Obstructions to Integrability of Nearly Integrable Dynamical Systems Near Regular Level Sets

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Abstract

We study the existence of real-analytic first integrals and real-analytic integrability for perturbations of integrable systems in the sense of Bogoyavlenskij, including non-Hamiltonian ones. In particular, we assume that there exists a family of periodic orbits on a regular level set of the first integrals having a connected and compact component and give sufficient conditions for nonexistence of the same number of real-analytic first integrals in the perturbed systems as the unperturbed ones and for their real-analytic nonintegrability near the level set such that the first integrals and commutative vector fields depend analytically on the small parameter. We compare our results with the classical results of Poincaré and Kozlov for systems written in action and angle coordinates and discuss their relationships with the subharmonic and homoclinic Melnikov methods for periodic perturbations of single-degree-of-freedom Hamiltonian systems. In particular, the latter discussion reveals that the perturbed systems can be real-analytically nonintegrable even if there exists no transverse homoclinic orbit to a periodic orbit. We illustrate our theory with three examples containing the periodically forced Duffing oscillator.

1. Introduction

In his famous memoir [23], which was related to a prize competition celebrating the 60th birthday of Oscar II, King of Sweden and Norway, Henri Poincaré studied two-degree-of-freedom Hamiltonian systems depending on a small parameter, say \( \varepsilon \), although he used the letter \( \mu \) instead, such that they are integrable when \( \varepsilon = 0 \), and showed the nonexistence of first integrals which are analytic in the state variables and parameter \( \varepsilon \) and functionally independent of Hamiltonians under some nondegenerate conditions. If there exists such a first integral, then the Hamiltonian systems are integrable for \( |\varepsilon| \geq 0 \) sufficiently small in the sense of Liouville [2,17]. The result was improved significantly in the first volume of his collected works [24],
published 2 years later, so that greater-degree-of-freedom Hamiltonian systems can be treated. Using these results, he discussed the nonexistence of such first integrals in the restricted planar and spacial three-body problem there. See also [7] for an account of his work from mathematical and historical perspectives. Subsequently, his results were sophisticated and generalized to non-Hamiltonian systems [13,14]. In particular, KOZLOV [14] treated multi-dimensional systems of the form
\[\dot{I} = \varepsilon h(I, \theta; \varepsilon), \quad \dot{\theta} = \omega(I) + \varepsilon g(I, \theta; \varepsilon), \quad (I, \theta) \in \mathbb{R}^\ell \times \mathbb{T}^m,\]
where \(\varepsilon\) is a small parameter such that \(|\varepsilon| \ll \mathbb{1}, \mathbb{T}^m = \prod_{j=1}^m \mathbb{S}^1\) with \(\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}\) is an \(m\)-dimensional torus and \(h(I, \theta; \varepsilon), \omega(I)\) and \(g(I, \theta; \varepsilon)\) are analytic in \((I, \theta, \varepsilon)\). Note that the system (1.1) is Hamiltonian if \(\ell = m\) as well as \(\varepsilon = 0\), or
\[D_I h(I, \theta; \varepsilon) = -D_\theta g(I, \theta; \varepsilon),\]
and non-Hamiltonian if not. When \(\varepsilon = 0\), Eq. (1.1) becomes
\[\dot{I} = 0, \quad \dot{\theta} = \omega(I)\]
which we refer to as the unperturbed system for (1.1). We often use this terminology for other systems below. Here we state some of the details of his result.

We expand \(h(I, \theta; 0)\) in Fourier series as
\[h(I, \theta; 0) = \sum_{r \in \mathbb{Z}^m} \hat{h}_r(I) \exp(ir \cdot \theta),\]
where \(\hat{h}_r(I), r \in \mathbb{Z}^m\), are the Fourier coefficients and “." represents the inner product. We assume the following for (1.1):

(K1) The system (1.1) has \(s\) first integrals \(F_j(I, \theta; \varepsilon), j = 1, \ldots, s\), which are analytic in \((I, \theta, \varepsilon)\);
(K2) If \(r \in \mathbb{Z}^m\) and \(r \cdot \omega(I) = 0\) for any \(I \in \mathbb{R}^\ell\), then \(r = 0\).

If assumption (K2) holds, then we say that the unperturbed system (1.2) is non-degenerate. Under (K1) and (K2) we can show that \(F_j(I, \theta; 0), j = 1, \ldots, s\), are independent of \(\theta\) (see Lemma 1 in Section 1 of Chapter IV of [14]), and write \(F_{j0}(I) = F_j(I, \theta; 0)\) and \(F_0(I) = (F_{10}(I), \ldots, F_{s0}(I))\). We refer to \(\mathcal{P}_s \subset \mathbb{R}^\ell\) as a Poincaré set if for each \(I \in \mathcal{P}_s\) there exists linearly independent vectors \(r_j \in \mathbb{Z}^m, j = 1, \ldots, \ell - s\), such that
(i) \(r_j \cdot \omega(I) = 0, j = 1, \ldots, \ell - s\);
(ii) \(\hat{h}_{r_j}(I), j = 1, \ldots, \ell - s\), are linearly independent.

Let \(U\) be a domain in \(\mathbb{R}^\ell\). A set \(\Delta \subset U\) is called a key set (or uniqueness set) for \(C^\omega(U)\) if any analytic function vanishing on \(\Delta\) vanishes on \(U\). For example, any dense set in \(U\) is a key set for \(C^\omega(U)\). In this situation, we have the following theorem (see Section 1 of Chapter IV of [14] for its proof):

**Theorem 1.1.** (Kozlov) Suppose that assumptions (K1) and (K2) hold, the Jacobian matrix \(DF_0(I)\) has the maximum rank at a point \(I_0 \in \mathbb{R}^\ell\) and a Poincaré set \(\mathcal{P}_s \subset U\) is a key set for \(C^\omega(U)\), where \(U\) is a neighborhood of \(I_0\). Then the system (1.1) has no first integral which is real-analytic in \((I, \theta, \varepsilon)\) and functionally independent of \(F_j(I, \theta; \varepsilon), j = 1, \ldots, s\), near \(\varepsilon = 0\).
A version of Theorem 1.1 for the Hamiltonian case \( \ell = m \) was given in [13] (see also Theorem 7.1 of [3]). The Hamiltonian case of \( s = 1 \) with a dense Poincaré set in Theorem 1.1 was treated by Poincaré for \( \ell = m \geq 2 \) in [24]. When \( s = 0 \) in (K1), Theorem 1.1 means that under its hypotheses there exists no first integral which is analytic in \((I, \theta, \varepsilon)\). When \( s = 1 \) in (K1), which always occurs if the system (1.1) is Hamiltonian, it means that under its hypotheses there exists no first integral which is analytic in \((I, \theta) \in U \times \mathbb{T}^m \) and \( \varepsilon \) near \( \varepsilon = 0 \) and functionally independent of \( F_1(I, \theta, \varepsilon) \). When \( s = m - 1 \) in (K1), if besides (K1) and (K2) there exists a key set \( \Delta \) for \( C^\infty(U) \) such that \( r \cdot \omega(I) = 0 \) and \( h_r(I) \neq 0 \) on \( \Delta \) for some \( r \in \mathbb{Z}^m \setminus \{0\} \), there exists no additional first integral which is analytic in \((I, \theta, \varepsilon)\). For Hamiltonian systems, if they are not Liouville-integrable for any \( |\varepsilon| > 0 \) sufficiently small, then there does not exist such a first integral. For non-Hamiltonian systems this may be true in an appropriate sense, but some additional ingredients are needed, as seen below.

In this paper we study more general dynamical systems of the form

\[
\dot{x} = X_\varepsilon(x), \quad x \in \mathcal{M},
\]

where \( \varepsilon \) is a small parameter such that \( |\varepsilon| \ll 1 \), \( \mathcal{M} \) is an \( n \)-dimensional analytic manifold for \( n \geq 2 \) and the vector field \( X_\varepsilon \) is analytic in \( x \) and \( \varepsilon \). Let \( X_\varepsilon = X^0 + \varepsilon X^1 + O(\varepsilon^2) \) for \( \varepsilon \) sufficiently small. When \( \varepsilon = 0 \), the system (1.4) becomes

\[
\dot{x} = X^0(x),
\]

which is assumed to be \textit{analytically} \((q, n-q)\)-integrable in the following sense of BOGOYAVLJENSIK [8] for some positive integer \( q \leq n \):

**Definition 1.2.** (Bogoyavlenskij) The system (1.5) is called \((q, n-q)\)-integrable or simply \textit{integrable} if there exist \( q \) vector fields \( Y_1(x)(:= X^0(x)), Y_2(x), \ldots, Y_q(x) \) and \( n-q \) scalar-valued functions \( F_1(x), \ldots, F_{n-q}(x) \) such that the following two conditions hold:

(i) \( Y_1(x), \ldots, Y_q(x) \) are linearly independent almost everywhere and commute with each other, i.e., \([Y_j, Y_k](x) = 0\) for \( j, k = 1, \ldots, q \), where \([\cdot, \cdot]\) denotes the Lie bracket;

(ii) \( F_1(x), \ldots, F_{n-q}(x) \) are functionally independent, i.e., \( dF_1(x), \ldots, dF_{n-q}(x) \) are linearly independent almost everywhere, and \( F_1(x), \ldots, F_{n-q}(x) \) are first integrals of \( Y_1, \ldots, Y_q \), i.e., \( dF_k(Y_j) = 0 \) for \( j = 1, \ldots, q \) and \( k = 1, \ldots, n - q \).

If \( Y_1, Y_2, \ldots, Y_q \) and \( F_1, \ldots, F_{n-q} \) are analytic (resp. meromorphic), then Eq. (1.5) is said to be \textit{analytically} (resp. \textit{meromorphically}) \textit{integrable}.

Definition 1.2 is considered as a generalization of Liouville-integrability for Hamiltonian systems [2,17] since an \( m \)-degree-of-freedom Liouville-integrable Hamiltonian system with \( m \geq 1 \) has not only \( m \) functionally independent first integrals but also \( m \) linearly independent commutative (Hamiltonian) vector fields generated by the first integrals. The \((\ell + m)\)-dimensional system (1.2) is \((m, \ell)\)-integrable in the Bogoyavlenskij sense. A more general version of the Bogoyavlenskij integrability was also proposed in [15].
The system (1.4) is regarded as a perturbation of the analytically integrable system (1.5). We assume that there exists a $q$-parameter family of periodic orbits on a regular level set of the $n - q$ first integrals having a connected and compact component (see Section 2 for our precise assumptions containing this one) and give sufficient conditions for nonexistence of such $n - q$ real-analytic first integrals and for real-analytic nonintegrability of the perturbed system (1.4) near the regular level set such that the first integrals and commutative vector fields depend analytically on $\varepsilon$ near $\varepsilon = 0$. The persistence of such first integrals and commutative vector fields in the perturbed system (1.4) along with that of periodic and homoclinic orbits was previously discussed in [21]. Our approach is based on the technique of [21] and different from those of Poincaré [23,24] and Kozlov [13,14]. To prove the nonintegrability of (1.4), only their persistence is insufficient, and we have to show the nonexistence of linearly independent first integrals and commutative vector fields. We overcome this difficulty here. Recently, another sufficient condition for nonintegrability of nearly integrable dynamical systems near resonant periodic orbits was also obtained using a different approach in [31] and the theory was applied to prove the nonintegrability of the restricted three-body problem in [32], when the independent and state variables are extended to complex ones.

The unperturbed system (1.5) can be transformed to (1.2) under our assumptions, as stated in Proposition 2.1 below. However, Theorem 1.1 only stays in our setting for (1.1) where there exists no such additional first integral even if its hypotheses hold. In particular, this does not allow us to determine the nonintegrability of (1.4) in the Bogoyavlenskij sense when it is non-Hamiltonian. We also describe a consequence of our results to (1.1) and show how it improves the results of Poincaré [24] and Kozlov [13,14].

