Lepton masses, mixings and FCNC in a minimal $S_3$-invariant extension of the Standard Model.

A. Mondragón, M. Mondragón and E. Peinado
Instituto de Física, Universidad Nacional Autónoma de México,
Apdo. Postal 20-364, 01000 México D.F., México.
(Dated: February 1, 2008)

The mass matrices of the charged leptons and neutrinos, previously derived in a minimal $S_3$-invariant extension of the Standard Model, were reparameterized in terms of their eigenvalues. We obtained explicit, analytical expressions for all entries in the neutrino mixing matrix, $V_{PMNS}$, the neutrino mixing angles and the Majorana phases as functions of the masses of charged leptons and neutrinos in excellent agreement with the latest experimental values. The resulting $V_{PMNS}$ matrix is very close to the tri-bimaximal form of the neutrino mixing matrix. We also derived explicit analytical expressions for the matrices of the Yukawa couplings and computed the branching ratios of some selected flavour changing neutral current processes as functions of the masses of the charged leptons and the neutral Higgs bosons. We find that the $S_3 \times Z_2$ flavour symmetry and the strong mass hierarchy of the charged leptons strongly suppress the FCNC processes in the leptonic sector well below the present experimental upper bounds by many orders of magnitude.

Keywords: Flavour symmetries; Quark and lepton masses and mixings; Neutrino masses and mixings; Flavour changing neutral currents.

PACS numbers: 11.30.Hv, 12.15.Ff, 14.60.Pq

I. INTRODUCTION

The recent discovery of flavour oscillations of solar, atmospheric, reactor and accelerator neutrinos have irrefutably established that neutrinos have non-vanishing masses and mix among themselves much like the quarks, thereby providing the first conclusive evidence of new physics beyond the Standard Model [1, 2]. Neutrino oscillation observations and experiments, made in the past eight years, have allowed the determination of the differences of the neutrino masses squared and the flavour mixing angles in the leptonic sector. The solar [3, 4, 5, 6], atmospheric [7, 8] and reactor [9, 10] experiments produced the following results:

\[ 7.1 \times 10^{-5} (eV)^2 \leq \Delta^2 m_{12} \leq 8.9 \times 10^{-5} (eV)^2, \]

\[ 0.24 \leq \sin^2 \theta_{12} \leq 0.40, \]

\[ 1.4 \times 10^{-3} (eV)^2 \leq \Delta^2 m_{13} \leq 3.3 \times 10^{-3} (eV)^2, \]

\[ 0.34 \leq \sin^2 \theta_{23} \leq 0.68, \]

at 90% confidence level [11, 12]. For a recent review on the phenomenology of massive neutrinos, see [13]. The CHOOZ experiment [14] determined an upper bound for the flavour mixing angle between the first and the third generation:

\[ \sin^2 \theta_{13} \leq 0.046. \]

However, neutrino oscillation data are insensitive to the absolute value of neutrino masses and also to the fundamental issue of whether neutrinos are Dirac or Majorana particles. Hence, the importance of the upper bounds on neutrino masses provided by the searches that probe the neutrino mass values at rest: beta decay experiments [15], neutrinoless double beta decay [16] and precision cosmology [17, 18, 19].

In the Standard Model, the Higgs and Yukawa sectors, which are responsible for the generation of the masses of quarks and charged leptons, do not give mass to the neutrinos. Furthermore, the Yukawa sector of the Standard Model already has too many parameters whose values can only be determined from experiment. These two facts point to the necessity and convenience of extending the Standard Model in order to make a unified and systematic treatment of the observed hierarchies of masses and mixings of all fermions, as well as the presence or absence of CP violating phases in the mixing matrices. At the same time, we would also like to reduce drastically the number of free parameters in the theory. These two seemingly contradictory demands can be met by means of a flavour symmetry under which the families transform in a non-trivial fashion.

Recently, we introduced a Minimal $S_3$-invariant Extension of the Standard Model [20] in which we argued that such a flavour symmetry unbroken at the Fermi scale, is the permutational symmetry of three objects $S_3$. In this model, we imposed $S_3$ as a fundamental symmetry in the matter sector. This assumption led us necessarily to extend the concept of flavour and generations to the Higgs sector. Hence, going to the irreducible representations of $S_3$, we added to the Higgs $SU(2)_L$ doublet in the $S_3$-singlet representation two more Higgs $SU(2)_L$ doublets, which can only belong to the two components of
the $S_3$-doublet representation. In this way, all the matter
fields in the Minimal $S_3$-invariant Extension of the Standard Model - Higgs, quark and lepton fields, including the
right handed neutrino fields - belong to the three dimen-
sional representation $1 \oplus 2$ of the permutational group
$S_3$. The leptonic sector of the model was further con-
strained by an Abelian $Z_2$ symmetry. We found that the
$S_3 \times Z_2$ symmetry predicts the tri-bimaximal mixing and
an inverted mass hierarchy of the left handed neutrinos
in good agreement with experiment [20]. More recently,
we reparametrized the mass matrices of the charged lep-
tons and neutrinos, previously derived in [20], in terms
of their eigenvalues and derived explicit analytical ex-
pressions for the entries in the neutrino mixing matrix,
$V_{PMNS}$, and the neutrino mixing angles and Majorana
phases as functions of the masses of charged leptons and
neutrinos, in excellent agreement with the latest experi-
mental values [21].

The group $S_3 \times S_3 \times S_3 \times S_3 \times S_2 \times S_3$ and the product groups $S_3 \times S_3 \times S_3 \times S_3 \times S_3 \times S_3 \times S_3 \times S_3$ and $S_3 \times S_3 \times S_3 \times S_3 \times S_3 \times S_3 \times S_3 \times S_2 \times S_3$ have been considered by many authors
to explain successfully the hierarchical structure of quark
masses and mixings in the Standard Model. However, in
these works, the $S_3$, $S_3 \times S_3$ and $S_3 \times S_3 \times S_3$ symmetries are explicitly broken at the Fermi scale to give mass to
the lighter quarks and charged leptons, neutrinos are left
massless. Some other interesting models based on the
$S_3$, $S_3 \times S_3$ and $D_3$ flavour symmetry groups, unbroken
at the Fermi scale, have also been proposed [37, 38, 39, 40, 41, 42, 43, 44], but in those models, equality of the number of fields and the irreducible representations is
not obtained. The generic properties of mass textures of
quarks and leptons derived in the standard model and in
supersymmetric models with a Higgs sector with non-
trivial flavours and an $S_3$ flavour symmetry have been
discussed in [45, 46]. Recent flavour symmetry models are reviewed in [47, 48, 49, 50], see also the references
therein.

