Yangian and quantum universal solutions
of Gervais–Neveu–Felder equations

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Abstract

We construct universal Drinfel’d twists defining deformations of Hopf algebra structures
based upon simple Lie algebras and contragredient simple Lie superalgebras. In particular,
we obtain deformed and dynamical double Yangians. Some explicit realisations as evaluation
representations are given for $sl_N$, $sl(1|2)$ and $osp(1|2)$.

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1 Introduction

Several consistent deformations of Yangian algebras have been proposed in the past years, starting with the scaling limit, defined in \([1, 2]\), of vertex-type quantum elliptic algebras \(A_{q,p}(sl_2)\) \([3]\). Extension of these scaling limits to face-type (so-called “dynamical”) elliptic algebras \([4, 5]\), and clarification of their connections at the level of evaluation representations, were proposed in \([6]\) for structures based upon the Lie algebra \(sl_2\). Construction of several deformations of Yangian algebras at the universal level, and understanding thereof as Drinfel’d twists of the centrally extended double Yangian \(DY(sl_2)\) \([4, 5]\), was achieved in \([9]\), following the schemes developed in the elucidation as Drinfel’d twists of face and vertex affine elliptic algebras based upon \(sl_N\) \([10]\) and face finite quantum algebras based upon any simple (contragredient super) Lie algebra \([11]\). The deformed double Yangians were thus characterised as Quasitriangular Quasi-Hopf Algebra (QTQHA).

Our purpose here is first of all to extend these universal constructions to the case of deformations of the centrally extended double Yangians \(DY(g)\) where \(g\) is a simple Lie algebra of type \(sl_N\) or a contragredient simple Lie superalgebra of type \(sl(M|N)\) \((M \neq N)\). We will also construct, by the same techniques, consistent deformations of \(U(g)\) and \(U_q(\widehat{g})\), for any \(g\). As in the previous case, the existence of such deformations may be conjectured from considering suitable limits of vertex or face-type elliptic quantum affine algebras in their Lax matrix formulation \([4, 3]\) as a quadratic algebra (Yang–Baxter equation or RLL formalism \([12]\)). However this limit procedure yields \(R\)-matrices which are only interpreted as conjectural evaluation representations of hypothetical universal \(R\)-matrices for (quasi)-Hopf type algebraic structures and one must needs establish its actual existence. This will be achieved here by identification of these limits as evaluation representations of universal \(R\)-matrices for the deformations by particular Drinfel’d twists, known as “shifted-cocycle” twists \([13]\), of Hopf algebra structures. This construction systematically endows these deformations with a Gervais–Neveu–Felder type QTQHA structure. It is characterised by a particular form of the universal Yang–Baxter equation, to be explicited below.

We shall first of all construct a deformation \(DY_r(g)\), with \(g\) (super) unitary, along the derivation generator \(d\), of the centrally extended double Yangian \(DY(g)\). When \(g\) is taken to be \(sl_N\), the evaluation representation of the \(R\)-matrix for this QTQHA is identified, up to a gauge transformation, with the scaling limit of the \(R\)-matrix for the vertex-type elliptic quantum affine algebra \(A_{q,p}(\widehat{sl_N})\), obtained by sending \(q\), \(p\) and the spectral parameter \(z\) to 1 whilst keeping the ratio of their logarithms as finite parameters \([14]\). We then give as the simplest illustration of the superalgebra case the evaluation representation of \(DY_r(sl(1|2))\).

We will then propose different deformations of Hopf algebra structures, this time along the Cartan subalgebra of the underlying (finite) Lie algebra. For historical reasons, they are called dynamical deformations.

We first recall the previous construction \([11]\) of a twist of the finite quantum enveloping algebra \(U_q(g)\) to the dynamical algebra \(B_{q,\lambda}(g)\). A consistent semi-classical limit of this procedure then yields a universal twist of shifted-cocycle form \([13, 15]\), from the undeformed enveloping algebra \(U(g)\) to a dynamical deformation \(U_s(g)\) for any simple Lie (contragredient super) algebra \(g\), thereafter evaluated
for \( \mathfrak{g} = sl_N, \mathfrak{g} = osp(1|2) \) and \( \mathfrak{g} = sl(1|2) \).

Using now the Hopf algebra inclusion of \( \mathcal{U}_q(\mathfrak{g}) \) into \( \mathcal{U}_\hat{q}(\hat{\mathfrak{g}}) \), the same twist acts on \( \mathcal{U}_\hat{q}(\hat{\mathfrak{g}}) \) to yield the QTQHA \( \mathcal{U}_{q,\lambda}(\hat{\mathfrak{g}}) \), the \( R \)-matrix of which may be obtained in the \( sl_N \) case (under an evaluated form) as a trigonometric limit \( p \to 0 \) of \( R \)-matrix for the elliptic affine face-type algebra \( \mathcal{B}_{q,p,\lambda}(\hat{sl}_N) \).

Using finally the Hopf algebra inclusion of \( \mathcal{U}(\mathfrak{g}) \) into the extended double Yangian \( \mathcal{D}Y(\mathfrak{g}) \), the previous twist from \( \mathcal{U}(\mathfrak{g}) \) to \( \mathcal{U}_\hat{q}(\hat{\mathfrak{g}}) \) leads from \( \mathcal{D}Y(\mathfrak{g}) \) to the dynamical double Yangian \( \mathcal{D}Y_s(\mathfrak{g}) \). The \( R \)-matrix of this QTQHA may also be obtained in the \( sl_N \) case (under an evaluated form) as the scaling limit of \( R \)-matrix for the previous algebra \( \mathcal{U}_{q,\lambda}(\hat{sl}_N) \).

## 2 General setting

Let \( \mathfrak{g} \) be a simple Lie algebra (or a contragredient simple Lie superalgebra different from \( psl(N|N) \)) of rank \( r_\mathfrak{g} \), with symmetrised Cartan matrix \( A = (a_{ij}) \) and inverse \( A^{-1} = (d_{ij}) \). In the superalgebra case, we denote by \( [\cdot] \) its \( \mathbb{Z}_2 \) grading. We denote by \( \mathcal{H} \) the Cartan subalgebra of \( \mathfrak{g} \) with basis \( \{h_i\} \) and dual basis \( \{h_i^\vee\} \). Let \( \Pi^+ \) be the set of positive roots of \( \mathfrak{g} \) endowed with a normal ordering <, i.e. if \( \alpha, \beta, \alpha + \beta \in \Pi^+ \) with \( \alpha \prec \beta \), then \( \alpha \prec \alpha + \beta \prec \beta \). Let \( \rho \) be the half-sum of the positive roots (resp. even positive roots) for a simple Lie algebra (resp. superalgebra).

We consider the corresponding quantum universal enveloping (super) algebra \( \mathcal{U}_q(\mathfrak{g}) \). It is endowed with a Hopf structure. We use the following coproduct for the generators related to simple roots

\[
\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \quad \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i.
\]

With this choice, the corresponding universal \( R \)-matrix was given (up to \( q \leftrightarrow q^{-1} \)) in [10].

