Representations of finite number of quadratic forms with same rank

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Abstract
Let \(m, n\) be positive integers with \(m \leq n\). Let \(\kappa(m, n)\) be the largest integer \(k\) such that for any (positive definite and integral) quadratic forms \(f_1, \ldots, f_k\) of rank \(m\), there exists a quadratic form of rank \(n\) that represents \(f_i\) for any \(i\) with \(1 \leq i \leq k\). In this article, we determine the number \(\kappa(m, n)\) for any integer \(m\) with \(1 \leq m \leq 8\), except for the cases when \((m, n) = (3, 5)\) and \((4, 6)\). In the exceptional cases, it will be proved that \(1 \leq \kappa(3, 5), \kappa(4, 6) \leq 2\). We also discuss some related topics.

Keywords Quadratic forms · Representations

Mathematics Subject Classification 11E12 · 11E20 · 11E25

1 Introduction

For a positive integer \(m\), let \(\phi_m(X, Y)\) be the classical modular polynomial (for the definition of this, see [6]). For three positive integers \(m_1, m_2, m_3\), Gross and Keating [6] showed that the quotient ring \(R(m_1, m_2, m_3) = \mathbb{Z}[X, Y]/(\phi_{m_1}, \phi_{m_2}, \phi_{m_3})\) is finite if and only if there is no positive definite binary quadratic form \(Q(x, y) = ax^2 + bxy + cy^2\) over \(\mathbb{Z}\) which represents the three integers \(m_1, m_2, m_3\). Moreover, when \(m_1, m_2, m_3\) satisfy this condition, they found an explicit formula for the cardinality of \(R(m_1, m_2, m_3)\). Later, Görtz proved in [5] that there is no positive definite binary...
quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \) over \( \mathbb{Z} \) which represents \( m_1, m_2, m_3 \) if and only if every positive semi-definite half-integral quadratic form with diagonal \((m_1, m_2, m_3)\) is non-degenerate.

Motivated by the above, we consider the following problem: For positive integers \( m \) and \( n \), find the largest integer \( k \) such that for any positive definite quadratic forms \( f_1, f_2, \ldots, f_k \) with rank \( m \), there is a quadratic form of rank \( n \) that represents \( f_i \) for any \( i \) with \( 1 \leq i \leq k \). In this article, the largest integer \( k \) satisfying the above property will be denoted by \( \kappa(m, n) \). As a sample, if \( m = 1 \) and \( n = 2 \), which is exactly the above case, then one may easily show that there does not exist a binary quadratic form that represents 1, 2, and 15. Since there is always a binary quadratic form representing any two positive integers given in advance, we have \( \kappa(1, 2) = 2 \). It seems to be very difficult problem to determine the number \( \kappa(m, n) \) for arbitrary positive integers \( m \) and \( n \). The aim of this article is to determine the number \( \kappa(m, n) \) for any positive integer \( m \) with \( m \leq 8 \) except for the cases when \( (m, n) = (3, 5) \) and \( (4, 6) \). In the exceptional cases, we only have \( 1 \leq \kappa(3, 5), \kappa(4, 6) \leq 2 \). We also discuss some related topics.

The subsequent discussion will be conducted in the better adapted geometric language of quadratic spaces and lattices. A \( \mathbb{Z} \)-lattice \( L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_m \) of rank \( m \) is a free \( \mathbb{Z} \)-module equipped with non-degenerate symmetric bilinear form \( B \) such that \( B(x_i, x_j) \in \mathbb{Q} \) for any \( i, j \) with \( 1 \leq i, j \leq m \). The \( m \times m \) matrix \( (B(x_i, x_j)) \) is called the corresponding symmetric matrix to \( L \), and we write

\[
L \cong (B(x_i, x_j)).
\]

The corresponding quadratic map is defined by \( Q(x) = B(x, x) \) for any \( x \in L \). If \( B(x_i, x_j) = 0 \) for any \( i \neq j \), then we write \( L = (Q(x_1), \ldots, Q(x_m)) \).

We say a \( \mathbb{Z} \)-lattice \( L \) is positive definite if \( Q(x) > 0 \) for any non-zero vector \( x \in L \), and we say \( L \) is integral if \( B(x, y) \in \mathbb{Z} \) for any \( x, y \in L \). We say \( L \) is non-classic integral if \( Q(x) \in \mathbb{Z} \) for any \( x \in L \). Throughout this article, we always assume that a \( \mathbb{Z} \)-lattice is positive definite and integral, unless stated otherwise. In Sect. 2, we also consider representations of even integers by an even integral \( \mathbb{Z} \)-lattice, which corresponds to representations of integers by a non-classic integral \( \mathbb{Z} \)-lattice. Here we say a \( \mathbb{Z} \)-lattice \( L \) is even if \( Q(x) \in 2\mathbb{Z} \) for any \( x \in L \). Note that any even \( \mathbb{Z} \)-lattice is integral. For any positive integer \( j \) not greater than \( m \), the \( j \)-th successive minimum of \( L \) will be denoted by \( m_j(L) \). For the precise definition and various properties on the successive minima, one may consult Chapter 12 of [2].

For two \( \mathbb{Z} \)-lattices \( \ell \) and \( L \), we say \( \ell \) is represented by \( L \) if there is a linear map \( \sigma : \ell \rightarrow L \) such that

\[
B(\sigma(x), \sigma(y)) = B(x, y) \quad \text{for any } x, y \in \ell
\]

Such a linear map \( \sigma \) is called an isometry from \( \ell \) to \( L \). If \( \ell \) is (not) represented by \( L \), then we will use the notation

\[
\ell \rightarrow L \quad (\ell \not\rightarrow L, \text{respectively}).
\]
For any positive integer \( m \), a \( \mathbb{Z} \)-lattice \( L \) is said to be \( m \)-universal if \( L \) represents all \( \mathbb{Z} \)-lattices of rank \( m \). We define
\[
u_{m}(m) := \min \{ \text{rank}(L) : L \text{ is } m \text{-universal} \}.
\]

It is well known that \( \nu_{m}(m) \) should be greater than or equal to \( m + 3 \). In fact, \( \nu_{m}(m) = m + 3 \) for any integer \( m \) with \( 1 \leq m \leq 5 \), and \( \nu_{m}(m) = 13, 15, 16, 28, 30 \) for \( m = 6, 7, 8, 9, 10 \), respectively (see [10]). Furthermore, it is well known that there are only finitely many \( m \)-universal \( \mathbb{Z} \)-lattices of minimal rank \( \nu_{m}(m) \) up to isometry, and for any \( m \) with \( 1 \leq m \leq 8 \), the complete lists of candidates of \( m \)-universal \( \mathbb{Z} \)-lattices with rank \( \nu_{m}(m) \) can be found in \([1,3,7,10,13]\).

