DIRAC QUANTIZATION OF TWO-DIMENSIONAL DILATON GRAVITY MINIMALLY COUPLED TO N MASSLESS SCALAR FIELDS

by

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ABSTRACT
It is shown that the Callan-Giddings-Harvey-Strominger theory on $S^1 \times R$ can be consistently quantized (using Dirac’s approach) without imposing any constraints on the sign of the gravitational coupling constant or the sign (or value) of the cosmological constant. The quantum constraints in terms of the original geometrical variables are also derived.
Two-dimensional theories of gravity have been studied as simplified models that might provide some insight into the problems that appear in the quantization of the (3+1)-dimensional theory.

In a recent paper [1], E. Benedict, R. Jackiw and H.-J. Lee showed that the CGHS model [2] on the cylinder ($S^1 \times R$) can be quantized using Dirac’s approach in the functional Schrödinger representation. The open case ($R \times R$) was considered by K. Kuchar, J.D. Romano and M. Varadarajan in [3].

In [1], a constraint on the sign of $G/\lambda$ (where $G$ is the gravitational coupling constant and $\lambda$ the cosmological constant) was imposed in order to assure the invertibility of the transformation that brings the Dirac constraints to the simple form of the pure string-inspired gravity theory. In terms of these new variables, the exact physical quantum states of the model were obtained [1].

In the conclusions to [1] it was pointed out that the constraint on the sign of $G/\lambda$ arose from considering the particular form of the canonical transformation that was used and, therefore, it was not expected to be generic.

In this report we show how this problem can be solved by introducing a different canonical transformation. We also derive the quantum constraints in terms of the original geometrical variables.

Let us consider the CGHS action functional [1]:

$$S = - \int dt \int_0^{2\pi} dx \sqrt{-g} \left( \frac{1}{2G} (\phi R + 2\lambda) + \frac{1}{2} \sum_{i=1}^{N} g^{\alpha\beta} \nabla_\alpha f_i \nabla_\beta f_i \right) \quad (1)$$

In (1), $g_{\mu\nu}$ is the metric tensor, $R$ is the scalar curvature, $\phi$ is the dilaton and $f_i$ ($i = 1, 2, ..., N$) are minimally coupled (to the metric tensor) massless scalar matter fields.

As in [1], we assume space-time to be a cylinder ($S^1 \times R$).

The metric can be parametrized as

$$ds^2 = e^{2\rho} \left( -\sigma^2 dt^2 + (dx + M dt)^2 \right) \quad (2)$$
In this parametrization, the canonical Hamiltonian takes the form:

\[ H_c = \int_0^{2\pi} dx (M \mathcal{F} + \sigma \mathcal{G}) \]  

(3)

where,

\[ \mathcal{F} \equiv \rho' \Pi_\rho + \phi' \Pi_\phi - \Pi'_\rho + \sum_{i=1}^{N} f'_i \Pi_i \approx 0 \]  

(4)

\[ \mathcal{G} \equiv -\frac{\phi''}{G} + \frac{\phi'}{G} \rho' + G \Pi_\phi \Pi_\rho + \frac{\lambda}{G} e^{2\rho} + \frac{1}{2} \sum_{i=1}^{N} \left( \Pi_i^2 + (f'_i)^2 \right) \approx 0 \]  

(5)

are the secondary first-class constraints [5] of the system. \( \Pi_\phi, \Pi_\rho \) and \( \Pi_i \) are the momentum densities conjugate to \( \phi, \rho \) and \( f_i \). In (4,5) \( \approx \) denotes weak equality in Dirac’s sense [3]. The momentum densities conjugate to \( M \) and \( \sigma \) are the primary constraints, which are also first-class. No other constraints appear following Dirac’s algorithm.

