Multiphoton supercoherent states

Erik Díaz-Bautista*1,2 and David J. Fernández C.†1

1Physics Department, Cinvestav, P.O. Box 14-740, 07000 Mexico City, Mexico
2Department of Theoretical Physics, Atomic Physics and Optics of the University of Valladolid, Spain

Abstract

If for a given system we are able to identify an annihilation operator, then it is plausible to build its multiphoton coherent states as eigenstates of the $m$-th power of such an operator, and to analyze then their properties. Following this idea, in this paper we are going to build the multiphoton supercoherent states for the supersymmetric harmonic oscillator, as eigenstates of the $m$-th power of a special form of the supersymmetric annihilation operator. The Heisenberg uncertainty relation as well as some statistical properties for these states will be studied, and the geometric phase will be also evaluated.

1 Introduction

The harmonic oscillator is the simplest binding model which has been ever used in quantum mechanics. The operators used to determine the equidistant spectrum for this system generate the well-known Heisenberg-Weyl algebra (HWA). In addition, from them it is possible to define more general algebraic structures, as the polynomial Heisenberg algebras (PHA), which appear when replacing the standard first-order annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ by $m$-th order differential ones [1–7]. In particular, by choosing the generalized ladder operators as $\tilde{a} = \hat{a}^m$, $\tilde{a}^\dagger = (\hat{a}^\dagger)^m$, together with the harmonic oscillator Hamiltonian, the simplest realization of the PHA is identified. In addition, it is quite natural to find the corresponding coherent states as eigenstates of $\tilde{a}$, which are called multiphotonic coherent states (MCS) in the literature [7–13].

On the other hand, the supersymmetric harmonic oscillator is a system coming from supersymmetric quantum mechanics (SUSY QM) which combines both boson and fermionic oscillators [14–16]. In terms of the standard annihilation and creation operators, the supersymmetric Hamiltonian $\hat{H}_{\text{SUSY}}$ is expressed as

$$\hat{H}_{\text{SUSY}} = \omega \begin{pmatrix} \hat{a}^\dagger \hat{a} & 0 \\ 0 & \hat{a} \hat{a}^\dagger \end{pmatrix},$$

(1)
whose eigenstates, represented by spinors of two components, and eigenvalues are given by:

\[ E_n = n\omega, \quad |\Psi^+_n\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} = |n\rangle \otimes |0\rangle_f, \quad |\Psi^-_n\rangle = \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix} = |n-1\rangle \otimes |1\rangle_f, \quad (2) \]

\( n = 0, 1, \ldots \), where \( |\Psi^-_0\rangle \equiv 0, \quad |0\rangle_f = (1 \quad 0)^T \) and \( |1\rangle_f = (0 \quad 1)^T \) are fermionic basis vectors. For each natural \( n \neq 0 \), the associated eigenspace is doubly degenerated.

A very general form of the supersymmetric annihilation operator (SAO) for this system was proposed by Kornbluth and Zypman recently [17], which is given by:

\[ \hat{A}_{\text{SUSY}} = \begin{pmatrix} k_1 \hat{a} & k_2 \\ k_3 \hat{a}^2 & k_4 \hat{a} \end{pmatrix}, \quad (3) \]

\( k_i \in \mathbb{C} \) being arbitrary parameters. However, this form is not unique, as it has been discussed in [17–19].

Now, the so-called supercoherent states \( |Z\rangle \) are defined as eigenstates of the annihilation operator \( \hat{A}_{\text{SUSY}} \). As usual, they are built as linear combinations of the eigenstates of the SUSY harmonic oscillator in the form

\[ |Z\rangle = \sum_{n=0}^{\infty} a_n |\Psi^+_n\rangle + \sum_{n=1}^{\infty} c_n |\Psi^-_n\rangle = \sum_{n=0}^{\infty} a_n \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} + \sum_{n=1}^{\infty} c_n \begin{pmatrix} 0 \\ |n-1\rangle \end{pmatrix}. \quad (4) \]

These states have been studied in some previous works [16–19], and they can be used in the generalized Jaynes-Cummings model [20], in the construction of superalgebras [21] and Q-balls [22,23], among other applications.

Taking into account these ideas, our goal here is to extend the construction of the multiphoton coherent states for the supersymmetric harmonic oscillator, \textit{i.e.}, to generate the multiphoton supercoherent states for a special form of the SAO and to analyze their properties through some physical functions of interest. In addition, due to the equidistant spectrum for this Hamiltonian, we will also determine the geometric phases associated to the multiphoton supercoherent states, which turn out to be periodic, as well as their corresponding period.

This paper is organized as follows. In sect. 2 the polynomial Heisenberg algebras and the multiphoton coherent states for the harmonic oscillator will be quickly reviewed. The Heisenberg uncertainty relation, some statistical and non-classical properties, as well as the evolution loop for the MCS, will be also analyzed. In sect. 3 the multiphoton supercoherent states will be generated, as eigenstates of the \( m \)-th power of a special choice of \( \hat{A}_{\text{SUSY}} \), and they will be analyzed in the same way as their scalar counterparts of sect. 2. Our conclusions will be presented in sect. 4.

## 2 Multiphoton coherent states

In a quantum mechanical description of the harmonic oscillator the Hamiltonian \( \hat{H} \) is usually written as

\[ \hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}, \quad (5) \]
where \( \hat{q} \) and \( \hat{p} \) denote the position and momentum operators, respectively. Defining the ladder operators as
\[
\hat{a} = \frac{1}{\sqrt{2}}(i\hat{p} + \hat{q}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(-i\hat{p} + \hat{q}),
\]
also known as annihilation and creation operators, respectively, it turns out that the set of operators \( \{\hat{H}, \hat{a}, \hat{a}^\dagger\} \) satisfies the commutation relations
\[
[\hat{H}, \hat{a}] = -\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \hat{a}^\dagger, \quad [\hat{a}, \hat{a}^\dagger] = 1,
\]
which define the so-called Heisenberg-Weyl algebra (HWA).

### 2.1 Polynomial Heisenberg algebras

The previous Heisenberg-Weyl algebra can be deformed by replacing the ladder operators \( \hat{a} \) and \( \hat{a}^\dagger \) by \( m \)-th order differential ladder operators, \( \hat{L}_m \) and \( \hat{L}_m^\dagger \), respectively \([1, 24–31]\). This leads to the so-called polynomial Heisenberg algebras (PHA) \([1–6]\), which are defined by the following commutation relationships
\[
\hat{L}_m\hat{L}_m^\dagger = \omega \hat{L}_m^\dagger, \quad \hat{L}_m\hat{L}_m = -\omega \hat{L}_m, \quad [\hat{L}_m, \hat{L}_m^\dagger] = \hat{N}_m(\hat{H} + \omega 1) - \hat{N}_m(\hat{H}) \equiv P_{m-1}(\hat{H}),
\]
where the generalized number operator \( \hat{N}_m(\hat{H}) \equiv \hat{L}_m^\dagger \hat{L}_m - m \) is a polynomial in \( \hat{H} \) of degree \( m \), that can be factorized as
\[
\hat{N}_m(\hat{H}) = \prod_{i=1}^{m} (\hat{H} - \mathcal{E}_i),
\]
and \( P_{m-1}(\hat{H}) \) is a \((m - 1)\)-th degree polynomial in \( \hat{H} \).

Let us consider now the set of states \( |\psi\rangle \in K_{\hat{L}_m} \), where \( K_{\hat{L}_m} \) denotes the Kernel of the operator \( \hat{L}_m \), i.e.,
\[
\hat{L}_m|\psi\rangle = 0 \quad \implies \quad \hat{L}_m^\dagger \hat{L}_m|\psi\rangle = \prod_{i=1}^{m} (\hat{H} - \mathcal{E}_i)|\psi\rangle = 0.
\]
Since \( K_{\hat{L}_m} \) is invariant under the action of \( \hat{H} \), the set of states \( |\psi_{\mathcal{E}_i}\rangle \) which are simultaneously eigenstates of \( \hat{H} \) with eigenvalue \( \mathcal{E}_i \) can be selected as the basis of \( K_{\hat{L}_m} \), namely,
\[
\hat{L}_m|\psi_{\mathcal{E}_i}\rangle = 0, \quad \hat{H}|\psi_{\mathcal{E}_i}\rangle = \mathcal{E}_i|\psi_{\mathcal{E}_i}\rangle.
\]

The states \( |\psi_{\mathcal{E}_i}\rangle \) are called extremal states; the remaining eigenstates of \( \hat{H} \) can be constructed by acting \( \hat{L}_m^\dagger \) on \( |\psi_{\mathcal{E}_i}\rangle \), such that the spacing between energy levels is \( \Delta \mathcal{E} = \omega \). However, if just \( s \) extremal states are physically meaningful, equivalently are eigenstates of \( \hat{H} \), the iterated action of \( \hat{L}_m^\dagger \), onto \( |\psi_{\mathcal{E}_i}\rangle \), \( i = 1, 2, \ldots, s \) will produce at the end \( s \) infinite energy ladders.

In particular, by choosing \([6,7]\)
\[
\hat{L}_m \equiv \hat{a} = \hat{a}^m, \quad \hat{L}_m^\dagger \equiv \hat{a}^\dagger = \hat{a}^{\dagger m},
\]
where \( \hat{a} \) and \( \hat{a}^\dagger \) denote the position and momentum operators, respectively. Defining the ladder operators as
\[
\hat{a} = \frac{1}{\sqrt{2}}(i\hat{p} + \hat{q}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(-i\hat{p} + \hat{q}),
\]
also known as annihilation and creation operators, respectively, it turns out that the set of operators \( \{\hat{H}, \hat{a}, \hat{a}^\dagger\} \) satisfies the commutation relations
\[
[\hat{H}, \hat{a}] = -\hat{a}, \quad [\hat{H}, \hat{a}^\dagger] = \hat{a}^\dagger, \quad [\hat{a}, \hat{a}^\dagger] = 1,
\]
which define the so-called Heisenberg-Weyl algebra (HWA).

### 2.1 Polynomial Heisenberg algebras

The previous Heisenberg-Weyl algebra can be deformed by replacing the ladder operators \( \hat{a} \) and \( \hat{a}^\dagger \) by \( m \)-th order differential ladder operators, \( \hat{L}_m \) and \( \hat{L}_m^\dagger \), respectively \([1, 24–31]\). This leads to the so-called polynomial Heisenberg algebras (PHA) \([1–6]\), which are defined by the following commutation relationships
\[
[\hat{H}, \hat{L}_m^\dagger] = \omega \hat{L}_m^\dagger, \quad [\hat{H}, \hat{L}_m] = -\omega \hat{L}_m, \quad [\hat{L}_m, \hat{L}_m^\dagger] = \hat{N}_m(\hat{H} + \omega 1) - \hat{N}_m(\hat{H}) \equiv P_{m-1}(\hat{H}),
\]
where the generalized number operator \( \hat{N}_m(\hat{H}) \equiv \hat{L}_m^\dagger \hat{L}_m - m \) is a polynomial in \( \hat{H} \) of degree \( m \), that can be factorized as
\[
\hat{N}_m(\hat{H}) = \prod_{i=1}^{m} (\hat{H} - \mathcal{E}_i),
\]
and \( P_{m-1}(\hat{H}) \) is a \((m - 1)\)-th degree polynomial in \( \hat{H} \).

