Topological Yang-Mills cohomology in pure Yang–Mills Theory

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Abstract

Using the first order formalism (BFYM) of the Yang-Mills theory we show that it displays an embedded topological sector corresponding to the field content of the Topological Yang-Mills theory (TYM). This picture arises after a proper redefinition of the fields of BFYM and gives a clear representation of the non perturbative part of the theory in terms of the topological sector. In this setting the calculation of the vev of a YM observable is translated into the calculation of a corresponding (non topological) quantity in TYM. We then compare the topological observables of TYM with a similar set of observables for BFYM and discuss the possibility of describing topological observables in YM theory.

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1 Introduction

The search for topological quantities in field theory has been strongly related to the efforts to perform non perturbative calculations and in particular in gauge theories has been related to the so-called instantons calculus [1]. It is widely believed that topologically non trivial configurations play a dominant role in phenomena like the quark confinement in QCD but the proper quantitative framework in which such a dynamics should emerge is still missing.

Topological theories arose as the first models in which topological quantities can be explicitly computed [2]. The twisting procedure derives these models from N=2 supersymmetric gauge theories but there is no systematic relationship with bosonic ones. Recently some interesting papers by Anselmi [3, 4] tried to shed light on the relation between topological theories and physical ones. In particular he suggested that Topological Yang-Mills theory (TYM) could be embedded in the ordinary Yang-Mills theory (YM) to account for the non perturbative sectors of the theory.

More recently [5], along similar ideas, YM theory has been translated in the first order formalism as a deformation of a topological theory of BF type [6], named BFYM theory. The full equivalence of BFYM with the standard second order formulation has been proved both with path integral [5] and with algebraic methods [7] and also the $uv$-behaviour has been checked to be the same [8]. This formulation of YM theory has an enlarged symmetry and field content from which new observables (inherited from the pure topological theory) can be defined [9, 10] and is promising to start both a deeper understanding of the topological and geometrical structure underlying gauge theories [11] and a new hint to disclose the long range dynamics of QCD. A wide review of the present status of the work on BFYM can be found in ref. [12]. A similar proposal to regard YM as deformation of a topological theory has been very recently discussed in [13].

The formulation of YM as a deformation of a topological theory of BF type strongly suggests the existence of an embedded topological sector, which should be related with the non perturbative, topologically non trivial features of the theory.

In this paper we explicitly show how this topological sector arises in BFYM after a suitable field redefinition. Precisely the theory is decomposed in TYM plus the local quantum fluctuations and a clear representation of the non perturbative part of the theory in terms of the topological sector is given, thus explicitly realizing the conjectures of [3, 4]. This point is discussed in section 2. We then discuss in section 3 the observables of BFYM. TYM observables can be directly compared with a similar set in BFYM, and the role of local fluctuations to spoil their topological character is clearly displayed. More generally our framework translates the computation of the $vev$ of every YM observable to the computation of a related (non topological) quantity in TYM. In this way we can formally give a set of sufficient conditions to be met by a YM observable in order to be topological.
2 BFYM in the self-dual gauge and relation with TYM

The classical Lagrangian of BFYM theory is given by [5]

\[ \mathcal{L}_{BFYM} = \text{Tr} \left\{ iB \wedge F_A + g^2 (B - \frac{1}{\sqrt{2g}} d_A \eta) \wedge \ast (B - \frac{1}{\sqrt{2g}} d_A \eta) \right\} , \]  

(2.1)

where \( d_A \equiv d + [A, \cdot] \) is the covariant derivative, \( A \) is the gauge field and where antihermitean conditions have been chosen for the generators of the Lie algebra. (Wedge products will always be understood in the following).

The 2-form \( B \) and the 1-form \( \eta \) represent the extra field content of the first order formulation of YM theory; nevertheless the physical degrees of freedom of the theory are not changed and the theory is still physically equivalent to the standard second order formulation. Indeed an enlarged symmetry content corresponds to the extra degrees of freedom and requires a proper gauge fixing and ghost structure.

