INFLUENCE OF CONVective BOUNDARY CONDITION ON HEAT AND MASS TRANSFER IN A WALTERS’ B FLUID OVER A VERTICAL STRETCHING SURFACE WITH THERMAL-DIFFUSION EFFECT

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ABSTRACT

This paper presents heat and mass transport in the flow of Walters’ B fluid via a vertical stretching sheet with the thermal-diffusion impact as a function of Convective Boundary Condition. The coupled nonlinear partial differential equations governing the system are presented in the form of coupled ordinary differential equations via similarity transformation variables which then solved by the Homotopy Analysis Method. The effect of various parameters on velocity, temperature and concentration profiles as well as Local Skin-friction, Nusselt and Sherwood numbers are plotted and discussed. The findings outcome revealed among others that distinct values of thermal buoyancy parameter speed-up the movement of the fluid and cools the thermal layer while the surface heat transfer is boosted when the strength of Radiation improves. Also, large values of Biot number constitute strong convective heating which consequently maximizes the thickness of the associated boundary layer and enables the heat influence to break through into the quiescent fluid. Biot number is of great importance in the engineering field for drying of the materials.

Keywords: Thermal Radiation, Similarity Variables, Local Weissenberg Number, Thermal-Diffusion, Homotopy Technique

INTRODUCTION

The interaction of heat and mass transport by natural convection in laminar boundary layer flows has received significant attention in the years past and extensively studied in the literature for both steady and unsteady phenomenon of Newtonian fluid over its numerous appliances in the sphere of science and engineering domain. Among the early investigation revealed by Ali et al.[1] shows that thermal radiation interaction enhances the wall shear stress as well as the surface heat transfer rate while investigating the natural convection-radiation interaction in boundary layer flow over the horizontal surface. Arpaci [2] studied the effect of thermal radiation on the laminar free convection from a heated vertical plate. Recently, other researchers also made their contribution to the literature. Afify and El-Aziz [3] revealed that the heat transfer rates for both pseudoplastic and dilatant nanofluids are insensitive to change in viscosity for lower values of Biot number and declined by relatively strong convective heating with higher values of Biot number. Sarafras et al.[4] reported that the only influence of sub-cooling temperature is found to decrease the corresponding heat flux related to the onset of nucleate boiling. El-Aziz and Nabil [5] justified among others that the velocity slip leads to a faster rate of cooling of the stretching sheet only in the case of free convection flow regime. El-Aziz [6] investigated thermal-diffusion and diffusion-thermo effects on combined heat and mass transfer by hydromagnetic three-dimensional free convection over a permeable stretching surface with radiation. The result shows among others that the maximum effect of the thermal-diffusion and diffusion-thermo on the velocity occurs in the absence of the magnetic field when the plate is impermeable. El-Aziz [7] examined the radiation effect on the flow and heat transfer over an unsteady stretching sheet where it was pointed out that the effect of radiation parameter on the temperature distribution of a steady flow is more pronounced than that of unsteady flow. Abo-Eldahab and El Aziz [8] investigated the effect of blowing/suction on hydromagnetic heat transfer by mixed convection from an inclined continuously stretching surface with internal heat
generation/absorption. Oahimire and Olajuwon [9] examined the effect of radiation absorption and thermo-diffusion on MHD heat and mass transfer flow of a micropolar fluid in the presence of heat source. The result shows among others that in the presence of the uniform magnetic field, an increase in the strength of the applied magnetic field decelerates the fluid motion along the wall of the plate inside the boundary layer, whereas the micro-rotational velocity of the fluid along the wall of the plate increases. The influence of heat and mass transport on magnetohydrodynamic flow with a convective boundary condition was studied by Makind [10]. However, the law of Newtonian fluid has been proved to be in good agreement with Newton’s second law of motion which work very well for air, water and other fluid delineated with Navier-Stokes and conservation of energy equation but failed while dealing with more complex fluid, especially with the emergence of viscoelastic fluid or polymeric liquid. The deficiency encountered in the theory of Newtonian fluid and recent development in Science and Technology with its numerous biological and industrial applications, such as polymer solution, paint ink, and cake butter e.t.c had made its studies interesting to all and recently studied in the literature. El-Aziz [11] found that a viscoelastic fluid is more sensitive to the variable fluid properties effect than a Newtonian fluid. Labropulu et.al.[12] examined the stagnation-point flow of the Walters’ B fluid with slip where the effect of condition and the viscoelasticity were to increase the velocity near the wall. Shivakumara et al.[13] reported that the effect of thermal modulation disappears at large frequencies in all the cases of thermal modulation while investigating the effect of thermal modulation on the onset of convection in Walters B viscoelastic fluid-saturated porous medium. Rana et al.[14] and (Aggarwal and Verma[15]) reported that Walters’ (model B') visco-elastic fluid behaves like an ordinary Newtonian fluid due to the vanishing of the viscoelastic parameter. Pandey et.al.[16] investigated the characteristic of Walter’s B visco-elastic Nanofluid layer heated from below. It is reported among other that the Kinematic visco-elasticity parameter destabilizes the oscillatory convection and has no effects on stationary convection. Other authors like Thirumurugan and Vasanthakumari [17], Sharma et al.[18], Kango et al.[19], Rana [20-21] also contributed to the literature about Walters’ B fluid.

