The Kobayashi metric, extremal discs, and biholomorphic mappings

Steven G. Krantz*

Department of Mathematics, Washington University in St. Louis, St. Louis, Missouri 63130, USA

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We study extremal discs for the Kobayashi metric. Inspired by work of Lempert on strongly convex domains, we present results on strongly pseudoconvex domains. We also consider a useful biholomorphic invariant, inspired by the Kobayashi (and Carathéodory) metric, and prove several new results about biholomorphic equivalence of domains. Some interesting results about automorphism groups of complex domains are also established.

Keywords: Kobayashi metric; extremal disc; Carathéodory metric; pseudoconvexity

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1. Introduction

Throughout this article, a domain in \( \mathbb{C}^n \) is a connected, open set. Usually our domains will be bounded. It is frequently convenient to think of a domain \( \Omega \) (with smooth boundary) as given by

\[ \Omega = \{ z \in \Omega : \rho(z) < 0 \}, \]

where \( \rho \) is a \( C^k \) function and \( \nabla \rho \neq 0 \) on \( \partial \Omega \). We say in this circumstance that \( \rho \) is a \( C^k \) defining function for \( \Omega \). It follows from the implicit function theorem that \( \partial \Omega \) is a \( C^k \) manifold in a natural sense. See [1] for more details on these matters.

Throughout the article, \( D \) denotes the unit disc in the complex plane \( \mathbb{C} \) and \( B \) denotes the unit ball in complex space \( \mathbb{C}^n \). If \( \Omega_1, \Omega_2 \) are domains in complex space then we let \( \Omega_1(\Omega_2) \) denote the holomorphic mappings from \( \Omega_2 \) to \( \Omega_1 \). In case \( \Omega_2 \) is either \( D \) or \( B \) and \( z \in \Omega \) then we sometimes let \( \Omega^*(D) \) (resp. \( \Omega^*(B) \)) denote the elements \( \varphi \in \Omega(D) \) (resp. \( \varphi \in \Omega(B) \)) such that \( \varphi(0) = z \).

*Email: sk@math.wustl.edu
The infinitesimal \textit{Kobayashi metric} on $\Omega$ is defined as follows. Let $z \in \Omega$ and $\xi \in \mathbb{C}^n$. Then
\[
F^K_\Omega(z, \xi) = \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(D) \text{ with } f(0) = z, f'(0) = \xi/\alpha\}
= \inf\left\{\frac{|\xi|}{|f'(0)|} : f \in \Omega^2(D)\right\}.
\]
The infinitesimal \textit{Carathéodory metric} is given by
\[
F^C_\Omega(z, \xi) = \sup_{f \in \Omega(D)} |f'(z)|\xi|.
\]
In these definitions, $|\cdot|$ denotes Euclidean length. The definitions of both these metrics are motivated by the proof of the Riemann mapping theorem, and by the classical Schwarz lemma. Details may be found in \cite{1,2}.

Companion notions are the Kobayashi and Carathéodory volume elements. We define these as follows (see also \cite{3}). If $\Omega$ is a fixed domain and $z \in \Omega$ then set
\[
\mathcal{C}_\Omega(z) = \mathcal{C}(z) = \sup\{\det \varphi'(z) : \varphi : \Omega \to B, \varphi(z) = 0\}
\]
and
\[
\mathcal{K}_\Omega(z) = \mathcal{K}(z) = \inf\left\{\frac{1}{\det \psi'(z)} : \psi : B \to \Omega, \psi(0) = z\right\}.
\]
If $\varphi$ is a candidate mapping for $\mathcal{C}$ and $\psi$ is a candidate mapping for $\mathcal{K}$, then an examination of $\varphi \circ \psi$ using the Schwarz lemma \cite{4} shows that $\mathcal{C}(z) \leq \mathcal{K}(z)$ for any $z \in \Omega$. We set
\[
\mathcal{M}(z) = \frac{\mathcal{K}(z)}{\mathcal{C}(z)}.
\]
We call $\mathcal{M}$ the \textit{quotient invariant}. Of course $\mathcal{M}(z) \geq 1$ for all $z \in \Omega$. The following remarkable lemma of Bun Wong (see \cite{5} as well as the original source \cite{6}) is useful in the study of automorphism groups:

\textbf{Lemma 1} Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. If there is a point $z \in \Omega$ so that $\mathcal{M}(z) = 1$ then $\Omega$ is biholomorphic to the unit ball $B$ in $\mathbb{C}^n$.

We shall not prove this result here, but refer the reader instead to \cite{1}. It is worth stating the fundamental result of Bun Wong and Rosay (again see \cite{1} for the details) that is proved using Lemma 1.

\textbf{Theorem 2} Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and $P \in \partial \Omega$ a point of strong pseudoconvexity. Fix a point $X \in \Omega$ and suppose that there are biholomorphic mappings $\varphi_j : \Omega \to \Omega$ (automorphisms of $\Omega$) so that $\varphi_j(X) \to P$ as $j \to \infty$. Then $\Omega$ is biholomorphic to the unit ball $B$ in $\mathbb{C}^n$.