Let us consider now the set of states \( |\psi\rangle \in K_{\hat{L}_m} \), where \( K_{\hat{L}_m} \) denotes the Kernel of the operator \( \hat{L}_m \), i.e.,
\[
\hat{L}_m|\psi\rangle = 0 \quad \implies \quad \hat{L}_m^\dagger \hat{L}_m|\psi\rangle = \prod_{i=1}^{m} (\hat{H} - \mathcal{E}_i)|\psi\rangle = 0.
\]
Since \( K_{\hat{L}_m} \) is invariant under the action of \( \hat{H} \), the set of states \( |\psi_{\mathcal{E}_i}\rangle \) which are simultaneously eigenstates of \( \hat{H} \) with eigenvalue \( \mathcal{E}_i \) can be selected as the basis of \( K_{\hat{L}_m} \), namely,
\[
\hat{L}_m|\psi_{\mathcal{E}_i}\rangle = 0, \quad \hat{H}|\psi_{\mathcal{E}_i}\rangle = \mathcal{E}_i|\psi_{\mathcal{E}_i}\rangle.
\]
one realizes the PHA through the set of operators \( \{ \hat{H}, \hat{a}, \hat{a}^\dagger \} \) for the harmonic oscillator \[32\]. In fact, these operators satisfy the algebra defined in the Eqs. (8a), (8b) as follows:

\[
[\hat{H}, \hat{a}^\dagger] = m\hat{a}^\dagger, \quad [\hat{H}, \hat{a}] = -m\hat{a}, \tag{13a}
\]

\[
[\hat{a}, \hat{a}^\dagger] = \hat{N}_m(\hat{H} + m) - \hat{N}_m(\hat{H}) \equiv P_{m-1}(\hat{H}), \tag{13b}
\]

where

\[
\hat{N}_m(\hat{H}) = \prod_{j=0}^{m-1} (\hat{H} - E_j), \tag{14}
\]

and \( E_{j+1} = E_j = j + 1/2, \ j = 0, \ldots, m - 1 \) are the first \( m \) eigenvalues of the harmonic oscillator Hamiltonian, that correspond to the \( m \) extremal states of the system (see Figure 1).

The eigenvalues of the \((j + 1)\)-th ladder are

\[
E_{jn}^j = E_j + mn, \quad n = 0, 1, 2, \ldots, \quad j = 0, 1, \ldots, m - 1, \tag{15}
\]

with corresponding eigenstates

\[
|\psi_{jn}^j\rangle \equiv |mn + j\rangle = \sqrt{\frac{j!}{(mn + j)!}} (\hat{a}^\dagger)^n |j\rangle, \quad j = 0, 1, \ldots, m - 1. \tag{16}
\]

Hence, the spectrum of the Hamiltonian \( \hat{H} \) becomes:

\[
\text{Sp}(\hat{H}) = \{ E_0^0, E_1^0, \ldots \} \cup \{ E_0^1, E_1^1, \ldots \} \cup \cdots \cup \{ E_0^j, E_1^j, \ldots \} \cup \cdots \cup \{ E_0^{m-1}, E_1^{m-1}, \ldots \}. \tag{17}
\]
We conclude that the Hilbert space $H = \text{span}(|n\rangle, n = 0, 1, 2, \ldots)$ has been decomposed as the direct sum of $m$ orthogonal subspaces, $H = H_1 \oplus H_2 \oplus \cdots \oplus H_m$, where

$$H_{j+1} = \text{span}(|mn + j\rangle, n = 0, 1, 2, \ldots), \quad j = 0, 1, 2, \ldots, m - 1.$$  \hfill (18)

### 2.2 Multiphoton coherent states

Although without this name, the coherent states were introduced for the first time by Schrödinger in 1926 for the harmonic oscillator \[33\], as wave packets whose dynamics is similar to a classical particle in such a quadratic potential. These states have been useful in various branches of physics \[34, 35\]. Their importance is such that nowadays they are called standard coherent states (SCS) in the literature, and they have been generalized in several different ways \[36–40\].

One of these generalizations leads to the multiphoton coherent states (MCS), which can be constructed as eigenstates $|\tilde{\alpha}\rangle_m$ with complex eigenvalues $\tilde{\alpha}$ of the generalized (multiphoton) annihilation operator $\tilde{a} = \hat{a}^m$ \[9–13\]:

$$\tilde{a}|\tilde{\alpha}\rangle_m = \tilde{\alpha}|\tilde{\alpha}\rangle_m, \quad \tilde{\alpha} \in \mathbb{C}. \hfill (19)$$

The MCS turn out to be expressed explicitly as superpositions of Fock states $|n\rangle$ differing among themselves by a number of photons which is as a multiple of the power $m$:

$$|\tilde{\alpha}; j\rangle_m = \mathcal{N}_m^{j} \sum_{n=0}^{\infty} \frac{\tilde{\alpha}^n}{\sqrt{(mn+j)!}} |mn+j\rangle, \hfill (20)$$

where $\mathcal{N}_m^{j}$ are normalization constants. In the PHA approach, the index $j$ indicates that for a given $\tilde{\alpha}$ there is a quantum state in each subspace $H_{j+1}$ satisfying relation (19), and the Fock state with minimum energy contributing to such a state $|\tilde{\alpha}; j\rangle_m$ is the one associated to $E_{j+1}^0$.

If we choose now $\tilde{\alpha} = \alpha^m$, the MCS take the form

$$|\alpha; m, j\rangle \equiv |\alpha^m; j\rangle_m = \mathcal{N}_m^{j} \sum_{n=0}^{\infty} \frac{\alpha^{mn+j}}{\sqrt{(mn+j)!}} |mn+j\rangle. \hfill (21)$$

As we can see, the standard coherent states are just a particular case of the multiphoton coherent states for $m = 1, j = 0$:

$$|\alpha\rangle \equiv |\alpha; 1, 0\rangle = \exp \left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \tilde{\alpha} = \alpha. \hfill (22)$$

On the other hand, the multiphoton coherent states for $m > 1$ can be also expressed as superpositions of $m$ standard coherent states (see Eq. (22)) which are distributed uniformly on a circle of radius $|\alpha|$ \[41,42\]:

$$|\alpha; m, j\rangle = \frac{\mathcal{N}_m^{j}}{m} \sum_{n=0}^{m-1} \omega_{j}^n |\alpha \omega_j^{n/j}\rangle, \hfill (23)$$
where
\[
\left\{ \omega_j = \exp \left( 2\pi i \frac{j}{m} \right) ; \ j = 0, 1, \ldots, m-1 \right\}
\]
denotes the abelian group of \(m\)-th roots of the unit. Such decompositions have been studied in detail in [4,3,47].

For example, if we take \(m = 2\), \(j = 0, 1\) in Eq. (21), and the two squared roots of 1 \((\{\omega_0, \omega_1\} = \{1, e^{i\pi}\} = \{1, -1\})\) in Eq. (23), then the explicit forms for the eigenstates of \(\tilde{a} = \hat{a}^2\) become:
\[
|\alpha\rangle_+ \equiv |\alpha; 2, 0\rangle = \mathcal{N}_2^0 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle = \exp \left( \frac{|\alpha|^2}{2} \right) \frac{\mathcal{N}_2^0}{2} \left[ |\alpha\rangle + | -\alpha\rangle \right], \quad (24a)
\]
\[
|\alpha\rangle_- \equiv |\alpha; 2, 1\rangle = \mathcal{N}_2^1 \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle = \exp \left( \frac{|\alpha|^2}{2} \right) \frac{\mathcal{N}_2^1}{2} \left[ |\alpha\rangle - | -\alpha\rangle \right], \quad (24b)
\]
where
\[
[\mathcal{N}_2^0]^2 = \frac{1}{\cosh(|\alpha|^2)}, \quad [\mathcal{N}_2^1]^2 = \frac{1}{\sinh(|\alpha|^2)}. \quad (25)
\]
This indicates that the states \(|\alpha\rangle_{\pm}\) are linear combinations of the two SCS with eigenvalues \(\alpha\) and \(\alpha e^{i\pi} = -\alpha\). The states in Eqs. (24a) and (24b) are known either as Schrödinger cat states or even and odd coherent states, respectively, since only even (odd) Fock states contribute to the corresponding decomposition [48,52].

Similarly, for \(m = 3\), \(j = 0, 1, 2\) and \(\tilde{\alpha} = \alpha^3\) we have to consider the three cube roots of 1, \(\{\omega_0, \omega_1, \omega_2\} = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}\). Then, we have
\[
|\alpha; 3, 0\rangle = \mathcal{N}_3^0 \sum_{n=0}^{\infty} \frac{\alpha^{3n}}{\sqrt{(3n)!}} |3n\rangle = \exp \left( \frac{|\alpha|^2}{2} \right) \frac{\mathcal{N}_3^0}{3} \left[ |\alpha\rangle + |\alpha\omega_1\rangle + |\alpha\omega_2\rangle \right], \quad (26a)
\]
\[
|\alpha; 3, 1\rangle = \mathcal{N}_3^1 \sum_{n=0}^{\infty} \frac{\alpha^{3n+1}}{\sqrt{(3n+1)!}} |3n+1\rangle = \exp \left( \frac{|\alpha|^2}{2} \right) \frac{\mathcal{N}_3^1}{3} \left[ |\alpha\rangle + \omega_2 |\alpha\omega_1\rangle + \omega_1 |\alpha\omega_2\rangle \right], \quad (26b)
\]
\[
|\alpha; 3, 2\rangle = \mathcal{N}_3^2 \sum_{n=0}^{\infty} \frac{\alpha^{3n+2}}{\sqrt{(3n+2)!}} |3n+2\rangle = \exp \left( \frac{|\alpha|^2}{2} \right) \frac{\mathcal{N}_3^2}{3} \left[ |\alpha\rangle + \omega_1 |\alpha\omega_1\rangle + \omega_2 |\alpha\omega_2\rangle \right], \quad (26c)
\]
where
\[
\mathcal{N}_3^0 = \left[ \frac{1}{3} \left( e^{|\alpha|^2} + 2 e^{-|\alpha|^2} \cos \left( \frac{\sqrt{3}|\alpha|^2}{2} \right) \right) \right]^{-1/2}, \quad (27a)
\]
\[
\mathcal{N}_3^1 = \left[ \frac{1}{3} \left( e^{|\alpha|^2} - 2 e^{-|\alpha|^2} \sin \left( \frac{\pi}{6} - \frac{\sqrt{3}|\alpha|^2}{2} \right) \right) \right]^{-1/2}, \quad (27b)
\]
\[
\mathcal{N}_3^2 = \left[ \frac{1}{3} \left( e^{|\alpha|^2} - 2 e^{-|\alpha|^2} \sin \left( \frac{\pi}{6} + \frac{\sqrt{3}|\alpha|^2}{2} \right) \right) \right]^{-1/2}. \quad (27c)
\]
Figure 2: Heisenberg uncertainty relation $\sigma_q \sigma_p$ as a function of $\alpha$ for some MCS. Depending on the subspace $H_{j+1}$ to which the MCS belong, this uncertainty takes a minimum value equal to $j + 1/2$, $j = 0, 1, \ldots, m - 1$.

