Nonlocal Superposed Solutions II: 
Coupled Nonlinear Equations 

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Abstract: 
We obtain novel periodic as well as hyperbolic solutions of a n Ablowitz-Musslimani variant of the coupled nonlocal, nonlinear Schrödinger equation (NLS) as well as a coupled nonlocal modified Korteweg-de Vries (mKdV) equation which can be re-expressed as a linear superposition of the sum or the difference of two hyperbolic or two periodic kink or pulse solutions. Besides, we also discuss some of the other solutions admitted by these coupled equations. These results demonstrate that the notion of the superposed solutions extends to the coupled nonlocal nonlinear equations as well. 

1 Introduction 
Nonlinear theories are highly nontrivial because unlike the linear theories, the nonlinear theories do not satisfy the superposition principle. However, by now several nonlinear equations such as $\phi^4$, NLS, mKdV, KdV etc. have been shown to satisfy a kind of superposition principle [1,2]. In particular, we showed that if a nonlinear equation admits a periodic pulse solution in terms of the Jacobi elliptic functions $\text{dn}(x,m)$ and $\text{cn}(x,m)$, then the same equation will also admit a superposed periodic solution $\text{dn}(x,m) \pm \sqrt{m}\text{cn}(x,m)$ where $m$ is the modulus of the Jacobi elliptic function. Similarly, if a nonlinear equation admits a solution in terms of $\text{dn}^2(x,m)$ then it will also admit a solution in terms of $\text{dn}^2(x,m) \pm \sqrt{m}\text{dn}(x,m)\text{cn}(x,m)$. Further, in another publication [4] we showed that if a nonlinear equation
satisfies solutions in terms of \( \text{dn}(x, m) \) or \( \text{cn}(x, m) \), then the same nonlinear equation will also admit a complex PT-invariant solution in terms of \( \text{dn}(x, m) \pm i \sqrt{m} \text{sn}(x, m) \) or \( \text{cn}(x, m) \pm i \text{sn}(x, m) \). Similarly, if a nonlinear equation admits a periodic kink solution in terms of \( \text{sn}(x, m) \), then the same nonlinear equation also admits a complex PT-invariant solution in terms of \( \sqrt{m} \text{sn}(x, m) \pm i \text{dn}(x, m) \) as well as \( \text{sn}(x, m) \pm i \text{cn}(x, m) \). We further showed that if a nonlinear equation admits a solution in terms of \( \text{dn}^2(x, m) \) then the same nonlinear equation will also admit complex PT-invariant periodic solutions in terms of \( \sqrt{m} \text{dn}(x, m) \pm i \text{sn}(x, m) \) or \( \text{dn}^2(x, m) \pm i m \text{sn}(x, m) \text{cn}(x, m) \).

Recently, inspired by the work of [5, 6] as well as earlier work in condensed matter as well as field theory [7, 8, 9, 10], we have further extended the notion of superposition for several nonlinear equations. In particular, we showed that the symmetric and the asymmetric \( \phi^4 \) model, the NLS, quadratic-cubic NLS as well as mKdV and mKdV-KdV equations admit novel solutions which can be re-expressed as the superposition of the periodic kink and the periodic pulse solutions [11]. Further, in some cases we also obtained solutions which can be re-expressed as superposition of two hyperbolic kink solutions. In another publication, we further extended the notion of the superposition principle for the coupled nonlinear equations. In particular, we showed that the coupled \( \phi^4 \), coupled NLS and coupled mKdV equations admit novel solutions which can be re-expressed as the superposition of two (hyperbolic) kink and two (hyperbolic) pulse solutions [12].

At this stage we recall that during the last few years the nonlocal, nonlinear equations have received wide attention. The next obvious question is whether this notion can also be extended to nonlocal nonlinear equations. As a first step in that direction we recently showed [13] that two different nonlocal versions of the NLS [14, 15], nonlocal mKdV [17] as well as the nonlocal Hirota equation [18, 19] admit novel solutions which can be re-expressed as the superposition of two kink or two pulse solutions. It is then natural to enquire if the notion of superposition can also be extended to the coupled nonlocal equations. The purpose of this paper is to partially answer this question. In particular, we show that the Ablowitz-Musslimani variant [14, 16] of the coupled nonlocal NLS equations [2] as well as the coupled nonlocal mKdV equations admit novel solutions which can be re-expressed as the superposition of two hyperbolic as well as two periodic kink and pulse solutions.

The plan of the paper is the following. In Sec. II, we consider a coupled Ablowitz-Musslimani variant of the nonlocal NLS and show that it admits
a large number of novel solutions which can be re-expressed as the super-
position of two kink or two pulse solutions. In particular, while in Sec. IIA
we discuss the superposed hyperbolic kink or pulse solutions, in Sec. IIB we
discuss superposed periodic kink or the pulse solutions. In Sec. III we dis-
cuss a coupled nonlocal mKdV model and show that this model also admits
novel solutions which can be re-expressed as superposition of either two kink
or two pulse solutions. Finally, in Sec. IV we summarize our main results
and point out some of the open problems.

In Appendix A we discuss those solutions of the coupled Ablowitz-
Musslimani variant of the nonlocal NLS equation where the two fields are
proportional to each other. In Appendix B we discuss those solutions of the
coupled nonlocal mKdV model where the two fields are proportional to each
other.

2 A Coupled Ablowitz-Musslimani Variant of Non-
local NLS Model

Let us consider the following coupled Ablowitz-Musslimani variant of the
nonlocal NLS which we had introduced in [2]

\[ iu_{x,t} + u_{xx}(x,t) + [g_{11}u(x,t)u^*(-x,t) + g_{12}v(x,t)v^*(-x,t)]u(x,t) = 0, \]  
\[ iv_{x,t} + v_{xx}(x,t) + [g_{21}u(x,t)u^*(-x,t) + g_{22}v(x,t)v^*(-x,t)]v(x,t) = 0. \]  

Note that for comparison with the standard version of the Manakov system
[20], we have changed the notation used in [2]. In particular, we have re-
placed \( z \) with \( t \) and also instead of the couplings \( a, b, f, e \) as used in [2]
we have used the couplings \( g_{11}, g_{12}, g_{21} \) and \( g_{22} \), respectively. Notice that the
limit \( g_{11} = g_{12} = g_{21} = g_{22} \) in the local coupled NLS case is the Manakov
system while \( g_{11} = g_{21} = -g_{12} = -g_{22} \) in the local case is the integrable
MZS system [21, 22, 23].

In [2] we had obtained 34 solutions of the coupled equations, most of
which were periodic in terms of the Jacobi elliptic functions and a few in
terms of hyperbolic solutions. However, it turns out that we had missed sev-
eral interesting solutions of the coupled equations. The purpose of this paper
is to discuss these solutions and to show that they can be re-expressed as the
superposition of a kink and an antikink or two kink or two pulse solutions.
For this purpose we will make use of six identities for the Jacobi elliptic func-
tions which can be easily derived by using the well known addition theorems
for the Jacobi elliptic functions $\text{sn}(x, m), \text{cn}(x, m)$ and $\text{dn}(x, m)$ \[3\]

$$\text{sn}(a + b, m) = \frac{\text{sn}(a, m)\text{cn}(b, m)\text{dn}(b, m) + \text{sn}(b, m)\text{cn}(a, m)\text{dn}(a, m)}{1 - m\text{sn}^2(a, m)\text{sn}^2(b, m)}, \quad (3)$$

$$\text{cn}(a + b, m) = \frac{\text{cn}(a, m)\text{cn}(b, m) - \text{sn}(a, m)\text{dn}(a, m)\text{sn}(b, m)\text{dn}(b, m)}{1 - m\text{sn}^2(a, m)\text{sn}^2(b, m)}, \quad (4)$$

$$\text{dn}(a + b, m) = \frac{\text{dn}(a, m)\text{dn}(b, m) - m\text{sn}(a, m)\text{cn}(a, m)\text{sn}(b, m)\text{cn}(b, m)}{1 - m\text{sn}^2(a, m)\text{sn}^2(b, m)}. \quad (5)$$

In particular, on using Eq. (3) we obtain the identities for the sum of two periodic kink solutions, i.e.

$$\text{sn}(y + \Delta, m) + \text{sn}(y - \Delta, m) = \frac{2\text{sn}(y, m)\text{cn}(\Delta, m)}{\text{dn}(\Delta, m)[1 + B\text{cn}^2(y, m)]}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}, \quad (6)$$

or the sum of a periodic kink and an antikink solution, i.e.

$$\text{sn}(y + \Delta, m) - \text{sn}(y - \Delta, m) = \frac{2\text{cn}(y, m)\text{dn}(y, m)\text{sn}(\Delta, m)}{\text{dn}(\Delta, m)[1 + B\text{cn}^2(y, m)]}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}. \quad (7)$$

On the other hand, by using Eq. (1) we obtain the identities for the sum of the two periodic pulse solutions, i.e.

$$\text{cn}(y + \Delta, m) + \text{cn}(y - \Delta, m) = \frac{2\text{cn}(y, m)\text{cn}(\Delta, m)}{\text{dn}(\Delta, m)[1 + B\text{cn}^2(y, m)]}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}, \quad (8)$$

$$\text{cn}(y - \Delta, m) - \text{cn}(y + \Delta, m) = \frac{2\text{sn}(y, m)\text{dn}(y, m)\text{sn}(\Delta, m)}{\text{dn}(\Delta, m)[1 + B\text{cn}^2(y, m)]}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}. \quad (9)$$

Finally, on using the addition theorem (5) we obtain the identities for the sum of the two periodic pulse solutions, i.e.

$$\text{dn}(y + \Delta, m) + \text{dn}(y - \Delta, m) = \frac{2\text{dn}(y, m)}{\text{dn}(\Delta, m)[1 + B\text{cn}^2(y, m)]}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}, \quad (10)$$

$$\text{dn}(y - \Delta, m) - \text{dn}(y + \Delta, m) = \frac{2m\text{sn}(y, m)\text{cn}(y, m)\text{sn}(\Delta, m)\text{cn}(\Delta, m)}{\text{dn}^2(\Delta, m)[1 + B\text{cn}^2(y, m)]}, \quad B = \frac{m\text{sn}^2(\Delta, m)}{\text{dn}^2(\Delta, m)}. \quad (11)$$
In the limit $m = 1$, from the above identities we can deduce the corresponding four hyperbolic identities. In particular, in the limit $m = 1$, from Eq. (6) we obtain an identity for the sum of two (hyperbolic) kink solutions, i.e.

$$\tanh(y + \Delta) + \tanh(y - \Delta) = \frac{2 \sinh(y) \cosh(y)}{B + \cosh^2(y)}, \quad B = \sinh^2(\Delta), \quad (12)$$

while from Eq. (7) in the limit $m = 1$ we obtain an identity for the sum of a (hyperbolic) kink and an antikink solution, i.e.

$$\tanh(y + \Delta) - \tanh(y - \Delta) = \frac{\sinh(2\Delta)}{B + \cosh^2(y)}, \quad B = \sinh^2(\Delta). \quad (13)$$

On the other hand, from Eqs. (8) and (9) (or (10) and (11)), in the $m = 1$ limit, we obtain the identities for the sum of two (hyperbolic) pulse solutions

$$\text{sech}(y + \Delta) + \text{sech}(y - \Delta) = \frac{2 \cosh(y) \cosh(\Delta)}{B + \cosh^2(y)}, \quad B = \sinh^2(\Delta), \quad (14)$$

$$\text{sech}(y - \Delta) - \text{sech}(y + \Delta) = \frac{2 \sinh(y) \sinh(\Delta)}{B + \cosh^2(y)}, \quad B = \sinh^2(\Delta). \quad (15)$$

In the next two subsections, we obtain 7 hyperbolic and 19 periodic solutions, respectively, of the coupled Eqs. (1) and (2) which can be re-expressed as the superposition of either a kink and an antikink or two kink or two pulse solutions.

