Parabolic Verma Modules and Invariant Differential Operators

V. K. Dobrev*

Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, 1784 Bulgaria
*e-mail: vkdobrev@yahoo.com

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Abstract—In the present paper we continue the project of systematic classification and construction of invariant differential operators for non-compact semisimple Lie groups. This time we make the stress on one of the main building blocks, namely the Verma modules and the corresponding parabolic subalgebras. In particular, we start the study of the relation between the parabolic subalgebras of real semisimple Lie algebras and of their complexification. Two cases are given in more detail: the conformal algebra of 4D Minkowski space-time and the minimal parabolics of classical real semisimple Lie algebras.

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1. INTRODUCTION

Verma modules play crucial role in the classification and construction of invariant differential operators for non-compact semisimple Lie groups. In our approach we make important use of the relations between the parabolic subalgebras of a real semisimple Lie algebra \( \mathfrak{g} \) and those of its complexification \( \mathbb{C} \mathfrak{g} \). The present paper we make this relation more transparent briefly in general and in some detail in examples.

2. PRELIMINARIES

Let \( G \) be a semisimple non-compact Lie group, and \( K \) a maximal compact subgroup of \( G \). Then we have the Iwasawa decomposition \( G = KA_0N_0 \), where \( A_0 \) is Abelian simply connected vector subgroup of \( G \), \( N_0 \) is a nilpotent simply connected subgroup of \( G \) preserved by the action of \( A_0 \). Further, let \( M_0 \) be the centralizer of \( A_0 \) in \( K \). Then the subgroup \( P_0 = M_0A_0N_0 \) is a minimal parabolic subgroup of \( G \). A parabolic subgroup \( P = MAN \) is any subgroup of \( G \) which contains a minimal parabolic subgroup. In the general case \( A \subset A_0 \) is abelian, \( N \subset N_0 \) is a nilpotent simply connected subgroup of \( G \) preserved by the action of \( A \), \( M \supset M_0 \) is a maximal centralizer of \( A \) in \( G \).

Further, let \( \mathfrak{g}_0, \mathfrak{h}, \mathfrak{p}, \mathfrak{m}, \mathcal{A}, \mathcal{N} \), denote the Lie algebras of \( G, K, P, M, A, N \), resp.

We note also another extremal case: maximal parabolic subgroup when rank \( A \) = 1, maximal parabolic subalgebra when \( \dim \mathcal{A} = 1 \).

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Let \( V^\Lambda \) be the corresponding Verma module, then
\[
V^\Lambda = U(\mathcal{G}) \otimes U(\mathbb{B}) \ 1_\Lambda.
\] (3)

Using
\[
U(\mathcal{G}) = U(N^-) \otimes U(\mathbb{B})
\] (4)
one has
\[
V^\Lambda = U(N^-) \otimes 1_\Lambda.
\] (5)

Obviously \( V^\Lambda \) is a highest weight module with highest weight \( \Lambda \), and highest weight vector \( v_0 \), \( V^\Lambda \cong U(N^-) \) as vector spaces.

Actually, since our ERs are induced from finite-dimensional representations of \( \mathcal{M} \) (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use \textit{generalised Verma modules} \( \overset{\circ}{V}^\Lambda \) such that the role of the highest/lowest weight vector \( v_0 \) is taken by the (finite-dimensional) space \( V^\Lambda_\mu v_0 \).

It is important to note that the generalised Verma modules defined just above and related to ERs are special cases of \textit{parabolic Verma modules} (PVM) which are introduced in purely algebraic context. More precisely, their construction is as follows. (Below we define PVM using [6], except that there these modules are called generalized Verma modules (which in our approach is used differently, see above and [7]).

