Shuffling functors and spherical twists on $D^b(O_0)$

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Abstract For a semisimple complex Lie algebra $g$, the BGG category $O$ is of particular interest in representation theory. It is known that Irving’s shuffling functors $Sh_w$, indexed by elements $w \in W$ of the Weyl group, induce an action of the braid group $B_W$ associated to $W$ on the derived categories $D^b(O_\lambda)$ of blocks of $O$.

We show that for maximal parabolic subalgebras $p$ of $sl_n$ corresponding to the parabolic subgroup $W_p = S_{n-1} \times S_1$ of $S_n$, the derived shuffling functors $LSh_w$ are instances of Seidel and Thomas’ spherical twist functors. Namely, we show that certain parabolic indecomposable projectives $P_p(w)$ are spherical objects, and the associated twist functors are naturally isomorphic to $LSh_w[-1]$ as auto-equivalences of $D^b(O_p)$.

We give an overview of the main properties of the BGG category $O$, the construction of shuffling and spherical twist functors, and give some examples how to determine images of both. To this end, we employ the equivalence of blocks of $O$ and the module categories of certain path algebras.

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1 Introduction Consider a finite dimensional semisimple complex Lie algebra $\mathfrak{g}$ with respective Cartan and Borel subalgebras $\mathfrak{h}$ and $\mathfrak{b}$. Representations of $\mathfrak{g}$ are equivalent to modules over the universal enveloping algebra $U(\mathfrak{g})$ [Hum72, §V]. The BGG category $\mathcal{O}$ of $\mathfrak{g}$ is the full subcategory of $\mathcal{O}(\mathfrak{g})$ consisting of modules that (O1) are finitely generated, (O2) have a weight space decomposition $M = \bigoplus_{\lambda \in \mathcal{P}} M_{\lambda}$ and (O3) are locally $\mathfrak{n}$-finite; i.e., for every $v \in M$, the orbit $U(n^+) \cdot v$ is finite dimensional.

1.1 Category $\mathcal{O}$, blocks and shuffling functors The category $\mathcal{O}$ has a decomposition $\mathcal{O} = \bigoplus_{\lambda} \mathcal{O}_{\lambda}$ into blocks $\mathcal{O}_{\lambda}$, indexed by dominant weights $\lambda$. Denote the Weyl group of $\mathfrak{g}$ by $W$ and let $L \leq W$ be its stabiliser subgroup of a weight $\lambda$. Each block $\mathcal{O}_{\lambda}$ contains the simple modules $L(w, \lambda)$, the indecomposable projectives $P(w, \lambda)$ and the Verma modules $M(w, \lambda)$ indexed by $w \in W/W_{\lambda}$ with $W_{\lambda}$ the stabiliser subgroup of $\lambda$. If a block $\mathcal{O}_{\lambda}$ is fixed, we just write $L(w), P(w)$ and $M(w)$ for the respective objects therein. Each block $\mathcal{O}_{\lambda}$ is Morita equivalent to modules over a quasi-hereditary algebra [BGG76]. A weight $\lambda$ is called regular if $W_{\lambda}$ is trivial; i.e., if $\lambda$ does not lie on any reflection plane. All blocks $\mathcal{O}_{\lambda}$ associated to regular weights are equivalent as categories; in the following we shall thus work in the principal block $\mathcal{O}_{\mathbf{0}}$ which contains the trivial $\mathfrak{g}$-representation $L(e, 0) = C$.

Definition 1.1. A Coxeter system consists of a group $W$, a set $S$ of generators and a presentation $W = \langle s \in S \mid s^2 = e, sts \cdots = tst \cdots \rangle$ with $m_{st}$ factors $s$, $t$ on both sides. The $s \in S$ are called simple reflections. The matrix $(m_{st})_{s,t \in W}$ is called the Coxeter matrix of $W$. To $W$, there is the associated braid group $B_W = \langle s \in S \mid sts \cdots = tst \cdots \rangle$, such that there is a natural quotient map $B_W \twoheadrightarrow W$.

Example 1.2. The Weyl group of $\mathfrak{g}$ is a Coxeter group. In particular, the symmetric group $S_n$, which is the Weyl group of $\mathfrak{sl}_n$, is a Coxeter group, generated by the simple reflections $s_1, \ldots, s_{n-1}$, the Coxeter matrix has entries $m_{s_i s_j} = 2$ if $i = j$, $1$ if $|i - j| = 1$, $0$ otherwise, and $B_n = B_{S_n}$ is the well-known Artin braid group.

Definition 1.3. Let $s \in W$ be a simple reflection and $\mu$ be a weight with $W^{\mu} = \{ e, s \}$. The translation through the $s$-wall is the composition

$$\Theta_s : \mathcal{O}_{\mathbf{0}} \xrightarrow{T^\mu_s} \mathcal{O}_{\mu} \xrightarrow{T^\mu_s} \mathcal{O}_{\mathbf{0}}$$

of the two bi-adjoint translation functors $T^\mu_s$ and $T^\mu_s$; $\Theta_s$ is independent from the choice of $\mu$, and is an exact self-adjoint auto-equivalence of the block $\mathcal{O}_{\mathbf{0}}$ [Jan79, §2.10]. It is uniquely characterised by the existence of short exact sequences

$$0 \rightarrow M(w) \rightarrow \Theta_s M(w) \rightarrow M(ws) \rightarrow 0 \quad \text{and by} \quad \Theta_s M(w) \cong \Theta_s M(ws) \quad (1.1)$$

for $w < ws$ [Jan79, Satz 2.10]. Furthermore, $\Theta^2_s = \Theta_s \oplus \Theta_s$ [Hum08, cf. §7.14].

Definition 1.4. From the adjunctions $T^s_\mu : T^s_0$ and $T^s_\mu : T^s_\mu$ we get adjunction maps $\eta_s : \text{id} \Rightarrow \Theta_s$ and $\varepsilon_s : \Theta_s \Rightarrow \text{id}$. The shuffling and the coshuffling functor $S_h$ and $C_h$ are the respective cokernel and kernel

$$\text{Sh}_s := \text{coker}(\eta_s) : \mathcal{O}_{\mathbf{0}} \rightarrow \mathcal{O}_{\mathbf{0}}, \quad \text{Csh}_s := \ker(\varepsilon_s) : \mathcal{O}_{\mathbf{0}} \rightarrow \mathcal{O}_{\mathbf{0}},$$

see [Car86, §2; Irv93, §3]. By the snake lemma, $\text{Sh}_s$ is right and $\text{Csh}_s$ is left exact, and we consider the derived functors $L \text{Sh}_s \cong \text{cone}(\eta)$ and $R \text{Csh}_s \cong \text{cocone}(\varepsilon)$.  

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Note that for $w < ws$ we have

\[
\begin{align*}
L^i \text{Sh}_s &= 0 \text{ for } i \neq 0, 1, \quad L^1 \text{Sh}_s M(w) = 0, \\
R^j \text{Csh}_s &= 0 \text{ for } j \neq -1, 0, \quad R^1 \text{Csh}_s M(ws) = 0.
\end{align*}
\]

1.2 Actions of $W$ on $K_0(\mathcal{O}_0)$ and $D^b(\mathcal{O}_0)$

Definition 1.5. The Grothendieck group $K_0(\mathcal{C})$ of an abelian category $\mathcal{C}$ is the abelian group generated by $[M]$ of objects $M \in \mathcal{C}$, subject to the relation $[E] = [A] + [B]$ whenever $E$ is an extension of $A$ by $B$. For $\mathcal{T}$ a triangulated category, $K_0(\mathcal{T})$ is defined in the same way, with “extension” replaced by “distinguished triangle” in the obvious way.

Any exact and any triangulated functor induces a group homomorphism on $K_0(\mathcal{C})$ and on $K_0(\mathcal{T})$, respectively. Both definitions of $K_0$ are compatible in the sense that the inclusion $\mathcal{C} \hookrightarrow D^b(\mathcal{C})$ induces an isomorphism $K_0(\mathcal{C}) \cong K_0(D^b(\mathcal{C}))$ of abelian groups [Gro57].

Each of the collections \{L(w)\}_{w \in W}, \{M(w)\}_{w \in W} and \{P(w)\}_{w \in W} is a $\mathbb{Z}$-base of $K_0(\mathcal{O}_0)$. The shuffling functors thus induce a right action of $W$ on $K_0(\mathcal{O}_0)$, defined for simple reflections $s$ by the assignment

\[
W \to \text{Aut}(K_0(\mathcal{O}_0)), \quad s \mapsto \text{[Sh}_s]: [M(w)] \mapsto [M(ws)].
\]

Theorem 1.6 [Rou06, thm. 4.4]. The assignment

\[
B_W \to \text{Aut}(D^b(\mathcal{O}_0)), \quad s \mapsto L \text{Sh}_s, \quad s^{-1} \mapsto R \text{Csh}_s
\]

defines an action of $B_W$ on the derived category $D^b(\mathcal{O}_0)$, and the following square commutes:

\[
\begin{array}{ccc}
B_W & \xrightarrow{\text{can}} & W \\
L \text{Sh}_{(-)} \downarrow & & \downarrow \text{[Sh}_{(-)}] \\
\text{Aut}(D^b(\mathcal{O}_0)) & \xrightarrow{-1} & \text{Aut}(K_0(\mathcal{O}_0)).
\end{array}
\]

1.3 Seidel and Thomas’ spherical twist functors
Let $\mathcal{C}$ be a $k$-linear abelian category $\mathcal{C}$ of finite global dimension. Seidel and Thomas have constructed an action of the braid group $B_n := B_{S_n}$ of the symmetric group $S_n$ on $\mathcal{C}$ in terms of spherical objects [ST01]. We give a short summary of their construction.