In this paper, we consider the flavour changing neutral
current (FCNC) processes in the Minimal $S_3$-Invariant
Extension of the Standard Model [21]. After a short re-
sort of some relevant results on lepton masses and mix-
ings, we derive exact, explicit expressions for the matrices
of the Yukawa couplings in the leptonic sector expressed
as functions of the masses of the charged leptons and neu-
tral Higgs bosons. With the help of the Yukawa matrices
we compute the branching ratios of some selected FCNC
processes as functions of the masses of charged leptons and
neutral Higgs bosons. We find that the interplay of the
$S_3 \times Z_2$ flavour symmetry and the strong mass hier-
archy of charged leptons strongly suppresses the FCNC
processes in the leptonic sector well below the experimen-
tial upper bounds by many orders of magnitude.

II. THE MINIMAL $S_3$-INARIANT
EXTENSION OF THE STANDARD MODEL

In the Standard Model analogous fermions in different
generations have identical couplings to all gauge bosons of
the strong, weak and electromagnetic interactions. Prior to the introduction of the Higgs boson and mass
terms, the Lagrangian is chiral and invariant with respect
to permutations of the left and right fermionic fields.

The six possible permutations of three objects $(f_1, f_2, f_3)$ are elements of the permutational group
$S_3$. This is the discrete, non-Abelian group with the smallest
number of elements. The three-dimensional real repre-
sentation is not an irreducible representation of $S_3$. It
can be decomposed into the direct sum of a doublet $f_D$
and a singlet $f_s$, where

$$f_s = \frac{1}{\sqrt{3}}(f_1 + f_2 + f_3),$$
$$f_D = \left( \frac{1}{\sqrt{2}}(f_1 - f_2), \frac{1}{\sqrt{2}}(f_1 + f_2 - 2f_3) \right).$$

The direct product of two doublets $pD^T = (p_{D1}, p_{D2})$ and $qD^T = (q_{D1}, q_{D2})$ may be decomposed into the direct
sum of two singlets $r_s$ and $r_e$, and one doublet $r_D^T$ where

$$r_s = p_{D1}q_{D1} + p_{D2}q_{D2}, \quad r_e = p_{D1}q_{D2} - p_{D2}q_{D1},$$
$$r_D^T = (r_{D1}, r_{D2}) = (p_{D1}q_{D2} + p_{D2}q_{D1}, p_{D1}q_{D1} - p_{D2}q_{D2}).$$

The antisymmetric singlet $r_e$ is not invariant under $S_3$.

Since the Standard Model has only one Higgs $SU(2)_L$
doublet, which can only be an $S_3$ singlet, it can only give
mass to the quark or charged lepton in the $S_3$ singlet
representation, one in each family, without breaking the
$S_3$ symmetry.

Hence, in order to impose $S_3$ as a fundamental sym-
metry, unbroken at the Fermi scale, we are led to extend
the Higgs sector of the theory. The quark, lepton and
Higgs fields are

$$Q^T = (u_L, d_L), \quad u_R, \quad d_R,$$
$$L^T = (\nu_L, e_L), \quad e_R, \quad \nu_R \quad \text{and} \quad H,$$

in an obvious notation. All of these fields have three
species, and we assume that each one forms a reducible
representation $1_S \oplus 2$. The doublets carry capital indices $I$ and $J$, which run from 1 to 2, and the singlets are
denoted by $Q_3$, $u_{3R}$, $d_{3R}$, $L_3$, $e_R$, $\nu_{3R}$ and $H_S$. Note that the subscript 3 denotes the singlet representation
and not the third generation. The most general renor-
malizable Yukawa interactions of this model are given by

$$L_Y = L_{Y_D} + L_{Y_U} + L_{Y_L} + L_{Y_e},$$

where

$$L_{Y_D} = -Y_{1D}Q_1Hsd_{1R} - Y_{2D}Q_2Hsd_{3R} - Y_{3D}[Q_1\kappa_{1j}H_1d_{JR} + Q_2\kappa_{1j}H_2d_{JR} + Q_2Q_3H_1d_{JR} + \text{h.c.}]$$

(11)
\[\mathcal{L}_{Y_L} = -Y_L^T \overline{Q}_L (i\sigma_2) H_S^\dagger u_{1R} - Y_L^T \overline{Q}_L (i\sigma_2) H_S^\dagger u_{3R} - Y_L^T \overline{Q}_L (i\sigma_2) H_L^\dagger u_{2R} + \text{h.c.}\]

\[\mathcal{L}_{Y_E} = -Y_E^T \overline{L}_E H_S e_{1R} - Y_E^T \overline{L}_E H_S e_{3R} - Y_E^T \overline{L}_E H_L e_{1R} + \text{h.c.}\]

\[\mathcal{L}_{Y_\nu} = -Y_\nu^T \overline{L}_\nu (i\sigma_2) H_S^\dagger \nu_{1R} - Y_\nu^T \overline{L}_\nu (i\sigma_2) H_S^\dagger \nu_{3R} - Y_\nu^T \overline{L}_\nu (i\sigma_2) H_L^\dagger \nu_{2R} + \text{h.c.}\]

and

\[\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Furthermore, we add to the Lagrangian the Majorana mass terms for the right-handed neutrinos

\[\mathcal{L}_M = -M_1 \nu^T_{1R} C \nu_{1R} - M_3 \nu^T_{3R} C \nu_{3R}.\]

Due to the presence of three Higgs fields, the Higgs potential \(V_H(H_S, H_D)\) is more complicated than that of the Standard Model. This potential was analyzed by Pakvasa and Sugawara, who found that in addition to the S_3 symmetry, it has a permutational symmetry S_2': \(H_1 \leftrightarrow H_2\), which is not a subgroup of the flavour group S_3. In this communication, we assume that the vacuum respects the accidental S_2' symmetry of the Higgs potential and that

\[\langle H_1 \rangle = \langle H_2 \rangle.\]

With these assumptions, the Yukawa interactions, eqs. (11)-(14) yield mass matrices, for all fermions in the theory, of the general form \[[20]\]

\[M = \begin{pmatrix} \mu_1 + \mu_2 & \mu_2 & \mu_5 \\ \mu_2 & \mu_1 + \mu_2 & \mu_3 \\ \mu_4 & \mu_4 & \mu_3 \end{pmatrix}.
\]

The Majorana mass for the left handed neutrinos \(\nu_L\) is generated by the see-saw mechanism. The corresponding mass matrix is given by

\[M_\nu = M_{\nu R} \tilde{M}^{-1} (M_{\nu L})^T,
\]

where \(\tilde{M} = \text{diag}(M_1, M_1, M_3)\).

In principle, all entries in the mass matrices can be complex since there is no restriction coming from the flavour symmetry S_3. The mass matrices are diagonalized by bi-unitary transformations as

\[U_{d(u, e)}^\dagger M_{d(u, e)} U_{d(u, e)} = \text{diag}(m_{d(u, e)}, m_{s(c, \mu)}, m_{b(t, \tau)}),\]

\[U_{\nu}^T M_{\nu} U_{\nu} = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}).\]

The entries in the diagonal matrices may be complex, so the physical masses are their absolute values.

The mixing matrices are, by definition,

\[V_{CKM} = U_{uL}^\dagger U_{uL}, \quad V_{PMNS} = U_{eL}^\dagger U_{eL} K,\]

where K is the diagonal matrix of the Majorana phase factors.

