Properties of counterexample to Robin hypothesis

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Abstract

Let $G(n) = \frac{\sigma(n)}{(n \log \log n)}$. Robin made hypothesis that $G(n) < e^\gamma$ for all integer $n > 5040$. If there exists counterexample to Robin hypothesis, then there must exist finite number of counterexamples $n > 5040$ such that $G(n)$ attains largest value. This article studies various properties of such number.

Introduction

Robin made a hypothesis [Robin 1984] that the Robin’s inequality

$$\sigma(n) < e^\gamma n \log \log n,$$

holds for all integers $n > 5040$. Here $\sigma(n) = \sum_{d|n} d$ is the divisor sum function, $\gamma$ is the Euler-Mascheroni constant, log is the nature logarithm.

For calculation convenience, we define

$$\rho(n) := \frac{\sigma(n)}{n}.$$
Then Robin’s inequality can also be written as
\[ \frac{\rho(n)}{\log \log n} < e^\gamma. \]  

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\[ G(n) < e^\gamma. \]  

Let \( N > 5040 \) be an integer. Write the factorization of \( N \) as
\[ N = \prod_{i=1}^{r} p_i^{a_i}, \]
where \( p_i \) are in increasing orders, \( p_r \) is the largest prime factor of \( N \).

According to [Morrill; Platt 2018], (RI) holds for all integers \( n, 5040 < n \leq 10^{10^{13}} \). So, we assume \( N > 10^{10^{13}} \).

By Grönwall’s theorem, [Broughan 2017] Theorem 9.2, if there exist counterexamples of Robin hypothesis, then there must exist finite number of counterexamples \( n > 5040 \) such that \( G(n) \) attains largest value. We call such an \( n \) a largest \( G \)-value (abbreviate LG) number.

This article proves the following properties of LG numbers. Assume \( N \) is an LG number. Then

1) \( N \) is colossally abundant.
2) \( p_r < \log N \).
3) \( p_r \) is the largest prime below \( N \).
4) 
\[ a_i \leq \left\lfloor \frac{\log(kp_r)}{\log p_i} \right\rfloor \text{, when } ((k + 1)p_r)^{1/(k+1)} < p_i \leq (kp_r)^{1/k}, \forall k \geq 1. \]
5) 
\[ a_i \geq \left\lfloor \frac{\log p_r}{\log p_i} \right\rfloor \forall i \leq r. \]
6) 
\[ \log N > p_r + \frac{1}{2} \log p_r + \frac{1}{2} - \frac{1}{2 \log p_r}. \]
7) Let $p$ be the smallest prime above $\log N$, then

$$\log N < p - \frac{1}{2} \log p + \frac{1}{2} - \frac{1}{2} \log p + \frac{1}{\log p(\log p + 1)}.$$  

8) 

$$G(N) < e^\gamma + \frac{0.00995}{(\log \log N)^2}.$$  

9) 

$$p_r > \log N \left(1 - \frac{0.005587}{\log \log N}\right) \text{ and } \log N \leq p_r \left(1 + \frac{0.005589}{\log p_r}\right).$$

Version Notes:

2019-02-13 version 2. Added two theorems. They are reverse of theorems 6 and 7.

Theorem 10. $G(N) > G(N/p)$ if

$$\log N > p + \frac{\log p}{2} + \frac{1}{2} - \frac{1}{2 \log p} + \frac{1}{(\log p)(\log p + 1)}.$$  

Theorem 11. $G(N) > G(Np)$ if

$$\log N < p - \frac{1}{2} \log p + \frac{1}{2} - \frac{1}{\log p + 1}.$$  

Main Content

Theorem 1. Let $N$ be an LG number, then $N$ is colossally abundant.

Proof. By Proposition 1 of [Robin 1984], $N$ is between two adjacent colossally numbers $n_i$ and $n_{i+1}$ for some integer $i$. We have

$$G(N) \leq \max(G(n_i), G(n_{i+1}).$$

By maximality of $G(N)$, the equal sign must hold. By strict convexity of $x \rightarrow e^x - \log \log x$ ($x > 1$), we must have $N = n_i$ or $N = n_{i+1}$.  

Theorem 2. Let $N$ be an LG number. Then $p_r < \log N$.  

Proof. Write $p := p_r$. By Theorem 1, we know $N$ is colossally abundant, so the exponent of $p$ in $N$ is 1. We have

$$\frac{G(N)}{G(N/p)} = \frac{\rho(N) \log\log(N/p)}{\rho(N/p) \log\log N} = \frac{\log(\log N - \log p)}{\log\log N} \left( 1 + \frac{1}{p} \right)$$

$$= \frac{\log\log N + \log \left( 1 - \frac{\log p}{\log N} \right)}{\log\log N} \left( 1 + \frac{1}{p} \right)$$

$$< \left( 1 - \frac{\log p}{\log N \log\log N} \right) \left( 1 + \frac{1}{p} \right). \quad (2.1)$$

If $p \geq \log N$, we would have

$$\frac{G(N)}{G(N/p)} < 1 + \frac{\log N \log\log N - p \log p - \log p}{p \log N \log\log N} < 1. \quad (2.2)$$

That means $G(N) < G(N/p)$, which contradicts to the maximality of $N$. \hfill \Box

Theorem 3. Let $N$ be an LG number. Then $p_r$ must be the largest prime below $\log N$.