This theorem has been quite influential in the development of the theory of automorphism groups of smoothly bounded domains. See, for example, \cite{7–9}. It is common to call the point $P$ in the theorem a \textit{boundary orbit accumulation point} for the automorphism group action (or ‘orbit accumulation point’ for short).
2. The quotient invariant
Here we discuss in detail the invariant of Bun Wong described in Section 1. It has far-reaching implications beyond the basic application in the proof of the Bun Wong/Rosay theorem.

**Proposition 3** Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. If there is a point $P \in \Omega$ such that $\mathcal{M}(P) = 1$ then $\mathcal{M}(z) = 1$ for all $z \in \Omega$. Obversely, if there is a point $P \in \Omega$ with $\mathcal{M}(z) > 1$ for all $z \in \Omega$.

**Proof** If $\mathcal{M}(P) = 1$ for some $P$ then Bun Wong’s original lemma (Lemma 1) shows that $\mathcal{M}$ is a biholomorphic invariant. And $B$ has transitive automorphism group. It follows therefore that $\Omega$ has invariant $\mathcal{M}$ with value 1 at every point.

Obversely, if $\mathcal{M}(P) \neq 1$ at some point then, by contrapositive reasoning in the last paragraph, it cannot be that $\mathcal{M}$ equals 1 at any point.

That completes the proof of the proposition. ■

**Proposition 4** Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let $P \in \partial \Omega$ and suppose that $\partial \Omega$ is $C^2$ and strongly pseudoconvex near $P$. Then

$$\lim_{z \to P} \mathcal{M}(z) = 1.$$ 

**Proof** This follows from the asymptotics of Graham for the Carathéodory and Kobayashi metrics on such a domain. The main point is that $\partial \Omega$ is approximately a ball near $P$, so the asymptotic behaviour of $F^\Omega$, $F^\Omega_C$, $K^\Omega$ and $C^\Omega$ is the same as that on the domain $B$. ■

**Proposition 5** Let

$$E = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{2m_1} + |z_2|^{2m_2} + \cdots + |z_n|^{2m_n} < 1\}$$

be a domain in $\mathbb{C}^n$, with $m_1, m_2, \ldots, m_n$ positive integers. Often $E$ is called an egg or an ellipsoid. If some $m_j > 1$ then $E$ is not biholomorphic to the ball.

**Proof** This result was first proved by Webster [10] using techniques of differential geometry. Later, Bell [11] gave a very natural proof by showing that any biholomorphism of the ball to $E$ must extend smoothly to the boundary, and then noting that the Levi form is a biholomorphic invariant. Here we give a proof that uses $\mathcal{M}$.

For simplicity we shall take $n = 2$, $m_1 = 1$ and $m_2 > 1$. Seeking a contradiction, we let $\varphi : B \to E$ be a holomorphic mapping that takes 0 to 0. Thus $\varphi$ is a candidate mapping for the calculation of $K^\Omega$. Now set

$$\tilde{\varphi}(z_1, z_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(z_1 e^{i\theta_1}, z_2 e^{i\theta_2}) e^{-i\theta_1} e^{-i\theta_2} d\theta_1 d\theta_2.$$ 

Then one may calculate that (i) $\tilde{\varphi}$ still maps $B$ into $E$ and (ii) the first (holomorphic) derivatives of $\tilde{\varphi}$ at 0 are the same as the first (holomorphic) derivatives of $\varphi$ at 0. Also $\tilde{\varphi}$ is linear (since the higher order terms all average to 0).

As a result of the last paragraph, we may calculate $K$ at 0 for $E$ using only linear maps. A similar argument applies to maps $\psi : E \to B$. Of course it is obvious that there is no linear equivalence of $B$ and $E$ (e.g. the boundaries of the two domains...
have different curvatures). In particular, \( \mathcal{M}(0) > 1 \). It follows that \( \mathcal{M}(P) > 1 \) at all points \( P \) of \( E \). Thus \( E \) and \( B \) are biholomorphically inequivalent.

**Proposition 6** Let \( \Omega \subseteq \mathbb{C}^n \) be a bounded domain with \( C^2 \) boundary. If \( P \in \partial \Omega \) is a point of strong pseudoconcavity, let \( v \) be the unit outward normal vector at \( P \). Set \( P_\epsilon = P - \epsilon v \). Then \( \mathcal{M}_{\Omega}(P_\epsilon) \approx C \cdot \epsilon^{-3/4} \).

**Proof** It is a result of [12] that the Kobayashi metric \( F_{\mathcal{K}}(P, v) \) is of size \( C \cdot \epsilon^{-3/4} \).

It is also clear that the Kobayashi metric at \( P_\epsilon \) in complex tangential directions is of size \( C \), where \( C > 0 \) is some universal positive constant. Hence \( C \sim C \cdot \epsilon^{-3/4} \). On the other hand, the Hartogs extension phenomenon gives easily that \( C(P) \sim C \). It follows then that \( \mathcal{M} \approx C \cdot \epsilon^{-3/4} \).

