DISTINGUISHING EVERY FINITELY GENERATED FIELD OF CHARACTERISTIC ZERO BY A SINGLE FIELD AXIOM

THE STRONG ELEMENTARY EQUIVALENCE VS ISOMORPHY PROBLEM

FLORIAN POP

Abstract. We show that the isomorphy type of every finitely generated field $K$ with $\text{char}(K) = 0$ is encoded by a single explicit axiom $\vartheta_K$ in the language of fields, i.e., for all finitely generated fields $L$ one has: $\vartheta_K$ holds in $L$ if and only if $K \cong L$ as fields. This extends earlier results by Julia Robinson, Rumely, Poonen, Scanlon, the author, and others.

1. Introduction

We begin by recalling that a sentence, or an axiom in the language of fields is any formula in the language of fields which has no free variables. One denotes by $\mathfrak{Th}(K)$ the set of all the sentences in the language of fields which hold in a given field $K$. For instance, by pure definitions, the field axioms are part of $\mathfrak{Th}(K)$ for every field $K$; further the fact that $K$ is algebraically closed, as well as $\text{char}(K)$ are encoded in $\mathfrak{Th}(K)$. Namely, $K$ is algebraically closed iff $K$ satisfies the scheme of axioms of algebraically closed fields (asserting that every non-zero polynomial $p(T)$ over $K$ has a root in $K$); respectively one has $\text{char}(K) = p \geq 0$ iff $K$ satisfies the $\text{char} = p$ scheme of axioms (asserting: $\text{char} = p > 0$ iff $\sum_{i=1}^n 1 = 0$, respectively $\text{char} = 0$ iff $\sum_{i=1}^n 1 \neq 0$ for all $n$). On the other hand, if $K := \mathbb{Q}(t)$ is the rational function field in the variable $t$ over $\mathbb{Q}$, then the usual way to say that $t$ is transcendental over $\mathbb{Q}$, namely “$p(t) \neq 0$ for all non-zero polynomials $p(T)$ over $\mathbb{Q}$” is not a scheme of axioms in the language of fields (because $t$ is not part of the language of fields).

Two fundamental general type results in algebra are the following:

- Algebraically closed fields $K, L$ have $\mathfrak{Th}(K) = \mathfrak{Th}(L)$ iff $\text{char}(K) = \text{char}(L)$.
- Arbitrary fields $K, L$ have $\mathfrak{Th}(K) = \mathfrak{Th}(L)$ iff $K \cong L$ as fields.

Restricting to fields which are at the center of (birational) arithmetic geometry, namely the finitely generated fields $K$, which are the function fields of integral schemes of finite type, the elementary theory $\mathfrak{Th}(K)$ is both extremely rich and mysterious. The so called Elementary equivalence vs Isomorphism Problem (EEIP) is about five decades old, and asks whether $\mathfrak{Th}(K)$ encodes the isomorphy type of $K$ in the class of all the finitely generated fields.

2010 Mathematics Subject Classification. Primary 11G30, 14H25; Secondary 03C62, 11G99, 12G10, 12G20, 12L12, 13F30.

Key words and phrases. Elementary equivalence vs isomorphism, first order definability, e.g. of valuations, finitely generated fields, Milnor K-groups, Galois/étale cohomology, Kato’s higher local-global Principles.
fields; or equivalently, whether there exists a system of axioms in the language of fields which characterizes \( K \) among all the finitely generated fields. On the other hand, building on Julia Robinson [Ro1], [Ro2] methods and ideas, Rumely [Ru] showed at the end of the 1970’s that for every global field \( K \) there exits a sentence \( \vartheta_K^{Ru} \) which characterizes the isomorphism type of \( K \) as a global field, i.e., if \( L \) is a global field, then \( \vartheta_K^{Ru} \) holds in \( L \) iff \( K \cong L \) as fields. In other words, the isomorphism type of \( K \) as a global field is characterized by a \textit{single explicit axiom \vartheta_K^{Ru} in the language of fields}. This goes far beyond the EEIP in the class of global fields!

Arguably, it is the \textit{main open question} in the elementary (or first order) theory of finitely generated fields whether a fact similar to Rumely’s result [Ru] holds for all finitely generated fields \( K \), namely whether there is a field axiom \( \vartheta_K \) which characterizes the isomorphism type of \( K \) in the class of all finitely generated fields; this question is also called the \textit{strong EEIP}. We notice that the (strong) EEIP is open in general; see Pop [P2], [P3], for more details and references on the EEPI both over finitely generated fields and function fields over algebraically closed base fields. A first attempt towards tackling the strong EEIP was Scanlon [Sc], and that reduces the strong EEIP for each \( K \) to first order defining “sufficiently many” divisorial valuations of \( K \). Finally, Pop [P4] tackles the strong EEIP for finitely generated fields which are function fields of curves over global fields.

In the present note we show that the answer to the strong EEIP, hence in particular to EEIP for finitely generated fields of characteristic zero is positive.

**Main Theorem.** For every finitely generated field \( K \) with \( \text{char}(K) = 0 \), there exists a sentence \( \vartheta_K \) in the language of fields such that for all finitely generated fields \( L \) one has:

\[
\vartheta_K \text{ holds in } L \text{ if and only if } L \cong K \text{ as fields.}
\]

The Main Theorem above will be proved in Section 5. One can give three (by some standards similar) proofs. A first proof follows simply from Scanlon, by invoking Theorem 1.1 below for the definability of geometric prime divisors (thus circumventing the gap in the proof of defining divisorial valuations in Section 3 of loc.cit.). A second proof reduces the Main Theorem above to results by Aschenbrenner–Khélif–Naziazeno–Scanlon [AKNS], by showing that finitely generated integrally closed subdomains in finitely generated fields of characteristic zero are uniformly first order definable. Among other things, these proofs show that finitely generated fields of characteristic zero are \textit{bi-interpretable with arithmetic}, see e.g. [Sc], Section 2, and/or [AKNS], Section 2, for a detailed discussion of bi-interpretablility with arithmetic. Third, a more direct proof based on Pop [P2], Poonen [Po1], and consequences of Rumely [Ru] (namely that the number fields are bi-interpretable with arithmetic).

The main step and technical key point in the proof of the Main Theorem is to give formulae \( \text{val}_d \), all \( d > 0 \), in the language of fields, which uniformly first order define the \textit{geometric prime divisors} of finitely generated fields \( K \) with \( \text{char}(K) = 0 \) and \( \dim(K) = d \). That is the content of Theorem 1.1 below, which could be viewed as the main result of this note.

To make these assertions more precise, let us introduce notation and mention a few fundamental facts about finitely generated fields, to be used throughout the manuscript.

For arbitrary fields \( \Omega \), let \( \kappa_0 \subset \Omega \) denote their prime fields. Recall that the Kronecker dimension of \( \Omega \) is \( \dim(\Omega) = \dim(\kappa_0) + \text{td}(\Omega|\kappa_0) \), where \( \text{td}(\Omega|\kappa_0) \) denotes the transcendence degree, and \( \dim(\mathbb{F}_p) = 0, \dim(\mathbb{Q}) = 1 \). We denote by \( \kappa := \Omega^{\text{hs}} \) the \textit{constant subfield} of \( \Omega \),

\[1\text{See the discussion below for more about this.} \]
i.e., the elements of $\Omega$ which are algebraic over the prime field $\kappa_0 \subset \Omega$, and set

$$\tilde{\Omega} := \Omega[\sqrt{-1}].$$

For $a := (a_1, \ldots, a_r)$ with $a_i \in \Omega^x$ we consider the $r$-fold Pfister form $q_a(x)$ in the variables $x = (x_1, \ldots, x_{2^r})$ and for field extensions $\Omega' | \Omega$ define the image of $\Omega'$ under $q_a$ as being

$$q_a(\Omega') := \{q_a(x') \mid x' \in \Omega'^{2^r}, x' \neq 0\}.$$

Next we recall that, using among other things the Milnor Conjecture\footnote{See e.g. Pfister [Pf1], Ch. 2, for basic facts.} by Pop [P2] there are sentences $\varphi_d$, and by Poonen [Po1] there are a sentence $\psi_0$, a predicate $\psi_{abs}(x)$, and formulas $\psi_r(t)$ with free variables $t := (t_1, \ldots, t_r)$ such that for all finitely generated fields $K$ and $\kappa = K^{abs} \subset K$, setting $\tilde{K} := K[\sqrt{-1}]$, one has:

- $\dim(K) = d$ iff $\varphi_d$ holds in $K$. Actually, $\varphi_d \equiv ((\varphi_0^d \land 2 = 0) \lor (\varphi_{d+1}^0 \land 2 \neq 0))$, where $\varphi_0^d \equiv (\exists a = (a_1, \ldots, a_r) \text{ s.t. } 0 \not\in q_a(\tilde{K})) \& (\forall a = (a_1, \ldots, a_{r+1}) \text{ one has } 0 \in q_a(\tilde{K})).$

- $\psi_0$ holds in $K$ iff $\text{char}(K) = 0$ iff $\kappa$ is a number field.
- $\kappa$ is defined by $\psi_{abs}(x)$ inside $K$, i.e., one has $\kappa = \{x \in K \mid \psi_{abs}(x) \text{ holds in } K\}$.
- $t_1, \ldots, t_r \in K$ are algebraically independent over $\kappa$ iff $\psi_r(t_1, \ldots, t_r)$ holds in $K$.

Note that if $\text{char}(K) = 0$, then by [P2] one can consider:

$$\psi_r(t_1, \ldots, t_r) \equiv (\exists a, b, a_1, \ldots, a_r \in \kappa \text{ s.t. } a := (a, b, t_1 - a_1, \ldots, t_r - a_r) \Rightarrow 0 \not\in q_a(\tilde{K}))$$

In particular, for algebraically independent elements $t_r := (t_1, \ldots, t_r)$ of $K$, the relative algebraic closure $k_{t_r}$ of $\kappa(t_r)$ in $K$ is uniformly first order definable as follows:

$$k_{t_r} = \{u \in K \mid \neg\psi_{r+1}(u, t_1, \ldots, t_r) \text{ holds in } K\}$$

Hence the transcendence bases $T := (t_1, \ldots, t_{d_K})$ of $K[\kappa]$ are uniformly first order definable, and so are the (maximal) flags of relatively algebraically closed subfields of $K$

$$k_0 := K_0 \subset \cdots \subset K_{d_K} := K.$$ 

Finally, a prime divisor of a finitely generated field $K$ is (the valuation ring of) any valuation $v$ of $K$ whose residue field $Kv$ satisfies

$$\dim(Kv) = \dim(K) - 1.$$ 

It turns out that prime divisors $v$ of finitely generated fields are discrete valuations, and $Kv$ is a finitely generated fields as well. A prime divisor $v$ of $K$ is called an arithmetic prime divisor, if $v$ is non-trivial on $\kappa = K^{abs}$ — in particular $\kappa$ must be a number field, respectively a geometric prime divisor if $v$ is trivial on $\kappa$. Recall that Rumely [Ru] gives formulae val$_1$ which uniformly first order define the prime divisors of global fields, and Poonen [P4] gives formulae val$_2$ which uniformly first order define the geometric prime divisors in the case $\dim(K) = 2$. 

\footnote{3 Proved by Vojvodsky, Orlov–Vishik–Vojvodsky, and Rost, see e.g. the survey articles [Pf2], [Kh].}
The focus of this note is to give similar formulae \( \text{val}_d \) which work under the hypothesis:

\[(H) \quad K \text{ is finitely generated, } \quad d := \dim(K) > 2, \quad \text{char}(K) = 0 \]

**Theorem 1.1.** There is an explicit procedure that, given an integer \( d > 1 \), produces a first-order formula \( \text{val}_d \) that in any finitely generated field \( K \) of characteristic \( \text{char}(K) = 0 \) and Kronecker dimension \( \dim(K) = d \) defines all the geometric prime divisors of \( K \).

For the proof see Section 4, Theorem 4.2, and Recipe 4.7 for the concrete form of \( \text{val}_d \).

We conclude the Introduction with the following remarks.

First, key points of the methods of the present note do not work and/or apply straightforward in the case of finitely generated fields of positive characteristic. A main obstacle stems from the well known fact that in positive characteristic the notions of regular point and smooth point are not equivalent.

Second, although the formulae \( \text{val}_d \) are completely explicit, see Recipe 4.7, it is an open question whether these formulae are optimal in any concrete sense; in particular, the formulae \( \text{val}_d \) do not address the question about the complexity of (uniform) definability of (some or all) the prime divisors. The complexity of definability of valuations deserves further special attention, because among other things it ties in with previous first order definability results of valuations (of finitely generated fields and more general fields, in both characteristic zero and positive characteristic) by Eisenträger [Ei], Eisenträger–Shlapentokh [E-S], Kim-Roush [K-R], Köenigsmann [Ko1, Ko3], Miller-Shlapentokh [M-Sh], Poonen [Po2], Shlapentokh [Sh1], [Sh2], and others. The focus of the aforementioned results and research is yet another open problem in the theory of finitely generated fields and function fields, namely the generalized Hilbert Tenth Problem — which for the time being is open over all number fields, e.g. \( \mathbb{Q} \), and all function fields of \( \mathbb{C} \)-curves, e.g. \( \mathbb{C}(t) \).

Third, it is strongly believed that the (strong) EEIP should hold for the function fields \( K/k \) over “reasonable” base fields \( k \); in particular, since finitely generated fields of characteristic zero \( K \) are nothing but function fields \( K/k_0 \) over number fields \( k_0 \), the Main Theorem above asserts that the number fields are “reasonable.” If \( k \) is an algebraically closed field, facts proved by Durré [Du], Pierce [Pi], Vidaux [Vi] for \( \text{td}(K/k) = 1 \), respectively Poonen [P2] for \( \text{td}(K/k) \) arbitrary, are quite convincing partial results supporting the possibility that algebraically closed fields are “reasonable.” Finally, Köenigsmann [Ko2], Poonen-Pop [P-P] give evidence for the fact that the much more general large fields \( k \), as introduced in Poonen [P1], e.g. \( k = \mathbb{R}, \mathbb{Q}_p, \) PAC, etc., should be “reasonable” base fields. These partial/preliminary results over large fields (including the algebraically closed ones) do not involve prime divisors of \( K/k \).

Two fundamental open questions arise: First, is it possible to recover prime \( k \)-divisors of functions fields \( K/k \) over large fields \( k \), at least in the case of special classes of large fields, e.g. local fields, or quasi-finite fields? Second, are there alternative approaches (which do not involve prime divisors) for recovering the isomorphy type of \( K \) from \( \Theta(K) \)?

**Thanks:** I would like to thank the participants at several activities, e.g., AWS 2003, AIM Workshop 2004, INI Cambridge 2005, HIM Bonn 2009, ALANT III in 2014, MFO Oberwolfach in 2016, IHP Paris in 2018, for the debates on the topic and suggestions concerning this problem. Special thanks are due to Bjorn Poonen, Thomas Scanlon, Jakob Stix, and Michael Temkin for discussing technical aspects of the proofs, and to Uwe Jannsen and Moritz Kerz for discussions concerning Kato’s higher dimensional Hasse local-global principles.
2. Higher dimensional Hasse local-global principles

A) Notations and general facts

For a (possibly trivial) valuation $v$ of $K$, let $m_v \subset O_v \subset K$ be its valuation ideal and valuation ring ring, $vK := K^\times/U_v$ be its (canonical) value group, and $K_v := \kappa(v) := O_v/m_v$ be its residues field. We denote by $V_K$ the Riemann space of $K$, i.e., the space of all the (equivalence classes of) valuations of $K$.

Let $X$ be a scheme of finite type over either $\mathbb{Z}$ or a field $k$. For $x \in X$, let $X_x := \overline{\{x\}} \subset X$ be the closure of $x$ in $X$, and recall that $\dim(x) := \dim(X_x)$. Following Kato [Ka], define:

$$X_i := \{x \in X \mid \dim(x) = i\}, \quad X^i := \{x \in X \mid \text{codim}(x) = i\},$$

and recall that if $X$ is integral and projective, then for all $0 \leq i \leq \dim(X)$ one has:

$$\dim(X) = \text{codim}(x) + \dim(x), \quad \text{and therefore: } x \in X^i \iff x \in X_{\dim(X) - i}$$

Notations/Remarks 2.1. Let $K$ be a finitely generated field, and $k \subset K$ be a subfield.

