TEST SETS FOR TAUTOLOGIES IN MODULAR
QUANTUM LOGIC

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Quantum logic; modular ortholattice; continuous geometry; universal test set

Abstract. As defined by Dunn, Moss, and Wang, an universal test set in an ortholattice \( L \) is a subset \( T \) such that each term takes value 1, only, if it does so under all substitutions from \( T \). Generalizing their result for ortholattices of subspaces of finite dimensional Hilbert spaces, we show that no infinite modular ortholattice of finite dimension admits a finite universal test set. On the other hand, answering a question of the same authors, we provide a countable universal test set for the ortholattice of projections of any type \( \Pi_1 \) von Neumann algebra factor as well as for von Neumann’s algebraic construction of a continuous geometry. These universal test sets consist of elements having rational normalized dimension with denominator a power of 2.

1. Introduction

In their seminal paper, Birkhoff and von Neumann [4], suggested ortholattices \( L(H) \) of closed subspaces of Hilbert spaces \( H \) as structures to deal with uncertainty features of quantum mechanics, the set \( \text{QL}(L) \) of tautologies of an ortholattice \( L \) consisting of all terms \( t(\bar{x}) \) such that \( t(\bar{a}) = 1 \) for every substitution in \( L \). In particular, towards an axiomatization, they showed that, for fixed finite \( d \geq 4 \), the ortholattices \( L(H) \) of subspaces of \( d \)-dimensional anisotropic inner products spaces \( H \) of \( \text{dim} \ H = d \) are, up to isomorphism, the directly irreducible \( d \)-dimensional ortholattices satisfying Dedekind’s modular law. Concerning infinite dimension, von Neumann suggested to consider certain continuous geometries. These are modular, too.

In contrast, ortholattices of all subspaces of infinite dimensional Hilbert spaces satisfy the orthomodular law, only, which became focus of Quantum Logic. Interest in the modular case was renewed in the “third life” of quantum logic, inspired by quantum computing: In [5], Dunn, Hagge, Moss, and Wang discussed quantum logic tautologies of \( L(H) \) where \( H \) is a finite dimensional Hilbert space. They derived
decidability of tautologies, in fixed dimension, from Tarski’s decision procedure for elementary algebra and geometry and discussed how tautologies relate to dimensions. In the abstract setting of modular ortholattices the latter question has been studied by Giuntini, Freytes, and Sergioli [8].

In [6], Dunn, Moss, and Wang introduced the concept of a universal test set for an ortholattice $L$: A subset $T$ of $L$ such that $t(\bar{x}) \in \mathcal{QL}(L)$ if and only if $t(\bar{a}) = 1$ for every substitution $\bar{a}$ within $T$. They showed that $L(H)$ does not admit a finite universal test set if $2 \leq \dim H < \infty$ and posed the problem to establish a universal test set for the ortholattice of projections of the hyperfinite type II$_1$ von Neumann algebra factor.

The purpose of the present note is to show non-existence of finite universal test sets for any infinite modular ortholattice of finite dimension and to provide countable universal test sets for certain continuous geometries: The ortholattices of projections of type II$_1$ von Neumann algebra factors and von Neumann’s [16] algebraic construction of a continuous geometry (which is not isomorphic to any of the former, according to von Neumann [17]). These universal test sets consist of elements having rational normalized dimension with denominator a power of 2. Observe that $\mathcal{QL}(L)$ is decidable for each of these continuous geometries $L$ [10, 9].

2. Test sets in MOLs of finite dimension

An ortholattice is a lattice with bounds 0, 1 and an orthocomplementation $x \mapsto x'$, that is an order reversing involution such that $x \land x' = 0$ and $x \lor x' = 1$ (cf. Section 1.2 of [6]). A lattice or ortholattice $L$ is modular if $x \geq z$ implies $x \land (y \lor z) = (x \land y) \lor z$. Such $L$ is of finite dimension $d$ if some/any maximal chain in $L$ has $d + 1$ elements, we write $d = d(L)$. We also use MOL for ”modular ortholattice”. Any interval $[b, c]$ of a MOL $L$ is a MOL with the relative orthocomplement $x \mapsto (x' \land c) \lor b$. Observe that $[b, c]$ is isomorphic to $[0, a]$, where $a = b' \land c$, and that the latter is a homomorphic image of the subortholattice $[0, a] \cup [a', 1]$ of $L$. Thus, any tautology of $L$ is also one of $[b, c]$. Also recall that with any ortholattice identity $t_1 = t_2$ one can associate a term $t$ such that, within any MOL, $t_1 = t_2$ is equivalent to $t = 1$.

