Straggler-Resilient Differentially-Private Decentralized Learning

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Abstract—We consider the straggler problem in decentralized learning over a logical ring while preserving user data privacy. Especially, we extend the recently proposed framework of differential privacy (DP) amplification by decentralization by Cyffers and Bellet to include overall training latency—comprising both computation and communication latency. Analytical results on both the convergence speed and the DP level are derived for both a skipping scheme (which ignores the stragglers after a timeout) and a baseline scheme that waits for each node to finish before the training continues. A trade-off between overall training latency, accuracy, and privacy, parameterized by the timeout of the skipping scheme, is identified and empirically validated for logistic regression on a real-world dataset and for image classification using the MNIST and CIFAR-10 datasets.

I. INTRODUCTION

In distributed learning, a finite-sum optimization problem is solved across multiple nodes without exchanging the local datasets directly, thus improving user data privacy and reducing the communication cost. A popular instance of distributed learning is federated learning [2]–[4] in which there is a single central sever coordinating the training process. On the other hand, in fully decentralized learning, see, e.g., [5], [6], there is no such coordinating central server—the nodes maintain a local estimate of the optimal model and iteratively update it by averaging estimates obtained from neighbors corrected on the basis of their local datasets. There are two modes of operation—sequential and parallel—and theoretical studies show that the physical communication topology has a strong impact on the number of epochs needed to converge [7].

It is well-known by now that the computed partial (sub)gradients can leak information on the local datasets [8]. In order to circumvent this, a carefully selected noise term can be added to the computed partial (sub)gradients before they are transmitted to other nodes, referred to as local differential privacy (LDP) [9], [10]. In fully decentralized learning, nodes have only a local view of the system. Hence, Cyffers and Bellet [11] recently proposed a novel relaxation of LDP, referred to as network DP (NDP), to naturally capture this. Furthermore, they showed that the privacy-utility trade-off under NDP can be significantly improved upon compared to what is achievable under LDP, illustrating that formal privacy gains can be obtained from full decentralization, complementing previous notions of “amplifying” the privacy by shuffling, subsampling, and iteration [12]–[15]. Differentially-private decentralized learning has been considered in several previous works, see, e.g., [6], [16], [17].

The problem of straggling nodes, i.e., nodes that take a long time to finish their tasks due to random phenomena such as processes running in the background and memory access, has been broadly studied in the literature. The ignoring-stragglers strategy, i.e., ignoring results from the slowest nodes, see, e.g., [6], [18], is simple and popular, but can lead to convergence to a local optimum when the data is heterogeneous [19], [20]. To provide resiliency against straggling nodes, e.g., coded computing methods have recently been proposed [21]–[30].

In this work, we study the impact of stragglers and user data privacy in decentralized training. In particular, we assume an underlying physical full mesh topology, i.e., all nodes can physically communicate with each other, but sequential training along a logical ring on top of the physical topology where each node communicates only with its immediate neighbors upstream and downstream. In sequential training, nodes do not need to be active during the whole training period, which makes it suitable for scenarios where the nodes have limited resources, and therefore remain dormant unless they are triggered to do an update. See also [31], [32] for further motivation for this scenario. For this setting, we extend the recently proposed framework of privacy amplification by decentralization by Cyffers and Bellet [11] to include the overall latency—comprising both computation and communication latency—under LDP, illustrating that formal privacy gains can be obtained from full decentralization, complementing previous notions of “amplifying” the privacy by shuffling, subsampling, and iteration [12]–[15]. Differentially-private decentralized learning has been considered in several previous works, see, e.g., [6], [16], [17].

We study a skipping scheme (which ignores the stragglers after a timeout) and a baseline scheme that waits for each node to finish its computation before the training continues, for a fixed and a randomized ring topology, and derive analytical results on the convergence behavior (see Theorem 1 and the DP level (see Theorems 2 and 3), revealing a trade-off parameterized by the timeout of the skipping scheme. We show that the asymptotic convergence rate is equal to that of 33, Thm. 2. We note that the presented proofs in Appendices A and B require several nontrivial steps which can not be found in previous work, e.g., the asymptotic convergence analysis in Appendix A-D and the adaption to a decreasing learning rate in Appendix E. See also the first paragraph of Section IV. Moreover, we emphasize again that this
work studies the effect of stragglers, which by itself is
 novel for the considered scenario.

- The optimal timeout that minimizes the time between two
  consecutive updates of the token is determined, showing that
  skipping is beneficial for faster convergence for certain
  popular computational delay models considered
  in the literature (see Lemma 2 and Section VI-B).

- We show that randomizing the processing order of nodes
  on the ring yields an improvement in both convergence
  behavior and privacy in the long run (see Section VI-A),
  although the error and the privacy level show the same
  order-wise asymptotic behavior in the number of update
  steps with and without randomization (see Remark 1).

Finally, we present extensive empirical results for both
logistic regression on a binarized version of the UCI
housing dataset [33] and for image classification using both the
MNIST [35] and CIFAR-10 [36] datasets to validate our
theoretical findings.

II. PRELIMINARIES

A. Notation

We use uppercase and lowercase letters for random variables (RVs)
and their realization (both scalars and vectors), respectively, and italics
for sets, e.g., \( X, x, \) and \( X' \) represent a
RV, a scalar/vector, and a set, respectively. An exception to this
rule is \( \tau \) which denotes the model description, also referred to
as the token. Matrices are denoted by uppercase letters, their
distinction from RVs will be clear from the context. Vectors are
represented as row vectors and the transpose of a vector or a matrix
is denoted by \((\cdot)^T\). The expectation of a RV \( X \) is
denoted by \( \mathbb{E}[X] \). We define \( [n] \triangleq \{1, 2, \ldots, n\} \), while \( \mathbb{N} \)
denotes the set of natural numbers and \( \mathbb{R} \) the set of real numbers.
The (sub)gradient of a function \( f(x) \) is denoted by \( \nabla f(x) \),
while the \( p \)-norm of a length-\( n \) vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \)
\( \|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \) where \( |\cdot| \) denotes
absolute value. The base of the natural logarithm is denoted by \( e \),
while \( \log \) denotes natural logarithm. \( \mathcal{N}(\mu, \sigma I_d) \) denotes the
d-\( d \)-dimensional Gaussian (uncorrelated) distribution with mean \( \mu \)
and standard deviation \( \sigma \) of each component, where \( I_d \) is the
identity matrix of size \( d \). \( X \sim \mathcal{P} \) denotes that \( X \) is distributed
according to the distribution \( \mathcal{P} \), while \( x \sim \mathcal{P} \) denotes a sample
\( x \) taken from the distribution \( \mathcal{P} \). We denote by \( D \sim_u D' \)
the fact that datasets \( D = \bigcup_{v \in V} D_v \) and \( D' = \bigcup_{v \in V} D'_v \)
are the same except perhaps for the dataset of the user \( u \), i.e.,
\( D_v = D'_v \) for all \( v \neq u \), where \( V \) is some set of users. Standard
order notation \( O(\cdot) \) is used for asymptotic results.

B. Definitions

Definition 1 (\( k \)-Lipschitz continuity). A function \( f : \mathcal{W} \to \mathbb{R} \)
is \( k \)-Lipschitz continuous over the convex domain \( \mathcal{W} \subseteq \mathbb{R}^d \) if
\[ |f(w) - f(w')| \leq k\|w - w'\|_2 \text{ for all } w, w' \in \mathcal{W}. \]

Definition 2 (\( \beta \)-smoothness). A function \( f : \mathcal{W} \to \mathbb{R} \)
is \( \beta \)-smooth over the convex domain \( \mathcal{W} \subseteq \mathbb{R}^d \) if
\[ \|\nabla f(w) - \nabla f(w')\|_2 \leq \beta \|w - w'\|_2 \text{ for all } w, w' \in \mathcal{W}. \]

C. System Model

Consider a decentralized network of \( n \) honest-but-curious
nodes (users) \( V = \{v_1, \ldots, v_n\} \) with a decentralized dataset
\( D = \bigcup_{v \in V} D_v \) where \( D_v = \{(x_i^{(v)}, y_i^{(v)})\}_{i=1}^{\tau_v} \), \( (x_i^{(v)}, y_i^{(v)}) \in \mathcal{R} \subseteq \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \), for some set \( \mathcal{R} \) and \( d_x, d_y, \kappa \in \mathbb{N} \), is
the private dataset of node \( v \in V \).

The nodes want to compute some function together based on
their datasets but want to keep their datasets private. For that,
they employ a decentralized protocol where a token \( \tau \)
for some convex set \( \mathcal{W} \subseteq \mathbb{R}^d \), travels between the nodes
according to some predefined (but potentially randomized)
path. When receiving the token the \( r \)-th time and the global
time is \( h \), the node \( v \) updates it as \( \tau \leftarrow g^{(v)}(\tau; \text{state}_v(h)) \),
and sends it further. Here, \( \text{state}_v(h) \) encapsulates all the
information available to the node \( v \) at time \( h \), e.g., the available
data points and the results of previous calculations. It can
also include some source of randomness. We assume that
the computation in each node \( v \) during the \( r \)-th visit of the token
takes random time \( T^{(v)}_r \). Hence, the computation of \( g^{(v)}(\cdot, \cdot) \)
takes time at most \( T^{(v)}_r \) as the token may be updated before the
entire computation is finished. We consider a model where
\( T^{(v)}_r \) is comprised of a deterministic constant part (the time it
takes for an actual computation) and a random part. Also,
we assume that communication between any two nodes is
noiseless and takes constant time \( \chi \), and hence the constant
part of the computation time can be set to zero. At the end of
the protocol, the token \( \tau \) is distributed among the nodes, which
allows for calculating the desired result. This final distribution
takes constant overhead time and is therefore ignored.

For a decentralized protocol \( \mathcal{A} \), we denote by \( \mathcal{A}(D) \) the (random)
transcript of all messages sent
or received by all the users, i.e., \( \mathcal{A}(D) \) equals
\[ \{(u, w, v) : u \in V \text{ sent a message with content } w \text{ to } v \in V\}. \]
However, due to the decentralized nature of \( \mathcal{A} \), the user \( v \)
only has access to the subset of \( \mathcal{A}(D) \) consisting of the
messages she sent or received, and we denote this view by
\( \mathcal{O}_v(\mathcal{A}(D)) = \{(u, w, u') \in \mathcal{A}(D) : u = v \text{ or } u' = v\} \). Let \( \Omega \)
denote the set of all possible views, i.e., \( \mathcal{O}_v(\mathcal{A}(D)) \subseteq \Omega \)
for all possible parameters and realizations.

D. Network Differential Privacy

We accept the notion of NDP introduced in [11].