We regard the quantum affine universal enveloping (super) algebra \( \mathcal{U}_q(\hat{\mathfrak{g}}) \), with universal \( R \)-matrix, compatible with eqs. (2.1)-(2.3), given in [17]. In that case, the Cartan subalgebra is completed with the derivation and central charge generators \( d \) and \( c \) respectively.

We introduce the double Yangian \( \mathcal{D}Y(sl_N) \) following [8, 9] with the generators

\[
e^{\pm}_i(u) \equiv \pm \sum_{k \geq 0} e_{i,k} u^{-k-1}, \quad f^{\pm}_i(u) \equiv \pm \sum_{k \geq 0} f_{i,k} u^{-k-1}, \quad h^{\pm}_i(u) \equiv 1 \pm \sum_{k \geq 0} h_{i,k} u^{-k-1}.
\]

satisfying relations given in [8]. The generators related to non-simple roots are derived from suitable combinations of generators related to simple roots, using Chevalley type relations.

Its universal \( R \)-matrix, defined in [9], obeys the Yang–Baxter equation. Denoting by \( \pi_\mathfrak{g} \) the evaluation representation of \( \mathcal{D}Y(sl_N) \), the Lax matrix \( L = (\pi_\mathfrak{g} \otimes \mathbb{1})(\mathcal{R}) \) realises an FRT-type formalism of \( \mathcal{D}Y(sl_N) \) with an \( R \)-matrix defined by \( R = (\pi_\mathfrak{g} \otimes \pi_\mathfrak{g})(\mathcal{R}) \).

Our constructions regarding the double Yangian deformations will rely upon the following conjecture [7]: The universal \( R \)-matrix of the centrally extended double Yangian \( \mathcal{D}Y(\mathfrak{g}) \) is given, for any
Lie algebra $\mathfrak{g}$, by the general formula (5.3) in [4]. We will also assume that the evaluation of this $R$-matrix corresponds to the defining $R$-matrix of the Yangian $\mathcal{Y}(sl_N)$.

We will consider the double Yangian $\mathcal{DY}(sl(M|N))$. In this case, the universal $R$-matrix is supposed to be obtained similarly from the double of the corresponding super-Yangian.

Again, central extensions of $\mathcal{DY}(\mathfrak{g})$ will contain in addition the derivation $d$ and central charge $c$.

The classes of QTQHA we consider have the particular property to possess an $R$-matrix that satisfies the Dynamical Yang–Baxter equation (or Gervais–Neveu–Felder equation):

$$\mathcal{R}_{12}(\lambda + h^{(3)})\mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\lambda + h^{(1)}) = \mathcal{R}_{23}(\lambda)\mathcal{R}_{13}(\lambda + h^{(2)})\mathcal{R}_{12}(\lambda). \quad (2.5)$$

We denote by $\lambda$ a vector with coordinates $(s_1, \ldots, s_r)$ in the basis $\{h_i\}$ in the finite case, or with coordinates $(s_1, \ldots, s_{r'}, r, s')$ in the basis $\{h_i, d, c\}$ otherwise. Then $\lambda + h^{(1)}$ is the vector with coordinates $s_i + h_i^{(1)}$ in the finite case, and $(\overline{s_i}, r + c^{(1)}, s')$ in other cases. The coefficients $\overline{s}_i$ are given by $\overline{s}_i = 0$ if all $s_j$ are zero and $\overline{s}_i = s_i + h_i^{(1)}$ otherwise.

In the case of superalgebras, the tensor product is graded: $(a \otimes b)(c \otimes d) = (-1)^{b|c}(ac \otimes bd)$.

In representation the spectral parameter (if any) is explicit and the dynamical Yang–Baxter takes the forms

$$R_{12}(z, \lambda + h^{(3)})R_{13}(zz', \lambda)R_{23}(z', \lambda + h^{(1)}) = R_{23}(z', \lambda)R_{13}(zz', \lambda + h^{(2)})R_{12}(z, \lambda), \quad (2.6)$$

or

$$R_{12}(\beta, \lambda + h^{(3)})R_{13}(\beta + \beta', \lambda)R_{23}(\beta', \lambda + h^{(1)}) = R_{23}(\beta', \lambda)R_{13}(\beta + \beta', \lambda + h^{(2)})R_{12}(\beta, \lambda), \quad (2.7)$$

depending upon the multiplicative or additive nature of the spectral parameter.

In the case of superalgebras, the $R$-matrix obtained as the evaluation of the universal $R$-matrix satisfies the graded Yang–Baxter equation

$$(R_{12})^{ij, j_2}_{i_1, i_2} (R_{13})^{k_1, j_3}_{j_1, i_3} (R_{23})^{k_2, k_3}_{j_2, j_3} (-1)^{[i_1][i_2] + [i_3][j_1] + [j_2][j_3]} = (R_{23})^{j_2, j_3}_{j_2, j_3} (R_{13})^{j_1, k_3}_{j_1, k_3} (R_{12})^{i_1, i_2}_{i_1, i_2} (-1)^{[i_3][i_2] + [i_1][j_3] + [j_2][j_1]} \quad (2.8)$$

Redefining the $R$-matrix as

$$(\tilde{R})^{ij, j_2}_{i_1, i_2} = (R)^{ij, j_2}_{i_1, i_2} (-1)^{[i_1][i_2]} \quad (2.9)$$

the $R$-matrix $\tilde{R}$ satisfies now the ordinary (i.e. non-graded) Yang–Baxter equation (this notation will be used throughout the paper when dealing with superalgebras).

In the $os(1|2)$ case (resp. $sl(1|2)$), the basis vectors $v_1, v_2, v_3$ of the three-dimensional representation have $\mathbb{Z}_2$ gradings 1, 0, 0 (resp. 1, 0, 1).

3 Deformed double Yangian $\mathcal{DY}_d(\mathfrak{g})$
3.1 Universal form

Our aim is to construct a Drinfel’d twist $F$ from $\mathcal{DY}(\mathfrak{g})$ to a deformed double Yangian $\mathcal{DY}_r(\mathfrak{g})$, which is thereby endowed with a QTQHA structure.

Let $\omega = \exp\left(\frac{2\pi i}{g+1}\right)$ be a root of unity. Following [9, 11], we postulate for $F$ the linear identity inspired by [19]:

$$F \equiv F(r) = \text{Ad}(\phi^{-1} \otimes \mathbb{1})(F) \cdot C,$$  \hspace{1cm} (3.1)

with

$$\phi = \omega^{h_{0,\rho}} e^{(r+c)d},$$  \hspace{1cm} (3.2)

$$C \equiv e^{\frac{1}{2}(c \otimes d + d \otimes c)} R.$$  \hspace{1cm} (3.3)

We will use the following properties:

- The operator $d$ in the double Yangian $\mathcal{DY}(\mathfrak{g})$ is defined by $[d, e_\alpha(u)] = \frac{d}{du} e_\alpha(u)$ for any root $\alpha$ (see [8]).

- The operator $d$ satisfies $\Delta(d) = d \otimes 1 + 1 \otimes d$.