### 2.1 Superposed Hyperbolic kink and Pulse Solutions

There are two different kind of solutions to the coupled Eqs. (1) and (2) depending on whether $v(x,t) \propto u(x,t)$ or not. In this section we only discuss those solutions where the two fields $v(x,t)$ and $u(x,t)$ are distinct (and not proportional to each other) while in Appendix A we discuss those solutions where $v(x,t) \propto u(x,t)$. As we show there, when the two fields $u$ and $v$ are proportional to each other, effectively we only need to solve the uncoupled Ablowitz-Musslimani nonlocal NLS equation.

We start from Eqs. (1) and (2) and take the ansatz

$$u(x,t) = e^{i\omega_1(t+t_0)}u(x), \quad v(x,t) = e^{i\omega_2(t+t_0)}v(x). \quad (16)$$

On substituting this ansatz in Eqs. (1) and (2) we obtain

$$u_{xx}(x) = \omega_1 u(x) - [g_{11}u(x)u^*(-x) + g_{12}v(x)v^*(-x)]u(x), \quad (17)$$
\[ v_{xx}(x) = \omega_2 v(x) - [g_{21} u(x) u^*(-x) + g_{22} v(x) v^*(-x)] v(x). \]  \hspace{1cm} (18)

We now show that the coupled Eqs. (17) and (18) and hence Eqs. (1) and (2) admit 7 novel hyperbolic solutions which can be re-expressed as a superposition of either two kink or two pulse solutions.

One major difference between the local and the nonlocal case is that, unlike the local case, the solutions of the nonlocal NLS Eqs. (1) and (2) are not invariant with respect to the shift in \( x \). However, a shift in \( t \) is allowed. For simplicity, from now onward we will take \( t_0 \) to be zero even though such a shift is always allowed.

**Superposed Solution I**

It is easy to check that the coupled Eqs. (17) and (18) (and hence Eqs. (1) and (2)) admit the hyperbolic solution

\[
\begin{align*}
  u(x,t) &= e^{i\omega_1 t} \frac{A}{B + \cosh^2(\beta x)}, \\
v(x,t) &= e^{i\omega_2 t} \frac{D \cosh(\beta x) \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0,
\end{align*}
\]

provided

\[
\begin{align*}
  \omega_1 &= -2\beta^2, \quad g_{11} A^2 = -2B(2B + 1)\beta^2, \quad g_{12} D^2 = 6\beta^2, \\
  \omega_2 &= -2\beta^2, \quad g_{21} A^2 = -6B(B + 1)\beta^2, \quad g_{22} D^2 = 2\beta^2.
\end{align*}
\]

Thus for this solution \( g_{11}, g_{21} \) are negative while \( \omega_1 = \omega_2, g_{12}, g_{22} \) are positive.

On comparing the solution (19) with the identities (12) and (13), the solution (19) can be re-expressed as

\[
\begin{align*}
  u(x,t) &= e^{i\omega_1 t} \frac{\beta}{\sqrt{2|g_{11}|}} [\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)], \\
v(x,t) &= e^{i\omega_2 t} \frac{\sqrt{3}\beta}{\sqrt{2g_{12}}} [\tanh(\beta x + \Delta) + \tanh(\beta x - \Delta)],
\end{align*}
\]

where \( \sinh^2(\Delta) = B \). Note that here while \( u(x,t) \) corresponds to a superposition of a kink and an antikink solution, \( v(x,t) \) is a superposition of two kink solutions.

**Superposed Solution II**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[
\begin{align*}
  u(x,t) &= e^{i\omega_1 t} \frac{A}{B + \cosh^2(\beta x)}, \\
v(x,t) &= e^{i\omega_2 t} \frac{D \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0,
\end{align*}
\]

\hspace{1cm} (22)
provided
\[ \omega_1 = 4\beta^2, \quad g_{11}A^2 = 2(B + 1)(2B + 3)\beta^2, \quad g_{12}D^2 = -6(2B + 1)\beta^2, \]
\[ \omega_2 = \beta^2, \quad g_{21}A^2 = 6(B + 1)\beta^2, \quad g_{22}D^2 = -2(3 + 4B)\beta^2. \]  
(23)

Thus for this solution \( g_{11}, g_{21}, \omega_1, \omega_2 \) are all positive while \( g_{12}, g_{22} \) are negative.

On comparing the solution (22) with the identities (13) and (15), the solution (22) can be re-expressed as
\[
u(x, t) = e^{i\omega_1 t} \frac{\sqrt{3} \beta}{\sqrt{2} g_{21} \sinh(\Delta)} [\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)],
\]
\[
v(x, t) = e^{i\omega_2 t} \frac{\sqrt{3} \cosh(2\Delta) \beta}{\sqrt{2} |g_{12}| \sinh(\Delta)} [\text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta)], \]
(24)

where \( \sinh^2(\Delta) = B \). Note that here while \( u(x, t) \) corresponds to a superposition of a kink and an antikink solution, \( v(x, t) \) is a superposition of two pulse solutions.

**Superposed Solution III**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution
\[
u(x, t) = e^{i\omega_1 t} \frac{A}{B + \cosh^2(\beta x)}, \quad v(x, t) = e^{i\omega_2 t} \frac{D \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \]
(25)

provided
\[ \omega_1 = 4\beta^2, \quad g_{12}D^2 = 6(1 + 2B)\beta^2, \quad g_{11}A^2 = 2B(2B - 1)\beta^2, \]
\[ \omega_2 = \beta^2, \quad g_{21}A^2 = -6B\beta^2, \quad g_{22}D^2 = 2(1 + 4B)\beta^2. \]  
(26)

Thus for this solution while \( g_{12}, g_{22}, \omega_1, \omega_2 > 0, g_{21} < 0, g_{11} \geq 0 \) if either \( B \geq 1/2 \) or \( B \leq 0 \), and \( g_{11} < 0 \) if \( 0 < B < 1/2 \).

On comparing the solution (25) with the identities (13) and (14), the solution (25) can be re-expressed as
\[
u(x, t) = e^{i\omega_1 t} \frac{\sqrt{3} \beta}{\sqrt{2} |g_{21}| \cosh(\Delta)} [\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)],
\]
\[
v(x, t) = e^{i\omega_2 t} \frac{\sqrt{3} \tanh(\Delta) \beta}{\sqrt{2} |g_{12}|} [\text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta)], \]
(27)

where \( \sinh^2(\Delta) = B \). Note that here while \( u(x, t) \) corresponds to a superposition of a kink and an antikink solution, \( v(x, t) \) is a superposition of two pulse solutions.
Superposed Solution IV

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ u(x,t) = e^{i\omega_1 t} \frac{A \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D \sinh(\beta x) \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \]

provided

\[ (B + 1)\omega_1 = -(5 - B)\beta^2, \quad (1 + B)g_{11} A^2 = -2B(1 + 4B)\beta^2, \]
\[ (1 + B)g_{12} D^2 = 6\beta^2, \quad (1 + B)\omega_2 = 2(2B - 1)\beta^2, \]
\[ (B + 1)g_{21} A^2 = -6B(2B + 1)\beta^2, \quad (B + 1)g_{22} D^2 = -2(2B - 1)\beta^2. \]

Thus for this solution \( g_{11}, g_{21} < 0 \) while \( g_{12} > 0 \).

On making use of the identities (12) and (15), one can then re-express the solution (28) as

\[ u(x,t) = e^{i\omega_1 t} \frac{\beta}{\sqrt{2|g_{11}|}} \left[ \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \right], \]
\[ v(x,t) = e^{i\omega_2 t} \frac{\sqrt{3}\beta}{\sqrt{2|g_{12}|} \cosh(\Delta)} \left[ \tanh(\beta x + \Delta) + \tanh(\beta x - \Delta) \right], \]

where \( \sinh^2(\Delta) = B \). Note that here while \( u(x,t) \) corresponds to a superposition of two pulse solutions, \( v(x,t) \) is a superposition of two kink solutions.

Superposed Solution V

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ u(x,t) = e^{i\omega_1 t} \frac{A \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D \sinh(\beta x) \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \]

provided

\[ B\omega_1 = (6 + B)\beta^2, \quad Bg_{11} A^2 = 2(B + 1)(4B + 3)\beta^2, \]
\[ Bg_{12} D^2 = -6\beta^2, \quad B\omega_2 = 2(2B + 3)\beta^2, \]
\[ Bg_{21} A^2 = 6(B + 1)(2B + 1)\beta^2, \quad Bg_{22} D^2 = -2(3 + 2B)\beta^2. \]

Thus for this solution \( g_{11}, g_{21}, \omega_1, \omega_2 \) are all positive while \( g_{12}, g_{22} \) are negative.
On making use of the identities (12) and (14), one can then re-express the solution (97) as

\[ u(x,t) = e^{i\omega_1 t} \frac{\sqrt{3\cosh(2\Delta)}}{\sqrt{2|g_{11}| \sinh(\Delta)}} [\text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta)], \]

\[ v(x,t) = e^{i\omega_2 t} \frac{\sqrt{3\beta}}{\sqrt{2|g_{12}| \sinh(\Delta)}} [\tanh(\beta x + \Delta) + \tanh(\beta x - \Delta)], \]

where \( \sinh^2(\Delta) = B \). Note that here while \( u(x,t) \) corresponds to a superposition of two pulse solutions, \( v(x,t) \) is a superposition of two kink solutions.

**Superposed Solution VI**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ u(x,t) = e^{i\omega_1 t} \frac{A \sinh(\beta x)}{B + \cosh^2(\beta x)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D \cosh(\beta x)}{B + \cosh^2(\beta x)}, \quad B > 0, \]

provided

\[ \omega_1 = \beta^2, \quad g_{11} A^2 = -2B\beta^2, \quad g_{12} D^2 = 6(B + 1)\beta^2, \]

\[ \omega_2 = \beta^2, \quad g_{21} A^2 = -6B\beta^2, \quad g_{22} D^2 = 2(B + 1)\beta^2. \]

Thus for this solution \( g_{12}, g_{22}, \omega_1, \omega_2 \) are all positive while \( g_{11}, g_{21} \) are negative.

On making use of the identities (14) and (15), one can then re-express the solution (34) as

\[ u(x,t) = e^{i\omega_1 t} \frac{\beta}{\sqrt{2|g_{11}|}} [\text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta)], \]

\[ v(x,t) = e^{i\omega_2 t} \frac{\beta}{\sqrt{2|g_{12}|}} [\text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta)], \]

where \( \sinh^2(\Delta) = B \). Note that here both \( u(x,t), v(x,t) \) correspond to the superposition of two pulse solutions.

