Distributed Feedback Control of Multichannel Linear Systems

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Abstract—In this article it is established that any jointly controllable, jointly observable, multichannel, discrete or continuous time linear system with a strongly connected neighbor (communication) graph can be exponentially stabilized with any pre-specified convergence rate using a time-invariant distributed linear control. As an illustration of how this finding can be used to deal with certain distributed tracking problems, a solution is given to a distributed set-point control problem for a continuous-time, multichannel linear system in which each and every agent with access to the system is able to independently adjust its scalar-valued, controlled output to any desired set-point value. To better understand the constraints on controller design, the distributed control problem is recast as a classical decentralized control problem. Armed with the tools of decentralized control, including the notion of a “fixed spectrum,” it is possible to show quite surprisingly that if the only information each agent is allowed to share with its neighbors is its measured output, then distributed stabilization in some cases is impossible. Using well-known decentralized control concepts, lower bounds are derived on the dimensions of the shared sub-states of local controllers which, if satisfied, guarantee that there will be no fixed closed-loop system eigenvalues to contend with. The decentralized control perspective also enables one to assert definitively that without imposing a partitioning constraint, the closed-loop spectrum of any jointly controllable, jointly observable multichannel linear system with a strongly connected neighbor graph, can be freely assigned with distributed feedback control, even in the face of finite delays.

Index Terms—Distributed stabilization, fixed eigenvalues, multichannel linear systems, transmission delays.

I. INTRODUCTION

DISTRIBUTED control and estimation have been under active study for more than 20 years. The central aim of this article is to contribute to this technology by solving what is arguably the most fundamental problem in distributed control, namely to constructively prove that any “multichannel” linear system with a strongly connected “neighbor” (communication) graph can be exponentially stabilized with an arbitrarily fast convergence rate using a time-invariant distributed linear control. By an $n$-dimensional, multichannel, continuous-time, linear system with $m$ channels is meant a linear system of the form

$$
\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i, \quad y_i = C_i x, \quad i \in m
$$

where, $n$ and $m$ are positive integers, $m = \{1, 2, \ldots, m\}, x \in \mathbb{R}^n$, and for each $i \in m$, $u_i \in \mathbb{R}^p_i$ is the control input of channel $i$ and $y_i \in \mathbb{R}^q_i$ is the measured output of channel $i$. Here $A$, $B_i$, and $C_i$ are real-valued, constant matrices of appropriate sizes. The discrete-time counterpart of (1) is an $m$-channel linear system of the form

$$
x(t+1) = Ax(t) + \sum_{i=1}^{m} B_i u_i(t), \quad y_i(t) = C_i x(t), \quad i \in m
$$

where, $t \in \{0, 1, 2, \ldots\}$. It is presumed that either system (1) or (2) is to be controlled by $m$ agents, labeled 1 through $m$, with the understanding that each agent $i \in m$ can measure output signal $y_i$ and has access to control input $u_i$. It is further assumed that each agent $i$ can receive the measured output of each of its “neighbors” as well as some suitably defined substate of each of its neighbors’ controllers. The specification of who agent $i$’s neighbors are is part of the problem formulation. The set of labels of agent $i$’s neighbors, including itself, is denoted by $N_i$. The neighbor graph associated with either (1) or (2), written $\mathbb{N}$, is a directed graph on $m$ vertices, with an arc from vertex $j$ to vertex $i$ just in case agent $j$ is a neighbor of agent $i$. It is assumed
that each agent’s neighbors do not change with time. Thus, \( N \) is a stationary graph.

**A. Problem**

It is assumed throughout this article that the multichannel system under consideration is both jointly controllable and jointly observable; that is for the system defined by (1) or (2) the matrix pairs

\[
(A, [B_1 B_2 \cdots B_m]) \quad \text{and} \quad \left( [C_1' C_2' \cdots C_m']', A \right)
\]

are controllable and observable, respectively, where ′ denotes transposition. For simplicity, it is also assumed that for \( i \in \mathbb{m} \), both \( B_i \neq 0 \) and \( C_i \neq 0 \). Subject to these assumptions, the main problem to which this article is addressed is as follows.

**Basic distributed control problem:** For the \( m \)-channel system defined by (1) or (2), develop a systematic procedure for deciding which signals each agent is to receive from its neighbors and for constructing \( m \) linear time-invariant feedback controllers using these signals, one for each channel, so that the state of the resulting closed-loop system converges to zero exponentially fast at a preassigned rate.

**B. Summary**

In this article it is shown that any jointly controllable, jointly observable, multichannel, discrete, or continuous time linear system with a strongly connected neighbor (communication) graph can be exponentially stabilized with an arbitrarily fast convergence rate using a time-invariant distributed linear control. This is proved constructively in Section II for continuous-time systems using a distributed observer-based certainty equivalence control. Here is how this is done. The notion of certainty equivalence control is briefly overviewed in Section II-A along with explanations of what the "cancellation" and "substitution" rules are and why they are important. A brief review is then given in Section II-B of the "open-loop" structure of the distributed observer [3], which will be used. Finally it is explained in Section II-C how to modify this observer so that it can be used in a feedback configuration. What results is a distributed observer-based architecture, which solves the distributed control problem of interest for systems with strongly connected neighbor graphs. The same construction applies to discrete-time systems. As an illustration of how these ideas can be used to deal with certain distributed tracking problems, a solution is then given in Section III to a distributed set-point control problem for a continuous-time, multichannel linear system in which each and every agent with access to the system is able to independently adjust its scalar-valued, controlled output to any desired set-point value.

With the aim of understanding more clearly some of the implications on controller design of the distributional constraint, the distributed control problem of interest is recast in an algorithmically independent way, as a classical decentralized control problem (see Section IV). This is done by first reviewing in Section IV-A, needed results from classical decentralized control theory [4], [5] including the notion of a "fixed spectrum." Armed with these concepts it is possible to show that if the only information each agent is allowed to share with its neighbors is its measured output, then distributed stabilization in some cases is impossible (see Section IV-B). This surprising observation is universal. It holds for all possible linear time-invariant distributed controls one might consider for either discrete or continuous time systems. The concept of an "extended system" is then introduced in Section IV-C and used in Section IV-D to restate the basic distributed control problem for the original system as a decentralized control problem for the extended system. Doing this enables one to address the distributed control problem without explicitly appealing to any one particular type of distributed dynamic compensator, such as those discussed in this article and in [6], [7], [8], [9]. Viewing the distributed control problem as a decentralized control problem has several immediate payoffs. For example, it enables one to easily derive lower bounds on the dimensions of the shared substates of local controllers which, if satisfied, guarantee that there will be no fixed closed-loop system eigenvalues to contend with (cf., Theorem 2). It also enables one to derive a corollary to this theorem, which asserts definitively that without imposing a partitioning constraint, the closed-loop spectrum of any jointly controllable, jointly observable multichannel linear system with a strongly connected neighbor graph, can be freely assigned with distributed feedback control (see Section IV-E).

Finally, this article turns to the important and often overlooked design problem of dealing with the effects of transmission delays across the network. It is explained in Section V why in the face of finite delays, exponential stabilization at any prescribed convergence rate can still be achieved with distributed control. This unexpected observation applies to discrete-time multichannel linear systems.

**C. Background**

There are several papers directly concerned with the basic distributed control problem formulated in Section I-A. Among these is [6], which explains how to construct a time-invariant distributed controller capable of exponentially stabilizing a continuous-time multichannel system under the same assumptions made in this article. This appears to be the first result of its kind and is noteworthy. A similar time-invariant controller has recently been described in the first part of [7]. There are several significant differences between what is done in [6], [7] and what is done in this article. For example, while the controllers developed in Section II of this article are "observer-based," those developed in [6], [7] are not in that no state estimation is carried out and the certainty equivalence idea is not used. In contrast to the controller construction described in Section II of this article, the constructions given in [6], [7] do not provide a means for controlling convergence rate, although it is plausible that they can be so modified; also in contrast to this article, the design methodologies proposed in [6], [7] are of the "high-gain" type and do not have a discrete-time counterpart. High gain controllers can of course become problematic when issues, such as unmodeled dynamics and measurement noise are taken into account.

One of the biggest stumbling blocks encountered in trying to mimic centralized observer-based control in a distributed
setting is that the certainty equivalence “cancellation rule” (cf., Section II-A) cannot be satisfied without violating the distributed information pattern assumption. The cancellation rule is satisfied in a centralized setting by applying to the state estimator, the same feedback signal that is applied to the process. Satisfaction of the cancellation rule ensures that the dynamics of the state estimation error functions autonomously. To overcome this limitation, [8] proposed a new idea called the “substitution rule” (cf., Section II-A), which if satisfied also results in an autonomous error system. However, unlike the centralized case, in the distributed case the construction of a provably correct distributed observer compatible with this rule is an especially challenging problem. An effort to devise such a compatible observer is undertaken in [8] by posing the observer’s construction as an optimization problem; however, this work is preliminary in that no constructive results are presented. An effort is also made in [9] to construct a distributed observer compatible with the substitution rule. This work is also preliminary in that no provably correct procedure is described for constructing a compatible distributed observer except under restrictive assumptions and conditions.

II. DISTRIBUTED OBSERVER-BASED CERTAINTY EQUIVALENCE CONTROL

The aim of this section is to explain how to solve the basic distributed control problem using “certainty equivalence” and a suitably defined distributed observer. We begin with a brief discussion about certainty equivalence itself and about the difficulty in applying it in a distributed context.

