BLOW-UP CRITERIA FOR FRACTIONAL NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider the focusing fractional nonlinear Schrödinger equation

\[ i\partial_t u - (-\Delta)^s u = -|u|^\alpha u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \]

where \( s \in (1/2, 1) \) and \( \alpha > 0 \). By using localized virial estimates, we establish general blow-up criteria for non-radial solutions to the equation. As consequences, we obtain blow-up criteria in both \( L^2 \)-critical and \( L^2 \)-supercritical cases which extend the results of Boulenger-Himmelsbach-Lenzmann [Blowup for fractional NLS, J. Funct. Anal. 271 (2016), 2569–2603] for non-radial initial data.

1. Introduction

We consider the Cauchy problem for fractional nonlinear Schrödinger equations with focusing power-type nonlinearity

\[
\begin{cases}
i\partial_t u - (-\Delta)^s u = -|u|^\alpha u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\quad u(0) = u_0,
\end{cases}
\]

where \( u \) is a complex valued function defined on \( \mathbb{R}^+ \times \mathbb{R}^d \), \( d \geq 1 \), \( s \in (1/2, 1) \) and \( \alpha > 0 \). The operator \((-\Delta)^s\) is the fractional Laplacian which is defined by \( \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}) \) with \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) the Fourier transform and its inverse respectively. The equation (1.1) can be seen as a canonical model for a nonlocal dispersive PDE with focusing nonlinearity that can exhibit standing waves and wave collapse. The fractional Schrödinger equation was first discovered by Laskin [28] as a result of extending the Feynmann path integral, from the Brownian-like to Lévy-like quantum mechanical paths. The fractional Schrödinger equation also appears in the continuum limit of discrete models with long-range interactions (see e.g. [26]) and in the description of Boson stars as well as in water wave dynamics (see e.g. [15] or [23]).

In the last decade, the fractional nonlinear Schrödinger equation (1.1) has attracted a lot of interest in mathematics, numerics and physics (see e.g. [1, 2, 7, 8, 9, 17, 19, 22, 23, 25, 31, 32] and references therein). The local well-posedness in Sobolev spaces for non-radial data was established by Hong-Sire in [22] (see also [7]). The local well-posedness for radial \( H^s \) data was studied by the author in [8]. The existence of radial finite time blow-up \( H^s \) solutions was established recently by Boulenger-Himmelsbach-Lenzmann in [1]. Dynamics of finite time blow-up solutions were studied in [8, 9]. The sharp threshold of blow-up and scattering in the \( L^2 \)-supercritical and \( \dot{H}^s \)-subcritical case was first considered by Sun-Wang-Yao-Zheng in [32]. This result was then extended by Guo-Zhu in [19]. The orbital
stability as well as the orbital instability was proved by Peng-Shi in \([31]\). Recently, the author in \([10]\) proved the strong instability of standing waves for the equation in the \(L^2\)-supercritical case.

Before stating our main results, let us recall some basic facts of \((1.1)\). The equation \((1.1)\) has formally the conservation of mass and the energy:

\[
M(u(t)) = \int |u(t, x)|^2 dx = M(u_0), \tag{Mass}
\]
\[
E(u(t)) = \frac{1}{2} \int |(-\Delta)^{s/2} u(t, x)|^2 dx - \frac{1}{\alpha + 2} \int |u(t, x)|^{\alpha + 2} dx = E(u_0). \tag{Energy}
\]

The equation \((1.1)\) also enjoys the scaling invariance

\[u_\lambda(t, x) = \lambda^{\frac{d}{2}} u(\lambda^{2s} t, \lambda x), \quad \lambda > 0.\]

A calculation shows

\[\|u_\lambda(0)\|_{\dot{H}^s} = \lambda^{\frac{s}{2} - \frac{d}{2}} \|u_0\|_{\dot{H}^s}.\]

We thus define the critical exponent

\[s_c := \frac{d}{2} - \frac{2s}{\alpha}. \tag{1.2}\]

The local well-posedness for \((1.1)\) in Sobolev spaces for non-radial data was studied by Hong-Sire in \([22]\) (see also \([7]\)). Note that the unitary group \(e^{-it(-\Delta)^s}\) enjoys several types of Strichartz estimates: non-radial Strichartz estimates (see e.g. \([5]\) or \([7]\)); radial Strichartz estimates (see e.g. \([18]\), \([24]\) or \([4]\)); and weighted Strichartz estimates (see e.g. \([12]\)). For non-radial data, these Strichartz estimates have a loss of derivatives. This makes the study of local well-posedness more difficult and leads to a weak local theory comparing to the standard nonlinear Schrödinger equation (e.g. \(s = 1\) in \((1.1)\)). One can remove the loss of derivatives in Strichartz estimates by considering radially symmetric initial data. However, these Strichartz estimates without loss of derivatives require an restriction on the validity of \(s\), that is \(s \in \left[\frac{d}{d+2}, 1\right)\). Since we are interested in blow-up criteria for solutions of \((1.1)\) in general Sobolev spaces \(H^s\), we first need to establish the local well-posedness in such spaces. This will lead to a regularity condition on the nonlinearity, that is,

\[\lceil \gamma \rceil \leq \alpha + 1, \tag{1.3}\]

where \(\lceil \gamma \rceil\) is the smallest positive integer greater than or equal to \(\gamma\). We refer the reader to Section 3 for more details.

Recently, Boulenger-Himmelsbach-Lenzmann in \([1]\) established blow-up criteria for radial \(H^s\) solutions to \((1.1)\). More precisely, they proved the following:

**Theorem 1.1** (Radial blow-up criteria \([1]\)). Let \(d \geq 2\), \(s \in (1/2, 1)\) and \(\alpha > 0\). Let \(u_0 \in H^s\) be radial and assume that the corresponding solution to \((1.1)\) exists on the maximal time interval \([0, T)\).

- **Mass-critical case**, i.e. \(\alpha = \frac{4s}{d}\): If \(E(u_0) < 0\), then the solution \(u\) either blows up in finite time, i.e. \(T < +\infty\) or blows up infinite time, i.e. \(T = +\infty\) and

\[\|(-\Delta)^{s/2} u(t)\|_{L^2} \geq C t^s, \quad \forall t \geq t_0,\]

with some \(C > 0\) and \(t_0 > 0\) that depend only on \(u_0, s\) and \(d\).
\begin{itemize}

\item **Mass and energy intercritical case**, i.e. $\frac{4}{d} < \alpha < \frac{4s}{d-2s}$: If $\alpha < 4s$ and either $E(u_0) < 0$, or if $E(u_0) \geq 0$, we assume that
\begin{equation}
\begin{cases}
E^+(u_0)M^{s-s^*}(u_0) < E^+(Q)M^{s-s^*}(Q), \\
\|(-\Delta)^{s/2}u_0\|_{L^2}^2 \|u_0\|_{L^{2^*-s^*}}^2 > \|(-\Delta)^{s/2}Q\|_{L^2}^2 \|Q\|_{L^{2^*-s^*}}^2,
\end{cases}
\end{equation}
where $Q$ is the unique (up to symmetries) positive radial solution to the elliptic equation
\begin{equation}
(-\Delta)^{s}Q + Q - |Q|^\alpha Q = 0,
\end{equation}
then the solution blows up in finite time, i.e. $T < +\infty$.

\item **Energy-critical case**, i.e. $\alpha = \frac{4s}{d-2s}$: If $\alpha < 4s$ and either $E(u_0) < 0$, or if $E(u_0) \geq 0$, we assume that
\begin{equation}
\begin{cases}
E(u_0) < E(W), \\
\|(-\Delta)^{s/2}u_0\|_{L^2} > \|(-\Delta)^{s/2}W\|_{L^2},
\end{cases}
\end{equation}
where $W$ is the unique (up to symmetries) positive radial solution to the elliptic equation
\begin{equation}
(-\Delta)^{s}W - |W|^\frac{4s}{d-2s}W = 0,
\end{equation}
then the solution blows up in finite time, i.e. $T < +\infty$.

\end{itemize}

Note that the uniqueness (up to symmetries) of positive radial solution to (1.4) and (1.5) were proved in [13, 14].

The main purposes of this paper is to show blow-up criteria for non-radial $H^\gamma$ solutions for (1.1). Before entering some details of our results, let us recall known blow-up criteria for the focusing nonlinear Schrödinger equation
\begin{equation}
i\partial_t u + \Delta u = -|u|^\alpha u, \quad u(0) = u_0.
\end{equation}
The existence of finite time blow-up $H^1$ solutions for (1.6) was first proved by Glassey in [16]. More precisely, he proved that for any negative $H^1$ initial data satisfying $xu_0 \in L^2$, the corresponding solution blows up in finite time. Ogawa-Tsutsumi in [29, 30] showed the existence of blow-up solutions for negative radial data in dimensions $d \geq 2$ and for negative data (not necessary radially symmetry) in the one dimensional case. Holmer-Roudenko in [20] showed that in the mass and energy intercritical case, if initial data satisfies $E(u_0) \geq 0$ and
\begin{equation}
\begin{cases}
E^<(u_0)M^{1-\gamma}(u_0) < E^<(R)M^{1-\gamma}(R), \\
\|
abla u_0\|_{L^2}^2 u_0\|_{L^{2^{-\gamma}}}^2 > \|
abla R\|_{L^2}^2 R\|_{L^{2^{-\gamma}}}^2,
\end{cases}
\end{equation}
and in addition if $xu_0 \in L^2$ or $u_0$ is radial with $N \geq 2$ and $\alpha < 4$, then the corresponding solution blows up in finite time. Here $R$ is the ground state of (1.6) which is the unique (up to symmetries) positive radial solution of the elliptic equation
\begin{equation}
\Delta R - R + |R|^\alpha R = 0,
\end{equation}
and $\gamma_c = \frac{d}{2} - \frac{2}{\alpha} \in (0, 1)$. Later, Holmer-Roudenko in [21] showed that if $H^1$ initial data (not necessary finite-variance or radially symmetry) satisfies (1.7), then the corresponding solution either blows up in finite time or it blows up infinite time in the sense that there exists a sequence of times $t_n \to +\infty$ such that $\|
abla u(t_n)\|_{L^2} \to \infty$. Recently, Du-Wu-Zhang extended the result of [21] and proved a blow-up criterion for (1.6) with initial data (without finite-variance and radially symmetric assumptions) in the energy-critical and energy-supercritical cases.
Inspiring by the idea of Du-Wu-Zhang, we study the blow-up criteria for the focusing fractional nonlinear equation (1.1). The main difficulty is the appearance of the fractional order Laplacian \((-\Delta)^s\). When \(s = 1\), one can compute easily the time derivative of the virial action, which is

\[
\frac{d}{dt} \left( \int \varphi |u(t)|^2 dx \right) = 2 \text{Im} \int \nabla \varphi \cdot \nabla u(t) \overline{u}(t) dx.
\] (1.8)

