AFFINE RSK CORRESPONDENCE AND CRYSTALS OF LEVEL ZERO
EXTREMAL WEIGHT MODULES

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Abstract. We give an affine analogue of the Robison-Schensted-Knuth (RSK) correspondence, which generalizes the affine Robinson-Schensted correspondence by Chmutov-Pylyavskyy-Yudovina. The affine RSK map sends a generalized affine permutation of period \( (m, n) \) to a pair of tableaux \((P, Q)\) of the same shape, where \( P \) belongs to a tensor product of level one perfect Kirillov-Reshetikhin crystals of type \( A_{m-1}^{(1)} \), and \( Q \) belongs to a crystal of extremal weight module of type \( A_{n-1}^{(1)} \) when \( m, n \geq 2 \). We consider two affine crystal structures of types \( A_{m-1}^{(1)} \) and \( A_{n-1}^{(1)} \) on the set of generalized affine permutations, and show that the affine RSK map preserves the crystal equivalence. We also give a dual affine Robison-Schensted-Knuth correspondence.

1. Introduction

The Robinson-Schensted-Knuth (simply RSK) correspondence \[19\] is a fundamental algorithm in algebraic combinatorics with rich applications and generalizations in various areas. In representation theory, we may regard it as a combinatorial aspect of Howe duality \[11\], where the pair of groups \((GL_m, GL_n)\) acts as mutual centralizers on the symmetric or exterior algebra generated by the tensor product of their natural representations. From the viewpoint of crystal base theory, the bijection can be further understood as an isomorphism of \((gl_m, gl_n)\)-bicrystals \[21\].

The Robinson-Schensted (simply RS) correspondence, a special case of RSK correspondence restricted to \( n \times n \) permutation matrices, gives a bijection from the symmetric group \( S_n \) on \( n \) letters to the set of pairs of standard tableaux of the same shape with size \( n \), and it provides a combinatorial tool to study the left and right cells of \( S_n \) in Kazhdan-Lusztig theory \[18\].

An affine analogue of the RS correspondence has been introduced by Shi \[27, 28\] in the study of affine Kazhdan-Lusztig cells, which maps an affine permutation \( w \) to a pair of tabloids \((P(w), Q(w))\), but not necessarily in an injective way. Recently in \[4\], Chmutov-Pylyavskyy-Yudovina construct a bijection from the set of extended affine permutations \( w \) to the set of triples \((P(w), Q(w), \rho)\), which recovers the Shi’s algorithm and characterizes each fiber of \((P(w), Q(w))\), where \( \rho \) is an integral vector satisfying a condition called dominance. The main ingredient of the affine RS correspondence in \[4\] is the affine matrix-ball construction, which generalizes the matrix-ball construction in the usual RSK algorithm \[7\].

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In this paper we show that the affine RS correspondence in \cite{3} can be naturally extended to a bijection on the set of generalized affine permutations, which also admits an isomorphism of affine crystals.

Let $m$ and $n$ be positive integers greater than 1. Let $\hat{\mathcal{M}}_{m \times n}$ be the set of matrices $A = (a_{ij})_{i,j \in \mathbb{Z}}$ with non-negative integral entries such that $a_{i+m,j+n} = a_{ij}$ for all $i,j \in \mathbb{Z}$, and for each $j$, $a_{ij} = 0$ except for finitely many $i$’s. We show that there exists a bijection

$$
\kappa : \hat{\mathcal{M}}_{m \times n} \rightarrow \bigcup_{\lambda \in \mathcal{P}_m \cap \mathcal{P}_n} CSST_{[m]}(\lambda) \times \mathcal{B}_n(\lambda).
$$

Here the union is over the partitions $\lambda$ with length no more than $\min\{m,n\}$, $CSST_{[m]}(\lambda)$ is the set of column semistandard tableaux of shape $\lambda$ with entries from 1 to $m$, and $\mathcal{B}_n(\lambda)$ is the set of tableaux of shape $\lambda$ with entries in $\mathbb{Z}$, whose rectangular subtableaux are semistandard when $\lambda$ is decomposed into rectangles along its columns. The bijection is also well-defined when $m = 1$ or $n = 1$.

The bijection $\kappa$ recovers the affine RS correspondence in \cite{3} after a suitable standardization. A key observation is that the dominance condition on $\rho$ in \cite{3} fits nicely into description of the connected components in $\mathcal{B}_n(\lambda)$, and hence gives the image of $\kappa$ as in (1.1).

We next consider an affine crystal structure related to (1.1). We see that $CSST_{[n]}(\lambda)$ can be viewed as a finite affine crystal of type $A^{(1)}_{m-1}$ isomorphic to the tensor product of Kirillov-Reshetikhin crystals corresponding to fundamental representations (see, for example, \cite{6} and reference therein). Also, $\mathcal{B}_n(\lambda)$ has a structure of $A^{(1)}_{n-1}$-crystal isomorphic to the crystal of an extremal weight module associated to the level-zero extremal weight corresponding to $\lambda$ \cite{10} \cite{17}. It is shown in \cite{13} using the theory of semi-infinite LS paths \cite{14}, but an elementary and self-contained proof of this isomorphism for $\mathcal{B}(\lambda)$ is given in this paper. On the other hand, we consider two affine crystal structures of types $A^{(1)}_{m-1}$ and $A^{(1)}_{n-1}$ on $\hat{\mathcal{M}}_{m \times n}$, where the Kashiwara operators for $A^{(1)}_{m-1}$ commute with those for $A^{(1)}_{n-1}$. We expect that the affine crystal structures on $\hat{\mathcal{M}}_{m \times n}$ coincides with the ones in \cite{22}.

We show that the bijection $\kappa$ preserves the crystal equivalence for these crystal structures. Indeed, we show that $\kappa$ commutes with the Kashiwara operators for both $A_{m-1}$ and $A^{(1)}_{n-1}$, while the operators $\tilde{c}_0$ and $\tilde{f}_0$ on $A$ for $A^{(1)}_{m-1}$ may change the second component $Q$ in (1.1) under $\kappa$. As a corollary, we have an isomorphism of $(A_{m-1}, A^{(1)}_{n-1})$-bicrystals

$$
\hat{\mathcal{M}}_{m \times n} \cong \bigoplus_{\lambda \in \mathcal{P}_m \cap \mathcal{P}_n} CSST_{[m]}(\lambda) \times \mathcal{B}_n(\lambda),
$$

where $\oplus$ means a disjoint union of crystals. We remark that the Knuth equivalence for the affine RS correspondence \cite{8} can be recovered using the crystal structure here.

We also have a dual affine RSK correspondence, where $\hat{\mathcal{M}}_{m \times n}$ is replaced by the set of binary matrices and $CSST_{[m]}(\lambda)$ is replaced by the set of row semistandard tableaux of the conjugate shape $\lambda'$. This can be viewed as a level-zero analogue of the decomposition of the crystal \cite{5} \cite{9} associated to the higher-level $q$-deformed Fock space \cite{39}.
Finally, we remark that an affine RSK correspondence is also given by Imamura-Mucciconi-Sasamoto [12], where the algorithm is given by using dynamics of Sagan-Stanley’s skew RSK correspondence [29]. The algorithm looks different from the one by affine matrix-ball construction in this paper. It would be interesting to compare these two algorithms. A representation theoretic interpretation of the identity corresponding to the bijection in [12] is also recently given using representations of current Lie algebras [5].

The paper is organized as follows: In Section 2 we introduce necessary background and notions for the affine RSK algorithm, which are analogous to those introduced in [4]. In Section 3, we construct the affine RSK correspondence \( \kappa \) in (1.1) (Theorem 3.10), where the proof of its well-definedness and bijectivity is given in Section 5. In Section 4 we discuss affine crystal structure on both sides of (1.1), and its compatibility with \( \kappa \) (Theorem 4.15). The proof is given in Section 6. In Section 7, we give a dual affine RSK correspondence.

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2. Preliminaries

Let us introduce some necessary notations and terminologies following [4] with slight modification and generalization.

2.1. Generalized affine permutations. Throughout the paper, let \( m \) and \( n \) denote the positive integers. Let \( \mathbb{Z}_{\geq 0} \) denote the set of non-negative integers and \( [k] = \{1, 2, \ldots, k\} \) for \( k \geq 1 \) with the usual linear order.

Let

\[
\mathcal{M}_{m \times n} = \left\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid \begin{array}{l}
(1) a_{ij} \in \mathbb{Z}_{\geq 0} \text{ and } a_{i+m,j+n} = a_{ij} \text{ for all } i, j \in \mathbb{Z}, \\
(2) \text{for each } j, a_{ij} = 0 \text{ except for finitely many } i's
\end{array} \right\}.
\]

An extended affine permutation of \( n \) is a bijection \( w : \mathbb{Z} \rightarrow \mathbb{Z} \) satisfying \( w(i+n) = w(i) \) for each \( i \in \mathbb{Z} \). The set of extended affine permutations of \( n \) is the extended Weyl group of type \( A^{1}_{n-1} \) if \( n > 1 \). Suppose that \( m = n \) and a matrix \( A \in \mathcal{M}_{m \times n} \) satisfies \( \sum_{j \in \mathbb{Z}} a_{ij} = \sum_{i \in \mathbb{Z}} a_{ij} = 1 \) for each \( i \) and \( j \). Then \( A \) can be viewed as an extended affine permutation \( w_{A} \), which is given by \( w_{A}(i) = j \) for a pair \((i, j)\) such that \( a_{ij} = 1 \). In this sense, we call an element of \( \mathcal{M}_{m \times n} \) a generalized affine permutation of period \((m, n)\). Following [4], we call \( A \in \mathcal{M}_{n \times n} \) a partial (extended affine) permutation if \( \sum_{j \in \mathbb{Z}} a_{ij} = \sum_{i \in \mathbb{Z}} a_{ij} \leq 1 \) for each \( i \) and \( j \). For \( A \in \mathcal{M}_{m \times n} \), let

\[
\text{supp}(A) = \{ c = (i, j) \in \mathbb{Z}^{2} \mid a_{ij} > 0 \}
\]

be the support of \( A \in \mathcal{M}_{m \times n} \). We denote by \( \emptyset \) the zero matrix.

An element \( c = (i, j) \) in the lattice \( \mathbb{Z}^{2} \) will be often called a cell. We regard each cell as a position of the entry \( a_{ij} \) in \( A = (a_{ij}) \in \mathcal{M}_{m \times n} \), where the row index \( i \) (resp. the column index \( j \)) is increasing from top to bottom (resp. from left to right). With this convention, we define partial orders \( >_{NW}, \geq_{NW} \) and \( \leq_{NW} \) on \( \mathbb{Z}^{2} \) as follows:

1. \( c_{1} >_{NW} c_{2} \) if and only if \( i_{1} < i_{2} \) and \( j_{1} < j_{2} \),
2. \( c_{1} \geq_{NW} c_{2} \) if and only if \( i_{1} \leq i_{2} \) and \( j_{1} \leq j_{2} \),
(3) \( c_1 \leq c_2 \) if and only if \( i_1 \geq i_2 \) and \( j_1 \leq j_2 \).

for \( c_1 = (i_1, j_1), c_2 = (i_2, j_2) \in \mathbb{Z}^2 \). By convention, we use \( \mathbb{N} \) (or \( \mathbb{E}, \mathbb{W}, \mathbb{S} \)) to emphasize strict inequality, while \( \mathbb{N} \) (or \( \mathbb{E}, \mathbb{W}, \mathbb{S} \)) allows equality of the co-ordinates of cells.

Let \( \tau = \tau_{m,n} \) denote the bijection on \( \mathbb{Z}^2 \) given by

\[
\tau(i, j) = (i + m, j + n) \quad ((i, j) \in \mathbb{Z}^2).
\]

Example 2.1. The following is an example of a generalized affine permutation in \( \hat{M}_{4 \times 5} \).

The co-ordinate of \(*\) is \((1, 2)\) and hence the co-ordinates of the cells with red entries are \((-1, -6), (3, -1), (7, 4), (11, 9)\) from northwest to southeast, while those with blue entries are \((-2, 3), (2, 8), (6, 13), (10, 18)\).

2.2. Standardization of generalized affine permutations. Let \( a = (a_j)_{j \in \mathbb{Z}} \) be a single row matrix with \( a_j \in \mathbb{Z}_{\geq 0} \). If \( r = \sum_{j \in \mathbb{Z}} a_j < \infty \), we define \( [r] \times \infty \) matrix \( a^\circ = (a^\circ_{ij})_{i \in [r], j \in \mathbb{Z}} \) by

\[
a^\circ_{ij} = \begin{cases} 
1 & \sum_{s=j+1}^{\infty} a_s < i \leq \sum_{s=j}^{\infty} a_s, \\
0 & \text{otherwise}
\end{cases}
\]

for each \( j \in \mathbb{Z} \). For example, if \( a = (\ldots, 0, 1, 0, 3, 2, 0, \ldots) \) with \( \sum_{j \in \mathbb{Z}} a_j = 6 \), then

\[
a^\circ = \begin{pmatrix}
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\
\end{pmatrix}.
\]

Note that each row of \( a^\circ \) has exactly one non-zero entry 1. If \( \sum_{j \in \mathbb{Z}} a_j = 0 \), we regard \( a^\circ \) as an empty matrix with no row (or removing the matrix \( a \)).
For a generalized affine permutation $A = (a_{ij})_{i,j \in \mathbb{Z}} \in \hat{M}_{m \times n}$, we define $A^\circ$ to be the matrix obtained from $A$ by replacing each row $A_i = (a_{ij})_{j \in \mathbb{Z}}$ with $A_i^\circ$ for $i \in \mathbb{Z}$ as in (2.1). Similarly, we define $A'$ with respect to the columns of $A$, that is, $A' = ((A^t)^\circ)^t$, where $A^t$ denotes the transpose of $A$. We define the \textit{standardization} of $A$ to be

\begin{equation}
A^{st} = (A')^\circ.
\end{equation}

It is straightforward to see that $A^{st} = (A^\circ)' = (A')^\circ$. Note that $A^{st}$ is an extended affine permutation of $K$ if $A$ is non-zero and $K = \sum_{i=1}^{m} \sum_{j \in \mathbb{Z}} a_{ij} = \sum_{j=1}^{n} \sum_{i \in \mathbb{Z}} a_{ij}$. We regard $\emptyset^{st}$ as an empty matrix with no row and no column.

Let us describe more explicitly the index sets for the rows and columns in $A^{st}$. Let

\begin{equation}
\text{row}(A) = \left( \sum_{j \in \mathbb{Z}} a_{1j}, \ldots, \sum_{j \in \mathbb{Z}} a_{mj} \right), \quad \text{col}(A) = \left( \sum_{i \in \mathbb{Z}} a_{i1}, \ldots, \sum_{i \in \mathbb{Z}} a_{in} \right)
\end{equation}

be the row and column contents of $A$, respectively and write $\text{row}(A) = (\alpha_1, \ldots, \alpha_m)$, $\text{col}(A) = (\beta_1, \ldots, \beta_n)$. Let $K = \alpha_1 + \cdots + \alpha_m = \beta_1 + \cdots + \beta_n$ and assume that $K \geq 1$, i.e., $A$ is not the zero matrix. For $i \in [m]$ and $j \in [n]$, let

\begin{align*}
I_i &= \{ k \in [K] \mid \alpha_1 + \cdots + \alpha_{i-1} < k \leq \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i \}, \\
J_j &= \{ l \in [K] \mid \beta_1 + \cdots + \beta_{j-1} < l \leq \beta_1 + \cdots + \beta_{j-1} + \beta_j \},
\end{align*}

where we understand the empty sum is 0, and let

\begin{align*}
I_{i+sn} &= I_i + sK, \quad J_{j+sn} = J_j + sK \quad (s \in \mathbb{Z}).
\end{align*}

Then we have

\begin{equation}
[K] = \bigsqcup_{i \in [m]} I_i = \bigsqcup_{j \in [n]} J_j,
\end{equation}

\begin{equation}
\mathbb{Z} = \bigsqcup_{i \in \mathbb{Z}} I_i = \bigsqcup_{j \in \mathbb{Z}} J_j.
\end{equation}

**Example 2.2.** Let $m = 3, n = 4$. If $A \in \hat{M}_{3 \times 4}$ is a generalized affine permutation given by

\[
\begin{array}{ccc|ccc}
1 & 1 & 3 & 1 & 1 & 3 \\
& 2 & & 1 & 1 & 3 \\
& & & 2 & & 3 \\
1 & & & & & 3 \\
& & & & & \\
& & & & & \\
\end{array}
\]

then we have

\begin{align*}
I_1 &= \{ 1, 3, 4, 5 \}, \quad I_2 = \{ 6 \}, \quad I_3 = \{ 7, 8 \}, \\
J_1 &= \{ 1 \}, \quad J_2 = \{ 2 \}, \quad J_3 = \{ 3 \}, \quad J_4 = \{ 4, 5, 6, 7, 8 \},
\end{align*}

and $A^{st}$ is
Remark 2.3. For $c = (i, j) \in \text{supp}(A)$, we denote by $A^c_{st}$ the matrix in $M_{\mathbb{Z} \times \mathbb{Z}}$, which is equal to $A^c_{st}$ at the positions of $(k, l) \in I_i \times J_j$ and has zero entries elsewhere. Then $A^c_{st}$ has an $a_{ij} \times a_{ij}$ block submatrix at $I_i \times J_j$ with 1 on the antidiagonal, and zero entries elsewhere.

The following lemmas can be checked easily.

Lemma 2.4. Let $c_1, c_2 \in \text{supp}(A^c_{st})$ be given with $c_1 = (i_1, j_1)$ and $c_2 = (i_2, j_2)$.

1. If $i_1 < i_2$ and $c_1, c_2 \in I_i \times \mathbb{Z}$ for some $i \in \mathbb{Z}$, then $c_2 \preceq_{\text{ne}} c_1$.
2. If $j_1 < j_2$ and $c_1, c_2 \in \mathbb{Z} \times J_j$ for some $j \in \mathbb{Z}$, then $c_1 \preceq_{\text{ne}} c_2$.

Lemma 2.5. Let $c_1, c_2 \in \text{supp}(A^c_{st})$ be given with $c_i \in \text{supp}(A^c_{st})$ for some $c_i' \in \text{supp}(A)$ ($i = 1, 2$). Then we have

1. $c_2 \preceq_{\text{WY}} c_1$ if and only if $c_2' \preceq_{\text{WY}} c_1'$,
2. $c_2 = \tau_{K,K}(c_1)$ implies $c_2' = \tau_{m,n}(c_1')$.

Remark 2.6. Let $a = (a_j)_{j \in \mathbb{Z}}$ with $a_j \in \mathbb{Z}_{\geq 0}$ and assume that $r = \sum_{j \in \mathbb{Z}} a_j < \infty$. We define $[r] \times \mathbb{Z}$ matrix $a^* = (a^*_{ij})_{i \in [r], j \in \mathbb{Z}}$ by

$$a^*_{ij} = \begin{cases} 1 & \sum_{s=-\infty}^{j-1} a_s < i \leq \sum_{s=-\infty}^{j} a_s, \\ 0 & \text{otherwise.} \end{cases}$$
Then \( a^* \) is given by rearranging the rows of \( a^c \) in a reverse way. For example, if \( a = (\ldots, 0, 1, 0, 3, 2, 0, \ldots) \), we have

\[
a^* = \begin{pmatrix}
\cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots
\end{pmatrix},
\]

In general, for \( A \in \mathcal{M}_{m \times n} \), we define \( A^* \) and \( A^{*t} = ((A^t)^*)^t \). We may define other versions of standardization of \( A \) using \( c, c', \bullet \) and \( \bullet' \).

### 2.3. Proper numberings.

**Definition 2.7** (cf. [4, Definition 3.1]). A numbering on \( A \) is a function \( d : \text{supp}(A) \rightarrow \mathbb{Z} \). It is called **proper** if

1. \( d(c_2) < d(c_1) \) for \( c_1, c_2 \in \text{supp}(A) \) with \( c_2 >_\text{NW} c_1 \),
2. for \( c_1 \in \text{supp}(A) \), there exists \( c_2 \in \text{supp}(A) \) with \( c_2 >_\text{NW} c_1 \) and \( d(c_2) = d(c_1) - 1 \).

The conditions (1) and (2) are called **monotone** and **continuous**, respectively.

The notion of proper numbering on a partial permutation was introduced in [4]. It also plays an important role when we describe an affine analogue for the matrix-ball construction of the RSK correspondence in this paper.

Let \( d \) be a numbering on \( A \). Let \( d^{st} \) be the numbering on \( \text{supp}(A^{st}) \) given by

\[
d^{st}(c) = d(c') \quad \text{if} \quad c \in \text{supp}(A^{st}_c) \quad \text{for some} \quad c' \in \text{supp}(A).
\]

**Lemma 2.8.** We have the following:

1. \( d \) is a proper numbering on \( A \) if and only if \( d^{st} \) is a proper numbering on \( A^{st} \),
2. any proper numbering on \( A^{st} \) is given by \( d^{st} \) for a unique proper numbering \( d \) on \( A \).

**Proof.** (1) Since no two cells corresponding to non-zero entries in \( A^{st}_c \) are comparable with respect to \( >_\text{NW} \) (see Remark 2.3), it follows from Lemma 2.5(1) that \( d \) satisfies the conditions Definition 2.7(1) and (2) if and only if \( d^{st} \) does so.

(2) Let \( d' \) be a proper numbering on \( A^{st} \). We claim that \( d' = d^{st} \) for some proper numbering \( d \) on \( A \). By Lemma 2.5(1), it suffices to show that \( d' \) is constant on \( A^{st}_c \) for each \( c \in \text{supp}(A) \). Suppose that it does not hold. Then there exist \( c_1, c_2 \in A^{st}_c \) for some \( c \in \text{supp}(A) \) such that \( d'(c_1) < d'(c_2) \). By Definition 2.7(1), there exists \( c_3 \in A^{st}_c \) for some \( c' \in \text{supp}(A) \) such that \( c' >_\text{NW} c \) and \( d'(c_3) = d'(c_1) \). Since \( c_3 >_\text{NW} c_1 \) by Lemma 2.4, it is a contradiction. This proves the claim.

**Lemma 2.9** (cf. [4, Proposition 3.4]). Under the above hypothesis, the following hold.

1. For any proper numbering \( d \) on \( A \), there exists a positive integer \( \ell \), which we call the **period** of \( d \), such that \( d(\tau(c)) = d(c) + \ell \) for \( c \in \text{supp}(A) \).
(2) If $\ell$ and $\ell'$ are the periods of any two proper numberings $d$ and $d'$ on $A$, respectively, then we have $\ell = \ell'$, which we call the width of $A$.

**Proof.** The assertions are proved in [4] Proposition 3.4, when $A$ is a partial permutation. Therefore for arbitrary $A$, (1) and (2) follow from Lemmas 2.5(2) and 2.8. □

2.4. Streams and channels. Let us verify the existence of a proper numbering on $A \in \hat{\mathcal{M}}_{m \times n}$. This can be done by using standardization and the result in case of partial permutations in [4].

**Definition 2.10** (cf. [4] Definition 3.20). A _stream_ is an infinite collection of cells $s = \{c_i\}_{i \in \mathbb{Z}}$, which is invariant under $\tau$ and forms a chain with respect to $>_\mathbb{W}$, that is, $c_i >_\mathbb{W} c_{i+1}$ for $i \in \mathbb{Z}$. A _flow of a stream_ $s$ is the positive integer $l$ such that $\tau(c_i) = c_{i+l}$ for $i \in \mathbb{Z}$. A _defining data of a stream_ $s = \{c_i = (a_i, b_i)\}_{i \in \mathbb{Z}}$ of flow $l$ is the triple $(a, b, r)$, where

1. $a = (a_1 + r_1, \ldots, a_{l+r_1})$ with $1 \leq a_{1+r_1} < \cdots < a_{l+r_1} \leq m$,
2. $b = (b_1 + r_2, \ldots, b_{l+r_2})$ with $1 \leq b_{1+r_2} < \cdots < b_{l+r_2} \leq n$,
3. $r = r_1 - r_2$.

Suppose that a non-zero $A \in \hat{\mathcal{M}}_{m \times n}$ is given.

**Definition 2.11** (cf. [4] Definition 3.6]). A stream $s$ is called a _stream of A_ if $s \subset \text{supp}(A)$. A stream $s$ of $A$ is called a _channel of A_ if its flow is maximal among the streams of $A$.

