Linear Invariants for Linear Systems

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Abstract. A central question in verification is characterizing when a system has invariants of a certain form, and then synthesizing them. We say a system has a k linear invariant, k-LI in short, if it has a conjunction of k linear (non-strict) inequalities – equivalently, an intersection of k (closed) half spaces – as an invariant. We present a sufficient condition – solely in terms of eigenvalues of the A-matrix – for an n-dimensional linear dynamical system to have a k-LI. Our proof of sufficiency is constructive, and we get a procedure that computes a k-LI if the condition holds. We also present a necessary condition, together with many example linear systems where either the sufficient condition, or the necessary is tight, and which show that the gap between the conditions is not easy to overcome. In practice, the gap implies that using our procedure, we synthesize k-LI for a larger value of k than what might be necessary. Our result enables analysis of continuous and hybrid systems with linear dynamics in their modes solely using reasoning in the theory of linear arithmetic (polygons), without needing reasoning over nonlinear arithmetic (ellipsoids).

Keywords: Invariants · Linear Systems · Polyhedral Lyapunov Functions.

1 Introduction

Linear systems are extensively studied because they serve as a good modeling formalism. Even systems that have nonlinear dynamics often exhibit nice linear behavior in certain regions of the state space, and can be modeled using (piecewise) linear systems or hybrid systems with linear continuous dynamics. Furthermore, linear systems are easier to analyze and can be used to build analyzers for nonlinear, piecewise linear, and hybrid systems.

A linear system is simply a continuous-time and continuous-space dynamical system whose state space is the n-dimensional reals, and whose dynamics is specified by an ODE of the form $dx/dt = Ax$ \[1\]. A classical result in control says that this system is stable (around the origin) if the real parts of all eigenvalues of $A$ are negative. Moreover, for such stable $A$, there exist quadratic Lyapunov functions: functions that are decreasing along the system trajectories. A Lyapunov function $f$ gives rise to invariants: if a system starts in the region $f(x) < 1$, then it will continue to stay inside that region. Such Lyapunov (and
Lyapunov-like) functions have been used extensively in verification of linear, nonlinear, and hybrid systems [15,17,14].

There are, however, a few undesirable features if we just rely on ellipsoidal invariants for stable systems. First, we also want to generate invariants for unstable systems. Second, while we have made significant progress in reasoning with nonlinear real arithmetic [18], it is still much more scalable and desirable to have linear (or piecewise linear) functions $f$ defining the invariant. We are interested here in invariants of the form $f(x) < 1$ that can be expressed solely using linear expressions. In other words, we are interested in conjunction of linear inequalities as invariants, and want to know how to compute good quality invariants of this form. We study this question for the class of linear systems in this paper.

We use the term $k$ linear invariant, or $k$-LI in short, to denote an invariant that can be represented as a conjunction of $k$ linear inequalities. The parameter $k$ here provides a good tradeoff between strength of the invariant and the cost of reasoning with it: (1) a small value of $k$ makes the task of reasoning with these invariants easier, but it also restricts the strength of the invariant, whereas (2) a large value of $k$ can yield potentially strong invariants, but it also makes the reasoning task more complex. Ideally, we want $k$ to be large enough to allow polytopes as candidate invariants, but not much larger than needed. Polyhedral Lyapunov functions [13], for example, define such polytopes with $2s$ faces, where $s \geq n$. A special case is when the polytope in $n$-dimensional space has $2n$ faces, such as a zonotope (rotated box). As we will show, for stable systems, we generate $2s$-LI where $s \geq n$, whereas for unstable systems, we can still generate $2s$-LI where $s$ may be less-than $n$.

Our main result is a sufficient condition that guarantees existence of $k$ linear invariants ($k$-LI) for linear systems. A corollary of the main result is a sufficient condition for existence of polyhedral Lyapunov functions for linear systems. Our proof is constructive: we actually show how to generate the $k$-LI if our condition holds. We then go on to answer the question about whether the condition is also necessary. We present examples to show the condition is not necessary, and we present a slightly weaker necessary condition. It is an interesting open question if we can close the gap between the sufficient condition and the necessary condition. Intuitively, the sufficient condition gives us the number $k$ of half-spaces needed to guarantee existence of a bounded $k$-LI, while the necessary condition gives the number $k$ of half-spaces so that there is guaranteed no $k'$-LI for $k' < k$.

Our results on necessary and sufficient conditions for existence of linear invariants for linear systems are foundational: the real part of eigenvalues of the $A$-matrix being negative characterizes existence of ellipsoidal invariants, and here we present similar conditions for existence of linear invariants. There is a long line of work on trying to characterize linear invariants starting with Bitsoris and Kiendl [4,20]. Our results nontrivially extend these results, and will be of interest in (a) verification of linear and hybrid systems, as verification techniques can be strengthened by computing strong linear invariants for modes with linear dynamics, and (b) applications of infinity-norm, or polyhedral, Lyapunov func-
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2 Preliminaries

2.1 Polyhedron

We use the term polyhedron to mean any subset of \( \mathbb{R}^n \) defined by intersection of half-spaces, and the term polytope to mean a polyhedron that is also bounded. A polyhedron that strictly contains the origin can be written as a conjunction of linear inequalities

\[
a_{11}x_1 + \cdots + a_{1n}x_n \leq 1 \\
\vdots \\
a_{s1}x_1 + \cdots + a_{sn}x_n \leq 1
\]

In matrix notation, this can be written succinctly as, \( Fx \leq 1 \), where \( F \) is an \( s \times n \) matrix that contains \( a_{ij} \) in its \((i, j)\)-cell. This can be equivalently written as \( \max(Fx) \leq 1 \).

