REALIZATION BY A DIFFERENTIAL OPERATOR OF THE ANNIHILATION OPERATOR FOR GENERALIZED CHEBYSHEV OSCILLATOR

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We study a generalized Chebyshev oscillator associated with a point interaction for the discrete Schrödinger equation. Our goal is to find a realization of the annihilation operator for this oscillator by a differential operator. This realization can be used to obtain a differential equation for the corresponding generalized Chebyshev polynomials. Bibliography: 12 titles.

To the memory of our friend V. D. Lyakhovsky

1. Introduction

This work continues the research of generalized Heisenberg algebras [4] connected with several orthogonal polynomial systems. In [3], a general scheme for realization of annihilation operator for these algebras by differential operator was constructed. Note that, with the exception of the standard case of Hermite polynomials, the differential operator $A$ appearing in such realizations has infinite order.

In the work [3], one important special case of orthogonal polynomial systems, for which the matrix of the operator $A$ in $l^2(Z_+)$ has only off-diagonal elements on the first upper diagonal different from zero, was considered. Generalized Hermite polynomials [5,6] give us an example of such an orthonormal system. In this work we consider another important case of orthogonal polynomial systems for which the matrix of the operator $A$ has only diagonal elements different from zero on all odd upper diagonals. Generalized Chebyshev polynomials [2,7] give us an example of such an orthonormal system.

1.1. Generalized Chebyshev polynomials. The generalized Chebyshev polynomials $\text{Ch}_n(z;k;a)$, $k \geq 1$, are defined by the recurrent relations:

$$b_n \text{Ch}_{n+1}(z;k;a) + b_{n-1} \text{Ch}_{n-1}(z;k;a) = z \text{Ch}_n(z;k;a), \quad n \geq 0,$$

$$\text{Ch}_0(z;k;a) = 1, \quad \text{Ch}_{-1}(z;k;a) = 0,$$

where $b_1 = a$, and $b_n = 1$, for $n \neq k - 1$. Using the expression obtained in [3] for the polynomials connected with relation (1) (and the associated Jacobi matrix), we have

$$\text{Ch}_n(z;k;a) = \sum_{m=0}^{\text{Ent}(\frac{k}{2})} \frac{(-1)^m}{\sqrt{n}!} b_0^{2m-n} \beta_{2m-1,n-1} z^{n-m},$$

where $\beta_{1,n} = 1$, $n \geq 0$, and

$$\beta_{2m-1,n-1} = \sum_{k_1=2m-1}^{n-1} \frac{[k_1]!}{k_1-2} \sum_{k_2=2m-3}^{k_1-2} \frac{[k_2]!}{k_2-1} \cdots \sum_{k_m=1}^{k_{m-1}-2} \frac{[k_m]!}{k_m}.$$
for all \( m \geq 1 \). Here \( [s] = \frac{b^2 - 1}{b^0} \), and the integral part of \( x \) is denoted by \( \text{Ent}(x) \).

As an example, we give the last formulas for the case \( k = 1 \) (we set \( \Psi_n(z) = \text{Ch}_n(z; 1; a) \)):

\[
\begin{align*}
\Psi_0(z) &= 1, \quad \Psi_1(z) = \frac{z}{a}, \\
\Psi_n(z) &= \frac{z^n}{a} - \frac{n + (a^2 - 2)}{a}z^{n-2} \\
&\quad + \sum_{m=2}^{\text{Ent}(\frac{2}{a})} (-1)^m \frac{(n-m-1)!(n+m(a^2-2))}{(n-2m)!m!a} z^{n-2m}, \quad n \geq 2.
\end{align*}
\] (2)

In this paper, we will only consider the case of \( k = 1 \).

1.2. Generalized Chebyshev oscillators (for the case \( k=1 \)). Let \( a > 0, \mathcal{H}_a = L^2(\mathbb{R}; \mu_a) \) be a Hilbert space and \( \{\varphi_n(x)\}_{n=0}^{\infty} \) be a system of polynomials, which are orthonormal with respect to the measure \( \mu_a \), where

\[
d\mu_a(x) = \frac{1}{2\pi} \begin{cases} 
\frac{a^2\sqrt{4-x^2}}{a^2-(a^2-1)x^2} dx & \text{if } |x| \leq 2, \\
0 & \text{if } |x| > 2.
\end{cases}
\]

Then, as follows from [2] (see also [7]), the polynomials \( \varphi_n(x) \) are generalized Chebyshev polynomials \( \Psi_n(x) \) (for the case \( k=1 \)) and the recurrence relations (1) takes the following form:

\[
\begin{align*}
a\Psi_1(x) &= z\Psi_0(x), \quad \Psi_2(x) + a\Psi_0(x) = x\Psi_1(x), \\
\Psi_{n+1}(x) + \Psi_{n-1}(x) &= x\Psi_n(x), \quad n \geq 2, \\
\Psi_0(x) &= 1, \quad \Psi_{-1}(x) = 0.
\end{align*}
\]

In the work [4] it was shown, that one can construct an oscillator-like algebra \( \mathfrak{A}_\psi \) corresponding to this polynomial system. The polynomials \( \{\Psi_n(x)\}_{n=0}^{\infty} \) give the Fock basis for this algebra \( \mathfrak{A}_\psi \) in the Fock space \( \mathcal{H}_a \). The generators \( a^+_{\mu_a}, a^-_{\mu_a}, N_{\psi} \) of the algebra \( \mathfrak{A}_\psi \) in this Fock representation acts as follows

\[
a^+_{\mu_a} \Psi_n = \sqrt{2}b_n\Psi_{n+1}, \quad a^-_{\mu_a} \Psi_n = \sqrt{2}b_{n-1}\Psi_{n-1}, \quad N_{\psi} \Psi_n = n\Psi_n,
\]

where

\[
b_{-1} = 0, \quad b_0 = a, \quad b_n = 1, \quad n \geq 1.
\]

Let \( I \) be the identity operator in the Hilbert space \( \mathcal{H}_a \). We denote by \( B_{\psi}(N_{\psi}) \) the operator-valued function defined by the relations

\[
B_{\psi}(N_{\psi})\Psi_n = b^2_{n-1}\Psi_n, \quad B_{\psi}(N_{\psi} + I)\Psi_n = b^2_n\Psi_n, \quad n \geq 0.
\]

Then the generalized Chebyshev oscillator algebra \( \mathfrak{A}_\psi \) is generated by operators \( a^\pm_{\mu_a}, N_{\psi} \) and \( I \) satisfying the relations

\[
[a^-_{\mu_a} a^+_{\mu_a} \Psi_n] = 2B_{\psi}(N_{\psi} + I), \quad a^+_{\mu_a} a^-_{\mu_a} \Psi_n = 2B_{\psi}(N_{\psi}),
\]

and by commutators of these operators.
1.3. **Statement of a problem.** The main purpose of the present work is to find the coefficients \( a_{ls} \), such that the operator \( A \) defined by the relation \( a_{\mu a}^{-} = \sqrt{2} A \) takes the form

\[
A = \sum_{s=1}^{\infty} \sum_{l=0}^{s-1} a_{ls} z^l \frac{d^s}{dz^s}. \tag{4}
\]

Denote by \( A = \{a_{ls}\} \quad (1 \leq s < \infty, \quad 0 \leq l \leq (s-1)) \) the matrix of the operator \( A \). Using the definition of the operator \( A \) and (3) we obtain the relations

\[
A \Psi_n = b_{n-1} \Psi_{n-1}, \quad n \geq 1, \quad A \Psi_0 = 0. \tag{5}
\]

