THE $\ell$-ADIC DUALIZING COMPLEX ON AN EXCELLENT SURFACE WITH RATIONAL SINGULARITIES

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Abstract. In this article, we show that if $X$ is an excellent surface with rational singularities, the constant sheaf $\mathbb{Q}_\ell$ is a dualizing complex. In coefficient $\mathbb{Z}_\ell$, we also prove that the obstruction for $\mathbb{Z}_\ell$ to become a dualizing complex lying on the divisor class groups at singular points. As applications, we study the perverse sheaves and the weights of $\ell$-adic cohomology groups on such surfaces.

Introduction

In [9, I], the dualizing complexes in étale cohomology was considered by Grothendieck. O. Gabber made a breakthrough on this fields. He prove that every excellent schemes admits a dualizing complex; and on a regular excellent scheme, the constant sheaf $\mathbb{Z}_\ell$ is a dualizing complex. See [12] for the detail.

In this paper, we study the properties of dualizing complexes on surfaces with rational singularities. In [11], J. Lipman prove that a two-dimensional normal local ring $R$ has a rational singularity if and only if $R$ has a finite divisor class group. This makes me think that rational singularities only affect the torsion part of the étale cohomology with coefficient $\mathbb{Z}_\ell$, but leave the free part invariant. As an evidence, we prove that on an excellent surface $X$ with at most rational singularities, the potential dualizing complex $\mathcal{K}_X$ (determined by the dimension function $x \mapsto 2 - \dim O_{X,x}$) concentrates on the entries $-2$ and $-4$; in detail, $H^{-4}(\mathcal{K}_X) = \mathbb{Z}_\ell(2)$ and $H^{-2}(\mathcal{K}_X)(-1)$ is the $\ell$-torsion part of the divisor class groups at singular points (see Theorem 3.2). In particular, $\mathbb{Q}_\ell$ is a dualizing complex of such surface in coefficient $\mathbb{Q}_\ell$.

The paper is organized as follows. In §1, we briefly review the theory of “potential dualizing complex” in [12]. Because the dualizing complexes on a scheme are not unique, they vary by tensor products with invertible complexes. So we prefer the potential dualizing complex which is a dualizing complex unique determined by the dimension function. Basing on this theory, we reiterate the étale homology in [10]. Since lacking of suitable references related to étale homology both in adic coefficients and over arbitrary base schemes, it is necessary to spend a little more space on rewriting it. In §2, we calculate the étale homology on curves.

In §3, we prove the main results. First we calculate the étale homology on arbitrary surfaces. Basing on it, we obtain the main results about the dualizing complexes on an excellent surface with rational singularities; and prove the Poincaré Duality for such surfaces. In §4, we give two applications of above theory. First we study perverse sheaves on a surface with rational singularities. Basing on it, we prove that the cohomologies of smooth sheaves of punctually pure weights on such surfaces, are also punctually pure of weights.

1. Review of dualizing sheaves and étale homology

Let $\ell$ be a prime number, $A$ the integral closure of $\mathbb{Z}_\ell$ in a finite extension field of $\mathbb{Q}_\ell$.

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We consider the following conditions related to a scheme $X$:

(†) $X$ is Noetherian, excellent, of finite Krull dimension, $\ell$ is invertible on $X$ and $cd_{\ell}(X) < \infty$.

From the Gabber’s finiteness theorem for étale cohomology in [7], we may construct an adic formalism $D^b_{\text{ét}}(X, \Lambda)$ together with Grothendieck’s six operations on schemes satisfying (†). See also [4] for details. Moreover if $X$ is a scheme satisfying (†), then any scheme of finite type over $X$ satisfies (†).

Next, we introduce the “potential dualizing complex” defined in [12].

**Definition 1.1** ([12], 2.2). Let $X$ be a scheme satisfying (†), $\delta : X \to \mathbb{Z}$ a dimension function. A potential dualizing complex (in coefficient $\Lambda$) on $X$ consisting of

1. an object $\mathcal{H}$ in $D^+_{\text{ét}}(X, \Lambda)$,
2. an isomorphism $R\Gamma_x(\mathcal{H}) \sim \Lambda(\delta(x))[2\delta(x)]$ in $D^+_{\text{ét}}(x, \Lambda)$ for every point $x$ on $X$;

and these data satisfy that: for every immediate specialization $y \to x$ on $X$, the following diagram commutes,

$$
\begin{array}{ccc}
R\Gamma_y(\mathcal{H}) & \xrightarrow{\text{sp}_{y \to x}^X} & R\Gamma_x(\mathcal{H})(1)[2] \\
\cong & & \cong \\
R(\delta(y))[2\delta(y)] & = & R(\delta(x) + 1)[2(\delta(x) + 1)]
\end{array}
$$

where $\text{sp}_{y \to x}^X$ is the transition morphism of codimension 1 defined in [12, §1].

The potential dualizing complexes have the following properties.

**Proposition 1.2** ([12], 4.1 & 5.1). Every scheme $X$ satisfying (†) equipped with a dimension function $\delta$ has a potential dualizing complex, unique up to unique isomorphism, which we denoted by $\mathcal{K}_{X, \delta}$ or simply by $\mathcal{K}_X$. Moreover the potential dualizing complex is a dualizing complex.

**Proposition 1.3** ([12], 2.8). Let $X$ be a regular scheme satisfying (†). Then $\delta(x) := -\dim \mathcal{O}_{X, x}$ is a dimension function on $X$ and we have $\mathcal{K}_{X, \delta} = \Lambda$.