Notation 1.7. We denote the Hom-space in $D^b(\mathcal{C})$ by $\text{Hom}^*_D(\mathcal{C})$; it is a graded $k$-vector space whose degree $d$-part $\text{Hom}^d_D(\mathcal{C})$ consists of chain maps of homological degree $d$. In contrast, for chain complexes $X, Y \in \text{Ch}(\mathcal{C})$, we let $\text{hom}_D^*(X, Y)$ denote the chain complex of arbitrary graded morphisms $X \to Y$ (i.e., not necessarily chain maps); it is a chain complex of $\mathcal{C}$-vector spaces with differential $d_{\text{hom}}$ defined on elements in degree $k$ by $d_{\text{hom}}(f) = df - (-1)^k fd_X$. Note that $H^* \text{hom}_D^*(-, -) = H^* \text{hom}_K^*(-, -)$, i.e., the Hom-space in the bounded homotopy category [Wei94, 2.7.5].

Finally, for a complex $V$ of $\mathcal{C}$-vector spaces and $X \in D^b(\mathcal{C})$, we let $\text{lin}_D^*(V, X)$ denote the complex of $\mathcal{C}$-linear maps from $V$ to $X$, which is a chain complex in $D^b(\mathcal{C})$ with differential $d_{\text{lin}}(f)v := (-1)^{\text{deg}v}[dfv - fdv]$. For details, see [ST01].
Theorem 1.14 [ST01, thm. 2.17]

There is an adjunction $T_E \dashv T'_E$.

**Proposition 1.10** [ST01, prop. 2.10]. If $E$ is $d$-spherical, then $T_E$ and $T'_E$ are mutually inverse auto-equivalences of $D^b(C)$.

**Notation 1.11.** Given a double (or triple) complex $X^{\bullet\bullet}$, we denote its $\oplus$-total complex by \{X$^{\bullet\bullet}$\}. In particular, since we can regard a morphism $f: X \to Y$ of chain complexes trivially as a double complex with $Y$ laced in degree zero, the mapping cone can be written as cone($f$) = \{f: X \to Y\}.

**Remark 1.12.** Since for every $d$-spherical object $E$, $\hom^\bullet(E, E) \cong \langle \id_C \rangle_C \langle x \rangle_C [-d]$ as complexes vector spaces, we have

$T'_E E = \{ (\varphi_c): E \to E \oplus E[d] \} \cong E[1-d],$

where the left $E$ is in degree zero.

**Definition 1.13.** A collection \{E$_1$, \ldots, E$_n$\} of $d$-spherical objects is an $A_n$-configuration if $\dim \hom^\bullet D^b(C)(E_i, E_j) = \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise} \end{cases}$.

**Theorem 1.14** [ST01, thm. 2.17]. Given an $A_n$-configuration \{E$_1$, \ldots, E$_n$\} of $d$-spherical objects, the assignment

$B_n \to \text{Aut}(D^b(C)), \ s_i \mapsto T_{E_i}$

defines an action of the braid group.

**1.4 Objective** The category $O_0$ is a $C$-linear category of finite global dimension and hence satisfies the requirements of [ST01]. For $g = sl_n$, we want to understand whether there is an $A_n$-configuration in $D^b(O_0)$ such that the associated twist functors relate to the shuffling functors. We shall prove the following:

**Theorem 4.15.** For a maximal parabolic subalgebra $p$ of $sl_n$ corresponding to the parabolic subgroup $S_{n-1} \times S_1 \leq S_n$, there is an $A_{n-1}$-configuration of $0$-spherical objects in $D^b(O^p_0)$ such that the associated spherical twist functor and the restriction $LSh_{s_1} | D^b(O^p_0)$ are naturally isomorphic auto-equivalences of $D^b(O^p_0)$.

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1 If $d = 0$, we require that $o: \hom D^b(F, E) \otimes \hom D^{d-1}(F, E) \to \hom D^d(E, E) \langle \id_E \rangle \cong C$ be non-degenerate.
Outline Section 2 gives an overview of some of the most important properties of blocks of $O$ and the tools we employ. We do not require any prior knowledge about $O$. We include a short refresher on Kazhdan-Lusztig theory, quivers and graded algebras.

In Section 3, we explain how to compute images of the shuffling functors for the special case $g = \mathfrak{sl}_2$. Our proof for the respective version of 4.15 serves as a model for the general case, which is worked out in section 4.

2 Tools for $O$ In this section, we collect the most important properties of $O_\lambda$. For the following, it is not necessary to assume that $\lambda$ is a regular weight.

2.1 Composition series Every module $M \in O_\lambda$ admits a composition series

$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$

with subquotients $M_i/M_{i-1}$ isomorphic to the simple modules $L(w \cdot \lambda)$ [Jan79, Satz 1.13]. According to the Jordan-Hölder theorem, any composition series for $M$ involves the same isomorphism classes of the simple factors, up to their order of appearance. The multiplicity of $L(w \cdot \lambda)$ in any composition series for $M$ is denoted by $[M : L(w \cdot \lambda)]$.

Notation 2.1. We write a composition factors from the simple submodules at the bottom to the simple quotients at the top. Any permutation of factors on the same horizontal level occurs in some composition series for $M$.

Example 2.2. 1. For $g = \mathfrak{sl}_2$, the non-simple Verma module has a composition series

$M(e) = \frac{L(e)}{L(s)}$, which is just another way to say that there is a short exact sequence

$0 \to L(s) \to M(e) \to L(e) \to 0$.

2. For $g = \mathfrak{sl}_3$, the Verma module $M(s) = \frac{L(s)}{L(st)}\frac{L(ts)}{L(w_0)}$ has the simple quotient $L(s)$ and the simple submodule $L(w_0)$. Its non-trivial submodules are $L(w_0)$, $\frac{L(st)}{L(w_0)}$, and $\frac{L(ts)}{L(w_0)}$.

Remark 2.3. A morphism in $O$ is compatible with composition series, i.e., it takes factors from the top of its domain to the bottom of its codomain, preserving the order. The cokernel and kernel of a morphism consist of the simple factors that are not mapped onto and from, respectively. Since $\dim \text{Hom}_O(L(v), L(w)) = \delta_{vw}$, a morphism is determined by a scalar for each composition factor of the domain.

Caveat 2.4. Compatibility with the composition series is necessary but not sufficient for a morphism to exist in $O_0$; i.e., not every composition series diagram describes an actual morphism. Moreover, it is hard to tell which other factors of $M$ are contained in the submodule generated by a particular simple composition factor of $M$.

Certain modules, such as the indecomposable projectives $P(w \cdot \lambda)$, also admit a standard filtration whose subquotients are isomorphic to Verma modules $M(w \cdot \lambda)$; one writes $(M : M(w \cdot \lambda))$ for the respective (unique) multiplicity of $M(w \cdot \lambda)$ in any standard filtration of $M$.

Theorem 2.5 (BGG reciprocity) [BGG76]. $(P(v \cdot \lambda) : M(w \cdot \lambda)) = [M(w \cdot \lambda) : L(v \cdot \lambda)]$.
2.2 Kazhdan-Lusztig theory There is a \((W/W_\lambda)^2\)-parametrised collection \(\{p_{vw}\}\) of polynomials in \(\mathbb{Z}[q^{\pm 1}]\), called Kazhdan-Lusztig polynomials, which occur as base change coefficients between the standard basis and the Kazhdan-Lusztig basis of the Iwahori-Hecke algebra \(H_q(W)\) [KL79].

**Theorem 2.6** (Kazhdan-Lusztig) [BB81, §4; BK81, §8]. The composition factor multiplicities in \(\mathcal{O}\) are given by

\[
(P(v \cdot \lambda) : M(w \cdot \lambda)) = [M(w \cdot \lambda) : L(v \cdot \lambda)] = p_{vw}(1). \quad (2.1)
\]

**Remark 2.7.** We employ the convention from [Soe97; Lus03] for the bases of \(H_q(W)\) and thus for the normalisation of the \(p_{vw}\)'s. Another widespread convention yields the formula \([M(w \cdot \lambda) : L(v \cdot \lambda)] = p_{w_0v,w_0w}(1)\).

2.3 Gradings The following paragraphs summarise how to pass from \(\mathcal{O}_\lambda\) to a graded category \(\mathcal{O}_\lambda^g\); see [Str03a] for details. The following results are of fundamental importance:

**Theorem 2.8** (Struktursatz) [Soe90, Thm. 2]. The functor \(V_\lambda : \mathcal{O}_\lambda \to \text{Mod-End}_\mathcal{O}(P(w_0 \cdot \lambda))\), called the combinatoric functor, is fully faithful on projectives.

**Theorem 2.9** (Endomorphismensatz) [Soe90, Thm. 3]. Let \(C\) be the coinvariant algebra \(C := \mathbb{C}[h^*]/(\mathbb{C}[h^*]^W)\) of \(g\), with the ideal \((\mathbb{C}[h^*]^W)\) generated by strictly positively graded \(W\)-invariant polynomials. Then there is an isomorphism \(\text{End}_\mathcal{O}(P(w_0 \cdot \lambda)) \cong C^W\).

The category \(\mathcal{O}_\lambda\) has a projective generator \(P_\lambda := \bigoplus_{w \in W/W_\lambda} P(w \cdot \lambda)\). By Morita’s theorem [Bas68, Thm. II.1.3], there is an equivalence of categories

\[
\mathcal{O}_\lambda \xrightarrow{\cong} \text{Mod-End}_\mathcal{O}(P_\lambda), M \mapsto \text{Hom}_\mathcal{O}(P_\lambda, M). \quad (2.2)
\]

The algebra \(\text{End}_\mathcal{O}(P(w_0 \cdot \lambda))\) carries a natural non-negative grading exhibited by the Endomorphismensatz, so we can consider its category of graded modules. The fully-faithful functor \(V_\lambda\) then endows \(A_\lambda\) with a grading. The equivalence in (2.2) motivates:

**Definition 2.10.** Let \(\mathcal{O}_\lambda^g := \text{gMod-}A_\lambda\) be the category of graded \(A_\lambda\)-modules.