Let $C = \{c_i\}_{i \in \mathbb{Z}}$ be a channel of $A$ and let $c \in \text{supp}(A)$. Let $k$ be the maximal integer such that $c_k >_\mathbb{W} c$. The maximal property of channel forces that $c >_\mathbb{W} c_{k+1}$. This implies either $c_{k+1} \leq c$ or $c \leq c_{k+1}$. In other words, we have

$$\text{supp}(A) = C_n \cup C_{sv},$$

where

$$C_n = \{c \in \text{supp}(A) \mid c' \leq_n c \text{ for some } c' \in C\},$$

(2.5)

$$C_{sv} = \{c \in \text{supp}(A) \mid c \leq_{sv} c' \text{ for some } c' \in C\},$$

and $C_n \cap C_{sv} = C$. Let $\mathcal{C}_A$ denote the set of channels of $A$, which is finite. For $C_1, C_2 \in \mathcal{C}_A$, we define

$$C_1 \geq_{sv} C_2 \text{ if and only if } C_1 \subset (C_2)_{sv}.$$ (2.6)

Then $\geq_{sv}$ is a partial order on $\mathcal{C}_A$.

Let $s = \{c_i\}_{i \in \mathbb{Z}}$ be a stream of $A$. Let $s^{st} = \{c_i^{st}\}_{i \in \mathbb{Z}}$ be the collection of cells such that $c_i^{st}$ is the cell corresponding to the most southwest 1 in the block submatrix $A_c^{st}$ for $i \in \mathbb{Z}$.

On the other hand, let $t = \{c'_i\}_{i \in \mathbb{Z}}$ be a stream of $A^{st}$. Let $t' \subset t$ be the collection of cells obtained from $t$ by replacing each $c'_i$ by the cell corresponding to the most southwest non-zero entry in $A_c^{st}$ if $c'_i \in \text{supp}(A_c^{st})$ for $c \in \text{supp}(A)$. Also let $\mathcal{T} = \{t_i\}_{i \in \mathbb{Z}}$ be the collection of cells in $\text{supp}(A)$ such that $c_i \in \text{supp}(A_c^{st})$ for $i \in \mathbb{Z}$.

**Lemma 2.12.** Under the above hypothesis, we have the following:
(1) \text{s}^{\text{st}} is a stream of \text{A}^{\text{st}} with the same flow as \text{s},
(2) \text{t}' and \overline{t} are streams of \text{A}^{\text{st}} and \text{A} with the same flow as \text{t}, respectively,
(3) \text{t} is a channel if and only if \text{t}' is a channel, in which case we have \text{t}' \succeq_{\text{sv}} \text{t},
(4) \text{s} is a channel if and only if \text{s}^{\text{st}} is a channel,
(5) for any channel \text{s}' of \text{A}, \text{s} \succeq_{\text{sv}} \text{s}' and only if \text{s}^{\text{st}} \succeq_{\text{sv}} \text{s}'^{\text{st}}.

\textbf{Proof.} It is straightforward to check (1) and (2). The assertion (3) follows from the fact
that no two cells of non-zero entries in \text{A}_c^{\text{st}} for \text{c} \in \text{supp} (\text{A}) belong to a stream of \text{A}^{\text{st}}. The
assertion (4) follows from the fact that for any stream \text{t} of \text{A}^{\text{st}}, there exists a unique stream
\text{s} of \text{A} such that \text{s}^{\text{st}} = \text{t}''. This implies (5) immediately.

\textbf{Proposition 2.13 (cf. \cite{4} Proposition 3.13).} The set \mathcal{C}_A has a greatest element with respect
to \succeq_{\text{sv}}, which we denote by \text{C}^{\text{sv}}_A.

\textbf{Proof.} As in the proof of Lemma 2.9, we reduce the proof to the case when \text{A} is a partial
permutation. Let \text{C}_1 and \text{C}_2 be two channels of \text{A}. By Lemma 2.12, \text{C}_1^{\text{st}} and \text{C}_2^{\text{st}} are two channels of \text{A}^{\text{st}}. By \cite{4} Proposition 3.13, there exists a channel \text{C} of \text{A}^{\text{st}} such that \text{C} \succeq_{\text{sv}} \text{C}_1^{\text{st}}, \text{C}_2^{\text{st}}. Let \text{C}_0 be a channel of \text{A} such that \text{C}_0^{\text{st}} = \text{C}''. Hence we have \text{C}_0^{\text{st}} = \text{C}'', \text{C} \succeq_{\text{sv}} \text{C}_1^{\text{st}}, \text{C}_2^{\text{st}}, which implies that \text{C}_0 \succeq_{\text{sv}} \text{C}_1, \text{C}_2 by Lemma 2.12. This
completes the proof.

\textbf{Definition 2.14 (cf. \cite{4} Definition 3.14).} We call \text{C}^{\text{sv}}_A the \textit{southwest channel} of \text{A}.

\textbf{Corollary 2.15.} Under the above hypothesis, we have
\[(\text{C}^{\text{sv}}_A)^{\text{st}} = \text{C}^{\text{sv}}_A^{\text{st}}.]\]

\textbf{Proof.} It follows from Lemma 2.14 (4) and (5).

Let \text{C} = \{\text{c}_i\}_{i \in \mathbb{Z}} be a channel of \text{A} and let \text{d}_0 be a numbering on \text{C} given by \text{d}_0(\text{c}_i) = i
for \text{i} \in \mathbb{Z}. For \text{c} \in \text{supp} (\text{A}), we define
\begin{equation}
\text{d}_A^C (c) = \sup \left\{ \text{d}_0(c'_k) + k \right| c'_k \succ_{\text{sv}} \cdots \succ_{\text{sv}} c'_0 \text{ is a chain in supp} (\text{A}) \right. \\
\left. (k \geq 0) \text{ such that } c'_0 = c \text{ and } c'_k \in \text{C} \right\}.
\end{equation}

\textbf{Proposition 2.16 (cf. \cite{4} Proposition 3.10).} The numbering \text{d}_A^C on \text{A} is a well-defined
proper numbering. Moreover, we have \text{d}_A^C (c) = \text{d}_0(c) for \text{c} \in \text{C}.

\textbf{Proof.} Put \text{d} = \text{d}_A^C. We claim that
\begin{equation}
\text{d}^{\text{st}} = \text{d}^{\text{sv}}_A^{\text{st}},
\end{equation}
where \text{d}^{\text{sv}}_A^{\text{st}} is defined in the same way as in (2.7) with respect to \text{C}^{\text{st}}.

Let \text{c} \in \text{supp} (\text{A}^{\text{st}}) be given. Let \overline{c} \in \text{supp} (\text{A}) such that \text{c} \in \text{supp} (\text{A}^{\text{st}}). Suppose that we
are given a chain \text{c}_k \succ_{\text{sv}} \cdots \succ_{\text{sv}} \text{c}_0 in \text{supp} (\text{A}) from \text{c}_0 = \overline{c} to \text{c}_k \in \text{C}. Let \text{c}'_k be a cell
in \text{supp} (\text{A}^{\text{st}}) (1 \leq i \leq k) such that \text{c}'_k \in \text{C}^{\text{st}}. Then we have a chain \text{c}'_k \succ_{\text{sv}} \cdots \succ_{\text{sv}} \text{c}'_0
in \text{supp} (\text{A}^{\text{st}}) from \text{c}'_0 = c to \text{c}_k \in \text{C}^{\text{st}}. Conversely, if we have a chain of length \text{k} + 1 in
supp\((A^\text{st})\) from \(c\) to \(C^\text{st}\), then we can associate a chain of length \(k + 1\) in supp\((A)\) from \(\tau\) to 
\(C\) by taking representatives of the block submatrix to which each cell in the chain belongs to. This implies that 
\(d^\text{st}(c) = d_{A^\text{st}}^\text{st}(c)\), which proves the claim. Hence, \(d\) is also proper by 
Lemma \[2.8(1)\] since \(d_{A^\text{st}}^\text{st}\) is proper by [4, Proposition 3.10]. It follows from 
\(d^\text{st} = d_{A^\text{st}}^\text{st}\) and [4, Proposition 11.3] that 
\(d(c) = d_0(c)\) \((c \in C)\).

\textbf{Remark 2.17.} Note that the period of the channel numbering \(d_{A^t}\) or, equivalently, the 
width of \(A\) is equal to the flow of a channel \(C\) of \(A\).

\textbf{Definition 2.18.} When \(C = C_{A^t}^{\text{sw}}\), we write \(d_{A^t} = d_{A^t}^{C_{A^t}^{\text{sw}}}\) for short, and call it the \textit{southwest channel numbering on} \(A\).

\textbf{Example 2.19.} Let \(A\) be the generalized affine permutation in Example 2.1. The southwest 
channel of \(A\) is given by 
\[C_{A^t}^{\text{sw}} = \{ \cdots \triangleright_{\text{NW}} \tau^{-1}(4, 7) \triangleright_{\text{NW}} (2, 3) \triangleright_{\text{NW}} (3, 6) \triangleright_{\text{NW}} (4, 7) \triangleright_{\text{NW}} \tau(2, 3) \triangleright_{\text{NW}} \cdots \} \]
which has flow 3. Then the southwest channel numbering \(d_{A^t}^{C_{A^t}^{\text{sw}}}\) is represented as follows:

Here each circle represents a cell in supp\((A)\) and the double circles forms the southwest 
channel \(C_{A^t}^{\text{sw}}\). The number in each circle is the numbering \(d_{A^t}^{C_{A^t}^{\text{sw}}}\) of that cell, where \(k\) means 
\(-k\). For example, \(d_{A^t}^{C_{A^t}^{\text{sw}}}(2, 8) = d_{A^t}^{C_{A^t}^{\text{sw}}}(3, 6) = d_{A^t}^{C_{A^t}^{\text{sw}}}(7, 4) = 1\).

\textbf{Remark 2.20.} Consider the southwest channel \(C_{A^t}^{\text{sw}}\) of \(A^t\). Then 
\[C_{A^t}^{\text{ne}} = \{ (i, j) \mid (j, i) \in C_{A^t}^{\text{sw}} \} \]
is the minimal element in \(C_A\) with respect to \(\triangleright_{\text{ne}}\), which is called the \textit{northeast channel of} 
\(A\). Let \(d_{A^t}^{C_{A^t}^{\text{ne}}}\) be the channel numbering corresponding to \(C_{A^t}^{\text{ne}}\). Then it follows from definition 
that \(d_{A^t}^{C_{A^t}^{\text{ne}}}(j, i) = d_{A^t}^{C_{A^t}^{\text{sw}}}(i, j)\) for \((j, i) \in \text{supp}(A^t)\).

The following lemma gives a characterization of a channel numbering (cf. [4, Remark 11.8]).
Lemma 2.21. Let $C$ be a channel of $A$. Let $d$ be a proper numbering on $A$ such that $d(c) = d^C_A(c)$ for $c \in C$. Then the following are equivalent:

1. $d = d^C_A$,  
2. for $c \in \text{supp}(A)$, there exists a chain $c_0 >_{NW} \cdots >_{NW} c_0$ in $\text{supp}(A)$ such that $c_0 = c$, $c_k \in C$ and $d(c_i) = d(c) - i$ for $0 \leq i \leq k$,  
3. if $d'$ is a proper numbering such that $d'(c) = d^C_A(c)$ for $c \in C$, then we have $d(c) \leq d'(c)$ for every $c \in \text{supp}(A)$.

Proof. Suppose that (1) holds. Let $c_k >_{NW} \cdots >_{NW} c_0 = c$ be a chain which gives the maximum value $d_0(c_k) + k$ in (2.7). Since $d$ is monotone, we have

$$d(c_k) + k \leq d(c_{k-1}) + k - 1 \leq \cdots \leq d(c) = d_0(c_k) + k.$$  

Since $c_k \in C$, we have $d(c_k) = d_0(c_k)$ by Proposition 2.19. Thus all the inequalities in (2.9) are in fact equalities and hence, $d(c_i) = d(c) - i$ for $0 \leq i \leq k$. This implies (2).

Suppose that (2) holds. For $c \in \text{supp}(A)$, let $c_k >_{NW} \cdots >_{NW} c_0 = c$ be a chain satisfying the condition in (2). Let $d'$ be a proper numbering such that $d' = d^C_A$ on $C$. Along this chain, we have

$$d'(c_k) + k \leq d'(c_{k-1}) + k - 1 \leq \cdots \leq d'(c).$$  

from the monotonicity of $d'$. Since $d'(c_k) = d(c_k)$ we conclude that $d(c) = d(c_k) + k = d'(c_k) + k \leq d'(c)$. This implies (3).

Suppose that (3) holds. Then, in particular, we have $d(c) \leq d^C_A(c)$ for $c \in \text{supp}(A)$ by letting $d' = d^C_A$. Let $c_k >_{NW} \cdots >_{NW} c_0 = c$ be a chain which gives the maximal value $d^C_A(c)$. We have $d(c_k) + k \leq d(c_{k-1}) + k - 1 \leq \cdots \leq d(c)$ from the monotonicity of $d$. Then we see that $d^C_A(c) = d^C_A(c_k) + k = d(c_k) + k \leq d(c)$. Hence $d(c) = d^C_A(c)$. □

Remark 2.22. By Lemma 2.21, we may understand that the channel numbering $d^C_A$ is the proper numbering with minimal values among the proper numberings $d$ which coincide with $d_0$ on $C$.

Corollary 2.23. Let $C_1$, $C_2$ be channels of $A$ such that $C_1 \cap C_2 \neq \emptyset$. If $d^C_A = d^C_B$ on $C_1 \cap C_2$, then we have $d^C_A = d^C_B$.

Proof. Let $C_1 = \{ \cdots >_{NW} c_0 >_{NW} \cdots >_{NW} c_l = \tau(0) >_{NW} \cdots \}$, where $l$ is the common flow of $C_1$ and $C_2$. We may assume that $c_0 \in C_1 \cap C_2$. Along $C_1$, we have

$$d^C_A(c_0) + l \leq d^C_A(c_1) + (l - 1) \leq \cdots \leq d^C_A(c_l).$$  

By Lemma 2.21 we see that $d^C_A(c_1) = d^C_A(c_0) + l$. Hence all the inequalities above are equalities, and for $0 \leq i \leq l$  

$$d^C_A(c_i) = d^C_A(c_0) + i = d^C_A(c_0) + i = d^C_A(c_i).$$  

By Lemma 2.21 (3), we have $d^C_A(c) \leq d^C_A(c)$ for $c \in \text{supp}(A)$. The reverse inequality can be obtained in the same way. Hence we obtain $d^C_A = d^C_B$. □
2.5. Tableaux. Let $\mathcal{P}$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$. Following [7], we regard a partition $\lambda$ as its Young diagram. Let $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ be the conjugate of $\lambda$, $\ell(\lambda)$ the length of $\lambda$, and $|\lambda| = \sum_{i \geq 1} \lambda_i$. For $n \geq 1$, let $\mathcal{P}_n = \{ \lambda \in \mathcal{P} | \ell(\lambda) \leq n \}$.

Let $\mathcal{A}$ be $\mathbb{Z}$ or $[k]$ $(k \geq 1)$ with the usual linear orders. For $\lambda \in \mathcal{P}$, let $SST_\mathcal{A}(\lambda)$ be the set of semistandard (or $\mathcal{A}$-semistandard) tableaux of shape $\lambda$, that is, tableaux with entries in $\mathcal{A}$ such that (1) the entries in each row are weakly increasing from left to right, (2) the entries in each column are strictly increasing from top to bottom.

We denote by $CSST_\mathcal{A}(\lambda)$ the set of column semistandard tableaux of shape $\lambda$, that is, tableaux of shape $\lambda$ with entries in $\mathcal{A}$ which satisfy the condition (2). We also let $CST_\mathcal{A}(\lambda)$ be the set of $T \in CSST_\mathcal{A}(\lambda)$ such that all the entries in $T$ are distinct.

Similarly, we let $RSST_\mathcal{A}(\lambda)$ be the set of row semistandard tableaux of shape $\lambda$, that is, tableaux of shape $\lambda$ with entries in $\mathcal{A}$ which satisfy the condition (1), and let $RST_\mathcal{A}(\lambda)$ be the set of $T \in RSST_\mathcal{A}(\lambda)$ such that all the entries in $T$ are distinct.

3. Affine RSK correspondence

3.1. Proper numberings and zig-zags.

Definition 3.1. Let $z = \{c_i\}_{i \in \mathbb{Z}}$ be an infinite collection of cells such that $c_{i+1}$ is adjacent to $c_i$ for $i \in \mathbb{Z}$.

(1) We say that $z$ is a zig-zag when it satisfies the following:
   (i) $c_{i+1}$ is located to the east or north of $c_i$ for all $i$,
   (ii) $c_{i+1}$ is located to the east of $c_i$ for $i \geq 0$,
   (iii) $c_{i-1}$ is located to the south of $c_i$ for $i \leq 0$.

(2) A cell $c_i$ in a zig-zag $z$ is called an inner corner if $c_{i-1}$ is located to the south and $c_{i+1}$ is located to the east of $c_i$, and called an outer corner if $c_{i-1}$ is located to the west and $c_{i+1}$ is located to the north of $c_i$. Note that a zig-zag $z$ has at least one inner corner.

(3) A cell $c = (i_l, j_k)$ is called the back-post corner of a zig-zag $z$ when $c_k = (i_k, j_k)$ and $c_l = (i_l, j_k)$ are the leftmost and the rightmost inner corners of $z$, respectively.

Let $d$ be a proper numbering on a non-zero $A \in \overset{\rightarrow}{M}_{m \times n}$. For each $k \in \mathbb{Z}$, the level set $d^{-1}(k)$ forms a chain with respect to $\leq_{\text{ne}}$, by monotonicity of $d$ (Definition 2.7).

We associate a set of zig-zags $Z_d = \{z_k\}_{k \in \mathbb{Z}}$ to $d$, where the inner corners of $z_k$ are the set of maximal elements in $d^{-1}(k)$ with respect to $\geq_{\text{nw}}$. It is straightforward to see that $\{z_k\}_{k \in \mathbb{Z}}$ satisfies

(z.1) the inner corners of each $z_k$ are contained in $\text{supp}(A)$,
(z.2) $z_k$’s are mutually disjoint and $\text{supp}(A) \subseteq \bigsqcup_{k \in \mathbb{Z}} z_k$,
(z.3) $z_k$ is located to the southeast of $z_{k-1}$ for $k \in \mathbb{Z}$ in the sense that

(3.1) for each $c_1 \in z_k$, there exists $c_2 \in z_{k-1}$ such that $c_2 \geq_{\text{nw}} c_1$. 


Conversely, a set of zig-zags $Z = \{z_k\}_{k \in \mathbb{Z}}$ satisfying (z.1)-(z.3) determines a unique proper numbering $d^Z$ on $A$ given by

\[(3.2) \quad d^Z(c) = k \quad \text{if} \ c \in \text{supp}(A) \cap z_k,
\]
whose associated set of zig-zags is $Z$.

**3.2. Matrix-ball construction.** Now suppose that a non-zero $A \in \hat{\mathcal{M}}_{m \times n}$ is given and let

- $\{z_k\}_{k \in \mathbb{Z}}$: the set of zig-zags associated to $d^A_{\text{pr}}$.

Note that $\tau(z_k) = z_{k+\ell}$ ($k \in \mathbb{Z}$) if $\ell$ is the period of $d^A_{\text{pr}}$ (cf. Lemma 2.9).

**Example 3.2.** Let $A$ and $d^A_{\text{pr}}$ be as in Example 2.19. The period of $d^A_{\text{pr}}$ is 3. The zig-zags $z_k$ corresponding to the level sets $(d^A_{\text{pr}})^{-1}(k)$ for $k = 1, 2, 3$ are given as red lines below.

Let

- $A^\delta$: the matrix in $\hat{\mathcal{M}}_{m \times n}$ obtained from $A$ by
  (i) subtracting one at the inner corners in $z_k$ ($k \in \mathbb{Z}$),
  (ii) adding one at the outer corners in $z_k$ ($k \in \mathbb{Z}$),
- $A^{(t)}$ ($t \geq 0$): the matrices in $\hat{\mathcal{M}}_{m \times n}$ defined inductively as follows:
  \[A^{(0)} = A, \quad A^{(t)} = \left(A^{(t-1)}\right)^\delta.
\]
We have $A^{(t)} = \varnothing$ for a sufficiently large $t$, since the number of outer corners in each zig-zag $z_k$ is strictly smaller than the number of inner corners. Let $s \geq 1$ be the integer such that

\[A^{(s-1)} \neq \varnothing, \quad A^{(s)} = \varnothing\]

For $1 \leq t \leq s$, we consider

- $\{z_k^{(t)}\}_{k \in \mathbb{Z}}$: the set of zig-zags associated to $d^A_{\text{pr}}^{(t)}$,
- $s^{(t)}$: the stream consisting of the back-post corners of $z_k^{(t)}$ ($k \in \mathbb{Z}$),
- $\mu_t$: the flow of $s^{(t)}$, equivalently, the width of $A^{(t-1)}$,
- $\{a_t, b_t, \rho_t\}$: the defining data of $s^{(t)}$ (1 \leq t \leq s).
Lemma 3.3 (cf. [4, Lemma 14.9]). The flows of \( s^{(1)}, \ldots, s^{(s)} \) are weakly decreasing positive integers, that is, \( \mu_1 \geq \cdots \geq \mu_s > 0 \).

**Proof.** Let \( \ell \) and \( \ell' \) be the widths of \( A \) and \( A' \), respectively. Consider the zig-zags \( z_k \) (\( k \in \mathbb{Z} \)) associated to \( d^x_A \). By definition, \( \text{supp}(A') \) is contained in the union of the \( z_k \)’s. Suppose that \( \ell < \ell' \). Let \( C = \{ c_i \}_{i \in \mathbb{Z}} \) be a channel of \( A' \). Note that each cell \( c_i \) belongs to a different zig-zag \( z_k \). But the width of \( A \) is \( \ell \), which implies that at least two cells among \( \ell' \) cells \( c_1 >_w c_2 >_w \cdots >_w c_{\ell'} = \tau(c_0) \) belong to the same zig-zag \( z_k \) for some \( k \). This is a contradiction. □

Now we let

- \( \lambda = \mu' \): the conjugate partition of \( \mu = (\mu_1, \ldots, \mu_s) \in \mathcal{P}_s \),
- \( P_0 \): the tableau of shape \( \lambda \), whose \( t \)-th column from the left is \( a_t \) (\( 1 \leq t \leq s \)),
- \( Q_0 \): the tableau of shape \( \lambda \), whose \( t \)-th column from the left is \( b_t \) (\( 1 \leq t \leq s \)),
- \( \rho = (\rho_1, \ldots, \rho_s) \in \mathbb{Z}^s \).

Here we understand that the entries of the \( t \)-th column in \( P_0 \) and \( Q_0 \) are given by those in \( a_t \) and \( b_t \) respectively. Note that \( \lambda \in \mathcal{P}_m \cap \mathcal{P}_n \), and \( P_0, Q_0 \) are column semistandard tableaux.

Summarizing, we have a map

\[
(3.3) \quad \kappa_0 : \tilde{M}_{m \times n} \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_m \cap \mathcal{P}_n} CSST_{[m]}(\lambda) \times CSST_{[n]}(\lambda) \times \mathbb{Z}^{\lambda_1},
\]

\[
A \quad \longrightarrow \quad (P_0, Q_0, \rho)
\]

where we assume \( \kappa_0(\emptyset) = (\emptyset, \emptyset, (0, \ldots, 0)) \) with \( \emptyset \) the empty tableau.

**Example 3.4.** Let \( A \) be the generalized affine permutation in Example 2.1, i.e.,

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
where the red lines denote the zig-zags in Example 3.2. The stream consisting of the back-post corners of $z_k$ ($k \in \mathbb{Z}$) is

$$s^{(1)} = \{ \cdots >_{NW} (2, 4) >_{NW} (3, 6) >_{NW} (4, 7) >_{NW} \cdots \}.$$ 

By subtracting one at each inner corner and adding one at each outer corner of $z_k$, we obtain $A^p$ as follows:

Repeating this process, we see that

$$A^{(1)} =$$

$$A^{(2)} =$$
with $A^{(5)} = \emptyset$, and

\[ s^{(2)} = \{ \cdots >_{w} (1, 3) >_{w} (2, 6) >_{w} (4, 7) >_{w} \cdots \}, \]

\[ s^{(3)} = \{ \cdots >_{w} (2, 0) >_{w} (3, 2) >_{w} \cdots \}, \]

\[ s^{(4)} = \{ \cdots >_{w} (1, -1) >_{w} (3, 3) >_{w} \cdots \}, \]

\[ s^{(5)} = \{ \cdots >_{w} (2, 3) >_{w} (3, 4) >_{w} \cdots \}. \]

Hence $\kappa_0(A) = (P_0, Q_0, \rho)$, where

\[ R_0 = \begin{bmatrix} 2 & 1 & 2 & 1 & 2 \\ 3 & 2 & 3 & 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 4 & 4 \end{bmatrix}, \quad \rho = (2, 2, -1, -1, 0). \]

**Remark 3.5.** We have shown that $(d_A^{st})^\pi = d_{A^{st}}^{\pi}$ in (2.3). This implies that

\[ (A^{st})^b = (A^b)^{st}, \]

in the sense that the right-hand side is obtained by removing all zero rows and columns on the left-hand side (cf. Section 2.2 for the convention of row and column indices of $A^{st}$).