**Proposition 1.4** ([12], 4.3). Let $f : X \to Y$ be a compactifiable morphism of schemes satisfying (†). Let $\delta_Y$ be a dimension function on $Y$ and equip $X$ with the dimension function

$$
\delta_X(x) := \delta_Y(f(x)) + \text{tr. d. } k(x)/k(f(x)) .
$$

Then we have $\mathcal{K}_{X, \delta_X} = Rf^*\mathcal{K}_{Y, \delta_Y}$.

**Proposition 1.5** ([12], 4.2). Let $f : X \to Y$ be a regular morphism of schemes satisfying (†). Let $\delta_Y$ be a dimension function on $Y$ and equip $X$ with the dimension function

$$
\delta_X(x) := \delta_Y(y) - \dim \mathcal{O}_{X, y} , \quad (y := f(x))
$$

Then we have $\mathcal{K}_{X, \delta_X} = f^*\mathcal{K}_{Y, \delta_Y}$.

Basing on above theory, we may generalize the étale homology in [10] to arbitrary schemes satisfying (†). Since most proof are almost the same with [10], we only give these which need special care.

**Definition 1.6.** Let $X$ be a scheme satisfying (†) equipped with a dimension function $\delta$. For each $n \in \mathbb{Z}$, we define

$$
H_n(X, \delta, \Lambda) := H^{-n}(X_{\text{ét}}, \mathcal{K}_{X, \delta}) = \text{Hom}_{D^b_{\text{ét}}(X, \Lambda)}(\Lambda_X, \mathcal{K}_{X, \delta}[-n]) .
$$

We also use $H_n(X, \delta)$ or $H_n(X, \Lambda)$ or $H_n(X)$ to denote $H_n(X, \delta, \Lambda)$, if no confusion arise.
Let $f: X \to Y$ be a compactifiable morphism of schemes satisfying ($\dagger$), $\delta$ a dimension function on $Y$, and $\delta'$ the dimension function on $X$ induced by $\delta$ as in Proposition 1.4. Then we also use $H_n(X, \delta, \Lambda)$ or $H_n(X, \delta)$ to denote $H_n(X, \delta', \Lambda)$.

If $f: X \to Y$ is a proper morphism of schemes satisfying ($\dagger$), $\delta$ a dimension function on $Y$. For each $n \in \mathbb{Z}$, we define a homomorphism of $\Lambda$-modules

$$f_*: H_n(X, \delta) \to H_n(Y, \delta)$$

as follows. For each $\alpha \in H_n(X, \delta)$, regarding $\alpha: \Lambda_X \to \mathcal{X}_X,\delta[-n]$ as a morphism in $D^b_c(X_{\text{et}}, \Lambda)$, then $f_*(\alpha)$ is defined to be the composition

$$\Lambda_Y \to Rf_!\Lambda_X \xrightarrow{Rf_!(\alpha)} Rf_!\mathcal{X}_X,\delta[-n] \xrightarrow{\sim} Rf_! \circ f^* \mathcal{X}_Y,\delta[-n] \to \mathcal{X}_Y,\delta[-n],$$

where the first and the last morphisms are induced by the adjunctions.

It is easy to verify that if $f: X \to Y$ and $g: Y \to Z$ are two proper morphisms, then $(g \circ f)_* = g_* \circ f_*$.  

If $Y$ is the spectrum of a separably closed field, as $H_0(Y) = \Lambda$, we may write

$$\deg := f_*: H_0(X) \to \Lambda.$$

Proposition 1.7. Let $X$ be a scheme satisfying ($\dagger$) equipped with a dimension function $\delta$, $Y$ a closed subscheme of $X$ and $U := X \setminus Y$. Then there is a long exact sequence of $\Lambda$-modules:

$$\cdots \to H_{n+1}(U, \delta) \to H_n(Y, \delta) \to H_n(X, \delta) \to H_n(U, \delta) \to H_{n-1}(Y, \delta) \to \cdots.$$

Proposition 1.8 (Mayer-Vietoris Sequence). Let $X$ be a scheme satisfying ($\dagger$) equipped with a dimension function $\delta$, $X_1$ and $X_2$ two closed subschemes of $X$ such that $X = X_1 \cup X_2$ (as sets). Then we have a long exact sequence

$$\cdots \to H_n(X_1 \cap X_2, \delta) \to H_n(X_1, \delta) \oplus H_n(X_2, \delta) \to H_n(X, \delta) \to H_{n-1}(X_1 \cap X_2, \delta) \to \cdots.$$

Notation 1.9. Let $X$ be a Noetherian scheme, $\delta$ a dimension function on $X$. Then we define

$$\dim_\delta(X) := \sup_{x \in X} \delta(x).$$

If $\xi_1, \xi_2, \ldots, \xi_r$ are all generic points of irreducible components of $X$, then

$$\dim_\delta(X) = \max_{1 \leq i \leq r} \delta(\xi_i) < +\infty.$$

Proposition 1.10 (Vanishing). Let $X$ be a scheme satisfying ($\dagger$) equipped with a dimension function $\delta$. Then $H_n(X, \delta) = 0$ for all $n > 2 \dim_\delta(X)$.

Corollary 1.11. Let $X$ be a scheme satisfying ($\dagger$) equipped with a dimension function $\delta$, $Y$ a closed subscheme of $X$, $X' := X \setminus Y$. Then for each integer $n > 2 \dim_\delta(Y) + 1$, there is a canonical isomorphism $H_n(X, \delta) \sim H_n(X', \delta)$ of $\Lambda$-modules.