The corresponding BRST transformations are [3]:

\[
\begin{align*}
s A &= d_A c \\
s c &= -\frac{i}{2} [c, c] \\
s B &= [B, c] - d_A \psi + \frac{1}{\sqrt{2g}} [F_A, \rho] \\
s \psi &= -[\psi, c] + d_A \phi \\
s \phi &= [\phi, c] \\
s \eta &= [\eta, c] - \sqrt{2g} \psi + d_A \rho \\
s \rho &= -[\rho, c] + \sqrt{2g} \phi ,
\end{align*}
\]

(2.2)

and amount to both the gauge and the “topological” symmetries. The latter corresponds to the symmetry present in the pure topological BF theory [3] and requires a ghost of ghost structure due to its reducible character; in particular the ghosts \( \psi \) and \( \phi \) are exactly those of pure BF theory. Note, in comparison with the pure topological theory in which no local degrees of freedom are present, that a local dynamics for the BFYM theory is restored by the vector field \( \eta \) and the associated ghost \( \rho \).

Moreover the field equations of (2.1) can be arranged [8] as

\[
\begin{align*}
d_A^\dagger A &= 0 \\
B &= -\frac{i}{2g} \ast F_A \\
\eta &= 0 ,
\end{align*}
\]

(2.3)

from which the moduli space \( \mathcal{M} \) of BFYM clearly appears to be the same of that of YM, and from which the classical value of \( \eta \) turns out to be zero, the proper on-shell value for a quantum fluctuation.

Therefore we want to interpret \( \eta \) as a gluon quantum fluctuation, and we interpret \( A_0 \) defined as \( A_0 = A - \sqrt{2g} \eta \) as the whole gluon field \textit{minus} its quantum fluctuations, i.e.
the background connection. Consistently with the BRST transformations, \( c_0 = c - \sqrt{2}g\rho \) will be regarded in a similar manner as the background gauge ghost and \( \rho \) as its quantum fluctuation. These are the first steps of a redefinition of the fields which will display the embedded topological sector of the theory.

To quantize the theory we have also to specify the gauge fixings and in order to study the instanton sector of the theory we choose the self-dual gauge-fixing \( B^- = 0 \) for the topological symmetry and a covariant gauge-fixing for the remaining symmetries; explicitly, the conditions we choose are:

\[
\begin{align*}
\text{d}^\dagger_{A - \sqrt{2}g\eta} A &= 0 \\
B^- &= 0 \\
\text{d}^\dagger_{A - \sqrt{2}g\eta} \psi &= 0 \\
\text{d}^\dagger_{A - \sqrt{2}g\eta} \eta &= 0 ,
\end{align*}
\]

(2.4)

properly expressed in terms of the background connection \( A_0 \).

Moreover, we implement the conditions (2.4) introducing the following BRST doublets

\[
\begin{align*}
s \bar{c} &= h_A \\
s h_A &= 0 \\
s \bar{\chi} &= h_B \\
s h_B &= 0 \\
s \bar{\phi} &= h_\psi \\
s h_\psi &= 0 \\
s \bar{\rho} &= h_\eta \\
s h_\eta &= 0 ,
\end{align*}
\]

(2.5)

(in particular \( \bar{\chi} \) and \( h_B \) are anti–self–dual 2–forms) and the Landau gauge–fixing Lagrangian:

\[
L_{gf} = \text{Tr} \left\{ s \left( \bar{c} \star \text{d}^\dagger_{A_0} A + \bar{\chi} \star B^- + \bar{\phi} \star \text{d}^\dagger_{A_0} \psi + \bar{\rho} \star \text{d}^\dagger_{A_0} \eta \right) \right\}.
\]

(2.6)

We now complete the change of variables in order to study the formal relation between the BFYM and TYM. This change will isolate a subset of the new fields having the same BRST transformations as the fields appearing in TYM theory. These fields will be interpreted as background fields for the quantum fluctuations of the BFYM theory.

