ON TOTALLY INTEGRABLE MAGNETIC BILLIARDS ON
CONSTANT CURVATURE SURFACE

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Abstract. We consider billiard ball motion in a convex domain of a constant
curvature surface influenced by the constant magnetic field. We prove that if
the billiard map is totally integrable then the boundary curve is necessarily a
circle. This result shows that the so-called Hopf rigidity phenomenon which
was recently obtained for classical billiards on constant curvature surfaces holds
true also in the presence of constant magnetic field.

1. Introduction and the result

Let $S$ be a surface of constant curvature $K = 0, \pm 1$. Let $\gamma$ be a simple closed
convex curve on $S$ of class $C^2$. We shall denote by $k$ the geodesic curvature of $\gamma$ and
assume it is strictly positive everywhere. We consider the so-called magnetic billiard
inside $\gamma$ where the magnitude of the magnetic field is assumed to be constant $\beta \geq 0$.
This means that the billiard ball moves with a unit speed along the curves of con-
stant geodesic curvature $\beta$ between elastic reflections from the boundary. The model
of magnetic billiards was extensively studied (see [2][1][14][12][10][13][8][7][9]). Let
me summarize without proof the basic facts about the model of magnetic billiards.
The main geometric assumption which assures that the dynamics is well defined is
the following:

$$\beta < \min_{x \in \gamma} k(x),$$

which says that the field is not too large relative to the geodesic curvature of the
boundary. In this case, the magnetic billiard ball map defines a smooth map $T$ of
the phase cylinder cylinder $\Omega = \gamma \times (0, \pi)$, where we shall denote by $x \in (0, P)$ the
arc-length coordinate along $\gamma$ and $\phi \in (0, \pi)$ is the inward angle. Moreover, $T$ is a
symplectic twist map: the form $dx \wedge d(\cos \phi)$ is preserved. Remarkably, this is the
same form which appears for classical billiards. We shall denote by $d\mu = \sin \phi dx d\phi$
the invariant measure.

An important question starting from [2] is when such a model is integrable.
The only known example of integrable magnetic billiards is the circular billiard, in
contrast to the classical case where for any constant curvature surface, ellipses are
integrable also (see [15]).
We shall use the following definition which was suggested by Andreas Knauf for the case of geodesic flow on the torus ([11]):

**Definition 1.1.** The billiard ball map $T$ is called totally integrable if through every point of the phase cylinder $\Omega = \gamma \times (0, \pi)$ passes a closed non-contractible curve which is invariant under the map $T$.

We prove the following theorem:

**Theorem 1.2.** If the magnetic billiard map is totally integrable then $\gamma$ must be a circle.

Remark 1. A more general result can be proved using the notion of conjugate points of twist maps ([4],[3]). Namely, the more general statement is the following: any magnetic billiard on surface $S$ which has no conjugate points is circular.

Remark 2. In view of the previous remark, one can consider the result of Theorem 1.2 as a magnetic billiard analog of Hopf’s theorem on tori without conjugate points. It was proved in [5] that Hopf type rigidity holds true also for magnetic geodesic flows on tori, provided the metric is conformally flat. Notice that the magnetic field in [5] is not assumed to be constant. In higher dimensions it is not known if the conformal flatness assumption can be relaxed.

2. **Magnetic versions of Santalo and mirror formula**

The key observation for the proof of Theorem 1.2 is the fact that the classical Santalo formula for geodesics (see for example [6]) remains the same for magnetic geodesics (it was proven by Santalo for horocycles, most general case follows from [10] and [12]):

**Lemma 2.1.** ("Magnetic” Santalo formula) Let $l(x, \phi)$ be the length of the magnetic geodesic starting at the point $x \in \gamma$ having the angle $\phi \in (0, \pi)$ with the boundary. Then the integral over the phase cylinder with respect to the invariant measure $d\mu = \sin \phi dx d\phi$ satisfies

$$\int l(x, \phi)d\mu = 2\pi A,$$

independently of the magnitude of the magnetic field $\beta$. Here, $A$ is the area of the billiard domain.