Definition 3 (NDP [11]). A protocol \( \mathcal{A} \) satisfies (\( \varepsilon, \delta \))-NDP if
for all pairs of distinct users \( u, v \in V \), all pairs of neighboring
datasets \( D \sim_u D' \), and any \( \mathcal{S} \subseteq \Omega \), we have
\[ \Pr[\mathcal{O}_v(\mathcal{A}(D)) \in \mathcal{S}] \leq e^\varepsilon \Pr[\mathcal{O}_v(\mathcal{A}(D')) \in \mathcal{S}] + \delta, \]
\( ^2 \)The RVs \( T^{(v)}_r \) are assumed to be independent and identically distributed
(i.i.d.) which is in accordance with the literature, where typically stragglers
are generated uniformly at random, except for a few works, e.g., [23, 27]
that consider a model where a node that becomes a straggler tends to remain
a straggler for a long time, violating the i.i.d. assumption on the RVs \( T^{(v)}_r \).
where the notion of neighboring datasets $D \sim u$, $D'$ is defined in Section II-A.

NDP measures how much the information collected by node $v$ depends on the dataset of node $u$. In the special case that all nodes can observe all messages sent and received, i.e., $\mathcal{O}_v$ is the identity map, NDP boils down to conventional LDP \[38\]. When processing information in a decentralized manner with no central coordinating entity, and when there is no third party (on top of the topology) observing all messages sent, NDP is a more natural privacy measure than DP or LDP.

III. EMPIRICAL RISK MINIMIZATION

In this section, we consider the empirical risk minimization problem

$$\tau^* = \arg\min_{\tau \in \mathcal{W} \subseteq \mathbb{R}^d} \left[ f(\tau; D) \triangleq \frac{1}{n} \sum_{v \in \mathcal{V}} f_v(\tau; D_v) \right],$$

where $f_v(\tau; \cdot)$ is a $k$-Lipschitz continuous convex function in its first argument.

A. Skipping Scheme

We suggest the following protocol inspired by projected noisy stochastic gradient descent to solve [1]. The token $\tau$ keeps the current estimate of the optimal point $\tau^*$ and follows a possibly randomized path over the available nodes $\mathcal{V}$. However, to speed up the process, the token waits up to a threshold time $t_{\text{skip}}$ and, if the computation has not finished by that time, the token is forwarded further without an update.

In our notation, it means that the calculation in each node $v$ is

$$g_v^{(\tau)}(\tau; \text{state}_v(h)) = \begin{cases} \Pi_{\mathcal{W}}(\tau - \eta_h (\nabla f_v(\tau; D_v) + N_h)) & \text{if } T^{(\tau)} \leq t_{\text{skip}}, \\ \tau & \text{otherwise,} \end{cases}$$

where $\eta_h$ is the step size (learning rate), $\Pi_{\mathcal{W}}$ denotes the Euclidean projection onto the set $\mathcal{W}$, and $N_h$ is noise with zero mean and standard deviation $\sigma_h$. The noise $N_h$ is added in order to protect the privacy of the local datasets, and the standard deviation $\sigma_h$ is chosen so a certain level of NDP is ensured. In this work, we consider both the gamma distribution (including the exponential distribution) and the Pareto type II (also known as Lomax) distribution for $T^{(\tau)}$, which are well-established models in the literature, see, e.g., [37], [39], [40]. Since we assume that the RVs $T^{(\tau)}$ are i.i.d., we simplify the notation in the following by letting $T \equiv T^{(\tau)}$.

The algorithm stops when a predefined convergence requirement is fulfilled. We refer to the algorithm detailed above as the skipping scheme with parameter $t_{\text{skip}}$, which can be optimized in order to reduce either the convergence time and/or the privacy leakage. In the special case of $t_{\text{skip}} = \infty$, it reduces to a scheme for which the token always waits. We denote by $p = \Pr[T > t_{\text{skip}}]$ the probability of skipping a node. The formal algorithm is given in Algorithm 1 where the output $\ell$ denotes its execution latency and $\tau_{h_{\text{max}}}$ the final value of the token after $h_{\text{max}}$ steps.

We use Algorithm 1 in two special cases as outlined below and illustrated in Fig. 1 for both schemes, the noise variance is fixed throughout the algorithm, i.e., $\sigma_h = \sigma, \forall h$, and we assume, for simplicity, that $h_{\text{max}}$ is a multiple of $n$ in the rest of the paper.

- First, we consider an update schedule in which the nodes in $\mathcal{V}$ are processed along a logical ring, i.e., the node path sequence of Algorithm 1 is $(\nu^{(1)}, \ldots, \nu^{(h_{\text{max}})}) = ((v_1, \ldots, v_n), (v_1, \ldots, v_n), \ldots, (v_1, \ldots, v_n))$. The corresponding scheme is denoted by Skip-Ring.
- Second, we consider a randomized version of the logical

![Fig. 1. Illustrating the j-th round in which node $v_j$ is a straggler.](image)
ring above. In particular, each round over the ring can be seen as a random walk on the set of nodes, but without replacement. For each round, the random walk procedure is restarted. Hence, the node path sequence becomes \( \{v^{(1)}, \ldots, v^{(h_{\max})}\} \)
\[
= \left( \left( v_{\pi_1(1)}, \ldots, v_{\pi_1(n)} \right), \left( v_{\pi_2(1)}, \ldots, v_{\pi_2(n)} \right), \ldots, \left( v_{\pi_{h_{\max}/n}(1)}, \ldots, v_{\pi_{h_{\max}/n}(n)} \right) \right),
\]
where \( \pi_1, \ldots, \pi_{h_{\max}/n} \) are independent random permutations over \([n]\). The scheme is denoted by Skip-Rand-Ring.

### B. Computation and Communication Latency

The average total latency of the skipping scheme in Algorithm 1 is given by the following lemma.

**Lemma 1.** The expected total latency for the skipping scheme in Algorithm 1 is
\[
h_{\max} \left( \chi + \int_0^{t_{\text{skip}}} t d\Phi_T(t) + t_{\text{skip}}(1 - \Phi_T(t_{\text{skip}})) \right),
\]
where \( \Phi_T(t) \equiv \Pr[T \leq t] \) and \( \Phi_T(t_{\text{skip}}) = 1 - p \).

**Proof:** The time between two consecutive nodes (either straggling or not) consists of the constant communication latency, \( \chi \), and the random waiting time in a node, which has the expected value
\[
E[\min(T, t_{\text{skip}})] = \int_0^\infty \min(t, t_{\text{skip}}) d\Phi_T(t)
= \int_0^{t_{\text{skip}}} t d\Phi_T(t) + t_{\text{skip}} \int_{t_{\text{skip}}}^\infty d\Phi_T(t)
= \int_0^{t_{\text{skip}}} t d\Phi_T(t) + t_{\text{skip}}(1 - \Phi_T(t_{\text{skip}}))
\]
and the result follows.

If the number of hops \( h_{\max} \) is large enough, we would expect shorter times between token updates (all other properties being the same) to be beneficial for convergence. In other words, expected time between two consecutive visits to Line 7 in Algorithm 1 should be minimized.

**Lemma 2.** The value of \( t_{\text{skip}} \) that minimizes the average time between two consecutive updates of the token is given by the solution of the optimization problem
\[
\arg \min_{t_{\text{skip}}} \chi + \int_0^{t_{\text{skip}}} t d\Phi_T(t) + t_{\text{skip}}(1 - \Phi_T(t_{\text{skip}})) \frac{\Phi_T(t_{\text{skip}})}{\Phi_T(t_{\text{skip}})}.
\]

**Proof:** Denote by \( Z \) the number of stragglers between two consecutive nonstraggling nodes. It follows the geometric distribution with success probability \( 1 - p = \Phi_T(t_{\text{skip}}) \). Then, the average latency between two token updates is
\[
E[Z] (\chi + t_{\text{skip}}) + \chi + E[T \mid T \leq t_{\text{skip}}]
= 1 - \Phi_T(t_{\text{skip}}) \frac{\chi + t_{\text{skip}}}{\Phi_T(t_{\text{skip}})} (\chi + t_{\text{skip}}) + \chi + \int_0^{t_{\text{skip}}} t d\Phi_T(t).
\]

**IV. Convergence Analysis**

Here, we provide a convergence result for the two considered schemes by adapting the classical convergence result of [33] Thm. 2 to decentralized learning where nodes are processed according to a Markov chain and for which the (sub)gradient estimate in each step is biased, but converges to unbiased exponentially fast, which are the main two new technicalities of the proof.

Additionally, the number of token updates is random (depending on the skipping probability), and we need to average over it. Note that, as in [33] Thm. 2, \( f, v \in \mathcal{V} \), is not required to be \( \beta \)-smooth or even \( k \)-Lipschitz continuous, as we only need the (sub)gradients to be bounded (which follows from \( k \)-Lipschitzness), and also that our result provides a guarantee on the performance of the last update of the token instead of for the average of all token values.

**Theorem 1.** If the diameter of \( \mathcal{W} \) is \( d_\mathcal{W} \), the expected difference between the minimum value \( f(\tau^*; \cdot) \) and that from Algorithm 1 with an arbitrary learning rate parameter \( \zeta > 0 \) after \( h_{\max} \) steps is bounded as
\[
\mathbb{E} \left[ f(\tau_{h_{\max}}; \cdot) - f(\tau^*; \cdot) \right] \leq \sum_{h=0}^{h_{\max}} \left( \frac{h_{\max}}{h} \right) (1 - p)^h p^{h_{\max} - h} e_h
= O \left( \frac{\log(h_{\max})}{\sqrt{h_{\max}}} \right),
\]
where \( \forall h > 0 \),
\[
e_h \triangleq \left( d_\mathcal{W}^2 + \zeta^2 (2 + 2d_\mathcal{W}) (2 + \log(h + 1)) \right) \frac{\zeta}{h + 1} + d_\mathcal{W} k \sqrt{n} \left( \frac{1}{h + 1} + \sum_{i=1}^h \frac{1}{j(j + 1)} \right)
+ \sum_{i=h+1}^{h_{\max}} \left| \lambda_1 \right|^i \frac{1}{i-1} \frac{1}{h + 1 - i} \text{ and } 0 < p < 1

and \( e_0 \triangleq d_\mathcal{W} k, \left| \lambda_1 \right| = \frac{1 - p}{\sqrt{(1 + p^2)^2 - 2p \cos(\frac{\pi}{2})}} \) and \( 0 < p < 1 \) for Skip-Rand-Ring.

**Proof:** See Appendix A.

Note that the asymptotic convergence rate is the same as that of [33] Thm. 2, while being a \( \log(h_{\max}) \)-factor worse compared to [41] Thm. 1. The latter is due to 1) the assumption that \( \sigma_h \) decays to zero with \( h \) [41] Eq. (16), and 2) that convergence there is proved for the running average of the token.

Interestingly, the asymptotic behavior of the bound in Theorem 1 is the same for both \( \lambda_1 = 0 \) and \( \lambda_1 > 0 \). Hence, a biased (sub)gradient estimate that converges to unbiased exponentially fast does not influence the asymptotic convergence rate. Moreover, in Theorem 1 we do not allow the optimal value of \( t_{\text{skip}} \) can incorporate the probability of link failures and channel noise between nodes by changing the distribution of \( T \).

