An Algebra Model for the Higher Order Sum Rules

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Abstract

We introduce an algebra model to study higher order sum rules for orthogonal polynomials on the unit circle. We build the relation between the algebra model and sum rules, and prove an equivalent expression on the algebra side for the sum rules, involving a Hall-Littlewood type polynomial. By this expression, we recover an earlier result by Golinskii and Zlat̋os, and prove a new case - half of the Lukic conjecture in the case of a single critical point with arbitrary order.

1 Introduction

OPUC (orthogonal polynomials on the unit circle) theory is an important field in mathematics introduced by Szegő, which has not only an intrinsic interest, but also many applications in different fields such as Spectral Theory, Random Matrix Theory, Combinatorics and so on (See Chapter 1 in Simon [13]). It has long been known in the OPUC theory that there is a one-to-one map between probability measures on the unit circle and their Verblunsky coefficients. See [13] for the definition of Verblunsky coefficients and their related properties. Let $\mathbb{D}$ be the unit disk. Let $\mu$ be a probability measure on $\partial \mathbb{D}$ with infinitely many points in its support, and let $(\alpha_n)_{n \geq 0}$ be its Verblunsky coefficients. Write

$$d\mu(\theta) = w(\theta) \frac{d\theta}{2\pi} + d\mu_s,$$

where $d\mu_s$ is singular with respect to $d\theta$. Then Szegő’s Theorem (see Simon [13] and Szegő [17]) implies that

$$\int_0^{2\pi} \log (w(\theta)) \frac{d\theta}{2\pi} > -\infty \iff \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty. \quad (1.1)$$

Results such as (1.1) are called "spectral theory gems" by Simon [14]. There have been lots of studies on higher order sum rules (gems), that is, to find the necessary and sufficient conditions on the Verblunsky coefficients side for the event

$$\int_0^{2\pi} \prod_{j=1}^{K} (1 - \cos(\theta - \theta_j))^{m_j} \log(w(\theta)) d\theta > -\infty. \quad (1.2)$$

In Simon [13], a set of conditions were conjectured; later Lukic ([9]) found a counterexample to Simon’s conditions, and introduced a modified conjecture. See a discussion of these conditions in [1].
One thing to note is that Lukic’s conditions and Simon’s conditions agree when $K = 1$, and in this case both state that $(1.2)$ is equivalent to $(S$ is the shift operator $(S\alpha)_n := \alpha_{n+1})$

$$(S - e^{-i\theta_1})^{m_1} \alpha \in l^2 \quad \text{and} \quad \alpha \in l^{2m_1 + 2}. \quad (1.3)$$

A lot of work has been done to study higher order sum rules ([2],[3],[5],[6],[7],[8],[10],[16]). Golinskii and Zlatos [8] proved that Simon’s conjecture is correct under the assumption that $\alpha \in l^4$. Simon and Zlatos [16] showed that in the case $(m_1, m_2) = (1, 1)$ or $(2, 0)$, Simon’s conjecture holds. Lukic [10] proved that in the case $K = 1$, under the assumption $(S - e^{-i\theta_1})^\alpha \in l^2$, (1.2) is equivalent to $\alpha \in l^{2m_1 + 2}$. Gamboa, Nagel and Rouault [7] found the relationship between higher order sum rules and the large deviations for random matrix models. The large deviation method was further developed in [1], where the authors proved the correctness of Lukic’s conjecture in several cases, including the case $(m_1, m_2) = (2, 1)$, where Lukic’s conditions differ from Simon’s conditions. The previous work focused on special cases where the orders are low, or studied the general cases but with additional assumptions. The main challenge is that all available expressions of (1.2) in terms of the Verblunsky coefficients are difficult to analyze.

In this work, we design an algebra model for the study of higher order sum rules, and build the relation between them. Under our algebra model, we prove that the left hand side of (1.2) could be expressed by a Hall-Littlewood type polynomial. With this model, we recover the result of [8], and by defining a degree function $L$ which relates Lukic’s conditions to the algebra side, we show that in the case $K = 1$ with arbitrary $m_1$, (1.3) leads to (1.2). Moreover, since all Lukic’s conditions are related to the polynomial $H$ defined in (1.4) below, and that our polynomial (1.9) is also defined by $H$, we expect our representation to shed light on other cases of Lukic’s conjecture as well.

1.1 Review of Breuer, Simon and Zeitouni [1]

We begin by stating a theorem, which is a combination of results in [1]. Write $Z$ as the set of integers, $Z_+$ as the set of positive integers, and $N$ as the set of non-negative integers. For $\theta_1, ..., \theta_K$ distinct in $[0, 2\pi)$, $m_1, ..., m_K \in Z_+$, and $d = \sum_{1 \leq j \leq K} m_j$, write

$$H(e^{i\theta}) := \prod_{j=1}^{K} (1 - \cos(\theta - \theta_j))^{m_j} = \frac{1}{2^d} \prod_{j=1}^{K} (e^{i\theta} - e^{-i\theta_j})^{m_j} (e^{-i\theta} - e^{i\theta_j})^{m_j} = \sum_{l=-d}^{d} h_le^{i\theta}, \quad (1.4)$$

where $h_l$’s $\in \mathbb{C}$. Hereafter we regard $H(\cdot)$ as a polynomial. Define

$$Z_H := \frac{1}{2\pi} \int_{0}^{2\pi} H(e^{i\theta}) d\theta, \quad (1.5)$$

and

$$V(x) := \frac{-1}{Z_H} \left( \sum_{l=1}^{d} \frac{h_l}{|l|} x^{|l|} + \sum_{l=-d}^{-1} \frac{h_l}{|l|} x^{|l|} \right). \quad (1.6)$$
Set $\alpha_n = 0$ if $n < -1$ and $\alpha_{-1} := -1$. Let $U_N$ be the $N \times N$ top-left corner of the GGT matrix (Section 4 of [13]), that is, $\forall k, l \in \{0, 1, ..., N - 1\},$

$$(U_N)_{kl} := \begin{cases} -\alpha_{k-1} \sigma_j \Pi_{j=k}^{l-1} \rho_j & 0 \leq k \leq l \\ \rho_l & k = l + 1 \\ 0 & k \geq l + 2 \end{cases}. \quad (1.7)$$

**Theorem 1** (Breuer, Simon and Zeitouni [1]). The inequality (1.2) holds if and only if

$$\limsup_{N \to \infty} \left( \text{Tr}(V(U_N)) - \sum_{n=0}^{N-1} \log(1 - |\alpha_n|^2) \right) < \infty. \quad (1.8)$$