**Superposed Solution VII**

It is easy to check that the coupled Eqs. (1) and (2) admit the superposed hyperbolic solution

\[ u(x,t) = e^{i\omega_1 t} \left[1 - \frac{A}{B + \cosh^2(\beta x)}\right], \quad v(x,t) = e^{i\omega_2 t} \frac{D}{B + \cosh^2(\beta x)}, \]

(37)
where $A, B, D > 0$ provided

$$g_{11} = \omega_1 = -2\beta^2, \quad A = \frac{3(1 + 2B) + \sqrt{8 + (2B + 1)^2}}{4},$$

$$g_{12}D^2 = -\left[\sqrt{8 + (2B + 1)^2} - (2B + 1)\right] \frac{3A\beta^2}{2}, \quad \omega_2 - g_{21} = 4\beta^2,$$

$$g_{21}A = -3(2B + 1)\beta^2, \quad g_{21}A^2 + g_{22}D^2 = -8B(B + 1)\beta^2. \quad (38)$$

Thus for this solution $g_{11}, g_{12}, \omega_1, g_{21} < 0$, while $\omega_2, g_{22} > 0$.

On comparing the solution (37) with the identity (15), the solution (37) can be re-expressed as

$$u(x, t) = e^{i\omega_1 t} \left[1 - \frac{K_1}{\sinh(2\Delta)\sqrt{2|g_{11}|}}\right] \left[\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)\right],$$

$$v(x, t) = e^{i\omega_2 t} \frac{\sqrt{3K_1K_3}\beta}{2\sinh(2\Delta)\sqrt{2|g_{12}|}} \left[\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)\right], \quad (39)$$

where $\sinh^2(\Delta) = B$ and

$$K_1 = \left[3\cosh(2\Delta) + \sqrt{8 + \cosh^2(2\Delta)}\right], \quad (40)$$

$$K_3 = \left[\sqrt{8 + \cosh^2(2\Delta)} - \cosh(2\Delta)\right]. \quad (41)$$

Note that here both $u(x, t), v(x, t)$ correspond to the superposition of a kink and an antikink solution.

### 2.2 Superposed Periodic Kink and Pulse Solutions

We now show that in case $u$ and $v$ are distinct (i.e. not proportional to each other) then by starting from the ansatz (16) and hence using the coupled Eqs. (17) and (18), and thus the coupled Eqs. (1) and (2) admit not only the 7 superposed hyperbolic but also 15 superposed periodic kink (i.e. $\text{sn}(x, m)$) and periodic pulse (i.e. $\text{dn}(x, m)$ as well as $\text{cn}(x, m)$) solutions. We now present these 15 solutions one by one.

**Solution VIII**

It is easy to check that

$$u(x, t) = e^{i\omega_{1t}} \frac{A\csc(\beta x, m)}{1 + B\csc^2(\beta x, m)}, \quad v(x, t) = e^{i\omega_{2t}} \frac{D\csc(\beta x, m)\csc(\beta x, m)}{1 + B\csc^2(\beta x, m)}. \quad (42)$$
where $B > 0$, is an exact solution of the coupled Eqs. (11) and (2) provided
\[
\omega_1 = \omega_2 = (2m - 1)\beta^2, \quad g_{12}D^2 = 3g_{22}D^2 = -6B\beta^2, \\
g_{21}A^2 = 3g_{11}A^2 = 6(B + 1)(m - (1 - m)B)\beta^2. \quad (43)
\]

On using the identities (8) and (9), the coupled solution (42) can be re-expressed as
\[
u(x,t) = e^{i\omega_1 t} \sqrt{m/2g_{21}} \beta [cn(\beta x, m) + cn(\beta x + \Delta, m)],
\]
\[
u(x,t) = e^{i\omega_2 t} \sqrt{m/2|g_{12}|} \beta [cn(\beta x - \Delta, m) - cn(\beta x + \Delta, m)], \quad (44)
\]
where $B = msn^2(\Delta, m)/dn^2(\Delta, m)$.

**Solution IX**

It is easy to check that
\[
u(x,t) = e^{i\omega_1 t} \frac{A\text{dn}(\beta x, m)}{1 + Bcn^2(\beta x, m)}, \quad \nu(x,t) = e^{i\omega_2 t} \frac{D\text{sn}(\beta x, m)\text{cn}(\beta x, m)}{1 + Bcn^2(\beta x, m)}, \quad (45)
\]
where $B > 0$, is an exact solution of the coupled Eqs. (11) and (2) provided
\[
\omega_1 = \omega_2 = (2 - m)\beta^2, \quad g_{21}D^2 = 3g_{22}D^2 = -6B[m - (1 - m)B]\beta^2, \\
g_{21}A^2 = 3g_{11}A^2 = 6(1 + B)\beta^2. \quad (46)
\]

On using the identities (10) and (11), the coupled solution (45) can be re-expressed as
\[
u(x,t) = e^{i\omega_1 t} \frac{\beta}{\sqrt{2g_{11}}} [\text{dn}(\beta x + \Delta, m) + \text{dn}(\beta x - \Delta, m)],
\]
\[
u(x,t) = e^{i\omega_2 t} \frac{\beta}{\sqrt{2|g_{22}|}} [\text{dn}(\beta x - \Delta, m) - \text{dn}(\beta x + \Delta, m)], \quad (47)
\]
where $B = msn^2(\Delta, m)/dn^2(\Delta, m)$.

**Solution X**

It is straightforward to check that
\[
u(x,t) = e^{i\omega_1 t} \frac{A\text{cn}(\beta x, m)}{1 + Bc^2(\beta x, m)}, \quad \nu(x,t) = e^{i\omega_2 t} \frac{D\text{sn}(\beta x, m)\text{sn}(\beta x, m)}{1 + Bcn^2(\beta x, m)}, \quad (48)
\]
where $B > 0$, is an exact solution to the coupled Eqs. (11) and (2) provided

$$\omega_1 = [(2m - 1) - 6(1 - m)B]\beta^2, \quad g_{12}D^2 = -6[m + (1 - m)B^2]B\beta^2,$$
$$g_{11}A^2 = -2[3(1 - m)B^3 + 7(1 - m)B^2 + B(4 - 5m) - m]\beta^2,$$
$$\omega_2 = [(5m - 4) - 6(1 - m)B]\beta^2,$$
$$g_{22}D^2 = -2[3(1 - m)B^2 + 2(1 - m)B + m]B\beta^2,$$
$$b_{21}A^2 = -6[(1 - m)B^3 + 3(1 - m)B^2 + (2 - 3m)B - m]\beta^2. \quad (49)$$

On using the identities (8) and (11), the coupled solution (48) can be re-expressed as

$$u(x, t) = e^{i\omega_1 t} \frac{A\text{dn}^2(\Delta, m)}{2\text{cn}(\Delta, m)}[\text{cn}(\beta x + \Delta, m) + \text{cn}(\beta x - \Delta, m)], \quad (50)$$
$$v(x, t) = e^{i\omega_2 t} \frac{D\text{dn}(\Delta)}{2\text{msn}(\Delta)\text{cn}(\Delta)}[\text{dn}(\beta x - \Delta, m) - \text{dn}(\beta x + \Delta, m)], \quad (51)$$

where $A, D$ are as given by Eq. (49) while $B = \text{msn}^2(\Delta, m)/\text{dn}^2(\Delta, m)$.

**Solution XI**

It is easy to check that

$$u(x, t) = e^{i\omega_1 t} \frac{A\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \quad v(x, t) = e^{i\omega_2 t} \frac{D\text{sn}(\beta x, m)\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \quad (52)$$

where $B > 0$, is an exact solution of the coupled Eqs. (11) and (2) provided

$$[m - (1 - m)B]\omega_1 = [(1 - m)(4 + m)B + m(2 - m)]\beta^2,$$
$$[m - (1 - m)B]g_{11}A^2 = 2(B + 1)[3(1 - m)B^2 + 2(1 - m)B + m],$$
$$m[m - (1 - m)B]g_{12}D^2 = -6B[m - 2(1 - m)B - (1 - m)B^2]\beta^2,$$
$$[m - (1 - m)B]\omega_2 = [(1 - m)(4m + 1)B + m(5 - 4m)]\beta^2,$$
$$m[m - (1 - m)B]g_{22}D^2 = -B[2m^2 - 5m(1 - m)B - 3(1 - m^2)B^2]\beta^2,$$
$$[m - (1 - m)B]g_{21}A^2 = [6(1 - m)B^3 + 6(1 - m)B^2 + (m^2 + 4m + 1)B + 6m][38]$$

On using the identities (9) and (10), the coupled solution (52) can be re-expressed as

$$u(x, t) = e^{i\omega_1 t} \frac{A\text{dn}(\Delta, m)}{2}[\text{dn}(\beta x + \Delta, m) + \text{dn}(\beta x - \Delta, m)], \quad (54)$$
$$v(x, t) = e^{i\omega_2 t} \frac{D\text{dn}(\Delta, m)}{2\text{sn}(\Delta, m)}[\text{cn}(\beta x - \Delta, m) - \text{cn}(\beta x + \Delta, m)], \quad (55)$$
where \(A, D\) are given by Eq. (53) while \(B = m \text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m)\).

**Hyperbolic Limit**

In the limit \(m = 1\), all the four solutions VIII to XI go over to the hyperbolic solution VI, i.e. in this limit all four solutions can be re-expressed as superposition of two pulse solutions, i.e. \(\text{sech}(\beta x - \Delta) \pm \text{sech}(\beta x + \Delta)\).

**Solution XII**

It is straightforward to check that

\[
\begin{align*}
  u(x, t) &= e^{i\omega_1 t} \frac{A \text{cn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \\
  v(x, t) &= e^{i\omega_2 t} \frac{D \text{sn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad (56)
\end{align*}
\]

where \(B > 0\), is an exact solution of the coupled Eqs. (11) and (2) provided

\[
\begin{align*}
  B\omega_1 &= [(2m - 1)B + 6m]\beta^2, \quad Bg_{12}D^2 = -6[m + (1 - m)B]\beta^2, \\
  Bg_{11}A^2 &= 2[3m + 2mB + (5m - 1)B^2 - (1 - m)B^3]B\omega_2 = [(5m - 1)B + 6m]\beta^2, \\
  Bg_{22}D^2 &= -2[3m^2 + 2mB - (1 - m)B^2]\beta^2, \quad Bg_{21}A^2 = 6(B + 1)[m + 2mB - (1 - m)B^2].
\end{align*}
\]

On using the identities (6) and (8), the coupled solution (56) can be re-expressed as

\[
\begin{align*}
  u(x, t) &= e^{i\omega_1 t} \frac{A \text{dn}^2(\Delta, m)}{2\text{cn}(\Delta, m)}[\text{cn}(\beta x + \Delta, m) + \text{cn}(\beta x - \Delta, m)], \\
  v(x, t) &= e^{i\omega_2 t} \frac{D \text{dn}(\Delta, m)}{2\text{cn}(\Delta, m)}[\text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m)], \quad (58)
\end{align*}
\]

where \(A, D\) are given by Eq. (57) while \(B = m \text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m)\).