Moreover, we discuss a relationship between our results and the subharmonic and homoclinic Melnikov methods [11,26,28] for time-periodic perturbations of single-degree-of-freedom analytic Hamiltonian systems, which can be transformed to the form of (1.4) with $\ell = 1$ and $m = 2$, i.e., $(2, 1)$-integrable, and have a one-parameter family of periodic orbits when $\varepsilon = 0$. As is well known, if the subharmonic Melnikov functions have a simple zero, then the corresponding unperturbed periodic orbits persist in the perturbed systems. Morales-Ruiz [18] studied the Hamiltonian perturbation case in which the unperturbed systems have homoclinic orbits, and showed a relationship between their nonintegrability and a version due to Ziglin [34] of the homoclinic Melnikov method when the independent and state variables are extended to complex ones and the small parameter $\varepsilon$ is also regarded as a state variable. More concretely, under some restrictive conditions, based on the generalized version due to Ayoul and Zung [5] of the Morales–Ramis theory [17,19], which provides a sufficient condition for nonintegrability of autonomous dynamical systems, Morales-Ruiz essentially proved that they are meromorphically nonintegrable if the homoclinic Melnikov functions which are obtained as integrals along closed loops on the complex plane are not identically zero. Here the version of the Melnikov method enables us to detect transversal self-intersection of complex separatrices of periodic orbits and to prove its analytic nonintegrability, unlike the standard version [11,16,26]. Here we prove two variants of Morales-Ruiz [18] for periodic orbits: if the subharmonic Melnikov functions for a dense set of
the unperturbed periodic orbits are not identically zero, then there exists no first integral depending analytically on \( \varepsilon \) near \( \varepsilon = 0 \); and if the ‘standard’ homoclinic Melnikov functions \([11,16,26]\) are not identically zero, then the perturbed systems are not Bogoyavlenskij-integrable such that the commutative vector fields and first integrals depend analytically on \( \varepsilon \) near \( \varepsilon = 0 \).

We illustrate our theory with three examples: a simple pendulum with a constant torque, second-order coupled oscillators, and the periodically forced Duffing oscillator \([11,12,26]\). Real-analytic nonintegrability is discussed in special cases of the second and third examples while existence of real-analytic first integrals in the rest. In particular, the special case of the third one is shown to be nonintegrable in the above meaning even if it does not have transverse homoclinic orbits to a periodic orbit.

This paper is organized as follows: in Section 2, we state our precise assumptions and main theorems. In Section 3, we give proofs of the main theorems. We describe a consequence of our results to (1.1) in Section 4 and discuss a relationship between our results and the subharmonic and homoclinic Melnikov methods for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems in Section 5. Finally, we give the three examples in Section 6.

2. Main Results

In this section, we state our main results for (1.4). We first make the following assumptions on the unperturbed system (1.5):

(A1) For some positive integer \( q < n \), the system (1.5) is analytically \((q, n – q)\) Bogoyavlenskij-integrable, i.e., there exist \( q \) analytic vector fields \( Y_1(x)(:= X^0(x)), \ldots, Y_q(x) \) and \( n – q \) analytic scalar-valued functions \( F_1(x), \ldots, F_{n-q}(x) \) such that conditions (i) and (ii) of Definition 1.2 hold.

(A2) Let \( F(x) = (F_1(x), \ldots, F_{n-q}(x)) \). There exists a regular value \( c \in \mathbb{R}^{n-q} \) of \( F \), i.e., \( \text{rank} \, dF(x) = n – q \) when \( F(x) = c \), such that \( F^{-1}(c) \) has a connected and compact component and \( Y_1(x), \ldots, Y_q(x) \) are linearly independent on \( F^{-1}(c) \).

We say that the level set \( F^{-1}(c) \) is regular if \( c \in \mathbb{R}^{n-q} \) is a regular value of \( F \). Henceforth we assume, without loss of generality, that \( F^{-1}(c) \) is connected and compact itself in (A2), by reducing the domain of \( F \) if necessary. Under (A1) and (A2) we have the following result like a well-known theorem for Hamiltonian systems \([2]\) (see \([8,35,36]\) for the details):

**Proposition 2.1.** (Liouville–Miurer–Arnold–Jost) Suppose that assumptions (A1) and (A2) hold. Then we have the following:

(i) The level set \( F^{-1}(c) \) is analytically diffeomorphic to the \( q \)-dimensional torus \( \mathbb{T}^q \);

(ii) There exists an analytic diffeomorphism \( \varphi : U \times \mathbb{T}^q \to U \), where \( U \) and \( U \) are, respectively, neighborhoods of \( I = I_0 \) in \( \mathbb{R}^{n-q} \) and of \( F^{-1}(c) \) in \( \mathcal{M} \) for some \( I_0 \in \mathbb{R}^{n-q} \), such that
(iia) \( \varphi([I_0] \times \mathbb{T}^q) = F^{-1}(c) \);

(iib) \( F \circ \varphi(I, \theta) \) depends only on \( I, \theta \in U \times \mathbb{T}^q \);

(iic) The flow of \( X^0 \) on \( U \) is analytically conjugate to that of (1.2) with \( \ell = n - q \) and \( m = q \) on \( U \times \mathbb{T}^q \).

The variables \( I \) and \( \theta \) are called the action and angle variables as in Hamiltonian systems, and \( \omega(I) \) is referred to as the angular frequency vector. Let \( \omega_j(I) \) be the \( j \)th component of \( \omega(I) \) for \( j = 1, \ldots, q \). Henceforth \( U \) denotes the neighborhood of \( I = I_0 \) in Proposition 2.1. Moreover, we assume the following on (1.2) along with (K2):

(A3) There exists a key set \( D_R \) for \( C^\omega(U) \) such that for \( I \in D_R \) a resonance of multiplicity \( q - 1 \),
\[
\dim_{\mathbb{Q}}(\omega_1(I), \ldots, \omega_q(I)) = 1,
\]
occurs with \( \omega(I) \neq 0 \), i.e., there exists a positive constant \( \omega_0(I) \) depending on \( I \) such that
\[
\frac{\omega(I)}{\omega_0(I)} \in \mathbb{Z}^q \setminus \{0\}.
\]

We choose \( \omega_0(I) \) as large as possible below.

We easily see that if \( \text{rank} \, D\omega(I^*) = q \) for some \( I^* \in \mathbb{R}^{n-q} \), then both (K2) and (A3) hold in a neighborhood \( U \) of \( I^* \). Assumption (A3) also implies that if \( I \in D_R \), then the system (1.2) has a \( q \)-parameter family of periodic orbits
\[
(I, \theta) = (I, \omega(I)t + \tau), \quad \tau \in \mathbb{T}^q,
\]
with the period \( T^I = 2\pi/\omega_0(I) \), which is only defined for \( I \in D_R \). Note that the periodic orbits given by (2.1) are parameterized by \((q - 1)\) parameters essentially since two periodic orbits \((I, \omega(I)t + \tau)\) and \((I, \omega(I)t + \tau_0)\) represent the same orbit if \( \tau - \tau_0 = \omega(I)t_0 \) for some \( t_0 \in \mathbb{R} \). We also have a \( q \)-parameter (but essentially \((q - 1)\)-parameter) family of periodic orbits
\[
\gamma^I_\tau(t) = \varphi(I, \omega(I)t + \tau), \quad (I, \tau) \in D_R \times \mathbb{T}^q,
\]
with the period \( T^I \) in the unperturbed system (1.5). Define the integrals
\[
\mathcal{J}_{F_k}^I(\tau) := \int_0^{T^I} dF_j(X^1)_{\gamma^I_\tau(t)} \, dt, \quad k = 1, \ldots, n - q
\]
for \( I \in D_R \) and set \( \mathcal{J}_F^I(\tau) := (\mathcal{J}_{F_1}^I(\tau), \ldots, \mathcal{J}_{F_{n-q}}^I(\tau)) \). Note that
\[
\mathcal{J}_{F_k}^I(\tau + \omega(I)t) = \mathcal{J}_{F_k}^I(\tau)
\]
for \( \tau \in \mathbb{T}^q \) and \( t \in \mathbb{R} \). We state the first of our main results as follows,
Theorem 2.2. Suppose that assumptions (A1)–(A3) and (K2) hold. If there exists a key set $D \subset D_R$ for $C^ω(U)$ such that $J_τ^I(τ)$ is not identically zero for any $I \in D$, then the perturbed system (1.4) does not have $n − q$ real-analytic first integrals in a neighborhood of $F^{-1}(c)$ near $ε = 0$ such that they are functionally independent in $x$ for $|ε| ≠ 0$ and depend analytically on $ε$.

We prove Theorem 2.2 in Section 3.1.

We next consider a special case in which Eq. (1.4) is a two- or more-degree-of-freedom Hamiltonian system. For an integer $m ≥ 2$, let $(M,Ω_1)$ be an $m$-dimensional analytic symplectic manifold with a symplectic form $Ω_1$, and let $H_ε(x) = H_0(x) + εH_1(x) + O(ε^2)$ be an analytic Hamiltonian for (1.4) and depend analytically on $ε$ near $ε = 0$. We assume the following:

(A1') The unperturbed Hamiltonian system (1.5) with the Hamiltonian $H_0(x)$ is real-analytically Liouville-integrable, i.e., there exist $m$ functionally independent analytic first integrals $F_1(x) := H_0(x), F_2(x), \ldots, F_m(x)$ such that they are pairwise Poisson commutative, i.e., $\{F_j, F_k\} = 0$ for $j, k = 1, \ldots, m$, where $\{·, ·\}$ denotes the Poisson bracket for the symplectic form $Ω_1$. Assumption (A1') means (A1) with $q = m$ and $Y_j = X_{F_j}$ for $j = 1, \ldots, m$, where $X_{F_j}$ denotes the Hamiltonian vector field for the Hamiltonian $F_j(x)$. Thus we immediately obtain the following from Theorem 2.2:

Corollary 2.3. Suppose that (A1'), (A2), (A3) and (K2) hold. If there exists a key set $D \subset D_R$ for $C^ω(U)$ such that $I I F(τ)$ with $X_1 = X_{H_1}$ is not identically zero for any $I \in D$, then the perturbed Hamiltonian system for the Hamiltonian $H_ε(x)$ is not real-analytically Liouville-integrable in a neighborhood of $F^{-1}(c)$ near $ε = 0$ such that $m − 1$ additional first integrals also depend analytically on $ε$.

For integrability of non-Hamiltonian systems (1.4), we have to consider not only first integrals but also commutative vector fields. Therefore we also need the following assumption additionally:

(A4) For some $I^* ∈ U$, the Jacobian matrix $Dω(I^*)$ is injective, i.e.,

$$\text{rank } Dω(I^*) = n − q.$$ 

Note that (A4) holds only when $n − q ≤ q$. Finally, we state the second main result as follows:

Theorem 2.4. Suppose that assumptions (A1)–(A4) and (K2) hold. If there exists a key set $D \subset D_R$ for $C^ω(U)$ such that $J_τ^I(τ)$ is not constant for any $I \in D$, then for $|ε| ≠ 0$ sufficiently small the perturbed system (1.4) is not real-analytically integrable in the Bogoyavlenskij sense near $F^{-1}(c)$ such that the first integrals and commutative vector fields depend analytically on $ε$ near $ε = 0$: In particular there exists only such $n − q − 1$ first integrals and such $q − 1$ commutative vector fields at most.

Remark 2.5. (i) Theorems 2.2 and 2.4 say nothing about the nonexistence of real-analytic first integral and real-analytic integrability of (1.4) for $ε > 0$ fixed, as in Theorem 1.1: These are much harder problems. Also, they exclude even the possibility that the first integrals are functionally independent for $|ε| ≠ 0$ sufficiently small but not for $ε = 0$. 
(ii) Theorem 2.4 does not exclude the possibility that the system (1.4) is real-analytically integrable in the Bogoyavlenskij sense near \( F^{-1}(c) \) such that in a punctured neighborhood of \( \varepsilon = 0 \) there exist such \( n - q' (\geq 0) \) first integrals and \( q' \) commutative vector fields for some \( q' > q \).

We prove Theorem 2.4 in Section 3.2.

