Explicit constructions of unitary transformations between equivalent irreducible representations

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Abstract
Irreducible representations (irreps) of a finite group $G$ are equivalent if there exists a similarity transformation between them. In this paper, we describe an explicit algorithm for constructing this transformation between a pair of equivalent irreps, assuming that we are given an algorithm for computing the matrix elements of these irreps. Along the way, we derive a generalization of the classical orthogonality relations for matrix elements of irreps of finite groups. We give an explicit form of such unitary matrices for the important case of conjugated Young–Yamanouchi representations, when our group $G$ is the symmetric group $S(N)$.

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1. Introduction
Group representation theory is a powerful tool in physics for studying systems with symmetries. When performing numerical optimization or simulation of physical systems, representation theory can dramatically simplify the required calculations. There are examples of this for the many-electron problem in physics and quantum chemistry [1–3], in quantum information theory [4–13] and elsewhere.

Motivated by this wide range of possible applications, we focus in this paper on relations between irreducible but equivalent representations of a finite group $G$. The irreps of finite groups
can be calculated by using for example the program GAP [21]. Two equivalent representations of group $G$ can be mapped to each other by a similarity transformation determined by a nonsingular matrix $X$. So in fact two sets of matrices representing the group elements in each irrep are conjugated. A general method of solving the conjugacy problem for two arbitrary sets of elements of a finite-dimensional algebra over a large class of fields is described in the paper [14]. The method presented in that paper is based on the solution systems of linear equations, yielding some linear subspaces and subalgebras of the algebra, and it is shown that the generators of the linear subspace (which are in fact a subalgebra module), if they exist, are solutions of the conjugacy problem. In addition, in the case of the matrix algebra over a real algebraic field it is possible, by calculating a square root of some symmetric positive definite matrix, to construct an orthogonal matrix which is the solution of the matrix conjugacy problem. The algorithms that lead to the solution of the conjugacy problem are of polynomial time. In this paper we show that in the case where we have to deal with very special sets of matrices representing the group elements in two equivalent irreducible representations, it is possible to construct a unitary conjugation of these sets in a different way, using very particular properties of irreducible representations of finite groups, and in particular using the orthogonality relations for irreps. As a result we derive an explicit formula for the unitary matrix defining the similarity transformation. In our method, instead of solving the systems of linear equations, one has to find a nonzero normalization factor in the formula for the unitary matrix, which is also a problem of polynomial time, but it seems to be easier to deal with. This is because due to the use of the group properties, we have to deal with a smaller number of equations. We give several examples of how this method works in practice for the permutation group $S(N)$. We also analyze the similarity transforms for a class of equivalent pairs of permutation-group irreps and show that the transformation matrices have a very simple anti-diagonal form. Using general results from the construction, we also formulate a generalization of the well-known classical orthogonality relations for irreps of a finite group $G$.

This paper is organized as follows. In the section 2 we formulate the problem and recall basic statements from group representation theory, which play important roles in subsequent sections. In the section 3 we describe an explicit method for computing unitary transformation matrices connecting two irreducible but equivalent representations for some finite group $G$, and we present the full solution of the problem with details and discussion. In particular we show a few interesting facts regarding some properties of such unitary transformations (the doubly stochastic property, generalized orthogonality property for irreducible representations etc). In section 4 we apply results from section 3 to the symmetric group $S(N)$ and present examples for $N = 3, 4, 5, 6$ which show how our algorithm works in practice. Next, in the section 5 we state and prove theorem 5.16 and proposition 5.17 in which we say that the unitary matrix which maps conjugated Young–Yamanouchi irreps of $S(N)$ consists simply of $\pm 1$ entries along the anti-diagonal. Finally, in section 6 we present a mathematical application of our result to the projector onto a specific subspace suggested by Schur–Weyl duality [16].

2. Preliminaries

In this section we give some basic ideas regarding similarity transformations between irreducible and equivalent irreps of some finite group $G$. Most of the information in this section is taken from [15–18]. We start from the following definition:

**Definition 2.1.** We say that two different irreducible representations (irreps) $\vartheta$ and $\psi$ of the finite group $G$ are equivalent when
where \( n \) is the dimension of the irreps \( \vartheta \) and \( \psi \).

Our task is to find an explicit formula for the transformation matrices \( X \) from definition 2.1.

The form of the matrix \( X \in \text{GL}(n, \mathbb{C}) \) in equation (2.1) in definition 2.1 is strongly restricted by the group \( G \) and its representations \( \vartheta \) and \( \psi \). In fact we have:

**Proposition 2.2.** Suppose that the matrix representations \( \vartheta, \psi \) (which are not necessarily unitary) of finite group \( G \) are irreducible and equivalent; then the matrix \( X \in \text{GL}(n, \mathbb{C}) \) which satisfies

\[
\forall g \in G \quad X^{-1}\vartheta(g)X = \psi(g)
\]

is unique up to a nonzero scalar multiple.

This statement is a corollary of the following:

**Theorem 2.3.** Let \( \vartheta, \psi \) be equivalent matrix irreps of \( G \) (not necessarily unitary). Then the map

\[
\forall g \in G \quad \forall X \in \text{M}(n, \mathbb{C}) \quad \vartheta(g)(X) = \vartheta(g)X\psi(g^{-1})
\]

defines a representation of the group \( G \) in the linear space \( \text{M}(n, \mathbb{C}) \) over \( \mathbb{C} \). The representation \( \Psi \) is reducible and the one-dimensional identity representation of \( G \) is included in \( \Psi \) only once, i.e. there is only one, up to a scalar multiple, matrix \( X \) which satisfies

\[
\forall g \in G \quad \vartheta(g)X\psi(g^{-1}) = X.
\]

**Remark 2.4.** In the particular case where \( \vartheta = \psi \), the matrix \( X \) is up to a scalar multiple equal to \( I \), which in this case generates the identity irrep in the representation \( \Psi \); then the statement of the theorem 2.3 follows directly from Schur’s lemma.

Now let us come back to proposition 2.2. Equation (2.1) from definition 2.1 may be written in the form

\[
\forall g \in G \quad \vartheta(g)X\psi(g^{-1}) = X
\]

given in theorem 2.3 and this theorem states that such a matrix \( X \) is unique up to a scalar multiple.