**Notation 2.11.** We denote the grading shift on \(\text{gMod-}A_\lambda\) by \((\cdot)\), where \(M(1)_i := M_{i-1}\).

**Remark 2.12.** A module \(M \in \mathcal{O}_\lambda\) is gradable if there is a graded module \(\tilde{M} \in \mathcal{O}_\lambda^g\) such that forgetting the grading yields \(\tilde{M} = M\). In particular, simple, Verma and indecomposable projective modules are gradable [Str03a, §82f]. It will prove natural to define these modules to have lowest non-trivial degree 0. In the following, we shall not distinguish notationally between these modules and their graded lifts.

Since \(A_\lambda\) is non-negatively graded, the grading of modules reflects their submodule structure. The same data is encoded in the exponents of Kazhdan-Lusztig polynomials:

**Definition 2.13.** A Loewy filtration for a module \(M \in \mathcal{O}_0\) is a filtration of minimal length with semisimple subquotients \(M_i\). A module \(M\) is called rigid if it has a unique Loewy filtration; for instance, Verma modules are rigid.

**Remark 2.14.** The semisimple quotients are precisely the horizontal layers in the composition series diagrams.

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Theorem 2.15 (generalised KL-theorem) [Irv88, Thm. 1, 2, Cor. 7]. The composition factor multiplicities in the degrees \( M(v)_i \) of a Verma module satisfy
\[ p_{vw} = \sum_k [M(v)_i] q^k, \]
i.e., any summand \( q^k \) of \( p_{vw} \) corresponds to a factor \( L(w) \) in the \( k \)-th layer of the Loewy filtration for \( M(v) \), with the zeroth layer at the top.

A graded analogue of the BGG reciprocity theorem (2.1) holds.

Definition 2.16. On \( O^\Lambda \), the graded translation through the \( s \)-wall \( \Theta_s \) is uniquely defined by short exact sequences
\[ 0 \to M(w)(1) \xrightarrow{\partial_s} \Theta_s M(w) \to M(ws) \to 0 \quad \text{and} \quad \Theta_s M(ws) = \Theta_s M(w)(-1) \]
for \( w < ws \). The adjunction \( \eta: M \to \Theta_s M \) is a degree-1-map and \( \Theta_s^2 \cong \Theta_s(-1) \oplus \Theta_s(1) \). The graded (co)shuffling functors are defined analogously to the non-graded case.

Remark 2.17. The Grothendieck group \( K_0(O^\Lambda) \) becomes a \( \mathbb{Z}[q^\pm] \)-module by \( q[M] \coloneqq [M(1)] \). There is an isomorphism of \( \mathbb{Z}[q^\pm] \)-modules
\[ K_0(O^\Lambda(\mathfrak{a}_w)) \to H_q(S_n), \quad [M(w)(q)] \mapsto qH_w, \quad [P(w)(q)] \mapsto qC_w, \]
where \( H_q(S_n) \) denotes the Iwahori-Hecke algebra of \( S_n \) with standard basis \( \{ H_w \}_{w \in W} \) and Kazhdan-Lusztig-basis \( \{ C_w \}_{w \in W} \). Under this isomorphism, the action via \( [\text{Sh}_w] \) corresponds to the right multiplication \( \cdot H_s \).

2.4 Quivers

Definition 2.18. The Ext\(^1\)-quiver \( Q_\lambda \) associated to \( A_\lambda \) has isoclasses of simple \( A_\lambda \)-modules as vertices and their extensions as edges.

Let \( \mathfrak{a}_\lambda \) be the ideal of the path algebra \( CQ_\lambda \) generated by the relations of extensions of the simple \( A_\lambda \)-modules. Then \( CQ_\lambda / \mathfrak{a}_\lambda \cong A_\lambda \) according to Gabriel's theorem [Gab72; Gab73].

Notation 2.19. We denote the composition of morphisms \( v_2 \xrightarrow{\partial_a} v_1 \) and \( v_3 \xrightarrow{b} v_2 \) of a quiver by \( v_3 \xleftarrow{ba} v_1 \). We denote trivial path associated to a vertex \( v \) by \( e_v \).

Remark 2.20. By the equivalence (2.2), \( Q_\lambda \) is the quiver with vertices indexed by \( W \) and edges \( w \xleftarrow{v} \) given by the irreducible morphisms from \( \text{Hom}_{C}(P(v \cdot \lambda), P(w \cdot \lambda)) \), i.e., morphisms that cannot be factored non-trivially. The ideal \( \mathfrak{a}_\lambda \) is generated by relations of these morphisms.

The path algebra of any quiver \( Q_\lambda \) is non-negatively graded by the length of a path in terms of arrows. One can show that \( \mathfrak{a}_\lambda \) is a homogeneous ideal; hence \( A_\lambda \) is graded as well. This grading coincides with the grading induced by \( V_\lambda \) in section 2.3 [Str03a].

Remark 2.21. To summarise, we have that \( O_\lambda \cong \text{Mod-}A_\lambda \) and \( O^\Lambda \cong g\text{Mod-}A_\lambda \). The algebra \( A_\lambda \) has a complete set of primitive idempotents \( \{ e_w \}_{w \in W} \), given by the trivial paths in \( Q_\lambda \), which correspond to the identities of the \( P(w) \)'s. This equivalence maps indecomposable projectives \( P(w) \) in \( O_\lambda \) to the indecomposable projectives in \( g\text{Mod-}A_\lambda \); which are precisely the right ideals \( e_w A_\lambda \) of all paths ending in \( w \) [Bar15, cor. 4.18, rmk. 4.20].

A canonical \( C \)-basis of \( e_w A_\lambda \) is given by paths that are in one-to-one correspondence with the simple composition factors of \( P(w) \). Explicitly, each composition factor \( L(v)(i) \) (i.e., a factor \( L(v) \) residing in the \( i \)-th layer) corresponds to a basis vector of \( e_w A_\lambda \) from the degree \( i \)-part \( (e_w A_\lambda e_v)_i \).
From now on, we shall stick to the principal block $\mathcal{O}_0$ and omit all the subscript-$\lambda$’s from $A$ and $Q$.

2.5 Parabolic subalgebras Let $(W, S)$ be a Coxeter system and $S_p \subseteq S$ be any subset of the simple reflections of $W$.

Definition 2.22. The associated parabolic subgroup $W_p \leq W$ is the subgroup $W_p = \langle s_i \rangle_{i \in S_p}$ of $W$. Every left coset in $W_p/W$ has a unique representative of minimal length [Hum90, §1.10]. We denote the set of such representatives by $W^p$.

To the quiver $Q$ we associate the full subquiver $Q^p$ of $Q$ with vertex set $Q^p = W^p$ and define the respective algebra $A^p := A/(c_v)_{v \in W^p}$. The category $\mathcal{O}_0^p := \text{Mod-}A^p$, is equivalent to the smallest Serre subcategory of $\mathcal{O}_0^p$ containing all simple modules $L(w)$ for $w \in W^p$ [KM16, §2]. The quotient map $A \rightarrow A^p$ induces an induction-restriction-adjunction

$$\text{Ind}^p: \mathcal{O}_0 \xrightarrow{\sim}\mathcal{O}_0^p : \text{Res}_p,$$

where $\text{Res}_p$ is fully faithful and thus turns $\mathcal{O}_0^p$ into a subcategory of $\mathcal{O}_0$. The functor $\text{Res}_p$ has both left and right adjoints, where its left adjoint $Z^p := \text{Ind}^p = - \otimes_A A^p$, called Zuckerman functor and its right adjoint $Z_p$, called dual Zuckerman functor, assigns to a module $M$ its largest quotient and largest submodule with simple composition factors corresponding to words from $W^p$, respectively. [Maz12, Thm. 6.1]. Let $P^p(w) := Z^p(P(w)) = \varepsilon_w A^p$ and $M^p(v) := Z^p(M(v))$. As notation suggests, the $P^p(w)$’s are the indecomposable projectives in $\mathcal{O}_0^p$ [Maz12, §4.6].

Remark 2.23. The constructions and statements from sections 2.1 to 2.4 have projective analogues; namely:

(i) The category $\mathcal{O}_0^p$ has a projective generator $P^p = \bigoplus_{w \in W^p} P^p(w)$.

(ii) The analogously constructed graded version $\mathcal{O}_0^{p, \mathbb{Z}}$ of $\mathcal{O}_0^p$ contains simples, parabolic Vermas and indecomposable projectives.

(iii) The $P^p(w)$’s have standard filtrations with subquotients isomorphic to $M^p(v)$’s.

(iv) The respective multiplicities satisfy a parabolic BGG reciprocity theorem [Roc80, Prop. 4.5, Thm. 6.1].

(v) Parabolic Kazhdan-Lusztig polynomials $p^p_{vw} \in \mathbb{Z}[q^{\pm 1}]$ [Deo87, §3; Soe97] occur as base change coefficients for parabolic Hecke algebras.

(vi) The composition factor multiplicities can be computed using the parabolic generalised Kazhdan-Lusztig theorem [Irv90, Cor. 7.1.3; CC87, Thm. 1.3]:

$$p^p_{vw} = \sum_{k \geq 0} (P^p(w) : M^p(v)\langle k \rangle)q^k = \sum_{k \geq 0} [M^p(v) : L(w)\langle k \rangle]q^k, \quad (2.3)$$

Explicitly, this means that a summand $q^k$ in $p^p_{vw}$ means that the factor $M^p(v)$ occurs in the $k$-th layer of a Loewy series of $P^p(w)$, counted from the top [BGS96, Thm. 3.11.4].

