Non-Hermitian topological phases and exceptional lines in topolectrical circuits

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Abstract
We propose a scheme to realize various non-Hermitian topological phases in a topolectrical (TE) circuit network consisting of resistors, inductors, and capacitors. These phases are characterized by topologically protected exceptional points and lines. The positive and negative resistive couplings \( R_g \) in the circuit provide loss and gain factors which break the Hermiticity of the circuit Laplacian. By controlling \( R_g \), the exceptional lines of the circuit can be modulated, e.g. from open curves to closed ellipses in the Brillouin zone. In practice, the topology of the exceptional lines can be detected by the impedance spectra of the circuit. We also considered finite TE systems with open boundary conditions, the admittance spectra of which exhibit highly tunable zero-admittance states demarcated by boundary points (BPs). The phase diagram of the system shows topological phases that are characterized by the number of their BPs. The transition between different phases can be controlled by varying the circuit parameters and tracked via the impedance readout between the terminal nodes. Our TE model offers an accessible and tunable means of realizing different topological phases in a non-Hermitian framework and characterizing them based on their boundary point and exceptional line configurations.

1. Introduction
There is growing interest in studying topological states in various platforms such as topological insulators [1, 2], cold atoms [3, 4], photonics systems [5, 6], superconductors [7], and optical lattices [8, 9] due to their extraordinary properties such as topologically protected edge states and unconventional transport characteristics [10, 11]. Such topological states have been studied in Hermitian and lossless systems [12], where the eigenenergies are always real. However, Hermitian systems do not exhibit many interesting phenomena such as exceptional points (EPs) [13], the skin effect [14–16], biorthogonal bulk polarization [17], and wave amplification and attenuation [18, 19]. In the pursuit of more exotic characteristics in topological phases, researchers have shifted attention from Hermitian to non-Hermitian systems [20]. In contrast to Hermitian systems, non-Hermitian systems in general exhibit complex eigenvalues unless the system obeys some specific symmetries such as the \( \mathcal{PT} \) symmetry where \( \mathcal{P} \) and \( \mathcal{T} \) are the parity and time reversal operations, respectively. One iconic feature of non-Hermitian systems is the existence of EPs, where two or more eigenvectors coalesce and the Hamiltonian becomes non-diagonalizable. This feature leads to many novel transport phenomena, such as unidirectional transparency [21, 22], unconventional reflectivity [23], and super sensitivity [24]. Generally, the exchange of energy or particles between lattice points and the surrounding environment induces non-Hermiticity in the system. One way to induce non-Hermiticity is to add imaginary onsite potentials at different sublattices that represent gain or loss in the system depending on the sign of the potentials. In addition, asymmetric sublattice couplings may also induce non-Hermiticity in the system Hamiltonian [16]. However, realizing non-Hermitian systems in condensed matter [25], acoustic metamaterials [26], and optical structures [27, 28] is in practice difficult because of the limited control over sublattice couplings, instability of the complex eigenspectra [29], and limitations in...
experimental accessibility. In pursuit of alternative platforms to overcome the aforementioned experimental limitations, topoelectrical (TE) circuits [30–34] have emerged as an ideal platform to not only realize non-Hermitian systems, but also to investigate many emerging phenomena such as Chern insulators [35], the quantum spin Hall effect [36–39], higher-order topological insulators [40, 41], topological corner modes [32, 42], Klein tunnelling [10, 34], and perfect reflection [10, 11]. Appropriately designed TE circuits can emulate the topological properties of materials and offer unparalleled degrees of tunability and experimental flexibility through the conceptual shift from conventional materials system to artificial electrical networks. The freedom in design and control over lattice couplings allow us to investigate electronic structures beyond the limitations of condensed matter systems. Moreover, TE circuit networks are not constrained by the physical dimension or distance between two lattice (nodal) points but are described solely by the mutual connectivity of the circuit nodes. Besides, the freedom of choice in the connections at each node and long-range hopping makes TE systems easier to fabricate compared to real material systems. Therefore, it is also possible to design an equivalent circuit network in lower dimensions that resembles the characteristics of higher-dimensional circuits [34]. Unlike real material systems, an infinite real system can be mimicked by finite-sized circuit networks in TE circuits. Therefore, TE networks enable us to design a non-Hermitian system in an RLC circuit network with better measurement accessibility as all the characteristic variables, such as the admittance bandstructure and the density of states, can be evaluated through electrical measurements [43] (e.g. impedance, voltage, and current readings).