III. THE MASS MATRICES IN THE LEPTONIC SECTOR AND Z_2 SYMMETRY

A further reduction of the number of parameters in the leptonic sector may be achieved by means of an Abelian \(Z_2\) symmetry. A possible set of charge assignments of \(Z_2\), compatible with the experimental data on masses and mixings in the leptonic sector is given in Table I.

These \(Z_2\) assignments forbid the following Yukawa couplings

\[Y_1^\nu = Y_3^\nu = Y_2^\nu = Y_4^\nu = 0.\]

Therefore, the corresponding entries in the mass matrices vanish, i.e., \(\mu_2 = 0\) and \(\mu_4 = 0\).

The mass matrix of the charged leptons

The mass matrix of the charged leptons takes the form

\[M_\ell = m_\tau \begin{pmatrix} \tilde{\mu}_2 & \tilde{\mu}_4 & \tilde{\mu}_5 \\ \tilde{\mu}_2 & \tilde{\mu}_2 & \tilde{\mu}_5 \\ \tilde{\mu}_4 & \tilde{\mu}_4 & \tilde{\mu}_5 \end{pmatrix}.
\]

The unitary matrix \(U_{eL}\) that enters in the definition of the mixing matrix, \(V_{PMNS}\), is calculated from

\[U_{eL}^\dagger M_\ell U_{eL} = \text{diag}(m_{\nu_1}^2, m_{\mu}^2, m_{\tau}^2),\]

where \(m_\nu, m_\mu\) and \(m_\tau\) are the masses of the charged leptons, and

\[M_\ell = \frac{M_\ell^\dagger}{m_\ell^2} = \begin{pmatrix} 2|\tilde{\mu}_2|^2 + |\tilde{\mu}_5|^2 & |\tilde{\mu}_5|^2 & 0 \\ |\tilde{\mu}_5|^2 & 2|\tilde{\mu}_2|^2 + |\tilde{\mu}_5|^2 & 0 \\ 0 & 2|\tilde{\mu}_5||\tilde{\mu}_4| e^{i\delta_4} & 2|\tilde{\mu}_4|^2 \end{pmatrix}.
\]

Notice that this matrix has only one non-ignorable phase factor.

The parameters \(\tilde{\mu}_2\), \(\tilde{\mu}_4\) and \(\tilde{\mu}_5\) may readily be expressed in terms of the charged lepton masses. From the invariants of \(M_\ell M_\ell^\dagger\), we get the set of equations

\[Tr(M_\ell M_\ell^\dagger) = m_\nu^2 + m_\mu^2 + m_\tau^2 = m_\nu^2 [4|\tilde{\mu}_2|^2 + 2 (|\tilde{\mu}_4|^2 + |\tilde{\mu}_5|^2)] + m_\mu^2 [4|\tilde{\mu}_2|^2 + 2 (|\tilde{\mu}_4|^2 + |\tilde{\mu}_5|^2)] + m_\tau^2 [4|\tilde{\mu}_2|^2 + 2 (|\tilde{\mu}_4|^2 + |\tilde{\mu}_5|^2)],\]
\[
\chi(M_e M_e^\dagger) = m_e^2 (m_e^2 + m_\mu^2) + m_\mu^2 m_\tau^2 = 4m_e^4 [\tilde{\mu}_2]^4 + [\tilde{\mu}_2]^2 ([\tilde{\mu}_4]^2 + [\tilde{\mu}_5]^2) + [\tilde{\mu}_4]^2 [\tilde{\mu}_5]^2, \tag{27}
\]

\[
det(M_e M_e^\dagger) = m_e^2 m_\mu^2 m_\tau^2 = 4m_e^6 \tilde{\mu}_2^2 [\tilde{\mu}_4]^2 [\tilde{\mu}_5]^2, \tag{28}
\]

where \(\chi(M_e M_e^\dagger)\) is the smallest solution of the equation

\[
- \delta - \beta \approx - \beta - \frac{\beta}{2} (1 - 2y) \beta^2 - \frac{1}{4} (y - y^2 - 4z + 7z - 12 \frac{z^2}{y}) \beta - \frac{y}{8} - \frac{z^2}{4} + \frac{3}{4} \frac{z^2}{y^3} = 0, \tag{31}
\]

where \(y = (m_e^2 + m_\mu^2)/m_e^2\) and \(z = m_\mu^2 m_e^2/m_\mu^4\).

A good, order of magnitude, estimate for \(\beta\) is obtained from

\[
\beta \approx \frac{m_e^2 m_\mu^2}{2m_e^2 (m_e^2 + m_\mu^2)}, \tag{32}
\]

\[
M_e \approx m_\tau \begin{pmatrix}
\frac{1}{\sqrt{2}} \tilde{\mu}_e & \frac{1}{\sqrt{2}} \tilde{\mu}_\mu & \frac{1}{\sqrt{2}} \tilde{\mu}_\tau \\
\frac{1}{\sqrt{1 + x^2}} & -\frac{1}{\sqrt{1 + x^2}} & \frac{1}{\sqrt{1 + x^2}} \\
\frac{\tilde{\mu}_e (1 + x^2)}{\sqrt{1 + x^2}} e^{i \delta_e} & \frac{\tilde{\mu}_\mu (1 + x^2)}{\sqrt{1 + x^2}} e^{i \delta_e} & 0
\end{pmatrix}, \tag{33}
\]

where the orthogonal matrix \(O_{eL}\) in the right hand side of eq. (34), written to the same order of magnitude as \(M_e\), is

Solving these equations for \(|\tilde{\mu}_2|^2\), \(|\tilde{\mu}_4|^2\) and \(|\tilde{\mu}_5|^2\), we obtain

\[
|\tilde{\mu}_2|^2 = \frac{m_e^2}{2} \frac{1 + x^4}{1 + x^2} + \beta, \tag{29}
\]

and

\[
|\tilde{\mu}_4|^2 = 16 \beta^2
\]

Once \(M_e M_e^\dagger\) has been reparametrized in terms of the charged lepton masses, it is straightforward to compute \(M_e\) and \(U_{eL}\) also as functions of the charged lepton masses \(m_{eL}\). The resulting expression for \(M_e\), written to order \((m_\mu m_e/m_\mu^2)^2\) and \(x^4 = (m_e/m_\mu)^4\) is

\[
M_e \approx m_\tau \begin{pmatrix}
\frac{1}{\sqrt{2}} \tilde{\mu}_e & \frac{1}{\sqrt{2}} \tilde{\mu}_\mu & \frac{1}{\sqrt{2}} \tilde{\mu}_\tau \\
\frac{1}{\sqrt{1 + x^2}} & -\frac{1}{\sqrt{1 + x^2}} & \frac{1}{\sqrt{1 + x^2}} \\
\frac{\tilde{\mu}_e (1 + x^2)}{\sqrt{1 + x^2}} e^{i \delta_e} & \frac{\tilde{\mu}_\mu (1 + x^2)}{\sqrt{1 + x^2}} e^{i \delta_e} & 0
\end{pmatrix}, \tag{33}
\]

This approximation is numerically exact up to order \(10^{-9}\) in units of the \(\tau\) mass. Notice that this matrix has no free parameters other than the Dirac phase \(\delta_e\).