1) A model of $K$ is a separated scheme of finite type $\mathcal{X}$ with function field $\kappa(\mathcal{X}) = K$. And a $k$-model of $K$ is a $k$-variety, i.e., a separated $k$-scheme of finite type, with $k(X) = K$.

2) Let a model $\mathcal{X}$ of $K$, and $v \in V_K$ be given. We say that $v$ has center $x \in \mathcal{X}$ on $\mathcal{X}$, if $O_x \prec O_v$, that is, $O_x \subseteq O_v$, and $m_x = m_v \cap O_x$. By the valuation criterion one has: Since $\mathcal{X}$ is separated, every $v \in V_K$ has at most one center on $\mathcal{X}$, respectively that $\mathcal{X}$ is proper iff every valuation $v \in V_K$ has a center on $\mathcal{X}$ (which is then unique).

3) Let a $k$-model $X$ of $K$ and $v \in V_K$ be given. We say that $x \in X$ is the center of $v$ on $X$, if $O_x \prec O_v$. If so, then $v \in V_{K|k}$. By the valuation criterion one has: Since $X$ is separated over $k$, every $v$ has at most one center on $X$, respectively that $X$ is a proper $k$-variety iff every $k$-valuation $v \in V_{K|k}$ has a center on $X$ (which is then unique).

4) A prime divisor of $K$ is any $v \in V_K$ satisfying the following equivalent conditions:

i) $\dim(Kv) = \dim(K) - 1$.

ii) $v$ is discrete, and $K_v$ is finitely generated and has $\dim(Kv) = \dim(K) - 1$.

iii) $v$ is defined by a prime Weil divisor of a projective normal model $\mathcal{X}$ of $K$.

5) A prime $k$-divisor of $K$ is any $v \in V_{K|k}$ satisfying the following equivalent conditions:

i) $\text{td}(Kv|k) = \text{td}(K|k) - 1$.

ii) $v$ is a prime divisor of $K$ which is trivial on $k$.

iii) $v$ is defined by a prime Weil divisor of a projective normal model $X$ of $K|k$.

6) Let $\mathcal{D}_K \supset \mathcal{D}_X$ be the spaces of prime divisors of $K$, respectively the ones defined the prime Weil divisors of a quasi-projective normal model $\mathcal{X}$ of $K$. Further define $\mathcal{D}_{K|k} \supset \mathcal{D}_X$ correspondingly, where $X$ is a quasi-projective normal $k$-model of $K$.

7) Next, in the notation from point 6) above, suppose that $\mathcal{X}$ and $X$ are projective. Then by the discussion above there are canonical identifications:

$$\mathcal{D}_\mathcal{X} \leftrightarrow \mathcal{X}^1 = \mathcal{X}_{\dim(K) - 1}, \quad \mathcal{D}_X \leftrightarrow X^1 = X_{\text{td}(K|k) - 1}$$
B) Local-global principles (LGP)

Let us first recall the famous Hasse–Brauer–Noether LGP. Let \( k \) be a global field, \( \mathbb{P}(k) \) be the set of non-trivial places of \( k \), and for \( v \in \mathbb{P}(k) \), let \( k_v \) be the completion of \( k \) with respect to \( v \). Denoting by \( n(\cdot) \) the \( n \)-torsion in an Abelian group, e.g. \( n(\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n \), the Hasse–Brauer–Noether LGP asserts that one has a canonical exact sequence:

\[
0 \to n\text{Br}(k) \to \bigoplus_v n\text{Br}(k_v) \to \mathbb{Z}/n \to 0,
\]

where, the first map is the direct sum of all the canonical restriction maps \( n\text{Br}(k) \to n\text{Br}(k_v) \); thus implicitly, for every division algebra \( D \) over \( k \) there exist only finitely many \( v \) such that \( D \otimes_k k_v \) is not a matrix algebra; and the second map is the sum of the invariants \( \sum_v \text{inv}_v \).

It is a fundamental observation by Kato [Ka] that the above local-global principle has higher dimensional variants as follows: First, following Kato loc.cit, for every positive integer \( n \), say \( n = mp^r \) with \( p \) the characteristic, and an integer twist \( i \), one sets \( \mathbb{Z}/n(0) = \mathbb{Z}/n \), and defines in general \( \mathbb{Z}/n(i) := \mathbb{Z}^\otimes_m \oplus W_i \Omega_{\log}^{[n]} \), where \( W_i \Omega_{\log} \) is the logarithmic part of the de Rham–Witt complex on the étale site, see Illusie [Ill] for details. With these notations, for every (finitely generated) field \( K \) one has:

\[
H^1(K, \mathbb{Z}/n(0)) = \text{Hom}_{\text{cont}}(G_K, \mathbb{Z}/n), \quad H^2(K, \mathbb{Z}/n(1)) = n\text{Br}(K),
\]

where \( G_K \) is the absolute Galois group of \( K \). Thus the cohomology groups \( H^{d+1}(K, \mathbb{Z}/n(i)) \) have a particular arithmetical significance, and in these notation, the Hasse–Brauer–Noether LGP is a local-global principle for the cohomology group \( H^2(K, \mathbb{Z}/n(1)) \). Noticing that \( K \) is a global field iff \( \dim(K) = 1 \), Kato had the fundamental idea that for finitely generated fields \( K \) with \( \dim(K) = d \), there should exist similar local-global principles for \( H^{d+1}(K, \mathbb{Z}/n(d)) \).

- **The Kato cohomological complex (KC)**

We briefly recall Kato’s cohomological complex (similar to complexes defined by the Bloch–Ogus) which is the basis of the higher dimensional Hasse local-global principles, see Kato [Ka], §1, for details. Let \( L \) be an arbitrary field, and recall the canonical isomorphism (generalizing the classical Kummer Theory isomorphism) \( h^1 : L^\times/n \to H^1(L, \mathbb{Z}/n(1)) \).

As explained in [Ka], §1, the isomorphism \( h^1 \) gives rise canonically for all \( q \neq 0 \) to morphisms, which by the (now proven) Milnor–Bloch–Kato Conjecture are actually isomorphisms:

\[
h^q : K^m_q(L)/n \to H^q(L, \mathbb{Z}/n(q)), \quad \{a_1, \ldots, a_q\}/n \mapsto h^1(a_1) \cup \ldots \cup h^1(a_q) =: a_1 \cup \ldots \cup a_q.
\]

Further, let \( v \) be a discrete valuation of \( L \). Then one defines the boundary homomorphism

\[
\partial_v : H^{q+1}(L, \mathbb{Z}/n(q + 1)) \to H^q(Lv, \mathbb{Z}/n(q)),
\]

defined by \( a \overset{\partial_v}{\to} v(a) \) if \( q = 0 \), \( a \lor a_1 \lor \ldots \lor a_q \overset{\partial_v}{\to} v(a) \cdot a_1 \lor \ldots \lor a_q \) for \( a \in L^\times, a_1, \ldots, a_q \in U_v \) if \( q > 0 \).

Now let \( X \) be an excellent integral scheme, with generic point \( \eta_X \), and recall the notations \( X_1, X^i \subset X \); hence \( X_0 \subset X \) are the closed points, and \( X_{\dim(X)} = \{\eta_X\} \). By mere definitions, for every \( x_{i+1} \in X_{i+1} \), one has that \( X_{x_{i+1}} \subset X \) consists of all the points \( x_i \in X_i \) which lie in the closure of \( X_{x_{i+1}} := \{x_{i+1}\} \). Since \( X \) is excellent, the normalization \( \tilde{X}_{x_{i+1}} \to X_{x_{i+1}} \) of \( X_{x_{i+1}} \) is a finite morphism. Hence for every \( x_i \in X_{x_{i+1}} \), there is a finitely many \( \tilde{x} \in \tilde{X}_{x_{i+1}} \) such that \( \tilde{x} \mapsto x_i \) under \( \tilde{X}_{x_{i+1}} \to X_{x_{i+1}} \), and the following hold: The local rings \( \mathcal{O}_{\tilde{x}} \) of all \( \tilde{x} \mapsto x_i \) are discrete valuations rings of the residue field \( \kappa(x_{i+1}) \), say with valuation \( v_{\tilde{x}} \), and
the residue field extensions $\kappa(\tilde{x}|\kappa(x_i)$ are finite field extensions. Then for every integer $n > 1$, which is invertible on $X$, letting $0 \leq i < \dim(X)$, one gets a sequence of the form:

(KC) \[ \cdots \to \oplus_{x_{i+1} \in X_{i+1}} H^{i+2}(\kappa(x_{i+1}), \mathbb{Z}/n(i+1)) \to \oplus_{x_i \in X} H^{i+1}(\kappa(x_i), \mathbb{Z}/n(i)) \to \cdots \]

where the component $H^{i+2}(\kappa(x_{i+1}), \mathbb{Z}/n(i+1)) \to \oplus_{x_i \in X} H^{i+1}(\kappa(x_i), \mathbb{Z}/n(i))$ is defined by

$$\sum_{\tilde{x} \in X_{i+1}, \tilde{x} \to x} \text{Cor}_{\kappa(\tilde{x})/\kappa(x_i)} \circ \delta_{\tilde{x}}$$

**Theorem 2.2** (Kato [Ka], Proposition 1.7). Suppose that $X$ is an excellent scheme such that for all $p$ dividing $n$ and $x_i \in X$, one has: If $p = \text{char}(\kappa(x_i))$, then $[\kappa(x_i) : \kappa(x_i)]^p \leq p^i$. Then (KC) is a complex. In particular, if $n$ is invertible on $X$, then (KC) is a complex.

That being said, the Kato Conjectures are about aspects of the fact that in arithmetically significant situations, the complex (KC) above is exact, excepting maybe for $i = 0$, where the homology of (KC) is perfectly well understood. And Kato proved himself several forms of the above local-global Principles in the case $X$ is an arithmetic scheme of dimension $\dim(X) = 2$ and having further properties. Among other things, one has:

**Theorem 2.3** (Kato [Ka], Corollary, p.145). Let $X$ be a proper regular integral $\mathbb{Z}$-scheme, $\dim(X) = 2$, and $K = \kappa(X)$ having no orderings. Then one has an exact sequence:

$$0 \to H^2(K, \mathbb{Z}/n(2)) \to \oplus_{x_1 \in X_1} H^2(\kappa(x_1), \mathbb{Z}/n(1)) \to \oplus_{x_0 \in X_0} H^1(\kappa(x_0), \mathbb{Z}/n) \to \mathbb{Z}/n \to 0.$$ 

Finally, notice that in Theorem 2.3 above, $K$ is finitely generated with $\dim(K) = 2$. Unfortunately, for the time being, the above result is not known to hold in the same form in higher dimensions $d := \dim(K) > 2$, although it is conjectured to be so. There are nevertheless partial results concerning the local-global principles involving $H^{d+1}(K, \mathbb{Z}/n(d))$. From those results, we pick and choose only what is necessary for our goals, see below.

**Notations/Remarks 2.4.** Let $K$ be a finitely generated field with constant field $\kappa$. We supplement Notations/Remarks 2.1 as follows:

1) $n > 1$ is a positive integer not divisible by $\text{char}(K)$.
2) $k_0 \subset K$ is a relatively algebraically closed subfield with $\dim(k_0) = 1$.
   Notice that $k_0$ is a global field, and $k_0$ is a number field iff $k_0 = \kappa$ iff $\text{char}(K) = 0$.
3) Let $\mathbb{P}(k_0)$ be the set of places of $k_0$. For $v \in \mathbb{P}(k_0)$, consider/denote the following:
   - Let $R_v$ be the Henselization of the valuation ring $\mathcal{O}_v$.
   - In particular, $R_v$ is an excellent Henselian DVR with finite residue field.
   - Let $k_0v = \text{Quot}(R_v)$ be the corresponding Henselianization of $k_0$ at $v$.

- **Localizing the global field $k_0$**

In the above notations, for every $v \in \mathbb{P}(k_0)$, consider the compositum $K_v := Kk_0v$ of $K$ and $k_0v$ (in some fixed algebraic closure $\overline{K}$). Then via the restriction functor(s) in cohomology, one gets canonical localization maps

$$H^{d+1}(K, \mathbb{Z}/n(d)) \to H^{d+1}(K_v, \mathbb{Z}/n(d)).$$

**Theorem 2.5** (Jannsen [Ja], Theorem 0.4). In the above notations, suppose that $\text{char}(K)$ does not divide $n$. Then the localization maps give rise to an embedding

$$H^{d+1}(K, \mathbb{Z}/n(d)) \to \oplus_{v \in \mathbb{P}(k_0)} H^{d+1}(K_v, \mathbb{Z}/n(d)).$$
• **Local-global principles over** $R_v$, $v \in \mathbb{P}(k_0)$

In the above notations, for every non-archimedean place $v \in \mathbb{P}(k_0)$, let $R_v \subset k_{0v}$ be the (unique) Henselization of the valuation ring $O_v$ inside $k_{0v}$, hence recall that $R_v$ is a Henselian discrete valuation ring with residue field $\kappa(v)$ finite. This being said, one has the following:

**Theorem 2.6** (Kerz–Saito [K–S], Theorem 8.1). Suppose that $R$ is either (i) a finite field, or (ii) a Henselian discrete valuation ring with finite residue field, such that $n$ is invertible in $R$, and $\mu_n \subset R$. Then for every projective regular flat $R$-scheme $X$, the complex $(KC)$ for $X$ is exact, with the only exception of the homology group $H_0(KC) = \mathbb{Z}/n$ in the case (i).

3. Consequences/Applications of the Local-Global Principles

In this section we give a few consequences of the higher Hasse local-global principles mentioned above, as well as an arithmetical interpretation of these consequences.

A) A local-global principle for $H_{U_0}$

We begin by supplementing the above Notations 2.1, 2.4 as follows:

**Notations/Remarks 3.1.** Recall that $K$ is a finitely generated field, $n$ is a positive integer not divisible by $\text{char}(K)$, and $k_0 \subset K$ is a relatively algebraically closed subfield with $\dim(k_0) = 1$, hence a global field, and $K$ separably generated over $k_0$. Set $d := \dim(K)$.

1) We let $S_0$ be the canonical model of $k_0$, that is:
   - $S_0 = \text{Spec} \mathcal{O}_K$, if $k_0 = K$ is a number field.
   - $S_0$ is the unique projective smooth $K$-curve with $\kappa(S_0) = k_0$, if $K$ is a finite field.

2) Notice that for every non-empty finite subset $\Sigma_0' \subset \mathbb{P}_{fin}(k_0)$, the complement $U_0' := S_0 \setminus \Sigma_0'$ is actually an affine open subset of $S_0$.

3) In these notations, consider the subgroups:

\[ \Delta_{U_0} := \{ u_0' \in k_0^\times \mid \nu(u_0' - 1) > 2 \cdot \nu(n) \ \forall \nu \in \Sigma_0 \} \subset k_0^\times, \]

\[ H_{U_0} := \langle u_0' \mid a_1, \ldots, a_d \in k_0^\times \rangle \subset H^{d+1}(K, \mathbb{Z}/n(d)). \]

4) In the Notations/Remarks 2.1 and 2.4, by general scheme theoretical non-sense, every proper model of $K$ is dominated by projective normal flat $S_0$-models $\mathcal{X}$ of $K$, with $\mathcal{X} \to S_0$ defining $k_0 \hookrightarrow K$. In particular, $\dim(\mathcal{X}) = d$, and for such $\mathcal{X}$, one has:
   a) The base change $\mathcal{X}_{U_0} := \mathcal{X} \times_{S_0} U_0 \to U_0$ is a projective normal flat $U_0$-model of $K$.
   b) The generic fiber $X = \mathcal{X}_{U_0} \times_{U_0} k_0 = \mathcal{X} \times_{S_0} k_0$ is a projective model $k_0$-model of $K|k_0$.

5) Finally, the sets of points of codimension one $\mathcal{X}_{U_0} \supset X^1$ are in canonical bijection with the Weil prime divisors $\mathcal{D}^1_{\mathcal{X}_{U_0}} \supset \mathcal{D}^1_X$ of $\mathcal{X}_{U_0}$, respectively $X$. Thus $\mathcal{D}^1_{\mathcal{X}_{U_0}} \supset \mathcal{D}^1_X$ are sets prime divisors of $K$, respectively of geometric prime $k_0$-divisors of $K|k_0$.