In this paper, we investigate the exceptional lines, i.e. loci of EPs, in a non-Hermitian TE system consisting of electrical components such as resistors, inductors and capacitors as a function of the non-Hermitian parameter (i.e. resistance). We show that by introducing non-Hermiticity to the circuit Laplacian through the insertion of positive or negative resistances between the voltage nodes and the ground to induce imaginary onsite potentials in the lattice sites and tuning the non-Hermitian parameters appropriately by using appropriate resistance values, the loci of the EPs may be easily switched to take the form of either a line or closed curves such as ellipses in the Brillouin zone (BZ). We show that a unique property of the TE platform over earlier works [44–46], viz the impedance spectrum, is that the exceptional lines can be detected and easily tracked by measuring the impedance spectrum of the circuit. We further investigate finite systems with open boundary conditions along one direction, and find that these finite systems possess a rich phase diagram with different phases possessing up to four pairs of BPs, depending on the circuit parameters. We also show that edge states in Hermitian or pure LC circuits become hybridized with the bulk modes in non-Hermitian RLC circuits. In summary, our TE model provides an experimentally accessible means to investigate the phase diagram and various topological phases of non-Hermitian systems.

2. Theoretical model

We consider a two-dimensional TE circuit, shown in figure 1, which is composed of capacitors, inductors, positive resistors (loss elements) and negative resistors (gain elements). The circuit has a unit cell consisting of two sublattice nodes A and B. Each node is connected by a capacitor of $C_1$ and $C_y$ to its nearest neighbour in the x and y-direction, respectively, and a parallel combination of a common capacitor $C$ and inductor $L$ to the ground. The inductance $L$ can be varied to modulate the resonance condition. To explore the effects of loss and gain in the TE network, we consider an imaginary onsite potential $iR_g$ on the A nodes and $-iR_g$ on the B nodes. This onsite potential can be realized by connecting resistors of resistance $r_a$ ($-r_a$) between each A (B) node and the ground. The onsite potential is related to the resistors by $R_g = 1/\omega r_{a}$, where $\omega$ denotes the frequency of the driving alternating current. The negative imaginary onsite potential at the B-type nodes can be obtained by using op-amp-based negative resistance converters with current inversion (INRC) (see supplemental section 1 for details (https://stacks.iop.org/NJP/23/033014/mmedia)). Alternatively, similar gain and loss terms can obtained in our TE model via using only two unequal positive resistances connected to ground for each type of nodes instead of using negative resistance converters, which yields a mathematically similar Hamiltonian to equation (1) except for a global shift of the imaginary part of all the admittance eigenenergies (see supplementary note 1 for details). The advantage of using grounding resistors with different values of (positive) resistance is the dynamic stability of the circuit because the absence of op-amps avoids the possibility that the voltage profiles may get over-amplified by the negative resistance. The Laplacian of the TE circuit over the $x$–$y$ plane can be expressed as

$$H_{2D}(k_x, k_y) = -(C_1(1 + \cos k_x) + 2C_y \cos k_y)\sigma_x - C_1 \sin k_x \sigma_y + iR_g \sigma_z,$$  

(1)

where the $\sigma$ denote the Pauli matrices in the sublattice space. In the absence of resistances, equation (1) exhibits both chiral symmetry, i.e. $CH_{2D}(k_x, k_y)C^{-1} = -H_{2D}(k_x, k_y)$ and inversion symmetry, i.e.
Equation (1) can still have real eigenvalues despite its non-Hermiticity [47]. More specifically, a non-zero $i$ of band-touching EPs. For instance, in figure 2(b), the splitting of the degeneracy points is most evident in the $y$-direction, nodes of different sublattices are connected diagonally by a capacitor $C_v$. The EPs occur at $\pm \arccos(\cos k_y)$ respectively. However, a finite $R_g$ terms on the diagonal of the Laplacian, which preserves the commutation with the $PT$ operator [48]. In contrast to the Hermitian case, $PT$ symmetry in the non-Hermitian TE system eigenmodes can be broken depending on the model parameters, i.e. the eigenmodes of equation (1) are not necessarily the eigenstates of the $PT$ operator [29] even when the Laplacian itself respects $PT$ symmetry. In this situation, EPs, where both the hole- and particle-like admittance bands coalesce, emerge in the complex admittance spectra. Therefore, the complex admittance dispersion for the circuit model takes the form of

$$E_{2D}(k_x, k_y) = \pm \sqrt{2C_y^2(1 + \cos k_y) + 4C_y^2 \cos^2 k_y + 4C_y C_y' \cos k_y(1 + \cos k_y) - R_g^2}$$

