Two quantum spin models on the checkerboard lattice with an exact two-fold degenerate Shastry-Sutherland ground state

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Two quantum spin models with bilinear-biquadratic exchange interactions are constructed on the checkerboard lattice. It is proved that, under certain sufficient conditions on the exchange parameters, their ground states consist of two degenerate Shastry-Sutherland singlet configurations. The constructions are studied for arbitrary spin-S. The sufficient conditions for the existence of ferromagnetic ground state are also found exactly. The approximate quantum phase diagrams are presented using the exact results, together with a variational estimate for the Néel antiferromagnetic phase. A two-leg spin-1/2 ladder model, based on one of the above constructions, is considered which admits exact solution for a large number of eigenstates. The ladder model is shown to have exact level-crossing between the rung-singlet state and the AKLT state in the singlet ground state. Also introduced is the notion of perpendicularity for quantum spin vectors, which appears in the discussion on one of the two checkerboard models, and is discussed in the Appendix.

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I. INTRODUCTION

The area of frustrated quantum magnetism is of great current interest. The increasing number of real materials with frustrated spin interactions requires us to have better insight into the nature of possible spin quantum states that may arise due to frustration, and the excitations thereof. While the phenomenological studies are guided by experimental findings, there is also a formal interest in the subject with an aspiration for constructing exactly solvable models, and for investigating possibly realizable new physics. The Shastry-Sutherland (SS) model of frustrated quantum spins, with a surprising realization in SrCu$_2$(BO$_3$)$_2$, is an interesting example of formal studies making sense in real systems.

The SS model is described by an antiferromagnetic bilinear spin exchange Hamiltonian on square lattice with the nearest neighbor (nn) interaction, and with the next nearest neighbor interaction along a select choice of diagonal bonds (see the arrangement of either solid or dashed diagonal bonds in Fig. 2). The lattice with such a specific topology of connections is called the SS lattice. When the diagonal exchange is twice as big or bigger than the nn exchange, then the configuration consisting of dimer-singlets on the diagonal bonds of the SS lattice is an exact ground state of the model. We call this ground state as the SS singlet state (or simply the SS state). The SS state, however, continues to be the ground state for lower values of the diagonal exchange, approximately up to 1.48 times the nn exchange, below which the system undergoes a quantum phase transition into a new state (and finally into the Néel antiferromagnetic state), as suggested by various numerical studies.

In recent times, there has also been a renewed interest in the studies of spin models with multiple exchange interactions (that is, more than bilinear exchange), particularly due to some suggestions that such higher order exchange interactions may be relevant in the insulating, antiferromagnetic phase of the high-T$_C$ cuprates, and also in the spin ladders. However, the models with multiple exchange have been studied for various independent reasons. Historically, the physical importance of multiple exchange was first realized in the studies of magnetism in solid $^3$He, starting effectively with the work of Thouless. Even for strongly correlated electronic systems, from the point of view of the Hubbard model, the presence of multiple exchange spin interactions is rather apparent from the fourth order perturbation theory in $t/U$. The actual physical relevance of such higher order exchange interactions, however, may differ in different materials.

In spin systems, a rather well known example of the models with higher order exchange is the Affleck, Kennedy, Lieb and Tasaki (AKLT) construction. These are exactly solvable models with valence-bond-solid ground states. In one dimension, the AKLT model is of particular interest, as it explicitly demonstrates the Haldane spin-gap conjecture for a spin-1 chain. There have also been studies of the models with multiple exchange interactions on the plaquettes of a two-leg spin ladder, some of which allow exact solution for the ground state (notably that of the AKLT type, besides the rung-singlet and the ferromagnetic ground states).

In this paper, we construct two quantum spin models with bilinear and biquadratic exchange interactions on the checkerboard lattice, and show that they have an exact ground state consisting of two SS singlet states. In Section II we motivate and define these models. In Sections III and IV we describe the proofs of their exact ground states, and also discuss their quantum phase diagrams. Inspired by our discussion on one of the two models, we introduce the notion of perpendicularity for quantum spin vectors, which is discussed in the Appendix. We also study a two-leg spin-1/2 ladder model in Section V which incidentally allows the exact solution of a large
number of eigenstates. Interestingly, the AKLT singlet state (as for a spin-1 chain) becomes an exact ground state of the ladder model which shows level-crossing transition between the rung-singlet and the AKLT states in the singlet ground state. The ladder model is shown to have another ground state with alternating singlet-triplet arrangement on rungs, which gives rise to extensive entropy in this ground state. Finally, we conclude with a summary, and with some general remarks.

II. THE MODELS

The construction of models, that are presented in this paper, is inspired by a simple observation that a SS state is also a zero energy eigenstate of the \( nn \) Heisenberg model on square lattice, but it is not the ground state. This fact is already contained in the proof that the SS state is always an eigenstate of the SS model. Since the \( nn \) Heisenberg model on square lattice is same as the SS model with zero diagonal exchange interaction, there are four degenerate SS eigenstates for the \( nn \) Heisenberg model, two for each of the two sets of checkerboard plaquettes (say, black or white plaquettes in Fig. 1).

Now the question is, "how do we make these SS states also the ground state?" One simple answer is the SS model itself. But the SS lattice has lower translational symmetry as compared to the square lattice (or even the checkerboard lattice), and therefore it can have only one of the four SS states as the ground state. We wish to keep at least the checkerboard (if not the full square lattice) symmetry in our model, and therefore allow the possibility of two degenerate SS singlet configurations in the ground state (though the original motivation was to find a model with four-fold degenerate SS ground state).

For our discussion, we write the Hamiltonian of the \( nn \) Heisenberg model as: 

\[
H = J \sum_{R} h(R, R + x, R + y, R + x + y),
\]

where the summation over \( R \) is done for one sublattice only, and the block-Hamiltonian \( h(R, R + x, R + y, R + x + y) \) acts on a plaquette of the corresponding checkerboard lattice. For the convenience of writing, on a given plaquette, we denote the site indices \( R, R + x, R + x + y \) and \( R + y \) as 1, 2, 3, and 4 respectively (see Fig. 1). In this notation, we define a term \( h(R, R + x, R + y, R + x + y) \equiv h = (S_1 \cdot S_2 + S_3 \cdot S_4 + S_1 \cdot S_4 + S_2 \cdot S_3) \). Denoting the total spin on the diagonal bonds of a plaquette as \( S_{13} = S_1 + S_3 \) and \( S_{24} = S_2 + S_4 \), we have \( h = S_{13} \cdot S_{24} \).

