GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS FOR A REACTIVE POLYMERIC FLUID NEAR EQUILIBRIUM

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ABSTRACT. In this paper, we study a new micro-macro model for a reactive polymeric fluid, which is derived recently in [Y. Wang, T.-F. Zhang, and C. Liu, J. Non-Newton. Fluid Mech. 293 (2021), 104559, 13 pp], by using the energetic variational approach. The model couples the breaking/reforming reaction scheme of the microscopic polymers with other mechanical effects in usual viscoelastic complex fluids. We establish the global existence of classical solutions near the global equilibrium, in which the treatment on the chemo-mechanical coupling effect is the most crucial part. In particular, a weighted Poincaré inequality with a mean value is employed to overcome the difficulty that arises from the non-conservative number density distribution of each species.

Keywords: Global existence; Viscoelastic fluids; Energetic variational approach; A priori estimate; Weighted Poincaré inequality.

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1. INTRODUCTION

Complex fluids are fluids with complicated rheological phenomena, arising from different “elastic” effects, such as elasticity of deformable particles, interaction between charged ions and bulk elasticity endowed by polymer molecules. These complicated elastic effects can

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usually be realized by the coupling between the dynamics of macroscopic fluids and the induced elastic stress at the microscopic level. The micro-macro models have been widely used to describe the dynamics of complex fluids, including especially for polymeric fluids [1, 6, 22, 25–27]. Beyond that, for many other materials and complex fluids model like living polymers and reptation model, the mathematical theory are almost unexplored [25].

In this paper, we consider a reactive complex fluids model proposed firstly in [48]:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla x u + \nabla x p &= \mu \nabla x u + \lambda \nabla x \left[ \int_{\mathbb{R}^3} \left( \nabla q U_A \Psi_A + \nabla q U_B \Psi_B \right) \otimes dq \right], \\
\nabla x \cdot u &= 0, \\
\partial_t \Psi_A + \nabla x \cdot (u \Psi_A) + \nabla q \cdot (\nabla x u q \Psi_A) &= \nabla q \cdot (\nabla q U_A \Psi_A) - (k_1 \Psi_A - k_2 \Psi_B^2), \\
\partial_t \Psi_B + \nabla x \cdot (u \Psi_B) + \nabla q \cdot (\nabla x u q \Psi_B) &= \nabla q \cdot (\nabla q U_B \Psi_B) + 2(k_1 \Psi_A - k_2 \Psi_B^2).
\end{aligned}
\]  

(1.1)

Here the unknowns \( \Psi_A(t, x, q) \) and \( \Psi_B(t, x, q) \) are number density distribution functions for two species \( A \) and \( B \) with different chain of lengths, depending on time \( t \geq 0 \), macroscopic position \( x \in \Omega \subset \mathbb{R}^3 \), and microscopic configuration variable \( q \in D \) (\( D \) is a bounded or unbounded domain in \( \mathbb{R}^3 \)). \( U_A(q) \) and \( U_B(q) \) denote the spring potential functions. This is a two-species model for a wormlike micellar solution (also called living polymers), in which linear chain polymers that can break and recombine continuously [2, 45, 48]. Both species of polymer particles are considered to be transported along with an incompressible fluid flow, whose evolution is described by its velocity field \( u(t, x) \). From a viewpoint of mathematics, this model couples the incompressible Navier-Stokes equations and kinetic Fokker-Planck equations through the induced elastic stress term in (1.1) (the last term on the right-hand side), and the drift terms in (1.1)\(_3\)–(1.1)\(_4\) (the third terms on the left-hand side).

This two-species micro-macro system (1.1) is inspired by the macroscopic phenomenological Vasquez-Cook-McKinley (VCM) model [45], where the longer chains can break at the middle to form shorter chains, and at the same time shorter chains can also recombine to form into longer chains. We denote the two species by \( A \) and \( B \), respectively, corresponding to the system (1.1). The breakage/reforming procedure between the two species is modeled as the chemical reaction scheme: \( A \overset{k_1}{\underset{k_2}{\rightleftharpoons}} 2B \). According to the law of mass action (LMA) in chemical kinetics theory [46], which indicates that the rate of a reaction process is proportional to the concentrations of the reactants, the total reaction rate can be expressed as follows:

\[
r = k_1 \Psi_A - k_2 \Psi_B^2,
\]  

(1.2)

where \( k_1 \) and \( k_2 \) are the rate coefficients for breakage and reforming process respectively [48].

It can be checked that the micro-macro model (1.1) obeys the following entropy-entropy production law:

\[
\frac{d}{dt} \left\{ \int_{\Omega} \frac{1}{2} |u|^2 \, dx + \lambda \int_{\Omega \times \mathbb{R}^3} \left[ \Psi_A (\ln \Psi_A + U_A - 1) + \Psi_B (\ln \Psi_B + U_B - 1) \right] \, dq \, dx \right\} = - \int_{\Omega} \mu |\nabla x u|^2 \, dx - \lambda \int_{\Omega \times \mathbb{R}^3} \left[ \Psi_A |\nabla q (\ln \Psi_A + U_A)|^2 + \Psi_B |\nabla q (\ln \Psi_B + U_B)|^2 \right] \, dq \, dx
\]

\[
- \lambda \int_{\Omega \times \mathbb{R}^3} \left( k_1 \Psi_A - k_2 \Psi_B^2 \right) (\ln \frac{\Psi_A}{\Psi_B^2} + U_A - 2U_B) \, dq \, dx.
\]  

(1.3)

Indeed, the PDE system (1.1) can be derived from this energy-dissipation law due to its variational structure, by using the energetic variational approach (EnVarA). A detailed derivation will be given in section 2, and we also refer interested readers to [48] for details on modeling and numerical simulations. Our main concern in this present paper is the global well-posedness around the global equilibrium.

**Brief reviews.** The study of viscoelastic fluids, as one of the most important types of complex fluids, has been an active research field that attracts more and more attentions from
different areas. There are two important and famous models for dilute polymeric fluids: the FENE (Finite Extensible Nonlinear Elastic) model and the Hookean dumbbell model, up to different choices for potential functions. In the former case, the potential is chosen as $U(q) = -k \ln(1 - \frac{|q|^2}{b_0^2})$ for some constant $k > 0$, in which it is commonly assumed that the polymer elongation vector belongs in a bounded ball (finite extensibility), i.e., $q \in B(0, b_0)$ with radius $b_0 > 0$. In the latter Hookean case, $U(q) = \frac{1}{2} |q|^2$ with $q \in \mathbb{R}^3$. By considering the equations of second moments $\tau = \int_{\mathbb{R}^3} \nabla q U \otimes q \Psi dq$ via an approximate closure process, one can recover the Oldroyd-B model \cite{38}. There are a huge of literatures in those fields, we just review here some related works of mathematical treatment in rigorous analysis, especially on well-posedness problems, and we refer interested readers to the surveys \cite{17, 37} and references therein for more issues.

As for the local existence, there are many researches from different settings, see \cite{8, 20, 24, 42, 51}, for instance. Concerning the global existence, Chemin-Masmoudi \cite{3} proved the local and global well-posedness in critical Besov spaces. Lions-Masmoudi \cite{29} considered the global weak solutions. We emphasize that Masmoudi proved in \cite{36} the global weak solutions to the FENE model by finding some new a priori estimates to justify the compactness. Hu-Lin proved recently in \cite{15, 16} the global existence of weak solution for incompressible and compressible case, respectively, by assuming some small conditions on initial velocity and initial perturbated deformation tensor. As for global existence of classical solutions with small data, we refer to, for example, Lei-Liu-Zhou \cite{23}, Lin-Zhang-Zhang \cite{28} and Masmoudi \cite{35}. Lin-Liu-Zhang \cite{27} used the EnVarA to derive rigorously a micro-macro polymeric system and proved global existence near the global equilibrium with some assumptions on the potential $U$ (including the Hookean spring case). Corresponding to the incompressible model in \cite{27}, Jiang-Liu-Zhang \cite{19} derived the compressible model by EnVarA and proved an analogous global result, where the two flow maps of both macroscopic spatial variables $x$ and microscopic configuration $q$ must be considered explicitly in the the derivation process.

The two-species micro-macro model (1.1) is a typical example of involves kinetic-fluid/micro-macro coupling, viscoelastic effect and chemical reaction process. Precisely speaking, (1.1) contains a micro-macro coupling between each kinetic equation of $\Psi_\alpha$ and the fluid equation of $u$, and the chemical reversible breaking/reforming reaction which satisfies the microscopic law of mass action. Furthermore, such a reversible reaction effect couples with viscoelastic effect under the presence of an extensional fluid flow. This also causes some inhomogeneous flow structures like shear banding. These above coupling effects and intrinsic features cause many difficulties for us in analyzing the well-posedness of solutions in an appropriate sense.

In addition, we mention that one can obtain from a kinetic equation the macroscopic constitutive equations by performing an approximate moment closure process. The macroscopic equations are usually referred as viscoelastic fluid model. The typical examples are the studies for classic FENE model and liquid crystal polymeric model in complex fluids theory, see \cite{18, 50}, for instance. Associated to the two-species micro-macro model (1.1), a viscoelastic system has been obtained from modelling and simulation perspectives in our another paper \cite{48}. The method is mainly based on the maximum entropy principle, and some different approaches of closure approximations are also discussed there. This in turn raises some new topics worthy of attentions, such as the well-posedness of the viscoelastic system, and the asymptotic relation between the micro-macro model and the macroscopic viscoelastic model with respect to some scaled parameter. We will study these topics in forthcoming papers.

**Equilibrium and perturbation system.** Our main concern in the present paper lies in the dynamic stability analysis on the two-species micro-macro model (1.1). Precisely speaking, we will study the global existence of classical solution near a global equilibrium in a perturbative framework.
The global equilibrium state of each species $\alpha = A, B$ is defined by a Maxwellian function $M_\alpha(q)$ (with a normalized coefficient $c_\alpha$ being fixed later):

$$M_A(q) = c_Ae^{-U_A}, \quad M_B(q) = c_Be^{-U_B}. \quad (1.4)$$

On the other hand, the detailed balance condition for the reaction scheme requires

$$K_{eq} = \frac{k_2}{k_1} = \frac{M_A}{M_B}.$$  

By assuming $c_A = c_B^2$ for simplicity, we have

$$K_{eq} = \frac{k_2}{k_1} = e^{-(U_A-2U_B)}. \quad (1.5)$$

As a beginning of this program, we consider here a simplified case that $K_{eq} = \frac{k_2}{k_1} = 1$ keeps as a constant, which implies $U_A = 2U_B$. Therefore, the dissipation on the reaction part in the right-hand side of (1.3) can be rewritten as,

$$D_R = \int_{\Omega \times \mathbb{R}^3} (k_1 \Psi_A - k_2 \Psi_B^2)(\ln \frac{\Psi_A}{\Psi_B} + U_A - 2U_B) \, dq \, dx \quad (1.6)$$

$$= \int_{\Omega \times \mathbb{R}^3} (k_1 \Psi_A - k_2 \Psi_B^2) \ln \frac{k_1 \Psi_A}{k_2 \Psi_B^2} \, dq \, dx \geq 0.$$

Furthermore, we assume $k_1 = k_2 = 1$, to simplify the exposition.

One of the most important features of the two-species micro-macro model (1.1) lies in the constraints on conservation of matter during the reversible breakage/reforming process $A \xrightleftharpoons[k_2]{k_1} B$. This mass conservation is only satisfied by the quantity of total concentration $2\Psi_A + \Psi_B$ but not any individual species, i.e., $\frac{d}{dt} \int_{\Omega \times \mathbb{R}^3} (2\Psi_A + \Psi_B) \, dq \, dx = 0$. Consequently, assuming initially that $\int_{\mathbb{R}^3} (2\Psi_A + \Psi_B)|_{t=0}(x, q) \, dq = 1$, we formally get,

$$\int_{\mathbb{R}^3} (2\Psi_A + \Psi_B)(t, x, q) \, dq = 1. \quad (1.7)$$

This will enable us to solve the constants $c_A$ and $c_B$ by combining the relation $c_A = c_B^2$.

Define the fluctuations of $\Psi_A$ and $\Psi_B$ around their global Maxwellian as follows,

$$\Psi_A = M_A + \sqrt{M_A}f_A, \quad \Psi_B = M_B + \sqrt{M_B}f_B. \quad (1.8)$$

Due to the above conservation of total mass, we have:

$$\int_{\mathbb{R}^3} (2\Psi_A + \Psi_B) \, dq = 1 = \int_{\mathbb{R}^3} (2M_A + M_B) \, dq,$$

which immediately implies

$$\int_{\mathbb{R}^3} (2f_A\sqrt{M_A} + f_B\sqrt{M_B}) \, dq = 0. \quad (1.9)$$

However, due to the non-conservation for each individual perturbations $f_\alpha$, it follows,

$$\int_{\mathbb{R}^3} f_\alpha\sqrt{M_\alpha} \, dq \neq 0. \quad (1.10)$$

This causes some difficulties from a perspective of mathematical analysis. Speaking precisely, (1.10) brings a mean value term in the (weighted) Poincaré inequality when we try to establish a closed $a$-priori estimate. This point will be specified in §1.2 later.

Inserting the perturbation scheme (1.8) into the original system (1.1) will lead us to get a perturbative system. Direct calculations imply that

$$r = k_1\Psi_A - k_2\Psi_B^2 = k_1(M_A + \sqrt{M_A}f_A) - k_1\frac{M_A}{M_B}(M_B + \sqrt{M_B}f_B)^2$$

$$= k_1\sqrt{M_A}(f_A - 2\sqrt{M_B}f_B - f_B^2),$$

$$= k_1\sqrt{M_A}f_A - k_1\sqrt{M_B}f_B - k_1f_B^2.$$
where we have used the fact \(K_{eq} = \frac{M_A}{M_B} = 1\). We then write
\[
\begin{align*}
    r_A &= -\frac{r}{\sqrt{M_A}} = -k_1(f_A - 2\sqrt{M_B}f_B - f_B^2), \\
    r_B &= 2\frac{r}{\sqrt{M_B}} = 2k_1\sqrt{M_B}(f_A - 2\sqrt{M_B}f_B - f_B^2).
\end{align*}
\]  
\tag{1.12}

Note that \(k_1 = 1\) has been assumed before. On the other hand, by noticing \(\nabla_q \cdot (\nabla_q \Psi_\alpha + \nabla_q U_\alpha \Psi_\alpha) = \nabla_q \cdot (\alpha \nabla_q (\frac{f}{\sqrt{M_A}}))\) for \(\alpha = A, B\), we can define
\[
\mathcal{L}_\alpha f_\alpha = \frac{1}{\sqrt{M_\alpha}} \nabla_q \cdot \left[ \alpha \nabla_q (\frac{f}{\sqrt{M_\alpha}}) \right] = \Delta_q f_\alpha + \frac{1}{2} \Delta_q U_\alpha f_\alpha - \frac{1}{2} |\nabla_q U_\alpha|^2 f_\alpha.  
\]  
\tag{1.13}

Therefore, the final perturbative system for \((u, f_A, f_B)\) can be written as
\[
\begin{align*}
    \partial_t u + u \cdot \nabla_x u + \nabla_x p &= \mu \Delta_x u + \lambda \text{div}_x \int_{\mathbb{R}^3} (\nabla_q U_A \otimes q f_A \sqrt{M_A} + \nabla_q U_B \otimes q f_B \sqrt{M_B}) dq, \\
    \text{div}_x u &= 0, \\
    \partial_t f_A + u \cdot \nabla_x f_A + \nabla_x u q \nabla_q f_A &= \mathcal{L}_A f_A + r_A + \nabla_x u q \nabla_q U_A (\sqrt{M_A} + \frac{1}{2} f_A), \\
    \partial_t f_B + u \cdot \nabla_x f_B + \nabla_x u q \nabla_q f_B &= \mathcal{L}_B f_B + r_B + \nabla_x u q \nabla_q U_B (\sqrt{M_B} + \frac{1}{2} f_B).
\end{align*}
\]  
\tag{1.14}

Here and throughout the paper we will use the divergence symbol \(\text{div}_x\) to avoid some possible confusion between divergences on variables \(x\) and \(q\).