There are several previous works that provide convergence results for Markov chain (noisy) stochastic gradient descent, e.g., [41], [23]. However, all of these works require that \( \sigma_h \) decays to zero with \( h \), which means a significantly higher leakage of private data.
for $p = 0$ in the Skip-Ring scheme as in this case the stochastic (sub)gradient is biased, even asymptotically, and hence a different proof technique is required. The asymptotic convergence rate in this special corner case is left open. Note that the proof of [33 Thm. 2] cannot be adapted to this scenario as it requires an unbiased stochastic (sub)gradient.

V. PRIVACY ANALYSIS

In this section, we present results on the privacy level of the skipping scheme for both updating schedules of the token outlined in Section III-A i.e., for both a fixed and a randomized logical ring on the set of nodes $V$. We highlight here that compared to [11], that only considers a constant learning rate and also a different randomized path (and no fixed path), our results apply to a decreasing learning rate of the form $\eta_t = \zeta/\sqrt{t}$ (as specified in Algorithm 1).

The full proof that can be found in Appendix [2] evolves around upper bounding the Rényi divergence between $\mathcal{O}_v(A(D))$ and $\mathcal{O}_v(A(D'))$, $D \sim_u D'$, for any distinct pair of users $u, v$, using tools (including a composition theorem for Rényi DP [43 Prop. 1]) from the framework of privacy amplification by iteration [13]. The resulting bound can be transformed into a bound on NDP using [43 Prop. 3] and further optimized. Allowing for a decreasing learning rate constitutes the main technical contribution of the proof.

**Theorem 2.** Assume $f_v, v \in V$, is $\beta$-smooth, and let $0 < \varepsilon \leq 1$ and $0 < \delta \leq 1$. Then, the Skip-Ring scheme on a ring with $n$ nodes and with learning rate parameter $0 < \zeta \leq 2/\beta$ achieves $(\varepsilon_{\text{skip}}, \delta')$-NDP for all $\delta' \in (0, 1]$ with

$$\varepsilon_{\text{skip}} = \varepsilon \sqrt{\frac{h \log(1/\delta)}{\log(1/\delta')}} + \varepsilon^2 \sqrt{\frac{h}{\log(1/\delta')}},$$

where $n \triangleq \frac{h_{\text{max}}(1-p)}{n} + \frac{3h_{\text{max}}(1-p)}{n \log(1/\delta')}$, $0 \leq p < 1$ is the probability of skipping a node.

The following theorem characterizes the privacy level $\varepsilon_{\text{skip}}$ of the Skip-Rand-Ring scheme.

**Theorem 3.** Assume $f_v, v \in V$, is $\beta$-smooth, and let $0 < \varepsilon \leq 1$ and $0 < \delta \leq 1$. Then, the Skip-Rand-Ring scheme on a ring with $n$ nodes and with learning rate parameter $0 < \zeta \leq 2/\beta$ achieves $(\varepsilon_{\text{skip}}, \delta')$-NDP for all $\delta' \in (0, 1]$ with

$$\varepsilon_{\text{skip}} = \frac{\epsilon^2 \alpha \zeta}{2 \log(1/\delta')} + \frac{\log(1/\delta')}{\alpha - 1},$$

where

$$a \triangleq \frac{1}{n-1} \sum_{r=0}^{d} \sum_{d=1}^{\lfloor h \rfloor} h_{\text{max}}(1-p)^h \gamma_{r,h},$$

$$\gamma_{r,h} \triangleq \frac{4(1 + r \cdot h)}{h^{r+1}},$$

$$\bar{h} \triangleq \frac{h_{\text{max}}(1-p)/n + \sqrt{3h_{\text{max}}(1-p)/n \log(1/\delta')}}{\sqrt{2 \log(1/\delta') \log(1/\delta')}},$$

$$\alpha \triangleq \min \left(\frac{1 + \sqrt{\frac{10 \log(1/\delta')}{2 \pi}}}{\epsilon}, \frac{1}{\epsilon \sqrt{a}} + 1\right),$$

and $0 \leq p < 1$ is the probability of skipping a node.

**Remark 1.** It follows from Theorems 2 and 3 that the asymptotic behavior of the privacy level $\varepsilon_{\text{skip}}$ for both Skip-Ring and Skip-Rand-Ring is linear in $h_{\text{max}}$, i.e., $\varepsilon_{\text{skip}} = O(h_{\text{max}})$, for $0 \leq p < 1$.

**Proof:** The result for the Skip-Ring scheme follows directly from the expression for $\varepsilon_{\text{skip}}$ given in Theorem 2 as $\bar{h}$ is linear in $h_{\text{max}}$, while for Theorem 3 it follows by lower-bounding $\gamma_{r,h}$ by $\gamma_0$ for all $0 \leq r \leq h - 1$ as $\gamma_{r,h}$ is a strictly increasing function in $r$ for a fixed $h > 0$ (see Appendix C). Then, $a$ is of order $O(h_{\text{max}})$ as 1 the double inner summation in the expression for $a$ can be upper-bounded by a constant, and 2) $\bar{h}$ is linear in $h_{\text{max}}$. Moreover, the value of $\alpha$ will approach 1 as $h_{\text{max}}$ increases as $a$ is of order $O(h_{\text{max}})$ (the first term in the minimization for $\alpha$ gives the minimum). The second term in the expression for $\varepsilon_{\text{skip}}$ is of order $O(\sqrt{h_{\text{max}}})$ as $\alpha - 1$ is of order $O(1/\sqrt{h_{\text{max}}})$, while the first term is of order $O(h_{\text{max}})$ (as $\alpha$ approaches 1), from which the result follows.

As a final remark we note the following. The privacy analysis relies on the exact number of updates performed. Skipping introduces uncertainty on which nodes participated and can be seen as a way to realize subsampling [12] on the fly.

VI. NUMERICAL RESULTS

Here, we first perform a comparison based on the analytical results from Sections V and VI before turning to training a logistic regression model using the dataset in [34] and a deep neural network for image classification using the MNIST [35] and CIFAR-10 [36] datasets.

A. Convergence Versus Privacy and Average Latency

We fix $n = 10$, $\varepsilon = 1$, $\delta = 10^{-6}$, $\delta' = 1/10$, $d = 8$, $d_V = 10$, $k = 1$, $\zeta = 3/100$, and $\chi = 1/100$. The three characteristics we are interested in are: average latency, expected error bound, and privacy level $\varepsilon_{\text{skip}}$. The top row of Fig. 2 plots expected error bound (left $y$-axis) and privacy level (right $y$-axis) versus average latency, and the bottom row shows privacy level versus expected error bound, illustrating the inherent trade-off between average latency, expected error bound, and privacy level. The plots are for the three latency models: exponential with mean $1/4$ and scale 1, and Pareto type II with shape 3 and scale 2 (as used in [40]). The probability of skipping $p = \Pr[T > t_{\text{skip}}] \in \{0.1, 0.2, 0.3\}$, since $p = 10^{-4}$ and $7/10$ are close to the values of $p$ corresponding to the optimal values of $t_{\text{skip}}$ given by Lemma 2 respectively 0/0.710/0.737 for the exponential/gamma/Pareto delay models, while $p = 1/2$ is a value in between.

As can be seen from the figure, $p = 10^{-4}$ (virtually, no skipping) gives the worst expected error bound but the best privacy level for the same average latency for all considered latency models, for both schemes. Hence, there is a trade-off between privacy and accuracy of the algorithm (cf. the bottom
row of figures in Fig. 2], and one needs to choose the skipping probability based on a particular optimization problem.

The privacy-versus-error trade-offs look similar for all latency models considered. Skip-Rand-Ring gives better trade-off curves (especially for \( p = 10^{-4} \)) for smaller values of expected error bound, while the situation changes for higher values of error (i.e., at the initial stages of Algorithm 1’s execution). Hence, path randomization improves the trade-off in the long run, but might be harder to realize in a real-world implementation as it would require a full mesh topology.

On the contrary, the Skip-Ring curve for \( p = 10^{-4} \) is the worst, which means that skipping helps. Also, there is not much difference between the Skip-Ring curves for \( p = 1/2 \) and \( p = 7/10 \) (they are almost on top of each other and hence difficult to distinguish). On the other hand, Skip-Rand-Ring favors smaller values of \( p \) (i.e., larger timeout) at the expense of a higher training latency as shown in the next subsection.

B. Empirical Results

We consider both training a logistic regression model and image classification trained on the MNIST [35] and CIFAR-10 [36] datasets.

1) Logistic Regression: For logistic regression the local loss functions are \( f_\ell(\tau; D_v) = \frac{1}{|D_v|} \sum_{(x,y) \in D_v} \log(1 + e^{-y \tau v x}) \), where \( x \in \mathbb{R}^d \) (\( d_x = d \)) and \( y \in \{-1, 1\} \) (\( d_y = 1 \)). We use a binarized version of the UCI housing dataset [34], trying to predict binary variable \( y \) (whether house price is above a threshold) from other features, \( x \). The features are standardized and we further normalize each data point to have unit \( \ell_2 \)-norm so that the loss functions \( f_\ell(\tau; D_v) \) are 1-Lipschitz continuous (i.e., \( k = 1 \)). The dataset is split uniformly at random into a training set with 80% of the data points and a test set with 20% of the points. Moreover, the training dataset is further randomly split across the \( n = 10 \) nodes in \( V \). We used the Skip-Rand-Ring scheme (similar results are obtained with the Skip-Ring scheme) with the same parameters as in Section VI-A but using a mini-batch implementation with batches of size 100 in order to speed up the learning. The chosen mini-batch size is a compromise between the two corner cases: a mini-batch size of 1 is difficult to parallelize, whereas a large mini-batch size may exceed the nodes’ limited parallelization capabilities.

The results of the training are shown in the top plots in Fig. 3 which show the prediction error rate, i.e., the ratio of incorrect predictions on the test set, versus average latency from Lemma 1 for the same skipping probabilities as in the corresponding plots in Fig. 2. We observe that skipping achieves a clear speed-up compared to no skipping, except for the exponential delay model (as predicted well by Lemma 2).
which suggests an optimal $t_{\text{skip}} = +\infty$ for the exponential model). This rhymes well with theoretical expected error bounds (dashed curves at the upper row of Fig. 2). As can be seen from the bottom plots, no skipping in general provides a slightly higher privacy for Skip-Rand-Ring. In order to have smooth curves the average of 100 independent runs is presented.

2) Image Classification: We consider both the MNIST and CIFAR-10 datasets. Both datasets are commonly-used benchmarks and are comprised of 10 classes of images; MNIST being comprised of $28 \times 28$ pixels grayscale images of handwritten digits from 0 to 9, while CIFAR-10 being comprised of $32 \times 32$ pixels color images. The number of training samples is 60000 (6000 for each digit) and 50000 (5000 for each class) for the MNIST and CIFAR-10 datasets, respectively. As for logistic regression in Section VI-A, the training dataset is further randomly split across a number of nodes $\tilde{n}$ in $\mathcal{V}$. While we used $n = 10$ nodes in Section VI-A, we use $n = 60$ and $n = 50$ nodes, respectively, for the MNIST and CIFAR-10 datasets. As for logistic regression, we use the Skip-Rand-Ring scheme with the same parameters as in Section VI-A but with a smaller initial learning rate of $\zeta = \frac{3}{1000}$ (MNIST) and $\zeta = \frac{7}{10000}$ (CIFAR-10), and a batch size of 500, which is half the number of data samples in each node. Moreover, we use a cross-entropy loss function and a Lipschitz parameter of $k = 1$.