If the irreps \( \vartheta, \psi \) are unitary, then the matrix \( X \) may be chosen to be unitary; in fact we have:

**Lemma 2.5.** If \( \vartheta \) and \( \psi \) are two different but equivalent unitary and irreducible matrix representations, in \( \text{M}(n, \mathbb{C}) \), of a finite group \( G \), then

\[
\exists U \in U(n); \quad U^\dagger\vartheta(g)U = \psi(g), \quad \forall g \in G.
\]
Proof. We have
\[ X^{-1} \vartheta(g) X = \psi(g), \quad \forall \, g \in G \iff \vartheta(g) X = X \psi(g), \quad \forall \, g \in G, \]  
and from the unitarity of \( \vartheta \) and \( \psi \) we get
\[ X^\dagger \vartheta \left( g^{-1} \right) = \psi \left( g^{-1} \right) X^\dagger, \quad \forall \, g \in G, \]  
and
\[ X^\dagger X = \psi \left( g^{-1} \right) X^\dagger \psi \left( g \right), \quad \forall \, g \in G \iff \psi \left( g \right) X^\dagger X = X^\dagger \psi \left( g \right), \quad \forall \, g \in G. \]  
The irreducibility of the representation \( \psi \) and the Schur’s lemma imply
\[ X^\dagger X = \alpha \mathbf{1}_n; \quad \alpha > 0 \Rightarrow X^\dagger = \alpha X^{-1}. \]  
Define
\[ U = \frac{1}{\sqrt{\alpha}} X \Rightarrow U^\dagger U = \frac{1}{\alpha} X^\dagger X = \mathbf{1}_n; \]  
then the matrix \( U \) is unitary and satisfies
\[ U^\dagger \vartheta \left( g \right) U = \sqrt{\alpha} X^{-1} \vartheta \left( g \right) \frac{1}{\sqrt{\alpha}} X = \psi \left( g \right), \quad \forall \, g \in G. \]  
In the following we will assume that irreps \( \vartheta \) and \( \psi \) are unitary, and our task is to find an explicit formula for the unitary matrix \( U \), which gives the similarity transformation between the representations \( \vartheta \) and \( \psi \). Such a matrix \( U \) is not unique, and we have.

Lemma 2.6. If \( U \) is such that
\[ U^\dagger \vartheta \left( g \right) U = \psi \left( g \right), \quad \forall \, g \in G, \]  
then \( U' = e^{\mu U}; \quad \mu \in \mathbb{R} \) also defines a unitary similarity between \( \vartheta \) and \( \psi \)
\[ U'^\dagger \vartheta \left( g \right) U' = \psi \left( g \right), \quad \forall \, g \in G. \]  
In the next section we show how to construct unitary transformation matrices \( U \) which map between two equivalent irreps of some finite group \( G \).

3. The general method of construction

In this section we present an explicit construction method for obtaining the unitary matrices which represent similarity transformations between irreducible but equivalent representations of some finite group \( G \).

In order to derive the formula for the matrix \( U \), we consider equation (2.13) which contains all of the conditions on \( U \). The RHS of equation (2.13) for \( U = (u_{ij}) \) may be written in the following way
\[ \sum_{m} u_{im}^\dagger \vartheta\left( u_{ij} \right) u_{ji} = \sum_{m} \pi_{ib} \pi_{ia} \vartheta\left( g \right) = \sum_{m} \left( \mathcal{U} \otimes \mathcal{U} \right)_{mb,jb} \vartheta\left( g \right), \]  
\[ \]
therefore the equation for $U = (u_{ij})$ takes the form
\[\sum_{st} (U \otimes U)_{st,ij} \theta_{st}(g) = \sum_{st} (U \otimes U)_{st,ij} \theta_{st}(g) = \psi_{ij}(g), \quad \forall \; g \in G, \tag{3.2}\]
where we have used
\[X = (x_{ij}), \quad Y = (y_{kl}) \Rightarrow (X \otimes Y)_{jk,il} = x_{jk}y_{il}. \tag{3.3}\]
In the next step we use the orthogonality relations for the irreducible representations of finite groups, which may be formulated as follows:

**Proposition 3.1.** Suppose that $\vartheta$ and $\psi$ are two irreducible matrix representations of a finite group $G$. Then
\[\sum_{g \in G} \psi_{ij}(g) \theta_{kl}(g^{-1}) = \begin{cases} 0 & \text{if } \psi \text{ and } \vartheta \text{ are inequivalent}, \\ \frac{|G|}{n} \delta_{jk} \delta_{il} & \text{if } \psi \text{ and } \vartheta \text{ are equal}, \end{cases} \tag{3.4}\]
where by $|G|$ we denote the cardinality of the group $G$. Equation (3.4) does not apply if $\psi$ and $\vartheta$ are equivalent but not equal.

Using this proposition, we get
\[\sum_{jk} \sum_{g \in G} (\overline{U} \otimes U)^{j}_{bj,ki} \theta_{st}(g) = \sum_{g \in G} \psi_{ij}(g) \theta_{st}(g^{-1}) = \sum_{g \in G} \psi_{ij}(g) \theta_{st}(g^{-1}) = \frac{|G|}{n} \sum_{g \in G} \psi_{ij}(g) \theta_{st}(g^{-1}) = \frac{|G|}{n} \sum_{g \in G} \psi_{ij}(g) \theta_{st}(g^{-1}), \tag{3.5}\]
and finally we get the equation wherein the desired matrix $U$ and the given representations $\vartheta$ and $\psi$ are separated:
\[\overline{U} \otimes U = A_{ai, bj}, \quad \overline{U} \otimes U = A. \tag{3.7}\]

Now we have to extract the matrix $U$ from this equation.

This equation is, in fact, the matrix equation in $M(n^2, \mathbb{C})$ and on the LHS we have a tensor product block structure where the blocks are of the form
\[A_{ab} = U_{ab} \in M(n^2, \mathbb{C}) \tag{3.8}\]
and if $\pi_{ab} = r_{ab} e^{i\omega_{ab}} \neq 0$, then
\[e^{i\omega_{ab}}U_{ij} = \frac{1}{r_{ab}} \frac{n}{|G|} \sum_{g \in G} \theta_{ai}(g) \psi_{bj}(g), \tag{3.9}\]
where, from lemma 2.6, the matrix $U' = e^{i\omega_{ab}}U$ also gives the similarity transformation that we are looking for. Thus in order to get the explicit formula for a unitary matrix connecting $\vartheta$ and $\psi$ by a similarity transformation, we have to know for which indices $(a, b)$ the weight $r_{ab}$ is not equal to zero. From equation (3.8) we get
which obviously shows that if $r_{ab} = 0$, then the corresponding block $A_{ab}$ in $A$ is a zero matrix. On the other hand, direct calculation gives

$$\|A_{ab}\|_{tr} = \frac{n}{|G|} \left( \sum_{g \in G} \delta_{ab}(g) \psi_{bb}(g^{-1}) \right)^\frac{1}{2},$$

\( (3.11) \)

and therefore

$$r_{ab} = \frac{n}{|G|} \left( \sum_{g \in G} \delta_{ab}(g) \psi_{bb}(g^{-1}) \right)^\frac{1}{2}. \quad (3.12)$$

The weight $r_{ab}$, as a function of indices $(a, b)$, indicates which elements $u_{ab}$ of the matrix $U$ are nonzero and consequently which blocks $A_{ab} \in M(n, \mathbb{C})$ in the block matrix $A$ are nonzero.