It is evident that \( h \) annihilates any state with a singlet on either of the two diagonal bonds of the corresponding plaquette. This fact gives rise to four degenerate SS eigenstates with zero energy for the square lattice \( nn \) Heisenberg model. If we suitably modify our block-Hamiltonian such that the zero energy states of \( h \) become the lowest energy states of the new block-Hamiltonian, then the lower bound to the ground state energy of the corresponding Hamiltonian on the checkerboard will also be zero. This will give us a ground state with two degenerate SS states. We realize that there are at least two simple ways of achieving our purpose on the checkerboard lattice. These are described below.

A. Definition of Model-1

One way of constructing a model with the ground state consisting of two degenerate SS states, is to introduce an additional term proportional to \((S_{13} \cdot S_{24})^2\) in the previously discussed block-Hamiltonian \( h \), and put these new block-Hamiltonians on the plaquettes of a checkerboard to make a lattice Hamiltonian. To be explicit, the new block-Hamiltonian, \( h_1 \), is given as:

\[
h_1 = J \ h + K \ h^2 \quad \text{where} \quad h = S_{13} \cdot S_{24} \quad (1)
\]

Evidently, the eigenstates of \( h_1 \) are completely given by the eigenstates of \( h \), and specified by the quantum numbers \( S_{13} \) and \( S_{24} \) of the spins on the diagonals and the total block-spin, \( S_{tot} \). The corresponding Hamiltonian on

FIG. 1: The checkerboard lattice. The models considered in the text consist of four-spin block-Hamiltonians sitting on the dark plaquettes of the lattice. Also shown is the convention of spin labeling on a single plaquette.

FIG. 2: Two SS singlet states on a checkerboard. A solid (or dashed) line joining two diagonally opposite sites denotes the singlet state. The solid dimer-singlets form one of the SS states, and the dashed dimers form the other. Thin dotted lines just mark the underlying square lattice.
TABLE I: Eigen-spectrum spectrum of \( h_1 \) for \( S=1/2 \).

| \( S_{13} \) | \( S_{24} \) | Energy eigenvalues |
|---------|---------|-----------------|
| 0       | 0       | 0               |
| 0       | 1       | 0               |
| 1       | 0       | 0               |
| 1       | 1       | \(-2J+4K : S_{tot} = 0\) |

The checkerboard lattice is given as:

\[
H_1 = \sum_R h_1(R, R + x, R + x + y, R + y)
\]

Since the eigenvalues of \( h^2 \) will always be greater than or equal to zero, for sufficiently +ve \( K \), \( h_1 \) will like to have zero energy eigenstates of \( h \) as the ground state (see Table I). This, for the lattice model \( H_1 \), will give us two-fold degenerate SS ground state. The details will be discussed in Section III.

B. Definition of Model-II

The second model is constructed using the following choice of the block-Hamiltonian.

\[
h_{II} = J h + \frac{K}{4} S_{13}^2 S_{24}^2
\]

Again, the block-states are specified by \( S_{13}, S_{24} \) and \( S_{tot} \). In this case, the term proportional to \( K \) is the interaction between the non-singlet states of the diagonal bonds of a plaquette. The lattice model is constructed by assigning some plaquettes to have two diagonal-singlets, then it will leave some other plaquettes unsatisfied. Consider a plaquette on the checkerboard with singlets on both the diagonal bonds. Each of the four plaquettes that share a corner with this plaquette can attain zero energy state only by having a singlet on the diagonal bond that does not have the shared corner (spin). This implies that another plaquette, which shares corners with two of these (four plaquettes), will not be able to form a singlet along any of its two diagonals. Thus, the two SS configurations are the only ones where each plaquette is simultaneously satisfied, and hence a two-fold degenerate ground state.

Under this condition, each plaquette of the checkerboard can simultaneously attain its lowest energy (i.e., zero) eigenstate by forming a singlet on at least one of the two diagonal bonds. We may call the singlet on a diagonal bond a *diagonal-singlet*. The prescription of having one diagonal-singlet per plaquette generates two degenerate SS singlet configurations on a checkerboard (see Fig. 2), which in the present case will form zero energy eigenstates of \( H_1 \). These two SS states will also form the ground state, because the lower bound on the eigenvalues of \( H_2 \) is zero for \( K > |J| \).

If we try to construct a singlet configuration by allowing some plaquettes to have two diagonal-singlets, then it will leave some other plaquettes unsatisfied. Consider a plaquette on the checkerboard with singlets on both the diagonal bonds. Each of the four plaquettes that share a corner with this plaquette can attain zero energy state only by having a singlet on the diagonal bond that does not have the shared corner (spin). This implies that another plaquette, which shares corners with two of these (four plaquettes), will not be able to form a singlet along any of its two diagonals. Thus, the two SS configurations are the only ones where each plaquette is simultaneously satisfied, and hence a two-fold degenerate ground state.

For \( J < 0 \) and \( K < -J \), \( H_I \) has an exact ferromagnetic (FM) ground state, with \( K = |J| \) as the line of level-crossing between the SS and the FM ground states. This condition for the existence of the FM ground state can again be inferred by demanding that the energy of \( S_{tot} = 2 \) in (1,1) sector of Table II be smaller than rest of the energies. Again each plaquette on the checkerboard can attain its lowest energy state by polarizing all the spins in the same direction, thus giving rise to the FM ground state. Next we find the exact conditions for the SS and the FM ground states.

III. DISCUSSION ON MODEL-I

Ground State for Spin-1/2 Case: In order to find the exact ground state of the Hamiltonian \( H_1 \), consider the eigen-spectrum of the block-Hamiltonian \( h_1 \). For spin-1/2, the eigenstates of \( h_1 \) are given in Table I. It is evident that the zero energy eigenstates of \( h_1 \) will become the lowest in energy for \( K > |J| \). We get this condition on \( J \) and \( K \) by demanding that the eigenvalues in (\( S_{13}, S_{24} \))=\((1,1)\) sector are greater than zero.

