Polynomial bound for the partition rank vs the analytic rank of tensors

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Abstract

A tensor defined over a finite field $\mathbb{F}$ has low analytic rank if the distribution of its values differs significantly from the uniform distribution. An order $d$ tensor has partition rank 1 if it can be written as a product of two tensors of order less than $d$, and it has partition rank at most $k$ if it can be written as a sum of $k$ tensors of partition rank 1. In this paper, we prove that if the analytic rank of an order $d$ tensor is at most $r$, then its partition rank is at most $f(r, d, |\mathbb{F}|)$, where, for fixed $d$ and $\mathbb{F}$, $f$ is a polynomial in $r$. This is an improvement of a recent result of the author, where he obtained a tower-type bound. Prior to our work, the best known bound was an Ackermann-type function in $r$ and $d$, though it did not depend on $\mathbb{F}$. It follows from our results that a biased polynomial has low rank; there too we obtain a polynomial dependence improving the previously known Ackermann-type bound.

A similar polynomial bound for the partition rank was obtained independently and simultaneously by Milićević.

1 Introduction

1.1 Bias and rank of polynomials

For a finite field $\mathbb{F}$ and a polynomial $P : \mathbb{F}^n \to \mathbb{F}$, we say that $P$ is unbiased if the distribution of the values $P(x)$ is close to the uniform distribution on $\mathbb{F}$; otherwise we say that $P$ is biased. It is an important direction of research in higher order Fourier analysis to understand the structure of biased polynomials.
Note that a generic degree $d$ polynomial should be unbiased. In fact, as we will see below, if a degree $d$ polynomial is biased, then it can be written as a function of not too many polynomials of degree at most $d - 1$. Let us now make this discussion more precise.

**Definition 1.1.** Let $\mathbb{F}$ be a finite field and let $\chi$ be a nontrivial character of $\mathbb{F}$. The bias of a function $f : \mathbb{F}^n \to \mathbb{F}$ with respect to $\chi$ is defined to be $\text{bias}_\chi (f) = \mathbb{E}_{x \in \mathbb{F}^n} \chi(f(x))$. (Here and elsewhere in the paper $\mathbb{E}_{x \in G} h(x)$ denotes $\frac{1}{|G|} \sum_{x \in G} h(x)$.)

**Remark 1.2.** Most of the previous work is on the case $\mathbb{F} = \mathbb{F}_p$ with $p$ a prime, in which case the standard definition of bias is $\text{bias}(f) = \mathbb{E}_{x \in \mathbb{F}^n} \omega f(x)$ where $\omega = e^{2\pi i/p}$.

**Definition 1.3.** Let $P$ be a polynomial $\mathbb{F}^n \to \mathbb{F}$ of degree $d$. The rank of $P$ (denoted $\text{rank}(P)$) is defined to be the smallest integer $r$ such that there exist polynomials $Q_1, \ldots, Q_r : \mathbb{F}^n \to \mathbb{F}$ of degree at most $d - 1$ and a function $f : \mathbb{F}^r \to \mathbb{F}$ such that $P = f(Q_1, \ldots, Q_r)$.

As discussed above, it is known that if a polynomial has large bias, then it has low rank. The first result in this direction was proved by Green and Tao [4] who showed that if $\mathbb{F}$ is a field of prime order and $P : \mathbb{F}^n \to \mathbb{F}$ is a polynomial of degree $d$ with $d < |\mathbb{F}|$ and bias($P$) $\geq \delta > 0$, then $\text{rank}(P) \leq c(\mathbb{F}, \delta, d)$. Kaufman and Lovett [8] proved that the condition $d < |\mathbb{F}|$ can be omitted. In both results, $c$ has Ackermann-type dependence on its parameters. Finally, Bhowmick and Lovett [1] proved that if $d < \text{char}(\mathbb{F})$ and bias($P$) $\geq |\mathbb{F}|^{-s}$, then $\text{rank}(P) \leq c'(d, s)$. The novelty of this result is that $c'$ does not depend on $\mathbb{F}$. However, it still has Ackermann-type dependence on $d$ and $s$.

One of our main results is the following theorem, which improves the result of Bhowmick and Lovett, unless $|\mathbb{F}|$ is very large.

**Theorem 1.4.** Let $\mathbb{F}$ be a finite field and let $\chi$ be a nontrivial character of $\mathbb{F}$. Let $P$ be a polynomial $\mathbb{F}^n \to \mathbb{F}$ of degree $d < \text{char}(\mathbb{F})$. Suppose that $\text{bias}_\chi (P) \geq \epsilon > 0$ where $\epsilon \leq 1/|\mathbb{F}|$. Then

$$\text{rank}(P) \leq (c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)} + 1$$

where $c$ is an absolute constant and $c'(d) = 4^{d^d}$.

Recall that if $G$ is an Abelian group and $d$ is a positive integer, then the Gowers $U^d$ norm (which is only a seminorm for $d = 1$) of $f : G \to \mathbb{C}$ is defined to be

$$\|f\|_{U^d} = \left| \mathbb{E}_{x,y_1,\ldots,y_d \in G} \prod_{S \subset [d]} C_{|S|}^{d-|S|} f(x + \sum_{i \in S} y_i) \right|^{1/2^d},$$

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where $C$ is the conjugation operator. It is a major area of research to understand the structure of functions $f$ whose $U^d$ norm is large. Our next theorem is a result in this direction.

**Theorem 1.5.** Let $\mathbb{F}$ be a finite field and let $\chi$ be a nontrivial character of $\mathbb{F}$. Let $P$ be a polynomial $\mathbb{F}^n \to \mathbb{F}$ of degree $d < \text{char}(\mathbb{F})$. Let $f(x) = \chi(P(x))$ and assume that $\|f\|_{U^d} \geq \epsilon > 0$ where $\epsilon \leq 1/|\mathbb{F}|$. Then

$$\text{rank}(P) \leq (c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)} + 1$$

where $c$ is an absolute constant and $c'(d) = 4^d$.

Our result implies a similar improvement to the bounds for the quantitative inverse theorem for Gowers norms for polynomial phase functions of degree $d$.

**Theorem 1.6.** Let $\mathbb{F}$ be a field of prime order and let $P$ be a polynomial $\mathbb{F}^n \to \mathbb{F}$ of degree $d < \text{char}(\mathbb{F})$. Let $f(x) = \omega^{P(x)}$ where $\omega = e^{2\pi i}$ and assume that $\|f\|_{U^d} \geq \epsilon > 0$ where $\epsilon \leq 1/|\mathbb{F}|$. Then there exists a polynomial $Q : \mathbb{F}^n \to \mathbb{F}$ of degree at most $d - 1$ such that

$$|\mathbb{E}_{x \in \mathbb{F}^n} \omega^{P(x)} \overline{\omega^{Q(x)}}| \geq |\mathbb{F}|^{-c(2^d \cdot \log(1/\epsilon))^{c'(d)-1}}$$

where $c$ is an absolute constant and $c'(d) = 4^d$.

Theorems 1.4 and 1.6 easily follow from Theorem 1.5.

**Proof of Theorem 1.4.** Note that when $f(x) = \chi(P(x))$, then $\|f\|_{U^1}^2 = |\mathbb{E}_{x \in \mathbb{F}^n} f(x + y)| = |\mathbb{E}_{x \in \mathbb{F}^n} f(x)|^2$, so $\|f\|_{U^1}^2 = |\mathbb{E}_{x \in \mathbb{F}^n} f(x)| = |\text{bias}_1(P)|$. However, $\|f\|_{U^d}$ is increasing in $k$ (see eg. Claim 6.2.2 in [6]), therefore $\|f\|_{U^d} \geq |\text{bias}_d(P)| \geq \epsilon$. The result is now immediate from Theorem 1.5.

**Proof of Theorem 1.6.** By Theorem 1.5, there exists a set of $r \leq (c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)} + 1$ polynomials $Q_1, \ldots, Q_r$ such that $P(x)$ is a function of $Q_1(x), \ldots, Q_r(x)$. Then $\omega^{P(x)} = g(Q_1(x), \ldots, Q_r(x))$ for some function $g : \mathbb{F}^r \to \mathbb{C}$. Let $G = \mathbb{F}^r$. Note that $|g(y)| = 1$ for all $y \in G$, therefore $|\hat{g}(\chi)| \leq 1$ for every character $\chi \in \hat{G}$. Now $\omega^{P(x)} = \sum_{\chi \in \hat{G}} \hat{g}(\chi) \chi((Q_1(x), \ldots, Q_r(x)))$, so

$$1 = \mathbb{E}_{x \in \mathbb{F}^n} |\omega^{P(x)}|^2 = \sum_{\chi \in \hat{G}} \overline{\hat{g}(\chi)} \mathbb{E}_{x \in \mathbb{F}^n} \omega^{P(x)} \chi((Q_1(x), \ldots, Q_r(x))).$$

Thus, there exists some $\chi \in \hat{G}$ with $|\mathbb{E}_{x \in \mathbb{F}^n} \omega^{P(x)} \chi((Q_1(x), \ldots, Q_r(x)))| \geq 1/|G| = 1/|\mathbb{F}^r|$. But $\chi$ is of the form $\chi(y_1, \ldots, y_r) = \omega^{\sum_{i \leq r} \alpha_i y_i}$ for some $\alpha_i \in \mathbb{F}$. Then $\chi(Q_1(x), \ldots, Q_r(x)) = \omega^{Q_\alpha(x)}$, where $Q_\alpha$ is the degree $d - 1$ polynomial $Q_\alpha(x) = \sum_{i \leq r} \alpha_i Q_i(x)$. So $Q = Q_\alpha$ is a suitable choice.
1.2 Analytic rank and partition rank of tensors

Related to the bias and rank of polynomials are the notions of analytic rank and partition rank of tensors. Recall that if \( \mathbb{F} \) is a field and \( V_1, \ldots, V_d \) are finite dimensional vector spaces over \( \mathbb{F} \), then an order \( d \) tensor is a multilinear map \( T : V_1 \times \cdots \times V_d \to \mathbb{F} \). (In this subsection, assume that \( d \geq 2 \).) Each \( V_k \) can be identified with \( \mathbb{F}^{n_k} \) for some \( n_k \), and then there exist \( t_{i_1, \ldots, i_d} \in \mathbb{F} \) for all \( i_1 \leq n_1, \ldots, i_d \leq n_d \) such that \( T(v^1, \ldots, v^d) = \sum_{i_1 \leq n_1, \ldots, i_d \leq n_d} t_{i_1, \ldots, i_d} v_1^{i_1} \cdots v_d^{i_d} \) for every \( v^1 \in \mathbb{F}^{n_1}, \ldots, v^d \in \mathbb{F}^{n_d} \) (where \( v_k \) is the \( k \)th coordinate of the vector \( v \)). Indeed, \( t_{i_1, \ldots, i_d} \) is just \( T(e^{i_1}, \ldots, e^{i_d}) \), where \( e^i \) is the \( i \)th standard basis vector.

The following notion was introduced by Gowers and Wolf [3].

**Definition 1.7.** Let \( \mathbb{F} \) be a finite field, let \( V_1, \ldots, V_d \) be finite dimensional vector spaces over \( \mathbb{F} \) and let \( T : V_1 \times \cdots \times V_d \to \mathbb{F} \) be an order \( d \) tensor. Then the analytic rank of \( T \) is defined to be \( \text{arank}(T) = -\log_{|\mathbb{F}|} \text{bias}(T) \), where \( \text{bias}(T) = \mathbb{E}_{v^1 \in V_1, \ldots, v^d \in V_d} [\chi(T(v^1, \ldots, v^d))] \) for any nontrivial character \( \chi \) of \( \mathbb{F} \).