Substituting the relations (2) and (4) in (5), we get the main relation for finding the elements \( a_{ls} \) of the matrix \( A \):

\[
\sum_{s=1}^{n} \sum_{l=0}^{s-1} a_{ls} \left( \frac{n!}{(n-s)!} z^{n+l-s} - (n-2 + a^2) \frac{(n-2)!}{(n-2-s)!} z^{n+l-s-2} \right)
\]

\[
+ \sum_{m=2}^{\text{Ent}(\frac{4}{3})} (-1)^m \frac{(n-m-1)!((n+m(a^2-2)))}{(n-2m-s)!m!} z^{n+l-2m-s})
\]

\[
= z^{-1} - (n-3+a^2)z^{-3}
\]

\[
+ \sum_{m=2}^{\text{Ent}(\frac{4}{3})} (-1)^m \frac{(n-m-2)!((n-1+m(a^2-2)))}{(n-2m-1)!m!} z^{n-1-2m}.
\]

By equating the coefficients for \( z^{-2k} \) in the left and right parts of the identity (6), we obtain the following equations for finding the coefficients \( a_{s-2t,s} \), under the condition \( s \geq 2t \geq 2 \):

\[
\sum_{m=1}^{k} (-1)^{m-1} \frac{n + (m-1)(a^2-2)}{(m-1)!} \sum_{s=2(k-m+1)}^{n-2(m-1)} a_{s-2(k-m+1),s} \frac{(n-m)!}{(n-s-2(m-1))!} = 0. \tag{7}
\]

Similarly, using the coefficients for degrees \( z^{-2k-1} \), we obtain the equations for determining the coefficients \( a_{s-2t-1,s} \), under the condition \( s \geq 1, \ t \geq 0 \):

\[
\sum_{m=1}^{k+1} (-1)^{m-1} \frac{n + (m-1)(a^2-2)}{(m-1)!} \times \sum_{s=2(k-m+1)+1}^{n-2(m-1)} a_{s-2(k-m+1)-1,s} \frac{(n-m)!}{(n-s-2(m-1))!} \]

\[
= (-1)^k \frac{(n-k-2)!(n-1-2k+ka^2)}{(n-1-2k)!k!}. \tag{8}
\]

2. **Calculating elements of the matrix \( A \) for the annihilation operator**

2.1. **Calculating elements of even overdiagonals of the matrix \( A \).** In this paragraph we will prove that the elements \( a_{n-2k,n} \) of even overdiagonals of the annihilation operator matrix \( A \) equal to zero:

\[
a_{n-2k,n} = 0, \quad k \geq 1, \quad n \geq 2k. \tag{9}
\]

First, we show that

\[
a_{0,2q} = 0, \quad q \geq 1. \tag{10}
\]
To do this, we use equality (7) for \( n = 2p, k = p : \)
\[
a_{0,2p} = \frac{2p!}{0!} - a_{0,2p-2}(2p - 2 + a^2)\frac{(2p - 2)!}{1!} + a_{0,2p-4}(2p - 4 + 2a^2)\frac{(2p - 4)!}{2!} + \cdots + (-1)^{p-1}a_{0,2a^2} = 0.
\]

Consistently using the last equality for all \( n = 2p, k = p : \)
\[
a_{1,1+2q} = 0, \quad q \geq 1. \tag{11}
\]

We use equality (7) for \( n = 2p + 1, k = p : \)
\[
a_{0,2p} = \frac{(2p + 1)!}{1!} + a_{1,2p+1}(2p + 1)! \frac{0!}{0!} - (2p - 1 + a^2)\frac{(2p - 1)!}{1!} + a_{1,2p-1}(2p - 1)! \frac{0!}{0!} \]
\[
+ (2p - 3 + 2a^2)\frac{(2p - 2)!}{1!2!} + a_{1,2p-3}(2p - 2)! \frac{0!}{0!2!} \]
\[
+ \cdots + (-1)^{p-1}(1 + pa^2)\frac{p!}{1!(p-1)!} + a_{1,1+1}(p-1)! \frac{0!}{0!} = 0.
\]

Consistently using the last equality for all \( p = 1, 2, \ldots, q, \) we get (11).

Finally, we consider the general case
\[
a_{t,t+2q} = 0, \quad q \geq 1, \quad t \geq 0. \tag{12}
\]

It is obvious that \( a_{t,t+2q} \) passes to \( a_{n-2k,n} \) at \( q = k, t = n-2k. \) It is easy to see from formula (7) that all coefficients \( a_{t,t+2q} \) are expressed only in terms of coefficients \( a_{s,s+2p} \) under conditions
\( 0 \leq s < t, p \leq q \) and \( s = t, p < q. \) Then, using the proven formulas (10), (12) and gradually increasing the indices \( p \) and \( s \) it is not difficult to prove the validity of equality (9).

### 2.2. Calculating elements of odd overdiagonals of the matrix \( A. \)

The main result of this work is the following formula:
\[
a_{l,t+2k+1} = \frac{(-1)^{l+1}}{(l + 2k + 1)!}((1 - \delta_{l,0})C_kC_{l+2k} - C_{l+2k+1}^2 P_{k,2k+2}(a)), \quad l, k \geq 0, \tag{13}
\]

where \( C_k = \frac{C_{2k}^k}{k + 1} \) are Catalan numbers, \( \delta_{0,0} = 1 \) and \( \delta_{l,0} = 0 \) for \( l > 0. \) The polynomials \( P_{k,2k+2}(a) \) are defined as follows
\[
P_{0,2}(a) = a^2; \quad P_{1,4}(a) = a^4; \quad P_{2,6}(a) = a^6 + a^4;
\]
\[
P_{k,2k+2}(a) = a^{2k+2} + \sum_{i=1}^{k-1} \beta_{k,i} a^{2(k-i+1)}, \quad k \geq 3;
\]
\[
\beta_{k,1} = k - 1, \quad k \geq 2; \quad \beta_{k,i} = \frac{(k + 1)(k + 2)\ldots(k + i - 1)(k - i)}{i!}, \quad 2 \leq i \leq k - 2; \quad \beta_{k,k-1} = \beta_{k,k-2} = C_{k-1}, \quad k \geq 4. \tag{14}
\]
From (8) we have

\[ a_{n-(2k+1), n} = \frac{1}{n!} \left( - \sum_{s=2k+1}^{n-1} a_{s-(2k+1), s} \frac{n!}{(n-s)!} \right) \]

\[ + \sum_{m=2}^{k+1} (-1)^{m-2} \sum_{s=(2k+1)-2(m-1)}^{n-2(m-1)} a_{s-(2k+1-2(m-1), s}} \frac{(n-m)!(n+(m-1)(a^2-2))}{(n-s-2(m-1)!(m-1)!} \]

\[ + (-1)^k \frac{(n-k-2)!(n-1-2k+ka^2)}{(n-1-2k)!k!}. \]

Replacing the indices \( l = n - (2k+1), t = s - (2k+1) + 2(m-1) \), we get

\[ a_{l,t+(2k+1)} = \frac{1}{(l + 2k + 1)!} \left( - \sum_{t=0}^{l-1} a_{t,t+(2k+1)} \frac{(l + 2k + 1)!}{(l-t)!} \right) \]

\[ + \sum_{m=2}^{k+1} (-1)^{m-2} \sum_{t=0}^{l} a_{t,t+(2k+1-2(m-1))} \]

\[ \times \frac{(l + 2k + 1 - m)!(l + 2k + 1 + (m-1)(a^2-2))}{(l-t)!(m-1)!} \times (-1)^k \frac{(l + k - 1)!(l+ka^2)}{l!k!}. \] (15)

In the remaining part of this article, we will prove the formula (13) recursively, using the relation (15). As a base of recursion, we take the case \( l = 0, k = 0 \). Obviously, the formula (13) is valid, since

\[ a_{0,0+1} = a^2 = P_{0,2}. \] (16)

We note that the coefficients \( a_{l,t+(2k+1)} \) are expressed using the formula (15) by elements \( a_{u,v} \) with "smaller" numbers, i.e. for \( u \leq l, v \leq l + (2k+1) \) and \( u + v < 2l + 2k + 1 \). Then the induction transition consists in the fact that assuming the correctness of formula (13) for all coefficients \( a_{u,v} \) standing on the right side of equality (15), we must prove the fulfillment of formula (13) also for the left side of equality (15). To prove this statement, it is sufficient to check the equality of all coefficients for the same powers of \( a^2 \) polynomials in both parts of the relation (15), which arise when substituting in (15) expressions for all \( a_{l,t+(2s+1)} \) by the formula (13).

