Perfect Space-Time Codes with Minimum and Non-Minimum Delay for Any Number of Antennas

Petros Elia, B. A. Sethuraman and P. Vijay Kumar

Abstract—Perfect space-time codes were first introduced by Oggier et. al. to be the space-time codes that have full rate, full diversity-gain, non-vanishing determinant for increasing spectral efficiency, uniform average transmitted energy per antenna and good shaping of the constellation. These defining conditions jointly correspond to optimality with respect to the Zheng-Tse D-MG tradeoff, independent of channel statistics, as well as to near optimality in maximizing mutual information. All the above traits endow the code with error performance that is currently unmatched. Yet perfect space-time codes have been constructed only for 2, 3, 4 and 6 transmit antennas. We construct minimum and non-minimum delay perfect codes for all channel dimensions.

I. INTRODUCTION

A. Definition of Perfect Codes

In [1, Definition 1], the concept of perfect codes is introduced to describe the \( n \times n \) space-time codes that satisfy all of the following criteria:

- **Full rate.** This corresponds to the ability of the code to transmit \( n \) symbols per channel use from a discrete constellation such as the QAM or the HEX constellation.
- **Full diversity.** This corresponds to having all \( \Delta X \Delta X^\dagger \), \( \Delta X \) a difference matrix of the code, be non-singular.
- **Non vanishing determinant for increasing spectral efficiency.** The determinant of any difference matrix, prior to SNR normalization, is lower bounded by a constant that is greater than zero and independent of the spectral efficiency.
- **Good shaping of the constellation.** When the code is based on cyclic division algebras (CDA) ([8], [14], [1], [2]), the condition requires that the signalling set, in the form of the layer-by-layer vectorization of the code-matrices, be isomorphic to QAM\(^n \) or HEX\( ^n \), where the isomorphism is given strictly by some unitary matrix.
- **Uniform average transmitted energy per antenna.** The condition requires that the expected value of the transmitted power is the same for all antennas. In fact we will see that the structure of the code will allow for equal average power across the antennas as well as across time.

The term **perfect**, coined in [1], was in reference to the ability of the codes to satisfy all the above criteria, as well as in reference to the codes having the best observed performance.

B. New results

1) Additional properties satisfied by perfect codes: In addition to the defining properties, perfect codes also satisfy

- **Approximate universality.** This property was introduced in [18] to describe a code that is D-MG optimal [3] irrespective of channel statistics. Such codes exhibit high-SNR error performance that is given by the high-SNR approximation of the probability of outage of any given channel, and allow for the probability of decoding error given no outage to vanish faster than the probability of outage. Currently, the only family of approximately universal codes is that of cyclic-division algebras with non-vanishing determinant, the generalization of which was presented in [2]. Consequently perfect codes are also approximately universal.
- **D-MG optimality for any spacial correlation of the fading or the additive noise.** This can be immediately concluded by post-multiplying the received signal matrix with the covariance matrix of the additive noise vectors and then using the approximate universality property of the code over channels with uncorrelated additive white noise.
- **Residual approximate universality.** Any truncated code version resulting from deletion of rows maintains approximate universality over the corresponding channel.
- **Gaussian-like signalling.** This is an empirical observation and it relates to a Gaussian-like signalling set with a covariance matrix that tends to maximize mutual information.
- **Information losslessness.** This property relates to having full rate as well as unitary linear dispersion matrices [22], [20], and guarantees that the mutual information is not reduced as a result of the code’s structure.
- **Scalable sphere decoding complexity.** This property guarantees that as the number of receive antennas becomes smaller, the structure of the code allows for substantial reductions in sphere decoding complexity without essential loss in performance. For MISO channels, the structure of the code will allow for reduction of sphere decoding complexity, from \( O(n^2) \) to \( O(n) \).

2) Summary of presented contributions: In this paper we introduce explicit constructions of minimum-delay perfect space-time codes for any number \( n \) of transmit antennas and any number \( n_r \), of receive antennas. Non-minimum delay perfect codes are constructed for any delay \( T \) that is a multiple of \( n \). We also generalize the defining conditions in [1] to be independent of the channel topology, thus directly providing for joint maximization of the mutual information and approximate...
universality. For channels with a smaller number of receive antennas, this information theoretic approach allows for the construction of efficient variants of perfect codes which exhibit almost the same error performance as standard perfect codes but with substantially reduced sphere decoding complexity.

II. SATISFYING THE PERFECT-CODE CONDITIONS

A. The general CDA structure

Our constructions of the perfect \( n \times n \) space-time block codes will be based on cyclic division algebras.

As shown in other related works such as [8], [14], [1], [2], the basic elements of a CDA space-time code are the number fields \( \mathbb{F} \), and \( \mathbb{L} \), with \( \mathbb{L} \) a finite, cyclic Galois extension of \( \mathbb{F} \) of degree \( n \). For \( \sigma \) being the generator of the Galois group \( \text{Gal}(\mathbb{L}/\mathbb{F}) \), we let \( z \) be some symbol that satisfies the relations

\[
\ell z = z\sigma(\ell) \quad \forall \ell \in \mathbb{L} \quad \text{and} \quad z^n = \gamma \tag{1}
\]

for some ‘non-norm’ element \( \gamma \in \mathbb{F}^* := \mathbb{F} \setminus \{0\} \) such that the smallest positive integer \( t \) for which \( \gamma^t \) is the relative norm \( N_{\mathbb{L}/\mathbb{F}}(u) \) of some element \( u \) in \( \mathbb{L}^* \) is \( n \). The cyclic division algebra is then constructed as a right \( \mathbb{L} \) space

\[
D = \mathbb{L} \oplus z\mathbb{L} \oplus \ldots \oplus z^{n-1}\mathbb{L}. \tag{2}
\]

A space-time code \( \mathcal{X} \) can be associated to \( D \) by selecting the set of matrices corresponding to the representation by left multiplication on \( D \) of elements from a finite subset of \( D \).

For an arbitrary choice of scaled integral basis \( \{\beta_i\}_{i=0}^{n-1} \) of \( \mathbb{L} \) over \( \mathbb{F} \) (by a scaled integral basis we mean a set of numbers obtained by multiplying all the elements of an integral basis by the same nonzero real number), the \( n \)-tuple \( \{f_{i,j}\}_{i=0}^{n-1} \) maps to

\[
\ell_j = \sum_{i=0}^{n-1} f_{i,j}\beta_i, \quad \ell_j \in \mathbb{L}, \quad f_{i,j} \in \mathcal{O}_\mathbb{F} \tag{3}
\]

where \( \mathcal{O}_\mathbb{F} \) is the ring of integers of \( \mathbb{F} \). Consequently, prior to SNR normalization, the code-matrix \( X \) representing the division algebra element \( x = \sum_{j=0}^{n-1} \ell_j \ell_j, \quad \ell_j \in \mathbb{L} \) is given by the defining equations (12) to be

\[
X = \begin{bmatrix}
\ell_0 & \gamma \sigma(\ell_{n-1}) & \gamma^2 \sigma(\ell_{n-2}) & \cdots & \gamma^{n-1} \sigma(\ell_1)
\ell_1 & \gamma \sigma(\ell_{n-1}) & \gamma^2 \sigma(\ell_{n-2}) & \cdots & \gamma^{n-1} \sigma(\ell_2)
\vdots & \vdots & \vdots & \ddots & \vdots
\ell_{n-2} & \sigma(\ell_{n-3}) & \sigma(\ell_{n-4}) & \cdots & \gamma^{n-2} \sigma(\ell_{n-1})
\ell_{n-1} & \sigma(\ell_{n-2}) & \sigma(\ell_{n-3}) & \cdots & \gamma^{n-1} \sigma(\ell_0)
\end{bmatrix}
\tag{4}
\]

or equivalently

\[
X = \sum_{j=0}^{n-1} \Gamma^j \left( \text{diag}(f_{j} \cdot G) \right) \tag{5}
\]

with \( \Gamma^j = [f_{j,0} \ f_{j,1} \ \cdots \ f_{j,n-1}] \) where

\[
G = \begin{bmatrix}
\beta_0 & \cdots & \sigma^{n-1}(\beta_0)
\vdots & \ddots & \vdots
\beta_{n-1} & \cdots & \sigma^{n-1}(\beta_{n-1})
\end{bmatrix} \tag{6}
\]

\[
\Gamma^j = \begin{bmatrix}
0 & 0 & \cdots & 0 & \gamma
1 & 0 & \cdots & 0 & 0
0 & 1 & \cdots & 0 & 0
\vdots & \vdots & \ddots & \vdots & \vdots
0 & 0 & \cdots & 1 & 0
\end{bmatrix}. \tag{7}
\]

For this setup, it was shown in [14][9][2], that if \( \gamma \) is independent of SNR and the \( f_{i,j} \) are from a discrete constellation such as QAM, then the code achieves

1) full diversity
2) full-rate
3) non-vanishing determinant.

1) Requirements for achieving equal power sharing and good constellation: Based on the above setup, we now show that using a unit-magnitude, algebraic, non-norm element \( \gamma \), and a unitary \( G \), renders the code perfect.

