The Scattering amplitude for Rationally extended shape invariant Eckart potentials

Rajesh Kumar Yadav \(a^\ast\), Avinash Khare\(b^\dagger\) and Bhabani Prasad Mandal\(a^\ddagger\)

May 22, 2014

\(a\) Department of Physics, Banaras Hindu University, Varanasi-221005, INDIA.
\(b\) Raja Ramanna Fellow, Indian Institute of Science Education and Research (IISER), Pune-411021, INDIA.

Abstract

We consider the rationally extended exactly solvable Eckart potentials which exhibit extended shape invariance property. These potentials are isospectral to the conventional Eckart potential. The scattering amplitude for these rationally extended potentials is calculated analytically for the generalized \(m\)th \((m = 1, 2, 3, \ldots)\) case by considering the asymptotic behavior of the scattering state wave functions which are written in terms of some new polynomials related to the Jacobi polynomials. As expected, in the \(m = 0\) limit, this scattering amplitude goes over to the scattering amplitude for the conventional Eckart potential.

1 Introduction

The ideas of supersymmetric quantum mechanics (SQM) and shape invariant potentials (SIP) have not only enriched our understanding of the exactly solvable systems but have helped in substantially increasing the list of exactly solvable potentials \([1]\). With the recent discovery \([2, 3]\) of the exceptional Laguerre and Jacobi polynomials, new shape invariant potentials with translation related to the radial oscillator, Scarf I and generalized Pöschl-Teller potentials were discovered \([4, 5]\) whose solution is in terms of \(X_1\) Laguerre or \(X_1\) Jacobi polynomials. Subsequently, Odake and Sasaki generalized this construction and for all three cases obtained one parameter families of these SIPs (with translation) whose bound state eigenfunctions are in terms of \(X_m\) EOPs \([6, 7]\). In addition to this SQM

\(\ast\) e-mail address: rajeshastrophysics@gmail.com
\(\dagger\) e-mail address: khare@iiserpune.ac.in
\(\ddagger\) e-mail address: bhabani.mandal@gmail.com
approach, there are several equivalent approaches such as Darboux-Crum transformation, Darboux-Bäcklund transformation, prepotential methods etc, which can also be used to get the extension of the conventional SIPs. However, till today, these three remain the only real potentials for which rationally extended SIPs with unbroken SUSY solutions are written in terms of EOPs. It is worth mentioning that all three new SIPs (with translation) are isospectral to the well known conventional SIPs.

Using the concept of first order SQM but broken SUSY, recently Quesne has been able to construct rational extensions of the Eckart potential [8] and has shown that in this case the SI property is no more valid, rather they exhibit an unfamiliar extended SI property in which the partner potential is obtained by translating both the potential parameter \(A\) (as in the conventional case) and \(m\), the degree of the polynomial arising in the denominator.

Out of these newly extended SI potentials, the potential which is isospectral to the generalized Pöschl-Teller (GPT) potential and rationally extended SI Eckart potentials have both the bound as well as the continuum spectrum. In recent publication, we have obtained the scattering amplitude for the potential isospectral to the GPT and whose solutions are in terms of \(X_m\) Jacobi polynomials [9] [10]. To the best of our knowledge the scattering amplitude for the rationally extended SI Eckart potentials have not been obtained so far. The purpose of this note is to fill this gap by calculating the s-wave scattering amplitude \((S_{l=0}^{m}(k))\) for this extended class of SI Eckart potentials whose solutions are in terms of some new type of polynomial \(y_n(x)\) (which can be expressed in terms of combination of Jacobi polynomials) for general values of \(m(= 1, 2, ..., )\). As a check on our calculation, we also show that in the limit \(m = 0\), we recover the scattering amplitude for the conventional Eckart potential.

The plan of the paper is as follows: In Sec. 2, we briefly review the work of Quesne regarding the bound states for the rationally-extended SI Eckart Potentials which are isospectral to the conventional Eckart potential. The calculation of the scattering amplitude for these potentials is discussed in Sec. 3 while Sec. 4 is reserved for summary and discussion.