Proof. We know $p_r < \log N$ by Theorem 2. Assume there exists a prime $p$ such that $p_r < p < \log N$. We will derive a contradiction. Compare $G(N)$ and $G(Np)$, we have

$$\frac{G(N)}{G(Np)} = \frac{\rho(N) \log\log(Np)}{\rho(Np) \log\log N} = \frac{\log(\log N + \log p)}{\log\log N} \left( \frac{p}{p + 1} \right)$$

$$= \frac{\log\log N + \log \left( 1 + \frac{\log p}{\log N} \right)}{\log\log N} \left( \frac{p}{p + 1} \right)$$

$$< \left( 1 + \frac{\log p}{\log N \log\log N} \right) \left( \frac{p}{p + 1} \right). \quad (3.1)$$

Since $p < \log N$, we have

$$\frac{G(N)}{G(Np)} < \left( 1 + \frac{\log p}{p \log p} \right) \left( \frac{p}{p + 1} \right) = 1. \quad (3.2)$$

That means $G(N) < G(Np)$, which contradicts to the maximality of $N$. \hfill \Box
Recall the construction of a colossally abundant number $N_\epsilon$ from a given parameter $\epsilon > 0$, cf. [EN 1975] Proposition 4 or [Broughan 2017] Section 6.3. Define

$$N_\epsilon := \prod_p p^{a_p(\epsilon)}, \quad a_p(\epsilon) := \left\lfloor \frac{\log((p^{1+\epsilon} - 1)/(p^\epsilon - 1))}{\log p} \right\rfloor - 1.$$  

Let $k \geq 1$ be an integer, $x_k$ be the solution of

$$F(x,k) := \frac{\log(1 + 1/(x + x^2 + \cdots + x^k))}{\log x} = \epsilon.$$  

Then one can show that

$$a_p(\epsilon) = \begin{cases} k, & \text{if } x_{k+1} < p \leq x_k, \ k \geq 1 \\ 0, & \text{if } p > x_1. \end{cases}$$

**Theorem 4.** Let $\epsilon > 0$ be a parameter, $N_\epsilon$ be the colossally number constructed from $\epsilon$, $p \geq 3299$ be the largest prime factor of $N_\epsilon$. Then

$$x_k < (kp)^{1/k}, \quad \forall \ k \geq 2. \quad (4.1)$$

**Proof.** This is an improvement based on Lemma 1 of [CNS 2012], which proved $x_k < (kx_1)^{1/k}, \forall k \geq 2$. Since the function $t \to F(t,k)$ is strictly decreasing on $1 < t < \infty$, to prove that $x_k < z := (kp)^{1/k}$, it suffices to show $F(z,k) < F(x_k,k)$. Since $F(x_k,k) = \epsilon = F(x_1,1)$, this reduces to showing $F(z,k) < F(x_1,1)$.

$$F(z,k) = \log \left( 1 + \frac{1}{z + z^2 + \cdots + z^k} \right) \frac{1}{\log z}$$

$$< \frac{1}{(z + z^2 + \cdots + z^k) \log z} < \frac{k}{(k - 1 + z^k) \log kp}$$

$$\leq \frac{\log kp}{(k + \frac{1}{2}) \log kp}$$

$$< \log \left( 1 + \frac{1}{p} \right) \frac{1}{\log kp}.$$  

$$\quad (4.2)$$
We need to show
\[
\log \left( 1 + \frac{1}{p} \right) \frac{1}{\log kp} < F(x_1, 1) = \log \left( 1 + \frac{1}{x_1} \right) \frac{1}{\log x_1},
\] (4.3)
that is
\[
\frac{\log \left( 1 + \frac{1}{p} \right)}{\log \left( 1 + \frac{1}{x_1} \right)} < \frac{\log kp}{\log x_1}. 
\] (4.4)

Write
\[
g(t) := t \log \left( 1 + \frac{1}{t} \right).
\]
Take derivative
\[
g'(t) = \log \left( 1 + \frac{1}{t} \right) + \frac{t}{1 + \frac{1}{t}} \cdot -\frac{1}{t^2} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{jt^j} - \frac{1}{t + 1} > \frac{t - 1}{2t^2(t + 1)} > 0,
\] (4.5)
for \( t > 1 \). Hence \( g(t) \) strictly increases, and
\[
\frac{\log \left( 1 + \frac{1}{p} \right)}{\log \left( 1 + \frac{1}{x_1} \right)} < \frac{x_1}{p}.
\] (4.6)
So in view of (4.4), it suffices to prove
\[
\frac{x_1}{p} < \frac{\log kp}{\log x_1}.
\] (4.7)
By Proposition 5 of [Dusart 1998], for all \( j \geq 463 \), \((p_{463} = 3299)\), we have
\[
p_{j+1} \leq p_j \left( 1 + \frac{1}{2(\log p_j)^2} \right).
\] (4.8)
Theorem assumes \( p \geq 3299 \). Since \( p \) is the largest prime \( \leq x_1 \), we must have
\[
x_1 < p \left( 1 + \frac{1}{2(\log p)^2} \right).
\] (4.9)
(4.7) becomes
\[
\frac{p \log kp}{x_1 \log x_1} > \frac{p(\log p + \log k)}{p \left( 1 + \frac{1}{2(\log p)^2} \right) \log \left( p \left( 1 + \frac{1}{2(\log p)^2} \right) \right)}
\]
\[
\frac{\log p + \log k}{\left(1 + \frac{1}{2(\log p)^2}\right) \left(\log p + \log \left(1 + \frac{1}{2(\log p)^2}\right)\right)} > \frac{\log p + \log k}{\log p + \frac{1}{2 \log p + \frac{1}{2 \log p} + \frac{1}{4 \log p}}}.
\]

(4.10)

Since
\[
\frac{1}{2 \log p} + \frac{1}{2(\log p)^2} + \frac{1}{4(\log p)^4} < \log k \quad \forall \ p \geq 5, \ k \geq 2,
\]
we have
\[
\frac{p \log kp}{x_1 \log x_1} > 1,
\]
(4.12)
i.e. (4.7) holds.

