ON ASSOCIATED VARIETY FOR LIE SUPERALGEBRAS

MICHEL DUFLO AND VERA SERGANOVA

Abstract. We define the associated variety $X_M$ of a module $M$ over a finite-dimensional superalgebra $g$, and show how to extract information about $M$ from these geometric data. $X_M$ is a subvariety of the cone $X$ of self-commuting odd elements.

For finite-dimensional $M$, $X_M$ is invariant under the action of the underlying Lie group $G_0$. For simple superalgebra with invariant symmetric form, $X$ has finitely many $G_0$-orbits; we associate a number (rank) to each such orbit. One can also associate a number (degree of atypicality) to an irreducible finite-dimensional representation.

We prove that if $M$ is an irreducible $g$-module of degree of atypicality $k$, then $X_M$ lies in the closure of all orbits on $X$ of rank $k$. If $g = gl(m|n)$ we prove that $X_M$ coincides with this closure.

Contents

1. Introduction 1
2. Definition and basic properties 2
3. Localization and projective modules 3
4. The structure of $X$ for contragredient simple Lie superalgebras 5
5. Central character and the main theorems 8
6. The structure of a generic fiber and the proof of Theorem 5.3 10
7. Application to supercharacters 13
8. Translation functor 14
9. Reduction to the stable case 16
10. Proof of Theorem 5.4 for $gl(m|n)$ 17
11. Application to $H(g(-1); M)$ for $gl(m|n)$ 20

References 21

1. Introduction

In this paper we introduce a notion of associated variety for a module over a Lie superalgebra. This is a superanalogue of an associated variety of Harish-Chandra
modules. Associated varieties have many interesting applications in classical representation theory (see, for example, [7, 11, 13]).

The associated variety for a Lie superalgebra is a subvariety of a cone $X \subset \mathfrak{g}_1$ of self-commuting odd elements. This cone $X$ was studied by Caroline Gruson, see [14, 5, 15]. She used geometric properties of $X$ to obtain important results about cohomology of Lie superalgebras.

While the associated variety in classical representation theory is trivial if a module is finite-dimensional, finite-dimensional modules over classical Lie superalgebras have interesting associated varieties. Since finite-dimensional representation theory of superalgebras still has many open problems, we hope that the associated variety will have some application in this theory. In particular, it should help to describe analytic properties of supercharacters and cohomology groups. Some simple applications are given in Sections 3 and 7.

Let us outline the results of this paper. In Section 2 we give a definition and formulate simple properties of associated variety. In Section 3 we construct a coherent sheaf on $X$ associated with $M$ and prove a criterion of projectivity for modules over certain Lie superalgebras. In Section 4 we discuss geometry of $X$. Section 5 contains main theorems (Theorem 5.3 and Theorem 5.4) about the associated varieties for simple classical contragredient superalgebras. In Section 6 we prove Theorem 5.3. In Section 7 we give some applications of Theorem 5.3 to supercharacters. Sections 8, 9 and 10 contain a proof of Theorem 5.4.

2. Definition and basic properties

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional complex Lie superalgebra, $G_0$ denote a simply-connected connected Lie group with Lie algebra $\mathfrak{g}_0$. Let

$$ X = \{ x \in \mathfrak{g}_1 \mid [x, x] = 0 \}. $$

It is clear that $X$ is $G_0$-invariant Zariski closed cone in $\mathfrak{g}_1$. Let $M$ be a $\mathfrak{g}$-module. For each $x \in X$ put $M_x = \text{Ker} \ x/\mathfrak{g}_0 M$ and define

$$ X_M = \{ x \in X \mid M_x \neq 0 \}. $$

We call $X_M$ the associated variety of $M$.

**Lemma 2.1.** If $M$ is a finite-dimensional $\mathfrak{g}$-module, then $X_M$ is Zarisky closed $G_0$-invariant subvariety.

**Proof.** Since $M$ is finite-dimensional, $M$ is a $G_0$-module. For each $g \in G_0$ and $x \in M$ one has

$$ M_{\text{Ad}_g(x)} = gM_x, $$

that implies Lemma. □

**Lemma 2.2.**

1. If $M = U(\mathfrak{g}) \otimes U(\mathfrak{g}_0) M_0$ for some $\mathfrak{g}_0$-module $M_0$, then $X_M = \{ 0 \}$;
2. If $M = \mathbb{C}$ is trivial, then $X_M = X$;
(3) For any \( g \)-modules \( M \) and \( N \), one has \( X_{M \otimes N} = X_M \cup X_N \);

(4) For any \( g \)-modules \( M \) and \( N \), one has \( X_{M \otimes N} = X_M \cap X_N \);

(5) For any finite-dimensional \( g \)-module \( M \), \( X_M^* = X_M \);

(6) For any finite-dimensional \( g \)-module \( M \) and any \( x \in X \), \( \text{sdim} M = \text{sdim} M_x \).

**Proof.** Properties 2,3,5 follow directly from definition. To prove 1, let \( x \in X \) and \( x \neq 0 \). Let \( \{v_j\}_{j \in J} \) be a basis of \( M_0 \) and \( x_1, \ldots, x_m \) be a basis of \( g_1 \) such that \( x = x_1 \).

Then by PBW for Lie superalgebras \( x_{i_1} x_{i_2} \ldots x_{i_k} \otimes v_j \) for all \( 1 \leq i_1 < i_2 < \cdots < i_k \leq m \), \( j \in J \) form a basis of \( M \). The action of \( x = x_1 \) in this basis is easy to write and it is clear that \( \text{Ker } x = xM \) is spanned by the vectors \( x_1 x_{i_2} \ldots x_{i_k} \otimes v_j \).

Now let us show (4). We will prove that \( M_x = 0 \) implies \((M \otimes N)_x = 0\). Indeed, \( M_x = 0 \) implies that \( M \) is a free \( \mathbb{C}[x] \)-module. Tensor product of a free \( \mathbb{C}[x] \)-module with any \( \mathbb{C}[x] \)-module is free. Therefore \( M \otimes N \) is free over \( \mathbb{C}[x] \) and \((M \otimes N)_x = 0\).

Finally we will prove (6). Let \( \Pi(N) \) stand for the superspace isomorphic to \( N \) with switched parity. Since \( M/\text{Ker } x \) is isomorphic to \( \Pi(xM) \), then

\[
\text{sdim } M = \text{sdim Ker } x + \text{sdim } \Pi(xM) = \text{sdim Ker } x - \text{sdim } xM = \text{sdim } (\text{Ker } x/xM) = \text{sdim } M_x.
\]

\( \square \)

### 3. Localization and Projective Modules

Let \( O_X \) denote the structure sheaf of \( X \). Then \( O_X \otimes M \) is the sheaf of sections of a trivial vector bundle with fiber isomorphic to \( M \). Let \( \partial : O_X \otimes M \rightarrow O_X \otimes M \) be the map defined by

\[
\partial \varphi(x) = x \varphi(x)
\]

for any \( x \in X \), \( \varphi \in O_X \otimes M \). Clearly \( \partial^2 = 0 \) and the cohomology \( \mathcal{M} \) of \( \partial \) is a quasi-coherent sheaf on \( X \). If \( M \) is finite-dimensional, then \( \mathcal{M} \) is coherent.

For any \( x \in X \) denote by \( O_x \) the local ring of \( x \), by \( I_x \) the maximal ideal. Then the fiber \( \mathcal{M}_x \) is the the cohomology of \( \partial : O_x \otimes M \rightarrow O_x \otimes M \). The evaluation map \( j_x : O_x \otimes M \rightarrow M \) satisfies \( j_x \circ \partial = x \circ j_x \). Hence we have the maps

\[
j_x : \text{Ker } \partial \rightarrow \text{Ker } x, \quad j_x : \text{Im } \partial \rightarrow xM.
\]

One can easily check that the latter map is surjective. Therefore \( j_x \) induces the map \( \tilde{j}_x : \mathcal{M}_x \rightarrow M_x \), and \( \text{Im } \tilde{j}_x \cong \mathcal{M}_x/I_x \mathcal{M}_x \).

**Lemma 3.1.** Let \( M \) be a finite-dimensional \( g \)-module. The support of \( \mathcal{M} \) is contained in \( X_M \). The map \( \tilde{j}_x \) is surjective for a generic point \( x \in X \). In particular, if \( X_M = X \), then \( \text{supp } \mathcal{M} = X \).

**Proof.** First, we will show that for any \( x \in X \setminus X_M \) there exists a neighborhood \( U \) of \( x \) such that \( \mathcal{M}(U) = 0 \). Indeed, there exists a map \( i_x : M \rightarrow M \) such that \( x \circ i_x = \text{id} \).

Therefore in some neighborhood \( U \) of \( x \) there exists a map \( i : O(U) \otimes M \rightarrow O(U) \otimes M \) such that \( \partial \circ i = \text{id} \) and \( i(x) = i_x \), hence \( \mathcal{M}(U) = 0 \). Thus, \( x \) does not belong to the support of \( \mathcal{M} \) and we have obtained that \( \text{supp } \mathcal{M} \subset X_M \).
To prove the second statement let $x \in X$ be such that $\dim xM$ is maximal possible. Let $m \in \ker x$. Then there exists some neighborhood $U$ of $x$ and $\varphi \in \mathcal{O}(U) \otimes M$ such that $\partial \varphi = 0$ and $\varphi(x) = m$. By definition $\varphi \in \mathcal{M}_x$ and $\hat{j}_x(\varphi) = m$. □

**Corollary 3.2.** Let $x \in X$ be a generic point, then in some neighborhood $U$ of $x$, the sheaf $\mathcal{M}_U$ coincides with the sheaf of section of a vector bundle with fiber $M_x$.