As can be seen from the definition (4) of operator \( A \), the matrix of this operator is an upper triangular matrix. It is convenient to start the proof of formula (13) by considering the "boundary" non-zero elements of the matrix \( A \), that is, elements \( a_{l,t+1} \) \((k = 0, l > 0)\) standing on the first overdiagonal of the matrix \( A \) and elements \( a_{0,2k+1} \) \((k \geq 0, l = 0)\) standing on the first row of the matrix \( A \).

3. Two special cases of the formula (13)

3.1. The case \( k = 0, l > 0 \). From (15) \((k = 0, l > 0)\) we get

\[ (l + 1)! a_{l,t+1} = - \sum_{t=0}^{l-1} a_{t,t+1} \frac{(l + 1)!}{(l-t)!} + (-1)^0 \frac{(l-1)!l}{l!0!}. \] (17)

Note that the formula (13) for \( k = 0, l > 0 \) has the form

\[ a_{l,t+1} = \frac{(-1)^{l+1}}{(l+1)!} (C_l^1 - C_{l+1}^1 P_{0,2}(a)). \] (18)
Substituting (18) in (17) and using (16), we get

\[ (-1)^{l+1} \frac{(l+1)!}{(l+1)!} - a^2 \frac{(l+1)!}{l!} = - \sum_{t=0}^{l-1} (-1)^{t+1} tC_{l+1}^t + a^2 \sum_{t=0}^{l-1} (-1)^{t+1} (t+1)C_{l+1}^{t+1} + 1. \]

Moving the first and second summands from the right part to the left part of the last equality, we get

\[ \sum_{t=0}^{l} (-1)^{t+1} tC_{l+1}^t - a^2 \sum_{t=0}^{l} (-1)^{t+1} (t+1)C_{l+1}^{t+1} = 1. \] (19)

For proof (19), it is sufficient to check the fulfillment of the following two equalities

\[ \sum_{t=0}^{l} (-1)^{t+1} tC_{l+1}^t = 1 \quad \text{and} \quad \sum_{t=0}^{l} (-1)^{t+1} (t+1)C_{l+1}^{t+1} = 0. \] (20)

The first of these equalities follows from the identity (44) proved below for \( q = 0 \). To prove the second equality, write the left part of it as a sum

\[ \sum_{t=0}^{l} (-1)^{t+1} tC_{l+1}^t + \sum_{t=0}^{l} (-1)^{t+1} C_{l+1}^{t+1} = 1 - 1 = 0. \]

Thus, the formula (13) for the case \( k = 0, l > 0 \) is proved.

**3.2. The case \( l = 0, k \geq 0 \)** *(proof of the formula (14)).* First of all, consider the special case \( l = 0, k \geq 0 \). Note that for \( l = 0 \), the first term in brackets in the right part of equality (13) disappear, i.e. equality (13) takes the form

\[ a_{0,2k+1} = \frac{1}{(2k+1)!} C_{2k+1}^{2k+1} P_{2k+2}(a), \quad k \geq 0. \] (21)

The formula is valid for \( k = 0 \), since as already mentioned (see (16)) \( a_{0,1} = a^2 \) if we assume that

\[ P_{0,2}(a) = a^2. \]

Next, we can rewrite formula (15) for the case \( l = 0, k \geq 1 \) (Note that for \( l = 0 \), the first term in brackets in the right part of equality (15) disappear)

\[ a_{0,2k+1} = \frac{1}{(2k+1)!} \left( \sum_{m=2}^{k+1} (-1)^{m-2} a_{0,2k+1-2(m-1)} \right. \]

\[ \left. \times \frac{(2k+1-m)!(2k+1+(m-1)(a^2-2))}{0!(m-1)!} \right) + (-1)^k \frac{(k-1)!ka^2}{0!k!}. \] (22)

To check formula (22), it is sufficient to check the equality of all coefficients with the same powers of \( a^2 \) of the polynomials that arise when substituting in both parts of the relation (22) expressions for all \( a_{0,2k+1-2(m-1)} \) by formula (21). In other words, we need to check the following identity.

\[ P_{2k+2}(a) = \sum_{m=2}^{k+1} (-1)^{m-2} P_{k-m+1,2(k-m+1)+2}(a) \]

\[ \times \frac{(2k+1-m)!(2k+1+(m-1)(a^2-2))}{(2(k-m+1)+1)!(m-1)!} \] + \((-1)^ka^2, \quad k \geq 1. \] (23)

Consider the cases \( k = 1 \) and \( k = 2 \). When \( k = 1 \) we have

\[ P_{1,4}(a) = (-1)^2 P_{0,2} \frac{1!(3+a^2-2)}{1!} - a^2 = a^2(1 + a^2) - a^2 = a^4. \] (24)
When \( k = 2 \) we have
\[
P_{2,6}(a) = P_{1,4}(a) \frac{3!(5 + a^2 - 2)}{3!} - P_{0,2}(a) \frac{2!(5 + 2a^2 - 4)}{1!} + a^2 = a^6 + a^4. \tag{25}
\]

Note that, using (24) and (25), it is not difficult to obtain from (23) the following polynomial
\[ P_{3,8}(a) \]
\[
P_{3,8}(a) = a^8 + 2(a^6 + a^4). \tag{26}
\]

Finally, for \( k \geq 2 \), we will look for an expression for the polynomial \( P_{k,2k+2}(a) \) in the following form
\[
P_{k,2k+2}(a) = a^{2k+2} + \sum_{i=1}^{k-1} \beta_{k,i} a^{2(k-i+1)}. \tag{27}
\]

Comparing the expressions (25) and (26) for the polynomials \( P_{2,6}(a) \) and \( P_{3,8}(a) \) with the corresponding expressions obtained from (27) for \( k = 2 \) and \( k = 3 \), respectively, we have
\[
\beta_{2,1} = C_1 = 1, \quad \beta_{3,1} = \beta_{3,2} = C_2 = 2. \tag{28}
\]

Next, we will use (27) for \( k \geq 3 \). To check formula (23) for \( k \geq 3 \), we need to rewrite it in the following form
\[
P_{k,2k+2}(a) = \sum_{m=2}^{k-2} \left[ (-1)^{m-2} P_{k-m+1,2(k-m+1)+2}(a) \right.
\times \frac{(2k + 1 - m)!(2k + 1 + (m - 1)(a^2 - 2))}{(2(k - m + 1) + 1)!(m - 1)!} \]
\[
+ \left. (-1)^{k-3} P_{2,6}(a) \frac{(2k + 1 - k + 1)!(2k + 1 + (k - 2)(a^2 - 2))}{(4 + 1)!(k - 2)!} \right]
\]
\[
+ \left. (-1)^{k-2} P_{1,4}(a) \frac{(k + 1)!(2k + 1 + (k - 1)(a^2 - 2))}{(2 + 1)!(k - 1)!} \right]
\]
\[
+ \left. (-1)^{k-1} P_{0,2}(a) \frac{k!(2k + 1 + k(a^2 - 2))}{1!k!} \right] + (-1)^{k}a^2, \quad k \geq 1. \tag{29}
\]

