SOME NEW SYMMETRIC HADAMARD MATRICES

DRAGOMIR Ž. ĐOKOVIĆ

Abstract. The first examples of symmetric Hadamard matrices of orders $4 \cdot 127$ and $4 \cdot 191$ are presented. The systematic computer search for symmetric Hadamard matrices based on the so called propus array was carried out recently for all orders $4v$ with odd $v \leq 53$. Hypothetically, such matrices exist for all odd $v$ and all propus parameter sets $(v; k_1, k_2, k_3, k_4; \lambda)$, $k_2 = k_3$, $\sum k_i = \lambda + v$, apart from the exceptional cases when all the $k_i$ are equal. In this note the search has been extended further to cover the cases $v = 55, 57, 59, 61, 63$.

1. Introduction

We fix some notation which will be used throughout this note. Let $X_i$, $i = 1, 2, 3, 4$, be a difference family (DF) in a finite abelian group $G$ (written additively) and let

$$(v; k_1, k_2, k_3, k_4; \lambda)$$

be its parameter set (PS). Thus $v = |G|$, $|X_i| = k_i$ and $\sum k_i(k_i - 1) = \lambda(v - 1)$, where $|X|$ denotes the cardinality of a finite set $X$. If $\sum k_i = \lambda + v$ we say that this PS is a Goethals-Seidel parameter set (GSPS) and that this DF is a Goethals-Seidel difference family (GSDF). If the $X_i$ form a GSDF and we replace one of the blocks by its set-theoretic complement in $G$, we again obtain a GSDF although the parameter set may change. For that reason we shall always assume that all the $k_i$ are $\leq v/2$.

Each GSDF in $G$ gives a Hadamard matrix $H$ of order $4v$. For more details about this construction see e.g. [7] [10]. Briefly, each $X_i$ provides a $G$-invariant matrix $A_i$ of order $v$, and $H$ is obtained by plugging the $A_i$ into the well known Goethals-Seidel array

$$GSA = \begin{bmatrix}
A_1 & A_2R & A_3R & A_4R \\
-A_2R & A_1 & -RA_4 & RA_3 \\
-A_3R & RA_4 & A_1 & -RA_2 \\
-A_4R & -RA_3 & RA_2 & A_1
\end{bmatrix}. $$

We recall that a matrix $A = (a_{x,y})$ with indices $x, y \in G$ is $G$-invariant if $a_{x+y,z} = a_{x,y}$ for all $x, y, z \in G$. The matrix $R = (r_{x,y})$ may be defined by the formula $r_{x,y} = \delta_{x+y,0}$, $x, y \in G$, where $\delta$ is the Kronecker symbol.

For a subset $X$ of $G$, we say that it is symmetric if $-X = X$, and we say that it is skew if $G$ is a disjoint union of $X$, $-X$ and $\{0\}$. If at least one of the blocks $X_i$ of a GSDF is skew then, after rearranging the $X_i$ so to have $X_1$ skew, the Hadamard matrix $H$ will be skew Hadamard, i.e. such that $H + H^T = 2I_{4v}$. (T denotes the matrix transposition, and $I_k$ is the identity matrix of order $k$.)

In order to obtain a symmetric Hadamard matrix $H$ we require that two of the blocks $X_i$ are the same and that one of the other two blocks is symmetric. A propus
parameter set (PPS) is a GSPS having \( k_i = k_j \) for some \( i \neq j \). By permuting the \( k_i \)'s we may assume that \( k_2 = k_3 \) and \( k_1 \geq k_4 \). In that case we say that this PPS is normalized. Note that these conditions in general do not specify the \( k_i \)'s uniquely. For instance the PPSs \((5; 1, 2, 2, 1; 1)\) and \((5; 2, 1, 1, 2; 1)\) are both normalized but they become the same if we ignore the ordering of the \( k_i \)'s.

We say that a GSDF is a propus difference family (PDF) if \( X_i = X_j \) for some \( i \neq j \) and one of the other two blocks is symmetric. If the \( X_i \)'s form a PDF then, after rearranging the blocks we may assume that \( X_2 = X_3 \) and that \( X_1 \) is symmetric. Then we plug the corresponding matrix blocks \( A_i \) into the so called propus array (PA) to construct a symmetric Hadamard matrix of order \( 4v \). This construction is known as the propus construction. It has been first introduced in [9]. For the reader’s convenience we display the propus array

\[
PA = \begin{bmatrix}
-A_1 & A_2R & A_3R & A_4R \\
A_3R & RA_4 & A_1 & -RA_2 \\
A_4R & A_1 & -RA_4 & RA_3 \\
-A_4 & RA_3 & RA_2 & A_1
\end{bmatrix}.
\]

Note that PA is obtained from GSA by multiplying the first column by \(-1\) and interchanging the second and the third rows.

From now on we assume that \( G = \mathbb{Z}_v \), a cyclic group of order \( v \), and that \( v \) is odd. Under this assumption, the matrix blocks \( A_i \) will be circulants. All PPSs for \( v \leq 41 \) are listed in [2] together with the corresponding PDF’s. There was only one case of a PPS having no PDF, namely \((25; 10, 10, 10; 15)\). Similarly, the cases \( 41 < v \leq 51 \) were handled in [4], and the case \( v = 53 \) in [1]. Again there was one exceptional case, \((49; 21, 21, 21, 21; 35)\). In the present note, for each PPS with \( 51 < v \leq 63 \) we exhibit at least one PDF. For more information on the exceptional cases see [4].

2. Symmetric Hadamard matrices of orders 508 and 764

The symmetry symbol \((abc)\) written immediately after a PPS shows the symmetry types of the three blocks \( X_1 \), \( X_2 \) and \( X_4 \). More precisely, the letter \( s \) means that we require that the corresponding block be symmetric, the letter \( k \) is used if we require that block to be skew, and the symbol \( * \) is used otherwise. In particular \( a = s \) means that we require \( X_1 \) to be symmetric, \( a = k \) means that we require \( X_1 \) to be skew, and \( a = * \) means that no symmetry condition is imposed on \( X_1 \).

