Sampling methods for multistage robust convex optimization problems

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Received: date / Accepted: date

Abstract In this paper, probabilistic guarantees for constraint sampling of multistage robust convex optimization problems are derived. The dynamic nature of these problems is tackled via the so-called scenario-with-certificates approach. This allows to avoid the conservative use of explicit parametrizations through decision rules, and provides a significant reduction of the sample complexity to satisfy a given level of reliability. An explicit bound on the probability of violation is also given. Numerical results dealing with a multistage inventory management problem show the efficacy of the proposed approach.

Keywords convex multistage robust optimization · constraint sampling · scenario with certificates · randomized algorithms

1 Introduction

In many practical situations, the decision process is affected by uncertainty. In such cases, the so-called uncertainty set where all realizations of the random parameters lie is considered, and then optimize an objective function protecting against the worst possible uncertainty realization. This is the key philosophy behind the robust optimization modeling paradigm. The original robust optimization models deal with static problems, where all the decision variables have to be determined
before the uncertain parameters are selected. A vast literature focused on uncertainty structure to obtain computationally tractable problems is available, see for instance [18] and [34] for polyhedral uncertainty sets and [5] for ellipsoidal uncertainty sets, respectively.

However, this approach cannot directly handle problems that are multiperiod in nature, where a decision at any period should take into account data realizations in previous periods, and the decision maker needs to adjust his/her strategy on the information revealed over time. This means that some of the variables (non-adjustable variables) must be determined before the realization of the uncertain parameters, while the other variables (adjustable variables) have to be chosen after the uncertainty realization. For a recent overview of multiperiod robust optimization, we refer to [7], [17] and [22]. In order to describe such a situation, and extend robust optimization to a dynamic framework, the concept of Adjustable Robust Counterpart (ARC) has been first introduced and analyzed in [4]. This approach opened up several new research directions, such as portfolio optimization [10], [33], [37], inventory management [3], [10], scheduling [24], [39], facility location [2], revenue management [32] and energy generation [40]. ARC is clearly less conservative than the static robust approach, but in most cases it turns out to be computationally intractable. One of the most recent methods to cope with this difficulty is obtained by approximating the adjustable decisions by decision rules, i.e. linear combinations of given basis functions of the uncertainty. A particular case is the Affinely Adjustable Robust Counterpart (AARC) [4], where the adjustable variables are affine functions of the uncertainty. The decision rule approximation often allows to obtain a formulation which is equivalent to a tractable optimization problem (such as linear, quadratic and second-order conic [4], or semidefinite [23]), transforming the original dynamic problem into a static robust optimization problem whose decision variables are the coefficients of the linear combination.

However, in many practical cases, also the static robust optimization problem ensuing from the decision rule approximation is still numerically intractable. In these situations, one can recur to approximate solutions based on constraint sampling, which consists in taking into account only a finite set of constraints, chosen at random among the possible continuum of constraint instances of the uncertainty. The attractive feature of this method is to provide explicit bounds on the measure of the original constraints of the static problem that are possibly violated by the randomized solution. The properties of the solutions provided by this approach, called scenario approach have been studied in [11], [15], [20], where it has been shown that most of the constraints of the original static problem are satisfied provided the number of samples sufficiently large. The constraint sampling method has been also extensively studied within the chance constraint approach through different directions by [19], [25], [29] and [31].

In [8], [12], [38], multistage convex robust optimization problems are solved by combining general nonlinear decision rules and constraint sampling techniques. This means that the dynamic robust optimization problem is transformed into a static robust optimization problem through decision rules approximation and then solved via a scenario counterpart. In practice, the novelty of [38] is to introduce, besides polynomial decision rules, also trigonometric monomials and basis functions based on sigmoidal and Gaussian radial functions, thus allowing more flexibility. A rigorous convergence proof for the optimal value, based on the decision rule approximation and of the constraint randomization approach is also investigated. Convergence is proved when both the complexity parameter (number of basis in the decision rule approximation) and the number of samples tends to infinity.

In the context of randomized methods for uncertain optimization control problems, the scenario with certificates approach has been proposed in [21], based on an original idea of [30]. This approach has been then extended and exploited for anti-windup augmentation problems [21]. The main idea of
this approach is to distinguish between design variables (corresponding to non-adjustable variables) and certificates (corresponding to adjustable variables).

In this paper, we consider randomized methods for robust convex multistage optimization problems. We treat the dynamic nature of the problem via the scenario with certificates approach, thus avoiding the conservative use of parametrization through decision rules. This implies a significant reduction of the number of samples required to satisfy the level of reliability of the constraints. In particular, we show that a multistage robust linear optimization problem $\text{RO}_H$, is equivalent to a linear robust optimization problem with certificates $\text{RwC}_H$, and a bound on the probability of violation is provided for the scenario with certificates problem $\text{SwC}_N^H$ based on $N$ instances (or scenarios) of the uncertain constraints and $H$ stages. The analysis has been extended to the convex case. Furthermore, upper and lower bounds obtained by relaxing the nonanticipativity constraints are also provided.

The rest of the paper is as follows. Section 2 discusses the formulations of two-stage, multistage robust linear and convex programs and provides a result on the probability of violation of constraints. Bounds on the number of scenarios needed to obtain a user-prescribed guarantee of violation is given. Section 3 provides a chain of inequalities among lower bounds on the multistage robust optimization problem. Section 4 presents several numerical results dealing with a multistage inventory management problem. The conclusions follow.

2 Problem formulation

2.1 Notation

In this paper, the uncertainty is described by a discrete random process $\xi^t$, $t = 1, \ldots, H$, defined on a probability space $(\Xi^t, \mathcal{A}^t, \Pr)$ with marginal support $\Xi^t \subseteq \mathbb{R}^{n^t}$ and given probability distribution $\Pr$ on the $\sigma$-algebra $\mathcal{A}^t$ (with $\mathcal{A}^t \subseteq \mathcal{A}^{t+1}$). The process $\xi^t$ is revealed gradually over discrete times in $H$ periods, and $\xi^t := (\xi^1, \ldots, \xi^t)$, $t = 1, \ldots, H - 1$ denotes the history of the process up to time $t$.

The decision variable at each discrete time is indicated with $x^t \in \mathbb{R}^{n^t}$, $t = 1, \ldots, H$. The decision $x^1$ is selected at time (stage) 1 before the future outcome of $\xi^1$ is revealed, the decision $x^t$ at stage $t = 2, \ldots, H$ is $\mathcal{A}^{t-1}$-measurable and it depends on the information up to time $t$. More precisely the decision process is nonanticipative, i.e. it has the form

\[
\text{decision}(x^1) \rightarrow \text{observation}(\xi^1) \rightarrow \text{decision}(x^2) \rightarrow \text{observation}(\xi^2) \rightarrow \ldots \\
\ldots \rightarrow \text{decision}(x^{t-1}) \rightarrow \text{observation}(\xi^{t-1}) \rightarrow \text{decision}(x^t) \rightarrow \ldots \\
\ldots \rightarrow \text{observation}(\xi^{H-1}) \rightarrow \text{decision}(x^H). 
\]

In the following $X$ denotes the Cartesian product among sets, and the Binomial distribution with parameters $\epsilon \in \mathbb{R}$, $N, n \in \mathbb{N}$, $N > n$, is denoted as $B(N, \epsilon, n + 1)$. 
2.2 Two-stage robust linear case

To simplify our exposition, we first analyze a simple two-stage robust linear program, formally defined as follows:

\[
\text{RO}_2 := \min_{x^1} c^T x^1 + \sup_{\xi^1 \in \Xi^1} \left[ \min_{x^2(\xi^1)} c^2^T (\xi^1) x^2(\xi^1) \right]
\] (1)

\[\text{s.t. } Ax^1 = h^1, \quad x^1 \geq 0 \]
\[
T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1), \quad x^2(\xi^1) \geq 0, \quad \forall \xi^1 \in \Xi^1,
\]

where \(c^1 \in \mathbb{R}^{m_1}\) and \(h^1 \in \mathbb{R}^{m_1}\) are known vectors and \(A \in \mathbb{R}^{m_1 \times n_1}\) is a given (known) matrix. The uncertain parameters vectors and matrices affected by the random process \(\xi^1\) are then given by \(h^2 \in \mathbb{R}^{m_2}, c^2 \in \mathbb{R}^{m_2}, T^1 \in \mathbb{R}^{m_2 \times n_1},\) and \(W^2 \in \mathbb{R}^{m_2 \times m_1} \).

The goal is to find a sequence of decisions \((x^1, x^2(\xi^1))\) that minimizes the cost function in the worst-case realization of \(\xi^1 \in \Xi^1\). The decision \(x^1\) is selected at time 1, before the future outcome of \(\xi^1\) has been revealed. The decision \(x^2\) at stage \(t = 2\) is \(\mathcal{F}^1\)-measurable and it depends only on the information up to time 2.

We first remark that problem (1) can equivalently be rewritten as follows:

\[
\text{RO}_2 = \min_{x^1} \left\{ c^1^T x^1 + \mathcal{Q}(x^1) \text{ s.t. } A x^1 = h^1, x^1 \geq 0 \right\},
\]

where \(\mathcal{Q}\) is the worst-case recourse function

\[
\mathcal{Q}(x^1) := \sup_{\xi^1 \in \Xi^1} \mathcal{Q}(x^1, \xi^1),
\]

being \(\mathcal{Q}(x^1, \xi^1)\) the (uncertain) recourse function

\[
\mathcal{Q}(x^1, \xi^1) := \min_{x^2(\xi^1)} \left\{ c^2^T(\xi^1) x^2(\xi^1) \text{ s.t. } T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1), x^2(\xi^1) \geq 0 \right\}.
\]

Our key observation is that problem RO\(_2\) can be restated in the form of a so-called robust with certificates RW\(_2\) problem, where we distinguish between design variables \((x^1, \gamma)\) and certificates \(x^2(\xi^1)\). This observation, which represents a first result of the paper, is crucial for our successive developments and it is proved in the following Theorem.

