Combinatorics/Lie algebras

The degree of the Hilbert–Poincaré polynomial of PBW-graded modules

Le degré du polynôme de Hilbert–Poincaré pour les modules PBW-gradués

Teodor Backhaus, Lara Bossinger, Christian Desczyk, Ghislain Fourier

Abstract

In this note, we study the Hilbert–Poincaré polynomials for the associated PBW-graded modules of simple modules for a complex Lie algebra. The computation of their degree can be reduced to modules of fundamental highest weight. We provide these degrees explicitly.

Résumé

Nous étudions les polynômes de Hilbert–Poincaré pour les modules PBW-gradués associés aux modules simples d’une algèbre de Lie simple complexe. Le calcul de leur degré peut être restreint aux modules de plus haut poids fondamental. Nous donnons une formule explicite pour ces degrés.

1. Introduction

Let $g$ be a simple complex finite-dimensional Lie algebra with triangular decomposition $g = n^+ \oplus h \oplus n^-$. Then the PBW filtration on $U(n^-)$ is given as $U(n^-)_k := \text{span}\{x_1 \cdots x_k \mid x_i \in n^-, i \leq k\}$. The associated graded algebra is isomorphic to $S(n^-)$. Let $V(\lambda)$ be a simple finite-dimensional module of highest weight $\lambda$ and $\nu_\lambda$ a highest weight vector. Then we have an induced filtration on $V(\lambda) = U(n^-)\nu_\lambda$, denoted $V(\lambda)_k := U(n^-)_k\nu_\lambda$. The associated graded module $V(\lambda)^G$ is an $S(n^-)$-module generated by $\nu_\lambda$.

These modules have been studied in a series of papers. Monomial bases of the graded modules and the annihilating ideals have been provided for the $\mathfrak{sl}_n, \mathfrak{sp}_n$ [8,9,11], for cominuscule weights and their multiples in other types [1], for certain Demazure modules in the $\mathfrak{sl}_n$-case in [13,12]. In type $G_2$, there is a monomial basis provided by [14].

The degenerations of the corresponding flag varieties have been studied in [6,10,3,4]. Further, it turned out [12] that these PBW degenerations have an interesting connection to fusion product for current algebras. The study of the characters of PBW-graded modules has been initiated in [5,7].

In the present paper, we will compute the maximal degree of PBW-graded modules in full generality (for all simple complex Lie algebras), where there have been partial answers in the above series of paper for certain cases.
We denote the Hilbert–Poincaré series of the PBW-graded module, often referred to as the \( q \)-dimension of the module, by

\[ p_\lambda(q) = \sum_{s=0}^\infty (\dim V(\lambda)_s/V(\lambda)_{s-1})q^s. \]

Since \( V(\lambda) \) is finite-dimensional, this is obviously a polynomial in \( q \). In this note we want to study further properties of this polynomial. We see immediately that the constant term of \( p_\lambda(q) \) is always 1 and the linear term is equal to

\[ \dim(n^-) - \dim \ker(n^- \rightarrow \text{End}(V(\lambda))}. \]

Our main goal is to compute the degree of \( p_\lambda(q) \) and the first step is the following reduction [5, Theorem 5.3 ii)]:

**Theorem.** Let \( \lambda_1, \ldots, \lambda_s \in P^+ \) and set \( \lambda = \lambda_1 + \ldots + \lambda_s \). Then

\[ \deg p_\lambda(q) = \deg p_{\lambda_1}(q) + \ldots + \deg p_{\lambda_s}(q). \]

It remains to compute the degree of \( p_\lambda(q) \) where \( \lambda \) is a fundamental weight. We have done this for all fundamental weights of simple complex finite-dimensional Lie algebras:

**Theorem 1.** The degree of \( p_{\omega_i}(q) \) is equal to the label of the i-th node in the following diagrams:

\[
\begin{align*}
A_n & \quad 1 \quad 2 \quad 3 \quad \ldots \quad 3 \quad 2 \quad 1 \\
B_n & \quad 2 \quad 2 \quad 4 \quad 4 \quad 6 \quad \ldots \\
C_n & \quad 1 \quad 2 \quad \ldots \quad \ldots \quad n-2 \quad n-1 \quad \ldots \\
D_n & \quad 2 \quad 2 \quad 4 \quad 4 \quad 6 \quad \ldots \\
E_6 & \quad 2 \quad 4 \quad \ldots \quad 6 \quad 8 \quad 10 \quad 12 \\
E_7 & \quad 2 \quad 6 \quad \ldots \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \\
E_8 & \quad 4 \quad 8 \quad \ldots \quad 12 \quad 16 \quad 20 \quad 24 \\
F_4 & \quad 2 \quad 6 \quad \ldots \quad 10 \quad 14 \quad \ldots \\
G_2 & \quad 2 \quad \ldots \\
\end{align*}
\]

The paper is organized as follows: in Section 2 we introduce definitions and basic notations, in Section 3 we prove Theorem 1.

2. Preliminaries

Let \( \mathfrak{g} \) be a simple Lie algebra of rank \( n \). We fix a Cartan subalgebra \( \mathfrak{h} \) and a triangular decomposition \( \mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \). The set of roots (resp., positive roots) of \( \mathfrak{g} \) is denoted \( R \) (resp. \( R^+ \)), \( \theta \) denotes the highest root. Let \( \alpha_i, \alpha_k, i = 1, \ldots, n \) be the simple roots and the fundamental weights. Let \( W \) be the Weyl group associated with the simple roots and \( w_0 \in W \) the longest element. For \( \alpha \in R^+ \) we fix an \( sI_2 \) triple \([e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]]\). The integral weights and the dominant integral weights are denoted \( P \) and \( P^+ \).

Let \( \{x_1, x_2, \ldots\} \) be an ordered basis of \( \mathfrak{g} \), then \( U(\mathfrak{g}) \) denotes the universal enveloping algebra of \( \mathfrak{g} \) with PBW basis \( \{x_1, \ldots, x_m \mid m \in \mathbb{Z}_{\geq 0}, i_1 \leq i_2 \leq \ldots \leq i_m\} \).

2.1 Modules

For \( \lambda \in P^+ \), we consider the irreducible \( \mathfrak{g} \)-module \( V(\lambda) \) with highest weight \( \lambda \). Then \( V(\lambda) \) admits a decomposition into \( \mathfrak{h} \)-weight spaces, \( V(\lambda) = \bigoplus_{\xi \in P} V(\lambda)_{\xi} \) with \( V(\lambda)_{\lambda} \) and \( V(\lambda)_{w_0(\lambda)} \), the highest and lowest weight spaces, being one dimensional. Let \( v_\lambda \) denote the highest weight vector, \( v_{w_0(\lambda)} \) denote the lowest weight vector satisfying:

\[ e_\alpha v_\lambda = 0, \quad \forall \alpha \in R^+; \quad f_\alpha v_{w_0(\lambda)} = 0, \quad \forall \alpha \in R^+. \]

We have \( U(n^-)v_\lambda \cong V(\lambda) \cong U(n^+)v_{w_0(\lambda)} \).

The comultiplication \( (x \mapsto x \otimes 1 + 1 \otimes x) \) provides a \( \mathfrak{g} \)-module structure on \( V(\lambda) \otimes V(\mu) \). This module decomposes into irreducible components, where the Cartan component generated by the highest weight vector \( v_\lambda \otimes v_\mu \) is isomorphic to \( V(\lambda + \mu) \).
2.2. PBW-filtration

The Hilbert–Poincaré series of the PBW-graded module \( V(\lambda)^q : = \bigoplus_{s \geq 0} V(\lambda)_s / V(\lambda)_{s-1} \) is the polynomial:

\[
p_q(\lambda) = \sum_{s \geq 0} \dim(V(\lambda)_s / V(\lambda)_{s-1}) q^s
\]

\[
= 1 + \dim(V(\lambda)_1 / V(\lambda)_0) q + \dim(V(\lambda)_2 / V(\lambda)_1) q^2 + ... \]

and we define the PBW-degree of \( V(\lambda) \) to be \( \deg(p_q(\lambda)) \).