Summarizing, we get:

**Theorem 3.2.** Suppose that $\theta$ and $\psi$ are two different but equivalent unitary and irreducible matrix representations, in $M(n, \mathbb{C})$, of a finite group $G$. Then

1. there exist indices $a, b = 1, \ldots, n = \dim \psi$ such that

   $$\sum_{g \in G} \delta_{ab}(g) \psi_{bb}(g^{-1}) > 0, \quad (3.13)$$

2. the matrix $U = (u_{ij})$ that determines the similarity transformation

   $$U^\dagger \theta(g) U = \psi(g), \quad \forall g \in G, \quad (3.14)$$

has the following form

$$u_{ij} \equiv u_{ij}(ab) = \frac{1}{r_{ab} |G|} \sum_{g \in G} \delta_{ab}(g) \psi_{bb}(g), \quad (3.15)$$

where

$$r_{ab} = \frac{n}{|G|} \left( \sum_{g \in G} \delta_{ab}(g) \psi_{bb}(g^{-1}) \right)^\frac{1}{2}. \quad (3.16)$$

and the $(a, b)$ are chosen in such a way that $r_{ab} > 0$, which is possible from statement (1).

**Remark 3.3.** In point (1) of theorem 3.2 there are maximally $n$ equations that we need to check to find a nonzero factor $r_{ab}$. The problem of finding a nonzero coefficient $r_{ab}$ can be realized in polynomial time.

**Remark 3.4.** If the representations $\theta$ and $\psi$ are orthogonal, then the matrix $U$ is also orthogonal.

**Remark 3.5.** For fixed values $a, b = 1, \ldots, n$ the unitary matrix $U = U(ab)$ in equation (3.15) in theorem 3.2 is determined in a unique way by irreps $\psi, \theta$. From lemma
2.6 it follows that for arbitrary \( \alpha \in \mathbb{R} \) the unitary matrix \( U' = e^{i\alpha} U(ab) \) also gives the similarity transformation equation (3.14).

**Remark 3.6.** From equation (3.15) and the orthogonality relations (proposition 3.1) it follows that if the irreducible representations \( \vartheta \) and \( \psi \) were not equivalent, then we would have \( U = 0 \), in agreement with Schur lemma.

**Remark 3.7.** From the unitarity of the matrix \( U = (u_{ab}) \) it follows that

\[
\forall \ a = 1, \ldots, n \quad \sum_b r_{ab}^2 = 1, \quad \forall \ b = 1, \ldots, n \quad \sum_a r_{ab}^2 = 1,
\]  

(3.17)

so the matrix with elements \( r_{ab}^2 \) is doubly stochastic.

From theorem 3.2 and in particular from equation (3.15) one can deduce the following corollary which is a generalization of the classical orthogonality relation for irreps of finite group \( G \) given in proposition 3.1, which plays a very important role in the theory of group representation.

**Corollary 3.8.** Let \( \psi, \vartheta \) be unitary, different but equivalent irreps of \( G \). Then there exists a unitary matrix \( U = (u_{ab}) \): \( r_{ab} = |u_{ab}| \) such that

\[
r_{ab}u_{ij} = \frac{n}{|G|} \sum_{g \in G} \vartheta_{ia}(r^{-1}) \psi_{bj}(g).
\]

(3.18)

In particular, when \( \vartheta = \psi \), then \( U = 1 \) and thus formula (3.18) takes the form of a classical orthogonality relation for irrep \( \vartheta \)

\[
\frac{n}{|G|} \sum_{g \in G} \vartheta_{ia}(r^{-1}) \delta_{ij} = \delta_{ab}\delta_{ij}.
\]

(3.19)

4. Examples relating to symmetric groups

In this section we show a few examples of unitary matrices obtained by application of theorem 3.2 (up to a global phase) for the symmetric group \( S(N) \) for some small \( N \). We start from the simplest examples for \( S(3) \).

**Example 4.1.** Consider two different but equivalent representations of the group \( S(3) \)

\[
\psi^\epsilon(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \psi^\epsilon(13) = \begin{pmatrix} 0 & \epsilon \\ \epsilon^{-1} & 0 \end{pmatrix}, \quad \psi^\epsilon(23) = \begin{pmatrix} 0 & \epsilon^{-1} \\ \epsilon & 0 \end{pmatrix},
\]

(4.1)

\[
\psi^\epsilon(123) = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \psi^\epsilon(132) = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}.
\]

(4.2)
where \( \epsilon^3 = 1 \), and

\[
\varphi(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varphi(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \varphi(23) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.
\]

(4.3)

\[
\varphi(123) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \varphi(132) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.
\]

(4.4)

Applying the theorem, we get

\[
n_1 = \sqrt{\frac{n}{|G|} \sum_{g \in S(3)} \varphi_{11}(g) \varphi_{11}(g^{-1})} = \frac{1}{\sqrt{2}},
\]

(4.5)

and

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \frac{1}{\sqrt{3}} (\epsilon - \bar{\epsilon}) & -\frac{1}{\sqrt{3}} (\epsilon - \bar{\epsilon}) \end{pmatrix}.
\]

(4.6)

**Example 4.2.** It is clear that the representations \( \psi^\epsilon \) and \( \varphi = \psi^\epsilon \) are equivalent. In this case the theorem gives

\[
n_1 = 0, \quad n_2 = 1,
\]

(4.7)

and

\[
U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

(4.8)

which are obvious without applying the theorem.

**Example 4.3.** It is also known that for \( S(3) \) the representations \( \psi^\epsilon \) and \( \varphi = \text{sgn} \psi^\epsilon \) are equivalent\(^4\). Again applying the theorem, we get

\[
n_1 = 1
\]

(4.9)

and

\[
U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(4.10)

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\(^4\) Here we use the following convention: by \( \text{sgn} \) we understand the so-called signum representation for the group \( S(N) \); by \( \text{sgn} (\sigma) \) for some \( \sigma \in S(n) \) we understand parity of the permutation \( \text{sgn} \) \( S(n) \rightarrow \{-1, 1\} \), which is defined as follows:

\[
\forall \sigma \in S(n) \quad \text{sgn} (\sigma) = (-1)^{N(\sigma)},
\]

where \( N(\sigma) \) is the number of inversions in \( \sigma \).
Now we present a few examples of unitary matrices from lemma 2.5 for irreducible representations of symmetric groups $S(N)$ for some small $N$ using directly formulas 3.15 and 3.16 from theorem 3.2. These matrices map conjugated irreps calculated in the Young–Yamanouchi basis which we describe further in this section. To obtain these results we wrote code in Mathematica 7, and examples 4.4, 4.5 and 4.13 are calculated using the Young–Yamanouchi formalism (here we refer the reader to [15–20] or further parts of this paper).