From Theorem 3.2 in [1], we can write

$$\text{Tr}(V(U_N)) = bdy + \sum_{j=0}^{N-1-d} G(\alpha_j, ..., \alpha_{j+d}),$$

where $G$ is a degree $2d$ polynomial, and $bdy$ stands for boundary terms whose absolute value is bounded by an $N$-independent constant $C$. So the main focus is the part $\sum_{j=0}^{N-1-d} G(\alpha_j, ..., \alpha_{j+d})$. Note that $G(\alpha_j, ..., \alpha_{j+d})$ is not unique as mentioned in [1], because we can do shifts on the indices, such as replacing $\alpha_{j+1} \alpha_{j+2}$ by $\alpha_j \alpha_{j+1}$. In [1], a particular $G$ is calculated in several simple cases. Note that any choice of $G$ consists of even degree terms (see the remark above Theorem 3.3 in [1]), which allows us to write $G = \sum_{k=1}^{d} G_{2k}$, where $G_{2k}$ is a degree $2k$ homogeneous polynomial. In the next subsection, we introduce an algebra for studying $G$ and provide a corresponding expression for it.

### 1.2 An algebra model for gems

For $k \in \mathbb{Z}_+$, we consider the polynomial ring $A_{2k} := \mathbb{C}[x_1, y_1, ..., x_k, y_k]$. Given $\alpha_n \in \mathbb{D}^\infty$, we define a linear map $\phi_{2k} : A_{2k} \to \mathbb{D}^\infty$, such that

$$[\phi_{2k}(\prod_{i=1}^{k} x_i^{\beta_i} y_i^{\gamma_i})]_n = \prod_{i=1}^{k} \alpha_{n+\beta_i}^{\alpha_{n+\gamma_i}} \text{ for all } \beta_i, \gamma_i \in \mathbb{N}, i \in [k].$$

Let $\bar{A}_{2k} := \mathbb{C}[x_1, y_1, ..., x_k, y_k, \frac{1}{x_1}, \frac{1}{y_1}, ..., \frac{1}{x_k}, \frac{1}{y_k}]$. Define the factor rings $B_{2k} := A_{2k}/(\prod_{i=1}^{k} x_i y_i - 1)$, $\bar{B}_{2k} := \bar{A}_{2k}/(\prod_{i=1}^{k} x_i y_i - 1)$, and write $\psi_{2k}$ ($\bar{\psi}_{2k}$) as the natural homomorphism from $A_{2k}$ ($\bar{A}_{2k}$) to $B_{2k}$ ($\bar{B}_{2k}$). Then we have

**Lemma 1.** For each $k \in \mathbb{Z}_+$, if two polynomials $G^{(1)}$ and $G^{(2)} \in A_{2k}$ have the same image under $\psi_{2k}$, then there exists $C < \infty$ (might depend on $G^{(1)}$ and $G^{(2)}$, but is $N$-independent) such that for any $N \in \mathbb{Z}_+$, we have

$$\left| \sum_{n=0}^{N} [\phi_{2k}(G^{(1)})]_n - \sum_{n=0}^{N} [\phi_{2k}(G^{(2)})]_n \right| < C.$$
The motivation to consider the factor ring $B_{2k}$ is that it describes the fact that we can do indices shifts for $G$ when considering gems. For example, as mentioned in the previous subsection, one can replace $\alpha_j+1\alpha_j+2$ by $\alpha_j\alpha_j+1$ in $G(\alpha_j,\ldots,\alpha_{j+d})$. Note that the preimages of $\alpha_j+1\alpha_j+2$ and $\alpha_j\alpha_j+1$ in $A_2$ are $x_i^{j+1}y_i^{j+2}$ and $x_i^{j}y_i^{j+1}$ respectively, which have the same image under $\psi_2$. The next theorem provides us with an alternative way to show that $G^{(1)}, G^{(2)}$ have the same image under $\psi_{2k}$.

**Lemma 2.** For each $k \in \mathbb{Z}_+$, if $G^{(1)}, G^{(2)} \in A_{2k}$ have the same image under $\psi_{2k}$, then they also have the same image under $\psi_{2k}$.

Let $a_{k,p} := \prod_{s=p}^{k} y_s \prod_{s=p+1}^{k} x_s$, $b_{k,p} := \prod_{s=1}^{p} x_s y_s$ for $p \in [k]$. Following is the main theorem of our algebra model.

**Theorem 2.** The polynomial $G'_{2k} \in \tilde{A}_{2k}$ and $G_{2k}$ has the same image under $\psi_{2k}$, where

$$G'_{2k} := \frac{(-1)^{k+1}}{kZ_{H}} \sum_{1 \leq p,q \leq k} \frac{H(a_{k,p}b_{k,q})}{\prod_{s \in [k]\setminus \{p\}} (1 - \frac{a_{k,s}}{a_{k,p}}) \prod_{t \in [k]\setminus \{q\}} (\frac{b_{k,t}}{b_{k,s}} - 1)} - \frac{1}{k}, \quad (1.9)$$

**Remark 1.** While each element in the summation of $G'_{2k}$ might not be in $\tilde{A}_{2k}$, $G'_{2k}$ is in $\tilde{A}_{2k}$. It is a Hall-Littlewood type polynomial (see Section 3.2 in [11]).

**Remark 2.** Note that $\phi_{2k}$ is not a one-to-one map. For example, $x_1y_1x_2^2y_2^2$ has the same image as $x_1^2y_1^2x_2y_2$. Indeed, for any polynomial, one can apply permutations on $\{x_1,\ldots,x_k\}$ and $\{y_1,\ldots,y_k\}$, without changing its image under $\phi_{2k}$. However, $a_{k,p}$ and $b_{k,p}$ are defined for a fixed order of $x_i$’s and $y_i$’s. Therefore one can get a set of polynomials having the same image with $G_{2k}$ under $\tilde{\psi}_{2k}$, by changing the order of $\{x_1,\ldots,x_k\}$ and $\{y_1,\ldots,y_k\}$, and defining $a_{k,p}, b_{k,p}$ and $G'_{2k}$ correspondingly. One can also do averaging of these polynomials to get a bipartite symmetric polynomial in $\{x_1,\ldots,x_k\}$ and $\{y_1,\ldots,y_k\}$.

**Remark 3.** Observing that Lukic’s conditions are all related to $H$, an advantage of (1.9) is that it directly relates the left hand side of (1.2) to the polynomial $H$. Thus we expect that for the general cases of the Lukic conjecture, after a deeper analysis of (1.9), or other polynomials mentioned in Remark 2, one can find a way to decompose the polynomial such that each component in the decomposition is controlled by Lukic’s conditions.

With Theorem 2 we get the following corollary.

