Spin operator in the Dirac theory

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We find all spin operators for a Dirac particle satisfying the following very general conditions: (i) spin does not convert positive (negative) energy states into negative (positive) energy states, (ii) spin is a pseudo-vector, and (iii) eigenvalues of the projection of a spin operator on an arbitrary direction are independent of this direction (isotropy condition). We show that there are four such operators and all of them fulfill the standard su(2) Lie algebra commutation relations. Nevertheless, only one of them has a proper non-relativistic limit and acts in the same way on negative and positive energy states. We show also that this operator is equivalent to the Newton-Wigner spin operator and Foldy-Wouthuysen mean-spin operator. We also discuss another operators proposed in the literature.

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I. INTRODUCTION

In recent years one can notice a renewal of interest in the long-standing problem of the definition of a proper relativistic spin operator [1–5]. One of the reasons of this renewal is the rapid development for relativistic quantum information theory [6–41]. In this context, especially important is the simplest case, i.e., the spin operator for a Dirac particle. Many such operators have been proposed in the literature (see, e.g., [42–46]). Nevertheless, it seems that the question which one of these is the best is still open.

In this paper we address the problem. We formulate very general and physically justified conditions which should be fulfilled by a relativistic spin operator and classify all operators satisfying these conditions. Our requirements are the following: (i) spin does not convert positive (negative) energy states into negative (positive) energy ones, (ii) spin is a pseudo-vector, and (iii) eigenvalues of the projection of a spin operator in an arbitrary direction are independent of this direction (isotropy). They are motivated by fundamental physical reasons: requirement (i) follows from the fact that spin is an inner degree of freedom, therefore it commutes with translations; requirement (ii) is a consequence of the demand that spin should transform in the same way as the total angular momentum; and requirement (iii) is implied by the isotropy of space. We show that there are four operators fulfilling the above conditions.

Note that we do not require any specific commutation relations for components of a spin operator. However, it turns out that all four operators satisfying our requirements fulfill the standard su(2) Lie algebra commutation relations.

Nonetheless, only one of those four operators has a proper non-relativistic limit and satisfies the charge symmetry condition (acts in the same way on positive and negative energy states). This operator turns out to be equivalent to the Newton-Wigner spin operator and Foldy-Wouthuysen mean-spin operator. In our opinion it is the best candidate for a relativistic spin operator for a Dirac particle.

We also compare operators we have found to various spin operators presented in the literature.

The paper is organized as follows. In Sec. II we review briefly the abstract Dirac formalism and its connection with the spin-1/2 unitary representation of the Poincaré group. In Sec. III we discuss the relativistic spin operator in the framework of the enveloping algebra of the Lie algebra of the Poincaré group. We analyze in this context the influence of the overcompleteness of the covariant basis on the identification of different forms of the same operator. In Secs. IV and V we find all spin operators satisfying our requirements in the Bargmann-Wigner and Dirac bases, respectively. Section VI is devoted to a comparison of various spin operators presented in the literature. Conclusions are given in Sec. VII.

We use natural units with \( \hbar = c = 1 \), the Minkowski metric tensor \( g^{\mu \nu} = \text{diag}(1, -1, -1, -1) \), and adopt the convention \( \epsilon^{0123} = 1 \).

II. THE SETUP: ABSTRACT DIRAC FORMALISM

A free spin-1/2 particle can be described in one of two equivalent frameworks: a unitary representation of the Poincaré group or with the help of the Dirac formalism. To establish the notation we review here briefly basic facts concerning those two approaches. For the details we refer the reader to, e.g., our previous paper [47].
A. Bargmann-Wigner basis

The space of states of a spin-1/2 particle, \( \mathcal{H} \), is the carrier space of the unitary representation of the Poincaré group. To allow negative energies one takes as \( \mathcal{H} \) the direct sum of two carrier spaces of unitary, irreducible representations of the Poincaré group, \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \), corresponding to positive and negative energies, respectively. The space \( \mathcal{H}_\epsilon \) (\( \epsilon = \pm 1 \)) is spanned by eigenvectors of the four-momentum operators

\[
\hat{P}^\mu |p, \sigma\rangle = ep^\mu |p, \sigma\rangle, \tag{1}
\]

where the spin index \( \sigma = \pm 1/2 \).

We assume that vectors \( |p, \sigma\rangle \) are normalized covariantly

\[
\langle \epsilon' \epsilon' | p, \sigma\rangle = 2p^0 \delta(\epsilon' - \epsilon) \delta_{\epsilon' \epsilon}, \tag{2}
\]

where

\[
p^0 = \sqrt{p^2 + m^2}. \tag{3}
\]

Moreover, under Lorentz group action vectors \( |p, \sigma\rangle \) transform according to Wigner rotation

\[
U(\Lambda) |p, \sigma\rangle = D(R(\Lambda, p))_{\lambda \sigma} |\epsilon \Lambda p, \lambda\rangle, \tag{4}
\]

where \( D \) stands for the unitary spin-1/2 representation of the rotation group, \( R(\Lambda, p) = L^{-1}_m \Lambda \Lambda_p L_p \) is a standard Lorentz transformation defined by: \( L_p q = p \), \( L_q = I \) with \( q = (m, 0) \). The basis defined by Eq. (4) is called the Bargmann-Wigner or spin basis.