Let \( \Delta \) be the root system of \((\mathcal{G}, \mathcal{H}), \Delta_+, \Delta_- \) denote the positive, negative, roots. Let \( \alpha_\ell \in \mathcal{H}^* \) be the simple roots where \( i = 1, \ldots, \ell = \text{dim} \ \mathcal{H} \). Let \( e_i \) (resp. \( f_i \)) be a non-zero element of the root space \( \mathcal{G}_{\alpha_i} \) (resp. \( \mathcal{G}_{-\alpha_i} \)) for all \( i = 1, \ldots, \ell \) normalized so that \( [e_i, f_i] = h_i \), and \( \alpha_i(h_i) = 2 \). Let \( \Sigma \) be a subset of the set \( \{ 1, \ldots, \ell \} \). Let \( \mathcal{G}_S \) be the subalgebra of \( \mathcal{G} \) generated by \( \{ h, e_1, f_1 \}_{i \in \Sigma} \), \( \mathcal{H}_S \) the span of \( \{ h \}_{i \in \Sigma} \) : \( \Delta^S = \Delta \cap \bigcup_{i \in \Sigma} \mathcal{Z} \alpha_i, \Delta^S = \Delta \cap \bigcup_{i \in \Sigma} \mathcal{Z} \alpha_i \). Let \( \Delta^S = \Delta_\pm \cap \Delta^S \), \( \Delta(S)_\pm = \Delta_\pm - \Delta^S \).

Further, we define the following subalgebras of \( \mathcal{G} \):
\[
\mathcal{N} = \bigoplus_{\beta \in \Delta^+} \mathcal{G}_\beta; \quad \mathcal{N}^- = \bigoplus_{\beta \in \Delta^-} \mathcal{G}_\beta \quad \text{(the latter two were already used above)};
\mathcal{N}_S = \bigoplus_{\beta \in \Delta^+_S} \mathcal{G}_\beta; \quad \mathcal{N}^- = \bigoplus_{\beta \in \Delta^-_S} \mathcal{G}_\beta;
\mathcal{U}_S = \bigoplus_{\beta \in \Delta(S)_+} \mathcal{G}_\beta; \quad \mathcal{U}^- = \bigoplus_{\beta \in \Delta(S)_-} \mathcal{G}_\beta;
\mathcal{R}_S = \mathcal{G}_S \oplus \mathcal{H}; \quad \mathcal{P}_S = \mathcal{R}_S \oplus \mathcal{U}_S.
\]

Then we have:
\[
\mathcal{G} = \mathcal{N} \oplus \mathcal{H} \oplus \mathcal{N}^-; \quad \mathcal{G}_S = \mathcal{N}_S \oplus \mathcal{H}_S \oplus \mathcal{N}^-; \quad \mathcal{N} = \mathcal{N}_S \oplus \mathcal{U}_S; \quad \mathcal{N}^- = \mathcal{N}^- \oplus \mathcal{U}_S;
\mathcal{R}_S = \mathcal{N}_S \oplus \mathcal{H} \oplus \mathcal{N}^-; \quad \mathcal{G} = \mathcal{U}_S \oplus \mathcal{P}_S.
\]

Further, \( \mathcal{G}_S \) is a split semisimple Lie algebra with splitting Cartan subalgebra \( \mathcal{H}_S; \mathcal{R}_S \) is a reductive Lie algebra with commutator subalgebra \( \mathcal{G}_S \) and centre a subalgebra of \( \mathcal{H} \).

As \( S \) varies among the subsets of \( J \), \( \mathcal{P}_S \) varies among the parabolic subalgebras of \( \mathcal{G} \) containing the Borel subalgebra \( \mathcal{B} = \mathcal{H} \oplus \mathcal{N} \). The reductive part of \( \mathcal{P}_S \) is \( \mathcal{R}_S \) and the nilpotent part of \( \mathcal{P}_S \) is \( \mathcal{U}_S \). We note that if \( S = \emptyset \), then \( \mathcal{P}_\emptyset = \mathcal{B} \) (since \( \mathcal{G}_\emptyset = 0, \mathcal{R}_\emptyset = \mathcal{H}, \mathcal{U}_\emptyset = \mathcal{N} \).)

Now let \( P \) index the set of (equivalent classes of) finite-dimensional irreducible \( \mathcal{G} \)-modules in the usual way—via the highest weight. Let \( P_S = \{ \lambda \in \mathcal{H}_\ast \mid \lambda(h) \in \mathbb{Z}_+ \text{ for all } i \in S \} \). Then it is clear that there is a natural bijection, \( \lambda \mapsto M(\lambda) \), between \( P_S \) and the set of (equivalent classes of) finite-dimensional irreducible \( \mathcal{R}_S \)-modules which are irreducible as \( \mathcal{G}_S \)-modules.