The results are depicted in Fig. 3 showing the prediction error rate on the test set (comprising 10000 images for both datasets) versus average latency from Lemma [1]. For both MNIST (the middle plots) and CIFAR-10 (the bottom plots), we can make the same observations as for the top plots (logistic regression); skipping achieves a speed-up compared to no skipping, except for the exponential delay model, as predicted by Lemma [2]. Moreover, the order of the curves stays the same across the datasets for a given computational delay model. Note, however, that there is some loss in accuracy due to privacy; the accuracy achieved with the MNIST dataset is close to 90%, while with no privacy requirement an accuracy of around 99% can be reached. For the CIFAR-10 dataset, the accuracy decreases from around 70% to around 43%. This aligns well with results in the literature, showing a reduction in accuracy due to privacy, which is particularly significant for CIFAR-10, see, e.g., [43]. Compared to the case of logistic regression, the average of only 30 (MNIST) and 6 (CIFAR-10)
independent runs is presented due to the much more complex learning task. The corresponding architectures of the trained deep neural networks are detailed in Table I.

VII. CONCLUSION

We have studied a skipping scheme for straggler resiliency in decentralized learning over a logical ring under NDP by extending the framework of privacy amplification by decentralization to include overall training latency—comprising both computation and communication latency. Analytical derivations on both the convergence speed and the DP level were presented, showing a trade-off between overall training latency, accuracy, and user data privacy. The theoretical findings were validated for logistic regression on a real-world dataset and for image classification using the MNIST and CIFAR-10 datasets.

APPENDIX A
PROOF OF THEOREM I

A. Notation

Define $[a : b] \triangleq \{a, \ldots, b\}$ for integers $a \leq b$. Moreover, $U^*$ denotes the conjugate transpose of a matrix $U$, while $U^{-1}$ denotes its inverse (for a full-rank square matrix $U$). $\text{diag}(a_1, \ldots, a_l)$ denotes an $l \times l$ diagonal matrix with $a_1, \ldots, a_l$ along the diagonal.

B. Preliminaries

For the convergence, what matters is only the nodes that actually contributed to the token updates (nonstragglers, i.e., those that reached Line 7 of Algorithm I). Let $H = \{0 : h_{\text{max}}\}$ be the RV denoting the number of nonstragglers when running Algorithm I and let the corresponding nodes visited by the token be denoted by $V^{(1)}, V^{(2)}, \ldots, V^{(h)}, \ldots, V^{(H)}$. If $H = 0$, then all nodes are straggling, no nodes are visited by the token, and Algorithm I simply returns $\tau_0 \triangleq 0$ (i.e., $\tau_{\text{max}} = \tau_0$). Otherwise (i.e., when $H > 0$), according to Algorithm I the token updates are (with some abuse of notation)

$$
\tau_h \leftarrow \Pi_W (\tau_{h-1} - \eta_h (\nabla f_{V^{(h)}}(\tau_{h-1}; D_{V^{(h)}}) + N_h)),
$$

for all $h \in [H]$. Note also that $\eta_h = \sqrt{\tau}$. In the rest of this subsection, we assume $H > 0$.

For Skip-Rand-Ring, the marginal distribution of a node $V^{(h)}$ is uniform over $V$ for any $h$. For Skip-Ring, the sequence of nodes $V^{(1)}, V^{(2)}, \ldots$ forms a Markov chain with state transition probability matrix

$$
Q = \begin{pmatrix}
\frac{1-p}{1-p^n} & \frac{1}{p} & \frac{p^2}{p^n} & \cdots & \frac{p^{n-2}}{p^n} & \frac{p^{n-1}}{p^n} \\
\frac{p-1}{1-p^n} & \frac{1}{p} & \frac{p^2}{p^n} & \cdots & \frac{p^{n-2}}{p^n} & \frac{p^{n-1}}{p^n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \frac{p-1}{p^n} & \frac{p^2}{p^n} & \cdots & \frac{p^{n-2}}{p^n} & \frac{p^{n-1}}{p^n}
\end{pmatrix},
$$

where the entries $Q_{ij} \triangleq \Pr[V^{(h)} = v_j \mid V^{(h-1)} = v_i]$, $1 \leq i, j \leq n$, $h \geq 1$, and, as we show in Lemma 3 below, the marginal distributions of $V^{(h)}$ converge to the uniform distribution exponentially fast when $h \to \infty$.

The uniform distribution of $V^{(h)}$ for Skip-Rand-Ring leads to an unbiased estimate of the real (sub)gradient for any fixed $\tau$, i.e.,

$$
\mathbb{E}_{V^{(h)}} [\nabla f_{V^{(h)}}(\tau; D_{V^{(h)}})] = \nabla f(\tau; D),
$$

while for Skip-Ring we have that

$$
\mathbb{E}_{V^{(h)}} [\nabla f_{V^{(h)}}(\tau; D_{V^{(h)}})] \xrightarrow{h \to \infty} \nabla f(\tau; D).
$$

Unbiasness of the (sub)gradient estimate at each step is a known condition used to prove convergence of (conventional) stochastic gradient descent. In this appendix, we will show that having asymptotically unbiased estimates is sufficient for the convergence of Algorithm I too. More precisely, we will adapt a proof from [33, Thm. 2] to our scenario.

First, we present some technical results used in the main part of the proof (next subsection).

Lemma 3. For $n \geq 2$, let $\{V^{(h)}\}$, $V^{(h)} \in V$, $h \geq 1$, be a homogeneous Markov chain with state transition probability matrix (3) with $0 < p < 1$. If we denote by $\pi^{(h)}$ the probability vector of the marginal distribution of $V^{(h)}$ (i.e., $\Pr[V^{(h)} = v_i] = \pi_i^{(h)}$, then $\pi^{(h)} \to \pi^{(\infty)} = (1/n, 1/n, \ldots, 1/n)^T$, as $h \to \infty$, and for all $h$,

$$
\left\|\pi^{(h)} - \pi^{(\infty)}\right\|_1 \leq \sqrt{n}|\lambda_1|^{h}, \tag{4}
$$

where $|\lambda_1| = \frac{1-p}{\sqrt{1+p^2-2p\cos \frac{2\pi}{n}}}$.

Proof: Since the state transition probability matrix $Q$ is circulant (cf. [45]), its eigenvalues $\lambda_j$ and corresponding eigenvectors $e_j$ are

$$
\lambda_j = \frac{(1-p)\omega^j}{1-p\omega^j}, \quad j \in [0 : n-1],
$$

$$
e_j = \frac{1}{\sqrt{n}} (1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j})^T, \quad j \in [0 : n-1],
$$

where $\omega = e^{2\pi i/n}$ is a primitive $n$-th root of unity and $i = \sqrt{-1}$ is the imaginary unit. The absolute values of the eigenvalues are

$$
|\lambda_j| = \frac{1-p}{\sqrt{1+p^2-2p\cos \frac{2\pi}{n}}}, \quad j \in [0 : n-1],
$$

and $|\lambda_j| \leq |\lambda_1| < \lambda_0 = 1$ for $j \in [n-1]$. Let $U$ be a matrix whose columns are $e_0, e_1, \ldots, e_{n-1}$. The matrix $U$ is unitary (i.e., $U^* = U^{-1}$) and we can diagonalize $Q$ as

$$
Q = U \text{diag}(1, \lambda_1, \ldots, \lambda_{n-1}) U^*.
$$

Next,

$$
\lim_{h \to \infty} Q^h = \lim_{h \to \infty} U \text{diag}(1, \lambda_1^h, \ldots, \lambda_{n-1}^h) U^* = U \text{diag}(1, 0, \ldots, 0) U^* = \left(\begin{array}{ccc}
\frac{1}{n} & \frac{1}{n} & \cdots \\
\frac{1}{n} & \frac{1}{n} & \cdots \\
\frac{1}{n} & \frac{1}{n} & \cdots 
\end{array}\right) \triangleq \mathbb{P}.\n$$
Finally,  
\[
\left\| \pi^{(h)} - \pi^{(\infty)} \right\|_1 \leq \sqrt{n} \left\| \pi^{(h)} - \pi^{(\infty)} \right\|_2^a \leq \sqrt{n} \left\| (Q^h)^\top \pi^{(0)} - (Q^\infty)^\top \pi^{(0)} \right\|_2^b \leq \sqrt{n} \left\| (Q^h)^\top - (Q^\infty)^\top \right\|_2 \left\| \pi^{(0)} \right\|_2^c \leq \sqrt{n} \left\| (Q^h)^\top \right\|_2 \left\| \pi^{(0)} \right\|_2^d \leq \sqrt{n} \left\| Q^h - Q^\infty \right\|_2^e \leq \sqrt{n} \left\| U \text{diag}(\lambda_1^h, \ldots, \lambda_{n-1}^h)U^\top \right\|_2^f \left\| \text{diag}(\lambda_1^h, \ldots, \lambda_{n-1}^h) \right\|_2^g \leq \sqrt{n} |\lambda_1^h|_h^h.
\]

Here, \((a)\) is from a general relation between the \(\ell_1\) and \(\ell_2\) norms in \(\mathbb{R}^n\); \((b)\) is a standard expression for Markov chains; \((c)\) is by the definition of induced norm; \((d)\) is because \(\left\| \pi^{(0)} \right\|_2 \leq \left\| \pi^{(0)} \right\|_1 = 1\) as \(\pi^{(0)}\) is a probability vector; \((e)\) is because the spectral norm is invariant under transposition; \((f)\) is a simple matrix manipulation; \((g)\) is because multiplying by a unitary matrix does not change the spectral norm; and \((h)\) is direct calculation of the spectral norm.

\section*{Remark 2.} For convenience, we also define the value \(\lambda_1 \triangleq 0\) for Skip-Rand-Ring (and any \(0 \leq p < 1\)). With this notation, \([4]\) holds in both cases.

\section*{Remark 3.} For any probability vector \(\pi\), it holds that \(\left\| \pi - \pi^{(\infty)} \right\|_1 \leq \sqrt{n}\), and, thus, Lemma \([3]\) technically holds also for \(h = 0\).

\section*{Lemma 4.} Let \(N \sim N(0, \sigma^2 I_d)\). Then, \(\mathbb{E} \left[ \left\| N \right\|_2 \right] < \sigma \sqrt{d}\) and \(\mathbb{E} \left[ \left\| N \right\|_2^2 \right] = d \sigma^2\).