The unitary matrix \(U_{eL}\) that diagonalizes \(M_e M_e^\dagger\) and enters in the definition of the neutrino mixing matrix \(V_{PMNS}\) may be written as

\[
U_{eL} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i \delta_e}
\end{pmatrix} \begin{pmatrix}
O_{11} & -O_{12} & O_{13} \\
-O_{21} & O_{22} & O_{23} \\
-O_{31} & -O_{32} & O_{33}
\end{pmatrix}, \tag{34}
\]
where, as before, $m_\mu = m_\mu/m_\tau$, $m_e = m_e/m_\tau$ and $x = m_e/m_\mu$.

The mass matrix of the neutrinos

According to the $Z_2$ selection rule eq. (22), the mass matrix of the Dirac neutrinos takes the form

$$M_{\nu D} = \begin{pmatrix} \mu_1^2 & \mu_2^2 & 0 \\ \mu_1^2 & \mu_2^2 & 0 \\ \mu_1^2 & \mu_2^2 & \mu_3^2 \end{pmatrix}.$$  \hspace{1cm} (36)

Then, the mass matrix for the left-handed Majorana neutrinos, $M_{\nu}$, obtained from the see-saw mechanism, $M_{\nu} = M_{\nu D} M^{-1}(M_{\nu D})^T$, is

$$M_{\nu} = \begin{pmatrix} 2(\rho_3')^2 & 0 & 2(\rho_3')^2 \\ 0 & 2(\rho_3')^2 & 0 \\ 2(\rho_3')^2 & 0 & 2(\rho_3')^2 \end{pmatrix},$$  \hspace{1cm} (37)

where $\rho_3' = (\mu_3')/M_3^{1/2}$, $\rho_4' = (\mu_4')/M_4^{1/2}$ and $\rho_5' = (\mu_5')/M_5^{1/2}$; $M_1$ and $M_2$ are the masses of the right handed neutrinos appearing in (18).

The non-Hermitian, complex, symmetric neutrino mass matrix $M_{\nu}$ may be brought to a diagonal form by a bi-unitary transformation, as

$$U_\nu^T M_\nu U_\nu = \text{diag}\left(|m_{\nu_1}| e^{i\delta_1}, |m_{\nu_2}| e^{i\delta_2}, |m_{\nu_3}| e^{i\delta_3}\right),$$  \hspace{1cm} (38)

where $U_{\nu}$ is the matrix that diagonalizes the matrix $M_{\nu}^T M_{\nu}$. In order to compute $U_{\nu}$, we notice that $M_{\nu}^T M_{\nu}$ has the same texture zeroes as $M_{\nu}$

$$M_{\nu}^T M_{\nu} = \begin{pmatrix} |A|^2 + |B|^2 & 0 & A^*B + B^*D \\ 0 & |A|^2 & 0 \\ AB^* + BD^* & 0 & |B|^2 + |D|^2 \end{pmatrix},$$  \hspace{1cm} (39)

$$M_{\nu} = \begin{pmatrix} m_{\nu_3} & 0 & \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_3} - m_{\nu_2})} e^{-i\delta_\nu} \\ 0 & m_{\nu_2} - m_{\nu_1} & 0 \\ \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_3} - m_{\nu_2})} e^{-i\delta_\nu} & 0 & m_{\nu_3} - m_{\nu_1} \end{pmatrix},$$  \hspace{1cm} (40)

where $A = 2(\rho_3')^2$, $B = 2\rho_3'\rho_4'$, and $D = 2(\rho_4')^2 + (\rho_5')^2$. Furthermore, notice that the entries in the upper right corner and lower left corner are complex conjugates of each other, all other entries are real. Therefore, the matrix $U_{\nu L}$ that diagonalizes $M_{\nu}^T M_{\nu}$, takes the form

$$U_{\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_\nu} \end{pmatrix} \begin{pmatrix} \cos \eta & \sin \eta & 0 \\ 0 & 0 & 1 \\ -\sin \eta & \cos \eta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_\nu} \end{pmatrix}.$$  \hspace{1cm} (41)

If we require that the defining equation (35) be satisfied as an identity, we get the following set of equations:

$$2(\rho_3')^2 = m_{\nu_3},$$

$$2(\rho_3')^2 = m_{\nu_1} \cos^2 \eta + m_{\nu_2} \sin^2 \eta,$$

$$2\rho_3' \rho_4' = \sin \eta \cos \eta (m_{\nu_2} - m_{\nu_1}) e^{-i\delta_\nu},$$

$$2(\rho_4')^2 + (\rho_5')^2 = (m_{\nu_1} \sin^2 \eta + m_{\nu_2} \cos^2 \eta) e^{-2i\delta_\nu}.$$  \hspace{1cm} (41)

Solving these equations for $\sin \eta$ and $\cos \eta$, we find

$$\sin^2 \eta = \frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_1}},$$

$$\cos^2 \eta = \frac{m_{\nu_3} - m_{\nu_1}}{m_{\nu_2} - m_{\nu_1}}.$$  \hspace{1cm} (41)

Hence, the matrices $M_{\nu}$ and $U_{\nu}$, reparametrized in terms of the complex neutrino masses, take the form [21]

$$U_{\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta_\nu} \end{pmatrix},$$

$$M_{\nu} = \begin{pmatrix} m_{\nu_3} & 0 & \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_3} - m_{\nu_2})} e^{-i\delta_\nu} \\ 0 & m_{\nu_2} - m_{\nu_1} & 0 \\ \sqrt{(m_{\nu_3} - m_{\nu_1})(m_{\nu_3} - m_{\nu_2})} e^{-i\delta_\nu} & 0 & m_{\nu_3} - m_{\nu_1} \end{pmatrix}.$$  \hspace{1cm} (43)

The unitarity of $U_{\nu}$ constrains $\sin \eta$ to be real and thus
\[ |\sin \eta| \leq 1, \] this condition fixes the phases \( \phi_1 \) and \( \phi_2 \) as
\[ |m_{\nu_1}| \sin \phi_1 = |m_{\nu_2}| \sin \phi_2 = |m_{\nu_3}| \sin \phi_3. \quad (45) \]
The only free parameters in these matrices, are the phase \( \phi_\nu \), implicit in \( m_{\nu_1} \), \( m_{\nu_2} \) and \( m_{\nu_3} \), and the Dirac phase \( \delta_\nu \).