Going by the previous effort of other researchers in the literature, much attention has not been given to the impact of the Boit number on Walters’ B fluid. On that note, this work is set to examine the influence of Convective Boundary Condition on heat and mass transfer in a Walters’ B fluid over a vertical stretching surface with thermal-diffusion effect, having considered unaddressed to the best of our knowledge in the literature. The boundary layer equations governing the system are then solved via the Homotopy Analysis Method being a modern method for solving both linear and nonlinear differential equations.

**MATHEMATICAL FORMULATION**

In this article, the steady flow of heat and mass transfer over a vertical surface of Walters’ B viscoelastic fluid is considered. We presumed that the plate experienced heat by convection at \( T_f \), resulting in a heat transfer coefficient of \( h_f \). In the \( y \)-direction, a magnetic field \( B_0 \) of uniform strength is applied while the induced magnetic field is not taken into account due to the magnetic Reynolds number that is really small in the most fluid used in industries and the joule heating is ignored because it is too little to affect free convection motion. The \( x \)-axis is considered alongside the main flow, and the \( y \)-axis is perpendicular to it. The temperature as well as concentration of the fluid is respectively considered as \( T \) and \( C \), \( C_w \) is the plate surface concentration while \( T_\infty \) and \( C_\infty \) respectively denote the ambient temperature and concentration. The transfer of heat and mass characteristics is considered via non-uniform heat generation/absorption and thermal-diffusion effect. (See Fig. 1). The velocity of the stretching is denoted by \( u_\eta(x) = ax \) where \( a > 0 \)

On the account of elastic properties of the fluid which are important in extensional behaviors of polymer, the stress tensor \( S^* \) for Walters’ B fluid is expressed as (See Nadeem et.al.[22]):

\[
S^* = 2\eta_0 - 2k_0 \frac{\delta e}{\delta t}
\]  

(1)
Figure 1. Flow configuration and coordinate system

where \( e \) is the strain tensor rate and the convected derivative of a tensor quantity in terms of material motion is \( \frac{\delta e}{\delta t} \), defined by

\[
\frac{\delta e}{\delta t} = e - e \nabla v - (\nabla v)^T . e, \eta_0 = \int_0^\infty N(\tau) d\tau
\]
denotes the limiting viscosity at small shear rates, \( k_0 = \int_0^\infty \tau N(\tau) d\tau \) represents the short memory coefficient, while \( N(\tau) \) denotes the relaxation time \( \tau \) of the distribution function. The term involving \( k_0 = \int_0^\infty \tau N(\tau) d\tau \) (at \( n \geq 2 \)) is ignored because of the short memory.

The governing equation for Walters’ B fluid based on the above assumption and approximated Boussinesq’s approximation in agreement with Mihra et al. [23] is given by

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - k_0 \left[ u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \right]
- \frac{\sigma B^2}{\rho} u + g \beta_T (T - T_\infty) + g \beta_c (C - C_\infty)
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} + Q_0
\]

\[
u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_m \frac{\partial^2 C}{\partial y^2} + \frac{D_m K_T}{T_m} \frac{\partial^2 T}{\partial y^2}
\]

The appropriate boundary conditions for the problem are expressed as

\[
u(x, 0) = u_w(x) = ax, \quad v(x, 0) = 0, \quad -k \frac{\partial T(x, 0)}{\partial y} = h_f [T_f - T(x, 0)], \quad C(x, 0) = C_w
\]

\[
U(x, \infty) = 0, \quad T(x, \infty) = T_\infty, \quad C(x, \infty) = C_\infty
\]
The velocity components acting in \(x\) and \(y\) directions are respectively denoted as \(u\) and \(v\), \(q_r\) is the radiation heat flux, the specific heat at constant pressure is denoted by \(C_p\), \(\nu\) connotes kinematic viscosity, \(\beta_e\) stands for concentration expansion coefficient, \(D_m\) is the mass diffusivity, \(\alpha\) is the thermal diffusivity, \(g\) is the acceleration due to gravity, \(\rho\) is the density, \(T_m\) is the mean fluid temperature, \(\sigma\) is the fluid electrical conductivity, \(K_f\) is the thermal diffusion ratio, \(Q_0\) is the non-uniform heat generation/absorption coefficient defined by \(Q_0 = \frac{k_{mw}(x)}{\rho c_p x} [A(T_w - T_\infty) \theta' + (T - T_\infty)B]\), where \(A\) connotes the space-dependent and \(B\) stands for the temperature-dependent heat generation/absorption (see Hayat et al.[24]) while \(\beta_e\) is the thermal expansion coefficient. Keeping in mind that the boundary layer is optically thick, therefore, the Rosseland approximation for heat transfer is considered (See Uddin et al.[25]), hence, the radiative heat flux is modeled as

\[
q_r = -\frac{4\sigma^* \partial T^4}{3k^* \partial y}
\]  