Let \( B := \{\beta_0, \ldots, \beta_{n-1}\} \) be a basis for \( \mathbb{L}/\mathbb{F} \) such that the matrix

\[
G(B) = \begin{bmatrix}
\beta_0 & \cdots & \sigma^{n-1}(\beta_0)
\vdots & \ddots & \vdots
\beta_{n-1} & \cdots & \sigma^{n-1}(\beta_{n-1})
\end{bmatrix}
\]

is unitary. Furthermore, we assume that \(|\gamma| = 1\).

Let \( \mathbb{F} = \mathbb{Q}(i) \) and let the \( \{f_{i,j}\} \) be restricted to belong to the \( M^2 \)-QAM constellation:

\[
f_{i,j} \in \mathcal{A}_{\text{QAM}} = \{a+ib \mid -(M-1) \leq a, b \leq M-1, \ a, b \text{ both odd}\}.
\]

For

\[
\tilde{f} := [f_{0,0} \ f_{0,1} \ \cdots \ f_{0,n-1} \ f_{1,0} \ \cdots \ f_{1,n-1} \ \cdots \ f_{n-1,n-1}]
\]

we denote the code-matrix \( X \) in (3) as \( X(\tilde{f}) \) to emphasize that it is a function of the QAM vector \( \tilde{f} \). If \( g \) is any function of the \( \{f_{i,j}\} \), we use \( \mathbb{E}[g(\{f_{i,j}\})] \) to denote the average of \( \frac{1}{|\mathcal{A}_{\text{QAM}}|} \sum_{f_{i,j} \in \mathcal{A}_{\text{QAM}}} g(\{f_{i,j}\}) \). Then for \( k_0 \in \{0,1\} \), it is the case that

\[
\mathbb{E}[|\gamma^{k_0} \sigma^t(\ell_j)|^2] = \mathbb{E}[(\sum_{i,j} f_{i,j}\sigma^t(\beta_i))(\sum_k f_{k,j}^*\sigma^t(\beta_i^*)]^2] = \mathbb{E}[(\sum_{i,j} |f_{i,j}|^2|\sigma^t(\beta_i)|^2] = \mathbb{E}[(|f_{i,j}|^2) \sum_i |\sigma^t(\beta_i)|^2] = \mathbb{E}[(|f_{i,j}|^2)\].

It follows that the average energy per transmitted element of the code matrix is the same.
With respect to the constellation shaping, let us now denote the layer-by-layer vectorization of \( X(f) \) as:

\[
\text{vec}(X(f)) := \begin{bmatrix}
I_{n \times n} & 0_{n \times n} & \ldots & 0_{n \times n} \\
0_{n \times n} & \Gamma^{(1)} & \ldots & 0_{n \times n} \\
\vdots & \vdots & \ddots & \vdots \\
0_{n \times n} & 0_{n \times n} & \ldots & \Gamma^{(n-1)} \\
\end{bmatrix}
\begin{bmatrix}
\ell_0 \\
\sigma(\ell_0) \\
\vdots \\
\ell_{n-1} \\
\sigma^{-1}(\ell_{n-1}) \\
\end{bmatrix}
\]

where

\[
\Gamma^{(1)} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad \Gamma^{(2)} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}
\]

We observe that each vector resulting from the layer-by-layer vectorization of any code-matrix, prior to SNR normalization, is exactly the linear transformation of the \( n^2 \)-tuple \( f \) from QAM\(^{n^2} \) or HEX\(^{n^2} \), by the unitary matrix

\[
R_v = \begin{bmatrix} G & 0_{n \times n} & 0_{n \times n} & \ldots & 0_{n \times n} \\
0_{n \times n} & G \cdot \Gamma^{(1)} & 0_{n \times n} & \ldots & 0_{n \times n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0_{n \times n} & 0_{n \times n} & \ldots & G \cdot \Gamma^{(n-1)} \\
\end{bmatrix}
\]

As a result the signalling set prior to SNR normalization is from the lattice

\[
\Lambda_{\text{signal}} = \{ fR_v, f \in \text{QAM}^{n^2} \}
\]

or \( \Lambda_{\text{signal}} = \{ fR_v, f \in \text{HEX}^{n^2} \} \). Modifying \( f \) by horizontally stacking its real and imaginary parts, \( f_{\text{Re}} \) and \( f_{\text{Im}} \) respectively, and transforming the resulting vector \( f' \) by \( [f_{\text{Re}} \ f_{\text{Im}}] \in \mathbb{Z}^{2n^2} \) by

\[
R'_v = \begin{bmatrix} R_{v,\text{Re}} & R_{v,\text{Im}} \\
-R_{v,\text{Im}} & R_{v,\text{Re}} \end{bmatrix}
\]

we get the real and imaginary stacking of the signalling set to be from

\[
\Lambda'_{\text{signal}} = \{ f' R'_v, f' \in \mathbb{Z}^{2n^2} \}, \quad R'_v R'^T_v = I_{2n^2 \times 2n^2}
\]

Equivalently, for any set of \( k_i \in \{0, 1\} \),

\[
\begin{bmatrix}
\gamma^k_0 \ell_j \\
\gamma^k_1 \sigma(\ell_j) \\
\vdots \\
\gamma^k_{n-1} \sigma^{n-1}(\ell_j) \\
\end{bmatrix} = \begin{bmatrix}
\ell_j \\
\sigma(\ell_j) \\
\vdots \\
\sigma^{n-1}(\ell_j) \\
\end{bmatrix} = \begin{bmatrix}
f_0,j \\
f_1,j \\
\vdots \\
f_{n-1,j} \\
\end{bmatrix}
\]

allowing for

\[
Tr(X^\dagger X) = ||\text{vec}(X(f))||^2 = ||f||^2
\]

and making the collection

\[
\{ \text{vec}(X(f)) \mid f \in \text{QAM}^{n^2} \}
\]

represent a cubic constellation in \( n^2 \)-dimensional complex space that is isometric to \( \text{QAM}^{n^2} \).

We proceed to find a proper unit-magnitude non-norm element \( \gamma \), and then a proper unitary matrix \( G \).

### B. Uniform average transmitted energy per antenna

We now provide algebraic, unit-magnitude ‘non-norm’ elements \( \gamma \) for suitable cyclic Galois extensions \( L/F \), independent of SNR. We will henceforth denote the \( i \)-th primitive root of unity as \( \omega_i \), i.e. \( \omega_i = e^{2\pi i / q} \), and by \( k^* \) we denote the complex conjugate of \( k \in \mathbb{C} \). We directly state the construction method for the different cases of interest, depending on whether the base field is \( F = \mathbb{Q}(i) \) (QAM) or \( F = \mathbb{Q}(\omega_3) \) (HEX).

**Proposition 1:** (Construction of the non-norm element for the QAM code) Let \( n = 2^s n_1 \) where \( n_1 \) is odd. Then there exists a prime \( p \) congruent to 1 mod \( n_1 \). Furthermore, there exists a prime \( q \) that is congruent to \( 1 \mod 4 \), as well as congruent to \( 5 \mod 2^n + 2 \), and which has order \( \text{ord}(q)|_{\mathbb{Z}_{p^a}} = n_1 \) and splits in \( \mathbb{Z}[i] \) as \( q = \pi_1 \pi_1^* \) for a suitable prime \( \pi_1 \in \mathbb{Z}[i] \). The fields \( \mathbb{Q}(\omega_p) \) and \( \mathbb{Q}(i) \) are linearly disjoint over \( \mathbb{Q} \). Let \( K \) be the unique subfield of \( \mathbb{Q}(i)(\omega_p) \) of degree \( n_1 \) over \( \mathbb{Q}(i) \) and let \( L = K \cdot \mathbb{Q}(\omega_{2^s + 2}) \). Then \( L \) is a cyclic extension of \( \mathbb{Q}(i) \), and the element

\[
\gamma = \frac{\pi_1}{\pi_1^*}
\]

is an (algebraic) unit-magnitude element that is a non-norm element for the extension \( L/\mathbb{Q}(i) \) and is independent of SNR.

When \( n_1 = 1 \), we take \( L = \mathbb{Q}(\omega_{2^s + 2}) \) and \( \gamma = \frac{1 + 2i}{1 - 2i} \).

The \( n \times n \) matrices arising from equations (5), (6) and (7) with the above choice of \( F, L \) and \( \gamma \), and with any choice of scaled integral basis \( \{ \beta_i \}_{i=0}^{n-1} \) hence yield a full-diversity, full-rate code over QAM with non-vanishing determinant satisfying the additional equal power sharing constraint.

**Proof:** See Appendix.

**Proposition 2:** (Construction of the non-norm element for the HEX code) Let \( n = 2^s n_1 \), \( s \in \{0, 1\} \), where \( n_1 \) is odd. Then there exists a prime \( p > 3 \) congruent to 1 mod \( n_1 \). Furthermore, there exists a prime \( q \) that is congruent to \( 1 \mod 3 \) and which has order \( \text{ord}(q)|_{\mathbb{Z}_{p^a}} = n_1 \) and splits in \( \mathbb{Z}[\omega_3] \) as \( q = \pi_1 \pi_1^* \) for a suitable prime \( \pi_1 \in \mathbb{Z}[\omega_3] \). If \( s = 1 \) then \( q \) should also be congruent to \( 3 \mod 4 \). The fields \( \mathbb{Q}(\omega_p) \) and \( \mathbb{Q}(\omega_3) \) are linearly disjoint over \( \mathbb{Q} \). Let \( K \) be the unique
subfield of \( \mathbb{Q}(\omega_3) \) of degree \( n_1 \) over \( \mathbb{Q}(\omega_3) \) and let \( L = K \cdot \mathbb{Q}(\omega_{2^s+1}) \). Then \( L \) is a cyclic extension of \( \mathbb{Q}(\omega_3) \), and the element

\[
\gamma = \frac{\pi_1}{\pi_1^s}
\]

is an (algebraic) unit-magnitude element that is a non-norm element for the extension \( L/\mathbb{Q}(\omega_3) \) and is independent of SNR. (When \( n_1 = 1 \), so \( s = 1 \), we take \( L = \mathbb{Q}(\omega_3) \).)