**Proposition 3.2.** Suppose that $k_0$ has no orderings, $n = \ell^\nu$ is a power of a prime number, and $\mu_n \subset K$. Then for every projective flat $S_0$-model $\mathcal{X} \to S_0$ the following hold:

1) Suppose that $\mathcal{X}_{U_0}$ is regular. Then the canonical map from the Kato complex

\[ H_{U_0} \hookrightarrow H^{d+1}(K, \mathbb{Z}/n(d)) \to \bigoplus_{x_1 \in \mathcal{X}_{U_0}} H^d(\kappa(x_1), \mathbb{Z}/n(d-1)) \]
is injective, i.e., for every $0 \neq \alpha \in H_{U_0}$ there are $x_1 \in X^1_{U_0}$ such that $\partial x_1(\alpha) \neq 0$.

2) Suppose that $X_{U_0} \to U_0$ is smooth. Then the canonical map from the Kato complex

$$H_{U_0} \hookrightarrow H^{d+1}(K, \mathbb{Z}/n(d)) \to \bigoplus_{x_1 \in X^1} H^d(\kappa(x_1), \mathbb{Z}/n(d-1))$$

is injective, i.e., for every $0 \neq \alpha \in H_{U_0}$ there are $x_1 \in X^1$ such that $\partial x_1(\alpha) \neq 0$.

**Proof.** To 1): Let $\alpha \neq 0$ be an element in $H_{U_0}$. 

**Step 1.** First, since $\alpha \neq 0$, by Jannsen’s result Theorem 2.5 mentioned above, there exists some $v \in \mathbb{P}(k_v)$ such that $\alpha \neq 0$ over $K_v$.

**Claim.** $v \in U_0$, hence $v$ is non-archimedean, and $n$ is invertible in $\mathcal{O}_v$.

Indeed, one has the following: First, by contradiction, suppose that $v$ is an archimedean place of $k_v$. Then $\kappa$ is a number field, hence $k_v = \kappa$, and since $k_v$ allows no orderings, $k_{0v}$ is by mere definitions algebraically closed. Hence $k_{0v}$ being algebraically closed, $K_v$ has cohomological dimension equal to $\text{td}(K_v[k_{0v}]) = \dim(K) - 2$, see e.g., Serre [Se]; thus concluding that $H^{d+1}(K_v, \mathbb{Z}/n(d)) = 0$, contradiction! Second, by contradiction, suppose that $v \not\in U_0$, or equivalently, $v \in \Sigma_0 = S_0 U_0$. Then by the definition of $\Delta_{U_0}$, one has $v(u' - 1) > 2 \cdot v(n)$ for all $u' \in \Delta_{U_0}$, thus $u' \in U_1^1$. Hence $p(T) := T^n - u' \in \mathcal{O}_v[T]$ and its derivative $p'(T) = nT^{n-1}$ evaluated at $T = 1$ satisfy: $p(1) = 1 - u'$, $p'(1) = n$, and therefore

$$v(p(1)) = v(1 - u') > 2 \cdot v(n) = 2 \cdot v(p'(1)).$$

Hence by Helsel’s Lemma, $p(T) \in \mathcal{O}_v[T]$ has a root in the henselian field $k_{0v}$, or equivalently, $u'$ is an $n^{th}$ power in $k_{0v}$. Hence setting $u' = u^n$ for some $u \in k_{0v}$, one has

$$u'_n u_1 \ldots u_n = u^n u_1 \ldots u_n = n \cdot (u u_1 \ldots u_n) = 0 \quad \text{over} \quad K_v.$$ 

Since this holds for all the generators $u'_n u_1 \ldots u_n$ of $H_{U_0}$, it follows that the image of $H_{U_0}$ in $H^{d+1}(K_v, \mathbb{Z}/n(d))$ is trivial, contradiction! We thus finally conclude that $v \in U_0$. On the other hand, by the definition of $U_v$, one has that $n$ is invertible on $U$, hence since $v \in U_0$, it follows that $n$ is invertible in $\mathcal{O}_v$. The Claim is proved.

**Step 2.** Consider any $v \in \mathbb{P}(k_v)$ such that $\alpha \neq 0$ over $K_v$, where $K_v = K k_{0v}$ with $k_{0v}$ the Henselization of $k_{0v}$ at $v$. In particular, by the discussion above, it follows that $v \in U_0$. Letting $R_v \subset k_{0v}$ be the Henselization of $\mathcal{O}_v$ at $v$, consider the base change defined by the canonical morphism $\text{Spec} R_v \to \text{Spec} \mathcal{O}_v \to U_0$

$$X_{R_v} := X_{U_0} \times_{U_0} R_v \to X_{U_0},$$

and the corresponding field extension

$$K_v = \kappa(X_{R_v}) \leftarrow \kappa(X) = K.$$

Since $X_{U_0}$ is a projective regular $U_0$-scheme, and $\text{Spec} R_v \to U_0$ is a pro-étale morphism, it follows that $X_{R_v}$ is a projective regular $R_v$-scheme, and $X_{R_v} \to X_{U_0}$ is a pro-étale morphism.

Coming back to the proof of the assertion 1), let $\alpha_v \neq 0$ be the image of $\alpha$ over $K_v$, and recall that $X_{R_v}$ is a projective regular flat $R_v$-scheme with $\kappa(X_{R_v}) = K_v$. Since $\alpha_v \neq 0$, by the Kerz–Saito’s Theorem 2.6 mentioned above, there exist points $x_v \in X^1_{R_v} = X_{R_v, d-1}$ such that $\partial x_v(\alpha_v) \in H^{d+1}(\kappa(x_v), \mathbb{Z}/n(d))$ is non-trivial. On the other hand, since $X_{R_v}$ is regular, the local ring $\mathcal{O}_{x_v}$ is a DVR with residue field $\kappa(x_v)$ satisfying

$$\dim(\kappa(x_v)) = \dim(X_{R_v}) - 1 = \dim(K_v) - 1 = \dim(K) - 1.$$
Further, recalling that the canonical projection $\mathcal{X}_{R_v} \to \mathcal{X}_{U_0}$ is a pro-étale morphism, it follows that letting $x_v \mapsto x_1 \in \mathcal{X}_{U_0}$ be the image of $x_v$ under $\mathcal{X}_{R_v} \to \mathcal{X}_{U_0}$, one has: The corresponding $U_0$-embedding $\mathcal{O}_{x_1} \hookrightarrow \mathcal{O}_{x_v}$ is a pro-étale extension of discrete valuation rings. In particular, since $\kappa(x_1) \hookrightarrow \kappa(x_v)$ is a separable algebraic field extension, one has
\[
dim(\kappa(x_1)) = \dim(\kappa(x_v)) = \dim(K) - 1 = \dim(\mathcal{X}_{U_0}) - 1,
\]
hence $x_1 \in \mathcal{X}_{U_0,d-1} = \mathcal{X}_{U_0}^{1}$. Since $\mathcal{X}_{U_0}$ is a projective regular $U_0$-scheme, $x_1 \in \mathcal{X}_{U_0}$ is a regular point. Thus the (pro-étale) extension of discrete valuation rings $\mathcal{O}_x \hookrightarrow \mathcal{O}_{x_v}$ and the resulting canonical embedding of residue fields $\kappa(x_1) \hookrightarrow \kappa(x_v)$ give rise to a commutative diagram:
\[
\begin{array}{ccc}
H^{d+1}(K, \mathbb{Z}/n(d)) & \xrightarrow{\text{res}} & H^{d+1}(K_v, \mathbb{Z}/n(d)) \\
\downarrow \partial_{x_1} & & \downarrow \partial_{x_v} \\
H^{d}(\kappa(x_1), \mathbb{Z}/n(d-1)) & \xrightarrow{\text{res}} & H^{d}(\kappa(x_v), \mathbb{Z}/n(d-1))
\end{array}
\]
Since $\partial_{x_v}(\alpha_v) \in H^{d}(\kappa(x_v), \mathbb{Z}/n(d-1))$ is non-trivial, it follows by mere definitions that $\partial_{x_1}(\alpha)$ is non-trivial. This concludes the proof of assertion 1) of Proposition 3.2.

To 2): In the notation and context of the proof of assertion 1) above, and the hypothesis of assertion 2), let us denote $\text{Spec}(R_v) = \{\eta_0, \mathfrak{m}_v\}$, with $\eta_0$ the generic point, $\mathfrak{m}_v$ the closed point of $\text{Spec} R_v$, and $\kappa(v) = R_v/\mathfrak{m}_v$ be the (finite) residue field of $v$. Then one has:

**Case 1.** $x_v$ maps to $\eta_0$ under $\mathcal{X}_{R_v} \to \text{Spec} R_v$. Equivalently, one has $\kappa(\eta_0) = \kappa_0 \subset \mathcal{O}_{x_v}$, thus concluding that $k_0 \subset \mathcal{O}_{x_v} \cap K = \mathcal{O}_x$. Equivalently, the image $x_v \mapsto x_1$ lies in the generic fiber $X \subset \mathcal{X}_{U_0}$ of $\mathcal{X}_{U_0} \to U_0$, thus finally $x_1 \in X_1$, and assertion 2) is proved.

**Case 2.** $x_v$ maps to $\mathfrak{m}_v$ under $\mathcal{X}_{R_v} \to \text{Spec} R_v$. Since $\mathcal{X}_{U_0} \to U_0$ is projective smooth, $\mathcal{X}_{R_v}$ is a projective smooth $R_v$-scheme, hence the special fiber
\[
\mathcal{X}_v := \mathcal{X}_{R_v} \times_{R_v} \kappa(v) = \mathcal{X} \times_{U_0} \kappa(v)
\]
is a projective smooth $\kappa(v)$-variety; thus a projective smooth $\kappa(v)$-model of $\kappa(x_v) = \kappa(\mathcal{X}_v)$, and $\partial_{x_v}(\alpha_v) \in H^{d}(\kappa(x_v), \mathbb{Z}/n(d-1))$ is non-trivial. Hence by the Kato-Saito’s Theorem 2.6 mentioned above, there exist points of codimension one $y_v \in \mathcal{X}_v^1$ of $\mathcal{X}_v$ such that
\[
0 \neq \partial_{y_v}(\partial_{x_v}(\alpha_v)) \in H^{d-1}(\kappa(y_v), \mathbb{Z}/n(d-2))
\]
Since $\dim(\mathcal{X}_v) = \dim(\mathcal{X}_{R_v}) - 1 = d - 1$, one has $\mathcal{X}_v^1 = \mathcal{X}_{v,d-2} \subset \mathcal{X}_{R_v,d-2} = \mathcal{X}_v^2$, hence $y_v \in \mathcal{X}_v$ is a regular point of $\mathcal{X}_{R_v}$ with codim($y_v$) = 2, such that $y_v \mapsto \mathfrak{m}_v$ under $\mathcal{X}_{R_v} \to \text{Spec} R_v$.

Let $\mathcal{X}(y_v) \subset \mathcal{X}_{R_v}^1$ be the points $x_v' \in \mathcal{X}_{R_v}$ with $y_v \in \overline{\{x_v'\}}$. Invoking the Kato complex (KC) for the projective regular flat $R_v$-scheme $\mathcal{X}_{R_v}$, it follows that
\[
\sum_{x_v' \in \mathcal{X}(y_v)} \partial_{y_v}(\partial_{x_v'}(\alpha)) = 0.
\]
On the other hand, since $x_v \in \mathcal{X}(y_v)$ and $\partial_{y_v}(\partial_{x_v}(\alpha)) \neq 0$, it follows that there must exist points $x_v' \in \mathcal{X}(y_v)$ satisfying the following:
\[
x_v' \neq x_v, \text{ and further, } \partial_{y_v}(\partial_{x_v'}(\alpha)) \in H^{d-1}(\kappa(y_v), \mathbb{Z}/n(d-2)) \text{ is non-trivial.}
\]
**Claim.** If $x_v' \in \mathcal{X}(y_v)$, $x_v' \neq x_v$, then $x_v' \mapsto \eta_0 \in \text{Spec } R_v$ under $\mathcal{X}_{R_v} \to \text{Spec } R_v$.

Indeed, by contradiction, suppose that $x_v' \mapsto \mathfrak{m}_v$ under $\mathcal{X}_{R_v} \to \text{Spec } R_v$, hence $x_v'$ lies in the special fiber $x_v' \in \mathcal{X}_v$ of $\mathcal{X}_{R_v} \to \text{Spec } R_v$. Since $\mathcal{X}_{R_v} \to \text{Spec } R_v$ is flat, and $x_v'$ has codimension
one in $\mathcal{X}_{R_v}$, it follows that $x'_v$ is a generic point of $\mathcal{X}_v$. Since $\mathcal{X}_v$ is integral, and $x_v \in \mathcal{X}_v$ is a generic point, one has $x'_v = x_v$, contradiction! The claim is proved.

Coming back to the proof of assertion 2), by the discussion above we have that $x'_v \in \mathcal{X}_v^1$ satisfies the hypothesis from Case 1 above. Hence by the discussion there, conclude that the image $x'_1 \in \mathcal{X}_{v_0}^1$ of $x'_v$ under $\mathcal{X}_v^1 \to \mathcal{X}_{v_0}^1$ lies in $X^1$, thus $x'_1$ does the job. \hfill $\square$

B) A few technical results for later use

Let $K$ be a field satisfying Hypothesis (H) from the Introduction. Set $e := \text{td}(K|k_0) - 1 > 0$, and suppose that $n$ is a prime number such that $\mu_{2n} \subseteq K$.

Let $t = \{t_1, \ldots, t_e\}$ be $e$ algebraically independent elements $t_i \in K$, and $k := k_t \subseteq K$ be the relative algebraic closure of $k_0(t)$ in $K$. Then $\text{td}(K|k) = 1$, and there exists a unique projective smooth geometrically integral $k$-curve $C$ such that $K = k(C)$. Conversely, if $k \subseteq K$ is any field with $\text{td}(K|k) = 1$ and $k$ relatively algebraically closed in $K$, then one has: $\text{td}(k|k_0) = e$, and $k = k_t$ for every set of $e$ algebraically independent elements $t = \{t_1, \ldots, t_e\}$ of $k$. Further, $K = k(C)$ for a unique projective smooth geometrically integral $k$-curve $C$.

Recall that the closed points $P \in C$ are in canonical bijection with the prime divisors $v$ of $K|k$ via $O_P = O_v$. Let $f \in K[K]$ be given. Our aim in this subsection is to give a criterion — which for $n = 2$ turns out to be first order expressible — for the following:

There exist $P \in C$ such that $v_P(f)$ is not divisible by $n$ in $v_P(K)$.

We begin by supplementing Notations/Remarks 241 243 344 as follows. Some/all of these facts might be well known to experts, but I cannot give precise references.\footnote{I would like to thank Dan Abramovich and Michael Temkin for helping me clarify this.}

Notations/Remarks 3.3. In the above notation, let $\text{div}_C : K^\times \to \text{Div}(C)$ be the divisor map, and for every $f \in K[K]$, let $(f) := \text{div}_C(f)$ be the divisor of $f$, and $(f)_0, (f)_{\infty}$ be the zero, respective pole, divisor of $f$. The degree formula asserts that

$$\deg(f) := |K : k(f)| = \deg(f)_0 = \deg(f)_{\infty}.$$ 

As explained above, our aim is to give criteria for the fact that the set below is nonempty:

$$D_f := \{P \in C \mid v_P(f) \not\in n \cdot vK\} \subset |\text{div}_C(f)|$$

We proceed by discussing two geometric aspects of the situation.

Part I.

In the above discussion, for non-empty subsets $I \subset I_e := \{1, \ldots, e\}$, set $t_I := \{t_i\}_{i \in I}$, and let $k_I \subseteq K$ be the relative algebraic closure of $k_0(t_I)$ in $K$. Then $k_0(t_I) \hookrightarrow k_I$ are finite field extensions, and if $I \subset J \subset I_e$, there are canonical field embeddings $k_I(t_J) \hookrightarrow k_I(t_J)$ and $k_J \hookrightarrow k_I$. To simplify notations, if $I = \{i\}$, we set $t_I = t_i$ and denote $k_I = k_{t_i}$.