(7)

where \(k^*\) is the means of absorption coefficient and \(\sigma^*\) is the Stefan-Boltzmann constant. We assumed that the temperature variation within the flow is such that the term \(T^4\) may be simplified as a linear function of temperature by expanding \(T^4\) in a Taylor series about \(T_\infty\) and neglecting higher-order terms, gives

\[
T^4 \approx 4T_\infty^3 T - 3T_\infty^4
\]

invoking (7) and (8) in equation (3), gives a modified equation of the form

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{16\sigma^* T_\infty^3}{3k^* \rho C_p} \frac{\partial^2 T}{\partial y^2} + Q_0
\]

(9)

The continuity equation (2) is automatically satisfied by the application of the stream function \(\psi\) defined by

\[
\psi = \frac{\partial \theta}{\partial \eta} \quad \text{and} \quad \nu = -\frac{\partial \theta}{\partial x}
\]

(10)

In accordance with Almakki et al.[26], the similarity solution for momentum, energy and concentration equations are obtained by the application of the appropriate transformation method defined as

\[
\eta = \frac{y}{\sqrt{\nu}}, \quad \psi = x\sqrt{\nu} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \quad \phi(\eta) = \frac{c - c_\infty}{c_w - c_\infty}
\]

(11)

Here, \(\eta\) denotes independent similarity variable, the dimensionless temperature as well as concentration are represented by \(\theta(\eta)\) and \(\phi(\eta)\), respectively.

Introducing (10) and (11) in (3), (9) and (5), result in

\[
f'''(\eta) + f(\eta)f''(\eta) - (f'(\eta))^2 + \beta \left[ (f''(\eta))^2 - 2f'(\eta)f'''(\eta) + f(\eta)f(iv)(\eta) \right]
\]

\[-Mn f'(\eta) + \lambda_T \theta(\eta) + \lambda_M \phi(\eta) = 0\]

\[
\left(1 + \frac{4}{3} Ra\right) \theta''(\eta) + Pr f(\eta) \theta'(\eta) + A f'(\eta) + B \theta(\eta) = 0
\]

\[
\phi''(\eta) + Sc f(\eta) \phi'(\eta) + Sr \theta''(\eta) = 0
\]

(12)

(13)

(14)
satisfying the following boundary conditions

\[
\begin{align*}
    f(0) &= 0, \quad f'(0) = 1, \quad \theta'(0) = Bi[\theta(0) - 1], \quad \phi(0) = 1 \\
    f'(\infty) &= 0, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0
\end{align*}
\]  

(15)

(16)

where the prime symbol denotes the derivative with respect to \( \eta \), \( Mn = \frac{a B_i}{\rho a} \) is the magnetic field, \( \beta = \frac{a k_o}{v} \) is the local Weissenberg Number, \( \lambda_T = \frac{Gr_x}{(Re_x)^2} \) is the thermal buoyancy parameter, \( \lambda_M = \frac{Gr_x}{(Re_x)^2} \) is the mass buoyancy parameter, \( Gr_x = \frac{\beta \alpha (T_w - T_\infty)x^3}{v^2} \) is the thermal Grashof Number, \( Gr_x = \frac{\beta \alpha (C_w - C_\infty)x^3}{v^2} \) is the solutal Grashof Number, \( Bi = \frac{h_L}{k} \sqrt{\frac{a}{\nu}} \) is the Biot number, \( Ra = \frac{u(x)}{v} \) symbolizes Relyolds Number, \( Pr = \frac{\nu C_p}{k} \) illustrates prandtl number, \( Ra = \frac{4 \sigma T_\infty^3}{kk^2} \) is the radiation parameter, \( Sr = \frac{K_T(T_w - T_\infty)}{T_m(C_w - C_\infty)} \) expresses Soret number, \( Sc = \frac{\nu}{D_m} \) connotes Schmidt number.