**Theorem 1** The robust two-stage linear program RO\(_2\) is equivalent to the following robust with certificates RW\(_2\) problem

\[
\text{RW}_2 := \min_{x^1, \gamma} \gamma
\]

\[\text{s.t. } A x^1 = h^1, x^1 \geq 0 \]
\[
\forall \xi^1 \in \Xi^1, \exists x^2(\xi^1) \in \mathbb{R}^{n_2} \text{ satisfying }
\]
\[
c^1^T x^1 + c^2^T(\xi^1) x^2(\xi^1) \leq \gamma
\]
\[
x^2(\xi^1) \geq 0, \quad T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1).
\]

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1 We adopt the convention of putting as pedices the number of stages of the problem, e.g. RO\(_2\) denotes a two-stage robust linear problem.
Proof. We first note that Problem RO₂ can be rewritten in epigraph form, by introducing the additional variable \( \gamma \), as follows

\[
\begin{align*}
\text{RO}_2 &= \min_{x^1, \gamma} \gamma \\
\text{s.t. } Ax^1 &= h^1, \quad x^1 \geq 0 \\
&\quad \quad \left[ \min_{x^2(\xi^1)} c^{2T}(\xi^1) x^2(\xi^1) \right] \leq \gamma, \quad \forall \xi^1 \in \Xi^1,
\end{align*}
\]

or, noting that \( c^{1T} x^1 \) does not depend on \( \xi^1 \), as

\[
\begin{align*}
\text{RO}_2 &= \min_{x^1, \gamma} \gamma \\
\text{s.t. } Ax^1 &= h^1, \quad x^1 \geq 0 \\
&\quad \quad \left[ \min_{x^2(\xi^1)} \left( c^{1T} x^1 + c^{2T}(\xi^1) x^2(\xi^1) \right) \right] \leq \gamma, \quad \forall \xi^1 \in \Xi^1,
\end{align*}
\]

or, equivalently, as

\[
\begin{align*}
\text{RO}_2 &= \min_{x^1, \gamma} \gamma \\
\text{s.t. } Ax^1 &= h^1, \quad x^1 \geq 0 \\
&\quad \quad (x^1, \gamma) \in \mathcal{X}_{\text{RO}_2}(\xi^1), \quad \forall \xi^1 \in \Xi^1,
\end{align*}
\]

where the set \( \mathcal{X}_{\text{RO}_2}(\xi^1) \) is defined as

\[
\mathcal{X}_{\text{RO}_2}(\xi^1) := \left\{ (x^1, \gamma) \in \mathbb{R}_+^{n_1+1} \mid \begin{array}{l}
\min_{x^2(\xi^1)} \left( c^{1T} x^1 + c^{2T}(\xi^1) x^2(\xi^1) \right) \\
\text{s.t. } x^2(\xi^1) \geq 0, \quad T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1)
\end{array} \leq \gamma \right\}.
\]

Similarly, Problem RwC₂ rewrites

\[
\begin{align*}
\text{RwC}_2 &= \min_{x^1, \gamma} \gamma \\
\text{s.t. } Ax^1 &= h^1, \quad x^1 \geq 0 \\
&\quad \quad (x^1, \gamma) \in \mathcal{X}_{\text{RwC}_2}(\xi^1), \quad \forall \xi^1 \in \Xi^1,
\end{align*}
\]

where the set \( \mathcal{X}_{\text{RwC}_2}(\xi^1) \) is defined as

\[
\mathcal{X}_{\text{RwC}_2}(\xi^1) := \left\{ (x^1, \gamma) \in \mathbb{R}_+^{n_1+1} \mid \begin{array}{l}
\exists x^2(\xi^1) \in \mathbb{R}_+^{n_2} \text{ satisfying} \\
\min_{x^2(\xi^1)} \left( c^{1T} x^1 + c^{2T}(\xi^1) x^2(\xi^1) \right) \leq \gamma \\
T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1)
\end{array} \right\}.
\]

So, we just need to prove that \( \mathcal{X}_{\text{RO}_2}(\xi^1) \equiv \mathcal{X}_{\text{RwC}_2}(\xi^1) \) for the minimum value of \( \gamma \).

- We prove that if \( (x^1, \gamma) \in \mathcal{X}_{\text{RO}_2} \), then \( (x^1, \gamma) \in \mathcal{X}_{\text{RwC}_2} \). If \( (x^1, \gamma) \in \mathcal{X}_{\text{RO}_2} \), then \( \exists x^2(\xi^1) \in \mathbb{R}_+^{n_2} \) such that \( T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1) \) is satisfied and

\[
\min_{x^2(\xi^1)} \left( c^{1T} x^1 + c^{2T}(\xi^1) x^2(\xi^1) \right) \leq \gamma,
\]

for the minimum value of \( \gamma \). Consequently \( (x^1, \gamma) \in \mathcal{X}_{\text{RwC}_2} \).
Conversely if \((x^1, \gamma) \in \mathcal{X}_{\text{RwC}_2}\), then we need to prove that \((x^1, \gamma) \in \mathcal{X}_{\text{RO}_2}\). If \((x^1, \gamma) \in \mathcal{X}_{\text{RwC}_2}\) then \(\exists x^2(\xi^1) \in \mathbb{R}^n_+\) such that \(T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1)\) and \(c^1^\top x^1 + c^2^\top (\xi^1)x^2(\xi^1) \leq \gamma\) for the minimum value of \(\gamma\). This implies that \(x^2(\xi^1)\) is the minimum of \(c^1^\top x^1 + c^2^\top (\xi^1)x^2(\xi^1)\).

By contradiction if \(x^2(\xi^1)\) were not be the minimum then \(\gamma\) would not be at the minimum of problem RwC_2.

Based on the result of Theorem 1 we are now ready to formulate the scenario with certificates counterpart of problem RO_2. To this end, we exploit the probabilistic information about the uncertainty and, similarly to what proposed in [38], we adopt a sampling approach, based on the extraction of \(N\) independent identically distributed (iid) samples

\[
\xi^{1(i)}, \ldots, \xi^{1(N)}
\]

of the random variable \(\xi^1\). The samples are extracted according to the probability distribution of the uncertainty over \(\Xi^1\). Let \(T^1(\xi^{1(i)}), h^2(\xi^{1(i)}), c^2(\xi^{1(i)})\) be the realization of \(T^1(\xi^1), h^2(\xi^1)\) and \(c^2(\xi^1)\) under scenario \(\xi^{1(i)}\), \(i = 1, \ldots, N\), and let \(x^2_i\) be the certificate variables created for the samples \(\xi^{1(i)}\), \(i = 1, \ldots, N\). These samples are used to construct the following *scenario with certificates* SwC_2^N problem based on \(N\) instances (scenarios) of the uncertain constraints

\[
\text{SwC}_2^N := \min_{x^1, \gamma, x^2_1, \ldots, x^2_N} \gamma
\]
\[\text{s.t. } Ax^1 = h^1, \quad x^1 \geq 0
\]
\[c^1^\top x^1 + c^2^\top (\xi^{1(i)}), x^2_i \leq \gamma
\]
\[T^1(\xi^{1(i)})x^1 + W^2(\xi^{1(i)})x^2_i = h^2(\xi^{1(i)}), \quad x^2_i \geq 0, \quad i = 1, \ldots, N.
\]

The solution of problem SwC_2^N is denoted with \((\hat{x}^2_N, \hat{\gamma}_N)\). We note that in problem SwC_2^N, a different *certificate* \(x^2\) is constructed for any sample \(\xi^{1(i)}\). The rationale behind this approach is the following: we are not interested in the explicit knowledge of the function \(x^2(\xi^1)\), what we are content with is that for every possible value of the uncertainty there exists a possible choice of \(x^2\) compatible with the ensuing realization of the constraints. In the SwC approach, this requirement is relaxed by asking that this is true only for the sampled scenario. Note that this represents a key difference with respect to other sampling based approaches. In particular, in [38] different explicit parameterizations of the function \(x^2(\xi^1)\) are introduced, of the form

\[
x^2(\xi^1) = \sum_{k=1}^{M} c_k \phi^2_k(\xi^1),
\]

where \(\phi^2_1, \ldots, \phi^2_M\) are specific basis functions, which can be for instance algebraic polynomials, trigonometric polynomials, sigmoidal or gaussian radial basis functions and \(c_k\) represent the coefficients of the linear combinations, which become the new decision variables. It is easy to infer how this latter approach is bound of being more conservative, since the existence of a solution with a pre-specified form is required.

It is clear that the approximate solution returned by problem SwC_2^N is optimistic, since it considers only a subset of possible scenarios. That is, the following bound holds for all \(N\):

\[
\text{SwC}_2^N \leq \text{RO}_2.
\]
Hence, we have derived a lower bound, which by construction is better than bounds derived using wait-and-see approaches, as discussed in Section 3. Moreover, it is easy to show that the formulation is consistent, that is

$$\lim_{N \to \infty} \text{SwC}_2^N = \text{RO}_2.$$ 

More importantly, we note that, by exploiting recent results in [21], it is possible to provide a formal assessment about its probabilistic properties. To this end, let formally introduce the violation probability \( V_2(x^1, \gamma) \) of \((x^1, \gamma)\) as follows

$$V_2(x^1, \gamma) := \Pr \left\{ \exists \xi^1 \in \Xi^1 \text{ for which } \exists x^2(\xi^1) \in \mathbb{R}^{n_2}_+ : \begin{bmatrix} c^1 x^1 + c^2(\xi^1) x^2(\xi^1) \leq \gamma \\ T^1(\xi^1) x^1 + W^2(\xi^1) x^2(\xi^1) = h^2(\xi^1) \end{bmatrix} \right\}.$$ 

The interpretation of the violation probability of the solution \( x^1 \) is as follows: if we select as first stage solution \( x^1 \), then \( V_2(x^1, \gamma) \) is the probability that at stage two we encounter an uncertainty realization \( \xi^1 \) for which there does not exist a feasible recourse decision \( x^2(\xi^1) \). Clearly, the smaller is \( V_2(x^1, \gamma) \), the higher is the probability that the solution at stage one will lead to a feasible stage two problem. We are in the position of providing a rigorous result connecting the violation probability to the number of samples \( N \) adopted in the construction of the \( \text{SwC}_2^N \) problem. The following theorem holds.

**Theorem 2 (two-stage robust linear case)** Assume that, for any multisample extraction, the problem \( \text{SwC}_2^N \) is feasible and attains a unique optimal solution. Then, given an accuracy level \( \epsilon \in (0, 1) \), the solution \((\hat{x}_N^1, \hat{\gamma}_N)\) of the problem (2) satisfies

$$\Pr \left\{ V_2(\hat{x}_N^1, \hat{\gamma}_N)_{\text{SwC}_2^N} > \epsilon \right\} \leq B(N, \epsilon, n_1 + 1),$$

where \( B(N, \epsilon, n_1 + 1) := \sum_{k=0}^{n_1} \binom{N}{k} \epsilon^k (1 - \epsilon)^{N-k} \).

The proof of Theorem 2 follows the same lines of the results presented in [21], and is reported in the Appendix. The theorem provides a way to a priori bound the probability of violation of the solution of \( \text{SwC}_2^N \). We remark that, in the literature, the minimum number of samples for which (4) holds for given \( \epsilon \in (0, 1) \) and \( \beta \in (0, 1) \) is referred to as sample complexity, see [36]. Several are the results derived in literature to bound sample complexity. In particular, in Lemma 1 and 2 in [1], it is proved that given \( \epsilon \in (0, 1) \) and \( \beta \in (0, 1) \)

$$N(\epsilon, \beta) \geq \frac{1}{e} \frac{\epsilon}{\epsilon - 1} \left( \ln \frac{1}{\beta} + n_0 + 1 \right),$$

where \( e \) is the Euler constant. This bound represents a (numerically) significant improvement upon other bounds available in the literature [13,14].

It is important to highlight that the number of samples \( N \) in formula (5) depends only on the dimension of non-adjustable variables (or design variables); thus it reduces the number of samples needed to satisfy a prescribed level of violation with respect to that proposed in [58]. Indeed, in the proof of Corollary 1 in [58], \( N \) depends on the size of the basis \( M \) and on the number of decision variables at each stage.