The full change of variables is the following\footnote{This change of variables was suggested in \[5\].}

\[
\begin{align*}
A_0 &= A - \sqrt{2}g\eta \\
c_0 &= c - \sqrt{2}g\rho \\
\phi_0 &= -\phi + \frac{1}{2} [\rho, \bar{\rho}] \\
\psi_0 &= \psi - [\eta, \bar{\rho}] \\
\eta' &= \eta \\
\rho' &= \rho .
\end{align*}
\]

(2.7)
The first two equations correspond to the previous expansion of the fields $A$ and $c$ in a background and a quantum fluctuation part. Note that also the other transformations are entirely given in terms of the fluctuations $\eta$ and $\rho$. Moreover the Jacobian of the transformation being 1, the functional measure is

$$\mathcal{D}A \mathcal{D}c \mathcal{D}\psi \mathcal{D}\phi \mathcal{D}\eta \mathcal{D}\rho = \mathcal{D}A_0 \mathcal{D}c_0 \mathcal{D}\psi_0 \mathcal{D}\phi_0 \mathcal{D}\eta' \mathcal{D}\rho'. \quad (2.8)$$

The BRST algebra in terms of the new variables becomes

$$sA_0 = dA_0 c_0 + 2g^2 \psi_0,$$

$$sc_0 = -\frac{1}{2} [c_0, c_0] + 2g^2 \phi_0,$$

$$s\psi_0 = - [\psi_0, c_0] - dA_0 \phi_0,$$

$$s\phi_0 = [\phi_0, c_0] \quad (2.9)$$

$$sB = [B, c_0] - dA_0 \psi_0 - \sqrt{2}g [\eta, \psi_0] + \frac{1}{\sqrt{2}g} [F_{A_0}, \rho] + [\eta, dA_0 \rho] + \sqrt{2}g [B, \rho],$$

$$s\eta = [\eta, c_0] - \sqrt{2}g \psi_0 + dA_0 \rho + \sqrt{2}g [\eta, \rho],$$

$$s\rho = - [\rho, c_0] - \sqrt{2}g \phi_0 - \frac{1}{\sqrt{2}g} [\rho, \rho].$$

This is the key result of the present paper: the fields $(A_0, c_0, \psi_0, \phi_0)$ correspond to the field content of TYM and their BRST transformations are exactly those of TYM (modulo the rescaling $2g^2 \psi_0 \longrightarrow \psi_0, 2g^2 \phi_0 \longrightarrow \phi_0$). These fields clearly display an embedded topological sector in the YM theory; moreover a direct comparison among the observables of YM and TYM is now available and will be discussed in the next section.

Following the previous interpretation we separate the fields into two classes, the “background” fields $\varphi_0$ (including those of TYM) and the local fluctuations $\varphi_q$, and express the action as the sum of a background action $S_0 = S_0[\varphi_0]$ containing all the terms depending only on $\varphi_0$ and a fluctuation action $S_q = S_q[\varphi_q; \varphi_0]$ made of the remaining terms. With this decomposition the functional integral becomes:

$$Z = \int \mathcal{D}\varphi_0 e^{-S_0[\varphi_0]} \int \mathcal{D}\varphi_q e^{-S_q[\varphi_q; \varphi_0]}.$$ \quad (2.10)

Note that the two actions aren’t decoupled. Following the physical interpretation, in an explicit computation we could make a saddle point expansion of the second integral over the background $\varphi_0$ and then make the integration on $\mathcal{D}\varphi_0$. The idea is that this separation of fluctuations from backgrounds could lead to a deeper understanding of the role of the topological sector (i.e. the content of TYM) in the non perturbative calculations of YM theory.

Now we turn to the $B$-field: how have we to treat it? We should decompose it into a background and a fluctuation part transverse with respect to the former,

$$B = B_0 \oplus B_q,$$ \quad (2.11)

such that the functional measure over $B$ factorize:

$$\mathcal{D}B = \mathcal{D}B_0 \mathcal{D}B_q.$$ \quad (2.12)

\footnote{In the following we will omit the primes.}
A natural choice in the self-dual gauge-fixing is:

$$B_0 = B^+ \quad B_q = B^- .$$  \hfill (2.13)

The decomposition (2.13) has the advantage that the gauge-fixing conditions set $B_q = 0$. Other decompositions are available: for example we could assign $s B_0 = [B_0, c_0] + d_{A_0} \psi_0$ and obtain the algebra of the topological BF theory with cosmological term. Such a decomposition should be useful in a covariant gauge, but is more difficult to implement in the functional integral.

