Renormalization of four-quark operators, effective theory
and the role of evanescent operators

Kassa Adel
Department of Physics, McGill University
Montreal, QUE H3A 2T8, Canada

York-Peng Yao
Randall Laboratory of Physics, University of Michigan
Ann Arbor, MI 48109 U. S. A.

Abstract

We present, in the context of dimensional regularization, a prescription to
renormalize Feynman diagrams with an arbitrary number of external fermions. This
prescription, which is based on the original t’Hooft-Veltman proposal to keep exter-
nal particles in four dimensions, is particularly useful to define the ‘renormalization’
(in the context of effective Lagrangian) of physical four-quark operators without in-
troducing any evanescent operator. The results obtained for \( b \to s \) processes agree
with those from the so-called naive prescription, but disagree with the ones with
the introduction of evanescent operators in a renormalization group analysis. We
also present an explicit two loop calculation of the mixing of the evanescent oper-
ators with the physical dimension five operators for the same processes. Particular
attention is paid to the unboundedness nature of such mixing and how a formal
finite transformation is effected to decouple. The inevitable mass dependence of
one of these schemes in the literature is pointed out as the cause for the difference
mentioned.

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1. Introduction

There are at least three independent complete calculations of $b \to s$ processes which disagree with each other Refs.[1,2,3]. Although the difference is numerically insignificant for $b \to s\gamma$ (within 2%), it is theoretically important to obtain the correct Wilson coefficients which contain all the heavy top and W-boson effects. The disagreement comes mainly from the introduction of *evanescent* four quark operators in Ref.[2,3]. These were not included in the analysis of Ref.[1], the contributions of which were subsequently given in Ref.[4], which brought Ref.[1] and Ref.[3] into agreement. In this article, we shall attempt to give some partial answer to where and when one should introduce these evanescent operators. We shall argue that if one is interested in constructing an effective Lagrangian to a particular *finite* order in the strong QCD coupling constant $g_s$, based on direct Feynman diagram evaluation of processes, then inclusion or exclusion of these operators is an arbitrary choice. We shall present a scheme for this purpose, in which there is no need to include any evanescent operators. It is based on the original t’Hooft-Veltman (HV) prescription to keep the external particles in four dimensions. However, if one is to perform a renormalization group analysis, then one has to be concerned also with the closure of the operator basis under multiplicative renormalization and the light mass independence aspect of the Wilson coefficients. Although we have not been able to pinpoint, we hope that the following presentation of our concern with respect to these last two items may spur further investigation.

Before we proceed any further into the technical details, we now give a brief exposition of the issues, so that an interested reader who may not be a practitioner can have an overview.

A singular development in theoretical physics in the past twenty years or so has to do with the successes of formalisms in short distance analysis. In many physical
situations, it has been shown that scales far different from the ones under consideration can be, very loosely speaking, 'integrated out' and their residual effects can be parameterized in a series of terms made of effective coefficients and local products of low excitations. Particularly in particle physics, let $\psi$ be the fields which are being prepared and detected experimentally. One is interested here in expanding a theory capable of more detailed description at a higher energy scale $\Lambda$ into terms schematically as

$$L^{\text{full}} = L^{\text{low energy}} + \frac{1}{\Lambda^2} \sum_i \bar{C}_i O_i + \cdots.$$  \hspace{1cm} (i)

If $L^{\text{full}}$ is a renormalizable theory, then $L^{\text{low energy}}$ will include all the renormalizable terms after such an ‘integrating out’ of the heavy particles. $\bar{C}_i$ are the Wilson coefficients, which are functions of $\log \Lambda$ and perhaps some other variables and $O_i$ are made of local functions of $\psi$. We have put a bar over $C_i$ to indicate that their $\frac{1}{\Lambda^2}$ factor has been taken out. This is a well-defined construct under the name of effective Lagrangian.

Let us discuss some aspects of the actual construction of an effective Lagrangian. Specifically, we consider only the case when $L^{\text{full}}$ is known and renormalizable. One can perform a loop expansion with it, from which the right hand side of Eq. (i) can be extracted. The most systematic way to do this to a particular order is by using Zimmermann oversubtraction identities to rearrange the integrand for any diagram in which there is at least one heavy internal particle. Terms are organized or discarded according to dimensional counting. Since the role of the identities is merely to add and to subtract the same terms to a given integrand to allow power counting down to parts of it, there are many possibilities. The common tread is an expansion in external momenta and internal masses of these subdiagrams relative to $\Lambda$. A point which needs to be emphasized, however, is that after the required wave function and parameter renormalizations inherent in $L^{\text{full}}$, all these parts are
finite quantities. If the regulator used for the said renormalizations is dimensional continuation, then the limit to go to four dimensions can be taken at this stage and each term remains finite. The advantage here is that there are fewer operators $O_i$, which need to be introduced. They are called dimensionally physical. On the other hand, one may choose to continue working slightly away from four dimensions after each finite order, because the algebraic aspect of the operators may be more systematic. This comes about because the mechanics of constructing an effective Lagrangian is such that one needs to reinsert the lower order pieces into a diagram in order to obtain from it the higher order pieces. In so doing, one may find it convenient to introduce extra operators, which are termed evanescent because they vanish formally in the four dimension limit. These are some of the common choices one is free to make. They all should produce exactly the same S-matrix elements, which are dictated by $L^{full}$, to the accuracy in any inverse power of $\Lambda$ demanded by us.

We now turn to the determination of $\bar{C}_i$ in infinite partial sums. As it is well-known, when we include QCD, we should view $\bar{C}_i$ as functions of $g_s^2$ and $g_s^2 \log \Lambda^2$, and perhaps of other variables also as we shall see. Because $g_s^2 \log \Lambda^2$ is not small, an improved perturbation expansion is to sum infinite series in leading powers, subleading powers, etc. of it. The tool for this is the renormalization group analysis. Essential to the procedure is the introduction of a dimensionful parameter $\mu$, which in dimensional continuation is associated with the regularization program. As our earlier discussion reveals, terms such as $\sum_i \bar{C}_i O_i$ are finite quantities, upon expressing them in renormalized fields, masses and coupling constants. It should also be noted that once a choice is made of the Zimmermann rearrangement, there ensues a particular renormalization prescription for $O_i$. Schematically, using the closure
property

\[ O_i = Z_{ij} O_j^{\text{bare}}, \quad (ii) \]

we write

\[ \mu \frac{d}{d \mu} O_i + \gamma_{ij} O_j = 0, \quad (iii) \]

with

\[ \gamma_{ij} = -\left( \mu \frac{d}{d \mu} Z_{ik} \right) Z_{kj}^{-1}, \quad (iv) \]

and then it follows that

\[ \mu \frac{d}{d \mu} \bar{C}_i = \bar{C}_j \gamma_{ji}. \quad (v) \]

It is obvious that a preferential scheme is one which is \textit{mass-independent}, such that \( \gamma_{ij} \) are functions only of \( g_s(\mu) \) and \( \bar{C}_i \) only of \( g_s(\mu) \) and \( \log(\Lambda/\mu) \). This allows one to perform infinite sums over leading logarithm, subleading logarithm, etc more conveniently.

We shall develop this program in much greater details in the sections to come. Thus, the article is organized as follows. In section 2, we review again the general formalism for calculating directly the Wilson coefficients from Feynman diagrams and explain how the effective theory is derived from the full theory (Standard Model). We also discuss the scheme-dependence of various renormalized quantities in the effective theory. In section 3, we present the renormalization group equations (RGE) for the Green’s functions and the Wilson coefficients and discuss the meaning of the leading logarithm approximation (LLA) and the next-to-leading logarithm approximation (NLLA). In section 4, we present a new prescription to obtain the physical four quark operators without introducing any evanescent operators. This scheme was implicitly used in Ref.[1] and therefore, if adhered to for (RGE) analysis, all the results in the LLA will be the same as those of Ref.[1]. In section 5, we follow others in introducing evanescent operators and perform a two loop calculation of
the mixing of these operators with the physical dimension five operators. We shall discuss the potential difficulty in dealing with unbounded infinite matrices. We shall give the similarity transformation which formally decouples these two sets of operators at any running scale $\mu$. It is the same transformation which makes the matrix elements of these evanescent operators to go to $b \to s\gamma$, $sG$ vanish at one loop. In section 6, we comment on the results and give conclusions.

2. Effective theory

Consider the process $b \to s\gamma$ in the Standard Model (SM). The diagrams contributing to this process to lowest order are shown in Fig. 1. Suppose we want to calculate the value of each renormalized diagram in the limit $m_t \sim m_w \gg p_{ext}, m_b, m_s, m_u, m_c$ to an accuracy $\mathcal{O}(1/m_w^2)$, where $p_{ext}$ represents here the momenta of the external particles. We use the first diagram, redrawn in Fig. 2, to explain how the effective theory is built to this order. We do this by partitioning the diagram into several pieces to extract individually the $1/m_w^2$ effects, an approach which is explained in details in Ref. [5]. Let $\Gamma$ denote the unrenormalized diagram on the left of Fig. 2 and let us define:

$$\tau_{\gamma}^{(e)} \equiv \text{Pole Part of } \gamma,$$

$$\tau_{\gamma}^{(r)} \equiv \gamma(p = 0) + p \frac{\partial}{\partial p} \gamma(p = 0) + \ldots + \frac{1}{r!} p^r \frac{\partial^r}{\partial p^r} \gamma(p = 0).$$

(1)

where $p$ represents the external momenta of the graph (or subgraph) $\gamma$, and $\epsilon = n-4$, $n$ being the number of space-time dimensions. The renormalized $\Gamma$ is obtained by subtracting $\Gamma$ at $p_{ext} = 0$,

$$\Gamma^{ren} = \left(1 - \tau_{\Gamma}^{(0)}\right) \Gamma.$$  

(2)

Let $\gamma_0$ be the tree subgraph of $\Gamma$ which contains the $W$-boson line as illustrated in
Fig. 3. We can now rewrite Eq. (2) as:

\[
\Gamma^{(ren)} = \left(1 - \tau^{(2)}_{\Gamma}\right)\tau^{(0)}_{\gamma_0}\Gamma \\
+ \left(\tau^{(2)}_{\Gamma} - \tau^{(0)}_{\Gamma}\right)\Gamma \\
+ \left(1 - \tau^{(2)}_{\Gamma}\right)\left(1 - \tau^{(0)}_{\gamma_0}\right)\Gamma.
\]

(3)

The first and second terms on the right in Eq. (3) correspond to the first and second diagrams on the right in Fig. 2. The last term in Eq. (3) is of order \(1/m^4_w\) and is discarded. Eq. (3) is nothing but an algebraic rearrangement, first used by Zimmermann [6]. We would like to point out that every term is finite and well defined; in particular when computing the quantities in Eq. (3), the Dirac algebra can be done with no ambiguity in four dimensions. This rearrangement or partitioning of graphs tells us how to compute the heavy effects in the full theory. It also generates an effective theory, order by order in perturbation theory.

