A note on Hilbert Space Representation of Quantum Mechanics with Minimal Length

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Abstract

We study some fundamental issues related to the Hilbert space representation of quantum mechanics in the presence of a minimal length and maximal momentum. In this framework, the maximally localized states and quasi-position representation introduced by Kempf et al. are reconsidered and modified. We show that all studies in recent years, including [15] and [16] need serious modification in order to be a consistent framework for quantum mechanics in Planck scale.

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1. Introduction

It seems that a natural ultraviolet cutoff, a minimum distance $\ell_{\text{min}}$, is an inevitable prediction of all approaches to quantum gravity proposal [1]-[7]. This is due to powerful gravitational effects on the very structure of spacetime when we aim to resolve very small distances. In the context of the doubly special relativity it is shown that the presence of a minimal measurable length will follow the appearance of a maximum measurable momentum for test particles [8]-[11]. This is in fact in complete agreement with the notion of the uncertainty principle. Therefore, in quantum gravity regime there are a lower bound for position measurements and consequently an upper bound for momentum measurements. Minimal length and maximal momentum modify the Heisenberg Uncertainty Principle (HUP) to the so-called Gravitational/Generalised Uncertainty Principle (GUP) and therefore, a revision of the standard Heisenberg algebra is inevitable. A generalized Heisenberg algebra in the presence of both minimal length and maximal momentum can be formulated as [12]-[14]

$$[x_i, p_j] = i\hbar \left( \delta_{ij} - \alpha(p\delta_{ij} + \frac{p_ip_j}{p}) + \alpha^2(p^2\delta_{ij} + 3p_ip_j) \right)$$  \hspace{1cm} (1)

where $\alpha = \frac{\alpha_0}{M_{Pl}c}$ is the GUP parameter and $M_{Pl} \sim 10^{-8}$ Kg is the Planck mass. Note that dimensionally $[\alpha] = (\text{momentum})^{-1}$, and $\alpha_0$ is a dimensionless quantity. It is normally assumed that $\alpha_0$ is of the order of unity, $\alpha_0 \sim 1$. In this case the $\alpha$-dependent terms are important only for energies near the Planck scale $\sim 10^{19}$ GeV. In one dimension, Eq.(1) up to $O(\alpha^2)$ terms follows the GUP

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 - 2\alpha \langle p \rangle + 4\alpha^2 \langle p^2 \rangle \right]$$

$$\geq \frac{\hbar}{2} \left[ 1 + \left( \frac{\alpha}{\sqrt{\langle p^2 \rangle}} + 4\alpha^2 \right)(\Delta p)^2 + 4\alpha^2 (\langle p \rangle)^2 - 2\alpha \sqrt{\langle p^2 \rangle} \right],$$  \hspace{1cm} (2)

where unlike the ordinary Heisenberg Uncertainty Principle, one can no longer make $\Delta x$ arbitrarily small by letting $\Delta p$ to grow arbitrarily. The GUP obtained in Eq.(2) implies a minimum measurable length and also a maximum measurable momentum as

$$\Delta x_{\text{min}} \approx \alpha_0 \ell_{Pl}$$

$$\Delta p_{\text{max}} \approx \frac{M_{Pl}c}{\alpha_0}$$

where $\ell_{Pl} \sim 10^{-35}$ m is the Planck length. Due to the existence of these two natural cutoffs, several modifications would be appeared in the very basics of the standard quantum mechanics and these modifications lead one to a generalized quantum mechanics in Planck.
scale. These types of studies are phenomenological in essence since there is no completely formulated quantum theory of gravity. Albeit, recently it has been shown in [20] that these natural cutoffs are actually global (topological) properties of compact symplectic manifolds much in the same way as gravity is a global property of curved spacetime. Many formalisms have been proposed in recent years, but despite all these efforts, yet there is no complete framework indicating a concrete picture of Planck scale modified quantum mechanics. One of the most famous approaches which has been the basis of many researches in recent years, is the KMM formalism [15], that is presented by respecting a GUP with just a minimal measurable length. In this streamline, the authors of the present paper have generalized the KMM formalism to a more general case in which the maximal momentum is taken into account too [16]. But, none of these two frameworks have been able thoroughly to provide a proper playground of quantum mechanics in quantum gravity regime. In fact, the main shortcoming of the mentioned two studies is that they cannot recover the standard results in the limit of energies much less than the Planck scale energy, and hence, in the language of the correspondence principle these studies need to be modified severely. Now, in this paper, using a generalized Heisenberg algebra defined as