As we can see, the states $|\alpha, 3, j\rangle$ turn out to be linear combinations of three SCS with eigenvalues $\alpha$, $\alpha \omega_1$ and $\alpha \omega_2$, which are located at the vertices of an equilateral triangle in the complex plane $\alpha$. The case with $m = 3$ has been considered also in [53], while the multiphoton coherent states for $m = 4$ have been studied in [13, 54–57].

2.2.1 Heisenberg uncertainty relation

We can find now joint expressions for the uncertainties associated to the MCS if we define an operator $\hat{s}$ as follows

$$\hat{s} = \frac{1}{\sqrt{2i}}(\hat{a} + (-1)^k \hat{a}^\dagger), \quad \hat{s}^2 = \frac{1}{2}(2\hat{N} + \hat{1} + (-1)^k(\hat{a}^2 + \hat{a}^{\dagger 2})), \quad (28)$$

such that

$$\langle \hat{s} \rangle|_{k=0} = \langle \hat{q} \rangle, \quad \langle \hat{s} \rangle|_{k=1} = \langle \hat{p} \rangle, \quad (29a)$$

$$\langle \hat{s}^2 \rangle|_{k=0} = \langle \hat{q}^2 \rangle, \quad \langle \hat{s}^2 \rangle|_{k=1} = \langle \hat{p}^2 \rangle, \quad (29b)$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively.

We thus get:

$$\langle \hat{s} \rangle_\alpha = \frac{1}{\sqrt{2i}}(\alpha + (-1)^k \alpha^*) \delta_{1m}, \quad (30a)$$

$$\langle \hat{s}^2 \rangle_\alpha = |\alpha|^2 \left[ \frac{N_{j_{m}}}{N_{m_{j-1}}} \right]^2 + \frac{1}{2} + (-1)^k(\text{Re}(\alpha))^2 \left[ \text{Im}(\alpha) \right]^2 \Delta, \quad (30b)$$
where

\[ m_j = m\delta_{0j} + j, \quad \Delta = \begin{cases} 1, & m = 1, 2, \\ 0, & \text{otherwise}, \end{cases} \tag{31} \]

and \( \delta_{mn} \) is the Kronecker delta.

In general, the MCS are not minimum uncertainty states, since it is fulfilled the Heisenberg uncertainty relation (HUR)

\[ \sigma_{q,\alpha}\sigma_{p,\alpha} \geq \frac{1}{2}. \tag{32} \]

Figure 2 shows the HUR for the states \( |\alpha\rangle_\pm \) and \( |\alpha; 3, j\rangle, j = 0, 1, 2 \). We can see that in the limit \( \alpha \to 0 \), this uncertainty achieves a minimum, equal to \( j + 1/2, j = 0, 1, \ldots, m - 1 \), which coincides with the HUR for the eigenstate \( |j\rangle \) of the harmonic oscillator since the Fock state with minimum energy appearing in the linear decomposition for the MCS in the subspace \( \mathcal{H}_{j+1} \) is precisely \( |j\rangle \) (see Eq. (21)).

### 2.2.2 Non-classicality criteria

There are some criteria in the literature allowing to investigate the non-classical nature of quantum states. For such a purpose, in this paper we will focus in the analysis of the sub-Poissonian statistics and the negativity of the Wigner function on phase space. In particular, Mandel’s Q-parameter brings information about the statistics of a quantum system.
Figure 4: Wigner function $W_\alpha(q,p)$ for the even (a) and odd (b) coherent states with $|\alpha| = 2.5$.

The Mandel’s $Q$-parameter is defined as $^{58,59}$

$$Q = \frac{\langle \sigma_N^2 \rangle - \langle \hat{N} \rangle}{\langle \hat{N} \rangle} = \frac{\langle \hat{a}^\dagger \hat{a} \rangle^2 - \langle (\hat{a}^\dagger \hat{a})^2 \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle},$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$ is the number operator for the harmonic oscillator.

Negative values of $Q$ correspond to quantum states with a sub-Poissonian statistics, i.e., the variance in the photon number is less than its mean. However, if $Q$ takes positive values, no concrete conclusion can be established about the non-classicality of the states. Finally, the standard coherent states have a Poisson distribution for which $Q = 0$.

For example, for the even and odd coherent states the Mandel’s $Q$-parameter turns out to be (see Figure 3a)

$$Q_+ = |\alpha|^2 \left[ \coth(|\alpha|^2) - \tanh(|\alpha|^2) \right],$$

$$Q_- = |\alpha|^2 \left[ \tanh(|\alpha|^2) - \coth(|\alpha|^2) \right],$$

while for the states $|\alpha; 3, j\rangle$, $j = 0, 1, 2$ it is found that (see Figure 3b):

$$Q_3^0 = |\alpha|^2 \left[ \frac{\mathcal{N}_3^2}{\mathcal{N}_3^1} \right]^2 - \left[ \frac{\mathcal{N}_3^0}{\mathcal{N}_3^1} \right]^2,$$

$$Q_3^1 = |\alpha|^2 \left[ \frac{\mathcal{N}_3^1}{\mathcal{N}_3^0} \right]^2 - \left[ \frac{\mathcal{N}_3^1}{\mathcal{N}_3^0} \right]^2,$$

$$Q_3^2 = |\alpha|^2 \left[ \frac{\mathcal{N}_3^2}{\mathcal{N}_3^0} \right]^2 - \left[ \frac{\mathcal{N}_3^2}{\mathcal{N}_3^0} \right]^2.$$

9
Figure 5: Wigner function $W_\alpha(q,p)$ for the MCS $|\alpha; 3, 0\rangle$ (a), $|\alpha; 3, 1\rangle$ (b) and $|\alpha; 3, 2\rangle$ (c) with $|\alpha| = 2.5$.

Figure 3 also shows that Mandel’s Q-parameter tends asymptotically to zero when $|\alpha| \rightarrow \infty$ [60]. On the other hand, the Wigner function on phase space is defined as [61–64]

$$W(q,p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle q - \frac{y}{2} | \hat{\rho} | q + \frac{y}{2} \rangle e^{ipy/\hbar} dy,$$

(36)

where $\hat{\rho}$ is the density operator and $|q \pm y\rangle$ are eigenkets of the position operator $\hat{q}$. If the state under analysis is pure, then $\hat{\rho} = |\psi\rangle\langle\psi|$ and hence (by making $\hbar = 1$):

$$W_\psi(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi^*(q + y)\psi(q - y)e^{2ipy} dy,$$

(37)
are also considered non-classical states. Negative values of the Wigner function indicate non-classicality of a state, which is interpreted as a sign of quantumness \cite{62,65,73}.

By taking now the standard coherent state $|\alpha\rangle$ of Eq. (22) and its corresponding wavefunction in the coordinates representation \cite{47}, it is straightforward to find the Wigner function for two SCS with different complex labels $\alpha$, $\beta$:

$$W_{\alpha,\beta}(q,p) = \frac{1}{\pi} \exp \left( - \left[ q - \frac{(\beta + \alpha^*)}{\sqrt{2}} \right]^2 - \left[ p - \frac{\sqrt{2} \beta^*}{\sqrt{2} i} \right]^2 + \alpha^* \beta - \frac{1}{2} \left[ |\alpha|^2 + |\beta|^2 \right] \right). \tag{38}$$

In particular, $W_{\alpha}(q,p)$ will appear from the previous expression for $\beta = \alpha$.

By using Eq. (38), it is possible to find compact expressions for the Wigner functions of the even and odd coherent states, respectively (see Figure 4):

$$W^+_{\alpha}(q,p) = \exp \left( |\alpha|^2 \right) \left[ \frac{N^0_2}{4} \right] \left[ W_{\alpha}(q,p) + W_{-\alpha}(q,p) + 2\text{Re}[W_{\alpha,-\alpha}(q,p)] \right], \tag{39a}$$

$$W^-_{\alpha}(q,p) = \exp \left( |\alpha|^2 \right) \left[ \frac{N^1_2}{4} \right] \left[ W_{\alpha}(q,p) + W_{-\alpha}(q,p) - 2\text{Re}[W_{\alpha,-\alpha}(q,p)] \right]. \tag{39b}$$

The first two terms in Eqs. (39a), (39b) correspond to the Gaussian functions of two standard coherent states centered in $\pm(q_0,p_0)$. The last one is an interference term that oscillates quickly as the distance between the two standard coherent states grows \cite{74}. These oscillating terms induce negative values in the Wigner function, thus the even and odd coherent states are considered non-classical states \cite{75}.

On the other hand, Eq. (38) allows also to find simply the Wigner function for the MCS with $m = 3$, $j = 0, 1, 2$ (see Figure 5):

$$W^0_{\alpha,\beta}(q,p) = \exp \left( |\alpha|^2 \right) \left[ \frac{N^0_3}{9} \right] \left[ W_{\alpha}(q,p) + W\omega_1(q,p) + W\omega_2(q,p) \right. \left. + 2\text{Re}[W_{\alpha,\omega_1}(q,p) + W_{\alpha,\omega_2}(q,p) + W_{\omega_1,\omega_2}(q,p)] \right], \tag{40a}$$

$$W^1_{\alpha,\beta}(q,p) = \exp \left( |\alpha|^2 \right) \left[ \frac{N^1_3}{9} \right] \left[ W_{\alpha}(q,p) + W\omega_1(q,p) + W\omega_2(q,p) \right. \left. + 2\text{Re}[\omega_2 W_{\alpha,\omega_1}(q,p) + \omega_1 W_{\alpha,\omega_2}(q,p) + \omega_1^* W_{\omega_1,\omega_2}(q,p)] \right], \tag{40b}$$

$$W^2_{\alpha,\beta}(q,p) = \exp \left( |\alpha|^2 \right) \left[ \frac{N^2_3}{9} \right] \left[ W_{\alpha}(q,p) + W\omega_1(q,p) + W\omega_2(q,p) \right. \left. + 2\text{Re}[\omega_1 W_{\alpha,\omega_1}(q,p) + \omega_2 W_{\alpha,\omega_2}(q,p) + \omega_1 W_{\omega_1,\omega_2}(q,p)] \right]. \tag{40c}$$

As for the even and odd coherent states, Eqs. (40a)-(40c) include once again interference terms, contributing mainly in the zone between the Gaussian functions of the three standard coherent states, the last being placed in the vertices of an equilateral triangle in phase space. Due to these terms, negative values for the Wigner function appear, thus the states in Eqs. (26a)-(26c) are also considered non-classical states.
2.2.3 Evolution loop

The dynamics of a quantum system is determined by its evolution operator, a unitary operator \( \hat{U}(t) \) which satisfies
\[
\frac{d\hat{U}(t)}{dt} = -i\hat{H}(t)\hat{U}(t), \quad \hat{U}(0) = \hat{1},
\]
where \( \hat{H}(t) \) is the system Hamiltonian and \( \hat{1} \) represents the identity operator.

Of special interest in this approach are the so-called evolution loops (EL), i.e., dynamical processes such that \( \hat{U}(t) \) becomes the identity operator (up to a phase factor) at a certain time \( \tau > 0 \):
\[
\hat{U}(\tau) = \exp(i\phi)\hat{1},
\]
where \( \tau \) is the loop period and \( \phi \in \mathbb{R} \) [76,77]. The EL, introduced by Mielnik in 1977 [76,77], are important since they can be used as the starting point to implement control techniques and to perform dynamical manipulation of quantum systems.