Recall also the Mirror formula for usual billiards on a surface $S$ of constant curvature $K$ reads

$$\frac{Y'(a)}{Y(a)} + \frac{Y'(b)}{Y(b)} = \frac{2k(x)}{\sin \phi},$$

where $Y$ denotes the orthogonal Jacobi field along geodesics on the surface $S$ satisfying initial conditions $Y(0) = 0, Y'(0) = 1$. Here, $x$ is a point on the mirror, $a$ is a distance from a point $A$ inside the domain to the point $x$ along the shortest ray, and $b$ is a distance along the reflected ray to the point $B$ where the focusing of the reflected beam occurs. $\phi$ is the angle of reflection.

It is well known that the presence of the magnetic field results in adding to the curvature $K$ at $\beta^2$ so that the Jacobi field $Y$ should be changed in the formula to
\[
Y_\beta = \begin{cases} 
\frac{1}{\sqrt{K+\beta^2}} \sin(\sqrt{K+\beta^2} \ t) & \text{for } K + \beta^2 > 0 \\
\frac{1}{\sqrt{- (K+\beta^2)}} \sinh(\sqrt{- (K+\beta^2)} t) & \text{for } K + \beta^2 < 0 \\
t & \text{for } K + \beta^2 = 0 
\end{cases}
\]

There is also a change on the right hand side of the Mirror formula so that for any \( K = 0, \pm 1 \), the formula reads as follows:

\[
\frac{Y_\beta'}{Y_\beta}(a) + \frac{Y_\beta'}{Y_\beta}(b) = \frac{2(k(x) - \beta \cos \phi)}{\sin \phi}.
\]

Another ingredient for the proof is the following statement, exactly as in ([4]) for usual billiards:

**Theorem 2.2.** If the billiard is totally integrable or, more generally, has no conjugate points, then there exists a measurable function on the phase cylinder \( a : \Omega \to \mathbb{R} \) with \( 0 < a(x, \phi) < l(x, \phi) \), which satisfies the mirror equation:

\[
\frac{Y_\beta'}{Y_\beta}(a(x, \phi)) + \frac{Y_\beta'}{Y_\beta}(l(x-1, \phi-1) - a(x-1, \phi-1)) = \frac{2(k(x) - \beta \cos \phi)}{\sin \phi}.
\]

The proof of this theorem is analogous to the non-magnetic case and is therefore omitted.

In the sequel, we shall distinguish between the cases of the Plane, \( K = 0 \); of the Sphere, \( K = 1 \) and the Hyperbolic plane, \( K = -1 \). In the last case, three sub-cases naturally appear: \( \beta > 1; \beta = 1 \) and \( \beta \in (0,1) \).

3. Planar and Spherical magnetic billiards. Reduction to the non-magnetic case

In this section we prove Theorem 1.2 for the Plane and the Sphere by a reduction to a non-magnetic case.

For the Plane and the Sphere, the mirror equation (3) reads:

\[
\cot(\sqrt{K + \beta^2} a(x, \phi)) + \cot(\sqrt{K + \beta^2} (l(x-1, \phi-1) - a(x-1, \phi-1))) = \frac{2(k(x) - \beta \cos \phi)}{\sqrt{K + \beta^2} \sin \phi}.
\]

Notice that the geometric assumption \( \beta < \min_{x \in \gamma} k(x) \) implies that the right hand side in (4) is positive and hence,

\[
\sqrt{K + \beta^2} a(x, \phi) + l(x-1, \phi-1) - a(x-1, \phi-1) < \pi
\]

so that the lemma of [4] can be applied to get the inequality

\[
\sqrt{K + \beta^2} \cot \frac{\sqrt{K + \beta^2} a(x, \phi) + l(x-1, \phi-1) - a(x-1, \phi-1)}{2} \leq \frac{k(x) - \beta \cos \phi}{\sin \phi}.
\]