For any \( \lambda \in P_S \), denote by \( V^{M(\lambda)}_\emptyset \) the corresponding \textit{parabolic Verma module} (PVM): the \( \mathcal{G} \)-module \( \text{ind}(M(\lambda), \mathcal{G}) \) induced by the \( P_S \)-module \( M(\lambda) \), viewed as an \( \mathcal{R}_S \)-module in the natural way and as a trivial \( \mathcal{U}_S \)-module.

Note that for \( S = \emptyset \) the parabolic Verma modules coincide with the usual Verma modules: \( V^{M(\lambda)}_\emptyset = V^\Lambda \).

Further we discuss the reducibility of Verma modules.

A classic result of [8] states that a Verma module \( V^\Lambda \) is reducible iff
\[
(\Lambda + \rho, \beta') = m, \quad m \in \mathbb{N}, \quad \beta \in \Delta^+,
\] (6)
where \( \beta' \equiv 2\beta/(\beta, \beta), \rho \) is half the sum of the positive roots of \( \mathcal{G} \).

The same criterion of reducibility is valid for general- ized Verma modules, though it is trivially satisfied for the \( \mathcal{M} \)-compact roots, and is essential only for \( \mathcal{M} \)-non-compact roots. (Recall that \( \mathcal{M} \)-compact roots are those elements of \( \Delta \) that belong to the root system of \( \mathcal{M}^\ell \), the latter being identified as a subset of \( \Delta \).)

The same criterion of reducibility is valid for a parabolic Verma module \( V^{M(\lambda)}_S \) though it is trivially satisfied for \( \beta \in \Delta^+_S \), and is essential only for \( \beta \in \Delta_+(S) \).

When (6) holds then the Verma module with shifted weight \( V^{\Lambda-m\beta} \) (or \( V^{\Lambda-m\beta}_S \) for PVM and \( \beta \in \Delta(S) \)) is embedded in the Verma module \( V^\Lambda \) (or \( V^\Lambda_S \)).

The above embedding is realized by a singular vector \( v_\beta \) determined by a polynomial \( P_{m, \beta}(\mathcal{N}_S) \) in the universal enveloping algebra \( U(\mathcal{N}_S) \). More explic-
4. CONFORMAL ALGEBRA

The conformal algebra in four-dimensional space-
time is \( \mathcal{G}_0 = su(2,2) \) \((\cong so(4,2))\). It has three noncon-
jugate parabolic subalgebras \((\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N})\):

\[
\mathcal{P}_0 = so(2) \oplus \mathcal{A}_0 \oplus \mathcal{N}_0,
\dim \mathcal{A}_0 = 2, \quad \dim \mathcal{N}_0 = 6,
\mathcal{P}_1 = so(2) \oplus sl(2,\mathbb{R}) \oplus \mathcal{A}_1 \oplus \mathcal{N}_1,
\dim \mathcal{A}_1 = 1, \quad \dim \mathcal{N}_1 = 5,
\mathcal{P}_2 = so(3,1) \oplus \mathcal{A}_2 \oplus \mathcal{N}_2,
\dim \mathcal{A}_2 = 1, \quad \dim \mathcal{N}_2 = 4,
\]

where \( \mathcal{P}_0 \) is the minimal parabolic, \( \mathcal{P}_1 \) is maximal
cuspidal, \( \mathcal{P}_2 \) is maximal noncuspidal.

4.1. Maximal Non-Cuspidal Case

We consider the following Bruhat decomposition
[9] (consistent with the maximal non-cuspidal para-
bolic subalgebra \( \mathcal{P}_2 \)):

\[
\mathcal{G}_0 = \mathcal{G}^+ \oplus \mathcal{M}_2 \oplus \mathcal{A}_2 \oplus \mathcal{G}^-,
\]

where \( \mathcal{M}_2 \) is the six-dimensional Lorentz subalgebra
\( so(3,1) \), \( \mathcal{A}_2 \) is the dilatation subalgebra, \( \mathcal{G}^+_0 \), \( \mathcal{G}^-_0 \), is the
four-dimensional isomorphic translation subalgebra, resp., special conformal transformations subalgebra.