Using this identity, Du-Wu-Zhang \([1]\) derive an \(L^2\)-estimate in the exterior ball. Thanks to this \(L^2\)-estimate and the virial estimates, they prove the result. In the case \(s \in (1/2, 1)\), the identity (1.8) does not hold. However, by exploiting the idea of \([1]\) with the use of the Balakrishnan’s formula, namely

\[
(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m} dm, \quad s \in (0, 1),
\]

we are able to compute the time derivative of the virial action (see Lemma 4.5):

\[
\frac{d}{dt} \left( \int \varphi |u(t)|^2 dx \right) = -i \int_0^\infty m^s \int \Delta \varphi |u_m(t)|^2 dx dm
\]

\[
-2i \int_0^\infty m^s \int \nabla u_m(t) \cdot \nabla \varphi dx dm,
\]

where \(u_m(t)\) is an auxiliary function defined by

\[
u_m(t) = \sqrt{\frac{\sin \pi s}{\pi}} \frac{1}{-\Delta + m} u(t).
\]

This identity plays a similar role as in (1.8), and we can show the blow-up criteria for (1.1) with non-radial initial data.

Denote

\[
K(u(t)) := \frac{s}{2} \|(-\Delta)^{s/2} u(t)\|_{L^2}^2 - \frac{d\alpha}{4(\alpha + 2)} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2}.
\] (1.9)

We will see in (4.16) that the quantity \(K(u(t))\) is related to the following virial identity

\[
\frac{d}{dt} \left( 4 \text{Im} \int \overline{u}(t) x \cdot \nabla u(t) dx \right) = 16K(u(t)).
\]

**Theorem 1.2 (Blow-up criteria).** Let \(d \geq 1\), \(s \in (1/2, 1)\) and \(\gamma \geq \max\{s, s_c\}\). Let \(\alpha > 0\) be such that if \(\alpha\) is not an even integer, then (1.3) holds. Let \(u_0 \in H^\gamma\) be such that the corresponding (not necessary radial) solution \(u\) to (1.1) exists on the maximal time \([0, T)\). If there exists \(\delta > 0\) such that

\[
\sup_{t \in [0, T)} K(u(t)) \leq -\delta < 0,
\] (1.10)

then one of the following statements holds true:

- \(u(t)\) blows up in finite time in the sense \(T < +\infty\) must hold;
- \(u(t)\) blows up infinite time and

\[
\sup_{t \in [0, +\infty)} \|u(t)\|_{L^q} = \infty,
\] (1.11)

for any \(q > \alpha + 2\). In particular, there exists a time sequence \((t_n)\) such that \(t_n \to +\infty\) and

\[
\lim_{n \to \infty} \|u(t_n)\|_{L^q} = \infty,
\]
for any \( q > \alpha + 2 \).

**Remark 1.3.**
- The condition \( \gamma \geq s \) ensures the solution enjoys the conservation of mass and energy.
- It is still possible (see e.g. [3, Remark 6.5.9]) that there exists a solution which blows up in finite positive time is global in negative time and vice versa.
- In the case \( T < +\infty \), we learn from the local theory that if \( \gamma > s_c \), then
  \[
  \lim_{t \uparrow T} \|u(t)\|_{H^s} = \infty.
  \]

The following result gives blow-up criteria for solutions with negative energy initial data.

**Corollary 1.4.** Let \( d \geq 1 \), \( s \in (1/2, 1) \) and \( \gamma \geq \max\{s, s_c\} \). Let \( \alpha \geq \frac{4s}{d} \) be such that if \( \alpha \) is not an even integer, then (1.3) holds. Let \( u_0 \in H^\gamma \) be such that the corresponding (not necessary radial) solution to (1.1) exists on the maximal time \([0, T)\). If \( E(u_0) < 0 \), then one of the following statements holds true:
- \( u(t) \) blows up in finite time in the sense \( T < +\infty \) must hold;
- \( u(t) \) blows up infinite time and
  \[
  \sup_{t \in [0, +\infty)} \|u(t)\|_{L^q} = \infty,
  \]
  for any \( q > \alpha + 2 \). In particular, there exists a time sequence \((t_n)_n\) such that \( t_n \to +\infty \) and
  \[
  \lim_{n \to \infty} \|u(t_n)\|_{L^q} = \infty,
  \]
  for any \( q > \alpha + 2 \).

This corollary follows directly from Theorem 1.2 with the fact

\[
K(u(t)) = sE(u(t)) - \frac{d\alpha - 4s}{4(\alpha + 2)} \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \leq sE(u_0) < 0.
\]

Let us now consider \( \gamma = s \). Note that in this case the regularity condition (1.3) is no longer needed. We firstly have the following blow-up criteria in the mass-critical case.

**Proposition 1.5** (Mass-critical blow-up criteria). Let \( d \geq 1 \) and \( s \in (1/2, 1) \). Let \( u_0 \in H^s \) be such that the corresponding (not necessary radial) solution to the mass-critical (1.1), i.e. \( \alpha = \frac{4s}{d} \) exists on the maximal time \([0, T)\). If \( E(u_0) < 0 \), then one of the following statements holds true:
- \( u(t) \) blows up in finite time, i.e. \( T < +\infty \) and
  \[
  \lim_{t \uparrow T} \|(-\Delta)^{s/2} u(t)\|_{L^2} = \infty;
  \]
- \( u(t) \) blows up infinite time and
  \[
  \sup_{t \in [0, +\infty)} \|u(t)\|_{L^q} = \infty,
  \]
  for any \( q \geq \frac{4s}{d} + 2 \). In particular, there exists a time sequence \((t_n)_n\) such that \( t_n \to +\infty \) and
  \[
  \lim_{n \to \infty} \|(-\Delta)^{s/2} u(t_n)\|_{L^2} = \infty.
  \]
Now, let \( \alpha_* < \alpha < \alpha^* \), where
\[
\alpha_* := \frac{4s}{d}, \quad \alpha^* := \begin{cases} 
\infty & \text{if } d = 1, \\
\frac{4s}{d-2s} & \text{if } d \geq 2.
\end{cases}
\]

In the mass and energy intercritical case, we have the following blow-up criteria.

**Proposition 1.6** (Mass and energy intercritical blow-up criteria). Let \( d \geq 1 \) and \( s \in (1/2, 1) \). Let \( u_0 \in H^s \) be such that the corresponding (not necessary radial) solution to the mass and energy intercritical (1.1), i.e. \( \alpha_* < \alpha < \alpha^* \) exists on the maximal time \([0, T)\). If either
\[
E(u_0) < 0,
\]
or if \( E(u_0) \geq 0 \), we assume that
\[
\begin{aligned}
E_{sc}(u_0)M^{s-s_c}(u_0) &< E_{sc}(Q)M^{s-s_c}(Q), \\
\|(-\Delta)^{s/2}u_0\|_{L^2}^s \|u_0\|_{L^2}^{s-s_c} &> \|(-\Delta)^{s/2}Q\|_{L^2}^s \|Q\|_{L^2}^{s-s_c},
\end{aligned}
\]
where \( Q \) is the unique (up to symmetries) positive radial solution to (1.4), then one of the following statements holds true:
- \( u(t) \) blows up in finite time, i.e. \( T < +\infty \) and
  \[
  \lim_{t \uparrow T} \|(-\Delta)^{s/2}u(t)\|_{L^2} = \infty;
  \]
- \( u(t) \) blows up infinite time and
  \[
  \sup_{t \in [0, +\infty)} \|u(t)\|_{L^q} = \infty,
  \]
for any \( q \geq \alpha + 2 \). In particular, there exists a time sequence \((t_n)\) such that \( t_n \to +\infty \) and
\[
\lim_{n \to \infty} \|(-\Delta)^{s/2}u(t_n)\|_{L^2} = \infty.
\]

Finally, we have the following blow-up criteria in the energy-critical case.

**Proposition 1.7** (Energy-critical blow-up criteria). Let \( d \geq 2 \) and \( s \in (1/2, 1) \). Let \( u_0 \in H^s \) be such that the corresponding (not necessary radial) solution to the energy-critical (1.1), i.e. \( \alpha = \frac{4s}{d-2s} \) exists on the maximal time \([0, T)\). If either
\[
E(u_0) < 0,
\]
or if \( E(u_0) \geq 0 \), we assume that
\[
\begin{aligned}
E(u_0) &< E(W), \\
\|(-\Delta)^{s/2}u_0\|_{L^2} &> \|(-\Delta)^{s/2}W\|_{L^2},
\end{aligned}
\]
where \( W \) is the unique (up to symmetries) positive radial solution to (1.5), then one of the following statements holds true:
- \( u(t) \) blows up in finite time, i.e. \( T < +\infty \);
- \( u(t) \) blows up infinite time and
  \[
  \sup_{t \in [0, +\infty)} \|u(t)\|_{L^q} = \infty,
  \]
for any \( q > \frac{2d}{d-2s} \). In particular, there exists a time sequence \((t_n)\) such that \( t_n \to +\infty \) and
\[
\lim_{n \to \infty} \|u(t_n)\|_{L^q} = \infty,
\]
for any \( q > \frac{2d}{d-2s} \).
The paper is organized as follows. In Section 2, we recall some preliminaries related to the fractional nonlinear Schrödinger equation such as Strichartz estimates and nonlinear estimates. In Section 3, we recall the local well-posedness for (1.1) in general Sobolev spaces $H^s$ with non-radial and radial initial data. In Section 4, we prove various virial-type estimates related to the equation. The blow-up criteria for non-radial solutions of (1.1) will be proved in Section 5.