**Corollary 7** Let \( \Omega \subseteq \mathbb{C}^n \) be a bounded domain with \( C^2 \) boundary. If \( P \in \partial \Omega \) is a point of strong pseudoconcavity, then \( P \) cannot be a boundary orbit accumulation point.

**Proof** Seeking a contradiction, we suppose that \( P \) is a boundary orbit accumulation point. So there is a point \( X \in \Omega \) and there are automorphisms \( \phi_j \) of \( \Omega \) so that \( \phi_j(X) \to P \). But of course \( \mathcal{M}(X) \) is some positive constant \( C \) that exceeds 1. And the invariant \( \mathcal{M}(z) \) blows up like \( \text{dist}(z, \partial \Omega)^{-3/4} \) as \( z \to P \). This is impossible.

**Remark 8** It is a result of [13] that if \( \Omega \) is any domain and \( P \in \partial \Omega \) a point of non-pseudoconvexity (even in the weak sense of Hartogs) then \( P \) cannot be a boundary orbit accumulation point. The last corollary captures a special case of this result using the idea of the quotient invariant.

**Proposition 9** Let \( \Omega \subseteq \mathbb{C}^2 \) be a smoothly bounded domain that is of finite type (in the sense of Kohn/D’Angelo/Catlin – see [1]) at every boundary point. Let \( P \in \partial \Omega \). Then

\[
0 < C_1 \leq \liminf_{z \to P} \mathcal{M}(z) \leq \limsup_{z \to P} \mathcal{M}(z) \leq C_2
\]

for some universal, positive constants \( C_1, C_2 \).

**Proof** This follows from the estimates in [14].

**Proposition 10** Let \( \Omega \subseteq \mathbb{C}^2 \) be a smoothly bounded, convex domain of finite type. Let \( P \in \partial \Omega \). Then

\[
0 < C_1 \leq \liminf_{z \to P} \mathcal{M}(z) \leq \limsup_{z \to P} \mathcal{M}(z) \leq C_2
\]

for some positive constants \( C_1, C_2 \).

**Proof** Fix a point \( z \in \Omega \) near \( P \) and \( \xi \) a tangent direction at \( z \). Certainly any mapping \( \varphi : D \to \Omega \), \( \varphi(0) = z \) with \( \varphi'(0) = \lambda \xi \) for some \( \lambda > 0 \) is a candidate for the Kobayashi metric at \( z \) in the direction \( \xi \), and the reciprocal of its derivative gives an upper bound for the Kobayashi metric at that point in that direction. In particular, we may take \( \varphi \) to be the obvious linear embedding of the disc \( D \) into \( \Omega \) pointing in the direction \( \xi \) (with image having diameter \( \delta \), the distance from \( z \) to \( \partial \Omega \) in the direction \( \xi \)) and with \( \varphi(0) = z \).

Thanks to work of McNeal [15], we know that the type of a convex point of finite type can be measured with the order of contact by complex lines. If, after a rotation and translation, we take \( P \) to be the point \( (1, 0) \) and \( (1, 0) \) the real normal direction, then the complex line of greatest contact will of course be \( \xi \mapsto (1, \xi) \). Let that order
of contact be $2m$ for some positive integer $m$. Then it is clear, after shrinking $\Omega$ if necessary, that an ellipsoid of the form
\[ E = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + K|z_2|^{2m} < 1\} \]
will osculate $\partial \Omega$ at $(1, 0)$ and will contain $\Omega$. So, in particular $F_C^\Omega(z, \xi) \geq F_E^\Omega(z, \xi)$ for any $z \in \Omega$ and $\xi$ any tangent vector.

We calculate that, for $z = (\alpha, 0) \in E$, the mappings
\[ (\xi_1, \xi_2) \mapsto \frac{\xi_1 - \alpha}{1 - \bar{\alpha}z_1} \]
and
\[ (\xi_1, \xi_2) \mapsto \frac{\sqrt{1 - |\alpha|^2}z_1}{1 - \bar{\alpha}z_1} \]
are candidate maps for the Carathéodory metric at the point $z$. The first one gives a favourable lower bound for the Carathéodory metric in the normal direction $(1, 0)$ at $z$ and the second gives a favourable lower bound for the Carathéodory metric in the tangential direction $(0, 1)$ at $z$. Of course these are also lower bounds for the Carathéodory metric on $\Omega$.

It is easy to see that the given upper bounds for the Kobayashi metric and the given lower bounds for the Carathéodory metric are comparable. Since $F_C^\Omega \leq F_K^\Omega$ always [1], it follows that $\mathcal{M} \approx C$ (a constant) on a smoothly bounded, convex domain of finite type in $\mathbb{C}^2$.