In the language of effective theory, the graphs on the right in Fig. 2 are equal to the sum of the products

\[
\sum_i C_i \Gamma^{ren}_{light} (O^{ren}_i)
\]

(4)

where the \(C_i\) are the Wilson coefficients, and contain all the heavy W-boson and/or top quark effects (also called the short distance effects) and the \(O^{ren}_i\) are local renormalized operators of dimension equal to at most six. The subscript \(light\) is there to remind us to use light fields \((u, d, s, c, b, G, A)\) only for the internal as well as external lines of the graphs when computing the Green functions with insertions of operators. In this case, \(\tau^{(0)}_{\gamma_0}\) gives us the tree vertex of Fig. 3, while \((1 - \tau^{(2)}_{\Gamma})\tau^{(0)}_{\gamma_0}\Gamma\) is the one loop renormalized matrix element of this same vertex, which is the first graph on the right hand side of Fig. 2. \((\tau^{(2)}_{\Gamma} - \tau^{(0)}_{\Gamma})\Gamma\) denoted by the second graph is a part of the effective Lagrangian induced at one loop order, where \(C_i\) have been evaluated to one loop and \(\Gamma(O_i)\) have tree values. A general remark is in order. Such evaluations of \(C_i\) give us the boundary conditions for RGE. The matching
between the high energy theory and the low energy theory has been automatically done.

We would like to digress for a moment and discuss the renormalization of the Feynman graphs in the full theory and the renormalization of the operators in the effective theory. In Eq. (2) we have used momentum subtraction for the graphs in the full theory. In Eq. (3) we have also used momentum subtraction to renormalize the operators: this is shown in the first term on the right in Eq. (3), where

\[
\left( -\tau^{(2)}_{\Gamma} \right) \tau^{(0)}_{\gamma_0} \Gamma
\]

is the counterterm to the bare graph \( \tau^{(0)}_{\gamma_0} \Gamma \). Actually, however, the renormalization prescription of the operators in the effective theory may be taken to be different from that of the graphs in the full theory. For example, we may switch to the minimal subtraction to define the renormalized operators. This is achieved by rewriting Eq. (3) as (neglecting \( 1/m^4_w \) effects)

\[
\Gamma^{(\text{ren})} = \left( 1 - \tau^{(e)}_{\Gamma} \right) \tau^{(0)}_{\gamma_0} \Gamma \\
+ \left[ \tau^{(2)}_{\Gamma} - \tau^{(0)}_{\Gamma} - \left( \tau^{(2)}_{\Gamma} - \tau^{(e)}_{\Gamma} \right) \tau^{(0)}_{\gamma_0} \right] \Gamma .
\]

(5)

What we learned from this exercise is that one can choose any scheme to renormalize the operators. This freedom of choice is also reflected in the dependence of the Wilson coefficients: the second term on the right in Eq. (1) and that in Eq. (5) depend on the renormalization chosen for the corresponding first term. However, the sum \( \sum_i C_i \Gamma^{\text{ren}}_{\text{light}}(O^{\text{ren}}_i) \) is uniquely given once the renormalization of the graphs in the full theory is fixed: it equals \( \Gamma^{\text{ren}} \). One may wonder why we have used momentum subtraction for the full theory instead of minimal subtraction. The reason is that power counting is easy in momentum subtraction. If we had used the MS scheme for the full theory, then some of the \( C_i \) would have been of order \( m^0_w \), and we would not have had explicit decoupling of the W-boson. The effective theory obtained would have necessarily required multiple insertions of the associated
operators, because they would not have been suppressed by inverse powers of $m_w$. In the end, all these multiple insertions could be reabsorbed into the redefinition of the fields and parameters in the full theory and one would obtain the same on-shell results. On the other hand, using momentum subtraction for the heavy graphs insures us that all the Wilson coefficients are of order $1/m_w^2$, and therefore we have explicit decoupling. The operators are inserted once only. Also the derivation of renormalization group equations is simpler when there is explicit decoupling.

Let us now consider another example which creates "problems" in the effective theory. This is illustrated in Fig. 4, where again we want to work to an accuracy of $1/m_w^2$. Let $\Gamma$ be the graph on the left in Fig. 4, and $\gamma_0$ be the tree subgraph of $\Gamma$ which contains the $W$-boson line, similar to Fig. 3. $\Gamma$ is finite and needs no renormalization. The Zimmermann algebraic rearrangement gives

$$\Gamma = \left(1 - \tau^{(0)}_\Gamma\right) \tau^{(0)}_{\gamma_0} \Gamma + \tau^{(0)}_\Gamma \Gamma + \left(1 - \tau^{(0)}_\Gamma\right) \left(1 - \tau^{(0)}_{\gamma_0}\right) \Gamma$$

(6)

The first and second terms on the right in Eq. (6) correspond to the first and second diagrams on the right in Fig. 4 respectively. The last term in Eq. (6) is of order $1/m_w^4$ and will be neglected. Here we have again tentatively used momentum subtraction to define the renormalized Green functions with $O^\text{ren}_i$ inserted. In a moment we will rewrite Eq. (6) in the MS scheme, but first we would like to make a remark. The first and second terms on the right in Eq. (6) are well defined and the Dirac algebra can be done in four dimensions safely since each term is finite. This means that $\Gamma^{\text{ren}}(O^\text{ren}_i)$ is unambiguously defined, as long as we evaluate the terms in the combinations as shown. However, we may want to compute the 'counterterms' $\left(-\tau^{(0)}_\Gamma \tau^{(0)}_{\gamma_0}\right) \Gamma$ separately by themselves, then we encounter in dimensional
regularization expressions like

\[ S \equiv \frac{1}{\epsilon} \left[ \left( \gamma^\mu \gamma^\nu \gamma^\alpha P_L \right)_{s_1 s_2} \left( \gamma^\mu \gamma^\nu \gamma^\alpha P_L \right)_{s_3 s_4} ight. 
\left. - 16 \left( \gamma^\mu P_L \right)_{s_1 s_2} \left( \gamma^\mu P_L \right)_{s_3 s_4} \right], \tag{7} \]

where \( P_{L,R} \equiv (1 \pm \gamma^5)/2 \). These 'counterterms' are just the unrenormalized operator \( (\tau_{\gamma_0}^{(0)}) \) inserted matrix elements, which form the basis for operator analysis that enters into RGE. In four dimensions the above expression in square brackets gives zero, but in \( n \neq 4 \) dimensions, it has to be defined: is it a pole which must be subtracted, or is it a finite quantity which should not require a subtraction if one uses the MS scheme to renormalize the operators? This is the main difference between the naive scheme of Ref.[1] and the scheme of Refs.[2,3]. In Ref.[2,3], the authors looked upon such an expression as new structures and introduced evanescent operators for renormalization, while in Ref.[1] we did not. We shall come back to this shortly.

Let us go back to Eq. (6) and rewrite it such that the operators are renormalized in the MS scheme.

\[ \Gamma = \left( 1 - \tau_{\Gamma}^{(e)} \right) \tau_{\gamma_0}^{(0)} \Gamma 
+ \left[ \tau_{\Gamma}^{(0)} - \left( \tau_{\Gamma}^{(0)} - \tau_{\Gamma}^{(e)} \right) \tau_{\gamma_0}^{(0)} \right] \Gamma \tag{8} \]

The \( 1/m^4 \) terms have been discarded. In this scheme too, we encounter expressions like those given in Eq. (7). Again, if one’s intent is just to use the Zimmermann indentities as a technique to calculate Fig. 4, then this is not a real problem: the expression \( \left( \tau_{\Gamma}^{(0)} - \tau_{\Gamma}^{(e)} \right) \tau_{\gamma_0}^{(0)} \Gamma \), which contains evanescent structures, is found in the first and second terms on the right in Eq. (8) with opposite signs. This makes the sum of the two terms independent of the definition (i.e. prescription) of that evanescent expression. One has various choices to define the expression in Eq. (7), as long as one uses the same choice to calculate the first and second terms on the right in Eq. (8). Two possible prescriptions are:
a) Formally, one can think of Eq. (7) as a pole that multiplies evanescent structures, and therefore treat them as counterterms for the operators (e.g. $\tau^{(0)}_{\gamma_0}$) which need renormalization. In this scheme, one introduces evanescent operators to carry out the renormalization of the operators, but some extra work is needed to obtain physical results. The renormalized Green functions of these evanescent operators must be included or be transformed away in the limit $n \to 4$.

b) One can think of the expression in Eq. (7) as being finite and give a prescription to define its value. For instance, we can use the following definition:

$$S = a \, (\gamma^\mu P_L)_{s_1 s_2} \, (\gamma^\mu P_L)_{s_3 s_4}$$

where $a$ is arbitrary, but otherwise finite in the limit $\epsilon \to 0$. In this scheme, there is no need to introduce any evanescent operator. Only physical operators need be included in the analysis. We will present in section 4 a consistent scheme to evaluate the coefficient $a$.

Let us summarize what we have said so far. The effective theory can be derived from the full theory with the use of the Zimmermann rearrangement: all the heavy effects in the full theory can be extracted by partitioning in momentum space the heavy graphs of the full theory. This is an alternative to the operator product expansion to construct the effective theory. In deriving the effective theory, one encounters evanescent structures which must be defined in the context of dimensional regularization. This makes the renormalization of the physical operators artificially prescription dependent, and as a consequence the Wilson coefficients become prescription dependent as well. However, provided one stays being consistent in dealing with these evanescent structures, one is guaranteed that $\sum_i C_i \Gamma_{\text{light}}(O_i^{\text{ren}})$ is independent of the prescription used to renormalize the various operators. One is then free to either include evanescent operators (assuming there are no inconsistencies within such a scheme) or not include them, in which case one must be ready to
present a consistent procedure in dealing with the evanescent structures. In section 5, we will treat in some details the processes $b \rightarrow s$ in the LLA, and explain why one has to be careful when evanescent operators are first introduced and then transformed away by a finite renormalization.

