Deformation theory of $G_2$ conifolds

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Abstract

We consider the deformation theory of asymptotically conical (AC) and of conically singular (CS) $G_2$ manifolds. In the AC case, we show that if the rate of convergence $\nu$ to the cone at infinity is generic in a precise sense and lies in the interval $(-4, -\frac{5}{2})$, then the moduli space is smooth and we compute its dimension in terms of topological and analytic data. For generic rates $\nu < -4$ in the AC case, and for generic positive rates of convergence to the cones at the singular points in the CS case, the deformation theory is in general obstructed. We describe the obstruction spaces explicitly in terms of the spectrum of the Laplacian on the link of the cones on the ends, and compute the virtual dimension of a finite-codimensional subspace of the moduli space, which is often the full moduli space.

We also present many applications of these results, including: the local rigidity of the Bryant–Salamon AC $G_2$ manifolds; an extension of our deformation theory in the AC case to higher rates under certain natural assumptions; the cohomogeneity one property of AC $G_2$ manifolds asymptotic to homogeneous cones; the smoothness of the CS moduli space if the singularities are modeled on particular $G_2$ cones; and the proof of existence of a “good gauge” needed for desingularization of CS $G_2$ manifolds. Finally, we discuss some open problems for future study.

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1 Introduction

In this paper we study the deformation theory of certain G₂ manifolds that are modeled on cones, which we call conifolds. Specifically, we consider the asymptotically conical (AC) G₂ manifolds, which are noncompact manifolds with one end that is asymptotic (in an appropriate sense) to a G₂ cone at infinity. We also consider conically singular (CS) G₂ manifolds. These are compact topological spaces such that after removing a finite set of points \{x_1, \ldots, x_n\} they are noncompact manifolds of G₂ holonomy with n ends that are asymptotic (in an appropriate sense) to n possibly distinct G₂ cones at their vertices. The precise definitions will be given in Section 3.

This paper is a sequel to [20], which studied the desingularization of CS G₂ manifolds by gluing in AC G₂ manifolds. We adopt the notation and conventions from [20], and as such it may be useful for the reader of the present paper to begin by familiarizing themselves with Sections 1, 2, and 4.
of [20], although we do review all of the important definitions and restate all the results from [20] that are needed to keep the present paper as self-contained as possible.

The deformation theory of CS or AC manifolds in the context of special holonomy and calibrated geometry has been studied by Joyce in [15] for CS special Lagrangian submanifolds, by Marshall [32] and Pacini [43] for AC special Lagrangian submanifolds of $\mathbb{C}^m$, and by the second author [27, 28, 29, 30] for coassociative AC and CS submanifolds and associative AC submanifolds of $G_2$ manifolds. Nordström [40] considered the deformation theory of asymptotically cylindrical $G_2$ manifolds. Finally, Pacini [44, 45, 46] has a series of papers on the analysis of special Lagrangian conifolds, allowing for a mixture of both AC and CS ends.

It is interesting to compare the study of moduli spaces of conifolds which are submanifolds, such as special Lagrangian, coassociative, or associative, to the moduli spaces of conifolds which are the ambient special holonomy manifolds themselves. The results in both cases are similar in spirit, but there are some notable exceptions in the details. One key point is the issue of gauge-fixing: in the submanifold case this is solved in a trivial way by essentially considering deformations defined using normal vector fields, but in the manifold case one has to work much harder leading to numerous analytic difficulties. Another key point is that in the submanifold case one can easily identify certain deformations (in the AC case) and obstructions (in the CS case) in a simple concrete way by using the fact that on the ends the submanifold may be viewed as a graph over a cone in Euclidean space, whereas in the manifold setting no such easy interpretation is usually possible.

**Main results and applications**

The main theorem we prove in this paper is the following. The notation and terminology used in this theorem is defined in Section 3 and Section 5.1.

**Main Theorem** (Theorem 5.1) Let $(M, \varphi)$ be a $G_2$ conifold, asymptotic to particular $G_2$ cones on the ends, at some rate $\nu$. Let $M_\nu$ be the moduli space of all torsion-free $G_2$ structures on $M$, asymptotic to the same cones on the ends, at the same rate $\nu$, modulo the appropriate notion of equivalence that preserves these conditions. Then for generic $\nu$ (in a sense made precise later), we have

- In the AC case, if $\nu \in (-4, -\frac{5}{2})$, the space $M_\nu$ is a smooth manifold whose dimension consists of topological and analytic contributions, given precisely in Corollary 5.21.

- In the AC case if $\nu < -4$, or in the CS case for any $\nu > 0$, the space $M_\nu$ is in general only a topological space, and the deformation theory may be obstructed. There is a subspace $\tilde{M}_\nu$ of $M_\nu$, called the reduced moduli space, which omits a finite-dimensional space (often trivial) of deformations. The virtual dimension of $\tilde{M}_\nu$ again consists of topological and analytic contributions, given precisely in Corollary 5.21.

Here the “appropriate notion of equivalence” is by the action of diffeomorphisms which preserve the convergence, at the same rate, to the $G_2$ cones at the ends. In particular, we consider only deformations of $G_2$ conifolds that are asymptotic to fixed $G_2$ cones on the ends. The reason for this is that the link of a $G_2$ cone is a compact strictly nearly Kähler 6-manifold, also known as a Gray manifold. The infinitesimal deformations of Gray manifolds were considered by Moroianu–Nagy–Semmelmann in [36], but there is no evidence that such objects have smooth, unobstructed deformations. Indeed, there are at present still only three known examples of Gray manifolds, other than the round $S^6$, all of which are homogeneous. They are diffeomorphic to $\mathbb{C}P^3$, $SU(3)/T^2$, and $S^3 \times S^3$, and are described in more detail in Section 3.1.
Remark 1.1. The restriction to $\nu < -\frac{5}{2}$ in the AC case may be artificial and could possibly be relaxed to $\nu < 0$ in general, with some more effort. This restriction comes from the relations between various gauge-fixing conditions in Section [4.7]. See also Remark 5.9 for more details. However, if the asymptotic cone comes from any of the explicit examples of $G_2$ cones that are known, we show in Section [6] that we can extend our main theorem in the AC case all the way up to $\nu < 0$ anyway. See also Remark 5.5.

Applications and corollaries

Perhaps even more interesting than our main theorem are its several applications, which are stated precisely in Section [6]. In particular, we use Theorem 5.1 and its ingredients to establish the following corollaries:

- Under certain natural conditions, which we show hold for all known examples, we extend our deformation theory of AC $G_2$ conifolds from rate $\nu < -\frac{5}{2}$ all the way to $\nu < 0$.
- The AC moduli space, when it is smooth, is always at least 1-dimensional because it contains deformations that are asymptotic to rescaling of the $G_2$ cone at infinity. As a corollary of this observation and our moduli space dimension formula, we prove the local rigidity of the Bryant–Salamon AC $G_2$ manifolds, not just as AC manifolds of rate $\nu + \varepsilon$, where $\nu$ is $-3$ or $-4$ in these cases, but in fact as AC manifolds of rate $\lambda$ all the way to any $\lambda < 0$.
- A consequence of our main theorem is that when $\nu \in (-4.3 + \varepsilon)$, the moduli space $M_\nu$ of AC $G_2$ manifolds, which is smooth in this case, is determined purely by the topology of the underlying $G_2$ manifold $M$. Moreover, we show that for $\nu = -3 + \varepsilon$ this moduli space $M_{-3+\varepsilon}$ can be naturally immersed into the vector space $H^3(M, \mathbb{R}) \times H^4(M, \mathbb{R})$.
- We prove that an AC $G_2$ manifold that is asymptotic to a homogeneous $G_2$ cone must be of cohomogeneity one. Combining this result with work of Cleyton–Swann [9] establishes that $\Lambda^2(S^4)$ and $\Lambda^2(\mathbb{C}P^2)$ are the unique AC $G_2$ manifolds asymptotic to the $G_2$ cones over $\mathbb{C}P^3$ and $SU(3)/T^2$, respectively.
- We argue that CS $G_2$ manifolds with particular types of conical singularities, including those modelled on the $G_2$ cones over $\mathbb{C}P^2$ or $S^3 \times S^3$, have unobstructed deformations and thus admit a smooth reduced moduli space $\tilde{M}_\nu$. In fact, we show that this holds for an extension $\tilde{M}_\mu$ of the reduced moduli space.
- We explicitly compare the dimensions of the extended reduced moduli space $\tilde{M}_\mu$ of CS $G_2$ manifolds with one singularity (when it is smooth) to the moduli space of the compact smooth $G_2$ manifolds obtained by the desingularization construction in [20]. This observation gives evidence of the likelihood that CS $G_2$ manifolds are the dominant contributor to the “boundary” of the moduli space of compact smooth $G_2$ manifolds.
- We prove that an appropriate gauge-fixing condition on AC $G_2$ manifolds, which is required for the desingularization theorem in [20], can in fact always be achieved.

To prove our main theorem and describe the deformation theory of $G_2$ conifolds, we follow in spirit the approach of Joyce [13], who considered the deformation theory of compact $G_2$ manifolds. However, almost all of the steps in his proof use compactness in a crucial way, so we need to make nontrivial extensions to establish our results in noncompact setting of conifolds.

One technical issue is that we use weighted Banach spaces of sections that decay at some rate $\lambda$ on the ends of the manifold, but these Banach spaces are not always subspaces of $L^2$. Indeed, the
rate at which the transition occurs between being in $L^2$ or not, specifically $\lambda = -\frac{7}{2}$, lies precisely between the rates $-4$ and $-3$ that together encompass all known examples of AC $G_2$ manifolds. As a result, we need to delicately work right on the edge of where certain analytic results actually hold, and in several cases we need to find subtle ways to enable us to extend the range where these results hold. This is in contrast, for example, to the asymptotically cylindrical case where one can always essentially work with $L^2$ sections. Another issue is that in the non-$L^2$ case, we are forced to use the Dirac operator on $G_2$ manifolds to prove our “slice theorem”. This is similar to Nordström [10]. Finally, the noncompactness of the manifolds (and thus the nonavailability of classical Hodge theory) makes it more natural to consider the Fredholm theory of the operator $d + d^*$ rather than the Laplacian $\Delta$ to study the moduli space. Some of the issues highlighted here are purely analytic technical problems but others are geometrically relevant.

**Remark 1.2.** There are at present no known examples of CS $G_2$ manifolds. The first author and Dominic Joyce have a new construction [16] of smooth, compact $G_2$ manifolds that may be generalizable to produce the first examples of CS $G_2$ manifolds. In these examples the singular cones would all be cones over the nearly Kähler $\mathbb{CP}^3$. This possible generalization is currently being investigated by the authors of the present paper.

**Organization of the paper**

We now discuss the organization of our paper. Section 2 reviews some aspects of $G_2$ geometry that we require, including the spinor bundle and the Dirac operator for $G_2$ manifolds. More details about $G_2$ structures can be found in Bryant [5] and Joyce [13]. Section 3 is a review of some of the material from [20] about $G_2$ conifolds. In Section 4, we begin with a brief review and summary of the relevant results that we need from the Lockhart–McOwen theory of weighted Sobolev spaces on conifolds, including some Hodge-theoretic results in this setting. We then discuss in great detail many analytic results about $G_2$ conifolds. In particular, this includes: a special index-change theorem; topological results about $G_2$ conifolds; parallel tensors on $G_2$ conifolds; various results concerning our gauge-fixing condition on moduli spaces of conifolds; and some material on analytic aspects of the Dirac operator on $G_2$ cones that we require. In Section 5 we consider the deformation theory of $G_2$ conifolds, and prove our main theorem in four steps. Finally, in Section 6 we present many applications of our results, as described above, and discuss some open problems.

**Conventions**

We use single vertical bars $|·|$ or angle braces $⟨·,·⟩$ for a pointwise inner product on sections of some vector bundle, and we use double vertical bars $||·||$ or angle braces $⟨⟨·,·⟩⟩$ for a global ($L^2$) inner product on sections. Since all of our manifolds are Riemannian, we often use the metric $g$ to identify vector fields and 1-forms. This will always be clear by the context.

There are two sign conventions in $G_2$ geometry. The convention we choose is the one used in Bryant–Salamon [6] and in Harvey–Lawson [12], but differs from the convention used in Bryant [5] or Joyce [13]. A detailed discussion of sign conventions and orientations in $G_2$ geometry can be found in the first author’s note [18].
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2 Preliminaries on $G_2$ manifolds

2.1 $G_2$ structures

A $G_2$ structure on a smooth 7-manifold $M$ is a smooth 3-form $\varphi$ satisfying a certain “nondegeneracy” condition. Various approaches to describing this nondegeneracy condition can be found, for example, in [5, 13, 19] but we will not explicitly need these. A $G_2$ structure $\varphi$ determines a Riemannian metric $g_\varphi$ and an orientation $\text{vol}_\varphi$ in a nonlinear way. Thus $\varphi$ determines a Hodge star operator $\ast_\varphi$, and $\psi = \ast_\varphi \varphi$ is the dual 4-form. When a $G_2$ structure exists, there is an open subbundle $\Omega_3^+$ of the bundle of 3-forms consisting of nondegenerate 3-forms, also called positive or stable 3-forms.

Definition 2.1. A $G_2$ manifold is a connected manifold with a $G_2$ structure $(M, \varphi)$ such that $\varphi$ is parallel with respect to the Levi-Civita connection $\nabla$ determined by $g_\varphi$. That is, $\nabla g_\varphi \varphi = 0$. Such a $G_2$ structure is also called torsion-free. In this case the Riemannian holonomy $\text{Hol}_{g_\varphi}(M)$ of $(M, g_\varphi)$ is contained in the group $G_2 \subseteq \text{SO}(7)$.

Remark 2.2. A $G_2$ manifold is always Ricci-flat, and a $G_2$ structure $\varphi$ is torsion-free if and only if it is both closed and coclosed: $d\varphi = 0$ and $d\psi = 0$.

On a manifold with $G_2$ structure, there is a decomposition into representations of $G_2$ of the bundle $\Lambda^k T^* M$ of $k$-forms. The space $\Omega^3$ of 3-forms decomposes as

$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},$$

into irreducible $G_2$ representations. Similarly we have a decomposition of the space $\Omega^2$ as

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14},$$

and isomorphic splittings of $\Omega^4$ and $\Omega^5$ given by the Hodge star of the above decompositions: $\Omega^k_l = \ast_\varphi(\Omega^k_{7-l})$. The bundle $\Omega^k_l$ has fibre dimension $l$ and these decompositions of $\Omega^k$ are orthogonal with respect to the metric $g_\varphi$. The explicit descriptions of these spaces that we will need are as follows:

$$\Omega^2_7 = \{ \ast (\alpha \wedge \psi); \alpha \in \Omega^1 \} = \{ \beta \in \Omega^2; \ast (\varphi \wedge \beta) = -2 \beta \},$$

$$\Omega^2_{14} = \{ \beta \in \Omega^2; \beta \wedge \psi = 0 \} = \{ \beta \in \Omega^2; \ast (\varphi \wedge \beta) = \beta \},$$

$$\Omega^3_1 = \{ f\varphi; f \in \Omega^0 \},$$

$$\Omega^3_7 = \{ \ast (\alpha \wedge \varphi); \alpha \in \Omega^1 \},$$

$$\Omega^3_{27} = \{ \eta \in \Omega^3; \eta \wedge \varphi = 0 \text{ and } \eta \wedge \psi = 0 \}.$$  

Remark 2.3. One can thus decompose $d\varphi = \pi_1(d\varphi) + \pi_7(d\varphi) + \pi_{27}(d\varphi)$ and $d\psi = \pi_7(d\psi) + \pi_{14}(d\psi)$ for any $G_2$ structure. It is a nontrivial but well known fact that $\pi_7(d\varphi)$ vanishes if and only if $\pi_7(d\psi)$ vanishes. See, for example, [19] for a direct verification of this fact. In particular, the implication of this that we will require is that for a closed $G_2$ structure, we have $d\psi \in \Omega^3_{27}$.
Remark 2.4. When the \( G_2 \) structure is torsion-free, these decompositions of the spaces of forms are preserved by the Hodge Laplacian \( \Delta = dd^* + d^*d \). The essential aspect of this fact that we will need is the following. Suppose \( f \) is any function and \( X \) is any 1-form on a \( G_2 \) manifold \( M \). Then

\[
\Delta(f\phi) = (\Delta f)\phi, \quad \Delta(f\psi) = (\Delta f)\psi, \quad (8)
\]
\[
\Delta(X \wedge \phi) = (\Delta X) \wedge \phi, \quad \Delta(X \wedge \psi) = (\Delta X) \wedge \psi. \quad (9)
\]

The identities in (8) can be proved using just the fact that \( \phi \) and \( \psi \) are parallel, while the identities in (9) also require the fact that \( G_2 \) manifolds have vanishing Ricci curvature.

The next three propositions are results about the decompositions of 2-forms and 3-forms in the torsion-free case.

Proposition 2.5. Let \( \phi \) be a torsion-free \( G_2 \) structure and suppose \( \mu \in \Omega^2_{14} \) is coclosed: \( d^*\mu = 0 \). Then \( d\mu \in \Omega^3_{27} \).

Proof. By equation (4) the condition \( \mu \in \Omega^2_{14} \) is equivalent to each of the following conditions:

\[
\mu \wedge \psi = 0, \quad \mu \wedge \phi = \ast \mu.
\]

Taking the exterior derivative of these equations and using \( d\phi = 0 \), \( d\psi = 0 \), and \( d^*\mu = \ast(d^*\mu) = 0 \), we find

\[
(d\mu) \wedge \psi = 0, \quad (d\mu) \wedge \phi = 0.
\]

The conclusion now follows from (7).

Proposition 2.6. Let \( \phi \) be a torsion-free \( G_2 \) structure and suppose \( \eta \in \Omega^3_{27} \). Then \( \pi_7(d\eta) = 0 \) if and only if \( \pi_7(d^*\eta) = 0 \).

Proof. This is justified in [5] using representation theory. Here we present an explicit computational proof using the local coordinate identities for \( G_2 \) structures and the descriptions of \( \Omega^3_{27} \) that can be found in [19]. In particular we will repeatedly use identities for contractions of \( \phi \) with itself given in the appendix of [19]. A 3-form \( \eta \) of type \( \Omega^3_{27} \) corresponds uniquely to a symmetric traceless 2-tensor \( h \) on \( M \), with the correspondence given in local coordinates (using the summation convention) by

\[
\eta_{ijk} = h_{ip}g^{pq}\varphi_{qjk} + h_{jp}g^{pq}\varphi_{qki} + h_{kp}g^{pq}\varphi_{qij}.
\]

A short computation now gives

\[
h_{ia} = \frac{1}{4}\varphi_{ijk}\eta_{abc}g^{jb}g^{kc}. \quad (10)
\]

Next, we note that \( \pi_7(d^*\eta) = 0 \) if and only if \( \langle d^*\eta, Y \cdot \varphi \rangle = 0 \) for all vector fields \( Y \), which in local coordinates becomes

\[
\pi_7(d^*\eta) = 0 \iff g^{p\ell}(\nabla_p\eta_{ijk})\varphi_{abc}g^{jb}g^{kc} = 0 \quad \text{for all } a. \quad (11)
\]

Similarly, we have \( \pi_7(d\eta) = 0 \) if and only if \( \langle d\eta, Y \wedge \varphi \rangle = 0 \) for all vector fields \( Y \). Using the fact that \( \eta \in \Omega^3_{27} \), which implies that \( \eta_{ijk}\varphi_{abc}g^{ia}g^{jb}g^{kc} = 0 \), in local coordinates this condition becomes

\[
\pi_7(d\eta) = 0 \iff g^{p\ell}(\nabla_p\eta_{ijk})\varphi_{abc}g^{jb}g^{kc} = 0 \quad \text{for all } i. \quad (12)
\]

Substituting (10) into (11) and (12) show that both conditions are equivalent to \( g^{p\ell}\nabla_p h_{jk} = 0 \) for all \( k \). That is, the divergence of the symmetric traceless 2-tensor \( h \) must vanish. \( \square \)
Proposition 2.7. Consider the operator \( \pi_7 d^* d : \Omega^2_3 \to \Omega^2_3 \). Under the identification \( \Omega^2_3 \cong \Omega^1_4 \), this operator \( \pi_7 d^* d \) corresponds to the operator \( \Delta \), where \( \Delta X = dd^* X + \frac{2}{3} d^* dX \), and is therefore elliptic.

Proof. Let \( X \varphi \in \Omega^2_3 \) for \( X \in \Omega^1_4 \), where as usual we use the metric \( g \) to identify vector fields and 1-forms. Then \( \Delta X = Y \), where \( Y \varphi = \pi_7 d^* d(X \varphi) \). We now compute in local coordinates:

\[
\begin{align*}
(X \varphi)_{jk} &= X^m \varphi_{mjk}, \\
(d(X \varphi))_{ijk} &= (\nabla_i X^m) \varphi_{mjk} + (\nabla_j X^m) \varphi_{mki} + (\nabla_k X^m) \varphi_{mij}, \\
(d^* d(X \varphi))_{jk} &= -g^{pi} \nabla_p (d(X \varphi))_{ijk} \\
&= -g^{pi} (\nabla_p \nabla_i X^m) \varphi_{mjk} - g^{pi} (\nabla_p \nabla_j X^m) \varphi_{mki} - g^{pi} (\nabla_p \nabla_k X^m) \varphi_{mij},
\end{align*}
\]

for some \( \mu \in \Omega^2_{14} \). Now we again use the various local coordinate identities for \( G_2 \) structures that can be found in [19], for example. Contracting both sides of the last equation above with \( \varphi \) on two indices, we find

\[
6 Y_l = \left( -g^{pi} (\nabla_p \nabla_i X^m) \varphi_{mjk} - g^{pi} (\nabla_p \nabla_j X^m) \varphi_{mki} - g^{pi} (\nabla_p \nabla_k X^m) \varphi_{mij} \right) \varphi_{lab} g^{ja} g^{kb}
\]

\[
= -6 g^{pi} (\nabla_p \nabla_i X_l) + 2 g^{pi} (\nabla_p \nabla_j X^m) (g_{ml} g_{ia} - g_{ma} g_{il} - \psi_{mla}) g^{ja}
\]

\[
= 6 (\Delta X)_l + 2 (\Delta X)_j + 2 (dd^* X)_l + (\nabla_p \nabla_j X_n - \nabla_j \nabla_p X_n) g^{pi} g^{ja} g^{nm} \psi_{iaml}
\]

Using the Ricci identities, the last term becomes \(-R_{pmlc} X^c g^{pi} g^{jm} \psi_{iaml} = -2 R_{mlnc} X^c g^{jm} = 0\), where we have used the fact that the Riemann tensor is in \( \Omega^2_{14} \) with respect to its first and last pair of indices, and that the Ricci curvature vanishes. Thus we conclude that

\[
6 Y = 4 \Delta X + 2 dd^* X = 4 d^* dX + 6 dd^* X,
\]

which is what we wanted to show.

We close this section with a discussion of the nonlinear map \( \Theta : \Omega^3_3 \to \Omega^4_3 \) which associates to any \( G_2 \) structure \( \varphi \), the dual 4-form \( \psi = \Theta(\varphi) = *\varphi \varphi \) with respect to the metric \( g_\varphi \) associated to \( \varphi \). One result which will be crucial is the following. This is Proposition 10.3.5 in Joyce [13], adapted to suit our present purposes.

Lemma 2.8. Suppose that \( \varphi \) is a torsion-free \( G_2 \) structure with induced metric \( g_\varphi \), and dual 4-form \( \psi = *\varphi \varphi \). Let \( \eta \) be a 3-form which has sufficiently small \( C^0 \) norm with respect to \( g_\varphi \), so that \( \varphi + \eta \) is still nondegenerate. Then we have

\[
\Theta(\varphi + \eta) = \psi + *\varphi \left( \frac{4}{3} \pi_1(\eta) + \pi_7(\eta) - \pi_2(\eta) \right) + Q_\varphi(\eta),
\]

where \( \pi_k \) is the projection onto the subspace \( \Omega^3_3 \) with respect to the \( G_2 \) structure \( \varphi \). The nonlinear map \( Q_\varphi : \Omega^3_3 \to \Omega^4_3 \) satisfies

\[
Q_\varphi(0) = 0, \quad |Q_\varphi(\eta)| \leq C|\eta|^2, \quad |\nabla Q_\varphi(\eta)| \leq C|\eta||\nabla| \eta|, \quad (14)
\]

for some \( C > 0 \), where the norms and the covariant derivatives are taken with respect to \( g_\varphi \).
We will denote the second term on the right hand side of (13), which is the term linear in $\eta$, by $L_\varphi(\eta)$. That is,

$$L_\varphi(\eta) = \ast_\varphi \left( \frac{4}{3} \pi_1(\eta) + \pi_7(\eta) - \pi_{27}(\eta) \right).$$

The map $L_\varphi : \Omega^3 \to \Omega^4$ is the linearization of the nonlinear map $\Theta$ at $\varphi$, and is therefore a key ingredient for understanding the infinitesimal deformations of torsion-free $G_2$ structures.

Suppose that $\varphi$ is a torsion-free $G_2$ structure, so that in particular it is coclosed: $d\psi = 0$. Take the exterior derivative of (13) to obtain:

$$d(\Theta(\varphi + \eta)) = d(L_\varphi(\eta)) + d(Q_\varphi(\eta))$$

and hence

$$\ast_\varphi d(\Theta(\varphi + \eta)) = -d^* \ast_\varphi (L_\varphi(\eta)) - d^* \ast_\varphi (Q_\varphi(\eta)).$$

We will use (16) in Section 5.2.2 when we establish a one-to-one correspondence between torsion-free “gauge-fixed” $G_2$ structures and solutions to a nonlinear partial differential equation.

### 2.2 The spinor bundle and the Dirac operator on $G_2$ manifolds

A $G_2$ structure on a manifold $M$ induces a spin structure, and therefore $M$ admits an associated Dirac operator $\mathcal{D}$ on its spinor bundle $\mathcal{S}(M)$. When the $G_2$ structure is torsion-free this Dirac operator squares to the Hodge Laplacian, after identifying spinors with forms. These facts are explained in detail in the first author’s note [18]. Here we only review the facts that are needed in the present paper. The $G_2$ structure $\varphi$ is always understood to be torsion-free in this section. Also, we will make repeated use of the identities relating the interior product, the wedge product, and the star operator for $G_2$ structures, which can be found in [17, Lemma 2.23]. (Note that since we are using the opposite orientation convention from [17], equation (2.13) in that paper should have a factor of $-2$ instead of $+2$.)

**Definition 2.9.** We define the curl of a vector field $X$ to be the vector field \( \text{curl} \, X \) given by

$$\text{curl} \, X = \ast(dX \wedge \psi)$$

In other words, up to $G_2$-equivariant isomorphisms, the vector field $\text{curl} \, X$ is the projection onto the $\Omega^2_\mathbb{R}$ component of the 2-form $dX$.

There is a natural identification of the spinor bundle $\mathcal{S}(M)$, a rank 8 real vector bundle, with the bundle $\mathbb{R} \oplus TM$ whose sections lie in $\Omega^0 \oplus \Omega^1_\mathbb{R}$.

**Definition 2.10.** The Dirac operator $\mathcal{D}$ is a first order differential operator from $\mathcal{S}(M)$ to $\mathcal{S}(M)$ defined as follows. Let $s = (f, X)$ be a section of $\mathcal{S}(M)$. Then

$$\mathcal{D}(f, X) = (d^* X \wedge df, \text{curl} X).$$

The Dirac operator is formally self-adjoint: $\mathcal{D}^* \mathcal{D} = \mathcal{D}^2$ to the Hodge Laplacian $\Delta$.

**Proposition 2.11.** Under the identification of the spinor bundle $\mathcal{S}(M)$ with the bundle $\Omega^0 \oplus \Omega^1_\mathbb{R}$, the Dirac Laplacian $\mathcal{D}^2$ and the Hodge Laplacian $\Delta$ are equal:

$$\mathcal{D}^2(f, X) = (\Delta f, \Delta X).$$
Remark 2.12. Proposition 2.11 is proved in [18]. The facts that are needed are that \( d^* (\text{curl } Y) = 0 \) for any vector field \( Y \), that \( \text{curl}(df) = 0 \) for any function \( f \), and that \( \text{curl}(\text{curl } Y) = -dd^* Y + \Delta Y = d^* dY \) for any vector field (1-form) \( Y \). These facts will be important in Proposition 2.13 below.

For our present purposes, we actually require a slight modification of the Dirac operator as follows. The spinor bundle \( \tilde{S}(M) \) is isomorphic to \( \Omega^0_1 \oplus \Omega^2_1 \) and hence, via a \( G_2 \)-equivariant isomorphism, it is also isomorphic to \( \Omega^3_1 \oplus \Omega^1_1 \). Now consider the map

\[
\tilde{\mathcal{D}} : \Omega^0_1 \oplus \Omega^1_1 \rightarrow \Omega^3_1 \oplus \Omega^1_1
\]

\[
(f, X) \mapsto \frac{1}{2} \ast (df \wedge \varphi) + \pi_{1+7}(d(X \lrcorner \varphi))
\]

where \( \pi_{1+7} \) denotes orthogonal projection onto \( \Omega^0_1 \oplus \Omega^2_1 \). This is a first order linear differential operator. Using a particular \( G_2 \)-equivariant isomorphism that identifies the codomain of \( \tilde{\mathcal{D}} \) with \( \Omega^0_1 \oplus \Omega^2_1 \), we can compare this operator \( \tilde{\mathcal{D}} \) with the usual Dirac operator \( \mathcal{D} \) from Definition 2.10. The result is summarized in the following proposition.

**Proposition 2.13.** The “modified Dirac operator” \( \tilde{\mathcal{D}} \) of equation (20), when considered as a linear operator on \( \Omega^0_1 \oplus \Omega^1_1 \) via the \( G_2 \)-equivariant isomorphism

\[
\Omega^0_1 \oplus \Omega^1_1 \cong \Omega^3_1 \oplus \Omega^1_1
\]

\[
(f, X) \leftrightarrow \left( -\frac{3}{7} f \varphi, \frac{1}{2} \ast (X \wedge \varphi) \right)
\]

is the usual Dirac operator

\[
\mathcal{D} : (f, X) \mapsto (d^* X, df + \text{curl } X).
\]

Hence the operator \( \tilde{\mathcal{D}} : \Omega^0_1 \oplus \Omega^1_1 \rightarrow \Omega^3_1 \oplus \Omega^1_1 \) is essentially the same as \( \mathcal{D} : \Omega^0_1 \oplus \Omega^1_1 \rightarrow \Omega^0_1 \oplus \Omega^1_1 \) and is in particular elliptic.

**Proof.** We first compute \( \pi_1 \) and \( \pi_7 \) of \( d(X \lrcorner \varphi) \). We have

\[
\pi_1(d(X \lrcorner \varphi)) = h \varphi \quad \text{for some } h \in \Omega^0_1.
\]

Using the fact that \( \Omega^3_2 \oplus \Omega^2_7 \) lies in the kernel of wedge product with \( \psi \), we compute

\[
d((X \lrcorner \varphi) \wedge \psi) = d(X \lrcorner \varphi) \wedge \psi = \pi_1(d(X \lrcorner \varphi)) \wedge \psi = h \varphi \wedge \psi = 7 h \text{vol}.
\]

Hence, we find that

\[
d(3 \ast X) = d((X \lrcorner \varphi) \wedge \psi) = 7 h \text{vol},
\]

and thus \( h = \frac{1}{7} \ast d(*X) = -\frac{3}{7} d^* X \). Similarly, we have

\[
\pi_7(d(X \lrcorner \varphi)) = \ast(Y \wedge \varphi) \quad \text{for some } Y \in \Omega^1_7.
\]

Using the fact that \( \Omega^3_2 \oplus \Omega^2_7 \) lies in the kernel of wedge product with \( \varphi \), we compute

\[
d((X \lrcorner \varphi) \wedge \varphi) = d(X \lrcorner \varphi) \wedge \varphi = \pi_7(d(X \lrcorner \varphi)) \wedge \varphi = \ast(Y \wedge \varphi) \wedge \varphi = -4 \ast Y.
\]

Hence, we find that

\[
-4 \ast Y = d((X \lrcorner \varphi) \wedge \varphi) = d(-2 \ast (X \lrcorner \varphi)) = -2 d(X \wedge \psi) = -2 d(X) \wedge \psi.
\]
and thus \( Y = \frac{1}{2} \ast ((dX) \wedge \psi) = \frac{1}{2} \text{curl} X \). Thus we have shown that
\[
\tilde{\mathcal{D}}(f, X) = \frac{1}{2} \ast (df \wedge \varphi) - \frac{3}{7} (d^* X) \varphi + \frac{1}{2} \ast (\text{curl} X \wedge \varphi)
\]
\[
= \left( -\frac{3}{7} (d^* X) \varphi, \frac{1}{2} \ast ((df + \text{curl} X) \wedge \varphi) \right) \subseteq \Omega^3_1 + \Omega^3_7,
\]
which is what we wanted to show.