B. Dirac basis

The main disadvantage of the Bargmann-Wigner basis is the transformation law, (4), which implies that this basis is not manifestly covariant. Nevertheless, one can define another basis of space \( \mathcal{H} \),

\[
|\alpha, \epsilon p\rangle = \sum_\sigma v^\sigma_{\alpha \sigma}(p) |p, \sigma\rangle, \tag{5}
\]

where we demand the following transformation rule:

\[
U(\Lambda) |\alpha, \epsilon p\rangle = S^{-1}(\Lambda)_{\alpha \beta} |\epsilon \Lambda p, \beta\rangle. \tag{6}
\]

Here \( S(\Lambda) \) designates the bispinor representation of the Lorentz transformation \( \Lambda \) and \( \alpha \) is a bispinor index.

One can show that there exist coefficients \( v^\sigma_{\alpha \sigma}(p) \) such that Eq. (5) holds; for details see, e.g., [47] and [4]. Intertwining matrices \( v^\sigma_{\alpha \sigma}(p) \), related to the Dirac amplitudes, fulfill relations

\[
v^\sigma(p) \tilde{v}^\sigma(p) = \epsilon \Lambda_\epsilon(p), \tag{7a}
\]

\[
v^\sigma(p) \tilde{v}^{-\sigma}(p) = -\epsilon \Lambda_\epsilon(p) \gamma^5, \tag{7b}
\]

\[
\tilde{v}^\sigma(p) v^\sigma(p) = \epsilon \delta_v I_2, \tag{7c}
\]

where \( \tilde{v}(p) \equiv v^\dagger(p) \gamma^0 \), \( \gamma^\mu \) are Dirac matrices and the projectors \( \Lambda_\epsilon(p) \) have the standard form

\[
\Lambda_\epsilon(p) = \frac{m \mathbf{I} + \epsilon p \gamma}{2m}. \tag{8}
\]

We give the explicit form of the intertwining matrices \( v^\sigma_{\alpha \sigma}(p) \) and other useful formulas in Appendix A.

Let us note that the spin basis \( \{|\epsilon p, \sigma\rangle\} \) is complete, while the covariant basis is overcomplete; vectors \( \{|\alpha, \epsilon p\rangle\} \) constrained by the Dirac condition. Indeed, the Dirac operator

\[
\hat{D}|\alpha, \epsilon p\rangle = (\hat{P} \gamma - m I)_{\alpha \beta} |\beta, \epsilon p\rangle = (\epsilon p \gamma - m I)_{\alpha \beta} |\beta, \epsilon p\rangle, \tag{9}
\]

we get from Eqs. (5) and (7)

\[
\hat{D}|\alpha, \epsilon p\rangle = 0; \tag{10}
\]

i.e.,

\[
\epsilon (p \gamma - m I)_{\alpha \beta} |\beta, \epsilon p\rangle = 0. \tag{11}
\]

This simple fact has very important consequences: the same abstract operator can be represented by distinct matrices in Dirac theory (cf. [4] and [18]). We discuss this point in detail below.

Now, let \( \hat{\Omega} \) be an \( \hat{P}^\mu \)-dependent operator acting in \( \mathcal{H} \). In the Dirac formalism \( \hat{\Omega} \) is represented by a \( 4 \times 4 \) matrix with matrix elements being functions of four-momentum operators \( \hat{P}^\mu \): \( \hat{\Omega} = [\Omega(\hat{P})]_{\alpha \beta} \). To determine the action of \( \hat{\Omega} \) on basis vectors \( \{|\alpha, \epsilon p\rangle\} \) we must take into account that \( \hat{\Omega} \) in general can convert states with positive energy into states with negative energy (and vice versa). Therefore, in the abstract Dirac formalism we have

\[
\hat{\Omega}|\alpha, \epsilon p\rangle = \sum_\epsilon \Omega^\epsilon_{\alpha \beta}(\epsilon p) |\beta, \epsilon p\rangle, \tag{12}
\]

where

\[
\Omega^\epsilon_{\alpha \beta}(\epsilon p) = \Lambda_\epsilon(p) \Omega(\epsilon p) \Lambda_\epsilon(p), \tag{13}
\]

with projectors \( \Lambda_\epsilon(p) \) defined in Eq. (8). Note that in the above equation matrix elements of \( \Omega \) are functions of \( \epsilon p \), because \( \hat{P}^\mu \) acting on state \( |\alpha, \epsilon p\rangle \) gives \( \epsilon \hat{p}^\mu |\alpha, \epsilon p\rangle \).