The local skin friction coefficient, the local Nusselt number, and the local Sherwood number are reported in the following format, keeping in mind the engineering importance of the study.

\[
C_f = \frac{2 \tau_w}{p u_w^2}, \quad Nu = \frac{x q_w}{k(T_w - T_\infty)}, \quad Sh = \frac{x q_m}{D_m(C_w - C_\infty)}
\]  

(17)

Where

\[
\tau_w = \left[ \frac{\partial u}{\partial y} - k_0 \left( u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) \right]_{y=0}, \quad q_w = \left[ -k \frac{\partial T}{\partial y} \right]_{y=0} = \left[ -\frac{4 \sigma}{3 K^*} \frac{\partial T^4}{\partial y} \right]_{y=0},
\]

\[
q_m = \left[ -D_m \frac{\partial C}{\partial y} \right]_{y=0}
\]  

(18)

By the introduction of (18) in (17) with the above transformation technique, an expression for local Skin-friction, the local Nusselt number, and the local Sherwood number formulated and given as

\[
Re_x^{\frac{1}{2}} C_f = (1 - \beta) f''(0), \quad Re_x^{\frac{1}{2}} Nu = -\left( 1 + \frac{4}{3} Ra \right) \theta'(0), \quad Re_x^{\frac{1}{2}} Sh = -\phi'(0)
\]  

(19)

Along with the plate, \( \tau_w \) body forth shear stress and \( q_w \) denotes the surface mass while \( Re_x = u_w(x)/v \) maintained the same name as Reynolds number as expressed above

**HOMOTOPY ANALYSIS METHOD**

The solution of the differential equation has been the utmost priority of every researchers, particularly in the area of modeling which can be tackled by diverse methods like Shooting Techniques with Runge-Kutta method, Variation iteration method and Weighted Residual Method e.t.c. Being a recent method and particularly effective in executing confined and unbounded interval of differential equations, the Homotopy Analysis Method is chosen and employed over other methods. In reference to (15) – (16), we adopt the initial guess (See Hayat et al.[24], and Liao [27])
\[ f_0(\eta) = 1 - \exp(-\eta), \quad \theta_0(\eta) = \frac{Bi \exp(-\eta)}{(1+Bi)}, \quad \phi_0(\eta) = \exp(-\eta) \]  

(20)

Here, \( f(\eta), \theta(\eta) \) and \( \phi(\eta) \) are the initial linear approximations. \( L_f, L_\theta, \) and \( L_\phi \) are auxiliary linear operations.

\[ L_f[f(\eta; r)] = \frac{\partial^3 f(\eta; r)}{\partial \eta^3} - \frac{\partial f(\eta; r)}{\partial \eta}, \quad L_\theta[\theta(\eta; r)] = \frac{\partial^2 \theta(\eta; r)}{\partial \eta^2} - \theta(\eta; r), \]

\[ L_\phi[\phi(\eta; r)] = \frac{\partial^2 \phi(\eta; r)}{\partial \eta^2} - \phi(\eta; r) \]  

(21)

Satisfied the following properties

\[ L_f[C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta)] = 0, \quad L_\theta[C_4 + C_5 \exp(-\eta)] = 0, \]

\[ L_\phi[C_6 + C_7 \exp(-\eta)] = 0 \]  

(22)

where \( C_1, C_2, \ldots, C_7 \) are constants.

**ZERO ORDER DEFORMATION PROBLEM.**

\[ (1 - r)L_f[f(\eta; r) - f_0(\eta)] = rh_fH_f(\eta)N_f[f(\eta; r), \theta(\eta; r), \phi(\eta; r)] \]  

(23)

\[ (1 - r)L_\theta[\theta(\eta; r) - \theta_0(\eta)] = rh_\thetaH_\theta(\eta)N_\theta[f(\eta; r), \theta(\eta; r)] \]  

(24)

\[ (1 - r)L_\phi[\phi(\eta; r) - \phi_0(\eta)] = rh_\phiH_\phi(\eta)N_\phi[f(\eta; r), \theta(\eta; r), \phi(\eta; r)] \]  

(25)

where \( L \) and \( N \) are called Linear and non-linear function respectively (for Algebra Equation) or Linear and Non-linear operators (for differential Equations) and the embedding parameter is \( r \in [0,1] \), while the boundary conditions are as follows (See Akinbo and Olajuwon [28]).

\[ f(\eta = 0; r) = 0, \quad \frac{\partial f(\eta; r)}{\partial \eta} \bigg|_{\eta=0} = 1, \quad \frac{\partial \theta(\eta; r)}{\partial \eta} \bigg|_{\eta=0} = Bi[\theta(\eta = 0; r) - 1], \]

\[ \phi(\eta = 0; r) = 1 \]  

(26)

\[ \frac{\partial f(\eta; r)}{\partial \eta} \bigg|_{\eta \to \infty} = 0, \quad \theta(\eta \to \infty; r) = 0 = \phi(\eta \to \infty; r) \]  

(27)

Nonlinear operators \( N_f, N_\theta, \) and \( N_\phi \) are defined as

\[ \frac{\partial^3 f(\eta; r)}{\partial \eta^3} + f(\eta; r) \frac{\partial^2 f(\eta; r)}{\partial \eta^2} - \beta \left[ 2 \frac{\partial f(\eta; r)}{\partial \eta} \frac{\partial^2 f(\eta; r)}{\partial \eta^2} - f(\eta; r) \frac{\partial^4 f(\eta; r)}{\partial \eta^4} - \left( \frac{\partial^2 f(\eta; r)}{\partial \eta^2} \right)^2 \right] 
- \left( \frac{\partial f(\eta; r)}{\partial \eta} \right)^2 - Mn \frac{\partial f(\eta; r)}{\partial \eta} + \lambda_1 \theta(\eta; r) + \lambda_m \phi(\eta; r) = 0 \]  