A. Certainty Equivalence Control

The idea of certainty equivalence has its roots extending as far back in 1957 [10]. The concept has been broadened over time to encompass not just linear control but also much of parameter adaptive control and even certain classes of nonlinear control. For a process modeled by a linear system of the form $y = Cx$, $\dot{x} = Ax + Bu$, which is to be regulated by a centralized control, the certainty equivalence approach amounts to first devising an appropriate candidate control $u = Fx$ assuming the process state $x$ is available for feedback, and then using instead of $x$ in the definitions of $u$, a suitably defined estimate $\hat{x}$ of $x$ (i.e., $u = F\hat{x}$) even though the estimate may not be correct. The estimate is typically generated by an estimator of the form $\hat{x} = (A + KC)\hat{x} - Ky + BF\hat{x}$, which may be either an observer or a Kalman filter. It is important to note that the signal $BF\hat{x}$ is included as an input to the estimator to ensure that “cancellation rule” is satisfied. That is, in forming the dynamical system, which models the evolution over time of the state estimation error $e = \hat{x} - x$, the term $BF\hat{x}$ is canceled and what results is the autonomous error system $\dot{e} = (A + KC)e$. This has several important consequences. First, the designs of the feedback matrix $F$ and the estimator matrix $K$ can be dealt with separately; this is a manifestation of the so-called “separation principal” of stochastic optimal control [11]. This is important because the closed-loop spectrum of the process equals the disjoint union of the spectrum of $A + BF$ and the spectrum of $A + KC$. Second, the fact that $u = Fx + Fe$ means if the estimation error is “small,” the behavior of the process is approximately the same as that, which would have resulted in had the actual candidate control $u = Fx$ been used. For the distributed control problem under consideration in this article, certainty equivalence has different consequences. The certainty equivalence approach for distributed control starts off mimicking the centralized case; i.e., a candidate distributed control of the form $u_i = F_i x$ for each agent $i \in m$ is crafted using standard techniques. Next, instead of using $x$ in the definitions of the $u_i$, in accordance with certainty equivalence agent $i$ uses a suitably defined estimate $x_i$ of $x$ (i.e., $u_i = F_i x_i$, $i \in m$). To carry out this program, it is of course necessary for there to be available a provably correct local agent estimator for generating each $x_i$. For continuous-time systems there are a handful different kinds of “open-loop” distributed observers which might be considered for this purpose [3], [9], [12], [13], [14], [15]. No matter which one is considered for agent $i$, for the cancellation rule to be satisfied, the estimator would have to include as an input, the input to the process, just as in the centralized case. However, in the distributed case, the process input is $\sum_{m=1}^{m} B_j F_j x_j$ and adding $\sum_{m=1}^{m} B_j F_j x_j$ as an input to agent $i$’s estimator clearly cannot be done without violating distributional assumptions, except in the very special case when every agent $j$ for which $F_j \neq 0$, is a neighbor of agent $i$. Thus, if all $F_j \neq 0$, the cancellation rule cannot be satisfied unless the neighbor graph is complete. One way to circumvent this problem is to add to agent $i$’s estimator the signal $\sum_{m=1}^{m} B_j F_j x_j$ rather than $\sum_{m=1}^{m} B_i F_j x_j$. This simple but clever idea, which seems to have been first suggested in [8], is henceforth called the substitution rule. Its motivation is pretty clear. If each $x_i$ estimates $x$ as intended, then each $x_i$ also estimates every $x_j$, $j \in m$, so there is not much lost in using $x_i$ instead of $x_j$ in agent $i$’s local estimator. What is nice about the substitution rule is that it causes the error system modeling the dynamics of the estimation errors $e_i = x_i - x$, $i \in m$, to be autonomous, just as the error model is in the centralized case. Since agent $i$’s input to the process can be written as $u_i = F_i x + F_i e_i$, it means that the local estimators can be designed to function as intended, $e_i$ will become small and $u_i$ will be approximately the same as agent $i$’s candidate control input $F_i x$, which is ultimately what is desired. But there is a catch. Although adherence to the substitution rule causes the error system to be autonomous, it also means that the dynamical model for each local estimation error $e_i$ will have as an input the signal $\sum_{j=1}^{m} B_j F_j (e_i - e_j)$ (cf., Section II-C). No matter which distributed observer is used, these signals will always be present in the error equations and their presence makes the design of the observer an especially challenging problem. As mentioned earlier, [8] and [9] both focus on this problem but neither paper presents a constructive solution.

One of the main contributions of this article is to explain why despite this challenge, it is possible to correctly design the type of closed-loop observer needed based on the open-loop estimator proposed in [3], [12]. The construction works no matter how the $F_i$’s are defined and is essentially the same as the construction in the open-loop case when there are no terms of the form $\sum_{j=1}^{m} B_j F_j (e_i - e_j)$ to contend with. Moreover the construction for the discrete-time case involves exactly the
same steps as the continuous-time case. It is unlikely that the observers discussed in [13, 14, 15] can be modified for this purpose because certain key structural properties upon which the open-loop versions of these observers depend are lost when the signals $\sum_{j=1}^{m} B_j F_j (e_i - e_j)$ are present.

### B. Distributed State Estimation for a Process Without Feedback

A variety of distributed estimators have been proposed in the literature for estimating the state of (1) or (2) assuming $u_i = 0$, $i \in \mathbf{m}$ [3], [12], [13], [14], [15], [16], [17], [18], [19], [20]. Among these, so far only the distributed observers discussed in [3], [12], seem to be amenable to application in a feedback loop. For this reason, the distributed observer studied in [3] will be used in this article. Assuming $\mathbb{N}$ is strongly connected, this particular observer is described by the equations:

$$
\dot{x}_i = (A + K_i C_i) x_i - K_i y_i + \sum_{j \in N_i} H_{ij} (x_i - x_j) + \delta_{iq} \bar{C} z
$$

$$
\dot{z} = \bar{A} z + \bar{K} C_q x_q - \bar{K} y_q + \sum_{j \in N_q} \bar{H}_j (x_q - x_j)
$$

where, for $i \in \mathbf{m}$, $x_i \in \mathbb{R}^n$ is the estimated state, $z \in \mathbb{R}^{m-1}$ is the state of channel controller (see the following), which is the part of agent $q$’s controller not shared with other agents, and the $K_i$, $H_{ij}$, $A$, $\bar{A}$, $\bar{K}$, $\bar{C}$ are matrices of appropriate sizes. Here, $\delta\{\}$ is the Kronecker delta and $q$ is any preselected integer in $\mathbf{m}$. The subsystem consisting of (4) and the signal $\delta_{iq} \bar{C} z$ is called a channel controller of (3). Its function will be explained in a moment.

The error system for this observer is described by the equations:

$$
\dot{e}_i = (A + K_i C_i) e_i + \sum_{j \in N_i} H_{ij} (e_i - e_j) + \delta_{iq} \bar{C} z
$$

$$
\dot{z} = \bar{A} z + \bar{K} C_q e_q + \sum_{j \in N_q} \bar{H}_j (e_q - e_j)
$$

where, for each $i \in \mathbf{m}$, $e_i$ is the $i$th state estimation error $e_i = x_i - x$. Note that (5) and (6) together form an $(mn + m - 1)$-dimensional, unforced linear system. It is known that if $\mathbb{N}$ is strongly connected, this error system’s spectrum can be freely assigned by appropriately picking the matrices $K_i$, $H_{ij}$, $A$, $\bar{A}$, $\bar{H}_{ij}$, and $\bar{C}$ [3]. Thus, by so choosing these matrices, all of the $e_i$ and $z$ can be made to converge to zero exponentially fast at a preassigned rate.

There are several steps involved in picking the matrices $K_i$, $H_{ij}$, $A$, $\bar{A}$, $\bar{C}$, and $\bar{C}$. First $q$ is chosen; any value of $q \in \mathbf{m}$ suffices. The next step is to temporarily ignore the channel controller (6) and to choose matrices $\bar{K}_i$ and the $\bar{H}_{ij}$ so that the open-loop error system

$$
\dot{e}_i = (A + \bar{K}_i C_i) e_i + \sum_{j \in N_i} \bar{H}_{ij} (e_i - e_j) + \delta_{iq} \bar{u}_q
$$

is controllable by $\bar{u}_q$ and observable through

$$
\dot{\bar{y}}_q = \begin{bmatrix} C_q e_q \\ e_q - e_{j_1} \\ \vdots \\ e_q - e_{j_m} \end{bmatrix}
$$

where, $\{j_1, j_2, \ldots, j_m\} = N_q$. In fact, the set of $\bar{K}_i$ and $\bar{H}_{ij}$, $j \in N_q$, for which these properties hold is the complement of a proper algebraic set in the linear space of all such matrices [3]. Thus, almost any choice for these matrices will accomplish the desired objective.

The next step is to pick matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ so that the closed-loop spectrum of the system consisting of (7), (8), and the channel controller

$$
\bar{u}_q = \bar{C} z + \bar{D} \bar{y}_q, \quad \dot{z} = \bar{A} z + \bar{B} \bar{y}_q
$$

has the prescribed spectrum. One technique for choosing these matrices can be found in [21]. Of course since the system defined by (7) and (8) is controllable and observable, there are many ways to define a channel controller and thus the matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$. In any event, once these matrices are chosen, the $\bar{K}_i$ and $\bar{H}_{ij}$ are defined so that for all $i \neq q$, $K_i \triangleq \bar{K}_i$ and $H_{ij} \triangleq \bar{H}_{ij}$, $j \in N_i$; while for $i = q$, $K_q \triangleq \bar{K}_q + \bar{K}_i$ and $H_{jq} \triangleq \bar{H}_{ij} + \bar{H}_{ij}$, $j \in N_q$, where

$$
\begin{bmatrix} \bar{K} & \bar{H}_{jj} & \cdots & \bar{H}_{jn} \end{bmatrix} = \bar{D}.
$$

Finally $\bar{K}$ and the $\bar{H}_j$, $j \in N_q$, are defined so that $\begin{bmatrix} \bar{K} & \bar{H}_{jj} & \cdots & \bar{H}_{jn} \end{bmatrix}$ is $\bar{B}$.

### C. Distributed State Estimation for a Process With Feedback

In analogy with the centralized case, the first step in the development of a distributed observer-based feedback system is to devise state feedback laws $u_i = F_i x$, $i \in \mathbf{m}$, which endow the closed-loop system $\dot{x} = (A + \sum_{i=1}^{m} B_i F_i) x$ with prescribed properties, such as stability and/or optimality with respect to some performance index. This is essentially a centralized design problem since for any $F$ of appropriate size it is easy to construct matrices $F_i$ such that $BF_i = \sum_{i=1}^{m} B_i F_i$ where $B = \text{row}\{B_1, B_2, \ldots, B_m\}$. Thus, centralized control techniques, such as spectrum assignment, can be used to construct $F_i$ so that the convergence of the state transition matrix of $A + \sum_{i=1}^{m} B_i F_i$ to zero is as fast as desired.

Having chosen the $F_i$, the next step is to invoke certainty equivalence by replacing the controls $u_i = F_i x$, $i \in \mathbf{m}$, with the controls $u_i = F_i \hat{x}_i$, $i \in \mathbf{m}$, where each $x_i$ is an estimate of $x$ generated by the observer discussed in Section II-B, modified to take into account the control inputs $u_i$, $i \in \mathbf{m}$. Doing this results in the system

$$
\dot{x} = Ax + \sum_{i \in \mathbf{m}} B_i F_i \hat{x}_i.
$$

Modifying the open-loop distributed observer described in Section II-B in accordance with the substitution rule, results in
the distributed observer described by
\[
\dot{x}_i = (A + K_i C_i) x_i - K_i y_i + \sum_{j \in \mathcal{N}_i} H_{ij} (x_i - x_j) \\
+ \delta_{iq} \bar{C} z + \left( \sum_{j=1}^{m} B_j F_j \right) x_i, \quad i \in \mathbf{m} 
\]
and
\[
\dot{z} = \bar{A} z + \bar{K} C_q x_q - \bar{K} y_q + \sum_{j \in \mathcal{N}_q} \bar{H}_j (x_q - x_j). 
\]
(10)

This is the observer that will be considered. In the sequel it will be shown that even with this modification, this distributed observer can still provide the required estimates of $x$.

