Individual ergodic theorem for intuitionistic fuzzy observables using IF-probability

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Abstract: The aim of this paper is to formulate the individual ergodic theorem for intuitionistic fuzzy observables using \(P\)-almost everywhere convergence, where \(P\) is an intuitionistic fuzzy probability. Since the intuitionistic fuzzy probability can be decomposed to two intuitionistic fuzzy states, we can use the results holding for intuitionistic fuzzy states.

Keywords: Intuitionistic fuzzy event, Intuitionistic fuzzy observable, Intuitionistic fuzzy probability, Product, \(P\)-almost everywhere convergence, \(P\)-preserving transformation, Individual ergodic theorem.

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1 Introduction

In [1, 2], K. T. Atanassov introduced the notion of intuitionistic fuzzy sets. Later P. Grzegorzewski and E. Mrówka defined the probability on the family of intuitionistic fuzzy events

\[ \mathcal{N} = \{ (\mu_A, \nu_A) ; \mu_A, \nu_A \text{ are } \mathcal{S} \text{-measurable and } \mu_A + \nu_A \leq 1 \Omega \} \]

as a mapping \(\mathcal{P}\) from the family \(\mathcal{N}\) to the set of all compact intervals in \(R\) by the formula

\[ \mathcal{P}(\mu_A, \nu_A) = \left[ \int_{\Omega} \mu_A dP, 1 - \int_{\Omega} \nu_A dP \right], \]

where \((\Omega, \mathcal{S}, P)\) is the probability space, see [7]. This intuitionistic fuzzy probability was axiomatically characterized by B. Riečan (see [10]).
In this paper, we formulate the Individual ergodic theorem for intuitionistic fuzzy observables, using \( \mathcal{P} \)-almost everywhere convergence, where \( \mathcal{P} \) is an intuitionistic fuzzy probability. Recall that the formulation of the individual ergodic theorem for intuitionistic fuzzy events with product first appeared in the paper [3]. There we used a separating intuitionistic fuzzy probability. Since the intuitionistic fuzzy probability \( \mathcal{P} \) can be decomposed to two intuitionistic fuzzy states, we can use the results holding for intuitionistic fuzzy states, which were proved in [6].

Remark that in a whole text we use a notation IF as an abbreviation for intuitionistic fuzzy.

2 IF-events, IF-states, IF-observables and IF-mean value

In this section we explain the basic notions from IF-probability theory, see [1, 2, 13, 14, 15].

**Definition 2.1.** Let \( \Omega \) be a nonempty set. An IF-set \( A \) on \( \Omega \) is a pair \((\mu_A, \nu_A): \Omega \rightarrow [0, 1]\) such that \( \mu_A + \nu_A \leq 1_{\Omega} \).

**Definition 2.2.** Start with a measurable space \((\Omega, S)\). Hence \( S \) is a \( \sigma \)-algebra of subsets of \( \Omega \). An IF-event is called an IF-set \( A = (\mu_A, \nu_A) \) such that \( \mu_A, \nu_A : \Omega \rightarrow [0, 1] \) are \( S \)-measurable.

The family of all IF-events on \((\Omega, S)\) will be denoted by \( \mathcal{F} \), \( \mu_A : \Omega \rightarrow [0, 1] \) will be called the membership function, \( \nu_A : \Omega \rightarrow [0, 1] \) will be called the non-membership function.