For every $t_i$ above, let $\mathbb{P}_{t_i} := \mathbb{P}^1_{k_0}$ be the $k_0$-projective line with parameter $t_i$. Further, for every non-empty $I \subset I_e$, we set $\mathbb{P}_{t_I} := \times_{i \in I} \mathbb{P}_{t_i}$, and consider the normalization $\tilde{\pi}_I : S_I \to \mathbb{P}_{t_I}$ of $\mathbb{P}_{t_I}$ in the field extension $k_0(t) \hookrightarrow k_I$. Then $\dim(S_I) = |I|$, thus $S_I := S_{\{i\}}$ are projective smooth geometrically integral $k_0$-curves, and further one has:

$$(*)_{S_I} \tilde{\pi}_I : S_I \to \mathbb{P}_{t_I} \text{ is a finite dominant morphism of geometrically integral } k_0\text{-varieties.}$$
Since \( \mathbb{P}_{t_i} \) is smooth, the branch locus of \( \tilde{\pi}_I \) is a divisor \( D_{t_i} \subset \mathbb{P}_{t_i} \). Hence if \( U_{t_i} \subset \mathbb{P}_{t_i} \setminus D_{t_i} \) is any open dense subset, and \( U_I \subset S_I \) is its preimage under \( \tilde{\pi}_I \), one has:

\[
(*)_{U_I} \quad \tilde{\pi}_I : U_I \rightarrow U_{t_i} \text{ is a finite étale morphism of smooth } k_0\text{-varieties.}
\]

To simplify notation, if \( I = I_e \), we set \( \mathbb{P}_t := \mathbb{P}_{t_e} / D_t, D_t := D_{t_e}, U_t := \mathbb{P}_t / D_t, \) and \( S := S_{t_e}, U := U_{t_e}, \) and denote the corresponding morphisms \( \pi_I : S \rightarrow S_I, \pi_I : S \rightarrow S_I, \) and \( \tilde{\pi} : S \rightarrow \mathbb{P}_t \).

Notice that if \( I \subset J \), the \( k_0\)-embeddings \( k_0(I) \rightarrow k_0(J) \), \( k_l \rightarrow k_J \) give rise canonically to dominant \( k_0\)-maps \( \pi_{J,I} : S_I \rightarrow S_J, \tilde{\pi}_{J,I} : \mathbb{P}_{t_I} \rightarrow \mathbb{P}_{t_J} \) fitting into the commutative diagram:

\[
\begin{array}{ccc}
S_I & \xrightarrow{\tilde{\pi}_{J,I}} & \mathbb{P}_{t_J} \\
\downarrow \pi_{J,I} & & \downarrow \tilde{\pi}_{J,I} \\
S_I & \xrightarrow{\tilde{\pi}_I} & \mathbb{P}_{t_I}
\end{array}
\]

Moreover, one can choose dense open subsets \( U_{t_i} \subset \mathbb{P}_{t_i} \setminus D_{t_i} \) such that their preimages \( U_I \subset S_I \setminus D_I \) are compatible with the maps \( \pi_{J,I} : S_I \rightarrow S_J \) for \( I \subset J \) as follows.

**Lemma 3.4.** In the above notation, let \( U_t \subset \mathbb{P}_t \setminus D_t \) be a fixed dense open subset, and \( U \subset S \) be the preimage of \( U_t \) under \( \tilde{\pi} : S \rightarrow \mathbb{P}_t \). Further, for every non-empty \( I \subset I_e \) define:

\[
U_{t_i} := \tilde{\pi}_I(U_t), \quad U_I := \pi_I(U).
\]

Then \( \pi_I : U_I \rightarrow U_{t_i} \) is a finite étale map, and \( \tilde{\pi}_{J,I}(U_{t_i}) = U_{t_i}, \pi_{J,I}(U_I) = U_I \) for all \( I \subset J \).

**Proof.** For every non-empty proper subset \( I \subset I_e \), set \( I' := I_e \setminus I. \) Then \( \mathbb{P}_t = \mathbb{P}_{t_I} \times \mathbb{P}_{t_{I'}} \), thus the projection map \( \tilde{\pi}_I : \mathbb{P}_t \rightarrow \mathbb{P}_{t_I} \) is an open map, hence \( U_{t_I} \subset \mathbb{P}_{t_I} \) is open dense. Further, \( \tilde{\pi} : S \rightarrow \mathbb{P}_t \) factors through \( \tilde{\pi}_I \times \text{id}_{t_{I'}} : S_I \times \mathbb{P}_{t_{I'}} \rightarrow \mathbb{P}_{t_I} \times \mathbb{P}_{t_{I'}}, = \mathbb{P}_t, \) and therefore one has:

First, \( U_I \) equals the preimage of \( U_{t_I} \) under \( \pi_I : S_I \rightarrow \mathbb{P}_{t_I} \), and therefore \( U_I \subset S_I \) is open.

Second, \( \tilde{\pi}_I : S_I \rightarrow \mathbb{P}_{t_I} \) is étale above \( U_{t_I} \subset \mathbb{P}_{t_I} \), hence \( \tilde{\pi}_I : U_I \rightarrow U_{t_i} \) is a finite étale map of smooth geometrically integral \( k_0\)-varieties. Finally, from this also follows that \( \pi_{J,I} : S_J \rightarrow S_I \) are open maps, and \( \pi_{J,I}(U_I) = U_I \). \( \square \)

Next let \( I \subset J \) be given. Recall that by mere definitions, \( k_I = k_0(S_I) \) is relatively algebraically closed in \( K \), hence in \( k_J = k_0(S_J) \subset K \). Therefore, the generic fiber \( S_{J,k_J} \) of \( \pi_{J,I} : S_J \rightarrow S_I \) is a projective geometrically integral \( k_I\)-variety, and \( U_{I,k_I} \subset S_{J,k_J} \) is an open dense smooth \( k_I\)-subvariety. Hence using Sard’s Lemma, after properly shrinking \( U_t \) and correspondingly all the \( U_I \), we can and will suppose that the following hold:

i) \( U_t \subset \text{Spec} k_0[\mathbf{t}, \mathbf{t}^{-1}] =: \mathbb{G}_t \subset \text{Spec} \subset \text{Spec}[\mathbf{t}] =: \mathbb{A}_t. \)

ii) The fibers \( U_{I,s} \subset S_{I,s} \) of \( \pi_{J,I} : S_J \rightarrow S_I \) at points \( s \in U_I \) are geometrically integral \( \kappa(s)\)-varieties, and \( U_{I,s} \) is smooth. Further, if \( s \in U_I \) is a closed point, then

\[
\dim(U_{I,s}) = \dim(S_{I,s}) = |J| - |I|.
\]

**Part II.**

In the above context and notation, notice that every proper model of \( K/k_0 \) is dominated by projective smooth models \( X \) of \( K/k_0 \) which have the following two main features:

- The \( k_0\)-embedding \( k_0(\mathbf{t}) \hookrightarrow K \) is defined by a dominant \( k_0\)-morphism \( \phi_0 : X \rightarrow \mathbb{P}_t. \)

In particular, for every \( I \) one gets a dominant \( k_0\)-morphism \( \phi_{t_I} : X \rightarrow \mathbb{P}_{t_I} \).
In the above context, for every over $k$ of $\phi$.

Finally, the generic fiber of $X$:

$$(*)_{X_I}$$

$$(*)_{X_I}$$

In particular, if $I \subseteq J$, one has that $\phi_I = \pi_{IJ} \circ \phi_J$.

Recalling that $k := k_0(S)$ is the function field of $S$, we notice that the generic fiber of $\phi : X \to S$ is the projective smooth $k$-curve $C = X \times_S k \subset X$ with function field $K|k$, i.e., $K = k(C)$ canonically. For closed points $P \in C$, let $X_P \subset X$ be the closure of $P$ in $X$, and notice the following: Since $P \in C$ is in the generic fiber of $\phi : X \to S$, thus of $\phi_I : X \to S_I$, it follows that $\phi(P) = \eta_S$ and $\phi_I(P) = \eta_{S_I}$ are the generic points of $S$, respectively $S_I$. In particular, $\phi$ and $\phi_I$ give rise to dominant surjective maps:

$$\phi : X_P \to S, \quad \phi_I : X_P \to S_I.$$ 

We require the projective smooth $k_0$-varieties $X$ introduced above have the property:

The Zariski closures $X_P \subset X$ of all the $P \in |\text{div}(f)|$ are smooth disjoint $k_0$-subvarieties.

In particular, since $X$ and $X_P$ are projective smooth $k_0$-varieties, their generic fibers $X_{k_I}$ and $X_{P,k_I}$ of $\phi_I : X \to S_I$ and $\phi_I : X_P \to S_I$ are projective smooth varieties over $k_I$. Further, since $k_I$ is relatively algebraically closed in $k \subset K$, the special fibers of $\phi_I : X \to S_I$ and $\pi_I : S \to S_I$ have geometrically integral generic fibers $X_{k_I}$ and $S_{k_I}$. Finally, the generic fiber of $X_{k_I} \to S_{k_I}$ is nothing but $C = X \times_S k = X_{k_I} \times_{S_{k_I}} k$. Hence, invoking among other things Sard’s Lemma, the following hold:

1) For all “sufficiently small” open non-empty subsets $U_I \subset S_I$, and all points $s \in U_I$, the fibers $\phi_s : X_s \to S_s$ of $\phi_I : X \to S_I$, and $\phi_s : X_{P,s} \to S_s$ of $\phi_I : X_P \to S_I$ at $s \in U_I$ satisfy:

a) $\phi_s : X_s \to S_s$ is a morphism of projective geometrically integral $\kappa(s)$-varieties.

Further, $X_s$ is smooth, and if $s \in S_I$ is a closed point, then

$$\dim(S_s) = \dim(S) - |I|, \quad \dim(X_s) = \dim(X) - |I|.$$ 

b) The generic fiber $C_s$ of $\phi_s : X_s \to S_s$ is a projective smooth $k_s$-curve, where $k_s$ is the function field $k_s = k_0(S_s)$ of $S_s$. Denote $K_s := k_0(C_s) = k_0(X_s)$.

c) $\phi_s : X_{P,s} \to S_s$ are dominant morphisms, and $X_{P,s} \subset X_s$ are disjoint projective smooth $k_0$-varieties having $\dim(X_{P,s}) = \dim(X_P) - |I| = \dim(X) - |I| - 1$.

d) The restriction $f_s$ of $f$ to $C_s$ is non-constant, and one has:

$$\deg(f) = [K : k(f)] = [K_s : k_s(f_s)] = \deg(f_s).$$ 

Further, denoting by $P_{s,\alpha} \in C_s$ the generic points of $X_{P,s} \subset X_s$, one has:

$$v_P(f) = v_{P_{s,\alpha}}(f_s) \text{ for all } P \in |\text{div}_C(f)|.$$ 

e) Hence recalling the set $D_f \subset |\text{div}_C(f)|$ introduced above, one has:

Corollary. $P \in D_f$ iff $P_{s,\alpha} \in D_{f_s}$ for all $\alpha$ iff $P_{s,\alpha} \in D_{f_s}$ for some $\alpha$.

2) In the above context, for every $I \in I$, choose some “sufficiently small” open dense subset $U_I' \subset S_I$, and let $U_I \subset S$ be the preimage of $U_I'$ under $\pi_I : S \to S_I$. Then $U_I$ are
open dense subsets in $S$, hence so is $U^0 := \cap_I U^I$. Finally, since $\tilde{\pi} : S \to \mathbb{P}_t$ is finite dominant, there exist open dense subsets $U_t \subset \mathbb{P}_t$ satisfying:

i) $U_t \subset \mathbb{P}_t \setminus D_t$ is contained in the complement of the branch locus of $\tilde{\pi} : S \to \mathbb{P}_t$.

ii) The preimage $U \subset S$ of $U_t$ under $\tilde{\pi} : S \to \mathbb{P}_t$ is contained in $U^0$.

Therefore, the sets $U_I := \pi_I(U)$ satisfy: First, by mere definitions one has that $U_I \subset U^I$.

Second, $U_{t_I} := \tilde{\pi}_I(U_t) \subset \mathbb{P}_{t_I}$, and by Lemma 3.4 it follows that $\pi_I : U_I \to U_{t_I}$ is a finite étale morphism, and $\pi_{II}(U_I) = U_I$, $\tilde{\pi}_{II}(U_{t_I}) = U_{t_I}$, for every $I \subset J$.

3) For $U_t$ as above, and $a = (a_1, \ldots, a_e)$, $b := (b_1, \ldots, b_e) \in U_t(k_o)$ with $a_i \neq b_i$ for all $i$, we set $u_i := (t_i - a_i)/(t_i - b_i)$, and denote $u := (u_1, \ldots, u_e)$. We notice that $k_t = k_u$.

Finally, for $S_0 = \text{Spec} \mathcal{O}_{k_0}$ and every $U_0 \subset S_0$ as in Notations/Remarks 3.1, consider

$$H_{U_0, u, f} := \langle u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_e \cup f \mid u'_0 \in \Delta_{U_0}, u_0 \in k^e_0 \rangle \subset H_{U_0} \subset H^{d+1}(K, \mathbb{Z}/n(d))$$

**Proposition 3.5.** In the above notations, the following are equivalent:

i) $D_f$ is non-empty.

ii) There exist $a, b \in U_t(k_o)$ such that $H_{U_0, u, f} \neq 0$ for all $U_0$.

**Proof.** To i) $\Rightarrow$ ii): The proof of this implication is more or less “easy” and requires just standard facts. Let $P \in D_f$ be given, hence by mere definitions we have $v_P(f) = m \in \mathbb{Z}$, and $m$ is not divisible by $n$ in $\nu_P K$. Therefore, setting $\kappa_P := \kappa(P)$, the residue map

$$\partial_P : H^{d+1}(K, \mathbb{Z}/n(d)) \to H^d(\kappa_P, \mathbb{Z}/n(d - 1))$$

restricted to $H_{U_0, u, f}$ is simply

$$\partial_P(u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_e \cup f) = (-1)^d m \cdot u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_e. \quad \text{In particular, if } u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_e \neq 0, \text{ then } u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_e \cup f \neq 0, \text{ thus } H_{U_0, u, f} \neq 0.$$

Recall that $k = k_o(S) \hookrightarrow \kappa_P$ is a finite field extension. Hence the normalization $S_P \to S$ of $S$ in the function field extension $k \hookrightarrow \kappa_P$ is a finite cover, and $S_P \to S \to \mathbb{P}_t$ is étale above a dense open subset $U_P \subset U_t$. Thus choosing $a \in U_P(k_o)$, its preimages $s_p \in S_P$ are regular points; and let $s_p \in S_P$ be a fixed preimage of $a$. Then $u = (u_1, \ldots, u_e)$ is a system of regular parameters at $s_p$, and if $\kappa_{s_p} := \kappa(s_p)$ is the residue field at $s_p$, then for every $u_i := (u_i, \ldots, u_e)$, $1 \leq i \leq e$, let $R_i = \mathcal{O}_{s_p}/(u_i)$. Then $R_i$ is a local regular ring with maximal ideal $m_i := m_{s_p}/(u_i)$. And setting $K_i := \text{Quot}(R_i)$, the residue map

$$\partial_i : H^{d+1}(K_i, \mathbb{Z}/n(d - i)) \to H^d(K_{i-1}, \mathbb{Z}/n(d - i - 1))$$

satisfies $\partial_i(u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_i) = u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_{i-1}$. Hence reasoning as above, it follows inductively that $u'_0 \cup u_0 \cup u_1 \cup \ldots \cup u_e \cup f \neq 0$, thus $H_{U_0, u, f} \neq 0$, provided $u'_0 \cup u_0 \neq 0$ in $H^2(\kappa_{s_p}, \mathbb{Z}/n(1))$. Hence is is left to prove that the image of $H_{U_0} := \langle u'_0 \cup u_0 \mid u'_0 \in U_0, u_0 \in k^e_0 \rangle$ under the restriction map

$$\text{res} : H^2(k_o, \mathbb{Z}/n(1)) \to H^2(\kappa_{s_p}, \mathbb{Z}/n(1))$$

is non-trivial. Since $k_o \hookrightarrow \kappa_{s_p}$ is an extension of number fields, this fact is well known.