**Remark 1.8.** This is well-defined. Indeed, if \( \chi \) is a nontrivial character of \( \mathbb{F} \), then

\[
\mathbb{E}_{v^1 \in V_1, \ldots, v^d \in V_d} [\chi(T(v^1, \ldots, v^d))] = \mathbb{E}_{v^1 \in V_1, \ldots, v^{d-1} \in V_{d-1}} [\mathbb{E}_{v^d \in V_d} \chi(T(v^1, \ldots, v^d))] \\
= \mathbb{E}_{v^1 \in V_1, \ldots, v^{d-1} \in V_{d-1}} [T(v^1, \ldots, v^{d-1}, x) \equiv 0],
\]

where \( T(v^1, \ldots, v^{d-1}, x) \) is viewed as a function in \( x \). The second equality holds because
\[
\mathbb{E}_{v^d \in V_d} \chi(T(v^1, \ldots, v^d)) = 0 \text{ unless } T(v^1, \ldots, v^{d-1}, x) \equiv 0, \text{ in which case it is } 1.
\]

Thus, \( \mathbb{E}_{v^1 \in V_1, \ldots, v^d \in V_d} [\chi(T(v^1, \ldots, v^d))] \) does not depend on \( \chi \), and is always positive. Moreover, it is at most 1, therefore the analytic rank is always nonnegative.

A different notion of rank was defined by Naslund [13].

**Definition 1.9.** Let \( T : V_1 \times \cdots \times V_d \to \mathbb{F} \) be a (non-zero) order \( d \) tensor. We say that \( T \) has partition rank 1 if there is some \( S \subset [d] \) with \( S \neq \emptyset, S \neq [d] \) such that \( T(v^1, \ldots, v^d) = T_1(v^i : i \in S) T_2(v^i : i \notin S) \) where \( T_1 : \prod_{i \in S} V_i \to \mathbb{F}, T_2 : \prod_{i \notin S} V_i \to \mathbb{F} \) are tensors. In general, the partition rank of \( T \) is the smallest \( r \) such that \( T \) can be written as the sum of \( r \) tensors of partition rank 1. This number is denoted \( \text{prank}(T) \).

Kazhdan and Ziegler [9] and Lovett [11] proved that \( \text{arank}(T) \leq \text{prank}(T) \). In the other direction, it follows from the work of Bhowmick and Lovett [1] that if an order \( d \) tensor \( T \) has \( \text{arank}(T) \leq r \), then \( \text{prank}(T) \leq f(r, d) \) for some function \( f \). Note that \( f \) does not depend on \( |\mathbb{F}| \) or the dimension of the vector spaces \( V_k \). However, \( f \) has an Ackermann-type dependence on \( d \)
and $r$. For $d = 3, 4$, better bounds were established by Haramaty and Shpilka [5]. They proved that for $d = 3$ we have $\text{prank}(T) = \Theta(r^4)$, and that for $d = 4$ we have $\text{prank}(T) = \exp(\Theta(r))$.

The main result of our paper is a polynomial upper bound, which holds for general $d$.

**Theorem 1.10.** Let $T : V_1 \times \cdots \times V_d \to \mathbb{F}$ be an order $d$ tensor with $\text{arank}(T) \leq r$ and assume that $r \geq 1$. Then

$$\text{prank}(T) \leq (c \cdot \log |\mathbb{F}|)^{c'(d)} \cdot r^{c'(d)}$$

for some absolute constant $c$, and $c'(d) = 4^{d^2}$.

We remark that a very similar result was obtained independently and simultaneously by Milićević [12]. Moreover, in the special case $d = 4$, a similar bound was proved independently by Lampert [10].

It is not hard to see that Theorem 1.10 implies Theorem 1.5. Indeed, let $P$ be a polynomial $\mathbb{F}^n \to \mathbb{F}$ of degree $d < \text{char}(\mathbb{F})$, let $f(x) = \chi(P(x))$ and assume that $\|f\|_{U_d} \geq \epsilon > 0$, where $\epsilon \leq 1/|\mathbb{F}|$. Define $T : (\mathbb{F}^n)^d \to \mathbb{F}$ by $T(y_1, \ldots, y_d) = \sum_{S \subset [d]}(-1)^{|S|}P(\sum_{i \in S} y_i)$. By Lemma 2.4 from [3], $T$ is a tensor of order $d$. Moreover, by the same lemma, we have $T(y_1, \ldots, y_d) = \sum_{S \subset [d]}(-1)^{|S|}P(x + \sum_{i \in S} y_i)$ for any $x \in \mathbb{F}^n$. Thus,

$$\text{bias}(T) = \mathbb{E}_{y_1, \ldots, y_d \in \mathbb{F}^n} \chi(T(y_1, \ldots, y_d)) = \mathbb{E}_{y_1, \ldots, y_d \in \mathbb{F}^n} \prod_{S \subset [d]} C^{d-|S|} f(x + \sum_{i \in S} y_i)$$

for any $x \in \mathbb{F}^n$. By averaging over all $x \in \mathbb{F}^n$, it follows that $\text{bias}(T) = \|f\|_{U_d}^d \geq \epsilon^{2^d}$. Thus, $\text{arank}(T) \leq 2^d \log_{|\mathbb{F}|}(1/\epsilon)$. Note that $2^d \log_{|\mathbb{F}|}(1/\epsilon) \geq 1$. Therefore, by Theorem 1.10 with $r = 2^d \log_{|\mathbb{F}|}(1/\epsilon)$, we get

$$\text{prank}(T) \leq (c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)}. \quad (1)$$

Note that $T(y_1, \ldots, y_d) = D_{y_1} \ldots D_{y_d} P(x)$ where $D_x g(x) = g(x + y) - g(x)$. Thus, by Taylor’s approximation theorem, since $d < \text{char}(\mathbb{F})$, we get $P(x) = \frac{1}{d!} D_x \ldots D_x P(0) + W(x) = \frac{1}{d!} T(x, \ldots, x) + W(x)$ for some polynomial $W$ of degree at most $d - 1$.

By equation (1), $T$ can be written as a sum of at most $(c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)}$ tensors of partition rank 1. Hence, $\frac{1}{d!} T(x, \ldots, x)$ can be written as a sum of at most $(c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)}$ expressions of the form $Q(x)R(x)$ where $Q, R$ are polynomials of degree at most $d - 1$ each. Thus, $P - W$ has rank at most $(c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)}$, and therefore $P$ has rank at most

$$(c \cdot 2^d \cdot \log(1/\epsilon))^{c'(d)} + 1.$$
the author in [7], but the argument is improved locally at a few places.

2 The proof of Theorem 1.10

2.1 Notation and preliminaries

In the rest of the paper, we identify $V_i$ with $\mathbb{F}^{n_i}$. Thus, the set of all tensors $V_1 \times \cdots \times V_d \to \mathbb{F}$ is the tensor product $\mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_d}$, which will be denoted by $\mathcal{G}$ throughout this section. Also, $\mathcal{B}$ will always stand for the multiset $\{u_1 \otimes \cdots \otimes u_d : u_i \in \mathbb{F}^{n_i} \text{ for all } i\}$. The elements of $\mathcal{B}$ will be called pure tensors. Note that $\mathcal{G} = \mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_d}$ can be viewed as the set of $d$-dimensional $(n_1, \ldots, n_d)$-arrays over $\mathbb{F}$ which in turn can be viewed as $\prod_{i=1}^d \mathbb{F}^{n_i}$, equipped with the entry-wise dot product.

For $I \subset [d]$, we write $\mathbb{F}^I$ for $\bigotimes_{i \in I} \mathbb{F}^{n_i}$ so that we naturally have $\mathcal{G} = \mathbb{F}^I \otimes \mathbb{F}^I$, where $I^c$ always denotes $[d] \setminus I$.

If $r \in \mathbb{F}^{[d]} = \mathcal{G}$ and $s \in \mathbb{F}^k$ (for some $k \leq d$), then we define $rs$ to be the tensor in $\mathbb{F}^{[k+1,d]}$ with coordinates $(rs)_{i_1,\ldots,i_d} = \sum_{i_1 \leq n_1, \ldots, i_k \leq n_k} r_{i_1,\ldots,i_d} s_{i_1,\ldots,i_k}$. If $k = d$, then $rs$ is the same as the entry-wise dot product $r \cdot s$. Also, note that viewing $r$ as a $d$-multilinear map $R : \mathbb{F}^{n_1} \times \cdots \times \mathbb{F}^{n_d} \to \mathbb{F}$, we have $R(v^1, \ldots, v^d) = \sum_{i_1 \leq n_1, \ldots, i_d \leq n_d} r_{i_1,\ldots,i_d} v^1_{i_1} \cdots v^d_{i_d} = r(v^1 \otimes \cdots \otimes v^d)$.

Finally, we use a non-standard notation and write $kB$ to mean the set of elements of $\mathcal{G}$ which can be written as a sum of at most $k$ elements of $B$, where $B$ is some fixed (multi)subset of $\mathcal{G}$, and similarly, we write $kB - lB$ for the set of elements that can be obtained by adding at most $k$ members and subtracting at most $l$ members of $B$.

We will use the next result several times in our proofs. It is a version of Bogolyubov’s lemma, due to Sanders.

Lemma 2.1 (Sanders [14]). There is an absolute constant $C$ with the following property. Let $A$ be a subset of $\mathbb{F}^n$ with $|A| \geq \delta |\mathbb{F}^n|$. Then $2A - 2A$ contains a subspace of $\mathbb{F}^n$ of codimension at most $C(\log(1/\delta))^4$.

Throughout the paper, $C$ stands for the constant appearing in the previous lemma. Clearly we may assume that $C \geq 1$. Logarithms are base 2.

2.2 The main lemma and some consequences

Theorem 1.10 will follow easily from the next lemma, which is the main technical result of this paper. See [2] for an application of a qualitative version of this lemma.
Lemma 2.2. Let $d \geq 1$ be an integer and let $\delta \leq 1/2$. Let $f_1(d) = 2^{3^d-1}$, $f_2(d) = 2^{-3^d+3}$ and $G(d, \delta, \mathbb{F}) = ((\log |\mathbb{F}|)c_1(d)(\log 1/\delta)^{\gamma(d)}$ where $c_1(d) = C \cdot 2^{3^d+6}$ and $c_2(d) = 4^d$. If $B' \subset B$ is a multiset such that $|B'| \geq \delta|B|$, then there exists a multiset $Q$ whose elements are pure tensors chosen from $f_1(d)B' - f_1(d)B'$ (but with arbitrary multiplicity) with the following property. The set of arrays $r \in G$ with $r.q = 0$ for at least $(1 - f_2(d))|Q|$ choices $q \in Q$ is contained in $\sum_{I \subset [d], I \neq \emptyset} V_I \otimes \mathbb{F}^{d_I}$ for subspaces $V_I \subset \mathbb{F}^l$ of dimension at most $G(d, \delta, \mathbb{F})$.

Throughout the paper, the functions $G, c_1, c_2$ will refer to the functions introduced in the previous lemma. In fact, as $\mathbb{F}$ is fixed, we will write $G(d, \delta)$ to mean $G(d, \delta, \mathbb{F})$.

In this subsection we deduce Theorem 1.10 from Lemma 2.2.

The notion introduced in the next definition is closely related to the partition rank, but will be somewhat more convenient to work with.

