Electro-vortex flow simulation using coupled meshes

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Abstract
A numerical model for simulating electro-vortical flows in OpenFOAM is developed. Electric potential and current are solved in coupled solid-liquid conductors by a parent-child mesh technique. The magnetic field is computed using a combination of Biot-Savart’s law and induction equation. Further, a PCG solver with special regularisation for the electric potential is derived and implemented. Finally, a performance analysis is presented and the solver is validated against several test cases.

Keywords: electro-vortex flow, OpenFOAM, coupled parent child mesh

1. Introduction

Electro-vortex flow is highly relevant in many industrial processes. Possible applications span from electromagnetic stirring [1] for grain size reduction in solidification [2, 3] over electrode welding [4], electroslag welding, electroslag (re-)melting [5, 6], vacuum arc melting [7] to electrolytic reduction (of e.g. aluminium [8]). Further, many technical devices, as liquid fuses [9], electric jet engines, arc furnaces [10] and liquid metal batteries [11-13] involve or rely on electro-vortex flows. For an overview about such flows, see [14-16].

Electro-vortex flow is not an instability. It develops at (or near) a changing cross-section of a (liquid) conductor. Radial currents produce, together with their own magnetic field, a Lorentz force, which is non-conservative, i.e. its curl is not equal to zero. This force cannot be compensated totally by a pressure
gradient and therefore drives a flow. For an illustrative example, see Shercliff [17].

Numerical simulation of electro-vortex flow is easy when modelling only the fluid, or a non-conducting obstacle inside a fluid. However, in most realistic cases, electric current passes from solid to liquid conductors and vice versa. The electric potential in these regions must therefore be solved in a coupled way. The classical, segregated approach means solving an equation in each region, and coupling the potential only at the interfaces by suitable boundary conditions [11]. While that is easy to implement, convergence is rather poor. An implicit coupling of the different regions by block matrices is a sophisticated alternative for increasing convergence [18]. However, it is memory-intensive and by no means easy to implement.

In this article we will present an alternative effective option for region coupling in OpenFOAM. We solve global variables (electric potential, current density) on a global mesh with a variable electric conductivity according to the underlying material. We then map the current density to the fluid regions and compute the electromagnetic induced flow there. This parent-child mesh technique was already used for the similar problem of thermal conduction [19, 20] and just recently for the solution of eddy-current problems with the finite volume method [21].

2. Mathematical and numerical model

2.1. Overview

The presented multi-region approach is based on a single phase incompressible magnetohydrodynamic (MHD) model [11, 22]. The flow in the fluid is described by the Navier-Stokes equation (NSE)

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \nu \Delta \mathbf{u} + \frac{J \times B}{\rho},
\end{align*}
\]

with \( \mathbf{u} \) denoting the velocity, \( t \) the time, \( p \) the modified pressure, \( \nu \) the kinematic viscosity and \( \rho \) the density. The fluid flow is modelled as laminar only; adding
a turbulence model is planned for the future. We split the electric potential \( \phi \),
the current density \( J \) and the magnetic field \( B \) into a constant (subscript 0)
and induced part (lower case) as

\[
\phi = \phi_0 + \varphi \\
J = J_0 + j \\
B = B_0 + b.
\]

In order to determine the distribution of the constant part of the electric potential \( \phi_0 \) we solve a Laplace equation for the electric potential

\[
\nabla \cdot \sigma \nabla \phi_0 = 0
\]

on the global mesh. The above equation is obtained starting from the Kirchhoff
law of charge conservation (\( \nabla \cdot J_0 = 0 \)) and \( J_0 = -\sigma \nabla \phi_0 \). Note that the
conductivity \( \sigma \) is a field and not a constant, because the equation is solved on
the full geometry. During mesh generation, it is ensured that the border between
two materials always coincide with a face between two neighbouring cells. The
global current density is then calculated as

\[
J_0 = -\sigma \nabla \phi_0
\]

and mapped to the fluid region. Afterwards, the constant magnetic field is
determined as described in section 2.1.1 only in the fluid.