**3.3. Semistandard tableaux of rectangular shape.** Let $\lambda \in \mathcal{P}$ be given. For $T \in SST_2(\lambda)$, let us write $T = (T^s, \ldots, T^2, T^1)$, where $T^j$ denotes the $j$-th column of $T$ from the right. For any $1 \leq j < k \leq s$, we may regard $(T^k, \ldots, T^{j+1}, T^j)$ as a semistandard subtableau of $T$ with the columns $T^k, \ldots, T^{j+1}, T^j$.

For $a \geq 1$ and $1 \leq b \leq n$, let

\[ R = (a^b) = (a, \ldots, a) \in \mathcal{P}_n \]
be a Young diagram of rectangular shape. Let
\[ \mathcal{B}(R) = \left\{ T \in \text{SST}_Z(R) \mid T^j(b) - T^j(1) < n \ (1 \leq j \leq a) \right\}, \]
where \( T^j(i) \) denotes the entry of \( T^j \) at the \( i \)-th row from the top.

For \( T \in \mathcal{B}((1^b)) \), let \( \tau_n(T) = \tau(T) \) be the column semistandard tableau obtained from \( T \) by replacing its entries \( T(1) < T(2) < \cdots < T(b) \) with \( T(2) < T(3) < \cdots < T(b) < T(1) + n \). Hence \( \tau \) defines a bijection \( \mathcal{B}((1^b)) \) and \( \tau^{-1} \) denotes its inverse. In general, for \( \alpha = (\alpha_1, \ldots, \alpha_a) \in \mathbb{Z}^a \) and \( (T_1, \ldots, T_a) \in \mathcal{B}((1^b))^a \), we define
\[ \tau^\alpha(T_1, \ldots, T_a) = (\tau^{\alpha_1}(T_1), \ldots, \tau^{\alpha_a}(T_a)), \]
which gives a bijection \( \mathcal{B}((1^b))^a \rightarrow \mathcal{B}((1^b))^a \). Regarding \( \mathcal{B}(R) \subset \mathcal{B}((1^b))^a \), we may define \( \tau^\alpha \) on \( \mathcal{B}(R) \). Let
\[ \mathcal{B}(R)_0 = \left\{ T \in \mathcal{B}(R) \mid (T^j+1, \tau^{-1}(T^j)) \text{ is not semistandard for } 1 \leq j \leq a - 1 \right\}. \]

**Lemma 3.6.** We have a bijection
\[ \mathcal{B}(R)_0 \times \mathcal{P}_{a-1} \rightarrow \mathcal{B}(R), \]
\[ (T, \nu) \mapsto \tau^{\nu_{rev}}(T) \]
for \( T = (T^a, \ldots, T^1) \) and \( \nu = (\nu_1, \ldots, \nu_{a-1}) \in \mathcal{P}_{a-1} \), where \( \nu_{rev} = (0, \nu_{a-1}, \ldots, \nu_2, \nu_1) \).

Let us also regard \( \text{CSST}_{[n]}(R) \subset \mathcal{B}((1^b))^n \), and define \( \tau^\alpha : \text{CSST}_{[n]}(R) \rightarrow \mathcal{B}((1^b))^a \) for \( \alpha = (\alpha_1, \ldots, \alpha_a) \in \mathbb{Z}^a \) as in (3.4).

**Definition 3.7.** Let \( T \in \text{CSST}_{[n]}(R) \) be given. For \( 1 \leq j \leq a - 1 \), let \( r_j \) be the minimal non-negative integer such that
\[ (T^j+1, \tau^{r_j}(T^j)) \]
is \( \mathbb{Z} \)-semistandard, and put \( \eta_j = r_j + r_{j+1} + \cdots + r_{a-1} \). We call \((r_1, \ldots, r_{a-1})\) the offset vector and \( \eta = (\eta_1, \ldots, \eta_{a-1}) \) the symmetrized offset vector of \( T \).

The symmetrized offset vector is the unique \( \eta \in \mathcal{P}_{a-1} \) such that \( \tau^{\eta_{rev}}(T) \in \mathcal{B}(R)_0 \). The following is immediate from the definition of \( \eta \).

**Lemma 3.8.** Let \( T \in \text{CSST}_{[n]}(R) \) and \( \alpha = (\alpha_1, \ldots, \alpha_a) \in \mathbb{Z}^a \) be given. Then \( \tau^{\alpha_{rev}}(T) \in \mathcal{B}(R) \) with \( \alpha_{rev} = (\alpha_a, \ldots, \alpha_1) \) if and only if
\[ \alpha_a \leq \alpha_{a-1} - \eta_{a-1} \leq \cdots \leq \alpha_1 - \eta_1. \]

By Lemma 3.8 we have a bijection
\[ \text{CSST}_{[n]}(R) \times \mathcal{P}_a \rightarrow \mathcal{B}(R), \]
\[ (T, \nu) \mapsto \tau^{\nu_{rev} + \eta_{rev}}(T) \]
where \( \mathcal{P}_a = \{ \nu = (\nu_1, \ldots, \nu_a) \in \mathbb{Z}^a \mid \nu_1 \geq \cdots \geq \nu_a \} \) is the set of generalized partitions of length \( a \) and \( \eta \) is the symmetrized offset vector of \( T \). Let

\[
(3.6) \quad \tau : \mathcal{B}(R) \longrightarrow \mathcal{B}(R)
\]

\[
T = (T^a, \ldots, T^1) \quad \longrightarrow \quad (\tau(T^a), \ldots, \tau(T^1))
\]

be the bijection given by applying \( \tau \) to each column of the tableaux in \( \mathcal{B}(R) \), which induces a \( \mathbb{Z} \)-action on \( \mathcal{B}(R) \) and \( \mathcal{B}(R)_0 \). Let \( \mathcal{B}(R)_0/\mathbb{Z} \) denote the set of equivalence classes under this \( \mathbb{Z} \)-action. We may identify \( \mathcal{B}(R)_0/\mathbb{Z} \) with the set of \( T \in \mathcal{B}(R)_0 \) such that the first column has entries in \([n]\). Hence, we have another bijection

\[
(3.7) \quad CSST_{[n]}(R) \longrightarrow \mathcal{B}(R)_0/\mathbb{Z} ,
\]

\[
T \quad \longmapsto \quad [\tau^{rev}(T)]
\]

where \( [T] \) denotes the equivalence class of \( T \) and \( \eta \) is the symmetrized offset vector of \( T \).

### 3.4. Rectangular decomposition

Let \( \lambda \in \mathcal{P}_a \) be given. We decompose \( \lambda \) (its Young diagram) into diagrams of rectangular shapes \( R_i \) defined by

\[
R_i = (m_i, \ldots, m_i) \quad (1 \leq i \leq l),
\]

where \( m_i \) is the number of occurrences of \( i \) in \( \lambda \) and \( l = \ell(\lambda) \). Here we assume that \( R_i \) is empty when \( m_i = 0 \). For example, if \( \lambda = (6, 4, 1, 1) \), then see that \( R_1 = (1) \), \( R_2 = (3^2) \), \( R_3 = \emptyset \) and \( R_4 = (2^4) \) as illustrated in the following figure.

\[\text{Diagram of \( \lambda \) with rectangular components.}\]

Let

\[
(3.8) \quad \mathcal{B}_n(\lambda)_0 = \mathcal{B}(\lambda)_0 = \mathcal{B}(R_1)_0 \times \cdots \times \mathcal{B}(R_l)_0,
\]

\[
\mathcal{B}_n(\lambda) = \mathcal{B}(\lambda) = \mathcal{B}(R_1) \times \cdots \times \mathcal{B}(R_l).
\]

If we put \( \mathcal{P}(\lambda) = \mathcal{P}_{m_1-1} \times \cdots \times \mathcal{P}_{m_l-1} \), where we take the product over \( m_i \geq 1 \), then we have a bijection

\[
\mathcal{B}(\lambda)_0 \times \mathcal{P}(\lambda) \longrightarrow \mathcal{B}(\lambda),
\]

\[
\left( (T^{(i)})_{1 \leq i \leq l}, (\nu^{(i)})_{1 \leq i \leq l} \right) \longrightarrow \left( \tau^{rev}(T^{(i)}) \right)_{1 \leq i \leq l}
\]

by applying Lemma 3.6 to each component, where \( T^{(i)} \in \mathcal{B}(R_i)_0 \) and \( \nu^{(i)} \in \mathcal{P}_{m_i-1} \).

Similarly, if we let \( \mathcal{P}(\lambda) = \mathcal{P}_{m_1} \times \cdots \times \mathcal{P}_{m_l} \), where we take the product over \( m_i \geq 1 \), and regard

\[
CSST_{[n]}(\lambda) = CSST_{[n]}(R_1) \times \cdots \times CSST_{[n]}(R_l),
\]
then by (3.6) we have a bijection

\[
CSST_{[\lambda]}(\lambda) \times P(\lambda) \longrightarrow \mathcal{B}(\lambda),
\]

\[
\left( (T^{(i)})_{1 \leq i \leq l}, (\nu^{(i)})_{1 \leq i \leq l} \right) \longrightarrow \left( \tau_{rev}^{(i)}(T^{(i)}) + \eta^{(i)}_c(T^{(i)}) \right)_{1 \leq i \leq l}
\]

where \( \eta^{(i)} \in \mathcal{P}_{m_i-1} \) is the symmetrized offset vector of \( T^{(i)} \)

**Example 3.9.** Let \( P_0 \in CSST_{[\lambda]}(\lambda) \) be the tableau in Example 3.4, i.e.,

\[
P_0 = \begin{bmatrix}
2 & 1 & 2 & 1 & 2 \\
3 & 2 & 3 & 3 & 2 \\
4 & 4 & & & \\
\end{bmatrix}
\]

where \( \lambda = (5,5,2) \). Then \( \lambda \) is decomposed into \( R_2 = (3,3) \) and \( R_3 = (2,2,2) \), and the corresponding decompositions of \( P_0 \) are

\[
P_0^{(3)} = \begin{bmatrix}
2 & 1 \\
3 & 2 \\
4 & 4 \\
\end{bmatrix}, \quad P_0^{(2)} = \begin{bmatrix}
2 & 1 & 2 \\
3 & 3 & 3 \\
\end{bmatrix}
\]

Since

\[
\tau^{(0,1)}(P_0^{(3)}) = \begin{bmatrix}
2 & 2 \\
3 & 4 \\
4 & 1 \\
\end{bmatrix} \in \mathcal{B}(R_3)_0,
\]

the symmetrized offset vector of \( P_0^{(3)} \) is \( \eta^{(3)} = (1) \in \mathcal{P}_1 \). Similarly, we have \( \eta^{(2)} = (1,1) \in \mathcal{P}_2 \). For \( \nu = ((1,-1),((2,1,0)) \in P(\lambda) \), the image of \( (P_0,\nu) \) under the bijection (3.10) is

\[
\begin{bmatrix}
0 & 4 & 2 & 5 & 7 \\
2 & 5 & 3 & 7 & 10 \\
3 & 6 \\
\end{bmatrix} \in \mathcal{B}(\lambda).
\]

Let

\[
\mathcal{B}(\lambda)_0/\mathbb{Z}^l = \mathcal{B}(R_1)_0/\mathbb{Z} \times \cdots \times \mathcal{B}(R_l)_0/\mathbb{Z},
\]

where each \( \mathcal{B}(R_i)_0/\mathbb{Z} \) is the set of equivalence classes under the \( \mathbb{Z} \)-action (3.6). Then we also have a bijection

\[
CSST_{[\alpha]}(\lambda) \longrightarrow \mathcal{B}(\lambda)_0/\mathbb{Z}^l,
\]

\[
\left( (T^{(i)})_{1 \leq i \leq l} \right) \longrightarrow \left( \left[ \tau^{(i)}_c(T^{(i)}) \right] \right)_{1 \leq i \leq l}
\]

where \( \eta^{(i)} \in \mathcal{P}_{m_i-1} \) is the symmetrized offset vector of \( T^{(i)} \).
3.5. Affine RSK correspondence. Suppose that \( A \in \tilde{\mathcal{M}}_{m \times n} \) is given. We keep the notations in Sections 3.2 and 3.4.

Let \((P_0, Q_0, \rho)\) be given as in (3.3). For \(1 \leq i \leq l\) with \(m_i \geq 1\), let
- \(P_i^{(i)}, Q_i^{(i)}\): the subtableaux of \(P_i\) and \(Q_i\) corresponding to \(R_i\), respectively,
- \(\rho^{(i)} \in \mathbb{Z}^{m_i}\): the subsequence of \(\rho\) corresponding to the columns of \(R_i\),
- \(\eta^{(i)} \in \mathcal{P}_{m_i-1}\): the symmetrized offset vector of \(P_i^{(i)}\).

Then we define
\[
Q = \left( \tau^{\rho^{(1)} + \eta^{(1)}}(Q_0^{(1)}), \ldots, \tau^{\rho^{(1)} + \eta^{(1)}}(Q_0^{(1)}) \right) = \left( Q^{(1)}, \ldots, Q^{(1)} \right).
\]

Note that the action of \(\tau\) on \(Q_0\) should be understood as \(\tau_{n}\). The following is one of the main results in this paper. The proof will be given in Section 5.

**Theorem 3.10.** We have a bijection
\[
\kappa : \tilde{\mathcal{M}}_{m \times n} \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_m \cap \mathcal{P}_n} \text{CSST}[n](\lambda) \times \mathcal{B}_n(\lambda).
\]

**Remark 3.11.** Applying the bijection (3.10) to \(B_m^{(n)}\), we have a bijection
\[
\kappa' : \tilde{\mathcal{M}}_{m \times n} \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_m \cap \mathcal{P}_n} \text{CSST}[m](\lambda) \times \text{CSST}[n](\lambda) \times \mathcal{P}(\lambda).
\]

Again applying the inverse of (3.10) to \(\text{CSST}[m](\lambda) \times \mathcal{P}(\lambda)\), we have a bijection
\[
\kappa'' : \tilde{\mathcal{M}}_{m \times n} \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_m \cap \mathcal{P}_n} \mathcal{B}_m(\lambda) \times \text{CSST}[m](\lambda).
\]

**Example 3.12.** Let \(\kappa_0(A) = (P_0, Q_0, \rho)\) be as in Example 3.4 \((m = 4, n = 5)\). The rectangular decompositions of \(P_0, Q_0\) and \(\rho\) are

\[
P_0^{(2)} = \begin{array}{cc}
2 & 1 \\
3 & 3 \\
\end{array}, \quad Q_0^{(2)} = \begin{array}{ccc}
2 & 3 & 3 \\
5 & 4 & 4 \\
\end{array}, \quad \rho^{(2)} = (-1, -1, 0),
\]

\[
P_0^{(3)} = \begin{array}{c}
2 \\
3 \\
\end{array}, \quad Q_0^{(3)} = \begin{array}{cc}
1 & 1 \\
2 & 2 \\
4 & 3 \\
\end{array}, \quad \rho^{(3)} = (2, 2),
\]

where \(R_2\) and \(R_3\) are the only non-trivial rectangles in this decomposition. The symmetrized offset vectors of \(P_0^{(2)}\) and \(P_0^{(3)}\) are

\[
\eta^{(2)}_{\text{rev}} = (0, 1, 1), \quad \eta^{(3)}_{\text{rev}} = (0, 1),
\]

and hence
\[
Q = \begin{array}{cccc}
4 & 6 & 0 & 3 \\
6 & 7 & 2 & 4 \\
7 & 8 \\
\end{array}.
\]
Remark 3.13. Let us give some comments on $\kappa_0$ and the bijection in [12]. Let $M_{m \times n}$ be the set of $M = (M_{j,i}(k))$ ($i \in [m], j \in [n], k \in \mathbb{Z}$) with $M_{j,i}(k) \in \mathbb{Z}_{\geq 0}$ and $M_{j,i}(k) = 0$ for $|k| > 0$ [12 (2.5)] and let $M_{m \times n}^{+}$ be the subset of $M_{m \times n}$ consisting of $M$ such that $M_{j,i}(k) = 0$ for $k < 0$. For $A = (a_{ij})_{i,j \in \mathbb{Z}} \in \hat{M}_{m \times n}$, we define $M_A = (M_{j,i}(k))$ by

$$M_{j,i}(k) = a_{i-km,n+1-j}.$$ 

Then the map sending $A$ to $M_A$ gives a bijection from $\hat{M}_{m \times n}$ to $M_{m \times n}$.

Let $\Upsilon$ denote the bijection $M_{m \times n}^{+} \rightarrow \bigcup_{\lambda \in \mathcal{P}_{m} \cap \mathcal{P}_{n}} \text{CSST}_{|m|}^{(\lambda)} \times \text{CSST}_{|n|}^{(\lambda)} \times \mathcal{K}(\lambda)$, given in [12, Corollary 8.2].

Let $A = (a_{ij})_{i,j \in \mathbb{Z}} \in \hat{M}_{m \times n}$ be given such that $M_A \in M_{m \times n}^{+}$. Applying $\kappa_0$ and $\Upsilon$ directly to $A$ and $M_A$, respectively, do not seem to give the same result in general. This may happen due to conventions for generalized affine permutations. For example, let $A \in \hat{M}_{5 \times 6}$ be as follows:

```
\begin{tabular}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{tabular}
```

Then we have $\kappa_0(A) = (P_0, Q_0, \rho)$, where

$$P_0 = \begin{pmatrix}
1 & 2 & 2 & 1 \\
4 & 3 & 3 & 2 \\
5 & 4 & 5 & 3
\end{pmatrix}, \quad Q_0 = \begin{pmatrix}
2 & 1 & 3 & 1 \\
3 & 2 & 4 & 4 \\
6 & 4 & 5 & 6
\end{pmatrix}$$

On the other hand, we have $\Upsilon(M_A) = (V, W, \xi)$, where

$$V = \begin{pmatrix}
1 & 2 & 2 & 1 \\
4 & 3 & 3 & 2 \\
5 & 4 & 5 & 3
\end{pmatrix}, \quad W = \begin{pmatrix}
1 & 2 & 3 & 1 \\
3 & 3 & 5 & 4 \\
6 & 4 & 6 & 5
\end{pmatrix}.$$
Hence $P_0 = V$ but $Q_0 \neq W$, while we observe that $W^j(i) = 7 - Q_0^{j-i}(4 - i)$ for $1 \leq i \leq 3$ and $1 \leq j \leq 4$. Recall that $W^j$ and $Q_0^j$ denote the $j$-th columns from the right.

In general, one may expect that $P_0 = V$ and $Q_0 = e_n W$, where $e_n$ is an operator on $\text{CSST}_{[n]}(\lambda)$ given by

$$(e_n W)^j(i) = n + 1 - W^{a+1-j}(b+1-i) \quad (1 \leq i \leq b, 1 \leq j \leq a),$$

in case of a rectangular shape $\lambda = (a^b)$. The operator $e_n$ can be viewed as a generalization of the affine evacuation in [2]. We do not know yet a precise relation between $\rho$ and $\xi$.

4. AFFINE CRYSTALS

4.1. Crystals. Let us give a brief review on crystals (see [10] for more details). Let $g$ be the Kac-Moody algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ indexed by a set $I$. Let $P^\vee$ be the dual weight lattice, $P = \text{Hom}_Z(P^\vee, Z)$ the weight lattice, $\Pi^\vee = \{ h_i | i \in I \} \subset P^\vee$ the set of simple coroots, and $\Pi = \{ \alpha_i | i \in I \} \subset P$ the set of simple roots of $g$ such that $\langle \alpha_j, h_i \rangle = a_{ij}$ for $i, j \in I$.

A $g$-crystal (or crystal if there is no confusion on $g$) is a set $B$ together with the maps $\text{wt} : B \longrightarrow P$, $\varepsilon_i, \varphi_i : B \longrightarrow Z \cup \{ -\infty \}$ and $\tilde{\varepsilon}_i, \tilde{\varphi}_i : B \longrightarrow B \cup \{ 0 \}$ for $i \in I$ such that for $b \in B$ and $i \in I$

1. $\varphi_i(b) = \langle \text{wt}(b), h_i \rangle + \varepsilon_i(b),$
2. $\varepsilon_i(\tilde{\varepsilon}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{\varepsilon}_i b) = \varphi_i(b) + 1$, $\text{wt}(\tilde{\varepsilon}_i b) = \text{wt}(b) + \alpha_i$ if $\tilde{\varepsilon}_i b \neq 0,$
3. $\varepsilon_i(\tilde{\varphi}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{\varphi}_i b) = \varphi_i(b) - 1$, $\text{wt}(\tilde{\varphi}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{\varphi}_i b \neq 0,$
4. $\tilde{\varphi}_i b = b'$ if and only if $b = \tilde{\varepsilon}_i b'$ for $b, b' \in B,$
5. $\tilde{\varepsilon}_i b = \tilde{\varphi}_i b = 0$ if $\varphi_i(b) = -\infty,$

where $0$ is a formal symbol. Here we assume that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. A crystal $B$ can be viewed as an $I$-colored graph where $b \xrightarrow{i} b'$ if and only if $\tilde{\varepsilon}_i b = b'$ for $b, b' \in B$ and $i \in I$.

A crystal $B$ is called normal if $\varepsilon_i(b) = \max \{ k | \tilde{\varepsilon}_i^k b \neq 0 \}$ and $\varphi_i(b) = \max \{ k | \tilde{\varphi}_i^k b \neq 0 \}$ for $b \in B$ and $i \in I$.

Let $B_1$ and $B_2$ be crystals. A morphism $\psi : B_1 \longrightarrow B_2$ is a map from $B_1 \cup \{ 0 \}$ to $B_2 \cup \{ 0 \}$ such that

1. $\psi(0) = 0,$
2. $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ if $\psi(b) \neq 0,$
3. $\psi(\tilde{\varepsilon}_i b) = \tilde{\varepsilon}_i \psi(b)$ if $\psi(b) \neq 0$ and $\psi(\tilde{\varphi}_i b) \neq 0,$
4. $\psi(\tilde{\varphi}_i b) = \tilde{\varphi}_i \psi(b)$ if $\psi(b) \neq 0$ and $\psi(\tilde{\varphi}_i b) \neq 0,$

for $b \in B_1$ and $i \in I$. We call $\psi$ an embedding when $\psi$ is injective, and call $\psi$ strict if $\psi : B_1 \cup \{ 0 \} \longrightarrow B_2 \cup \{ 0 \}$ commutes with $\tilde{\varepsilon}_i$ and $\tilde{\varphi}_i$ for all $i \in I$, where we assume that $\tilde{\varepsilon}_i 0 = \tilde{\varphi}_i 0 = 0$. When $\psi$ is a bijection, it is called an isomorphism.

Given $b_1 \in B_1$ and $b_2 \in B_2$, we say that $b_1$ is $g$-crystal equivalent to $b_2$ if there is an isomorphism of crystals $C(b_1) \longrightarrow C(b_2)$ mapping $b_1$ to $b_2$, where $C(b_i)$ denotes the connected component of $b_i$ in $B_i$ for $i = 1, 2.$
The tensor product $B_1 \otimes B_2$ is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where
\[
\begin{align*}
\text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_1), h_i \rangle\}, \\
\varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_1) + \langle \text{wt}(b_2), h_i \rangle, \varphi_i(b_2)\}, \\
\end{align*}
\]
(4.1)
\[\hat{e}_i(b_1 \otimes b_2) = \begin{cases} \\
\hat{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
b_1 \otimes \hat{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\
b_1 \otimes \hat{e}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2),
\end{cases}\]
\[\hat{f}_i(b_1 \otimes b_2) = \begin{cases} \\
\hat{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
b_1 \otimes \hat{f}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\
b_1 \otimes \hat{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2),
\end{cases}\]
for $i \in I$. Here, we assume that $0 \otimes b_2 = b_1 \otimes 0 = 0$. Then $B_1 \otimes B_2$ is a crystal.

Let us recall a well-known combinatorial rule, which is often used to describe a crystal structure on the tensor product of crystals, and which will be used later in this paper. Suppose that $\sigma = (\ldots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots)$ is a sequence (not necessarily finite) where $\sigma_k \in \{+,-,\cdot\}$ and $\sigma_k = \cdot$ except for finitely many $k$. In the sequence $\sigma$, we replace a pair $(\sigma_s, \sigma_{s'}) = (+,-)$, where $s < s'$ and $\sigma_s = \cdot$ for $s < t < s'$, with $(\cdot, \cdot)$, and repeat this process as far as possible until we get a sequence with no $-\text{crystal}$ for simplicity. For $\Lambda \in \mathcal{P}$, let $B(\Lambda)$ denote the crystal of the extremal weight module generated by an extremal vector of weight $\Lambda$ \[16\].