The neutrino mixing matrix

The neutrino mixing matrix \( V_{PMNS} \), is the product
\[ U_{PMNS}^\dagger U_\nu K, \]
where \( K \) is the diagonal matrix of the Majorana phase factors, defined by
\[ \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) = K \text{diag}(|m_{\nu_1}|, |m_{\nu_2}|, |m_{\nu_3}|) K^\dagger. \quad (46) \]

\[ V_{PMNS}^{th} = \begin{pmatrix} O_{11} \cos \eta + O_{31} \sin \eta e^{i\delta} & O_{12} \cos \eta + O_{32} \sin \eta e^{i\delta} & O_{13} \cos \eta - O_{33} \sin \eta e^{i\delta} \\ -O_{12} \cos \eta + O_{32} \sin \eta e^{i\delta} & O_{21} \cos \eta - O_{23} \sin \eta e^{i\delta} & O_{23} \cos \eta + O_{33} \sin \eta e^{i\delta} \\ O_{13} \cos \eta - O_{33} \sin \eta e^{i\delta} & O_{23} \cos \eta + O_{33} \sin \eta e^{i\delta} & O_{31} \cos \eta + O_{33} \sin \eta e^{i\delta} \end{pmatrix} \]

where \( \cos \eta \) and \( \sin \eta \) are given in eq (42), \( O_{ij} \) are given in eq (33) and (34), and \( \delta = \delta_\nu - \delta_\mu \).

To find the relation of our results with the neutrino mixing angles we make use of the equality of the absolute values of the elements of \( V_{PMNS}^{th} \) and \( V_{PMNS}^{PDG} \) [51], that is
\[ |V_{PMNS}^{th}| = |V_{PMNS}^{PDG}|. \quad (49) \]

This relation allows us to derive expressions for the mixing angles in terms of the charged lepton and neutrino masses
\[ |\sin \theta_{13}| = O_{21}, \quad |\sin \theta_{23}| = \frac{O_{22}}{\sqrt{O_{22}^2 + O_{23}^2}} \quad (50) \]
and
\[ |\tan \theta_{12}|^2 = \cot^2 \eta \frac{O_{11}^2 \cot^2 \eta + O_{31}^2 - 2O_{31}O_{11} \cot \eta \cos \delta}{O_{11}^2 \cot^2 \eta + O_{31}^2 + 2O_{31}O_{11} \cot \eta \cos \delta}. \quad (51) \]
The magnitudes of the reactor and atmospheric mixing angles, \( \theta_{13} \) and \( \theta_{23} \), are determined by the masses of the charged leptons only. Keeping terms up to order \( (m_\mu^2/m_\nu^2) \) and \( (m_\mu/m_\tau)^4 \), we get
\[ \sin \theta_{13} \approx \frac{1}{\sqrt{2}} \frac{1 + 4x^2 - \hat{m}_\mu^4}{\sqrt{1 + \hat{m}_\mu^2 + 5x^2 - \hat{m}_\mu^4}}, \quad \sin \theta_{23} \approx \frac{1}{\sqrt{2}} \frac{1 - 2\hat{m}_\mu^2 + \hat{m}_\mu^4}{\sqrt{1 - 4\hat{m}_\mu^2 + x^2 + 6\hat{m}_\mu^4}}. \quad (52) \]
Except for an overall phase factor \( e^{i\phi_1} \), which can be ignored, \( K \) is
\[ K = \text{diag}(1, e^{i\alpha}, e^{i\beta}), \quad (47) \]
where \( \alpha = 1/2(\phi_1 - \phi_2) \) and \( \beta = 1/2(\phi_1 - \phi_3) \) are the Majorana phases.

Therefore, the theoretical mixing matrix \( V_{PMNS}^{th} \), is given by

Substitution of the small numerical values \( \hat{m}_\mu = 5.94 \times 10^{-2} \) and \( x = m_e/m_\mu = 4.84 \times 10^{-3} \hat{m}_\mu = m_\mu/m_\tau = 5.95 \times 10^{-2} \) for the leptonic mass ratios \( \hat{m}_\mu \) and \( x \) in the right hand side of (52) yields the numerical values of \( \sin \theta_{13} \) and \( \sin \theta_{23} \)
\[ \sin \theta_{13} = 0.0034 \]
\[ \sin \theta_{23} = \frac{1}{\sqrt{2}} - 8.4 \times 10^{-6}. \quad (53) \]

From these numbers, it is evident that the theoretical values of \( \sin \theta_{13} \) and \( \sin \theta_{23} \) are very close to the corresponding tri-bimaximal mixing values \( \sin \theta_1^{tri} = 0 \) and \( \sin \theta_2^{tri} = 1/\sqrt{2} \) [52].

The dependence of \( \tan \theta_{12} \) on the Dirac phase \( \delta \), see (51), is very weak, since \( O_{31} \sim 1 \) but \( O_{11} \sim 1/\sqrt{2} \). Hence, we may neglect it when comparing (51) with the data on neutrino mixings.

The dependence of \( \tan \theta_{12} \) on the phase \( \phi_\nu \) and the physical masses of the neutrinos enters through the ratio of the neutrino mass differences under the square root sign, it can be made explicit with the help of the unitarity constraint on \( U_\nu \), eq. (43),
\[ m_{\nu_2} - m_{\nu_3} = \left( |m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_\nu \right)^{1/2} - |m_{\nu_1}| \cos \phi_\nu \]
\[ m_{\nu_3} - m_{\nu_1} = \left( |m_{\nu_3}|^2 - |m_{\nu_1}|^2 \sin^2 \phi_\nu \right)^{1/2} + |m_{\nu_1}| \cos \phi_\nu. \quad (54) \]
Similarly, the Majorana phases are given by

$$\sin 2\alpha = \sin(\phi_1 - \phi_2) = \frac{|m_{\nu_3}| \sin \phi_{\nu}}{|m_{\nu_1}| |m_{\nu_2}|} \times \left( \sqrt{|m_{\nu_2}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu}} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu}} \right),$$

(55)

$$\sin 2\beta = \sin(\phi_1 - \phi_2) = \frac{\sin \phi_{\nu}}{|m_{\nu_1}|} \left( |m_{\nu_3}| \sqrt{1 - \sin^2 \phi_{\nu}} + \sqrt{|m_{\nu_1}|^2 - |m_{\nu_3}|^2 \sin^2 \phi_{\nu}} \right).$$

(56)

A more complete and detailed discussion of the Majorana phases in the neutrino mixing matrix \( V_{PMNS} \) obtained in our model is given by J. Kubo \[53\].

**Neutrino masses and mixings**

In the present model, \( \sin^2 \theta_{13} \) and \( \sin^2 \theta_{23} \) are determined by the masses of the charged leptons in very good agreement with the experimental values \[11, 12, 54\],

\( \langle \sin^2 \theta_{13} \rangle^{th} = 1.1 \times 10^{-5}, \quad \langle \sin^2 \theta_{13} \rangle^{exp} \leq 0.046, \)

and

\( (\sin^2 \theta_{23})^{th} = 0.499, \quad (\sin^2 \theta_{23})^{exp} = 0.5^{+0.06}_{-0.05}. \)

In this model, the experimental restriction \(|\Delta m_{21}^2| < |\Delta m_{13}^2|\) implies an inverted neutrino mass spectrum, \( |m_{\nu_3}| < |m_{\nu_1}| < |m_{\nu_2}| \) [20].