**Remark 11** The elementary comparison of the domains $\Omega$ and $E$ that we exploited in the last proof will not work in higher dimensions. The matter in that context is more subtle.

### 3. More on the quotient invariant

It is natural to wonder about the role of the ball $B$ in the definition of the quotient invariant $\mathcal{M}$. We define $K$ in terms of mappings from the ball $B$ to the given domain $\Omega$ and we define $C$ in terms of mappings from the given domain $\Omega$ to the ball $B$. What if the ball $B$ were to be replaced by some other 'model domain'?

Let $B$ be some fixed, bounded domain in $\mathbb{C}^n$. Fix a point $P_0 \in B$. Let $\Omega$ be some other bounded domain, and let $z \in \Omega$. Define new invariants
\[ \widehat{C}_\Omega(z) = \widehat{C}(z) = \sup\{|\det \psi'(z)| : \psi : \Omega \to B, \psi(z) = P_0\} \]
and
\[ \widehat{K}_\Omega(z) = \widehat{K}(z) = \inf\left\{\frac{1}{|\det \psi'(z)|} : \psi : B \to \Omega, \psi(P_0) = z\right\} \]
and a new quotient invariant
\[ \widehat{\mathcal{M}}_\Omega(P) = \widehat{\mathcal{M}}(P) = \frac{\widehat{K}_\Omega(P)}{\widehat{C}_\Omega(P)} \]

Now we have the following proposition.
PROPOSITION 12 Let $\omega$ be any given bounded domain in $\mathbb{C}^n$. Suppose that there is a point $P \in \omega$ such that $\tilde{M}_\omega(P) = 1$. Then $\omega$ is biholomorphic to the model domain $\mathcal{B}$.

Proof The argument is just the same as in the classical case of $\mathcal{B} = \mathcal{B}$, the unit ball of $\mathbb{C}^n$, see [1, Ch. 11]. It is a relatively straightforward normal families argument. We shall not repeat the details.

It is no longer the case in general (see our Proposition 3) that $\tilde{M}$ equals 1 at one point if and only if $M$ equals 1 at all points – unless the model domain $\mathcal{B}$ has transitive automorphism group. See more on this point in what follows.

Now of course one of the great classical applications of Proposition 11, when $\mathcal{B}$ is the unit ball $\mathcal{B}$, is to prove the Bun Wong/Rosay theorem (our Theorem 2). One might now ask whether a similar sort of result could be proved with the new quotient invariant $\tilde{M}$. The answer is that the proof requires that the model domain has transitive automorphism group (see the details in [1, Ch. 11]).

Thus we may only consider models $\mathcal{B}$ chosen from among the bounded symmetric domains of Cartan [16]. Let us concentrate here on the case when $\mathcal{B}$ is the unit polydisc. The following result is similar to one proved in [17]:

THEOREM 13 Let $\omega \subseteq \mathbb{C}^2$ be a smoothly bounded, convex domain. Let $P \in \partial \omega$ and assume that $\partial \omega$ in a neighbourhood $U$ of $P$ coincides with a real hyperplane in $\mathbb{C}^n$. In suitable local coordinates we may say that
\[
\partial \omega \cap U = \{z \in U : \text{Re} z_1 = 0\}.
\]

If $P$ is a boundary orbit accumulation point for $\omega$ then $\omega$ is biholomorphic to the bidisc.

Sketch of proof The key fact in the proof of this result when $P$ is a strongly pseudoconvex point (our Theorem 3) is that the geometry localizes at $P$. This means that if $X \in \omega$ and $\varphi_j$ are automorphisms of $\omega$ such that $\varphi_j(X) \to P$ then $\varphi_j$ converges uniformly on any compact set $K$ to $P$.

Such is not the case in our present situation. But the automorphisms $\varphi_j$ and the point $X$ still exist (by a classical lemma of Cartan [18]). As indicated in line (†), assume that the real normal direction at $P$ is the $\text{Re} z_1$ direction. If $K \subseteq \omega$ is any compact set then we may compose $\varphi_j$ for $j$ large with a dilation in the tangential directions $z_2, z_3, \ldots, z_n$ to localize the geometry near $P$, just as in the classical case. The rest of the proof goes through as in the classical case described in [1]. Instead of localizing to an image of the ball, one localizes to a bidisc.

Remark 14 In [17], Kim uses the method of scaling to obtain his result. This is a powerful technique that has wide applicability in this subject (see e.g. [9]). The argument that we sketch here is similar in spirit to scaling.

Perhaps another point worth considering is stability results for the quotient invariant $M$ (i.e. the original invariant modelled on the unit ball $\mathcal{B}$). We have the following result:

THEOREM 15 Let $\omega, \omega_j \subseteq \mathbb{C}^n$ be bounded domains with $C^2$ boundary and suppose that $\omega_j \to \omega$ in the $C^2$ topology on domains (see [19,20] for this concept). Then
\[
M_{\omega_j} \to M_\omega
\]
uniformly on compact subsets of $\omega$ as $j \to \infty$. 