Proof. We have only to apply Proposition 1.7. \qed

We define the pull-back maps as follows. Let $Y$ be a scheme satisfying ($\dagger$) equipped with a dimension function $\delta$, $f: X \to Y$ is a compactifiable flat morphism of relative dimension $d$. For each $n \in \mathbb{Z}$, we define a homomorphism of $\Lambda$-modules

$$f^*: H_n(Y, \delta) \to H_{n+2d}(X, \delta)(-d)$$

as follows. For each $\beta \in H_n(Y, \delta)$, $f^*(\beta)$ is defined to be the composition

$$\Lambda_X \xrightarrow{t_f} Rf^!\Lambda_X[-2d](-d) \xrightarrow{Rf^!(\beta)} Rf^!\mathcal{X}_Y[-n-2d](-d) \sim \mathcal{X}_X[-n-2d](-d),$$

where $t_f$ is the morphism dual to the trace morphism $\text{Tr}_f$ (see [8, XVIII (3.2.1.2)].
Notation 1.12. Let $X$ be a scheme satisfying (†) equipped with a dimension function $\delta$, $d := \dim_\delta(X)$.

(1) First we assume that $X$ is regular and integral. Then we use $t_X : \Lambda \sim \mathcal{K}_{X,\delta}(-d)[-2d]$ to denote the canonical morphism defined by Proposition 1.3, and use $\text{cl}(X)$ to denote the element in $H_{2d}(X, \delta)(-d)$ corresponding to $t_X$.

(2) Second we only assume that $X$ is integral. As $X$ is excellent, $U := X_{\text{reg}}$ is an open dense subset of $X$. Obviously $\dim(X \setminus U) < d$. By Corollary 1.11, there is a canonical isomorphism

$$H_{2d}(X, \delta)(-d) \sim H_{2d}(U, \delta)(-d).$$

Let $\text{cl}(X)$ be the inverse image of $\text{cl}(U)$ under above isomorphism.

(3) In general case, we let $X_1, X_2, \ldots, X_r$ be all irreducible components of $X$ with $\dim_\delta(X_i) = d$. For each $i$, regard $X_i$ as a reduced subscheme of $X$, let $\iota_i : X_i \hookrightarrow X$ be the inclusion and let $\xi_i$ be the generic point of $X_i$. Then we define

$$\text{cl}(X) := \sum_{i=1}^r \text{length}(\mathcal{O}_{X,\xi_i}) \cdot \iota_{i,*} \left( \text{cl}(X_i) \right) \in H_{2d}(X, \delta)(-d).$$

We also use $t_X : \Lambda \sim \mathcal{K}_{X,\delta}(-d)[-2d]$ to denote the morphism in $D^b_c(X, \Lambda)$ corresponding to $\text{cl}(X)$.

Notation 1.13. Let $X$ be a scheme satisfying (†) equipped with a dimension function $\delta$, $Y$ a closed subscheme of $X$, $d := \dim_\delta(Y)$. Then we define

$$\iota_* \left( \text{cl}(Y) \right) \in H_{2d}(X, \delta)(-d),$$

where $\iota : Y \hookrightarrow X$ is the inclusion.

Next, we study the étale homology under birational morphisms. First we need two lemmas.

Lemma 1.14. Let $X$ be a scheme satisfying (†), $Y$ a closed subscheme of $X$, $U := X \setminus Y$, $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$ the inclusions. Then for each object $\mathcal{F}$ in $D^b_c(X_{\text{ét}}, \Lambda)$, there are two distinguished triangles in $D^b_c(X_{\text{ét}}, \Lambda)$:

1. $j_*j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to j_*j^*\mathcal{F}[1]$,
2. $i_*R^1\mathcal{F} \to \mathcal{F} \to R^1j_*j^*\mathcal{F} \to i_*R^1\mathcal{F}[1]$.

Lemma 1.15. Let $X$ be a scheme satisfying (†), $Y$ a closed subscheme of $X$, $U := X \setminus Y$, $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$ the inclusions, $\mathcal{F}^1 \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}^n$ a sequence of objects in $D^b_c(X_{\text{ét}}, \Lambda)$. Assume that one of the following two conditions holds:

1. $j^*(\varphi) : j^*\mathcal{F}^1 \sim j^*\mathcal{F}$ is an isomorphism, $j^*\mathcal{F}^n = 0$, and for all $n \in \mathbb{Z}$,
   $$0 \to H^n(i^*\mathcal{F}) \to H^n(i^*\mathcal{F}) \to H^n(i^*\mathcal{F}^n) \to 0$$
   is a short exact sequence;
2. $j^*(\psi) : j^*\mathcal{F} \sim j^*\mathcal{F}^n$ is an isomorphism, $j^*\mathcal{F}^1 = 0$, and for all $n \in \mathbb{Z}$,
   $$0 \to H^n(R^1i^*\mathcal{F}) \to H^n(R^1i^*\mathcal{F}) \to H^n(R^1i^*\mathcal{F}^n) \to 0$$
   is a short exact sequence.
Then there exists a morphism \( \delta : \mathcal{F}' \to \mathcal{F}'[1] \) which makes
\[
\mathcal{F}' \xleftarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}' \xrightarrow{\delta} \mathcal{F}'[1]
\]
a distinguished triangle in \( D^b_c(X_{\text{et}}, \Lambda) \).

**Proof.** The morphism \( \varphi \) extends to a distinguished triangle
\[
\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'[1]
\]
in \( D^b_c(X_{\text{et}}, \Lambda) \). Note that \( 0 \to \mathcal{F}' \xrightarrow{\text{id}} \mathcal{F}' \to 0[1] \) is also a distinguished triangle. Hence there exists a morphism \( \alpha : \mathcal{G} \to \mathcal{F}' \) such that the triple \((0, \psi, \alpha)\) is a morphism of distinguished triangles, i.e., the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{F}' & \xrightarrow{\varphi} & \mathcal{F} \\
\downarrow{0} & & \downarrow{\psi} \\
\mathcal{F}'[1] & \xrightarrow{\delta} & 0[1]
\end{array}
\]

Now if (1) holds, then both \( j^*(\alpha) \) and \( i^*(\alpha) \) are isomorphic; and if (2) holds, then both \( j^*(\alpha) \) and \( R\psi_!(\alpha) \) are isomorphic. In either case, by Lemma 1.14, the morphism \( \alpha : \mathcal{G} \xrightarrow{\sim} \mathcal{F}' \) is an isomorphism. Therefore we may let \( \delta := \rho \circ \alpha^{-1} \).