**Corollary 1.** The inequality (1.2) holds if and only if the following limit superior $< \infty$:

$$\limsup_{N \to \infty} \sum_{n=0}^{N-1} \sum_{k=1}^{d} \frac{\phi_{2k}(-1)^{k+1}}{kZ_{H}} \sum_{1 \leq p,q \leq k} \frac{H(a_{k,p}b_{k,q})}{\prod_{s \in [k]\setminus \{p\}} (1 - \frac{a_{k,s}}{a_{k,p}}) \prod_{t \in [k]\setminus \{q\}} (\frac{b_{k,t}}{b_{k,s}} - 1)} - \log(1 - |\alpha_n|^2) - \sum_{k=1}^{d} \frac{|\alpha_n|^{2k}}{k}, \quad (1.10)$$

**Remark 4.** The additional term $(\prod_{i \in [k]} x_i y_i)^{2k}$ is to make sure that the polynomial inside $\phi_{2k}$ is in $A_{2k}$. 

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Remark 5. From this corollary we see that the term $-1/k$ in (1.9) is not redundant but rather surprising, since $[\phi_{2k}(-1/k)]_n$ perfectly matches the kth order expansion of $\log(1-|\alpha_n|^2)$. Thus when $\|\alpha\|_d < \infty$ (for example under Lukic’s conditions), these terms $-1/k$ for $k \in [d]$ perfectly cancel the log terms in (1.8) up to a constant.

1.3 Higher order sum rules

With the above theorems, we can recover the following theorem.

**Theorem 3** (Golinskii and Zlatôs [8]). If $\alpha \in l^4$, then (1.2) is equivalent to

$$
\prod_{j=1}^{K} (S - e^{-i\theta_j})^{m_j} \alpha \in l^2.
$$

(1.11)

Then we prove that

**Theorem 4.** In the case that $K = 1$ and arbitrary $m_1$, (1.3) implies (1.2).

Theorem 4 is a new result in higher order sum rules. Note that it is similar to Lukic [10] but different, since in [10] there is an assumption $(S - e^{-i\theta_1})\alpha \in l^2$ which is stronger than (1.3).

As we can see, the analysis focuses on three parts: the Verblunsky coefficients part, the algebra part, and the sum rules part. In the rest of this paper we provide the proofs in the three parts separately. In Section 2, we show how to derive Theorem 1 based on [1], and we provide Lemma 3, which is a key step for Theorem 2. In Section 3, we give the proofs for all statements of Subsection 1.2. In Section 4, we show how to get Theorem 3, and prove Theorem 4 by the discrete Gagliardo-Nirenberg inequality and defining a degree function $L_{2k}$.

2 Proofs of the Verblunsky coefficients part

In this section, first we show how to combine the results of [1] to get Theorem 1, and then provide Lemma 3, which is an important step for Theorem 2.

**Proof of Theorem 1.** Substituting equations [1, (3.6) and (3.7)] into [1, (3.4)], after simple calculations we can verify that the function $V(\cdot)$ defined in (1.6) is the same as the $V(\cdot)$ in equation [1, (3.4)] with $C = 0$. With equations [1, (1.3) and (3.6)], we see that $H(\eta | \mu)$ is the integral we care about. With [1, Theorem 3.2 and Theorem 3.5], we see that Theorem 1 holds if $U_N$ is an $N \times N$ unitary CMV matrix. With the discussions above equation [1, (9.14)], it is easy to see that it also holds for $U_N$ the $N \times N$ top-left corner of the GGT matrix.

Next, we prove the following lemma, which analyzes the degree $2k$ terms in $\text{Tr}((U_N)^d)$. Write

$$
\bar{D}_{2k,l} := \{(i_1,j_1,\ldots,i_k,j_k) : \sum_{s=1}^{k} (j_s - i_s) = l, \forall s \in [k], i_s, j_s \in \mathbb{N}, j_s \geq i_s, j_s > i_{s+1}, i_{k+1} := i_1\},
$$

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and 
\[ D_{2k,l} := \bar{D}_{2k,l} \cap \{(i_1, j_1, ..., i_k, j_k) : i_1 = 0\}. \]

For F a polynomial in \( \mathbb{C}[\alpha_n, n \geq -1] \), denote by \( g_{2k}(F) \) the degree 2k terms in \( F \). Then we have

**Lemma 3.** For each \( l \in \mathbb{Z}_+ \), there exists \( C_l < \infty \) such that \( \forall N \in \mathbb{Z}_+ \),

\[ \left| g_{2k} \left( \text{Tr}((U_N)^l) \right) - (-1)^k \frac{l}{k} \sum_{n=0}^{N} \sum_{p=1}^{k} \alpha_{n+p} \alpha_{n+j_p} \right| < C_l. \]

Similarly, for \( l \in \mathbb{Z}_- \), there exists \( C_l < \infty \) such that \( \forall N \in \mathbb{Z}_+ \),

\[ \left| g_{2k} \left( \text{Tr}((U_N)^l) \right) - (-1)^k \frac{l|l|}{k} \sum_{n=0}^{N} \sum_{p=1}^{k} \alpha_{n+p} \alpha_{n+j_p} \right| < C_l. \]

**Proof of Lemma 3.** Fix \( l \in \mathbb{Z}_+ \), and write \( \rho_l := \sqrt{1 - |\alpha_l|^2} \). Define an \( N \times N \) matrix \( \tilde{U}_N \) as follows: \( \forall k, j \in \{0, 1, ..., N - 1\} \),

\[ (\tilde{U}_N)_{kj} := \begin{cases} -\alpha_{k-j} & 0 \leq k \leq l \\ \rho_l^2 & k = l + 1 \\ 0 & k \geq l + 2 \end{cases}. \]

We claim that \( \text{Tr}((U_N)^l) = \text{Tr}((\tilde{U}_N)^l) \). To see this, define

\[ F_l := \{(i_1, ..., i_l) : \forall s \in [l], \ i_s \in \{0, 1, ..., N - 1\}, i_{s+1} \geq i_s - 1, i_{s+1} := i_1\}, \]

then \( \text{Tr}((U_N)^l) = \sum_{(i_1, ..., i_l) \in F_l} \text{Tr}((U_N)_{i_1i_2}(U_N)_{i_2i_3}... (U_N)_{i_li_{i+1}}) \), and the same equation holds for \( \tilde{U}_N \). For each \( (i_1, ..., i_l) \in F_l \), by the form of the GGT matrix in (1.7) we observe that

\[ (U_N)_{i_1i_2}(U_N)_{i_2i_3}... (U_N)_{i_{i+1}} = \prod_{s=1}^{l} \left( 1 \{i_{s+1} = i_s - 1\} \rho_{i_{s+1}} + 1 \{i_{s+1} > i_s - 1\} (-\alpha_{i_s-1} \alpha_{i_{s+1}}) \prod_{q=i_s}^{i_{s+1}-1} \rho_q \right). \]