3. Proofs of the Main Theorems

In this section, we prove Theorems 2.2 and 2.4. The proofs are based on the results of [21] for persistence of first integrals and commutative vector fields in a neighborhood of \( \varepsilon = 0 \) but we need to show the nonexistence of linearly independent first integrals and commutative vector fields in a punctured neighborhood of \( \varepsilon = 0 \). Henceforth we mean real-analyticity when saying analyticity.

3.1. Proof of Theorem 2.2

We first prove Theorem 2.2. We begin with the following two lemmas on (1.2):

Lemma 3.1. Condition (K2) is equivalent to the following condition:

(K2’) There exists a dense subset \( D_N \) of \( \mathbb{R}^\ell \) such that for \( I \in D_N \) the nonresonance condition holds, i.e.,

\[
\dim_{\mathbb{Q}}(\omega_1(I), \ldots, \omega_m(I)) = m.
\]

Proof. Assume that condition (K2’) holds. We see that if \( r \in \mathbb{Z}^m \) satisfies \( r \cdot \omega(I) = 0 \) for \( I \in D_N \), then \( r = 0 \). Hence, condition (K2) holds.

Conversely, assume that condition (K2) holds. Let

\[
\mathcal{L}_r = \{I \in \mathbb{R}^\ell \mid r \cdot \omega(I) = 0\}
\]

for each \( r \in \mathbb{Z}^m \). By condition (K2), we have \( \mathcal{L}_r \neq \mathbb{R}^\ell \) if \( r \neq 0 \). Since \( \omega(I) \) is analytic, \( \mathcal{L}_r \) is nowhere dense in \( \mathbb{R}^\ell \) if \( r \neq 0 \). Let

\[
D_N = \bigcap_{r \in \mathbb{Z}^m \setminus \{0\}} \mathcal{L}_r^c,
\]

where \( \mathcal{L}_r^c \) is the complement of \( \mathcal{L}_r \). We see that \( r \cdot \omega(I) \neq 0 \) for any \( I \in D_N \) if \( r \neq 0 \). Since \( \mathcal{L}_r \) is nowhere dense in \( \mathbb{R}^\ell \), \( \mathcal{L}_r^c \) is open dense, so that \( D_N \) is dense in \( \mathbb{R}^\ell \) by the Baire category theorem. Hence, condition (K2’) holds. \( \square \)

In the next lemma we weaken the regularity of \( \omega(I) \) and first integrals from analyticity to continuity.

Lemma 3.2. Suppose that \( \omega : \mathbb{R}^\ell \to \mathbb{R}^m \) is continuous and condition (K2’) holds. If \( F(I, \theta) \) is a continuous first integral of (1.2), then it depends only on \( I \).
Proof. Assume that condition (K2') holds, and fix $I_0 \in D_N$ and $\theta_0 \in \mathbb{T}^m$. We see that the orbit $(I_0, \omega(I_0)t + \theta_0)$ is dense in $[I_0] \times \mathbb{T}^m$. Since $F(I, \theta)$ is constant on the orbit, so is it on $[I_0] \times \mathbb{T}^m$. Let $F^*(I) := F(I, \theta_0)$. Then $F(I, \theta) = F^*(I)$ on $D_N \times \mathbb{T}^m$. Since $D_N \times \mathbb{T}^m$ is dense in $\mathbb{R}^\ell \times \mathbb{T}^m$, and $F(I, \theta)$ and $F^*(I)$ are continuous, we have $F(I, \theta) = F^*(I)$ on $\mathbb{R}^\ell \times \mathbb{T}^m$. Thus, $F(I, \theta)$ depends only on $I$. \hfill \Box

Remark 3.3. In Lemma 1 of Sect. 1 in Chapter IV of [14], the statement of Lemma 3.2 was proven by using the Fourier expansion under stronger conditions that $\omega(I)$ is analytic and $F(I, \theta)$ is $C^1$.

We turn to the perturbed or unperturbed system (1.4) or (1.5), and assume that conditions (A1)–(A3) and (K2) hold. We have the following lemma on the unperturbed system (1.5):

Lemma 3.4. Suppose that $G_1(x), \ldots, G_k(x)$ are analytic first integrals of the unperturbed system (1.5) near $F^{-1}(c)$ for $k \geq 1$ and they may be functionally dependent. Then there exists an analytic map $\psi : F(U) \to \mathbb{R}^k$ such that $G = \psi \circ F$ in a neighborhood $U$ of $F^{-1}(c)$ in $\mathcal{M}$, where $G(x) = (G_1(x), \ldots, G_k(x))$.

Proof. We first transform (1.5) to (1.2) with $\ell = n - q$ and $m = q$, based on Proposition 2.1. In particular, we write $F = F(I)$. On the other hand, we use Lemmas 3.1 and 3.2 to show that $G$ depends only on $I$, so we have

$$G = (G \circ F^{-1}) \circ F,$$

and take $\psi = G \circ F^{-1}$ to obtain the desired result. \hfill \Box

Using Lemma 3.4 and Theorem 2.2 of [21], we obtain the following result, which allows us to check the persistence of first integrals in the perturbed system (1.4) with $\varepsilon \neq 0$ and plays a key role in our proof of Theorem 2.2:

Lemma 3.5. Let $I \in D_\mathcal{R}$. Suppose that near $\varepsilon = 0$ the perturbed system (1.4) has $n - q$ analytic first integrals $G_1^\varepsilon, \ldots, G_{n-q}^\varepsilon$ near $\tilde{T}_I = \{\gamma_t^I | \tau \in \mathbb{T}^q\}$ in $\mathcal{M}$ such that they are functionally independent in $x$ on $\tilde{T}_I$ and depend analytically on $\varepsilon$. Then $\mathcal{J}_F^I(\tau)$ must be identically zero.

Proof. Assume that the hypothesis of the lemma holds. By Lemma 3.4, $G^0 := (G_1^0, \ldots, G_{n-q}^0)$ is expressed as $G^0 = \psi \circ F$ for some analytic map $\psi : F(U) \to \mathbb{R}^{n-q}$, where $U$ is a neighborhood of $F^{-1}(c)$. Since $F$ is constant along the periodic orbit $\gamma_t^I(t)$, we have $dG^0_{\gamma_t^I(t)} = d\psi_{F(\gamma_t^I(0))}dF_{\gamma_t^I(t)}$, so that

$$\mathcal{J}_{G^0}^I(\tau) = \int_0^{T_l} d\psi_{F(\gamma_t^I(0))}dF_{\gamma_t^I(t)}(X_{\gamma_t^I(t)}) \, dt = d\psi_{F(\gamma_t^I(0))}\mathcal{J}_F^I(\tau).$$

On the other hand, using Theorem 2.2 of [21], we have

$$\mathcal{J}_{G^0}^I(\tau) = \int_0^{T_l} dG^0(X^1)_{\gamma_t^I(t)} \, dt = 0.$$

Since $dG^0$ and $dF$ have the maximum rank on $\gamma_t^I(t)$ so that $\text{rank } d\psi_{F(\gamma_t^I(0))} = n - q$, we obtain $\mathcal{J}_F^I(\tau) = 0$ for any $\tau \in \mathbb{T}^q$. \hfill \Box
For the proof of Theorem 2.2 we also need the following result, which is an extension of Ziglin’s lemma [4, 10, 33] (the former deals with analytic first integrals near the regular level set \( F^{-1}(c) \) while the latter meromorphic ones near a point on it; see Appendix A for its proof):

**Proposition 3.6.** Let \( k \leq n - q \) be a positive integer. Suppose that in a neighborhood of \( F^{-1}(c) \) the perturbed system (1.4) has \( k \) first integrals that are analytic in \((x, \varepsilon)\) near \( \varepsilon = 0 \). If they are functionally independent in \( x \) for \( \varepsilon \neq 0 \), then in a neighborhood of \( F^{-1}(c) \) there exist \( k \) first integrals that are analytic in \((x, \varepsilon)\) and functionally independent in \( x \) near \( \varepsilon = 0 \).

**Remark 3.7.** From Proposition 3.6 we see that the statement of Theorem 1.1 also holds if \( D F(I, \theta; \varepsilon) \) has a maximum rank near \( F^{-1}(c) \) in a punctured neighborhood of \( \varepsilon = 0 \) instead of \( D F_0(I) \) having at a point \( I_0 \in \mathbb{R}^\ell \).

**Proof of Theorem 2.2.** Suppose that \( I F(T) \) is not identically zero for any \( I \in D \).

Using Lemma 3.5 for each \( I \in D \), we see that the perturbed system (1.4) does not have \( n - q \) analytic first integrals near \( \mathcal{T}_I \) such that they are functionally independent on \( \mathcal{T}_I \) and depend analytically on \( \varepsilon \) near \( \varepsilon = 0 \). Note that if \( \mathcal{J}^I_T(\tau) \) was not identically zero only for some \( I \in D \), then there might exist \( n - q \) first integrals which are functionally independent almost everywhere but dependent on \( \mathcal{T}_I = \{ y^*_\ell | \tau \in \mathbb{T}^q \} \).

Additionally, suppose that there are \( n - q \) analytic first integrals such that they are functionally independent for \( |\varepsilon| \neq 0 \) sufficiently small but not at \( \varepsilon = 0 \) and depend analytically on \( \varepsilon \) near \( \varepsilon = 0 \). Then, by Proposition 3.6, there exist \( n - q \) analytic first integrals \( G^0_1, \ldots, G^0_{n-q} \) which are functionally independent and depend analytically on \( \varepsilon \) near \( \varepsilon = 0 \). Hence, \( dG^0_1, \ldots, dG^0_{n-q} \) are linearly dependent on \( \mathcal{T}_I \) for \( I \in D \). Here we consider the transformed system (1.2) and write \( G^0(I) = (G^0_1(I), \ldots, G^0_{n-q}(I)) \).

We see that the determinant of the Jacobi matrix of \( G^0(I) \) is zero for \( I \in D \), so that it is identically zero on \( U \) since \( D \) is a key set for \( C^\omega(U) \). This yields a contradiction. 

\[ \Box \]

### 3.2. Proof of Theorem 2.4

We turn to the proof of Theorem 2.4. Let \( \ell = n - q \) and \( m = q \). We begin with the following lemmas on (1.2):

**Lemma 3.8.** Condition (A4) is equivalent to the following condition:

\[(A4') \text{ There exists an open dense subset } U^* \text{ of } U \text{ such that for each } I \in U^* \text{ the Jacobian matrix } D \omega(I) \text{ is injective, i.e.,} \]

\[ \text{rank } D \omega(I) = n - q. \]

**Proof.** We only prove that (A4) implies (A4’) because the converse is trivial. Assume that condition (A4) holds. Recall that \( n - q \leq q \). Let \( \tilde{q} = \binom{q}{n-q} \) and let...
\( \Delta_j(I), \ j = 1, \ldots, \tilde{q}, \) be the \((n-q)\)th-order minor determinants of \(D\omega(I)\). Define

\[
K = \bigcap_{j=1}^{\tilde{q}} \{ I \in U \mid \Delta_j(I) = 0 \}.
\]

Obviously, \(I^* \not\in K\), and \(I \in K\) if and only if \(D\omega(I)\) is not injective. Hence, \(K\) is closed and nowhere dense in \(U\) since \(\Delta_j(I)\) is analytic. So \(U^* := U \setminus K\) is an open dense subset in \(U\) and \(\Delta_j(I)\) is not identically zero for some \(j = 1, \ldots, \tilde{q}\), i.e., \(D\omega(I)\) is injective on \(U^*\). Thus, we obtain the desired result. \(\square\)

In the following lemma we weaken the regularity of \(\omega(I)\) and commutative vector fields from analyticity to \(C^1\):

**Lemma 3.9.** Let \(\omega\) be \(C^1\) and let \(Y(I, \theta)\) be a \(C^1\) commutative vector field of (1.2). Suppose that conditions \((K2')\) and \((A4')\) hold. Then \(Y(I, \theta)\) is written in the form

\[
Y(I, \theta) = \sum_{i=1}^{q} \mu_i(I, \theta) \frac{\partial}{\partial \theta_i}
\]

on \(U \times \mathbb{T}^q\) where \(\mu_i : U \to \mathbb{R}, i = 1, \ldots, q\) are \(C^1\) maps.