Remark 2.24. The functor $Z^p$ commutes with projective functors, in particular with $\Theta_\lambda$ [Maz12, Thm. 6.1]. This implies that both $\Theta_\lambda$ and $\text{Sh}_{\lambda}$ restrict to $\mathcal{O}_0^p$, and $\Theta_\lambda$ is uniquely characterised by short exact sequences

$$0 \rightarrow M^p(w)\langle 1 \rangle \xrightarrow{\delta_{w}} P^p(w) \rightarrow M^p(ws) \rightarrow 0 \quad \text{and} \quad \Theta_\lambda M(ws) = \Theta_\lambda M(w)\langle -1 \rangle \quad (2.4)$$

for $w < ws$.

Caveat 2.25. The inclusion $\mathcal{O}_0^p \subset \mathcal{O}_0$ does not preserve projectives.
3 $B_n$-actions for $\mathfrak{sl}_2$

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ and its Weyl group $S_2 = \{e, s\}$. The Verma modules and the indecomposable projectives have composition series

$$P(e) = M(e) = \begin{pmatrix} L(e) \\ L(s) \end{pmatrix}, \quad P(s) = \begin{pmatrix} M(s) \\ M(e) \end{pmatrix} = \begin{pmatrix} L(s) \\ L(e) \end{pmatrix}, \quad M(s) = L(s); \quad (3.1)$$

these can be computed using Kazhdan-Lusztig polynomials. Therefore, $\mathcal{O}_0$ is equivalent to $\text{Mod-}A$ for the path algebra $A = CQ/(ba)$ of the quiver $Q = e \rightarrow s$. The arrows $a, b$ of $Q$ and their relations correspond to the unique (up to scalars) non-trivial morphisms $a: P(e) \hookrightarrow P(s)$ and $b: P(s) \rightarrow P(e)$ [Str03a].

3.1 The action of $\text{Sh}_s$

We are interested in the images $\text{Sh}_s M$ for the following modules $M$:

1. $M(e) = P(e)$: from (1.1) we get $\Theta_s M(e) = \begin{pmatrix} M(s) \\ M(e) \end{pmatrix} = P(s)$ and $\text{Sh}_s M(e) = M(s)$.

2. $M(s)$: Up to scalars, there is a unique morphism $M(s) \hookrightarrow M(e)$; hence we obtain that $\text{Sh}_s M(s) \cong \begin{pmatrix} M(s) \\ M(e) \end{pmatrix} = M'(s)$.

3. $P(s)$: using $\Theta_s^2 \cong \Theta_s \oplus \Theta_s$, it follows from $P(s) = \Theta_s M(e)$ that $\text{Sh}_s P(s) \cong P(s)$.

Similar arguments show that $\mathbf{R} \text{Csh}_s P(s) \cong P(s)$ and $\mathbf{R} \text{Csh}_s P(e)$ is the mapping cone $\mathbf{R} \text{Csh}_s P(e) \cong \{ P(s) \rightarrow P(e) \}$ (recall the notation of mapping cones from Notation 1.11); to summarise, we have

$$\begin{align*}
\mathbf{L} \text{Sh}_s P(e) & = \{ P(e) \rightarrow P(s) \} \cong M(s) & \mathbf{R} \text{Csh}_s P(s) & = \{ P(s) \rightarrow P(e) \} \\
\mathbf{L} \text{Sh}_s P(s) & = P(s) & \mathbf{R} \text{Csh}_s P(s) & = P(s). 
\end{align*} \quad (3.2)$$

From the naturality diagram

$$\begin{array}{c}
M(e) \xrightarrow{a} P(s) \xrightarrow{b} P(e) \\
\eta M(e) \downarrow \quad \eta P(s) \downarrow (1, 1) \quad \eta M(s) \downarrow (0, 1) \\
P(s) \xrightarrow{M(e)} P(s) \oplus P(s) \xrightarrow{(0, 1)} 0
\end{array} \quad (3.3)$$

of $\eta: \text{id} \Rightarrow \Theta_s$ we get that the morphisms $a: P(e) \hookrightarrow P(s)$ and $b: P(s) \rightarrow P(e)$ have images

$$\text{Sh}_s a: M(s) \hookrightarrow P(s) \quad \text{and} \quad \text{Sh}_s b: P(s) \twoheadrightarrow M(s). \quad (3.4)$$

Since $D^b(\mathcal{O}_0)$ is generated as a triangulated category by $P(e)$ and $P(s)$, this datum suffices to describe the behaviour of $\mathbf{L} \text{Sh}_s$: thus we have shown:

**Proposition 3.1.** The Artin braid group $B_1$ acts via $\text{Sh}_s$ on $D^b(\mathcal{O}_0(\mathfrak{sl}_2))$ by

$$B_1 \cong \mathbf{Z} \rightarrow \text{Aut}(D^b(\mathcal{O}_0))$$

$$n \mapsto P(s) \mapsto P(s),$$

$$P(e) \mapsto \begin{cases} \{ P(e) \hookrightarrow P(s) \rightarrow P(s) \rightarrow \cdots \rightarrow P(s) \} & \text{if } n \geq 0, \\
\{ P(s) \twoheadrightarrow P(s) \rightarrow \cdots \rightarrow P(s) \rightarrow P(e) \} & \text{if } n \leq 0
\end{cases}$$

with homological degree 0 as indicated.
3.2 Spherical objects $D^b(\mathcal{O}_0)$ contains two spherical objects:

1. $P(s)$ is 0-spherical since $\text{End}_\mathcal{O}(P(s)) \cong \mathbb{C}[x]/(x)^2$ with $x = ab$.

2. $L(e)$ is 2-spherical; too see this, consider the projective resolution $L(e) \cong \{P(e) \rightarrow P(s) \rightarrow P(e)\}$.

Remark 3.2. A simple module never can be 0 or 1-spherical since simples do not extend themselves non-trivially.

3.3 Spherically twisting by $P(s)$ The non-trivial endomorphism $x := ab$ of $P(s)$ factors through $P(e)$. Since $P(e)$ and $P(s)$ generate $D^b(\mathcal{O}_0)$, this already implies that composition pairing from the Calabi-Yau-property (definition (S3)) is non-degenerate, so $P(s)$ is spherical. The images of projectives under the cotwisting functor are

$$T_{P(s)}'P(s) = \{(\frac{1}{x}) : P(s) \rightarrow P(s) \oplus P(s)\} \cong P(s)[-1] = \mathcal{S}_a P(s)[-1],$$

$$T_{P(s)}'P(e) = \{a : P(e) \rightarrow P(s)\} \cong M(s)[-1] = \mathcal{S}_a P(e)[-1].$$

This proves half of the following:

Theorem 3.3. There is a natural isomorphism $L\mathcal{S}_a[-1] \cong T_{P(s)}'$ of autoequivalences of $D^b(\mathcal{O}_0(\mathfrak{s}l_2))$.

Lemma 3.4 (Morita). Let $A$ and $B$ be rings. Any right exact functor $F \colon \text{Mod-}A \rightarrow \text{Mod-}B$ that preserves arbitrary direct sums is isomorphic to tensoring with the $A$-$B$-bimodule $FA$ [Bas68, Thm. II.2.3].

Corollary 3.5. For abelian categories $\mathcal{A}$ and $\mathcal{B}$ with projective generators $P_A$ and $P_B$, Morita’s theorem (2.2) allows us to identify $\mathcal{A}$ and $\mathcal{B}$ with $\text{Mod-End}_{\mathcal{A}}(P_A)$ and $\text{Mod-End}_{\mathcal{B}}(P_B)$ respectively. Then for any right exact functor $F : \mathcal{A} \rightarrow \mathcal{B}$ commuting with arbitrary direct sums there is a natural isomorphism of functors

$$F \cong \otimes_{\text{End}_{\mathcal{A}}(P_A)} \text{Hom}_{\mathcal{B}}(P_B, FP_A) : \mathcal{A} \rightarrow \mathcal{B},$$

where $M_F$ becomes an $\text{End}_{\mathcal{A}}(P_A)\text{-End}_{\mathcal{B}}(P_B)$-bimodule by $a \cdot f \cdot b = Fa \circ f \circ b$ for $a \in \text{End}_{\mathcal{A}} P_A, b \in \text{End}_{\mathcal{B}}(P_B)$ and $f \in M_F$.

Proof of theorem 3.3. By the corollary there are natural isomorphisms

$$L\mathcal{S}_a[-1] \overset{\text{def}}{=} \{\text{id}_\mathcal{O} \Rightarrow \Theta_s\} \cong - \otimes_{\mathcal{A}} \{A \rightarrow M_{\Theta_s}\},$$

$$T_{P(s)}' \overset{\text{def}}{=} \{\text{id}_\mathcal{O} \Rightarrow \mathcal{S}_{P(s)}\} \cong - \otimes_{\mathcal{A}} \{A \rightarrow M_{\mathcal{S}}\},$$

such that it suffices to show $M_{\Theta_s} \cong M_{\mathcal{S}}$. Recall that $\mathcal{S}_{P(s)} = \text{lim}_C(\text{hom}_\mathcal{O}(\mathcal{O}_C, P(s)), P(s))$. By finite dimensionality, there is an isomorphism $M_{\mathcal{S}} \cong P(s)^* \otimes_C P(s)$ of $A$-$A$-bimodules. Consider the canonical vector space basis $\{\varepsilon, s \leftarrow e, s \leftarrow e \leftarrow s\}$ of $P(s) = \varepsilon s A$ and the
dual basis of \( P(s)^* \). Then, a basis of \( M_{\Xi} \) is given by all pairwise tensor products in the schematic

\[
M_{\Xi} \cong P(s)^* \otimes CP(s):
\begin{pmatrix}
\varepsilon_s \leftarrow e \leftarrow s^* \\
\varepsilon_s \leftarrow (s \leftarrow e)^* \\
\varepsilon^*_s \leftarrow (s \leftarrow e)^*
\end{pmatrix}
\otimes
\begin{pmatrix}
e_s \leftarrow (s \leftarrow e)^* \\
\theta(s \leftarrow e) \\
\theta(s \leftarrow e)
\end{pmatrix}
\]

with the indicated action on basis vectors. To describe the \( A-A \)-bimodule action on \( M_{\Theta} = \text{Hom}_A(P, \Theta P) \) in terms of a vector space basis, we introduce the following notation. Recall that \( P = P(e) \oplus P(s) \) and \( \Theta_s P \cong P(s)^{\otimes 3} \). We enumerate the summands of \( \Theta_s P = P(s)^3 \). We then abbreviate, e.g., the morphism \( \begin{pmatrix} 0 & 0 \\ 0 & 0 & x \end{pmatrix} \in \text{Hom}_A(P, \Theta_s P) \), by \( P(s)^3 \overset{\xi}{\longrightarrow} P(s) \). For \( \Theta_s \), the naturality diagram (3.3) of \( \eta: \text{id} \Rightarrow \Theta_s \) shows that the images under \( \Theta_s \) of the morphisms \( a \) and \( b \) generating \( \text{End}_C(P) \) are

\[
P(e) \oplus P(s) \rightarrow P(s) \rightarrow P(e) \oplus P(s)
\]