The \( n \times n \) matrices arising from equations \( \text{[5.], [6.]} \) and \( \text{[7.]} \) with the above choice of \( F, L \) and \( \gamma \), and with any choice of scaled integral basis \( \{\beta_i\}_{i=0}^{n-1} \) hence yield a full-diversity, full-rate code over HEX with non-vanishing determinant satisfying the additional equal power sharing constraint.

**Proof:** See Appendix \( \text{[\!]} \)

Consequently, we have constructed algebraic, unit-magnitude ‘non-norm’ elements, valid for use in perfect codes, for any \( n \). Some examples are given in Table \( \text{[\!]} \)

### TABLE I
### NON-NORM ELEMENTS

| No. of Antennas | Non-norm \( \gamma \) |
|-----------------|----------------------|
| 2               | \( (2 + 3)/(1 + 2; 1 + 4) \) |
| 3               | \( (3 + \omega_3)/(3 + \omega_3^2) \) |
| 4               | \( (2 + 1)/(2 - 1) \) |
| 5               | \( (3 + 2)/(3 - 2) \) |
| 6               | \( (3 + \omega_3)/(3 + \omega_3^2) \) |
| 7               | \( (8 + 3\alpha)/(5 + 3\alpha) \) |
| 8               | \( (2 + 1)/(2 + 1) \) |
| 9               | \( (3 + \omega_3)/(1 + 3\omega_3) \) |

C. Good Constellation shaping

For any dimension \( n \), we now proceed to describe the construction of the unitary matrix \( G \) that complies with the cyclic Galois requirements of the division algebra. The construction method will involve the embedding of the scaled integral basis of a submodule of \( OL \) over the ring of integers \( \mathbb{Q}(\omega) \). Where \( \mathbb{F}, L \) are as in Propositions \( \text{[\!]} \) and \( \text{[\!]} \)

We proceed to first construct lattices for any odd dimension \( n_1 \), then lattices of dimension \( 2^s, s \in \mathbb{Z}^+ \), and then proceed to combine them in order to give the final desired lattices for any dimension \( n \) over \( \mathbb{Q}(\omega) \) or any dimension that is not a multiple of \( 4 \) over \( \mathbb{Q}(\omega_3) \). Without any loss of generality, we will be analyzing the QAM case, corresponding to \( \mathbb{F} = \mathbb{Q}(\omega) \). Unless we state otherwise, the same results will also hold for \( \mathbb{F} = \mathbb{Q}(\omega_3) \).

1) Orthogonal Lattices in a Cyclic Galois Extension over \( \mathbb{Q} \) of Odd Degree \( \gamma \). Recently, the authors in \( \text{[\!]} \) Section \( \text{V.} \) give a detailed exposition of a previous result in \( \text{[\!]} \) of an explicit construction of \( q \)-dimensional orthogonal lattices that belong in a \( q \)-degree cyclic Galois extension \( \mathbb{K} \) over \( \mathbb{Q} \), with the restriction that \( q \) be an odd prime integer. We here show that the same construction actually gives \( n_1 \)-dimensional orthogonal lattices in \( \mathbb{O}(\mathbb{K}) \), for any odd integer \( n_1 \). Moreover, the field \( \mathbb{K}(\omega) \) will be precisely the field \( \mathbb{K} \) of Proposition \( \text{[\!]} \)

The steps for constructing an \( n_1 \)-dimensional orthogonal lattice in \( \mathbb{O}(\mathbb{K}) \) are as follows:

- pick a guaranteed to exist odd prime \( p \equiv 1 \pmod{n_1} \)
- let \( \omega = \omega_p = e^\frac{2\pi}{p} \)
- find a guaranteed to exist primitive element \( r \) of \( \mathbb{Z}_p^* \)
- for \( m = \frac{p-1}{2} \), create \( \alpha = \prod_{k=0}^{m-1}(1 - \omega^k) \) where \( r^{p-1} = 1 \)
- Find a guaranteed to exist \( \lambda \) such that \( \lambda(r - 1) \equiv 1 \pmod{p} \) and let \( z = \omega^\lambda(1 - \omega) \)
- For \( (\omega) = \omega^r \), let \( x = \sum_{k=1}^{\frac{p-1}{2}} \sigma_\omega zn_1(z) \)

The element \( x \) is hence in the field \( \mathbb{K} \), the subfield of \( \mathbb{Q}(\omega) \) fixed by \( \sigma^{n_1} \), of degree \( n_1 \) over \( \mathbb{Q} \). It is then the case that the following lattice generator matrix \( G_{n_1} \) is unitary and the resulting lattice (which arises from the canonical embedding of the free \( \mathbb{Z} \) module generated by \( x/\sqrt{\mathbb{Q}}, \sigma(x)/\sqrt{\mathbb{Q}}, \ldots, \sigma^{n_1-1}(x)/\sqrt{\mathbb{Q}} \) in \( \mathbb{R}^n \)) is orthogonal.\

Since \( (\omega) \) and \( (\omega) \) are linearly disjoint over \( \mathbb{Q} \), the field \( \mathbb{K} \) will be cyclic over \( \mathbb{Q} \), and the elements \( x/\sqrt{\mathbb{Q}}, \sigma(x)/\sqrt{\mathbb{Q}}, \ldots, \sigma^{n_1-1}(x)/\sqrt{\mathbb{Q}} \) will be a scaled integral basis for \( \mathbb{K}/\mathbb{Q} \).

**Proof:** See Appendix \( \text{[\!]} \)

More specifically the first row of \( G_{n_1} \) is given by

\[
G_{n_1}(0, j) = \frac{1}{p} \omega\lambda \alpha \sum_{k=1}^{\frac{p-1}{2}} (-1)^{kn_j+1}(1 - \omega^{-kn_j+1}) \]

and the rest of the circulant matrix by:

\[
G_{n_1}(i, j + 1) = G_{n_1}(i, j + 1 \pmod{n_1}), \quad i = 0, \ldots, n_1 - 1
\]

Example 1: The first row of the 9-dimensional \( G_9 \) is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}^T (\mathbb{Z}_{20000})
\]

and every next row is obtained by a single left cyclic shift of the previous row. The matrix was obtained by setting \( n_1 = 9, p = 19, r = 3 \) and \( \lambda = 10 \). Similarly the first row of the 15-dimensional \( G_{15} \) is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}^T (\mathbb{Z}_{20000})
\]

obtained by setting \( n_1 = 15, p = 31, r = 3 \) and \( \lambda = 16 \).

2) Lattices of dimension \( m = 2^s \) \([\!]\): For when the information set is QAM, then \( \mathbb{F} = \mathbb{Q}(\omega) \) and we consider \( \mathbb{K} = \mathbb{Q}(\omega_M) \) where \( M = 2^{s+2} \) and \( \omega_M = \omega^{e\pi i/M} \) the \( M^{th} \) primitive root of unity. \( \mathbb{Q}(\omega) \) is a cyclic Galois extension over \( \mathbb{Q} \). Considering the order of 5 in \( \mathbb{Z}_p^* = \text{Gal}(\mathbb{K}/\mathbb{Q}) \) is \( m = 2^s = \frac{\phi(M)}{2} \), we see that for \( \sigma \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) such that \( \sigma(\omega) = \omega^5 \), it is the case that \( \sigma(i) = \sigma(\omega^2) = \omega^{2^2} = \omega^4 = \omega^2 = i \) which gives that \( \text{Gal}(\mathbb{K}(\omega)/\mathbb{Q}) = \langle \sigma \rangle \). Taking \( \{0, \omega^1, \omega^2, \ldots, \omega^{m-1}\} \) to
be the integral basis over \( \mathbb{Q}(i) \), the canonical embedding then gives the lattice generator matrix

\[
G_e = \frac{1}{\sqrt{m}} \begin{bmatrix} \sigma^i(\omega^j) \end{bmatrix}_{i,k} = \frac{1}{\sqrt{m}} \begin{bmatrix} \omega^{j,5^k} \end{bmatrix}_{i,k} \tag{13}
\]

The fact that the lattice corresponds to the ring of integers of the \( m \)-dimensional cyclic Galois extension \( \mathbb{K}_i \) over \( \mathbb{Q}(i) \), allows for \( G_e \) to be directly used in (5) to construct the \( m \times m \) space-time code.