A polyhedron is negative-closed if, whenever a point \( c \) occurs in the polyhedron, then \( -c \) also occurs in it. A negative-closed polyhedron can be represented as \( \max(Fx) \leq 1 \), \( \min(Fx) \geq -1 \), which can be equivalently written as \( \max(\|Fx\|) \leq 1 \), where \( \|\cdot\| \) denotes the infinity-norm of a vector. Note that the infinity-norm of a vector is the maximum of the absolute value of the components of the vector.

If we define a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( f(x) = \|Fx\|_\infty \), then the negative-closed polyhedron \( \|Fx\|_\infty \leq 1 \) can be written as \( f(x) \leq 1 \). We are interested in finding functions \( f \) that define “invariants” for a given linear system.

If \( F \) is a \( s \times n \) matrix, then in general the region \( \|Fx\|_\infty \leq 1 \) could be unbounded. However, if the matrix \( F \) has full column rank, then this region is bounded, and represents a polytope (a closed, bounded polyhedron). A specific case is when \( F \) is a \( n \times n \) full rank matrix. In this case, the set \( \|Fx\|_\infty \leq 1 \) is a zonotope [16]. Invariant sets are closely related to Lyapunov functions, and the special case when \( F \) is \( s \times n \) with \( s \geq n \) and \( F \) has full column rank has been studied under the name of “infinity-norm Lyapunov functions” (also called polyhedral, or piecewise-linear, Lyapunov functions) [20].

Note that, in logical notation, the set \( \|Fx\|_\infty \leq c \) can also be described by the formula

\[
\bigwedge_{i=1}^{s} (-c \leq F_{i,\cdot}x \leq c)
\]

\[1\] In general, the \( l \)-norm \( \|b\|_l \) of the vector \( b \) is \( \|b\|_l = (|b_1|^l + \cdots + |b_n|^l)^{\frac{1}{l}} \) where \( b_1, \ldots, b_n \) are the \( n \) components of the vector \( b \) and \( |b_i| \) denotes the absolute value of \( b_i \).
where $F_{i,*}$ denotes the $i$-th row of the matrix $V$. The formula above is just a conjunction of 2s linear (non-strict) inequalities, and can potentially be, what we will call, a 2s-LI.

### 2.2 Invariant Sets

Consider a (continuous-time continuous-space) dynamical system with state space $\mathbb{R}^n$. Assume that its trajectories are given by a function $g : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $g(t, c)$ denotes the state reached at time $t$ starting from $c \in \mathbb{R}^n$ at time 0. Note that $g(0, c) = c$.

**Definition 1 (Invariant Set).** A set of states $f(x) \leq 1$ is an invariant set for (a system whose trajectories are defined by) $g$ if, for every $c$ s.t. $f(c) \leq 1$, it is the case that $f(g(t, c)) \leq 1$ for all $t \geq 0$.

The notion of invariance considers states reached at all future times $t \geq 0$ – this has also been called positive invariance [7]. For dynamical systems where $g$ is a continuous function of the time $t$ and the initial state $c$, invariance as defined above is equivalent to inductive invariance defined below, where we only consider states reached in some small time interval.

**Definition 2 (Inductive Invariant Set).** A set of states $f(x) \leq 1$ is an inductive invariant set for $g$ if, for every $c$ s.t. $f(c) \leq 1$, there exists a $\delta > 0$ s.t. for all $t \in [0, \delta)$, $f(g(t, c)) \leq 1$.

A consequence of the equivalence between invariance and inductive invariance for continuous dynamics is that for checking invariance, we only need to look at points $c$ on the boundary, that is, $f(c) = 0$, and not worry about trajectories starting from points $c$ in the interior ($f(c) < 1$). This fact has been used extensively in safety verification of continuous and hybrid dynamical systems [27,30,25,28].

We are interested in linear dynamical systems whose dynamics is specified as $\frac{dx}{dt} = Ax$, where $A$ is an $n \times n$ matrix. In this case, we know that there exists a function $g$ that satisfies the equations $\frac{dg(t,c)}{dt} = Ag(t,c)$ and $g(0,c) = c$, and that $g$ is continuous in both its first and second arguments.

We will use the term 2s linear invariant, in short 2s-LI, to denote an invariant for such a dynamical system that is of the form $\|Fx\|_\infty \leq 1$ where $F$ is an $s \times n$ matrix.

### 2.3 Invariants and Weak Lyapunov Functions

Linear invariants for linear systems are closely related to weak polyhedral Lyapunov functions.

Let us fix a linear system $\frac{dx}{dt} = Ax$, and the function $g$ specifying its trajectories. A (weak) Lyapunov function is (non-increasing) decreasing along all possible system trajectories.
Definition 3 (PLF). Let $f : \mathbb{R}^n \mapsto \mathbb{R}^0^+$ be the function $f(x) := \|Fx\|_\infty$ where $F$ is a full column-rank $s \times n$ matrix. The function $f$ is a polyhedral Lyapunov function (PLF) for $g$ if $f(g(t, x)) < f(x)$ for all $t > 0$ and for all $x$ s.t. $f(x) > 0$. The function $f$ is a weak polyhedral Lyapunov function (weak PLF) for $g$ if $f(g(t, x)) \leq f(x)$ for all $t > 0$ and for all $x$ s.t. $f(x) > 0$.

Existence of weak PLF is equivalent to existence of bounded linear invariants.

**Proposition 1.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^0^+$ be the function $f(x) := \|Fx\|_\infty$ where $F$ is a full column-rank $s \times n$ matrix. The function $f$ is a weak polyhedral Lyapunov function (weak PLF) for $g$ if, and only if, for every constant $c > 0$, the set $\{x \mid f(x) \leq c\}$ is an (inductive) invariant set for $g$.

