THE BAR INVOLUPTION FOR QUANTUM SYMMETRIC PAIRS
– HIDDEN IN Plain SIGHT

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To Jasper Stokman on his 50th birthday in gratitude and friendship

ABSTRACT. We show that all quantum symmetric pair coideal subalgebras $B_c$ of Kac-Moody type have a bar involution for a suitable choice of parameters $c$. The proof relies on a generalized notion of quasi $K$-matrix. The proof does not involve an explicit presentation of $B_c$ in terms of generators and relations.

1. Introduction

Let $g$ be a symmetrizable Kac-Moody algebra and $\theta : g \to g$ an involutive Lie algebra automorphism of the second kind. The theory of quantum symmetric pairs, as developed by G. Letzter in the finite setting [Let99], studies certain coideal subalgebras $B_c$ of the quantized enveloping algebra $U = U_q(g)$. The coideal subalgebras $B_c$ depend on a family of parameters $c$. They are quantum group analogs of $U(g^\theta)$ where $g^\theta = \{ x \in g | \theta(x) = x \}$.

In March 2012, C. Stroppel asked the author if there exists a bar involution for $B_c$. She conjectured that $B_c$ should allow an algebra automorphism $-B : B_c \to B_c$ which is the identity on the Letzter generators $B_i \in B_c$ and maps $q$ to $q^{-1}$. Unfortunately, back in 2012, this author did not sufficiently appreciate the question.

This changed drastically with the appearance of the preprint versions of [BW18] and [ES18] in October 2013. Both these papers contained a bar involution for an explicit example of $B_c$ of type AIII. For this example, H. Bao and W. Wang used the bar involution to construct an element $X$ (denoted by $\Upsilon$ in [BW18]), now called the quasi $K$-matrix for $B_c$, which intertwines the Lusztig bar involution $-B$ for $U$ and the new bar involution $-B$ for $B_c$ as follows

$$b^B X = X b^B \quad \text{for all } b \in B_c.$$ (1.1)

Moreover, Bao and Wang used the quasi $K$-matrix to construct a natural family of $B_c$-module automorphisms $\mathcal{F}$ on the category of finite-dimensional $U$-modules. At this point it became clear that the family $\mathcal{F}$ was a hot candidate for a universal $K$-matrix which the author had been chasing previously in joint work with J. Stokman [KS09]. Subsequently, this was worked out jointly with M. Balagović [BK19].

The line of argument, first proposed in [BW18] and then performed in [BK15], [BK19] in a general Kac-Moody setting, was as follows. First the existence of a bar involution on $B_c$ was established via an explicit presentation of $B_c$ in terms of

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generators and relations. Then it was proved that the defining relation (1.1) of the quasi $K$-matrix has an essentially unique solution.

Writing down $B_c$ in terms of generators and relations is a difficult problem in the general Kac-Moody setting. The quest for the bar involution for $B_c$ was often cited as an important motivation to find such a presentation of $B_c$, see for example the introductions of the papers [CLW20], [dC19]. However, G. Lusztig asked the author in January 2019 if there is a construction of the bar involution for $B_c$ which does not rely on a presentation of $B_c$ in terms of generators and relations. Lusztig pointed out that the construction of the bar involution for the positive part $U^+$ in [Lus94, 1.2.12] does not involve explicit knowledge of the quantum Serre relations for $U$.

The quasi $K$-matrix is the quantum symmetric pair analog of the quasi $R$-matrix for $U$. The quasi $R$-matrix for $U$ has a description which does not involve Lusztig’s bar involution, see [Lus94, Theorem 4.1.2 (b)]. Hence it is natural to ask for a bar involution free description of the quasi $K$-matrix. Such a construction was given in the quasi-split case (and in a more general setting of Nichols algebras of diagonal type) in [KY20]. For suitable parameters $c$ there exists an element $X$ satisfying a relation similar to (1.1), and the map

$$b \mapsto \overline{X}bX^{-1} \quad \text{for all } b \in B_c$$

is a bar involution on $B_c$. The element $X$ in [KY20] is given by $X = (\varepsilon \otimes \text{id})(\Theta^q)$ where $\varepsilon$ is the counit and the element $\Theta^q$ is defined by [KY20] (6.1)], see also [KY20, Remark 6.3].