Now for \( r_i = \begin{bmatrix} 1 & \omega^5 & \omega^{5^2} & \omega^{5^3} & \cdots & \omega^{5^{(n-1)}} \end{bmatrix}, i = 0, 1, \ldots, m-1 \), being the \( i^{th} \) row of \( \sqrt{m}G^T \) in (13), we have that \( r_ir_j^T = \sum_{k=0}^{m-1} \omega^{j,5^k}k = \sum_{k=0}^{m-1} \omega^{k(5^i-5^j)} \). Since \( 5 \) has order \( \frac{m}{5} = \frac{2^s(M)}{2} \) in \( \mathbb{Z}_M^\times \), then \( 5^i \neq 5^j \) \( \forall i \neq j \), \( i, j = 0, 1, \ldots, \frac{m}{5} - 1 \). This combines with the fact that \( k(5^i-5^j) = k5^i(5^{j-1}) \equiv 0 \) (mod 4) so that each summand pairs with another summand in the summation so that their ratio is \( 5^4 \). This symmetry, the fact that the complex conjugate of \( \omega \) is \( \omega^{-1} \), results in \( r_ir_j^T = m\delta_{ij} \) and in the desired orthogonality \( G_eG_e^T = I \). The lattices apply only for codes over QAM.

3) Combining lattices: We will need the following, which is an easy modification of Proposition 6 in [12] and which eventually guarantees for the creation of lattices over a cyclic Galois extension for any dimension \( n \) over \( \mathbb{Q}(i) \), and any dimension that is not a multiple of 4 over \( \mathbb{Q}(\omega_5) \).

**Lemma 3:** Let \( \mathbb{L} \) be the compositum of \( l \) Galois extensions \( \mathbb{K}_i \) over \( \mathbb{Q} \) of co-prime degrees \( n_i \). Assuming that there exists an orthogonal \( O_{\mathbb{L}} \)-lattice generator matrix \( G_l \), for all \( i = 1, 2, \ldots, l \), then the Kronecker product of these matrices is a unitary generator matrix of an \( n \)-dimensional lattice in \( O_{\mathbb{L}} \), \( n = \prod_{i=1}^{l} n_i \).

For this, the discriminants are not required to be coprime since the involved fields already have coprime degrees, so their composite is their tensor product over \( \mathbb{Q} \). Specifically, for \( F = \mathbb{Q}(i) \), for any \( n = n_12^s \), \( n_1 \) odd, the orthogonal lattice generator matrix \( G \) is the Kronecker product of the generator matrix of the \( n_1 \)-dimensional lattice from Section II.C.1 and that of the cyclotomic lattice of dimension \( 2^s \). For \( \mathbb{F} = \mathbb{Q}(\omega_5) \), for \( n = n_1 \) odd we again use the \( n_1 \)-dimensional lattice from Section II.C.1 and for \( n = 2n_1, n_1 \) odd, the orthogonal lattice generator matrix \( G \) is the Kronecker product of the generator matrix of the \( n_1 \)-dimensional lattice from Section II.C.1 and matrix \( C_2 = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \) coming from the field \( \mathbb{Q}(i) \).

The above orthogonal lattice generator matrices correspond to a suitable scaled integral basis of the \( n \)-dimensional cyclic Galois extension \( \mathbb{L}/\mathbb{F} \), defined (respectively) in Propositions 1 and 2. As discussed above, these matrices allow for good constellation shaping. Consequently, with this choice of lattice generator matrix and the choice of \( \mathbb{L}/\mathbb{F} \) and \( \gamma \) as in Propositions 1 and 2, the code defined by equations (12) form perfect codes satisfying full-diversity, full-rate, non-vanishing determinant, equal power sharing, and good constellation shaping.

III. INFORMATION THEORETICAL INTERPRETATION AND GENERALIZATION OF THE PERFECT CODE CONDITIONS

The D-MG tradeoff [3] bounds the optimal performance of a space-time code \( \mathcal{X} \) operating at rate \( \nu \) bpcu, corresponding to a multiplexing gain

\[
r = \frac{R}{\log_2(\text{SNR})}.
\]

The diversity gain corresponding to a given \( r \), is defined by

\[
d(r) = \lim_{\text{SNR} \to \infty} \frac{\log(P_e)}{\log(\text{SNR})}.
\]

where \( P_e \) denotes the probability of codeword error. For the Rayleigh fading channel, Zheng and Tse [3] described the optimal tradeoff between these two gains by showing that for a fixed integer multiplexing gain \( r \), the maximum achievable diversity gain is

\[
d(r) = (n - 1)(n - r). \tag{14}
\]

The function for non-integral values is obtained through straight-line interpolation.

We use this D-MG approach as a basis for interpreting and generalizing the conditions that define perfect-codes.

4) Full rate condition: Consider an \( n \times T \) code \( \mathcal{X} \) where each code-matrix carries \( m \) information symbols per channel use from a discrete constellation \( \mathcal{A} \) such as QAM. It is then the case that

\[
|\mathcal{X}| = 2^{2RT} = 2^{\nu T \log_2 \text{SNR}} = \text{SNR}^{\nu T} = |\mathcal{A}|^{mT}
\]

which implies that \( |\mathcal{A}| = \text{SNR}^{\nu T} \) and since the constellation is discrete, we have that \( \mathbb{E}[||\alpha||^2] = |\mathcal{A}| \). The fact that each element \( X_{i,j} \) of a code matrix is a linear combination of elements of \( \mathcal{A} \), gives that

\[
\mathbb{E}[||X_{i,j}||^2] = |\mathcal{A}| = \text{SNR}^{\nu T}.
\]

The SNR normalizing factor \( \nu \) that guarantees that \( \mathbb{E}[\nu (\nu X)^2] = \mathbb{E}[||\nu X||^2] = \text{SNR}^{\nu T} \) is then given by

\[
\nu^2 = \text{SNR}^{1-\nu T}. \tag{15}
\]

Without loss of generality we can assume that there exist two code-matrices \( X_1, X_2 \in \mathcal{X} \), with each \( X_i \) mapping the information \( nm \)-tuple \( \{\alpha_i, 0, 0, \ldots, 0\} \), where \( \alpha_i \in \mathbb{SNR}^{0} \). As a result, the determinant and trace of the difference matrix \( \Delta X \), is a polynomial of degree less than \( n \) over \( \alpha = \alpha_1 - \alpha_2 \neq SNR^0 \), with coefficients independent of \( \text{SNR} \), i.e.

\[
\det(\Delta X \Delta X^T) = \text{Tr}(\Delta X \Delta X^T) \neq \text{SNR}^0
\]

and thus with all its eigenvalues

\[
l_i \neq \text{SNR}^0.
\]
The corresponding pairwise error probability \( P_{e} \), in the Rayleigh fading channel \([4], [7]\), then serves as a lower bound to the codeword error probability \( P_{e} \), i.e.,

\[
P_{e} \geq \text{PEP}(X_1 \rightarrow X_2) = \frac{1}{\prod_{j=1}^{n}[1 + \frac{\sigma^2}{T_f}]^{n_r}}
\]

which results in a diversity gain of

\[
d(r) \leq n_r n \left(1 - \frac{r}{m}\right).
\]

What this means is that \( m \) discrete information symbols per channel use can potentially sustain reliable communication for up to rate \( R_{\text{max}} \approx m \log_2(\text{SNR}) \).

For large SNR, the outage capacity over an \( n \times n_r \) Rayleigh fading channel is given by \( C_{\text{out}} \approx \min\{n, n_r\} \log_2(\text{SNR}) \), implying a maximum achievable multiplexing gain of \( r_{\text{max}} = \min\{n, n_r\} \). Consequently the relation between \( R_{\text{max}} \) and \( C_{\text{out}} \), allows for the interpretation that the full rate defining condition is necessary for reliable transmission at rates close to the outage capacity of the Rayleigh fading channel, independent of the channel topology. Equivalently, given some rate \( R \), the full rate defining condition is necessary for reliable transmission at the smallest allowable SNR

\[
\text{SNR}_{\text{min}} \doteq \frac{2^{2R(m/n_r+m/n)}}{\min\{n, n_r\}}.
\]

again independent of the channel topology. Let us now re-examine the full-rate condition, in conjunction with the determinant condition.

5) Non-vanishing determinant condition: We consider the \( n \times T \) truncated code \( X \), \( T \geq n \), constructed by deleting the same \( T - n \) rows from all the code-matrices \( X' \) of a \( T \times T \) perfect code \( X' \). We have seen that for an \( n \times n \) code mapping \( n^2 \) information symbols from a discrete constellation \( n \) information symbols per channel use), the standard \( n \)-dimensional ‘folding’ \( |X'| = |A^{[\kappa]}| \) forces a normalizing factor of \( \nu^2 = \text{SNR}^{1-\tau} \), whereas in the truncated \( n \times T \) code mapping \( T^2 \) information symbols (\( \frac{T^2}{n} \) information symbols per channel use), the constellation is folded in \( T \) dimensions \( |X' = |A^{[T]}| \), requiring for

\[
\nu^2 = \text{SNR}^{1-\tau}.
\]

This scenario accentuates the fact that in essence, we are limited by a lower bound on the determinant of the energy-normalized difference matrix \( \nu^2 \Delta X \Delta X^\dagger \). As a result, for the \( n \)-dimensional case, the defining condition of non-vanishing determinant for the non-normalized matrix \( \Delta X \Delta X^\dagger \geq \text{SNR}^0 \), translates to

\[
\det[\nu^2 \Delta X \Delta X^\dagger] \geq (\nu^2)^n \text{SNR}^0 = (\text{SNR}^{1-\tau})^n = \text{SNR}^{n-\tau}
\]

which, for the \( n \times T \) case with \( T \)-dimensional folding, translates back to the determinant bound

\[
\det(\Delta X \Delta X^\dagger) \geq \text{SNR}^{-\frac{\tau}{2}(T-n)}
\]

for the non-normalized code-matrices. But from \([2], [16]\), we see that the above determinant bound is the best that any code can attain, thus allowing us to generalize the full-rate, full-diversity and non-vanishing determinant perfect code conditions, to the general condition of having

\[
\det[\nu^2 \Delta X \Delta X^\dagger] \geq \text{SNR}^{n-\tau}, \quad 0 \leq \tau \leq \min(n, n_r). \quad (16)
\]

In regards to non-minimum delay perfect codes, let us briefly note that codes resulting from row deletion of perfect codes essentially maintain all the conditions of the original minimum-delay perfect code constructions except that now the vectorization of the code-matrices is not isometric to QAM\(^{T^2}\). Non-minimum delay perfect codes can be constructed though for delays \( T = nk, \ k \in \mathbb{Z}^+ \) that are multiples of \( n \), by the horizontal stacking construction found in \([2], [16]\) which maintains the non-vanishing determinant property as well as the isometry of the code matrices with QAM\(^{n^2k}\).

