Divergences in the Effective Action for Acausal Spacetimes

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Abstract

The 1–loop effective Lagrangian for a massive scalar field on an arbitrary causality violating spacetime is calculated using the methods of Euclidean quantum field theory in curved spacetime. Fields of spin $\frac{1}{2}$, spin 1 and twisted field configurations are also considered. In general, we find that the Lagrangian diverges to minus infinity at each of the $n$th polarised hypersurfaces of the spacetime with a structure governed by a DeWitt–Schwinger type expansion.
1 Introduction

If one attempts to quantise fields on acausal spacetimes, one inevitably runs into awkward problems of interpretation. Quantities that are well defined in globally hyperbolic spacetimes can become ambiguous in geometries where strong causality is violated. For example, the first attempt to construct an interacting quantum field theory on a Morris, Thorne, Yurtsever\cite{1} type wormhole spacetime found that the $S$ matrix was nonunitary when the state evolved through the region of closed timelike curves (CTCs)\cite{2}.

One also encounters problems with the definition of a suitable Green function. Consider the (normally well defined) commutator of two free field operators $iD(x,y) = [\phi(x),\phi(y)]$. In an acausal spacetime, even if $x$ and $y$ are locally spacelike separated, it is not clear that $D = 0$ because there may be a large timelike loop connecting the two points, due to the nontrivial homotopy of the spacetime.

Various attempts to circumvent these problems in a consistent way have been suggested\cite{3,4,5,6,7,8}. This paper is concerned with the Euclidean approach, proposed in a recent paper by Hawking\cite{3}. Motivation for this proposal comes from the simple observation that in Euclidean space, there are no CTCs and in particular, no closed or self-intersecting null geodesics. Therefore, if one considers a Euclidean space $M_E$ which has some acausal Lorentzian section $M_L$, then the appropriate analytic continuation of quantities that are well defined on $M_E$ should give unambiguous results valid on the chronology violating section.

The object of this paper is to apply the methods of Euclidean quantum field theory in curved spacetime to derive a 1–loop effective action for fields of arbitrary mass and spin in a typical causality violating spacetime. Accordingly, section 2 is devoted to a discussion of multiply connected Euclidean spaces and their universal covering spaces. Section 3 reviews all the necessary theory of the heat operator, most notably its divergence structure and asymptotic expansion for multiply connected spacetimes. This expansion is then used in section 4 to derive the 1–loop effective Lagrangian for a massive scalar field, renormalised by the point splitting method. The corresponding expressions for
twisted configurations and fields of spin $\frac{1}{2}$ and spin 1 are also obtained. In section 4, these results are applied to a number of interesting chronology violating spacetimes, including Grant’s generalisation of Misner space [9] and the wormhole spacetime studied by Kim and Thorne [11]. The relevance of these results for chronology protection [12] is discussed in section 6.

2 Multiply connected Euclidean spaces

Consider an arbitrary multiply connected Euclidean spacetime, $M_E$. This spacetime is just the quotient space

$$M_E = \frac{\bar{M}_E}{\Gamma}$$

(1)

where $\bar{M}_E$ is the simply connected universal covering space and $\Gamma$ is a properly discontinuous, discrete group of isometries of $\bar{M}_E$. $\Gamma$ is isomorphic to the fundamental group of $M_E$, $\pi_1(M_E)$ and $M_E$ is obtained from $\bar{M}_E$ by identifying points equivalent under $\Gamma$. If $\pi_1(M_E) = \mathbb{Z}_\infty$, then the fundamental domains (i.e. the copies of $M_E$ in $\bar{M}_E$) can be labelled by a single integer $n$, usually interpreted as a winding number. Copies of the point $p \in M_E$ in the covering space are labelled $\bar{p}_n \in \bar{M}_E$, where the points $\bar{p}_n$ are obtained by repeated application of $\Gamma$ to the right, i.e. $\bar{p}_n = \bar{p}_0 \gamma_n$.