**Example 4.4.** In this example we present unitary transformations between conjugated Young–Yamanouchi irreps for the symmetric group $S(4)$. We restrict our attention to partitions $\lambda_1 = (3, 1)$ and $\lambda_2 = (2, 2)$, so this means that our unitary matrices transform irreps on partitions $\lambda_i$ to irreps on $\text{sgn} \lambda_i$, where $i = 1, 2$. We will use this convention also in the next example.

$$U_{\lambda_1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad U_{\lambda_2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (4.11)$$

**Example 4.5.** In this example we present unitary transformations between conjugated Young–Yamanouchi irreps for the symmetric group $S(5)$. We restrict our attention to partitions $\lambda_1 = (4, 1)$, $\lambda_2 = (3, 2)$ and $\lambda_3 = (3, 1, 1)$.

$$U_{\lambda_1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad U_{\lambda_2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad U_{\lambda_3} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (4.12)$$

**Example 4.6.** In this example we present unitary transformations between conjugated Young–Yamanouchi irreps for the symmetric group $S(6)$. We restrict our attention to partitions $\lambda_1 = (5, 1)$, $\lambda_2 = (4, 2)$, $\lambda_3 = (4, 2)$, $\lambda_4 = (4, 1, 1)$, $\lambda_5 = (3, 3)$ and $\lambda_6 = (3, 2, 1)$.

$$U_{\lambda_1} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad U_{\lambda_2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad U_{\lambda_3} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (4.13)$$
These examples calculated in the Young–Yamanouchi basis suggest that all unitary matrices representing similarity relations between Young–Yamanouchi conjugated irreps have, quite simply, anti-diagonal form with entries ±1 only. In the next section we prove this conjecture. As we mentioned in the introduction, our method from section 3 can be applied to any finite group $G$ for which we have a characterization of its irreps. To obtain the desired matrix elements of irreps one can use for example the program GAP [21], which returns a list of representatives of the irreducible matrix representations of $G$ over field $F$, up to equivalence.

5. Analytical formulas for similarity relations

The main goal of this section is to prove that all unitary matrices which map Young–Yamanouchi conjugated irreps (labeled with $\lambda$ and $\lambda'$, where $\lambda$ is a partition of $N \in \mathbb{N}$; see equation (5.3) below) have anti-diagonal form with entries ±1. Namely we will show that the unitary matrix $U$ which transforms the irreducible representation $D^{\lambda}$ of $S(N)$ calculated in the Young–Yamanouchi basis into the equivalent irreducible representation $\text{sgn} D^{\lambda'}$ has anti-diagonal form with entries ±1 only. In fact the matrix $U$ is of the form...
\[
U = \begin{pmatrix}
0 & 0 & \cdots & \text{sgn } (\sigma_1) \\
0 & \text{sgn } (\sigma_2) & 0 & \vdots \\
0 & \cdots & \vdots & 0 \\
\text{sgn } (\sigma_{d-1}) & \cdots & 0 & 0 \\
\text{sgn } (\sigma_d) & 0 & \cdots & 0
\end{pmatrix},
\]  

(5.1)

where the permutations \(\sigma_i\) are described in proposition 5.13, equation (5.19), below. We show also proposition 5.17 which states that our unitary transformation can be written as

\[
U = \sum_{T} \text{sgn } (T_{\lambda}^\sigma)|T_{\lambda}^\sigma\rangle\langle T_{\lambda}^\sigma|,
\]

(5.2)

where \(\text{sgn } (T_{\lambda}^\sigma) = \text{sgn } (\sigma)\), and \(\sigma\) is the permutation that transforms an arbitrarily chosen, fixed SYT \(T_{\lambda}^\sigma\) into \(T_{\lambda}\) (see proposition 5.5 and remark 5.6). We also argue that for different choices of \(\lambda_{\lambda}\), the corresponding \(U\) may differ by a global sign.

In the next part of this section we will prove the above statements. In order to do this, firstly we have to introduce briefly the concept of irreducible representations of the group \(S(N)\) based on the concepts of the natural Young representation and Yamanouchi symbols; therefore we call such representations Young–Yamanouchi representations.

As is known, any irreducible representation of the group \(S(N)\) is uniquely determined by a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\), where

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0, \quad \sum_{i=1}^{k} \lambda_i = n.
\]

(5.3)

With each partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) there is associated a Young diagram (YD), also called a Young frame (see example 5.1), with \(\lambda_i\) boxes in the \(i\)th row, and the rows of boxes lined up on the left. The partition \(\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_k)\) conjugate to the partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) is defined by interchanging rows and columns of the Young diagram \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\). For example, if \(\lambda = (3, 2, 2, 1)\), then \(\lambda' = (4, 3, 1)\). In general, for an arbitrary partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\), we can obtain the conjugate partition \(\lambda'\) using the formula

\[
(\lambda_1, \lambda_2, \ldots, \lambda_k)' = (k^{\lambda_k}, (k-1)^{\lambda_{k-1}-\lambda_k}, \ldots, 2^{\lambda_2-\lambda_1}, 1^{\lambda_1-\lambda_1}),
\]

(5.4)

where the notation \(j^m\) denotes that integer \(j\) is to be repeated \(m\) times with \(m = 0\) meaning no occurrence.

**Example 5.1.** In this example we show explicitly all Young diagrams for \(N = 4\) together with the corresponding partitions \(\lambda\).

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\]

\(\lambda = (1, 1, 1, 1), \quad \lambda = (2, 1, 1), \quad \lambda = (2, 2), \quad \lambda = (3, 1), \quad \lambda = (4)\)

**Definition 5.2.** A Young tableau (YT) of a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) is a Young diagram \(\lambda\) in which the boxes are filled bijectively with numbers \(\{1, 2, \ldots, n\}\). The Young tableau will
be denoted as $T_i = (t_{ij}^i)$, where $t_{ij}^i \in \{1, 2, \ldots, n\}$ denotes the entry of $T_i$ in the position $(i, j)$. There are a total of $n!$ YT.

A standard Young tableau (SYT) is a Young tableau for which the numbers $\{1, 2, \ldots, n\}$ appear in the rows of the tableau in the sequences increasing to the right and in the columns of the tableau in the sequences increasing from the top to downwards. An SYT will be denoted as $T_i = (t_{ij}^i)$. The standard Young tableau $T^T_{ij}$ conjugate to the standard Young tableau $T_i$ is defined by interchanging the rows and columns (together with the numbers contained in them) of the standard Young tableau $T_i$. Thus the conjugated standard Young tableau $T^T_{ij}$ is a standard Young tableau for the conjugated partition $\lambda^t = (\lambda_1^t, \lambda_2^t, \ldots, \lambda_l^t)$.