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Proof. Simply use the Carathéodory and Kobayashi stability results established in [21].

4. Extremal discs and chains for the Kobayashi metric

In the remarkable paper [21], Lempert shows that, on a convex domain $\Omega \subseteq \mathbb{C}^n$, the integrated Kobayashi distance on $\Omega$ may be calculated using a Kobayashi chain of length one disc (see [1,23] for the concept of Kobayashi chain). This is done as a prelude to developing his profound theory of extremal discs on strongly convex domains.

Lempert comments that such a result is not true for general pseudoconvex domains, and he provides the following example:

Example 1

Let

$$\Omega_\epsilon = \{(z,w) \in \mathbb{C}^2 : |z| < 2, |w| < 2, |zw| < \epsilon\}.$$ 

Let $P = (1,0) \in \Omega_\epsilon$ and $Q = (0,1) \in \Omega_\epsilon$. Then the Kobayashi one-disc distance of $P$ to $Q$ tends to infinity as $\epsilon \to 0^+$. Just to be perfectly clear, we note that the one-disc Kobayashi distance of two points $P$ and $Q$ in a domain $\Omega$ is defined to be

$$d(P,Q) = \inf\{\rho(a,b) : \varphi : D \to \Omega, \varphi \text{ holomorphic}, \varphi(a) = P, \varphi(b) = Q\},$$

where $\rho$ is the classical Poincaré metric on the disc $D$.

Lempert’s reasoning in this example (private communication) is as follows: suppose not. Then there are mappings $\varphi_\epsilon : D \to \Omega_\epsilon$ with $\varphi_\epsilon(a_\epsilon) = P$ and $\varphi_\epsilon(b_\epsilon) = Q$ and $\rho(a_\epsilon, b_\epsilon)$ bounded above as $\epsilon \to 0^+$. Thus we have that $a_\epsilon, b_\epsilon$ remain in a compact subset $K$ of $D$. Passing to a normal limit (with Montel’s theorem), we find a holomorphic function $\varphi_0 : D \to \{(z,w) : |z| \leq 2, |w| \leq 2, |zw| = 1\}$ and points $a_0, b_0 \in K$ such that $\varphi_0(a_0) = P, \varphi_0(b_0) = Q$. Of course this is impossible, since it must be that either the image of $\varphi_0$ lies in $\{(z,w) : z = 0\}$ or in $\{(z,w) : w = 0\}$.

It is useful, and instructive, to have a more constructive means of seeing that this example works. We thank John E. McCarthy for the following argument.

Take

$$\varphi = (f_1,f_2) : D \to \Omega_\epsilon$$

holomorphic. We assume that

- $\varphi(0) = (1,0)$;
- $\varphi(r) = (0,1)$.

We shall show constructively that, as $\epsilon \to 0^+$, it must follow that $r \to 1^-$. This is equivalent to what is claimed for the domains $\Omega_\epsilon$.

Now use the inner-outer factorization for holomorphic functions on the disc (see, e.g. [24]) to write $f_1 = F_1 \cdot I_1$ and $f_2 = F_2 \cdot I_2$. Here each $F_j$ is outer and each $I_j$ is inner. Since $|f_1 \cdot f_2| < \epsilon$, we may be sure that

$$|F_1 \cdot F_2| < \epsilon.$$ (1)
Now certainly
\[ |F_1(0)| \geq |f_1(0)| = 1 \]
and hence
\[ |F_2(0)| < \epsilon. \]

Certainly \( \log |F_1| + \log |F_2| \) is harmonic, and by line \((*)\) is majorized by \( \log \epsilon \).

Let \( h \) denote the harmonic function \( \log |F_2| \). We can be sure that

1. \( h \leq \log 2 \);
2. \( h(0) \leq \log \epsilon \);
3. \( h(r) \geq 0 \).

Let \( h^+ \) be the positive part of \( h \) and \( h^- \) the negative part. Of course \( h^+ \geq 0 \) and \( h^- \geq 0 \). Then the mean-value property for harmonic functions tells us that

\[ \frac{1}{2\pi} \int_0^{2\pi} h^+(e^{i\theta})d\theta - \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta = h(0) \leq \log \epsilon, \]

hence

\[ \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} h^+(e^{i\theta})d\theta + \log \frac{1}{\epsilon}. \quad (***) \]

Let \( P_r(e^{i\theta}) \) denote the Poisson kernel for the unit disc \( D \). Then

\[ h(r) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta})P_r(e^{i\theta})d\theta. \]

But Harnack’s inequalities tell us that

\[ \frac{1 - r}{1 + r} \leq P_r(e^{i\theta}) \leq \frac{1 + r}{1 - r}. \]

As a result, using (3) above,

\[ 0 \leq h(r) \leq \frac{1 + r}{1 - r} \cdot \frac{1}{2\pi} \int_0^{2\pi} h^+(e^{i\theta})d\theta - \frac{1 - r}{1 + r} \cdot \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta. \quad (****) \]