- The generator $h_{0,\rho}$ of $\mathcal{DY}(\mathfrak{g})$ is such that

$$h_{0,\rho} e_\alpha(u) = e_\alpha(u) (h_{0,\rho} + (\rho|\alpha)),$$

$$h_{0,\rho} f_\alpha(u) = f_\alpha(u) (h_{0,\rho} - (\rho|\alpha)),$$

$$[h_{0,\rho}, h_\alpha(u)] = 0,$$  \hspace{1cm} (3.4)

and hence $\tau = \text{Ad} \left( \omega^{h_{0,\rho}} \right)$ is idempotent.

Equation (3.1) can be solved by

$$F(r) = \prod_k F_k(r), \quad F_k(r) = \phi_k^i C_{12}^{-1} \phi_k^{-i}.$$  \hspace{1cm} (3.5)

The solution of the linear equation (3.1) given by the infinite product (3.3) satisfies the following shifted cocycle relation:

$$F^{(12)}(r)(\Delta \otimes \text{id})(F(r)) = F^{(23)}(r + e^{(1)})(\text{id} \otimes \Delta)(F(r)).$$  \hspace{1cm} (3.6)

The proof follows the same lines as in [9], inspired by [10].

The twist $F(r)$ defines a QTQHA denoted $\mathcal{DY}_r(\mathfrak{g})$ with $R$-matrix $R_{\mathcal{DY}_r}(r) = F_{21}(r)R_{\mathcal{DY}}F_{12}(r)^{-1}$.

3.2 In representation for $\mathfrak{g} = \mathfrak{sl}_N$

A deformed double Yangian $\mathcal{DY}_r(\mathfrak{sl}_N)$ with $R$-matrix $\overline{R}$ was obtained in [14] (and in [2] for $\mathfrak{sl}_2$) by taking the scaling limit of the $RLL$ representation of $A_{q,p}(\mathfrak{sl}_N)$. We now characterise $\overline{R}$, up to gauge transformation, as an evaluation representation of the action of the Drinfel’d twist (3.5) on the $R$-matrix of $\mathcal{DY}(\mathfrak{sl}_N)$. In this way, we identify the structure in [14] as a QTQHA.
3.2.1 Gauge transformation of $\mathcal{DY}_r(sl_N)$

The $R$-matrix of $\mathcal{DY}_r(sl_N)$ is given by

$$
\overline{R}^{c,a+b-c}_{a,b}(u,r) = -\frac{1}{N} \rho_{\mathcal{DY}_r}(u) \frac{\sin \frac{\pi u}{r} \sin \frac{\pi}{r}}{\sin \frac{\pi}{r} \sin \frac{\pi}{r}} S^{c,a+b-c}_{a,b}(u,r)
$$

where

$$
S^{c,a+b-c}_{a,b}(u,r) = \frac{\sin \frac{\pi}{N r} (u + (b - a)r)}{\sin \frac{\pi}{N r} (u + (b - c)r) \sin \frac{\pi}{N r} (1 - (a - c)r)}
$$

The normalisation factor $\rho_{\mathcal{DY}_r}(u)$ is defined by

$$
\rho_{\mathcal{DY}_r}(u) = \frac{S_2(-u|r,N) S_2(1+u|r,N)}{S_2(u|r,N) S_2(1-u|r,N)},
$$

where $S_2(x|\omega_1,\omega_2)$ is Barnes’ double sine function of periods $\omega_1$ and $\omega_2$.

We perform the gauge transformation

$$
R := (V \otimes V) \overline{R}(V \otimes V)^{-1}
$$

$$
S := (V \otimes V) \overline{S}(V \otimes V)^{-1}
$$

with

$$
V^j_i = N^{-1/2} \omega^{(i-1)j}.
$$

A similar transformation was recently exhibited in [20]. (3.10) leads to the following expression of the matrix elements of $S$

$$
S^{j_1 j_2}_{i_1 i_2}(u) = \delta^{j_1+1}_{i_1+i_2} \left( \delta_{i_2}^{j_2} \Omega_{i_2-i_1} \left( \frac{u}{r} \right) + \delta_{i_1}^{j_2} \Omega_{i_1-i_2} \left( \frac{1}{r} \right) \right)
$$

where the function $\Omega_n(x)$ is defined by

$$
\Omega_n(x) := \sum_{k=0}^{N-1} \omega^{nk} \cot \frac{\pi}{N} (x + k)
$$

that is

$$
\Omega_n(x) = N \left( \frac{e^{i\pi x}}{\sin \pi x} e^{-2i\pi n x/N} - i \delta_n^0 \right) \quad \text{for } n \in \{0, ..., N-1\}
$$

5
Note that in (3.13) the integer \( n \) has to be taken in the interval \([0, \ldots, N - 1]\) since the expression (3.15) is not explicitly \( N \)-periodic in \( n \).

In particular, all the non zero entries of \( S \) are

\[
S_{aa}^{\alpha}(u) = N \left( \cot \frac{\pi u}{r} + \cot \frac{\pi}{r} \right) \quad (3.16)
\]
\[
S_{ab}^{\alpha}(u) = N \frac{e^{i\pi/r}}{\sin \frac{\pi}{r}} e^{-2i\pi(b-a)/Nr} \quad \text{for } b - a \in \{1, \ldots, N - 1\} \quad (3.17)
\]
\[
S_{ab}^{\beta}(u) = N \frac{e^{r\pi u/r}}{\sin \frac{\pi u}{r}} e^{-2i\pi b/uNr} \quad \text{for } b - a \in \{1, \ldots, N - 1\} \quad (3.18)
\]

### 3.2.2 Twist from \( DY(sl_N) \) to \( DY_c(sl_N) \)

We start with the \( R \)-matrix of \( DY(sl_N) \): its non-vanishing matrix elements \( R_{ijk}^{1j2} \) are given by \((1 \leq a, b \leq N)\)

\[
F_{aa}^{\alpha} = \rho_{DY}(u) \quad F_{ab}^{\alpha} = \rho_{DY}(u) \frac{u}{u + 1} \quad F_{ab}^{\beta} = \rho_{DY}(u) \frac{1}{u + 1} \quad (3.19)
\]

\[
\rho_{DY}(u) = \frac{\Gamma_1(u|N) \Gamma_1(u + N|N)}{\Gamma_1(u + 1|N) \Gamma_1(u + N - 1|N)} \quad (3.20)
\]

We then solve the evaluated linear equation, that is:

\[
F(u + r) = (\phi \otimes \mathbb{I})^{-1} F(u)(\phi \otimes \mathbb{I}) R(u + r) \quad (3.21)
\]

where

\[
\phi = \text{diag}(\omega^{-a}) \quad (3.22)
\]

and \( F \) is now a \( N^2 \times N^2 \) matrix which reduces to 1×1 blocks \( F_{aa}^{\alpha} = 1 \) and to 2×2 blocks:

\[
\begin{pmatrix}
F_{ab}^{\alpha}(u) & F_{ba}^{\alpha}(u) \\
F_{ab}^{\beta}(u) & F_{ba}^{\beta}(u)
\end{pmatrix}
= \begin{pmatrix}
b_{ab}(u) & c_{ab}(u) \\
c_{ab}'(u) & b_{ab}'(u)
\end{pmatrix} \quad (3.23)
\]

The linear equation (3.21) is then equivalent to:

\[
\omega^{b-a}(u + 2r + 1) b_{ab}(u + 2r) - (u + r + \omega^{b-a}(u + 2r)) b_{ab}(u + r) + (u + r - 1) b_{ab}(u) = 0 \quad (3.24)
\]
\[
(u + 2r + 1) c_{ab}(u + 2r) - (u + r + \omega^{a-b}(u + 2r)) c_{ab}(u + r) + \omega^{a-b}(u + r - 1) c_{ab}(u) = 0 \quad (3.25)
\]

together with similar equations for \( b_{ab}'(u) \) and \( c_{ab}'(u) \), deduced from (3.24) by the change \( \omega \rightarrow \omega^{-1} \).