2. Preliminaries

2.1. Strichartz estimates. In this subsection, we recall Strichartz estimates for the fractional Schrödinger equation. Let $I \subset \mathbb{R}$ and $p, q \in [1, \infty]$. We define the Strichartz norm

$$
\|f\|_{L^p(I, L^q)} := \left( \int_I \left( \int_{\mathbb{R}^d} |f(t, x)|^q \, dx \right)^{\frac{p}{q}} \right)^{\frac{1}{q}},
$$

with a usual modification when either $p$ or $q$ are infinity. Let $\chi_0$ be a bump function supported in $\{x \in \mathbb{R}^d : |x| \leq 2\}$ and $\chi_0(x) = 1$ for $|x| \leq 1$. Set $\chi(x) = \chi_0(x) - \chi_0(2x)$. We denote the Littlewood-Paley projections $P_0 := \chi_0(D), P_N := \chi(N^{-1}D)$ with $N = 2^k, k \in \mathbb{Z}$, where $\chi_0(D)f = \mathcal{F}^{-1}[\chi_0\mathcal{F}(f)]$ and similarly for $\chi(N^{-1}D)$ with $\mathcal{F}$ and $\mathcal{F}'$ the Fourier transform and its inverse respectively. Given $\gamma \in \mathbb{R}$ and $1 \leq q \leq \infty$, one defines the Besov space $B^\gamma_q$ as

$$
B^\gamma_q := \left\{ u \in \mathcal{S}': \|u\|_{B^\gamma_q} := \|P_0u\|_{L^q} + \left( \sum_{N \in 2^\mathbb{N}} N^{2\gamma} \|P_Nu\|_{L^q}^2 \right)^{\frac{1}{2}} < \infty \right\},
$$

where $\mathcal{S}'$ is the space of tempered distributions. There are several types of Strichartz estimates for the Schrödinger operator $e^{-i(-\Delta)^s}$. We recall below two-types of Strichartz estimates for the fractional Schrödinger equation:

For general data (see e.g. [5] or [7]): the following estimates hold for $d \geq 1$ and $s \in (0, 1) \setminus \{1/2\}$,

$$
\|e^{-i(-\Delta)^s}\psi\|_{L^p(R, L^q)} \lesssim \|\nabla^{\gamma_{p,q}}\psi\|_{L^2},
$$

$$
\left| \int_0^t e^{-i(t-\tau)(-\Delta)^s}f(\tau) \, d\tau \right|_{L^p(R, L^q)} \lesssim \|\nabla^{\gamma_{p,q} - \gamma_{a,b} - 2s}f\|_{L^{s'}(R,L')} ,
$$

where $(p, q)$ and $(a, b)$ are Schrödinger admissible, i.e.

$$
p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2},
$$

and

$$
\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{2s}{p},
$$

similarly for $\gamma_{a,b'}$. Here $(a, a')$ and $(b, b')$ are conjugate pairs. It is worth noticing that for $s \in (0, 1) \setminus \{1/2\}$ the admissible condition $\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}$ implies $\gamma_{p,q} > 0$ for all admissible pairs $(p, q)$ except $(p, q) = (\infty, 2)$. This means that the above Strichartz estimates have a loss of derivatives. In the local theory, this loss of derivatives makes the problem more difficult, and leads to a weak local well-posedness result comparing to the nonlinear Schrödinger equation (see Section 3).

For radially symmetric data (see e.g. [24], [18] or [4]): the estimates (2.1) and (2.2) hold true for $d \geq 2, s \in (0, 1) \setminus \{1/2\}$ and $(p, q), (a, b)$ satisfy the radial
Schrödinger admissible condition:

\[ p \in [2, \infty], \quad q \in [2, \infty], \quad (p, q) \neq \left( \frac{2d - 2}{2d - 3} \right), \quad \frac{2d - 1}{p} + \frac{2d - 1}{q} \leq \frac{2d - 1}{2}. \]

Note that the admissible condition \( \frac{2}{p} + \frac{2d - 1}{q} \leq \frac{2d - 1}{2} \) allows us to choose \( (p, q) \) so that \( \gamma_{p,q} = 0 \). More precisely, we have for \( d \geq 2 \) and \( \frac{d}{2d - 1} \leq s < 1 \)

\[
\|e^{-it(-\Delta)^s}\psi\|_{L^p(\mathbb{R}, L^q)} \lesssim \|\psi\|_{L^2}, \tag{2.3}
\]

\[
\left| \int_0^t e^{-i(t-\tau)(-\Delta)^s} f(\tau) d\tau \right|_{L^p(\mathbb{R}, L^q)} \lesssim \|f\|_{L^{q'}(\mathbb{R}, L^{q'})}, \tag{2.4}
\]

where \( \psi \) and \( f \) are radially symmetric and \( (p, q), (a, b) \) satisfy the fractional admissible condition,

\[ p \in [2, \infty], \quad q \in [2, \infty], \quad (p, q) \neq \left( \frac{2d - 2}{2d - 3} \right), \quad \frac{2s}{p} + \frac{d}{q} = \frac{d}{2}. \tag{2.5} \]

These Strichartz estimates with no loss of derivatives allow us to give a similar local well-posedness result as for the nonlinear Schrödinger equation (see again Section 3).

2.2. Nonlinear estimates. We recall the following fractional chain rule which is needed in the local well-posedness for (1.1).

**Lemma 2.1** (Fractional chain rule [6]). Let \( F \in C^1(\mathbb{C}, \mathbb{C}) \) and \( \gamma \in (0, 1) \). Then for \( 1 < q \leq q_2 < \infty \) and \( 1 < q_1 \leq \infty \) satisfying \( \frac{1}{q} = \frac{1}{q_1} + \frac{d}{q_2} \),

\[ \|\nabla^\gamma F(u)\|_{L^q} \lesssim \|F(u)\|_{L^{q_1}} \|\nabla^\gamma u\|_{L^{q_2}}. \]

3. Local well-posedness

In this section, we recall the local well-posedness for (1.1) in Sobolev spaces. The proof is based on the contraction mapping argument using Strichartz estimates. Due to the loss of derivatives in Strichartz estimates, we thus consider separately two cases: non-radial initial data and radially symmetric initial data.

3.1. Non-radial initial data. We have the following local well-posedness for (1.1) in Sobolev spaces due to [22] (see also [7]). Let us start with the local well-posedness in the sub-critical case, i.e. \( \gamma > s_c \).

**Proposition 3.1** (Non-radial local theory I [22, 7]). Let \( d \geq 1 \), \( s \in (1/2, 1) \) and \( \alpha > 0 \). Let \( \gamma \geq 0 \) be such that

\[ \gamma > \begin{cases} 1/2 - 2s / \max(\alpha, 4) & \text{if } d = 1, \\
                    d/2 - 2s / \max(\alpha, 2) & \text{if } d \geq 2, \end{cases} \]

and also, if \( \alpha \) is not an even integer, (1.3) holds. Then for any \( u_0 \in H^\gamma \), there exist \( T \in (0, +\infty) \) and a unique solution to (1.1) satisfying

\[ u \in C([0, T), H^\gamma) \cap L^p_{\text{loc}}([0, T), L^\infty), \]

for some

\[ p \geq \begin{cases} \max(\alpha, 4) & \text{if } d = 1, \\
                     \max(\alpha, 2) & \text{if } d \geq 2. \end{cases} \]

Moreover, the following properties hold:

---

1This condition follows by plugging \( \gamma_{p,q} = 0 \) to \( \frac{2}{p} + \frac{2d - 1}{q} \leq \frac{2d - 1}{2} \).
If \( T < \infty \), then \( \lim_{t \to T} \| u(t) \|_{H^s} = \infty \);
There is conservation of mass, i.e. \( M(u(t)) = M(u_0) \) for all \( t \in [0, T) \);
If \( \gamma \geq s \), then the energy is conserved, i.e. \( E(u(t)) = E(u_0) \) for all \( t \in [0, T) \).

We refer the reader to \([7]\) (see also \([22]\)) for the proof of above result. The proof is based on Strichartz estimates and the contraction mapping argument. Note that in the non-radial case, there is a loss of derivatives in Strichartz estimates. Fortunately, this loss of derivatives can be compensated for by using the Sobolev embedding. However, there is still a gap between \( s_c \) and \( 1/2 - 2s/(\max(\alpha, 4)) \) when \( d = 1 \), and \( d/2 - 2s/(\max(\alpha, 2)) \) when \( d \geq 2 \). We also have the local well-posedness in the critical case, i.e. \( \gamma = s_c \).

**Proposition 3.2** (Non-radial local theory II \([22, 7]\)). Let \( d \geq 1 \), \( s \in (1/2, 1) \) and

\[
\alpha > \begin{cases} 
4 & \text{if } d = 1, \\
2 & \text{if } d \geq 2,
\end{cases}
\]

be such that \( s_c \geq 0 \), and also, if \( \alpha \) is not an even integer, \((1.3)\) holds. Then for any \( u_0 \in H^{s_c} \), there exist \( T \in (0, +\infty) \) and a unique solution to \((1.1)\) satisfying

\[
u \in C([0, T), H^{s_c}) \cap L^p_{\text{loc}}(0, T), B^{s_c-\gamma}_{p,q}),
\]

where

\[
\begin{cases} p = 4, q = \infty & \text{if } d = 1, \\
2 < p < \alpha, q = 2p/(p-2) & \text{if } d = 2, \\
p = 2, q = 2d/(d-2) & \text{if } d \geq 3,
\end{cases}
\]

and

\[
\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{2s}{p}.
\]

Moreover, the following properties hold:

- There is conservation of mass, i.e. \( M(u(t)) = M(u_0) \) for all \( t \in [0, T) \);
- If \( s_c \geq s \), then the energy is conserved, i.e. \( E(u(t)) = E(u_0) \) for all \( t \in [0, T) \).

We refer the reader to \([22]\) (see also \([7]\)) for the proof of this result. Unlike the sub-critical case, the Sobolev embedding does not help us to overcome the loss of derivatives. It needs a delicate estimate on \( L^p_t L^\infty_x \) to overcome this loss of derivatives.

### 3.2. Radial initial data

In this subsection, we show the local well-posedness for \((1.1)\) with radial initial data in Sobolev spaces. The proof is again based on the contraction mapping argument via Strichartz estimates. Thanks to Strichartz estimates without loss of derivatives in the radial case, we have better local well-posedness comparing to the non-radial case. Let us start with the local well-posedness in the subcritical case.

