Electron acceleration by an intense laser pulse with
echelon phase modulation

Zheng-Mao Sheng¹,³, Lun-Wu Zhu¹,², M Y Yu¹ and
Zhi-Meng Zhang¹

¹ Institute for Fusion Theory and Simulation, Department of Physics,
Zhejiang University, Hangzhou 310027, People’s Republic of China
² Department of Science, Zhejiang University of Science and Technology,
Hangzhou 310023, People’s Republic of China
E-mail: zmsheng@zju.edu.cn

New Journal of Physics 12 (2010) 013011 (8pp)
Received 14 September 2009
Published 15 January 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/1/013011

Abstract. Electron acceleration, in vacuum, mediated by an intense echelon
phase-modulated long Gaussian laser pulse, is investigated here. Theoretical
and numerical analyses show that this pulse can accelerate an electron to much
higher energies than an unmodulated pulse. The staircase-like phase structure
of the laser field encourages electron trapping in the favorable wave phase, greatly
increasing the effective acceleration distance. Also, the electron motion is now
more in the laser propagation direction. The conditions for efficient acceleration
are obtained. Since the slower electrons can also be efficiently accelerated to
high energies and the accelerated electrons are more axially oriented, the average
energy gain by an initially Maxwellian electron bunch can be more than 13 times
that obtained without the echelon phase modulation. This is at the expense of
somewhat increased spatial and thermal spread, and the accelerated bunch is
also more collimated.

Acceleration of electrons by intense laser pulses has been of much interest in view of potential
applications in modeling astrophysical phenomena, realizing table-top particle accelerators
and fast ignition in inertial confinement fusion [2], [5]–[15]. Most schemes, such as direct
ponderomotive force acceleration, plasma and beat wave acceleration, etc, are based on
interactions between lasers and plasmas. On the other hand, a suitable intense electromagnetic
wave pulse can also accelerate electrons in vacuum with or without a static external magnetic
field, and there has been much progress recently [16]–[25], [28]. A properly injected fast

³ Author to whom any correspondence should be addressed.
electron can be captured by the laser field and accelerated to high energies when the laser is of intensity \( a_0 = eE_0/m_e c \omega c \geq 100 \), where \( e \) and \( m_e \) are the charge and mass of the electron, respectively, and \( E_0 \) and \( \omega \) are the electrical field and frequency of the laser light, respectively [23]–[25].

An electron in vacuum accelerated by a laser field will in general phase slip with respect to the latter because it is slower than the speed of light \( c \). Thus, even if the electron is initially optimally placed in the laser field, it will quickly become out of phase and will decelerate until it is trapped again in the next suitable wave phase. On average the acceleration and deceleration cancel each other out, resulting in very little or no net energy gain by the electron [16, 27]. To extract the oscillation energy of an accelerated electron, one can use an external magnetic field to excite cyclotron resonance, or several overlapping pulses, a tailored asymmetrical laser field, a light-wave-perturbing boundary such as placing a dilute gas in its path, and so on [16, 19, 20], [22]–[25]. To convert laser energy into electron kinetic energy efficiently, the initial speed of the electron should be as great as possible, so that it can be continuously accelerated in the favorable laser phase for a longer time. The higher its speed becomes, the longer the electron can remain in the accelerating light-wave phase. In this paper, we show that greatly enhanced acceleration can be accomplished by using a radially staircase-like, or echelon, phase-modulated laser pulse [1]–[3]. Numerical and analytical studies show that an electron with a relatively low initial velocity can be accelerated to high energies by a laser pulse of relatively low intensity.

Echelon, or staircase, structured laser pulses have been used to measure fast chemical processes [1, 3] and as polarizing laser-beam splitters [4] because the echelon stairs of the phase profile lead to different optical path lengths, resulting in the required time delays. They have also been used to significantly improve the profile of high-power lasers used in inertial confinement fusion by slicing the intense laser beam into an array of beamlets, so that more uniform illumination of the target can be achieved [2]. In our scheme of electron acceleration in vacuum, the phase of the laser field has a radially symmetric echelon profile with the phase increment in each stair step being less than that of the induced spatial incoherence. It is shown that there are two regimes for electron acceleration, one of low energy and the other of high energy. With appropriate slice and phase increments of the echelon laser field, the two regimes partially overlap, so that an initially slow electron can first be accelerated in the low-energy regime to an energy that is high enough to enter the high-energy regime. That is, it can be trapped in the laser pulse for a longer time and can be accelerated to higher energy. It follows that a thermal electron bunch can also be accelerated to high average energy, even when most of the electrons are not optimally injected.