# 2 Rationally extended Eckart potentials: bound states

In this section we briefly review the work of Quesne [8] regarding rationally-extended SI Eckart potentials and its bound states. The conventional Eckart potential is given by [11] [12]

\[
V_{A,B}(r) = A(A - 1)cosech^2r - 2B \coth r, \quad 0 < r < \infty ,
\]

where \(A > 1\) and \(B > A^2\), has a finite number of bound states. The energy eigenvalues and the eigenfunctions are given by

\[
E^{(A,B)}_{\nu} = -(A+\nu)^2 - \frac{B^2}{(A+\nu)^2}, \quad \nu = 0, 1, 2, \ldots, \nu_{max}, \sqrt{B-A}-1 \leq \nu_{max} < \sqrt{B-A},
\]
and

\[ \psi_{(A,B)}^{(\pm)}(r) = (z - 1)^{- \frac{1}{2} (A + v - \frac{B}{A + v})} (z + 1)^{- \frac{1}{2} (A + v + \frac{B}{A + v})} P_n^{(-A - v + \frac{B}{A + v}, -A - v - \frac{B}{A + v})}(z), \]

where \( z = \coth r \) and \( P_n^{(A,B)}(z) \) is the Jacobi polynomial. The rational extension of this potentials has been done by Quesne by determining all possible polynomials-type, nodeless solutions \( \phi(r) \) (see Ref. [8]) of the Schrödinger equation

\[ - \frac{d^2 \phi(r)}{dr^2} + V_{A,B}(r) \phi(r) = E \phi(r), \]

with the factorization energy \( E < E_0^{(A,B)} = -A^2 - \frac{B^2}{A^2} \).

Out of all the possible solutions of \( \phi(r) \), two independent polynomial type solutions \( \phi_1(r) \) and \( \phi_2(r) \) with the energy \( E_1 \) and \( E_2 \) respectively have been constructed. On putting some restrictions on the parameters \( A \) and \( B \), four polynomial type solutions (one corresponding to \( \phi_1(r) \) and three corresponding to \( \phi_2(r) \)) have been obtained. Out of these four possibilities, there exist three acceptable polynomial-type nodeless solutions (one corresponding to \( \phi_1(r) \) and two corresponding to \( \phi_2(r) \)) of the Eckart potentials.

Each of the above factorization function (\( \phi(r) \)) gives rise to a pair of partner potentials through the superpotential \( W(r) = -(\ln \phi(r))' \), i.e.

\[ V^{(\pm)}(r) = W^2(r) \mp W'(r) + E. \]

One can now define the raising and lowering operators

\[ \hat{A}^\dagger = -\frac{d}{dr} + W(r), \quad \hat{A} = \frac{d}{dr} + W(r). \]

The factorized Hamiltonians \( \hat{H}^{(\pm)} = \hat{A}^\dagger \hat{A} \) and \( \hat{H}^{(-)} = \hat{A} \hat{A}^\dagger \), can then be expressed as

\[ \hat{H}^{(\pm)} = -\frac{d^2}{dr^2} + V^{(\pm)}(r) - E, \]

and satisfy the intertwining relations \( \hat{A} \hat{H}^{(\pm)} = \hat{H}^{(-)} \hat{A} \) and \( \hat{A}^\dagger \hat{H}^{(-)} = \hat{H}^{(\pm)} \hat{A}^\dagger \). As shown by Quesne, in this way the factorization functions \( \phi(r) \) yield three partners potentials \( V^{(-)}(r) \), out of which two are isospectral since their inverses are not normalizable, while the third partner has an additional bound state below the spectrum of \( V^{(-)}(r) \), corresponding to its normalizable inverse.

The rationally-extended Eckart potential \( V^{(-)}(r) \) with given \( A \) and \( B \) is obtained from a conventional Eckart potential \( V_{A,B}(r) \) by shifting the parameters \( A \) as

\[ V^{(+)}(r) = V_{A',B}(r), \quad V^{(-)}(r) \equiv V_{A,B,ext}(r) = V_{A,B}(r) + V_{A,B,ext}(r), \]

where

\[ V_{A,B,ext}(r) = 2(1 - z^2) \left[ 2 \frac{\dot{g}_{m}^{(A,B)}(z)}{g_{m}^{(A,B)}(z)} - (1 - z^2) \left( \frac{\ddot{g}_{m}^{(A,B)}(z)}{g_{m}^{(A,B)}(z)} - \left( \frac{g_{m}^{(A,B)}(z)}{\dot{g}_{m}^{(A,B)}(z)} \right)^2 \right) \right], \]

and

\[ \psi_{(A,B)}^{(\pm)}(r) = (z - 1)^{- \frac{1}{2} (A + v - \frac{B}{A + v})} (z + 1)^{- \frac{1}{2} (A + v + \frac{B}{A + v})} P_n^{(-A - v + \frac{B}{A + v}, -A - v - \frac{B}{A + v})}(z), \]
here dot denotes a derivative with respect to $z$.