As can be seen from eqs. \[51\] and \[54\], the solar mixing angle is sensitive to the neutrino mass differences and the phase \( \phi_{\nu} \) but is only very weakly sensitive to the charged lepton masses. If we neglect the small terms proportional to \( O_{11} \) and \( O_{12}^2 \) in \[51\], we get

\[ \tan^2 \theta_{12} = \frac{(\Delta m_{21}^2 + \Delta m_{13}^2 + |m_{\nu_3}|^2 |\cos \phi_{\nu}|^{1/2} - |m_{\nu_2}| |\cos \phi_{\nu}|)}{(\Delta m_{13}^2 + |m_{\nu_3}|^2 |\cos \phi_{\nu}|^{1/2} + |m_{\nu_2}| |\cos \phi_{\nu}|^{1/2})}. \]

(57)

From this expression, we may readily derive expressions for the neutrino masses in terms of \( \tan \theta_{12} \) and \( \phi_{\nu} \) and the differences of the squared masses

\[ |m_{\nu_3}| = \frac{\sqrt{\Delta m_{13}^2}}{2 \cos \phi_{\nu} \tan \theta_{12}} \frac{1 - \tan^4 \theta_{12} + r^2}{\sqrt{1 + \tan^2 \theta_{12}} \sqrt{1 + \tan^2 \theta_{12} + r^2}} \]

(58)

in a similar way, we obtain

\[ |m_{\nu_1}| = \frac{\sqrt{\Delta m_{13}^2}}{2 \cos \phi_{\nu} \tan \theta_{12}} \frac{1}{2 \cos \phi_{\nu} \tan \theta_{12}} (1 - \tan^2 \theta_{12}). \]

(60)

\[ |m_{\nu_1}| + |m_{\nu_2}| + |m_{\nu_3}| \approx \frac{\sqrt{\Delta m_{13}^2}}{2 \cos \phi_{\nu} \tan \theta_{12}} \left( 1 + 2 \sqrt{1 + 2 \tan^2 \theta_{12} (2 \cos^2 \phi_{\nu} - 1) + \tan^4 \theta_{12} - \tan^2 \theta_{12}} \right). \]

(61)

The most restrictive cosmological upper bound for this sum is \[17\]

\[ \sum |m_{\nu_i}| \leq 0.17 eV. \]

(62)

From this upper bound and the experimentally determined values of \( \tan \theta_{12} \) and \( \Delta m_{ij}^2 \), we may derive a lower bound for \( \cos \phi_{\nu} \)

\[ \cos \phi_{\nu} \geq 0.55 \]  \[ (63) \]

or \( 0 \leq \phi_{\nu} \leq 57^\circ \). The neutrino masses \( |m_{\nu_i}| \) assume their minimal values when \( \cos \phi_{\nu} = 1 \). When \( \cos \phi_{\nu} \) takes values in the range \( 0.55 \leq \cos \phi_{\nu} \leq 1 \), the neutrino masses change very slowly with \( \cos \phi_{\nu} \), see Figure 1. In the
absence of experimental information we will assume that \( \phi_\nu \) vanishes. Hence, setting \( \phi_\nu = 0 \) in our formula, we find

\[
\begin{align*}
|m_{\nu_1}| & \approx 0.056 \text{eV}, \\
|m_{\nu_2}| & \approx 0.055 \text{eV}, \\
|m_{\nu_3}| & \approx 0.022 \text{eV},
\end{align*}
\]

(64)

where we used the values \( \Delta m_{13}^2 = 2.6 \times 10^{-3} \text{eV}^2 \), \( \Delta m_{23}^2 = 7.9 \times 10^{-5} \text{eV}^2 \) and \( \tan \theta_{12} = 0.667 \), taken from [13].

**V_{PMNS} and the tri-bimaximal form**

Once the numerical values of the neutrino masses are determined, we may readily verify that the theoretical mixing matrix, \( V_{PMNS} \), is very close to the tri-bimaximal form of the mixing matrix,

\[
V_{PMNS}^{th} = \left( \begin{array}{ccc} 
\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
-\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}} \\
-\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} 
\end{array} \right) + \delta V_{PMNS}^{tri}
\]

(65)

where \( \delta V_{PMNS}^{tri} = V_{PMNS}^{th} - V_{PMNS}^{tri} \). From eqs. (55), (10), (42), (48), (52) and (64), the correction term to the tri-

bimaximal form of the mixing matrix comes out as

\[
\delta V_{PMNS}^{tri} \approx \begin{pmatrix}
1.94 \times 10^{-2} & -2.84 \times 10^{-2} & -3.4 \times 10^{-3} \\
2.21 \times 10^{-2} & 1.5 \times 10^{-2} & -8.2 \times 10^{-6} \\
1.8 \times 10^{-2} & 1.24 \times 10^{-2} & 3.1 \times 10^{-10}
\end{pmatrix}.
\]

(66)

**IV. Flavour Changing Neutral Currents (FCNC)**

Models with more than one Higgs \( SU(2) \) doublet have tree level flavour changing neutral currents. In the Minimal \( S_3 \)-invariant Extension of the Standard Model considered here, there is one \( Higgs \) \( SU(2) \) doublet per generation coupling to all fermions. The flavour changing Yukawa couplings may be written in a flavour labelled, symmetry adapted weak basis as

\[
\mathcal{L}_{Y}^{\text{FCNC}} = (\overline{E}_{aL}Y_{ab}^{ES}E_{bR} + \overline{U}_{aL}Y_{ab}^{US}U_{bR} + \overline{D}_{aL}Y_{ab}^{DS}D_{bR})H_{S}^0
\]

\[
+ (\overline{E}_{aL}Y_{ab}^{E1}E_{bR} + \overline{U}_{aL}Y_{ab}^{U1}U_{bR} + \overline{D}_{aL}Y_{ab}^{D1}D_{bR})H_{1}^0 + (\overline{E}_{aL}Y_{ab}^{E2}E_{bR} + \overline{U}_{aL}Y_{ab}^{U2}U_{bR} + \overline{D}_{aL}Y_{ab}^{D2}D_{bR})H_{2}^0 + \text{h.c.}
\]

(67)

where the entries in the column matrices \( E' \), \( U' \), and \( D' \) are the left and right fermion fields and \( Y_{ab}^{(e,u,d)} \), \( Y_{ab}^{(e,u,d)1} \), \( Y_{ab}^{(e,u,d)2} \) are \( 3 \times 3 \) matrices of the Yukawa couplings of the fermion fields to the neutral Higgs fields \( H_{1}^0 \) and \( H_{2}^0 \) in the the \( S_3 \)-singlet and doublet representations, respectively.