1.1. Statement of main results. Before we state the main results, we firstly introduce some notations.

**Notations.** For simplicity, we denote by \(|\cdot|_{L^2_x}\) the usual \(L^2\)-norm over spatial variables \(x\), and by \(|\cdot|_{H^s_x}\) the higher-order spatial derivatives \(H^s\)-norm. Besides, we denote by \(|\cdot|_{L^2_{x,q}}\) the mixed \(L^2\)-norm with respect to both spatial variables \(x\) and microscopic configuration variables \(q\), and by \(|\cdot|_{H^s_{x,q}}\) the higher-order mixed derivatives. The notation \(\langle \cdot, \cdot \rangle\) will stand for the inner product in a specific space marked by a subscript. Moreover, we will omit the subscript of space marks when it is with respect to only \(x\) or both \(x\) and \(q\). Furthermore, the angle bracket \(\langle \cdot \rangle\) denotes the integral over the configurational space, i.e., \(\langle f \rangle = \int_{\mathbb{R}^3} f dq\).

It is convenient to introduce the weighted Sobolev spaces with the weighted \(L^2\)-inner product defined by
\[
\langle f, g \rangle_M \triangleq \langle f, g M \rangle = \int_{\Omega \times \mathbb{R}^3} f g M \, dq \, dx,
\]
for any pairs \(f(x, q), g(x, q) \in L^2_{x,q}\). We also use the notation \(|\cdot|_{M}\) to denote a norm with respect to the weighted space \(L^2(M dq dx)\). Moreover, the subscript \(M_\alpha\) in above norm is usually simplified as a single subscript \(M\) when no possible confusion arises.

Let \(\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3\) be a multi-index with its length defined as \(|\alpha| = \sum_{i=1}^3 \alpha_i\). We define here the multi-derivative operator \(\nabla^\alpha_x = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}\), and we also denote \(\nabla^k_l\) for \(|\alpha| = k\) simplicity. In the following texts, we will use frequently the notation \(\nabla^k_l \triangleq \nabla^k_x \nabla^l_q\) to stand for the mixed derivatives over the variables \(x\) and \(q\).

In addition, the notation \(A \lesssim B\) means that there exists some positive constant \(C > 0\) such that \(A \leq CB\). The notation \(A \sim B\) stands for the equivalence between both sides up to a constant, i.e., some \(C > 0\) exists such that \(C^{-1}B \leq A \leq CB\). Sometimes we will also omit the integral domain symbol for simplicity.

**Main results.** We employ here some assumptions on the potentials \(U_\alpha\) with \(\alpha = A, B\), which are same as that of [27]. More precisely, we assume
\[
\begin{align*}
    |q| &\lesssim (1 + |\nabla_q U_\alpha|), \quad \text{(sometimes we just assume } |q| \lesssim |\nabla_q U_\alpha| \text{ for simplicity)}, \\
    \Delta_q U_\alpha &\leq C + \delta |\nabla_q U_\alpha|^2 \quad \text{for } \delta < 1,
\end{align*}
\]
\begin{align}
\int_{\mathbb{R}^3} |\nabla q U_\alpha|^2 M \, dq & \leq C, \quad \int_{\mathbb{R}^3} |q|^4 M \, dq \leq C, \tag{1.15}
\end{align}
and
\begin{align}
|\nabla^k q (q \nabla U_\alpha)| & \lesssim (1 + |q| |\nabla q U_\alpha|), \\
\int_{\mathbb{R}^3} |\nabla^k q (q \nabla U_\alpha \sqrt{M})|^2 \, dq & \leq C, \\
|\nabla^k q (\Delta U_\alpha - \tfrac{1}{2} |\nabla q U_\alpha|^2)| & \lesssim (1 + |\nabla q U_\alpha|)^2. \tag{1.16}
\end{align}

Define energy and energy-dissipation functionals that,
\begin{equation}
E_s(t) = |u|^2_{H^s_x} + \|(f_A, f_B)\|^2_{H^s_{x,q}} + \|q(f_A, f_B)\|^2_{H^{s-1}_{x,q}}, \tag{1.17}
\end{equation}
\begin{align}
D_s(t) &= |\nabla u|^2_{H^s_x} + |\rho|_{H^s_x} + \sum_{k+l \leq s} \left[ \left\| q \nabla \left( \frac{\nabla^k f_A}{\sqrt{M_A}} \right) \right\|_M^2 + \left\| q \nabla \left( \frac{\nabla^k f_B}{\sqrt{M_B}} \right) \right\|_M^2 \right] \\
&\quad + \sum_{k=0}^{s-1} \left[ \left\| q \nabla \left( \frac{\nabla^k f_A}{\sqrt{M_A}} \right) \right\|_M^2 + \left\| q \nabla \left( \frac{\nabla^k f_B}{\sqrt{M_B}} \right) \right\|_M^2 \right] + \sum_{k+l \leq s} \left\| \nabla^l f_A - 2 \nabla^l f_B \sqrt{M_B} \right\|_{H^s_{x,q}}^2. \tag{1.18}
\end{align}
Here \(\rho_\alpha = \langle f_\alpha \sqrt{M_\alpha} \rangle = \int_{\mathbb{R}^3} f_\alpha \sqrt{M_\alpha} \, dq\) denote the macroscopic number density fluctuations.

**Theorem 1.1** (Global existence). Let \(s \geq 5\). Let \((u, f_A, f_B)\) be the fluctuation near the global equilibrium \((0, M_A, M_B)\) of the two-species micro-macro model (1.1), with their initial data \((u_0, f_{A,0}, f_{B,0})\) satisfying
\begin{equation}
\Psi_{\alpha,0} = M_\alpha + \sqrt{M_\alpha} \rho_\alpha > 0, \quad \text{and} \quad \int_{\mathbb{R}^3} (2\Psi_{A,0} + \Psi_{B,0}) \, dq = 1. \tag{1.19}
\end{equation}

Then, there exists some constant \(\varepsilon\) sufficiently small, such that, if the initial fluctuation satisfies
\begin{equation}
E_s(0) = |u_0|^2_{H^s_x} + \|(f_{A,0}, f_{B,0})\|^2_{H^s_{x,q}} + \|(q f_{A,0}, q f_{B,0})\|^2_{H^{s-1}_{x,q}} \leq \varepsilon, \tag{1.20}
\end{equation}
\begin{equation}
\frac{1}{2} \int_{\Omega} |u_0|^2 \, dx + \int_{\Omega \times \mathbb{R}^3} \sum_{\alpha=A,B} (\Psi_{\alpha,0} \ln \frac{\Psi_{\alpha,0}}{M_\alpha} - \Psi_{\alpha,0} + M_\alpha) \, dq \, dx \leq \varepsilon. \tag{1.21}
\end{equation}
Then system (1.1) admits a unique global classical solution \((u, \Psi_A, \Psi_B)\) with \(\Psi_\alpha = M_\alpha + \sqrt{M_\alpha} \rho_\alpha > 0\), and moreover,
\begin{equation}
\sup_{t \in [0, +\infty)} E_s(t) + \int_0^{+\infty} D_s(t) \, dt \leq C \varepsilon, \tag{1.22}
\end{equation}
where the constant \(C\) is independent of \(\varepsilon\).

**Remark 1.2.** In fact, as pointed out in §4.3 below, the requirement of Sobolev index for first-order moments \((q f_A, q f_B)\) in (1.17)-(1.18) does not have to be exactly \(s - 1\). An index of \(s' \in [4, s - 1]\) suffices to close the same a-priori estimate, see (4.3).

Before explaining our proof, we may explain briefly the motivation for this paper by considering the property of global equilibrium state \((0, M_A, M_B)\) of the system (1.1). Denote by \(D(u, \Psi_A, \Psi_B)\) the dissipative part of the right-hand side in (1.3), then we can infer by noticing the assumptions \(k_1 = k_2 = 1\) and \(U_A = 2U_B\), that
\begin{align}
D(u, \Psi_A, \Psi_B) &= \int_{\mathbb{R}^3} \mu |\nabla u|^2 \, dx + \lambda \sum_{\alpha=A,B} \int_{\Omega \times \mathbb{R}^3} \Psi_{\alpha} \left| \nabla q \left( \ln \frac{\Psi_{\alpha}}{M_\alpha} \right) \right|^2 \, dq \, dx \\
&\quad + \lambda \int_{\Omega \times \mathbb{R}^3} (\Psi_A - \Psi_B^2) \left( \ln \frac{\Psi_A}{\Psi_B^2} \right) \, dq \, dx.
\end{align}
Apparently it holds \(D(0, M_A, M_B) = 0\). Moreover, the global equilibrium \((0, M_A, M_B)\) is a critical point of energy dissipation function \(D(u, \Psi_A, \Psi_B)\).
Indeed, performing a perturbative analysis on the function \( D(u, \Psi_A, \Psi_B) \) around the global equilibrium \((0, M_A, M_B)\) yields, for a small perturbative parameter \( \epsilon > 0 \), that
\[
\frac{d}{d\epsilon}\bigg|_{\epsilon=0} D(\epsilon v, M_A + \epsilon \psi_A, M_B + \epsilon \psi_B) = 0, \tag{1.24}
\]
and
\[
\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} D(\epsilon v, M_A + \epsilon \psi_A, M_B + \epsilon \psi_B) = 2 \int_{\Omega} \mu |\nabla_x v|^2 \, dx + 2\lambda \sum_{\alpha=A,B} \int_{\Omega \times \mathbb{R}^3} \left| \nabla_q \left( \ln \frac{\psi_{\alpha}}{M_{\alpha}} \right) \right|^2 \, M_{\alpha} \, dq \, dx \tag{1.25}
\]
\[
+ 2\lambda \int_{\Omega \times \mathbb{R}^3} \left( \frac{\psi_A}{\sqrt{M_A}} - 2\psi_B \right)^2 \, dq \, dx,
\]
which keeps positive for non-trivial perturbative values.

The above analysis provides our motivation on studying the global in time existence of classical solutions near the global equilibrium state \((0, M_A, M_B)\), with small initial assumptions on perturbations, as stated in Theorem 1.1 before.

We point out that, the right-hand side of (1.25) contains three non-negative terms: fluid dissipation term, microscopic dissipation term and an additional dissipation term coming from chemical reactions. They correspond to three terms in perturbative system (1.14): the macroscopic diffusion of Laplacian operator \( \Delta_x u \) on velocity field, the kinetic diffusion of Fokker-Planck operator \( L_{\alpha} f_{\alpha} \), and the linear reaction part \((f_A - 2\sqrt{M_B} f_B)\), respectively. This observation is very useful in constructing the dissipation functionals (1.18) for energy estimates.

1.2. Strategy of the proof. The most important part in proving the global existence Theorem 1.1 is to establish a closed a-priori estimate. The a-priori estimate in Proposition 4.1, together with the basic energy-dissipation law (1.3) (or exactly, the refined energy-dissipation law (5.8) defined below in §5), and a standard bootstrap principle of continuity argument, will enable us to get the global-in-time existence result Theorem 1.1, based on the local existence of solution under same assumptions.

Proving the a-priori estimate requires more concerns on higher-order energy estimates for perturbations \((u, f_A, f_B)\). We firstly notice that both of energy and dissipation functionals defined in (1.17)-(1.18) contain following parts: pure spatial derivatives, spatial derivatives on first-order moment, and mixed spatial-configurational derivatives. In particular, the dissipation functionals \( D_{\alpha}(t) \) in (1.18) is constructed with additional contributions coming from the reversible reaction scheme, which involves the linear reaction rate part \((f_A - 2\sqrt{M_B} f_B)\), and the number density term \( \rho_A \).

The whole proof is correspondingly decomposed into four steps:

1. The higher-order purely spatial derivatives for the whole perturbative system (1.14), which contains mainly two parts of estimates, one is for macroscopic perturbative distribution functions \((f_A, f_B)\).

   The key point is to deal with the linear terms arising in the micro-macro coupling: \( \nabla_x \nabla_x^s u q \nabla_q U_{\alpha} \) and \( \text{div}_x \int_{\mathbb{R}^3} \nabla_q U_{\alpha} \otimes q \nabla_x f_{\alpha} \sqrt{M_{\alpha}} \, dq \) with respect to the perturbation functions, due to the reason that a linear term is usually the worst term in the near-equilibrium-framework with small data. Fortunately, there exists a cancellation relation between the micro-macro coupling, as noticed in [27], i.e.
   \[
   \langle \nabla_x^{s+1} u q \nabla_q U_{\alpha} \sqrt{M_{\alpha}}, \nabla_x^s f_{\alpha} \rangle_{L^2_{x,q}} + \langle \text{div}_x \int_{\mathbb{R}^3} \nabla_q U_{\alpha} \otimes q \nabla_x^s f_{\alpha} \sqrt{M_{\alpha}} \, dq, \nabla_x^s u \rangle_{L^2_x} = 0.
   \]

   However, the above step is not sufficient to close the energy estimates since some configuration moment and mixed derivatives terms are involved and hence need to be controlled (see Remark 4.2), as in the following steps:

2. The higher-order spatial derivatives on first-order moment \( q f_{\alpha} \): as pointed out in Remark 4.2, the highest order derivative on moment \( q f_{\alpha} \) can be viewed only as a
dissipative part, so it suffices to estimate the higher-not-highest order derivatives on first-order moment, in Sobolev space with index less than s.

(3) The higher-order derivatives on mixed spatial-configurational variables \((x,q)\) is performed in a similar but more complicated way.

In fact, our proof in this present paper relies heavily on the Poincaré inequality, in a formulation of that with a Maxwellian integral weight, and with a mean value function due to the above mentioned non-conservation of perturbation, see the weighted Poincaré inequalities stated in Lemma 3.2. This type of lower-order mean value function should be viewed only as an energy contribution, rather than a dissipation contribution equipped with one more higher-order derivatives. Notice from the simple fact that the mean value function defines actually the number density \(\rho_\alpha\) of each species in a macroscopic level (see, for example, (3.6)), so we are required to search in the last step some potential dissipations on perturbative number density \(\rho_\alpha\):

(4) The higher-order spatial derivatives on perturbative number density \(\rho_\alpha\). By performing integration with respect to weighted measure \(\sqrt{M_\alpha} \, dq\), we can get equation (4.65) of perturbative number density \(\rho_\alpha\) which represents almost pure reaction contributions without any other micro-macro coupling effects, see §4.5 for more details. Roughly speaking, this process enables us to separate the chemical reaction effect from a complex micro-macro coupling structure.

At last, an additional dissipation term \((4 \langle M_A \rangle + 1) \| \nabla_x^2 \rho_A \|^2_{L^2_x}\) is obtained in (4.73), due mainly to the linear reaction rate term \((f_A - 2\sqrt{M_B} f_B)\) and the total conservation of perturbations expressed in a macroscopic formulation \(2\rho_A + \rho_B = 0\), as a direct consequence of (1.9).

In summary, combining the above four steps leads to the desired a-priori estimate, see Proposition 4.1. As a common sense, the linear terms and lower-order terms in energy estimates are usually hard to deal with in establishing the global existence in a “small solution” theory. Both types of difficulties are encountered here and need more concerns. There are two types of linear terms in the perturbative system (1.14), one of which comes from the micro-macro coupling, as we mentioned in Step (1). Our treatment is similar as that in Lin-Liu-Zhang [27] and Jiang-Liu-Zhang [19], and the cancellation relation will play an important role.

The other type of linear terms arise from the linear part of LMA-rate reaction contributions: \((f_A - 2\sqrt{M_B} f_B)\) appearing in the last third term \(r_\alpha\) of (1.14). This provides a contribution to dissipation in two different scales. At a microscopic level, this exhibits an explicit formulation of a positive quadratic dissipation term \(\| \nabla_x^k \nabla_q^l f_A - 2\sqrt{M_B} \nabla_x^k \nabla_q^l f_B \|^2_{L^2_x}\), as appeared in the last term in the definition of dissipation functional (1.18). At a macroscopic level, there is an implicit formulation through the dissipation of number density \((4 \langle M_A \rangle + 1) \| \nabla_x^s \rho_A \|^2_{L^2_x}\), as we have stated in above Step (4). Note that, this linear reaction dissipation contribution seems more important on a macroscopic level rather than that on a microscopic level. Besides, one can find in [48] some similar treatment on the macroscopic number density, from a viewpoint of modeling.

At the end, we point out that the treatment for number density in step (4) can also be understood in a terminology of kinetic theory, especially of the hydrodynamic limit theory for the Boltzmann equation, see for instance Guo [12–14] and Lin-Yu-Yang [32, 33]. Because the number density lies actually in the kernel of Fokker-Planck operator which has only one dimension spanned by constant variable (up to the weighted measure), the above integral process can be viewed as the macroscopic projection on kernel space of Fokker-Planck operator. On the other hand, the Poincaré inequality applied to the pure microscopic part will show a clear formulation without a mean value. There is a close relation between our treatment here and the so-called macro-micro decomposition in hydrodynamic limit theory. However,
the non-conservative kinematics (2.1) below and the LMA-rate reaction appeared in the basic energy-dissipation law (1.3) make an effort in energy estimates, both of which are very different from the classical kinetic theory.