As stated above, the aim of this article is to determine \( \kappa(m, n) \) for any integer \( m \) with \( 1 \leq m \leq 8 \). If \( n \geq \nu_{m}(m) \), then there is an \( m \)-universal \( \mathbb{Z} \)-lattice. Hence we may naturally define \( \kappa(m, n) = \infty \) in this case. Therefore, we always assume that
\[
m \leq n \leq \nu_{m}(m) - 1.
\]

For an integer \( m \), we define \( \mathcal{P}(m) \) the set of prime divisors of \( m \). For any integers \( a \) and \( b \), if \( ab^{-1} \in (\mathbb{R}^{\times})^{2} \), then we write \( a \sim b \) over \( R \), where \( R \) is either the \( p \)-adic integer ring \( \mathbb{Z}_{p} \) or \( p \)-adic field \( \mathbb{Q}_{p} \) for some prime \( p \).

Any unexplained notation and terminology can be found in [8] or [11].

2 Quadratic forms representing finite number of positive integers

Recall that for any positive integers \( m, n \), we define \( \kappa(m, n) \) the largest integer \( k \) such that for any \( \mathbb{Z} \)-lattices \( \ell_{1}, \ell_{2}, \ldots, \ell_{k} \) of rank \( m \), there always is a \( \mathbb{Z} \)-lattice of rank \( n \) that represents \( \ell_{i} \) for any \( i \) with \( 1 \leq i \leq k \). In this section, we determine \( \kappa(1, n) \) for any positive integer \( n \). Since there is a universal \( \mathbb{Z} \)-lattice of rank \( 4 \), it suffices to consider the case when \( 1 \leq n \leq 3 \). To begin with, we start proving the following general properties on \( \kappa(m, n) \).

**Theorem 2.1** Let \( m \) and \( n \) be positive integers such that \( m \leq n \), and let \( \kappa(m, n) \) be the integer defined above. Then we have the following properties:

(i) \( \kappa(m, m) = 1 \) for any positive integer \( m \);
(ii) \( \kappa(m + 1, n + 1) \leq \kappa(m, n) \leq \kappa(m, n + 1) \);
(iii) \( \kappa(1, 2) = 2 \) and \( \kappa(m, m + 1) = 1 \) for any \( m \geq 2 \).

**Proof** For the proof of (i), note that there does not exist a \( \mathbb{Z} \)-lattice of rank \( m \) that represents both \( I_{m} \) and \( I_{m-1} \perp \langle 2 \rangle \) simultaneously, where \( I_{m} \) is the \( \mathbb{Z} \)-lattice of rank \( m \) whose corresponding symmetric matrix is the identity matrix.

To prove the first inequality of (ii), let \( k = \kappa(m + 1, n + 1) \). Let \( \ell_{1}, \ldots, \ell_{k} \) be any \( \mathbb{Z} \)-lattices of rank \( m \). From the definition of \( \kappa(m + 1, n + 1) \), there exists a \( \mathbb{Z} \)-lattice \( L \) of rank \( n + 1 \) which represents \( \langle 1 \rangle \perp \ell_{i} \) for any \( 1 \leq i \leq k \). Since the \( \mathbb{Z} \)-lattice \( L \) represents \( 1 \), there is a \( \mathbb{Z} \)-sublattice \( L' \) of \( L \) such that \( L = \langle 1 \rangle \perp L' \). Note that the \( \mathbb{Z} \)-lattice \( L' \) represents \( \ell_{i} \) for any \( i = 1, 2, \ldots, k \). Therefore, we have \( \kappa(m + 1, n + 1) \leq \kappa(m, n) \). The second inequality is almost trivial.
Now, we prove (iii). Note that for any positive integers $a$ and $b$, the unary $\mathbb{Z}$-lattices $\langle a \rangle$ and $\langle b \rangle$ are represented by the binary $\mathbb{Z}$-lattice $\langle a, b \rangle$. One may easily check that there does not exist a binary $\mathbb{Z}$-lattice representing 1, 2, and 15. Therefore, we have $\kappa(1, 2) = 2$. To prove the second assertion, suppose that there is a $\mathbb{Z}$-lattice, say $L$, of rank $m + 1$ that represents both $I_m$ and $I_{m-2}$, i.e., $\langle a, b \rangle$ for some positive integer $a$. However, $(3, 3)$ is not represented by $I_2 \perp \langle a \rangle$ over $\mathbb{Q}_2$ for any positive integer $a$. Therefore, we have $\kappa(m, m + 1) = 1$ for any positive integer $m \geq 2$. \hfill \Box

In fact, we have the following proposition more general than the first part of (iii) in Theorem 2.1.

**Proposition 2.2** For any positive integers $a$ and $b$ which are not contained in the same square class, there are infinitely many positive integers $c$ such that no binary $\mathbb{Z}$-lattice represents $a$, $b$, and $c$ simultaneously.

**Proof** Assume that a binary $\mathbb{Z}$-lattice $L$ represents both $a$ and $b$, and assume that $Q(x) = a$ and $Q(y) = b$ for some $x, y \in L$. Then the two vectors $x$ and $y$ are linearly independent by the hypothesis of the lemma. Hence, we have $m_1(L) \leq \min(a, b)$ and $m_2(L) \leq \max(a, b)$. Therefore, we have $dL \leq m_1(L) \cdot m_2(L) \leq ab$, and hence there are only finitely many $\mathbb{Z}$-lattices of rank 2, up to isometry, that represents both $a$ and $b$.

Now, let $L_1, \ldots, L_t$ be all such binary $\mathbb{Z}$-lattices up to isometry. Let $p$ be any prime such that

$$\left(\frac{-dL_i}{p}\right) = -1 \quad \text{for any } 1 \leq i \leq t,$$

where $(\cdot)$ denotes the Legendre symbol. Then, $p$ is not represented by $(L_i)_p \cong \langle 1, -\Delta_p \rangle$ over $\mathbb{Z}_p$ for any $i$, where $\Delta_p$ is a non-square unit in $\mathbb{Z}_p^\times$. Hence it is not represented by $L_i$ over $\mathbb{Z}$ for any $i = 1, 2, \ldots, t$. The lemma follows from this and the fact that there are infinitely many such primes $p$ by the Chinese Remainder Theorem and the Dirichlet’s theorem on arithmetic progressions. \hfill \Box