In this basis, the Yukawa couplings of the Higgs fields to each family of fermions may be written in terms of matrices \( M_{Y}^{(e,u,d)} \), which give rise to the corresponding mass matrices \( M_{Y}^{(e,u,d)} \) when the gauge symmetry is spontaneously broken. From this relation we may calculate the flavour changing Yukawa couplings in terms of the fermion masses and the vacuum expectation values of the neutral Higgs fields. For example, the matrix \( M_{Y} \) is written in terms of the matrices of the Yukawa couplings of the charged leptons as

\[
M_{Y}^{c} = Y_{w}^{E1}H_{1}^0 + Y_{w}^{E2}H_{2}^0,
\]

(68)

in this expression, the index \( w \) means that the Yukawa matrices are defined in the weak basis.

\[
Y_{w}^{E1} = \frac{m_{\tau}}{v_{1}} \begin{pmatrix}
0 & \frac{\tilde{m}_{\mu}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \sqrt{1 + \tilde{m}_{\mu}^2 - \beta}\n\frac{1}{\sqrt{2}} \sqrt{1 + \tilde{m}_{\mu}^2 - \beta} & 0 & 0
\end{pmatrix}
\]

\[
Y_{w}^{E2} = \frac{1}{\sqrt{2}} \sqrt{1 + \tilde{m}_{\mu}^2 - \beta}\]

(69)
and

\[
Y_w^E = \frac{m_\tau}{v_2} \begin{pmatrix}
\frac{1}{\sqrt{2}} \sqrt{1+x^2} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} \sqrt{1+x^2} & \frac{1}{\sqrt{2}} \sqrt{1+x^2-m_r^2} \\
0 & \frac{1}{\sqrt{2}} \sqrt{1+x^2-m_r^2} e^{i8_x} & 0 \\
\end{pmatrix},
\]

(70)

The Yukawa couplings of immediate physical interest in the computation of the flavour changing neutral currents are those defined in the mass basis, according to the computation of the flavour changing neutral currents

\[
\text{Y}_{\text{E}} = U_{\text{EL}}^* Y_{\text{E}}^U U_{\text{ER}},
\]

where \(U_{\text{EL}} \) and \(U_{\text{ER}} \) are the matrices that diagonalize the charged lepton mass matrix defined in eqs. (20) and (34). We obtain

\[
\hat{Y}_{\text{E}1} \approx \frac{m_\tau}{v_1} \begin{pmatrix}
2\bar{m}_e & -\frac{1}{2} \bar{m}_e & \frac{1}{2} x \\
-\bar{m}_\mu & \frac{1}{2} \bar{m}_\mu & -\frac{1}{2} \\
\frac{1}{2} \bar{m}_\mu x^2 & -\frac{1}{2} \bar{m}_\mu & \frac{1}{2} \\
\end{pmatrix},
\]

(71)

and

\[
\hat{Y}_{\text{E}2} \approx \frac{m_\tau}{v_2} \begin{pmatrix}
\bar{m}_e & \frac{1}{2} \bar{m}_e & -\frac{1}{2} x \\
\bar{m}_\mu & \frac{1}{2} \bar{m}_\mu & \frac{1}{2} \\
\frac{1}{2} \bar{m}_\mu x^2 & \frac{1}{2} \bar{m}_\mu & \frac{1}{2} \\
\end{pmatrix},
\]

(72)

where \(\bar{m}_\mu = 5.94 \times 10^{-2}, \bar{m}_e = 2.876 \times 10^{-4} \) and \(x = m_\tau / m_\mu = 4.84 \times 10^{-3} \). All the non-diagonal elements are responsible for tree-level FCNC processes. The actual values of the Yukawa couplings in eqs. (71) and (72) still depend on the VEV’s of the Higgs fields \(v_1 \) and \(v_2 \), and, hence, on the Higgs potential. If the \(S^*_2 \) symmetry in the Higgs sector is preserved \([23], (H^0_1) = (H^0_2) = v \). To make an order of magnitude estimate of the Yukawa couplings in the Yukawa matrices, \(m_\tau / v \), we may further assume that the VEV's for all the Higgs fields are comparable, that is, \(\langle H^0_1 \rangle = \langle H^0_2 \rangle = \langle H^0_3 \rangle = \sqrt{2} \sqrt{2} m_\tau \), then, \(m_\tau / v = \sqrt{2} \sqrt{2} m_\tau / M_W \) and we may estimate the numerical values of the Yukawa couplings from the numerical values of the lepton masses. For instance, the amplitude of the flavour violating process \(\tau^- \rightarrow \mu^- e^+ e^- \), is proportional to \(Y_{\text{E}} Y_{\text{E}}^\dagger \) [56]. Then, the leptonic branching ratio,

\[
\text{Br}(\tau \rightarrow \mu e^+ e^-) = \frac{\Gamma(\tau \rightarrow \mu e^+ e^-)}{\Gamma(\tau \rightarrow e\nu\bar{\nu}) + \Gamma(\tau \rightarrow \mu \nu\bar{\nu})}
\]

(73)

and

\[
\Gamma(\tau \rightarrow \mu e^+ e^-) \approx \frac{m_\tau^5}{3 \times 2^{10/3} \pi^3} \frac{(Y_{\text{E}} Y_{\text{E}}^\dagger)^2}{M_{\text{H},1,2}^4}
\]

(74)

which is the dominant term, and the well known expressions for \(\Gamma(\tau \rightarrow e\nu\bar{\nu}) \) and \(\Gamma(\tau \rightarrow \mu \nu\bar{\nu}) \) [51], give

\[
\text{Br}(\tau \rightarrow \mu e^+ e^-) \approx \frac{9}{4} \left( \frac{m_\tau m_\mu}{m_\tau} \right)^2 \left( \frac{m_\tau}{M_{\text{H},1,2}} \right)^4,
\]

(75)

taking for \(M_{H,1,2} \sim 120 \text{ GeV} \), we obtain

\[
\text{Br}(\tau \rightarrow \mu e^+ e^-) \approx 3.15 \times 10^{-17}
\]

well below the experimental upper bound for this process, which is \(2.7 \times 10^{-7} \) [57]. Similar computations give the following estimates

\[
\begin{align*}
\text{Br}(\tau \rightarrow e\gamma) & \approx \frac{3 \alpha}{8 \pi} \left( \frac{m_\mu}{M_H} \right)^4, \\
\text{Br}(\tau \rightarrow \mu \gamma) & \approx \frac{3 \alpha}{128 \pi} \left( \frac{m_\mu}{m_\tau} \right)^2 \left( \frac{m_\tau}{M_H} \right)^4, \\
\text{Br}(\tau \rightarrow 3\mu) & \approx \frac{9}{64} \left( \frac{m_\mu}{M_H} \right)^4, \\
\text{Br}(\mu \rightarrow 3e) & \approx 1 \left( \frac{m_\mu}{m_\tau} \right)^2 \left( \frac{m_\tau}{M_H} \right)^4, \\
\text{Br}(\mu \rightarrow e\gamma) & \approx \frac{27 \alpha}{64 \pi} \left( \frac{m_\mu}{m_\tau} \right)^4 \left( \frac{m_\tau}{M_H} \right)^4.
\end{align*}
\]