Definition 2.3. Let $k$ be a positive integer. We say that $r \in G$ is $k$-degenerate if for every $I \subset [d], I \neq \emptyset, I \neq [d]$, there exists a subspace $H_I \subset \mathbb{F}^l$ of dimension at most $k$ such that $r \in \sum_{I \subset [d], I \neq \emptyset} H_I \otimes H_I$.

If $r \in H_I \otimes \mathbb{F}^{d_I}$ with $\dim(H_I) \leq k$, then $r \in H_I \otimes H_F$ for some $H_F \subset \mathbb{F}^{d_F}$ of dimension at most $k$. (This follows by writing $r$ as $\sum_{j \leq m} s_j \otimes t_j$ with $\{s_j\}$ a basis for $H_I$ and letting $H_F$ be the span of all the $t_j$.) Thus, $r$ is $k$-degenerate if and only if $r \in \sum_{I \subset [d-1], I \neq \emptyset} H_I \otimes \mathbb{F}^{d_I}$ for some $H_I \subset \mathbb{F}^l$ of dimension at most $k$, or equivalently, if and only if $r \in \sum_{I \subset [d-1], I \neq \emptyset} \mathbb{F}^{d_I} \otimes H_F$ for some $H_F \subset \mathbb{F}^{d_F}$ of dimension at most $k$. Moreover, note that if $r$ is $k$-degenerate, then $\text{prank}(r) \leq 2^{d-1}k$. This is because if $I \neq \emptyset, I \subset [d-1]$ and $w \in H_I \otimes H_F$ for subspaces $H_I \subset \mathbb{F}^l$ and $H_F \subset \mathbb{F}^{d_F}$ of dimension at most $k$, then $w = \sum_{i \in k} s_i \otimes t_i$ for some $s_i \in H_I, t_i \in H_F$. But clearly, $s_i \otimes t_i$ has partition rank 1.

Lemma 2.4. Let $\delta \leq 1/2$ and $d \geq 2$. Suppose that Lemma 2.2 has been proved for $d' = d-1$. Let $r \in G$ be such that $r(v_1 \otimes \cdots \otimes v_{d-1}) = 0 \in \mathbb{F}^{d'}$ for at least $\delta|\mathbb{F}|^{n_1-\cdots-n_{d-1}}$ choices $v_1 \in \mathbb{F}^{n_1}, \ldots, v_{d-1} \in \mathbb{F}^{n_{d-1}}$. Then $r$ is $f$-degenerate for $f = G(d-1, \delta)$.

Proof. Write $r = \sum_i s_i \otimes t_i$ where $s_i \in \mathbb{F}^{[d-1]}$ and $\{t_i\}$ is a basis for $\mathbb{F}^{n_i}$. Let $D$ be the multiset $\{u_1 \otimes \cdots \otimes u_{d-1} : u_1 \in \mathbb{F}^{n_1}, \ldots, u_{d-1} \in \mathbb{F}^{n_{d-1}}\}$ and let $D' = \{w \in D : rw = 0\}$. Since $|D'| \geq \delta|D|$, by Lemma 2.2 there is a multiset $Q$ with elements from $2^{3^{d-2}}D' - 2^{3^{d-2}}D'$ such that the set of arrays $r' \in \mathbb{F}^{[d-1]}$ with $r'.q = 0$ for all choices $q \in Q$ is contained in some $\sum_{I \subset [d-1], I \neq \emptyset} V_I \otimes \mathbb{F}^{d-1}|I|$, where $\dim(V_I) \leq G(d-1, \delta)$. Note that for every $i$ we have $s_i.w = 0$ for all $w \in D'$ and so also $s_i.q = 0$ for all $q \in Q$. Thus, $r \in \sum_{I \subset [d-1], I \neq \emptyset} V_I \otimes \mathbb{F}^{d_I}$.

Now we are in a position to prove Theorem 1.10 conditional on Lemma 2.2.
Proof of Theorem 1.10. Let \( T : \mathbb{F}^{p_1} \times \cdots \times \mathbb{F}^{p_d} \to \mathbb{F} \) be an order \( d \) tensor with \( \text{arank}(T) \leq r \). By Remark 1.8, we have \( \mathbb{P}_{v_1 \in \mathbb{F}^{p_1}, \ldots, v_{d-1} \in \mathbb{F}^{p_{d-1}}} [ T(v_1, \ldots, v_{d-1}, x) \equiv 0 ] \geq \mathbb{F}^{-r} \). Writing \( t \) for the element in \( G \) corresponding to \( T \), we get that \( t(v_1 \otimes \cdots \otimes v_{d-1} \otimes x) \equiv 0 \) as a function of \( x \) for at least \( \delta \mathbb{P}^{p_1 \cdot \cdots \cdot p_{d-1}} \) choices \( v_1 \in \mathbb{F}^{p_1}, \ldots, v_{d-1} \in \mathbb{F}^{p_{d-1}} \), where \( \delta = |\mathbb{F}|^{-r} \). But \( t(v_1 \otimes \cdots \otimes v_{d-1} \otimes x) = (t(v_1 \otimes \cdots \otimes v_{d-1})) \cdot x \), so we have \( t(v_1 \otimes \cdots \otimes v_{d-1}) = 0 \) for all these choices of \( v_i \). The condition \( r \geq 1 \) implies \( \delta \leq 1/2 \), therefore by Lemma 2.4, \( t \) is \( f \)-degenerate for \( f = G(d-1, \delta) \). Hence,

\[
\text{prank}(T) \leq 2^{d-1} G(d-1, \delta) = 2^{d-1} (\log |\mathbb{P}| \cdot c_1(d-1) \cdot \log(|\mathbb{P}|))^{c_2(d-1)} = 2^{d-1} (\log |\mathbb{P}|)^2 \cdot c_1(d-1) \cdot r^{c_2(d-1)} \leq (\log |\mathbb{P}|)^2 \cdot c_1(d) \cdot r^{c_2(d)}
\]

But there exists some absolute constant \( c \) such that \( c_1(d)^{c_2(d-1)} \leq c_2(d) \) holds for all \( d \). Moreover, \( 2c_2(d-1) \leq c_2(d) \). Thus, \( \text{prank}(T) \leq (c \cdot \log |\mathbb{P}|)^{c_2(d)} \cdot r^{c_2(d)} = (c \cdot \log |\mathbb{P}|)^{c_2(d)} \cdot r^{c_2(d)} \). \( \square \)

2.3 The overview of the proof of Lemma 2.2

The proof of the lemma goes by induction on \( d \). In what follows, we shall prove results conditional on the assumption that Lemma 2.2 has been verified for all \( d' < d \). Eventually, we will use these results to prove the induction step.

In this subsection, we give a detailed sketch of the proof in the \( d = 3 \) case. At the end of the subsection, we also briefly sketch the \( d > 3 \) case.

2.3.1 The high-level outline in the case \( d = 3 \)

We assume that Lemma 2.2 has been proven for \( d \leq 2 \) and use this assumption to show that it holds for \( d = 3 \). We will take \( Q = Q_{(1,2,3)} \cup Q_{(1)} \cup Q_{(2)} \cup Q_{(3)} \) with elements chosen from \( 2^{3^{d+1}} \mathcal{B}' - 2^{3^{d+1}} \mathcal{B}' \) such that the \( Q_I \) have roughly equal size. This implies that if for some \( r \in G \) we have \( r.q = 0 \) for almost all \( q \in Q \), then \( r.q = 0 \) holds for almost all \( q \in Q_I \) for every \( I = \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \). We define \( Q_{(1,2,3)} \) first, in a way that if \( r.q = 0 \) for almost all \( q \in Q_{(1,2,3)} \), then \( r = x + y \) where \( x \in V_{(1,2,3)} \) for a vector space \( V_{(1,2,3)} \) which is independent of \( r \) and have small dimension, and \( y \) has small partition rank. This already implies that any array \( r \in G \) with \( r.q = 0 \) for almost all \( q \in Q \) is contained in \( V_{(1,2,3)} + \mathbb{F}^{p_1} \otimes H_{(2,3)}(r) + \mathbb{F}^{p_2} \otimes H_{(1,3)}(r) + \mathbb{F}^{p_3} \otimes H_{(1,2)}(r) \) for some subspaces \( H_f(r) \subset \mathbb{F}^d \) depending on \( r \) and of small dimension. We then find \( Q_{(1)} \) such
that if $r \in V_{(1,2,3)} + \mathbb{F}^m \otimes H_{(1,3)}(r) + \mathbb{F}^{m_2} \otimes H_{(1,2)}(r)$ has $r.q = 0$ for almost all $q \in Q_{(1)}$, then $r \in V_{(1,2,3)} + V_{(1)} \otimes \mathbb{F}^{[2,3]} + \mathbb{F}^m \otimes V_{(2,3)} + \mathbb{F}^{m_2} \otimes K_{(1,3)}(r) + \mathbb{F}^{m_3} \otimes K_{(1,2)}(r)$, where $V_{(1)} \subset \mathbb{F}^m$ and $V_{(2,3)} \subset \mathbb{F}^{[2,3]}$ are subspaces independent of $r$ and have small dimension, and $K_{(r)} \subset \mathbb{F}^d$ are subspaces of small dimension (although quite a bit larger than dim($H_{(r)}))$. Then we find $Q_{(2)}$ such that if $r \in V_{(1,2,3)} + V_{(1)} \otimes \mathbb{F}^{[2,3]} + \mathbb{F}^m \otimes V_{(2,3)} + \mathbb{F}^{m_2} \otimes K_{(1,3)}(r) + \mathbb{F}^{m_3} \otimes K_{(1,2)}(r)$ has $r.q = 0$ for almost all $q \in Q_{(2)}$, then $r \in V_{(1,2,3)} + V_{(1)} \otimes \mathbb{F}^{[2,3]} + \mathbb{F}^m \otimes V_{(2,3)} + V_{(2)} \otimes \mathbb{F}^{[1,3]} + \mathbb{F}^{m_2} \otimes V_{(1,3)} + \mathbb{F}^{m_3} \otimes L_{(1,2)}(r)$, where $V_{(2)} \subset \mathbb{F}^{m_2}$ and $V_{(1,3)} \subset \mathbb{F}^{[1,3]}$ are subspaces independent of $r$ and have small dimension, and $L_{(1,2)}(r) \subset \mathbb{F}^{(1,2)}$ is a subspace of small dimension. Finally, we find $Q_{(3)}$ such that if $r \in V_{(1,2,3)} + V_{(1)} \otimes \mathbb{F}^{[2,3]} + \mathbb{F}^m \otimes V_{(2,3)} + V_{(2)} \otimes \mathbb{F}^{[1,3]} + \mathbb{F}^{m_2} \otimes V_{(1,3)} + \mathbb{F}^{m_3} \otimes L_{(1,2)}(r)$ has $r.q = 0$ for almost all $q \in Q_{(3)}$, then $r \in V_{(1,2,3)} + V_{(1)} \otimes \mathbb{F}^{[2,3]} + \mathbb{F}^m \otimes V_{(2,3)} + V_{(2)} \otimes \mathbb{F}^{[1,3]} + \mathbb{F}^{m_2} \otimes V_{(1,3)} + V_{(3)} \otimes \mathbb{F}^{(1,2)} + \mathbb{F}^{m_3} \otimes V_{(1,2)}$, where $V_{(3)} \subset \mathbb{F}^{m_3}$ and $V_{(1,2)} \subset \mathbb{F}^{(1,2)}$ are subspaces independent of $r$ and have small dimension.