Equation (12) determines uniquely the action of an operator \( \hat{\Omega} \) on the basis vectors \( \{|\alpha, \epsilon p\rangle\} \). To do this, let us define the operator

\[
\hat{\Omega} + \hat{\Lambda} \hat{D}, \tag{14}
\]

where \( \hat{\Lambda} \) is an arbitrary \( \hat{P}^\mu \)-dependent operator and \( \hat{D} \) is given in Eq. (10). Then, by virtue of constraint (10), we get

\[
\Lambda_\epsilon(p) \Omega(\epsilon p) \Lambda_\epsilon(p) = \Lambda_\epsilon(p) \left[ \Omega(\epsilon p) + A(\epsilon p)(\epsilon p \gamma - m I) \right] \Lambda_\epsilon(p), \tag{15}
\]
meaning that operators $\hat{\Omega}$ and $\hat{\Omega} + \hat{A} \hat{D}$ represent the same abstract endomorphism. In other words, any abstract endomorphism is represented by the whole class of operators [Eq. (14)] acting in the same way on basis vectors.

Knowing the action of an operator $\hat{\Omega}$ on basis vectors $|\alpha, \epsilon p\rangle$, i.e., having Eq. (12), we can determine the matrix representing $\hat{\Omega}$. To this end, let us define the matrix

$$\hat{\Omega}(p) = \sum_{\epsilon \epsilon} \Omega^{\epsilon, \epsilon'}(\epsilon p).$$

(16)

Of course, it holds that

$$\Lambda_{\epsilon}(p) \Omega(\epsilon p) \Lambda_{\epsilon}(p) = \Lambda_{\epsilon}(p) \hat{\Omega}^{\epsilon}(\epsilon p) \Lambda_{\epsilon}(p).$$

(17)

Now, to obtain the matrix operator with matrix elements being functions of four-momentum operators, we can use the equation

$$\hat{\rho}^0/|\rho^0| = \Omega(\epsilon p),$$

(18)

and write

$$\hat{\Omega}(p) = \hat{\Omega}(\epsilon p) \to \hat{\Omega} = \left( \frac{\hat{\rho}^0}{|\rho^0|} \right).$$

(19)

The operator $\hat{\Omega}$ given by the matrix $\hat{\Omega} = \left( \frac{\rho^0}{|\rho^0|} \right)$ belongs to the class of (14).

As a simple illustration of the above discussion let us consider the energy operator

$$\hat{H} = \hat{\rho}^0.$$

(20)

For this operator we have

$$\hat{H}(\epsilon p) = \epsilon \rho^0 I,$$

(21)

and, according to Eq. (13)

$$H^{\epsilon, \epsilon'}(\epsilon p) = \rho^0 \Lambda_{\epsilon},$$

(22)

$$H^{\epsilon, -\epsilon'}(\epsilon p) = 0.$$  

(23)

Now, Eq. (16) implies that

$$\hat{\rho}^0 = \rho^0 m p \gamma = \rho^0 I + \frac{\rho^0}{m} (p \gamma - m I),$$

(24)

and from Eq. (19) we get

$$\hat{\rho}^0 = \hat{\rho}^0 + \frac{\rho^0}{m} \hat{D}.$$  

(25)

It is also easy to see that the Dirac Hamiltonian is equivalent to $\hat{H}$. The corresponding $\hat{P}^{\mu}$-dependent matrix has the form

$$H_D(\hat{P}) = \gamma^0 (\hat{P} \cdot \gamma + m I) = \hat{P}^0 I - \gamma^0 (P \gamma - m I).$$

(26)

C. Interrelation between operators in the spin and Dirac bases

Now, let $\hat{\Omega}$ be an operator acting in space $\mathcal{H}$. The action of $\hat{\Omega}$ on the vector $|\epsilon p, \lambda\rangle$ in the Bargmann-Wigner basis can be written in the form

$$\hat{\Omega}|\epsilon p, \lambda\rangle = \sum_{\lambda} \omega_{\lambda\lambda'}^{\epsilon, \epsilon'}(p) |\epsilon p, \sigma\rangle.$$  

(27)

Thus, by virtue of Eqs. 7 we obtain in the Dirac basis

$$\hat{\Omega}|\alpha, \epsilon p\rangle = \sum_{\epsilon} \omega^{\alpha, \epsilon}(p, p^\epsilon) \hat{\Omega}|\alpha, \epsilon p\rangle.$$  

(28)

Therefore, by means of Eq. (16), the matrix $\hat{\Omega}$ representing operator $\hat{\Omega}$ in the Dirac basis can be obtained from the following formula:

$$\hat{\Omega}(p) = \sum_{\epsilon} \omega^{\epsilon, \epsilon'}(p) \hat{\Omega}(p, p^\epsilon).$$

(29)

D. Charge conjugation and parity

One can also define the charge conjugation and parity operators. On the level of quantum mechanics the charge conjugation operator $\hat{C}$ is antiunitary [49]. Thus, assuming that $\hat{C}$ commutes with Poincaré group transformations, i.e.,

$$[e^{ia_\mu} \hat{P}^\mu, \hat{C}] = 0,$$

(30a)

$$[U(\Lambda), \hat{C}] = 0,$$

(30b)

we get from Eq. (30a)

$$\hat{C} \hat{P}^\mu = - \hat{P}^\mu \hat{C}.$$  

(31)

Therefore, $\hat{C}$ converts vectors with four-momentum $p$ into vectors with four-momentum $-p$. Thus, antiunitarity of $\hat{C}$ and Eqs. (4) and (30b) implies

$$\hat{C}|\alpha, \epsilon p\rangle = \epsilon \xi(\alpha_2)_{\alpha \beta} |\alpha, \epsilon p\rangle,$$

(32)

with $|\xi| = 1$.