If \( A = (\mu_A, \nu_A) \in \mathcal{F} \), \( B = (\mu_B, \nu_B) \in \mathcal{F} \), then we define the Łukasiewicz binary operations \( \oplus, \odot \) on \( \mathcal{F} \) by

\[
A \oplus B = ((\mu_A + \mu_B) \land 1_{\Omega}, (\nu_A + \nu_B - 1) \lor 0_{\Omega})),
\]

\[
A \odot B = ((\mu_A + \mu_B - 1) \lor 0_{\Omega}, (\nu_A + \nu_B) \land 1_{\Omega})).
\]

and the partial ordering is given by

\[
A \leq B \iff \mu_A \leq \mu_B, \nu_A \geq \nu_B.
\]

In the paper, we use max–min connectives defined by

\[
A \lor B = (\mu_A \lor \mu_B, \nu_A \land \nu_B),
\]

\[
A \land B = (\mu_A \land \mu_B, \nu_A \lor \nu_B)
\]

and the de Morgan rules

\[
(a \lor b)^* = a^* \land b^*,
\]

\[
(a \land b)^* = a^* \lor b^*,
\]

where \( a^* = 1 - a \).

**Example 2.3.** A fuzzy set \( f: \Omega \rightarrow [0, 1] \) can be regarded as an IF-set if we put

\[
A = (f, 1_{\Omega} - f).
\]
If \( f = \chi_A \), then the corresponding IF-set has the form
\[
A = (\chi_A, 1 - \chi_A) = (\chi_A, \chi_A').
\]
In this case \( A \oplus B \) corresponds to the union of sets, \( A \odot B \) to the intersection of sets and \( \leq \) to the set inclusion.

Consider a probability space \((\Omega, \mathcal{S}, P)\). Then in [7] the IF-probability \( \mathcal{P}(A) \) of an IF-event \( A = (\mu_A, \nu_A) \in \mathcal{F} \) has been defined as a compact interval by the equality
\[
\mathcal{P}(A) = \left[ \int_\Omega \mu_A \, dP, 1 - \int_\Omega \nu_A \, dP \right].
\]
Let \( J \) be the family of all compact intervals. Then the mapping \( \mathcal{P} : \mathcal{F} \to J \) can be defined axiomatically similarly as in [10].

**Definition 2.4.** Let \( \mathcal{F} \) be the family of all IF-events in \( \Omega \). A mapping \( \mathcal{P} : \mathcal{F} \to J \) is called an IF-probability if the following conditions hold:

(i) \( \mathcal{P}((1_\Omega, 0_\Omega)) = [1, 1], \mathcal{P}((0_\Omega, 1_\Omega)) = [0, 0] \);

(ii) If \( A \odot B = (0_\Omega, 1_\Omega) \), then \( \mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B) \);

(iii) If \( A_n \nrightarrow A \), then \( \mathcal{P}(A_n) \nrightarrow \mathcal{P}(A) \).

(Recall that \([\alpha_n, \beta_n] \nrightarrow [\alpha, \beta]\) means that \( \alpha_n \nrightarrow \alpha, \beta_n \nrightarrow \beta \), but \( A_n = (\mu_{A_n}, \nu_{A_n}) \nrightarrow A = (\mu_A, \nu_A) \).)

IF-probability \( \mathcal{P} \) is called separating, if
\[
\mathcal{P}((\mu_A, \nu_A)) = [\mathcal{P}^b(\mu_A), 1 - \mathcal{P}^d(\nu_A)],
\]
where the functions \( \mathcal{P}^b, \mathcal{P}^d : \mathcal{T} \to [0, 1] \) are probabilities.

Of course, each \( \mathcal{P}(A) \) is an interval, denote it by \( \mathcal{P}(A) = [\mathcal{P}^b(A), \mathcal{P}^d(A)] \). By this way we obtain two functions
\[
\mathcal{P}^b : \mathcal{F} \to [0, 1], \mathcal{P}^d : \mathcal{F} \to [0, 1]
\]
and some properties of \( \mathcal{P} \) can be characterized by some properties of \( \mathcal{P}^b, \mathcal{P}^d \), see [11].

**Theorem 2.5.** Let \( \mathcal{P} : \mathcal{F} \to J \) and \( \mathcal{P}(A) = [\mathcal{P}^b(A), \mathcal{P}^d(A)] \) for each \( A \in \mathcal{F} \). Then \( \mathcal{P} \) is an IF-probability if and only if \( \mathcal{P}^b \) and \( \mathcal{P}^d \) are IF-states.