\section*{Proof:} The random variable \(X = \frac{1}{\pi} \left\| N \right\|_2\) is distributed according to a \(\chi\) distribution with \(d\) degrees of freedom. Then  
\[
\mathbb{E} \left[ \left\| N \right\|_2 \right] = \sigma \mathbb{E} [X] = \sigma \sqrt{2 \Gamma \left( \frac{d+1}{2} \right)} \leq \sigma \sqrt{d},
\]

where \(\Gamma(\cdot)\) denotes the gamma function and \((a)\) follows from the logarithmic convexity of \(\Gamma\). Also,  
\[
\mathbb{E} \left[ \left\| N \right\|_2^2 \right] = \sigma^2 \mathbb{E} [X^2] = 2 \sigma^2 \Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d}{2} \right) \leq d \sigma^2.
\]

\section*{Lemma 5 (\([46]\) Lem. 2).} If the domain \(\mathcal{W} \subset \mathbb{R}^d\) is convex and closed, then for any \(x, y \in \mathbb{R}^d\), we have \(\left\| x - y \right\|_2 \geq \left\| \Pi_{\mathcal{W}(x)} - \Pi_{\mathcal{W}(y)} \right\|_2\).

\section*{Lemma 6.} For any \(x, y \in \mathbb{R}^d\), \( \left\| x \pm y \right\|_2 = \left\| x \right\|_2 + \left\| y \right\|_2 \pm 2x^\top y\).

\section*{C. Main Part of the Proof of Theorem \([7]\)}

We first consider the case of \(H \geq 1\). For convenience, define  
\[
\begin{align*}
gh \triangleq & \nabla f(\tau_{h-1}; D), \\
\bar{g}_h \triangleq & \nabla f_{V(\cdot)}(\tau_{h-1}; D_{V(\cdot)}) + N_h
\end{align*}
\]

as a shorthand notation for \(h \in [H]\). With this notation, the token is updated as \(\tau_h \leftarrow \Pi_{\mathcal{W}}(\tau_{h-1} - \eta h \bar{g}_h)\).

If \(V(h)\) is uniformly distributed over \(\mathcal{V}\), we have that \(\mathbb{E} [\bar{g}_h] = g_h\) for any fixed \(\tau_{h-1}\), and in both schemes,  
\[
\begin{align*}
\mathbb{E} \left[ \left\| \bar{g}_h \right\|_2^2 \right] &= \mathbb{E} \left[ \left\| \nabla f_{V(\cdot)}(\tau_{h-1}; D_{V(\cdot)}) \right\|_2^2 \right] + \mathbb{E} \left[ \left\| N_h \right\|_2^2 \right] \\
&\geq 2 \mathbb{E} \left[ \left\| \nabla f_{V(\cdot)}(\tau_{h-1}; D_{V(\cdot)}) \right\|_2 \right] + \mathbb{E} \left[ \left\| N_h \right\|_2 \right] \\
&\leq c \left( k^2 + d \sigma^2 \right),
\end{align*}
\]

where \((a)\) is from Lemma \([3]\), \((b)\) is because \(N_h\) is independent of other RVs and has the mean of 0, and \((c)\) follows from the \(k\)-Lipschitz property of \(f\) and Lemma \([4]\).

Now, we prove the main statement of Theorem \([1]\). In the proof, if it is not mentioned explicitly, the norm of a vector is the \(\ell_2\)-norm. Also, we assume the same dataset \(D\) everywhere and thus omit it for brevity.

Assume \(H \geq 1\) is fixed (i.e., we condition on it). For any \(\tau \in \mathcal{W}\), by Lemma \([5]\),  
\[
\mathbb{E} \left[ \left\| \Pi_{\mathcal{W}}(\tau_{h-1} - \eta h \bar{g}_h) - \Pi_{\mathcal{W}}(\tau) \right\|_2^2 \right] \leq \mathbb{E} \left[ \left\| \tau_{h-1} - \eta h \bar{g}_h - \tau \right\|_2^2 \right].
\]

Thus,  
\[
\begin{align*}
\mathbb{E} \left[ \left\| \tau_h - \tau \right\|_2^2 \right] &= \mathbb{E} \left[ \left\| \Pi_{\mathcal{W}}(\tau_{h-1} - \eta h \bar{g}_h) - \Pi_{\mathcal{W}}(\tau) \right\|_2^2 \right] \\
&\leq \mathbb{E} \left[ \left\| \tau_{h-1} - \tau \right\|_2^2 \right] + \eta h^2 \mathbb{E} \left[ \left\| \bar{g}_h \right\|_2^2 \right] - 2 \eta h \mathbb{E} \left[ \left( \tau_{h-1} - \tau \right)^\top \bar{g}_h \right] \\
&\leq \mathbb{E} \left[ \left\| \tau_{h-1} - \tau \right\|_2^2 \right] + \eta h^2 (k^2 + d \sigma^2) - 2 \eta h \mathbb{E} \left[ \left( \tau_{h-1} - \tau \right)^\top \bar{g}_h \right] \\
&\leq \mathbb{E} \left[ \left\| \tau_{h-1} - \tau \right\|_2^2 \right] - 2 \eta h \mathbb{E} \left[ \left( \tau_{h-1} - \tau \right)^\top \bar{g}_h \right] + \eta h^2 (k^2 + d \sigma^2) + 2 \eta h d_w k \sqrt{|\lambda_1^h|}^h,
\end{align*}
\]

where the term \(d_w k \sqrt{|\lambda_1^h|}^h\) appears because of the difference between the distributions of \(\bar{g}_h\) and \(g_h\) (cf. Lemma \([3]\) and Remark \([2]\)). Then,  
\[
\begin{align*}
\mathbb{E} \left[ \left\| \tau_{h-1} - \tau \right\|_2^2 \right] &\leq \mathbb{E} \left[ \left\| \tau_{h-1} - \tau \right\|_2^2 \right] - \frac{2 \eta h}{2 \eta h} \left( \frac{2 \eta h}{2 \eta h} \right) + \eta h^2 (k^2 + d \sigma^2) + d_w k \sqrt{n} |\lambda_1^h| \]
\]

\footnote{Note that the matrix norm \(\left\| \cdot \right\|_2\) induced by the vector norm \(\left\| \cdot \right\|_2\) is the spectral norm, which is different from the entry-wise norm.}
Let \( j \) be an arbitrary element in \([H - 1]\). Then, summing up and re-arranging, we get
\[
\begin{align*}
\sum_{h=H-j}^{H} \mathbb{E} \left[ (\tau_{h-1} - \tau)^{\top} g_h \right] & \leq \frac{\mathbb{E} \left[ \|\tau_{H-j-1} - \tau\|^2 \right]}{2\eta_{H-j}} \\
& + \frac{H-1}{\sum_{h=H-j}^{H} \mathbb{E} \left[ \|\tau_h - \tau\|^2 \right]} \left( \frac{1}{\eta_{h+1}} - \frac{1}{\eta_h} \right) \\
& + \frac{k^2 + d\sigma^2}{2} \sum_{h=H-j}^{H} \mathbb{E} \left[ \|\tau_{H-j-1} - \tau\|^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} \mathbb{E} \left[ \|\tau_{H-j-1} - \tau\|^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} \mathbb{E} \left[ \|\tau_{H-j-1} - \tau\|^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\end{align*}
\]

Since \( \tau_h \in \mathcal{W} \), we have that \( \|\tau_h - \tau\|^2 \leq d_{W}^2 \). We also substitute \( \eta_h \) with \( \sqrt{\pi} \), which results in
\[
\sum_{h=H-j}^{H} \mathbb{E} \left[ (\tau_{h-1} - \tau)^{\top} g_h \right] \\
\leq \frac{d_{W}^2}{2\zeta} \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \frac{k^2 + d\sigma^2}{2} \sum_{h=H-j}^{H} \mathbb{E} \left[ \|\tau_{H-j-1} - \tau\|^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} \mathbb{E} \left[ \|\tau_{H-j-1} - \tau\|^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

Here, we can upper bound the sum of inverse square roots as
\[
\sum_{h=H-j}^{H} \mathbb{E} \left[ \frac{\zeta_h}{h} \right] \leq \int_{H-j-1}^{H} \mathbb{E} \left[ \frac{\zeta_h}{h} \right] dh = 2\zeta \left( \sqrt{H} - \sqrt{H-j-1} \right).
\]

Next, by convexity of \( f \), we can lower bound \( (\tau_{h-1} - \tau)^{\top} g_h \) by \( f(\tau_{h-1}) - f(\tau) \). Hence,
\[
\sum_{h=H-j}^{H} \mathbb{E} \left[ (\tau_{h-1} - \tau)^{\top} g_h \right] \\
\leq \sum_{h=H-j}^{H} \mathbb{E} \left[ (\tau_{h-1} - \tau)^{\top} g_h \right] \\
\leq \frac{\mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right]}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h \\
+ \frac{d_{W}^2}{2\zeta} \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h \\
+ \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h \\
+ \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

By setting \( \tau = \tau_{H-j-1} \) in (5), we get
\[
\sum_{h=H-j}^{H} \mathbb{E} \left[ f(\tau_{h-1}) - f(\tau_{H-j-1}) \right] \\
\leq \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

Next, for simplicity, let \( S_j \) denote the average of the following \( j + 1 \) iterates, i.e., \( S_j = \frac{1}{j+1} \sum_{h=H-j}^{H} f(\tau_{h-1}) \). Then,
\[
(j+1)\mathbb{E} \left[ S_j \right] - (j+1)\mathbb{E} \left[ f(\tau_{H-j-1}) \right] \\
\leq \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

Hence,
\[
(\tau_{H-j-1}) \leq \mathbb{E} \left[ f(\tau_{H-j-1}) \right] \\
\leq \mathbb{E} \left[ S_j \right] + \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

Using this, we have
\[
\mathbb{E} \left[ S_{j-1} \right] = \mathbb{E} \left[ S_j \right] - \mathbb{E} \left[ f(\tau_{H-j-1}) \right] \\
\leq \mathbb{E} \left[ S_j \right] + \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

In the following, to simplify notation, define
\[
a_j \equiv \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

as a shorthand. Then,
\[
\mathbb{E} \left[ f(\tau_{H-1}) \right] = \mathbb{E} \left[ S_0 \right] \leq \mathbb{E} \left[ S_1 \right] + a_1 \leq \mathbb{E} \left[ S_2 \right] + a_1 + a_2 \leq \cdots \leq \mathbb{E} \left[ S_{H-1} \right] + \sum_{j=1}^{H-1} a_j.
\]

Next, we bound a part of the sum on the right hand side as
\[
\sum_{j=1}^{H-1} \left( \frac{d_{W}^2}{2\zeta} + \zeta (k^2 + d\sigma^2) \right) \left( \sqrt{H} - \sqrt{H-j} \right) \\
+ \mathbb{E} \left[ ||\tau_{H-j-1} - \tau||^2 \right] \frac{H-j}{2\zeta} + d_{W}k\sqrt{n} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]
\[
\leq \sum_{j=1}^{H-1} \left( \frac{d_W^2}{\zeta} + \zeta(k^2 + d\sigma^2) \right) \frac{1}{j\sqrt{H}} \\
\leq \frac{d_W^2 + \zeta^2(k^2 + d\sigma^2)}{\zeta\sqrt{H}} (1 + \log H)
\]
and obtain
\[
\mathbb{E}[f(\tau_{H-1})] \leq \mathbb{E}[S_{H-1}] + \frac{d_W^2 + \zeta^2(k^2 + d\sigma^2)}{\zeta\sqrt{H}} (1 + \log H) \\
+ \sum_{j=1}^{H-1} \frac{d_W k\sqrt{n}}{j(j+1)} \sum_{h=H-j}^{H} |\lambda_1|^h.
\]