Under this condition, each plaquette of the checkerboard can simultaneously attain its lowest energy (i.e., zero) eigenstate by forming a singlet on at least one of the two diagonal bonds. We may call the singlet on a diagonal bond a *diagonal-singlet*. The prescription of having one diagonal-singlet per plaquette generates two degenerate SS singlet configurations on a checkerboard (see Fig. 2), which in the present case will form zero energy eigenstates of \( H_1 \). These two SS states will also form the ground state, because the lower bound on the eigenvalues of \( H_2 \) is zero for \( K > |J| \).

If we try to construct a singlet configuration by allowing some plaquettes to have two diagonal-singlets, then it will leave some other plaquettes unsatisfied. Consider a plaquette on the checkerboard with singlets on both the diagonal bonds. Each of the four plaquettes that share a corner with this plaquette can attain zero energy state only by having a singlet on the diagonal bond that does not have the shared corner (spin). This implies that another plaquette, which shares corners with two of these (four plaquettes), will not be able to form a singlet along any of its two diagonals. Thus, the two SS configurations are the only ones where each plaquette is simultaneously satisfied, and hence a two-fold degenerate ground state.

For \( J < 0 \) and \( K < -J \), \( H_I \) has an exact ferromagnetic (FM) ground state, with \( K = |J| \) as the line of level-crossing between the SS and the FM ground states. This condition for the existence of the FM ground state can again be inferred by demanding that the energy of \( S_{tot} = 2 \) in (1,1) sector of Table II be smaller than rest of the energies. Again each plaquette on the checkerboard can attain its lowest energy state by polarizing all the spins in the same direction, thus giving rise to the FM ground state. Next we find the exact conditions for the SS and the FM ground states for spin-S system.

Ground State for Spin-S Case: Let us denote
an eigenvalue of $h$ as $\varepsilon$ and that of $h_1$ as $\varepsilon_1$. From the
definition of $h_1$ (as given in Eq.1), $\varepsilon_1 = J - K \varepsilon^2$. And as
usual, all the states of $h_1$ are labeled by spin quantum numbers $S_{13}$, $S_{24}$ and $S_{tot}$. Let $S_{13}$, $S_{24}$ and $S_{tot}$ take
the values $m$, $n$, and $l$ respectively, where $m$ and $n$ take integer values between 0 and 2S, and $l$ is an integer such that
$|m-n| \leq l \leq m+n$. For a given choice of $l$, $m$, and
$n$, $\varepsilon = \{l(l+1) - m(m+1) - n(n+1)\}/2$, which in turn
gives us the value of $\varepsilon_1$. Notice that $\varepsilon$ will always be an integer
(since all the terms inside \{ \} are even integers).

Now we ask when the states with a diagonal-singlet ($m$
or $n$ = 0) become the ground state of $h_1$. For $m$ (or $n$) = 0, $\varepsilon_1 = 0$ (since $l$ is same as $n$ (or $m$)). Therefore, we
need to find the conditions on $J$ and $K$ such that $\varepsilon_1 \geq 0$.
There are two things to be done. Since any state with
$\varepsilon = 0$ gives $\varepsilon_1 = 0$, it is important to find integer triples
($l$, $m$, $n$) for which $\varepsilon = 0$ (see Appendix). Secondly, we
need to find the condition when all the states with non-zero values of $\varepsilon$
correspond to $+ve$ definite values of $\varepsilon_1$. For $K < 0$, we will always have $+ve$ values of $\varepsilon_1$, and hence
will never satisfy the desired condition. Thus we
need to consider the case when $K > 0$.

For $K > 0$ and $J > 0$, all $+ve$ values of $\varepsilon$ will always
give $+ve$ values for $\varepsilon_1$. For all $+ve$ values of $\varepsilon$, the condition $-J+K|\varepsilon| > 0$ should be satisfied for $\varepsilon_1$ to have only
$+ve$ values. This implies $K > J/|\varepsilon|$. Since the smallest
non-zero value of $|\varepsilon|$ is 1 ($\varepsilon = -1$ for ($l$, $m$, $n$) = (1, 1, 1),
and $\varepsilon = 1$ for ($l$, $m$, $n$) = (2, 1, 1)), $K > J$ is the condition
for the $+ve$ definiteness of $\varepsilon_1$ corresponding to all non-zero values of $\varepsilon$. Hence, for $J > 0$ and $K > J$, the zero energy
eigenstates will form the ground state of $h_1$. For
$K > 0$ and $J < 0$, we need only to consider the states
with $\varepsilon > 0$. The requirement of $\varepsilon_1 > 0$ implies $K > |J|/\varepsilon$,
which in turn implies that the condition $K > |J|$ guarantees $\varepsilon_1 > 0$ for all non-zero $\varepsilon$ states. And hence, the
desired condition for $-ve$ values of $J$ is $K > |J|$. The
two conditions together imply that for $K > |J|$, the states
with $\varepsilon_1 = 0$ will form the ground state of $h_1$.

Thus $K > |J|$ is also a sufficient condition for a zero
energy eigenstate of $H_1$ to be the ground state. We can
therefore try to construct such zero energy states by having
all the plaquettes simultaneously in their zero energy states. As discussed above, unlike spin-1/2 case, now there are two kinds of zero energy states for $h_1$: (1) trivial,
when $S_{13}$ (or $S_{24}$) = 0; and (2) non-trivial, when both
$S_{13}$ and $S_{24}$ are non-singlets. The trivial plaquette states
(i.e., with diagonal-singlets) will generate two degenerate
SS states on the checkerboard, with zero eigenvalue.
And for $K > |J|$, the two SS states will form the exact
ground state of $H_1$.

The first non-trivial plaquette state occurs for spin-1
system, when $S_{13} = 2$, $S_{24} = 2$ and $S_{tot} = 3$. The occurrence
of such states is very sparse in general. See Appendix for further discussion, and for the related notion of
dertermining the quantum phase diagram of $H_1$. We have
disproved the possibility of any other zero energy eigenstates for $H_1$ in a strict mathematical sense, but we
believe that this will be the case from the arguments
above.

In the FM state where spins on each plaquette are
polarized in the same way, each block Hamiltonian simul-
taneously attains maximum $S_{tot}$. Therefore, we can
find the sufficient condition for the FM ground state by
considering the situation when $S_{tot} = 4S$ block-state is
the lowest energy state of $h_1$. We find that $H_1$ can
have FM ground state only for $J < 0$. The sufficient
condition for the existence of the FM ground state is:
$J/2S < K < J/2S(4S-1)$. We derive this by comparing
the energy of $S_{tot} = 4S$ block-state with the energies
of the states ($S_{tot} = 4S-1$, $S_{13} = 2S$, $S_{24} = 2S-1$) for
$K > 0$, and ($S_{tot} = 0$, $S_{13} = 2S$, $S_{24} = 2S$) for $K < 0$
(because these block-states compete against $S_{tot} = 4S$
state for the lowest energy).