**Proof.** See [11, Theorem 2.3] \( \square \)

Recall that by an intuitionistic fuzzy state (IF-state) \( m \) we understand each mapping \( m : \mathcal{F} \to [0, 1] \) which satisfies the following conditions (see [12]):

(i) \( m((1_\Omega, 0_\Omega)) = 1, m((0_\Omega, 1_\Omega)) = 0 \);

(ii) if \( A \odot B = (0_\Omega, 1_\Omega) \) and \( A, B \in \mathcal{F} \), then \( m(A \oplus B) = m(A) + m(B) \);

(iii) if \( A_n \nrightarrow A \) (i.e. \( \mu_{A_n} \nrightarrow \mu_A, \nu_{A_n} \searrow \nu_A \)), then \( m(A_n) \nrightarrow m(A) \).
Now we introduce the notion of an observable. Let $J$ be the family of all intervals in $\mathbb{R}$ of the form $[a,b) = \{ x \in \mathbb{R} : a \leq x < b \}$.

Then the $\sigma$-algebra $\sigma(J)$ is denoted by $\mathcal{B}(\mathbb{R})$ and it is called the $\sigma$-algebra of Borel sets, its elements are called Borel sets (see [16]).

**Definition 2.6.** By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ satisfying the following conditions:

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) If $A \cap B = \emptyset$ and $A, B \in \mathcal{B}(\mathbb{R})$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) If $A_n \nrightarrow A$ and $A_n, A \in \mathcal{B}(\mathbb{R})$, $n \in \mathbb{N}$, then $x(A_n) \nrightarrow x(A)$.

Similarly, we can define the notion of $n$-dimensional IF-observable.

**Definition 2.7.** By an $n$-dimensional IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(\mathbb{R}^n) \to \mathcal{F}$ satisfying the following conditions:

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) If $A \cap B = \emptyset$ and $A, B \in \mathcal{B}(\mathbb{R}^n)$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) If $A_n \nrightarrow A$ and $A_n, A \in \mathcal{B}(\mathbb{R}^n)$, $n \in \mathbb{N}$, then $x(A_n) \nrightarrow x(A)$.

Similarly, as in the classical case the following theorem can be proved ([9, 15]).

**Theorem 2.8.** Let $x : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \to [0,1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(\mathbb{R}) \to [0,1]$ by the formula $\mathbf{m}_x(C) = \mathbf{m}(x(C))$.

Then $\mathbf{m}_x : \mathcal{B}(\mathbb{R}) \to [0,1]$ is a probability measure.

Since $\mathbf{m}_x : \mathcal{B}(\mathbb{R}) \to [0,1]$ plays now an analogous role as $P_\xi : \mathcal{B}(\mathbb{R}) \to [0,1]$, we can define the **IF-expected value** $\mathbf{E}(x)$ by the same formula (see [9]).

**Definition 2.9.** We say that an IF-observable $x$ is an integrable IF-observable if the integral $\int_{\mathbb{R}} t \, d\mathbf{m}_x(t)$ exists. In this case, we define the IF-expected value $\mathbf{E}(x)$ by the integral $\int_{\mathbb{R}} t \, d\mathbf{m}_x(t)$. If the integral $\int_{\mathbb{R}} t^2 \, d\mathbf{m}_x(t)$ exists, then we define the **IF-dispersion** $D^2(x)$ by the formula $D^2(x) = \int_{\mathbb{R}} t^2 \, d\mathbf{m}_x(t) - (\mathbf{E}(x))^2 = \int_{\mathbb{R}} (t - \mathbf{E}(x))^2 \, d\mathbf{m}_x(t)$. 

40
3 Product operation, joint IF-observable and function of several IF-observables

In [8] we introduced the notion of product operation on the family of IF-events \( \mathcal{F} \), and showed an example of this operation.