We let
\[P^0 = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_{n-1} \oplus \mathbb{Z}\delta, \quad P_{cl} = P/\mathbb{Z}\delta, \quad P^0_{cl} = P^0/\mathbb{Z}\delta,\]
where $\varpi_i = \Lambda_i - \Lambda_0$ is the $i$-th level zero fundamental weight for $i \in \{1, \ldots, n-1\}$. Let $\text{cl} : P \rightarrow P_{cl}$ be the canonical projection. Define $\epsilon_1 = \varpi_1$ and $\epsilon_i = \varpi_i - \varpi_{i-1}$ for $i \in \{2, \ldots, n-1\}$. Put $\epsilon_n = -\epsilon_1 - \cdots - \epsilon_{n-1}$. We have $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in \{1, \ldots, n-1\}$ and $\alpha_0 = \epsilon_n - \epsilon_1 + \delta$.

In this paper, we also need to consider an $A_{n-1}^{(1)}$-crystal $B$ with wt defined on $P^0$ and $P^0_{cl}$ instead of $P$. In this case, we call $B$ an $A_{n-1}^{(1)}$-crystal with $P^0$-weights and $P^0_{cl}$-weights, respectively.

**Example 4.1.** Let $\mathcal{F}$ be the set of sequences $\nu = (v_k)_{k \in \mathbb{Z}}$ such that $v_k \in \{0,1\}$, $v_k = 1$ for $k < 0$ and $v_k = 0$ for $k \geq 0$. For $i \in I$, consider a sequence
\[\sigma_{\nu,i} = (\ldots, \sigma_{i-n}, \sigma_{i+1-n}, \sigma_i, \sigma_{i+1}, \sigma_{i+n}, \sigma_{i+1+n}, \ldots),\]
where we assign for \( k \equiv i \pmod{n} \)

\[
(\sigma_k, \sigma_{k+1}) = \begin{cases} 
(+, \cdot) & \text{if } (v_k, v_{k+1}) = (1, 0), \\
(\cdot, -) & \text{if } (v_k, v_{k+1}) = (0, 1), \\
(\cdot, \cdot) & \text{if } (v_k, v_{k+1}) = (0, 0) \text{ or } (1, 1).
\end{cases}
\]

We define \( \tilde{e}_i \mathbf{v} \) to be the sequence obtained from \( \mathbf{v} \) by replacing \((v_k, v_{k+1}) = (0, 1) \) with \((1, 0) \), where \( k + 1 \equiv i + 1 \pmod{n} \) is the index corresponding to the rightmost \(-\) in the reduced sequence \( \widetilde{\sigma_{\mathbf{v},i}} \). We define \( \tilde{e}_i \mathbf{v} = \mathbf{0} \) if there is no such \( k \). Similarly, we define \( \tilde{f}_i \mathbf{v} \) to be the sequence obtained from \( \mathbf{v} \) by replacing \((v_k, v_{k+1}) = (1, 0) \) with \((0, 1) \), where \( k \equiv i \pmod{n} \) is the index corresponding to the leftmost \(+\) in \( \widetilde{\sigma_{\mathbf{v},i}} \). We define \( \tilde{f}_i \mathbf{v} = \mathbf{0} \) if there is no such \( k \).

For \( j \in I \), let \( \mathbf{v}_{\Lambda_j} = (a_k)_{k \in \mathbb{Z}} \) be such that \( v_k = 1 \) if and only if \( k \leq j \). It is well-known that the connected component of \( \mathbf{v}_{\Lambda_j} \) in \( \mathcal{F} \) with respect to \( \tilde{e}_i \) and \( \tilde{f}_i \) for \( i \in I \) is isomorphic to \( B(\Lambda_j) \), where we put \( \text{wt}(\mathbf{v}_{\Lambda_j}) = \Lambda_j \) [23]. Moreover, if we put \( \Lambda_j = \Lambda_{j+n} \) for \( j \in \mathbb{Z} \), we have \( \mathcal{F} \cong \bigoplus_{j \in \mathbb{Z}} B(\Lambda_j) \).

Let \( \mathcal{F}^\vee \) be the set of sequences \( \mathbf{v} = (v_k)_{k \in \mathbb{Z}} \) such that \( v_k \in \{0, 1\} \), \( v_k = 0 \) for \( k < 0 \) and \( v_k = 1 \) for \( k > 0 \) (which is the dual of \( \mathcal{F} \)).

Let \( \mathbf{v}_{-\Lambda_j} = (v_k)_{k \in \mathbb{Z}} \) be such that \( a_k = 0 \) if and only if \( k \leq j \). Then the connected component of \( \mathbf{v}_{-\Lambda_j} \) in \( \mathcal{F}^\vee \) with respect to \( \tilde{e}_i \) and \( \tilde{f}_i \) for \( i \in I \) is isomorphic to \( B(-\Lambda_j) \), where \( \text{wt}(\mathbf{v}_{-\Lambda_j}) = -\Lambda_j \).

### 4.3. Bicrystal structure on \( \hat{\mathbb{M}}_{m \times n} \)

In this subsection, we define two crystal structures on \( \hat{\mathbb{M}}_{m \times n} \) of type \( A_{m-1}^{(1)} \) and \( A_{n-1}^{(1)} \).

For \( i, j \in \mathbb{Z} \), let \( E_{ij} \) denote the elementary matrix in \( \hat{\mathbb{M}}_{m \times n} \) with 1 at the \((i, j)\)-position and 0 elsewhere and put

\[
\hat{E}_{ij} = \sum_{k \in \mathbb{Z}} E_{i+kj, j+kn} \in \hat{\mathbb{M}}_{m \times n}.
\]

Let us first describe an \( A_{m-1}^{(1)} \)-crystal structure on \( \hat{\mathbb{M}}_{m \times n} \) for \( m \geq 2 \). Suppose that \( A = (a_{ij})_{i,j \in \mathbb{Z}} \in \hat{\mathbb{M}}_{m \times n} \) is given. For \( i \in \{0, 1, \ldots, m-1\} \), we define \( \tilde{e}_i A \) and \( \tilde{f}_i A \) as follows:

1. Let \( \sigma \) be a sequence of \( \{+, -\} \) given by

\[
\sigma = (\cdots, +, \cdots, -, \cdots, +, \cdots, -, \cdots, +, \cdots),
\]

and let \( \tilde{\sigma} \) be the reduced one, which is well-defined since \( \sigma \) has only finitely many +’s and −’s.

2. If \( \tilde{\sigma} \) has at least one −, then we define

\[
\tilde{e}_i A = A + \hat{E}_{ij_0} - \hat{E}_{i+1, j_0},
\]

where \( j_0 \) is the column index of \( A \) corresponding the rightmost − in \( \tilde{\sigma} \). If \( \tilde{\sigma} \) has no −, then we define \( \tilde{e}_i A = 0 \). Similarly, if \( \tilde{\sigma} \) has at least one +, then we define

\[
\tilde{f}_i A = A - \hat{E}_{ij_1} + \hat{E}_{i+1, j_1},
\]

where \( j_1 \) is the column index of \( A \) corresponding the leftmost + in \( \tilde{\sigma} \). If \( \tilde{\sigma} \) has no +, then we define \( \tilde{f}_i A = 0 \).
Put
\begin{equation}
\text{wt}^0(A) = \sum_{k \in \mathbb{Z}} \left( \sum_{j=1}^{n} a_{1+k m \cdot j} \right) (\epsilon_1 - k \delta) + \cdots + \sum_{k \in \mathbb{Z}} \left( \sum_{j=1}^{n} a_{m+k m \cdot j} \right) (\epsilon_m - k \delta) \in P^0,
\end{equation}

\begin{equation}
\text{wt}^0_{\text{cl}}(A) = \text{cl}(\text{wt}^0(A)) = \left( \sum_{j \in \mathbb{Z}} a_{1j} \right) \text{cl}(\epsilon_1) + \cdots + \left( \sum_{j \in \mathbb{Z}} a_{mj} \right) \text{cl}(\epsilon_m) \in P^0_{\text{cl}},
\end{equation}

\[ \varepsilon_i(A) = \max\{ k \mid \varpi^0_i A \neq 0 \}, \quad \varphi_i(A) = \max\{ k \mid \varpi_i A \neq 0 \}, \quad (i \in I). \]

Both \( \varepsilon_i(A) \) and \( \varphi_i(A) \) are finite since \( \sigma \) has only finitely many \( + \)'s and \( - \)'s. Moreover, we have
\[ \varphi_i(A) - \varepsilon_i(A) = \sum_{j \in \mathbb{Z}} a_{i+1,j} - \sum_{j \in \mathbb{Z}} a_{ij} = \langle \text{wt}^0(A), h_i \rangle = \langle \text{wt}^0_{\text{cl}}(A), h_i \rangle. \]

Hence we have the following lemma.

**Lemma 4.2.** The set \( \widehat{\mathcal{M}}_{n \times n} \) is a normal \( A_{n-1}^{(1)} \)-crystal with \( P^0 \) and \( P^0_{\text{cl}} \)-weights with respect to \( \text{wt}, \tilde{\varepsilon}_i, \tilde{\varphi}_i \) for \( i \in \{0, 1, \ldots, m-1\} \), where \( \text{wt} = \text{wt}^0 \) and \( \text{wt}^0_{\text{cl}} \), respectively.

**Example 4.3.** Let \( A \) be the generalized affine permutation in Example 2.1. For \( i = 2 \), the associated sequence \( \sigma \) and its reduced one \( \tilde{\sigma} \) are
\[ \sigma = (+, -, -, +, -, +, +, -), \]
\[ \tilde{\sigma} = (\cdot, \cdot, -, -, \cdot, +, +, \cdot, \cdot). \]

Hence the cell \((2, 8)\) is the position corresponding to the leftmost \( + \) in \( \tilde{\sigma} \), and \( \tilde{f}_2 A = A - \tilde{E}_{28} + \tilde{E}_{38} \). The followings are the submatrices of \( A \) and \( \tilde{f}_2 A \) with the row indices in \([4]\).

Next, we define an \( A_{n-1}^{(1)} \)-crystal structure on \( \widehat{\mathcal{M}}_{m \times n} \) for \( n \geq 2 \), say \( \text{wt}^t, \varepsilon_i^t, \varphi_i^t, \tilde{\varepsilon}_i^t, \tilde{\varphi}_i^t \) for \( i \in \{0, 1, \ldots, n-1\} \), by applying the \( A_{n-1}^{(1)} \)-crystal structure on \( \widehat{\mathcal{M}}_{n \times m} \) to the transpose \( A^t \) of \( A \in \widehat{\mathcal{M}}_{m \times n} \).}

\[ \text{wt}^t(A) = \text{wt}(A^t), \]
\[ \tilde{\varepsilon}_i^t A = (\tilde{\varepsilon}_i A^t)^t, \quad \tilde{\varphi}_i^t A = (\tilde{\varphi}_i A^t)^t, \]
\[ \varepsilon_i^t(A) = \varepsilon_i(A^t), \quad \varphi_i^t(A) = \varphi_i(A^t). \]
Proposition 4.4. The operators $\tilde{e}_i$ and $\tilde{f}_j$ for $i \in \{0, 1, \ldots, m-1\}$ commute with $\tilde{e}_j$ and $\tilde{f}_j$ for $j \in \{0, 1, \ldots, n-1\}$ on $\hat{M}_{m \times n} \cup \{0\}$.

Proof. Let $A \in \hat{M}_{m \times n}$ be given. Suppose that for each $1 \leq i \leq m$, we have $a_{ij} = 0$ unless $1 \leq j \leq n$. Then $A$ can be viewed as an $m \times n$ matrix and it is well known that the proposition holds for $A$ when $i, j \neq 0$ (see for examples [20, Lemma 3.4] or [31, Lemma 1.4.7]). For an arbitrary $A \in \hat{M}_{m \times n}$ and $i, j$, we may apply the same argument. \hfill \Box

Remark 4.5. If we assume that

$$\text{wt}(A) = \text{wt}_0^\delta(A), \quad \text{wt}^I(A) = \text{wt}^\delta_0(A^I),$$

for $A \in \hat{M}_{m \times n}$, then $\hat{M}_{m \times n}$ is an $(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$-bicrystal or $(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$-crystal. In other words, $\tilde{e}_i$ and $\tilde{f}_j$ on $\hat{M}_{m \times n} \cup \{0\}$ are strict morphisms of $A^{(1)}_{m-1}$-crystals and $A^{(1)}_{n-1}$-crystals, respectively for $x, y \in \{e, f\}$, $i \in \{0, 1, \ldots, m-1\}$, and $j \in \{0, 1, \ldots, n-1\}$. On the other hand, if we assume that $\text{wt} = \text{wt}^0$ on $\hat{M}_{m \times n}$ as an $A^{(1)}_{m-1}$-crystal, then $\tilde{e}_j, \tilde{f}_j$ preserves the weights only when $j \neq 0$. In case of $\tilde{e}_0, \tilde{f}_0$, we have

$$\text{wt}(\tilde{e}_0^j A) = \text{wt}(A) - \delta, \quad \text{wt}(\tilde{f}_0^j A) = \text{wt}(A) + \delta,$$

for $A \in \hat{M}_{m \times n}$. The same holds for $\tilde{e}_i$ and $\tilde{f}_i$.

4.4. Crystals of level zero extremal weight modules. Let $n \geq 2$. In this subsection, we describe a structure of $A^{(1)}_{n-1}$-crystal with $P^0$-weights on $B(\lambda) = B_n(\lambda)$ for $\lambda \in \mathcal{P}_n$.

First, consider $B((1)) = SST_{\mathbb{Z}}((1))$, which can be identified with $\mathbb{Z}$. For $[k] \in B((1))$ and $i \in I = \{0, 1, \ldots, n-1\}$, we define

$$\text{wt}^0([k]) = \epsilon_r - q \delta, \quad \text{where } k = nq + r \text{ for } q \in \mathbb{Z} \text{ and } 1 \leq r \leq n,$$

$$\tilde{e}_i[k] = \begin{cases} \frac{k-1}{r} & \text{if } k \equiv i + 1 \pmod{n}, \\ 0 & \text{otherwise}, \end{cases} \quad \tilde{f}_i[k] = \begin{cases} \frac{k+1}{r} & \text{if } k \equiv i \pmod{n}, \\ 0 & \text{otherwise}. \end{cases}$$

Then $B((1))$ is a normal $A^{(1)}_{n-1}$-crystal with $P^0$-weights with respect to $\text{wt}^0$, $\tilde{e}_i$, $\tilde{f}_i$ for $i \in I$.

Next, consider $B((1^b))$ for $2 \leq b \leq n$. Let $T \in B((1^b))$ be given with entries $T(1) < T(2) < \cdots < T(b)$. For $i \in I$, we define $\tilde{e}_i T$ as follows:

1. If $T(k) \equiv i + 1 \pmod{n}$ and $T(k-1) < T(k) - 1$ for some $2 \leq k \leq b$, then we define $\tilde{e}_i T$ to be the tableau obtained by replacing $T(k)$ with $T(k) - 1$.
2. If $T(1) \equiv i + 1 \pmod{n}$ and $T(b) < T(1) - 1 + n$, then we define $\tilde{e}_i T$ to be the tableau obtained by replacing $T(1)$ with $T(1) - 1$.
3. Otherwise, we define $\tilde{e}_i T = 0$.

We define $\tilde{f}_i T$ in a similar way. We put $\text{wt}^0(T) = \sum_{k=1}^b \text{wt}^0(T(k))$, where we regard $T(k)$ as an element in $B((1^b))$.

Proposition 4.6. For $1 \leq b \leq n$, $B((1^b))$ is a connected normal $A^{(1)}_{n-1}$-crystal with $P^0$-weights. For $1 \leq b \leq n - 1$, $B((1^b))$ is isomorphic to $B(\varpi_b)$. 

Proof. It is easy to see that $B((1^b))$ is a connected normal $A_{n-1}^{(1)}$-crystal. In particular, $B((1^n))$ is the crystal consisting of single element of weight 0. Suppose that $1 \leq b \leq n-1$. Let us keep the notations in Example 4.1. For $i \in \mathbb{Z}$, let $e_i = (\delta_{ik})_{k \in \mathbb{Z}}$. Let $T \in B((1^b))$ be given with entries $T(1) < \cdots < T(b)$. Let $k$ be the positive integer such that $1 \leq k \leq b - 1$ and

$$T(1) < \cdots < T(k) < 0 < T(k+1) < \cdots < T(b).$$

In particular, $k = 0$ if $0 < T(1)$ and $k = b$ if $T(b) \leq 0$. Let

$$v_1 = v_{A_k} + e_{T(k+1)} + \cdots + e_{T(b)},$$

$$v_2 = v_{-A_k} + e_{T(1)} + \cdots + e_{T(k)}.$$

Recall that $u_{(1^b)} \in B((1^b))$ is the column tableau having $i$ in its $i$-th row. Then the map sending $T$ to $v_1 \otimes v_2$ gives a strict embedding

$$B((1^b)) \longrightarrow B(A_k) \otimes B(-A_0),$$

and it maps $u_{(1^b)}$ to $v_{A_k} \otimes v_{-A_0}$. This implies that $B((1^b))$ is isomorphic to the connected component of $v_{A_k} \otimes v_{-A_0}$ in $B(A_k) \otimes B(-A_0)$. Hence $B((1^b))$ is isomorphic to $B(\varpi_b)$ by [16] and [17] Proposition 5.4.

□

Recall the bijection $\tau$ on $B((1^b))$, defined in Section 3.3. Then we have the following.

Corollary 4.7. For $T \in B((1^b))$, we have $\tau(\tilde{e}_iT) = \tilde{e}_i\tau(T)$ and $\tau(\tilde{f}_iT) = \tilde{f}_i\tau(T)$ for $i \in I$, where $\text{wt}^0(\tau(T)) = \text{wt}^0(T) - \delta$.

Proof. It follows directly from the crystal structure on $B((1^b))$. □

Suppose that $R = (a^b) \in \mathcal{P}_n$ is given for $a \geq 1$ and $1 \leq b \leq n$. Define a map

$$\psi: \begin{array}{ccc}
B(R) & \longrightarrow & B((1^b))^\otimes a \\
T = (T^1, \ldots, T^b) & \longrightarrow & T^1 \otimes \cdots \otimes T^a
\end{array}$$

Let $u_R$ denote the unique element of $B(R)$ such that the $i$-th row is filled with $i$ for $1 \leq i \leq b$.

Lemma 4.8. Under the above hypothesis, we have the following.

1. The image of $B(R) \cup \{0\}$ under $\psi$ is closed under $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I$. Hence $B(R)$ is an $A^{(1)}_{n-1}$-crystal with $P^0$-weights such that $\psi$ is a strict embedding.

2. The connected component of $u_R$ in $B(R)$ is $B(R)_0$ when $b < n$.

3. The map

$$\begin{array}{ccc}
B(R)_0 \times \mathcal{P}_{a-1} & \longrightarrow & B(R) \\
\end{array}$$

in Lemma 4.6 is an isomorphism of $A^{(1)}_{n-1}$-crystals with $P^0$-weights. Here we understand $B(R)_0 \times \mathcal{P}_{a-1}$ as $B(R)_0 \otimes \mathcal{P}_{a-1}$ and $\mathcal{P}_{a-1}$ as an $A^{(1)}_{n-1}$-crystal with $P^0$-weights, where $\text{wt}(\nu) = -|\nu|\delta$, $\tilde{e}_i\nu = \tilde{f}_i\nu = 0$, and $\varepsilon_i(\nu) = \varphi_i(\nu) = 0$ for $\nu \in \mathcal{P}_{a-1}$ and $i \in I$. 

Proof. (1) Since $\mathcal{B}((1^b)) \cup \{0\}$ is closed under $\mathcal{C}_i$ and $\mathcal{F}_i$ for $i \in I$, it is enough to show that $\mathcal{C}_iT$ and $\mathcal{F}_iT$ are semistandard for $T \in \mathcal{B}(R)$. The proof is similar to the case of type $A_{n-1}$ (cf. [19]).

(2) Let $T = (T^a, \ldots, T^1) \in \mathcal{B}(R)_0$ be given.

Suppose that $\mathcal{C}_iT \notin \mathcal{B}(R)_0 \cup \{0\}$ for some $i$. Then $\mathcal{C}_iT = (\ldots, \mathcal{C}_iT^j, \ldots)$ for some $1 \leq j \leq a$, and $\mathcal{C}_iT^j$ is obtained from $T^j$ by replacing $T^j(k)$ with $T^j(k) - 1$ for some $1 \leq k \leq b$. Since $T \in \mathcal{B}(R)_0$ but $\mathcal{C}_iT \notin \mathcal{B}(R)_0 \cup \{0\}$, at least one of $(\mathcal{C}_iT^j, \tau^{-1}(T^j-1))$ or $(T^{j+1}, \tau^{-1}(\mathcal{C}_iT^j))$ is semistANDARD. Suppose that $(\mathcal{C}_iT^j, \tau^{-1}(T^j-1))$ is semistandard. Considering $\mathcal{C}_iT \in \mathcal{B}(R)$, it is straightforward to see that $(T^j, \tau^{-1}(T^j-1))$ is also semistandard, which is a contradiction. For the other cases, we have similar contradiction. By the same arguments, we also have $\mathcal{F}_iT \notin \mathcal{B}(R)_0 \cup \{0\}$. Hence $\mathcal{B}(R)_0 \cup \{0\}$ is closed under $\mathcal{C}_i$ and $\mathcal{F}_i$ for $i \in I$.

Conversely, we claim that any $T \in \mathcal{B}(R)_0$ is connected to $u_R$. Let $T = (T^a(1))$. We first show that $T \in \mathcal{B}(R)_0$ is connected to $T_0 = u_R^{(0)}$, where $u_R^{(t)}$ is the element of $\mathcal{B}(R)$ such that the $i$-th row is filled with $t + i - 1$ for $1 \leq i \leq b$. Suppose that $T \neq T_0$. Let $d(T) = \sum_{j=1}^a \sum_{k=1}^b (T^j(k) - T_0^j(k)) > 0$. Since $T \in \mathcal{B}(R)_0$ and $\mathcal{B}(R)_0 \cup \{0\}$ is closed under $\mathcal{C}_i$ and $\mathcal{F}_i$ for $i \in I$, we have $0 \leq \delta(\mathcal{C}_iT) < d(T)$ for $i$ such that $\mathcal{C}_iT \neq 0$. Note that there exists at least one $i$ such that $\mathcal{C}_iT \neq 0$. For example, choose the smallest $k$ such that $T^j(k) \neq T_0^j(k)$ and then $i = T^j(k) - 1$. By induction on $d(T)$, we conclude that $T$ is connected to $T_0$.

Next, we have

$$\begin{align*}
\tilde{c}_{i}^{a+1} \cdots \tilde{c}_{i-1}^{a} T_0^{(t-1)} & = u_R^{(t-1)} \quad (t > 1), \\
\tilde{f}_{i}^{a} \cdots \tilde{f}_{i}^{a+1} \tilde{f}_{i}^{a+2} \tilde{f}_{i}^{a+3} T_0^{(t+1)} & = u_R^{(t+1)} \quad (t < 1).
\end{align*}$$

Repeating this step, we conclude that $T_0$ is connected to $u_R = u_R^{(0)}$. Therefore, $\mathcal{B}(R)_0$ is the connected component of $u_R$ in $\mathcal{B}(R)$.

(3) It follows from Corollary 4.7. 

Remark 4.9. Suppose that $b = n$. In this case, $\mathcal{B}(R)_0 = \{ \tau^t(u_R) \mid t \in \mathbb{Z} \}$, where each $\{ \tau^t(u_R) \}$ forms a trivial crystal of weight $-at\delta$.

Proposition 4.10. For $R = (a^b)$ with $a \geq 1$ and $1 \leq b < n$, $\mathcal{B}(R)$ is isomorphic to $B(a\varpi_b)$ as an $A_{n-1}^{(1)}$-crystal with $P^0$-weights.

Proof. It follows from Proposition 4.6, Lemma 4.8 and [1] Theorem 4.16(a). 

Remark 4.11. The isomorphism in Lemma 4.8(3) is also proved in [24] Theorem 2] for an affine Lie algebra $\mathfrak{g}$ in terms of Lakshmibai-Seshadri paths.

