$, $I_{\nu}(z)$ is the modified Bessel function of the first kind and $F_2(a_1, a_2; b_1, b_2; z)$ is the generalized hypergeometric function.

In Fig. 1 the evaluation of expression for $x_F = 2.2$ and for different initial positions $x_0$ (as sketched in the legend) is reported. Two different regimes can be observed, depending on the initial state of the particle: (i) the NES effect for all initial conditions $x_0 > x_c$, and (ii) the divergent regime for $x_u < x_0 < x_c$, where $x_u = 1$ is the location of the relative maximum of the potential profile.

In this latter regime, for all initial positions $x_0$ smaller than (but sufficiently close to) $x_c$, the MFPT displays a non monotonic behavior with a minimum and a maximum. For very low noise intensities, the Brownian particle is trapped into the potential well, as a consequence of the divergence of the MFPT in the limit $D \to 0$. For increasing noise intensity, the particle can escape out more easily, and the MFPT decreases. As the noise intensity reaches a value $D \approx \Delta U = 0.1$ (corresponding to the potential barrier height), the concavity of the MFPT curves changes. Close to such a noise intensity, the escape process of the Brownian particle is slowed down, due to the fact that the probability to reenter the well is increased. At higher noise intensities one recovers a monotonic decreasing behavior of the MFPT. In summary, from inspection of Fig. 1, the behavior of MFPT $\tau$ vs. $D$ goes with continuity from a monotonic divergent behavior to a non monotonic finite behavior (typical NES effect), passing through a non monotonic divergent behavior with a minimum and a maximum.

In the following we will focus on this last dynamical regime. First of all, we compare the theoretical results of Fig. 1 with direct numerical simulations of Eq. 1. Namely, we numerically integrate Eq. 1 with different initial conditions $x_0 \lesssim x_c$. For each initial condition (and for each value of the noise intensity $D$), the integration is performed over an ensemble of $N_R = 350,000$ different realizations of the white noise process $\xi(t)$. The target position $x_F = 2.2$ is selected, and the MFPT (calculated as the ensemble average of the first passage times through $x_F$ for different noise realizations) is reported in Fig. 2 together with the corresponding theoretical curves.
agreement between theoretical predictions and numerical simulations is very good. For low values of \( D \), however, we cannot reproduce the theoretical curves because of the finiteness of the integration time (\( T_{\text{max}} = 20,000 \text{ a.u.} \)) and of the number of realization. The MFPT evaluated in this range of noise intensities, therefore, tends to the deterministic escape time.

The numerical study of Eq. (1) allows also to calculate the standard deviation \( \sigma \) of the set of first passage times obtained for different noise realizations. The results are shown in the inset of Fig. 2 for the same values of \( x_0 \), where a divergent behavior of \( \sigma(D) \) is visible in the limit \( D \to 0 \). Such a feature confirms that the only information on the MFPT is not sufficient to fully unravel the statistical properties of this dynamical regime, and motivated our search for a complementary approach.

This is done by monitoring the properties of the mean growth rate coefficient \( \Lambda \). Let \( \delta x_0 = \delta x(t = 0) \ll 1 \) be the initial separation of two neighboring Brownian particles subjected to the same noise process \( \xi(t) \). By linearization of Eq. (1), the evolution of the particle separation \( \delta x(t) \) is given by

\[
\delta x(t) = -\frac{d^2U(x)}{dx^2} \delta x(t) = \lambda_i(x, t) \delta x(t),
\]

and allows for the definition of an instantaneous growth rate \( \lambda_i(x, t) \). It is important to stress that, in Eq. (4), \( \frac{d^2U(x)}{dx^2} \) is calculated onto (and as so, it is a function of) the noisy trajectory \( x[\xi(t)] \). The growth rate coefficient \( \Lambda_i \) (for the \( i \)th noise realization), is then defined as the long-time average of the instantaneous \( \lambda_i \) coefficient over \( \tau \), 

\[
\Lambda_i = \frac{1}{\tau(x_0, x_F)} \int_0^\tau(x_0, x_F) \lambda_i(x, s) ds.
\]

Notice that, in the limit \( \tau(x_0, x_F) \to \infty \), Eq. (3) coincides formally with the definition of the maximum Lyapunov exponent, and that, therefore, our \( \Lambda_i \) coefficient has the meaning of a finite-time Lyapunov exponent, since we are interested in characterizing a transient dynamics. The mean growth rate coefficient (MGRC) \( \Lambda \) is then obtained by ensemble averaging the \( \Lambda_i \) coefficients over the \( N_R \) different noise realizations (\( \Lambda = \frac{1}{N_R} \sum_{i=1}^{N_R} \Lambda_i \)).

