Clique covers of $H$-free graphs

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Abstract

It takes $n^2/4$ cliques to cover all the edges of a complete bipartite graph $K_{n/2,n/2}$, but how many cliques does it take to cover all the edges of a graph $G$ if $G$ has no $K_{t,t}$ induced subgraph? We prove that $O(n^{2-1/(2t)})$ cliques suffice for every $n$-vertex graph; and also prove that, even for graphs with no stable set of size four, we may need more than linearly many cliques. This settles two questions discussed at a recent conference in Lyon.
1 Introduction

A clique $X$ of a graph $G$ covers an edge $uv$ of $G$ if $u, v \in X$, and a clique cover of $G$ is a collection of cliques of $G$ that together cover all the edges. The size of a clique cover is the number of cliques in the collection. What can we say about the sizes of clique covers?

The complete bipartite graph $K_{[\frac{n^2}{4}], [\frac{n^2}{4}]}$ shows that, for an $n$-vertex graph, we may need as many as $\lceil \frac{n^2}{4} \rceil$ cliques for a clique cover. In fact every graph $G$ has a clique cover of size at most $\lceil \frac{|G|^2}{4} \rceil$, where $|G|$ denotes the number of vertices of $G$. (To see this, note that if $x, y$ are adjacent, we can cover all edges incident with $x$ or $y$ with at most $|G| - 1$ cliques, so we may delete $x, y$ and use induction on $|G|$.) But what if we restrict to $H$-free graphs? (A graph is $H$-free if it does not contain an induced copy of $H$.) To make a difference, $H$ must be complete bipartite, or else $K_{[\frac{n^2}{4}], [\frac{n^2}{4}]}$ is $H$-free; but what happens when $H$ is complete bipartite?

Indeed, what happens if $H = K_{s,0}$? Thus a graph $G$ is $H$-free if and only if $\alpha(G) < s$. (The sizes of the largest stable set and the largest clique in $G$ are denoted by $\alpha(G), \omega(G)$ respectively.) The minimum size of clique covers in graphs $G$ with $\alpha(G)$ bounded already involves interesting questions. For example, there is a long-standing conjecture that:

1.1 Conjecture. If $\alpha(G) \leq 2$, there is a clique cover of size at most $|G|$.

("Long-standing", but we do not know the source. Seymour recalls working on it many years ago, possibly in the 1980’s.) Which other graphs $H$ have the property that every $H$-free graph has a clique cover of size at most $|G|$? It turns out that $H$ must be an induced subgraph of $K_{1,3}$. (To see this, observe that every such graph $H$ must be an induced subgraph of a complete bipartite graph, and of the graph obtained from $K_{2,3}$ by subdividing two disjoint edges.) This leads us to the case when $H = K_{1,3}$, and for that there is a remarkable result of Javadi and Hajebi [4]:

1.2 Theorem. If $G$ is connected and $K_{1,3}$-free, and has a stable set of cardinality three, then $G$ admits a clique cover of size at most $|G|$.

Thus, if 1.1 is true, then every $H$-free graph has a clique cover of size at most $|G|$ if and only if $H$ is an induced subgraph of $K_{1,3}$.

We have nothing to contribute to 1.1 itself, but what if we increase the bound on $\alpha(G)$? Javadi and Hajebi [4] asked whether all graphs $G$ with $\alpha(G)$ at most a constant admit clique covers of size $O(|G|)$, but we will disprove this. We will show that:

1.3 Theorem. There exists $C > 0$ such that for infinitely many $n$, there is a graph $G$ on $n$ vertices with $\alpha(G) \leq 3$ that requires $\frac{Cn^{6/5}}{(\log n)^2}$ cliques in any clique cover.

And as an upper bound, we will show:

1.4 Theorem. For every integer $s \geq 3$, if $G$ is a graph with no stable set of size $s$, then $G$ admits a clique cover of size at most $O(|G|^{2 - \frac{1}{s-1}})$.

At the other extreme, what happens if we exclude $K_{t,t}$? Sepehr Hajebi [3] recently proposed the following:

1.5 Conjecture. For every integer $t \geq 1$ there exists $\varepsilon > 0$ such that every $K_{t,t}$-free $G$ has a clique cover of size $O(|G|^{2-\varepsilon})$. 