It will be useful to have a concrete example to refer to throughout the paper. Therefore, we shall consider the Euclidean section of Grant space [4]. Grant space is just flat Minkowski space with points identified under a combined boost in the $(x, t)$ plane and translation in the $y$ direction. It is the universal covering space of the Gott spacetime [10], which describes two cosmic strings passing each other with a constant velocity. The appropriate Euclidean section is flat Euclidean space with points identified under a combined rotation plus a translation in the orthogonal direction, as before. In other words, for an arbitrary point $\bar{q} = (\tau, r, \theta, z)$, the effect of acting on $\bar{q}$ by $\gamma_n$ is just

$$\bar{q}_n = \bar{q} \gamma_n = (\tau + n\beta, r, \theta + n\alpha, z).$$

(2)

One recovers the Lorentzian Grant space by analytically continuing the rotation parameter to a boost ($\alpha \rightarrow a = i\alpha$).
A fundamental quantity of interest is $\sigma_n(p, \{\beta_E\}) = \sigma(p, \bar{p}_n)$, which gives the squared distance along the spacelike geodesic connecting $p \in M_E$ to itself with winding number $n$. $\{\beta_E\}$ collectively denotes various metric parameters, which relate equivalent points in the covering space. On the Euclidean section of Grant space, one would have

$$\sigma(q, q'_{n'}) = (\tau - \tau' - n\beta)^2 + r^2 + r'^2 - 2rr' \cos(\theta - \theta' - n\alpha) + (z - z')^2$$

so that

$$\sigma_n(q, \{\alpha, \beta\}) = 2r^2 \left(1 - \cos(n\alpha)\right) + n^2 \beta^2.$$  \hspace{1cm} (4)

On the Euclidean section $M_E$, provided the parameters $\{\beta_E\}$ are nonzero, the equation

$$\sigma_n(p, \{\beta_E\}) = 0$$  \hspace{1cm} (5)

can (in general) only be satisfied if $n = 0$. However, if one considers analytically continuing any of the metric parameters to imaginary values, thus obtaining a Lorentzian spacetime $M_L$ with parameters $\{\beta_L\}$, then one may be able to find solutions to the equivalent of (3) for all values of $n$. In that case, the point $p$ would be joined to itself by a (self-intersecting) null geodesic with winding number $n$. One can define the $n$th polarised hypersurface as the set of points $\{p \in M_L : \sigma_n(p, \{\beta_L\}) = 0\}$. The Cauchy horizon is just the limit of this family of surfaces as $n \to \infty$. One may easily verify that by setting equation (4) equal to zero and then analytically continuing $\alpha \to a = i\alpha$, one obtains the criterion for polarised hypersurfaces in Grant space (see 3).

3 The Heat Operator

As a first step towards obtaining the effective Lagrangian, we need to examine the structure of the heat operator defined on $M_E$. Heat operators on Riemannian manifolds have been extensively studied, so here we only review the most relevant properties. For further technical details, the reader is referred to the paper by Wald 15 and its associated references. Quantum field theory on multiply connected spacetimes is discussed in Dowker 13 and Dowker and Banach 14.
We begin by considering the 'wave operator' \( A = -\nabla^2 + m^2 \), defined on the dense domain \( C_0^\infty(M_E) \) of smooth \( (C^\infty) \) functions of compact support. \( A \) is a symmetric operator on \( L^2(M_E) \), the Hilbert space of square integrable functions on \( M_E \). However, to do quantum theory we need a self-adjoint operator so we must extend this domain of definition in an appropriate way. Since \( A \) is positive on its initial domain, standard theory states that positive self-adjoint extensions must exist. The only problem is that \( A \) is unbounded, so there may be more than one possible extension. However, if \( M_E \) is a complete manifold without boundary, then Gaffney [18] has shown that \( A \) has a unique self-adjoint extension, defined as the closure of \( A \) and denoted by \( \mathcal{A} \). The domain of \( \mathcal{A} \) is just the Cauchy completion of \( \text{dom}(A) \) in the norm \( \| \psi \|_2 + \| A\psi \|_2 \), for \( \psi \in L^2(M_E) \). This property of the wave operator is known as essential self-adjointness and also holds for some incomplete manifolds, such as Euclidean space with a point removed. It does not hold for manifolds with boundaries or most manifolds with singularities. In this paper, we shall always assume that \( A \) is essentially self-adjoint.