**Proposition 3.3** (Radial local theory I). Let \( d \geq 2 \) and \( s \in \left(\frac{d}{4d-1}, 1\right) \). Let \( \gamma \in [0, \frac{d}{2}) \) be such that \( \gamma > s_c \), and also, if \( \alpha \) is not an even integer, \((1.3)\) holds. Let

\[
p = \frac{4s(\alpha + 2)}{\alpha(d - 2\gamma)}, \quad q = \frac{d(\alpha + 2)}{d + \alpha \gamma}.
\]
Then for any \( u_0 \in H^\gamma \) be radial, there exists \( T \in (0, +\infty) \) and a unique solution to (1.1) satisfying
\[
u \in C([0, T), H^\gamma) \cap L^p_{\text{loc}}([0, T), W^{\gamma, q}).
\]
Moreover, the following properties hold:
- If \( T < +\infty \), then \( \lim_{t \to T} \| u(t) \|_{H^\gamma} = \infty \);
- There is conservation of mass, i.e. \( M(u(t)) = M(u_0) \) for all \( t \in [0, T) \);
- If \( \gamma \geq \alpha \), then the energy is conserved, i.e. \( E(u(t)) = E(u_0) \) for all \( t \in [0, T) \).

**Proof.** It is easy to check that \((p, q)\) satisfies the fractional admissible condition (2.5). We choose \((m, n)\) so that
\[
\frac{1}{p'} = \frac{1}{p} + \frac{\alpha}{m}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{\alpha}{n}.
\]
We see that
\[
\frac{\alpha}{m} - \frac{\alpha}{p} = 1 - \frac{\alpha(d - 2\gamma)}{4s} = : \theta > 0, \quad q \leq n = \frac{dq}{d - \gamma q}.
\]
The later fact gives the Sobolev embedding \( \dot{W}^{\gamma, q} \hookrightarrow L^n \). Let us now consider
\[
X := \left\{ C(I, H^\gamma) \cap L^p(I, W^{\gamma, q}) : \| u \|_{L^\infty(I, \dot{H}^\gamma)} + \| u \|_{L^p(I, \dot{W}^{\gamma, q})} \leq M \right\},
\]
equipped with the distance
\[
d(u, v) := \| u - v \|_{L^\infty(I, L^2)} + \| u - v \|_{L^p(I, L^2)} + \| u - v \|_{L^p(I, L^q)},
\]
where \( I = [0, \zeta] \) and \( M, \zeta > 0 \) to be chosen later. By Duhamel’s formula, it suffices to prove that the functional
\[
\Phi(u)(t) := e^{-it(-\Delta)^s} u_0 - i\mu \int_0^t e^{-i(t-\tau)(-\Delta)^s} |u(\tau)|^\alpha u(\tau) d\tau
\]
is a contraction on \((X, d)\). By radial Strichartz estimates (2.3) and (2.4),
\[
\| \Phi(u) \|_{L^\infty(I, \dot{H}^\gamma)} + \| \Phi(u) \|_{L^p(I, \dot{W}^{\gamma, q})} \lesssim \| u_0 \|_{\dot{H}^\gamma} + \| u \|_{L^p(I, \dot{W}^{\gamma, q})},
\]
\[
\| \Phi(u) - \Phi(v) \|_{L^\infty(I, L^2)} + \| \Phi(u) - \Phi(v) \|_{L^p(I, L^q)} \lesssim \| u - v \|_{L^p(I, L^q)} + \| u \|_{L^p(I, L^q)}.
\]
The fractional chain rule given in Lemma 2.1 and the Hölder inequality give
\[
\| u \|_{L^{\alpha'}} \lesssim \| u \|_{L^\infty(I, L^\alpha')} \lesssim \| u \|_{L^{\alpha} = (I, L^\alpha)} \lesssim |I|^{\theta} \| u \|_{L^p(I, L^\alpha)} \lesssim |I|^{\theta} \| u \|_{L^p(I, \dot{W}^{\gamma, q})}.
\]
Similarly,
\[
\| u - v \|_{L^{\alpha'}} \lesssim \left( \| u \|_{L^\infty(I, L^\alpha)} + \| v \|_{L^\infty(I, L^\alpha)} \right) \| u - v \|_{L^p(I, L^\alpha)} \lesssim |I|^{\theta} \left( \| u \|_{L^p(I, \dot{W}^{\gamma, q})} + \| v \|_{L^p(I, \dot{W}^{\gamma, q})} \right) \| u - v \|_{L^p(I, L^\alpha)}.
\]
This shows that for all \( u, v \in X \), there exists \( C > 0 \) independent of \( T \) and \( u_0 \in L^\gamma \) such that
\[
\| \Phi(u) \|_{L^\infty(I, \dot{H}^\gamma)} + \| \Phi(u) \|_{L^p(I, \dot{W}^{\gamma, q})} \leq C \| u_0 \|_{\dot{H}^\gamma} + C \zeta^\theta M^{\alpha + 1},
\]
\[
d(\Phi(u), \Phi(v)) \leq C \zeta^\theta M^\alpha d(u, v).
\]
If we set $M = 2C\|u_0\|_{H^\gamma}$ and choose $\zeta > 0$ so that
\[ C\zeta^\delta M^\alpha \leq \frac{1}{2}, \]
then $\Phi$ is a strict contraction on $(X,d)$. This proves the existence of solution $u \in C(I,H^\gamma) \cap L^p(I,W^{\gamma,q})$. The blow-up alternative follows easily since the existence time depends only on the $\dot{H}^\gamma$-norm of initial data. The conservation of mass and energy follow by a standard approximation procedure. The proof is complete. \(\square\)

Finally, we have the local well-posedness with radial initial data in the critical case.

**Proposition 3.4** (Radial local theory II). Let $d \geq 2$ and $s \in \left[ \frac{d}{2d-1}, 1 \right)$. Let $\alpha > 0$ be such that $s_c \geq 0$, and also, if $\alpha$ is not an even integer, (1.3) holds. Let
\[ p = \alpha + 2, \quad q = \frac{2d(\alpha + 2)}{d(\alpha + 2) - 4s}. \]
Then for any $u_0 \in H^{s_c}$ radial, there exist $T \in (0, +\infty]$ and a unique solution to (1.1) satisfying
\[ u \in C([0,T), H^{s_c} \cap L^p_{\text{loc}}([0,T), W^{s_c,q})]. \]
Moreover, the following properties hold:

- There is conservation of mass, i.e. $M(u(t)) = M(u_0)$ for all $t \in [0,T)$;
- If $s_c \geq s$, then the energy is conserved, i.e. $E(u(t)) = E(u_0)$ for all $t \in [0,T)$.

**Proof.** It is easy to check that $(p,q)$ satisfies the fractional admissible condition. We next choose $n$ so that
\[ \frac{1}{q'} = \frac{1}{q} + \frac{\alpha}{n} \quad \text{or} \quad n = \frac{dq}{d - s_c q}. \]
The last condition ensures the Sobolev embedding
\[ \|u\|_{L^p(I,L^s)} \lesssim \|u\|_{L^p(I,W^{s_c,q})}. \tag{3.8} \]
Let us consider
\[ X := \left\{ u \in L^p(I,W^{s_c,q}) : \|u\|_{L^p(I,W^{s_c,q})} \leq M \right\}, \]
equipped with the distance
\[ d(u,v) = \|u - v\|_{L^p(I,L^s)}, \]
where $I = [0,\zeta]$ and $M, \zeta > 0$ to be chosen later. We will show that the functional $\Phi$ is a contraction on $(X,d)$, where
\[ \Phi(u)(t) = e^{-it(-\Delta)^{s_c}}u_0 - i\mu \int_0^t e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^\alpha u(\tau) d\tau =: u_{\text{hom}}(t) + u_{\text{inh}}(t). \]
By radial Strichartz estimate (2.3), we have
\[ \|u_{\text{hom}}\|_{L^p(I,W^{s_c,q})} \lesssim \|u_0\|_{\dot{H}^{\gamma_c}}. \]
This shows that $\|u_{\text{hom}}\|_{L^p(I,W^{s_c,q})} \leq \epsilon$ for some $\epsilon$ small enough provided that $\zeta$ is small or $\|u_0\|_{\dot{H}^{\gamma_c}}$ is small. Similarly, by (2.4), we have
\[ \|u_{\text{inh}}\|_{L^p(I,W^{s_c,q})} \lesssim \|u|^\alpha u\|_{L^{p'}(I,W^{-s_c,q'})} \]

Thus, for all \( u, v \), we have
\[
||| u^{\alpha} u |||_{L^p(I, \dot{W}^{s, \alpha}_q)} \lesssim \| u \|_{L^p(I, \dot{W}^{s, \alpha}_q)} \| u \|_{L^p(I, \dot{W}^{s, \alpha}_q)} \lesssim \| u \|_{L^p(I, \dot{W}^{s, \alpha}_q)}^{\alpha + 1}.
\]

Similarly, we have
\[
||| u^{\alpha} u - |v|^{\alpha} v |||_{L^p(I, \dot{W}^{s, \alpha}_q)} \lesssim \left( \| u \|_{L^p(I, \dot{W}^{s, \alpha}_q)}^{\alpha} + \| v \|_{L^p(I, \dot{W}^{s, \alpha}_q)}^{\alpha} \right) \| u - v \|_{L^p(I, \dot{W}^{s, \alpha}_q)}.
\]

Thus, for all \( u, v \in X \), there exists \( C \) depending only on \( u_0 \in H^\infty \) such that
\[
\| \Phi(u) \|_{L^p(I, \dot{W}^{s, \alpha}_q)} \leq \epsilon + CM^{\alpha + 1},
\]
\[
d(\Phi(u), \Phi(v)) \leq CM^\alpha d(u, v).
\]

If we choose \( \epsilon, M > 0 \) small so that
\[
CM^\alpha \leq 1, \quad \epsilon + M^2 \leq M,
\]
then \( \Phi \) is a contraction on \((X, d)\). This shows the existence of solutions. The conservation of mass and energy are standard and we omit the details. The proof is complete. \( \square \)

4. Virial estimates

In this section, we recall and prove some virial estimates related to (1.1) which are in the same spirit as in [1, Section 2]. Let us start with the following estimates.

Lemma 4.1 ([1]). Let \( d \geq 1 \) and \( \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \) be such that \( \nabla \varphi \in W^{1, \infty} \). Then for any \( u \in H^{1/2} \), it holds that
\[
\left| \int \nabla \varphi \cdot \nabla u(x) dx \right| \leq C \| \nabla \varphi \|_{W^{1, \infty}} \left( \| \nabla^{1/2} u \|_{L^2}^2 + \| u \|_{L^2} \| \nabla^{1/2} u \|_{L^2} \right),
\]
for some constant \( C > 0 \) depending only on \( d \).