Before we close this section, let us complete this last example. For the process $b + \bar{c} \rightarrow \bar{c} + s$, in addition to Fig. 4 there are five more diagrams, plus external wave function renormalizations. If we follow the procedure to calculate the second term of Eq.(8) and its corresponding ones for the other diagrams, we obtain the effective Lagrangian to $g_s^2$

$$L_{b + \bar{c} \rightarrow \bar{c} + s}^{\text{eff}} = -G_c \bar{c} \gamma_\mu P_L \bar{s} \gamma_\mu P_L b + G_c \frac{g_s^2}{16\pi^2} \left[ \bar{c} \gamma_\mu P_L \frac{\lambda_\alpha}{2} b \bar{s} \gamma_\mu P_L \frac{\lambda_\alpha}{2} c \left( 6 \log \frac{m_w^2}{\mu^2} - 11 \right) \right. \right.$$  

\hspace{1cm} \left. - \bar{c} R_\kappa \gamma_\mu P_L \frac{\lambda_\alpha}{2} b \bar{s} \gamma_\mu R^\lambda_\kappa P_L \frac{\lambda_\alpha}{2} c \left( \frac{1}{2} \log \frac{m_w^2}{\mu^2} - \frac{1}{4} \right) \right], \hfill (10)

in which the evanescent operator is

$$R_\kappa \gamma_\mu P_L \otimes \gamma^\mu R^\lambda_\kappa P_L \equiv \gamma_\kappa \gamma_\lambda \gamma_\mu P_L \otimes \gamma^\mu \gamma^\lambda \gamma^\kappa P_L - \gamma_\kappa \gamma_\lambda \gamma_\mu P_L \gamma^\lambda \gamma^\kappa \otimes \gamma^\mu P_L - \gamma^\mu P_L \otimes \gamma_\kappa \gamma_\lambda \gamma_\mu P_L \gamma^\lambda \gamma^\kappa + \gamma_\mu \gamma_\kappa \gamma_\lambda P_L \otimes \gamma^\lambda \gamma^\kappa \gamma^\mu P_L, \hfill (11)$$

$G_c = \frac{g_s}{2m_w} V^*_c V_b$, and the V’s are the relevant Cabbibo-Kobayashi-Maskawa matrix elements. This agrees with the result given by Buras and Weisz in Ref.[7]. The constant -11 in the first term of Eq.(10) is the one loop matching condition for the physical Wilson coefficient and it depends on the choice of the evanescent operator of Eq.(11). It is needed for subleading logarithm approximation in solving RGE. Please note that the first term of Eq. (8) is also dependent on evanescent operators. It is only the S-matrix elements which should be and will be basis independent.

We have also explicitly checked that the method is internally consistent up to two loop order in Feynman diagram evaluation of the processes $b \rightarrow s \gamma$ and $b \rightarrow sG$. The results can be found in Ref. [4].
3. Renormalization Group Equations (RGE)

We want to give a quick review in this section, so that we can establish notation. We will define the renormalized operators and their anomalous dimension matrix. Then we will write down the RGE for the Wilson coefficients and that for the renormalized Green functions with the insertion of an operator. We will also discuss the finite transformations that take us from one renormalization scheme to another, and the advantage of using a mass independent renormalization scheme (MIRS) for the operators. The meaning of the leading logarithmic approximation (LLA) and the next-to-leading logarithmic approximation (NLLA) will be briefly reviewed in such a scheme. We will also say a few words about the matching of the effective theory with the full theory. It is understood that we use dimensional continuation as regularization.

The renormalized operators are defined as linear combinations of a complete set of bare operators

\[ O_i^{ren} \equiv Z_{ij} O_j^{bare} \]  \hspace{1cm} (12)

where the matrix \( Z \) is an infinite square matrix if we introduce evanescent operators, otherwise it is finite. It is determined order by order in perturbation theory by requiring that \( \Gamma^{ren}(O_i^{ren}) \) be finite in the limit \( n \to 4 \). We would like to mention that the infinite matrices we will encounter have entries which eventually are unbounded (this will be discussed in section 5); we can therefore treat these matrices very formally.

The \( \mu \)-independence of bare quantities allows us to derive a RGE for the Green functions with the insertion of a renormalized operator:

\[
\mu \frac{d}{d\mu} \Gamma^{bare}(O_i^{bare}) = 0 \quad \implies \quad \left[ (\mu \frac{d}{d\mu} + N_\phi \gamma_\phi) \delta_{ij} + \gamma_{ij} \right] \Gamma^{ren}(O_j^{ren}) = 0
\]  \hspace{1cm} (13)

13
where $N_\phi \gamma_\phi$ denotes symbolically the anomalous dimension of the external fields, and $\gamma_{ij}$ is the anomalous dimension mixing matrix of the operators given by

$$\gamma_{ij} = -\left( \mu \frac{d}{d\mu} Z_{ik} \right) Z_{kj}^{-1}. \quad (14)$$

The same mixing matrix also governs the evolution of the Wilson coefficients $C_i$

$$\mu \frac{dC_i}{d\mu} = C_j \gamma_{ji}, \quad (15)$$

which is a result of the $\mu$-independence of $C_i O_i$. Eqs. (12-15) are valid for all schemes, including mass dependent renormalization schemes. They are also valid (at least formally) if we include evanescent operators in the analysis.

A change in the renormalization of the operators must be compensated by a change in the Wilson coefficients. This amounts to making a finite transformation such that the sum of products $C_i O_i^{ren}$ is invariant. We shall drop the superscript $\text{ren}$ from now on. Under a finite renormalization $(1+A)$, various quantities are transformed into the corresponding primed ones

$$O' = (1 + A) O = (1 + A) ZO^{(bare)} \equiv Z'O^{(bare)},$$

$$C' = C (1 + A)^{-1},$$

$$\gamma' \equiv -\mu \frac{dZ'}{d\mu} Z'^{-1} = - \left[ \mu \frac{d}{d\mu} (1 + A) \right] (1 + A)^{-1} + (1 + A) \gamma (1 + A)^{-1}, \quad (16)$$

where the matrix $A$ is assumed finite but otherwise arbitrary, and may depend explicitly on $\mu$.

We will now discuss the advantages of using a mass independent scheme. In such a scheme, the Wilson coefficient vector $C$ is assumed to have the following expansion:

$$C = \sum_{n=0}^{\infty} \left( g^2 \log \frac{m^2}{\mu^2} \right)^n \left[ c_n^{(0)} + c_n^{(1)} g^2 + \ldots + c_n^{(p)} g^{2p} + \ldots \right] \quad (17)$$
where $m_h$ stands for the mass of the heavy field which is integrated out, and $g \equiv g(\mu)$. In Eq. (17) we have shown the explicit dependence of $C$ on $g$, $m_h$ and $\mu$; the $c_n^{(p)}$ do not depend explicitly on $g$, $m_h$ and $\mu$. In the LLA we keep $c^{(0)}$ only, in the NLLA we keep both $c^{(0)}$ and $c^{(1)}$, etc ...

$$C^{LLA} = \sum_{n=0}^{\infty} c_n^{(0)} \left( g^2 \log \frac{m_h^2}{\mu^2} \right)^n,$$

$$C^{NLLA} = \sum_{n=0}^{\infty} \left( c_n^{(0)} + g^2 c_n^{(1)} \right) \left( g^2 \log \frac{m_h^2}{\mu^2} \right)^n,$$

(18)

To solve the RGE of Eq. (15) for $C$ we need the initial (or boundary) conditions $C(\mu)|_{\mu=m_h}$. In a mass independent scheme, the initial conditions are given by $C(m_h) = c_0^{(0)}$ for the LLA, and $C(m_h) = c_0^{(0)} + g(m_h)^2 c_0^{(1)}$ for the NLLA. These are all very simple and clean in principle. As a contradistinction, in a mass dependent scheme where the $c^{(p)}$ have an explicit dependence on $\mu$, the boundary conditions become $C(m_h) = \sum_{p=0}^{\infty} g(m_h)^{2p} c_0^{(p)}(m_h)$, and one cannot truncate the series, say to the first term to obtain the LLA result. This is because each term $g(m_h)^{2p} c_0^{(p)}(m_h)$ contains $\left[ g^2 \log (m_h^2) \right]^n$ and therefore has to be taken into account to fulfill a consistent partial sum for the LLA.

As we saw, in a mass-independent scheme for the NLLA, we need to compute $c_0^{(1)}$ as well to obtain the boundary conditions. We would like to repeat that these NLLA initial conditions ($c_0^{(1)}$) are scheme-dependent, i.e. they depend on the particular choice of a mass independent scheme one uses to renormalize the operators. Therefore the $C(\mu)$ we obtain in the NLLA are scheme-dependent as well. Again, the sum $C_i(\mu)O_i(\mu)$ is scheme-independent. For instance, for four fermion processes, the NLLA results are given by:

$$[C_i \Gamma(O_i)]^{NLLA} = C_i^{NLLA} \Gamma(O_i)^{(tree)} + C_i^{LLA} \Gamma(O_i)^{(1-loop)}.$$  (19)

This is illustrated in Fig. 5. In this regard, we differ from a remark in Ref.[8] by Dugan and Grinstein. In Eq. (19) the scheme-dependence of the $C_i^{NLLA}$ will
be precisely canceled by that of $\Gamma(O_i)^{(1-\text{loop})}$. We have already given an explicit example at the end of the last section in this regard.

In the renormalization group analysis of $b \rightarrow s$ processes, all authors have assumed that their schemes are mass-independent. A closer examination seems warranted, however.

4. Evanescent structures in the t’Hooft-Veltman scheme

In this section, we present in the t’Hooft-Veltman scheme certain equations which are needed to define the renormalized graphs with an arbitrary number of external fermions, with or without insertions of operators. We are particularly interested in defining the expression in Eq. (7) without introducing any evanescent operator.