Let $\lambda \in \mathcal{O}_n$ be given and let $\mathcal{B}(\lambda)$ as given in (3.9). We regard $\mathcal{B}(\lambda)$ as an $A_{n-1}^{(1)}$-crystal with $P^0$-weights by identifying

$$\mathcal{B}(\lambda) = \mathcal{B}(R_1) \otimes \cdots \otimes \mathcal{B}(R_n).$$

Let

$$\varpi_\lambda = m_1 \varpi_1 + \cdots + m_{n-1} \varpi_{n-1},$$

where $m_i$ is the multiplicity of $i$ in $\lambda$. By [1] Theorem 4.16 and [17] Conjectures 13.1, 13.2 (see also [1] Remark 4.17), we have the following.
Proposition 4.12. For \( \lambda \in P_{n-1} \), \( B(\lambda) \) is isomorphic to \( B(\varpi_\lambda) \).

Remark 4.13. (1) In [13], another proof of Proposition 4.12 is given using the standard monomial theory for semi-infinite Lakshmibai–Seshadri paths [14].

(2) Let \( u_\lambda = u_{R_1} \otimes \cdots \otimes u_{R_r} \). Then \( B(\lambda)_0 = B(R_1)_0 \otimes \cdots \otimes B(R_r)_0 \) is the connected component of \( u_\lambda \) in \( B(\lambda) \) (see [1] Remark 4.17). In case of \( A_{n-1}^{(1)} \), we can prove it directly using the crystal structure described here.

(3) Suppose that \( \lambda \in P_n \) with \( n = \ell(\lambda) \). We have \( B(\lambda) = B(\mu) \otimes B(R_n) \), where \( \mu = (\lambda_1, \ldots, \lambda_{n-1}) \) (see Remark 4.9).

(4) We may also regard \( B(\lambda) \) as an \( A_{n-1}^{(1)} \)-crystal with \( P_{cl}^0 \)-weights by using \( \text{wt}^0_{cl} = \text{cl}(\text{wt}^0_{cl}) \).

Remark 4.14. Let \( R = (a^b) \). Then the set \( B(R)_0/\mathbb{Z} \cup \{0\} \) is closed under \( \hat{e}_i \) and \( \hat{f}_i \) for \( i \in I \) since \( \tau \) commutes with \( \hat{e}_i \) and \( \hat{f}_i \), and \( \text{wt}^0_{cl} \) is constant on any equivalence class. Hence by (3.7), we may regard \( CSST_{[n]}(R) \) as a normal \( A_{n-1}^{(1)} \)-crystal with \( P_{cl}^0 \)-weights. Then \( CSST_{[n]}(R) \) is isomorphic to \( SST_{[n]}((1^b)^{\otimes a}) \). Note that \( SST_{[n]}((1^b)^{\otimes a}) = CSST_{[n]}((1^b)) \) and it is isomorphic to the crystal of Kirillov-Reshetikhin module or level one perfect crystal associated to \( cl(\varpi_\lambda) \) [14].

Furthermore, if \( (r_1, \ldots, r_{a-1}) \) is the offset vector of \( T \in CSST_{[n]}(R) \), then \( -r_j \) is equal to the value of the local energy function on \( SST_{[n]}((1^b)^{\otimes 2}) \) at \( (T^{j+1}, T^j) \) or \( T^j \otimes T^{j-1} \) for \( 1 \leq j \leq a-1 \) (see [25], Section 3).

In general, for \( \lambda \in P_{m-1} \) we regard as an \( A_{n-1}^{(1)} \)-crystal with \( P_{cl}^0 \)-weights

\[ CSST_{[n]}(\lambda) = CSST_{[n]}(R_1) \otimes \cdots \otimes CSST_{[n]}(R_r). \]

Hence if we regard \( B(\lambda) \) as an \( A_{n-1}^{(1)} \)-crystal with \( P_{cl}^0 \)-weights, then the bijection (3.10)

\[ CSST_{[n]}(\lambda) \times P(\lambda) \longrightarrow B(\lambda) \]

is an isomorphism, where the left-hand side is understood as a disjoint union of \( CSST_{[n]}(\lambda) \) parametrized by \( P(\lambda) \).

4.5. Isomorphism of crystals. Let

\[ \mathcal{I}_{m \times n} = \bigsqcup_{\lambda \in P_m \cap P_n} CSST_{[n]}(\lambda) \times B_n(\lambda). \]

We first assume that \( \mathcal{I}_{m \times n} \) is a normal \( A_{n-1}^{(1)} \)-crystal with \( P_{cl}^0 \)-weights with respect to \( \hat{e}_i, \hat{f}_i \) for \( i \in \{0,1,\ldots,m-1\} \) for \( m \geq 2 \), where \( \hat{e}_i, \hat{f}_i \) are the Kashiwara operators on \( CSST_{[n]}(\lambda) \) (see Remark 4.14). Also we assume that \( \mathcal{I}_{m \times n} \) is a normal \( A_{n-1}^{(1)} \)-crystal with \( P^0 \)-weights with respect to \( \hat{e}_j \), \( \hat{f}_j \) for \( j \in \{0,1,\ldots,n-1\} \) for \( n \geq 2 \), where \( \hat{e}_j, \hat{f}_j \) denote the Kashiwara operators on \( B_n(\lambda) \).

The following is the second main result in this paper. The proof is given in Section 6.

Theorem 4.15. The bijection

\[ \kappa : \widehat{\mathcal{M}}_{m \times n} \longrightarrow \mathcal{I}_{m \times n} \]

commutes with \( \hat{e}_i, \hat{f}_i \) for \( i \in \{1,\ldots,m-1\} \) and \( \hat{e}_j, \hat{f}_j \) for \( j \in \{0,1,\ldots,n-1\} \).
We remark that the map $\kappa$ does not commute with $\tilde{c}_0$ and $\tilde{f}_0$, but $\kappa_1 := \pi_1 \circ \kappa$ does, where $\pi_1$ is the projection of $T_{m \times n}$ along the first component (see Remark 6.10). Since $\text{wt}_t^0(A) = \text{wt}_t^0(P_0)$, $\kappa_1$ induces the following.

**Corollary 4.16.** A generalized affine permutation $A \in \hat{M}_{m \times n}$ is $A_{m-1}^{(1)}$-crystal equivalent to $P_0$, where $\kappa(A) = (P_0, Q)$.

Note that $\kappa$ is an isomorphism of $A_{n-1}^{(1)}$-crystals with $P_{\text{cl}}^0$-weights, but it does not preserve $P^0$-weights. More precisely, for $A \in \hat{M}_{m \times n}$ with $\kappa(A) = (P_0, Q)$, we see from the definition of $\kappa$ that

$$\text{wt}^t(A) = \text{wt}^0(A^t) = \text{wt}^0(Q) - \left( \sum_{i=1}^{l} |\eta^{(i)}| \right) \delta,$$

where $\eta^{(i)}$ is the symmetrized offset vectors of $P_0^{(i)}$ in Section 3.5. So in order to have a morphism of $A_{n-1}^{(1)}$-crystals with $P^0$-weights, we may modify the weight function on $\hat{M}_{m \times n}$ in $P^0$ by

$$\text{wt}^t(A) = \text{wt}^0(A^t) + H_m(A) \delta,$$

where $H_m(A) = \sum_{i=1}^{l} |\eta^{(i)}|$. Then we have the following.

**Corollary 4.17.** If we regard $\hat{M}_{m \times n}$ as an $A_{m-1}$-crystal with $P_{\text{cl}}^0$-weights and as an $A_{n-1}^{(1)}$-crystal with $P^0$-weights with respect to $\text{wt}^t$, then it is an $(A_{m-1}, A_{n-1}^{(1)})$-bicrystal and $\kappa$ is an isomorphism of $(A_{m-1}, A_{n-1}^{(1)})$-bicrystals. In particular, a generalized affine permutation $A \in \hat{M}_{m \times n}$ is $A_{m-1}^{(1)}$-crystal equivalent to $Q$, where $\kappa(A) = (P_0, Q)$.

We remark that both $m$ and $n$ do not need to be greater than 1 for Theorem 4.15 and its corollaries. In particular, Corollary 4.16 holds for $n = 1$ and Corollary 4.17 holds for $m = 1$.

**Remark 4.18.** The function $H_m(\cdot)$ in (4.5) is related to the intrinsic energy function on $A_{m-1}^{(1)}$-crystals with $P_{\text{cl}}^0$-weights as follows. Let $T \in CSST_{[m]}(R)$ be given where $R = (a^b)$. Let $r = (r_1, \ldots, r_{n-1})$ be the offset vector and $\eta = (\eta_1, \ldots, \eta_{n-1})$ the symmetrized offset vector of $T$. Then we have

$$\mathcal{H}_m(T) = |\eta| - a|r|,$$

where $|r| = r_1 + \cdots + r_{n-1}$ and $\mathcal{H}_m(\cdot)$ is the intrinsic energy function on $CSST_{[m]}(R)$ with $\mathcal{H}_m(u_R) = 0$ (cf. 29 for its definition). Hence for $A \in \hat{M}_{m \times n}$ with $\kappa(A) = (P_0, Q)$, we have

$$H_m(A) = \sum_{i=1}^{l} \left( \mathcal{H}_m(P_0^{(i)}) + m_i |r^{(i)}| \right),$$

where $R_i = (a_i^{m_i})$ is the shape and $r^{(i)}$ is the offset vector of $P_0^{(i)}$ in Section 3.5, respectively.

5. **Proof of Theorem 3.10**

5.1. **Standardization and $\kappa_0$**. In this subsection, we show the map $\kappa_0$ in (3.3) is compatible with the standardization (2.2).
Lemma 5.1. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_0$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$ be given such that $|\alpha| = |\beta|$, where $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Let

$$\hat{M}_{m \times n}(\alpha, \beta) = \{ A \mid A \in \hat{M}_{m \times n}, \text{row}(A) = \alpha, \text{col}(A) = \beta \},$$

where row$(A)$ and col$(A)$ are the row and column contents of $A$ given in (2.4). We have $\hat{M}_{m \times n} = \bigsqcup_{\alpha, \beta} \hat{M}_{m \times n}(\alpha, \beta)$.

For $K \geq 1$, recall that $A = (a_{ij})_{i,j \in \mathbb{Z}} \in \hat{M}_{K \times K}$ is an extended affine permutation of $K$ if $\sum_{j \in \mathbb{Z}} a_{ij} = \sum_{i \in \mathbb{Z}} a_{ij} = 1$ for each $i$ and $j$. Let $\hat{W}_K \subset \hat{M}_{K \times K}$ denote the set of extended affine permutations of $K$. Let $B \in \hat{W}_K$ be given. We say that $i \in [K]$ is a descent of $B$ if $(i, j), (i + 1, j') \in \text{supp}(B)$ and $j > j'$. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_0$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$ be given with $|\alpha| = |\beta| = K$. We say that $B \in \hat{W}_K$ is $\alpha$-descending if for any $k \in [m]$ and $i$ with $\alpha_1 + \cdots + \alpha_{k-1} < i < \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k$,

$i$ is a descent of $B$, where we understand the empty sum is 0. We say that $B$ is $(\alpha, \beta)$-descending if $B$ is $\alpha$-descending and $B^T$ is $\beta$-descending. Let $\hat{W}_K(\alpha, \beta)$ denote the set of extended affine permutations which are $(\alpha, \beta)$-descending.

**Lemma 5.1.** Under the above hypothesis, we have a bijection

$$\hat{M}_{m \times n}(\alpha, \beta) \quad \longleftrightarrow \quad \hat{W}_K(\alpha, \beta).$$

**Proof.** For $A \in \hat{M}_{m \times n}(\alpha, \beta)$, we have $A^{\alpha\beta} \in \hat{W}_K(\alpha, \beta)$ by Lemma 2.4, hence the map is well-defined.

Let us prove the existence of its inverse. Let $B = (b_{kl})_{k,l \in \mathbb{Z}} \in \hat{W}_K(\alpha, \beta)$ be given. Consider the partition $\{ I_i \times J_j \}_{i,j \in \mathbb{Z}}$ of $\mathbb{Z} \times \mathbb{Z}$ associated to $\alpha$ and $\beta$ in (2.4). Then we define $A = (a_{ij})_{i,j \in \mathbb{Z}}$ to be a $\mathbb{Z} \times \mathbb{Z}$ matrix where

$$a_{ij} = \sum_{(k,l) \in I_i \times J_j} b_{kl}$$

Since $B \in \hat{W}_K(\alpha, \beta)$, we see from the definition that $I_i \times J_j$ submatrix of $B$ contains an $a_{ij} \times a_{ij}$ matrix with 1 on the antidiagonal and 0 elsewhere. Then it is easy to see that $A$ is the unique matrix in $\hat{M}_{m \times n}(\alpha, \beta)$ such that $A^{\alpha\beta} = B$. This proves the bijectivity. \(\square\)

Now, let us consider standardization on the side of tableaux in (3.3).

Let $\lambda \in \mathcal{P}_n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$ be given. Let $\text{CSST}_{[n]}(\lambda)_{\alpha}$ be the set of $T \in \text{CSST}_{[n]}(\lambda)$ with content $\alpha$, that is, $\alpha_k$ is the number of occurrences of $k \in [n]$ in $T$. For $T \in \text{CSST}_{[n]}(\lambda)_{\alpha}$, we define $T^{\alpha\beta}$ to be a tableau obtained from $T$ by replacing each $k \in [n]$ in $T$ with $\alpha_k$ numbers

$$\alpha_1 + \cdots + \alpha_{k-1} + 1 < \cdots < \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k$$


Lemma 5.3. Then we can check the following. \( \alpha \) is a descent of \( i \) if for any \( \tau \) from the definition of \( T \) if the letter \( T \) be the smallest integer satisfying \( p \). Let \( \nu \). It follows from \([4, Proposition 3.6]\).

Proof. Let \( T = (T^a, \ldots, T^1) \) and \( T^{st} = S = (S^a, \ldots, S^1) \). For \( 1 \leq j \leq a - 1 \), let \( r_j \) be the smallest integer satisfying \((T^{j+1}, \tau^{r_j}_n(T^{j+1}))\) is semistandard. It suffices to show that \((S^{j+1}, \tau^{r_j}_K(S^{j+1}))\) is semistandard but \((S^{j+1}, \tau^{r_j-1}_K(S^{j+1}))\) is not. It is straightforward to see from the definition of \( T^{st} \).

Let \( T \in CST_{[K]}(\lambda) \) be given with \(|\lambda| = K\). We say that \( i \in [K] \) is a (column) descent of \( T \) if the letter \( i+1 \) appears to the right of \( i \) in \( T \) \((i+1 \) lies in the column with a smaller index than that of \( i \)). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+ \) with \( K = |\alpha| \), we say that \( T \in CST_{[K]}(\lambda) \) is \( \alpha \)-descending if for any \( k \in [n] \) and \( i \) with

\[
\alpha_1 + \cdots + \alpha_{k-1} < i < \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k,
\]

\( i \) is a descent of \( T \). Let \( CST_{[K],\alpha}(\lambda) \) be the set of \( T \in CST_{[K]}(\lambda) \) which are \( \alpha \)-descending. Then we can check the following.

Lemma 5.3. Under the above hypothesis, we have a bijection

\[
\begin{array}{ccc}
CSST_{[n]}(\lambda)_\alpha & \longrightarrow & CST_{[K],\alpha}(\lambda).
\end{array}
\]

The following generalizes a well-known property of the usual RS correspondence (cf. \([7, Section 1.1]\)).

Lemma 5.4. Let \( B \in \widetilde{W}_K \) be given with \( \kappa_0(B) = (P_0, Q_0, \rho) \). Then \( i \in [K] \) is a descent of \( B \) if and only if \( i \) is a descent of \( P_0 \). Similarly, \( j \in [K] \) is a descent of \( B^t \) if and only if \( j \) is a descent of \( Q_0 \).

Proof. It follows from \([4, Proposition 3.6]\).

Corollary 5.5. Let \( B \in \widetilde{W}_K \) be given with \( \kappa_0(B) = (P_0, Q_0, \rho) \), where \( P_0, Q_0 \in CST_{[K]}(\lambda) \) and \( \rho \in \mathbb{Z}^n_{\geq 0} \) with \( \lambda \in \mathcal{P}_m \cap \mathcal{P}_n \). For \( \alpha \in \mathbb{Z}^n_{\geq 0} \) and \( \beta \in \mathbb{Z}^n_{\geq 0} \) with \(|\alpha| = |\beta| = K\), the following are equivalent:

1. \( B \in \widetilde{W}_{K,(\alpha,\beta)} \),
2. \( P_0 \in CST_{[K],\alpha}(\lambda) \) and \( Q_0 \in CST_{[K],\beta}(\lambda) \).
It is clear that \( \kappa_0 \) in (5.3) preserves contents, that is, if \( A \in \hat{M}_{m \times n}(\alpha, \beta) \) with \( \kappa_0(A) = (P_0, Q_0, \rho) \), then \( P_0 \in CSST_{[m]}(\lambda) \) and \( Q_0 \in CSST_{[n]}(\lambda) \). The following proposition shows that \( \kappa_0 \) is compatible with standardizations.

**Proposition 5.6.** For \( A \in \hat{M}_{m \times n} \) with \( \kappa_0(A) = (P_0, Q_0, \rho) \), we have

\[
\kappa_0(A^{st}) = (P_0^{st}, Q_0^{st}, \rho).
\]

**Proof.** Assume that \( A \in \hat{M}_{m \times n}(\alpha, \beta) \) for some \( \alpha \in \mathbb{Z}_{\geq 0}^m \) and \( \beta \in \mathbb{Z}_{\geq 0}^n \) with \( |\alpha| = |\beta| = K \).

Let \( \kappa_0(A^{st}) = (P_1, Q_1, \rho) \). By Lemma 5.1 and Corollary 5.5, we have

\[
P_1 \in CST_{[K], \alpha}(\lambda), \quad Q_1 \in CST_{[K], \beta}(\lambda)
\]

for some \( \lambda \in \mathcal{P} \).

We claim that \( P_0 \) (resp. \( Q_0 \)) is the image of \( P_1 \) (resp. \( Q_1 \)) under the inverse of the bijection in Lemma 5.3 and \( \rho = \rho_0 \). Let \( \{I_i \times J_j\}_{i,j \in \mathbb{Z}} \) be the partition of \( \mathbb{Z} \times \mathbb{Z} \) associated to \( \alpha \) and \( \beta \) in (2.4). Let \( s \) and \( s^{st} \) be the stream given from the the back-post corners of the sets of zig-zags \( \{z_k\}_{k \in \mathbb{Z}} \) and \( \{z_k^{st}\}_{k \in \mathbb{Z}} \) associated to \( d_A^{st} \) and \( d_A^{st} \), respectively. Since we have \( (d_A^{st})^{st} = d_A^{st} \) by (2.8), the flow of \( s \) is equal to that of \( s^{st} \), and \( s \) is obtained from \( s^{st} \) by replacing each \( (k, l) \) in \( s^{st} \) with \( (i, j) \) when \( (k, l) \in I_i \times J_j \). Since \( (A^{st})^{st} = (A^{st}) \) (see Remark 3.5), we use induction on \( K \) to prove the claim. Hence we have \( P_0^{st} = P_1 \) and \( Q_0^{st} = Q_1 \). \( \square \)

### 5.2. Dominance condition and \( \kappa \)

In this subsection, we characterize the image of \( \kappa_0 \), and then prove Theorem 3.10.

**5.2.1. Offset constants.** Let \( \lambda = (2^l) \) be of rectangular shape with two columns for \( 1 \leq l \leq \min\{m, n\} \).

Suppose that \( P_0 = (P_0^2, P_0^1) \in CSST_{[m]}(\lambda) \) and \( Q_0 = (Q_0^2, Q_0^1) \in CSST_{[n]}(\lambda) \) are given. Let \((0, \eta)\) and \((0, \theta)\) be the offset vectors for \( P_0 \) and \( Q_0 \), that is,

\[
\tau^{(0, \eta)}(P_0) = (P_0^2, \tau^\eta(P_0^1)) \in \mathcal{B}_m(\lambda)_0, \quad \tau^{(0, \theta)}(Q_0) = (Q_0^2, \tau^\theta(Q_0^1)) \in \mathcal{B}_n(\lambda)_0,
\]

and put

\[
r = \theta - \eta.
\]

We claim that \( r \) is equal to the offset constant introduced in [4, Section 5.2], which plays an important role in characterizing the image of \( \kappa_0 \) in the case of extended affine permutations.

Write \( P_0 = (a_1, a_2) \) and \( Q_0 = (b_1, b_2) \). For \((\rho_1, \rho_2) \in \mathbb{Z}^2\), let

\[
s = \{c_{i} = (a_i, b_i)\}_{i \in \mathbb{Z}}, \quad s' = \{c'_{i} = (a'_i, b'_i)\}_{i \in \mathbb{Z}}
\]

be the streams of flow \( l \), whose defining data are \((a_1, b_1, \rho_1)\) and \((a_2, b_2, \rho_2)\), respectively. We may assume that

\[
a_1 = (a_1, \ldots, a_l), \quad b_1 = (b_{1-\rho_1}, \ldots, b_{l-\rho_1}), \\
a_2 = (a'_1, \ldots, a'_l), \quad b_2 = (b'_{1-\rho_2}, \ldots, b'_{l-\rho_2}).
\]

Let $k$ be the smallest integer such that $c_i \geq_{\mathbb{W}} c'_{i+k}$ for all $i \in \mathbb{Z}$, and then consider the two streams
\[ t = \{ d_i = (a'_{i+k}, b_i) \}_{i \in \mathbb{Z}}, \quad t' = \{ d'_i = (a_i, b'_{i+k}) \}_{i \in \mathbb{Z}}. \]

Let $A = (a_{ij})_{i,j \in \mathbb{Z}} \in \hat{\mathbb{M}}_{m \times n}$, where $a_{ij}$ is the number of occurrence of $(i,j)$ in $t \cup t'$. It is easy to observe that $l$ is the width of $A$ and $t$ is the southwest corner of $A$. Let $d$ be a proper numbering on $A$ defined by $d(d_i) = d(d'_i) = i$ for $i \in \mathbb{Z}$.

**Lemma 5.7.** Under the above hypothesis, we have $\rho_2 - \rho_1 \geq r$ if and only if $d = d_A^r$.

**Proof.** We observe that in order to show $d = d_A^r$, it suffices to find $i$ such that $d_i \geq_{\mathbb{W}} d'_{i+1}$ by the condition (2) in Lemma 2.21. Furthermore, $d_{i-1} \geq_{\mathbb{W}} d'_i$ for some $i$ is equivalent to $a'_{i-1+k} \leq a_i$, since $b_{i-1} < b'_{i+k}$ is redundant by our choice of $k$.

By (5.1), the constants $\eta$ and $\theta$ are the smallest integers such that
\[ a_i \leq a'_{i+\eta}, \quad b_i \leq b'_{i+\rho_1-\rho_2+\theta} \]
for all $i \in \mathbb{Z}$ respectively. By the minimality of $\eta$, $\theta$, we have $\eta \leq k$ and $\theta \leq \rho_2 - \rho_1 + k$.

Suppose that $d = d_A^r$. Then by the above observation, there exists $i$ with $a_i > a'_{i+k-1}$. In this case, we have $k - 1 < \eta \leq k$ by the minimality of $\eta$. Hence $\eta = k$ and
\[ \rho_2 - \rho_1 \geq \theta - k = \theta - \eta = r. \]

Note that we have $\eta \geq \rho_1 - \rho_2 + \theta$, which implies that $b_i \leq b'_{i+\eta}$.

Conversely, suppose that $\rho_2 - \rho_1 \geq \theta - k$. Since $\rho_2 - \rho_1 + \eta \geq \theta$, we have
\[ b_i \leq b'_{i+\rho_1-\rho_2+(\rho_2-\rho_1+\eta)} = b'_{i+\eta} \quad (i \in \mathbb{Z}). \]

Since $a_i \leq a'_{i+\eta}$, we have $k \leq \eta$ by the minimality of $k$, which implies $a'_{i+k-1} < a_i$ for some $i$. Hence $d = d_A^r$. \hfill $\square$

The following is another characterization of $r$.

**Corollary 5.8.** Under the above hypothesis, we have $\rho_2 - \rho_1 \geq r$ if and only if
\[ \tau(\rho_1, \rho_2 + \eta)(Q_0) \in \mathcal{B}_n(\lambda). \]

**Proof.** We have shown in the proof of Lemma 5.7 that $\rho_2 - \rho_1 \geq r$ is equivalent to $b_i \leq b'_{i+\eta}$ (5.4). On the other hand, by (5.4) it is equivalent to
\[ \tau(\rho_1, \rho_2 + \eta)(Q_0) = (\tau^{\rho_1} b_1, \tau^{\rho_2 + \eta} b_2) \in \mathcal{B}_n(\lambda). \] \hfill $\square$

**Remark 5.9.** It is not difficult to see that the constants $\eta$ and $\theta$ are equal to the local charges of $P_0$ and $Q_0$, respectively by Definition 5.3. Hence by Theorem 5.10, the constant $r = \theta - \eta$ in (5.2) is equal to the offset constant of $(P_0, Q_0)$ in (4) when $P_0, Q_0 \in CST_{K}((2^l))$ for some $K$. 