Lemma 4.2. Let \( d \geq 1 \), \( s \in (1/2, 1) \) and \( \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \) be such that \( \nabla \varphi \in W^{1, \infty} \). Then for any \( u \in L^2 \), it holds that
\[
\left| \int_0^\infty m^s \int \Delta \varphi |u_m|^2 dx dm \right| \leq C \| \Delta \varphi \|_{L^{\infty}}^{2s-1} \| \nabla \varphi \|_{L^{\infty}}^{2-2s} \| u \|_{L^2}^2,
\]
for some constant \( C > 0 \) depending only on \( s \) and \( d \). Here
\[
u_m(x) = c_s \frac{1}{1 - \Delta + m} u(x) = c_s F^{-1} \left( \frac{\hat{u}(\xi)}{|\xi|^2 + m} \right), \quad m > 0,
\]
where
\[
c_s := \sqrt{\frac{\sin \pi s}{\pi}}.
\]

Proof. The proof is essentially given in [1, Lemma A.2]. For the reader’s convenience, we give some details. We split the \( m \)-integral into \( \int_0^\tau \cdots \) and \( \int_{\tau}^\infty \cdots \) with
\( \tau > 0 \) to be chosen later. By integration by parts and Hölder’s inequality, we learn that
\[
\left| \int_0^T m^s \int \Delta \varphi |u_m|^2 \, dx \right| = \left| \int_0^T m^s \int \nabla \varphi \cdot (\nabla u_m \nabla \varphi + u_m \nabla \varphi) \, dx \right|
\]
\[
= \| \nabla \varphi \|_{L^\infty} \int_0^T m^s \| \nabla u_m \|_{L^2} \| u_m \|_{L^2} \, dx \, dm
\]
\[
\lesssim \| \nabla \varphi \|_{L^\infty} \| u \|_{L^2}^2 \left( \int_0^T m^{s-3/2} \, dx \right)
\]
\[
\lesssim \tau^{s-1/2} \| \nabla \varphi \|_{L^\infty} \| u \|_{L^2}^2.
\]

Here we use the fact
\[
\| \nabla u_m \|_{L^2} \lesssim m^{-1/2} \| u \|_{L^2}, \quad \| u_m \|_{L^2} \lesssim m^{-1} \| u \|_{L^2},
\]
which follows directly from the definition of \( u_m \). On the other hand, we find that
\[
\left| \int_T^\infty m^s \int \Delta \varphi |u_m|^2 \, dx \right| \lesssim \| \Delta \varphi \|_{L^\infty} \left( \int_T^\infty m^s \| u_m \|_{L^2}^2 \, dx \right)
\]
\[
\lesssim \| \Delta \varphi \|_{L^\infty} \| u \|_{L^2}^2 \left( \int_T^\infty m^{s-2} \, dx \right)
\]
\[
\lesssim \tau^{s-1} \| \Delta \varphi \|_{L^\infty} \| u \|_{L^2}^2.
\]

Collecting the above estimates, we obtain
\[
\left| \int_0^\infty m^s \int \Delta \varphi |u_m|^2 \, dx \right| \lesssim \left( \tau^{s-1/2} \| \nabla \varphi \|_{L^\infty} + \tau^{s-1} \| \Delta \varphi \|_{L^\infty} \right) \| u \|_{L^2}^2,
\]
for arbitrary \( \tau > 0 \). Minimizing the right hand side with respect to \( \tau \), i.e. choosing \( \tau = \frac{(1-s)^2 \| \Delta \varphi \|_{L^\infty}^2}{2(1-s/2) \| \nabla \varphi \|_{L^\infty}^2} \), we complete the proof. \( \square \)

**Lemma 4.3.** Let \( d \geq 1 \), \( s \in (1/2, 1) \) and \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be such that \( \nabla \varphi \in W^{1, \infty} \). Then for any \( u \in H^{1/2} \), it holds that
\[
\left| \int_0^\infty m^s \int \nabla \varphi \cdot \nabla u_m \, dx \right| \leq C \| \nabla \varphi \|_{W^{1, \infty}} \| u \|_{H^{1/2}}^2,
\]
for some constant \( C > 0 \) depending only on \( d \), where \( u_m \) is given in (4.3).

**Proof.** As in the proof of Lemma 4.2, we split the \( m \)-integral into two parts \( \int_0^T \cdots \) and \( \int_T^\infty \cdots \) with \( \tau > 0 \) to be chosen shortly. By Hölder’s inequality, we estimate the first term as
\[
\left| \int_0^T m^s \int \nabla \varphi \cdot \nabla u_m \, dx \right| \lesssim \| \nabla \varphi \|_{L^\infty} \int_0^T m^s \| u_m \|_{L^2} \| \nabla u_m \|_{L^2} \, dx \, dm
\]
\[
\lesssim \| \nabla \varphi \|_{L^\infty} \| u \|_{L^2}^2 \left( \int_0^T m^{s-3/2} \, dx \right)
\]
\[
\lesssim \tau^{s-1/2} \| \nabla \varphi \|_{L^\infty} \| u \|_{L^2}^2.
\]
For the second term, we use Lemma 4.1 to get
\[
\left| \int_{\tau}^{\infty} m^s \int \pi_m \nabla \varphi \cdot \nabla u_m \, dxdm \right| \\
\lesssim \| \nabla \varphi \|_{W^{1, \infty}} \int_{\tau}^{\infty} m^s \left( \| |\nabla|^{1/2} u_m \|_{L^2}^2 + \| u_m \|_{L^2} \| |\nabla|^{1/2} u_m \|_{L^2} \right) \, dm \\
\lesssim \| \nabla \varphi \|_{W^{1, \infty}} \left( \| |\nabla|^{1/2} u \|_{L^2}^2 + \| u \|_{L^2} \| |\nabla|^{1/2} u \|_{L^2} \right) \left( \int_{\tau}^{\infty} m^{s-2} \, dm \right) \\
\lesssim \tau^{s-1} \| \nabla \varphi \|_{W^{1, \infty}} \left( \| |\nabla|^{1/2} u \|_{L^2}^2 + \| u \|_{L^2} \| |\nabla|^{1/2} u \|_{L^2} \right).
\]
Collecting two terms, we get
\[
\left| \int_{0}^{\infty} m^s \int \pi_m \nabla \varphi \cdot \nabla u_m \, dxdm \right| \lesssim \left( \tau^{s-1/2} \| \nabla \varphi \|_{L^\infty} + \tau^{s-1} \| \nabla \varphi \|_{W^{1, \infty}} \right) \| u \|_{H^{1/2}}^2,
\]
for any \( \tau > 0 \). Taking \( \tau = 1 \), we prove (4.4).

\textbf{Lemma 4.4 (1).} Let \( d \geq 1, s \in (0, 1) \) and \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be such that \( \Delta \varphi \in W^{2, \infty} \). Then for any \( u \in L^2 \), it holds that
\[
\left| \int_{0}^{\infty} m^s \int \Delta^2 \varphi |u_m|^2 \, dxdm \right| \leq C \| \Delta^2 \varphi \|_{L^\infty} \| \Delta \varphi \|_{L^\infty}^{-s} \| u \|_{L^2}^2,
\]
for some constant \( C > 0 \) depending only on \( s \) and \( d \), where \( u_m \) is given in (4.3).

We refer the reader to [1, Appendix A] for the proofs of Lemmas 4.1 and 4.4. We also note that by using the fact
\[
\frac{\sin \pi s}{\pi} \int_{0}^{\infty} \frac{m^s \, dm}{(|\xi|^2 + m)^2} = s|\xi|^{2s-2},
\]
the Plancherel’s and Fubini’s theorems imply the following useful identity
\[
\int_{0}^{\infty} m^s \int |\nabla u_m|^2 \, dxdm = \int \left( \frac{\sin \pi s}{\pi} \int_{0}^{\infty} \frac{m^s \, dm}{(|\xi|^2 + m)^2} \right) |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi \\
= \int (s|\xi|^{2s-2})|\xi|^2 |\hat{u}(\xi)|^2 \, d\xi = s \| (-\Delta)^{s/2} u \|_{L^2}^2,
\]
for any \( u \in H^s \).

Now, let \( d \geq 1, 1/2 < s < 1 \) and \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be such that \( \varphi \in W^{2, \infty} \). Assume \( u \in C((0, T), H^s) \) is a solution to (1.1). Note that in [1], the authors derive virial estimates by assuming that the solution \( u(t) \) belongs to \( H^{2s} \) for any \( t \in [0, T) \). This regularity assumption is necessary due to the lack of local theory at the time. By the local theory given in Section 3, one can extend virial estimates to \( u \in C((0, T), H^s) \) by an approximation argument. The type-I localized virial action of \( u \) associated to \( \varphi \) is defined by
\[
V_\varphi(u(t)) := \int \varphi(x)|u(t, x)|^2 \, dx.
\]

\textsuperscript{2} The smoothing operator \((-\Delta + m)^{-1}\) implies that \( u_m \in H^{s+2} \) whenever \( u \in H^s \). Hence the hypotheses of Lemma 4.1 are satisfied.
Lemma 4.5 (Localized virial identity I). Let $d \geq 1, 1/2 < s < 1$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ be such that $\varphi \in W^{2, \infty}$. Assume that $u \in C([0, T), H^s)$ is a solution to (1.1). Then for any $t \in [0, T)$, it holds that
\begin{equation}
\frac{d}{dt} V_\varphi(u(t)) = -i \int_0^\infty m^s \int \Delta \varphi |u_m(t)|^2 dx dm - 2i \int_0^\infty m^s \int \nabla \varphi \cdot \nabla u_m(t) dx dm,
\end{equation}
where $u_m(t) = c_s(-\Delta + m)^{-1} u(t)$.