Assume \( i_{n_1} < i_{n_2} < ... < i_{n_p} \) such that \( \{i_{n_t}, t \in [p]\} = \{s : s \in [l], i_{s+1} > i_s - 1\} \). Let \( n_{p+1} = n_1 \), and define \( N^{(1)}_q := \# \{t \in [p] : i_{n_t} \leq q \leq i_{n_t+1} - 1\} \), \( N^{(2)}_q := \# \{t \in [p] : i_{n_t+1} \leq q \leq i_{n_{t+1}} - 1\} \). We claim that \( N^{(1)}_q = N^{(2)}_q \). This is because, if we draw a graph such that \( f(2t - 1) = i_{n_t} - 1/2 \) for \( t \in [p+1] \), \( f(2t) = i_{n_t+1} - 1/2 \) for \( t \in [p] \), and connect adjacent pairs of points by lines, then \( N^{(1)}_q, N^{(2)}_q \) are respectively the numbers of upcrossings and downcrossings of \( f \) w.r.t. the level \( p \), which must be equal by the fact that \( f(2p + 1) = f(1) \). From this we see that

\[ \prod_{t=1}^{p} \left( \prod_{q=i_{n_t}}^{i_{n_t+1}-1} \rho_q \right) = \prod_{q} \rho_q^{N^{(1)}_q} = \prod_{q} \rho_q^{N^{(2)}_q} = \prod_{t=1}^{p} \prod_{q=i_{n_t+1}}^{i_{n_t+1}-1} \rho_q, \]

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We define the map \( \varphi \) from \( \mathbb{N} \) to a length-1, and uniformly distributes its weight to the starting position. For \( n \in \mathbb{N} \), \( M^{-U} \) has the form

\[
\prod_{t=1}^{p} \left( -\alpha_{i_{nt}+1} \prod_{q=i_{nt}}^{i_{nt+1}-1} \rho_q \prod_{q=i_{nt+1}}^{\rho} \rho_q \right)
\]

Therefore \( \text{Tr}(\mathcal{N}^k) = \text{Tr}(\mathcal{N}^{k})^{l} \). For \( k \in \mathbb{Z}_+ \), we consider the terms in \( g_{2k} \left( \text{Tr}(\mathcal{N}^{k})^{l} \right) \). Each term has the form \( u_{i_{1}i_{2}i_{3}...u_{i_{k}i_{l}+1}} \), where \( (i_{1},...,i_{k}) \in F_{l} \), \( u_{i_{k}i_{l}+1} = -\alpha_{i_{k}+1} \) or 1 if \( i_{k}+1 = i_{k} - 1 \), \( u_{i_{k}i_{l}+1} = -\alpha_{i_{k}+1} \) if \( i_{k}+1 > i_{k} - 1 \), and \# \{s \in [l] : u_{i_{k}i_{l}+1} \neq 1\} = k \). Let

\[
\{n_{1},n_{2},...,n_{k}\} = \{s \in [l] : u_{i_{k}i_{l}+1} \neq 1\}, n_{1} < n_{2} < ... < n_{k}.
\]

We define the map \( \varphi \) from \( \mathcal{D}_{2k,l} \) to a degree \( 2k \) monomial, such that \( \varphi((i_{1},j_{1},...,i_{k},j_{k})) = \prod_{p=1}^{k} \alpha_{i_{p}j_{p}} \).

Write \( \Lambda := \{(1,2,...,k),(2,3,...,k),(3,4,...,k),(1,2),...,k,1,2,...,k-1\} \).

Consider the following weight distributing operation: assume that each \( u_{i_{1}i_{2}i_{3}...u_{i_{k}i_{l}+1}} \) has weight 1, and uniformly distributes its weight to \( k \) objects: \( (i_{n_{1}(1)} - 1, i_{n_{1}(1)+1}, i_{n_{2}(1)} - 1, i_{n_{2}(2)+1}, ..., i_{n_{k}(k)+1}) \) for \( \pi \in \Lambda \). This operation corresponds to the following identity

\[
u_{i_{1}i_{2}i_{3}...u_{i_{k}i_{l}+1}} = \frac{1}{k} \sum_{\pi \in \Lambda} \varphi((i_{n_{1}(1)} - 1, i_{n_{1}(1)+1}, i_{n_{2}(1)} - 1, i_{n_{2}(2)+1}, ..., i_{n_{k}(k)+1})).
\]

(2.1)

It is easy to verify that \( \forall \pi \in \Lambda, (i_{n_{1}(1)} - 1, i_{n_{1}(1)+1}, i_{n_{2}(1)} - 1, i_{n_{2}(2)+1}, ..., i_{n_{k}(k)+1}) \in \mathcal{D}_{2k,l} \).

Conversely, we claim that for each \( n \in [2d,N-2d] \) and \( (i_{1},j_{1},...,i_{k},j_{k}) \in \mathcal{D}_{2k,l} \), the term \( (n+i_{1},n+j_{1},...,n+i_{k},n+j_{k}) \) receives weight \( l/k \). To see this, clockwisely choose \( l \) positions on a circle. Put \( -\alpha_{n+i_{1}} \alpha_{n+j_{1}} \) at the 1st position, and put \( j_{1} - i_{2} - 1 \) number of 1’s in the next positions, then put \( -\alpha_{n+i_{1}+1} \alpha_{n+j_{1}} \) at the \( (j_{1} - i_{2}) \)th position. Continue doing this until all the \( l \) positions are filled. Then we can observe that, each preimage of \( (n+i_{1},n+j_{1},...,n+i_{k},n+j_{k}) \) corresponds to a length-\( l \) consecutive sequence on the circle, which has \( l \) choices depending on how to choose the starting position. For \( n \in [2d,N-2d] \), since the \( \ell_{\infty} \) norm of the elements in \( \mathcal{D}_{2k,l} \) is bounded by \( l \leq d \), it is easy to verify that all the \( l \) choices correspond to \( l \) terms in \( g_{2k} \left( \text{Tr}(\mathcal{N}^{k})^{l} \right) \). Thus \( (n+i_{1},n+j_{1},...,n+i_{k},n+j_{k}) \) receives weight \( l/k \) (in some cases, due to symmetry these \( l \) choices of sequences are not all different. For example, it is possible that there are just \( M \) different choices with \( M\) divides \( l \). However this makes no influence: in this case, when we start from each one of these \( M \) choices and distribute their weights, there are just \( M \) different elements in \( \mathcal{D}_{2k,l} \) receiving weights, with each of them getting \( M/k \) weights. Thus we can easily verify that each element in \( \mathcal{D}_{2k,l} \) receives \( l/k \) weights). Interpreting the weights as the coefficients in the summation as (2.1), and noting that the contribution of other terms with \( i_{1} \notin [2d,N-2d] \) is controlled by a constant
independent of $N$, we complete the proof for $l \in \mathbb{Z}_+$. Similar applies to $l \in \mathbb{Z}_-$. 

