Comment

Is the phase of plane waves a frame-independent quantity?
Comment on “The invariance of the phase of waves among inertial frames is questionable” by Huang Young-Sea

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In a recent letter by Huang [1], the invariance of the phase of plane waves among inertial frames has been challenged for the case of a “superluminal” motion of the medium, when the wave propagation in a stationary optical medium is observed from a frame traveling in the direction of the wave at a speed larger than the speed of the wave with respect to the medium. Apparently, by using the form-invariance of the phase of waves \( \Phi = \mathbf{k} \cdot \mathbf{r} - \omega t \) under a Lorentz transformation, Huang obtained negative values for the angular frequency \( \omega \) of the wave with respect to the frame in which the medium is in motion. To overcome this difficulty, Huang introduced two types of relativistic transformation for the wave four-vector \( \mathbf{k}^\mu = (\mathbf{k}, \omega/c) \) based on differential Lorentz transformation.

In this letter, we argue that the reason that eventually led Huang to this apparent non-invariance of the phase of waves is the ignorance of the effect of relativistically induced optical anisotropy [2] in the analysis of the problem.

Let us consider a three-dimensional plane-wave disturbance with a harmonic wave form [3,4]. We choose the profile of the wave to vary in space sinusoidally, namely \( \psi(\mathbf{r}, t = 0) = A \sin(\mathbf{k} \cdot \mathbf{r}) \). We observe that \( \psi(\mathbf{r}, t = 0) \) is constant when \( \mathbf{k} \cdot \mathbf{r} = \text{const} \), which is an equation defining a set of planes perpendicular to the vector \( \mathbf{k} \). The value of \( \psi(\mathbf{r}, t = 0) \) repeats itself in space after a displacement \( n\lambda \) in the direction of \( \mathbf{k} \), where \( \lambda \) is the wavelength of the wave, and \( n \) is an integer. If the profile of the wave moves at a velocity \( \mathbf{u} \), it is described by a harmonic wave function in the form
\[
\psi(\mathbf{r}, t) = A \sin \mathbf{k} \cdot (\mathbf{r} - \mathbf{u}t) = A \sin \Phi,
\]
which we obtain by merely replacing \( \mathbf{r} \) in \( \psi(\mathbf{r}, t = 0) \) with \( \mathbf{r} - \mathbf{u}t \). The expression
\[
\Phi = \mathbf{k} \cdot \mathbf{r} - \mathbf{k} \cdot \mathbf{u}t
\]
is the phase of the wave, and \( \mathbf{k} \) is the wave vector, having a magnitude \( 2\pi/\lambda \) and pointing in the direction orthogonal to the planes \( \Phi = \text{const} \). For a fixed stationary observer at \( \mathbf{r} = \mathbf{r}_0 \), the wave disturbance \( \psi(\mathbf{r}_0, t) \) repeats itself in time after a temporal period \( T = 2\pi/|\mathbf{k} \cdot \mathbf{u}| \), the inverse of which is the frequency \( f \) of the wave. Here, the absolute value of the dot product between the wave vector and the velocity of the wave is the angular frequency of the wave, commonly denoted by \( \omega = |\mathbf{k} \cdot \mathbf{u}| \). In the case of isotropic media (including vacuum), the wave vector \( \mathbf{k} \) and the wave velocity \( \mathbf{u} \) are parallel and pointing in the same direction, allowing the expression for the phase of the wave in eq. (2) to be recasted in the frequently-used form
\[
\Phi = \mathbf{k} \cdot \mathbf{r} - \omega t.
\]

If the plane wave disturbance in eq. (1) is observed from a different reference frame, the phase of the wave \( \Phi \) should remain invariant [4,5]. By using the form-invariance of the expression for the phase \( \Phi \) in eq. (3) to Lorentz-transform the wave characteristics, Huang obtained negative angular frequencies in the frame where the medium moves at “superluminal” speeds against the wave. What has not been taken into account in Huang’s analysis is that the dot product between the wave vector and the velocity of the wave changes its sign from positive to negative when switching between the medium’s rest frame and the frame in which the medium moves “superluminal”. In simple words, the expression for the phase in eq. (3) is generally not form-invariant. Let us clarify this point more explicitly. The approach by using eq. (3) as a form-invariant expression for the phase will work efficiently if the wave is propagating in vacuum. In the vacuum case,
the wave vector and the velocity of the wave will remain parallel and unidirectional with respect to any inertial frame. Hence, $k \cdot u = |k \cdot u| = \omega$ and $k' \cdot u' = |k' \cdot u'| = \omega'$, and the form-invariance $k \cdot r - k \cdot u t = k' \cdot r' - k' \cdot u' t'$ between the frames $S$ and $S'$ would imply $k \cdot r - \omega t = k' \cdot r' - \omega' t'$. However, when the wave propagates in a material medium, the form-invariance of the phase given in eq. (3) is generally not preserved even if the optical medium is isotropic in its rest frame. This is due to the fact that an optical medium that is optically isotropic in its rest frame $S'$ will possess an effective optical anisotropy in the frame $S$ in which it is moving at a constant velocity $V$. Hence, while the wave vector and the velocity of the wave are parallel and unidirectional with respect to the rest frame $S'$ of the medium, this may not be the case with respect to some reference frame $S$ in which the medium is in motion. Consequently, the angle between the wave vector and the velocity of the wave in a medium depends on the reference frame from where the observation is made. Since the dot product between these vectors depends on the cosine of the angle between them, we may conclude that the sign of $k \cdot u = |k \cdot u| = -\omega < 0$ when $k' \cdot u' = |k' \cdot u'| = \omega' > 0$. In this sense, the expression for the phase of the wave in eq. (3) is not form-invariant under a Lorentz transformation.