Quantum Phase Diagram: In Figs. 3 and 4, we show
the approximate quantum phase diagrams of spin-
1/2 and spin-S cases of Model-I. There are three main
phases in both cases: (1) the two-fold degenerate SS
phase, (2) the FM phase, and (3) the Néel AFM phase.
We have estimated the boundaries of the AFM phase
by taking the Néel state as a variational choice. The
variational energy (per plaquette) of Model-I in the Néel
state is: $\varepsilon_1$ (Néel) = $-4S^2J + 4S^2(4S^2+1)K$.

For $J > 0$, comparing $\varepsilon_1$(Néel) with zero gives $K = J/(4S^2+1)$, which is the variational phase boundary
between the Néel and the SS phases. For $K > J/(4S^2+1)$,
the SS states have lower energy than the Néel state. How-
ever, the exact sufficient condition for the existence of the SS ground state is $K \geq |J|$. But it is not a necessary condition. This means that the SS ground state may continue even for $K < |J|$. There is an interesting point here. The proof for the SS ground state is based on the fact that the lower bound to the ground state energy of $H_I$ is zero for $K \geq |J|$. This does not imply that the zero energy states can not be the ground state for $K < |J|$. Besides, the SS states are always exact eigenstates of $H_I$. Therefore, the actual phase boundary, where the SS ground state gives way to another state, depends upon how the energy of the excited states in the SS phase vary as a function of $J$ and $K$. It is likely that the SS ground state extends further into the region, $K < |J|$. For $J > 0$, therefore, the actual phase boundary for the SS ground state will lie somewhere in the shaded region (we call it a transition region) given by $J/(4S^2 + 1) < K < |J|$, which needs to be found numerically. We believe that, in the transition region, the system will pass through some intermediate complex states before going into the Néel state.

For $J < 0$, the variational boundary, $K = 2J$, for the Néel phase is obtained by comparing $\varepsilon_1(\text{Néel})$ with the FM ground state energy (the FM energy per plaquette: $\varepsilon_1(\text{FM}) = 4S^2J + K(4S^2)^2$). For $K > 2J$, the FM state has lower energy than the Néel state. Again, the actual phase boundary for the FM ground state can be found only numerically, which is bound to lie in the region, $2J < K < J/2S$.

The transition region between the FM and the SS phases is different for spin-1/2 and spin-S ($S \geq 1$) cases. For spin-1/2 system, as shown in Fig. 3, $K = |J|$ is the exact phase boundary for the level-crossing transition between the SS and the FM ground states. This we obtain from the exact sufficient conditions on the existence of the SS and the FM ground states which, for spin-1/2 case, leave no room for uncertainty.

For $S > 1/2$, the SS phase may further extend into the region, $K < |J|$, before giving way to a non-singlet ground state for the $-ve$ values of $J$. The FM phase boundary, $K = -J/2S(4S-1)$, is however an exact first order phase boundary in the present case. We can show this by looking at the exact one-magnon dispersion above the FM ground state. The one-magnon dispersion is given as:

$$\varepsilon_{\text{magnon}}^{\pm}(k) = \varepsilon_0 + 4KS^2 \cos k_x \cos k_y$$

$$\pm \sqrt{\varepsilon_1(k)^2 + |4KS^2 \sin k_x \sin k_y|^2}$$

where $\varepsilon_0 = -4S[J + KS(8S-3)]$ and $\varepsilon_1(k) = 2S[J + 4KS(2S-1)] \cos k_x \cos k_y$. The superscript $\pm$ denotes two branches of the magnon dispersion on the checkerboard lattice, and $k$ takes the values within the half-Brillouin zone (same as the Néel AFM Brillouin zone on the square lattice). What one finds (for $J < 0$) from Eq. 5 is that around $k = 0$, $\varepsilon_{\text{magnon}}^{-}(k) \approx S[J - 2KS(4S-1)](k_x^2 + k_y^2)/2$, which implies that for $K > |J|/2S(4S-1)$, the FM ground state becomes unstable against magnon excitations. Across this line, for $S > 1/2$, we expect the system to undergo a series of transitions into successive non-singlet phases (with decreasing total spin), and finally going into the SS phase. Along the line $K = J/2S$, for $J < 0$, the FM state is stable against the magnons. Therefore, the FM phase is likely to extend further into the shaded region between the FM and AFM phases, before undergoing a first order transition into the AFM state.

### IV. DISCUSSION ON MODEL-II

**Ground State of Spin-1/2 Case:** Consider the block-Hamiltonian, $h_{II}$, whose eigen-spectrum for spin-1/2 case is given in Table II. Just as in the previous discussion, we need to find the conditions for the zero energy states of $h_{II}$ to become the lowest in energy so that $h_{II}$ can realize the SS ground state. It is clear from Table II that the zero energy states of $h_{II}$ become the lowest in energy when $K > 2J$ for $J > 0$, and $K < -J$ for $J < 0$. Since the zero energy states of $h_{II}$ also have at least one diagonal-singlet, just as discussed in the case of $H_I$, we can again construct two degenerate SS singlet states with zero energy which form the exact ground state of $H_{II}$ under the abovementioned conditions. For $J < 0$ and $K < -J$, $S_{\text{tot}} = 2$ state in (1,1) sector is the lowest energy state of $h_{II}$. This gives the exact FM ground state for $H_{II}$.

**Ground State of Spin-S Case:** Let us denote the eigenvalue of $h_{II}$ by $\varepsilon_{II}$. Let the quantum numbers $S_{\text{tot}}$, $S_{13}$, and $S_{24}$ take the values $l$, $m$, and $n$ respectively. Same as in the previous section, $m$ and $n$ take integer values between 0 and $2S$, and the integer $l$ is such that $|m - n| \leq l \leq m + n$. In this notation, $\varepsilon_{II} = \frac{J}{2}(l(l + 1) - m(m + 1) - n(n + 1)) + \frac{K}{2}m(m + 1)n(n + 1)$. Again, when either of the diagonal bonds on a plaquette is in the singlet state ($m = n = 0$), then $\varepsilon_{II} = 0$. Now we try to find the condition for which $\varepsilon_{II} \geq 0$.