**Example 5.3.** Here we present Young tableaux (YT) and standard Young tableaux (SYT) for $N = 3$ and the partition $\lambda = (2, 1)$.

- **All Young tableaux:**
  
  \[
  \begin{array}{ccc}
  1 & 2 & 3 \\
  3 & 2 & 1 \\
  
  \end{array}
  \]

- **All standard Young tableaux:**
  
  \[
  \begin{array}{cc}
  1 & 2 \\
  3 & 1 \\
  
  \end{array}
  \]

Now one can define, in a natural way, the action of the group $S(N)$ on the set of all YT.

**Definition 5.4.**

\[\forall \sigma \in S(N) \quad \sigma \left( T_i \right) = \sigma \left( t_{ij}^i \right) \equiv \left( \sigma \left( t_{ij}^i \right) \right), \quad (5.5)\]

i.e., a permutation $\sigma \in S(N)$ acts on each entry of the Young tableau $T_i$.

Note that this action of the group $S(N)$ is well defined on the set of all YT and it is not well defined on the subset of SYT because the action of $\sigma \in S(N)$ on the standard Young tableau $T_i$ may give a YT which is not an SYT. For a given standard Young tableau $T_i$, only the particular permutations $\sigma \in S(N)$ are such that $\sigma (T_i)$ is an SYT. From definition 5.4 it follows that $S(N)$ acts on the set of YT transitively and, moreover, we have:

**Proposition 5.5.** Choosing YT $T^T_{ij}$ which is in fact an SYT with the canonical embedding\(^5\), we establish a bijective correspondence between $S(N)$ and the set of YT given by the following relation:

\[\forall T_i \quad \exists ! \sigma \in S(N) \quad \sigma T^T_{ij} = T_i. \quad (5.6)\]

In particular we have

\[\forall T_i \quad \exists ! \sigma \in S(N) \quad \sigma T^T_{ij} = T_i, \quad (5.7)\]

where the symbol $\exists !$ means ‘there exists a unique’.

---

\(^5\) By an SYT with canonical row embedding we understand a Young tableau of the shape $\lambda$ filled with numbers $1, \ldots, N$ in such a way that starting from the top left corner we put $1$ into the first box, then we put $2$ into the second one on the right in the same row. We continue this procedure up to $N$. The reader may see example 5.3, where all SYT for $\lambda = (2, 1)$ are presented (second row). The canonical embedding is presented by the first SYT from the left. In a similar way we can define canonical column embedding.
**Remark 5.6.** Obviously, in general, for a given $T_i$ (or $T_j$) the permutation $\sigma$ in equations 5.6, 5.7 depends on the choice of $T_i^1$, but as we will see, this dependence is not important for the properties of the matrix $U$.

In our further considerations, the SYT will be the most important because, as we will see, they will label the bases of the Young–Yamanouchi irreducible representations of the symmetric group $S(N)$.

In the Young–Yamanouchi irreducible representations of the group $S(N)$ the concept of axial distance plays an important role.

**Definition 5.7.** The axial distance $\rho(T_i; i, j)$ between the boxes $i, j$ in the standard Young tableau $T_i$ is the number of horizontal or vertical steps needed to get from $i$ to $j$. Each step is counted as $+1$ if it goes down or to the left and its counts as $-1$ if it goes up or to the right.

The axial distance has the following properties, which follow directly from its definition:

**Proposition 5.8.**

$$\rho(T_i; i, j) = -\rho(T_j; j, i), \quad \rho(T_j; i, j) = -\rho(T_i; i, j).$$

The SYT can be characterized in simple way using so-called Yamanouchi symbols, as follows.

**Definition 5.9.** For any standard Young tableau $T_i$ we define a row Yamanouchi symbol (RYS) as a row of $n$ numbers

$$M_i = (M_1(\lambda), M_2(\lambda), \ldots, M_n(\lambda)), \quad (5.9)$$

where $M_i(\lambda)$ is the number of the row in the standard Young tableau $T_i$ in which the number $i$ is contained. Similarly a column Yamanouchi symbol (CYS) is defined also as a row of $n$ numbers

$$N_i = (N_1(\lambda), N_2(\lambda), \ldots, N_n(\lambda)), \quad (5.10)$$

where now $N_i(\lambda)$ is the number of the column in $T_i$ in which the number $i$ appears.

From the definitions of the standard Young tableau and of the Yamanouchi symbols it follows that for a given Young diagram $\lambda$ the row Yamanouchi symbol $M_i$ characterizes uniquely the corresponding standard Young tableau $T_i$. Similarly we have a bijective correspondence between the column Yamanouchi symbols $N_i$ and the standard Young tableau $T_i$. The symbols $M_i$ and $N_i$ both characterize in a unique way the corresponding Young diagram and the standard Young tableau, and in the notation of definition 5.2 we have

$$T_i = \left( t_{iM_i(i)N_i(i)}(\lambda) \right). \quad (5.11)$$

Directly from the definition of RYS and CYS, we get:

**Proposition 5.10.** Let $T_i$ be an SYT, with $M_i = (M_1(\lambda), M_2(\lambda), \ldots, M_n(\lambda))$ and $N_i = (N_1(\lambda), N_2(\lambda), \ldots, N_n(\lambda))$ the corresponding RYS and CYS respectively. For the
conjugated standard Young tableau $T^k_\lambda$ we denote by $M^k_\lambda = (M_1(\lambda'), M_2(\lambda'), \ldots, M_n(\lambda'))$ and $N^k_\lambda = (N_1(\lambda'), N_2(\lambda'), \ldots, N_n(\lambda'))$ the corresponding RYS and CYS. Then we have

\[ M^k_\lambda = N^k_\lambda, \quad N^k_\lambda = M^k_\lambda, \quad (5.12) \]

i.e., the RYS (respectively CYS) for the conjugated standard Young tableau $T^k_\lambda$ is equal to the CYS (respectively RYS) of $T_\lambda$.

The advantage of the description of the SYT in terms of Yamanouchi symbols is that one can easily introduce the linear (lexicographic) ordering in the set of all RYS (respectively CYS) symbols for a given Young diagram $\lambda$. In fact we have:

**Definition 5.11.** Let $M_\lambda = (M_1(\lambda), M_2(\lambda), \ldots, M_n(\lambda))$ and $M'_\lambda = (M'_1(\lambda), M'_2(\lambda), \ldots, M'_n(\lambda))$ be two RYS; then $M_\lambda$ is smaller than $M'_\lambda$, which will be denoted by $M_\lambda < M'_\lambda$, if

\[ \exists j \in \{1, 2, \ldots, n\}: M_i(\lambda) = M'_i(\lambda) \quad i < j \quad \land \quad M_j(\lambda) < M'_j(\lambda), \quad (5.13) \]

and similarly for the CYS.