**Corollary 2.14.** Suppose that \( s = (f, X) \) lies in the kernel of \( \tilde{\mathcal{D}} \) or \( \tilde{\mathcal{D}}^* \). Then \( \Delta f = 0 \) and \( \Delta X = 0 \).

**Proof.** This is immediate from Proposition 2.13, \( \tilde{\mathcal{D}}^* = \mathcal{D} \) and \( \tilde{\mathcal{D}}^2 (f, X) = (\Delta f, \Delta X) \).

**Corollary 2.15.** Let \( \mu = X \llcorner \varphi = *(X \wedge \psi) \in \Omega^3_7 \). Then \( \pi_1(d\mu) = 0 \) if and only if \( d^* X = 0 \); and \( \pi_7(d\mu) = 0 \) if and only if \( \text{curl} X = 0 \).

**Proof.** In the proof of Proposition 2.13, we showed \( \pi_1(d\mu) = -\frac{3}{7} d^* X \) and \( \pi_7(d\mu) = \frac{1}{2} \ast (\text{curl} X \wedge \varphi) \). The result follows since wedge product with \( \varphi \) is injective on 1-forms.

## 3 G\(_2\) conifolds

In this section we reproduce material from [20] on G\(_2\) cones, and asymptotically conical (AC) and conically singular (CS) G\(_2\) manifolds that will be needed in the present paper. Any results stated in this section without proof can be found in [20, Section 2]. Proofs are given for any results in this section that are not from [20].

### 3.1 G\(_2\) cones

Let \( \Sigma^6 \) be a compact, connected, smooth 6-manifold. An SU(3) structure on \( \Sigma \) is described by a Riemannian metric \( g_\Sigma \), an almost complex structure \( J \) which is orthogonal with respect to \( g_\Sigma \), the associated 2-form \( \omega(u, v) = g_\Sigma(Ju, v) \) which is real and of type (1,1) with respect to \( J \), and a nowhere vanishing complex (3,0)-form \( \Omega \). The two forms are related by the normalization condition
\[
\text{vol}_\Sigma = \frac{1}{6} \omega^3 = \frac{i}{8} \Omega \wedge \bar{\Omega} = \frac{1}{4} \text{Re}(\Omega) \wedge \text{Im}(\Omega).
\]

A manifold \( \Sigma^6 \) with SU(3) structure is called strictly nearly Kähler if the following equations are satisfied:
\[
d\omega = -3 \text{Re}(\Omega), \quad d\text{Im}(\Omega) = 2 \omega^2. \tag{23}
\]

Such manifolds are also called Gray manifolds. The Riemannian metric of a Gray manifold is always Einstein with positive Einstein constant [47].

**Definition 3.1.** Let \( \Sigma^6 \) be nearly Kähler. Then there exists a torsion-free G\(_2\) structure \((\varphi_c, \psi_c, g_c)\) on \( C = (0, \infty) \times \Sigma \) defined by
\[
\varphi_c = r^3 \text{Re}(\Omega) - r^2 dr \wedge \omega,
\]
\[
\psi_c = -r^3 dr \wedge \text{Im}(\Omega) - r^4 \frac{\omega^2}{2},
\]
\[
g_c = dr^2 + r^2 g_\Sigma.
\]
where \( r \) is the coordinate on \((0, \infty)\). The space \( C \) is a \( G_2 \) cone, and \( \Sigma \) is called the link of the cone. We choose the orientation on \( C \) so that \( \text{vol}_C = r^6 dr \wedge \text{vol}_C \) is the volume form on \( C \).

It is known for a Riemannian cone \( C \) with holonomy contained in \( G_2 \), that the holonomy is either trivial, in which case \( \Sigma \) is the standard round sphere \( S^6 \) and \( C \) is the Euclidean \( \mathbb{R}^7 \), or else the holonomy is exactly equal to \( G_2 \), in which case the link \( \Sigma \) is nearly Kähler, but not equal to \( S^6 \). (See Bär [3] for more details.) We reiterate that for us, a \( G_2 \) cone will always have holonomy exactly \( G_2 \), thus we will exclude the case where the link is \( S^6 \).

There are three known compact nearly Kähler manifolds (other than the round \( S^6 \)), and thus three known \( G_2 \) cones. These are discussed in detail in Bär [3], but we summarize them here. They are all obtained by taking the bi-invariant metric on a compact Lie group \( G \) and descending this to the normal metric on \( G/H \) for an appropriate Lie subgroup \( H \). In particular, all these examples are homogeneous spaces, and there is a proof by Butruille [7] that these examples are the only homogeneous nearly Kähler compact 6-manifolds. In [47, 48] Podestà–Spiro obtain some classification results about compact examples of cohomogeneity one. The three known examples (as smooth manifolds) are: \( \mathbb{CP}^3 \cong \text{Sp}(2)/(\text{Sp}(1) \times \text{U}(1)) \), the flag manifold \( F_{1,2} \cong \text{SU}(3)/T^2 \), and \( S^3 \times S^3 \cong (S^3 \times S^3 \times S^3)/S^3 \) where we embed \( S^3 \) into \( S^3 \times S^3 \times S^3 \) as the diagonal subgroup. We note for later use some of the Betti numbers for these examples:

\[
\begin{align*}
&b_2^{\mathbb{CP}^3} = 1, & b_3^{\mathbb{CP}^3} = 0, \\
&b_2^2(F_{1,2}) = 2, & b_3^2(F_{1,2}) = 0, \\
&b_2^3(S^3 \times S^3) = 0, & b_3^3(S^3 \times S^3) = 2.
\end{align*}
\]  

(24)

For any \( t > 0 \), we have a dilation map \( t : C \to C \) defined by

\[
t(0) = 0, \quad t(r, \sigma) = (tr, \sigma).
\]

It is easy to see that

\[
\begin{align*}
&t^* (\varphi_c) = t^3 \varphi_c, & t^* (\psi_c) = t^4 \psi_c, \\
&t^* (g_c) = t^2 g_c, & t^* (\text{vol}_c) = t^7 \text{vol}_c,
\end{align*}
\]

and hence we say that the conical \( G_2 \) structure is dilation-equivariant. Since \( g_c(r, \sigma) = t^{-2} g_c(tr, \sigma) \), we have

\[
|t^* (\gamma)(r, \sigma)|_{g_c(tr, \sigma)} = t^k |\gamma(tr, \sigma)|_{g_c(tr, \sigma)}
\]

whenever \( \gamma \) is a contravariant tensor of degree \( k \).

Let \( \alpha \) be a \((k - 1)\)-form on \( \Sigma \) and \( \beta \) be a \( k \)-form on \( \Sigma \). Then we have

\[
|r^{k-1} dr \wedge \alpha + r^k \beta|_{g_c}^2 = |\alpha|_{g_Z}^2 + |\beta|_{g_Z}^2.
\]

For this reason, we will always write a \( k \)-form on \( C \) as

\[
\gamma = r^{k-1} dr \wedge \alpha + r^k \beta
\]

for some \( \alpha \) and \( \beta \), which are forms on \( \Sigma \) possibly depending on the parameter \( r \). Note that if \( \alpha \) and \( \beta \) were independent of \( r \), then \( \gamma \) would be dilation-equivariant, as defined above.

**Definition 3.2.** We say that a smooth \( k \)-form \( \gamma \) on \( C \) is homogeneous of order \( \lambda \) if

\[
\gamma = r^\lambda (r^{k-1} dr \wedge \alpha + r^k \beta)
\]

where \( \alpha \) and \( \beta \) are forms on \( \Sigma \), independent of \( r \). Then we see that

\[
|\gamma(tr, \sigma)|_{g_c(tr, \sigma)} = |t^{\lambda+k} \gamma(r, \sigma)|_{g_c(tr, \sigma)} = t^{\lambda+k} |\gamma(r, \sigma)|_{g_c(r, \sigma)}.
\]
which we can write more concisely as
\[ t^*|\gamma|_{gc} = t^*|\gamma|_{gc}, \]
so the function $|\gamma|_{gc}$ on $C$ is homogeneous of order $\lambda$ in the variable $r$ in the usual sense.

Let $\ast$, $\nabla_C$, $d_C$, $d_C^\ast$, and $\Delta_C$ denote the Hodge star, Levi-Civita connection, exterior derivative, coderivative, and Hodge Laplacian on $\Sigma$, respectively. Similarly $\ast_c$, $\nabla_c$, $d_c$, $d_c^\ast$, and $\Delta_c$ will denote the corresponding operators on the cone $C$.

For a homogenous $k$-form $\gamma = r^\lambda (r^{k-1}dr \wedge \alpha + r^k\beta)$ of order $\lambda$, it is trivial to calculate that:
\[ d_C^\ast \gamma = r^{\lambda+k-1}dr \wedge ((\lambda + k)\beta - d_C^\ast \alpha) + r^{\lambda+k}d_C^\ast \beta, \tag{25} \]
\[ d_C^\ast \gamma = r^{\lambda+k-3}dr \wedge (-d_C^\ast \alpha) + r^{\lambda+k-2}(-\lambda - k + 7)\alpha + d_C^\ast \beta, \]
\[ \Delta_C \gamma = r^{\lambda+k-3}dr \wedge (\Delta_C \alpha - (\lambda + k - 2)(\lambda - k + 7)\alpha - 2d_C^\ast \beta) \]
\[ + r^{\lambda+k-2}(-\lambda - k + 5)\beta - 2d_C^\ast \alpha). \tag{26} \]

The next lemma is a special case of [20, Lemma 2.12].

**Lemma 3.3.** Let $\gamma$ be a smooth closed 3-form on $C$. Suppose that either
\begin{align*}
  i) & \quad |\gamma|_{gc} = O(r^\lambda) \text{ on } (0, \varepsilon) \times \Sigma, \text{ for } \lambda > -3 \text{ or} \\
  ii) & \quad |\gamma|_{gc} = O(r^\lambda) \text{ on } (R, \infty) \times \Sigma, \text{ for } \lambda < -3.
\end{align*}
for some small $\varepsilon$ or some large $R$. Then for each case respectively we have that
\begin{align*}
  i) & \quad \gamma = d\zeta \text{ for some 2-form } \zeta \text{ on } (0, \varepsilon) \times \Sigma, \text{ or} \\
  ii) & \quad \gamma = d\zeta \text{ for some 2-form } \zeta \text{ on } (R, \infty) \times \Sigma.
\end{align*}

We will need to consider the possible order $\lambda$ of a homogeneous $k$-form $\gamma_k$ on a cone $C$ which is in the kernel of $\Delta_C$, or of a mixed degree form $\gamma = \sum_{k=0}^7 \gamma_k$ which is in the kernel of $d_C + d_C^\ast$.

**Proposition 3.4.** Let $\gamma$ be a homogeneous $k$-form of order $\lambda$ which is harmonic on the cone: $\Delta_C \gamma = 0$. Then we have:
\begin{align*}
  \text{For } k = 0, 7, & \quad \gamma = 0 \text{ if } \lambda \in (-5, 0), \tag{27} \\
  \text{For } k = 1, 6, & \quad \gamma = 0 \text{ if } \lambda \in (-4, -1), \tag{28} \\
  \text{For } k = 2, 5, & \quad \gamma = 0 \text{ if } \lambda \in (-3, -2). \tag{29}
\end{align*}

**Proposition 3.5.** Suppose that $\gamma$ is a homogeneous $k$-form of order $\lambda$ which is closed and coclosed: $d_C \gamma = 0$ and $d_C^\ast \gamma = 0$. Then we have:
\begin{align*}
  \text{For } k = 0, 7, & \quad \gamma = 0 \text{ if } \lambda \in (-7, 0), \tag{30} \\
  \text{For } k = 1, 6, & \quad \gamma = 0 \text{ if } \lambda \in (-6, -1), \tag{31} \\
  \text{For } k = 2, 5, & \quad \gamma = 0 \text{ if } \lambda \in (-5, -2), \tag{32} \\
  \text{For } k = 3, 4, & \quad \gamma = 0 \text{ if } \lambda \in (-4, -3). \tag{33}
\end{align*}
Propositions 3.4 and 3.5 use only the fact that we have a 7-dimensional Riemannian cone. However, the fact that the link $\Sigma^6$ is a compact Einstein manifold of positive scalar curvature allows us to slightly extend the results of these propositions, for functions and 1-forms. This is the content of the next result and the remark that follows. These extended ranges of excluded rates for harmonic functions and 1-forms are used, for example, in Theorem 5.3 to establish that theorem in the AC case all the way to $\nu \leq -1$, rather than just $\nu < -2$.

**Proposition 3.6.** Let $f$ be a harmonic function on a $G_2$ cone, homogeneous of order $\lambda$. Then

$$f = \begin{cases} 0 & \text{if } \lambda \in [-6, 1] \setminus \{-5, 0\}, \\ C & \text{if } \lambda = 0, \\ Cr^{-5} & \text{if } \lambda = -5, \end{cases}$$

where $C$ is a constant.

Let $\omega$ be a harmonic 1-form on a $G_2$ cone, homogeneous of order $\lambda$. Then

$$\omega = 0, \quad \text{if } \lambda \in [-5, 0].$$

**Proof.** A theorem of Obata [42] states that on a compact Einstein 6-manifold $\Sigma$ with positive scalar curvature $R$, the first nonzero eigenvalue of the Laplacian on functions is not less than $\frac{R}{6}$, with equality if and only if $\Sigma$ is isometric to the round $S^6$. The Einstein metric on the link $\Sigma$ of our $G_2$ cones has been scaled so that $R = 30$ (see [36, 37]) and we always exclude the case of $\Sigma = S^6$, so if $\Delta_c h = \mu h$ for $h \in C^\infty(\Sigma)$, then we must have $\mu > 6$. We will use this result repeatedly in what follows.

As in Definition 3.2, we can write $f = r^\lambda \beta$ for some function $\beta$ on $\Sigma$, independent of $r$. From (26) we have $\Delta_c f = 0$ if and only if $\Delta_c \beta = \lambda(\lambda + 5) \beta$. Applying the result of Obata we can say that if $f$ is nonzero and $\mu = \lambda(\lambda + 5) \neq 0$, then $\mu > 6$. Thus if $\lambda \neq -5, 0$, then for $f$ to be nonzero we must have $\lambda(\lambda + 5) > 6$ which easily implies that $\lambda > 1$ or $\lambda < -6$. This proves the result for functions.

Now consider a 1-form $\omega = r^\lambda (dr \wedge \alpha + r \beta)$, where $\alpha$ is a function on $\Sigma$ and $\beta$ is a 1-form on $\Sigma$. From (26) we find that $\Delta_c \omega = 0$ if and only if

$$\Delta_c \alpha = (\lambda - 1)(\lambda + 6) \alpha + 2d_\Sigma^\ast \beta, \quad (34)$$

$$\Delta_c \beta = (\lambda + 1)(\lambda + 4) \beta + 2d_\Sigma \alpha. \quad (35)$$

Using (34) and (35), an easy computation yields

$$\Delta_c (d_\Sigma^\ast \beta + (\lambda - 1) \alpha) = (\lambda + 1)(\lambda + 6)(d_\Sigma^\ast \beta + (\lambda - 1) \alpha).$$

Thus, for the function $h = d_\Sigma^\ast \beta + (\lambda - 1) \alpha$, we have $\Delta_c h = (\lambda + 1)(\lambda + 6) h$. So if $\lambda \neq -6, -1$, then the Obata result says that $h = 0$ if $(\lambda + 1)(\lambda + 6) \leq 6$, which corresponds to $\lambda \in [-7, 0] \setminus \{-6, -1\}$.

Suppose then, that $\lambda \in [-7, 0] \setminus \{-6, -1\}$. We have $d_\Sigma^\ast \beta = -(\lambda - 1) \alpha$, and equation (34) becomes

$$\Delta_c \alpha = (\lambda - 7)(\lambda + 4) \alpha.$$

Hence once again, we can say that $\alpha$ is zero if $\lambda \neq -4, 1$ and $(\lambda - 1)(\lambda + 4) \leq 6$, which corresponds to $\lambda \in [-5, 2] \setminus \{-4, 1\}$. But since our initial assumption was $\lambda \in [-7, 0] \setminus \{-6, -1\}$, we conclude that $\alpha = 0$ if $\lambda \in [-5, 0] \setminus \{-4, -1\}$, and that $\alpha$ is a constant if $\lambda = -4$.

Therefore, if $\lambda \in [-5, 0] \setminus \{-1\}$, then equation (35) becomes

$$\Delta_c \beta = (\lambda + 1)(\lambda + 4) \beta,$$
and \((\lambda + 1)(\lambda + 4) \leq 4\). But the Bochner formula gives
\[
(\Delta \beta, \beta) = \langle \nabla^*_E \nabla^*_E \beta, \beta \rangle + \text{Ric}_{E}(\beta, \beta) = \langle \nabla^*_E \nabla^*_E \beta, \beta \rangle + 5|\beta|^2,
\]
since \(\text{Ric}_E = \frac{R}{6} g_E = 5 g_E\). Integrating the above equation over \(\Sigma\), we find that if \(\Delta \beta = c \beta\), then we must have \(c \geq 5\) for nonzero \(\beta\). Thus in all cases of \(\lambda \in [-5, 0] \setminus \{-1\}\), we have \(\beta = 0\). We have already shown that for these rates, \(\alpha = 0\) except when \(\lambda = -4\), in which case \(\alpha = C\) is a constant. But substituting this information back into equation (34) gives \(C = 0\) in this case as well. Thus we have shown that \(\omega = 0\) for \(\lambda \in [-5, 0] \setminus \{-1\}\).

All that remains is to consider the case \(\lambda = -1\). In this case (35) says that \(\Delta \beta = 2d_E \alpha\) is exact, so by the Hodge theorem, \(\beta = \beta_H + G_E \Delta_E(\beta) = \beta_H + d_E(2G_E \alpha)\) is closed, where \(\beta_H\) is the harmonic part of \(\beta\) and \(G_E\) is the Green’s operator for \(\Delta_E\). Using the fact that \(d_E \beta = 0\) in this case, taking \(d_E\) of equation (34), using equation (35), and substituting \(\lambda = -1\), gives
\[
\Delta_E d_E \alpha = -10 d_E \alpha + 2 \Delta \beta = -10 d_E \alpha + 2d_E \alpha = -8d_E \alpha.
\]
Hence we have \(d_E \alpha = 0\). Now equation (35) says that \(\beta\) is harmonic, hence \(\beta = 0\) again by Myers’ Theorem. Now (34) says \(\Delta_E \alpha = -10 \alpha\), so \(\alpha = 0\) as well.

**Proposition 3.7.** Let \(\omega\) be a 1-form on a \(G_2\) cone, homogeneous of order \(\lambda\), satisfying
\[
d_E d_E^* \omega + \frac{2}{3} d_E^* d_E \omega = 0.
\]

Then \(\omega = 0\), if \(\lambda \in (-4, 0]\).

**Proof.** The proof is entirely analogous to the proof of Proposition [3.6]. First, it is easy to check that if \(d_E d_E^* \omega + \frac{2}{3} d_E^* d_E \omega = 0\), where \(\omega = r^\lambda (dr \wedge \alpha + r \beta)\), then
\[
\frac{2}{3} \Delta_E \alpha = (\lambda - 1)(\lambda + 6)\alpha - \frac{1}{3} (\lambda - 5) d_E^* \beta, \tag{36}
\]
\[
\Delta_E \beta = \frac{2}{3} (\lambda + 1)(\lambda + 4) \beta + \frac{1}{3} (\lambda + 10) d_E \alpha. \tag{37}
\]

Using (36) and (37), an easy computation yields
\[
\Delta_E (d_E^* \beta - (\lambda + 6) \alpha) = (\lambda - 1)(\lambda + 4)(d_E^* \beta - (\lambda + 6) \alpha).
\]

Thus, for the function \(h = d_E^* \beta - (\lambda + 6) \alpha\), we have \(\Delta_E h = (\lambda - 1)(\lambda + 4) h\). So if \(\lambda \neq -4, 1\), then the Obata result says that \(h = 0\) if \((\lambda - 1)(\lambda + 4) \leq 6\), which corresponds to \(\lambda \in [-5, 2] \setminus \{-4, 1\}\). Thus, for this range of rates we have \(d_E^* \beta = (\lambda + 6) \alpha\). Substituting this into (36) gives
\[
\Delta_E \alpha = (\lambda + 1)(\lambda + 6) \alpha.
\]

Thus we conclude that \(\alpha = 0\) if \(\lambda \notin \{-6, -1\}\) and \((\lambda + 1)(\lambda + 6) \leq 6\), which corresponds to \(\lambda \in [-7, 0] \setminus \{-6, -1\}\). If \(\lambda = -1\), then \(\alpha\) is harmonic, and \(d_E^* \beta = (\lambda + 6) \alpha = 5 \alpha\) says that \(\alpha\) is coexact, hence it is again zero. Hence, at least if \(\lambda \in (-6, 0]\) then \(\alpha = 0\), and substituting into (37) yields
\[
\Delta_E \beta = \frac{2}{3} (\lambda + 1)(\lambda + 4) \beta.
\]

Therefore if \(\lambda \in (-4, 0]\) then \(\Delta_E \beta = c \beta\) for \(c \leq \frac{2}{3} < \frac{4}{3}\). We now use the Bochner formula argument in the proof of Proposition [3.6] to conclude that \(\beta = 0\) as well. \(\Box\)
Remark 3.8. We clearly did not present the most general result possible in Proposition 3.7. The statement given will be sufficient for our purposes. This result is used in the proof of Theorem 5.3 because by Proposition 2.7, the operator \(dd^* + \frac{2}{3}d^*d\), is essentially \(\pi_7 d^*d : \Omega^2_7 \rightarrow \Omega^2_7\), which is the operator that is related to our gauge-fixing condition.

The next lemma is similar to [20, Proposition 2.22]. It is more general in that it is valid for any rate \(\lambda\), but less general in that it is stated only for forms of pure degree. The method of proof, however, is different and follows the discussion immediately preceding [29, Proposition 5.6]. This result is needed to compute the dimension of the moduli space in Section 5.2.4 because the dimension is computed using Theorem 4.20, and the spaces \(K(\lambda)\) a priori could involve log terms.

Lemma 3.9. Let \(m \geq 0\), and let \(\gamma = \sum_{l=0}^{m} (\log r)^l \gamma_l\) be a \(k\)-form in the kernel of \(d + d^*_C\), where each \(\gamma_l\) is homogeneous of order \(\lambda\), and \(\gamma_m \neq 0\). Then necessarily \(m = 0\). That is, \(\gamma = \gamma_0\) has no log terms.

Proof. Each \(\gamma_l\) is homogeneous of order \(\lambda\), so it can be written as

\[
\gamma_l = r^{k-1+\lambda}dr \wedge \alpha_l + r^{k+\lambda}\beta_l
\]

where \(\alpha_l\) and \(\beta_l\) are \((k-1)\)-forms and \(k\)-forms on \(\Sigma\), respectively, independent of \(r\). For any \(k\)-form \(\gamma_l\) on \(C\), it is easy to check that

\[
(d + d^*_C)((\log r)^l \gamma_l) = (\log r)^l(d_c + d^*_c)\gamma_l + \frac{l}{r} (\log r)^l-1 (dr \wedge \gamma_l) - \frac{l}{r} (\log r)^l-1 \left( \frac{\partial}{\partial r} \gamma_l \right).
\]

Using this identity, we see that

\[
(d + d^*_C) \left( \sum_{l=0}^{m} (\log r)^l \gamma_l \right) = (\log r)^m (d_c + d^*_c) \gamma_m
\]

\[
+ \sum_{l=0}^{m-1} (\log r)^l \left( (d + d^*_C)\gamma_l + \frac{l+1}{r} dr \wedge \gamma_{l+1} - \frac{l+1}{r} \left( \frac{\partial}{\partial r} \gamma_{l+1} \right) \right).
\]

The above expression must vanish as a polynomial in \(\log r\). Setting the coefficient of \((\log r)^m\) equal to zero, and decomposing into forms of pure degree, we obtain \(d\gamma_m = 0\) and \(d^*_C \gamma_m = 0\), which from (38) can be simplified to

\[
d\alpha_m = (\lambda + k) \beta_m, \quad d^*_C \alpha_m = 0, \\
d\beta_m = 0, \quad d^*_C \beta_m = (\lambda + 7 - k) \alpha_m.
\]

Similarly the coefficient of \((\log r)^{m-1}\) gives \(d\gamma_{m-1} + \frac{2}{3} dr \wedge \gamma_m = 0\) and \(d^*_C \gamma_{m-1} - \frac{2}{3} \left( \frac{\partial}{\partial x} \right) \gamma_m = 0\), which simplify to

\[
d\alpha_{m-1} = (\lambda + k) \beta_{m-1} + m \beta_m, \quad d^*_C \alpha_{m-1} = 0, \\
d\beta_{m-1} = 0, \quad d^*_C \beta_{m-1} = (\lambda + 7 - k) \alpha_{m-1} + m \alpha_m.
\]
Using the systems of equations (39) and (40) on $\Sigma$ and taking $L^2$ inner products, we find
\[
m||\alpha_m||^2 + m||\beta_m||^2 = \langle\langle ma_m, \alpha_m\rangle\rangle + \langle\langle m\beta_m, \beta_m\rangle\rangle = \langle\langle d^c_*\beta_{m-1} - (\lambda + 7 - k)\alpha_{m-1}, \alpha_m\rangle\rangle + \langle\langle da_{m-1} - (\lambda + k)\beta_{m-1}, \beta_m\rangle\rangle = \langle\langle \beta_{m-1}, da_m\rangle\rangle - (\lambda + 7 - k)\langle\langle \alpha_{m-1}, \alpha_m\rangle\rangle + \langle\langle \alpha_{m-1}, d^c_*\beta_m\rangle\rangle - (\lambda + k)\langle\langle \beta_{m-1}, \beta_m\rangle\rangle = 0.
\]
Since $\gamma_m \neq 0$, we conclude that $m = 0$. \hfill \Box

**Remark 3.10.** A generalization of Lemma 3.9 to mixed degree forms is possible, using the same techniques, and in any dimension. We do not state it because we will not have occasion to use it. This means in particular, that in the published version of [20], the last sentence in the proof of Proposition 2.23] is incorrect. That is, there are never any “log terms” for the operator $d + d^c_\ast$.

The next proposition is useful for analyzing the critical rates of the operator $d + d^c_\ast$ in Section 4.2.

**Proposition 3.11.** Let $\gamma = \sum_{k=0}^7 \gamma_k$ be a mixed degree form on the cone, homogeneous of order $\lambda$, and suppose that $(d + d^c_\ast)\gamma = 0$.

- If $\lambda = -3$, then $\gamma = \beta + dr \wedge \alpha$, where $\beta$ and $\alpha$ are both harmonic 3-forms on $\Sigma$.
- If $\lambda = -4$, then $\gamma = r^{-2}dr \wedge \alpha + \beta + (r^{-2}\sigma - r^{-1}dr \wedge d\sigma) + (dr \wedge \mu - r^{-1}d^c_\ast\mu)$, where $\alpha$ is a harmonic 2-form on $\Sigma$, $\beta$ is a harmonic 4-form on $\Sigma$, $\sigma$ is a coexact 2-form on $\Sigma$ satisfying $\Delta_\Sigma \sigma = 2\sigma$, and $\mu$ is an exact 4-form on $\Sigma$ satisfying $\Delta_\Sigma \mu = 2\mu$.
- If $\lambda = -2$, then $\gamma = \alpha + r^2dr \wedge \beta + (dr \wedge \sigma + r d\sigma) + (rdr \wedge d^c_\ast\mu + r^2\mu)$, where $\alpha$ is a harmonic 2-form on $\Sigma$, $\beta$ is a harmonic 4-form on $\Sigma$, $\sigma$ is a coexact 2-form on $\Sigma$ satisfying $\Delta_\Sigma \sigma = 2\sigma$, and $\mu$ is an exact 4-form on $\Sigma$ satisfying $\Delta_\Sigma \mu = 2\mu$.

**Proof.** The even-degree case of the first statement is exactly Proposition 2.21 in [20]. The odd-degree case, and the second and third statements, are proved in essentially the same way. \hfill \Box

**Remark 3.12.** Proposition 3.11 is used several times in Section 4.3 to explicitly describe the change in the space of closed and coclosed 3-forms on a $G_2$ conifold at rates $-3$ and $-4$. In fact we will mainly need this proposition for 3-forms. If $\gamma$ is a closed and coclosed 3-form on $C$, homogeneous of order $\lambda$, then Proposition 3.11 says that: when $\lambda = -3$, then $\gamma = \beta$ is a harmonic 3-form on $\Sigma$ and when $\lambda = -4$, then $\gamma = r^{-2}dr \wedge \alpha$ where $\alpha$ is a harmonic 2-form on $\Sigma$, because in this case $\mu = 0$ implies $d^c_\ast\mu = 0$.

Finally, we need to consider the excluded range of orders of homogeneity for elements in the kernel of the modified Dirac operator $\tilde{D}_C$ defined in equation (20). This result does not appear in [20], so we include the proof.

**Proposition 3.13.** Let $\tilde{D}_C : \Omega^0_1 \oplus \Omega^1_1 \rightarrow \Omega^3_1 \oplus \Omega^2_2$ be the modified Dirac operator on a $G_2$ cone. Let $s = (f, X)$ in the domain of $\tilde{D}_C$ be homogeneous of order $\lambda$ such that $\tilde{D}_C(s) = 0$. Then we have:
\[
s = 0 \text{ if } \lambda \in (-5, 0).
\]
Proof. Suppose that $\tilde{D}_c s = 0 \in \Omega^2_c \oplus \Omega^3_c$. Corollary 2.14 tells us that $\Delta_c f = 0$ and $\Delta_c X = 0$. The statement now follows immediately from Proposition 3.6.

Remark 3.14. We can possibly obtain a larger excluded range of orders of homogeneity for elements of the kernel of the modified Dirac operator, by exploiting the second half of equation (22), namely that $df = \frac{1}{2} \text{curl} X$. However, for our purposes the result of Proposition 3.13 will be good enough.

3.2 Asymptotically conical (AC) $G_2$ manifolds

Let $M$ be a noncompact, connected smooth 7-dimensional manifold.

Definition 3.15. The manifold $M$ is called an asymptotically conical $G_2$ manifold with cone $C$ and rate $\nu < 0$ if all of the following holds:

- The manifold $M$ is a $G_2$ manifold with torsion-free $G_2$ structure $\varphi_M$ and metric $g_M$.
- There is a $G_2$ cone $(C, \varphi_C, g_C)$ with connected link $\Sigma$.
- There is a compact subset $L \subset M$, an $R > 1$, and a map $h : (R, \infty) \times \Sigma \to M$ that is a diffeomorphism of $(R, \infty) \times \Sigma$ onto $M \setminus L$.
- The pullback $h^*(\varphi_M)$ is a torsion-free $G_2$ structure on the subset $(R, \infty) \times \Sigma$ of $C$. We require that this approach the torsion-free $G_2$ structure $\varphi_C$ in a $C^\infty$ sense, with rate $\nu < 0$. This means that
  \[ |\nabla^j_C (h^*(\varphi_M) - \varphi_C)|_{g_C} = O(r^{\nu-j}) \quad \forall j \geq 0 \]  
  in $(R, \infty) \times \Sigma$. Note that all norms and derivatives are computed using the cone metric $g_C$. It follows immediately from (42) and Taylor’s theorem that the metric on $M$ is asymptotic to the cone metric at the same rate:
  \[ |\nabla^j_C (h^*(g_M) - g_C)|_{g_C} = O(r^{\nu-j}) \quad \forall j \geq 0 \]

It is clear that an AC $G_2$ manifold of rate $\nu_0$ is also an AC $G_2$ manifold for all $\nu > \nu_0$.

Remark 3.16. The link $\Sigma$ of an AC $G_2$ manifold $M$ must be connected because $M$ can have only one end. This follows from the Cheeger–Gromoll splitting theorem, which says that a complete noncompact Ricci-flat manifold with more than one end isometrically splits into a Riemannian product, and thus the holonomy would be reducible.