**Proof.** Suppose that conditions \((K2')\) and \((A4')\) hold and take \(I_0 \in U^*\) and \(\theta_0 \in \mathbb{T}^q\). We first consider the variational equation of (1.2) along the solution \((I, \theta) = (I_0, \omega(I_0)t + \theta_0)\),

\[
\begin{align*}
\dot{\xi} &= 0, \\
\dot{\eta} &= D_I \omega(I_0) \xi, \\
(\xi, \eta) &\in \mathbb{R}^{n-q} \times \mathbb{R}^q.
\end{align*}
\]

The general solution to (3.2) is given by

\[
\begin{align*}
\xi(t) &= \xi_0, \\
\eta(t) &= D_I \omega(I_0) \xi_0 t + \eta_0,
\end{align*}
\]

where \(\xi_0 \in \mathbb{R}^{n-q}\) and \(\eta_0 \in \mathbb{R}^q\) are constants. Since condition \((A4')\) holds, the solution (3.3) is bounded only if \(\xi_0 = 0\), so that Eq. (3.2) has \(q\) linearly independent bounded solutions at most.

Here we assume that \(Y(I, \theta)\) is linearly independent of \(\partial/\partial \theta_j, j = 1, \ldots, q\), on some open ball \(B\) intersecting \(I = I^*\) in \(U \times \mathbb{T}^q\). Since \(U^*\) is dense in \(U\), the intersection \(B \cap (U^* \times \mathbb{T}^q)\) is not empty. Taking a point \((I_0, \theta_0) \in B \cap (U^* \times \mathbb{T}^q)\), we see that \((\xi, \eta) = Y(I_0, \omega(I_0)t + \theta_0)\) gives \(q + 1\) linearly independent bounded solutions along with \((\xi, \eta) = (0, \eta_0), \eta_0 \in \mathbb{R}^q\). This contradicts the fact that Eq. (3.2) possesses only \(q\) linearly independent bounded solutions at most. Therefore, \(Y(I, \theta)\) is linearly dependent of \(\partial/\partial \theta_j, j = 1, \ldots, q\), on \(U \times \mathbb{T}^q\).

Let

\[
Y(I, \theta) = \sum_{i=1}^{q} \mu_i(I, \theta) \frac{\partial}{\partial \theta_i}
\]
on $U \times \mathbb{T}^q$, where $\mu_i(I, \theta) : U \times \mathbb{T}^q \to \mathbb{R}$, $i = 1, \ldots, q$, are $C^1$ maps. Since $Y(I, \theta)$ commutes with the vector field of (1.2), we compute the Lie bracket

$$
\left[ \sum_{j=1}^q \omega_j(I) \frac{\partial}{\partial \theta_j}, \sum_{i=1}^q \mu_i(I, \theta) \frac{\partial}{\partial \theta_i} \right] = \sum_{i=1}^q \left( \sum_{j=1}^q \omega_j(I) \frac{\partial}{\partial \theta_j} \mu_i(I, \theta) \right) \frac{\partial}{\partial \theta_i} = 0.
$$

Thus, $\mu_i(I, \theta)$, $i = 1, \ldots, q$, are first integrals of (1.2). From Lemma 3.2 we see that $\mu_i(I, \theta)$, $i = 1, \ldots, q$ depend only on $I$, and obtain (3.1).

\[\square\]

\textbf{Remark 3.10.} In Lemma 1 of Section 3 in Chapter IV of [14], the statement of Lemma 3.9 was proven by using the Fourier expansion under stronger conditions that both of $\omega(I)$ and $Y(I, \theta)$ are analytic.

We turn to the perturbed or unperturbed system (1.4) or (1.5), and assume that conditions (A1)–(A4) and (K2) hold. Recall that $Y_j(x)$, $j = 1, \ldots, q$, are commutative vector fields for the unperturbed system (1.5) and $n - q \leq q$.

\textbf{Lemma 3.11.} An analytic vector field $Z(x)$ commutes with the vector field $X^0(x)$ if and only if it can be written as

$$Z(x) = \sum_{j=1}^q \rho_j(F(x))Y_j(x), \quad (3.4)$$

where $\rho_j : \mathbb{R}^{n-q} \to \mathbb{R}$, $j = 1, \ldots, n-q$, are analytic. In particular, the vector field $X^0(x)$ has only $q$ commutative vector fields which are linearly independent almost everywhere.

\textbf{Proof.} The sufficiency is obvious. We prove the necessity. Assume that $Z(x)$ is a commutative vector field of $X^0(x)$. Consider the transformed system (1.2) and the commutative vector fields $Y_j = Y_j(I, \theta)$, $j = 1, \ldots, q$ given by condition (A1). Using Lemmas 3.1, 3.8 and 3.9, we see that the vector fields $Y_1(I, \theta), \ldots, Y_q(I, \theta)$ and $Z = Z(I, \theta)$ must be of the form (3.1). It follows from the linear independence of $Y_1(I, \theta), \ldots, Y_q(I, \theta)$ on $U \times \mathbb{T}^q$ that the vector fields $\partial/\partial \theta_j$, $j = 1, \ldots, q$, are linear combinations of $Y_1(I, \theta), \ldots, Y_q(I, \theta)$ in which the coefficients depend on $I$ analytically, and so is $Z(I, \theta)$. This yields the desired result. \[\square\]

Using Lemma 3.11 and Theorem 3.5 of [21], we obtain the following result, which allows us to check the persistence of commutative vector fields and plays a key role in our proof of Theorem 2.4:

\textbf{Lemma 3.12.} Let $I \in D_R$. Suppose that near $\varepsilon = 0$ the perturbed system (1.4) has $q$ analytic commutative vector fields $Z_1^0(x), \ldots, Z_q^0(x)$ near $\mathcal{F}_I = \{ \gamma_\tau^I | \tau \in \mathbb{T}^q \}$ in $\mathcal{M}$ such that they are linearly independent on $\mathcal{F}_I$ and depend analytically on $\varepsilon$. Then $\mathcal{F}_I^\varepsilon(\tau)$ must be constant.

\textbf{Proof.} Assume that the hypothesis of the lemma holds. Then by Lemma 3.11, $Z^0_j(I, \theta)$ has the form (3.4) with $\rho_j(I) = \rho_{ij}(I)$, $j = 1, \ldots, q$, for $i = 1, \ldots, q$. By the construction of action and angle variables (see, e.g., Theorem 2.1 of [36]), the
$q$-parameter family of periodic orbits $\gamma_t^\ell$, $\tau \in \mathbb{T}^q$, is generated by the commutative vector fields $Y_1, \ldots, Y_q$, such that

$$\frac{\partial \gamma_t}{\partial \tilde{\tau}_i} = Y_i(\gamma_t(t)), \quad i = 1, \ldots, q,$$

where $\tau \mapsto \tilde{\tau} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_q)$ is a diffeomorphic parameterization. We compute

$$\frac{\partial J^I}{\partial \tilde{\tau}_i}(\tau) = \int_0^T \frac{\partial}{\partial \tilde{\tau}_i} \left( dF(X^1)_{\gamma_t(t)} \right) dt = \int_0^T Y_i \left( dF(X^1)_{\gamma_t(t)} \right) dt$$

$$= \int_0^T dF([Y_i, X^1])_{\gamma_t(t)} dt, \quad i = 1, \ldots, q.$$ 

On the other hand, using Theorem 3.5 of [21], we have

$$0 = \int_0^T dF \left( \left[ Z_0^q, X^1 \right] \right)_{\gamma_t(t)} dt = \sum_{j=1}^q \left( \int_0^T \left( (\rho_{ij} \circ F) dF([Y_j, X^1]) \right)_{\gamma_t(t)} dt \right)$$

$$= \sum_{j=1}^q \rho_{ij} \circ F(\gamma_t(0)) \frac{\partial J^I}{\partial \tilde{\tau}_j}(\tau), \quad i = 1, \ldots, q.$$ 

Since $Z_0^q, \ldots, Z_0^q$ are linearly independent on $\mathcal{F}_I$, the $q \times q$ matrix $(\rho_{ij})_{i,j=1,\ldots, q}$ is invertible, so that by $n - q \leq q$, $(\partial J^I / \partial \tau)(\tau) = 0$, i.e., $J^I(\tau)$ is constant. 

**Proof of Theorem 2.4.** Suppose that $J^I(\tau)$ is not constant for any $I \in D$. From Theorem 2.2 we see that there exist only $n - q - 1$ first integrals at most such that they are functionally independent and depend analytically on $\epsilon$ near $\epsilon = 0$.

Assume that there exist $q$ analytic commutative vector fields $Z_1^\epsilon, \ldots, Z_q^\epsilon$ such that for $\epsilon = 0$ they are linearly independent almost everywhere and depend analytically on $\epsilon$. Applying Lemma 3.12 for each $I \in D$, we see that the perturbed system (1.4) does not have $q$ analytic commutative vector fields near $\mathcal{F}_I$ such that they are linearly independent on $\mathcal{F}_I$ and depend analytically on $\epsilon$ near $\epsilon = 0$.

Hence, $Z_1^\epsilon, \ldots, Z_q^\epsilon$ are linearly dependent on $\mathcal{F}_I$ for $I \in D$ at $\epsilon = 0$ if they depend analytically on $\epsilon$ near $\epsilon = 0$. As in the proof of Lemma 3.12, we consider the transformed system (1.2) and use Lemma 3.11 to write $Z_0^q$ in the form (3.1) with $\mu_i(I) = \mu_{ij}(I), i = 1, \ldots, q$, for $j = 1, \ldots, q$. We see that the determinant of the matrix $(\mu_{ij}(I))_{i,j=1,\ldots,q}$ is zero for $I \in D$, so that it is identically zero on $U$ since $D$ is a key set for $C^0(U)$. This yields a contradiction, so we have obtained the desired result.

**4. Consequences of the Theory to (1.1)**

In this section, we consider nearly integrable systems of the form (1.1) written in the action-angle coordinates and describe consequences of Theorems 2.2 and 2.4 to it. The unperturbed system (1.2) is $(m, \ell)$-integrable in the Bogoyavlenskij sense and has $\ell$ first integrals $I_1, \ldots, I_\ell$ and $m$ commutative vector fields

q-parameter family of periodic orbits $\gamma_t^\ell$, $\tau \in \mathbb{T}^q$, is generated by the commutative vector fields $Y_1, \ldots, Y_q$, such that

$$\frac{\partial \gamma_t}{\partial \tilde{\tau}_i} = Y_i(\gamma_t(t)), \quad i = 1, \ldots, q,$$

where $\tau \mapsto \tilde{\tau} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_q)$ is a diffeomorphic parameterization. We compute

$$\frac{\partial J^I}{\partial \tilde{\tau}_i}(\tau) = \int_0^T \frac{\partial}{\partial \tilde{\tau}_i} \left( dF(X^1)_{\gamma_t(t)} \right) dt = \int_0^T Y_i \left( dF(X^1)_{\gamma_t(t)} \right) dt$$

$$= \int_0^T dF([Y_i, X^1])_{\gamma_t(t)} dt, \quad i = 1, \ldots, q.$$ 

On the other hand, using Theorem 3.5 of [21], we have

$$0 = \int_0^T dF \left( \left[ Z_0^q, X^1 \right] \right)_{\gamma_t(t)} dt = \sum_{j=1}^q \left( \int_0^T \left( (\rho_{ij} \circ F) dF([Y_j, X^1]) \right)_{\gamma_t(t)} dt \right)$$

$$= \sum_{j=1}^q \rho_{ij} \circ F(\gamma_t(0)) \frac{\partial J^I}{\partial \tilde{\tau}_j}(\tau), \quad i = 1, \ldots, q.$$ 

Since $Z_0^q, \ldots, Z_0^q$ are linearly independent on $\mathcal{F}_I$, the $q \times q$ matrix $(\rho_{ij})_{i,j=1,\ldots, q}$ is invertible, so that by $n - q \leq q$, $(\partial J^I / \partial \tau)(\tau) = 0$, i.e., $J^I(\tau)$ is constant. 