1.3. Organization of the paper. The rest of this paper is as follows: a formal derivation of this two-species micro-macro model (1.1) will be given, by using the EnVarA in the following section §2, containing derivations for the mechanical and chemical reaction parts.

In §3, some useful lemmas are presented, involving the structure of the Fokker-Planck operator and the weighted Poincaré inequalities. §4 is devoted to closing the a priori estimate, see Proposition 4.1, for that we need to derive closed estimates for fluctuations, including purely spatial higher-order derivatives on fluctuations themselves and fluctuation moments, mixed spatial-configurational derivatives, and furthermore, additional dissipation on number densities at the macroscopic level. In the last section §5, we complete the proof of global existence result (Theorem 1.1), by combining Proposition 4.1, the basic energy-dissipation law, and a standard bootstrap principle of continuity argument together.

2. A Formal Energetic Variational Derivation

In this section, we provide a formal derivation of our model (1.1) by the Energetic Variational Approach (EnVarA), in the spirit of [48]. We present it here in a self-contained way, for the sake of readability of this paper. The framework of EnVarA, which was developed from seminal work of Rayleigh [43] and Onsager [39, 40], has been a powerful tool to deal with the couplings and competitions between different mechanisms in different scales, such the competition between the microscopic interactions and the macroscopic dynamics in complex fluids (see the surveys [10, 30], for example).

Starting from the energy-dissipation law and the kinematic relations, EnVarA provides a unique, well-defined way to derive the dynamics of the systems (which is usually represented as a coupled system of nonlinear partial differential equations). The main ingredients include the Least Action Principle (LAP) and the Maximum Dissipation Principle (MDP), which derive the conservative force and the dissipative force respectively. The force balance condition will lead to the final PDE system. The EnVarA has been successfully applied to model many systems, especially those in complex fluids, such as liquid crystals, polymeric fluids, phase field and ion channels [10, 30]. Since these models are derived based on the basic thermodynamic laws and hence are thermodynamically consistent. Besides, this approach can also be used to study problems with boundary, especially for dynamical boundary conditions problems for the Cahn-Hilliard equation [21, 31].

Recently, EnVarA has been applied in the study of reaction-diffusion systems driven by the law of mass action (LMA), see Wang-Liu-Liu-Eisenberg [47]. The authors applied a generalized notion of EnVarA to a non-equilibrium reaction-diffusion system, by finding a way to couple successfully the chemical reaction with other effects. It is worthy to point out that their treatment is different from the linear response theory in which the dissipation rate function of total energy is of a quadratic formulation, as a common assumption in non-equilibrium thermodynamic theory [39, 40]. The authors showed in [47] that the dynamics of system is determined by the choice of the dissipation.

In this paper, we consider a micro-macro model for such wormlike micellar solutions involving the chemical reaction of breakage/reforming scheme. The kinematic relations for the two species can be written as follows:

\[
\begin{align*}
    \partial_t \Psi_A + \nabla_x \cdot (u_A \Psi_A) + \nabla_q \cdot (V_A \Psi_A) &= -r, \\
    \partial_t \Psi_B + \nabla_x \cdot (u_B \Psi_B) + \nabla_q \cdot (V_B \Psi_B) &= 2r.
\end{align*}
\] (2.1)

where \( u_\alpha \) and \( V_\alpha \) are the corresponding macroscopic and microscopic velocities in spatial and configurational spaces, respectively. Since we consider the case the molecule orientations \( q \) transporting along with the environmental fluid velocity \( u(t, x) \), so \( u_A = u_B = u \). Besides, the
fluid velocity field $u$ is considered as an incompressible flow. Note that the left-hand side of each equation represents a formulation of Smoluchowski equation [1, 6], while the right-hand side represents the chemical reaction of breakage/reforming process between the two species, and moreover, $r = k_1\Psi_A - k_2\Psi_B^2$ by recalling (1.2).

Based on the kinematics (2.1), the two-species micro-macro model can be derived from the following basic energy-dissipation law:

$$\frac{d}{dt}\left\{ \int_{\Omega} \frac{1}{2}|u|^2 \, dx + \lambda \sum_{\alpha=A,B} \int_{\Omega \times \mathbb{R}^3} \Psi_\alpha (\ln \Psi_\alpha + U_\alpha - 1) \, dq \, dx \right\}$$

$$= - \int_{\Omega} \mu |\nabla u|^2 \, dx - \lambda \sum_{\alpha=A,B} \int_{\Omega \times \mathbb{R}^3} \Psi_\alpha |V_\alpha - \nabla_x uq|^2 \, dq \, dx$$

$$- \lambda \int_{\Omega \times \mathbb{R}^3} (k_1\Psi_A - k_2\Psi_B^2)(\ln \frac{\Psi_A}{\Psi_B} + U_A - 2U_B) \, dq \, dx,$$

(2.2)

by employing the EnVarA [10, 30]. The second dissipative term in the right-hand side stands for the relative friction of microscopic polymer particles to the environmental macroscopic flow, in which the expression $\nabla_x uq$ comes from the Cauchy-Born rule $q = FQ$ with $F$ being the deformation tensor and $Q$ being the initial configuration in Lagrangian coordinates.

### 2.1. The mechanical part.

Noticing the variational structure of above energy-dissipation law, it is convenient to employ the EnVarA, which has the advantage of describing mathematically the competitions between the kinetic energy and free energy (including both entropy and internal energy, or say, elastic energy here). We firstly point out that both Lagrangian and Eulerian coordinates are used in this approach, where the flow map for macroscopic position $x = x(t, X)$, associated to the Lagrangian coordinate of initial position $X$ and a given velocity field $u(t, x)$, is defined as

$$\begin{cases}
\frac{d}{dt} x = u(t, x(t, X)), \\
x(t, X)|_{t=0} = X.
\end{cases}$$

(2.3)

Similarly, associated to its Lagrangian coordinate $Q$ mentioned earlier, the flow map $q = q(t, X, Q)$ in the configuration space, generated by a given microscopic velocity $V = V(t, x, q)$, is defined through the formula $\frac{d}{dt} q = V(t, x(t, X), q(t, X, Q))$. We also introduce the deformation tensor $F(t, X) = \frac{\partial V}{\partial X}$ (in Lagrangian coordinate) and $F(t, x(t, X)) = F(t, X)$ (in Eulerian coordinate).

To apply the EnVarA, we firstly write the energy functional in Lagrangian coordinates, corresponding to its kinetic energy and free energy part, that,

$$\mathcal{K} + \mathcal{F} = \int_{\Omega'} \frac{1}{2} |\dot{x}|^2 \, dx + \lambda \sum_{\alpha=A,B} \int_{\Omega' \times \mathbb{R}^3} \Psi_\alpha (\ln \Psi_\alpha + U_\alpha(q) - 1) \, dq \, dx$$

$$= \int_{\Omega'} \frac{1}{2} |\dot{x}|^2 \, dX + \lambda \sum_{\alpha=A,B} \int_{\Omega' \times \mathbb{R}^3} \Psi_{\alpha,0}(X, Q) \left( \ln \frac{\Psi_{\alpha,0}(X, Q)}{\det \frac{\partial x}{\partial X}} + U_\alpha(q) - 1 \right) \, dQ \, dX,$$

(2.4)

where we have used the fact $\det F = 1$ due to the incompressibility of fluid flow. This formulation will allow us to define the least action functional $\mathcal{A} = \int_0^T (\mathcal{K} - \mathcal{F}) \, dt$.

Note that the notion of “separation of scales” plays a crucial role in the derivation process. In fact, the microscopic configuration variable $q$ and $V$ should be viewed independent from macroscopic variable $x$, which means roughly that the fluid flow is like a “static” background. By the LAP, performing the variation with respect to the configuration variable $q$,
the variation on $\int_0^T K \, dt$ will vanish and we get
\[
\delta_q \int_0^T \mathcal{F} \, dt = \iiint \left\{ \Psi_{\alpha,0} \frac{\text{det} \frac{\partial q}{\partial x}}{\Psi_{\alpha,0}} \cdot \left[ -\frac{\Psi_{\alpha,0}}{(\text{det} \frac{\partial q}{\partial x})^2} \cdot \text{tr} \left( \frac{\partial q}{\partial q} \cdot \frac{\partial q}{\partial q} \right) \right] + \Psi_{\alpha,0} \nabla q \Delta u \cdot \delta q \right\} \, dq \, dX \, dt
= \int_0^T \iiint_{\Omega^\text{e} \times \mathbb{R}^3} \left[ -\Psi_{\alpha,0} (\nabla q \cdot \delta q) + \Psi_{\alpha,0} \nabla q \Delta u \cdot \delta q \right] \, dq \, dX \, dt \quad (2.5)
= \int_0^T \iiint_{\Omega^\text{e} \times \mathbb{R}^3} (\nabla q \Psi_{\alpha} \cdot \delta q + \Psi_{\alpha} \nabla q \Delta u \cdot \delta q) \, dq \, dx \, dt
= \langle \nabla q \Psi_{\alpha} + \nabla q \Delta u \Psi_{\alpha}, \delta q \rangle_{L^2_{t,x,q}}.
\]

On the other hand, the MDP, meaning to perform variation on dissipation part with respect to the microscopic "velocity" $V = \dot{q}$, yields,
\[
\delta_V D_q = \int \Psi_{\alpha} (V_{\alpha} - \nabla_x uq) \cdot \delta V_{\alpha} \, dq \, dx = \langle \Psi_{\alpha} (V_{\alpha} - \nabla_x uq), \delta V_{\alpha} \rangle_{L^2_{t,x,q}}. \quad (2.6)
\]
So we can obtain the force balance
\[
\Psi_{\alpha} (V_{\alpha} - \nabla_x uq) = - (\nabla q \Psi_{\alpha} + \nabla q \Delta u \Psi_{\alpha}), \quad (2.7)
\]
i.e.,
\[
V_{\alpha} - \nabla_x uq = - \nabla q (\ln \Psi_{\alpha} + U_{\alpha}). \quad (2.8)
\]

With this relation in hand, we are able to consider the energetic variation on macroscopic level to get the momentum equation. The LAP implies, by taking variation over $x$, that
\[
\delta_x \int K \, dt = \int \dot{\mathbf{x}} \cdot \frac{d}{dt} \delta \mathbf{x} \, dX \, dt = - \int \frac{d}{dt} u \cdot \delta x \, dx \, dt = \langle \partial_t u + u \cdot \nabla_x u, \delta x \rangle_{L^2_{t,x}}, \quad (2.9)
\]
while $\delta_x \int F \, dt$ vanishes because of the notion "separation of scales". Meanwhile, the MDP yields, by taking variation over $u = \dot{x}$, that
\[
\delta_u D_u = \int \mu \nabla x u \nabla x (\delta u) \, dx + \lambda \sum_{\alpha = A,B} \int \Psi_{\alpha} (V_{\alpha} - \nabla_x uq) \cdot (-\nabla x \delta u) \, q \, dq \, dx
= \langle -\mu \Delta x u, \delta u \rangle_{L^2_{t,x}} + \lambda \sum_{\alpha = A,B} \langle \nabla \cdot \int \Psi_{\alpha} (V_{\alpha} - \nabla_x uq) \otimes q \, dq, \delta u \rangle_{L^2_{t,x}}
= \langle -\mu \Delta x u, \delta u \rangle_{L^2_{t,x}} - \lambda \sum_{\alpha = A,B} \langle \nabla \cdot \left( \int \nabla q \Psi_{\alpha} + \nabla q \Delta u \Psi_{\alpha} \right) \otimes q \, dq, \delta u \rangle_{L^2_{t,x}}
= \langle -\mu \Delta x u, \delta u \rangle_{L^2_{t,x}} - \lambda \sum_{\alpha = A,B} \langle \nabla \cdot \left( \int \nabla q U_{A} \otimes q \Psi_{A} \, dq \right) - \nabla x n_{\alpha}, \delta u \rangle_{L^2_{t,x}}, \quad (2.10)
\]
where we have used the above microscopic force balance (2.8) in the penultimate line. Therefore, we get the momentum equation that,
\[
\partial_t u + u \cdot \nabla_x u + \nabla_x p = \mu \Delta x u + \lambda \nabla_x \left[ \int_{\mathbb{R}^3} (\nabla q U_{A} \Psi_{A} + \nabla q U_{B} \Psi_{B}) \otimes q \, dq \right], \quad (2.11)
\]
with the pressure $p$ being the Lagrangian multiplier.

**Remark 2.1.** We point out that, the above induced elastic stress (Kramer’s stress), namely, \( \int_{\mathbb{R}^3} \nabla q U_{\alpha} \otimes q \Psi_{\alpha} \, dq \), is able to be derived either by the MDP (see (2.10)), or by the LAP by virtue of the Cauchy-Born rule, as did in [19, 27]. This means that the elastic stress can be viewed as dissipative or conservative. The same idea can be found in the study of liquid crystal, for example, [49].
2.2. The chemical reaction part. In order to take the special reaction dissipation into account, here we need a generalized EnVarA, by employing the reaction trajectory $R$ defined through
\[
\frac{d}{dt} R = r ,
\] (2.12)
here $r$ is the LMA-rate defined in (1.2). We refer readers to [41, 47] for more detailed discussion. Note that the reaction trajectory $R$ in the chemical reaction system is analogy to the flow map $x(X, t)$ in the mechanical system. Due to the “separation of scales” again, the chemical reaction of reversible breakage/reforming process $A \frac{k_1}{k_2} 2B$ can be expressed as,
\[
\Psi_A = -R + \Psi_{A,0}, \quad \Psi_B = 2R + \Psi_{B,0}.
\] (2.13)
Actually, this relation may be regarded as the kinematics for the above breakage/reforming reaction process.

The new state variable of reaction trajectory enables us to rewrite the free energy $F$ as,
\[
F(R) = F(\Psi_A(R), \Psi_B(R)).
\] (2.14)
Meanwhile, we also write the reaction dissipation part (1.6) as $\mathcal{D}_R = \mathcal{D}_R(R, \dot{R})$. So the energy-dissipation law for a pure chemical reaction process can be rewritten as
\[
\frac{d}{dt} F(R) = -\mathcal{D}_R(R, \dot{R}).
\] (2.15)
We assume that the nonnegative reaction dissipation $\mathcal{D}_R$ takes the form of
\[
\mathcal{D}_R(R, \dot{R}) = \left< \mathcal{G}_R(R, \dot{R}), \dot{R} \right>,
\] (2.16)
this, combining with the fact $\frac{d}{dt} F(R) = \left< \frac{\delta F}{\delta R}, \dot{R} \right>$, yields that,
\[
\mathcal{G}_R(R, \dot{R}) = -\frac{\delta F}{\delta R}.
\] (2.17)
We refer this as a general gradient flow. The exact expression of LMA-rate $r$ will be revisited by choosing
\[
\mathcal{D}_R(R, \dot{R}) = \dot{R} \ln \left( \frac{\dot{R}}{k_2 \Psi_B^2} + 1 \right),
\] (2.18)
Indeed, direct calculations imply that
\[
\frac{\delta F}{\delta R} = \sum_{\alpha} \frac{\delta F}{\delta \Psi_{\alpha}} \cdot \frac{\partial \Psi_{\alpha}}{\partial R} = -(\ln \Psi_A + U_A) + 2(\ln \Psi_B + U_B)
\]
\[
= -(\ln \frac{\Psi_A}{\Psi_B^2} + U_A - 2U_B) = -\ln \frac{k_1 \Psi_A}{k_2 \Psi_B^2},
\] (2.19)
where we have used the equilibrium relation (1.5). As a result, the above general gradient flow (2.17) implies,
\[
r = \dot{R} = k_1 \Psi_A - k_2 \Psi_B^2,
\] (2.20)
which is exactly the same formulation as (1.2).

Therefore, inserting the above LMA-rate formulation, and the microscopic force balance relation (2.8) into the kinematics (2.1) enables us to get, for the number density distribution functions $(\Psi_A, \Psi_B)(t, x, q)$, that
\[
\begin{align*}
\partial_t \Psi_A + \nabla_x \cdot (u \Psi_A) + \nabla_q \cdot (\nabla_x u q \Psi_A) &= \nabla_q \cdot (\nabla_q \Psi_A + \nabla_q U_A \Psi_A) - (k_1 \Psi_A - k_2 \Psi_B^2), \\
\partial_t \Psi_B + \nabla_x \cdot (u \Psi_B) + \nabla_q \cdot (\nabla_x u q \Psi_B) &= \nabla_q \cdot (\nabla_q \Psi_B + \nabla_q U_B \Psi_B) + 2(k_1 \Psi_A - k_2 \Psi_B^2).
\end{align*}
\] (2.21)

Finally, we have derived the whole two-species micro-macro model (1.1) for wormlike micelles by combining the momentum equation (2.11) and above equations for the number
density distribution functions. We refer the readers to [48] for a more detailed derivation on model (1.1) and its moment closure model.