The results presented in this section can be readily extended to the more general case of dynamic multistage (\( H \)-stages) robust linear decision problem under uncertainty. This is done in the next section.
2.3 Multistage robust linear case

We consider the following robust linear program over $H$ stages

$$ \text{RO}_H := \min_{x^1, \ldots, x^H(\xi^{H-1}) \in \Xi^{H-1}} \sup_{\xi} \left[ (x^1, \ldots, x^H(\xi^{H-1}), \xi^{H-1}) \right] $$

$$ = \min_{x^1} c^1 x^1 + $$

$$ + \sup_{\xi} c^2(\xi) x^2(\xi) + \cdots $$

$$ = \min_{x^1} c^1 x^1 + $$

$$ + \sup_{\xi} c^2(\xi) x^2(\xi) + \cdots $$

$$ + \sup_{\xi} \left[ \cdots + \sup_{\xi^{H-1}} \left[ \cdots \right] \right] $$

$$ \text{s.t. } Ax^1 = h^1, x^1 \geq 0 $$

$$ T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1), \forall \xi^1 \in \Xi^1 $$

$$ \vdots $$

$$ T^{H-1}(\xi^{H-1})x^{H-1}(\xi^{H-2}) + W^H(\xi^{H-1})x^H(\xi^{H-1}) = h^H(\xi^{H-1}), \forall \xi^{H-1} \in \Xi $$

where $c^1 \in \mathbb{R}^{n_1}$ and $h^1 \in \mathbb{R}^{m_1}$ are known vectors and $A \in \mathbb{R}^{m_1 \times n_1}$ is known matrix. The uncertain parameter vectors and matrices affected by the random process $\xi^t$ are then given by $h^t \in \mathbb{R}^{m_1}$, $c^t \in \mathbb{R}^{n_t}$, $T^{t-1} \in \mathbb{R}^{m_{t-1} \times n_t}$, and $W^t \in \mathbb{R}^{m_t \times m_{t-1}}$, $t = 2, \ldots, H$.

The aim of the problem $\text{RO}_H$ is to find a sequence of decisions $(x^1, \ldots, x^H)$ that minimizes a cost function in the worst-case realization of $\xi^{H-1} \in \Xi = \bigotimes_{t=1}^{H-1} \Xi^t$. The decision process is nonanticipative and depends on the information up to time $t$ as described in Section 2.1.

Similarly to the two-stage case, we first rewrite problem (6) as the multistage robust optimization problem with certificates $\text{RwC}_H$, where we distinguish between design variables $x^1, \gamma$ and nonanticipative certificates $(x^2(\xi^1), \ldots, x^H(\xi^{H-1}))$ as follows

$$ \text{RwC}_H := \min_{x^1, \gamma} \gamma $$

$$ \text{s.t. } \forall \xi^{H-1} \in \Xi, \exists x^t(\xi^{t-1}) \in \mathbb{R}^{n_t}, t = 2, \ldots, H \text{ satisfying} $$

$$ c^1 x^1 + c^2(\xi^1) x^2(\xi^1) + \cdots + c^H(\xi^{H-1}) x^H(\xi^{H-1}) \leq \gamma $$

$$ Ax^1 = h^1, x^1 \geq 0 $$

$$ T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1) $$

$$ \vdots $$

$$ T^{H-1}(\xi^{H-1})x^{H-1}(\xi^{H-2}) + W^H(\xi^{H-1})x^H(\xi^{H-1}) = h^H(\xi^{H-1}). $$

The equivalence of problems $\text{RO}_H$ and $\text{RwC}_H$ is formally stated in the following theorem, which represents a generalization of Theorem 1 to the multistage case. The proof follows the same lines and is reported in Appendix B.

**Theorem 3** The multistage linear program $\text{RO}_H$ (6) is equivalent to the robust with certificates $\text{RwC}_H$ (7) problem.
Again, the previous theorem is very important in that it allows to reformulate problem RO$_H$ using the scenario with certificates approach. For this purpose, we extract $N$ iid samples $\xi^{H-1(1)}, \ldots, \xi^{H-1(N)}$ according to the probability distribution of the uncertainty over $\Xi$, where

$$\xi^{H-1(i)} = (\xi^{1(i)}, \ldots, \xi^{H-1(i)}), \quad i = 1, \ldots, N.$$

Let $T^{t-1}(\xi^{t-1(i)})$, $h^{t}((\xi^{t-1(i)}))$, $c^{t}(\xi^{t-1(i)})$ be the realization of $T^{t-1}(\xi^{t-1})$, $h^{t}(\xi^{t-1})$ and $c^{t}(\xi^{t-1})$ under scenario $\xi^{t-1(i)}$, $i = 1, \ldots, N$, $t = 2, \ldots, H$, and let $x^t_i$ be the certificate $x^t(\xi^{t-1(i)})$ created for the sample $\xi^{H-1(i)}$, $i = 1, \ldots, N$ taking into account the history of the process until period $t-1$. That is

$$x^t_i = x^t(\xi^{t-1(i)}), \quad t = 2, \ldots, H,$$

which means that the decision process is still nonanticipative. These samples are used to construct the following multistage scenario with certificates SwC$_H^N$ problem based on $N$ instances (scenarios) of the uncertain constraints

$$\text{SwC}_H^N := \min_{x^1, \gamma, x^1_1, \ldots, x^H} \gamma \tag{8}$$

s.t. $c^{1\top} x^1 + c^{2\top} (\xi^{1(i)}) x^2_1 + \cdots + c^{H\top} (\xi^{H-1(i)}) x^H_1 \leq \gamma, \quad i = 1, \ldots, N$

$$Ax^1 = h^1, \quad x^1 \geq 0$$

$$T^1(\xi^{1(i)}) x^1 + W^2(\xi^{1(i)}) x^2_1 = h^2(\xi^{1(i)}), \quad i = 1, \ldots, N$$

$$\vdots$$

$$T^{H-1}(\xi^{H-1(i)}) x^{H-1}_1 + W^H(\xi^{H-1(i)}) x^H_1 = h^H(\xi^{H-1(i)}), \quad i = 1, \ldots, N$$

$$x^t_i \geq 0, \quad t = 2, \ldots, H, \quad i = 1, \ldots, N.$$

The solution of problem SwC$_H^N$ is denoted with $(\hat{x}_N^1, \hat{\gamma}_N)$.

**Remark 1 (Scenario construction)** Note that the type of scenario construction proposed by the implementation of problem SwC$_H^N$ differs from the classical scenario trees proposed in literature. Indeed, instead of generating a few possible “leaves” at every stage, and considering the tree obtained from all possible combinations, we sample $N$ different “paths”. This procedure is illustrated in Figure 1 which shows the construction of SwC$_H^N$ from RwC$_H$ in the case of a three-stage robust optimization problem in which the first and second period uncertainties are discrete and can take a finite number of possible values. This allows to visualize the tree of all possible solution (left figure). The figure on the right shows the paths generated by a scenario with certificates SwC$_H^3$, based on $N = 4$ samples (thick lines) of the uncertain constraints in the initial problem. $(\hat{x}_3^1, \hat{\gamma}_3)$ represent the design variables solution of SwC$_H^3$, and $(x^2(\xi^{1(1)}), x^3(\xi^{2(1)}))$ the certificates over the samples $i = 1, 2, 3, 4$.

Notice that, the nonanticipativity constraints have to be imposed, which in our case translate in requiring that $x^2(\xi^{1(2)}) = x^2(\xi^{1(4)})$.

Again, by construction, the following bounds hold

$$\text{SwC}_H^{N_1} \leq \text{SwC}_H^{N_2} \leq \text{RO}_H, \quad 1 \leq N_1 \leq N_2, \tag{9}$$
Fig. 1 Example of three-stage robust optimization problem solved through a scenario with certificates approach. In this case, the first and second period uncertainties $\xi_1$ and $\xi_2$ can assume the values \{1, 2, 3, 4, 5\}, with equal probability. On the left, the complete (robust) tree for problem RO is shown. On the right, the SwC, based on the extraction of $N = 4$ samples (thick lines) of the uncertain constraints in the initial problem is shown. In the example, the sampled uncertainties extracted are $\xi_2^{(1)} = (3, 4)$; $\xi_2^{(2)} = (5, 2)$; $\xi_2^{(3)} = (2, 1)$; $\xi_2^{(4)} = (5, 5)$. The quantities $x^2(\xi_1(i))$, $x^3(\xi_2(i))$ represent the certificates over the samples $i = 1, 2, 3, 4$. Notice that the extracted samples $\xi_1^{(2)} = \xi_1^{(4)} = 5$ coincide, and the scenario with certificates SwC is constructed accordingly so to satisfy the non-anticipativity constraint $x^2(\xi_1^{(2)}) = x^2(\xi_1^{(4)})$. 
where we explicitly highlight that the lower bound improves for increasing values of $N$. In particular, it can be shown that
\[
\lim_{N \to \infty} \text{SwC}^N_H = \text{RO}_H.
\]

Moreover, similarly to the two-stage case, we can formally investigate the probabilistic properties of the approximate solution returned by problem SwC$^N_H$. To this end, we introduce the reliability $R_H(x^1, \gamma)$ and violation probability of the scenario with certificates problem as follows

\[
V_H(x^1, \gamma) := \Pr \left\{ \exists \xi^{H-1} \in \Xi \text{ for which } \exists x^t(\xi^{t-1}) \in \mathbb{R}^n_t, t = 2, \ldots, H : \right. \\
\epsilon_1 x^1 + c^2(\xi^1)x^2(\xi^1) + \ldots + c^H(\xi^{H-1})x^H(\xi^{H-1}) \leq \gamma \\
T^{t-1}(\xi^{t-1})x^{t-1}(\xi^{t-2}) + W^t(\xi^{t-1})x^t(\xi^{t-1}) = h^t(\xi^{t-1}), \\
t = 2, \ldots, H \right\}.
\]

We provide now a sample complexity result for the multistage robust linear case which extends Theorem 2 for the two-stage robust linear case. The proof is given in Appendix B.

**Theorem 4 (multistage robust linear case)** Assume that, for any multisample extraction, problem SwC$^N_H$ is feasible and attains a unique optimal solution. Then, given an accuracy level $\epsilon \in (0, 1)$, the solution $(\hat{x}^1, \hat{\gamma})$ of problem (8) satisfies

\[
\Pr \left\{ V(\hat{x}^1, \hat{\gamma}) > \epsilon \right\} \leq B(N, \epsilon, n_0 + 1),
\]

where $B(N, \epsilon, n_0 + 1) := \sum_{k=0}^{n_0} \binom{N}{k} \epsilon^k (1 - \epsilon)^{N-k}$.

We note that the sample complexity for guaranteeing with high probability $(1 - \beta)$ that the solution of problem SwC$^N_H$ has a violation probability bounded by $\epsilon$ can be computed by (8). It is important to remark again that, also in the multistage case, the necessary number of samples $N$ does not depend on the number of stages $H$. This is in sharp contrast with the setup in [38], in which $N$ depends on $\sum_{i=0}^H n_i \times M_i$, that is on the number of decision variables at each stage multiplied by the number of basis functions chosen for each stage. On the other hand, problem SwC$^N_H$ introduces an increment in the number of variables, since new variables are introduced for each stage. This growth can be easily handled in the case of linear programs, which constitute the main focus of this paper. We observe however that the SwC setup can be easily extended to the general context of convex multistage problems. This is briefly outlined in the next section.