**Definition 3.1.** We say that a binary operation \( \cdot \) on \( \mathcal{F} \) is product if it satisfies the following conditions:

(i) \( (1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2) \) for each \( (a_1, a_2) \in \mathcal{F} \);

(ii) The operation \( \cdot \) is commutative and associative;

(iii) If \( (a_1, a_2) \circ (b_1, b_2) = (0_\Omega, 1_\Omega) \) and \( (a_1, a_2), (b_1, b_2) \in \mathcal{F} \), then

\[
(c_1, c_2) \cdot ((a_1, a_2) \oplus (b_1, b_2)) = (c_1, c_2) \oplus ((c_1, c_2) \cdot (b_1, b_2))
\]

and

\[
((c_1, c_2) \cdot (a_1, a_2)) \circ ((c_1, c_2) \cdot (b_1, b_2)) = (0_\Omega, 1_\Omega)
\]

for each \( (c_1, c_2) \in \mathcal{F} \);

(iv) If \( (a_{1n}, a_{2n}) \nearrow (0_\Omega, 1_\Omega), (b_{1n}, b_{2n}) \nearrow (0_\Omega, 1_\Omega) \) and \( (a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F} \), then

\[
(a_{1n} \cdot b_{1n}) \cdot b_{2n} \nearrow (0_\Omega, 1_\Omega).
\]

The following theorem defines the product operation for IF-events.

**Theorem 3.2.** The operation \( \cdot \) defined by

\[
(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)
\]

for each \( (x_1, y_1), (x_2, y_2) \in \mathcal{F} \) is product operation on \( \mathcal{F} \).

**Proof.** See [8, Theorem 1].

In [13] B. Riečan defined the notion of a joint IF-observable and he proved its existence.

**Definition 3.3.** Let \( x, y : \mathcal{B}(R) \rightarrow \mathcal{F} \) be two IF-observables. The joint IF-observable of the IF-observables \( x, y \) is a mapping \( h : \mathcal{B}(R^2) \rightarrow \mathcal{F} \) satisfying the following conditions:

(i) \( h(R^2) = (1_\Omega, 0_\Omega), h(\emptyset) = (0_\Omega, 1_\Omega) \);

(ii) If \( A, B \in \mathcal{B}(R^2) \) and \( A \cap B = \emptyset \), then \( h(A \cup B) = h(A) \oplus h(B) \)

and \( h(A) \circ h(B) = (0_\Omega, 1_\Omega) \);

(iii) If \( A, A_1, \ldots \in \mathcal{B}(R^2) \) and \( A_n \nrightarrow A \), then \( h(A_n) \nrightarrow h(A) \);

(iv) \( h(C \times D) = x(C) \cdot y(D) \) for each \( C, D \in \mathcal{B}(R) \).
**Theorem 3.4.** For each two IF-observables $x, y : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ there exists their joint IF-observable.

**Proof.** See [13, Theorem 3.3].

**Remark 3.5.** The joint IF-observable of the IF-observables $x, y$ from Definition 3.3 is a two-dimensional IF-observable.

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. About this says the following definition.

**Definition 3.6.** Let $x_1, \ldots, x_n : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ be IF-observables, $h_n$ be their joint IF-observable and $g_n : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function. Then, we define the IF-observable $g_n(x_1, \ldots, x_n) : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ by the formula

$$g_n(x_1, \ldots, x_n)(A) = h_n(g_n^{-1}(A)).$$

for each $A \in \mathcal{B}(\mathbb{R})$.

4 Lower and upper limits, $\mathcal{P}$-almost everywhere convergence

In [4] we defined the notions of lower and upper limits for a sequence of IF-observables.