The heat operator is defined as
\[
e^{-\tau A} = \int e^{-\tau \lambda} dE_\lambda
\] (6)
where \( E_\lambda \) is the spectral family of \( A \). Once \( e^{-\tau A} \) has been constructed, one can apply the functional calculus of self-adjoint operators [19] to obtain mathematically well-defined expressions for quantities of physical interest. If one considers the 1-parameter family of integrals
\[
K(s) = \int_0^\infty e^{-\tau A} \tau^{s-1} d\tau,
\] (7)
then \( K(1) \) and \( K(0) \) are particularly interesting. \( K(1) = A^{-1} \) defines the Feynman propagator and \( K(0) \) is related to \( \ln A \) (the effective Lagrangian) by
\[
\ln A = \lim_{\epsilon \to 0} \left( -\int_\epsilon^{\infty} e^{-\tau A} \frac{d\tau}{\tau} + (\gamma - \ln \epsilon)I \right)
\] (8)
where \( I \) is the identity operator and Euler’s constant \( \gamma = \int_0^{\infty} e^{-x} \ln x dx \).

It is well known that for \( \tau > 0 \), the heat operator is given by a smooth integral kernel \( H(\tau, x, x') \). Thus the only possible divergences in (7) which could prevent \( K(s) \)
from being given by a smooth integral kernel $K(s, x, x')$ are those which could arise as $\tau \to \infty$ (infra–red divergences) or $\tau \to 0$ (ultra–violet). Infra–red divergences can only occur if the field mass $m = 0$ and $M_E$ is noncompact. Since we are considering the massive scalar field, we shall not worry about these divergences. We shall be more concerned with the ultra–violet divergence structure, which is completely determined by the asymptotic expansion of $H(\tau, x, x')$ about $\tau = 0$

$$H(\tau, x, x') = \frac{\Delta^{\frac{1}{2}}(x, x')}{(4\pi \tau)^\frac{d}{2}} e^{-m^2 \tau} e^{-\sigma(x, x')/4\tau} \sum_{j=0}^{N} a_j(x, x') \tau^j$$

(9)

The coefficients $a_j(x, x')$ are recursively obtained and depend on local geometric quantities. $\sigma(x, x')$ was defined earlier as the square of the geodesic distance between $x$ and $x'$ and $d = \dim(M)$. $\Delta(x, x') = -\det(-\sigma_{\mu\nu})$ is the Van–Vleck determinant.

One can see that if $x \neq x'$, the factor $e^{-\sigma(x, x')/4\tau}$ ensures that $H$ vanishes as $\tau \to 0$ faster than any power of $\tau$. Therefore, provided there are no infra–red divergences, $K(s)$ is given by an integral kernel $K(s, x, x')$ which can only be singular when $x = x'$.

In a multiply connected spacetime, one can express $H(\tau, x, x')$ in terms of $\overline{H}$, the heat kernel defined on the universal covering space. The most general relation is

$$H(\tau, x, x') = \sum_{\gamma} a(\gamma) \overline{H}(\tau, x, x'\gamma)$$

(10)

where $a(\gamma)$ is a unitary, 1–dimensional representation of $\Gamma$ (i.e. $a(\gamma_1)a(\gamma_2) = a(\gamma_2\gamma_1)$). Note that from now on, points in the covering space have no bars on them as the distinction between $x \in M_E$ and $x \in \overline{M}_E$ should be clear. If $\Gamma = \pi_1(M_E) = Z_\infty$, one can write

$$H(\tau, x, x') = \sum_{n=-\infty}^{\infty} a(\gamma_n) \overline{H}(\tau, x, x'\gamma_n)$$

(11)

where $a(\gamma_n) = e^{2\pi i n \delta}$ and $0 \leq \delta \leq \frac{1}{2}$. In general, for real fields $a(\gamma_n)$ must be real also, so that $a(\gamma_n)$ can only take the values $\pm 1$, where the negative value would correspond to a twisted field configuration [16, 17]. For the moment, we take $a(\gamma_n) = 1$ for simplicity, and note that this decomposition can be suitably extended to the family of integrals
\( K(s) \), so that
\[
K(s, x, x') = \sum_{n=-\infty}^{\infty} K(s, x, x_0 \gamma_n)
\]