**Theorem 2.3** We have $\kappa(1, 3) = 6$.

**Proof** Consider the following seven ternary $\mathbb{Z}$-lattices:

$$L(1) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L(2) = \langle 1 \rangle \perp \begin{pmatrix} 3 & 1 & 0 \\ 1 & 5 & 0 \end{pmatrix}, \quad L(3) = \langle 1, 1, 5 \rangle,$$

$$L(4) = \langle 1 \rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad L(5) = \langle 1, 2, 3 \rangle, \quad L(6) = \langle 1, 1, 1 \rangle, \quad \text{and} \quad L(7) = \langle 1, 1, 2 \rangle.$$

Note that each of these seven $\mathbb{Z}$-lattices has class number 1, and one may easily check that $L(1), \ldots, L(7)$ represent all positive integers except for the integers of the form

$$2^{2s}(8t + 1), \quad 2^{2s+1}(8t + 1), \quad 2^{2s}(8t + 3),$$

$$2^{2s}(8t + 5), \quad 2^{2s+1}(8t + 5), \quad 2^{2s}(8t + 7), \quad \text{and} \quad 2^{2s+1}(8t + 7),$$

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respectively, where \( s \) and \( t \) run over all non-negative integers. Therefore, for any set \( S \) of six positive integers, at least one of \( L(1), \ldots, L(7) \) represents all integers in the set \( S \). This proves that \( \kappa(1,3) \geq 6 \).

On the other hand, we claim that there is no ternary \( \mathbb{Z} \)-lattice that represents \( 1, 2, 3, 5, 10, 14, \) and \( 15 \) simultaneously. Assume to the contrary that there is a ternary \( \mathbb{Z} \)-lattice, say \( L \), which represents all of these seven integers. Since \( L \) represents 1, \( L = \langle 1 \rangle \perp L' \) for some binary \( \mathbb{Z} \)-lattice \( L' \). In order for \( L \) to represent 2, the first successive minimum \( m_1(L') \) of \( L' \) should be less than or equal to 2. Furthermore, in order for \( L \) to represent 3 or 5, \( L' \) should be isometric to one of the following 11 binary \( \mathbb{Z} \)-lattices:

\[
(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (2, 5),
\]

\[
\left( \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right), \left( \begin{array}{cc} 2 & 1 \\ 1 & 3 \end{array} \right), \left( \begin{array}{cc} 2 & 1 \\ 1 & 4 \end{array} \right), \text{ and } \left( \begin{array}{cc} 2 & 1 \\ 1 & 5 \end{array} \right).
\]

One may easily check that none of the 11 \( \mathbb{Z} \)-lattices of the form \( \langle 1 \rangle \perp L' \) represents integers 1, 2, 3, 5, 10, 14, and 15 simultaneously. Therefore, we have \( \kappa(1,3) = 6 \). \( \Box \)

The following proposition says that in many cases, there is a ternary \( \mathbb{Z} \)-lattice that represents all of seven integers given in advance.

**Proposition 2.4** Let \( A = \{a_1, \ldots, a_7\} \) be a set of positive integers such that there does not exist a ternary \( \mathbb{Z} \)-lattice representing all of integers in the set \( A \). Then, we have the following:

(i) For any \( i \neq j \), \( a_i \sim a_j \) over \( \mathbb{Q}_2 \), and \( a_i \sim 6 \) over \( \mathbb{Q}_2 \) for any \( i = 1, 2, \ldots, 7 \);

(ii) For some \( i \) and \( j \), \( a_i \sim 3 \) and \( a_j \sim 6 \) over \( \mathbb{Q}_3 \).

(iii) For some \( i \) and \( j \), \( a_i \sim 5 \) and \( a_j \sim 10 \) over \( \mathbb{Q}_5 \).

(iv) For some \( i \), \( a_i \) is odd. Moreover, under the GRH, it belongs to \( \{3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719\} \).

**Proof** If the set \( A \) does not satisfy the condition (i), then for some \( i = 1, 2, \ldots, 7 \) the ternary \( \mathbb{Z} \)-lattice \( L(i) \) in Theorem 2.3 represents all integers in the set \( A \).

Now, consider the following four ternary \( \mathbb{Z} \)-lattices:

\[
M(1) = \langle 1, 1, 6 \rangle, \ M(2) = \langle 1, 1, 3 \rangle, \ M(3) = \langle 1 \rangle \perp \left( \begin{array}{cc} 2 & 1 \\ 1 & 3 \end{array} \right), \text{ and } M(4) = \langle 1, 2, 5 \rangle.
\]

Each of them has class number 1, and one may easily check that \( M(1), M(2), M(3), \) and \( M(4) \) represents all positive integers except for the integers of the form

\[
3^{2s+1}(3t + 1), 3^{2s+1}(3t + 2), 5^{2s+1}(5t + 1), 5^{2s+1}(5t + 2),
\]

respectively. This proves the assertions (ii) and (iii). To prove the assertion (ii) and (iii), let us consider the ternary \( \mathbb{Z} \)-lattice \( N = \langle 1, 1, 10 \rangle \) which corresponds to the Ramanujan’s ternary quadratic form. Note that \( N \) represents all positive even integers.
except for those of the form $2^{2s+1}(8t + 3)$. Since $A$ does not contain an even integer of the form $2^{2s+1}(8t + 3)$ by (i), $N$ represents all even integers in the set $A$. Moreover, under the GRH, all odd integers which are not represented by $N$ are those integers given above (see [12]). This completes the proof of (iv). \hfill $\square$

In fact, the infinitude of the ring $R(m_1, m_2, m_3)$ defined in the introduction comes from the existence of a non-classical integral binary $\mathbb{Z}$-lattice representing $m_1, m_2,$ and $m_3$. Note that an integer $a$ is represented by a non-classical integral $\mathbb{Z}$-lattice $L$ if and only if $2a$ is represented by the even $\mathbb{Z}$-lattice $L^2$. Here $L^2$ is the $\mathbb{Z}$-lattice obtained from $L$ by scaling 2. Furthermore, for any integral $\mathbb{Z}$-lattice $M$, an even integer $2b$ is represented by $M$ if and only if $2b$ is represented by $M(e) := \{x \in L : Q(x) \equiv 0 \pmod{2}\}$, which is an even integral $\mathbb{Z}$-lattice. Therefore, to deal with a non-classical integral case, we consider the representations of even integers by an integral $\mathbb{Z}$-lattice.

**Theorem 2.5** (1) For any subset $A = \{a_1, \ldots, a_7\}$ of even positive integers, there is a ternary $\mathbb{Z}$-lattice that represents all integers in the set $A$.