**Solution XIII**

It is easy to check that

\[
\begin{align*}
  u(x, t) &= e^{i\omega_1 t} \frac{A \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \\
  v(x, t) &= e^{i\omega_2 t} \frac{D \text{sn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad (60)
\end{align*}
\]

where \(B > 0\), is an exact solution of the coupled Eqs. (11) and (2) provided

\[
\begin{align*}
  B\omega_1 &= [(5m - 4)B + 6]\beta^2, \quad B\omega_2 = [(5m - 1)B + 6m]\beta^2, \\
  Bg_{12}D^2 &= -6[(1 - m)^2B^3 + (1 - m)(2 - 3m)B^2 - 3m(1 - m)B + m^2]\beta^2, \\
  Bg_{11}A^2 &= 2(B + 1)[3m + 2(3m - 1)B - 3(1 - m)B^2]\beta^2, \beta^2, \\
  Bg_{21}A^2 &= 6(B + 1)[m + 2mB - (1 - m)B^2]\beta^2, \\
  Bg_{22}D^2 &= -2[3m^2 + m(9m - 7)B + (4 - 9m)(1 - m)B^2 + 3(1 - m)^2].
\end{align*}
\]
On using the identities (6) and (10), the coupled solution (6.0) can be re-expressed as

\[ u(x,t) = e^{i\omega_1 t} \frac{A \text{dn}(\Delta, m)}{2} \left[ \text{dn}(\beta x + \Delta, m) + \text{dn}(\beta x - \Delta, m) \right], \quad (62) \]

\[ v(x,t) = e^{i\omega_2 t} \frac{D \text{dn}(\Delta, m)}{2 \text{cn}(\Delta, m)} \left[ \text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m) \right], \quad (63) \]

where \( A, D \) are given by Eq. (61) while \( B = m \text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m) \).

**Hyperbolic Limit**

In the limit \( m = 1 \), the two solutions XII and XIII go over to the hyperbolic solution V, i.e. in this limit both the solutions can be re-expressed as superposition of pulse (i.e. \( \text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta) \)) and kink (i.e. \( \text{tanh}(\beta x + \Delta) + \text{tanh}(\beta x - \Delta) \)) solutions, respectively.

**Solution XIV**

It is straightforward to check that

\[ u(x,t) = e^{i\omega_1 t} \frac{A \text{cn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D \text{cn}(\beta x, m) \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad (64) \]

where \( B > 0 \), is an exact solution of the coupled Eqs. (1) and (2) provided

\[ \omega_1 = \left[ (2m - 1) - 6(1 - m)B \right] \beta^2, \quad mg_{12}D^2 = -6B[m - (1 - m)B^2] \beta^2, \]
\[ mg_{11}A^2 = 2[m^2 + m(5m - 1)B - 7m(1 - m)B^2 - 3(1 - m)^2 B^3], \]
\[ \omega_2 = \left[ ((5m - 1) - 6(1 - m)B \right] \beta^2, \quad mg_{22}D^2 = 6[m + 2mB + 3(1 - m)B^2] \beta^2, \]
\[ mg_{21}A^2 = 2B[-m^2 + (3m - 1)mB - 3m(1 - m)B^2 + (1 - m)^2 B^3]) \beta^2. \quad (65) \]

On using the identities (7) and (8), the coupled solution (64) can be re-expressed as

\[ u(x,t) = e^{i\omega_1 t} \frac{A \text{dn}(\Delta, m)}{2 \text{cn}(\Delta, m)} \left[ \text{cn}(\beta x + \Delta, m) + \text{cn}(\beta x - \Delta, m) \right], \quad (66) \]

\[ v(x,t) = e^{i\omega_2 t} \frac{D \text{dn}(\Delta, m)}{2 \text{sn}(\Delta, m)} \left[ \text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m) \right], \quad (67) \]

where \( A, D \) are given by Eq. (65) while \( B = m \text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m) \).

**Solution XV**

It is easy to check that

\[ u(x,t) = e^{i\omega_1 t} \frac{A \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D \text{cn}(\beta x, m) \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad (68) \]
where $B > 0$, is an exact solution of the coupled Eqs. (11) and (2) provided

\[
\begin{align*}
[m - (1 - m)B] \omega_1 &= [m(2 - m) + (1 - m)(4 + m)B] \beta^2, \\
[m - (1 - m)B] g_{11} A^2 &= 2[m + 2(1 + m)B - (1 - m)B^2] \beta^2, \\
[m - (1 - m)B] g_{12} D^2 &= -B[6m^2 - 12m(1 - m)B + 4(1 - m)(2 - m)B^2] \beta^2, \\
[m - (1 - m)B] \omega_2 &= [(5 - m)m + (1 - m^2)B] \beta^2, \\
[m - (1 - m)B] g_{22} A^2 &= 6[m + 2mB - (1 - m)B^2] \beta^2, \\
[m - (1 - m)B] g_{21} D^2 &= -2B[m - 2(2 - m)B - 2(1 - m)B^2] \beta^2.
\end{align*}
\]

(69)

On using the identities (7) and (10), the coupled solution (68) can be re-expressed as

\[
\begin{align*}
u(x, t) &= e^{i \omega t} \frac{A \text{dn}(\Delta, m)}{2} \left[ \text{dn}(\beta x + \Delta, m) + \text{dn}(\beta x - \Delta, m) \right], \\
v(x, t) &= e^{i \omega t} \frac{D \text{sn}(\beta x + \Delta, m)}{2 \text{sn}(\Delta, m)} \left[ \text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m) \right],
\end{align*}
\]

(70), (71)

where $A, D$ are given by Eq. (69) while $B = m \text{sn}^2(\Delta, m) / \text{dn}^2(\Delta, m)$.

**Hyperbolic Limit**

In the limit $m = 1$, the two solutions XIV and XV go over to the hyperbolic solution III, i.e. in this limit both the solutions can be re-expressed as superposition of one pulse (i.e. $\text{sech}(\beta x + \Delta) + \text{sech}(\beta x - \Delta)$) and one kink (i.e. $\tanh(\beta x + \Delta) - \tanh(\beta x - \Delta)$) solution.

**Solution XVI**

It is easy to check that

\[
\begin{align*}
u(x, t) &= e^{i \omega t} \frac{4 \text{cn}(\beta x, m) \text{sn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \\
v(x, t) &= e^{i \omega t} \frac{D \text{cn}(\beta x, m) \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)},
\end{align*}
\]

(72)

where $B > 0$, is an exact solution of the coupled Eqs. (11) and (2) provided

\[
\begin{align*}
\omega_1 &= [(5m - 4) - 6(1 - m)B] \beta^2, \\
g_{12} D^2 &= 6(B + 1)[m - 2(1 - m)B + (1 - m)B^2] \beta^2, \\
g_{11} A^2 &= -2[3m^2 + (9m - 5)mB - (1 - m)(9m - 2)B^2 + 3(1 - m^2)B^3] \beta^2, \\
\omega_2 &= (5m - 1) - 6(1 - m)B \beta^2, \\
g_{22} D^2 &= 2m(B + 1)[3 + 2(3m - 2)B - 3(1 - m^2)B^2] \beta^2, \\
g_{21} A^2 &= -6[m^2 + m(3m - 1)]B^2 + 3(1 - m)B^2 + (1 - m^2)B^3] \beta^2.
\end{align*}
\]

(73)

On using the identities (7) and (11), the coupled solution (72) can be re-expressed as

\[
u(x, t) = e^{i \omega t} \frac{A \text{sn}^2(\Delta, m)}{2m \text{cn}(\Delta, m) \text{sn}(\Delta, m)} \left[ \text{dn}(\beta x + \Delta, m) - \text{dn}(\beta x - \Delta, m) \right],
\]

(74)
\[ v(x,t) = e^{i\omega_1 t} \frac{D\text{dn}^2(\Delta, m)}{2\text{sn}(\Delta, m)} [\text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m)] , \]  

where \( A, D \) are given by Eq. (73) while \( B = m\text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m) \).

**Solution XVII**

It is easy to check that

\[ u(x,t) = e^{i\omega_1 t} \frac{A\text{cn}(\beta x, m)\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)} , \quad v(x,t) = e^{i\omega_2 t} \frac{D\text{sn}(\beta x, m)\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)} , \]  

where \( B > 0 \), is an exact solution of the coupled Eqs. (1) and (2) provided

\[
\begin{align*}
[m - (1 - m)B] \omega_1 &= [(5 - m)m + (1 - m^2)B]\beta^2, \\
[m - (1 - m)B] \omega_2 &= [(1 - m)(1 + 4m)B + m(5 - 4m)]\beta^2, \\
[m - (1 - m)B] g_{12} D &= -6[m + 2mB - (1 - m)B^2] \beta^2, \\
[m - (1 - m)B] g_{11} A &= [6m + 2(1 + 4m)B + 2(1 + m)mB^2 + (1 - m)B^3]\beta^2, \\
[m - (1 - m)B] g_{22} D &= -2[3m + 4mB - (1 - m)B^2] \beta^2, \\
[m - (1 - m)B] g_{21} A &= [6m + 6mB - 2(1 - m)(8m - 1)B^2 - 4(1 - m)(2m - 1)B^3] \beta^2. 
\end{align*}
\]

On using the identities (7) and (9), the coupled solution (76) can be re-expressed as

\[ u(x,t) = e^{i\omega_1 t} \frac{A\text{dn}^2(\Delta, m)}{2\text{sn}(\Delta, m)} [\text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m)] , \]  

\[ v(x,t) = e^{i\omega_2 t} \frac{D\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)} [\text{cn}(\beta x - \Delta, m) - \text{cn}(\beta x + \Delta, m)] , \]

where \( A, D \) are given by Eq. (77) while \( B = m\text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m) \).

**Hyperbolic Limit**

In the limit \( m = 1 \), the two solutions XVI and XVII go over to the hyperbolic solution II, i.e. in this limit both the solutions can be re-expressed as superposition of one pulse (i.e. \( \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \)) and one kink (i.e. \( \text{tanh}(\beta x + \Delta) - \text{tanh}(\beta x - \Delta) \)) solution.

**Solution XVIII**

It is straightforward to check that

\[ u(x,t) = e^{i\omega_1 t} \frac{A\text{sn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)} , \quad v(x,t) = e^{i\omega_2 t} \frac{D\text{sn}(\beta x, m)\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)} , \]  

(80)
where $B > 0$, is an exact solution of the coupled Eqs. (11) and (2) provided

\[
(1 + B)\omega_1 = [(5 - m)B - (1 + m)]\beta^2, \quad m(1 + B)g_{12}D^2 = -6B[m + 2mB - (1 - m)B^2]\beta^2, \\
m(1 + B)g_{11}A^2 = 2[m^2 - m(m + 1)B + 5m(1 - m)B^2 - 3(1 - m)^2B^3]\beta^2, \\
(1 + B)\omega_2 = [(5 - 4m)B - (4m + 1)]\beta^2, \quad m(1 + B)g_{22}D^2 = 2[m + 4mB - 3(1 - m)B^2]B\beta^2, \\
m(1 + B)g_{21}A^2 = 6[m^2 - m(1 - m)B + m(1 - m)B^2 - (1 - m)^2B^3]\beta^2.
\]

(81)

On using the identities (6) and (9), the coupled solution (80') can be re-expressed as

\[
u(x, t) = e^{i\omega_1 t} \frac{A\text{dn}(\Delta, m)}{2\text{cn}(\Delta, m)}[\text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m)], \quad (82)\\
\]

\[
v(x, t) = e^{i\omega_2 t} \frac{D\text{dn}(\Delta, m)}{2\text{sn}(\Delta, m)}[\text{cn}(\beta x - \Delta, m) - \text{cn}(\beta x + \Delta, m)], \quad (83)
\]

where $A, D$ are given by Eq. (81) while $B = m\text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m)$.