\[ [x, p] = i\hbar(1 - m\alpha p + n\alpha^2 p^2) \]

and also taking a maximal momentum as \( P_{\text{max}} = \frac{1}{\kappa\alpha} \), we focus on and reconsider the basic results obtained in the Refs.[15] and [16]. Then, by a novel analysis we derive a precise framework for quantum mechanics in extremely high energy regime near the Planck scale. In this direction, the generalized relations for plane wave, Dirac \( \delta \)-function, Fourier transformation, de Broglie equation and Planck relation would be obtained or modified. These are, though very simple, some basic and important achievements leading us to a phenomenologically reliable gravitational quantum mechanics. These results open also some new windows on the issue of special relativity in quantum gravity domain (Planck scale).

2. Minimum Length and Maximal Localization states

As a common feature of all quantum gravity candidate theories, there is a fundamental length of the order of the Planck length in which one cannot probe distances smaller than this natural cutoff. This means that the very notion of localizability should be restricted to a lower bound (of the order of the Planck length) and there is no further localization possible in essence. Hence, we are forced to introduce the maximally localized states [15]-[18] with \( \Delta x_{\text{min}} = \ell_{\text{min}} \) instead of the usual absolute localized states with \( \Delta x_{\text{min}} = 0 \). As has been
mentioned in Ref. [15], due to the presence of a nonzero minimum measurable distance, the ordinary position space representation is no longer applicable in quantum gravity regime. But, there still exists a continuous momentum space representation in which we can explore the physical implications of the minimal length scenario.

We start by defining the operators \( \hat{\mathcal{P}} \) and \( \hat{\mathcal{X}} \) as (see [15] and [16])

\[
\hat{\mathcal{P}} \psi(p) = p \psi(p)
\]
\[
\hat{\mathcal{X}} \psi(p) = (1 - m\alpha p + n\alpha^2 p^2) \hat{x} \psi(p) = (1 - m\alpha p + n\alpha^2 p^2) i\hbar \partial_p \psi(p)
\]

where \( \hat{x} \) and \( \hat{p} \) satisfy the canonical commutation relation \([\hat{x}, \hat{p}] = i\hbar\), and \( \hat{\mathcal{P}} \) and \( \hat{\mathcal{X}} \) satisfy (3). It is shown that, the scalar product in this representation should be modified as

\[
\langle \Psi | \Phi \rangle = \int_{-P_{\text{max}}}^{+P_{\text{max}}} \Psi^*(p) \Phi(p) \frac{dp}{1 - m\alpha p + n\alpha^2 p^2}.
\]

Further, the generalized identity operator and the generalization of the scalar product of momentum eigenstates would be represented as

\[
1 = \int_{-P_{\text{max}}}^{+P_{\text{max}}} |p\rangle \langle p| \frac{dp}{1 - m\alpha p + n\alpha^2 p^2}
\]

and

\[
\langle p | p' \rangle = (1 - m\alpha p + n\alpha^2 p^2) \delta(p - p').
\]

respectively. In order to calculate the states \(|\zeta^{\text{ml}}\rangle\) of maximal localization around a position \(\zeta \geq \ell_{\text{min}}\)

\[
\langle \zeta^{\text{ml}} | \mathcal{X} | \zeta^{\text{ml}} \rangle = \zeta
\]

we can use the positivity of the norm [15]

\[
\| (\mathcal{X} - \langle \mathcal{X} \rangle + \frac{[\mathcal{X}, \mathcal{P}]}{2(\Delta \mathcal{P})^2} (\mathcal{P} - \langle \mathcal{P} \rangle)) |\varphi\rangle \| \geq 0.
\]