(2) For the set $P = \{7, 11, 13, 17, 23, 29, 31, 37, 39\}$ of prime numbers, let

$$N = N_\alpha = 4\alpha \cdot \prod_{p \in P} p,$$

where $\alpha$ is a positive integer satisfying

$$\alpha \equiv - \prod_{p \in P} p \pmod{8} \quad \text{and} \quad \left(\frac{\alpha}{p}\right) = \begin{cases} \prod_{q \in P - \{p\}} \left(\frac{q}{p}\right) & \text{if } p \in \{11, 17\}, \\ - \prod_{q \in P - \{p\}} \left(\frac{q}{p}\right) & \text{if } p \in P - \{11, 17\}. \end{cases}$$

Then, there does not exist a ternary $\mathbb{Z}$-lattice representing 2, 4, 6, 10, 12, 14, 20, and $N$ simultaneously.

**Proof** The assertion (1) follows immediately from (iv) of Proposition 2.4. To prove the assertion (2), assume that there is a ternary $\mathbb{Z}$-lattice $L$ representing all of the integers given above. Let $x_1 \in L$ be a vector such that $Q(x_1) = m_1(L)$. Since $L$ represents 2, we have $Q(x_1) \leq 2$.

If $Q(x_1) = 1$, then $\mathbb{Z}x_1$ does not represent 2. Hence, $L$ contains a vector $x_2 \notin \mathbb{Z}x_1$ such that $Q(x_2) = 2$. If $Q(x_1) = 2$, then $\mathbb{Z}x_1$ does not represent 4, and hence $L$ contains a vector $x_2 \notin \mathbb{Z}x_1$ such that $Q(x_2) = 4$. Therefore, $L$ contains a binary $\mathbb{Z}$-lattice $\mathbb{Z}x_1 + \mathbb{Z}x_2$ which is isometric to one of the following binary $\mathbb{Z}$-lattices:

$$(1 0 2), \quad (1 1 2), \quad (2 0 4), \quad (2 1 4), \quad \text{and} \quad (2 2 4).$$

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Note that each of the above five binary \( \mathbb{Z} \)-lattices does not represent 10, 6, 10, 6, and 6, respectively. Hence, \( L \) contains a vector \( x_3 \not\in \mathbb{Z}x_1 + \mathbb{Z}x_2 \) such that \( Q(x_3) = 6 \) or 10. We define a sublattice \( L' = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \) of \( L \). For example, if \( L \) contains the binary \( \mathbb{Z} \)-lattice \( \langle 1, 2 \rangle \), then

\[
L' = \begin{pmatrix}
1 & 0 & a \\
0 & 2 & b \\
a & b & 10
\end{pmatrix}
\]

for some integers \( a \) and \( b \).

Since \( L' \) is positive definite, \( dL' = 20 - 2a^2 - b^2 > 0 \). Therefore, all possible candidates for \( (a, b) \) are, up to isometry,

\[
(a, b) = (0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (1, 0), (1, 1), (1, 2),
(1, 3), (1, 4), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1).
\]

By considering all possible cases, one may show that there are exactly 52 candidates for \( L' \) up to isometry. Note that if the discriminant of \( L' \) is square-free, then we have \( L = L' \). However, if the discriminant of \( L' \) is not square-free, \( L \) could be a ternary \( \mathbb{Z} \)-lattice properly containing \( L' \).

Among those 52 candidates, there are exactly 34 lattices which do not represent one of the integers 10, 12, 14, and 20. Furthermore, one may show that any \( \mathbb{Z} \)-lattice properly containing one of those 34 lattices does not represent the same integer except for the following the only one case:

\[
L' \cong \langle 2 \rangle \perp \begin{pmatrix}
4 & 2 \\
2 & 8
\end{pmatrix} \quad \text{and} \quad L \cong \langle 2 \rangle \perp \begin{pmatrix}
2 & 1 \\
1 & 4
\end{pmatrix}.
\]

In this exceptional case, the \( \mathbb{Z} \)-lattice \( L \) does represent 14, though \( L' \) does not represent 14. However, \( L \) does not represent \( N \) over \( \mathbb{Z}_7 \) and therefore it does not represent \( N \) over \( \mathbb{Z} \).

Among the remaining 18 (= 52 – 34) \( \mathbb{Z} \)-lattices, exactly nine \( \mathbb{Z} \)-lattices are sublattices of \( \langle 1, 1, 1 \rangle \). Those nine \( \mathbb{Z} \)-lattices do not represent the integer \( N \), for \( \langle 1, 1, 1 \rangle \) does not represent any integer of the form \( 4 \cdot (8k + 7) \). Finally, the remaining 9 (= 18 – 9) \( \mathbb{Z} \)-lattices are one of the following:

\[
\langle 2 \rangle \perp \begin{pmatrix}
2 & 1 \\
1 & 4
\end{pmatrix}, \quad \langle 2 \rangle \perp \begin{pmatrix}
2 & 1 \\
1 & 6
\end{pmatrix}, \quad \begin{pmatrix}
2 & 1 & 0 \\
1 & 4 & 1
\end{pmatrix}, \quad \begin{pmatrix}
2 & 1 & 1 \\
1 & 4 & 2
\end{pmatrix}, \quad \langle 2 \rangle \perp \begin{pmatrix}
4 & 1 \\
1 & 6
\end{pmatrix},

\begin{pmatrix}
2 & 0 & 1 \\
0 & 4 & 1
\end{pmatrix}, \quad \langle 2 \rangle \perp \begin{pmatrix}
4 & 1 \\
1 & 8
\end{pmatrix}, \quad \begin{pmatrix}
2 & 0 & 1 \\
0 & 4 & 1
\end{pmatrix}, \quad \begin{pmatrix}
2 & 0 & 1 \\
0 & 4 & 1
\end{pmatrix}, \quad \langle 2 \rangle \perp \begin{pmatrix}
4 & 1 \\
1 & 10
\end{pmatrix}.
\]

One may directly check that all these \( \mathbb{Z} \)-lattices do not represent the integer \( N \) locally. This completes the proof. \( \square \)
3 Binary case

In this section, we consider the binary case. We also introduce the notion that a pair of \( \mathbb{Z} \)-lattices of rank \( m \) is “buried in rank \( n \)” for some positive integers \( m \) and \( n \) with \( m \leq n \), and deal with some related problems. In the previous section, we have proved that \( \kappa(2, 2) = \kappa(2, 3) = 1 \). To compute the value \( \kappa(2, 4) \), we need some lemmas.

**Lemma 3.1** For any binary \( \mathbb{Z} \)-lattice \( \ell \), there are infinitely many isometry classes of binary \( \mathbb{Z} \)-lattices \( \ell' \) such that the number quaternary \( \mathbb{Z} \)-lattices representing both \( \ell \) and \( \ell' \) is finite up to isometry.