In the Dirac basis this operator acts as follows:

$$\hat{C}|\alpha, \epsilon p\rangle = \xi \gamma_5 \alpha_2 \beta |\alpha, -\epsilon p\rangle.$$  

(33)

We also use the parity operator $\hat{P}$ for which $\hat{P}(p^0, \hat{p}) = (p^0, -p) \equiv p^\pi$. The action of parity on vectors of the spin basis reads [47]

$$\hat{P}|\epsilon p, \lambda\rangle = \epsilon \xi |p^\pi, \lambda\rangle,$$

(34)

where $|\xi| = 1$.

The action of the parity operator on vectors of the Dirac basis reads

$$\hat{P}|\alpha, \epsilon p\rangle = \xi \gamma_5 \alpha_2 |\beta, \epsilon p^\pi\rangle.$$  

(35)
III. RELATIVISTIC SPIN OPERATOR

There are different approaches to the definition of a relativistic spin observable. First, one can try to split the total angular momentum, $\hat{J}$, into the orbital part $\hat{L}$ and spin part $\hat{S}$:

$$\hat{J} = \hat{L} + \hat{S}. \quad (36)$$

The total angular momentum is well defined via generators of the Lorentz group as $\hat{J}^i = \frac{1}{2} \varepsilon_{ijk} \hat{J}^j k$. However, to find $\hat{L} = \hat{X} \times \hat{P}$ one needs to know the position operator $\hat{X}$. But a uniquely defined relativistic position operator does not exist (in the literature); different choices of $\hat{X}$ lead to different spin observables.

On the other hand, it is well known that in the unitary representation of the Poincaré group there exists a well-defined spin-square operator,

$$\hat{S}^2 = -\frac{1}{m^2} \hat{W}^\mu \hat{W}_\mu, \quad (37)$$

where $\hat{W}^\mu$ is the Pauli-Lubanski four-vector

$$\hat{W}^\mu = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\mu} \hat{P}_\gamma \hat{J}_{\alpha\beta} \quad (38)$$

and $\hat{J}_{\alpha\beta}$ are generators of the Lorentz group. Thus, one can naturally try to define a spin operator as a linear function of components of the Pauli-Lubanski four-vector. Of course, for such a function to constitute a spin operator it has to fulfill some conditions which are believed to be the most important properties of the spin observable. These conditions are the following: (i) spin commutes with the four-momentum operators (this means that spin is an inner degree of freedom)

$$[\hat{S}^i, \hat{P}^j] = 0, \quad (39)$$

(ii) spin components fulfill the standard $su(2)$ Lie algebra commutation relations

$$[\hat{S}^i, \hat{S}^j] = i \varepsilon_{ijk} \hat{S}^k, \quad (40)$$

and (iii) spin transforms like a (pseudo)vector under rotations

$$[\hat{J}^i, \hat{S}^j] = i \varepsilon_{ijk} \hat{S}^k. \quad (41)$$

One can show [4] that the only operator from the enveloping algebra of the Lie algebra of the Poincaré group that is a linear function of the components of $\hat{W}^\mu$ and has the properties (39) and (40) has the following form:

$$\hat{S}_{NW} = \frac{1}{m} \left( \frac{\hat{P}^0}{P^0} \hat{W} - \hat{W}^0 \frac{\hat{P}}{P^0 + m} \right). \quad (42)$$

We use the notation $\hat{S}_{NW}$ because it appears that the above operator can be obtained from Eq. (36) if we take the Newton-Wigner operator [51] as a position operator $\hat{X}$ (see, e.g., [47] and [4]). In the case where one considers only positive energies, the spin operator, (42), takes the form

$$\hat{S}_{NW} = \frac{1}{m} \left( \hat{W} - \hat{W}^0 \frac{\hat{P}}{P^0 + m} \right). \quad (43)$$

Let us stress that the spin operator, (42), was discussed for the first time by Pryce in [42]. This spin operator naturally arises also in quantum field theory (see, e.g., [51]).

Operator (42) transforms under Lorentz-group action according to an operator Wigner rotation [4].

The $\hat{S}_{NW}$ operator is defined in the enveloping algebra of the Lie algebra of the Poincaré group. Therefore, it can be used for a particle with arbitrary spin. In the following we discuss this operator for a Dirac particle and compare it with various operators proposed in the literature.