Now, recall (3). Set there \( j = H - 1 \) (i.e., \( H - j = 1 \)), \( \tau = \tau^* \), and bound all norms by \( d_W^2 \), which results in
\[
\sum_{h=1}^{H} \mathbb{E}[f(\tau_{h-1}) - f(\tau^*)] \\
\leq \frac{d_W^2}{2\zeta} + d_W k\sqrt{n} \sum_{h=1}^{H} |\lambda_1|^h + \left( \frac{d_W^2}{2\zeta} + \zeta(k^2 + d\sigma^2) \right) \sqrt{H} \\
\leq \left( \frac{d_W^2}{\zeta} + \zeta(k^2 + d\sigma^2) \right) \sqrt{H} + d_W k\sqrt{n} \sum_{h=1}^{H} |\lambda_1|^h.
\]
Therefore,
\[
\mathbb{E}[S_{H-1}] - f(\tau^*) = \mathbb{E} \left[ \frac{1}{H} \sum_{h=1}^{H} (f(\tau_{h-1}) - f(\tau^*)) \right] \\
\leq \frac{d_W^2 + \zeta^2(k^2 + d\sigma^2)}{\zeta\sqrt{H}} + \frac{d_W k\sqrt{n}}{H} \sum_{h=1}^{H} |\lambda_1|^h.
\]
Finally, combining (6) and (7), we obtain
\[
\mathbb{E}[f(\tau_{H-1}) - f(\tau^*)] \\
\leq \left( \frac{d_W^2}{\zeta} + \zeta(k^2 + d\sigma^2) \right) (2 + \log H) \\
+ d_W k\sqrt{n} \left( \frac{1}{H} \sum_{h=1}^{H} |\lambda_1|^h + \frac{1}{j(j+1)} \sum_{h=H-j}^{H} |\lambda_1|^h \right).
\]
Then,
\[
\mathbb{E}[f(\tau_{H}) - f(\tau^*)] \\
\leq \left( \frac{d_W^2}{\zeta} + \zeta(k^2 + d\sigma^2) \right) (2 + \log(H + 1)) \\
+ d_W k\sqrt{n} \left( \frac{1}{H} \sum_{h=1}^{H} |\lambda_1|^h + \frac{1}{j(j+1)} \sum_{h=H-j+1}^{H+1} |\lambda_1|^h \right).
\]
The corner case of \( H = 0 \) (and thus, \( \tau_{h_{\text{max}}} = \tau_0 = 0 \)) can be bounded as \( |f(0) - f(\tau^*)| \leq k\|\cdot - \tau^*\| \leq kd_W \).

As a final step, we need to take expectation conditioned on the distribution of \( H \), which is binomial with \( h_{\text{max}} \) independent trials and success probability \( 1 - p \), i.e.,
\[
\Pr[H = h] = \binom{h_{\text{max}}}{h} (1 - p)^h p^{h_{\text{max}} - h},
\]
which concludes the proof.

D. Asymptotic Convergence Rate
Define a positive binomial RV \( B \) with parameters \( n = h_{\text{max}} \) and success probability \( q = 1 - p \) by
\[
\Pr[B = h] = \frac{1}{1 - p^n} \binom{n}{h} q^h (1 - q)^{n-h}, \quad h \in [1 : n].
\]
We are interested in the asymptotic behavior of
\[
(1 - p^n) \mathbb{E} \left[ \frac{\log B}{\sqrt{B}} \right] = \sum_{h=1}^{n} \frac{\log h}{\sqrt{h}} \binom{n}{h} q^h (1 - q)^{n-h}.
\]
Since \( p \) is fixed, we can focus on the quantity \( \mathbb{E} \left[ \frac{\log B}{\sqrt{B}} \right] \). We first use the following inequality for the logarithm function, which simply states that the concave function \( \log(x) \) lies below its tangent line at a point \( a > 0 \).

**Proposition 1.** For any \( a > 0 \) and \( x > 0 \), we have
\[
\log x \leq \frac{x}{a} + \log \frac{a}{e}.
\]
Hence, we have
\[
\mathbb{E} \left[ \frac{\log B}{\sqrt{B}} \right] \leq \mathbb{E} \left[ \frac{\sqrt{B}}{\mathbb{E}[\sqrt{B}]} \right] + \mathbb{E} \left[ \frac{\mathbb{E}[\sqrt{B}]}{\mathbb{E}[\sqrt{B}]} \right] \log \frac{\mathbb{E}[\sqrt{B}]}{\mathbb{E}[\sqrt{B}]} e
\]
and, with \( a \triangleq \frac{\mathbb{E}[\sqrt{B}]}{\mathbb{E}[\sqrt{B}]} > 0 \), it becomes
\[
\mathbb{E} \left[ \frac{\log B}{\sqrt{B}} \right] \leq \mathbb{E} \left[ \frac{\sqrt{B}}{\mathbb{E}[\sqrt{B}]} \right] + \mathbb{E} \left[ \frac{\mathbb{E}[\sqrt{B}]}{\mathbb{E}[\sqrt{B}]} \right] \log \frac{\mathbb{E}[\sqrt{B}]}{\mathbb{E}[\sqrt{B}]} e
\]
\[
= \mathbb{E} \left[ \frac{\sqrt{B}}{\mathbb{E}[\sqrt{B}]} \right] + \mathbb{E} \left[ \frac{\log \mathbb{E}[\sqrt{B}]}{\mathbb{E}[\sqrt{B}]} \right] e
\]
\[
= \mathbb{E} \left[ \frac{\sqrt{B}}{\mathbb{E}[\sqrt{B}]} \right] \log \frac{\mathbb{E}[\sqrt{B}]}{\mathbb{E}[\sqrt{B}]}.
\]
Since the function \( f = \sqrt{x} \) is concave and \( q = \frac{1}{a} \) is convex for \( x > 0 \), we can use Jensen’s inequality [48, Ch. 2.6] and get
\[
\mathbb{E} \left[ \frac{\sqrt{B}}{\mathbb{E}[\sqrt{B}]} \right] \leq \sqrt{\mathbb{E} \left[ \frac{\log B}{\sqrt{B}} \right]} = \sqrt{\frac{\log n}{1 - p^n}} \mathbb{E} \left[ \frac{1}{\sqrt{B}} \right] q^h (1 - q)^{n-h}
\]
and
\[
\mathbb{E} \left[ \frac{1}{\sqrt{B}} \right] \geq \frac{1}{\mathbb{E} \left[ \sqrt{B} \right]} = \frac{1}{\mathbb{E} \left[ \sqrt{B} \right]}.
\]
Hence, (8) can be further bounded from above by
\[
\mathbb{E} \left[ \frac{\log B}{\sqrt{B}} \right] \leq \mathbb{E} \left[ \frac{1}{\sqrt{B}} \right] \log O(n).
\]

\footnote{For a formal proof, see, e.g., [47, Lem. 2.29] for the case of \( a = 1 \). The general case for an arbitrary \( a > 0 \) can be proved in the same way.}
Finally, we use the asymptotic result from [49, Thm. 1], which indicates that
\[ \mathbb{E} \left[ \frac{1}{\sqrt{n}} \right] = O \left( \frac{uq}{(uq + p)^{1+\frac{1}{2}}} \right) = O \left( \frac{1}{\sqrt{n}} \right). \]

**APPENDIX B**

**Proof of Theorems 1 and 5**

The main tool of the proofs is the concept of privacy amplification by iteration [13], and Theorem 22 therein. The setting in [13] is projected noisy stochastic gradient descent, in which noise is added for every gradient update step. The main technical tool is Rényi divergence and the proof evolves around upper bounding it for a single view of a node. In particular, based on Lemma 8 for any distinct pair of users \( u, v \), we can derive an upper bound on the Rényi divergence between the views of user \( u \) when the token visits for the \((r+1)\)-th time, excluding received and sent messages observed up to and including the \( r \)-th visit, for two neighboring datasets of user \( u \) (Lemma 12). By maximizing this upper bound over all pairs of distinct users \( u, v \) and by using a composition theorem for Rényi DP (RDP) [43, Prop. 1] (Lemma 9), we can derive an upper bound on the RDP level of Algorithm 1, which can be transformed into an upper bound on the DP level using [43, Prop. 3] (Lemma 11). In order to get the best (lowest) upper bound, the Rényi divergence parameter \( \alpha \) can be optimized. Finally, since the number of visits to a node is not a constant, but instead follows a binomial distribution, a standard Chernoff bound in combination with Lemma 10 can be used to derive the final result. We start by defining Rényi divergence and RDP and then state some important results from the privacy amplification by iteration literature. In particular, definitions and results from [13, 43].

### A. Important Results From [13, 43]

We start by stating and adapting some important definitions and results from [13, 43]. Central to the arguments in [13] is the concept of Rényi divergence and shifted Rényi divergence.

**Definition 4** (Rényi divergence). For two probability distributions \( \mu \) and \( \nu \) defined over the same common set \( \mathcal{Z} \), the Rényi divergence of order \( \alpha \neq 1 \) is
\[ D_\alpha (\mu \| \nu) \triangleq \frac{1}{\alpha - 1} \log \int_{z \in \mathcal{Z}} (\mu(z) / \nu(z)) \alpha \nu(z) \, dz. \]

**Definition 5** (Shifted Rényi divergence [13, Def. 8]). For two probability distributions \( \mu \) and \( \nu \) defined on a complete normed vector space \( (\mathcal{Z}, \| \cdot \|) \). Let \( u > 0 \) and \( \alpha > 1 \), the \( \alpha \)-shifted Rényi divergence of order \( \alpha > 1 \) between \( \mu \) and \( \nu \) is
\[ D_\alpha^{(u)} (\mu \| \nu) \triangleq \inf_{\mu' : d_W(\mu, \mu') \leq u} D_\alpha (\mu' \| \nu), \]
where \( d_W(\cdot, \cdot) \) denotes the \( \infty \)-Wasserstein distance [13, Def. 6] between distributions \( \mu \) and \( \nu \) on \( (\mathcal{Z}, \| \cdot \|) \).