Considering (3) and \( P_{\text{max}} = \frac{1}{\kappa\alpha} \), on the boundary of the physically allowed region, we obtain the states of maximal localization \(\varphi^{\text{ml}}_{\zeta}(p)\) as

\[
\varphi^{\text{ml}}_{\zeta}(p) = \mathcal{N} (1 - m\alpha p + n\alpha^2 p^2) - \frac{n\alpha^2}{m} e^{-\frac{1}{\sqrt{4n-m}} \left( \frac{m\alpha(n+1)}{2n} + i\frac{\zeta}{\alpha\hbar}\right) \left( \tan^{-1}\left( \frac{2n\alpha p - m}{\sqrt{4n-m^2}} \right) + \tan^{-1}\left( \frac{m}{\sqrt{4n-m^2}} \right) \right)}.
\]

in which \( \mathcal{N} \) is the normalization factor (one can find the details of computations in [16]). Note that, these states are obtained for \( \langle p \rangle = 0 \) and \( \Delta p = \frac{1}{\kappa\alpha} \) that gives the states of absolutely maximal localization and critical momentum uncertainty (corresponding to the
maximal momentum) respectively. In the language of Dirac notation, $\varphi^ml(\zeta)$ can be written as $\langle p|\zeta^ml \rangle$ which presents the probability amplitude for the particle with the momenta $p$, being maximally localized around the “position” $\zeta$. Thus $\varphi^ml(\zeta)$ (or $\varphi^ml(\zeta)^* = \langle \zeta^ml|p \rangle$) gives the generalized concept of change of basis or the ordinary translation function $\langle p|x \rangle$ (or $\langle x|p \rangle$).

Now the critical point which has been the basis of mistakes in previous studies ([15] and [16]) shows itself: in these studies the authors have used the relation $\langle \zeta^ml|\zeta^ml \rangle = 1$ for normalization of the maximally localized states. Unfortunately, this procedure has led the authors of these papers to a normalization factor that vanishes in the limit of the standard quantum mechanics. As a result, KMM in [15] found a divergent energy for a test particle in the limit of the standard quantum mechanics, which is obviously impossible! This is more or less in the same manner in Ref. [16], though one more step has been taken toward the complete framework. Here, we focus on this issue and present a deeper argument on this issue to see the essence of the problem and its possible solution. For this purpose, we use the completeness of the set of maximally localized eigenbasis $\{\zeta^ml\}$ (for proof of completeness, see the appendix of Ref. [18]). By using the completeness relation in the left hand side of the generalized relation of the scalar product of momentum eigenstates (7), we obtain

$$\int_{-\infty}^{+\infty} \varphi^ml(\zeta) \varphi^ml^*(p') d\zeta = (1 - m\alpha p + na^2p^2) \delta(p - p'). \quad (11)$$

Then, from Eq. (10) it follows that

$$\mathcal{N} \mathcal{N}^* e^{-\frac{ma(n+\kappa^2)}{4n}} \left( \frac{\tan^{-1} \left( \frac{2m\alpha p - m}{\sqrt{4n-m^2}} \right) + \tan^{-1} \left( \frac{2m\alpha p' - m}{\sqrt{4n-m^2}} \right) + 2 \tan^{-1} \left( \frac{m}{\sqrt{4n-m^2}} \right)}{1 - m\alpha p + na^2p^2} \right) \times$$

$$\left( \frac{1}{1 - m\alpha p + na^2p^2} \right) \int_{-\infty}^{+\infty} d\zeta \left( \frac{\tan^{-1} \left( \frac{2m\alpha p - m}{\sqrt{4n-m^2}} \right) - \tan^{-1} \left( \frac{2m\alpha p' - m}{\sqrt{4n-m^2}} \right)}{1 - m\alpha p + na^2p^2} \right) d\zeta = \delta(p - p'). \quad (12)$$