This can be written in equivalent way:
\[ a(x, \phi) + l(x_{-1}, \phi_{-1}) - a(x_{-1}, \phi_{-1}) \geq \frac{2}{\sqrt{K + \beta^2}} \arctan \frac{\sqrt{K + \beta^2} \sin \phi}{k(x) - \beta \cos \phi} \]

Integrating (5) with respect to the invariant measure \( d\mu = \sin \phi \, dx \, d\phi \) yields

\[ \int l \, d\mu \geq \int_0^P dx \int_0^\pi \frac{2}{\sqrt{K + \beta^2}} \arctan \frac{\sqrt{K + \beta^2} \sin \phi}{k(x) - \beta \cos \phi} \sin \phi \, d\phi. \]

By the “magnetic” Santalo formula, the integral on the left hand side equals \( 2\pi A \), independently of the magnetic field \( \beta \). Without any additional calculations, this fact implies that the inner integral on the right hand side of (6) is also independent of \( \beta \). Indeed, assume for a moment that the boundary curve is a circle of constant geodesic curvature \( k \) on \( S \). Then one can easily see that there is equality in (5) and hence also in (6). Moreover due to the rotational symmetry of the circular billiard, (6) leads to the following equality for the circle of curvature \( k \):

\[ 2\pi A = P \int_0^\pi \frac{2}{\sqrt{K + \beta^2}} \arctan \frac{\sqrt{K + \beta^2} \sin \phi}{k - \beta \cos \phi} \sin \phi \, d\phi. \]

Therefore, the inner integral in (6) equals \( 2\pi A/P \) independently of \( \beta \). This proves the claim. (Of course one could compute the integral for any \( \beta \), but this is lengthy.) By the independence on \( \beta \), we can compute the right hand side of (6) by setting \( \beta = 0 \). But then the inequality becomes identical to the one obtained in a non-magnetic case [4]. Namely, consider first the case of the Sphere, \( K = 1 \). We have from (6) putting \( \beta = 0 \):

\begin{align*}
2\pi A & \geq \int_0^P dx \int_0^\pi 2 \arctan \left( \frac{\sin \phi}{k(x)} \right) \sin \phi \, d\phi \\
& = 4 \int_0^P k(x) \int_0^{\pi/2} \frac{\cos^2 \phi}{k^2(x) + \sin^2 \phi} \, d\phi \\
& = 2\pi \int_0^P \left( \sqrt{k^2(x) + 1} - k(x) \right) dx.
\end{align*}

This inequality implies ([4]) that \( \gamma \) must be a circle. Indeed, use the Gauss-Bonnet formula to write (7) in the form

\[ A \geq \int_0^P \sqrt{k^2(x) + 1} \, dx - (2\pi - A), \]

which leads to

\[ \int_0^P \sqrt{k^2(x) + 1} \, dx \leq 2\pi. \]

Denote this integral by \( I \). On the other hand, by Cauchy Schwartz, one has

\[ \int_0^P \left( \sqrt{k^2(x) + 1} + 1 \right) \, dx \cdot \int_0^P \left( \sqrt{k^2(x) + 1} - 1 \right) \, dx \geq \left( \int_0^P k(x) \, dx \right)^2 = (2\pi - A)^2. \]

This can be rewritten as

\[ (I - P)(I + P) \geq (2\pi - A)^2, \]
and since $I \leq 2\pi$ then
$$4\pi^2 \geq I^2 \geq P^2 + A^2 - 4\pi A + 4\pi^2.$$ 

Thus we end with the inequality
$$0 \geq P^2 + A^2 - 4\pi A,$$

which contradicts the isoperimetric inequality on the sphere. Therefore $\gamma$ must be a circle.

For the Plane, $K = 0$, the inequality (6) looks even simpler when one passes to the limit $\beta \to 0$:
$$2\pi A \geq \pi \int_0^P \frac{1}{k(x)} \, dx,$$

which is possible only for circles as was observed in [16]. Once again, by Cauchy-Schwartz, one has
$$\int_0^P \frac{1}{k(x)} \, dx \geq \frac{P^2}{\int_0^P k(x) \, dx} = \frac{P^2}{2\pi},$$

contradicting the isoperimetric inequality in the plane, unless $\gamma$ is a circle.