(2)

where the $\pm$ refers to the two admittance bands, respectively. By tuning the circuit parameters, we can obtain different numbers of real solutions for $E_{2D} = 0$ in equation (2), which translates into different number of EPs in the BZ. The EPs occur at $\omega_{ex} = (k_x, k_y) = (\pi, \pm \arccos(R_g/2C_y))$ and $\omega_{ex} = (\pi, \mp \pi \pm \arccos(R_g/2C_y))$. To illustrate the effect of non-Hermitian gain or loss, we plot the admittance dispersion as a function of wavevector $k_y$ and fix $k_x = \pi$ for three representative values of $R_g$. In the absence of gain or loss (i.e. $R_g = 0$), the admittance spectrum becomes purely real with two Dirac points (see figure 2(a)). Because the Laplacian in equation (1) obeys chiral symmetry, for a given $k_y$, the admittance eigenvalues always come in pairs with equal magnitude but opposite signs. For non-zero values of $R_g$, the admittance dispersion becomes complex and the band-touching degeneracy points split into pairs of band-touching EPs. For instance, in figure 2(b), the splitting of the degeneracy points is most evident in

Figure 1. (a) Schematic of the two-dimensional TE lattice in the $x,y$ plane. The blue and magenta circles represent sublattices $A$ and $B$ respectively. The alternating sublattice sites $A$ and $B$ are connected to each other in the $x$-direction by an inductor $-C_v$ and capacitor $C_v$ for the intracell and intercell connections, respectively (the dashed rectangle delineates a unit cell). Along the $y$-direction, nodes of different sublattices are connected diagonally by a capacitor $C_v$. (b) Schematic circuit of a negative resistance converter, which introduces a $\pi$-phase difference and therefore converts a loss resistive term $\tau_a$ to a gain term $-\tau_a$. The combination of two resistors having the same resistance $R_1$, along with an ideal operational amplifier with supply voltages $+V_{dd}$ and $-V_{dd}$ results in current inversion and hence acts as a negative resistance converter. (c) Grounding mechanism of the TE circuit. All nodes are connected to ground via a parallel combination of a common capacitor $(C)$ and inductor $(L)$. Furthermore, in parallel to these, a positive resistor $\tau_a$ (loss term) and negative resistor $-\tau_a$ (gain term) is connected to ground from the $A$ and $B$ nodes, respectively. The negative resistor $-\tau_a$ is implemented by means of INRC depicted in (b).
Figure 2. Absolute value, real part, and imaginary part of the complex admittance as a function of $k_y$ with the parameters $C_1 = 0.78 \text{ mF}$, $C_y = 0.39 \text{ mF}$, and $k_x = \pi$. We consider three representative values of the grounding resistance, i.e. (a) $R_g = 0 \Omega^{-1} \text{ Hz}^{-1}$, (b) $R_g = 0.5 \Omega^{-1} \text{ Hz}^{-1}$, and (c) $R_g = 0.78 \Omega^{-1} \text{ Hz}^{-1}$. Note that case (c) corresponds to the critical value of the non-Hermitian parameter $R_g = 2C_y = 0.78 \Omega^{-1} \text{ Hz}^{-1}$, beyond which the admittance spectrum becomes purely imaginary. All the EPs are represented by open red circles.