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Our main result is a proof of 1.5. We will show that:

1.6 Theorem.

- For all integers $s, t$ with $s \geq 3$ and $t \geq 2$, every $K_{s,t}$-free graph $G$ with sufficiently many vertices has a clique cover of size at most $\frac{3}{2}|G|^{2-1/(s+t)}$.
- For all integers $s$ with $s \geq 3$, every $K_{s,1}$-free graph $G$ (and a fortiori, every $K_{s,0}$-free graph) has a clique cover of size at most
  \[ O\left(\left(\frac{|G|}{\log |G|}\right)^{\frac{s-2}{s-1}}|G|\right). \]
- Every $K_{2,2}$-free graph $G$ has a clique cover of size $O(|G|^{3/2})$;
- Every $K_{2,3}$-free graph $G$ has a clique cover of size $O(|G|^{3/2}(\log |G|)^{1/2})$.

The second bullet implies 1.4. We observe that the third bullet here is asymptotically sharp, since there are bipartite $K_{2,2}$-free graphs $G$ with $\Omega(|G|^{3/2})$ edges.

2 Subquadratic clique covers

In this section we will prove 1.6. We begin with some lemmas. Ajtai, Komlós and Szemerédi [1] showed (logarithms in this paper are to base two):

2.1 Lemma. For every integer $s \geq 2$ there exists $d > 0$ such that, for all integers $a \geq 2$, the Ramsey number

\[ R(a, s) \leq \frac{da^{s-1}}{(\log a)^{s-2}}; \]

that is, every graph with at least $\frac{da^{s-1}}{(\log a)^{s-2}}$ vertices has either a clique of size $a$ or a stable set of size $s$.

Let us rewrite 2.1 in a form more convenient for us:

2.2 Lemma. For every integer $s \geq 2$ there exists $c > 0$ such that if $w > 1$ is some real number, and $G$ is a graph with $\alpha(G) < s$ and $\omega(G) \leq w$, then

\[ |G| < \frac{cw^{s-1}}{(\log w)^{s-2}}. \]

Proof. Choose $d$ satisfying 2.1, and let $c = 2^{s-1}d$; we claim that $c$ satisfies 2.2. Let $w \geq 1$, and let $G$ be a graph with $\alpha(G) < s$ and $\omega(G) \leq w$. Let $a = \lceil w \rceil + 1$. Then $a \geq 2$ is an integer, and $\omega(G) < a$. By 2.1,

\[ |G| < \frac{da^{s-1}}{(\log a)^{s-2}}. \]

But $a \leq 2w$ since $w \geq 1$, and so $da^{s-1} \leq cw^{s-1}$, and since $(\log a)^{s-2} \geq (\log w)^{s-2}$, this proves 2.2. \[ \blacksquare \]
A theorem of Erdős and Hajnal [2], in support of their well-known conjecture, implies that for all \( s, t \) there exists \( c > 0 \) such that if \( G \) is \( K_{s,t} \)-free then \( G \) has a clique or stable set of cardinality at least \( |G|^c \). But we want to make the result as sharp as we can, so we give a different proof.

**2.3 Lemma.** Let \( s, t \) be integers with \( s \geq t \geq 2 \), and let \( c \geq s \) satisfy 2.2. If \( G \) is \( K_{s,t} \)-free and \( w > 1 \) is a real number with \( w \geq \omega(G) \), then

\[
|G| \leq \frac{cw^{s-1}}{(\log w)^{s-2}}.
\]

**Proof.** We may assume that \( \alpha(G) \geq 2 \), because otherwise

\[
|G| = \omega(G) \leq \frac{cw^{s-1}}{(\log w)^{s-2}}.
\]

(since \( c \geq 1 \) and \( s \geq 2 \), and \( \omega(G) \leq w \), and \( w \geq \log w \)), and the theorem holds. We first prove the following:

(1) \( V(G) \) is the union of at most \( \alpha(G)^t \) sets each including no stable set of cardinality \( s \).