Example 3.17. There are three known examples of asymptotically conical $G_2$ manifolds, whose asymptotic cones have links given by the three known nearly Kähler manifolds. They are all total spaces of vector bundles over a compact base. These manifolds were discovered by Bryant–Salamon [6] and were the first examples of complete $G_2$ manifolds. Specifically, they are described in the following list, where the metric on the base manifold is the one induced from the Bryant–Salamon metric by restriction.

- $\Lambda^2(S^4)$, the bundle of anti-self-dual 2-forms over the 4-sphere. This is a non-trivial rank 3 vector bundle over the standard round $S^4$. This AC $G_2$ manifold is asymptotic to the cone over the nearly Kähler $\mathbb{CP}^3$, with rate $\nu = -4$.
- $\Lambda^2(\mathbb{CP}^2)$, the bundle of anti-self-dual 2-forms over the complex projective plane. This is a non-trivial rank 3 vector bundle over the standard Fubini-Study $\mathbb{CP}^2$. This AC $G_2$ manifold is asymptotic to the cone over the nearly Kähler flag manifold $F_{1,2}$, also with rate $\nu = -4$.  

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\begin{itemize}
  \item $\mathcal{S}(S^3)$, the spinor bundle of the 3-sphere. This is a trivial rank 4 vector bundle over the standard round $S^3$, hence is topologically $S^3 \times \mathbb{R}^4$. This AC $G_2$ manifold is asymptotic to the cone over the nearly Kähler $S^3 \times S^3$, with rate $\nu = -3$.

\end{itemize}

**Remark 3.18.** Explicit formulas for these asymptotically conical $G_2$ structures, as well as the fact that their rates are $-4$, $-4$, and $-3$, respectively, can be found in Bryant–Salamon [6], and also in Atiyah–Witten [2]. We will not have need for these explicit formulas.

### 3.3 Conically singular (CS) $G_2$ manifolds

Let $\overline{M}$ be a compact, connected topological space, and let $x_1, \ldots, x_n$ be a finite set of isolated points in $\overline{M}$. Assume that $M = \overline{M}\setminus\{x_1, \ldots, x_n\}$ is a smooth noncompact 7-dimensional manifold that we will call the \textit{smooth part} of $\overline{M}$ and $\{x_1, \ldots, x_n\}$ will be called the \textit{singular points} of $\overline{M}$.

**Definition 3.19.** The space $\overline{M}$ is called a $G_2$ manifold with isolated conical singularities, with cones $C_1, \ldots, C_n$ at $x_1, \ldots, x_n$ and rates $\nu_1, \ldots, \nu_n$, where each $\nu_i > 0$, if all of the following holds:

\begin{itemize}
  \item The smooth part $M$ is a $G_2$ manifold with torsion-free $G_2$ structure $\varphi_M$ and metric $g_M$.
  \item There are $G_2$ cones $(C_i, \varphi_{C_i}, g_{C_i})$ with links $\Sigma_i$ for $i = 1, \ldots, n$.
  \item There is a compact subset $K \subset M$ such that $M \setminus K$ is a union of open sets $S_1, \ldots, S_n$ whose closures $\overline{S}_1, \ldots, \overline{S}_n$ in $\overline{M}$ are all disjoint in $\overline{M}$. There is an $\varepsilon \in (0, 1)$, and for each $i = 1, \ldots, n$, there is a map $h_i: (0, \varepsilon) \times \Sigma_i \to M$ that is a diffeomorphism of $(0, \varepsilon) \times \Sigma_i$ onto $S_i$.
  \item The pullback $h_i^*(\varphi_M)$ is a torsion-free $G_2$ structure on the subset $(0, \varepsilon) \times \Sigma_i$ of $C_i$. We require that this approach the torsion-free $G_2$ structure $\varphi_{C_i}$ in a $C^\infty$ sense, with rate $\nu_i > 0$. This means that

$$|\nabla^2 h_i^*(\varphi_M) - \varphi_{C_i}|_{g_{C_i}} = O(r^{\nu_i-1}) \quad \forall j \geq 0$$

in $(0, \varepsilon) \times \Sigma_i$. Note that all norms and derivatives are computed using the cone metric $g_{C_i}$.

It follows immediately from (43) and Taylor’s theorem that the metric on $M$ is asymptotic to the cone metric at the same rate:

$$|\nabla^2 h_i^*(g_M) - g_{C_i}|_{g_{C_i}} = O(r^{\nu_i-1}) \quad \forall j \geq 0$$

It is clear that a CS $G_2$ manifold of rate $\nu_0$ is also a CS $G_2$ manifold for all $\nu < \nu_0$. We also note that $\overline{M}$ is the closure of $M$ in $\overline{M}$. We will often abbreviate the phrase “compact $G_2$ manifold with isolated conical singularities” as conically singular or CS $G_2$ manifold.

There are at present still no examples of conically singular $G_2$ manifolds, although they are expected to exist in abundance. The main theorem in [20] can be interpreted as evidence for the likelihood of their existence, in the sense that they should arise as ‘boundary points’ in the moduli space of smooth compact $G_2$ manifolds. Moreover, the discussion in Section 6.5 of the present paper, which is a corollary of our main theorem, can also be interpreted as saying that CS $G_2$ manifolds should in fact make up a large part of the boundary of the moduli space of smooth compact $G_2$ manifolds. The first author, in collaboration with Dominic Joyce, has a new construction of compact $G_2$ manifolds [16] that may be generalizable to produce the first examples of compact $G_2$ manifolds with isolated conical singularities, which would all be modelled on the cone over the nearly Kähler $\mathbb{C}F^3$.  

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4 Analysis on $G_2$ conifolds

In this section we collect a plethora of analytic results, some general and some specific to $G_2$ conifolds. We begin in Sections 4.1 and 4.2 by summarizing the essential aspects of the Lockhart–McOwen theory for AC and CS manifolds that we will require. This theory originally appeared in Lockhart–McOwen [25] and Lockhart [26]. A very detailed exposition can also be found in Marshall [32]. Then in Section 4.3 we use this theory to establish Hodge-theoretic results for weighted Sobolev spaces of forms. These are all combined in Section 4.4 to establish a special index change theorem for an operator $D$ that plays the key role in our deformation theory. In Section 4.5 we consider some topological results on $G_2$ conifolds. These are important ingredients in computing the (virtual) dimension of the moduli spaces later. Finally, in Sections 4.6, 4.7, and 4.8 we discuss parallel tensors, gauge-fixing conditions, and other analytic results particular to $G_2$ conifolds that will be needed to prove our main theorem in Section 5.

4.1 Weighted Banach spaces on conifolds

The essential idea is as follows. By introducing appropriately weighted Banach spaces of sections of vector bundles over an AC manifold or over the smooth, noncompact part of a CS manifold, one generically obtains a nice Fredholm theory for an elliptic operator $P: V \to W$ such as the Laplacian or the Dirac operator. Basically, as long as we stay away from certain “critical rates”, which form a discrete set, these operators will be Fredholm and we can write $W = \text{im}(P) \oplus C$ for some finite-dimensional complement $C$ which is isomorphic to $\ker(P^*)$. The precise details are explained below.

We will mostly use this theory for weighted Sobolev spaces. However, it applies equally well to weighted Hölder spaces, and we will require the relations between these spaces (the Sobolev embedding theorems) in order to deal with some regularity issues, in particular to ensure that the sections are at least twice continuously differentiable.

Throughout this section, we use $M$ to denote a $G_2$ conifold, which is either an asymptotically conical (AC) $G_2$ manifold, as in Definition 3.15, or the smooth part of a conically singular (CS) $G_2$ manifold, as in Definition 3.19. Many, but not all, of the results are valid for any Riemannian conifold, although the results are always stated in the particular dimension 7 for convenience.

The analytic results for AC manifolds hold equally well for CS manifolds with minor differences. The most significant difference is that all the inequalities involving rates must be reversed, since the noncompact ends correspond to $r \to 0$ instead of $r \to \infty$. Also, on a CS manifold we can have $n$ ends as opposed to just one.

In order to be able to define sensible “weighted” Banach spaces on $M$, we need the concept of a radius function.

**Definition 4.1.** A radius function $\rho$ is a smooth function on $M$ satisfying the following conditions.

- **AC case:** On the compact subset $L$ of $M$, we define $\rho \equiv 1$. If $x = h(r, p)$ for some $r \in (2R, \infty)$ and $p \in \Sigma$, then set $\rho(x) = r$. In the region $h((R, 2R) \times \Sigma)$, the function $\rho$ is defined by interpolating smoothly between its definition near infinity and its definition in the compact subset $L$, in a decreasing fashion.

- **CS case:** On the compact subset $K = M \setminus \sqcup_{i=1}^{n} S_i$, we define $\rho \equiv 1$. If $x = h_i(r, p)$ for some $r \in (0, \frac{1}{2} \epsilon)$ and $p \in \Sigma_i$, then set $\rho(x) = r$. In the regions $h_i((\frac{1}{2} \epsilon, \epsilon) \times \Sigma_i)$, the function
$g$ is defined by interpolating smoothly between its definitions near the singularities and its definition in the compact subset $K$, in an increasing fashion.

We can now define the weighted Sobolev spaces on $M$. Let $E$ be a vector bundle over $M$ with a fibrewise metric. In all instances in this paper, $E$ will either be the bundle $\Lambda^k(T^*M)$ of $k$-forms on $M$, or the space of all forms $\Lambda^\bullet(T^*M) = \sum_{k=0}^{\infty} \Lambda^k(T^*M)$ on $M$, or the spinor bundle $S(M)$ over $M$, described in Section 2.2. The fibrewise metric in all these cases is naturally induced from the Riemannian metric on $M$, and the Levi-Civita connection $\nabla_M$ naturally induces a connection on $E$ which we continue to denote by $\nabla_M$.

We want to define the weighted Sobolev space of sections of $E$ with rate $\lambda$. In the AC case, we let $\lambda \in \mathbb{R}$. In the CS case, we let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. We can add such $n$-tuples and multiply them by real numbers using the vector space structure of $\mathbb{R}^n$. We also define $\lambda + j = (\lambda_1 + j_1, \ldots, \lambda_n + j_n)$ for any $j \in \mathbb{R}$, and we say that $\lambda > \lambda'$ if $\lambda_i > \lambda_i'$ for all $i = 1, \ldots, n$. Finally we define $\rho^\lambda$ to equal $\rho^{\lambda_i}$ on $h_i((0, \epsilon) \times \Sigma_i)$ and to equal 1 on the compact subset $K$. Then $\rho^\lambda$ is a smooth function on $M$ which equals $r^{\lambda_i}$ on the neighbourhood $h_i((0, \frac{1}{2}\epsilon) \times \Sigma_i)$ of the singular point $x_i$.

**Definition 4.2.** Let $l \geq 0$, $p > 1$, and let $\lambda$ be as above. We define the weighted Sobolev space $L^p_{l,\lambda}(E)$ of sections of $E$ over $M$ as follows. Consider the space $C^\infty(E)$ of smooth compactly supported sections of $E$. For such sections the quantity

$$||\gamma||_{L^p_{l,\lambda}} = \left(\sum_{j=0}^{l} \int_{M'} |\rho^{\lambda+j} \nabla_M^j \gamma|^p \rho^{-\lambda} \text{vol}_M \right)^{\frac{1}{p}}$$

is clearly finite, and is a norm. We define the Banach space $L^p_{l,\lambda}(E)$ to be the completion of $C^\infty(E)$ with respect to this norm.

**Remark 4.3.** We make a few important remarks about this definition.

(a) As a topological vector space, $L^p_{l,\lambda}(E)$ is independent of the choice of radius function $\rho$, and any two such choices lead to equivalent norms.

(b) We clearly have $L^p_{l,\lambda}(E) \subseteq L^p_{l,\lambda'}(E)$ if $\lambda < \lambda'$ in the AC case or if $\lambda > \lambda'$ in the CS case.

(c) An element $\gamma$ in $L^p_{l,\lambda}(E)$ can be intuitively thought of as a section of $E$ that is $l$ times weakly differentiable such that near each end, the tensor $\nabla_M^j \gamma$ is growing at most like $r^{\lambda-j}$.

(d) The space $L^2_{l,\lambda}(E)$ is a Hilbert space, with inner product coming from the polarization of the norm in (44). Because of the factor $\rho^{-\lambda}$ in (44), we have

$$L^2_{l,\lambda}(E) = L^2(E),$$

where $L^2(E)$ is the usual space of $L^2$ sections of $E$. Here and henceforth it is understood that in the CS case $\frac{7}{2}$ denotes the ‘constant’ $n$-tuple $(\frac{7}{2}, \ldots, \frac{7}{2})$.

**Remark 4.4.** We will almost always just take $p = 2$ in this paper. The only time we will need to consider $p \neq 2$ is in Lemma 5.12, which uses the general Sobolev embedding Theorem 4.6 below.

The following proposition about dual spaces is easy to see from Definition 4.2.

$$...$$
Proposition 4.5. There is a Banach space isomorphism

\[(L^2_{0,\lambda}(E))^* \cong L^2_{0,-\lambda-7}(E),\]
given by the \(L^2\) inner product pairing.

We will likewise have need of the analogous weighted Hölder spaces. Their definition is a bit more involved. See, for example [27, 32] for the precise definition. However, all that we will require from the weighted Hölder spaces is that elements in them have some degree of differentiability with control on their growth rate on the ends, and that these spaces are related to the weighted Sobolev spaces by the embedding theorems, which we will state precisely. The embedding theorems are used implicitly in the sketch proof of Theorem 4.10 below to explain why elements in the kernel of a uniformly elliptic operator are in fact \(C^\infty\).

Let \(m \geq 0\) and \(\alpha \in (0,1)\). Then the weighted Hölder space \(C^{m,\alpha}_\lambda(E)\) is a Banach space of sections of \(E\), whose elements have \(m\) continuous derivatives.

Theorem 4.6 (Weighted Sobolev embedding theorem). Let \(l, m \geq 0\) and let \(\alpha \in (0,1)\).

- If \(l \geq m\), \(l - \frac{7}{2} \geq m - \frac{7}{4}\), and \(p \leq q\), then there is a continuous embedding
  \[L^p_{l,\lambda}(E) \hookrightarrow L^q_{m,\lambda}(E)\].

- If \(l - \frac{7}{2} \geq m + \alpha\), then there is a continuous embedding
  \[L^p_{l,\lambda}(E) \hookrightarrow C^{m,\alpha}_\lambda(E)\].

Proof. See Marshall [32, Theorem 4.17] for a proof. We have only stated some special cases, which are all that we will require.

Corollary 4.7. If \(l \geq 6\), then any section \(\gamma \in L^2_{l,\lambda}(E)\) is twice continuously differentiable.

Proof. This follows immediately from Theorem 4.6 by taking \(m = 2\).

We will always assume that \(l \geq 6\) without explicit mention, so that in particular any second order differential operators on such sections are unambiguously defined.

4.2 Fredholm and elliptic operators on conifolds

Many standard facts will be stated without proof in this section. The reader can consult [25, 26, 32] for details. To make some of the equations easier to read, we will often use the following shorthand notation:

\[
\Omega_{t,\lambda}^\bullet = L^2_{t,\lambda}(\Lambda^\bullet(T^*M)),
\]
\[
\Omega_{t,\lambda}^k = L^2_{t,\lambda}(\Lambda^k(T^*M)), \quad 0 \leq k \leq 7,
\]
\[
\mathcal{S}_{t,\lambda} = L^2_{t,\lambda}(\mathcal{S}(M)).
\]

We will be interested in the following three differential operators:

\[
(d + d^*_M)_{l+1,\lambda} : \Omega_{l+1,\lambda}^\bullet \to \Omega_{l,\lambda}^\bullet - 1, \quad (45)
\]
\[
(\Delta_M)_{l+2,\lambda} : \Omega_{l+2,\lambda}^k \to \Omega_{l,\lambda}^k - 2, \quad (46)
\]
\[
(\mathcal{D}_M)_{l+1,\lambda} : \mathcal{S}_{l+1,\lambda} \to \mathcal{S}_{l,\lambda} - 1. \quad (47)
\]
They are defined by extending the operators $d + d^*_r$, $\Delta_m$, and $\mathcal{D}_m$ from smooth compactly supported sections to the Sobolev spaces. Note that the Laplacian $\Delta_m$ preserves the degree $k$ of forms, so strictly speaking we should include the degree $k$ as an extra label on the left hand side of (46), but we will not do this, to avoid the proliferation of notation. We will let $r$ denote the order of the differential operator, which is 1, 2, and 1, respectively, in these three cases. Using the symbol $P$ to denote one of these operators, and $E$ to denote the vector bundle on which it acts, the above three operators can all be written as

$$P_{l+r, \lambda} : L^2_{l+r, \lambda}(E) \to L^2_{l, \lambda-r}(E).$$

(48)

In fact, we will also be interested in the modified Dirac operator $\tilde{\mathcal{D}}_m$ defined in Section 2.2 as well as in the restriction of $d + d^*_m$ to the space $\Omega^k_{l+1, \lambda}$ of 3-forms, which we will denote simply by $(\mathcal{D}_m)^k_{l+1, \lambda}$. That is,

$$\mathcal{D}_m^k = (d + d^*_m)|_{\Omega^k}.$$  

(49)

It is a standard fact that the operators $d + d^*_r$, $\Delta_m$, and $\mathcal{D}_m$ are elliptic, and in Proposition 2.13 we proved that $\mathcal{D}_m$ is also elliptic. In fact these operators are also all uniformly elliptic in the sense that near infinity, they approach the elliptic operators $d + d^*_r$, $\Delta_c$, $\mathcal{D}_c$, and $\tilde{\mathcal{D}}_c$ on the cone $C$, respectively. See Marshall [32, Chapter 4] for the precise definition of uniform ellipticity in this context. We note here that the operator $\mathcal{D}_m^k$, being the restriction of $d + d^*_r$ to the space of $k$-forms, is not elliptic, but for suitable rates $\lambda$ and suitably redefined codomain, it will be Fredholm. This is discussed in Section 4.4. The following result is an elliptic regularity statement for uniformly elliptic operators.

**Theorem 4.8.** Let $P$ be a uniformly elliptic operator. Suppose that $\gamma$ and $\nu$ are both locally integrable sections of $E$, and that $\gamma$ is a weak solution of the equation $P(\gamma) = \nu$. If $\gamma \in L^2_{0, \lambda}(E)$ and $\nu \in L^2_{l, \lambda-r}(E)$, then $\gamma \in L^2_{l+r, \lambda}(E)$, and $\gamma$ is a strong solution of $P(\gamma) = \nu$. Furthermore, we have

$$||\gamma||_{L^2_{l+r, \lambda}} \leq C \left( ||P(\gamma)||_{L^2_{l, \lambda-r}} + ||\gamma||_{L^2_{d, \lambda}} \right)$$

(50)

for some constant $C > 0$ independent of $\gamma$. That is, $\gamma$ has at least $r$ more derivatives worth of regularity than $\nu = P(\gamma)$.

We will need to use some results about the kernels and indices of linear operators. To this end we must first define the critical rates for these operators, which depend on the geometry of the links of the cones on each end.

**Definition 4.9.** Let $C$ be a $G_2$ cone. Let $P_C$ be one of the operators $d + d^*_r$, $\Delta_c$, $\mathcal{D}_c$, $\tilde{\mathcal{D}}_c$, or $\mathcal{D}_m^k$ acting on sections of some vector bundle $E$ over $C$. The set $\mathcal{D}_{P_C}$ of critical rates of the operator $P_C$ on sections of $E$ is defined as follows:

$$\mathcal{D}_{P_C} = \{ \lambda \in \mathbb{R} ; \exists \text{ a nonzero section } \gamma \text{ of } E, \text{ homogeneous of order } \lambda, \text{ with } P_C(\gamma) = 0 \}.$$  

(51)

The definition of ‘homogeneous of order $\lambda$’ for a $k$-form on a cone was given in Definition 3.2. If $\gamma$ is a mixed degree form or a spinor in $\mathcal{S}(C)$ (which from Section 2.2 consists of a function and a 1-form), homogeneous means that each graded component is homogeneous. The set $\mathcal{D}_{P_C}$ is a countable, discrete subset of $\mathbb{R}$, and has finite intersection with any closed bounded interval of $\mathbb{R}$.

In the AC case a rate $\lambda \in \mathbb{R}$ is a critical rate of $P$ if it is a critical rate of the corresponding operator $P_C$ on its asymptotic cone $C$. In the CS case we have $n$ ends which are modeled on cones,
and a rate \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \) will be a critical rate for \( P \) if any of its components \( \lambda_i \) lie in the corresponding critical set \( \mathcal{D}_{P_{C_i}} \) for the cone \( C_i \). We say that the “interval” \([\lambda, \lambda']\) does not contain any critical rates for \( P \) on \( M \) if each interval \([\lambda_i, \lambda'_i]\) contains no critical rates for \( P \) on the cone \( C_i \).  

**Theorem 4.10.** The kernel of \( P_{l+r,\lambda} \) is independent of \( l \). Hence we can denote it unambigiously as \( \ker(P)_\lambda \). This kernel is also invariant as we change the rate \( \lambda \), as long as we do not hit any critical rates. That is, if the interval \([\lambda, \lambda']\) is contained in the complement of \( \mathcal{D}_P \), then

\[
\ker(P)_\lambda'' = \ker(P)_\lambda.
\]

Proof. The invariance of the kernel in the absence of critical rates is explained in [25, 26]. We present a sketch of the proof of the independence on \( l \). The Sobolev embedding Theorem 4.6 says that for large enough \( l \), we can embed the Sobolev space \( \mathcal{L}_{l,\lambda}^2(E) \) into an appropriate H"older space \( C_{m,\alpha}^m(E) \) having \( m \) continuous derivatives. It follows from this theorem and the elliptic regularity of Theorem 4.8 that elements in the kernel of \( P \) are smooth, and the independence of the kernels on \( l \) follows from this.

Recall that a linear map between Banach spaces is called Fredholm if it has closed image, finite-dimensional kernel, and finite-dimensional cokernel. The main significance of the critical rates \( \mathcal{D}_P \) is that they are related to the rates \( \lambda \) for which the operator \( P_{l+r,\lambda} \) of equation (48) is Fredholm, by the following theorem.

**Theorem 4.11.** The map \( P_{l+r,\lambda} : \mathcal{L}_{l+r,\lambda}^2(E) \to \mathcal{L}_{l+r,\lambda}^2(E) \) is Fredholm if and only if \( \lambda \notin \mathcal{D}_P \), where the set of critical rates \( \mathcal{D}_P \) is as given in Definition 4.9.

Now consider the formal adjoint of the map \( P_{l+r,\lambda} : \mathcal{L}_{l+r,\lambda}^2(E) \to \mathcal{L}_{l+r,\lambda}^2(E) \).

By Proposition 4.5, the formal adjoint is a map

\[
P^*_{m+\tau,\lambda} : \mathcal{L}_{m+\tau,\lambda}^2(E) \to \mathcal{L}_{m+\tau,\lambda}^2(E),
\]

where \( l, m \geq 0 \).

**Remark 4.12.** Here we are being slightly sloppy, in the following sense. Technically, we really have \( (\mathcal{L}_2^l)^* = \mathcal{L}_2^{l-\tau} \), but we would like to avoid having to consider the meaning of \( \mathcal{L}_2^l \) for \( l < 0 \). Fortunately, we will only ever be interested in the kernel of the formal adjoint \( P^* \) on spaces of the form \( \mathcal{L}_{m+\tau,\lambda}^2 \), which by Theorem 4.10 is independent of \( m \), so it is safe to assume that \( m \geq 0 \).

The next result is the version of the ‘Fredholm Alternative’ for conifolds.

**Theorem 4.13.** Suppose that \( \lambda \) is not in \( \mathcal{D}_P \), so that by Theorem 4.11, the map

\[
P_{l+r,\lambda} : \mathcal{L}_{l+r,\lambda}^2(E) \to \mathcal{L}_{l+r,\lambda}^2(E)
\]

is Fredholm, and also uniformly elliptic. Then:

(a) We can choose a finite-dimensional subspace \( \mathcal{W}_{\lambda-r} \) of \( \mathcal{L}_{l,\lambda-r}^2(E) \) such that

\[
\mathcal{L}_{l,\lambda-r}^2(E) = P(\mathcal{L}_{l+r,\lambda}^2(E)) \oplus \mathcal{W}_{\lambda-r},
\]

such that

\[
\mathcal{W}_{\lambda-r} \cong \ker(P^*)_{-\tau+r,\lambda}.
\]
(b) Furthermore, if \( \ker(P^*)_{-7+r-\lambda} \) lies in \( L^2_{r,-\lambda}(E) \), then we can take
\[
W_{r,-\lambda} = \ker(P^*)_{-7+r-\lambda}.
\]

By Remark [4.14], this happens whenever \( \lambda > -\frac{7}{2} + r \) in the AC case, and whenever \( \lambda < -\frac{7}{2} + r \) in the CS case.

**Remark 4.14.** Equation (54) is a consequence of general Banach space theory and Proposition 4.5, since whenever a subspace \( W \) of a Banach space \( V \) is closed, any direct complement of it will be isomorphic to its annihilator in the dual space.

**Remark 4.15.** Because of Remark 4.3 (b) and equation (54), we see that \( \ker(P)_\lambda \) and \( \text{coker}(P)_\lambda \) are always finite-dimensional, even if \( \lambda \) is a critical rate. Thus the failure of \( P \) to be Fredholm at a critical rate is due only to \( \text{im}(P) \) not being closed.

As we will be using both the Fredholm theory of \( \Delta_M \) and that of \( d + d^*_M \), we will need to know how elements in the kernels of these two operators are related. In particular, a \( k \)-form which is closed and coclosed is always harmonic, but the converse will only be true for certain rates. Also, when a mixed degree form \( \gamma = \sum_{k=0}^7 \gamma_k \) is closed and coclosed, it will not always be the case that each graded component \( \gamma_k \) is independently closed and coclosed. Before presenting this result, we give a proof of “integration by parts” for weighted spaces.

**Lemma 4.16.** Let \( \alpha \in \Omega^k_{r-1} \) and \( \beta \in \Omega^m_{\mu} \). If \( \lambda + \mu < -6 \) (AC) or \( \lambda + \mu > -6 \) (CS), then we have
\[
\langle \langle \alpha, \beta \rangle \rangle = \langle \langle \alpha, d^*_M \beta \rangle \rangle.
\]

**Proof.** We give the proof in the AC case. The CS case is identical except that there are \( n \) ends instead of just one, and \( \rho \to 0 \) on each end instead of \( \rho \to \infty \). Let \( M_R = \{ x \in M; g(x) \leq R \} \), and observe that \( \partial(M_R) = \{ R \} \times \Sigma \). Hence, by Stokes’s Theorem and the fact that \( d\alpha \land *\beta - \alpha \land *d^*_M \beta = d(\alpha \land *\beta) \), we find
\[
\int_{M_R} \langle \alpha, \beta \rangle \text{vol}_M - \int_{M_R} \langle \alpha, d^*_M \beta \rangle \text{vol}_M = \int_{\{ R \} \times \Sigma} (\alpha \land *\beta).
\]

The proof will be complete if we can establish that the integral on the right hand side above goes to zero as \( R \to \infty \). But since \( |\alpha| \leq CR^\lambda \) and \( |\beta| \leq CR^\mu \) on the end, we have
\[
\left| \int_{\{ R \} \times \Sigma} (\alpha \land *\beta) \right| \leq \int_{\{ R \} \times \Sigma} |\alpha \land *\beta| \text{vol}_E \leq CR^{\lambda+\mu+6}.
\]

This goes to zero as \( R \to \infty \) since \( \lambda + \mu < -6 \).

**Proposition 4.17.** Let \( \gamma = \sum_{k=0}^7 \gamma_k \in \Omega^*_{r-1,\lambda} \), where each \( \gamma_k \in \Omega^k_{r-1,\lambda} \), and suppose \( (d + d^*_M)\gamma = 0 \). If \( \lambda < -\frac{7}{2} \) (AC) or \( \lambda > -\frac{7}{2} \) (CS), then in fact \( (d + d^*_M)\gamma_k = 0 \) for each \( k \).

**Proof.** Decomposing the equation \( (d + d^*_M)\gamma = 0 \) into graded components, we have
\[
d\gamma_{k-1} = -d^*_M \gamma_{k+1},
\]
where \( \gamma_{-1} = \gamma_8 = 0 \). Since \( \gamma_k \in \Omega^k_{r-1,\lambda} \) and \( d\gamma_k \in \Omega^k_{r-1,\lambda-1} \), the hypothesis on \( \lambda \) and Lemma 4.16 then give
\[
|d\gamma_k|^2 = \langle \langle d\gamma_k, d\gamma_k \rangle \rangle = -\langle \langle (d\gamma_k, d^*_M \gamma_{k+2}) \rangle \rangle = -\langle \langle \gamma_k, d^*_M \gamma_{k+2} \rangle \rangle = 0,
\]
and hence \( d\gamma_k = d^*_M \gamma_k = 0 \) for all \( k \).
Corollary 4.18. Suppose that $\gamma \in \Omega^k_{i+2,\lambda}$ and that $\Delta_\lambda \gamma = 0$. Then we have:

For $k = 0, 7$, if $\lambda < 0$ (AC) or $\lambda > -5$ (CS), then $d^*_\lambda \gamma = 0$ and $d\gamma = 0$. \hfill (55)

For $k = 1, 6$, if $\lambda < -1$ (AC) or $\lambda > -4$ (CS), then $d^*_\lambda \gamma = 0$ and $d\gamma = 0$. \hfill (56)

For $k = 2, 5$, if $\lambda < -2$ (AC) or $\lambda > -3$ (CS), then $d^*_\lambda \gamma = 0$ and $d\gamma = 0$. \hfill (57)

Proof. In the AC setting, by Proposition 3.4 we see that in all three cases, as we decrease $\lambda$ there are no critical rates until at the earliest $\lambda = -3$. So by Theorem 4.10, in all three cases we can say that $\gamma$ actually lies in $\ker(\Delta_\lambda)\mu$ for some $\mu < -\frac{3}{2}$. In particular we conclude that $d\gamma$ and $d^*_\lambda \gamma$ are both in $\Omega^*_{i+1,\mu-1}$. Then using Lemma 4.16 we find

\[ 0 = \langle \langle \Delta_\lambda \gamma, \gamma \rangle \rangle = \langle \langle dd^*_\lambda \gamma + d^*_\lambda d\gamma, \gamma \rangle \rangle = ||d\gamma||^2 + ||d^*_\lambda \gamma||^2, \]

so $d^*_\lambda \gamma = 0$ and $d\gamma = 0$.

In the CS setting, by Proposition 3.4 we see that in all three cases, as we increase each $\lambda$, there are no critical rates until at the earliest $\lambda_i = -2$. Therefore by Theorem 4.10, in all three cases we can say that $\omega_k$ actually lies in $\ker(\Delta_\lambda)\mu$ for some $\mu > -\frac{3}{2}$. The claims now follow just as in the AC case. \hfill \Box

Recall that the index of a Fredholm operator $P$ is given by $\ind(P) = \dim(\ker P) - \dim(\ker P^*)$. In order to compute the dimension of the moduli space, we will need to understand how the index of $P$ changes as we cross a critical rate. To this end we require the following definition. Let $P_C$ denote the operator corresponding to $P$ on the cone.

Definition 4.19. Let $C$ be a $G_2$ cone. For $\lambda \in \mathbb{R}$, we define the space $K(\lambda)_{P_C}$ to be

\[ K(\lambda)_{P_C} = \left\{ \gamma = \sum_{j=0}^m (\log r)^j \gamma_j; \text{ such that } P_C \gamma = 0, \text{ where each } \gamma_j \text{ is a section of } E \text{ that is homogeneous of order } \lambda \right\}. \]

That is, $K(\lambda)_{P_C}$ consists of the sections of $E$ over $C$ in the kernel of $P_C$, that are polynomials in $\log r$ whose coefficients are sections of $E$ over $C$ that are homogeneous of order $\lambda$. These spaces are all finite-dimensional. This follows from the ellipticity of $P_C$ and is discussed in [25].

The importance of the $K(\lambda)_{P_C}$ spaces is that their dimensions tell us how the index of $P$ changes when we cross a critical rate. The following crucial result appears in general in Lockhart–McOwen [25, §8], and can also be found explicitly for AC manifolds in [27] [6.3.2].