Substituting in (29) instead of the polynomials \( P_{p,2p+2}(a) \) their expressions by the formula (27) for \( p \geq 3 \), we have
\[
a^{2k+2} + \sum_{i=1}^{k-1} \beta_{k,i} a^{2(k-i+1)} = \sum_{m=2}^{k-2} \left[ (-1)^{m-2} a^{2(k-m+1)+2}
\right.
\times \frac{(2k + 1 - m)!(2k + 1 + (m - 1)(a^2 - 2))}{(2(k - m + 1) + 1)!(m - 1)!} \]
\[
+ \left. (-1)^{k-3} P_{2,6}(a) \frac{(2k + 1 - k + 1)!(2k + 1 + (k - 2)(a^2 - 2))}{(4 + 1)!(k - 2)!} \right]
\]
\[
+ \left. (-1)^{k-2} P_{1,4}(a) \frac{(k + 1)!(2k + 1 + (k - 1)(a^2 - 2))}{(2 + 1)!(k - 1)!} \right]
\]
\[
+ \left. (-1)^{k-1} P_{0,2}(a) \frac{k!(2k + 1 + k(a^2 - 2))}{1!k!} \right] + (-1)^{k}a^2, \quad k \geq 1. \tag{30}
\]

Equating the coefficients at the same powers of \( a^2 \) in both parts of the relation (30) we will see further the validity of the formulas (14) for \( \beta_{k,i} \).
Let’s start by finding $\beta_{k,k-2}$ (coefficient for $a^6$) and $\beta_{k,k-1}$ (coefficient for $a^4$):

$$\beta_{k,k-2} = (-1)^{k-2}C_{k+1}^{k-2} + \sum_{t=1}^{k-2}(-1)^{k-2-t}C_t C_{k+2+t}^{k-1-t} = C_{k-1}, \quad (31)$$

$$\beta_{k,k-1} = (-1)^{k-1}C_k^{k-1} + \sum_{t=0}^{k-2}(-1)^{k-2-t}C_t C_{k+1+t}^{k-1-t} = C_{k-1}. \quad (32)$$

For the proof of formulas (31), see Appendix 1. Thus, we proved that the following equalities are valid for $k \geq 2$

$$\beta_{k,k-2} = \beta_{k,k-1} = C_{k-1}. \quad (33)$$

By equating the coefficients for $a^{2(k-i+1)}$ for $k \geq 3$ and $1 \leq i \leq k-2$, we get the following system of equations for the quantities $\beta_{p,q}$

$$\beta_{k,i} = (-1)^{i-1}C_{2k-i}^{i} + (-1)^{i}C_{2k-i-1}^{i} + (-1)^{i-1}\beta_{k-1,i}C_{2k-i}^{i-1} + \sum_{m=2}^{i}(-1)^{m-2}(\beta_{k-m+1,i-m+1}C_{2k-m+1}^{m-1} + \beta_{k-m+1,i-m+2}C_{2k-m+1}^{m-2}) \quad (34)$$

Let’s start by calculating the value of $\beta_{k,1}$. We will prove the following formula

$$\beta_{k,1} = k - 1. \quad (35)$$

by induction. The above equation $\beta_{3,1} = 2$ serves as the base of recursion. From the formula (33) we have

$$\beta_{k,1} = -C_{2k-1}^1 - C_{2k-2}^1 + \beta_{k-1,1}C_{2k-1}^0 = 1 + \beta_{k-1,1}. \quad (36)$$

The last equality provides an inductive transition: if $\beta_{k-1,1} = k-2$, then $\beta_{k,1} = 1+k-2 = k-1$. Thus the formula (34) is proved.

Consider also the case $\beta_{k,2}$. From (33) for $k \geq 3$ and $i = 2$ we have

$$\beta_{k,2} = -C_{2k-2}^2 + C_{2k-3}^2 + (\beta_{k-1,1}C_{2k-1}^2 + \beta_{k-1,2}C_{2k-1}^0) - \beta_{k-2,1}C_{2k-2}^1 = \beta_{k-1,2} + k - 1. \quad (37)$$

The last equality provides an inductive transition and above equation $\beta_{3,2} = 2$ serves as the base of induction. Then

$$\beta_{k,2} = (k - 1) + (k - 2) + \cdots + 2 = \frac{(k+1)(k-2)}{2!}, \quad (38) \quad k \geq 3.$$

This coincides with the formula (14) for $i = 2$.

Taking into account (16), (24), (25), (27), (32), and (34), to complete the proof of formulas (14) it remains to check that for coefficients $\beta_{k,i}$ defined by the equality (33) the following relation is true.

$$\beta_{k,i} = C_{k+i-1}^{k-1} - C_{k+i-1}^k = \frac{k-i}{i}C_{k+i-1}^{k-1} = \frac{(k+1)(k+2)\ldots(k+i-1)(k-i)}{i!}. \quad (39)$$

We will prove (36) by induction. In this case, (28) and (35) are considered as the base of induction. The inductive transition means that the coefficients $\beta_{k,i}$, calculated by the formula (33) must satisfy the equality (36), provided that all the coefficients $\beta_{p,q}$ included in the right part (33) satisfy the equality (36).
So, to prove (36) it is sufficient to check the validity of the following equality

\[ (-1)^i \binom{m}{i} C_{2k-i-1} + (-1)^{i-1} C_{2k-i} + (-1)^{i-1} C_{k-i-1} C_{2k-i} \]

\[ + \sum_{m=2}^{i} (-1)^{m-2} \left[ (C_{k+i-m+1} - C_{k+i-2m+1}) \right] C_{2k-m+1} \]

\[ + \left( C_{k+i-2m+2} - C_{k+i-2m+1} \right) C_{2k-m+1} \]

\[ = C_{k+i-1} - C_{k+i-1}^2, \quad 2 \leq i \leq k - 2. \quad (37) \]

Denote by \( S(m; k, i) \) the general term of the sum in (37):

\[ S(m; k, i) = (-1)^{m-2} \left[ (C_{k+i-m+1} - C_{k+i-2m+1}) C_{2k-m+1} \right. \]

\[ + \left( C_{k+i-2m+2} - C_{k+i-2m+1} \right) C_{2k-m+1} \].

It is easy to see that

\[ S(1; k, i) = -C_{k+i-1}^2 + C_{k+i-1}^2, \]
\[ S(i+1; k, i) = (-1)^{i-1} (C_{2k-i} + C_{k-i-1} C_{2k-i}), \]
\[ S(i+2; k, i) = (-1)^i C_{2k-i-1}. \]

Then the relation (36) can be rewritten as

\[ \sum_{m=1}^{i+2} S(m; k, i) = 0. \quad (38) \]

The validity of this equality was tested in the program of symbolic computation "Mathematica". Thus, the the relation (36) is proved, and hence the formula (14) is proved.