In this case the ERs of \( su(2,2) \) are parametrized by
triples: \( \chi = [j_1, j_2; d] \), where \( j_1, j_2 \in \frac{1}{2} \mathbb{Z} \), parametrize
the finite-dimensional representations of \( \mathcal{M}_2 \), while the number \( d \) parametrizing the representations of \( \mathcal{A}_2 \)
called the conformal weight or energy.

The complexification of \( \mathcal{G}_0 \) is \( \mathcal{G} = sl(4) \). The root
system of \( \mathcal{G} \) is given by:

\[
\Delta^+ = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_{12} = \alpha_1 + \alpha_2, \alpha_{13} = \alpha_1 + \alpha_3, \}
\alpha_{23} = \alpha_2 + \alpha_3, \alpha_{13} = \alpha_1 + \alpha_2 + \alpha_3, \}
\]

where \( \{ \alpha_1, \alpha_2, \alpha_3 \} \) are the simple roots.

Note that when relating the root systems of \( \mathcal{G} \) to \( \mathcal{G}_0 \)
relative to the Bruhat decomposition (10) the roots
\( \{ \alpha_i, \alpha_j \} \) are \( \mathcal{M}_2 \)-compact, the rest are \( \mathcal{M}_2 \)-non-
compact.

The reducibility conditions (6) of a Verma module
\( V^\chi \) over \( \mathcal{G} \) are written explicitly as follows:

\[
\begin{align*}
m_1 &= (\Lambda + \rho, \alpha_i) \in \mathbb{N}, \\
m_2 &= (\Lambda + \rho, \alpha_2) \in \mathbb{N}, \\
m_3 &= (\Lambda + \rho, \alpha_3) \in \mathbb{N}, \\
m_{12} &= (\Lambda + \rho, \alpha_{12}) = m_1 + m_2 \in \mathbb{N}, \\
m_{23} &= (\Lambda + \rho, \alpha_{23}) = m_2 + m_3 \in \mathbb{N}, \\
m_{13} &= (\Lambda + \rho, \alpha_{13}) = m_1 + m_3 + m_2 \in \mathbb{N}.
\end{align*}
\]

We want to apply these conditions to the signatures
\( \chi \) of the ERs. In these terms we have [7]:

\[
\begin{align*}
m_1 &= 2j_1 + 1 \in \mathbb{N}, \\
m_2 &= 2d - j_1 - j_2 \in \mathbb{N}, \\
m_3 &= 2j_2 + 1 \in \mathbb{N}, \\
m_{12} &= 3 - d + j_1 - j_2 \in \mathbb{N}, \\
m_{23} &= 3 - d - j_1 + j_2 \in \mathbb{N}, \\
m_{13} &= 4 - d + j_1 + j_2 \in \mathbb{N}.
\end{align*}
\]

Note that (12a), (12c) are fulfilled always since
\( 2j_1 + 1 \in \mathbb{N}, 2j_2 + 1 \in \mathbb{N} \), as expected for the \( \mathcal{M}_2 \)-com-
 pact roots. On the other hand the expressions in the
other cases depend on \( \chi \) and may be arbitrary.

Note that \( m_i \) considered abstractly are called Dyn-
kin labels, while together with \( m_j \) they are called Har-
ish–Chandra parameters [10]:

\[
m_{\beta} \equiv (\Lambda + \rho, \beta),
\]

where \( \beta \) is any positive root of \( \mathcal{G} \). These parameters
are redundant, since they are expressed in terms of the
Dynkin labels, however, some statements are best
formulated in their terms.

4.2. Maximal Cuspidal Case

We consider again the conformal algebra
\( \mathcal{G}_0 = su(2,2) \) \((\cong so(4,2))\). Here we consider the
following Bruhat decomposition (consistent with the
maximal cuspidal parabolic subalgebra):

\[
\mathcal{G}_0 = \mathcal{G}^+_1 \oplus \mathcal{M}_1 \oplus \mathcal{A}_1 \oplus \mathcal{G}^-_1,
\]

where \( \mathcal{M}_1 = so(2) \oplus so(2,1) \), \( \mathcal{A}_1 \) is one-dimensional,
\( \mathcal{G}^+_1 \), \( \mathcal{G}^-_1 \) are five-dimensional isomorphic subalgebras.