We then choose (2.13) and redefine the (2.5) as:

$$\bar{c}_0 = \bar{c} \quad h_{A_0} = h_A \quad \bar{\phi}_0 = \bar{\phi} \quad h_{\psi_0} = h_{\psi} \quad \bar{\chi}' = \bar{\chi} \quad h_{\eta}' = h_{\eta} + \sqrt{2} g h_A ,$$  \hfill (2.14)

in such a way that they remain BRST doublets. Then we set

$$\varphi_0 = \left( A_0, c_0, \psi_0, \phi_0, B^+, \bar{c}_0, h_{A_0}, \bar{\phi}_0, h_{\psi_0}, \bar{\chi}' \right) \quad \varphi_q = \left( \eta, \rho, B^-, \bar{\rho}', h_{\eta}', h_{B}' \right) ,$$

and discuss first the classical action and then the gauge-fixing one.

By substituting (2.7) and (2.11,2.13) into the classical Lagrangian (2.1) and imposing the $B^- = 0$ condition we get

$$\mathcal{L}_{BFYM} \equiv \mathcal{L}_0 + \mathcal{L}_q$$  \hfill (2.15)

with

$$\mathcal{L}_0 = i B^+ F_{A_0} + g^2 B^+ \ast B^+$$

$$\mathcal{L}_q = i B^+ \left( \sqrt{2} g d_{A_0} \eta + g^2 [\eta, \eta] \right) - B^+ \ast \left( \sqrt{2} g d_{A_0} \eta + 2 g^2 [\eta, \eta] \right) + \frac{1}{2} d_{A_0} \eta \ast d_{A_0} \eta + \sqrt{2} g d_{A_0} \eta \ast [\eta, \eta] + g^2 [\eta, \eta] \ast [\eta, \eta] ,$$  \hfill (2.17)

where $\mathcal{L}_0$ is the classical Baulieu-Singer TYM action written in the first order formalism \cite{2} and $\mathcal{L}_q$ is the action on the quantum fluctuations and corresponds to the classical YM action expanded around a background connection $A_0$ with quantum fluctuation $\sqrt{2} g \eta$. The gauge-fixing Lagrangian becomes instead:

$$\mathcal{L}^{gf} \equiv \mathcal{L}_0^{gf} + \mathcal{L}_q^{gf}$$  \hfill (2.18)
with

\[ \mathcal{L}_{0}^{gf} = \text{Tr} \left\{ h_{A0} \cdot d_{A0}^\dagger A_{0} + h_{\psi_0} \cdot d_{A0}^\dagger \psi_0 - \bar{\psi}_0 \cdot d_{A0}^\dagger \psi_0 - 2g^2 \bar{\psi}_0 \cdot d_{A0}^\dagger \psi_0 + \right. \]

\[ - \bar{\phi}_0 \cdot \left[ d_{A0}^\dagger \phi_0, c_0 \right] - 2g^2 \bar{\phi}_0 \cdot \left[ \phi_0, A_0 \right] + \bar{\phi}_0 \cdot d_{A0}^\dagger \phi_0 - 2g^2 \bar{\phi}_0 \cdot \left[ \phi_0, \psi_0 \right] + \]

\[ + \bar{\phi}_0 \cdot \left( d_{A0}^\dagger \phi_0 \right) \right\} \right. \]

\[ \mathcal{L}_{q}^{gf} = \text{Tr} \left\{ h_{\psi_0} \cdot \left[ d_{A0}^\dagger \eta, \rho \right] + h_{\eta} \cdot d_{A0}^\dagger \eta \right. \]

\[ + \sqrt{2}g \bar{\phi}_0 \cdot \left[ d_{A0}^\dagger \eta, \phi_0 \right] - \frac{1}{2} \bar{\phi}_0 \cdot d_{A0}^\dagger \eta, d_{A0} \cdot \left[ \rho, \rho \right] - \bar{\phi}_0 \cdot \left[ c_0, d_{A0}^\dagger \eta, \rho \right] + \]