Let $X_M \neq X$. Then $\mathcal{M}$ is the extension by zero of the sheaf $\mathcal{M}_{X_M}$. If we denote by $\mathcal{M}(x)$ the image of $\hat{j}_x$, then $\mathcal{M}_{X_M}$ locally is the sheaf of sections of the vector bundle with fiber $\mathcal{M}(x)$ for a generic $x \in X_M$. Note that $\mathcal{M}(x) \subset M_x$, but usually $\mathcal{M}(x) \neq M_x$, as one can see from the following example.

**Example 3.3.** Let $\mathfrak{g} = \mathfrak{sl}(1|n)$. Then $\mathfrak{g}_1 = \mathfrak{g}(-1) \oplus \mathfrak{g}(1)$, where $\mathfrak{g}(-1)$ and $\mathfrak{g}(1)$ are abelian superalgebras. Assume that $\mathfrak{g}$ is a typical irreducible $\mathfrak{g}$-module. Then $X_M = \{0\}$, $M_0 = M$ and $\mathcal{M}(0) = M^{(1)} \oplus M^{(-1)}$.

Let $\mathcal{F}$ be the category of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$. The latter condition is automatic if $\mathfrak{g}_0$ is semisimple.

**Theorem 3.4.** Assume that $\mathfrak{g}_0$ is a reductive Lie algebra and elements of $X$ span $\mathfrak{g}_1$. Then $M \in \mathcal{F}$ is projective iff $X_M = \{0\}$.

**Proof.** Let $M$ be projective. Since $M$ is a quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M$, then $M$ is a direct summand of $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M$. By Lemma 2.2 (1) and (3) $X_M = \{0\}$.

To prove the assertion in opposite direction we need the following lemma. Let $H_{\text{red}}^1(\mathfrak{g}, M)$ denote the cohomology of $\mathfrak{g}$, induced by cocycles trivial on the center of $\mathfrak{g}_0$.

**Lemma 3.5.** Let $\mathfrak{g}$ satisfy the condition of Theorem, $M \in \mathcal{F}$ and $X_M = \{0\}$. Then $H_{\text{red}}^1(\mathfrak{g}, M) = \{0\}$.

**Proof.** Let $\varphi \in \mathfrak{g}^* \otimes M$ be a 1-cocycle. Then for any $x \in X$ we have $x \varphi(x) = 0$. Thus, $\varphi$ induces a global section of $\mathcal{M}$. Since $\varphi(0) = 0$ and the supp $\mathcal{M} = \{0\}$ by Lemma 3.1, this global section must be zero. Therefore there exists $\psi(x)$ such that $x \psi(x) = \varphi(x)$ for all $x \in X$. But $\varphi$ is a linear function, therefore $\psi$ is constant. If $d$ is the differential in the cohomology complex, $\eta = \varphi - d \psi$ is a 1-cocycle homologically equivalent to $\varphi$. On the other hand, $\eta(x) = 0$ for any $x \in X$, and since elements of $X$ span $\mathfrak{g}_1$, we have $\eta(\mathfrak{g}_1) = 0$. The restriction $\eta$ on $\mathfrak{g}_0$ is a 1-cocycle for a Lie algebra $\mathfrak{g}_0$. But $\mathfrak{g}_0$ is reductive, hence $H_{\text{red}}^1(\mathfrak{g}_0, M) = 0$ and therefore $\eta = d \nu$. We have shown that $\varphi$ induces the trivial cohomology class. Thus, $H_{\text{red}}^1(\mathfrak{g}, M) = \{0\}$. □

Now, assume that $X_M = \{0\}$. We have to show that $\text{Ext}^1(M, N) = \{0\}$, the latter is equivalent to $H_{\text{red}}^1(\mathfrak{g}, M^* \otimes N) = \{0\}$. By Lemma 2.2 (4), (5) we have $X_{M^* \otimes N} = \{0\}$. Therefore $H_{\text{red}}^1(\mathfrak{g}, M^* \otimes N) = \{0\}$ and $M$ is projective. □

**Remark 3.6.** Note that the conditions of Theorem 3.4 hold for any simple classical superalgebra except $\mathfrak{osp}(1|2n)$. In case $\mathfrak{g} = \mathfrak{osp}(1|2n)$, $X = \{0\}$ and $\mathcal{F}$ is semisimple, hence every finite-dimensional module is projective. In general, however,
Theorem 3.4 is not true if we drop the assumption that $X$ spans $\mathfrak{g}_1$. Indeed, let $\mathfrak{g} = \mathfrak{q}(1)$, in other words $\mathfrak{g}$ has a basis of an even element $C$ and an odd element $T$ such that $[T, T] = C$. Then $X = \{0\}$, but not every module in $\mathcal{F}$ is projective. For example, the trivial one-dimensional $\mathfrak{g}$-module is not projective.

4. The structure of $X$ for contragredient simple Lie superalgebras

Let $\mathfrak{g}$ be a contragredient finite-dimensional Lie superalgebra with indecomposable Cartan matrix, i.e. $\mathfrak{g}$ is isomorphic to one from the following list: $\mathfrak{sl}(m|n)$ if $m \neq n$, $\mathfrak{gl}(n|n)$, $\mathfrak{osp}(m|2n)$, $D(\alpha)$, $F_4$ or $G_3$ (for definitions see [1]).

Remark 4.1. The Lie superalgebras we consider are simple except one case. For a simple Lie superalgebra $\mathfrak{psl}(n|n)$ the Cartan matrix is degenerate and we consider the corresponding Kac-Moody Lie superalgebra which is isomorphic to $\mathfrak{gl}(n|n)$. Later we will do the proofs for $\mathfrak{gl}(m|n)$ even if $m \neq n$, in this case $\mathfrak{gl}(m|n) \cong \mathfrak{sl}(m|n) \oplus \mathbb{C}$.

We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In this case the Cartan subalgebra of $\mathfrak{g}$ coincides with a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{g}$ has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

each root space $\mathfrak{g}_\alpha$ is one dimensional. The parity of $\alpha \in \Delta$ by definition is equal to the parity of the root space $\mathfrak{g}_\alpha$. The invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ is not positive definite and some of odd roots are isotropic. For a non-isotropic $\beta$ we denote by $\beta^\vee$ the element of $\mathfrak{h}$ such that $\alpha (\beta^\vee) = \frac{2(\alpha, \beta)}{\langle \beta, \beta \rangle}$. Let $S$ denote the set of subsets of mutually orthogonal linearly independent isotropic roots of $\Delta_+$, i.e. an element of $S$ is $A = \{\alpha_1, \ldots, \alpha_k \mid \langle \alpha_i, \alpha_j \rangle = 0\}$. The Weyl group $W$ of $\mathfrak{g}_0$ acts on $S$ in the obvious way. Put $S_k = \{A \in S \mid |A| = k\}$, here $S_0 = \{\emptyset\}$. 

Theorem 4.2. There are finitely many $G_0$-orbits on $X$. These orbits are in one-to-one correspondence with $W$-orbits in $S$.

Proof. We define the map $\Phi : S \rightarrow X/G_0$ in the following way. Let $A = \{\alpha_1, \ldots, \alpha_k\} \in S$, choose a non-zero $x_i \in \mathfrak{g}_{\alpha_i}$ and put $x = x_1 + \cdots + x_k \in X$. By definition $\Phi(A) = G_0 x$. To see that $\Phi(A)$ does not depend on a choice of $x_i$ note that since $\alpha_1, \ldots, \alpha_k$ are linearly independent, for any other choice

$$x' = \Sigma x'_i = \Sigma_{c_i} x_i$$

there is $h \in \mathfrak{h}$ such that $c_i = e^{\alpha_i(h)}$ and therefore

$$x' = \exp (\text{ad} (h)) (x).$$

If $B = w (A)$ for some $w \in W$, then clearly $\Phi (B)$ and $\Phi (A)$ belong to the same orbit. Therefore $\Phi$ induces the map $\Phi : S/W \rightarrow X/G_0$. We check case by case that $\Phi$ is injective and surjective.

If $\mathfrak{g}$ is $\mathfrak{sl}(m|n)$ or $\mathfrak{gl}(n|n)$, $\mathfrak{g}$ has a natural $\mathbb{Z}$ grading $\mathfrak{g} = \mathfrak{g} (-1) \oplus \mathfrak{g} (0) \oplus \mathfrak{g} (1)$ such that $\mathfrak{g}_0 = \mathfrak{g} (0)$, $\mathfrak{g}_1 = \mathfrak{g} (1) \oplus \mathfrak{g} (-1)$. The orbits of $W$ on $S$ are enumerated by the
pairs of numbers \((p,q)\), where \(p = |A \cap \Delta (g(1))|\), \(q = |A \cap \Delta (g(-1))|\). The orbits of \(G_0\) on \(X\) are enumerated by the same pairs of numbers \((p,q)\) in the following way. If \(x = x^+ + x^-\), where \(x^\pm \in g(\pm 1)\), then \(p = \text{rank}(x^+)\), \(q = \text{rank}(x^-)\). We can see by the construction of \(\Phi\) that \(\Phi\) maps \((p,q)\)-orbit on \(S\) to the \((p,q)\)-orbit on \(X\).

Let \(g = \mathfrak{osp}(m|2n)\). If \(m = 2l + 1\) or \(m = 2l\) with \(l > n\), then the \(W\)-orbits on \(S\) are in one-to-one correspondence with \(\{0,1,2,\ldots, \min(l,n)\}\). Namely, \(A\) and \(B\) are on the same orbit if they have the same number of elements. As it was shown in [14], \(X\) can be identified with the set of all linear maps \(x : \mathbb{C}^m \rightarrow \mathbb{C}^{2n}\), such that \(\text{Im} \: x\) is an isotropic subspace in \(\mathbb{C}^{2n}\) and \(\text{Im} \: x^*\) is an isotropic subspace in \(\mathbb{C}^m\). Furthermore, \(x, y \in X\) belong to the same \(G_0\)-orbit iff \(\text{rank}(x) = \text{rank}(y)\). One can see that \(\Phi(A) = |A|\).