In our simulations of Eq. (1), an integration time step of \( dt = 0.0001 \), and an initial condition \( \delta x_0 = 0.01 \) are used. The instantaneous growth rate \( \lambda_i(x, t) \) is calculated over \( N_r \) small subintervals of the trajectory, each of them corresponding to a time interval \( \tau = 5dt \).

In Fig. 3, we report the behavior of \( \lambda_i(x, t) \) as a function of time for \( D = 1 \). The \( \lambda_i(x, t) \) coefficient is negative for most of the time, due to noise-induced local stability of metastable state. As the Brownian particle reaches the point at which the potential changes its concavity, \( \lambda_i(x, t) \) becomes positive. As for the \( \Lambda \) coefficient, fixing \( x_F = 20.0 \), it displays a non monotonic behavior with a minimum as a function of noise intensity (see Fig. 4). Though being always positive, this reflects the fact that the Brownian particles always escape in average from the metastable state, but the non monotonic behavior of the MGRC marks the presence of the NES effect. In particular, the closer is the initial position of the particles to \( x_u \), the smaller is the MGRC. Notice that \( \lambda_i(x, t) \) is proportional to the trajectory of the Brownian particle because the potential is a cubic one. Any metastable state, however, can be described through a local cubic potential even if the real potential has other local or global stable states.

Fig. 4 is a comparative plot of the numerically calculated MFPT (a) and the corresponding MGRC (b) vs. \( D \) for various initial positions of the Brownian particles. The MFPT’s plot [Fig. 4 (a)] shows the divergent behavior as a maximum, which is shifted towards lower
external noise, and, as so, they can be alternatively used to track the lifetime of metastable states in the presence of activation processes in complex systems characterized by energy landscape $\mathcal{V}$. These latter ones contribute to lower down the value of $\Lambda$, due to the negative contributions of the $\lambda_i$’s that corresponds to such positions (see Fig. 3).

In conclusion, we investigated the average escape time from a metastable state in a cubic potential profile for different initial unstable positions of the Brownian particles. The two introduced measures (the behavior of the MFPT and of the MGRC) furnish suitable tools to detect the lifetime of metastable states in the presence of external noise, and, as so, they can be alternatively used in many relevant circumstances, such as understanding activation processes in complex systems characterized by energy landscape $\mathcal{V}$.

This work was supported by MIUR-FIRB and INFM-CNR.

FIG. 4: (a) Numerical estimation of the MFPT for various initial positions (as in the legend) and (b) the corresponding MGRC vs. the noise intensity $D$. The average has been performed over 20,000 realizations. The absorbing boundary is $x_F = 20.0$, and the maximum waiting time used in the simulations is $T_{max} = 20,000$.

values of the noise intensity. Because of the finiteness of the ensemble of particles considered in our numerical experiments, observation of the divergence in time is prevented in our simulations, and for very low noise intensities ($D \to 0$) the deterministic escape time is retrieved. As it can be seen in Fig. 4 (b), the maximum of the MFPT curve is reflected by a local minimum in the MGRC shape, which is however slightly shifted towards lower values of the noise intensity. This is because, at low noise intensities, not all the particles coming back into the potential well reach positions around the metastable state with positive concavity, that instead will be attained by particles experiencing larger noise driving. These latter ones contribute to lower down the value of $\Lambda$, due to the negative contributions of the $\lambda_i$’s that corresponds to such positions (see Fig. 3).

In conclusion, we investigated the average escape time from a metastable state in a cubic potential profile for different initial unstable positions of the Brownian particles. The two introduced measures (the behavior of the MFPT and of the MGRC) furnish suitable tools to detect the lifetime of metastable states in the presence of external noise, and, as so, they can be alternatively used in many relevant circumstances, such as understanding activation processes in complex systems characterized by energy landscape $\mathcal{V}$.

This work was supported by MIUR-FIRB and INFM-CNR.