How will we find $Q_{(1,2,3)}$, $Q_{(1)}$, $Q_{(2)}$ and $Q_{(3)}$? In this outline we will only explain how to find $Q_{(2)}$ (but finding $Q_{(1)}$ and $Q_{(3)}$ is very similar). We take $Q_{(2)} = \bigcup_{u \in U} u \otimes Q_u$ where $U \subset \mathbb{F}^{m_2}$ is a subspace of low codimension, and for each $u \in U$, $Q_u \subset \mathbb{F}^{[1,3]}$ is a multiset consisting of pure tensors such that if for some $x \in \mathbb{F}^{[1,3]}$, we have $x.t = 0$ for almost all $t \in Q_u$, then $x \in W_{(1,3)}(u) + \mathbb{F}^m \otimes W_{(1)}(u) + W_{(1)}(u) \otimes \mathbb{F}^{m_3}$ for some subspaces $W_{(i)}(u) \subset \mathbb{F}^d$ not depending on $x$ and of small dimension. Let us call a $Q_u$ with this property forcing. We will also make sure that all the $Q_u$ have roughly the same size.

### 2.3.2 Why does this $Q_{(2)}$ work?

In what follows, we will sketch why this choice is suitable. We remark that in the general case this is done in Lemma 2.15. Let $R$ consist of those

$$r \in V_{(1,2,3)} + V_{(1)} \otimes \mathbb{F}^{[2,3]} + \mathbb{F}^m \otimes V_{(2,3)} + \mathbb{F}^{m_2} \otimes K_{(1,3)}(r) + \mathbb{F}^{m_3} \otimes K_{(1,2)}(r)$$

such that $r.q = 0$ for almost all $q \in Q_{(2)}$. Let $r \in R$. Write $r = r_2 + r_3 + r_4$ where

$$r_2 \in V_{(1)} \otimes \mathbb{F}^{[2,3]} + \mathbb{F}^m \otimes V_{(2,3)} + \mathbb{F}^{m_2} \otimes K_{(1,2)}(r), \quad r_3 \in V_{(1,2,3)}, \quad r_4 \in \mathbb{F}^{m_2} \otimes K_{(1,3)}(r).$$

It is enough to prove that

$$r_4 \in V_{(2)} \otimes \mathbb{F}^{[1,3]} + \mathbb{F}^{m_2} \otimes V_{(1,3)} + \mathbb{F}^{m_3} \otimes L'_{(1,2)}(r)$$

for some small subspaces $V_{(2)} \subset \mathbb{F}^{m_2}$, $V_{(1,3)} \subset \mathbb{F}^{[1,3]}$ and $L'_{(1,2)}(r) \subset \mathbb{F}^{(1,2)}$ (in fact, we will be able to take $V_{(2)} = U^\perp$).
First note that $r_2u$ has small (partition) rank for every $u \in U$. Indeed, $r_2u \in V_{(1)} \otimes \mathbb{F}^{n_3} + \mathbb{F}^{n_3} \otimes V_{(2,3)}u + \mathbb{F}^{n_1} \otimes K_{(1,3)}(r)u$, where, for a vector space $L$ of tensors, $Lu$ denotes the space $\{su : s \in L\}$.

Moreover, since the $Q_u$ all have roughly the same size, for almost every $u \in U$ we have that $r(u \otimes t) = 0$ holds for almost every $t \in Q_u$. But $r(u \otimes t) = (ru)t$, therefore as $Q_u$ is forcing, it follows that for any such $u$

$$ru \in W_{(1,3)}(u) + \mathbb{F}^{n_1} \otimes W_{(3)}(u) + W_{(1)}(u) \otimes \mathbb{F}^{n_3}$$

for some subspaces $W_i(u) \subset \mathbb{F}^l$ not depending on $r$ and of small dimension. Since any element of $\mathbb{F}^{n_1} \otimes W_{(3)}(u) + W_{(1)}(u) \otimes \mathbb{F}^{n_3}$ has small partition rank, it follows that for almost every $u \in U$,

$$r_4u = ru - r_2u - r_3u \in W_{(1,3)}(u) + V_{(1,2,3)}u + s(u)$$

(3)

where $s(u)$ is a tensor of small partition rank.

Define a sequence $0 = Z(0) \subset Z(1) \subset \ldots \subset Z(m) \subset \mathbb{F}^{l,3}$ of subspaces recursively as follows. Given $Z(j)$, if there is some $r \in R$ such that $r_4u$ is far from $Z(j)$ for many $u \in U$, then set $Z(j + 1) = Z(j) + K_{(1,3)}(r)$. What we mean by $r_4u$ being far from $Z(j)$ is that there is no $z \in Z(j)$ such that $r_4u - z$ has small partition rank. For suitably chosen parameters, one can show that this procedure cannot go on for too long, i.e. that for some not too large $m$ we have that for every $r \in R$, for almost all $u \in U$ there is some $z \in Z(m)$ with $r_4u - z$ having small partition rank.

Now let $r \in R$. Let $X(r)$ be the set consisting of those $x \in K_{(1,3)}(r)$ which are close to $Z(m)$. Then $r_4u \in X(r)$ for almost every $u \in U$. Let $t_1, \ldots, t_n$ be a maximal linearly independent subset of $X(r)$ and extend it to a basis $t_1, \ldots, t_a, t'_1, \ldots, t'_{\beta}$ for $K_{(1,3)}(r)$. Now if a linear combination of $t_1, \ldots, t_a, t'_1, \ldots, t'_{\beta}$ is in $X(r)$, then the coefficients of $t'_1, \ldots, t'_{\beta}$ are all zero. Write $r_1 = \sum_{i \leq a} s_i \otimes t_i + \sum_{j \leq \beta} s'_j \otimes t'_j$ for some $s_i, s'_j \in \mathbb{F}^{n_2}$. Since $r_4u \in X(r)$ for almost all $u \in U$, we have, for all $j$, that $s'_j, u = 0$ for almost all $u \in U$. Since these hold for more than half of $u \in U$, we obtain $s'_j \in U^\perp$ for every $j$, therefore $\sum_{j \leq \beta} s'_j \otimes t'_j \in U^\perp \otimes \mathbb{F}^{l,3}$.

Since $t_i \in X(r)$ for every $i$, we may choose $z_i \in Z(m)$ such that $t_i = z_i + y_i$ where $y_i \in \mathbb{F}^{l,3}$ has small partition rank. Now $\sum_{i \leq a} s_i \otimes t_i \in \mathbb{F}^{n_2} \otimes Z(m) + \sum_{i \leq a} s_i \otimes y_i$. Moreover, as $\alpha$ is small and each $y_i$ has small partition rank, we have $\sum_{i \leq a} s_i \otimes y_i \in L_{(1,2)}'(r) \otimes \mathbb{F}^{n_3}$ for some small $L_{(1,2)}'(r) \subset \mathbb{F}^{l,2}$. So we have proved (2) with $V_{(2)} = U^\perp$ and $V_{(1,3)} = Z(m)$.

2.3.3 Why can we find such a $Q_{(2)}$ inside $2^{3^{i+3}}B' - 2^{3^{i+3}}B''$?

Now we describe why there must exist $Q_{(2)}$ with elements chosen from $2^{3^{i+3}}B' - 2^{3^{i+3}}B''$ and having the required properties. We remark that in the general case this is done in Lemma 2.14.
We want to find a subspace $U \subset \mathbb{F}^{p_2}$ of low codimension, and forcing multisets $Q_u \subset \mathbb{F}^{(1,3)}$ ($u \in U$) consisting of pure tensors such that for every $u \in U$, $u \otimes Q_u \subset 2^{3^{k+1}}B' - 2^{3^{k+1}}B'$. Let $\mathcal{D}$ be the multiset $\{v \otimes w : v \in \mathbb{F}^{p_1}, w \in \mathbb{F}^{p_3}\}$. Notice that if some set $R$ is dense in $\mathcal{D}$, then by the induction hypothesis we can find a forcing set in $2^{3^{k+1}}R - 2^{3^{k+1}}R$ consisting of pure tensors. Therefore it is enough to find a low codimensional subspace $U$ and dense sets $R_u \subset \mathcal{D}$ (for every $u \in U$) such that $u \otimes R_u \subset 32B' - 32B'$. As $B'$ is dense in $B$, we have a dense subset $S \subset \mathbb{F}^{p_2}$ and dense subsets $T_s \subset \mathcal{D}$ ($s \in S$) such that $s \otimes T_s \subset B'$ for every $s \in S$. By Bogolyubov’s lemma (Lemma 2.1), there is a low codimensional subspace $U$ contained in $2S - 2S$. To establish the existence of a dense $R_u \subset \mathcal{D}$ with $u \otimes R_u \subset 32B' - 32B'$ for every $u \in U$, it is enough to prove the following lemma.

**Lemma 2.5.** Let $T_1, T_2, T_3, T_4$ be dense subsets of $\mathcal{D}$. Then $\mathcal{D} \cap \bigcap_{i \leq 4}(8T_i - 8T_i)$ is dense in $\mathcal{D}$.

Indeed, once we have this lemma, it follows that for any $s_1, s_2, s_3, s_4 \in S$, the set $\mathcal{D} \cap \bigcap_{i \leq 4}(8T_{s_i} - 8T_{s_i})$ is dense in $\mathcal{D}$. But if $u \in U$, then we can write $u = s_1 + s_2 - s_3 - s_4$ for some $s_i \in S$, and then $u \otimes \bigcap_{i \leq 4}(8T_{s_i} - 8T_{s_i})$ contains $s_1 \otimes \bigcap_{i \leq 4}(8T_{s_i} - 8T_{s_i}) + s_2 \otimes \bigcap_{i \leq 4}(8T_{s_i} - 8T_{s_i}) - s_3 \otimes \bigcap_{i \leq 4}(8T_{s_i} - 8T_{s_i}) - s_4 \otimes \bigcap_{i \leq 4}(8T_{s_i} - 8T_{s_i}) \subset 32B' - 32B'$.

**Lemma 2.6.** Let $A$ be a dense subset of $\mathcal{D}$. Then there exist a dense subspace $V \subset \mathbb{F}^{p_1}$ and for each $v \in V$ a dense subspace $W_v \subset \mathbb{F}^{p_3}$ such that $v \otimes W_v \subset 8A - 8A$ for every $v \in V$.

**Proof.** There exist a dense subset $B \subset \mathbb{F}^{p_1}$ and dense subsets $C_b \subset \mathbb{F}^{p_3}$ for each $b \in B$ such that $b \otimes C_b \subset A$. By Bogolyubov’s lemma, $2B - 2B$ contains a dense subspace $V \subset \mathbb{F}^{p_1}$, and for every $b \in B$, $2C_b - 2C_b$ contains a dense subspace $L_b \subset \mathbb{F}^{p_3}$. For any $v \in V$, choose $b_1, b_2, b_3, b_4 \in B$ with $v = b_1 + b_2 - b_3 - b_4$ and set $W_v = \bigcap_{i \leq 4}L_{b_i}$. Note that $b_i \otimes w \in 2A - 2A$ for every $i \leq 4$ and $w \in W_v$, therefore $v \otimes w \in 8A - 8A$. \(\square\)

**Lemma 2.7.** Suppose that we have dense subspaces $V, V' \subset \mathbb{F}^{p_1}$, for each $v \in V$ a dense subspace $W_v \subset \mathbb{F}^{p_3}$, and for each $v' \in V'$ a dense subspace $W_{v'} \subset \mathbb{F}^{p_3}$. Then $(\bigcup_{v \in V}v \otimes W_v) \cap (\bigcup_{v' \in V'}v' \otimes W'_{v'}) = \bigcup_{v \in V \cap V'} v \otimes (W_v \cap W'_{v'})$. In particular, this intersection is a dense subset of $\mathcal{D}$.