In a geometric context, if a system performs an evolution loop and we take an arbitrary state \( |\psi\rangle \in \mathcal{H} \) as an initial condition,
\[
|\psi(0)\rangle \equiv |\psi\rangle,
\]
then a cyclic state of period \( \tau \) is produced, namely:
\[
|\psi(\tau)\rangle = \hat{U}(\tau)|\psi\rangle = \exp(i\phi)|\psi\rangle.
\]
The total phase \( \phi \) has a geometric component \( \beta \), which in general is not zero. If the system’s Hamiltonian is time-independent, such a geometric phase becomes [78]
\[
\beta = \phi + \frac{1}{\hbar} \int_0^\tau \langle \psi(0)|\hat{U}(t)\hat{H}(t)|\psi(0)\rangle dt = \phi + \frac{\tau}{\hbar} \langle \psi|\hat{H}|\psi\rangle.
\]

Going back to the harmonic oscillator, let us note that in the Hilbert space \( \mathcal{H} \) it is produced a global evolution loop of period \( T = 2\pi \). Moreover, when \( \mathcal{H} \) is decomposed as in section 2.1, \( \mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \), it turns out that in each subspace a partial evolution loop of period \( \tau = 2\pi/m \) appears. This implies that any state in each subspace is cyclic with period \( \tau = 2\pi/m \). In particular, this is valid for the MCS, inducing then a geometric phase \( \beta_j \) which is different for each subspace \( \mathcal{H}_{j+1} \) and is given by:
\[
\beta_j = -\frac{2\pi}{m} E_0 + \frac{2\pi}{m} \left[ |\alpha|^2 \left[ \frac{N_j^m}{N_{m_j}^m} \right]^2 + \frac{1}{2} \right], \quad j = 0, 1, \ldots, m - 1.
\]
This result was recently obtained in [7].

3 Multiphoton supercoherent states

We are going to build now the coherent states \( |Z\rangle_m \) for the supersymmetric harmonic oscillator as eigenstates of the \( m \)-th power of the SAO. For simplicity, let us choose the parameters \( k_i \) as \( k_1 = k_4 = 1, k_3 = 0 \) while \( k_2 \in \mathbb{R} \) is left arbitrary, so the annihilation operator to be used is:
\[
\hat{A}^m_{\text{SUSY}} = \begin{pmatrix}
\hat{a}^m & m k_2 \hat{a}^{m-1} \\
0 & \hat{a}^m
\end{pmatrix} = \hat{a}^m \otimes I + m k_2 \hat{a}^{m-1} \otimes \hat{f}^\dagger,
\]

12
where the fermionic operators $I$, $\hat{f}$ and $\hat{f}^\dagger$ denote the identity, annihilation and creation operators respectively, which satisfy the algebra

$$\{\hat{f}, \hat{f}^\dagger\} = I, \quad \{\hat{f}, \hat{f}\} = \{\hat{f}^\dagger, \hat{f}^\dagger\} = 0.$$  \hspace{1cm} (47)

This choice will allow us to analyze the effects of the free parameter $k_2$ on the properties of the multiphoton supercoherent states. Before doing this, however, let us sketch first the result of acting the supersymmetric ladder operators $\hat{A}^m_{\text{SUSY}}$, $\hat{A}^\dagger_m_{\text{SUSY}}$ on the system’s Hilbert space.

As in the previous section, the Hilbert space $\mathcal{H}_{\text{SUSY}} = \text{span}(|\Psi^+_n\rangle, |\Psi^-_n\rangle, n = 0, 1, \ldots)$ is again decomposed as the direct sum of $m$ orthogonal subspaces $\mathcal{H}_{\text{SUSY}} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m$, where each subspace

$$\mathcal{H}_{j+1} = \text{span}(|\Psi^+_{mn+j}\rangle, |\Psi^-_{mn+j}\rangle, n = 0, 1, 2, \ldots), \quad j = 0, 1, 2, \ldots, m - 1,$$

is invariant under the action of $\hat{A}^m_{\text{SUSY}}$, $\hat{A}^\dagger_m_{\text{SUSY}}$, while the spectrum of the Hamiltonian $\hat{H}_{\text{SUSY}}$ is expressed as

$$\text{Sp}(\hat{H}_{\text{SUSY}}) = \{E^0_j, E^0_k, \ldots \} \cup \{E^1_j, E^1_k, \ldots \} \cup \cdots \cup \{E^{m-1}_j, E^{m-1}_k, \ldots \},$$

whose spacing between neighbor energy levels $E^j_n = mn + E_j$ in each subspace $\mathcal{H}_{j+1}$ is $\Delta E^j_n = m$.

Now, taking into account the coherent states definition,

$$\hat{A}^m_{\text{SUSY}}|Z\rangle_m = \alpha|Z\rangle_m, \quad \alpha \in \mathbb{C},$$

and using the expansion in Eq. (4), the following relations for the coefficients $a_n$ and $c_n$ are obtained:

$$c_{mn+m_j} = \sqrt{(m_j-1)!} \left(\frac{mn+j-1}{mn+j}!\right)^{\alpha^n} c_{mj}, \quad j = 0, 1, \ldots, m - 1,$$  \hspace{1cm} (51a)

$$a_{mn+j} = \sqrt{\frac{j!}{(mn+j)!}} \alpha^n a_j - \sqrt{(m_j-1)!} \left(\frac{mn+j}{mn+j}!\right)^{\alpha^n} k_2 c_{mj}, \quad j = 0, 1, \ldots, m - 1,$$  \hspace{1cm} (51b)

where $a_j$ and $c_{mj}$ are $2m$ free parameters.

If we make now $\alpha = z^m$ in Eqs. (51a), (51b), the multiphoton supercoherent states become:

$$|Z; m, j\rangle = \tilde{a}_{mj} |Z; m, j\rangle_f + \tilde{c}_{mj} |\tilde{Z}; m, j\rangle_s,$$  \hspace{1cm} (52)

where

$$|Z; m, j\rangle_f = \begin{pmatrix} |z; m, j\rangle \\ 0 \end{pmatrix}, \quad |\tilde{Z}; m, j\rangle_s = \begin{pmatrix} -k_2 |z'; m, j\rangle \\ |z; m, m_j - 1\rangle \end{pmatrix},$$  \hspace{1cm} (53)

with

$$\tilde{a}_{mj} = \sqrt{j!} z^{-j} a_j + jk_2 z^{-1} c_{mj},$$  \hspace{1cm} (54a)

$$\tilde{c}_{mj} = \sqrt{(m_j-1)!} z^{-(m_j-1)} c_{mj},$$  \hspace{1cm} (54b)

$$|z; m, j\rangle = \sum_{n=0}^{\infty} \frac{z^{mn+j}}{\sqrt{(mn+j)!}} |mn+j\rangle,$$  \hspace{1cm} (54c)

$$|z'; m, j\rangle = \frac{d}{dz} |z; m, j\rangle = \hat{a}^\dagger |z; m, m_j - 1\rangle.$$  \hspace{1cm} (54d)
Figure 6: Heisenberg uncertainty relation \((\sigma_q)_Z(\sigma_p)_Z\) as function of \(z\) for the states \(|Z\rangle\) with \(a_0 = c_1 = 1\) and different values of \(k_2\).

The states in Eqs. (54c), (54d) are not normalized and \(m_j\) is again given by Eq. (31). By taking now \(m = 1, j = 0\), the supercoherent states for the simplest diagonal SAO are recovered for \(k_2 = 0\) [17, 18], while the results for a nondiagonal SAO, which mixes both bosonic and fermionic components, are obtained for \(k_2 = 1\) [16].

According to Eq. (53), the multiphoton supercoherent states \(|Z; m, j\rangle\) in general are expressed in terms of (scalar) multiphoton coherent states \(|z; m, j\rangle\) which belong to different Hilbert subspaces \(\mathcal{H}_{j+1}\). It is possible to introduce also a new set of states \(|Z; m, j\rangle_s\) belonging to the supercoherent two-dimensional subspace, Eq. (52). Since they are also eigenstates of the operator \(\hat{A}_S^{m}\), without loss of generality we can take as multiphoton supercoherent states those given by:

\[
|\widetilde{Z}; m, j\rangle = \chi_1 |Z; m, j\rangle_f + \chi_2 |Z; m, j\rangle_s, \quad \chi_1, \chi_2 \in \mathbb{C},
\]

where

\[
|Z; m, j\rangle_s = \frac{1}{\sqrt{2}} \left( k_2 z^* |z; m, s_j\rangle - k_2 |z'; m, s_j\rangle \right),
\]

with

\[
|z'; m, s_j\rangle = \hat{a}^\dagger |z; m, j\rangle, \quad s_j = (j + 1)(1 - \delta(m-1)j).
\]

The multiphoton supercoherent states of Eq. (55) with \(m = 1, j = 0\) and \(k_2 = 1\) have been studied extensively in [16], while those with \(m = 2, j = 0, 1\) have been built recently in [80], as even and odd superpositions of supercoherent states.

In the next sections, we will analyze only the multiphoton supercoherent states \(|Z; m, j\rangle\) of Eq. (52), which were obtained directly as eigenstates of the annihilation operator \(\hat{A}_S^{m}\).
3.1 Heisenberg uncertainty relation

In order to analyze the HUR for the multiphoton supercoherent states, let us consider an extension of the operator $\hat{s}$ as $\hat{s} \rightarrow \hat{s} \otimes I$, i.e., we take the following matrix operator:

$$
\hat{s} = \frac{1}{\sqrt{2^k}} \begin{pmatrix}
\hat{a} + (-1)^k \hat{a}^\dagger & 0 \\
0 & \hat{a} + (-1)^k \hat{a}^\dagger
\end{pmatrix}
$$

(58)

For the multiphoton supercoherent states $|Z;m,j\rangle$ of Eq. (52), the mean value of the observable $\hat{s}$ is

$$
\langle \hat{s} \rangle = \frac{\langle Z;m,j|\hat{s}|Z;m,j\rangle}{\langle Z;m,j|Z;m,j\rangle},
$$

(59)

where

$$
\langle Z;m,j|\hat{s}|Z;m,j\rangle = |\tilde{a}_{m,j}|^2 \langle z;m,j|\hat{s}|z;m,j\rangle + |\tilde{c}_{m,j}|^2 \langle z;m,m_j-1|\hat{s}|z;m,m_j-1\rangle \\
+ |c_{m_j}|^2 k_2^2 \langle z';m,j|\hat{s}|z';m,j\rangle - k_2 \left( \tilde{a}_{m_j} \tilde{c}_{m,j}^* \langle z';m,j|\hat{s}|z;m,j\rangle + \tilde{a}_{m_j}^* \tilde{c}_{m_j} \langle z;m,j|\hat{s}|z';m,j\rangle \right).
$$

(60)

3.1.1 Heisenberg uncertainty relation for $m = 1$.

First of all, let us express the multiphoton supercoherent states with $m = 1$, $j = 0$ in terms of the normalized standard coherent states $|z\rangle$ of Eq. (22):

$$
|Z\rangle \equiv |Z;1,0\rangle = \mathcal{N} \exp \left( \frac{|z|^2}{2} \right) \left[ a_0 \left( \frac{|z\rangle}{0} \right) + c_1 \left( -k_2 \sqrt{|z|^2 + 1} \hat{a}^\dagger |z\rangle \right) \right],
$$

(61)
where \( \mathcal{N} \) is the normalization constant given by

\[
\mathcal{N}^2 = \exp(-|z|^2) \left[ |a_0|^2 + |c_1|^2 + |c_1|^2 k_2^2(|z|^2 + 1) - 2k_2 \text{Re}[a_0^* c_1 z^*] \right]^{-1}.
\] (62)

By taking now \( m = 1, j = 0 \) in Eq. (59) it is found that:

\[
(\sigma_q)^2 = \mathcal{N}^2 \exp(|z|^2) \left[ (|a_0|^2 + |c_1|^2)(2|z|^2 + 1 + 2(-1)^k([\text{Re}(z)]^2 - [\text{Im}(z)]^2)) + k_2 \text{Re}[a_0 c_1^* (2|z|^2 + 3) z + (-1)^k((|z|^2 + 2)z^* + z^3)] + (-1)^{k+1} \mathcal{N}^2 \exp(|z|^2) \right]
\]

\[
\times \left[ (|a_0|^2 + |c_1|^2)(z + (-1)^k z^*) + |c_1|^2 k_2^2(|z|^2 + 2) z + (-1)^k z^*) - k_2 (a_0 c_1^* (|z|^2 + 1 + (-1)^k z^2) + a_0 c_1^* z^2 + (-1)^k ((|z|^2 + 1))) \right]^2.
\] (63)

Let us note that \((\sigma_q)^2 = (\sigma_s)^2 |_{k=0}\) and \((\sigma_p)^2 = (\sigma_s)^2 |_{k=1}\).