If \( \alpha(G) < s \), the claim holds, so we may assume that \( \alpha(G) \geq s \). Let \( S \) be a stable set of cardinality \( \alpha(G) \geq s \). For \( i \in \{t-1, t\} \), let \( A_i \) be the set of all subsets of \( S \) of cardinality \( i \). For each \( X \in A_{t-1} \), let \( R_X \) be the set of all \( v \in V(G) \) such that all neighbours of \( v \) in \( S \) belong to \( X \) (thus, \( X \subseteq R_X \)). Since \( S \) is a largest stable set of \( G \), it follows that \( \alpha(G|R_X|) \leq t - 1 \leq s - 1 \), because if there were a larger stable set in \( R_X \), its union with \( S \setminus X \) would be a stable set larger than \( S \). For each \( X \in A_t \), let \( R_X \) be the set of all \( v \in V(G) \setminus S \) that are adjacent to every vertex in \( X \). Then \( \alpha(G|R_X|) \leq s - 1 \) since \( G \) is \( K_{s,t} \)-free. But every vertex with at most \( t - 1 \) neighbours in \( S \) belongs to \( R_X \) for some \( X \in A_{t-1} \) (here we use that \( |S| \geq t - 1 \geq 1 \)), and every vertex with at least \( t \) neighbours in \( S \) belongs to \( R_X \) for some \( X \in A_t \), and so \( V(G) \) is the union of the sets \( R_X \) (\( X \in A_{t-1} \cup A_t \)). Moreover,

\[
|A_{t-1}| + |A_t| = \binom{\alpha(G)}{t-1} + \binom{\alpha(G)}{t} = \binom{\alpha(G) + 1}{t} \leq \alpha(G)^t.
\]

This proves (1).

From the choice of \( c \), if \( R \subseteq V(G) \) includes no stable set of size \( s \), then

\[
|R| \leq \frac{cw^{s-1}}{(\log w)^{s-2}}.
\]

By (1), it follows that

\[
|G| \leq \frac{cw^{s-1}}{(\log w)^{s-2}}.
\]

This proves 2.3.  

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We may assume that $G$ has edge set the uncovered edges. Then at least one unhappy vertex. Let $H$ be the subgraph of $G$ with vertex set the unhappy vertices and edge set the uncovered edges. Then at least one unhappy vertex. Let $H$ be the subgraph of $G$ with vertex set the unhappy vertices and edge set the uncovered edges. Then $H$ has minimum degree at least $n^{1-d}$. Furthermore, no clique of $G$ covers at least $n^d$ edges of $H$, or we could have added it to our maximal sequence at the first step.

2.3 implies that if $G$ is $K_{s,t}$-free then $\max(\alpha(G),\omega(G)) \geq O(|G|^{1/(s+t-1)})$. A similar proof shows that for every complete multipartite graph $H$, there exists $\varepsilon$ such that if $G$ is $H$-free then

$$\max(\alpha(G),\omega(G)) \geq \varepsilon |G|^{1/(|H|-1)}.$$  

(We omit the proof, since we shall not use the result.)

Let us prove the first statement of 1.6, the following:

**2.4 Theorem.** Let $s \geq 3$ and $t \geq 2$ be integers. There exists $N$ such that every $K_{s,t}$-free graph $G$ with at least $N$ vertices admits a clique cover of size at most $\frac{3}{2}|G|^{2-1/(s+t)}$.

**Proof.** We may assume that $s \geq t$, by exchanging them if necessary. Let $c$ satisfy 2.2. Since $s \geq 3$ we may choose $N$ such that

$$(\log N)^{s-2} \geq c(2t)^t(s+t)^{s-2}$$

(this is the only place in the proof that we need $s \geq 3$). Let $d = 1/(s+t)$, and let $G$ be a $K_{s,t}$-free graph with $n \geq N$ vertices. We must show that $G$ has a clique cover of size at most $\frac{3}{2}n^{2-d}$, and so we may assume that $\omega(G) \geq 2$. We begin by choosing a maximal sequence of cliques in $G$, such that each clique covers at least $n^d$ edges not covered by previous cliques. Thus, so far we have used at most $\frac{1}{2}n^{2-d}$ cliques.

Next, if there is any vertex $v$ that is incident with at most $n^{1-d}$ edges that have not yet been covered, we take copies of $K_2$ to cover all the uncovered edges incident with $v$. Repeat this process until no such vertices remain. Note that this step uses at most $n^{2-d}$ cliques in total, so altogether we have used at most $\frac{3}{2}n^{2-d}$ cliques.

We claim that all edges of $G$ have now been covered; so, for a contradiction, suppose not. Call a vertex $x$ happy if all edges incident with $x$ have been covered and unhappy otherwise; thus there is at least one unhappy vertex. Let $H$ be the subgraph of $G$ with vertex set the unhappy vertices and edge set the uncovered edges. Then $H$ has minimum degree at least $n^{1-d}$. Furthermore, no clique of $G$ covers at least $n^d$ edges of $H$, or we could have added it to our maximal sequence at the first step.

Fix an unhappy vertex $v$, and let $D$ be the set of its neighbours in $H$, so $|D| \geq n^{1-d}$. There is no clique $K$ of $G[D]$ with size at least $n^d$, since adding $v$ to $K$ would give a clique of $G$ that covers $n^d$ edges of $H$ (all the edges from $v$ to $K$). So by 2.3, taking $w = n^d$, it follows that $G[D]$ contains a stable set $S$ where

$$\frac{c|S|^t(n^d)^{s-1}}{(\log n^{d})^{s-2}} \geq |D| \geq n^{1-d},$$

that is,

$$|S| \geq d^{(s-2)/t}c^{-1/t}n^{(1-d)/t}(\log n)^{(s-2)/t} = d^{(s-2)/t}c^{-1/t}n^{d}(\log n)^{(s-2)/t}$$

(since $d = (1 - ds)/t$).

By a copy of $K_{1,t}$ we mean an induced subgraph of $G$ isomorphic to $K_{1,t}$, and a leaf of a graph means a vertex with degree one. We count copies of $K_{1,t}$ in $H$ with all their leaves in $S$. Let $L \subseteq S$ with $|L| = t$, and let $M$ be the set of vertices in $V(H) \setminus S$ that are adjacent in $H$ to every vertex in $L$. Since $L$ is stable in $G$ (as it is a subset of $S$), and $G$ is $K_{s,t}$-free, it follows that $M$ does not contain a stable set (of $G$) of size $s$. Moreover, $M$ contains no clique (of $G$) of size at least $n^d$, since adding any vertex of $L$ to such a clique would give a clique in $G$ covering at least $n^d$ edges from $H$.  

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From 2.2, \(|M| \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}}\). Since this holds for each choice of \(L\), and there are only \((|S|)\) choices of \(L\), it follows that there are at most
\[
\frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}} \left(\frac{|S|}{t}\right) \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}} \frac{|S|^t}{t!}
\]
copies of \(K_{1,t}\) with all leaves in \(S\).

On the other hand, there are at least \(n^{1-d}|S|\) edges of \(H\) with an end in \(S\). For each \(y \in V(H)\), let \(r(y)\) be the set of vertices in \(S\) adjacent in \(H\) to \(y\), and let \(r\) be the average of the \(r(y)\) over \(y \in V(H)\). Thus
\[
r \geq n^{1-d}|S|/|H| \geq n^{-d}|S| \geq d(s-2)/t c^{-1/2} (\log n)^{(s-2)/t}.
\]
Moreover, since \(n \geq N\), it follows that
\[
(r(\log n))^{s-2} \geq c(2t)^d 2^{-s}
\]
and so
\[
r \geq d(s-2)/t c^{-1/2} (\log n)^{(s-2)/t} \geq 2t.
\]
The number of copies of \(K_{1,t}\) with all leaves in \(S\) is at least
\[
\sum_{y \in V(H)} \binom{r(y)}{t}
\]
(taking \((a/b) = 0\) when \(a < b\)); and hence at least
\[
|H| \binom{r}{t} \geq |H| \frac{(r-t)^t}{t!} \geq |H| \frac{(r/2)^t}{t!}
\]
by convexity and since \(r \geq 2t \geq t\). Consequently
\[
|H| \binom{r/2}{t} \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}} \frac{|S|^t}{t!},
\]
that is,
\[
|H| (r/2)^t \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}} |S|^t.
\]
Since \(|S| \leq r|H| n^{d-1}\) and \(d(s+t) = 1\), the right side of the above is at most
\[
\frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}} r^t |H|^t n^{t(d-1)} = \frac{c}{d^{-2}(\log n)^{s-2}} r^t |H|^t n^{1-d-t} \leq \frac{c}{d^{-2}(\log n)^{s-2}} r^t |H| n^{-d}.
\]
Consequently
\[
|H| (r/2)^t \leq \frac{c}{d^{-2}(\log n)^{s-2}} r^t |H| n^{-d},
\]
that is,
\[
(\log n)^{s-2} n^d s^{-2} \leq c 2^t,
\]
contradicting that \(n \geq N\). This proves 2.4.
A simplified version of the same argument yields a weakened form of the second statement of 1.6:

**2.5 Proposition.** For every integer \( s \geq 3 \) and \( t \in \{0, 1\} \), if \( G \) is a \( K_{s,t} \)-free graph, then \( G \) admits a clique cover of size at most \( O(|G|^{2-1/s}) \).

**Proof.** Let \( c \) be as in the proof of 2.4, let \( d = 1/s \), and choose \( N \) such that \( (\log N)^{s-2} > cs^{s-2} \). We will show that if \( |G| = n \geq N \) and \( G \) is \( K_{s,t} \)-free then \( G \) admits a clique cover of size at most \( \frac{3}{2}|G|^{2-1/s} \), which implies the result. As in the proof of 2.4, we may assume that \( G \) has a nonnull subgraph \( H \) with minimum degree at least \( n^{1-d} \), such that no clique of \( G \) covers \( n^d \) edges of \( H \). Choose \( v, D \) as before; then \( G[D] \) has no clique of size \( n^d \), and no stable set of size \( s \) (because \( G \) is \( K_{s,t} \)-free and \( t \leq 1 \)), and so by 2.2,

\[
|D| < \frac{cn^{d(s-1)}}{(d \log n)^{s-2}}.
\]

But \( |D| \geq n^{1-d} \), and so

\[
n^{1-d} \leq \frac{cn^{d(s-1)}}{(d \log n)^{s-2}}.
\]

Since \( 1 - d = d(s - 1) \), it follows that \( (\log n)^{s-2} \leq cs^{s-2} \), contradicting that \( n \geq N \). This proves 2.5.

But we can do a little better. To prove the second statement of 1.6 as stated, we use a consequence of 2.2:

**2.6 Lemma.** For all integers \( s \geq 2 \) there exists \( c > 0 \) such that if \( G \) is a graph with \( \alpha(G) < s \) and \( |G| \geq 2 \), then

\[
|G| < \frac{c \omega(G)^{s-1}}{(|G|)^{s-2}}.
\]

**Proof.** Choose \( d \) such that 2.2 holds with \( c = d \). We may assume that \( d \geq s^{1/2} \) by increasing \( d \).

Choose \( c \) such that \( c \geq d^2(2 \log d)^{s-2} \) and \( c > d^2(2(s-1))^{s-2} \); we will show that \( c \) satisfies the lemma.

Let \( G \) be a graph with \( \alpha(G) < s \) and \( |G| \geq 2 \). If \( \log |G| \leq 2 \log d \), then \( |G| \leq d^2 \), and so

\[
|G|(|G|)^{s-2} \leq d^2(2 \log d)^{s-2} \leq c \leq \omega(G)^{s-1}
\]

as required. Thus we may assume that \( \log |G| > 2 \log d \). In particular \( |G| \geq d^2 \geq s \) and so \( G \) has an edge. By 2.2,

\[
|G| < \frac{d \omega(G)^{s-1}}{(\log \omega(G))^{s-2}} \leq \omega(G)^{s-1}
\]

and so \( \log |G| \leq \log(d) + (s-1) \log(\omega(G)) \). Since \( \log |G| > 2 \log(d) \), it follows that \( \log |G| < 2(s-1) \log(\omega(G)) \). Hence

\[
|G| < \frac{d \omega(G)^{s-1}}{(\log \omega(G))^{s-2}} \leq \frac{d(2(s-1))^{s-2} \omega(G)^{s-1}}{(\log |G|)^{s-2}} \leq \frac{c \omega(G)^{s-1}}{(\log |G|)^{s-2}}
\]

as required. This proves 2.6.
We deduce:

2.7 Lemma. For all integers $s \geq 3$ there exists $c > 0$ such that for every graph $G$ with $\alpha(G) < s$ and $|G| \geq 2$, $V(G)$ is the union of at most

$$c \left( \frac{|G|}{\log |G|} \right)^{\frac{s-2}{s-1}}$$

cliques.

Proof. Choose $d$ such that 2.6 holds (with $c$ replaced by $d$). Choose $f$ such that $2df^{s-1} \geq (1/2)^{s-2}$. Choose $N \geq 4$ such that $\log N > (1 - 2^{-\frac{1}{s-2}})^{-1}$, and choose $c$ such that $c \geq 2/f$, and $c(n/\log n)^{\frac{1}{s-2}} \geq n$ for all nonzero integers $n \leq N$. We will show that $c$ satisfies 2.7. Let $G$ be a graph with $\alpha(G) < s$. We prove that the statement of the theorem is true for $G$, by induction on $|G|$. If $|G| \leq N$, then $V(G)$ is the union of $|G| \leq c(|G|/\log |G|)^{\frac{s-2}{s-1}}$ cliques and the theorem holds, so we may assume that $|G| > N$.

Choose as many pairwise disjoint cliques as possible that each have cardinality at least $f|G|^{\frac{1}{s-1}} (\log |G|)^{\frac{s-2}{s-1}}$, say $A_1 \ldots A_k$. Let $G' = G \setminus (A_1 \cup \ldots \cup A_k)$. Thus

$$\omega(G') \leq f|G|^{\frac{1}{s-1}} (\log |G|)^{\frac{s-2}{s-1}}.$$

We claim:

(1) $|G'| \leq |G|/2$.

Suppose not; then by 2.6,

$$|G'| \leq \frac{d\omega(G')^{s-1}}{(\log |G'|)^{s-2}},$$

and so

$$(|G'|/2)(\log(|G'|/2))^{s-2} \leq |G'|(\log |G'|)^{s-2} \leq d\omega(G')^{s-1} \leq df^{s-1}|G|(\log |G|)^{s-2}.$$

Thus

$$(\log |G| - 1)^{s-2} \leq 2df^{s-1}(\log |G|)^{s-2}.$$ But $\log |G| - 1 \geq \frac{1}{2} \log |G|$ (because $|G| \geq N \geq 4$), and so $(1/2)^{s-2} \leq 2df^{s-1}$, a contradiction. This proves (1).

Since $A_1 \ldots A_k$ all have cardinality at least $f|G|^{\frac{1}{s-1}} (\log |G|)^{\frac{s-2}{s-1}}$, it follows that

$$k \leq \frac{f^{-1}|G|^{\frac{s-2}{s-1}}}{(\log |G|)^{\frac{s-2}{s-1}}} = f^{-1}\left( \frac{|G|}{\log |G|} \right)^{\frac{s-2}{s-1}} \leq (c/2)\left( \frac{|G|}{\log |G|} \right)^{\frac{s-2}{s-1}}.$$

From the inductive hypothesis, if $|G'| \geq 2$ then $V(G')$ is the union of

$$c \left( \frac{|G'|}{\log |G'|} \right)^{\frac{s-2}{s-1}}$$
cliques. Hence, $V(G')$ is the union of at most
\[ c \left( \frac{|G|/2}{\log |G| - 1} \right)^{\frac{s-2}{s-1}} \]
cCliques, by (1), even if $|G'| \leq 1$. But
\[ c \left( \frac{|G|/2}{\log |G| - 1} \right)^{\frac{s-2}{s-1}} \leq (c/2) \left( \frac{|G|}{\log |G|} \right)^{\frac{s-2}{s-1}} \]
since
\[ 2^{\frac{s-2}{s-1}} \geq \frac{\log |G|}{\log |G| - 1}. \]
Thus, both $A_1 \cup \cdots \cup A_k$ and $V(G')$ are the union of at most
\[ (c/2) \left( \frac{|G|}{\log |G|} \right)^{\frac{s-2}{s-1}} \]
cCliques. Adding, this proves 2.7.

We use this to show the second statement of 1.6, the following:

**2.8 Theorem.** For every integer $s \geq 3$, let $c$ be as in 2.7. If $G$ is a $K_{s,1}$-free graph with $|G| \geq 2$, then $G$ admits a clique cover of size at most
\[ \frac{c|G|^{2-\frac{1}{s-1}}}{(\log |G|)^{\frac{s-2}{s-1}}}. \]

**Proof.** By 2.7, there is a set $\mathcal{A}$ of cliques of $G$ with union $V(G)$ and with
\[ |\mathcal{A}| \leq c \left( \frac{|G|}{\log |G|} \right)^{\frac{s-2}{s-1}}. \]

For each $v \in V(G)$ and $A \in \mathcal{A}$, let $A_v$ be the clique consisting of $v$ and the set of neighbours of $v$ that belong to $A$. Then the set of all the cliques $A_v$ is a clique cover satisfying the theorem. This proves 2.8.

Now we prove the third and fourth statements of 1.6. We will need the following, which is implied by 2.2 with $s = 3$:

**2.9 Lemma.** There exists $k > 0$ such that every graph $G$ with no stable set of size three has a clique of size at least $k|G|^{1/2} \sqrt{\log |G|}$.

We will show:

**2.10 Theorem.**

- Every $K_{2,2}$-free graph $G$ has a clique cover of size $O(|G|^{3/2})$. 


Every $K_{2,3}$-free graph $G$ has a clique cover of size $O(|G|^{3/2}(\log |G|)^{1/2})$.

Proof. The proofs for both statements are much the same, and we will do them at the same time. Let $G$ be either $K_{2,2}$-free or $K_{2,3}$-free, let $v$ be a vertex of minimum degree, and let $D$ be the set of its neighbours. We will show that $D$ is the union of a small number of cliques. Adding $v$ to each of these cliques, we see that the edges incident with $v$ can be covered by the same small number of cliques; thus we may delete $v$ and argue by induction. It remains to show that $D$ is the union of an appropriately small number of cliques.

First we need:

(1) Let $M \subseteq D$. Then either:

- there is a set $J_1 \subseteq M$ with $|J_1| \geq |M|^2/(4n)$, and two nonadjacent vertices $x, y \in V(G) \setminus J_1$, both adjacent to every vertex in $J_1$; or

- there is a clique $J_2 \subseteq M$ with $|J_2| \geq |M|/4$.

Let $A \subseteq M$ be the set of vertices in $M$ with at least $|M|/3$ neighbours outside $D \cup \{v\}$, and let $B = M \setminus A$. Suppose first that $|A| \geq 3|M|/4$. Then the number of edges from $A$ to $V(G) \setminus (D \cup \{v\})$ is at least $|M|^2/4$, and so some vertex $x \in V(G) \setminus (D \cup \{v\})$ has a set $J$ of at least $|M|^2/(4n)$ neighbours in $M$, and the first bullet of (1) holds (taking $y = v$).

Otherwise $|B| \geq |M|/4$. If $B$ is a clique then the second bullet holds. Otherwise there are nonadjacent vertices $x, y \in B$; and as $x, y$ each have at most $|M|/3$ non-neighbours in $D$ (because $v$ was chosen with minimum degree, and $x, y \in B$), there are at most $2|M|/3$ vertices in $M$ nonadjacent to one of $x, y$ (counting $x, y$ themselves); and so $x, y$ have at least $|M|/3$ common neighbours in $M$, and the first bullet holds. This proves (1).

We deduce:

(2) If $G$ is $K_{2,2}$-free, then for every $M \subseteq D$, there is a clique in $M$ with size at least $|M|^2/(4n)$.

This is immediate from (1), because the set $J_1$ in (1) must be a clique, since $G$ is $K_{2,2}$-free, and the set $J_2$ satisfies $|J_2| \geq |M|/4 \geq |M|^2/(4n)$. This proves (2).

(3) Let $k$ satisfy 2.9, and let $\beta = \min(k/2, 1/4)$. If $G$ is $K_{2,3}$-free, then for every $M \subseteq D$ with $|M|^2 \geq 4n$, there is a clique in $M$ with size at least

$$\frac{\beta|M|}{\sqrt{n}} \sqrt{\log(|M|^2/(4n))}.$$ 

By (1), one of the sets $J_1, J_2$ of (1) exist. If $J_1$ exists, then it contains no stable triple of vertices, and so by 2.9, it contains a clique of size at least

$$k(|J_1| \log(|J_1|))^{1/2} \geq \frac{k|M|}{2\sqrt{n}} (\log(|M|^2/(4n)))^{1/2},$$

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and the claim holds. If $J_2$ exists then again the claim holds since

$$|M|/4 \geq \beta|M| \geq \frac{\beta|M|}{\sqrt{n}} \sqrt{\log(|M|^2/(4n))}.$$ 

This proves (3).

Now we will use (2) or (3) to show that the vertices in $D$ can be covered by an appropriately small collection $C$ of cliques. We choose $C$ by choosing greedily a largest clique among the uncovered vertices of $D$ until at most $4\sqrt{n}$ vertices remain, and then covering the remaining vertices by singletons. To bound the total number of cliques, we track the process, writing $M$ for the set of uncovered vertices at each stage. We divide the values of $|M|$ into ranges $[1, 4\sqrt{n})$ and $[2^i \sqrt{n}, 2^{i+1} \sqrt{n})$ for $i \geq 2$.

We assume first that $G$ is $K_{2,2}$-free. Thus by (2), if $|M|$ is in the range $[2^i \sqrt{n}, 2^{i+1} \sqrt{n})$, then the size of the clique we obtain is at least $|M|^2/(4n) \geq 2^{2i-2}$, and so there will be at most

$$\frac{2^{i+1} \sqrt{n}}{2^{2i-2}} = \frac{8\sqrt{n}}{2^i}$$

cliques chosen for $|M|$ in this range. The total number of cliques in $C$ is therefore at most

$$\sum_{i \geq 2} \frac{8\sqrt{n}}{2^i} + 4\sqrt{n} = O(\sqrt{n}).$$

Consequently the first bullet of the theorem follows by induction.

Now we assume that $G$ is $K_{2,3}$-free, and use (3) in place of (2). If $|M|$ is in the range $[2^i \sqrt{n}, 2^{i+1} \sqrt{n})$ where $i \geq 2$, then the size of the clique we obtain is at least

$$\frac{\beta|M|}{\sqrt{n}} \sqrt{\log(|M|^2/(4n))} \geq \beta 2^i \sqrt{\log(2^{2i-2})} = \beta 2^i \sqrt{2^{2i-2}} - 2.$$ 

Consequently, at most

$$\frac{2^{i+1} \sqrt{n}}{\beta 2^i \sqrt{2^{2i-2}} - 2} = \frac{2\sqrt{n}}{\beta \sqrt{2^{2i-2}}}$$

cliques will be chosen during this range. Thus the total number of cliques is at most

$$\sum_{i=2}^{\log n} \frac{2\sqrt{n}}{\beta \sqrt{2^{2i-2}}} + 4\sqrt{n} = O(\sqrt{n \log n}).$$

Hence the second bullet of the theorem follows by induction. This proves 2.10.

## 3 Lower bounds

What can we say from the other side? For $K_{s,t}$-free graphs, the result of this section, with 1.6, shows that (roughly speaking) the answer is somewhere between $n^{2-4/(s+1)}$ and $n^{2-1/s-t}$. We need the following result of Spencer (theorem 2.2 of [5]):
3.1 Lemma. For all integers \( s \geq 3 \), there exists \( c > 0 \) such that for all integers \( t \geq 3 \), the Ramsey number \( R(s, t) \) is at least \( c(t/\log t)^{\frac{s+1}{s+2}} \). Consequently, for all \( s \geq 3 \) there exists \( C > 0 \) such that for infinitely many \( n \), there is a graph \( J \) with \( n \) vertices such that \( \omega(G) < Cn^{\frac{2}{s+1}} \log n \) and \( \alpha(G) < s \).

3.2 Theorem. For all \( s \geq 3 \), there exists \( c > 0 \) such that for infinitely many \( n \), there is a graph with \( n \) vertices and no stable set of size \( s \), such that every clique cover has size at least \( cn^{2-4/(s+1)}/(\log n)^2 \).

Proof. Choose \( C \) as in the second statement of 3.1, and let \( c \) satisfy \( c^{-1} = C^{2}2^{2-4/(s+1)} \). Now choose \( m > 0 \) such that there is a graph \( J \) with \( m \) vertices, and with \( \omega(J) < Cn^{\frac{2}{s+1}} \log m \) and \( \alpha(J) < s \). Let \( n = 2m \). Take two vertex-disjoint copies \( J_1, J_2 \) of \( J \), and make every vertex of \( J_1 \) adjacent to every vertex of \( J_2 \), forming \( G \); thus \( |G| = n \). Then \( G \) has no stable set of size \( s \); and every clique of \( G \) covers at most \( C^2m^{\frac{4}{s+1}}(\log m)^2 \) of the edges between \( V(J_1) \) and \( V(J_2) \). Since there are \( m^2 \) such edges, every clique cover of \( G \) has size at least

\[
C^{-2}m^{2-\frac{4}{s+1}}/(\log m)^2 \geq cn^{2-\frac{4}{s+1}}/(\log n)^2.
\]

This proves 3.2.

Taking \( s = 4 \), this proves 1.3.

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