Taking into account the general property of Dirac $\delta$-function $\delta(\Omega(z)) = \frac{1}{\Omega'(z_0)} \delta(z - z_0)$, it would be obtained that

$$\int_{-\infty}^{+\infty} e^{-\frac{i}{\alpha \sqrt{4n-m^2}} \left( \tan^{-1} \left( \frac{2m\alpha p - m}{\sqrt{4n-m^2}} \right) - \tan^{-1} \left( \frac{2m\alpha p' - m}{\sqrt{4n-m^2}} \right) \right)} d\zeta = 2\pi \hbar (1 - m\alpha p' + na^2p^2) \delta(p - p'). \quad (13)$$

By putting this in (12) we obtain the normalization factor as

$$\mathcal{N} = \frac{1}{\sqrt{2\pi \hbar}} \left( \frac{1}{1 - m\alpha p + na^2p^2} \right)^{\frac{\alpha(n+\kappa^2)}{4n}} \left( \frac{\tan^{-1} \left( \frac{2m\alpha p - m}{\sqrt{4n-m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n-m^2}} \right)}{1 - m\alpha p + na^2p^2} \right). \quad (14)$$

Therefore, the momentum space wave functions $\varphi^ml(\zeta)$ which are maximally localized around a position $\zeta$, would be achieved as

$$\varphi^ml(\zeta) = \frac{1}{\sqrt{2\pi \hbar}} e^{-\frac{i}{\alpha \sqrt{4n-m^2}} \left( \tan^{-1} \left( \frac{2m\alpha p - m}{\sqrt{4n-m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n-m^2}} \right) \right)}. \quad (15)$$
This is completely different with the results obtained in previous studies (see [15], Eq. (37) and [16], Eq. (35)). Now, there is a correct limiting result for \( \alpha \to 0 \) in the favor of correspondence principle. Note that the normalization factors in previous studies were vanishing in this limit which cannot be the case based on the correspondence principle. Now equation (15) gives the generalized profile of the plane wave solution as

\[
e^{-\frac{2\sqrt{\zeta - m}}{\alpha \sqrt{4n - m^2}}} \left( \tan^{-1}\left( \frac{2\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1}\left( \frac{m}{\sqrt{4n - m^2}} \right) \right).
\]

One can easily check that, in the limit of \( \alpha \to 0 \) the ordinary plane wave profile and momentum space wave function would be exactly recovered

\[
\lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi \hbar}} e^{-\frac{2\sqrt{\zeta - m}}{\alpha \sqrt{4n - m^2}}} \left( \tan^{-1}\left( \frac{2\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1}\left( \frac{m}{\sqrt{4n - m^2}} \right) \right) \Rightarrow \frac{1}{\sqrt{2\pi \hbar}} e^{-\frac{p^2}{2\hbar}}
\]

or

\[
\lim_{\alpha \to 0} \langle p|\zeta^m \rangle \Rightarrow \langle p|x \rangle.
\]

In comparison with ordinary wave mechanics, the generalized relations of the plane wave and momentum space wave function lead us to a significant outcome; the modified wavenumber in quantum gravity regime \( \mathcal{K}_{QG} \) as

\[
\mathcal{K}_{QG} = \frac{2}{\alpha \hbar \sqrt{4n - m^2}} \left( \tan^{-1}\left( \frac{2\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1}\left( \frac{m}{\sqrt{4n - m^2}} \right) \right).
\]

Therefore, the modified form of the corresponding wavelength would be resulted as

\[
\lambda_{QG} = \frac{\pi \alpha \hbar \sqrt{4n - m^2}}{\tan^{-1}\left( \frac{2\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1}\left( \frac{m}{\sqrt{4n - m^2}} \right)},
\]