**Solution XIX**

It is easy to check that

\[
u(x, t) = e^{i\omega_1 t} \frac{A\text{sn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \quad v(x, t) = e^{i\omega_2 t} \frac{D\text{sn}(\beta x, m)\text{cn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)},
\]

(84)

where $B > 0$ is an exact solution of the coupled Eqs. (11) and (2) provided

\[
(1 + B)\omega_1 = [(5 - m)B - (1 + m)]\beta^2, \\
(1 + B)g_{12}D^2 = -6B[m + 2mB - (1 - m)B^2]\beta^2, \\
(1 + B)g_{11}A^2 = 2[m^2 - m(2 - m)B - (1 - m)B^2]\beta^2, \\
(1 + B)\omega_2 = [(2 - m)B - (4 + m)]\beta^2, \\
(1 + B)g_{22}D^2 = -2B[m + 2(1 + m)B - (1 - m)B^2]\beta^2.
\]

(85)

On using the identities (6) and (11), the coupled solution (84) can be re-expressed as

\[
u(x, t) = e^{i\omega_1 t} \frac{A\text{dn}(\Delta, m)}{2\text{cn}(\Delta, m)}[\text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m)], \quad (86)\\
\]

\[
v(x, t) = e^{i\omega_2 t} \frac{D\text{dn}(\Delta, m)}{2\text{cn}(\Delta, m)\text{sn}(\Delta, m)}[\text{dn}(\beta x - \Delta, m) - \text{dn}(\beta x + \Delta, m)], \quad (87)
\]

where $A, D$ are as given by Eq. (85) while $B = m\text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m)$. 

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Hyperbolic Limit

In the limit \( m = 1 \), the two solutions XVIII and XIX go over to the hyperbolic solution IV, i.e. in this limit both the solutions can be re-expressed as superposition of one pulse (i.e. \( \text{sech}(\beta x - \Delta) - \text{sech}(\beta x + \Delta) \)) and one kink (i.e. \( \tanh(\beta x + \Delta) + \tanh(\beta x - \Delta) \)) solution.

Solution XX

It is straightforward to check that

\[
 u(x,t) = e^{i\omega_1 t} \frac{A \text{sn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D \text{cn}(\beta x, m) \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad B > 0, \tag{88}
\]

is an exact solution of the coupled Eqs. (1) and (2) provided

\[
 \omega_1 = \omega_2 = -(1 + m)\beta^2, \quad g_{12} D^2 = 3g_{22} D^2 = -6B(B + 1)\beta^2, \\
g_{21} A^2 = 3g_{11} A^2 = 6[m - (1 - m)B]\beta^2. \tag{89}
\]

On using the identities (11) and (7), the coupled solution (88) can be re-expressed as

\[
 u(x,t) = e^{i\omega_1 t} \sqrt{\frac{m\beta}{2g_{11}}} \left[ \text{sn}(\beta x + \Delta, m) + \text{sn}(\beta x - \Delta, m) \right], \tag{90}
\]

\[
 v(x,t) = e^{i\omega_2 t} \sqrt{\frac{m\beta}{2g_{22}}} \left[ \text{sn}(\beta x + \Delta, m) - \text{sn}(\beta x - \Delta, m) \right], \tag{91}
\]

where \( B = m\text{sn}^2(\Delta, m)/\text{dn}^2(\Delta, m) \).

Hyperbolic Limit

In the limit \( m = 1 \), the solution XX goes over to the hyperbolic solution I, i.e. in this limit the solution can be re-expressed as superposition of two kink solutions (i.e. \( \tanh(\beta x + \Delta) \pm \tanh(\beta x - \Delta) \)).

Solution XXI

It is easy to check that

\[
 u(x,t) = e^{i\omega_1 t} \frac{A \text{cn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D \text{dn}(\beta x, m)}{1 + B \text{cn}^2(\beta x, m)}, \tag{92}
\]

where \( B > 0 \), is an exact solution of the coupled Eqs. (1) and (2) provided

\[
 \omega_1 = [(2m - 1) + \frac{6m}{B}]\beta^2, \quad (1 - m)g_{12} D^2 = -6[(1 - m)B + \frac{m}{B}]\beta^2, \\
g_{11} A^2 = 2\left[ (1 - m)^2 B^2 - (4m - 3)(1 - m)B - 7m(1 - m)B + \frac{3m^2}{B} \right]\beta^2, \\
\omega_2 = [(5m - 4) + \frac{6m}{B}]\beta^2, \quad (1 - m)g_{22} D^2 = 2[2(1 - m) - (1 - m)B - \frac{3m}{B}]\beta^2, \\
(1 - m)g_{21} A^2 = \frac{6\beta^2}{B}[m - (1 - m)B][1 - (1 - m)(B + 1)^2]. \tag{93}
\]
Note that this solution is only valid if $0 < m < 1$.

On using the identities (9) and (10), the coupled solution (92) can be re-expressed as

$$u(x,t) = e^{i\omega_1 t} \frac{A\text{dn}^2(\Delta, m)}{2\text{cn}(\Delta, m)} [\text{cn}(\beta x + \Delta, m) + \text{cn}(\beta x - \Delta, m)],$$

(94)

$$v(x,t) = e^{i\omega_2 t} \frac{D\text{dn}(\Delta, m)}{2} [\text{dn}(\beta x + \Delta, m) + \text{dn}(\beta x - \Delta, m)],$$

(95)

where $A, D$ are as given by Eq. (93) while $B = \text{msn}^2(\Delta, m)/\text{dn}^2(\Delta, m)$.

**Solution XXII**

It is straightforward to check that

$$u(x,t) = e^{i\omega_1 t} \frac{A\text{cn}(\beta x, m)\text{sn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)}, \quad v(x,t) = e^{i\omega_2 t} \frac{D\text{sn}(\beta x, m)\text{dn}(\beta x, m)}{1 + B\text{cn}^2(\beta x, m)},$$

(96)

where $B > 0$, is an exact solution of the coupled Eqs. (1) and (2) provided

$$(B + 1)\omega_1 = [(2 - m)B - (4 + m)]\beta^2, \quad (1 - m)(B + 1)g_{11}A^2$$

$$= -[(1 - m)(2 - m)B^3 + (1 - m)(5m - 4)B^2 + 12m(1 - m)B - 6m^2]\beta^2,$$

(1 - m)(B + 1)g_{12}D^2 = -6[1 - (1 - m)(1 + B)^2]\beta^2,

(B + 1)\omega_2 = [(5 - 4m)B - (4m + 1)]\beta^2,

(1 - m)(B + 1)g_{21}A^2 = [1 - 15m + 20m^2 - (1 - m)(5 + 2m)B$$

+ (1 - m)(2m - 1)B^2 + (1 - m)(2m - 1)B^3]\beta^2,

(1 - m)(B + 1)g_{22}D^2 = -2[3 - 8(1 - m)B - (1 - m)B^2]\beta^2.

(97)

Note that this solution is only valid for $0 < m < 1$.

On using the identities (9) and (11), the coupled solution (96) can be re-expressed as

$$u(x,t) = e^{i\omega_1 t} \frac{A\text{dn}^2(\Delta, m)}{2\text{cn}(\Delta, m)\text{sn}(\Delta, m)} [\text{dn}(\beta x - \Delta, m) - \text{dn}(\beta x + \Delta, m)],$$

(98)

$$v(x,t) = e^{i\omega_2 t} \frac{D\text{dn}(\Delta, m)}{2\text{sn}(\Delta, m)} [\text{cn}(\beta x - \Delta, m) - \text{cn}(\beta x - \Delta, m)],$$

(99)

where $A, D$ are as given by Eq. (97) while $B = \text{msn}^2(\Delta, m)/\text{dn}^2(\Delta, m)$. 

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3 Superposed Solutions of a Coupled Nonlocal mKdV Model

Let us consider the following model of coupled nonlocal mKdV equations

\[ u_t(x, t) + u_{xxx} + 6(g_{11}u(x, t)u(-x, -t) + g_{12}v(x, t)v(-x, -t))u_x(x, t) = 0, \]  
\[ (100) \]

\[ v_t(x, t) + v_{xxx} + 6(g_{21}u(x, t)u(-x, -t) + g_{22}v(x, t)v(-x, -t))v_x(x, t) = 0. \]  
\[ (101) \]

We now show that these coupled nonlocal equations admit a large number of exact solutions. In particular, there are two types of solutions to these coupled equations depending on if \( v(x, t) \) is proportional to \( u(x, t) \) or not which we discuss one by one. In this section we only discuss those solutions where the two coupled fields \( u \) and \( v \) are distinct (and not proportional to each other) while in Appendix B we discuss those solutions of the nonlocal coupled mKdV model in which the two fields \( v \) and \( u \) are proportional to each other.

One major difference between the local and the nonlocal case is that, unlike the local case, the solutions of the nonlocal mKdV Eqs. \( (100) \) and \( (101) \) are not invariant with respect to shifts in \( x \) and \( t \).

3.1 Solutions When \( u(x, t) \) and \( v(x, t) \) are Distinct

It turns out that in case \( v(x, t) \) and \( u(x, t) \) are distinct then the coupled Eqs. \( (100) \) and \( (101) \) admit a number of exact solutions which we now present one by one.

Solution I

It is easy to show that

\[ u(x, t) = Adn(\xi, m), \quad v(x, t) = B\sqrt{m}\text{sn}(\xi, m), \quad \xi = \beta(x - ct), \]  
\[ (102) \]

is an exact solution to the coupled Eqs. \( (100) \) and \( (101) \) provided

\[ g_{11}A^2 + g_{12}B^2 = g_{21}A^2 + g_{22}B^2 = \beta^2, \]  
\[ (103) \]

\[ c = (2 - m)\beta^2 - 6g_{12}B^2, \]  
\[ (104) \]

\[ 2(g_{22} - g_{12})B^2 = \beta^2. \]  
\[ (105) \]

Solution II

It is straightforward to show that

\[ u(x, t) = A\sqrt{m}\text{cn}(\xi, m), \quad v(x, t) = B\sqrt{m}\text{sn}(\xi, m), \quad \xi = \beta(x - ct), \]  
\[ (106) \]
is an exact solution to the coupled Eqs. (100) and (101) provided Eqs. (103) and (105) are satisfied and if further

\[ c = (5 - 4m)\beta^2 - 6[g_{12}B^2 + (1 - m)g_{11}A^2]. \] (107)

In the limit \( m = 1 \), both the solutions XXIII and XXIV go over to the hyperbolic solution

\[ u(x, t) = \text{Asech}(\xi), \quad v(x, t) = B \tanh(\xi), \quad \xi = \beta(x - ct), \] (108)

provided Eqs. (103) and (105) are satisfied and if further

\[ c = \beta^2 - 6[g_{12}B^2 + (1 - m)g_{11}A^2]. \] (109)

**Solution III**

It is easy to show that

\[ u(x, t) = A\text{dn}(\xi, m), \quad v(x, t) = B\sqrt{m}\text{cn}(\xi, m), \quad \xi = \beta(x - ct), \quad 0 < m < 1, \] (110)

is an exact solution to the coupled Eqs. (100) and (101) provided Eqs. (103) and (105) are satisfied and if further

\[ c = (5 - 4m)\beta^2 - 6[g_{12}B^2 + (1 - m)g_{11}A^2]. \] (111)

Note that if \( m = 1 \), then \( v(x, t) = u(x, t) \) and that solution has been discussed in Appendix B.