4 The Effective Action

The effective action of the quantum field, \( S \), is related to the operator \( A \) by
\[
e^{-S} = (\det A)^{-\frac{1}{2}} = e^{-\frac{1}{2} \text{tr}(\ln A)}.
\]

One would like to represent \( \ln A \) by an integral kernel \( L(x, x') \), so that
\[
S = \frac{1}{2} \int L(x, x) g^{\frac{1}{2}} d^4 x.
\]

One could then obtain the energy–momentum tensor by functionally differentiating the effective Lagrangian \( \mathcal{L}(x) = \frac{1}{2} L(x, x) \) with respect to the metric \( g_{\mu\nu} \). However, from the above discussion of ultra–violet divergences in \( K(s, x, x') \), it is clear that \( \ln A \) will be singular at \( x = x' \). We must therefore adopt some renormalisation prescription.

In the point–splitting approach \([21, 22]\), one first considers the quantity
\[
L(x, x') = -\int_{0}^{\infty} H(\tau, x, x') \frac{d\tau}{\tau}
\]
which is well defined for \( x \neq x' \). In 4 dimensions, the divergences which occur as the limit \( x' \to x \) is taken are governed entirely by the first 3 terms of the asymptotic expansion \([9]\). We therefore obtain a finite, renormalised \( L(x, x') \) by subtracting this divergent part from \( L(x, x') \) before taking the coincidence limit.

For a multiply connected spacetime \( M_E \), we have
\[
L(x, x') = -\sum_{n=-\infty}^{\infty} \int_{0}^{\infty} \mathcal{H}(\tau, x, x'_n) \frac{d\tau}{\tau}.
\]

The \( \tau \) integration is performed with the help of the definite integral \([20]\)
\[
\int_{0}^{\infty} x^{\nu-1} e^{-\frac{\beta}{x} - \gamma x} \, dx = 2 \left( \frac{\beta}{\gamma} \right)^{\frac{\nu}{2}} K_{\nu} \left( 2 \sqrt{\beta \gamma} \right)
\]
where \( K_\nu \) is the modified Hankel function \([20]\). The contribution to \( L(x, x') \) from the first 3 terms in the series is

\[
L(x, x') = - \frac{1}{(4\pi)^2} \sum_{n=-\infty}^{\infty} \Delta^\frac{1}{2}(x, x_n) \left[ 8 \left( \frac{m^2}{\sigma(x, x_n')} \right) K_2 \left( m\sqrt{\sigma(x, x_n')} \right) \right.
\]

\[
+ 4a_1(x, x'_n) \left( \frac{m^2}{\sigma_n} \right)^{\frac{1}{2}} K_1 \left( m\sqrt{\sigma_n} \right) + a_{2n} \int_0^\mu e^{-m^2\tau - \frac{\sigma_n}{4\tau}} d\tau + O(\sigma_n) \bigg] \tag{18}
\]

where the cutoff \( \mu \) is to prevent infra–red divergences in the massless limit. The first coefficient \( a_0 = 1 \) for scalar fields.

Recalling the discussion in section 2, the only value of \( n \) for which \( \sigma(x, x_n) \to 0 \) as \( x' \to x \) is \( n = 0 \). To renormalise therefore, one drops the \( n = 0 \) term in \( L(x, x') \) and takes the coincidence limit to obtain

\[
-2\mathcal{L}(x) = -L(x, x) = \frac{1}{(4\pi)^2} \sum_{n=-\infty \atop n \neq 0}^{\infty} \Delta^\frac{1}{2}(x, x_n) \left[ 8 \left( \frac{m^2}{\sigma_n} \right) K_2 \left( m\sqrt{\sigma_n} \right) \right. \\
\]

\[
+ 4a_1(x, x'_n) \left( \frac{m^2}{\sigma_n} \right)^{\frac{1}{2}} K_1 \left( m\sqrt{\sigma_n} \right) + a_{2n} \int_0^\mu e^{-m^2\tau - \frac{\sigma_n}{4\tau}} d\tau + O(\sigma_n). \bigg] \tag{19}
\]

It is now clear what happens if one can analytically continue any of the metric parameters to obtain an acausal Lorentzian section. The quantity \( \sigma(x, x_n) \) goes to zero at each of the \( n \)th polarised hypersurfaces and hence the renormalised effective Lagrangian diverges.