The solution is expressed in terms of hypergeometric functions \( 2F_1 \): \( (3.26) \):

\[
b_{ab}(u) = \frac{\Gamma \left( \frac{u-1}{r} + 1 \right)}{\Gamma \left( \frac{u}{r} + 1 \right)} 2F_1 \left( -\frac{1}{r} \left\{ \frac{u-1}{r} + 1 \right\} ; \omega^{b-a} \right) \quad (3.26)
\]

\[
c_{ab}(u) = -\frac{\omega^{b-a}}{r} \frac{\Gamma \left( \frac{u-1}{r} + 1 \right)}{\Gamma \left( \frac{u}{r} + 2 \right)} 2F_1 \left( -\frac{1}{r} + 1 \left\{ \frac{u-1}{r} + 1 \right\} ; \omega^{b-a} \right) \quad (3.27)
\]
Similarly, \( b'_{ab}(u) \) and \( c'_{ab}(u) \) are obtained from \( b_{ab}(u) \) and \( c_{ab}(u) \) by changing \( \omega^{b-a} \) into \( \omega^{a-b} \).

The twist \( F(u) \) given by the collection of \( 2 \times 2 \) blocks (3.23) and \( 1 \times 1 \) blocks \( F_{aa}^{aa} = 1 \) applied to the \( R \)-matrix of \( DY(sl_N) \), Eq. (3.19), i.e.

\[
R^F(u) = F_{21}(-u)R(u)F^{-1}_{12}(u)
\]

provides the \( R \)-matrix \( R^F(u) \) of the deformed double Yangian \( DY_r(sl_N) \), Eq. (3.10), the non vanishing entries of which are expressed in terms of (3.16)-(3.18). The proof follows by a direct computation using the properties of the hypergeometric functions \(_2F_1\).

3.3 In representation for \( g = sl(1|2) \)

We conjecture the existence of a double Yangian of \( sl(1|2) \) endowed with a Hopf superalgebra structure. The evaluated \( R \)-matrix of this double Yangian is assumed to have the canonical Yang-type simplest rational form:

\[
\tilde{R}(u) = \frac{u}{u + 1} \sum_{a,b} (-1)^{[a][b]} E_{aa} \otimes E_{bb} + \frac{1}{u + 1} \left( \sum_{a \neq b} E_{ab} \otimes E_{ba} + \sum_a E_{aa} \otimes E_{aa} \right)
\]

where the gradation is \([1] = [3] = 1\), \([2] = 0\) (the conventions used for \( sl(1|2) \) are those of [21] with the fermionic basis).

A similar evaluation of the twist (3.5) now leads to the following expression for the \( R \)-matrix of the QTQHA \( DY_r(sl(1|2)) \) (with \( N = 3 \)):

\[
\tilde{R}(u) = -\frac{\sin \frac{\pi u}{r}}{\sin \frac{\pi (u+1)}{r}} (E_{11} \otimes E_{11} + E_{33} \otimes E_{33}) + E_{22} \otimes E_{22} + \sum_{a<b} (-1)^{[a][b]} \left( e^{i\pi/r + 2i\pi(a-b)/Nr} E_{aa} \otimes E_{bb} + e^{-i\pi/r - 2i\pi(b-a)/Nr} E_{bb} \otimes E_{aa} \right)
\]

4 Twist from \( U_q(g) \) to \( B_{q,\lambda}(g) \): a summary

4.1 Universal form

The universal \( R \)-matrix for \( U_q(g) \) takes the following form

\[
\hat{R} = \hat{R} \mathcal{K} ; \quad \hat{R} = \prod_{\gamma \in \Pi^+} \hat{R}_\gamma
\]

where the product is ordered with respect to \( > \), the reversed normal order on \( \Pi^+ \). The objects \( \hat{R}_\gamma \) and \( \mathcal{K} \) are given by

\[
\hat{R}_\gamma = \exp_{q^2}(-(q - q^{-1})e_\gamma \otimes f_\gamma)
\]
and

$$K = q^{-\sum_{ij} d_{ij} h_{i} \otimes h_{j}}.$$  \hspace{1cm} (4.3)

$e_\gamma$, $f_\gamma$ are the root generators and $h_i$ the Cartan generators in the Serre-Chevalley basis. In (4.2) the $q$-exponential is defined by

$$\exp_q(x) \equiv \sum_{n \in \mathbb{Z}^+} \frac{x^n}{(n)_q!} \quad \text{where} \quad (n)_q! \equiv (1)_q(2)_q \ldots (n)_q \quad \text{and} \quad (k)_q \equiv \frac{1 - q^k}{1 - q} \hspace{1cm} (4.4)$$

Expanding the product formula (4.1) with respect to a Poincaré–Birkhoff–Witt basis ordered with $<$, $\hat{R}$ reads

$$\hat{R} = RK - 1 = 1 \otimes 1 + \sum_{m \in \mathbb{Z}^*} \sigma_m e^m \otimes f^m$$  \hspace{1cm} (4.5)

where $Z = \text{Map}(\mathbb{Z}, \mathbb{Z}^+)$, and $Z^* = \mathbb{Z} \setminus \{0, \ldots, 0\}$. The term $e^m$, (resp. $f^m$) denotes an element of the PBW basis of the deformed enveloping nilpotent subalgebra $U_q(N^+)$ (resp. $U_q(N^-)$).

In [10, 11], it has been shown that there exists a twist $F$ from $U_q(g)$ to $B_{q,\lambda}(g)$ which can be expressed as:

$$F_{[U_q(g) \to B_{q,\lambda}(g)]} = K^{-1} \hat{F}K, \quad \hat{F} = \prod_{k \geq 1} \text{Ad}(\phi \otimes 1)^k \left( \hat{R}^{-1} \right)$$ \hspace{1cm} (4.6)

with

$$\phi \equiv q^\chi \equiv q^{\sum_{ij} d_{ij} h_i h_j + 2 \sum_i s_i h_i} \quad s_i \in \mathbb{C}$$  \hspace{1cm} (4.7)

It obeys the cocycle condition

$$F_{12}(w)(\Delta \otimes 1)F(w) = F_{23}(wq^{h_\vee (1)}) (1 \otimes \Delta)F(w)$$  \hspace{1cm} (4.8)

with $w = (w_1, \ldots, w_r), q^a = (q^{a_1}, \ldots, q^{a_r}) \in \mathbb{C}^r, wq^{h_\vee} = (w_1 q^{h_1 \vee}, \ldots, w_r q^{h_r \vee})$ and $h_i \vee = \sum_j d_{ij} h_j$.