**Proof of Theorem 2.4.** Suppose that $J^I(\tau)$ is not constant for any $I \in D$. From Theorem 2.2 we see that there exist only $n - q - 1$ first integrals at most such that they are functionally independent and depend analytically on $\epsilon$ near $\epsilon = 0$.

Assume that there exist $q$ analytic commutative vector fields $Z_1^\epsilon, \ldots, Z_q^\epsilon$ such that for $\epsilon = 0$ they are linearly independent almost everywhere and depend analytically on $\epsilon$. Applying Lemma 3.12 for each $I \in D$, we see that the perturbed system (1.4) does not have $q$ analytic commutative vector fields near $\mathcal{F}_I$ such that they are linearly independent on $\mathcal{F}_I$ and depend analytically on $\epsilon$ near $\epsilon = 0$.

Hence, $Z_1^\epsilon, \ldots, Z_q^\epsilon$ are linearly dependent on $\mathcal{F}_I$ for $I \in D$ at $\epsilon = 0$ if they depend analytically on $\epsilon$ near $\epsilon = 0$. As in the proof of Lemma 3.12, we consider the transformed system (1.2) and use Lemma 3.11 to write $Z_0^q$ in the form (3.1) with $\mu_i(I) = \mu_{ij}(I), i = 1, \ldots, q$, for $j = 1, \ldots, q$. We see that the determinant of the matrix $(\mu_{ij}(I))_{i,j=1,\ldots,q}$ is zero for $I \in D$, so that it is identically zero on $U$ since $D$ is a key set for $C^0(U)$. This yields a contradiction, so we have obtained the desired result.

**4. Consequences of the Theory to (1.1)**

In this section, we consider nearly integrable systems of the form (1.1) written in the action-angle coordinates and describe consequences of Theorems 2.2 and 2.4 to it. The unperturbed system (1.2) is $(m, \ell)$-integrable in the Bogoyavlenskij sense and has $\ell$ first integrals $I_1, \ldots, I_\ell$ and $m$ commutative vector fields
\[ \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_m}. \]  
Thus, conditions (A1) and (A2) with \( n = \ell + m \) and \( q = m \) already hold. In particular, the level set of \( I = c \) given by \( \{ c \} \times \mathbb{T}^m \) is connected and compact. Take some \( I_0 \in \mathbb{R}^\ell \) and let \( U \) be its neighborhood in \( \mathbb{R}^\ell \), as in the preceding sections.

We first discuss consequences of Theorem 2.2 to (1.1) and assume that conditions (K2) and (A3) with \( n = \ell + m \) and \( q = m \) hold. For \( I \in D_R \) the unperturbed system (1.2) has an \( m \)-parameter family of periodic orbits given by (2.1) with \( q = m \). The integrals given by (2.2) for the \( \ell \) first integrals \( I = (I_1, \ldots, I_\ell) \) become

\[
\mathcal{I}_I^I(\tau) = \int_0^{T I} h(I, \omega(I)t + \tau; 0) \, dt,
\]

where \( \tau \in \mathbb{T}^m \).

Assume that \( m > 1 \). Using the Fourier expansion of \( h(I, \theta; 0) \) given in (1.3), we rewrite the above integral as

\[
\mathcal{I}_I^I(\tau) = \int_0^{T I} \sum_{r \in \mathbb{Z}^m} \hat{h}_r(I) \exp(i r \cdot (\omega(I)t + \tau)) \, dt = T I \sum_{r \in \Lambda_I} \hat{h}_r(I) e^{ir \cdot \tau}, \tag{4.1}
\]

where \( \Lambda_I = \{ r \in \mathbb{Z}^m \mid r \cdot \omega(I) = 0 \} \). Applying Theorem 2.2, we obtain the following result for (1.1) (recall that \( \hat{h}_r(I), r \in \mathbb{Z}^m \), represent the Fourier coefficients of \( h(I, \theta; 0) \) (see Eq. (1.3))):

**Theorem 4.1.** Let \( m > 1 \), and suppose that assumptions (K2) and (A3) with \( n = \ell + m \) and \( q = m \) hold. If there exists a key set \( D \subset D_R \) for \( C^\omega(U) \) such that \( \hat{h}_r(I) \neq 0 \) for some \( r \in \Lambda_I \) with \( I \in D \), then the perturbed system (1.1) does not have \( \ell \) real-analytic first integrals in a neighborhood of the level set \( \{ c \} \times \mathbb{T}^m \) near \( \varepsilon = 0 \) such that they are functionally independent in \( x \) for \( |\varepsilon| \neq 0 \) and depend analytically on \( \varepsilon \).

**Remark 4.2.**

(i) From the proof given in [14] we see that the conclusion of Theorem 1.1 also holds even if the zero vector is taken as one of \( r_j \in \mathbb{Z}^m, j = 1, \ldots, \ell - s \), in the definition of a Poincaré set. This fact was overlooked in [14].

(ii) If a Poincaré set \( \mathcal{P}_{\ell - 1} \subset U \) modified as stated in part (i) is a key set for \( C^\omega(U) \), then condition (A3) holds. Moreover, there exists a key set \( D \subset D_R \) for \( C^\omega(U) \) such that \( \hat{h}_r(I) \neq 0 \) with some \( r \in \Lambda_I \) for \( I \in D \) and only if such a Poincaré set \( \mathcal{P}_{\ell - 1} \subset U \) is a key set for \( C^\omega(U) \).

(iii) The hypothesis of Theorem 4.1 holds if both of \( \omega(I) \) and \( \hat{h}_0(I) \) are not identically zero in \( U \).

(iv) If the system (1.1) is Hamiltonian, then \( \hat{h}_0(I) \equiv 0 \).

(v) From Theorem 2.2 of [21] and Eq. (4.1) we see that the first integrals \( I_1, \ldots, I_m \) do not persist in (1.1) near the resonant torus \( \{ I \} \times \mathbb{T}^m \) if \( \hat{h}_r(I) \neq 0 \) for some \( r \in \Lambda_I \).
Let \( m = 1 \) and assume that \( \omega(I) \neq 0 \). Then the integral (4.1) becomes

\[
I = \int_0^{2\pi/\omega(I)} h(I, \omega(I)t + \tau; 0) \, dt = \frac{2\pi\hat{h}_0(I)}{\omega(I)},
\]

(4.2)
since \( \Lambda_1 = \{0\} \subset \mathbb{Z} \). Noting that assumptions (K2) and (A3) hold if \( \omega(I) \neq 0 \) for some \( I \in U \), we obtain

**Theorem 4.3.** Let \( m = 1 \). If \( \omega(I) \neq 0 \) and \( \hat{h}_0(I) \neq 0 \) for some \( I \in U \), then the perturbed system (1.1) does not have \( \ell \) real-analytic first integrals in a neighborhood of \( \{c\} \times S^1 \) near \( \epsilon = 0 \) such that they are functionally independent in \( x \) for \( |\epsilon| \neq 0 \) and depend analytically on \( \epsilon \).

Assuming the existence of \( \ell - 1 \) functionally independent first integrals in the perturbed system (1.1) and taking Remarks 4.2(i) and (ii) into account, we obtain the same result as Theorems 4.1 and 4.3 from Theorem 1.1 (see also Remark 3.7). Moreover, when the existence of such only \( s (\ell-1) \) first integrals is assumed, Theorem 1.1 guarantees the nonexistence of additional first integrals if a Poincaré set \( \mathcal{P}_s \) modified in Remark 4.2(i) is a key set for \( C^\omega(U) \), in particular \( \hat{h}_{r_j}(I) \), \( j = 1, \ldots, \ell - s (> 1) \), are linearly independent. Note that such a Poincaré set \( \mathcal{P}_s \) does not exist when \( m = 1 \) and \( s < \ell - 1 \).

We next apply Theorem 2.4 to (1.1). When \( m = 1 \), the integral \( \mathcal{I}_1^I(\tau) \) is constant by (4.2), so that Theorem 2.4 does not apply. Thus, we obtain

**Theorem 4.4.** Let \( m > 1 \), and suppose that assumptions (K2), (A3) and (A4) hold. If there exists a key set \( D \subset D_R \) for \( C^\omega(U) \) such that \( \hat{h}_r(I) \neq 0 \) for some \( r \in \Lambda_1 \setminus \{0\} \) with \( I \in D \), then for \( |\epsilon| \neq 0 \) sufficiently small the perturbed system (1.1) is not real-analytically integrable in the meaning of Theorem 2.4 near \( \{c\} \times \mathbb{T}^m \).

### 5. Relationships with the Melnikov Methods

In this section, we discuss relationships of our main results in Section 2 with the subharmonic and homoclinic Melnikov methods for time-periodic perturbations of single-degree-of-freedom Hamiltonian systems; see \([11, 16, 26, 28]\) for the details of the Melnikov methods. A concise review of the methods was also given in Section 4.1 of \([21]\).

Consider systems of the form

\[
\dot{x} = JDH(x) + \epsilon u(x, \nu t), \quad x \in \mathbb{R}^2,
\]

(5.1)

where \( \epsilon \) is a small parameter as in the preceding sections, \( \nu > 0 \) is a constant, \( H : \mathbb{R}^2 \to \mathbb{R} \) and \( u : \mathbb{R}^2 \times S \to \mathbb{R}^2 \) are analytic, and \( J \) is the \( 2 \times 2 \) symplectic matrix,

\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Equation (5.1) represents a time-periodic perturbation of the single-degree-of-freedom Hamiltonian system

\[ \dot{x} = JDH(x), \]  

with the Hamiltonian \( H(x) \). Letting \( \phi = \nu t \mod 2\pi \) such that \( \phi \in S^1 \), we rewrite (5.1) as an autonomous system,

\[ \dot{x} = JDH(x) + \varepsilon u(x, \phi), \quad \dot{\phi} = \nu. \]  

We easily see that assumptions (A1) and (A2) hold in (5.3) with \( \varepsilon = 0 \): \( H(x) \) is a first integral and \((0, 1) \in \mathbb{R}^2 \times \mathbb{R} \) is a commutative vector field. We make the following assumptions on the unperturbed system (5.2):

(M1) There exists a one-parameter family of periodic orbits \( x^\alpha(t) \) with period \( \hat{T}^\alpha > 0, \alpha \in (\alpha_1, \alpha_2) \), for some \( \alpha_1 < \alpha_2 \). Moreover, \( \hat{T}^\alpha \) is not constant as a function of \( \alpha \).

(M2) \( x^\alpha(t) \) is analytic with respect to \( \alpha \in (\alpha_1, \alpha_2) \).

Note that in (M1) \( x^\alpha(t) \) is automatically analytic with respect to \( t \) since the vector field of (5.2) is analytic.

We assume that at \( \alpha = \alpha l/n \)

\[ \frac{2\pi}{\hat{T}^\alpha} = \frac{n}{l} \nu, \]  

where \( l \) and \( n \) are relatively prime integers. We define the subharmonic Melnikov function as

\[ M^{l/n}(\phi) = \int_0^{2\pi/l\nu} DH(x^{\alpha}(t)) \cdot u(x^{\alpha}(t), \nu t + \phi)dt, \]  

where \( \alpha = \alpha^{l/n} \). Let \( T^\alpha = n\hat{T}^\alpha = 2\pi l/\nu \) for \( \alpha = \alpha^{l/n} \). If \( M^{l/n}(\phi) \) has a simple zero at \( \phi = \phi_0 \) and \( d\hat{T}^\alpha / d\alpha \neq 0 \) at \( \alpha = \alpha^{l/n} \), then for \( |\varepsilon| > 0 \) sufficiently small there exists a \( T^\alpha \)-periodic orbit near \((x, \phi) = (x^{\alpha}(t), \nu t + \phi_0) \) in (5.3). See Theorem 3.1 of [28]. A similar result is also found in [11, 26]. The stability of the periodic orbit can also be determined easily [28]. Moreover, several bifurcations of periodic orbits when \( d\hat{T}^\alpha / d\alpha \neq 0 \) or not were discussed in [28–30].