Now let \(g = \mathfrak{osp}(2l|2n)\) where \(l \leq n\). If \(A, B \in S\) and \(|A| = |B| < l\), then \(A\) and \(B\) are on the same \(W\)-orbit. In the same way if \(\text{rank}(x) = \text{rank}(y) < l\), then \(x\) and \(y\) are on the same \(G_0\)-orbit. However, the set of all \(x \in g_1\) of maximal rank splits in two orbits, since the Grassmannian of maximal isotropic subspaces in \(\mathbb{C}^{2l}\) has two connected components. In the same way \(S_1\) splits in two \(W\)-orbits. Hence in this case again \(\Phi\) is a bijection.

If \(g\) is one of exceptional Lie superalgebras \(D(\alpha), G_3\) or \(F_4\), then the direct calculation shows that \(X\) has two \(G_0\)-orbits: \(\{0\}\) and the orbit of a highest vector in \(g_1\). The set \(S\) also consists of two \(W\)-orbits: \(\emptyset\) and the set of all isotropic roots in \(\Delta\).

**Remark 4.3.** Note that in our situation the representation of \(G_0\) in \(g_1\) is symplectic and multiplicity free (see [8]). The cone \(X\) is the preimage of 0 under the moment map \(g_1 \rightarrow g_0^*\).

We use the notation \(\Phi : S \rightarrow X/G_0\) introduced in the proof of Theorem 4.2. Using the explicit description of \(G_0\)-orbits on \(X\) and the description of roots systems, which can be found in [1], one can check the following statements case by case. We omit this checking here.

**Lemma 4.4.** Let \(A, B \in S\).

1. If \(\alpha \in \Delta\) is a linear combination of roots from \(A\), then \(\alpha \in A \cup -A\);
2. If \(|A| \leq |B|\), then there exists \(w \in W\) such that \(w(A) \subset B \cup -B\);
3. \(\Phi(A)\) lies in the closure of \(\Phi(B)\) iff \(w(A) \subset B\) for some \(w \in W\).

By \(A^\perp\) we denote the set of all weights orthogonal to \(A\) with respect to the standard form on \(\mathfrak{h}^*\).

**Theorem 4.5.** Let \(A \in S\). Then \(\dim \Phi(A) = \frac{|\Delta| \setminus A_\perp|}{2} + |A|\).

*Proof. Let \(A = \{\alpha_1, \ldots, \alpha_k\}, \: x = x_1 + \cdots + x_k\) for some choice of \(x_i \in g_{\alpha_i}, \: y = y_1 + \cdots + y_k\) for some \(y_i \in g_{-\alpha_i}\). Let \(h = [x, y], \: h_i = [x_i, y_i]\). Clearly, \(h = h_1 + \cdots + h_k\) and \(h, x, y\) generate the \(\mathfrak{sl}(1|1)\)-subalgebra in \(g\). With respect to this subalgebra \(g\) has a decomposition \(g = \bigoplus_{\mu} g^\mu\).*
where
\[ g^\mu = \{ g \in g \mid [h, g] = \mu g \} . \]

Note that
\[ \dim [g, x] = \sum_{\mu} \dim [g^\mu, x] , \]
and from the description of irreducible \( \mathfrak{sl}(1|1) \)-modules for \( \mu \neq 0 \)
\[ \dim [g^\mu, x] = \frac{\dim g^\mu}{2} . \]

On the other hand, for \( \mu \neq 0 \) \( \text{sdim} g^\mu = 0 \), and therefore
\[ \dim g^\mu = 2 \dim g^\mu_t . \]

Observe that for a generic choice of \( x_i \in g_{\alpha_i} \), \( g_{\beta} \subset g^0 \) iff \( (\beta, \alpha_i) = 0 \) for all \( i \leq k \).
Indeed, for generic choice of \( x_i \) the condition \( \beta (h) = 0 \) implies \( \beta (h_i) = 0 \) for all \( i \), and therefore \( (\beta, \alpha_i) = 0 \) for all \( i \). Hence
\[ \bigoplus_{\mu \neq 0} g^\mu_t = \bigoplus_{\alpha \in \Delta \cap A^\perp} g_{\alpha} \]
and
\[ \sum_{\mu \neq 0} \dim [g^\mu, x] = \sum_{\mu \neq 0} \dim g^\mu_t = |\Delta \cap A^\perp| . \]

To calculate \( \dim [g^0, x] \) note that
\[ g^0 = h \oplus \bigoplus_{\beta \in \Delta \cap A^\perp} g_{\beta} . \]

We claim that
\[ [g^0, x] = \bigoplus_{i=1}^k C h_i \oplus \bigoplus_{i=1}^k g_{\alpha_i} , \]
hence \( \dim [g^0, x] = 2k \). Indeed, if \( (\beta, \alpha_i) = 0, \beta \neq \pm \alpha_i \) then \( \beta \pm \alpha_i \notin \Delta \). Therefore \( [x, g_{\beta}] = 0 \) for any \( \beta \in \Delta \cap A^\perp, \beta \neq -\alpha_i \). Furthermore, \( [x, g_{-\alpha_i}] = C h_i \) and \( [x, h] = \bigoplus_{i=1}^k g_{\alpha_i} . \) Thus, we obtain
\[ (4.1) \quad \dim [g, x] = |\Delta \cap A^\perp| + 2k . \]

Now the statement will follow from the lemma.

**Lemma 4.6.** \( \text{sdim} [g, x] = 0 \).

**Proof.** Define the odd skew-symmetric form on \( g \) by
\[ \omega (y, z) = (x, [y, z]) . \]

Obviously the kernel of \( \omega \) coincides with centralizer \( C_g (x) \). Thus, \( \omega \) is non-degenerate odd skew-symmetric form on \( C_g (x) \). Hence \( \text{sdim} g/C_g (x) = 0 \). But \( [g, x] \cong \Pi (g/C_g (x)) \), which implies the lemma.

Lemma implies that \( \dim [g^0, x] = 1/2 \dim [g, x] \). Since \( \dim G_0 x = \dim [g_0, x] \), the theorem follows from (4.1).
Corollary 4.7. If $|A| = |B|$, then $\dim \Phi (A) = \dim \Phi (B)$.

Proof. Follows from Theorem 4.2 and Lemma 4.4 (2).

The maximal number of isotropic mutually orthogonal linearly independent roots is called the defect of $g$. This notion was introduced in [16]. One can see that the defect of $g$ is equal to the dimension of maximal isotropic subspace in $h^*$. All exceptional Lie superalgebras has defect 1. The defect of $\mathfrak{sl}(m|n)$ is $\min(m, n)$, the defect $\mathfrak{osp}(2l+1|2n)$ and $\mathfrak{osp}(2l|2n)$ is $\min(l, n)$.

Corollary 4.8. Let $d$ be the defect of $g$. Then the irreducible components of $X$ are in bijection with $W$-orbits on $S_d$. If all odd roots of $g$ are isotropic, then the dimension of each component equals $\dim g_1 = |\Delta_1|/2$.

Proof. As follows from Theorem 4.2 and Lemma 4.4 (3), each irreducible component is the closure of $\Phi (A)$ for a maximal $A \in S_d$. By Lemma 4.4 (2) $|A| = d$. Hence the first statement. Theorem 4.5 immediately implies the statement about dimension.

Corollary 4.9. If all odd roots of $g$ are isotropic, then the codimension of $\Phi (A)$ in $X$ equals $|\Delta_1 \cap A^\perp|/2 - |A|$.

Proof. The codimension equals $\dim X - \dim \Phi (A)$. Using Theorem 4.5 and Corollary 4.8

$$\text{codim } \Phi (A) = |\Delta_1| - |\Delta_1 \setminus A^\perp| - |A| = |\Delta_1 \cap A^\perp|/2 - |A|.$$ 

5. Central character and the main theorems

Let us fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ by choosing a decomposition $\Delta = \Delta^+ \cup \Delta^-$. Note that this choice is not unique but our consideration will not depend on it. Later we will use different Borel subalgebras in some proofs.

Let $Z$ denote the center of the universal enveloping algebra $U(g)$. One can see that any $z \in Z$ acts as a scalar $\chi_z(z)$ on $M_\lambda$ and $L_\lambda$. We say that $\lambda \in h^*$ is integral dominant if $L_\lambda$ is finite-dimensional. We denote by $\Sigma$ the set of all integral dominant weights.

Let $\chi_\lambda : Z \to \mathbb{C}$.

$$\mathfrak{h}_\chi = \{ \mu \in h^* \mid \chi_\mu = \chi \}.$$
Lemma 5.1. Let $\chi = \chi_{\lambda}$, $A \in S$ be a maximal set of linearly independent mutually orthogonal isotropic roots orthogonal to $\lambda + \rho$ and $t_{\lambda} = \lambda + \oplus_{\alpha \in A} C \alpha$. Then
\[ h_{\chi} = \bigcup_{w \in W} t_{\lambda}^w. \]

Proof. Easily follows from the description of the $Z$ formulated in [12] and proven [4] and in [9].

Let us fix a central character $\chi$. For each $\lambda \in h_{\chi}$ define $S_{\lambda} \subset S$ by the following
\[ S_{\lambda} = \left\{ A \in S \mid A \subset (\lambda + \rho)^\perp \right\}. \]
Put
\[ S_{\chi} = \cup_{\lambda \in h_{\chi}} S_{\lambda}. \]

Lemma 5.2. There exists a number $k$ such that $S_{\chi} = \bigcup_{i \leq k} S_i$.

