Predicate and relation liftings for coalgebras with side effects

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**Motivation**

- Hennessy-Milner logic
- Bisimilarity
- Coalgebraic modal logic
- Behavioural equivalence

**Running example:** Conditional transition systems (CTSs).
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- Hennessy-Milner logic
- Bisimilarity
- Coalgebraic modal logic
- Behavioural equivalence

- Predicate liftings
- Relation liftings

Predicate and relations liftings for coalgebras living in (co)Kleisli categories and Eilenberg-Moore categories.
Motivation

Predicate liftings

Relation liftings

- Predicate and relations liftings for coalgebras living in (co)Kleisli categories and Eilenberg-Moore categories.
- Running example: Conditional transition systems (CTSs).
Conditional transition systems (CTSs)

To model a software family

- CTSs are compact representation of a family of LTSs.
- Transitions are labelled with conditions $K$ and action labels $A$. For simplicity, we let $A$ to be singleton.
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- CTSs are compact representation of a family of LTSs.
- Transitions are labelled with conditions $K$ and action labels $A$. For simplicity, we let $A$ to be singleton.
Coalgebraic modelling of CTSs

- \( C = \text{coKl}(G) \) where \( G = K \times \_ : \)

\[
\begin{align*}
X \in \text{Set} & \quad \Rightarrow \quad X \in \text{coKl}(G) \\
GX \xrightarrow{f} Y \in \text{Set} & \quad \Rightarrow \quad X \xrightarrow{f} Y \in \text{coKl}(G)
\end{align*}
\]

- Branching type is given by \( \text{coKl}(G) \xrightarrow{\bar{P}} \text{coKl}(G) \), which maps \( X \mapsto P X \) and \( GX \xrightarrow{f} Y \in \text{Set} : \)

\[
G\mathcal{P}X \xrightarrow{\bar{P}f} \mathcal{P}Y \quad \bar{P} f(k, U) = \{ f(k, x) | x \in U \}.
\]

- More abstractly, \( \bar{P} \) is a coKleisli extension of \( \mathcal{P} \).

- A CTS is a coalgebra \( X \xrightarrow{\alpha} \bar{P}X \in \text{coKl}(G) \).
LIFTING OF PREDICATE LIFTINGS

- Systems are coalgebras living in $\mathcal{C}$, where $\mathcal{C}$ can be (co)Kleisli or Eilenberg-Moore category induced by some set (co)monad.

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\bar{F}} & \mathcal{C} \\
\downarrow\text{\textbullet} & & \downarrow\text{\textbullet} \\
\text{Set} & \xrightarrow{F} & \text{Set}
\end{array}
\]

- Predicates on sets are given by $\text{Set}^{\text{op}} \xrightarrow{\hat{\mathcal{P}}} \text{Cat}$:

\[X \mapsto (\mathcal{P}X, \subseteq) \text{ and } f \mapsto f^{-1}.\]

- So predicates on $\mathcal{C}$ is given by $\Phi$ defined as the composition

\[\mathcal{C}^{\text{op}} \xrightarrow{\text{\textbullet}} \text{Set}^{\text{op}} \xrightarrow{\hat{\mathcal{P}}} \text{Cat}.\]
LIFTING OF PREDICATE LIFTINGS

- Systems are coalgebras living in \( C \), where \( C \) can be (co)Kleisli or Eilenberg-Moore category induced by some set (co)monad.

\[
\begin{array}{ccc}
C & \xrightarrow{\bar{F}} & C \\
\downarrow{\|} & & \downarrow{\|} \\
\text{Set} & \xrightarrow{F} & \text{Set}
\end{array}
\]

Research questions

- Predicate liftings (Jacobs 2010) are indexed morphisms of type \( \Phi \to \Phi \bar{F} \).
- Can we lift indexed morphisms of type \( \widehat{\mathcal{P}} \to \widehat{\mathcal{P}}F \) to \( \Phi \to \Phi \bar{F} \)?
LIFTING OF PREDICATE LIFTINGS

When \( C = \text{coKl}(G) \) or \( C = \text{EM}(T) \)

Let \( \bar{F} \) be a coKleisli extension/Eilenberg-Moore lifting of \( F \).

\[
\begin{array}{ccc}
\text{C} & \xrightarrow{\bar{F}} & \text{C} \\
\downarrow \gamma & & \downarrow \gamma \\
\text{Set} & \xrightarrow{F} & \text{Set}
\end{array}
\]

Suppose \( \hat{\mathcal{P}} \xrightarrow{\sigma} \hat{\mathcal{P}}F \) is given.