We consider a long linearly polarized echelon phase-modulated laser pulse satisfying Maxwell’s equations to the order \( 1/kw_0 \ll 1 \), where \( w_0 \) is the pulse waist and \( k = 2\pi/\lambda \) is the wave number. The electric and magnetic fields of the laser are given by

\[
E_x = E_0 G \cos(\phi),
\]

\[
E_z = -E_0 G \left[ \frac{x \cos(\phi)}{R(z)} + \frac{2x \sin(\phi)}{kw(z)^2} \right],
\]

\[
B_z = -E_0 G \left[ \frac{y \cos(\phi)}{cR(z)} + \frac{2y \sin(\phi)}{\omega w(z)^2} \right],
\]

and \( B_y = E_x/c \), where \( w(z) = w_0[1 + (z/z_R)^2]^{1/2} \), \( r = \sqrt{x^2 + y^2} \), \( \phi = \omega t - kz + \phi_e - \phi_r + \Delta\phi(z) \), \( \phi_e = \tan^{-1}(z/z_R) \), \( \phi_r = kr^2/2R(z) \), \( R(z) = z[1 + (z_R/z)^2] \), \( z_R = kw_0^2/2 \) is the Rayleigh

New Journal of Physics 12 (2010) 013011 (http://www.njp.org/)
length, $G = [w_0/w(z)] \exp[-r^2/w(z)^2 - (\omega t - k z)^2/\tau_0^2 \omega^2]$ and $\tau_0 \gg 1/\omega$ is the pulse duration. The echelon modulated phase increment is $\Delta \phi(z) = \text{floor}[r w_0/\Delta r w(z)] s \Delta r$, where the floor function is defined as $\text{floor}(x) = \max(n)$ and $n \leq x$ is an integer [26]; $\Delta r w(z)/rw_0$ is the $z$-dependent slice step and $s$ is the gradient of the echelon structure. The wave phase has, thus, a radially symmetric echelon structure, with a phase increment $\Delta \phi$ for each slice increment $\Delta r w(z)/rw_0$, which necessarily depends on $w(z)$ in order that equations (1)–(3) be valid within our assumptions. For our purpose, $s \sim O(0.01)$ and $\Delta r \sim O(\lambda)$. Since the floor function is of order unity, $\Delta \phi$ represents a small modulation of the phase in most regions of the pulse.

The electron trajectory can be obtained numerically by solving the relativistic equation of motion

$$d_t p = -e(E + v \times B),$$

where $v$, $p$ and $\gamma = (1 - v^2/c^2)^{-1/2}$ are the velocity, momentum and normalized (by $m_ec^2$) energy of the electron. In the following, the space, time and electron momentum shall be normalized by $\lambda$, $1/\omega$ and $m_ec$, respectively.

Equations (4) can be solved numerically. We shall first consider the motion of a single test electron. The electron is injected with initial speed $v_0 = 0.1c$ at an angle $\alpha = 21.8^\circ$ with respect to the propagation direction $+z$ of the laser. The laser parameters are $a_0 = 4$, $w_0 = 20 \lambda$, $\tau_0 c = 32 \lambda$ and $\Delta r = 5 \lambda$. Figure 1(a) shows that the electron energy gain $\Delta \gamma = \gamma - \gamma_0$ is about 2.2 MeV for an unmodulated laser pulse. In contrast, figure 1(b) shows that when the latter is echelon phase modulated an electron can gain 192.5 MeV, an increase by almost 90 times.

In order to see why the echelon phase-modulated laser pulse can enhance the electron acceleration, we first Fourier decompose the phase increment

$$\Delta \phi = sr + \sum_{n=1}^{\infty} a_n \sin \frac{2\pi n(r \cos \beta + \Delta \phi \sin \beta)}{\Delta r}$$

which, to the lowest order, is $\phi = t - z - sr$ or $\phi'(t) = 1 - v_z - sv_r$, where the prime symbol denotes the time derivative. The temporal increment ($t$) in the phase $\phi$ is partly canceled by the axial ($z$) as well as the radial ($sr$) increments. More importantly, if $sv_r \sim 1 - v_z$, we have $\phi'(t) \sim 0$, so that $\phi$ changes very slowly, and the electron will remain in the same (favorable)