**Lemma 7** (Weak convexity Rényi divergence [13, Lem. 25]). Let \( \mu_1, \ldots, \mu_n \) and \( \nu_1, \ldots, \nu_n \) be probability distributions defined on a complete normed vector space \( (\mathcal{Z}, \| \cdot \|) \) such that \( \forall i \in [n] \), \( D_\alpha (\mu_i \| \nu_i) \leq \hat{b}/(n-1) \) for some \( b \in (0, 1] \). Let \( \rho \) be a probability distribution over \( [n] \) and denote by \( \mu_\rho \) the probability distribution over \( \mathcal{Z} \) obtained by sampling \( i \) from \( \rho \) and then outputing a random sample from \( \mu_i \) (respectively, \( \nu_i \)). Then
\[ D_\alpha (\mu_\rho \| \nu_\rho) \leq (1 + b) \cdot \mathbb{E}_{i \sim \rho} D_\alpha (\mu_i \| \nu_i). \]

**Definition 6** ([13, Def. 10]). Consider a distribution \( \zeta \) over \( (\mathcal{Z}, \| \cdot \|) \) and any \( a \geq 0 \), the magnitude of noise is the largest Rényi divergence of order \( \alpha \) between \( \zeta \) and the same distribution \( \zeta \) shifted by a vector of length at most \( a \), i.e.,
\[ R_\alpha (\zeta, a) \triangleq \sup_{z : \| z \| \leq a} D_\alpha (\zeta + z \| \zeta). \]

**Remark 4.** Consider the standard Gaussian distribution over \( \mathbb{R}^d \) with variance \( \sigma^2 \), denoted by \( N(0, \sigma^2 I_d) \). Then, it is known that \( \forall \zeta \in \mathbb{R}^d, \sigma > 0 \) (see, e.g., [50, Ex. 3]), we have
\[ D_\alpha (N(x, \sigma^2 I_d) \| N(0, \sigma^2 I_d)) = \frac{\| x \|^2}{2\sigma^2}, \]
\[ R_\alpha (N(0, \sigma^2 I_d), \sigma) = \frac{\sigma^2}{2a^2}. \]

**Definition 7** (Contractive noisy iteration (CNI) [13, Def. 19]). Given an initial random state \( Z_0 \in \mathcal{Z} \), a sequence of contractive maps \( \{\psi_h\}_{h=1}^m \), and a sequence of noise distributions \( \{\zeta_h\}_{h=1}^m \), the contracive noisy iteration after \( m \) steps, denoted by \( \text{CNI}_m \{Z_0, \{\psi_h\} \{\zeta_h\} \} \), is defined by the following update process:
\[ Z_h \triangleq \psi_h (Z_{h-1}) + N_h, \text{ where } N_h \sim \zeta_h, h \in [m]. \]

The following lemma is taken from [13, Thm. 22].

**Lemma 8** ([13, Thm. 22]). Let \( Z_m \) and \( Z_{m}' \) represent the outputs of \( \text{CNI}_m \{Z_0, \{\psi_h\} \{\zeta_h\} \} \) and \( \text{CNI}_m \{Z_0, \{\psi_h'\} \{\zeta_h'\} \} \), respectively. Define \( s_h \triangleq \sup_{z \in \mathcal{Z}} \| \psi_h(z) - \psi_h'(z) \|, \quad \{a_h\}_{h=1}^m \) a sequence of nonnegative reals, and \( u_h \triangleq \sum_{i=1}^h (s_i - a_i) \). If \( u_h \geq 0, \forall h \in [m] \), then \( \mathcal{D}_\alpha \{Z_m \| Z_{m}'\} \leq \sum_{h \in [m]} \mathcal{D}_\alpha (\zeta_h, a_h). \)

Now, we review some results from Rényi DP [43].

**Definition 8** ((\( \alpha, \varepsilon \))-RDP). For any \( \alpha \geq 1 \) and \( \varepsilon \geq 0 \), a (randomized) protocol \( \mathcal{A} \) is said to satisfy \((\alpha, \varepsilon)-\text{RDP}\), if for all neighboring datasets \( \mathcal{D}, \mathcal{D}' \) and for all \( S \) in the output space \( \Omega \), we have \( D_\alpha (\mathcal{A}(\mathcal{D}) \in S \mid \mathcal{A}(\mathcal{D}') \in S) \leq \varepsilon. \)

Next, we state the composition theorem for RDP.

**Lemma 9** ([43, Prop. 1]). Let \( r \in \mathbb{N} \). If \( \{\mathcal{A}_i\}_{i=1}^r \) are protocols satisfying, respectively, \((\alpha_1, \varepsilon_1)-\text{RDP}\), \ldots, \((\alpha_r, \varepsilon_r)-\text{RDP}\), then their composition defined as \((\mathcal{A}_1, \ldots, \mathcal{A}_r)\) satisfies \((\alpha, \sum_{i=1}^r \varepsilon_i)-\text{RDP}\).

The DP (RDP) level with a random number of entries in the composition can be bounded as follows.

**Lemma 10.** Let \( R \) denote a RV with range \([1, 2, \ldots] \) that satisfies \( \Pr (R > r) \leq \delta \). If \( \{\mathcal{A}_i\}_{i=1}^R \) are protocols satisfying, respectively, \((\varepsilon_1, \delta_1)-\text{DP}\), \ldots, \((\varepsilon_R, \delta_R)-\text{DP}\), then their composition defined as \((\mathcal{A}_1, \ldots, \mathcal{A}_R)\) satisfies \((\varepsilon_c, \delta_c + \delta)-\text{DP}\), where \((\varepsilon_c, \delta_c)\) is the differential privacy guarantee under \( r\)-fold composition for DP.
**Proof:** Denote the \(i\)-fold composition \((A_1, \ldots, A_i)\) by \(A^{(i)}\), and its privacy guarantee by \((\varepsilon^{(i)}, \delta^{(i)})\), \(i \in [r]\). Let \(S\) be any set, we have

\[
\Pr \left[ A^{(R)}(D) \in S \right] = \Pr \left[ \{A^{(R)}(D) \in S \} \cap \{R \leq r\} \right] + \Pr \left[ \{A^{(R)}(D) \in S \} \cap \{R > r\} \right] \leq \Pr(R > r)
\]

\[
\leq \sum_{i=1}^{r} \Pr[R = i] \Pr \left[ A^{(R)}(D) \in S \mid R = i \right] + \Pr(R > r)
\]

\[
\leq \sum_{i=1}^{r} \Pr[R = i] \left[ \varepsilon^{(i)} + \Pr \left[ A^{(R)}(D') \in S \mid R = i \right] + \delta^{(i)} \right] + \Pr(R > r)
\]

\[
\leq \sum_{i=1}^{r} \Pr[R = i] \left[ \varepsilon^{(r)} + \Pr \left[ A^{(R)}(D') \in S \mid R = i \right] + \delta^{(r)} \right] \quad \text{for } \alpha > 1, \text{ where } D \sim_{v} D', \xi_{u,v}^{(r+1)} \triangleq \xi_{u,v}^{(r+1)} - \epsilon_{u,v}^{(r+1)} + 1 \text{ and } \epsilon_{u,v}^{(r+1)} \in [\xi_{u,v}^{(r+1)}] \text{ is the index of } \xi_{u,v}^{(r+1)} = u \text{ for } u \in \{v_1^{(r+1)}, \ldots, v_{r+1}^{(r+1)}\}, \text{i.e., } u = v_{i}^{(r+1)} \text{. Otherwise, if } u \notin \{v_1^{(r+1)}, \ldots, v_{r+1}^{(r+1)}\}, \text{ then } \xi_{u,v}^{(r+1)} \triangleq \infty \text{.}
\]

For simplicity of notation, we omit the superscript \((r + 1)\) from \(\ell, c, v_1, \ldots, v_{k+1}\) in the following.

**Proof:** Consider the case when \(u \in \{v_1, \ldots, v_{k+1}\}\). Otherwise, \(O_v^{(r+1)}(A(D)) = O_v^{(r+1)}(A(D'))\) and it follows directly that \(\mathcal{D}_\alpha \left( O_v^{(r+1)}(A(D)) \left\| O_v^{(r+1)}(A(D')) \right\| \right) = 0\).

By assumption, the learning rate \(\eta_h\) is upper-bounded by \(2/\beta\), and hence the update rule \(g_v^{(r)}(\tau; \text{state}_{e_i}(h))\) in Algorithm 1 constitutes a CNI (see [13, Prop. 18]). Consider now the CNI from Definition 2 with \(\psi_i(\tau) = \Pi_{W}(\tau - \eta_h \nabla f_{v_i}(\tau; D_{v_i})) = \Pi_{W}(\tau - \eta_h \nabla f_{v_i}(\tau; D_{v_i})) = \Pi_{W}(\tau - \eta_h \nabla f_{v_i}(\tau; D_{v_i})) = \Pi_{W}(\tau - \eta_h \nabla f_{v_i}(\tau; D_{v_i}))\), corresponding to \(g_v^{(r)}(\tau; \text{state}_{e_i}(h))\) in (2). It follows that

\[
\sup_{\tau} \left\| \psi_i(\tau) - \psi_i(\tau) \right\|_2 = \sup_{\tau} \left\| \eta_h \nabla f_{v_i} \left( \Pi_{W}(\tau; D_{v_i}) \right) - \eta_h \nabla f_{v_i} \left( \Pi_{W}(\tau; D_{v_i}) \right) \right\|_2 \leq \left\{ \begin{array}{ll}
0 & \text{if } i \neq c, \\
2\eta_h & \text{otherwise},
\end{array} \right.
\]

since by assumption \(f_{v_i}\) is \(k\)-Lipschitz continuous.

We can now apply Lemma 8 with \(a_i = 0, \forall i \in [c-1]\), and \(a_i = 2\eta_h, k/\epsilon_{u,v}^{(r+1)}, \forall i \in [c, \ell]\), where

\[
\eta_{u,v}^{(r+1)} = \sum_{i \in [c: \ell]} \eta_h \frac{\epsilon_{u,v}^{(r+1)}}{\eta_h}.
\]

Clearly, \(z_i = s_i - a_i \geq 0, \forall i \in [\ell]\), and \(z_i = 0\). Hence, using Remark 4

\[
\mathcal{D}_\alpha \left( O_v^{(r+1)}(A(D)) \left\| O_v^{(r+1)}(A(D')) \right\| \right) \leq \alpha \sum_{i \in [c: \ell]} \frac{4\eta_h^2 k^2}{\eta_h^2 \sigma_i^2} = \alpha \sum_{i \in [c: \ell]} \frac{2k^2}{\epsilon_{u,v}^{(r+1)}} \leq 2\alpha |c: \ell| k^2 \frac{1}{\sqrt{\epsilon_{u,v}^{(r+1)}}} \leq \leq 2\alpha k^2 \frac{1}{\sigma^2}.
\]

Now, if \(c = \ell\), i.e., \(u = v_{\ell}\) and \(\xi_{u,v}^{(r+1)} = 1\), then from (9) it follows that \(\eta_{u,v}^{(r+1)} = 1\) and therefore

\[
\mathcal{D}_\alpha \left( O_v^{(r+1)}(A(D)) \left\| O_v^{(r+1)}(A(D')) \right\| \right) \leq \leq 2\alpha k^2 \frac{1}{\sigma^2}.
\]

Otherwise, i.e., when \(\ell > c\) and \(1 < \xi_{u,v}^{(r+1)} < \infty\),

\[
\eta_{u,v}^{(r+1)} = \frac{\sum_{i \in [c: \ell]} 1}{\sqrt{\epsilon_{u,v}^{(r+1)}}}.
\]
expression in (11) is strictly increasing in
composition theorem for RDP (Lemma 9) and Lemma 10,
in the formulation of the theorem. Then, it follows from a
\( r = 1 \) (for every \( r \)-distribution with parameters
\( h_{\Xi} \)). Lemma 12 that

\[ \xi_{u,v}(r) = 1 \] for all \( r \), and it follows from Lemma 12 that

\[ \max_{u,v \in V, u \neq v} D_{\alpha}(O_v^r(A(D))) \leq \frac{2\alpha k^2}{\sigma^2} \cdot \hat{h} \] (12)

The number of visits of the token to a node \( v \) during the execution of the algorithm, denoted by \( \Xi_v \), follows a binomial distribution with parameters \( h_{\Xi} / n \) (number of independent trials) and \( 1 - p \) (success probability). Let \( \hat{h} \) be defined as in
the formulation of the theorem. Then, it follows from a standard Chernoff bound that \( \Pr(\Xi_v \geq \hat{h}) \leq \delta' \), for some \( \delta' \in (0,1] \). Now,

\[ \max_{u,v \in V, u \neq v} D_{\alpha}(O_v^r(A(D))) \leq \frac{2\alpha k^2}{\sigma^2} \cdot \hat{h} \] (13)

for every \( \alpha > 1 \), where \( D \sim_d D' \). (a) follows from the
composition theorem for RDP (Lemma 9) and Lemma 10,
(b) from swapping the order of maximization and summation,
and (c) from (12).