Since $\varepsilon_{II}$ is symmetric under the exchange of $m$ and $n$, we will consider $n \geq m$ only. Now consider a sector of states for a given $m$ and $n$. Different states in this sector correspond to different values of $l$. For $K \leq 0$, $h_{II}$ will always have some $-ve$ energy states. In order to make $\varepsilon_{II} \geq 0$, we need $K$ to be sufficiently positive. For $K = 0$, the minimum energy state in a given sector corresponds to $l = n - m$ for $J > 0$, and for $J < 0$,

| $S_{13}$ | $S_{24}$ | Energy eigenvalues |
|---------|---------|-------------------|
| 0       | 0       | 0                 |
| 0       | 1       | 0                 |
| 1       | 0       | 0                 |
| 1       | 1       | $-2J + K : S_{\text{tot}} = 0$ |
|         |         | $-J + K : S_{\text{tot}} = 1$ |
|         |         | $J + K : S_{\text{tot}} = 2$ |
it corresponds to \( l = n + m \). The order (according to energy) of different states in a sector remains the same for any non-zero value of \( K \). This is because the slope of \( \varepsilon_{11} \) with respect to \( K \) depends only on \( m \) and \( n \), and not on \( l \). Thus it is sufficient to consider the variation of the lowest energy state with respect to \( K \), in a given sector. By making \( K \) sufficiently +ve, we can raise these lowest energy states above zero.

For \( J > 0 \), the minimum energy for a given \( m \) and \( n \) is, 
\[
\varepsilon_{11}^{\text{min}}(m, n) = -Jm(n+1) + \frac{K}{4}m(m+1)(n+1).
\]
Demanding \( \varepsilon_{11}^{\text{min}}(m, n) \geq 0 \) gives us \( K \geq 4J/(m+1) \).
The lower bound to the values of \( K \), for which the above-mentioned condition is simultaneously satisfied for all the sectors, is given by \( n, m = 1 \). Thus, \( \varepsilon_{11} \geq 0 \) for \( J > 0 \) and \( K \geq 2J \).

For \( J < 0 \), the minimum energy in a sector is, 
\[
\varepsilon_{11}^{\text{min}}(m, n) = Jmn + \frac{K}{4}m(m+1)n(n+1).
\]
Demanding again that \( \varepsilon_{11}^{\text{min}}(m, n) \geq 0 \) gives us \( K \geq -4J/(m+1) \). This inequality implies that the lower bound to \( K \) again corresponds to \( m, n = 1 \). Thus, \( \varepsilon_{11} \geq 0 \) for \( J < 0 \) and \( K \geq -J \). This condition and the condition in the previous paragraph together provide the exact conditions for the diagonal-singlet block-states to be the lowest in energy. This further implies that two zero energy SS eigenstates form the exact ground state of \( H_{11} \).

As noticed earlier, in a given sector, the lowest energy block-state for \( J < 0 \) corresponds to \( l = n + m \) when \( K = -ve \), then \( m, n = 2S \) is the lowest of them all. However, making \( K +ve \) would, at some point, make \( l = 4S \) state cross above some other state. Remember that states belonging to same sector don’t cross as a function of \( K \), because they have same slope. Thus, for \( J < 0 \), demanding \( \varepsilon_{11}(l^* = 4S, m^* = 2S, n^* = 2S) \) to be less than \( \varepsilon_{11}(l = n + m, m, n) \) gives us the inequality: 
\[
K < -4J/\left\{m^*n^*+mn+n^*+m^*m+2mn(n+m)\right\}.
\]
The lower upper bound on \( K \) for the above inequality gives the sufficient condition for \( l = 4S \) state to be the lowest in energy. The lowest upper bound for \( K \) is achieved when \( m = 2S-1 \) and \( n = 2S \). In other words, it is a condition for level-crossing between the lowest energy state of \( m, n = 2S \) sector and the same of \( m = 2S-1, n = 2S \) sector. Since \( l = 4S \) state on a plaquette gives the FM state on the lattice, we have an exact sufficient condition for the existence of the FM ground state. That is, for \( J < 0 \) and \( K < -J/(2S+1) \), \( H_{11} \) has the FM ground state.

**Quantum Phase Diagram**: The approximate quantum phase diagrams for spin-1/2 and spin-\( S \) cases of Model-II are shown in Figs. 5 and 6. The variational energy per plaquette of \( H_{11} \) in the Néel AFM state is: 
\[
\varepsilon_{11}(\text{Néel}) = -4S^2J + KS^2(2S+1)^2.
\]
Comparing \( \varepsilon_{11}(\text{Néel}) \) with the FM ground state energy \( (\varepsilon_{11}(\text{FM}) = 4S^2J + KS^2(2S+1)^2) \) gives variational boundary, \( J = 0 \), in the region, \( K < 0 \). Incidentally, \( J = 0 \) (for \( K < 0 \)) is also an exact sufficient bound for the FM ground state. Therefore, \( J = 0 \) is an exact first order phase boundary between the FM and the AFM ground states for \( K < 0 \). This is unlike what we saw for Model-I, where we could only give bounds on the transition region between the FM and and AFM phases.

For \( J > 0 \), the variational phase boundary between the SS and the Néel states is given by \( K = J/(S+1/2)^2 \). The exact sufficient condition for the existence of the SS ground state, for \( J > 0 \), implies that the actual line of transition below which the SS state gives way to some other singlet (or total \( S_z = 0 \) state) has to lie in the region \( J/(S+1/2)^2 < K < 2J \), similar to the case of Model-I.

Situation for \( J < 0 \) is again similar to that for Model-I. The line \( K = -J/(2S+1) \), is the exact first order transition line for the FM ground state. The one-magnon dispersion for Model-I is given below.

\[
\varepsilon_{\text{magnon}}(k) = \varepsilon_2 + 2KS^2(2S+1)\cos k_x \cos k_y \pm 2S \sqrt{\varepsilon_3(k^2) + [KS(2S+1)\sin k_x \sin k_y]^2}
\]
TABLE III: Eigen-spectrum of $\tilde{h}_1$ for spin-1/2.

| $S_1$ | $S_2$ | Energy eigenvalues |
|-------|-------|-------------------|
| 0     | 0     | $2G - 2J + 4K : S_{tot} = 0$ |
| 0     | 1     | $2G - J + K : S_{tot} = 1$ |
| 1     | 0     | $2G + J + K : S_{tot} = 2$ |

where $\varepsilon_2 = -4JS - 2KS^2(2S+1)$, and $\varepsilon_3(k) = J[\cos k_x + \cos k_y]$. Again, one finds that the FM ground state becomes unstable against magnons for $J > 0$, and for $K > -J/S(2S+1)$ when $J < 0$. Thus the phase boundaries for the FM ground state of Model-II are exact first order transition lines.