The group of units \( \mathbb{Z}_v^* \) acts on \( \mathbb{Z}_v \) by multiplication. It may happen that there is a nontrivial subgroup \( H \) of \( \mathbb{Z}_v^* \) such that some block \( X_i \) of a PDF is a union of orbits of \( H \). In such case we may specify \( X_i \) by writing it as \( HY_i \), where \( Y_i \) is a set of representatives of the \( H \)-orbits contained in \( X_i \).

For \( v = 127 \) we give five nonequivalent PDFs and for \( v = 191 \) only one.
\[ v = 127, \ H = \{1, 19, 107\} \]
\[(127; 57, 61, 61, 55; 107) (**s)\]
\[X_1 = H\{4, 5, 6, 9, 12, 15, 23, 24, 30, 33, 36, 39, 45, 52, 58, 59, 60, 64, 66\}\]
\[X_2 = H\{0, 4, 5, 6, 13, 15, 17, 26, 30, 32, 40, 46, 51, 53, 58, 59, 60, 64, 65, 66, 72\}\]
\[X_4 = H\{0, 2, 4, 8, 9, 12, 15, 23, 24, 26, 30, 33, 40, 46, 51, 52, 53, 65, 71\}\]

\[(127; 60, 60, 60, 54; 107) (**s)\]
\[X_1 = H\{1, 5, 6, 11, 13, 15, 16, 17, 20, 23, 24, 29, 32, 45, 46, 52, 58, 66, 71, 72\}\]
\[X_2 = H\{2, 4, 5, 11, 12, 15, 16, 18, 22, 23, 29, 33, 36, 39, 46, 51, 52, 53, 60, 71\}\]
\[X_4 = H\{6, 8, 17, 20, 22, 23, 30, 33, 36, 39, 45, 51, 58, 59, 60, 64, 66, 71\}\]

\[(127; 60, 60, 60, 54; 107) (**s)\]
\[X_1 = H\{1, 2, 3, 4, 5, 6, 9, 11, 12, 13, 15, 17, 23, 24, 32, 33, 39, 46, 64, 65\}\]
\[X_2 = H\{2, 5, 9, 10, 12, 13, 15, 16, 17, 29, 33, 36, 39, 40, 45, 51, 53, 58, 60, 66\}\]
\[X_4 = H\{1, 5, 6, 10, 11, 13, 16, 17, 20, 23, 30, 32, 45, 58, 64, 65, 66, 71\}\]

\[(127; 60, 60, 60, 55; 106) (**s)\]
\[X_1 = H\{0, 2, 3, 4, 5, 12, 13, 16, 17, 18, 20, 22, 29, 30, 46, 51, 53, 58, 59, 71\}\]
\[X_2 = H\{8, 9, 10, 16, 20, 22, 23, 24, 26, 29, 32, 36, 45, 46, 51, 52, 59, 60, 65, 78\}\]
\[X_4 = H\{0, 3, 5, 10, 11, 17, 18, 22, 24, 29, 32, 39, 45, 52, 58, 59, 60, 64, 72\}\]

\[(127; 60, 57, 57, 58; 105) (**s)\]
\[X_1 = H\{2, 4, 6, 10, 12, 13, 15, 17, 23, 24, 26, 36, 40, 46, 51, 52, 58, 64, 71, 78\}\]
\[X_2 = H\{1, 2, 3, 4, 10, 16, 17, 18, 20, 23, 29, 30, 45, 51, 52, 58, 64, 66, 72\}\]
\[X_4 = H\{0, 2, 5, 8, 9, 10, 11, 13, 15, 17, 18, 30, 39, 40, 46, 53, 58, 60, 66, 78\}\]

\[v = 191, \ H = \{1, 39, 49, 109, 184\} \]
\[(191; 91, 90, 90, 85; 165) (**s)\]
\[X_1 = H\{0, 1, 3, 4, 7, 9, 16, 17, 18, 21, 22, 28, 31, 36, 57, 61, 62, 68, 112\}\]
\[X_2 = H\{1, 4, 14, 16, 18, 19, 22, 23, 28, 29, 31, 32, 34, 36, 38, 61, 62, 68\}\]
\[X_4 = H\{1, 2, 9, 11, 12, 17, 18, 22, 28, 29, 31, 32, 38, 41, 56, 61, 66\}\]

3. Small orders of symmetric Hadamard matrices

There are several known infinite series of PDFs \[6\ [9].\ We shall use only two of them. The first one is essentially the Turyn series \[11\] with \(v = (q + 1)/2, q\) a prime power \(\equiv 1 \pmod 4\), and all four blocks \(X_i\) symmetric. The second one is essentially the series constructed in \[13\] (see also \[6\]) to which we refer as the
XXSW-series. In this case \( v = (q + 1)/4 \), \( q \) a prime power \( \equiv 3 \) (mod 4), and we may arrange the blocks so that \( X_1 \) is skew, \( X_2 = X_3 \) and \( X_4 \) is symmetric.

In the handbook \cite{3} published in 2007 it is indicated (see Table 1.52, p.277) that, for odd \( v < 200 \), no symmetric Hadamard matrices of order \( 4v \) are known for

\[
v = 23, 29, 39, 43, 47, 59, 65, 67, 73, 81, 89, 93, 101, 103, 107, 119, 127, 133, 149, 151, 153, 163, 167, 179, 183, 189, 191, 193.
\]

The cases \( v = 23 \) and \( v = 81 \) should not have been included. For the case \( v = 23 \) see \cite{4}. For \( v = 81 \) note that symmetric Hadamard matrices of orders \( 4 \cdot 9^k \), \( k \geq 1 \) integer, were constructed by Turya \cite{12} back in 1984. Moreover, the Bush-type Hadamard matrix of order \( 4 \cdot 81 = 324 \) constructed in 2001 \cite{8} is also symmetric.