\[ - 2g^2 \bar{\phi}_0 \cdot \left[ \phi_0, \left[ \eta, \rho \right] \right] - \bar{\eta} \cdot \left[ d_{A0}^\dagger \eta, c_0 \right] - 2g^2 \bar{\rho} \cdot \left[ \phi_0, \left[ \eta \right] \right] - \bar{\eta} \cdot d_{A0}^\dagger \eta, d_{A0} \rho + \]

\[ + \sqrt{2}g \bar{\rho} \cdot d_{A0}^\dagger \psi_0 - \sqrt{2}g \bar{\rho} \cdot \left[ d_{A0}^\dagger \eta, \rho \right] + \]

\[ + \sqrt{2}g \bar{\chi} \cdot \left[ \eta, \psi_0 \right] - \frac{1}{\sqrt{2}g} \bar{\chi} \cdot \left[ F^{\cdots}_{A0}, \rho \right] - \bar{\chi} \cdot \left[ \eta, d_{A0} \rho \right] \right\} \right. \]

\[ \text{The Lagrangian } \mathcal{L}_{0}^{gf} \text{ is quite equal to the gauge-fixing Lagrangian of TYM quantized in the gauge } d_{A0}^\dagger A_{0} = 0, \quad F^{\cdots}_{A0} = 0 \text{ and } d_{A0}^\dagger \psi_0 = 0. \text{ The only difference is that the terms } \]

\[ - \bar{\chi} \cdot \left[ F^{\cdots}_{A0}, c_0 \right] \text{ and } h_{\rho} \cdot F^{\cdots}_{A0} \text{ are lacking and we have } - \bar{\chi} \cdot \left[ B^{-}, c_0 \right] \text{ and } h_{B} \cdot B^{-} \text{ instead, all these terms vanishing in the Landau gauge. Thus we can read } \mathcal{L}_{0}^{gf} \text{ as the TYM self–dual gauge–fixing Lagrangian in the first order formalism and, after the } h_{B} \text{ and } B^{-} \text{ integrations,} \]

\[ \mathcal{L}_{0}^{gf} \equiv \mathcal{L}_{TYM} \quad . \]

\[ \text{In summary the partition function is:} \]

\[ Z = \int \mathcal{D}A_0 \mathcal{D}B^+ \mathcal{D}c_0 \mathcal{D}\psi_0 \mathcal{D}\phi_0 \mathcal{D}\bar{\phi}_0 \mathcal{D}h_{A0} \mathcal{D}h_{\psi_0} \cdot e^{- \int \text{d}^4x (iB^+ F_{A0} + \phi^2 B^+ \cdot B^+)} + S_0^{gf} \cdot \]

\[ \int \mathcal{D}\eta \mathcal{D}\rho \mathcal{D}\bar{\rho} \mathcal{D}\bar{\chi} \mathcal{D}h_{\eta} \cdot e^{-S_q[\varphi_q; \varphi_0]} \]

\[ \text{where } S_0^{gf} = \int \text{d}^4x \mathcal{L}_{0}^{gf} \text{ and } S_q = \int \text{d}^4x \left( \mathcal{L}_q + \mathcal{L}_q^{gf} \right) . \text{ Moreover if we would like to compute a YM observable, i.e. an observable not containing } B^+, \text{ or an observable linear in } B \text{ then we}

\[ \text{could integrate over } B^+ \text{ in } \mathcal{L}_0 + \mathcal{L}_0^{gf} \text{ and obtain exactly the standard second order YM action plus fluctuations. Therefore we see that the YM theory has been recast in the theory given by eq.}(2.22) \text{ where the nested integration corresponds to the contribution of local fluctuations on a background described by a topological theory. In the next section we discuss the observables of BFYM theory in this framework.} \]