4. Proof of Theorem 1.2 for Hyperbolic Plane

On the Hyperbolic plane, $K = -1$, we shall also make a reduction to the non-magnetic case. However, we shall distinguish between the following cases where the mirror formula (2) looks differently depending on the magnitude of the magnetic field:

Case 1. Assume here that $\beta > 1$. Using the lemma of [4] exactly as in the case of the Sphere, we get from (3)

$$\sqrt{\beta^2 - 1} \cot \frac{\sqrt{\beta^2 - 1}[a(x, \phi) + l(x-1, \phi-1) - a(x-1, \phi-1)]}{2} \leq \frac{k(x) - \beta \cos \phi}{\sin \phi},$$

or equivalently

$$a(x, \phi) + l(x-1, \phi-1) - a(x-1, \phi-1) \geq \frac{2}{\sqrt{\beta^2 - 1}} \arctan \frac{\sqrt{\beta^2 - 1} \sin \phi}{k(x) - \beta \cos \phi}.$$

Integrating (9) with respect to the invariant measure $d\mu = \sin \phi \, dx d\phi$ yields

$$\int l \, d\mu \geq \int_0^P dx \int_0^\pi \frac{2}{\sqrt{\beta^2 - 1}} \arctan \frac{\sqrt{\beta^2 - 1} \sin \phi}{k(x) - \beta \cos \phi} \sin \phi \, d\phi.$$

Denote by $I_1$ the inner integral of (10):

$$I_1 = \int_0^\pi \frac{2}{\sqrt{\beta^2 - 1}} \arctan \frac{\sqrt{\beta^2 - 1} \sin \phi}{k(x) - \beta \cos \phi} \sin \phi \, d\phi.$$

Case 2. Here we assume $\beta = 1$. Therefore the effective curvature $K + \beta^2$ vanishes and formula (3) implies:

$$\frac{1}{a(x, \phi)} + \frac{1}{l(x-1, \phi-1) - a(x-1, \phi-1)} = \frac{2(k(x) - \cos \phi)}{\sin \phi}.$$
Using the convexity of the function $\frac{1}{x}$, we get

$$\frac{2}{a(x, \phi) + l(x_{-1}, \phi_{-1}) - a(x_{-1}, \phi_{-1})} \leq \frac{(k(x) - \cos \phi)}{\sin \phi}$$

so that

$$a(x, \phi) + l(x_{-1}, \phi_{-1}) - a(x_{-1}, \phi_{-1}) \geq \frac{2 \sin \phi}{k(x) - \cos \phi}. \tag{11}$$

Integrating with respect to the invariant measure, we end up with the inequality:

$$\int l \, d\mu \geq \int_0^P dx \int_0^\pi \frac{2 \sin \phi}{k(x) - \cos \phi} \sin \phi d\phi. \tag{12}$$

Denote by $I_2$ the inner integral of (12):

$$I_2 = \int_0^\pi \frac{2 \sin \phi}{k(x) - \cos \phi} \sin \phi d\phi.$$

**Case 3.** Lastly, suppose $\beta \in (0, 1)$. By formula (3), we have

$$\frac{\coth(\sqrt{1 - \beta^2} a(x, \phi)) + \coth(\sqrt{1 - \beta^2} l(x_{-1}, \phi_{-1}) - a(x_{-1}, \phi_{-1}))}{2} = \frac{k(x) - \beta \cos \phi}{\sqrt{1 - \beta^2 \sin \phi}} \tag{13}$$

One can see that for given $x$, the minimum of the right hand side of (13) equals $\sqrt{k^2(x) - \beta^2}$, which is attained for some angle $\phi$. Comparing this value with the left hand side, which is obviously strictly greater than 1, we get

$$\sqrt{1 - \beta^2} < \sqrt{k^2(x) - \beta^2}.$$

Thus, $k(x) \geq 1$ for all $x$, so that the curve $\gamma$ must be convex with respect to horocycles.