In the t’Hooft-Veltman scheme, the external particles are kept in four dimensions: their momenta have the form $p_{\text{ext}} = (p_1, p_2, p_3, p_4, 0, \ldots, 0)$, and the Lorentz indices attached to them run from 1 to 4. The spinors attached to these external particles will be elaborated. We will first solve the Dirac equation in even integer $n$ dimensions. Among the solutions we obtain, we will choose two to describe the two physical degrees of freedom of an external particle and two others for the anti-particle. These particular solutions will have the property that only the first four components of are non-zero. This will allow us to write for the spinors of the external particles as

$$u(p) = \mathcal{P} u(p) \quad , \quad \bar{u}(p) = \bar{u}(p) \mathcal{P}, \quad (20)$$

where $\mathcal{P}$ is some projector which will be defined in a moment. This projector, which was first introduced in Ref [9], will be used to define expressions like those given in Eq. (7).
We start with \( n \) hermitian matrices \( \gamma^\mu (\mu = 1, 2, \ldots, n = \text{even}) \), satisfying
\[
\{ \gamma^\mu, \gamma^\nu \} = 2 \delta_{\mu\nu}.
\] (21)
These are \( 2^{n/2} \times 2^{n/2} \) matrices. Their construction is inductive and is explained in Ref. [10]. Let us briefly review the main ideas. The induction starts at \( m = 2 \) (four dimensional Dirac matrices). We then assume that we have constructed \( 2m \) hermitian matrices \( \gamma^{\mu}_{(2m)} \), of dimension \( 2^m \times 2^m \), which satisfy Eq. (21). This is extended to \( 2m + 2 \) hermitian matrices \( \gamma^{\mu}_{(2m+2)} \), of dimension \( 2^{m+1} \times 2^{m+1} \), which satisfy Eq. (21) as follows. We choose
\[
\gamma^{\mu}_{(2m+2)} = \begin{pmatrix} \gamma^{\mu}_{(2m)} & 0 \\ 0 & \gamma^{\mu}_{(2m)} \end{pmatrix}
\] if \( 1 \leq \mu \leq 2m \),
and
\[
\gamma^{2m+1}_{(2m+2)} = \begin{pmatrix} 0 & \hat{\gamma}_{(2m)} \\ \hat{\gamma}_{(2m)} & 0 \end{pmatrix}, \quad \gamma^{2m+2}_{(2m+2)} = \begin{pmatrix} 0 & i\hat{\gamma}_{(2m)} \\ -i\hat{\gamma}_{(2m)} & 0 \end{pmatrix},
\]
where
\[
\hat{\gamma}_{(2m)} = -i^m \gamma^1_{(2m)} \ldots \gamma^{2m}_{(2m)}.
\]
One can easily check that all the matrices \( \gamma^{\mu}_{(2m+2)} \) are hermitian, and satisfy Eq. (21).

From now on, we will write \( \gamma^\mu \) instead of \( \gamma^{\mu}_{(n)} \). In this representation, the first four matrices \( \gamma^i, i = 1, \ldots, 4 \), have the special form:
\[
\gamma^i = \begin{pmatrix} \gamma^i_{(4)} & \\ & \ddots & \\ & & \gamma^i_{(4)} \end{pmatrix},
\] (22)
where \( \gamma^i_{(4)} \) are the \( 4 \times 4 \) Dirac \( \gamma \)-matrices, chosen such that \( \gamma^4_{(4)} \) is diagonal:
\[
\gamma^4_{(4)} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.
\] (23)
The non-trivial solutions of the Dirac equation, i.e. those which do not vanish identically, which satisfy
\[(i\gamma.p + m) u(p) = 0,\]
\[p^2 + m^2 = 0,\]
are easily found to be

\[
\begin{align*}
\{ u^{(i)}(p) &\sim (-i\gamma.p + m) V^{(i)}, \\
[V^{(i)}]_{j} &\equiv \delta_{ij}; \quad i, j = 1, 2, 3, 4, \ldots, 2^{n/2}. \}
\end{align*}
\]

Those with \(p_4 = +iE, \ E = \sqrt{\vec{p}^2 + m^2},\) will correspond to particles, while others with \(p_4 = -iE\) will be for anti-particles. For concreteness we will discuss the particle-solutions only, where the \(2^{n/2-1}\) independent spinors have been affixed with a normalizing factor,

\[
\begin{align*}
\{ u^{(i)}(p) &= \frac{1}{\sqrt{2E(m + E)}} (-i\gamma.p + m) V^{(i)}, \\
i &= 1, 2, 5, 6, 9, 10, \ldots, 4\ell + 1, 4\ell + 2, \ldots \\
0 &\leq \ell \leq 2^{n/2-2} - 1. \}
\end{align*}
\]

The normalization of the spinors is:

\[
\left( u^{(i)} \right)^\dagger u^{(i)} = 1, \quad \text{for each } i
\]

and one can easily derive:

\[
\overline{u}^{(i)} u^{(i)} = \frac{m}{E}, \quad \text{for each } i,
\]
\[
\sum_i u^{(i)} \overline{u}^{(i)} = \frac{-i\gamma.p + m}{2E},
\]
where \(i\) takes the values \(1, 2, 5, 6, \ldots, 4\ell + 1, 4\ell + 2, \ldots.\) In deriving the above equation, we have used

\[
\sum_i V^{(i)} V^{(i)\dagger} = \frac{1 + \gamma^4}{2},
\]
where the upperscript \(^\dagger\) stands for transpose.
If we go to the rest frame of the particle ($\vec{p} = 0$ and $p^4 = +im$), the spinors in Eq. (26) become:

\[ u^{(1)} = (1, 0, 0, 0, 0, 0, 0, 0, \ldots, 0), \]
\[ u^{(2)} = (0, 1, 0, 0, 0, 0, 0, 0, \ldots, 0), \]
\[ u^{(5)} = (0, 0, 0, 0, 1, 0, 0, 0, \ldots, 0), \]
\[ u^{(6)} = (0, 0, 0, 0, 1, 0, 0, 0, \ldots, 0), \]
and so on.

In the real world where $n = 4$, a spin-$\frac{1}{2}$ particle has only two degrees of freedom. This nomenclature is kept in $n$ dimensions where we choose to designate the spinors $u^{(1)}$ and $u^{(2)}$ given in Eq. (29) as the two physical degrees of freedom of a particle at rest: $u^{(1)}$ for a particle with spin up, and $u^{(2)}$ with spin down. The other spinors will describe non-physical degrees of freedom which must disappear when the limit $n \to 4$ is taken. Similarly, we will choose $u^{(3)}(p)$ and $u^{(4)}(p)$, $\vec{p} = 0$, $p_4 = -im$, to describe the anti-particle at rest, with spin up and down, respectively.

Therefore, when computing Feynman diagrams in $n$ dimensions, because the external fermions are kept in four dimensions, they will be described by states (or spinors) which are linear combinations of $u^{(1)}(p)$ and $u^{(2)}(p)$. Such states, which we will call physical spinors, have the form:

\[ u(p)_{phys} = \alpha u^{(1)}(p) + \beta u^{(2)}(p) = (x, x, x, x, 0, \ldots, 0) \quad \text{(30)} \]

i.e. only the first four components of $u_{phys}$ can be non-zero. This can easily be checked if we note that when $p$ is the momentum of a particle in four dimensions, then the matrix $\gamma \cdot p$ is of the form

\[ \gamma \cdot p = \sum_{\mu=1}^{4} \gamma^{\mu} p^{\mu} = \begin{pmatrix} \gamma(4) \cdot p & \ldots \\ \ldots \\ \ldots \end{pmatrix} \quad \text{(31)} \]

and therefore the vectors $(-i\gamma \cdot p + m)V^{1,2}$ are of the form shown in Eq. (30). We
now define a $2^{n/2} \times 2^{n/2}$ matrix $P$

$$P \equiv \begin{pmatrix} I_4 & 0 \\ 0 & 0 \end{pmatrix}.$$  \hfill (32)

In the above equation $I_4$ is the $4 \times 4$ identity matrix. $P$ is a projector ($PP = P$) which satisfies:

$$u(p)_{phys} = P u(p)_{phys}, \quad \bar{u}(p)_{phys} = \bar{u}(p)_{phys} P.$$  \hfill (33)

We can now state our prescription in handling gamma matrices: "when computing Feynman diagrams in the t’Hooft-Veltman scheme, we simply attach a factor $P$ to each external fermion" (see Eq. (32)). For instance, for four quark processes, Eq. (7) will be rewritten as:

$$S = \frac{1}{\epsilon} \left[ (P \gamma^\mu \gamma^\nu \gamma^\alpha L P)_{s_1 s_2} (P \gamma^\mu \gamma^\nu \gamma^\alpha L P)_{s_3 s_4} - 16 (P \gamma^\mu L P)_{s_1 s_2} (P \gamma^\mu L P)_{s_3 s_4} \right].$$  \hfill (34)

We will see later that the presence of $P$ will make the above expression collapse into Eq. (9), in fact with $a=0$. In other words, the gamma algebra in the first term may as well have been done in four dimensions.

One can check that for $n$ even, $P$ can be written as:

$$P = \left( \frac{1-i\gamma_5 \gamma_6}{2} \right) \left( \frac{1-i\gamma_7 \gamma_8}{2} \right) \cdots \left( \frac{1-i\gamma_{n-1} \gamma_n}{2} \right)$$

$$= \prod_{p=3}^{n/2} \left( \frac{1-i\gamma_{2p-1} \gamma_{2p}}{2} \right).$$  \hfill (35)

This is easily seen if we observe that the products of matrices $\gamma_{2p-1} \gamma_{2p}$, $p \geq 3$ are diagonal; if $I_m$ and $0_m$ denote identity and the null $2^m \times 2^m$ matrices, respectively, we have:

$$\frac{1-i\gamma_{n-1} \gamma_n}{2} = \begin{pmatrix} I_{2^{n/2-1}} & 0_{2^{n/2-1}} \\ 0_{2^{n/2-1}} & I_{2^{n/2-1}} \end{pmatrix},$$

$$\frac{1-i\gamma_{n-3} \gamma_{n-2}}{2} = \begin{pmatrix} I_{2^{n/2-2}} & 0_{2^{n/2-2}} \\ 0_{2^{n/2-2}} & I_{2^{n/2-2}} \end{pmatrix},$$  \hfill (36)

$$\cdots$$
Using Eq. (35), we easily derive:

\[
\begin{align*}
[P, \gamma^\mu] &= 0 \quad \text{if } \mu = 1, 2, 3, 4, \\
\mathcal{P} \gamma^\mu \mathcal{P} &= 0 \quad \text{if } \mu > 4, \\
[P, \gamma^5] &= 0.
\end{align*}
\]

The last identity in Eq. (37) holds if \(\gamma^5\) anticommutes with all \(\gamma^\mu\). It also holds in the t’Hooft-Veltman definition, where \(\gamma^5\) anticommutes with the first four \(\gamma\) matrices but commutes with the remaining \(n - 4\) matrices. This matrix should not be confused with the matrix \(\gamma^\mu\), when \(\mu = 5\). Moreover, whether \(\gamma^5\) is defined as the product of the first four \(\gamma\)-matrices, or as the product of the \(n\) \(\gamma\)-matrices (\(n\) even), we have:

\[
\mathcal{P} \gamma^5 \mathcal{P} = \mathcal{P} \gamma^1 \gamma^2 \gamma^3 \gamma^4 \mathcal{P} = \frac{\epsilon \bar{\mu}_1 \bar{\mu}_2 \bar{\mu}_3 \bar{\mu}_4}{4!} \mathcal{P} \gamma^{\bar{\mu}_1} \gamma^{\bar{\mu}_2} \gamma^{\bar{\mu}_3} \gamma^{\bar{\mu}_4} \mathcal{P},
\]

where the barred indices take on the values 1, 2, 3 and 4 only.