Obviously the linear order in the RYS or in the CYS induces linear order of the SYT, but these orders are not the same. In fact, using this definition as well as the definitions of the RYS and CYS, it is not difficult to prove the following statement:

**Proposition 5.12.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a Young diagram and suppose that all RYS describing all SYT for $\lambda$ are ordered in the following way

\[ M^1_\lambda < M^2_\lambda < \ldots < M^k_\lambda, \quad (5.14) \]

then,

\[ N^1_\lambda > N^2_\lambda > \ldots > N^k_\lambda, \quad (5.15) \]

where $M^j_\lambda$ and $N^j_\lambda$ are respectively the RYS and CYS of the standard Young tableau $T^j_\lambda$. Thus the ordering of the SYT induced by the linear ordering of the RYS is opposite to the ordering of the SYT induced by the linear ordering of the CYS. One can see that in 5.14, the symbol $M^j_\lambda$ corresponds to the canonical row embedding, while $M^k_\lambda$ corresponds to the canonical column embedding.

It is clear that the action of the group $S(N)$ on the SYT induces the action of $S(N)$ on the RYS and CYS of the SYT, and if the standard Young tableaux $T_j$ (with $M_j$ and $N_j$), $T'_j$ (with $M'_j$ and $N'_j$) with $\sigma \in S(N)$ are such that

\[ \sigma(T_j) = T'_j, \quad (5.16) \]

then

\[ \sigma(M_j) = M'_j, \quad \sigma(N_j) = N'_j. \quad (5.17) \]

From proposition 5.5 and proposition 5.12 we have the following:

**Proposition 5.13.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a Young diagram and suppose that all RYS describing all SYT for $\lambda$ are linearly ordered

\[ M^1_\lambda < M^2_\lambda < \ldots < M^k_\lambda, \quad (5.18) \]
where $M_i^j$ corresponds to $T_j^i$. Then for any $M_i^j$, $i = 1, 2, \ldots, k$, there exists a unique permutation $\sigma_i \in S(N)$ such that

$$M_i^j = \sigma_i \left( M_i^j \right), \quad \sigma_i \equiv \text{id.} \hspace{1cm} (5.19)$$

So we have a bijective concordance between the set of all SYT of a given Young diagram $\lambda$ and a subset of permutations in $S(N)$.

Now we describe a construction of Young–Yamanouchi irreducible representations of the group $S(N)$. The dimension of an irreducible representation of $S(N)$ indexed by a partition $\lambda$ is determined by the corresponding Young diagram and it will be denoted as $d_\lambda$. The construction of irreducible representations of $S(N)$ is based on the fact that the basis vectors of the representation space may be indexed by the set of SYT for the Young diagram $\lambda$. Because, for a given $\lambda$, any standard Young tableau $T_j$ may be described uniquely by the corresponding row Yamanouchi symbol $M_j$, it is the case that the basis vectors in the representation space may be labeled with $M_j$, which additionally introduces a linear order in the set of basis vectors in the representation space (see definition 5.11 and proposition 5.12). So the orthonormal basis of the representation space for the Young diagram $\lambda$ will be denoted in the following way

$$\{ e_{M_j}^i : i = 1, 2, \ldots, d_j \}, \hspace{1cm} (5.20)$$

and the order of the basis is induced by the order of the RYS: $M_1^j < M_2^j < \ldots < M_{d_j}^j$.

It is known that the symmetric group $S(N)$ is generated by transpositions of the form $(k, k + 1)$, $k = 1, 2, \ldots, n - 1$; thus in order to define a representation of $S(N)$

$$D^j : S(N) \rightarrow \text{Hom} \left( \text{span}_C \left\{ e_{M_j}^i : i = 1, 2, \ldots, d_j \right\} \right) \hspace{1cm} (5.21)$$

it is enough to define the representation operators for the generators $(k, k + 1)$ only. By definition, these generators act on the basis vectors $\{ e_{M_j}^i : i = 1, 2, \ldots, d_j \}$ in the following way

$$D^j (k, k + 1) \left( e_{M_j}^i \right) = \rho^{-1} (T_j; k + 1, k) e_{M_j}^i + \sqrt{1 - \rho^{-2} (T_j; k + 1, k)} e_{M_j}^i, \hspace{1cm} (5.22)$$

where the second term in the RHS appears only if $(k, k + 1)T_j$ is an SYT—in this case $(k, k + 1)M_j = M_j^j$, $j = 1, \ldots, d_j$.

It is known that for any irreducible representations $D^\lambda$ of $S(N)$ the composition of representations $\text{sgn} D^\lambda$ is also a representation of $S(N)$ and, moreover, we have the following:

**Lemma 5.14.** [18] Suppose that we are given two inequivalent irreducible representations $D^\lambda$ and $\text{sgn} D^\lambda$, where $\lambda^\dagger$ denotes the partition dual to $\lambda$. Then the irreducible representations $\text{sgn} D^\lambda$ and $D^\lambda$ are isomorphic.

From this lemma it follows that

$$\exists \ U \in U \left( d_j \right) \quad \forall \ \sigma \in S(N) \quad D^\sigma (\sigma) = \text{sgn} (\sigma) UD^\lambda (\sigma) U^\dagger. \hspace{1cm} (5.23)$$

The examples calculated in the previous section suggest that this matrix $U$ has a very simple anti-diagonal form with $\pm 1$ on the anti-diagonal as in equation (5.1). Now we are ready to prove the hypothesis.
We have two irreducible representations $D^\lambda$ and $D^{\lambda'}$ of the group $S(N)$ acting respectively in the representation spaces $\mathfrak{g}_i : i = 1, 2, \ldots, d_i$ and $\mathfrak{g}_i : i = 1, 2, \ldots, d_i$. From proposition 5.10 we get
\[ e_{M_i}^{\lambda'} = e_{N_i}^{\lambda} : i = 1, 2, \ldots, d_i \] (5.24)
and from proposition 5.12 it follows that the base $\{ e_{M_i}^{\lambda'} : i = 1, 2, \ldots, d_i \}$ has an opposite order with respect to the order of the basis $\{ e_{M_i}^{\lambda} : i = 1, 2, \ldots, d_i \}$. Now let us consider a unitary transformation between these bases:
\[ U\left( e_{M_i}^{\lambda} \right) \equiv \text{sgn} (\sigma_i) e_{N_i}^{\lambda} = \text{sgn} (\sigma_i) e_{M_i}^{\lambda'}, \] (5.25)
where the $\sigma_i$ are defined in proposition 5.13.