Proof. It follows easily from Lemma 5.1 that $S_{\chi}$ is $W$-invariant. Furthermore, if $A \in S_{\chi}$ and $A'$ is obtained from $A$ by multiplication of some roots in $A$ on $-1$, then $A' \in S_{\chi}$. Hence the statement follows from Lemma 4.4 (1) and (2).

The number $k$ is called the degree of atypicality of $\chi$. The degree of atypicality of $\lambda$ is by definition the degree of atypicality of $\chi_{\lambda}$. If $k = 0$, then $\chi$ is called typical. It is clear that the degree of atypicality of $\chi$ is not bigger than the defect of $g$.

Let $X_k = \Phi (S_k)$, $\bar{X}_k$ denote the closure of $X_k$. Lemma 4.4 (3) implies that
\[ \bar{X}_k = \bigcup_{i=0}^k \Phi (X_i). \]

Theorem 5.3. Let $g$ be a contragredient simple Lie superalgebra, $M$ be a $g$-module which admits central character $\chi$, the degree of atypicality of $\chi$ be equal to $k$. Then $X_M \subset \bar{X}_k$.

Theorem 5.4. Let $g = gl(m|n)$ or $sl(m|n)$. For any integral dominant $\lambda \in h^*$ with degree of atypicality $k$, $X_{L_{\lambda}} = \bar{X}_k$.

Conjecture 5.5. Let $g$ be a contragredient simple Lie superalgebra. For any integral dominant $\lambda \in h^*$ with degree of atypicality $k$, $X_{L_{\lambda}} = \bar{X}_k$.

First, observe that the conjecture is true for the typical character.

Theorem 5.6. If $\lambda$ is typical, then $X_{L_{\lambda}} = \{0\}$.

Proof. If $\lambda$ is typical, then $L_{\lambda}$ is a direct summand of some induced module $U(g) \otimes U(g_0) M_0$ (see [3]). Therefore Theorem follows from Lemma 2.2 (1) and (3).
6. The structure of a generic fiber and the proof of Theorem 5.3

In this section we discuss properties of the fiber $M_x$ over a point $x \in X_M$. Let $C_x = C_m(x)/[x, g]$.  

**Lemma 6.1.** The subspace $[x, g]$ is an ideal in $C_m(x)$. Let $m^\perp$ denote the orthogonal complement to $m$ with respect to the invariant form on $g$. Then $[x, g]^\perp = C_m(x)$.  

**Proof.** Let $u \in C_m(x)$, $v \in [x, g]$. Then $v = [x, z]$ and 
\[
[u, [x, z]] = (-1)^{p(u)} [x, [u, z]] \in [x, g].
\]
The second statement follows from the identity 
\[
(u, [x, z]) = -([u, x], z).
\]

**Lemma 6.2.** $M_x$ is a $C_m(x)$-module trivial over $[x, g]$.  

**Proof.** Let $m \in \ker \mu$, $v = [x, z] \in [x, g]$. Then 
\[
vm = xzm - (-1)^{p(x)} zxm = xzm \in xM,
\]

In the case of contragredient finite-dimensional superalgebra we can describe $g_x$ precisely. Let $A = \{\alpha_1, \ldots, \alpha_k\} \in S$, $x \in X$, and $x = x_1 + \cdots + x_k$, where $x_i \in g_{\alpha_i}$, 
$h_{\alpha} = [g_{\alpha}, g_{-\alpha}]$. Define $A' = A' \cap \Delta \setminus (\Delta \cup -\Delta)$, $h_A = h_{\alpha_1} + \cdots + h_{\alpha_k}$.  

**Lemma 6.3.** If $g$ is finite-dimensional contragredient superalgebra, $A = \{\alpha_1, \ldots, \alpha_k\} \in S$, $x \in X$, and $x = x_1 + \cdots + x_k$, where $x_i \in g_{\alpha_i}$. Then $C_m(x)$ can be decomposed in a semidirect sum $g_x \oplus [x, g]$, where $g_x$ is spanned by the root spaces $g_{\alpha}$ for all $\alpha \in A'$ and $h_x \subset h_A$ is such that $h_x \oplus h_A = h_A$. Furthermore, $\text{def} g_x = \text{def} g - k$.  

**Proof.** We use the same argument as in the proof of Theorem 4.5. Let $h$ and $g^\mu$ be as in this proof. First, there is an isomorphism 
\[
g_x \cong g^0 \cap C_m(x)/g^0 \cap [x, g].
\]
Then we notice that 
\[
g^0 \cap C_m(x) = h_A \oplus \alpha_{\alpha \in A'} \cap \Delta \setminus -\Delta \oplus g_{\alpha} \cdot g^0 \cap [x, g] = h_A \oplus \alpha_{\alpha \in A} \cdot g_{\alpha}.
\]
Choose $h_x$ in such a way that $g_x = (h_x \oplus \oplus_{\alpha \in A} g_{\alpha})$ is a subalgebra, then 
\[
g^0 \cap C_m(x) = g_x \oplus g^0 \cap [x, g].
\]

**Remark 6.4.** If $g = gl(m|n)$, then $g_x \cong gl(m - k|n - k)$. If $g = osp(m|2n)$, then $g_x \cong osp(m - 2k|2n - 2k)$. If $g = D(\alpha)$ and $x \neq 0$, then $g_x \cong C$. For $g = G_3$ or $F_4$ for a non-zero $x \in X$, $g_x$ is isomorphic to $sl(2)$ and $sl(3)$ respectively.
Lemma 6.5. Let $x : V \to V$ be an odd linear operator such that $x^2 = 0$. Assume that $V = W \oplus U$, where $W$ is a trivial $\mathbb{C}[x]$-submodule and $U$ is a free $\mathbb{C}[x]$-module. Let $S(V)^x$ denote the space of $x$-invariants in $S(V)$. Then $S(V)^x = S(W) \otimes S(U)^x$ and $S(U)^x \subset S(U) U^x$.

Let $U(\mathfrak{g})^{ad(x)}$ denote the subalgebra of $ad(x)$-invariants in $U(\mathfrak{g})$. $I_x$ be the left ideal in $U(\mathfrak{g})$ generated by $[x, \mathfrak{g}]$. One has the following sequence

$$U(\mathfrak{g}) \xrightarrow{\iota} U(\mathfrak{g})^{ad(x)} \xrightarrow{\pi} U(\mathfrak{g})^{ad(x)}/I_x \cap U(\mathfrak{g})^{ad(x)}.$$  

Let $\phi = \pi \circ \iota$.

Lemma 6.6. The map $\phi : U(\mathfrak{g}_x) \to U(\mathfrak{g})^{ad(x)}/I_x \cap U(\mathfrak{g})^{ad(x)}$ is an isomorphism of vector spaces.

Proof. Since $I_x \cap U(\mathfrak{g}_x) = \{0\}$, $\phi$ is injective. To prove surjectivity of $\phi$ use PBW and the corresponding sequence for symmetric algebras

$$S(\mathfrak{g}_x) \to S(\mathfrak{g})^{ad(x)} \to S(\mathfrak{g})^{ad(x)}/J_x \cap S(\mathfrak{g})^{ad(x)},$$  

where $J_x = [x, \mathfrak{g}] S(\mathfrak{g})$. Apply Lemma 6.5 with $V = \mathfrak{g}$. Then $W = \mathfrak{g}_x, U^x = [x, \mathfrak{g}],$ and we obtain $S(\mathfrak{g})^{ad(x)} = S(\mathfrak{g}_x) \otimes S(U)^x$ and $S(U)^x \subset [x, \mathfrak{g}] S(U)$. Thus, gr$\phi$ is an isomorphism. Hence $\phi$ is an isomorphism. □

Define the map $\eta : U(\mathfrak{g})^{ad(x)} \to U(\mathfrak{g}_x)$ by putting $\eta = \phi^{-1} \circ \pi$. As follows from Lemma 6.2 for any $u \in U(\mathfrak{g})^{ad(x)}, m \in M_x$

$$\tag{6.1} u m = \eta(u) m$$

Note that $\iota, \pi$ are homomorphisms of $\mathfrak{g}_x$-modules (with respect to the adjoint action). The center $Z$ of $U(\mathfrak{g})$ obviously is a subalgebra in $U(\mathfrak{g})^{ad(x)}$. Let $Z(\mathfrak{g}_x)$ be the center of $U(\mathfrak{g}_x)$. Since $\eta$ is a homomorphism of $\mathfrak{g}_x$-modules, $\eta(Z) \subset Z(\mathfrak{g}_x)$. We are going to describe the dual map

$$\eta^* : \text{Hom}(Z(\mathfrak{g}_x), \mathbb{C}) \to \text{Hom}(Z, \mathbb{C}).$$

Choose a borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\alpha_1, \ldots, \alpha_k$ are simple roots.

Lemma 6.7. Let $\lambda \in \mathfrak{h}^*$ satisfy $(\lambda + \rho, \alpha_1) = \cdots = (\lambda + \rho, \alpha_k) = 0$. Then $(L_\lambda)_x \neq 0$. In particular the highest vector $v$ belongs to $(L_\lambda)_x$.

Proof. Clearly, $v \in \text{Ker} x$. If $v = xw$, then one can choose $w$ with weight $\lambda - \alpha_i$ for some $i$. However, $L_\lambda$ does not have vectors of such weight. □

Corollary 6.8. Let $\lambda$ be as in Lemma 6.7 and $\mu$ be the restriction of $\lambda$ to $\mathfrak{h}_x$. Let $\chi_\mu \in \text{Hom}(Z(\mathfrak{g}_x), \mathbb{C})$ be induced by $\mu$ and $\chi_\lambda \in \text{Hom}(Z, \mathbb{C})$ be induced by $\lambda$ via Harish-Chandra homomorphism. Then $\eta^*(\chi_\mu) = \chi_\lambda$.