Along a slightly different line, A. Appel and B. Vlaar observed in [AV20] that the construction of the quasi $K$-matrix in [BK19] can be reformulated without the bar involution. Recall that $B_c$ is defined for a (generalized) Satake diagram $(X, \tau)$, where $X$ is a subset of the nodes $I$ of the Dynkin diagram of $g$, and $\tau : I \to I$ is an involutive diagram automorphism. The pair $(X, \tau)$ needs to satisfy certain compatibility conditions, see Section 2.2. The Letzter generators of $B_c$ are given by $B_i = F_i$ for $i \in X$ and $B_i = F_i - c_i T_{wx}(E_{\tau(i)}) K_i^{-1}$ for $i \in I \setminus X$, see Section 3.1. With this notation, condition (3.9) is equivalent to the condition

$$B_iX = X\overline{B'_i} \quad \text{for all } i \in I$$

where $B_i$ are the Letzter generators corresponding to the parameter family $c$, and $B'_i$ are Letzter generators corresponding to a different parameter family $c'$. One then obtains an algebra isomorphism $\Psi : B_{c'} \to B_c$ defined by $\Psi(b) = \overline{X}bX^{-1}$. This isomorphism satisfies $\Psi(B'_i) = B_i$ and $\Psi(q) = q^{-1}$.

In [AV20] Appel and Vlaar are mostly concerned with the bar involution free formulation of the quasi $K$-matrix $X$. The main point of the present note, which we still believe has novelty to it, is the observation that the above argument provides a proof of the existence of the bar involution for all symmetrizable Kac-Moody algebras and all (generalized) Satake diagrams if the parameters are chosen such that $c' = c$, see Corollary 4.2. Although Appel and Vlaar get very close to this observation, they stop just short of the formulation of the existence of the bar involution in [AV20, Remark 7.2]. In particular, with the new perspective on the quasi $K$-matrix, the study of the quantum Serre relations for $B_c$ mentioned in [AV20, Remark 7.2] is not necessary anymore.
This note is organised as follows. In Section 2 we fix notation for quantized enveloping algebras and recall the notion of a generalized Satake diagram as formulated in [RV20] which generalizes the notion of an admissible pair in [Kol14, Definition 2.3]. We promote the view that future papers on quantum symmetric pairs should be written in the setting of generalized Satake diagrams. In Section 3 we recall the definition of $\mathcal{B}_k$ and reprore the existence of the quasi $K$-matrix $X$. While this result is contained in [AV20, Theorem 7.4], we feel that it is worthwhile to reproduce it in the precise notations and setting of [BK19]. Section 4 contains the existence of the isomorphism $\Psi$. As a consequence we obtain the main message of this note, namely the general, relation-free existence proof for the bar involution on $\mathcal{B}_k$, see Corollary 4.2.

2. Preliminaries

2.1. Quantum groups. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with general-ized Cartan matrix $(a_{ij})_{i,j \in I}$ where $I$ is a finite set and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\{d_i | i \in I\}$ be a set of relatively prime positive integers such that the matrix $(d_i a_{ij})$ is symmetric. Let $\Pi = \{\alpha_i | i \in I\}$ be the set of simple roots for $\mathfrak{g}$, let $\Phi$ be the root system, and let $Q = \mathbb{Z}I$ be the root lattice. For $\beta = \sum_{i \in I} n_i \alpha_i \in Q$ we write $\text{ht}(\beta) = \sum_{i \in I} n_i$. Consider the symmetric bilinear form $(\cdot, \cdot) : Q \times Q \to \mathbb{Z}$ defined by $(\alpha_i, \alpha_j) = d_i a_{ij}$ for all $i, j \in I$. We denote $Q^+ = \mathbb{N}_0 \Pi$ where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Let $W$ be the Weyl group of $\mathfrak{g}$ which is generated by the simple reflections $s_i$ for $i \in I$.

Throughout this paper let $k$ be a field of characteristic zero and let $\mathbb{K} = k(q)$ be the field of rational functions in a variable $q$. We consider the quantized enveloping algebra $U = U_q(\mathfrak{g})$ as the $\mathbb{K}$-algebra with generators $E_i, F_i, K_i^{\pm 1}$ for $i \in I$ subject to the relations given in [Lus94, 3.1.1]. For $\beta = \sum_{i \in I} n_i \alpha_i \in Q$ we write $K_\beta = \prod_{i \in I} K_i^{n_i}$. Let $U^+$ and $U^-$ denote the subalgebras of $U$ generated by $\{E_i | i \in I\}$ and $\{F_i | i \in I\}$, respectively. For any $\mu \in Q^+$ we write $U^+_{\mu} = \text{span}_K \{E_i \ldots E_i | \sum_{j=1}^n \alpha_i = \mu\}$ where $\text{span}_K$ denotes the $K$-linear span. Moreover, define $U^-_{-\mu} = \omega(U^+_{\mu})$ where $\omega : U \to U$ denotes the algebra automorphism given in [Lus94, 3.1.3]. For any $i \in I$ let $T_i : U \to U$ denote the algebra automorphism denoted by $T_1, \alpha_i$ in [Lus94, 37.1]. By [Lus94, 39.4.3] the automorphisms $T_i$ satisfy braid relations. Hence, for $w = s_{i_1} \cdots s_{i_t}$ there is a well-defined automorphism $T_w = T_{i_t} \cdots T_{i_1}$ of $U$. For any $i \in I$ let $r_i, i^* : U^+ \to U^+$ denote the Lusztig-Kashiwara skew-derivations which are defined uniquely by the relation