2.3. Comments on the derived model. We make some comments here, on some general case of equilibriums and on related models including especially the VCM model.

**General cases.** In a general setting, the chemical equilibrium “constant” $K_{eq}$ might be possibly dependent upon the variables $q$ (and moreover, upon the variables $x$). This requires more complex treatment both from analysis and simulation viewpoints.

Indeed, taking the classic Hookean elastic springs for example, namely, $U_\alpha = \frac{1}{2} H_\alpha |q|^2$ for $\alpha = A, B$. As mentioned above, the simplest case $K_{eq} = 1$ combined with equation (1.5) yields $U_A = 2U_B$, and hence $H_A = 2H_B$. This stands for the parallel connection when two shorter polymeric particles $B$ recombine themselves to form one longer $A$. Another physical case interested is the series connection, i.e., $H_B = 2H_A$, as studied in VCM model [45]. For that, we have to deal with a non-trivial dependence of $K_{eq} = K_{eq}(q)$ on variables $q$, which needs more treatment for the LMA reaction rate term (1.2), $r = k_1 \Psi_A - k_2 \Psi_B$, with non-trivial rate coefficients $k_1(q)$ and $k_2(q)$.

Furthermore, it is possible that rate coefficients $k_1$, $k_2$ (and the equilibrium “constants” $K_{eq}$) depend not only on variables $q$ but also on variables $x$, when the local equilibrium states can also be considered, see [48] for details.

We also mention that, a compatibility condition had been employed, requiring that the chemical reaction equilibrium state is consistent with the equilibrium of each species themselves, see (1.4)-(1.5). For now, more general inconsistent cases still remain open.

**Related models.** There exist various models describing wormlike micellar solutions. One famous model is Cates’ living polymer theory. In [2], Cates studied reptation dynamics of the reversible breaking/reforming reaction process of the micellar chains, in which the polymers with chain of length $L$ is assumed to be continuous and can break anywhere into polymers of shorter lengths. Based on a discrete version of Cates’ model, Vasquez-Cook-McKinley (VCM) [45] proposed a simplified model where only two different species are incorporated to describe the flow behavior of wormlike micellar solutions coupled of a viscoelastic fluid rheology. Later, German-Cook-Beris [9] revisited VCM model via a general bracket approach from a viewpoint of non-equilibrium thermodynamics.

The two-species micro-macro model (1.1) in this paper is mainly inspired by the VCM model [45]. The main difference between our model and VCM model lies on the assumption for the microscopic reaction mechanism, which enables us to derive the model with a clear variational structure. To our best knowledge, there is no any statement about this energy/entropy structure in VCM model. The VCM model assumes a convolution form of reaction rate for the reforming process $\Psi_B * \Psi_B$, which leads to the kinematics involving LMA-rate at a macroscopic level:

$$\begin{align*}
\partial_t n_A + \nabla_x \cdot (u_A n_A) &= -k_1 n_A + k_2 n_B^2, \\
\partial_t n_B + \nabla_x \cdot (u_B n_B) &= 2(k_1 n_A - k_2 n_B^2),
\end{align*}$$

(2.22)

where $n_\alpha = \int_{\mathbb{R}^3} \Psi_\alpha \, dq$, $(\alpha = A, B)$, is the number density for each species. So the VCM model is indeed a macroscopic model, due to the macroscopic LMA formulation $\tilde{r} = k_1 n_A - k_2 n_B^2$ on the right-hand side. In contrast of the VCM model, the LMA-rate in our model, $r = k_1 \Psi_A - k_2 \Psi_B^2$ in (1.2), is assumed in a microscopic level.

On the other hand, we also mention there are some other related models concerning similar chemical reaction process, such as coagulation/fragmentation, merging/splitting model in different setting, see [5, 11] and references therein.
3. PRELIMINARIES AND LEMMAS

3.1. The kernel structure of Fokker-Planck operator. Let us begin with the linear Fokker-Planck operator $\mathcal{L}_\alpha f = \frac{1}{\sqrt{\alpha}} \nabla_v \cdot \left[ M_\alpha \nabla_q \left( \frac{f}{\sqrt{\alpha}} \right) \right]$, which naturally defines the inner product, (the subscripts $A$, $B$ are omitted here)

$$
\langle \mathcal{L}f, f \rangle_{L^2_q} = \left\langle \frac{1}{\sqrt{\alpha}} \nabla_v \cdot \left[ M_\alpha \nabla_q \left( \frac{f}{\sqrt{\alpha}} \right) \right], f \right\rangle_{L^2_q}
= \left\langle M_\alpha \nabla_q \left( \frac{f}{\sqrt{\alpha}} \right), \nabla_q \left( \frac{f}{\sqrt{\alpha}} \right) \right\rangle_{L^2_q} = - \int \left| \nabla_q \left( \frac{f}{\sqrt{\alpha}} \right) \right|^2 M dq.
$$

This fact can be recast, by denoting $f = g\sqrt{M}$ and employing a new weighted inner product $\langle F, G \rangle_{L^2_M} = \langle F, GM \rangle_{L^2_q}$, as follows,

$$
\langle A g, g \rangle_{L^2_M} = \langle \nabla_q g, \nabla_q g \rangle_{L^2_M} = - \int |\nabla_q g|^2 M dq = - |\nabla_q g|^2_{L^2_M},
$$

where the operator $A$ is defined as $Ag = A(\mathcal{P}_0 g + \mathcal{P}_0^+ g) = A\mathcal{P}_0^+ g$, and we have used $L^2_q$ and $L^2_M$ as a shorthand for $L^2(dq)$ and $L^2(Mdq)$, respectively.

As a well-known property of Fokker-Planck operator, we state the following lemma (see [4, 7] for instance):

**Lemma 3.1.** Note that we have

1. Both of $A$ and $\mathcal{L}$ are self-adjoint operators with respect to their scalar products on $L^2(Mdq)$ and $L^2(dq)$, respectively;
2. Kernel of $A$ is 1-dimensional that $\text{Ker}A = \text{span}\{1\}$ on $L^2(Mdq)$.

Next we define the macroscopic quantities of number density and first-order moment as

$$
\rho^0(t, x) = \int g(t, x, q) M dq = \langle g \rangle_M, \quad j^0(t, x) = \int g(t, x, q) q M dq = \langle qg \rangle_M.
$$

The coercivity estimates:

Define the projections on the kernel of Fokker-Planck operator $\mathcal{P}_0 g = \rho^0$. The fact $A g = A(\mathcal{P}_0 g + \mathcal{P}_0^+ g) = A\mathcal{P}_0^+ g$, (3.4) together with the self-adjoint property, yields

$$
\langle A\mathcal{P}_0^+ g, \mathcal{P}_0^+ g \rangle_{L^2_M} = - \left| \nabla_q \mathcal{P}_0^+ g \right|^2_{L^2_M}.
$$

On the other hand, recall the weighted Poincaré inequality with a mean value, that

$$
\int |\nabla_q h|^2 M dq \geq \lambda_0 \int |h - (\overline{h})_M|^2 M dq,
$$

with the mean value defined as $\overline{(h)}_M = \frac{\int h M dq}{\int M dq}$. Then we can derive, for a kernel orthogonal function $h = \mathcal{P}_0^+ g$, the following partial coercivity estimate that

$$
\langle -A g, g \rangle_M = \int \int |\nabla_q \mathcal{P}_0^+ g|^2 M dq dx \geq \lambda_0 \int \int |\mathcal{P}_0^+ g|^2 M dq dx.
$$

Due to the commutative property of $\nabla_x$ and $A$, we have the similar higher-order coercivity estimate:

$$
\langle -A \nabla_x^2 g, \nabla_x^2 g \rangle_M = \int \int |\nabla_x \mathcal{P}_0^+ (\nabla_x^2 g)|^2 M dq dx \geq \lambda_0 \int \int |\mathcal{P}_0^+ (\nabla_x^2 g)|^2 M dq dx.
$$

We mention here that the coercivity estimates are of crucial importance in establishing additional dissipation effect for perturbative number density, see §4.5 for more details.
3.2. The weighted Poincaré inequality with mean value. We state here the following lemma describing the weighted Poincaré inequality with a mean value. The key part of its proof is to transfer the considered space $L^2(dqdx)$ into a weighted one $L^2(Mdqdx)$ with respect to a Maxwellian weight $M = M(q)$. The proofs are similar as that for Lemmas 3.2–3.3 in [27] (or Lemma 1.6 in [19]), except that an additional mean value term appeared in the present case. We thus omit the proofs here.

**Lemma 3.2 (Weighted Poincaré Inequality).** We have the weighted Poincaré inequalities with mean value over both variables $x, q$, that

\[
\|f\|_{L^2_{x,q}}^2 \lesssim \int \int |\nabla q \left( \frac{f}{\sqrt{M}} \right)|^2 M dq \, dx + |\rho|_{L^2_x}^2, \quad (3.9)
\]

\[
\|\nabla_q U f\|_{L^2_{x,q}}^2 \lesssim \int \int |\nabla q \left( \frac{f}{\sqrt{M}} \right)|^2 M dq \, dx + |\rho|_{L^2_x}^2, \quad (3.10)
\]

\[
\|q \nabla_q U f\|_{L^2_{x,q}}^2 \lesssim \int \int \langle q \rangle \nabla q \left( \frac{f}{\sqrt{M}} \right)|^2 M dq \, dx + |\rho|_{L^2_x}^2, \quad (3.11)
\]

where we have used the notation $\langle q \rangle = (1 + |q|^2)^{1/2}$.

**Remark 3.3.** We make two remarks here:

1. The proof relies on weighed Poincaré inequalities (such as (3.6)), in which we will use the fact for the mean value function, that

\[
\left| \frac{1}{\langle q \rangle_M} \right|_{L^2_x}^2 = \left[ \frac{\int gM dq}{\int M dq} \right]_{L^2_x}^2 = \langle M \rangle^{-1} |\rho|_{L^2_x}^2.
\]

2. Note that these inequalities can be directly extended to the case of pure spatially higher-order derivatives, by replacing $f$, $g$ and $\rho$ by $\nabla_x f$, $\nabla_x g$ and $\nabla_x \rho$, respectively.

We next state the following lemma for similar inequalities involving higher-order mixed derivatives:

**Lemma 3.4.** We have, for higher-order mixed derivatives with $l \geq 1$, that

\[
\left\| \nabla_q U \nabla_x^k \nabla_q^l f \right\|_{L^2_{x,q}}^2 \lesssim \sum_{m=0}^{l} \int \int |\nabla q \left( \frac{\nabla_x^k \nabla_q^l f}{\sqrt{M}} \right)|^2 M dq \, dx + \left\| \nabla_x^k \rho \right\|_{L^2_x}^2, \quad (3.12)
\]

\[
\left\| \nabla_x^k \nabla_q^l f \right\|_{L^2_{x,q}}^2 \lesssim \sum_{m=0}^{l-1} \int \int |\nabla q \left( \frac{\nabla_x^k \nabla_q^l f}{\sqrt{M}} \right)|^2 M dq \, dx + \left\| \nabla_x^k \rho \right\|_{L^2_x}^2. \quad (3.13)
\]

4. The A-Priori Estimates

4.1. The a-priori estimates. Define energy functionals and energy-dissipation functionals, for pure spatial derivatives, that,

\[
E_s(t) = \|u\|_{H^s_x}^2 + \|(f_A, f_B)\|_{H^s_x L^2_t}^2 = E_{s,u} + E_{s,f}, \quad (4.1)
\]

\[
D_s(t) = |\nabla_x u|^2_{H^s_x} + \sum_{k=0}^{s} \left[ \int \int |\nabla q \left( \frac{\nabla_x^k f_A}{\sqrt{M_A}} \right)|^2 M_A \, dq \, dx + \int \int |\nabla q \left( \frac{\nabla_x^k f_B}{\sqrt{M_B}} \right)|^2 M_B \, dq \, dx \right] \quad (4.2)
\]

\[
+ \left\| f_A - 2f_B \sqrt{M_B} \right\|_{H^s_x L^2_t}^2 = D_{s,u} + D_{s,f} + D_{s,f,v},
\]

for spatial derivatives on first-order moment $j = \langle qf \rangle$ with index $s' = s - 1$, that,

\[
E_{s',j}(t) = \|q(f_A, f_B)\|_{H^{s'}_x L^2_t}^2, \quad (4.3)
\]
\[ D_{s',j}(t) = \sum_{k=0}^{s'} \left[ \int \int |q \nabla_q \left( \frac{\nabla^k f_A}{\sqrt{M_A}} \right) |^2 M_A \, dq \, dx + \int \int |q \nabla_q \left( \frac{\nabla^k f_B}{\sqrt{M_B}} \right) |^2 M_B \, dq \, dx \right] \]
\[ + \left\| f_A - 2f_B \sqrt{M_B} \right\|^2_{H^s \times L^q} \]  
\quad \text{and, for mixed spatial-configurational derivatives, that,}
\[ E_{s,\text{mix}}(t) = \sum_{k+l \leq s} \frac{1}{l!} \left\| \nabla^l \left( f_A \right) \right\|^2_{H^s \times L^q} \]  
\[ D_{s,\text{mix}}(t) = \sum_{k+l \leq s} \frac{1}{l!} \left[ \int \int |\nabla_q \left( \frac{\nabla^k f_A}{\sqrt{M_A}} \right) |^2 M_A \, dq \, dx + \int \int |\nabla_q \left( \frac{\nabla^k f_B}{\sqrt{M_B}} \right) |^2 M_B \, dq \, dx \right] \]
\[ + \sum_{k+l \leq s} \frac{1}{l!} \left\| \nabla^l f_A - 2\nabla^l f_B \sqrt{M_B} \right\|^2_{L_{x,q}^2} \]  
By introducing the similar notations for number densities that
\[ E_{s,\rho}(t) = |\rho_A|_{H^s_x}, \quad D_{s,\rho}(t) = |\rho_A|^2_{H^s_x}, \]  
we will prove the following a-priori estimate, as follows:

**Proposition 4.1** (A-priori estimate). Denote that
\[ \mathcal{E}_s^\eta = E_s + \eta E_{s,\rho} + \eta E_{s,\text{mix}} + \eta^2 E_{s',j}, \]
\[ \mathcal{D}_s^\eta = D_s + \eta D_{s,\rho} + \eta D_{s,\text{mix}} + \eta^2 D_{s',j}. \]  
Assuming that \( \mathcal{E}_s^\eta \leq \varepsilon \), we can get, for some small fixed constant \( \eta \), that
\[ \frac{d}{dt} \mathcal{E}_s^\eta + \mathcal{D}_s^\eta \lesssim (\eta^2 + \eta^{-2} \varepsilon^2) \mathcal{D}_s^\eta. \]  

Note that we have the equivalence between the energy functionals defined in a way with or without the small constant parameter \( \eta \), namely, \( \mathcal{E}_s^\eta \sim \mathcal{E}_s \). Indeed, it is an easy matter to check \( \eta^{-(s+1)} \mathcal{E}_s \leq \mathcal{E}_s^\eta \leq 2 \mathcal{E}_s \). Similar equivalence holds for dissipation functionals \( \mathcal{D}_s^\eta \sim \mathcal{D}_s \).

4.2. **Higher-order spatial estimates for coupled system.** We will perform energy estimates of higher-order derivatives over variables \( x \) on fluid velocity equation (4.11), and microscopic equations of two species \( (A, B) \) (4.29), respectively, then combine them together to give the pure spatial estimates (4.31).