To ii) $\Rightarrow$ i): The proof of this implication is much more involved, and it is based on the technical preparation we made above and relies in an essential way on the higher Hasse local-global principles mentioned above. We make induction on $e = \text{td}(K[k_o]) - 1$, which was supposed to be positive. Notice that the assertion for $e = 0$, i.e., the case when $K = k_o(C)$
is the function field of a curve was dealt with Pop [P4]. Precisely, it was shown there that $H_{U_0,f}$ is non-trivial for all $U_0 \subset S_0$ iff $D_f$ is non-empty.

Here we show that given $e > 0$, one has: If the implication ii) $\Rightarrow$ i) holds in all cases for all $e' < e$, then the implication ii) $\Rightarrow$ i) holds for $e$ in all cases.

First, in the context of Proposition 3.5, since $X$ is a projective smooth geometrically integral $k_0$-variety, there exists an open subset $U_{0,X} \subset S_0$ such that $X$ has a projective regular (actually even smooth) $U_{0,X}$-model $\mathcal{X}_{U_{0,X}}$, thus $X = \mathcal{X}_{U_{0,X}} \times_{U_{0,X}} k_0$ is the generic fiber of $\mathcal{X}_{U_{0,X}} \to U_{0,X}$. Therefore, for every open subset $U_0 \subset U_{0,X}$, the base change

$$\mathcal{X}_{U_0} := \mathcal{X}_{U_{0,X}} \times_{U_{0,X}} U_0 \to U_0$$

is a regular (and even smooth) $U_0$-model of $X$. Hence by Proposition 3.5, the canonical map

$$H^{d+1}(K, \mathbb{Z}/n(d)) \to \bigoplus_{w \in D_X} H^d(\kappa(x), \mathbb{Z}/n(d-1))$$

is injective on $H_{U_0}$, thus on $H_{U_0,u,f} \subset H_{U_0}$.

For $\alpha = u_0' \cup u_0 \cup u_1 \cup \ldots \cup u_e \cup f \neq 0$, let $w \in D^1_X$ satisfy $\partial_w(\alpha) \neq 0$ in $H^d(\kappa(x), \mathbb{Z}/n(d-1))$. Then by mere definitions, not all the values $w(u_1), \ldots, w(u_e), w(f)$ are divisible by $n$ in $wK$.

**Case 1.** $w$ is trivial on $k$, i.e., $w = v_P$ for some $P \in C$. Then

$$0 \neq \partial_w(\alpha) = (-1)^d w(f) \cdot u_0' \cup u_0 \cup u_1 \cup \ldots \cup u_e$$

hence $w(f)$ is not divisible by $n$, and $P \in D_f$, concluding that $D_f$ is non-empty.

**Case 2.** $u_1, \ldots, u_e$ are $w$-units. Then one must have $w(f) \neq 0$ and not divisible by $n$. Further, letting $\overline{u}_i \in Kw$ be the residue of $u_i$, it follows that $\partial_w(\alpha) = u_0' \cup u_0 \cup \overline{u}_1 \cup \ldots \cup \overline{u}_e$. We claim that $\overline{u}_1, \ldots, \overline{u}_e$ must be algebraically independent over $k_0$. Indeed, otherwise the field $L := k_0(\overline{u}_1, \ldots, \overline{u}_e)$ has dim($L$) = 1 + $td(L/k_0) < e + 1$; and since $k_0$ has no orderings, it follows that $H^{e+2}(L, \mathbb{Z}/n(e+1))$ is trivial, thus $\partial_w(\alpha) \in H^{e+2}(L, \mathbb{Z}/n(e+1))$ must be trivial, contradiction! Hence we conclude that $\overline{u}_1, \ldots, \overline{u}_e$ must be algebraically independent over $k_0$, and therefore $w$ is trivial on $k_0(u_1, \ldots, u_e) = k_0(\hat{t})$; thus finally implying that $w$ is trivial on $k$ — which is a finite, thus algebraic, extension of $k_0(\hat{t})$. Conclude by applying Case 1.

**Case 3.** $u_i$ is not a $w$-unit. We first claim that $t_i$ is a $w$-unit. Indeed, by contradiction, if $w(t_i) \neq 0$, then a case-by-case computation shows: If $w(t_i) > 0$, then $w(t_i - a_i) = w(a_i) = 0$, and similarly, $w(t_i - b_i) = w(b_i) = 0$, thus $w(u_i) = 0$, contradiction! If $w(t_i) < 0$, then $w(t_i - a_i) = w(t_i) < 0$, and similarly, $w(t_i - b_i) = w(t_i) < 0$, thus $w(u_i) = 0$, contradiction! Since $t_i$ and $a_i, b_i \in k_0^\times$ are $w$-units, one has that $w(t_i - a_i), w(t_i - b_i) \geq 0$. Hence since $0 \neq w(u_i) = w(t_i - a_i) - w(t_i - b_i)$, and $(t_i - a_i) - (t_i - b_i) = a_i - b_i \in k_0^\times$ is a $w$-unit, one finally gets: Either $w(t_i - a_i) > 0 \& w(t_i - b_i) = 0$, or $w(t_i - a_i) = 0 \& w(t_i - b_i) > 0$. Hence we conclude that the restriction $w_i$ of $w$ to $k_0(t_i)$ is the zero of either $t_i - a_i$ or of $t_i - b_i$.

To fix notations, w.l.o.g., we can suppose that the former is the case (otherwise on can work with $t_i - b_i$ in the same way). Let $\eta_w \in X$ be the center of $w$ on $X$, i.e., $\eta_w \in X$ is the generic point of the Weil prime divisor of $X$ defining $w$. Recalling the Notations/Remarks 3.3 in the case $I = \{1\}$, consider/recall the dominant morphisms introduced/defined there, and the corresponding images of $\eta_w$ under those morphisms:

$$X \xrightarrow{\phi} S \xrightarrow{\phi_i} \mathbb{P}_{t_i} \quad \eta_w \mapsto s \mapsto a_i$$
Then by Notations/Remarks 3.3 in the case $I = \{i\}$, it follows that $s_i \in S_i$ is one of the finitely preimages of $a_i$ under the finite morphism $\pi_{t_i} : S_i \to \mathbb{P}_{t_i}$. Moreover, since $a = (a_1, \ldots, a_e) \in U_e(k_o)$, by mere definitions it follows that $s_i \in U_i$, and that $s_i$ is a closed point. Hence by Notations/Remarks 3.3 especially 1), we have that $X_{s_i}$ is a projective smooth geometrically integral $\kappa(s_i)$-variety. Since $\phi_t(a_i) = s_i$, one has $\eta_i \in X_{s_i}$, and since $\eta_i \in X^1$ has codimension one, $\eta_i$ is the generic point of $X_{s_i}$. To fix notations, let $w_{s_i} \in D_X$ be the prime divisor defined by the Weil prime divisor $\eta_i$ for every $w_{s_i} \in D_X$.

Repeating this argument for smaller an smaller open subsets $U_0 \subset U_{0,X}$, one has: Since for every $i = 1, \ldots, e$ there are only finitely many points $s_i \in U_i$ above $a_i$, and each such point $s_i$ defines a unique $w_{s_i} \in D_X$, one has:

**There exist $i \in I_e$ and $s_i \in U_i$, fixed once an for all, such that $w_{s_i}$ satisfies:**

*For every $U_0 \subset U_{0,X}$ there is $\alpha \in HU_{0,U_{i,s}}$ with $\partial_{w_{s_i}}(\alpha) \neq 0$ in $\text{H}(\kappa(w_{s_i}), \mathbb{Z}/n(n))$.

In order to simplify further notation, after renumbering $\{t_1, \ldots, t_e\}$, we can and will suppose that $i = e$. We set $e' := e - 1$, $I_{e'} := \{1, \ldots, e'\}$. For subsets $I \subset I_{e'}$ with $e \in I$, set $I' := I \cap I_{e'}$, and further denote $t' := (t_1, \ldots, t_{e'})$, $t'_e = \{t_i\}_{i \in I'}$. Then considering/applying Notations/Remarks 3.3 in the special case $I := \{e\}$, $s := s_e \in S_e$, $k'_e := \kappa(s)$, and subsequently, for all subset $I \subset J \subset I$ containing $e$, we get the following picture:

1) $P_{t,e'} \rightarrow P_{t_i} \rightarrow P_{t'_i}$ are defined by $k_0[t'_i] \rightarrow k_0[t] \rightarrow k_0[t]/(t_e - a_e) \cong k_0[t'_i]$, the latter being the canonical isomorphism. Hence $P_{t,e'} \rightarrow P_{t_i} \rightarrow P_{t'_i}$ identifies $P_{t,e'} \cong P_{t'_i}$ canonically. We further denote by $U_{t'_i} \subset P_{t'_i}$ the fiber of $U_{t_i}$ at $a_e$.

2) For subsets $I \subset J \subset I_{e'}$ with $e \in I$, the special fiber at $s \mapsto a_e$ gives: First a commutative diagram in which the rows are finite dominant $k_0$-morphisms and the columns are open $k_0$-morphisms, a second diagram in which the rows are finite étale morphisms, and the third one of the corresponding diagram of function field $k_0$-embeddings:

$$\begin{array}{ccc}
S_{I,s} & \xrightarrow{\tilde{\pi}_{I,s}} & P_{t'_i} \\
\downarrow \pi_{I,s} & & \downarrow \pi_{I,s} \\
S_{I,s} & \xrightarrow{\tilde{\pi}_{I,s}} & P_{t'_i} \\
\end{array}$$

We set $U' := U_s \subset S'_s := S'_t$, and let $k' = k'_0(S'_s)$ be the function field of $S'_s$. Recalling that $U_I = \tilde{\pi}_I(U)$ for all $I$, the functoriality of the base change gives:

$$\pi_I(U') = U'_I, \quad \text{hence } \pi_{J'}(U'_{J'}) = U'_{J'} \text{ for all } I \subset J \text{ with } e \in I.$$
φ_2: X' → S_t,s factors through S'_t → S_t,s, that is, there exists a dominant morphism

φ'_2: X' → S'_t

such that φ_2 = (S'_t → S_t,s) ∘ φ'_2,

and by the functoriality of normalization, one has φ'_2 = π'_t ∘ φ'_t for I ⊆ J with e ∈ I. Further, since s ∈ S_e is a closed point, one has:

\[ \dim(S'_t) = \dim(S_I) - 1, \quad \dim(X') = \dim(X) - 1. \]

4) The generic fiber C' of φ'_2: X' → S' is a projective smooth k'_1-curve, where k'_1 = k'_0(S'_t).

Denote K' := k'_1(C') = k'_1(X') the corresponding function field.

5) The k'_0-subvarieties X'_P := X_{P,s} ⊂ X' are disjoint projective smooth k'_0-subvarieties having dim(X'_P) = dim(X_P) - 1 = dim(X' - 1) for all P ∈ |div(f)|. Further, the restriction of φ': X' → S' to each X'_P is a dominant k'_0-morphisms φ'_P : X'_P → S'_t.

6) The restriction f' := f_e of f to C' = C_s is non-constant, and one has:

\[ \deg(f) = [K : k(f)] = [K' : k'(f')] = \deg(f'). \]

Further, denoting by P'_α ∈ C' the generic points of X'_P ⊂ X', one has:

\[ v_P(f) = v'_P(f') \quad \text{for all } P \in |\text{div}_C(f)| \quad \text{and } P'_α. \]

In particular, D_f is non-empty iff D_f' is non-empty.

7) Recall that P'_t := P_t is the fiber of P_t → P_{t,e} at (t_e - a_e). By mere definitions one has:

First, a k'_0-rational point c := (c_1, ..., c_e) ∈ P_t(k_0) lies in the fiber P'_t := P_t if c_e = a_e, and if so, then c is the image under \P'_t := \P_t of \c := (a_1, ..., a_e) and c' is the unique preimage of c under \P'_t := \P_t. Hence a' := (a_1, ..., a_e), b' := (b_1, ..., b_e) ∈ \P'_t(k_0) are the preimages of a, b ∈ \U_t(k_0) ⊂ P_t(k_0) under \U'_t := U_t. Further, u' = (u_1, ..., u_e) is actually the restriction of u to P'_t, and moreover, k_o(t') = k_o(u').

8) Recall also that for every open subset U_0 ⊂ U_{o,X} ⊂ S_0, one has that

\[ (†) \quad 0 \neq H_{U_0,u',f'} := \langle u'_0 ∪ u_0 ∪ u_1 ∪ ... ∪ u_n ∪ f' \mid u'_0 ∈ Δ_{U_0}, u_0 ∈ k'_0 \rangle \subset H^d(K'/\mathbb{Z}/n(d - 1)). \]

Clearly, by the Chebotarev Density Theorem, the condition (†) implies that the corresponding condition holds as well if one replaces U_0 ⊂ S_0 by U'_0 ⊂ S'_0, where S'_0 → S_0 is the normalization of S in the finite extension of number fields k_o → k'_0, and U'_0 is the preimage of U_0 in S'_0.

Hence the hypothesis ii) from Proposition 3.3.5 is satisfied by the projective smooth k'_0-model X' of K' together with X' → S' → P'_t, u'_t := U_{t,a_e}, u' := (u_1, ..., u_e), and the resulting H_{U'_0,u',f'} for all U'_0 ⊂ S'_0. Since e' = td(K'/k'_0) - 1 < td(K/k_0) - 1 = e, by the induction hypothesis we have that D_f is non-empty. Thus by point 6) above, D_f is non-empty as well. □

C) A criterion for D_f to be non-empty

Notations/Remarks 3.6. Recalling that v_o denotes the finite places of k_o, we supplement the previous Notations/Remarks 3.3.3 as follows. For every δ, δ_1, ..., δ_e ∈ k'_0 we define:

1) U_δ := \{ a ∈ k'_0 \mid v_o(δ) ≠ 0 or v_o(δ) > 0 \Rightarrow v_o(a - 1) > 2 \cdot v_o(n) \forall v_o \}, the n,δ-units.

2) Σ_δ := \{ a ∈ k'_0 \mid v_o(δ) ≠ 0 \Rightarrow v_o(a) ≠ 0 \}, the non-δ-units.

We notice the following:
a) \( U_\delta \) is a subgroup of \( k_0^\times \), and \( \Sigma_\delta \) is a \( U_\delta \)-set, i.e., \( U_\delta \cdot \Sigma_\delta = \Sigma_\delta \).

b) For every finite set \( A \subset k_0 \) there exist “many” \( \delta \in k_0^\times \) such that \( A \cap \Sigma_\delta = \emptyset \).

3) \( U^*_\delta := U_\delta \times k_0^\times \), \( \Sigma^*_\delta := \{(a, b) \mid a, b \in \Sigma_\delta, a \neq b\} \), and \( \Sigma^*_\delta := \times \Sigma^*_\delta \) for \( \delta := (\delta_1, \ldots, \delta_e) \).

4) In the above context, for \( (a_i, b_i) \in \Sigma^*_\delta \), set \( u_i := (t_i - a_i)/(t_i - b_i) \), \( u := (u_1, \ldots, u_e) \), and for \( \delta_0 \in k_0^\times \) and \( u_0 := (u_0^0, u_0) \in U^*_\delta \), define

\[
\alpha_{u_0, u, f} := u_0^0 u_0 u_1 \cdots u_{e-1} \cup H^{d+1}(K, \mathbb{Z}/n(d))
\]

and consider the subgroup:

\[
H_{\delta_0, u, f} := \langle \alpha_{u_0, u, f} \mid u_0 \in U^*_\delta \rangle \subset H^{d+1}(K, \mathbb{Z}/n(d))
\]

**Key Lemma 3.7.** In the above notations, the following are equivalent:

i) \( D_f \) is non-empty.

ii) \( \exists \delta \forall (a_1, b_1) \in \Sigma^*_\delta \ldots \exists \delta_e \forall (a_e, b_e) \in \Sigma^*_\delta \exists \delta_0 \exists (u_0^0, u_0) \in U^*_\delta \) such that \( \alpha_{u_0, u, f} \neq 0 \).