We conclude that

\[ 0 \leq h(r) \leq \frac{1 + r}{1 - r} \cdot \log 2 - \frac{1 - r}{1 + r} \cdot \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta. \]

Therefore

\[ \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta \leq \left(\frac{1 + r}{1 - r}\right)^2 \frac{1}{2\pi} \int_0^{2\pi} h^+(e^{i\theta})d\theta \]

\[ \leq \left(\frac{1 + r}{1 - r}\right)^2 \left[ \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta + \log \epsilon \right], \]
where we have use (**) in the last inequality. Now certainly
\[
\log \frac{1}{\epsilon} \leq |h(0)|
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} h^+(e^{i\theta})d\theta
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta + \log \epsilon
\]
\[
\leq \frac{1}{2\pi} \int_0^{2\pi} h^-(e^{i\theta})d\theta
\]
\[
\leq \left(\frac{1+r}{1-r}\right)^2 \frac{1}{2\pi} \int_0^{2\pi} h^+(e^{i\theta})d\theta
\]
\[
\leq \left(\frac{1+r}{1-r}\right)^2 \cdot \log 2.
\]
As \( \epsilon \rightarrow 0^+ \), this last inequality can only be true if \( r \rightarrow 1^- \). That is what we wished to prove.

There has been some interest, since Lempert’s paper, in developing an analogous theory on strongly pseudoconvex domains. Sibony [25] has shown that certain aspects of such a programme are impossible.

It is natural to reason as follows:

- Near the boundary of a strongly pseudoconvex domain, the domain is well-approximated by the biholomorphic image of \( B \), the unit ball. It is easy to verify directly (or by invoking Lempert) that Kobayashi distance on the ball can be realized with Kobayashi chains of length 1.

- In the interior of the domain – away from the boundary – things should be trivial. After all, if \( \Omega \) is strongly pseudoconvex and \( P \in \Omega \) is in the interior – away from the boundary – then the infinitesimal Kobayashi metric \( F^K_\Omega(P, \xi) \) for one Euclidean unit vector \( \xi \) ought to be roughly the same as the infinitesimal Kobayashi metric \( F^K_\Omega(P, \xi') \) for any other Euclidean unit vector \( \xi' \). Also the Kobayashi metric on a compact subset \( K \) of \( \Omega \) is comparable to the Euclidean metric. So one should be able to check directly that chains in the interior behave like chains for the Euclidean metric.

Unfortunately the expectation enunciated in the second bulleted item above is not true.

**Example 2** Let \( N > 0 \) be a large positive integer and set
\[
B_N = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2/N|^2 < 1\}.
\]
Of course \( B_N \) is biholomorphic to the unit ball \( B \) via the biholomorphism
\[
\Psi : B \rightarrow B_N
\]
\[
(z_1, z_2) \mapsto (z_1, Nz_2).
\]
And one calculates readily, using the mapping $\Psi$, that

$$F_{K}^{\mathcal{O}}((0,0),(1,0)) = 1$$

while

$$F_{K}^{\mathcal{O}}((0,0),(0,1)) = N.$$  

So the two different infinitesimal Kobayashi metric measurements at the base point $0 = (0,0)$ – in two different Euclidean unit directions – are very different.

Interestingly, the following contrasting result is true for the Carathéodory metric.

**Proposition 16** Let $\Omega$ be a fixed, bounded domain in $\mathbb{C}^n$. Let $K \subseteq \Omega$ be a fixed compact subset. There is a positive constant $C_0$ so that, if $P \in K$ and $\xi_1, \xi_2$ are Euclidean unit vectors then

$$\|F_C^{\mathcal{O}}(P, \xi_1) - F_C^{\mathcal{O}}(P, \xi_2)\| \leq C_0.$$  

**Proof** Let $r > 0$ be a small number. Let $\gamma$ be a $C^\infty_c$ function that satisfies:

(a) $\gamma$ is radial.

(b) $\gamma$ is supported in the Euclidean ball with centre at $P$ and radius $r$.

(c) $\gamma$ is identically equal to 1 on the Euclidean ball with centre at $P$ and radius $r/2$.

Now let $\mu$ be a unitary rotation of $\mathbb{C}^n$ that takes $\xi_1$ to $\xi_2$. Fix a point $P \in K$ and vectors $\xi_1, \xi_2$ as in the statement of the proposition. Let $\psi$ be an element of $(\Omega, D)$ with $\psi(P) = 0$ and $\psi'(P)$ a positive, real multiple of $\xi$ – say that $\psi'(P) = \kappa \xi$. Set

$$\widetilde{\psi}(z) = \gamma(z) \cdot [\psi \circ \mu^{-1}(z)] + [1 - \gamma(z)] \cdot \psi(z) + h(z). \quad (*)$$

Of course $\widetilde{\psi}$ will not be *a priori* holomorphic – because we have constructed the function using cutoff functions – but we hope to use the $\overline{\partial}$ problem to select $h$ so that $\widetilde{\psi}$ will be holomorphic.