**Proof.** (Sketch) If $f$ is a weak PLF, then $f$ does not increase along trajectory starting from any point $x$ s.t. $f(x) > 0$, and hence $f(x) \leq c$ is clearly an invariant for every $c > 0$. Conversely, if $f$ does increase along a trajectory starting from some point $x_0$, then the set $f(x) \leq d$ is not an invariant, where $d = f(x_0) > 0$. \hfill \Box

For linear systems, the function $g$ defining the trajectory has a special linearity property: $g(t, \alpha x + \beta y) = \alpha g(t, x) + \beta g(t, y)$. Using this property, invariance of $f(x) \leq c$ for all $c$ is equivalent to invariance of $f(x) \leq 1$.

**Proposition 2.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^0^+$ be as in Proposition[1]. The set $\{x \mid f(x) \leq c\}$ is an (inductive) invariant set for $g$ for all $c > 0$ if and only if it is an (inductive) invariant set for $c = 1$.

**Proof.** (Sketch) Given a point $x$, if $\alpha = f(x)$, then $f(x/\alpha) = 1$. Hence, if a trajectory starting from $x_0$ exits the set $f(x) \leq f(x_0)$, then the trajectory starting from $x_0/f(x_0)$ exits the set $f(x) \leq 1$. \hfill \Box

The two propositions above together show that the ability to compute linear invariants will also give us the ability to obtain weak PLFs for linear systems. Next, we show that we can restrict our focus to negative-closed polyhedron without loss of any generality.

Recall that a negative-closed polyhedron is written as $\|Fx\|_\infty \leq 1$. The next lemma, Lemma[1], states that if there is a polyhedral invariant then there also exists a negative-closed polyhedral invariant. Its proof relies on the interesting observation that (1) given a polyhedron $P$, we can get a smaller negative-closed polyhedron $Q$ contained inside $P$, and (2) whenever a point $c$ is on the boundary of $Q$, then we can prove that trajectories are pointing inwards by either looking at where $c$ lies on $P$ and using PLF property of $P$, or looking at $-c$ and then using PLF property of $P$.

**Lemma 1.** If $\max(Fx) \leq 1$ is an invariant for a linear system $\dot{x} = Ax$, then $\|Fx\|_\infty \leq 1$ is also an invariant.
Using Fact 1, we will show that

\[ F_{i,*} g(t, -c) \leq 1 \quad \forall i, \forall t \geq 0 \forall c \in Q \]  

Using Fact 1 we will show that \( g(t, c) \) is in \( Q \) for all \( t \geq 0 \).

We prove by contradiction. Suppose \( g(t, c) \) is not in \( Q \) for all \( t \geq 0 \). Define 
\[ t_{\text{escape}} = \inf \{ t \mid g(t, c) \notin Q \} \]. Consider the point \( d = g(t_{\text{escape}}, c) \). Point \( d \) is in \( Q \), but points reached from \( d \), that is, \( g(t_{\text{escape}}, c) \), are not in \( Q \), but they are in \( P \); hence, there exists some index \( i \in \{1, \ldots, s\} \) s.t. \( F_{i,*} d = -1 \) and \( F_{i,*} g(0^+, d) < -1 \).

Consider the point \(-d\). We note that \( F_{i,*}(-d) = -(-1) = 1 \) by linearity. From Fact 1 we get \( F_{i,*} g(0^+, -d) \leq 1 \). By linearity, this implies \( F_{i,*} g(0^+, d) \geq -1 \). This contradicts \( F_{i,*} g(0^+, d) < -1 \). This completes the proof. \( \square \)

2.4 Necessary and Sufficient Check for Invariance

Before we can say anything interesting about (positive) invariant sets, we need a way to establish when some set is a (positive) invariant set and when it is not. The Lie derivative helps here: given the vector field \( f \) (of the dynamical system), and a function \( g : \mathbb{X} \rightarrow \mathbb{R} \), the Lie derivative \( \mathcal{L}_f(g) \) is defined as the dot-product \( \nabla g \cdot f \) of the gradient of \( g \) with \( f \); that is, \( \mathcal{L}_f(g)(x) := \sum_{i=1}^{n} \frac{\partial g(x)}{\partial x_i} f_i(x) \). Note that \( \mathcal{L}_f(g) \) is just the time derivative of \( g \), and the gradient \( \nabla g \) is the vector \( [\partial g/\partial x_1, \ldots, \partial g/\partial x_n] \) of the real-valued function \( g : \mathbb{R}^n \to \mathbb{R} \). We will assume \( \nabla g \) is a row vector, and so \( \nabla g \cdot f \) is just matrix multiplication, which we will denote by juxtaposition (and not use \( \cdot \)).

Proposition 3 shows that checking invariance is equivalent to checking inductive invariance. When checking inductive invariance of the set \( f(x) \leq 1 \), we first note that if point \( c \) is strictly in the interior, that is, \( f(c) < 1 \), then \( f(g(t, c)) \leq 1 \) for all sufficiently small \( t > 0 \). Hence, to check if \( f(x) \leq 1 \) is an inductive invariant, we only need to worry about points \( c \) s.t. \( f(c) = 1 \); that is, the so-called boundary points, and prove that the vector field points “inwards” at these points. Necessary and sufficient condition for checking if a vector field points “inwards” were discussed in [29]. A sufficient condition for checking that the vector field \( f \) that maps \( x \) to \( A x \) is pointing “inwards” into the region \( g(x) \leq 1 \) at the point \( c \), where \( g(c) = 1 \) is that \( \mathcal{L}_f(g)(c) < 0 \). In general, a necessary, but not sufficient, condition is that \( \mathcal{L}_f(g)(c) \leq 0 \). However, for linear dynamics and linear invariant sets, this necessary condition is also sufficient. This fact was implicitly stated in [29], and we make it explicit in Proposition 3. We first note that if the vector field \( f \) is given by \( A x \), then \( \mathcal{L}_f(g)(x) \) is simply \( \nabla g(x)A x \).
Proposition 3. A polyhedral region \( Fx \leq c \) is positively invariant for a linear system \( \dot{x} = Ax \) iff for each \( i \in \{1, \ldots, n\} \), it is the case that

\[
 Fx \leq c \land F_{i,*}x = c \Rightarrow F_{i,*}Ax \leq 0
\]  

(2)

where \( F_{i,*} \) is the \( i \)-th row of \( F \).