Often it is sufficient to calculate only the constant current and magnetic field.
Nevertheless, our solver also allows to compute their induced counterparts, e.g.
for simulating the Tayler instability \[23–31\]. The scheme is similar to that
described above: in a first step, the induced electric potential \( \varphi \) is determined
by solving a Poisson equation

\[
\nabla \cdot \sigma \nabla \varphi = \nabla \cdot \sigma (u \times B)
\]

after mapping the source term \( u \times B \) to the global mesh. The induced current
can be computed taking into account Ohm’s law

\[
\dot{j} = \sigma (-\nabla \varphi + u \times B).
\]
After mapping $j$ to the fluid mesh we determine the induced magnetic field as described in section 2.1.1.

Our model is not capable of describing AC currents, because we use the quasi-static approximations by neglecting the temporal derivation of the vector potential ($da/dt = 0$) and magnetic field ($db/dt = 0$) [32]. For a detailed flowchart of the model, please refer to figure 1.
2.1.1. Computation of the magnetic field

For the computation of both, the constant part of the magnetic field $B_0$ and its induced counterpart $b$ we use the inversion of Ampère’s law, the Biot-Savart integral

$$ B(r) = \frac{\mu_0}{4\pi} \int \frac{J(r') \times (r - r')}{|r - r'|^3} dV' $$

(9)

to determine both from the current density $J$. This integro-differential approach was proposed by Meir and Schmidt [33–38] and later used for describing dynamos [39–41] and the Tayler instability [22].

In order to obtain the magnetic field in one single cell (at the position $r$), the electric current densities of all other cells (at the position $r'$) have to be integrated. The number of operations is therefore equal to the number of cells squared. This way of computation is extremely costly. We will explain here several ways for a speed up of the procedure. Solving Biot-Savart’s integral on a coarser grid, recalculating it every $n$th time step, and an appropriate parallelisation [22] are the most simple ways.

The parallelisation is implemented in OpenFOAM using MPI. Basically, each processor contains only the current density of its local cells. With this, it computes the magnetic field for the full geometry (see figure 2). Finally, the field $B$ of each cell has to be summed up over all processors. This might be done using the MPI function ALLREDUCE, resulting in a correct and global $B$ on all processors. However, this is not necessary, because a single processor needs only its local $B$ for further computation. Therefore, each processor receives only its local magnetic field from all other processors and adds up all contributions given. The communication process is illustrated in figure 2.

Increasing the speed-up considerably is possible by computing Biot-Savart’s integral only on the boundaries and solving the induction equations [42, 43]

$$ 0 = \Delta B_0 $$

(10)

$$ 0 = \frac{1}{\sigma \mu_0} \Delta b + \nabla \times (u \times B_0) + \nabla \times (u \times b) $$

(11)

for the constant and induced magnetic field in the quasi-static limit [32].
An even faster alternative is shifting the problem from the magnetic field $\mathbf{B}$ to the vector potential $\mathbf{A}$ using the relation $\mathbf{B} = \nabla \times \mathbf{A}$. Similar to Biot-Savart’s law for $\mathbf{B}$, the vector potential can be determined by Green’s identity [44]:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.$$

Please note that this formula is much cheaper to compute than Biot-Savart’s law (equation 9) [45, 46].