**\( H^s_x \)-estimate for fluid velocity \( u \):** Applying higher-order derivative operator \( \nabla^s_x \) with index \( s \geq 5 \) on the third equation of (1.14), we have
\[ \frac{1}{2} \frac{d}{dt} |\nabla^s_x u|_{L^2_x}^2 + \mu |\nabla^{s+1}_x u|_{L^2_x}^2 \lesssim |u|_{H^5_x} |\nabla^s_x u|_{H^{s-1}_x} |\nabla^s_x u|_{L^2_x} \]
\[ + \left\langle \nabla_q U_A \otimes q \nabla^s_x f_A \sqrt{M_A}, \nabla^{s+1}_x u \right\rangle + \left\langle \nabla_q U_B \otimes q \nabla^s_x f_B \sqrt{M_B}, \nabla^{s+1}_x u \right\rangle. \]  
The process is direct, by noticing the
\[ \left\langle \nabla^s_x (u \cdot \nabla_x u), \nabla^s_x u \right\rangle = \left\langle (\nabla^s_x u, u \cdot \nabla_x u), \nabla^s_x u \right\rangle + \left\langle u \cdot \nabla_x (\nabla^s_x u), \nabla^s_x u \right\rangle \]
\[ \lesssim |u|_{H^5_x} |\nabla^s_x u|_{H^{s-1}_x} \cdot |\nabla^s_x u|_{L^2_x} \]  
where the latter term in first line vanishes due to the incompressibility condition \( \text{div}_x u = 0 \), while the commutator term is bounded by using the Moser-type inequality (see [34, 44]):
\[ ||\nabla^s_x u \cdot \nabla_x v||_{L^2_x} \lesssim |u|_{H^5_x} |\nabla^s_x v|_{L^2_x} + |\nabla_x u|_{L^\infty_x} |\nabla_x v|_{H^{s-1}_x}, \]  
and the Sobolev embedding inequalities together.
$H^s_L\nu^2$-estimate for microscopic perturbations $(f_A, f_B)$: Apply higher-order spatial derivative operator $\nabla_x^s$ with index $s \geq 5$ on the first and second equation of (1.14), we then get, respectively,

$$\partial_t \nabla_x^s f_A + \nabla_x^s (u \cdot \nabla_x f_A) + \nabla_x^s (\nabla_x u q \nabla_q f_A) = \mathcal{L}_A (\nabla_x^s f_A)$$

$$+ \nabla_x^s [\nabla_x u q \nabla_q U_A (\sqrt{M_A} + \tfrac{1}{2}f_A)] + \nabla_x^s r_A, \quad (4.12)$$

and

$$\partial_t \nabla_x^s f_B + \nabla_x^s (u \cdot \nabla_x f_B) + \nabla_x^s (\nabla_x u q \nabla_q f_B) = \mathcal{L}_B (\nabla_x^s f_B)$$

$$+ \nabla_x^s [\nabla_x u q \nabla_q U_B (\sqrt{M_B} + \tfrac{1}{2}f_B)] + \nabla_x^s r_B. \quad (4.13)$$

Here we actually have used the commutative property between the spatial derivative operator $\nabla_x^s$ and the Fokker-Planck operator $\mathcal{L}_\alpha$ with $\alpha = A, B$.

Take an inner product with $(\nabla_x^s f_A, \nabla_x^s f_B)$, respectively, in space $L^2_x q$, then we can get some similar estimates for above two equations, except the last reaction terms. We take species $A$ for example, and write the estimates as follows:

$$\langle \partial_t \nabla_x^s f_A, \nabla_x^s f_A \rangle = \frac{1}{2} \frac{d}{dt} ||\nabla_x^s f_A||^2_{L^2_x q}, \quad (4.14)$$

and

$$\langle \nabla_x^s (u \cdot \nabla_x f_A), \nabla_x^s f_A \rangle = \langle \nabla_x^s (u \cdot \nabla_x f_A), \nabla_x^s f_A \rangle + \langle u \cdot \nabla_x (\nabla_x^s f_A), \nabla_x^s f_A \rangle, \quad (4.15)$$

where the latter product will vanish due to the incompressibility condition $\text{div}_x u = 0$. The commutator in the former can be decomposed as:

$$\|\nabla_x^s u \cdot \nabla_x f_A\|_{L^2_x q} = \sum_{s_1 + s_2 = s \geq 1} ||\nabla_x^{s_1} u \nabla_x (\nabla_x^{s_2} f_A)\|_{L^2_x q}. \quad (4.16)$$

In the case $s_1 = s$, the Hölder inequality yields that,

$$\|\nabla_x^s u \nabla_x f_A\|_{L^2_x q} \leq \|\nabla_x^s u\|_{L^2_x} \|\nabla_x f_A\|_{H^s_x L^2_q} \lesssim \|\nabla_x^s u\|_{L^2_x} \|\nabla_x f_A\|_{H^s_x L^2_q}, \quad (4.17)$$

where we have used the Sobolev embedding inequality in the last line.

In the case $s_1 = s - 1$, we get similarly, that

$$\|\nabla_x^{s-1} u (\nabla_x^2 f_A)\|_{L^2_x q} \leq \|\nabla_x^{s-1} u\|_{L^2_x} \|\nabla_x^2 f_A\|_{H^s_x L^2_q} \lesssim \|\nabla_x^{s-1} u\|_{H^s_x} \|\nabla_x^2 f_A\|_{H^s_x L^2_q}, \quad (4.18)$$

where we have used the Hölder inequality and the Sobolev embedding inequality again.

For the left cases $1 \leq s_1 \leq s - 2$ (and hence $s_2 \leq s - 1$), we can get the following control,

$$\sum_{1 \leq s_1 \leq s - 2} ||\nabla_x^{s_1} u \nabla_x (\nabla_x^{s_2} f_A)\|_{L^2_x q} \leq \sum_{1 \leq s_1 \leq s - 2} ||\nabla_x^{s_1} u\|_{H^s_x} \|\nabla_x^{s_2+1} f_A\|_{L^2_x L^4_q} \lesssim \|u\|_{H^s_x} \|f_A\|_{H^s_x L^2_q}. \quad (4.19)$$

Combining three above cases together gives the bound on the commutator

$$\|\nabla_x^s f_A \cdot u \cdot \nabla_x f_A\|_{L^2_x q} \lesssim \|u\|_{H^s_x} \|f_A\|_{H^s_x L^2_q}, \quad (4.20)$$

and consequently gives the control on the term:

$$\langle \nabla_x^s (u \cdot \nabla_x f_A), \nabla_x^s f_A \rangle \lesssim \|u\|_{H^s_x} \|f_A\|_{H^s_x L^2_q} \|\nabla_x^s f_A\|_{L^2_x q}. \quad (4.21)$$

For the third term in left-hand side of equation (4.12), by noticing the fact $\nabla_q \cdot (\nabla_x u q) = \text{div}_x u = 0$, we have the following decomposition,

$$\langle \nabla_x^s (\nabla_x u q \nabla_q f_A), \nabla_x^s f_A \rangle = \langle \nabla_x^s (\nabla_x u q \nabla_q f_A) - \nabla_x u q \nabla_q (\nabla_x^s f_A), \nabla_x^s f_A \rangle$$

$$= \sum_{s_1 + s_2 = s \geq 1} \langle \nabla_x^{s_1} (\nabla_x u) \nabla_q \nabla_x^{s_2} f_A, q \nabla_x^s f_A \rangle. \quad (4.22)$$
The summation can be bounded in a similar discussion way by cases as above, that,

$$\sum_{s_1+s_2=s \geq 1} \|\nabla_{x}^{s_1} (\nabla_{u} f_A) \nabla_{q}^{s_2} f_A \|_{L_{x,q}^{2}}$$

(4.23)

$$= \|\nabla_{x}^{s_1+1} u \nabla_{q} f_A \|_{L_{x,q}^{2}} + \|\nabla_{x}^{s_1} u \nabla_{q} \nabla_{x} f_A \|_{L_{x,q}^{2}} + \sum_{1 \leq s_1 \leq s-2} \|\nabla_{x}^{s_1+1} u \nabla_{q}^{s_2} f_A \|_{L_{x,q}^{2}}$$

$$\leq |\nabla_{x}^{s_1+1} u|_{L_{x}^{2}} \|\nabla_{q} f_A \|_{L_{x,q}^{2}} + |\nabla_{x} u|_{H_{x}^{1}} \|\nabla_{q} \nabla_{x} f_A \|_{L_{x,q}^{2}} + \sum_{1 \leq s_1 \leq s-2} |\nabla_{x}^{s_1+1} u|_{L_{x,q}^{2}} \|\nabla_{q}^{s_2} f_A \|_{L_{x,q}^{2}}$$

(4.24)

where we have used again the Hölder inequality and the Sobolev embedding inequalities. Hence, we get,

$$\langle \nabla_{x}^{s} (\nabla_{x} u q \nabla_{q} f_A), \nabla_{x}^{s} f_A \rangle \lesssim |\nabla_{x} u|_{H_{x}^{s}} \|\nabla_{q} f_A \|_{H_{x}^{s-1} L_{q}^{2}} \|\nabla_{q} U_{A} \nabla_{x}^{s} f_A \|_{L_{x,q}^{2}}.$$

(4.25)

We now turn to consider the right-hand side of equation (4.12). Firstly we have,

$$\langle \mathcal{L}_{A} \nabla_{x}^{s} f_A, \nabla_{x}^{s} f_A \rangle = - \iint \nabla_{q} \left( \frac{\nabla_{x} f_A}{M_{A}} \right) \|_{M_{x,q}}^{2} \mathrm{d}q \mathrm{d}x = - \left\| \nabla_{q} \left( \frac{\nabla_{x} f_A}{M_{A}} \right) \right\|_{M_{x,q}}^{2}.$$

(4.26)

For the penultimate term in the right-hand side, we get,

$$\langle \nabla_{x}^{s} (\nabla_{x} u q \nabla_{q} U_{A} f_A), \nabla_{x}^{s} f_A \rangle = \sum_{s_1+s_2=s} \langle \nabla_{x}^{s_1+1} u q \nabla_{x}^{s_2} f_A, \nabla_{q} U_{A} \nabla_{x}^{s} f_A \rangle$$

(4.27)

$$\lesssim \langle |\nabla_{x} u|_{H_{x}^{s}} \|q f_A\|_{H_{x}^{s-1} L_{q}^{2}} + |u|_{H_{x}^{s}} \|q f_A\|_{H_{x}^{s-1} L_{q}^{2}} \rangle \left\|\nabla_{q} U_{A} \nabla_{x}^{s} f_A \right\|_{L_{x,q}^{2}},$$

where we have used a similar discussion by cases as above, that

$$\sum_{s_1+s_2=s} \left\|\nabla_{x}^{s_1+1} u q \nabla_{x}^{s_2} f_A \right\|_{L_{x,q}^{2}}$$

(4.28)

where the Hölder inequality and the Sobolev embedding inequalities are used again.

Since the estimates for equation (4.13) of species $B$ are similar as above process for species $A$, then we are left to deal with the reaction contribution now, by combining both equations for two species $A, B$ together. We calculate that

$$\langle \nabla_{x}^{s} (f_A - 2 f_B \sqrt{M_B} - f_B^2), \nabla_{x}^{s} f_B \rangle$$

(4.29)

$$= - \langle \nabla_{x}^{s} (f_A - 2 f_B \sqrt{M_B} - f_B^2), \nabla_{x}^{s} f_B \rangle + 2 \left\| M_B \nabla_{x}^{s} (f_A - 2 f_B \sqrt{M_B} - f_B^2), \nabla_{x}^{s} f_B \right\|_{L_{x,q}^{2}}$$

$$= - \left\| \nabla_{x}^{s} (f_A - 2 f_B \sqrt{M_B}) \right\|_{L_{x,q}^{2}}^{2} + \iint \nabla_{x}^{s} (f_B) \cdot \nabla_{x}^{s} (f_A - 2 f_B \sqrt{M_B}) \mathrm{d}q \mathrm{d}x$$

$$\lesssim - \left\| \nabla_{x}^{s} (f_A - 2 f_B \sqrt{M_B}) \right\|_{L_{x,q}^{2}}^{2} + \|f_B\|_{H_{x}^{s} L_{q}^{2}} \|f_B\|_{H_{x}^{s} L_{q}^{2}} \left\|\nabla_{x}^{s} (f_A - 2 f_B \sqrt{M_B}) \right\|_{L_{x,q}^{2}},$$
where we have used the product formula in Sobolev spaces
\[ |\nabla_x^s (fg)|_{L^2_q} \lesssim |f|_{L^\infty} |g|_{H^2} + |f|_{H^2} |g|_{L^\infty}. \]

Therefore, combining all the above estimates together enable us to get the higher-order purely spatial estimates on microscopic perturbations \((f_A, f_B)\), as follows,
\[
\frac{1}{2t} \frac{d}{dt} \left( |\nabla_x^s (f_A, f_B)|_{L^2_q}^2 + \left| \nabla_q \left( \frac{\nabla_x^s (f_A)}{M} \right) \right|_{L^2_q}^2 + \left| \nabla_q \left( \frac{\nabla_x^s f_B}{M} \right) \right|_{L^2_q}^2 \right)
\]
\[
\lesssim \left( \nabla_x^{s+1} u q \nabla_q U A \sqrt{M_A}, \nabla_x^s f_A \right) + \left( \nabla_x^{s+1} u q \nabla_q U B \sqrt{M_B}, \nabla_x^s f_B \right) \tag{4.29}
\]
\[
+ |u|_{H^2}^2 |f_A|_{H^2 L^2_q}^2 |\nabla_x^s f_A|_{L^2_q}^2 + |\nabla_x u|_{H^2}^2 |\nabla_q^2 f_A|_{H^2 \to 1 L^2_q}^2 |\nabla_q^2 U A \nabla_x^s f_A|_{L^2_q}^2
\]
\[
+ \left( |u|_{H^2}^2 |q f_A|_{H^2 L^2_q}^2 + |\nabla_x u|_{H^2}^2 |q f_A|_{H^2 \to 1 L^2_q}^2 \right) \left| \nabla_q^2 U A \nabla_x^s f_A \right|_{L^2_q}^2
\]
\[
+ |u|_{H^2}^2 |f_B|_{H^2 L^2_q}^2 |\nabla_x^s f_B|_{L^2_q}^2 + |\nabla_x u|_{H^2}^2 |\nabla_q^2 f_B|_{H^2 \to 1 L^2_q}^2 |\nabla_q^2 U B \nabla_x^s f_B|_{L^2_q}^2
\]
\[
+ \left( |u|_{H^2}^2 |q f_B|_{H^2 L^2_q}^2 + |\nabla_x u|_{H^2}^2 |q f_B|_{H^2 \to 1 L^2_q}^2 \right) \left| \nabla_q^2 U B \nabla_x^s f_B \right|_{L^2_q}^2
\]
\[
+ |f_B|_{H^2 L^2_q}^2 |f_B|_{H^2 L^2_q}^2 |\nabla_x \left( f_A - 2 f_B \sqrt{M_B} \right) |_{L^2_q}^2.
\]

**Higher-order spatial estimates for coupled system:** At the end of this subsection, by noticing the cancellation relations between micro-macro coupling:
\[
\begin{align*}
\left( \nabla_q U A \otimes q \nabla_x^s f_A \sqrt{M_A}, \nabla_x^{s+1} u \right) + \left( \nabla_x^{s+1} u q \nabla_q U A \sqrt{M_A}, \nabla_x^s f_A \right) = 0, \\
\left( \nabla_q U B \otimes q \nabla_x^s f_B \sqrt{M_B}, \nabla_x^{s+1} u \right) + \left( \nabla_x^{s+1} u q \nabla_q U B \sqrt{M_B}, \nabla_x^s f_B \right) = 0,
\end{align*}
\]
we can conclude the higher-order spatial estimates for coupled perturbative system (1.14), that
\[
\frac{1}{2t} \frac{d}{dt} \left( |\nabla_x^s u|_{L^2_q}^2 + \left| \nabla_x^s (f_A, f_B) \right|_{L^2_q}^2 \right) + \mu |\nabla_x^{s+1} u|_{L^2_q}^2
\]
\[
+ \left| \nabla_q \left( \frac{\nabla_x^s f_A}{M} \right) \right|_{L^2_q}^2 + \left| \nabla_q \left( \frac{\nabla_x^s f_B}{M} \right) \right|_{L^2_q}^2 \right)
\]
\[
\lesssim |u|_{H^2}^2 |\nabla_x u|_{H^2}^2 |\nabla_x^s u|_{L^2_q}^2 + |u|_{H^2}^2 |f_A|_{H^2 L^2_q}^2 |\nabla_x^s f_A|_{L^2_q}^2 + |\nabla_x u|_{H^2}^2 |q f_A|_{H^2 \to 1 L^2_q}^2 |\nabla_q^2 U A \nabla_x^s f_A|_{L^2_q}^2
\]
\[
+ \left( |u|_{H^2}^2 |q f_A|_{H^2 L^2_q}^2 + |\nabla_x u|_{H^2}^2 |q f_A|_{H^2 \to 1 L^2_q}^2 \right) \left| \nabla_q^2 U A \nabla_x^s f_A \right|_{L^2_q}^2
\]
\[
+ |u|_{H^2}^2 |f_B|_{H^2 L^2_q}^2 |\nabla_x^s f_B|_{L^2_q}^2 + |\nabla_x u|_{H^2}^2 |\nabla_q^2 f_B|_{H^2 \to 1 L^2_q}^2 |\nabla_q^2 U B \nabla_x^s f_B|_{L^2_q}^2
\]
\[
+ \left( |u|_{H^2}^2 |q f_B|_{H^2 L^2_q}^2 + |\nabla_x u|_{H^2}^2 |f_B|_{H^2 \to 1 L^2_q}^2 \right) \left| \nabla_q^2 U B \nabla_x^s f_B \right|_{L^2_q}^2
\]
\[
+ |f_B|_{H^2 L^2_q}^2 |f_B|_{H^2 L^2_q}^2 |\nabla_x \left( f_A - 2 f_B \sqrt{M_B} \right) |_{L^2_q}^2.
\]

By virtue of notations for energy and dissipation functionals defined in subsection §4.1, we can write
\[
\frac{d}{dt} E_s + D_s \lesssim E_{s,u}^\frac{1}{2} D_{s,u} + E_{s,u}^\frac{1}{2} (D_{s,f} + D_{s,\rho}) + D_{s,u}^\frac{1}{2} |\nabla_q (f_A, f_B)|_{H^2 \to 1 L^2_q}^2 \left( D_{s,f}^\frac{1}{2} + D_{s,\rho}^\frac{1}{2} \right) \tag{4.32}
\]
\[
+ \left[ E_{s,u}^\frac{1}{2} |q (f_A, f_B)|_{H^2 L^2_q}^2 + D_{s,u}^\frac{1}{2} |q (f_A, f_B)|_{H^2 L^2_q}^2 \right] \left( D_{s,f}^\frac{1}{2} + D_{s,\rho}^\frac{1}{2} \right)
\]
\[
+ |f_B|_{H^2 L^2_q}^2 |f_B|_{H^2 L^2_q}^2 D_{s,f,r}^\frac{1}{2}.
\]

**Remark 4.2.** Indeed, this suggests us to estimate higher-order mixed derivatives for \((f_A, f_B)\) and higher-order purely spatial derivatives for first-order moments \((q f_A, q f_B)\). In fact, for
the latter, it suffices to control higher but not highest order derivatives, since the highest order quantities should be viewed as dissipative part.