So, for massless particles we can infer a generalized frequency as follows

\[
\nu_{QG} = \frac{c}{\pi \alpha \hbar \sqrt{4n - m^2}} \left( \tan^{-1}\left( \frac{2\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1}\left( \frac{m}{\sqrt{4n - m^2}} \right) \right).
\]

Now, by keeping \( \hbar \) as a subatomic-scale constant which describes the relationship between energy and frequency as

\[
\frac{\text{Energy}}{\text{Frequency}} = \hbar
\]

we arrive at the generalized Planck relation in the domain of quantum gravity as \( \mathcal{E}_{QG} = \hbar \nu_{QG} \). The presence of a maximum measurable momentum concludes that there is no wavelength smaller than \( \lambda_{QG}(P_{\text{max}}) \), or equivalently no frequency larger than \( \nu_{QG}(P_{\text{max}}) \). Hence, the highest energy for a massless particle would be

\[
\mathcal{E}_{QG}(P_{\text{max}}) = \hbar \nu_{QG}(P_{\text{max}})
\]

\[
= \frac{2c}{\alpha \sqrt{4n - m^2}} \tan^{-1}\left( \frac{\sqrt{4n - m^2}}{2\kappa - m} \right).
\]
Further, Eqs. (18) and (19) together with the generalized Planck relation lead us to the “generalized de Broglie relation” $\mathcal{P}_{QG}$ as

$$\mathcal{P}_{QG} = \frac{2}{\alpha \sqrt{4n - m^2}} \left( \tan^{-1} \left( \frac{2n\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n - m^2}} \right) \right). \quad (22)$$

We call this quantity as “quasimomentum” in what follows, since it has the dimension of momentum. The results obtained so far contain a crucial point that in extremely high energy regime the role of momentum should be reconsidered essentially in comparison with the standard situation. Indeed, the quantity quasimomentum does not mean just a modified momenta here. It can be interpreted as the modified method of the momentum arrangement in the related equations. As a result, one encounters $\mathcal{P}_{QG}(p)$ instead of $p$ in field equations. Accordingly, one concludes the modified kinetic energy for particles as follows

$$E_{QG}^{\text{kin}}(p) = \frac{[\mathcal{P}_{QG}(p)]^2}{2M} = \frac{2}{\alpha^2 M(4n - m^2)} \left( \tan^{-1} \left( \frac{2n\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n - m^2}} \right) \right)^2. \quad (23)$$

This is a new approach and completely different from the considerations adopted in previously proposed formalisms. This result leads us to a new formulation of energy in quantum gravity regime. Since there is an upper bound for momentum, $P_{\max}$, so the most energetic particles would have the kinetic energy as $E_{QG}^{\text{kin}}(P_{\max})$. It is easy to check that all these relations in the limit of $\alpha \to 0$ recover the corresponding ordinary relations.

For the expectation value of energy $E_{QG}^{\text{kin}}$ in this setup we have

$$\langle \zeta^m_l | [\mathcal{P}_{QG}(p)]^2 | \zeta'^m_l \rangle = \frac{2}{\alpha^2 M(4n - m^2)} \int_{-P_{\max}}^{+P_{\max}} \frac{\left( \tan^{-1} \left( \frac{2n\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n - m^2}} \right) \right)^2 dp}{1 - m\alpha p + n\alpha^2 p^2}$$

$$= \frac{8}{3\alpha^3 M(4n - m^2)^{3/2}} \left( \tan^{-1} \left( \frac{\sqrt{4n - m^2}}{2\kappa - M} \right) \right)^3. \quad (24)$$

which indicates that in contrast to ordinary states, the maximal localization states are proper physical states with finite energy. We note that this relation in the limit of $\alpha \to 0$ goes to infinity and this is not surprising since now there is no restriction on momentum values in the same way as the standard quantum mechanics. The scalar product of the maximally localized states now is given by

$$\langle \zeta^m_l | \zeta'^m_l \rangle = \frac{1}{2\pi \hbar} \int_{-P_{\max}}^{+P_{\max}} e^{iK_{QG}(\zeta - \zeta')} \frac{\left( \tan^{-1} \left( \frac{2n\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n - m^2}} \right) \right)}{1 - m\alpha p + n\alpha^2 p^2} dp. \quad (25)$$