The first point to note is that equation (19) should be understood as a formal expression only. For the purposes of practical calculation, the first term gives the Gaussian approximation \([23]\) to the effective action which is only exact in special cases, namely flat space and the Einstein universe. In a more general spacetime, one has to consider the coefficients \( a_i \) for \( i > 0 \). The covariant expansion of these coefficients in terms of the geodetic interval can be found in the papers by Christensen \([21, 22]\). Consider the expansion of the first nontrivial coefficient \( a_1 \) for a scalar field

\[
a_1(x, x') = \left( \frac{1}{6} - \xi \right) R - \frac{1}{2} \left( \frac{1}{6} - \xi \right) R_{\alpha\sigma} + (\ldots) \sigma^\alpha_{\alpha\beta} + \ldots \tag{20}
\]
where $\sigma_\alpha = \partial_\alpha \sigma$. Normally, one would have $\sigma^\alpha \to 0$ as the coincidence limit is taken, which leaves a simple finite expression for the coefficient of interest. However, if identifications have been made, the quantity $\sigma^\alpha$ does not necessarily go to zero as the points are brought together. Hence we are left with an infinite number of terms which may or may not converge. However, for most purposes one would only be interested in the strongest divergence, which is given by the Gaussian approximation

$$-\mathcal{L}(x) = \frac{a_0}{2\pi^2} \sum_{n=1}^{\infty} \left( \frac{m^2}{\sigma_n} \right) \Delta_n \frac{1}{4} K_2 \left( m \sqrt{\sigma_n} \right).$$  \hspace{1cm} (21)

A twisted real scalar field configuration can be considered by including a factor of $(-1)^n$ in the Lagrangian. In this case, the contributions from twisted and untwisted fields cancel at odd numbered polarised hypersurfaces, but reinforce at even numbered ones.

For higher spin fields, the only factor which changes in this expression is the coefficient $a_0$. For spin $\frac{1}{2}$, $a_0$ is given by the unit spinor, whose trace is just the number of spinor components (*i.e.* the dimension of the gamma matrices used). For spin 1 fields, $a_0$ has four components and is just the metric tensor $g_{\mu\nu}$ (in the Feynman gauge). The ghost Lagrangian is given by minus twice the scalar Lagrangian, because one would have to consider two anticommuting scalar ghost fields. Here, the ghost contribution would cancel with two of the vector field components so overall, the spin $\frac{1}{2}$ and spin 1 Lagrangians would still diverge to minus infinity at the polarised hypersurfaces.

### 5 Examples

In flat space, we can obtain an exact result. The Van–Vleck determinant $\Delta(x, x') = 1$ and the only nonzero coefficient is $a_0 = 1$, so for Euclidean space identified under a combined rotation and orthogonal translation, one obtains

$$-\mathcal{L}(x) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \left( \frac{m^2}{\sigma_n} \right) K_2 \left( m \sqrt{\sigma_n} \right)$$  \hspace{1cm} (22)
which in the massless limit becomes

\[- \mathcal{L}(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{2r^2 \left(1 - \cos(n\alpha)\right) + n^2 \beta^2}.\]  

(23)

Analytically continuing $\alpha \to a = i\alpha$ yields the Grant space result which as stated above, diverges at each of its polarised hypersurfaces.