To obtain the correct relativistic transformations of the wave characteristics between the frames $S$ and $S'$, we will utilize the form-invariance of the phase in eq. (2). This expression can be re-written as

$$\Phi = k^\mu x^\mu,$$

(4)

where $x^\mu = (r, ct) = (x, y, z, ct)$ is the displacement 4-vector, and

$$k^\mu = \left( \frac{k_0, k_y, k_z, k_x u_x + k_y u_y + k_z u_z}{c} \right),$$

(5)

is the wave 4-vector. Suppose that in $S'$-frame the wave is travelling in the direction determined by its velocity $u'$, and the wavefront normal is described by the wave 3-vector $k'$. The components of the wave 4-vector in $S'$ are then given by $k^\mu = (k', k' \cdot u'/c)$. Taking the frames $S$ and $S'$ to be in standard configuration, and $S'$ moving at a constant speed $V$ along the $x$-direction with respect to $S$, we apply the Lorentz transformation to $k^\mu$ and obtain its components in the $S$-frame:

$$k_0 = \gamma \left( k'_0 + \frac{V}{c^2} k' \cdot u' \right), \quad k_y = k'_y, \quad k_z = k'_z,$$

(6)

$$k \cdot u = \gamma \left( \frac{k'_0 V}{c^2} + \frac{k' \cdot u'}{c} \right),$$

(7)

where $\gamma = (1 - V^2/c^2)^{-1/2}$. If we put eqs. (6) into eq. (7), and compare the terms for arbitrary $k'_x$, $k'_y$ and $k'_z$, we also obtain

$$u_x = \frac{u'_x + V}{1 + u'_x V/c^2}, \quad u_y = \frac{u'_y}{1 + u'_y V/c^2}, \quad u_z = \frac{u'_z}{1 + u'_z V/c^2}$$

(8)

which are the relativistic velocity transformation formulas for the wave. To describe the relativistic transformation of the wave characteristics from $S'$ to $S$, in addition to eqs. (6)–(8) we also need the formulas for the angular frequencies of the wave in the corresponding frames

$$\omega' = |k' \cdot u'|, \quad \omega = |k \cdot u|.$$  

(9)

We will use the above analysis to explain the situation discussed in the afore-mentioned letter [1]. In the rest frame $S'$ of the medium, a plane wave propagates in the direction of the positive $x'$-axis at a speed $u'$. Assuming that the medium is homogeneous, isotropic and nondispersive in $S'$, the components of the wave 4-vector are $k^\mu = (k', 0, 0, k'u'/c)$, and the angular frequency of the wave is $\omega' = |k' \cdot u'| = k'u'$. Evidently, the velocity of the wave and the wavefront normal in $S'$-frame are parallel and unidirectional. Transforming to $S$-frame in which the medium moves at a velocity $V = (-V, 0, 0)$, from eqs. (6)–(9) we obtain $k_0 = \gamma k'_0 (1 - u'V/c^2)$, $k_y = k'_y = 0$, $u_x = (u' - V)/(1 - u'V/c^2)$, $u_y = u_z = 0$ and $\omega = |k \cdot u| = \gamma |\omega' - k'V|$. In the case of “superluminal” motion of the medium, $u' < c$, which implies $k_0 > 0$ and $u_x < 0$. Hence, the dragging of the wave by the medium is overwhelming, and the wave will propagate in the negative $x$-axis as the medium. That the wave 3-vector and the velocity of the wave in $S$-frame remain parallel, but in opposite directions, is a drastic example of the relativistically induced optical anisotropy at work. The reader may notice that the frequency of the wave remains positive by definition, in spite of the fact that $k \cdot u < 0$.

In conclusion, when analyzing the wave propagation in a medium with respect to different inertial reference frames, one should use the correct expression for the wave four-vector: $k^\mu = (k, k \cdot u/c)$, where $k$ is the wave three-vector, and $u$ is the velocity of the wave. Employing the less general, but widely used expression for the four-vector in the form $k^\mu = (k, \omega/c)$ as in the recent letter [1] one may be tempted into a spurious conclusion that the invariance of the phase of waves among inertial frames is questionable.

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