Applying the $\overline{\partial}$ operator to both sides of Equation (*), we find that

$$\overline{\partial} h = -\overline{\partial} \gamma \cdot [\psi \circ \mu^{-1}] + \overline{\partial} \gamma \cdot \psi.$$  

Now it is essential to notice the following properties:

- $|\overline{\partial} \gamma|$ is of size $\approx 1/r$;
- $\psi(P) = 0$, so that, on the support of $\overline{\partial} \gamma$, $|\psi|$ of size $r$;
- $\overline{\partial} h$ is supported on the ball with centre $P$ and radius $r$;
- $\overline{\partial} h$ is $\overline{\partial}$-closed.

We see therefore that $\overline{\partial} h$ is of size $\mathcal{O}(1)$ (in Landau’s notation) and supported in a Euclidean ball of radius $r$. Hence $h$ has $L^3$ norm on any one-dimensional complex slice of space not exceeding $C \cdot [r^2 \cdot 1]^{1/3} = C \cdot r^{2/3}$.

Now we may solve the equation $\overline{\partial} u = h$ using the solution

$$h(z) = -\frac{1}{\pi} \int \int \frac{\tau_j(z_1,\ldots,z_{j-1},\xi,z_{j+1},\ldots,z_n)}{\xi - z_j} \, dA(\xi).$$
Here $\bar{\partial}h = \tau_1 dz_1 + \tau_2 dz_2$ (see [1, p. 16] for a discussion of this idea). Then we see that

$$\|u\|_{\sup} \leq \| \tau_j (z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n) \|_{L^1} \cdot \left\| \frac{1}{z - z_j} \right\|_{L^{1/2}} \leq r^{2/3} \cdot r^{1/2} = r^{7/6}.$$ 

In summary, $h$ is small in uniform norm if $r$ is small, and we may choose $r$ in advance to be as small as we please.

Now what is more essential for our purposes is that we may likewise estimate the size of $\|\nabla h\|_{\sup}$. For we may write

$$h(z) = -\frac{1}{\pi} \int \tau_j (z_1, \ldots, z_{j-1}, \xi - z_j, z_{j+1}, \ldots, z_n) \, dA(\xi)$$

and hence

$$\nabla h(z) = -\frac{1}{\pi} \int \nabla z \tau_j (z_1, \ldots, z_{j-1}, \xi - z_j, z_{j+1}, \ldots, z_n) \, dA(\xi) \tag{\dagger}$$

But now it is essential to notice that

- $|\nabla \psi|$ is of size $r^{-2}$,
- $\nabla \psi$ is of size $O(1)$.

It follows then that $\nabla \bar{\partial} h$ is of size $r^{-1}$ and is still supported on a Euclidean ball of radius $r$. Thus we may estimate (\dagger) again using H"{o}lder’s inequality. The result is that $\|\nabla h\|_{\sup} \leq C \cdot r^{1/6}$.

We conclude that the corrected candidate function $\widetilde{\psi}$ is near $P$ uniformly close to being just a rotation of $\psi$. We also see that

$$\widetilde{\psi}(P) = \psi(P) \circ \mu + h'(P).$$

Thus $\widetilde{\psi}(P)$ is as close as we like to equalling $\xi'$. Now taking a normal limit (again using Montel’s theorem) as $r \to 0^+$ yields a function $\psi_0 : \Omega \to B$ with $\psi_0(P) = 0$ and $\psi_0'(P) = \kappa \cdot \xi'$. So we find a candidate for the Carathéodory metric at $P$ in the direction $\xi'$ that is comparable to the original candidate $\psi$ in the direction $\xi$.

We would like to explore here the nature of Kobayashi chains on a strongly pseudoconvex domain. In principle, the Kobayashi chains on a given domain $\Omega$ can have any number of discs. We shall prove, however, that on a strongly pseudoconvex domain there is an a priori upper bound for the length of chains. This result may be thought of as a prelude to the development of a Lempert-type theory on strongly pseudoconvex domains.

**Proposition 17** Let $\Omega \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with $C^2$ boundary. Let $f : D \to \Omega$ and $g : D \to \Omega$ be holomorphic mappings of the disc into $\Omega$. We assume that $\sup_{z \in D} |f(\xi) - g(\xi)| < \delta$ for some small $\delta > 0$. Further, following Lempert’s notation [22, pp. 430–431], we let $\xi, \omega, \omega'$ and $\sigma \in D$ satisfy

$$f(\xi) = z, \quad f(\omega) = g(\omega') = w, \quad g(\sigma) = s.$$

Then there is a holomorphic mapping

$$h : D \to \Omega$$
with \( h(z) = z \), \( h(s) = s \). It follows then that, in the calculation of the Kobayashi metric using chains, we may replace the two discs \( f, g \) with the single disc \( h \).