The Gaussian approximation is also known to be exact in the Einstein universe, which has topology $\mathbb{R} \times S^3$. If one identifies points on the Euclidean section under a combined rotation plus translation, then the effective Lagrangian can be calculated as before. In this case, however, one must also sum over contributions from geodesics which loop around the three–sphere more than once, so one has to sum over two winding numbers $n$ and $m$. If the metric is written as

\[ds^2 = d\tau^2 + r^2 \left( d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right)\]  

(24)

and the points $(\tau, \chi, \theta, \phi)$ and $(\tau + m\beta, \chi, \theta, \phi + m\alpha)$ are identified, then the geodetic interval is given by

\[\sigma_{nm}(x, x') = (\tau - \tau' - m\beta)^2 + (s_m + 2\pi nr)^2\]  

(25)

where

\[\cos \left( \frac{s_m}{r} \right) = \cos \chi \cos \chi' + \sin \chi \sin \chi' (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi' - m\alpha)).\]  

(26)

One therefore obtains

\[- L(x, x') = \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{s_m + 2\pi n}{\sin \left( \frac{s_m}{r} \right)} \frac{1}{\left( (\tau - \tau' - m\beta)^2 + (s_m + 2\pi nr)^2 \right)^2}.\]  

(27)

for the integral kernel in the massless limit, where the factor $\left( \frac{s_m}{r} + 2\pi n \right) / \sin \left( \frac{s_m}{r} \right)$ is the Van–Vleck determinant for this spacetime. This expression can be written in an alternative form by combining terms of positive and negative $n$.

\[\frac{1}{2\pi^2 r} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{s_m y^4 + 2y^2(x + n)(x - n) + (x^2 + 3n^2)(x + n)(x - n)}{16\pi^4 r^4(n + z_1)^2(n - z_1)^2(n + z_1^*)^2(n - z_1^*)^2}\]  

(28)
where the complex quantity
\[ z_1 = x + iy = \frac{s_m + i(\tau - \tau' - m\beta)}{2\pi r} \]  
(29)

The sum over \( n \) can be evaluated using the method of residues to obtain finally

\[ -L(x, x') = \frac{1}{4\pi^2 r^4} \sum_{m=-\infty}^{\infty} \sum_{m\neq 0} \left( \frac{\tau - \tau' - m\beta}{r} \right)^{-1} \sinh \left( \frac{\tau - \tau' - m\beta}{r} \right) \frac{1}{(\cosh \left( \frac{\tau - \tau' - m\beta}{r} \right) - \cos \left( \frac{s_m \omega}{r} \right))^2} \]  
(30)

If one analytically continues the parameter \( \alpha \to a = i\alpha \) in this case, the spacetime that one obtains is the product of three dimensional de Sitter space and the real line, periodically identified under a combined boost and translation. The condition for polarised hypersurfaces in this spacetime is given by

\[ \cosh \left( \frac{m\beta}{r} \right) - 1 + \sin^2 \chi \sin^2 \theta \left( 1 - \cosh(ma) \right) = 0. \]  
(31)

This criterion and the Lagrangian both reduce to the Grant space expressions in the limit as \( r \to \infty \), if one defines a new radial coordinate by \( r' = r \sin \chi \sin \theta \).

One can also try to calculate the effective Lagrangian for the Anti–de Sitter analogue of Grant space. The Gaussian approximation is exact for conformally invariant fields in this case also, due to the fact that Anti–de Sitter space can be conformally mapped into half of the Einstein static universe. The Euclidean section of Anti–de Sitter space can be realised as the 4–dimensional hyperboloid

\[ -\left( \omega^0 \right)^2 + \left( \omega^1 \right)^2 + \left( \omega^2 \right)^2 + \left( \omega^3 \right)^2 + \left( \omega^4 \right)^2 = r^2 \]  
(32)

in the 5–dimensional space with metric

\[ ds^2 = -\left( d\omega^0 \right)^2 + \left( d\omega^1 \right)^2 + \left( d\omega^2 \right)^2 + \left( d\omega^3 \right)^2 + \left( d\omega^4 \right)^2. \]  
(33)

If one defines

\[ \omega^0 = \frac{1}{r} \cosh \tau \sec \rho \]
\[ \omega^1 = \frac{1}{r} \tan \rho \cos \theta \]
\[ \omega^2 = \frac{1}{r} \tan \rho \sin \theta \cos \phi \]
\[ \omega^3 = \frac{1}{r} \tan \rho \sin \theta \sin \phi \]
\[ \omega^4 = \frac{1}{r} \sinh \tau \sec \rho \]  

then the metric takes the form
\[ ds^2 = \frac{\sec^2 \rho}{r^2} \left( d\tau^2 + dp^2 + \sin^2 \rho \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right). \]  