**Theorem 1.** An infinite MOL of finite dimension does not admit a finite universal test set.

*Proof.* First, we consider a directly irreducible MOL $L$ of $\dim L = d < \infty$. Recall that the lattice $L$ is isomorphic to the subspace lattice of an
irreducible $d-1$-dimensional projective space. In particular, $L$ is finite if so is some of its intervals $[0, a]$ of dimension 2. Coordinate systems, that is $d+1$ points in general position, correspond to non-trivial $d$-frames in $L$.

Here, a $d$-frame $a$ in a modular lattice $M$ is given by $a_0, \ldots, a_d, a_\perp, a_\top$ in $M$ such that $a_\top = \bigvee_{i=0}^d a_i$ and $a_\perp = a_j \land \bigvee_{i \neq j} a_i$ for all $j$. A $d$-frame $a$ is trivial if $a_\perp = a_\top$; otherwise, any set $\{a_i \mid i \neq j\}$ is independent. In particular, any $d$-frame in $M$ of $\dim M < d$ has to be trivial.

Huhn [12] has provided a tuplet $\bar{a}^d(z^d)$ of lattice terms which yields a $d$-frame $\bar{a}^d(b)$ for every substitution $\bar{b}$ in a modular lattice and such that $\bar{a}^d(b) = \bar{b}$ if $b$ is a $d$-frame. For the equivalent concept of von Neumann normalized frames of order $d$ the analogous result has been obtained by Freese [7].

Now, observe that there is an ortholattice term $s(y_0, y_1, y_2, y_3)$ such that for any $c_1, c_2 \in [c_0, c_3]$ in a MOL one has $t(\bar{c})$ a complement of $c_2$ in $[c_0, c_3]$ and such that $s(\bar{c}) = c_1$ if, in addition, $c_1$ is a complement of $c_2$ in $[c_0, c_3]$. Namely, choosing $d_1$ the relative orthocomplement of $c_1 \land c_2$ in $[c_0, c_3]$ one obtains a complement of $c_2$ in $[c_0, c_3]$ by the relative orthocomplement of $c_1 \lor c_2$ in $[d_1, c_3]$.

Finally, put $\bar{x}_n = (x_1, \ldots, x_n)$, where $x_1, x_2, \ldots$ are pairwise distinct variables, and define

$$
\tilde{x}^d_i := s(z^d_1, (x_i \land (z^d_0 \lor z^d_1)) \lor z_\perp, z^d_0 \lor z^d_1)
$$

$$
t^d_n(\tilde{z}, \bar{x}_n) := (a_\top^d)' \lor \bigvee_{k=1}^{d-1} z^d_k \lor \bigvee_{1 \leq i < j \leq n} (\tilde{x}_i^d \land \tilde{x}_j^d).
$$

Observe that $z^d_0 \leq \tilde{x}^d_i \leq z^d_0 \lor z^d_1$ holds in any MOL $M$. Thus, given a substitution $\bar{b}, \bar{c}$ one has $t^d_n(\bar{b}, \bar{c}) = 1$ if the $d$-frame $\bar{a} := \bar{a}^d(\bar{b})$ is trivial. Otherwise, if in addition $M$ is directly irreducible and $\dim M = d$, then $a_\perp = 0$, $a_\top = 1$, and the $a_k$ and $p_i := \tilde{x}_i^d(\bar{b}, \bar{c})$ are atoms of $M$, the latter also complements of $a_1$ in $[0, a_0 \lor a_1]$. Therefore, $t^d_n(\bar{b}, \bar{c}) = 1$ if and only if $p_i = p_j$ for some $i < j$. It follows that $t^d_n(\bar{b}, \bar{c}) = 1$ for all substitutions in a set $T$ with $|T| < n$. On the other hand, if $M$ is infinite then there is a substitution such that $t^d_n(\bar{b}, \bar{c}) < 1$, namely $\bar{b}$ some non-trivial $d$-frame and the $c_i$ pairwise distinct atoms in $[0, b_0 \lor b_1]$, $c_i \neq b_1$.