It is easy to see that \( n^+ \cdot (U(n^+) \cdot v_s) \subseteq U(n^+) \cdot v_s \) \( \forall s \geq 0 \) (see also [8]) and hence \( U(n^+) \cdot V(\lambda)_s \subseteq V(\lambda)_s \). Let \( s_\lambda \) be minimal such that \( v_{w_0(\lambda)} \in V(\lambda)_{s_\lambda} \). Then \( V(\lambda) = U(n^+) \cdot v_{w_0(\lambda)} \subseteq V(\lambda)_{s_\lambda} \) and we have the following corollary:

**Corollary 1.** \( s_\lambda = \deg(p_q(\lambda)) \) and 

\[
V(\lambda) = V(\lambda)_{s_\lambda} .
\]

2.3. Graded-weight spaces

The PBW filtration is compatible with the decomposition into \( \mathfrak{h} \)-weight spaces:

\[
\dim V(\lambda)_\tau = \sum_{s \geq 0} \dim(V(\lambda)_s / V(\lambda)_{s-1}) \cap V(\lambda)_\tau .
\]

So we can define for every weight \( \tau \) the Hilbert–Poincaré polynomial:

\[
p_{\lambda, \tau}(q) = \sum_{s \geq 0} \dim(V(\lambda)_s / V(\lambda)_{s-1}) q^s \quad \text{and then } p_{\lambda}(q) = \sum_{\tau \in P} p_{\lambda, \tau}(q).
\]

A natural question is, if we can extend our results to these polynomials? If the weight space \( V(\lambda)_\tau \) is one-dimensional, then \( p_{\lambda, \tau}(q) \) is a power of \( q \). For \( \tau = \lambda \) this is constant 1, for \( \tau = w_0(\lambda) \), the lowest weight, this is \( q^{\deg p_{\lambda}(q)} \), as we have seen in Corollary 1. A first approach to study these polynomials can be found in [5].

2.4. Graded Kostant partition function

For the readers’ convenience, we recall here the graded Kostant partition function (see [16]), which counts the number of decompositions of a fixed weight into a sum of positive roots, and how it is related to our study. We consider the power series and its expansion:

\[
\prod_{\alpha > 0} \frac{1}{(1 - q e^{\alpha})}, \quad \sum_{\nu \in P} P_{\nu}(q) e^{\nu} .
\]

We have immediately \( \text{char } S(n^-) = \sum_{\nu \in P} P_{\nu}(q) e^{-\nu} \).

**Remark.** For a polynomial \( p(q) = \sum_{i=0}^{n} a_i q^i \), we denote \( \min \deg p(q) \) the minimal \( j \) such that \( a_j \neq 0 \). Then we have obviously:

\[
\min \deg p_{\lambda, \nu}(q) \geq \min \deg p_{\lambda, -\nu}(q) . \tag{2.1}
\]

We will use this inequality for the very special case \( \nu = w_0(\lambda) \) in the proof of Theorem 1.

We see from Theorem 1 that this inequality is a proper inequality for certain cases in exceptional type as well as \( B_n, D_n \) (this has been noticed also in [5]).

3. Proof of Theorem 1

In this section, we will provide a proof of Theorem 1. For a fixed \( 1 \leq i \leq \text{rank } \mathfrak{g} \), we will give a monomial \( u \in U(n^-) \) of the predicted degree mapping the highest weight vector \( v_{\alpha_i} \) to the lowest weight vector \( v_{w_0(\alpha_i)} \). We then show that there is no monomial of smaller degree satisfying this.