Proof. We only verify (4.8) for $u \in C_0^\infty(\mathbb{R}^d)$. The general case follows by an approximation argument (see [1, Lemma 2.1]). By definition, we see that
\begin{equation}
V_\varphi(u(t)) = \langle u(t), \varphi u(t) \rangle,
\end{equation}
where $\langle u, v \rangle$ is the scalar product in $L^2$. Taking the time derivative and using that $u(t)$ solves (1.1), we have
\begin{equation}
\frac{d}{dt} V_\varphi(u(t)) = i \langle u(t), [(-\Delta)^s, \varphi] u(t) \rangle,
\end{equation}
where $[X, Y] = XY - YX$ denotes the commutator of $X$ and $Y$. To study $[(-\Delta)^s, \varphi]$, we recall the following Balakrishnan’s formula
\begin{equation}
(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m} dm.
\end{equation}
This formula follows from spectral calculus applied to the self-adjoint operator $-\Delta$ and the identity
\begin{equation}
x^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{x}{x + m} dm
\end{equation}
which is available for any $x > 0$ and $s \in (0, 1)$. The Balakrishnan’s representation formula (4.10) for the fractional Laplacian $(-\Delta)^s$ was firstly used in [27] to study the nonlinear half-wave equation, i.e. (1.1) with $s = 1/2$. We also have the following commutator identity
\begin{equation}
\left[ \frac{A}{A + m}, B \right] = \left[ 1 - \frac{m}{A + m}, B \right] = -m \left[ \frac{1}{A + m}, B \right] = m \frac{1}{A + m} [A, B] \frac{1}{A + m},
\end{equation}
for operators $A \geq 0$ and $B$, where $m > 0$ is any positive real number. Using (4.10), we apply (4.11) with $A = -\Delta$ to get
\begin{equation}
[(-\Delta)^s, B] = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{1}{-\Delta + m} [-\Delta, B] \frac{1}{-\Delta + m} dm.
\end{equation}
Applying the above identity with $B = \varphi$ and using the fact
\begin{equation}
[-\Delta, \varphi] = -\Delta \varphi - 2 \nabla \varphi \cdot \nabla,
\end{equation}

We have the following virial identity (see □).

The proof is complete.

A direct consequence of Lemmas 4.2, 4.3, 4.5 and the fact \( \| \nabla \varphi \|_{W^{1,\infty}} \sim \| \nabla \varphi \|_{L^\infty} + \| \Delta \varphi \|_{L^\infty} \) is the following estimate.

**Corollary 4.6.** Let \( d \geq 1, 1/2 < s < 1 \) and \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be such that \( \varphi \in W^{2,\infty} \). Assume that \( u \in C([0,T), H^s) \) is a solution to (1.1). Then for any \( t \in (0,T) \),

\[
\left| \frac{d}{dt} V_F(u(t)) \right| \leq C \| \nabla \varphi \|_{W^{1,\infty}} \| u(t) \|_{H^s}^2,
\]

(4.13)

for some constant \( C > 0 \) depending only on \( s \) and \( d \).

We next define the type-II localized virial action of \( u \) associated to \( \varphi \) by

\[
M_\varphi(u(t)) := 2\text{Im} \int \overline{\varphi(t,x)} \nabla \varphi(x) \cdot \nabla u(t,x) dx.
\]

(4.14)

Thanks to Lemma 4.1, the quantity \( M_\varphi(u(t)) \) is well-defined. Indeed, by (4.1),

\[
|M_\varphi(u(t))| \leq C(\| \nabla \varphi \|_{L^\infty}, \| \Delta \varphi \|_{L^\infty}) \| u(t) \|_{H^{s/2}}^2 \lesssim C(\varphi) \| u(t) \|_{H^s}^2 < \infty.
\]

We have the following virial identity (see [1, Lemma 2.1]).

**Lemma 4.7** (Localized virial identity II [1]). Let \( d \geq 1, 1/2 < s < 1 \) and \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be such that \( \nabla \varphi \in W^{3,\infty} \). Assume that \( u \in C([0,T), H^s) \) is a solution to (1.1). Then for any \( t \in (0,T) \), it holds that

\[
\frac{d}{dt} M_\varphi(u(t)) = -\int_0^\infty m^s \int \Delta^2 \varphi |u_m(t)|^2 dx dm
\]

\[
+ 4 \sum_{j,k=1}^d \int_0^\infty m^s \int \partial_{jk}^2 \varphi \partial_{\alpha} \varphi \partial_{\alpha} \varphi \partial_j \varphi \partial_k u_m(t) dx dm
\]

\[
- \frac{2\alpha}{\alpha + 2} \int \Delta \varphi |u(t)|^{\alpha+2} dx,
\]

(4.15)

where \( u_m(t) = c_s(-\Delta + m)^{-1} u(t) \).

**Remark 4.8.** If we make the formal substitution and take the unbounded function \( \varphi(x) = |x|^2 \), then by (4.6), we obtain the virial identity

\[
\frac{d}{dt} \left( 4\text{Im} \int \overline{\varphi(t)} x \cdot \nabla u(t) dx \right) = 8s \| (-\Delta)^{s/2} u(t) \|_{L^2}^2 - \frac{4\alpha}{\alpha + 2} \| u(t) \|_{L^{\alpha+2}}^2
\]

\[
= 4\alpha E(u(t)) - 2(\alpha - 4s) \| (-\Delta)^{s/2} u(t) \|_{L^2}^2.
\]

(4.16)
This identity can be proved rigorously by integrating (1.1) against \( i(x \cdot \nabla + \nabla \cdot x) \pi(t) \) on \( \mathbb{R}^d \).

5. Blow-up criteria

In this section, we give the proof of Theorem 1.2 and its applications. We follow closely the argument of [11].

5.1. Proof of Theorem 1.2. If \( T < +\infty \), then we are done. If \( T = +\infty \), we show (1.11). By contradiction, we assume that the solution exists globally in time and there exists \( q > \alpha + 2 \) such that

\[
\sup_{t \in [0, +\infty)} \| u(t) \|_{L^q} < \infty. \tag{5.1}
\]

Interpolating between \( L^2 \) and \( L^q \), the conservation of mass implies

\[
\sup_{t \in [0, +\infty)} \| u(t) \|_{L^{\alpha+2}} < \infty. \tag{5.2}
\]

By the conservation of mass and energy, we get

\[
\sup_{t \in [0, +\infty)} \| u(t) \|_{H^s} < \infty. \tag{5.3}
\]

As in [11], the first step is to control \( L^2 \)-norm of the solution outside a large ball. To do so, we introduce \( \vartheta : [0, +\infty) \to [0, 1] \) a smooth function satisfying

\[
\vartheta(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq \frac{1}{2}, \\
1 & \text{if } r \geq 1.
\end{cases}
\]

Given \( R > 1 \), we denote the radial function

\[
\psi_R(x) = \psi(r) := \vartheta(r/R), \quad r = |x|.
\]

It is easy to check that

\[
\nabla \psi_R(x) = \frac{x}{r R} \vartheta'(r/R), \quad \Delta \psi_R(x) = \frac{1}{R^2} \vartheta''(r/R) + \frac{(d-1)}{r R} \vartheta'(r/R).
\]

We thus get

\[
\| \nabla \psi_R \|_{W^{1,\infty}} \sim \| \nabla \psi_R \|_{L^\infty} + \| \Delta \psi_R \|_{L^\infty} \lesssim \frac{1}{R} + \frac{1}{R^2} \lesssim \frac{1}{R}. \tag{5.3}
\]

We next define

\[
V_{\psi_R}(u(t)) := \int \psi_R(x) |u(t, x)|^2 \, dx.
\]

By the fundamental theorem of calculus, we have

\[
V_{\psi_R}(u(t)) = V_{\psi_R}(u_0) + \int_0^t \frac{d}{d\tau} V_{\psi_R}(u(\tau)) \, d\tau \leq V_{\psi_R}(u_0) + \left( \sup_{\tau \in [0, t]} \left| \frac{d}{d\tau} V_{\psi_R}(u(\tau)) \right| \right) t.
\]

Using Corollary 4.6,(5.3) and (5.2), we get

\[
\sup_{\tau \in [0, t]} \left| \frac{d}{d\tau} V_{\psi_R}(u(\tau)) \right| \lesssim \| \nabla \psi_R \|_{W^{1,\infty}} \sup_{\tau \in [0, t]} \| u(\tau) \|^2_{H^s} \leq CR^{-1},
\]

for some constant \( C \) independent of \( R \). We thus obtain

\[
V_{\psi_R}(u(t)) \leq V_{\psi_R}(u_0) + CR^{-1} t.
\]
By the choice of \( \theta \), the conservation of mass yields
\[
V_{\psi_R}(u_0) = \int \psi_R(x)|u_0(x)|^2dx \leq \int_{|x|>R/2} |u_0(x)|^2dx \to 0,
\]
as \( R \to \infty \) or \( V_\psi(u_0) = o_R(1) \). Using the fact
\[
\int_{|x| \geq R} |u(t,x)|^2dx \leq V_{\psi_R}(u(t)) ,
\]
we obtain the following control on the \( L^2 \)-norm of \( u \) outside a large ball.

**Lemma 5.1** (\( L^2 \)-norm outside a large ball). Let \( \varepsilon > 0 \) and \( R > 1 \). Then there exists a constant \( C > 0 \) independent of \( R \) such that for any \( t \in [0,T_0] \) with \( T_0 := \frac{\varepsilon R}{2|\omega|} \),
\[
\int_{|x| \geq R} |u(t,x)|^2dx \leq o_R(1) + \varepsilon . \tag{5.4}
\]

Next, let us choose \( \theta : [0,\infty) \to [0,\infty) \) a smooth function such that
\[
\theta(r) = \begin{cases} r^2 & \text{if } 0 \leq r \leq 1, \\ 2 & \text{if } r \geq 2, \end{cases} \quad \text{and } \theta''(r) \leq 2 \text{ for } r \geq 0.
\]

Given \( R > 1 \), we define the radial function
\[
\varphi_R(x) = \varphi_R(r) := R^2 \theta(r/R), \quad r = |x|. \tag{5.5}
\]

We readily verify that
\[
2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi''_R(r)}{r} \geq 0, \quad 2d - \Delta \varphi_R(x) \geq 0, \quad \forall r \geq 0, \quad \forall x \in \mathbb{R}^d. \tag{5.6}
\]

Moreover,
\[
\|\nabla^k \varphi_R\|_{L^\infty} \lesssim R^{2-k}, \quad k = 0, \ldots, 4,
\]
and
\[
\text{supp}(\nabla^k \varphi_R) \subset \begin{cases} \{|x| \leq 2R\} & \text{for } k = 1, 2, \\ \{R \leq |x| \leq 2R\} & \text{for } k = 3, 4. \end{cases}
\]

