Progress towards $2 \to 2$ scattering at two loops

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We discuss the two-loop integrals necessary for evaluating massless $2 \to 2$ scattering amplitudes. As a test process, we consider the leading colour two-loop contribution to $q \bar{q} \to q' \bar{q}'$. We show that for physical scattering processes the two Smirnov-Veretin planar box graphs $I_1$ and $I_2$ are accompanied by factors of $1/(D-4)$ thereby necessitating a knowledge of both $I_1$ and $I_2$ to $O(\epsilon)$. Using an alternative basis $I_1$ and the irreducible numerator integral $I_3$, the factors of $1/(D-4)$ disappear.

1. Introduction

Two-to-two scattering processes are well known to be one of the most basic probes of the fundamental interactions of nature. In hadron-hadron collisions, parton-parton scattering to form a large transverse momentum jet tests the point-like nature of the partons down to distance scales of $10^{-17}$ m. However, extracting useful results from experimental data requires both plentiful data and accurate theoretical calculations. For example, the single jet inclusive transverse energy distribution observed by the CDF collaboration in Run I at the TEVATRON indicated possible new physics at large transverse energy $[1]$. Data obtained by the D0 collaboration $[2]$ was more consistent with theoretical next-to-leading order expectations, however, because of both theoretical and experimental uncertainties no definite conclusion could be drawn. The experimental situation may be clarified in the forthcoming high statistic Run II starting in 2001. The theoretical prediction may be improved by including the next-to-next-to-leading order perturbative predictions. This has the effect of (a) reducing the renormalisation scale dependence and (b) improving the matching of the parton level theoretical jet algorithm with the hadron level experimental jet algorithm because the jet structure can be modelled by the presence of a third parton. Varying the renormalisation scale up and down by a factor of two about the jet transverse energy leads to a 20% (10%) renormalisation scale uncertainty at leading order (next-to-leading order) for jets with $E_T \sim 100$ GeV. The improvement in accuracy expected at next-to-next-to-leading order can be estimated using the renormalisation group equations together with the known leading and next-to-leading order coefficients and is at the 1-2% level. Of course, the full next-to-next-to-leading order prediction requires a knowledge of the two-loop $2 \to 2$ matrix elements as well as the contributions from the one-loop $2 \to 3$ and tree-level $2 \to 4$ processes.

In this talk, we wish to review the recent progress that has been made towards the analytic evaluation of the two-loop matrix elements relevant for massless $2 \to 2$ scattering. As can be seen from Table 1, the number of Feynman diagrams contributing to the basic parton scattering processes increases dramatically with the number of loops. The one-loop graphs are those computed by Ellis and Sexton $[3]$ in 1986. The much more numerous two-loop graphs may be either products of one-loop graphs, self-energy insertions or genuinely new topologies. It is the latter class which has proved to be a major stumbling block. However, in the last twelve months, all of the necessary integrals have been computed and a complete basis set of master integrals now exists. We note in passing that a particular two-loop helicity amplitude for $gg \to gg$ scattering has been calculated by Bern, Dixon and Kosower $[4]$. 

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Table 1. Numbers of Feynman diagrams contributing to 2 → 2 parton scattering processes

| Process   | Tree | One loop | Two loops |
|-----------|------|----------|-----------|
| $gg \to gg$ | 4    | 72       | 1531      |
| $q\bar{q} \to gg$ | 3    | 29       | 563       |
| $q\bar{q} \to q'\bar{q}'$ | 1    | 10       | 186       |

2. Master Integrals

The complete set of massless master integrals comprises the trivial topologies of single scale integrals which can be written as products of Gamma functions,

\( \begin{array}{c}
\begin{array}{c}
\text{(s)} \\
\text{(s)} \\
\text{(s)}
\end{array}
\end{array} \)

the less trivial non-planar triangle graph \( \begin{array}{c}
\begin{array}{c}
\text{(s)}
\end{array}
\end{array} \),\n
two scale integrals that are related to the one-loop box graphs \( \begin{array}{c}
\begin{array}{c}
\text{[ ]}
\end{array}
\end{array} \),