the admittance plots in the left-most and middle columns. Here, the degeneracy point with zero admittance at $k_y \approx -1.6$ in figure 2(a) splits into two EPs at $k_y \approx -2.3$ and $k_y \approx -0.9$ in figure 2(b). The admittance spectrum is then either real (for some range of $k_y$) with $P'T$-symmetrical eigenmodes or purely imaginary (in the complementary range of $k_y$), in which case the eigenmodes break the $P'T$ symmetry. The boundaries between the real and imaginary admittances are defined by the EPs, where all the eigenmodes coalesce at the eigenvalue of zero. Two of the four EPs are located at $k_y = \pm \pi \pm \arccos(R_g/2C_y)$ while the other two are at $k_y = \pm \arccos(R_g/2C_y)$. As the magnitude of $R_g$ increases, the range of $k_y$ corresponding to the real (imaginary) part of the admittance spectrum shrinks (expands). At some critical value given by $R_c = 2C_y$, the whole spectrum becomes purely imaginary with three EPs (see figure 2(c)). When $R_g$ exceeds $R_c$, the two admittance bands will become gapped and no EP exists in the BZ (not shown in figure 2). In this case, the admittance eigenmodes break the $P'T$ symmetry for the entire range of $k_y$ in the BZ. As can be seen from figure 2(c), the real part of the admittance spectrum vanishes at the critical resistance $R_c$. In summary, we can obtain a variable number of EPs depending on the $R_g$ parameter, i.e. two, four, three, and zero EPs for $R_g = 0$, $R_g < 2C_y$, $R_g = 2C_y$, and $R_g > 2C_y$, respectively.

To obtain the exceptional lines (the loci of the EPs), we use the equation for the degeneracy points of the admittance spectrum:

$$\left(C_1(1 + \cos k_x) + 2C_y \cos k_y\right)^2 + \left(C_1 \sin k_x\right)^2 = R_g^2.$$  \hspace{1cm} (3)

Equation (3) governs the loci of the EPs in the $k_x$-$k_y$ plane. A finite $R_g$ will transform a single pair of band-touching points into exceptional or nodal lines on the $k_x$-$k_y$ plane characterized by equation (3). On these exceptional lines, both the real and imaginary parts of the eigenvalues vanish. The top panels of figures 3(a) and (b) show the exceptional lines for the parameter sets in figures 2(b) and (c), respectively. The EPs shown in figures 2(b) and (c) then correspond to the $k_y$ cross sections of the exceptional lines in figure 3 at $k_x = \pi$. In addition to the above analysis of the exceptional lines based on the admittance spectrum, an alternative visual representation of the exceptional lines can also be obtained from the impedance spectrum of the TE circuit. In general, the impedance between any two arbitrary nodes $p$ and $q$ in the circuit can be measured by connecting an external current source providing a fixed current $I_{pq}$ to the
two nodes and measuring the resulting voltages at the two nodes $V_p$ and $V_q$. The impedance in the circuit is then given by

$$Z_{pq} = \frac{V_p - V_q}{I_{pq}} = \sum_{i} \frac{|\psi_{ip} - \psi_{iq}|^2}{\lambda_i},$$

(4)

where $\psi_{ia}$ and $\lambda_i$ are the voltage at node $a$ and the eigenvalue of the $i$th eigenmode of the (finite-width) Laplacian. One of the key characteristics of equation (4) is that the impedance diverges (increases to a large value) in the vicinity of the zero-admittance modes ($\lambda_i = 0$) for non-zero eigen-mode voltages $\psi_{ip}$ and $\psi_{iq}$. Therefore, the locus of the high impedance readout can be used to mark out the exceptional lines in momentum space. The lower panels of figures 3(a) and (b) depict the corresponding impedance spectra for the parameter sets in figures 2(b) and (c) respectively. The plotted impedance is that across the two nodes in a unit cell (i.e. with nodes $p$ and $q$ chosen to be terminal points at either end of the circuit). (For the case of $C_1 = 2C_y = R_y$ plotted in figure 3(b), the $k_y = \pm \pi$ lines are also exceptional lines.) For both resistive values, the locus of high impedance readouts coincides exactly with the exceptional lines in the admittance spectrum (compare upper and lower panels of figure 3). This suggests the possible electrical detection of exceptional lines in the TE system via impedance measurements.
3. Zero-admittance states in finite system

To gain further insight into the zero-admittance states of a non-Hermitian system, we will study a 2D dissipative TE system described by equation (1), which is finite along the x-direction, i.e. having open boundary conditions along that direction, but is infinite in the y-direction. Before investigating the properties of the finite system, we will first explain some properties of the \textit{infinite-sized} 2D system. For this system, Kirchoff's current law at the A and B nodes at resonance can be written as

\[
- EV^A_{x,y} = - C_1 V^B_{x,y} + C_1 V^B_{x-1,y} - C_y(V^B_{x,y+1} + V^B_{x,y-1}) + iR_y V^A_{x,y},
\]

and

\[
- EV^B_{x,y} = - C_1 V^A_{x,y} + C_1 V^A_{x+1,y} - C_y(V^A_{x,y+1} + V^A_{x,y-1}) - iR_y V^B_{x,y}.
\]