Let us now incorporate all the perfect-code defining conditions in order to provide an information theoretic interpretation that spans both the high and the low SNR regimes.

6) Approximate universality, information losslessness and Gaussian-like signalling: We begin with:

**Theorem 4:** Perfect codes are both approximately universal as well as information lossless.

**Proof:** See Appendix [IV]

The code’s information losslessness, shown in the proof to be the result of the CDA structure and the last two conditions, essentially allows for the code to maintain the maximum mutual information corresponding to the channel and signalling set statistics. This mutual information is empirically related to the Gaussian-like signalling set and its good covariance properties, observed in Figure 1 The expedited rate with which the signalling becomes Gaussian, relates to the high-dimensional and orthogonal nature of the lattice generator matrix which together with a unit-magnitude non-norm element, jointly allow equal magnitudes for the diagonal elements of the covariance matrix of the signalling set.

Let us now draw from the information theoretic interpretation of the defining conditions and provide variants of perfect codes that are specifically tailored for channels with a smaller number of receive antennas, and which manage to maintain good performance at a considerably reduced sphere decoding complexity.

### A. Channel topology and efficient variants of perfect codes

We have seen that \( n \times n \) perfect codes utilize \( n \) different layers to achieve approximate universality for all \( n_r \). Each layer has non-vanishing product distance and maps \( n \)-elements from

![Fig. 1. Gaussian nature of the signalling set of the 3 × 3 perfect code, compared to QAM and random Gaussian signalling (left). Covariance of columns of perfect codes (center) compared to covariance of random Gaussian vectors (right).](image-url)
a discrete constellation, thus maintaining two properties that
were shown in [18, Theorem 4.1] to guarantee for optimality
over the statistically symmetric parallel channel, i.e., a channel
with a diagonal fading coefficient matrix, as well as potentially
allowing for optimality over the statistically symmetric $n \times 1$
MISO channel. To offer intuition, we observe that the sum-
capacity of the $n_r$ independent MISO channels relates to the
full rate condition, whereas the achieved full diversity relates
to the CDA structure and the discreteness of the powers of $\Gamma$
which manage to translate the non-vanishing product distance
to an overall non-vanishing determinant, and thus to keep
the different layers independent and at some non-vanishing
distance from each other. The full rate condition comes with
a sphere decoding complexity of $O(n^2)$, but as the number of
MISO channels reduces with $n_r$, so does the required decoding
complexity. Codes over such channels can have the form

$$X = \sum_{j=0}^{n_r-1} \Gamma^j \left( \text{diag}(f_j, G) T \right)$$

where $T(i, j) = 1, i = j \in [1, \ldots, n_r]$ and $T(i, j) = 0$
otherwise, or can have the form

$$X = \sum_{j=0}^{n_r-1} \Gamma^j \left( \text{diag}(f_j, G) \right).$$

Note here that the above codes have not been proven to be
D-MG optimal.

Motivated by the down-link requirements and by the coopera-
tive diversity uses of space-time coding in wireless networks,
we will concentrate on the MISO case ($n_r = 1$), for which a
D-MG optimal perfect code variant

$$\mathcal{X}_d = \{ \text{diag}(\mathcal{L}) = \text{diag}(f \cdot G), \forall f \in \text{QAM}^n \}. \quad (17)$$

with sphere decoding complexity of $O(n)$ was recently con-
structed in [49], [50] for all $n$, together with the code

$$\mathcal{X}_{ir} = \{ X = \sum_{k=0}^{n-1} f_k \Gamma^k, \forall f_k \in \text{QAM-HEX} \} \quad (18)$$

that corresponds to the center of the division algebra. With
the exception of $n = 2$, $\mathcal{X}_{ir}$ has not yet been proven to be
D-MG optimal. Figure 2 provides a performance comparison
between the single dimensional perfect code variant with the
equivalent standard perfect-code.

IV. RECENT DEVELOPMENTS INVOLVING PERFECT CODES

The proposed high-dimensional perfect codes have had an
impact on establishing outage-based optimality expressions
for wireless networks where independently distributed users
utilize different parts of space-time schemes to relay messages
for one another, hence improving the overall quality of service
([43] etc). Up to now, outage-based optimality results were
known only for infinite time duration networks in which
the assisting relays required full knowledge of the channel.
Encoding was based on random Gaussian codes. Using perfect
codes as an information theoretic tool, it was shown in [47],
[49] that the same optimality can be achieved, for finite
and minimum delay, and without requiring knowledge of the
channel at the intermediate relays. This was achieved for the
most general network topology and statistical characterization.

Perfect codes and some perfect-code variants, provided for
the first ever optimal encoding method [47], [48], [49], [50]
in several cooperative-diversity schemes such as the non-
dynamic linear-processing (receive-and-forward) scheme [44],
the non-dynamic selection-decode-and-forward scheme [45]
and finally for the dynamic receive-and-forward scheme [46].

V. EXAMPLES OF NEW PERFECT CODES AND SIMULATIONS

A. Examples of new perfect codes

- A $2 \times 2$ perfect code can be chosen to have code-matrices
  which prior to SNR normalization, are of the form

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} f_{0,0} + f_{1,0} \omega_8 \gamma (f_{1,0} + f_{1,1} \sigma (\omega_8^2)) & f_{0,1} + f_{1,1} \sigma (\omega_8^2) \\
f_{0,0} + f_{1,0} \omega_8 \gamma (f_{1,0} + f_{1,1} \sigma (\omega_8^2)) & \gamma f_{0,1} + f_{1,1} \omega_8^2 \end{bmatrix}$$

where $f_{i,j}$ are from the desired QAM constellation, $\omega_8 := e^{2\pi i / 8}$
and $\gamma = \gamma_{QAM}$. Matrices map $n^2 = 4$ information elements
from QAM. Furthermore the signalling set, in the form of the
layer-by-layer vectorization of the code-matrices, before SNR
normalization, comes from the lattice

$$\Lambda = \{ [f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1}] R_v : \forall [f_{0,0}, f_{0,1}, f_{1,0}, f_{1,1}] \in \text{QAM}^{n^2} \}$$

where

$$R_v = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\
\omega_8 & \omega_8 & 0 & 0 \\
0 & 0 & 1 & \gamma \\
0 & 0 & \omega_8 & \gamma \omega_8^2 \end{bmatrix}$$

satisfying the defining condition of

$$R_v R_v^\dagger = I_4.$$
which is larger than some previously constructed $2 \times 2$ perfect codes. The code’s performance improves if the existing
\[ G = \begin{bmatrix} 1 & 1 \\ \omega^3 & \omega \end{bmatrix} \]
taken from [12].

Other examples:

- The $5 \times 5$ perfect space-time code is given by
\[ \mathcal{X} = \left\{ X = \sum_{j=0}^{4} \Gamma^j (\text{diag}(f_j \cdot G_5)), \; f_j \in \text{QAM}^5 \right\} \]
for $\Gamma$ given in [4] based on $\gamma = \frac{3+2i}{1+2i}$, and generator matrix
\[ G_5 = \begin{bmatrix} -0.3260 & 0.5485 & -0.4557 & -0.5969 & -0.1699 \\ 0.5485 & -0.4557 & -0.5969 & -0.1699 & -0.3260 \\ -0.4557 & -0.5969 & -0.1699 & -0.3260 & 0.5485 \\ -0.5969 & -0.1699 & -0.3260 & 0.5485 & -0.4557 \\ -0.1699 & -0.3260 & 0.5485 & -0.4557 & -0.5969 \end{bmatrix} \]

- The $7 \times 7$ perfect space-time code is given by
\[ \mathcal{X} = \left\{ X = \sum_{j=0}^{6} \Gamma^j (\text{diag}(f_j \cdot G_7)), \; f_j \in \text{QAM}^7 \right\} \]
for $\Gamma$ based on $\gamma = \frac{8+5i}{8-5i}$, and generator matrix $G_7 =$
\[ \begin{bmatrix} -0.681 & 0.163 & -0.449 & 0.077 & 0.082 & 0.276 & -0.469 \\ 0.163 & -0.449 & 0.077 & 0.082 & 0.276 & -0.469 & -0.681 \\ -0.449 & 0.077 & 0.082 & 0.276 & -0.469 & -0.681 & 0.163 \\ 0.077 & 0.082 & 0.276 & -0.469 & -0.681 & 0.163 & -0.449 \\ 0.082 & 0.276 & -0.469 & -0.681 & 0.163 & -0.449 & 0.077 \\ 0.276 & -0.469 & -0.681 & 0.163 & -0.449 & 0.077 & 0.082 \\ -0.469 & -0.681 & 0.163 & -0.449 & 0.077 & 0.082 & 0.276 \end{bmatrix} \]

- The $25 \times 25$ integral restriction code is given by
\[ \mathcal{X}_d = \left\{ X = \sum_{k=0}^{24} s_k \Gamma^k, \; s_k \in \text{QAM} \right\} \]
with $\gamma = \frac{3+2i}{2+3i}$. This code has the same sphere decoding complexity of $O(25)$ as the $5 \times 5$ standard perfect code in the example above, and is expected to have the same performance, when $n_r = 1$, as the $25 \times 25$ perfect code whose sphere decoding complexity is $O(625)$.