The phase boundary, $K = -J$, for spin-1/2 case is the exact level-crossing line between the FM and the SS ground states. Similar to Model-I, for the case of $S > 1/2$, there is a transition region between the FM and the SS phases, which is also expected to have a series of low total spin phases, finally giving way to the SS ground state. Beyond these intuitive expectations, various shaded regions in the quantum phase diagrams need to be investigated numerically.

V. A TWO-LEG SPIN-1/2 LADDER MODEL WITH EXACT AKLT GROUND STATE

Here, we briefly present a two-leg spin-1/2 ladder model using $h_1$ block-Hamiltonian. The idea is to show an apparent relation between the AKLT construction for spin-1 chain and our $h_1$ construction for a two-leg spin-1/2 ladder. The Hamiltonian for our ladder model is given as: $H_{ladder} = \sum_{r=1}^{L-1} \tilde{h}_1(r, r+1)$, where $r$ is the rung index and $L$ is the total number of rungs. We assume periodic boundary condition. The block-Hamiltonian $\tilde{h}_1$ is a slightly generalized form of $h_1$, as given below. The notation is suitably chosen for a two-leg ladder.

$$\tilde{h}_1(r, r+1) = \frac{G}{2} (S^2_{r} + S^2_{r+1}) + JS_r \cdot S_{r+1} + K(S_r \cdot S_{r+1})^2$$  \hspace{1cm} (7)

where $S_r = S_{r,a} + S_{r,b}$ is the total spin on a rung, and $S_{r,a}$ and $S_{r,b}$ are the two spins of a rung (coming from the two legs, denoted by $a$ and $b$ respectively). For $G = 0$, the form of $\tilde{h}_1$ is same as $h_1$. The eigen-spectrum of $\tilde{h}_1(1,2)$, is given in Table III We can construct a large number of exact eigenstates for this model, but we consider the ground state first.

For $G > 0$, the block-states $(S_1, S_2) = (0, 1)$ or $(1, 0)$ will always cost energy. Thus the block-states that compete for the lowest energy, come from $(0, 0)$ and $(1, 1)$ sector. The state $(0, 0)$ will give rise to the rung singlet (RS) state for $H_{ladder}$, which is a rather commonly occurring ground state in various two-leg ladder models. The sufficient conditions for the RS ground state are: (1) $(J < -G$ and $K > -J - 2G)$, (2) $(-G < J < 3G$ and $K > (J - G)/2)$, and (3) $(J > 3G$ and $K > J - 2G)$. These conditions ensure that the energies of all the block-states in $(1,1)$ sector are greater than zero. The sufficient condition for the FM ground state is : $J < K < -J - 2G$. The other well-known ground state which can be exactly realized for this model is the AKLT singlet state. This will be described in the following paragraph. The RS and the AKLT states can be written as:

$$|RS\rangle = \otimes_{r=1}^{L} |s, r\rangle$$  \hspace{1cm} (8)

$$|AKLT\rangle = Tr \prod_{r=1}^{L} |\Psi, r\rangle$$  \hspace{1cm} (9)

where $|s, r\rangle = (|\uparrow\rangle_{r}^a |\downarrow\rangle_{r}^b + |\downarrow\rangle_{r}^a |\uparrow\rangle_{r}^b)/\sqrt{2}$, and $|\Psi, r\rangle$ is a matrix wavefunction in the triplet sector of the total spin on $r$th rung. It can be written as:

$$|\Psi, r\rangle = \left(\begin{array}{c}
|0, r\rangle \\
\sqrt{2} |\uparrow\rangle, r\rangle \\
|0, r\rangle
\end{array}\right)$$

Here $|1, r\rangle$, $|0, r\rangle$ and $|\uparrow, r\rangle$ are the spin-1 states on a rung. In Eq. 9, $Tr$ denotes the trace of the matrix-product, which gives the AKLT state for a closed chain. Without trace, the matrix-product will give the AKLT state for an open spin-1 chain (equivalently for the ladder in the present case).

For $J = 3K$, $S_{tot} = 0$ and 1 states in (1,1) sector are degenerate while $S_{tot} = 2$ has different energy. Thus for $J = 3K$, $\tilde{h}_1$ actually acts like the AKLT projection operator in (1,1) sector. Therefore, when $G > 0$, the model has the following three ground states corresponding to three different segments on $J = 3K$ line. It has the FM ground state for $K < -G/2$, the RS ground state for $-G/2 < K < G$, and the AKLT ground state for $K > G$.

The point $K = -G/2$ is the level-crossing point between the FM and the RS ground states, and $K = G$ is the same for the RS and the AKLT ground states. At both these points, the ground state is two-fold degenerate, and a domain wall is an exact localized excitation of energy $G$. This is because the block-state on the plaquette at the domain wall (whose one rung belongs to the RS state and the other to the AKLT or FM states), is either $(0,1)$ or $(1,0)$.

For $G < 0$, the block-states, $(0,1)$ and $(1,0)$, are $-ve$ energy states. Their competition with the states in (1,1) sector gives the sufficient conditions for various exact ground states. For $K - |G| > |J|$, the $(0,1),(1,0)$ states are lowest in the energy, and the exact ground state of $H_{ladder}$ consists of configurations with singlets on alternate rungs, and the same for triplets. Let us call it as the Alternating-Singlet-Triplet (AST) state. It can be written explicitly as:

$$|\text{AST}_1\rangle = |s, 1\rangle \otimes |t, 2\rangle \otimes |s, 3\rangle \otimes |t, 4\rangle \otimes \cdots$$  \hspace{1cm} (10)

$$|\text{AST}_2\rangle = |t, 1\rangle \otimes |s, 2\rangle \otimes |t, 3\rangle \otimes |s, 4\rangle \otimes \cdots$$  \hspace{1cm} (11)
where $|s, r\rangle$ and $|t, r\rangle$ denote the singlet and a triplet state on $r^{th}$ rung. Since there are $2 \times (L/2)^3$ degenerate AST states, this ground state has an extensive entropy. The sufficient condition for the FM ground state, when $G < 0$, is: $J < 0$ and $J < K < -J + |G|$. Again, for $J = 3K$, the AKLT state is an exact ground state of $H_{\text{ladder}}$ when $J > 0$, and for $J < 0$, the FM state is the exact ground state. This is still more closer to the spin-1 AKLT model, because depending upon the sign of $J$, the ground state is either FM or AKLT, exactly like the AKLT chain. In fact in the limit of $G \rightarrow -\infty$, $H_{\text{ladder}}$ gets completely projected onto the spin-1 AKLT model.