The signatures of the ERs in this case are [11]:

\[
\chi_i = \{ n', k, \varepsilon, \nu' \},
\]

where \( n' \in \mathbb{Z} \) is a character of \( so(2) \), \( \nu' \in \mathbb{C} \) is a character
of \( \mathcal{A}_1 \), \( k \) fix a discrete series representation
of \( so(2,1), k \in \mathbb{N}, \varepsilon = \pm 1 \), or a limit thereof when \( k = 0 \).
The relation with the \( sl(4) \) Dynkin labels is as follows [11]:

\[
m_1 = \frac{1}{2}(k - \nu' + n'), \\
m_2 = -k, \\
m_3 = \frac{1}{2}(k - \nu' - n').
\] (17)

For the analysis we need the additional Harish–Chandra parameters:

\[
m_{12} = \frac{1}{2}(n' - k - \nu'), \\
m_{23} = -\frac{1}{2}(k + \mu' + n'), \\
m_{13} = -\nu'.
\] (18)

We see that if \( \nu' \notin \mathbb{Z} \) then no Harish–Chandra parameter can be a positive integer, thus, the ERs would be irreducible.

Thus, we consider the case \( \nu' \in \mathbb{Z} \). Actually, we shall use the analysis of the partially equivalent ERs in this case done in [11]. Thus, we use a parametrization taken from there (up to change of sign) by three positive integers: \( p, \nu, n \), so that we have:

\[
\lambda_{p,\nu,n} = (m_1, m_2, m_3)_{p,\nu,n} = (-p - \nu, -n - \nu), \\
m_{12} = -p, \\
m_{23} = -n, \\
m_{13} = -p - n - \nu.
\] (19)

It is known that when relating the root systems of \( \mathcal{G} \) to \( \mathcal{G}_0 \) relative to the Bruhat decomposition (15) the root \( \alpha_2 \) is compact, the rest are non-compact, and the above parametrization is consistent with this.

5. PARABOLIC VERMA MODULES FOR \( sl(4) \)

Here we enumerate the parabolic Verma modules for \( sl(4) \). For this we need to produce the list of the various \( \Delta^\mathbb{C}_\pm \) and the corresponding parabolic subalgebras. We have:

\[
\Delta^\mathbb{C}_+ = \{0\}, \\
\Delta^\mathbb{C}_- = \{\pm \alpha_1\}, \\
\Delta^\mathbb{C}_0 = \{\pm \alpha_2\}, \\
\Delta^\mathbb{C}_- = \{\pm \alpha_2, \pm \alpha_3\}, \\
\Delta^\mathbb{C}_0 = \{\pm \alpha_1, \pm \alpha_2, \pm \alpha_3\}.
\] (20)

\[
P_{\mathcal{G}_0} = \mathcal{B}, \\
P_{(1)} = \mathcal{G}_1 \prod \mathcal{B}, \\
P_{(12)} = \mathcal{G}_2 \prod \mathcal{G}_1 \prod \mathcal{B}, \\
P_{(123)} = \mathcal{G}_3 \prod \mathcal{G}_2 \prod \mathcal{G}_1 \prod \mathcal{B}.
\] (21)

Now we can make connection with some generalized Verma modules.

In order to compare the parabolic subalgebras of the real form with the parabolic subalgebras of the complexification \( sl(4, \mathbb{C}) \) we need the complexification of (9). We have:

\[
\mathcal{P}^\mathbb{C}_0 = so(2, \mathbb{C}) \oplus \mathcal{A}^\mathbb{C}_0 \oplus \mathcal{N}^\mathbb{C}_0, \\
\mathcal{P}^\mathbb{C}_1 = so(2, \mathbb{C}) \oplus sl(2, \mathbb{C}) \oplus \mathcal{A}^\mathbb{C}_1 \oplus \mathcal{N}^\mathbb{C}_1, \\
\mathcal{P}^\mathbb{C}_2 = so(4, \mathbb{C}) \oplus \mathcal{A}^\mathbb{C}_2 \oplus \mathcal{N}^\mathbb{C}_2.
\] (22a, 22b, 22c)

—First we note that \( \mathcal{H} \equiv so(2, \mathbb{C}) \oplus \mathcal{A}^\mathbb{C}_0, \mathcal{N}^\mathbb{C}_0 \equiv \mathcal{N} \), thus: \( \mathcal{P}^\mathbb{C}_0 = \mathcal{B} \).