To write down these monomials explicitly, let us denote \( \theta_{X_n} \) the highest root of a Lie algebra of type \( X_n \). We set further (using the indexing from [15]):

- in the \( A_{n+1} \)-case, \( Y_{n+1} \) the type of the Lie algebra generated by the simple roots \( \{\alpha_2, \ldots, \alpha_{n+1}\} \);
- in the \( B_n, D_n \)-case, \( Y_{n-k} \) the type of the Lie algebra generated by the simple roots \( \{\alpha_{k+1}, \ldots, \alpha_n\} \);
\[ X_n \omega_0 = 0_{X_n} \]
\[ A_n \omega_1 \]
\[ C_n \omega_1 \]
\[ B_n \omega_{21} \]
\[ B_n \omega_{2i+1} \]
\[ B_n n \ \text{even}, \omega_h \]
\[ B_n n \ \text{odd}, \omega_h \]
\[ D_n \omega_{21} \]
\[ D_n \omega_{2i+1} \]
\[ D_n \ n \ \text{even}, \omega_h, i = n - 1, n \]
\[ D_n \ n \ \text{odd}, \omega_h, i = n - 1, n \]
\[ E_6 \omega_1, \omega_6 \]
\[ E_6 \omega_{21}, \omega_5 \]
\[ E_6 \omega_4 \]
\[ E_7 \omega_2 \]
\[ E_7 \omega_3 \]
\[ E_7 \omega_4 \]
\[ E_7 \omega_7 \]
\[ E_8 \omega_1 \]
\[ E_8 \omega_2 \]
\[ E_8 \omega_4 \]
\[ E_8 \omega_5 \]
\[ E_8 \omega_6 \]
\[ E_8 \omega_7 \]
\[ F_4 \omega_2 \]
\[ F_4 \omega_3 \]
\[ F_4 \omega_4 \]
\[ G_2 \omega_1 \]

Fig. 1. Minimal monomials.

- in the exceptional and symplectic cases, \( \theta_{X_n} = c_k \omega_k \) for some \( k \), \( Y_{n-1} \) the type of the Lie algebra generated by the simple roots \( \{ \alpha_1, \ldots, \alpha_n \} \).}

Let \( u \in U(n^-) \) be one of the monomials in Fig. 1. It can be seen easily from Fig. 1 that \( u = f_{\theta_{X_0}} u_1 \), where \( a_i^w = w_i(h_{\omega_0}) \) and \( u_1 \) is the monomial in Fig. 1 corresponding to the restriction of \( \omega_0 \) to the Lie subalgebra of type \( Y_{n-1} \). If we denote \( n^{-} \) the lower part in the triangular decomposition of the Lie subalgebra of type \( Y_{n-1} \), then \( u_1 \in U(n^-) \).

Let \( u = f_{\theta_1}^i f_{\theta_2}^j \cdots f_{\theta_k}^h \). Note that all \( f_{\theta_j} \) commute and it is easy to see that \( \theta_j(h_{\omega_0}^{j+p}) = 0 \), \( \forall p \geq 0 \) (since \( \theta_j \) is a sum of fundamental weights, which are all orthogonal to the simple roots of the Lie algebra with highest root \( \theta_{j+p} \) and \( b_j = \omega_i(h_{\theta_j}) \)).

The Weyl group \( W \) acts on \( V(\omega_0) \) and if \( v \) is an extremal weight vector of weight \( \mu \), then \( w \cdot v \) is a nonzero extremal weight vector of weight \( w(\mu) \). Further, if \( w = s_\alpha \) (reflection at a root \( \alpha \)) and \( \mu(h_{\alpha}) \geq 0 \), then \( w \cdot v = c^* f_{h_{\omega_0}}^\mu(\omega_{\alpha}) \cdot v \) for some \( c^* \in \mathbb{C}^* \).