As far as four fermion processes are concerned, we need to evaluate the following quantities:

\[
\mathcal{T}_{ij} \equiv (\mathcal{P} \mathcal{S}_i \mathcal{P})_{s_1 s_2} (\mathcal{P} \mathcal{S}_j \mathcal{P})_{s_3 s_4}
\]

where the \(\mathcal{S}_i\) are products of \(\gamma\)-matrices, and possibly \(\gamma^5\)’s. The algorithm to compute such expressions is the following: we first use the sum splitting notation

\[
(\gamma^\mu)_{s_1 s_2} (\gamma^\mu)_{s_3 s_4} = (\gamma^{\bar{\mu}})_{s_1 s_2} (\gamma^{\bar{\mu}})_{s_1 s_2} + (\gamma^{\hat{\mu}})_{s_1 s_2} (\gamma^{\hat{\mu}})_{s_1 s_2},
\]

where

\[
\begin{align*}
\mu &= 1, \ldots, n, \\
\bar{\mu} &= 1, \ldots, 4, \\
\hat{\mu} &= 5, \ldots, n.
\end{align*}
\]

Then we move all \(\gamma^{\bar{\mu}_i}\), to the left with the use of Eq. (21). If \(\hat{\gamma}\)’s with the t’Hooft-Veltman prescription are present, they are also moved to the left. [For that matter,
one could use a completely anti-commuting $\gamma^5$ if there are no traces involving $\gamma^5$.

At this point, $\mathcal{T}_{ij}$ has the form

$$
\mathcal{T}_{ij} \sim \left[ \gamma^5 \cdots \gamma^5 \gamma^{i_1} \gamma^{\bar{\mu}_1} \cdots \gamma^{i_m} (P \gamma^{\hat{\nu}_1} \gamma^{\hat{\bar{\nu}}_2} \cdots \gamma^{\hat{\bar{\nu}}_p} P) \right]_{s_1 s_2}
$$

$$
\left[ \gamma^5 \cdots \gamma^5 \gamma^{i_1} \gamma^{\bar{\alpha}_1} \cdots \gamma^{i_l} (P \gamma^{\hat{\bar{\nu}}_1} \gamma^{\hat{\nu}_2} \cdots \gamma^{\hat{\nu}_p} P) \right]_{s_3 s_4}.
$$

In the expression above, we do not have any hatted index which is not summed, because all the external particles, and hence free indices, are in four dimensions. The following identity is then used to reduce Eq. (42):

$$
\left( P \gamma^{[\hat{\nu}_1} \gamma^{\hat{\nu}_2} \cdots \gamma^{\hat{\nu}_m]} P \right)_{s_1 s_2} \left( P \gamma^{[\hat{\bar{\nu}}_1} \gamma^{\hat{\bar{\nu}}_2} \cdots \gamma^{\hat{\bar{\nu}}_p]} P \right)_{s_3 s_4} =
$$

$$
\begin{cases}
(P)_{s_1 s_2} (P)_{s_3 s_4} \frac{(-1)^{m/2} m! (n/2 - 2)!}{(n/2 - 2 - m/2)! (m/2)!} & \text{if } m \text{ even}, \\
0 & \text{if } m \text{ odd}
\end{cases}
$$

(43)

where the square brackets denote antisymmetrization in all $m$ enclosed indices (including the usual $1/m!$ factor). At this stage, the remaining Dirac $\gamma$-algebra is performed in four dimensions to obtain $\mathcal{T}_{ij}$.

Let us be reminded that Eq. (43) has been obtained for even integer, and is used as a definition for complex values of $n$. It is easily obtained if we observe that:

$$
P (-i \gamma_{2p-1} \gamma_{2p}) P = P \quad \text{for } p \geq 3
$$

(44)

As an example, an application of Eq. (43) gives

$$
\left( P \gamma^{\hat{\nu}_1} \gamma^{\hat{\bar{\nu}}_2} P \right)_{s_1 s_2} \left( P \gamma^{\hat{\bar{\nu}}_1} \gamma^{\hat{\nu}_2} P \right)_{s_3 s_4} = 0
$$

(45)

where there is no antisymmetrization over $\hat{\nu}_1$ and $\hat{\bar{\nu}}_2$. The final result for $\mathcal{T}_{ij}$ must then have the form:

$$
\mathcal{T}_{ij} \sim \sum_{A,B=1}^{5} t_{AB} (\Gamma_A)_{s_1 s_2} (\Gamma_B)_{s_3 s_4},
$$

(46)

where

$$
\Gamma_A \equiv P, \ P \gamma^{\hat{\bar{\nu}}} P, \ P \sigma^{\hat{\bar{\mu}} \hat{\bar{\nu}}} P, \ P \gamma^{\hat{\bar{\bar{\mu}}} \hat{\bar{\nu}}} P, \ P \gamma^5 P
$$

(47)
This is obvious because any $\gamma$-string $\mathcal{P}\mathcal{S}\mathcal{P}$ can be written as

$$\mathcal{P}\mathcal{S}\mathcal{P} = \begin{pmatrix} S_4 & 0 \\ 0 & 0 \end{pmatrix},$$  \hspace{1cm} (48)

where $S_4$ is the projected $4 \times 4$ submatrix. Thereupon, it can be written as a linear combination of the $\Gamma_A$ listed above. One can also perform "four dimensional Fierz transformation" for the $\Gamma_A$’s if that is necessary.

Before we close this section, we would like to make a few more remarks.

1. The final result for $\mathcal{T}$ is not fully Lorentz invariant (although it is invariant under a "four dimensional" Lorentz transformation). One could however rewrite Eq. (46) with the use the following relations:

$$\mathcal{P}\gamma^{\mu}\mathcal{P} \otimes \mathcal{P}\gamma^{\bar{\mu}}\mathcal{P} = \mathcal{P}\gamma^{\mu}\mathcal{P} \otimes \mathcal{P}\gamma^{\bar{\mu}}\mathcal{P}$$

$$\mathcal{P}\sigma^{\bar{\mu}\bar{\nu}} \mathcal{P} \otimes \mathcal{P}\sigma^{\bar{\mu}\bar{\nu}} \mathcal{P} = \mathcal{P}\sigma^{\mu\nu} \mathcal{P} \otimes \mathcal{P}\sigma^{\mu\nu} \mathcal{P} + a_0 \mathcal{P} \otimes \mathcal{P}$$

where $a_0$ is proportional to $\epsilon$. This apparently "restores" the full Lorentz invariance (apart from the projector $\mathcal{P}$ which can then be reabsorbed into the external spinors: $\mathcal{P}u(p) = u(p)$).

2. The spinors $u^{(1)}(p)$ and $u^{(2)}(p)$ which describe external particles, satisfy:

$$2 \sum_{i=1}^2 u^{(i)}(p) \bar{u}^{(i)}(p) = \frac{1}{2E(m+E)}(-i\gamma.p + m) \mathcal{P} \frac{1 + \gamma_4}{2} \mathcal{P} (-i\gamma.p + m)$$

$$= \mathcal{P} \frac{1}{2E}(-i\gamma.p + m) \mathcal{P}$$

The above relation is easily obtained if we observe that $p = (p_1, p_2, p_3, p_4, 0, \ldots, 0)$, and

$$2 \sum_{i=1}^2 V^{(i)} V^{(i)*} = \mathcal{P} \frac{1 + \gamma_4}{2} \mathcal{P}$$

Eq. (50) is used to calculate cross-sections, as an example.

3. If we have more than four external fermions, we have to generalize Eq. (43). For example, expressions like

$$\left(\mathcal{P}\gamma^{[\bar{\mu}_1} \ldots \gamma^{\bar{\mu}_2]}\mathcal{P}\right) \otimes \left(\mathcal{P}\gamma^{[\bar{\nu}_1} \ldots \gamma^{\bar{\nu}_2]}\mathcal{P}\right) \otimes \left(\mathcal{P}\gamma^{[\bar{\alpha}_1} \ldots \gamma^{\bar{\alpha}_2]}\mathcal{P}\right)$$

(52)
are first calculated in even integer \( n \) dimensions. The result is then taken as the
definition of that expression for complex values of \( n \). The Lorentz indices in the
above equation must of course come in pair. Here are a few examples:

\[
\left( P \gamma[^{\tilde{\mu}_1} \gamma^{\tilde{\mu}_2} \gamma^{\tilde{\mu}_3} \gamma^{\tilde{\mu}_4}] \right) \otimes \left( P \gamma[^{\mu}_1 \gamma^{\tilde{\mu}_2} \gamma^{\tilde{\mu}_3} \gamma^{\tilde{\mu}_4}] \right) \otimes \left( P \gamma[^{\tilde{\mu}_1} \gamma^{\tilde{\mu}_2} \gamma^{\mu}_1 \gamma^{\mu}_2] \right) = -3(n-4)(n-6)(n-8) \; P \otimes P \otimes P,
\]

\[
\left( P \gamma[^{\tilde{\mu}_1} \gamma^{\tilde{\mu}_2} \gamma^{\tilde{\mu}_3} \gamma^{\tilde{\mu}_4}] \right) \otimes \left( P \gamma[^{\mu}_1 \gamma^{\tilde{\mu}_2}] \right) \otimes \left( P \gamma[^{\tilde{\mu}_3} \gamma^{\mu}_4] \right) = (n-4)(n-6) \; P \otimes P \otimes P,
\]

\[
\left( P \gamma[^{\tilde{\mu}_1} \gamma^{\tilde{\mu}_2}] \right) \otimes \left( P \gamma[^{\tilde{\mu}_2} \gamma^{\tilde{\mu}_3}] \right) \otimes \left( P \gamma[^{\tilde{\mu}_3} \gamma^{\mu}_1] \right) = 0.
\]