**Proof.** The identity is trivial. Since the subspaces $V \cap V'$ and $W_v \cap W'_{v'}$ are dense, the second assertion follows. \(\square\)

### 2.3.4 How can this be extended to $d > 3$?

Now we briefly sketch what the main difficulties are in the $d > 3$ case and how we can address them. The underlying strategy is similar: we take an ordering $<$ of the set of non-empty subsets
\( I \subset [d - 1] \), and for each such \( I \) we choose \( Q_I \) such that any array

\[
    r \in W_{[d]} + \sum_{J \neq I} (W_J \otimes \mathbb{R}^J + \mathbb{R}^J \otimes W_{J'}) + \sum_{J \neq I} \mathbb{R}^J \otimes H_{J'}(r)
\]

with \( r.q = 0 \) for almost all \( q \in Q_I \) has

\[
    r \in W_{[d]} + \sum_{J \neq I} (U_J \otimes \mathbb{R}^J + \mathbb{R}^J \otimes U_{J'}) + \sum_{J \neq I} \mathbb{R}^J \otimes K_{J'}(r)
\]

where \( U_J, U_{J'}, K_{J'}(r) \) can have dimension slightly larger than those of \( W_J, W_{J'} \) and \( H_{J'} \), but they are still low dimensional. In the \( d = 3 \) case, we have made use of a decomposition \( r = r_2 + r_3 + r_4 \) where \( r_4 \in \mathbb{R}^J \otimes H_{J'}(r), r_2u \) has small partition rank and \( r_3u \) is in a small subspace independent of \( r \) for every \( u \in \mathbb{R}^J \). In general, such a decomposition need not exist. For example, when \( d = 4 \) and \( I = \{1, 2\} \), then an array in \( W_{[1]} \otimes \mathbb{R}^{[2,3,4]} \) (or in \( \mathbb{R}^{t_1} \otimes H_{[2,3,4]}(r) \) if we were to take \( \{1, 2\} \prec \{1\} \)), when multiplied by some pure tensor \( u \in \mathbb{R}^{[1,2]} \), yields a tensor which need not have small partition rank and need not lie in a small space independent of \( r \). However, by restricting the possible choices for \( u \), we can make sure that the product is always zero. So we will take a decomposition \( r = r_1 + r_2 + r_3 + r_4 \) such that \( r_4 \in \mathbb{R}^J \otimes H_{J'}(r) \); for every pure tensor \( u \in \mathbb{R}^J \), \( r_2u \) has small partition rank and \( r_3u \) lies in a small space depending only on \( u \); and crucially, for every \( q \in Q_I, r_1.q = 0 \). To achieve this, we need to insist that \( J \prec I \) whenever \( J \subseteq I \) and that \( Q_I \) is orthogonal to certain subspaces. To see this, note that in the above example where \( d = 4 \) and \( I = \{1, 2\} \) we need that \( \{1\} \prec \{1, 2\} \) and \( Q_{\{1,2\}} \) is orthogonal to \( W_{[1]} \otimes \mathbb{R}^{[2,3,4]} \). If we had \( \{1, 2\} \prec \{1\} \), then in (4) we would have a term \( \mathbb{R}^{t_1} \otimes H_{[2,3,4]}(r) \) rather than \( W_{[1]} \otimes \mathbb{R}^{[2,3,4]} \), which we could not control.

We also need to generalise Lemma 2.5 to the case \( d > 3 \). Instead of using \( \bigcup_{v \in V} v \otimes W_v \) as in Lemma 2.6, we need to define an object in \( B \) such that

1. an instance of the object can be found in \( kB' -kB' \) for some small \( k \) whenever \( B' \) is dense in \( B \) (generalising Lemma 2.6)

2. the intersection of few instances of this object is a dense subset of \( B \) (generalising Lemma 2.7)

In the next subsection we describe this object and show that it has the required properties.
2.4 Construction of some auxiliary sets

**Definition 2.8.** Suppose that we have a collection of vector spaces as follows. The first one is $U \subset \mathbb{F}^m$, of codimension at most $l$. Then, for every $u_1 \in U$, there is some $U_{u_1} \subset \mathbb{F}^n$. In general, for every $2 \leq k \leq d$ and every $u_1 \in U, u_2 \in U_{u_1}, \ldots, u_{k-1} \in U_{u_1, \ldots, u_{k-2}}$, there is a subspace $U_{u_1, \ldots, u_{k-1}} \subset \mathbb{F}^n$. Assume, in addition, that the codimension of $U_{u_1, \ldots, u_{k-1}}$ in $\mathbb{F}^n$ is at most $l$ for every $u_1 \in U, \ldots, u_{k-1} \in U_{u_1, \ldots, u_{k-2}}$. Then the multiset $Q = \{u_1 \otimes \cdots \otimes u_d : u_1 \in U, \ldots, u_d \in U_{u_1, \ldots, u_{d-1}}\}$ is called an $l$-system.

The next lemma is the generalisation of Lemma 2.7 from the previous subsection.

**Lemma 2.9.** Let $Q$ be an $l$-system and let $Q'$ be an $l'$-system. Then $Q \cap Q'$ contains an $(l + l')$-system.

**Proof.** Let $Q$ have spaces as in Definition 2.8 and let $Q'$ have spaces $U'_{u_1, \ldots, u_{k-1}}$. We define an $(l + l')$-system $P$ contained in $Q \cap Q'$ as follows. Let $V = U \cap U'$. Suppose we have defined $V_{v_1, \ldots, v_{j-1}}$ for all $j \leq k$. Let $V_{v_1, v_2} \in V_{v_1, \ldots, v_{k-1}} \cap U'_{v_1, \ldots, v_{k-1}}$. This is well-defined and has codimension at most $l + l'$ in $\mathbb{F}^n$. Let $P$ be the $(l + l')$-system with spaces $V_{v_1, \ldots, v_{k-1}}$. □

The next lemma is the generalisation of Lemma 2.6 from the previous subsection.

**Lemma 2.10.** Let $\mathcal{B}' \subset \mathcal{B}$ be a multiset such that $|\mathcal{B}'| \geq \delta|\mathcal{B}|$. Then there exists an $f_1$-system whose elements are chosen from $f_2 \mathcal{B}' - f_2 \mathcal{B}'$ with $f_1 = C \cdot 4^d(\log(2^d/\delta))^4$ and $f_2 = 4^d$.

**Proof.** The proof is by induction on $d$. The case $d = 1$ is a direct consequence of Lemma 2.1. Suppose that the lemma has been proved for all $d' < d$ and let $\mathcal{B}' \subset \mathcal{B}$ be a multiset such that $|\mathcal{B}'| \geq \delta|\mathcal{B}|$. Let $\mathcal{D}$ be the multiset $\{v_2 \otimes \cdots \otimes v_d : v_2 \in \mathbb{F}^n, \ldots, v_d \in \mathbb{F}^n\}$. For each $u \in \mathbb{F}^m$, let $\mathcal{B}'_u = \{s \in \mathcal{D} : u \otimes s \in \mathcal{B}'\}$ and let $T = \{u \in \mathbb{F}^m : |\mathcal{B}'_u| \geq \frac{\delta}{2} |\mathcal{D}|\}$. By averaging, we have that $|T| \geq \frac{\delta}{2} |\mathcal{D}|$. Now by the induction hypothesis, for every $t \in T$, there exists a $g_1$-system in $\mathbb{F}^n \otimes \cdots \otimes \mathbb{F}^n$ (whose definition is analogous to the definition of a system in $\mathbb{F}^m \otimes \cdots \otimes \mathbb{F}^m$), called $P_t$, contained in $g_2 \mathcal{B}'_t - g_2 \mathcal{B}'$ where $g_1 = C \cdot 4^{d-1}(\log(2^d/\delta))^4$ and $g_2 = 4^{d-1}$. By Lemma 2.1, $2T - 2T$ contains a subspace $U \subset \mathbb{F}^m$ of codimension at most $C(\log(2^d/\delta))^3$. For each $u \in U$, write $u = t_1 + t_2 - t_3 - t_4$ arbitrarily with $t_i \in T$, and let $Q_u = P_{t_1} \cap P_{t_2} \cap P_{t_3} \cap P_{t_4}$, which is a $g_3$-system with $g_3 = 4g_1 = C \cdot 4^d(\log(2^d/\delta))^4$, by Lemma 2.9. Thus, $Q = \bigcup_{u \in U} (u \otimes Q_u)$ is indeed an $f_1$-system. Moreover, for any $u \in U$, $s \in Q_u$, we have $u \otimes s = t_1 \otimes s + t_2 \otimes s - t_3 \otimes s - t_4 \otimes s$ for some $t_i \in T$ and $s \in \bigcap_{i \leq 4} P_{t_i}$. Then $t_i \otimes s \in g_2 \mathcal{B}' - g_2 \mathcal{B}'$, therefore $u \otimes s \in 4g_2 \mathcal{B}' - 4g_2 \mathcal{B}'$, so the elements of $Q$ are indeed chosen from $f_2 \mathcal{B}' - f_2 \mathcal{B}'$. □
The next lemma describes a property of systems which was not needed for us in the $d = 3$ case, but is crucial in the general case. It is required for finding a suitable decomposition $r = r_1 + r_2 + r_3 + r_4$ described at the end of the previous subsection. Indeed, we need a set $Q_l$ which is orthogonal to certain spaces of the form $W_f \otimes \mathbb{P}^F$ (i.e. is contained in $W_f^\perp \otimes \mathbb{P}^F$) to make sure that $r_1.q = 0$ for every $q \in Q_l$. We will use the following lemma to guarantee the existence of such a set $Q_l$.

**Lemma 2.11.** Let $Q$ be a $k$-system and for every non-empty $I \subseteq [d]$, let $L_I \subseteq \mathbb{P}^l$ be a subspace of codimension at most $l$. Let $T = \bigcap I (L_I \otimes \mathbb{P}^F)$. Then $Q \cap T$ contains an $f$-system for $f = k + 2^d l$.

**Proof.** Let the spaces of $Q$ be $U_{u_1,\ldots,u_{j-1}}$. It suffices to prove that for every $1 \leq j \leq d$, and every $u_1 \in U_{j}$, the codimension of $(u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1,\ldots,u_{j-1}}) \cap \bigcap I (L_I \otimes \mathbb{P}^F)$ in $u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1,\ldots,u_{j-1}}$ is at most $2^d l$. Thus, it suffices to prove that for every $I \subseteq [j]$ with $j \in I$, the codimension of $(u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1,\ldots,u_{j-1}}) \cap (L_I \otimes \mathbb{P}^F)$ in $u_1 \otimes \cdots \otimes u_{j-1} \otimes U_{u_1,\ldots,u_{j-1}}$ is at most $l$. But this is equivalent to the statement that $((\bigotimes \in I^{(j)} u_i) \otimes U_{u_1,\ldots,u_{j-1}}) \cap L_I$ has codimension at most $l$ in $(\bigotimes \in I^{(j)} u_i) \otimes U_{u_1,\ldots,u_{j-1}}$, which clearly holds. \hfill \square

### 2.5 The proof of Lemma 2.2

We now turn to the proof of Lemma 2.2. As described in the outline, the first step is to find a $Q[d]$ such that if $r.q = 0$ for almost all $q \in Q[d]$, then $r = x + y$ where $x \in V[d]$ for a small space $V[d]$ independent of $r$, and $y$ has low partition rank.