Proof. The result in [29] showed that the necessary check \( F_{i,*}Ax \leq 0 \) also sufficient if we can prove that the gradient is not zero on the boundary points. This is clearly the case for linear invariants. \( \Box \)

The reason why \( \mathcal{L}_fg(c) \leq 0 \) is not sufficient is that, in general, if the “first-derivative” is zero, we need to check the sign of the “second-derivative”, and if that is zero, then the sign of the “third-derivative”, and so on [29]. For linear dynamics and linear invariants, it is also possible to prove by first principles that these additional checks are implied by the necessary condition. Note that the sufficient check based on strict inequalities on the right-hand side are often used for the general case, for example, in definition of Barrier certificates [27].

2.5 Existence and Synthesis Problem

We now formally state the problem we solve in this paper.

Definition 4 (2s-LI Decision Problem). Given a linear system \( \frac{dx}{dt} = Ax \) and a natural number \( s > 0 \), determine if there exists an \( s \times n \) matrix \( F \) such that \( \|Fx\|_\infty \leq 1 \) is an invariant of the linear system.

Remark 1. The 2s-LI decision problem insists on finding a negative-closed polyhedron. There is no loss of generality here, since Lemma 1 showed that if there is any polyhedral linear invariant for a linear system, then there is one that is negative-closed. \( \Box \)

The synthesis problem asks us to generate the invariant for the given linear system.

Definition 5 (2s-LI Synthesis Problem). Given a linear system \( \frac{dx}{dt} = Ax \), find a natural number \( s > 0 \) and a matrix \( F \in \mathbb{R}^{s \times n} \) such that \( \|Fx\|_\infty \leq 1 \) is an invariant.

Remark 2. While the formulation of the synthesis problem leaves open the choice of \( s \), ideally we want \( s \) to be as close to \( n \) as possible. Note that there are systems that have an 2s-LI with \( s > n \), but that have no 2s-LI for \( s \leq n \). One such example is shown in Example 1.

Example 1. Consider the linear system

\[
\frac{dx}{dt} = -x - \sqrt{3}y, \quad \frac{dy}{dt} = \sqrt{3}x - y
\]
This models a spiral converging to the origin. It has a 2s-LI for $s = 3$, but not for $s = 2$; that is, it has a hexagon-shaped invariant (with 6 edges), but no parallelogram-shaped one (with 4 edges). For example, consider the following hexagon

$$|2y| \leq 1, \quad |y + \sqrt{3}x| \leq 1, \quad |y - \sqrt{3}x| \leq 1$$

Using Proposition 3, we can verify that the hexagon is indeed an inductive invariant. For example, the check in Equation (2) when instantiated for $i = 3$ gives

$$|2y| \leq 1 \land |y + \sqrt{3}x| \leq 1 \land y - \sqrt{3}x = 1 \Rightarrow 2\sqrt{3}x + 2y \leq 0$$

which is actually a valid formula (because $2\sqrt{3}x + 2y = -2(y - \sqrt{3}x) + 4y \leq -2+2 \leq 0$). We state without proof that this system can not have a parallelogram as an invariant, but we provide some intuition in the next example.

**Example 2.** Generalizing from Example 1, it is instructive to consider the family of linear systems:

$$\frac{dx}{dt} = -x - ay, \quad \frac{dy}{dt} = ax - y$$

where $a > 0$ is a positive real number. As $a$ becomes large and tends to infinity, the dynamics gets closer and closer to circular motion. For circular motion, there is no 2s-LI (as we would need a polyhedron with infinitely many edges to properly contain a circle). When $a = 1$, we need 4 edges to construct an invariant, and when $a = \sqrt{3}$, we need 6 edges (as in Example 1). Intuitively, as $a$ increases, we need more and more edges in the polyhedron to get an invariant.

If we solve the 2s-LI decision and synthesis problems, then we also get a solution for the corresponding problems for PLFs since the only additional check needed there is to ensure that the matrix $F$ has full column rank.

### 3 Motivation and Related Work

The decision and synthesis problems for 2s-LI have a long history. The motivation mainly comes from interest in *polyhedral Lyapunov functions* (PLFs) [510] to achieve robust control. There is also work that shows that quadratic Lyapunov functions are insufficient for establishing structural stability of basic motifs of biochemical networks, whereas polyhedral Lyapunov functions are enough to do so (of course, the motifs here have nonlinear dynamics) [11121910]. Robust control and structural stability are both related to absolute stability – where the goal is to prove stability for not a single system, but for a whole class of systems obtained by varying some parameters in the system definition [20]. Informally, PLFs are a useful tool for performing such analysis, and there are also results that show they are complete, whereas quadratic Lyapunov functions are not, in some cases [240]. Our immediate motivation comes more from the use of linear invariants for analysis and verification.
Blanchini [8,13] mentions that checking if a given polyhedron is an invariant (or defines a PLF) is simple, but synthesis of such an invariant (or PLF) is not so simple. There are iterative procedures that have been suggested for synthesis or procedures based on somehow “guessing” vertices of the polygon [21,22,23,26]. However, we are more interested in direct methods based on looking at eigenvectors of the $A$ matrix. Bitsoris [3,4] presented some of the earliest results in this direction by giving sufficient conditions for existence of linear invariants, and so did Kiendl [20]. However, those works covered just the case when $|b| < |a|$ for a complex eigenvalue $a + bi$. There were no known results for existence of linear invariants when $|b| > |a|$, which is a gap we fill in this paper.