**Definition 1.** Now we construct a lower bound curve \(L\) for the exponents. Define
\[
L(p_i) = L_{p_r}(p_i) := \left\lfloor \frac{\log p_r}{\log p_i} \right\rfloor \text{ for } i \leq r.
\]
(D1.1)

**Theorem 5.** Let \(N > 10^{(10^{13})}\) be an LG number. Then \(a_i \geq L(p_i)\).

**Proof.** As \(N\) being a colossally abundant number, we know \(a_r = 1 = L(p_r)\).
Assume \(a_s < L(p_i)\) for some index \(s < r\). We will derive a contradiction.
Define
\[
N_1 := (p_s/p_r)N.
\]
Then \(\log N - \log N_1 = \log p_r - \log p_s\). \(p_s < p_r\) means \(N_1 < N\). \(a_s < L(p_s) = \left\lfloor \frac{\log p_r}{\log p_s} \right\rfloor\) means \(a_s + 1 \leq \left\lfloor \frac{\log p_r}{\log p_s} \right\rfloor \leq \frac{\log p_r}{\log p_s}\). Hence \(p_s^{a_s+1} \leq p_r\) and
\[
\log p_s \leq \frac{1}{a_s + 1} \log p_r.
\]
(5.1)
It is easy to deduce
\[
\frac{G(N)}{G(N_1)} = \frac{\rho(N) \log \log N_1}{\rho(N_1) \log \log N}
= \frac{\log(p_s - p_s^{-a_s})}{\log \log N} \left(\frac{p_s - p_s^{-a_s}}{p_s - p_s^{-a_s-1}}\right) \left(\frac{p_r + 1}{p_r}\right)
\]
\[
\leq \left(1 - \frac{\log p_r - \frac{1}{a_s + 1} \log p_r}{\log N \log \log N}\right) \left(\frac{p_s - p_s^{-a_s}}{p_s - p_s^{-a_s - 1}}\right) \left(1 + \frac{1}{p_r}\right)
= \left(1 - \left(\frac{a_s}{a_s + 1}\right) \frac{\log p_r}{\log N \log \log N}\right) \left(\frac{p_s - p_s^{-a_s}}{p_s - p_s^{-a_s - 1}}\right) \left(1 + \frac{1}{p_r}\right).
\]
(5.2)

\[
\frac{p_s - p_s^{-a_s}}{p_s - p_s^{a_s - 1}} = 1 - \frac{1}{p_s^{a_s + 1} + p_s^{a_s} + \cdots + 1}.
\]
(5.3)

By Proposition 5 of [Dusart 1998], for all \(j \geq 463\), \((p_{463} = 3299)\), we have

\[
p_j + 1 \leq p_j \left(1 + \frac{1}{2(\log p_j)^2}\right).
\]

By Theorem 3, \(p_r\) is the largest prime below \(\log N\), so

\[
p_r > \log N \left(1 - \frac{1}{2(\log p_r)^2}\right).
\]
(5.4)

We have, noting \(N > 10^{(10^{13})}\),

\[
\log N < c p_r, \quad c := \left(1 - \frac{1}{2(\log(2.3 \times 10^{13}))^2}\right) = 1.000528 \cdots.
\]
(5.5)

Since \(\log(c p_r) < c \log p_r\), (5.2) can be simplified to

\[
\frac{G(N)}{G(N_1)} < \left(1 - \left(\frac{a_s}{a_s + 1}\right) \frac{\log p_r}{(c p_r) \log(c p_r)}\right) \left(\frac{p_s - p_s^{-a_s}}{p_s - p_s^{-a_s - 1}}\right) \left(1 + \frac{1}{p_r}\right)
< \left(1 - \left(\frac{a_s}{a_s + 1}\right) \frac{1}{c^2 p_r}\right) \left(\frac{p_s - p_s^{-a_s}}{p_s - p_s^{-a_s - 1}}\right) \left(1 + \frac{1}{p_r}\right).
\]
(5.6)

Now we split the proof into two cases.

**Case 1**) \(a_s = 1\). We have in this case

\[
1 - \left(\frac{a_s}{a_s + 1}\right) \frac{1}{c^2 p_r} < 1 - \frac{1}{2c^2 p_r} < 1 - \frac{0.49}{p_r}
\]
(5.7)

\[
p_s^{a_s + 1} + p_s^{a_s} + \cdots + 1 = p_s^2 + p_s + 1 \leq \frac{7}{4} p_s^2,
\]
(5.8)

\[
\frac{p_s - p_s^{-a_s}}{p_s - p_s^{-a_s - 1}} = 1 - \frac{1}{p_s^2 + p_s + 1} \leq 1 - \frac{4}{7 p_s^2} < 1 - \frac{0.57}{p_r}.
\]
(5.9)
Substitute (5.7) and (5.9) in to (5.6), we get
\[
\frac{G(N)}{G(N_1)} < \left(1 - \frac{0.49}{p_r}\right) \left(1 - \frac{0.57}{p_r}\right) \left(1 + \frac{1}{p_r}\right) < 1, \tag{5.10}
\]
which contradicts to the maximality of N.

**Case 2)** \(a_s > 1\). We have
\[
1 - \left(\frac{a_s}{a_s + 1}\right) \frac{1}{c^2 p_r} < 1 - \frac{2}{3c^2 p_r} < 1 - \frac{0.66}{p_r}. \tag{5.11}
\]

\[
\frac{p_s - p_s^{-a_s}}{p_s - p_s^{-a_s-1}} = 1 - \frac{1}{p_s^{a_s+1} + p_s^{a_s} + \cdots + 1} < 1 - \frac{1}{2p_s^{a_s+1}} < 1 - \frac{0.50}{p_r}. \tag{5.12}
\]
Substitute (5.11) and (5.12) in to (5.6), we get
\[
\frac{G(N)}{G(N_1)} < \left(1 - \frac{0.66}{p_r}\right) \left(1 - \frac{0.50}{p_r}\right) \left(1 + \frac{1}{p_r}\right) < 1, \tag{5.13}
\]
which contradicts to the maximality of N.