The transport equations for the vector potential are derived from Ampère’s law, $\mathbf{B} = \nabla \times \mathbf{A}$, Ohm’s law [47] and using the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ as

$$0 = \frac{1}{\sigma \mu_0} \Delta \mathbf{A}_0 - \nabla \phi_0$$

$$0 = \frac{1}{\sigma \mu_0} \Delta \mathbf{a} + \mathbf{u} \times \mathbf{B}_0 + \mathbf{u} \times (\nabla \times \mathbf{a}) - \nabla \varphi.$$

Basically all mentioned approaches of determining $\mathbf{B}$ based on the equations [9] till [14] are equal from a physical point of view. But due to the way they are discretised and numerically solved, there will be differences in both accuracy and calculation time. While being the most expensive method, calculating the magnetic field by means of Biot-Savart’s law also gives the most accurate result. This stems from the fact that the integral equation [9] represents an exact solution for $\mathbf{B}$ which is only numerically integrated for a finite number of cells. As already mentioned, a computationally less expensive evaluation can be achieved with the help of the magnetic vector potential $\mathbf{A}$ and Green’s identity.
where the complexity of the integrand is reduced compared to equation (9). Despite of equation (12) also being an exact solution, the subsequent calculation of $B = \nabla \times A$ introduces an additional layer of discretisation errors from cell averaging and face interpolation.

As outlined above, Biot-Savart’s law may be used also in combination with equations (10) and (11) or Green’s identity (12) combined with equations (13) and (14), while only boundary values of $B$ or $A$ are evaluated using the exact integral equations. Internal values are then recovered from solving the related differential equations. This drastically improves computational efficiency at the cost of some accuracy. However, from figure 1 one can comprehend that it is sufficient to calculate $B_0$ or $A_0$ only once at the beginning of a simulation, whereas $b$ or $a$ needs to be updated recurringly while marching in time. The most promising way of determining the total magnetic field $B$ is thus to compute its static part $B_0$ once and solely using Biot-Savart’s law with the current density $J_0$ and the induced part $b$ mediately from $b = \nabla \times a$, whereby the solution of the induced magnetic vector potential $a$ is in turn based on the transport equation (14) for which Dirichlet boundary conditions are derived from evaluating Green’s identity with the current density $j$. This approach has been used for all following calculations.

In this way, $B_0$ is most accurate and $b$ is repeatedly calculated with minimum computational effort. Another important advantage of this realisation is that the solenoidal nature of $B$ is implicitly satisfied, as $B_0$ results from an exact solution in shape of Biot-Savart’s law and $b$ is calculated from the definition of the induced vector potential $a$ with $\nabla \cdot b = \nabla \cdot (\nabla \times a) \equiv 0$. Numerically, Gauss’s law $\nabla \cdot B = 0$ is of course only met approximatively due to discretisation errors. With linear interpolation, the corresponding finite volume approximation is second order accurate. Additionally, for small magnetic Reynolds numbers, which are typical for most liquid metal MHD flows on laboratory scale, $b$ is usually small compared to $B_0$.

As opposed to using $b = \nabla \times a$, a direct solution of the induction equation (11) for $b$ would in general require additional steps to ensure its solenoidal
property. This is particularly true in ideal or plasma MHD for time-dependent problems at high magnetic Reynolds numbers, where the induction equation is dominated by convection \cite{48}. For such cases it is usually necessary to adopt a special correction. An overview of possible divergence cleaning methods can be found in \cite{49}. The OpenFOAM standard solver mhdFoam for example uses the projection method, which is well known from the pressure-velocity coupling of the Navier-Stokes equations.

Accordingly, also the solution of equation (14) would generally require some correction to maintain the solenoidal property of the magnetic vector potential $a$. However $\nabla \cdot b = 0$ is a physical requirement, whereas $\nabla \cdot a = 0$ is just a gauge. If the Helmholtz decomposition $a = \nabla \times \Pi + \nabla \Psi$ is consulted, it can be easily seen that $\nabla \cdot a = \nabla \cdot \nabla \Psi$ does not influence the magnetic field $b = \nabla \times a = \nabla \times \nabla \times \Pi$. Thus, even a weakly satisfied Coulomb gauge $\nabla \cdot a \approx 0$ would suffice to correctly represent $b$. Fixing the gauge is merely important to achieve a unique solution for $a$. Moreover, for diffusively dominated cases the Coulomb gauge may be incorporated directly into Ampère's law according to $\nabla \times \nabla \times a - \nabla (\nabla \cdot a) = -\nabla \cdot \nabla a = \mu_0 j$ as long as charge conservation $\nabla \cdot j = 0$ is satisfied \cite{50}. This requirement is met with the solution of $7$. In equation (14), the sum $u \times B_0 + u \times (\nabla \times a) - \nabla \phi$ corresponds to the induced current density $j/\sigma$ \cite{8}. If we explicitly discretise these terms, they can be regarded as one source term for a Poisson equation, whose system matrix is symmetric. For cases like this, it was demonstrated in \cite{21} that indeed no additional divergence cleaning for $a$ is required.