IV. SPIN OPERATOR FOR A DIRAC PARTICLE IN THE BARGMANN-WIGNER BASIS

Many different spin operators have been proposed for a Dirac particle. We review them in Sec. [7] but first we try to determine the most general spin operator which fulfills very general physical requirements. Namely, we only assume that spin is a pseudo-vector, eigenvalues of the spin projection on an arbitrary direction $\mathbf{a}$ are independent of $\mathbf{a}$ (isotropy condition), and spin does not mix positive and negative energy states. On the level of quantum field theory this requirement is a consequence of the superselection rule forbidding the superpositions of particle and anti-particle states.

Because a matrix representation of an abstract operator in the covariant (Dirac) basis is not unique it is much more convenient to perform our analysis in the Bargmann-Wigner basis.

A spin operator which does not mix positive and negative energy states has the form:

$$\hat{S}|cp, \sigma\rangle = s^{ce}(p)|cp, \sigma\rangle, \quad (44)$$

[because $s^{-ce}(p) = 0$, compare Eq. (27)]. We assume that $\hat{S}$ is a three-vector and does not change under parity operation (i.e. $\hat{S}$ is a pseudo-vector). We have at our disposal only four independent three-vectors which can be used for the construction of the matrix $s^{ce}(p)$:

$$\sigma, \quad (\sigma \cdot p)p, \quad I p, \quad \sigma \times p. \quad (45)$$

Now, the condition that $\hat{S}$ does not change under parity

$$\hat{\sigma} \hat{S} \hat{\sigma} = \hat{S} \quad (46)$$

together with Eqs. (11) and (39) implies

$$s^{ce}(p) = s^{ce}(-p). \quad (47)$$
Thus, the most general form of the spin which is a pseudo-vector and does not mix positive and negative energy states in the Bargmann-Wigner basis is

\[ \hat{S}^{\alpha \beta}(p) = \alpha(p, \epsilon) \sigma + \beta(p, \epsilon) (p \cdot \sigma)p, \]  

where \( \alpha(p, \epsilon) \) and \( \beta(p, \epsilon) \) are scalar functions of \( p \). Now, eigenvalues of \( a \cdot \hat{S} \) are independent of \( a \) iff

\[ \beta(p, \epsilon) = 0 \quad \text{or} \quad p^2 \beta(p, \epsilon) + 2 \alpha(p, \epsilon) = 0. \]

Therefore, we arrive at two distinct possibilities:

\[ \hat{S}_I |p, \lambda\rangle = \alpha(p, \epsilon) \sigma_T^{\alpha \lambda} |p, \sigma\rangle, \]  

and

\[ \hat{S}_II |p, \lambda\rangle = \alpha(p, \epsilon) \left[ \sigma_T^{\alpha \lambda} - \frac{2}{p^2} (p \cdot \sigma_T)p \right] |p, \sigma\rangle. \]  

Equations (50) and (51) and the antiunitarity of \( \hat{C} \) give

\[ \hat{C} \hat{S} = -\hat{S} \hat{C}. \]  

Hence, the charge symmetry implies that spin should act in the same way on positive and negative energy states. Assuming that eigenvalues of \( a \cdot \hat{S} \) are equal to \( \pm 1/2 \), we have \( \alpha(p, \epsilon) = \pm 1/2 \). Therefore, we finally get two operators

\[ \hat{S}_I |p, \lambda\rangle = \frac{1}{2} \sigma_T^{\alpha \lambda} |p, \sigma\rangle, \]  

\[ \hat{S}_II |p, \lambda\rangle = \frac{1}{2} \left[ \sigma_T^{\alpha \lambda} - \frac{2}{p^2} (p \cdot \sigma_T)p \right] |p, \sigma\rangle, \]

where we have chosen \( \alpha(p, \epsilon) = 1/2 \) for the spin operator given in Eq. (50) and \( \alpha(p, \epsilon) = -1/2 \) for the operator given in Eq. (51). Under these choices both operators, (50) and (51), fulfill the standard commutation relations (10). This statement is obvious for operator (50); for operator (51) it is a consequence of the relation

\[ \left[ \frac{2}{p^2} (p \cdot \sigma_T)p - \sigma_T^{\alpha \lambda} \right] |p, \sigma\rangle = R_{ij}(p) \sigma_j, \]

where the matrix

\[ R(p) = \frac{2p \otimes p^T}{p^2} - I_3 \]

is a proper rotation.

It is worth stressing that we did not require any commutation relations for the spin components. These relations have been received as by-product of more fundamental assumptions.