Theorem 4.20. Let $\nu < \mu$ be two noncritical rates for $P$. By Theorem 4.11 the maps

\[ P_{l+r,\nu} : L^2_{l+r,\nu}(E) \to L^2_{l+r,\nu}(E) \]

and

\[ P_{l+r,\mu} : L^2_{l+r,\mu}(E) \to L^2_{l+r,\mu}(E) \]

are both Fredholm. The difference in their indices is given by

\[ \ind(P_{l+r,\mu}) - \ind(P_{l+r,\nu}) = \sum_{\lambda \in \mathcal{D}_{P_C} \cap (\mu, \nu]} \dim K(\lambda)_{P_C}. \] \hfill (AC) \hfill (59)

\[ \ind(P_{l+r,\mu}) - \ind(P_{l+r,\nu}) = -\sum_{i=1}^n \sum_{\lambda \in \mathcal{D}_{P_{C_i}} \cap (\nu_i, \mu_i]} \dim K(\lambda)_{P_{C_i}}. \] \hfill (CS) \hfill (60)
That is, the index of \( P \) jumps precisely by the dimension of the space \( \mathcal{K}(\lambda)_{P_{C_i}} \) as we cross each critical rate in the interval \((\nu, \mu)\). This sum is finite because the set \( \mathcal{D}_{P_{C_i}} \) of critical rates has only finitely many points in any bounded interval.

Let \( \mathcal{K}(\lambda)_{P_{C_i}} \) be as in Definition 4.19. The following result can be deduced from the Lockhart–McOwen theory, and appeared in a less general form in \([20, \text{Proposition 4.27}]\).

**Proposition 4.21.** Let \((M, \varphi)\) be a \( G_2 \) conifold of rate \( \nu \). Suppose that \( \beta_1, \beta_2 \) are two noncritical rates for \( P \), and that \( \beta_1 > \beta_2 \) (AC) or \( \beta_1 < \beta_2 \) (CS). Suppose there exists a single critical rate for \( P \) between \( \beta_1 \) and \( \beta_2 \). This means that for some end of the manifold indexed by \( i \in \{1, \ldots, n\} \), there is a critical rate \( \lambda_0 \) for \( P_i \) on the cone \( C_i \) between \( \beta_1 \) and \( \beta_2 \). Suppose that \( \gamma_1 \in \ker(P)_{\beta_1} \). Then there exists some \( v \in \mathcal{K}(\lambda_0)_{P_{C_i}} \) and some \( \tilde{\gamma} \) with \( |\tilde{\gamma}|_{g_M} = O(q^{3_1 + \nu}) \) such that, when restricted to the \( i \)th end of \( M \), we have

\[
\gamma_1 - (h^{-1}_v)(v) - \tilde{\gamma} = \gamma_2 \in \ker(P)_{\beta_2}. \tag{61}
\]

*Note that the term \( \gamma_2 \), which is in the kernel of \( P \) with noncritical rate \( \beta_2 \), decays faster on the end.*

**Remark 4.22.** The \( O(q^{3_1 + \nu}) \) term arises from comparing a solution of \( P \gamma = 0 \) to a solution of \( P_w \gamma = 0 \), using the relation between the metrics \( g_C \) and \( g_M \) on the end. Thus in particular the term \( \tilde{\gamma} \) in \((61)\) will vanish if and only if the term \( v = 0 \).

**Remark 4.23.** Recall Theorem 4.10 says that the kernel will only change as we cross a critical rate. The essential content of Proposition 4.21 is that, when the kernel does indeed change as we cross a critical rate \( \lambda \), any section which is added or removed from the kernel must be asymptotic at the \( i \)th end to an element of \( \mathcal{K}(\lambda)_{P_{C_i}} \).

**Definition 4.24.** Let \( 0 \leq k \leq 7 \). We define the space \( \mathcal{H}_\lambda^k \) to be the space of closed and coclosed \( k \)-forms of rate \( \lambda \) on the ends. Explicitly, we have

\[
\mathcal{H}_\lambda^k = \{ \gamma \in \Omega^k_{l, \lambda}; d\gamma = 0, d^*\gamma = 0 \}.
\]

This definition makes sense for any \( l \geq 0 \), since \( \mathcal{H}_\lambda^k \) is a subspace of \( \ker(d + d^*_{M})_\lambda \), where \( d + d^*_{M} \) is acting on \( \Omega^*_{l, \lambda} \), and thus by Theorem 4.10 and Remark 4.15 the space \( \mathcal{H}_\lambda^k \) is independent of \( l \) and finite-dimensional.

Given a form \( \gamma \in \Omega^k \), its *pure-type* components are the components in the decompositions \((1)\) and \((2)\) into \( G_2 \) representations.

**Lemma 4.25.** Suppose that \( \lambda < -\frac{5}{2} \) (AC) or \( \lambda > -\frac{5}{2} \) (CS). Then the pure-type components of an element \( \gamma \) in \( \mathcal{H}_\lambda^k \) are each closed and coclosed.

*Proof.* An element \( \gamma \) of \( \mathcal{H}_\lambda^k \) is closed and coclosed, hence harmonic. By Remark 2.4 the projections onto the pure-type forms commute with the Laplacian, and thus the pure-type components of \( \gamma \) are each harmonic. Then it follows from Corollary 4.18 that the pure-type components of \( \gamma \) are each closed and coclosed.

For any \( 0 \leq k \leq 7 \), the space \( \mathcal{H}_\lambda^k \) is always a subspace of \( \ker(d + d^*_{M})_\lambda \). Consider varying the rate \( \lambda \) in the direction in which the space \( \mathcal{H}_\lambda^k \) potentially gets larger. If new elements are added to \( \mathcal{H}_\lambda^k \), then new elements are added to \( \ker(d + d^*_{M})_\lambda \), and thus this can only happen when we cross a rate \( \lambda_0 \) which is critical for \( d + d^*_{M} \). The next lemma says that new elements are added to \( \mathcal{H}_\lambda^k \) only when there exist closed and coclosed \( k \)-forms of rate \( \lambda_0 \) on the asymptotic cones.

**Lemma 4.26.** Let \( \lambda_0 \) be a critical rate for \( d + d^*_{M} \). For \( \varepsilon > 0 \) define

\[
\lambda_{+} = \begin{cases} 
\lambda_0 + \varepsilon & \text{ (AC)}, \\
\lambda_0 - \varepsilon & \text{ (CS)},
\end{cases} \quad \lambda_{-} = \begin{cases} 
\lambda_0 - \varepsilon & \text{ (AC)}, \\
\lambda_0 + \varepsilon & \text{ (CS)}.
\end{cases}
\]

\[27\]
Thus in either case $\lambda_+$ is a slower rate of decay and $\lambda_-$ is a faster rate of decay on the ends. Further choose $\varepsilon$ small enough so that $2\varepsilon = |\lambda_+ - \lambda_-| < |\nu|$, where $\nu$ is the rate of the $G_2$ conifold, and such that the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ contains no other critical rates for $d + d^*_M$. Let $\gamma_+ \in \mathcal{H}^k_{\lambda_+}$. Equation (61) says that

$$\gamma_+ - (h^{-1})^*(v) - \tilde{\gamma} = \gamma_- \in \ker(d + d^*_M)_{\lambda_-}$$

for some $v \in \mathcal{K}(\lambda_0)_{d + d^*_M}$. Then the form $v$ is a closed and coclosed $k$-form on the cone $C_i$.

**Proof.** We give the proof in the AC case. The CS case is identical with the inequalities reversed. Since there is only one end, we drop the index $i$. Let $\gamma_+ \in \mathcal{H}^k_{\lambda_+}$. By Proposition 4.21, we know that in equation (62) the term $\tilde{\gamma}$ satisfies $|\tilde{\gamma}|_{g_M} = O(\rho^{\lambda_+ + \nu})$. Let $v_m$ and $\tilde{\gamma}_m$ denote the degree $m$ components of $v$ and $\tilde{\gamma}$, for $m \neq k$. Since $\gamma_+$ is a pure degree $k$-form, we find that

$$(h^{-1})^*(v_m) + \tilde{\gamma}_m \text{ is at most } O(\rho^{\lambda_-}),$$

and $\tilde{\gamma}_m$ is at most $O(\rho^{\lambda_+ + \nu})$. But our hypothesis that $\lambda_+ + \nu < \lambda_-$ allows us to absorb the $\tilde{\gamma}_m$ term on the right hand side, to conclude that $v_m$ is at most $O(\rho^{\lambda_-})$. However, because the form $v$ lies in $\mathcal{K}(\lambda_0)_{d + d^*_M}$, we know in fact that if $v_m \neq 0$ then it is at least $O(\rho^{\lambda_0})$, and $\lambda_0 > \lambda_-$. Thus we must have $v_m = 0$ for all $m \neq k$. Since $v \in \mathcal{K}(\lambda_0)_{d + d^*_M}$, we conclude that $v$ is a closed and coclosed $k$-form on the cone $C_i$. \hfill $\Box$

**Corollary 4.27.** Let $0 \leq k \leq 7$. Let $\lambda, \mu$ be two noncritical rates for $d + d^*_M$. If there are no closed and coclosed homogeneous $k$-forms on the asymptotic cones of $M$ of any rates between $\lambda$ and $\mu$, then $\mathcal{H}^k_{\lambda} = \mathcal{H}^k_{\mu}$. In particular, we have

$$\mathcal{H}^k_{\lambda} = \mathcal{H}^k_{\mu} \quad \text{if } \lambda, \mu \in (-4, -3).$$

**Proof.** We observe first that Lemma 4.26 together with Remark 4.22 and Lemma 3.9 says that the space $\mathcal{H}^k_{\lambda}$ will only change when we cross a rate $\lambda_0$ for which there exists a homogeneous closed and coclosed $k$-form of rate $\lambda_0$ on some asymptotic cone $C_i$ of $M$. This proves the first statement. Equation (63) now follows from Proposition 3.5 which says that there are no nontrivial homogeneous closed and coclosed $k$-forms of any rate in $(-4, -3)$ for any $G_2$ cone $C_i$. \hfill $\Box$

### 4.3 Hodge theoretic results for $k$-forms

In order to avoid the proliferation of too much notation, from now on all the $M$ subscripts will be dropped. It will be understood that the Hodge star operator $\ast$, the covariant derivative $\nabla$, the Hodge Laplacian $\Delta$, the codervivative $d^*$, the projections maps $\pi_k$, and the maps $L_{\varphi}$ and $Q_{\varphi}$ defined in Lemma 2.8 will all be taken with respect to a fixed $G_2$ structure $\varphi = \varphi_M$ on $M$. Furthermore, we often have to deal with sections of a vector bundle $E$ that are smooth, but have particular growth on the ends. Therefore we use the following notation:

$$C_*^\infty(E) = \{ \gamma \in C^\infty(E); |\nabla \gamma| = O(\rho^{\lambda-j}) \forall j \geq 0 \}.$$

In this section we use Fredholm theory of the operator $d + d^*$ to determine Hodge theoretic results for the spaces of $k$-forms with specified rates of decay on the ends. These results are used repeatedly throughout the sequel. Recall from Definition 4.24 that $\mathcal{H}^k_{\lambda}$ is the space of closed and coclosed $k$-forms of rate $\lambda$ on the ends.
Lemma 4.28. Suppose \( \lambda + 1 \) is a noncritical rate for \( d + d^* \). Then
\[
d(\Omega_{t+1,\lambda+1}^{k-1}) + d^*(\Omega_{t+1,\lambda+1}^{k+1}) \subseteq \Omega_{t,\lambda}^k
\]
is a closed subspace of finite codimension.

Proof. Since \( \lambda + 1 \) is noncritical, by Theorem 4.11 we know that
\[
(d + d^*)(\Omega_{t+1,\lambda+1}^k) \subseteq \Omega_{t,\lambda}^k
\]
is a closed subspace of finite codimension. Since
\[
(d + d^*)(\Omega_{t+1,\lambda+1}^k) \subseteq d(\Omega_{t+1,\lambda+1}^k) + d^*(\Omega_{t+1,\lambda+1}^k),
\]
we see that the latter space must also have finite codimension in \( \Omega_{t,\lambda}^k \). Therefore,
\[
d(\Omega_{t+1,\lambda+1}^{k-1}) + d^*(\Omega_{t+1,\lambda+1}^{k+1}) \subseteq \Omega_{t,\lambda}^k
\]
has finite codimension. However, the former space is the image of the continuous map \((\alpha, \beta) \mapsto d\alpha + d^*\beta\) from the Banach space \(\Omega_{t+1,\lambda+1}^{k-1} \oplus \Omega_{t+1,\lambda+1}^{k+1}\) to the Banach space \(\Omega_{t,\lambda}^k\) and hence by [24, Chapter XV, Corollary 1.8] it is closed.

We will see in Section 5 that several of the ingredients in the proof of our main theorem have two distinct flavours, depending on which of the following two situations we are considering:

- In the \( L^2 \) setting (when \( \nu < -\frac{7}{2} \) for the AC case or for any \( \nu > 0 \) in the CS case), many of the analytic arguments are simple, but this is precisely the regime in which obstructions occur.
- In the AC case when \( \nu > -\frac{7}{2} \), we are not in \( L^2 \), and because of this most of the analytic arguments are more delicate. For example, we need to use the surjectivity of the Dirac operator to prove our infinitesimal slice theorem. However, this regime has the nice feature of having an unobstructed deformation theory.

The Hodge theory results we establish are slightly different for these two settings, so we state and prove them separately.

We begin with the \( L^2 \) setting. Recall that \( \Omega_{t,\lambda}^k = L^2_{t,\lambda}(\Lambda^k(T^*M)) \) is a Hilbert space.

Proposition 4.29. Suppose \( \lambda + 1 \) is noncritical for \( d + d^* \). Let \( 0 \leq k \leq 7 \). In the \( L^2 \) setting (when \( \lambda < -\frac{7}{2} \) for the AC case or when \( \lambda > -\frac{7}{2} \) for the CS case), there exists a decomposition
\[
\Omega_{t,\lambda}^k = d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1}) \oplus \mathcal{H}_{t,\lambda}^k \oplus W_{t,\lambda}^k.
\]
Here \( W_{t,\lambda}^k \) is a finite-dimensional space. Moreover, the spaces \( d(\Omega_{t+1,\lambda+1}^{k-1}) \), \( d^*(\Omega_{t+1,\lambda+1}^{k+1}) \), and \( \mathcal{H}_{t,\lambda}^k \) are \( L^2 \)-orthogonal to each other, and \( W_{t,\lambda}^k \) can be defined as the \( L^2_{t,\lambda} \) orthogonal complement of \( d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1}) \oplus \mathcal{H}_{t,\lambda}^k \).

Proof. Since we are in \( L^2 \), integration by parts is valid so if \( \gamma \in \mathcal{H}_{t,\lambda}^k \) we have that
\[
\langle \langle d\alpha, d^*\beta \rangle \rangle_{L^2} = \langle \langle d\alpha, \gamma \rangle \rangle_{L^2} = \langle \langle d^*\beta, \gamma \rangle \rangle_{L^2} = 0.
\]
Thus the spaces $d(\Omega_{t+1,\lambda+1}^{k-1})$, $d^*(\Omega_{t+1,\lambda+1}^{k+1})$, and $\mathcal{H}_\lambda^k$ are $L^2$-orthogonal to each other. By Lemma 4.28 we know that

$$d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1}) \oplus \mathcal{H}_\lambda^k$$

is a closed subspace of $\Omega_{t+1,\lambda}^k$ of finite codimension. Hence a finite dimensional complement $W_{t+1,\lambda}^k$ exists. Since $\Omega_{t+1,\lambda}^k$ is a Hilbert space, we know that the orthogonal complement of a closed subspace with respect to the Hilbert space inner product is a direct complement, so we can uniquely define $W_{t+1,\lambda}^k$ as claimed.

**Corollary 4.30.** The dimension of the space $W_{t+1,\lambda}^k$ is given by

$$\dim W_{t+1,\lambda}^k = \dim \mathcal{H}_{t-\lambda}^k - \dim \mathcal{H}_\lambda^k.$$  \hspace{1cm} (65)

In particular, the dimension of $W_{t+1,\lambda}^k$ is independent of $l \geq 0$.

**Proof.** By Remark 4.14 since the subspace $d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})$ of $\Omega_{t+1,\lambda}^k$ is closed, any direct complement of it will be isomorphic to its annihilator in the dual space. It is trivial to see that this annihilator is the space $\mathcal{H}_{t-\lambda}^k$ of closed and coclosed forms of the dual rate $7-\lambda$, using the fact that the dual space of $\Omega_{t+1,\lambda}^k$ is $\Omega_{-t-1,-\lambda}$ and Remark 4.12. Thus $\dim(\mathcal{H}_\lambda^k \oplus W_{t+1,\lambda}^k) = \dim \mathcal{H}_{t-\lambda}^k$, and hence equation (65) follows immediately.

The next result is the analogue to Proposition 4.29 for the AC case when $\lambda > -4$.

**Proposition 4.31.** Suppose $\lambda + 1$ is noncritical for $d + d^*$. Let $0 \leq k \leq 7$. In the AC case when $\lambda > -4$, there exists a decomposition

$$\Omega_{t,\lambda}^k = A_{t,\lambda}^k \oplus B_{t,\lambda}^k \oplus \mathcal{H}_{t+\lambda}^k = A_{t,\lambda}^k \oplus \mathcal{H}_\lambda^k,$$  \hspace{1cm} (66)

where $B_{t,\lambda}^k$ is the intersection of $\mathcal{H}_\lambda^k$ with the Banach space $d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})$, and $A_{t,\lambda}^k$ is a topological complement of $B_{t,\lambda}^k$ in $d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})$, and thus a closed subspace. Moreover, in $A_{t,\lambda}^k$ the intersection of the images of $d$ and $d^*$ is zero.

**Proof.** We prove this proposition for $\lambda \geq -\frac{7}{2}$ and explain in Remark 4.32 below why it actually holds for $\lambda > -4$. We begin by noting that by Lemma 4.28 we have that $d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})$ is closed in $\Omega_{t+1,\lambda}^k$. The argument from the proof of Corollary 4.30 applies here, so any topological complement of $d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})$ will be isomorphic to $\mathcal{H}_{t+\lambda}^k$. However, since $\lambda \geq -\frac{7}{2}$ is equivalent to $-7 - \lambda \leq \lambda$, we see that $\mathcal{H}_{t+\lambda}^k$ actually lies in $\Omega_{t+1,\lambda}^k$ and thus we can write

$$\Omega_{t+1,\lambda}^k = (d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})) \oplus \mathcal{H}_{t+\lambda}^k.$$ 

Let $B_{t,\lambda}^k$ be the intersection of $\mathcal{H}_\lambda^k$ with $d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})$ and let $A_{t,\lambda}^k$ be a topological complement of the finite-dimensional space $B_{t,\lambda}^k$ in the Banach space $d(\Omega_{t+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{t+1,\lambda+1}^{k+1})$. We thus have

$$\Omega_{t,\lambda}^k = A_{t,\lambda}^k \oplus B_{t,\lambda}^k \oplus \mathcal{H}_{t+\lambda}^k,$$

with $\mathcal{H}_\lambda^k = B_{t,\lambda}^k \oplus \mathcal{H}_{t+\lambda}^k$. Finally, suppose $\eta = d\alpha = d^*\beta$ in $A_{t,\lambda}^k$. Then $\eta$ is both closed and coclosed, and thus lies in $\mathcal{H}_\lambda^k$. Since $A_{t,\lambda}^k \cap \mathcal{H}_\lambda^k = \{0\}$, we have $\eta = 0$. \qed
Remark 4.32. Suppose we are in the AC case. If \( \lambda \in (-4, -\frac{7}{2}) \), then \(-7 - \lambda \in (-\frac{7}{2}, -3)\), and hence Corollary 4.27 and Corollary 4.30 tell us that \( W_{\lambda}^k = 0 \). Thus Proposition 4.29 then says

\[
\Omega_{i,\lambda}^k = d(\Omega_{i+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{i+1,\lambda+1}^{k+1}) \oplus H_{\lambda}^k.
\]

Similarly, if \( \lambda \in (-\frac{7}{2}, -3) \), then \(-7 - \lambda \in (-4, -\frac{7}{2})\), and hence Corollary 4.27 and \( H_{\lambda}^k = B_{i,\lambda}^k \oplus H_{\lambda}^{k-7-\lambda} \) tell us that \( B_{i,\lambda}^k = 0 \). Therefore in this case Proposition 4.31 says \( A_{i,\lambda}^k = d(\Omega_{i+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{i+1,\lambda+1}^{k+1}) \), and

\[
\Omega_{i,\lambda}^k = d(\Omega_{i+1,\lambda+1}^{k-1}) \oplus d^*(\Omega_{i+1,\lambda+1}^{k+1}) \oplus H_{\lambda}^k.
\]

Since \( H_{\lambda}^k \) is independent of \( \lambda \in (-4, -3) \), we have established that in the interval \((-4, -3)\) the two decompositions (64) and (66) are identical. Thus we conclude that Proposition 4.31 actually holds for all \( \lambda > -4 \).

We are now ready for the main result of this section, which is a Hodge-type decomposition for \( k \)-forms on \( G_2 \) conifolds.

Theorem 4.33. Let \( (M, \varphi) \) be a \( G_2 \) conifold of rate \( \nu \), and suppose that \( \nu + 1 \) is a noncritical rate for \( d + d^* \). Let \( \eta \in \Omega_{i,\nu}^k \).

- In the \( L^2 \) setting (when \( \nu < -\frac{7}{2} \) for the AC case or for any \( \nu > 0 \) in the CS case), we can express the \( k \)-form \( \eta \), in a unique way, as

\[
\eta = d\alpha + d^*\beta + \kappa + \gamma,
\]

where \( \alpha \in \Omega_{i+1,\nu+1}^{k-1}, \beta \in \Omega_{i+1,\nu+1}^{k+1}, \kappa \in H_{\nu}^k \), and \( \gamma \) is in \( W_{\nu}^k \). Moreover, if \( d\eta = 0 \), then we can express \( \eta \) in a unique way as

\[
\eta = d\alpha + \kappa + \delta
\]

where \( \alpha \in \Omega_{i+1,\nu+1}^{k-1}, \kappa \in H_{\nu}^k \), and

\[
\delta \in U_{\nu}^k = \{ d\beta + \gamma : \beta \in \Omega_{i+1,\nu+1}^{k+1}, \gamma \in W_{\nu}^k, d(d^*\beta + \gamma) = 0 \}.
\]

Moreover, \( U_{\nu}^k \) is finite-dimensional and \( \dim U_{\nu}^k \leq \dim W_{\nu}^k \).

- In the AC case when \( \nu \in (-4, 0) \), we can express the \( k \)-form \( \eta \), in a unique way, as

\[
\eta = d\alpha + d^*\beta + \kappa,
\]

where \( d\alpha + d^*\beta \in A_{i,\nu}^k \) and \( \kappa \in H_{\nu}^k \). Moreover, if \( d\eta = 0 \), then we can actually write

\[
\eta = \tilde{\alpha} + \kappa
\]

for some \( \tilde{\alpha} \in H_{\nu}^k \).

Proof. Equations (67) and (69) are immediate from Propositions 4.29 and 4.31. Now suppose \( d\eta = 0 \).

In the \( L^2 \) setting, Proposition 4.29 says that we can write \( \eta \) uniquely as

\[
\eta = d\alpha + d^*\beta + \kappa + \gamma
\]

for \( \alpha \in \Omega_{i+1,\nu+1}^{k-1}, \beta \in \Omega_{i+1,\nu+1}^{k+1}, \kappa \in H_{\nu}^k \), and \( \gamma \in W_{\nu}^k \). Since \( d\eta = 0 \) and \( d\alpha + \kappa \) is closed we see that \( d(d^*\beta + \gamma) = 0 \). Moreover, if \( dd^*\beta = dd^*\beta' = -d\gamma \) then \( d^*(\beta - \beta') \) is closed and coclosed and
thus lies in $H^k$, which implies that $d^* \beta = d^* \beta'$ since $d^*(\Omega^k_{0+1\nu+1})$ is $L^2$-orthogonal to $H^k$. Thus any $\gamma \in W^k_{\nu}$ can be paired with at most one $d^* \beta$ so that $\gamma + d^* \beta$ is closed. We have established (68).

In the AC case when $\nu \in (-4, 0)$ corresponding to equation (69) we get $d(d^* \beta) = 0$, so $\kappa = d^* \beta + \kappa$ is both closed and coclosed, and thus lies in $H^k$, establishing (70).

The decomposition of the space of $k$-forms in Theorem 4.33 is an essential tool, which we will use repeatedly in the rest of this paper.

### 4.4 A special index-change theorem

In our study of the moduli space of $G_2$ conifolds in Section 5, we will need to consider the operator

$$\mathcal{D}^k_{l,\nu} = (d + d^*)_{l,\nu}|_{\Omega^k_{l,\nu}} : \Omega^k_{l,\nu} \rightarrow d(\Omega^k_{l,\nu}) + d^*(\Omega^k_{l,\nu}).$$

(71)

For simplicity we will often use the symbol $\mathcal{D}^k_{l,\nu}$ to denote this map, which is just $(d + d^*)_{l,\nu}$ with domain restricted to $\Omega^k_{l,\nu}$ and codomain restricted to $d(\Omega^k_{l,\nu}) + d^*(\Omega^k_{l,\nu})$. One of the principal results we will need is a refined version of the “index-change” formula of Theorem 4.20 for the operator $\mathcal{D}^3_{l,\nu}$ defined in (71) for $k = 3$. Note that Theorem 4.20 does not directly apply to this operator $\mathcal{D}^3_{l,\nu}$, because although (for generic rates $\lambda$) we show in Proposition 4.41 that it is Fredholm, it is clearly not elliptic.

**Definition 4.34.** Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be an $n$-tuple of rates, with $n = 1$ in the AC case as usual. Suppose there exists a nontrivial closed and coclosed $k$-form $\nu_i$ on the cone $C_i$, homogeneous of order $\lambda_i$ for some $i = 1, \ldots, n$. Then we say $\lambda$ is a critical rate for the operator

$$\mathcal{D}^k_{l,\nu} = (d + d^*)_{l,\nu}|_{\Omega^k_{l,\nu}} : \Omega^k_{l,\nu} \rightarrow d(\Omega^k_{l,\nu}) + d^*(\Omega^k_{l,\nu})$$

on the conifold $M$. The critical rates for $\mathcal{D}^k$ are thus a subset of the critical rates for the operator $d + d^* : \Omega^*_{l,\nu} \rightarrow \Omega^*_{l-1,\nu-1}$. From Lemma 3.9, we know that there are no log-$r$ terms for the operator $d + d^*$ on the cone, so we can use the notation of Definition 4.19 to define the space $\mathcal{K}(\lambda_i)_{\mathcal{D}^k_{C_i}}$ to be exactly the space of such forms $\nu_i$. That is,

$$\mathcal{K}(\lambda_i)_{\mathcal{D}^k_{C_i}} = \{ \gamma \in \Gamma(\Lambda^k(T^* C_i)); d\gamma = 0, d^*\gamma = 0, \gamma \text{ is homogeneous of order } \lambda_i \}. \hspace{1cm} (72)$$

**Example 4.35.** Consider the operator $\mathcal{D}^3_{1,\nu}$ on the AC $G_2$ manifolds of Bryant–Salamon discussed in Example 3.17. By Remark 3.12, we see that $\lambda = -3$ is a critical rate for $\mathcal{D}^3_{1,\nu}$ if and only if $b^1(\Sigma)$ is nonzero, which by (24) occurs only for $S^3$. Similarly, $\lambda = -4$ is a critical rate for $\mathcal{D}^3_{1,\nu}$ if and only if $b^1(\Sigma) = b^2(\Sigma)$ is nonzero, which by (24) occurs only for $\Lambda^2(\mathbb{CP}^2)$ and $\Lambda^2(S^4)$. Hence, in all three cases the rate $\nu$ of convergence at infinity to the asymptotic cone is a critical rate for $\mathcal{D}^3_{1,\nu}$.

The next lemma shows that elements in the space $\mathcal{K}(\lambda)_{\mathcal{D}^k_{C_i}}$ correspond to solutions to a certain system of eigenvalue equations on the link $\Sigma_i$ of the cone $C_i$.

**Lemma 4.36.** Let $\gamma = r^\lambda (r^{k-1} \alpha \wedge \alpha_{k-1} + r^k \alpha_k)$ be a $k$-form on the cone $C = (0, \infty) \times \Sigma$, homogeneous of order $\lambda$, where $\alpha_{k-1} \in \Omega^{k-1}(\Sigma)$ and $\alpha_k \in \Omega^k(\Sigma)$. Then $(d + d^*)\gamma = 0$ if and only if

$$d^* \alpha_{k-1} = (\lambda + k) \alpha_k, \hspace{1cm} d^* \alpha_k = 0, \hspace{1cm} d^* \alpha_{k-1} = 0, \hspace{1cm} d^* \alpha_k = (\lambda - k + 7) \alpha_{k-1}. \hspace{1cm} (73)$$

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Recall from Theorem 4.33 that any $k$-obstruction space.

**Proof.** This is immediate from (25).

We will now proceed to formally define a space $\hat{O}^k_{l,\lambda}$ that will turn out to be very closely related to the obstruction space $O^k_{l,\lambda}$ for our deformation problem, which will be defined later in Definition 4.39. Recall from Theorem 4.33 that any $k$-form $\eta$ in $O^k_{l,\lambda}$ can be written uniquely as $\eta = \kappa + d\alpha + d^*\beta + \gamma$, where $\kappa \in H^k_\lambda$, and $\gamma \in W^k_{l,\lambda}$. We also know that the space $W^k_{l,\lambda} = \{0\}$ in the AC case when $\lambda > -4$. However, when $W^k_{l,\lambda}$ is nonzero, there could be forms in $W^k_{l,\lambda}$ that are either closed or coclosed, but not both, since otherwise they would be in $H^k_\lambda$ which is transverse to $W^k_{l,\lambda}$. We will need to split off such forms, if there are any, to define our obstruction space.

**Definition 4.37.** We define the space $\hat{O}^k_{l,\lambda}$ for rate $\lambda$ as follows:

- In the $L^2$ setting (when $\lambda < -\frac{7}{2}$ for the AC case or for any $\lambda > 0$ in the CS case), define $(W_c)^k_{l,\lambda}$ to be the subspace of $W^k_{l,\lambda}$ consisting of closed forms, and similarly let $(W_{cc})^k_{l,\lambda}$ be the subspace of coclosed forms. We have $(W_c)^k_{l,\lambda} \cap (W_{cc})^k_{l,\lambda} = \{0\}$. Define $\hat{O}^k_{l,\lambda}$ to be the $L^2$-orthogonal complement in $W^k_{l,\lambda}$ of the subspace $(W_c)^k_{l,\lambda} \oplus (W_{cc})^k_{l,\lambda}$. That is,

$$W^k_{l,\lambda} = ((W_c)^k_{l,\lambda} \oplus (W_{cc})^k_{l,\lambda}) \oplus \hat{O}^k_{l,\lambda}. \quad (74)$$

The second $\oplus$ symbol above is an orthogonal direct sum, but the sum $(W_c)^k_{l,\lambda} \oplus (W_{cc})^k_{l,\lambda}$ need not be orthogonal. Hence any $\gamma \in W^k_{l,\lambda}$ can be written uniquely as $\gamma = \gamma_c + \gamma_{cc} + \gamma_o$ where $d\gamma_c = 0$ and $d^*\gamma_{cc} = 0$ and $\gamma_o \in \hat{O}^k_{l,\lambda}$ is neither closed nor coclosed.

- In the AC case when $\lambda > -4$, we set $\hat{O}^k_{l,\lambda} = \{0\}$.

Recall we are considering the operator

$$D^k_{l,\lambda} = (d + d^*)|_{\Omega^k_{l,\lambda}} : \Omega^k_{l,\lambda} \to d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda}).$$

Thus, the space $d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})$ is the codomain of $D^k_{l,\lambda}$ and the image of $D^k_{l,\lambda}$ is

$$\text{im}(D^k_{l,\lambda}) = (d + d^*)(\Omega^k_{l,\lambda}). \quad (75)$$

We need the following crucial result.