Basing on above lemma, we have the following two propositions.

**Proposition 1.16.** Let
\[
\begin{array}{ccc}
Y' & \xleftarrow{\iota} & X' \\
\downarrow{q} & & \downarrow{p} \\
Y & \xrightarrow{i} & X
\end{array}
\]
be a Cartesian square of schemes satisfying \((\dagger)\), \( r := p \circ \iota' = i \circ q \). Assume that \( p \) is proper, \( i \) is a closed immersion; and there exists an open subset \( U \) of \( X \) such that \( i(Y) = X \setminus U \) and \( p \) induces an isomorphism \( p^{-1}(U) \xrightarrow{\sim} U \). Then for each object \( \mathcal{F} \) in \( D^b_c(X_{\text{et}}, \Lambda) \), there are two distinguished triangles:
\[
\mathcal{F} \xrightarrow{\delta_Y \oplus \delta} i_* \circ \iota^* \mathcal{F} \oplus R\mathcal{P} \circ p^* \mathcal{F} \xrightarrow{\delta_Y \circ \delta} R\mathcal{P}_* \circ r^* \mathcal{F} \to \mathcal{F}[1],
\]
\[
R\mathcal{P}_* \circ R\psi_! \mathcal{F} \xrightarrow{\varphi_Y \oplus \varphi} i_* \circ R\iota_! \mathcal{F} \oplus R\mathcal{P} \circ R\mathcal{P}' \mathcal{F} \xrightarrow{-\delta_Y \circ \delta \circ \varphi} R\mathcal{F} \to (R\mathcal{P}_* \circ R\iota_! \mathcal{F})[1],
\]
in \( D^b_c(X_{\text{et}}, \Lambda) \), where \( \delta \) and \( \varphi \) are induced by the adjunctions.

Apply the second triangle in above proposition to \( \mathcal{K} \_X \), we obtain the following proposition.

**Proposition 1.17.** Let the assumptions and the notations be as in Proposition 1.16; and let \( \delta \) be a dimension function on \( X \). Then there is a long exact sequence of \( \Lambda \)-modules:
\[
\cdots \to H_{n+1}(X, \delta) \to H_n(Y', \delta) \xrightarrow{q_+ \circ \iota'} H_n(Y, \delta) \oplus H_n(X', \delta) \xrightarrow{-i_* \circ \mathcal{P}_1 \circ \mathcal{P}_2 \circ \mathcal{P}_2} H_n(X, \delta) \to \cdots.
\]

1. **Calculation of étale homology on the exceptional divisor**

In this section, we calculate the étale homology of exceptional curves.

**Proposition 2.1.** Let \( C \) be a 1-equidimensional proper algebraic scheme over a separably closed field \( k \). Let \( C^{(1)}, C^{(2)}, \ldots, C^{(r)} \) be all connected components of \( C \); and let \( C_1, C_2, \ldots, C_n \) be all irreducible components of \( C \). Assume that \( H^1(C, \mathcal{O}_C) = 0 \). Then we have

1. As to homology, we have
Proposition 1.8, we obtain an isomorphism:

\[ Z^r = \bigoplus_{i=1}^r H_0(C^{(i)}) \xrightarrow{\oplus \deg_l} \Lambda^{\oplus r} . \]

(b) \( H_1(C) = 0 \).

(c) \( H_2(C)(-1) \) is a free \( \Lambda \)-module with basis \( \mathrm{cl}(C_1), \ldots, \mathrm{cl}(C_n) \).

(2) As to cohomology, we have

(a) There is a canonical isomorphism:

\[ H^0(C, \Lambda) = \bigoplus_{i=1}^r H^0(C^{(i)}) \xrightarrow{\sim} \Lambda^{\oplus r} . \]

(b) \( H^1(C, \Lambda) = 0 \).

(c) There is a canonical isomorphism \( H^2(C, \Lambda(1)) \xrightarrow{\sim} \Lambda^{\oplus n} \) which sends \( c_1(\mathcal{L}) \) to \((\deg(\mathcal{L}_1|_{C_1}), \ldots, \deg(\mathcal{L}_r|_{C_r}))\) for each invertible \( \mathcal{O}_C \)-module \( \mathcal{L} \).

**Proof.** (1). Let \( \bar{k} \) be the algebraic closure of \( k \). After replacing \( k \) with \( \bar{k} \) and \( X \) with \( (X \otimes_k \bar{k})_{\mathrm{red}} \), we may assume that \( k \) is algebraically closed and \( X \) is reduced. Now we use induction on the number \( n \) of irreducible components of \( C \).

First consider the case \( n = 1 \). In this case, \( C \) is integral. Let \( \bar{C} \) be the normalization of \( C \). Then there is a long exact sequence

\[ 0 \to \Gamma(C, \mathcal{O}_C) \to \Gamma(C, \mathcal{O}_{\bar{C}}) \to \Gamma(C, \mathcal{O}_{\bar{C}}/\mathcal{O}_C) \to H^1(C, \mathcal{O}_C) . \]

Note that \( H^1(C, \mathcal{O}_C) = 0 \), \( \Gamma(C, \mathcal{O}_C) = k \) and \( \Gamma(C, \mathcal{O}_{\bar{C}}) = \Gamma(C, \mathcal{O}_{\bar{C}}/\mathcal{O}_C) = k \). Hence \( \Gamma(C, \mathcal{O}_{\bar{C}}/\mathcal{O}_C) = 0 \). Since \( \mathcal{O}_{\bar{C}}/\mathcal{O}_C \) is a coherent \( \mathcal{O}_C \)-module whose support is of dimension \( \leq 0 \), we have \( \mathcal{O}_{\bar{C}}/\mathcal{O}_C = 0 \), i.e., \( C = \bar{C} \) is normal. So we have \( C = P^1_k \).