**Lemma 2.12.** Let $d \geq 2$ and suppose that Lemma 2.2 has been proved for $d' = d - 1$. Let $B' \subseteq B$ be such that $|B'| \geq \delta|B|$ for some $\delta > 0$. Then there exist some $Q \subseteq 2B' - 2B'$ consisting of pure tensors and a subspace $V[d] \subseteq \mathbb{P}^d$ of dimension at most $4C(\log(2/\delta))^4$ with the following property. Any array $r$ with $r.q = 0$ for at least $\frac{1}{8}|Q|$ choices $q \in Q$ can be written as $r = x + y$ where $x \in V[d]$ and $y$ is $f$-degenerate for $f = G(d - 1, \frac{\delta}{4C(\log(2/\delta))^4})$.

**Proof.** Let $D$ be the multiset $\{u_1 \otimes \cdots \otimes u_{d-1} : u_1 \in \mathbb{F}^{m_1}, \ldots, u_{d-1} \in \mathbb{F}^{m_{d-1}}\}$ and let $D' = |t \in D : t \otimes u \in B'$ for at least $\frac{\delta}{4} |B'|$ choices $u \in \mathbb{F}^{m_d}$}. Clearly, we have $|D'| \geq \frac{\delta}{2} |D|$. Moreover, by Lemma 2.1, for every $t \in D'$, there exists a subspace $U_t \subseteq \mathbb{P}^{n_d}$ of codimension at most $C(\log(2/\delta))^4$ such that $t \otimes U_t \subseteq 2B' - 2B'$. After passing to suitable subspaces, we may assume that all $U_t$ have the same codimension $k \leq C(\log(2/\delta))^4$. Now let $Q = \bigcup_{t \in D'} (t \otimes U_t)$.

Write $R$ for the set of arrays $r$ with $r.q = 0$ for at least $\frac{7}{8} |Q|$ choices $q \in Q$.

We now define a sequence of subspaces $0 = V(0) \subset V(1) \subset \ldots \subset V(m) \subset \mathbb{P}^d$ recursively as follows.
Given $V(j)$, if for every $r \in R$ there are at least $\frac{|\mathcal{D}|}{4}$ choices $t \in \mathcal{D}'$ with $rt \in V(j)t$, then we set $m = j$ and terminate. (Here and below, for a subspace $L \subset \mathcal{G}$ and an array $s \in \mathbb{F}^J$, we write $Ls$ for the subspace $\{rs : r \in L\} \subset \mathbb{F}^R$.)

Else, we choose some $r \in R$ such that there are at most $\frac{|\mathcal{D}|}{4}$ choices $t \in \mathcal{D}'$ with $rt \in V(j)t$.

We set $V(j + 1) = V(j) + \text{span}(r)$. Note that $r.t \otimes s = (rt).s$ for each $s \in U_1\[I\]$. If $rt \notin U_1^{\perp}$, then $(rt).s = 0$ holds for only a proportion $1/|\mathcal{F}| \leq 1/2$ of all $s \in U_1$. Thus, as $r \in R$, we have $rt \in U_1^{\perp}$ for at least $\frac{2}{4}|\mathcal{D}'|$ choices $t \in \mathcal{D}'$. Moreover, since $rt \in V(j)t$ holds for at most $\frac{|\mathcal{D}|}{4}$ choices $t \in \mathcal{D}'$, it follows that for at least $\frac{|\mathcal{D}|}{4}$ choices $t \in \mathcal{D}'$ we have $rt \in U_1^{\perp} \setminus V(j)t$. Thus, we have $\dim(U_1^{\perp} \cap V(j + 1)t) > \dim(U_1^{\perp} \cap V(j)t)$ for at least $\frac{|\mathcal{D}|}{4}$ choices $t \in \mathcal{D}'$.

However, for any $j$ we have $\sum_{t \in \mathcal{D}} \dim(U_1^{\perp} \cap V(j)t) \leq \sum_{t \in \mathcal{D}} \dim U_1^{\perp} \leq |\mathcal{D}'|(\log(2/\delta))^4$. Thus, we get $m \leq 4C(\log(2/\delta))^4$. Set $V_{[d]} = V(m)$. Then $\dim V_{[d]} \leq 4C(\log(2/\delta))^4$, as claimed.

Now let $r \in R$ be arbitrary. By definition, there are at least $|\mathcal{D}'|/2$ choices $t \in \mathcal{D}'$ with $rt \in V_{[d]}t$. Then there is some $v \in V_{[d]}$ such that $rt = vt$ for at least $\frac{|\mathcal{D}'|}{4|V_{[d]}|}$ choices $t \in \mathcal{D}'$, and hence also for at least $\frac{\delta |\mathcal{D}'|}{4|V_{[d]}|}$ choices $t \in \mathcal{D}$. Note that $\frac{\delta}{4|V_{[d]}|} \geq \frac{\delta}{4|\mathbb{F}|^{2\log(2/\delta)^4}}$, therefore by Lemma 2.4, $r - v$ is $f$-degenerate. 

**Definition 2.13.** Let $k$ be a positive integer and let $0 \leq \alpha \leq 1$. Let $Q$ be a multiset with elements chosen from $\mathcal{G}$ (with arbitrary multiplicity). We say that $Q$ is $(k, \alpha)$-forcing if the set of all arrays $r \in \mathcal{G}$ with $r.q = 0$ for at least $\alpha|Q|$ choices $q \in Q$ is contained in a set of the from $\sum_{I \subset \{1, \ldots, d\}, I \neq \emptyset} V_I \otimes \mathbb{F}^k$ for some $V_I \subset \mathbb{F}^J$ of dimension at most $k$.

We now turn to the main part of the proof of Lemma 2.2. For each non-empty $I \subset \{1, \ldots, d\}$ we will construct $Q_I$ as defined in the next result, and (roughly) we will take $Q = Q_{[d]} \cup \bigcup_{I \subset \{1, \ldots, d\}, I \neq \emptyset} Q_I$, where $Q_{[d]}$ is provided by Lemma 2.12. The properties that $Q_I$ has are generalisations of the properties that $Q_{[2]}$ had in Subsection 2.3. Accordingly, the next lemma is the generalisation of the discussion in Subsubsection 2.3.3.

**Lemma 2.14.** Let $d \geq 2$ and suppose that Lemma 2.2 has been proved for every $d' < d$. Let $\mathcal{B}' \subset \mathcal{B}$ have $|\mathcal{B}'| \geq \delta|\mathcal{B}|$ for some $0 < \delta \leq 1/2$. Let $k \geq G(d - 1, \delta)$ be arbitrary, let $I \subset \{d - 1\}$, $I \neq \emptyset$, and let $W_J \subset \mathbb{F}^J$ be subspaces of dimension at most $k$ for every $J \subset I$, $J \neq I$. Then there exist a multiset $Q'$, and a multiset $Q_s$, for each $s \in Q'$ with the following properties.

1. The elements of $Q'$ are pure tensors chosen from $\bigcap_{J \subset I, J \neq \emptyset} (W_J \otimes \mathbb{F}^{0,J}) \subset \mathbb{F}^I$

2. $Q'$ is $(f_1, 1 - f_2)$-forcing with $f_1 = G(|J|, |\mathbb{F}|^{2^{d+1}d})$, $f_2 = 2^{-d+2}$

3. For each $s \in Q'$, the elements of $Q_s$ are pure tensors chosen from $\mathbb{F}^k$
(4) For each $s \in Q'$, $Q_s$ is $(f_3, 1 - f_4)$-forcing with $f_3 = G(d - |I|, |P| - 2^{3d_4} C (\log (2d_1 - 1/\delta))^4)$, $f_4 = 2^{-3d_4}$.

(5) $\max_{s \in Q'} |Q_s| \leq 2 \min_{s \in Q'} |Q_s|$.

(6) The elements of the multiset $Q_I := \{ s \otimes t : s \in Q', t \in Q_s \} = \bigcup_{s \in Q'} (s \otimes Q_s)$ are chosen from $f_5 B' - f_5 B'$ with $f_5 = 2^{3d_3}$.

**Proof.** By symmetry, we may assume that $I = [a]$ for some $1 \leq a \leq d - 1$. Let $C$ be the multiset $\{u_1 \otimes \ldots \otimes u_d : u_i \in \mathbb{F}_m\}$ and let $D$ be the multiset $\{u_{a+1} \otimes \ldots \otimes u_d : u_i \in \mathbb{F}_m\}$. For each $s \in C$, let $D_s = \{ t \in D : s \otimes t \in B' \}$. Also, let $C' = \{ s \in C : |D_s| \geq \frac{2}{3} |D| \}$. Clearly, $|C'| \geq \frac{2}{3} |C|$.

By Lemma 2.10, there exists a $g_1$-system $R$ (with respect to $\mathbb{F}^l$) with elements chosen from $g_1 C' - g_2 C'$ with $g_1 = C \cdot 4^d (\log (2d - 1/\delta))^4$ and $g_2 = 4^d$. By Lemma 2.11, $R \cap \bigcap_{l \in I, l' \neq 0} (P_l \otimes \mathbb{F}^{|l'|})$ contains a $g_1$-system $T'$ for $g_1 = C \cdot 4^d (\log (2d - 1/\delta))^4 + 2^{dk}$. Now $|T'| \geq |\mathbb{F}|^{-d_3} |C|$. By Lemma 2.2 (applied to $a$ in place of $d$), it follows that there exists a multiset $Q'$ whose elements are pure tensors chosen from $g_1 T' - g_4 T'$ and which is $(g_5, 1 - g_6)$-forcing for $g_4 = 2^{3d_3} \leq 2^{3d_2}$, $g_5 = G(a, \mathbb{F}^{-d_3})$ and $g_6 = 2^{-3d_3} \geq 2^{-3d_2}$. Note that since $\delta \leq 1/2$, we have $C \cdot 4^d (\log (2d - 1/\delta))^4 = C \cdot 4^d (d - 1 + \log (1/\delta))^4 \leq C \cdot 4^d (d \log (1/\delta))^4$. But this is at most as $G(d - 1, \delta) \leq k$, so $g_3 \leq 2 \cdot 2^dk$, therefore $Q'$ satisfies (1) and (2) in the statement of this lemma.

By Lemma 2.10, for each $s \in C'$ there exists a $g_7$-system $R_s$ (with respect to $\mathbb{F}^l$) contained in $g_8 D_s - g_8 D_s$, where $g_7 = C \cdot 4^d (\log (2d - 1/\delta))^4$ and $g_8 = 4^d$. For every $s \in Q'$, choose $s_1, \ldots, s_{l+\ell} \in C'$ with $l, l' \leq 2^{3d_3}$ such that $s = s_1 + \cdots + s_l - s_{l+1} - \cdots - s_{l+\ell}$ (this is possible, since the elements of $Q'$ are chosen from $2g_2 g_4 C' - 2g_2 g_4 C'$ and $2g_2 g_4 \leq 2^{3d_3}$), and let $P_s = \bigcap_{l \in [a, l]} R_s$.

By Lemma 2.9, $P_s$ contains a $g_9$-system $g_9 = 2 \cdot 2^{3d_3} \cdot C \cdot 4^d (\log (2d - 1/\delta))^4$, therefore $|P_s| \geq g_{10} |D|$ for $g_{10} = |\mathbb{F}|^{-d_9} \geq |\mathbb{F}|^{-3d_4} (\log (2^{3d_3} - 1/\delta))^4$. By Lemma 2.2 (applied to $d - a$ in place of $d$), for every $s \in Q'$ there exists a multiset $Q_s$ consisting of pure tensors with elements chosen from $g_{11} P_s - g_{11} P_s$ which is $(g_{12}, 1 - g_{13})$-forcing for $g_{11} = 2^{3d_3} \leq 2^{3d_2}$, $g_{12} = G(d - a, |\mathbb{F}|^{-d_9}) \leq G(d - a, |\mathbb{F}|^{-3d_4} (\log (2^{3d_3} - 1/\delta))^4)$ and $g_{13} = 2^{-3d_3} \geq 2^{-3d_2}$. Notice that if we repeat every element of $Q_s$ the same number of times, then the multiset obtained is still $(g_{12}, 1 - g_{13})$-forcing, so we may assume that $\max_{s \in Q'} |Q_s| \leq 2 \min_{s \in Q'} |Q_s|$. Thus, the $Q_s$ satisfy (3), (4) and (5).