Here, we define the generalized Dirac $\delta$-function in the modified quantum mechanics as

$$\delta(\zeta - \zeta') = \frac{1}{2\pi} \int_{-P_{\max}}^{+P_{\max}} e^{iK_{QG}(\zeta - \zeta')} dK_{QG}. \quad (26)$$
whence we obtain
\[ \langle \zeta^{ml} | \zeta'^{ml} \rangle = \delta (\zeta - \zeta') . \] (27)

So, unlike the previous studies ([15] and [16]), now there is mutual orthogonality of the maximal localization states! Indeed, the maximal localization state \( |\zeta^{ml}\rangle \) and its momentum space counterpart \( \varphi^{ml}_\zeta (p) \), provide a proper background for describing the behavior of particles near the Planck scale. In a similar fashion, we need to change and modify our viewpoint on the very notion of space too. That is to say, we need a modified position space which realizes the existence of a minimum distance in its very structure from the beginning. Actually, in order to work with the maximal localization states, one needs a generalized space that treats the minimal length as an ultimate limit for the resolution of the spacetime points or nearby particles.

3. Quasiposition Space

In ordinary quantum mechanics one has the position and momentum space representations in Hilbert space with position and momentum wave functions given as \( \psi(x) = \langle x | \psi \rangle \) and \( \psi(p) = \langle p | \psi \rangle \) respectively. But, in extreme situations such as the Planck scale, this framework would be drastically disturbed because of the presence of a nonzero minimum measurable length. When there exists a minimal length, it means that there is a nonzero uncertainty in position measurements as
\[ (\Delta x)^2_{|\psi>} = \langle \psi | (X - \langle \psi | X | \psi \rangle)^2 | \psi \rangle \geq \Delta x_{\text{min}} . \] (28)

So, in one hand, we should change the ordinary concept of absolute localizability \( \Delta x_{\text{min}} = 0 \) to the modified concept of maximal localization, i.e. the states that are localized just up to the minimal length \( \ell_{\text{min}} \). On the other hand, we can no longer build a Hilbert space on the usual position wave function, and thus, the ordinary position space has no sense in this respect [16]-[19]. Hence, in extremely high energy regimes we need to reformulate quantum mechanics in a generalized space with minimal length. In order to work with maximally localized states \( |\zeta^{ml}\rangle \), we need a space in which the concept of point or localizability is modified in the presence of a minimal length. In this sense, the quasiposition space introduced by KMM formalism [15] would be the proper representation. In fact, quasiposition space is the modified notion of the ordinary position space which treats the existence of a minimal length in a realistic manner.

Taking \( |\phi\rangle \) as an arbitrary state, we can define \( \langle \zeta^{ml} | \phi \rangle \) as the state’s quasiposition wave function \( \phi(\zeta) \) [15], [16]. That is, \( \phi(\zeta) \) projects the probability amplitude for the particle.
being maximally localized around the position \( \zeta \) in the quasiposition space. So, from Eqs. (6) and (15), the transformation for a state wave function in the momentum representation, \( \phi(p) = \langle p|\phi \rangle \), to its quasiposition wave function is

\[
\phi(\zeta) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-P_{\text{max}}}^{+P_{\text{max}}} e^{\frac{2i}{\hbar\sqrt{4n-m^2}} \left( \tan^{-1}\left(\frac{2m\alpha p-m}{\sqrt{4n-m^2}}\right) + \tan^{-1}\left(\frac{m}{\sqrt{4n-m^2}}\right) \right)} \phi(p) \frac{dp}{1 - m\alpha p + n\alpha^2 p^2}. \tag{29}
\]