### 2.4 Extension to the multistage robust convex case

In this section we further generalize the formulation given in Section 2.3 to a dynamic multistage ($H$-stages) robust convex decision problem under uncertainty, which can be formulated as follows

\[
\text{CRO}_H := \min_{x^1, x^2(\xi^1), \ldots, x^H(\xi^{H-1}) \in \Xi} \sup_{\xi^{H-1} \in \Xi} f(x^1, x^2(\xi^1), \ldots, x^H(\xi^{H-1}), x^{H-1}) \\
\text{s.t. } g(x^1, x^2(\xi^1), \ldots, x^H(\xi^{H-1}), x^{H-1}) \leq 0, \forall \xi^{H-1} \in \Xi \\
x^1 \geq 0, \quad x^t(\xi^{t-1}) \geq 0, \quad t = 2, \ldots, H,
\]
where \( f : \mathbb{R}^{\sum_{i=1}^{n_t} n_t} \times \Xi \to \mathbb{R} \) and \( g : \mathbb{R}^{\sum_{i=1}^{n_t} n_t} \times \Xi \to \mathbb{R} \) are convex in \( x^t \in \mathbb{R}^{n_t}, t = 1, \ldots, H \) and continuous in \((x^t, \xi^{H-1})\). Again we assume that the decision process in nonanticipative according to the description given in Section 2.1.

The aim of problem \( \text{CRO}_H \) is to find a sequence of decisions \((x^1, \ldots, x^H)\) that minimizes a cost function \( f \) in the worst-case realization of \( \xi^{H-1} \in \Xi = \bigcup_{i=1}^{H-1} \Xi' \). First, we observe that problem \( \text{CRO}_H \) can be rewritten as the following convex robust optimization problem with certificates \( \text{CRwC}_H \), where again we distinguish between design variables \( x^1, \gamma \) and certificates \((x^2(\xi^1), \ldots, x^H(\xi^{H-1}))\) as follows

\[
\text{CRwC}_H := \min_{x^1, \gamma} \gamma \\
\text{subject to} \quad \forall \xi^{H-1} \in \Xi, \exists x^t(\xi^{t-1}) \in \mathbb{R}^{n_t}, t = 2, \ldots, H \text{ satisfying} \\
f(x^1, x^2(\xi^1), \ldots, x^H(\xi^{H-1}), \xi^{H-1}) \leq \gamma \\
g(x^1, x^2(\xi^1), \ldots, x^H(\xi^{H-1}), \xi^{H-1}) \leq 0 \\
x^1 \geq 0, \quad x^t(\xi^{t-1}) \geq 0, \quad t = 2, \ldots, H.
\]

Then, we extract \( N \) iid samples \( \xi^{H-1(1)}, \ldots, \xi^{H-1(N)} \), and denote by \( x^t_i \) the certificate \( x^t(\xi^{t-1(i)}) \) created for sample \( \xi^{H-1(i)}, i = 1, \ldots, N \). This means that the decision process is still nonanticipative. These samples are used to construct the following \textit{multistage convex scenario with certificates} \( \text{CSwC}_H^N \) problem

\[
\text{CSwC}_H^N := \min_{x^1, \gamma; x^2_1, \ldots, x^H_1, \ldots, x^2_N, \ldots, x^H_N} \gamma \\
\text{subject to} \quad f(x^1, x^2_1, \ldots, x^H_1, \xi^{i(1)}) \leq \gamma, \quad i = 1, \ldots, N \\
g(x^1, x^2_1, \ldots, x^H_1, \xi^{i(1)}) \leq 0, \quad i = 1, \ldots, N \\
x^1 \geq 0, \quad x^t_i \geq 0, \quad t = 2, \ldots, H, \quad i = 1, \ldots, N.
\]

The solution of problem \( \text{CSwC}_H^N \) is denoted with \((\hat{x}^1_N, \gamma_N)\). In order to investigate probabilistic properties of the approximate solution returned by problem \( \text{CSwC}_H^N \) we introduce the violation probability \( V_H(x^1, \gamma) \) of its solution \((x^1, \gamma)\)

\[
V_H(x^1, \gamma) := \Pr \left\{ \exists \xi^{H-1} \in \Xi \text{ for which } \exists x^t(\xi^{t-1}) \in \mathbb{R}^{n_t}, t = 2, \ldots, H : \\
\left\{ f(x^1, x^2(\xi^1), \ldots, x^H(\xi^{H-1}), \xi^{H-1}) \leq \gamma \\
g(x^1, x^2(\xi^1), \ldots, x^H(\xi^{H-1}), \xi^{H-1}) \leq 0 \right\} \right\}.
\]

Then, the following sample complexity result for the multistage robust convex programs holds true:

\textbf{Corollary 1 (multistage robust convex case)} Assume that, for any multisample extraction, problem \( \text{CSwC}_H^N \) is feasible and attains a unique optimal solution. Then, given an accuracy level \( \epsilon \in (0, 1) \), the solution \((\hat{x}^1, \hat{\gamma})\) of problem \( \text{CSwC}_H^N \) satisfies

\[
\Pr \left\{ V_H(\hat{x}^1, \hat{\gamma}) > \epsilon \right\} \leq B(N, \epsilon, n_0 + 1)
\]

where \( B(N, \epsilon, n_0 + 1) := \sum_{k=0}^{n_0} (N \choose k) \epsilon^{k}(1 - \epsilon)^{N-k} \).

\textbf{Proof} The proof works similarly to the one of Theorem 4 for the multistage robust linear case and is omitted for brevity. \( \square \)
3 Lower Bounds for Multistage Linear Robust Optimization Problems

In this section, we present the robust counterpart of different lower bounds known in the context of stochastic programming, see for instance [20], [21] and [22]. To the best of our knowledge such relaxations, while frequently encountered when facing stochastic multistage problems, have never been formally stated in the context of robust programming. In particular, we here introduce and compare them in terms of optimal objective function values for the case of robust multistage linear programs. Similarly lower bounds for multistage convex robust programs can be defined.

First, we introduce the robust multistage wait-and-see problem RWS$^H$, where the realizations of all the random parameters are assumed to be known at the first stage, which takes the form

$$
RWS^H := \sup_{\xi^H} \min_{(x^1(\xi^H), \ldots, x^H(\xi^H))} z \left[ (x^1(\xi^H), \ldots, x^H(\xi^H)), \xi^H \right] 
$$

where with $z \left[ (x^1(\xi^H), \ldots, x^H(\xi^H)), \xi^H \right]$ we denote in a compact way the objective function and constraints of problem (12). Notice that, in the above setup, the minimum and supremum have been exchanged. Hence, the decision process is anticipative, since the decisions $x^1, x^2, \ldots, x^H$ depend on a given realization of $\xi^H$. We introduce the following definition, which is an immediate extension of the concept of Expected Value of Perfect Information for stochastic programs.

**Definition 1** The difference

$$
RVPI^H := RO^H - RWS^H, 
$$

denotes the Robust Value of Perfect Information and compares robust multistage wait-and-see RWS$^H$ and robust multistage RO$^H$.

The RVPI$^H$ can be interpreted as a measure of the advantage of reaching perfect information: a small RVPI$^H$ indicates a small advantage for reaching the perfect information since all possible realizations have similar costs. In particular, the following inequality can be proven.

**Proposition 1 (lower bound for RO$^H$)** Given the robust multistage linear optimization problem $RO^H$ defined in (9), and the robust multistage wait-and-see problem $RWS^H$ defined in (12), the following inequality holds true

$$
RWS^H \leq RO^H. 
$$

**Proof** For every realization, $\xi^H$, we have the relation

$$
z \left[ (x^1(\xi^H), \ldots, x^H(\xi^H)), \xi^H \right] \leq z \left[ (x^{1*}, \ldots, x^{H*}(\xi^H)), \xi^H \right],
$$

where $x^{1*}, \ldots, x^{H*}(\xi^H)$ denote the optimal solutions for the corresponding deterministic problems.
where, \( (x^1, \ldots, x^H) (\xi^{H-1}) \) denotes an optimal solution to the RO\(_H\) problem (6) and \( (\bar{x}^1(\xi^{H-1}), \ldots, \bar{x}^H(\xi^{H-1})) \) denotes the optimal solution for each realization of \( \xi^{H-1} \). Taking the supremum of both sides yields the required inequality. \( \square \)

A second lower bound for problem RO\(_H\) can be obtained by relaxing the nonanticipativity constraints only in stages 2, \ldots, H (see [26]). The ensuing program is the so-called robust two-stage relaxation RT\(_H\). Formally, consider the discrete random process as follows

\[
\tilde{\xi}^t := (\xi^1, \xi^2, \ldots, \xi^t), \quad t = 2, \ldots, H - 1,
\]

where \( \tilde{\xi}^t \) is a deterministic realization of the random process \( \xi^t \). For instance, for long processes, \( \tilde{\xi}^t, t = 2, \ldots, H - 1 \) can be chosen as the expected value of the random variable \( \xi^t \). We denote the robust two-stage relaxation problem RT\(_H\), as follows

\[
RT_H := \min_{\tilde{x}^1} c_1^T \tilde{x}^1 + \sup_{\xi^t \in \Xi^t} \left[ \min_{\tilde{x}^1, \ldots, \tilde{x}^H} c_2^T \tilde{x}^2 (\xi^1) + c_3^T \tilde{x}^3 (\xi^2) + \ldots + c_H^T \tilde{x}^H (\xi^{H-1}) \right]
\]

s.t. \( A \tilde{x}^1 = h^1, \tilde{x}^1 \geq 0 \)

\[
T^1 (\xi^1) x^1 + W^2 (\xi^1) x^2 (\xi^1) = h^2 (\xi^1), \quad \forall \xi^1 \in \Xi^1
\]

\[
\vdots
\]

\[
T^{H-1} (\xi^{H-1}) x^{H-1} (\xi^{H-1}) + W^H (\xi^{H-1}) x^H (\xi^{H-1}) = h^H (\xi^{H-1}), \quad \forall \xi^1 \in \Xi^1
\]

\[
x^t (\tilde{\xi}^{t-1}) \geq 0, \quad t = 2, \ldots, H, \quad \forall \xi^1 \in \Xi^1.
\]

Following reasonings similar to those in the proof Proposition 1, based on relaxation of constraints respectively in the first stage and in the following ones, the following bounds can be proven.