$$x F_i - F_i x = \frac{1}{q_i - q_i^{-1}} (r_i(x) K_i - K_i^{-1} r_i(x)) \quad \text{for all } x \in U^+$$

where $q_i = q^{d_i}$, see [Lus94, 3.1.6]. The maps $r_i$ and $i^*$ satisfy the relations $r_i(E_j) = i^*(E_j) = \delta_{i,j}$ and the skew-derivation properties

$$r_i(x y) = q^{\alpha_i(x)} r_i(x) y + x r_i(y), \quad i^*(x y) = i^*(x) y + q^{\alpha_i(n)} x i^*(y)$$

for all $x \in U^+_{\mu}$, $y \in U^+_{\nu}$.

Recall that the bar involution for $U$ is the $K$-algebra automorphism $\overline{\cdot} : U \to U$, $x \mapsto \overline{x}$ defined by

$$\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{K_i} = K_i^{-1}, \quad \overline{q} = q^{-1}.$$
for all $i \in I$.

To formulate the properties of the quasi $K$-matrix we need to work in an over-
algebra $\mathcal{W}$ of $U$. Let $\mathcal{O}_{\text{int}}$ denote the category of integrable $U$-modules in category
$\mathcal{O}$ and let $\text{For} : \mathcal{O}_{\text{int}} \to \text{Vect}$ be the forgetful functor into the category of $\mathbb{K}$-vector
spaces. We define $\mathcal{W} = \text{End}(\text{For})$ to be the set of natural transformations from $\text{For}$ to itself. By construction, $\mathcal{W}$ is an algebra which contains $U$ as a subalgebra. See [BK19 Section 3] for precise definitions and further details. Consider the elements
of the product $\hat{U}^+ = \prod_{\mu \in Q^+} U^+_i$ as additive infinite sums of elements in $U^+$. Then
$\hat{U}^+$ is a subalgebra of $\mathcal{W}$, see [BK19 Example 3.2].

2.2. Generalized Satake diagrams. In [Kol14] quantum symmetric pairs were defined
for involutive automorphisms of $\mathfrak{g}$ of the second kind. Such involutions are
classified in terms of Satake diagrams $(X, \tau)$ where $X \subset I$ is a subset of finite type
and $\tau : I \to I$ is an involutive diagram automorphism satisfying the compatibility
conditions given in [Kol14 Definition 2.3]. Associated to the finite type subset $X$ is
a finite type root system $\Phi_X \subset \Phi$ with parabolic Weyl group $W_X \subset W$. Let $w_X \in W_X$ be the
longest element and let $\rho_X^\vee \in \mathfrak{h}$ be the half sum of positive coroots of $\Phi_X$. To be a Satake diagram, the pair $(X, \tau)$ has to satisfy the two conditions

1. $\tau(\alpha_j) = -w_X(\alpha_j)$ for all $j \in X$;
2. If $i \in I \setminus X$ and $\tau(i) = i$ then $\alpha_i(q_X^\vee) \in \mathbb{Z}$.

Condition 1 implies that the map $\Theta = -w_X \circ \tau : Q \to Q$ is an involutive auto-
morphism of the root lattice. It was observed by V. Regelskis and B. Vlaar that
condition 2 can be replaced by the weaker condition

2'. If $\tau(i) = i$ and $\alpha_{ji} = -1$ then $\Theta(\alpha_i) \neq -\alpha_i - \alpha_j$.

Following [RV20], we call pairs $(X, \tau)$ as above, satisfying conditions 1, 2' general-
ized Satake diagrams. As explained in [RV20 Section 4] the construction of quantum
symmetric pairs and the structure theory developed in [Kol14] remains valid for generalized Satake diagrams.