3. Discretisation

Special attention must be paid to the discretisation of the Laplace term $\nabla \cdot (\sigma \nabla \phi)$ of equation (5) and (7) because of the sharp jump in conductivity between different materials. This jump is not smeared, but exactly reproduced in our model. A linear interpolation of $\sigma$ would lead to a wrong potential near the interface.
For a consistent application of the Gauss theorem to discretise the equations (see \cite{51}), the electric conductivity is interpolated harmonically. Knowing that the potential $\phi_f$ and the normal current $(j \cdot n)_f$ must be continuous from a cell $P$ to its neighbour $N$, we find the conductivity at the face $f$ to be

$$\sigma_f = \left( \frac{\delta P / \delta}{\sigma_P} + \frac{\delta N / \delta}{\sigma_N} \right)^{-1}$$

with $\delta$ denoting the distance cell centre - face and $\delta$ the distance between both cell centres. In the quasi-static limit, this exactly matches the embedded discretisation scheme which was derived in \cite{21} to get a proper discretisation of the Laplacian. For a more detailed discussion and similar discretisation of the thermal conductivity, see \cite{52–55}.

Secondly, care must be taken when computing the gradient of the potential to determine the current density as $J = -\sigma \nabla \phi$. In order to be able to use the Gauss theorem for discretisation, the electric potential on the faces must be determined. Using the same assumptions as for the harmonic interpolation described above, we identify the electric potential at the face as

$$\phi_f = w\phi_P + (1 - w)\phi_N$$

with the interpolation weight

$$w = \frac{\delta_N \sigma_P}{\delta_P \sigma_N + \delta_N \sigma_P}.$$  

As before, this interpolation scheme corresponds to the embedded discretisation of the gradient from \cite{21} in case of the quasi-static assumption. All other discretisation schemes do not need special attention. We use backward differencing for time discretisation, a mixed linear-upwind scheme for the convective term in equation (1) and second order linear discretisation for all other schemes.

4. Equation solvers

The solution procedure of our model is illustrated in figure 1. As the Navier-Stokes equation is discretised and solved by means of the PISO-algorithm \cite{54},
four different Poisson equations need to be addressed. This comprises the Laplace equation for the static potential $\phi_0$, one Poisson equation for the vector potential $a$, one Poisson equation for the potential $\varphi$ and another Poisson equation for the fluid pressure $p$. Especially the latter two are most commonly solved for Neumann boundary conditions. To improve the overall robustness of the solution process in connection with the employed parent-child mesh approach, we have implemented an alternative regularisation technique for the iterative equation solvers in OpenFOAM, which is briefly explained in the following.

The discretisation of a Poisson equation leads to a linear equation system

$$M\psi = \mathbf{r}, \quad (18)$$

where $M \in \mathbb{R}^{n \times n}$ is a symmetric positive semi-definite matrix, $\psi \in \mathbb{R}^n$ is the discrete solution vector for either $\varphi$ or $p$, and the right-hand side $\mathbf{r} \in \mathbb{R}^n$ mainly represents the inhomogeneous part. Each row of the system (18) is related to one of $n$ cells. In case of a Neumann problem, the system matrix will be singular and the solution is only defined up to an additive constant vector. More specifically, the one-vector $\mathbf{1} = (1, 1, \ldots, 1, 1)^T$ lies in the null space of the linear map $M\psi$. In other words, $\mathbf{v}_1 = 1/\sqrt{n}$ is a normalized eigenvector corresponding to the eigenvalue $\lambda_1 = 0$ in accordance with the identity $(M - \lambda_1 I)v_1 = 0$.