A vector space basis of \( M_{\Theta} \) is given by irreducible morphisms from one summand of \( P \) to another in the schematic

\[
M_{\Theta}:
\begin{pmatrix}
\Theta_s(-) \overset{\xi}{\longrightarrow} P(s)^3 \\
\Theta_s(-) \overset{\xi}{\longrightarrow} P(s)^3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
P(s) \leftarrow P(e) \\
P(s) \leftarrow P(s) \leftarrow P(s)
\end{pmatrix}
\]

with the bimodule action as indicated. Comparison of (3.5) and (3.7) shows that the obvious isomorphism \( M_{\Theta} \cong M_{\Xi} \) of vector spaces is an isomorphism of \( A-A \)-bimodules. \( \square \)

### 3.4 Spherically twisting by \( L(e) \)
Recall that \( L = L(e)[1] \). From remark 1.12 it follows that if there is any isomorphism \( T_{L(e)} \cong L \), then the shift \( ? \) must be zero. Recall the projective resolution \( L(e) \simeq \{ P(e) \rightarrow P(s) \rightarrow P(e) \} \). We obtain

\[
\text{hom}^\bullet(P(e), L(e)) = \begin{pmatrix}
\text{id} & P(e) & P(e) & P(e) \\
P(e) & P(e) & P(e) & P(e)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
P(e) & P(e) & P(e) \\
P(e) & P(e) & P(e) & P(e)
\end{pmatrix}
\]

\[
\text{hom}^\bullet(P(s), L(e)) = \begin{pmatrix}
\text{id} & P(s) & P(s) & P(s) \\
P(s) & P(s) & P(s) & P(s)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
P(s) & P(s) & P(s) \\
P(s) & P(s) & P(s) & P(s)
\end{pmatrix}
\]

\[
\simeq 0,
\]

\[
\simeq 0,
\]

\[
\simeq 0,
\]

\[
11
\]
with the rightmost entry in degree zero. The angle brackets denote vector spaces generated by
the indicated morphisms. From (3.9) it follows that \( T'_{\mathcal{L}(e)} P(s) = P(s) \cong T_{\mathcal{L}(e)} P(s) \). For \( P(e) \), the
images under \( T'_{\mathcal{L}(e)} \) and \( T_{\mathcal{L}(e)} \) are the respective total complexes of the triple complexes

\[
T'_{\mathcal{L}(e)} P(e) = \left\{ \begin{array}{c}
P(e)_{\text{id}} \overset{a}{\rightarrow} P(s)_{\text{id}} \overset{b}{\rightarrow} P(e)_{\text{id}} \\
P(e)_{a} \overset{a}{\rightarrow} P(s)_{a} \overset{b}{\rightarrow} P(e)_{a} \\
P(e)_{\text{id}} \overset{id}{\rightarrow} P(s)_{\text{id}} \overset{b}{\rightarrow} P(e)_{\text{id}} \\
\end{array} \right\}
\]

\[
T_{\mathcal{L}(e)} P(e) = \left\{ \begin{array}{c}
P(e)_{\text{id}} \overset{a}{\rightarrow} P(s)_{\text{id}} \overset{b}{\rightarrow} P(e)_{\text{id}} \\
P(e)_{a} \overset{a}{\rightarrow} P(s)_{a} \overset{b}{\rightarrow} P(e)_{a} \\
P(e)_{\text{id}} \overset{id}{\rightarrow} P(s)_{\text{id}} \overset{b}{\rightarrow} P(e)_{\text{id}} \\
\end{array} \right\}
\]

with the gray \( P(e) \) in degree 0. We write out the double complexes \( \text{lin} \cdot (\text{hom} \circ (-,-),-) \) and \( \text{hom} \circ (-,-) \otimes - \):

\[
\begin{align*}
\begin{array}{c}
P(e)_{\text{id}} \overset{a}{\rightarrow} P(s)_{\text{id}} \overset{b}{\rightarrow} P(e)_{\text{id}} \\
\end{array} & \quad \begin{array}{c}
P(e)_{a} \overset{a}{\rightarrow} P(s)_{a} \overset{b}{\rightarrow} P(e)_{a} \\
\end{array} & \quad \begin{array}{c}
P(e)_{\text{id}} \overset{id}{\rightarrow} P(s)_{\text{id}} \overset{b}{\rightarrow} P(e)_{\text{id}} \\
\end{array} \\
\end{align*}
\]

(3.10)

where the \( P(-) \)'s are indexed by basis elements from (3.8) (i.e., by morphisms from \( P(e) \)). The adjunction maps comprise the three gray morphisms. Gauß elimination (i.e., the
removal of an identity together with the two identical summands at its ends) along the
dashed identities yields

\[
\begin{align*}
\simeq \{ P(e) \overset{a}{\twoheadrightarrow} P(s) \}, & \quad \simeq \{ P(s) \overset{b}{\twoheadrightarrow} P(e) \} \\
\simeq \text{L} \text{Sh}_a P(e), & \quad \simeq \text{R} \text{Csh}_a P(e),
\end{align*}
\]

(3.11)

with \( P(s) \) in degree zero. Comparing this to (3.2) shows half of the following:

**Theorem 3.6.** There are natural isomorphisms isomorphisms \( T'_{\mathcal{L}(e)} \cong \text{L} \text{Sh}_a \).

**Proof.** To show that both functors are isomorphic we have yet to show that the morphisms
\( \{ \text{id}_{P(e)}, \text{id}_{P(s)}, a, b \} \) generating \( \text{End}_{\mathcal{O}}(P) \) have identical images under both functors. Recall
the images \( \text{Sh}_a, a: M(s) \Rightarrow P(s) \) and \( \text{Sh}_b, b: P(s) \Rightarrow M(s) \) from (3.4).

To compute \( T'_{\mathcal{L}(e)} b \) consider the triple complex (3.10) with only the bottom three identities
eliminated. A triple complex representing \( T'_{\mathcal{L}(e)} P(s) \simeq P(s) \) with partial cancellation is
obtained similarly from (3.9). We thus obtain that \( T'_{\mathcal{L}(e)} b \) is the map

\[
\begin{align*}
T_{\mathcal{L}(e)} P(s) \simeq & \quad \begin{array}{c}
P(b) \rightarrow P(s) \rightarrow P(e) \\
P(s) \rightarrow P(e) \\
P(e) \\
\end{array} \\
T'_{\mathcal{L}(e)} P(e) \simeq & \quad \begin{array}{c}
P(e)_{\text{id}} \rightarrow P(s)_{\text{id}} \rightarrow P(e)_{\text{id}} \\
P(e)_{a} \rightarrow P(s)_{a} \rightarrow P(e)_{a} \\
P(e)_{\text{id}} \rightarrow P(s)_{\text{id}} \rightarrow P(e)_{\text{id}} \\
\end{array} \\
\end{align*}
\]

(3.12)
between the complexes representing $T'_{L(e)}P(s)$ and $T'_{L(e)}P(e)$. We have indicated the elements of $\text{hom}_O^*(P(-), L(e))$ by which the summands of $T'_{L(e)}P(-)$ are indexed. We pass to the total complexes of (3.12) and choose quasi-isomorphic replacements

$$
\begin{align*}
P(s) &\rightarrow P(e) \oplus P(s) \oplus P(s) \oplus P(e) \oplus P(e) \oplus P(e) \\
T'_{L(e)}P(s) &\cong \left\{ P(e) \left( \begin{array}{c}
-1 \\
0 \\
0
\end{array} \right), P(e) \right\} \\
T'_{L(e)}P(e) &\cong \left\{ P(e) \left( \begin{array}{c}
0 \\
1 \\
0
\end{array} \right), P(e) \right\}
\end{align*}
$$

as indicated. This shows that $T'_{L(e)}b = L\text{Sh}_s b : P(s) \rightarrow M(s)$ indeed is the canonical quotient map. A similar argument shows that $T'_{L(e)}p = \text{Sh}_s a : M(s) \hookrightarrow P(s)$ is the canonical inclusion. Since all morphisms in $D^b(O_0)$ are generated by $a$ and $b$, $T'_{L(e)}$ and $L\text{Sh}_s$ are naturally isomorphic as auto-equivalences of $D^b(O_0)$.