(28)


\[
\left[1 + \frac{4}{3} Ra \right] \frac{\partial^2 \theta(\eta; r)}{\partial \eta^2} + Pr \frac{\partial \theta(\eta; r)}{\partial \eta} f(\eta; r) + A \frac{\partial f(\eta; r)}{\partial \eta} + B \theta(\eta; r) = 0 \quad (29)
\]

\[
\frac{\partial^2 \phi(\eta; r)}{\partial \eta^2} + Scf(\eta; r) \frac{\partial \phi(\eta; r)}{\partial \eta} + Sr \frac{\partial^2 \theta(\eta; r)}{\partial \eta^2} = 0 \quad (30)
\]

The change of the function \(f(\eta; r), \theta(\eta; r)\) and \(\phi(\eta; r)\) from \(f_0(\eta), \theta_0(\eta)\) and \(\phi_0(\eta)\) to become a solutions \(f(\eta), \theta(\eta)\) and \(\phi(\eta)\) corresponds to a changes in \(r\) from Zero-One. We have a Taylor series with respect to \(r\) as

\[
f(\eta; r) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) r^m, \quad \theta(\eta; r) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) r^m,
\]

\[
\phi(\eta; r) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) r^m \quad (33)
\]

where \(f_m(\eta) = \frac{1}{m!} \frac{\partial^m f(\eta; r)}{\partial \eta^m}, \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \theta(\eta; r)}{\partial \eta^m}, \phi_m(\eta) = \frac{1}{m!} \frac{\partial^m \phi(\eta; r)}{\partial \eta^m}\)

Obviously, the auxiliary parameter influences the convergence of the series (33). Assuming that \(h\) considered in such a way that the series (33) converge at \(r = 1\), we have with respect to \(r\)

\[
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \quad \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \quad \phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) \quad (34)
\]

**Mth-ORDER DEFORMATION PROBLEM**

Following Hayat et al.[29], the mth-order deformation are considered as follow

\[
L_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h R^f_m(\eta), \quad L_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h R^\theta_m(\eta),
\]

\[
L_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = h R^\phi_m(\eta) \quad (35)
\]

\[
f_m(\eta = 0; 0) = 0, \quad \frac{\partial f_m(\eta = 0; 0)}{\partial \eta} = 0, \quad \frac{\partial \theta_m(\eta = 0; 0)}{\partial \eta} = Bi[\theta_m(\eta = 0; 0)],
\]

\[
\phi_m(\eta = 0; 0) = 0 \quad (36)
\]

\[
\frac{\partial f_m(\eta \rightarrow \infty)}{\partial \eta} = 0, \quad \theta_m(\eta \rightarrow \infty) = 0 = \phi_m(\eta \rightarrow \infty) \quad (37)
\]

Where
\[ R_m^f(\eta) = \frac{d^3 f_{m-1}(\eta)}{d\eta^3} + \sum_{n=0}^{m-1} f_n(\eta) \frac{d^2 f_{m-1-n}(\eta)}{d\eta^2} - \sum_{n=0}^{m-1} \frac{d f_n(\eta) d f_{m-1-n}(\eta)}{d\eta} \]

\[-2\beta \eta \frac{d^2 f_{m-1-n}(\eta)}{d\eta^2} + \beta \sum_{n=0}^{m-1} f_n(\eta) \frac{d^2 f_{m-1-n}(\eta)}{d\eta^2} + \beta \sum_{n=0}^{m-1} \frac{d^2 f_n(\eta) d^2 f_{m-1-n}(\eta)}{d\eta^2} \]

\[-Mn \frac{d f_{m-1}(\eta)}{d\eta} + \lambda_T \theta_{m-1}(\eta) + \lambda_M \phi_{m-1}(\eta) \tag{38} \]

\[ R_m^\theta(\eta) = \left[ 1 + \frac{4}{3} Ra \right] \frac{d^2 \theta_{m-1}(\eta)}{d\eta^2} + Pr \sum_{n=0}^{m-1} f_n(\eta) \frac{d \theta_{m-1-n}(\eta)}{d\eta} \]

\[ + A \frac{d f_{m-1}(\eta)}{d\eta} + Q \theta_{m-1}(\eta) \tag{39} \]

\[ R_m^\phi(\eta) = \frac{d^2 \phi_{m-1}(\eta)}{d\eta^2} + Sc \sum_{n=0}^{m-1} f_n(\eta) \frac{d \phi_{m-1-n}(\eta)}{d\eta} + Sr \frac{d^2 \theta_{m-1}(\eta)}{d\eta^2} \tag{40} \]

and

\[ \chi_m = 0 \text{ for } m \leq 1, \]

\[ \chi_m = 1 \text{ for } m > 1 \]