**Solution IV**

It is straightforward to show that

\[ u(x, t) = A\text{dn}(\xi, m), \quad v(x, t) = B\sqrt{m}\text{sn}(\xi, m), \quad \xi = \beta(x - ct), \] (112)

is an exact solution to the coupled Eqs. (100) and (101) provided

\[ g_{11}A^2 - g_{12}B^2 = g_{21}A^2 - g_{22}B^2 = (1 - m)\beta^2, \] (113)

\[ c = (2 - m)\beta^2 + 6g_{12}B^2, \] (114)

\[ 2(g_{22} - g_{12})B^2 = (1 - m)\beta^2. \] (115)

**Solution V**

It is easy to show that

\[ u(x, t) = A\text{dn}(\xi, m), \quad v(x, t) = B\sqrt{m}\text{cn}(\xi, m), \quad \xi = \beta(x - ct), \] (116)
is an exact solution to the coupled Eqs. (100) and (101) provided Eq. (114) is satisfied and further

\[ g_{11}A^2 - (1 - m)g_{12}B^2 = g_{21}A^2 - (1 - m)g_{22}B^2 = (1 - m)\beta^2, \]  
\[ 2(g_{22} - g_{12})B^2 = -\beta^2. \]  

**Solution VI**

It is straightforward to show that

\[ u(x, t) = A\sqrt{m}\text{sn}(\xi, m)\text{dn}(\xi, m), \quad v(x, t) = B\sqrt{m}\text{cn}(\xi, m)\text{dn}(\xi, m), \quad \xi = \beta(x - ct), \]  

is an exact solution to the coupled Eqs. (100) and (101) provided

\[ g_{11}A^2 + (1 - m)g_{12}B^2 = g_{21}A^2 + (1 - m)g_{22}B^2 = -(1 - m)\beta^2, \]  
\[ c = (5 - 4m)\beta^2 + 6[g_{11}A^2 + g_{12}B^2], \]  
\[ 2(g_{11} - g_{21})A^2 - 2(g_{12} - g_{22})B^2 = m\beta^2. \]

**Solution VII**

The coupled nonlocal MKdV Eqs. (100) and (101) admit the solution

\[ u(x, t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad v(x, t) = \frac{\alpha A}{B + \cosh^2(\xi)}, \quad A, B > 0, \]  

provided

\[ c = 4\beta^2 + 6g_{11}, \quad g_{21} = g_{11} < 0, \quad g_{22} = g_{12}, \quad Ag_{11} = -(2B + 1)\beta^2, \]  
\[ (g_{11} + \alpha^2g_{12})A^2 = -4B(B + 1)\beta^2. \]  

On comparing it with the identity (7), we can re-express solution (123) as

\[ u(x, t) = 1 - \frac{\beta}{\sqrt{|g_{11} + \alpha^2g_{22}|}}[\tanh(\xi + \Delta) - \tanh(\xi - \Delta)], \]  
\[ v(x, t) = \frac{\alpha \beta}{\sqrt{|g_{11} + \alpha^2g_{22}|}}[\tanh(\xi + \Delta) - \tanh(\xi - \Delta)], \]  

where \( B = \sinh^2(\Delta) \).

**Solution VIII**
Yet another hyperbolic superposed solution to the coupled Eqs. (100) and (101) is
\[ u(x,t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad v(x,t) = -b - \frac{\alpha A}{B + \cosh^2(\xi)}, \quad A, B, b > 0 \] (127)
provided
\[ c - 4\beta^2 = g_{12} + b^2 g_{22} = g_{11} + b^2 g_{12}, \]
\[ A(g_{11} - g_{12} \alpha b) = A(g_{21} - g_{22} \alpha b) = -(2B + 1)\beta^2, \]
\[ (g_{11} + \alpha^2 g_{12}) A^2 = (g_{21} + \alpha^2 g_{22}) = -4B(B + 1)\beta^2. \] (128)
On comparing it with the identity (11), we can re-express solution (127) as

\[ u(x,t) = 1 - \frac{\beta}{\sqrt{|g_{11} + \alpha^2 g_{22}|}} \left[ \tanh(\xi + \Delta) - \tanh(\xi - \Delta) \right], \] (129)

and

\[ v(x,t) = -b - \frac{\alpha \beta}{\sqrt{|g_{11} + \alpha^2 g_{22}|}} \left[ \tanh(\xi + \Delta) - \tanh(\xi - \Delta) \right], \] (130)

where \( B = \sinh^2(\Delta) \).

4 Conclusion and Open Problems

In this paper we have partially extended the notion of superposition to the coupled nonlocal equations. In particular, we showed that the Ablowitz-Musslimani variant of the coupled nonlocal NLS equations as well as the coupled nonlocal mKdV equations admit solutions which can be re-expressed as the superposition of two hyperbolic as well as two periodic kink and pulse solutions. In particular, in Sec. II, for the Ablowitz-Musslimani NLS case we obtained seven hyperbolic superposed solutions as well as fifteen periodic superposed solutions. On the other hand, for the nonlocal mKdV case, in Sec. III we obtained two superposed hyperbolic solutions. Besides, in case the two coupled fields are proportional to each other then in the Ablowitz-Musslimani case we obtained four superposed periodic solutions while in the mKdV case we obtained four superposed periodic and one superposed hyperbolic superposed solution. In addition, for completeness we have also presented some other solutions admitted by these models.

This paper raises several important questions which are still not understood. Some of these questions are
1. Unlike the coupled nonlocal Ablowitz-Musslimani case, we have not been able to obtain superposed periodic solutions in the nonlocal mKdV case when the two fields are not proportional to each other. It would be worthwhile looking for such solutions.

2. For the nonlocal mKdV case, in the hyperbolic case we have only been able to obtain solutions which can be re-expressed as a superposition of a kink and an antikink solution but not as a superposition of two kink or two pulse solutions. It is clearly of interest to obtain such solutions.

3. It is clearly of interest to discover similar superposed solutions in the case of other nonlocal equations such as the nonlocal KdV equation, the nonlocal Hirota equation, etc. As a first step in that direction one needs to construct models with coupled nonlocal KdV and coupled nonlocal Hirota equations.

4. Finally, what is the interpretation of such superposed solutions? Do they correspond to bound state or merely an excitation of two kink or two pulse solutions?

Hopefully, one can find answer to some of the questions raised above.

Acknowledgment

One of us (AK) is grateful to Indian National Science Academy (INSA) for the award of INSA Senior Scientist position at Savitribai Phule Pune University. The work at Los Alamos National Laboratory was carried out under the auspices of the U.S. DOE and NNSA under Contract No. DEAC52-06NA25396.

Appendix A: Solutions of a Coupled Ablowitz-Musslimani Nonlocal NLS Model when $v(x,t) \propto u(x,t)$

In case $v(x,t) = \alpha u(x,t)$ where $\alpha$ is a number, in that case the two Eqs. (1) and (2) are consistent with each other provided

$$g_{11} + \alpha^2 g_{12} = g_{21} + \alpha^2 g_{22}, \quad (131)$$

and we only need to solve the single equation

$$i u_{x,t} + u_{xx}(x) + gu^2(x,t)u^*(-x,t) = 0, \quad g = g_{11} + \alpha^2 g_{12}, \quad (132)$$

which is effectively the uncoupled nonlocal Ablowitz-Musslimani variant of the NLS for which we have already obtained a large number of solutions.
However, it turns out that we have missed several solutions which we now present one by one.

As mentioned in Sec. II, one major difference between the local and the nonlocal case is that, unlike the local case, the solutions of the nonlocal mKdV Eq. (132) are not invariant with respect to shift in \(x\).

**Solution I**

In [2] we had shown that \(e^{i\omega t}dn(\beta x, m)\) as well as \(e^{i\omega t}dn[\beta(x+K[m]), m]\) are the exact solutions of the nonlocal Eq. (132). Remarkably, it turns out that not only \(e^{i\omega t}dn(\beta x, m)\) and \(e^{i\omega t}dn[\beta(x+K[m]), m]\) but even their superposition is a solution of the nonlocal NLS Eq. (132). In particular, it is straightforward to check that

\[
u(x,t) = e^{i\omega t}[A dn(\beta x, m) + B \sqrt{1-m} dn(\beta x, m)], \tag{133}
\]

is also an exact solution of the nonlocal Eq. (132) provided

\[
g > 0, \quad gA^2 = \beta^2, \quad B = \pm A, \quad \omega = [2 - m \pm 6\sqrt{1-m}] \beta^2, \tag{134}
\]

where \(g\) is as given by Eq. (132) while the \(\pm\) sign in \(B = \pm A\) and in \(\omega\) are correlated.

**Solution II: Peregrine Soliton**

It is easy to check that the celebrated Peregrine soliton solution [24, 25] of the local NLS is also a solution of the nonlocal Eq. (132), i.e. in particular, it is straightforward to check that

\[
u(x,t) = \frac{1}{\sqrt{2g}} \left[ 1 - \frac{4(1+2it)}{(1+2x^2+4t^2)} \right] e^{it}, \quad g > 0, \tag{135}
\]

is an exact solution of Eq. (132).

**Solution III: Akhmediev-Eleonskii-Kulagin Breather Solution**

It is easy to check that even the celebrated Akhmediev-Eleonskii-Kulagin breather solution [25, 26] of the local NLS, i.e.

\[
u(x,t) = \sqrt{\frac{a^2}{2g}} e^{ia^2t} \left[ \frac{b^2 \cosh(\theta) + ib\sqrt{2-b^2}}{\sqrt{2\cosh(\theta) - \sqrt{2-b^2}\cos(abx)}} \right], \quad g > 0, \tag{136}
\]

where \(\theta = a^2b\sqrt{2-b^2}t\), is also a solution of the nonlocal Eq. (132).

**Solution IV: Kuznetsov-Ma Soliton**

Remarkably, even the celebrated Kuznetsov-Ma soliton solution [25, 27, 28] of the local NLS is also a solution of the nonlocal Eq. (132). In particular, it is easy to check that

\[
u(x,t) = \frac{a}{\sqrt{2g}} e^{ia^2t} \left[ 1 + \frac{2m(m\cos(\theta) + in\sin(\theta))}{n\cosh(2ax) + \cos(\theta)} \right], \quad g > 0, \tag{137}
\]
where \( n^2 = 1 + m^2 \), \( \theta = 2mna^2t \), is an exact solution of the nonlocal Eq. (132).

We now show that unlike the local NLS or the Yang version of the nonlocal NLS [29], the Ablowitz-Musslimani variant of the nonlocal NLS admits complex PT-invariant periodic and hyperbolic superposed solutions.

**Solution V**

It is straightforward to show that the nonlocal Eq. (132) admits the complex PT-invariant periodic solution

\[
\begin{align*}
  u(x, t) &= e^{i\omega t} \left[ A \text{dn}(\beta x, m) + iB \sqrt{m} \text{sn}(\beta x, m) \right], \\
  \omega &= -\frac{(2m - 1)}{2} \beta^2, \quad g > 0.
\end{align*}
\]

**Solution VI**

Another complex PT-invariant periodic solution of the nonlocal Eq. (132) is

\[
\begin{align*}
  u(x, t) &= e^{i\omega t} \left[ A \sqrt{m} \text{cn}(\beta x, m) + iB \sqrt{m} \text{sn}(\beta x, m) \right], \\
  \omega &= -\frac{(2m - 1)}{2} \beta^2, \quad g > 0.
\end{align*}
\]

**Solution VII**

In the limit \( m = 1 \), both the solutions V and VI go over to the complex PT-invariant hyperbolic solution

\[
\begin{align*}
  u(x, t) &= e^{i\omega t} \left[ A \text{sech}(\beta x) + iB \tanh(\beta x) \right], \\
  \omega &= \frac{-\beta^2}{2}, \quad g > 0.
\end{align*}
\]

**Solution VIII**

Remarkably, the nonlocal Eq. (132) also admits the complex PT-invariant periodic solution

\[
\begin{align*}
  u(x, t) &= e^{i\omega t} \left[ A \sqrt{m} \text{sn}(\beta x, m) + iB \text{dn}(\beta x, m) \right], \\
  \omega &= -\frac{\beta^2}{2}.
\end{align*}
\]
Another complex PT-invariant periodic solution of the nonlocal Eq. (1)
is
\[ u(x, t) = e^{i\omega t} [A\sqrt{m} \operatorname{sn}(\beta x, m) + iB\sqrt{m} \operatorname{cn}(\beta x, m)] , \tag{145} \]
provided the same relations as in Eq. (141) are satisfied.