For \( k_2 = 0 \) the Heisenberg uncertainty relation as function of \( z \) for the states \(|Z\rangle\) can be drawn as a constant plane, since then \((\sigma_q)_Z (\sigma_p)_Z = 1/2\). However, for \( k_2 \neq 0 \) it appears a maximum for this uncertainty, which tends to the limit value \( \sim 1.5 \) as \(|k_2|\) grows (see Figure 6); on the other hand, when \(|z| \to \infty\) the HUR decreases quickly, approaching asymptotically its lowest possible value \((1/2)\).

### 3.1.2 Heisenberg uncertainty relation for \( m = 2 \).

Let us express now the multiphoton supercoherent states with \( m = 2, j = 0, 1 \) in terms of the normalized even and odd coherent states \(|z\rangle_\pm\) of Eqs. (24a), (24b):

\[
|Z\rangle_+ \equiv |Z;2,0\rangle = \mathcal{N}_+ \sqrt{\cosh(|z|^2)} \left[ \tilde{a}_{20} \left( \begin{array}{c} |z|_+ \\ 0 \end{array} \right) + \tilde{e}_{20} \left( \begin{array}{c} -k_2 \sqrt{|z|^2 + \tanh(|z|^2)} \tilde{a}^\dagger |z|_+ \\ \sqrt{\tanh(|z|^2)} |z|_+ \end{array} \right) \right],
\] (64a)

\[
|Z\rangle_- \equiv |Z;2,1\rangle = \mathcal{N}_- \sqrt{\sinh(|z|^2)} \left[ \tilde{a}_{21} \left( \begin{array}{c} |z|_- \\ 0 \end{array} \right) + \tilde{e}_{21} \left( \begin{array}{c} -k_2 \sqrt{|z|^2 + \coth(|z|^2)} \tilde{a}^\dagger |z|_- \\ \sqrt{\coth(|z|^2)} |z|_- \end{array} \right) \right],
\] (64b)

where the normalization constants \( \mathcal{N}_\pm \) are given by

\[
\mathcal{N}_+^2 = \frac{[|\tilde{a}_{20}|^2 + |\tilde{e}_{20}|^2 \tanh(|z|^2) + |\tilde{c}_{20}|^2 k_2^2(|z|^2 + \tanh(|z|^2)) - 2k_2 \tanh(|z|^2) \text{Re}[\tilde{a}_{20}^* \tilde{e}_{20} z]\]^{-1}}{\cosh(|z|^2)},
\] (65a)

\[
\mathcal{N}_-^2 = \frac{[|\tilde{a}_{21}|^2 + |\tilde{c}_{21}|^2 \coth(|z|^2) + |\tilde{e}_{21}|^2 k_2^2(|z|^2 + \coth(|z|^2)) - 2k_2 \coth(|z|^2) \text{Re}[\tilde{a}_{21}^* \tilde{e}_{21} z]\]^{-1}}{\sinh(|z|^2)},
\] (65b)
(a) $k_2 = -2$ (purple), $k_2 = 0$ (gray) and $k_2 = 1$ (cyan).

(b) $k_2 = -2$ (purple), $k_2 = 0$ (gray) and $k_2 = 1$ (cyan).

(c) $k_2 = -2$ (purple), $k_2 = 0$ (gray) and $k_2 = 1$ (cyan).

Figure 8: Heisenberg uncertainty relation $\langle \sigma_q \rangle_Z \langle \sigma_p \rangle_Z$ as function of $z$ for the states $|Z; 3, 0\rangle$ with $a_0 = c_3 = 1$ (a), $|Z; 3, 1\rangle$ with $a_1 = c_1 = 1$ (b), and $|Z; 3, 2\rangle$ with $a_2 = c_2 = 1$ (c), for different values of $k_2$.

with

\[\begin{align*}
\tilde{a}_{2_0} &= a_0, & \tilde{c}_{2_0} &= z^{-1}c_2, \\
\tilde{a}_{2_1} &= z^{-1}a_1 + k_2z^{-1}c_1, & \tilde{c}_{2_1} &= c_1.
\end{align*}\]  

(66a)  

(66b)
By taking now \( m = 2, j = 0, 1 \) in Eq. (59) we arrive at:

\[
\sigma_{3+}^2 = N_+^2 \frac{\cosh(|z|^2)}{2} \left[ |\tilde{a}_{20}|^2 (2|z|^2 \tanh(|z|^2) + 1 + 2(-1)^k (|\text{Re}(z)|^2 - |\text{Im}(z)|^2)) + |\tilde{c}_{20}|^2 k_2^2 \times \left( 2|z|^4 \tanh(|z|^2) + 7|z|^2 + 3 \tanh(|z|^2) + 2(-1)^k (|z|^2 + 3 \tanh(|z|^2)) (|\text{Re}(z)|^2 - |\text{Im}(z)|^2) \right) - k_2 \left( 2|z|^2 + 3 \tanh(|z|^2) + (-1)^k (|z|^2 \tanh(|z|^2) + 2) \right) \text{Re}[\tilde{a}_{20} \tilde{c}_{20} z] + \tanh(|z|^2) \text{Re}[\tilde{a}_{20}^* \tilde{c}_{20}^* z^3] \right],
\]

(67a)

\[
\sigma_{3-}^2 = N_-^2 \frac{\sinh(|z|^2)}{2} \left[ |\tilde{a}_{21}|^2 (2|z|^2 \coth(|z|^2) + 1 + 2(-1)^k (|\text{Re}(z)|^2 - |\text{Im}(z)|^2)) + |\tilde{c}_{21}|^2 k_2^2 \times \left( 2|z|^4 \coth(|z|^2) + 7|z|^2 + 3 \coth(|z|^2) + 2(-1)^k (|z|^2 + 3 \coth(|z|^2)) (|\text{Re}(z)|^2 - |\text{Im}(z)|^2) \right) - k_2 \left( 2|z|^2 + 3 \coth(|z|^2) + (-1)^k (|z|^2 \coth(|z|^2) + 2) \right) \text{Re}[\tilde{a}_{21} \tilde{c}_{21} z] + \coth(|z|^2) \text{Re}[\tilde{a}_{21}^* \tilde{c}_{21}^* z^3] \right].
\]

(67b)

Let us note once again that \( \sigma_{q}^2_{Z} = \sigma_{p}^2_{Z} |k=0 \) and \( \langle \sigma_{p}^2_{Z} |k=1 \rangle \).

Figures 7a and 7b show the Heisenberg uncertainty relation as function of \( z \) for \( |Z\rangle_+ \) and \( |Z\rangle_- \), respectively, and several values of the parameter \( k_2 \). As we can observe, such a behavior is similar to that obtained for the cat states (see Figure 2a). It can be seen also that the minimum value that \( \langle \sigma_{q}^2_{Z} (\sigma_{p})^2_{Z} \rangle \) can take for both states \( |Z\rangle_{\pm} \) grows as \( |k_2| \) does.

### 3.1.3 Heisenberg uncertainty relation for \( m = 3 \).

Finally, for \( m = 3, j = 0, 1, 2 \), we get the following multiphoton supercoherent states expressed in terms of the normalized multiphoton coherent states \( |z; 3, j\rangle \) of Eqs. (26a)-(26c):

\[
|Z; 3, 0\rangle = N_0 |\mathcal{N}_0|^{-1} \left[ |\tilde{a}_{30} z; 3, 0\rangle + |\tilde{c}_{30} z; 3, 0\rangle \right],
\]

(68a)

\[
|Z; 3, 1\rangle = N_1 |\mathcal{N}_1|^{-1} \left[ |\tilde{a}_{31} z; 3, 1\rangle + |\tilde{c}_{31} z; 3, 1\rangle \right],
\]

(68b)

\[
|Z; 3, 2\rangle = N_2 |\mathcal{N}_2|^{-1} \left[ |\tilde{a}_{32} z; 3, 2\rangle + |\tilde{c}_{32} z; 3, 2\rangle \right],
\]

(68c)

where the normalization constants \( \mathcal{N}_j \) are

\[
|\mathcal{N}_0|^2 = \left( |\tilde{a}_{30}|^2 |\mathcal{N}_3|^{-2} + |\tilde{c}_{30}|^2 |\mathcal{N}_3|^{-2} + |\tilde{c}_{30}|^2 k_2^2 (|z|^2 |\mathcal{N}_3|^{-2} + |\mathcal{N}_3|^2)^{-2} - 2k_2 |\mathcal{N}_3|^2 |\mathcal{N}_3|^{-2} \text{Re}[\tilde{a}_{30} \tilde{c}_{30} z^*] \right)^{-1},
\]

(69a)

\[
|\mathcal{N}_1|^2 = \left( |\tilde{a}_{31}|^2 |\mathcal{N}_3|^{-2} + |\tilde{c}_{31}|^2 |\mathcal{N}_3|^{-2} + |\tilde{c}_{31}|^2 k_2^2 (|z|^2 |\mathcal{N}_3|^{-2} + |\mathcal{N}_3|^2)^{-2} - 2k_2 |\mathcal{N}_3|^2 |\mathcal{N}_3|^{-2} \text{Re}[\tilde{a}_{31} \tilde{c}_{31} z^*] \right)^{-1},
\]

(69b)

\[
|\mathcal{N}_2|^2 = \left( |\tilde{a}_{32}|^2 |\mathcal{N}_3|^{-2} + |\tilde{c}_{32}|^2 |\mathcal{N}_3|^{-2} + |\tilde{c}_{32}|^2 k_2^2 (|z|^2 |\mathcal{N}_3|^{-2} + |\mathcal{N}_3|^2)^{-2} - 2k_2 |\mathcal{N}_3|^2 |\mathcal{N}_3|^{-2} \text{Re}[\tilde{a}_{32} \tilde{c}_{32} z^*] \right)^{-1},
\]