In OpenFOAM such a singular matrix $M$ is regularised by means of adding the equation

$$c_R \psi_P = c_R \psi_R \quad (19)$$

to the row which belongs to cell $P$, where $c_R$ is initially an arbitrary coefficient, $\psi_P$ is the unknown solution and $\psi_R$ is a reference solution for that cell. In order to slightly increase diagonal dominance of $M$, $c_R$ is usually set to the diagonal coefficient of the matrix before adding the equation: $c_R = m_P$. By specifying the reference value $\psi_R$, the solution gets locally constrained in a weak sense. This approach is however extremely sensitive to the smallest errors in the corresponding compatibility condition of the Neumann problem. Such numerical errors may arise from the data exchange between child and parent mesh due to interpolation.
A much more robust regularisation can be achieved by inverting the idea of the so called Hotelling deflation \[57\], which is actually a simple technique to solve eigenproblems by selectively shifting single known eigenvalues of a matrix to zero. Conversely, we may use the same procedure to shift them also from zero to an arbitrary value, thus inflating the matrix.

According to the spectral theorem for symmetric matrices \[58\], it is possible to decompose $M$ based on its eigenvalues $\lambda_k$ and orthonormal eigenvectors $v_k$:

$$M = \sum_{k=1}^{n} \lambda_k v_k v_k^T = \lambda_1 v_1 v_1^T + \sum_{k=2}^{n} \lambda_k v_k v_k^T. \quad (20)$$

Using this decomposition we may then create a non-singular matrix $\tilde{M}$ using only $v_1$ from above:

$$\tilde{M} = M + \tilde{\lambda}_1 v_1 v_1^T = M + \frac{1}{n} \mathbf{1} \mathbf{1}^T, \quad (21)$$

where $\tilde{\lambda}_1$ is any non-zero eigenvalue replacing $\lambda_1$. It is important to note that $\tilde{M}$ does not preserve the original sparsity pattern of $M$, which is usually undesired. Hence, a direct manipulation would not only mean a waste of memory, but also a contraction in terms of the face addressing of OpenFOAM. However, we may include the modification indirectly when computing the matrix-vector product:

$$\tilde{M}\psi = M\psi + \tilde{\lambda}_1 \frac{1}{n} \mathbf{1} \mathbf{1}^T \psi = M\psi + \frac{1}{n} \sum_{k=1}^{n} \psi_k \mathbf{1}, \quad (22)$$

which is essentially the kernel of any iterative equation solver \[59\]. Furthermore parallelisation is straight-forward as the exchange of the rightmost sum does only require little communication.

Taking the properties of $M$ into consideration, it can be shown that all of its eigenvalues are smaller or equal to twice the maximum of its diagonal coefficients. Therefore we use the diagonal mean as modified eigenvalue $\tilde{\lambda}_1 = (m_P)$, thus preserving the spectral radius of $M$. Tests with the preconditioned CG-method \[59\] showed that the smoothness of the numerical solution is preserved even if errors in the compatibility condition exist. It is exactly this preservation of smoothness which distinguishes our method from the original regularisation technique in OpenFOAM, and which makes our method superior.
5. Results

5.1. Test case 1: speed-up of Biot-Savart’s law

In this section we present a performance analysis of the magnetic field computation in a cylindrical geometry with an imposed current density $J$ (the other parts of the solver are switched off). The speedup and scaling analysis is carried out on a cluster with Intel 8-Core Xeon 3.3 GHz CPUs cross linked with 40 Gbit/s Infiniband. The solvers are compiled with OpenFOAM 2.2.0 and MPI 1.6.3.