4. CHECKING THE EQUALITY OF FIRST-DEGREE POLYNOMIALS

Now we consider the case \( l > 0, k > 0 \). Equating polynomials of the first degree from \( a^2 \), arising when replacing in (15) expressions for all \( a_{t,t+(2s+1)} \) by the formula (13), we get

\[ \frac{(-1)^{l+1}}{(l+2k+1)!} C_k C_{t+2k} = \frac{1}{(l+2k+1)!} \]

\[ \times \left\{ \sum_{t=0}^{l-1} \left[ (\delta_{t,0} \Delta_{t+1}) + (-1)^{i+1} \frac{(l+2k+1)! C_k C_{t+2k}}{(l-t)!} \sum_{m=2}^{k+1} (-1)^{m-2} \right] \right\}. \quad (39) \]

So, we need to proof the validity of this equality. To do this, it is enough to check the following two relations:

\[ \sum_{m=2}^{k+1} (-1)^{m-2} (l+2k+1-m)! (m-1)! \sum_{t=1}^{l} C_{k-m+1} C_{t+2k-m+1} \]

\[ \times \frac{(-1)^{l+1}}{(l-t)! (t+2k-m+1)!} = \frac{(-1)^{l+1} (l+2k-m+1)!}{l! k!}. \quad (40) \]
and also

\[-1^{t+1}(1 - \delta_{t,0})C_k C_{t+2k}^{2k+1} = \sum_{t=0}^{l-1} (\delta_{t,0} - 1) \frac{(-1)^{t+1}}{(t + 2k + 1)!} C_k C_{t+2k}^{2k+1} \frac{(l + 2k + 1)!}{(l-t)!} \]

\[+ \sum_{m=2}^{k+1} (-1)^{m-2} \sum_{l=0}^{l} (1 - \delta_{t,0}) \sum_{t=1}^{l} \frac{(-1)^{t+1}}{(t + 2(k-m + 1) + 1)!} C_{k-m+1} C_{l+2(k-m+1)}^{2(k-m+1)+1} \]

\[\times \frac{(l + 2k + 1 - m)! (l + 2k + 1 + (m-1)(-2))}{(l-t)!(m-1)!} + (-1)^k (l + k - 1)! l! k! \]

First, we will check the equality (40), which means that in this case the coefficient for \( a^2 \) on the right side of (13) is zero. We rewrite equality (40) by replacing the summation index \( m = k + 1 - q \) \((m > 1)\)

\[-1^{k-1} \sum_{q=0}^{k-1} (-1)^{k-q-1} \frac{(l+k+q)!}{(k-q-1)!} \sum_{l=1}^{l} C_q C_{t+2q}^{2q+1} \frac{(-1)^{t+1}}{(l-t)!(l+2q+1)!} \]

\[= (-1)^{k-1} C_{t+k-1}^{k-1}. \]

Next, we compare equality (42) for \( s = l + k \) with the identity (which will be proved in Appendix 2):

\[-1^{k-1} \sum_{q=0}^{k-1} (-1)^{k-q-1} C_q C_{s+q}^{k-1-q} = (-1)^{k-1} C_{s-1}^{k-1}, \quad 1 \leq k \leq s. \]

Obviously, to prove equality (40), it is sufficient to check that the coefficients of all \( C_q \) in the left part of the relations (42) and (43) coincide, that is,

\[-1^{t+1} \sum_{l=1}^{l} C_{t+2q}^{l+2q} C_{t+2q+1}^{l+2q+1} \]

\[= 1. \]

Replacing the variable \( \tau = t + 2q + 1 \), we get

\[-1^{t+2q+1} \sum_{\tau=2q+2}^{\tau} (-1)^{\tau-2q} C_{\tau-1}^{1+2q} C_{\tau+2q+1}^{\tau} \]

\[= 1. \]

Since

\[-1^{t+2q} C_{\tau-1}^{\tau} C_{\tau+2q+1}^{\tau} = C_{\tau-2q}^{\tau} \sum_{\tau}^{\tau} (-1)^{\tau-2q} C_{\tau}^{\tau-2q-2} \]

\[(45) \] can be rewritten as

\[-1^{t+2q+1} \sum_{\tau=2q+2}^{\tau} (-1)^{\tau-2q} C_{\tau}^{\tau-2q-2} \]

\[= 1. \]

Changing the summation index again \( s = \tau - 2(q + 1) \), we get

\[-1^{t+2q+1} \sum_{s=0}^{l-1} (-1)^{s} \frac{C_{t-1}^{s}}{s + 2(q + 1)} \]

\[= 1. \]

Using the formula (4.2.2.45) from [8], we have

\[-1^{t+2q} \frac{(l-1)!}{\prod_{s=0}^{l} (s + 2q + 2)} = \frac{(l + 2q + 1)!}{(l + 2q + 1)!} = 1, \]

so equality (45) is true. This proves the validity of the relation (40).
Let’s go to the equality check (41). For convenience, we will rewrite equality (41) in the following form.

\[ A = B_1 + B_2, \tag{46} \]

where

\[ A = (-1)^{l+1}(1 - \delta_{l,0})C_kC_{2k+1}^{2} + \sum_{t=0}^{l-1}(1 - \delta_{t,0})\frac{(-1)^{t+1}}{(t + 2k + 1)!}C_kC_{2k+1}^{2}(l + 2k + 1)!/(t - l)!, \]

and

\[ B_1 = \sum_{m=2}^{k+1}(-1)^{m-2}\sum_{t=0}^{l}(1 - \delta_{t,0})\left[\frac{(-1)^{t+1}}{(t + 2(k - m + 1) + 1)!} \times C_{k-m+1}C_{2(k-m+1)}^{2}(l + 2k + 1 - m)!/(l - t)!(m - 1)! \right]; \tag{47} \]

\[ B_2 = \sum_{m=2}^{k+1}(-1)^{m-2}\sum_{t=0}^{l}(1 - \delta_{t,0})\left[\frac{(-1)^{t+1}}{(t + 2(k - m + 1) + 1)!} \times C_{k-m+1}C_{2(k-m+1)}^{2}(l + 2k + 1 - m)!/(l - t)!(m - 1)! \right] + (-1)^k(l + k - 1)!!/l!!k!. \]

Let’s start by calculating the value of A. Note that the first term can be included in the sum, i.e.

\[ A = \sum_{t=0}^{l}(1 - \delta_{t,0})(-1)^{t+1}C_kC_{2k+1}^{2}C_t^{2} + \sum_{t=0}^{l}(1 - \delta_{t,0})(-1)^{t+1}C_t^{2} \frac{t}{t + 2k + 1} \]

\[ = C_{2k+1}^{2}C_k\left[\sum_{t=1}^{l}(-1)^{t+1}C_t^{2} - (2k + 1)\sum_{t=1}^{l}(-1)^{t+1}\frac{C_t^{2}}{t + 2k + 1} \right]. \tag{48} \]

From the properties of binomial coefficients, it follows that the first term in square brackets is equal to 1, and the second, after replacing \( t = \tau + 1 \) takes the form

\[ -l\left(\sum_{\tau=0}^{l-1}\frac{(-1)^{\tau}}{\tau + 1}C_{l-\tau}^{2} - \sum_{\tau=0}^{l-1}\frac{(-1)^{\tau}}{\tau + 2(k + 1)}C_{l-\tau}^{2} \right). \]

Using the formula (4.2.2.45) from [8], we can continue the equality (for the second term in square brackets)

\[-l\left(\frac{(l - 1)!}{l!!} - \frac{(l - 1)!}{(2k + 2)(2k + 3)\ldots(2k + 2 + l - 1)} \right). \]

Substituting the resulting expression instead of the second term in the right side of the expression (48) for the value A, we get

\[ A = -C_kC_{2k+1}^{2}C_{l+2k+1}^{2} + C_kC_{2k+1}^{2}C_{l+2k+1}^{2} + C_kl!\frac{(2k + l + 1)!}{(2k + 1)!(2k + 2)\ldots(2k + l + 1)!} = C_k. \tag{49} \]

Let’s move on to calculating \( B_1 \). Replacing the index \( m = k + 1 - q \) on the right hand side of the expression for \( B_1 \) (see (47)), just as in (40) when getting (42), we get the following expression for \( B_1 \).