**Proof** By the Fornaess embedding theorem, there is a strongly convex domain \( \Omega' \) with \( C^2 \) boundary, \( \Omega' \subseteq \mathbb{C}^N \) with \( N > n \) in general, and a proper holomorphic embedding

\[
\Phi : \overline{\Omega} \to \overline{\Omega}'.
\]

We refer the reader to [26] for the details of the domain and the mapping. Let \( \widehat{\Omega} \subseteq \Omega' \) be the image of \( \Omega \) under the mapping \( \Phi \). According to the Docquier–Grauert theorem [27,28], there is a neighbourhood \( U \) of \( \widehat{\Omega} \) and a holomorphic retraction \( \pi : U \to \widehat{\Omega} \).

Of course \( \Phi(f(D)) \) and \( \Phi(g(D)) \) both lie in \( \widehat{\Omega} \). We may apply Lempert’s Theorem 1 to obtain a convex combination \( \lambda(z) \) of \( \Phi(f(D)) \) and \( \Phi(g(D)) \). Now we may not conclude that the image of \( \lambda \) lies in \( \widehat{\Omega} \). But it certainly lies in the strongly convex domain \( \Omega' \). And, if \( \delta \) is sufficiently small, then we know that the image of \( \lambda \) lies in \( U \). Thus we may consider the analytic disc \( \widehat{\lambda} = \pi \circ \lambda \), whose image does lie in \( \Omega' \). Now \( \Phi^{-1} \) makes sense on \( \Omega' \), so we may define

\[
h(z) = \Phi^{-1} \circ \widehat{\lambda}.
\]

Tracing through the logic shows that this \( h \) is the one that we seek. \( \Box \)

**Theorem 18** Let \( \Omega \subseteq \mathbb{C}^n \) be a strongly pseudoconvex domain with \( C^2 \) boundary. Then there is an \( \epsilon > 0 \) and an a priori constant \( K = K(\Omega) \) so that if \( P, Q \in \Omega \) then there is a Kobayashi chain with elements \( \varphi_1, \ldots, \varphi_K \) so that the integrated Kobayashi distance of \( P \) to \( Q \) is within \( \epsilon \) of the length given by the Kobayashi chain.

**Proof** Since \( \Omega \) is a bounded domain, it is contained in a large Euclidean ball. By elementary comparisons, [1], we know that the Kobayashi metric in \( \Omega \) is not less than the Kobayashi metric in the ball. In particular, we get an a priori upper bound on derivatives of extremal discs for the Kobayashi metric in \( \Omega \). As a result, there is an \( \eta > 0 \) and a finite net of points \( \mathcal{P} \subseteq \Omega \) so that

(i) every point of \( \Omega \) is Euclidean distance not more than \( \eta \) from some point of \( \mathcal{P} \);

(ii) there is an a priori integer \( M > 0 \) so that if \( \psi : D \to \Omega \) is a Kobayashi extremal disc then there is a collection of elements \( \mathcal{Q}_\psi \) of at most \( M \) points in \( \mathcal{P} \) so that every point in the image \( \psi(D) \) is Euclidean distance at most \( \eta \) from some point of \( \mathcal{Q}_\psi \). More importantly, there is a finite net of points \( \mathcal{K}_\psi \) in the disc \( D \) – of cardinality at most \( M \) – so that every element of \( \mathcal{Q}_\psi \) is the approximate image (within distance \( \eta \)) under \( \psi \) of some element of \( \mathcal{K}_\psi \) (in fact one can conveniently take \( \mathcal{K}_\psi \) to be a net in the disc \( D \) that has unit distance \( \eta' \), for some small \( \eta' > 0 \), in the Poincaré metric). Thus we associate with \( \psi \) the set \( \mathcal{Q}_\psi, \mathcal{K}_\psi \).

Of course there are only finitely many possible sets \( \mathcal{K}_\psi, \mathcal{Q}_\psi \) (indeed \( 2^M \) is an upper bound on the cardinality of \( \{ \mathcal{Q}_\psi \} \), and there is a similar upper bound \( 2^M \) for the \( \{ \mathcal{K}_\psi \} \)). If \( \mathcal{T} \) is a Kobayashi chain in \( \Omega \) with more than \( (2^M)^{2M} \) discs, then two of those discs will share the same \( \mathcal{K}_\psi \) and \( \mathcal{Q}_\psi \). As a result, if \( \eta \) and \( \eta' \) are fixed small
enough (depending on $\delta$ in the last proposition), then the two corresponding extremal discs in the chain will be close enough that the last proposition applies. And those two discs may be replaced by a single disc.

This shows that our a priori constant $K$ exists and does not exceed $\left(2^M\right)^{2M}$. $

5. Concluding remarks

In the past 40 years or more, the Carathéodory and Kobayashi metric constructions have proved to be powerful tools in both geometry and function theory. Their role in the study of automorphism group is more recent, but is equally significant. We trust that the contributions of this article will point in some new directions in the subject. What lies in the future can only be a topic for omphaloskepsis.

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