**Lemma 1.** Let \(N\) be an integer, \(p\) be a prime factor of \(N\) with exponent 1. Write \(\log N = p + \frac{1}{2} \log p + d\). Then \(G(N) > G(N/p)\) if and only if
\[
\frac{1}{2} \log p^2 + d \log p + d + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(k+1)p^k} \left(\frac{1}{2} \log p + d\right)^{k+1} > \frac{1}{2} \log p + \frac{p \log p^2}{2 \log N} + \frac{(\log p)^2}{2 \log N} + (p + 1) \sum_{k=1}^{\infty} \frac{(\log p)^{k+2}}{(k + 2)(\log N)^{k+1}}. \tag{L1.1}
\]

**Proof.** Substitute \(\log N\)
\[
\log N \log \log N = \left(p + \frac{1}{2} \log p + d\right) \log \left(p + \frac{1}{2} \log p + d\right) = \left(p + \frac{1}{2} \log p + d\right) \left(\log p + \log \left(1 + \frac{1}{2} \log p + d\right)\right) = p \log p + \frac{1}{2} (\log p)^2 + d \log p + \cdots
\]
\[
= p \log p + \frac{1}{2}(\log p)^2 + d \log p \\
+ \left( p + \frac{1}{2} \log p + d \right) \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left( \frac{1}{2} \log p + d \right)^k}{kp^k}
\]

\[
= p \log p + \frac{1}{2}(\log p)^2 + d \log p \\
+ \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left( \frac{1}{2} \log p + d \right)^k}{kp^{k-1}} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left( \frac{1}{2} \log p + d \right)^{k+1}}{kp^k}
\]

\[
= p \log p + \frac{1}{2}(\log p)^2 + d \log p + \frac{1}{2} \log p + d \\
+ \sum_{k=1}^{\infty} \left( \frac{(-1)^k \left( \frac{1}{2} \log p + d \right)^{k+1}}{(k+1)p^k} + \frac{(-1)^{k-1} \left( \frac{1}{2} \log p + d \right)^{k+1}}{kp^k} \right)
\]

\[
= p \log p + \frac{1}{2}(\log p)^2 + d \log p + \frac{1}{2} \log p + d \\
+ \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left( \frac{1}{2} \log p + d \right)^{k+1}}{k(k+1)p^k}
\]

(L1.2)

Compare \( G(N) \) and \( G(N/p) \), we have

\[
\frac{G(N)}{G(N/p)} = \frac{\rho(N) \log \log (N/p)}{\rho(N/p) \log \log N} = \frac{\log(\log N - \log p)}{\log \log N} \left( 1 + \frac{1}{p} \right)
\]

\[
= \left( 1 + \frac{\log \left( 1 - \frac{\log p}{\log N} \right)}{\log \log N} \right) \left( 1 + \frac{1}{p} \right).
\]

(L1.3)

Therefore,

\[
G(N) > G(N/p) \\
\iff \left( 1 + \frac{\log \left( 1 - \frac{\log p}{\log N} \right)}{\log \log N} \right) \left( 1 + \frac{1}{p} \right) > 1
\]

\[
\iff 1 + \frac{\log \left( 1 - \frac{\log p}{\log N} \right)}{\log \log N} > \left( 1 + \frac{1}{p} \right)^{-1} = 1 - \frac{1}{p + 1}
\]

\[
\iff \frac{1}{\log \log N} \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\log p}{\log N} \right)^k > \frac{1}{p + 1}
\]
\[
\frac{\log p}{\log N \log \log N} \left( 1 + \sum_{k=2}^{\infty} \frac{1}{k} \left( \frac{\log p}{\log N} \right)^{k-1} \right) < \frac{1}{p+1}
\]

\[
\iff (p+1) \log p \left( 1 + \sum_{k=1}^{\infty} \frac{1}{k+1} \left( \frac{\log p}{\log N} \right)^k \right) < \log N \log \log N \quad (L1.4)
\]

Compare (L1.2) and (L1.4), we see that \(G(N) > G(N/p)\) if and only if

\[
p \log p + \frac{1}{2} (\log p)^2 + d \log p + \frac{1}{2} \log p + d + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left( \frac{1}{2} \log p + d \right)^{k+1}}{k(k+1)p^k}
\]

\[
> (p+1) \log p \left( 1 + \frac{\log p}{2 \log N} + \sum_{k=2}^{\infty} \frac{1}{k+1} \left( \frac{\log p}{\log N} \right)^k \right)
\]

\[
= p \log p + \log p + \frac{p(\log p)^2}{2 \log N} + \frac{(\log p)^2}{2 \log N} + (p+1) \sum_{k=1}^{\infty} \frac{(\log p)^{k+2}}{(k+2)(\log N)^{k+1}} \quad (L1.5)
\]

\[\Box\]

**Theorem 6.** Let \(N > 10^{10^{13}}\) be an LG number. Then

\[
\log N > p_r + \frac{1}{2} \log p_r + \frac{1}{2} - \frac{1}{2 \log p_r}.
\] (6.1)

**Proof.** Write \(p := p_r\). By Theorem 3, \(p\) is the largest prime below \(\log N\).

Write \(\log N = p + \frac{1}{2} \log p + d\), where \(d\) is a to-be-determined expression. By Lemma 1, \(G(N) > G(N/p)\) if and only if

\[
\frac{1}{2} (\log p)^2 + d \log p + d + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \left( \frac{1}{2} \log p + d \right)^{k+1}}{k(k+1)p^k}
\]

\[
> \frac{1}{2} \log p + \frac{p(\log p)^2}{2 \log N} + \frac{(\log p)^2}{2 \log N} + (p+1) \sum_{k=1}^{\infty} \frac{(\log p)^{k+2}}{(k+2)(\log N)^{k+1}} \quad (6.2)
\]

This implies

\[
\frac{1}{2} (\log p)^2 + d \log p + d + \frac{(\frac{1}{2} \log p + d)^2}{2p} > \frac{\log p}{2} + \frac{p(\log p)^2}{2 \log N}. \quad (6.3)
\]
Since $p$ is the largest prime below $\log N$, by Proposition 5.4 of [Dusart 2018], for $p \geq 89693$ we have

$$p > \log N \left(1 - \frac{1}{(\log p)^3}\right), \quad (6.4)$$

$$\frac{p}{2 \log N} > \frac{1}{2} \left(1 - \frac{1}{(\log p)^3}\right). \quad (6.5)$$