4 $B_n$-actions for $\mathfrak{sl}_3$ and $\mathfrak{sl}_n$. The Lie algebra $\mathfrak{sl}_3$ has as its Weyl group the symmetric group $S_3 = \{e, s, t, st, ts, w_0\}$. A quiver $Q_{\mathfrak{sl}_3}$ and a homogeneous ideal $\mathfrak{a}_{\mathfrak{sl}_3}$ of $\mathcal{C}Q_{\mathfrak{sl}_3}$ such that $O_0(\mathfrak{sl}_3) \cong \text{Mod-}A_{\mathfrak{sl}_3}$ for the path algebra $A_{\mathfrak{sl}_3} = Q_{\mathfrak{sl}_3}/\mathfrak{a}_{\mathfrak{sl}_3}$ is provided in [Str03b; Mar06]. One sees that $Q_{\mathfrak{sl}_2}$ is a full subquiver of $Q_{\mathfrak{sl}_3}$ and $\mathfrak{a}_{\mathfrak{sl}_2} \cap \mathcal{C}Q_{\mathfrak{sl}_3} = \mathfrak{a}_{\mathfrak{sl}_2}$. The inclusion $A(\mathfrak{sl}_2) \hookrightarrow A(\mathfrak{sl}_3)$ thus induced gives rise to an adjoint pair of functors

$$
\begin{align*}
\text{Res}_{\mathfrak{sl}_3}^{\mathfrak{sl}_2} & : O_0(\mathfrak{sl}_3) \rightarrow O_0(\mathfrak{sl}_2) \equiv \text{Ind}_{\mathfrak{sl}_2}^{\mathfrak{sl}_3}, \\
P(e), P(t) & \mapsto P(e), \quad P(e) \mapsto P(e), \\
P(s), P(st), P(ts), P(w_0) & \mapsto P(s), \quad P(s) \mapsto P(s),
\end{align*}
$$

which turns $O_0(\mathfrak{sl}_3)$ into a full subcategory of $O_0(\mathfrak{sl}_2)$. In particular, $\text{End}_{O_0(\mathfrak{sl}_3)}(P(s)) \cong \mathbb{C}[x]/(x^2)$ and $P(s)$ is 0-spherelike also in $O_0(\mathfrak{sl}_3)$.

Caveat 4.1. The Calabi-Yau property from definition (S3) is not “local”, in the sense that an object can lose this property in a larger ambient category. For instance, there are non-trivial morphisms $P(s) \rightarrow P(t)$ and $P(t) \rightarrow P(s)$ in $O_0(\mathfrak{sl}_3)$ whose composition is zero, so $P(s)$ cannot be spherical. We shall present two possible remedies in this section.

4.1 Spherical subcategories Consider a $k$-linear triangulated category $\mathcal{T}$.

Definition 4.2. An object $E \in \mathcal{T}$ is has a Serre dual $SE$ if the covariant functor $\text{Hom}_{\mathcal{T}}(-, E)^*$ is representable by $SE$ (here, the star indicates vector space dual). If a Serre dual can be chosen functorially and this functor is an auto-equivalence, $\mathcal{T}$ is said to have a Serre functor $S$.

Remark 4.3. The Calabi-Yau condition from definition (S3) requires that $E[d]$ be a Serre dual for $E$. 

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Assume that $E \in \mathcal{T}$ is a $d$-spherelike but not necessarily spherical object that has some Serre dual $S_E$. Since, in particular, $\text{End}_\mathcal{T}(E)^* \cong \text{Hom}_\mathcal{T}(E, S_E[-d])$, there is a morphism $x^* : E \to S_E[-d]$, which corresponds to the dual of the non-trivial endomorphism $x$ of $E$.

**Definition 4.4.** The **asphericality** of a spherelike object $E$ is $Q(E) := \text{cone}(x^*)$. Its left complement $\perp Q(E) := \{ X \in \mathcal{T} : \text{Hom}_\mathcal{T}(X, Q(E)) = 0 \}$ is a full triangulated subcategory of $\mathcal{T}$.

**Theorem 4.5** (Hochengeg, Kalck, Ploog) [HKP16, Thm. 4.4, Appendix A]. The **spherical subcategory** $\text{Sph}(E) := \perp Q(E)$ of $E$ is the largest triangulated subcategory of $D^b(\mathcal{T})$ in which $E$ is spherical.

**Example 4.6.** For $\mathfrak{g}$ a semisimple complex Lie algebra and $\lambda$ a regular weight (for instance, $\lambda = 0$), the auto-equivalence $S := \text{LSh}^{\lambda}_{\mathfrak{g}_0}$ is a Serre functor of $D^b(\mathcal{O}_\lambda)$ [MS08, Prop. 4.1].

**Proposition 4.7.** The 0-spherelike module $P(s) \in D^b(\mathcal{O}_0(\mathfrak{g}_3))$ has Serre dual $S P(s) \cong P(s)^\vee$.

**Proof.** For this proof, we take the graded structure on $\mathcal{O}_0^\mathfrak{g}$ into account. Recall from definition 2.16 that $\Theta_s^2 \cong \Theta_s \langle -1 \rangle \oplus \Theta_s \langle 1 \rangle$, and hence $\text{Sh}_s \Theta_s \cong \Theta_s \langle -1 \rangle$. Recall that for $w < ws$, $\text{Sh}_s$ maps standard factors $M(w)$ to $M(ws)$ and $M(ws)$ to $\left( M(w)/M(ws) \right) \langle -1 \rangle$. For $P(s) \in D^b(\mathcal{O}_0(\mathfrak{g}_3))$, one can, therefore, compute

$$P(s) = \Theta_s M(s) \quad \text{and} \quad \text{Sh}_s P(s) \langle -1 \rangle = \left( M(s)/M(e) \right) \langle -1 \right)$$

$$= \left( \begin{array}{ccc} M(t) \\ M(t)/M(w) \end{array} \right) \langle -2 \rangle \quad \text{Sh}_s \left( \begin{array}{ccc} M(\langle -1 \rangle) \\ M(t)/M(w) \end{array} \right) \langle -3 \rangle$$

The morphisms dual to the endomorphisms of $P(s)$ span $\text{End}(P(s))^* \cong \text{Hom}(P(s), S P(s))$; these are

$$x^* : L(s) \to L(w)$$

where the grey composition factors belong to the kernel and cokernel of $x^*$. \qed

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Theorem 4.8. The inclusion $D^b(O_0(sl_3)) \hookrightarrow D^b(O_0(sl_3))$ factors through $Sph(P(s))$.

Proof. Consider the projective resolution $P(s)^\vee \simeq \{ P(s) \rightarrow P(w_0)(-2) \rightarrow P(w_0)(-4) \}$. The asphericity $Q(P(s)) := \text{cone}(x^*)$ of $P(s)$ is the total complex

$$Q \simeq \begin{cases} & P(s)^{(-2)} \\ P(s) & \rightarrow P(w_0)^{(-2)} & \rightarrow P(w_0)^{(-4)} \end{cases}$$

with the bottom right $P(w_0)$ in homological degree 0. We claim that

$$\text{Hom}_{D^b(O(sl_3))}(P(w), Q) \begin{cases} = 0 & \text{if } w \in \{ e, s \}, \\ \neq 0 & \text{if } w \in \{ t, st, ts, w_0 \}. \end{cases}$$

Consider composition series for the modules involved in the double complex above. The $P(w_0)(-4)$ in degree 0 has composition series

$$P(w_0)(-4) = \begin{pmatrix} L(st) & L(e) & L(t) \\ L(st) & L(e) & L(t) \\ L(st) & L(e) & L(t) \end{pmatrix} (-4),$$

where $\text{im}(P(s)(-2) \xrightarrow{x_s} P(w_0)(-4)) + \text{im}(P(w_0)(-2) \rightarrow P(w_0)(-4))$ consists of the gray factors. Any map $P(e), P(s) \rightarrow P(w_0)$ must factor through $P(s)$ and hence is null-homotopic. The black factors $L(w)$ generate images of morphisms $P(w)(-) \rightarrow (-4)$ that cannot factor through $P(s)(-2)$ or $P(w_0)(-2)$ and thus are not null-homotopic. We see that the triangulated subcategory $Sph(P(s))$ of $D^b(O_0(sl_3))$ is generated by $P(s)$ and $P(e)$, which proves the claim.

We shall address another remedy for failure of the Calabi-Yau property of $P(s)$ in $O_0(sl_3)$.

### 4.2 Maximal parabolic subalgebras

Consider the category $O_0^P$ corresponding to the parabolic subgroup $W_p \cong (t) \cong S_2 \times S_1$ of $W = S_3$ with minimal-length representatives $W_0 = \{ e, s, st \}$ of cosets in $W/W_p$. An algebra $A_p$ such that $O_0^P \simeq A_p$-Mod is given by the path algebra $A_p := A/\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle = C[e \rightarrow s \rightarrow st] / \langle e \rightarrow s \rightarrow e = 0, s \rightarrow e \rightarrow s = 0, s \rightarrow e \rightarrow s \rightarrow e \rangle$, see section 2.5. The parabolic Verma modules and projectives have the following composition series:

| $MP^P(e)$ | $MP^P(s)$ | $MP^P(st)$ | $PP^P(e)$ | $PP^P(s)$ | $PP^P(st)$ |
|----------|-----------|-----------|-----------|-----------|-----------|
| $L(e)$   | $L(s)$    | $L(st)$   | $M(e)$    | $M(s)$    | $M(st)$   |
| $L(e)$   | $L(s)$    | $L(st)$   | $L(e)$    | $L(s)$    | $L(st)$   |

Since $\Theta_s$ and $LSh_s$ commute with $Z_P$, we immediately obtain the following images under translation and shuffling:

| $M$   | $\Theta_s M$ | $LSh_s M$ | $\Theta_t M$ | $LSh_t M$ |
|-------|---------------|------------|---------------|------------|
| $PP^P(e)$ | $PP^P(s)$     | $\{0 \rightarrow MP^P(s)\}$ | 0             | $\{PP^P(e) \rightarrow 0\}$ |
| $PP^P(s)$ | $PP^P(s) \oplus PP^P(s)$ | $\{0 \rightarrow PP^P(s)\}$ | $PP^P(st)$   | $\{PP^P(s) \rightarrow PP^P(st)\}$ |
| $PP^P(st)$ | $PP^P(st)$    | $\{PP^P(st) \rightarrow PP^P(st)\}$ | $PP^P(st) \oplus PP^P(st)$ | $\{0 \rightarrow PP^P(st)\}$ |
All chain complexes have the right entry in degree zero.