Corollary 6.9. Let $\chi \in \text{Hom}(Z(\mathfrak{g}_x), \mathbb{C})$ and have the degree of atypicality $s$. Then the degree of atypicality of $\eta^*(\chi)$ equals $s + k$. 

Corollary 6.9 implies Theorem 5.3. It also implies the following

**Theorem 6.10.** Let $M$ admit a central character with degree of atypicality $k$, and $x \in X_k$. Then $g_x$-module $M_x$ admits a typical central character. In particular, if $M_x$ is finite dimensional, it is semi-simple over $g_x$, and therefore over $C_g(x)$.

**Theorem 6.11.** If $g \neq \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, then $\eta^*$ is injective, and therefore $\eta$ is surjective. If $g = \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, then a preimage of $\eta^*$ has at most two elements.

**Proof.** Let $A = \{\alpha_1, \ldots, \alpha_k\}$, $x = x_1 + \cdots + x_k$, $x_i \in g_{\alpha_i}$, $b$ be such that $\alpha_1, \ldots, \alpha_k$ are simple. Let

$$W' = \{w \in W \mid w(A) \subseteq A \cup -A\},$$

and $W(g_x)$ denote the Weyl group of $g_x$. Clearly, $W(g_x) \subseteq W$. One can show that if $g \neq \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, then $W' = W(g_x) \times W''$, where $W''$ consists of all elements which act trivially on $g_x$.

Let $\lambda \in h^*$, $(\lambda + \rho, \alpha_i) = 0$ for all $i = 1, \ldots, k$. Then

$$h_{\lambda \lambda} \cap (h_{\lambda}^*)^* = \bigcup_{w \in W'} t_{\lambda}^w, h_{\lambda \lambda} \cap h_x = \bigcup_{w \in W'} (t_{\lambda}^w \cap h_x^*).$$

Let $g \neq \mathfrak{osp}(2l|2n)$ or $D(\alpha)$ and $\mu$ be the restriction of $\lambda$ on $h_x$. Then

$$h_{\lambda \lambda} \cap h_x = \bigcup_{w \in W(g_x)} (t_{\lambda} \cap h_x^*)^w = \bigcup_{w \in W(g_x)} t_{\mu}^w = (h_x)_{\lambda \lambda}^\mu,$$

that shows $(\eta^*)^{-1}(\lambda \lambda) = \lambda \mu$.

In case $g = \mathfrak{osp}(2l|2n)$ or $D(\alpha)$, $W(g_x) \times W''$ has index 2 in $W'$. Take $u \in W'$, $u \not\in W(g_x) \times W''$, let $\mu$ be the restriction of $\lambda$ on $h_x$ and $\mu'$ be the restriction of $\lambda^u$ on $h_x$. Then

$$h_{\lambda \lambda} \cap h_x = \bigcup_{w \in W'} (t_{\lambda} \cap h_x^*)^w = \bigcup_{w \in W(g_x)} (t_{\mu}^w \cup t_{\mu'}^w) = (h_x)_{\lambda \lambda}^\mu \cup (h_x)_{\lambda \lambda}^\mu' .$$

Therefore $(\eta^*)^{-1}(\lambda \lambda) = \{\lambda \mu, \lambda \mu'\}$.

Assume that $M$ is finite-dimensional and has central character $\chi$ with degree of atypicality $k$. Let $x \in \tilde{X}_k$. Let

$$Y_x = \{y \in (g_x)_1 \mid [y, y] = 0\} .$$

Then

$$x + Y_x \subseteq X$$

(6.2)

We also use the fact that $\rho_x = \frac{1}{2} \sum_{\alpha \in \Delta_0(g_x)} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1(g_x)} \alpha = \rho_{h_x}$. Hence the shifted action of $W(g_x)$ is the same.
Define the coherent sheaf \( \mathcal{N} \) on \( Y_x \) as the cohomology of
\[
\partial : \mathcal{O}_{Y_x} \otimes M_x \to \mathcal{O}_{Y_x} \otimes M_x.
\]
Let \( \mathcal{N}(x) \) be the image of the fiber \( \mathcal{N}_x \) in \( M_x \) under the evaluation map.

**Theorem 6.12.** \( M(x) = \mathcal{N}(0) \).

**Proof.** Obviously \( M(x) \subset \mathcal{N}(0) \). We have to show that \( M(x) = \mathcal{N}(0) \). Let \( m \in \mathcal{N}(0) \). There exists an open \( V \subset Y_x \), \( 0 \in V \), \( \varphi \in \mathcal{O}(V) \otimes M_x \) such that \( \partial \varphi = 0 \) and \( \varphi(0) = m \). We have to extend \( \varphi \) to some open set \( U \subset X \). Let \( g = C_0(x) \oplus I \) as \( g_x \)-module. Define the map
\[
\tau : l_0 \times Y_x \to X
\]
by the formula
\[
\tau(l, y) = \exp \text{ad}(l)(x + y),
\]
for any \( y \in Y_x \), \( l \in l_0 \). Then \( \tau \) is a local isomorphism. Hence in some neighborhood \( U \subset X \), \( x \in U \), \( x = \tau(l, y) \) and one can define
\[
\psi(\tau(l, y)) = \exp l \varphi(y).
\]
Then \( \partial \psi = 0 \) and \( \psi(x) = m \). Theorem is proven. \( \square \)

7. **Application to supercharacters**

The properties of \( M_x \) allow one to say something about the superdimension and supercharacter of \( M \). First, we recall that \( \text{sdim} M_x = \text{sdim} M \). Therefore

**Lemma 7.1.** If \( X_M \neq X \), then \( \text{sdim} M = 0 \). In particular, if a finite-dimensional module \( M \) admits a central character whose degree of atypicality is less than the defect of \( g \), then \( \text{sdim} M = 0 \).

Now let \( M \) be a finite-dimensional \( g \)-module and \( h \in \mathfrak{h} \). Write
\[
\text{ch}_M(h) = \text{str}_M(e^h).
\]
Obviously, \( \text{ch}_M \) is \( W \)-invariant analytic function on \( \mathfrak{h} \). We can write Taylor series for \( \text{ch}_M \) at \( h = 0 \)
\[
\text{ch}_M(h) = \sum_{i=0}^{\infty} p_i(h),
\]
where \( p_i(h) \) is a homogeneous polynomial of degree \( i \) on \( \mathfrak{h} \). The order of \( \text{ch}_M \) at zero is by definition the minimal \( i \) such that \( p_i \neq 0 \).

**Theorem 7.2.** Assume that all odd roots of \( g \) are isotropic. Let \( M \) be a finite-dimensional \( g \)-module, \( s \) be the codimension of \( X_M \) in \( X \). The order of \( \text{ch}_M \) at zero is greater or equal than \( s \). Moreover, the polynomial \( p_s(h) \) in Taylor series for \( \text{ch}_M \) is determined uniquely up to proportionality.

**Proof.** The proof is based on the following Lemma, the proof of this Lemma is similar to the proof of Lemma 2.2 (6). We leave it to the reader.
Lemma 7.3. Let $x \in X$, $h \in \mathfrak{g}_0$ and $[h, x] = 0$. Then $\text{Ker } x$ and $xM$ are $h$-invariant and $\text{str}_M h = \text{str}_{M_x} h$.

If $X_M = X$, the statement of theorem is trivial. Let $X_M \neq X$. By Theorem 5.3 there exists $k$ less than the defect of $\mathfrak{g}$ such that

$$X_M \subset \bigcup_{A \in S, |A| \leq k} \Phi(A).$$

Let $A = \{\alpha_1, \ldots, \alpha_{k+1}\} \subset S$, $x = x_1 + \cdots + x_{k+1}$ for some nonzero $x_i \in \mathfrak{g}_{\alpha_i}$. Then $M_x = \{0\}$. If $h \in \mathfrak{h}$ satisfies $\alpha_1(h) = \cdots = \alpha_{k+1}(h) = 0$, then $[h, x] = 0$. Hence by Lemma 7.3 $\text{str}_M h = \text{str}_{M_x} h = 0$. Hence we just have proved the following property

\begin{equation}
\text{ch}_M (\mathfrak{h}_B^1) = 0 \text{ for all } A \in S, |A| = k + 1.
\end{equation}

Let $p_i$ be the first non-zero polynomial in the Taylor series for $\text{ch}_M$ at zero. Then $p_i$ also satisfies (7.1). Let $B = \{\alpha_1, \ldots, \alpha_k\} \subset S$ and $\bar{p}_i$ be the restriction of $p_i$ to $\mathfrak{h}_B^1$. If $\bar{p}_i \neq 0$, then degree of $\bar{p}_i$ is $i$. Since $p_i (\mathfrak{h}_B^1 \alpha_i) = 0$ for any $\alpha \neq \pm \alpha_i$, $\alpha \in B^\perp$, then $\alpha$ divides $\bar{p}_i$. That gives the estimate on $i$. Indeed, $i$ is not less the number of all possible $\alpha$ up to proportionality, i.e. $\frac{|\Delta \cap B^\perp|}{2} - |B|$. By Corollary 4.9 the latter number is the codimension $s$ of $X_M$ in $X$. Hence $i \geq s$.

To prove the second statement we need to show that if two homogeneous $W$-invariant polynomials $p$ and $q$ of degree $s$ satisfy (7.1), then $p = cq$ for some $c \in \mathbb{C}$. After restriction on $\mathfrak{h}_B^1$ we have

\begin{equation}
\bar{p} = a \Pi_{\alpha \in (\Delta + \cap B^\perp) \setminus B^\alpha} \alpha, \quad \bar{q} = b \Pi_{\alpha \in (\Delta^+ \cap B^\perp) \setminus \pm B^\alpha}
\end{equation}

for some constants $a$ and $b$. Therefore there exists $f = p - cq$ such that $f (\mathfrak{h}_B^1) = 0$. Thus, $f$ satisfies (7.1) for $k$ instead of $k + 1$. Then the degree of $f$ is bigger than $s$, which implies $f = 0$. \hfill \Box

8. Translation functor

In this section we introduce translation functors, we use these functors in the proof of Theorem 5.4. A translation functor is a superanalogue of similar functor in category $\mathcal{O}$ (see [6]). For superalgebras translation functors were used in [17] and [10].