Since $N > 10^{10^{10^{13}}}$, $\log N > (\log 10) \times 10^{13}$, the last term on left of (6.3) is in order of $10^{-13}(\log p)^2$ and can be absorbed by rounding: the numerator $1$ in (6.4) was rounded from 0.998. We can concentrate on main terms. $G(N) > G(N/p)$ implies

$$\frac{1}{2}(\log p)^2 + d \log p + d > \frac{\log p}{2} + \frac{(\log p)^2}{2} \left(1 - \frac{1}{(\log p)^3}\right). \quad (6.6)$$

Hence

$$d(\log p + 1) > \frac{1}{2} \log p - \frac{1}{2 \log p},$$

$$d > \frac{\log p - \frac{1}{\log p}}{2(\log p + 1)} = \frac{1 - \frac{1}{(\log p)^3}}{2 \left(1 + \frac{1}{\log p}\right)} = \frac{1}{2} \left(1 - \frac{1}{\log p}\right). \quad (6.7)$$

\Box

**Lemma 2.** Let $N > 5040$ be an integer. $p < N$ be a prime. Assume $p$ does not divide $N$. Write $\log N = p - \frac{1}{2} \log p + d$. Then $G(N) > G(Np)$ if and only if

$$p \sum_{k=1}^{\infty} \frac{(-1)^k (\log p)^{k+1}}{(k+1)(\log N)^k}$$

$$> -\frac{1}{2}(\log p)^2 + d \log p - \frac{1}{2} \log p + d + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2} \log p - d\right)^{k+1}}{k(k+1)p^k}. \quad (L2.1)$$

**Proof.** Substitute $\log N$

$$\log N \log \log N = \left(p - \frac{1}{2} \log p + d\right) \log \left(p - \frac{1}{2} \log p + d\right)$$
\[
\begin{align*}
= & \left( p - \frac{1}{2} \log p + d \right) \left( \log p + \log \left( 1 - \frac{1}{2} \log p - d \right) \right) \\
= & p \log p - \frac{1}{2} (\log p)^2 + d \log p \\
- & \left( p - \frac{1}{2} \log p + d \right) \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{1}{2} \log p - d \right)^k \left( \frac{k}{p} \right)
\end{align*}
\]

\[= p \log p - \frac{1}{2} (\log p)^2 + d \log p
\]

\[\begin{align*}
- & \left( \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{1}{2} \log p - d \right)^k \left( \frac{k}{p} \right) - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{1}{2} \log p - d \right)^{k+1} \right)
\end{align*}\]

\[= p \log p - \frac{1}{2} (\log p)^2 + d \log p - \frac{1}{2} \log p + d
\]

\[\begin{align*}
- & \left( \sum_{k=2}^{\infty} \frac{1}{k} \left( \frac{1}{2} \log p - d \right)^k \left( \frac{k}{p} \right) - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{1}{2} \log p - d \right)^{k+1} \right)
\end{align*}\]

\[= p \log p - \frac{1}{2} (\log p)^2 + d \log p - \frac{1}{2} \log p + d
\]

\[+ \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{1}{2} \log p - d \right)^{k+1} \frac{k}{k(k+1)p^k} \quad (L2.2)
\]

Compare \( G(N) \) and \( G(Np) \), we have

\[
\frac{G(N)}{G(Np)} = \frac{\rho(N) \log \log(Np)}{\rho(Np) \log \log N}
\]

\[
= \frac{\log(\log N + \log p)}{\log \log N} \left( \frac{p}{p+1} \right)
\]

\[
= \left( 1 + \frac{\log \left( 1 + \frac{\log p}{\log N} \right)}{\log \log N} \right) \left( \frac{p}{p+1} \right) . \quad (L2.3)
\]

Therefore,

\[
G(N) > G(Np)
\]

\[
\iff \left( 1 + \frac{\log \left( 1 + \frac{\log p}{\log N} \right)}{\log \log N} \right) \left( \frac{p}{p+1} \right) > 1
\]

\[
\iff 1 + \frac{\log \left( 1 + \frac{\log p}{\log N} \right)}{\log \log N} > \left( \frac{p}{p+1} \right)^{-1} = 1 + \frac{1}{p}
\]
\[
\begin{align*}
\Leftrightarrow & \quad \frac{\log p}{\log N \log \log N} \left( 1 + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \left( \frac{\log p}{\log N} \right)^{k-1} \right) > \frac{1}{p} \\
\Leftrightarrow & \quad p \log p \left( 1 + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \left( \frac{\log p}{\log N} \right)^{k-1} \right) > \log N \log \log N. \quad \text{(L2.4)}
\end{align*}
\]

Combine (L2.2) and (L2.4), we get \( G(N) > G(Np) \) if and only if
\[
\begin{align*}
p \log p \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k + 1} \left( \frac{\log p}{\log N} \right)^{k+1} \right) & > p \log p - \frac{1}{2} \left( \frac{\log p}{2} \right)^{2} + d \log p - \frac{1}{2} \log p + d \\
& \quad + \sum_{k=1}^{\infty} \frac{\left( \frac{1}{2} \log p - d \right)^{k+1}}{k(k+1)p^{k}}. \quad \text{(L2.5)}
\end{align*}
\]

**Theorem 7.** Let \( N > 5040 \) be an integer, \( p \) be the prime just above \( \log N \). Assume \( G(N) > G(Np) \) Then
\[
\log N < p - \frac{1}{2} \log p + \frac{1}{2} - \frac{1}{2} \log p + \frac{1}{2} \log p \left( \log p \right)^{2} + \frac{1}{2} \log p \left( \log p \right)^{3} > d \log p + d. \quad \text{(7.1)}
\]