**Proposition 2 (Chain of lower bounds for RO\(_H\))** Given the robust multistage linear optimization problem RO\(_H\) (7), the robust multistage wait-and-see problem RWS\(_H\) (11) and the robust two-stage relaxation problem TP\(_H\), the following inequalities hold true

\[
RWS_H \leq RT_H \leq RO_H.
\]

We remark that, in the general case, both problems RWS\(_H\) and RT\(_H\) may be hard to solve. In such case, one can recur to sampled versions of them. In particular, we can introduce the sampled wait-and-see problem SWS\(_H^N\), based on the extraction of \( N \) iid samples \( \xi^{H-1(1)}, \ldots, \xi^{H-1(N)} \)

\[
SWS_H^N := \sup_{i=1, \ldots, N} \min_{x^1(\xi^{H-1(i)}), \ldots, x^H(\xi^{H-1(i)})} c_1^T x^1 (\xi^{H-1(i)}) + \ldots + c_H^T x^H (\xi^{H-1(i)})
\]

s.t. \( A \tilde{x}^1 = h^1, \tilde{x}^1 \geq 0 \)

\[
T^1 (\xi^{H-1(i)}) x^1 (\xi^{H-1(i)}) + W^2 (\xi^{H-1(i)}) x^2 (\xi^{H-1(i)}) = h^2 (\xi^{H-1(i)})
\]

\[
\vdots
\]

\[
T^{H-1} (\xi^{H-1(i)}) x^{H-1} (\xi^{H-1(i)}) + W^H (\xi^{H-1(i)}) x^H (\xi^{H-1(i)}) = h^H (\xi^{H-1(i)})
\]

\[
x^t (\xi^{H-1(i)}) \geq 0, \quad t = 2, \ldots, H.
\]
We note that probabilistic guarantees of the solution returned by problem SWS\(_H^N\) can be directly derived using the maximization bound in [35]. Similarly, one can extract \(N\) iid samples of the first period random process \(\xi^{1(i)}\), \ldots, \(\xi^{1(N)}\)
and construct the scenario with certificates version of the RT\(_H^N\) problem

\[
\text{SwCT}_{H}^{N} := \min_{x^1, \gamma, x^2_1, \ldots, x^N_0} \gamma
\]

\[
\text{s.t. } c^1^T x^1 + c^2^T x^1_1 + c^3^T x^1_i + \ldots + c^H^T x^H_i \leq \gamma, \quad i = 1, \ldots, N
\]

\[
A x^1 = h^1, \quad x^1 \geq 0
\]

\[
T^1(\xi^{1(i)}) x^1 + W^2(\xi^{1(i)}) x^2_1 = h^2(\xi^{1(i)}), \quad i = 1, \ldots, N
\]

\[
\vdots
\]

\[
T^{H-1}(\xi^{H-1-(i)}) x^{H-1}_i + W^H(\xi^{H-1-(i)}) x^H_i = h^H(\xi^{H-1-(i)}), \quad i = 1, \ldots, N
\]

\[
x^i_t \geq 0, \quad i = 1, \ldots, N, \quad t = 2, \ldots, H,
\]

where \(\xi^{H-1-(i)}\), \(i = 1, \ldots, N\) are iid samples of \(\xi^{H-1-}\). Again, probabilistic guarantees of the solution of problem SwCT\(_H^N\) being also a solution of RT\(_H^N\) can be obtained on the same lines of Theorem 4. We conclude this section by providing the following proposition, which shows the relationship between the various lower bounds based on sampling presented in this paper.

**Proposition 3 (Chain of sampling-based lower bounds for RO\(_H\))** Given the robust multistage linear optimization problem RO\(_H\) [9], the scenario with certificates relaxation SwC\(_H^N\) [8], the sampled multistage wait-and-see problem SWS\(_H^N\) [10], and the scenario with certificates two-stage relaxation SwCT\(_H^N\) [17], the following chain of inequalities holds true

\[
SWS_H^N \leq SwCT_H^N \leq SwC_H^N \leq RO_H.
\]

### 4 Numerical Results: Inventory Management with Cumulative Orders

In this section, to show the effectiveness of the proposed approach, we consider a problem from inventory management which was originally considered in [3], describing the negotiation of flexible contracts between a retailer and a supplier in the presence of uncertain orders from customers. In particular, we analyze the performance of the approach proposed in this paper on a simplified version discussed in [9] and in [35]. We remark that the considered numerical problem is such that the optimal solution of the multistage robust optimization problem can be assessed: this allows to evaluate the performance of the scenario with certificate approach.

The problem setting can be summarized as follows: a retailer received orders \(\xi^t\) at the beginning of each time period \(t \in \mathbb{T} = \{1, \ldots, H-1\}\), \(\xi^t\) represents the demand history up to time \(t\). The demand needs to be satisfied from an inventory with filling level \(s^t_{\text{inv}}\) by means of orders \(x^t\) at a cost \(d^t\) per unit of product. Unsatisfied demand may be backlogged at cost \(p^t\) and inventory may be held in the warehouse with a unitary holding cost \(h^t\). Lower and upper bounds on the orders \(\underline{x}^t\) and \(\overline{x}^t\) at each period as well as on the cumulative orders \(s^t_{\text{co}}\) and \(s^t_{\text{co}}\) up to period \(t\) are imposed. We assume that there is no demand at time \(t = 1\) and that the demand at time \(t\) lies within an interval centered around a nominal value \(\tilde{\xi}^t\) and uncertainty level \(\rho \in [0, 1]\) resulting in a
box uncertainty set as follows: $\Xi = \times_{t \in T} \{ \xi^t \in \mathbb{R} : |\xi^t - \bar{\xi}^t| \leq \rho \bar{\xi}^t \}$. Denoting with $x^t_c$ the retailer’s cost at stage $t$, the problem can be modeled as follows.

\[
\text{RO}_H(COC) := \min_{x_1^t, x_t^t, s_1^{co}, s_t^{co}} \left[ x_1^t + \max_{\xi \in \Xi} \sum_{t \in T} x^{t+1}_c(\xi^t) \right] \tag{19a}
\]

s.t. $x_1^t \geq d^1 x_1^t + \max \{ h^1 s^{inv}_1, -p^1 s^{inv}_1 \}$ \tag{19b}

$x^{t+1}_c(\xi^t) \geq d^{t+1} x^{t+1}_o(\xi^t) + \max \{ h^{t+1} s^{inv}_{t+1}(\xi^t), -p^{t+1} s^{inv}_{t+1}(\xi^t) \}$, $t = 1, \ldots, H-2$ \tag{19c}

$x^H_c(\xi^{H-1}) \geq \max \{ h^H s^{inv}_H(\xi^{H-1}), -p^H s^{inv}_H(\xi^{H-1}) \}$ \tag{19d}

$s^{inv}_1(\xi^1) = s^{inv}_1 + x^{1}_o - \xi^1$ \tag{19e}

$s^{inv}_{t+1}(\xi^t) = s^{inv}_t(\xi^{t-1}) + x^{t}_o(\xi^{t-1}) - \xi^t$, $t = 2, \ldots, H-1$ \tag{19f}

$s^{co}_1(\xi^1) = s^{co}_1 + x^{1}_o$ \tag{19g}

$s^{co}_{t+1}(\xi^t) = s^{co}_t(\xi^{t-1}) + x^{t}_o(\xi^{t-1})$, $t = 2, \ldots, H-1$ \tag{19h}

$x^{1}_o \leq x^{1}_o \leq \bar{x}^{1}_o$, $s^{1}_o \leq s^{1}_o \leq \bar{s}^{1}_o$ \tag{19i}

$x^{t}_o \leq s^{t}_o(\xi^{t-1}) \leq \bar{x}^{t}_o$, $s^{t}_o \leq s^{t}_o(\xi^{t-1}) \leq \bar{s}^{t}_o$, $t = 2, \ldots, H$. \tag{19j}

The objective function (19a) corresponds to minimize the worst-case cumulative cost. Constraints (19b)-(19d) define the stage-wise costs $x^t_c(\xi^t)$, $t = 1, \ldots, H$ while constraints (19e)-(19f) and (19g)-(19h) respectively define the dynamics of the inventory level and cumulative orders. Finally, constraints (19i)-(19j) denote the lower and upper bounds on the instantaneous and cumulative orders. Notice that the decision process is nonanticipative.
The corresponding multistage scenario with certificates formulation, based on the extraction of \( N \) samples of the uncertainty, becomes as follows

\[
\text{SwC}_H^N(COC) := \min_{x_{0,i}^t, x_{o,i}^t, x_{co}^t, s_{inv,i}^t} \gamma \\
\text{s.t. } \gamma \geq x_c^1 + \sum_{t \in T} x_{c,i}^{t+1}(\xi^{t(i)}) , \quad i = 1, \ldots, N \\
x_c^1 \geq d^1 x_0^1 + \max \{ h^1 s_{inv}^1, -p^1 s_{inv}^1 \} \\
x_{c,i}^{t+1}(\xi^{t(i)}) \geq d^{t+1} x_{o,i}^{t+1}(\xi^{t(i)}) + \\
+ \max \{ h^{t+1} s_{inv,i}^{t+1}(\xi^{t(i)}), -p^{t+1} s_{inv,i}^{t+1}(\xi^{t(i)}) \} , \quad t = 1, \ldots, H-2, \quad i = 1, \ldots, N \\
x_{c,i}^H(\xi^{H-1(i)}) \geq \max \{ h^H s_{inv,i}^H(\xi^{H-1(i)}), -p^H s_{inv,i}^H(\xi^{H-1(i)}) \} , \quad i = 1, \ldots, N \\
\quad s_{inv,i}^1(\xi^{i(i)}) = s_{inv}^1 + x_0^1 - \xi^1 , \quad i = 1, \ldots, N \\
\quad s_{inv,i}^{t+1}(\xi^{t(i)}) = s_{inv,i}^t(\xi^{t-1(i)}) + x_{o,i}^t(\xi^{t-1(i)}) - \xi^{t(i)} , \quad t = 2, \ldots, H-1, \quad i = 1, \ldots, N \\
\quad s_{co,i}^2(\xi^{i(i)}) = s_{co}^1 + x_{o,i}^1 , \quad i = 1, \ldots, N \\
\quad s_{co,i}^{t+1}(\xi^{t(i)}) = s_{co,i}^t(\xi^{t-1(i)}) + x_{o,i}^t(\xi^{t-1(i)}) , \quad t = 2, \ldots, H-1, \quad i = 1, \ldots, N \\
\quad \underline{x}_0^t \leq x_{o,i}^t \leq \bar{x}_0^t , \quad \underline{x}_0^1 \leq s_{co}^1 \leq \bar{x}_0^1 \\
\quad \underline{x}_o^t \leq x_{o,i}^t(\xi^{t-1(i)}) \leq \bar{x}_o^t , \quad t = 2, \ldots, H, \quad i = 1, \ldots, N \\
\quad \underline{s}_{co}^t \leq s_{co,i}^t(\xi^{t-1(i)}) \leq \bar{s}_{co}^t , \quad t = 2, \ldots, H, \quad i = 1, \ldots, N.
\]

We consider specific instances of problem RO\(_H(COC)\) as summarized in Table 1 under the assumption of two-stage \((H = 2)\) and a five-stage \((H = 5)\) time horizons. The data presents some slight modifications of the data presented in [38].

| Parameters | RO\(_H(COC)\) |
|------------|----------------|
| \(H\) | 2 / 5 |
| \((p^t, d^t, h^t)\) | \((11, 1, 10)\) |
| \(s_{inv}^1\) | 0 |
| \(\xi, t = 1, \ldots, H - 1\) | \(100 \left( 1 + \frac{1}{2} \sin \left( \frac{\pi (t-2)}{6} \right) \right) = (75, 100, 125, 143.3013)\) |
| \(\rho\) | 30% |

| Parameters | RO\(_H(COC)\) |
|------------|----------------|
| \((\xi^{t-1(i)}), \xi^{t(i)}\) | \((47, 134, 188, 429)\) |
| \((\bar{s}_{co}^1, \bar{s}_{co}^2, \bar{s}_{co}^H)\) | \((94, 248, 370, 586)\) |

Table 1  Input data for the inventory management problem.