Assume that (1) is valid for all integers \( < n \). Let \( C' \) be an irreducible component \( C \) and let \( C'' \) be the union of all other irreducible components. Regard \( C' \) and \( C'' \) as reduced closed subschemes of \( C \). Then there is a short exact sequence:

\[ 0 \to \mathcal{O}_C \xrightarrow{c\mapsto(e,\bar{e})} \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \xrightarrow{(a,b)\mapsto\bar{a}-\bar{b}} \mathcal{O}_{C'\cap C''} \to 0 , \]

which induces a long exact sequence:

\[ 0 \to \Gamma(C, \mathcal{O}_C) \to \Gamma(C, \mathcal{O}_{C'}) \oplus \Gamma(C, \mathcal{O}_{C''}) \to \Gamma(C, \mathcal{O}_{C'\cap C''}) \to H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_{C'}) \oplus H^1(C, \mathcal{O}_{C''}) \to 0 . \]

Let \( r \) and \( s \) be the numbers of connected components of \( C \) and \( C'' \) respectively; and let \( Z_1, Z_2, \ldots, Z_m \) be all connected components of \( C'' \) which intersect with \( C' \). Then \( m = s - r + 1 \). On the other hand, we have

\[ \dim_k \Gamma(C, \mathcal{O}_C) = r, \quad \dim_k \Gamma(C, \mathcal{O}_{C'}) = 1, \quad \dim_k \Gamma(C, \mathcal{O}_{C''}) = s . \]

As \( H^1(C, \mathcal{O}_C) = 0 \), we have \( H^1(C, \mathcal{O}_{C'}) = H^1(C, \mathcal{O}_{C''}) = 0 \), and

\[ \dim_k \Gamma(C, \mathcal{O}_{C'\cap C''}) = s - r + 1 = m . \]

This shows that for each \( i \), \( Z_i \cap C' \) contains only one point \( P_i \), and \( C' \cap C'' = \{P_1, P_2, \ldots, P_m\} \).

Using induction on \( Z_i \), \( C' \) and \( C'' \), we obtain that (1) is valid on all \( Z_i \), \( C' \) and \( C'' \). Applying Proposition 1.8, we obtain an isomorphism \( H_2(C') \oplus H_2(C'') \xrightarrow{\sim} H_2(C) \) and an exact sequence

\[ 0 \to H_1(C) \to H_0(C' \cap C'') \xrightarrow{\sim} H_0(C') \oplus H_0(C'') \to H_0(C) \to 0 . \]
Note that the map \( l \) has an inverse which is defined by the composition
\[
H_0(C') \oplus H_0(C'') \xrightarrow{pr} H_0(C'') \xrightarrow{pr} \bigoplus_{i=1}^{m} H_0(Z_i) \xrightarrow{\oplus \phi_i} \bigoplus_{i=1}^{m} H_0(P_i) = H_0(C' \cap C''),
\]
where the isomorphism \( \phi_i \), which maps \( cl_{Z_i}(P_i) \) to \( cl(P_i) \), is obtained by applying (a) on \( Z_i \). Therefore (1) are valid for \( C \).

(2) Let \( \pi : C \to \text{Spec} k \) be the structural morphism. Note that
\[
\text{Verifies (c) is also by [8, IX, 4.7].}\]

2.2 Remark
In other words, all entries of \( D \) are valid for \( \text{Verifies (1) are valid for} \).

\[
\text{where the isomorphism} \phi \text{ is defined by the composition.}
\]

\[
\text{Note that the map} \phi \text{ has an inverse which is defined by the composition}
\]

\[
\text{There is an exact sequence of} \Lambda \text{-modules:}
\]

\[
0 \to H_2(E) \to H^2(E, \Lambda(2)) \to H_2(X) \to H_1(E) \to 0,
\]

Moreover the kernel of the map \( H_2(X) \to H_1(E) \) is a finite group.

(4) There is an exact sequence of \( \Lambda \)-modules:
\[
0 \to H_1(X) \to H_0(E) \xrightarrow{\text{deg}} \Lambda \to 0.
\]
(5) $H_q(X) = 0$ for $q > 4$ or $q < 1$.

Proof. After applying Proposition 1.17 to the following Cartesian square,

$$
\begin{array}{ccc}
E' & \xrightarrow{i} & \tilde{X} \\
\tau \downarrow & & \downarrow \pi \\
P & \xrightarrow{i} & X
\end{array}
$$

(3.1)

we obtain a long exact sequence

$$
\cdots \to H_{n+1}(X) \to H_n(E) \to H_n(P) \oplus H_n(\tilde{X}) \to H_n(X) \to \cdots .
$$

(3.2)

For each $q \in \mathbb{Z}$, there is a canonical isomorphism of $\Lambda$-modules:

$$
H_q(\tilde{X}) \xrightarrow{\sim} H^{4-q}(\tilde{X}, \Lambda(2)) \xrightarrow{\sim} H^{4-q}(E, \Lambda(2)) ,
$$

where the first isomorphism is induced by Proposition 1.3 (since $\tilde{X}$ is regular), and the last isomorphism is from the base change theorem via the square (3.1). Thus $H_q(\tilde{X}) = 0$ for $q < 2$ and $q > 4$.