The diagram automorphism $\tau$ lifts to an Hopf-algebra automorphism of $U$. As in
[Lus94 3.1.3] let $\sigma : U \to U$ denote the $\mathbb{K}$-algebra antiautomorphism defined by

$$\sigma(E_i) = E_i, \quad \sigma(F_i) = F_i, \quad \sigma(K_i) = K_i^{-1}$$

for all $i \in I$.

The following proposition was conjectured in [BK15 Conjecture 2.7] and was proved in
[BK15 Proposition 2.5] up to a sign. The sign was confirmed by H. Bao and
W. Wang in [BW21] by a subtle argument involving canonical bases for $U$. The
arguments in [BK15, BW21] do not involve condition (2) above, and hence also
hold in the setting of generalized Satake diagrams, see also [AV20 Lemma 7.14].

**Proposition 2.1.** Let $(X, \tau)$ be a generalized Satake diagram for the symmetrizable
Kac-Moody algebra $\mathfrak{g}$. Then the relation

$$\sigma \circ \tau r_i(T_{w_X}(E_i)) = r_i(T_{w_X}(E_i))$$

holds in $U$ for all $i \in I \setminus X$.

3. The quasi $K$-matrix, revisited

3.1. Quantum symmetric pairs. Let $(X, \tau)$ be an generalized Satake diagram. Set $M_X = \mathbb{K}(E_j, F_j, K_j^{\pm 1} \mid j \in X)$ and set $U_0^\beta = \mathbb{K}(K_\beta \mid \beta \in Q, \Theta(\beta) = \beta)$. Define
a set of parameters

\[ C = \{ c = (c_i)_{i \in I \setminus X} \in (K^\times)^{I \setminus X} \mid c_i = c_{\tau(i)} \text{ if } (\alpha_i, \Theta(\alpha_i)) = 0 \}. \]

For any \( c = (c_i)_{i \in I \setminus X} \in C \) we consider the subalgebra \( B_c \) of \( U_q(g) \) generated by \( M_X, U_0^0 \) and the elements

\[ B_i = F_i - c_i T_{wx}(E_\tau(i)) K_i^{-1} \text{ for all } i \in I \setminus X. \]

To unify notation, we set \( c_i = 0 \) for \( i \in X \) and extend (3.1) by writing \( B_i = F_i \) if \( i \in X \). Following [Let99, Kol14] we call \( B_c \) the quantum symmetric pair coideal subalgebra of \( U_q(g) \) corresponding to the generalized Satake diagram \( (X, \tau) \).

**Remark 3.1.** The parameters \( c_i \) were denoted by \( c_i s(\tau(i)) \) in [BK19]. The additional parameters \( s(\tau(i)) \) appeared in the construction of involutive automorphisms of \( g \) of the second kind corresponding to the Satake diagram \( (X, \tau) \). For the construction of quantum symmetric pairs, however, it is advantageous to suppress the parameters \( s(\tau(i)) \) in the notation.

**Remark 3.2.** In [Kol14], quantum symmetric pairs depend on a second family of parameters \( s \) in a certain subset \( S \subseteq K^\times \). The corresponding coideal subalgebras are then denoted by \( B_{c,s} \). By [Kol14, Theorem 7.1] the algebra \( B_c \) is isomorphic to \( B_{c,s} \) as an algebra for all \( s \). As we only aim to establish the existence of a bar involution for \( B_{c,s} \), it suffices to consider the case where \( s = (0, 0, \ldots, 0) \). Moreover, it was noted in [BK19, 3.5], that for the construction of the quasi \( K \)-matrix there is no loss of generality in the restriction to the case \( s = (0, 0, \ldots, 0) \).

### 3.2. The quasi \( K \)-matrix \( \mathfrak{X} \)

In [AV20, Section 7] A. Appel and V. Vlaar gave a reformulation of the results of [BK19, Section 6] which does not rely on the existence of a bar involution for \( B_c \). As notations and conventions in [AV20] somewhat differ from those of the present note, we feel it is beneficial to recall this reformulation in the present setting. We begin with a reformulation of [BK19, Proposition 6.1]. Let \( 2p_{\rho_X} \) be the sum of positive roots in \( \Phi_X \).

**Proposition 3.3.** Let \( \mathfrak{X} = \sum_{\mu \in Q^+} \mathfrak{X}_\mu \in \tilde{U}^+ \) with \( \mathfrak{X}_\mu \in U_\mu^+ \). The following are equivalent:

1. For all \( i \in I \) the relation

\[ B_i \mathfrak{X} = \mathfrak{X} F_i - (-1)^{2\alpha_i(\rho_X)} q^{-\alpha_i(\Theta(\alpha_i)) - 2p_{\rho_X}} C_{\tau(i)} T_{wx}(E_\tau(i)) K_i \]

holds in \( \mathfrak{U} \).