**Proof** Choose a prime \( p \) such that

\[
p \equiv 3 \pmod{4} \quad \text{and} \quad \left( \frac{d\ell}{p} \right) = 1.
\]

Note that there are infinitely many primes satisfying these properties. Let \( L \) be a quaternary \( \mathbb{Z} \)-lattice representing both \( \ell \) and \( \ell' \). Let \( \{ x_i \} \) be a Minkowski reduced basis for \( L \) such that \( \mathcal{Q}(x_i) = m_i(L) \) for each \( i = 1, 2, \ldots, 4 \). Note that such a basis always exists (for this, see [14]). If \( \ell \) is not represented by the \( 3 \times 3 \) section \( L' = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \) of \( L \), then we have \( m_4(L) \leq m_2(\ell) \) by Lemma 2.1 of [9]. Hence the number of possible quaternary \( \mathbb{Z} \)-lattices \( L \) is finite up to isometry. Suppose on the contrary that \( \ell \) is represented by \( L' \). Then, we have \( L'_p \cong (1, 1, \alpha) \) for some \( \alpha \in \mathbb{Z}_p \). Since \( \langle p, p \rangle \) is not represented by \( L' \) over \( \mathbb{Z}_p \), we have \( m_4(L) \leq m_2(\ell'(p)) = p \). This completes the proof. \( \square \)

**Lemma 3.2** For any finite number of quaternary \( \mathbb{Z} \)-lattices \( L_1, \ldots, L_t \), there are infinitely many isometry classes of binary \( \mathbb{Z} \)-lattices \( \ell \) which are not represented by \( L_i \) for any \( i \) with \( 1 \leq i \leq t \).

**Proof** Without loss of generality, we may assume that there is an integer \( j \) with \( 0 \leq j \leq t \) such that

\[
dL_i = \begin{cases} u_i^2 & \text{for } 1 \leq i \leq j, \\ u_i^2 \cdot v_i & \text{for } j + 1 \leq i \leq t, \end{cases}
\]

where \( u_i \)'s are integers for any \( i \) with \( 1 \leq i \leq t \), and \( v_i \)'s are square-free integers greater than 1 for any \( i \) with \( j + 1 \leq i \leq t \). Choose a prime \( p \) such that

(i) \( \left( \frac{u_i}{p} \right) = -1 \) for any \( i \) with \( j + 1 \leq i \leq t \),
(ii) \( (p, 2u_1 \cdots u_i) = 1 \),
(iii) \( p \equiv 1 \pmod{4} \).

For each \( 1 \leq i \leq j \), note that if the Hasse symbol \( S_2(\mathbb{Q}_2L_i) = 1 \), then \( \mathbb{Q}_2L_i \) is the anisotropic space. Hence \( (L_i)_2 \) does not represent any binary isotropic \( \mathbb{Z}_2 \)-lattice in this case. On the other hand, if \( S_2(\mathbb{Q}_2L_i) = -1 \), then by the Hilbert Reciprocity Law, there exists an odd prime \( q = q_i \) such that \( S_q(\mathbb{Q}_qL_i) = -1 \). Hence \( (L_i)_q \) is anisotropic, and therefore \( (L_i)_q \) does not represent any binary isotropic \( \mathbb{Z}_q \)-lattice.
For any $i$ with $j + 1 \leq i \leq t$, we have $(L_i)_p \cong (1, 1, 1, \Delta_p)$. Note that the binary $\mathbb{Z}_p$-lattice $\langle p, -p\Delta_p \rangle$ is not represented by $(L_i)_p$.

Now, choose a positive integer $\alpha$ satisfying

$$\alpha \equiv 7 \pmod{8}, \left(\frac{-\alpha}{p}\right) = -1, \text{ and } \left(\frac{-\alpha}{q_i}\right) = 1,$$

for any $i$ with $1 \leq i \leq j$ such that $S_2(\mathbb{Q}_2L_i) = 1$. Note that there are infinitely many such integers $\alpha$. We define a binary $\mathbb{Z}$-lattice $\ell = \ell(\alpha) = \langle p, p\alpha \rangle$. Then from the construction of $p$ and $\alpha$, the binary $\mathbb{Z}$-lattice $\ell$ is not represented by any of $L_i$’s for any $i$ with $1 \leq i \leq t$, because

\[
\begin{cases}
\ell_2 \cong \langle p, -p \rangle_2 & \Rightarrow (L_i)_2 \text{ if } 1 \leq i \leq j \text{ and } S_2(\mathbb{Q}_2L_i) = 1, \\
\ell_{q_i} \cong \langle p, -p q_i \rangle & \Rightarrow (L_i)_{q_i} \text{ if } 1 \leq i \leq j \text{ and } S_2(\mathbb{Q}_2L_i) = -1, \\
\ell_p \cong \langle p, -p\Delta_p \rangle_p & \Rightarrow (L_i)_p \text{ if } j + 1 \leq i \leq t.
\end{cases}
\]

This completes the proof. \hfill \Box

**Theorem 3.3** For any binary $\mathbb{Z}$-lattice $\ell$, there are infinitely many pairs $(\ell_1, \ell_2)$ of isometry classes of binary $\mathbb{Z}$-lattices such that there does not exist a quaternary $\mathbb{Z}$-lattice representing $\ell$ and $\ell_1, \ell_2$ simultaneously. In particular, we have $\kappa(2, 4) = 2$.

**Proof** For any two binary $\mathbb{Z}$-lattices $\ell_1$ and $\ell_2$, they are represented by the quaternary $\mathbb{Z}$-lattice $\ell_1 \perp \ell_2$. Hence we have $\kappa(2, 4) \geq 2$. Now, the theorem follows directly from Lemmas 3.1 and 3.2. \hfill \Box

**Remark 3.4** As a concrete example of the above theorem, one may easily show that there does not exist a quaternary $\mathbb{Z}$-lattice representing $(1, 1)$, $(3, 3)$, and $(7, 161)$ simultaneously.