The posn construction has been used in several recent papers \cite{1} \cite{2} \cite{3} \cite{4} \cite{6} \cite{9} to construct symmetric Hadamard matrices of new orders. By taking into account these results and those from Sect. \cite{2} the above list of undecided cases reduces to

\[
v = 65, 89, 93, 101, 107, 119, 133, 149, 153, 163, 167, 179, 183, 189, 193.
\]

4. List of PPSs and PDFs for odd \( v \), \( 53 < v \leq 63 \)

The following conventions and notation will be used in the listings below. We have \( Z_v = \{0, 1, 2, ..., v - 1\} \) and recall that \( v \) is odd. Let \( X \subseteq Z_v \) and \( k = |X| \). Define \( X' = X \cap \{1, 2, ..., (v - 1)/2\} \). In particular \( Z_v' = \{1, 2, ..., (v - 1)/2\} \).

If \( X \) is skew then \( k = (v - 1)/2 \) and

\[
X = X' \cup (Z_v' \setminus X').
\]

If \( X \) is symmetric then

\[
X = \begin{cases} 
X' \cup (-X'), & \text{for } k \text{ even;} \\
\{0\} \cup X' \cup (-X'), & \text{for } k \text{ odd.}
\end{cases}
\]

Hence, a skew \( X \) can be recovered uniquely from \( X' \). This is also true for symmetric \( X \) provided we know the parity of \( k \).

For a PDF \( X_i \), \( i = 1, 2, 3, 4 \), with normalized PPS \( (v; k_1, k_2, k_3, k_4; \lambda) \) we always assume that \( X_2 = X_3 \). Thus it suffices to specify only the blocks \( X_1, X_2 \) and \( X_4 \). We say that a PPS is exceptional if all the \( k_i \) are equal. The following conjecture is implicit in \cite{1} \cite{2} \cite{3}. It has been verified there for odd \( v \leq 53 \).

**Conjecture 1.** For each normalized and non-exceptional PPS \( (v; k_1, k_2, k_3, k_4; \lambda) \) there exist PDFs with symmetry symbols \( (***) \) and \( (**s) \).

The list below shows that the conjecture is true also for \( v = 55, 57, 59, 61, 63 \).

If a block \( X_i \) is symmetric or skew, in order to save space we record only \( X_i' \). As the \( k_i \) are specified by the PPS, \( X_i \) can be recovered uniquely from \( X_i' \).

**Example 1.** For the first PDF below, the symmetry symbol \( (***) \) shows that \( X_1 \) must be symmetric. As \( X_1' = \{5, 6, 7, 9, 10, 13, 15, 16, 19, 21, 23, 25, 27\} \) we have \( -X_1' = \{28, 30, 32, 34, 36, 39, 40, 42, 45, 46, 48, 49, 50\} \). As \( k_1 = 27 \) is odd we have

\[
X_1 = \{0\} \cup X_1' \cup (-X_1').
\]
\[ v = 55 \]
\[ (55; 27, 25, 25, 21; 43) (*** \]
\[ X'_1 = \{5, 6, 7, 9, 10, 13, 15, 16, 19, 21, 23, 25, 27\} \]
\[ X_2 = \{0, 1, 2, 3, 4, 6, 9, 10, 14, 17, 19, 24, 26, 29, 30, 34, 37, 38, 39, 40, 41, 47, 48, 52, 53\} \]
\[ X_4 = \{0, 3, 4, 5, 10, 11, 12, 14, 16, 17, 18, 19, 21, 22, 24, 30, 36, 43, 46, 47\} \]

\[ (55; 27, 25, 25, 21; 43) (**s \]
\[ X_1 = \{0, 4, 5, 6, 7, 9, 10, 12, 15, 20, 21, 24, 25, 26, 28, 32, 33, 34, 38, 39, 41, 44, 45, 51, 52, 53, 54\} \]
\[ X_2 = \{0, 3, 5, 7, 8, 9, 10, 19, 23, 25, 28, 31, 32, 33, 34, 36, 37, 40, 41, 43, 44, 45, 47, 48, 53\} \]
\[ X_4 = \{2, 4, 6, 9, 12, 13, 18, 19, 20, 23\} \]

\[ (55; 27, 24, 24, 22; 42) (*** \]
\[ X'_1 = \{1, 4, 6, 7, 9, 11, 14, 15, 18, 20, 23, 24, 27\} \]
\[ X_2 = \{0, 4, 6, 7, 12, 14, 15, 16, 18, 23, 25, 26, 30, 31, 33, 34, 35, 36, 37, 40, 41, 46, 47, 53\} \]
\[ X_4 = \{0, 1, 4, 6, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27, 29, 31, 33, 36, 42, 43, 44, 44\} \]

\[ (55; 27, 24, 24, 22; 42) (**s \]
\[ X_1 = \{0, 1, 5, 8, 10, 12, 15, 16, 17, 24, 25, 26, 29, 30, 34, 37, 39, 40, 41, 42, 44, 47, 48, 50, 52, 53, 54\} \]
\[ X_2 = \{0, 10, 13, 14, 15, 16, 17, 20, 21, 22, 24, 26, 30, 31, 33, 34, 35, 41, 43, 44, 47, 49, 50, 53\} \]
\[ X_4 = \{1, 6, 11, 12, 13, 16, 19, 20, 22, 24, 27\} \]

\[ (55; 26, 23, 23, 24; 41) (*** \]
\[ X'_1 = \{1, 2, 5, 8, 10, 11, 13, 14, 15, 19, 21, 23, 27\} \]
\[ X_2 = \{0, 2, 3, 4, 5, 6, 7, 10, 11, 14, 18, 24, 25, 30, 35, 37, 39, 40, 41, 42, 50, 51, 52\} \]
\[ X_4 = \{0, 2, 5, 6, 8, 14, 16, 19, 20, 21, 22, 24, 25, 28, 31, 32, 33, 37, 38, 40, 45, 47, 49, 52\} \]

\[ (55; 26, 23, 23, 24; 41) (**s \]
\[ X_1 = \{0, 2, 7, 11, 14, 15, 17, 18, 23, 24, 25, 28, 29, 30, 31, 36, 37, 38, 39, 42, 44, 45, 46, 48, 50, 53\} \]
\[ X_2 = \{0, 3, 8, 12, 13, 14, 18, 19, 23, 26, 33, 34, 36, 38, 42, 45, 46, 47, 48, 49, 50, 51, 52\} \]
\[ X_4 = \{1, 2, 7, 10, 11, 13, 17, 19, 21, 24, 26, 27\} \]
(55; 24, 27, 27, 21; 44) (s**)

\[ X'_1 = \{6, 8, 10, 13, 15, 16, 18, 19, 20, 22, 25, 26\} \]
\[ X_2 = \{0, 2, 4, 5, 8, 13, 14, 16, 17, 19, 25, 26, 27, 32, 33, 34, 37, 38, 39, 40, 41, 42, 44, 49, 50, 53, 54\} \]
\[ X_4 = \{0, 4, 5, 8, 11, 12, 13, 16, 18, 20, 22, 24, 31, 33, 34, 36, 37, 41, 42, 44, 51\} \] 

(55; 24, 27, 27, 21; 44) (**s)

\[ X_1 = \{0, 3, 11, 12, 13, 14, 16, 17, 23, 24, 26, 27, 29, 30, 35, 38, 39, 40, 41, 42, 47, 48, 49, 50\} \]
\[ X_2 = \{0, 1, 3, 4, 5, 6, 8, 9, 14, 16, 18, 21, 22, 27, 28, 32, 35, 36, 37, 39, 43, 47, 49, 51, 52, 53, 54\} \]
\[ X'_4 = \{3, 5, 9, 10, 14, 16, 17, 20, 23, 25\} \] 

(55; 24, 25, 25, 22; 41) (s**)

\[ X'_1 = \{1, 3, 5, 6, 7, 11, 14, 15, 16, 18, 21, 25\} \]
\[ X_2 = \{0, 9, 11, 12, 14, 17, 20, 22, 23, 24, 25, 26, 27, 30, 31, 33, 37, 38, 42, 46, 47, 48, 49, 52, 54\} \]
\[ X_4 = \{0, 4, 5, 7, 8, 11, 12, 16, 18, 19, 21, 24, 25, 26, 28, 34, 35, 39, 41, 43, 53, 54\} \] 

(55; 24, 25, 25, 22; 41) (**s)

\[ X_1 = \{0, 2, 3, 5, 13, 14, 16, 17, 21, 22, 23, 26, 29, 32, 37, 38, 42, 43, 44, 45, 46, 48, 49, 51\} \]
\[ X_2 = \{0, 1, 2, 3, 4, 10, 11, 15, 17, 18, 19, 21, 22, 24, 25, 27, 28, 30, 32, 34, 35, 39, 40, 44, 46\} \]
\[ X'_4 = \{2, 3, 5, 7, 10, 11, 15, 21, 23, 25, 26\} \] 

(55; 23, 26, 26, 22; 42) (sss), Turyn series

\[ X'_1 = \{6, 7, 10, 11, 15, 17, 18, 19, 21, 24, 26\} \]
\[ X'_2 = \{1, 2, 4, 8, 14, 16, 17, 18, 19, 23, 24, 25, 27\} \]
\[ X'_4 = \{6, 7, 10, 11, 15, 17, 18, 19, 21, 24, 26\} \]

\[ v = 57, \ H = \{1, 7, 49\} \]

(57; 28, 28, 28, 21; 48) (s**)

\[ X'_1 = \{1, 2, 6, 8, 10, 11, 14, 16, 17, 18, 21, 22, 25, 27\} \]
\[ X_2 = \{0, 2, 3, 5, 6, 7, 11, 12, 13, 15, 18, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 39, 40, 42, 46, 49, 50, 55\} \]
\[ X_4 = \{0, 1, 2, 3, 5, 6, 8, 9, 12, 13, 14, 15, 19, 23, 30, 31, 32, 39, 45, 48, 51\} \]
(57; 28, 28, 28, 21; 48) (k*s), XXSW − series

\[ X'_1 = \{2, 4, 12, 13, 15, 21, 23, 24, 25, 27, 28\} \]
\[ X_2 = \{1, 3, 5, 8, 9, 12, 15, 20, 23, 24, 26, 27, 29, 31, 32, 33, 35, 36, 37, 41, 42, 45, 49, 50, 51, 52, 53, 55\} \]
\[ X'_4 = \{1, 4, 6, 13, 14, 15, 19, 20, 21, 26\} \]

(57; 27, 26, 26, 22; 44) (s**), all \( X_i \) are \( H \) − invariant

\[ X'_1 = \{3, 6, 9, 10, 11, 13, 15, 19, 20, 21, 23, 24, 26\} \]
\[ X_2 = \{4, 6, 8, 9, 15, 16, 19, 23, 25, 28, 30, 31, 37, 38, 39, 42, 43, 44, 45, 46, 47, 48, 50, 51, 55, 56\} \]
\[ X_4 = \{2, 4, 6, 9, 10, 11, 13, 14, 20, 22, 25, 26, 28, 30, 34, 38, 39, 40, 41, 42, 45, 52\} \]

(57; 27, 26, 26, 22; 44) (**s)

\[ X_1 = \{0, 8, 9, 10, 12, 13, 15, 18, 19, 20, 25, 30, 33, 34, 37, 40, 41, 43, 47, 48, 49, 51, 52, 53, 54, 55, 56\} \]
\[ X_2 = \{0, 4, 8, 9, 11, 16, 17, 18, 22, 24, 25, 26, 27, 28, 30, 31, 33, 34, 37, 38, 42, 43, 51, 53, 54, 55\} \]
\[ X'_4 = \{2, 4, 7, 9, 11, 12, 14, 17, 21, 23, 28\} \]

(57; 27, 25, 25, 23; 43) (s**), all \( X_i \) are \( H \) − invariant

\[ X'_1 = \{3, 6, 9, 10, 11, 13, 15, 19, 20, 21, 23, 24, 26\} \]
\[ X_2 = \{2, 3, 4, 14, 16, 21, 22, 24, 25, 28, 29, 30, 32, 33, 36, 38, 39, 40, 41, 43, 45, 49, 52, 53, 54, 55\} \]
\[ X_4 = \{1, 2, 5, 7, 10, 13, 14, 17, 19, 24, 29, 30, 32, 34, 35, 36, 38, 39, 41, 45, 49, 53, 54\} \]

(57; 27, 25, 25, 23; 43) (**s)

\[ X_1 = \{0, 1, 7, 11, 15, 16, 17, 19, 20, 21, 24, 25, 26, 28, 30, 35, 36, 37, 38, 40, 44, 46, 47, 48, 49, 51, 54\} \]
\[ X_2 = \{0, 1, 2, 3, 7, 10, 14, 17, 19, 20, 23, 26, 31, 32, 34, 36, 37, 38, 41, 42, 43, 44, 45, 46, 49\} \]
\[ X'_4 = \{2, 4, 6, 9, 10, 11, 15, 16, 18, 23, 26\} \]

(57; 25, 25, 25, 24; 42) (sss), Turyn series

\[ X'_1 = \{2, 3, 8, 9, 10, 18, 20, 22, 23, 24, 26, 27\} \]
\[ X'_2 = \{6, 7, 9, 10, 14, 16, 19, 21, 24, 25, 27, 28\} \]
\[ X'_4 = \{2, 3, 8, 9, 10, 18, 20, 22, 23, 24, 26, 27\} \]
The third and the sixth PDF below are taken from [1].

\[ v = 59 \]
\[ (59; 28, 29, 22, 49) \quad (***) \]
\[ X'_1 = \{1, 3, 5, 8, 10, 11, 13, 15, 16, 20, 21, 22, 26, 29\} \]
\[ X_2 = \{0, 1, 3, 7, 8, 9, 10, 12, 13, 15, 16, 19, 21, 22, 25, 27, 29, 34, 35, 36, 37, 38, 39, 40, 44, 51, 54, 55, 58\} \]
\[ X_4 = \{0, 3, 4, 5, 7, 8, 14, 15, 16, 17, 19, 22, 24, 27, 28, 33, 39, 49, 53, 54, 55, 56\} \]

\[ (59; 28, 29, 22, 49) \quad (***) \]
\[ X_1 = \{0, 5, 6, 7, 8, 9, 10, 12, 17, 18, 20, 25, 26, 27, 34, 35, 39, 42, 44, 45, 47, 48, 49, 50, 51, 54, 55, 58\} \]
\[ X_2 = \{0, 1, 2, 3, 4, 5, 6, 9, 11, 13, 14, 18, 20, 24, 25, 26, 29, 31, 32, 35, 37, 44, 45, 47, 48, 51, 55, 56, 58\} \]
\[ X'_4 = \{1, 5, 7, 11, 15, 16, 18, 20, 21, 23, 29\} \]

\[ (59; 27, 25, 26, 44) \quad (***) \]
\[ X'_1 = \{2, 4, 7, 8, 12, 13, 15, 16, 17, 18, 20, 23, 29\} \]
\[ X_2 = \{1, 2, 4, 5, 12, 13, 17, 19, 20, 21, 22, 23, 26, 27, 31, 35, 37, 38, 40, 44, 47, 49, 50, 55, 57\} \]
\[ X_4 = \{3, 7, 12, 13, 14, 16, 18, 19, 20, 22, 23, 24, 25, 26, 31, 32, 33, 34, 36, 38, 43, 45, 46, 50, 51, 53\} \]

\[ (59; 27, 25, 26, 44) \quad (***) \]
\[ X_1 = \{0, 1, 3, 5, 6, 7, 8, 9, 12, 15, 18, 20, 28, 29, 31, 33, 34, 35, 38, 42, 44, 47, 48, 49, 55, 56, 58\} \]
\[ X_2 = \{0, 3, 4, 5, 7, 10, 16, 21, 22, 24, 25, 26, 28, 29, 32, 33, 34, 38, 39, 40, 41, 43, 48, 49, 52\} \]
\[ X'_4 = \{1, 3, 5, 7, 10, 12, 13, 15, 18, 19, 20, 27, 28\} \]

\[ (59; 26, 28, 23, 46) \quad (***) \]
\[ X'_1 = \{4, 6, 10, 12, 13, 15, 17, 21, 22, 24, 25, 27, 29\} \]
\[ X_2 = \{0, 1, 4, 5, 6, 7, 10, 11, 12, 13, 14, 17, 21, 22, 23, 27, 33, 34, 36, 37, 39, 41, 42, 45, 52, 55, 56, 57\} \]
\[ X_4 = \{0, 7, 9, 12, 13, 14, 22, 25, 26, 27, 28, 31, 33, 35, 39, 40, 46, 47, 49, 51, 55, 57, 58\} \]
(59; 26, 28, 28, 23; 46) (***)

\[ \begin{array}{l}
X_1 = \{2, 3, 10, 12, 13, 14, 16, 18, 19, 26, 28, 29, 36, 38, 39, 40, 42, 44, 46, 47, 49, \\
50, 53, 54, 55, 57\}
X_2 = \{4, 5, 7, 11, 12, 16, 17, 24, 25, 26, 27, 28, 29, 33, 34, 37, 39, 40, 42, 43, 44, 45, \\
47, 49, 51, 53, 56, 58\}
X_4 = \{1, 4, 5, 7, 8, 11, 14, 20, 25, 28, 29\}
\end{array} \]

\[ v = 61, \ H_1 = \{1, 13, 47\}, \ H_2 = \{1, 9, 20, 34, 58\} \]

(61; 30, 29, 29, 23; 50) (***)

\[ \begin{array}{l}
X_1' = \{2, 3, 8, 10, 11, 13, 14, 16, 18, 20, 22, 23, 24, 28, 30\}
X_2' = \{0, 1, 2, 4, 9, 10, 11, 13, 18, 19, 22, 24, 26, 27, 37, 38, 39, 40, 42, 43, 44, 48, 49, \\
51, 52, 55, 56, 58, 59\}
X_4' = \{0, 6, 7, 8, 11, 12, 14, 16, 17, 23, 26, 30, 31, 33, 34, 35, 37, 38, 40, 43, 45, 49, 50\}
\end{array} \]

(61; 30, 29, 29, 23; 50) (***)

\[ \begin{array}{l}
X_1 = \{1, 2, 5, 6, 7, 8, 11, 12, 14, 15, 19, 20, 21, 22, 24, 27, 28, 30, 31, 32, 38, 39, 40, \\
41, 43, 45, 46, 55, 58, 60\}
X_2 = \{0, 2, 4, 8, 9, 10, 13, 14, 17, 18, 20, 23, 24, 25, 26, 27, 32, 37, 38, 44, 45, 47, \\
49, 53, 55, 56, 58, 59, 60\}
X_4 = \{0, 3, 5, 6, 21, 23, 24, 26, 27, 30, 32, 33, 39, 41, 43, 44, 45, 46, 48, 50, 51, 52, \\
53, 54, 59, 60\}
\end{array} \]

(61; 30, 26, 26, 26; 47) (**)  all \(X_i\) are \(H_2\) - invariant

\[ \begin{array}{l}
X_1' = \{1, 3, 4, 5, 9, 12, 13, 14, 15, 16, 19, 20, 22, 25, 27\}
X_2' = \{0, 1, 6, 8, 9, 11, 12, 20, 21, 25, 26, 28, 30, 32, 34, 37, 38, 42, 43, 44, 47, 51, \\
54, 57, 58, 59\}
X_4' = \{0, 3, 5, 6, 21, 23, 24, 26, 27, 30, 32, 33, 39, 41, 43, 44, 45, 46, 48, 50, 51, 52, \\
53, 54, 59, 60\}
\end{array} \]

(61; 30, 26, 26, 26; 47) (***)

\[ \begin{array}{l}
X_1 = \{0, 2, 4, 5, 6, 8, 9, 10, 12, 18, 19, 21, 22, 23, 25, 26, 27, 28, 30, 32, 34, 36, 37, \\
42, 49, 55, 56, 57, 58, 59\}
X_2 = \{0, 2, 5, 8, 9, 10, 15, 18, 24, 27, 28, 31, 33, 35, 38, 39, 44, 45, 46, 47, 49, 50, 51, \\
52, 59, 60\}
X_4' = \{2, 3, 6, 9, 11, 14, 18, 19, 21, 23, 24, 25, 29\}
\end{array} \]

(61; 30, 25, 25, 30; 49) (***)  Turyn series

\[ \begin{array}{l}
X_1' = \{1, 2, 6, 8, 9, 12, 13, 14, 15, 16, 17, 19, 24, 25, 28\}
X_2' = \{1, 5, 6, 8, 10, 11, 12, 14, 20, 24, 27, 29\}
X_4' = \{3, 4, 5, 7, 10, 11, 18, 20, 21, 22, 23, 26, 27, 29, 30\}
\end{array} \]
(61; 30, 25, 25, 30; 49) (k**) \(\text{XXSW series,}
\)

\[
X_1' = \{3, 4, 6, 13, 14, 15, 16, 19, 21, 22, 23, 24, 26, 27, 30\}
\]

\[
X_2 = \{5, 6, 8, 12, 14, 15, 17, 20, 31, 32, 33, 36, 40, 44, 45, 46, 48, 49, 51, 53, 54, 55, 56, 59, 60\}
\]

\[
X_4' = \{1, 3, 5, 9, 10, 13, 15, 16, 17, 20, 22, 26, 27, 29, 30\}
\]

(61; 28, 28, 28, 24; 47) (s**)

\[
X_1' = \{1, 4, 6, 7, 8, 10, 11, 14, 19, 20, 22, 26, 28, 30\}
\]

\[
X_2 = \{0, 1, 6, 10, 12, 13, 17, 21, 22, 23, 26, 27, 29, 32, 34, 35, 36, 39, 40, 42, 43, 50, 52, 54, 57, 58, 59, 60\}
\]

\[
X_4 = \{0, 5, 6, 8, 18, 23, 24, 28, 32, 34, 37, 42, 43, 44, 48, 49, 50, 51, 52, 54, 55, 56, 57, 58\}
\]

(61; 28, 28, 28, 24; 47) (***), all \(X_i\) are \(H_1\) – invariant

\[
X_1 = \{0, 1, 3, 4, 5, 9, 12, 13, 14, 15, 18, 19, 27, 32, 34, 39, 40, 46, 47, 48, 49, 50, 51, 52, 53, 56, 57, 60\}
\]

\[
X_2 = \{0, 1, 2, 4, 5, 7, 11, 13, 14, 21, 22, 24, 26, 29, 30, 31, 33, 36, 37, 41, 42, 45, 47, 48, 52, 54, 58, 60\}
\]

\[
X_4' = \{1, 3, 11, 13, 14, 16, 19, 20, 21, 22, 25, 29\}
\]

(61; 28, 28, 27, 25; 46) (s**)

\[
X_1' = \{1, 2, 3, 4, 5, 7, 9, 11, 17, 18, 20, 24, 27, 29\}
\]

\[
X_2 = \{0, 4, 6, 7, 11, 12, 15, 18, 19, 21, 26, 30, 31, 32, 33, 35, 36, 41, 43, 44, 45, 48, 49, 51, 52, 53, 54\}
\]

\[
X_4 = \{0, 4, 10, 11, 13, 16, 17, 21, 22, 27, 30, 32, 37, 38, 40, 42, 43, 44, 46, 47, 51, 53, 54, 55, 56\}
\]

(61; 28, 27, 27, 25; 46) (***), all \(X_i\) are \(H_1\) – invariant

\[
X_1 = \{0, 7, 11, 18, 21, 22, 23, 24, 28, 29, 30, 31, 32, 35, 36, 37, 40, 41, 42, 44, 45, 50, 51, 53, 54, 55, 58, 59\}
\]

\[
X_2 = \{1, 3, 7, 8, 9, 10, 13, 19, 23, 24, 27, 28, 30, 31, 35, 37, 39, 43, 44, 46, 47, 49, 54, 55, 56, 57, 59\}
\]

\[
X_4' = \{1, 3, 8, 10, 13, 14, 16, 18, 19, 20, 22, 25\}
\]
(61; 25, 30, 30, 25; 49) (s**), all $X_i$ are $H_1$ – invariant

$X'_1 = \{1, 3, 8, 10, 13, 14, 16, 18, 19, 20, 22, 25\}$

$X_2 = \{2, 6, 7, 14, 17, 18, 22, 23, 24, 26, 27, 28, 30, 33, 35, 36, 38, 41, 42, 44, 45, 46, 48, 49, 51, 53, 55, 58, 59, 60\}$

$X_4 = \{0, 1, 2, 3, 6, 13, 14, 17, 19, 26, 27, 31, 32, 33, 37, 38, 39, 40, 46, 47, 49, 50, 54, 60\}$

$v = 63, \ H_1 = \{1, 4, 16\}, \ H_2 = \{1, 25, 58\}$

(63; 31, 26, 26, 30; 50) (sss), Turyn series, all $X_i$ are $H_2$ – invariant

$X'_1 = \{1, 3, 4, 5, 7, 12, 14, 15, 19, 20, 25, 26, 28, 29, 31\}$

$X'_2 = \{7, 8, 9, 11, 14, 18, 19, 21, 23, 27, 28, 29, 31\}$

$X'_4 = \{1, 3, 4, 5, 7, 12, 14, 15, 19, 20, 25, 26, 28, 29, 31\}$

(63; 30, 30, 30, 24; 51) (s**), all $X_i$ are $H_1$ – invariant

$X'_1 = \{1, 2, 4, 8, 9, 11, 13, 16, 18, 19, 22, 25, 26, 27, 31\}$

$X_2 = \{7, 9, 11, 13, 14, 15, 18, 19, 22, 25, 26, 27, 28, 30, 35, 36, 37, 38, 39, 41, 44, 45, 49, 50, 51, 52, 54, 56, 57, 60\}$

$X_4 = \{5, 9, 13, 15, 17, 18, 19, 20, 22, 23, 25, 26, 29, 31, 36, 37, 38, 41, 51, 52, 53, 55, 60, 61\}$

(63; 30, 30, 30, 24; 51) (**s), all $X_i$ are $H_1$ – invariant

$X_1 = \{3, 5, 6, 9, 10, 12, 13, 14, 17, 18, 19, 20, 23, 24, 26, 29, 30, 33, 34, 35, 36, 38, 39, 40, 41, 48, 52, 53, 56, 57\}$

$X_2 = \{3, 5, 7, 9, 10, 12, 13, 15, 17, 18, 19, 20, 23, 26, 27, 28, 29, 34, 36, 38, 40, 41, 45, 48, 49, 51, 52, 53, 54, 60\}$

$X'_4 = \{5, 7, 9, 10, 14, 17, 18, 20, 23, 27, 28, 29\}$

(63; 30, 27, 27, 27; 48) (s**), all $X_i$ are $H_2$ – invariant

$X'_1 = \{1, 5, 8, 9, 11, 16, 17, 18, 19, 22, 23, 25, 27, 29, 31\}$

$X_2 = \{3, 4, 7, 12, 15, 16, 17, 20, 22, 26, 27, 28, 29, 32, 37, 41, 43, 44, 45, 46, 47, 48, 49, 51, 54, 59, 60\}$

$X_4 = \{4, 6, 9, 10, 13, 17, 18, 19, 24, 27, 29, 31, 32, 33, 34, 36, 37, 40, 41, 43, 44, 45, 47, 52, 54, 55, 61\}$
(63; 30, 27, 27, 27; 48) (**), all \( X_i \) are \( H_1 \) – invariant

\[
X_1 = \{3, 6, 11, 12, 13, 19, 22, 23, 24, 25, 26, 27, 29, 30, 33, 37, 38, 39, 41, 43, 44, 45, 46, 48, 50, 52, 53, 54, 57, 58\}
\]

\[
X_2 = \{3, 11, 12, 13, 14, 15, 19, 22, 25, 26, 31, 35, 37, 38, 41, 43, 44, 46, 48, 50, 51, 52, 55, 56, 58, 60, 61\}
\]

\[
X_4 = \{1, 4, 7, 9, 14, 16, 18, 21, 22, 25, 26, 27, 28\}
\]

(63; 29, 31, 31, 24; 52) (***)

\[
X_1 = \{2, 3, 5, 8, 9, 10, 12, 15, 16, 22, 24, 26, 28, 31\}
\]

\[
X_2 = \{0, 1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 15, 20, 23, 24, 28, 29, 30, 31, 34, 40, 42, 43, 45, 49, 50, 52, 58, 59, 60, 61\}
\]

\[
X_4 = \{0, 2, 8, 9, 10, 12, 15, 16, 20, 25, 26, 30, 33, 34, 37, 39, 45, 46, 48, 50, 57, 60, 61, 62\}
\]

(63; 29, 31, 31, 24; 52) (***)

\[
X_1 = \{1, 2, 3, 11, 15, 18, 19, 21, 22, 26, 27, 28, 29, 30, 33, 34, 35, 36, 38, 39, 45, 48, 49, 51, 52, 55, 59, 60, 61\}
\]

\[
X_2 = \{0, 2, 3, 5, 10, 11, 17, 18, 19, 21, 23, 24, 28, 30, 32, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 48, 51, 52, 53, 56, 62\}
\]

\[
X_4 = \{2, 6, 10, 11, 12, 14, 17, 20, 22, 25, 27, 29\}
\]

