On the Impossibility of Dipole Modulation in E and B Mode Polarization Fields of CMB

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Abstract

Cosmic Microwave Background Radiation is characterized by $T$, $Q$ and $U$ fields. A dipole modulation in these fields has been studied in different contexts. $E$ and $B$ are derived from $Q$ and $U$ fields with the help of the so called 'eth' operator. In this short write-up, I do a systematic analysis to demonstrate that a dipole modulation in $E$ mode polarization can’t be introduced. Although, the analysis has been done for the $E$ mode, a similar exercise can be repeated for $B$ mode as well. It has been explicitly demonstrated that the introduction of a dipole modulation leads to a contradiction and hence such a modulation isn’t allowed.

1 Introduction

The Cosmic Microwave Background Radiation (from now on to be referred as CMB) is characterized by $T$, $Q$ and $U$ fields. $T$ field is invariant under a rotation in the tangent plane perpendicular to the given direction of observation $\hat{n}$ [1], so that it can be expanded in terms of the ‘usual’ spherical harmonics (i.e., spin zero spherical harmonics) as

$$T(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} T_{lm} Y_{lm}. \tag{1.1}$$

It is to be noted that in general $T_{lm} \neq 0$ for a given $l$. Using orthogonality properties of spherical harmonics Equation (1.1) can be inverted to obtain

$$T_{lm} = \int T(\hat{n}) Y_{lm}^{*} d\Omega. \tag{1.2}$$

The temperature field satisfies the conditions of statistical isotropy until a potential violation in it was reported in 2003 and was termed as hemispherical power asymmetry [2]. A dipole modulation of the following form

$$\bar{T}(\hat{n}) = T(\hat{n}) \left[ 1 + \hat{\lambda} \cdot \hat{n} \right], \tag{1.2}$$

has been studied in this context [3–6]. It turns out that this type of modulation leads to a correlation between $l$ and $l \pm 1$ multipole coefficients [7].

The fields $Q$ and $U$ on the other hand are not spin 0. The combination $Q \pm iU$ behaves in a specific manner upon rotation in the tangent plane, due to which it turns out to be spin $\pm 2$. Thus they can be expanded in terms of spin $\pm 2$ spherical harmonics as (here $sY_{lm} \equiv Y_{s,lm}$):

$$Q \pm iU = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{\pm2,lm} Y_{\pm2,lm}. \tag{1.3}$$

Using the orthogonality properties of the spin $s$ spherical harmonics, Equation (1.3) can be inverted to obtain

$$a_{\pm2,lm} = \int (Q \pm iU) Y_{\pm2,lm}^{*} d\Omega. \tag{1.4}$$

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Although Equation (1.3) looks similar to Equation (1.1), it turns out that the sum over $l$ in Equation (1.3) must start from $l = 2$.

Spin 0 fields are easier to work with as compared to spin ±2 fields. By applying the $\bar{\partial}$ operator [8] appropriately, we can obtain a scalar (i.e., a spin 0 field) from $Q$ and $U$ fields [1]:

$$E = \frac{1}{2} [\bar{\partial}^2 (Q - iU) + \bar{\partial}^2 (Q + iU)].$$  \hfill (1.5)

The field $E$ being a scalar (i.e. spin 0 field) will have the following harmonic decomposition similar to Eq. (1.1).

$$E = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} E_{lm} Y_{lm}. \hfill (1.6)$$

In spite of similarity between Equations (1.1) and (1.6), $E$ field is fundamentally different from $T$ field. On account of properties of spin weighted spherical harmonics $Y_{\pm 2lm}$, it turns out that $E_{lm} = 0$ when $l < 2$. In Section (2) this very fact is used to demonstrate the impossibility of dipole modulation in the $E$ field.

## 2 Mathematical Analysis

The basic strategy to prove the impossibility of dipole modulation in the polarization fields ($E$ and $B$) is as follows. First of all on account of the properties of the spin ±2 spherical harmonics [12, 13] it follows that $Y_{\pm 2lm} = 0$ when $l < 2$, thus the sum must start from $l = 2$. Using this fact, Theorem (1) concludes $E_{lm} = 0$ when $l < 2$ and finally in Corollary (1) which is also the main result of this write up, shows the impossibility of dipole modulation in $E$ mode polarization. First of all I demonstrate that $E_{lm} = 0$ when $l < 2$.

**Theorem 1.** Given that $E$ field as per Equation (1.5) can be obtained by an application of $\bar{\partial}$ operator on the $Q$ and $U$ fields, the spherical harmonic coefficients $E_{lm} = 0$ when $l < 2$.