\section{3 Observables}

\subsection{3.1 YM observables}

We define a \textit{YM functional} a functional of the form

\[ \mathcal{O} = \mathcal{O}[A, c, \bar{c}] = \mathcal{O}[\eta, \rho; A_0, c_0, \bar{c}_0] \quad (3.1) \]
i.e. a functional constructed out of the fields that are naively identified with the YM ones
(all the other fields are added by gaussian integration to the YM action to obtain the
action of BFYM).
Then we consider the YM amplitude $\langle O \rangle$ and recall that for $B$–independent amplitudes
we can integrate out $B^+$ and obtain:

$$
\langle O \rangle = \int D\varphi_0 e^{-S_{TYM}[\varphi_0]} \int D\varphi_q e^{-S_q[\varphi_q; \varphi_0]} O[\eta, \rho; A_0, c_0, \bar{c}_0]
$$

$$
\equiv \int D\varphi_0 e^{-S_{TYM}[\varphi_0]} A_{O}[\varphi_0] ,
$$

where we can think the amplitude $A_{O}$ computed for instance perturbatively.
Note that we have reduced the evaluation of an YM amplitude in BFYM theory to the
computation of an amplitude in TYM theory; as it is well known, if the amplitude were
topological the only contribution to it would come from the moduli space (or in other
words the semiclassical approximation would be exact). In our case this will be not true
because the amplitude $A_{O}$ will not be in general an observable in the TYM sense, i.e. it
will not be topological.

The ordinary perturbation theory corresponds to the the $k = 0$ sector of the BFYM
in which case the zero instantons moduli space is $M_0 = \{A_0 = 0\}$. In general only the
terms which saturate the ghost anomaly contribute to the amplitude and in this case
the computation of (3.2) is performed expanding $e^{-S_q[\varphi_q; \varphi_0]}$ and $O$ in powers of the
background ghosts and setting $A_0 = 0$; since $M_0$ has no ghost anomaly we can retain
only the zero order terms. When considering YM amplitudes in the higher instantons
sectors we are similarly led to a computation of a certain amplitude in TYM. In this case
we have however to saturate the fermionic anomaly of the moduli space $M_k$. We can do
it in two ways.

- We can expand $A_{O}[\varphi_0]$ in powers of the background ghosts and retain only the
terms of ghost number equal to the moduli space dimension (the other terms give
a zero contribution when evaluated in TYM).

- We can consider a YM amplitude with the insertion of an observable $G$ of background
ghost number equal to $\text{dim} M_k$. Then

$$
\langle G O \rangle = \int D\varphi_0 \mathcal{A}_{O}[A_0] e^{-S_{TYM}[\varphi_0]}
$$

where $\mathcal{A}_{O}[A_0]$ is the vacuum expectation value of the YM functional $O[A, c, \bar{c}]$ in
YM theory expanded over the background connection $A_0$. Therefore in this case
we can interpret $D\varphi_0 G$ as the measure over the instanton moduli space of the YM
theory.

3.2 Topological observables

Having translated the computation of correlators in YM theory into that of related quanti-
ties in TYM, the interesting question arises whether topological amplitudes in YM theory
exist. Moreover in our framework we can directly consider the set of topological observables of TYM which, after a proper identification of the fields $\varphi_0$, is given by

$$
\begin{align*}
T_4^0 &= \text{Tr} \left( \frac{1}{8g^2} F_{A0} F_{A0} \right) \\
T_3^1 &= \text{Tr} \left( -\frac{1}{2g^2} F_{A0} \psi_0 \right) \\
T_2^2 &= \text{Tr} \left( -\frac{1}{2g^2} F_{A0} \phi_0 + \frac{1}{2} \psi_0 \psi_0 \right) \\
T_1^3 &= \text{Tr} \left( \psi_0 \phi_0 \right) \\
T_0^4 &= \text{Tr} \left( \frac{1}{2} \phi_0 \phi_0 \right).
\end{align*}
$$

They satisfy the following descent equations:

$$
\begin{align*}
&s T_4^0 + d T_3^1 = 0 \\
&s T_3^1 + d T_2^2 = 0 \\
&s T_2^2 + d T_1^3 = 0 \\
&s T_1^3 + d T_0^4 = 0 \\
&s T_0^4 = 0.
\end{align*}
$$

Apart from some field rescaling, they are the same that have been studied in [3, 4], where it is shown that in TYM they give multi-link invariants of submanifolds of $\mathbb{R}^4$.