In a first step we solve only Biot-Savart’s law (equation 9) for all cells and boundary faces – on a changing number of processors. The test case contains 352 000 cells. Figure 3a shows a good scaling up to 64 processors. The communication time is 28% when using all 64 processors. In that case a single processor contains only 5500 cells.

In a second test case, we use the same configuration again and compare the full Biot-Savart integral with the method of solving the induction equation (10). For the latter, we compute Biot-Savart’s law only on the patches in order to obtain the correct boundary conditions. Figure 4 shows the relative computation times (total cpu time/(cpu time for simulation in one processor)·100%) for one to 16 processors. The method of using the Biot-Savart law on the boundary regions only together with the solution of the corresponding induction equation...
in the inner region scales very well, too; it is approximately 13.5 times faster than the volume Biot-Savart method. Note that this factor will probably increase for larger problems with more cells.

In a third case we use a mesh with 63 200 cells and compare the magnetic field with the vector potential approach. In both cases we firstly compute the boundary conditions and solve then a transport equation for $A$ or $B$ on a single processor 50 times. The fastest result we obtain by using Biot-Savart for the vector potential (equation 12 and 13). Computing the magnetic field on the boundary and solving the induction equation (9 and 10) is five times slower. The volume-based Biot-Savart is 84 times slower. Of course this holds only for the Biot-Savart calculation; the differences for the whole solver, where the flow simulation is included, will be smaller.

5.2. Test case 2: current distribution in 2D

In a second test case the discretisation schemes for electric conductivity and potential are validated by comparison with the commercial software Opera. We simulate a simple two-dimensional geometry ($1 \times 2 \times 0.1$ m), consisting of two conductors of very different conductivity with an inclined surface (inclination 45°) – see figure 5a. A vertical current of 1 A is applied. Figure 5b shows the equipotential lines, figure 5c the current lines and 5d the disturbed current. As
expected, the current lines concentrate in the area of high conductivity. Figure 5 shows the conductivity distribution (a), electric potential (b), complete current density (c) and disturbed current density (d). The applied electrical current of 1 A if flowing upwards.

Figure 6 shows the electric potential and the currents along a vertical and horizontal centred line. The result of OpenFOAM and Opera match very well. Obviously, Opera uses Dirichlet boundary conditions for the electric potential (i.e. an equipotential surface) – so the same was done in OpenFOAM.

5.3. Test case 3: electro-vortex flow in a cylindrical geometry

Several model experiments \cite{60,62} and similar analytical solutions \cite{63,64} of electro-vortex flow are known from literature with most of them unfortunately lacking detailed information. Here we will study the well reviewed example of a
thin electrode touching a cylindrical bath of liquid metal \[65,67\]. The experiment was conducted at the Institute of Physics in Riga and published by Zhilin et al. \[68\]. Figure 7 illustrates the setup: a horizontal current passes through a cylindrical bath of liquid mercury (colored in blue). One copper electrode covers the whole surface, the other is reduced to a small rod. The whole experiment is embedded into a steel pipe; two mercury filled “buffer zones provide for a smooth current transition” between external wires and the experiment.

The axial velocity along the cylinder axis is measured with a spacing of 1 mm in \(x\)-direction at \(y = 0\). The current is increased to up to 1500 A.

Unfortunately, the article does not provide any details about the external current leads. They are therefore assumed to be infinitely long. The measurements colored in red (fig. 7) were not quoted by Zhilin et al. \[68\], but estimated from the sketch. Similarly, the material properties were not given; they may vary considerably depending on the exact material/alloy. We assume the copper conductivity to be \(\sigma_{Cu} = 58.5 \cdot 10^6\) S/m, the conductivity of mercury as \(\sigma_{Hg} = 1.04 \cdot 10^6\) S/m, its density as \(\rho_{Hg} = 13534\) kg/m\(^3\) and its kinematic viscosity as \(\nu = 1.2 \cdot 10^{-7}\) m\(^2\)/s \[69,70\]. The tube is made of “stainless steel”; we assume therefore an electric conductivity of \(\sigma_{St} = 1.4 \cdot 10^6\) S/m which is typical for X5CrNi18-10. The tube works as potential divider – only a part of the current passes through the mercury/copper.