This twist satisfies the linear equation

$$F = \text{Ad}(\phi^{-1} \otimes 1)(F) K^{-1} \hat{R} K$$ \hspace{1cm} (4.9)

the solution of which is uniquely defined [11] once one imposes (i) $F \in (U_q(B^+) \otimes U_q(B^-))^c$, (ii) its projection on $(U_q(S)^{\otimes 2})^c$ is $1 \otimes 1$ where the superscript $c$ denotes a suitable completion. Under these assumptions, one therefore obtains

$$\hat{F} = 1 \otimes 1 + \sum_{(p, r) \in (\mathbb{Z}^*)^2} \varphi_{pr}(w) e^p \otimes f^r$$  \hspace{1cm} (4.10)
where the $\varphi_{pr}(w)$ belong to $\mathbb{C}[[s_1, \ldots, s_r, s_1^{-1}, \ldots, s_r^{-1}, h]] \otimes (U_q(\mathfrak{sl}_2))^c$. They are defined recursively (using (4.9)) by

$$
\left(1 - q^{(-2h^{(1)} + \gamma_p - s|\gamma_p)}\right) \varphi_{pr}(w) = \sum_{k+m=p \atop l+m=r \atop m \neq 0} (-1)^{[l][m]} d_p^{km} t_r^{lm} \sigma_m q^{(-2h^{(1)} + \gamma_k - s|\gamma_k)} \varphi_{kl}(w). \tag{4.11}
$$

In the above equation, $\gamma_p$ is the element of the root lattice associated to $e^p$. The scalar product $(.|.)$ is given by $(x|y) \equiv \sum_{i,j} a_{ij} x_i y_j$. The numbers $a_{km}^p$ and $b_{lm}^r$ are defined by

$$
e^ke^m = \sum_{p \in \mathbb{Z}} a_{km}^p e^p \quad \text{and} \quad f^lf^m = \sum_{r \in \mathbb{Z}} b_{lm}^r f^r. \tag{4.12}
$$

The formulas (4.10), (4.11) taken from [11] will be used in Section 5 in their $q \to 1$ limit.

## 4.2 In representation for $\mathfrak{g} = \mathfrak{sl}_N$

In the fundamental representation for $\mathfrak{g} = \mathfrak{sl}_N$, we get

$$
R = q^{1/N} \left( \mathbb{I} \otimes \mathbb{I} + (q^{-1} - 1) \sum_a E_{aa} \otimes E_{aa} + (q^{-1} - q) \sum_{a < b} E_{ab} \otimes E_{ba} \right) \tag{4.13}
$$

the $N \times N$ matrices $E_{ab}$ being the usual elementary matrices with entry 1 in position $(a, b)$ and 0 elsewhere.

The twist is represented by

$$
\hat{F} = \prod_{k \geq 1} (B \otimes \mathbb{I})^k \hat{R}^{-1} (B \otimes \mathbb{I})^{-k} \tag{4.14}
$$

$$
F_{[U_q(\mathfrak{sl}_N) \to B_q, \lambda(\mathfrak{sl}_N)]} = \mathbb{I} \otimes \mathbb{I} + (q - q^{-1}) \sum_{a < b} \frac{w_{ab}}{1 - w_{ab}} E_{ab} \otimes E_{ba} \tag{4.15}
$$

with $B = q^{N-1} \text{diag}(q^{x_a})$, $w_{ab} = q^{x_a - x_b}$ and $x_a = 2s_a - 2s_{a-1}$ with $s_0 = s_N = 0$.

The non vanishing elements $R_{i_1 i_2}^{j_1 j_2}$ of the $R$-matrix of $B_q, \lambda(\mathfrak{sl}_N)$ are then given by ($1 \leq a, b \leq N$)

$$
R_{aa}^{aa} = q^{1/N} \frac{1}{q} 
$$

$$
R_{ab}^{ab} = q^{1/N} \begin{cases} 
1 - q^2 w_{ab} & \text{if } b > a \\
(1 - q^{-2} w_{ab}) & \text{if } b < a \\
1 & \text{if } b = a
\end{cases} \tag{4.16}
$$

$$
R_{ab}^{ba} = q^{1/N} (q^{-1} - 1) \frac{1}{w_{ab} - 1}
$$
4.3 In representation for $g = osp(1|2)$

The universal $R$-matrix of $\mathcal{U}_q(osp(1|2))$ was initially obtained in \cite{22, 23}. As indicated in Section 2, we use the finite $R$-matrix obtained as the evaluation of the universal $R$-matrix given in \cite{10} (changing $q$ to $q^{-1}$):

$$\tilde{R}_{12} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & q^2 - 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & q^{-1} - q & 0 & 0 \\ 0 & q^{-1} - q & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

The twist from $\mathcal{U}_q(osp(1|2))$ to $\mathcal{B}_{q,\lambda}(osp(1|2))$ is represented by

$$F = 1 \otimes 1 - \frac{w(q-q^{-1})}{w-q} (E_{13} \otimes E_{31} - qE_{13} \otimes E_{12})$$

$$- \frac{qw(q-q^{-1})}{qw-1} (E_{21} \otimes E_{12} - q^{-1}E_{21} \otimes E_{31}) - \frac{w^2(q-q^{-1})(q+1)}{(qw-1)(w+1)} (E_{23} \otimes E_{32}) \quad (4.17)$$

The resulting $R$-matrix of $\mathcal{B}_{q,\lambda}(osp(1|2))$ is given by $F_{21}R_{12}F^{-1}_{12}$. The corresponding matrix $\tilde{R}$ has the following expression

$$\tilde{R} = \begin{pmatrix} r_{1111} & 0 & 0 & 0 & 0 & (q^2-1)qw \frac{q-w}{q-w} & 0 & r_{1132} & 0 \\ 0 & \frac{(q^3-1)(w-q)}{q(qw-1)^2} & 0 & \frac{(q^2-1)w}{1-qw} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & q^{-1} - q & 0 & 0 \\ 0 & q^{-1} - q & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{q-q^{-1}}{qw-1} & 0 & 0 & 0 & 0 & q & 0 & (q^2-1)(q+1) \frac{(w-q)(w+1)}{(w-q)(w+1)} & 0 \\ 0 & 0 & \frac{(q^{-1})w}{q-w} & 0 & 0 & 0 & \frac{(q^2-1)(q+1)}{q(q-w)^2} & 0 & 0 \\ \frac{q-q^{-1}}{qw-1} & 0 & 0 & 0 & 0 & \frac{(q^2-1)(q+1)w^2}{(1-qw)(w+1)} & 0 & \frac{(q^2w+1)(q^{-1}+w)}{q(w+1)^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(q^2w+1)(q^{-1}+w)}{q(w+1)^2} & q^{-1} \end{pmatrix}$$

where $r_{1111} = \frac{q^2(w-1)^2+w(q-1)(q^2-1)}{q(qw-1)(q-w)}$, $r_{1132} = \frac{(q^2-1)(q+1)(1-qw)}{(q-w)^2(w+1)}$ and $r_{3211} = \frac{(q^2w+1)(q^{-1}+w)(w-q)}{q^2(qw-1)^2(w+1)}$. It obeys the ordinary dynamical Yang–Baxter equation.