Note that the error system for (10) and (11) is described by the linear system
\[
\dot{e}_i = (A + K_i C_i) e_i + \sum_{j \in \mathcal{N}_i} H_{ij} (e_i - e_j) + \delta_{iq} \bar{C} z \\
+ \sum_{j=1}^{m} B_j F_j (e_i - e_j), \quad i \in \mathbf{m} 
\]
and
\[
\dot{z} = \bar{A} z + \bar{K} C_q e_q + \sum_{j \in \mathcal{N}_q} \bar{H}_j (e_q - e_j). 
\]
(12)

Since (12) and (13) are the closed-loop error system defined by (9), (10), and (11) can be made to converge to zero with any prescribed rate. We now explain how to assign the error system’s spectrum.

In the sequel, it will be explained how to choose the $K_i, H_{ij}, K, H_j, \bar{A}$, and $\bar{C}$ for any given $F_i$, so that the spectrum of (12) and (13) coincides with a prescribed symmetric set of complex numbers. Assuming $\mathbb{N}$ is strongly connected. To achieve this, attention will first be focused on the properties of the open-loop error system described by
\[
\dot{e}_i = (A + K_i C_i) e_i + \sum_{j \in \mathcal{N}_i} H_{ij} (e_i - e_j) \\
+ \sum_{j=1}^{m} B_j F_j (e_i - e_j) + \delta_{iq} \bar{u}_q, \quad i \in \mathbf{m} 
\]
and (8) and (14). This system is what results when the channel controller appearing in (13) is removed. The main technical result of this is as follows.

**Proposition 1:** Suppose $\mathbb{N}$ is strongly connected. There are matrices $K_i$ and $H_{ij}, i \in \mathbf{m}, j \in \mathcal{N}_i$, such that for all $q \in \mathbf{m}$, the open-loop error system described by (14) and (8) is observable through $\bar{u}_q$ and controllable by $\bar{u}_q$ with controllability index $m$.

The implication of this proposition is clear. It is possible to choose the coefficient matrices $\bar{K}, \bar{H}_j, \bar{A}$, and $\bar{C}$ which define the channel controller to freely assign the closed-loop spectrum of (12) and (13). We are led to the following result.

**Theorem 1:** Suppose $\mathbb{N}$ is strongly connected. For any set of feedback matrices $F_i, i \in \mathbf{m}$, any integer $q \in \mathbf{m}$, and any symmetric set $\mathcal{A}$ of $mn + m - 1$ complex numbers, there are matrices $K_i, H_{ij}, \bar{K}, \bar{H}_j, \bar{A}$, and $\bar{C}$ for which the spectrum of the closed-loop error system defined by (12) and (13) is $\mathcal{A}$.

**Remark 1:** Before proceeding, it is worth noting that while the local agent controllers can be implemented in a distributed manner, they collectively require a centralized design in which all coefficient matrices are constructed using a single centralized design procedure. Centralized designs are implicitly assumed in almost all decentralized control and distributed control algorithms involving feedback, such as the work in [4], [5] as well as the work referred to earlier in [7], [8]. Preliminary efforts have also been made in [7], [8] to develop distributed controls relying on distributed designs. Whether or not the inevitable complexities and operational limitations introduced by these generalizations lead to practical feedback algorithms, which exhibit at least some degree of noise tolerance and robustness to unmodeled dynamics, remains to be seen. For sure, in situations where centralized design is acceptable, it is difficult to imagine a situation where one would want to opt for a distributed design. Of course there are some distributed algorithms, such as those studied in [22], [23], which do not call for centralized designs, but they are not feedback control algorithms.

**D. Analysis**

We will now proceed to justify Proposition 1. Toward this end, note that by introducing the joint estimation error $\epsilon = \text{column} \{ e_1, e_2, \ldots, e_m \}$, the open-loop error system dynamics (14) can be written compactly as
\[
\dot{\epsilon} = \left( \bar{A} + \sum_{i=1}^{m} \bar{B}_i (K_i \bar{C}_i + H_i \bar{C}_i) \right) \epsilon + \bar{B}_q \bar{u}_q 
\]
(15)

where, $\bar{A} = I_{mn \times m} \circ (A + \sum_{j=1}^{m} B_j F_j)$ and for $i \in \mathbf{m}$, $\bar{B}_i = b_i \circ I_{n \times n}$, $b_i$ is the $i$th unit vector in $\mathbb{R}^n$, $\bar{C}_i = C_i \bar{B}_i$ and $\circ$ is the Kronecker product. Here $Q$ is the $nm \times nm$ block partitioned matrix of $m^2$ square blocks whose $ij$th block is $B_j F_j$, $H_i = [H_{ij_1}^i H_{ij_2}^i \cdots H_{ij_m}^i]$ where $\{j_1^i, j_2^i, \ldots, j_m^i\} = \mathcal{N}_i$, $\bar{C}_i = \text{column} \{ C_{ij_1}^i, C_{ij_2}^i, \ldots, C_{ij_m}^i \}$ where $C_{ij} = c_{ij} \circ I_{n \times n}, j \in \mathcal{N}_i, i \in \mathbf{m}$, and $c_{ij} = b'_j - b'_j$.

Next observe that (15) is what results when in the distributed feedback law $\tilde{v}_i = [K_i H_i] \tilde{y}_i + \delta_{iq} \bar{u}_q, i \in \mathbf{m}$, is applied to the $m$-channel linear system
\[
\dot{\epsilon} = \dot{\epsilon} + \sum_{j=1}^{m} \bar{B}_j \tilde{v}_j, \quad \tilde{y}_i = \left[ \begin{array}{c} \tilde{C}_i \\ \bar{C}_i \end{array} \right] \epsilon, \quad i \in \mathbf{m}. 
\]
(16)

The proof of Proposition 1 depends on the following lemmas.

**Lemma 1:** The $m$-channel linear system described by (16) is jointly controllable and jointly observable.
Proof of Lemma 1: In view of the definitions of the $\tilde{B}_i$, it is clear that $\begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 & \cdots & \tilde{B}_m \end{bmatrix}$ is the $nm \times nm$ identity. Therefore, (16) is jointly controllable.

To establish joint observability, suppose that $\tilde{v}$ is an eigenvector of $\tilde{A}$ for which $\begin{bmatrix} C_i \ C_i \end{bmatrix}^{\top} \tilde{v} = 0, i \in m$. From the relations $C_i \tilde{v} = 0$, the definitions of the $C_i$, and the assumption that $N$ is strongly connected, it follows that $C_i v = 0, i \in m$. Moreover, from the definition of $\tilde{A}$ and the structure of $\tilde{v}$, it is clear that $\tilde{A}\tilde{v} = (I_{m \times m} \otimes A)\tilde{v} = \text{column}[Av, Av, \ldots, Av]$. This and the hypothesis that $\tilde{v}$ is an eigenvector of $\tilde{A}$ imply that $v$ must be an eigenvector of $A$. But this is impossible because of joint observability of (1) and the fact that $C_i v = 0, i \in m$. Thus, (16) has no unobservable modes through the combined outputs $\tilde{y}_i, i \in m$, which means that the system is jointly observable. $\blacksquare$

Lemma 2: Suppose $N$ is strongly connected. There are matrices $H_i, i \in m$, for which $(\sum_{i=1}^m \tilde{B}_i H_i C_i, \tilde{B}_i)$ is a controllable pair with controllability index $m$ for every choice of $q \in m$.

Proof of Lemma 2: With $\overline{N}_i$ denoting the complement of $\{i\}$ in $N_i$, it is shown in the proof of Proposition 1 in [3] that there are scalars $\phi_{ij}$ for which the matrix pair $(\sum_{j \in \overline{N}_i} \sum_{j \in \overline{N}_i} b_{ij} c_{ij}, b_q)$ is controllable for all $q \in m$. Fix any such $\phi_{ij}$ and any set of scalars $\phi_{ij}, i \in m$. Next, define $D = \sum_{i \in m} \sum_{j \in \overline{N}_i} b_{ij} c_{ij}$ and note that $D = \sum_{i \in m} \sum_{j \in \overline{N}_i} b_{ij} c_{ij}$ because $c_{ii} = 0, i \in m$. Therefore, $(D, b_q)$ is also a controllable pair for each $q \in m$.

Define $\tilde{H}_i = [\phi_{ij1} \ \phi_{ij2} \ \cdots \ \phi_{ijm}] \otimes I_n$, whereas before $\{j_1, j_2, \ldots, j_{m-1}\} = \overline{N}_i$. Note that for each $i \in m$, $\tilde{B}_i H_i C_i = (\sum_{j \in \overline{N}_i} b_{ij} c_{ij}) \otimes I_n$. Thus, the matrix $D \triangleq \sum_{i=1}^m \tilde{B}_i H_i C_i = D \otimes I_n$. Hence, for all $q \in m$,

$$\begin{bmatrix} \tilde{B}_1 \tilde{D} & \tilde{B}_2 \tilde{D} & \cdots & \tilde{B}_m \tilde{D} \end{bmatrix} = \begin{bmatrix} b_q \ D b_q & \cdots & D^{m-1} b_q \end{bmatrix} \otimes I_n.$$ Since each pair $(D, b_q)$ is controllable

rank $\begin{bmatrix} \tilde{B}_1 \tilde{D} & \tilde{B}_2 \tilde{D} & \cdots & \tilde{B}_m \tilde{D} \end{bmatrix} = mn$.

Therefore, for each $q \in m$, $(\tilde{D}, \tilde{B}_q)$ is a controllable pair with controllability index $m$. $\blacksquare$

Lemma 3: Let $(A_{n \times n}, B_{n \times r})$ be a real-valued controllable matrix pair with controllability index $m$. For each real matrix $M_{n \times n}$, the matrix pair $(M \otimes M_{n \times n})$ is controllable with controllability index no greater than $m$ for all but at most a finite number of values of the real scalar gain $g$. Moreover if $m = n$, then $m$ is the controllability index of $(M \otimes M_{n \times n})$ for all but a finite number of values of $g$.

Proof of Lemma 3: The assumed properties of the pair $(A, B)$ imply that $m \geq n$ and that there must be a minor of order $n$ of the matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1} B \end{bmatrix}$, which is nonzero. Let, 1, 2, \ldots, $p$ be a labeling of the $nth$ order minors of $B$ $AB \cdots A^{n-1} B$ and suppose that the $kth$ such minor is nonzero. Let $\mu : \mathbb{R}^{n \times n} \oplus \mathbb{R}^{n \times r} \rightarrow \mathbb{R}$ denote the function which assigns to any matrix pair $(A_{n \times n}, B_{n \times r})$ the value of the $kth$ minor of $\begin{bmatrix} B & AB & \cdots & A^{n-1} B \end{bmatrix}$. Thus, $\mu(A, B) \neq 0$ and if $(\tilde{A}, \tilde{B})$ is a matrix pair for which $\mu(\tilde{A}, \tilde{B}) \neq 0$, then $(\tilde{A}, \tilde{B})$ is a controllable pair with controllability index no greater than $m$.