Now consider \( w = s_{\theta_1} \cdots s_{\theta_j} \), where \( s_{\theta_j} \) is the reflection at the root \( \theta_j \). Then we have \( w \cdot v_{\omega_0} = v_{w_0(\omega_0)} = u \cdot v_{\omega_0} \neq 0 \) in \( V(\omega_0) \). So we obtain an upper estimate for the degree.

In general, the degree of \( u \) is bigger than the minimal degree coming from Kostant’s graded partition function (2.1). For \( A_n, C_n, \) the degrees coincide and hence we are done in these cases.

We will prove Theorem 1 for the remaining cases \( X_n \) by induction on the rank of the Lie algebra. So we want to prove that if \( p \in U(n^-) \) with \( p \cdot v_{\omega_0} = v_{w_0(\omega_0)} \), then \( \deg(p) \geq \deg(u) \), where \( u \) is from Fig. 1.

Consider the induction start, e.g., \( \omega_0 = \theta_{X_n} \), then the minimal degree is obviously 2. The maximal non-vanishing power of \( f_{\theta_{X_0}} \) is certainly \( a_p^w \) and \( f_{\theta_{X_0}}^{a_p^w} v_{\omega_0} \) is the highest weight vector of a simple module of fundamental weight for the Lie algebra \( Y_{n-1} \) defined as above. By induction, we know that if \( q \in U(n^-) \), with \( q f_{\theta_{X_0}}^{a_p^w} v_{\omega_0} = v_{w_0(\omega_0)} \) then \( \deg(q) \geq \deg(u_1) \).
First we suppose \( f_{\theta \alpha}^\gamma \cdot \nu_{\omega} \) is a factor of \( p \), so \( p = f_{\theta \alpha}^\gamma \cdot p' \) and then by weight considerations \( p' \in U(\mathfrak{n}^-) \). Then \( p' = (f_{\theta \alpha}^\gamma \cdot \nu_{\omega}) = \nu_{\omega}(\mathfrak{n}^-) \) (the lowest weight vector in \( V(\omega) \) as well as in the simple submodule). Therefore \( \deg(p') \geq \deg(\nu) \), which implies \( \deg(p) \geq \deg(\nu) \).

Suppose now the maximal power of \( f_{\theta \alpha} \) in \( p \) is \( f_{\theta \alpha}^{\gamma - k} \), \( k \geq 0 \) and \( \deg(p) < \deg(\nu) \). Let \( X_n \) be of type \( B_n, D_n \) or exceptional, then \( \theta_{X_n} = \omega_j \), and we denote:

\[
R^+ = \{ \alpha \in R^+ | w_j(h_\alpha) = s \}.
\]

Then \( R^+ = \{ \theta_{X_n} \} \) and if \( \beta \in R^+ \) then \( \theta_{X_n} - \beta \in R^+_1 \). By weight reasons, \( p = f_{\theta \alpha}^\gamma \cdot f_{\beta_1} \cdots f_{\beta_k} \cdot p_1 \) for some \( \beta_1, \ldots, \beta_k \in R_1^+ \) and some polynomial \( p_1 \) in root vectors of roots in \( R_0^+ \). We have to show that \( p \cdot \nu_{\omega} = 0 \in V(\omega) \) and we will use an induction on \( k \) for that: the induction start is \( k = 0 \). The induction step is for \( k \geq 1 \):

\[
0 = p_1 \cdot f_{\theta \alpha}^{\gamma - k} \cdot \nu_{\omega} = (e_{\theta \alpha} - \beta_1) \cdots (e_{\theta \alpha} - \beta_2) \cdot p_1 \cdot f_{\theta \alpha}^{\gamma - k} \cdot \nu_{\omega} = c f_{\theta \alpha}^{\gamma - k} \cdot f_{\beta_1} \cdots f_{\beta_2} \cdot p_1 \cdot \nu_{\omega} + \sum_{\ell > 0} f_{\theta \alpha}^{\gamma - k + \ell} \cdot \beta_1 \cdots \beta_\ell \cdot \nu_{\omega}
\]