**Remark 5.15.** Using the isomorphism $V^* \otimes V \cong \text{End}(V)$, where $V$ is a linear space and $V^*$ is a dual of $V$, the unitary transformation $U$ may be written in the following operator form:
\[ U = \sum_{M_i} \text{sgn} (\sigma_i) e_{M_i}^{\lambda^*} \otimes e_{M_i}^{\lambda'} = \sum_{T_k} \text{sgn} \left( T_k \right) \left| T_k \right|^\dagger \left( T_k \right), \] (5.26)
where $\text{sgn} (T_k) = \text{sgn} (\sigma_i)$ (props. 5.5, 5.13) and in the last equation we have introduced a physical 'bra, ket' notation: $e_{M_i}^{\lambda^*} \equiv \left| T_k \right|^\dagger$. Note also that if we chose another SYT as $T_k^\dagger$ in proposition 5.5, then the corresponding matrix $U$ will differ from the initial one by a global sign, but the similarity transformations defined by these matrices will be the same.

The action of $U$ on both sides of equation (5.22) gives
\[ UD^\lambda (k + 1) U^\dagger \text{sgn} (\sigma_i) \left( e_{M_i}^{\lambda'} \right) \]
\[ = \rho^{-1} \left( T_k; k + 1 \right) \text{sgn} (\sigma_i) e_{M_i}^{\lambda'} \]
\[ + \sqrt{1 - \rho^{-2} \left( T_k; k + 1 \right)} \text{sgn} \left( (k + 1)\sigma_i \right) e_{M_i}^{\lambda'} \] (5.27)
Using the properties of the representation $\text{sgn}$ and proposition 5.8 we get
\[ UD^\lambda (k + 1) U^\dagger \left( e_{M_i}^{\lambda'} \right) = -\rho^{-1} \left( T_k; k + 1 \right) e_{M_i}^{\lambda'} \]
\[ - \sqrt{1 - \rho^{-2} \left( T_k; k + 1 \right)} e_{M_i}^{\lambda'} \] (5.28)
which means that
\[ -UD^\lambda (k + 1) U^\dagger = D^\lambda (k + 1), \quad k = 1, 2, \ldots, n - 1, \] (5.29)
and consequently we get
\[ \forall \sigma \in S(N) \quad D^\lambda (\sigma) = \text{sgn} (\sigma) UD^\lambda (\sigma) U^\dagger. \] (5.30)
In the bases $\{ e_{M_i}^{\lambda} : i = 1, 2, \ldots, d_i \}$ and $\{ e_{M_i}^{\lambda'} : i = 1, 2, \ldots, d_i \}$ the operator $U$ takes a matrix form as in equation (5.1), so one can state:

**Theorem 5.16.** The unitary operator $U$ which defines a similarity transformation between two conjugated Young–Yamanouchi irreducible representations of $S(N)$ with bases $\{ e_{M_i}^{\lambda} : i = 1, 2, \ldots, d_i \}$ and $\{ e_{M_i}^{\lambda'} : i = 1, 2, \ldots, d_i \}$ may be written in the following way:
\[ U = \sum_{M_i} \text{sgn}(\sigma_i) e^{\lambda_i}_{M_i} \otimes e^{\lambda_j}_{M_{ij}} = \sum_{T_1} \text{sgn}(T_1) \langle T_1 \rangle \langle T_1 \rangle, \quad (5.31) \]

where \( \text{sgn}(T_1) = \text{sgn}(\sigma) \) (see proposition 5.13) and the relation between the Yamanouchi symbols \( M_i \) and the SYT \( \lambda_1 \) is described in definition 5.9 (see also remark 5.15). If the bases \( \{ e^{\lambda_i}_{M_i} \} \) and \( \{ e^{\lambda_i}_{M_{ij}} \} \) are ordered according to the lexicographic order (see definition 5.11 and proposition 5.12), then the matrix of \( U \) has the following form

\[
U = \begin{pmatrix}
0 & 0 & \cdots & \text{sgn}(\sigma_1) \\
0 & \text{sgn}(\sigma_2) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\text{sgn}(\sigma_{d_1}) & 0 & \cdots & 0
\end{pmatrix}. \quad (5.32)
\]

One can see that we can also reformulate the above statement without referring to a particular ordering of the Young–Yamanouchi basis, namely we can write:

**Proposition 5.17.** The similarity matrix \( U \) can also be written in the following form

\[
U = \sum_{T_1} \text{sgn}(T_1) \langle T_1 \rangle \langle T_1 \rangle, \quad (5.33)
\]

where \( \text{sgn}(T_1) = \text{sgn}(\sigma) \), and \( \sigma \) is the permutation that transforms an arbitrary, chosen, fixed SYT \( \lambda_1 \) into \( \lambda_1 \) (see proposition 5.5 and remark 5.6). For different choices of \( \lambda_1 \), the corresponding \( U \) may differ by a global sign.

### 6. Applications

In this section we present a mathematical application of the isomorphism between \( D^3 \) and \( \text{sgn} D^4 \), which recovers the known results given in [1, 2]. Let us consider an \( n \)-particle system on the Hilbert spaces \( \mathcal{H}_A^{\otimes n} \) and \( \mathcal{H}_B^{\otimes n} \), then the full Hilbert space of such a system is of course \( \mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n} \equiv \mathcal{H}_{AB}^{\otimes n} \). Our goal here is to study the antisymmetric projector \( P_a \) and to decompose it as a direct sum of some smaller projectors, which can be studied separately. Suppose that \( \mathcal{H}_{AB}^{\otimes n} = (\mathbb{C}^d)^{\otimes n} \); then we can define a special class of operators called permutation operators \( V \) in the following way:

**Definition 6.1.** \( V: S(n) \to \text{Hom}( (\mathbb{C}^d)^{\otimes n} ) \) and

\[
\forall \sigma \in S(n) \quad V_\sigma: e_1 \otimes e_2 \otimes \cdots \otimes e_n = e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)}, \quad (6.1)
\]

where \( d \in \mathbb{N} \) and \( \{ e_i \}_{i=1}^d \) is an orthonormal basis of the space \( \mathbb{C}^d \).

Now using the above definition we are ready to formulate the following:
Fact 6.2. The antisymmetric projector $P_{as}$ acting on $\mathcal{H}^{\otimes n}_{AB}$ can be written in the form

$$P_{as} = \bigoplus_{\lambda \vdash n} 1^A_{\lambda} \otimes 1^B_{\lambda} \otimes W_{AB}^{\lambda}$$

(6.2)

where the identities and operator $W_{AB}^{\lambda}$ act on the unitary and symmetric parts respectively. By $\lambda^t$ we denote the conjugate Young diagram, and the operator $\psi_{\lambda} = \langle \psi \mid \psi \rangle^{\lambda^t}$, where

$$\psi_{\lambda} = \sum_{T_{ij}} \operatorname{sgn}(T_{ij}) |T_{ij}\rangle |T_{ij}\rangle$$

(6.3)

The interpretation of $\lambda^t \operatorname{sgn}(\lambda)$ is given in proposition 5.17.