Note that $\mu(\lambda M + A, B)$ is a polynomial in the scalar variable $\lambda$. Since $\mu(\lambda M + A, B)_{|\lambda = 0} \neq 0, \mu(\lambda M + A, B)$ is not the zero polynomial. It follows that there are at most a finite number of values of $\lambda$ for which $\mu(\lambda M + A, B)$ vanishes and $\lambda = 0$ is not one of them. Let $g$ be any number for which $\mu(\lambda M + A, B) \neq 0$. Since $\mu(M + g A, g B) = g^p \mu(\lambda M + A, B)$ for some integer $j \geq 0$, it must be true that $\mu(M + g A, g B) \neq 0$ and thus $\mu(M + g A, B) \neq 0$. Therefore, $(M + g A, B)$ is a controllable pair with controllability index no greater than $m$.

Let $m_g$ denote the controllability index of $(M + g A, B)$; then $m_g \geq n$. Suppose that $m_r \geq n$. It follows that $m_r \geq m_r$ and thus that $m_g \geq m$. But for all but at most a finite set of values of $g, m_g \leq n$. Therefore, $m_g = m$ for all but at most a finite set of values of $g$.

Lemma 4: Suppose $N$ is strongly connected. For any given set of appropriately sized matrices $K_i, i \in m$, there exist matrices $H_i$ for which the matrix pair $(\tilde{A} + \sum_{i=1}^m \tilde{B}_i (K_i \tilde{C}_i + H_i \tilde{C}_i), \tilde{B}_i)$ is controllable with controllability index $m$ for every $q \in m$.

Proof of Lemma 4: As an immediate consequence of Lemmas 2 and 3, it is clear that for any given $K_i, i \in m$, and for all but a finite number of values of $g$, the matrix pair $(\tilde{A} + \sum_{i=1}^m \tilde{B}_i (K_i \tilde{C}_i + g H_i \tilde{C}_i), \tilde{B}_i)$ is controllable with controllability index $m$ for every $q \in m$. Setting $H_i = g H_i$ thus gives the desired result.

Proof of Proposition 1: Let $L$ denote the collection of all lists $\{K_i, H_i\}, i \in m$, where for each $i, (K_i, H_i)$ is a pair of matrices sized so that the products $\tilde{B}_i K_i \tilde{C}_i$ and $\tilde{B}_i H_i \tilde{C}_i$ are defined. Then $L$ is a linear space. Let $C$ denote the subset of lists in $L$ for which $(\tilde{A} + \sum_{i=1}^m \tilde{B}_i (K_i \tilde{C}_i + H_i \tilde{C}_i), \tilde{B}_i)$ is controllable with controllability index $m$ for every $q \in m$. In view of Lemma 4, $C$ is nonempty. Moreover, since the complement of $C$ in $L$ coincides with the set of solutions to a finite set of algebraic equations, $C$ is the complement of a proper algebraic set in $L$.

By Lemma 1, (16) is jointly controllable. Moreover $C$ is nonempty. Therefore, by Theorem 1 of [5], each complementary subsystem of (16) is complete.

By Lemma 1, (16) is also jointly observable. But (16) is also complete. Therefore, by Corollary 1 of [5], the set of lists $O \subset L$ for which (16) is observable through $\tilde{y}_q$ for all $q \in m$, is nonempty and thus the complement of a proper algebraic set in $L$. Since the union of proper algebraic sets in a linear space is also proper, $C \cap O$ is also the complement of a proper algebraic set in $L$. Therefore, any list in the nonempty set $C \cap O$ has the required property.

Construction: The steps involved in constructing the matrices $F_j, K_i, H_j, A, \hat{K}, H_j$, and $C$, which define the observer-based distributed control of interest, can be summarized as follows.

In [3] the symbols $f_{ij}$ and $p$ are used instead of the symbols $\phi_{ij}$ and $q$, respectively, used here.
First, state feedback matrices $F_i$, $i \in \mathbb{m}$ are chosen as explained at the beginning of Section II-C to ensure, among other things, that the state transition matrix of $A + \sum_{i \in \mathbb{m}} B_i F_i$ has a prescribed convergence rate. The remaining matrices $K_i$, $H_i$, $\bar{A}$, $\bar{K}$, $\bar{H}_i$, and $\bar{C}$ are then chosen to endow the autonomous error system (12), (13) with a prescribed convergence rate; this is done in exactly the same way as these matrices would have been chosen (as is explained at the end of Section II-B), had the objective been to provide the error system (5), (6) with the same convergence rate—except of course, in the current situation one would use $A + \sum_{i \in \mathbb{m}} B_i F_i$ in the calculations rather than just $A$.

III. DISTRIBUTED SET-POINT CONTROL

Although the substitution rule discussed in Section II can handle candidate control inputs of the forms $u_i = F_i x$, $i \in \mathbb{m}$, the rule is of no help in dealing with exogenous reference signals $r_i$ which might appear as additive terms in the candidate control signals; i.e., $u_i = F_i x + r_i$, $i \in \mathbb{m}$. Nonetheless certain classes of exogenous reference signals, such as steps, ramps, and sinusoids, can in fact be dealt with without violating distributional constraints, if the goal is signal tracking. The aim of this section is to illustrate this by explaining how the ideas discussed in the preceding section can be used to solve the “distributed set-point control problem.” This problem will be formulated assuming that each agent $i$ senses a scalar output $y_i = c_i x$ with the goal of adjusting $y_i$ to a prescribed number $r_i$, which is agent $i$’s desired set-point value. The distributed set-point control problem is then to develop a distributed feedback control system for a process modeled by the multichannel system (1) which, when applied, will enable each and every agent to independently adjust its output to any desired set-point value.

To construct such a control system, each agent $i$ will make use of integrator dynamics of the form

$$\dot{w}_i = y_i - r_i, \quad i \in \mathbb{m} \quad (17)$$

where, $r_i$ is the desired (constant) value to which $y_i$ is to be set. The combination of these integrator equations plus the multichannel system described by (1), is thus a system of the form

$$\dot{x} = \bar{A} x + \sum_{i=1}^{m} \bar{B}_i u_i - \bar{r}, \quad w_i = c_i x, \quad i \in \mathbb{m} \quad (18)$$

where, $\bar{c} = \text{column}\{x, w_1, w_2, \ldots, w_m\}$.

$$\bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad i \in \mathbb{m}, \quad \bar{r} = \begin{bmatrix} 0 \\ r \end{bmatrix}.$$  

$C = \text{column}\{c_1, c_2, \ldots, c_m\}_{m \times n}$, $r = \text{column}\{r_1, r_2, \ldots, r_m\}$, and $\bar{c}_i = \begin{bmatrix} 0 & e_i \end{bmatrix}$, $e_i$ being the $i$th unit vector in $\mathbb{R}^m$.

Thus, (18) is an $(n + m)$-dimensional, $m$-channel system with measurable outputs $w_i$, $i \in \mathbb{m}$, control inputs $u_i$, $i \in \mathbb{m}$, and constant exogenous input $\bar{r}$. Note that any linear time-invariant feedback control, distributed or not, which stabilizes this system, will enable each agent to attain its desired set-point value. The reason for this is simple. First note that any such control will bound the state of the resulting closed-loop system and cause the state to tend to a constant limit as $t \rightarrow \infty$. Therefore, since each $w_i$ is a state variable, each must tend to a finite limit. Similarly, $x$ and thus each $y_i$ must also tend to a finite limit. In view of (17), the only way this can happen is if each $y_i$ tends to agent $i$’s desired set-point value $r_i$.

To solve the distributed set-point control problem, it is enough to devise a distributed controller, which stabilizes (18). This can be accomplished using the ideas discussed earlier in Section II, provided that (18) is both jointly controllable by the $u_i$ and jointly observable through the $w_i$. According to Hautus’ lemma [24], the condition for joint observability is that

$$\text{rank} \begin{bmatrix} sI - \bar{A} \\ \bar{C} \end{bmatrix} = n + m$$

for all complex number $s$, where $\bar{C} = \text{column}\{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m\}$. In other words, what is required is that

$$\text{rank} \begin{bmatrix} sI_n - A & 0 \\ -C & sI_m \\ 0 & I_m \end{bmatrix} = n + m. \quad (19)$$

But $(C, A)$ is an observable pair because (1) is a jointly observable system. From this, the Hautus condition, and the structure of the matrix pencil appearing in (19), it is clear that the required rank condition is satisfied and thus that (18) is a jointly observable system.

To establish joint controllability of (18), it is enough to show that

$$\text{rank} \begin{bmatrix} sI_n - \bar{A} & \bar{B} \\ 0 & I_m \end{bmatrix} = n + m$$

for all complex number $s$, where $\bar{B} = [\bar{B}_1 \bar{B}_2 \cdots \bar{B}_m]$. In other words, what is required is that

$$\text{rank} \begin{bmatrix} sI_n - A & 0 & B_1 & B_2 & \cdots & B_m \\ -C & sI_m & 0 & 0 & \cdots & 0 \end{bmatrix} = n + m. \quad (20)$$

But since (1) is jointly controllable, $\text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} = n$ for all $s$, where $B = [B_1 \ B_2 \cdots \ B_m]$. Thus, (20) holds for all $s \neq 0$. For $s = 0$, (20) will also hold provided

$$\text{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + m. \quad (21)$$

That is, (21) is the condition for (18) to be jointly controllable and thus stabilizable with distributed control.

It is possible to give a simple interpretation of condition (21) for the case when each $\bar{B}_i$ is a single column. In this case the transfer matrix $C(sI - A)^{-1}B$ is square and condition (21) is equivalent to the requirement that its determinant has no zeros at $s = 0$ [25]. Note that if the transfer matrix were nonsingular but had a zero at $s = 0$, this would lead to a pole-zero cancellation at $s = 0$ because of the integrators.

Suppose condition (21) is satisfied. In order to stabilize (18), the first step would be to construct a distributed observer-based
control as outlined in Section II, which stabilizes the reference-signal-free system
\[
\dot{x} = \tilde{A}x + \sum_{i=1}^{m} \tilde{B}_i u_i, \quad w_i = \tilde{c}_i \dot{x}, \quad i \in m.
\]

Application of such a distributed control to (18) would then stabilize (18) and thus provide a solution to the distributed set-point control problem, despite the fact that the signals \(x_i\) would not be asymptotically correct estimates of \(\dot{x}\).

IV. DISTRIBUTED CONTROL RECAST AS DECENTRALIZED CONTROL

The main result of [6] states that every jointly controllable, jointly observable continuous-time multichannel linear system with an associated strongly connected neighborhood graph, can be exponentially stabilized with a distributed dynamic compensator. Meanwhile results of Section II make it clear that under the same conditions, exponentially stabilized with a prescribed convergence rate can be achieved with a distributed observer-based control. These findings prompt a number of questions. For example, is exponential stabilization still possible if there is no message passing across the network? Is exponential stabilization at a prescribed convergence rate possible with message passing but without a distributed observer? If yes, what messages should be passed among neighboring agents? Under what conditions can distributed control be used to freely assign the closed-loop spectrum of a multichannel linear system without the spectral separation restriction imposed by certainty equivalence observer-based control? These are some of the questions to which this section is addressed.