Once again, we identify the points \((\tau, \rho, \theta, \phi)\) and \((\tau + n\beta, \rho, \theta, \phi + n\alpha)\). In Anti–de Sitter space, the chief problem encountered when trying to construct quantum field theoretic quantities comes from the fact that information can be lost to, or gained from, spatial infinity in a finite coordinate time. Appropriate boundary conditions need to be imposed at infinity, so that the field (or its gradient) vanishes there \cite{24}. If one thinks of the Einstein universe as a cylinder, then Anti–de Sitter spatial infinity is the timelike surface at \(\chi = \frac{\pi}{2}\) obtained by slicing the cylinder with a vertical plane wave. Thus, the Anti–de Sitter Lagrangian is obtained from the Einstein expression by adding in the image charge at the antipodal point and inserting the appropriate conformal weighting factor, to obtain
\[ -L(x, x') = \frac{\cos^2 \rho \cos^2 \rho'}{4\pi^2} \sum_{m=-\infty}^{\infty} \sum_{m \neq 0} \left[ \frac{(\tau - \tau' - m\beta)}{r} \right]^{-1} \sinh \left( \frac{(\tau - \tau' - m\beta)}{r} \right) \left( \cosh \left( \frac{(\tau - \tau' - m\beta)}{r} \right) - \cos \left( \frac{2\pi m}{r} \right) \right)^2 \]
\[ \pm \left( \pi - \rho', \pi - \theta', \pi + \phi' \right) \text{ image charge}, \]  

where the upper (lower) sign refers to Dirichlet (Neumann) boundary conditions.

As a final example, we consider a massless scalar field in the wormhole spacetime originally studied by Kim and Thorne \cite{11}, who calculated the (divergent) behaviour of its renormalised energy–momentum tensor. One constructs this spacetime by removing two 3–spheres of radius \(b\) from Minkowski space and identifying the resulting world tubes which form when one sets the right hand mouth moving towards the left with speed \(\beta\). Initially the two mouths are separated by a shortest distance \(D\). Kim and Thorne
have calculated the Van–Vleck determinant and geodetic interval for this spacetime. Combining their results with our expression, one immediately obtains

$$- \mathcal{L}(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{D \left( \frac{b}{2D} \right)^{n-1}} \left( \frac{1}{1 - \xi^n} \left( \frac{1}{\lambda(x, x')} + \frac{1}{\lambda(x', x)} \right) \xi^{2n} (1 - \xi) \right)^2,$$

(37)

for the Lagrangian, where \( \xi = \left( \frac{1-\beta}{1+\beta} \right)^{\frac{1}{2}} \) is the inverse Doppler blueshift suffered by a ray passing along the X axis and through the wormhole, and the quantity \( \lambda \) is defined by

$$\lambda(x, x') = 2 \left( b - \sqrt{b^2 - \rho^2} \right) + X - T - (X' - T') \xi^n$$

(38)

where a point \( x \) has coordinates \((T, X, Y, Z)\) and \( \rho = \sqrt{Y^2 + Z^2} \) measures the transverse shift of \( x \) from the axis of symmetry.

6 Discussion

One cannot dispute the fact that in many causality violating spacetimes, the renormalised expectation value \( \langle T_{\mu\nu} \rangle \) diverges as the Cauchy horizon is approached. Indeed, the original chronology protection conjecture [12] was motivated heavily by this fact, and it was therefore proposed that the back reaction induced by this divergent energy–momentum would distort the spacetime geometry sufficiently to prevent the formation of CTCs. Recently, however, examples have been presented which lead one to question the universal validity of this basic mechanism and it is now known that \( \langle T_{\mu\nu} \rangle \) does not necessarily diverge for all initial quantum states as the Cauchy horizon is approached. Sushkov, who considered automorphic fields on four dimensional Misner space [25], gave an example of a Hadamard state for which \( \langle T_{\mu\nu} \rangle \) vanishes everywhere on the initially globally hyperbolic region (see also Krasnikov [26]). Actually, one does not even need to consider automorphic fields, as one can readily find a simple counterexample from inspecting the closed form of the scalar field energy–momentum tensor on Misner space, obtained by Euclidean methods. Recall that Misner space is just Minkowski space with points identified under a boost in the \( x \) direction. The appropriate Euclidean section of this spacetime, therefore, is flat Euclidean space identified under a rotation, \( \alpha \). This
space also happens to be the analytic continuation of the Lorentzian spacetime produced by an infinitely long cosmic string. The energy–momentum tensor for a massless conformally coupled scalar field in the cosmic string spacetime is well known, and is given on the Euclidean section (in \((\tau, r, \theta, z)\) coordinates) by