Let $\ell_1$ and $\ell_2$ be $\mathbb{Z}$-lattices of rank $m$. We say the pair $(\ell_1, \ell_2)$ of $\mathbb{Z}$-lattices is *buried in rank* $n$ if there is a $\mathbb{Z}$-lattice $L$ of rank $n$ representing both $\ell_1$ and $\ell_2$. Similarly, we define the pair $(\ell_1, \ell_2)$ is buried in rank $n$ over $\mathbb{Z}_p$ ($\mathbb{Q}_p$) if there is a $\mathbb{Z}_p$-lattice $L_p$ ($\mathbb{Q}_p$-space $V_p$) representing both $\ell_1$ and $\ell_2$ over $\mathbb{Z}_p$ ($\mathbb{Q}_p$, respectively). From the definition, if the pair $(\ell_1, \ell_2)$ is buried in rank $n$, then it is buried in rank $r$ for any integer $r \geq n$. Clearly, any pair of $\mathbb{Z}$-lattices of rank $m$ is buried in rank $2m$. For any binary $\mathbb{Z}$-lattice $\ell$, there are infinitely many isometry classes of binary $\mathbb{Z}$-lattices $\ell'$ such that $(\ell, \ell')$ is not buried in rank 3 by Lemma 3.1.

Note that if the pair $(\ell_1, \ell_2)$ of $\mathbb{Z}$-lattices is buried in rank $n$ then it should be buried in rank $n$ over $\mathbb{Z}_p$ for any prime $p$.

**Proposition 3.5** Let $\ell_1$ and $\ell_2$ be even $\mathbb{Z}$-lattices of rank $m$. Then for any prime $p$, the followings are equivalent.

(i) The pair $(\ell_1, \ell_2)$ is buried in rank $n$ over $\mathbb{Z}_p$.
(ii) The pair $(\ell_1, \ell_2)$ is buried in rank $n$ over $\mathbb{Q}_p$.
(iii) As quadratic $\mathbb{Q}_p$-spaces, $\mathbb{Q}_p \ell_1 \cong \mathbb{Q}_p \ell_2$ or $d(\mathbb{Q}_p \ell_1) \neq d(\mathbb{Q}_p \ell_2)$ and $n = m + 1$, or $n \geq m + 2$. 

\[\text{Springer}\]
Proof} One may easily show that (i) implies (ii), and (ii) is equivalent to (iii). To show that (ii) implies (i), let \( V_p \) be a quadratic space that represents \( \mathbb{Q}_p \ell_1 \) and \( \mathbb{Q}_p \ell_2 \) over \( \mathbb{Q}_p \). Choose any \( 2\mathbb{Z}_p \)-maximal \( \mathbb{Z}_p \)-lattice \( L_p \) on \( V_p \). From the definition of the maximal lattice, \( L_p \) represents both \( \ell_1 \) and \( \ell_2 \) over \( \mathbb{Z}_p \).

\[ \exists \]

Remark 3.6 If \( \ell_1 \) or \( \ell_2 \) is not an even \( \mathbb{Z} \)-lattice, then the above proposition does not hold for \( p = 2 \). For example, let \( \ell_1 = (1, 28) \) and \( \ell_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \). Then the pair \((\ell_1, \ell_2)\) is buried in rank 3 over \( \mathbb{Q}_2 \). In fact, the quadratic \( \mathbb{Q}_2 \)-space \( \ell_2 \perp \langle 2 \rangle \) of rank 3 represents \( \langle 1, 28 \rangle \) over \( \mathbb{Q}_2 \). However, there is no \( \mathbb{Z}_2 \)-lattice of rank 3 representing both \( \ell_1 \) and \( \ell_2 \).

Let \( \ell_1 \) and \( \ell_2 \) be \( \mathbb{Z} \)-lattices of rank \( m \). We say the pair \((\ell_1, \ell_2)\) is buried in a genus of rank \( n \) if there is a \( \mathbb{Z} \)-lattice \( L \) of rank \( n \) such that \( \ell_1 \rightarrow L \) and \( \ell_2 \rightarrow L' \) for some \( L' \in \text{gen}(L) \). Note that if the pair \((\ell_1, \ell_2)\) of \( \mathbb{Z} \)-lattices is buried in rank \( n \), then it is buried in a genus of rank \( n \). The following example shows that the converse does not hold in general.

Example 3.7 Let \( \ell_1 = (1, 23) \) and \( \ell_2 = (2, 3) \). One may easily show that there is no ternary \( \mathbb{Z} \)-lattice representing both \( \ell_1 \) and \( \ell_2 \). However, we see that

\[
\ell_1 \rightarrow L_1 = \langle 1 \rangle \perp \begin{pmatrix} 5 \\ 1 \\ 23 \end{pmatrix} \quad \text{and} \quad \ell_2 \rightarrow L_2 = \langle 2, 3, 19 \rangle,
\]

and \( L_2 \in \text{gen}(L_1) \). Hence the pair \((\ell_1, \ell_2)\) is buried in \( \text{gen}(L_1) \) of rank 3.

Proposition 3.8 Let \( \ell_1, \ell_2 \) be even \( \mathbb{Z} \)-lattices of rank \( m \). Then the pair \((\ell_1, \ell_2)\) is buried in rank \( n \) over \( \mathbb{Q}_p \) for any prime \( p \) if and only if it is buried in a genus of rank \( n \).

Proof} Note that the “if” part is trivial. To prove the “only if” part, we first prove that there is a quadratic \( \mathbb{Q} \)-space \( V \) representing both \( \mathbb{Q} \ell_1 \) and \( \mathbb{Q} \ell_2 \). If \( n = m \) or \( n \geq m + 2 \), then the existence of such a quadratic \( \mathbb{Q} \)-space follows directly from Proposition 3.5. Assume that \( n = m + 1 \). Let \( \alpha \) be a positive integer. Note that by the Local–Global principle, we have

\[
\mathbb{Q} \ell_1 \rightarrow \mathbb{Q} \ell_2 \perp \langle \alpha \rangle \quad \iff \quad (d \ell_1 d \ell_2, \alpha)_p = S_p \ell_1 \cdot S_p \ell_2 \cdot (d \ell_1 d \ell_2, d \ell_2)_p \text{ for any prime } p,
\]

where \( S_p(\cdot) \) is the Hasse symbol over \( \mathbb{Q}_p \). One may easily show that there is a subset \( \mathcal{P}_0 \subset \mathcal{P}(2d \ell_1 d \ell_2) \) and a prime \( q \not\in \mathcal{P}(2d \ell_1 d \ell_2) \) such that

\[
\left( d \ell_1 d \ell_2, \prod_{r \in \mathcal{P}_0} r \cdot q \right)_p = S_p \ell_1 \cdot S_p \ell_2 \cdot (d \ell_1 d \ell_2, d \ell_2)_p \text{ for any prime } p \in \mathcal{P}(2d \ell_1 d \ell_2).
\]

Note that if \( d \ell_1 d \ell_2 \sim 1 \) over \( \mathbb{Q}_p \), then the above equality holds, for \( \mathbb{Q}_p \ell_1 \cong \mathbb{Q}_p \ell_2 \) from the assumption. For any prime \( p \not\in \mathcal{P}(2d \ell_1 d \ell_2) \cup \{q\} \), one may easily check that

\[
\mathbb{Q}_p \ell_1 \rightarrow \mathbb{Q}_p \ell_2 \perp \left( \prod_{r \in \mathcal{P}_0} r \cdot q \right).
\]
Since 3.1 holds for any prime $p \neq q$, $\mathbb{Q}\ell_1$ is represented by $V = \mathbb{Q}\ell_2 \perp \langle \prod_{r \in P_0} r \cdot q \rangle$ by the Hilbert Reciprocity Law and the Local–Global principle. Now, one may easily show that the pair $(\ell_1, \ell_2)$ is buried in the genus of a $2\mathbb{Z}$-maximal lattice on $V$ of rank $n + 1$.