5 Twist from $\mathcal{U}(g)$ to $\mathcal{U}_s(g)$
5.1 Universal form and cocycle condition

We now construct the twist from $\mathcal{U}(g)$ to $\mathcal{U}_s(g)$ as a limit $q \to 1$, (i.e. $\hbar \to 0$ with $q = e^{\hbar}$) of the twist ($4.10$). The consistency of this procedure follows from the well-known Hopf algebra identification $\mathcal{U}_h(g)/ (h \mathcal{U}_h(g)) \simeq \mathcal{U}(g)$ $^{[24]}$.

In this quotient, the twist $\mathcal{F}$ coincides with $\hat{\mathcal{F}}$. It is given by a formula analogous to ($4.10$):

$$\mathcal{F}_{[\mathcal{U}(g) \to \mathcal{U}_s(g)]} = \mathbb{I} \otimes \mathbb{I} + \sum_{(p,r) \in (\mathbb{Z}^*)^2} \varphi_{pr}(s) \, e^p \otimes f^r$$

the functions $\varphi_{pr}(s)$ being the representatives of those appearing in ($4.10$).

It follows from the Hopf algebra identification that $\mathcal{F}_{[\mathcal{U}(g) \to \mathcal{U}_s(g)]}$ obeys in $\mathcal{U}(g) \otimes 2$ the cocycle equation

$$\mathcal{F}_{12}(s)(\Delta \otimes \mathbb{I})\mathcal{F}(s) = \mathcal{F}_{23}(s + h^{\nu(1)})(\mathbb{I} \otimes \Delta)\mathcal{F}(s)$$

defining a QTQHA denoted $\mathcal{U}_s(g)$ with $R$-matrix $\mathcal{R}(s) = \mathcal{F}_{21}(s)\mathcal{F}_{12}(s)^{-1}$.

The functions $\varphi_{pr}(s)$ satisfy recursion equations obtained as the leading order in $\hbar$ of ($4.11$). This procedure is well-defined since the coefficient of $\varphi$ in the left hand side of ($4.11$) is of order $\hbar$ and the coefficients in the right hand side are at least of order 1 in $\hbar$ (due to the presence of $\sigma_m$). The leading order in $\hbar$ of ($4.11$) can be expressed as the following equation in $\mathcal{F}$:

$$[\mathcal{X}, \mathcal{F}] = \mathcal{F} \hat{\mathcal{R}}$$

where $\mathcal{X}$ was defined in ($4.7$) and $\hat{\mathcal{R}} = \mathbb{I} \otimes \mathbb{I} + h \hat{\mathcal{R}} + o(\hbar)$. Note that the equation ($5.3$) is also the first non trivial term in the expansion in $\hbar$ of ($4.9$).

Under a similar hypothesis on $\mathcal{F}$ as in Section $4$, replacing $q$-deformed enveloping algebras by classical enveloping algebras, equation ($5.3$) has a unique solution expressed either by ($5.1$) or as the infinite product

$$\mathcal{F}_{[\mathcal{U}(g) \to \mathcal{U}_s(g)]} = \prod_{k \geq 1} (\mathcal{X} \otimes \mathbb{I})^{-k} \left(\mathbb{I} \otimes \mathbb{I} + (\mathcal{X} \otimes \mathbb{I})^{-1} \hat{\mathcal{R}}\right)(\mathcal{X} \otimes \mathbb{I})^k$$

5.2 In representation for $g = sl_N$

In the fundamental representation for $g = sl_N$, the evaluated infinite product expression for $\mathcal{F}$ reads

$$F_{[\mathcal{U}(sl_N) \to \mathcal{U}_s(sl_N)]} = \prod_{k \geq 1} (X \otimes \mathbb{I})^{-k} \, Y \, (X \otimes \mathbb{I})^k$$

with

$$X = \frac{N-1}{N} \mathbb{I} + \text{diag}(x_a) \quad \text{where} \quad x_a = 2s_a - 2s_{a-1}, \quad s_0 = s_N = 0 \ ,$$

$$Y = \mathbb{I} \otimes \mathbb{I} + (X \otimes \mathbb{I})^{-1} \hat{\mathcal{R}} = \mathbb{I} \otimes \mathbb{I} + (X \otimes \mathbb{I})^{-1} \sum_{a<b} -2E_{ab} \otimes E_{ba} \ ,$$

$$\hat{\mathcal{R}} = \mathbb{I} \otimes \mathbb{I} + \sum_{a<b} -2E_{ab} \otimes E_{ba} \ .$$

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\[ \hat{r} = -2 \sum_{a<b} E_{ab} \otimes E_{ba} \] being the classical $\bar{R}$ matrix of $\mathcal{U}(sl_N)$.

Computing the product leads to

\[ F = \mathbb{1} \otimes \mathbb{1} - \sum_{a<b} \frac{2}{x_a - x_b} E_{ab} \otimes E_{ba} \]  \hspace{1cm} (5.8)

Regarding the $R$-matrix of $\mathcal{U}_s(sl_N)$, its non vanishing matrix elements $R_{ij}^{ij,j_2}$ are given by $(1 \leq a, b \leq N)$

\[
R_{aa}^{aa} = 1 \\
R_{ab}^{ab} = \begin{cases} 
1 & \text{if } b > a \\
1 - \frac{4}{(x_a - x_b)^2} & \text{if } b < a 
\end{cases} \\
R_{ba}^{ba} = \frac{2}{x_a - x_b} \quad \text{for } a \neq b
\]  \hspace{1cm} (5.9)

which indeed satisfies the dynamical Yang–Baxter equation (2.5).

### 5.3 In representation for $g = osp(1|2)$

Using again the infinite product expression (5.4), the twist from $\mathcal{U}(osp(1|2))$ to $\mathcal{U}_s(osp(1|2))$ is represented by

\[ F = \mathbb{1} \otimes \mathbb{1} - \frac{2}{s - 1} (E_{13} \otimes E_{31} - E_{13} \otimes E_{12}) - \frac{2}{s + 1} (E_{21} \otimes E_{12} + E_{23} \otimes E_{32} - E_{21} \otimes E_{31}) \]  \hspace{1cm} (5.10)

The $R$-matrix of $\mathcal{U}_s(osp(1|2))$ is then straightforwardly $R(s) = F_{21}(s)F_{12}^{-1}(s)$. The corresponding matrix $\bar{R}$ satisfies the ordinary dynamical Yang–Baxter equation.