3 Proofs of the algebra part

In this section we prove all the statements in Section 1.2. First we show Lemma 1.

Proof of Lemma 1. Since $\phi_{2k}$ is linear, it is enough to show that if $G(3) \in A_{2k}$ and $\psi_{2k}(G(3)) = 0$ in $\tilde{B}_{2k}$, then there exists an $N$-independent $C < \infty$ such that $|\sum_{n=0}^{N} [\phi_{2k}(G(3))]_n| < C$ for all $N$. Since $\psi_{2k}(G(3)) = 0$, we can write $G(3) = (\prod_{i=1}^{k} x_i y_i - 1) \left(\sum_{s=1}^{M} c_s \prod_{i=1}^{k} x_i^{p_{i,s}} y_i^{q_{i,s}}\right)$ where $c_s \neq 0, p_{i,s}, q_{i,s} \in \mathbb{N}$ for $s \in [M], i \in [k]$. By the linearity of $\phi_{2k}$ it suffices to show that $\forall s \in [M]$ there exists an $N$-independent $C_s < \infty$ such that $|\sum_{n=0}^{N} [\phi_{2k}(\prod_{i=1}^{k} x_i y_i - 1) c_s \prod_{i=1}^{k} x_i^{p_{i,s}} y_i^{q_{i,s}})_n| < C_s$ for all $N$, which is implied by the fact that $\forall N \in \mathbb{Z}_+$

$$\left|\sum_{n=0}^{N} [\phi_{2k}(\prod_{i=1}^{k} x_i y_i - 1) c_s \prod_{i=1}^{k} x_i^{p_{i,s}} y_i^{q_{i,s}})_n\right| = c_s \prod_{i=1}^{k} \alpha_{N+1+\beta_i \alpha_{N+1+\gamma_i}} - \prod_{i=1}^{k} \alpha_{\beta_i \gamma_i} \leq 2c_s,$$

where in the rightmost inequality we use the fact that $\|\alpha\|_{\infty} \leq 1$. 

Next we show Lemma 2.

Proof of Lemma 2. Since $\phi_{2k}$ is linear, it is enough to show that if $G(3) \in A_{2k}$ and $\tilde{\psi}_{2k}(G(3)) = 0$ in $\tilde{B}_{2k}$, then $\psi_{2k}(G(3)) = 0$. We can write $G(3) = (\prod_{i=1}^{k} x_i y_i - 1) \left(\sum_{s=1}^{M} c_s \prod_{i=1}^{k} x_i^{p_{i,s}} y_i^{q_{i,s}}\right)$ where $c_s \neq 0, p_{i,s}, q_{i,s} \in \mathbb{Z}$ for $s \in [M], i \in [k]$, and $(p_{i,s}, q_{i,s})$ are different for different $s$. We claim that we must have $p_{i,s}, q_{i,s} \in \mathbb{N}$ for $s \in [M], i \in [k]$. Otherwise, without loss of generality we assume $p_{1,1} < 0$. Consider the set $\Gamma := \{s_0 \in [M] : s_0 = \arg \min \{p_{1,s}\}\}$. Since $(p_{i,s}, q_{i,s})$ are different for $s \in \Gamma$, $\sum_{s \in \Gamma} c_s q_{i,s} \prod_{i=2}^{k} x_i^{p_{i,s}} y_i^{q_{i,s}} \neq 0$. Therefore after expanding $G(3)$ according to the degree of $x_1$ as an element in $C[y_1, \ldots, x_k, y_k, \frac{1}{y_1}, \ldots, \frac{1}{x_k - x_1}][x_1, \frac{1}{x_1}]$, there is a term $-x_1^{-\min\{p_{i,s}\}} \sum_{s \in \Gamma} c_s q_{i,s} \prod_{i=2}^{k} x_i^{p_{i,s}} y_i^{q_{i,s}}$. Since $\min\{p_{i,s}\} < 0$, we see that $G(3) \notin A_{2k}$, leading to a contradiction. So $p_{i,s}, q_{i,s} \in \mathbb{N}$ for all $s \in [M], i \in [k]$, and it completes the proof. 

Based on Lemma 3, we prove Theorem 2 in the following.

Proof of Theorem 2. For each $l \in \mathbb{Z}_+$, note that $[\phi_{2k}(\sum_{D_{2k,l}} \prod_{p=1}^{k} x_p^{v_p} y_p^{J_p})]_n = \sum_{D_{2k,l}} \prod_{p=1}^{k} \alpha_{n+i_p \alpha_{n+j_p}}$. With Lemma 3 it suffices to calculate $\sum_{D_{2k,l}} \prod_{p=1}^{k} x_p^{v_p} y_p^{J_p}$. Write

$$E_{k,l} := \{(v_1, \ldots, v_k) : \forall p \in [k], v_p \in \mathbb{N}, \sum_{p=1}^{k} v_p = l\},$$

$$\tilde{E}_{k,l} := \{\tilde{v}_1, \ldots, \tilde{v}_k) : \forall p \in [k], \tilde{v}_p \in \mathbb{Z}_+, \sum_{p=1}^{k} \tilde{v}_p = l\}.$$
It is not hard to see that there is a one-to-one map between $E_{k,l} \times \tilde{E}_{k,l}$ and $D_{2k,l}$ as follows:

$$i_p = \sum_{s=1}^{p-1} (v_s - \bar{v}_s), \quad j_p = i_p + v_p. \quad (3.1)$$

For $\forall p \in [k]$, we define

$$a_{k,p} := \prod_{s \geq p} y_s \prod_{s \geq p+1} x_s, \quad b_{k,p} := \prod_{1 \leq s \leq p} x_s y_s.$$

With the one-to-one map (3.1), after some algebra we can see that in $\tilde{B}_{2k}$

$$\sum_{D_{2k,l} \ p=1}^k \prod x_i y_j = \left( \sum_{E_{k,l} \ p=1}^k a_{k,p}^{v_p} \right) \left( \sum_{\tilde{E}_{k,l} \ p=1}^k b_{k,p}^{\tilde{v}_p} \right). \quad (3.2)$$