\( \begin{array}{c}
\begin{array}{c}
\text{(s,t)} \\
\text{(s,t)}
\end{array}
\end{array} \)

the planar double boxes \( \begin{array}{c}
\begin{array}{c}
\text{[ ]}
\end{array}
\end{array} \),

\( \begin{array}{c}
\begin{array}{c}
\text{(s,t)} \\
\text{(s,t)}
\end{array}
\end{array} \)

which we denote $I_1$ and $I_2$ respectively together with the non-planar double boxes \( \begin{array}{c}
\begin{array}{c}
\text{[ ]}
\end{array}
\end{array} \),

\( \begin{array}{c}
\begin{array}{c}
\text{(s,t)} \\
\text{(s,t)}
\end{array}
\end{array} \).

The Mandelstam variables $s$ and $t$ represent the kinematic scales involved in the integral while the blobs on the propagators represent an additional power of that propagator. The latter blobbed graphs are necessary to evaluate tensor integrals. In other words, starting from a planar or non-planar box tensor integral, it is not possible to reduce the powers of all propagators to unity and the second master integral is required \( \begin{array}{c}
\begin{array}{c}
\text{[ ]}
\end{array}
\end{array} \). The scalar planar \( \begin{array}{c}
\begin{array}{c}
\text{[ ]}
\end{array}
\end{array} \) and non-planar \( \begin{array}{c}
\begin{array}{c}
\text{[ ]}
\end{array}
\end{array} \) integrals themselves were evaluated as multiple Mellin-Barnes integrals and represent significant achievements in the field of Feynman diagramology.

3. Application: $q\bar{q} \to q'\bar{q}'$

As a first step in carrying out the analytic evaluation of all massless $2 \to 2$ scattering matrix elements let us concentrate on the specific process $q(p_1)\bar{q}(p_2) \to q'(p_3)\bar{q}'(p_4)$ where the lightlike momentum assignments are in parentheses.

The amplitude $M$ has the perturbative expansion,

$$M = g_s^2 M_0 + g_s^4 M_1 + g_s^6 M_2 + O(g_s^8),$$

in terms of the tree level ($M_0$), one-loop ($M_1$) and two-loop ($M_2$) amplitudes. The squared and summed tree-level amplitude is given by,

$$|M_0|^2 = 2(N^2 - 1) \left( \frac{t^2 + u^2}{s^2} - \epsilon \right)$$

where $s = (p_1+p_2)^2$, $t = (p_2-p_3)^2$ and $u = -s-t$. The one-loop amplitude $M_1$ was first calculated by Ellis and Sexton \( \begin{array}{c}
\begin{array}{c}
\text{[ ]}
\end{array}
\end{array} \) and contributes to the cross section at $O(g_s^4)$, or next-to-leading order. The two-loop amplitude $M_2$ first contributes at $O(g_s^6)$ through its interference with the tree level amplitude $M_0$ and has the following colour structure,

$$M_2 M_0^\dagger + M_0^\dagger M_2 = (N^2 - 1) \left( AN^2 + B + C \frac{1}{N^2} + DN_F N + E \frac{N_F}{N} \right)$$

where $N$ is the number of colours and $N_F$ the number of light fermions. The leading colour amplitude $A$ is gauge invariant and contains only planar diagrams such as those shown in Figure 1. The amplitudes suppressed by powers of $N$ ($B, C, D$ and $E$) contain the non-planar graphs. As a first step in carrying out the analytic evaluation of the two loop graphs, we therefore focus on the leading colour amplitude $A$. 
4. Auxiliary diagram

To handle all possible permutations of planar diagrams it is convenient to work with the auxiliary diagram shown in Figure 2,

\[
\int \frac{d^Dk_1}{i\pi^{d/2}} \int \frac{d^Dk_2}{i\pi^{d/2}} \frac{1}{A_1^{\nu_1}A_2^{\nu_2}A_3^{\nu_3}A_4^{\nu_4}A_5^{\nu_5}A_6^{\nu_6}A_7^{\nu_7}A_8^{\nu_8}A_9^{\nu_9}}
\]

where \(A_1 = k_1^2\), \(A_2 = (k_1 + p_1)^2\), \(A_3 = (k_1 + p_1 + p_2)^2\), \(A_4 = (k_1 + p_1 + p_2 + p_3)^2\), \(A_5 = k_2^2\), \(A_6 = (k_2 + p_1 + p_2)^2\), \(A_7 = (k_2 + p_1 + p_2 + p_3 + p_4)^2\), \(A_8 = (k_2 + p_1 + p_2 + p_3 + p_4 + p_5)^2\), \(A_9 = (k_2 + p_1)^2\) and \(p_{ij} = p_i + p_j\) and \(p_{ijk} = p_i + p_j + p_k\). The \(i\)th propagator is raised to the power \(\nu_i\). Scalar integrals have all \(\nu_i = 1\) or 0. For example,

\[
\begin{align*}
\Box (s, t) & = I^D(1, 1, 1, 0, 1, 0, 1, 1, 1) \\
\Box (s, t) & = I^D(1, 1, 0, 0, 0, 1, 1, 1) \\
\Box (s, t) & = I^D(1, 1, 1, 0, 0, 0, 1, 1, 1)
\end{align*}
\]

By permuting the arguments, we obtain the other orientations. For example,

\[
\Box (s, t) = I^D(1, 1, 1, 0, 1, 1, 1, 1, 1).
\]

For the interference of tree-level with two-loop graphs, loop momenta in the numerator are always contracted with either external or loop momenta. These dot-products can always be written as combinations of the 9 propagators so that tensor integrals appear as generalised scalar integrals. Negative values of \(\nu_i\) correspond to irreducible numerators. For example, the planar box integral with one irreducible numerator on the left hand loop can be written,

\[
\Box (s, t) = I^D(1, 1, 1, -1, 1, 0, 1, 1, 1).
\]

5. General procedure

The general procedure for computing the two loop graphs is as follows:

1. Use QGRAF \[11\] to generate the Feynman diagrams
2. Multiply by tree-level and compute traces
3. Identify combinations of scalar and tensor auxiliary integrals
4. Exchange tensor integrals (\(\nu_i < 0\)) for auxiliary integrals in higher dimension with higher powers of propagators (\(\nu_i \geq 0\)) \[12\]
5. Apply integration-by-parts (IBP) identities \[13\] to reduce general auxiliary integrals to combinations of the Master Topologies (\(\nu_i \neq 1\))
6. Apply specific IBP identities to reduce Master Topologies to Master Integrals (\(\nu_i = 1\))

6. Planar box graphs

However, because the specific IBP identities for the planar box are quite complicated \[9\], we choose an alternative method for the tensor planar box graphs. In particular we try to stay in \(D \sim 4\). To do this we adopt the approach of
Gehrmann and Remiddi [14]. The idea is very simple both in concept and in implementation. We characterise a loop integral with three numbers, \( t \) the number of different propagators in the denominator, \( r \) the sum of powers of propagators in the denominator and \( s \) the sum of powers of propagators in the numerator. For example, \( (s, t) = (r = 7, s = 0) \)

\[
\begin{align*}
\text{(s,t)} & \quad t = r = 7, \quad s = 0 \\
\text{(s,t)} & \quad t = r = 7, \quad s = 1 \\
\text{(s,t)} & \quad t = 7, \quad r = 8, \quad s = 0
\end{align*}
\]

When acting on a loop integral \( I_{t,r,s} \), the IBP (and Lorentz invariance [14]) identities produce more complicated integrals with the same topology \( I_{t,r+1,s} \) and \( I_{t,r+1,s+1} \), simpler integrals with the same topology \( I_{t,r-1,s} \) and \( I_{t,r-1,s-1} \) as well as simpler topologies \( I_{t-1,r,s} \). By applying each identity to each \( I_{t,r,s} \), we can form a linear system of equations from which the more complicated integrals can be eliminated.

For example, when \( t = 7 \), the number of integrals for a given value of \( r \) and \( s \) is shown in Table 2. The two master integrals [7, 8] have

\[
\begin{align*}
\text{Table 2: Numbers of integrals with different values of } r \text{ and } s \text{ for } t = 7 \text{ (taken from [14]).}
\end{align*}
\]

\[
\begin{array}{c|cccc}
    r & 0 & 1 & 2 & 3 \\
    \hline
    0 & \text{7} & \text{1} & \text{2} & \text{3} \text{ 4} \\
    8 & \text{7} & \text{14} & \text{21} & \text{28} \\
    9 & \text{28} & \text{56} & \text{84} & \text{112} \\
    10 & \text{84} & \text{168} & \text{252} & \text{336} \\
\end{array}
\]

\[t = 7\] and \[r = 7\] and \[r = 8\] respectively. Tensor integrals (corresponding to \( r = 7 \) and \( s > 0 \)) lie on the first row of Table 2. Using only these integrals as seeds for the IBP identities and eliminating the unknowns using linear algebra, we immediately obtain all tensor integrals for the planar box in \( D \sim 4 \) in terms of \( I_1 \) and the irreducible numerator graph

\[
I_3 = \begin{array}{c}
\end{array}
\]

(rather than \( I_1 \) and \( I_2 \)) together with simpler pinched integrals that can be straightforwardly simplified. If we denote the \( s = i + j \), \( t = r = 7 \) planar box integral as

\[
\begin{align*}
I^D(1,1,1, -i, 1, -j, 1, 1, 1) = \begin{array}{c}
\end{array}
\end{align*}
\]

then,

\[
\begin{align*}
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array},
\end{align*}
\]

and, for example, the second rank tensor integrals are given by,

\[
\begin{align*}
\begin{array}{c}
\end{array} = \frac{st}{2} \begin{array}{c}
\end{array} (s,t) - \frac{3s}{2} \begin{array}{c}
\end{array} (s,t) + \frac{8(D-3)}{(D-4)} \begin{array}{c}
\end{array} (s,t) - \frac{(7s + 9t)}{s} \begin{array}{c}
\end{array} (s,t) + \frac{17(D-3)(3D - 10)}{2s(D-4)^2} \begin{array}{c}
\end{array} (s) - \frac{2(3D-8)(3D - 10)}{s^2(D-4)^3} \begin{array}{c}
\end{array} (s) + \frac{9(3D - 10)(3D - 8)(D-3)}{st(D-4)^3} \begin{array}{c}
\end{array} (t) + \frac{2(D-3)((2D-5)s + 2(D-3)t)}{s^2(D-4)^2} \begin{array}{c}
\end{array} (s)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\end{array} = \frac{(D-4)st}{2(D-3)} \begin{array}{c}
\end{array} (s,t) - \frac{(3(D-4)s - 2t)}{2(D-3)} \begin{array}{c}
\end{array} (s,t) + 8 \begin{array}{c}
\end{array} (s,t)
\end{align*}
\]
We see that the coefficients of both $I_1$ and $I_3$ are always finite as $D \to 4$ and that the IBP identities have not introduced fake singularities.

7. Leading Colour Matrix Elements

In $D$ dimensions, the leading colour two-loop amplitude $A$ for $q\bar q \to q'\bar q'$ scattering has the following structure,

$$A = -\frac{2}{(4\pi)^D} \left( a_1 \quad (s, t) + a_2 \quad (s, t) + a_3 \quad (t, s) + a_4 \quad (t, s) + a_5 \quad (s, t) + a_6 \quad (t, s) + a_7 \quad (s, t) + a_8 \quad (t, s) + a_9 \quad (s) + a_{10} \quad (t) + a_{11} \quad (s) + a_{12} \quad (t) + a_{13} \quad (s) + a_{14} \quad (t) \right).$$

We have calculated the coefficients $a_1 - a_{14}$ in arbitrary dimension. For example, the first two coefficients of the planar box graphs are given by,

$$a_1 = -\frac{(7s + 9t)(D - 4)}{s(D - 3)} \frac{1}{(s, t)} - \frac{(13s - 2t)(3D - 10)}{2s^2(D - 4)} \frac{1}{(s, t)} + \frac{2(s + 2t)(3D - 8)(3D - 10)}{2s^3(D - 3)(D - 4)^2} \frac{1}{(s)} + \frac{9(3D - 8)(3D - 10)}{st(D - 4)^2} \frac{1}{(t)} + \frac{2((2D - 7)s + 2(D - 4)t)}{s^2(D - 4)} \frac{1}{(s)}$$

We see that both $a_1$ and $a_2$ are well behaved as $D \to 4$ indicating again that the IBP identities have not introduced fake singularities.

8. Relationship between Master Integrals

We can also use the Gehrmann-Remiddi approach [4] to find a relation between the irreducible numerator master integral $I_3$ and those of Smirnov and Veretin ($I_1$ and $I_2$) [3, 5]. Suppressing the simpler pinched integrals, we find,

$$\begin{align*}
\frac{1}{D-4} & \quad \frac{1}{D-5} \quad (s, t) + \frac{(D - 6)s}{2(D - 4)(D - 5)} \quad \frac{1}{(s, t)} \\
& - \frac{(3D - 14)s}{2(D - 4)} \quad \frac{1}{(s, t)} + \text{pinchings}
\end{align*}$$

The presence of the $D - 4$ factor in the denominator immediately indicates a problem. Both $I_1$ and $I_2$ have $1/\epsilon^4$ leading poles and it would appear that $I_3 \sim 1/\epsilon^5$. This is not the case as close examination of $I_1$ and $I_2$ shows that in this combination the $1/\epsilon^4$ poles and descendents cancel completely so that

$$\frac{1}{D-4} \sim \frac{1}{\epsilon^4}$$

as we expect. However, the finite parts of $I_3$ are controlled by the $O(\epsilon)$ parts of $I_1$ and $I_2$.

$I_3$ can also be written in terms of derivatives of $I_1$,

$$\begin{align*}
I_3 &= -\frac{(s(D - 5) - t)}{(D - 4)} \quad \quad \quad \frac{1}{(s, t)} \\
&+ \frac{ut}{(D - 4)} \quad \frac{1}{(s, t)} + \text{pinchings}
\end{align*}$$

which again indicates that the $O(\epsilon)$ part of $I_1$ is necessary to determine the $O(\epsilon^0)$ part of $I_2$. In fact this additional part is not very difficult to
obtain either by considering the differential equations for $I_1$ and $I_2$ at $t = -s$ or by explicit evaluation of the Mellin-Barnes integrals. As expected, the $\epsilon$ expansion for $I_3$ through to $O(\epsilon^0)$ contains quadrilogarithms at worst.

9. Results

Using the analytic expansions around $D \sim 4$ we can evaluate the leading colour two-loop amplitudes for $q\bar{q} \rightarrow q'\bar{q}'$. Expanding both the integrals and coefficients $a_1 - a_{14}$ we find that the leading singularities are proportional to tree level,

$$M_2 M_0^1 + M_0^2 M_0 =$$

$$\frac{4N^2}{(4\pi)^D} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)\Gamma(1-2\epsilon)\Gamma(1-4\epsilon)}{\epsilon^4} \left(-\frac{t}{\mu^2}\right)^{-2\epsilon} |M_0|^2$$

$$+ O\left(\frac{1}{\epsilon^4}\right).$$

The leading pole has the same coefficient as the square of the one-loop amplitude. The remaining pole and finite contributions are in the process of being checked.

10. Summary

We have made a study of two-loop amplitudes of massless $2 \rightarrow 2$ scattering by considering $q\bar{q} \rightarrow q'\bar{q}'$ as a trial process. At leading colour, only planar graphs contribute and we have expressed the amplitude as a sum over the basis set of two-loop master integrals. It turns out that for this process, in reducing the tensor integrals to scalars, factors of $1/(D - 4)$ are generated multiplying both of the Smirnov-Veretin planar box graphs, $I_1$ and $I_2$. These factors do not cancel in the physical process thereby necessitating a knowledge of both $I_1$ and $I_2$ to $O(\epsilon)$. Alternatively, if one uses the basis $I_1$ and the irreducible numerator integral $I_3$, the factors of $1/(D - 4)$ disappear. Of course, evaluating $I_3$ also requires the $O(\epsilon)$ of $I_1$ but this has now been calculated. We therefore have the ingredients to evaluate the leading colour two-loop amplitude for $q\bar{q} \rightarrow q'\bar{q}'$ for $D \sim 4$ and the leading singularities agree with expectations.

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