The first question just raised, namely “Is exponential stabilization still possible if there is no message passing across the network?” actually is a question in the area of “decentralized control.” Accordingly, some of the main results from classical decentralized control will be reviewed next.

A. Decentralized Control

The basic distributed control problem formulated in Section I-A, but without message passing, is precisely the classical decentralized control problem studied years ago in [4], [5]. Thus, the classical decentralized control problem is to find \(m\) linear time-invariant controllers, one for each channel of (1), which stabilize the resulting closed-loop system with any prescribed convergence rate. Much is known about this problem. Perhaps the most fundamental is the fact that the closed-loop spectrum, which results when time-invariant linear controllers are applied to the channels of (1), contains a uniquely determined subset of eigenvalues (or “fixed modes” [4]), called the fixed spectrum of (1), which is the same for all possible linear, time-invariant channel controllers which might be applied. This subset, denoted by \(\Lambda_{\text{fixed}}\), is defined by the formula

\[
\Lambda_{\text{fixed}} = \bigcap_{F_i \in \mathbb{R}^{n \times n}, i \in m} \sigma \left( A + \sum_{i=1}^{m} B_i F_i C_i \right)
\]

where, \(\sigma(\cdot)\) denotes the spectrum of a matrix. Thus, \(\Lambda_{\text{fixed}}\) is the set of eigenvalues of the matrix \(A + B_1 F_1 C_1 + B_2 F_2 C_2 + \cdots + B_m F_m C_m\) which do not change as the real-valued matrices \(F_1, F_2, \ldots, F_m\) range over all possible values. It is shown in [4] that the classical decentralized control problem for (1) is solvable if and only if \(\Lambda_{\text{fixed}}\) is an empty set. Identical definitions and analogous conclusions hold for the discrete-time multichannel linear system defined by (2).

There is a more explicit characterization of the elements in \(\Lambda_{\text{fixed}}\) [26]. To explain what it is, use will be made of the following notation. For each subset \(s = \{i_1, i_2, \ldots, i_s\} \subseteq m\) whose elements are ordered as \(i_1 < i_2 < \ldots < i_s\), let \(B_s\) and \(C_s\) denote the matrices

\[
B_s = [B_{i_1} \ B_{i_2} \ \cdots \ B_{i_s}] \quad \text{and} \quad C_s = [C'_{i_1} \ C'_{i_2} \ \cdots \ C'_{i_s}]
\]

respectively. A proof of the following result can be found in [26].

**Proposition 2:** Let \(A, B_i,\) and \(C_i\), \(i \in m\), be the coefficient matrices of the \(m\)-channel linear system defined by either (1) or (2). A complex number \(\lambda \in \Lambda_{\text{fixed}}\) if and only if for some nonempty, proper subset \(s \subseteq m\)

\[
\text{rank} \left[ \begin{array}{cc} \lambda I_n - A & B_{m-s} \\ C_s & 0 \end{array} \right] < n \tag{22}
\]

where, \(m - s\) is the complement of \(s\) in \(m\).

Obviously the condition for (1) or (2) to have no fixed eigenvalues is that the rank of the matrix pencil in (22) is greater than or equal to \(n\) for all \(\lambda \in \sigma(A)\) and all proper subsets \(s \subseteq m\). Use will be made of this characterization of a multichannel system’s fixed eigenvalues at several different places in this article.

Also of importance in decentralized control is the concept of the “transfer graph” of a multichannel linear system. By the **transfer graph** of an \(m\)-channel linear system (1) or (2) is meant the directed graph with \(m\) vertices which has an arc from vertex \(j\) to vertex \(i\) whenever the transfer matrix \(C_i(sI - A)^{-1} B_j \neq 0\) [5]. It is known that if multichannel linear system (1) or (2) has both a strongly connected transfer graph and no fixed eigenvalues (i.e., \(\Lambda_{\text{fixed}}\) is an empty set), then there exist matrices \(F_i, i \in m\), of appropriate sizes such that the closed-loop system, which results under application of the decentralized feedback laws \(u_i = F_i y_i + v_i, i \in m\), is controllable by any one input \(v_j\) and observable through any one output \(y_k\) [5]. Thus, with a suitably defined feedback controller from \(y_i\) to \(v_i\) for any one channel index \(i \in m\), the resulting closed-loop spectrum of the multichannel system under consideration can be freely assigned; such a controller can be crafted using standard techniques, such as those found in [21]. If, on the other hand, (1) or (2) has no fixed eigenvalues and its transfer graph is not strongly connected, “free” spectrum assignment can only be achieved with certain constraints on the symmetry of subsets of the spectrum. These constraints are that the closed-loop spectrum must be partitioned into \(\eta\) symmetric sets of complex numbers, where \(\eta\) is the number of strongly connected components in the system’s transfer graph, and that the cardinality of each symmetric set must

\[\text{Note that the rank of the matrix pencil in (22) is } n \text{ if } s \text{ is either empty or equal to } m \text{ because of joint controllability and joint observability, respectively, of either (1) or (2).}\]
equal the dimension of the corresponding closed-loop “strongly connected subsystem” [5].

B. Transmission of Just Measured Outputs Across a Network is Not Sufficient for Stabilization

The aim of this section is to demonstrate that stabilization of a multichannel linear system with distributed control can not necessarily be achieved if the only signal, which each agent transmits to its followers is its measured output. The following example illustrates this.

Example: Consider the $m = 3$ channel, jointly controllable, jointly observable linear system (1) with coefficient matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

and the cyclic neighbor graph; i.e., $\mathcal{N}_1 = \{1, 2\}$, $\mathcal{N}_2 = \{2, 3\}$, and $\mathcal{N}_3 = \{3, 1\}$. Suppose each agent $i$ receives only the measured output $y_j$ of neighbor $j \in \mathcal{N}_i$ and nothing more. Then for $i \in \mathbf{m}$, the signal agent $i$ can input to its channel controller is the augmented output $\bar{y}_i = \text{column}\{y_i, y_j\}$. Clearly this distributed control problem is the same as the classical decentralized control problem for the three-channel system

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i, \quad \bar{y}_i = C \nu_i, \quad i \in \mathbf{m} \tag{23}$$

where, $C \nu_i = \text{column}\{C_1, C_2, C_3\}$, $C \nu_i = \text{column}\{C_2, C_3\}$, and $C \nu_i = \text{column}\{C_3, C_1\}$. It is easy to check that

$$\text{rank} \begin{bmatrix} I - A & B_1 & B_3 \\ C_{\mathcal{N}_2} & 0 & 0 \end{bmatrix} = 2.$$ 

Thus, by Proposition 2, (23) has a fixed eigenvalue at $\lambda = 1$. Therefore, (23) cannot be stabilized by decentralized control, which means that (1) cannot be stabilized by distributed control in which the only signal each agent $i$ receives from its neighbor $j$ is $y_j$.

What this example illustrates is something a little counter intuitive, namely that for distributed stabilization to be possible it is not in general enough for each agent to share just its measured output with its followers. Meanwhile, the results of Section II-B clearly demonstrate that to achieve stabilization with distributed control it is enough for each agent $i$ to share its controller state with all of its followers, at least when that state is the state $x_i$ of agent $i$’s local state estimator. In the sequel, we expand on this observation, while being mindful of the limitation the preceding example illustrates.

C. Extended System

Examination of the observer-based agent controllers described by (10) and (11) shows that all agents employ state vectors $x_i$, which are shared across the network and, in addition, one agent uses an additional vector $z$, which is not shared. Prompted by this, we now consider a configuration in which each agent’s controller state consists of two components, namely a substate $x_i \in \mathbb{R}^{n_i}$, which agent $i$ communicates to its followers, and an additional substate $z_i \in \mathbb{R}^{k_i}$, which is not shared across the network. From this perspective, agent $i$’s controller is an $(n_i + k_i)$-dimensional linear system with output $u_i$, state column $\{x_i, z_i\}$, and inputs consisting of the measured output $y_j$ and shared substate $x_j$ of each of its neighbors $j \in \mathcal{N}_i$. The state dynamics for each $x_i$ and $z_i$ are thus of the forms

$$\dot{x}_i = v_i, \quad i \in \mathbf{m} \tag{24}$$

and $\dot{z}_i = \nu_i$ for $i \in \mathbf{m}$, respectively, where $v_i$ and $\nu_i$ are appropriately chosen linear functions of $z_i$ and the signals $y_j$, $x_j$ for all $j \in \mathcal{N}_i$. It will become clear soon that the original $m$-channel system (1) and each $x_i$’s dynamics (24) can be combined into a new open-loop $m$-channel system called the “extended system,” which incorporates the message passing required by distributed control. Then the basic distributed control problem for the original system (1) can be recast as a decentralized control problem for the extended system defined in the following. To effectively control the extended system, a decentralized dynamic output feedback controller may be needed at each channel. From this point of view, each $z_i$ is the state of the decentralized dynamic controller for channel $i$ of the extended system, that is why $z_i$ is not shared across the network. Note that it is possible that the decentralized output feedback for some channel $i$ of the extended system is static, in that case $k_i = 0$, so there will be no $z_i$ in agent $i$’s controller.

There is a convenient way to describe this configuration. Toward this end, let $\bar{x}$ denote the extended state

$$\bar{x} = \text{column}\{x, x_1, \ldots, x_m\}$$

and for $i \in \mathbf{m}$, define

$$\bar{u}_i = \text{column}\{u_i, v_i\}$$

$$\bar{y}_i = \text{column}\{y_i, y_1, \ldots, y_{i-1}, x_1, x_2, \ldots, x_{i-1}\}$$

where $\{i_1, i_2, \ldots, i_m\}$ is the set of labels of agent $i$’s neighbors in $\mathcal{N}_i$ arranged in ascending order. By the extended system determined by (1) and (24) is meant the $\bar{n}$-dimensional, $m$-channel linear system

$$\dot{\bar{x}} = \bar{A}\bar{x} + \sum_{i=1}^{m} \bar{B}_i \bar{u}_i, \quad \bar{y}_i = \bar{C}_i \bar{x}, \quad i \in \mathbf{m} \tag{25}$$

where

$$\bar{A} = \text{diagonal} \{A, 0_{(\bar{n}-n) \times (\bar{n}-n)}\}$$

$$\bar{B}_i = \text{diagonal} \{B_i, E_i\}, \quad i \in \mathbf{m}$$

$$\bar{C}_i = \text{diagonal} \{C_{\mathcal{N}_i}, E_{\mathcal{N}_i}\}, \quad i \in \mathbf{m}$$

where $E_i = \text{column} \{0_{g^+_i \times n_i}, I_{n_i \times n_i}, 0_{g_i^+ \times n_i}\}$

and $g^+_i = n_1 + \cdots + n_{i-1}$ and $g^+_i = n_{i+1} + \cdots + n_m$. It is straightforward to verify that this system is jointly controllable and jointly observable regardless of the connectivity of $\mathcal{N}$. 

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D. Distributed Control Problem Restated

With the concept of an extended system at hand, it is possible to restate the basic distributed control problem for (1) posed at the beginning of Section I-A in the following way.

Basic Distributed Control Problem—Restated: For the m-channel extended system described by (25), develop a systematic procedure for picking integers \( n_i, i \in \mathbb{m} \), and constructing m time-invariant feedback controllers with \( \hat{y}_i \) and \( u_i \) the input to and output from controller \( i \), respectively, so that the state of the resulting closed-loop system converges to zero exponentially fast at a preassigned rate.

It is obvious that the originally stated basic distributed control problem and its restated version are entirely equivalent. One virtue of the revised formulation is that it enables one to easily address the basic distributed control problem without being compelled to use a distributed observer. The reason for this stems from the fact that for fixed \( n_i \), the restated version is mathematically a decentralized control problem. Thus deciding whether or not the distributed control problem for (1) can be solved amounts to deciding what must be true of the \( n_i \) for the extended system (25) to have no fixed eigenvalues. In the sequel this will be done assuming \( N \) is strongly connected. In light of the results of Section II and Proposition 2, it is reasonable to expect that when the dimensions \( n_i \) of all local controller's shared substates are large enough to compensate for the rank deficiency induced by the fixed eigenvalues of (1), the corresponding extended system (25) will have no fixed eigenvalues. This is indeed the case.

**Theorem 2:** Suppose that \( N \) is strongly connected and that

\[
n_i \geq n - \min_{s \in \mathbb{m}} \text{rank}_{\lambda \in \sigma(A)} \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_{N_s} & 0 \end{bmatrix}, \quad i \in \mathbb{m}
\]  

(26)

where, \( \mathbb{m} \) is the set of all nonempty proper subsets of \( \mathbb{m} \). Then the extended system defined by (25) has no fixed eigenvalues.

Theorem 2 thus gives a lower bound on the dimensions of the local controllers' shared substates needed for the extended system (25) to have no fixed eigenvalues. It is easy to check that in the special case when (1) has no fixed eigenvalues, this lower bound is zero as it should be. Another observation is that the lower bound is reached for every \( i \in \mathbb{m} \) if and only if for each \( i \in \mathbb{m} \), there exists a subset \( s \subset \mathbb{m} \) such that \( N_s \cap (\mathbb{m} - s) = \{i\} \) and

\[
\text{rank}_{\lambda \in \sigma(A)} \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_{N_s} & 0 \end{bmatrix} = \min_{s \in \mathbb{m}} \text{rank}_{\lambda \in \sigma(A)} \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_{N_s} & 0 \end{bmatrix}.
\]

Whether or not this lower bound is tight remains to be seen.

**Remark 2:** It is straightforward to verify that if the matrices \( \tilde{C}_i = \text{diagonal}\{C_{N_s}, E'_{N_i}\} \) appearing in the definition of the extended system in (25), are replaced with the matrices

\[
\tilde{C}_i \text{modified} = \text{diagonal}\{C_i, E'_{N_i}\}
\]

the resulting modified extended system will still be jointly observable. Moreover, as the proof of Theorem 2 in the following readily reveals, if hypothesis (26) in the statement of Theorem 2 is replaced with the hypothesis

\[
n_i \geq n - \min_{s \in \mathbb{m}} \text{rank}_{\lambda \in \sigma(A)} \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_{s} & 0 \end{bmatrix}, \quad i \in \mathbb{m}
\]  

(27)

the modified extended system will have no fixed eigenvalues. While (27) demands more of the \( n_i \) than (26) does, because

\[
\text{rank}_{\lambda \in \sigma(A)} \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_{N_s} & 0 \end{bmatrix} \geq \text{rank}_{\lambda \in \sigma(A)} \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_s & 0 \end{bmatrix}
\]

the virtue of this modification is that only the signals \( x_i \) and not the signals \( y_i \) need be transmitted across the network.

**Proof of Theorem 2:** Suppose that (26) holds. Fix \( s \in \mathbb{m} \) and \( \lambda \in \sigma(A) \). Let

\[
\tilde{M} = \begin{bmatrix} \lambda I - \tilde{A} & B_{m-s} \\ \tilde{C}_s & 0 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} \lambda I - A & B_{m-s} \\ C_{N_s} & 0 \end{bmatrix}.
\]

In view of the structure of the coefficient matrices defining (25)

\[
\text{rank} \tilde{M} = \text{rank} M + \text{rank} E_{m-s} + \text{rank} E_{N_s},
\]

(28)

Clearly

\[
\text{rank} E_{m-s} = \sum_{i \in \mathbb{m} - s} \text{rank} E_i = \sum_{i \in \mathbb{m} - s} n_i
\]

(29)

and

\[
\text{rank} E_{N_s} = \sum_{j \in N_i} \sum_{i \in s} \text{rank} E_j = \sum_{j \in N_i} \sum_{i \in s} n_j
\]

(30)

By hypothesis, \( N \) is strongly connected. This implies that \( N_s \cap (\mathbb{m} - s) \) is nonempty so for some \( k \in \mathbb{m} - s \), there must be a \( j \in s \) such that \( k \in N_j \); clearly \( k \neq j \). But \( i \in N_i \), \( i \in s \). It follows that:

\[
\sum_{i \in \mathbb{m}} \sum_{j \in N_i} n_j \geq \sum_{i \in \mathbb{m}} n_i + n_k.
\]

From this, and (28)–(30) it follows that:

\[
\text{rank} \tilde{M} \geq \text{rank} M + \sum_{i \in \mathbb{m}} n_i + n_k.
\]

But from (26)

\[
n_k \geq n - \text{rank} M
\]

so

\[
\text{rank} \tilde{M} \geq n + \sum_{i \in \mathbb{m}} n_i = \bar{n}.
\]

Since this holds for all \( s \in \mathbb{m} \) and all \( \lambda \in \sigma(A) \), the hypothesis of Proposition 2 is satisfied, so the extended multichannel system (25) has no fixed eigenvalues.

Note that the only place in the proof of Theorem 2 where strong connectivity of \( N \) is used is to ensure that \( N_s \cap (\mathbb{m} - s) \) is a nonempty set. It is easy to verify that \( N \) is strongly connected if and only if \( N_s \cap (\mathbb{m} - s) \) is nonempty for all \( s \in \mathbb{m} \).

The example in Section IV-B and Theorem 2 imply that to avoid the fixed eigenvalues, it is necessary and sufficient to transmit shared substates of appropriate dimensions of the local controllers.
E. Free Spectrum Assignability

The results of Section II make it clear that the closed-loop spectrum of a multichannel linear system can be freely assigned with distributed observer-based control provided the spectrum consists of the disjoint union of two symmetric sets, one for the distributed observer and the other for the closed-loop process under state feedback. It will now be explained how this partitioning constraint can be avoided.

As discussed at the end of Section IV-A and established in [5], the conditions for free spectrum assignability with decentralized control are that the multichannel system to be controlled must have a strongly connected transfer graph and no fixed eigenvalues. To make use of this result here, it is necessary to relate the transfer graph of the extended system (25) to the transfer graph of the original system (1). The following proposition, which utilizes the concept of the “union” of two directed graphs,\(^3\) accomplishes this.

Proposition 3: Suppose that \( n_i > 0 \) for \( i \in \mathbb{m} \), that \( \mathbb{N} \) and \( \mathbb{G} \) are the neighbor and transfer graphs, respectively, of the original system (1), and that \( \mathbb{G} \) is the transfer graph of the extended system (25). Then, \( \mathbb{N} \cup \mathbb{G} \subset \mathbb{G} \).

Proof of Proposition 3: Suppose, \((j, i)\) is an arc in \( \mathbb{N} \cup \mathbb{G} \) in which case \((j, i)\) is an arc in \( \mathbb{N} \) or an arc in \( \mathbb{G} \). If \((j, i)\) is an arc in \( \mathbb{N} \), then \( j \in \mathbb{N}_i \), so \( E_{\mathbb{N}} \neq 0 \). On the other hand, if \((j, i)\) is an arc in \( \mathbb{G} \), then \( C_{\mathbb{G}}(\lambda I - A)^{-1} B_j \neq 0 \), so \( C_{\mathbb{G}}(\lambda I - A)^{-1} B_j \neq 0 \). Thus, the definition of the coefficient matrices defining (25). Therefore, \((j, i)\) is an arc in \( \mathbb{G} \).

The proposition implies that if the neighbor graph \( \mathbb{N} \) is strongly connected, then so is the transfer graph of (25). We are led to the following corollary to Theorem 2.

Corollary 1: Under the hypotheses of Theorem 2, the closed-loop spectrum of (1) can be freely assigned with distributed control.

More specifically, under the hypotheses of Theorem 2, there are output feedback laws\(^4\) \( \bar{u}_i = \bar{F}_i \bar{y}_i \), \( i \in \mathbb{m} \), such that the system

\[
\dot{x} = (\bar{A} + \sum_{i=1}^{m} \bar{B}_i \bar{F}_i \bar{C}_i) x + \bar{B}_q \bar{u}_q, \quad \bar{y}_q = \bar{C}_q \bar{x}
\]

(31)
is controllable and observable through any channel \( q \in \mathbb{m} \) [5]. Suppose the minimum of the controllability index and the observability index of (31) is \( k_q + 1 \). Then, with suitable matrices \( \bar{A}, \bar{B}, \bar{C}, \) and \( \bar{D} \), the closed-loop system consisting of (31) and agent \( q \)'s channel controller \( z_q \in \mathbb{R}^{k_q} \), which is not shared on the network

\[
\dot{z}_q = \bar{A} z_q + \bar{B} \bar{y}_q, \quad \bar{u}_q = \bar{C} z_q + \bar{D} \bar{y}_q
\]
can be exponentially stabilized at any prescribed convergence rate using a linear time-invariant control [21]. In this way, the total dimension of distributed controllers for the original system (1) is \( k_q + \sum_{i=1}^{m} n_i \).

While the results derived thus far mean that the basic distributed control problem can be addressed with the tools from classical decentralized control theory, other yet to be developed design techniques may possibly also be applied.

V. Transmission Delays

The aim of this section is to broaden the distributed feedback problem to account for possible transmission delays across the network. To avoid the technical challenges associated with delay differential equations, the investigation is restricted exclusively to discrete-time systems.

In Section V-A, it will be shown that the techniques discussed so far in Section IV can be used to obtain an extended system whose only fixed eigenvalues are at 0. Fixed eigenvalues at 0 do not affect convergence rate for discrete-time systems. Having completed this discussion, attention will be turned to the problem of achieving completely free spectrum assignment. This will be done in Section V-B by developing a new extended system, which has no fixed eigenvalues at all, not even eigenvalues at zero. To accomplish this, two modifications of the approach used in Section V-A are needed. The first is to transmit just the signals \( x_i \) and not the signals \( y_i \); the second is to introduce “holding,” which comes with the price of increasing the dimensions of local controllers.

A. Controlled Convergence Rate

Prompted by the control structure studied in the continuous-time case, we consider a controller for the discrete-time multichannel system described by (2) in which each agent's controller state consists of two components, namely a substate \( x_i \in \mathbb{R}^{m_i} \) which agent \( i \) communicates to its follower with possible delay, and an additional substate \( z_i \in \mathbb{R}^{k_i} \) which is not shared across the network. Agent \( i \)'s controller is a \((n_i + k_i)\)-dimensional linear system with output \( u_i \), state column \( \{x_i, z_i\} \), and inputs consisting of delayed versions of the measured output \( y_j \) and shared substate \( x_j \) of each of its neighbors \( j \in N_i \). We assume that each pair \((y_j, x_j)\) undergoes a delay of \( d_{ij} \) integer-valued time units when transmitted from agent \( j \) to follower \( i \); it is assumed that \( d_{ii} = 0 \) for \( i \in \mathbb{m} \). Thus, the signals agent \( i \) receives from its neighbors are the pairs \((y_j(t - d_{ij}), x_j(t - d_{ij})) \), \( j \in N_i \). The state dynamics for each \( x_i \) and \( z_i \) are of the forms

\[
x_i(t + 1) = v_i(t), \quad i \in \mathbb{m}
\]

(32)

and \( z_i(t + 1) = v_i(t) \), respectively, where \( v_i \) and \( v_i \) are appropriately chosen linear functions of \( z_i \) and the signals \( y_j(t - d_{ij}), x_j(t - d_{ij}) \) for all \( j \in N_i \).

To model the delayed signals within the framework of finite-dimensional linear systems, it is necessary to use lifting. Toward this end, for each \( j \in \mathbb{m} \), let \( \mathcal{F}_j \) denote the set of labels of agent \( j \)'s followers and let

\[
d_j = \max_{i \in \mathcal{F}_j} d_{ij}.
\]

\(^3\)By the union of two directed graphs \( \mathbb{G}_1 \) and \( \mathbb{G}_2 \) with the same vertex set \( \mathcal{V} \) is meant the graph \( \mathbb{G}_1 \cup \mathbb{G}_2 \) with vertex set \( \mathcal{V} \) whose arc set is the union of the arc sets of \( \mathbb{G}_1 \) and \( \mathbb{G}_2 \).

\(^4\)Almost all matrices \( \bar{F}_i \), \( i \in \mathbb{m} \), of appropriate sizes will do [5].
For any \( j \in m \) for which \( d_j > 0 \), the \( d_j(n_j + q_j) \)-dimensional linear system\(^5\) is called a lift, which is described by the equations

\[
\begin{align*}
    w_j(t + 1) &= L_j w_j(t) + H_j \begin{bmatrix} x_j(t) \\ y_j(t) \end{bmatrix} \\
\end{align*}
\]

where

\[
L_j = \text{diagonal} \{ S_{d_j} \otimes I_{n_j}, S_{d_j} \otimes I_{q_j} \}
\]

\[
H_j = \text{diagonal} \{ e_{d_j}^T \otimes I_{n_j}, e_{d_j}^T \otimes I_{q_j} \}
\]

and for any positive integer \( d \), \( S \) is the \( d \)-unit shift matrix

\[
S_d = \begin{bmatrix} 0 & 0 \\ I_{d-1} & 0 \end{bmatrix}_{d \times d}
\]

Here, \( e_d^i \) is the \( i \)th unit vector in \( \mathbb{R}^d \). It is easy to see that if \( w_j \) is properly initialized

\[
\begin{bmatrix} x_j(t - d_j) \\ y_j(t - d_j) \end{bmatrix} = C_{ij} w_j(t) + D_{ij} \begin{bmatrix} x_j(t) \\ y_j(t) \end{bmatrix}, \quad i \in F_j
\]

where

\[
C_{ij} = \text{diagonal} \left( (e_{d_j}^T)^{\otimes I_{n_j}}, ((e_{d_j}^T)^{\otimes I_{q_j}}) \right)
\]

\[
D_{ij} = 0
\]

if \( d_{ij} > 0 \), and \( C_{ij} = 0 \) and \( D_{ij} = I_{n_j} + q_j \) if \( d_{ij} = 0 \). Note that the lift described by (33) is introduced for modeling purposes only; it is not implemented as a component subsystem of agent \( j \)'s controller.

In contrast to the definition of an extended system used in Section IV, in the present setting, the extended system must not model only the original process (2) and the shared state dynamics (32), but also all of the lifts described by (33) for all \( j \) for which \( d_j > 0 \). Toward this end, let \( \tilde{x} \) denote the extended state

\[
\tilde{x} = \text{column} \{ x, x_1, w_1, x_2, w_2, \ldots, x_m, w_m \}
\]

and for \( i \in m \), define

\[
\begin{align*}
    \tilde{u}_i(t) &= \text{column} \{ u_i(t), v_i(t) \} \\
    \tilde{y}_i(t) &= \text{column} \{ y_{ij}(t - d_{ij}), y_{ij}(t - d_{ij}), \ldots, y_{ijm_i}(t - d_{ijm_i}), x_j(t - d_{ij}), \ldots, x_{jnm_i}(t - d_{ijm_i}) \}
\end{align*}
\]

where, \( \{ i_1, i_2, \ldots, i_m \} = N_i \). By the extended system determined by (2), (32), and (33) is meant the \( \tilde{n} \)-dimensional, \( m \)-channel linear system

\[
\tilde{x}(t + 1) = \tilde{A} \tilde{x}(t) + \sum_{i=1}^{m} \tilde{B}_i \tilde{u}_i(t)
\]

\[
\tilde{y}_i(t) = \tilde{C}_i \tilde{x}(t), \quad i \in m, \quad t \in \{ 0, 1, 2, \ldots \}
\]

where, \( \tilde{n} = n + \sum_{i=1}^{m} (n_i + d_i n_i + d_i q_i) \)

\[
\tilde{A} = \begin{bmatrix} A & 0 \\ G & K \end{bmatrix}
\]

\[
\tilde{B}_i = \text{diagonal} \{ B_i, H_{i0} \}, \quad i \in m
\]

\[
\tilde{C}_i = \text{diagonal} \{ C_{N_i}, \text{column} \{ Q_{N_i}, H_{N_i} \} \}, \quad i \in m
\]

where

\[
G = \text{column} \{ G_1, G_2, \ldots, G_m \}
\]

\[
G_i = \text{column} \{ 0_{(d_i+1)n_i \times n_i}, C_i, 0_{(d_i-1)q_i \times n_i} \}
\]

\[
K = \text{diagonal} \{ S_{d_i+1} \otimes I_{n_i}, S_{d_i} \otimes I_{q_i}, S_{d_i+1} \otimes I_{n_i}, S_{d_i} \otimes I_{q_i} \}
\]

\[
H_{i0} = \text{column} \{ 0_{r_i \times n_i}, I_{n_i}, 0_{r_i \times q_i} \}, \quad 0 \leq \delta \leq d_i
\]

where, \( \delta n_i = \sum_{j=1}^{i} (n_j + d_j n_j + d_j q_j) \), \( \delta n_i = (d_i - \delta) n_i + d_i q_i \), \( \delta n_i + d_i q_i + \sum_{j=i+1}^{m} (n_j + d_j n_j + d_j q_j) \); for \( N_i = \{ j_1, j_2, \ldots, j_m \} \)

\[
H_{N_i} = \begin{bmatrix} H_{j_1d_{ij_1}} & H_{j_2d_{ij_2}} & \cdots & H_{j_md_{ij_m}} \end{bmatrix}
\]

Finally, \( N_{0+}, N_{i+} \subset N_i \) so that neighbor \( j \in N_{0+} \) if \( d_{ij} = 0 \) and \( j \in N_{i+} \) if \( d_{ij} > 0 \). Clearly, \( N_{0+} \cup N_{i+} = N_i \). Let

\[
Q_{i} = \begin{bmatrix} 0_{q_i \times \alpha_i} & I_{q_i} \end{bmatrix}, \quad 1 \leq \delta \leq d_i
\]

where, \( \alpha_i = (d_i + 1)n_i + (\delta - 1)q_i + \sum_{j=1}^{i-1} (n_j + d_j n_j + d_j q_j) \), \( \alpha_i = (d_i - \delta) q_i + \sum_{j=i+1}^{m} (n_j + d_j n_j + d_j q_j) \). Then for \( N_{i+} = \{ j_1, j_2, \ldots, j_n \} \)

\[
Q_{N_{i+}} = \text{column} \{ Q_{j_1d_{ij_1}}, Q_{j_2d_{ij_2}}, \ldots, Q_{j_{n-1}d_{ij_{n-1}}} \}
\]

Similar to Proposition 3, it is not hard to show that when \( n_i > 0 \) for all \( i \in m \), the union of the neighbor and transfer graphs of the original system (2) is a subgraph of the transfer graph of the abovementioned extended system (34). The following theorem asserts that in the face of transmission delays, the extended system does not have any nonzero fixed eigenvalues. By Theorem 4 of [5], apart from the fixed eigenvalues at 0, the rest of the closed-loop spectrum of the extended system can be freely assigned through any channel under the hypotheses of Theorem 3. Since the convergence rate for a discrete-time system is determined only by the system’s spectral radius, the implication of the theorem (see Corollary 2) is that the closed-loop extended system (and thus the original system with delays) can be exponentially stabilized at any prescribed convergence rate using distributed controllers, even though the system may have fixed eigenvalues at 0.

**Theorem 5:** Suppose that \( N \) is strongly connected, that the transmission delays across \( N \) are finite, and that the lower bound \( (26) \) on \( n_i \) holds for \( i \in m \). Then the extended system defined by (34) has no fixed, nonzero eigenvalues.

---

\(^5\) Here, the state \( w_j(t) = \text{column} \{ x_j(t-1), x_j(t-2), \ldots, x_j(t-d_j), y_j(t-1), y_j(t-2), \ldots, y_j(t-d_j) \} \).
The discussion above Theorem 3 is summarized into the corollary as follows.

**Corollary 2.** Under the hypotheses of Theorem 3, the closed-loop original system (2) can be exponentially stabilized at any prescribed convergence rate with distributed control.

**Proof of Theorem 3.** For simplicity, let $e_i$ denote the $i$th unit vector with 1 in its $i$th entry and 0 in all other entries, in which case the dimension of the space will become clear from the context. Let

$$M_k \triangleq \begin{bmatrix}
-1 & \lambda & 0 & \ldots & 0 \\
0 & -1 & \lambda & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & -1 & \lambda \\
\end{bmatrix}_{k \times (k+1)}$$

where $\lambda \neq 0$. It is claimed that

$$\text{rank} \left[ M_k \right]_{(k+1) \times (k+1)} = k + 1, \quad i \in \{1, 2, \ldots, k + 1\}.$$  (36)

So with proper elementary row and column operations, matrix

$$\begin{bmatrix}
-C_{q \times n} & \lambda e_i' \otimes I_q \\
0 & M_k \otimes I_q \\
0 & e_i' \otimes I_q \\
\end{bmatrix}, \quad i \in \{1, 2, \ldots, k + 1\}$$

can be transformed into matrix

$$\begin{bmatrix}
0 & \lambda I_{(k+1)q} \\
C & 0 \\
\end{bmatrix}.$$

It is easy to check that the extended system (34) is jointly observable but may not be jointly controllable and that the uncontrollable spectrum of $\left( \tilde{A}, [\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m] \right)$ consists of zeros only.