Also note that $H_q(E) = 0$ for $q < 0$ and $q > 2$; and $H_q(P) = \begin{cases} 0 & q \neq 0, \\ \Lambda & q = 0. \end{cases}$ Hence we may divide (3.2) to three exact sequences

1. $0 \to H^0(E, \Lambda(2)) \to H_4(X) \to 0$,

(3.3)

2. $0 \to H^1(E, \Lambda(2)) \to H_3(X) \to H_2(E) \to H^2(E, \Lambda(2)) \to H_2(X) \to H_1(E) \to 0$,

(3.4)

3. $0 \to H_1(X) \to H_0(E) \xrightarrow{\deg} \Lambda \to H_0(X) \to 0$.

(3.5)

Now (1) is by the exact sequence (3.3).

From the sequence (3.4), to prove (4) & (5) we have only to prove the map $H_0(E) \xrightarrow{\deg} \Lambda$ is epimorphic. Since $X$ is strictly local, the field $k := k(P)$ is separably closed. Let $Q$ be a closed point on $E$. Then $\deg cl_E(Q) = [k(Q), k]$. Put $p := \text{char } k$. Since $k$ is separably closed, the number $[k(Q), k]$ is either 1 or a power of $p$; in both cases, it is invertible on $\Lambda$ (as $\ell \neq p$). Hence $\deg$ is surjective.

From the sequence (3.4), to prove (2) & (3) we have only to prove that the homomorphism $H_2(E)(-1) \to H^2(E, \Lambda(1))$ is injective and its cokernel is a torsion group. Let $E_1, E_2, \ldots, E_n$ be all irreducible components of $E$. Then $H_2(E)(-1)$ is a free $\Lambda$-Module with a basis $\{cl_E(E_i)\}_{i=1}^n$.

Note that the composite map

$$
H_2(E)(-1) \to H_2(\tilde{X})(-1) \xrightarrow{\sim} H^2(\tilde{X}, \Lambda(1)) \xrightarrow{\sim} H^2(E, \Lambda(1)) \xrightarrow{\rho} \Lambda^\oplus n
$$

(3.6)

sends each $cl_E(E_i)$ to $\sum_{j=1}^n a_{ij} e_j$, where the isomorphism $\rho$ is defined in [8, IX, 4.7], $e_1, e_2, \ldots, e_n$ is the standard basis for $\Lambda^\oplus n$, and

$$
a_{ij} = \deg (\mathcal{L}_{\tilde{X}}(E_i)|_{E_j}) = (E_i, E_j),
$$

where $\mathcal{L}_{\tilde{X}}(E_i)$ means the invertible sheaf on $\tilde{X}$ associated to the divisor $E_i$. In other words, the map (3.6) is given by the intersection matrix $((E_i, E_j))$, which is negative-definite by [11, (14.1)]. Therefore the map (3.6) is injective with torsion cokernel.

\[\square\]

**Theorem 3.2.** Let the assumptions and the notations be as in Proposition 3.1. We further assume that $X$ has a rational singularity at $P$. Then we have
(1) Let $\text{Cl}(X)$ denote the Weil divisor class group of X. Then there is a canonical isomorphism

$$\text{cl} : \text{Cl}(X) \otimes_{\mathbb{Z}} \Lambda \overset{\sim}{\rightarrow} H_2(X)(-1)$$

which sends each $[Y] \otimes 1$ to $c_{1}(Y)$.

(2) The map $H^0(t_X) : \Lambda \overset{\sim}{\rightarrow} H_4(X)(-2)$ is an isomorphism.

(3) $H_q(X) = 0$ for $q \neq 2, 4$.

Proof. By [11, (4.1)], any desingularization of a surface with rational singularities is a product of quadratic transformations. Since $X$ is the spectrum of a local ring, the exceptional divisor $E$ is connected. Thus the map $\text{deg} : H_0(E) \rightarrow \Lambda$ is an isomorphism by Proposition 2.1 (1a). Then (2) & (3) are by Proposition 3.1 and Proposition 2.1. So we have only to prove (1).

We adopt the assumptions and notation in [11, §14]. Let $E_1, E_2, \ldots, E_n$ be all irreducible components of $E$. For each $i$, let $d_i > 0$ be the greatest common divisor of all the degrees of invertible sheaves on $E_i$. Put $k := k(P)$ and $p := \text{char} k$. For each $i$, let $Q_i$ be a closed point on $\text{Reg}(E_i)$. Then $Q_i$ defines an invertible sheaf of degree $[k(Q_i), k]$ on $E_i$. Since the number $[k(Q_i), k]$ is invertible on $\Lambda$, so is $d_i$.

Let $E$ be the additive group of divisors on $\bar{X}$ generated by $\{E_i\}_{i=1}^n$. Then $E$ is a free abelian group with basis $E_1, E_2, \ldots, E_n$. Put $E^* := \text{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$ and let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the dual basis. We define a homomorphism of groups:

$$\partial : E \rightarrow E^*, \quad E_i \mapsto \sum_{j=1}^n \frac{1}{d_j}(E_i, E_j) \cdot \varepsilon_j.$$  

By [11, (14.3), (14.4) & (17.1)], there is an exact sequence

$$0 \rightarrow E \overset{\partial}{\rightarrow} E^* \rightarrow \text{Cl}(X) \rightarrow 0.$$ 