**Definition 4.1.** We shall say that a sequence $(x_n)_n$ of IF-observables has $\limsup_{n \to \infty}$ if there exists an IF-observable $x : \mathcal{B}(\mathbb{R}) \to \mathcal{F}$ such that

$$x((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n((-\infty, t - \frac{1}{p}))$$

for every $t \in \mathbb{R}$. We write $\overline{x} = \limsup_{n \to \infty} x_n$.

Note that if another IF-observable $y$ satisfies the above condition, then $m \circ y = m \circ \overline{x}$.

**Definition 4.2.** A sequence $(x_n)_n$ of IF-observables has $\liminf_{n \to \infty}$ if there exists an IF-observable $\underline{x}$ such that

$$\underline{x}((-\infty, t)) = \bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n((-\infty, t - \frac{1}{p}))$$

for all $t \in \mathbb{R}$. Notation: $\underline{x} = \liminf_{n \to \infty} x_n$.

In paper [5] we showed the connection between two kinds of $\mathcal{P}$-almost everywhere convergence.

**Definition 4.3.** Let $(x_n)_n$ be a sequence of IF-observables on an IF-space $(\mathcal{F}, \mathcal{P})$. We say that $(x_n)_n$ converges $\mathcal{P}$-almost everywhere to 0, if

$$\mathcal{P}\left(\bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left((-1/p, 1/p)\right)\right) = \lim_{p \to \infty} \lim_{k \to \infty} \lim_{i \to \infty} \mathcal{P}\left(\bigwedge_{n=k}^{k+i} x_n\left((-1/p, 1/p)\right)\right) = [1, 1] = 1.$$
Remark 4.4. The defining formula is equivalent to the following equality
\[ \mathcal{P} \left( \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left( R \setminus \left( - \frac{1}{p}, \frac{1}{p} \right) \right) \right) = [0,0] = 0. \]

Theorem 4.5. A sequence \( (x_n)_n \) of IF-observables converges \( \mathcal{P} \)-almost everywhere to 0 if and only if it converges \( \mathcal{P}^\flat \)-almost everywhere and \( \mathcal{P}^\sharp \)-almost everywhere to 0.

Proof. See [5, Theorem 5].

Proposition 4.1. A sequence \( (x_n)_n \) of IF-observables converges \( \mathcal{P} \)-almost everywhere to 0 if and only if
\[ \mathcal{P} \left( \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left( \left( - \infty, t - \frac{1}{p} \right) \right) \right) = \mathcal{P} \left( \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left( \left( - \infty, t - \frac{1}{p} \right) \right) \right) = \mathcal{P} \left( 0_{\mathcal{F}}((\infty, t)) \right), \]
for every \( t \in \mathbb{R} \).

Proof. See [5, Proposition 2].

In accordance to Proposition 4.1, we can extend the notion of \( \mathcal{P} \)-almost everywhere convergence in the following way.

Definition 4.6. A sequence \( (x_n)_n \) of IF-observables converges \( \mathcal{P} \)-almost everywhere to an IF-observable \( x \), if
\[ \mathcal{P} \left( \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left( \left( - \infty, t - \frac{1}{p} \right) \right) \right) = \mathcal{P} \left( \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left( \left( - \infty, t - \frac{1}{p} \right) \right) \right) = \mathcal{P} \left( x((\infty, t)) \right), \]
for every \( t \in \mathbb{R} \).

Sometimes we need to work with a sequence of IF-observables induced by a Borel measurable function.