2. The element \( \mathfrak{X} \) satisfies the relations

\[ r_i(\mathfrak{X}_\mu) = (q_{i^{-1}} - q_i)^{2\alpha_i(\rho_X)} q^{-\alpha_i(\Theta(\alpha_i)) - 2p_{\rho_X}} C_{\tau(i)} X_{\mu + \Theta(\alpha_i) - \alpha} T_{wx}(E_{\tau(i)}) K_i \]

and

\[ i^r(\mathfrak{X}_\mu) = (q_{i^{-1}} - q_i)^{\alpha_i(\Theta(\alpha_i))} c_i T_{wx}(E_{\tau(i)}) \mathfrak{X}_{\mu + \Theta(\alpha_i) - \alpha} \]

for all \( i \in I \).

If the equivalent conditions (1) and (2) hold, then the following also hold:

3. For all \( x \in U_0^0 M_X \) one has \( x \mathfrak{X} = \mathfrak{X} x \).

4. For all \( \mu \in Q^+ \) such that \( \mathfrak{X}_\mu \neq 0 \) one has \( \Theta(\mu) = -\mu \).

**Proof.** The equivalence of (1) and (2) follows from the property (2.1) of the skew derivations \( r_i \) and \( r^r \). Property (4) follows by induction over the height of \( \mu \) and in turn implies property (3) just as in the proof of [BK19, Proposition 6.1]. □
Next we need \([BK19]\) Proposition 6.3. This proposition is a general statement about solving recursive relations of the type \((3.2), \ (3.3)\) simultaneously, and hence holds in our setting. Let \(\langle \, , \rangle : U^- \otimes U^+ \to K\) be the nondegenerate pairing considered in \([BK19]\) Section 2.3.

**Proposition 3.4.** (\([BK19]\) Proposition 6.3) Let \(\mu \in Q^+\) with \(ht(\mu) \geq 2\) and fix elements \(A_i, \ iA \in U^-_{\mu-\alpha_i}\) for all \(i \in I\). Then the following are equivalent:

1. There exists an element \(\underline{X} \in U^+_{\mu}\) such that

   \[
   r_i(\underline{X}) = A_i \quad \text{and} \quad i r(\underline{X}) = i A \quad \text{for all} \quad i \in I.
   \]

2. The elements \(A_i, \ iA\) have the following two properties:
   
   a. For all \(i, j \in I\) one has
   
   \[
   r_i(j A) = j r(A_i).
   \]
   
   b. For all \(i \neq j \in I\) one has
   
   \[
   -\frac{1}{q_i - q_j^{-1}} \sum_{s=1}^{1-a_{ij}} \left[ 1 - \frac{a_{ij}}{s} \right] \cdot \left( -1 \right)^s \langle F_i^{1-a_{ij}} F_j \rangle^{-s} A_j
   \]
   
   \[
   - \frac{1}{q_j - q_i^{-1}} \langle F_j^{1-a_{ij}}, A_j \rangle = 0.
   \]

Moreover, if the system of equation \((3.4)\) has a solution \(\underline{X}\) then this solution is uniquely determined.

We now translate \([BK19]\) Section 6.4 into our setting. This is the crucial step where we also need Proposition 2.21. We hence give all the details. Fix \(\mu \in Q^+\) and assume that a collection \((X_{\mu'})_{\mu' < \mu}\) with \(X_{\mu'} \in U^+_{\mu'}\) and \(X_0 = 1\) has already been constructed, and that this collection satisfies the relations \((3.2), \ (3.3)\) for all \(\mu' < \mu\) and all \(i \in I\). Define

\[
A_i = (q_i - q_i^{-1})(-1)^{2\alpha_i(\rho_X)} q^{-(\alpha_i, \Theta(\alpha_i)-2\rho_X)} c_{r(i)} X_{\mu' + \Theta(\alpha_i)-\alpha_i} T_{wx}(E_{r(i)}) X_{\mu + \Theta(\alpha_i)-\alpha_i}
\]

\[
i A = (q_i - q_i^{-1})^{-1}(q_i - q_i^{-1})(-1)^{2\alpha_i(\rho_X)} q^{-(\alpha_i, \Theta(\alpha_i)-2\rho_X)} c_{r(i)} c_j c_{r(i)} c_{r(i)} T_{wx}(E_{r(i)}) X_{\mu + \Theta(\alpha_i)-\alpha_i}.
\]

We have the following analog of \([BK19]\) Lemma 6.7.