**Figure 1.** Evolution of the energy of an electron with initial speed $v_0 = 0.1c$ and incident angle $\alpha = 21.8^\circ$. The laser is focused at $z_f = 7\tau_0 c$ and its parameters are $a_0 = 4$, $w_0 = 20 \lambda$, $\tau_0 c = 32 \lambda$ and $\Delta r = 5 \lambda$. (a) For $s = 0$, $r = (-4.2, 0, 128) \lambda$, and (b) for $s = 0.06$, $r = (-31.8, 0.32, 36.1) \lambda$. Both the cases are under optimum conditions.
Figure 2. The electric fields (a) $E_x$ and (b) $E_z$ as seen by the electron as it moves in the laser field. (c) Trajectories of an electron with $s = 0$ and $s = 0.06$. (d) Transverse momentum of an electron with $s = 0.06$. The other parameters are the same as in figure 1(b).

phase region of the light pulse for a long time. In fact, figure 2(a), illustrating the electrical field as seen by the electron, shows that the interaction distance is significantly increased, namely from $z = 700$ to 2700. In this range, the sign of $E_x$ remains almost unchanged. The electron can thus be accelerated continuously to a very high energy. Thereafter the field (now it may be unfavorable) experienced by the electron is very small. That is, the echelon phase-modulated laser pulse leads to prolonged trapping of the electron in the accelerating phase of the laser field. From figure 2(b) and equation (2) we see that the longitudinal field $E_z$ experienced by the electron is on average two orders less than the transverse electric field $E_x$. That is, the transverse electric field $E_x$ is mainly responsible for the relativistic electron acceleration. The trajectory and transverse momentum of the electron are shown in figure 2. From figure 2(c) it can be seen that the electron is accelerated continuously by each beamlet, without which the electron would have experienced both the accelerating and decelerating phases of the laser field.

From the laser-field equations (1)–(3) and the above discussion, one can see that the problem of electron acceleration in a linearly polarized echelon phase-modulated laser pulse is essentially two dimensional. For an analytical model of the present acceleration scheme, we can thus consider an electron moving in the $(x, z)$ plane and set $sr = sx$ (the case $x < 0$ can be accounted for by changing the sign of $s$). For simplicity, we neglect the much weaker longitudinal electrical field. That is, the beam waist $w_0$ and the pulse duration $\tau_0$ are much larger than the other space and time scales. The electron dynamics can then be described by the equations of motion $d_t p_x = -(1 - v_z)E_0 \cos(t - z - sx)$, $d_t p_z = -v_x E_0 \cos(t - z - sx)$ and $d_t \gamma = -v_x E_0 \cos(t - z - sx)$. Since $\gamma - p_z = \text{const} = q_0$ [16] and $\gamma^2 = 1 + p_x^2 + p_z^2$, one
can write
\[ \gamma = \frac{q_0}{2} + \frac{1}{2q_0} + \frac{p_x^2}{2q_0}. \]  
Using \( \phi(t) = t - z - sx \) as the independent variable, we obtain \((q_0 - sp_x)dx \) which is in good agreement with the value \( q_0 \). Thus, the minimum of \( s \) is only \( u_c = - (q_0 - sp_0 \sin \alpha)^2 / 2a_0q_0s \)

is determined by \( (q_0 - sp_0 \sin \alpha)^2 + 2a_0q_0su_c = 0 \). Accordingly, there are two solution regimes. When \( u = u_c \), the only solution of equation (7) is \( p_x = q_0/s \). That is, the two regimes of \( p_x \) merge at \( u = u_c \) (see figure 3(a)). Together they form a complete trajectory of the electron. From equation (7), when \( u = 2 + u_c \) for \( s > 0 \) and \( u = u_c - 2 \) for \( s < 0 \), we see that the momentum reaches a peak value

\[ p_{x, \text{max}} = (q_0 + 2\sqrt{a_0q_0s})/s, \]  
from which, together with equation (6), the maximum energy gain can be calculated.

Using the parameters of the numerical simulation, from equation (10) we obtain \( s_c = 0.0567 \), which is in good agreement with the value \( s = 0.06 \) from the simulation. However, for \( s = 0.06 \), equation (9) yields a peak energy \( \gamma = 520.84 \), which is higher than that \( (\gamma = 391) \) from the numerical simulation. The discrepancy can be attributed to the fact that our rough model neglects the higher-order and higher-dimensional effects, which allow for larger electron scattering.

