Perturbations of Collective Hamiltonian Systems generated by Lie Algebra contractions

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Abstract. In this article we study perturbed Hamiltonian dynamics in the class of collective Hamiltonian systems associated to Lie algebra Hamiltonian actions. For perturbations of collective systems generated by contractions of a compact Lie algebra \( g \) into a limiting algebra \( g_0 \), we give general conditions to transform such perturbations coming from deformations of the action into perturbations produced by corrections in the energy function of the collective system associated to the action by the limiting Lie algebra \( g_0 \). We give an application to perturbations of collective systems generated by contraction of \( \text{so}(4) \) into \( \epsilon(3) \).

1. Introduction
A Hamiltonian action \( \Sigma \) by a Lie algebra \( g \) on a symplectic manifold \( M \) is a Lie algebra homomorphism between \( g \) and the Lie algebra of Hamiltonian vector fields on \( M \). Such actions always possesses a momentum map whose components in a given dual basis \( B^\ast \) for the coalgebra \( g^\ast \), are given by the hamiltonian functions of the Hamiltonian vector fields which are the images of the basis \( B \) for \( g \). Associated to such kind of infinitesimal actions, we have two main Lie algebras of Hamiltonian systems, one formed by the so called invariant systems which commute with each infinitesimal generator of the action and the second one formed by the so called collective systems whose Hamiltonian function depends only on the components of the momentum map. For any action, the collective Hamiltonian systems commute with the invariant Hamiltonian systems.

Now, suppose we have a contraction of the Lie algebra \( g \) into a Lie algebra \( g_0 \) by an \( \varepsilon \)-family of linear endomorphisms \( A_\varepsilon \) in the usual sense of Inönü-Wigner [1] and consider the \( \varepsilon \)-family of Hamiltonian actions \( \Sigma_\varepsilon \) generated by the \( \varepsilon \)-family \( g_\varepsilon \) of Lie algebras defined in the contraction process. Under the above circumstances, each collective Hamiltonian system with respect the action by the initial Lie algebra, induces for each value \( \varepsilon \) of the contraction parameter, a collective Hamiltonian system with respect the action of \( g_\varepsilon \). Our goal is to study the \( \varepsilon \)-family of collective systems as a perturbation of the collective system with respect the limiting Lie algebra \( g_0 \) and transform such perturbations generated by the contraction into perturbations of the limiting system produced by variations in its Hamiltonian function. A similar problem has been studied in [3] and [4], for perturbations of Hamiltonian systems generated by deformations of the Poisson bracket.

In this article, we show that if the initial Lie algebra \( g \) is compact and the Lie-Poisson structures \( \Psi_\varepsilon \) defined on the coalgebra \( g_\varepsilon^\ast \) are locally trivial deformations of the limiting Lie-Poisson structure \( \Psi_0 \) on \( g_0^\ast \), then on some compact sets of the symplectic manifold \( M \), it is
possible to transform canonically, the perturbed collective system generated by the deformation of the action into a collective system for the limiting action by \( g_0 \) with a perturbation in its Hamiltonian function.

In the first section, we present definitions and well known facts on Hamiltonian actions by Lie algebras and its momentum map. In the second one, we study the family of momentum maps induced by a contraction of the Lie algebra and using techniques based on the solvability of a homological equation expressed in terms of the Schouten bracket for contravariant antisymmetric tensors, we transform such family into a family of momentum maps of Hamiltonian actions by the limiting Lie algebra \( g_0 \). In this situation, extending results by Palais [10] and Miranda et al [11] on the rigidity of symplectic actions by compact groups, we show that the family of momentum maps relative to actions by \( g_0 \) is a trivial deformation of the limiting momentum map \( \mu^0 \).

In the third section, we apply the previous results to perturbations of collective Hamiltonian systems generated by contraction of the Lie algebra and show how to transform them by symplectomorphisms into perturbations of the limiting collective system coming from corrections in its Hamiltonian function. Finally, in the last section, we present an application to perturbations of collective systems generated by a contraction of the Lie algebra \( so(4) \) into \( e(3) \). For this case, we give explicit formulas for the energy correction to the limiting collective system.