—Further, we record the triangular decomposition:

\( sl(2, \mathbb{C}) = sl(2, \mathbb{C})^+ \oplus sl(2, \mathbb{C})_h \oplus sl(2, \mathbb{C})^- \), where \( sl(2, \mathbb{C})_h \) is a Cartan subalgebra of \( sl(2, \mathbb{C}) \). Then we note that \( \mathcal{H} \equiv so(2, \mathbb{C}) \oplus sl(2, \mathbb{C})_h \oplus \mathcal{A}^\mathbb{C}_1, \mathcal{N} \equiv sl(2, \mathbb{C})^+ \oplus \mathcal{N}^\mathbb{C}_1 \), and thus we have:

\[
\mathcal{P}^\mathbb{C}_1 = sl(2, \mathbb{C})^- \oplus \mathcal{B} \equiv \mathcal{P}_{(2)}. \\
\mathcal{P}^\mathbb{C}_2 = so(4, \mathbb{C}) \oplus \mathcal{B} \equiv \mathcal{P}_{(1,3)}. \\
\mathcal{P}^\mathbb{C}_3 = so(4, \mathbb{C}) \oplus \mathcal{B} \equiv \mathcal{P}_{(1,2,3)}.
\] (23, 24, 25)

Finally, we note that the GVM \( \mathcal{V}^{M_{\chi}} \) with \( \chi = [j_1, j_2; d] \) is isomorphic to \( \mathcal{P}^\mathbb{C}_3 \mathcal{V}^{M_{\chi}} \), where

\[
\lambda(h_1, h_2, h_3) = (m_1 - 1, m_2 - 1, m_3 - 1) \\
= (-p - \nu - 1, \nu - 1, -\nu - 1).
\] (26)

6. MINIMAL PARABOLICS VS. COMPLEX PARABOLIC SUBALGEBRAS

Here we briefly discuss the relation of minimal parabolics \( \mathcal{P}_0 = \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 \) of classical real Lie algebras to the parabolic subalgebras of their complexification. (For the minimal parabolic subalgebras and the enumeration of simple roots we use info from [7, 12].)

In the case of split real Lie algebras \( \mathcal{G}_r \) we have the general fact that \( \mathcal{M}_0 = 0 \) and then the complexification of the minimal parabolics of \( \mathcal{G}_r \) is isomorphic to the Borel subalgebra of \( \mathcal{G}^\mathbb{C}_r \). (We recall also that in this
case $\mathcal{A}_0^C \cong \mathcal{H}(\mathcal{G}_r^C)$. For completeness we list the classical split real Lie algebras: $sl(n, \mathbb{R})$, $so(r, r)$, $so(r+1, r)$, $sp(n, \mathbb{R})$.

There are cases of non-split real Lie algebras $\mathcal{G}_r$ when the minimal parabolic is isomorphic to $\mathcal{B}(\mathcal{G}_r^C)$. That is when the subalgebra $\mathcal{M}_0$ is abelian. Then $\mathcal{M}_0 \oplus \mathcal{A}_0^C \cong \mathcal{H}(\mathcal{G}_r^C)$. In the classical case this is the real Lie algebra $su(n, n)$, $(n > 1)$, where we have: $\mathcal{M}_0 = u(1) \oplus \cdots \oplus u(1)$, $(n - 1)$ entries, $\dim_{\mathbb{R}} \mathcal{A}_0 = n$, $\dim_{\mathbb{R}} \mathcal{N}_0 = n(2n - 1)$. Then, $\mathcal{P}_0^C \cong \mathcal{B}(sl(2n, \mathbb{C}))$.

Next we consider the rest of the real Lie algebras where the relation of the minimal parabolic to the complex parabolics is more involved.