Moreover, by the convexity of $\coth$

$$\sqrt{1 - \beta^2} \coth \frac{\sqrt{1 - \beta^2} [a(x, \phi) + l(x_{-1}, \phi_{-1}) - a(x_{-1}, \phi_{-1})]}{2} \leq \frac{k(x) - \beta \cos \phi}{\sin \phi}, \tag{14}$$

or equivalently

$$a(x, \phi) + l(x_{-1}, \phi_{-1}) - a(x_{-1}, \phi_{-1}) \geq \frac{2}{\sqrt{1 - \beta^2}} \arctanh \frac{\sqrt{1 - \beta^2} \sin \phi}{k(x) - \beta \cos \phi}. \tag{15}$$

Integrating (15) with respect to the invariant measure $d\mu = \sin \phi \, dx \, d\phi$ yields

$$\int l \, d\mu \geq \int_0^P dx \int_0^\pi \frac{2}{\sqrt{1 - \beta^2}} \arctanh \frac{\sqrt{1 - \beta^2} \sin \phi}{k(x) - \beta \cos \phi} \sin \phi d\phi. \tag{16}$$

Denote by $I_3$ the inner integral of (16):

$$I_3 = \int_0^\pi \frac{2}{\sqrt{1 - \beta^2}} \arctanh \frac{\sqrt{1 - \beta^2} \sin \phi}{k(x) - \beta \cos \phi} \sin \phi d\phi.$$
Remarkably, the following lemma holds true:

**Lemma 4.1.** All three integrals $I_1, I_2, I_3$ are independent of $\beta$ and

$$I_1 = I_2 = I_3 = 2\pi(k(x) - \sqrt{k^2(x) - 1}).$$

We postpone the proof of the lemma until the end of the section. All three inequalities of the Cases 1,2,3 (10),(12),(16) lead—by the “magnetic” Santalo formula and by Lemma 4.1—to the same inequality, which does not contain $\beta$:

$$A \geq \int_0^P (k(x) - \sqrt{k^2(x) - 1}) \, dx.$$  

We proceed like in [4]. Use the Gauss-Bonnet formula to write it in the form

$$A \geq 2\pi + A - \int_0^P \sqrt{k^2(x) - 1} \, dx$$

and therefore

$$\int_0^P \sqrt{k^2(x) - 1} \, dx \geq 2\pi.$$  

On the other hand, the last integral can be estimated from above by the Cauchy-Schwartz inequality

$$\int_0^P \sqrt{k^2(x) - 1} \, dx \leq \left( \int_0^P (k(x) - 1) \, dx \int_0^P (k(x) + 1) \, dx \right)^{\frac{1}{2}} =$$

$$= ((A + 2\pi - P)(A + 2\pi + P))^{\frac{1}{2}},$$

where we have applied the Gauss-Bonnet formula again. Thus, we have the inequality

$$((A + 2\pi - P)(A + 2\pi + P))^{\frac{1}{2}} \geq 2\pi,$$

which is equivalent to

$$A^2 + 4\pi A - P^2 \geq 0.$$  

But this contradicts the isoperimetric inequality on the Hyperbolic plane, unless $\gamma$ is a circle. This completes the proof for the Hyperbolic plane.

**Proof of the lemma.** This goes exactly like in the Spherical case. Indeed, for the circle of curvature $k$, both inequalities (10), (16) become equalities. Using rotational symmetry, the integrals $I_1, I_3$ can easily be shown to equal $\frac{2\pi k P}{P}$, which shows independence of $\beta$. Moreover, it is clear that for $\beta \to 1$ both integrals $I_1, I_3$ tend to $I_2$. But the integral $I_2$ can also be easily computed. This completes the proof of the lemma.  

\[ \square \]

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