B. Simulations

All the simulations assume $\mathbb{CN}(0, 1)$ fading and thermal noise. A sphere decoder was used. We begin with Figure 3 to indicate the performance improvement as the different defining conditions are satisfied one-by-one. The first curve from the top corresponds to satisfying the full-diversity condition (commutative CDA code - orthogonal design). The second curve now includes the full-rate condition (random, full-rate, linear-dispersion codes). The third curve corresponds to the family of D-MG optimal but not information lossless CDA codes presented in [2], which achieve the first three criteria of full-diversity, full-rate, and non-vanishing determinant. The performance transition from the CDA codes to perfect codes is described by the next two curves. Figures 4 and 5 show a comparison of the $2 \times 2$ unified perfect code presented here, with some perfect codes from [1] and with the Alamouti code ($n_r = 2$). The Golden code [15] performs best among all existing $2 \times 2$ perfect codes. When rates are lower, the unified perfect and the Golden code perform better than the orthogonal design whereas one of the perfect codes does not always do so. For higher rates, all considered perfect codes perform substantially better than the orthogonal design. At all rates and all SNR, the perfect code constructed here has performance very close to that of the Golden code. In Figure 6, we show the performance of the newly constructed 5-dimensional perfect code and compare that with the corresponding $5 \times 5$ single-dimensional commutative perfect code [18]. As expected, the former utilizes fully the $n_r = n = 5$ channel and is thus able to transmit with a small probability of error at high rates and low SNR.

VI. Conclusion

We have explicitly constructed perfect space-time codes for any number $n$ of transmit antennas, any number $n_r$ of receive antennas and any delay $T$ that is a multiple of $n$. Achieving all the defining conditions from [1], allows for perfect codes to exhibit performance that is currently unmatched. The information theoretic interpretation of the exhibited good performance both for low and high SNR, is that the defining conditions jointly endow the code with approximate universality and the ability to provide for near optimal mutual information.

High dimensional perfect codes cover a much needed requirement for optimal codes in multi-user cooperative diversity wireless networks, where each user acts as a transmit antenna. Specifically, perfect codes have already been used to establish
the high-SNR outage region of unknown channels, and have provided the first ever optimal schemes for a plethora of cooperative diversity methods.

**APPENDIX I**

**PROOF OF CONSTRUCTION METHODOLOGY FOR NON-NORM ELEMENTS**

We will prove here Propositions [12].

For future reference, we first recall three results that relate to identifying a “non-norm” element $\gamma$, i.e. an element $\gamma \in \mathbb{F}^*$ satisfying $\gamma^i \notin \mathbb{N}_{L/F}(L)$, $0 < i < n$ for some $n$-dimensional field extension $L$ of $\mathbb{F}$.

**Lemma 5:** [9] Let $L$ be a degree $n$ Galois extension of a number field $\mathbb{F}$ and let $p$ be a prime ideal in the ring $\mathcal{O}_L$ below the prime ideal $\mathfrak{p} \subset \mathcal{O}_L$ with norm given by $\|\mathfrak{p}\| = \|p\|^f$, where $f$ is the inertial degree of $\mathfrak{p}$ over $p$. If $\gamma$ is any element of $p \setminus p^2$, then $\gamma^i \notin \mathbb{N}_{L/F}(L)$ for any $i = 1, 2, \ldots, f - 1$.

In order to find a “non-norm” element $\gamma$ in $\mathbb{F} = \mathbb{Q}(i)$ ($\mathbb{F} = \mathbb{Q}(\omega_3)$), it is sufficient to find a prime ideal in $\mathbb{Q}[i]$ ($\mathbb{Q}[\omega_3]$) whose inertial degree $f$ in $\mathbb{L}/\mathbb{F}$ is $f = [\mathbb{L} : \mathbb{F}] = n$. Such an ideal is said to be inert in $\mathbb{L}/\mathbb{F}$.

**Lemma 6:** [38] Let $p$ be any odd prime. Then for any $k \in \mathbb{Z}$, $Z_{p^k}^*$ is cyclic of order $\phi(p^k)$. For any integer $f$ dividing $\phi(p^k)$ there exists an $a \in Z_{p^k}^*$ such that $a$ has order $f$ in $Z_{p^k}^*$.

**Theorem 7:** (Dirichlet’s theorem) Let $a, m$ be integers such that $1 \leq a \leq m, \gcd(a, m) = 1$. Then the progression $\{a, a + m, a + 2m, \ldots, a + km, \ldots\}$ contains infinitely many primes.

We will now proceed to establish the exact methodology that will give unit-magnitude non-norm elements $\gamma$, for the different cases of interest.

a) Unit-magnitude, non-norm elements for $\mathbb{F} = \mathbb{Q}(i)$:

Let

$$n = 2^s \prod_{i=1}^{r} p_i^{e_i} = 2^s n_1$$

(say)

where $p_i$ are distinct odd primes. Assume first that $n_1 > 1$. Let $p$ be the smallest odd prime $p$ such that $n_1 \mid (p - 1)$. The cyclic group $\mathbb{Z}_p^*$ contains an element whose order equals $(p - 1)$. Let $a$ denote this element. Our first goal is to find a prime $q$ such that

$$q = 5 \quad \text{(mod } 2^{s+2}\text{)}$$

$$q = a \quad \text{(mod } p\text{)}.$$

Note that

$$q = 1 \quad \text{(mod } 4\text{)}.$$

Since $(2^{s+2}, p) = 1$, we can, by the Chinese Remainder Theorem, find an integer $b$ such that

$$b = 5 \quad \text{(mod } 2^{s+2}\text{)}$$

$$b = a \quad \text{(mod } p\text{)}.$$

Note that such an integer $b$ is relatively prime to $2^{s+2}p$. Consider the arithmetic progression

$$b + l(2^{s+2}p), \quad l = 0, 1, 2, \ldots$$

By Dirichlet’s theorem, this arithmetic progression is guaranteed to contain a prime $q$ having the desired properties. Now let us verify that this leads to a CDA.

Let $K$ be the subfield of $\mathbb{Q}(\omega_p)$ that is a cyclic extension of $\mathbb{Q}$ of degree $n_1$. Let $\mathbb{K}$ be the compositum of $K$ and $\mathbb{Q}(i)$ and let $\mathbb{L}$ be the compositum of the fields $K$ and $\mathbb{Q}(\omega_{2^{s+2}})$.

Note that $\mathbb{L}$ is cyclic over $\mathbb{Q}(i)$, since it is a composite of the cyclic extension $\mathbb{Q}(\omega_{2^{s+2}})/\mathbb{Q}(i)$ of degree $2^s$ and the cyclic extension $K/\mathbb{Q}(i)$ of $n_1$ (note that $2^s$ and $n_1$ are relatively prime). Now consider the decomposition of the prime ideal $(q)$ in the extension $\mathbb{L}/\mathbb{Q}$.

Since $q = 1 \pmod{4}$ we have that $q$ splits completely in $\mathbb{Q}(i)/\mathbb{Q}$. Since $q$ has order $(p - 1)$ in $\mathbb{Z}_p$, it follows that $q$ remains inert in $\mathbb{Q}(\omega_p)/\mathbb{Q}$. Since $q = 5 \pmod{2^{s+2}}$ and $5$ has order $2^s$ in $\mathbb{Z}_{2^{s+2}}$, it follows that in the extension $\mathbb{Q}(\omega_{2^{s+2}})/\mathbb{Q}$, $q$ splits completely in $\mathbb{Q}(i)/\mathbb{Q}$ but remains inert thereafter.

Let $q$ split in $\mathbb{Q}(i)/\mathbb{Q}$ according to

$$q = \pi_1 \pi_1^*$$

where $\pi_1 = (a + ib)$ and $\pi_1^* = (a - ib)$. Now by using the fact that in a field tower $[\mathbb{E} : \mathbb{K} : \mathbb{F}]$ of field extensions, $f_{E/F} = f_{E/K} f_{K/F}$, $g_{E/F} = g_{E/K} g_{K/F}$, $[\mathbb{E} : \mathbb{F}] = f_{E/F} g_{E/F}$, it follows that $\pi_1$ remains inert in the extension $\mathbb{L}/\mathbb{Q}(i)$.