2. Hamiltonian actions and its momentum map

Let \( (M, \Omega) \) be a symplectic manifold and \( g \) an \( r \)-dimensional real Lie algebra. For each smooth function \( f \in C^\infty(M) \) denote by \( X_f \) the Hamiltonian vector field of \( M \) with Hamiltonian function \( f \). A Hamiltonian action of \( g \) on \( M \), is a homomorphism of Lie algebras
\[
\Sigma : g \rightarrow \mathcal{X}_{Ham}(M) \tag{2.1}
\]
where \( \mathcal{X}_{Ham}(M) \) denotes the Lie algebra of Hamiltonian vector fields on \( M \). By definition, \( \Sigma \) has the following properties
\[
\Sigma(\lambda \alpha + \beta) = \lambda \Sigma(\alpha) + \Sigma(\beta), \tag{2.2}
\]
\[
[\Sigma(\alpha), \Sigma(\beta)] = \Sigma([\alpha, \beta]), \quad \forall \lambda \in \mathbb{R}, \quad \alpha, \beta \in g. \tag{2.3}
\]

For a basis \( \{e_1, e_2, ..., e_r\} \) of \( g \), let us choose smooth functions \( f_i \), \( i = 1, ..., r \) with
\[
\Sigma(e_i) = X_{f_i}, \quad i = 1, ..., r \tag{2.4}
\]
and define the momentum map
\[
\mu : M \rightarrow g^* \tag{2.5}
\]
\[
\mu(x) = \sum_{i=1}^r f_i(x) e_i^*
\]
where \( B^* = \{e_1^*, ..., e_r^*\} \) denotes the dual basis for the coalgebra \( g^* \) corresponding to the basis \( B \). Then, for each \( \alpha \in g \) the smooth function
\[
\mu_\alpha(x) = \langle \mu(x), \alpha \rangle, \quad x \in M \tag{2.6}
\]
is a Hamiltonian function for \( \Sigma(\alpha) \) and the correspondence \( \alpha \rightarrow \mu_\alpha \) establishes a Lie algebra homomorphism between \( g \) and the Poisson algebra of smooth function on \( M \), i.e.
\[
\mu_{\lambda \alpha + \beta} = \lambda \mu_\alpha + \mu_\beta \tag{2.7}
\]
\[
\{\mu_\alpha, \mu_\beta\} = \mu_{[\alpha, \beta]} \quad \forall \alpha, \beta \in g. \tag{2.8}
\]
where \(\{,\}\) denotes the Poisson bracket on \(M\) defined as
\[
\{f,g\} = \langle dg, X_f \rangle, \ \forall f, g \in C^\infty(M)
\] (2.9)
Moreover, the momentum map \(\mu\) is a Poisson map between the symplectic structure of \(M\) and the Lie-Poisson structure on \(\mathfrak{g}^*\), i.e. for all \(F,G \in C^\infty(\mathfrak{g}^*)\) we have
\[
\{F \circ \mu, G \circ \mu\} = \{F,G\}_\mathfrak{g} \circ \mu,
\] (2.10)
where \(\{,\}_\mathfrak{g}\) denotes the Lie-Poisson bracket on \(\mathfrak{g}^*\) defined by
\[
\{F,G\}_\mathfrak{g}(\xi) = \langle \xi, [\delta F(\xi), \delta G(\xi)] \rangle, \ \xi \in \mathfrak{g}^*
\] (2.11)
and \(\delta F(\xi)\) is for each \(F \in C^\infty(\mathfrak{g}^*)\) the unique element of \(\mathfrak{g}\) such that
\[
\langle \delta F(\xi), \eta \rangle = dF(\xi)(\eta), \ \forall \eta \in \mathfrak{g}^*
\] (2.12)
For each Hamiltonian action \(\Sigma\) on \(M\), we have two important Poisson algebras, the Poisson algebra \(I(\mu)\) of \(\mathfrak{g}\)-invariant smooth functions \(f\) defined by the condition
\[
\{f,\mu_\alpha\} = 0, \ \forall \alpha \in \mathfrak{g}
\] (2.13)
and the Poisson algebra \(C(\mu)\) of collective Hamiltonian functions defined as those smooth functions \(F\) on \(M\) having the form
\[
F = K \circ \mu \tag{2.14}
\]
with \(K: \mathfrak{g}^* \to \mathbb{R}\) a smooth function. Note that each \(\mathfrak{g}\)-invariant function commutes with each collective Hamiltonian function.