We claim that

$$\sum_{E_{k,l} \ p=1}^k a_{k,p}^{v_p} = \sum_p \prod_{s \neq p} (1 - a_{k,s}/a_{k,p}), \quad \sum_{\tilde{E}_{k,l} \ p=1}^k b_{k,p}^{\tilde{v}_p} = \sum_p \prod_{s \neq p} \left( b_{k,p}/b_{k,s} - 1 \right). \quad (3.3)$$

One way to prove (3.3) is by induction on $l$ (also see (2.9) and (2.10) in [11]. Letting $t = 0$ in (2.10) and expanding the generating function, we get (3.3). Combining (3.2) and (3.3), we get

$$\sum_{D_{2k,l} \ p=1}^k x_i y_j = \sum_{1 \leq p,q \leq k} (a_{k,p} b_{k,q})^l \prod_{s \neq p} (1 - a_{k,s}/a_{k,p}) \prod_{s \neq q} \left( b_{k,q}/b_{k,s} - 1 \right). \quad (3.4)$$

Next we consider $l \in \mathbb{Z}_-$. Note that $[\phi_{2k} \left( \sum_{D_{2k,l} \ p=1}^k y_j x_i \right)]_n = \sum_{D_{2k,l} \ p=1}^k \prod_{p=1}^k \frac{\alpha_{n+i_p} \alpha_{n+j_p}}{\alpha_{n+i_p} \alpha_{n+j_p}}$ where $y_0 := y_k$. Define

$$c_{k,p} := \prod_{s \geq p} x_s \prod_{s \geq p} y_s, \quad d_{k,p} := \prod_{1 \leq s \leq p} y_{s-1} x_s.$$

With the similar analysis to the $l > 0$ case, we get that in $\tilde{B}_{2k}$,

$$\sum_{D_{2k,l} \ p=1}^k y_j x_i = \left( \sum_{E_{k,l} \ p=1}^k c_{k,p} \right) \left( \sum_{\tilde{E}_{k,l} \ p=1}^k d_{k,p} \right).$$
Let \( e_{k,p} = c_{k,p} \) for \( 2 \leq p \leq k \), and \( e_{k,k+1} = e_{k,1} = c_{k,1}/(\prod_{i=1}^{k} x_i y_i) \). In \( \tilde{B}_{2k} \) we have

\[
\left( \sum_{E_{k,i} \neq 1}^{k} c_{k,p} \right) \left( \sum_{E_{k,i} \neq 1}^{k} \tilde{c}_{k,p} \right) = \left( \sum_{E_{k,i} \neq 1}^{k} e_{k,p} \right) \left( \sum_{E_{k,i} \neq 1}^{k} \tilde{e}_{k,p} \right).
\]

It is easy to verify that \( \forall p, q \in [k] \),

\[
d_{k,p} e_{k,q+1} = \left( \prod_{i=1}^{k} x_i y_i \right)^2/(a_{k,p} b_{k,q}),
\]

\[
\prod_{s \neq p} (1 - a_{k,s}/a_{k,p}) \prod_{t \neq q} (b_{k,q}/b_{k,t} - 1) = \prod_{s \neq q} \prod_{t \neq p} (1 - e_{k,s+1}/e_{k,q+1})(d_{k,p}/d_{k,t} - 1). \tag{3.5}
\]

With (3.3), (3.5) and some algebra, we get that in \( \tilde{B}_{2k} \)

\[
\left( \sum_{E_{k,i} \neq 1}^{k} e_{k,p} \right) \left( \sum_{E_{k,i} \neq 1}^{k} \tilde{e}_{k,p} \right) = \sum_{1 \leq p, q \leq k} \prod_{s \neq p} \prod_{t \neq q} (1 - a_{k,s}/a_{k,p})(b_{k,q}/b_{k,t} - 1)^{-1} \]. \tag{3.6}

which is in \( \tilde{A}_{2k} \) by (3.3). Now, combining (1.6), (3.4), (3.6) and Lemma 3, we see that the following polynomial has the same image as \( G_{2k} \) under \( \tilde{\psi}_{2k} \):

\[
(-1)^{k+1} \frac{1}{kZ_H} \sum_{1 \leq p, q \leq k} \frac{H(a_{k,p} b_{k,q}) - h_0}{\prod_{s \neq p} \prod_{t \neq q} (1 - a_{k,s}/a_{k,p})(b_{k,q}/b_{k,t} - 1)}. \tag{3.7}
\]

Finally we show where the term \(-1/k\) comes from. According to the definition of \( Z_H \), it is easy to verify that \( h_0 = Z_H \). Because

\[
\sum_{1 \leq p, q \leq k} \prod_{s \neq p} \prod_{t \neq q} (1 - a_{k,s}/a_{k,p})(b_{k,q}/b_{k,t} - 1) = \sum_{p} \prod_{s \neq p} (1 - a_{k,s}/a_{k,p}) \sum_{p} \prod_{s \neq p} (b_{k,p}/b_{k,s} - 1) = (-1)^{k},
\]

which could be proved by induction, combined with (3.7), \(-1/k\) appears and the proof is completed.

\[\square\]

### 4 Proofs of the sum rules part

In this section we first show that for any \((\theta_j, m_j)_{1 \leq j \leq K}\), the degree 2 term \( G_2 \) matches the condition (1.11). This match recovers the result in [8]. We then provide the proof of Theorem 4, by the discrete Galiardo-Nirenberg Inequality and a degree function \( L_{2k} \) which relates Lukic’s conditions to the algebra model.
Proof of Theorem 3. Under the assumption \( \alpha \in l^1 \), we have \( \limsup_{N \to \infty} \sum_{n=0}^{N-1} (-|\alpha_n|^2 - \log(1 - |\alpha_n|^2)) < \infty \) (for example see Proposition 4.1 in [1]). Note that in \( \tilde{B}_2 \),

\[
H(a_{1,1}b_{1,1}) = \frac{1}{2\pi} \prod_{j=1}^{K} (x_1y_1^2 - e^{i\theta_j})^{m_j} (1/(x_1y_1^2) - e^{-i\theta_1})^{m_1} = \frac{1}{2\pi} \prod_{j=1}^{K} (y_1 - e^{i\theta_j})^{m_j} (x_1 - e^{-i\theta_1})^{m_1}.
\]

Since

\[
[\phi_2(\frac{1}{2\pi} \prod_{j=1}^{K} (y_1 - e^{i\theta_j})^{m_j} (x_1 - e^{-i\theta_1})^{m_1})]_n = \left| (\prod_{j=1}^{K} (S - e^{-i\theta_j})^{m_j})_n \right|^2,
\]

applying Lemma 1 and Lemma 2 to \( H(a_{1,1}b_{1,1}) / Z_H \) and \( \prod_{j=1}^{K} (y_1 - e^{i\theta_j})^{m_j} (x_1 - e^{-i\theta_1})^{m_1} / 2^d \), with (4.1) and Corollary 1, the proof is completed.