Polyhedral invariants are essentially box invariants [1,2] but with a change of coordinates. A box invariant is an invariant of the form $\max(|x_1|, \ldots, |x_n|) \leq 1$. It is a $2n$-LI. While it is easy to characterize existence of box invariants based on simple checkable properties of the $A$-matrix, we are not aware of any such characterization for polyhedral invariants. The difficulty comes from the fact that we have to also discover the transformation that we can apply to the system to turn it into one that has box invariants.

4 Synthesizing Linear Invariants

Based on our formulation of the decision and synthesis problem for linear invariants, we henceforth restrict ourselves to invariants of the form $||Fx||_\infty \leq 1$, where $F$ is an $(s \times n)$-matrix.

Our main result is a sufficient condition for the existence of $2n$-LI. The result can be used to both determine if a linear system has an invariant, and also synthesize it. We formulate the result for the case when $s = n$, but the proof will show that it can be used to synthesize invariants with $s \neq n$.

**Theorem 1 (Sufficient condition for existence of $2n$-LI).** The linear system $dx/dt = Ax$, where $A$ is a $n \times n$ matrix, has a $2n$-LI if

(a) $\lambda \leq 0$ for every real eigenvalue $\lambda$ of $A$
(b) $\lambda < 0$ for every real eigenvalue $\lambda$ of $A$ such that $\text{geom}(\lambda) < \text{algm}(\lambda)$
(c) $a + |b| \leq 0$ for every complex eigenvalue $(a + ib)$ of $A$, and
(d) $a + |b| < 0$ for every complex eigenvalue $(a + ib)$ of $A$ such that $\text{geom}(a + ib) < \text{algm}(a + ib)$

Here $\text{algm}(l)$ denotes the algebraic multiplicity of the eigenvalue $l$ (that is, multiplicity of $l$ as a root of the characteristic polynomial of $A$), and $\text{geom}(l)$ denotes the geometric multiplicity of the eigenvalue $l$ (that is, the maximum number of linearly independent eigenvectors corresponding to $l$; equivalently, the dimension of the kernel of the matrix $A - lI$). Note that $\text{geom}(l) \leq \text{algm}(l)$.

Given a rational matrix $A$, the conditions (a)–(d) are decidable, and hence it follows that we have a sound check for existence of $2n$-LIs. Our proof of sufficiency is constructive – it will explicitly generate the $2n$-LI when the condition holds. We will prove the above result by proving it for special cases in different lemmas and finally we will put it all together.
First we consider the case when $A$ has only real eigenvalues and prove sufficiency for such $A$.

**Lemma 2.** The linear system $dx/dt = \lambda x$, where $\lambda \leq 0$, has a $2$-LI.

**Proof.** The invariant is given by $|x| \leq 1$. \hfill $\square$

An $(A, I)$-Jordan block is a matrix with a submatrix $A$ on all its diagonal positions and identity matrix $I$ on its top off-diagonal and 0 elsewhere; that is, a Jordan block is of the form

$$
\begin{pmatrix}
A & I & 0 & 0 & 0 \\
0 & A & I & 0 & 0 \\
0 & 0 & A & I & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & A \\
0 & 0 & 0 & 0 & 0 & A
\end{pmatrix}
$$

**Lemma 3.** The linear system $dx/dt = Jx$, where $J$ is a $m \times m$ $(\lambda, 1)$-Jordan block with diagonal $\lambda < 0$, has a $2m$-LI.

**Proof.** Let $x_1, \ldots, x_m$ denote the $m$ variables. We show that

$$
\max(|x_m|, |\lambda x_{m-1}|, |\lambda^2 x_{m-2}|, \ldots, |\lambda^{m-1} x_1|) \leq 1
$$

is a $2n$-LI. Let us denote the above formula by $\phi$. Consider the polyhedron defined by $\phi$, and consider the face $|\lambda^i x_{m-i}| = 1$ on this polyhedron. We want to prove that the direction of flow at points on this face is inwards; that is,

$$
\phi \land |\lambda^i x_{m-i} = 1 \Rightarrow \frac{d|\lambda^i x_{m-i}}{dt} \leq 0
$$

For points on the face, we note that

$$
\frac{d|\lambda^i x_{m-i}}{dt} = |\lambda^i|(|\lambda x_{m-i} + x_{m-i+1})
\leq \lambda(|\lambda^i x_{m-i} + |\lambda||\lambda^{i-1} x_{m-i+1}|
\leq \lambda(1) + |\lambda|(1) \leq \lambda + |\lambda| = 0
$$

The above proof shows that the flow points inwards on all faces, and hence $\phi$ is a $2m$-LI. \hfill $\square$

We next move to the case when $A$ has complex eigenvalues. First we start with a 2-d case that has one complex eigenvalue.

**Lemma 4.** The linear system $dx/dt = Ax$, where $A = [a, b; -b, a]$ and $a + |b| \leq 0$, has a $4$-LI.

**Proof.** It is easily verified that the polyhedron $\max(|x_1|, |x_2|) \leq 1$ is a $4$-LI for this system. \hfill $\square$
In the next lemma, we consider complex eigenvalues in the diagonal of a Jordan block.

**Lemma 5.** The linear system \( \frac{dx}{dt} = Jx \), where \( J \) is a \( 2m \times 2m \) \((A, I)\)-Jordan block with \( A = [a, b; -b, a] \) such that \( a + |b| < 0 \), and \( I = [1, 0; 0, 1] \), has a \( 4m \)-LI.