**Lemma 3.5.** The relation \(r_i(j A) = j r(A_i)\) holds for all \(i, j \in I\).

**Proof.** Using the skew-derivation properties \((2.2)\) and the assumptions on \(iA, A_i\) we calculate

\[
r_i(j A) = (q_j - q_j^{-1}) q^{-(\Theta(\alpha_i), \alpha_j)} c_j T_{wx}(E_{r(j)}) r_i(X_{\mu' + \Theta(\alpha_i)-\alpha_i})
\]

\[
+ (q_j - q_j^{-1}) q^{-(\Theta(\alpha_i), \alpha_j)} c_j q^{-(\alpha_i, \mu + \Theta(\alpha_i)-\alpha_j)} r_i(T_{wx}(E_{r(j)})) X_{\mu' + \Theta(\alpha_i)-\alpha_i}
\]

\[
= (q_j - q_j^{-1})(q_i - q_i^{-1})(-1)^{2\alpha_i(\rho_X)} q^{-(\alpha_i, \Theta(\alpha_i)-2\rho_X)} c_j c_{r(i)}
\]

\[
\cdot T_{wx}(E_{r(j)}) X_{\mu' + \Theta(\alpha_i)-\alpha_i} + \Theta(\alpha_i)-\alpha_i, T_{wx}(E_{r(j)})
\]

\[
+ (q_j - q_j^{-1}) q^{-(\alpha_i, \alpha_j)} c_j q^{-(\alpha_i, \mu + \Theta(\alpha_i)-\alpha_j)} r_i(T_{wx}(E_{r(j)})) X_{\mu' + \Theta(\alpha_i)-\alpha_i}
\]

\[
\cdot c_{r(i)} c_{r(i)} T_{wx}(E_{r(i)}) X_{\mu + \Theta(\alpha_i)-\alpha_i}.
\]
and similarly
\[ j^r(A_i) = (q_i - q_i^{-1})(-1)^{2\alpha_i(p^\vee)} q^{-((\alpha_i, \Theta(\alpha_i)) - 2p\chi)} c_{r(i)} j^{r}(\mathbf{x}_{\mu + \Theta(\alpha_i) - \alpha_i}) T_{w_X}(E_{r(i)}) \]
\[ + (q_i - q_i^{-1})(-1)^{2\alpha_i(p^\vee)} q^{((\alpha_i, \mu + \Theta(\alpha_i) - \alpha_i) - (\alpha_i, \Theta(\alpha_i) - 2p\chi)) \times} \]
\[ \times c_{r(i)} x_{\mu + \Theta(\alpha_i) - \alpha_i} j^{r}(T_{w_X}(E_{r(i)})) \]
\[ = (q_i - q_i^{-1})(q_j - q_j^{-1})(-1)^{2\alpha_i(p^\vee)} q^{-((\alpha_i, \Theta(\alpha_i)) - 2p\chi)} q^{-(\Theta(\alpha_i), \alpha_j)} c_{r(i)} c_{r(j)} \times} \]
\[ \times T_{w_X}(E_{r(j)}) \mathbf{x}_{\mu + \Theta(\alpha_i) - \alpha_i + \Theta(\alpha_j) - \alpha_j} T_{w_X}(E_{r(i)}) \]
\[ + (q_i - q_i^{-1})(-1)^{2\alpha_i(p^\vee)} q^{((\alpha_i, \mu + \Theta(\alpha_i) - \alpha_i) - (\alpha_i, \Theta(\alpha_i) - 2p\chi)) \times} \]
\[ \times c_{r(i)} x_{\mu + \Theta(\alpha_i) - \alpha_i} j^{r}(T_{w_X}(E_{r(i)})). \]
Comparing the above two expressions we see that the relation \( r_i(j^r A) = j^r(A_i) \) is equivalent to the relation
\[ (3.8) \quad (q_j - q_j^{-1})q^{-(\Theta(\alpha_i), \alpha_j)} c_j q^{((\alpha_i, \mu + \Theta(\alpha_i) - \alpha_j) - r_i(T_{w_X}(E_{r(j)}))) \mathbf{x}_{\mu + \Theta(\alpha_i) - \alpha_j} \times} \]
\[ \times c_{r(i)} x_{\mu + \Theta(\alpha_i) - \alpha_i} j^{r}(T_{w_X}(E_{r(i)})). \]
By [Lus94] 1.2.14 we have \( j^r(T_{w_X}(E_{r(i)})) = q^{((\alpha_i, \mu + \Theta(\alpha_i) - \alpha_j) - r_i(T_{w_X}(E_{r(i)}))). \) Hence both sides of (3.8) vanish unless \( \tau(i) = j \in I \setminus X \), and in this case (3.8) can be rewritten as
\[ q^{((\alpha_i, \Theta(\alpha_i) - \alpha_j) - (\alpha_i, \mu + \Theta(\alpha_i) - \alpha_j)) + (\alpha_i, \mu) r_i(T_{w_X}(E_{r(i)}))) \mathbf{x}_{\mu + \Theta(\alpha_i) - \alpha_j} \]
\[ = (-1)^{2\alpha_i(p^\vee)} q^{((\alpha_i, \mu + \alpha_i - \alpha_j) - (\alpha_i, \Theta(\alpha_i) - 2p\chi)) \times} \mathbf{x}_{\mu + \Theta(\alpha_i) - \alpha_j} j^{r}(T_{w_X}(E_{r(i)})). \]
From now on we assume \( \tau(i) = j \) and hence \( \Theta(\alpha_i) = \alpha_i = \Theta(\alpha_j) = \alpha_j \). Hence we may assume that \( \mathbf{x}_{\mu + \Theta(\alpha_i) - \alpha_j} \neq 0 \) and in this case the above equation is equivalent to the equation
\[ q^{((\alpha_i, \Theta(\alpha_i) - \alpha_j)) r_i(T_{w_X}(E_{r(i)})) = (-1)^{2\alpha_i(p^\vee)} q^{((\alpha_j, \mu - \alpha_j) - (\alpha_j, \Theta(\alpha_j) - 2p\chi)) r_{r(i)}(T_{w_X}(E_{r(i))}). \]
By Proposition (3.5) (4) we have \( (\alpha_i - \alpha_j, \mu) = 0 \) and hence the above equation can be rewritten as
\[ r_i(T_{w_X}(E_{r(i)})) = (-1)^{2\alpha_i(p^\vee)} q^{((\alpha_i, \alpha_i - w_X(\alpha_i) - 2p\chi)) r_{r(i)}(T_{w_X}(E_{r(i)})). \]
By [BK15] Lemma 2.9, which also holds for generalized Satake diagrams, and by Proposition (2.1) the above equation does indeed hold. \( \square \)