**Corollary 3.9** Let $\ell_1, \ell_2$ be $\mathbb{Z}$-lattices of rank $m$ which are not necessarily even. Then the pair $(\ell_1, \ell_2)$ is buried in rank $n$ over $\mathbb{Z}_p$ for any prime $p$ if and only if it is buried in a genus of rank $n$.

**Proof** Since the proof is quite similar to the above, we only provide the proof of the “only if” part in the case when $n = m + 1$. Assume that $K(2)$ is a $\mathbb{Z}_2$-lattice representing both $\ell_1$ and $\ell_2$ over $\mathbb{Z}_2$. Then one may suitably choose an integer $\alpha$ in Proposition 3.8 so that the quadratic $\mathbb{Q}$-space $V$ defined there represents $K(2)$ over $\mathbb{Q}_2$. Let $K$ be any $2\mathbb{Z}$-maximal lattice on $V$. Then both $\ell_1$ and $\ell_2$ are represented by $K$ over $\mathbb{Z}_p$ for any prime $p \neq 2$. Now, define a $\mathbb{Z}$-lattice $L$ such that

$$L_p = \begin{cases} K(2) & \text{if } p = 2, \\ K_p & \text{otherwise.} \end{cases}$$

Then clearly, the pair $(\ell_1, \ell_2)$ is buried in the genus $L$ of rank $n + 1$.

We say a $\mathbb{Z}$-lattice $L$ is primitive if there is no integral $\mathbb{Z}$-lattice properly containing $L$ on the quadratic $\mathbb{Q}$-space $Q_L$.

**Theorem 3.10** Let $\ell_1$ and $\ell_2$ be primitive binary $\mathbb{Z}$-lattices such that the pair $(\ell_1, \ell_2)$ is not buried in rank 2. Then the followings are equivalent.

(i) The pair $(\ell_1, \ell_2)$ is buried in rank 3. 
(ii) There exists a positive integer $a$ primitively represented by both $\ell_1$ and $\ell_2$ as well as the following open interval contains an integer:

$$I_{\ell_1, \ell_2, a} := \left( \frac{b_1 b_2 - \sqrt{d_{\ell_1} d_{\ell_2}}}{a}, \frac{b_1 b_2 + \sqrt{d_{\ell_1} d_{\ell_2}}}{a} \right),$$

where $\ell_1 = \begin{pmatrix} a & b_1 \\ b_1 & c_1 \end{pmatrix}$ and $\ell_2 = \begin{pmatrix} a & b_2 \\ b_2 & c_2 \end{pmatrix}$.

**Proof** First, we prove that (i) implies (ii). Let $L$ be a ternary $\mathbb{Z}$-lattice containing both $\ell_1$ and $\ell_2$. Let us write

$$\ell_1 = \mathbb{Z}u_1 + \mathbb{Z}v_1 \text{ and } \ell_2 = \mathbb{Z}u_2 + \mathbb{Z}v_2.$$ 

Since the vectors $u_1, u_2, v_1$, and $v_2$ are linearly dependent in $L$, there are integers $a, b, c, d$ such that

$$x := au_1 + bv_1 = cu_2 + dv_2 \quad \text{and} \quad (a, b, c, d) = 1.$$
We claim that \((a, b) = (c, d) = 1\). Suppose on the contrary that, without loss of generality, there exists a prime \(p\) such that \(p \mid (a, b)\) but \(p \nmid g := (c, d)\). Then, we have
\[
\frac{x}{g} = \left(\frac{c}{g}\right) u_2 + \left(\frac{d}{g}\right) v_2 = \frac{p}{g} \left[ \left(\frac{a}{p}\right) u_1 + \left(\frac{b}{p}\right) v_2 \right] \in \ell_2 \cap pL.
\]
Hence \(\frac{x}{g}\) is a primitive vector in \(\ell_2\), whereas it is not a primitive vector in \(L\). This is a contradiction to the assumption that \(\ell_2\) is primitive. Therefore, we have \((a, b) = (c, d) = 1\), and hence \(x\) is a primitive vector in both \(\ell_1\) and \(\ell_2\). Moreover, \(x\) is also a primitive vector in \(L\). Let
\[
\ell_1 = \mathbb{Z}x + \mathbb{Z}x_1 \approx \begin{pmatrix} a & b_1 \\ b_1 & c_1 \end{pmatrix} \quad \text{and} \quad \ell_2 = \mathbb{Z}x + \mathbb{Z}x_2 \approx \begin{pmatrix} a & b_2 \\ b_2 & c_2 \end{pmatrix}.
\]
If we consider a ternary \(\mathbb{Z}\)-lattice \(L' := \mathbb{Z}x + \mathbb{Z}x_1 + \mathbb{Z}x_2 \subseteq L\), then one may easily verify that \(B(x_1, x_2)\) belongs to the open interval \(I_{\ell_1, \ell_2, a}\) from the fact that \(dL'\) is positive.

To prove that (ii) implies (i), let \(t\) be an integer in the open interval \(I_{\ell_1, \ell_2, a}\). Consider a ternary \(\mathbb{Z}\)-lattice
\[
L(t) = \begin{pmatrix} a & b_1 & b_2 \\ b_1 & c_1 & t \\ b_2 & t & c_2 \end{pmatrix}.
\]
Clearly, \(L(t)\) represents both \(\ell_1\) and \(\ell_2\), and the condition \(t \in I_{\ell_1, \ell_2, a}\) implies \(dL(t) > 0\), that is, \(L(t)\) is positive definite. \(\square\)

**Corollary 3.11** Let \(\ell_1\) and \(\ell_2\) be binary \(\mathbb{Z}\)-lattices. If there is a positive integer \(a\) with \(a^2 \leq 4d\ell_1 d\ell_2\) that is primitively represented by both \(\ell_1\) and \(\ell_2\), then the pair \((\ell_1, \ell_2)\) is buried in rank 3.