**Proof.** We can assume without loss of generality that \( a < 0 \) and \( b \geq 0 \). Define \( \lambda = -a - b \). By assumption, \( \lambda > 0 \). Let \( x_1, y_1, \ldots, x_m, y_m \) denote the \( 2m \) variables. Consider the polyhedron defined by the following constraints:

\[
|\lambda x_{m-1}| \leq 1 \quad |\lambda y_{m-1}| \leq 1 \\
|\lambda^m x_1| \leq 1 \\
|\lambda^m y_1| \leq 1
\]

Let \( \phi \) denote the conjunction of the above formulas. We need to prove that flows point inwards on the faces of the polyhedron \( \phi \). Consider the face \( \lambda^i y_{m-i} = 1 \) for an arbitrary \( i \). We need to show that

\[
\phi \land \lambda^i y_{m-i} = 1 \Rightarrow \frac{d(\lambda^i y_{m-i})}{dt} \leq 0
\]

Assuming \( \phi \) and assuming \( \lambda^i y_{m-i} = 1 \), we note that

\[
\frac{d(\lambda^i y_{m-i})}{dt} = \lambda^i (-bx_{m-i} + ay_{m-i} + y_{m-i+1}) \\
\leq -b\lambda^i x_{m-i} + a + \lambda^i y_{m-i+1} \\
\leq -b\lambda^i x_{m-i} + a + \lambda \\
\leq b + a + \lambda \leq 0
\]

It can be similarly verified that the flow points inwards for all other faces, and thus the formula \( \phi \) is an invariant.

Now, we can put all the above lemmas together to get a proof of Theorem 1.

**Proof (Theorem 1).** Transform \( A \) into Jordan normal form \( J \) and say \( A = UJU^{-1} \). If we define \( y = U^{-1}x \), then \( y = U^{-1}AX = U^{-1}AUy = Jy \). Let \( J_1, \ldots, J_k \) be the \( k \) different Jordan blocks in \( J \). For each Jordan block \( J_i \), we generate an invariant \( \phi_i \), and then we get an invariant for \( J \) by putting together the \( \phi_i \)’s. Specifically,

1. if \( J_i \) is a \( 1 \times 1 \) matrix, then we use Lemma 2 to construct an invariant \( \phi_i \) for \( J_i \);  
2. if \( J_i \) is a \( m \times m \) \((\lambda, 1)\)-Jordan block \( (m > 1) \), then we use Lemma 3 to construct an invariant \( \phi_i \) for \( J_i \);  
3. if \( J_i \) is a \( 2 \times 2 \) matrix and corresponds to complex eigenvalue \( a + ib \), then we use Lemma 4 to construct an invariant \( \phi_i \) for \( J_i \); and finally,  
4. if \( J_i \) is a \( 2m \times 2m \) \((A, I)\)-Jordan block corresponding to a complex eigenvalue \( a + ib \) with \( m > 1 \), then we use Lemma 5 to construct an invariant \( \phi_i \) for \( J_i \).
Consider $\phi := \phi_1 \land \phi_2 \land \cdots \land \phi_k$. It is easy to see that $\phi$ is a 2n-LI for $\dot{y} = Jy$.

Now, we can get an invariant for our original system by transforming $\phi$ back to original coordinates, namely, we transform $\phi(y)$ to $\phi(U^{-1}x)$ to get the required invariant. □

The proof of Theorem 1 shows how one could synthesize linear invariants: every eigenvalue that satisfies one of the conditions in Theorem 1 gives rise to a conjunct in the invariant. We do not need every eigenvalue to satisfy one of those conditions.

**Theorem 2 (Sufficient condition for existence of 2s-LI).** The linear system $dx/dt = Ax$, where $A$ is a $n \times n$ matrix, has a 2s-LI for a value $s$ obtained by adding $\text{num}(\lambda)$ for each distinct eigenvalue $\lambda$ of $A$, where:

(a) $\text{num}(\lambda) = \text{algm}(\lambda)$ if $\lambda$ is real and $\lambda < 0$
(b) $\text{num}(\lambda) = \text{geom}(\lambda)$ if $\lambda$ is real and $\lambda = 0$
(c) $\text{num}(\lambda) = \text{algm}(\lambda)$ if $\lambda := a + ib$ is complex and $a + |b| < 0$
(d) $\text{num}(\lambda) = \text{geom}(\lambda)$ if $\lambda := a + ib$ is complex and $a + |b| = 0$
(e) $\text{num}(\lambda) = 0$ otherwise

The proof of Theorem 1 also serves as a proof of Theorem 2.

5 Generalized Sufficient Condition and a Necessary Condition

The sufficient condition in Theorem 1 for existence of 2n-LI intuitively appears to be necessary too. However, it is not.

**Example 3.** Consider the linear system whose $A$ matrix is

$$A = \begin{pmatrix}
-0.5, & \sqrt{3}/2, & 0 \\
-\sqrt{3}/2, & -0.5, & 0 \\
0, & 0, & -2
\end{pmatrix} \quad (3)$$

The matrix $A$ has one real eigenvalue, $-2$, and a pair of complex conjugate eigenvalues, $-0.5 \pm \sqrt{3}/2i$. The complex eigenvalues do not satisfy the condition $a + |b| \leq 0$ because $-0.5 + \sqrt{3}/2 > 0$. Now, apply a “change of coordinates” transformation by defining new variables $y_1, y_2, y_3$ in terms of the old variables $x_1, x_2, x_3$ as follows:

$$y_1 = -\sqrt{3}x_1/6 - x_2/2 - \sqrt{3}x_3/3,$$
$$y_2 = \sqrt{3}x_1/6 - x_2/2 + \sqrt{3}x_3/3,$$
$$y_3 = \sqrt{3}x_1/3 - \sqrt{3}x_3/3$$

If we transform the dynamical system into these new coordinates, the new $A$ matrix will be given by $FAF^{-1}$, which turns out to be

$$A' = \begin{pmatrix}
-1, & 1, & 0 \\
0, & -1, & 1 \\
-1, & 0, & -1
\end{pmatrix} \quad (4)$$
This new system has the invariant \( \max(|y_1|, |y_2|, |y_3|) \leq 1 \). Thus, we can get a 2n-LI on the \( x_i \)'s by replacing each \( y_i \) in this invariant by its definition.