Denote
\[
M_{\varphi_R}(u(t)) := 2\text{Im} \int \bar{\varphi}(t,x)\nabla \varphi_R(x) \cdot \nabla u(t,x)dx.
\]

Applying Lemma 4.7 with \( \varphi(x) = \varphi_R(x) \), we have
\[
\frac{d}{dt}M_{\varphi_R}(u(t)) = -\int_0^\infty m^s \int \Delta^2 \varphi_R|u_m(t)|^2dxdm
+ 4 \sum_{j,k=1}^d \int_0^\infty m^s \int \partial_j \varphi_R \partial_j \overline{\varphi}(t) \partial_k u_m(t)dxdm
- \frac{2\alpha}{\alpha + 2} \int \Delta \varphi_R |u(t)|^{\alpha+2}dx,
\]
where \( u_m(t) = c_s (-\Delta + m)^{-1} u(t) \). Since \( \text{supp}(\Delta^2 \varphi_R) \subset \{|x| \geq R\} \), we use Lemma 4.4 to have
\[
\left| \int_0^\infty m^s \int \Delta^2 \varphi_R |u_m(t)|^2dxdm \right| \lesssim \|\Delta^2 \varphi_R\|_{L^\infty} \|\Delta \varphi_R\|_{L^\infty} \|\frac{1}{t} u(t)\|_{L^2(|x| \geq R)}^2 \lesssim R^{-2s} \|u(t)\|_{L^2(|x| \geq R)}^2.
\]
Since $\varphi_R$ is radial, we use the fact
\[ \partial^2_{jk} = \left( \frac{\delta_{jk}}{r} - \frac{x_j x_k}{r^3} \right) \partial_r + \frac{x_j x_k}{r^2} \partial_r^2 \]
to write
\[ \sum_{j,k=1}^d \int_0^\infty \! m^s \int \partial^2_{jk} \varphi_R \partial_j \pi_m(t) \partial_k u_m(t) dx dm = \int_0^\infty \! m^s \int \frac{\varphi'_R}{r} \nabla u_m(t)^2 dx dm \]
\[ + \int_0^\infty \! m^s \int \left( \frac{\varphi''_R}{r^2} - \frac{\varphi'_R}{r^3} \right) |x \cdot \nabla u_m(t)|^2 dx dm. \]

Thanks to the identity (4.6), we have
\[ \int_0^\infty \! m^s \int \frac{\varphi'_R}{r} |\nabla u_m(t)|^2 dx dm = 2s \|(-\Delta)^{s/2} u(t)\|_{L^2}^2 \]
\[ + \int_0^\infty \! m^s \int \left( \frac{\varphi''_R}{r^2} - \frac{\varphi'_R}{r^3} \right) |\nabla u_m(t)|^2 dx dm. \]

We next use the fact $\varphi'_R \leq 2$ and the Cauchy-Schwarz estimate $|x \cdot \nabla u_m| \leq r |\nabla u_m|$ to see that
\[ \int_0^\infty \! m^s \int \left( \frac{\varphi'_R}{r} - 2 \right) |\nabla u_m(t)|^2 dx dm \]
\[ + \int_0^\infty \! m^s \int \left( \varphi''_R - \frac{\varphi'_R}{r^2} \right) |x \cdot \nabla u_m(t)|^2 dx dm \leq 0. \]

We next write
\[ -\frac{2\alpha}{\alpha + 2} \int \Delta \varphi_R |u(t)|^{\alpha+2} dx = -\frac{4d\alpha}{\alpha + 2} \|u(t)\|_{L^{\alpha+2}}^2 + \frac{2\alpha}{\alpha + 2} \int \left( 2d - \Delta \varphi_R \right) |u(t)|^{\alpha+2} dx. \]

Collecting the above estimates, we obtain
\[ \frac{d}{dt} M_{\varphi_R}(u(t)) \leq 8s \|(-\Delta)^{s/2} u(t)\|_{L^2}^2 - \frac{4d\alpha}{\alpha + 2} \|u(t)\|_{L^{\alpha+2}}^2 + CR^{-2s} \|u(t)\|_{L^2(|x| \geq R)}^2 \]
\[ + \frac{2\alpha}{\alpha + 2} \int \left( 2d - \Delta \varphi_R \right) |u(t)|^{\alpha+2} dx. \]

Since $\text{supp}(2d - \Delta \varphi_R) \subset \{|x| \geq R\}$ and $\|2d - \Delta \varphi_R\|_{L^\infty} \lesssim 1$, we interpolate between $L^2$ and $L^\eta$ and use (5.1) to get
\[ \int \left( 2d - \Delta \varphi_R \right) |u(t)|^{\alpha+2} dx \lesssim \|u(t)\|_{L^{(1-\eta)(\alpha+2)}} \|u(t)\|_{L^\eta(|x| \geq R)} \|u(t)\|_{L^{(\alpha+2)}} \lesssim \|u(t)\|_{L^{\alpha+2}} \]
for some $0 < \eta < 1$. Note that the condition $q > \alpha + 2$ is neccessary in the above estimate. We thus obtain the following estimate.

**Lemma 5.2.** Let $R > 1$ and $\varphi_R$ be as in (5.5). There exist a constant $C > 0$ independent of $R$ and $0 < \eta < 1$ such that
\[ \frac{d}{dt} M_{\varphi_R}(u(t)) \leq 16K(u(t)) + CR^{-2} \|u(t)\|_{L^2(|x| \geq R)}^2 + C \|u(t)\|_{L^\eta(|x| \geq R)}^{\eta+2}, \] (5.7)
for any $t \in [0, T)$, where $K(u(t))$ is given in (1.9).
We now complete the proof of Theorem 1.2. Applying Lemma 5.1 and Lemma 5.2, we see that for any \( \varepsilon > 0 \) and any \( R > 1 \), there exists a constant \( C > 0 \) independent of \( R \) such that for any \( t \in [0, T_0] \) with \( T_0 := \frac{c R}{\varepsilon} \),

\[
\frac{d}{dt} M_{\varphi_R}(u(t)) \leq 16K(u(t)) + CR^{-2}(o_R(1) + \varepsilon)^2 + C(o_R(1) + \varepsilon)^{\eta(\alpha+2)} \\
\leq -16\delta + CR^{-2}(o_R(1) + \varepsilon)^2 + C(o_R(1) + \varepsilon^{\eta(\alpha+2)}) .
\]

Note that the constant \( C \) may change from lines to lines but is independent of \( R \). We now choose \( \varepsilon > 0 \) so that

\[
C\varepsilon^{\eta(\alpha+2)} = 4\delta.
\]

We see that for \( R \gg 1 \) large,

\[
\frac{d}{dt} M_{\varphi_R}(u(t)) \leq -\delta < 0,
\]

for any \( t \in [0, T_0] \) with \( T_0 = \frac{c R}{\varepsilon} \). Note also that since \( \varepsilon > 0 \) is fixed, we can take \( T_0 \) as large as we want by increasing \( R \) accordingly. From (5.8), we infer that

\[
M_{\varphi_R}(u(t)) \leq -ct,
\]

for all \( t \in [t_0, T_0] \) with some sufficiently large time \( t_0 \in [0, T_0] \) and some constant \( c > 0 \) depending only on \( \delta \). On the other hand, by Lemma 4.1 and the conservation of mass, we see that for \(^3\) any \( t \in [0, +\infty) \),

\[
|M_{\varphi_R}(u(t))| \lesssim C(\varphi_R) \left( \|\nabla|^{1/2}u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}\|\nabla|^{1/2}u(t)\|_{L^2} \right) \\
\lesssim C(\varphi_R) \left( \|\nabla|^{1/2}u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \right) \\
\lesssim C(\varphi_R) \left( \|\nabla|^{1/2}u(t)\|_{L^2}^2 + 1 \right) \\
\lesssim C(\varphi_R) \left( \|(-\Delta)^{s/2}u(t)\|_{L^2}^2 + 1 \right) .
\]

Here we use the interpolation estimate \( \|\nabla^{1/2}u\|_{L^2} \lesssim \|(-\Delta)^{s/2}u\|_{L^2}^{1-s/2} \|u\|_{L^2}^{1+s/2} \) for \( s > 1/2 \). This combined with (5.9) yield

\[
ct \leq -M_{\varphi_R}(u(t)) = |M_{\varphi_R}(u(t))| \lesssim C(\varphi_R) \left( \|(-\Delta)^{s/2}u(t)\|_{L^2}^2 + 1 \right) ,
\]

for any \( t \in [t_0, T_0] \). This shows that

\[
\|(-\Delta)^{s/2}u(t)\|_{L^2} \geq Ct^s ,
\]

for any \( t \in [\overline{t_0}, T_0] \) with some sufficiently large time \( t_0 \leq \overline{t}_0 \leq T_0 \). Taking \( t \uparrow T_0 = \frac{c R}{\varepsilon} \), we see that

\[
\|u(t)\|_{H^s} \rightarrow \infty \text{ as } R \rightarrow \infty ,
\]

which contradicts to (5.2). The proof is complete. \( \square \)

\(^3\)The solution is assumed to exist on \([0, +\infty)\).
5.2. Mass-critical blow-up criteria. In this short subsection, we give the proof of Proposition 1.5. This result follows directly from Corollary 1.4 with \( \gamma = s \). Moreover, if \( T < +\infty \), the limit

\[
\lim_{t \uparrow T} \| (-\Delta)^{s/2} u(t) \|_{L^2} = \infty
\]

follows from the blow-up alternative (see Section 3). In the case \( T = +\infty \), the Sobolev embedding, namely \( H^s(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) \) for any \( q \in [2, \infty) \) satisfying \( \frac{1}{q} \geq \frac{1}{2} - \frac{s}{d} \) together with the conservation of mass show

\[
\sup_{t \in [0, +\infty)} \| (-\Delta)^{s/2} u(t) \|_{L^2} = \infty.
\]

The conservation of energy then yields

\[
\sup_{t \in [0, +\infty)} \| u(t) \|_{L^{\frac{4}{s}+2}} = \infty.
\]

This proves Proposition 1.5. \( \square \)