Then we can define \( \Phi \xrightarrow{\lambda} \Phi \bar{F} \) as follows:

\[
\Phi X = \hat{\mathcal{P}}|X| \xrightarrow{\sigma|X|} \hat{\mathcal{P}}F|X| \xrightarrow{\gamma^*|X|} \hat{\mathcal{P}}|FX| = \Phi \bar{F}X.
\]
When \( C = \text{Kl}(T) \)

Let \( \tilde{F} \) be a Kleisli extension of \( F \).

\[
\begin{array}{ccc}
C & \xrightarrow{\tilde{F}} & C \\
\downarrow & & \uparrow \\
\text{Set} & \xrightarrow{F} & \text{Set}
\end{array}
\]
When $C = \text{Kl}(T)$

- Let $\tilde{F}$ be a Kleisli extension of $F$.

\[
\begin{array}{ccc}
C & \xrightarrow{\tilde{F}} & C \\
\downarrow \| & & \downarrow \| \\
\text{Set} & \xrightarrow{G} & \text{Set}
\end{array}
\]

- Suppose $\hat{P} \xrightarrow{\sigma} \hat{P}G$ is given.

Then we can define $\Phi \xrightarrow{\lambda} \Phi \tilde{F}$ as follows:

\[
\Phi X = \hat{P}|X| \xrightarrow{\sigma|X|} \hat{P}G|X| \xrightarrow{\gamma_X^*} \hat{P}|FX| = \Phi \tilde{F}X.
\]
When \( C = \text{Kl}(T) \)

- Let \( \bar{F} \) be a Kleisli extension of \( F \).

\[
\begin{array}{ccc}
\text{Set} & \xrightarrow{G} & \text{Set} \\
\downarrow \lower{\gamma} & & \downarrow \\
\text{C} & \xrightarrow{\bar{F}} & \text{C}
\end{array}
\]

As an example, \( G \) is typically associated with the branching type of deterministic version of the system of interest. I.e., in the context of NDA, \( G = \_^A \times 2 \) when \( F = A \times \_ + 1 \).
Box Modality for CTSs

\[
\text{coKl}(G) \xrightarrow{\bar{\mathcal{P}}} \text{coKl}(G)
\]

\[
\text{Set} \xrightarrow{\mathcal{P}} \text{Set}
\]

- We need first a predicate lifting for \( F = \mathcal{P} \) on \( \text{Set} \). To this end, take \( \hat{\mathcal{P}} \xrightarrow{\sigma} \hat{\mathcal{P}}F \) for box modality.
- Our \( \gamma \) is given by the distributive law:

\[
GFX = \mathbb{K} \times \mathcal{P}X \xrightarrow{\gamma_X} \mathcal{P}(\mathbb{K} \times X) = FGX
\]

that maps a pair \( (k, U) \mapsto \{k\} \times U \).
- Now upon computing \( \lambda \) we find that:

\[
|\alpha|^{-1} \lambda_X \bar{U} = \{(k, x) \mid \forall x' \ x \xrightarrow{k} x' \implies (k, x') \in \bar{U}\},
\]

where \( \bar{U} \subseteq \mathbb{K} \times X \) or \( \bar{U} \in \Phi X \).
Under some restrictions, relation liftings can be lifted in the same manner as predicate liftings.
Relation liftings

Under some restrictions, relation liftings can be lifted in the same manner as predicate liftings.

\[ \begin{array}{c}
\mathcal{C} \\
\downarrow \quad \downarrow \quad \downarrow
\end{array} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
**Relation liftings**

Under some restrictions, relation liftings can be lifted in the same manner as predicate liftings.

\[ C \xymatrix{ \ar[r]^F & C \ar[d]^-{|_|} }\]

\[ \xymatrix{ \text{Set} & \text{Set} \ar[l]_G }\]