We see that FCNC processes in the leptonic sector are strongly suppressed by the small values of the mass ratios \(m_e / m_\tau, m_\mu / m_\tau, \text{ and } m_\tau / M_H \). The numerical estimates of the branching ratios and the corresponding experimental upper bounds are shown in Table 11. It may be seen that, in all cases considered, the numerical values for the branching ratios of the FCNC in the leptonic sector are well below the corresponding experimental upper bounds. The matrices of the quark Yukawa couplings may be computed in a similar way. Numerical values for the Yukawa couplings for u and d-type quarks are given in our previous paper [20]. There, it was found that, due to the strong hierarchy in the quark masses and the corresponding small or very small mass ratios, the numerical values of all the Yukawa couplings in the quark sector are small or very small. Kubo, Okada and Sakamaki [62] have investigated the breaking of the gauge symmetry in the present \(S_3\)-invariant extension of the Standard Model with the \(S_3\)-invariant Higgs potential \(V_H(H_S, H_2) \) analyzed by Pakvasa and Sugawara [23]. They found that it is possible that all physical Higgs bosons, except one neutral one, could become sufficiently heavy \((M_H \sim 10 \text{ TeV}) \) to suppress all the flavour changing neutral current processes in the quark sector of the theory without having a problem with triviality.

V. CONCLUSIONS

By introducing three Higgs fields that are \(SU(2)_L \) doublets in the theory, we extended the concept of
flavour and generations to the Higgs sector and formulated a Minimal $S_3$-Invariant Extension of the Standard Model [20]. A well defined structure of the Yukawa couplings is obtained, which permits the calculation of mass and mixing matrices for quarks and leptons in a unified way. A further reduction of redundant parameters is achieved in the leptonic sector by introducing a $Z_2$ symmetry. The flavour symmetry group $Z_2 \times S_3$ relates the mass spectrum and mixings. This allowed us to derive explicit, analytical expressions for all entries in the neutrino mixing matrix, $V_{PMNS}$ as functions of the masses of the charged leptons and neutrinos and two phases $\delta$ and $\phi_\nu$ [21]. In this model, the tri-bimaximal mixing structure of $V_{PMNS}$ and the magnitudes of the three mixing angles are determined by the interplay of the flavour $S_3 \times Z_2$ symmetry, the seesaw mechanism and the charged lepton mass hierarchy. We also found that $V_{PMNS}$ has three CP violating phases, namely, one Dirac phase $\delta = \delta_\nu - \delta_\tau$ and two Majorana phases, $\alpha$ and $\beta$, which are functions of the neutrino masses and the phase $\phi_\nu$ which is independent of the Dirac phase. The numerical values of the reactor, $\theta_{13}$, and the atmospheric, $\theta_{23}$, mixing angles are determined by the masses of the charged leptons only, in very good agreement with the experiment. The solar mixing angle $\theta_{12}$ is almost insensitive to the values of the masses of the charged leptons, but its experimental value allowed us to fix the scale and origin of the neutrino mass spectrum, which has an inverted hierarchy, with the values $|m_{\nu_1}| = 0.056eV$, $|m_{\nu_2}| = 0.055eV$ and $|m_{\nu_3}| = 0.022eV$. In the present work, we obtained explicit expressions for the matrices of the Yukawa couplings of the lepton sector parametrized in terms of the charged lepton masses and the VEV's of the neutral Higgs bosons in the $S_3$-doublet representation. These Yukawa matrices are closely related to the fermion mass matrices and have a structure of small and very small entries reflecting the observed charged lepton mass hierarchy. With the help of the Yukawa matrices, we computed the branching ratios of a number of FCNC processes and found that the branching ratios of all FCNC processes considered are strongly suppressed by powers of the small mass ratios $m_e/m_\tau$ and $m_\mu/m_\tau$, and by the ratio $(m_\tau/M_{H_{1,2}})^4$, where $M_{H_{1,2}}$ is the mass of the neutral Higgs bosons in the $S_3$-doublet. Taking for $M_{H_{1,2}}$ a very conservative value ($M_{H_{1,2}} \approx 120$ GeV), we found that the numerical values of the branching ratios of the FCNC in the leptonic sector are well below the corresponding experimental upper bounds by many orders of magnitude. We may add that although the theoretical values of the branching ratios of FCNC processes computed in this work are much smaller than their experimental upper bounds measured in terrestrial laboratories, they still are larger than the vanishing or nearly vanishing values allowed by the Standard Model, and could be important in astrophysical processes [63]. It has already been argued that small FCNC processes mediating non-standard quark-neutrino interactions could be important in the theoretical description of the gravitational core collapse and shock generation in the explosion stage of a supernova [64, 65].

VI. ACKNOWLEDGEMENTS

We are indebted to Prof. S. Pakvasa for useful comments on this work. This work was partially supported by CONACYT México under contract No 51554-F and by DGAPA-UNAM under contract PAPIIT-IN115207-2.

| FCNC processes | Theoretical BR | Experimental BR | References |
|----------------|----------------|----------------|------------|
| $\tau \to 3\mu$ | $8.43 \times 10^{-14}$ | $2 \times 10^{-7}$ | B. Aubert et. al. [52] |
| $\tau \to \mu e^+ e^-$ | $3.15 \times 10^{-17}$ | $2.7 \times 10^{-7}$ | B. Aubert et. al. [52] |
| $\tau \to \nu \gamma$ | $9.24 \times 10^{-15}$ | $6.8 \times 10^{-8}$ | B. Aubert et. al. [58] |
| $\tau \to e \gamma$ | $5.22 \times 10^{-16}$ | $1.1 \times 10^{-11}$ | B. Aubert et. al. [50] |
| $\mu \to 3e$ | $2.53 \times 10^{-16}$ | $1 \times 10^{-12}$ | U. Bellgardt et al. [60] |
| $\mu \to e \gamma$ | $2.42 \times 10^{-20}$ | $1.2 \times 10^{-11}$ | M. L. Brooks et al. [61] |

[1] C. K. Jung, C. Mc Grew, T. Kajita and T. Mann, Annu. Rev. Nucl. Part. Sci, 51 (2001) 451.
[2] R. N. Mohapatra and A. Y. Smirnov, Annu. Rev. Nucl. Part. Sci. 56, 569 (2006) [arXiv:hep-ph/0603118]
[3] M. Altmann et al. [GNO collaboration], Phys. Lett. B 616, (2005), 174.
[4] M. B. Smy et al. [SK collaboration], Phys. Rev. D 69 (2004), 011104.
[5] Q. R. Ahmad et al. [SNO collaboration], Phys. Rev. Lett. 89 (2002) 011301.
[6] B. Aharmim et al. [SNO collaboration], Phys. Rev. C72 (2005), 055502. arxiv:nucl-ex/0502028
[7] S. Fukuda et al. [SK collaboration] Phys. Lett. B539 (2002) 179.
[8] Y. Ashie et al., Phys. Rev. Lett. 93 (2004) 101801; hep-ex/0404034. 