for some \( c \in \mathbb{C}^*, \beta_\ell \in U(\mathfrak{n}^-) \). For \( 0 \leq \ell < k \), all the summands are zero to equal by induction (on \( k \)). For \( \ell = k \), we recall our assumption \( \deg(p) < \deg(\nu) \) and so \( \deg(\beta_k) < \deg(\nu) \), which implies \( f_{\theta \alpha}^{\gamma - k} \cdot \beta_k \cdot \nu_{\omega} = 0 \). So we can conclude \( f_{\theta \alpha}^{\gamma - k} \cdot f_{\beta_1} \cdots f_{\beta_2} p_1 \cdot \nu_{\omega} = 0 \).

Acknowledgements

T.B. was funded by the DFG-priority program 1388 “Representation Theory”, grant “LI 990/10-1”, G.F. was partially funded the grant “FO 867/1-1”, L.B. and C.D. were partially funded within the framework of this program. The main work of this article has been conducted during a workshop organized at the University of Glasgow, and all authors would like to thank the Glasgow Mathematical Journal Trust Fund, the Edinburgh Mathematical Journal and especially the University of Glasgow for this opportunity.

References

[1] T. Backhaus, C. Dersch, PBW filtration: Feigin–Fourier–Littelmann modules via Hasse diagrams, arXiv:1407.7366, 2014.
[2] R. Biswal, G. Kostant, M. Kostant, Schubert varieties: poset polytopes, PBW-degenerated Demazure modules, and Kogan faces, Preprint, arXiv:1410.1126, 2014.
[3] G. Cerulli Irelli, M. Lanini, Degenerate flag varieties of type A and C are Schubert varieties, Preprint, arXiv:1403.2889, 2014.
[4] G. Cerulli Irelli, M. Lanini, P. Littelmann, Degenerate flag varieties and Schubert varieties, Preprint, 2014.
[5] I. Cherednik, E. Feigin, Extremal part of the PBW-filtration and E-polynomials, Preprint, arXiv:1306.3146, 2013.
[6] E. Feigin, \( \mathbb{G}_M \) degeneration of flag varieties, Sel. Math. New Ser. 18 (3) (2012) 513–537.
[7] E. Feigin, M. Makedonski, Nonsymmetric Macdonald polynomials, Demazure modules and PBW filtration, Preprint, arXiv:1407.6316, 2014.
[8] E. Feigin, G. Fourier, P. Littelmann, PBW filtration and bases for PBW-irreducible modules in type \( A_n \), Transform. Groups, 16 (1) (2011) 71–89.
[9] E. Feigin, G. Fourier, P. Littelmann, PBW filtration and bases for symplectic Lie algebras, Int. Math. Res. Not. 1 (24) (2011) 5760–5784.
[10] E. Feigin, G. Fourier, P. Littelmann, Favourable modules: filtrations, polytopes, Newton–Okounkov bodies and flat degenerations, arXiv:1306.1292v3, 2013.
[11] E. Feigin, G. Fourier, P. Littelmann, PBW-filtration over \( \mathbb{Z} \) and compatible bases for \( \mathfrak{g}(\mathbb{L}) \) in type \( A_n \) and \( C_n \), Springer Proc. Math. Stat. 40 (2013) 35–63.
[12] G. Fourier, New homogeneous ideals for current algebras: filtrations, fusion products and Pieri rules, Preprint, arXiv:1403.4758, 2014.
[13] G. Fourier, PBW-degenerated Demazure modules and Schubert varieties for triangular elements, arXiv:1408.6939, 2014.
[14] A. Grothendieck, Esquisse d’un programme, Exp. No. 355, 1984.
[15] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1972.
[16] B. Kostant, A formula for the multiplicity of a weight, Trans. Amer. Math. Soc. 93 (1959) 53–73.