We define optimality gaps of the problem SwC\(_H^N(COC)\) as

\[
\text{optimality gap} := \frac{\inf \text{SwC}_H^N(COC) - \inf \text{RO}_H(COC)}{\inf \text{RO}_H(COC)}.
\]
Table 2  Vertices of $\Xi$ for the management inventory problem in the two-stage case ($H = 2$).

|   | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ |
|---|----------|----------|----------|----------|
| 1 | 52.5     | 70       | 87.5     | 100      |
| 2 | 52.5     | 70       | 87.5     | 186      |
| 3 | 52.5     | 70       | 163      | 100      |
| 4 | 52.5     | 70       | 163      | 186      |
| 5 | 52.5     | 130      | 87.5     | 100      |
| 6 | 52.5     | 130      | 87.5     | 186      |
| 7 | 52.5     | 130      | 163      | 100      |
| 8 | 52.5     | 130      | 163      | 186      |
| 9 | 97.5     | 70       | 87.5     | 100      |
| 10| 97.5     | 70       | 87.5     | 186      |
| 11| 97.5     | 70       | 163      | 100      |
| 12| 97.5     | 70       | 163      | 186      |
| 13| 97.5     | 130      | 87.5     | 100      |
| 14| 97.5     | 130      | 87.5     | 186      |
| 15| 97.5     | 130      | 163      | 100      |
| 16| 97.5     | 130      | 163      | 186      |

Table 3  Vertices of $\Xi$ for the management inventory problem in the five-stage case ($H = 5$).

We note that the optimality gap in (20) can be computed, since problem $RO_H(COC)$ can be solved exactly by using a scenario tree that consists of the vertices of $\Xi$ reported in Tables 2 and 3.

To assess the performance of our approach, we compute the empirical violation probability of a given solution $(x^1, \hat{\gamma})$, defined as follows:

$$\hat{V}_H(x^1, \hat{\gamma}) := \frac{1}{L^H} \sum_{\ell=1}^{L} \left\{ \xi_{t}^{H-1} \in \Xi \text{ s.t.} \right. $$

$$\left. \begin{array}{l}
\frac{1}{N} \sum_{i=1}^{N} 1 \left[ x^t(\xi_{t}^{H-1(i)}) \in \mathbb{R}^n_t, \ t = 2, \ldots, H \right. \text{satisfying} \\
\left. c_{1}^{T} x^1 + c_{2}^{T} (\xi_{t}^{2(i)}) x^2 (\xi_{t}^{1(i)}) + \cdots + c_{H}^{T} (\xi_{t}^{H-1(i)}) x^H (\xi_{t}^{H-1(i)}) \leq \gamma \right] \\
T_{t-1}(\xi_{t}^{H-1(i)}) x^{t-1}(\xi_{t}^{H-1(i)}) + W_{t} x^t (\xi_{t}^{H-1(i)}) = h^t(\xi_{t}^{H-1(i)}), \ t = 2, \ldots, H \} \right\}, 
\right. $$

where $1$ is the indicator function counting the number of scenarios where the constraints are not satisfied and $\{\xi_{t}^{H-1(i)}\}_{\ell=1,\ldots,L}$ is a sequence of $L$ independent sets distributed according to $Pr$.

The $\ell$-th-sample is composed by scenarios $\xi_{t}^{H-1(1)}, \ldots, \xi_{t}^{H-1(N)}$. Notice that these samples are independent of the $N$ samples $\xi_{t}^{H-1(i)}$ used in problem (8) to obtain solution $(\hat{x}^1, \hat{\gamma})$.

The numerical results are obtained as follows:
- we fix a confidence level of $\beta = 0.1\%$ for the constraint sampling;
- we select the target violation probability $\epsilon = 0.00025, 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.2, 0.3,$ and compute the corresponding sample size $N = N(\epsilon, \beta)$ according to formula (5) rounded up to the next integer;
- we solve 100 instances of problem $SwC_N$ each based on a different number $N$ of independent samples as computed in the previous point;
- for each instance, we compute the optimality gap given in formula (20) and empirical violation probability given in formula (4) with $L = 1000$;
- we compute statistics over the 100 instances.

The problems derived from the case study have been formulated and solved under AMPL environment along CPLEX 12.5.1.0 solver. All computations have been performed on a 64-bit machine with 12 GB of RAM and a Intel Core i7-3520M CPU 2.90 GHz processor.

First, we evaluate the performance of the scenario with certificates $\text{SwC}_N^H(COC)$ approach. Figures 2 and 3 display the optimality gaps of problems $\text{RO}_H(COC)$ with respect to inf $\text{RO}_H(COC)$ for different values of violation probability $\epsilon$ (%) ranging from 30% up to 0.025% respectively for the two ($H = 2$) and five-stage ($H = 5$) cases. Number of samples $N$, constraints and variables of the corresponding optimization models are reported in Table 4.

| $\epsilon$ (%) | $N$ | # of const. ($H = 2$) | # of var. ($H = 2$) | # of const. ($H = 5$) | # of var. ($H = 5$) |
|----------------|-----|----------------------|---------------------|----------------------|---------------------|
| 30             | 63  | 756                  | 442                 | 2772                 | 1198                |
| 20             | 95  | 1140                 | 666                 | 8136                 | 1806                |
| 10             | 189 | 2268                 | 1324                | 8316                 | 3592                |
| 5              | 377 | 4524                 | 2649                | 16588                | 7164                |
| 1              | 1884| 22608                | 13189               | 82896                | 35797               |
| 0.5            | 3768| 45216                | 26377               | 165792               | 71593               |
| 0.1            | 18838| 226056               | 131867              | 414433               | 339085              |
| 0.05           | 37676| 452112               | 263733              | 828869               | 678169              |
| 0.025          | 73552| 904224               | 527465              | 1657741              | 1356337             |

Table 4 Number of samples $N$, constraints and variables for decreasing values of $\epsilon$ (%) both in the two-stage (columns 3 and 4) and five stage (columns 5 and 6) cases.

From the results shown in Figure 2 and 3 we can observe that the variance of $\text{SwC}_N^H(COC)$ decreases substantially as $\epsilon$ decreases as well as the optimality gaps passing from $-2\%$ (in average) to $-10^{-3}\%$ for the two-stage case and from $-34\%$ (in average) to $-21\%$ for the five-stage case. It should be observed that, for the same given level of allowed violation $\epsilon$, the $\text{SwC}_N^H(COC)$ cost will always be lower than the solution returned by the approach in [28] (the reader is referred to the example proposed in that paper for comparison). This is due to the conservatism introduced in [28] by the fact that the solution is constrained to variables for stages 2, . . . , 5 with a special structure, and it is the reason why we have larger optimality gaps. We stress that this is a desirable feature, since we find a better result using the same level of probability.

We also note that, since the uncertainty lies in continuous intervals, we have a probability close to zero to get twice the same sample. Consequently, the nonanticipativity constraints in problem $\text{SwC}_5^H(COC)$ are not required to be satisfied by our data, with resulting lower costs in term of objective function values. This was not the case with $H = 2$. Based on this observation, we performed a second computational test, shown in Figure 4 where the demand $\xi_t$ $t = 1, \ldots, 4$ is assumed to take only integer values in the intervals $[52.5, 97.5], [70, 130], [87.5, 163], [100, 186]$. In this way, we increase the probability of having repeated samples, thus enforcing nonanticipativity constraints. Results show that the optimality gaps are now reduced passing from $-33\%$ (on average) to $-15\%$ for the five-stage case.

In Figures 5 and 6, we plot the distribution of the empirical violation probability as function of $\epsilon$, for the two-stage ($H = 2$) case and the five-stage ($H = 5$) case. As expected, as $\epsilon$ decreases, the violation converges to 0. We also note that the empirical violation probability is smaller than $\epsilon$ in all the considered cases.
Fig. 2 Optimality gaps for $\text{SwC}_2^N(\text{COC})$ (boxes and whiskers) for decreasing values of $\epsilon$ for the two-stage ($H = 2$) case.

Fig. 3 Optimality gaps for $\text{SwC}_5^N(\text{COC})$ (boxes and whiskers) for decreasing values of $\epsilon$ for the five-stage ($H = 5$) case.

Finally, Figures [7 and 8 show the average solver time (solid lines) and number of samples (dashed lines) for problems $\text{SwC}_2^N(\text{COC})$ and $\text{SwC}_5^N(\text{COC})$ as a function of $1/\epsilon$. Notice that the required
Sampling methods for multistage robust convex optimization problems

![5-stage case (H=5), integer demand](image1)

Fig. 4 Optimality gaps for SwC₅(\text{COC}) (boxes and whiskers) for decreasing values of \( \epsilon \) for the five-stage \((H = 5)\) case with integer demand over stages.

![2-stage case (H=2)](image2)

Fig. 5 Empirical violation probability for SwC₂(\text{COC}) (boxes and whiskers) for increasing values of \( \epsilon \) for the two-stage \((H = 2)\) case.

The number of samples obtained using formula (5), corresponding to a prescribed level of violation probability does not depend on the number of stages and dimension of the certificates variables.
Consequently, the number of samples shown in Table 4 are the same both for the two and five stage cases. In particular, they are considerably lower than those used in [38], where the number of samples depends on the size of the basis and on the number of decision variables at each stage. On the other hand, we should remark that the number of variables used in our approach is larger, due to the introduction of sample-dependent certificates.

4.1 Bounds for the Inventory management with cumulative orders

In this section, we evaluate possible relaxations to problem $\text{RO}_H(COC)$ as described in Section 3. In particular we consider the multistage wait-and-see problem $\text{RWS}_H(COC)$ for problem $\text{RO}_H(COC)$, and the robust two-stage relaxation problem $\text{RT}_H(COC)$ where the nonanticipativity constraints are relaxed in stages 2, ..., $H$. Again, we remark that for the case at hand these two problems can be computed exactly by considering only the vertices of $\Xi$. Similarly to formula (20), we define optimality gaps of the problem $\text{RWS}_H(COC)$ as

$$ (\text{optimality gap})_{\text{RWS}_H(COC)} := \frac{\inf \text{RWS}_H(COC) - \inf \text{RO}_H(COC)}{\inf \text{RO}_H(COC)}. $$

(21)

and similarly for $\text{RT}_H(COC)$. Both optimality gaps of $\text{RWS}_5(COC)$ and of $\text{RT}_5(COC)$ turned out to be equal to $-0.170171593$, passing from an objective function value of 2207.554108 for $\text{RO}_5(COC)$ to 1831.891109. Consequently the Robust Value of Perfect Information $\text{RVP}_5$ is 375.66299.
Fig. 7 Mean solver times (solid lines) and number of samples (dashed lines) as a function of $\ln(1/\epsilon)$ for problem SwC$^N_C$ for the two-stage ($H = 2$) case.