5.2.2. Proof of Theorem 3.10. We keep the notations in Section 3.5. Suppose that \((P_0, Q_0, \rho) \in \text{CSST}_m(\lambda) \times \text{CSST}_n(\lambda) \times \mathbb{Z}^{\lambda}\) is given for \(\lambda \in \mathcal{P}_m \cap \mathcal{P}_n\).

For \(1 \leq i \leq l\) with \(m_i \geq 1\), we let

- \(\rho^{(i)} \in \mathbb{Z}^{m_i}\) : the subsequence of \(\rho\) corresponding to the columns of \(R_i\),
- \(\eta^{(i)} \in \mathcal{P}_{m_i-1}\) : the symmetrized offset vector of \(P_0^{(i)}\),
- \(\theta^{(i)} \in \mathcal{P}_{m_i-1}\) : the symmetrized offset vector of \(Q_0^{(i)}\),
- \(\zeta^{(i)} = \theta^{(i)} - \eta^{(i)} \in \mathbb{Z}^{m_i}\) with the last component being zero.

We write \(\zeta = (\zeta^{(1)}, \ldots, \zeta^{(l)})\).

**Definition 5.10.** Under the above hypothesis, we say that \((P_0, Q_0, \rho)\) is dominant if

\[
 \rho_{rev} - \zeta \in \mathcal{P}(\lambda).
\]

**Lemma 5.11.** Under the above hypothesis, we have \((P_0, Q_0, \rho)\) is dominant if and only if \((P_0^{st}, Q_0^{st}, \rho)\) is dominant.

**Proof.** For \(1 \leq i \leq l\) with \(m_i \geq 1\), let \(\eta_{st}^{(i)}\) and \(\theta_{st}^{(i)}\) be the symmetrized offset vectors of \((P_0^{st})^{(i)}\) and \((Q_0^{st})^{(i)}\) respectively. It is immediate from Lemma 5.2 that \(\eta^{(i)} = \eta_{st}^{(i)}\) and \(\theta^{(i)} = \theta_{st}^{(i)}\). Hence the assertion follows. \(\square\)

**Remark 5.12.** By Remark 5.9 we see that \((P_0^{st}, Q_0^{st}, \rho)\) is dominant if and only if \(\rho\) is dominant in the sense of [4] Definition 5.8.

Now, we can describe the image of \(\kappa_0\) as follows.

**Proposition 5.13.** Let

\[
 \Omega_{\text{dom}} = \left\{ (P_0, Q_0, \rho) \mid \begin{array}{l}
 (1) \ P_0 \in \text{CSST}_m(\lambda), \ Q_0 \in \text{CSST}_n(\lambda) \ (\lambda \in \mathcal{P}_m \cap \mathcal{P}_n), \\
 (2) \ \rho \in \mathbb{Z}^{\lambda}, \\
 (3) \ (P_0, Q_0, \rho) \text{ is dominant}
 \end{array} \right\}.
\]

Then the map \(\kappa_0\) gives a bijection

\[
 \hat{M}_{m \times n} \longrightarrow \Omega_{\text{dom}}.
\]

**Proof.** Suppose that \(A \in \hat{M}_{m \times n}\) is given. We have \(\kappa_0(A) = (P_0, Q_0, \rho)\) for some \(P_0 \in \text{CSST}_m(\lambda), \ Q_0 \in \text{CSST}_n(\lambda)\) with \(\lambda \in \mathcal{P}_m \cap \mathcal{P}_n\), and \(\rho \in \mathbb{Z}^{\lambda}\).

Then we have \(\kappa_0(A^{st}) = (P_0^{st}, Q_0^{st}, \rho)\) by Proposition 5.6. Since \(A^{st}\) is an extended affine permutation, \((P_0^{st}, Q_0^{st}, \rho)\) is dominant by [4] Theorem 5.11] (see Remark 5.12), and hence \(\kappa_0(A) = (P_0, Q_0, \rho)\) is dominant by Lemma 5.11.

Conversely, suppose that \((P_0, Q_0, \rho) \in \Omega_{\text{dom}}\) is given. We claim that there exists unique \(A \in \hat{M}_{m \times n}\) with \(\kappa_0(A) = (P_0, Q_0, \rho)\).

Let \(\alpha\) and \(\beta\) be the contents of \(P_0\) and \(Q_0\) respectively, and let \(K = |\alpha| = |\beta| = |\lambda|\). By Lemma 5.11 \((P_0^{st}, Q_0^{st}, \rho)\) is dominant. Then it follows from [4] Theorem 5.1, Theorem 5.12] that there exists a unique extended affine permutation \(B \in \hat{M}_{K \times K}\) such that \(\kappa_0(B) = (P_0^{st}, Q_0^{st}, \rho)\). By Corollary 5.5 \(B\) is \((\alpha, \beta)\)-descending, and hence by Lemma 5.3 there exists a unique \(A \in \hat{M}_{m \times n}(\alpha, \beta)\) such that \(A^{st} = B\). Finally, we have \(\kappa_0(A) = (P_0, Q_0, \rho)\) by Lemma 5.3. \(\square\)
Corollary 5.14. The map $\kappa$ is well-defined.

Proof. It follows from Lemma 5.8. \hfill \Box

Lemma 5.15. For $\lambda \in \mathcal{P}_m \cap \mathcal{P}_n$, let $\Omega_{\text{dom}}(\lambda)$ denote the set of $(P_0, Q_0, \rho) \in \Omega_{\text{dom}}$ such that the shape of $P_0$ and $Q_0$ is $\lambda$. Then we have a bijection

$$
\Omega_{\text{dom}}(\lambda) \longrightarrow CSST_{[m]}(\lambda) \times B_n(\lambda),
$$

$$(P_0, Q_0, \rho) \longmapsto (P_0, Q),$$

where $Q$ is given as in (3.11).

Proof. Consider the bijection

$$
\Omega_{\text{dom}}(\lambda) \longrightarrow CSST_{[m]}(\lambda) \times CSST_{[n]}(\lambda) \times \mathcal{P}(\lambda).
$$

Then the map $(P_0, Q_0, \rho) \longmapsto (P_0, Q)$ is a composition of (5.5) followed by the bijection (3.10) to the last two components. Hence the bijectivity follows. \hfill \Box

Since $\Omega_{\text{dom}} = \bigsqcup_{\lambda \in \mathcal{P}_m \cap \mathcal{P}_n} \Omega_{\text{dom}}(\lambda)$, Theorem 3.10 follows from Proposition 5.13 and Lemma 5.15. This completes the proof.

Remark 5.16. The well-definedness of $\kappa$ also follows directly from Proposition 5.13 and Lemma 5.15.

6. Proof of Theorem 4.15

In this section, we prove

$$
\kappa(\tilde{x}_i A) = \tilde{x}_i \kappa(A), \quad \kappa(\tilde{y}_j^\vee A) = \tilde{y}_j^\vee \kappa(A),
$$

for $A \in \hat{M}_{m \times n}$, $i \in \{1, \ldots, m - 1\}$, $j \in \{0, 1, \ldots, n - 1\}$ and $x, y \in \{e, f\}$.

6.1. Notations. Let $s$ be a stream of flow $l$ with defining data $(a, b, r)$. We regard $s$ as an element of $\mathcal{T}_{m \times n}$ in (4.4), which is both an $A_{m-1}^{(1)}$ and $A_{n-1}^{(1)}$ crystal with $P_{\text{cl}}$-weights, given by

$$
s = (a, r^s b) \in CSST_{[m]}((1^l)) \times B_n((1^l)) \subset \mathcal{T}_{m \times n}.
$$

For $A \in \hat{M}_{m \times n}$ with the corresponding streams $s^{(1)}, \ldots, s^{(e)}$ in Section 3.2 we identify $\kappa(A)$ with $s^{(s)} \otimes \cdots \otimes s^{(1)}$.

Define a map

$$
\Psi : \hat{M}_{m \times n} \longrightarrow \bigsqcup_{l \geq 0} \hat{M}_{m \times n} \otimes (CSST_{[m]}((1^l)) \times B_n((1^l))).
$$

$$
A \longmapsto A^s \otimes s^{(1)}
$$

Since

$$
((\Psi \otimes \text{id}^{\otimes s - 1}) \circ \cdots \circ (\Psi \otimes \text{id}) \circ \Psi)(A) = \otimes \otimes s^{(s)} \otimes \cdots \otimes s^{(1)} = \otimes \otimes \kappa(A),
$$
where id is the identity morphism, it suffices to show that $\Psi$ commutes with $\tilde{x}_i$ and $\tilde{y}_j$ for the proof of (6.1).

In order to simplify the description of $\tilde{x}_i$ and $\tilde{y}_j$ on $A^p \otimes s$ (see (6.6)), let us introduce some additional notations and conventions. Let $Z^* = \mathbb{Z} \cup \{\infty\}$, where we understand that $a < \infty$ and $a + \infty = \infty$ for $a \in \mathbb{Z}$. Let $A \in \tilde{M}_{m \times n}$ be given. We define $A^* = (a^*_{ij})_{i,j \in \mathbb{Z}^*}$ by

$$a^*_{ij} = \begin{cases} a^*_{ij} & \text{if } (i,j) \in \mathbb{Z} \times \mathbb{Z}, \\ 1 & \text{if } i = \infty \text{ and } (k,j) \in s^{(1)} \text{ for some } k \in \mathbb{Z}, \\ 1 & \text{if } j = \infty \text{ and } (i,k) \in s^{(1)} \text{ for some } k \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}$$

where $A^p = (a^p_{ij})_{i,j \in \mathbb{Z}}$. In other words, $A^*$ is an augmented matrix obtained from $A^p$ by

$$A^* = A^p + \sum_{(i,j) \in s^{(1)}} (E_{i\infty} + E_{\infty j}).$$

Note that $A^*$ satisfies $a^*_{i+m,j+n} = a^*_{ij}$ for $(i,j) \in \mathbb{Z}^* \times \mathbb{Z}^*$.

Let $z$ be a zig-zag of $A$ with the back-post corner $(i,j)$. Let $z^* = z \cup \{(\infty,j), (i,\infty)\}$ and regard $(\infty,j)$ and $(i,\infty)$ as outer corners of $z^*$. Then we may understand $\Psi(A) = A^*$ as a $\mathbb{Z}^* \times \mathbb{Z}^*$-matrix obtained by

- identifying $A = (a_{ij})_{i,j \in \mathbb{Z}}$ with $(a_{ij})_{i,j \in \mathbb{Z}^*}$ where $a_{i\infty} = a_{i\infty} = 0$ for $i,j \in \mathbb{Z}^*$,
- applying the same rules for $\phi$ in Section 3.2 to $A$ along $z^*_k$ instead of $z_k$,

where $\{z_k\}_{k \in \mathbb{Z}}$ is the set of zig-zags associated to $d^*_A$. Note that we can recover $A^p$ and $s^{(1)}$ from $A^*$ and $\{z^*_k\}_{k \in \mathbb{Z}}$. From now on, we assume that a matrix is a $\mathbb{Z}^* \times \mathbb{Z}^*$-matrix and a zig-zag is of the form $z^*$.

**Example 6.1.** Let $A$ be the generalized affine permutation in Example 2.1. We regard $A$ as a $\mathbb{Z}^* \times \mathbb{Z}^*$ matrix as follows:
where the red lines denote the zig-zags \( z_1, z_2, z_3 \) associated to \( \sigma_{ij}^A \). Then \( A^* \) is given as follows.

6.2. **Tensor product rule.** From now on, we fix \( A \in \tilde{M}_{m \times n} \) and \( j \in [n] \). If there is no confusion, let us write \( \tilde{f}_j, \varepsilon_j, \) and \( \varphi_j \) instead of \( \tilde{f}_j^0, \varepsilon_j^0, \) and \( \varphi_j^0 \) for simplicity. In the remainder of this section, we will focus on the proof of

\[
\tilde{f}_j \Psi(A) = \Psi(\tilde{f}_j A).
\]

Let

\[
\sigma = (\cdots, -\cdots-, \cdots+, \cdots-, \cdots+, \cdots),
\]

\[
\sigma^* = (\cdots, -\cdots-, \cdots+, \cdots-, \cdots+, \cdots, (\cdots+, \cdots),
\]

where \( \sigma^* \) is a concatenation of two sequences. By tensor product rule (4.1), we see that

\[
\tilde{f}_j \left( A^* \otimes s \right) = \begin{cases} 
A^* \otimes \tilde{f}_j s & \text{if the leftmost } + \text{ in } \sigma^* \text{ corresponds to } (\infty, j), \\
(\tilde{f}_j A^*) \otimes s & \text{if the leftmost } + \text{ in } \sigma^* \text{ corresponds to } (i, j) \text{ for some } i < \infty, \\
0 & \text{if } \sigma^* \text{ has no } +.
\end{cases}
\]

In terms of \( A^* \), this can be simplified as

\[
\tilde{f}_j A^* = \begin{cases} 
A^* - \hat{E}_{ij} + \hat{E}_{i+} & \text{if the leftmost } + \text{ in } \sigma^* \text{ corresponds to } (i, j), \\
0 & \text{if } \sigma^* \text{ has no } +,
\end{cases}
\]

where we define \( \hat{E}_{xj} = \sum_{k \in \mathbb{Z}} E_{x,j+k} \).

**Lemma 6.2.** We have \( \tilde{f}_j A \neq 0 \) if and only if \( \tilde{f}_j \Psi(A) \neq 0 \).
Proof. We may assume that there exists a non-zero cell in the $j$-th column. Otherwise, we have $\tilde{f}_j A = \tilde{f}_j \Psi(A) = 0$.

Let $\{z_k\}_{k \in \mathbb{Z}}$ be the set of zig-zags associated to $d_{\Psi A}^w$. Let $k_0$ (resp. $k_1$) be the minimal (resp. maximal) value of $d_{\Psi A}^w$ in the $j$-th column. For $k_0 \leq k \leq k_1$, let $i_k$ be the minimal row index with $(i_k, j) \in z_k$. Note that $(i_k, j + 1) \in z_k$.

Put

$$\sigma_k = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_k j} & a_{i_k j + 1} & a_{i_{k+1} j} & a_{i_{k+1} j + 1} \end{array} \right),$$

$$\sigma_k^* = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_k j}^* & a_{i_k j + 1}^* & a_{i_{k+1} j}^* & a_{i_{k+1} j + 1}^* \end{array} \right)$$

for $k_0 \leq k < k_1$, and

$$\sigma_{-\infty} = \left( \begin{array}{cccc} \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{k-1} j + 1} & a_{i_k j} & a_{i_{k+1} j} & a_{i_{k+1} j + 1} \end{array} \right),$$

$$\sigma_{\infty} = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{k-1} j} & a_{i_k j} & a_{i_{k+1} j} & a_{i_{k+1} j + 1} \end{array} \right),$$

$$\sigma_{-\infty}^* = \left( \begin{array}{cccc} \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{k-1} j}^* & a_{i_k j}^* & a_{i_{k+1} j}^* & a_{i_{k+1} j + 1}^* \end{array} \right),$$

$$\sigma_{\infty}^* = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_{k-1} j}^* & a_{i_k j}^* & a_{i_{k+1} j}^* & a_{i_{k+1} j + 1}^* \end{array} \right).$$

Then $\sigma$ and $\sigma^*$ in (6.4) decompose as follows:

$$\sigma = \sigma_{-\infty} \cdot \sigma_{k_0} \cdots \cdot \sigma_{k_1-1} \cdot \sigma_{\infty},$$

$$\sigma^* = \sigma_{-\infty}^* \cdot \sigma_{k_0}^* \cdots \cdot \sigma_{k_1-1}^* \cdot \sigma_{\infty}^*.$$

Suppose that $k$ is given with $k_0 \leq k < k_1$. Let $u$ be the maximal row index with $(u, j) \in z_k$, and let $v$ be the minimal row index with $(v, j + 1) \in z_{k+1}$. Note that $i_k \leq u$ and $v \leq i_{k+1}$. Suppose first that $u < v$. Then we have

$$\sigma_k = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_k j} & a_{i_k j + 1} & a_{i_{k+1} j} & a_{i_{k+1} j + 1} \end{array} \right),$$

$$\sigma_k^* = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_k j} & a_{i_k j + 1} & a_{i_{k+1} j} & a_{i_{k+1} j + 1} \end{array} \right) = \sigma_k,$$

and hence $\sigma_k = \sigma_k^*$. Next, suppose that $u \geq v$. In this case, we have $i_k < v \leq u < i_{k+1}$. Therefore, $(i_k, j), (v, j + 1)$ are inner corners, and $(u, j), (i_{k+1}, j + 1)$ are outer corners. Then we have

$$\sigma_k = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_k j} & a_{i_k j + 1} & a_{i_{k+1} j} & a_{i_{k+1} j + 1} \end{array} \right)$$

and

$$\sigma_k^* = \left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_k j} & a_{i_k j + 1} & a_{i_{k+1} j} & a_{i_{k+1} j + 1} \end{array} \right).$$

Since one cancelling pair $(+, -)$ of $\sigma_k$ in

$$\left( \begin{array}{cccc} + & \cdots & + & - \\ \cdot & \cdot & \cdot & \cdot \\ a_{i_k j} & a_{i_k j + 1} \end{array} \right)$$
is moved to a pair $(+, -)$ of $\sigma_k^*$ in
\[
\left(\underbrace{+, \cdots, +}, \underbrace{-, \cdots, -}\right), \quad a_{i,j+1}
\]
we conclude that $\widetilde{\sigma}_k = \sigma_k^*$. By similar argument, we see that $(\sigma_{\pm \infty})^\sim = (\sigma_{\pm \infty}^*)^\sim$, where $(\cdot)^\sim$ means $(\cdot)$. Hence we have
\[
\widetilde{\sigma}_k = \sigma_k^* \quad (-\infty \leq k \leq \infty).
\]
Since reducing a sequence does not depend on the order of cancelling $(+, -)$, we have
\[
\tilde{\sigma} = (\sigma_{-\infty}, \cdots, \sigma_{\infty})^\sim = ((\sigma_{-\infty})^\sim, \cdots, (\sigma_{\infty})^\sim)^\sim
\]
\[
= ((\sigma_{-\infty}^*)^\sim, \cdots, (\sigma_{\infty}^*)^\sim)^\sim = (\sigma_{-\infty}^*, \cdots, \sigma_{\infty}^*)^\sim = \sigma^*.
\]
This shows that $\varphi_j(A) = \varphi_j(\Psi(A))$ and $\varepsilon_j(A) = \varepsilon_j(\Psi(A))$, and hence the lemma follows. \(\square\)

From now on, we assume that $\hat{f}_j A \neq 0$ and $\hat{f}_j \Psi(A) \neq 0$. We also adopt the following notations:
- $u$ : the row index corresponding to the leftmost $+$ in $\tilde{\sigma}$,
- $u^*$ : the row index corresponding to the leftmost $+$ in $\sigma^*$,
- $s = d^A(u, j)$,
- $\tilde{A} = \hat{f}_j A = (\tilde{a}_{ij})_{i,j \in \mathbb{Z}^*}$.

We have
\[
\tilde{A} = \hat{f}_j A = A - \hat{E}_{u,j} + \hat{E}_{u,j+1}, \quad \tilde{f}_j A^* = A^* - \hat{E}_{u^*,j} + \hat{E}_{u^*,j+1}.
\]
Note that the leftmost $+$ in $\sigma^*$ also appears in $(\sigma^*)^\sim$ by (6.7). More explicitly, $u^*$ is the minimal row index such that $u \leq u^*$ and $a^*_{u,j} \neq 0$. We have
\[
\begin{cases}
  u < u^* \text{ if } (u, j) \text{ is an inner corner of } z, \text{ with } a_{u,j} = 1, \\
  u = u^* \text{ otherwise.}
\end{cases}
\]

\subsection*{6.3. Southwest channel numberings on $A$ and $\tilde{A}$}
In this subsection, we discuss the relation between the southwest channel numberings on $A$ and $\tilde{A}$.

For $(x, y) \in \mathbb{Z}^2$, let $\langle x, y \rangle = \{ \tau^k(x, y) \mid k \in \mathbb{Z} \}$. We have
\[
\supp(\tilde{A}) - \supp(A) \subseteq (u, j+1)^\sim, \quad \supp(A) - \supp(\tilde{A}) \subseteq (u, j)^\sim,
\]
where the equalities hold when $a_{u,j+1} = 0$ and $a_{u,j} = 1$ respectively.

We first give an example, where a proper numbering on $\tilde{A}$ is induced by a proper numbering on $A$.

**Example 6.3.** Let $A$ be the generalized affine permutation given in Example 2.1. It is easily checked that $\hat{f}_2 A = A - \hat{E}_{5,2} + \hat{E}_{5,3}$. Consider the set of zig-zags $Z = \{ z_k \}_{k \in \mathbb{Z}}$ corresponding to the southwest channel numbering $d^A_{\text{sw}}$ on $A$. If we draw the zig-zags $z_1$, $z_2$ and $z_3$ over $\hat{f}_2 A$ as follows:
then one can see that $Z$ satisfies the conditions (z.1)-(z.3) in Section 3.2 with respect to $\tilde{f}_2 A$. Here, the dashed and solid circles are the positions where $\tilde{f}_2 A$ differs from $A$. We conclude that $Z$ induces a proper numbering on $\tilde{f}_2 A$.

It is also easily checked that $\tilde{f}_3 \tilde{f}_2 A = \tilde{f}_2 A - \tilde{E}_{23} + \tilde{E}_{24}$. However, $Z$ does not give a proper numbering on $\tilde{f}_3 \tilde{f}_2 A$, since an inner corner $(2, 3)$ of $z_3$ is not a non-zero cell of $\tilde{f}_3 \tilde{f}_2 A$. If we modify a segment $(9, 8), (8, 8), (7, 8), (6, 8), (6, 9)$ of $z_3$ by $(9, 8), (9, 9), (8, 9), (7, 9), (6, 9)$ (see below), it remedies the failure of the condition (z.1). The modified zig-zag, which is denoted by $z'_3$, does not intersect with $z_4$ as follow,

where dashed red line is the original part of $z_3$, and the dashed circle and solid circle are the positions where $\tilde{f}_3 \tilde{f}_2 A$ differs from $\tilde{f}_2 A$. Hence the modified set of zig-zags $Z' = \{ \cdots, z'_0, z_1, z_2, z_3', z_4, \cdots \}$ give a proper numbering on $\tilde{f}_3 \tilde{f}_2 A$.

In the remainder of this section, we will see that the induced numberings on $\tilde{f}_2 A$ and $\tilde{f}_3 \tilde{f}_2 A$ are, in fact, the southwest channel numberings.
The following lemma describes how to construct a proper numbering on $\tilde{A}$ from a given numbering $d$ on $A$ in general.

**Lemma 6.4.** We have the following.

1. Let $d$ be a proper numbering on $A$ with the associated zig-zags $Z = \{z_k\}_{k \in \mathbb{Z}}$, and $d(u, j) = s$. Then there exists a proper numbering $d^-$ on $\tilde{A}$ satisfying

$$d^-(c) = d(c) \quad \text{if } c \in \text{supp}(A) \cap \text{supp}(\tilde{A}),$$

$$d^-(u, j + 1) = \begin{cases} d(u, j) & \text{if } u \text{ is minimal such that } (u, j) \in z_s, \\ d(u, j) + 1 & \text{otherwise.} \end{cases}$$

2. Let $d$ be a proper numbering on $\tilde{A}$ with the associated zig-zags $Z = \{z_k\}_{k \in \mathbb{Z}}$, and $d(u, j + 1) = t$. Then there exists a proper numbering $d^+$ on $A$ satisfying

$$d^+(c) = d(c) \quad \text{if } c \in \text{supp}(A) \cap \text{supp}(\tilde{A}),$$

$$d^+(u, j) = \begin{cases} d(u, j) - 1 & \text{if } (u, j) \in z_t - 1, \\ d(u, j) & \text{otherwise.} \end{cases}$$

In particular, the widths of $A$ and $\tilde{A}$ are the same.

**Proof.** (1) We construct a set of zig-zags $Z^-$ (by adjusting $Z$) which satisfies the conditions (z.1)-(z.3) in Section 3.2 with respect to $\tilde{A}$ and hence gives a proper numbering $d^-$ on $\tilde{A}$.