**Proof.** To i) \( \Rightarrow \) ii): The proof of this implication is a kind of standard, and similar to the proof of the implication i) \( \Rightarrow \) ii) from Proposition 3.5. Recalling the usual notation as at Notations/Remarks 2.4, 3.1 above, let \( P \in C \) be such that \( m := v_P(f) \) is not divisible by \( n \) in \( vK \). Then the residue field \( \kappa_P := \kappa(P) \) is a finite extension of \( k = k_0(S) \), hence if \( S_P \rightarrow S \rightarrow \mathbb{P}_t \) is the normalization of \( \mathbb{P}_t \) in the finite function field extension \( k_0(t) \rightarrow \kappa_P \), it follows that there exists a dense open subset

\[
U_t \subset G_t := \text{Spec} k_0[\mathcal{t}] \subset A_t := \text{Spec} \{t\} \subset \mathbb{P}_t
\]

such that \( S \rightarrow \mathbb{P}_t \) is étale above \( U_t \). Equivalently, letting \( U_P \subset S_P \) be the preimage of \( U_t \) under \( S_P \rightarrow \mathbb{P}_t \), then \( U_P \rightarrow U_t \) is a finite étale cover. Hence if \( \alpha := (a_1, \ldots, a_e) \in U_t(k_0) \) is a \( k_0 \)-rational point, then \( t_\alpha := (t_1 - a_1) \) is a regular system of parameters at \( \alpha \). Further, if \( x_\alpha \in U_P \) is any preimage of \( \alpha \) under \( S_P \rightarrow \mathbb{P}_t \), then \( x_\alpha \) is regular as well, and \( t_\alpha \) is a regular system of parameters at \( x_\alpha \) as well. Then proceeding as in the proof of the implication i) \( \Rightarrow \) ii) from Proposition 3.5, it follows that for every \( \delta_0 \in k_0^\times \) there exist \( u_0^0, u_0 \in U^*_\delta \) such that \( u_0^0 u_0 u_1 \cdots u_{e-1} \cup H^{d+1}(K, \mathbb{Z}/n(d)) \neq 0 \).

To conclude the proof of i) \( \Rightarrow \) ii), we proceed as follows: Recalling that \( U_t \subset G_t \subset A_t \), we identify \( A_t(k_0) = k_0^\times \) canonically via the functions \( \mathcal{t} = (t_1, \ldots, t_e) \), thus \( U_t(k_0) \subset k_0^\times \) is a Zariski open dense subset of \( k_0^\times \). Hence setting \( t_i := (t_1, \ldots, t_i) \) for \( 1 \leq i \leq e \), and letting \( \varpi_i : A_t \rightarrow A_{t_i} \) be the canonical projections defined by \( k_0[\mathcal{t}] \rightarrow k_0[t_i] \), one has: The map defined by \( \varpi_i \) on \( k_0 \)-points is the projection on the first \( i \) coordinates \( \varpi_i : k_0^\times \rightarrow k_0^\times \). Further, the following hold:

a) \( U_i := \varpi_i(U_t(k_0)) \subset k_0^\times \) is a Zariski open dense subset.

b) If \( \alpha_i \in U_i \), and \( U_{t, \alpha_i} \subset A_{t, \alpha_i} \rightarrow A_t \) are the fibers of \( U_t \subset A_t \) at \( \alpha_i \) under \( \varpi_i : A_t \rightarrow A_{t_i} \), then at the level of \( k_0 \)-rational points, one has canonical identifications

\[
U_{t, \alpha_i}(k_0) \subset A_{t, \alpha_i}(k_0) = \varpi_i^{-1}(\alpha_i) = \alpha_i \times k_0^{(e-i)} \subset k_0^\times = A_t(k_0)
\]

c) In particular, \( U_{t, \alpha_i}(k_0) \subset A_i \times k_0^{(e-i)} \) is a Zariski open dense subset for all \( \alpha_i \in U_i \).

Proceed by induction on \( i = 1, \ldots, e \) as follows:

**Step 1.** \( i = 1 \): Since \( \Sigma_1 := \varpi_1(U_t(k_0)) \subset k_0 \) is Zariski open dense, its complement \( A_1 \subset k_0 \) is finite. Hence \( \exists \delta_1 \in k_0^\times \) such that \( \Sigma_0, \delta_1 \cap A_1 = \emptyset \), thus \( \Sigma_0, \delta_1 \subset \Sigma_1 \). Then \( \forall (a_1, b_1) \in \Sigma_\delta \), set \( \alpha_1 := (a_1), b_1 := (b_1) \), and proceed.
Step 2. Induction step: Suppose that \(a_i = (a_1, \ldots, a_i)\), \(b_i = (b_1, \ldots, b_i)\) are dense subsets; in particular, w.l.o.g., suppose that condition ii) of Key Lemma 3.7 is satisfied. Let \(U_a := U_t(a_i) \subset a_i \times k_0^{(e-i)}\) be Zariski open dense, and therefore \(U_{a<i} \subset a_i \times k_0\) is Zariski open dense, and hence there exists a cofinite set \(\Sigma_{a} \cap k_0\) such that \(U \times \Sigma_{a} \subset U_{a<i}\). Accordingly, there exists a cofinite set \(\Sigma_{b} \subset k_0\) such that \(U \times \Sigma_{b} \subset U_{b<i}\). Hence \(U \times \Sigma_{a} \cap U \times \Sigma_{b} \subset k_0\) is cofinite and

\[
{a_i} \times \Sigma_{a} \subset U_{a<i} \subset a_i \times k_0, \quad {b_i} \times \Sigma_{b} \subset U_{b<i} \subset b_i \times k_0
\]

are Zariski open dense subsets. In particular, \(a_{i+1} := k_0 \setminus \Sigma_{a+i}\) is finite. Therefore, \(\exists \delta_{a+i+1} \in k_0^\times\) such that \(\Sigma_{a+i+1} \cap A_{i+1} = \emptyset\), hence in particular, \(\Sigma_{a+i+1} \subset \Sigma_{a+i}\). Then \(\forall (a_{i+1}, b_{i+1}) \in \Sigma_{a+i+1}\), and setting \(a_{i+1} := (a_i, a_{i+1})\), \(b_{i+1} := (b_i, b_{i+1})\), the following hold:

\[
{a_{i+1}} \in \Sigma_{a+i+1}(U_{a+i}(k_0)) \subset \Sigma_{a+i}(U_{a+i}(k_0)), \quad {b_{i+1}} \in \Sigma_{b+i+1}(U_{b+i}(k_0)) \subset \Sigma_{b+i}(U_{b+i}(k_0)).
\]

This completes the proof of the induction step, thus of the implication i) \(\Rightarrow\) ii).

To ii) \(\Rightarrow\) i): Let \(X \overset{\phi}{\rightarrow} S \overset{\phi}{\rightarrow} \mathbb{P}_t, U \rightarrow U_t\), etc., as in the Proposition 3.5. We prove that condition ii) from our Key Lemma 3.7 is satisfied. Suppose that condition ii) of Key Lemma 3.7 is satisfied. Let \(U_t \subset \mathbb{P}_t\) be an arbitrary open dense subset; in particular, w.l.o.g., suppose that \(U_t \subset G_t \subset A_t\). Then arguing as at the end of the proof of the implication i) \(\Rightarrow\) ii) above, it follows that condition ii) implies the existence of points \(a, b \in U_t(k_0)\) such that for every \(\delta_0 \in k_0^\times\), there are nontrivial symbols \(u_0 \cup u_0 \cup u_1 \cup \ldots \cup u_0 \cup f \neq 0\) with \((u_0, u_0) \in U_0^\times\). Given \(U_0 \subset S_0\) an open subset, choose any \(\delta_0 \in k_0^\times\) such that for all \(v_0 \in S_0 \setminus U_0\) one has: \(v_0(\delta_0) \neq 0\). Then by mere definitions, see Notations/Remarks 3.1, 3), it follows that \(U_0 \subset \Delta_0\). Therefore,

\[
0 \neq u_0 \cup u_0 \cup u_1 \cup \ldots \cup u_n \cup f \in H_{U_0,u,f} \subset H^{d+1}(K, \mathbb{Z}/n(d))
\]

thus \(H_{U_0,u,f}\) is a non-trivial subgroup. Hence condition ii) from Proposition 3.5 is satisfied, thus concluding that \(D_f\) is non-empty. \(\square\)

4. Uniform definability of the geometric prime divisors of \(K\)

In this section we work in the context and notation of the previous sections, but supposing \(n = 2\).

In particular, for \(a := (a_1, \ldots, a_r)\) with \(a_i \in K^\times\), by the Milnor Conjecture one has:

\[
a_1 \cup \ldots \cup a_r \in H^r(K, \mathbb{Z}/n(r - 1)) \text{ is trivial if } 0 \in q_a(K),
\]

and therefore, the fact \(a_1 \cup \ldots \cup a_r = 0\) is first order expressible.

Next let \(K\) be a field satisfying Hypothesis (H). Hence \(\text{char}(K) = 0\) and the constant field \(\kappa =: k_0\) is a number field, and \(e = \text{td}(K/k_0) - 1 > 0\). Further, \(k = k_t\) denotes the relative algebraic closure of \(k_0(t)\) in \(K\), where \(t = (t_1, \ldots, t_e)\) are \(e\) algebraically independent functions \(t_i \in K\). Further, \(K = k(C)\) is the function field of a projective geometrically integral \(k\)-curve \(C\). Recalling the context of the Key Lemma 3.7 for \(t = (t_1, \ldots, t_e)\) and
$f \in K \setminus k$, we let $x := (x_1, \ldots, x_{2^{d+1}}) \neq (0, \ldots, 0)$ be a system of $2^{d+1}$ variables, and consider the following uniform first order formula:

$$\varphi(f, t) \equiv \exists \delta_1 \forall (a_1, b_1) \in \Sigma_\delta^* \ldots \exists \delta_e \forall (a_e, b_e) \in \Sigma_\delta^* \exists \delta_0 \exists (u_0', u_0) \in U_{\delta_0}^* \forall x : q_{u_0, u_0, f}(x) \neq 0.$$  

Then the Key Lemma 3.7 can be reformulated uniformly first order in the following way:

**Key Lemma (revisited) 4.1.** In the above notation, the following are equivalent:

i) $D_f$ is non-empty.

ii) $\varphi(f, t)$ holds in $K$.

**Proof.** As explained above, this is just a reformulation of Key Lemma 3.7. 

Our final aim in this section is to show that the prime divisors of $K|k$ are uniformly first order definable. Precisely, we will give formulæ

$$\text{val}_d(x; f, t, \xi, \delta, (a_i, b_i), \in \Sigma_\delta^*, \delta_0, u_0 \in U_{\delta_0}^*, q_{u_0, u_0, f}(x))$$

such that evaluating all the variables but $x$ in $K$, the resulting predicates in the variable $x$ define all the valuation rings $O_{u_0} \subset K$ of prime $k$-divisors $K|k$. Here, $q_{u_0, u_0, f}$ is the $(d+1)$-fold Pfister form defined $u_0, u, f$. These formulæ involve — among other things — RUMELY’s [Ru] formulæ $\text{val}_1$ which uniformly define the prime divisors of number fields; further, in the case of finitely generated fields of characteristic zero and Kronecker dimension two, Pop [P4] gives formulæ $\text{val}_2$ which uniformly define the geometric prime divisors.

**Theorem 4.2.** There exist explicit formulæ $\text{val}_d$ which uniformly define the geometric prime divisors of finitely generated fields $K$ with $\text{char}(K) = 0$ and $\text{dim}(K) > 1$ as follows

$$\text{val}_d(x; f, t, \xi, \delta, (a_i, b_i), \in \Sigma_\delta^*, \delta_0, u_0 \in U_{\delta_0}^*, q_{u_0, u_0, f}(x)).$$

The proof of Theorem 4.2 follows from the Recipe 4.7 below.

A) The uniformly definable subsets $\Theta_{f, t}, \overline{\Theta}_{f, t}$ and semi-local subrings $\mathfrak{a}_{f, t} \subset R_{f, t}$ of $K$.

We recall the formula $\varphi(f, t)$ and its negation $\neg \varphi(f, t)$ to be used often later

$$\varphi(f, t) \equiv \exists \delta_1 \forall (a_1, b_1) \in \Sigma_\delta^* \ldots \exists \delta_e \forall (a_e, b_e) \in \Sigma_\delta^* \exists \delta_0 \exists (u_0', u_0) \in U_{\delta_0}^* \forall x : q_{u_0, u_0, f}(x) \neq 0$$

$$\neg \varphi(f, t) \equiv \forall \delta_1 \exists (a_1, b_1) \in \Sigma_\delta^* \ldots \forall \delta_e \exists (a_e, b_e) \in \Sigma_\delta^* \exists \delta_0 \exists (u_0', u_0) \in U_{\delta_0}^* \forall x : q_{u_0, u_0, f}(x) = 0$$

and interpret them over $K$ and its quadratic extensions $K_{\eta} := K[\sqrt{\eta}], \eta \in K$, and $\tilde{K}, \tilde{K}_{\eta}$. Furthermore, in order to simply notation and language, for fixed $f, t$ and $\varepsilon, \xi$, we denote:

- $\varphi(f, t, \varepsilon) \equiv (\varphi(f, t) \text{ holds in } K_{\varepsilon})$.

- $\overline{\varphi}(f, t, \xi) \equiv (\neg \varphi(f, t) \text{ holds in both } K_{\xi} \text{ and } K_{\xi-1})$.

**Notations/Remarks 4.3.** We supplement Notations/Remarks 3.1 3.3 3.6 as follows.

1) For $\varepsilon \in K$ and $K_{\varepsilon} := K[\sqrt[d]{\varepsilon}]$ as above, consider the restriction map

$$\text{res}_{\varepsilon} : H^{d+1}(K, Z/n(d)) \to H^{d+1}(K_{\varepsilon}, Z/n(d)).$$

2) Let $C_{\varepsilon} \to C$, $P_{\varepsilon} \to P$, be the normalization of $C$ in the field extension $K_{\varepsilon} \leftrightarrow K$, and denote by $D_{f, \varepsilon}$ the set of all $P_{\varepsilon} \subset C_{\varepsilon}$ such that $v_{P_{\varepsilon}}(f) \not\in n \cdot v_{P_{\varepsilon}}(K_{\varepsilon})$.

3) For every $P \subset C$, let $U_P := \mathcal{O}_P^*$ be the $P$-units, and let $U_P \subset \mathcal{O}_{v_P}$ be the $v_P$-units. Then for $\varepsilon \in K^\times$ the following are equivalent:

...
In particular,

\[ v_P(\varepsilon) \in n \cdot v_P(K) \]

Indeed, i) \( \Rightarrow \) ii) is clear. For ii) \( \Rightarrow \) i), let \( v_P(\varepsilon) \in n \cdot v_P(K) \), and \( \eta \in K \) be such that \( v_P(\varepsilon) = n \cdot v_P(\eta) = v_P(\eta^n) \). Then \( \varepsilon/\eta^n \in U_P \), hence one has \( \varepsilon \in U_P \cdot K^n \).

**Lemma 4.4.** In the above notations, for given \( t, f \), the assertions below are equivalent:

i) \( \varepsilon \in \bigcup_{P \in D_f} U_P \cdot K^n \).

ii) \( \varphi(f, t, \varepsilon) \), i.e., \( \varphi(f, t) \) holds in \( K_\varepsilon \).

In particular, \( \Theta_{f, t} := \bigcup_{P \in D_f} U_P \cdot K^n \subset K \) are uniformly first order definable in \( K \) as follows:

\[ \Theta_{f, t} = \{ \varepsilon \in K \mid \varphi(f, t, \varepsilon) \} \]

**Proof.** i) \( \Rightarrow \) ii): Let \( \varepsilon \in \bigcup_{P \in D_f} U_P \cdot K^n \) be given, and \( P \in D_f \) be such that \( \varepsilon \in U_P \cdot K^n \). Then \( P \) is unramified in the extension \( K_\varepsilon[K] \). Hence if \( C_\varepsilon \rightarrow C \) is the normalization of \( C \) in the field extension \( K_\varepsilon \leftrightarrow K \), it follows that \( v_P(\Delta) = v_P(f) \) is prime to \( n \). Hence \( D_{f, \varepsilon} \neq \emptyset \), and therefore, the hypothesis ii) is satisfied over \( K_\varepsilon \). Let \( k_{\varepsilon} = k_\varepsilon \cap K_\varepsilon \) be the field of constants of \( K_\varepsilon \). Then \( k_{\varepsilon} \cap K_\varepsilon \) is a finite field extension, and therefore, for every \( \delta, \varepsilon \in k_{\varepsilon} \), there is \( \delta \in k_{\varepsilon}^{\times} \) such that for all \( v_\varepsilon \) and \( v_\eta := v_\varepsilon|_{k_\varepsilon} \), one has: \( v_\varepsilon(\delta, \varepsilon) = v_\varepsilon(\delta) \neq 0 \). In particular, \( \Sigma_{\varepsilon} \cap k_\varepsilon = \Sigma_\delta \). Therefore, condition ii) for \( \varepsilon \) implies condition ii) for \( K_\varepsilon \). This concludes the proof of the implication i) \( \Rightarrow \) ii).