This transformation explicitly exhibits the generalization of the Fourier transformation. By inverse Fourier transform, we have the transformation of a quasiposition wave function into a momentum space wave function as

\[
\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-\frac{2i}{\hbar\sqrt{4n-m^2}} \left( \tan^{-1}\left(\frac{2m\alpha p-m}{\sqrt{4n-m^2}}\right) + \tan^{-1}\left(\frac{m}{\sqrt{4n-m^2}}\right) \right)} \phi(\zeta) \, d\zeta. \tag{30}
\]

The remarkable note is that, here unlike the prior formalisms, in the limit of \( \alpha \to 0 \) we exactly recover the corresponding ordinary transformations (this is not the case for KMM framework for instance). Now, using Eqs. (5) and (30), we can calculate the scalar product of two arbitrary states \(|\phi\rangle\) and \(|\psi\rangle\) in terms of the quasiposition wave functions \(\phi(\zeta)\) and \(\psi(\zeta)\) as

\[
\langle \phi|\psi \rangle = \int_{-P_{\text{max}}}^{+P_{\text{max}}} \phi^*(p)\psi(p) \frac{dp}{1 - m\alpha p + n\alpha^2 p^2} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{2i}{\hbar\sqrt{4n-m^2}} \left( \tan^{-1}\left(\frac{2m\alpha p-m}{\sqrt{4n-m^2}}\right) + \tan^{-1}\left(\frac{m}{\sqrt{4n-m^2}}\right) \right)} \phi^*(\zeta)\psi(\zeta') d\zeta d\zeta' \frac{dp}{1 - m\alpha p + n\alpha^2 p^2} \tag{31}
\]

where we have used the generalized Dirac \(\delta\)-function as given by (26).

From Eq. (4) we have the operators \(\hat{P} = \frac{\hbar}{i} \partial_\zeta\) and \(\hat{X} = (1 - m\alpha p + n\alpha^2 p^2) i\hbar \partial_p\). By applying these operators on the generalized plane wave (15) we can derive

\[
\hat{P} \varphi^m_\zeta(p) = \mathcal{P}_{\text{QQ}} \varphi^m_\zeta(p) \\
\hat{X} \varphi^m_\zeta(p) = \zeta \varphi^m_\zeta(p) \tag{32}
\]

Since \(\partial_p \equiv \frac{1}{(1 - m\alpha p + n\alpha^2 p^2)} \partial_{\mathcal{P}_{\text{QQ}}}\), so we can represent operator \(\hat{X}\) as \(i\hbar \partial_{\mathcal{P}_{\text{QQ}}}\) which can be called as ”quasiposition operator”. Therefore, in this generalized framework momentum and quasiposition operators operate as

\[
\hat{P} \phi(\zeta) = \frac{\hbar}{i} \partial_\zeta \phi(\zeta) \\
\hat{X} \phi(\zeta) = \zeta \phi(\zeta) \tag{33}
\]
in quasiposition representation, and also as

\[ \hat{\mathcal{P}} \phi(p) = \mathcal{P}_{QG} \phi(p) \]
\[ \hat{\mathcal{X}} \phi(p) = i\hbar \partial_{\mathcal{P}_{QG}} \phi(p) \]  \hspace{1cm} (34)

in momentum space representation. These are novel achievement in comparison with the corresponding results obtained in [15] and also [16]. Here we obtained the appropriate operators \( \hat{\mathcal{P}} \) and \( \hat{\mathcal{X}} \) which act straightforwardly on the wave functions in momentum and quasiposition representations. As it is obtained in Ref. [16], we see that here there is not noncommutativity in the structure of quasiposition space \([\mathcal{X}_i, \mathcal{X}_j] = 0\). Therefore, quasiposition space now is a proper space in order to study the maximal localization states in Planck scale quantum mechanics.