The constraints (4,5) satisfy the following Poisson-bracket (PB) relations:

\[ \{ \mathcal{F}(x), \mathcal{F}(y) \} = \{ \mathcal{G}(x), \mathcal{G}(y) \} = (\mathcal{F}(x) + \mathcal{F}(y)) \frac{\partial}{\partial x} \delta(x - y) \]  

(6)

\[ \{ \mathcal{F}(x), \mathcal{G}(y) \} = \{ \mathcal{G}(x), \mathcal{F}(y) \} = (\mathcal{G}(x) + \mathcal{F}(y)) \frac{\partial}{\partial x} \delta(x - y) \]  

(7)

It is convenient to consider the following equivalent set of first-class constraints:

\[ C_- \equiv \frac{1}{2}(\mathcal{F} - \mathcal{G}) \]  

(8)

\[ C_+ \equiv \frac{1}{2}(\mathcal{F} + \mathcal{G}) \]  

(9)

We introduce a canonical transformation defined as follows:

\[ (X^-)' = -\frac{1}{\alpha} e^{\Pi_A} \]  

(10)
\[ P_+ = \alpha \left( -(Ae^{-\Pi_A})' + \frac{\lambda}{2G}e^{-\Pi_B} \right) \quad (11) \]

\[ P_+ = \beta \left( -(Be^{\Pi_B})' + \frac{\lambda}{2G}e^{\Pi_A} \right) \quad (13) \]

where,

\[ A = \frac{1}{2} \left( \phi' - \Pi_\rho \right) \]

\[ \Pi'_A = \rho' - G\Pi_\phi \]

\[ B = \frac{1}{2} \left( \phi' + \Pi_\rho \right) \]

\[ \Pi'_B = -\rho' - G\Pi_\phi \]

The positive real numbers \( \alpha \) and \( \beta \) are defined as:

\[ \alpha = \frac{1}{2\pi} \int_0^{2\pi} dze^{\Pi_A} \]

\[ \beta = \frac{1}{2\pi} \int_0^{2\pi} dze^{-\Pi_B} \]

In terms of these new variables, the constraints (8, 9) take the simple form:

\[ C_- \equiv (X^-)'P_- - \frac{1}{4} \sum_{i=1}^{N} (\Pi_i - f'_i)^2 \approx 0 \quad (20) \]

\[ C_+ \equiv (X^+)'P_+ + \frac{1}{4} \sum_{i=1}^{N} (\Pi_i + f'_i)^2 \approx 0 \quad (21) \]
This is exactly the form obtained in [1, 3, 4, 6], however, our canonical transformation (10 - 17) is different from the ones presented in those papers.

Notice that from (10, 12) it follows that:

\[(X^-)' < 0 \quad (22)\]

\[(X^+)' > 0 \quad (23)\]

This result follows solely from the canonical redefinition (10 - 17) without imposing any constraints on the sign of the gravitational coupling constant $G$ or on the sign of the cosmological constant $\lambda$. Notice also that (22, 23) are valid in general for any field configuration, regardless of whether or not it satisfies the constraints. Finally, notice that no restriction has been imposed on the value of the cosmological constant ($\lambda$ can take any value, including zero).

From (10, 12, 18, 19) it also follows that

\[X^- (2\pi) - X^- (0) = -2\pi \quad (24)\]

\[X^+ (2\pi) - X^+ (0) = 2\pi \quad (25)\]

This (24 - 25) allows us to use the results of [4].

So far we have considered only the classical theory. Let us now turn our attention to the quantization of the model (1).

At the quantum level, the constraints (24, 25) need to be modified [1, 4] in order to satisfy Dirac’s consistency conditions [3]. The quantum constraints $C_-$ and $C_+$ in terms of Kuchar’s variables can be written as [1]:

\[C_-(X^-)' P_- - \frac{N\hbar}{48\pi} \left(\ln(-(X^-)')\right)'' - \frac{1}{4} \sum_{i=1}^{N} (\Pi_i - f_i')^2 + \frac{N\hbar}{48\pi} \quad (26)\]
\[ C_+ = (X^+)'P_+ + \frac{N\hbar}{48\pi} \left( \ln((X^+)'\right)'' + \frac{1}{4} \sum_{i=1}^{N} (\Pi_i + f_i')^2 - \frac{N\hbar}{48\pi} \]  \quad (27)