Now, consider any infinite MOL $L$ of finite dimension. Then, up to isomorphism, $L$ is a direct product of finitely many directly irreducibles. Let $L_0$ be a factor of maximal dimension $d$. One has $d \geq 2$ since $d = 1$ would imply that $L$ is distributive and finite. If $d = 2$, all factors are of dimension $\leq 2$ and we may assume $L_0$ infinite. If $d > 2$ then $L_0$ is infinite due to [11]. As observed, above, for each $n$ there is an assignment in $L_0$, such that $t^d_n$ does not evaluate to 1. It follows that $t^d_n$ is not a
tautology of $L$. Now, given any finite $T \subseteq L$, $t_n^d$ with $n > |T|$ witnesses that $T$ is not a universal test set. \hfill \Box

Within modular lattices, the identity $a_1^d = a_1^-d$ is equivalent to Huhn’s $d - 1$-distributive law. Given $n < \infty$, with [13] Theorem 2.12 it follows that a subdirectly irreducible MOL $L$ is of dim $L \leq n$ if and only if $L$ is $n$-distributive – whence of dim $L = n$ if and only if, in addition, $L$ is not $n - 1$-distributive. In particular, this applies to directly irreducible complete atomic MOLs, cf. [8, Theorem 4.10].

3. Test sets in continuous geometries

Recalling von Neumann’s [16] algebraic construction of continuous geometries, let $F$ be one of the following fields with conjugation as involution: $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{F}$, the real algebraic numbers. Let the vector space $F^d$ be endowed with the canonical scalar product and the inner product spaces $V_n(F)$ recursively defined by $V_0(F) = F^1$ and $V_{n+1}(F) = V_n(F)^\perp \oplus V_n(F)$. For an anisotropic inner product space $V$ of dim $V < \infty$, denote by $L(V)$ its MOL of subspaces. Then $L(V_n(F))$ embeds into $L(V_{n+1}(F))$ via $U \mapsto U \oplus U$ and one may form the direct limit $L_\infty(F)$. The latter admits a normalized dimension function giving rise to a metric. The MOL $CG(F)$ is the metric completion of $L_\infty(F)$ cf. [9]. For a type II$_1$ von Neumann algebra factor $A$ let $L(A)$ denote the MOL of all projections in $A$, cf. [11].

**Theorem 2.** Let $L = CG(\mathbb{C})$ or $L = L(A)$ for a type II$_1$ von Neumann algebra factor. Then $L_\infty(\mathbb{F})$ embeds into $L$ such that the image is a universal test set for $L$ and consist of elements having rational normalized dimension with denominator a power of 2. In particular, $QL(L) = QL(L_\infty(\mathbb{F})).$

**Proof.** We begin with some observations. $L(V_n(\mathbb{F}))$ is isomorphic to $L(F^{2^n})$. $QL(L(\mathbb{F}^{2^n})) \subseteq QL(L(\mathbb{F}^m))$ if $m \leq 2^n$ since $L(\mathbb{F}^m)$ is isomorphic to an interval of $L(\mathbb{F}^{2^n})$. $QL(L(\mathbb{R}^m)) = QL(L(\mathbb{R}^n))$ since $\mathbb{F}$ and $\mathbb{R}$ are elementarily equivalent and since the ortholattices can be interpreted within $\mathbb{F}$ and $\mathbb{R}$, respectively. Finally, $QL(L(\mathbb{R}^m)) \subseteq QL(L(\mathbb{C}^k))$ if $m = 2k$ since $L(\mathbb{C}^k)$ embeds into $L(\mathbb{R}^m)$ considering $\mathbb{C}^k$ the complexification of $\mathbb{R}^k$. It follows that $QL(L_\infty(\mathbb{F})) \subseteq \cap_{n<\infty} QL(L(\mathbb{C}^n))$.