\[ B_1 = (l + 2k + 1)\sum_{q=k-1}^{l}(-1)^{l+1}(l + k + q)!/(k - q)!\sum_{t=1}^{l}C_qC_{2q+1}^{2}C_{t+2q}^{2} \frac{(-1)^{t+1}}{(t + 2q + 1)!(l - t)!}. \tag{50} \]
The coefficient for \( C_q \) in this sum has the form

\[
(-1)^{k-q-1}C_{l+k+q}^{k-q-1} \frac{l + 2k + 1}{k-q} \sum_{t=1}^{l} (-1)^{t+1} C_{t+2q}^{t+2q+1}.
\]

Note that the last sum is equal to 1 (see (44)). Substituting the found values of coefficients for \( C_q \) into the formula (50), we find (denoting \( s = l + k \))

\[
B_1 = \sum_{q=0}^{k-1} (-1)^{k-q-1} C_q C_{s+q}^{k-q-1} \frac{s + k + 1}{k-q} = \sum_{q=0}^{k-1} (-1)^{k-q-1} C_q C_{s+q}^{k-q-1}
\]

\[
+ \sum_{q=0}^{k-1} (-1)^{k-q-1} C_q C_{s+q+1}^{k-q} = \sum_{q=0}^{k-1} (-1)^{k-q-1} C_q C_{s+q}^{k-q-1} - \sum_{q=0}^{k} (-1)^{k-q} C_q C_{s+q+1} + C_k.
\]

Using the identity (43) (and its analog) for the first and second sums, we obtain for \( B_1 \) the final expression

\[
B_1 = (-1)^{k-1} C_{s-k}^{k-1} + (-1)^{k-1} C_s^k + C_k. \tag{51}
\]

Now we calculate \( B_2 \). From the comparison of the formulas (47) and (42), it is obvious that the first sum in the right part of (47) coincides with the left part of (42), multiplied by a factor (-2). Then from (47) and (42) follows the equality

\[
B_2 = (-1)^{k-1} C_{s-1}^{k-1}(-2) + (-1)^k C_{s-1}^k = (-1)^k (2C_{s-1}^k + C_{s-1}^k) = (-1)^k C_s^k + (-1)^k C_{s-1}^{k-1}.
\]

Let’s check the validity of the equality (46) using (51), (49) and (42). We have

\[
A = C_k, \quad B_1 + B_2 = (-1)^{k-1} C_{s-1}^{k-1} + (-1)^{k-1} C_s^k + C_k + (-1)^k C_s^k + (-1)^k C_{s-1}^{k-1} = C_k.
\]

Hence, the validity of the equality of (46), and therefore the equality of (39) and (41) is proved.

5. Checking the equality of polynomials of degree higher than the first

Assuming \( l > 0 \) and equating polynomials of degree higher than the first by \( a^2 \), arising when replacing in (15) expressions for all \( a_{t,t+2(s+1)} \) by the formula (13), we get

\[
\frac{(-1)^{l+1}}{(2k+1)!!} P_{k,2k+2}(a) \frac{1}{(l+2k+1)!} \left\{ \sum_{t=0}^{l} (-1)^{t+1} \frac{1}{(2k+1)!} P_{k,2k+2}(a) \frac{(l+2k+1)!}{(l-t)!} \right.
\]

\[
+ \sum_{m=2}^{k+1} (-1)^{m-2} \sum_{t=0}^{l} \left[ \frac{(-1)^{t+1}}{(2(k-m+1)+1)!!} P_{k-m+1,2(k-m+1)+2}(a) \right.
\]

\[
\left. \times \frac{(l+2k+1-m)!}{(l-t)!(m-1)!} \right\}
\]

\[
\times \frac{(l+2k+1-m)!}{(l+2k+1+m-1)(a^2-2))!} \frac{1}{(l-t)!(m-1)!}.
\]

So, we need to test the validity of this equality. For convenience, we will rewrite equality (52) in the following form

\[
\tilde{A} = \tilde{B}, \tag{53}
\]

where

\[
\tilde{A} = \frac{(-1)^{l+1}}{(2k+1)!!} P_{k,2k+2}(a) + \sum_{t=0}^{l} \frac{(-1)^{t+1}}{(2k+1)!} P_{k,2k+2}(a) \frac{(l+2k+1)!}{(l-t)!},
\]

\[
\tilde{B} = \frac{(-1)^{l+1}}{(2k+1)!!} P_{k,2k+2}(a) + \sum_{t=0}^{l} \frac{(-1)^{t+1}}{(2k+1)!} P_{k,2k+2}(a) \frac{(l+2k+1)!}{(l-t)!}.
\]
and
\[
\tilde{B} = \sum_{m=2}^{k+1} (-1)^{m-2} \sum_{t=0}^{l} \frac{(-1)^{t+1}}{(2(k - m + 1) + 1)!l!} P_{k-m+1,2(k-m+1)+2}(a) \\
\times \frac{(l + 2k + 1 - m)!((l + 2k + 1 + (m - 1)(a^2 - 2))}{(l - t)!(m - 1)!}.
\]

Let’s start by calculating the value of \( \tilde{A} \). It is obvious that the first term can be included in the second sum, i.e.
\[
\tilde{A} = \sum_{t=0}^{l} \frac{(-1)^{t+1}}{(2k + 1)!l!} P_{k,2k+2}(a) = \frac{P_{k,2k+2}(a)}{l!(2k + 1)!} \sum_{t=0}^{l} (-1)^{t+1} C_t^l = 0.
\]

Next, find \( \tilde{B} \). To do this, we calculate the coefficient for the polynomial \( P_{k-m+1,2(k-m+1)+2}(a) \):
\[
(-1)^{m-2} \frac{(l + 2k + 1 - m)!((l + 2k + 1 + (m - 1)(a^2 - 2))}{l!(2k - m + 1 + 1)!} \sum_{t=0}^{l} (-1)^{t+1} C_t^l = 0.
\]

So, the equality (53) holds, and hence the identity (52) is true. This proves the validity of formula (15) for all \( k > 0 \) and \( l > 0 \). Therefore, the formula (13) is fully proved.

6. Concluding remarks

Recently, linear differential operators of arbitrary order are increasingly used in the theory of orthogonal polynomials. Let us give some examples.

First, there are systems of orthogonal polynomials that satisfy only a linear differential equation of infinite order [10].

Second, it is well known [11] that any linear transformation
\[
T : \mathbb{C}[x] \rightarrow \mathbb{C}[x]
\]
can be represented by the differential operator
\[
T = \sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} D^n,
\]
where \( D \) denotes the differentiation \( D^n f(x) = f^{(n)}(x) \) and where the \( Q_n(x) \) – complex polynomials. Such transformations that preserve or shrink the location of the complex zeros of polynomials are a recent object of study motivated by the Riemann Hypothesis.

Further examples, as well as an extensive bibliography, are available in the monograph [12]. Thus the importance of linear differential operators of infinite order in the study of orthogonal polynomials needs no further emphasis.