3. Deformations of Hamiltonian actions generated by Lie algebra contractions
A contraction of a Lie algebra \(\mathfrak{g} = (\mathbb{R}^r, [,])\) into a "limiting" algebra \(\mathfrak{g}_0 = (\mathbb{R}^r_0, [,])\) is given by a smooth \(\varepsilon\)-family of linear transformations
\[
A_\varepsilon: \mathbb{R}^r \to \mathbb{R}^r
\] (3.1)
such that: i) \(A_\varepsilon\) is invertible for \(\varepsilon > 0\), ii) \(A_0\) is a singular linear transformation, iii) the limit \([\alpha, \beta]_0 = \lim_{\varepsilon \to 0} A_\varepsilon^{-1}[A_\varepsilon(\alpha), A_\varepsilon(\beta)]\), exists \(\forall \alpha, \beta \in \mathbb{R}^r\), iv) the \(\varepsilon\)-family of commutators \([\alpha, \beta]_\varepsilon = A_\varepsilon^{-1}[A_\varepsilon(\alpha), A_\varepsilon(\beta)]\), \(\forall \alpha, \beta \in \mathbb{R}^r\) depends smoothly on \(\varepsilon\).
For more complete information on contraction of Lie algebras see [5].
For \(\varepsilon \geq 0\), the \(\varepsilon\)-family of Lie algebras \(\mathfrak{g}_\varepsilon = (\mathbb{R}^r_\varepsilon, [,])\) define an \(\varepsilon\)-family of Hamiltonian actions on \(M\) through
\[
\Sigma_\varepsilon: \mathfrak{g}_\varepsilon \to \mathcal{X}_{Ham}(M) \tag{3.2}
\]
\[
\Sigma_\varepsilon(\alpha) = \Sigma(A_\varepsilon(\alpha)), \ \forall \alpha \in \mathfrak{g}_\varepsilon
\]
with momentum map
\[
\mu_\varepsilon: M \to \mathfrak{g}_\varepsilon^* \tag{3.3}
\]
\[
\mu_\varepsilon = A^T_\varepsilon \mu
\]
For each \(\varepsilon \geq 0\), we denote by \(\Psi_\varepsilon\) the Lie-Poisson tensor on the coalgebra \(\mathfrak{g}_\varepsilon^*\)
\[
\Psi_\varepsilon(dF, dG)(\xi) = \langle \xi, [\delta F(\xi), \delta G(\xi)]_\varepsilon \rangle \tag{3.4}
\]
and
\[ \Psi_0 = \lim_{\epsilon \to 0} \Psi_\epsilon \]  
(3.5)

In very important cases, the Poisson tensor \( \Psi_\epsilon \) is a locally trivial deformation on regular domains of \( \Psi_0 \). We recall that \( D \) is a regular domain if \( D \) is the union of symplectic leaves of constant rank of \( \Psi_0 \) and this foliation is a fibration. The Poisson tensors \( \Psi_\epsilon \) define a locally trivial deformation of \( \Psi_0 \) on \( D \) if there exists on \( D \) a near identity family of diffeomorphisms \( \gamma_\epsilon \) such that
\[ \gamma_\epsilon^* \Psi_\epsilon = \Psi_0 \]  
(3.6)