On the other hand one has $\cap_{n<\infty} QL(L(\mathbb{C}^n)) \subseteq QL(L)$ by [9] [10]. Thus, given an embedding $\varepsilon : L_\infty(\mathbb{F}^m)) \rightarrow L$, one can conclude that $QL(L) = QL(L_\infty(\mathbb{F})))$ whence $\varepsilon$ is a universal test set for $L$; moreover, the normalized dimension function $\delta$ of $L$ restricts to one on $\varepsilon(L(V_n(\mathbb{F})))$ and elements $x$ of the latter have $\delta(x) = \frac{r}{2^n}$ with integer $r$, $0 \leq r \leq 2^n$. 
In case $CG(F)$, the embedding $\varepsilon$ results from the embeddings $U \mapsto U \otimes C$ of $L(V_n(F))$ into $L(V_n(C))$. It remains to establish $\varepsilon$ in case $L = L(A)$.

Recall, e.g. from [2], the notion of $\ast$-regular ring: A ring $R$ with unit, endowed with an involution $x \mapsto x^\ast$ such that $xx^\ast = 0$ only for $x = 0$, and such that for any $a$ there is $x$ with $a = axa$. A $\ast$-regular ring $R$ admits a unique function $x \mapsto x^+$ satisfying the equations axiomatizing Moore-Penrose-Rickart pseudo-inverse. In particular, the pseudo-inverse on a $\ast$-regular sub-$\ast$-ring $S$ is that inherited from $R$.

For $F$ as above, the $m \times m$-matrices over $F$ form a $\ast$-regular ring $M_m(F)$ where $X^\ast$ is the conjugate transpose. Observe that $M_m(F)$ embeds into $M_{2m}(F)$ mapping $X$ onto the block diagonal matrix having 2 diagonal blocks $X$. The direct limit $M_\infty(F)$ of the $M_{2^n}(F)$, $n \to \infty$, is also $\ast$-regular and the embeddings $M_{2^n}(F) \to M_{2^n}(C)$ result into an embedding $\iota : M_\infty(F) \to M_\infty(C)$.

The projections of a $\ast$-regular ring $R$ form a MOL $L(R)$, ordered by $e \leq f \iff fe = e$, the fundamental operations of which are defined by terms in the language of $\ast$-rings with pseudo-inversion. Thus, any embedding $\iota : S \to R$ of $\ast$-regular ring restricts to an embedding $L(S) \to L(R)$. Therefore, the isomorphisms $L(F^{2^n}) \to L(M_{2^n}(F))$ result into an isomorphism $\omega : L_\infty(F) \to L(M_\infty(F))$.

Recall from [15, Definition 4.1.1] the notion of approximately finite type II$_1$ factor. All these are isomorphic as $\ast$-rings [15, Theorem XII] and each type II$_1$ factor contains one of them as sub-$\ast$-ring [15, Theorem XIII]. Finally, by [15, Lemma 4.1.1], the type $[2^n, n \to \infty]$ approximately finite type II$_1$ factor contains a sub-$\ast$-ring isomorphic to $M_\infty(C)$. Thus any type II$_1$ factor $A$ contains a sub-$\ast$-ring isomorphic to $M_\infty(F)$.

On the other hand, according to [14, Theorem XV] any type II$_1$ von Neumann algebra factor $A$ extends to a $\ast$-ring $U(A)$ having the same projections, that is $L(A) = L(U(A))$; moreover, $U(A)$ is $\ast$-regular [18, Part II, Ch.II, App.2(IV)] and [19, p.191]; cf. [3] and [10, Theorem 4.2]. Now, $M_\infty(F)$ embeds into $A$ whence into $U(A)$ and it follows that $L_\infty(F)$ embeds into $L(A)$. $\square$

To see that $CG(C)$ is not isomorphic to any $L(A)$, $A$ a type II$_1$ factor, observe that a lattice isomorphism would induce a ring isomorphism of the ring constructed in [17] onto $U(A)$, contradicting von Neumann’s result, cf. the introduction to [18].
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