Let $i_s$ be the minimal row index with $(i_s, j) \in z_s$. If $i_s = u$, then $(u, j + 1) \in z_s$ by definition of zig-zag. Suppose that $i_s < u$. Then $(i_s, j)$ is an inner corner of $z_s$, and $a_{i_s, j} > 0$. Consider a subsequence of $\sigma$ in (6.4)

$$\begin{pmatrix} + & \cdots & + \\ \cdots & a_{i_s, j} & \cdots \\ \cdots & a_{i_s, j + 1} & \cdots \\ \cdots & a_{i_s + 1, j} & \cdots \\ a_{u, j} & \cdots & + \end{pmatrix}.$$

Since $(u, j)$ is the cell corresponding to the leftmost $+$ in $\tilde{\sigma}$, there exists no $+$ in the reduced form of (6.9). This implies that there exists some $a_{v, j + 1} > 0$ for some $v$ with $i_s < v < u$ so that $+$ in the cell $(i_s, j)$ is paired with $-$ in $(v, j + 1)$. It is easy to see that $d(v, j + 1) = s + 1$, and hence $(u, j + 1) \in z_{s + 1}$.

Hence we see that $\text{supp}(\tilde{A}) \subseteq \bigsqcup_{k \in \mathbb{Z}} z_k$ and $Z$ satisfies the conditions in (z.2) and (z.3) for $\tilde{A}$. Note that the condition (z.1) fails if and only if $(u, j)$ is an inner corner of $z_s$ with $a_{u, j} = 1$.

**Case 1.** Suppose that $(u, j)$ is not an inner corner of $z_s$ or $(u, j)$ is an inner corner of $z_s$ with $a_{u, j} > 1$. Then $Z^- := Z$ satisfies the condition (z.1), and induces a proper numbering $d^-$ on $\tilde{A}$ as given in (3.2). Hence $d^-$ satisfies

$$d^-(u, j + 1) = \begin{cases} s & \text{if } u = i_s, \\ s + 1 & \text{if } i_s < u, \end{cases}$$

and $d^-(c) = d(c)$ for $c \in \text{supp}(A) \cap \text{supp}(\tilde{A})$. 
Case 2. Suppose that \((u, j)\) is an inner corner of \(Z_s\) with \(a_{u,j} = 1\). In this case, the condition (z.1) fails since the inner corner \((u, j)\) of \(Z_s\) does not lie in \(\text{supp}(\bar{A})\). Now let us modify \(Z_s\) as follows: Let \((v, j)\) be an outer corner of \(Z_s\) and let

\[
w = \min\{ i \in \mathbb{Z} \mid u < i \leq v \text{ and } a_{ij} > 0 \}.
\]

Note that \(w = u^*\) when \(d = d_\bar{A}^w\). Consider a subsequence of \(\sigma\)

\[
(6.10) \quad (\overbrace{+ \cdots +}, - \cdots -, \cdots + \cdots +, - \cdots -).
\]

We see that \(a_{ij} = 0\) for \(u < i \leq w - 1\) by definition of \(w\), and moreover \(a_{ij+1} = 0\) for \(u < i \leq w\) since the reduced form of \((6.10)\) is \((+). Indeed the sequence in \((6.10)\) is \((+). According to this observation, we define a zig-zag \(Z_{s}^-\) by replacing the cells

\[
(w - 1, j), \cdots, (u + 1, j), (u, j)
\]

in \(Z_s\) with the following cells

\[
(w, j + 1), (w - 1, j + 1), \cdots, (u + 1, j + 1).
\]

Then each inner corner of \(Z_{s}^-\) lies in \(\text{supp}(\bar{A})\).

Let \(Z^-\) be the set of zig-zags obtained from \(Z\) by replacing \(\{ \tau^kZ_s \mid k \in \mathbb{Z} \}\) with \(\{ \tau^kZ_{s}^- \mid k \in \mathbb{Z} \}\). Then \(Z^-\) satisfies the conditions (z.1)-(z.3) for \(\bar{A}\), and hence induces a proper numbering \(d^-\) on \(\bar{A}\). It is easy to see that \(d^- (u, j + 1) = s\) since \(u = i_s\), and that \(d^- (c) = d(c)\) for \(c \in \text{supp}(A) \cap \text{supp}(\bar{A})\).

By definition, the proper numberings \(d\) and \(d^-\) have the same flow, which implies that \(A\) and \(\bar{A}\) have the same width. This proves (1).

(2) As in (1), we construct a set of zig-zags from \(Z\) to which a proper numbering \(d^+\) on \(A\) is associated.

Case 1. Suppose first that \((u, j) \in Z_{t-1}\). Then there exists an inner corner \((v, j)\) of \(Z_{t-1}\) and an inner corner \((w, j + 1)\) of \(Z_t\) with \(v < w \leq u\). In particular, we have \(\tilde{a}_{v,j} > 0\). Consider a subsequence of \(\sigma\)

\[
(\overbrace{+ \cdots +}, - \cdots -, \cdots + \cdots +, - \cdots -).
\]

Since \(\tilde{a}_{v,j} > 0\) and \((u, j + 1)\) is the cell corresponding to the rightmost \(-\), we have

\[
\tilde{a}_{w,j+1} + \tilde{a}_{w+1,j+1} + \cdots + \tilde{a}_{u,j+1} \geq 2.
\]

This shows that \(w < u\) and the inner corner \((w, j + 1)\) of \(Z_t\) lies in \(\text{supp}(A)\). Hence \(Z^+ := Z\) satisfies the conditions (z.1)-(z.3), and it induces a proper numbering \(d^+\) on \(A\). It is obvious that \(d^+ (u, j) = t - 1\) and \(d^+ (c) = d(c)\) for \(c \in \text{supp}(A) \cap \text{supp}(\bar{A})\).

Case 2. Suppose that \((u, j) \notin Z_{t-1}\). Let \(i_t\) be the maximal row index with \((i_t, j + 1) \in Z_t\). Since \((u, j) \notin Z_{t-1}\), we have \(\tilde{a}_{ij} = 0\) for \(u \leq i < i_t\). Consider a subsequence of \(\sigma\)

\[
(6.11) \quad \overbrace{\cdots , + \cdots +}, \cdots, \overbrace{+ \cdots +}, \overbrace{- \cdots -} = \overbrace{\cdots , + \cdots +}, \cdots, \overbrace{- \cdots -}.
\]
Since the rightmost $-\in\sigma$ corresponding to position $(u, j + 1)$ is the one in (6.11), we see that $\tilde{a}_{i,j+1} = 0$ for $u < i \leq i_t$. We define $Z_i^+$ to be a zig-zag by replacing the cells 

$$(i_t, j + 1), (i_t - 1, j + 1), \ldots, (u + 1, j + 1)$$

in $Y_i$ with the following cells 

$$(i_t - 1, j), \ldots, (u - 1, j), (u, j).$$

Then $(u, j)$ is an inner corner of $Z_i^+$, and $(u, j) \in \text{supp}(A)$. Let $Z^+$ be the set of zig-zags obtained from $Z$ by replacing $\{\tau^kZ_i \mid k \in \mathbb{Z}\}$ with $\{\tau^kZ_i^+ \mid k \in \mathbb{Z}\}$. Then $Z^+$ satisfies the conditions (z.1)-(z.3) for $A$, and it induce a proper numbering $d^+$ on $A$. We have $d^+(u, j) = t$ and $d^+(c) = d(c)$ for $c \in \text{supp}(A) \cap \text{supp}(\tilde{A})$. This proves (2). 

\[\square\]

**Remark 6.5.** Let $d_1$ be a proper numbering on $A$. If follows from the construction of $d^\pm$ in the proof of Lemma 6.2 that 

1. $(d_1^-)^+ = d_1$, 
2. if $d_2$ is another proper numbering on $A$ such that $d_1(c) \leq d_2(c)$ for $c \in \text{supp}(A)$, then $d_1^-(c) \leq d_2^-(c)$ for $c \in \text{supp}(\tilde{A})$. 

The similar properties hold for a proper numbering on $\tilde{A}$.

**Lemma 6.6.** We have the following. 

1. Let $C$ be a channel of $A$. Then there exists a channel $C^-$ of $\tilde{A}$ given by 

$$
\begin{cases}
    (C - (u, j)^\wedge) \cup (u, j + 1)^\wedge & \text{if } (u, j) \in C, \ a_{u,j} = 1 \text{ and } (i, j + 1) \notin C \text{ for all } i, \\
    (C - (u, j)^\wedge) \cup (v, j)^\wedge & \text{if } (u, j) \in C, \ a_{u,j} = 1 \text{ and } (i, j + 1) \in C \text{ for some } i, \\
    C & \text{otherwise},
\end{cases}
$$

where $v$ is the minimal row index such that $u < v$ and $a_{v,j} > 0$ if it exists.

2. Let $C$ be a channel of $\tilde{A}$. Then there exists a channel $C^+$ of $A$ given by 

$$
\begin{cases}
    (C - (u, j + 1)^\wedge) \cup (u, j)^\wedge & \text{if } (u, j + 1) \in C, \ a_{u,j+1} = 1 \text{ and } (i, j) \notin C \text{ for all } i, \\
    (C - (u, j + 1)^\wedge) \cup (w, j + 1)^\wedge & \text{if } (u, j + 1) \in C, \ a_{u,j+1} = 1 \text{ and } (i, j) \in C \text{ for some } i, \\
    C & \text{otherwise},
\end{cases}
$$

where $w$ is the maximal row index such that $w < u$ and $a_{w,j+1} > 0$ if it exists.

**Proof.** Let us prove (1) only, since the proof of (2) is similar.

First, suppose $(u, j) \notin C$ or $a_{u,j} > 1$. Then $C \subseteq \text{supp}(\tilde{A})$ and $C$ is a channel of $\tilde{A}$. We put $C^- = C$ in this case.

Now suppose that $(u, j) \in C$ and $a_{u,j} = 1$. Then we have $C \notin \text{supp}(\tilde{A})$. Let us write 

$$C = \{\cdots >_{\text{NW}} c_{s-1} >_{\text{NW}} c_s >_{\text{NW}} c_{s+1} >_{\text{NW}} \cdots\}$$

with $c_s = (u, j)$. We have two cases.

**Case 1.** Suppose that $(i, j + 1) \notin C$ for all $i$. Let $C^-$ be a set obtained from $C$ by replacing $c_s^+ = (u, j)^\wedge \subset C$ with $(u, j + 1)^\wedge$, which is clearly a stream of $\tilde{A}$ by assumption.
Case 2. Suppose that \((i,j+1) \in C\) for some \(i\). Then we have \(c_{s+1} = (u',j+1)\) for some \(u' > u\). Consider a subsequence of \(\sigma\) in (6.14)

\[
(6.12)
\begin{align*}
&\left(\begin{array}{ccc}
& a_{u,j} & 
& \vdots & 
& \vdots & 
& a_{u+1,j+1} \\
\end{array}\right) \quad \left(\begin{array}{ccc}
& \vdots & 
& \vdots & 
& \vdots & 
& a_{u+1,j+1} \\
\end{array}\right).
\end{align*}
\]

Since \(-\) in \(c_{s+1}\) is paired with \(+\) in (6.12) other than \(+\) in \(c_s = (u,j)\), we have \(a_{ij} > 0\) for some \(u < i < u'\). Let \(v\) be the minimal such one. Then we have

\[
(6.13)
c_{s-1} >_{SW} (v,j+1) >_{SW} c_{s+1}.
\]

Let \(C^-\) be a set obtained from \(C\) by replacing \(c_s^\wedge = (u,j)^\wedge \subset C\) with \((v,j+1)^\wedge\), which is a stream of \(\tilde{A}\) by (6.13).

By definition, \(C^-\) has the same flow as \(C\). Since \(A\) and \(\tilde{A}\) have the same width by Lemma 6.4, \(C^-\) is a stream of maximal flow, and hence a channel of \(\tilde{A}\) (cf. Remark 2.17).

**Remark 6.7.** Let \(C\) be a channel of \(A\). Under the above hypothesis, we have

\[
(C^-)^+ - C = \begin{cases} 
(v,j) & \text{if } (u,j) \in C, a_{uj} = 1 \text{ and } (i,j+1) \in C \text{ for some } i, \\
(u,j+1) & \text{if } (u,j) \in C, a_{uj} = 1, a_{u,j+1} > 0 \text{ and } (i,j+1) \notin C \text{ for all } i, \\
\emptyset & \text{otherwise.}
\end{cases}
\]

Note that there exists no cell in \(\text{supp}(A)\) between \((u,j)\) and \((v,j)\) and between \((u,j)\) and \((u,j+1)\). Hence it follows that

\[
C \geq_{SW} (C^-)^+ \quad \text{or} \quad (C^-)^+ \geq_{SW} C,
\]

and there exists no other channel between \(C\) and \((C^-)^+\). It is also easy to check that if \(C'\) is another channel of \(A\) with \(C \geq_{SW} C'\), then we have

\[
C^- \geq_{SW} (C')^-.
\]

The similar properties also hold with respect to channels of \(\tilde{A}\). 

**Lemma 6.8.** We have the following.

1. Let \(C\) be a channel of \(A\). If \(d\) is the channel numbering on \(A\) associated to \(C\), then \(d^-\) is the channel numbering on \(\tilde{A}\) associated to \(C^-\).

2. Let \(C\) be a channel of \(\tilde{A}\). If \(d\) is the channel numbering on \(\tilde{A}\) associated to \(C\), then \(d^+\) is the channel numbering on \(A\) associated to \(C^+\).

**Proof.** Let us prove (1) only since the proof of (2) is similar.

Let \(d'\) be the channel numbering on \(\tilde{A}\) associated to \(C^-\). Since the widths of \(A\) and \(\tilde{A}\) coincide, we may assume that \(d^-\) coincides with \(d'\) on the channel \(C^-\) by adding a constant to \(d^-\). Hence we have \(d' \leq d^-\) by Lemma 2.21 and \((d')^+ \leq (d^-)^+ = d\) by Remark 6.5.

Let \(\ell\) be the common width of \(A\) and \(\tilde{A}\). If \(\ell > 1\), then we see from Lemma 6.1(1) that there exists \(c \in C \cap C^-\) such that \(c \in \text{supp}(A) \cap \text{supp}(\tilde{A})\). We have

\[
d(c) = d^-(c) = d'(c) = (d')^+(c).
\]

So \((d')^+\) and \(d\) also coincide on \(C\) (cf. Remark 6.7), and \(d \leq (d')^+\) by Lemma 2.21. Therefore, we have \(d = (d')^+\), and \(d^- = d'\) by Remark 6.5.
If \( \ell = 1 \), then it is not possible to have \( d^-(u, j + 1) = d(u, j) + 1 \) or \( C^- = (C - (u, j)^+) \cup (v, j)^- \) since we must have another cell \((i, j + 1)\) with \((u, j) >_{\bowtie} (i, j + 1) \geq_{\bowtie} (u, j) + (m, n)\). Hence we see directly that \((d')^+(u, j) = d(u, j) = d^-(u, j + 1) = d'(u, j + 1)\). By similar arguments as in the above case, we conclude that \(d^- = d'\). \(\square\)

Now we can describe the southwest channel numbering on \(\tilde{A}\) in terms of the one on \(A\).

**Proposition 6.9.** Let \(d\) be the southwest channel numbering on \(A\). Then \(d^-\) is the southwest channel numbering on \(\tilde{A}\). Equivalently, let \(d\) be the southwest channel numbering on \(\tilde{A}\). Then \(d^+\) is the southwest channel numbering on \(A\).

**Proof.** Let \(C_1 = C_A^-\) and \(C_2 = C_A^+\). Let \(d'\) be the southwest channel numbering on \(\tilde{A}\). By Lemma 6.8, we have

\[
d^- = d_{\tilde{A}}^-, \quad (d')^+ = d_{\tilde{A}}^+.
\]

Thus it suffices to show that either \(C_1 = C_2^+\) or \(C_1^- = C_2\), which implies that \(d = (d')^+\) or \(d^- = d'\), respectively (see also Remark 6.5).

Since \(C_1\) and \(C_2\) are the southwest channels, we have

\[
(6.15) \quad C_1 \geq_{sv} C_2^+, \quad C_2 \geq_{sv} C_1^-.
\]

By Remark 6.7 (cf. (6.14)), we get from (6.15)

\[
(6.16) \quad C_1 \geq_{sv} C_2^+ \geq_{sv} (C_1^-)^+, \quad C_2 \geq_{sv} C_1^- \geq_{sv} (C_2^+)^-.
\]

We claim that \(C_1 = C_2^+\) if \(C_2 >_{sv} C_1^-\). By (6.16), we have \(C_2 >_{sv} (C_2^+)^-\). By Remark 6.7, we see that \(C_2 >_{sv} (C_2^+)^-\) occurs only when

\[
C_2^+ = (C_2 - (u, j + 1)^+) \cup (w, j + 1)^-.
\]

Then we have \(C_2^+ = (A)\).

On the other hand, we have \(C_1^- = (C_2^+)^-\) since there is no other channel between \(C_2\) and \((C_2^-)^-\). Hence we get \(C_1^- = (C_2^-)^- = C_2^+\), and in particular \((w, j + 1) \in C_1^-\). Since we have \((u, j) \notin C_1\) by Lemma 6.6 (1), it follows that \(C_1 = C_1^- = C_2^+\). This proves the claim. \(\square\)

### 6.4. Proof of (6.3)

Now we are in a position to prove (6.3). Let \(d = d_A^+\) and let \(Z = \{z_k\}_{k \in \mathbb{Z}}\) be the set of zig-zags associated to \(d\). Let \(Z^-\) be the set of zig-zags associated to \(d^-\) (see the proof of Lemma 6.3 (1)). Note that \(d^- = d_A^+\) by Proposition 6.9.

**Case 1.** Suppose that \((u, j)\) is not an inner corner of \(z_s\) or \((u, j)\) is an inner corner of \(z_s\) with \(a_{u,j} > 1\). Since \(Z = Z^-\) and the cells corresponding the leftmost + in \(A\) and \(A^*\) coincide in this case, we have \(A^* = (\tilde{A})^*\).

**Case 2.** Suppose that \((u, j)\) is an inner corner of \(z_s\) with \(a_{u,j} = 1\).

Let us first compare \(z_s\) with the modified zig-zag \(z_s^*\). Let \((v_0, j)\) be an outer corner of \(z_s^*\) and let \(v_1\) be the minimal row index with \((v_1, j + 1) \in z_s^*\). Note that the inner and outer corners of \(Z\) and \(Z^-\) always coincide other than the following cells (more precisely, their orbits under \(\tau^{\pm 1}\)):

\[
(6.17) \quad (u, j), (u^*, j), (v_0, j), (v_1, j + 1), (u, j + 1), (u^*, j + 1),
\]
where \( v_1 \leq u < u^* \leq v_0 \leq \infty \). For the reader’s convenience, we summarize the positions of the cells in (6.17) as follows:

| \( (u, j) \) | \( (u^*, j) \) | \( (v_1, j + 1) \) | \( (u, j + 1) \) | \( (u^*, j + 1) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| inner corner    | outer corner if \( u^* = v_0 \) | inner corner if \( u^* < v_0 \) | outer corner     | inner corner if \( u^* < v_0 \) |

Hence we may write

\[
A^* = A - \hat{E}_{u,j} + \hat{E}_{v_0,j} - \hat{E}_{v_1,j+1} + \hat{E}_{u,j+1} + B,
\]

\[
(\hat{A})^* = \hat{A} - \hat{E}_{u}*j + \hat{E}_{v_0,j} - \hat{E}_{v_1,j+1} + \hat{E}_{u}*j+1 + B,
\]

where \( B \) is a finite linear combination of \( \hat{E}_{kl} \)'s over the cells \((k, l)\) not belonging to (6.17).

Combining (6.8) and (6.18), we have

\[
\hat{A}^* = A^* - \hat{E}_{u}*j + \hat{E}_{u}*j+1
\]

\[
= \left(A - \hat{E}_{u,j} + \hat{E}_{v_0,j} - \hat{E}_{v_1,j+1} + \hat{E}_{u,j+1} + B\right) - \hat{E}_{u}*j + \hat{E}_{u}*j+1
\]

\[
= \left(A - \hat{E}_{u,j} + \hat{E}_{u,j+1} - \hat{E}_{u}*j + \hat{E}_{v_0,j} - \hat{E}_{v_1,j+1} + \hat{E}_{u}*j+1 + B\right)
\]

\[
= \hat{A} - \hat{E}_{u}*j + \hat{E}_{v_0,j} - \hat{E}_{v_1,j+1} + \hat{E}_{u}*j+1 + B
\]

By Case 1 and Case 2, we have \( \hat{A}^* = (\hat{A})^* \). Let us write \( \Psi(\hat{f}_j A) = (\hat{f}_j A)^p \otimes s' \). From \( \hat{A}^* = (\hat{A})^* \) and Proposition 6.9 we see that

\[
(\hat{f}_j A)^p \otimes s' = \begin{cases} A^p \otimes \hat{f}_j s & \text{if } u^* = \infty, \\ (\hat{f}_j A^p) \otimes s & \text{if } u^* < \infty. \end{cases}
\]

Comparing this with (6.9), we have (6.3).

By (6.2), \( \kappa \) commutes with \( \hat{f}_j \), and hence \( \kappa \) commutes with \( \hat{c}_j \) for \( j \in \{0, 1, \ldots, n - 1\} \).

6.5. **Proof of Theorem 4.15.** We have proved that \( \kappa \) commutes with \( \hat{c}_j^i \) and \( \hat{f}_j^i \) for \( j \in \{0, 1, \ldots, n - 1\} \). Let us finish the proof of Theorem 4.15 by showing that \( \kappa \) commutes with \( \hat{c}_i \) and \( \hat{f}_i \) for \( i \in \{1, \ldots, m - 1\} \).

First, it is not difficult to see that Proposition 6.9 still holds if we replace the southwest channel numberings with the northeast channel numberings (cf. Remark 2.20). Hence by the same arguments as Section 6.4 we have

\[
\hat{x}_i A^* = (\hat{x}_i A)^*
\]

for \( i \in \{0, 1, \ldots, m - 1\} \) and \( x \in \{c, f\} \). This implies that \( \Psi \) commutes with \( \hat{c}_i \) and \( \hat{f}_i \) for \( i \in \{1, \ldots, m - 1\} \) (see Remark 6.10 for \( i = 0 \)). Hence \( \kappa \) commutes with \( \hat{c}_i \) and \( \hat{f}_i \) for \( i \in \{1, \ldots, m - 1\} \). This completes the proof of Theorem 4.15. \( \square \)
Remark 6.10. We should remark that $r_0$ and $f_0$ may not commute with $\Psi$. Let $A \in \hat N_{m \times n}$ be given such that $x_0A \neq 0$ for $x \in \{e, f\}$. Suppose that $\Psi(A) = A^\rho \otimes s$ and $s = (a, b, r)$.

If $\tilde x_0(A^\rho \otimes s) = A^\rho \otimes \tilde x_0s$, then it follows from (6.19) that $\Psi(\tilde x_0A) = A^\rho \otimes s'$ and $s' = (\tilde x_0a, b, r')$, where $r' = r + 1$ (resp. $r - 1$) if $x = e$ (resp. $x = f$). Since $\tilde x_0s = (\tilde x_0a, b, r) \neq s'$, we have $\Psi(\tilde x_0A) \neq \tilde x_0\Psi(A)$. If $\kappa(A) = (P_0, Q)$, then by applying $\Psi$ repeatedly we have

$$\kappa(\tilde x_0A) = (\tilde x_0P_0, Q'),$$

for some $Q' \in B_n(\lambda)$ with $Q' \neq Q$.

7. Dual affine RSK correspondence

In this section, we construct a dual analogue of Theorem 3.10.

7.1. Generalized dual affine permutations. Let

$$\hat N_{m \times n} = \left\{ A = (a_{ij})_{i,j \in \mathbb{Z}} \mid \begin{array}{ll}
(1) & a_{ij} \in \{0, 1\} \text{ and } a_{i+m,j+n} = a_{ij} \text{ for all } i, j \in \mathbb{Z}, \\
(2) & \text{for each } j, \ a_{ij} = 0 \text{ except for finitely many } i \text{'s.} \end{array} \right\}.$$

For $A \in \hat N_{m \times n}$, we define a (dual) standardization of $A$ to be

$$A^{\ast_{\rho'}} = (A^\ast)^{\rho'},$$

(see Remark 2.6 for the definition of $A^\ast$). We may adopt the same notations in Section 2.2 for $A^{\ast_{\rho'}}$. Note that a cell $c$ in $\text{supp}(A)$ corresponds to a unique cell in $\text{supp}(A^{\ast_{\rho'}})$.

Example 7.1. Let $A \in \hat N_{3 \times 4}$ be given as follows.