### 5.4 In representation for $g = sl(1|2)$

Similarly, the twist from $\mathcal{U}(sl(1|2))$ to $\mathcal{U}_s(sl(1|2))$ is represented by

\[ F = \mathbb{1} \otimes \mathbb{1} + \frac{1}{s_2 - 1} E_{21} \otimes E_{12} - \frac{1}{s_1 + s_2} E_{31} \otimes E_{13} + \frac{1}{s_1 + 1} E_{32} \otimes E_{23} \]  \hspace{1cm} (5.11)

The $R$-matrix of $\mathcal{U}_s(sl(1|2))$ is then given by $R(s) = F_{21}(s)F_{12}^{-1}(s)$. Again, the corresponding matrix $\bar{R}$ satisfies the ordinary dynamical Yang–Baxter equation.

### 6 Twist from $\mathcal{U}_q(\hat{g})$ to $\mathcal{U}_{q,\lambda}(\hat{g})$

#### 6.1 Universal form

The expression of a universal twist from $\mathcal{U}_q(\hat{g})$ to $\mathcal{B}_{q,\lambda}(\hat{g})$ was recalled in Section 4. Using the fact that $\mathcal{U}_q(\hat{g})$ is a Hopf subalgebra of $\mathcal{U}_q(\hat{g})$, we use this twist to construct the dynamical $\mathcal{U}_{q,\lambda}(\hat{g})$. 

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Indeed, \( \mathcal{F}_{U_q(\mathfrak{g}) \to \mathcal{B}_{\mathfrak{s}l_N}} \) seen as an element of \( U_q(\mathfrak{g})^\otimes 2 \) satisfies the shifted cocycle condition, yielding a dynamical \( R \)-matrix

\[
\mathcal{R}_{U_q,\lambda}(\mathfrak{g})(w) = \mathcal{F}_{21}(w) \mathcal{R}_{U_q}(\mathfrak{g}) \mathcal{F}_{12}^{-1}(w).
\]

(6.1)

This defines \( U_q,\lambda(\mathfrak{g}) \) as a QTQHA. We now identify it, by computing its evaluation representation for \( \mathfrak{g} = \mathfrak{sl}_N \), to a realisation of the \( p \to 0 \) limit of \( \mathcal{B}_{q,p,\lambda}(\mathfrak{sl}_N) \) defined in Appendix B.

### 6.2 In representation for \( \mathfrak{g} = \mathfrak{sl}_N \)

One gets the matrix elements \( R_{ab}^{ij} \) of the \( R \)-matrix of \( U_q,\lambda(\mathfrak{sl}_N) \) \((1 \leq a, b \leq N):\)

\[
\begin{align*}
R_{aa}^{aa} &= \rho U_{q,\lambda}(z) \\
R_{ab}^{ab} &= \rho U_{q,\lambda}(z) \begin{cases} 
\frac{q(1-z)}{1-q^2 z} & \text{if } b > a \\
\frac{q(1-z)}{1-q^2 z} \frac{(1-w_{ab}q^2)(1-w_{ab}q^{-2})}{(1-w_{ab})^2} & \text{if } b < a
\end{cases} \\
R_{ab}^{ba} &= \rho U_{q,\lambda}(z) \frac{(1-q^2)(1-w_{ab}z)}{(1-q^2 z)(1-w_{ab})}
\end{align*}
\]

(6.2)

the normalisation factor being given by

\[
\rho U_{q,\lambda}(z) = q^{\frac{N-1}{N}} \frac{(q^2 z; q^{2N})_{\infty} (q^{2N-2} z; q^{2N})_{\infty}}{(z; q^{2N})_{\infty} (q^{2N} z; q^{2N})_{\infty}}.
\]

(6.3)

This \( R \)-matrix satisfies the Dynamical Yang–Baxter equation (2.6). It is indeed the limit \( p \to 0 \) of the \( R \)-matrix (B.3).

### 6.3 In representation for \( \mathfrak{g} = \mathfrak{osp}(1|2) \)

We first construct a represented \( R \)-matrix of \( \mathfrak{osp}(1|2) \) through a Baxterisation procedure [25]. We get two \( R \)-matrices with spectral parameter constructed from \( \tilde{R}_{12}, \tilde{R}_{21}^{-1} \) and \( \tilde{P}_{12} \) (the non-graded permutation) which obey the non-graded Yang–Baxter equation (with spectral parameter). They read:

\[
\tilde{R}(z) = \frac{1 - z}{(1 - za)(1 - zq^2)} \tilde{R}_{12} - \frac{aq^2 z(1 - z)}{(1 - za)(1 - zq^2)} \tilde{R}_{21}^{-1} + \frac{z(1 - a)(1 - q^2)}{(1 - za)(1 - zq^2)} \tilde{P}_{12} \text{ with } a = -q \ \text{or } q^3
\]

(6.4)
where the normalisations are such that \( \tilde{R}(0) = \tilde{R}_{12} \), \( \tilde{R}(\infty) = q^{-2} \tilde{R}_{21}^{-1} \) and \( \tilde{R}(1) = q^{-1} \tilde{P}_{12} \). Explicitly, for \( a = -q \), we get:

\[
\tilde{R}(z) = \begin{pmatrix}
    r_{1111} & 0 & 0 & 0 & 0 & \frac{z(1-z)(q^2-1)}{(1-zq^2)(1+zq^2)} & 0 & \frac{(1-z)(q^2-1)}{(1-zq^2)(1+zq^2)} & 0 \\
    0 & \frac{1-z}{1-zq^2} & 0 & \frac{(q^{-1}-q)z}{1-zq^2} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \frac{q^{-1}}{1-zq^2} & 0 & 0 & 0 \\
    0 & 0 & \frac{(q^{-1}-q)z}{1-zq^2} & 0 & 0 & 0 & \frac{(1-z)(z+q)}{(1-zq^2)(1+zq^2)} & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \frac{(1-z)(q^{-1}-q)}{(1-zq^2)(1+zq^2)} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \frac{z(1-z)(q^{-1}-q)}{(1-zq^2)(1+zq^2)} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \frac{(q^{-1}-q)}{1-zq^2} & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \frac{q^{-1}}{1-zq^2} & 0 & 0 \\
\end{pmatrix}
\]

with \( r_{1111} = \frac{z-1}{1-zq^2} + \frac{z(q+1)(q^{-1}-q)}{(1-zq^2)(1+zq^2)} \).

Applying to the above matrix the twist given by formula (\ref{4.17}) allows one to get the explicit evaluated \( R \)-matrix for the QTQHA \( U_{q, \lambda}(osp(1|2)) \). Due to its cumbersome nature, we omit this explicit form here.

7 Twist from \( \mathcal{DY}(\mathfrak{g}) \) to \( \mathcal{DY}_s(\mathfrak{g}) \)

7.1 Universal form

We similarly use the fact that \( \mathcal{U}(\mathfrak{g}) \) is a Hopf subalgebra of \( \mathcal{DY}(\mathfrak{g}) \), \( \mathfrak{g} \) belonging to the (super) unitary series. Hence the cocycle identity (\ref{5.2}) is also an identity in \( \mathcal{DY}(\mathfrak{g})^{\otimes 3} \) for \( \mathcal{F} \) defined as in (\ref{5.1}), now considered as an element of \( \mathcal{DY}(\mathfrak{g})^{\otimes 2} \). This twist, applied to the universal \( R \)-matrix of \( \mathcal{DY}(\mathfrak{g}) \) (given in (\ref{7})), yields a dynamical \( R \)-matrix

\[
\mathcal{R}(s) = \mathcal{F}_{21}(s) \mathcal{R} \mathcal{F}_{12}^{-1}(s)
\]

which characterise \( \mathcal{DY}_s(\mathfrak{g}) \) as a QTQHA.