(63; 27, 31, 25; 51) (***)

\[
X_1 = \{2, 3, 7, 8, 10, 12, 14, 15, 21, 23, 28, 29, 31\}
\]

\[
X_2 = \{3, 6, 7, 11, 12, 13, 14, 19, 22, 23, 24, 25, 26, 28, 29, 33, 35, 37, 38, 41, 42, 43, 44, 46, 48, 49, 50, 52, 53, 56, 58\}
\]

\[
X_4 = \{3, 5, 12, 13, 15, 17, 19, 20, 22, 23, 25, 26, 29, 30, 37, 38, 39, 41, 42, 48, 51, 52, 53, 57, 60\}
\]

(63; 27, 31, 25; 51) (***)

\[
X_1 = \{0, 1, 2, 3, 4, 6, 7, 10, 11, 12, 15, 19, 23, 25, 28, 31, 32, 39, 40, 47, 49, 51, 52, 53, 58, 59, 62\}
\]

\[
X_2 = \{0, 1, 6, 7, 9, 10, 13, 14, 16, 18, 19, 23, 24, 28, 30, 35, 37, 38, 40, 41, 43, 44, 46, 47, 48, 49, 50, 54, 59, 61, 62\}
\]

\[
X_4 = \{1, 2, 4, 6, 10, 14, 16, 17, 19, 21, 27, 28\}
(63; 27, 29, 29, 26; 48) (***), all $X_i$ are $H_2$ invariant

\[
\begin{align*}
X_1 &= \{1, 2, 3, 5, 10, 12, 13, 15, 19, 21, 25, 29, 31\} \\
X_2 &= \{2, 7, 9, 10, 13, 14, 18, 20, 21, 26, 27, 28, 29, 32, 35, 36, 40, 42, 44, 45, 49, 50, \\
&\quad 52, 53, 54, 55, 56, 59, 61\} \\
X_4 &= \{4, 7, 8, 9, 11, 16, 17, 18, 20, 21, 22, 23, 26, 28, 29, 32, 36, 37, 41, 42, 43, 44, \\
&\quad 46, 47, 49, 59\}
\end{align*}
\]

\[
(63; 27, 29, 29, 26; 48) (***)
\]

\[
\begin{align*}
X_1 &= \{0, 1, 3, 5, 6, 8, 9, 12, 15, 17, 21, 25, 26, 27, 28, 29, 31, 35, 36, 41, 43, 46, 48, 53, \\
&\quad 54, 57, 61\} \\
X_2 &= \{0, 2, 3, 5, 9, 10, 21, 25, 26, 27, 28, 30, 31, 33, 34, 35, 41, 42, 43, 44, 45, 46, 49, \\
&\quad 53, 54, 55, 57, 59, 60\} \\
X_4 &= \{1, 3, 4, 10, 11, 14, 16, 18, 20, 23, 26, 27, 31\}
\end{align*}
\]

In the case $v = 57$ our list contains two PDFs having different parameter sets and sharing the same symmetric block. The same is true for $v = 61$.

5. Acknowledgements

This research was enabled in part by support provided by SHARCNET (http://www.sharcnet.ca) and Compute Canada (http://www.computecanada.ca).

References

[1] L. V. Abuzin, N. A. Balonin, D. Ž. Doković, I. S. Kotsireas, Hadamard matrices from Goethals-Seidel difference families with a repeated block, Informatsionno-upravliaushchie sistemy [Information and Control Systems], 2019, no. 5, pp. 2–9. doi:10.31799/1684-8853-2019-5-2-9

[2] N. A. Balonin, Y. N. Balonin, D. Ž. Doković, D. A. Karbovskiy, and M. B. Sergeev, Construction of symmetric Hadamard matrices, Informatsionno-upravliaushchie sistemy [Information and Control Systems], 2017, no. 5, pp. 2-11. doi:10.15217/issn1684-8853.2017.5.2

[3] N. A. Balonin and D. Ž. Doković, Symmetric Hadamard matrices of orders 268, 412, 436 and 604. Informatsionno-upravliaushchie sistemy [Information and Control Systems], 2018, no. 4, pp. 2–8. doi:10.31799/1684-8853-2018-4-2-8

[4] N. A. Balonin, D. Ž. Doković, and D. A. Karbovskiy, Construction of symmetric Hadamard matrices of order $4v$ for $v = 47, 73, 113$, Spec. Matrices 6 (2018), 11-22.

[5] R. Craigen and H. Kharaghani, Hadamard matrices and Hadamard designs, in Handbook of Combinatorial Designs, 2nd ed. C. J. Colbourn, J. H. Dinitz (eds) pp. 273–280. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007.

[6] O. Di Mateo, D. Ž. Doković, I. S. Kotsireas, Symmetric Hadamard matrices of order 116 and 172 exist, Spec. Matrices 3 (2015), 227–234.

[7] Dragomir Ž. Doković, Ilias S. Kotsireas, Computational methods for difference families in finite abelian groups, Spec. Matrices 2019; 7:127–141.

[8] Zvonimir Janko, Hadi Kharaghani, Vladimir Tonchev, The existence of a Bush-type Hadamard matrix of order 324 and two new infinite classes of symmetric designs, Designs, Codes and Cryptography 24 (2001), issue 2, pp. 225–232.
[9] J. Seberry and N. A. Balonin, Two infinite families of symmetric Hadamard matrices, Australas. J. Combin. 69(3) (2017), 349–357.
[10] J. Seberry, M. Yamada, Hadamard matrices, sequences, and block designs. In Contemporary design theory, 431-560, Wiley-Intersci. Ser. Discrete Math. Optim., Wiley, New York, 1992.
[11] R. J. Turyn, An infinite class of Williamson matrices, J. Combinatorial Theory Ser. A 12 (1972), 319–321.
[12] R. Turyn, A special class of Williamson matrices and difference sets, JCT (A) 36 (1984), 111–115.
[13] M. Xia, T. Xia, J. Seberry and J. Wu, An infinite series of Goethals-Seidel arrays, Discrete Applied Mathematics 145 (2005), 498–504.

University of Waterloo, Department of Pure Mathematics and Institute for Quantum Computing, Waterloo, Ontario, N2L 3G1, Canada

Email address: djokovic@uwaterloo.ca