4. Implications for Special Relativity

Now by having the results obtained in previous sections in hand, we focus on the possible implications of these results on special relativity. In special relativity one has

\[ E = \sqrt{p^2c^2 + E_0^2}. \]

For massless particles, \( M_0 = 0 \), and therefore \( E = pc \). In our case the generalized de Broglie relation leads to the following relation for the generalized relativistic energy of photons and other massless particles

\[ \mathcal{E}_{QG}^{Rel} = \mathcal{P}_{QG} c \]
\[ = \frac{2c}{\alpha \sqrt{4n - m^2}} \left( \tan^{-1} \left( \frac{2n\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n - m^2}} \right) \right), \]  \hspace{1cm} (35)

where recovers the standard relation \( E = pc \) in the limit of \( \alpha \rightarrow 0 \). For massive particles we have

\[ \mathcal{E}_{QG}^{Rel} = \sqrt{\mathcal{P}_{QG}^2 c^2 + E_0^2} \]
\[ = \sqrt{\frac{4c^2}{\alpha^2(4n - m^2)} \left( \tan^{-1} \left( \frac{2n\alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n - m^2}} \right) \right)^2 + E_0^2}, \]  \hspace{1cm} (36)

where

\[ \lim_{\alpha \rightarrow 0} \mathcal{E}_{QG}^{Rel} = \sqrt{p^2c^2 + E_0^2}. \]
in the favor of the correspondence principle. To proceed further, we look at the modification of the Lorentz factor \( \gamma(v) = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \) in our framework. From the relativistic momentum equation \( p(v) = \frac{M_0 v}{\sqrt{1-(\frac{v}{c})^2}} \), one can obtain easily

\[ \gamma(p) = \sqrt{1 + \left( \frac{p}{M_0 c} \right)^2}. \]  

(37)

Then, putting the quasimomentum in this relation we deduce the generalized Lorentz factor in our framework as follows

\[ \gamma_{QG}(p) = \sqrt{1 + \frac{4}{M_0^2 c^2 \alpha^2 (4n - m^2)} \left( \tan^{-1} \left( \frac{2n \alpha p - m}{\sqrt{4n - m^2}} \right) + \tan^{-1} \left( \frac{m}{\sqrt{4n - m^2}} \right) \right)^2}. \]  

(38)

One could reach this generalized form of the Lorentz factor from the energy-momentum relation \( E = \gamma(p) E_0 \) too. As the most essential factor in all relativistic formulae, this generalized factor would modify the basic relations of special relativity. Thus, in extremely high energy regimes one has generalized relativistic equations such as the time dilation as \( T_{QG} = \gamma_{QG}(p) T_0 \) and length contraction as \( L_{QG} = \frac{L_0}{\gamma_{QG}(p)} \). While in the standard special relativity for velocities near the speed of light one has

\[ \lim_{v \to c} \gamma(v) = \lim_{p \to \infty} \gamma(p) = \infty, \]  

(39)

in the generalized framework presented here, due to the presence of a maximal momentum, one has

\[ \lim_{p \to P_{\text{max}}} \gamma_{QG}(p) = \sqrt{1 + \frac{4}{M_0^2 c^2 \alpha^2 (4n - m^2)} \left( \tan^{-1} \left( \frac{\sqrt{4n - m^2}}{2 \kappa - M} \right) \right)^2}. \]  

(40)

In the energies much less than the Planck scale, \( \alpha \to 0 \), the ordinary special relativity would be exactly recovered.

5. Summary

In this note we have shown that there is a serious flaw in the renowned paper [15] and then we have provided a strategy to overcome this flaw. In this framework the generalized relations for plane wave profile, Dirac \( \delta \)-function, Fourier transformation, the de Broglie equation and the Planck relation are obtained or modified. We have also derived some new and important relations for special relativity in quantum gravity domain.

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