To now find a non-norm element of unit magnitude, we note that since the units of $\mathbb{Z}[i]$ belong to the set $\{-1, -1\}$, the associates of

$$\pi_1 = a + ib$$

belong to the set

$$\{a, a + m, a + 2m, \ldots, a + km, \ldots\}.$$
\[ \{a + ib, \ -a - ib, \ \pm(a + ib), \ -i(a + ib)\}. \]

It follows that since \( ab \neq 0 \), \( a - ib \) does not belong to the set of associates of \( a + ib \). Our goal now is to show that

\[ \gamma = \frac{\pi_1}{\pi_1} \]

is a non-norm element, i.e., that the smallest exponent \( k \) for which \( \gamma^k \) is the norm of an element in \( \mathbb{L} \), is \( n \). This is the case since if

\[ \gamma^k = N_{\mathbb{L}/\mathbb{F}}(\ell) \text{ some } \ell \in \mathbb{L} \]

then

\[ \pi_1^k = \pi_1^{k \sum a^\ell(\ell)} \]

where \( \sigma \) is the generator of the cyclic Galois group of \( \mathbb{L}/\mathbb{F} \).

For \( \ell = \frac{4}{3}, \ a, b \in \mathcal{O}_L \), we have, in terms of ideals of \( \mathcal{O}_L \),

\[ (\pi_1)^k \prod_{l=0}^{n-1} (\sigma^l(b)) = (\pi_1^*)^k \prod_{l=0}^{n-1} (\sigma^l(a)). \]

Since \( \sigma(\pi_1) = \pi_1 \) we have that if \( (\pi_1) \) divides \( (\sigma^l(x)) \) for some \( l \) and \( x \in \mathcal{O}_L \), it must divide \( (\sigma^l(x)) \), for all \( l \). This in turn implies that the power of \( (\pi_1) \) in the prime decomposition of \( (\pi_1)^k \prod_{l=0}^{n-1} (\sigma^l(b)) \) is \( k \mod n \) whereas the power of \( (\pi_1) \) in the prime decomposition of \( (\pi_1^*)^k \prod_{l=0}^{n-1} (\sigma^l(a)) \) is a multiple of \( n \). Equivalently \( k \) must be a multiple of \( n \).

When \( n_1 = 1 \), it is sufficient to take \( q \) to equal 5, and \( \mathbb{L} = \mathbb{Q}(\omega_{2,3}) \). The prime 5 splits in \( \mathbb{Q}(\iota) \) as \((1 + 2\iota)(1 - 2\iota)\) and then each of \((1 + 2\iota) \) and \((1 - 2\iota) \) remain inert in the extension \( \mathbb{L}/\mathbb{Q}(\iota) \). The element

\[ \gamma = \frac{1 + 2\iota}{1 - 2\iota} \]

is then a non-norm element for this extension, for the same reasons as above. This concludes the proof of Proposition 2

\[ \square \]

**APPENDIX II**

**ORTHOGONAL LATTICES IN \( \mathbb{Q}_8 \), WHERE \( \mathbb{K}/\mathbb{Q} \) IS CYCLIC GALOIS OF ODD DEGREE**

We here show that the construction in [10] (of which a detailed exposition has been provided in [12, Section 5]) of lattices that belong in a cyclic Galois extension \( \mathbb{K} \) of prime degree \( q \) over \( \mathbb{Q} \), actually gives without any modification orthogonal lattices for any odd degree \( n \). We will follow the exposition in [12] closely, retaining even the notation in [12], and show that the proofs there only use the fact that \( n \) is odd, and not that it is an odd prime.

To this end, let \( n \geq 3 \) be a given odd integer, and fix a prime \( p \equiv 1 \) (mod 3). Note that the existence of such a \( p \) is guaranteed since the sequence \( \{1 + dn, \ d = 1, 2, \cdots\} \), as shown by Dirichlet, contains infinitely many primes. Let \( \omega \) be a primitive \( p \)-th root of unity. Thus, \( \mathbb{Q}(\omega) \) is cyclic of degree \( p - 1 \) over \( \mathbb{Q} \), and contains the real subfield \( \mathbb{Q}(\omega + \omega^{-1}) \) which is cyclic of degree \( (p - 1)/2 \) over \( \mathbb{Q} \). Since \( n \) divides \( p - 1 \), there is a unique field \( \mathbb{K} \) contained in \( \mathbb{Q}(\omega) \) which is cyclic of degree \( n \) over \( \mathbb{Q} \). This is the field we will work with. Note that since \( n \) is odd, \( n \) divides \( (p - 1)/2 \) as well, so \( \mathbb{K} \) is contained in the real subfield \( \mathbb{Q}(\omega + \omega^{-1}) \).

Recall that we are following the notation in [12]. Let \( G = \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \), with generator \( \sigma \), chosen so that \( \sigma(\omega) = \omega^r \),

\[ \\]
Proposition 2 of [12], gives us the key to constructing the

\[ \lambda(r - 1) \equiv 1 \pmod{p}. \]

We define \( \alpha = \prod_{k=0}^{m-1} (1 - \omega^k) \). The following result is just a combination of Lemmas 3 and 4 of [12], and since they have to do purely with the cyclotomic extension \( \mathbb{Q}(\omega)/\mathbb{Q} \) and have nothing to do with \( n \), their proofs remain valid:

**Lemma 8:** The following equalities hold:

1. \( \sigma(\alpha) = -\omega^{p-1} \alpha \)
2. \( \sigma(\omega^\alpha) = -\omega^\alpha \)
3. \( (\omega^\alpha)^2 = (1)^mp \)

We now define \( z = \omega^\alpha \alpha(1 - \omega) \in \mathcal{O}_Q(\omega) \), and

\[ x = \text{Tr}_{\mathbb{Q}(\omega)/\mathcal{O}}(z) = \sum_{j=1}^{(p-1)/n} \sigma^j(z). \]

Note that \( x \) is in \( \mathcal{O}_\mathcal{K} \), as \( z \) is in \( \mathcal{O}_\mathcal{Q}(\omega) \). Observing that

\[ G_N Q_N^T(i,j) = \text{Tr}_{\mathbb{K}/\mathcal{Q}}(\sigma^j(x)\sigma^i(x)), \]

we are interested in \( \text{Tr}_{\mathbb{K}/\mathcal{Q}}(x\sigma^t(x)) \). The following, which is Proposition 2 of [12], gives us the key to constructing the orthogonal lattice.

**Proposition 9:** \( \text{Tr}_{\mathbb{K}/\mathbb{Q}}(x\sigma^t(x)) = p^2 \delta_{0, t} \), for \( t = 0, \ldots, n - 1 \).

Remark 1: Note that \( \text{Tr}_{\mathbb{K}/\mathcal{Q}}(\sigma^t(x)\sigma^i(x)) = \text{Tr}_{\mathbb{K}/\mathcal{Q}}(x\sigma^{t-i}(x)) \). Thus, if we embed \( \mathcal{O}_\mathcal{K} \) in \( \mathbb{R}^n \) via \( a \rightarrow v(a) = [a, \sigma(a), \ldots, \sigma^{n-1}(a)] \) (note that \( \mathcal{K} \) is a real field), this Proposition says that the vectors \([v(x), v(\sigma(x)), \ldots, v(\sigma^{n-1}(x))]\) are orthogonal to one another.

**Proof:** For \( n \) odd, we have

\[ \text{Tr}_{\mathbb{K}/\mathbb{Q}}(x\sigma^t(x)) = \sum_{a=0}^{n-1} \sigma^t(x\sigma^i(x)) = \sum_{a=0}^{n-1} \sum_{m=0}^{n-1} \sigma^{t+a} \omega^{n+cn}(z) \sum_{j=1}^{(p-1)/n} \sigma^j(z) \]

and from Lemma 8

\[ \text{Tr}_{\mathbb{K}/\mathbb{Q}}(x\sigma^t(x)) = \sum_{a=0}^{n-1} \sum_{m=0}^{n-1} \omega^{a+cn}(1 - \omega^{x+cn}) \cdot \sum_{j=1}^{(p-1)/n} \sigma^j(z) \]

We observe that since \( n \) is odd, \( (-1)^{cn} = (-1)^c \) and \( (-1)^m = (-1)^t \). Moreover, \( (-1)^t \cdot (-1)^a = 1 \), and \( (-1)^t \) is common to the sums above. By Lemma 8 we may replace \( (\omega^\lambda)^2 \) by \( (-1)^mp \). Thus we find, after rearranging the sums, that

\[ \text{Tr}_{\mathbb{K}/\mathbb{Q}}(x\sigma^t(x)) = (-1)^t(-1)^mp \sum_{c=1}^{(p-1)/n} (-1)^c \cdot \sum_{a=0}^{n-1} \sum_{m=0}^{n-1} (-1)^j(1 - \omega^{x+cn}) \cdot \sum_{j=1}^{(p-1)/n} (-1)^j(\omega^{x+cn} - \omega^{x+cn+1}) \]

Now the term \( \sum_{j=1}^{(p-1)/n} (-1)^j(1 - \omega^{x+cn}) \) can be rewritten as \( (1 - \omega^{x+cn}) - \sum_{j=1}^{(p-1)/n} (-1)^j(\omega^{x+cn+1}) \) becomes zero: this is because the terms in \( \sum_{a=0}^{n-1} \sum_{j=1}^{(p-1)/n} (-1)^j(\omega^{x+cn}) \) are independent of \( c \), while the term \( \sum_{c=1}^{(p-1)/n} (-1)^c = 0 \) as \( (p-1)/n \) is even and there as many positive as negative terms. We thus find

\[ \text{Tr}_{\mathbb{K}/\mathbb{Q}}(x\sigma^t(x)) = (-1)^t(-1)^mp \sum_{c=1}^{(p-1)/n} (-1)^c \cdot \sum_{a=0}^{n-1} \sum_{m=0}^{n-1} (-1)^j(\omega^{x+cn+1}) \cdot \sum_{j=1}^{(p-1)/n} (-1)^j(\omega^{x+cn+1}) \]