Finally, about a large number of other exact eigenstate of $H_{\text{ladder}}$. Start with the RS state as the reference eigenstate. Changing the singlet on a rung into a triplet gives a new eigenstate with energy $2G$. We may call these states as one-triplet states. These are localized eigenstates. Next, we can have two rungs having triplets. This case has two different situations depending upon whether the triplet-rungs are nearest neighbors or not. If they are not, then it’s like two independent one-triplet states. If yes then, two neighboring triplet-rungs will form plaquette eigenstates of $(1,1)$ sector. Thus we have two types of eigenstates in two-triplet sector (with reference to the RS state). This is similar to, say, two independent quasi-particles and to their bound-states in a given system. For a state with three rungs having triplets, we have three one-triplet states, two ‘one-triplet + a bound state’ states, and third possibility of a three site open boundary spin-1 bilinear-biquadratic problem (which becomes AKLT only for $J = 3K$).

In simple terms, $H_{\text{ladder}}$ is a one dimensional model, with site variables which are either spin-0 or spin-1, with different local energies (0 or $2G$, respectively). If the site variable is spin-0, then it is inert, because the Hamiltonian does nothing to it. If it’s spin-1, then it can interact with neighboring site variables, and hence an active variable. Therefore, for a given number of active sites, the problem can be resolved into different sectors of a set of independent open boundary spin-1 problems separated by inert sites. For example, the N-active site problem can be resolved into N one-active site problems, or ‘(N-1) one-active site problems + 1 two-active site problem’, and so on, finally into an N-active site problem. For spins greater than 1/2, singlet-rungs will still be inert, but now the number of active rungs (or sites) becomes larger. One can still find the exact conditions for the FM and the RS ground states, but the problem otherwise becomes more involved.

VI. CONCLUSION

We, now, conclude this work with a summary, and with some general remarks. We have presented two exactly solvable quantum spin models, Models-I and II, with bilinear-biquadratic exchange interactions on checkerboard lattice, for arbitrary spin-S. We have shown that two degenerate SS singlet states form an exact ground state of these models. We have also found the sufficient conditions for the existence of both the SS and the FM ground states, and presented approximate quantum phase diagrams. Inspired by our discussion on Model-I, we have also introduced the notion of perpendicularity for quantum spin vectors, which is discussed in the Appendix.

We have, further, discussed a simple two-leg spin-1/2 ladder model, and pointed out its apparent connection with a general bilinear-biquadratic spin-1 chain. Particularly interesting, for a certain ratio of the exchange couplings, is its similarity with the AKLT spin-1 chain, for which it can realize the exact AKLT singlet ground state (which shows level-crossing transition with the FM or the RS ground states). It can also have highly entropic, exact ground state of AST type (see Eqs. (10) & (11)).

It is important to note that the block-construction scheme of Models-I and II, is directly applicable to the models of corner-sharing octahedra, of which the pyrochlore lattice could be an interesting case. A plausible case of further study, for Models-I and II, with a realistic interest, is the coupling to phonons. It is possible that a suitable spin-phonon interaction will favor one of the two SS states by inducing antiferro-distortive arrangement of plaquettes on the checkerboard, though there may still be two such degenerate states. Therefore, the requirement of $K/J$ for the SS ground state may not be a limitation, in a realistic situation, if at all. This, in a way, would be similar to the relation between SrCu$_2$(BO$_3$)$_2$ and the SS model. A brief remark about the classical spins. In this case (with an appropriate rescaling of $K \rightarrow K/S^2$), there is no SS phase. Instead, the zero energy sector has a huge classical degeneracy (which is typical of a frustrated classical antiferromagnet). This degeneracy arises due to the fact that each dimer (in an SS like arrangement) requires one of the two SS states by inducing antiferro-distortive arrangement of plaquettes on the checkerboard, though there may still be two such degenerate states. Therefore, the requirement of $K/J$ for the SS ground state may not be a limitation, in a realistic situation, if at all. The shaded area between the SS and the Néel state (in Figs. 4 and 5) will become the region of zero energy states in the classical limit.

Finally, it is interesting to mention that the exact ground states of the models, that are presented here, are superstable eigenstates. The superstability of an eigenstate of an operator signifies the stability of its being eigenstate against the inclusion of extra interactions which do not commute with the original operator. The dimer-singlet states, of which the SS states are one example, manifest this property rather well. There is also a further stability against the exchange disorder. For example, the existence of the SS ground state for Model-I requires one to have $K > |J|$ on each plaquette. However, $J$ and $K$ on different plaquettes can still be different (say, $J$ having random sign), as long as each plaquette fulfills the sufficient condition. This is because the sufficient condition is a local (plaquette) condition. By the same argument, the FM ground state is also stable against the disorder in $J$ and $K$, as long as the sufficient conditions
are respected. We remark that such a stability against constrained disorder is an important physical aspect of the superstability property.

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APPENDIX: PERPENDICULAR STATES FOR QUANTUM SPINS

The discussion on Model-I led us to the following question. For a system of two quantum spins, \( S_A \) and \( S_B \), what are the states for which \( S_A \cdot S_B = 0 \)? To be precise, it’s about finding an eigenstate \( |\psi\rangle \) of \( S_A \cdot S_B \) such that \( S_A \cdot S_B |\psi\rangle = 0 \). This naturally leads us to formulate the notion of perpendicularity of quantum spin vectors. We develop and elaborate upon this concept in the following.