**Remark 4.9.** A module $M$ is $\text{Sh}_a$-acyclic if and only if the entry for $\text{LSh}_a$ is concentrated in its 0-th degree. These results are, therefore, examples for Caveat 2.25: albeit projective in the category $\mathcal{O}_0^p$, the objects $P^p(-)$ are not $\text{Sh}_a$-acyclic and hence in particular not projective in $\mathcal{O}_0$.

**Lemma 4.10.** The set $\{P^p(s), P^p(st)\}$ is an $\mathcal{A}_2$-collection of 0-spherical objects in $D^b(\mathcal{O}_0^p)$.

**Proof.** From the composition series above, one sees we get that $P^p(s)$ and $P^p(st)$ have endomorphism algebras isomorphic to $\mathbb{C}[x]/(x^2)$. In the following, all Hom-spaces are one-dimensional, and we see that the composition pairings

$$o : \text{Hom}(P^p(e), P^p(s)) \otimes \text{Hom}(P^p(s), P^p(e)) \rightarrow \text{Hom}(P^p(e), P^p(s))$$

$$o : \text{Hom}(P^p(st), P^p(s)) \otimes \text{Hom}(P^p(s), P^p(st)) \rightarrow \text{Hom}(P^p(e), P^p(s))$$

are non-degenerate; hence $P^p(s)$ and $P^p(st)$ are 0-spherical objects in $D^b(\mathcal{O}_0^p)$. Furthermore, we see from the composition series that the dimensions of the spaces $\text{Hom}(P^p(-), P^p(-))$'s are as required; thus the set $\{P^p(s), P^p(st)\}$ is indeed an $\mathcal{A}_2$-configuration.

**Proposition 4.11.** For $p$ as above, there are natural isomorphisms $T'_{p^p(s)} \cong \text{LSh}_a[-1]$ and $T'_{p^p(st)} \cong \text{LSh}_a[-1]$ of autoequivalences of $\mathcal{O}_0^p$.

**Proof.** The above shows that the respective images under $\Theta_s$ and $\Theta_t$ and $T'_{p^p(s)}$ and $T'_{p^p(st)}$ of the $P^p(-)$'s are isomorphic; hence $T'_{p^p(s)} M \cong \text{LSh}_a[-1]M$ and $T'_{p^p(st)} M \cong \text{LSh}_a[-1]M$ for all $M \in D^b(\mathcal{O}_0^p)$.

The proof that these isomorphisms of images form a natural isomorphism of functors is carried out analogously to theorem 3.3. The proof for $\Theta_s \cong T'_{p^p(s)}$ is carried out, m. m., the same way as for $\Theta_s \cong T'_{p^p(st)}$; therefore, we only show the latter. To that end, we show that $M_{\Theta_s} \cong M_{\Theta_s}$, where the notation is the same as in theorem 3.3.
The $A_p$-$A_p$-bimodule $M_{\Xi_{p^p}(s)} = P^p(s)^* \otimes C P^p(s)$ has a $C$-basis given by all pairwise tensor products

\begin{align}
(s \leftarrow e \leftarrow s) & \rightarrow (e \leftarrow s) \rightarrow (s \leftarrow e) \rightarrow e^* \\
(s \leftarrow e) & \rightarrow (e \leftarrow s) \rightarrow (s \leftarrow e) \rightarrow e^* \\
(e \leftarrow s) & \rightarrow (s \leftarrow e) \rightarrow (e \leftarrow s) \rightarrow e^*
\end{align}

with the indicated $A_p$-$A_p$-bimodule action. The algebra $\text{End}_C(P^p)$ is generated by the following four elements; a diagram chase of morphisms through the relevant naturality diagrams shows that $\Theta_s$ acts on these by

$$
P^p = P(e) \oplus P(s) \oplus P(st) \quad P(s) = P(s) \oplus P(s)_2 \oplus P(s)_4 = \Theta_s P^p
$$

A vector space basis of $M_{\Theta_s}$ = $\text{Hom}(P^p, \Theta_s P^p)$ is given by the morphisms between the summands of $P^p$ in the following schematic. Recall that the left $A^p$-action on $M_{\Theta_s}$ is given by $\phi \cdot m = \Theta_s(\phi) \circ m$ for $\phi \in A^p$ and $m \in M_{\Theta_s}$, so it acts on basis elements as follows:

we obtain that the $A^p$-$A^p$-bimodule action on its $C$-basis is

$$
\Theta_s(s \leftarrow e \leftarrow s) \rightarrow \Theta_s(e \leftarrow s) \rightarrow \Theta_s(s \leftarrow e) \rightarrow \Theta_s(e) \rightarrow \Theta_s(s) \rightarrow e^* \\
\Theta_s(s \leftarrow e) \rightarrow \Theta_s(e \leftarrow s) \rightarrow \Theta_s(s \leftarrow e) \rightarrow \Theta_s(e) \rightarrow \Theta_s(s) \rightarrow e^* \\
\Theta_s(e \leftarrow s) \rightarrow \Theta_s(s \leftarrow e) \rightarrow \Theta_s(e \leftarrow s) \rightarrow \Theta_s(e) \rightarrow \Theta_s(s) \rightarrow e^*
$$

Comparing (4.7) and (4.9) shows that the isomorphism $M_{\Xi_{p^p}(s)} \cong M_{\Theta_s}$ of vector spaces is an isomorphism of $A^p$-$A^p$-bimodules. It follows that $T_{\Xi_{p^p}(s)} \simeq \{1 \rightarrow - \otimes M_{\Xi_{p^p}(s)}\}$ and $\text{Sh}_s = \{1 \rightarrow - \cong M_{\Theta_s}\}$ are naturally isomorphic functors.

We now transfer results for $\mathcal{O}_0^p$ from $\mathfrak{sl}_k$ to $\mathfrak{sl}_n$. Consider the parabolic subalgebra $p$ of $\mathfrak{sl}_n$ corresponding to the subgroup $W_p = \{s_2, \ldots, s_{n-1}\} = S_{n-1} \times S_1 \leq S_n$ and the coset representatives $W^p = \{e, s_1 s_2, \ldots, s_1 \cdots s_{n-1}\}$; we abbreviate these representatives in $W^p$ by $s_i := s_1 \cdots s_i$.

Lemma 4.12. The category $\mathcal{O}_0^p(\mathfrak{sl}_n)$ is equivalent to $\text{Mod-}A^p(\mathfrak{sl}_n)$, where $A^p(\mathfrak{sl}_n)$ is the path algebra quotient $A^p = C[e \leftarrow \sigma_1 \leftarrow \cdots \leftarrow \sigma_{n-1}] / \left( e \leftarrow \sigma_1 \leftarrow e \leftarrow 0, \sigma_i \leftarrow \sigma_{i+1} \leftarrow e \leftarrow \sigma_i \right) \right.$ for $1 \leq i \leq n-2$.

Proof. We compute the composition series of Verma modules and indecomposable projectives in $\mathcal{O}_0^p$ using the generalised Kazhdan-Lusztig theorem [Irv90, Cor. 7.1.3; CC87, Thm. 1.3]. For parabolic subgroups of the form $W_p = S_k \times S_{n-k} \leq S_n$ there is a graphical calculus for computing parabolic Kazhdan-Lusztig polynomials [BS11, §5; LS13]. The composition series...
thus obtained are:

\[
\begin{array}{cccccc}
& e & s_1 & s_1s_2 & \cdots & s_1\cdots s_{n-1} \\
M^p(-) & L(e) & L(\sigma_1) & L(\sigma_2) & \cdots & L(\sigma_{n-1}) \\
& L(s_1) & L(s_2) & L(s_3) & \cdots & L(s_{n-1}) \\
P^p(-) & n & L(\sigma_1) & L(\sigma_2) & L(\sigma_3) & \cdots & L(\sigma_{n-1}) \\
& L(s_1) & L(s_2) & L(s_3) & \cdots & L(s_{n-1}) \\
\end{array}
\]  

\[(4.10)\]

The arrows in \(Q^p(s_{1,n})\) are given by irreducible morphisms between \(P^p(-)\)'s, so the relations of \(Q^p(s_{1,n})\) follow from the composition series in (4.10).

\[\square\]

Remark 4.13. Every Verma module \(M^p(\sigma_i)\) fits uniquely into a short exact sequence

\[M^p(\sigma_i) \hookrightarrow P^p(\sigma_{i+1}) \twoheadrightarrow M^p(\sigma_{i+1});\]

in particular, \(\Theta_{s_{1,n}}M^p(\sigma_i) = P^p(\sigma_i)\) is always projective, which is not true in the non-parabolic category \(O_0\). We obtain the following list of images under translation and shuffling functors:

\[
\begin{array}{cccccccc}
M & \Theta_{s_1} M & \text{L Sh}_{s_1} M & \Theta_{s_2} M & \text{L Sh}_{s_2} M & \Theta_{s_3} M & \text{L Sh}_{s_3} M & \cdots \\
\hline
P^p(e) & P^p(\sigma_1) & M^p(\sigma_1) & 0 & P^p(\sigma_2) & 0 & P^p(e)[1] & \\
P^p(\sigma_1) & P^p(\sigma_1) & P^p(\sigma_2) & P^p(\sigma_2) & P^p(\sigma_2) & 0 & P^p(\sigma_1)[1] & \\
M^p(\sigma_1) & P^p(\sigma_1) & \{M^p(\sigma_1) \to P^p(\sigma_1)\} & P^p(\sigma_2) & M^p(\sigma_2) & 0 & M^p(\sigma_1)[1] & \\
P^p(\sigma_2) & P^p(\sigma_1) & \{M^p(\sigma_2) \to P^p(\sigma_1)\} & P^p(\sigma_2) & P^p(\sigma_2) & P^p(\sigma_2) & \{P^p(\sigma_2) \to P^p(\sigma_3)\} & \\
M^p(\sigma_2) & 0 & M^p(\sigma_2)[1] & P^p(\sigma_2) & \{M^p(\sigma_2) \to P^p(\sigma_2)\} & P^p(\sigma_3) & M^p(\sigma_2) & \\
P^p(\sigma_3) & 0 & P^p(\sigma_3)[1] & P^p(\sigma_3) & \{P^p(\sigma_3) \to P^p(\sigma_2)\} & P^p(\sigma_3) & P^p(\sigma_3) & \\
M^p(\sigma_3) & 0 & M^p(\sigma_3)[1] & 0 & M^p(\sigma_3)[1] & P^p(\sigma_3) & \{M^p(\sigma_3) \to P^p(\sigma_3)\} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