The implication ii) \( \Rightarrow \) i): Consider any \( \varepsilon \not\in \bigcup_{P \in D_f} U_P \cdot K^n \) that is, \( \varepsilon \not\in U_P \cdot K^n \) for all \( v \in D_f \). Then by Notations/Remarks 4.3, it follows that \( v_P(\varepsilon) \not\in n \cdot v_P(K) \) for all \( P \in D_f \). Therefore, for all \( P \in D_f \), and any prolongation \( P_\varepsilon \) to \( K_\varepsilon \) one has: \( e(\varepsilon|_{P_\varepsilon}) = e(\varepsilon|_{P}) \cdot v_P(\varepsilon) \in n \cdot v_P(\varepsilon) \). Further, since \( v_P(\varepsilon) \in n \cdot v_P(K) \) for \( P \not\in D_f \), one has \( v_P(\varepsilon) \in n \cdot v_P(K) \) for all \( P \in D_f \). Thus we finally conclude that \( v_P(\varepsilon) \in n \cdot v_P(K) \) for all \( P \in D_f \), and hence for the discussion at Notations/Remarks 4.3, it follows that \( \varepsilon \not\in U_P \cdot K^n \).

**Notations/Remarks 4.5.** In the notations from Lemma 4.4 above, we have the following:

1. Let \( \eta \in K \setminus \Theta_{f, t} \) be given. Then by Notations/Remarks 4.3, it follows that for all \( P \in D_f \) one has: \( v_P(\eta) \not\in n \cdot v_P(K) \). In particular, \( v_P(\eta) \neq 0 \), and therefore one has:
   - If \( v_P(\eta) > 0 \), then \( \eta - 1 \in m_P - 1 \subset U_P \subset \Theta_{f, t} \), hence finally \( \eta - 1 \in \Theta_{f, t} \).
   - If \( v_P(\eta) < 0 \), then \( v_P(\eta - 1) = v_P(\eta) \neq n \cdot v_P(K) \). Therefore, by the discussion at Notations/Remarks 4.3, it follows that \( \eta - 1 \not\in U_P \cdot K^n \).

2. Conclude that for \( \eta \in K \) the conditions (i), (ii) below are equivalent:
   - (i) \( \eta, \eta - 1 \not\in \Theta_{f, t} \);
   - (ii) \( v_P(\eta) < 0 \) for all \( P \in D_f \) and \( v_P(\eta) \not\in n \cdot v_P(K) \) for all \( P \in D_f \).

3. Hence the subsets \( \overline{\Theta}_{f, t} \subset K \) below are uniformly definable:
   \[ \overline{\Theta}_{f, t} := \{ \xi \in K \mid [\xi, \xi'] = 1, [\xi', \xi] = 1 \} \subset \{ \xi \in K \mid \overline{\varphi}(f, t, \xi) \} \]

4. Finally notice: \( \xi \in \overline{\Theta}_{f, t} \) iff \( \forall P \in D_f \) one has: \( v_P(\xi) > 0 \), \( v_P(\xi) \not\in n \cdot v_P(K) \).

**Lemma 4.6.** In the above notation, one has

\[ a_{f, t} := \bigcap_{P \in D_f} m_P = \overline{\Theta}_{f, t} \subset K \]

Hence \( a_{f, t} \subset K \) is uniformly definable, and therefore so is the subring \( R_{f, t} \) below:
\[ R_{f,t} = \cap_{P \in D_f} \mathcal{O}_P = \{ r \in K \mid r \cdot a_{f,t} \subset a_{f,t} \} \]
\[ = \{ r \in K \mid \forall \xi', \xi'' \in K \text{ s.t. } \widetilde{\varphi}(f,t,\xi'), \widetilde{\varphi}(f,t,\xi'') \exists \tilde{\xi}', \tilde{\xi}'' \in K \text{ s.t. } \overline{\varphi}(f,t,\tilde{\xi}'), \overline{\varphi}(f,t,\tilde{\xi}'') \land r(\xi' - \xi'') = \tilde{\xi}' - \tilde{\xi}'' \} \]

**Proof.** We first prove the equality \( \cap_{P \in D_f} m_P = \overline{\mathcal{O}_{f,t}} - \overline{\mathcal{O}_{f,t}} \). For the inclusion \( \subset \), notice that \( \overline{\mathcal{O}_{f,t}} \subset m_P, P \in D_f \) by Notations/Remarks [4.3, 4] above. Hence \( \overline{\mathcal{O}_{f,t}} - \overline{\mathcal{O}_{f,t}} \subset m_P - m_P = m_P, P \in D_f \), thus finally one has \( \overline{\mathcal{O}_{f,t}} - \overline{\mathcal{O}_{f,t}} \subset a_{f,t} \). For the converse inclusion \( \supset \), let \( \xi \in a_{f,t} \) be arbitrary. Since \( a_{f,t} = \cap_{P \in D_f} m_P \), it follows by Notations/Remarks [4.3, 3], above that \( v_P(\xi) > 0 \) for all \( P \in D_f \). Hence by the weak approximation lemma, it follows that there exists \( \xi' \in K \) such that both \( \xi' \) and \( \xi'' := \xi' - \xi \) satisfy \( v_P(\xi') = v_P(\xi'') = 1 \). In particular, by Notations/Remarks [4.3, 4], one has \( \xi', \xi'' \in \overline{\mathcal{O}_{f,t}}, \xi = \xi' - \xi'' \in \overline{\mathcal{O}_{f,t}} - \overline{\mathcal{O}_{f,t}} \).

Concerning the assertions about \( R_{f,t} \), the first row equalities are well known basic valuation theoretical facts (which follow, e.g. using the weak approximation lemma), whereas the second row equality is simply the definition of \( \{ r \in K \mid r \cdot a_{f,t} \subset a_{f,t} \} \) using the explicit definition of \( a_{f,t} = \overline{\mathcal{O}_{f,t}} - \overline{\mathcal{O}_{f,t}} \); this also shows/implies the uniform definability of \( R_{f,t} \). □

**B) Defining the k-valuation rings of \( K|k \)**

In the notations and hypotheses the previous sections, recall that \( K = k(C) \) for some projective smooth \( k \)-curve \( C \). By Riemann-Roch we have: For every closed point \( P \in C \) and \( m \gg 0 \), there exist functions \( f \in K \) such that \( (f)_\infty = mP \). Hence choosing \( m \) to be prime to \( n = 2 \), we have \( P \in D_f \). Thus by Lemma 4.6 it follows that \( P \in D_f \) and \( R_{f,t} = \mathcal{O}_P \cap R^0_{f,t} \), where \( R^0_{f,t} = \cap_{P \in D_f} \mathcal{O}_P \) with \( P \neq P \) the zeros of \( f \) which lie in \( D_f \).

For \( f \) as above, we set \( g := f + 1 \), and notice that \( (g)_\infty = mP = (f)_\infty \), etc., and obviously, \( f \) and \( g \) have no common zeros. We repeat the constructions above with \( f \) replaced by \( g \), and get \( R_{g,t} = \mathcal{O}_P \cap R^0_{g,t} \), where \( R^0_{g,t} = \cap_{Q \in D_g} \mathcal{O}_Q \) with \( Q \neq P \) the zeros of \( g \) from \( D_g \). Since \( |\text{div}(f)| \cap |\text{div}(g)| = \{ P \} \), by the weak approximation lemma one has:
\[
\mathcal{O}_P = R_{f,t} \cdot R_{g,t} := \{ r_1r_2 \mid r_1 \in R_{f,t}, r_2 \in R_{g,t} \}
\]
Therefore, setting \( f_1 := f \) and \( f_2 := g = f + 1 \), we have the following:
\[
\mathcal{O}_P = R_{f_1,t} \cdot R_{f_2,t} = \{ r \in K \mid \exists r_1, r_2 \in R_{f_1,t} \text{ s.t. } r = r_1r_2 \}
\]
\[
= \{ r \in K \mid \exists r_1, r_2 \in K \text{ s.t. } r = r_1r_2 \text{ and for } i = 1,2 \text{ one has: } \forall \xi'_i, \xi''_i \in K \text{ s.t. } \overline{\varphi}(f_i,t,\xi'_i), \overline{\varphi}(f_i,t,\xi''_i) \exists \tilde{\xi}'_i, \tilde{\xi}''_i \in K \text{ s.t. } \overline{\varphi}(f_i,t,\tilde{\xi}'_i), \overline{\varphi}(f_i,t,\tilde{\xi}''_i) \text{ and } r_i(\xi'_i - \xi''_i) = \tilde{\xi}'_i - \tilde{\xi}''_i \}
\]
and finally one recovers \( m_P \) as being
\[
| m_P | = \{ r \in K \mid r \in \mathcal{O}_{P}, r^{-1} \notin \mathcal{O}_{P} \}
\]
Hence we have the following **uniform first order recipe** to define the prime \( k_0 \)-divisors of \( K|k \).

**Recipe 4.7.** Recalling \( \varphi_d, \psi_0, \psi_{abs}(x) \), \( \psi_{l}(t_1, \ldots, t_r) \) as introduced in the Introduction, consider/recall the following: First, \( \text{char}(K) = 0 \) iff \( \psi_0 \) holds in \( K \), and \( \text{dim}(K) = d \) iff \( \varphi_d \) holds in \( K \). Further, the constant subfield \( k_0 := k \subset K \) is \( k_0 = \{ x \in K \mid \psi_{abs}(x) \text{ holds in } K \} \).

• From now on suppose that \( \text{char}(K) = 0 \).

Recall that if \( \text{dim}(K) = 1 \), then \( K = k_0 \) is a number field, and the prime divisors of \( K \) are uniformly first order definable by the formulae \( \text{val}_1 \) given by Rümel [Ru]. Second, if \( \text{dim}(K) = 2 \), the geometric prime divisors of \( K|k_0 \) are uniformly first order definable by the formulae \( \text{val}_2 \) given by Pop [P4].
We next consider the case $\dim(K) > 2$. Setting $e := \dim(K) - 2 > 0$, we construct

$$\text{val}_d(x; f, t, \xi, \delta, (a_i, b_i)) \in \Sigma^*_d, \delta_0, u_0 \in U^*_d, q_{u_0, u, f}(x_\xi)$$

in a concrete way along the following steps:

1) The systems $t := (t_1, \ldots, t_e)$ of algebraically independent elements of $K$ are uniformly definable using the algebraic (in)dependence formula $\psi_e(t_1, \ldots, t_e)$ over $k_0$. Further, the relative algebraic closure $k := k_t$ of $k_0(t)$ in $K$ is uniformly first order definable by

$$k = k_t = \{ u \in K \mid \neg \psi_e+1(u, t) \text{ holds in } K \}.$$  

2) The sets $P \subset K \times K$ of all the pairs $(f, t)$ with $k_t \subset K$ as above and $f \in K \setminus k_t$ such that $H_{\delta_0, u, f} \neq 0$ are uniformly first order definable, being defined as follows:

$$P = \{(f, t) \mid \varphi(f, t) \text{ holds in } K \}$$

Hence so is $P_1 := \{(f, t) \in P \mid (f+1, t) \in P \} = \{(f, t) \mid \varphi(f, t) \land \varphi(f+1, t) \text{ holds in } K \}.$

3) Therefore, the sets $P_{\text{val}} \subset K \times K$ below are uniformly first order definable:

$$P_{\text{val}} := \{(f, t) \in P_1 \mid O_{f, t} := R_{f, t} \cdot R_{f+1, t} \text{ is a proper valuation ring of } K \}$$

4) Finally note that the above $O_{f, t}$ are valuation rings of prime $k_0$-divisors of $K|k_0$, and conversely, for every prime $k_0$-divisor $w$ of $K|k_0$ there are pairs $(f, t) \in P_{\text{val}}$ such that the valuation ring $O_w$ is of the form $O_w = O_{f, t}$.

Conclude that the prime divisors of $K|k_0$ are uniformly first order definable via the set $P_{\text{val}}$.

5. Proof of the Main Theorem

A) First proof: Using Scanlon [Sc]

A first proof follows simply from Scanlon, Theorem 4.1 and Theorem 5.1, applied to the case of characteristic zero, using Theorem [11] for the definability of valuations (which is essential in both Theorem 4.1 and Theorem 5.1 of loc.cit.). Note that char($K$) = 0 is singled out by enhancing the sentences/formulas from these theorems by Poonen’s sentence $\psi_0$, to fulfill the characteristic zero requirement. In particular, this proof also shows that finitely generated fields of characteristic zero are bi-interpretable with the arithmetic.

B) Second proof: Using Aschenbrenner–Khé利夫–Naziazeno–Scanlon [AKNS]

Recall that one of the main results of [AKNS] asserts that the finitely generated infinite domains $R$ are bi-interpretable with arithmetic, see Theorem in the Introduction of loc.cit. In particular, the isomorphy type of any such domain is encoded by a sentence $\vartheta_R$. Thus the Main Theorem from the Introduction follows from the following stronger assertion:

**Theorem 5.1.** Let $T = (t_1, \ldots, t_r)$ be independent variables. Then the integral closures $R \subset K$ of $\mathbb{Z}[T]$ in finite field extensions $\mathbb{Q}(T) \hookrightarrow K$ are uniformly first order definable finitely generated domains.

**Proof.** Since $\mathbb{Z}[T]$ is Noetherian, and $\mathbb{Q}(T) \hookrightarrow K$ are finite separable extensions, the Finiteness Lemma asserts that $R$ is a finite $\mathbb{Z}[T]$-module, hence finitely generated as ring. The uniform definability of $R$ is though more involved, and uses the uniform definability of generalized geometric prime divisors of $K|k_0$ combined with Rumely [Ru].
Lemma 5.2. Let $A$ be an integrally closed domain, and $\mathcal{V}$ be a set of valuations of the fraction field $K_A := \text{Quot}(A)$ such that $A = \cap_{v \in \mathcal{V}} \mathcal{O}_v$. Let $B$ be the integral closure of $A$ in an algebraic extension $K_B|K_A$, and $\mathcal{W}$ be the prolongation of $\mathcal{V}$ to $K_B$. Then $B = \cap_{w \in \mathcal{W}} \mathcal{O}_w$.

Proof. Klar, left to the reader. □

Let $R_0$ be an integrally closed domain, $L_0 := \text{Quot}(R_0)$, and $\mathcal{V}_0$ be a set of valuations of $L_0$ such that $R_0 = \cap_{v \in \mathcal{V}_0} \mathcal{O}_v$. Let $L_1|L_0(t)$ be a finite field extension, $R_1 \subset \tilde{R}_1 \subset L_1$ be the integral closures of $R_0[t] \subset L_0[t]$ in $L_1$. Then $\kappa(P)$ are finite field extensions of $L_0$, $P \in \text{Max}(\tilde{R}_1)$ and let $\mathcal{V}_p$ be the prolongation of $\mathcal{V}_0$ to $\kappa(P)$. Finally let $\mathcal{V}_1$ be the set of all the valuations of the form $v_1 := v^p \circ v_p$ with $v_p$ the valuation of $P \in \text{Max}(\tilde{R}_1)$, and $v^p \in \mathcal{V}_p$. Then $v^p = v_1/v_p$, $v_1L_1 = v^pL_1 \times \mathbb{Z}$ lexicographically ordered, and $L_1v_1 = L^p v^p$. Further, the canonical restriction map $\text{Val}(L_1) \to \text{Val}(L_0)$ gives rise to a well defined surjective maps:

$$\mathcal{V}_1 \to \mathcal{V}_p \to \mathcal{V}, \quad v_1 \mapsto v^p \mapsto v := v^p|_{L_0} = v_1|_{L_0}.$$

Lemma 5.3. In the above notation, one has $R_1 = \cap_{v_1 \in \mathcal{V}_1} \mathcal{O}_{v_1}$.