Both operators, \( \hat{S}_I \) and \( \hat{S}_II \), fulfill relation (52). However, \( \hat{S}_I \) seems to be more advantageous because there is a problem with the limit \( p \to 0 \) of the operator \( \hat{S}_II \) defined in Eq. (54). Namely, in this limit the first term on the right-hand side of Eq. (54) does not vanish. Nevertheless, we consider \( \hat{S}_II \) for completeness. For the same reasons we also discuss spin operators breaking the charge symmetry, i.e. operators which act in a different way on positive and negative energy states. Taking into account Eqs. (50) and (51) we get two such operators,

\[ \hat{S}_III |p, \lambda\rangle = \frac{1}{2} \sigma_T^{\alpha \lambda} |p, \sigma\rangle, \]  

\[ \hat{S}_III |p, \lambda\rangle = \frac{1}{2} \left[ \sigma_T^{\alpha \lambda} - \frac{2}{p^2} (p \cdot \sigma_T)p - \sigma_T^{\alpha \lambda} \right] |p, \sigma\rangle. \]

Obvious, these operators do not satisfy the charge symmetry condition, Eq. (52). We conclude that the operator \( \hat{S}_I \) is the best one.

V. SPIN OPERATOR IN THE DIRAC BASIS

Now, we find the form of the discussed spin operators in the Dirac basis. If the action of the spin operator \( \hat{S} \) in the spin basis is given in Eq. (14), then Eq. (28) implies, in the Dirac basis,

\[ \hat{S} |\alpha, \epsilon\rangle = S^{\alpha \beta}(\epsilon) |\alpha, \beta, \epsilon\rangle, \]

with

\[ S^{\alpha \beta}(\epsilon) = \epsilon \sigma^{\alpha \beta}(p) S^{\epsilon \gamma_T}(p) \tilde{\epsilon}(p). \]

By virtue of Eq. (60), we get, for the spin operators defined in Eqs. (50) and (51),

\[ S^I_\gamma(p) = \frac{\gamma^5}{4m} \left\{ \epsilon \left[ m \gamma - \frac{p \cdot \gamma_0 + \frac{p}{p^0 + m} (p \cdot \gamma)}{p^0 + m} \right] \right. \]

\[ + \left. \gamma(p \gamma) - p \frac{p}{p^0 + m} \gamma_0 (p \cdot \gamma) \right\}, \]  

\[ S^I_{II}(\epsilon) = \frac{\gamma^5}{4m} \left\{ \epsilon \left[ \frac{p}{p^0 - m} (p \cdot \gamma) - m \gamma - \gamma_0 (p \cdot \gamma) \right] \right. \]

\[ \left. + \gamma(p \gamma) - p \frac{p}{p^0 - m} \gamma_0 (p \cdot \gamma) \right\}. \]

Furthermore, according to Eq. (110), we obtain the spin matrix via the formula

\[ \hat{S}(p) = \sum_{\alpha} S^{\alpha \beta}(\epsilon). \]

Thus, spin matrices representing operators \( \hat{S}_I \) and \( \hat{S}_II \) derived by means of the procedure (10) are

\[ S_I(\hat{P}) = \frac{\gamma^5}{2m} \left\{ \frac{\hat{P}^0}{|P|} \left[ \gamma(\hat{P} \gamma) - \hat{P} \right] + \frac{\hat{P} \gamma_0 (\hat{P} \cdot \gamma)}{|P^0| + m} \right\}. \]
and
\[
S_H(\hat{p}) = -\frac{\gamma}{2m} \left\{ \frac{\hat{p} \gamma}{|\hat{p}|^2 - m^2} \left( \gamma(\hat{p}\gamma) - \hat{p} \right) + \frac{\hat{p} \gamma(\hat{p}\gamma)}{|\hat{p}|^2 - m^2} \right\},
\]
respectively.

In the same way we find the matrix representation of the spin operators given in Eqs. (57) and (58). We get
\[
S_{II}(\hat{p}) = \frac{\gamma}{2} \left[ \gamma - \frac{\hat{p}}{|\hat{p}|^2 - m^2} (\gamma^0 + I)(\hat{p} \cdot \gamma) \right],
\]
and
\[
S_{IV}(\hat{p}) = \frac{\gamma^5}{2} \left[ -\gamma - \frac{\hat{p}}{|\hat{p}|^2 - m^2} (\gamma^0 - I)(\hat{p} \cdot \gamma) \right],
\]
respectively.

Evidently, when we restrict ourselves to positive energy states, operators \( S_I \) and \( S_{II} \) coincide. The same statement holds for operators \( S_{II} \) and \( S_{IV} \).

VI. REVIEW OF SPIN OPERATORS DISCUSSED IN THE LITERATURE

In this section we review and compare various relativistic spin operators for a Dirac particle discussed in the literature. We have collected their definitions in Table I. A few remarks are in order about the notation we used.

The first operator presented in Table I, \( \hat{S}_D \), is the standard Dirac spin operator.

The second operator in Table I is called the Newton-Wigner operator and denoted \( \hat{S}_{NW} \) because it can be obtained from Eq. (59) under the assumption that the position operator is the Newton-Wigner operator [54]. However, the abstract form of the operator \( \hat{S}_{NW} \) [Eq. (13)] was discussed for the first time by Pryce in 1935 [42]. This spin operator has also been used in quantum information theory (see, e.g., [11, 17, 20, 30]).

The third operator in Table I is called the Foldy-Wouthuysen mean-spin, \( \hat{S}_{FW} \). These operators do not convert positive (negative) energy states into negative (positive) ones \( (\hat{s}^+_{NW} = \hat{s}^+_{FW} = 0) \) and fulfill the fundamental isotropy condition. Moreover, both of these operators act in the same way on positive and negative energy states \( (\hat{s}^+_{NW} = \hat{s}^+_{FW} (p) = 0) \) and this action in the Bargmann-Wigner basis is given by the standard Pauli matrices. In our classification \( \hat{S}_{NW} \) and \( \hat{S}_{FW} \) are equivalent to the operator \( \hat{S}_I \) [Eqs. (52) and (54)].

All other operators presented in Table I have some disadvantages.

The standard Dirac spin operator, \( \hat{S}_D \), is excluded because it can convert positive energy states into negative ones, and vice versa. Even its projection to positive energy states does not satisfy the isotropy condition.

The Frenkel operator, \( \hat{S}_F \), is in fact the sum of (block-diagonal) projections of the Dirac spin operator on positive and negative energy sectors. For this reason it does not convert positive (negative) energy states into negative (positive) energy states. However, it does not satisfy the isotropy condition. The same flaw has the spin operator \( \hat{S}_C \).

The Chakrabarti spin operator, \( \hat{S}_{Ch} \), for positive energy states reduces to the Newton-Wigner operator: \( \hat{s}_{Ch}^+ = \sigma/2, \hat{s}_{Ch}^- = 0 \). However, for negative energy states its action is different. Moreover, it can convert negative energy states into positive ones.

The last operator, \( \hat{S}_P \), is simply equal to our spin operator \( \hat{S}_{II} \). Therefore, it does not satisfy the charge symmetry condition. Moreover, its nonrelativistic limit is ill defined.

VII. CONCLUSIONS

We have formulated very general physical conditions which should be fulfilled by a spin operator for a Dirac particle. These conditions are the following: (i) spin converts positive energy states into positive energy states and negative energy states into negative energy states; (ii) spin is a pseudo-vector; and (iii) eigenvalues of the projection of the spin operator on arbitrary direction \( \hat{a} \) are independent of \( \hat{a} \) (isotropy condition).

We have found all spin operators fulfilling the above conditions; there exist four such operators. We have also shown that components of all four of these opera-
form of spin operators given in the literature. In the right column we present the equivalent form used in our calculations.

Table I. Summary of the properties of various relativistic spin operators defined in Tab. I.

| Projections in the Dirac basis | Projections in the spin basis |
|--------------------------------|-------------------------------|
| 1. $\hat{S}^{(+)}(ep) = \frac{\hbar}{2m} \gamma(p^0 \gamma - \gamma^0 p) \Lambda_e$ | $s^{(+)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| $s^{(+)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ | $s^{(+)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| 2. $\hat{S}^{(-)}(ep) = \frac{\hbar}{2m} (\gamma - \frac{p^0}{m+p^0} [e^0 + 1]) \Lambda_e$ | $s^{(-)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| $s^{(-)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ | $s^{(-)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| 3. $\hat{S}^{(0)}(ep) = \frac{\hbar}{2m} (\gamma - \frac{p^0}{m+p^0} [e^0 + 1]) \Lambda_e$ | $s^{(0)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| $s^{(0)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ | $s^{(0)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| 4. $\hat{S}^{(++)}(ep) = \frac{\hbar}{2m} (e_1 \gamma - \frac{p^0}{m+p^0} [e_1 + 1]) \Lambda_e$ | $s^{(++)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| $s^{(++)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ | $s^{(++)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| 5. $\hat{S}^{(+-)}(ep) = \frac{\hbar}{2m} (e_1 \gamma - \frac{p^0}{m+p^0} [e_1 + 1]) \Lambda_e$ | $s^{(+-)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| $s^{(+-)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ | $s^{(+-)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| 6. $\hat{S}^{(--)}(ep) = \frac{\hbar}{2m} (e_1 \gamma - \frac{p^0}{m+p^0} [e_1 + 1]) \Lambda_e$ | $s^{(--)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| $s^{(--)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ | $s^{(--)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| 7. $\hat{S}^{(++)}(ep) = \frac{\hbar}{2m} (e_1 \gamma - \frac{p^0}{m+p^0} [e_1 + 1]) \Lambda_e$ | $s^{(++)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |
| $s^{(++)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ | $s^{(++)}_D(p) = \frac{\hbar}{2m} (\sigma - \frac{p \cdot [p \cdot \sigma]}{p^2})$ |

Table II. Summary of the properties of various relativistic spin operators defined in Tab. I.

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Appendix A: Gamma matrices and amplitudes

In explicit calculations we use the following representation of Dirac gamma matrices:

\[ \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \]

(A1)

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) and \( \sigma_i \) are the standard Pauli matrices. The explicit form of amplitudes \( v^\rho(p) \) reads (cf. [1, 41])

\[ v^\rho(p) = \frac{1}{2\sqrt{1 + \frac{m^2}{p^2}\sigma_2}} \left( I_2 + \frac{1}{m}p^\rho\sigma_\mu \right) \sigma_\rho, \]

(A2)

where \( \sigma_\rho = I_2 \). Amplitudes (A2) and gamma matrices (A1) fulfill the following relations:

\[ \gamma^5 v^\rho(p) = v^{-\rho}(p), \]

(A3)

\[ \gamma^0 v^\rho(p) = \epsilon v^\rho(p^T), \]

(A4)

\[ \gamma^2 v^\rho(p) = -\epsilon v^{-\rho}(p)\sigma_2, \]

(A5)

\[ -\bar{v}^\rho(p) = \bar{v}^{-\rho}(p)\gamma^5. \]

(A6)

The following useful formulas can be proved by direct calculation:

\[ \bar{v}^\rho(p)\gamma^\mu v^\rho(p) = \frac{p^\mu}{m}I_2, \]

(A7)

\[ \bar{v}^\rho(p)\gamma^5 v^\rho(p) = 0, \]

(A8)

\[ \bar{v}^\rho(p)\gamma^0 v^\rho(p) = -\frac{\epsilon}{m}p \times \sigma^T, \]

(A9)

\[ \bar{v}^\rho(p)\gamma^0 (p \cdot \gamma) v^\rho(p) = 0, \]

(A10)

\[ \bar{v}^\rho(p)\gamma^0 (p \cdot \gamma) v^\rho(p) = -\frac{\epsilon i}{m}p \times \sigma^T, \]

(A11)

\[ \bar{v}^\rho(p)\gamma^5\gamma^0 (p \cdot \gamma) v^\rho(p) = 0, \]

(A12)

\[ \bar{v}^\rho(p)\gamma^5 v^\rho(p) = \frac{1}{m}(p \cdot \sigma^T), \]

(A13)

\[ \bar{v}^\rho(p)\gamma^0 v^\rho(p) = -c(p \cdot \sigma^T), \]

(A14)

\[ \bar{v}^\rho(p)\gamma^5 v^\rho(p) = \frac{1}{m}(mp + p^T), \]

(A15)

\[ \bar{v}^\rho(p)\gamma^5 (p \cdot \gamma) v^\rho(p) = \frac{1}{m}(p^2 - p(p \cdot \sigma^T)), \]

(A16)

\[ \bar{v}^\rho(p)\gamma^5\gamma^0 (p \cdot \gamma) v^\rho(p) = -\frac{1}{m}(p \cdot \sigma^T). \]

(A17)

The algebra of gamma matrices and Eq. (8) imply:

\[ \Lambda_\rho \gamma^\mu \Lambda_\rho = \frac{\epsilon p^\mu}{m} \Lambda_\rho, \]

(A18)

\[ \Lambda_\rho \gamma^0 \Lambda_\rho = \left( \gamma^\mu - \frac{\epsilon p^\mu}{m} \right) \Lambda_\rho, \]

(A19)

\[ \Lambda_\rho \gamma^5 \Lambda_\rho = \gamma^5 \left( \gamma - \frac{\epsilon p^\mu}{m} \right) \Lambda_\rho, \]

(A20)

\[ \Lambda_\rho \gamma^5 \gamma^0 \Lambda_\rho = \frac{p^\mu}{m} \gamma^5 \Lambda_\rho, \]

(A21)

\[ \Lambda_\rho \gamma^5 (p \cdot \gamma) \Lambda_\rho = \frac{p^\mu}{m} \gamma^5 \Lambda_\rho, \]

(A22)

\[ \Lambda_\rho \gamma^5 \gamma^0 (p \cdot \gamma) \Lambda_\rho = \frac{p^\mu}{m} \gamma^5 \Lambda_\rho, \]

(A23)

\[ \Lambda_\rho \gamma^5 \gamma^0 \gamma_\rho \Lambda_\rho = \frac{m}{\gamma^5} \gamma^5 (p - \gamma p^0) \Lambda_\rho, \]

(A24)

\[ \Lambda_\rho \gamma^5 \gamma^0 (p \cdot \gamma) \Lambda_\rho = \gamma^5 \gamma^0 (p \cdot \gamma) \Lambda_\rho, \]

(A25)

\[ \Lambda_\rho \gamma^5 (p \cdot \gamma) \Lambda_\rho = 0, \]

(A26)

\[ \Lambda_\rho \gamma^5 (p \cdot \gamma) \Lambda_\rho = 0, \]

(A27)

\[ \Lambda_\rho \gamma^5 (p \cdot \gamma) \Lambda_\rho = 0, \]

(A28)

\[ \Lambda_\rho \gamma^5 (p \cdot \gamma) \Lambda_\rho = 0, \]

(A29)

\[ \Lambda_\rho \gamma^5 (p \cdot \gamma) \Lambda_\rho = 0, \]

(A30)

\[ \Lambda_\rho \gamma^5 (p \cdot \gamma) \Lambda_\rho = 0. \]

(A31)

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