Let $V$ be a $\mathfrak{g}$-module, on which the center $Z$ of the universal enveloping algebra acts locally finitely. Then $V = \oplus V^\chi$, where

$$V^\chi = \left\{ v \in V \mid (z - \chi(z))^N v = 0, z \in Z \right\}.$$

Let $\mathcal{B}$ be the category of all finitely generated $\mathfrak{g}$-modules with finite $Z$-action. Then $\mathcal{B}$ has a decomposition

$$\mathcal{B} = \oplus \mathcal{B}^\chi,$$

where $\mathcal{B}^\chi$ denotes the subcategory of all $V \in \mathcal{B}$ such that $V^\chi = V$.

Let $E$ be a finite-dimensional $\mathfrak{g}$-module. A translation functor $T^\chi_E$ is a functor in the category $\mathcal{B}$, defined by

$$T^\chi_E (V) = (V \otimes E)^\chi.$$
To simplify the notation we also will write $T^\lambda_E$ instead of $T^{\lambda\chi}_E$.

**Lemma 8.1.** $T^\chi_E$ is an exact functor.

**Proof.** Both tensoring with finite-dimensional vector space and the projection on the component with a given central character are obviously exact functors. □

Denote by $P(E)$ the set of all weights of $E$ counted with multiplicities.

**Lemma 8.2.**

1. For the Verma module $M_\lambda$, $M_\lambda \otimes E$ has a finite filtration 
\[ \{0\} = F_0 \subset \cdots \subset F_q = (M_\lambda \otimes E) \] 
of length $q = \dim E$ such that $F_{i+1}/F_i$ is isomorphic to $M_{\lambda+\nu}$, $\nu \in P(E)$; 

2. If $V$ is a module generated by a highest vector of weight $\lambda$, then $T^\chi_E(V)$ has a finite filtration 
\[ \{0\} = V_0 \subset \cdots \subset V_r = T^\chi_E(V) \] 
such that $V_i/V_{i+1}$ is a highest weight module of weight $\lambda + \nu$ for some $\nu \in P(E)$.

**Proof.** The first statement can be found in [6]. The second one follows from the first and Lemma 8.1. □

Let $b$ be a Borel subalgebra of $g$, $V$ be a $g$-module. A vector $v \in V$ is $b$-primary if $bv \in Cv$.

**Lemma 8.3.** If $v$ is a $b$-primitive vector of $(L_\lambda \otimes E)$ then the weight of $v$ equals $\lambda + \nu$ for some $\nu \in P(E)$.

**Proof.** Introduce the order on $h^*$ by putting $\mu \leq \nu$ if $\nu = \mu + \Sigma n_\alpha \alpha$ for some $\alpha \in \Delta^+$ and $n_\alpha \in \mathbb{Z}_{\geq 0}$. Choose a maximal weight $\gamma$ of $L_\lambda$ such that
\[ v = v_1 \otimes w_1 + \cdots + v_r \otimes w_r + v'_1 \otimes w'_1 + \cdots + v'_t \otimes w'_t \]
for some linearly independent $v_1, \ldots, v_r \in L_\lambda$ of weight $\gamma$, linearly independent weight vectors $w_1, \ldots, w_r \in E$ and some linearly independent weight vectors $v'_1, \ldots, v'_t \in L_\lambda$ of weights different from $\gamma$, $w'_1, \ldots, w'_t \in E$. For any simple root element $e \in n$ the condition $ev = 0$ implies
\[ ev_1 \otimes w_1 + \cdots + ev_r \otimes w_r = 0. \]
Since $w_1, \ldots, w_r$ are linearly independent, we must have $ev_i = 0$. Therefore all $v_i$ are $b$-primary. But $L_\lambda$ has a unique up to proportionality $b$-primary vector. Therefore $\gamma = \lambda$, $r = 1$ and the weight of $v$ is the sum of $\lambda$ and the weight of $w_1$. □

For any $\lambda \in h^*$ put
\[ h_\lambda = h_{\chi\lambda}, \Sigma_\lambda = h_\lambda \cap \Sigma. \]

**Lemma 8.4.** Let $\lambda, \mu \in \Sigma$ satisfy the conditions
\[ (8.1) \quad (\lambda + P(E)) \cap \Sigma_\mu = \{\mu\} \]
\[ (8.2) \quad (\mu - P(E)) \cap \Sigma_\lambda = \{\lambda\} \]
and \( \lambda \) is minimal in \((\mu - P(E)) \cap \mathfrak{h}_{\lambda}\). Then
\[
T_E^\mu(L_\lambda) = L_\mu, \quad T_{E^*}^\mu(L_\mu) = L_\lambda.
\]

Proof. By Lemma 8.2 (2) and (8.1) \( T_E^\mu(L_\lambda) \) is a highest weight module with highest weight \( \mu \). By Lemma 8.3 and (8.1) \( T_E^\mu(L_\lambda) \) has a unique up to proportionality \( b \)-primitive vector. Therefore either \( T_E^\mu(L_\lambda) = L_\mu \) or \( T_{E^*}^\mu(L_\mu) = \{0\} \). In the same way either \( T_E^\lambda(L_\mu) = L_\lambda \) or \( T_{E^*}^\lambda(L_\mu) = \{0\} \).

Our next observation is
\[
(8.3) \quad \text{Hom}_g(M \otimes E^*, N) \cong \text{Hom}_g(M, N \otimes E),
\]
hence, in particular
\[
(8.4) \quad \text{Hom}_g(T_{E^*}^\lambda(M_\mu), L_\lambda) \cong \text{Hom}_g(L_\mu, T_E^\mu(L_\lambda)).
\]
Therefore \( T_E^\mu(L_\lambda) = \{0\} \) iff \( T_{E^*}^\mu(L_\mu) = \{0\} \). Let us prove that \( T_E^\mu(L_\lambda) \neq \{0\} \). Note that by Lemma 8.2 (1) and (8.2), \( T_{E^*}^\lambda(M_\mu) \) has a subquotient isomorphic to \( M_{\lambda} \). Moreover, since \( \lambda \) is a minimal weight in \((\mu - P(E)) \cap \mathfrak{h}_{\lambda}\), there is a quotient in \( T_{E^*}^\lambda(M_\mu) \) isomorphic to \( M_{\lambda} \), hence there is a quotient isomorphic to \( L_\lambda \). Therefore
\[
\text{Hom}_g(T_{E^*}^\lambda(M_\mu), L_\lambda) \neq \{0\}.
\]

But then using (8.3)
\[
\text{Hom}_g(M_\mu, T_E^\mu(L_\lambda)) \cong \text{Hom}_g(T_{E^*}^\lambda(M_\mu), L_\lambda) \neq 0.
\]
Therefore \( T_E^\mu(L_\lambda) \neq \{0\} \). Finally by (8.4) \( T_{E^*}^\lambda(L_\mu) \neq \{0\} \). \( \square \)

**Lemma 8.5.** Let \( M \) be a finite-dimensional \( \mathfrak{g} \)-module and \( N = T_E^\lambda(M) \). Then \( X_N \subset X_M \).

Proof. Let \( x \in X \setminus X_M \), then \( M \) is free over \( \mathbb{C}[x] \), and \( M \otimes E \) is also free over \( \mathbb{C}[x] \). Since \( N \) is a direct summand of \( M \otimes E \), then \( N \) is free over \( \mathbb{C}[x] \). That implies \( x \notin X_N \). \( \square \)

9. **Reduction to the stable case**

Fix a set of simple roots and the Borel subalgebra \( \mathfrak{b} \subset \mathfrak{g} \) generated by \( \mathfrak{h} \) and simple roots. We say that a subalgebra \( \mathfrak{q} \) is **admissible** if \( \mathfrak{q} \) is generated by \( \mathfrak{h} \), some subset of simple roots and their negatives. By \( \Delta(\mathfrak{q}) \) we denote the root system of \( \mathfrak{q} \). We call \( \lambda \) **stable** with respect to \( \mathfrak{q} \) if the following conditions hold for any isotropic \( \alpha \in \Delta \), \( (\lambda + \rho, \alpha) = 0 \) implies \( \alpha \in \Delta(\mathfrak{q}) \).

In this section we assume that \( \mathfrak{g} = \mathfrak{gl}(m|n) \). Then
\[
\Delta_0 = \{ \varepsilon_i - \varepsilon_j \mid i, j \leq m \} \cup \{ \delta_i - \delta_j \mid i, j \leq n \}, \quad \Delta_1 = \{ \pm(\varepsilon_i - \delta_j) \mid i \leq m, j \leq n \}.
\]
All odd roots are isotropic. The choice of the form on \( \mathfrak{h}^* \) is such that \( (\varepsilon_i, \varepsilon_j) = \delta_{ij} \), \( (\delta_i, \delta_j) = -\delta_{ij} \). The defect \( d = \min(m, n) \). We choose a Borel subalgebra \( \mathfrak{b} \) so that the simple roots are
\[
\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \ldots, \delta_{n-1} - \delta_n.
\]
If \( \lambda + \rho = a_1 \varepsilon_1 + \cdots + a_m \varepsilon_m + b_1 \delta_1 + \cdots b_n \delta_n \), then \( \lambda \in \Sigma \) iff \( a_i - a_{i+1}, b_j - b_{j+1} \in \mathbb{Z}_{>0} \) for all \( i < m, j < n \). In other words, \( \lambda \in \Sigma \) iff \( \langle \lambda + \rho, \gamma^\vee \rangle \in \mathbb{Z}_{>0} \) for all \( \gamma \in \Delta_0^+ \). Since we consider only atypical \( \lambda \) we may assume that \( a_i, b_j \in \mathbb{Z} \).