Recall, that the corresponding probability spaces are \( (\mathbb{R}^N, \sigma(\mathcal{C}), \mathcal{P}^\flat) \) and \( (\mathbb{R}^N, \sigma(\mathcal{C}), \mathcal{P}^\sharp) \), where \( \mathcal{C} \) is the family of all sets of the form
\[ \{ (t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n \}, \]
and \( \mathcal{P}^\flat, \mathcal{P}^\sharp \) are the probability measures determined by the equalities
\[ \mathcal{P}^\flat \left( \{ (t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n \} \right) = \mathcal{P}^\flat \left( x_1(A_1) \cdot \ldots \cdot x_n(A_n) \right), \]
\[ \mathcal{P}^\sharp \left( \{ (t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n \} \right) = \mathcal{P}^\sharp \left( x_1(A_1) \cdot \ldots \cdot x_n(A_n) \right). \]
The corresponding projections \( \xi_n : \mathbb{R}^N \rightarrow \mathbb{R} \) are defined by the equality
\[ \xi_n \left( (t_i)_{i=1}^{\infty} \right) = t_n. \]
Theorem 4.7. Let \((x_n)_n\) be a sequence of IF-observables, \((\xi_n)_n\) be the sequence of corresponding projections, \((g_n)_n\) be a sequence of Borel measurable functions \(g_n : \mathbb{R}^n \to \mathbb{R}\). If the sequence \((g_n(\xi_1, \ldots, \xi_n))_n\) converges \(P^\sharp\)-almost everywhere and \(P^\flat\)-almost everywhere, then the sequence \((g_n(x_1, \ldots, x_n))_n\) converges \(P\)-almost everywhere and

\[
\mathcal{P}\left(\limsup_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right) = \mathcal{P}\left(\liminf_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right)
\]

for each \(t \in \mathbb{R}\). Moreover

\[
\mathcal{P}\left(\limsup_{n \to \infty} g_n(x_1, \ldots, x_n)((-\infty, t))\right) = \left[P^\flat(E), P^\sharp(E)\right]
\]

for each \(t \in \mathbb{R}\), where \(E = \{u \in \mathbb{R}^N : \limsup_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\}\).

Proof. See [5, Theorem 6]. \(\square\)

5 Individual Ergodic Theorem

In paper [5] we proved the modification of the classical Individual Ergodic Theorem using \(m\)-almost everywhere convergence. Since the intuitionistic fuzzy probability \(\mathcal{P}\) can be decomposed to two intuitionistic fuzzy states \(m\) (see [11, 14]), then we try to formulate the modification of the classical Individual Ergodic Theorem using \(\mathcal{P}\)-almost everywhere convergence.

Now, we recall the modification of the Individual Ergodic Theorem for the IF-state (see [6]).

Theorem 5.1. (Individual Ergodic Theorem) Let \((\mathcal{F}, \cdot)\) be a family of IF-events with product, and \(m\) be an IF-state. Let \(x\) be an integrable IF-observable and \(\tau\) be an \(m\)-preserving transformation. Then there exists an integrable IF-observable \(x^*\) such that

(i) \(E(x) = E(x^*)\),

(ii) \(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^*\) \(m\)-almost everywhere.

Proof. See [6, Theorem 6.3]. \(\square\)

We defined the IF-mean value of an IF-observable and \(\mathcal{P}\)-almost everywhere convergence in the previous sections. Now we must define a transformation preserving an intuitionistic probability \(\mathcal{P}\).

Definition 5.2. Let \((\mathcal{F}, \cdot)\) be a family of IF-events with product, \(\mathcal{P}\) be an IF-probability. Then, a mapping \(\tau : \mathcal{F} \to \mathcal{F}\) is said to be a \(\mathcal{P}\)-preserving transformation if the following conditions are satisfied:

(i) \(\tau((1_{\Omega}, 0_{\Omega})) = (1_{\Omega}, 0_{\Omega})\);

(ii) If \(A, B \in \mathcal{F}\) and \(A \odot B = (0_{\Omega}, 1_{\Omega})\), then \(\tau(A) \odot \tau(B) = (0_{\Omega}, 1_{\Omega})\) and \(\tau(A \oplus B) = \tau(A) \oplus \tau(B)\);
(iii) If $A_n \not\supset A$, $A_n, A \in \mathcal{F}$, $n \in N$, then $\tau(A_n) \not\supset \tau(A)$;

(iv) $\mathcal{P}(\tau(A) \cdot \tau(B)) = \mathcal{P}(A \cdot B)$ for each $A, B \in \mathcal{F}$.