By substituting the ansatz \( V_{x,y} = \lambda e^{ik_x + ik_y} \) in equations (5) and (6), we obtain

\[
(E - iR_y \sigma_y)\lambda = (C_1(1 + \chi_x(\sigma_x - i\eta_x) + \chi_x^{-1}(\sigma_x + i\eta_x)) + 2C_2 \cos k_y \sigma_y)\lambda
\]

where \( \chi_x = e^{ik_x} \) and \( \lambda = (\lambda_x, \lambda_y)^T \). For a given \( E \) and \( k_y \), \( \chi_x \) can be solved from equation (7) as

\[
\chi_x = \frac{-(t^2 + C_1^2 - p^2) \pm \sqrt{\Delta^2}}{2tC_1},
\]

where \( t = C_1 + 2C_y \cos k_y \) and \( p^2 = E^2 + R_y^2 \), and

\[
\Delta^2 \equiv (t^2 + C_1^2 - p^2)^2 - (2tC_1)^2.
\]

When \( \Delta^2 < 0 \),

\[
\chi_x(\Delta^2 < 0) = \frac{-(t^2 + C_1^2 - p^2) \pm i\sqrt{|\Delta^2|}}{2tC_1}.
\]

It can be readily seen that \( |\chi_x(\Delta^2 < 0)| = 1 \), and because \( \chi_x \equiv \exp(i k_x) \), this indicates that the corresponding values of \( k_x \) would be real when \( \Delta^2 < 0 \). In this case, \( k_x = \arg(\chi_x(\Delta^2 < 0)) = \pm \arctan(\sqrt{(-\Delta^2)}/(t^2 + C_1^2 - p^2)) \). On the other hand, when \( \Delta^2 > 0 \) \( \chi_x(\Delta^2 > 0) \) is real, and, in general, \( |\chi_x(\Delta^2 > 0)| \neq 1 \). This indicates that the corresponding values of \( k_x \) would be imaginary. At the boundary between the two cases, we have

\[
\Delta^2 = 0 \Rightarrow (t^2 + C_1^2 - p^2)^2 = (2tC_1)^2
\]

\[
\Rightarrow \chi_x(\Delta^2 = 0) = -\sqrt{(2tC_1)^2}/(2tC_1) = -\text{sign}(2tC_1),
\]

so that the corresponding \( k_x = 0, \pi \) depending on the sign of \( 2tC_1 \). When \( \Delta^2 < 0 \), the \( \pm i\sqrt{|\Delta^2|} \) terms in equation (10) result in a finite separation along the \( k_x \) axis between two points on the same equal admittance contours (EACs) for a given \( k_y \) value (see figure 4). The two values of \( k_x \) meet when \( \Delta^2 = 0 \). Figure 4(a) depicts the case where \( \chi_x = +1 \) when \( \Delta^2 = 0 \). Here, for a given \( k_y \), \( k_x \) on the EAC becomes single-valued at \( k_x = 0 \). Figure 4(b) depicts the case where \( \chi_x = -1 \) when \( \Delta^2 = 0 \), and here \( k_x \) becomes single-valued at \( k_x = \pi \). For both cases, the values of \( k_x \) where \( \Delta^2 = 0 \) mark the boundaries for the existence of real \( k_x \). We now consider the nanoribbon geometry with a finite width along the x-direction.

We show in the Supplementary Note 2 that for the nanoribbon geometry with non-zero \( |R_y| \), the zero-admittance EPs do not occur within the bulk energy gaps, but in the bulk bands where real values of \( k_y \) exist for \( E = 0 \) in the infinite bulk system. Due to the finite width of the nanoribbons, these zero-admittance points occur as quantized bulk states. The points on the \( k_x \) axis that mark the threshold for the existence of the zero admittance states for the nanoribbon system coincide with the \( k_x \) values where the solutions of \( \chi_x \) at \( E = 0 \) in equation (8) are equal, i.e. when \( \Delta^2 = 0 \). These values of \( k_x \) would mark the ‘boundary points’ (BPs) of the system. The values of \( k_y \) corresponding to the BPs are given by the solutions of the following equation:

\[
C_1 + 2C_y \cos k_y + \eta C_1 = \zeta R_y,
\]

where \( \eta = \pm 1 \) and \( \zeta = \pm 1 \) independently. The four possible combinations of \( \eta \) and \( \zeta \) and the two possible signs of \( k_y \) in equation (11) therefore provide up to eight real solutions for \( k_y \). The BPs come in pairs, and hence, depending on the choice of the coupling capacitances \( C_1 \) and \( C_y \) and resistance \( R_y \), we can obtain up to a maximum of four pairs of BPs (see later discussions on phase diagram). Note that the BPs set the
boundaries for the existence of EPs. In a nanoribbon, the EPs do not necessarily appear exactly at the BPs, but in between alternating pairs of BPs. To illustrate the role of the capacitive and resistive coupling parameters in determining the number of BPs, we plot the admittance spectra for finite TE circuits with $N = 20$ unit cells along the $x$-direction in figure 5. In the Hermitian limit (i.e. $R_g = 0$), the admittance spectrum is purely real, and each pair of quantized bands is symmetric about the $E = 0$ axis. The Hermitian TE circuit can host two or four BPs depending upon the relative strength of the capacitive couplings. If $C_1 > C_y$, we only have a pair of BPs occurring at $k_y = \pm \pi/2$. However, an additional pair of BPs emerges at $k_y = \arccos(\pm C_1/C_y)$ in the spectrum if $C_1 < C_y$ (which is the case illustrated in figure 5(a)). The Hermitian system possesses chiral and time-reversal symmetries and belongs to the BDI Altland–Zimbauer class [49]. Treating $k_y$ as a parameter to an effectively one-dimensional model, the effective 1D model along the $x$ direction is mathematically identical to the SSH model, for which the topological invariant is the winding number. The band-touching points B1 to B4 will then correspond to the values of $k_y$ at which the bulk band gaps close and the system transits between topologically trivial phases and topologically non-trivial phases. The emergence of the non-trivial states in figure 5(a) span over $(-\pi/2$ to $-\arccos(C_1/C_y))$ and $(\pi/2$ to $\arccos(C_1/C_y))$ in the $k_y$ axis. (Although the bands appear to be flat at the scale of the figure, they actually disperse weakly and touch only at isolated points.)

The transition points between non-trivial (with edge states) and trivial regions (without edge states) are marked by the onset of high impedance states (shown in the rightmost plot of figure 5(a)). A finite non-Hermitian gain or loss term $R_g$ results in four different types of admittance regions for a given value of $k_y$. These admittance regions can host purely real, purely imaginary, complex spectra with EPs, and complex admittances without $E_{2D} = 0$ states, which are labelled as I, III, IIA and IIB respectively, as shown in figures 5(b) and (c). (For instance, the states between B2 and B3, and B6 and B7 in figure 5(c) have purely imaginary admittances.) Eigenstates with preserved and broken PT symmetry are present in regions I and III. Additionally, region IIA hosts EPs while IIB does not (see figures 5(b) and (c)). The difference between regions IIA and IIB can be explained via the existence and non-existence of real solutions of $\chi_x$ in equation (8). The transition points between any two successive regions are marked by the BPs. Moreover,
Figure 6. (a) Phase diagram of a 2D non-Hermitian TE model showing the effect of the capacitive coupling $C_1$ and the gain/loss parameter $R_g$ on the number of BPs, and hence the number of zero-admittance regions in the admittance spectrum. We consider a finite number of unit cells along $x$-direction, and set $C_y = 1.0 \text{ mF}$. (b) and (c): (Left) Dispersion relations of $N_x = 20$ unit cell-wide nanoribbons at (b). $C_1 = 2 \text{ mF}$ and $R_g = 0 \text{ mF}$ and (c). $C_1 = 1 \text{ mF}$ and $R_g = 1.5 \text{ mF}$ indicated by the two circles in panel (a); (right) the square of the voltage amplitudes corresponding to the EP with the minimum $k_y$ value, as indicated by the circles in the dispersion relations.

the zero-admittance states in region IIA and the $E_{2D} \neq 0$ states in the other regions are characterized by high and low impedances respectively (see impedance plots shown in the right-most column of figures 5(b) and (c)). For illustration, the EPs in the region IIA are marked by crosses in figure 5(b) in both the impedance and admittance plots. The impedance readout switches between the high and low impedance states at the BPs, which mark the boundaries between the IIA and other regions.