In this paper, we obtain an realization of the annihilation operator \( a_{\mu_o} = \sqrt{2}A \) for the oscillator-like system, associated with the system of generalized Chebyshev polynomials \( Ch_n(z;1;a) \), by a differential operator of infinite order. This operator has the form
\[
A = \sum_{s=1}^{\infty} \sum_{i=0}^{s-1} a_{ls} z^i \frac{d^s}{dz^s}.
\]

264
Formulas for calculating the coefficients $a_i$s are obtained. To illustrate, we write out the beginning of first few rows and columns of an infinite coefficient matrix

$$A = \begin{bmatrix}
0 & a^2 & 0 & \frac{a^4}{3!} & 0 & \frac{a^6 + a^4}{5!} & 0 & \frac{a^8 + 2(a^6 + a^4)}{7!} & \ldots \\
0 & 0 & \frac{1-2a^2}{2!} & 0 & \frac{1-4a^4}{4!} & 0 & \frac{2-6(a^6 + a^4)}{6!} & 0 & \ldots \\
0 & 0 & 0 & -\frac{2-3a^2}{3!} & 0 & -\frac{4-10a^4}{4!} & 0 & -\frac{12-21(a^6 + a^4)}{6!} & \ldots \\
0 & 0 & 0 & 0 & \frac{3-4a^2}{4!} & 0 & \frac{10-20a^4}{6!} & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & -\frac{4-5a^2}{5!} & 0 & \frac{20-25a^4}{7!} & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{5-6a^2}{6!} & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6-7a^2}{7!} & \ldots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ldots
\end{bmatrix}$$

We hope that similar representations of ladder operators may prove useful in the study of generalized Heisenberg algebras related to systems of orthogonal polynomials. For example, when obtaining differential equations for the corresponding polynomials by the method proposed in the work of the authors [3].

**Appendix 1**

1. Let’s prove the first of the relations (31), namely

$$(-1)^{k-2} C_{k+1}^{k-2} + \sum_{t=1}^{k-2} (-1)^{k-2-t} C_t C_{k+2+t}^{k-1-t} = C_{k-1}. \tag{54}$$

We transform this equality by entering the right hand side under the sign of the sum, and take into account that for $t = k - 1$ we have $(-1)^{k-2-t} = (-1)$ and $C_{k+2+t}^{k-1-t} = C_{2k+1}^0 = 1$. As a result, (54) goes to

$$(-1)^{k-2} C_{k+1}^{k-2} + \sum_{t=0}^{k-1} (-1)^{k-2-t} C_t C_{k+2+t}^{k-1-t} = 0$$

or, after dividing by $(-1)^{k-2}$,

$$C_{k+1}^{k-2} + \sum_{m=0}^{k-2} (-1)^{m+1} C_{m+1} C_{k+m+3}^{k-m-2} = 0, \tag{55}$$

where $m = t - 1$. Since for $m \geq k - 1$ we have $C_{k+m+3}^{k-m-2} = 0$, then after shortening by $(-1)$, we can rewrite (55) in the form

$$\sum_{m=0}^{\infty} (-1)^m C_{m+1} C_{k+m+3}^{2m+5} = C_{k+1}^3. \tag{56}$$

Denoting $n = k + 3$, we get

$$\sum_{m=0}^{\infty} (-1)^m C_{m+1} C_{n+m}^{2m+5} = C_{n-2}^{m-5}. \tag{57}$$

---

1When proving this and the following formulas we will use a variant “Snake Oil” of the generating function method [9].
This is the relation we will prove. To calculate the sum on the left side of the equality (57) consider the generating function

\[ \mathfrak{F}(x) = \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{\infty} (-1)^{m} C_{m+1} C_{2m+5}^{n+m} \right] x^{n} \]  

(58)

We rewrite \( \mathfrak{F}(x) \) as follows

\[ \mathfrak{F}(x) = \sum_{m=0}^{\infty} (-1)^{m} C_{m+1} \left( \sum_{n=0}^{\infty} C_{n+m+1}^{2m+5} x^{n} \right) = \sum_{m=0}^{\infty} (-1)^{m} C_{m+1}^{m+5} \left[ \sum_{n=m+5}^{\infty} C_{n+m}^{2m+5} x^{n-m-5} \right]. \]  

(59)

Taking into account that for \( m+5 > n \) we have \( C_{n+m}^{2m+5} = 0 \), we rewrite this equality as

\[ \mathfrak{F}(x) = \sum_{m=0}^{\infty} (-1)^{m} C_{m+1} x^{m+5} \left[ \sum_{n=m+5}^{\infty} C_{n+m}^{2m+5} x^{n-m-5} \right]. \]  

(60)

From the power series

\[ \frac{1}{(1-x)^{q}} = \sum_{r=0}^{\infty} C_{q-r-1}^{r} x^{r}, \]

for \( q = 2m+6 \) we obtain

\[ \sum_{r=0}^{\infty} C_{2m+r+5}^{2m+5} x^{r} = \frac{1}{(1-x)^{2m+6}}, \]  

(61)

or for \( r = n-m-5 \)

\[ \sum_{n=m+5}^{\infty} C_{n+m}^{2m+5} x^{n-m-5} = \frac{1}{(1-x)^{2m+6}}. \]  

(62)

Substituting (62) in (60), we get

\[ \mathfrak{F}(x) = \sum_{m=0}^{\infty} (-1)^{m} C_{m+1} x^{m+5} \left( \frac{1}{(1-x)^{2m+6}} \right) = \frac{x^{5}}{(1-x)^{6}} \sum_{m=0}^{\infty} (-1)^{m} C_{m+1} \frac{x^{m}}{(1-x)^{2m}} \]

\[ = \frac{x^{5}}{(1-x)^{6}} \frac{(1-x)^{2}}{4} \sum_{m=0}^{\infty} C_{m+1} \left[ \frac{(-x)}{(1-x)^{2}} \right]^{m+1} = -x^{4} \sum_{m=0}^{\infty} \frac{C_{m+1}}{(1-x)^{4}} \left[ \frac{(-x)}{(1-x)^{2}} \right]^{m+1}. \]  

(63)

Denoting \( p = m+1 \), we obtain

\[ \mathfrak{F}(x) = -x^{4} \frac{(1-x)^{2}}{(1-x)^{4}} \left[ \sum_{p=1}^{\infty} C_{p} \left( \frac{(-x)}{(1-x)^{2}} \right)^{p} + C_{0} - C_{0} \right], \]  

(64)

or, taking into account that \( C_{0} = 1 \),

\[ \mathfrak{F}(x) = -x^{4} \frac{(1-x)^{2}}{(1-x)^{4}} \left[ \sum_{p=0}^{\infty} C_{p} \left( \frac{(-x)}{(1-x)^{2}} \right)^{p} \right] + \frac{x^{4}}{(1-x)^{4}}. \]  

(65)

Using the generating function for Catalan numbers

\[ \sum_{p=0}^{\infty} C_{p} y^{p} = \frac{1 - \sqrt{1 - 4y}}{2y}, \]  

(66)
and the Maclaurin series expansion of the function \( \frac{1}{(1-x)^q} \) for \( q = 4 \), we have

\[
\mathfrak{F}(x) = \frac{-x^4}{(1-x)^4}(1-x) + \frac{x^4}{(1-x)^4}(1-1+x) = \frac{x^5}{(1-x)^4}
\]

\[
= x^5 \left[ \sum_{r=0}^{\infty} C_{r+3}^5 x^r \right] = \sum_{r=0}^{\infty} C_{r+3}^5 x^{r+5} = \sum_{n=5}^{\infty} C_{n-2}^{n-5} x^n = \sum_{n=0}^{\infty} C_{n-2}^{n-5} x^n,
\]

(67)

where \( n = r + 5 \) and we take into account that \( C_{n-2}^{n-5} = 0 \) for \( n < 5 \). Thus (57), and hence (54), are proved.

2. Let’s prove the second of the relations (31), namely

\[
(-1)^{k-1} C_k - 1 + \sum_{t=0}^{k-2} (-1)^{k-2-t} C_t C_{k+1+t} - 1 = C_k - 1.
\]

(68)

Note that for \( t = k - 1 \) we have \((-1)^{k-2-t} = (-1)\) and \( C_{k+t+1} = C_{2k+1} = 1 \). Thus we can rewrite (68) as

\[
\sum_{t=0}^{k-1} (-1)^{k-2-t} C_t C_{k+1+t} = (-1)^{-2} C_k - 1.
\]

(69)

Dividing by \((-1)^{k-2}\), and taking into account that \( C_{k+t+1} = C_{2k+2} \), we obtain

\[
\sum_{t=0}^{k-1} (-1)^{t} C_t C_{k+1+t} = C_{k-1}.
\]

(70)

Because for \( t \geq k \) we have \( C_{k+t+1} = 0 \), we can rewrite (70) in the form

\[
\sum_{t=0}^{\infty} (-1)^{t} C_t C_{k+1+t} = C_{k-1}.
\]

(71)

If we denote \( n = k + 1 \) we obtain

\[
\sum_{t=0}^{\infty} (-1)^{t} C_t C_{n+1} = C_{n-1}.
\]

(72)

So we must prove this formula. To this end, we introduce a generating function

\[
\mathfrak{F}(x) = \sum_{n=0}^{\infty} \left[ \sum_{t=0}^{n} (-1)^{t} C_t C_{n+1} \right] x^n = \sum_{n=0}^{\infty} (-1)^{t} C_t x^t \left[ \sum_{n=0}^{\infty} C_{n+1} x^{n-t} \right].
\]

(73)

Then we have

\[
\mathfrak{F}(x) = \sum_{t=0}^{\infty} (-1)^{t} C_t x^{t+2} \left[ \sum_{n=0}^{\infty} C_{n+1} x^{n-t-2} \right].
\]

(74)

Since for \( n < t + 2 \) we have \( \sum_{n=0}^{\infty} \Rightarrow \sum_{n=t+2} \), we have

\[
\mathfrak{F}(x) = \sum_{t=0}^{\infty} (-1)^{2+t} C_t x^{t+2} \left[ \sum_{n=t+2}^{\infty} C_{n+1} x^{n-t-2} \right].
\]

(75)

Then for \( q = 2t + 3 \) we have

\[
\frac{1}{(1-x)^q} = \sum_{r=0}^{\infty} C_{q+r-1}^{q-1} x^r = \sum_{r=0}^{\infty} C_{2t+r+2}^{2t+2} x^r.
\]

(76)
Replacement $r = n - t - 2$ gives
\[
\frac{1}{(1-x)^t} = \sum_{n=t+2}^{\infty} C_{n+t}^{2t+2} x^{n-t-2} = \frac{1}{(1-x)^{2t+3}}. \tag{77}
\]

Then
\[
\mathcal{F}(x) = \sum_{t=0}^{\infty} (-1)^t C_t x^{t+2} \frac{1}{(1-x)^{2t+3}} = \frac{x^2}{(1-x)^3} \sum_{t=0}^{\infty} C_t \left(\frac{-x}{(1-x)^2}\right)^t. \tag{78}
\]

Considering that (see (66))
\[
\sum_{t=0}^{\infty} C_t \left(\frac{-x}{(1-x)^2}\right)^t = 1 - x, \tag{79}
\]
we obtain
\[
\mathcal{F}(x) = \frac{x^2}{(1-x)^3} (1-x) = \frac{x^2}{(1-x)^2} = x^2 \sum_{r=0}^{\infty} C_{r+1}^1 x^r = \sum_{r=0}^{\infty} C_{r+1}^1 x^{r+2}. \tag{80}
\]
Replacing $n = r + 2$ we get
\[
\mathcal{F}(x) = \sum_{n=2}^{\infty} C_{n-1}^1 x^n = \sum_{n=0}^{\infty} C_{n-1}^1 x^n. \tag{81}
\]
So
\[
\sum_{t=0}^{\infty} (-1)^t C_t C_{n+t}^{2t+2} = C_{n-1}^1, \tag{82}
\]
which was exactly what we needed to prove.

**Appendix 2**

Let us prove (43):
\[
\sum_{q=0}^{k-1} (-1)^{k-q-1} C_q C_{s+q}^{k-1-q} = (-1)^{k-1} C_{s-1}^{k-1}, \quad 1 \leq k \leq s. \tag{83}
\]
or
\[
\sum_{q=0}^{k-1} (-1)^{k-q} C_q C_{s+q}^{k-1-q} = (-1)^k C_{s-1}^{k-1}. \tag{84}
\]
Note that for $k - 1 < q$ we have $C_{s+q}^{k-1-q} = 0$, we can rewrite (84) as
\[
\sum_{q=0}^{\infty} (-1)^{k-q} C_q C_{s+q}^{k-1-q} = (-1)^k C_{s-1}^{k-1}. \tag{85}
\]
Consider generating function
\[
\mathcal{F}(x) = \sum_{s=0}^{\infty} x^s \left[\sum_{q=0}^{\infty} (-1)^{k-q} C_q C_{s+q}^{k-1-q}\right]. \tag{86}
\]
We have
\[
\mathcal{F}(x) = \sum_{q=0}^{\infty} (-1)^{k-q} C_q \left[\sum_{s=0}^{\infty} C_{s+q}^{k-1-q} x^s\right] = \sum_{q=0}^{\infty} (-1)^{k-q} x^{k-2q-1} C_q \left[\sum_{s=0}^{\infty} C_{s+q}^{k-1-q} x^{s-k+2q+1}\right]. \tag{87}
\]
Because for $k - 1 - q < s + q$ we have $C_{s+q}^{k-1-q} = 0$ the last relation can be rewritten as

$$\mathcal{F}(x) = \sum_{q=0}^{\infty} (-1)^{k-q} x^{k-2q-1} C_q \left[ \sum_{s=k-1-2q}^{\infty} C_{s+q}^{k-1-q} x^{s-k+2q+1} \right]$$  \hspace{1cm} (88)$$

The sum in square brackets is the Maclaurin series expansion of the expression $(1 - x)^{-p}$ for $p = k - q$:

$$\frac{1}{(1 - x)^{p}} = \sum_{r=0}^{\infty} C_{p+r-1}^{r} x^{r}.$$  \hspace{1cm} (89)$$

So, we obtain

$$\mathcal{F}(x) = \sum_{q=0}^{\infty} (-1)^{k-q} x^{k-2q-1} C_q \frac{1}{(1 - x)^{k-q}} = (-1)^{k} \frac{x^{k-1}}{(1 - x)^{k}} \sum_{q=0}^{\infty} C_q \left( \frac{-1 - x}{x^{2}} \right)^{q},$$  \hspace{1cm} (90)$$

or for $y = \frac{1-x}{x^{2}}$

$$\mathcal{F}(x) = (-1)^{k} \frac{x^{k-1}}{(1 - x)^{k}} \sum_{q=0}^{\infty} C_q y^{q} = (-1)^{k} \frac{x^{k-1}}{(1 - x)^{k}} \frac{1 - \sqrt{1 - 4y}}{2y},$$  \hspace{1cm} (91)$$

where we use the generating function for Catalan numbers (66). Because for $y = -\frac{1-x}{x^{2}}$

$$\frac{1 - \sqrt{1 - 4y}}{2y} = x,$$  \hspace{1cm} (92)$$

we finally obtain

$$\mathcal{F}(x) = (-1)^{k} \sum_{q=0}^{\infty} C_{x+s}^{k-1-r}.$$  \hspace{1cm} (93)$$

Thus the coefficient at $x^{s}$ is equal

$$(-1)^{k} C_{k+s-k-1}^{k-1} = (-1)^{k} C_{s-1}^{k-1},$$  \hspace{1cm} (94)$$

that’s what we needed to prove.

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