**Lemma 9.1.** Let \( \lambda + \rho = a_1 \varepsilon_1 + \cdots + a_m \varepsilon_m + b_1 \delta_1 + \cdots b_n \delta_n \in \Sigma \). If \( a_i + b_j = 0 \) implies \( i > m - k \), then \( \lambda \) is stable for \( \mathfrak{q} \) with simple roots \( \varepsilon_{m-k+1} - \varepsilon_{m-k+2}, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \ldots, \delta_{n-1} - \delta_n \).

**Proof.** Trivial. \( \square \)

**Theorem 9.2.** Let \( \mathfrak{g} = \mathfrak{gl}(m|n) \). If \( \lambda \in \Sigma \) and has the degree of atypicality \( k \), then there exists a subalgebra \( \mathfrak{q} \subset \mathfrak{g} \) of defect \( k \), a stable \( \mu \in \Sigma \) of degree atypicality \( k \) and translation functors \( T_1, \ldots, T_r, T_{1^*}, \ldots, T_r^* \) such that

\[
L_\mu = T_1 \ldots T_r (L_\lambda), \quad L_\lambda = T_r^* \ldots T_1^* (L_\mu).
\]

**Proof.** Translation functors which we use are always related with \( E \) being the standard representation or its dual. We will provide a combinatorial algorithm, which constructs from a weight \( \lambda \in \Sigma \) a new weight \( \mu \in \Sigma_\lambda \) in such way that \( \lambda, \mu \) satisfy the conditions of Lemma 8.4 and therefore \( T_{E,\mu}^\mu (L(\lambda)) = L(\mu), T_{E,\lambda}^\mu (L(\mu)) = L(\lambda) \).

Applying this algorithm several times we obtain a sequence of weights \( \mu_1, \ldots, \mu_r \) such that \( \mu_r \) is stable. Let \( \lambda + \rho = a_1 \varepsilon_1 + \cdots + a_m \varepsilon_m + b_1 \delta_1 + \cdots b_n \delta_n \). Let \( g \) be maximal such that \( a_i + b_j \neq 0 \) for any \( i < g, j < n \). If \( g = m - k \), then \( \lambda \) is stable as in Lemma 9.1 and we can stop to apply the algorithm. Otherwise choose first \( i > g \) such that \( a_i + b_j \neq 0 \) for all \( j \leq n \). Construct \( \mu \) depending on the following

1. If \( b_j \neq -a_i - 1 \) for any \( j \leq n \), then put \( \mu = \lambda + \varepsilon_i \);
2. If \( b_j = -a_i - 1 \) for some \( j \) look at \( a_{i-1} \). If \( a_{i-1} = a_i + 1 \), put \( \mu = \lambda + \delta_j \).
   Otherwise go to the next step;
3. If \( b_j = -a_i - 1 \), \( a_{i-1} \neq a_i + 1 \) find the maximal \( p \) such that \( b_{j+p} = b_j - p \). If \( a_{i-1} + b_{j+p} > 0 \), put \( \mu = \lambda - \delta_{j+p} \). Otherwise go to the next step.
4. If \( a_{i-1} + b_{j+p} \leq 0 \), then there exists \( t \leq p \) such that \( a_{i-1} + b_{j+t} = 0 \). Put \( \mu = \lambda - \varepsilon_{i-1} \).

Note that at some point one always arrives to the case 2, that decreases \( i \) and eventually increases \( g \). In the end one will come to the stable weight. \( \square \)

Theorem 9.2 and Lemma 8.5 imply

**Theorem 9.3.** Let \( \mathfrak{g} = \mathfrak{gl}(m|n) \). For any \( \lambda \in \Sigma \) there exists a stable \( \mu \in \Sigma \) with the same degree of atypicality such that \( X_{L_\lambda} = X_{L_\mu} \).

10. **Proof of Theorem 5.4 for \( \mathfrak{gl}(m|n) \)**

In this section \( \mathfrak{g} = \mathfrak{gl}(m|n) \) and \( \lambda \) is an integral dominant weight with degree of atypicality \( k \). As Theorem 5.3 is already proven we have to show only that if \( A \in S, \ |A| = k \), then \( (L_\lambda)_x \neq \{0\} \) for any \( x \in \Phi(A) \). As follows from Theorem 9.3, we may assume that \( \lambda \) is stable with respect to \( \mathfrak{q} = \mathfrak{gl}(k|n) \). It is easy to check
that \( \Phi(A) \cap \mathfrak{q} \neq \emptyset \), and therefore one may assume that \( x \in \mathfrak{q} \). On the other hand, \( L_\lambda = L_\lambda(\mathfrak{q}) \oplus N \) as a module over \( \mathfrak{q} \). Thus, it is sufficient to prove that \( (L_\lambda(\mathfrak{q}))_x \neq \{0\} \). In other words, we reduce the theorem to the case of \( \mathfrak{gl}(k|n) \). Using the isomorphism \( \mathfrak{gl}(k|n) \cong \mathfrak{gl}(n|k) \) we can repeat the above argument and reduce the theorem to the case \( \mathfrak{g} = \mathfrak{gl}(k|k) \). Summing up, Theorem 5.4 is equivalent to the following Lemma.

**Lemma 10.1.** Let \( \mathfrak{g} = \mathfrak{gl}(k|k) \) and \( \lambda \) be an integral dominant weight with degree of atypicality \( k \). Then \( (L_\lambda)_x \neq \{0\} \) for any \( x \in X \).

We prove Lemma 10.1 in several steps. We use the fact that \( \mathfrak{g} \) has the \( \mathbb{Z} \)-grading \( \mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \) and \( \mathfrak{g}(-1) \), \( \mathfrak{g}(1) \) are irreducible components of \( X \). We have \( k + 1 \) open orbits on \( X \). Choose a representative \( x \) on each orbit in the following way:

\[
\begin{pmatrix}
0 & x^+ \\
-x^- & 0
\end{pmatrix},
\]

where \( x^+ \) is the block matrix

\[
\begin{pmatrix}
1_p & 0 \\
0 & 0
\end{pmatrix}
\]

and \( x^- \) is the block matrix

\[
\begin{pmatrix}
0 & 0 \\
0 & 1_q
\end{pmatrix};
\]

here \( p + q = k \).

If \( x \in \mathfrak{g}(1) \), then \( x^- = 0 \), if \( x \in \mathfrak{g}(-1) \), then \( x^+ = 0 \). In both cases the stabilizer \( K \) of \( x \) in \( G_0 \) is isomorphic to \( \text{GL}(k) \) embedded diagonally in \( G_0 = \text{GL}(k) \times \text{GL}(k) \). By \( \mathfrak{g} \) we denote the Lie algebra of \( K \).

**Lemma 10.2.** If \( x \in \mathfrak{g}(\pm 1) \) and \( M \) is a finite-dimensional \( \mathfrak{g} \)-module, then \( M_x \) is a trivial \( K \)-module.

*Proof.* Follows from the fact \( C_\mathfrak{g}(x) = [x, \mathfrak{g}] \) and Lemma 6.2. \( \square \)

**Lemma 10.3.** Let \( \mathfrak{g} = \mathfrak{gl}(m|n) \), \( \mathfrak{b} \) is the Borel subalgebra containing \( \mathfrak{g}(1) \), \( M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda \) be the Verma module. If \( \alpha \) is a negative isotropic root such that \( (\lambda + \rho, \alpha) = 0 \), then \( M_\lambda \) contains a \( \mathfrak{b} \)-primitive vector of weight \( \lambda + \alpha \).

*Proof.* Let \( I_\alpha \) be the set of all weights \( \lambda \in \mathfrak{h}^* \) such that \( M_\lambda \) has a \( \mathfrak{b} \)-primitive vector of weight \( \lambda + \alpha \). Then \( I_\alpha \) is Zariski closed, see for example [2]. Let

\[ H_\alpha = \{ \lambda \in \mathfrak{h}^* \mid (\lambda + \rho, \alpha) = 0 \}. \]

We want to show that \( H_\alpha \subset I_\alpha \). Consider

\[ H'_\alpha = \{ \lambda \in H_\alpha \mid (\lambda + \rho, \beta) \neq 0, \beta \neq \pm \alpha, \beta \in \Delta_{\mathfrak{g}_1} \}. \]

It suffices to show that \( H'_\alpha \subset I_\alpha \). Consider \( \mathfrak{b}' = \mathfrak{b}_0 + \mathfrak{g}(-1) \). If \( v \) is a highest vector of \( M_\lambda \) and \( X_\beta \in \mathfrak{g}_\beta \), then \( w = \Pi_{\beta \in \Delta(\mathfrak{g}(-1))} X_\beta v \) is a \( \mathfrak{b}' \)-primitive, and \( u = \)
\[ \Pi_{\beta \in \Delta(g(1)) \setminus \{a\}} X_\beta w \text{ is a } b\text{-primitive. Since the weight of } u \text{ equals } \lambda + \alpha, \text{ we obtain } H'_\alpha \subset I_\alpha \text{ as required.} \]

The \( \mathbb{Z} \)-grading on \( g \) induces the \( \mathbb{Z} \)-grading on an irreducible \( g \)-module \( M = M(0) \oplus M(-1) \oplus \cdots \oplus M(-k^2) \) in the following way

\[ M(0) = \text{Ker } g(1), \quad M_i = g(-1) M(i + 1). \]

Each \( M(i) \) is a \( g_0 \)-submodule of \( M \).