**Lemma 4.38.** The space $d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})$ is the vector space sum of the subspaces $(d + d^*)(\Omega^k_{l,\lambda})$ and $d(\hat{O}^k_{l,\lambda})$. That is, we have

$$d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda}) = (d + d^*)(\Omega^k_{l,\lambda}) + d(\hat{O}^k_{l,\lambda}). \quad (76)$$

**Proof.** It is clear that $(d + d^*)(\Omega^k_{l,\lambda}) + d(\hat{O}^k_{l,\lambda}) \subseteq d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})$, so we need only show the reverse inclusion. Let $\sigma, \tau \in \Omega^k_{l,\lambda}$. Applying the Hodge decomposition of Theorem 4.33 to $\tau - \sigma$, we can write $\tau - \sigma = \kappa + d\alpha + d^*\beta + \gamma$ where in particular $\kappa \in H^k_\lambda$ and $\gamma \in W^k_{l,\lambda}$. But then we find that

$$d\sigma + d^*\tau = (d + d^*)\sigma + d^*(\tau - \sigma) \quad = (d + d^*)\sigma + d^*(d\alpha + d^*\beta + \kappa + \gamma) \quad = (d + d^*)\sigma + d^*(d\alpha + \gamma).$$

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By (74) we can write $\gamma = \gamma_c + \gamma_{cc} + \gamma_o$ for some closed form $\gamma_c$, some coclosed form $\gamma_{cc}$, and some form $\gamma_o \in \mathcal{O}^k_{l,\lambda}$. Therefore we have

$$d\sigma + d^*\tau = (d + d^*)(\sigma + d\alpha + \gamma_c + \gamma_o)$$

$$= (d + d^*)((\sigma + d\alpha + \gamma_c + \gamma_o) + d(-\gamma_o))$$

$$\in (d + d^*)(\Omega^k_{l,\lambda}) + (d\mathcal{O}^k_{l,\lambda})$$

which is what we wanted to show.

We are now ready to define the spaces $\mathcal{O}^k_{l,\lambda}$, which are closely related to the spaces $\mathcal{O}^k_{l,\lambda}$, and in fact for $k = 3$ the space $\mathcal{O}^1_{l,\lambda}$ will be exactly the obstruction space for our deformation problem.

**Definition 4.39.** We define the space $\mathcal{O}^k_{l,\lambda}$ for rate $\lambda$ as follows:

- In the $L^2$ setting (when $\lambda < -\frac{3}{4}$ for the AC case or for any $\lambda > 0$ in the CS case), we define $\mathcal{O}^k_{l,\lambda}$ to be a subspace of $d(\mathcal{O}^k_{l,\lambda})$ that is a topological complement to $(d + d^*)(\Omega^k_{l,\lambda})$ in $d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})$. That is,

$$d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda}) = (d + d^*)(\Omega^k_{l,\lambda}) \oplus \mathcal{O}^k_{l,\lambda},$$

with $\mathcal{O}^k_{l,\lambda} \subseteq d(\mathcal{O}^k_{l,\lambda})$. For example, we can choose $\mathcal{O}^k_{l,\lambda}$ to be the intersection with $d(\mathcal{O}^k_{l,\lambda})$ of the $L^2$-orthogonal complement in $d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})$ of $((d + d^*)(\Omega^k_{l,\lambda}) \cap d(\mathcal{O}^k_{l,\lambda})$). In particular, we have that $\mathcal{O}^k_{l,\lambda}$ is isomorphic to the quotient

$$\mathcal{O}^k_{l,\lambda} \cong (d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})) / (d + d^*)(\Omega^k_{l,\lambda}).$$

$$\tag{77}$$

- In the AC case when $\lambda > -4$, we set $\mathcal{O}^k_{l,\lambda} = \{0\}$.

Since, from Definition 4.37 we have that $d$ is injective on $\mathcal{O}^k_{l,\lambda}$, it follows from Lemma 4.38 that the dimension of $\mathcal{O}^k_{l,\lambda}$ is less than or equal to the dimension of $\mathcal{O}^k_{l,\lambda}$, which is finite. Note that the space $\mathcal{O}^k_{l,\lambda}$ is isomorphic to the cokernel of the map $\mathcal{D}^k_{l,\lambda}$.

**Lemma 4.40.** For generic rates $\lambda$, the space $d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda}) = (d + d^*)(\Omega^k_{l,\lambda}) + d(\mathcal{O}^k_{l,\lambda})$ is a Banach space.

**Proof.** If we assume that $\lambda + 1$ is noncritical for $d + d^*$, the map

$$(d + d^*)_{l+1,\lambda+1} : \Omega^*_{l+1,\lambda+1} \rightarrow \Omega^*_{l,\lambda}$$

is Fredholm, and thus has closed image. In fact, the Lockhart–McOwen theory [25, Section 2] says that at a noncritical rate, for any $\eta \in \Omega^k_{l,\lambda}$ that is orthogonal (with respect to the $L^2_{l,\lambda}$ inner product) to the kernel of $d + d^*$, we have the estimate

$$||\eta||_{L^2_{l,\lambda}} \leq C ||(d + d^*)\eta||_{L^2_{l-1,\lambda-1}}. \tag{78}$$

But from this estimate, it is a standard result [1, Corollary 2.15] that $(d + d^*)(\Omega^k_{l,\lambda})$ is a closed subspace of $\Omega^*_{l-1,\lambda-1}$, and thus a Banach space. Since $\mathcal{O}^k_{l,\lambda}$ is finite-dimensional, so is $d(\mathcal{O}^k_{l,\lambda})$ and thus $d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda}) = (d + d^*)(\Omega^k_{l,\lambda}) + d(\mathcal{O}^k_{l,\lambda})$ is a Banach space.

$\square$
The next proposition establishes that generically, $\mathcal{D}^k_{l,\lambda}$ is Fredholm, and it gives a characterization of the cokernel of $\mathcal{D}^k_{l,\lambda}$, which is isomorphic to $\mathcal{O}^k_{l,\lambda}$.

**Proposition 4.41.** Let $\lambda$ be a noncritical rate for $d + d^*$ on $M$.

(a) The map

$$\mathcal{D}^k_{l,\lambda} : \Omega^k_{l,\lambda} \to d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})$$

is Fredholm. Furthermore, ker $\mathcal{D}^k_{l,\lambda}$ is a subspace of ker$(d + d^*)_{l,\lambda}$, and coker $\mathcal{D}^k_{l,\lambda}$ is a subspace of coker$(d + d^*)_{l,\lambda}$, in the following sense: there exists a particular topological complement of im$(\mathcal{D}^k_{l,\lambda})$ in $d(\Omega^k_{l,\lambda}) + d^*(\Omega^k_{l,\lambda})$, and thus isomorphic to coker $\mathcal{D}^k_{l,\lambda}$, that is a subspace of the orthogonal complement of im$(d + d^*)_{l,\lambda}$ in $\Omega^{*^1}_{l-1,\lambda-1}$, with respect to the Hilbert space inner product.

(b) The space coker $\mathcal{D}^k_{l,\lambda}$ is isomorphic to the quotient of the space ker$(d + d^*)_{-6-\lambda} \cap (\Omega^{k-1} + \Omega^{k+1})$ of closed and coclosed forms of degree $k-1$ plus degree $k+1$ of rate $-6-\lambda$ by the subspace $\mathcal{H}^{-k}_{-6-\lambda} \oplus \mathcal{H}^{k+1}_{-6-\lambda}$ of closed and coclosed $(k-1)$-forms plus closed and coclosed $(k+1)$-forms of rate $-6-\lambda$.

**Proof.** For any $\lambda$, we have ker $\mathcal{D}^k_{l,\lambda} = \mathcal{H}^k_{l,\lambda}$ is finite-dimensional. We know from Definition 4.39 that coker $\mathcal{D}^k_{l,\lambda}$ is finite-dimensional. Finally, if $\lambda$ is not critical for $d + d^*$ on $M$, then we proved in Lemma 4.40 that $\mathcal{D}^k_{l,\lambda}$ has closed image. Thus $\mathcal{D}^k_{l,\lambda}$ is Fredholm. Next, we will prove the statements about the kernel and cokernel of $\mathcal{D}^k_{l,\lambda}$. The arguments are identical in the CS case (except for the fact that we have $n$ ends instead of just one, and the inequalities are reversed) so we prove just the AC case.

It is clear from the definition of $\mathcal{D}^k_{l,\lambda}$ that ker $\mathcal{D}^k_{l,\lambda}$ is a subspace of ker$(d + d^*)_{l,\lambda}$. We need to establish the analogous result for coker $\mathcal{D}^k_{l,\lambda}$. To simplify notation, in this proof only, we will use $E$ to denote the subspace $(d + d^*)(\Omega^k_{l,\lambda})$ of $\Omega^{*^1}_{l-1,\lambda-1}$, which is closed if $\lambda$ is noncritical for $d + d^*$. Also, let $F$ denote the orthogonal complement of $E$ with respect to the Hilbert space inner product on $\Omega^k_{l,\lambda}$. Thus we have

$$\Omega^{*^1}_{l-1,\lambda-1} = (d + d^*)(\Omega^k_{l,\lambda}) \oplus \text{coker}(d + d^*)_{l,\lambda} = E \oplus F$$

where in fact by Remark 4.14 we know that

$$F \cong \text{Ann}(E) = \text{ker}(d + d^*)_{-6-\lambda}$$ (79)

where Ann$(E)$ denotes the annihilator of $E$ in the dual space.

Now consider the orthogonal projection $P$ of $E$ onto the closed subspace $\Omega^{k-1}_{l-1,\lambda-1} \oplus \Omega^{k+1}_{l-1,\lambda-1}$. We have that

$$P(E) = d(\Omega^{k-2}_{l-1,\lambda-1}) + (d + d^*)(\Omega^k_{l,\lambda}) + d^*(\Omega^{k+2}_{l,\lambda}) = E'$$

is closed in the Hilbert space $\Omega^{k-1}_{l-1,\lambda-1} \oplus \Omega^{k+1}_{l-1,\lambda-1}$. Thus we can write

$$\Omega^{k-1}_{l-1,\lambda-1} \oplus \Omega^{k+1}_{l-1,\lambda-1} = E' \oplus F'$$

where we take $F'$ to be the orthogonal complement of $E'$ with respect to the Hilbert space inner product on $\Omega^{k-1}_{l-1,\lambda-1} \oplus \Omega^{k+1}_{l-1,\lambda-1}$. By Remark 4.14 we have

$$F' \cong \text{Ann}(E').$$ (80)
It is trivial to compute that
\[
\text{Ann}(E') = \ker(d + d^*)_{-6-\lambda} \cap (\Omega^{k-1} \oplus \Omega^{k+1}).
\] (81)
That is, \(E'\) is isomorphic to the space of forms of degree \(k-1\) plus degree \(k+1\) of rate \(-6-\lambda\) in the kernel of \(d + d^*\).

From Lemma 4.38 and Lemma 4.40 we have that
\[
d(\Omega_{t,\lambda}^k) + d^*(\Omega_{t,\lambda}^k) = (d + d^*)(\Omega_{t,\lambda}^k) + d(\hat{\Omega}_{t,\lambda}^k) = (d + d^*)(\Omega_{t,\lambda}^k) + \mathcal{O}_\lambda^k
\]
is closed in \(\Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1}\), and \(\text{coker} (\partial^k_{t,\lambda}) \cong \mathcal{O}_\lambda^k\). Note that \(\mathcal{O}_\lambda^k\) is a subspace of \((k+1)\)-forms, and is thus always transverse to \(d(\Omega_{t,\lambda}^{k-2})\). In addition, it is transverse to \(d^*(\Omega_{t,\lambda}^{k+2})\), because in the \(L^2\) setting the images of \(d\) and \(d^*\) are orthogonal, and in the non-\(L^2\) setting we know that \(\mathcal{O}_\lambda^k = \{0\}\). These observations tell us that we can also write
\[
\Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1} = E'' \oplus F''
\]
where
\[
E'' = d(\Omega_{t,\lambda}^{k-2}) + d(\Omega_{t,\lambda}^k) + d^*(\Omega_{t,\lambda}^k) + d^*(\Omega_{t,\lambda}^{k+2})
\]

\[= E' \oplus \mathcal{O}_\lambda^k,
\]
and \(F''\) is the orthogonal complement of \(E''\) with respect to the Hilbert space inner product on \(\Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1}\). We therefore clearly have
\[
F' = \mathcal{O}_\lambda^k \oplus F''
\] (82)
and we note again that by Remark 4.14 we have
\[
F'' \cong \text{Ann}(E'').
\] (83)
In this case it is easy to see that
\[
\text{Ann}(E'') = \mathcal{H}_{-6-\lambda}^{k-1} \oplus \mathcal{H}_{-6-\lambda}^{k+1}.
\] (84)
We pause here to note that equations (80), (81), (82), (83), and (84) together imply part (b) of the proposition.

Now we have that \(\text{coker} (\partial^k_{t,\lambda}) \cong \mathcal{O}_\lambda^k\), which is a subspace of \(F'\). But we see that
\[
F' = \{\gamma \in \Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1}; \langle (P\alpha, \gamma) \rangle_{\Omega_{t-1,\lambda-1}} = 0, \forall \alpha \in E\}
\]
\[= \{\gamma \in \Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1}; \langle (\alpha, \gamma) \rangle_{\Omega_{t-1,\lambda-1}} = 0, \forall \alpha \in E\}
\]
\[= \{\gamma \in \Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1}; \langle (\alpha, \gamma) \rangle_{\Omega_{t-1,\lambda-1}} = 0, \forall \alpha \in E\} \cap \left(\Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1}\right)
\]
\[= \left(\Omega_{t-1,\lambda-1}^{k-1} \oplus \Omega_{t-1,\lambda-1}^{k+1}\right) \subseteq F
\]
and the proof is complete. \qed
Remark 4.42. By Proposition 4.17, the quotient of \( \ker(d + d^*)_{-6 - \lambda} \cap (\Omega^{k-1} + \Omega^{k+1}) \) by the subspace \( \mathcal{H}^{k-1}_{-6 - \lambda} \oplus \mathcal{H}^{k+1}_{-6 - \lambda} \) (and thus \( \text{coker} \mathcal{D}^k_{-6 - \lambda} \)) will be zero when \(-6 - \lambda < -\frac52\) in the AC case. This reconfirms the fact that \( \text{coker} \mathcal{D}^k_{-6 - \lambda} = \mathcal{O}^k_{-6} \) is zero in the non-\( L^2 \) setting of the AC case, corresponding to \( \lambda > -\frac72 \).

We pause here to state and prove an important result about homogeneous forms on a cone, namely Theorem 4.45 below, which relates closed and coclosed \( k \)-forms on \( C \), homogeneous of order \( \lambda \), to a particular subspace of forms on the cone \( C \) of degree \( k - 1 \) plus degree \( k + 1 \), homogeneous of order \(-6 - \lambda \), in the kernel of \( d + d^* \). This result will be used in Theorem 4.49 to establish an index change formula for \( \mathcal{D}^k_{-6} \). Before we can state the theorem, we need to define several spaces.

**Notation 4.43.** Consider a form \( \gamma \) of degree \( k - 1 \) plus degree \( k + 1 \) on the cone, homogeneous of order \(-6 - \lambda \). Using Definition 3.2, we can write

\[
\gamma = r^{-6 - \lambda} (r^{k-2} dr \wedge \beta_{k-2} + r^{k-1} \beta_{k-1} + r^k dr \wedge \beta_k + r^{k+1} \beta_{k+1})
\]

where each \( \beta_m \) is an \( m \)-form on \( \Sigma \). We will write this form as a 4-tuple \((\beta_{k-2}, \beta_{k-1}, \beta_k, \beta_{k+1})\). From the equations \(25\), it follows easily that \( \gamma \) is in the kernel of \( d + d^* \) if and only if

\[
\begin{align*}
d^*_s \beta_{k-2} &= 0, & d^*_s \beta_{k-1} &= - (\lambda + k - 2) \beta_{k-2}, \\
d_s \beta_k &= - (\lambda - k + 5) \beta_{k+1}, & d_s \beta_{k+1} &= 0, \\
d_s \beta_{k-2} + d^*_s \beta_k &= - (\lambda - k + 7) \beta_{k-1}, & d_s \beta_{k-1} + d^*_s \beta_{k+1} &= - (\lambda + k) \beta_k.
\end{align*}
\]

We will denote by \( A(\lambda) \) the space of solutions to the system of equations \(86\). Let \( B(\lambda) \) denote the subspace consisting of forms \( \gamma \in A(\lambda) \) of degree \( k - 1 \) plus degree \( k + 1 \), homogeneous of order \(-6 - \lambda \), such that each pure degree component \( \gamma_{k-1} = r^{-6 - \lambda} (r^{k-2} dr \wedge \beta_{k-2} + r^{k-1} \beta_{k-1}) \) and \( \gamma_{k+1} = r^{-6 - \lambda} (r^k dr \wedge \beta_k + r^{k+1} \beta_{k+1}) \) is independently closed and coclosed. Again using equations \(25\), we find that \( \gamma \) is in \( B(\lambda) \) if and only if, in addition to equations \(86\), we also have

\[
\begin{align*}
d_s \beta_{k-1} &= 0, & d^*_s \beta_k &= 0.
\end{align*}
\]

Finally, denote \( C(\lambda) \) to be the subspace of \( A(\lambda) \) consisting of forms of the type \(85\) with \( \beta_{k-2} = 0 \) and \( \beta_{k+1} = 0 \). That is, \( \gamma \) lies in \( C(\lambda) \) if and only if \( \gamma = r^{-6 - \lambda} (r^{k-1} \beta_{k-1} + r^k dr \wedge \beta_k) \) with

\[
\begin{align*}
d^*_s \beta_{k-1} &= 0, & d^*_s \beta_k &= 0, \\
d_s \beta_{k-1} &= -(\lambda - k + 7) \beta_{k-1}, & d_s \beta_{k-1} &= -(\lambda + k) \beta_k.
\end{align*}
\]

Remark 4.44. From Lemma 4.36, we note that \((0, \beta_{k-1}, \beta_k, 0) \in C(\lambda) \) if and only if the homogeneous \( k \)-form \( r^\lambda (r^{k-1} dr \wedge \beta_{k-1} + r^k dr \wedge \beta_k) \) is closed and coclosed. That is, the space \( C(\lambda) \) is isomorphic to the space of closed and coclosed \( k \)-forms on the cone, homogeneous of order \( \lambda \).

**Theorem 4.45.** We have \( C(-k) \subseteq B(-k) \) and \( C(k - 7) \subseteq B(k - 7) \). Furthermore, if \( \lambda \neq -k \) and \( \lambda \neq k - 7 \), then \( A(\lambda) = B(\lambda) \oplus C(\lambda) \), where the direct sum is orthogonal with respect to the \( L^2 \) inner product on forms on \( \Sigma \). That is, for \( \lambda \not\in \{-k, k - 7\} \), the subspace of forms on the cone of degree \( k - 1 \) plus degree \( k + 1 \), homogeneous of order \(-6 - \lambda \), in the kernel of \( d + d^* \), and \( L^2 \)-orthogonal to those forms which are independently closed and coclosed, is isomorphic to the space of closed and coclosed \( k \)-forms, homogeneous of order \( \lambda \).

**Proof.** Suppose \( \lambda = -k \), and that \((0, \beta_{k-1}, \beta_k, 0) \in C(-k) \). Then equations \(88\) say that \( \beta_{k-1} \) is a closed and coclosed (thus harmonic) \((k - 1)\)-form on \( \Sigma \), which is also coexact. By Hodge
theory, we get $\beta_{k-1} = 0$ and hence $\beta_k$ is a harmonic $3$-form on $\Sigma$. But then $(0, 0, \beta_k, 0)$ satisfies the
equations \([86]\) and \([87]\), and thus lies in $B(-k)$. The proof of $C(k - 7) \subseteq B(k - 7)$ is similar.

Next we show that if $\lambda \notin \{-k, -k - 7\}$, the subspaces $B(\lambda)$ and $C(\lambda)$ are $L^2$-orthogonal. Suppose
$(\beta_{k-2}, \beta_{k-1}, \beta_k, \beta_{k+1}) \in B(\lambda)$ and $(0, \gamma_{k-1}, \gamma_k, 0) \in C(\lambda)$. Then using equations \([86]\) and \([87]\) for
$(\beta_{k-2}, \beta_{k-1}, \beta_k, \beta_{k+1})$, and equations \([88]\) for $(0, \gamma_{k-1}, \gamma_k, 0)$ we compute that
\[
\langle \langle \beta_{k-1}, \gamma_{k-1} \rangle \rangle = \langle \langle -(\lambda - k + 7)^{-1}(d_\Sigma \beta_{k-2} + d_\Sigma^* \beta_k), -(\lambda - k + 7)^{-1}d_\Sigma^* \gamma_k \rangle \rangle
\]
\[
= (\lambda - k + 7)^{-2}\langle \langle d_\Sigma \beta_{k-2}, d_\Sigma^* \gamma_k \rangle \rangle = 0
\]
and similarly
\[
\langle \langle \beta_k, \gamma_k \rangle \rangle = \langle \langle -(\lambda + k)^{-1}(d_\Sigma \beta_{k-1} + d_\Sigma^* \beta_{k+1}), -(\lambda + k)^{-1}d_\Sigma \gamma_{k-1} \rangle \rangle
\]
\[
= (\lambda + k)^{-2}\langle \langle d_\Sigma \beta_{k-1}, d_\Sigma \gamma_{k-1} \rangle \rangle = 0.
\]
Thus we indeed have $B(\lambda) \perp C(\lambda)$.

Finally, we will complete the proof by showing that if $(\gamma_{k-2}, \gamma_{k-1}, \gamma_k, \gamma_{k+1}) \in A(\lambda)$ is $L^2$-
orthogonal to $B(\lambda)$, then it is in $C(\lambda)$. This would imply that $A(\lambda) = B(\lambda) \oplus C(\lambda)$, as claimed.
Define $(\beta_{k-2}, \beta_{k-1}, \beta_k, \beta_{k+1})$ by
\[
\beta_{k-2} = \gamma_{k-2}, \quad \beta_{k-1} = -(\lambda - k + 7)^{-1}d_\Sigma \gamma_{k-2},
\beta_k = -(\lambda + k)^{-1}d_\Sigma^* \gamma_{k-2}, \quad \beta_{k+1} = \gamma_{k+1}.
\]
Using the fact that $(\gamma_{k-2}, \gamma_{k-1}, \gamma_k, \gamma_{k+1})$ satisfies equations \([86]\), it follows that $(\beta_{k-2}, \beta_{k-1}, \beta_k, \beta_{k+1})$
satisfies equations \([86]\) and \([87]\), so $(\beta_{k-2}, \beta_{k-1}, \beta_k, \beta_{k+1})$ lies in $B(\lambda)$. Our hypothesis is that $(\gamma_{k-2}, \gamma_{k-1}, \gamma_k, \gamma_{k+1})$ is $L^2$-
orthogonal to the space $B(\lambda)$. Thus we have
\[
\langle \langle \gamma_{k-2}, \beta_{k-2} \rangle \rangle = ||\gamma_{k-2}||^2,
\langle \langle \gamma_{k-1}, \beta_{k-1} \rangle \rangle = \langle \langle \gamma_{k-1}, -(\lambda - k + 7)^{-1}d_\Sigma \gamma_{k-2} \rangle \rangle = -(\lambda - k + 7)^{-1}\langle \langle d_\Sigma^* \gamma_{k-1}, \gamma_{k-2} \rangle \rangle
\]
\[
= -(\lambda - k + 7)^{-1}(\lambda + k + 2)\langle \langle \gamma_{k-2}, \gamma_{k-2} \rangle \rangle = \frac{\lambda + k - 2}{\lambda - k + 7} ||\gamma_{k-2}||^2,
\langle \langle \gamma_k, \beta_k \rangle \rangle = \langle \langle \gamma_k, -(\lambda + k)^{-1}d_\Sigma^* \gamma_{k-1} \rangle \rangle = -(\lambda + k)^{-1}\langle \langle d_\Sigma \gamma_k, \gamma_{k+1} \rangle \rangle
\]
\[
= -(\lambda + k)^{-1}(\lambda + k + 5)\langle \langle \gamma_{k+1}, \gamma_{k+1} \rangle \rangle = \frac{\lambda + 5}{\lambda + k} ||\gamma_{k+1}||^2,
\langle \langle \gamma_{k+1}, \beta_{k+1} \rangle \rangle = ||\gamma_{k+1}||^2.
\]
Thus, since $(\gamma_{k-2}, \gamma_{k-1}, \gamma_k, \gamma_{k+1})$ is $L^2$-orthogonal to $(\beta_{k-2}, \beta_{k-1}, \beta_k, \beta_{k+1})$, we find that
\[
\sum_{m=k-2}^{k+1} \langle \langle \gamma_m, \beta_m \rangle \rangle = \left(1 + \frac{\lambda + k - 2}{\lambda - k + 7} \right) ||\gamma_{k-2}||^2 + \left(1 + \frac{\lambda - k + 5}{\lambda + k} \right) ||\gamma_{k+1}||^2 = 0. \quad \text{(89)}
\]
We have $\Delta \gamma_{k-2} = d_\Sigma d_\Sigma^* \gamma_{k-2} + d_\Sigma^* d_\Sigma \gamma_{k-2} = -(\lambda - k + 7)d_\Sigma^* \gamma_{k-1} = (\lambda + k - 2)(\lambda - k + 7)\gamma_{k-2}$. Thus by
the nonnegativity of the Hodge Laplacian, we have $\gamma_{k-2} = 0$ if $(\lambda + k - 2)(\lambda - k + 7) < 0$. Conversely, if $(\lambda + k - 2)(\lambda - k + 7) \geq 0$, then since $\lambda \neq -k + 7$ we must have $1 + \frac{\lambda + k - 2}{\lambda - k + 7} > 0$. Similarly we observe
that $\Delta \gamma_{k-1} = d_\Sigma d_\Sigma^* \gamma_{k-1} + d_\Sigma^* d_\Sigma \gamma_{k-1} = (\lambda + k)(\lambda - k + 5)\gamma_{k-1}$. Thus we have $\gamma_{k+1} = 0$ if $(\lambda + k)(\lambda - k + 5) < 0$. Conversely, if $(\lambda + k)(\lambda - k + 5) \geq 0$, then since $\lambda \neq -k$ we must
have \( 1 + \frac{\lambda_+ + \frac{5}{2}}{\lambda + k} > 0 \). Thus we conclude that in all cases when \( \lambda \neq -k \) and \( \lambda \neq k - 7 \), equation (89) tells us that \( \gamma_k - 1 = 0 \) and \( \gamma_{k+1} = 0 \). Thus indeed we have \( (\gamma_{k-2}, \gamma_{k-1}, \gamma_k, \gamma_{k+1}) \in C(\lambda) \), and the proof is complete.

**Remark 4.46.** Essentially, Theorem 4.45 and Remark 4.44 together say that on the cone, if \( \lambda \notin \{-k, k - 7\} \), then

\[
\mathcal{H}^{(k-1)+(k+1)}_{-6-\lambda} \cong \mathcal{H}^k_{-6-\lambda} \oplus \mathcal{H}^{k+1}_{-6-\lambda},
\]

where the notation should be self-explanatory.

The next two results give a “partial” converse to Lemma 4.26 that will be needed in Theorem 4.49, which is about the index change of the operator \( \mathcal{D} \) that controls infinitesimal deformations of \( G_2 \) conifolds.

**Lemma 4.47.** Let \( \lambda_- \) and \( \lambda_+ \) be two noncritical rates for \( d + d_M^* \), such that

\[
\lambda_- < \lambda_+ \quad \text{(AC)},
\]

\[
\lambda_- > \lambda_+ \quad \text{(CS)}.
\]

In either case \( \lambda_+ \) is a slower rate of decay and \( \lambda_- \) is a faster rate of decay on the ends.

Suppose further that the closed interval \([\lambda_-, \lambda_+]\) contains a single exceptional rate \( \lambda_0 \) for \( d + d_M^* \).

Let \( \gamma_- \in \ker(d + d_M^*)_{\lambda_-} \), and \( \gamma_+ \in \ker(d + d_M^*)_{\lambda_+} \). By Proposition 4.21, there exists a form \( v \in K(\lambda_0)_{d + d_M^* C_i} \) such that

\[
\gamma_+ - (h_i^{-1})^* v - \tilde{\gamma} = \gamma_-,
\]

for some form \( \tilde{\gamma} \), where in particular \((d + d_M^*)((h_i^{-1})^* v + \tilde{\gamma}) = 0\). Suppose that each pure degree piece \( (\gamma_-)_m \) of \( \gamma_- \) is individually closed and coclosed. If the form \( v \) is a pure degree \( k \) form on \( C_i \), then each pure degree piece \( (\gamma_+)_m \) of \( \gamma_+ \) is individually closed and coclosed.

**Proof.** First, we claim that we can assume that \( \tilde{\gamma}_m = 0 \) for \( m \neq k \). In other words, we can take \( \tilde{\gamma} \) to be pure degree if \( v \) is pure degree. This can be seen by adapting the techniques from Lockhart–McOwen [25 Section 3] to the operator \( \mathcal{D}_M^* : \Omega^k_M \to d(\Omega^k_M) + d^*(\Omega^k_M) \). Although this operator is not elliptic, by Proposition [4.44] it is Fredholm at noncritical rates, and because it is the restriction of a self-adjoint elliptic operator, its kernel and the kernel of its adjoint consist of smooth sections. These properties of elliptic regularity and Fredholmness are all that are required for the Lockhart–McOwen techniques in [25 Section 3] to apply.

Hence without loss of generality \((h_i^{-1})^* (v) + \tilde{\gamma}\) is of pure degree \( k \) and thus closed and coclosed. Therefore we have \((\gamma_+)_m = (\gamma_-)_m \) for all \( m \neq k \), so each \((\gamma_+)_l \) for \( l \neq k \) is individually closed and coclosed. But then

\[
(\gamma_+)_k = (h_i^{-1})^* v + \tilde{\gamma} + (\gamma_-)_k
\]

is also closed and coclosed.

**Corollary 4.48.** Let \( \lambda_- \) and \( \lambda_+ \) be as in the first paragraph of the statement of Lemma 4.47. Let \( \lambda_1, \ldots, \lambda_N \) be all of the critical rates of \( d + d_M^* \) in the interval \([\lambda_-, \lambda_+]\), for some \( N \geq 1 \).

Suppose that both of the following conditions hold:

1. We have \( \lambda_- < -\frac{5}{2} \) (AC) or \( \lambda_- > -\frac{5}{2} \) (CS).
2. The forms \( v_j \in K(\lambda_j)_{d + d_M^* C_i} \) for \( 1 \leq j \leq N \) from Proposition 4.21 are all pure degree \( k \).
Then for any \( \gamma_+ \in \ker(d + d^*)\lambda_+ \), each pure degree piece of \( \gamma_+ \) is individually closed and coclosed.

**Proof.** This is a simple induction from Lemma 4.47. The base case \( N = 1 \) holds because by condition (1) and Proposition 4.17 each pure degree piece \( (\gamma_+)_{\ell} \) of \( \gamma_- \) is individually closed and coclosed. \( \square \)

Finally, we use Theorem 4.49 and Proposition 4.41 to identify precisely when the index of \( D^k_{l,\lambda} \) can change.

**Theorem 4.49.** The index of \( D^k_{l,\lambda} \) can only change as we cross a rate \( \lambda \) if \( \lambda \) is a critical rate of \( D^k_{l,\lambda} \) in the sense of Definition 4.34. As we cross a critical rate, there exists a basis for each space \( K(\lambda_{i'})_{D^k_{l,\lambda}} \) of equation \((72)\), such that each basis element contributes to either change \( \ker D^k_{l,\lambda} \) or to change \( \text{coker} D^k_{l,\lambda} \).

**Proof.** Since \( \ker D^k_{l,\lambda} = H^k_{l,\lambda} \), Lemma 4.26 says that \( \ker D^k_{l,\lambda} \) can only change at a critical rate for \( D^k \). Now consider \( \text{coker} D^k_{l,\lambda} \). By Proposition 4.41 the cokernel will only change as we cross the rate \( \lambda \) if there exist new elements of \( \ker(d + d^*)_{-6 - \lambda} \) that are in \( \Omega^{k-1} \oplus \Omega^{k+1} \) but are transverse to the subspace \( H^{k-1}_{6-\lambda} \oplus H^{k-1}_{6-\lambda} \). However, by the contrapositive of Corollary 4.48, we know that such elements must be asymptotic at the \( i \)-th end to forms of degree \( k - 1 \) plus degree \( k + 1 \) on the cone \( C_i \), homogeneous of order \(-6 - \lambda \), which are in the kernel of \( d + d_{C_i}^* \), but which are transverse to the spaces of closed and coclosed \((k-1)\)-forms and closed and coclosed \((k+1)\)-forms on \( C_i \), homogeneous of order \(-6 - \lambda \). In the language of Notation 4.43 this corresponds to rates \( \lambda \) for which the quotient space \( A(\lambda)/B(\lambda) \) is nonzero for some asymptotic cone \( C_i \). If \( \lambda \not\in \{-k,k-7\} \), then Theorem 4.45 says that this quotient \( A(\lambda)/B(\lambda) \) is \( C(\lambda) \), and thus by Remark 4.44 such a \( \lambda \) is a critical rate of \( D^k_{l,\lambda} \). For \( \lambda \in \{-k,k-7\} \), Theorem 4.45 tells us that \( C(\lambda) \subseteq B(\lambda) \), so these rates cannot contribute to changes in coker \( D^k_{l,\lambda} \). [In fact, we will show exactly how these rates change ker \( D^k_{l,\lambda} = H^k_{l,\lambda} \) when \( k = 3 \) later in Proposition 4.61.]