The condition (3.6) is equivalent to the solvability of the homological equation
\[ [[X_\epsilon, \Psi_\epsilon]] = -\frac{d}{d\epsilon} \Psi_\epsilon \]  
(3.7)

where \([[],]\) is the Schouten bracket between contravariant antisymmetric tensor fields and \( X_\epsilon \) denotes the time-dependent vector field which generates the \( \epsilon \)-family \( \gamma_\epsilon \), see [6]. In general, the homological equation (3.7) is not solvable in the whole phase space because of the different topological type of the symplectic leaves for the perturbed tensor \( \Psi_\epsilon \) and the unperturbed \( \Psi_0 \).

In such a situation we can ask for solutions in some region or solutions up to some power of \( \epsilon \) as it is usual in perturbation problems.

If the homological equation is solvable in some regular domain \( D \) for some vector field \( X_\epsilon \) and \( \gamma_\epsilon \) denotes the solution of the Cauchy problem
\[ \frac{d}{d\epsilon} \gamma_\epsilon = X_\epsilon(\gamma_\epsilon), \; \gamma_0 = \text{Identity}, \]  
(3.8)

then, each \( \gamma_\epsilon \) is a Poisson map on \( D \) between the Poisson tensors \( \Psi_\epsilon \) and \( \Psi_0 \)
\[ \{ F \circ \gamma_\epsilon, G \circ \gamma_\epsilon \}_0 = \{ F, G \}_\epsilon \circ \gamma_\epsilon, \; \forall F, G \in C^\infty(\mathfrak{g}_\epsilon^*) \]  
(3.9)

and taking into account that the momentum map \( \mu_0 \) in (3.4) is a Poisson map between the Poisson bracket of \( M \) and \( \Psi_0 \), we have for \( F, G \in C^\infty(\mathfrak{g}_\epsilon^*) \)
\[ \{ F \circ \gamma_\epsilon \circ \mu_0, G \circ \gamma_\epsilon \circ \mu_0 \} = \{ F \circ \gamma_\epsilon, G \circ \gamma_\epsilon \}_0 \circ \mu_0 = \{ F, G \}_\epsilon \circ \gamma_\epsilon \circ \mu_0, \]  
(3.10)

and for each \( \epsilon \geq 0 \), we have proved that the map
\[ \gamma_\epsilon \circ \mu_0 : (\mu_0)^{-1}(\gamma_\epsilon(D)) \to \mathfrak{g}_\epsilon, \]  
(3.12)

is a momentum map associated to a Hamiltonian action by the perturbed Lie algebra \( \mathfrak{g}_\epsilon \).

Now, suppose the initial Lie algebra \( \mathfrak{g} \) is compact. Then \( \mathfrak{g}_\epsilon \) for \( \epsilon > 0 \) is also compact but the limiting Lie algebra \( \mathfrak{g}_0 \) is always not compact. In this situation, we can apply the following result by E. Miranda et al [11] on the equivalence and rigidity of symplectic actions by compact Lie groups on compact symplectic manifolds.

**Theorem 3.1** (Miranda et al [11]) Let \( \Phi_i : G \times M \to M, \; i = 1, 2 \) be two \( C^2 \)-close symplectic actions of a compact Lie group \( G \) on a compact symplectic manifold \( (M, \Omega) \). Then there exists a symplectomorphism \( \phi \) on \( M \) with
\[ \Phi_1(g, x) = \phi(\Phi_2(g, \phi^{-1}(x)), \; \forall g \in G, \; x \in M \]  
(3.13)
To apply the above theorem to our case, we note that condition (3.13) implies for Hamiltonian actions by a Lie algebra $g$ with momentum maps $\mu$ and $\rho$, the existence of a symplectomorphism $\phi$ such that
\[ \mu = \rho \circ \phi. \]
In such a case we say that both momentum maps are equivalent [9]. Note that for each $\alpha \in g$, we have $\mu_\alpha = \rho_\alpha \circ \phi$ and the associated Hamiltonian vector fields $X_{\mu_\alpha}$ and $X_{\rho_\alpha}$ have conjugate flows.