4. We now give an example at the two-loop order to show how the projector \( P \)
can be used to derive the effective theory to this order. This is illustrated in Fig.
6. Let \( \Gamma \) represent the two-loop diagram on the left, \( \gamma_0 \) its tree (heavy) subgraph,
and \( \gamma_1 \) its one-loop (heavy) subgraph. \( \gamma_0 \) is represented by the diagram on the left
in Fig. 3, and \( \gamma_1 \) by the diagram on the left in Fig. 4. Since \( \Gamma \) is finite, the Dirac
\( \gamma \)-algebra can be done in four dimensions with no ambiguities. We can also attach
a factor \( P \) to each external fermion without changing the value of the graph in the
limit \( n \to 4 \). The Zimmermann algebraic rearrangement for \( \Gamma \) gives:

\[
\Gamma = \left[ (1 - \tau_{\Gamma}^{(0)}) \left( 1 - \tau_{\Gamma}^{(0)} \right) \tau_{\gamma_0}^{(0)} \right] \Gamma \\
+ \left[ (1 - \tau_{\Gamma}^{(0)}) \tau_{\gamma_1}^{(0)} \right] \Gamma \\
+ \left[ \tau_{\Gamma}^{(0)} \right] \Gamma \\
+ \left[ (1 - \tau_{\Gamma}^{(0)}) \left( 1 - \tau_{\gamma_1}^{(0)} \right) \left( 1 - \tau_{\gamma_0}^{(0)} \right) \right] \Gamma
\]

(54)

A simple power counting argument shows that the last term in the above equation
is of order \( 1/m_w^4 \) and will be discarded. The first, second and third terms in Eq.
(54) can be identified to the first, second and third diagrams on the right in Fig. 6,
respectively. Note that Eq. (54) uses subtraction at zero momentum to renormalize
the four-quark operators. Also, each of the first three terms are finite: the Dirac
algebra can be done in four dimensions, and the external fermions carry implicitly a factor $\mathcal{P}$. Let us now rewrite Eq. (54) such that the four-quark operators are renormalized in the MS scheme:

$$\Gamma = \left(1 - \tau_{\Gamma}^{(e)}\right) \lim_{\epsilon \to 0} \left(1 - \tau_{\gamma_1}^{(e)}\right) \tau_{\gamma_0}^{(0)} \Gamma$$

$$+ \left(1 - \tau_{\Gamma}^{(e)}\right) \lim_{\epsilon \to 0} \left(\tau_{\gamma_1}^{(0)} - \tau_{\gamma_1}^{(e)}\right) \left(1 - \tau_{\gamma_0}^{(0)}\right) \Gamma$$

$$+ \left(\tau_{\Gamma}^{(0)} - \tau_{\Gamma}^{(e)}\right) \left[1 - \lim_{\epsilon \to 0} \left(\tau_{\gamma_1}^{(0)} - \tau_{\gamma_1}^{(e)}\right) \left(1 - \tau_{\gamma_0}^{(0)}\right) - \lim_{\epsilon \to 0} \left(1 - \tau_{\gamma_1}^{(e)}\right) \tau_{\gamma_0}^{(0)}\right] \Gamma$$

What we have done is simply to rearrange the terms in Eq. (54). In the above equation, " $\lim_{\epsilon \to 0}[...]$ " means precisely that we compute the finite subgraph " [...] " by first assuming its external particles are in four dimensions. The result is then reinserted in $n$ dimensions. The operation $\tau_{\gamma}^{(e)}$ extracts the pole of the graph (or subgraph) $\gamma$ in the same fashion. Let us recall that external particles carry a factor $\mathcal{P}$: when the result is reinserted in $n$ dimensions, the factor $\mathcal{P}$ must be removed.

When this projector method is used to renormalize the various four-quark operators, together with the use of an anti-commuting $\gamma^5$, the LLA results of $b \to s$ processes are identical to those of Ref.[1].

5. $b \to s$ processes in the LLA

In this section we discuss in the context of dimensional regularization and evanescent operators the processes $b \to s$, in the LLA. It is independent of section 4, and no projector is used. Here we present in the MS scheme a two loop calculation of the mixing of all the evanescent operators with the physical dimension five operators (also called the dipole moment operators). The results in other schemes can be obtained with the use of Eq. (16). We want to take note that we are now dealing with an infinite set of operators. It follows that the dimension of the mixing matrix is infinite. In order to make mathematical sense, in principle one needs
to address the issues of boundedness, convergence, asymptotic completeness, etc. These unfortunately we do not know how to deal with because some of the matrix elements are in fact unbounded in value. An example of inconsistency is given to show the potential pitfalls. On a formal basis, however, we shall show explicitly that there is a similarity transformation which will decouple the evanescent set from the physical operators we are interested in. Thus, based on this 'physical argument', one may give meaning to the mixing matrix as such.

For \( b \to s \) processes, the commonly used basis of physical operators is

\[
\begin{align*}
O_{61} &= (\bar{s}\gamma^\mu P_L c_\alpha)(\bar{e}\gamma^\nu P_L b_\beta), \\
O_{62} &= (\bar{s}\gamma^\mu P_L c_\beta)(\bar{e}\gamma^\nu P_L b_\alpha), \\
O_{63} &= (\bar{s}\gamma^\mu P_L u_\alpha)(\bar{u}\gamma^\nu P_L b_\beta), \\
O_{64} &= (\bar{s}\gamma^\mu P_L u_\beta)(\bar{u}\gamma^\nu P_L b_\alpha), \\
O_{65} &= (\bar{s}\gamma^\mu P_L b_\alpha)(\bar{u}\gamma^\nu P_L u_\beta + \ldots + \bar{b}\gamma^\mu P_L b_\beta), \\
O_{66} &= (\bar{s}\gamma^\mu P_L b_\beta)(\bar{u}\gamma^\nu P_L u_\alpha + \ldots + \bar{b}\gamma^\mu P_L b_\alpha), \\
O_{67} &= (\bar{s}\gamma^\mu P_L b_\alpha)(\bar{u}\gamma^\nu P_R u_\beta + \ldots + \bar{b}\gamma^\mu P_R b_\beta), \\
O_{68} &= (\bar{s}\gamma^\mu P_L b_\beta)(\bar{u}\gamma^\nu P_R u_\alpha + \ldots + \bar{b}\gamma^\mu P_R b_\alpha), \\
O_{51} &= \frac{1}{2} i g_s \bar{s} G_{\mu\nu} \sigma_{\mu\nu} (m_s P_L + m_b P_R) b, \\
O_{52} &= \frac{1}{2} i e Q_d \bar{s} F_{\mu\nu} \sigma_{\mu\nu} (m_s P_L + m_b P_R) b; \quad Q_d = -\frac{1}{3}.
\end{align*}
\]

\( \alpha \) and \( \beta \) are color indices.

A few words about this choice of basis are in order. When the operators \( O_{65}, \ldots, O_{68} \) are inserted in the Green functions corresponding to the processes \( b \to s\gamma, sG \), one encounters traces of Dirac matrices and \( \gamma^5 \). This has led some authors to use a scheme different from the naive dimensional regularization scheme (i.e. \( \gamma^5 \) anti-commuting), such as the t’Hooft-Veltman scheme or the dimensional reduction scheme. It should be observed that when we include evanescent operators,
this $\gamma^5$ problem is artificial and is created by the choice of the basis listed in Eq. (56). One can in fact choose a different basis of physical operators and use an anticommuting $\gamma^5$ unambiguously. For instance, one can replace the operators $O_{65}$, ..., $O_{68}$ by the following new operators

$$Q_{65} = (\bar{s}_\alpha \gamma^\mu P_L b_\alpha) (\bar{u}_\beta \gamma^\mu u_\beta + \ldots + \bar{b}_\beta \gamma^\mu b_\beta),$$

$$Q_{66} = (\bar{s}_\alpha \gamma^\mu P_L b_\beta) (\bar{u}_\beta \gamma^\mu u_\alpha + \ldots + \bar{b}_\beta \gamma^\mu b_\alpha),$$

$$Q_{67} = (\bar{s}_\alpha \gamma^{[\mu \nu \gamma \sigma]} P_L b_\alpha) (\bar{u}_\beta \gamma^{[\mu \nu \gamma \sigma]} u_\beta + \ldots + \bar{b}_\beta \gamma^{[\mu \nu \gamma \sigma]} b_\beta),$$

$$Q_{68} = (\bar{s}_\alpha \gamma^{[\mu \nu \gamma \sigma]} P_L b_\beta) (\bar{u}_\beta \gamma^{[\mu \nu \gamma \sigma]} u_\alpha + \ldots + \bar{b}_\beta \gamma^{[\mu \nu \gamma \sigma]} b_\alpha),$$

which do not generate any trace involving $\gamma^5$. Thus if we include the corresponding evanescent four-quark operators (see below), then the use of a $\gamma^5$ anticommuting with all the $\gamma^\mu$ is justified.

Following the notation of Ref. [8], we define the totally antisymmetric matrices

$$\gamma^{[\mu_1 \ldots \mu_p]},$$

with the understanding that $\gamma^{[\mu_0]}$ is the identity matrix. The basis of evanescent operators that can mix with $Q_{67,68}$ can be chosen as (below $p$ is odd, and $p \geq 5)$:

$$Q_{67}^p = (\bar{s}_\alpha \gamma^{[\mu_1 \ldots \mu_p]} P_L b_\beta) (\bar{u}_\beta \gamma^{[\mu_1 \ldots \mu_p]} u_\beta + \ldots + \bar{b}_\beta \gamma^{[\mu_1 \ldots \mu_p]} b_\beta),$$

$$Q_{68}^p = (\bar{s}_\alpha \gamma^{[\mu_1 \ldots \mu_p]} P_L b_\beta) (\bar{u}_\beta \gamma^{[\mu_1 \ldots \mu_p]} u_\alpha + \ldots + \bar{b}_\beta \gamma^{[\mu_1 \ldots \mu_p]} b_\alpha).$$

We have checked explicitly that these are the only evanescent operators that can mix with the dimension five operators $O_{51,52}$, in the MS scheme.

Let us first rewrite Eq. (12) as:

$$O_t^{\text{ren}} = \sum_j Z_{ij} \mu^{-\epsilon D_j} O_j^{\text{bare}}$$

where the matrix $Z$ is dimensionless, $D_i = 0$ if $i$ corresponds to $O_{51,52}$, and $D_i = 1$ if $i$ corresponds to a four quark operator (physical or evanescent). The factor $\mu^{-\epsilon D_i}$ is introduced to make the mass dimension of $O_t^{\text{ren}}$ constant: $\text{dim}(O_i) = n + 2$ for
all \iota. With this choice, all the Wilson coefficients have dimension \textit{mass}^{-2}. To order \(g_s^2\), we have in the MS scheme:

\[ Z = 1 + \frac{g_s^2}{8\pi^2} B, \]
\[ \gamma_{ij} = -\frac{g_s^2}{8\pi^2} \left( 1 + D_i - D_j \right) B_{ij}. \]  

(61)

We want to calculate the matrix \(B\). As discussed, because we will not encounter any traces involving \(\gamma^5\), we will take for definiteness \(\gamma^5\) anti-commuting. Also, the calculation will be done in the general linear covariant gauges \(\frac{1}{2\pi} (\partial_\mu G_\mu)^2\) for the gluons.