7.2 In representation for \( \mathfrak{g} = sl_N \)

We apply the twist (\ref{5.8}) to the \( R \)-matrix of \( \mathcal{DY}(sl_N) \), given in (\ref{3.19})-(\ref{3.20}), to get the \( R \)-matrix of \( \mathcal{DY}_s(sl_N) \). In the fundamental representation, it has the following non vanishing elements \( R_{i_1j_1}^{i_2j_2} \).
\[(1 \leq a, b \leq N)\]

\[
R_{\text{aa}}^{\text{aa}} = \rho_{DY_s}(u) \begin{cases} 
\frac{u}{u + 1} & \text{if } b > a \\
\left(1 - \frac{4}{(x_a - x_b)^2}\right) \frac{u}{u + 1} & \text{if } b < a
\end{cases}
\]

\[
R_{\text{ab}}^{\text{ab}} = \rho_{DY_s}(u) \left(1 + \frac{2u}{x_a - x_b}\right) \frac{1}{u + 1}
\]

(7.2)

the normalisation factor being \(\rho_{DY_s}(u) = \rho_{DY}(u)\). This matrix can also be obtained as the scaling limit of the \(R\)-matrix (6.2) of \(U_{q,\lambda}(\hat{sl}_N)\). It satisfies the dynamical Yang–Baxter equation (2.7).

7.3 In representation for \(g = sl(1|2)\)

Similarly, by applying the twist (5.11) to the \(R\)-matrix of \(DY(sl(1|2))\), one gets the evaluated \(R\)-matrix for the QTQHA \(DY_s(sl(1|2))\):

\[
\tilde{R} = \begin{pmatrix}
\frac{1-u}{1+u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{us_2(s_2-2)}{(1+u)(s_2-1)^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{s_2-1+u}{(1+u)(s_2-1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1+u}{1+u} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1+u}{1+u} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{us_1(s_1+2)}{(1+u)(s_1+1)^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{s_1+1+u}{(1+u)(s_1+1)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-u}{1+u}
\end{pmatrix}
\]

It satisfies the dynamical Yang–Baxter equation (2.7).

A Notations

Multiple Gamma functions are defined by

\[
\Gamma_r(x|\omega_1, \ldots, \omega_r) = \exp \left(\frac{\partial}{\partial s} \sum_{n_1,\ldots,n_r \geq 0} (x + n_1\omega_1 + \ldots + n_r\omega_r)^{-s}\bigg|_{s=0}\right).
\]  (A.1)

Barnes’ multiple sine function \(S_r(x|\omega_1, \ldots, \omega_r)\) of periods \(\omega_1, \ldots, \omega_r\) is defined by

\[
S_r(x|\omega_1, \omega_2) = \Gamma_r(\omega_1 + \ldots + \omega_r - x|\omega_1, \ldots, \omega_r)^{-1} \Gamma_r(x|\omega_1, \ldots, \omega_r)^{-1},
\]  (A.2)
They satisfy for each $i \in [1, \ldots, r]$
\[
\frac{\Gamma_r(x + \omega_i|\omega_1, \ldots, \omega_r)}{\Gamma_r(x|\omega_1, \ldots, \omega_r)} = \frac{1}{\Gamma_{r-1}(x|\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_r)}, \tag{A.3}
\]
\[
\frac{S_r(x + \omega_i|\omega_1, \ldots, \omega_r)}{S_r(x|\omega_1, \ldots, \omega_r)} = \frac{1}{S_{r-1}(x|\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_r)}. \tag{A.4}
\]

In particular,
\[
\Gamma_1(x|\omega_1) = \frac{\omega_1^{x/\omega_1}}{\sqrt{2\pi \omega_1}} \Gamma \left( \frac{x}{\omega_1} \right), \quad S_1(x|\omega_1) = 2 \sin \frac{\pi x}{\omega_1}. \tag{A.5}
\]

**B  Definition of $B_{q,p,\lambda}(\hat{sl}_N)$**

The quantum affine elliptic algebra $B_{q,p,\lambda}(\hat{sl}_N)$ was originally defined in [4] using the RLL formalism. The characteristic $R$-matrix takes the following form (1 $\leq a, b \leq N$):
\[
R(z) = \rho_{B_{q,p,\lambda}}(z) \left( \sum_a E_{aa} \otimes E_{aa} + \sum_{a \neq b} q^2 \frac{\Theta_p(q^{-2}w_{ab})}{\Theta_p(w_{ab})} \frac{\Theta_p(z)}{\Theta_p(q^2 z)} E_{ab} \otimes E_{ba} \right) + \sum_{a \neq b} \frac{\Theta_p(w_{ab}z)}{\Theta_p(w_{ab})} \frac{\Theta_p(q^2)}{\Theta_p(q^2 z)} E_{ab} \otimes E_{ba}, \tag{B.1}
\]
where $\Theta_p(z) = (z;p)_\infty (p^{-1};p)_\infty (p;p)_\infty$ and $w_{ab} = q^{x_a-x_b}$, the infinite multiple products being defined by $(z;p_1, \ldots, p_m)_\infty = \prod_{n_i \geq 0} (1-zp_1^{n_1} \ldots p_m^{n_m})$. The normalisation factor $\rho_{B_{q,p,\lambda}}(z)$ is given by
\[
\rho_{B_{q,p,\lambda}}(z) = q^{-\frac{z}{q}} \frac{(q^2 z; q^{2N}; p)_\infty (q^{2N} z^{-2}; q^{2N}; p)_\infty (p z^{-1}; q^{2N}, p)_\infty (pq^{2N} z^{-1}; q^{2N}, p)_\infty}{(q z^{-1}; q^{2N}, p)_\infty (q^{2N} z^{-1}; q^{2N}, p)_\infty (pq^{2N-2} z^{-1}; q^{2N}, p)_\infty}. \tag{B.2}
\]

It was proven in [10] that this quantum affine elliptic algebra was a QTQHA obtained by a Drinfel’d twist of shifted-cocycle type from $U_q(\hat{sl}_N)$. The $R$-matrix thus obtained is actually a gauge transform of (B.1) where the coefficients of $E_{aa} \otimes E_{bb}$ become
\[
R_{ab} = \rho_{B_{q,p,\lambda}}(z) \begin{cases} 
q \frac{(pw_{ab}^{-1} q^2; p)_\infty (pw_{ab}^{-1} q^{-2}; p)_\infty \Theta_p(z)}{(pw_{ab}^{-1}; p)_\infty^2} \Theta_p(q^2 z) & \text{if } b > a \\
q \frac{(w_{ab}^{-1} q^2; p)_\infty (w_{ab}^{-1} q^{-2}; p)_\infty \Theta_p(z)}{(w_{ab}^{-1}; p)_\infty^2} \Theta_p(q^2 z) & \text{if } b < a
\end{cases} \tag{B.3}
\]

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