Now we show the connection to the $m$-preserving transformation. Recall that by **m-preserving transformation** we understand each mapping $\tau: \mathcal{F} \to \mathcal{F}$ if the following conditions are satisfied:

(i) $\tau(((1\Omega, 0\Omega)) = (1\Omega, 0\Omega)$;

(ii) If $A, B \in \mathcal{F}$ and $A \odot B = (0\Omega, 1\Omega)$, then $\tau(A) \odot \tau(B) = (0\Omega, 1\Omega)$ and $\tau(A \oplus B) = \tau(A) \oplus \tau(B)$;

(iii) If $A_n \not\supset A$, $A_n, A \in \mathcal{F}$, $n \in N$, then $\tau(A_n) \not\supset \tau(A)$;

(iv) $m(\tau(A) \cdot \tau(B)) = m(A \cdot B)$ for each $A, B \in \mathcal{F}$.

See [6].

**Theorem 5.3.** Let $(\mathcal{F}, \cdot)$ be a family of IF-events with product, $\mathcal{P}$ be an IF-probability. The mapping $\tau: \mathcal{F} \to \mathcal{F}$ is the $\mathcal{P}$-preserving transformation if and only if the mapping $\tau$ is the $\mathcal{P}^\flat$-preserving transformation and the $\mathcal{P}^\sharp$-preserving transformation, where $\mathcal{P}^\flat, \mathcal{P}^\sharp$ are the IF-states.

**Proof.** “$\Rightarrow$” Let $\mathcal{P}$ be an IF-probability. Then by *Theorem 2.5* it can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$ such that $\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$ for each $A \in \mathcal{F}$. If the mapping $\tau: \mathcal{F} \to \mathcal{F}$ is the $\mathcal{P}$-preserving transformation, then by (iv) from *Definition 5.2* we have

$$[\mathcal{P}^\flat(A \cdot B), \mathcal{P}^\sharp(A \cdot B)] = \mathcal{P}(A \cdot B) = \mathcal{P}(\tau(A) \cdot \tau(B)) = [\mathcal{P}^\flat(\tau(A) \cdot \tau(B)), \mathcal{P}^\sharp(\tau(A) \cdot \tau(B))] .$$

Hence,

$$\mathcal{P}^\flat(\tau(A) \cdot \tau(B)) = \mathcal{P}^\flat(A \cdot B),$$

$$\mathcal{P}^\sharp(\tau(A) \cdot \tau(B)) = \mathcal{P}^\sharp(A \cdot B),$$

for each $A, B \in \mathcal{F}$. Therefore, $\tau$ is a $\mathcal{P}^\flat$-preserving transformation and a $\mathcal{P}^\sharp$-preserving transformation.

“$\Leftarrow$” The opposite direction can be proved similarly. 

**Theorem 5.4. (Individual Ergodic Theorem)** Let $(\mathcal{F}, \cdot)$ be a family of IF-events with product, $\mathcal{P}$ be an IF-probability. Let $x$ be an integrable IF-observable and $\tau$ be an $\mathcal{P}$-preserving transformation. Then there exists an integrable IF-observable $x^*$ such that

(i) $E^\flat(x) = E^\flat(x^*)$, $E^\sharp(x) = E^\sharp(x^*)$

(ii) $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^*$, $\mathcal{P}$-almost everywhere.
Proof. Let $\mathcal{P}$ be an IF-probability. By Theorem 2.5 it can be decomposed to two IF-states $\mathcal{P}^\flat, \mathcal{P}^\sharp$, such that $\mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$ for each $A \in \mathcal{F}$. Let $\tau$ be the $\mathcal{P}$-preserving transformation. Then from Theorem 5.3 we obtain that $\tau$ is the $\mathcal{P}^\flat$-preserving transformation and the $\mathcal{P}^\sharp$-preserving transformation, where $\mathcal{P}^\flat, \mathcal{P}^\sharp$ are the IF-states. Hence by Theorem 5.1 there exists an integrable IF-observable $x^*$ such that

(i) $E^\flat(x) = E^\flat(x^*), \quad E^\sharp(x) = E^\sharp(x^*)$

(ii) $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^*, \quad \mathcal{P}^\flat$-almost everywhere and $\mathcal{P}^\sharp$-almost everywhere.