(69c)
with the constants \( \mathcal{N}_3^j \), \( j = 0, 1, 2 \) given by Eqs. (27a)-(27c) and

\[
\begin{align*}
\tilde{a}_{30} &= a_0, & \tilde{c}_{30} &= \sqrt{2}! z^{-2} c_3, \\
\tilde{a}_{31} &= z^{-1} a_1 + k_2 z^{-1} c_1, & \tilde{c}_{31} &= c_1, \\
\tilde{a}_{32} &= \sqrt{2}! z^{-2} a_2 + 2 k_2 z^{-2} c_2, & \tilde{c}_{32} &= z^{-1} c_2.
\end{align*}
\] 

(70a)  

(70b)  

(70c)

If we take \( m = 3, j = 0, 1, 2 \) in Eq. (59) we will get the uncertainty relation associated to the operator \( \hat{s} \) for this case:

\[
\langle s \rangle_{jZ}^2 = \frac{\mathcal{N}_3^j}{2} \left[ |\tilde{a}_{3j}|^2 \left( 2 |z|^2 [\mathcal{N}_3^{3j} - 1]^{-2} + [\mathcal{N}_3^{3j}]^{-2} \right) + |\tilde{c}_{3j}|^2 \left( 2 |z|^2 [\mathcal{N}_3^{3j}]^{-2} + [\mathcal{N}_3^{3j}]^{-2} \right) \right. \\
\left. + k_2 \left( 2 |z|^4 [\mathcal{N}_3^{3j}]^{-2} + 7 |z|^2 [\mathcal{N}_3^{3j}]^{-2} + 3 [\mathcal{N}_3^{3j}]^{-1} \right) \right] \]

(71)

where \( 3_j = 3 \delta_{0j} + j \) and \( s_j = (j + 1)(1 - \delta_{2j}) \). It is clear once again that \( \langle s \rangle_{2Z}^2 = \langle s \rangle_{3Z}^2 |_{k=0} \), \( \langle s \rangle_{2Z}^2 = \langle s \rangle_{3Z}^2 |_{k=1} \), \( j = 0, 1, 2 \).

Figure 8 shows that, in general, the Heisenberg uncertainty relation as function of \( z \) for the states \( |Z; 3, j \rangle \) behaves qualitatively in the same way as the one obtained for the scalar states \( |\alpha; 3, j \rangle \) (compare Figure 2b).

Finally, according to the previous discussions, the case of \( k_2 = 0 \) constitutes a limit situation for the HUR, since the corresponding gray surfaces are below the other ones for most of the considered cases. This means that, in general, although the multiphoton supercoherent states are not minimum uncertainty states, their HUR reach the allowed minimum values for \( k_2 = 0 \).

### 3.2 Non-classicality criteria

In order to analyze in more detail the multiphoton supercoherent states, we are going to consider next the Mandel’s \( Q \)-parameter and the Wigner function \( W_{Z,m}(q, p) \).

For the multiphoton supercoherent states of Eq. (52), the \( Q \)-parameter is given by Eq. (33), while the corresponding Wigner function \( W_{Z,m}(q, p) \) is expressed as

\[
W_{Z,m}^{j}(q, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Psi_{Z,m}^{j}(q + y) \Psi_{Z,m}^{j}(q - y) e^{2ipy} dy = |\tilde{a}_{m,j}|^2 W_{Z,m}^{j}(q, p) + |\tilde{c}_{m,j}|^2 W_{Z,m}^{m-1}(q, p) \\
+ |\tilde{c}_{m,j}|^2 k_2^2 W_{Z,m}^{j}(q, p) - k_2 \left( \tilde{a}_{m,j} \tilde{c}_{m,j} W_{Z,m}^{j}(q, p) + \tilde{a}_{m,j} \tilde{c}_{m,j} W_{Z,m}^{j}(q, p) \right),
\] 

(72)

where \( \Psi_{Z,m}^{j}(q) \equiv |q; Z; m, j \rangle \), \( W_{Z,m}^{j}(q, p) \) denotes the Wigner function for the scalar wavefunction \( \psi_{Z,m}^{j}(q) \equiv |q; Z; m, j \rangle \), \( W_{Z,m}^{m}(q, p) \) represents the Wigner function for the scalar wavefunction \( \psi_{Z,m}^{m}(q) = \langle q; z'; m, j \rangle \), which is given by

\[
W_{\alpha,\beta}^{\prime}(q, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_{\alpha}^{*}(q + y) \psi_{\beta}^{*}(q - y) \exp(2ipy) dy \\
= 2 \left( \left[ q - \frac{(\beta + \alpha^*) \sqrt{2}}{\sqrt{2}i} \right]^2 + \left[ p - \frac{(\beta - \alpha^*) \sqrt{2}i}{\sqrt{2}i} \right]^2 - \frac{1}{2} \right) W_{\alpha,\beta}(q, p),
\] 

(73)
(a) $z \in \mathbb{R}$  
(b) $|z| = 1, k_2 = 1.6$  
(c) $|z| = 2, k_2 = 0.97561$  
(d) $|z| = 3, k_2 = 0.66298$

Figure 9: Mandel’s $Q_\mathbb{R}$-parameter (a) and Wigner function $W_\mathbb{R}(q,p)$ (b-d) for the states in Eq. (61) with $a_0 = c_1 = 1$ and different values of $|z|, k_2$. The red line for $Q_\mathbb{R}$ marks the values of $k_2$ for which a Poissonian statistics is observed.

While the Wigner functions involving the two scalar states of Eqs. (54c), (54d) turn out to be

$$W_{\alpha,\beta}^I(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_\alpha^*(q+y) \psi_\beta(q-y) \exp(2ipy) \, dy = \sqrt{2} \left[ q + ip - \frac{\beta}{\sqrt{2}} \right] W_{\alpha,\beta}(q,p), \quad (74a)$$

$$W_{\alpha,\beta}^{II}(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_\alpha^*(q+y) \psi_\beta^*(q-y) \exp(2ipy) \, dy = \sqrt{2} \left[ q - ip - \frac{\alpha^*}{\sqrt{2}} \right] W_{\alpha,\beta}(q,p). \quad (74b)$$
3.2.1 Non-classicality criteria for \( m = 1 \).

By considering first the case with \( m = 1, j = 0 \) we obtain

\[
Q_Z = |z|^2 \left[ (|a_0|^2 + |c_1|^2) |z|^2 + |c_1|^2 k_2^2 (|z|^2 + |z|^2) - 2k_2 (|z|^2 + 1) \Re[a_0^* c_1 z^*] \right]^{-1} \times \\
\left[ (|a_0|^2 + |c_1|^2) |z|^2 + |c_1|^2 k_2^2 (|z|^2 + 2) + |z|^2 - 2k_2 (|z|^2 + 2) \Re[a_0^* c_1 z^*] \right] \\
- \mathcal{N}^2 \exp(|z|^2) \left[ (|a_0|^2 + |c_1|^2) |z|^2 + |c_1|^2 k_2^2 (|z|^2 + 1) + |z|^2 - 2k_2 (|z|^2 + 1) \Re[a_0^* c_1 z^*] \right]^{-1} \times \\
\left[ (|a_0|^2 + |c_1|^2) W_z(q, p) + |c_1|^2 k_2^2 W_z'(q, p) - k_2 (a_0^* c_1 W_z''(q, p) + a_0 c_1 W_z''(q, p)) \right] \\
(75a)
\]

\[
W_Z(q, p) = \mathcal{N}^2 \left[ (|a_0|^2 + |c_1|^2) W_z(q, p) + |c_1|^2 k_2^2 W_z'(q, p) - k_2 (a_0^* c_1 W_z''(q, p) + a_0 c_1 W_z''(q, p)) \right]. \\
(75b)
\]
The Mandel's parameter $Q_Z$ as function of $k_2$ is shown in Figure 9a. Although the supercoherent states $|Z\rangle$ exhibit in general either sub-Poissonian or super-Poissonian statistics, nonetheless there are $k_2$-values for which $Q_Z = 0$ (red line), i.e., for the conditions defined by this line they can be considered as semi-classical states. On the other hand, the corresponding Wigner function $W_Z(q,p)$ acquires negative values for $k_2 \neq 0$, which is a sign of an intrinsically quantum nature for these supercoherent states [62]; however, these negative values tend to disappear as $k_2 \to 0$, i.e., the parameter $k_2$ affects substantially the quantum nature of the states $|Z\rangle$ (see Figures 9b-9d). In particular, for $k_2 = 0$ it is recovered the Wigner function for the SCS, up to constant factor.
3.2.2 Non-classicality criteria for $m = 2$.

If we take now $m = 2$, $j = 0, 1$, it is obtained:

$$Q_+ = |z|^2 \left[ (|\tilde{a}_{20}|^2 \tanh(|z|^2) + |\tilde{c}_{20}|^2) |z|^2 + |\tilde{c}_{20}|^2 k_2^2 \left( (|z|^4 + 1) \tanh(|z|^2) + 3|z|^2 \right) \right. \right.$$  
$$- 2k_2 \left( |z|^2 + \tanh(|z|^2) \right) \text{Re}[\tilde{a}_{20}^* \tilde{c}_{20} z^*] \right]^{-1} \left[ (|\tilde{a}_{20}|^2 + |\tilde{c}_{20}|^2 \tanh(|z|^2)) |z|^2 \right.$$  
$$+ |\tilde{c}_{20}|^2 k_2^2 \left( |z|^4 + 5|z|^2 \tanh(|z|^2) + 4 \right) - 2k_2 \left( |z|^2 \tanh(|z|^2) + 2 \right) \text{Re}[\tilde{a}_{20}^* \tilde{c}_{20} z^*] \right]$$  
$$- N_+ \cosh(|z|^2) \left[ (|\tilde{a}_{20}|^2 \tanh(|z|^2) + |\tilde{c}_{20}|^2) |z|^2 + |\tilde{c}_{20}|^2 k_2^2 \left( (|z|^4 + 1) \tanh(|z|^2) + 3|z|^2 \right) \right. \right.$$  
$$- 2k_2 \left( |z|^2 + \tanh(|z|^2) \right) \text{Re}[\tilde{a}_{20}^* \tilde{c}_{20} z^*] \right],$$  

(76a)
Figure 13: Mandel's $Q_3$ parameter (a) and Wigner function $W_{Z,3}(q,p)$ (b-d) for the states in Eq. (68b) with $a_1 = c_1 = 1$ and different values of $|z|, k_2$. The red line for $Q_3$ marks the values of $k_2$ for which a Poissonian statistics is observed.

\[
Q_- = |z|^2 \left( (|\tilde{a}_2|^2 \coth(|z|^2) + |\tilde{c}_2|^2) |z|^2 + |\tilde{a}_2|^2 k_2^2 \left( (|z|^4 + 1) \coth(|z|^2) + 3|z|^2 \right) \right.
\]
\[
-2k_2 \left( |z|^2 + \coth(|z|^2) \right) \text{Re}[\tilde{a}_2^* \tilde{c}_2, z^*] \right)^{-1} \left( (|\tilde{a}_2|^2 + |\tilde{c}_2|^2 \coth(|z|^2)) |z|^2 \right.
\]
\[
+ |\tilde{c}_2|^2 k_2^2 \left( |z|^4 + 5|z|^2 \coth(|z|^2) + 4 \right) - 2k_2 \left( |z|^2 \coth(|z|^2) + 2 \right) \text{Re}[\tilde{a}_2^*, \tilde{c}_2, z^*] \right)
\]
\[
- \mathcal{N}^2 \sinh(|z|^2) \left[ (|\tilde{a}_2|^2 \coth(|z|^2) + |\tilde{c}_2|^2) |z|^2 + |\tilde{c}_2|^2 k_2^2 \left( (|z|^4 + 1) \coth(|z|^2) + 3|z|^2 \right) \right.
\]
\[
-2k_2 \left( |z|^2 + \coth(|z|^2) \right) \text{Re}[\tilde{a}_2, \tilde{c}_2, z^*] \right]
\]
\[
W_{Z}^\pm(q,p) = \mathcal{N}^2 \left[ |\tilde{a}_2|^2 W_z^\pm(q,p) + |\tilde{c}_2|^2 W_z^\mp(q,p) + |\tilde{c}_2|^2 k_2^2 W_z^{\prime\pm}(q,p) \right.
\]
\[
- k_2 \left( \tilde{a}_2^* \tilde{c}_2 W_z^{H\pm}(q,p) + \tilde{a}_2 \tilde{c}_2^* W_z^{H\mp}(q,p) \right) \right].
\]
The Mandel’s $Q_{\pm}$-parameter as function of $k_2$ for the states $|Z\rangle_{\pm}$ is represented in Figures 10a and 11a. Once again, these multiphoton supercoherent states exhibit sub-Poissonian and super-Poissonian statistics, but there are also $k_2$-values for which $Q_{\pm} = 0$ (red line), which means that on this line these states have a semi-classical behavior.