A square matrix \( A \) is diagonally-dominant if, for each row, the (absolute value of the) diagonal is more than the sum of the (absolute values of the) other elements in the row; that is, for all \( i \), \( |a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \).

**Definition 6 (B-matrix).** A diagonally-dominant matrix \( A \) is a B-matrix if its diagonal elements are all not positive; that is, for all \( i \), \( a_{ii} \leq 0 \).

The matrix \( A' \) in Example 3 is a B-matrix.

Proposition 3 presented a necessary and sufficient condition for a linear system to have a polyhedral invariant. It said that \( \dot{x} = Ax \) has a polyhedral invariant iff there is an \((s \times n)\)-matrix \( F \) s.t. the \( s \) universally quantified formulas in Equation 2 in Proposition 3 are valid. We can turn these validity checks of for-all formulas into checks for exists formulas using Farkas Lemma to get the following result.

**Proposition 4 (Necessary and Sufficient Check).** A linear system \( \dot{x} = Ax \) has a polyhedral invariant \( ||Fx||_\infty \leq 1 \), where \( F \) is an \((s \times n)\)-matrix iff there exists an \((s \times s)\) B-matrix \( X \) such that \( FA = XF \).

**Proof.** (Sketch) Informally, the \( i \)-th row of \( FA \) represents the Lie derivative of \( F_i \cdot x \), and since Equation 2 requires that the derivative be non-positive (or non-negative), Farkas Lemma says that we should be able to write it as a linear combination of the rows of \( F \) in a certain way. The rows of \( X \) define this linear combination. The fact that \( X \) is B-matrix ensures that the Lie derivative will be non-positive (or non-negative) at appropriate points.

One way to interpret Proposition 4 is that \( F \) define a “change of variables”– it defines \( s \) new variables – each as a linear combination of the original \( n \) variables. On these new variables, call them \( y_1, \ldots, y_s \), we want the \( A \)-matrix of system dynamics to be given by a B-matrix. In that case, we get \( \max(|y_1|, \ldots, |y_s|) \leq 1 \) as an invariant. Invariant of this form have been called box invariants [1,2]. Thus, polyhedral invariants are just box invariants after a suitable transformation.

In the special case of 2n-LI (that is, \( s = n \)), Theorem 1 attempted to characterize linear systems that have 2n-LIs based on the eigenstructure of the \( A \) matrix. However, it only provided a sufficient condition. In this case, since \( s = n \), if \( F \) is full rank, then \( FA = XF \) is the equivalent to \( X = FAF^{-1} \), which implies that \( X \) is similar to \( A \). Finding a necessary and sufficient condition would be equivalent to the problem of characterizing when a matrix is similar to a B-matrix completely in terms of its eigenstructure. There is no known solution to this problem.

We can generalize Theorem 2 to also synthesize invariants when a complex eigenvalue \( a + bi \) violates \( a + |b| < 0 \) – it will involve adding more faces by increasing \( s \).
Let $A$ and $B$ be two adjacent vertices on a regular polygon with $2s$ sides and center on origin $O$. Let $C$ be the midpoint of $AB$, let $D$ be an arbitrary point on $CB$. Let $\text{len}(AB) = r_0$. If $r$ denotes $\text{len}(OD)$, and $\alpha$ is $\angle COD$, $r$ as a function of $\alpha$ is given by $r(\alpha) = r_0/\cos(\alpha)$. One way to ensure that vector fields point inwards on the boundary is by comparing $dr/d\alpha$. On the trajectories, $dr/d\alpha = (dr/dt)/(dt/d\alpha) = ar/b = ar_0/(b \cos(\alpha))$, and on the boundary, $dr/d\alpha = r_0 \sin(\alpha)/\cos^2(\alpha)$. Vector field points inwards iff $|ar_0/(b \cos(\alpha))| \geq |r_0 \sin(\alpha)/\cos^2(\alpha)|$, which simplifies to $|a/b| \geq \tan(\alpha)$ and maximum value of $\alpha$ is $360/4s = 90/s$, and we get $|a/b| \geq \tan(90/s)$.

**Theorem 3 (Generalized sufficient condition for 2s-LI).** The linear system $dx/dt = Ax$, where $A$ is a $n \times n$ matrix, has a 2s-LI for a value $s$ obtained by adding $\text{num}(\lambda)$ for each distinct eigenvalue $\lambda$ of $A$, where:

(a) $\text{num}(\lambda) = \text{algm}(\lambda)$ if $\lambda$ is real and $\lambda < 0$

(b) $\text{num}(\lambda) = \text{geom}(\lambda)$ if $\lambda$ is real and $\lambda = 0$

(c) $\text{num}(\lambda) = k \times \text{algm}(\lambda)$ if $\lambda := a + ib$ is complex, $a < 0$, and $k > \frac{90}{\tan^{-1}(|a/b|)}$

(d) $\text{num}(\lambda) = k \times \text{geom}(\lambda)$ if $\lambda := a + ib$ is complex, $a < 0$, and $k = \frac{90}{\tan^{-1}(|a/b|)}$

(e) $\text{num}(\lambda) = 0$ otherwise

**Proof.** (Sketch) The main difference with Theorem 2 is the complex eigenvalue case. In this case, we increase the number $s$ of rows in the polyhedral invariant depending on the ratio $|a/b|$. The important case to consider is the 2-dimensional case when the $A$-matrix is $A = [a,b; -b,a]$. Figure 1 illustrates this case and shows that if $k > \frac{90}{\tan^{-1}(|a/b|)}$, then any regular polygon with $2k$ sides whose center is the origin will be a 2k-LI for this 2-dimensional system.

We next present a necessary condition for existence of 2n-LI, and examples that show that the gap between the sufficient condition, given in Theorem 4, and the necessary condition, given below in Theorem 5, is hard to overcome.

**Theorem 4 (Necessary condition for existence of 2s-LI).** If the linear system $dx/dt = Ax$ has a bounded 2s-LI, then for every eigenvalue $\lambda$ of $A$, it is the case that either

(a) $\lambda$ is real and $\lambda < 0$, or

(b) $\lambda = 0$ and $\text{algm}(\lambda) = \text{geom}(\lambda)$, or

(c) $\lambda := a + ib$ is complex, $a < 0$, and $|a/b| > \tan(90/s)$,

(d) $\lambda := a + ib$ is complex, $a < 0$, $|a/b| = \tan(90/s)$, and $\text{algm}(\lambda) = \text{geom}(\lambda)$.

**Proof.** (Sketch) Since the 2s-LI is bounded, the 2s-LI also corresponds to a weak Lyapunov function. Hence, real eigenvalues have to be non-positive. Moreover, if an eigenvalue is zero, but its $\text{algm}(\lambda)$ is different from $\text{geom}(\lambda)$, then we can not get a weak Lyapunov function; for example, consider $A = [0,1;0,0]$, which has trajectories escaping any bounded region, and hence it has no weak Lyapunov function. For the complex case, note that a 2s-LI can intersect a 2-dimensional
plane in a polygon that can have at most 2s faces, and that gives rise to condition (c) and (d).

We now present some examples that show there might not be any simple condition that is both necessary and sufficient for existence of 2s-LI in terms of eigenvalues.

**Example 4.** Consider the 2-dimensional linear system from Example 1. This 2d system does not have a 4-LI, but it has a 6-LI. These results are predicted by our theorems: specifically, the sufficient condition from Theorem 3 implies that this 2d system will have a 6-LI, whereas the necessary condition from Theorem 4 implies that this 2d system will *not* have a 4-LI. So, our results are strong enough to make perfect predictions for this case.

Next, consider the linear system from Example 3. It extends the example from Example 1 to 3 dimensions by adding a 3rd dimension whose dynamics is given by \( \dot{x}_3 = -2x_3 \). This 3d system continues to have a 6-LI. Note that \( 90/\tan^{-1}(a/b) = 90/\tan^{-1}(1/\sqrt{3}) = 90/30 = 3 \). If we use the sufficient condition from Theorem 3 on the 3d system, we conclude that the 3d system would have a 8-LI, but it does not help us infer existence of a 6-LI. The necessary condition from Theorem 4 says that the 3d system can not have a 4-LI. Neither theorem says anything conclusive about existence or non-existence of a 6-LI.

**Example 5.** Consider the linear system whose \( A \) matrix is given by

\[
A = \begin{pmatrix}
-0.5 & b & 0 \\
-b & -0.5 & 0 \\
0 & 0 & -\lambda
\end{pmatrix}
\]  

(5)

where \( b \) and \( \lambda \) are two parameters. When \( b = \sqrt{3}/2 \), then setting \( \lambda = 2 \) gives us the 3d linear system from Example 3. In fact, when \( b = \sqrt{3}/2 \), then \( \lambda = 2 \) is the only choice for \( \lambda \) that allows the system to have a 6-LI. As \( b \) decreases toward 1/2, there are more choices of \( \lambda \) that imply existence of 6-LI, but these choices remain in a bounded range of values (always upper-bounded by 2). This simply shows that there is no simple property of individual eigenvalues that characterizes existence of 2n-LI.

**Example 6.** Consider the 6d linear system with \( A \) matrix whose element are given by

\[
a_{ij} = \begin{cases} 
1 & \text{if } i = 1, j = 2 \\
-1 & \text{elseif } i = j \lor j = i + 1 \lor i = 6, j = 1 \\
0 & \text{otherwise}
\end{cases}
\]  

(6)

This \( A \) matrix has eigenvalues \(-0.134 \pm 0.5i, -1.866 \pm 0.5i \) and \(-1 \pm i \). The value \( \tan^{-1}(|a/b|) \) for the eigenvalue \(-0.134 + 0.5i \) is roughly 15 degrees, so Theorem 4 tells us that this system can not have a 2s-LI where \( s < 90/15 = 6 \). It clearly has a 12-LI, one such invariant is given by \( \max(\|x_1\|, \ldots, \|x_6\|) \leq 1 \). However, our sufficient condition in Theorem 3 only guarantees existence of 2s-LI where \( s \) is at least \( 6 + 2 + 2 = 10 \).
If we replace 6 by 8 in the above definition of $a_{ij}$, we get an $A$ matrix for a 8d linear system. This matrix has an eigenvalue $-0.076 + 0.383i$, and $\tan^{-1}(0.076/0.383)$ is roughly 11.25 degrees, and thus Theorem 4 implies we need $s$ to be at least $90/11.25 = 8$ to get an 2s-LI. Indeed there is an 16-LI.  

6 Conclusion

We presented necessary and sufficient conditions for existence of linear invariants for linear dynamical systems. The proof of sufficiency is constructive and yields a procedure for synthesizing linear invariants that only needs computation of the eigenvalues and eigenvectors of the $A$ matrix. We also presented examples that show the conditions are tight when applied to specific linear systems.

Our first sufficient condition for existence of 2n-LI, which is given in Theorem 1, can be derived from the sufficient condition for existence of infinity-norm Lyapunov functions presented by Bitsoris and Kiendl [4,20]. However, since we are interested in invariants (and not contractive invariants or Lyapunov functions), we need to distinguish the cases when algebraic and geometric multiplicities of an eigenvalue are equal and when not. Our generalized sufficient condition in Theorem 3 is novel and has not been stated before. The same is also true for the necessary condition in Theorem 4 and the examples showing the gap between the necessary and sufficient conditions.

Apart from improving our understanding of linear systems and infinity-norm (weak) Lyapunov functions, the results can also be used to build verification tools for piecewise-linear and hybrid systems that just rely on reasoning over linear arithmetic. The question of characterizing matrices similar to some $B$-matrix (Definition 6) based on its spectral properties remains open for future work.

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