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the basis for the free $\Lambda$-module $H^2(E, \Lambda(1))$ defined by the isomorphism in Proposition 2.1 (2c). Since each $d_j$ is invertible on $\Lambda$, the homomorphism

$$\delta : E^* \otimes_{\mathbb{Z}} \Lambda \rightarrow H^2(E, \Lambda(1)),$$ 

$$\varepsilon_j \otimes 1 \mapsto d_j \varepsilon_j$$

is an isomorphism of $\Lambda$-modules. It is easy to check that the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & E \otimes_{\mathbb{Z}} \Lambda & \longrightarrow & E^* \otimes_{\mathbb{Z}} \Lambda & \longrightarrow & \text{Cl}(X) \otimes_{\mathbb{Z}} \Lambda & \longrightarrow & 0 \\
\downarrow & & \downarrow & \delta \otimes \text{id} & \downarrow & \delta & \downarrow \text{cl} & \downarrow & 0 \\
0 & \longrightarrow & H_2(E)(-1) & \longrightarrow & H^2(E, \Lambda(1)) & \longrightarrow & H_2(X)(-1) & \longrightarrow & 0
\end{array}$$

is commutative with both rows exact, where the isomorphism (i) sends each $E_i \otimes 1$ to $c_{1}(E_i)$. Therefore $\text{cl} : \text{Cl}(X) \otimes_{\mathbb{Z}} \Lambda \overset{\sim}{\rightarrow} H_2(X)(-1)$ is an isomorphism. □

By [11, (17.1)], the divisor class group $\text{Cl}(X)$ is a finite group. So we have:

Corollary 3.3. Let $X$ be a surface having only finitely many singular points, all of which are rational singularities. Then $t_X : Q_\ell(2)[d] \overset{\sim}{\rightarrow} \mathcal{X}_X$ is an isomorphism in $D^b_c(X_{\text{et}}, Q_\ell)$. In particular, $Q_\ell$ is a dualizing complex on $X_{\text{et}}$ in coefficient $Q_\ell$.

Corollary 3.4 (Poincaré Duality). Let $k$ be a separably closed field on which $\ell$ is invertible. And let $X$ be a surface over $k$. Assume that $X$ has only finitely many singular points, all of which are rational singularities. Then

(1) The trace map $\text{Tr}_X : H^1(X, Q_\ell(2)) \overset{\sim}{\rightarrow} Q_\ell$ is an isomorphism.
(2) For every $0 \leq r \leq 4$ and every object $\mathcal{F}$ in $D^b_c(X_\text{ét}, \mathbb{Q}_\ell)$, the pairing
\[ H^r_c(X, \mathcal{F}) \times \text{Ext}^{4-r}_X(\mathcal{F}, \mathbb{Q}_\ell(2)) \to H^4_c(X, \mathbb{Q}_\ell(2)) \xrightarrow{\text{Tr}_X} \mathbb{Q}_\ell \]

is nondegenerate.

(3) For every $0 \leq r \leq 4$, the pairing
\[ H^r_c(X, \mathbb{Q}_\ell) \times H^{4-r}(X, \mathbb{Q}_\ell(2)) \to H^4_c(X, \mathbb{Q}_\ell(2)) \xrightarrow{\text{Tr}_X} \mathbb{Q}_\ell \]

is nondegenerate.

**Proof.** (3) is a special case of (2). Let $p: X \to \text{Spec} \ k$ be the structural morphism. By Corollary 3.3 and Proposition 1.4, we have isomorphisms
\[ \mathbb{Q}_\ell(2)[4] \xrightarrow{t_X} \mathcal{K}_X \xrightarrow{\sim} R^1p_!\mathbb{Q}_\ell. \]
Hence we have
\[ \text{Ext}^{4-r}_X(\mathcal{F}, \mathbb{Q}_\ell(2)) \xrightarrow{\sim} \text{Hom}_{D^b_c(X_\text{ét}, \mathbb{Q}_\ell)}(\mathcal{F}[r], \mathbb{Q}_\ell(2)[4]) \]
\[ \xrightarrow{\sim} \text{Hom}_{D^b_c(X_\text{ét}, \mathbb{Q}_\ell)}(\mathcal{F}[r], R^1p_!\mathbb{Q}_\ell) \]
\[ \xrightarrow{\sim} \text{Hom}_{D^b_c(\mathbb{Q}_\ell)}(R^1p_!\mathcal{F}[r], \mathbb{Q}_\ell) \]
\[ \xrightarrow{\sim} H^r_c(X, \mathcal{F})^\vee. \]
So (2) is proved. In above isomorphism, letting $r = 4$ and $\mathcal{F} = \mathbb{Q}_\ell$, we obtain
\[ H^4_c(X, \mathbb{Q}_\ell)^\vee \xrightarrow{\sim} H^0(X, \mathbb{Q}_\ell(2)) = \mathbb{Q}_\ell(2). \]
So (1) is proved. 

**Corollary 3.5.** Let $X$ be a surface having only finitely many singular points, all of which are rational singularities. Assume further that for any singular point $P$, $\mathcal{O}^{sh}_{X, P}$ is factorial. Then $t_X : \mathbb{Z}_\ell(2)[4] \xrightarrow{\sim} \mathcal{K}_X$ is an isomorphism in $D^b_c(X_\text{ét}, \mathbb{Z}_\ell)$. In particular, $\mathbb{Z}_\ell$ is a dualizing complex on $X_\text{ét}$ in coefficient $\mathbb{Z}_\ell$.

**Remark 3.6.** By [11, (17.2)], the ring $\mathcal{O}^{sh}_{X, P}$ is factorial if and only if its completion $\widehat{\mathcal{O}}^{sh}_{X, P}$ is factorial. See [11, §25] for the complete list of two-dimensional factorial complete local rings with rational singularities.

4. Applications

In this section, we study perverse sheaves and the weights of smooth sheaves on surfaces with rational singularities. First we briefly introduce the t-structures on arbitrary schemes.