On the other hand, from Figures 10 and 11 it is seen that the Wigner function $W_{\pm}(q,p)$ for each multiphoton supercoherent state $|Z\rangle_{\pm}$ behaves qualitatively in the same way as its scalar counterpart, showing two localized Gaussian distributions which however interfere with each other, according to the value taken by $k_2$ (see Figures 10c, 10d, 11c, 11d).
3.2.3 Non-classicality criteria for $m = 3$.

Finally, if we take $m = 3$, $j = 0, 1, 2$, the corresponding Mandel’s parameter and Wigner function in this case are given by

$$Q_3^j = |z|^2 \left[ (|\tilde{a}_3^j|^2 [\mathcal{N}^j_3]^{-1} - 2 + |\tilde{c}_3^j|^2 [\mathcal{N}^j_3]^{-2}) |z|^2 + |\tilde{c}_3^j|^2 k_2^2 \left( |z|^4 [\mathcal{N}^j_3]^{-2} + 3|z|^2 [\mathcal{N}^j_3]^{-2} + [\mathcal{N}^j_3]^{-1} \right) \right] - 2k_2 \left( |z|^2 [\mathcal{N}^j_3]^{-2} + [\mathcal{N}^j_3]^{-1} \right) \text{Re}[\tilde{a}_3^j \tilde{c}_3^j z^*] - 2k_2 \left( |z|^2 [\mathcal{N}^j_3]^{-2} + 2|\mathcal{N}^j_3|^{-2} \right) \text{Re}[\tilde{a}_3^j \tilde{c}_3^j z^*]$$

$$W_{z,3}^j(q, p) = \mathcal{N}_j^2 \left[ |\tilde{a}_3^j|^2 W_{z,3}^j(q, p) + |\tilde{c}_3^j|^2 W_{z,3}^{j+1}(q, p) + |\tilde{c}_3^j|^2 k_2 W_{z,3}^j(q, p) \right] - k_2 \left( \tilde{a}_3^j \tilde{c}_3^j W_{z,3}^j(q, p) + \tilde{a}_3^j \tilde{c}_3^j \tilde{W}_{z,3}^j(q, p) \right).$$

The Mandel’s $Q_3^j$-parameter as function of $k_2$ for the states $|Z; 3, j \rangle$ shows sub-Poissonian and super-Poissonian statistics. Moreover, for some particular values of the parameter $k_2$ it also arises a Poissonian behavior (see red line in Figures 12a, 13a, and 14a). Thus, as mentioned previously these multiphoton supercoherent states could be considered as semi-classical for these $k_2$-values.

Concerning the Wigner functions $W_{3}^j(q, p)$ associated to the states $|Z; 3, j \rangle$, they behave similarly as their scalar counterparts of Eqs. (40a)-(40c), with three localized Gaussian distributions interfering to each other according to the value taken by $k_2$ (see Figures 12c, 12d, 13c, 13d, 14c, 14d).

3.3 Evolution loop and geometric phase

Let us apply now the evolution operator $\hat{U}(t) = \exp(-i\hat{H}t)$ to the states $|Z; m, j \rangle$, which are expressed as a linear combination of the eigenstates of $\hat{H}_{SUSY}$ (see Eq. (2)). We find that

$$\hat{U}(t)|Z; m, j \rangle = \sum_{n=0}^{\infty} \frac{\tilde{a}_m \tilde{a}^n}{\sqrt{(mn + j)!}} \left( -i \omega (mn + j) t \right) \left( \begin{array}{c} mn+j \\ 0 \end{array} \right)^{n-1} \left( -i \omega (mn + j) t \right) \left( \begin{array}{c} mn + j \\ 0 \end{array} \right)$$

$$+ \sum_{n=0}^{\infty} \frac{\tilde{c}_m \tilde{a}^n}{\sqrt{(mn + m_j)!}} \left( -i \omega (mn + m_j) t \right) \left( \begin{array}{c} 0 \\ mn + m_j - 1 \end{array} \right)$$

$$= \exp\left( -i\omega j t \right) \left( \begin{array}{c} \tilde{a}_m \left( |\tilde{z}(t); m, j \rangle \right) \\ 0 \end{array} \right) + \tilde{c}_m \left( -k_2 \exp(-i\omega t) |(\tilde{z}(t))^\prime; m, j \rangle \right)$$

$$\neq \exp\left( -i\omega j t \right) |Z(t); m, j \rangle.$$

where $\tilde{z}(t) = \tilde{a} \exp(-i\omega t) = [z \exp(-i\omega t)]^m = |z(t)|^m$. 

26
Figure 15: Geometric phase $\beta^j_m$ for the states $|Z; m, j\rangle$, $m = 1, 2, 3$ with different values of the parameter $k_2$: $k_2 = -4$ (purple), $k_2 = 0$ (gray) and $k_2 = 2$ (cyan). The case with $k_2 = 0$ produces the allowed minimum values for $\beta^j_m$. 
Although the scalar multiphoton coherent states evolve coherently, so that $\hat{U}(t)$ transforms any of these states into another of the same family, Eq. (78) indicates, however, that for the multiphoton supercoherent states in general this does not happen. In fact, only for $\tilde{c}_{m_j} = 0$ such a property is satisfied. Despite, the multiphoton supercoherent states turn out to be cyclic, since by taking $t \equiv \tau = 2\pi/\omega m$ in Eq. (78) we obtain

$$\hat{U}(\tau)|Z; m, j\rangle = \exp (i\phi)|Z; m, j\rangle, \quad \phi = -2\pi \frac{j}{m}. \quad (79)$$

Eq. (79) characterizes an important property that the states $|Z; m, j\rangle$ share with the multiphoton coherent states of Eq. (21): both states recover their initial condition after a time interval $\tau = \tau_{\text{classical}}/m$, where $\tau_{\text{classical}} = 2\pi/\omega$. This fact does not have any classical counterpart, except for $m = 1$ where both periods coincide.

Finally, from Eq. (44) the geometric phase $\beta^j_m$ associated to the multiphoton supercoherent states in each subspace $\mathcal{H}_{j+1}$ is given by (see also Figure 15):

$$\beta^j_m = -2\pi \frac{j}{m} + \frac{2\pi}{m} \omega^{-1} \frac{\langle Z; m, j | \hat{H}_{\text{SUSY}} | Z; m, j \rangle}{\langle Z; m, j | Z; m, j \rangle}, \quad j = 0, 1, 2, \ldots, m - 1, \quad (80)$$

where the mean energy value is given by

$$\omega^{-1}\langle Z; m, j | \hat{H}_{\text{SUSY}} | Z; m, j \rangle = |\tilde{a}_{m_j}|^2 \langle z; m, j | \tilde{N} | z; m, j \rangle + |\tilde{c}_{m_j}|^2 \langle z; m, m_j - 1 | \tilde{N} + 1 | z; m, m_j - 1 \rangle + |\tilde{c}_{m_j}|^2 k_2^2 \langle z'; m, j | \tilde{N} | z'; m, j \rangle - 2k_2 \text{Re}(\tilde{a}_{m_j} \tilde{c}_{m_j}^* \langle z'; m, j | \tilde{N} | z; m, j \rangle). \quad (81)$$

In particular, if we choose $m = 1$ and $k_2 = 0$ in Eq. (46), we will obtain the simplest supercoherent states, which share many properties with the SCS [16,18], as the Heisenberg uncertainty relation $(\sigma_q)_Z(\sigma_p)_Z$, the Mandel’s $Q$-parameter, the Wigner function $W_Z(q, p)$, and the evolution loop period $\tau = 2\pi/\omega$.

4 Conclusions

We have seen that a clever choice of annihilation and creation operators for the harmonic oscillator allows to define different algebraic structures for such a system, as the Heisenberg-Weyl algebras (HWA) or the polynomial Heisenberg algebras (PHA). The last ones are generalizations of the HWA which appear when replacing the standard annihilation and creation operators $\hat{a}, \hat{a}^\dagger$ by $m$-th order differential ones $\hat{L}_m^-, \hat{L}_m^\dagger$. In particular, if we take $\hat{L}_m^- \equiv \hat{a}^m, \hat{L}_m^\dagger \equiv \hat{a}^{*m}$, then the set of operators $\{\hat{H}, \hat{a}^m, \hat{a}^{*m}\}$ generates a PHA, allowing to construct $m$ infinite energy ladders for the harmonic oscillator Hamiltonian $H$. In each ladder there is an extremal state $|\psi^j_0\rangle$ associated to a minimum energy level $E^j_0, j = 0, 1, \ldots, m - 1$. As a consequence, the Hilbert space $\mathcal{H}$ decomposes as the direct sum of $m$ orthogonal subspaces $\mathcal{H}_{j+1}$, whose basis vectors are constructed by applying iteratively the operator $\hat{a}^{*m}$ to the extremal states $|\psi^j_0\rangle$.

Once the algebraic structure has been identified, it is straightforward to construct the so-called multiphoton coherent states (MCS) as eigenstates of the generalized annihilation operator
As it has been shown, these states can be expressed either as linear combinations of the energy eigenstates of each subspace $H_{j+1}$ or as superpositions of standard coherent states (SCS) \[33,44\], the last ones being also a particular case of the MCS with $m = 1$. Although the SCS are minimum uncertainty states, in general this is no longer valid for the MCS, mainly due to the fact that the extremal state contributing to each MCS is not the ground state of the oscillator anymore. Besides, according to the behavior of the Mandel’s $Q$-parameter and the Wigner function, the multiphoton coherent states can be considered as intrinsically quantum states for $m > 1$ \[62\]. Finally, since the energy spectrum for the harmonic oscillator is equidistant, a partial evolution loop on each subspace $H_{j+1}$ is produced, whose period turns out to be the fraction $1/m$ of the classical period. As a consequence, the MCS are cyclic states, with the same loop period, and the corresponding geometric phase has been explicitly calculated.