**Proof.** Write \( \log N = p - \frac{1}{2} \log p + d \), where \( d \) is a to-be-determined expression. By Lemma 2, \( G(N) > G(Np) \) if and only if
\[
\begin{align*}
p \sum_{k=1}^{\infty} \frac{(-1)^{k}(\log p)^{k+1}}{(k+1)(\log N)^{k}} & > -\frac{1}{2} \left( \frac{\log p}{2} \right)^{2} + d \log p - \frac{1}{2} \log p + d + \sum_{k=1}^{\infty} \frac{\left( \frac{1}{2} \log p - d \right)^{k+1}}{k(k+1)p^{k}}. \quad \text{(7.2)}
\end{align*}
\]
So the theorem assumption \( G(N) > G(Np) \) implies
\[
\begin{align*}
\frac{1}{2} \left( \frac{\log p}{2} \right)^{2} + \frac{1}{2} \log p - \frac{p(\log p)^{2}}{2 \log N} + \frac{p(\log p)^{3}}{3(\log N)^{2}} & > d \log p + d. \quad \text{(7.3)}
\end{align*}
\]
Since \( p > \log N \), we can replace \( \log N \) with \( p \) and get
\[
\begin{align*}
\frac{1}{2} \left( \frac{\log p}{2} \right)^{2} + \frac{1}{2} \log p - \frac{(\log p)^{2}}{2} + \frac{(\log p)^{3}}{3p} & > d \log p + d. \quad \text{(7.4)}
\end{align*}
\]
\[
\begin{align*}
\frac{\log p}{2(\log p + 1)} + \frac{(\log p)^3}{3p(\log p + 1)} < \frac{1}{2} - \frac{1}{2 \log p} + \frac{1}{\log p(\log p + 1)}. 
\end{align*}
\] (7.5)

**Lemma 3. (Mertens’ third theorem)** For any integer \(n > 7713133853\), we have
\[
\sum_{p \leq n} \log \left( \frac{p}{p-1} \right) = \log \log n + \gamma + R(n),
\] (L3.1)

where \(\gamma\) is the Euler-Mascheroni constant, \(R(n)\) is the remainder such that
\[
-0.005586 < R(n) < \frac{0.005586}{(\log n)^2}.
\] (L3.2)

**Proof.** By setting \(k = 2, \eta_2 = 0.01\) in Theorem 5.9 of [Dusart 2018], we have, for \(n > 7713133853\),
\[
|R(n)| < \frac{0.01}{2(\log n)^2} + \frac{4}{3} \cdot \frac{0.01}{(\log n)^3} = \frac{0.01}{(\log n)^2} \left( \frac{1}{2} + \frac{4}{3 \log n} \right) < \frac{0.005586}{(\log n)^2}.
\] (L3.3)

**Lemma 4.** Let \(g(x) = (\log x)e^{R(x)}\), where
\[
R(x) = \frac{0.005586}{(\log x)^2},
\]
then \(g(x)\) is strictly increasing in interval \((1.1115, \infty)\).

**Proof.** Take derivative, we get
\[
g'(x) = \frac{1}{x}e^{R(x)} + (\log x)e^{R(x)} \left( -2 \times \frac{0.005586}{x(\log x)^3} \right)
= \frac{1}{x(\log x)^2} e^{R(x)} \left( (\log x)^2 - 0.011172 \right).
\]
So, \(g'(x)\) has a zero at \(x = 1.1115\), and is positive on the right.
Theorem 8. Let \( N > 10^{10^{13}} \) be an LG number, then

\[
G(N) < e^\gamma + \frac{0.00995}{(\log \log N)^2} \quad (8.1)
\]

Proof. It is easy to see

\[
\rho(N) = \prod_{i=1}^{r} \frac{p_i - p_i^{-a_i}}{p_i - 1}. \quad (8.2)
\]

Because a part is smaller than total, we have

\[
\rho(N) < \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \leq \prod_{p \leq p_r} \frac{p}{p - 1} \quad (8.3)
\]

Substitute \( n \) by \( p_r \) in (L3.1) of Lemma 3, we get

\[
\sum_{p \leq p_r} \log \left( \frac{p}{p - 1} \right) = \log \log p_r + \gamma + R(p_r) \quad (8.4)
\]

here \( R(p_r) \) is the remainder. Take exponential of (8.4),

\[
\prod_{p \leq p_r} \left( \frac{p}{p - 1} \right) = e^{\gamma \log(p_r) + R(p_r)} \quad (8.5)
\]

We get by (8.3)

\[
\rho(N) < \prod_{p \leq p_r} \frac{p}{p - 1} = e^{\gamma \log(p_r) + R(p_r)} \quad (8.6)
\]

By Lemma 4, \( \log(p_r)e^{R(p_r)} \) is increasing, and by Theorem 2, \( p_r < \log N \), we can replace \( p_r \) with \( \log N \).

\[
G(N) = \frac{\rho(N)}{\log \log N} < \frac{e^{\gamma \log(p_r) + R(p_r)}}{\log \log N} \leq e^{\gamma e^{R(\log N)}} \quad (8.7)
\]

By Lemma 3,

\[
\exp(R(\log N)) < \exp \left( \frac{0.005586}{(\log \log N)^2} \right) = 1 + \sum_{k=1}^{\infty} \frac{(0.005586)^k}{k!(\log \log N)^{2k}} < 1 + \frac{0.005587}{(\log \log N)^2}. \quad (8.8)
\]
Theorem 9. \(\text{let } N > 10^{(10^{13})} \text{ be an LG number. Then}

1) \[ p_r > (\log N) \left(1 - \frac{0.005587}{\log \log N}\right). \] (9.1)

Conversely, 2) \[ \log N \leq p_r \left(1 - \frac{0.005589}{\log p_r}\right). \] (9.2)