**Lemma 10.4.** Let \( x \in g(\pm 1) \), \( M = L_\lambda \) for a dominant integral \( \lambda \) of degree atypicality \( k \). Then \( M(0) \) contains one trivial \( K \)-submodule and \( M(-1) \) does not have trivial \( K_x \)-submodules.

**Proof.** Since the degree of atypicality of \( \lambda \) is \( k \), one can write

\[ \lambda = a_1 \varepsilon_1 + \cdots + a_k \varepsilon_k - a_k \delta_1 - \cdots - a_1 \delta_1. \]

We denote by \( V(a_1, \ldots, a_k) \) the irreducible \( gl(k) \)-module with highest weight \( (a_1, \ldots, a_k) \) and by \( L_\lambda(g_0) \) the irreducible \( g_0 \)-module with highest weight \( \lambda \). Since \( M(0) \) is isomorphic to \( L_\lambda(g_0) \), then

\[ M(0) \cong V(a_1, \ldots, a_k) \otimes V^*(a_1, \ldots, a_k) \]

as \( K \)-module, which has exactly one trivial component. Hence the first statement is true.

Obviously \( M(-1) \) is a submodule in

\[ L_\lambda \otimes g(-1) \subset \oplus_{\alpha \in \Delta(g(-1))} L_{\lambda + \alpha}(g_0). \]

However, \( (\lambda + \rho, \varepsilon_i - \delta_{k+1-i}) = 0 \), therefore by Lemma 10.3 \( M(-1) \) does not contain the component \( L_{\lambda + \delta_k + \cdots + \delta_{k+1-i} - \varepsilon_i}(g_0) \) for all \( i = 1, \ldots, k \). Hence \( M(-1) \) is a \( K \)-submodule of the \( K \)-module

\[ \oplus_{i \neq j} V(a_1, \ldots, a_i - 1, \ldots, a_k) \otimes V^*(a_1, \ldots, a_j - 1, \ldots, a_k). \]

Therefore \( M(-1) \) does not contain \( K \)-trivial submodules. \( \square \)

**Lemma 10.5.** Let \( x \in g(\pm 1) \) belong to an open \( G_0 \)-orbit, \( M = L_\lambda \) for a dominant integral \( \lambda \) of degree atypicality \( k \) and \( N \) be a trivial \( K \)-submodule in \( M(0) \). Then \( N \subset M_x \) and therefore \( M_x \neq \{0\} \).

**Proof.** If \( x \in g(1) \), then \( xN = 0 \). Since \( x : M(-1) \to M(0) \) is a homomorphism of \( g_x \)-modules and \( M(-1) \) does not have trivial \( g_x \)-submodules, then \( N \) does not belong to \( \text{Im } x \). If \( x \in g(-1) \), then \( N \) clearly is not in \( \text{Im } x \). Since \( x : M(0) \to M(-1) \) is a homomorphism of \( g_x \)-modules and \( M(-1) \) does not contain trivial \( g_x \)-submodules, then \( xN = 0 \). \( \square \)
Lemma 10.5 shows that \( M_x \neq \{0\} \) in two special cases: \( x \in \mathfrak{g}(1) \) or \( x \in \mathfrak{g}(-1) \). Now we will show the same for each open \( G_0 \)-orbit on \( C \). Let \( y_c \) be an odd element in \( \mathfrak{g} \) given by
\[
\begin{pmatrix}
0 & y_c^+ \\
y_c^- & 0
\end{pmatrix},
\]
where \( y_c^+ \) is the block matrix
\[
\begin{pmatrix}
1_p & 0 \\
0 & c_1 q
\end{pmatrix}
\]
and \( y_c^- \) is the block matrix
\[
\begin{pmatrix}
c_1 p & 0 \\
0 & 1_q
\end{pmatrix};
\]
here \( p + q = k, c \in \mathbb{C} \). Note that \( y_c \notin X \) if \( c \neq 0 \), but \([y_c, y_c]\) lies in the center of \( \mathfrak{g} \). If \( M = L_\lambda \) has the degree atypicality \( k \), then the center of \( \mathfrak{g} \) acts by zero on \( M \). Hence \( M_{y_c} = \text{Ker} y_c/\text{Im} y_c \) is well defined. Lemma 10.5 implies \( M_{y_1} \neq \{0\} \). If \( c \neq 0 \), then there exists \( g \in G_0 \) such that \( y_c = c^{1/2} \text{Ad}_g(y_1) \). Therefore \( M_{y_c} \neq \{0\} \) for any \( c \neq 0 \). The continuity argument shows that \( M_{y_0} \neq \{0\} \). But \( y_0 \in X \) is an element on an open orbit. Therefore Lemma 10.1 and Theorem 5.4 are proven.

11. Application to \( H(\mathfrak{g}(-1); M) \) for \( \mathfrak{gl}(m|n) \)

Let \( \mathfrak{g} = \mathfrak{gl}(m|n) \), then \( \mathfrak{g}(-1) \) is an abelian subalgebra and the cohomology \( H(\mathfrak{g}(-1); M) \) determine the character of a finite-dimensional module \( M \). On the other hand, \( \mathfrak{g}(-1) \) is an irreducible component of \( X \). The complex calculating \( H(\mathfrak{g}(-1); M) \) is
\[
\partial : \mathcal{O}(\mathfrak{g}(-1)) \otimes M \to \mathcal{O}(\mathfrak{g}(-1)) \otimes M,
\]
where \( \partial \) is the same as for the sheaf \( \mathcal{M} \). One can consider the localization of this complex and the corresponding coherent sheaf \( \mathcal{H}_M \) is the restriction of \( \mathcal{M} \) on \( \mathfrak{g}(-1) \).

Theorem 5.4 and Theorem 6.12 imply the following

**Theorem 11.1.** Let \( M \) be an irreducible finite-dimensional module with central character \( \chi \) and the degree of atypicality of \( \chi \) equal \( k \). Then \( \text{supp} \mathcal{H}_M = \overline{X}_k \cap \mathfrak{g}(-1) \).

**Lemma 11.2.** Let \( M \) be a typical finite-dimensional module. Then \( \text{supp} \mathcal{H}_M = \{0\} \) and \( \mathcal{H}_M(0) = H^0(\mathfrak{g}(-1), M) \).

**Proof.** Since \( M \) is typical, then \( M \) is a free \( \mathfrak{g}(-1) \) module and \( H^i(\mathfrak{g}(-1), M) = 0 \) for \( i > 0 \). \( \square \)

**Theorem 11.3.** Let \( x \in X_k \cap \mathfrak{g}(-1), M = L_\lambda, \) the degree of atypicality of \( \lambda \) be \( k \), and \( Z = G_0 x \). Then \( \mathcal{H}_M(Z) \) is the sheaf of section of the \( G_0 \)-vector bundle induced by \( (\mathfrak{g}_x)_{G_0} \)-module \( H^0(\mathfrak{g}_x \cap \mathfrak{g}(-1); M_x) \).

**Proof.** Follows from Lemma 11.2, Theorem 6.12 and Theorem 6.10. \( \square \)
ON ASSOCIATED VARIETY FOR LIE SUPERALGEBRAS

REFERENCES

1. V. G. Kac, Lie superalgebras, Adv. Math. 26 (1977), 8–96.
2. J. Dixmier, Enveloping algebras, Graduate Studies in Mathematics, vol. 11, American Mathematical Society, Providence, RI, 1996. Revised reprint of the 1977 translation.
3. V. G. Kac, Representations of classical Lie superalgebras, Lecture Notes in Math. 676 (1978), 597–626.
4. A. Sergeev, The invariant polynomials on simple Lie superalgebras, Representation theory 3 (1999), 250–280.
5. C. Gruson, Sur la cohomologie des super algèbres de Lie étranges, Transform. Groups 5 (2000), no. 1, 73–84.
6. J. Bernstein, I. M. Gelfand, and S.I. Gelfand, Category of G–modules, Funct. Anal. and Appl. 10 (1976), 87–92.
7. David A. Vogan, Jr., Associated varieties and unipotent representations.
8. Friedrich Knopp, Invariant functions on symplectic representations, preprint, arXiv:math.AG/0506171, Jun 2005.
9. Maria Gorelik, The Kac construction of the centre of u(g) for Lie superalgebras, J. Nonlinear Math. Phys. (2004).
10. Jonathan Brundan, Kazhdan-lusztig polynomials and character formulae for the L.
11. Shohei Kato and Hiroyuki Ochiai, The degrees of orbits of the multiplicity-free actions, Astérisque (2001), no. 273, 139–158, Nilpotent orbits, associated cycles and Whittaker models for highest weight representations.
12. Victor G. Kac, Laplace operators of infinite-dimensional Lie algebras and theta functions, Proceedings of the National Academy of Sciences of the United States of America 81 (1984), no. 2, 645–647.
13. Kyo Nishiyama, Hiroyuki Ochiai, and Kenji Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules—Hermitian symmetric case, Astérisque (2001), no. 273, 13–80, Nilpotent orbits, associated cycles and Whittaker models for highest weight representations.
14. Caroline Gruson, Sur l’idéal du cône autocommutant des super algèbres de Lie basiques classiques et étranges, Ann. Inst. Fourier (Grenoble) 50 (2000), no. 3, 807–831.
15. , Cohomologie des modules de dimension finie sur la super algèbre de Lie osp(3, 2), J. Algebra 259 (2003), no. 2, 581–598.
16. V. G. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, Progress in Math. 123 (1994), 415–456.
17. V. Serganova, Kazhdan-Lusztig polynomials and character formula for the Lie superalgebra gl(m|n), Selecta Math. (N.S.) 2 (1996), no. 4, 607–651.

Dept. of Mathematics, University of California at Berkeley, Berkeley, CA 94720
E-mail address: serganov@math.berkeley.edu