For each nonempty proper subset $\mathfrak{s} \subset \mathfrak{m}$, there exists $j \in N_{\mathfrak{s}} \cap \mathfrak{m} - s$ for some $\beta \in \mathfrak{s}$. Then by (35), (36), and (26), with proper elementary row and column operations, it is not hard to verify that for any $\lambda \neq 0$

$$\text{rank} \left[ \lambda I - \tilde{A} B_{\mathfrak{m} - s} \right] \geq \text{rank} \left[ \lambda I - A \begin{array}{c}
C_{\mathfrak{n}} \\
0 \\
\end{array} \right] + \text{rank} \left[ M_{d_j} \otimes I_{n_j} \right]$$
$$+ d_j q_j + \sum_{i \in \mathfrak{m}, i \neq j} (n_i + d_i n_i + d_i q_i)$$
$$= \text{rank} \left[ \lambda I - A \begin{array}{c}
C_{\mathfrak{n}} \\
0 \\
\end{array} \right] + (d_j + 2)n_j + d_j q_j$$
$$+ \sum_{i \in \mathfrak{m}, i \neq j} (n_i + d_i n_i + d_i q_i)$$
$$\geq n + \sum_{i=1}^{m} (n_i + d_i n_i + d_i q_i) = \tilde{n}.$$  (37)

Thus, by Proposition 2, $\lambda \neq 0$ is not a fixed eigenvalue of the extended system.

Now the claim will be proved by induction on $k$. For $k = 1$, $M_1 = [-1, \lambda]$. As $\text{rank} \begin{bmatrix} -1 & \lambda \\ 1 & 0 \end{bmatrix} = 2$, the claim is true for $k = 1$. Suppose the claim holds for $k = j \geq 1$, then for $i \in \{1, 2, \ldots, j + 1\}$

$$\text{rank} \left[ M_{j+1} \right] = j + 2.$$  (38)

Apparently, $\text{rank} \left[ M_{j+1} \right] = j + 2$. So the claim also holds for $k = j + 1$. This completes the proof of the claim.  

Theorem 3 suggests that the introduction of transmission delays does not affect the bound on $n_i$, as long as fixed eigenvalues at 0 are acceptable. Similarly to Remark 2, it is straightforward to verify that when only the signals $x_i$ and not the signals $y_j$ are transmitted across the network, Theorem 3 still holds, except in this case that the bound on the $n_i$ is (27) rather than (26).

**B. Spectrum Assignment**

Next it will be demonstrated that with two modifications of the approach used in Section V-A, it is possible to avoid all the fixed eigenvalues including those at 0 and to assign the closed-loop spectrum freely in the presence of transmission delays.

It can be checked easily that when each agent $i \in \mathfrak{m}$ sends both signals $x_i$ and $y_j$ to its followers, the extended system (34) may not be jointly controllable and thus may have fixed eigenvalues at 0 in the face of transmission delays. So the first modification is that each agent $i$ sends only the signal $x_i$ and not the signal $y_j$ to its followers.

The second modification is that each agent $i \in \mathfrak{m}$ “holds” the signal $x_i$ by appropriate amount of time before sending it to each of its followers. To this end, each agent $i$ needs to know the transmission delay $d_{ij}$ from itself to each follower $j \in \mathcal{F}_i$. So each agent $i$ knows $d_{ij}$ as well.

Roughly speaking, the idea of holding is that instead of using the current state of itself, each agent $i$ tries to “hold” its state $x_i$ by appropriate amounts of time before deploying $x_i$ into its own measured output and sending $x_i$ to its other followers, so that a part of agent $i$’s own measured signal and the signals received from agent $i$ by its other followers are precisely $x_i(t - d_i)$, the maximally delayed state of agent $i$. In other words, each agent $i$ does the following. It holds its state $x_i(t)$ by $d_i$ units of time and then releases $x_i(t - d_i)$ as part of its measured output. For each of its other followers $j \in \mathcal{F}_i$, agent $i$ holds its own state $x_i(t)$ by exactly $(d_i - d_{ij})$ units of time and then transmits the state $x_i(t - d_i + d_{ij})$ to follower $j$. Taking into account the $d_{ik}$ units of delay in the transmission process, follower $j$ receives $x_i(t - d_i)$ and then makes this signal available for its own measurement. Thus, all of agent $i$’s followers have $x_i(t - d_i)$ in their measured signals.
The new extended system is defined as follows. As before, each agent \( i \in \mathbf{m} \) has an \( n_i \)-dimensional local open-loop controller
\[ x_i(t+1) = v_i(t). \]

Let \( \hat{x} \) denote the new extended state
\[ \dot{\hat{x}}(t) = \text{column } \{ x(t), x_1(t), x_1(t-1), \ldots, x_1(t-d_1) \}
\[ x_2(t), x_2(t-1), \ldots, x_2(t-d_2), \ldots \]
\[ x_m(t), x_m(t-1), \ldots, x_m(t-d_m) \}\]
and for \( i \in \mathbf{m} \), define
\[ \hat{u}_i(t) = \text{column } \{ u_i(t), v_i(t) \} \]
\[ \hat{y}_i(t) = \text{column } \{ y_i(t), x_j(t-d_j), x_j(t-d_{j^2}), \ldots \}
\[ x_{j_m}(t-d_{j_{m^2}}) \}\]
where, \( \{ j_1, j_2, \ldots, j_{m^2} \} = N_i \). Using lifting, the relationship between \( \hat{u}_i(t) \) and \( \hat{y}_i(t) \) can be described by a delay-free \( m \)-channel, \( \hat{n} \)-dimensional linear system of the form
\[ \dot{\hat{x}}(t+1) = \hat{A}\hat{x}(t) + \sum_{i=1}^{m} \hat{B}_i\hat{u}_i(t) \]
\[ \hat{y}_i(t) = \hat{C}_i\hat{x}(t), \quad i \in \mathbf{m}, \quad t \in \{ 0, 1, 2, \ldots \} \]
where, \( \hat{n} \triangleq n + \sum_{i=1}^{m} (n_i + d_i n_i) \)
\( \hat{A} = \text{diagonal } \{ A, S_d_{d+1} \otimes I_n, \ldots, S_d_{m+1} \otimes I_n \} \)
\( \hat{B}_i = \text{diagonal } \{ B_i, L_{i0} \}, \quad i \in \mathbf{m} \)
\( \hat{C}_i = \text{diagonal } \{ C_i, [L_{j_{d_{j^2}}}, \ldots, L_{j_{m^2}d_{j_{m^2}}} \} \}, \quad i \in \mathbf{m} \)
where
\[ L_{i\delta} = \text{column } \{ 0_{\gamma_i^\delta}, I_n, 0_{\gamma_i^\delta}, \ldots, 0_{\gamma_i^\delta} \}, \quad 0 \leq \delta \leq d_i \]
where, \( \gamma_i^\delta = \delta n_i + \sum_{j=1}^{i-1} (n_j + d_j n_j) \) and \( \gamma_i^{\delta+} = (d_i - \delta) n_i + \sum_{j=2}^{i+1} (n_j + d_j n_j) \).

**Theorem 4:** Suppose that \( \mathbb{N} \) is strongly connected, that the transmission delays across \( \mathbb{N} \) are finite, and that the modified lower bound \( (27) \) on \( n_i \) holds for \( i \in \mathbf{m} \), then the new extended system defined by \( (37) \) has no fixed spectrum.

**Proof of Theorem 4:** It is easy to check that the new extended system \( (37) \) is jointly controllable and jointly observable.

By \( (38) \) and \( (27) \), for each nonempty proper subset \( s \subset \mathbf{m} \) and for any \( \lambda \in \mathbb{C} \)
\[ \text{rank } \begin{bmatrix} \lambda I - \hat{A} & \hat{B}_{m-s} \\ \hat{C}_s & 0 \end{bmatrix} = \text{rank } \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_s & 0 \end{bmatrix} \]
\[ + \sum_{i=1}^{m} (n_i + d_i n_i) + \min_{i \in \mathcal{N}} n_i \]
\[ \geq \text{rank } \begin{bmatrix} \lambda I_n - A & B_{m-s} \\ C_s & 0 \end{bmatrix} \]
\[ + \sum_{i=1}^{m} (n_i + d_i n_i) + \min_{i \in \mathcal{N}} n_i \]
Therefore, by Proposition 2, the new extended system has no fixed spectrum.

Similar to Proposition 3, it is straightforward to verify that when \( n_i > 0 \) for all \( i \in \mathbf{m} \), the transfer graph of the new extended system \( (37) \) is strongly connected if and only if the union of the neighbor and transfer graphs of the original system \( (2) \) is strongly connected. Therefore, under the hypotheses of Theorem 4, the closed-loop spectrum of the new extended system can be freely assigned via a suitably designed channel controller \( z_q \) [21] for any selected channel \( q \in \mathbf{m} \) [5]. Other yet to be developed design techniques may possibly also be applied.

**VI. CONCLUDING REMARKS**

This article presents new systematic procedures for exponentially stabilizing a jointly controllable, jointly observable, multichannel linear system with any prescribed convergence rate using linear time-invariant distributed controllers. While the first procedure in Section II addresses the problem using a distributed observer, the second in Section IV casts the problem in a more general setting, which is broad enough to enable alternative and yet to be devised solutions. There is a large literature focusing on decentralized control from an optimization perspective (e.g., [27]), which may prove useful here.

This article also draws attention to the practical but challenging problem of dealing with transmission delays across a network. It would be interesting to see how ideas from the literature on control of linear systems over networks with limited data rates (e.g., [28], [29]) might be brought to bear on this problem. Other issues such as the effects of asynchronous operations and resilience to local failures would also be of interest to study.

Practical feedback control procedures have features not really possessed by the algorithms proposed in this article or by the algorithms described in [6], [7], [8], [9]. None of the algorithms address design metrics (other than convergence rate), potential problems with unmodeled dynamics, noise, the range over which a linear process model is valid, etc. In our view all of these algorithms should be thought of as preliminary ideas contributing to what might at some point become a bona fide practical feedback theory for distributed systems. It was with this type of thinking in mind that we were inspired to develop the material in Section IV to demonstrate how any solution to the basic distributed control problem could be viewed through the lens of decentralized control technology. There is considerable value in doing this. For example, Theorem 2 gives a clear indication of the dimensions any distributed controllers must satisfy in order to provide solutions to the basic problem. Moreover, this setup enables one to draw some conclusions about what type of shared signals will and will not enable solutions to the basic problem. An example of this is the example in Section IV-B, which shows quite surprisingly, that if the only signal each
agent transmits to its neighbors is its measured output $y_j$, then distributed stabilization may be impossible, no matter how much local dynamics each agent uses. It is results like this, which are needed to enable a distributed feedback control theory.

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