Fig. 8 Mean solver times (solid lines) and number of samples (dashed lines) as a function of $\ln(1/\epsilon)$ for problem SwC$^N_C$ for the five-stage ($H = 5$) case.
Figure 9 displays optimality gaps for the two-stage relaxation scenario with certificates problem SwCT$_5^N$ (COC) with respect to the robust two-stage relaxation problem RT$_5^N$ (COC) for different values of violation probability $\epsilon$ (%) ranging from 30% up to 0.01%. From the results we can observe that the variance of SwCT$_5^N$ (COC) decreases substantially as $\epsilon$ decreases as well as the optimality gaps passing from $-21\%$ (in average) to $-8\%$. Note that the smaller values of optimality gaps compared to the ones obtained in Figure 3 for problem SwC$_5^N$ (COC) are mainly due to the fact that in problem SwCT$_5^N$ (COC) the nonanticipativity constraints are relaxed. Finally Figure 10 refers to the empirical violation probability of SwCT$_5^N$ (COC) with respect to the robust two-stage relaxation problem RT$_5$ (COC). As expected as $\epsilon$ decreases it converges to 0. We again note that the empirical violation probability is smaller than $\epsilon$ in all the cases considered.

Fig. 9 Optimality gaps for SwCT$_5^N$ (COC) with respect to the robust two-stage relaxation problem RT$_5^N$ (COC), (boxes and whiskers) for decreasing values of $\epsilon$ for the five-stage $(H = 5)$ case.

5 Conclusions

In this paper probabilistic guarantees for constraint sampling in multistage convex robust optimization problems have been proposed. The scenario with certificates approach has been considered to treat the dynamic nature of convex multistage robust optimization problems. A multistage robust convex optimization problem has been proved to be equivalent to a convex robust optimization problem with certificates and a bound on the probability of violation for the scenario with certificates approach has been provided. The proposed approach has the important advantage to avoid the conservative use of parametrization through decision rules proposed in literature, implying a
We first prove the convexity of the set $X_{\text{RwC}_2}(\xi^1)$ defined above. Given $\hat{x}^1, \tilde{x}^1 \in X_{\text{RwC}_2}(\xi^1)$, then there exist $\hat{x}^2(\xi^1), \tilde{x}^2(\xi^1)$ such that

$$
\begin{align*}
T^1(\xi^1)\hat{x}^1 + W^2(\xi^1)\hat{x}^2(\xi^1) &= h^2(\xi^1), \\
T^1(\xi^1)\tilde{x}^1 + W^2(\xi^1)\tilde{x}^2(\xi^1) &
\leq \gamma
\end{align*}
$$

Consider now $x^{1\lambda} := \lambda\hat{x}^1 + (1-\lambda)\tilde{x}^1$, with $\lambda \in [0, 1]$, and let $x^{2\lambda} := \lambda\hat{x}^2 + (1-\lambda)\tilde{x}^2$. Then

$$
T^1(\xi^1)x^{1\lambda} + W^2(\xi^1)x^{2\lambda} = T^1(\xi^1)(\lambda\hat{x}^1 + (1-\lambda)\tilde{x}^1) + W^2(\xi^1)(\lambda\hat{x}^2 + (1-\lambda)\tilde{x}^2) \\
= \lambda\langle T^1(\xi^1)\hat{x}^1 + W^2(\xi^1)\hat{x}^2(\xi^1) \rangle + (1-\lambda)\langle T^1(\xi^1)\tilde{x}^1 + W^2(\xi^1)\tilde{x}^2(\xi^1) \rangle \\
= \lambda h^2(\xi^1) + (1-\lambda)h^2(\xi^1) \\
= h^2(\xi^1),
$$

and

$$
\begin{align*}
c^1^T x^{1\lambda} + c^2^T (\xi^1)x^{2\lambda} &= c^1^T (\lambda\hat{x}^1 + (1-\lambda)\tilde{x}^1) + c^2^T (\xi^1)(\lambda\hat{x}^2 + (1-\lambda)\tilde{x}^2) \\
&= \lambda\langle c^1^T \hat{x}^1 + c^2^T (\xi^1)\hat{x}^2(\xi^1) \rangle + (1-\lambda)\langle c^1^T \tilde{x}^1 + c^2^T (\xi^1)\tilde{x}^2(\xi^1) \rangle \\
&\leq \lambda \gamma + (1-\lambda)\gamma \\
&= \gamma,
\end{align*}
$$

By definition of the sets $X_{\text{RwC}_2}(\xi^1)$ and $X_{\text{RwC}_2}(\xi^1)$, we have

$$
\begin{align*}
\sum_{\lambda \in [0, 1]} \lambda h^2(\xi^1) + (1-\lambda)h^2(\xi^1) &\leq \gamma \\
&= \gamma
\end{align*}
$$

Thus, the set $X_{\text{RwC}_2}(\xi^1)$ is convex.

A Proof of Theorem 2

We first prove the convexity of the set $X_{\text{RwC}_2}(\xi^1)$ defined above. Given $\hat{x}^1, \tilde{x}^1 \in X_{\text{RwC}_2}(\xi^1)$, then there exist $\hat{x}^2(\xi^1), \tilde{x}^2(\xi^1)$ such that

$$
\begin{align*}
T^1(\xi^1)\hat{x}^1 + W^2(\xi^1)\hat{x}^2(\xi^1) &= h^2(\xi^1), \\
T^1(\xi^1)\tilde{x}^1 + W^2(\xi^1)\tilde{x}^2(\xi^1) &
\leq \gamma
\end{align*}
$$

Consider now $x^{1\lambda} := \lambda\hat{x}^1 + (1-\lambda)\tilde{x}^1$, with $\lambda \in [0, 1]$, and let $x^{2\lambda} := \lambda\hat{x}^2 + (1-\lambda)\tilde{x}^2$. Then

$$
T^1(\xi^1)x^{1\lambda} + W^2(\xi^1)x^{2\lambda} = T^1(\xi^1)(\lambda\hat{x}^1 + (1-\lambda)\tilde{x}^1) + W^2(\xi^1)(\lambda\hat{x}^2 + (1-\lambda)\tilde{x}^2) \\
= \lambda\langle T^1(\xi^1)\hat{x}^1 + W^2(\xi^1)\hat{x}^2(\xi^1) \rangle + (1-\lambda)\langle T^1(\xi^1)\tilde{x}^1 + W^2(\xi^1)\tilde{x}^2(\xi^1) \rangle \\
= \lambda h^2(\xi^1) + (1-\lambda)h^2(\xi^1) \\
= h^2(\xi^1),
$$

and

$$
\begin{align*}
c^1^T x^{1\lambda} + c^2^T (\xi^1)x^{2\lambda} &= c^1^T (\lambda\hat{x}^1 + (1-\lambda)\tilde{x}^1) + c^2^T (\xi^1)(\lambda\hat{x}^2 + (1-\lambda)\tilde{x}^2) \\
&= \lambda\langle c^1^T \hat{x}^1 + c^2^T (\xi^1)\hat{x}^2(\xi^1) \rangle + (1-\lambda)\langle c^1^T \tilde{x}^1 + c^2^T (\xi^1)\tilde{x}^2(\xi^1) \rangle \\
&\leq \lambda \gamma + (1-\lambda)\gamma \\
&= \gamma,
\end{align*}
$$

By definition of the sets $X_{\text{RwC}_2}(\xi^1)$ and $X_{\text{RwC}_2}(\xi^1)$, we have

$$
\begin{align*}
\sum_{\lambda \in [0, 1]} \lambda h^2(\xi^1) + (1-\lambda)h^2(\xi^1) &\leq \gamma \\
&= \gamma
\end{align*}
$$

Thus, the set $X_{\text{RwC}_2}(\xi^1)$ is convex.
which proves the convexity of $\mathcal{X}_{RwC_2}(\xi^1)$. From Theorem 1 we observe that the condition $(x^1, \gamma) \in \mathcal{X}_{RwC_2}(\xi^1)$ is equivalent to $(x^1, \gamma) \in \mathcal{X}_{RO_2}(\xi^1)$ and that the problem $RwC_2$ is equivalent to $RO_2$ given by

$$\begin{align*}
\min_{x^1, \gamma} & \quad \gamma \\
\text{s.t.} & \quad A x^1 = h^1, \quad x^1 \geq 0 \\
& \quad c^1 x^1 + \min_{x^2(\xi^1)} c_{x^2(\xi^1)} x^2(\xi^1) \leq \gamma \\
& \quad T^1(\xi^1) x^1 + W^2(\xi^1) x^2(\xi^1) = h^2(\xi^1), \quad x^2(\xi^1) \geq 0, \quad \forall \xi^1 \in \Xi^1.
\end{align*}$$

(22)

For the convexity of $\mathcal{X}_{RwC_2}(\xi^1)$ it follows that the following functions computed at the optimum of $x^2(\xi^1)$, say $x^2(\xi^1)$,

$$T^1(\xi^1) x^1 + W^2(\xi^1) x^2(\xi^1) = h^2(\xi^1),$$

and

$$c^1 x^1 + c_{x^2(\xi^1)} x^2(\xi^1) \leq \gamma$$

are convex in $x^1$ for given $\xi^1$. Hence, the problem (22) is a robust convex optimization problem. Then, we construct its scenario counterpart

$$\begin{align*}
\min_{x^1, \gamma} & \quad \gamma \\
\text{s.t.} & \quad A x^1 = h^1, \quad x^1 \geq 0 \\
& \quad c^1 x^1 + \min_{x^2(\xi^1)} c_{x^2(\xi^1)} x^2(\xi^1) \leq \gamma \\
& \quad T^i(\xi^1) x^1 + W^2(\xi^1) x^2(\xi^1) = h^2(\xi^1), \quad x^2(\xi^1) \geq 0, \quad i = 1, \ldots, N,
\end{align*}$$

where the subscript $i$ for the variables $x^2$ highlights that the different minimization problems are independent. Finally, we note that the problem (23) corresponds to the problem $SwC_2^N$ and the thesis follows from [15]. \hfill \square

B Proof of Theorem 3

We first note that Problem $RO_H$ can be rewritten in epigraph form, by introducing the additional variable $\gamma$, as follows

$$\begin{align*}
RO_H &= \min_{x^1, \gamma} \gamma \\
\text{s.t.} & \quad A x^1 = h^1, \quad x^1 \geq 0 \\
& \quad c^1 x^1 + \min_{x^2(\xi^1)} \left[ c_{x^2(\xi^1)} x^2(\xi^1) + \cdots + \min_{x^H(\xi^H-1)} c_{x^H(\xi^H-1)} x^H(\xi^H-1) \right] \\
& \quad T^i(\xi^1) x^1 + W^2(\xi^1) x^2(\xi^1) = h^2(\xi^1), \\
& \quad T^{H-1}(\xi^{H-1}) x^{H-1} + W^H(\xi^H) x^H(\xi^H) = h^H(\xi^H),
\end{align*}$$

(24)

or, equivalently, as

$$\begin{align*}
RO_H &= \min_{x^1, \gamma} \gamma \\
\text{s.t.} & \quad A x^1 = h^1, \quad x^1 \geq 0 \\
& \quad (x^1, \gamma) \in \mathcal{X}_{RO_H}(\xi^{H-1}), \quad \forall \xi^{H-1} \in \Xi,
\end{align*}$$