Proof. The proof is based on direct calculations and the Schur–Weyl duality described in for example [16]. First of all, let us write the projector on the antisymmetric subspace of $\mathcal{H}^{\otimes n}_{AB}$ as

$$P_{as} = \frac{1}{n!} \sum_{\sigma \in S(n)} \operatorname{sgn}(\sigma) V^A_{\sigma} \otimes V^B_{\sigma}$$

(6.4)

Thanks to the Schur–Weyl duality, we can decompose every permutation operator as a direct sum of irreducible components:

$$V^A_{\sigma} = \bigoplus_{\lambda \vdash n} 1^A_{\lambda} \otimes \left( V^A_{\lambda} \right)^{\lambda}, \quad V^B_{\sigma} = \bigoplus_{\lambda \vdash n} 1^B_{\lambda} \otimes \left( V^B_{\lambda} \right)^{\lambda}$$

(6.5)

Now putting equation (6.5) into equation (6.4), we obtain

$$P_{as} = \bigoplus_{\lambda \vdash n} 1^A_{\lambda} \otimes 1^B_{\lambda} \otimes \frac{1}{n!} \sum_{\sigma \in S(n)} \operatorname{sgn}(\sigma) \left( V^A_{\lambda} \right)^{\lambda} \otimes \left( V^B_{\lambda} \right)^{\lambda}$$

$$= \bigoplus_{\lambda \vdash n} 1^A_{\lambda} \otimes 1^B_{\lambda} \otimes P_{\lambda}^{A,B}$$

(6.6)

Let us define the following. For an arbitrary matrix $C$, the state $\Psi[C]$ is defined by taking the matrix elements $c_{ij}$ of $C$ and setting them as coefficients in the standard basis $|i\rangle \otimes |j\rangle$ on subsystems $A$ and $B$, i.e. $\Psi[C] = \sum_i c_{ij} |i\rangle \otimes |j\rangle$. Now thanks to the property

$$X \otimes Y \Psi[C] = \Psi[XCY^T], \quad X, Y - \text{arbitrary matrices},$$

(6.7)

we can write

$$P_{\lambda}^{A,B} \Psi[C] = \frac{1}{n!} \Psi \left[ \sum_{\sigma \in S(n)} \operatorname{sgn}(\sigma) \left( V^A_{\lambda} \right)^{\lambda} C \left( V^B_{\lambda} \right)^{\lambda} \right]$$

(6.8)

Using considerations from previous sections, we see that for $\lambda_B = \lambda_A^t$ there is always a nonsingular matrix $U$ such that $U \left( V^A_{\lambda} \right)^{\lambda} U^\dagger = \operatorname{sgn}(\lambda) \left( V^A_{\lambda} \right)^{\lambda}$. Because of this equivalence and the Schur lemma, we can write that $\sim C$. Now taking into account formula (6.7) we have $\Psi[U] = \sum_{ij} u_{ij} |i\rangle \otimes |j\rangle$, where the $u_{ij}$ are matrix elements of the unitary transformation $U$ (see proposition 5.17), so

$$P_{as} = \frac{1}{n!} \bigoplus_{\lambda} 1^A_{\lambda} \otimes 1^B_{\lambda} \otimes W_{\lambda}^{AB}$$

(6.9)

where the operator $W_{\lambda}^{AB}$ is given by equation (6.3). □
Finally, it is worth mentioning that a similar decomposition was carried out for the symmetric projector in [13], where the authors consider entanglement concentration for many copies of unknown pure states and propose a protocol which produces a perfect maximally entangled state.

7. Conclusions

In this paper we present and discuss an explicit method for constructing unitary maps between two arbitrary but equivalent irreducible representations of some finite group $G$ (lemma 2.5). We observe a few interesting properties in the general case, such as the doubly stochastic property (remark 3.7) and a generalization of the classical orthogonality relation for irreducible representations (corollary 3.8). In the next part we apply our method to the symmetric group $S(N)$ (examples 4.1, 4.2, 4.3, 4.4, 4.5 and finally 4.13), which gives us the clue that whenever we use as a basis the Young–Yamanouchi basis, our transformation matrices $U$ connecting Young–Yamanouchi conjugated irreps have an anti-diagonal form with entries $\pm 1$ (see theorem 5.16). We hope that our results will be useful for numerical work involving $S(N)$ and other group symmetries.

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References

[1] Pauncz R 1995 The Symmetric Group in Quantum Chemistry 1st edn (USA: CRC Press)
[2] Pauncz R 2000 The Construction of Spin Eigenfunction: An Exercise Book (New York: Springer)
[3] Pauncz R 1986 The Unitary Group in Quantum Chemistry (Amsterdam: Elsevier)
[4] Eggeling T and Werner R F 2000 Phys. Rev. A 63 042111
[5] Keyl M and Werner R F 2001 Annales Henri Poincare 2 1
[6] Cirac J I, Ekert A K and Macchiavello C 1999 Phys. Rev. Lett. 82 4344
[7] Bowles P, Guţă M and Adesso G 2011 Phys. Rev. A 84 022320
[8] Whitfield J D 2014 J. Chem. Phys. submitted (arXiv:1306.1147)
[9] Czechlewski M, Grudka A, Horodecki M, Mozrzymas M and Studziński M 2012 J. Phys. A: Math. Theor. 45 125303
[10] Ćwikliński P, Horodecki M and Studziński M 2012 Phys. Lett. A 32 2178–87
[11] Studziński M, Ćwikliński P, Horodecki M and Mozrzymas M 2014 Phys. Rev. A 89 052322
[12] Brandao F G S L, Ćwikliński P, Horodecki M, Horodecki P, Korbič J and Mozrzymas M 2012 Phys. Rev. E 86 031101
[13] Matsumoto K and Hayashi M 2007 Phys. Rev. A 75 062338
[14] Chistov A, Ivanovs G and Karpinski M 1997 Proc. Int. Symp. on Symbolic and Algebraic Computations (ISSAC) pp 68–74
[15] Fulton W and Harris J 1991 Representation Theory—A First Course (New York: Springer-Verlag)
[16] Goodman R and Wallach N R 2009 *Symmetry, Representations and Invariants* (New York: Springer)

[17] Chen J Q, Ping J and Wang F 2002 *Group Representation Theory for Physicists* (River Edge, NJ: World Scientific)

[18] Procesi C 2007 *Lie Groups: An Approach through Invariants and Representations* (New York: Springer)

[19] Yamanouchi T 1937 *Proc. Phys. Math. Soc. Japan* **18** 623

[20] Young A 1901 *Proc. London Math. Soc.* **33** 97–146

[21] Program GAP, www.gap-system.org/Manuals/doc/ref/chap71.html#X79BC08C6846718D9