A similar set of observables can be found also in BFYM. They are

$$
\begin{align*}
K_4^0 &= \text{Tr} \left( \frac{1}{\sqrt{2g}} (d_A B) \eta + \frac{1}{2} BB - \frac{1}{2g^2} F_A \eta \eta \right) \\
K_3^1 &= \text{Tr} \left( B \psi - \frac{1}{\sqrt{2g}} (d_A B) \rho \right) \\
K_2^2 &= \text{Tr} \left( -B \phi + \frac{1}{2} \psi \psi - \frac{1}{2g^2} F_A \rho \rho \right) \\
K_1^3 &= \text{Tr} \left( -\psi \phi \right) \\
K_0^4 &= \text{Tr} \left( \frac{1}{2} \phi \phi \right).
\end{align*}
$$

These observables satisfy the following descent equations:

$$
\begin{align*}
&s K_4^0 + d K_3^1 = 0 \\
&s K_3^1 + d K_2^2 = 0 \\
&s K_2^2 + d K_1^3 = 0 \\
&s K_1^3 + d K_0^4 = 0 \\
&s K_0^4 = 0.
\end{align*}
$$

The choice of the (3.6) is dictated by its formal ressemblance with those of TYM. Indeed, if we take $\eta = \rho = 0$ (i.e. vanishing fluctuations) and substitute $B$ with $F_A$, we obtain up to multiplicative constants exactly the TYM observables.

These observables are related in the following way:

$$
K_3^1 = -T_3^1 + s \text{Tr} \left\{ \frac{1}{2\sqrt{2g^2}} F_A \eta - \frac{1}{\sqrt{2g}} B \eta + \frac{1}{3\sqrt{2g}} \eta \eta \right\} + \ldots
$$
\[ +d \text{Tr} \left\{ \frac{1}{2\sqrt{2}g} F_A \rho + \frac{1}{\sqrt{2}g} B \rho - \frac{1}{\sqrt{2}g} \psi \eta - \frac{1}{\sqrt{2}g} \eta \eta \rho \right\} \]  \hspace{1cm} (3.8) \\
\[ K_2^2 = T_2^2 - s \text{Tr} \left\{ \frac{1}{2\sqrt{2}g} F_A \rho + \frac{1}{\sqrt{2}g} B \rho - \frac{1}{2g^2} (d_A \eta) \rho + \frac{1}{\sqrt{2}g} \eta \eta \rho \right\} + \\
- d \text{Tr} \left\{ \frac{1}{\sqrt{2}g} \eta \rho \rho \right\} \]  \hspace{1cm} (3.9) \\
\[ K_3^3 = T_3^3 + s \text{Tr} \left\{ \frac{1}{\sqrt{2}g} \eta \rho \rho \right\} - d \text{Tr} \left\{ \frac{1}{3\sqrt{2}g} \rho \rho \rho \right\} \]  \hspace{1cm} (3.10) \\
\[ K_0^4 = T_0^4 + s \text{Tr} \left\{ \frac{1}{3\sqrt{2}g} \rho \rho \rho \right\} . \]  \hspace{1cm} (3.11) \\

We see in general that the difference between the sets is given in terms of the dressing due to the quantum fluctuations \( \eta \) and \( \rho \). Note also that the \( K \)'s are equivalent to the \( T \)'s modulo \( d \)-exact and \( s \)-exact terms, hence the cohomology of YM includes that of TYM.

In order to study the topological properties of an observable \( T \) we have to consider the correlator 

\[ \langle T \rangle = \int \mathcal{D} \varphi_0 e^{-S_{TYM}[\varphi_0]} \int \mathcal{D} \eta \mathcal{D} \rho \mathcal{D} \bar{\rho} e^{-S_q[\eta, \rho, \bar{\rho}; \varphi_0]} T[\eta, \rho, \bar{\rho}, A_0, c_0, \bar{c}_0] \equiv \int \mathcal{D} \varphi_0 e^{-S_{TYM}[\varphi_0]} T_T[\varphi_0]. \]  \hspace{1cm} (3.12) 