Then, converting from RDP to DP using Lemma 11

gives that Algorithm 1 satisfies

\[ \left( \frac{2\beta}{\sigma^2} + \frac{\log(1/\delta)}{\alpha - 1}, \delta + \delta' \right) - \text{NPD}. \] (14)

Now, the Rényi divergence parameter \( \alpha \) can be optimized in order to minimize \( \frac{2\alpha k^2}{\sigma^2} + \frac{\log(1/\delta)}{\alpha - 1} \) by taking the derivative
with respect to \( \alpha \). Doing so, gives \( \alpha = 1 + \frac{\sigma \sqrt{\log(1/\delta)}}{k \sqrt{2 \beta}} > 1 \)
from which the result follows by substituting this value of \( \alpha \)
into (14) and setting \( \sigma = \frac{k \sqrt{8 \log(1.25/\delta)}}{e} \), where \( 0 < \varepsilon \leq 1 \) and \( 0 < \delta < 1 \).

**D. Proof of Theorem 3**

In contrast to the proof of Theorem 2, the distance between
two nodes \( u, v \) is random over the rounds of the algorithm. Hence, we have to resort to a weak form of
convexity for Rényn divergence as formulated in Lemma 7.

**Lemma 13.** The fraction \( \xi_{u,v}^{(r+1)}/(\xi_{u,v}^{(r+1)})^2 \) from the proof of
Lemma 7 (see (10)) is upper-bounded by one.

**Proof:** From (9),

\[ \left( \frac{\xi_{u,v}^{(r+1)}}{(\xi_{u,v}^{(r+1)})^2} \right)^2 = \sum_{i \in [0:\ell-1]} D_{\alpha} \left( O_v^r(A(D)) \right) \leq \frac{2\alpha k^2}{\sigma^2} \cdot \hat{h} \]

which completes the proof. \( \blacksquare \)

Now, let \( \Xi_{u,v}^{(r)} \) denote the actual number of noise terms added in between the \( (r-1) \)-th and \( r \)-th visit of the token at node \( v \) after visiting node \( u \). \( \Xi_{u,v}^{(r)} \) is a binomial RV with parameters \( d^{(r)}(u,v) \) and \( 1 - p \), where \( d^{(r)}(u,v) \) is the distance between \( u \) and \( v \) along the direction of the token over the ring.

From Lemma 7, it follows that

\[ D_{\alpha}(O_v^r(A(D))) \leq 1 + b \]

where \( D_{\alpha}(O_v^r(A(D))) \) is the Rényi divergence between the views \( O_v^r(A(D)) \) and \( O_v^r(A(D')) \) given that in between the \( i \)-th and \( i \)-th visit of the token at node \( v \), \( \xi_{u,v}^{(i)} \in [d^{(i)}(u,v)] \) nodes after node \( u \) (including) have been visited, and where \( 0 < b \leq 1 \) is a constant such that

\[ D_{\alpha}(O_v^r(A(D))) \leq (1 + b) \]

for all \( \xi_{u,v}^{(i)} \in [d^{(i)}(u,v)] \). By picking \( b = 1 \) and applying
Lemma 12 gives the expression in (15) at the top of the next page. As \( \xi_{u,v}^{(i)}(\xi_{u,v}^{(r+1)})^2 \leq 1 \) (see Lemma 13), in order to satisfy (16) (with \( b = 1 \)), we require that \( 2 \alpha (\alpha - 1) k^2 \leq \sigma^2 \) (see (10)), which is equivalent to \( 1 - \frac{1}{\sqrt{2 \beta + 1}} \leq \alpha \leq 1 + \frac{1}{\sqrt{2 \beta + 1}} \).

Since the lower bound on \( \alpha \) above is less than one,

\[ 1 < \alpha < 1 + \frac{2 \beta}{\sigma^2 + 1} = 1 + \frac{\sqrt{16 \log(1.25/\delta)}}{2}, \] (17)
\[
\max_{u,v\in V, u \neq v} \mathcal{D}_\alpha \left( \mathcal{O}_v^{(r)}(A(D)) \left\| \mathcal{O}_v^{(r)}(A(D')) \right\| \right) \\
\leq \max_{u,v\in V, u \neq v} (1 + 1) \frac{2a\alpha k^2}{\sigma^2} \mathbb{E}_{u,v} \left[ \sum_{i=1}^{r} \Xi(u,v) \right] \\
= \frac{1}{(n-1)^2} \sum_{d_1=1}^{n-1} \sum_{d_r=1}^{n-1} \sum_{d_1=1}^{d_r} g(h_1, \ldots, h_r) \\
	imes \frac{d_1}{h_1} \cdots \frac{d_r}{h_r} p^{d_1 + \cdots + d_r - (h_1 + \cdots + h_r)} (1-p)^{h_1 + \cdots + h_r} \\
= \frac{1}{(n-1)^2} \sum_{d_1=1}^{n-1} \sum_{h_1=1}^{d_1} \sum_{d_r=1}^{d_1} g(h_1, \ldots, h_r) \left( \frac{d_1}{h_1} \cdots \frac{d_r}{h_r} \right) p^{d-h} (1-p)^h \\
\leq \frac{4a\alpha k^2}{(n-1)^2} \sum_{d_1=1}^{n-1} \sum_{d_r=1}^{n-1} \sum_{h_1=1}^{d_1} g(h_1, \ldots, h_r) \left( \frac{d_1}{h_1} \cdots \frac{d_r}{h_r} \right) p^{d-h} (1-p)^h.
\]
TABLE I
Neural network architectures used for training for the MNIST and CIFAR-10 image datasets. The input pixel value is rescaled between $-1.0$ and $1.0$. FC stands for a fully connected layer. Conv stands for a convolutional layer and “st” is shorthand for stride. Size is the number of neurons or input size. As activation function we used SeLU and softmax.

| Dataset   | Detailed architecture (consecutive layers’ sizes and types) |
|-----------|-----------------------------------------------------------|
| MNIST     | $1 \times 28 \times 28 \times 1$ (Input), $1 \times 26 \times 26 \times 64$ (Conv, SeLU), $1 \times 12 \times 12 \times 64$ (Conv, SeLU), $1 \times 4 \times 4 \times 128$ (Conv, SeLU), $1 \times 10 \times 10 \times 128$ (Conv, SeLU), $1 \times 4 \times 4 \times 128$ (Conv, SeLU), $1 \times 8 \times 8 \times 128$ (Conv, SeLU), $1 \times 8 \times 8 \times 256$ (Conv, SeLU), $1 \times 8 \times 8 \times 256$ (Conv, SeLU), $1 \times 4 \times 4 \times 256$ (Conv, SeLU), $1 \times 4 \times 4 \times 512$ (Conv, SeLU), $1 \times 2 \times 2 \times 512$ (Conv, SeLU), $10$ (FC, softmax) |
| CIFAR-10  | $1 \times 32 \times 32 \times 3$ (Input), $1 \times 32 \times 32 \times 64$ (Conv, SeLU), $1 \times 16 \times 16 \times 64$ (Conv, SeLU), $1 \times 16 \times 16 \times 128$ (Conv, SeLU), $1 \times 8 \times 8 \times 128$ (Conv, SeLU), $1 \times 8 \times 8 \times 256$ (Conv, SeLU), $1 \times 8 \times 8 \times 256$ (Conv, SeLU), $1 \times 4 \times 4 \times 256$ (Conv, SeLU), $1 \times 4 \times 4 \times 512$ (Conv, SeLU), $1 \times 2 \times 2 \times 512$ (Conv, SeLU), $10$ (FC, softmax) |

APPENDIX D
NEURAL NETWORK ARCHITECTURES

In this appendix, the architectures of the deep neural networks used for training for the MNIST and CIFAR-10 datasets are detailed (see Table I). For all three considered computational models, the same neural network architecture was used with the same initial learning rate.

APPENDIX E
COMPUTATION LATENCY MODELS

The following computation latency models were used throughout this work.

1) Exponential with mean 1:

\[ \Phi_T(t) = 1 - e^{-t}, \]

\[ \Phi'_T(t) = e^{-t}. \]

2) Gamma with shape $1/4$ and scale 1:

\[ \Phi_T(t) = \frac{1}{(1/4)} \int_0^t x^{-1/4} e^{-x} dx, \]

\[ \Phi'_T(t) = \frac{t^{-1/4} e^{-t}}{(1/4)}. \]

3) Pareto type II (also known as Lomax) with shape 3 and scale 2:

\[ \Phi_T(t) = 1 - \frac{8}{(t+2)^4}, \quad t \geq 0, \]

\[ \Phi'_T(t) = \frac{24}{(t+2)^5}, \quad t \geq 0. \]

Using these and Lemma 2, we get the best values of $t_{\text{skip}}$ and $p$ as follows.

| Distribution | Best $t_{\text{skip}}$ | Best $p$ |
|--------------|------------------------|----------|
| Exponential  | +\infty                | 0        |
| Gamma        | 0.0048                 | 0.709857 |
| Pareto type II | 9.89047               | 0.737256 |

We use the following values for $p$ and $t_{\text{skip}}$ in Figs. 2 and 3.

\[
\begin{array}{|c|c|c|c|}
\hline
p & \text{Exponential} & \text{Gamma} & \text{Pareto type II} \\
\hline
10^{-4} & 9.21034 & 6.42831 & 41.0887 \\
1/2 & 0.693147 & 0.0436738 & 0.519842 \\
7/10 & 0.356675 & 0.0054913 & 0.252496 \\
\hline
\end{array}
\]
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