with the general solution

\[ f_m(\eta) = f_m^*(\eta) + C_1 + C_2 \exp(-\eta) + C_3 \exp(\eta) \tag{41} \]

\[ \theta_m(\eta) = \theta_m^*(\eta) + C_4 + C_5 \exp(\eta) \tag{42} \]

\[ \phi_m(\eta) = \phi_m^*(\eta) + C_6 + C_7 \exp(\eta) \tag{43} \]

where \( f_m^*(\eta), \theta_m^*(\eta), \) and \( \phi_m^*(\eta) \) denote the specific solution of equation (35). We analyze the the ergodicity coefficient rule as well as solution existence rule, and select the auxiliary-functions as Olubode et al. [30] suggested.

\[ H_f = H_\theta = H_\phi = 1 \]

**HAM CONVERGENCE OF SERIES SOLUTION**

The current investigation's convergence solution is in accordance with the recommendations of Liao [31], Akinbo, and Olajuwon [28]. The parameters \( h_f, h_\theta, \) and \( h_\phi \) which are non-zero help in monitoring and dictating the convergence of the model. By the application of \( Mn = 1, \lambda_T = 0.1, \lambda_M = 0.1, Pr = 0.72, Sc = 0.62, Bi = 0.1, Ra = 0.7, \beta = 0.1, R = 0.1, Ps = 1. \) The acceptable result of \( h_f, h_\theta, \) and \( h_\phi \) demonstrated in figures (2-4) are presented at the length where \( h \) curve form a parallel line which in turns give \(-1.2 \leq h_f \leq -0.3, -1.3 \leq h_\theta \leq -0.4 \) and \(-1.5 \leq h_\phi \leq -0.4 \) for \( h_f, h_\theta, \) and \( h_\phi \) respectively.
The exact approximate solution for the convergence of the governing equations which corresponds to momentum, energy and concentration equations are presented in table 1. The dimensionless equations meet the far-field domains and the iteration series converges at 20th – order for momentum and concentration equations while at the 22nd order of iterations, the energy equation converges, and table 2 agreed with Hayat et al. [24]. However, the results are validated using Galerkin Weighted Residual Method, which shows a good agreement with each other (See table 3).
Table 3. Validation/Numerical result for local Skin-friction coefficient, Local Nusselt number, and Local Sherwood number

| Parameters | Results with HAM | Results with Galerkin Weighted Residual Method |
|------------|------------------|-----------------------------------------------|
|            | Re_x^2 C_f  | Re_x^2 Nu | Re_x^2 Sh | Re_x^2 C_f  | Re_x^2 Nu | Re_x^2 Sh |
| β          | Mn   | λ_f | λ_M | Pr | Sc | Bi | Sr | Ra | A | B | 0.1 | 0.1 | 0.1 | 0.1 | 0.72 | 0.62 | 0.1 | 0.1 | 0.7 | 0.01 | 0.01 | Re_x^2 C_f | Re_x^2 Nu | Re_x^2 Sh | Re_x^2 C_f | Re_x^2 Nu | Re_x^2 Sh |
| 0.1        | 0.1   | 0.1  | 0.1  | 0.72 | 0.62 | 0.1  | 0.1 | 0.7 | 0.01 | 0.01 | -0.90956 | 0.14076 | 0.42049 | -0.90955 | 0.14075 | 0.42046 |
| 0.3        |       |      |      |     |     |     |     |     |      |      | 0.90862 | 0.13987 | 0.40152 | -0.80861 | 0.13895 | 0.40150 |
| 0.5        |       |      |      |     |     |     |     |     |      |      | 0.77412 | 0.13644 | 0.38047 | -0.77410 | 0.13643 | 0.38045 |
| 1.0        |       |      |      |     |     |     |     |     |      |      | 1.27062 | 0.13497 | 0.35813 | -1.27061 | 0.13495 | 0.35811 |
| 2.0        |       |      |      |     |     |     |     |     |      |      | 1.58216 | 0.13101 | 0.31603 | -1.58215 | 0.13100 | 0.31601 |
| 1.0        |       |      |      |     |     |     |     |     |      |      | 1.57565 | 0.14465 | 0.46315 | -1.57566 | 0.14463 | 0.46307 |
| 2.0        |       |      |      |     |     |     |     |     |      |      | 0.61330 | 0.14712 | 0.48913 | -0.61329 | 0.14711 | 0.48911 |
| 1.0        |       |      |      |     |     |     |     |     |      |      | -0.41109 | 0.14803 | 0.50317 | -0.41106 | 0.14801 | 0.50315 |
| 2.0        |       |      |      |     |     |     |     |     |      |      | 0.07701 | 0.15155 | 0.55698 | 0.07700 | 0.15148 | 0.55695 |
| 1.0        |       |      |      |     |     |     |     |     |      |      | -0.91448 | 0.14961 | 0.41747 | -0.91445 | 0.14960 | 0.41746 |
| 3.0        |       |      |      |     |     |     |     |     |      |      | -0.92416 | 0.17035 | 0.41203 | -0.92415 | 0.17040 | 0.41201 |
| 0.24       |       |      |      |     |     |     |     |     |      |      | -0.89268 | 0.14186 | 0.22676 | -0.89266 | 0.14185 | 0.22675 |
| 0.78       |       |      |      |     |     |     |     |     |      |      | -0.91384 | 0.14046 | 0.49116 | -0.91382 | 0.14045 | 0.49115 |
| 0.5        |       |      |      |     |     |     |     |     |      |      | -0.88312 | 0.35047 | 0.42254 | -0.88307 | 0.35045 | 0.42252 |
| 1.0        |       |      |      |     |     |     |     |     |      |      | -0.87333 | 0.43293 | 0.42290 | -0.87331 | 0.43291 | 0.42289 |
| 1.0        |       |      |      |     |     |     |     |     |      |      | -0.90632 | 0.14134 | 0.38096 | -0.90630 | 0.14132 | 0.38095 |
| 2.0        |       |      |      |     |     |     |     |     |      |      | -0.90287 | 0.14189 | 0.33731 | -0.90285 | 0.14186 | 0.33730 |
| 1.0        |       |      |      |     |     |     |     |     |      |      | -0.90693 | 0.16455 | 0.42211 | -0.90691 | 0.16448 | 0.42210 |
| 2.0        |       |      |      |     |     |     |     |     |      |      | -0.90107 | 0.24083 | 0.42567 | -0.90105 | 0.24081 | 0.42565 |
| 0.05       |       |      |      |     |     |     |     |     |      |      | -0.90490 | 0.13092 | 0.42321 | -0.90489 | 0.13091 | 0.42320 |
| 0.07       |       |      |      |     |     |     |     |     |      |      | -0.90253 | 0.12591 | 0.42459 | -0.90251 | 0.12590 | 0.42457 |
| 0.05       |       |      |      |     |     |     |     |     |      |      | -0.90599 | 0.13396 | 0.42250 | -0.90597 | 0.13394 | 0.42249 |
| 0.07       |       |      |      |     |     |     |     |     |      |      | -0.90362 | 0.12949 | 0.42378 | -0.90361 | 0.12947 | 0.42376 |