Let $X$ be a scheme satisfying (†) equipped with a dimension function $\delta$. For each point $x$ on $X$, the canonical morphism $i_x : \text{Spec} \ k(x) \to X$ factor as
\[ \text{Spec} \ k(x) \xrightarrow{j} \{x\} \xrightarrow{i} X. \]
Then we define two functors:
\[ i_x^* := j^* \circ i^* : D^b_c(X_\text{ét}, \overline{\mathbb{Q}_\ell}) \to D^b_c(x_\text{ét}, \overline{\mathbb{Q}_\ell}), \]
\[ i_x^! := j^* \circ R\xi^! : D^b_c(X_\text{ét}, \overline{\mathbb{Q}_\ell}) \to D^b_c(x_\text{ét}, \overline{\mathbb{Q}_\ell}). \]
Now we define a pair of full subcategories $(P^{b, \leq 0}(X_\text{ét}, \overline{\mathbb{Q}_\ell}), P^{b, > 0}(X_\text{ét}, \overline{\mathbb{Q}_\ell}))$ of $D^b_c(X_\text{ét}, \overline{\mathbb{Q}_\ell})$ as following: for every object $\mathcal{F}$ in $D^b_c(X_\text{ét}, \overline{\mathbb{Q}_\ell})$,

1. $\mathcal{F} \in P^{b, \leq 0}(X_\text{ét}, \overline{\mathbb{Q}_\ell})$ if and only if for any point $x$ on $X$, $i_x^*\mathcal{F} \in D^{b, \leq \delta(x)}(x_\text{ét}, \overline{\mathbb{Q}_\ell})$;
2. $\mathcal{F} \in P^{b, > 0}(X_\text{ét}, \overline{\mathbb{Q}_\ell})$ if and only if for any point $x$ on $X$, $i_x^!\mathcal{F} \in D^{b, > \delta(x)}(x_\text{ét}, \overline{\mathbb{Q}_\ell})$. 

By [5] or [6], this pair defines a $t$-structure on $D_c^b(X_{\text{ét}}, \overline{\mathbb{Q}_\ell})$. If $f : X \to Y$ is a quasi-finite morphism, then the functors $Rf_!$ and $f^*$ are right $t$-exact, and the functors $Rf^!$ and $Rf_*$ are left $t$-exact. Thus we may define the functor $f_!$.

**Proposition 4.1.** Let $X$ be a surface having only finitely many singular points, all of which are rational singularities; we may define the functor $\mathcal{F} \to i_!^* \mathcal{F} [2]$. If $f : X \to Y$ is a quasi-finite morphism, then the functors $Rf_!$ and $f^*$ are right $t$-exact, and the functors $Rf^!$ and $Rf_*$ are left $t$-exact. Thus we may define the functor $f_!$.

**Definition 4.2.** Let $X$ be an algebraic scheme over $k$. A sheaf $\mathcal{F}$ on $X$ is said to be punctually pure of weight $w$ if for any $Q$-embedding $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \overline{\mathbb{Q}}$, we have $|\iota(\alpha)| = q^w$.

**Definition 4.3.** Let $X$ be an algebraic scheme over $k$.

1. A constructible $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on $X$ is said to be
   - punctually pure of weight $n$ if for every closed point $x$ on $X$, all eigenvalues of the geometric Frobenius $F_x$ acting on $\mathcal{F}_x$ are pure of weight $w$;
   - mixed (of weights $\leq w$), if there exists a finite filtration
     
     $0 = \mathcal{F}_x \subset \mathcal{F}_{x-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{F}$

     consisting of constructible $\overline{\mathbb{Q}_\ell}$-sheaves, such that each $\mathcal{F}_i/\mathcal{F}_{i+1}$ is punctually pure (of weights $\leq n$).

2. An object $\mathcal{F}$ in $D^b_c(X_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ is said to be
mixed of weights \( \leq w \) if for every \( n \in \mathbb{Z} \), \( H^n(F) \) is mixed of weights \( \leq n + w \);
- mixed of weights \( \geq w \) if \( D(F) = R \mathcal{H}om(F, \mathcal{O}_X) \) is mixed of weights \( \leq -w \);
- pure of weight \( w \) if it is mixed of weights \( \leq w \) and \( \geq w \).

**Proposition 4.4.** Let \( X \) be a complete surface over \( k \), \( F \) a smooth \( \mathbb{Q}_\ell \)-sheaf on \( X_\text{ét} \). Assume that \( X \) has only finitely many singular points, all of which are rational singularities; and \( F \) is punctually pure of weight \( w \). Then for every \( n \in \mathbb{N} \), the \( \mathbb{Q}_\ell \)-module \( H^n(X_\text{ét}, F) \) is punctually pure of weight \( n + w \).

**Proof.** Put \( U := X_{\text{reg}} \) and let \( j: U \hookrightarrow X \) be the inclusion. Since \( U \) is smooth and \( j^* F \) is a smooth \( \mathbb{Q}_\ell \)-sheaf, \( j^* F[2] \) is pure of weight \( w + 2 \). By Proposition 4.1, \( F[2] \in \text{Perv}(X, \mathbb{Q}_\ell) \) and \( j_! j^* F[2] \cong F[2] \). So after applying [2, Corollaire 5.4.3], we obtain that \( j_! j^* F[2] \cong F[2] \) is pure of weight \( w + 2 \). Let \( \pi: X \to \text{Spec} k \) be the structural morphism. By [3, (3.3.1) & (6.2.3)], the functor \( R\pi_* = R\pi_! \) sends perverse sheaves pure of weight \( w + 2 \) on \( X \) to perverse sheaves pure of weight \( w + 2 \) on \( \text{Spec} k \). Hence we prove the proposition. \( \square \)

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