4.3. Higher-order spatial estimates on first-order moments. We will perform in this subsection higher-order spatial derivative estimates for first-order moments \( (qf_A, qf_B) \) with a lower Sobolev index \( s' = s - 1 \) (actually, \( 4 \leq s' \leq s - 1 \) suffices to get our result).

We consider to act higher-order spatial derivative operator \( \nabla_x^k \) with index \( 4 \leq k \leq s' \) on the first and second equation of (1.1), then by taking inner product of equation (4.12) with \( |q|^2 \nabla_x^k f_A \), we get,

\[
\left\langle \partial_t \nabla_x^k f_A, |q|^2 \nabla_x^k f_A \right\rangle = \frac{1}{2} \frac{d}{dt} \left\| \nabla_x^k f_A \right\|_{L^2_{x,q}}^2,
\]

and

\[
\left\langle \nabla_x^k (u \cdot \nabla_x f_A), |q|^2 \nabla_x^k f_A \right\rangle = \left\langle [\nabla_x^k u, \nabla_x] q f_A, q \nabla_x^k f_A \right\rangle + \left\langle u \cdot \nabla_x (q \nabla_x^k f_A), q \nabla_x^k f_A \right\rangle,
\]

where we have used the same discussion as that in subsection §4.3 and the incompressibility \( \text{div}_x u = 0 \) again.

We decompose the estimate for the third term in left-hand side into three parts, more precisely,

\[
\left\langle \nabla_x^k (\nabla_x uq \cdot \nabla_x f_A), |q|^2 \nabla_x^k f_A \right\rangle
\]

\[
= \left\langle \nabla_x^k (\nabla_x u) q \nabla_x f_A, |q|^2 \nabla_x^k f_A \right\rangle + \left\langle \nabla_x u q \nabla_x \nabla_x^k f_A, |q|^2 \nabla_x^k f_A \right\rangle
\]

\[
+ \sum_{k_1 + k_2 = k} \left\langle \nabla_x^{k_1} (\nabla_x u) q \nabla_x^{k_2} f_A, |q|^2 \nabla_x^k f_A \right\rangle.
\]

The first term in above expression can be controlled as follows,

\[
\left\langle \nabla_x^k (\nabla_x u) q \nabla_x f_A, |q|^2 \nabla_x^k f_A \right\rangle \lesssim |\nabla_x u|_{H^k_x} \left\| q \nabla_x f_A \right\|_{H^k_x L^2_{x,q}} \left\| q \nabla_x U_A \nabla_x^k f_A \right\|_{L^2_{x,q}}.
\]

Recalling the fact \( \nabla \cdot (\nabla_x u) = \text{div}_x u = 0 \), we have,

\[
\left\langle \nabla_x u q \nabla_x^k f_A, |q|^2 \nabla_x^k f_A \right\rangle = \int \nabla_x u q \left| q \nabla_x^k f_A \right|^2 dq dx
\]

\[
= -\int |\nabla_x u|_{L^\infty_x} \left| q \nabla_x^k f_A \right|^2 dq dx
\]

\[
\lesssim |\nabla_x u|_{L^\infty_x} \left\| q \nabla_x^k f_A \right\|_{L^2_{x,q}}^2.
\]

For the summation term, we get

\[
\sum_{k_1 + k_2 = k} \sum_{1 \leq k_1 \leq k - 1} \left\langle \nabla_x^{k_1} (\nabla_x u) q \nabla_x^{k_2} f_A, |q|^2 \nabla_x^k f_A \right\rangle
\]

\[
= \left\langle \nabla_x^k u q \nabla_x f_A, |q|^2 \nabla_x^k f_A \right\rangle + \left\langle \nabla_x^{k-1} u q \nabla_x^2 f_A, |q|^2 \nabla_x^k f_A \right\rangle
\]

\[
+ \sum_{k_1 + k_2 = k} \sum_{1 \leq k_1 \leq k - 3} \left\langle \nabla_x^{k_1} (\nabla_x u) q \nabla_x^{k_2} f_A, |q|^2 \nabla_x^k f_A \right\rangle \left\| q \nabla_x^2 f_A \right\|_{L^2_{x,q}} \left\| q \nabla_x^k f_A \right\|_{L^2_{x,q}}
\]

\[
\lesssim \left\| q \nabla_x f_A \right\|_{L^\infty_x L^2_{x,q}} \left\| q \nabla_x^2 f_A \right\|_{L^2_{x,q}} + |\nabla_x^{k-1} u|_{L^2_x} \left\| q \nabla_x^2 f_A \right\|_{L^2_{x,q}} \left\| q \nabla_x^k f_A \right\|_{L^2_{x,q}}
\]
\[
+ \sum_{1 \leq k_1 \leq k \leq 3} |\nabla^k_x \nabla^k_x u|_{L^\infty_x} \left\| q \nabla_q \nabla^k_x f_A \right\|_{L^2_x L^2_q} \left\| |q|^2 \nabla^k_x f_A \right\|_{L^2_x L^2_q} \\
\lesssim |u|_{H^k_x} \left\| q \nabla_q f_A \right\|_{H^{k-1} \tilde{H} \tilde{L}^2_x} \left\| q \nabla_q U A \nabla^k_x f_A \right\|_{L^2_x L^2_q}.
\]

As a consequence, we obtain that,
\[
\left\langle L_A \nabla^k_x f_A, |q|^2 \nabla^k_x f_A \right\rangle = - \left\langle q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right), q \nabla_q \left( |q|^2 \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\rangle \\
= - \int \int |q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right)|^2 M_A \, dq \, dx - \int \int q M_A \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right)^2 \, dq \, dx \\
= - \int \int |q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right)|^2 M_A \, dq \, dx + 3 \int \int \left| \nabla^k_x f_A \right|^2 M_A \, dq \, dx \\
= - \int \int q \nabla_q U A \left| \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right|^2 M_A \, dq \, dx \\
= - \left\| q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\|_{L^2_x L^2_q}^2 + 3 \left\| \nabla^k_x f_A \right\|_{L^2_x}^2 - \int \int q \nabla_q U A \left| \nabla^k_x f_A \right|^2 dq \, dx.
\]

Noting here the third term in the last line can be controlled by the Hölder inequality and Lemma 3.2 as follows,
\[
\int \int q \nabla_q U A \left| \nabla^k_x f_A \right|^2 dq \, dx \leq \left\| q \nabla_q U A \nabla^k_x f_A \right\|_{L^2_x L^2_q} \left\| \nabla^k_x f_A \right\|_{L^2_x L^2_q} \\
\lesssim \left( \left\| q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\|_{L^2_x} + \left\| \nabla^k_x \rho_A \right\|_{L^2_x} \right) \cdot \left( \left\| q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\|_{L^2_x} + \left\| \nabla^k_x \rho_A \right\|_{L^2_x} \right) \\
\leq \frac{1}{4} \left\| q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\|_{L^2_x}^2 + C \left( \left\| q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\|_{L^2_x}^2 + \left\| \nabla^k_x \rho_A \right\|_{L^2_x}^2 \right)^2,
\]

where we have used the simple fact \( \left\| \nabla^k_x \rho_A \right\|_{L^2_x} \leq \left\| \nabla^k_x f_A \right\|_{L^2_x L^2_q} \) due to Hölder’s inequality again. Thus, we are ready to obtain the estimates,
\[
\left\langle L_A \nabla^k_x f_A, |q|^2 \nabla^k_x f_A \right\rangle \geq - \frac{3}{4} \left\| q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\|_{L^2_x}^2 + C \left( \left\| q \nabla_q \left( \frac{\nabla^k_x f_A}{\sqrt{M_A}} \right) \right\|_{L^2_x}^2 + \left\| \nabla^k_x \rho_A \right\|_{L^2_x}^2 \right).
\]

As for the penultimate term in the right-hand side, we can infer from a discussion by cases that,
\[
\left\langle \nabla^k_x (\nabla_x u q \nabla_q U A f_A), |q|^2 \nabla^k_x f_A \right\rangle \\
= \sum_{k_1 + k_2 = k} \left\langle \nabla^{k_1+1}_x u q \nabla_q U A \nabla^k_x f_A, |q|^2 \nabla^k_x f_A \right\rangle \\
\lesssim \left( |\nabla_x u|_{H^k_x} \left\| q \nabla_q U A f_A \right\|_{H^{k-1} \tilde{H} \tilde{L}^2_x} + |u|_{H^k_x} \left\| q \nabla_q U A f_A \right\|_{H^{k-1} \tilde{H} \tilde{L}^2_x} \right) \left\| q \nabla_q U A \nabla^k_x f_A \right\|_{L^2_x L^2_q} \\
\text{and} \\
\left\langle \nabla^k_x (\nabla_x u q \nabla_q U A \sqrt{M_A}), |q|^2 \nabla^k_x f_A \right\rangle \leq |\nabla_x u|_{H^k_x} \left\| |q|^2 \sqrt{M_A} \right\|_{L^2_x} \left\| q \nabla_q U A \nabla^k_x f_A \right\|_{L^2_x L^2_q}.
\]
We next deal with the reaction contribution. Combining the two equations for both species \( A, B \) together implies,
\[
\left\langle \nabla_x^k f_A, \nabla_x^k f_B \right\rangle + \left\langle \nabla_x^k f_B, \nabla_x^k f_B \right\rangle = -\left| q \right|^2 \langle \nabla_x^k (f_A - 2f_B \sqrt{MB}) \rangle^2_{L^2, q} + \left( \nabla_x^k (|q|^2 f_B^2) \cdot \nabla_x^k (f_A - 2f_B \sqrt{MB}) \right) \, dq \, dx
\]
\[
\lesssim -\left| q \right|^2 \langle \nabla_x^k (f_A - 2f_B \sqrt{MB}) \rangle^2_{L^2, q} + \left\| q f_B \right\|_{H^s L^2} \left\| q f_B \right\|_{H^s L^2} \left\| \nabla_x^k (f_A - 2f_B \sqrt{MB}) \right\|_{L^2, q}.
\]
Therefore, together with previous estimates for species \( A \) and \( B \), this concludes spatial estimates on first order moments \( (q f_A, q f_B) \) with index \( 4 \leq k \leq s' = s - 1 \),
\[
\frac{1}{2} \frac{d}{dt} \left\| \nabla_x^k (f_A, f_B) \right\|_{L^2, q}^2 + \frac{\alpha}{2} \left( \left\| \nabla_x^k f_A \right\|_{L^2, q}^2 + \left\| \nabla_x^k f_B \right\|_{L^2, q}^2 \right) + \left| q \right|^2 \langle \nabla_x^k (f_A - 2f_B \sqrt{MB}) \rangle^2_{L^2, q} 
\]
\[
\lesssim \left( \left\| \nabla_x^k f_A \right\|_{L^2, q}^2 + \left\| \nabla_x^k f_B \right\|_{L^2, q}^2 \right) + \left\| q f_B \right\|_{H^s L^2} \left\| q f_B \right\|_{H^s L^2} \left\| \nabla_x^k (f_A - 2f_B \sqrt{MB}) \right\|_{L^2, q},
\]
whence we get
\[
\frac{d}{dt} E_{s'} + D_{s'} \lesssim \left( D_{s'}, f + D_{s'}, \rho \right) + E_{\frac{s'}{2}, u} \left( D_{s', f} + D_{s', \rho} \right)
\]
\[
+ \left( E_{\frac{s}{2}, u} \left\| q \nabla_x f_A, f_B \right\|_{H^{s'} - 1 L^2} + E_{\frac{s}{2}, u} \left\| q \nabla_x (f_A, f_B) \right\|_{H^s L^2} \right) \left( D_{\frac{s}{2}, j} + D_{\frac{s}{2}, f} + D_{\frac{s}{2}, \rho} \right)
\]
\[
+ \left( D_{s', u} + E_{\frac{s}{2}, u} \right) \left( D_{s', f} + D_{s', \rho} \right) + D_{\frac{s}{2}, u} \left( D_{\frac{s}{2}, j} + D_{\frac{s}{2}, f} + D_{\frac{s}{2}, \rho} \right)
\]
\[
+ \left\| q f_B \right\|_{H^{s'} L^2} \left\| q f_B \right\|_{H^s L^2} D_{s', f, r},
\]
where we have used \( D_{s', u} \leq E_{s', u} \) and \( \left\| q \nabla_x U \alpha \nabla_x f_A \right\|_{L^2, q} \lesssim D_{\frac{k}{2}, j} + D_{\frac{k}{2}, f} + D_{\frac{k}{2}, \rho} \).

4.4. Higher-order mixed estimates on microscopic equations.

We will perform in this subsection higher-order mixed derivative estimates for both microscopic equations. In fact, we consider here the higher-order mixed derivatives on \( \left\| \nabla_x^k \nabla_q^l (f_A, f_B) \right\|_{L^2, q} \), in which \( k + l = s, \ l \geq 1 \). Besides, we denote \( \nabla^k_1 = \nabla_x^k \nabla^l_1 \) for notational simplicity.