**Solution X**

In the limit \(m = 1\), both the solutions VIII and IX go over to the complex PT-invariant hyperbolic solution
\[ u(x, t) = e^{i\omega t} [A \tanh(\beta x) + iB \operatorname{sech}(\beta x)] , \tag{146} \]
provided the same relations as in Eq. (143) are satisfied. Thus the Ablowitz-Musslimani variant of the nonlocal NLS is rather unusual in the sense that the same model (i.e. with \(g > 0\)) admits not only the kink and the pulse solutions but also the complex PT-invariant pulse and kink solutions with the PT-eigenvalue \(+1\) as well as \(-1\).

We now show that Eq. (132) also satisfies four novel periodic solutions which can be re-expressed as the superposition of either the two periodic kink or the two periodic pulse solutions \(\operatorname{sn}(x, m)\) or \(\operatorname{dn}(x, m)\), respectively.

**Solution XI**

It is easy to check that the nonlocal NLS Eq. (132) admits the periodic solution
\[ u(x, t) = e^{i\omega t} \left[ \frac{A \operatorname{dn}(\beta x, m) \operatorname{cn}(\beta x, m)}{1 + B \operatorname{cn}^2(\beta x, m)} \right] , \quad B > 0 , \tag{147} \]
provided \(g < 0\) and further
\[
0 < m < 1 , \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}} , \\
\omega = -[1 + m + 6\sqrt{m}]\beta^2 < 0 , \quad g < 0 , \quad |g|A^2 = \frac{4\sqrt{m}\beta^2}{(1 - \sqrt{m})^2} . \tag{148}
\]
Note that this solution is not valid for \(m = 1\), i.e. the nonlocal NLS Eq. (1) does not admit a corresponding hyperbolic solution.

On comparing Eqs. (147) and the identity (7) and using Eq. (148), one can re-express the periodic solution (147) as the superposition of a periodic kink and a periodic antikink solution, i.e.
\[ u(x, t) = e^{i\omega t} \sqrt{\frac{m}{2g}} \beta [\operatorname{sn}(\beta x + \Delta, m) - \operatorname{sn}(\beta x - \Delta, m)] . \tag{149} \]
Here \(\Delta\) is defined by \(\operatorname{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4}\), where use has been made of the identity \[30\]
\[ \sqrt{m} \operatorname{sn}(y, m) = \operatorname{sn}(\sqrt{m} y, 1/m) . \tag{150} \]
Solution XII
Remarkably, Eq. (1) also admits another periodic solution

\[ u(x, t) = e^{i\omega t} \frac{Asn(\beta x, m)}{1 + Bcn^2(\beta x, m)}, \quad B > 0, \]  

providing

\[ 0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad g < 0, \]
\[ \omega = [6\sqrt{m} - (1 + m)]\beta^2, \quad |g|A^2 = 4\sqrt{m}\beta^2. \]  

(152)

Note that this solution too is not valid in the hyperbolic limit of \( m = 1 \).

On comparing the solution (151) and the novel identity (6) and using Eq. (152), the periodic solution XII given by Eq. (151) can be re-expressed as the superposition of the two periodic kink solutions

\[ u(x, t) = i e^{i\omega t} \sqrt{m} |g| \beta \left[ sn(\beta x + \Delta, m) + sn(\beta x - \Delta, m) \right]. \]  

(153)

Here \( \Delta \) is defined by \( sn(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity (150).

It is worth noting that for both the solutions XI and XII, not only the value of \( B \) is the same but even \( g < 0 \) for both the solutions.

Solution XIII
It is easy to check that the nonlocal NLS Eq. (132) admits another periodic solution

\[ u(x, t) = e^{i\omega t} \frac{Asn(\beta x, m)cn(\beta x, m)}{1 + Bcn^2(\beta x, m)}, \]  

(154)

provided

\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad g < 0, \]
\[ \omega = (2 - m - 6\sqrt{1 - m})\beta^2, \quad A^2 = \frac{2(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \]  

(155)

Note that whereas this solution is valid if \( g < 0 \), the same solution in the Yang’s nonlocal case, as well as in the local NLS case is valid only if \( g > 0 \).
On comparing the solution (154) with the identity (111) and using Eq. (155), we find that the solution as given by Eq. (154) can be re-expressed as a superposition of the two periodic pulse solutions, i.e.

\[ u(x,t) = e^{i\omega t} \beta \sqrt{\frac{1}{2g}} (\text{dn}[\beta(x)-K(m)/2,m]-\text{dn}[\beta(x)+K(m)/2,m]). \] (156)

**Solution XIV**

Remarkably, the nonlocal NLS Eq. (132) also admits another periodic solution

\[ u(x,t) = e^{i\omega t} \frac{A \text{dn}(\beta x,m)}{1+B\text{cn}^2(\beta x,m)}, \] (157)

provided

\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1-m}}{\sqrt{1-m}}, \quad g > 0, \]

\[ \omega = [2 - m + 6\sqrt{1-m}]\beta^2, \quad A^2 = \frac{4}{\sqrt{1-m}}\beta^2. \] (158)

Note that this solutions is also not valid for \( m = 1 \). Thus for this solution \( g > 0, \omega > 0 \).

On comparing the solution (157) and the identity (10) and using Eq. (158), the periodic solution (157) can be re-expressed as a superposition of the two periodic pulse solutions, i.e.

\[ u(x,t) = e^{i\omega t} \frac{\beta}{\sqrt{g}} [\text{dn}(\beta x + K(m)/2,m) + \text{dn}(\beta x - K(m)/2,m)], \] (159)

where \( \Delta = \pm K(m)/2 \)

It is worth noting that for both the superposed periodic pulse solutions XIII and XIV, while the value of \( B \) is the same but the value of \( g \) is opposite for the two solutions.

Note that out of the 14 solutions, the solutions XI to XIV are only valid if \( m \neq 1 \). Further, while the solutions XI, XII and XIII are valid if \( g < 0 \), the remaining eleven solutions are valid if \( g > 0 \).

**Appendix B: Solutions of a Coupled Nonlocal mKdV Model**

When \( v(x,t) \propto u(x,t) \)

In case \( v(x,t) = \alpha u(x,t) \) where \( \alpha \) is a number, then the two coupled Eqs. (100) and (101) are consistent with each other provided

\[ g_{11} + \alpha^2 g_{12} = g_{21} + \alpha^2 g_{22}. \] (160)

29
Remarkably, in this case we only need to solve an uncoupled nonlocal mKdV equation
\[ u_t(x,t) + u_{xxx} + 6gu(x,t)u(-x,-t)u_x(x,t) = 0, \quad g = g_{11} + \alpha^2 g_{12}. \quad (161) \]

In a recent paper [13] we have obtained several exact solutions to an uncoupled nonlocal mKdV equation using which we can immediately write down the exact solutions to the coupled nonlocal mKdV Eqs. (100) and (101) in case the two nonlocal fields \( v(x,t) \) and \( u(x,t) \) are proportional to each other.

**Solution I**

It is easy to show that one of the exact solution of the nonlocal mKdV Eq. (161) is
\[ u(x,t) = A \text{dn}[\beta(x - ct), m] , \quad (162) \]
provided
\[ g > 0, \quad gA^2 = \beta^2, \quad c = (2 - m)\beta^2. \quad (163) \]

**Solution II**

Similarly
\[ u(x,t) = A\sqrt{m} \text{cn}[\beta(x - ct), m] , \quad (164) \]
is an exact solution to Eq. (161) provided
\[ g > 1, \quad gA^2 = \beta^2, \quad c = (2m - 1)\beta^2. \quad (165) \]

**Solution III**

Remarkably, even a linear superposition of the solutions I and II is also a solution of Eq. (161), i.e.
\[ u(x,t) = A \text{dn}[\beta(x - ct), m] + B\sqrt{m} \text{cn}[\beta(x - ct), m] , \quad (166) \]
provided
\[ g > 0, \quad 4gA^2 = \beta^2, \quad B = \pm A, \quad c = \frac{(1 + m)}{2}\beta^2. \quad (167) \]

**Solution IV**

In the limit \( m = 1 \), the solutions I, II and III (with \( B = A \)) go over to the hyperbolic solution
\[ u(x,t) = A\text{sech}[\beta(x - ct)] , \quad (168) \]
provided
\[ g > 1, \quad gA^2 = \beta^2, \quad v = \beta^2, \quad (169) \]
while solution III with \( B = -A \) goes over to the vacuum solution \( u = 0 \).
Solution V
Remarkably, unlike the local mKdV, for the nonlocal case even
\[ u(x, t) = A\sqrt{m}\text{sn}[\beta(x - ct), m], \]  
(170)
is an exact solution to Eq. (161) provided
\[ g > 0, \ gA^2 = \beta^2, \ c = -(1 + m)\beta^2. \]  
(171)

Solution VI
In the limit \( m = 1 \), the solution V goes over to the hyperbolic solution
\[ u(x, t) = A\tanh[\beta(x - ct)], \]  
(172)
provided
\[ g > 0, \ gA^2 = \beta^2, \ c = -2\beta^2. \]  
(173)
Thus unlike the local mKdV, the nonlocal attractive mKdV Eq. (161) (i.e. with \( g = +1 \)) admits both the kink and the pulse solutions.

Unlike the local case, the solutions of the nonlocal mKdV Eq. (161) are not invariant with respect to shifts in \( x \) and \( t \). For example, while \( A\text{dn}[\beta(x - ct + x_0), m] \) is an exact solution of the local mKdV no matter what \( x_0 \) is, it is not an exact solution of the nonlocal Eq. (161). However, for the special values of \( x_0 \), \( \text{sn}(x, m) \), \( \text{cn}(x, m) \) and \( \text{dn}(x, m) \) are still the solutions of the nonlocal Eq. (161). In particular, we now show that when \( x_0 = K(m) \), where \( K(m) \) is the complete elliptic integral of the first kind, there are exact solutions of the nonlocal Eq. (161) in both the focusing \((g > 0)\) and the defocusing \((g < 0)\) cases. This is because of the relations [3]
\[ \text{dn}[x + K(m), m] = \frac{\sqrt{1 - m}}{\text{dn}(x, m)}, \ \text{sn}[x + K(m), m] = \frac{\text{cn}(x, m)}{\text{dn}(x, m)}, \]  
\[ \text{cn}[x + K(m), m] = -\frac{\sqrt{1 - m}\text{sn}(x, m)}{\text{dn}(x, m)}. \]  
(174)

Solution VII
It is easy to show that
\[ u(x, t) = A\frac{1}{\text{dn}[\beta(x - ct), m]}, \]  
(175)
is an exact solution to the Eq. (161) provided
\[ g > 0, \ gA^2 = (1 - m)\beta^2, \ c = (2 - m)\beta^2. \]  
(176)
Solution VIII
It is easy to show that

\[ u(x,t) = A \sqrt{m} \text{sn}[\beta(x - ct), m] \frac{\text{dn}[\beta(x - ct), m]}{\text{dn}[\beta(x - ct), m]} , \]  

(177)
is an exact solution to the nonlocal mKdV Eq. (161) provided

\[ g < 0, \quad |g|A^2 = (1 - m)\beta^2, \quad c = (2m - 1)\beta^2 . \]  

(178)

Solution IX
It is easy to show that

\[ u(x,t) = A \sqrt{m} \text{cn}[\beta(x - ct), m] \frac{\text{dn}[\beta(x - ct), m]}{\text{dn}[\beta(x - ct), m]} , \]  

(179)
is an exact solution to the Eq. (161) provided

\[ g < 0, \quad |g|A^2 = \beta^2, \quad c = -(1 + m)\beta^2, \quad m \neq 1 . \]  

(180)

Solution X
Remarkably, it turns out that not only \( \text{dn}[\beta(x - ct), m] \) and \( \text{dn}[\beta(x - ct + K[m]), m] \) but even their superposition is a solution of the nonlocal mKdV Eq. (161). In particular, it is easy to check that

\[ u(x,t) = e^{i\omega t} \left( A \text{dn}[\beta(x - ct), m] + \frac{B \sqrt{1 - m}}{\text{dn}[\beta(x - ct), m]} \right) , \]  

(181)
is also an exact solution of the nonlocal Eq. (161) provided

\[ g > 0, \quad gA^2 = \beta^2, \quad B = \pm A, \quad c = [2 - m \pm 6\sqrt{1 - m}]\beta^2 , \]  

(182)

where the ± sign in \( B = \pm A \) and in \( v \) are correlated.