**Proof.** First of all, apply the $\bar{\partial}$ operator on both sides of Equation (1.3) and knowing the operation of this operator on spin spherical harmonics [9], we obtain:

$$\bar{\partial}^2 (Q + iU) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \sqrt{\frac{(l+2)!}{(l-2)!}} a_{2lm} Y_{lm}, \right], \hfill (2.1)$$

$$\bar{\partial}^2 (Q - iU) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \sqrt{\frac{(l+2)!}{(l-2)!}} a_{-2lm} Y_{lm}. \right]. \hfill (2.2)$$

Here I can start the sum from $l = 0$ since the harmonic coefficients $a_{\pm 2lm} = 0$ anyway. Using Equations (2.1) and (2.2) in (1.5), we get

$$E = \frac{1}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \sqrt{\frac{(l+2)!}{(l-2)!}} (a_{2lm} + a_{-2lm}) Y_{lm}. \right]. \hfill (2.3)$$

Furthermore using, Equation (1.6) and orthogonality property of the spherical harmonics we finally get

$$E_{lm} = \frac{1}{2} \sqrt{\frac{(l+2)!}{(l-2)!}} (a_{2lm} + a_{-2lm}).$$

Now $a_{\pm 2lm} = 0$ for $l = 0$ and 1, therefore $E_{lm} = 0$ when $l = 0$ and 1, i.e., when $l < 2$. \hfill \Box

Thus since $E_{lm} = 0$ when $l < 2$, we can start the sum from $l = 2$ in Equation (1.6). Due to this very deduction, the $E$ field is fundamentally different from $T$ field, although both are scalars. Finally we reach to the main result of this article.

**Corollary 1.** Given that $E$ is a scalar field under a rotation in the tangent plane, a dipole modulation of the following form

$$\hat{E} = E \left( 1 + A \hat{\lambda} \cdot \hat{n} \right), \hfill (2.4)$$

in it is not possible. Here $\hat{\lambda}$ is given fixed direction and $A$ is magnitude of modulation.
Proof. Here we proceed with proof by contradiction. I’ll demonstrate that if we assume the modulated field \( \bar{E} \) as scalar and perform its harmonic decomposition as per Equation (1.6), then its harmonic coefficients \( \bar{E}_{lm} \) aren’t zero when \( l = 1 \) which thus contradicts Theorem (1). The most general direction for the two vectors would be

\[
\hat{\lambda} = (\cos \Phi \sin \Theta, \sin \Phi \sin \Theta, \cos \Theta), \quad \hat{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)
\]

Let us assume that \( \bar{E} \) is a scalar field, therefore its harmonic decomposition can be performed as

\[
\bar{E} = \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \bar{E}_{l'm'} Y_{l'm'}.
\]

Here I am not apriori assuming that \( \bar{E}_{lm} = 0 \) when \( l < 2 \). Now using Equations (2.5) and (1.6) in (2.4), we obtain

\[
\sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \bar{E}_{l'm'} Y_{l'm'} = [1 + A (\cos \Phi \sin \Theta \cos \phi \sin \theta + \sin \Phi \sin \Theta \sin \phi \sin \theta + \cos \Theta \cos \theta)] \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} E_{l'm'} Y_{l'm'}.
\]

We can use orthogonality property of spherical harmonics and properties of associated Legendre Polynomials [10], whence it can be shown that

\[
\bar{E}_{lm} = E_{lm} + \frac{A \sin \Theta}{2} e^{-i \Phi} [E_{l+1,m-1} g(l+1,-m+1,-m+1) - E_{l-1,m-1} g(l,m,m)]
\]

\[+ A \cos \Theta [g(l+1,-m+1,m) E_{l+1,m} + g(l+1,m,-m-1) E_{l-1,m}] + \]

\[
\frac{A \sin \Theta}{2} e^{i \Phi} [E_{l-1,m+1} g(l,-m,-m) - E_{l+1,m+1} g(l+1,m+1,m+1)],
\]

here the function \( g(l,m,n) \) is

\[
g(l,m,n) = \sqrt{\frac{(l + m - 1)(l + n)}{(2l + 1)(2l - 1)}},
\]

Now \( \bar{E}_{lm} \) for \( l = 1 \) and \( m = 0 \) is

\[
\bar{E}_{1,0} = A \sqrt{\frac{2}{5}} \left[ \cos \Theta \sqrt{\frac{2}{3}} E_{2,0} - \sin \Theta \Re(e^{i \Phi} E_{2,1}) \right] \neq 0,
\]

where \( \Re(z) \) denotes the real part of \( z \in \mathbb{C} \). This contradicts Theorem (1) and hence dipole modulation of the type given in Equation (2.4) isn’t possible.

3 Conclusion

This article systematically demonstrates that the dipole modulation in the polarization fields \( E \) and \( B \) isn’t possible. It must be pointed out that the result doesn’t mean that a difference in \( E \) mode power can’t/won’t be found, what this means is that the power difference can’t be accounted to a dipole modulation in \( E \) field. This is different from the temperature case where infact the power difference in the \( T \) mode was phenomenologically studied using such a modulation.

But no such problem arises when one tries to modulate \( Q \) and \( U \) fields. Interestingly, it too leads to a correlation between \( l \) and \( l \pm 1 \) multipoles [11], just like the temperature case.

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