On the other hand, since it is a single-degree-of-freedom Hamiltonian system, the unperturbed system (5.2) is integrable, so that it can be transformed into the form (1.2) with \( \ell, m = 1 \). So the perturbed system (5.3) is transformed into the form (1.1) with \( \ell = 1 \) and \( m = 2 \). Here we take \( I = \alpha \) unlike [28, 31], and have \( \omega(I) = (\Omega(I), \nu) \), where

\[ \Omega(\alpha) = \frac{2\pi}{\hat{T}^\alpha}. \]

We remark that the transformed system is not Hamiltonian even when \( \varepsilon = 0 \), unlike [28, 31]. Choose a point \( \alpha = \alpha_0 \in (\alpha_1, \alpha_2) \) such that \( d\hat{T}^\alpha / d\alpha \neq 0 \), and let \( U \) be a neighborhood of \( \alpha_0 \). We see that assumptions (K2) and (A3) hold for

\[ D_R = \{ \alpha^{l/n} \mid l, n \in \mathbb{N} \} \cap U. \]
Let $\alpha = \alpha^{1/n}$ and let $\gamma^\varepsilon_\alpha(t) = (x^\alpha(t + \tau_1), v(t + \tau_1) + \tau_2)$. We see that $\gamma^\varepsilon_\alpha(t)$ is a $T^\alpha$-periodic orbit in (5.3) with $\varepsilon = 0$. Note that $\gamma^\alpha(t)$ is essentially parameterized by a single parameter, say $\phi := \nu\tau_1 + \tau_2$. So we write $\gamma^\phi_\alpha(t) = (x^\alpha(t), \nu t + \phi)$. The integral (2.2) for $H(x)$ along $\gamma^\phi_\alpha(t)$ becomes

$$S^\alpha_H(\phi) = \int_0^{2\pi/\nu} DH(x^\alpha(t)) \cdot u(x^\alpha(t), \nu t + \phi) \, dt = M^{1/n}(\phi),$$

by (5.5). As stated above, if $M^{1/n}(\phi)$ has a simple zero at $\phi = \phi_0$, then there exists a $T^\alpha$-periodic orbit near $\gamma^\phi_\alpha(t)$. Applying Theorems 2.2 and 2.4, we have the following two results:

**Theorem 5.1.** Suppose that there exists a key set $D \subset D_R$ for $C^\alpha(U)$ such that $M^{1/n}(\phi)$ is not identically zero for $\alpha^{1/n} \in D$. Then for $|\varepsilon| \neq 0$ sufficiently small the system (5.3) has no real-analytic first integral in a neighborhood of $\{x^{\alpha_0}(t) \mid t \in [0, \tilde{T}^{\alpha_0})\} \times S^1$ such that it depends analytically on $\varepsilon$ near $\varepsilon = 0$.

**Theorem 5.2.** Suppose that there exists a key set $D \subset D_R$ for $C^\alpha(U)$ such that $M^{1/n}(\phi)$ is not constant for $\alpha^{1/n} \in D$. Then for $|\varepsilon| \neq 0$ sufficiently small the system (5.3) is not real-analytically integrable in the meaning of Theorem 2.4 in a neighborhood of $\{x^{\alpha_0}(t) \mid t \in [0, \tilde{T}^{\alpha_0})\} \times S^1$.

**Remark 5.3.** (i) If $D$ has an accumulation point, then it becomes a key set for $C^\alpha(U)$.

(ii) If the system (5.1) is Hamiltonian, then the hypothesis of Theorem 5.1 is equivalent to that of Theorem 5.2. Actually, letting

$$u(x, \phi) = JD_x H^1(x, \phi) = J \sum_{r \in \mathbb{Z}} D\hat{H}^1_r(x) e^{ir\phi},$$

we have

$$M^{1/n}(\phi) = \int_0^{2\pi/\nu} DH(x^\alpha(t)) \cdot JD_x H^1(x^\alpha(t), \nu t + \phi) \, dt$$

$$= \sum_{r \in \mathbb{Z}} e^{ir\phi} \int_0^{2\pi/\nu} DH(x^\alpha(t)) \cdot JD\hat{H}^1_r(x^\alpha(t)) e^{ir\nu t} \, dt$$

and

$$\int_0^{2\pi/\nu} DH(x^\alpha(t)) \cdot JD\hat{H}_0^1(x^\alpha(t)) \, dt$$

$$= - \int_0^{2\pi/\nu} D\hat{H}_0^1(x^\alpha(t)) \cdot JDH(x^\alpha(t)) \, dt$$

$$= - \int_0^{2\pi/\nu} D\hat{H}_0^1(x^\alpha(t)) \cdot \dot{x}^\alpha(t) \, dt = 0,$$

where $\hat{H}^1_r(x), r \in \mathbb{Z}$, represent the Fourier coefficients of $H^1(x, \phi)$. Thus, we obtain the claim.
We additionally assume the following on the unperturbed system (5.2):

(M3) There exists a hyperbolic saddle $x_0$ with a homoclinic orbit $x^h(t)$ such that

$$\lim_{\alpha \to \alpha_2} \sup_{t \in \mathbb{R}} d(x^\alpha(t), \Gamma) = 0,$$

where $\Gamma = \{x^h(t) | t \in \mathbb{R}\} \cup \{x_0\}$ and $d(x, \Gamma) = \inf_{y \in \Gamma} |x - y|$. See Fig. 1.

We define the homoclinic Melnikov function as

$$M(\phi) = \int_{-\infty}^{\infty} DH(x^h(t)) \cdot u(x^h(t), t + \phi) \, dt. \quad (5.7)$$

If $M(\phi)$ has a simple zero, then for $|\varepsilon| > 0$ sufficiently small there exist transverse homoclinic orbits to a periodic orbit near $\{x_0\} \times S^1$ in (5.3) [11,16,26]. The existence of such transverse homoclinic orbits implies that the system (5.3) exhibits chaotic motions by the Smale–Birkhoff theorem [11,26] and has no real-analytic (additional) first integral (see, e.g., Chapter III of [20]). However, we cannot exclude the possibility of analytical integrability of (5.3) since it may have two additional linearly independent commutative vector fields. We easily show that

$$\lim_{l \to \infty} M^{l/1}(\phi) = M(\phi) \quad (5.8)$$

for each $\phi \in S^1$ (see Theorem 4.6.4 of [11]). Let $U$ be a neighborhood of $\alpha = \alpha_2$. It follows from (5.8) that if $M(\phi)$ is not identically zero or constant, then for $l > 0$ sufficiently large neither is $M^{l/1}(\phi)$. Let $\hat{U} \subset \mathbb{R}^2$ be a region such that $\partial \hat{U} \supset \Gamma$ and $\hat{U} \supset \{x^\alpha(t) | t \in [0, \tilde{T}^\alpha]\}$ for some $\alpha \in (\alpha_1, \alpha_2)$. We obtain the following from Theorems 5.1 and 5.2.

**Theorem 5.4.** Suppose that $M(\phi)$ is not identically zero. Then for $|\varepsilon| \neq 0$ sufficiently small the system (5.3) has no real-analytic first integral in $\hat{U} \times S^1$ such that it depends analytically on $\varepsilon$ near $\varepsilon = 0$.

**Theorem 5.5.** Suppose that $M(\phi)$ is not constant. Then for $|\varepsilon| \neq 0$ sufficiently small the system (5.3) is not real-analytically integrable in the meaning of Theorem 2.4 in $U \times S^1$. 

---

**Fig. 1.** Assumption (M3)
Remark 5.6. (i) Theorems 5.4 and 5.5, respectively, mean that the system (5.3) has no first integral and is nonintegrable even if the Melnikov function $M(\phi)$ does not have a simple zero, i.e., there may exist no transverse homoclinic orbit to the periodic orbit, but it is not identically zero and constant. See Sect. 6.3. (ii) As in Remark 5.3(ii), if the system (5.1) is Hamiltonian, then the hypothesis of Theorem 5.4 is equivalent to that of Theorem 5.5. (iii) In the statements of Theorems 5.4 and 5.5, the region $\hat{U} \times S^1$ may be replaced with a neighborhood of $\Gamma \times S^1$ although they are weakened.

6. Examples

We now illustrate the above theory with three examples: a simple pendulum with a constant torque, second-order coupled oscillators, and the periodically forced Duffing oscillator [11,12,26].

6.1. Simple Pendulum with a Constant Torque

Consider a simple pendulum with a constant torque
\[ \dot{I} = \varepsilon(\beta \sin \theta + 1), \quad \dot{\theta} = I, \quad (I, \theta) \in \mathbb{R} \times \mathbb{T}, \]
where $\beta \in \mathbb{R}$ is a constant. Equation (6.1) is of the form (1.1) with $m = \ell = 1$. Using Theorem 4.3, we obtain the following:

Proposition 6.1. The system (6.1) has no real-analytic first integral depending analytically on $\varepsilon$ near $\varepsilon = 0$.

Remark 6.2. (i) The system (6.1) has the first integral
\[ F(I, \theta; \varepsilon) = \frac{1}{2}I^2 + \varepsilon(\beta \cos \theta - \theta) \]
and is $(1, 1)$-integrable as a system on $\mathbb{R} \times \mathbb{R}$, although $F(I, \theta; \varepsilon)$ is not even a function on $\mathbb{R} \times S^1$. (ii) Let $\beta = 0$. Then the system (6.1) is $(2, 0)$-integrable when $\varepsilon \neq 0$, where the vector fields $\varepsilon \frac{\partial}{\partial I} + I \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \theta}$ are commutative and linearly independent. However, when $\varepsilon = 0$, the two vector fields are linearly dependent and Eq. (6.1) is not $(2, 0)$-integrable; see also Remark 2.5(ii).

6.2. Second-Order Coupled Oscillators

Consider
\[ \dot{I}_j = \varepsilon \left( -\delta I_j + \Omega_j + \sum_{i=1}^{\ell} \sum_{k \in \mathbb{N}^2} a_k \sin(k_1 \theta_{j} - k_2 \theta_i) \right), \]
\[ \dot{\theta}_j = I_j, \quad j = 1, \ldots, \ell, \] (6.2)
where $\delta, \Omega_j \geq 0$, $j = 1, \ldots, \ell$, and $a_k, k = (k_1, k_2) \in \mathbb{N}^2$, are constants such that $|a_k| \leq M e^{-(k_1+k_2)\delta}$ for some $M, \delta > 0$. We see by the remark after Lemma 2 in Section 12 of Chapter 3 in [1] that the vector field of (6.2) is analytic. Equation (6.2) has the form (1.1) with $m = \ell$ and is rewritten in a system of second-order differential equations as
\[
\ddot{\theta}_j + \varepsilon \delta \dot{\theta} = \varepsilon \left( \Omega_j + \sum_{i=1}^{\ell} \sum_{k \in \mathbb{N}^2} a_k \sin(k_1 \theta_j - k_2 \theta_i) \right), \quad j = 1, \ldots, \ell,
\]
which reduces to the second-order Kuramoto model [25] when $a_k \neq 0$ for $k = (1, 1)$ and $a_k = 0$ for $k \neq (1, 1)$. Obviously, assumptions (K2), (A3) and (A4) hold. Using Theorems 4.1 and 4.4, we obtain the following:

**Proposition 6.3.** The following statements hold for (6.2):

(i) If at least one of $\delta$ and $\Omega_j$, $j = 1, \ldots, \ell$, is nonzero, then the system (6.2) does not have $\ell$ real-analytic first integrals near $\varepsilon = 0$ such that they are functionally independent in $(I, \theta)$ for $|\varepsilon| \neq 0$ and depend analytically on $\varepsilon$;

(ii) If $K_1 = \{k_1/k_2 : a_k, k_2 \neq 0\}$ or $K_2 = \{k_2/k_1 : a_k, k_1 \neq 0\}$ has an accumulation point, then for $|\varepsilon| \neq 0$ sufficiently small the system (6.2) is not real-analytically integrable in the meaning of Theorem 2.4.

**Proof.** Part (i) immediately follows from Theorem 4.1 and Remark 4.2(iii) since $\hat{h}_0(I)$ is not identically zero if at least one of $\delta$ and $\Omega_j$, $j = 1, \ldots, \ell$, is nonzero.