The idea to prove Theorem 4 is the following. For each \( k \in \mathbb{Z}_+ \), we define \( L_{2k} \), a map from \( A_{2k} \) to \( \mathbb{N} \), as follows. For \( F \in A_{2k} \), we first do the Taylor expansion of \( F \) at the point \((e^{-i\theta_1}, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_1})\), such that

\[
F = \sum_{s=1}^{M} C_s \prod_{p=1}^{k} (x_p - e^{-i\theta_1})^{\beta_{p,s}} (y_p - e^{i\theta_1})^{\gamma_{p,s}},
\]

where \( M \in \mathbb{Z}_+, \forall s \in [M], p \in [k], C_s \neq 0, \beta_{p,s}, \gamma_{p,s} \in \mathbb{N} \), and \((\beta_{p,s}, \gamma_{p,s})_{1 \leq p \leq k} \) are distinct for different \( s \). Then let

\[
L_{2k}(F) := \min_{1 \leq s \leq M} \left( \sum_{p=1}^{k} (\beta_{p,s} \wedge d + \gamma_{p,s} \wedge d) \right).
\]

Since the Taylor expansion is unique, \( L_{2k} \) is well-defined.

Lemma 4. Under condition (1.3), if \( F \in A_{2k} \) and \( L_{2k}(F) \geq 2(d + 1) - 2k \), then

\[
\limsup_{N \to \infty} \sum_{n=0}^{N} [\phi_{2k}(F)]_n < \infty.
\]

To this end, in order to prove Theorem 4, it suffices to show the following lemma.

Lemma 5. \( \forall k \in [d], \) there exists \( G''_{2k} \in A_{2k} \) such that \( G''_{2k} = G'_{2k} \) in \( \tilde{B}_{2k} \), and \( L_{2k}(G''_{2k}) \geq 2(d + 1) - 2k \).

To prove Lemma 4, we need the following discrete Gagliardo-Nirenberg Inequality. The references of this inequality are Gagliardo [4], Nirenberg [12], and also see the the remark of Theorem 2.5 in [1], Section 6.3 of Simon [15] and Taylor [18].

Lemma 6 (Discrete Gagliardo-Nirenberg Inequality). If \( (S - e^{-i\theta_1})^d \alpha \in l^2 \) and \( \alpha \in l^{2d+2} \), then for any \( j \in [d + 1] \) we have

\[
(S - e^{-i\theta_1})^j \alpha \in l^{\frac{2(d+1)}{j+1}}.
\]
Remark 6. Since in our case \( \|\alpha\|_\infty \leq 1 \), under the same conditions we have \( (S - e^{-i\theta_1})^q \alpha \in l^{2(d+1)} \) for any \( q \geq j \).

Now we prove Lemma 4.

Proof of Lemma 4. It suffices to prove it for \( F = \prod_{p=1}^{k} (x_p - e^{-i\theta_1})^{\beta_p} (y_p - e^{i\theta_1})^{\gamma_p} \) with

\[
\sum_{p=1}^{k} (\beta_p \wedge d + \gamma_p \wedge d) \geq 2(d+1) - 2k.
\]

Write \( \tilde{\beta}_p = \beta_p \wedge d \) and \( \tilde{\gamma}_p = \gamma_p \wedge d \). Let \( \lambda := \sum_{p=1}^{k} (\tilde{\beta}_p + 1 + \tilde{\gamma}_p + 1) \), then with the Hölder’s Inequality we have

\[
\sum_{n} \prod_{p=1}^{k} \left| (S - e^{-i\theta_1})^{\beta_p} \alpha \right| \left| (S - e^{-i\theta_1})^{\gamma_p} \alpha \right| \leq \prod_{p=1}^{k} \left| (S - e^{-i\theta_1})^{\beta_p} \alpha \right| \left| (S - e^{-i\theta_1})^{\gamma_p} \alpha \right| \frac{\lambda}{\lambda_p}. \tag{4.2}
\]

Recalling that \( \phi_{2k}(F)_n = \prod_{p=1}^{k} ((S - e^{-i\theta_1})^{\beta_p} \alpha)_n (S - e^{-i\theta_1})^{\gamma_p} \alpha)_n \), with the fact that \( \|\alpha\|_\infty \leq 1 \) and \( \lambda \geq 2(d+1) \), we finish the proof by (4.2) and Lemma 6.

Next we show Lemma 5. First we provide a method to calculate the polynomial \( G'_{2k} \). Define an operator \( D(x_1, ..., x_n)(\cdot) \) from \( \mathbb{C}[x, 1/x] \) to \( \mathbb{C}[x_1, ..., x_n, 1/x_1, ..., 1/x_n] \) as follows: for \( f(x) = \sum_{i=-d_1}^{d_2} c_i x^i \), let

\[
D(x_1, ..., x_n)(f) := \sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j \neq i}(x_j - x_i)}.
\]

We can observe that \( D(x_1, ..., x_n)(f) \) is a Hall-Littlewood type polynomial, and

\[
D(x_1, ..., x_n)(f) = \frac{D(x_1, x_3, ..., x_n)(f) - D(x_2, x_3, ..., x_n)(f)}{x_2 - x_1}. \tag{4.3}
\]

Proof of Lemma 5. Let

\[
f_1(b_{k,q}, x) := \left( xb_{k,q} - e^{i\theta_1} \right)^d ((\Pi_i x_i y_i)^2 / (xb_{k,q}) - e^{-i\theta_1})^d x^{k-1} b_{k,q}^{-1},
\]

\[
f_2(a_{k,1}, ..., a_{k,k}, x) := D(a_{k,1}, ..., a_{k,k})(f(x, \cdot)).
\]