Define $Q_I = \{ s \otimes t : s \in Q', t \in Q_s \} = \bigcup_{s \in Q'} (s \otimes Q_s)$. Note that as $R_s \subset g_8 D_s - g_8 D_s$ for all $s \in C'$, we have $s \otimes R_s \subset g_8 B' - g_8 B'$ for all $s \in C'$. But the elements of $Q'$ are chosen from $2g_2 g_4 C' - 2g_2 g_4 C'$, so $s \otimes P_s \subset 4g_2 g_4 g_8 B' - 4g_2 g_4 g_8 B'$ for all $s \in Q'$. Finally, the elements of $Q_s$ are chosen from $g_{11} P_s - g_{11} P_s$, so the elements of $s \otimes Q_s$ are chosen from $8g_2 g_4 g_8 g_{11} B' - 8g_2 g_4 g_8 g_{11} B'$ for every $s \in Q'$. Since $8g_2 g_4 g_8 g_{11} \leq 8 \cdot (4^d)^2 \cdot (2^{3d_2})^2 = 2^{3+4d+2.3d_2} \leq 2^{3d_3}$, property (6) is satisfied.
The next lemma is the last ingredient of the proof. It is a generalisation of the discussion in Subsubsection 2.3.2. Given a tensor \( r \in V_{[d]} + \sum_{t-[d-1], t \neq 0} \mathbb{F}^t \otimes H_t(r) \), we turn the terms \( \mathbb{F}^t \otimes H_t(r) \) one by one into terms \( V_t \otimes \mathbb{F}^t + \mathbb{F}^t \otimes V_t \) where \( V_t \) are small and do not depend on \( r \). (Note that this is not quite the same as our approach to the case \( d = 3 \).) As briefly explained in Subsubsection 2.3.4, the order in which the various \( I \) are considered is important: we define \( < \) to be any total order on the set of non-empty subsets of \( [d-1] \) such that if \( J \subseteq I \) then \( J < I \). It is worth noting that unlike in the \( d = 3 \) case, the subspaces \( V_t, V_J \) with \( J < I \) are allowed to change when \( V_I \) and \( V_J \) get defined (although in fact the \( V_J \) will not change, and the \( V_J \) change only for \( J \subseteq I \)). All we require is that they do not become much larger.

**Lemma 2.15.** Let \( d \geq 2, 0 < \delta \leq 1/2 \) and \( k \geq G(d-1, \delta)^2 \). Let \( I \in [d-1], I \neq \emptyset \) and let \( W_J \subset \mathbb{F}^J, W_F \subset \mathbb{F}^F \) be subspaces of dimension at most \( k \) for every \( J < I \). Moreover, let \( W_{[d]} \subset \mathbb{F}^{[d]} \) have dimension at most \( k \). Suppose that \( Q', Q_1 \) and \( Q_J \) have the six properties described in Lemma 2.14. Then any array

\[
r \in W_{[d]} + \sum_{J \subseteq I} (W_J \otimes \mathbb{F}^J + \mathbb{F}^J \otimes W_J) + \sum_{J \subseteq I} \mathbb{F}^J \otimes H_J(r)
\]

with \( \dim(H_J(r)) \leq k \) and the property that \( r.q = 0 \) for at least \( (1-\frac{1}{4}(2^{-3d+2})^2)|Q_I| \) choices \( q \in Q_I \) is contained in

\[
W_{[d]} + \sum_{J \subseteq I} (U_J \otimes \mathbb{F}^J + \mathbb{F}^J \otimes U_J) + \sum_{J \subseteq I} \mathbb{F}^J \otimes K_J(r)
\]

for some \( U_J \subset \mathbb{F}^J, U_F \subset \mathbb{F}^F \) not depending on \( r \) and some \( K_J(r) \subset \mathbb{F}^F \) possibly depending on \( r \), all of dimension at most \( k^2 \).

**Proof.** By (4) in Lemma 2.14, for every \( s \in Q' \) there exist subspaces \( V_J(s) \subset \mathbb{F}^J \) for every \( J \subset F, J \neq \emptyset \), with dimension at most \( g_1 = G(d-1, |\mathbb{F}|^{-3d+4}C(\log 2d-1/\delta)^4) \) such that the set of arrays \( t \in \mathbb{F}^F \) with \( t.q = 0 \) for at least \( (1-g_2)|Q_s| \) choices \( q \in Q_s \) is contained in \( \sum_{J \subset F, J \neq 0} V_J(s) \otimes \mathbb{F}^{F \setminus J} \), where \( g_2 = 2^{-3d+2} \). Note, for future reference, that

\[
g_1 = G(d-1, |\mathbb{F}|^{-3d+4}C(\log 2d-1/\delta)^4) = ((\log |\mathbb{F}|)^2c_1(d-1)2^{-3d+4}C\log 2d-1/\delta)^4c_2(d-1)
\]

\[
\leq ((\log |\mathbb{F}|)^2c_1(d-1)2^{-3d+4}C(d \log 1/\delta)^4)^2c_2(d-1) \leq ((\log |\mathbb{F}|)^2c_1(d-1))^2(\log 1/\delta)^4c_2(d-1)
\]

\[
\leq G(d-1, \delta)^4 \leq k^2.
\]

Let \( R \) consist of the set of arrays with \( r \in W_{[d]} + \sum_{J \subseteq I}(W_J \otimes \mathbb{F}^J + \mathbb{F}^J \otimes W_J) + \sum_{J \subseteq I} \mathbb{F}^J \otimes H_J(r) \) with \( \dim(H_J(r)) \leq k \) and the property that \( r.q = 0 \) for at least \( (1-\frac{1}{4}(2^{-3d+2})^2)|Q_I| \) choices \( q \in Q_I \).
Let \( r \in R \). Then by averaging and using (5) from Lemma 2.14, for at least \( (1 - g_3)|Q'| \) choices \( s \in Q' \) we have \( r.(s \otimes t) = 0 \) for at least \( (1 - g_2)|Q| \) choices \( t \in Q \), where \( g_3 = \frac{1}{2} 2^{-3d_2} \). Thus, (noting that \( r.(s \otimes t) = (r.s).t \)), \( rs \in \sum_{J \subseteq F} V_J(s) \otimes \mathbb{R}^{F \setminus J} \) holds for at least \( (1 - g_3)|Q'| \) choices \( s \in Q' \). Let \( Q'(r) \) be the submultiset of \( Q' \) consisting of those \( s \in Q' \) for which \( rs \in \sum_{J \subseteq F} V_J(s) \otimes \mathbb{R}^{F \setminus J} \). Then we have \(|Q'(r)| \geq (1 - g_3)|Q'| \).

Note that we can write \( r = r_1 + r_2 + r_3 + r_4 \) where

\[
\begin{align*}
r_1 &\in \sum_{J \subseteq I, J \neq I \neq 0} W_J \otimes \mathbb{R}^F, \\
r_2 &\in \sum_{J \subseteq I, J \neq I \neq 0} (W_J \otimes \mathbb{R}^F + \mathbb{R}^I \otimes W_{\mathbb{R}^F}) + \sum_{J \neq I} \mathbb{R}^I \otimes H_{\mathbb{R}^F}(r), \\
r_3 &\in W_{[d]} + \sum_{J \subseteq I, J \neq I \neq 0} \mathbb{R}^I \otimes W_{\mathbb{R}^F}, \\
r_4 &\in \mathbb{R}^I \otimes H_{\mathbb{R}^F}(r).
\end{align*}
\]

By (1) in Lemma 2.14, the elements of \( Q' \) belong to \( \bigcap_{J \subseteq I, J \neq I \neq 0} (W_{\mathbb{R}^F} \otimes \mathbb{R}^{F \setminus J}) \), so we have \( r_1 s = 0 \) for every \( s \in Q' \).

Note that for every pure tensor \( s \in \mathbb{R}^J \), \( r_2 s \) is \( 2^d k \)-degenerate. Indeed, for any \( J \subseteq [d - 1] \) with \( J \neq I \) there are some \( s_1 \in \mathbb{R}^{F \setminus J} \), \( s_2 \in \mathbb{R}^{F \setminus J} \) with \( s = s_1 \otimes s_2 \). Then \( (W_J \otimes \mathbb{R}^F) s \subset (W_J s_1) \otimes \mathbb{R}^{F \setminus J} \).

Since \( \dim(W_J s_1) \leq k \), \( J \neq I \) and \( d \in I' \setminus J \), any tensor in \((W_J s_1) \otimes \mathbb{R}^{F \setminus J}\) is \( k \)-degenerate. Similarly, any tensor in \((\mathbb{R}^J \otimes W_{\mathbb{R}^F}) s \) or \((\mathbb{R}^I \otimes H_{\mathbb{R}^F}(r)) s \) is also \( k \)-degenerate, so \( r_2 s \) is indeed \( 2^d k \)-degenerate. Since \( Q' \) consists of pure tensors, this holds for every \( s \in Q' \).

Also, \( r_3 s \in \sum_{J \subseteq I, J \neq I}(W_J \otimes W_{\mathbb{R}^F}) s \). It follows that for every \( s \in Q'(r) \), there exists some \( t(s) \in V_F(s) + \sum_{J \subseteq I, J \neq I}(W_J \otimes W_{\mathbb{R}^F}) s \) such that \( r_3 s - t(s) \) is \( g_4 \)-degenerate for \( g_4 = g_1 + 2^d k \) (we have used that \( \dim(V_J(s)) \leq g_1 \)). To ease the notation, write \( T(s) \) for the space \( V_F(s) + \sum_{J \subseteq I, J \neq I}(W_J \otimes W_{\mathbb{R}^F}) s \).

We claim that the dimension of \( T(s) \) is at most \( g_4 = g_1 + 2^d k \). Indeed, \( \dim(V_F(s)) \leq g_1 \), so it suffices to prove that \( \dim((W_J \otimes W_{\mathbb{R}^F}) s) \leq k \) for every \( J \subseteq I \), \( J \neq I \). Since \( s \in Q' \), \( s \) is a pure tensor, so for any such \( J \) we have \( s = s_1 \otimes s_2 \) for some \( s_1 \in \mathbb{R}^J \), \( s_2 \in \mathbb{R}^{F \setminus J} \). But then \((W_J \otimes W_{\mathbb{R}^F}) s \subset W_{J, s_2} \), which has dimension at most \( \dim(W_{J, s_2}) \leq k \).

Let us define a sequence of subspaces \( 0 = Z(0) \subset Z(1) \subset \ldots \subset Z(m) \subset \mathbb{R}^F \) recursively as follows. Given \( Z(j) \), if for all \( r \in R \) we have that for all but at most \( 2 g_3 |Q'| \) choices \( s \in Q' \) there is some \( z \in Z(j) \) such that \( r_4 s - z \) is \( (g_4 + 1) g_4 \)-degenerate, then set \( m = j \) and terminate.