5.3. Mass and energy intercritical blow-up criteria. We now give the proof of Proposition 1.6. By the same argument as in the previous subsection using the Sobolev embedding and the conservation of mass and energy, it remains to show (1.10) for some \( \delta > 0 \). The case \( E(u_0) < 0 \) follows as in Corollary 1.4. Let us now consider initial data \( u_0 \) with \( E(u_0) \geq 0 \) and (1.12). The assumption (1.12) implies

\[
\left\{ \begin{array}{ll}
E(u_0)M^\sigma(u_0) & < E(Q)M^\sigma(Q), \\
\| (-\Delta)^{s/2} u_0 \|_{L^2} \| u_0 \|_{L^2}^\sigma & > \| (-\Delta)^{s/2} Q \|_{L^2} \| Q \|_{L^2}^\sigma,
\end{array} \right.
\]

(5.11)

where

\[ \sigma := \frac{s - s_c}{s_c} = \frac{4s - (d - 2s)\alpha}{d\alpha - 4s}. \]

We next recall the sharp Gagliardo-Nirenberg inequality (see e.g. [1, Appendix])

\[
\| u(t) \|_{L^{\frac{4}{s}+2}}^{\sigma+2} \leq C_{GN} \| u(t) \|_{L^2}^{\frac{4s - (d - 2s)\alpha}{2s}} \| (-\Delta)^{s/2} u(t) \|_{L^2}^{\frac{d\alpha}{2s}},
\]

(5.12)

where the sharp constant is given by

\[ C_{GN} = \frac{\| Q \|_{L^{\frac{4}{s}+2}}^{\sigma+2}}{\| Q \|_{L^2}^{\frac{4s - (d - 2s)\alpha}{2s}} \| (-\Delta)^{s/2} Q \|_{L^2}^{\frac{d\alpha}{2s}}}, \]

(5.13)

with \( Q \) is the unique (up to symmetries) positive radial solution to (1.4). We also have the following Pohozaev’s identities

\[
\| (-\Delta)^{s/2} Q \|_{L^2}^2 = \frac{d\alpha}{2s(\alpha + 2)} \| Q \|_{L^{\alpha+2}}^{\alpha+2} = \frac{d\alpha}{4s - (d - 2s)\alpha} \| Q \|_{L^2}^2,
\]

(5.14)

A direct calculation shows

\[
C_{GN} = \frac{2s(\alpha + 2)}{d\alpha} \frac{1}{\left( \| (-\Delta)^{s/2} Q \|_{L^2} \| Q \|_{L^2}^\sigma \right)^{\frac{d\alpha}{2s}}},
\]

(5.15)

\[
E(Q)M^\sigma(Q) = \frac{d\alpha - 4s}{2d\alpha} \left( \| (-\Delta)^{s/2} Q \|_{L^2} \| Q \|_{L^2}^\sigma \right)^2.
\]

(5.16)
We now multiply both sides of $E(u(t))$ by $M^\sigma(u(t))$ and use the sharp Gagliardo-Nirenberg inequality to get
\[
E(u(t))M^\sigma(u(t)) = \frac{1}{2} \left( \|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} \right) - \frac{1}{\alpha + 2} \|u_0\|_{L^2}^{2s} \|u(t)\|_{L^2}^{2s} \\
\geq \frac{1}{2} \left( \|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} \right) - \frac{C_{GN}}{\alpha + 2} \|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} \\
= f \left( \|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} \right),
\]
where $f(x) := \frac{1}{2} x^2 - \frac{C_{GN}}{\alpha + 2} x^{\delta \frac{\alpha}{2}}$. It is easy to see that $f$ is increasing on $(0, x_0)$ and decreasing on $(x_0, \infty)$, where
\[
x_0 = \left( \frac{2s(\alpha + 2)}{d\alpha C_{GN}} \right)^{\frac{1}{2s}} = \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s}.
\]
Here the last equality follows from (5.15). By (5.15) and (5.16), we see that
\[
f \left( \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s} \right) = \frac{d\alpha}{2d\alpha} \left( \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s} \right)^2 \\
= E(Q)M^\sigma(Q).
\]
Thus the conservation of mass and energy together with the first condition in (5.11) imply
\[
f \left( \|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} \right) \leq E(u(t))M^\sigma(u(t)) = E(u_0)M^\sigma(u_0) \\
< E(Q)M^\sigma(Q) = f \left( \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s} \right).
\]
Using the second condition (5.11), the continuity argument shows that
\[
\|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} > \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s},
\]
for any $t \in [0, T)$. This implies that there exists $\delta > 0$ so that (1.10) holds. Indeed, since $E(u_0)M^\sigma(u_0) < E(Q)M^\sigma(Q)$, we pick $\rho > 0$ small enough so that
\[
E(u_0)M^\sigma(u_0) \leq (1 - \rho)E(Q)M^\sigma(Q).
\]
Multiplying $K(u(t))$ with the conserved quantity $M^\sigma(u(t))$ and using (5.17), (5.18) and (5.19), we obtain
\[
K(u(t))M^\sigma(u(t)) = \frac{d\alpha}{4} E(u(t))M^\sigma(u(t)) - \frac{d\alpha - 4s}{8} \left( \|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} \right)^2 \\
= \frac{d\alpha}{4} E(u_0)M^\sigma(u_0) - \frac{d\alpha - 4s}{8} \left( \|(-\Delta)^{s/2}u(t)\|_{L^2} \|u(t)\|_{L^2}^{2s} \right)^2 \\
\leq \frac{d\alpha}{4} (1 - \rho)E(Q)M^\sigma(Q) - \frac{d\alpha - 4s}{8} \left( \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s} \right)^2 \\
= \frac{(d\alpha - 4s)\rho}{8} \left( \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s} \right)^2,
\]
for any $t \in [0, T)$. This shows (1.10) with
\[
\delta = \frac{(d\alpha - 4s)\rho}{8} \|(-\Delta)^{s/2}Q\|_{L^2} \|Q\|_{L^2}^{2s} \left( \frac{M(Q)}{M(u_0)} \right)^2 > 0.
\]
The proof is complete. \(\square\)
5.4. Energy critical blow-up criteria. In this subsection, we give the proof of Proposition 1.7. The proof is similar to the one of Proposition 1.6. Instead of using the sharp Gagliardo-Nirenberg inequality, we make use of the sharp Sobolev embedding

\[ \|u\|_{L^{s^*}} \leq C_{SE}\|(-\Delta)^{s/2}u\|_{L^2}, \]

where \( s^* = \frac{2d}{d-2s} \) and the sharp constant

\[ C_{SE} = \frac{\|W\|_{L^{s^*}}}{\|(-\Delta)^{s/2}W\|_{L^2}}. \tag{5.20} \]

Here \( W \) is the unique (up to symmetries) positive radial solution to (1.5). The following identities are easy to check

\[ \|(-\Delta)^{s/2}W\|_{L^2}^2 = \|W\|_{L^{s^*}}^2 = \frac{1}{C_{SE}^2}, \tag{5.21} \]

\[ E(W) = \frac{1}{2}\|(-\Delta)^{s/2}W\|_{L^2}^2 - \frac{1}{s^*}\|W\|_{L^{s^*}}^{s^*} = \frac{s}{d}\frac{1}{C_{SE}^2}. \tag{5.22} \]

In particular, we have

\[ C_{SE} = \|(-\Delta)^{s/2}W\|_{L^2}^{-\frac{s}{s^*}} = \|W\|_{L^{s^*}}^{-\frac{s^*}{s}} = \left[ \frac{s}{dE(W)} \right]^{\frac{s}{s^*}}. \tag{5.23} \]

We now apply the sharp Sobolev embedding to get

\[ E(u(t)) = \frac{1}{2}\|(-\Delta)^{s/2}u(t)\|_{L^2}^2 - \frac{1}{s^*}\|u(t)\|_{L^{s^*}}^{s^*} \]
\[ \geq \frac{1}{2}\|(-\Delta)^{s/2}u(t)\|_{L^2}^2 - \frac{[C_{SE}]^{s^*}}{s^*}\|(-\Delta)^{s/2}u(t)\|_{L^2}^{s^*} = g\left(\|(-\Delta)^{s/2}u(t)\|_{L^2}\right), \]

where \( g(y) := \frac{1}{2}y^2 - \frac{[C_{SE}]^{s^*}}{s^*}y^{s^*} \). We see that \( g \) is increasing on \((0, y_0)\) and decreasing on \((y_0, \infty)\) with

\[ y_0 = \left( \frac{1}{[C_{SE}]^{s^*}} \right)^{\frac{4-2s}{s}} = \|(-\Delta)^{s/2}W\|_{L^2}. \]

Here we use (5.21) to have the second equality. We also have from (5.21) and (5.22) that

\[ g\left(\|(-\Delta)^{s/2}W\|_{L^2}\right) = \frac{s}{d}\frac{1}{C_{SE}^2} = E(W). \tag{5.24} \]

Thanks to the conservation of energy, the first condition in (1.13) yields

\[ g\left(\|(-\Delta)^{s/2}u(t)\|_{L^2}\right) \leq E(u(t)) = E(u_0) < E(W) = g\left(\|(-\Delta)^{s/2}W\|_{L^2}\right), \]

for any \( t \in [0, T) \). By the second condition in (1.13), the continuity argument implies that

\[ \|(-\Delta)^{s/2}u(t)\|_{L^2} > \|(-\Delta)^{s/2}W\|_{L^2}, \tag{5.25} \]

for any \( t \in [0, T) \). We next pick \( \rho > 0 \) small enough so that

\[ E(u_0) \leq (1 - \rho)E(W). \tag{5.26} \]
By the conservation of energy, (5.25), (5.26) and the fact $E(W) = \frac{1}{d} \|(-\Delta)^{s/2}W\|^2_{L^2}$, we learn that

$$K(u(t)) = \frac{ds}{d-2s} E(u(t)) - \frac{s^2}{d-2s} \|(-\Delta)^{s/2}u(t)\|^2_{L^2}$$

$$= \frac{ds}{d-2s} E(u_0) - \frac{s^2}{d-2s} \|(-\Delta)^{s/2}u(t)\|^2_{L^2}$$

$$\leq \frac{ds}{d-2s} (1 - \rho) E(W) - \frac{s^2}{d-2s} \|(-\Delta)^{s/2}W\|^2_{L^2}$$

$$= -\frac{\rho s^2}{d-2s} \|(-\Delta)^{s/2}W\|^2_{L^2},$$

for any $t \in [0, T)$. This shows (1.10) with

$$\delta = \frac{\rho s^2}{d-2s} \|(-\Delta)^{s/2}W\|^2_{L^2} > 0.$$

The proof is complete. \hfill \Box

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