After some algebra we can see that

\[
\sum_{1 \leq p, q \leq k} \frac{(a_{k,p} b_{k,q} - e^{i\theta_1})^d ((\Pi_i x_i y_i)^2 / (a_{k,p} b_{k,q}) - e^{-i\theta_1})^d}{1 - a_{k,s} / a_{k,p} \prod_{t \neq q}(b_{k,q} / b_{k,t} - 1)} = D(b_{k,1}, ..., b_{k,k})(f_2(a_{k,1}, ..., a_{k,k}, \cdot)) \prod_{t=1}^{k} b_{k,t}. \tag{4.4}
\]
Note that for any \( r_1, ..., r_M \in \mathbb{C} \) and \( \beta \in \mathbb{Z} \), we have
\[
\frac{x_1^\beta \prod_{i=1}^{M} (x_1 - r_i) - x_2^\beta \prod_{i=1}^{M} (x_2 - r_i)}{x_1 - x_2} = (\sum_{j=0}^{\beta-1} x_1^j x_2^{\beta-1-j} \prod_{i=1}^{M} (x_1 - r_i) + x_2^2 \sum_{i=1}^{M} \prod_{s<i}^{M} (x_1 - r_s) \prod_{t>i}^{M} (x_2 - r_t)),
\]
where in the right hand side each term contains a factor in the form of
\[
\prod_{k,q} a_{k,p} b_{k,q} - e^{i\theta_1} \left( \frac{\Pi_i x_i y_i}{x b_{k,q}} \right)^d e^{-i\theta_1} = \prod_{p,q} (a_{k,p} b_{k,q} - e^{i\theta_1}) (d_{p,q}^{(1)} e^{i\theta_1} - e^{-i\theta_1}) d_{p,q}^{(2)},
\]
where the right hand side is a multiplication of \( 2d \) terms. With (4.3), (4.5) and (4.7), it is not hard to observe that, we can express (4.4) as a summation, where each term in this summation contains a factor like
\[
\prod_{p,q} (a_{k,p} b_{k,q} - e^{i\theta_1}) (d_{p,q}^{(1)} e^{i\theta_1} - e^{-i\theta_1}) d_{p,q}^{(2)}.
\]
What’s more, each \((d_{p,q}^{(1)}, d_{p,q}^{(2)})_{p,q \in [k]}\) corresponds to some \((\tilde{d}_{p,q}^{(1)}, \tilde{d}_{p,q}^{(2)})_{p,q \in [k]}\) generated as follows. Put \( d \) white balls and \( d \) black balls alternately, that is, White, Black, ..., White, Black. Here White stands for \((xb_{k,q} - e^{i\theta_1})\), and Black stands for \((\Pi_i x_i y_i)/(xb_{k,q}) - e^{i\theta_1})\). Choose \( 0 = z_0 \leq z_1 \leq z_2 \leq ... \leq z_{k-1} \leq z_k = 2d \) where \( z_i \in \{0, 1, ..., 2d\} \forall i \in [k-1] \), and \( 0 = w_0 \leq w_1 \leq w_2 \leq ... \leq w_{k-1} \leq w_k = 2d \) where \( w_i \in \{0\} \cup \{0, 1, ..., d\} \setminus \{z_1, ..., z_{k-1}\} \forall i \in [k-1] \), then let
\[
\tilde{d}_{p,q}^{(1)} : = \#\{\text{White balls in } [z_{p-1}, z_p] \cap [w_{q-1}, w_q]\},
\]
\[
\tilde{d}_{p,q}^{(2)} : = \#\{\text{Black balls in } [z_{p-1}, z_p] \cap [w_{q-1}, w_q]\}.
\]
The correspondence between \((d_{p,q}^{(1)}, d_{p,q}^{(2)})_{p,q \in [k]}\) and \((\tilde{d}_{p,q}^{(1)}, \tilde{d}_{p,q}^{(2)})_{p,q \in [k]}\) could be observed by (4.6). Therefore, we see that \(d_{p,q}^{(1)}, \tilde{d}_{p,q}^{(1)} \in [d], \left|d_{p,q}^{(1)} - \tilde{d}_{p,q}^{(1)}\right| \leq 1\), and
\[
\sum_{p,q} (d_{p,q}^{(1)} + \tilde{d}_{p,q}^{(2)}) \geq 2d - 2(k-1),
\]
where the last inequality holds because in order to get (4.4), we need to do the operation like (4.3) for \(2(k-1)\) times, and each operation at most reduce degree 1 for these factors. Note that there are at most \(2k - 1\) pairs \((p, q)\) with \([z_{p-1}, z_p] \cap [w_{q-1}, w_q] \neq \emptyset\), thus \(\sum |d_{p,q}^{(1)} - \tilde{d}_{p,q}^{(2)}| \leq 2k - 1\), and with
(4.9) we see that
\[
\max_{i_{p,q} \in \{1,2\}} \forall_{p,q \in [k]} \sum_{p,q} d^{(i_{p,q})}_{p,q} \leq d. \tag{4.10}
\]
Now consider any \(f_3, f_4 \in A_{2k}\) where
\[
f_3 = \Pi_{p,q} \left( a_{k,p} b_{k,q} - e^{i\theta_1} \right) d^{(1)}_{p,q} \left( d_{k,p} e_{q+1} - e^{-i\theta_1} \right) d^{(2)}_{p,q} f_4.
\]
By the definition of \(a_{k,p} b_{k,q}\) and \(d_{k,p} e_{q+1}\), we can see that \(f_3\) has the same image with the following polynomial \(\tilde{f}_3\) under \(\phi_{2k}\):
\[
\tilde{f}_3 := \Pi_{p,q} \left( \Pi_{s \in I_{p,q}^{(1)}} x_s y_t - e^{i\theta_1} \right) d^{(1)}_{p,q} \left( \Pi_{s \in [k] \setminus I_{p,q}^{(1)}} x_s y_t - e^{-i\theta_1} \right) d^{(2)}_{p,q} f_4
\]
with \(I_{p,q}^{(1)}, I_{p,q}^{(2)} \subset [k]\) and \(|I_{p,q}^{(1)}| = |I_{p,q}^{(2)}| - 1\). Expand each \(\Pi_{s \in I_{p,q}^{(1)}} x_s y_t - e^{i\theta_1}\) as a Taylor series, whose degree 1 terms are exactly \(\sum_{s \in I_{p,q}^{(1)}} (x_s - e^{-i\theta_1}) + \sum_{t \in I_{p,q}^{(2)}} (y_t - e^{i\theta_1})\), and each higher degree term is divided by some degree 1 term. We claim that if we apply \(L_{2k}\) on each lowest degree term in the Taylor expansion of \(\tilde{f}_3/f_4\), the result \(\geq 2d - 2(k - 1)\), since for each such term, the degree of any \((x_s - e^{-i\theta_1})\) and \((y_t - e^{i\theta_1})\) is \(\leq d\) by (4.10), and the total degree is \(\geq 2d - 2(k - 1)\) by (4.9). Noting that each term in the Taylor expansion of \(\tilde{f}_3/f_4\) is divided by some lowest degree term in the expansion, we see that \(L_{2k}(\tilde{f}_3) \geq 2d - 2(k - 1)\), and the proof is completed by (4.8).

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