On the other hand, the analysis of the coherent states for the SUSY harmonic oscillator involves the corresponding coherent states of the scalar case \[16-19\]. Thus, the multiphoton coherent states were also explored for the supersymmetric harmonic oscillator. To achieve this goal, we considered a particular form of the supersymmetric annihilation operator $\hat{A}_{\text{SUSY}}$ and built then its $m$-th power, where $k_2$ was left as a free real parameter (see Eq. (46)). This choice allowed us to analyze the effect of the parameter $k_2$ on the quantum nature of the multiphoton supercoherent states, for at least three particular values of $m$.

For the simplest case, with $m = 1$ and $k_2 = 0$, the Heisenberg uncertainty relation and Wigner function are qualitatively the same as the corresponding results for the standard coherent states. Moreover, the Mandel’s $Q$-parameter vanishes ($Q_Z = 0$) for $k_2 = 0$ (see also Figure 9a). On the other hand, for the multiphoton supercoherent states with $m = 2, 3$, the minimum value of the uncertainty product $(\sigma_q)_Z(\sigma_p)_Z$ (which arises in the limit $\alpha \to 0$) changes as $|k_2|$ does (see Figures 7-8). Meanwhile, the states $|Z; m, j\rangle$ exhibit sub-Poissonian statistics for any value of $m$, meaning that these are in general non-classical states, as the corresponding Wigner functions also show for some values of $k_2$ (see Figures 9-14). However, there are also some values of $k_2$ for which $Q = 0$ (red line in Figures 9a-14a). This could be interpreted as the existence of a particular domain of $k_2$ for which the non-classical effects in the multiphoton supercoherent states disappear.

In addition, by taking into account that the SUSY harmonic oscillator has an equidistant spectrum, it has been possible to study both, the evolution loop of the system as well as the geometric phases $\beta_m^j$ associated to the multiphoton supercoherent states. Since both, the SCS and the MCS in the scalar case are cyclic states which recover their initial condition after the time interval $\tau = \tau_{\text{classical}}/m$, then for $m > 1$ they can be considered as states without any classical counterpart. Concerning the multiphoton supercoherent states, the same situation appears for $m = 1$ and $k_2 = 0$. Moreover, as Eq. (80) shows, the geometric phase $\beta_m^j$ for the states $|Z; m, j\rangle$ is similar to the one of its scalar counterpart (see Eq. (45)), having to subtract just the ground state energy. On the other hand, the case with $k_2 = 0$ defines a lower bound for the geometric phase associated to the multiphoton supercoherent states considered in this paper (see Figure 15).

Finally, it is important to remark that the form chosen for the supersymmetric annihilation operator allowed us to construct and then describe in a relatively simple way the multiphoton supercoherent states, as well as to study the effect of one of the parameters $k_i$ on their intrinsi-
cally quantum nature. It would be interesting to consider the most general case, i.e., to allow arbitrary values for the parameters $k_i$ and to obtain then the extended family of eigenstates of the SAO, which presumably will have a richer and more general structure.

Acknowledgments

The authors acknowledge the support of Conacyt. EDB also acknowledges the warm hospitality at Department of Theoretical Physics of the University of Valladolid.

References

[1] D J Fernández and V Hussin, *J Phys A: Math Gen* **32** 3603 (1999).
[2] D J Fernández, V Hussin and L M Nieto, *J Phys A: Math Gen* **27** 3547 (1994).
[3] D J Fernández, L M Nieto and O Rosas-Ortiz, *J Phys A: Math Gen* **28** 2693 (1995).
[4] J M Carballo, D J Fernández, J Negro and L M Nieto, *J Phys A: Math Gen* **37** 10349 (2004).
[5] D Bermudez and D J Fernández, *AIP Conf Proc* **1575** 50 (2014).
[6] M Castillo-Celeita and D J Fernández, *J Phys: Conf Ser* **698** 012007 (2016).
[7] M Castillo-Celeita, E Díaz-Bautista and D J Fernández C, preprint Cinvestav, 2018 *(arXiv:1804.08543).*
[8] A M Perelomov, *Comm Math Phys* **26** 222 (1972).
[9] A O Barut and L Giradello, *Comm Math Phys* **21** 41 (1971).
[10] V Bužek, I Jex and T Quang, *J Mod Optics* **37** 159 (1990).
[11] V Bužek, *J Mod Optics* **37** 303 (1990).
[12] J Sun, J Wang and C Wang, *Phys Rev A* **46** 1700-2 (1992).
[13] I Jex and V Bužek, *J Mod Opt* **40** 771-83 (1993).
[14] P Salomonson and J W van Holten, *Nucl Phys B* **196** 509 (1982).
[15] F Cooper and B Freedman, *Ann Phys B* **146** 262 (1983).
[16] C Aragone and F Zypman, *J Phys A: Math Gen* **19** 2267 (1986).
[17] M Kornbluth and F Zypman, *J Math Phys* **54** 012101 (2013).
[18] Y Bérubé-Lauzière and V Hussin, *J Phys A: Math Gen* **26** 6271 (1993).
[19] E Díaz-Bautista and D J Fernández, *Eur Phys J Plus* **131** 151 (2016).
[20] M Daoud and J A Douari, *Int J Mod Phys B* **17** 2473 (2003)
[21] O Kovras, *New Developments in Field Theory* (Nova Science, New York, 2006)
[22] A Kusenko and M Shaposhnikov, *Phys Lett B* **418** 46 (1998)
[23] A Kusenko et al, *Phys Rev Lett* **80** 3185 (1998)
[24] D J Fernández, *M Sc Thesis*, México DF, CINVESTAV (1984).
[25] S Y Dubov, V M Eleonsky and N E Kulagin, *Sov Phys JETP* **75** 446 (1992).
[26] V E Adler, *Physica D* **73** 335 (1994).
[27] U P Sukhatme, C Rasinariu and A Khare, *Phys Lett A* **234** 401 (1997).
[28] N Aizawa and H T Sato, *Prog Theor Phys* **98** 707 (1997).
[29] F Cannata, G Junker and J Trost, in *Particles, Fields and Gravitation*, ed J Rembielinski (AIP Conf Proc 453, Woodbury, 1998) p 209.
[30] M Arik, N M Atakishiyev and K B Wolf, *J Phys A: Math Gen* **32** L371 (1999).
[31] A A Andrianov, F Cannata, M Ioffe and D Nishnianidze, *Phys Lett A* **266** 341 (2000).
[32] R Dutt, A Gangopadhyaya, C Rasinariu and U P Sukhatme, *Phys Rev A* **60** 3482 (1999).
[33] E Schrödinger, *Naturwissenschaften* **23** 807-12, 823, 844 (1935).
[34] J R Klauder and B S Skagerstam, 1985, *Coherent States* (Singapore: World Scientific)
[35] W M Zhang, D H Feng and R G Gilmore, *Rev Mod Phys* **62** 867-927 (1990).
[36] L C Biedenharn, *J Phys A: Math Gen* **22** L873-8 (1989).
[37] A J Macfarlane, *J Phys A: Math Gen* **22** 4581-8 (1989).
[38] J R Klauder, *J Math. Phys* **4** 1055-8 (1963).
[39] A Perelomov, 1986, *Generalized Coherent States and Their Applications* (Berlin: Springer).
[40] M M Nieto and L M Simmons Jr, *Phys Rev Lett* **41** 207-10 (1978).
[41] O Castaños, R López-Peña and V I Man’ko, Russ Laser Research 16 477 (1995).
[42] O Castaños and J A Lópe-Saldívar, J Phys: Con Ser 380 012017 (2012).
[43] J Janszky, P Domokos and P Adam, Phys Rev A 48 2213-19 (1993).
[44] M J Gagen, Phys Rev A 51 2715-25 (1995).
[45] I Jeq, P Törmä and S Stenholm, J Mod Opt 42 1377-86 (1995).
[46] N B An, Phys Lett A 284 72-80 (2001).
[47] V V Dodonov, J Opt B: Quantum Semiclass Opt 4, R1-R33 (2002).
[48] C Moore, D M Meekhof, B E King and D J Wineland, Science 272 1131 (1996).
[49] M Brune, E Hagley, J Dreyer, X Maitre, A Maali, C Wunderlich, J M Raimond and S Haroche, Phys Rev Lett 77 4887 (1996).
[50] S Haroche, in New Perspectives on Quantum Mechanics: Latin-American School Physics, eds: S Hacyan, R Jaúregui and R López-Peña.
[51] V V Dodonov, I A Malkin and V I Man’ko, Physica 72 597 (1974).
[52] W P Schleich, 2001, Quantum optics in Phase Space (Weinheim: Wiley-VCH).
[53] J Sun, J Wang and C Wang, Phys Rev A 44 3369-72 (1991).
[54] E E Hach III and C C Gerry, J Mod Opt 39, 2501-17 (1992).
[55] R Lynch, Phys Rev A 49 2800-5 (1994).
[56] H Moya-Cessa, A Vidiella-Barranco and V Bužek, J Mod Optics 39 1441 (1992).
[57] V Bužek and B Hladky, J Mod Optics 40 in press.
[58] L Mandel, Opt Lett 4 205 (1979).
[59] L Mandel, Phys Rev Lett 49 136 (1982).
[60] A El Allati, S Robles-Pérez, Y Hassouni and P F González-Díaz, Quantum Inf Process 12 2587 (2013).
[61] E P Wigner, Phys Rev 40 794 (1932).
[62] A Kenfack and K Zyczkowski, J Opt B: Quantum Semiclass Opt 6 396-404 (2004).
[63] M Hillery, R F O’Connell, M O Scully and E P Wigner, Phys Rep. 106, 123 (1984).
[64] K E Cahill and R J Glauber, Phys Rev 177 1882 (1996).
[65] D T Smithey, M Beck, M G Raymer and A Faridani, *Phys Rev Lett* **70** 1244 (1993).

[66] T J Dunn, I A Walmsley and S Mukamel, *Phys Rev Lett* **74** 884 (1995).

[67] G Breitenbach, S Schiller and J Mlynek, *Nature* **387**, 471 (1997).

[68] K Banaszek, C Radzewicz and K W’odkiewicz, *Phys Rev A* **60** 674 (1999).

[69] P Lougovski, E Solano, Z M Zhang, H Walther, H Mack and P W Schleich, *Phys Rev Lett* **91** 010401 (2003).

[70] W E Lamb, *Phys Today* **22** 23 (1969).

[71] K W’odkiewicz, *Phys Rev Lett* **52** 1064 (1984).

[72] A Royer, *Phys Rev Lett* **55** 2745 (1985).

[73] K Banaszek and K W’odkiewicz, *Phys Rev Lett* **76** 4344 (1996).

[74] U Leonhardt, 1997, *Measuring the Quantum State of Light* (Cambridge: Cambridge University Press).

[75] B Braverman: The Wigner function and quantum state tomography (2012).

[76] B Mielnik, *Rep Math Phys* **12** 331 (1977).

[77] B Mielnik, *J Math Phys* **27** 2290 (1986).

[78] D J Fernandez, *Int J Theor Phys* **33** 2037-2047 (1994).

[79] D J Fernández, *SIGMA* **8** 041 (2012).

[80] D Afshar, A Motamedinasab, A Anbaraki and M Jafarpour, *Int J Mod Phys B* **30**, 1650026 (2016).