Applying the mixed derivative operator \( \nabla_x^k \) on the first and second equation of (1.14), we get respectively,
\[ \partial_t \nabla^k_f A + \nabla^k \left( u \cdot \nabla x \nabla f_A \right) + \nabla^k \left( \nabla_x u q \nabla q f_A \right) = \nabla^k \mathcal{L}_A f_A + \nabla^k \left[ \nabla_x u q \nabla q U_A \left( \sqrt{M_A} + \frac{1}{2} f_A \right) \right] + \nabla^k r_A, \quad (4.48) \]

and

\[ \partial_t \nabla^k f_B + \nabla^k \left( u \cdot \nabla x \nabla f_B \right) + \nabla^k \left( \nabla_x u q \nabla q f_B \right) = \nabla^k \mathcal{L}_B f_B + \nabla^k \left[ \nabla_x u q \nabla q U_B \left( \sqrt{M_B} + \frac{1}{2} f_B \right) \right] + \nabla^k r_B. \quad (4.49) \]

It follows that,

\[ \left\langle \partial_t \nabla^k f_A, \nabla^k f_A \right\rangle = \frac{1}{2} \frac{d}{dt} \left\| \nabla^k f_A \right\|_{L^2_{x,q}}^2, \quad (4.50) \]

and by a discussion by cases, that

\[ \left\langle \nabla^k \left( u \cdot \nabla x f_A \right), \nabla^k f_A \right\rangle \quad (4.51) \]

\[ = \left\langle \left[ \nabla^k_x, u \cdot \nabla x \right] \nabla^k f_A, q \nabla^k f_A \right\rangle + \left\langle u \cdot \nabla x \left( \nabla^k f_A \right), \nabla^k f_A \right\rangle, \quad (4.52) \]

\[ = \sum_{k_1 + k_2 = k, 1 \leq k_1 \leq k \leq s-1} \left\langle \nabla_x^{k_1} u \nabla x \nabla_x^{k_2} f_A, \nabla^k f_A \right\rangle \]

\[ \leq \left( \left\| \nabla_x^k u \right\|_{L^2_x} \left\| \nabla^k f_A \right\|_{L^2_x L^2_q} \right) + \sum_{k_1 + k_2 = k, 1 \leq k_1 \leq k \leq s-2} \left\| \nabla_x^{k_1} u \right\|_{L^\infty_x} \left\| \nabla_x^{k_2+1} f_A \right\|_{L^2_x L^2_q} \left\| \nabla^k f_A \right\|_{L^2_x L^2_q} \]

\[ \lesssim |u|_{H^2_x} \left\| \nabla^k f_A \right\|_{H^1_x L^2} \left\| \nabla^k f_A \right\|_{L^2_{x,q}}. \quad (4.53) \]

As for the third term in the left-hand side of (4.48), we notice that

\[ \left\langle \nabla^k \left( \nabla_x u q \nabla q f_A \right), \nabla^k f_A \right\rangle = \left\langle \nabla^k \left( \nabla_x u q \nabla_q f_A \right), \nabla^k f_A \right\rangle + \left\langle \nabla^k \left( \nabla_x \nabla_q f_A \right), \nabla^k f_A \right\rangle \quad (4.54) \]

A similar discussion by cases as above ensures that

\[ \left\langle \nabla^k \left( \nabla_x u q \nabla_q f_A \right), \nabla^k f_A \right\rangle \]

\[ = \left\langle \nabla_x^{k+1} u q \nabla_q f_A, \nabla^k q \nabla^k f_A \right\rangle + \left\langle \nabla^k u \nabla_q^{k+1} f_A, \nabla^k q \nabla^k f_A \right\rangle + \sum_{k_1 + k_2 = k, 1 \leq k_1 \leq k \leq s-2} \left\langle \nabla_x^{k+1} u q \nabla_q^{k+1} f_A, \nabla^k q \nabla^k f_A \right\rangle \quad (4.55) \]

\[ \leq \left( \left\| \nabla_x^{k+1} u \right\|_{L^2_x} \left\| \nabla^k q \nabla^k f_A \right\|_{L^\infty_x L^2_q} \right) + \left\| \nabla^k u \right\|_{L^2_{x,q}} \left\| \nabla^k q \nabla^k f_A \right\|_{L^2_{x,q}} \]

\[ \lesssim |u|_{H^2_x} \left\| \nabla^k q \nabla^k f_A \right\|_{H^1_x L^2} \left\| \nabla^k q \nabla^k f_A \right\|_{L^2_{x,q}}. \quad (4.56) \]

\[ \text{and} \quad \left\langle \nabla^k \left( \nabla_x \nabla_q f_A \right), \nabla^k f_A \right\rangle \]

\[ = \left\langle \nabla_x^{k+1} \nabla_q f_A, \nabla^k f_A \right\rangle + \left\langle \nabla_x^k \nabla_q^{k+1} f_A, \nabla^k f_A \right\rangle + \sum_{k_1 + k_2 = k, 1 \leq k_1 \leq k \leq s-2} \left\langle \nabla_x^{k+1} \nabla_q^{k+1} f_A, \nabla^k f_A \right\rangle \quad (4.57) \]
the potential assumption (1.16), $|\nabla_q^l (\Delta_q U_A - \frac{1}{2}|\nabla_q U_A|^2)| \lesssim |\nabla U_A|^2 + 1$, then this relation implies

$$\left< \nabla^k L A f_A, \nabla^k f_A \right> \geq - \left< \nabla_q (\nabla^k f_A) \right>^2_M$$

+ $C \sum_{0 \leq l_2 \leq l-1} \left( \left< \nabla_q U_A \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla_q U_A \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla^l_2 f_A \right>_{L^2_{x,q}} \right)$. 

As for the penultimate term in right-hand side of (4.48), the potential assumption (1.16) $|\nabla^l_1 (q \nabla q U_A)| \lesssim |q \nabla U_A|$ enables us to get

$$\left< \nabla^l_1 (\Delta_x u q \nabla q U_A f_A), \nabla^k f_A \right> \leq \sum_{l_2 \leq l} \left( \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}} \right) + \sum_{l_2 \leq l} \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}$$

and the potential assumption (1.16) $\left< \nabla^l_1 (q \nabla q U_A \sqrt{M_A}) \right>_{L^2_{q}} \leq C$ ensures that

$$\left< \nabla^l_1 (\Delta_x u q \nabla q U_A \sqrt{M_A}), \nabla^k f_A \right> \leq \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}, \left< \nabla^l_1 u \nabla^l_2 f_A \right>_{L^2_{x,q}}$$

Now we are in a position to deal with the reaction contributions, which can be reformulated as

$$\left< \nabla^l_1 f_A, \nabla^k f_A \right> + \left< \nabla^l_1 f_A, \nabla^k f_A \right>$$

$$= - \left< \nabla^l_1 (f_A - 2f_B \sqrt{M_B} - f^2_B), \nabla^k f_A \right> + 2 \left< \nabla^l_1 (f_A \sqrt{M_B} - 2f_B M_B - f^2_B \sqrt{M_B}), \nabla^k f_A \right>$$
\begin{equation}
\begin{aligned}
&= - \left\| \nabla^k f_A - 2 \nabla^k f_B \sqrt{M_B} \right\|_{L_{2,q}^2}^2 + \int \int \nabla_i^k (f_B^2) (\nabla^k f_A - 2 \nabla^k f_B \sqrt{M_B}) \, dq \, dx + \sum_{l_2 \geq 1} \sum_{i=1}^{4} \Re_{i},
\end{aligned}
\end{equation}

where the remainder terms \( \Re_i \)'s can be controlled by

\begin{equation}
|\Re_1| = 2 \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2} \left\| \nabla_i^l q \sqrt{M_B} \right\|_{L_{2,q}^\infty} \left\| \nabla_i^k f_A \right\|_{L_{2,q}^2},
\end{equation}

and similarly,

\begin{align*}
|\Re_2| &= 2 \left\| \nabla_i^k f_A \nabla_i^l q \sqrt{M_B} \nabla_i^k f_B \right\|_{L_{2,q}^2} dq \, dx \lesssim \left\| \nabla_i^k f_A \right\|_{L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2}, \\
|\Re_3| &= -4 \int \int \nabla_i^k f_B \nabla_i^l q M_B \nabla_i^k f_B dq \, dx \lesssim \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2}, \\
|\Re_4| &= -2 \int \int \nabla_i^k (f_B^2) \nabla_i^l q \sqrt{M_B} \nabla_i^k f_B dq \, dx \lesssim \left\| \nabla_i^k (f_B^2) \right\|_{L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2}.
\end{align*}

We then get

\begin{equation}
\langle \nabla_i^k r_A, \nabla_i^k f_A \rangle + \langle \nabla_i^k r_B, \nabla_i^k f_B \rangle
\end{equation}

\begin{align*}
&\lesssim - \left\| \nabla_i^k f_A - 2 \nabla_i^k f_B \sqrt{M_B} \right\|_{L_{2,q}^2}^2 + \left\| \nabla_i^k (f_B^2) \right\|_{L_{2,q}^2} \left\| \nabla_i^k f_A - 2 \nabla_i^k f_B \sqrt{M_B} \right\|_{L_{2,q}^2} \\
&\quad + \sum_{l_2 \leq l-1} \left( \left\| \nabla_i^l f_B \right\|_{L_{2,q}^2} \left\| \nabla_i^k f_A \right\|_{L_{2,q}^2} + \left\| \nabla_i^l f_A \right\|_{L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2} \\
&\quad \quad + \left\| \nabla_i^l f_B \right\|_{L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2} \right) \right) \\
&\lesssim - \left\| \nabla_i^k f_A - 2 \nabla_i^k f_B \sqrt{M_B} \right\|_{L_{2,q}^2}^2 + \| f_B \|_{H^2_2} \| f_B \|_{L_{2,q}^2} \left\| \nabla_i^k f_A - 2 \nabla_i^k f_B \sqrt{M_B} \right\|_{L_{2,q}^2} \\
&\quad + \left( \| f_B \|_{H^2_2} \| \nabla_i^k f_A \|_{L_{2,q}^2} + \| f_B \|_{H^2_2} \| \nabla_i^k f_B \|_{L_{2,q}^2} \right) \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2}.
\end{align*}

Therefore, we have derived the higher-order mixed estimates on microscopic equations, for index satisfying \( k + l = s, \ l \geq 1 \), that

\begin{equation}
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left\| \nabla_i^k (f_A, f_B) \right\|_{L_{2,q}^2}^2 + & \left\| \nabla_i^l \left( \frac{\nabla_i^k f_A}{\sqrt{M_B}} \right) \right\|_{M}^2 + \left\| \nabla_i^l \left( \frac{\nabla_i^k f_B}{\sqrt{M_B}} \right) \right\|_{M}^2 + \left\| \nabla_i^k f_A - 2 \nabla_i^k f_B \sqrt{M_B} \right\|_{L_{2,q}^2}^2 \\
\lesssim & |u|_{H^2_2} \left\| \nabla_i^k f_A \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^k f_A \right\|_{L_{2,q}^2} + |u|_{H^2_2} \left\| \nabla_i^l f_A \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^k f_A \right\|_{L_{2,q}^2} \\
&+ \sum_{0 \leq l_2 \leq l-1} \left( \left\| \nabla_i^l U_A \nabla_i^l q \left\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^l U_A \nabla_i^l f_A \right\|_{L_{2,q}^2} + \left\| \nabla_i^l f_A \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^k f_A \right\|_{L_{2,q}^2} \right) \\
&+ |u|_{H^2_2} \left\| \nabla_i^k f_A \right\|_{L_{2,q}^2} + |u|_{H^2_2} \sum_{0 \leq l_2 \leq l} \left\| q \nabla_i^l f_A \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^l U_A \nabla_i^l f_A \right\|_{L_{2,q}^2} \\
&+ |u|_{H^2_2} \left\| \nabla_i^l f_B \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2} + |u|_{H^2_2} \left\| \nabla_i^l f_B \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2} \\
&+ \sum_{0 \leq l_2 \leq l-1} \left( \left\| \nabla_i^l U_B \nabla_i^l q \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^l U_B \nabla_i^l f_B \right\|_{L_{2,q}^2} + \left\| \nabla_i^l f_B \right\|_{H^2_2 L_{2,q}^2} \left\| \nabla_i^k f_B \right\|_{L_{2,q}^2} \right)
\end{aligned}
\end{equation}
This yields that
\[
\frac{d}{dt} E_{s,\text{mix}} + D_{s,\text{mix}} \leq \eta E_{s,u}^2 (D_{s-1,mix} + D_{s-1,f} + D_{s-1,\rho}) \\
+ \eta \frac{1}{2} E_{s,u}^2 (D_{s,mix}^2 + \eta \frac{1}{2} D_{s-1,f}^2 + \eta \frac{1}{2} D_{s-1,\rho}^2) (D_{s-1,mix}^2 + D_{s-1,f}^2 + D_{s-1,\rho}^2) \\
+ \sum_{1 \leq l_2 < l_1 \leq 2} (\eta \frac{l_1 - l_2}{2} D_{s-2,mix}^2 + \eta \frac{1}{2} D_{s-1,f}^2 + \eta \frac{1}{2} D_{s-1,\rho}^2) (D_{s,mix}^2 + \eta \frac{1}{2} D_{s-1,f}^2 + \eta \frac{1}{2} D_{s-1,\rho}^2) \\
+ \sum_{1 \leq l_2 < l_1 \leq 2} \eta \frac{l_1 - l_2}{2} D_{s-2,mix}^2 + \eta \frac{1}{2} D_{s-1,f}^2 + \eta \frac{1}{2} D_{s-1,\rho}^2) (D_{s,mix}^2 + \eta \frac{1}{2} D_{s-1,f}^2 + \eta \frac{1}{2} D_{s-1,\rho}^2) \\
+ \sum_{k+l = s \atop l \geq 1} \eta \frac{l}{2} D_{s-2,mix}^2 + \eta \frac{l}{2} D_{s-1,f}^2 + \eta \frac{l}{2} D_{s-1,\rho}^2) (D_{s,mix}^2 + \eta \frac{l}{2} D_{s-1,f}^2 + \eta \frac{l}{2} D_{s-1,\rho}^2) \\
+ \sum_{k+l = s \atop l \geq 1} \eta \frac{l}{2} \|f \|_{H^2_{s}H^2_{\rho}} \|f \|_{H^s_{x,q}D_{s,mix,r}^2} \\
+ \sum_{k+l = s \atop l \geq 1} \eta \frac{l}{2} \|f \|_{H^2_{s}H^2_{\rho}} \|f \|_{H^s_{x,q}D_{s-1,mix}^2 + \eta \frac{l}{2} D_{s-1,f}^2 + \eta \frac{l}{2} D_{s-1,\rho}^2}).
\]  

4.5. **Additional dissipation on number densities.** In the above process of energy estimates, the weighted Poincaré inequalities are frequently used to help us control the energy terms by dissipation terms, in which the mean value functions of number densities \( \rho_\alpha(t, x) \) are involved. This causes a new difficulty in closing the estimates because these lower order terms seem to be pure energy terms without some natural dissipation effects. 

As we described before, the mean value terms are due to the non-conservation of each distribution function \( f_\alpha \), or speaking specifically, due to the reaction terms. This suggests us to focus on the reaction contributions by extracting them from the micro-macro structure of kinetic equations. 

Indeed, the micro-macro coupling will vanish by performing integrations over configuration space due to their nice divergence structure of Smoluchowski equation, as a correspondence to the conservation law part in the left-hand side of the non-conservative kinematics (2.1). More precisely, we perform an integration on both two kinetic equations in the perturbative system (1.14) with respect to the weighted measure \( \sqrt{M_\alpha} dq \), then get equations of perturbative number densities \( \rho_\alpha(t, x) = \int f_\alpha \sqrt{M_\alpha} dq = \langle g_\alpha \rangle_M \), involving (almost) pure reaction.
contributions, as follows,
\[
\begin{align*}
\partial_t \rho_A + u \cdot \nabla_x \rho_A &= -\langle r \rangle, \\
\partial_t \rho_B + u \cdot \nabla_x \rho_B &= 2 \langle r \rangle,
\end{align*}
\] (4.65)

where
\[
\langle r(q) \rangle (t, x) = \left( \sqrt{M_A} f_A - 2 \sqrt{M_B} f_B - f_B^2 \right). \quad (4.66)
\]

We point out there holds the conservation relation for the total perturbations,
\[
2 \rho_A + \rho_B = 0. \quad (4.67)
\]
which is actually a direct consequence of (1.9) revealing in a macroscopic formulation.

Now we consider the higher-order derivatives estimates for number density \( |\nabla_x^s \rho_A|_{L^2_x} \) (or equivalently, \( |\nabla^s_x \rho_B|_{L^2_x} \)). Applying higher-order derivative operator \( \nabla_x^s \) on the first equation of (4.65), and taking \( L^2_x \)-inner product with \( \nabla_x^s \rho_A \), we can get
\[
\frac{1}{2} \frac{d}{dt} |\nabla_x^s \rho_A|_{L^2_x}^2 = -\langle \nabla_x^s (u \cdot \nabla_x \rho_A), \nabla_x^s \rho_A \rangle - \langle \nabla_x^s r, \nabla_x^s \rho_A \rangle. \quad (4.68)
\]

Recall the definition of projection on the kernel of linear Fokker-Planck operator \( \mathcal{P}_0 g = \rho^q \), the linear part in above reaction contribution (4.66) can be recast as,
\[
\langle \sqrt{M_A} (f_A - 2 \sqrt{M_B} f_B) \rangle = \rho_A - 2 \langle (\mathcal{P}_0 g_B + \mathcal{P}_0^+ g_B) M_B^2 \rangle = (\rho_A - 2 \langle M_A \rangle \rho_B) + \langle (\mathcal{P}_0^+ g_B) M_B^2 \rangle = (1 + 4 \langle M_A \rangle) \rho_A + \langle (\mathcal{P}_0^+ g_B) M_B^2 \rangle.
\]

due to the above relation \( 2 \rho_A + \rho_B = 0 \). In fact, the expression will lead to an additional dissipation term, \( (1 + 4 \langle M_A \rangle) |\nabla_x^s \rho_A|_{L^2_x}^2 \), in energy estimate for number density (4.68).