Proof. Proof by contradiction. Assume \(p_r \leq \log N \left(1 - \frac{0.005587}{\log \log N}\right).\) It is easy to see

\[ \rho(N) = \prod_{i=1}^{r} \frac{p_i - p_{i-a_i}}{p_i - 1}. \] (9.3)

Because a part is smaller than total, we have

\[ \rho(N) < \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \leq \prod_{p \leq p_r} \frac{p}{p - 1} \] (9.4)

Substitute \(n\) by \(p_r\) in (L3.1) of Lemma 3, we get

\[ \sum_{p \leq p_r} \log \left(\frac{p}{p - 1}\right) = \log \log p_r + \gamma + R(p_r) \] (9.5)

here \(R(p_r)\) is the remainder. Take exponential of (9.5),

\[ \prod_{p \leq p_r} \left(\frac{p}{p - 1}\right) = e^{\gamma \log(p_r)e^{R(p_r)}} \] (9.6)

We get by (9.4)

\[ \rho(N) < \prod_{p \leq p_r} \frac{p}{p - 1} = e^{\gamma \log(p_r)e^{R(p_r)}} \] (9.7)
By Lemma 4, \( \log(p_r) e^{R(p_r)} \) is increasing and by assumption, \( p_r \leq C \log N \), where \( C := 1 - 0.005587/\log \log N \), we can replace \( p_r \) with \( C \log N \).

\[
\rho(N) < e^\gamma \log(p_r) e^{R(p_r)} \leq e^\gamma \log(C \log N)) e^{R(C \log N)}
\]  
(9.8)

To get a contradiction, we need to prove

\[
e^\gamma \log(C \log N)) e^{R(C \log N)} < e^\gamma \log \log N.
\]  
(9.9)

Cancel \( e^\gamma \) and substitute \( M := \log N \), the inequality looks simpler:

\[
\log(CM) e^{R(CM)} < \log M.
\]  
(9.10)

It suffices to prove

\[
f(M) := \log(CM) e^{R(CM)} - \log M < 0.
\]  
(9.11)

By Lemma 3,

\[
R(CM) < \frac{0.005586}{(\log(CM))^2}.
\]

Expand the exponential and substituting,

\[
f(M) = \log(CM) \left( \sum_{k=0}^{\infty} \frac{1}{k!} R(CM)^k \right) - \log M
\]

\[
= \log(CM) \left( 1 + \sum_{k=1}^{\infty} \frac{(0.005586)^k}{k!(\log(CM))^{2k}} \right) - \log M
\]

\[
= \log(C) + \log M + \sum_{k=1}^{\infty} \frac{(0.005586)^k}{k!(\log(CM))^{2k-1}} - \log M
\]

\[
= \log \left( 1 - \frac{0.005587}{\log M} \right) + \sum_{k=1}^{\infty} \frac{(0.005586)^k}{k!(\log(CM))^{2k-1}}
\]

\[
= -\sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{0.005587}{\log M} \right)^k + \sum_{k=1}^{\infty} \frac{(0.005586)^k}{k!(\log(CM))^{2k-1}}.
\]  
(9.12)

The summands for \( k \geq 2 \) are obviously negative. For \( k = 1 \), we have

\[
- \frac{0.005587}{\log M} + \frac{0.005586}{\log(CM)} = -\frac{0.005587 \log C - 0.000001 \log M}{(\log M) \log(CM)}.
\]  
(9.13)
The difference in numerator decreases when $M$ increases, so we need only to test at $M = (\log 10) \times 10^{13}$, and the difference is $-0.00003 < 0$. This proves $f(M) < 0$ and hence $N$ satisfies (RI) by (9.8), which contradicts to $N$ being LG.

2) Proof by contradiction. Assume $\log N \leq p_r \left(1 + \frac{0.005589}{\log p_r}\right)$. Substituting (9.2) into the right side of (9.1), we get

$$\log N \left(1 - \frac{0.005587}{\log \log N}\right) > p_r \left(1 + \frac{0.005589}{\log p_r}\right) \left(1 - \frac{0.005587}{\log p_r \left(1 + \frac{0.005589}{\log p_r}\right)}\right)$$

$$> p_r \left(1 + \frac{0.005589}{\log p_r}\right) \left(1 - \frac{0.005587}{\log p_r}\right) > p_r, \quad (9.14)$$

when $p_r > 2.3 \times 10^{13}$. Hence, $N$ satisfies (RI) by proof of 1). This contradicts to $N$ being LG.

\[\square\]

**Theorem 10.** Let $N > 10^{10^{13}}$ be an integer, $p$ be the largest prime factor of $N$. Assume $p$ is the largest prime below $\log N$. If

$$\log N > p_r + \frac{1}{2} \log p + \frac{1}{2} - \frac{1}{2 \log p} + \frac{1}{(\log p)(\log p + 1)}, \quad (10.1)$$

then $G(N) > G(N/p)$.

**Proof.** Write $\log N = p + \frac{1}{2} \log p + d$, where $d$ is a to-be determine expression. By Lemma 1, $G(N) > G(N/p)$ if and only if

$$\frac{1}{2} (\log p)^2 + d \log p + d + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (\frac{1}{2} \log p + d)^{k+1}}{k(k+1)p^k}$$

$$> \frac{1}{2} \log p + \frac{p(\log p)^2}{2 \log N} + \frac{(\log p)^2}{2 \log N} + (p + 1) \sum_{k=1}^{\infty} \frac{(\log N)^{k+2}}{(k + 2)(\log N)^{k+1}} \quad (10.2)$$

Since $p$ is the largest prime below $\log N$, by Proposition 5.4 of [Dusart 2018], for $p \geq 89693$ we have

$$p < \log N \left(1 + \frac{1}{(\log p)^3}\right), \quad (10.3)$$

$$\frac{p}{2 \log N} < \frac{1}{2} \left(1 + \frac{1}{(\log p)^2}\right). \quad (10.4)$$
Since \( N \geq 10^{10^{13}} \), \( \log N > (\log 10) \times 10^{13} \), the last terms on both sides of (10.2) are in order of \( 10^{-13}(\log p)^2 \) and can be absorbed by rounding: the numerator 1 in (10.3) was rounded from 0.998. We can concentrate on main terms. For \( G(N) > G(N/p) \) it suffices to have

\[
\frac{1}{2} (\log p)^2 + d \log p + d > \frac{\log p}{2} + \frac{(\log p)^2}{2} \left( 1 + \frac{1}{(\log p)^2} \right).
\]