Classically, the scalar product of two perpendicular vectors is zero. For quantum spins vectors, this is not an obvious proposition. This is because the states describing a quantum spin vector carry information only about the total length and the length of one of its components (say, \( z \)-component). Since the uncertainty about the other two components of a spin vector is inherent in the quantum description, there is no strict sense of direction for a quantum spin vector. The simplest demonstration of this fact can be seen in the relation: \( S \times S = i \mathbf{S} \), where the right hand side is a direct consequence of the quantum mechanical uncertainty in the transverse components (\( \hbar \) is taken to be 1). And hence, the notion of perpendicularity of two quantum spin vectors is not something straightforward. In fact, for the extreme quantum case of spin-1/2, the eigenvalues of the scalar product of two spins are \( 1/4 \) (triplet) or \(-3/4\) (singlet), both non-zero.

The question now is whether it is possible to find a zero eigenvalue state of the scalar product operator of two spins. The answer to this question lies in the integer (or half-integer) solutions of an equation that we may like to call as the quantum analog of the Pythagorean relation.

Let \( l, m \) and \( n \) be the spin quantum numbers for operators \( S_{A+B} = (S_A + S_B) \), \( S_A \) and \( S_B \) respectively. Since \( S_A \cdot S_B = \frac{1}{2} [S_{A+B}^2 - S_A^2 - S_B^2] \), the eigenvalue of a state specified by \( l, m \) and \( n \) is \( \varepsilon(l, m, n) = \frac{1}{2} [(l+1) - m(m+1) - n(n+1)] \). Our interest, here, is to find an eigenstate with \( \varepsilon(l, m, n) = 0 \). We call such an eigenstate as the perpendicular state, and denote it by \( |\perp\rangle \) (to symbolize its equivalence to the perpendicularity of classical vectors). This requires us to find the integer or half-integer solutions (because spin quantum numbers can be integer or half-integer) to the equation:

\[
I(l+1) = m(m+1) + n(n+1) \quad (A.1)
\]

We call Eq. (A.1) as the quantum analog of the Pythagorean equation: \( l^2 = m^2 + n^2 \). Taking \( l, m, n \) to be \( L/2, M/2 \), and \( N/2 \) respectively (where \( L, M, N \) are \(+ve\) integers), Eq. (A.1) can be re-written as:

\[
I(L+2) = M(M+2) + N(N+2)
\]

In this notation, the odd integers correspond to the half-integer spins and the even integers to the integer spins.

One interesting thing that we can say for sure is that, for odd integer values of \( M \) and \( N \), the above equation will never be satisfied for any integer value of \( L \). This means that two half-integer spins can never be in a perpendicular state! This is an exact proposition. And for one more time, it puts half-integer spins in a special place. However, the same is not true for other three cases, that is, \( (M,N) = (odd,even), (even,odd) \) and \( (even,even) \).

The first integer solution to the equation is \((L,M,N) = (7,3,6)\), which corresponds to the spin quantum numbers, \((l,m,n) = (7/2,3/2,3)\). By first we me the smallest value of \( m \) that satisfies Eq. (A.1) for some suitable values of \( n \) and \( l \). Note that for a given solution \((l,m,n)\) of Eq. (A.1), \((l,m,n)\) is also a solution, because Eq. (A.1) is symmetric under the exchange of \( m \) and \( n \). We will, therefore, consider solutions with \( \geq m \) only. This is just a convention for enumerating various perpendicular states. Thus, the first perpendicular state is \(|\perp\rangle = (7/2,3/2,3)\). Then next few perpendicular states are: \((l,m,n) = (3,2,2), (17/2,5/2,8), (6,3,5), (11/2,7/2,4), \ldots \). And, of course, there are many more. One can enumerate successive states by a one-line Mathematica program.

A perpendicular state, \(|\perp\rangle\), also satisfies many relations that hold true for classical perpendicularity, which otherwise will not be satisfied by an arbitrary quantum spin state. This happens to be true precisely because \( S_A \cdot S_B \perp = 0 \). We describe some of these identities below. First we consider the cross-product, \( S_A \times S_B \), of two quantum spins. One can show the following identity for the cross-product operator:

\[
(S_A \times S_B)^2 = S_A^2 S_B^2 - (S_A \cdot S_B)^2 - S_A \cdot S_B \quad (A.2)
\]

Interestingly, in the perpendicular state, \( (S_A \times S_B)^2 \perp = 0 \), which is same as for two perpendicular classical vectors. Of course, here \( S_A^2 = S_A(S_A +1) \), and similarly for \( S_B^2 \). In the following, we describe some more relations that are satisfied by \(|\perp\rangle\), in the spirit of classical perpendicularity.

For classical vectors, the identities, \( S_A \cdot (S_A \times S_B) = 0 \) and \( S_B \cdot (S_A \times S_B) = 0 \) are always true, regardless of the fact whether \( S_A \) is perpendicular to \( S_B \) or not. We will show that, for quantum spin vectors, these are not true in general. However, in the state \(|\perp\rangle\), it happens to be true. In other words, the cross-product operator of two quantum spins can be perpendicular to the spins
themselves only in the perpendicular state of the two spins. The proof is as follows. For quantum spin vectors, one can show that the following identity holds true.

$$S_A \cdot (S_A \times S_B) = -(S_A \times S_B) \cdot S_A = iS_A \cdot S_B$$  \hspace{1cm} (A.3)$$

There are two interesting facts about Eq. A.3. One is that the scalar product of $S_A$ and $S_A \times S_B$ is anti-commuting. We may like to call it as scalar product anti-commutation. And secondly, $S_A \cdot (S_A \times S_B)$ is non-zero, but proportional to $S_A \cdot S_B$. For the state $| \perp \rangle$, this second fact implies that $S_A \cdot (S_A \times S_B)| \perp \rangle = 0$, and similarly for $S_B \cdot (S_A \times S_B)$. We also find that $S_A \cdot (S_A \times (S_A \times S_B))| \perp \rangle = 0$ and $S_B \cdot (S_A \times (S_A \times S_B))| \perp \rangle = -S_A^2 S_B^2 | \perp \rangle$. This is as it should be in the classical sense. The above results follow from the identities:

$$S_A \cdot (S_A \times (S_A \times S_B)) = -S_A \cdot S_B,$$  \hspace{1cm} (A.4)$$

$$S_B \cdot (S_A \times (S_A \times S_B)) = -(S_A \times S_B)^2$$  \hspace{1cm} (A.5)$$

It is encouraging to see such close correspondence between the classical perpendicularity and the quantum state, $| \perp \rangle$. Though one may still like to further understand the physical import of a perpendicular state.