Then $A^{\ast_{\rho'}}$ is
We define partial orders $>_\text{NW}$ and $<_\text{NE}$ on $\mathbb{Z}^2$ as follows:

1. $c_1 >_\text{NW} c_2$ if and only if $i_1 \leq i_2$ and $j_1 < j_2$.
2. $c_1 <_\text{NE} c_2$ if and only if $i_1 > i_2$ and $j_1 \leq j_2$.

for $c_1 = (i_1, j_1)$ and $c_2 = (i_2, j_2)$ in $\mathbb{Z}^2$.

With respect to these partial orders, we have natural dual analogues of the notions and their properties in Section 2, the proofs of which are almost parallel to those in the case of $\hat{N}_{m \times n}$. Let us summarize them as follows: suppose that $A \in \hat{N}_{m \times n}$ is given.

- A proper numbering $d$ on $A$ is defined as in Definition 2.7 with respect to $>_\text{NW}$ instead of $>_\text{SW}$. Let $d^{st'}$ denote the proper numbering on $A^{st'}$ which corresponds to $d$. Lemmas 2.8 and 2.9 hold for proper numberings on $A$.
- A stream $s$ and a channel $C$ of $A$ are defined in the same way as in Definition 2.11 with respect to $>_\text{NW}$. Let $s^{st'}$ and $C^{st'}$ denote the stream and channel of $A^{st'}$ which naturally correspond to $s$ and $C$, respectively.
- Let $\mathcal{C}_A'$ be the set of channels of $A$. We define a partial order $\geq_{\text{sw}}$ on $\mathcal{C}_A'$ as in (2.6) by using $\leq_{\text{NE}}$ instead of $\leq_{\text{SE}}$ (see also (2.5)). Then $\mathcal{C}_A'$ has the greatest and the smallest elements with respect to $\geq_{\text{sw}}$. It can be proved by the same arguments as in Proposition 2.13 using $A^{st'}$.
- Let $C_{A}^{sw'}$ be the greatest element in $\mathcal{C}_A'$, which we call the southwest channel of $A$. We have $(C_{A}^{sw'})^{st'} = C_{A}^{sw'}$. 


For $C \in \mathcal{C}_A$, we define a channel numbering $d^C_A$ as in (2.7) with respect to $>_m$. Then $d^C_A$ is a well-defined proper numbering on $A$ (cf. Proposition 2.16), and
\[ (d^C_A)^{st'} = d^{Cst'}_A. \]

Let $d^{st'}_A = d^{Cst'}_A$ be the southwest channel numbering on $A$. Lemma 2.21 also holds.

7.2. Row semistandard tableaux and offset vectors. For $a \geq 1$, let $\mathcal{B}_n'(a(a)) = \mathcal{B}'(a(a))$ be the set of $T \in \text{SST}_Z((a(a)))$ such that $T(a) - T(1) \leq n$, where $T(j)$ is the $j$-th entry of $T$ from the left for $1 \leq j \leq a$. Let $\tau_n = \tau$ be a bijection on $\mathcal{B}'(a(a))$ where $\tau(T)$ is the row semistandard tableau obtained from $T \in \mathcal{B}'((a))$ by replacing its entries $T(1) \leq T(2) \leq \cdots \leq T(a)$ with $T(2) \leq T(3) \leq \cdots \leq T(a) \leq T(1) + n$.

Let $b \geq 1$. For $\alpha = (\alpha_1, \ldots, \alpha_b) \in \mathbb{Z}^b$ and $(T_1, \ldots, T_b) \in \mathcal{B}'((a)) \times \cdots \times \mathcal{B}'((a))$, we define
\[ \tau^\alpha((T_1, \ldots, T_b)) = (\tau^{\alpha_1}(T_1), \ldots, \tau^{\alpha_b}(T_b)). \]

Let $R = (a^b)$. Let us regard $T \in \text{RSST}_{[n]}(R)$ as $T = (T^1, \ldots, T^i) \in \mathcal{B}'((a)) \times \cdots \times \mathcal{B}'((a))$, where $T^i$ is the $i$-th row of $T$ from the bottom. For $1 \leq i \leq b - 1$, let $r_i$ be the minimal non-negative integer such that
\[ (T^{i+1}, \tau^{r_i}(T^{i})) \]
is a $\mathbb{Z}$-semistandard tableau of shape $(a^2)$, and put $r_i = r_i + r_{i+1} + \cdots + r_{b-1}$. We call $r = (r_1, \ldots, r_{b-1})$ an offset vector and $\eta = (\eta_1, \ldots, \eta_{b-1})$ a symmetrized offset vector of $T$. Note that $\tau^{\eta}(T) \in \text{SST}_Z(R)$.

7.3. Dual affine RSK correspondence. Let $d$ be a proper numbering on $A \in \tilde{N}_{m \times n}$. Note that each level set $d^{-1}(k)$ forms a chain with respect to $<_m$.

Let $\{z_k\}_{k \in \mathbb{Z}}$ be the set of zig-zags associated to $d$, where the inner corners of $z_k$ are the set of elements in $d^{-1}(k)$ maximal with respect to $>_m$ (cf. Section 3.2). Then $\{z_k\}_{k \in \mathbb{Z}}$ satisfies
\begin{enumerate}
\item[(z'.1)] the inner corners of each $z_k$ are contained in $\text{supp}(A)$,
\item[(z'.2)] $\text{supp}(A) \subseteq \bigcup_{k \in \mathbb{Z}} z_k$,
\item[(z'.3)] $z_k$ is located to the southeast of $z_{k-1}$ for $k \in \mathbb{Z}$ in the sense of (3.1) with respect to $>_m$.
\end{enumerate}

Remark 7.2. We should remark that no outer cell of $z_k$ belongs to $\text{supp}(A)$, and $z_k$’s are not always mutually disjoint. More precisely, two horizontal lines (or line segments) in $z_k$ and $z_l$ $(k < l)$ may have non-trivial intersection, while vertical lines (or line segments) in $z_k$ and $z_l$ $(k < l)$ are always disjoint.

Suppose that a non-zero $A \in \tilde{N}_{m \times n}$ is given and let
\begin{itemize}
\item $(z_k)_{k \in \mathbb{Z}}$: the set of zig-zags associated to $d^{sw}_A$,
\item $A'$: the matrix obtained from $A$ by the same rule as in $A^p$ with respect to $(z_k)_{k \in \mathbb{Z}}$,
\item $A^{(t)}$: the matrices in $\tilde{N}_{m \times n}$ defined inductively by
\[ A^{(0)} = A, \quad A^{(t)} = \left(A^{(t-1)}\right)^{st'} (t \geq 1). \]
\end{itemize}

Note that $A^{(s-1)} \neq \emptyset$ and $A^{(s)} = \emptyset$ for some $s \geq 1$. For $1 \leq t \leq s$, we let
The set of zig-zags associated to \( d_{\lambda(t-1)} \),

\[ s^{(t)} = (a_t, b_t, \rho_t) \] : the stream of the back-post corners of \( \{z_k^{(t)}\}_{k \in \mathbb{Z}} \) with flow \( \mu_t \),

where we can check that \( \mu = (\mu_1, \ldots, \mu_s) \in \mathcal{P}_s \) as in Lemma 3.3.

Now let

- \( P_0, Q_0 \) : the tableau of shape \( \lambda = \mu' \) defined as in Section 3.2,
- \( P_0^t \) : the tableau of shape \( \mu \) obtained by flipping \( P_0 \) with respect to the main diagonal,
- \( \rho = (\rho_1, \ldots, \rho_s) \in \mathbb{Z}^s \).

**Lemma 7.3.** Under the above hypothesis, \( P_0^t \) is row semistandard, while \( Q_0 \) is column semistandard.

**Proof.** It follows immediately from Remark 7.2. \( \square \)

Let \( l = \ell(\lambda) \) and let \( m_i \) be the number of occurrences of \( i \) in \( \lambda' \) for \( 1 \leq i \leq l \). Let

- \( R_i \) : the Young diagram given in (3.8),
- \( P_0^{(i)}, Q_0^{(i)} \) : the subtableaux of \( P_0 \) and \( Q_0 \) corresponding to \( R_i \), respectively,
- \( \rho^{(i)} \in \mathbb{Z}^{m_i} \) : the subsequence of \( \rho \) corresponding to the columns of \( R_i \),
- \( \eta^{(i)} \in \mathcal{P}_{m_i-1} \) : the symmetrized offset vector of \( \left( P_0^{(i)} \right)^t \).

for \( 1 \leq i \leq l \) with \( m_i \geq 1 \), and

- \( Q \) : the tableau defined as in (3.11).

Then we have the following, which is a dual analogue of Theorem 3.10.

**Theorem 7.4.** We have a bijection

\[
\kappa' : \tilde{\mathcal{N}}_{m \times n} \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_n} RSST_{[m]}(\lambda') \times \mathcal{B}_n(\lambda),
\]

\[ A \longrightarrow (P_0^t, Q) \]

where \( \kappa'(\emptyset) = (\emptyset, \emptyset) \).

**Example 7.5.** Let \( A \) be the generalized permutation in Example 7.1. Then
where the red lines denote the zig-zags associated to $d_{\Delta_{t-1}}^{\times}$ for $1 \leq t \leq 4$, and

$$P_0^t = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 2 & 2 \\ 3 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 \\ 4 & 3 \end{bmatrix}, \quad \rho = (2, 1, 0, 0).$$

In this case, $R_1$ and $R_3$ are the only non-trivial rectangles in the decomposition of the shape of $P_0$ and $Q_0$. It is easy to see that $\eta^{(3)}_{\text{rev}} = (0, 2)$, $\eta^{(1)}_{\text{rev}} = (0, 0)$, and hence
7.4. Proof of Theorem 7.4. The proof of Theorem 7.4 is almost parallel to that of Theorem 3.10 in Section 3. So we give an outline of its proof and leave the details to the reader.

Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_{\geq 0}^m \) and \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{\geq 0}^n \) be given. Let \( \hat{N}_{m \times n}(\alpha, \beta) = \{ A | A \in \hat{N}_{m \times n}, \text{row}(A) = \alpha, \text{col}(A) = \beta \} \), where \( \text{row}(A) \) and \( \text{col}(A) \) are given as in (2.3).

Let \( K \geq 1 \) be given. We say that \( i \in [K] \) is an ascent of \( B \in \hat{W}_K \) if \( (i, j), (i + 1, j') \in \text{supp}(B) \) and \( j < j' \). We say that \( B \in \hat{W}_K \) is \( \alpha \)-ascending if for any \( k \in \text{supp}(\alpha) \) and \( i \) with
\[
\alpha_1 + \cdots + \alpha_{k-1} - 1 < i < \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k,
\]
i is an ascent of \( B \). Let \( \hat{W}_{K,\alpha,\beta} \) denote the set of \( B \in \hat{W}_K \) such that \( B \) is \( \alpha \)-ascending and \( B^\tau \) is \( \beta \)-descending. As in Lemma 5.1, we have a bijection
\[
\hat{N}_{m \times n}(\alpha, \beta) \longrightarrow \hat{W}_{K,\alpha,\beta}.
\]

Let \( \lambda \in \mathcal{P} \) be given with \( |\lambda| = K \). Let \( \text{RSST}_{[m]}(\lambda)_{\alpha} \) be the set of \( T \in \text{RSST}_{[m]}(\lambda) \) with content \( \alpha \). For \( T \in \text{RSST}_{[m]}(\lambda)_{\alpha} \), we define \( T^{\tau^\prime} \) to be a tableau obtained from \( T \) by replacing each \( k \in [m] \) in \( T \) with \( \alpha_k \neq 0 \) by the consecutive numbers
\[
\alpha_1 + \cdots + \alpha_{k-1} + 1 < \cdots < \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k
\]
from the bottom row to the top one, and from left to right in each row. Then we have
\[
T^{\tau^\prime} \in \text{RST}_{[K]}(\lambda),
\]
We call \( T^{\tau^\prime} \) the standardization of \( T \). Suppose that \( R = (a^b) \in \mathcal{P} \) for some \( a, b \geq 1 \). As in Lemma 5.2 we have
\[
\tau_{m}^{\tau^\prime}(T) \in \text{SST}_{\mathbb{Z}}(R) \quad \text{if and only if} \quad \tau_{K}^{\tau^\prime}(T^{\tau^\prime}) \in \text{SST}_{\mathbb{Z}}(R),
\]
for \( T \in \text{RSST}_{[m]}(R) \) and \( \nu \in \mathcal{P}_{a-1} \). In particular, the symmetrized offset vector of \( T \) is the same as that of \( T^{\tau^\prime} \).

Let \( S \in \text{RST}_{[K]}(\lambda) \) be given. We say that \( i \in [K] \) is a (row) ascent of \( S \) if the letter \( i + 1 \) does not appear below \( i \) in \( T \). We define \( S \in \text{RST}_{[K]}(\lambda) \) to be \( \alpha \)-ascending in the same way as in the case of \( \text{CST}_{[K]}(\lambda) \) with respect to ascent. Let \( \text{RST}_{[K],\alpha}(\lambda) \) be the set of \( \alpha \)-ascending tableaux in \( \text{RST}_{[K]}(\lambda) \). As in Lemma 5.3 we have a bijection
\[
\text{RSST}_{[m]}(\lambda)_{\alpha} \longrightarrow \text{RST}_{[K],\alpha}(\lambda) \ .
\]

\[
Q = \begin{pmatrix} 4 & 5 & 2 & 3 \\ 5 & 6 \\ 0 & 7 \end{pmatrix}
\]
Let $B \in \tilde{W}_K$ be given with $\kappa_0(B) = (P_0, Q_0, \rho)$, where $\kappa_0$ is the map in (3.3). Suppose that the shape of $P_0$ and $Q_0$ is $\lambda$ so that $P'_0 \in \text{RST}_\lambda(\lambda')$, $Q_0 \in \text{CST}_\lambda(\lambda)$. By Lemma 5.4 we have as in Corollary 5.5

(7.2) $B \in \tilde{W}_{K,[a,b]}$ if and only if $P'_0 \in \text{RST}_{[\lambda],\alpha}(\lambda')$ and $Q_0 \in \text{CST}_{[\lambda],\beta}(\lambda)$.

We also have the following analogue of Proposition 5.6. For $A \in \hat{N}_{m \times n}$, let $\kappa'_0(A) = (P_0, Q_0, \rho)$ be given in Section 7.3. Then

(7.3) $\kappa_0(A^{\text{st}'}) = (P'_0, Q'_0, \rho)$,

where we understand $P'_0$ as $((P'_0)^{\text{st}'})^\lambda$.

Let $(P_0, Q_0, \rho)$ be a triple such that $(P'_0, Q_0, \rho) \in \text{RSST}_{[\lambda]}(\lambda') \times \text{CSST}_{[\lambda]}(\lambda) \times \mathbb{Z}^{\lambda_1}$ for some $\lambda \in \mathcal{P}_n$. Keeping the notations in Section 7.3 we let

$\gamma^{(i)} \in \mathcal{P}_{m-1}$ : the symmetrized offset vector of $Q^{(i)}_0$,

$\zeta^{(i)} = \gamma^{(i)} - n^{(i)} \in \mathbb{Z}^{m_1}$ with the last component being zero,

for $1 \leq i \leq l$ with $m_i \geq 1$. We say that $(P_0, Q_0, \rho)$ is dominant if $\rho_{\text{rev}} - \zeta \in \mathcal{P}(\lambda)$, where $
abla = (\zeta^{(1)}, \ldots, \zeta^{(l)}) \in \mathbb{Z}^{m_1} \times \cdots \times \mathbb{Z}^{m_l}$. Then as in Lemma 5.11 it follows from (7.1) that

(7.4) $(P_0, Q_0, \rho)$ is dominant if and only if $(P'_0, Q'_0, \rho)$ is dominant.

Furthermore, by Remark 5.12 (7.4) is equivalent to saying that $\rho$ is dominant in the sense of Definition 5.8.

Let $\Omega'_{\text{dom}}$ be the set of $(P_0, Q_0, \rho)$ which are dominant. Then it follows from (7.2), (7.3), (7.4) and Theorem 5.1, Theorem 5.11 that we have a bijection

$\kappa'_0 : \hat{N}_{m \times n} \rightarrow \Omega'_{\text{dom}}$.

Finally, by the same arguments as in 5.15 we conclude that

$\kappa' : \hat{N}_{m \times n} \rightarrow \bigsqcup_{\lambda \in \mathcal{P}_n} \text{RSST}_{[\lambda]}(\lambda') \times \mathcal{B}_n(\lambda),$

$A \rightarrow (P'_0, Q_0)$

is a well-defined bijection. This proves Theorem 7.3.

7.5. Affine crystal morphisms. We first remark that $\hat{N}_{m \times n}$ has a structure of normal $A^{(1)}_{m-1}$-crystal with $P^0$ and $P_{\text{cl}}^0$-weights with respect to wt, $\tilde{e}_i, \tilde{f}_i$ for $i \in \{0, 1, \ldots, m-1\}$ for $m \geq 2$, where wt = wt$^0$ in (4.2) and wt$^0_{\text{cl}}$ in (4.3) respectively. For $A \in \hat{N}_{m \times n}$, $\tilde{e}_i A$ and $\tilde{f}_i A$ are given as follows:

(1) Let

$\sigma = (\cdots, \underbrace{\tilde{+}, \tilde{+}, \tilde{+}, \tilde{+}, \cdots}_{a_{i+1,j+1}}, \tilde{+}, \tilde{-}, \tilde{-}, \cdots)$,

and let $\tilde{\sigma}$ be the reduced one.
(2) If $\tilde{e}$ has at least one $-$, then we have
\[ \tilde{e}_i A = A + \hat{E}_{i,j_0} - \hat{E}_{i+1,j_0}, \]
where $j_0$ is the column index of $A$ corresponding the rightmost $-$ in $\tilde{e}$. If $\tilde{e}$ has no $-$, then we have $\tilde{e}_i A = 0$. Similarly, if $\tilde{e}$ has at least one $+$, then we have
\[ \tilde{f}_i A = A - \hat{E}_{i,j_1} + \hat{E}_{i+1,j_1}, \]
where $j_1$ is the column index of $A$ corresponding the leftmost $+$ in $\tilde{e}$. If $\tilde{e}$ has no $+$, then we have $\tilde{f}_i A = 0$.

Similarly, $\tilde{N}_{m,n}^\dagger$ has a structure of normal $A_{n-1}^{(1)}$-crystal with $P_0^0$ and $P_0^0$-weights with respect to $\text{wt}^t, \hat{e}_j^t, \tilde{f}_j^t$ for $j \in \{0, 1, \ldots, n - 1\}$ for $n \geq 2$, where $\text{wt}^t(A) = \text{wt}(A^t)$, and $\hat{e}_j^t A, \tilde{f}_j^t A$ are given as follows:

1. Let
\[ \sigma' = (\cdots, \underbrace{\ + \ + \ + \ + \ + \ + \cdots}_{a_{i,j}}, \underbrace{- \ - \ - \ + \ - \ + \ + \cdots}_{a_{i,j+1}}, \underbrace{- \ + \ + \ - \ + \ + \cdots}_{a_{i,j+1}}, \cdots). \]
and let $\tilde{\sigma}'$ be the reduced one.

2. If $\tilde{\sigma}'$ has at least one $-$, then we have
\[ \tilde{e}_j^t A = A + \hat{E}_{i_{0},j} - \hat{E}_{i_{0},j+1}, \]
where $i_0$ is the row index of $A$ corresponding the rightmost $-$ in $\tilde{\sigma}'$. If $\tilde{\sigma}'$ has no $-$, then we have $\tilde{e}_j^t A = 0$. Similarly, if $\tilde{\sigma}'$ has at least one $+$, then we have
\[ \tilde{f}_j^t A = A - \hat{E}_{i_{1},j} + \hat{E}_{i_{1},j+1}, \]
where $i_1$ is the row index of $A$ corresponding the leftmost $+$ in $\tilde{\sigma}'$. If $\tilde{\sigma}'$ has no $+$, then we have $\tilde{f}_j^t A = 0$.

The following is an analogue of Proposition \[14\] which is straightforward to check.

**Proposition 7.6.** The operators $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in \{0, 1, \ldots, m - 1\}$ commute with $\hat{e}_j^t$ and $\tilde{f}_j^t$ for $j \in \{0, 1, \ldots, n - 1\}$ on $\tilde{N}_{m,n}^\dagger \cup \{0\}$.

For $r \geq 1$, the set $\text{SST}_{[m]}((r))$ has a structure of normal $A_{m-1}^{(1)}$-crystal with $P_0^0$-weights with respect to $\text{wt}^t, \tilde{e}_i, \tilde{f}_i$ for $i \in \{0, 1, \ldots, m - 1\}$, where for $T \in \text{SST}_{[m]}((r))$

1. $\text{wt}_{0}^t(T) = \sum_{j=1}^{r} \text{cl}(\epsilon_{T(j)})$,
2. $\tilde{f}_i T$ is the row semistandard tableau obtained by replacing an entry $i$ in $T$ with $i+1$ (mod $n$) if exists, and $0$ otherwise.

Recall that $\text{SST}_{[m]}((r))$ is isomorphic to the crystal of the Kirillov-Reshetikhin module corresponding to $r \in \mathcal{C}_n$ [13].

In general, for $\mu = (\mu_1, \ldots, \mu_l) \in \mathcal{P}$, we regard $\text{RSST}_{[m]}(\mu)$ as an $A_{m-1}^{(1)}$-crystal with $P_0^0$ by identifying $T \in \text{RSST}_{[m]}(\mu)$ with $T^t \otimes \ldots \otimes T^1 \in \text{SST}_{[m]}((\mu_1)) \otimes \ldots \otimes \text{SST}_{[m]}((\mu_l))$, where $T^i$ is the $i$-th row of $T$ from the bottom.

Let
\[ S_{m,n} = \bigsqcup_{\lambda \in \mathcal{P}_n} \text{RSST}_{[m]}((\lambda')) \times \mathcal{B}_n(\lambda), \]
We assume the normal $A_{n-1}^{(1)}$-crystal structure with $P_{cl}^0$-weights for $m \geq 2$ and $A_{n-1}^{(1)}$-crystal structure with $P^0$-weights for $n \geq 2$ on $S_{m\times n}$ in the same manner as in $\mathcal{T}_{m\times n}$.

**Theorem 7.7.** The bijection

$$\kappa': \widehat{N}_{m\times n} \longrightarrow S_{m\times n}$$

commutes with $\varepsilon_i, \tilde{f}_i$ for $i \in \{1, \ldots, m-1\}$ and $\varepsilon'_j, \tilde{f}'_j$ for $j \in \{0, 1, \ldots, n-1\}$.

**Proof.** The proof is almost parallel to that of Theorem 4.15, where we need to modify the arguments in Section 6 with respect to the notions in Section 7.1. We leave the details to the reader. □

Let $\kappa'_1 = \pi_1 \circ \kappa'$, where $\pi_1$ is the projection of $S_{m\times n}$ along the first component. Then $\kappa'_1$ commutes with $\varepsilon_i$ and $\tilde{f}_i$ for $i \in \{0, 1, \ldots, m-1\}$, and preserves $\text{wt}^0_{cl}$. Hence we have the following.

**Corollary 7.8.** A generalized dual affine permutation $A \in \widehat{N}_{m\times n}$ is $A_{m-1}^{(1)}$-crystal equivalent to $P^0_0$, where $\kappa'(A) = (P^0_0, Q)$.

Moreover, if we define $\text{wt}^t$ on $\widehat{N}_{m\times n}$ in the same way as in (4.5) with respect to $\kappa'$, then we have the following analogue of Corollary 4.17.

**Corollary 7.9.** If we regard $\widehat{N}_{m\times n}$ as an $A_{m-1}$-crystal with $P_{cl}^0$-weights and as an $A_{n-1}^{(1)}$-crystal with $P^0$-weights with respect to $\text{wt}^t$, then it is an $(A_{m-1}, A_{n-1}^{(1)})$-bicrystal and $\kappa'$ is an isomorphism of $(A_{m-1}, A_{n-1}^{(1)})$-bicrystals. In particular, a generalized dual affine permutation $A \in \widehat{N}_{m\times n}$ is $A_{n-1}^{(1)}$-crystal equivalent to $Q$, where $\kappa'(A) = (P^0_0, Q)$.

We remark that both $m$ and $n$ do not need to be greater than 1 in Theorem 7.7 and its corollaries. In particular, when $m = 1$ we have the following multiplicity-free decomposition

$$\widehat{N}_{1\times n} \cong \bigoplus_{\lambda \in \mathcal{P}_n} \mathcal{B}_n(\lambda),$$

since $\text{RSST}_{[1]}(\lambda')$ consists of single element for all $\lambda \in \mathcal{P}_n$.

**Declarations**

**Conflict of Interest.** The authors declare that they have no conflict of interest.

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