Proof. Lemma 5.2 reduces the problem to the case $L_1 = L_0(t)$, $R_1 = R_0[t]$. For $v_1 = v^p \circ v_p$, $\mathcal{O}_{v_1} \subset \mathcal{O}_{v_p}$, hence $\cap_{v_1 \in \mathcal{V}_1} \mathcal{O}_{v_1} \subset \cap_P \mathcal{O}_{v_P} = L_0[t]$. Thus it is left to prove that $f(t) \in L_0[t]$ satisfies: $v_1(f) \geq 0$ for all $v_1 \in \mathcal{V}_1$ iff $f \in R_0[t]$. This easy exercise is left to the reader. □

Lemma 5.4. Suppose that all the valuation rings $\mathcal{O}_P$, $P \in \text{Max}(\tilde{R}_1)$ and $\mathcal{O}_{v_P}$, $v^p \in \mathcal{V}_p$ are (uniformly) first order definable. Then so are $\mathcal{O}_{v_1}$, $v_1 \in \mathcal{V}_1$ and $R_1 = \cap_{v_1 \in \mathcal{V}_1} \mathcal{O}_{v_1}$.

Proof. For $v_1 = v^p \circ v_p$, one has $\mathcal{O}_{v_1} = \pi_P^{-1}(\mathcal{O}_{v_P})$, where $\pi_P : \mathcal{O}_P \to \kappa(P) =: L^P$, etc. □

Finally, all of the above can be performed inductively for systems of variables $T := (t_1, \ldots, t_r)$, $L_r$ finite field extension of $L_0(T)$, $R_r \subset \tilde{R}_r \subset L_r$ the integral closures of $R_{r-1}[t_r] \subset L_{r-1}[t_r]$, thus leading to the corresponding sets of all valuations $\mathcal{V}_r$ of $L_r$ the form $v_r = v_{r-1}^p \circ v_p$, where $P \in \text{Max}(\tilde{R}_r)$ and $v_{r-1}^p \in \mathcal{V}_{r-1}$ to $\kappa(P)$.

Lemma 5.5. In the above notation, one has $R_r = \cap_{v_r \in \mathcal{V}_r} \mathcal{O}_{v_r}$. Further, if all the valuation rings $\mathcal{O}_P$, $P \in \text{Max}(\tilde{R}_r)$ and $\mathcal{O}_{v_P}$, $v^p \in \mathcal{V}_p$ are (uniformly) first order definable, then so are the valuation rings $\mathcal{O}_{v_r}$, $v_r \in \mathcal{V}_r$ and $R_r = \cap_{v_r \in \mathcal{V}_r} \mathcal{O}_{v_r}$.

Proof. Induction on $r$ reduces everything to $r = 1$. Conclude by using Lemmas 5.3, 5.4. □

Coming back to the proof of Theorem 5.1, let $k_0 = \kappa \subset K$ be the constant subfield of $K$, and set $R_0 := \text{Spec} \mathcal{O}_{k_0}$. Then $R$ is the integral closure of $R_0[T]$ in $K$, and Theorem 5.1 above follows from Lemma 5.5 above.

C) Third proof: Using Ruély’s result [Ru]

We begin by mentioning that Pop [P4], Theorem 1.2 holds in the following more general form (which might be well known to experts, but we cannot give a precise reference). Namely, let $\mathcal{K}$ be a class of function fields of projective normal geometrically integral curves $K = k(C)$ such that $k \subset K$ and the $k$-valuations rings $\mathcal{O}_m$ of $K|k$ are (uniformly) first order definable. Then for every non-zero $t \in K$, $e > 0$, the sets

$$\Sigma_{t,e} := \{ \mathcal{O}, m \mid t \in m^e, t \notin m^{e+1} \}$$

are (uniformly) first order definable subset of the set of all the valuation rings $\mathcal{O}_m$. Hence given $N > 0$, a function $t \in K^\times$ has $\text{deg}(t) := [K : k(t)] = N$ iff the following hold:
i) \( \Sigma_{t,N+1} = \emptyset \) and \( |\Sigma_{t,e}| \leq N \) for all \( 0 < e \leq N \).

ii) \( \dim_k \mathcal{O}/\mathfrak{m} \leq N \) for all \( \mathcal{O}, \mathfrak{m} \in \Sigma_{t,e} \), and moreover: \( N = \sum_{0 < e \leq N} \sum_{\mathcal{O}, \mathfrak{m} \in \Sigma_{t,e}} e \dim_k \mathcal{O}/\mathfrak{m} \).

In particular, there exists a (uniform) first order formula \( \deg_N(t) \) such that for every \( K = k(C) \) as above, and non-constant \( t \in K \) one has:

- \( \deg_N(t) \) is true in \( K \) iff \( t \) has degree \( N \) as a function of \( K|k \), i.e., \([K:k(t)] = N\).

Now let \( K \) be a finitely generated field with \( \text{char}(K) = 0 \), and \( T = (\mathbf{t}_e, t) \) be a transcendence basis of \( K|k_0 \), where \( \mathbf{t}_e := (t_1, \ldots, t_e) \). Setting \( T := (T_1, \ldots, T_e) \), there exists an absolutely irreducible \( U \)-monic polynomial \( f_K \in k_0[T, T, U] \), and \( u \in K \) such that

\[
K = k_0(\mathbf{t}_e, t)[u], \quad f_K(\mathbf{t}_e, t, u) = 0.
\]

In the above notation, the isomorphy type of \( K \) is given by the following data:

a) a transcendence basis \( (\mathbf{t}_e, t) \) of \( K|k_0 \) and a non-constant function \( u \in K \),

b) an absolutely irreducible \( U \)-monic polynomial \( f_K \in k_0[T, T, U] \) such that \( f(\mathbf{t}_e, t, u) = 0 \),

c) letting \( k = k_{t_e} \subset K \) be the relative algebraic closure of \( k_0(\mathbf{t}_e) \) in \( K \), and \( K = k(C) \) with \( C \) a projective smooth geometrically integral \( k \)-curve \( C \), one has:

\[
\deg_C(u) := [K:k(u)] = \deg_U(f_K) =: N_{f_K}.
\]

Recall that Rumely [Ru] gives a uniform bi-interpretability of number fields with Peano arithmetic. In particular, for number fields \( k \), endowed with finite system \( \Sigma \) of \( n \) constants, there exists a sentence \( \vartheta_{k_0,\Sigma}^\text{abs} \) such that for any other global field \( l \) there is an isomorphism \( \iota : k_0 \rightarrow l \), thus endowing \( l \) with the finite system of \( n \) constants \( \iota(\Sigma) \). Recalling Poonen’s sentence \( \psi_0 \) and the predicate \( \psi_\text{abs}(x) \), consider the sentence:

\[
\vartheta_{k_0,\Sigma} \equiv \psi_0 \land (\vartheta_{k_0,\Sigma}^\text{abs} \text{ hols in } \kappa = \kappa(\psi_\text{abs}(x))).
\]

Then all finitely generated fields \( L \) one has: If \( \vartheta_{k_0,\Sigma} \text{ holds in } L \), and \( l \) is the field of constants of \( L \), then there is an isomorphism \( \iota : k_0 \rightarrow l \), which endows \( l \) with \( \iota(\Sigma) \).

A special case of this arises by starting with \( K = k_0(\mathbf{t}_e, t)[u], \ f_K(\mathbf{t}_e, t, u) = 0 \) as above, and letting \( \Sigma := \Sigma_{f_K} \) be the system of coefficients \( (a_{t_i})_i \) of \( f_K \). In particular, if \( L \) is a finitely generated field with field of constants \( \iota \) such that \( \vartheta_{k_0,\Sigma_{f_K}} \) holds in \( L \), then there is an isomorphism \( \iota : k_0 \rightarrow l \) which gives rise to a polynomial \( f_L := \iota(f_K) \); and notice that \( f_L \) is absolutely irreducible and \( U \)-monic, and obviously, \( \deg_U(f_L) = N_{f_K} \).

Next let \( \mathbf{t}_e = (t_i)_{1 \leq i \leq e}, \ t, u \) be variables, and consider “generic” polynomials \( f(\mathbf{t}_e, t, u) \) which are monic in \( u \) and have degree \( N_f := \deg_u(f) \). Further let \( \Sigma_f \) be the system of the coefficients of \( f \). Recalling the algebraic independence formula \( \psi_r(t_1, \ldots, t_r) \), in the above context, we denote by \( k_{t_e} \) the relative algebraic closure of \( \mathbb{Q}(\mathbf{t}_e) \) in finitely generated fields \( K \) in which \( \mathbf{t}_e \) are evaluated. In particular, by the discussion above, for every finitely generated field \( K \) with constants a number field \( k_0 \) and \( \text{td}(K) = e + 1 \), there is some \( f_K \) describing \( K \), and we think of \( f_K \) as being obtained by properly specializing the variables \( (\mathbf{t}_e, t, u) \mapsto (\mathbf{t}_e, t, u) \) and \( f \mapsto f_K \). In particular, \( \Sigma_f \mapsto \Sigma_{f_K} \), and \( \vartheta_{k_0,\Sigma_{f_K}} \) holds in \( K \). Finally, recalling the sentence \( \varphi_{d} \) defining \( d = \dim(K) = e + 2 \), consider the sentence

\[
\vartheta_K \equiv \varphi_{d} \land (\exists \mathbf{t}_e, t, u : \psi_{e+1}(\mathbf{t}_e, t) \land f_K(\mathbf{t}_e, t, u) = 0 \land \vartheta_{k_0,\Sigma_{f_K}} \land k = k_{t_e} \land \deg_N(f_K)(u))
\]

25
To conclude the second proof of the Main Theorem, let $L$ be a finitely generated field with constant field $l_0$ such that $\vartheta_K$ holds in $L$. Then one has the following:

a) First, $\dim(L) = d = e + 2 = \dim(K)$.

b) $\vartheta_{l_0,\Sigma_K}$ holds in $l_0$, hence one has an isomorphism $\iota : k_0 \to l_0$, thus $\text{td}(K) = e + 1 = \text{td}(L)$.

\[\text{Let } f_L := \iota(f_K) \text{ be the image of } f_K \text{ under } \iota, \text{ and notice that } f_L \text{ is absolutely irreducible.}\]

c) $\exists t'_1, \ldots, t'_e, t', u' \in L \text{ s.t. } \psi_{e+1}(t'_e, t') \text{ holds, hence } (t'_e, t') \text{ are algebraically independent.}$

Hence since $\text{td}(L) = e + 1$, it follows that $(t'_e, t')$ is a transcendence basis of $L$.

d) Setting $l := k_{t_p}$, one has: $f_L(t'_e, t', u') = 0$ and $\deg_{L|l}(u') = N_{f_K} = N_{f_L}$.

But then by the discussion above, it follows that $L = l_0(t'_e, t', u')$, and the map

$$\iota_K : K \to L, \quad (t_e, t, u) \mapsto (t'_e, t', u'), \quad \iota_K|_{k_0} = \iota$$

is an isomorphism of fields.

References

[AKNS] Aschenbrenner, M., Khélif, A., Naziazeno, E. and Scanlon, Th., *The logical complexity of finitely generated commutative rings*, Int. Math. Res. Notices (to appear).

[Du] Duret, J.-L., *Équivalence élémentaire et isomorphisme des corps de courbe sur un corps algébriquement clos*, J. Symbolic Logic 57 (1992), 808–923.

[Ei] Eisenträger, K., *Integrality at a prime for global fields and the perfect closure of global fields of characteristic $p > 2$*, J. Number Theory 114 (2005), 170–181.

[E-S] Eisenträger, K. and Shlapentokh, A., *Hilbert’s Tenth Problem over function fields of positive characteristic not containing the algebraic closure of a finite field*, JEMS 19 (2017), 2103–2138.

[Hi] Hironaka, H., *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. 79 (1964), 109–203; 205–326.

[II] Illusie, L., *Complexe de de Rham–Witt et cohomologie cristalline*, Ann. Sci. Ec. Norm. Sup. 12 (1979), 501–661.

[Ja] Jannsen, U., *Hasse principles for higher-dimensional fields*, Annals of Math. 183 (2016), 1–71.

[Kh] Kahn, E., *La conjecture de Milnor (d’après Voevodsky)*, Séminaire Bourbaki, Astérisque 245 (1997), 379–418.

[Ka] Kato, K., *A Hasse principle for two dimensional global fields*, J. reine angew. Math. 366 (1986), 142–180.

[K-S] Kerz, M. and Saito, Sh., *Cohomological Hasse principle and motivic cohomology for arithmetic schemes*, Publ. Math. IHES 115 (2012), 123–183.

[K-R] Kim, H. K. and Roush, F. W., *Diophantine undecidability of $\mathbb{C}(T_1, T_2)$*, J. Algebra 150 (1992), 35–44.

[Ko1] Koenigsmann, J., *Defining transcendental in function fields*, J. Symbolic Logic 67 (2002), 947–956.

[Ko2] Koenigsmann, J., *Defining $Z$ in $\mathbb{Q}$*, Annals of Math. 183 (2016), 73–93.

[Ko3] Koenigsmann, J., *Decidability in local and global fields*, Proc. ICM 2018 Rio de Janeiro, Vol. 2, 63–78.

[M-S] Merkurjev, A. S. and Suslin, A. A., *$K$-cohomology of Severi–Brauer variety and norm residue homomorphism*, Math. USSR Izvestia 21 (1983), 307–340.

[M-Sh] Miller, R. and Shlapentokh, A., *On existential definitions of $C.E.$ subsets of rings of functions of characteristic 0*, arXiv:1706.03302 [math.NT]

[Pf1] Pfister, A., *Quadratic Forms with Applications to Algebraic Geometry and Topology*, LMS LNM 217, Cambridge University Press 1995; ISBN 0-521-46755-1.

[Pf2] Pfister, A., *On the Milnor conjectures: history, influence, applications*, Jahresbericht DMV 102 (2000), 15–41.

[Pi] Pierce, D., *Function fields and elementary equivalence*, Bull. London Math. Soc. 31 (1999), 431–440.
[Po] Poonen, B., Uniform first-order definitions in finitely generated fields, Duke Math. J. 138 (2007), 1–21.

[P-P] B. Poonen and F. Pop, First-order characterization of function field invariants over large fields, in: Model Theory with Applications to Algebra and Analysis, LMS LNM Series 350, Cambridge Univ. Press 2007; pp. 255–271.

[P1] Pop, F., Embedding problems over large fields, Annals of Math. 144 (1996), 1–34.

[P2] Pop, F., Elementary equivalence versus isomorphism, Invent. Math. 150 (2002), 385–308.

[P3] Pop, F., Elementary equivalence of finitely generated fields, Course Notes Arizona Winter School 2003, see [http://swc.math.arizona.edu/oldaws/03Notes.html](http://swc.math.arizona.edu/oldaws/03Notes.html)

[P4] Pop, F., Elementary equivalence versus Isomorphisms II, Algebra & Number Theory 11 (2017), 2091-2111.

[Ro1] Robinson, Julia, Definability and decision problems in arithmetic, J. Symbolic Logic 14 (1949), 98–114.

[Ro2] Robinson, Julia, The undecidability of algebraic fields and rings, Proc. AMS 10 (1959), 950–957.

[Ru] Rumely, R., Undecidability and Definability for the Theory of Global Fields, Transactions AMS 262 No. 1, (1980), 195–217.

[Sc] Scanlon, Th., Infinite finitely generated fields are biinterpretable with N, JAMS 21 (2008), 893–908. Erratum, J. Amer. Math. Soc. 24 (2011), p. 917.

[Se] Serre, J.-P., Zeta and L-functions, in: Arithmetical Algebraic Geometry, Proc. Conf. Purdue 1963, New York 1965, pp. 82–92.

[Sh1] Shlapentokh, A., First Order Definability and Decidability in Infinite Algebraic Extensions of Rational Numbers, Israel J. Math. 226 (2018), 579–633.

[Sh2] Shlapentokh, A., On definitions of polynomials over function fields of positive characteristic, See [arXiv:1502.02714v1](http://arxiv.org/abs/1502.02714v1)

[V1] Vidaux, X., Équivalence élémentaire de corps elliptiques, CRAS Série I 330 (2000), 1-4.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA
DRL, 209 S 33RD STREET, PHILADELPHIA, PA 19104, USA
E-mail address: pop@math.upenn.edu
URL: [http://math.penn.edu/~pop](http://math.penn.edu/~pop)