\[
\langle T^{\mu\nu} \rangle = \frac{1}{1440\pi^2 r^4} \left( \left( \frac{2\pi}{\alpha} \right)^4 - 1 \right) \text{diag}(1, 1, -3, 1). \tag{39}
\]

Clearly, if one analytically continues the parameter \(\alpha \to a = i\alpha\) in this case, then the energy–momentum tensor vanishes everywhere if \(a = 2\pi\), so there will be no divergence in this case. \(\langle T_{\mu\nu} \rangle\) has also been shown to be bounded at the Cauchy horizon for (sufficiently) massive fields in Gott space [28] and Grant space [27]. Cramer and Kay [29] have replied to all of these examples by demonstrating that even though there is no divergence, \(\langle T_{\mu\nu} \rangle\) must always be ill defined on the Cauchy horizon itself. However, one is still left with the feeling that \(\langle T_{\mu\nu} \rangle\) does not quite tell the whole story.

In this paper, we have offered a new viewpoint by focusing on the effective Lagrangian and a general expression for the leading order divergence at the polarised hypersurfaces of a typical causality violating spacetime has been obtained. Immediately one can apply this result to the examples outlined in the preceding paragraph. In four dimensional Misner space, a quick inspection of (23) with the parameter \(\beta = 0\) shows that even though \(\langle T_{\mu\nu} \rangle\) can remain finite, \(\mathcal{L}\) always diverges to minus infinity at the Cauchy horizon \(r = 0\). Similarly, (22) implies that \(\mathcal{L}\) diverges at the Cauchy horizon in Grant space (and therefore Gott space), even though \(\langle T_{\mu\nu} \rangle\) can remain finite for massive fields at the Cauchy horizon.

Finally, consider the behaviour of a Euclidean path integral of the form

\[
\Psi = \int \mathcal{D}[g] \mathcal{D}[\phi] e^{-S[g, \phi]}, \tag{40}
\]

where \(S\) is obtained from a Lagrangian appropriate for some causality violating spacetime. If the metric parameters are adjusted so as to introduce CTCs into the spacetime, then we have already shown that the action diverges to minus infinity. If one now interprets this path integral according to the no–boundary proposal, then it seems that
causality violations will be strongly enhanced, rather then suppressed. However, as one might expect, there is a subtlety involved. We shall leave a full discussion of this problem to a future paper, but conclude with a few brief remarks. Basically, one is interested in constructing the density of states, or microcanonical partition function, as the squared amplitude $\Psi^2$. The problem is that the microcanonical partition function should be defined as a function of definite conserved quantities, such as energy and angular momentum. The amplitude $\Psi$ above, however, is generally given as a function of the metric parameters which relate equivalent points in the universal covering space, which could be inverse temperature or angular velocity, for example. In order to achieve the correct result, one must project the amplitude $\Psi$ on to states of definite ‘charge’ rather than the ‘potentials’ before constructing the microcanonical partition function. If one does this, then one finds that the corrected $\Psi^2$ tends to zero, rather than infinity, as the CTCs are introduced. The situation is rather similar to that encountered when trying to calculate the rate of pair production of electrically and magnetically charged black holes. In that case, the introduction of a projection operator is necessary to ensure that the pair production rates for both types of black holes are equal, as one would expect [30]. In this case, it means that causality violating amplitudes are strongly suppressed, in accordance with the chronology protection conjecture.

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