Finally, the second statement of the theorem follows from Theorem 4.20 since every critical rate for \( D^k \) is a critical rate for \( d + d^* \), and the Lockhart–McOwen index change formulas \((59)\) and \((60)\) show that only one of \( \ker(d + d^*) \) or \( \text{coker}(d + d^*) \) changes for each independent contribution to \( K(\lambda_{i'})_{D^k_{l,\lambda}} \).

We are now in a position to prove our index change formula for the operator \( D^k_{l,\lambda} \), away from the exceptional rates \(-k \) and \( k-7 \).

**Theorem 4.50.** Let \( \nu < \mu \) be two noncritical rates for \( D^k \) on \( M \). Suppose that both \(-k \) and \( k-7 \) are not in the set \( D^k_{0,\lambda} \cap (\nu, \mu) \) of critical rates between \( \nu \) and \( \mu \). Then the difference in the indices of \( D^k_{l,\nu} \) and \( D^k_{l,\mu} \) is given by

\[
\text{ind}(D^k_{l,\mu}) - \text{ind}(D^k_{l,\nu}) = \sum_{\lambda \in D^k_{l,\nu} \cap (\nu, \mu)} \dim K(\lambda)_{D^k_{l,\nu}}. \quad (AC)
\]

\[
\text{ind}(D^k_{l,\mu}) - \text{ind}(D^k_{l,\nu}) = -\sum_{i=1}^n \sum_{\lambda \in D^k_{l,\nu} \cap (\nu, \mu)} \dim K(\lambda)_{D^k_{l,\nu}}. \quad (CS)
\]

**Proof.** From Theorem 4.49, either the kernel or the cokernel of \( D^k_{l,\lambda} \) will change at each critical rate of \( D^k \). In the AC case, as \( \lambda \) increases, the kernel can either increase or the cokernel can decrease, and in each case the integer by which these dimensions change is exactly \( \dim K(\lambda)_{D^k_{l,\nu}} \). Thus the
Remark 4.55. The images of the maps $\Upsilon^\gamma$ are related to topological obstructions to the desingularization of CS $G_2$ conifolds, as discussed in [20, Section 5].
From [14 §2.4], any conifold \( M \) gives rise to a long exact sequence

\[
\cdots \to H^k_{cs}(M, \mathbb{R}) \xrightarrow{\mathcal{T}^k} H^k(M, \mathbb{R}) \xrightarrow{\mathcal{Y}^k} \bigoplus_{i=1}^n H^k(\Sigma_i, \mathbb{R}) \xrightarrow{\partial^k} H^{k+1}_{cs}(M, \mathbb{R}) \to \cdots
\]  

(94)

where \( \mathcal{Y}^k : H^k(M, \mathbb{R}) \to \bigoplus_{i=1}^n H^k(\Sigma_i, \mathbb{R}) \) is the map from Definition 4.54 and \( \mathcal{T}^k : H^k_{cs}(M, \mathbb{R}) \to H^k(M, \mathbb{R}) \) is the natural map induced from inclusion of the complex of compactly supported forms into the complex of all smooth forms. This is the long exact sequence for cohomology of \( M \) relative to its topological boundary \( \Sigma \).

Let \( b^k = \dim H^k(M, \mathbb{R}) \) and \( b^k_{cs} = \dim H^k_{cs}(M) \) be the ordinary and compactly supported \( k \)th Betti numbers of \( M \), respectively. Note that by Poincaré duality we have \( H^k(M, \mathbb{R}) \cong H^{7-k}_{cs}(M, \mathbb{R}) \) and thus \( b^k = b^7-k \). The next lemma contains results that will be used to compute the virtual dimension of the conifold moduli space in Section 5.2.4 and for the applications in Section 6.4.

**Lemma 4.56.** Let \( M \) be a G\(_2\) conifold. The following equations hold.

\[
b^k - \dim(\text{im } \mathcal{Y}^k) = \dim(\text{im } \mathcal{T}^k) = \dim(\text{im } (H^k_{cs} \to H^k)), \tag{95}
\]

\[
\dim(\ker \mathcal{Y}^k) = b^k - \dim(\text{im } \mathcal{Y}^k), \tag{96}
\]

\[
\dim(\ker \mathcal{Y}^k) = b^k_{cs} - \dim(\text{im } (H^k_{cs} \to H^k)), \tag{97}
\]

\[
\dim(H^k(\Sigma, \mathbb{R})) = \dim(\text{im } \mathcal{Y}^k) + \dim(\text{im } \mathcal{T}^k). \tag{98}
\]

**Proof.** Equation (96) is just the rank-nullity theorem. From (96) and the exactness of (94), we find

\[
b^k = \dim(\text{im } \mathcal{Y}^k) + \dim(\ker \mathcal{Y}^k) = \dim(\text{im } \mathcal{T}^k) \]

from which we immediately obtain (95).

Using the fact that the Hodge star operator takes compactly supported forms to compactly supported forms, it is easy to see that the Poincaré pairing between \( \ker \mathcal{Y}^k \) and \( \ker \mathcal{Y}^{7-k} \) given by \([\alpha], [\beta] \mapsto \int_M (\alpha \wedge \beta)\) is nondegenerate. Hence \( \dim(\ker \mathcal{Y}^k) = \dim(\ker \mathcal{Y}^{7-k}) \). Thus we have

\[
\dim(\ker \mathcal{Y}^k) = \dim(\ker \mathcal{Y}^{7-k})
= b^{7-k} - \dim(\text{im } \mathcal{Y}^{7-k})
= b^k_{cs} - \dim(\text{im } \mathcal{Y}^{7-k})
\]

which establishes (97). For equation (98), we apply repeatedly the long exact sequence (94) and rank-nullity, to obtain

\[
\dim(H^k(\Sigma, \mathbb{R})) - \dim(\text{im } \mathcal{Y}^k) = \dim(H^k(\Sigma, \mathbb{R})) - \dim(\ker \partial^k)
= \dim(\text{im } \partial^k)
= \dim(\ker \mathcal{T}^{k+1})
= b^{k+1}_{cs} - \dim(\text{im } \mathcal{T}^{k+1})
= b^{k+1}_{cs} - \dim(\ker \mathcal{Y}^{k+1})
= \dim(\text{im } \mathcal{Y}^{7-k})
\]

where we have used (97) in the last step. \( \square \)
Choosing \( \varepsilon > M \) inequality in the definition of \( d \) and hence parts is valid: by Corollary 4.27, we have \( \beta \) Thus we have

\[
\{(\im \Upsilon_r \cap \Sigma) \cup \Sigma \}\} = \{\} \text{ and } \{\Sigma \} = \{\} \text{ on the (im } \Upsilon_r \cap \Sigma\text{) such that } d\zeta = 0 \text{ and } (\im \Upsilon_r \cap \Sigma)(\zeta) = \gamma. \text{ Then } d^*\mu \zeta \in \Omega^2_{i-4+\varepsilon} \text{ for any } \varepsilon > 0. \text{ By Theorem 4.13, for }

\]

\[
\chi^3(\eta) = \eta \in H^3(M, \mathbb{R}),
\]

\[
\chi^4(\eta) = \eta \in H^4(M, \mathbb{R}).
\]

The next two lemmas are used in the proof of Proposition 4.61.

**Lemma 4.58.** Let \( M \) be an AC \( G_2 \) conifold. Let \( [\gamma] \) be a \( \Lambda^3 \) \( G_2 \) 3-form such that \( \gamma \) is a smooth 3-form \( \zeta \) on \( M \) such that \( |\zeta| = O(\varepsilon^3) \) on the ends with \( d\zeta = 0 \) and \( (\im \Upsilon_r \cap \Sigma)(\zeta) = \gamma \). Then \( d^*\mu \zeta \in \Omega^2_{i-4+\varepsilon} \text{ for any } \varepsilon > 0. \text{ By Theorem 4.13, for }

\]

\[
\Omega^2_{i-4+\varepsilon} = \Delta_M(\Omega^2_{i+2,2+\varepsilon}) = B^2_{-4+\varepsilon}
\]

where

\[
B^2_{-4+\varepsilon} = \{z \in \Omega^2_{i+2,2+\varepsilon} : \Delta_M z = 0\},
\]

\[
	ext{since } -7 - (-4 + \varepsilon) = -3 - \varepsilon. \text{ Now Corollary 4.18 implies that } d\beta = 0 \text{ and } d^*\mu \beta = 0, \text{ and therefore by Corollary 4.27, we have } \beta \in \mathcal{H}^2_{-4+\varepsilon} = H^2_{-4+\varepsilon}. \text{ We deduce that we can therefore write}
\]

\[
\Omega^2_{i-4+\varepsilon} = \Delta_M(\Omega^2_{i+2,2+\varepsilon}) \oplus H^2_{-4+\varepsilon}. \quad (99)
\]

Hence, there exist \( \alpha \in \Omega^2_{i+2,2+\varepsilon} \) and \( \beta \in H^2_{-4+\varepsilon} \) such that

\[
d^*\mu \zeta = \Delta_M \alpha + \beta = dd^* \alpha + d^* \mu \alpha + \beta.
\]

Choosing \( \varepsilon > 0 \) such that \(-4 + \varepsilon < -\frac{7}{2}\), we have by Lemma 4.16 that the following integration by parts is valid:

\[
||d^* \zeta - d^* \mu \alpha||^2 = \langle (d^* \zeta - d^* \mu \alpha), d^* \mu \alpha + \beta \rangle = \langle (\zeta - \mu \alpha), d(d^* \mu \alpha + \beta) \rangle = 0,
\]

and hence \( d^* \mu \zeta = d^* \mu \alpha \). Thus, if we let \( \eta = \zeta - \mu \alpha \), we have \( \eta \in H^3_{-3+\varepsilon} = H^3_{-4-\varepsilon} \text{ and } (\im \Upsilon_r \cap \Sigma)(\eta) = (\im \Upsilon_r \cap \Sigma)(\zeta) = \gamma \), as required.
Lemma 4.59. Let $M$ be a a CS $G_2$ conifold. Let $[\sigma] \in \text{im} \, \Upsilon^4 \subseteq H^4(\Sigma, \mathbb{R})$. There exists $\eta \in \mathcal{H}^4_{4-\epsilon}$ such that $(\Upsilon^4 \circ \chi^4)(*) = [\sigma]$.

Proof. In order to construct $\eta$, we need to first understand the precise nature of the index change of $d + d^*_M$ at rate $-4$. The operator $P_\lambda = d + d^*_M : \Omega^{even}_{k+1-\lambda} \to \Omega^{even}_{k-\lambda-1}$ is an elliptic operator with adjoint

$$P^*_{-6-\lambda} = d + d^*_M : \Omega^{even}_{k+1,-6-\lambda} \to \Omega^{odd}_{k,-7-\lambda}.$$

By Theorem 4.20 we know that the change of the index of $P_\lambda$ as we cross $\lambda = -4$ is given by the dimension of the space of homogeneous odd degree forms on the cone $C$ with rate $-4$ in the kernel of $d + d^*$. From Proposition 3.11 this space of forms decomposes as $U_1 \oplus U_2$ where

$$U_1 = \{ r^{-2} dr \wedge \alpha; \alpha \in \Omega^2(\Sigma), \Delta_3 \alpha = 0 \}$$

and

$$U_2 = \{ dr \wedge \mu - r^{-1} d^*_M \mu; \mu \in d(\Omega^3(\Sigma)), \Delta_3 \mu = 2 \mu \}.$$

We notice that $U_1$ consists of closed and coclosed 3-forms, but that $U_2$ consists of mixed degree $3 + 5$ forms. Again from Proposition 3.11 we determine the corresponding space of homogeneous even degree forms on the cone $C$ with rate $-2$ in the kernel of $d + d^*$ to be $V_1 \oplus V_2$, where

$$V_1 = \{ \alpha; \alpha \in \Omega^2(\Sigma), \Delta_3 \alpha = 0 \}$$

and

$$V_2 = \{ r dr \wedge d^*_M \mu + r^2 \mu; \mu \in d(\Omega^3(\Sigma)), \Delta_3 \mu = 2 \mu \}.$$

We observe that we have natural isomorphisms $U_j \cong V_j$ for $j = 1, 2$, and that $V_1$ consists of closed and coclosed 2-forms whereas $V_2$ consists of closed and coclosed 4-forms.

Thus, by Theorem 4.20 we have that

$$\text{ind}(P_{-4-\epsilon}) - \text{ind}(P_{-4+\epsilon}) = (\dim \ker P_{-4-\epsilon} - \dim \ker P_{4+\epsilon}) + (\dim \ker P_{-2-\epsilon} - \dim \ker P_{-2+\epsilon})$$

$$= \dim U_1 + \dim U_2.$$ 

Now from Proposition 4.21 we know that a change in the kernel of $P_\lambda$ occurs as $\lambda$ crosses $-4$ if and only if there exists $\beta$ in $\ker P_{-4-\epsilon}$ which is asymptotic to $\nu \in U_1 \oplus U_2$ on the end, and a similar change in the kernel of $P_{-6-\lambda}$ occurs if and only if there exists $\gamma$ in $\ker P_{-2-\epsilon}$ asymptotic to an element of $V_1 \oplus V_2$. We want to calculate the change in the closed and coclosed 3-forms $\mathcal{H}_3 \subseteq \ker P_\lambda$ as $\lambda$ crosses $-4$, since this will be the same as the change in the closed and coclosed 4-forms at rate $-4$ by taking the Hodge star.

If $\beta \in \mathcal{H}_3$ is asymptotic to some $\nu \in U_1 \oplus U_2$ on the end, then Lemma 4.26 implies that $\nu$ is a 3-form which is closed and coclosed with rate $-4$, so in fact $\nu \in U_1$. Conversely, suppose $\beta \in \ker P_{-4-\epsilon}$ is asymptotic to $\nu \in U_1$. Then by Lemma 4.26 we can write $\beta = \beta_3 + \tilde{\beta}$ where $\beta_3$ is a 3-form of order $O(\epsilon^{-4-\epsilon})$ and $\tilde{\beta} \in \ker P_{-4+\epsilon}$. Hence $\beta_3 \in \mathcal{H}_3$. We conclude that $\beta \in \ker P_{-4-\epsilon}$ is asymptotic to $v_1 \in U_1$ if and only if $\beta \in \mathcal{H}_3$.

We now need to determine which elements $\nu$ of $U_1$ do in fact generate elements $\beta$ of $\mathcal{H}_3$. Equivalently, we can study which elements of $V_1$ add to $\ker P_{-2-\epsilon}$. Notice that since we may take $-2 - \epsilon > -\frac{5}{2}$, by Proposition 4.17 the space of forms $\gamma$ such that each pure degree component is independently closed and coclosed. We also see that $V_1$ consists of 2-forms and $V_2$
consists of 4-forms, so if $\gamma \in \ker P_{-2-\varepsilon}$ is asymptotic to $\alpha \in V_1$, we may assume without loss of generality that $\gamma$ is a 2-form, since the other degree parts of $\gamma$ will be closed and coclosed and have slower decay and thus lie in $\ker P_{-2-\varepsilon}$. We also see that by Lemma 4.26 that on the end we can write

$$\gamma = (h^{-1}) \ast \alpha + \tilde{\gamma}$$

where $|\tilde{\gamma}| = O(\rho^{-2-\varepsilon+\nu})$. Moreover, $d\tilde{\gamma} = 0$ since $d\gamma = 0$ and $d\alpha = 0$. Therefore, choosing $\varepsilon > 0$ sufficiently small that $-2 - \varepsilon + \nu > -2$, we may apply [20, Lemma 2.12] to deduce that the 2-form part of $\tilde{\gamma}$ is exact, and thus $\Upsilon^2[\gamma] = [\alpha]$. Hence a necessary condition for $\gamma$ to be asymptotic to $\alpha$ is that $[\alpha] \in \im \Upsilon^2$. Conversely, take $\alpha \in V_1$ with $[\alpha] \in \im \Upsilon^2$. We can show, exactly as in the proof of Lemma 4.58, that there exists $\gamma \in \mathcal{H}_{2-\varepsilon} \subseteq \ker P_{-2-\varepsilon}$ asymptotic to $\alpha$ with $\Upsilon^2[\gamma] = [\alpha]$. Thus we conclude that $\alpha \in V_1$ defines a new element of $\ker P^*_{-\delta-\lambda}$ as $\lambda$ crosses $-4$ if and only if $[\alpha] \in \im \Upsilon^2$. [We pause here to remark that the reason this proof is more complicated than the proof of Lemma 4.58 is precisely because the argument in that proof will work in the CS case for rate $-2$ but not for $-4$. This is why we consider the dual 2-forms rather than the 4-forms in this proof.]

Now we have $V_1 \cong H^2(\Sigma, \mathbb{R}) \cong U_1$ and we know that all of $H^2(\Sigma, \mathbb{R})$ must contribute to changes in the kernel or cokernel of $P_3$ as $\lambda$ crosses $-4$ by the index change formula. Thus the $v \in U_1$ which contribute to changes in $\ker P_3$, and thus to $\mathcal{H}_3^\lambda$ at $\lambda = -4$, are those given by $v = r^{-2} d\sigma \wedge \alpha$ where $[\alpha] \in \im \Upsilon^2$. Hence by Corollary 4.57 we conclude that $v$ contributes to changes in the closed and coclosed 3-forms at rate $-4$ if and only if $[\ast_v \alpha] \in \im \Upsilon^2$. Therefore given $[\sigma] \in \im \Upsilon^2$, we have $\ast_v \eta \in \mathcal{H}_{4-\varepsilon}^\lambda$ asymptotic to $r^{-2} d\sigma \wedge (\ast_v \sigma)$, and thus $\eta \in \mathcal{H}_{4-\varepsilon}^\lambda$ asymptotic to $\ast_v (r^{-2} d\sigma \wedge (\ast_v \sigma)) = \sigma$, and hence $\Upsilon^4[\eta] = [\sigma]$ as required.

**Remark 4.60.** The results of Lemmas 4.58 and 4.59 have almost certainly been argued before in greater generality. For example they are very likely consequences of [35, Proposition 6.18].

We can now use Lemmas 4.58 and 4.59 to establish the main result of this section.

**Proposition 4.61.** Let $\lambda_0$ be a critical rate for $d + d_M^*$ (understood to be a “constant” n-tuple in the CS case), and let $\varepsilon > 0$ be chosen so that there are no other critical rates in $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Then for $\lambda_0 = -3$ we have

$$\dim \mathcal{H}_{3+\varepsilon}^\lambda - \dim \mathcal{H}_{3-\varepsilon}^\lambda = \dim(\im \Upsilon^3) \quad (AC),$$

$$\dim \mathcal{H}_{3+\varepsilon}^\lambda - \dim \mathcal{H}_{3-\varepsilon}^\lambda = -\dim(\im \Upsilon^3) \quad (CS),$$

and for $\lambda_0 = -4$ we have

$$\dim \mathcal{H}_{4+\varepsilon}^\lambda - \dim \mathcal{H}_{4-\varepsilon}^\lambda = \dim(\im \Upsilon^4) \quad (AC),$$

$$\dim \mathcal{H}_{4+\varepsilon}^\lambda - \dim \mathcal{H}_{4-\varepsilon}^\lambda = -\dim(\im \Upsilon^4) \quad (CS).$$

**Proof.** Consider first the AC case. By Lemma 4.26, the space $\mathcal{H}_3^\lambda$ changes by the addition of forms that are asymptotic to closed and coclosed 3-forms in $\mathcal{K}(-3)_{d+d_M^*}$, as we cross $\lambda_0 = -3$. Also, by Lemma 3.9 and Remark 3.12, a 3-form $\nu$ in $\mathcal{K}(-3)_{d+d_M^*}$ must be of the form $\nu = \beta$ for some harmonic 3-form $\beta$ on $\Sigma$. Explicitly, we have that if $\gamma_1 \in \mathcal{H}_{-3+\varepsilon}$, then on the end we have

$$\gamma_1 = (h^{-1})^\ast \beta + \tilde{\gamma} + \gamma_2,$$

where $\tilde{\gamma} + \gamma_2 = O(\rho^{-3+\varepsilon+\nu}) + O(\rho^{-3-\varepsilon})$. If $\varepsilon$ is sufficiently small so that $-3 + \varepsilon + \nu < -3$, then Lemma 3.3 tells us that the 3-form component of $\tilde{\gamma} + \gamma_2$ is exact on the end. Hence, we find that
\[ \gamma^3[\gamma_1] = [\beta], \] so a necessary condition for \( \beta \) to define a 3-form on \( M \) which adds to \( \mathcal{H}_3^3 \) as \( \lambda \) crosses \( \lambda_0 = -3 \) is that \([\beta] \in \text{im} \gamma^3\). Sufficiency follows from Lemma \[4.58\]. This establishes (100).

In exactly the same way, the space \( \mathcal{H}_4^3 \) changes by the addition of forms that are asymptotic to closed and coclosed 3-forms in \( \mathcal{K}(-4d_d\gamma) \), as we cross \( \lambda_0 = -4 \). This time, Lemma \[3.9\] and Remark \[3.12\] says that a 3-form \( v \) in \( \mathcal{K}(-4d_d\gamma) \) must be of the form \( v = r^{-2}dr \wedge \alpha \) for some harmonic 2-form \( \alpha \) on \( \Sigma \). But \( *_{\Sigma} \mathcal{H}_4^3 = \mathcal{H}_4^4 \), and \( *_{\Sigma} v = *_{\Sigma}(r^{-2}dr \wedge \alpha) = *_{\Sigma} \alpha \), a harmonic 4-form on \( \Sigma \). Thus the previous argument can be repeated to conclude that changes in \( \mathcal{H}_4^4 \) (and hence to \( \mathcal{H}_4^3 \)) as we cross \( \lambda_0 = -4 \) correspond to elements of \( \text{im} \gamma^4 \). (Necessity follows exactly as above, but this time sufficiency is easier, since it follows directly from the isomorphism \( \chi^4 : \mathcal{H}^{3,4+\varepsilon}_4 \cong H^4(M, \mathbb{R}) \) given in Proposition \[4.51\].) This establishes (101).

To prove the CS case, the arguments for \( \lambda = -3, -4 \) are analogous to the \( \lambda = -4, -3 \) arguments of the AC case, respectively, but we need to use Lemma \[4.59\] instead of Lemma \[4.58\] \( \square \)

We pause here to note that we can reinterpret Proposition \[4.61\] in terms of cohomology, as follows.

**Proposition 4.62.** Suppose we are in the AC case. Then

\[ \mathcal{H}^{3,4-\varepsilon}_3 \subseteq \mathcal{H}^{3+\varepsilon}_3 = \mathcal{H}^{3-\varepsilon}_3 \subseteq \mathcal{H}^{3+\varepsilon}_3 \]

and

\begin{align*}
(TAC1) 
\chi^3 : \mathcal{H}^{3,3-\varepsilon}_3 &\to H^3(M, \mathbb{R}) \text{ is surjective,} \\
(TAC2) \ker \gamma^3 &\subseteq \chi^3(\mathcal{H}^{3,3-\varepsilon}_3), \\
(TAC3) \chi^4 : \mathcal{H}^{3,4+\varepsilon}_4 &\to H^4(M, \mathbb{R}) \text{ is an isomorphism,} \\
(TAC4) \ker \gamma^4 &\subseteq \chi^4(\mathcal{H}^{3,4-\varepsilon}_4).
\end{align*}

Suppose we are in the CS case. Then

\[ \mathcal{H}^{3,4-\varepsilon}_3 \supseteq \mathcal{H}^{3+\varepsilon}_3 = \mathcal{H}^{3-\varepsilon}_3 \supseteq \mathcal{H}^{3+\varepsilon}_3 \]

and

\begin{align*}
(TCS1) \chi^4 : \mathcal{H}^{3,4-\varepsilon}_3 &\to H^4(M, \mathbb{R}) \text{ is surjective,} \\
(TCS2) \ker \gamma^4 &\subseteq \chi^4(\mathcal{H}^{3,4+\varepsilon}_3), \\
(TCS3) \chi^3 : \mathcal{H}^{3,3-\varepsilon}_3 &\to H^3(M, \mathbb{R}) \text{ is an isomorphism,} \\
(TCS4) \ker \gamma^3 &\subseteq \chi^3(\mathcal{H}^{3,3+\varepsilon}_3).
\end{align*}

**Proof.** We prove the AC case. The CS case is essentially the same. The relation \( \mathcal{H}^{3,4-\varepsilon}_3 \subseteq \mathcal{H}^{3+\varepsilon}_3 = \mathcal{H}^{3-\varepsilon}_3 \subseteq \mathcal{H}^{3+\varepsilon}_3 \) is just Corollary \[4.27\]. Statement (TAC1) is Lemma \[4.58\]. Statement (TAC2) follows from Proposition \[4.53\] which says \( \text{im} \chi^3 = \text{im}(H^{3,3}_3(M, \mathbb{R}) \to H^3(M, \mathbb{R})) \), and the long exact sequence \[61\]. Statement (TAC3) is part of Proposition \[4.51\]. Finally, (TAC4) follows from the proof of equation \[101\], statement (TAC3), and the rank-nullity theorem applied to the map \( \gamma^4 : H^4(M, \mathbb{R}) \to \oplus_{i=1}^n H^7(\Sigma_i, \mathbb{R}) \). \( \square \)

**Remark 4.63.** We see that to go between the AC and CS cases, we effectively switch 3 with 4 in the maps and cohomology groups, and \( +\varepsilon \) with \(-\varepsilon\).

The results of this section will be used in Section \[5.2.4\] to compute the virtual dimension of the moduli space, which will have both topological and analytic components.
4.6 Parallel tensors on $G_2$ conifolds

The holonomy $H(\nabla)$ of a connection $\nabla$ on the tangent bundle $TM$ of a connected manifold $M$ is contained in (and often equal to) the subgroup of the general linear group whose action fixes all parallel tensors on $M$. See, for example, Joyce [13, Chapter 2] for more details. In particular, the holonomy of the Levi-Civita connection on an oriented Riemannian manifold $M^n$ is reduced from $SO(n)$ by each additional parallel tensor. On an irreducible $G_2$ manifold (one whose holonomy is exactly $G_2$) the only parallel tensors are the metric $g$, the volume form $\text{vol}$, the $G_2$ structure $\varphi$, and the dual 4-form $\psi = *\varphi$. Since $G_2$ conifolds are all irreducible, they admit no nontrivial parallel 1-forms.

We now recall the Bochner–Weitzenböck formula, valid for any Riemannian manifold $(M, g)$. Let $X$ be a 1-form. Then

$$\langle \Delta X, X \rangle = \langle \nabla^* \nabla X, X \rangle + \text{Ric}(X, X)$$

where $\Delta$ is the Hodge Laplacian, $\nabla$ is the Levi-Civita covariant derivative, and $\text{Ric}$ is the Ricci tensor of $(M, g)$, with indices raised by the metric to become a symmetric bilinear form on 1-forms. Since all $G_2$ manifolds are Ricci-flat, the last term above vanishes, and we have

$$\langle \Delta X, X \rangle = \langle \nabla^* \nabla X, X \rangle. \quad (102)$$

Lemma 4.64. Let $M$ be a $G_2$ conifold. Let $X$ be a harmonic 1-form, and let $f$ be a harmonic function on $M$. If

$$X = O(g^\lambda) \quad \text{for some } \lambda < -\frac{5}{2} \ (AC) \text{ or } \lambda > -\frac{5}{2} \ (CS), \quad (103)$$

then $X = 0$. If

$$f = O(g^\lambda) \quad \text{for some } \lambda < -\frac{5}{2} \ (AC) \text{ or } \lambda > -\frac{5}{2} \ (CS), \quad (104)$$

then $f$ is constant.

Finally, if the 1-form $X$ satisfies $dd^*X + \frac{2}{3}d^*dX = 0$, and the estimate $\langle \nabla^* \nabla X, X \rangle \leq CR^2\lambda$, then we can again conclude that $X = 0$.

Proof. We give the proof in the AC case. The CS case is identical except that there are $n$ ends instead of just one, and $g \to 0$ on each end instead of $g \to \infty$. We want to integrate both sides of (102) over $M$. Note that

$$\langle \nabla^* \nabla X, X \rangle = -g^{ij}(\nabla_i \nabla_j X_k)X_mg^{km} = -g^{ij}\nabla_i((\nabla_j X_k)X_mg^{km}) + g^{ij}g^{km}(\nabla_j X_k)(\nabla_i X_m) = d^*Y + |\nabla X|^2$$

for the vector field $Y = \langle \nabla X, X \rangle$. Since $X \in L^1_{r, \lambda}$, we have $\nabla X \in L^1_{r, -1, \lambda - 1}$ and thus the vector field $Y = \langle \nabla X, X \rangle$ is $O(r^{2\lambda-1})$ as $r \to \infty$. Let $M_R = \{ x \in M : g(x) \leq R \}$, and observe that $\partial(M_R) = \{ R \} \times \Sigma$. Hence, by Stokes’s Theorem and the fact that $Y = O(r^{2\lambda-1})$, for $R$ sufficiently large we have

$$\left| \int_{M_R} (d^*Y) \text{vol}_{M_R} \right| \leq \left| \int_{\partial(M_R)} (Y d^*\text{vol}_{M_R}) \right| \leq CR^{2\lambda-1} \left| \int_{\{ R \} \times \Sigma} \text{vol}_{\{ R \} \times \Sigma} \right| = CR^{2\lambda+5}$$
which goes to zero as $R \to \infty$, since $\lambda < -\frac{5}{2}$. Therefore, since $\Delta X = 0$, when we integrate both sides of (102) over $M$, we obtain
\[ 0 = ||\nabla X||^2_{L^2}. \] (105)

Hence $\nabla X = 0$, so $X$ is a parallel 1-form. But the hypothesis that $M$ is a $G_2$ conifold then implies that $X = 0$. The statement about functions follows in the same way by integrating the equation $\langle \Delta f, f \rangle = \langle \nabla^* \nabla f, f \rangle$.

The case when $X$ satisfies $dd^* X + \frac{2}{3} d^* dX = 0$ is similar, using $dd^* X + \frac{2}{3} d^* dX = \Delta X - \frac{1}{3} d^* dX$ and the fact that $|dX| \leq |\nabla X|$.

4.7 A gauge-fixing condition on moduli spaces of $G_2$ conifolds

In this section we discuss a gauge-fixing condition on moduli spaces of $G_2$ conifolds and some related results. At first our discussion is quite general, to motivate the definition of the gauge-fixing condition that we choose.

Let $(M, \varphi)$ be a $G_2$ manifold, which is not necessarily compact. Let $T$ be the space of all torsion-free $G_2$ structures on $M$. Then the space $D$ of diffeomorphisms of $M$ acts on $T$ by pullback. If $F \in D$, then
\[ F : \varphi \mapsto F^* \varphi. \]

Consider a smooth curve $F_t = \exp(tX)$ in $D$, where $X$ is a smooth vector field on $M$. This path passes through the identity diffeomorphism $F_0 = \text{Id}_M$ at $t = 0$. Therefore, the tangent space $T_\varphi(D \cdot \varphi)$ at $\varphi$ to the orbit $D \cdot \varphi$ is spanned by elements of the form $\frac{d}{dt}|_{t=0}(F_t^* \varphi) = \mathcal{L}_X \varphi = d(X \varphi)$. Thus we have
\[ T_\varphi(D \cdot \varphi) = d(\Omega^2). \] (106)

Let $\tilde{\varphi}$ be another torsion-free $G_2$ structure on $M$, such that $\tilde{\varphi} = \varphi + \eta$ for some smooth 3-form $\eta$. Since both $\varphi$ and $\tilde{\varphi}$ are torsion-free and thus closed, we must have $d\eta = 0$. In order to break the diffeomorphism invariance, we want to consider those new $G_2$ structures for which $\tilde{\varphi} - \varphi = \eta$ is transverse to the infinitesimal diffeomorphisms which, as explained above, are all of the form $\mathcal{L}_X \varphi = d(X \varphi)$ for a smooth vector field $X$. Suppose that $\eta$ lies in $L^2(\Lambda^3(T^* M))$. Then the condition that $\eta$ is actually $L^2$-orthogonal to $d(\Omega^2)$ is that
\[ 0 = \langle d(X \varphi), \eta \rangle = \langle (X \varphi, d^* \eta) \rangle. \]

Notice that this condition is always implied by the stronger condition that $\pi_7(d^* \eta) = 0$ pointwise. This observation motivates the following definition.

**Definition 4.65.** Suppose $\tilde{\varphi} = \varphi + \eta$ is another torsion-free $G_2$ structure on $M$, for some closed 3-form $\eta$. We say that $\tilde{\varphi}$ satisfies the gauge-fixing condition (with respect to $\varphi$) if
\[ \pi_7(d^* \eta) = 0. \]

Here $\pi_7$ and $d^*$ are taken with respect to the $G_2$ structure $\varphi$.

In fact, we will often have $\eta = \kappa + \zeta$ for some closed and coclosed form $\kappa$. So in that case, the gauge-fixing condition becomes $\pi_7(d^* \zeta) = 0$.

For the rest of this section, $M$ is a $G_2$ conifold. The next two results relate this gauge-fixing condition to a slightly different condition.
Lemma 4.66. Let \( \zeta \) be a smooth 3-form such that \( d\zeta = 0 \) and \( \pi_7(d^*\zeta) = 0 \). Let
\[
\zeta = \pi_1\zeta + \pi_7\zeta + \pi_{27}\zeta = f\varphi + *(X \wedge \varphi) + \pi_{27}\zeta
\]  
(107)
for some function \( f \) and some 1-form \( X \). Then \( \Delta f = 0 \) and \( \Delta X = 0 \). In addition, if \( X \) and \( f \) satisfy \((103)\) and \((104)\), respectively, then \( f = c \) is constant and \( X = 0 \), so \( \zeta = c\varphi + \pi_{27}\zeta \).

Proof. By Proposition 2.5 and the hypothesis \( d^*\zeta \in \Omega^2_{\mathbb{H}} \), we have \( dd^*\zeta \in \Omega^3_{\mathbb{H}} \). The other hypothesis \( d\zeta = 0 \) now gives \( \Delta \zeta = dd^*\zeta + d^*d\zeta \in \Omega^3_{\mathbb{H}} \), so \( \pi_1(\Delta \zeta) = 0 \) and \( \pi_7(\Delta \zeta) = 0 \). Since \( \varphi \) is torsion-free, the Laplacian \( \Delta \) commutes with the projections, so we have \( \Delta(\pi_1(\zeta)) = 0 \) and \( \Delta(\pi_7(\zeta)) = 0 \). But \( \pi_1(\zeta) = f\varphi \) and \( \pi_7\zeta = *(X \wedge \varphi) \), so equations \((8)\) and \((9)\) yield that \( \Delta f = 0 \) and \( \Delta X = 0 \), where we have used the fact that wedge product with \( \psi \) is injective on 1-forms and that \( * \) commutes with \( \Delta \).

The result now follows immediately from Lemma 4.64. \( \square \)

Lemma 4.67. Let \( \zeta \) be a smooth 3-form such that \( d\zeta = 0 \). Let
\[
\zeta = \pi_1\zeta + \pi_7\zeta + \pi_{27}\zeta = f\varphi + *(X \wedge \varphi) + \pi_{27}\zeta
\]  
(108)
for some function \( f \) and some 1-form \( X \). If \( df = 0 \) and \( \text{curl} X = 0 \), then we have \( \pi_7(d^*\zeta) = 0 \).

Proof. We begin by showing that \( \text{curl} X = 0 \) if and only if \( \pi_7(d^*\zeta) = 0 \). To see this, we observe that \( \pi_7(d\pi_7\zeta) = 0 \) if and only if, for all vector fields \( Y \), we have
\[
0 = \langle d(\pi_7\zeta), Y \wedge \varphi \rangle = \langle d*(X \wedge \varphi), Y \wedge \varphi \rangle = -\langle d(X \rfloor \psi), Y \wedge \varphi \rangle.
\]
In local coordinates, using the fact that \( \psi \) is parallel and that \( d \) is the skew-symmetrization of \( \nabla \), this condition can be shown to be equivalent to
\[
0 = (\nabla_a X^p)\psi_{pbcd}(Y_i\varphi_{jkl} - Y_j\varphi_{ikl} + Y_k\varphi_{ijl} - Y_l\varphi_{ijk})g^{ai}g^{bj}g^{ck}g^{dl}.
\]
Using the identities in the appendix of \((19)\) for the contractions of \( \varphi \) with \( \psi \), this simplifies to
\[
0 = 12(\nabla_i X^p)\varphi_{ipb}Y^b.
\]
But the above equation holds for any \( Y \) if and only if \( \text{curl} X = 0 \), since \( (\text{curl} X)_b = (\nabla^i X^p)\varphi_{ipb} \). Thus the claim is proved.

Now because \( df = 0 \), we have \( d(\pi_1\zeta) = 0 \), and thus \( d\zeta = d(\pi_7\zeta) + d(\pi_{27}\zeta) = 0 \). Projecting to \( \Omega^2_{\mathbb{H}} \), we find that \( \pi_7(d(\pi_{27}\zeta)) = -\pi_7d(\pi_7\zeta) = 0 \). Proposition 2.6 now tells us that \( \pi_7(d^*\pi_{27}\zeta) = 0 \). Since \( f = c \) is a constant, we have \( d^*(\pi_1\zeta) = 0 \), and thus
\[
\pi_7(d^*\zeta) = \pi_7d^*(\pi_7\zeta) + \pi_7d^*(\pi_{27}\zeta) = -\pi_7d*(X \wedge \varphi) + 0
\]
\[
= -\pi_7d(X \wedge \varphi).
\]
However from \((3)\) we have that \( *\pi_7(dX \wedge \varphi) = -2\pi_7(dX) \), and from Definition 2.9 this is just \( \text{curl} X \) as an element of \( \Omega^2_{\mathbb{H}} \), which we assume to be zero. Thus we have indeed that \( \pi_7(d^*\zeta) = 0 \). \( \square \)

The final result in this section applies to a special case of the gauge-fixing condition when \( \zeta = d\beta \) is exact. This will be needed in Theorem 5.3 to establish the infinitesimal version of the slice theorem for the AC case.
Lemma 4.68. Let $\beta \in \Omega^2$ with $\beta = \pi_7 \beta + \pi_{14} \beta$ the decomposition into types, where $\pi_7 \beta = *(Y \wedge \psi) = Y \wedge \varphi$ for some vector field $Y$. Suppose that $d^* \beta = 0$ and $\pi_7 (d^* d\beta) = 0$. Then $\Delta Y = 0$.

Proof. Since $d^* \beta = 0$, we have

$$0 = \pi_7 (d^* d\beta) = \pi_7 (dd^* \beta + d\beta) = \pi_7 (\Delta \beta) = \Delta \pi_7 \beta,$$

where we have used the fact that the Hodge Laplacian $\Delta$ commutes with the projection $\pi_7$ when the $G_2$ structure is torsion-free, as explained in Remark 2.4. But using (9) we see that

$$\Delta \pi_7 \beta = \Delta * (Y \wedge \psi) = * \Delta (Y \wedge \psi) = * ((\Delta Y) \wedge \psi).$$

Since the Hodge star operator is always an isomorphism, and wedge product with $\psi$ on 1-forms is injective, we conclude that $\Delta Y = 0$. \hfill \Box

Remark 4.69. In fact, it is clear that Lemmas 4.67 and 4.68 apply to any $G_2$ manifolds, as we never used the conifold condition. It is only Lemma 4.66 that may not be true in general.

Remark 4.70. The gauge-fixing condition of Definition 4.65 is used by Joyce [13] to study the moduli space of compact $G_2$ manifolds. We use the same gauge-fixing condition for the moduli space of $G_2$ conifolds, but the arguments in the proofs of several results need to be modified. In particular, in the AC case when the rate of convergence to the cone at infinity is not too negative, we have $\zeta = d\sigma$ is exact, as in the compact case, but other complications arising from noncompactness require us to work a little harder to obtain some results that are analogous to the compact case, specifically Theorem 5.3 and Theorem 5.8. Moreover, in the AC case when the rate is sufficiently negative, or for any positive rate in the CS case, we do not have that $\zeta$ is exact. Indeed, this failure is related to the obstructions to the deformation theory. Our Lemmas 4.66 and 4.67 are therefore by necessity more general than the analogous results of Joyce [13].

Remark 4.71. A slightly different gauge-fixing condition for AC $G_2$ manifolds is discussed in [20, Definition 3.3]. The relation between these two conditions is explained in Section 6.6.

4.8 Two more analytic results particular to $G_2$ conifolds

In this section we present two more analytic results that are particular to $G_2$ conifolds. The first result is just a special case of [13 Proposition 10.3.4], with essentially the same proof, except that it has been adapted to the setting of conifolds. Therefore we need to make assumptions that some forms have a certain decay rate on the ends, and for this reason we give the proof for completeness.

Lemma 4.72. Let $(M, \varphi)$ be a $G_2$ conifold, so in particular $d\varphi = d\Theta(\varphi) = 0$. Suppose further that $\tilde{\varphi}$ is another closed $G_2$ structure on $M$ such that

$$d(\Theta(\tilde{\varphi})) = \theta \wedge \psi + dX \wedge \varphi$$

for some 1-forms $\theta$ and $X$ on $M$. Further assume that

$$d(\Theta(\tilde{\varphi})) = O(g^\lambda)$$

and

$$X = O(g^{\lambda+1})$$

for some $\lambda < -\frac{7}{2}$ (AC) or $\lambda > -\frac{7}{2}$ (CS).

Note that this says, in particular, that both $d(\Theta(\tilde{\varphi}))$ and $dX$ are in $L^2$. There is a universal constant $\varepsilon$ such that if $\tilde{\varphi}$ is within $\varepsilon$ of $\varphi$ in the $C^0$ norm on $M$, then $\theta = 0$ and $dX = 0$, so $d(\Theta(\tilde{\varphi})) = 0$ and thus $\tilde{\varphi}$ is also torsion-free.
Proof. We give the proof in the AC case. The CS case is identical except that there are $n$ ends instead of just one, and $g \to 0$ on each end instead of $g \to \infty$. Let $V$ be a 7-dimensional vector space, with two $G_2$ structures $\varphi$ and $\tilde{\varphi}$. It follows from simple linear algebra that if $\tilde{\varphi}$ and $\varphi$ are close with respect to the metric $g_{\varphi}$ induced by $\varphi$, then the decompositions $\Lambda^5(V) = \Lambda_2^5 \oplus \Lambda_1^5$ and $\Lambda^5(V) = \Lambda_2^5 \oplus \Lambda_1^5$ with respect to $\tilde{\varphi}$ and $\varphi$, respectively, will also be close. In particular there exists a universal constant $\varepsilon$ such that if $|\varphi - \varphi| < \varepsilon$, using the metric $|\cdot |$ from $\varphi$, then an element $\xi \in \Lambda^5(V)$ for which $\pi_7(\xi) = 0$ will also have $\pi_7(\xi)$ small enough so that $|\pi_7(\xi)| \leq |\pi_{14}(\xi)|$.

Unless stated otherwise, all our projections and inner products will be taken with respect to the $G_2$ structure $\varphi$. To simplify notation, we will sometimes write $\zeta = d(\Theta(\tilde{\varphi}))$. We take the decomposition of $\Omega^5 = \Omega_5^2 \oplus \Omega_1^5$:

$$
\zeta_7 = [d(\Theta(\tilde{\varphi}))]_7 = [\theta \wedge \psi]_7 + [dX \wedge \varphi]_7 = \theta \wedge \psi - 2 \ast \pi_7(dX),
$$

$$
\zeta_{14} = [d(\Theta(\tilde{\varphi}))]_{14} = [\theta \wedge \psi]_{14} + [dX \wedge \varphi]_{14} = 0 + \ast \pi_{14}(dX),
$$

(111)

(112)

where we have used the Hodge stars of equations (3) and (4). Because $\tilde{\varphi}$ is closed, we know by Remark 2.3 that $\zeta = d(\Theta(\tilde{\varphi}))$ lies in the space $\Omega^5_{14}$, where the tilde denotes the decomposition with respect to $\tilde{\varphi}$. Hence, if $|\tilde{\varphi} - \varphi|_{C^0} < \varepsilon$, by the above remarks we have that $|\zeta_7| \leq |\zeta_{14}|$. Since we assume that $\zeta = d(\Theta(\tilde{\varphi}))$ is in $L^2$, we can integrate over $M$ to conclude that

$$
||\zeta_7|| \leq ||\zeta_{14}||.
$$

(113)

Now consider the 7-form $dX \wedge dX \wedge \varphi$, which is exact since $\varphi$ is closed. By equations (3) and (4), we have

$$
dX \wedge dX \wedge \varphi = dX \wedge (-2 \ast \pi_7(dX) + \ast \pi_{14}(dX)) = (-2|\pi_7(dX)|^2 + |\pi_{14}(dX)|^2) \text{vol}.
$$

(114)

The integral over $M$ of the right hand side is finite because $dX$ is assumed to be in $L^2$. To compute the integral over $M$ of the left hand side, let $M_R = \{x \in M; \varrho(x) \leq R\}$, and observe that $\partial(M_R) = \{R\} \times \Sigma$. Hence, by Stokes’s Theorem and the hypothesis that $X = O(\varrho^{\lambda+1})$, for $R$ sufficiently large we have

$$
\left| \int_{M_R} dX \wedge dX \wedge \varphi \right| = \left| \int_{\partial(M_R)} X \wedge dX \wedge \varphi \right| \leq CR^{2\lambda+1} \left| \int_{\{R\} \times \Sigma} \text{vol}_{\{R\} \times \Sigma} \right| = CR^{2\lambda+7}
$$

which goes to zero as $R \to \infty$, since $\lambda < -\frac{7}{2}$. Therefore, integrating both sides of (114) over $M$, we obtain

$$
2||\pi_7(dX)||^2 = ||\pi_{14}(dX)||^2.
$$

(115)

Similarly, $d(\Theta(\tilde{\varphi})) \wedge dX$ is an exact 7-form, and

$$
d(\Theta(\tilde{\varphi})) \wedge dX = \zeta \wedge \ast dX = \langle \zeta_7, \ast \pi_7(dX) \rangle \text{vol} + \langle \zeta_{14}, \ast \pi_{14}(dX) \rangle \text{vol}.
$$

Since both $\zeta = d(\Theta(\tilde{\varphi}))$ and $dX$ are in $L^2$, and since $\zeta = O(\varrho^\lambda)$ and $X = O(\varrho^{\lambda+1})$, we can argue exactly as before to integrate both sides over $M$ to conclude that

$$
\langle \langle \zeta_7, \ast \pi_7(dX) \rangle \rangle = -\langle \langle \zeta_{14}, \ast \pi_{14}(dX) \rangle \rangle = -||\zeta_{14}||^2,
$$

(116)

using the fact that $\zeta_{14} = \ast \pi_{14}(dX)$ from equation (112).
Now we use the Cauchy–Schwarz inequality, and equations (113), (115), and (116) to compute
\[
||ζ|| ||π_7(dX)|| = ||ζ|| ||π_7(dX)||
\geq -(⟨⟨ζ, π_7(dX)⟩⟩)
= +⟨⟨ζ_14, π_14(dX)⟩⟩
= ||ζ_14|| ||π_14(dX)||
= |\sqrt{2}||ζ_14|| ||π_14(dX)||
\geq \sqrt{2}||ζ|| ||π_7(dX)||.
\]

Therefore we have concluded that
\[
||ζ|| ||π_7(dX)|| \geq \sqrt{2}||ζ|| ||π_7(dX)||.
\]

We have two cases. If \(π_7(dX) = 0\), then by (115) we have \(π_14(dX) = 0\), so \(ζ_14 = 0\) by (112), and thus \(ζ = 0\) by (113). If, on the other hand, we have \(ζ = 0\), then (116) forces \(ζ_14 = 0\), and then (112) and (115) together give \(dX = 0\). In either case we also get \(θ ∧ ψ = 0\) from (111), which implies that \(θ = 0\), since wedge product with \(ψ\) is injective on 1-forms.

Remark 4.73. The reason that Lemma 4.72 is true is because of the representation theory of \(G_2\). Essentially, equations (3) and (4) and Stokes’s theorem force the very powerful restrictions (115) and (116) on the forms \(ζ = d(Θ( \tilde{ϕ}))\) and \(dX\). The remaining ingredients are the \(C^0\) proximity of \(\tilde{ϕ}\) and \(ϕ\), together with the facts that \(\tilde{ϕ}\) is closed and \(ϕ\) is torsion-free, which force the \(Ω^5_7\) component of \(ζ\) to be controlled by the \(Ω^5_{14}\) component.

Remark 4.74. We also remark that in the simplified setting that we consider here, we do not need to assume that \(θ\) lies in \(L^2\). In order to adapt the full generality of Joyce [13, Proposition 10.3.4] to the noncompact setting, one would need to assume that \(θ\) is in \(L^2\).

The are two more results in this section, concerning the modified Dirac operator \(\tilde{D}\) defined in Section 2.2 and the operator \(dd^* + \frac{7}{2}d^*d\) which appears often in relation to gauge-fixing. The first result, Lemma 4.75, is used to prove one case of the infinitesimal slice theorem in Section 5.2.1. The second result, Lemma 4.76, is used in Section 6.2 to extend our AC deformation theory to higher rates and in Section 6.3 to establish smoothness of the CS moduli space under certain conditions.

Suppose that \(λ + 1\) is a noncritical rate for \(\tilde{D}\). Thus the operator
\[
\tilde{D}_{l+1,λ+1} : L^2_{l+1,λ+1}(Λ_1^0 ⊕ Λ_7^1) \rightarrow L^2_{l,λ}(Λ_3^0 ⊕ Λ_3^7)
\]
is Fredholm, and therefore by Theorem 4.13 we have
\[
L^2_{l,λ}(Λ_3^0 ⊕ Λ_3^7) = \tilde{D}(L^2_{l+1,λ+1}(Λ_1^0 ⊕ Λ_7^1)) ⊕ V_λ,
\]
where \(V_λ\) is a finite-dimensional subspace of \(L^2_{l,λ}(Λ_3^0 ⊕ Λ_3^7)\), such that
\[
V_λ ≅ \ker(\tilde{D}^*)_{-7-λ}.
\]

Lemma 4.75. Let \((M, ϕ_M)\) be an AC \(G_2\) manifold. The map \(\tilde{D}_{l+1,λ+1}\) is surjective for all \(λ > -6\).
Proof. Suppose that $s = (f, X)$ lies in $\ker(\tilde{\nabla}^*)_{-7-\lambda}$. Corollary 2.14 tells us that $\Delta f = 0$ and $\Delta X = 0$. Hence $f$ is a harmonic function and $X$ is a harmonic 1-form on $M$. By the maximum principle, if $f$ is bounded on $M$ then it must be constant. Since the rate of decay of $f$ at infinity is $-7 - \lambda$, it will be bounded if $-7 - \lambda < 0$, so it is a constant if $\lambda > -7$. But $f$ decays to zero at infinity, so we actually have $f = 0$. Now $X$ is a harmonic 1-form on $M$, so by Lemma 4.64 if $X = O(r^{-\frac{7}{2}-\epsilon})$, then $X = 0$. Since $X$ is smooth of rate $-7 - \lambda$ on the end, we find that $X = 0$ if $-7 - \lambda < -\frac{5}{2}$, or $\lambda > -\frac{9}{2}$. Hence, we have shown that $\ker(\tilde{\nabla}^*)_{-7-\lambda} = 0$ for $\lambda > -\frac{9}{2}$.

Now, if $\lambda + 1$ is noncritical for $\tilde{\nabla}$, then the space $V_\lambda \cong \ker(\tilde{\nabla}^*)_{-7-\lambda} = 0$ is zero if $\lambda > -\frac{9}{2}$. Thus, for such rates, (117) says that $\tilde{\nabla}_{t+1,\lambda+1}$ is surjective. But by Proposition 3.13 there are no critical rates for $\lambda + 1 \in (-5, 0)$, or equivalently if $\lambda \in (-6, -1)$, and hence $\tilde{\nabla}_{t+1,\lambda+1}$ is surjective for all $\lambda > -6$.

Similarly, suppose that $\lambda + 1$ is a noncritical rate for $dd^* + \frac{2}{3} d^* d$. Thus the operator

$$(dd^* + \frac{2}{3} d^* d)_{t+1,\lambda+1} : L^2_{t+1,\lambda+1}(\Lambda^2_7) \to L^2_{t,\lambda}(\Lambda^2_7)$$

is Fredholm, and thus

$$L^2_{t,\lambda}(\Lambda^2_7) = \ker(\tilde{\nabla}^*)_{-7-\lambda}. $$

Lemma 4.76. Let $(M, \varphi_M)$ be a $G_2$ conifold. Consider the map $(dd^* + \frac{2}{3} d^* d)_{t+1,\lambda+1}$. In the AC case it is surjective for all $\lambda > -5$, and in the CS case it is injective for all $\lambda > -5$.

Proof. In the AC case, suppose $\omega \in \ker(\tilde{\nabla}^*)_{-7-\lambda}$. By Lemma 4.64, arguing exactly as in the proof of Lemma 4.75, we conclude that $\omega = 0$ if $\lambda > -\frac{9}{2}$. Thus $(dd^* + \frac{2}{3} d^* d)_{t+1,\lambda+1}$ is surjective for all $\lambda > -\frac{9}{2}$. In the CS case, suppose $\omega \in \ker(\tilde{\nabla}^*)_{-7-\lambda}$. Again by Lemma 4.64, we conclude that $\omega = 0$ if $\lambda + 1 > -\frac{5}{2}$, or equivalently $\lambda > -\frac{7}{2}$. But by Proposition 3.7, there are no critical rates for the operator $(dd^* + \frac{2}{3} d^* d)_{t+1,\lambda+1}$ in the range $\lambda + 1 \in (-4, 0)$, or equivalently if $\lambda \in (-5, -1)$, and hence the result follows.

5 The deformation theory of $G_2$ conifolds

In this section we study the deformation theory of $G_2$ conifolds, and state and prove our main theorem. Recall that for us, a $G_2$ conifold $(M, \varphi_M)$ is either an AC $G_2$ manifold of some rate $\nu < 0$ as in Definition 3.15 or a CS $G_2$ manifold of some rates $(\nu_1, \ldots, \nu_n) > (0, \ldots, 0)$ as in Definition 3.19. There are some analytic difficulties that are encountered if one attempts to study deformations of AC $G_2$ manifolds with rate $\nu \geq -\frac{3}{2}$, which will become evident below. For this reason we will always assume that $\nu < -\frac{3}{2}$ in the AC case. Note that this includes all the known examples of AC $G_2$ manifolds.
5.1 The $G_2$ conifold moduli space

In this section we define the $G_2$ conifold moduli space, and then give some informal arguments to motivate how we will proceed to prove our main theorem.

Let $\mathcal{T}_\nu$ denote the set of all torsion-free conifold $G_2$ structures on $M$ which converge at the same rate $\nu$ on the ends to the same $G_2$ cones as the original conifold $G_2$ structure $\varphi$. Explicitly,

$$\mathcal{T}_\nu = \{ \tilde{\varphi} \in \Omega^*_+(M); \tilde{\varphi} - \varphi \in C^\infty_0(\Lambda^3 T^* M), d\tilde{\varphi} = 0, d\Theta(\tilde{\varphi}) = 0 \}.$$ 

In order to define the moduli space of torsion-free conifold $G_2$ structures on $M$, we need to take the quotient of $\mathcal{T}_\nu$ by an appropriate equivalence relation. The torsion-free condition is diffeomorphism invariant, but arbitrary diffeomorphisms do not preserve the convergence condition on the ends. We need to quotient out by those diffeomorphisms for which pullbacks preserve the condition $\tilde{\varphi} - \varphi \in C^\infty_0(\Lambda^3 T^* M)$. As mentioned in the introduction, we only consider deformations that fix the asymptotic $G_2$ cones on the ends, because the deformation theory of compact strictly nearly Kähler manifolds is still not completely understood. Thus we are interested in diffeomorphisms isotopic to the identity, which are generated by vector fields that decay to zero on the ends. For such diffeomorphisms to preserve the rate of convergence at the ends to the asymptotic cones, their infinitesimal generators (vector fields) must be of rate $\nu + 1$ on the ends. Specifically, we define $\mathcal{D}_\nu$ to be the group generated by the set

$$\{ \exp(X); X \in \Gamma(TM), X \in C^\infty_{\nu+1}(TM) \}.$$ 

This is a connected component of the identity in the space of all diffeomorphisms of $M$, and hence a subgroup of $\text{Diff}^0(M)$, the diffeomorphisms isotopic to the identity.

It is clear that $\mathcal{D}_\nu$ acts on $\mathcal{T}_\nu$ by pullback. We now define the $G_2$ conifold moduli space $\mathcal{M}_\nu$ of rate $\nu$ on $M$ to be the quotient space $\mathcal{M}_\nu = \mathcal{T}_\nu/\mathcal{D}_\nu$. This defines $\mathcal{M}_\nu$ as a topological space. We want to describe the structure of $\mathcal{M}_\nu$ more precisely. In the AC case, we will see that for generic rates that lie in a certain range, the space $\mathcal{M}_\nu$ is actually a finite-dimensional smooth manifold. For other rates in the AC case, and in general for the CS case, the deformation theory will be obstructed and we will describe the obstruction spaces explicitly.

Both $\mathcal{T}_\nu$ and the orbits of $\mathcal{D}_\nu$ are infinite-dimensional smooth manifolds. Their tangent spaces are described as follows. Consider a smooth curve $F_t = \exp(tX)$ in $\mathcal{D}_\nu$, where $X \in C^\infty_{\nu+1}(TM)$. This path passes through the identity diffeomorphism $F_0 = \text{Id}_M$ at $t = 0$. Therefore, the tangent space $T_{\varphi}(\mathcal{D}_\nu \cdot \varphi)$ at $\varphi$ to the orbit $\mathcal{D}_\nu \cdot \varphi$ is spanned by elements of the form $\frac{d}{dt}_{|t=0}(F_t^* \varphi) = L_X \varphi = d(X \rfloor \varphi)$. Thus we have

$$T_{\varphi}(\mathcal{D}_\nu \cdot \varphi) = d \left( C^\infty_{\nu+1}(\Lambda^3(T^*M)) \right).$$

Similarly, let $\varphi_t$ be a smooth path in $\mathcal{T}_\nu$ passing through $\varphi$ at $t = 0$. Thus $\varphi_t$ is a torsion-free $G_2$ structure for all $t$, and therefore by Lemma 2.8 we have that the $3$-form $\eta = \frac{d}{dt}_{|t=0} \varphi_t$ satisfies $d\eta = 0$ and $\frac{1}{3} d^* \pi_1(\eta) + d^* \pi_7(\eta) - d^* \pi_{27}(\eta) = 0$, where the projections are taken with respect to the $G_2$ structure $\varphi = \varphi_0$. Hence we have shown that

$$T_{\varphi} \mathcal{T}_\nu \subseteq \{ \eta \in C^\infty_\nu(\Lambda^3 T^* M); d\eta = 0, d(L_\varphi(\eta)) = 0 \},$$

where

$$L_\varphi(\eta) = \frac{4}{3} \ast \pi_1(\eta) + \ast \pi_7(\eta) - \ast \pi_{27}(\eta).$$

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is the linearization of $\Theta$ at $\varphi$ defined in equation \((15)\). For the purpose of motivation, let us assume that the subspace inclusion in \((119)\) is actually an equality. Then if $\mathcal{M}_\nu = T_\nu D_{\nu+1}$ were indeed a smooth manifold, we would have that

$$T_{[\varphi]}\mathcal{M}_\nu \oplus T_{\varphi}(D_{\nu+1} \cdot \varphi) = T_{\varphi}T_\nu.$$  \hspace{1cm} \text{(120)}$$

Thus, one of our goals will be to use \((118)\) and \((119)\) to find a direct complement of $T_{\varphi}(D_{\nu+1} \cdot \varphi)$ in $T_{\varphi}T_\nu$. This will tell us what the “tangent space” at $[\varphi]$ to $\mathcal{M}_\nu$ would have to be. Then we will use the Banach space implicit function theorem to describe the structure of $\mathcal{M}_\nu$. We will prove our main theorem without requiring any assumption about equality in \((119)\).

Theorem 5.1. Let $(M, \varphi)$ be a $G_2$ conifold, asymptotic to particular $G_2$ cones on the ends, at some rate $\nu$. Let $\mathcal{M}_\nu$ be the moduli space of all torsion-free $G_2$ structures on $M$, asymptotic to the same cones on the ends, at the same rate $\nu$, modulo the appropriate notion of equivalence that preserves these conditions. Then for generic $\nu$ (in a sense made precise later), we have

- In the AC case, if $\nu \in (-4, -\frac{5}{2})$, the space $\mathcal{M}_\nu$ is a smooth manifold whose dimension consists of topological and analytic contributions, given precisely in Corollary 5.21.

- In the AC case if $\nu < -4$, or in the CS case for any $\nu > 0$, the space $\mathcal{M}_\nu$ is in general only a topological space, and the deformation theory may be obstructed. There is a subspace $\hat{\mathcal{M}}_\nu$ of $\mathcal{M}_\nu$, called the reduced moduli space, which omits a finite-dimensional space (often trivial) of deformations. The virtual dimension of $\hat{\mathcal{M}}_\nu$ again consists of topological and analytic contributions, given precisely in Corollary 5.21.

5.2 Proof of the main theorem

In this section we prove Theorem 5.1 on the deformation theory of $G_2$ conifolds.

Recall that in the AC case, we always assume that $\nu < -\frac{5}{2}$. Some of the intermediate results that will be needed actually work up to rates greater than $-\frac{5}{2}$. Whenever possible, we state results with their most general range of validity. However, the upper bound for which all the results that we need will be true is $\nu < -\frac{5}{2}$. In the CS case, we always assume that $\nu = (\nu_1, \ldots, \nu_n) > (0, \ldots, 0)$. Some of the analytic results will hold if each $\nu_i > -\frac{5}{2}$, but we do not always state this explicitly, since such rates are unreasonable because the $G_2$ structure $\varphi$ will not converge to the cone structure $\varphi_{C_i}$ at the $i^{th}$ end unless $\nu_i > 0$.

The proof of Theorem 5.1 is broken up into the following four steps.

**Step 1:** We prove a slice theorem, showing that the space of torsion-free gauge-fixed $G_2$ structures with the correct asymptotics on the ends is homeomorphic to the $G_2$ conifold moduli space.

**Step 2:** We demonstrate that a subset $\hat{\mathcal{M}}_\nu$ of the moduli space, called the reduced moduli space, is locally isomorphic to the zero set of a smooth nonlinear map.

**Step 3:** We use the Banach space implicit function theorem to describe the structure of this zero set, and explain when it is a smooth manifold.

**Step 4:** We compute the (expected) dimension of the reduced $G_2$ conifold moduli space in terms of topological and analytic data.
5.2.1 Step 1: Gauge-fixing and the slice theorem

In order to break the diffeomorphism invariance, we need to prove a “slice theorem” that establishes a local homeomorphism between: (i) the space of $G_2$ structures satisfying a particular condition modulo diffeomorphisms which preserve this condition, and (ii) a space of solutions to a system of differential equations. Ideally, we would like to prove this slice theorem directly for torsion-free $G_2$ structures which have prescribed cone-like behaviour on the ends. However, the fact the the torsion-free condition is nonlinear makes it difficult to do this directly. Instead, we first prove a slice theorem for the space of closed $G_2$ structures with prescribed cone-like behaviour on the ends, which is a linear condition, and then in Section 5.2.2 we impose the torsion-free condition to describe a smaller subset of this space.

Our approach to the slice theorem for closed $G_2$ structures is very similar to that of Nordström [40], who considers the asymptotically cylindrical case. A more detailed treatment is in [39], to which we will occasionally refer. To begin, we need to find a direct complement of the tangent space to the space of diffeomorphisms that preserve the cone-like behaviour of the appropriate rate on the ends, within the space of closed 3-forms with the same decay at the ends. In order to later apply the Banach space implicit function theorem to determine the structure of the moduli space, we will need to consider (weighted) Sobolev spaces of forms, and thus we actually need to establish a “slice theorem” for forms in such weighted Sobolev spaces.

In the space $\Omega^3_{l,\nu}$, the analogue of the space of infinitesimal diffeomorphisms defined in equation (118) is the space of 3-forms that are the exterior derivative of a 2-form of type $\Lambda^2_7$ in the appropriate Sobolev space. Explicitly, we define $D_{l+1,\nu+1}$ to be the group generated by the set

$$\{\exp(X); X \in \Gamma(TM), X \in L^2_{l+1,\nu+1}(T^*M)\}.$$ 

and thus the tangent space to the orbit $D_{l+1,\nu+1} \cdot \varphi$ at $\varphi$ is given by

$$T_\varphi(D_{l+1,\nu+1} \cdot \varphi) = d\left(L^2_{l+1,\nu+1}(\Lambda^2_7(T^*M))\right) \subseteq \Omega^3_{l,\nu}.$$ 

Similarly, in $\Omega^3_{l,\nu}$ the tangent space at $\varphi$ to the closed $G_2$ structures $C_{l,\nu}$ that are asymptotic to $\varphi$ with rate $\nu$ is given by

$$T_\varphi C_{l,\nu} = \{\eta \in \Omega^3_{l,\nu}; d\eta = 0\} = C_{l,\nu}.$$ 

It is clear that $T_\varphi(D_{l+1,\nu+1} \cdot \varphi)$ is a subspace of $T_\varphi C_{l,\nu} = C_{l,\nu}$.

**Definition 5.2.** Given a rate $\nu$, define $(G_\varphi)_{l,\nu}$ to be the following subspace of $\Omega^3_{l,\nu}$:

$$(G_\varphi)_{l,\nu} = \{\eta \in \Omega^3_{l,\nu}; d\eta = 0, \pi_7(d^*\eta) = 0\}.$$ 

Thus $(G_\varphi)_{l,\nu}$ is a proper subspace of $C_{l,\nu}$, corresponding to the “gauge-fixed” (with respect to $\varphi$) infinitesimal deformations of closed $G_2$ structures, given by Definition 4.65.

We can now state the first result of this subsection, which is an infinitesimal version of our slice theorem.

**Theorem 5.3.** Suppose that $\nu < 0$ (AC) or $\nu > 0$ (CS).

1. In the $L^2$ setting: AC when $\nu < -\frac{7}{2}$, or CS for any $\nu > 0$. There exists a finite-dimensional subspace $(E_\varphi)_{l,\nu}$ of $C_{l,\nu}$ such that

$$C_{l,\nu} = T_\varphi(D_{l+1,\nu+1} \cdot \varphi) \oplus (G_\varphi)_{l,\nu} \oplus (E_\varphi)_{l,\nu}.$$ 

(121)
[2] In the non-$L^2$ setting: AC when $\nu \in (-4, -0)$, there are two subcases.

- When $\nu \in (-4, -1)$, we have
  \[ \mathcal{C}_{l, \nu} = T_\varphi(D_{l+1, \nu+1} \cdot \varphi) \oplus (\mathcal{G}_\varphi)_{l, \nu}. \]  
  \[ \tag{122} \]

- When $\nu \in (-1, 0)$ there is a closed subspace $(\mathcal{G}_\varphi')_{l, \nu}$ of $(\mathcal{G}_\varphi)_{l, \nu}$, of finite codimension, such that
  \[ \mathcal{C}_{l, \nu} = T_\varphi(D_{l+1, \nu+1} \cdot \varphi) \oplus (\mathcal{G}_\varphi')_{l, \nu}. \]  
  \[ \tag{123} \]

**Proof.** The proof has a very different flavour in the two cases.

Case [1]: In the $L^2$ setting we can integrate by parts. We first observe that $T_\varphi(D_{l+1, \nu+1} \cdot \varphi)$ and $(\mathcal{G}_\varphi)_{l, \nu}$ are $L^2$-orthogonal, since if $Y \in L^2_{l+1, \nu+1}(TM)$ and $\eta \in (\mathcal{G}_\varphi)_{l, \nu}$ then
  \[ \langle \langle d(Y \cdot \varphi), \eta \rangle \rangle = \langle \langle Y \cdot \varphi, \pi_7(d^* \eta) \rangle \rangle = 0. \]

Next, we claim that there exists a finite-dimensional space $Q_{l-1, \nu-1} \subseteq (\Omega^2_l)_{l-1, \nu-1}$ such that
  \[ \pi_7 d^* (\mathcal{C}_{l, \nu}) = \pi_7 d^* (\langle \Omega^2_l \rangle_{l+1, \nu+1}) \oplus Q_{l-1, \nu-1}. \]  
  \[ \tag{124} \]

To see this, we note by Proposition 2.7 that the map $\pi_7 d^*$ acting from $\Omega^2_l$ to itself is elliptic and thus for generic rates $\nu$, the map $\pi_7 d^* : (\Omega^2_l)_{l+1, \nu+1} \to (\Omega^2_l)_{l-1, \nu-1}$ is Fredholm. Therefore for these rates it has a closed image with finite-dimensional complement. Now the image of the map $\pi_7 d^*$ acting on $\mathcal{C}_{l, \nu}$ contains the image of $\pi_7 d^*$ and hence has finite-dimensional complement in $(\Omega^2_l)_{l-1, \nu-1}$. Since $\pi_7 d^*$ is the image of a continuous map acting on the Banach space $\mathcal{C}_{l, \nu}$, we have that its image is closed by [24] Chapter XV, Corollary 1.8]. The existence of the decomposition in equation (124) is now clear.

Thus, for each $\chi \in Q_{l-1, \nu-1}$ there exists $\zeta \in \mathcal{C}_{l, \nu}$ such that $\pi_7 (d^* \zeta) = \chi$, where $\zeta$ is unique up to elements in $(\mathcal{G}_\varphi)_{l, \nu}$ and must be transverse to $d(\Omega^2_l)_{l+1, \nu+1}$ by definition. Hence we can choose $\zeta$ transverse to $d(\Omega^2_l)_{l+1, \nu+1} \oplus (\mathcal{G}_\varphi)_{l, \nu}$, where we have already shown that the sum is direct by $L^2$ orthogonality. In summary, we have shown that there is a subspace $(\mathcal{E}_\varphi)_{l, \nu}$ of $\mathcal{C}_{l, \nu}$ which is isomorphic to $Q_{l-1, \nu-1}$ and transverse to $d(\Omega^2_l)_{l+1, \nu+1} \oplus (\mathcal{G}_\varphi)_{l, \nu}$.

Therefore we deduce that given any $\eta \in \mathcal{C}_{l, \nu}$ there exists $Y \cdot \varphi \in (\Omega^2_l)_{l+1, \nu+1}$ and a unique $\zeta \in (\mathcal{E}_\varphi)_{l, \nu}$ such that
  \[ \pi_7 d^* \eta = \pi_7 d^* (Y \cdot \varphi) + \pi_7 d^* \zeta. \]

Thus we may decompose
  \[ \eta = d(Y \cdot \varphi) + (\eta - d(Y \cdot \varphi) - \zeta) + \zeta \]
where $\eta - d(Y \cdot \varphi) - \zeta \in (\mathcal{G}_\varphi)_{l, \nu}$ and the decomposition is unique. This completes the proof in the $L^2$ case.

Case [2]: Let $\eta$ be in $T_\varphi \mathcal{C}_{l, \nu}$. We will first show that $\eta$ can be written as the sum of an element of $T_\varphi(D_{l+1, \nu+1} \cdot \varphi)$ and an element of $(\mathcal{G}_\varphi)_{l, \nu}$, and then show, if $\nu \leq -1$, that the intersection of these two subspaces is trivial. Since $\eta$ is closed, Theorem 4.33 says that we can write $\eta = k + d\alpha$ for some $k \in \mathcal{H}_\varphi$ and some 2-form $\alpha$. By Lemma 4.75, the modified Dirac operator $\tilde{D}$ of equation (20) is surjective, so there exists a pair $(2h, Y) \in L^2_{l+1, \nu+1}(\Lambda^0 \oplus \Lambda^1)$ such that
  \[ \pi_{1+7}(d\alpha) = * (d\alpha \wedge \varphi) + \pi_{1+7}(d(Y \cdot \varphi)). \]
Since \(* (d\mathbf{h} \wedge \varphi)\) is pointwise of type \(\Lambda^3\), the above equation says that
\[
dX - d(Y \lrcorner \varphi) = *(d\mathbf{h} \wedge \varphi) + \eta_{27}
\]
for some \(\eta_{27}\) pointwise of type \(\Lambda^3\). Define \(\sigma = \alpha - (Y \lrcorner \varphi)\), which is an element of \(\Omega^2_{l+1,\nu+1}\). Let \(\zeta = \sigma \mathbf{r}\) denote the right hand side of (125). We have \(d\zeta = 0\), and \(\zeta\) is of the form \(f\mathbf{r}\) with \(f = 0\) and \(X = d\mathbf{h}\). Because \(\text{curl} X = \text{curl}(d\mathbf{h}) = 0\), we can apply Lemma 4.67 to conclude that \(\pi \tau d^* \zeta = 0\).

Since \(\kappa\) is coclosed, we find that \(\eta = d(Y \lrcorner \varphi) + \kappa + d\sigma\) with \(\pi \tau d^*(\kappa + d\sigma) = 0\). But \(d(Y \lrcorner \varphi)\) is in \(T^*_h(D_{l+1,\nu+1} \cdot \varphi)\) and thus the difference \(\eta - d(Y \lrcorner \varphi) = \kappa + d\sigma\) is in \((\mathcal{G}_{\varphi})_{l,\nu}\), as required.

To complete the proof of case [2] we need to consider the intersection \(T^*_h(D_{l+1,\nu+1} \cdot \varphi) \cap (\mathcal{G}_{\varphi})_{l,\nu}\).

Let \(d(X \lrcorner \varphi)\) lie in this intersection. Let \(\mu = X \lrcorner \varphi\). We have \(\pi \tau d^* d\mu = 0\). By Proposition 2.7, we have \(d d^* X + \frac{1}{2} d^* d X = 0\). But \(\mu \in L^2_{l+1,\nu+1}(\Lambda^2_7(T^*M))\), so \(X \in L^2_{l+1,\nu+1}(TM)\). Now using Proposition 3.7, on the excluded range of \(1\)-forms \(X\) satisfying \(d d^* X + \frac{1}{2} d^* d X = 0\) and Theorem 4.10 on the invariance of the kernel we conclude that if \(\nu + 1 \leq 0\), then in fact we can say that \(X\) is actually \(O(\nu^{\nu+1})\) where \(\nu' + 1 = -4 + \varepsilon < -\frac{7}{2}\). Thus we can use Lemma 4.65 to conclude that \(X = 0\) and thus \(d(X \lrcorner \varphi) = 0\) when \(\nu \leq -1\). If \(\nu \in (-1, 0)\), then all we have shown is that \(X\) is in the kernel of \(\pi \tau d^* d : \Omega^2_7 \to \Omega^2_7\). Since this operator is elliptic, there is only a finite-dimensional space \(\mathcal{J}_\nu\) of such \(1\)-forms. Choosing a topological complement \((\mathcal{G}_{\varphi}')_{l,\nu}\) of the finite-dimensional space \(\{d(X \lrcorner \varphi); X \in \mathcal{J}_\nu\}\) in \((\mathcal{G}_{\varphi})_{l,\nu}\) completes the proof.

\[\square\]

**Remark 5.4.** In the CS case we cannot always conclude that the space \((\mathcal{E}_{\varphi})_{l,\nu}\) vanishes. The effect of this is that we will only actually be able to say something about a subspace \(\mathcal{M}_{\nu}\) of the full moduli space \(\mathcal{M}_{\nu}\), which we will call the reduced moduli space. When \((\mathcal{E}_{\varphi})_{l,\nu}\) does indeed vanish, we will have \(\mathcal{M}_{\nu} = \mathcal{M}_{\nu}\). It is possible to obtain estimates on the dimension of \((\mathcal{E}_{\varphi})_{l,\nu}\) in terms of the kernels of certain elliptic operators on the conifold, but we did not see any value in including such estimates in the present paper.

**Remark 5.5.** In the AC case when \(\nu > -4\), Theorem 5.3 says that the space \((\mathcal{G}_{\varphi})_{l,\nu}\) is only a good infinitesimal slice when \(\nu \leq -1\). When \(\nu \in (-1, 0)\), not all gauge-fixed infinitesimal deformations are actually transverse to the orbit of the diffeomorphism action. Hence we need to consider a smaller slice whose tangent space if \((\mathcal{G}_{\varphi}')_{l,\nu}\). Note that \(\nu \leq -1\) is satisfied by all currently known examples. We note here that the arguments in Section 5.2.2 will require a restriction to \(\nu < -\frac{7}{2}\). See also Remark 1.1.

Now let \((\mathcal{G}_{\varphi}')_{l,\nu}\) be as in Theorem 5.3 in the AC case when \(\nu > -1\), and set \((\mathcal{G}_{\varphi})_{l,\nu} = (\mathcal{G}_{\varphi}')_{l,\nu}\) in the AC case when \(\nu \leq -1\) or in the CS case. Furthermore, let \((\mathcal{E}_{\varphi})_{l,\nu}\) be as in Theorem 5.3 in the CS case, and set \((\mathcal{E}_{\varphi})_{l,\nu} = \{0\}\) in the AC case. If we now define a map \(\exp_{\varphi} : (\mathcal{G}_{\varphi}')_{l,\nu} \oplus (\mathcal{E}_{\varphi})_{l,\nu} \to \mathcal{C}_{l,\nu}\) by affine translation, namely
\[
\exp_{\varphi}(\eta) = \varphi + \eta,
\]
then it is clear that the image \(S_{l,\nu} = \exp_{\varphi}((\mathcal{G}_{\varphi}')_{l,\nu} \oplus (\mathcal{E}_{\varphi})_{l,\nu})\) is an (infinite-dimensional) smooth submanifold of \(\mathcal{C}_{l,\nu}\) whose tangent space at \(\varphi\) is exactly \(T_{\varphi}S_{l,\nu} = (\mathcal{G}_{\varphi}')_{l,\nu} \oplus (\mathcal{E}_{\varphi})_{l,\nu}\). We would like to conclude that, near the point \(\varphi\), the space \(S_{l,\nu}\) contains exactly one representative from each orbit of the action of the diffeomorphisms in \(D_{l+1,\nu+1}\). More precisely, we want to establish that, near \(\varphi\), the space \(S_{l,\nu}\) is homeomorphic to \(\mathcal{C}_{l,\nu}/D_{l+1,\nu+1}\). In fact, what we can actually prove is that near \(\varphi\), the space of torsion-free elements in \(S_{l,\nu}\) is homeomorphic to our moduli space \(\mathcal{M}_{\nu}\). This result will be sufficient for our purposes. The details of this argument are discussed in Nordström 39, Section 3.1. Our situation admits several nice features that allow us to use the simplifications that Nordström explains in 39, Section 3.1.3, which we now briefly discuss.
We define $R_{l,v} = S_{l,v} \cap T$ to be the torsion-free $G_2$ structures in $S_{l,v}$. The content of Corollary 5.10 in the next section is that $R_{l,v}$ consists of smooth elements, so we can drop the subscript $l$ on $R_{l,v} = R_v$ and we are able to use [39, Theorem 3.1.4]. Thus, we can conclude as in [39, page 51], that the space $R_v$ is locally homeomorphic to an open neighbourhood of $[\varphi]$ in $M_v$. So the problem of understanding the local structure of $M_v$ reduces to understanding $R_v$. We summarize the preceding discussion in the following theorem.

**Theorem 5.6.** Let $R_{l,v} = \{ \eta + \zeta \in (G'_v)_{l,v} \oplus (E_v)_{l,v}; |\eta + \zeta|_{C^0} < \varepsilon, d(\Theta(\varphi + \eta + \zeta)) = 0 \}$ be the space parametrizing the torsion-free gauge-fixed $G_2$ structures close to $\varphi$. Then for $\varepsilon$ sufficiently small, any element $\eta + \zeta \in R_{l,v}$ has $\eta$ smooth and hence independent of $l$. Moreover $R_{l,v}$ is homeomorphic to an open neighbourhood of the point $D_{\nu + 1} \cdot \varphi$ in $M_v$.

In the situations where the finite-dimensional space $(E_v)_{l,v}$ is not necessarily trivial, we need to define a reduced moduli space $\hat{M}_v$, as follows.

**Corollary 5.7.** Let $\hat{S}_v = \exp_{\varphi}((G'_v)_{l,v} \oplus \{ 0 \})$ and define $\hat{R}_{l,v} = \hat{S}_{l,v} \cap T$ to be the torsion-free $G_2$ structures in $\hat{S}_{l,v}$. As before, this space is independent of $l$. Then

$$\hat{R}_{l,v} = \{ \eta \in (G'_v)_{l,v}; |\eta|_{C^0} < \varepsilon, d(\Theta(\varphi + \eta)) = 0 \}$$

is the space parametrizing the torsion-free gauge-fixed $G_2$ structures close to $\varphi$ that do not involve a deformation in the $(E_v)_{l,v}$ direction. It is clear that $\hat{R}_{l,v}$ is a topological subspace of $R_{l,v}$, and under the homeomorphism that identifies $R_{l,v}$ with $M_v$, the space $\hat{R}_{l,v}$ is identified with a topological subspace $\hat{M}_v$ of $M_v$, called the reduced moduli space. Note that $\hat{M}_v = M_v$ whenever $(E_v)_{l,v} = \{ 0 \}$. In particular by Theorem 5.3 we always have $\hat{M}_v = M_v$ in the AC case when $\nu > -4$.

**Proof.** This follows immediately from the discussion leading to the proof of Theorem 5.6. 

### 5.2.2 Step 2: Local one-to-one correspondence with solutions of an elliptic PDE

In this section we establish a (local) one-to-one correspondence between (i) gauge-fixed torsion-free $G_2$ structures with the same conical asymptotics on the ends as $\varphi$ that are sufficiently $C^0$-close to $\varphi$; and (ii) solutions of a nonlinear elliptic partial differential equation on $M$.

Let $(M, \varphi)$ be a $G_2$ conifold of rate $\nu$. As always, we assume $\nu > 0$ in the CS case. One direction of Theorem 5.8 below in the AC case requires that we restrict to $\nu < -\frac{2}{3}$, but the other direction works for all $\nu < 0$. This will have very important applications in Section 6.

Let $\varepsilon > 0$ be the constant from Lemma 4.72. Consider the set of $G_2$ structures $\hat{\varphi}$ that are asymptotic, at the same rate $\nu$, to the same $G_2$ cones on the ends, which are $\varepsilon$-close to $\varphi$ in the $C^0$ norm, such that the difference $\hat{\varphi} - \varphi$ lies in $\Omega^3_\nu$. Using Theorem 4.33 we can write $\eta = \hat{\varphi} - \varphi$ as

$$\eta = \hat{\varphi} - \varphi = \kappa + d\alpha + \delta,$$

where $\kappa \in \mathcal{H}_\nu^3$, the 2-form $\alpha$ is in $\Omega^2_{l+1,\nu+1}$, and $\delta$ is a closed form in $U^3_\nu$. We also know that $\delta = 0$ in the AC case when $\nu > -4$. For convenience we will often write

$$\eta = \hat{\varphi} - \varphi = \kappa + \zeta, \quad \text{where} \quad \zeta = d\alpha + \delta.$$

The next result should be compared with [13, Theorem 10.3.6]. It is both a generalization to the conifold setting, and a reformulation in terms of the first order operator $d + d^*$ rather than the Laplacian $\Delta$. 

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**Theorem 5.8.** Consider the subset of torsion-free and gauge-fixed \( G_2 \) structures \( \tilde{\varphi} \) that are \( \varepsilon \)-close to \( \varphi \) in the \( C^0 \) norm and such that \( \tilde{\varphi} - \varphi \) lies in \( \Omega^3_{l,\nu} \). Specifically, these are the \( G_2 \) structures \( \tilde{\varphi} \) expressed in the form (126) such that \( \eta = \tilde{\varphi} - \varphi \) satisfies

\[
d\eta = 0 \quad \text{and} \quad d\Theta(\varphi + \eta) = 0 \quad \text{and} \quad \pi_7(d^*\eta) = 0.
\]

In particular, this includes all \( \eta \in \tilde{\mathcal{R}}^7_{l,\nu} \) as defined in Corollary 5.7.

Consider also the following nonlinear equation:

\[
(d + d^*)\eta = d^* (Q_\varphi(\eta)).
\]

In the CS case (\( \nu > 0 \)) and in the AC case for all rates \( \nu < 0 \), we always have that small (\( |\eta|_{C^0} < \varepsilon \)) 3-forms \( \eta \in \Omega^3_{l,\nu} \) that satisfy (128) are solutions in \( \Omega^3_{l,\nu} \) of (129). That is, we always have, for any \( G_2 \) conifolds, that (128) implies (129).

Moreover, in the CS case, and in the AC case if \( \nu < -\frac{5}{7} \), there is a one-to-one correspondence between small (\( |\eta|_{C^0} < \varepsilon \)) 3-forms \( \eta \in \Omega^3_{l,\nu} \) that satisfy (128) and solutions in \( \Omega^3_{l,\nu} \) of (129).

**Proof.** First we establish an identity (130) that will be used to prove both directions of this theorem. If we substitute \( \eta = \kappa + \zeta \) into equation (16) and simplify using equation (15), we obtain:

\[
*d(\Theta(\tilde{\varphi})) = -d^* (L_\varphi(\kappa + \zeta)) - d^* (Q_\varphi(\eta))
\]

\[
= -d^* \left( 4\pi_1(\kappa + \zeta) + \pi_7(\kappa + \zeta) - \pi_{27}(\kappa + \zeta) \right) - d^* (Q_\varphi(\eta)).
\]

By Lemma 4.25 we know that the pure-type components \( \pi_1(\kappa) \), \( \pi_7(\kappa) \), and \( \pi_{27}(\kappa) \) are each closed and coclosed, and thus all the linear terms involving \( \kappa \) vanish. Therefore if we add and subtract \( d^*\zeta = d^*\eta \) we find that

\[
*d(\Theta(\tilde{\varphi})) = -\frac{7}{3} d^*\pi_1(\zeta) + 2d^*\pi_7(\zeta) + d^*\eta - d^* (Q_\varphi(\eta)).
\]

Now suppose that (128) holds. The third equation of this set is exactly the gauge-fixing condition \( \pi_7(d^*\eta) = 0 \). Since \( \kappa \) is coclosed, this is equivalent to \( \pi_7(d^*\zeta) = 0 \). Write \( \zeta \) in the form (107). Since \( \zeta \in \Omega^3_{l,\nu} \), we have that \( \nabla f \) and \( \nabla X \) are of rate \( \nu - 1 \). In the CS case, this means both \( \nabla f \) and \( \nabla X \) are in \( L^2 \). Thus we can apply Lemma 4.66 to conclude that \( f = c \) is a constant and \( X = 0 \). Hence we have \( \zeta = c\varphi + \pi_{27}\zeta \). In particular, we get

\[
d^*\pi_1(\zeta) = 0 \quad \text{and} \quad d^*\pi_7(\zeta) = 0.
\]

In the AC case, we have to argue slightly differently. Since \( d^*\zeta \) is in \( \Omega^3_{14} \) and is coclosed, we can use Proposition 2.5 and the fact that \( d\zeta = 0 \), to conclude that \( dd^*\zeta = \Delta\zeta \in \Omega^3_{27} \). Thus because the Laplacian commutes with the projections, we have

\[
\Delta(\pi_1(\zeta)) = 0 \quad \text{and} \quad \Delta(\pi_7(\zeta)) = 0.
\]

Since \( \pi_1(\zeta) = f\varphi \) for some function \( f \), equation 8 tells us that \( f \) is harmonic. We have \( f = O(r^\nu) \) on the ends and \( \nu < 0 \). Hence \( f \to 0 \) at infinity, and therefore the maximum principle forces \( f = 0 \). Now \( \pi_7(\zeta) = *(X \wedge \varphi) \) for some 1-form \( X \) which by equation 9 is harmonic. But Proposition 3.6 says that there are no changes in the space of harmonic 1-forms in the interval \([-5,0] \), so in fact
we can say that \( X = O(r^{-5}) \), and then Lemma 4.64 says that \( X = 0 \). So in the AC case as well, equations (131) both hold. These two equations, together with the second equation of (128), when substituted into (130), yield \( d^* \eta = d^* (Q_\varphi(\eta)) \). Now this expression together with the first equation of (128) gives equation (129).

Conversely, suppose that equation (129) holds. We thus have \( d\eta = 0 \), which is one of the three equations in (128), and we also have \( d^* \eta = d^* (Q_\varphi(\eta)) \). Substituting this into equation (130) and taking the Hodge star, we obtain

\[
d(\Theta(\tilde{\varphi})) = \frac{7}{3} d \ast \pi_1(\zeta) + 2 d \ast \pi_7(\zeta).
\]

(132)

Now we note that \( \pi_1(\zeta) = f \varphi \) and \( \pi_7(\zeta) = *(X \wedge \varphi) \) for some function \( f \) and some 1-form \( X \). Therefore (132) can be written as

\[
d(\Theta(\tilde{\varphi})) = \frac{7}{3} df \wedge \psi + 2 dX \wedge \varphi,
\]

(133)

using the fact that \( \varphi \) is closed and coclosed. Now by the \( C^0 \)-closeness assumption, we can apply Lemma 4.72 to (133) to conclude that all three terms of (132) vanish, provided we can show that \( d(\Theta(\tilde{\varphi})) = O(\varphi') \) and \( X = O(\lambda^{\lambda+1}) \) for some \( \lambda < -\frac{2}{7} \) (AC) or \( \lambda > -\frac{7}{2} \) (CS).

Since \( \tilde{\varphi} - \varphi \) is \( O(\varphi') \), equation (10) and Lemma 2.8 give us that \( d(\Theta(\tilde{\varphi})) = O(\varphi'^{-1}) + O(\varphi'^{2\nu-1}) \). But \( \nu \) is not known in the AC case and \( \nu > 0 \) in the CS case, so we do not consider the first term vanishing on the ends, and thus \( \nu = 1 \). Certainly in the CS case we have \( \nu - 1 > -\frac{7}{2} \). In the AC case we need \( \nu - 1 < -\frac{7}{2} \), that is \( \nu < -\frac{5}{2} \), which is our hypothesis. Meanwhile \( X \wedge \varphi = \ast \pi_7(\zeta) = O(\varphi'^{\lambda}) \), so \( X = O(\varphi') \) since \( \varphi') = O(1) \). (Recall it is the difference \( \varphi - \varphi_c \) that is \( O(\varphi') \). The \( G_2 \) structure \( \varphi \) is \( O(\varphi') \) because \( \varphi_c \) is.) Therefore \( X = O(\lambda^{\lambda+1}) \) for some \( \lambda < -\frac{7}{2} \) (AC) or \( \lambda > -\frac{7}{2} \) (CS), which both hold. Thus we can indeed apply Lemma 4.72 to (133) to conclude that all three terms of (132) vanish.

All that remains to do in order to prove that (129) implies (128) is to show that \( \pi_7(d^* \eta) = 0 \), the gauge-fixing condition. Since \( \kappa \) is coclosed, this is equivalent to \( \pi_7(d^* \zeta) = 0 \). But we have just concluded that \( \zeta = \pi_1(\zeta) + \pi_7(\zeta) + \pi_7(\zeta) \) with \( \pi_7(\zeta) = f \varphi \) for some function \( f \) with \( df = 0 \) and \( \pi_7(\zeta) = \ast(X \wedge \varphi) \) for some 1-form \( X \) such that \( dX \wedge \varphi = 0 \). From Definition 2.9 we see that \( \text{curl} \, X = 0 \), so we can apply Lemma 4.67 to conclude that \( \pi_7(d^* \zeta) = 0 \), as required.

\[\Box\]

Remark 5.9. The reader will have noticed that in the AC case, we needed the crucial hypothesis \( \nu < -\frac{5}{2} \) for one direction of this theorem to be able to apply the various gauge-fixing results of Section 4.7. Without this assumption, we do not have a one-to-one correspondence. All we would know is that solutions to (128) give solutions to (129), but not conversely. Also, one direction in the proof of Theorem 5.8 did not require the assumption of \( C^0 \)-closeness. See also Remark 1.1.

Corollary 5.10. After possibly making \( \varepsilon > 0 \) smaller, the space \( \mathcal{R}_{t,\nu}^\varepsilon \) is equal to the set of smooth forms \( \eta \) with \( |\eta|_{C^0} < \varepsilon \) that satisfy equation (129).

\[\Box\]
5.2.3 Step 3: Applying the Banach space implicit function theorem

In this section we will apply the Banach space implicit function theorem to study the local structure of the reduced moduli space \( \tilde{M}_\nu \) of \( G_2 \) conifolds of rate \( \nu \), which in many cases will be shown to equal the full moduli space \( M_\nu \).

For completeness, we explicitly state here the Banach space implicit function theorem that we will use. Its proof can be found, for example, in Lang [23, Theorem 6.2.1]. The hats on \( \hat{U} \) and \( \hat{F} \) are employed to match notation with the eventual use of this theorem later in this section.

**Theorem 5.11** (Banach space implicit function theorem). Let \( X \) and \( Y \) be Banach spaces, and let \( \hat{U} \subseteq X \) be an open neighbourhood of 0. Let \( \hat{F} : \hat{U} \to Y \) be a \( C^k \)-map (with \( k \geq 1 \)) such that \( \hat{F}(0) = 0 \). Suppose that the differential \( \hat{D}\hat{F}(0) : X \to Y \) is surjective, with kernel \( K \) such that \( X = K \oplus Z \) for some closed subspace \( Z \) of \( X \). Then there exist open sets \( V \subseteq K \) and \( W \subseteq Z \), both containing 0, with \( V \times W \subseteq \hat{U} \), and a unique \( C^k \)-map \( \hat{G} : V \to W \) such that

\[
\hat{F}^{-1}(0) \cap (V \times W) = \{ (x, \hat{G}(x)) : x \in V \}
\]

in \( X = K \oplus Z \). That is, the zero set of \( \hat{F} \) near the origin in \( X \) is parametrized by a neighbourhood of the origin in the space \( K \).

From now on we assume \( \nu < -\frac{5}{2} \) in the AC case so that Theorem 5.11 gives a one-to-one correspondence. To begin, we define a nonlinear map

\[
F : U \subseteq \Omega^3_{l,\nu} \to \Omega^2_{l-1,\nu} \oplus \Omega^4_{l-1,\nu-1}
\]

by the rule

\[
F(\eta) = (d + d^*)\eta - d^* \ast (\Theta(\eta)). \tag{134}
\]

We show that this map is well defined below, in Lemma 5.12. The motivation for this definition is that, by Theorem 5.11 the zero set of \( F \) is precisely the space \( \mathcal{R}_{l,\nu}^{c} \) which by Corollary 5.7 is homeomorphic to a subspace \( \tilde{M}_\nu \) of (and is often equal to) the moduli space \( M_\nu \).

Let \( \varepsilon > 0 \) be the constant from Lemma 4.5 and Theorem 5.8. Let \( U \) denote the open subset of \( \Omega^3_{l,\nu} \) consisting of 3-forms which are within \( \varepsilon \) of \( \varphi \) in the \( C^0 \) norm.

**Lemma 5.12.** For \( \eta \in U \), we have \( Q_\varphi(\eta) \in \Omega^3_{l,\nu} \), and so \( d^* \ast (Q_\varphi(\eta)) \in \Omega^2_{l-1,\nu} \).

**Proof.** This argument is very similar to [15, Proposition 6.4] and [31, Proposition 5.7], with some minor differences. We present it here for completeness. From Lemma 2.8 we have \( Q_\varphi(0) = 0 \) and \( |Q_\varphi(\eta)| \leq C|\eta|^2 \) for some positive constant \( C \). That is, the smooth function \( Q_\varphi \) is the quadratic term in the second order Taylor polynomial for the smooth function \( \Theta_\varphi \) of \( \varphi \) in \( L^2(T^*_x M) \), for fixed \( x \in M \). More precisely, if we let \( x^1, \ldots, x^7 \), be local coordinates for a trivialization of the bundle \( \Lambda^3(T^*_x M) \), then we can regard \( Q_\varphi \) locally as a smooth function

\[
R(x) = Q_\varphi(x, y(x))
\]

such that, for fixed \( x \) and for \( |y| \leq \varepsilon \), we have

\[
(\nabla_x)^a (\partial_y)^b Q(x, y) = O(|y|^{\max(0,2-b)}). \tag{135}
\]

We need to modify (135) by inserting the appropriate function of \( x \) as a multiplier for such an estimate to hold uniformly on \( M \). Since \( \varphi \) is asymptotic to a \( G_2 \) cone at each end, it is clear that the appropriate uniform estimate is

\[
|t(\partial_x)^a (\partial_y)^b Q(x, y)| \leq C|y|^{\max(0,2-b)}, \quad \forall a, b \geq 0, \tag{136}
\]
where \( \varrho \) is a radius function on \( M \). