NOTE ON THE STABILITY OF FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we will prove the generalized Hyers-Ulam stability of the linear differential equation of the form $y'(x) + f(x)y(x) + g(x) = 0$ under some additional conditions.

Keywords: fixed point method, differential equation, Hyers-Ulam stability.

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1. INTRODUCTION

The study of the stability functional equations is strongly related to Ulam’s question concerning the stability of group homomorphisms. We mention that the concept of stability for a functional equation appears when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question for functional equations shows “how the solutions of the inequality differ from those of the given functional equation.” D.H. Hyers [3] excellently answered the question of Ulam and proved the following result:

Theorem 1.1 (Hyers, [3]). Let $E$ and $E'$ be two Banach spaces and $f : E \rightarrow E'$ a given function such that there exists $\delta \geq 0$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta, \quad \forall x, y \in X. \quad (1.1)$$

Then the limit $L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$, $L$ is an additive function and the inequality

$$\|f(x) - L(x)\| < \delta \quad (1.2)$$

is true for all $x \in E$. Moreover, $L(x)$ is the only additive function which satisfies the inequality (1.2).
Since Hyers’ result, a great number of papers on the subject have been published, extending and generalizing the Ulam’s problem and the Hyers’ theorem in various directions (see [3,9,10]).

In [9] V. Radu proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative and this theorem is:

**Theorem 1.2** (The fixed point alternative). Suppose we are given a complete generalized metric space \((Ω,d)\) and a strictly contractive mapping \(T: Ω → Ω\) with the Lipschitz constant \(a\). Then, for each given element \(x ∈ Ω\), either

\[
d(T^n x, T^{n+1} x) = \infty, \; \forall n ≥ 0,
\]

or there exists a natural number \(n_0\) such that:

(i) \(d(T^n x, T^{n+1} x) < \infty\) for all \(n ≥ n_0\).

(ii) The sequence \((T^n x)_{n≥0}\) is convergent to a fixed point \(y^*\) of \(T\).

(iii) \(y^*\) is the unique fixed point of \(T\) in the set \(Δ = \{y ∈ Ω | d(T^{n_0} x, y) < \infty\}\).

(iv) \(d(y, y^*) ≤ \frac{1}{1-a}d(y, Ty)\) for all \(y ∈ Δ\).

Let \(a_0, a_1, \ldots, a_{n-1}\) be real numbers and let \(I\) be an interval. For \(y ∈ C^n(I, \mathbb{R})\), \(ε > 0\) and \(φ ∈ C(I, \mathbb{R}_+)\) we consider the following equation:

\[
y^{(n)}(t) = \sum_{k=0}^{n-1} a_k y^{(k)}(t), \quad t ∈ I
\]

and the following inequations

\[
\left| y^{(n)}(t) - \sum_{k=0}^{n-1} a_k y^{(k)}(t) \right| ≤ ε, \quad t ∈ I
\]

and

\[
\left| y^{(n)}(t) - \sum_{k=0}^{n-1} a_k y^{(k)}(t) \right| ≤ φ(t), \quad t ∈ I.
\]

**Definition 1.3.** The equation (1.3) is Hyers-Ulam stable if there exists a real number \(c > 0\) such that for each \(ε > 0\) and for each solution \(s ∈ C^{(n)}(I, \mathbb{R})\) of (1.4) there exists a solution \(y ∈ C^{(n)}(I, \mathbb{R})\) of (1.3) with

\[
|s(t) - y(t)| ≤ c ⋅ ε, \; \forall t ∈ I.
\]

**Definition 1.4.** The equation (1.3) is Hyers-Ulam-Rassias stable, with respect to \(φ\), if there exists a real number \(c_φ > 0\) such that for each solution \(s ∈ C^{(n)}(I, \mathbb{R})\) of (1.5) there exists a solution \(y ∈ C^{(n)}(I, \mathbb{R})\) of (1.3) with

\[
|s(t) - y(t)| ≤ c_φ ⋅ φ(t), \; \forall t ∈ I.
\]
Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of differential equations. In 1998, they proved in [1] the stability of differential equation

$$y'(t) = y(t).$$ (1.6)

Following the same approach as in [1], Miura [8] proved the Hyers-Ulam stability of differential equation

$$y'(t) = \lambda y(t).$$ (1.7)

S.M. Jung [4–7] applied the fixed point method for proving the Hyers-Ulam-Rassias stability of a Volterra integral equation of the second kind and for differential equations of first order. Using the same technique we prove the Hyers-Ulam-Rassias stability and Hyers-Ulam stability of differential equation

$$y'(x) + f(x)y(x) + g(x) = 0$$ (1.8)

under some conditions, others than the conditions from [4].

2. MAIN RESULTS

In this paper, by using the idea of Cădariu and Radu [2], we will prove the Hyers-Ulam-Rassias stability for the equation (1.8) on the intervals $$I = [a, b)$$, where $$-\infty < a < b \leq \infty$$.

**Theorem 2.1.** Let $$f, g : I \rightarrow \mathbb{R}$$ be continuous functions and let for a positive constant $$M$$, $$|f(x)| \geq M$$ for all $$x \in I$$. Assume that $$\psi : I \rightarrow [0, \infty)$$ is an integrable function with the property that there exists $$P \in (0, 1)$$ such that

$$\int_a^x |f(t)|\psi(t)dt \leq P\psi(x)$$ (2.1)

for all $$x \in I$$. If a continuously differentiable function $$y : I \rightarrow \mathbb{R}$$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x)$$ (2.2)

for all $$x \in I$$, then there exists a unique solution $$S : I \rightarrow \mathbb{R}$$ of the equation (1.8) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x)$$ (2.3)

for all $$x \in I$$ and $$S(a) = y(a)$$.

**Proof.** Let us consider the set $$\Omega = \{h : I \rightarrow \mathbb{R} \mid h \text{ is continuous and } h(a) = y(a)\}$$ and the generalized metric $$d(h_1, h_2)$$ defined on $$\Omega$$ as

$$d(h_1, h_2) = d_\psi(h_1, h_2) = \inf\{k > 0 \mid |h_1(x) - h_2(x)| \leq k\psi(x), \forall x \in I\}.$$
Then \((\Omega, d)\) is a generalized complete metric space (see [4]). We define the operator 
\[ T : \Omega \to \Omega, \]
\[ Th(x) = y(a) - \int_a^x (f(t)h(t) + g(t)) \, dt \quad x \in I, \]
for all \(h \in \Omega\). Indeed \(Th\) is a continuously differentiable function on \(I\), since \(f\) and \(g\) are continuous function and \(Th(a) = y(a)\).

Now, let \(h_1, h_2 \in \Omega\). Then we have
\[ |Th_1(x) - Th_2(x)| = \left| \int_a^x f(t)(h_1(t) - h_2(t)) \, dt \right| \leq \int_a^x |f(t)||h_1(t) - h_2(t)| \, dt \leq d(h_1, h_2) \int_a^x |f(t)|\psi(t) \, dt \leq P\psi(x)d(h_1, h_2) \]
for all \(x \in I\). Therefore,
\[ d(Th_1, Th_2) \leq Pd(h_1, h_2), \quad (2.4) \]
thus the operator \(T\) is a contraction with the constant \(P\).

Now integrating the both sides of the relation (2.2) on \([a, x]\) we obtain
\[ \left| y(x) - y(a) + \int_a^x (f(t)y(t) + g(t)) \, dt \right| \leq \frac{P}{M}\psi(x) \quad (2.5) \]
for all \(x \in I\), which means \(d(y, Ty) \leq \frac{P}{M} < \infty\). By the fixed point alternative there exists an element \(S = \lim_{n \to \infty} T^n y\) and \(S\) is unique fixed point of \(T\) in the set \(\Delta = \{h \in \Omega | d(T^{n_0}y, h) < \infty \}\). It may be proved that
\[ \Delta = \{h \in \Omega | d(y, h) < \infty \}. \]
Therefore the set \(\Delta\) is independent of \(n_0\). To prove that the function \(S\) is a solution to the equation (1.8), we derive with respect to \(x\) the both sides of the relation
\[ S(x) = TS(x), \quad x \in I. \quad (2.6) \]
Thus
\[ S'(x) = -f(x)S(x) - g(x) \quad (2.7) \]
for all \(x \in I\) which implies that the function \(S\) is a solution to the equation (1.8) and verifies the relation \(S(a) = y(a)\).

Applying again the fixed point alternative we obtain
\[ d(h, S) \leq \frac{1}{1-P}d(h, Th) \quad \text{for all} \quad h \in \Delta. \]
Since \( y \in \Delta \), we have
\[
d(y, S) \leq \frac{1}{1 - P} d(y, Ty) \leq \frac{P}{M(1 - P)},
\]
whence
\[
|y(x) - S(x)| \leq \frac{P}{M - MP} \psi(x)
\]
for all \( x \in I \). This inequality proves the relation (2.3).

In the same manner it can be proved the following theorem of the Hyers-Ulam-Rassias stability of the equation (1.8) on the interval \( J = (b, a) \), where \(-\infty \leq b < a < \infty\).

**Theorem 2.2.** Let \( f, g : J \rightarrow \mathbb{R} \) be continuous functions and let for some positive constant \( M \), \( |f(x)| \geq M \) for all \( x \in J \). Assume that \( \psi : J \rightarrow [0, \infty) \) is an integrable function with the property that there exists \( P \in (0, 1) \) such that
\[
\left| \int_x^a |f(t)| \psi(t) dt \right| \leq P \psi(x) \tag{2.8}
\]
for all \( x \in J \). If a continuously differentiable function \( y : J \rightarrow \mathbb{R} \) verifies the relation:
\[
|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.9}
\]
for all \( x \in J \), then there exists a unique solution \( S : J \rightarrow \mathbb{R} \) of the equation (1.8) which verifies the following relations:
\[
|y(x) - S(x)| \leq \frac{P}{M - MP} \psi(x) \tag{2.10}
\]
for all \( x \in J \) and \( S(a) = y(a) \).

The Hyers-Ulam-Rassias stability equation (1.8) on \( \mathbb{R} \) will be proved by Theorem 2.1 and Theorem 2.2.

**Corollary 2.3.** Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be continuous functions and let for some positive constant \( M \), \( |f(x)| \geq M \) for all \( x \in \mathbb{R} \). Assume that \( \psi : \mathbb{R} \rightarrow [0, \infty) \) is an integrable function with the property that there exists \( P \in (0, 1) \) such that
\[
\left| \int_0^x |f(t)| \psi(t) dt \right| \leq P \psi(x) \tag{2.11}
\]
for all \( x \in \mathbb{R} \). If a continuously differentiable function \( y : \mathbb{R} \rightarrow \mathbb{R} \) verifies the relation:
\[
|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.12}
\]
for all \( x \in \mathbb{R} \), then there exists a unique solution \( S : \mathbb{R} \to \mathbb{R} \) of equation (1.8) which verifies the following relations:

\[
|y(x) - S(x)| \leq \frac{P}{M - MP} \psi(x)
\]  

(2.13)

for all \( x \in \mathbb{R} \) and \( S(0) = y(0) \).

Proof. By the relation (2.11) we have

\[
\int_{0}^{x} |f(t)| \psi(t) dt \leq P \psi(x)
\]  

(2.14)

for all \( x \geq 0 \). Applying Theorem 2.1, there exists a solution of equation (1.8), \( S_{1} : [0, \infty) \to \mathbb{R} \) which verifies the relations (2.3) and \( S_{1}(0) = y(0) \).

From (2.11) we also obtain

\[
\int_{x}^{0} |f(t)| \psi(t) dt \leq P \psi(x)
\]  

(2.15)

for all \( x \leq 0 \). Applying Theorem 2.2, there exists a solution of equation (1.8), \( S_{2} : (-\infty, 0] \to \mathbb{R} \) which verifies (2.10) and \( S_{2}(0) = y(0) \). It is easy to check that the function

\[
S(x) = \begin{cases} 
S_{1}(x), & x \geq 0, \\
S_{2}(x), & x < 0,
\end{cases}
\]  

(2.16)

is a continuously differentiable function on \( \mathbb{R} \), a solution of equation (1.8) on \( \mathbb{R} \) and it verifies relation (2.13).

Using Theorem 2.1 it can be shown the Hyers-Ulam stability for the equation (1.8) on \( I = [a, b) \), where \( -\infty < a < b \leq \infty \).

Corollary 2.4. Let \( \varepsilon, M > 0 \) and let \( f : I \to [M, \infty) \) and \( g : I \to \mathbb{R} \) be continuous. If a continuously differentiable function \( y : I \to \mathbb{R} \) verifies the relation

\[
|y'(x) + f(x)y(x) + g(x)| \leq \varepsilon
\]  

(2.17)

for all \( x \in I \), then there exists a unique solution \( S : I \to \mathbb{R} \) of equation (1.8) which verifies the relations:

\[
|y(x) - S(x)| \leq \frac{\varepsilon}{M (2q - 1)}
\]  

(2.18)

for all \( x \in I \), where \( q \in \left( \frac{1}{2}, 1 \right) \) and \( S(a) = y(a) \).

Proof. Let \( q \in \left( \frac{1}{2}, 1 \right) \). Multiplying relation (2.17) by \( e^{q \int_{a}^{x} f(t) dt} \), and denoting

\[
z(x) := y(x)e^{q \int_{a}^{x} f(t) dt}, \quad x \in I
\]  

(2.19)
we have
\[
|z'(x) + (1 - q) f(x) z(x) + g(x) e^{q \int_a^x f(t) dt}| \leq \varepsilon \cdot e^{q \int_a^x f(t) dt}
\] (2.20)
for all \( x \in I \). Then the function \( F(x) = (1 - q) f(x) \), where \( x \in I \), is continuous on \( I \)
and satisfies the relation \( |F(x)| > (1 - q) M \) for all \( x \in I \).

Let \( \psi(x) = \varepsilon \cdot e^{q \int_a^x f(t) dt} \), when \( x \in I \). We see that
\[
\int_a^x |F(t)| \psi(t) dt = (1 - q) \varepsilon \int_a^x f(t) e^{q \int_a^x f(u) du} dt \leq \frac{1 - q}{q} \psi(x)
\] (2.21)
for all \( x \in I \), thus the function \( \psi : I \rightarrow [0, \infty) \) verifies relation (2.1) with
\( P = \frac{1 - q}{q} \in (0, 1) \).

By Theorem 2.1, there exists \( s \in C^1 (I, \mathbb{R}) \), which is a unique solution for the
equation
\[
z'(x) + (1 - q) f(x) z(x) + g(x) e^{q \int_a^x f(t) dt} = 0
\] (2.22)
and verifies the relations
\[
|z(x) - s(x)| \leq \frac{1}{M (2q - 1)} \cdot \varepsilon \cdot e^{q \int_a^x f(t) dt}
\] (2.23)
for all \( x \in I \) and \( s (a) = z (a) \).

Then the function \( S(x) = s(x) e^{-q \int_a^x f(t) dt} \) is a solution of equation (1.8) and
verifies relation (2.18).

Equation (1.8) is not Hyers-Ulam stable on the intervals \( J = (-\infty, a] \) in general,
as we can see in the following example.

**Example 2.5.** Let us consider equation (1.8) where \( f(x) = x^2 \) and \( g(x) = 0 \). The
solution of this equation \( S : J \rightarrow \mathbb{R} \) which verifies the condition \( S (a) = p \) is
\[
S(x) = p \cdot e^{\frac{x^3 - a^3}{3}}.
\] (2.24)
A continuously differentiable function \( y : J \rightarrow \mathbb{R} \) which verifies inequality (2.17) is
\[
y(x) = p \cdot e^{\frac{x^3 - a^3}{3}} + \varepsilon \cdot e^{\frac{x^3}{3}} \int_a^x e^{\frac{u^3}{3}} du.
\] (2.25)
Considering equation (1.8) being Hyers-Ulam stable, there exists \( k > 0 \) such that
\[
|y(x) - S(x)| \leq \varepsilon \cdot k
\] (2.26)
for all \( x \in J \). By substitution, we have
\[
\int_a^x e^{\frac{u^3}{3}} du \leq k e^{\frac{x^3}{3}}
\] (2.27)
for all \( x \in J \). Now letting \( x \rightarrow -\infty \) it generates a contradiction. So equation (1.8) is
not Hyers-Ulam stable.
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