(25)

where the set $\mathcal{X}_{RO_H}(\xi^{H-1})$ is defined as
\( X_{\text{RO}_H}(\xi^{H-1}) := \)
\[
\begin{cases} 
(x^1, \gamma) \in \mathbb{R}_{+}^{n_1+1} \text{ s.t.} \\
\min_{x^2(\xi^t), \ldots, x^H(\xi^{H-1})} c^1^T x^1 + c^2^T x^2(\xi^t) + \cdots + c^H^T x^H(\xi^{H-1}) \text{ s.t.} \\
T^1(\xi^1)x^1 + W^2(x^2(\xi^t)) = h^2(\xi^1) \\
\vdots \\
T^{H-1}(\xi^{H-1})x^{H-1}(\xi^{H-2}) + W^H(\xi^{H-1})x^H(\xi^{H-1}) = h^H(\xi^{H-1})
\end{cases}
\leq \gamma.
\]

Similarly, problem \( \text{Rw}_C \) rewrites
\[
\text{Rw}_C = \min_{x^1, \gamma} \\
\text{s.t. } Ax^1 = h^1, \ x^1 \geq 0 \\
(x^1, \gamma) \in X_{\text{Rw}_C}(\xi^{H-1}), \ \forall \xi^{H-1} \in \Xi,
\]
where the set \( X_{\text{Rw}_C}(\xi^{H-1}) \) is defined as
\[
X_{\text{Rw}_C}(\xi^{H-1}) := \\
\begin{cases} 
(x^1, \gamma) \in \mathbb{R}_{+}^{n_1+1} \text{ s.t.} \\
\exists \gamma \in \mathbb{R}_{+}^{n_1+1} \text{ satisfying} \\
T^1(\xi^1)x^1 + W^2(x^2(\xi^t)) = h^2(\xi^1) \\
\vdots \\
T^{H-1}(\xi^{H-1})x^{H-1}(\xi^{H-2}) + W^H(\xi^{H-1})x^H(\xi^{H-1}) = h^H(\xi^{H-1})
\end{cases}
\]

So, we just need to prove that \( X_{\text{RO}_H}(\xi^{H-1}) \equiv X_{\text{Rw}_C}(\xi^{H-1}) \) for the minimum value of \( \gamma \).

* We prove that if \((x^1, \gamma) \in X_{\text{RO}_H} \), then \((x^1, \gamma) \in X_{\text{Rw}_C} \). If \((x^1, \gamma) \in X_{\text{RO}_H} \), then \( \exists \gamma(x^1, t = 2, \ldots, H) \text{ satisfying} \\
T^t(\xi^t)x^t + W^H(x^H(\xi^{H-1})) = h^H(\xi^{H-1}) \), \( t = 2, \ldots, H \) are satisfied for the minimum value of \( \gamma \). Conversely if \((x^1, \gamma) \in X_{\text{Rw}_C} \), then we need to prove that \((x^1, \gamma) \in X_{\text{RO}_H} \). If \((x^1, \gamma) \in X_{\text{Rw}_C} \), then \( \exists \gamma(x^1, t = 2, \ldots, H) \text{ such that} \\
T^{t-1}(\xi^{t-1})x^{t-1}(\xi^{t-1}) + W^t(x^t(\xi^t)) = h^t(\xi^{t-1}), \ t = 2, \ldots, H \) are satisfied for the minimum value of \( \gamma \).

This implies that \( x^t(\xi^{t-1}), t = 2, \ldots, H \) is the minimum of \( c^t x^t + c^2^T x^2(\xi^t) + \cdots + c^H^T (\xi^{H-1}) x^H(\xi^{H-1}) \leq \gamma \).

By contradiction if \( x^t(\xi^{t-1}), t = 2, \ldots, H \) were not the minimum then \( \gamma \) would not be at the minimum of problem \( \text{Rw}_C \). \( \square \)

**C Proof of Theorem 4**

We first prove the convexity of the set \( X_{\text{Rw}_C}(\xi^{H-1}) \) defined above. Given \( \tilde{x}^1, \tilde{x}^2 \in X_{\text{Rw}_C}(\xi^{H-1}) \), then there exist \( \tilde{x}^t(\xi^{t-1}), \ t = 2, \ldots, H \) such that
\[
\begin{align*}
T^{t-1}(\xi^{t-1})\tilde{x}^{t-1}(\xi^{t-2}) + W^t(\xi^{t-1})\tilde{x}^t(\xi^{t-1}) = h^t(\xi^{t-1}), & \ t = 2, \ldots, H \\
T^{t-1}(\xi^{t-1})\tilde{x}^{t-1}(\xi^{t-2}) + W^t(\xi^{t-1})\tilde{x}^t(\xi^{t-1}) = h^t(\xi^{t-1}), & \ t = 2, \ldots, H \\
c^t_1 \tilde{x}^1 + c^2_2(\xi^t) \tilde{x}^2(\xi^t) + \cdots + c^H_H(\xi^{H-1}) \tilde{x}^H(\xi^{H-1}) \leq \gamma \\
c^t_1 \tilde{x}^1 + c^2_2(\xi^t) \tilde{x}^2(\xi^t) + \cdots + c^H_H(\xi^{H-1}) \tilde{x}^H(\xi^{H-1}) \leq \gamma.
\end{align*}
\]
Consider now \( x^\lambda := \lambda \hat{x}^1 + (1 - \lambda) \hat{x}^1 \), with \( \lambda \in [0, 1] \), and let \( x^{t\lambda} := \lambda \hat{x}^t(\xi^{t-1}) + (1 - \lambda) \hat{x}^t(\xi^{t-1}) \), \( t = 2, \ldots, H \). We have

\[
T^{t-1}(\xi^{t-1})x^{t-1\lambda} + W^t(\xi^{t-1})x^{t\lambda} = T^{t-1}(\xi^{t-1}) (\lambda \hat{x}^{t-1}(\xi^{t-2}) + (1 - \lambda) \hat{x}^{t-1}(\xi^{t-2}))
+ W^{t}(\xi^{t-1}) (\lambda \hat{x}^{t-1}(\xi^{t-2}) + (1 - \lambda) \hat{x}^{t-1}(\xi^{t-2}))
\]

\[
= \lambda (T^{t-1}(\xi^{t-1})\hat{x}^{t-1}(\xi^{t-2}) + W^{t}(\xi^{t-1})\hat{x}^{t}(\xi^{t-1}))
+ (1 - \lambda) (T^{t-1}(\xi^{t-1})\hat{x}^{t-1}(\xi^{t-2}) + W^{t}(\xi^{t-1})\hat{x}^{t}(\xi^{t-1}))
\]

\[
= \lambda h^t(\xi^{t-1}) + (1 - \lambda) h^t(\xi^{t-1})
= h^t(\xi^{t-1}), \ t = 2, \ldots, H.
\]

and

\[
e_1^T x^{1\lambda} + e_2^T(\xi^1) x^{2\lambda} + \ldots + e_H^T(\xi^{H-1}) x^{H\lambda}
= e_1^T (\lambda \hat{x}^1 + (1 - \lambda) \hat{x}^1)
+ e_2^T(\xi^1) (\lambda \hat{x}^2(\xi^1) + (1 - \lambda) \hat{x}^2(\xi^1)) + \ldots
+ e_H^T(\xi^{H-1}) (\lambda \hat{x}^H(\xi^{H-1}) + (1 - \lambda) \hat{x}^H(\xi^{H-1}))
\]

\[
= \lambda e_1^T \hat{x}^1 + e_2^T(\xi^1) \hat{x}^2(\xi^1) + \ldots + e_H^T(\xi^{H-1}) \hat{x}^H(\xi^{H-1})
+ (1 - \lambda) e_1^T \hat{x}^1 + e_2^T(\xi^1) \hat{x}^2(\xi^1) + \ldots + e_H^T(\xi^{H-1}) \hat{x}^H(\xi^{H-1})
\]

\[
\leq \lambda \gamma + (1 - \lambda) \gamma = \gamma,
\]

which proves the convexity of the set \( \mathcal{X}_{RwC_H}(\xi^{H-1}) \). From Theorem 3 we observe that the condition \((x^1, \gamma) \in \mathcal{X}_{RwC_H}(\xi^{H-1})\) is equivalent to \((x^1, \gamma) \in \mathcal{X}_{RO_H}(\xi^{H-1})\) and that the problem \(RwC_H\) is equivalent to \(RO_H\) given by

\[
\begin{align*}
\min_{x^1, \gamma} \quad & \gamma \\
\text{s.t.} \quad & Ax^1 = h^1, \quad x^1 \geq 0 \\
& e_1^T x^1 + \min_{x^2(\xi^1)} \left[ e_2^T(\xi^1) x^2(\xi^1) + \ldots + \min_{x_H(\xi^{H-1})} e_H^T(\xi^{H-1}) x^H(\xi^{H-1}) \right] \leq \gamma \\
& T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1), \ \forall \xi^1 \in \Xi^1 \\
& \vdots \\
& T^{H-1}(\xi^{H-1})x^{H-1}(\xi^{H-2}) + W^H(\xi^{H-1})x^H(\xi^{H-1}) = h^H(\xi^{H-1}), \ \forall \xi^{H-1} \in \Xi \\
& x^t(\xi^{t-1}) \geq 0, \ t = 2, \ldots, H, \ \forall \xi^{t-1} \in \mathcal{X}_{\text{RO}_H}^{t-1} \Xi \\
\end{align*}
\]

For the convexity of \( \mathcal{X}_{RwC_H}(\xi^{H-1}) \) it follows that the following functions computed at the optimal sequence of \((x^2(\xi^1), \ldots, x^H(\xi^{H-1}))\), say \((x_2^*,(\xi^1), \ldots, x_H^*(\xi^{H-1}))\),

\[
T^{t-1}(\xi^{t-1})x_1^{t-1*}(\xi^{t-2}) + W^t(\xi^{t-1})x_1^*(\xi^{t-1}) = h^t(\xi^t),
\]

and

\[
e_1^T x^1 + e_2^T(\xi^1) x_2^*(\xi^1) + \ldots + e_H^T(\xi^{H-1}) x_H^*(\xi^{H-1}) \leq \gamma,
\]
are convex in $x^i$ for given $\xi^{H-1}$. Hence, the problem (27) is a robust convex optimization problem. Then, we construct its scenario counterpart

$$\min_{x^1, \gamma} \quad \gamma$$

s.t. $Ax^1 = h^1$, $x^1 \geq 0$

$$c^T x^1 + \min_{x^2} \left[ c^T \left( \xi^{(1)} \right) x^2 + \ldots + \min_{x^H} \left( \xi^{H-1(i)} \right) x^H \right] \leq \gamma$$

$$T^1(\xi^{(1)}) x^1 + W^2(\xi^{(1)}) x^2 = h^2(\xi^{(1)}), \quad x^2 \geq 0, \quad i = 1, \ldots, N$$

$$\vdots$$

$$T^{H-1}(\xi^{H-1(i)}) x^{H-1} + W^H(\xi^{H-1(i)}) x^H = h^H(\xi^{H-1(i)}), \quad i = 1, \ldots, N$$

$$x^1 \geq 0, \quad x^i \geq 0, \quad t = 2, \ldots, N, \quad i = 1, \ldots, N,$$

where the subscript $i$ for the variables $x^i$ highlights that the different minimization problems are independent. Finally, we note that the problem (28) corresponds to the problem $SwC^N_H$ and the thesis follows from [13]. □

Acknowledgements The authors would like to thank Daniel Kuhn and Phebe Vayanos for helpful discussions on the inventory management with cumulative orders problem.

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