**Proof** From the hypothesis, we may assume that
\[
\ell_1 = \begin{pmatrix} a & b_1 \\ b_1 & c_1 \end{pmatrix} \quad \text{and} \quad \ell_2 = \begin{pmatrix} a & b_2 \\ b_2 & c_2 \end{pmatrix},
\]
for some integers \(b_1, b_2, c_1, \) and \(c_2\). Let \(L(t)\) be the \(\mathbb{Z}\)-lattice defined in the proof of Theorem 3.10. Then \(L(t)\) represents both \(\ell_1\) and \(\ell_2\) for any integer \(t\), though it is not necessarily positive definite. If we choose an integer \(t\) such that \(|t - b_1 b_2/a| \leq 1/2\), then
\[
dL(t) = -a(t - b_1 b_2)^2 + d\ell_1 d\ell_2/a \geq -a/4 + d\ell_1 d\ell_2/a \geq 0.
\]
Therefore, \(L(t)\) is a positive semi-definite \(\mathbb{Z}\)-lattice of rank 3. The corollary follows directly from this. \(\square\)

**Example 3.12** In general, the converse of Corollary 3.11 does not hold. Let us consider the following two binary \(\mathbb{Z}\)-lattices:
\[
\ell_1 = \begin{pmatrix} 21 & 5 \\ 5 & 64 \end{pmatrix} \quad \text{and} \quad \ell_2 = \begin{pmatrix} 24 & 1 \\ 1 & 55 \end{pmatrix}.
\]
They are in the same genus and $d\ell_1 = d\ell_2 = 1319$. The smallest positive integer $a$ that is primitively represented by both $\ell_1$ and $\ell_2$ is 3080. Note that $a^2 > 4d\ell_1d\ell_2$ and

$$\ell_1 \cong \begin{pmatrix} 3080 & 1321 \\ 1321 & 567 \end{pmatrix} \quad \text{and} \quad \ell_2 \cong \begin{pmatrix} 3080 & 1409 \\ 1409 & 645 \end{pmatrix}.$$

Although $\ell_1$ and $\ell_2$ do not satisfy the condition of Corollary 3.11, the ternary $\mathbb{Z}$-lattice $L(604)$ defined in the proof of Theorem 3.10, that is,

$$L(604) = \begin{pmatrix} 3080 & 1321 & 1409 \\ 1321 & 567 & 604 \\ 1409 & 604 & 645 \end{pmatrix},$$

represents both $\ell_1$ and $\ell_2$.

**Conjecture 3.13** Let $\ell_1, \ell_2$ be any binary $\mathbb{Z}$-lattices with $d\ell_1 = d\ell_2$. Moreover, assume that the pair $(\ell_1, \ell_2)$ is buried in a genus of rank 3. Then we conjecture that the pair $(\ell_1, \ell_2)$ is buried in rank 3. We checked that this conjecture is true when $d\ell_1 = d\ell_2 \leq 3000$.

### 4 Lower rank cases

In this section, we compute the value $\kappa(m, n)$ for any integer $m$ with $3 \leq m \leq 8$.

**Theorem 4.1** We have the followings:

(i) $\kappa(m, n) = 1$ for any $(m, n)$ with $m \leq n < u_{\mathbb{Z}}(m) - 1$ and $3 \leq m \leq 8$;

(ii) $\kappa(m, n)$ is given as follows when $n = u_{\mathbb{Z}}(m) - 1$.

| $(m, n)$ | (3, 5) | (4, 6) | (5, 7) | (6, 12) | (7, 14) | (8, 15) |
|----------|--------|--------|--------|---------|---------|---------|
| $\kappa(m, n)$ | 1 or 2 | 1 or 2 | 1      | 2       | 2       | 1       |

**Proof** Note that if $\kappa(5, 7) = 1$, then the results for $3 \leq m \leq 4$ follows immediately by Theorems 2.1 and 3.3. Therefore, we first show that $\kappa(5, 7) = 1$. In fact, we will show that there is no $\mathbb{Z}$-lattice of rank 7 that represents both

$$\ell_1 := A_5 \quad \text{and} \quad \ell_2 := I_2 \perp K \perp \langle 105 \rangle,$$

where $K = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$.

Assume to the contrary that a $\mathbb{Z}$-lattice $L$ of rank 7 represents the above two quinary $\mathbb{Z}$-lattices. Let $L = I_k \perp M$, where $M$ is a $\mathbb{Z}$-lattice of rank $7 - k$ such that $m_1(M) \geq 2$. Since $L$ represents $\ell_2$, we have $k \geq 2$. Furthermore, since $L$ represents the indecomposable lattice $\ell_1 = A_5$, it should be isometric to either $I_2 \perp A_5$ or $I_6 \perp \langle t \rangle$ for
some positive integer \( t \). However, one may easily check that \( \ell_2 \not\rightarrow I_2 \perp A_5 \), which implies that \( L = I_6 \perp \langle t \rangle \) for some positive integer \( t \). Now, since \( \ell_2 \) is represented by \( L \), the binary \( \mathbb{Z} \)-lattice \( K \) is represented by \( I_4 \perp \langle t \rangle \). Since \( K \) is not represented by \( I_4 \), we have \( 1 \leq t \leq 3 \). Under the assumption that \( K \) is a sublattice of \( I_4 \perp \langle t \rangle \), the orthogonal complement \( K^\perp \) of \( K \) in \( I_4 \perp \langle t \rangle \) is isometric to
\[
\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \perp \langle 3 \rangle, \quad \langle 1, 3, 10 \rangle, \quad \text{or} \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \langle 15 \rangle,
\]
according as \( t = 1, 2, \) or \( 3 \), respectively. However, \( \langle 105 \rangle \) is not represented by \( K^\perp \) in any cases, which contradicts to the assumption that \( \ell_2 \rightarrow L \).

Now, assume that \( m = 6, 7, \) or \( 8 \). Note that if a \( \mathbb{Z} \)-lattice \( L \) represents both \( I_m \) and the root lattice \( E_m \) of rank \( m \), then \( L \) should represent \( I_m \perp E_m \). This implies that \( \kappa(m, n) = 1 \) for any integer \( n \) with \( m \leq n \leq 2m - 1 \). On the other hand, since
\[
A_6 \not\rightarrow I_6 \perp E_6 \quad \text{and} \quad A_6 \begin{bmatrix} 2 \cr 1 \end{bmatrix} \rightarrow I_7 \perp E_7,
\]
we have \( \kappa(6, 12) = \kappa(7, 14) = 2 \). For the definition of the above \( \mathbb{Z} \)-lattice, see [4]. This completes the proof. \( \square \)

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