First, we insert the evanescent operators in a four quark process: this gives the mixing of these operators with themselves. There are 6 one loop diagrams which give:

\[
\begin{align*}
B_{Q_{67}^p, Q_{67}^q} &= \frac{1}{6} \delta_{p+2,q} + (8 - \frac{32}{3} p + \frac{8}{3} p^2) \delta_{p,q} \\
&\quad + (-5 p + \frac{41}{6} p^2 - 2 p^3 + \frac{1}{6} p^4) \delta_{p-2,q}, \\
B_{Q_{68}^p, Q_{68}^q} &= -\frac{1}{2} \delta_{p+2,q} + (15 p - \frac{41}{2} p^2 + 6 p^3 - \frac{1}{2} p^4) \delta_{p-2,q}, \\
B_{Q_{68}^p, Q_{67}^q} &= -\frac{1}{4} \delta_{p+2,q} + (3 - 6 p + \frac{3}{2} p^2) \delta_{p,q} \\
&\quad + (\frac{15}{2} p - \frac{41}{4} p^2 + 3 p^3 - \frac{1}{4} p^4) \delta_{p-2,q}, \\
B_{Q_{68}^p, Q_{68}^q} &= -\frac{7}{12} \delta_{p+2,q} + (-1 + \frac{22}{3} p - \frac{11}{6} p^2) \delta_{p,q} \\
&\quad + (\frac{35}{2} p - \frac{287}{12} p^2 + 7 p^3 - \frac{7}{12} p^4) \delta_{p-2,q}.
\end{align*}
\]  

(62)

The above results will be needed when we insert the evanescent operators in \(b \to s\gamma\) and \(b \to sG\), to order \(g_s^2\). They have been obtained with the help of the following
identities:
\[ \gamma^\alpha [\mu_p] \gamma^\alpha = (-1)^p (n - 2p) \gamma^{[\mu_p]}, \]
\[ \gamma^\alpha [\mu_p] \otimes \gamma^\alpha [\mu_p] = \gamma^{[\mu_p]} \gamma^\alpha \otimes \gamma^{[\mu_p]} \gamma^\alpha = \gamma^{[\mu_p+1]} \otimes \gamma^{[\mu_p+1]} + p(n + 1 - p) \gamma^{[\mu_p-1]} \otimes \gamma^{[\mu_p-1]}, \]
\[ (63) \]
\[ \gamma^\alpha [\mu_p] \otimes \gamma^\alpha [\mu_p] \gamma^\alpha = ( -1)^p \left( \gamma^{[\mu_p+1]} \otimes \gamma^{[\mu_p+1]} - p(n + 1 - p) \gamma^{[\mu_p-1]} \otimes \gamma^{[\mu_p-1]} \right), \]

which were also derived in Ref. [8].

Next, we insert \( Q_{67,68}^p \) in \( b \to s \gamma \) and \( b \to s G \) to the lowest order. This is illustrated in Fig. 7. The results are finite, and will be used (at the end of the calculation) to "renormalize away" these evanescent operators; in other words, we will redefine them such that they vanish at the lowest order (for the LLA). For this, we define a finite matrix \( A \) such that if \( O_i \) is evanescent, then
\[ \Gamma(O_i + A_{ij} O_j) \rightarrow 0 \quad \text{when} \quad \epsilon \to 0. \]  
\[ (64) \]

The above equation defines \( A \) at lowest order in the strong coupling constant, and \( O_j = O_{51,52} \).

Before we present the results, we would like to give the various identities which are used to simplify the \( n \)-dimensional Dirac \( \gamma \)-algebra. We can easily derive, for arbitrary \( p \),
\[ \gamma^{[\mu_p]} \gamma^{[\mu_p]} = (-1)^{p(p-1)/2} \frac{n!}{(n-p)!}. \]  
\[ (65) \]
For \( p \geq 5 \) and \( n \to 4 \), Eq. (65) becomes
\[ \gamma^{[\mu_p]} \gamma^{[\mu_p]} = -24 \epsilon \left( -1 \right)^{p(p+1)/2} \frac{(p-5)!}{(p-5)!} + \mathcal{O} \left( \epsilon^2 \right). \]  
\[ (66) \]
More generally, we have (see also Ref.[11]):
\[ \gamma^{[\mu_p]} \gamma^{[\nu_q]} \gamma^{[\mu_p]} = \gamma^{[\mu_p]} \gamma^{[\mu_p]} \frac{\beta^{(p)}}{\gamma^{[\nu_q]} \gamma^{[\nu_q]}} \beta_q^{(p)} \gamma^{[\nu_q]}, \]  
\[ (67) \]
where the parameters $\beta^{(p)}_q$ are given by the following recursion relation:

\[
\begin{align*}
\beta^{(p)}_0 &= 1, \\
\beta^{(p)}_1 &= (-1)^p(n - 2p), \\
\beta^{(p)}_q &= (-1)^{p+q-1}(n - 2p) \beta^{(p)}_{q-1} + (q - 1)(n - q + 2) \beta^{(p)}_{q-2}
\end{align*}
\]

(68)

For the present calculation, the values of $\beta^{(p)}_q$, for $q = 0, 1, \ldots, 5$ only, are needed.

The above recursion relation is obtained with the help of the following identity:

\[
\gamma^{[\mu_1]} \otimes \gamma^{[\mu_p]} = \sum_{r=0}^{p} \alpha^{(p)}_r \gamma^{\mu_1} \cdots \gamma^{\mu_r} \otimes \gamma^{\mu_r} \cdots \gamma^{\mu_1};
\]

\[
\alpha^{(0)}_0 = \alpha^{(1)}_1 = 1, \quad \alpha^{(1)}_0 = 0,
\]

(69)

\[
\alpha^{(p)}_r = 0 \quad \text{if} \quad r < 0 \quad \text{or} \quad r > p,
\]

\[
\alpha^{(p+1)}_r = p(n + 1 - p) \alpha^{(p-1)}_r + (-1)^p \alpha^{(p)}_{r-1}.
\]

Also, the following relation holds:

\[
\beta^{(p)}_q = \sum_{r=0}^{q} \alpha^{(q)}_r \left[ (-1)^p(n - 2p) \right]^r.
\]

(70)

Our results for the matrix $A$ are, when $p \geq 5$:

\[
A_{Q^p_{67}, O_{51}} = A_{Q^p_{67}, O_{52}} = \frac{1}{3} A_{Q^p_{68}, O_{52}} = \frac{h(p)}{\pi^2},
\]

(71)

\[
A_{Q^p_{68}, O_{51}} = 0,
\]

where

\[
h(p) \equiv (-1)^{p+1/2} (p - 5)! (p - 1)(p - 3).
\]

(72)

Next, we insert these evanescent operators in $b \rightarrow s\gamma$ and $b \rightarrow sG$ to order $g_s^2$. This is illustrated in Figs. 8, 9 and 10. To extract the infinities of the two loop diagrams, we need to calculate the double pole $(1/\epsilon^2)$ only; this is because the Dirac $\gamma$-algebra always generates an extra $\epsilon$, as can be seen from Eq. (66). In the MS scheme, it is a well known fact that the double pole of a two-loop diagram is a polynomial in the external momenta and masses. One can therefore expand each
diagram in a power series in the external momenta and masses, to order mass$^2$. The calculation is straightforward and is done with the help of the symbolic manipulation program Schoonschip Ref.[12]. The results for $\gamma_{ij}$ (see Eq. (72)) are:

\[
\begin{align*}
\gamma_{Q_{67},O_{51}} &= \frac{g^2_s}{8\pi^4} h(p) \left( -\frac{40}{3} + \frac{28}{3} p - \frac{7}{3} p^2 \right), \\
\gamma_{Q_{68},O_{51}} &= \frac{g^2_s}{8\pi^4} h(p) \left( -2 + 8 p - 2 p^2 \right) \\
\gamma_{Q_{67},O_{52}} &= \frac{g^2_s}{8\pi^4} h(p) \left( -\frac{40}{3} + \frac{64}{3} p - \frac{16}{3} p^2 \right) \\
\gamma_{Q_{68},O_{52}} &= \frac{g^2_s}{8\pi^4} (-8) h(p).
\end{align*}
\]

Let us imagine that we solve Eq. (13), which formally has the solution

\[
C(\mu) = C(m_w) T \exp \left( \int_{m_w}^{\mu} \frac{d\mu}{\mu} \gamma(\mu) \right),
\]

where $T$ stands for normal ordering of the matrices $\gamma(\mu)$. For the LLA results, the mixing matrices $\gamma(\mu)$ commute for different $\mu$ and therefore the $T$-product can be omitted. Thus We obtain:

\[
C(\mu) = C(m_w) \exp \left( \frac{\tilde{\gamma}}{2 \beta_F} \log(\eta_w) \right),
\]

where

\[
\begin{align*}
\gamma &\equiv \frac{g^2_s}{8\pi^2} \tilde{\gamma}, \\
\eta_w &\equiv \frac{g^2_s(\mu)}{g^2_s(m_w)} \equiv 1 + \frac{\beta_F}{16\pi^2} g^2_s(\mu) \log \frac{m_w^2}{\mu^2}.
\end{align*}
\]

In the above equation, the only non-vanishing boundary conditions needed in the LLA are $C_{O_{61,63}}(m_w)$ and $C_{O_{51,52}}(m_w)$ (see Eq. (56)).

The matrix $\tilde{\gamma}$ has entries that become large as $p$ becomes large (see Eqs. (62 and 73). This tells us that the evanescent Wilson coefficients may become large as $p \to \infty$, and the convergence of

\[
\sum_{i=\text{evan}} C_{O_i} \gamma_{O_i,O_{51}},
\]
is immediately brought into question. For example, we do not know how to evaluate numerically $C_{Q_{67}^p}$ at say $\mu = 5$ GeV.