Next we obtain an analog of [BK19] Lemma 6.8. We include the proof which is simplified along the lines of [BK19] Remark 6.9.

**Lemma 3.6.** For all \( i \neq j \in I \) the elements \( A_i, A_j \) given by (3.6) satisfy the relation (3.5).

**Proof.** As \( A_i \in U^{+}_{\mu, \alpha} \), we may assume that \( \mu = (1 - a_{ij})\alpha_i + \alpha_j \) and \( \Theta(\mu) = -\mu \) as otherwise both terms of (3.5) vanish. By [BK19] Lemma 6.4 it suffices to consider the following two cases:

**Case 1:** \( \Theta(\alpha_i) = -\alpha_j \) and \( a_{ij} = 0 \). In this case \( \mu = \alpha_i + \alpha_j \) and \( (\alpha_i, \alpha_k) = (\alpha_j, \alpha_k) = 0 \) for all \( k \in X \), and hence
\[ A_i = (q_i - q_i^{-1})c_j E_j, \quad A_j = (q_i - q_i^{-1})c_i E_i. \]
Hence the left hand side of (3.5) vanishes.
Case 2: \( \Theta(\alpha_i) = -\alpha_i \) and \( \Theta(\alpha_j) = -\alpha_j \). In this case induction on \( \mu' < \mu \) shows that \( \bar{X}_{\mu'} \neq 0 \) implies that \( \mu' \in \text{span}_{\mathbb{N}_0} \{2\alpha_i, 2\alpha_j\} \), see [BK19 Lemma 6.5]. Hence for \( \mu = (1 - \alpha_j)\alpha_i + \alpha_j \) we have \( \bar{X}_{\mu + \Theta(\alpha_i) - \alpha_i} = \bar{X}_{\mu - 2\alpha_i} = 0 \) and similarly \( \bar{X}_{\mu + \Theta(\alpha_j) - \alpha_j} = 0 \). Hence \( A_i = A_j = 0 \) which implies (3.5) also in this case. \( \Box \)

With the above preparations the following theorem is proved just as in [BK19 Theorem 6.10] using Proposition 3.4.