The observable \( T \) in YM corresponds to \( T_T \) in TYM. Then we can give implicitly on \( T_T \) the sufficient conditions for \( T \) to be topological. In TYM topological quantities must be closed under TYM-BRST transformations and their metric dependence must be TYM-BRST exact. These are precisely the requirements on \( T_T \); when fullfilled, the corresponding \( T \) should be a topological operator in YM. The TYM-BRST transformations in our case are precisely the nihilpotent subalgebra of (2.9):

\[ s_0 A_0 = d_{A_0} c_0 + 2g^2 \psi_0 \]  \hspace{1cm} (3.13) \\
\[ s_0 c_0 = -\frac{1}{2} [c_0, c_0] + 2g^2 \phi_0 \]  \hspace{1cm} (3.14) \\
\[ s_0 \psi_0 = -[\psi_0, c_0] - d_{A_0} \phi_0 \]  \hspace{1cm} (3.15) \\
\[ s_0 \phi_0 = [\phi_0, c_0] \]  \hspace{1cm} (3.16) \\
\[ s_0 \text{other} = 0. \]  \hspace{1cm} (3.17) \\

This is the only nihilpotent subalgebra which includes the topological ghosts. We can then formally rewrite the “topological” conditions as

\[ sT = 0 \]  \hspace{1cm} (3.18) \\
\[ s_0 T_T = 0 \]  \hspace{1cm} (3.19) \\
\[ \delta_\eta T_T = s_0 f, \]  \hspace{1cm} (3.20) 

with arbitrary \( f \), where \( \delta_\eta \) is the derivative with respect to the metric. It would be very important to rewrite these conditions directly in terms of \( T \) to work out the possible topological quantities. Clearly also the pure observables of TYM in this framework are no more topological, owing to the local contribution of the action on the quantum fluctuations.
4 Conclusions

In this letter we have displayed the topological sector embedded in Yang–Mills theory. Starting from the first–order (BFYM) formulation and performing a suitable field redefinition we find that a subset of the fields of BFYM represents the field content of Topological Yang–Mills theory; moreover the BRST algebra of YM theory on these fields reduces to that of TYM. The whole theory can then be split in the integration over these “topological” degrees of freedom and over the remaining fields, which are correctly identified as the local quantum fluctuations that restore a local dynamics on the topological theory. The non perturbative sector of the theory should be naturally related with this topological sector; in any case we give an explicit realization of the relationship of topological theories with physical ones in the bosonic case. In this framework the calculation of correlators in YM theory is translated in the calculation of non topological correlators in TYM. We also explicitly compare the observables of TYM with a similar set in BFYM theory; the two sets are cohomologically equivalent and differ only by the dressing due to the local quantum fluctuations, thus showing the inclusion of the cohomology of TYM in that of YM theory. In this framework we have also discussed the possibility to have topological correlators in YM and found a set of sufficient criteria to identify them, provided that any solution exists.

The rich structure exhibited by the first order formulation of Yang-Mills theory is currently investigated in several respects [14]. First of all, we think that our decomposition of YM into TYM theory plus fluctuations could improve the old problem of finding a well defined bosonic measure over the instanton moduli space. In this case the “topological” symmetry and the related gauge fixing procedure should correspond to a symmetry in the instanton moduli space requiring a proper treatment in order to give a finite integration volume.

A second very interesting point is to clarify the relationship with the Seiberg-Witten analysis of N=2 susy YM theories [15]. The twisting operation links N=2 susy YM to TYM theory and should therefore have some counterpart also in the YM case, the ghost of ghost structure present in BFYM providing the field content corresponding to that of susy multiplets. Such a relationship is clearly relevant to the confinement issues in QCD.

Finally, new nonlocal observables directly inherited from the topological BF theory can be considered in BFYM theory [3, 10, 11] and generalize the Wilson loop operator and the linking observables of knot theory. Again their analysis should be relevant for the long range features of the theory and indeed an area law behaviour for the \( vev \) of the Wilson loop has been derived in this framework [10, 11]. We believe that these non local observables constitute the bridge which should connect the microscopical description of QCD to the language of long range hadron physics.

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