**Lemma 4.14.** \(\{P^p(\sigma_1), \ldots, P^p(\sigma_{n-1})\}\) is an \(A_{n-2}\)-configuration of 0-spherical objects.

Proof. The composition series exhibit that \(\text{Hom}_{\mathcal{O}_0^0}(P^p(\sigma_i), P^p(\sigma_i)) \cong \mathbb{C}_x/(x^2)\), where the nontrivial endomorphism \(x\) is the degree 2-map

\[x : P^p(\sigma_i) \to \text{hd} P^p(\sigma_i) = \text{soc} P^p(\sigma_i) \hookrightarrow P^p(\sigma_i),\]

and that

\[
\dim \text{Hom}_{\mathcal{O}_0^0}(P^p(\sigma_j), P^p(\sigma_i)) = \begin{cases} 
2 & \text{if } i = j, \\
1 & \text{if } |i - j| = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

It is sufficient to check non-degeneracy of the composition pairing for indecomposable projectives \(P^p(\sigma_i \pm 1)\) that are connected by an arrow of \(Q^p(s_{1,n})\). In these two cases, the picture is analogous to (4.4–4.6):

\[
\circ : \text{Hom}(P^p(\sigma_{i \pm 1}), P^p(\sigma_i)) \otimes \text{Hom}(P^p(\sigma_i), P^p(\sigma_{i \pm 1})) \equiv \langle x_{P^p(\sigma_i)} \rangle \quad (4.11)
\]
for $0 < i < n - 2$, and similar for $i = 0$ and $n - 1$. This shows that $\circ$ gives a non-degenerate pairing.

**Theorem 4.15.** for the parabolic subalgebra $p \subseteq sl_n$ corresponding to $W_p = S_{n-1} \times S_1 < S_n$, the autoequivalences $L_{Sh_n}[-1]$ and $T_{sp}^p(\sigma_i)$ of $D^b(O^p_0)$ are naturally isomorphic.

**Proof.** Let $A^p_n := A^p(sl_n)$. Assume that the assertion holds for the respective subalgebra of $sl_{n-1}$. One checks that the assignment $Q_{sl_{n-1}} \to Q_{sl_n}, \sigma_i \to \sigma_{i-1}$ of quivers gives rise to an isomorphism $A^p_n/(\varepsilon_{\sigma_i}) \to A^p_{n-1}$ of path algebras. The maps

\[
p: A^p_n \to A^p_n/(\varepsilon_{\sigma_i}) \cong A^p_{n-1}, \quad i: A^p_{n-1} \hookrightarrow A^p_n
\]

thus defined induce fully faithful functors

\[
p^*: O^p_0(sl_{n-1}) \to O^p_0(sl_n), \quad i_*: O^p_0(sl_{n-1})-Proj \to O^p_0(sl_n)-Proj, \quad P^p(e) \mapsto M^p(\sigma_1), \quad P^p(\sigma_i) \mapsto P^p(\sigma_i + 1)
\]

for $1 \leq k < n - 1$ and $0 \leq l < n - 1$. By induction, this shows that the assertion holds for the functors restricted to the triangulated subcategories $(P^p(\sigma_2), \ldots, P^p(\sigma_{n-1}))$ (via $p^*$) and $(P^p(e), P^p(\sigma_1), P^p(\sigma_2))$ (via $i_*$) of $D^b(O^p_0(sl_n))$. Hence, the assertion holds on $O^p_0(sl_n)$. □

**Remark 4.16.** We know that the object $P^p(s) \in O^p_0(sl_3)$ is spherical, so one might ask whether $O^p_0(sl_3)$ arises as the spherical subcategory $Sph(P^p(s))$ of $P^p(s) \in O_0(sl_3)$. However, $P^p(s)$ is not spherical in $D^b(O(sl_3))$, i.e., we cannot assign a meaningful spherical subcategory to it.

**Proof.** Consider the projective resolution $P^p(s) \simeq \{P(s) \to P(ts) \to P(s)\}$ in $D^b(O(sl_3))$. Using this resolution, we obtain the chain complex

\[
\text{hom}_{D^b(O_0)}(P^p(s), P^p(s)) \cong \left\{ \begin{array}{c}
P(s) \\ \downarrow \\ P(s)
\end{array} \to \begin{array}{c}
P(ts) \\ \downarrow \\ P(ts)
\end{array} \to \begin{array}{c}
P(s)
\end{array} \to \begin{array}{c}
P(s) \\ \downarrow \text{id}_{x}
\end{array} \to \begin{array}{c}
P(s)
\end{array} \to \begin{array}{c}
P(ts) \\ \downarrow \\ P(ts)
\end{array} \to \begin{array}{c}
P(s)
\end{array} \to \begin{array}{c}
P(s)
\end{array} \right\},
\]

i.e., $\text{hom}_{D^b(O_0)}(P^p(s), P^p(s))$ has total dimension 3, so $P^p(s)$ is not spherical. As a side note, we notice that the inclusion $D^b(O^p_0) \subset D^b(O_0)$, given by mapping projectives to projectives, is not full. □

**Remark 4.17.** The object $L(\sigma_{n-2}) \in D^b(O^p_0(sl_n))$ is 2-spherical. However, its induced spherical twist functor does not yield an $L_{Sh_n}[-\cdot]$ for $n > 2$.

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References

[Bar15] M. Barot. *Introduction to the Representation Theory of Algebras*. Springer, 2015. ISBN: 978-3-319-11474-3. DOI: 10.1007/978-3-319-11475-0.

[Bas68] H. Bass. *Algebraic K-Theory*. 1st ed. New York, Amsterdam: W. A. Benjamin, 1968.

[BB81] A. Beilinson and J. Bernstein. ‘Localisation de g-modules’. In: *C. R. Acad. Sci. Paris. Série I*: Sciences mathématique 292.1 (1981), pp. 15–18. URL: http://gallica.bnf.fr/ark:/12148/bpt6k6226873r/f32.image.

[BGG76] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand. ‘A certain category of g-modules’. In: *Funkcional. Anal. i Prilozen*. 10.2 (1976), pp. 1–8.

[BGS96] A. Beilinson, V. Ginzburg and W. Soergel. ‘Koszul duality patterns in representation theory’. In: *J. Amer. Math. Soc.* 9.2 (1996), pp. 473–527. DOI: 10.1090/S0894-0347-96-00192-0.

[BK81] J.-L. Brylinski and M. Kashiwara. ‘Kazhdan-Lusztig conjecture and holonomic systems’. In: *Invent. Math.* 64.3 (1981), pp. 387–410. DOI: 10.1007/BF01389272.

[BS11] J. Brundan and C. Stroppel. ‘Highest Weight Categories Arising from Khovanov’s Diagram Algebra I: Cellularity’. In: *Mosc. Math. J.* 11.4 (2011), pp. 685–722, 821–822. arXiv: 0806.1532.

[Car86] K. J. Carlin. ‘Extensions of Verma modules’. In: *Trans. Amer. Math. Soc.* 294.1 (1986), pp. 29–43. DOI: 10.2307/2000116.

[CC87] D. H. Collingwood and L. G. Casian. ‘The Kazhdan-Lusztig Conjecture for Generalized Verma Modules’. In: *Math. Z.* 195 (1987), pp. 581–600. DOI: 10.1007/bf01166705. EuDML: 183714.

[Deo87] V. Deodhar. ‘On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials’. In: *J. Algebra* 111.2 (1987), pp. 483–506. DOI: 10.1016/0021-8693(87)90232-8.

[Gab72] P. Gabriel. ‘Unzerlegbare Darstellungen I.’ In: *Manuscripta mathematica* 6 (1972), pp. 71–104. EuDML: 154087.

[Gab73] P. Gabriel. ‘Indecomposable representations II’. In: *Symposia Mathematica*. Convegno di Algebra Commutativa (Rome, 1971). Vol. XI. London: Academic Press, 1973, pp. 81–104.

[Gro57] A. Grothendieck. ‘Sur quelques points d’algèbre homologique’. In: *Tohoku Mathematical Journal*. 2nd ser. 9 (1957), pp. 119–221. DOI: 10.2748/tmj/1178244839.

[HKP16] A. Hochencogger, M. Kalck and D. Ploog. ‘Spherical subcategories in algebraic geometry’. In: *Math. Nachr.* 289.11-12 (2016), pp. 1450–1465. DOI: 10.1002/mana.201400232. arXiv: 1208.4046.

[Hum08] J. E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category O*. Graduate Studies in Mathematics 94. American Mathematical Society, 2008. ISBN: 978-0-8218-4676-0.

[Hum72] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics 9. New York: Springer, 1972. ISBN: 978-1-4612-6398-2.

[Hum90] J. E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge Studies in Advanced Mathematics 29. Cambridge University Press, 1990.

[Irv88] R. S. Irving. ‘The socle filtration of a Verma module’. In: *Ann. Sci. ENS*. 4th ser. 21 (1988), pp. 47–65. DOI: 10.24033/asens.1550.

[Irv90] R. S. Irving. *A filtered category Os and applications*. Vol. 1. 6 vols. Memoirs of the American Mathematical Society 419. Providence, RI: Amer. Math. Soc., 1990. ISBN: 978-0-8218-2482-5.