By choosing $A' = A - 1$, and the other parameters as given below

$$g_m^{(A,B)}(z) = P_m^{(\alpha_m, \beta_m)}(z),$$

$$\alpha_m = -A + 1 - m + \frac{B}{A - 1 + m}, \quad \beta_m = -A + 1 - m - \frac{B}{A - 1 + m},$$

$m = 1, 2, 3, \ldots, A > 2, \quad (A - 1)^2 < B < (A - 1)(A - 1 + m), \quad (10)$

one obtains the rationally extended Eckart potentials, $V^{(-)}(r) (= V_{A,B,ext}(r))$ isospectral to the potentials $V^{(+)}(r)$ with a bound state spectrum

$$E^{(+)}_{\nu} = E^{(-)}_{\nu} = -(A - 1 + \nu)^2 - \frac{B^2}{(A - 1 + \nu)^2}, \quad \nu = 0, 1, 2, \ldots, \nu_{\text{max}},$$

$$\sqrt{B} - A \leq \nu_{\text{max}} < \sqrt{B} - A - 1. \quad (11)$$

The corresponding bound state eigenfunctions of $V^{(+)}(r)$ are

$$\psi^{(+)}_{\nu}(r) \propto (z - 1)^{\alpha_{\nu}}(z + 1)^{\beta_{\nu}} P_{\nu}^{(\alpha_{\nu}, \beta_{\nu})}(z), \quad \nu = 0, 1, 2, \ldots, \nu_{\text{max}},$$

$$\alpha_{\nu} = -A + 1 - \nu + \frac{B}{(A - 1 + \nu)}, \quad \beta_{\nu} = -A + 1 - \nu - \frac{B}{(A - 1 + \nu)}, \quad (12)$$

and those of $V^{(-)}(r)$ are obtained by applying the operator $\hat{A}$ (as given by Eq. (10)) (in terms of $z$ variable)

$$\hat{A} = (1 - z^2) \frac{d}{dz} + \frac{B}{A - 1 + m} - (A - 1 + m)z - (1 - z^2) \frac{g_m^{(A,B)}(z)}{g_m^{(A,B)}(z)},$$

$$= (1 - z^2) \frac{d}{dz} + \frac{B}{A - 1} - (A - 1)z - \frac{2(m + \alpha_m)(m + \beta_m)}{2m + \alpha_m + \beta_m} \frac{g_{m+1}^{(A,B)}(z)}{g_m^{(A,B)}(z)}, \quad (13)$$

on the bound state eigenfunctions of $V^{(+)}(r)$. The bound state eigenfunctions of $V^{(-)}(r)$ are then given by

$$\psi^{(-)}_{\nu}(r) \propto (z - 1)^{\alpha_{\nu}}(z + 1)^{\beta_{\nu}} y_n^{(A,B)}(z), \quad n = m + \nu - 1, \quad \nu = 0, 1, 2, \ldots, \nu_{\text{max}}, \quad (14)$$

where $y_n^{(A,B)}(z)$ is some $n$th-degree polynomial in $z$, defined by

$$y_n^{(A,B)}(z) = \frac{2(\nu + \alpha_{\nu})(\nu + \beta_{\nu})}{2\nu + \alpha_{\nu} + \beta_{\nu}} g_m^{(A,B)}(z) P_{\nu-1}^{(\alpha_{\nu}, \beta_{\nu})}(z)$$