**Research question**

- When can we lift a relation lifting \( \hat{P}(\_ \times \_) \longrightarrow \hat{P}G(\_ \times \_) \) to an indexed morphism \( \Psi \longrightarrow \Psi \bar{F} \)?
Prerequisite on lifting a relation lifting

**Recall that \( \hat{\mathcal{P}} \) is a bifibration**

So for any \( X \xrightarrow{f} Y \in \text{Set} \) we have

\[
\begin{align*}
\hat{\mathcal{P}} X & \quad \xrightarrow{f!} \quad \hat{\mathcal{P}} Y. \\
\end{align*}
\]

**Recall that \( \mathcal{C} \) has product \( \otimes \)**

Moreover, projection functions \( X \otimes X \xrightarrow{\pi_1^X, \pi_2^X} X \in \mathcal{C} \) induce:

\[
|X \otimes X| \xrightarrow{\langle |\pi_1^X|, |\pi_2^X| \rangle} |X| \times |X| \in \text{Set}.
\]

Therefore, for any \( X \in \mathcal{C} \), we have

\[
\Psi X = \hat{\mathcal{P}} |X \otimes X| \xrightarrow{\langle |\pi_1^X|, |\pi_2^X| \rangle!} \hat{\mathcal{P}}(|X| \times |X|).
\]
Our recipe of lifting a relation lifting

Given a diagram

\[
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\bar{F}} & \mathbf{C} \\
\downarrow \gamma & & \downarrow \gamma \\
\text{Set} & \xrightarrow{G} & \text{Set}
\end{array}
\]

then we can lift an indexed morphism \( \hat{\mathcal{P}}(\_ \times \_) \xrightarrow{\sigma} \hat{\mathcal{P}}G(\_ \times \_): \)

\[
\begin{array}{ccc}
\hat{\mathcal{P}}(|X| \times |X|) & \xrightarrow{\sigma|X|} & \hat{\mathcal{P}}(G|X| \times G|X|) \\
& & \xrightarrow{(\gamma_X \times \gamma_X)^{-1}} \\
\Psi X & & \Psi \bar{F}X.
\end{array}
\]
RESULT

If the following commutative square

\[
\begin{array}{ccc}
|X \otimes X| & \langle |\pi_1^X|, |\pi_2^X| \rangle & |X| \times |X| \\
\downarrow |f \otimes f| & & \downarrow |f| \times |f| \\
|Y \otimes Y| & \langle |\pi_1^Y|, |\pi_2^Y| \rangle & |Y| \times |Y|
\end{array}
\]

is a weak pullback in \textbf{Set} (for every \(X \xrightarrow{f} Y \in \mathcal{C}\)) then our recipe indeed defines an indexed morphism.

COROLLARY

When \(\mathcal{C}\) is either a Kleisli or an Eilenberg-Moore category, then the above property is always satisfied.
Behavourial equivalence fibrationaly

Given $X \xrightarrow{\alpha} \bar{F}X \in C$ and $\text{C}^{\text{op}} \xrightarrow{\Psi} \text{Cat}$:

$$\text{Coalg}(\bar{F}) \xrightarrow{} C \xrightarrow{\bar{F}} C$$
**Behavioural equivalence fibrationally**

Given $X \xrightarrow{\alpha} \bar{F}X \in C$ and $C^{\text{op}} \xrightarrow{\Psi} \text{Cat}$:

$$\int \Psi$$

$p$

$\text{Coalg}(\bar{F}) \to C \to \bar{F} \to C$

**$\Psi$ gives rise to a fibration $\int \Psi \xrightarrow{p} C$**

$X \in C \land R \in \Psi X \quad \frac{X \xrightarrow{f} Y \in C \land R \subseteq f^* S \in \Psi X}{(X, R) \in \int \Psi}$

$(X, R) \xrightarrow{f} (Y, S) \in \int \Psi$
**Behavoural Equivalence Fibrationally**