We turn to the proof of part (ii). Let $D_1 = \{I \in \mathbb{R}^\ell : k_1 I_j - k_2 I_i = 0, i, j = 1, \ldots, \ell, k_1/k_2 \in K_1\}$ and $D_2 = \{I \in \mathbb{R}^\ell : k_1 I_j - k_2 I_i = 0, i, j = 1, \ldots, \ell, k_2/k_1 \in K_2\}$. If $K_1$ (resp. $K_2$) has an accumulation point, then $D_1$ (resp. $D_2$) is a key set for $C^\omega(\mathbb{R}^\ell)$. Their claim is shown as follows. Assume that $K_1$ has an accumulation point. Let $f(I) \in C^\omega(\mathbb{R}^\ell)$ be an analytic function which vanishes on $D_1$, and take a line $L_B := \{I \in \mathbb{R}^\ell : (I_1, \ldots, I_{\ell-1}) = b\}$ for $b = (b_1, \ldots, b_{\ell-1}) \in \mathbb{R}^{\ell-1}$ fixed. Then $(b_1, \ldots, b_{\ell-1}, k_1 b_1/k_2) \in L_B \cap D_1$ for all $k_1/k_2 \in K_1$, so that $f(I)$ is identically zero on $L_B$. This means that $f(I)$ is identically zero in $\mathbb{R}^\ell$, and consequently $D_1$ is a key set for $C^\omega(\mathbb{R}^\ell)$. Similarly, we see that the claim is true for $K_2$ and $D_2$. Applying Theorem 4.4, we obtain the desired result. \(\square\)

### 6.3. Periodically Forced Duffing Oscillator

Consider the periodically forced Duffing oscillator
\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = ax_1 - x_1^3 + \varepsilon (\beta \cos \nu t - \delta x_2),
\]
where $\nu > 0$ and $\beta, \delta \geq 0$ are constants, and $a = -1$ or 1. The system (6.3) has the form (5.1) with
\[
H = -\frac{1}{2} ax_1^2 + \frac{1}{4} x_1^4 + \frac{1}{2} x_2^2
\]
and the autonomous system (5.3) becomes

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = ax_1 - x_3 + \varepsilon (\beta \cos \theta - \delta x_2), \quad \dot{\phi} = \nu, \]

where \((x, \phi) \in \mathbb{R}^2 \times S\). See Fig. 2 for the phase portraits of (6.3) with \(\varepsilon = 0\).

We begin with the case of \(a = 1\). When \(\varepsilon = 0\), in the phase plane there exist a pair of homoclinic orbits

\[ x^h_{\pm}(t) = (\pm \sqrt{2} \text{sech } t, \mp \sqrt{2} \text{sech } t \tanh t), \]

a pair of one-parameter families of periodic orbits

\[ x^k_{\pm}(t) = \left( \pm \sqrt{2} \frac{\text{sn}}{\sqrt{2-k^2}} \left( \frac{t}{\sqrt{2-k^2}} \right), \right. \]

\[ \left. \mp \frac{\sqrt{2}k^2}{2-k^2} \frac{\text{cn}}{\sqrt{2-k^2}} \left( \frac{t}{\sqrt{2-k^2}} \right) \frac{\text{dn}}{\sqrt{2-k^2}} \left( \frac{t}{\sqrt{2-k^2}} \right) \right), \quad k \in (0, 1), \]

inside each of them, and a one-parameter periodic orbits

\[ \tilde{x}^k(t) = \left( \sqrt{2} k \frac{\text{cn}}{\sqrt{2-k^2}} \left( \frac{t}{\sqrt{2-k^2}} \right), \right. \]

\[ \left. - \sqrt{2} k \frac{\text{sn}}{2k^2-1} \frac{\text{sn}}{\sqrt{2-k^2}} \left( \frac{t}{\sqrt{2-k^2}} \right) \frac{\text{dn}}{\sqrt{2-k^2}} \left( \frac{t}{\sqrt{2-k^2}} \right) \right), \quad k \in (1/\sqrt{2}, 1), \]

outside of them, as shown in Fig. 2(a), where \(\text{sn}, \text{cn}\) and \(\text{dn}\) represent the Jacobi elliptic functions with the elliptic modulus \(k\). The periods of \(x^k_{\pm}(t)\) and \(\tilde{x}^k(t)\) are given by \(\hat{T}^k = 2K(k)\sqrt{2-k^2}\) and \(\tilde{T}^k = 4K(k)\sqrt{2-k^2}-1\), respectively, where \(K(k)\) is the complete elliptic integral of the first kind. Note that \(x^k_{\pm}(t)\) approaches \(x^h_{\pm}(t)\) as \(k \to 1\); see [11,26]. See also [9] for general information on elliptic functions.

Assume that the resonance conditions

\[ i\hat{T}^k = \frac{2\pi n}{\nu}, \quad \text{i.e.,} \quad \nu = \frac{2\pi n}{2l K(k)\sqrt{2-k^2}}, \]

FIG. 2. Phase portraits of (6.3) with \(\varepsilon = 0\): \(a\) \(a = 1\); \(b\) \(a = -1\)
and
\[
|\tilde{T}^k| = \frac{2\pi n}{\nu}, \quad \text{i.e.,} \quad \nu = \frac{2\pi n}{4lK(k)\sqrt{2k^2 - 1}},
\]
(6.6)
hold for \(x^k_\pm(t)\) and \(\tilde{x}^k(t)\), respectively, with \(l, n > 0\) relatively prime integers. Then the subharmonic Melnikov function (5.5) for \(x^k_\pm(t)\) and \(\tilde{x}^k(t)\) are computed as
\[
M_{\pm}^{n/l}(\tau) = -\delta J_1(k, l) \pm \beta J_2(k, n, l) \sin \tau
\]
and
\[
\tilde{M}_{\pm}^{n/l}(\tau) = -\delta \tilde{J}_1(k, l) + \beta \tilde{J}_2(k, n, l) \sin \tau,
\]
respectively, where
\[
J_1(k, l) = \frac{4l[(2 - k^2)E(k) - 2k^2K(k)]}{3(2 - k^2)^{3/2}},
\]
\[
J_2(k, n, l) = \begin{cases} \sqrt{2}\pi \nu \sech\left(\frac{n\pi K(k')}{K(k)}\right) & \text{(for } l = 1) ; \\ 0 & \text{(for } l \neq 1) , \end{cases}
\]
\[
\tilde{J}_1(k, l) = \frac{8l[(2k^2 - 1)E(k) + k^2K(k)]}{3(2k^2 - 1)^{3/2}},
\]
\[
\tilde{J}_2(k, n, l) = \begin{cases} 2\sqrt{2}\pi \nu \sech\left(\frac{n\pi K(k')}{2K(k)}\right) & \text{(for } l = 1 \text{ and } n \text{ odd)} ; \\ 0 & \text{(for } l \neq 1 \text{ or } n \text{ even}) . \end{cases}
\]
Here \(E(k)\) is the complete elliptic integral of the second kind and \(k' = \sqrt{1 - k^2}\) is the complimentary elliptic modulus. When \(\delta \neq 0\), the subharmonic Melnikov functions \(M_{\pm}^{n/l}(\tau)\) and \(\tilde{M}_{\pm}^{n/l}(\tau)\) are not identically zero for any relatively prime integers \(n, l > 0\) since \(J_1(k, l)\) and \(\tilde{J}_1(k, l)\) are not zero. Moreover, the homoclinic Melnikov function (5.7) for \(x^h_\pm(t)\) becomes
\[
M_{\pm}(\tau) = -\frac{4}{3}\delta \pm \sqrt{2}\pi \nu \beta \csch\left(\frac{\pi \nu}{2}\right) \sin \tau,
\]
which is not identically zero for \(\beta \neq 0\). See [11,26] for the computations of the Melnikov functions.

Let
\[
R = \{k \in (0, 1) \mid k \text{ satisfies (6.5) for some } n, l \in \mathbb{N}\},
\]
\[
\tilde{R} = \{k \in (1/\sqrt{2}, 1) \mid k \text{ satisfies (6.6) for some } n, l \in \mathbb{N}\},
\]
and let
\[
S^k_\pm = \{(x^k_\pm(t), \theta) \in \mathbb{R}^2 \times S^1 \mid t \in [0, \hat{T}^k), \theta \in S^1\},
\]
\[
\hat{S}^k = \{\tilde{x}^k(t), \theta) \in \mathbb{R}^2 \times S^1 \mid t \in [0, \hat{T}^k), \theta \in S^1\},
\]
\[
\Gamma_\pm = \{x^h_\pm(t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\} \cup \{0\}.
\]
Noting that
\[ \lim_{n \to \infty} \tilde{M}_{2n+1/2}(\tau) = M_+(\tau) + M_-(\tau) \]
and applying Theorems 5.1, 5.4 and 5.5 and their slight extensions, we have the following:

**Proposition 6.4.** The system (6.4) with \( a = 1 \) has no real-analytic first integral depending analytically on \( \varepsilon \) near \( \varepsilon = 0 \) in neighborhoods of \( S^k_\pm \) for \( k \in \mathbb{R} \), of \( \tilde{S}^k_\pm \) for \( k \in \tilde{\mathbb{R}} \), and of \( S_{h\pm} \) if \( \delta \neq 0 \).

**Proposition 6.5.** Let \( \hat{U}_\pm \) (resp. \( \tilde{U} \)) be regions (resp. a region) in \( \mathbb{R}^2 \) such that \( \partial \hat{U}_\pm \supset \Gamma_\pm \) (resp. \( \partial \tilde{U}_\pm \supset \Gamma_+ \cup \Gamma_- \)) and \( \hat{U}_\pm \supset \{ x^k_\pm(t) \mid t \in [0, \hat{T}_k^\alpha) \} \) (resp. \( \tilde{U} \supset \{ \tilde{x}^k(t) \mid t \in [0, \tilde{T}_k^k) \} \) for some \( k \in (0, 1) \) (resp. \( k \in (1/\sqrt{2}, 1) \)). For \( |\varepsilon| \neq 0 \) sufficiently small the system (6.4) with \( a = 1 \) is not real-analytically integrable in the regions \( \hat{U}_\pm \times \mathbb{S}^1 \) (resp. \( \tilde{U} \times \mathbb{S}^1 \)) in the meaning of Theorem 2.4 if \( \beta \neq 0 \).

If \( \beta \neq 0 \) and
\[ \frac{\delta}{\beta} < \frac{3}{4} \sqrt{2}\pi \nu \operatorname{csch} \left( \frac{\pi \nu}{2} \right), \] (6.7)
then \( M_{\pm}(\tau) \) has a simple zero so that for \( |\varepsilon| > 0 \) sufficiently small, there exist transverse homoclinic orbits to a periodic orbit near the origin and chaotic dynamics may occur in (6.4) with \( a = 1 \), as stated in Section 5. From Proposition 6.5 we see that the system (6.4) is nonintegrable in the meaning of Theorem 2.4 even if condition (6.7) does not hold, i.e., there may exist no transverse homoclinic orbit to the periodic orbit, as stated in Remark 5.6(i). On the other hand, when the system (6.3) is Hamiltonian, i.e., \( \delta = 0 \), condition (6.7) always holds and such inconsistency does not occur. See also Remark 5.6(ii).