Else, choose some \( r \in R \) such that for at least \( 2 g_3 |Q'| \) choices \( s \in Q' \) there is no \( z \in Z(j) \) such that \( r_4 s - z \) is \( (g_4 + 1) g_4 \)-degenerate, and set \( Z(j + 1) = Z(j) + H_F(r) \). Recall that for
every \( s \in Q'(r) \), and in particular, for at least \((1 - g_3)|Q'|\) choices \( s \in Q' \), there exists some \( t(s) \in T(s) \) such that \( r_3s - t(s) \) is \( g_4 \)-degenerate. So for at least \( g_3|Q'| \) choices \( s \in Q' \) there is some \( t(s) \in T(s) \) such that \( r_4s - t(s) \) is \( g_4 \)-degenerate, but there is no \( z \in Z(j) \) such that \( r_4s - z \) is \((g_4 + 1)g_4\)-degenerate. In this case there is no \( z \in Z(j) \) such that \( z - t(s) \) is \( g_4^2 \)-degenerate. On the other hand, since \( r_4s \in H_F(r) \subset Z(j + 1) \), there is some \( z \in Z(j + 1) \) such that \( z - t(s) \) is \( g_4 \)-degenerate. For any \( i \), let \( K(i, s) \) be the subspace of \( T(s) \) spanned by those \( t \in T(s) \) for which there is some \( z \in Z(i) \) with \( z - t \) being \( g_4 \)-degenerate. Since the dimension of \( T(s) \) is at most \( g_4 \), we have \( t(s) \notin K(j, s) \), else there would exist some \( z \in Z(j) \) such that \( z - t(s) \) is \( g_4^2 \)-degenerate. On the other hand, \( t(s) \in K(j + 1, s) \). Thus, \( \dim K(j + 1, s) > \dim K(j, s) \). This holds for at least \( g_3|Q'| \) choices \( s \in Q' \), so

\[
\sum_{s \in Q'} \dim K(j + 1, s) \geq g_3|Q'| + \sum_{s \in Q'} \dim K(j, s).
\]

Since \( K(m, s) \subset T(s) \), we have \( \dim K(m, s) \leq g_4 \). Thus,

\[
|Q'|g_4 \geq \sum_{s \in Q'} \dim K(m, s) \geq mg_3|Q'|,
\]

so \( m \leq \frac{g_4}{g_3} \) and \( \dim Z(m) \leq \frac{kg_4}{g_3} \). Write \( Z = Z(m) \).

Now let \( r \in R \). Let \( X(r) \) be the set consisting of those \( x \in H_F(r) \) for which there is some \( z \in Z \) with \( x - z \) being \( (g_4 + 1)g_4 \)-degenerate. Then \( r_4s \in X(r) \) apart from at most \( 2g_3|Q'| \) choices \( s \in Q' \). Let \( t_1, \ldots, t_\alpha \) be a maximal linearly independent subset of \( X(r) \) and extend it to a basis \( t_1, \ldots, t_\alpha, t'_1, \ldots, t'_\beta \) for \( H_F(r) \). Now if a linear combination of \( t_1, \ldots, t_\alpha, t'_1, \ldots, t'_\beta \) is in \( X(r) \), then the coefficients of \( t'_1, \ldots, t'_\beta \) are all zero. Write \( r_4 = \sum s_i \otimes t_i + \sum s'_j \otimes t'_j \) for some \( s_i, s'_j \in F \).

Since \( r_4q \in X(r) \) for at least \((1 - 2g_3)|Q'| = (1 - 2^{-3^{h_2}})|Q'| \) choices \( q \in Q' \), we have, for all \( j \), that \( s'_j, q = 0 \) for at least \((1 - 2^{-3^{h_2}})|Q'| \) choices \( q \in Q' \). Thus, by (2) in Lemma 2.14 there exist subspaces \( L_j \subset F^l \ (J \subset I, J \neq \emptyset) \) not depending on \( r \), and of dimension at most \( G(|I|, |F|^{-2^{d_1}dk}) \) such that \( s'_j \in \sum_{J \subset I, J \neq \emptyset} L_j \otimes F^{I/J} \) for all \( j \). Thus, \( r_4 \in \sum s_i \otimes t_i + \sum_{J \subset I, J \neq \emptyset} L_j \otimes F^r \). Moreover, for every \( i \leq \alpha \), we have \( t_i \in X(r) \), so there exist \( z_i \in Z \) such that \( t_i - z_i \) is \((g_4 + 1)g_4\)-degenerate. It follows that \( r_4 \in F^l \otimes Z + \sum_{J \subset I, J \neq \emptyset} L_j \otimes K_{r'(r)} + \sum_{J \subset I, J \neq \emptyset} L_j \otimes F^r \) for some \( K_{r'}(r) \subset F^r \) of dimension at most \( \alpha \cdot (g_4 + 1)g_4 \leq k \cdot (g_4 + 1)g_4 \).

We claim that \( \dim(Z), \dim(K_{r'}) \) and \( \dim(L_j) \) are all bounded by \( k^{2\epsilon_2(|I|)} - k \).

Firstly, note that \( g_4 = g + 2^dk \leq k^2 + 2^dk \leq 2k^2 \).
Now \( \dim(K'_p) \leq k(g_4 + 1)g_4 \leq k^6 \leq k^{2c_2(|I|)} - k \). Also, \( \dim(Z) \leq \frac{k^{2c_2(|I|)}}{g_3} \leq k^4 \leq k^{2c_2(|I|)} - k \). Finally, \( \dim(L_j) \leq G(|I|, |\mathbb{P}|^{-2d+1}dk) = (\log |\mathbb{P}|)^2 c_1(|I|)(2^{d+1}dk)^{c_2(|I|)} \leq (\log |\mathbb{P}|)^2 c_1(d - 1)^2 k^{c_2(|I|)} \leq G(d - 1, \delta)^2 k^{c_2(|I|)} \leq k^{c_2(|I|)+1} \leq k^{2c_2(|I|)} - k \).

This completes the proof of the claim and the lemma. \(\square\)

**Proof of Lemma 2.2.** As stated earlier, the proof goes by induction on \( d \). For \( d = 1 \), by Lemma 2.1 there is a subspace \( U \subset \mathbb{P}^n \) of codimension at most \( C(\log 1/\delta)^4 \) contained in \( 2B' - 2B' \).

Choose \( Q = U \). Now if \( r.q = 0 \) for at least \( (1 - 2^{-3^{d+1}})|Q| \) choices \( q \in Q \) then the same holds for all \( q \in Q \), therefore \( r \in U^\perp \), but \( \dim(U^\perp) \leq C(\log 1/\delta)^4 \), so the case \( d = 1 \) is proved.

Now let us assume that \( d \geq 2 \). Extend the total order \(<\) defined above such that it now contains \( \emptyset \) which has \( \emptyset < I \) for every non-empty \( I \subset [d-1] \). Say \( \emptyset = I_0 < I_1 < I_2 < \cdots < I_{2^{d-1}-1} \) where \( \{I_0, \ldots, I_{2^{d-1}-1}\} = P([d-1]) \).

**Claim.** For every \( 0 \leq i \leq 2^{d-1} - 1 \) there exists a multiset \( Q_i \) of pure tensors with elements chosen from \( 2^{3^{d+1}}B' - 2^{3^{d+1}}B' \), and subspaces \( W_j(i) \subset \mathbb{P}^{l_j}, W_{(i),r}(i) \subset \mathbb{P}^{(l_j) r} \) for every \( j \leq i \) (for \( j = 0 \), we only require \( W_{(i)}(i) \) and not \( W_0(i) \)) with the following properties. The dimension of each of these spaces is at most \( g_1(i) = G(d - 1, \delta)^{g_2(i)} \), where \( g_2(i) = 4 \cdot \Pi_{1 \leq j \leq 2c_2(l_j)} \). Moreover, if \( r \in G \) has \( r.q = 0 \) for at least \( (1 - 1/2^{3^{d+1}})|Q| \) choices \( q \in Q_l \) for all \( j \leq i \), then \( r \in W_{(i)}(i) + \sum_{1 \leq j \leq 1} W_{(i)}(i) \otimes \mathbb{P}^{l_j} + \mathbb{P}^{l_j} \otimes W_{(i),r}(i) \) holds for some \( H_{(i),r}(i, r) \) possibly depending on \( r \) and of dimension at most \( g_1(i) \).

**Proof of Claim.** This is proved by induction on \( i \). For \( i = 0 \), by Lemma 2.12, there exist \( Q_0 \subset 2B' - 2B' \) consisting of pure tensors and \( V_{(d)} \subset \mathbb{P}^{d(d)} \) of dimension at most \( 4C(\log 2/\delta)^4 \leq 4C(2 \log 1/\delta)^4 \leq G(d - 1, \delta)^4 \) such that if \( r.q = 0 \) for at least \( 1/2 |Q_0| \) choices \( q \in Q_0 \), then \( r \) can be written as \( r = x + y \) where \( x \in V_{(d)} \) and \( y \) is \( g_2 \)-degenerate for \( g_2 = G(d - 1, \frac{\delta}{4 \log(2/\delta)^4}) \). Since \( g_2 \leq G(d - 1, |\mathbb{P}|^{-5C(\log 2/\delta)^4}) = (\log |\mathbb{P}|)^2 c_1(d - 1)5C(\log 2\delta)^4 c_2(d - 1) \leq G(d - 1, \delta)^4 \), we can take \( W_{(d)}(0) = V_{(d)} \).

Once we have found suitable sets \( W_{(i)}(i-1) \) and \( W_{(i),r}(i-1) \) for all \( j \leq i - 1 \), we can apply Lemmas 2.14 and 2.15 with \( I = I_i \) and \( k = g_1(i - 1) \) to find a suitable \( Q_i \), \( W_{(i)}(i) \) and \( W_{(i),r}(i) \) for all \( j \leq i \), and the claim is proved, since \( g_1(i) = g_1(i - 1)2^{c_2(|I_i|)} \).

Now, after taking several copies of each \( Q_i \), we may assume that additionally \( \max_I |Q_i| \leq 2 \min_I |Q_i| \). Let \( Q = \bigcup_{j \leq d-1} Q_i \) and suppose that \( r.q = 0 \) for at least \( (1 - 2^{-3^{d+1}})|Q| \) choices \( q \in Q \).
Since $2^{-3d^2} \leq \frac{1}{2^{2d}} \cdot \frac{1}{4}(2^{-3d^2+2})^2$, it follows that for every $I \subset [d-1]$ we have $r,q = 0$ for at least $(1 - \frac{1}{4}(2^{-3d^2})^2)|Q_I|$ choices $q \in Q_I$. By the Claim with $i = 2d-1$, we get that $r \in \sum_{I \subset [d], I \neq \emptyset} V_I \otimes \mathbb{F}_2$ for some $V_I \subset \mathbb{F}_2$ not depending on $r$, and of dimension at most $g_1(2d-1) = G(d-1, \delta) \alpha(2^d)$. Note that
\[ \alpha(2^d) = 4 \cdot 2^{2d-1} \cdot \prod_{1 \leq i \leq d-1} c_2(i)^{2i-1}. \]
But
\[ \prod_{1 \leq i \leq d-1} c_2(i)^{2i-1} \leq 4 \sum_{1 \leq i \leq d-1} (i)^{d-1} \leq 4(d-1)^{d-1} = 4^{d-1}. \]
Thus, $\alpha(2^d) \leq 4^{d^2}$. This completes the proof of the lemma. 

\section*{Acknowledgment}

I would like to thank Timothy Gowers for helpful discussions. I am also grateful to him and the anonymous referee for their valuable comments on a previous version of this paper.

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