Given \( X \xrightarrow{\alpha} \bar{F}X \in C \) and \( C^{\text{op}} \xrightarrow{\Psi} \text{Cat} \):

\[
\int \Psi \xrightarrow{\bar{F}_\lambda} \int \Psi
\]

\[
\downarrow p \quad \downarrow p
\]

\[
\text{Coalg}(\bar{F}) \xrightarrow{} C \xleftarrow{\bar{F}} C
\]

**Indexed Morphism Gives Rise to a Map of Fibrations**

An indexed morphism \( \Psi \xrightarrow{\lambda} \Psi \bar{F} \) induces \( \bar{F}_\lambda \):

\[
\bar{F}_\lambda(X, R) = (\bar{F}X, \lambda_X R) \quad \bar{F}_\lambda f = \bar{F}f
\]
**BEHAVIOURAL EQUIVALENCE FIBRATIONALLY**

Given $X \xrightarrow{\alpha} FX \in C$ and $C^{\text{op}} \xrightarrow{\Psi} \text{Cat}$:

\[
\text{Coalg}(\bar{F}_\lambda) = \int \Psi \bar{F} \xrightarrow{p} \int \Psi \bar{F}_\lambda \xrightarrow{p} \int \Psi
\]

\[
\text{Coalg}(\bar{F}) \xrightarrow{p} C \xrightarrow{\bar{F}} C
\]

**Coalg($\bar{F}_\lambda$) is again a fibration (Jacobs 2010)**

Moreover, there is an indexed category $\text{Coalg}(\bar{F})^{\text{op}} \xrightarrow{\Psi \bar{F}_\lambda} \text{Cat}$ that $(X, \alpha) \mapsto \text{Coalg}(\alpha^* \circ \lambda_X)$. In the context of a CTS $X \xrightarrow{\alpha} \bar{P}X$,

$R$ is a conditional bisimulation $\iff R \subseteq \alpha^* \lambda_X R.$
Behavoural equivalence fibrationally

Given $X \xrightarrow{\alpha} \bar{F}X \in C$ and $C^{\text{op}} \xrightarrow{\Psi} \text{Cat}$:

$$\text{Coalg}(\bar{F}_\lambda) = \int \Psi \bar{F} \xrightarrow{p} \int \Psi \bar{F}_\lambda \xrightarrow{p} \int \Psi$$



The behavoural conformance functor $1^\lambda$

- It exists when $\Psi \bar{F}_\lambda$ has indexed final objects.
- E.g., in the case of CTS, $1^\lambda(X, \alpha) = (X, \alpha, \equiv_X)$. 
CONCLUSIONS

Objects of our study are *coalgebras with side effects*, i.e., those living in (co)Kleisli/Eilenberg-Moore categories.

- Recipe to construct predicate/relation liftings.
- Conditions when behavioural equivalence can be characterised as coalgebras living in the fibre of ‘abstract’ relations.
- Extended the dual adjunction framework for fibrations by (Kupke and Rot 2020) to coalgebras with side effects.

FUTURE WORK

- Coalgebraic games (Mika-Michalski 2022) / *Chase’s talk on behavioural equivalence games*.
- Shift from behavioural equivalences to distances.
B. P. F. Jacobs. *Predicate Logic for Functors and Monads*. Available from author’s website. 2010. url: http://www.cs.ru.nl/~bart/PAPERS/predlift-indcat.pdf.

C. Kupke and J. Rot. ‘Expressive Logics for Coinductive Predicates’. In: *28th EACSL Annual Conference on Computer Science Logic (CSL 2020)*. Ed. by M. Fernández and A. Muscholl. Vol. 152. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2020, 26:1–26:18.

Christina Mika-Michalski. ‘System Verification Via Generic Games: Behavioural Equivalence and Model Checking Games’. PhD thesis. Feb. 2022.
An extension semantics $\gamma$ that connects the KL-law $\vartheta : FT \to TF$ and the EM-law $\psi' : TG \to GT$ is a natural transformation $\gamma : TF \to GT$ satisfying the following two commutative diagrams:
RESULT

If $\mathcal{T} = \text{Eq} \circ \mathcal{T}$ has a left adjoint $\mathcal{S}$, the logic $(L, \delta)$ is adequate. Moreover it is expressive if $|\delta_A|$ is injective.