DISCUSSION OF RESULTS

In this study, computation analysis is carried out via the Homotopy Analysis Method (HAM) at 20th – order to meet the far-field boundary conditions. This is done by holding $Bi = \lambda_f = Sr = \lambda_M = \beta = 0.1$, $Sc = 0.62$, $Pr = 0.72$, $Ra = 0.7$, $Mn = 1$, $A = B = 0.01$ constant for each varying parameter. We observed that almost all the values of the local Skin-Friction $Re_x^2 C_f$ quantitatively displayed negative as shown in Table 3. This agreed with the expectation as the negative values justify that a drag force is exerted on the fluid by the plate which in turn impede the flow. However, the surface heat transfer significantly improve as a result of the higher values of thermal buoyancy parameter ($\lambda_f$), Prandtl number ($Pr$), Radiation Parameter ($Ra$) and Boit number ($Bi$) and this consequently enhances the rate of heat transfer while the rate of mass transfer gain more strength for large values of mass buoyancy parameter ($\lambda_M$) and Schmidt number ($Sc$) (see Table 3).
In figures (5-8), we observed that increase in thermal-mass buoyancy parameters ($\lambda_T, \lambda_M$) boost the buoyancy forces and accelerate the flow of which its aftermath effect improves the velocity of the fluid (as well as its layer thickness). The results are not the same for temperature field as well as concentration field where both the layers of the thicknesses decline for large values of $\lambda_T$ and $\lambda_M$. In that case, $\lambda_T > 0$ corresponds to the cooling problem while $\lambda_M > 0$ justify that the concentration on the surface of the plate is greater than the concentration in the free stream.
In figures (9), the influence of Prandtl number \((Pr)\) were captured between 0.72(Air) to 7.1(Liquid). It is observed from the figures that higher values of \(Pr\) due to the low thermal diffusivity diminish the average temperature within the boundary layer whose aftermath portrays reduction on the thickness of momentum layer. In that wise, lower range of \(Pr\) improve thermal conductivity and allow heat to diffuse from the heated surface more quickly than larger values.

![Figure 10. Velocity profile for different values of Mn](image)

![Figure 11. Temperature profile for different values of Mn](image)

Figures (10-11) reveal the influence of magnetic interaction \((Mn)\) on velocity-temperature profiles. It is noticed from fig.10 that higher values of \(Mn\) pioneer resistive forces called Lorentz force that resist the run of the fluid and reduces the velocity profile and the thickness of its associated layer. On the other hand, a reverse behaviors is detected in the temperature of the fluid. An increase in \(Mn\) causes frictional heating inside the boundary layer whose physical outcome intensified the temperature field of which its aftermath increases the thermal boundary layer thickness.