Figure 6(a) shows the phase diagram of the 2D non-Hermitian TE model as a function of the resistive coupling $R_g$ and the capacitive coupling $C_1$. The different phases are characterized by different numbers of BPs and hence different numbers of zero-admittance regions. These phases are separated from one another by phase boundaries that are defined by the existence and number of real solutions in equation (11). Note that BPs always occur and annihilate in pairs and they connect two admittance bands together. To study the effect of non-Hermitian term $R_g$ on the zero-admittance states, we consider a 2D TE system with open-boundary conditions in the $x$-direction. In the Hermitian condition, i.e. $R_g = 0$, the system hosts zero-energy modes in which the square of the voltage amplitudes are localized at its edges, as shown in top panel of figure 6(b). This corresponds to the usual non-trivial SSH edge state. However, in the non-Hermitian case, i.e. with the introduction of a finite $R_g$, we would instead obtain bulk states at $E = 0$ rather than edge states (see figure 6(c)). This is indicated by the square of the voltage amplitude being no longer localized at an edge.

4. Conclusion

In conclusion, we proposed a TE circuit model with resistive elements which provide loss and gain factors that break the Hermiticity of the circuit to model and realize various non-Hermitian topological phases. By varying the resistive elements, the loci of the EPs (or exceptional lines) of the circuit can be modulated. We showed that the topology of the exceptional lines in the BZ can be traced by the impedance spectra of the circuit. Additionally, we studied a finite TE system where open boundary conditions apply in one of the dimensions. In these finite circuits, we demonstrated the tunability of both the number of EPs corresponding to zero-admittance states, as well as that of BPs which delineate the circuit parameter range where these EPs exist. The regions separated by the BPs are characterized by high and low values of impedance differing by several orders of magnitude, which are detectable in a practical circuit. We also derived a phase diagram of the finite TE system which delineates different between topological phases that are characterized by different number of BP pairs (up to a maximum of four). The edge state character of zero-admittance states of Hermitian LC circuits are transformed into EPs which are hybridized with the bulk modes in the non-Hermitian RLC circuits. In summary, we have proposed a tunable electrical framework consisting of RLC circuit networks as a means to realize different topological phases of non-Hermitian systems, and characterize them based on their impedance output, as well as their BP and exceptional line configurations.
Code availability

The computer codes used in the current study are accessible from the corresponding author upon reasonable request.

Author contributions

SMR-U-I, ZBS and MBAJ initiated the primary idea. SMR-U-I and ZBS contributed to formulating the analytical model and developing the code, under the kind supervision of MBAJ. All the authors contributed to the data analysis and the writing of the manuscript.

Conflicts of interests

The authors declare no conflict of interests.