If \( s < s_c \), the two solutions for \( p_x \) do not overlap and they represent two completely separate electron acceleration regimes (figure 3(a)). In this case, an initially slow electron will remain in the low-energy regime and its acceleration will be limited. For example, for \( s = 0.05 \) and when the other parameters are the same as that of the numerical calculation, the maximum energy gain is only \( \gamma = 79.43 \), much less than the case with \( s \) slightly larger than \( s_c \). However, an initially fast electron starting in the high-energy regime (corresponding to the second solution of \( p_x \)) can still be accelerated to high energy.

The initial phase \( \phi_0 \) of the electron is crucial for efficient acceleration. As mentioned, when \( s = s_c \) the two solutions join at \( u_c = -2 \). This case corresponds to \( \sin \phi_0 = 1 \), or \( \phi_0 = \pi/2 \), and is the only value of \( \phi_0 \) that allows the two solutions to join, so that an electron in the low-energy regime of acceleration can be accelerated and also enter the high-energy regime. Clearly, it is desirable to extend the range of \( \phi_0 \) for which the high- and low-energy regimes can join or overlap. In fact, one can show that overlap of the two regimes occurs if \( \phi_0 \) satisfies...
Figure 3. (a) $\gamma$ versus $u$ from equation (7). (b) Average energy $\gamma_a$ and peak energy $\gamma_p$ of an electron bunch versus $s$. Here, $a_0 = 4$, $v_0 = 0.1c$ and $\alpha = 21.8^\circ$. (c–f) Energy and scattering-angle spectra of 1000 electrons initially Maxwell distributed at 300 K, for (c, e) $s = 0$ and (d, f) $s = 0.11$. The other parameters are the same as in figure 2.

$-1 - u_c \leq \sin \phi_0 \leq 1$ for $s > 0$, and $-1 \leq \sin \phi_0 \leq 1 - u_c$ for $s < 0$. That is, for $s > 0$, when the initial phase of a slow electron lies in $[\phi_c, \pi - \phi_c]$, where $\phi_c = \arcsin(-1 - u_c)$, it can always be accelerated from low to high energy. It can be seen from equation (8) that this can be achieved by increasing the value of $u_c$, or $s$. However, equation (9) indicates that increasing $s$ leads to reduction of the maximum energy gain. On the other hand, the acceleration now is not as initial phase sensitive, so that electrons with less than optimum initial phases can also be accelerated to high energies.

From the above result one can conclude that when a thermal electron bunch is accelerated by an echelon-modulated laser pulse, there should be an optimal value of $s$ so that the average energy gain of the bunch is maximized. To verify this, we inject an electron bunch into the laser pulse. The bunch contains 1000 electrons that are Maxwell distributed at the temperature of 300 K. The electrons are injected at an angle of $\sim 21.8^\circ$ with an emittance of $5^\circ$. The initial
positions of the electrons are randomly distributed within a diameter of $\lambda$. The average energy gain and the peak energy of the 1000 electrons for different $s$ are shown in figure 3(b), where the peak energy is smoothed by averaging the energy of the 50 most energetic electrons. It can be seen from figure 3(b) that the peak value of average energy gain (solid line) occurs for $s \sim 0.11$, and the peak energy (dashed line) occurs for $s \sim 0.06$.

The final energy and scattering-angle spectra of the 1000 electrons for the Gaussian and echelon phase-modulated laser pulses are shown in figures 3(c)–(f). It should be noted that a large portion of the electrons have been accelerated to high energies, so that the energy spread of the bunch is nonthermal. The average energy gain is $\Delta \gamma = 47.52$, which is 13.85 times higher than that ($\Delta \gamma = 3.43$) from a pulse without modulation. Furthermore, from figures 3(e)–(f), it can be seen that the scattering angle of the electrons is also significantly decreased. More energetic electrons can be generated by increasing the initial energy of the electrons and/or the intensity of the laser. For example, with $s = 0.03$ phase modulation, an $a_0 = 7.5$ pulse can accelerate an electron with initial energy $\gamma_0 = 1.0044$ to the GeV level. This is much higher than that achieved by a laser pulse without echelon phase modulation [23].