In the case of $su^*(2n)$ $(n > 1)$ the minimal parabolic subalgebra is given by: $\mathcal{M}_0 = su(2) \oplus \cdots \oplus su(2)$, $(n)$ entries, $\dim_{\mathbb{R}} \mathcal{A}_0 = n - 1$, $\dim_{\mathbb{R}} \mathcal{N}_0 = 2n(n - 1)$. Thus, $\mathcal{P}_0^C \cong P_{1,3, \ldots, 2n-1} \cong \prod_{i=1}^{n} \mathcal{G}_{i-1}^{-1} \prod \mathcal{B}$.

In the case of $su(p, r)$ $(p > r \geq 1)$ the minimal parabolic subalgebra is given by: $\mathcal{M}_0 = su(p - r) \oplus u(1) \oplus \cdots \oplus u(1)$, $(r)$ entries, $\dim_{\mathbb{R}} \mathcal{A}_0 = r$, $\dim_{\mathbb{R}} \mathcal{N}_0 = r(2p - 1)$. Thus, $\mathcal{P}_0^C \cong P_{r+1, \ldots, p-1} \cong \prod_{i=r+1}^{p-1} \mathcal{G}_{i-1}^{-1} \prod \mathcal{B}$.

In the case of $so(r, r)$ $(p > r + 1)$ the minimal parabolic subalgebra is given by: $\mathcal{M}_0 = so(p - r)$, $\dim_{\mathbb{R}} \mathcal{A}_0 = r$, $\dim_{\mathbb{R}} \mathcal{N}_0 = r(p - 1)$. Thus, $\mathcal{P}_0^C \cong P_{r+1, \ldots, (p+r)/2} \cong \prod_{i=2r+1}^{(p+r)/2} \mathcal{G}_{i-1}^{-1} \prod \mathcal{B}$.

Next we consider $sp(p, r)$ $(p \geq r)$. The minimal parabolic subalgebra is given by: $\mathcal{M}_0 = sp(p - r) \oplus sp(1) \oplus \cdots \oplus sp(1)$, $(r)$ factors, $\dim_{\mathbb{R}} \mathcal{A}_0 = r$, $\dim_{\mathbb{R}} \mathcal{N}_0 = r(4p - 1)$. In the case $p = r$ we have $\mathcal{P}_0^C \cong P_{1,3, \ldots, 2r-1} \cong \prod_{i=1}^{r} \mathcal{G}_{i-1}^{-1} \prod \mathcal{B}$. In the case $p > r$ we have $\mathcal{P}_0^C \cong P_{1,3, \ldots, 2r-1,2r+1,2r+2, \ldots, r+r} \cong \prod_{i=r+1}^{r+r} \mathcal{G}_{i-1}^{-1} \prod \mathcal{B}$.

Finally, we consider $so^*(2n)$. First we suppose $n = 2r$. Then $\mathcal{M}_0 = so(3) \oplus \cdots \oplus so(3)$, $(r)$ factors, $\dim_{\mathbb{R}} \mathcal{A}_0 = r$, $\dim_{\mathbb{R}} \mathcal{N}_0 = r(4r - 3)$. Note also that $\mathcal{M}_0 \oplus \mathcal{A}_0 \cong \mathcal{H}(so(2n, \mathbb{C}))$.

Thus, we have $\mathcal{P}_0^C \cong P_{1,3, \ldots, 2r-1} \cong \prod_{i=1}^{r} \mathcal{G}_{i-1}^{-1} \prod \mathcal{B}$.

Next we suppose $n = 2r + 1$. Then $\mathcal{M}_0 = so(2) \oplus so(3) \oplus \cdots \oplus so(3)$, $(r)$ factors, $\dim_{\mathbb{R}} \mathcal{A}_0 = r$, $\dim_{\mathbb{R}} \mathcal{N}_0 = r(4r + 1)$. Thus, we have $\mathcal{P}_0^C \cong P_{1,3, \ldots, 2r} \cong \prod_{i=1}^{r} \mathcal{G}_{i-1}^{-1} \prod \mathcal{B}$.

For the lack of space we leave consideration of the exceptional real Lie algebras for a subsequent publication [13].

**DISCUSSION**

Another main ingredient of our approach as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [3]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines* (arrows) between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation. The notion of multiplets was introduced in [14] and then it was applied to (infinite-dimensional) (super-)algebras, quantum groups and other symmetry objects. For a current summary of these developments, see [7, 15], for further developments [13].

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