We now have the following:

**Lemma 10:**

\[ \sum_{c=1}^{(p-1)/n} (-1)^c \sum_{a=0}^{n-1} \sum_{m=0}^{n-1} (-1)^j(\omega^{x+cn+1}) \cdot \sum_{j=1}^{(p-1)/n} (-1)^j(\omega^{x+cn+1}) \]

**Proof:** See Appendix [3]

As in [12], we write

\[ \sum_{d=1}^{(p-1)/n} (-1)^d \sum_{a=0}^{n-1} \sum_{k=1}^{n-1} (\omega^{x+cn})(\omega^{x+cn+k}) \]

where \( \omega_{d,t} = \omega^{x+cn+k} \), and of course,

\[ \sum_{s=1}^{(p-1)/n} \omega^s_{d,t} = \begin{cases} p\! - \! 1 & \text{if } \omega_{d,t} = 1, \\ -1 & \text{otherwise} \end{cases} \]

To determine when \( \omega_{d,t} = 1 \), note that this happens (as in [12]) when \( t = nd - m + k_1(p-1) \). Since \( n \) is odd, \( m \) divides \( n \), so \( m \) must divide \( t \). This forces \( t = 0 \).

We now have \( \omega_{d,t} = 1 \) implies \( \omega^{nd} = -1 \pmod{p} \), and writing \( -1 = p^m \), yields \( nd = m + k_1(p-1) \) for some \( l \). This then gives \( d = (p-1)(2l + 1)/2n \), which we may write as \( (2l + 1) \) times \( (p-1)/2n \) (note again that since \( n \) is odd, \( n \) divides \( (p-1)/2 \)). Since \( d \) varies in the range \( 1, \ldots, (p-1)/n \), we find that \( l \) must be zero, that is, \( d = (p-1)/2n \). Thus, \( \omega_{d,t} = 1 \) precisely when \( t = 0 \) and \( d = (p-1)/2n \).
In particular, when \( t \neq 0 \) then \( \omega_{d,t} \neq 1 \) and we have that

\[
\text{Tr}_K/Q(x^\sigma(x)) = (-1)^{t^m} p \sum_{d=1}^{(p-1)/n} (-1)^d \sum_{s=1}^{(p-1)/n} \omega_d^s
\]

\[
= (-1)^{t^m} p \sum_{d=1}^{(p-1)/n} (-1)^d \langle -1 \rangle
\]

Once again, since \( n \) is odd, \( (p-1)/n \) is even, so the term \( \sum_{d=1}^{(p-1)/n} (-1)^d = 0 \). Thus, for \( t \neq 0, \text{Tr}_K/Q(x^\sigma(x)) = 0 \).

When \( t = 0 \), we find

\[
\text{Tr}_K/Q(x^\sigma(x)) = (-1)^m p \sum_{d=1, d \neq (p-1)/2n}^{(p-1)/n} \left[ (-1)^d \langle -1 \rangle \right]
\]

and the right side then yields \( p + p(p - 1) = p^2 \). To see this last fact, consider first the case where \( (p - 1)/2 \) is even (i.e., \( p \equiv 1 \mod 4 \)), Then, since \( n \) is odd, \( (p - 1)/2n \) is also even.

The sum \( \sum_{d=1}^{(p-1)/n} (-1)^d \) equals \( (p-1)/2n \), and we have already seen that, again because \( n \) is odd, \( \sum_{d=1}^{(p-1)/n} (-1)^d = 0 \). Thus the right hand side in the equation above for \( \text{Tr}_K/Q(x^\sigma(x)) \) indeed yields \( p^2 \) in this case.

We can similarly deal with the case when \( (p-1)/2 \) is odd (i.e., \( p \equiv 3 \mod 4 \)), to find that in both cases, indeed \( \text{Tr}_K/Q(x^\sigma(x)) = p^2 \) when \( t = 0 \). This proves the Proposition. \( \square \)

**APPENDIX III**

**PROOF OF LEMMA 10**

We wish to prove:

\[
\sum_{c=1}^{n-1} \sum_{a=0}^{n-1} (-1)^j \sum_{j=1}^{n-1} (-1)^j \omega^{a+c+n+j} = \sum_{c=1}^{n-1} (-1)^d \sum_{a=0}^{n-1} \sum_{k=1}^{n-1} \omega^{c+2a+n+j}.
\]

Set \( m = \frac{p-1}{n} \) and denote \( \mathbb{Z}/m\mathbb{Z} \) by \( \mathbb{Z}_m \). In the above equation, the dependence on \( c, j, d, k \) is only through their values (mod \( m \)) or through their values (mod \( 2 \)). If we assume \( 2|m \), which follows from the assumption that \( n \) is odd, we can then treat \( c, j, d, k \) as elements of \( \mathbb{Z}_m \). We thus have

\[
\sum_{c=1}^{n-1} (-1)^j \sum_{a=0}^{n-1} \sum_{j=1}^{n-1} (-1)^j \omega^{a+c+n+j}
\]

\[
= \sum_{c=1}^{n-1} \sum_{a=0}^{n-1} \sum_{k=1}^{n-1} \omega^{c+2a+n+k}.
\]

We now make the change of variables: \( c = d + k \) (mod \( m \)) which implies, since \( 2|m \), that \( c = d + k \) (mod \( 2 \)) and hence \( d = c - k = c + k \pmod{2} \). As the pair \((c, k)\) varies over all of \((\mathbb{Z}_m \times \mathbb{Z}_m)\), so does the pair \((d, k)\). We thus have

\[
\sum_{c=1}^{n-1} \sum_{a=0}^{n-1} \sum_{k=1}^{n-1} \omega^{c+2a+n+k} = \sum_{c=1}^{n-1} \sum_{a=0}^{n-1} \sum_{k=1}^{n-1} \omega^{c+2a+n+k}.
\]

**APPENDIX IV**

**PROOF OF APPROXIMATE UNIVERSALITY AND INFORMATION LOSSLESSNESS OF PERFECT CODES**

(THM 11)

The approximate universality part of the proof, is based on the derivation of the approximate universality conditions in [18]. It is reproduced here for completeness.

**Proof:** Let \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) and \( l_1 \geq l_2 \geq \ldots \geq l_n \) be the ordered eigenvalues of \( H^T H \) and \( \Delta X \Delta X^T \) respectively.

Irrespective of the statistics of the channel, in the high-SNR regime, the probability of no-outage at multiplexing gain \( r \), is shown in [3] to satisfy

\[
Pr(\text{no-outage}) = Pr \left\{ \sum l'_i > \ln(1 + \text{SNR}_r) \right\},
\]

where \( l' = \min(n, n_\mu) \). Through the Lagrange multiplier technique we determine

\[
d^2_{E, \text{worst}} = \inf_{\lambda_i} \sum_{i=1}^{n'} l_i \lambda_i
\]

by writing the functional as

\[
J(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n'} l_i \lambda_i + \mu \sum_{i=1}^{n'} \ln(1 + \text{SNR}_r \lambda_i) - \mu r \ln \text{SNR}
\]

and differentiating w.r.t. \( \lambda_i \), we obtain \( \lambda_i = (\mu / l_i - \text{SNR}^{-1})^{-1} \).

We then use the Kuhn-Tucker conditions to verify that the solution \( \lambda_i = (\mu / l_i - \text{SNR}^{-1})^{-1} \) is what gives the worst possible \( d^2_{E, \text{worst}} \) for \( \mu \) such that

\[
\sum_{i=1}^{n'} \ln(1 + \text{SNR}(\mu / l_i - \text{SNR}^{-1})^{-1}) = r \ln \text{SNR}.
\]

Solving the above, we obtain that

\[
\mu = \text{SNR}^{-(1 + \psi)} \prod_{i=1}^{n'} l_i^{-\frac{1}{\psi}} \text{ and thus } \lambda_i = \frac{\psi G}{l_i} - \frac{1}{\text{SNR}}.
\]

Substituting this value of \( \lambda_i \) in \( d^2_{E, \text{worst}} \) and setting \( d^2_{E, \text{worst}} > \text{SNR}' \) for some \( \epsilon > 0 \), we obtain a condition on the smallest \( n' \) eigenvalues of the code \( \prod_{i=1}^{n'} l_i > \text{SNR}^{-\frac{n'}{\epsilon}} \), a condition satisfied by CDA codes with non-vanishing determinant. [16].
Finally create

Starting with where the information losslessness and the entire theorem. This property, together with the full rate condition, establish to show that the linear dispersion matrices are unitary. It is easy to see that the unitary nature of makes each of the \(A_u\) unitary

\[
A_u = \begin{bmatrix}
A_{u,0} \\
\vdots \\
A_{u,n-1}
\end{bmatrix}
\]

Finally \(A_u\) is constructed as

It is easy to see that the unitary nature of \(\Gamma\) and \(G\) makes each of the \(A_u\) unitary

\[
A_u^\dagger A_u = I_n.
\]

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