Notice that we have established the partial coercivity estimates on the kernel orthogonal part of Fokker-Planck operator in (3.7), or in other words, the Poincaré inequality without a mean value function, which enables us to get
\[
\int \int (\mathcal{P}_0^+ g_B) M_B^2 \cdot \rho_A \, dq \, dx \leq \langle M_A^3 \rangle^{\frac{1}{2}} \left( \int \int |\mathcal{P}_0^+ g_B|^2 M_B \, dq \, dx \right)^{\frac{1}{2}} |\rho_A|_{L^2_x} \quad (4.70)
\]
\[
\leq \lambda_0^{-1} \langle M_A^3 \rangle^{\frac{1}{2}} \langle -A g_B, g_B \rangle^\frac{1}{2} \cdot |\rho_A|_{L^2_x} 
\]
\[
\lesssim (\mathcal{L} f_B, f_B)^{\frac{1}{2}} \cdot |\rho_A|_{L^2_x}, \quad (4.71)
\]

and similarly, for higher-order derivatives case,
\[
\int \int (\mathcal{P}_0^+ \nabla_x^s g_B) M_B^2 \cdot \nabla_x^s \rho_A \, dq \, dx \lesssim (\mathcal{L} \nabla_x^s f_B, \nabla_x^s f_B)^{\frac{1}{2}} \cdot |\nabla_x^s \rho_A|_{L^2_x}. \quad (4.71)
\]

Since we have
\[
\langle \nabla_x^s (u \cdot \nabla_x \rho_A), \nabla_x^s \rho_A \rangle = \langle [\nabla_x^s, u \cdot \nabla_x] \rho_A \rangle, \quad (4.72)
\]
\[
\lesssim |u|_{H^2_x} |\rho_A|_{H^2_x} |\nabla_x^s \rho_A|_{L^2_x},
\]
we can derive finally the higher-order estimates for number density, that
\[
\frac{1}{2} \frac{d}{dt} |\nabla_x^s \rho_A|_{L^2_x}^2 + (4 \langle M_A \rangle + 1) |\nabla_x^s \rho_A|_{L^2_x}^2 
\lesssim |u|_{H^2_x} |\rho_A|_{H^2_x} |\nabla_x^s \rho_A|_{L^2_x} + |\nabla_x^s \rho_A|_{L^2_x} \| \nabla_x^s f_B \|_{M} + |\nabla_x^s \rho_A|_{L^2_x} \| f_B \|_{H^2_x} \| f_B \|_{H^2_x L^2}. \quad (4.73)
\]
This can also be written as
\[ \frac{d}{dt} E_{s,\rho} + D_{s,\rho} \lesssim \frac{1}{2} E_{s,\rho}^2 + D_{s,\rho}^{\frac{1}{2}} D_{s,f}^{\frac{1}{2}} + \|fB\|_{L^2_\rho} \|fB\|_{L^2_\rho} \|fB\|_{L^2_\rho} \] (4.74)

4.6. Closing the a-priori estimate. Before combining all the above estimates to obtain the closed a-priori estimate, we are required firstly to control product terms appeared in the form \( \|fB\|_{L^2_\rho} \|fB\|_{X} \) with \( X \) denoting another norm.

**Necessary bound on quadratic terms:**

We consider the following products involving \( \|fB\|_{L^2_\rho} \), which is viewed as a dissipation part. More precisely, for \( k \leq 2 \),
\[ \left\| \nabla_x fB \right\|_{L^2_\rho}^2 = \left\| \nabla_x fB \right\|_{L^2_\rho}^2 + \left\| \nabla_x fB \right\|_{L^2_\rho}^2 + \left\| \nabla_x fB \right\|_{L^2_\rho}^2 \] (4.75)
\[ \lesssim \left\| \nabla_x \left( \nabla_x fB \right) \right\|_{L^2_\rho}^2 + \left\| \nabla_x \left( \nabla_x fB \right) \right\|_{L^2_\rho}^2 + \left\| \nabla_x \rho B \right\|_{L^2_\rho}^2 , \]

as a consequence, it holds,
\[ \eta \left\| fB \right\|_{L^2_\rho}^2 \lesssim D_{3,\text{mix}} + \eta D_{2,f} + \eta D_{2,\rho} . \] (4.76)

This enables us to get
\[ \eta \left\| fB \right\|_{L^2_\rho}^2 \lesssim E_{s,f}^{\frac{1}{2}} (D_{3,\text{mix}}^{\frac{1}{2}} + \eta^2 D_{2,f}^{\frac{1}{2}} + \eta^2 D_{2,\rho}^{\frac{1}{2}}) . \] (4.77)

Similar process implies that
\[ \eta \left\| fB \right\|_{L^2_\rho}^2 \lesssim E_{s,f}^{\frac{1}{2}} (D_{4,\text{mix}}^{\frac{1}{2}} + \eta D_{2,f}^{\frac{1}{2}} + \eta D_{2,\rho}^{\frac{1}{2}}) , \] (4.78)
\[ \eta^{\frac{1}{2}} \left\| fB \right\|_{L^2_\rho}^2 \lesssim E_{4,\text{mix}} \left( D_{s-1,\text{mix}}^{\frac{1}{2}} + \eta^2 D_{s,f}^{\frac{1}{2}} + \eta^2 D_{s,\rho}^{\frac{1}{2}} \right) . \] (4.79)

**Closing the estimates:**

We are now ready to close the desired estimates. Recall the definitions of energy and dissipation functionals introduced in Proposition 4.1
\[ E_s^n = E_s + \eta E_{s,\rho} + \eta E_{s,\text{mix}} + \eta^2 E_{s',j} , \] (4.80)
\[ D_s^n = D_s + \eta D_{s,\rho} + \eta D_{s,\text{mix}} + \eta^2 D_{s',j} , \] (4.81)

then we can infer, respectively, for pure spatial estimates, that
\[ \frac{d}{dt} E_s + D_s \lesssim \eta^{-1} (E_{s,u}^{\frac{1}{2}} + E_{s,f}^{\frac{1}{2}}) D_s^n + \eta^{-\frac{1}{2}} \left( \eta^2 E_{s',j} \right) \frac{1}{2} + \left( \eta E_{s,\text{mix}} \right) \frac{1}{2} D_s^n . \] (4.82)

for pure spatial estimates on first order moment, that
\[ \frac{d}{dt} \left( \eta^2 E_{s',j} \right) + \left( \eta^2 D_{s',j} \right) \lesssim \eta D_s^n + \eta D_{s',u} (D_s^n) \frac{1}{2} + E_{s,u} D_s^n + \eta^{-\frac{1}{2}} \left( \eta^2 E_{s',j} \right) \frac{1}{2} D_s^n . \] (4.83)

for mixed spatial-configuration estimates, that
\[ \frac{d}{dt} \eta E_{s,\text{mix}} + (\eta D_{s,\text{mix}}) \lesssim \eta^{\frac{1}{2}} D_s^n + \sum_{k+l=s, l \geq 1} \eta^{\frac{1}{2}} D_{k,u}^s (D_s^n) \frac{1}{2} \] (4.84)
\[ + E_{s,u} D_s^n + \eta^{-\frac{l+1}{2}} \left( \eta E_{s,\text{mix}} \right) \frac{1}{2} D_s^n , \]

and for additional dissipation estimates on number density, that
\[ \frac{d}{dt} D_s^n + D_s^n \lesssim \eta^{\frac{1}{2}} D_s^n + E_{s,u} D_s^n + \eta^{-\frac{1}{2}} (E_{s,f}) \frac{1}{2} D_s^n . \] (4.85)

Finally, we can derive by combining all the above estimates together, that
\[ \frac{d}{dt} E_s^n + D_s^n \lesssim \eta^{\frac{1}{2}} D_s^n + \eta^{-\frac{1}{2}} (E_s^n) \frac{1}{2} D_s^n + \eta D_{s',u} (D_s^n) \frac{1}{2} + \sum_{k+l=s, l \geq 1} \eta^{\frac{1}{2}} D_{k,u}^s (D_s^n) \frac{1}{2} , \] (4.86)
which yields immediately the desired \textit{a-priori} estimate (4.10) that,
\[ \frac{d}{dt} \mathcal{E}_s^\eta + \mathcal{D}_s^\eta \leq (\eta^\frac{1}{2} + \eta^{-\frac{1}{2}} + \varepsilon^2) \mathcal{E}_s^\eta, \]  
provided \( \mathcal{E}_s^\eta \leq \varepsilon \). This completes the whole proof of Proposition 4.1. \( \square \)

**Remark 4.3.** Note that, we will use in the following texts the formulation (4.86) for the \textit{a-priori} estimate, in which the last two terms in the right-hand side are kept. This is very important to help justify the global-in-time existence by a continuity argument, as we will see below.

5. COMPLETION OF THE PROOF FOR GLOBAL EXISTENCE

To prove Theorem 1.1, a local existence result is required. We state it through the following lemma, the proof of which relies on a standard iterating scheme and compactness method and is thus omitted here. The readers are referred to, e.g. [19, 24, 42].

**Lemma 5.1** (Local existence). Let \( s \geq 5 \) and \( \epsilon_0 > 0 \) be a small positive constant. Assume \( \mathcal{E}(0) \leq \epsilon/2 \) for any constant \( \epsilon \in (0, \epsilon_0) \), then there exists a time \( T_* > 0 \) such that the perturbative system (1.14) of two-species micro-macro model for wormlike micellar solutions admits a unique local classical solution \((u, f_A, f_B) \in L^\infty(0, T_*; H_x^s \times H_{x,q}^2 \times H_{x,q}^2)\), satisfying,
\[ \mathcal{E}(t) \leq \epsilon. \]  
Moreover, the positivity \( \Psi_\alpha = M_\alpha + \sqrt{M_\alpha} f_\alpha > 0 \) is preserved on \([0, T_*]\) if it is valid for initial data \( \Psi_{\alpha,0} = M_\alpha + \sqrt{M_\alpha} f_{\alpha,0} > 0 \).

We now in a position of proving our main result Theorem 1.1. By the equivalence of energy functionals \( \mathcal{E}_s \sim \mathcal{E}_s^\eta \), it follows from (1.20) that, \( \mathcal{E}_s^\eta(0) \leq 2 \mathcal{E}_s(0) \leq 2 \epsilon \). Lemma 5.1 then ensures that, there exists a local solution with the maximal existing time \( T_* > 0 \), moreover, for \( t \in [0, T_*] \),
\[ \mathcal{E}_s^\eta(t) \leq 4 \epsilon. \]  
We claim that \( T_* = +\infty \). For this, we prove by contradiction.

Recall the \textit{a-priori} estimate (4.86) obtained in Proposition 4.1, then there exists some positive constant \( C > 0 \), such that the following inequality holds
\[ \frac{d}{dt} \mathcal{E}_s^\eta + \mathcal{D}_s^\eta \leq C(\eta^{\frac{1}{2}} + \eta^{-\frac{1}{2}} + \varepsilon^2) \mathcal{D}_s^\eta + C \eta |\nabla_x u|_{H_x^{s'}} (\mathcal{D}_s^\eta)^{\frac{1}{2}} + C \sum_{k+l=s} \eta^{\frac{1}{2}} |\nabla_x u|_{H_x^k} (\mathcal{D}_s^\eta)^{\frac{1}{2}}, \]  
for any \( t \in [0, T_*] \).

As for the last two terms in above expression, the Hölder inequality yields that
\[ \eta |\nabla_x u|_{H_x^{s'}} (\mathcal{D}_s^\eta)^{\frac{1}{2}} + \sum_{k+l=s} \eta^{\frac{1}{2}} |\nabla_x u|_{H_x^k} (\mathcal{D}_s^\eta)^{\frac{1}{2}} \leq C \eta^{\frac{1}{2}} \mathcal{D}_s^\eta + C \eta^{\frac{1}{2}} |\nabla_x u|_{L_x^2}^2, \]  
where we have used the interpolation inequality of Sobolev spaces, for \( k \leq s' = s - 1 \) and some small parameter \( \delta > 0 \), that
\[ |\nabla_x u|_{H_x^k}^2 \leq \delta |\nabla_x u|_{H_x^{s'}}^2 + C_\delta |\nabla_x u|_{L_x^2}^2, \]  
and the simple fact \( |\nabla_x u|_{L_x^2}^2 \leq \mathcal{D}_s^\eta \).

Therefore, by virtue of the small assumptions on \( \varepsilon \) and \( \eta \), it follows from (5.3) that,
\[ \frac{d}{dt} \mathcal{E}_s^\eta + \frac{1}{2} \mathcal{D}_s^\eta \leq C \eta^{\frac{1}{2}} |\nabla_x u|_{L_x^2}^2, \]  
which implies immediately, for \( t \in [0, T_*] \),
\[ \mathcal{E}_s^\eta(t) + \int_0^t \frac{1}{2} \mathcal{D}_s^\eta(\tau) \, d\tau \leq \mathcal{E}_s^\eta(0) + C \eta^{\frac{1}{2}} \int_0^t |\nabla_x u|_{L_x^2}^2 \, d\tau. \]  

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We now turn back to the basic energy-dissipation law (1.3) by employing the notion of relative entropy. We introduce the (relative) energy density functions, for $\alpha = A, B$,

$$ E_\alpha(\Psi_\alpha|M_\alpha) = \Psi_\alpha \ln \frac{\Psi_\alpha}{M_\alpha} - \Psi_\alpha + M_\alpha. \quad (5.7) $$

It is easy to check its non-negativity that $E_\alpha(\Psi_\alpha|M_\alpha) = M_\alpha h(\frac{\Psi_\alpha}{M_\alpha}) \geq 0$, by considering a convex function $h(z) = (1 + z) \ln(1 + z) - z$ which is defined on $(-1, +\infty)$.

As a consequence, replacing in (1.3) the original energy density $\tilde{E}_\alpha = \Psi_\alpha (\ln \Psi_\alpha + U_\alpha - 1)$ by the relative energy density $E_\alpha(\Psi_\alpha|M_\alpha)$, we get the following refined energy-dissipation law for the two-species micro-macro model (1.1):

$$ \frac{d}{dt} \left\{ \int_\Omega \frac{1}{2} |u|^2 \, dx + \lambda \sum_{\alpha = A, B} \int_{\Omega \times \mathbb{R}^3} E_\alpha(\Psi_\alpha|M_\alpha) \, dq \, dx \right\} = - \int_\Omega \mu |\nabla_x u|^2 \, dx - \lambda \sum_{\alpha = A, B} \int_{\Omega \times \mathbb{R}^3} \Psi_\alpha |\nabla_q (\ln \Psi_\alpha + U_\alpha)|^2 \, dq \, dx $$

$$ - \lambda \int_{\Omega \times \mathbb{R}^3} (k_1 \Psi_A - k_2 \Psi_B) (\ln \frac{\Psi_A}{\Psi_B} + U_A - 2U_B) \, dq \, dx, $$

which immediately yields

$$ \frac{d}{dt} \left\{ \int_\Omega \frac{1}{2} |u|^2 \, dx + \lambda \sum_{\alpha = A, B} \int_{\Omega \times \mathbb{R}^3} E_\alpha(\Psi_\alpha|M_\alpha) \, dq \, dx \right\} \leq 0. \quad (5.9) $$

By integrating over $[0, t]$, this inequality implies

$$ \frac{1}{2} |u|^2_{L^2}(t) + \int_0^t \mu |\nabla_x u|^2_{L^2} (\tau) \, d\tau $$

$$ \leq \frac{1}{2} |u_0|^2_{L^2} + \lambda \int_{\Omega \times \mathbb{R}^3} [E_A(\Psi_A,0|M_A) + E_B(\Psi_B,0|M_B)] \, dq \, dx \leq \varepsilon, \quad (5.10) $$

due to the small assumption on initial data (1.21). Then (5.6) reduces to

$$ E^\alpha_s(t) + \int_0^t \frac{1}{2} D^\alpha_s(\tau) \, d\tau \leq 2\varepsilon + C\eta^{\frac{1}{2}} \varepsilon \leq 3\varepsilon, $$

which is valid for all $t \in [0, T_*)$ and especially for $t = T_*$. This contradicts (5.2) unless it holds that $T_* = +\infty$.

Recalling the equivalence between energy functionals with and without the fixed parameter $\eta$, the above energy inequality in turn yields the global energy bound (1.22). This completes the whole proof of Theorem 1.1.

\[ \square \]

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