We turn to the case of \( a = -1 \). When \( \varepsilon = 0 \), in the phase plane there exists a one-parameter family of periodic orbits
\[ \hat{x}^k(t) = \left( \frac{\sqrt{2}k}{\sqrt{1 - 2k^2}} \right) \operatorname{cn} \left( \frac{t}{\sqrt{1 - 2k^2}} \right), \]
\[ - \frac{\sqrt{2}k}{1 - 2k^2} \operatorname{sn} \left( \frac{t}{\sqrt{1 - 2k^2}} \right) \operatorname{dn} \left( \frac{t}{\sqrt{1 - 2k^2}} \right), \quad k \in (0, 1/\sqrt{2}) \]
as shown in Fig. 2(b), and their period is given by \( \hat{T}_k = 4K(k)\sqrt{1 - 2k^2} \). See [27,28]. Assume that the resonance conditions
\[ l \hat{T}_k = \frac{2\pi n}{\nu}, \quad \text{i.e.,} \quad \nu = \frac{\pi n}{2lK(k)\sqrt{1 - 2k^2}} \] (6.8)
hold for \( l, n > 0 \) relatively prime integers. We compute the subharmonic Melnikov function (5.5) for \( \hat{x}^k(t) \) as
\[ \hat{M}^{n/l}(\tau) = -\delta \hat{J}_1(k, l) \pm \beta \hat{J}_2(k, n, l) \sin \tau, \]
where

\[
\hat{J}_1(k, l) = \frac{8l[2k^2 - 1]E(k) + k^2 K(k)]}{3(1 - 2k^2)^{3/2}},
\]

\[
\hat{J}_2(k, n, l) = \begin{cases} 
\sqrt{2\pi^2 n} & \text{for } l = 1 \text{ and } n \text{ odd}; \\
0 & \text{for } l \neq 1 \text{ or } n \text{ even}.
\end{cases}
\]

See also [27,28] for the computations of the Melnikov function. When \( \delta \neq 0 \), the subharmonic Melnikov function \( \hat{M}^{n/l}(\tau) \) is not identically zero for any relatively prime integers \( n, l > 0 \) since \( \hat{J}_1(k, l) \) is not zero.

Let

\[
\hat{R} = \{ k \in (0, 1/\sqrt{2}) \mid k \text{ satisfies (6.8) for } n, l \in \mathbb{N} \},
\]

and let

\[
\hat{S}^k = \{ (\hat{x}^k(t), \theta) \in \mathbb{R}^2 \times S^1 \mid t \in [0, \hat{T}^k), \theta \in S^1 \}.
\]

Applying Theorem 5.1, we obtain

**Proposition 6.6.** The system (6.4) with \( a = -1 \) has no real-analytic first integral depending analytically on \( \varepsilon \) in a neighborhood of \( \hat{S}^k \) for \( k \in \hat{R} \) near \( \varepsilon = 0 \) if \( \delta \neq 0 \).

**Remark 6.7.**

(i) Since the subharmonic Melnikov function \( \hat{M}^{n/l}(\tau) \) is constant for \( l \neq 1 \), Theorem 5.2 is not applicable to (6.4) with \( a = -1 \). So we cannot exclude the possibility that the system (6.4) with \( a = -1 \) is (3, 0)-integrable in the meaning of Theorem 2.4 when \( \beta, \delta \neq 0 \).

(ii) It was shown in [31] that the system (6.4) with \( a = -1 \) is meromorphically nonintegrable in a meaning similar to that of Theorem 2.4 when the independent and state variables are extended to complex ones.

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Appendix A. Proof of Proposition 3.6

In this Appendix, we prove Proposition 3.6, which is an extension of Ziglin’s lemma [4,10,33]. Our approach is to reduce the functional independence of first integrals near the level set $F^{-1}(c)$ to that near a point on it using action and angle variables. We begin with the following lemma:

**Lemma A.1.** Let $\Omega$ be an open subset of $\mathbb{R}^k$ and let $\chi_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, m$, be analytic, where $k, m \in \mathbb{N}$. Let $\chi(x) = (\chi_1(x), \ldots, \chi_m(x))$. If rank $d\chi$ is constant on $\Omega$ and less than $m$, then for any $x \in \Omega$ there exists a neighborhood $V$ of $x$ on which $\chi_1, \ldots, \chi_m$ are analytically dependent, i.e., there exist an open set $\Omega' \subset \mathbb{R}^m$ and a non-constant analytic map $\xi : \Omega' \to \mathbb{R}$ such that $\chi(V) \subset \Omega'$ and $\xi(\chi(y)) = 0$ for any $y \in V$.

**Proof.** Using Theorem 1.3.14 of [22] and an argument in the proof of Theorem 1.4.15 of [22], we can immediately obtain the desired result as follows. The theorem says that there exist a neighborhood $V$ (resp. $V'$) of $x$ (resp. $\chi(x)$), a cube $Q$ (resp. $Q'$) in $\mathbb{R}^k$ (resp. in $\mathbb{R}^m$) and analytic isomorphisms $u : Q \to V$ and $u' : V' \to Q'$ such that the composite map $u' \circ \chi \circ u$ has the form $(x_1, \ldots, x_k) \to (x_1, \ldots, x_m, 0, \ldots, 0)$, where $x_j$ is the $j$th element of $x$ for $j = 1, \ldots, k$ and $m' = \text{rank } d\chi < m$. Here a *cube* in $\mathbb{R}^k$ is an open set of the form

$$\{x \mid |x_j - a_j| < r_j, j = 1, \ldots, k\}$$

for some $a_j \in \mathbb{R}$ and $r_j > 0$, $j = 1, \ldots, k$. Letting $u' = (u'_1, \ldots, u'_m)$ and $\xi = u'_m$, we have $\xi(\chi(y)) = 0$ for every $y \in V$. \qed

Let $f_\varepsilon : \mathcal{M} \to \mathbb{R}$ be an analytic function such that it depends on $\varepsilon$ analytically. We expand it near $\varepsilon = 0$ as $f_\varepsilon(x) = \sum_{j=0}^{\infty} f^j(x) \varepsilon^j$, where $f^j(x)$, $j \in \mathbb{Z}_0 := \mathbb{N} \cup \{0\}$, are analytic functions on $\mathcal{M}$. Define the order function $\sigma(f_\varepsilon)$ by

$$\sigma(f_\varepsilon) := \min\{j \in \mathbb{Z}_0 \mid f^j(x) \neq 0\}$$

if $f_\varepsilon \neq 0$ and $\sigma(0) := +\infty$, as in [6].

**Lemma A.2.** Suppose that $f_\varepsilon(x)$ is a nonconstant analytic first integral of (1.4) depending analytically on $\varepsilon$ near $\varepsilon = 0$. Then there exists an analytic first integral $f_\varepsilon(x) = f^0(x) + O(\varepsilon)$ depending analytically on $\varepsilon$ near $\varepsilon = 0$ such that $f^0(x)$ is not constant.

**Proof.** Since $f_\varepsilon$ is not constant, $\sigma(df_\varepsilon)$ takes a finite value. Let $k = \sigma(df_\varepsilon)$ and $f_\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j f^j(x)$. Define

$$\tilde{f}_\varepsilon(x) := \frac{1}{\varepsilon^k} \left( f_\varepsilon(x) - \sum_{j=0}^{k-1} \varepsilon^j f^j(x) \right).$$

Then $\tilde{f}_\varepsilon(x) = f^k(x) + O(\varepsilon)$ and $\tilde{f}^0(x) = f^k(x)$ is not constant. Moreover, $\tilde{f}_\varepsilon(x)$ is a first integral of (1.4) since $X_\varepsilon(\tilde{f}_\varepsilon) = 0$ and $\sum_{j=0}^{k-1} \varepsilon^j f^j$ is constant. \qed
We are now in a position to give a proof of Proposition 3.6.

**Proof of Proposition 3.6.** For \( k = 1 \) the statement of the proposition holds by Lemma A.2. Let \( k > 1 \) and suppose that it is true up to \( k - 1 \). Let \( G^e_k(x), \ldots, G^e_k(x) \) be analytic first integrals of (1.4) in a neighborhood of \( F^{-1}(c) \) near \( \varepsilon = 0 \) such that they are functionally independent for \( \varepsilon \neq 0 \) and depend analytically on \( \varepsilon \).

Without loss of generality, we assume that \( G^e_1(x), \ldots, G^e_k(x) \) are functionally independent near \( F^{-1}(c) \). Letting \( G^e(x) = (G^e_1(x), \ldots, G^e_k(x)) \), we see that \( G^e(\varepsilon(I, \theta)) \) depends only on \( I \), where \( \varphi \) denotes the analytic diffeomorphism in Proposition 2.1(ii). Let \( \tilde{G}_j(I) = G^e_j(\varepsilon(I, \theta)) \) for \( j = 1, \ldots, k \). Note that, if \( d\tilde{G}_1(I), \ldots, d\tilde{G}_k(I) \) are linearly independent at \( I = I_0 \in U \), then so are \( dG^e_1(x), \ldots, dG^e_k(x) \) on \( \varphi([I_0] \times \mathbb{T}^q) \subset U \).

Assume that \( G^e_1(I), \ldots, G^e_{k-1}(I), \tilde{G}_k(I) \) are functionally dependent in an open set \( U' \subset U \). So \( \Omega := \{ p \in U' \mid \text{rank } d_p G = k - 1 \} \) contains a dense open set in \( U' \) since \( d\tilde{G}_1(I), \ldots, d\tilde{G}_{k-1}(I) \) are functionally independent on \( U \). By Lemma A.1, there exist an open set \( \Omega' \subset \mathbb{R}^k \) and a nonzero analytic function \( \zeta : \Omega' \to \mathbb{R} \) such that \( \tilde{G}(V) = (\tilde{G}_1(V), \ldots, \tilde{G}_k(V)) \subset \Omega' \) and

\[
\hat{\zeta}(\tilde{G}_1(I), \ldots, \tilde{G}_k(I)) = 0
\]

in a neighborhood \( V \) of \( p \in \Omega \). Moreover, there is a positive integer \( s \) such that \( (\partial^s \zeta/\partial y^s_k)(\tilde{G}(I)) \neq 0 \), since if not, then \( \hat{\zeta}(\tilde{G}_1(I), \ldots, \tilde{G}_{k-1}(I), y_k) \) depends on \( y_k \) near \( \tilde{G}(V) \) and consequently \( \tilde{G}_1(I), \ldots, \tilde{G}_{k-1}(I) \) are functionally dependent. Let \( s \) be the smallest one of such integers and let \( \hat{\zeta}(y) = (\partial^s \zeta/\partial y^s_k)(y) \). Then \( \hat{\zeta} \) satisfies

\[
\hat{\zeta}(\tilde{G}_1(I), \ldots, \tilde{G}_k(I)) = 0,
\]

and \( (\partial \hat{\zeta}/\partial y_k)(\tilde{G}(I)) \neq 0 \) on \( V \). Hence,

\[
\hat{\zeta}(G^e(\varepsilon)) = 0,
\]

and \( (\partial \hat{\zeta}/\partial y_k)(G^e(\varepsilon)) \neq 0 \) on \( \varphi(V \times \mathbb{T}^q) \).

Let \( \hat{G}^e_k(x) = \hat{\zeta}(G^e(x))/\varepsilon \). By (A1) \( \hat{G}^e_k \) is an analytic first integral depending analytically on \( \varepsilon \). We have

\[
d\hat{G}^e_k = \varepsilon^{-1} d(\hat{\zeta}(G^e(x))) = \varepsilon^{-1} \sum_{j=1}^k \frac{\partial \hat{\zeta}}{\partial y_j}(G^e(x))dG^e_j
\]

and

\[
N(\hat{G}^e) := dG^e_1 \wedge \ldots \wedge dG^e_{k-1} \wedge d\hat{G}^e_k = \varepsilon^{-1} \frac{\partial \hat{\zeta}}{\partial y_k}(G^e(x))dG^e_1 \wedge \ldots \wedge dG^e_k.
\]

Since \( (\partial \zeta/\partial y_k)(G^e(\varepsilon)) \neq 0 \) on \( \varphi(V \times \mathbb{T}^q) \), we have \( \sigma((\partial \zeta/\partial y_k)(G^e(\varepsilon))) = 0 \), so that

\[
\sigma(N(\hat{G}^e)) = \sigma(\varepsilon^{-1}N(G^e)) + \sigma \left( \frac{\partial \zeta}{\partial y_k}(G^e(x)) \right) = \sigma(N(G^e)) - 1.
\]

Repeating this procedure till \( \sigma(N(\hat{G}^e)) = 0 \), we obtain

\[
dG^e_1 \wedge \ldots \wedge dG^e_0 \wedge d\hat{G}^e_1 \neq 0,
\]

which gives the desired result. \( \square \)
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