![Figure 12. Velocity profile for different values of β](image)

![Figure 13. Temperature profile for different values of β](image)

Figures (12-13) present the influence of velocity-temperature fields on variation in local Weissenberg number \((β)\) on velocity and temperature profiles. It is noticed from the fig. 12 that the fluid velocity decline for the large values of \(β\). This result agreed with the expectation as higher values of \(β\) are to improve the viscoelasticity through the tensile stress which opposes the fluid velocity and reduces its layer thickness. The effect of improving
viscoelasticity generates more heat within the thermal boundary layer which in turn intensifies the thickness of thermal layer.

**Figure 14.** Concentration profile for different values of $Sc$

**Figure 15.** Temperature profile for different values of $Sr$

The effect of Schmidt number for most encountered chemical in the application is varied in Figure 14 on concentration field. At higher values of $Sc$, the fluid diffusion characteristics experienced downfall which in turn falls the concentration field near the boundary layer as well as concentration boundary layer thickness (See Akinbo and Olajuwon [32]).

Figures (15-16) present the dynamics of Soret Number ($Sr$) via temperature-concentration field. We noticed with reference to figure 15 that variation in values of $Sr$ contributes to the falling of the temperature whose direct impact eventually abates the thickness thermal layer. However, the opposite phenomenon is observed in the concentration field which ultimately boosts concentration boundary layer thickness (see Fig. 16).

**Figure 16.** Concentration profile for different values of $Sr$

**Figure 17.** Temperature profile for different values of $Ra$

Figure 17 addresses the influence of the radiation parameter ($Ra$) on the temperature profile. It is noticed that the temperature of the fluid improves due to the increasing values of $Ra$. This is true as higher values of $Ra$ magnify the conduction of heat transfer to thermal heat transfer of which its after-effect strengthens the thickness of the thermal layer.
Figures (18-19) body forth the significance of the internal heat generation/absorption parameter $(A, B) > 0$ on the temperature field. As expected, a rise in $(A, B)$ corresponds to the enhancement of more heat in the layer, whose outcome improves the temperature field and grow the thermal layer thickness.

![Figure 18](image1.png)  ![Figure 19](image2.png)

**Figure 18.** Temperature profile for different values of $A$

**Figure 19.** Temperature profile for different values of $B$

Figure (20) presents the influence of Boit Number $(Bi)$ on the temperature profile. An increase in $Bi$ constitutes strong convective heating within the boundary which eventually maximizes the thickness of the thermal layer. However, this enhancement enables the heat influence to break through into the quiescent fluid.

![Figure 20](image3.png)

**Figure 20.** Temperature profile for different values of $Bi$

**CONCLUSION**

In this work, Homotopy Analysis Method is employed to solve the three dimensionless equations corresponding to momentum, energy, and concentration which describe the influence of Convective Boundary Condition on heat and mass transfer in Walters’ B fluid over a vertical plate with thermal-diffusion effect. The results of various embedded parameters are analyzed through graphs and tables. The following conclusions were drawn from the results obtained:

- Setting $\beta = 0$, Walters’ B model behaves like an ordinary Newtonian fluid.
- The skin-friction quantitatively displayed negative, indicating that the drag forces are exerted on the fluid by the plate which in turns impede the flow.
Higher values of \((A, B)\) enhance the temperature which in turn pioneer the lightening of the surface and enable the fluid to flow faster.

The tensile stress effect is magnified for large values of Weissenberg number which ultimately declines the velocity boundary layer thickness.

The temperature distribution and surface heat transfer are boosted when Radiation intensity improves.

Large values of Biot number magnifies thermal effect and allow its penetration to the quiescent fluid. The strength of the Biot number contributes to the drying of the materials component in the Engineering field.

Compliance with ethical standards
Conflict of interest: The authors declare that they no conflict of interest.

NOMENCLATURE

- \(Mn\): Magnetic field
- \(\beta\): local Weissenberg Number
- \(\lambda_T, \lambda_M\): Thermal and buoyancy
- \(Bi\): Biot number
- \(Pr\): Prandtl number,
- \(Ra\): Radiation parameter
- \(Sc\): Schmidtl number
- \(Q_0\): non-uniform heat generation/absorption Coefficient
- \(D_m\): Mass diffusivity
- \(\alpha\): thermal diffusivity
- \(\beta_c\): Concentration expansion coefficient
- \(\eta\): Similarity variable
- \(\theta\): Dimensionless temperature
- \(\nu\): kinematic viscosity
- \(\psi\): Stream Function
- \(\rho\): Density
- \(q_r\): Radiation heat flux
- \(C_p\): Specific heat at constant pressure
- \(\beta_T\): Temperature expansion coefficient
- \(g\): Acceleration due to gravity
- \(\sigma\): Fluid electrical conductivity
- \(h_f\): Heat transfer coefficients

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