In this paper, we have considered electron acceleration by a linearly polarized laser pulse with echelon phase modulation. Numerical and analytical results show that a low-energy electron can be accelerated to high energies, and a thermal electron bunch can also be accelerated efficiently. Significant improvement in the bunch divergence is also obtainable. The acceleration can be attributed to the fact that under appropriate laser and injection conditions, the echelon modulated phase partially cancels out the phase increment, or phase slipping, by the electron, so that it can remain in the favorable acceleration phase for longer. In particular, an initially slow electron is accelerated to high (relativistic) energies by the modulated transverse electric field of the laser and propelled forward by the Lorentz force of the intense laser field. For effective acceleration or prolonged trapping of the electron in the favorable phase of the light wave, the radial gradient $s$ of the echelon modulation should be larger than a critical value. However, large $s$ leads to reduction of the peak energy gain, together with less dependence on the initial phase of the electron. It should also be noted that the effectiveness of the present acceleration scheme can be further enhanced by using a multistage design in which the gradients of the echelon phase structures in consecutive laser pulses are appropriately modulated. On the other hand, it should be emphasized that the echelon phase-modulated Gaussian laser pulse discussed here is an approximate solution of the Maxwell equations for long pulses containing a large number of oscillation periods. As it propagates in vacuum the pulse will slowly evolve self-consistently; hence, applying the present scheme to very long-distance acceleration or very short pulses should be done only with great caution [24, 29].

Acknowledgments

This work was supported by the National Natural Science Foundation of China under grant nos 10675102 and 10835003 and the National Hi-Tech Inertial Confinement Fusion Committee of China.

References

[1] Topp M R, Rentzepis P M and Jones R P 1971 Chem. Phys. Lett. 9 1
[2] Lehmberg R H and Obenschain S P 1983 Opt. Commun. 46 27
[3] Weinkauf R, Lehr L, Georgiev D and Schlag E W 1997 Appl. Phys. B 64 515

New Journal of Physics 12 (2010) 013011 (http://www.njp.org/)
[4] Liu S and Chen Y 1995 Opt. Lett. 20 1832
[5] Tajima T and Dawson J M 1979 Phys. Rev. Lett. 43 267
[6] Huang Y C, Zheng D, Tulloch W M and Byer R L 1996 Appl. Phys. Lett. 68 753
[7] Panosky W K H and Breidenbach M 1999 Rev. Mod. Phys. 71 121
[8] Pukhov A, Sheng Z-M and Meyer-ter-Vehn J 1999 Phys. Plasmas 6 2848
[9] Yu W et al 2000 Phys. Rev. Lett. 85 570
[10] Wang P X et al 2001 Appl. Phys. Lett. 78 2253
[11] Sheng Z M et al 2002 Phys. Rev. Lett. 88 055004
[12] Pukhov A and Meyer-ter-Vehn J 2002 Appl. Phys. B 74 355
[13] Yu M Y et al 2003 Phys. Plasmas 10 2468
[14] Liu H, He X T and Chen S G 2004 Phys. Rev. E 69 066409
[15] Leemans W P et al 2006 Nat. Phys. 2 696
[16] Clemmow P C and Daugherty J P 1969 Electrodynamics of Particles and Plasmas (Reading, MA: Addison-Wesley)
[17] Quesnel B and Mora P 1998 Phys. Rev. E 58 3719
[18] Wang J X et al 1998 Phys. Rev. E 58 6575
[19] Wang J X et al 1999 Phys. Rev. E 60 7473
[20] Salamin Y I 2006 Phys. Rev. A 73 043402
[21] Plettner T and Byer R L 2005 Phys. Rev. Lett. 95 134801
[22] Sohbatzadeh F, Mirzanejhad A S and Ghasemi M 2006 Phys. Plasmas 13 123108
[23] Singh K P 2004 Phys. Plasmas 11 1164
[24] Xie B S et al 2007 Appl. Phys. Lett. 91 011118
[25] Chen F C et al 2007 Phys. Scr. 75 340
[26] Sullivan M 2007 Precalculus 8th edn (Englewood Cliffs, NJ: Prentice-Hall) p 86
[27] Lawson J D 1979 IEEE. Trans. Nucl. Sci. 26 4217
[28] Hartemann F V et al 2000 Astrophys. J. 127 347
[29] an der Brügge D and Pukhov A 2009 Phys. Rev. E 79 016603

New Journal of Physics 12 (2010) 013011 (http://www.njp.org/)