Figure 8a shows the general flow structure (grid resolution in Hg 0.5 mm). Assuming infinitely long lateral current leads and neglecting external magnetic

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Figure 7: Sketch of the experiment of Zhilin et al. \[68\]. The experiment is modelled with thick lateral current collectors which are 3 m long (but not shown in the image). The red dimensions (in mm) are estimated. The working section (blue) with the symmetry axis \(x\) is filled with liquid mercury.
fields, we expect exactly such a symmetric flow. Further we expect the velocity along the cylinder axis to be approximately uniform in the middle of the test section (see fig. 8b) as long as the current is not extremely low. The simulated curves for $I = 150 \ldots 300$ A fit very well to the measured velocity values (dotted). A certain deviation can be explained by the many unknown experimental parameters; especially the length of the rod has a certain influence on the magnitude of the flow (for details, see Appendix A).

6. Summary and outlook

We have developed a solver for electro-vortical flow, using a mesh mapping method. Arbitrary solid and fluid conductors are fully coupled. Electric potential and current density are solved on a global mesh, and copied to the fluid mesh. This parent-child mesh technique is much faster than the classical segregated approach. An improved regularisation technique for the solution of the
The Poisson equation of the electric potential is presented. The magnetic field is computed fully parallelly using Biot-Savart’s law. This was shown to be efficient at least up to 64 processors. Calculating Biot-Savart’s law only on the boundaries and solving a corresponding induction equation in the fluid region speeds up the magnetic field computation drastically. A first validation of the solver was done using the commercial software Opera and by comparison with experimental data.

The solver presented can easily cope with up to 1 million cells. For larger simulations, a multigrid method or a coarser grid for the magnetic field computation might be necessary. Further, the solver shall be enhanced by a turbulence model and will be further validated using recent experimental data. For a meaningful comparison to experimental data, all dimensions of the setup and all conductivities of the conductors as well the placement of the feeding lines and possible magnetic background fields must be known. Only in that case a computation of the experimentally investigated case can successfully be performed. We aim to use the solver to study electro-vortex flow in liquid metal batteries [13, 71, 72] and aluminium reduction cells [73] as well as for related experiments [74, 75]. In the long term, a comparison between a segregated, a block matrix coupled (not existing in OpenFOAM yet) and the here presented electro-vortex flow solver is planned.

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Appendix A. Detailed simulation of test case 3

In this section we present further details of test case 3. Figure A.9 shows the grid study: at a current of 200 A, 60 cells on the diameter show convergence of the jet velocity. Figure A.10 shows the steady state jet velocity at 200 A using an old segregated [11] and the here developed solver. The results match perfectly; the coupled solver was more than 10 times faster than the segregated one.

Finally, we illustrate in figure A.11 the jet velocity at 200 A for a variation of the copper and steel electrical conductivity and the length of the rod. These
properties / measurements are not exactly known from the experiment. While the exact conductivity of the copper conductors is negligible, the conductivity of the steel tube and the length of the rod change the steady state velocity. We illustrate further the influence of the induced current and the Earth magnetic field. We see, that the induced current and magnetic field are negligible. However, the (vertical) Earth magnetic field changes not only the speed of the jet, but also its shape. We suspect, that other vertical magnetic fields (from the feeding lines) may have an additional influence on the jet.
Figure A.11: Steady state flow velocity along the jet for $I = 200$ A. In (a) the conductivity of the copper electrodes is varied, in (b) the conductivity of the steel tube. A variation of the length of the rod (c) has a significant influence on the flow speed. Finally, (d) illustrates the influence of the Earth magnetic field and the induced current.