Hence

\[
d(\log p + 1) > \frac{1}{2} \log p + \frac{1}{2 \log p}.
\]

\[
d > \frac{\log p + 1}{2(\log p + 1)} = \frac{1 + \frac{1}{(\log p)^2}}{2 \left( 1 + \frac{1}{\log p} \right)} = \frac{1}{2} - \frac{1}{2 \log p} + \frac{1}{(\log p)(\log p + 1)}. \tag{10.6}
\]

**Theorem 11.** Let \( N > 5040 \) be an integer, \( p > \log N \) be a prime and \( p \) is not a factor of \( N \). If

\[
\log N < p - \frac{1}{2} \log p + \frac{1}{2} - \frac{1}{\log p + 1}, \tag{11.1}
\]

then \( G(N) > G(Np) \).

**Proof.** We divide the proof into two cases.

Case 1. \( p \geq N \).

\[
\frac{G(N)}{G(Np)} = \frac{\rho(N) \log \log(Np)}{\rho(Np) \log \log N}
\]

\[
= \frac{\log(\log N + \log p)}{\log \log N} \left( \frac{p}{1 + p} \right)
\]

\[
\geq \frac{\log(2 \log N)}{\log \log N} \left( 1 - \frac{1}{1 + p} \right)
\]

\[
= \left( 1 + \frac{\log 2}{\log \log N} \right) \left( 1 - \frac{1}{1 + p} \right)
\]

\[
= 1 + \frac{\log 2}{\log \log N} - \frac{1}{1 + p} - \frac{\log 2}{(1 + p) \log \log N}
\]

\[
= 1 + \frac{p \log 2 - \log \log N}{(1 + p) \log \log N} > 1. \tag{11.2}
\]
Case 2. $p < N$.

Write $\log N = p - \frac{1}{2} \log p + d$, where $d$ is a to-be-determined expression. By Lemma 2, $G(N) > G(Np)$ if and only if

$$p \sum_{k=1}^{\infty} \frac{(-1)^k (\log p)^{k+1}}{(k+1)(\log N)^k} > \frac{1}{2} (\log p)^2 + d \log p - \frac{1}{2} \log p + d + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2} \log p - d\right)^{k+1}}{k(k+1)p^k}. \quad (11.3)$$

That is, if and only if

$$d(\log p + 1) < \left(\frac{\log p}{2}\right)^2 - \frac{\log p}{2} - \sum_{k=1}^{\infty} \frac{(\frac{1}{2} \log p - d)^{k+1}}{k(k+1)p^k} + p \sum_{k=1}^{\infty} \frac{(-1)^k (\log p)^{k+1}}{(k+1)(\log N)^k}. \quad (11.4)$$

Since $p > \log N$, we can replace $\log N$ with $p$ for all terms with $k \geq 2$.

$$-\sum_{k=1}^{\infty} \frac{(\frac{1}{2} \log p - d)^{k+1}}{k(k+1)p^k} + \sum_{k=1}^{\infty} \frac{(-1)^k p(\log p)^{k+1}}{(k+1)(\log N)^k} > -\frac{p(\log p)^2}{2 \log N} - \sum_{k=1}^{\infty} \frac{(\frac{1}{2} \log p - d)^{k+1}}{k(k+1)p^k} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\log p)^{k+2}}{(k+2)p^k}$$

$$= -\frac{p(\log p)^2}{2(p - \frac{1}{2} \log p + d)} + \sum_{k=1}^{\infty} \left(\frac{(\frac{1}{2} \log p - d)^{k+1}}{k(k+1)p^k} + \frac{(-1)^{k-1}(\log p)^{k+2}}{(k+2)p^k}\right). \quad (11.5)$$

Consider the sum for $k = 2j - 1$ and $k = 2j$,

$$-\frac{(\frac{1}{2} \log p - d)^{2j}}{(2j-1)(2j)p^{2j-1}} + \frac{p(\log p)^{2j+1}}{(2j+1)p^{2j-1}} - \frac{(\frac{1}{2} \log p - d)^{2j+1}}{(2j)(2j+1)p^{2j-1}} = \frac{p(\log p)^{2j}}{p^{2j-1}} \left(\frac{1}{2j-1} + \frac{\log p}{2j+1} - \frac{1}{2j} \left(\frac{1}{2} - \frac{d}{\log p}\right)^{2j} \log p - \frac{\log p}{(2j+2)p}\right)$$

$$> \frac{(\log p)^{2j}}{p^{2j-1}} \left(-\frac{1}{2j} \left(\frac{1}{2} - \frac{d}{\log p}\right)^{2j} \log p - \frac{\log p}{(2j+2)p}\right) > 0. \quad (11.6)$$
So for $G(N) > G(Np)$, it suffices to have
\[ d \log p + d < \frac{1}{2}(\log p)^2 + \frac{1}{2} \log p - \frac{p(\log p)^2}{2(p - \frac{1}{2} \log p + d)}. \] (11.7)

Since
\[ 1 - \frac{p}{p - \frac{1}{2} \log p + d} = -\frac{\log p + 2d}{2p - \log p + 2d} > -\frac{\log p}{2p} > -\frac{1}{(\log p)^2}, \] (11.8)

it suffices to have
\[ d \log p + d < \frac{1}{2} \log p - \frac{1}{2}. \] (11.9)

That is
\[ d < \frac{\log p - 1}{2(\log p + 1)} = \frac{1}{2} - \frac{1}{\log p + 1}. \] (11.10)

\[ \square \]

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