Solution XI: The Bion Solution
The well known breather (also called bion) solution of the attractive local mKdV Equation

\[ u_t + u_{xxx} + 6gu^2u_x = 0 , \]  

(183)
is \[ 25 \]

\[ u(x,t) = -\frac{2}{\sqrt{g}} \frac{d}{dx} \tan^{-1} \left[ \frac{c \sin(ax + bt + a_0)}{a \cosh(cx + dt + c_0)} \right] , \]  

(184)
provided

\[ g > 0, \quad b = a(a^2 - 3c^2), \quad d = c(3a^2 - c^2) . \]  

(185)
Here $a_0, c_0$ are arbitrary constants. It is then clear that the bion solution as given by Eq. (184) is also the bion solution of the nonlocal mKdV Eq. (161) provided $a_0 = c_0 = 0$.

Solution XII: Periodic Generalization of the Bion Solution

It has been shown [31] that the periodic generalization of the bion solution of the local attractive mKdV Eq. (183) is

$$u(x,t) = -\frac{2}{\sqrt{g}} \frac{d}{dx} \tan^{-1}\left[ \alpha \text{sn}(ax + bt + a_0, k) \text{dn}(cx + dt + c_0, m) \right], \quad (186)$$

provided

$$a^4 k = c^4 (1 - m), \quad \alpha = \frac{c}{a},$$

$$b = a[a^2 (1 + k) - 3c^2 (2 - m)], \quad d = c[3a^2 (1 + k) - (2 - m)c^2]. \quad (187)$$

As expected, in the limit $m \to 1, k \to 0$, the periodic bion solution (186) goes over to the bion solution (184) and the relations between $c$ and $d$ as well as between $a$ and $b$ as given by Eq. (187) go over to the one given in Eq. (185).

It is then clear that the periodic bion solution as given by Eq. (186) is also the periodic bion solution of the nonlocal mKdV Eq. (161) provided $a_0 = c_0 = 0$.

Solution XIII: Rational Solution

It is easy to check that following the rational solution of the local mKdV Eq. (183) [25], the solution of the nonlocal mKdV Eq. (161) is

$$\sqrt{g}u(x,t) = c - \frac{4c}{4c(x - 6c^2 t)^2 + 1}. \quad (188)$$

Solution XIV

It has been shown [32] that there is another periodic solution of the attractive “local” mKdV Eq. (183) and it is easy to check that it is also the solution of the nonlocal mKdV Eq. (161) and is given by

$$u(x,t) = -\frac{2}{\sqrt{g}} \frac{d}{dx} \tan^{-1}\left[ \alpha \text{sn}(ax + bt, k) \text{dn}(cx + dt, m) \right], \quad (189)$$

provided

$$g > 0, \quad (1 - k)a^4 = (1 - m)c^4, \quad \alpha^2 = -\frac{c}{a},$$

$$b = -a[a^2 (2 - k) - 3c^2 (2 - m)], \quad d = -c[3a^2 (2 - k) + (2 - m)c^2]. \quad (190)$$
Solution XV
Following [32] it is easy to see that
\[ u(x,t) = -\frac{2}{\sqrt{|g|}} \frac{d}{dx} \tan^{-1} \left[ \alpha \text{sn}(ax + bt, k) \text{sn}(cx + dt, m) \right], \quad (191) \]
is an exact solution of the nonlocal mKdV Eq. (161) provided
\[ g > 0, \quad ka^4 = mc^4, \quad \alpha^2 = \sqrt{km}, \quad b = a[a^2(1 + k) + 3c^2(1 + m)], \quad d = c[3a^2(1 + k) + (1 + m)c^2]. \quad (192) \]

Solution XVI
Following [32], it is easy to see that another periodic solution of the nonlocal mKdV Eq. (161) is
\[ u(x,t) = -\frac{2}{\sqrt{|g|}} \frac{d}{dx} \tan^{-1} \left[ \alpha \text{cn}(ax + bt, k) \text{cn}(cx + dt, m) \right], \quad (193) \]
provided
\[ g < 0, \quad k(1 - k)a^4 = m(1 - m)c^4, \quad \alpha^2 = \frac{km}{(1 - k)(1 - m)}, \quad b = a[a^2(1 - 2k) + 3c^2(1 - 2m)], \quad d = c[3a^2(1 - 2k) + (1 - 2m)c^2]. \quad (194) \]

Recently, we [13] have obtained four novel periodic and one hyperbolic solutions of local mKdV Eq. (183) and shown that they can be re-expressed as superposed kink or pulse solutions. We now show that the nonlocal mKdV Eq. (161) also admits these five superposed solutions.

Solution XVII
Following [13] it is easy to see that
\[ u(x,t) = \frac{A}{1 + B\text{cn}^2(\xi, m)}, \quad B > 0, \quad \xi = \beta(x - ct), \quad (195) \]
is an exact solution of the nonlocal mKdV Eq. (161) provided
\[ g < 0, \quad 0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \quad c = -[1 + m + 6\sqrt{m}]\beta^2 < 0, \quad |g|A^2 = \frac{4\sqrt{m}\beta^2}{(1 - \sqrt{m})^2}. \quad (196) \]

Note that this solution is not valid for \( m = 1 \), i.e. the nonlocal mKdV Eq. (161) does not admit a corresponding hyperbolic solution. Notice that for this solution \( v < 0 \).
On using the identity (7) one can then rewrite the periodic pulse solution (195) as a superposition of a periodic kink and a periodic antikink solution, i.e.

\[ u(x, t) = \sqrt{\frac{2m}{g}} \beta \left[ \text{sn}(\xi + \Delta, m) - \text{sn}(\xi - \Delta, m) \right], \quad \xi = \beta(x - ct). \tag{197} \]

Here \( \Delta \) is defined by \( \text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity (150).

**Superposed Solution XVIII**

Following [13] it is easy to show that the nonlocal mKdV Eq. (161) admits the periodic kink solution

\[ u(x, t) = \frac{A}{1 + B\text{cn}^2(\xi, m)}, \quad B > 0, \tag{198} \]

provided

\[ g < 0, \quad 0 < m < 1, \quad B = \frac{\sqrt{m}}{1 - \sqrt{m}}, \]

\[ c = [6\sqrt{m} - (1 + m)]\beta^2, \quad |g|A^2 = 4\sqrt{m}\beta^2. \tag{199} \]

Note that this solution does not exist for \( m = 1 \), i.e. the corresponding hyperbolic solution does not exist.

On using the identity (6), the periodic solution (198) can be reexpressed as

\[ u(x, t) = i \sqrt{\frac{m}{g}} \beta \left[ \text{sn}(\xi + \Delta, m) + \text{sn}(\xi - \Delta, m) \right], \quad \xi = \beta(x - ct). \tag{200} \]

Here \( \Delta \) is defined by \( \text{sn}(\sqrt{m}\Delta, 1/m) = \pm m^{1/4} \), where use has been made of the identity (150).

It is worth noting that for both the superposed solutions XVIII and XIX of the nonlocal repulsive mKdV Eq. (161), the value of \( B \) as well as \( g \) are the same.

**Solution XIX**

Following [13] it is easy to show that another periodic solution to the nonlocal mKdV Eq. (161) is

\[ u(x, t) = \frac{A\text{sn}(\xi, m)\text{cn}(\xi, m)}{1 + B\text{cn}^2(\xi, m)}, \tag{201} \]
provided

\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad c = (2 - m - 6\sqrt{1 - m})\beta^2, \]

\[ g < 0, \quad |g|A^2 = \frac{4(1 - \sqrt{1 - m})^2\beta^2}{\sqrt{1 - m}}. \quad (202) \]

On using the identity (11), the periodic solution (201) can be re-expressed as a superposition of the two periodic pulse solutions, i.e.

\[ u(x,t) = \beta \left( \text{dn} \left[ \xi - \frac{K(m)}{2}, m \right] - \text{dn} \left[ \xi + \frac{K(m)}{2}, m \right] \right), \quad \xi = \beta(x - ct). \quad (203) \]

**Solution XX**

Following [13], yet another periodic solution to the nonlocal mKdV Eq. (161) is

\[ u(x,t) = \frac{A \text{dn}(\xi, m)}{1 + B \text{cn}^2(\xi, m)}, \quad (204) \]

provided

\[ 0 < m < 1, \quad B = \frac{1 - \sqrt{1 - m}}{\sqrt{1 - m}}, \quad g > 0, \]

\[ c = [2 - m + 6\sqrt{1 - m}]\beta^2, \quad |g|A^2 = \frac{4}{\sqrt{1 - m}}\beta^2. \quad (205) \]

Thus for this solution \( c > 0 \).

On using the identity (10) the periodic solution (204) can be re-expressed as superposition of two periodic pulse solutions, i.e.

\[ u(x,t) = \beta \left[ \text{dn}(\beta x + K(m)/2, m) + \text{dn}(\beta x - K(m)/2, m) \right]. \quad (206) \]

**Superposed Solution XXI**

Following [5, 13] it is easy to check that the nonlocal mKdV Eq. (161) admits a hyperbolic pulse solution

\[ \psi(x,t) = 1 - \frac{A}{B + \cosh^2(\xi)}, \quad B > 0, \quad \xi = \beta(x - ct), \quad (207) \]

provided

\[ g < 0, \quad \sqrt{|g|}A = 2\sqrt{B(B + 1)}\beta, \quad \beta^2 = \frac{4(B + 1)}{(B + 2)^2} < 1, \quad c = 4\beta^2 - 6. \quad (208) \]
On comparing with the hyperbolic identity (13), the solution (207) can be re-expressed as the superposition of a (hyperbolic) kink and an antikink solution, i.e.

$$u(x, t) = 1 - \frac{\beta}{\sqrt{|g|}} \tanh(\xi + \Delta) - \tanh(\xi - \Delta),$$  \hspace{1cm} (209)

where $\xi = \beta(x - ct)$ while $\sinh(\Delta) = \sqrt{B}$.

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