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Data availability

The data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

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References

[1] Fu L and Kane C L 2007 Phys. Rev. B 76 045302
[2] Moore J E and Balents L 2007 Phys. Rev. B 75 121306
[3] Liu X-J, Law K-T, Ng T-K and Lee C-S 2013 Phys. Rev. Lett. 111 120402
[4] Zhang D-W, Zhu Y-Q, Zhao Y X, Yan H and Zhu S-L 2018 Adv. Phys. 67 253
[5] Lu L, Ioannopoulos J D and Soljačić M 2014 Nat. Photon. 8 821
[6] Khamkhae A B, Hosseini M, Varghese S, Tse W-K, Kargarian M, MacDonald A H and Shvets G 2013 Nat. Mater. 12 233
[7] Sato M and Ando Y 2017 Rep. Prog. Phys. 80 076501
[8] Goldman N, Budich J C and Zoller P 2016 Nat. Phys. 12 639
[9] Lang I-J, Cai X and Chen S 2012 Phys. Rev. Lett. 108 220401
[10] Rafi-Ul-Islam S M, Siu Z B and Jalil M B A 2020 Appl. Phys. Lett. 116 111904
[11] Rafi-Ul-Islam S, Siu Z B, Sun C and Jalil M B 2020 Phys. Rev. Appl. 14 034007
[12] Roy R 2009 Phys. Rev. B 79 195322
[13] Alvarez V M, Vargas I B and Torres L F 2018 Phys. Rev. B 97 121401
[14] Yao S and Wang Z 2018 Phys. Rev. Lett. 121 086803
[15] Lee C H and Thomale R 2019 Phys. Rev. B 99 201103
[16] Rafi-Ul-Islam S, Siu Z B and Jalil M B 2021 Phys. Rev. B 103 035420
[17] Kunst F K, Edvardsson E, Budich J C and Bergholtz E J 2018 Phys. Rev. Lett. 121 026808
[18] El-Ganainy R, Makris K G, Khajavikhan M, Musslimani Z H, Rotter S and Christodoulides D N 2018 Nat. Phys. 14 11
[19] Jin L and Song Z 2019 Phys. Rev. B 99 081103
[20] Liu T, Zhang Y-R, Ai Q, Gong Z, Kawabata K, Ueda M and Nori F 2019 Phys. Rev. Lett. 122 076801
[21] Lin Z, Ramezani H, Eichfeldraut T, Kottos T, Cao H and Christodoulides D N 2011 Phys. Rev. Lett. 106 213901
[22] Feng L, Xu Y-J, Fagadolin W S, Lu M-H, Oliveira J E B, Almeida V R, Chen Y-F and Scherer A 2012 Phys. Rev. Lett. 108 213901
[23] Zhu W, Fang X, Li D, Sun Y, Li Y, Jing Y and Chen H 2018 Phys. Rev. Lett. 121 124501
[24] Liu Z-Z et al 2016 Phys. Rev. Lett. 117 110802
[25] Alvarez V M, Vargas I B, Berdakin M and Torres L F 2018 Eur. Phys. J. Spec. Top. 227 1295
[26] Achilleos V, Theocharis G, Richoux O and Pagneux V 2015 Phys. Rev. B 91 044303
[27] Zhang Z, Zhang Y, Sheng J, Yang L, Mini M-A, Christodoulides D N, He B, Zhang Y and Xiao M 2016 Phys. Rev. Lett. 117 123601
[28] Jin L 2017 Phys. Rev. A 96 032103
[29] Yuce C and Oztas Z 2018 Sci. Rep. 8 1
[30] Rafi-Ul-Islam S M, Siu Z B, Sun C and Jalil M B A 2020 New J. Phys. 22 023025
[31] Lee C H, Imhof S, Berger C, Bayer F, Brehm J, Molenkamp L W, Kiessling T and Thomale R 2018 Commun. Phys. 1 1
[32] Imhof S et al 2018 Nat. Phys. 14 923
[33] Helbig T, Hofmann T, Lee C H, Thomale R, Imhof S, Molenkamp L W and Kiessling T 2019 Phys. Rev. B 99 161114
[34] Rafi-Ul-Islam S, Siu Z B and Jalil M B 2020 Commun. Phys. 3 1
[35] Haenel R, Branch T and Franz M 2019 Phys. Rev. B 99 235110
[36] Zhu W, Long Y, Chen H and Ren J 2019 Phys. Rev. B 99 115410
[37] Sun C, Rafi-Ul-Islam S, Yang H and Jalil M B 2020 Phys. Rev. B 102 214419
[38] Zhu W, Hou S, Long Y, Chen H and Ren J 2018 Phys. Rev. B 97 075310
[39] Sun C, Deng J, Rafi-Ul-Islam S, Liang G, Yang H and Jalil M B 2019 Phys. Rev. Appl. 12 034022
[40] Ezawa M 2018 Phys. Rev. B 98 201402
[41] Peterson C W, Bernalazar W A, Hughes T L and Bahl G 2018 Nature 555 346
[42] Serra-Garcia M, Süssstrunk R and Huber S D 2019 Phys. Rev. B 99 020304
[43] Hofmann T, Helbig T, Lee C H, Greiter M and Thomale R 2019 Phys. Rev. Lett. 122 247702
[44] Rui W, Hirschmann M M and Schnyder A P 2019 Phys. Rev. B 100 245116
[45] Choi Y, Hahn C, Yoon J W and Song S H 2019 Nat. Commun. 10 1
[46] Stegmaier A et al 2020 arXiv:2011.14836
[47] Lee T E 2016 Phys. Rev. Lett. 116 135903
[48] Weimann S, Kremer M, Plotnik Y, Lumer Y, Nolte S, Makris K G, Segev M, Rechtsman M C and Szameit A 2017 Nat. Mater. 16 433
[49] Kawabata K, Shiozaki K, Ueda M and Sato M 2019 Phys. Rev. X 9 041015