The handling of such an infinite set of operators is quite delicate. To illustrate, suppose one chooses instead a new basis $\{U_i\}$ of evanescent operators, defined by:

$$
U_{67}^5 \equiv Q_{67}^5, \quad U_{68}^5 \equiv Q_{68}^5, \\
U_{67}^{p+2} \equiv Q_{67}^p + \frac{1}{(p+1)(p-4)} Q_{67}^{p+2} \text{ for } p \geq 5, \\
U_{68}^{p+2} \equiv Q_{68}^p + \frac{1}{(p+1)(p-4)} Q_{68}^{p+2} \text{ for } p \geq 5.
$$

This new basis has the property that in the limit $\epsilon \to 0$, the renormalized Green functions of $U_{67,68}^p$, $p \geq 7$, vanish at the lowest order for $b \to s\gamma$ and $b \to sG$. Only $U_{67,68}^5$ have a non-vanishing value. Because the basis $\{U_i\}$ is infinite, one may query whether the operators $U_{67,68}^5$ are in fact linearly independent of the other $U_{67,68}^p$, $p \geq 7$. One may be flippant and formally writes

$$
U_{67}^{(5)} = \sum_{p=5}^{\infty} d_p U_{67}^{p+2}
$$

where the coefficients $d_p$ are obtained by the following recursion relation:

$$
d_{p+2} = -\frac{1}{(p+1)(p-4)} d_p; \quad d_5 = 1.
$$

Is one allowed then to choose a reduced basis $\{U_i\}$ which does not contain the two evanescent operators $U_{67,68}^5$? Formally, it seems that nothing should prevent one to start with this reduced new basis to renormalize all the physical and evanescent operators. The question is then: do the full basis and the reduced basis give the same on-shell results? If one uses Eq. (78), then the Green functions to $b \to s\gamma$ of $U_{67}^5$ at the lowest order are not well defined, since the term on the left in Eq. (78) gives non-zero, and each term on the right gives formally zero. This "problem" is obviously related to the fact that one has an infinite number of evanescent operators. It would not exist if the number of evanescent operators were finite. Unfortunately,
as we have seen, one cannot carry out the program of renormalization with a finite number of evanescent operators, except for a prescription in which no evanescent operator is introduced altogether.

One may skirt this potential difficulty by the following: the LLA result for the amplitude of $b \rightarrow s\gamma$ is

$$[C_i \Gamma(O_i)]^{\text{LLA}} = C_i^{\text{LLA}} \Gamma(O_i)^{\text{lowest order}}$$

where the sum is over all physical and evanescent operators. If we use Eq. (64), then we can rewrite Eq. (80) as

$$[C_i \Gamma(O_i)]^{\text{LLA}} = \sum_{i=\text{phys}} \left[ C_i^{\text{LLA}} - \sum_{j=\text{evan}} C_j^{\text{LLA}} A_{ji} \right] \Gamma(O_i)^{\text{lowest order}}$$

The above result was obtained in the MS scheme. It is however the same in all schemes (including mass-dependent schemes). One has redefined the evanescent operators such that their physical matrix elements vanish at the lowest order. This leads to

$$Z \rightarrow Z' = 1 + A + \frac{g_s^2}{8 \pi^2} B',$$

$$= (1 + A) Z.$$  

This manipulation was implicitly used in Ref. [2,3] and the anomalous dimension matrix becomes (see Eq. (16)):

$$\gamma \rightarrow \gamma' = (1 + A) \gamma (1 - A),$$

where $\gamma$ is the mixing matrix obtained in the MS scheme. To the order, we have used $(1 + A)^{-1} = 1 - A$, $A^2 = 0$. One can easily check that

$$\gamma_{Q_{67},O_{51}}' = 0,$$

$$\gamma_{Q_{66},O_{51}}' = 0,$$

$$\gamma_{Q_{67},O_{52}}' = 0,$$

$$\gamma_{Q_{66},O_{52}}' = 0.$$
Effectively, the mixing matrix between the four quark and the dipole moment operators is collapsed into a finite dimensional one, which has elements from physical to physical.

We can now finish constructing the mixing matrix between the physical four quark operators and the dipole moment operators. They agree with the results by Ciuchini et. al. Parenthetically, we remark again that in view of our previous discussion this requires the assumption that \( \lim_{\epsilon \to 0} \sum_{i=\text{evanes}} C_i \Gamma(O_i) = 0 \). In our opinion, this has not been firmly established because, as we have seen, \( C_{Q_{67}} \) may become unbounded as \( p \) approaches infinity.

The renormalized physical operators in the above scheme (where evanescent operators are first introduced then renormalized away) are related to those in the naive scheme in which there are no such operators by a finite transformation which depends explicitly on \( \mu \):

\[
[O_{\text{phys}}]^{\text{ev. scheme}} = (1 + g_s^2 \log(\mu/m_{\text{light}}) F + \ldots) [O_{\text{phys}}]^{\text{naive scheme}},
\]

where the finite matrix \( F_{ij} \sim \gamma_{ik}A_{kj} \). This means that the Wilson coefficients cannot be of the form given by Eq.(17) in both schemes. One of the schemes must be mass-dependent. This is what we referred to at the beginning of this article as the light-mass dependence of the \( C_i \). Thus, the problem has been shifted to a problem of factorization, namely, finding a prescription in which the Wilson coefficients do not depend on the masses of the light particles. This is our present understanding to account for the difference in results between Ref.[1] and Ref.[3]. One has to go to at least three loop order to settle the issue.

6. Concluding Remarks

In this article, we have attempted to locate the cause for the difference in the short distance analysis of \( b \to s\gamma \) between the scheme in Ref.[1,4] in which no
evanescent operators are introduced and those schemes in Ref.[2,3], which contain such operators. We have argued that if one follows a systematic formulation to arrive at an effective Lagrangian, which for our preference is the Zimmermann rearrangement, then there is ample freedom in dividing different schemes. We have explained how the Wilson coefficients $C_i$ depend on the scheme used to renormalize the operators. The boundary conditions also depend on the procedure. This can be easily understood if we observe that the identification and the subtraction of the ”infinities” are done differently in different schemes. Consequently, the $C_i$ to that order must be different, which affect the matching conditions. Such difference will exactly cancel out the difference in the anomalous mixing between various schemes. It is the S-matrix elements of the product $\sum C_i O_i$ which is independent of the scheme used. We have also argued that the problem of $\gamma^5$ is artificial, since all quantities are finite to begin with; provided one is consistent all prescriptions for handling $\gamma^5$ are equivalent.

In section 4, we have elaborated on a version of the t’Hooft Veltman scheme, in which all external particles are kept in four dimensions and we have developed an algorithm to handle the Dirac algebra for processes with more than two fermions, in particular when four-quark operators are introduced into the analysis. This scheme as laid out so far may, however, be incomplete, since it breaks Lorentz invariance in $n$ dimensions. It would be instructive to construct a Lagrangian which is Lorentz invariant only in four dimensions; for instance, we could replace every fermion field $\psi$ by $\mathcal{P}\psi$ (and $\bar{\psi}$ by $\bar{\psi}\mathcal{P}$) at the lagrangian level. This would have the advantage that in the evaluation of Feynman graphs derived from such a Lagrangian, the Dirac $\gamma$-algebra could be handled in four dimensions. We do not know if such a theory (or scheme) is consistent, but from a theoretical point of view we believe it is worth investigating.

In section 5, we have followed the approach in Ref.[2,3] in introducing evanes-
cent operators. We have shown that some entries in the mixing matrix are unbounded. However, by a formal finite transformation, which is induced by imposing a requirement that the matrix elements of the evanescent operators vanish at the lowest order, the mixing matrix elements of the new evanescent basis with the magnetic operators formally vanish also. The results of Ref.[3] (as far as the anomalous dimension mixing matrix of the physical operators) have been reproduced. We nonetheless are concerned that this scheme lacks certain justification in some formal manipulations which are related to the convergence of the infinite series with coefficients that eventually go to infinity.

We have also stressed that in going from finite order analysis to LLA with the help of RGE, we need to make sure that mass independence of $C_i$ is valid to facilitate the summation of leading logarithms. By this, we mean that the only $\log\mu$ dependence in $C_i$ can only be in the form $\log(M_{\text{heavy}}/\mu)$. We have not been able to establish this property in either scheme. We however believe that this is the most important aspect which needs to be clarified: it is our contention that this should account for the difference between the results of Ref.[1] and Ref.[3] and an unambiguous way to settle this is to perform an explicit higher loop Feynman diagram calculation, where the discrepancies first show up. Further work needs to be done.

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FIGURE CAPTION

Figure 1. Diagrams contributing to $b \to s\gamma$ at lowest order in the Standard Model.

Figure 2. The Zimmermann rearrangement for a one-loop graph contributing to $b \to s\gamma$.

Figure 3. The Zimmermann rearrangement for four-quark processes at the tree level. The effective vertex on the right is reinserted in one-loop diagrams contributing to $b \to s\gamma$, as shown on the first graph on the right in Fig. 2.

Figure 4. An example of the Zimmermann rearrangement for four-quark processes to order $g_w^2 g_s^2$.

Figure 5. Graphical representation of the next-to-leading logarithmic approximation final results for four-quark processes. In this approximation we are summing all terms of the forms $g_s^{2n} \log^n(m_w^2/\mu^2)$ and $g_s^{2n+2} \log^n(m_w^2/\mu^2)$. The final results are prescription independent since the "ambiguities" (concerning $\gamma^5$ and the inclusion or exclusion of evanescent operators) in the graphs on the right exactly cancel.

Figure 6. Example of the Zimmermann rearrangement to order $g_w^2 g_s^4$ for $b\bar{c} \to s\bar{c}$. In the LLA, one keeps the third graph on the right only; in the NLLA, one keeps the second graph on the right as well; in the next-to-next-to-leading logarithmic approximation, one must also include the contribution of the first graph on the right. Again, all the ambiguities we encounter in the graphs on the right cancel exactly, since the graph on the left is finite and prescription independent.

Figure 7. Contribution of four-quark operators to $b \to s\gamma$, $sG$ at lowest order. These diagrams also give the finite mixing matrix $A$ which is used to "renormalize away" the evanescent operators.

Figure 8. Two-loop graphs contributing to the mixing of four-quark operators with the dipole moment operators. The external wavy line represents a photon or a
gluon, and the black square represents an insertion of a four-quark operator.

Figure 9. More two-loop graphs contributing to $b \to sG$ with an insertion of a four-quark operator.

Figure 10. Graphs needed to remove the subdivergences of the two-loop graphs in Figs. 8,9. The crosses represent the necessary one-loop counterterms.
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