From (26, 27), using the canonical transformation (10 - 17) and the definitions (8, 9), we can obtain the quantum constraints \( \mathcal{F} \) and \( \mathcal{G} \) in terms of the original geometrical variables (1, 2):

\[ \mathcal{F} \equiv \rho'\Pi_\rho + \phi'\Pi_\phi - \Pi_\rho' + \frac{G\Pi_\phi}{24\pi}\rho' + \sum_{i=1}^{N} f_i'\Pi_i \]  \quad (28)

\[ \mathcal{G} \equiv -\frac{\phi''}{G} + \frac{\phi'}{G}\rho' + G\Pi_\phi\Pi_\rho + \frac{\lambda}{G} e^{2\rho} + \frac{\hbar}{24\pi}\rho'' + \frac{1}{2} \sum_{i=1}^{N} \left( \Pi_i^2 + (f_i')^2 \right) - \frac{N\hbar}{24\pi} \]  \quad (29)

The terms \( \frac{G\Pi_\phi}{24\pi}\rho' \) and \( \frac{\hbar}{24\pi}\rho'' \), added to \( \mathcal{F} \) and \( \mathcal{G} \) respectively, cancel the Schwinger terms (in the commutators between \( \mathcal{F} \) and \( \mathcal{G} \)) arising from the presence of \( N \) matter scalar fields in the model. The term \( \frac{\hbar}{24\pi} \) substracted from the Hamiltonian constraint \( \mathcal{G} \) originates from the fact that space is assumed to be a circle (Casimir effect).

The quantum constraints (28, 29) satisfy the commutation relations:

\[ [\mathcal{F}(x), \mathcal{F}(y)] = [\mathcal{G}(x), \mathcal{G}(y)] = i\hbar (\mathcal{F}(x) + \mathcal{F}(y)) \frac{\partial}{\partial x}\delta(x - y) \]  \quad (30)

\[ [\mathcal{F}(x), \mathcal{G}(y)] = [\mathcal{G}(x), \mathcal{F}(y)] = i\hbar (\mathcal{G}(x) + \mathcal{G}(y)) \frac{\partial}{\partial x}\delta(x - y) \]  \quad (31)

Therefore, Dirac’s quantization [5] can be carried out in a consistent way.

The quantum canonical transformation [1, 4]:

\[ X^\pm = X^\pm \]  \quad (32)

\[ (X^\pm)'P_\pm = C_\pm \]  \quad (33)

\[ Q_n^{(i)\pm} = \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} dz \cos(nX^\pm)(\Pi_i \pm f_i') \]  \quad (34)
\[ \Pi_n^{(i)\pm} = \frac{1}{\sqrt{2\pi n}} \int_0^{2\pi} dz \sin(nX^\pm)(\Pi_i \pm f'_i) \] (35)

\[ q_i = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} dz \Pi_i \] (36)

\[ p_i = \frac{1}{2\sqrt{2\pi}} \int_0^{2\pi} dz \left( X^-(\Pi_i - f'_i) + X^+(\Pi_i + f'_i) \right) \] (37)

brings the constraints (26, 27) to the simple form of the pure \((N = 0)\) string-inspired gravity theory.

In (34, 35) \(n = 1, 2, \ldots\) are positive integers.

The invertibility of the transformation (32 - 37) is guaranteed by the conditions (22 - 25). Notice that we assume \(f(0) = f(2\pi)\) (periodicity condition).

The operators (34 - 37) commute with the quantum constraints (26, 27). Therefore, (34 - 37) are quantum observables in Dirac’s sense 5.

In this representation, the physical quantum states are given by the wave functions [1, 4]:

\[ \chi(Q_n^{(j)\pm}, q_j) \] (38)

**Conclusions**

A canonical transformation which relates the original geometrical variables for the CGHS model on \(S^1 \times R\) with variables similar to the ones used by Kuchar 4 and more recently by E. Benedict, R. Jackiw and H.-J. Lee 1 was presented. The new variables \(X^\pm\) automatically satisfy the conditions \((X^+)' > 0, (X^-)' < 0\) and \(X^\pm(2\pi) - X^\pm(0) = \pm 2\pi\). These conditions guarantee that the (classical and quantum) canonical redefinitions that bring the constraints to the simple form of the pure dilaton gravity theory are invertible. The quantization can then proceed as in 1, 4, without imposing any constraints on the sign of the gravitational coupling constant \(G\) or on the
sign (or value) of the cosmological constant $\lambda$. The (modified) quantum constraints in terms of the geometrical variables were obtained.

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