$$- \frac{2(m + \alpha_m)(m + \beta_m)}{2m + \alpha_m + \beta_m} \frac{g_{m+1}^{(A,B)}(z)}{g_m^{(A,B)}(z)} P_{\nu}^{(\alpha_{\nu}, \beta_{\nu})}(z), \quad (15)$$
and satisfying a second order differential equation
\[
\left[(1 - z^2) \frac{d^2}{dz^2} - \left\{ \alpha_\nu - \beta_\nu + (\alpha_\nu + \beta_\nu + 2)z + 2(1 - z^2) \frac{g_m^{(A,B)}(z)}{g_m^{(A,B)}(z)} \right\} \frac{d}{dz} + \right.
\]
\[
\left. \begin{array}{c}
+ (\nu - 1)(\alpha_\nu + \beta_\nu + \nu) - m(\alpha_m + \beta_m + m - 1) \\
+ \left[ \alpha_\nu - \beta_\nu + \alpha_m - \beta_m + (\alpha_\nu + \beta_\nu + \alpha_m + \beta_m)z \frac{g_m^{(A,B)}(z)}{g_m^{(A,B)}(z)} \right] \frac{y_{m+\nu-1}(z)}{g_m^{(A,B)}(z)} \right) = 0,
\end{array}
\]
\[
\nu = 0, 1, 2, \ldots, \nu_{\text{max}}.
\] (16)

From the Eqs. (11), (12) and (14), we see that the energy eigenvalues of the extended shape invariant potentials \( V^{(+)}(r) = V_{A+1, B}(r) \) and \( V^{(-)}(r) = V_{A, B, \text{ext}}(r) \) are isospectral (broken SUSY), while the eigenfunctions are different.

### 3 Scattering amplitude for extended shape invariant Eckart potentials

For obtaining the scattering amplitude of the new rationally-extended SI Eckart potentials, we have to first obtain the scattering wave functions for these potentials. On using Eq. (15) and Eq. (14) the bound state solutions for the rationally extended Eckart potentials are given by

\[
\psi_{\nu}^{(-)}(r) = (\text{Const.}) \times (z - 1)^{\frac{\alpha_\nu}{2}} (z + 1)^{\frac{\beta_\nu}{2}} \left[ \frac{2(\nu + \alpha_\nu)(\nu + \beta_\nu)}{2\nu + \alpha_\nu + \beta_\nu} P^{(\alpha_\nu, \beta_\nu)}_{\nu-1}(z) - \right.
\]
\[
\left. \frac{2(m + \alpha_m)(m + \beta_m)}{2m + \alpha_m + \beta_m} \frac{g_m^{(A+1,B)}}{g_m^{(A,B)}}(z) P^{(\alpha_m, \beta_m)}(z) \right],
\] (17)

where \( m = 1, 2, \ldots \), \( \nu = 0, 1, 2, \ldots, \nu_{\text{max}} \).

To get the scattering wave function for this rationally extended new Eckart potential, two modifications of the bound state wavefunctions have to be made [11]: (i) The second solution of the Schrödinger equation must be retained - it has been discarded for bound state problems since it diverged asymptotically. (ii) Instead of the parameter \( \nu \) labeling the number of nodes, one must use the wavenumber \( k \) so that we get the asymptotic behavior in terms of \( e^{\pm ikr} \) as \( r \to \infty \).

The Jacobi polynomial in terms of the hypergeometric function \( {}_2F_1 \) is given by [13]

\[
P^{(\alpha_\nu, \beta_\nu)}(z) = (-1)^\nu \frac{\Gamma(\nu + \beta_\nu + 1)}{\nu! \Gamma(1 + \beta_\nu)} {}_2F_1(\nu + \alpha_\nu + \beta_\nu + 1, -\nu, 1 + \beta_\nu; \frac{1 - z}{2})
\] (18)

After considering the second solution of the Schrödinger equation related to the bound state wave function \( \psi^{(-)}(r) \) (i.e., the second solution of the hypergeometric differential
equation), the above equation becomes.

\[
P_\nu^{(\alpha_\nu,\beta_\nu)}(\coth r) = (-1)^\nu \frac{\Gamma(\nu + \beta_\nu + 1)}{\nu! \Gamma(1 + \beta_\nu)} \left[ C_1 \ 2F_1(\nu + \alpha_\nu + \beta_\nu + 1, -\nu, 1 + \beta_\nu; \frac{1 + \coth r}{2}) + C_2 \left( \frac{1 + \coth r}{2} \right)^{-\beta_\nu} 2F_1(\nu + \alpha_\nu + 1, -\nu - \beta_\nu, 1 - \beta_\nu; \frac{1 + \coth r}{2}) \right], \tag{19}
\]

where \(C_1\) and \(C_2\) are constants assumed to be independent of \(\nu, \alpha_\nu,\) and \(\beta_\nu\).

Considering the boundary condition, i.e as \(r \to 0\), \(\psi_\nu^{(-)}(r) \to 0\), and hence \(C_2 \to 0\), thus the allowed solution is

\[
P_\nu^{(\alpha_\nu,\beta_\nu)}(\coth r) = C_1 (-1)^\nu \frac{\Gamma(\nu + \beta_\nu + 1)}{\nu! \Gamma(1 + \beta_\nu)} 2F_1(\nu + \alpha_\nu + \beta_\nu + 1, -\nu, 1 + \beta_\nu; \frac{1 + \coth r}{2}). \tag{20}
\]

Similarly by replacing \(\nu\) by \(\nu - 1\) keeping in mind that \(\alpha_\nu\) and \(\beta_\nu\) are constants, we get the expression for \(P_{\nu-1}^{(\alpha_\nu,\beta_\nu)}(\coth r)\).

As \(r \to \infty\) the new potential \(V_{A,B,ext}(r) \to -2B\), hence let us define

\[
E_\nu^{(+)}(or E_\nu^{(-)}) - V_{A,B,ext}(x \to \infty) = E_\nu^{(+)}(or E_\nu^{(-)}) + 2B = - \left( A - 1 + \nu - \frac{B}{A - 1 + \nu} \right)^2 = k^2 \quad \text{(say)}, \tag{21}
\]

therefore \(\alpha_\nu = -ik\), where \(k\) is wavenumber.

Also the polynomial \(g_{m}^{(A,B)}(z)\) in terms of usual Jacobi polynomial is defined as

\[
g_m^{(A,B)}(z) = P_m^{(\alpha_m,\beta_m)}(z) = \frac{\Gamma(\alpha_m + m + 1)}{m! \Gamma(\alpha_m + \beta_m + m + 1)} \times \sum_{p=0}^{m} \binom{m}{p} \frac{\Gamma(\alpha_m + \beta_m + m + p + 1)}{\Gamma(\alpha_m + p + 1)} \left( \frac{z - 1}{2} \right)^p,
\]

where \(m = 1, 2, 3, \ldots\). \tag{22}

Similarly by replacing \(m \to m - 1\) and then changing \(A \to A + 1\), we get \(g_{m-1}^{(A+1,B)}(z)\).

Note that the Jacobi polynomial as given by Eq. \[20\] is valid for any complex number \(\nu \in \mathbb{C}\), hence it is easy to see that the Schrödinger equation corresponding to Eq. \[14\] holds for any complex number \(\nu \in \mathbb{C}\) and the energy \(E^{(+)\nu}(or E^{(-)\nu})\) is real for \(\alpha_\nu = -ik(k \in \mathbb{R})\). So, on replacing \(\alpha_\nu\) by \(-ik\), and using Eq.\[22\], a property related to the hypergeometric function \[15\] i.e.,

\[
2F_1(a, b; c; z) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} 2F_1(a, b; a + b - c + 1; 1 - z) + (1 - z)^{(c-a-b)} \times \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} 2F_1(c - a, c - b; c - a - b + 1; 1 - z), \tag{23}
\]

\[\]
in Eq. (17), then taking the asymptotic behavior of the wavefunction as \( r \to \infty \), and by using the fact that the hypergeometric function \( _2F_1(\alpha, \beta, \gamma; 0) \to 1 \), the scattering state wavefunction (17) is given by

\[
\lim_{r \to \infty} \psi_k(r) = (\text{Const.}) \times (S^m_{l=0}(k)e^{ikr} + e^{-ikr}).
\]  

(24)

In this way, we find that the expression for the scattering amplitude \( S^m_{l=0}(k) \) is given by

\[
S^m_{l=0}(k) = S^{(+)}_{l=0}(k) \times \left[ \frac{(A - ik + (m - 1) - \frac{B}{A+m-1})}{(A + ik + (m - 1) - \frac{B}{A+m-1})} \right].
\]  

(25)

Here \( S^{(+)}_{l=0}(k) \) is the scattering amplitude for the potential \( V^{(+)}(r) \) isospectral to \( V^{(-)}(r) = V_{A,B,ext}(r) \), given by

\[
S^{(+)}_{l=0}(k) = \frac{\Gamma(ik)\Gamma(-A + 2 - \frac{ik}{2} - (B - \frac{k^2}{4})^{1/2})\Gamma(A - 1 - \frac{ik}{2} - (B - \frac{k^2}{4})^{1/2})}{\Gamma(-ik)\Gamma(-A + 2 + \frac{ik}{2} - (B - \frac{k^2}{4})^{1/2})\Gamma(A - 1 + \frac{ik}{2} - (B - \frac{k^2}{4})^{1/2})}.
\]  

(26)

After simplifying \( S^{(+)}_{l=0}(k) \), Eq. (25) is written as

\[
S^m_{l=0}(k) = S^{\text{Eckart}}_{l=0}(k) \times \left[ \frac{(A + ik - 1 - \frac{B}{A-1})(A - ik + (m - 1) - \frac{B}{A+m-1})}{(A - ik - 1 - \frac{B}{A-1})(A + ik + (m - 1) - \frac{B}{A+m-1})} \right],
\]  

(27)

where \( S^{\text{Eckart}}_{l=0}(k) \) is the scattering amplitude for the conventional Eckart potential \( (V^{(A,B)}(r)) \), given by

\[
S^{\text{Eckart}}_{l=0}(k) = \frac{\Gamma(ik)\Gamma(A - \frac{ik}{2} + (B - \frac{k^2}{4})^{1/2})\Gamma(A - \frac{ik}{2} - (B - \frac{k^2}{4})^{1/2})}{\Gamma(-ik)\Gamma(A + \frac{ik}{2} + (B - \frac{k^2}{4})^{1/2})\Gamma(A + \frac{ik}{2} - (B - \frac{k^2}{4})^{1/2})}a_k,
\]

here \( a_k \) is a phase factor.

As a check on our calculations, by starting from Eq. (17), we have also explicitly calculated the scattering amplitudes in the special cases of \( m = 1, 2 \) and 3 and checked that we get the same expressions as obtained from Eq. (27) with \( m = 1, 2 \) and 3 respectively. Further, in the limit \( m = 0 \), the scattering amplitude as given by Eq. (27) reduces to \( S^{\text{Eckart}}_{l=0}(k) \), providing a further check on our calculations.

### 4 Summary and discussion

Recently the bound state eigenfunction of the rationally-extended Eckart potential, which is isospectral to the conventional Eckart potential, has been written by Quesne in terms of some new polynomials. These new polynomials are not EOPs, but are related to the usual Jacobi polynomials. In this paper we have considered the scattering problem for the one parameter family of such rationally-extended Eckart potentials and have obtained
the scattering amplitude for these new rationally extended Eckart potentials. We have shown that the scattering amplitude for the general \( m \)th case (\( m \) is the order of the new polynomials) is related to the scattering amplitude for the conventional Eckart potential. In the special case of \( m = 0 \), as expected, we recover the scattering amplitude for the usual Eckart potential.

Acknowledgment

One of us (RKY) acknowledges financial support from UGC under the FIP Scheme.

References

[1] F. Cooper, A. Khare, U. Sukhatme Phys. Rep. 251 (1995) 267; ”SUSY in Quantum Mechanics” World Scientific (2001).

[2] D. Gomez-Ullate, N. Kamran and R. Milson, J. Math. Anal.Appl. 359 (2009) 352.

[3] D. Gomez-Ullate, N. Kamran and R. Milson, J. Phys. A 43 (2010) 434016.

[4] C. Quesne, J.Phys.A 41 (2008) 392001.

[5] B. Bagchi, C. Quesne and R. Roychoudhary, Pramana J. Phys. 73(2009) 337, C. Quesne, SIGMA 5 (2009) 84; A. Khare, Unpublished.

[6] C-L. Ho, S ODAKE and R Sasaki, SIGMA 7 (2011) 107.

[7] S. Odake and R. Sasaki, Phys. Lett. B, 684 (2010) 173; ibid 679 (2009) 414. J. Math. Phys, 51, 053513 (2010).

[8] C. Quesne, SIGMA 8, (2012) 080 .

[9] R. K. Yadav, A. Khare and B. P. Mandal, Annals of Physics 331 (2013) 313.

[10] R. K. Yadav, A. Khare and B. P. Mandal, Physics Letter B 723 (2013) 433.

[11] A. Khare and Uday P Sukhatme J. Phys. A: Math. Gen 21 (1988) L501.

[12] G. Levai J. Phys. A: Math. Gen 22 (1989) 689.

[13] I.S. Gradshteyn, I.M.Ryzhik, and Alan Jeffrey, “Table of Integrals, Series and Products” Academic Press (1991).