Applying the above result to the $g_\varepsilon$-Hamiltonian actions with momentum maps $\gamma_\varepsilon \circ \mu^0$ and $\mu^\varepsilon$, we have the following

**Proposition 3.2** Let $\gamma_\varepsilon : D \to g_\varepsilon^*$ with $\gamma_\varepsilon^* \Psi_\varepsilon = \Psi_0$ and $C \subset M$ a compact set with $\mu^0(C) \subset D$. Then for sufficiently small $\varepsilon$ there exist a near identity symplectomorphism $\phi_\varepsilon : C \to M$ such that
\[ \gamma_\varepsilon \circ \mu^0 = \mu^\varepsilon \circ \phi_\varepsilon \text{ on } C. \] (3.14)

From (3.14) we also have
\[ \mu^0 \circ \phi_\varepsilon^{-1} = \gamma_\varepsilon^{-1} \circ \mu^\varepsilon \text{ on } \phi_\varepsilon(C) \] (3.15)
and then, the $\varepsilon$-family of momentum maps $\gamma_\varepsilon^{-1} \circ \mu^\varepsilon$ are for each $\varepsilon$ equivalent to the limiting momentum map $\mu^0$ on the compact set $\phi_\varepsilon(C)$. Note that the $g_0$-Hamiltonian action $\Lambda^\varepsilon : g_0 \to \mathcal{X}_{\text{Ham}}$ with momentum map $\gamma_\varepsilon^{-1} \circ \mu^\varepsilon$ takes the form
\[ \Lambda^\varepsilon(\alpha) = X_{h_\alpha}, \text{ with } \]
\[ h_\alpha(x) = \langle \gamma_\varepsilon^{-1} \circ \mu^\varepsilon(x), \alpha \rangle = \mu^0_\alpha(\phi_\varepsilon^{-1}(x)) \] (3.16)
We can summarize the above discussion in terms of Hamiltonian actions in the following proposition

**Proposition 3.3** If under the contraction (3.1) of a compact Lie algebra $g$ into a limiting algebra $g_0$, the homological equation (3.7) is solvable on some domain $D \subset g_0^*$, then for a compact set $C \subset M$ with $\mu^0(C) \subset D$ and sufficiently small $\varepsilon$, the momentum map $\mu^\varepsilon$ associated to the Hamiltonian action (3.2) can be transformed by a near identity diffeomorphism in $g_\varepsilon^*$ into a momentum map equivalent to $\mu^0$.

4. Perturbations of collective Hamiltonian systems
A Hamiltonian system $X_F$ on $M$ is called a Collective Hamiltonian system with respect some $g$-Hamiltonian action $\Sigma$ if its Hamiltonian function has the form
\[ F = K \circ \mu \] (4.1)
where $\mu$ is the momentum map and $K : g^* \to \mathbb{R}$ is a smooth function. To the function $K$ we called the collective part of function $F$.

If $x(t)$ is an integral curve of the Collective Hamiltonian system $X_F$, then $\xi(t) = \mu(x(t))$ is an integral curve for the Euler equation on $g^*$ with Hamiltonian vector field $Y_K$ given by
\[ Y_K(\xi) = ad^*_{\delta K(\xi)} \xi \] (4.2)
Moreover, if $\{e_1, e_2, ..., e_r\}$ is a basis for $g$, the collective Hamiltonian system $X_F$ belongs at each point of $M$ to the tangent subspace spanned by the generator vector fields $X_{e_i}$ with Hamiltonian functions $\mu e_i$ for $i = 1, ..., r$. In fact, expressing each $\xi \in g^*$ in the form $\xi = \sum_{i=1}^r \xi_i e_i^*$, we have
\[ X_F(x) = \sum_{i=1}^r \frac{\partial K}{\partial \xi_i}(\mu(x))X_{e_i}(x) \] (4.3)
with \( \xi = \sum_{i=1}^{r} \xi_i e_i \).

From the above remarks we have: each collective Hamiltonian vector field is projectable. Moreover, the integral curve \( x(t) \) of \( X_F \) with \( x(0) = x_0 \) satisfies the time dependent system on \( M \)

\[
\frac{dx}{dt} = \sum_{i=1}^{r} \frac{\partial K}{\partial \xi_i}(\xi(t)) X_{e_i}(x)
\]

(4.4)

where \( \xi(t) \) is the integral curve with \( \xi(0) = \mu(x_0) \) of the Euler equation on \( g^* \)

\[
\frac{d\xi}{dt} = ad^*_\delta K(\xi)\xi
\]

(4.5)

Remark 4.1 note that the system (4.4) is a time dependent system where the right term is a linear combination of vector fields closing under the Lie bracket into the algebra \( g \). This systems are called Lie systems and has very special properties, see [8].

Remark 4.2 If the function \( K \) is a Casimir function for the Lie-Poisson structure associated to \( g \), then \( \xi(t) = \mu(x_0) \) in (4.5) and the vector field in the equation (4.4) is the Hamiltonian vector field \( \Sigma(\delta K(\mu(x_0))) \). For more complete information on collective Hamiltonian dynamics see [7].

Consider now the contraction of \( g \) into \( g_0 \) and the \( \varepsilon \)-family of collective Hamiltonian functions

\[ H_\varepsilon = K \circ \mu^\varepsilon, \varepsilon \geq 0 \]

corresponding to the Hamiltonian actions \( \Sigma_\varepsilon \). Using the expression (3.15) we can write

\[ H_\varepsilon = K \circ \gamma_\varepsilon \circ \gamma_\varepsilon^{-1} \circ \mu^\varepsilon = K \circ \gamma_\varepsilon \circ \mu^0 \circ \phi_\varepsilon^{-1} \]

(4.6)

where \( \gamma_\varepsilon \) satisfies (3.8) and \( \phi_\varepsilon \) is a symplectomorphism.

From the previous discussion we have the following

**Proposition 4.3** Under the assumptions of previous proposition, for each \( \varepsilon > 0 \), the collective Hamiltonian system on \( M \) with Hamiltonian function \( H_\varepsilon \) can be canonically transformed by a near identity symplectomorphism \( \phi_\varepsilon \) into a collective Hamiltonian system with respect the limiting action by \( g_0 \) with Hamiltonian function

\[ H_\varepsilon^0 = H_\varepsilon \circ \phi_\varepsilon = K \circ \gamma_\varepsilon \circ \mu^0 \]

(4.7)

To the function \( K_\varepsilon = K \circ \gamma_\varepsilon \) we call the corrected collective part

**Remark 4.4** If in the homological equation (3.7) we have a solution \( X_\varepsilon = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + O(\varepsilon^3) \), we can write the correction to the collective part of the limiting collective system in the form

\[ K_\varepsilon = K \circ \gamma_\varepsilon = K + \varepsilon L_X K + \frac{\varepsilon^2}{2} L_X K + O(\varepsilon^2) \]

(4.8)
5. Perturbations of collective Hamiltonian systems generated by contractions of \(so(4)\) into \(e(3)\)

Consider the 6-dimensional Lie algebra \(so(4)\) with commutation relations given on a basis \(\{e_1, e_2, e_3, v_1, v_2, v_3\}\)

\[
[e_i, e_j] = \epsilon_{ijk} e_k, \\
[e_i, v_j] = \epsilon_{ijk} v_k, \\
v_1, v_2 = \epsilon_{ijk} e_k, \ i, j, k = 1, 2, 3.
\]

and the \(\epsilon\)-family of linear transformations \(A_\epsilon\) defined on the above basis by

\[
A_\epsilon(e_i) = e_i, \ i = 1, 2, 3 \quad (5.1)
\]

\[
A_\epsilon(v_i) = \epsilon v_i, \ i = 1, 2, 3 \quad (5.2)
\]

For each \(\epsilon > 0\) we have the new commutator

\[
[e_i, e_j]_\epsilon = \epsilon_{ijk} e_k \\
[e_i, v_j]_\epsilon = \epsilon_{ijk} v_k \\
v_1, v_2 = \epsilon^2 \epsilon_{ijk} e_k
\]

and for \(\epsilon = 0\), the limiting algebra \(e(3)\) with commuting relations

\[
[e_i, e_j]_0 = \epsilon_{ijk} e_k \\
[e_i, v_j]_0 = \epsilon_{ijk} v_k \\
v_1, v_2 = 0
\]

On \(\mathbb{R}^6 = (y, z)\), \(y \in \mathbb{R}^3\), \(z \in \mathbb{R}^3\) associated to \(so(4)\) and \(e(3)\) we have following Lie-Poisson brackets

\[
\Psi_\epsilon(f, g) = \left\langle y \times \frac{\partial f}{\partial y} + z \times \frac{\partial f}{\partial z}, \frac{\partial g}{\partial y} \right\rangle + \left\langle z \times \frac{\partial f}{\partial y} + \epsilon^2 y \times \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \right\rangle \quad (5.3)
\]

\[
\Psi_0(f, g) = \left\langle y \times \frac{\partial f}{\partial y} + z \times \frac{\partial f}{\partial z}, \frac{\partial g}{\partial y} \right\rangle + \left\langle z \times \frac{\partial f}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \quad \text{for} \ \epsilon = 0 \quad (5.4)
\]

We can write

\[
\Psi_\epsilon = \Psi_0 + \epsilon^2 \Phi \quad \text{with}
\]

\[
\Phi(f, g) = \left\langle y \times \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z} \right\rangle
\]

The corresponding Casimir functions for \(\Psi_\epsilon\) for \(\epsilon \geq 0\) are

\[
k_1(y, z) = y \cdot z
\]

\[
k_2(y, z) = \frac{\epsilon^2}{2} y \cdot y + \frac{1}{2} z \cdot z
\]

and for \(\epsilon \geq 0\), the set \(R_\epsilon\) of points with maximum rank (= 4) for \(\Psi_\epsilon\) are

\[
R_\epsilon = \{(y, z) \in \mathbb{R}^6 \text{ with } \epsilon z \neq \pm y\}
\]

\[
R_0 = \{(y, z) \in \mathbb{R}^6 \text{ with } z \neq 0\}
\]
The regular symplectic leaves $O_{\lambda_1, \lambda_2}$ for $\Psi_\varepsilon$ have the form
\[ O_{\lambda_1, \lambda_2 > 0}^0 = T^* S^2 \]
\[ O_{\lambda_1, \lambda_2 > 0} = S^2 \times S^2, \quad \varepsilon > 0 \]
and consequently because its different topological type, there is no global homotopy between the $\Psi_\varepsilon$ and $\Psi_0$.

The homological equation (3.7) takes the form
\[ [[X_\varepsilon, \Psi_0 + \varepsilon^2 \Phi]] = -2\varepsilon \Phi \quad (5.5) \]
and the general solution is given by the vector fields
\[ X_\varepsilon(y, z) = -\varepsilon y^2 \| y \times z \|^2 y \times (z \times y) \frac{\partial}{\partial z} + Z \quad (5.6) \]
where $Z$ is any Poisson vector field of $\Psi_0 + \varepsilon^2 \Phi$.

Taking $Z = 0$, the solution to the Cauchy problem
\[ \frac{d\gamma_\varepsilon}{d\varepsilon} = X_\varepsilon(\gamma_\varepsilon), \quad \gamma_0 = \text{Identity} \quad (5.7) \]
takes the form
\[ \gamma_\varepsilon(y, z) = (y, \lambda(\varepsilon)z + \frac{y \cdot z}{y^2} (1 - \lambda(\varepsilon))y) \quad (5.8) \]
where
\[ \lambda(\varepsilon) = (1 - \frac{\varepsilon^2 y^4}{2 \| y \times z \|^2})^{\frac{1}{2}} \quad (5.9) \]

Consider now a Hamiltonian action $\Sigma$ by so(4) on a compact symplectic manifold $M$ with momentum map $\mu$ and the $\varepsilon$-family of Hamiltonian actions generated by the contraction (5.1)-(5.2) of so(4) into $e(3)$. Consider a collective Hamiltonian system on $M$ with Hamiltonian function $F = K \circ \mu$ and the $\varepsilon$-family of collective Hamiltonian systems with Hamiltonian function $F_\varepsilon = K \circ \mu^\varepsilon$ where $\mu^\varepsilon$ is the momentum map $\mu^\varepsilon = A_\varepsilon^T \circ \mu$ associated to the action $\Sigma_\varepsilon = \Sigma \circ A_\varepsilon$ by the Lie algebra $\mathfrak{g}_\varepsilon$. Then, on any compact $C \subset M$ such $\mu^0(C)$ belongs to the set $\{(y, z) \in \mathbb{R}^6, \| y \times z \| \neq 0 \}$ and for $\varepsilon$ sufficiently small, $F_\varepsilon$ is transformed by some symplectomorphism $\phi_\varepsilon$ on into a collective Hamiltonian system with Hamiltonian function $H_\varepsilon^0 = H_\varepsilon \circ \phi_\varepsilon = K_\varepsilon \circ \mu^0$ where
\[ K_\varepsilon = K - \frac{\varepsilon^2}{2 \| y \times z \|^2} \left( y \times (z \times y), \frac{\partial K}{\partial z} \right) + O(\varepsilon^3) \quad (5.10) \]

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References
[1] Inonu E and Wigner E P 1953 On contraction of groups and their representations Proc. Nat. Acad. Sci. US 39 510-524
[2] Libermann P and Marle Ch-M 1987 Symplectic Geometry and Analitical Mechanics (D. Reidel Publ. Co.)
[3] Flores-Espinoza R, and Vorobjev Y  On perturbations of Hamiltonian systems generated by contractions of Lie algebras in New Trends for Hamiltonian Systems and Celestial Mechanics, ed Lacomba and J Llibre (London Advanced Series in Nonlinear dynamics V.8 World Scientific) pp 357-374

[4] Flores-Espinoza R and Vorobjev Y M 2000  Relativistic corrections to elementary Galilean dynamics and deformations of Poisson brackets, in Hamiltonian Systems and Celestial Mechanics (HAMSYS-98) ed J. Delgado, E Lacomba, E Perez-Chavela and J Llibre (World Scientific Monograph Series in Mathematics V. 6. World Scientific Publ. Co.) pp 161-173

[5] Gilmore R 1974 Lie groups, Lie algebras, and some of their applications (New York: Wiley and Sons)

[6] Maslov V P and Karasev M V 1993 Nonlinear Poisson brackets, Geometry and Quantization Transl. Math. Monographs. 110 Am. Math. Soc. Providence R.I.

[7] Guillemin V and Sternberg S 1984 Symplectic Techniques in Physics (Cambdrige: Univ. Press.)

[8] Cariñena J F, Grabowski J and Marmo G 2007 Superposition rules, Lie theorem and partial differential equations, Rep. Math. Phys. 60 237-258

[9] Palais R S and Stewart T E 1960 Deformations of compact differentiable Transformation groups American Journal of Mathematics 82 4 935-937

[10] Palais R 1961 Equivalence of nearby differentiable actions of a compact group Bull. Amer. Math. Soc. 67 362-364

[11] Miranda E, Monnier P and Nguyen Tien Zung 2011 Rigidity of Hamiltonian actions on Poisson manifolds arXiv:1102.0175v1 [Math.SG]