**Theorem 3.7.** There exists a uniquely determined element \( \bar{X} = \sum_{\mu \in Q^+} \bar{X}_{\mu} \in \widehat{U}^+ \) with \( \bar{X}_0 = 1 \) and \( \bar{X}_{\mu} \in U^+_{\mu} \) such that the equality
\[
(3.9) \quad B_i \bar{X} = \bar{X}(F_i - (-1)^{2\alpha_i(q^\rho)}q^{-(\alpha_i, \Theta(\alpha_i) - 2\rho X)})c_{\rho X}T_{\mu X}(E_{\tau(i)}K_i)
\]
holds in \( \mathcal{U} \) for all \( i \in I \). Moreover, the element \( \bar{X} \) commutes with all elements of \( U_0^+M_+X \).

**Remark 3.8.** We call the element \( \bar{X} \in \mathcal{U} \) from the above theorem the quasi \( K \)-matrix corresponding to the quantum symmetric pair coideal subalgebra \( \mathcal{B}_c \). The quasi \( K \)-matrix is invertible in \( \widehat{U}^+ = \prod_{\mu \in Q^+} U^+_{\mu} \) because \( \bar{X}_0 = 1 \neq 0 \).

4. **The bar involution for quantum symmetric pairs, revisited**

Recall that \( \mathbb{K} = k(q) \). Following [AV20 Section 7.3] for any \( c = (c_i)_{i \in I \setminus X} \in \mathbb{K}^{I \setminus X} \) we define \( c' = (c'_i)_{i \in I \setminus X} \in \mathbb{K}^{I \setminus X} \) by
\[
c'_i = (-1)^{2\alpha_i(q^\rho)}q^{-(\alpha_i, \Theta(\alpha_i) - 2\rho X)}c_{\rho X}T_{\mu X}(E_{\tau(i)}K_i).
\]
Observe that \( c \in \mathcal{C} \) if and only if \( c' \in \mathcal{C} \). For \( c \in \mathcal{C} \) let \( \mathcal{B}_{c'} \) denote the quantum symmetric pair coideal subalgebra corresponding to the parameters \( c' \). We denote the generators (3.1) for \( \mathcal{B}_{c'} \) by \( B'_i \) to distinguish them from the corresponding generators \( B_i \) of \( \mathcal{B}_c \). We immediately obtain the following consequence of Theorem 3.7.

**Corollary 4.1.** For any \( c \in \mathcal{C} \) there exists a \( k \)-algebra isomorphism \( \Psi : \mathcal{B}_{c'} \rightarrow \mathcal{B}_c \) such that
\[
\Psi|_M X U_0^+ = -|_M X U_0^+ \quad \text{and} \quad \Psi(B'_i) = B_i \quad \text{for all } i \in I \setminus X.
\]
In particular, we have \( \Psi(q) = q^{-1} \) and \( \Psi(K_\beta) = K_{-\beta} \) for all \( \beta \in Q^{\Theta} \).

**Proof.** With the above notation, Equation (3.9) can be rewritten as
\[
(4.1) \quad B_i \bar{X} = \bar{X}B'_i \quad \text{for all } i \in I \setminus X
\]
Hence there is a well-defined \( k \)-algebra homomorphism \( \Psi : \mathcal{B}_{c'} \rightarrow \mathcal{B}_c \) such that \( \Psi(b) = \bar{X}b\bar{X}^{-1} \) for all \( b \in \mathcal{B}_{c'} \). Equation (4.1) implies that \( \Psi(B'_i) = B_i \) for all \( i \in I \setminus X \). Property (3) in Proposition 3.3 implies that \( \Psi|_M X U_0^+ = -|_M X U_0^+ \). \( \Box \)

In the special case that \( c = c' \) the above corollary provides the desired bar involution for \( \mathcal{B}_c \).

**Corollary 4.2.** Assume that the parameters \( c = (c_i)_{i \in I \setminus X} \in \mathbb{K}^{I \setminus X} \) satisfy the relation
\[
(4.2) \quad \bar{c}_i = (-1)^{2\alpha_i(q^\rho)}q^{-(\alpha_i, \Theta(\alpha_i) - 2\rho X)}c_{\rho X}T_{\mu X}(E_{\tau(i)}K_i)
\]
for all \( i \in I \setminus X \). Then there exists a \( k \)-algebra automorphism

\[
\overline{B} : \mathcal{B}_c \to \mathcal{B}_c, \quad b \mapsto \overline{b}
\]

such that \( \overline{B} |_{U^0_{\Theta M_X}} = \overline{U^0_{\Theta M_X} B_i} = B_i \) for all \( i \in I \setminus X \). In particular, the automorphism \( \overline{B} \) of \( \mathcal{B}_c \) satisfies the relation \( \overline{q}^B = q^{-1} \).

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