Finally by Theorem 5.3 we obtain that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\tau^i \circ x) = x^*, \quad \mathcal{P}$-almost everywhere. \qed$

6 Conclusion

The paper is concerned in ergodic theory for family of intuitionistic fuzzy events. We proved the Individual ergodic theorem for intuitionistic fuzzy observables using $\mathcal{P}$-almost everywhere convergence, where $\mathcal{P}$ is an intuitionistic fuzzy probability. The results are a generalization of results given in [3].

References

[1] Atanassov, K. (1999). Intuitionistic Fuzzy Sets: Theory and Applications, Springer Physica Verlag, Heidelberg.

[2] Atanassov, K. (2012). On Intuitionistic Fuzzy Sets, Springer, Berlin.

[3] Čunderlíková, K. (2010). The individual ergodic theorem on the IF-events with product, Soft Computing - A Fusion of Foundations, Methodologies and Applications, Springer - Verlag, 24 (3), 229–234.

[4] Čunderlíková, K. (2018). Upper and lower limits and $m$-almost everywhere convergence of intuitionistic fuzzy observables, Notes on Intuitionistic Fuzzy Sets, 24 (4), 40–49.

[5] Čunderlíková, K. (2019). Intuitionistic fuzzy probability and almost everywhere convergence, Proceedings of IWIFSGN 2019, submitted.

[6] Čunderlíková, K. (2019). Individual ergodic theorem for intuitionistic fuzzy observables using IF-state, Iranian Journal of Fuzzy Sets, submitted.

[7] Grzegorzewski, P., & Mrózwa, E. (2002). Probability of intuitionistic fuzzy events, Soft Methods in Probability, Statistics and Data Analysis, P. Grzegorzewski et al. eds., Physica Verlag, New York, 105–115.
[8] Lendelová, K. (2006). Conditional IF-probability, Advances in Soft Computing: Soft Methods for Integrated Uncertainty Modelling, 275–283.

[9] Lendelová K., & Riečan B. (2004). Weak law of large numbers for IF-events, Current Issues in Data and Knowledge Engineering, Bernard De Baets et al. eds., EXIT, Warszawa, 309–314.

[10] Riečan, B. (2003). A descriptive definition of the probability on intuitionistic fuzzy sets, EUSFLAT '2003, M. Wagenacht, R. Hampet eds., Zittau-Goerlitz Univ. Appl. Sci., 263–266.

[11] Riečan, B. (2005). On the probability on IF-sets and MV-algebras, Notes on intuitionistic fuzzy sets, 11 (6), 21–25.

[12] Riečan, B. (2006). On a problem of Radko Mesiar: general form of IF-probabilities, Fuzzy Sets and Systems, 152, 1485–1490.

[13] Riečan, B. (2006). On the probability and random variables on IF events, Applied Artificial Intelligence, Proc. 7th FLINS Conf. Genova, D. Ruan et al. eds., 138–145.

[14] Riečan, B. (2007). Probability theory on intuitionistic fuzzy events, Algebraic and Proof-theoretic aspects of Non-classical Logics, Papers in honour of Daniele Mundici’s 60th birthday, Lecture Notes in Computer Science, Vol. 4460, 290–308.

[15] Riečan, B. (2012). Analysis of fuzzy logic models, Intelligent systems (V. Koleshko ed.), INTECH, 219–244.

[16] Riečan, B., & Neubrunn, T. (1997). Integral, Measure, and Ordering, Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava.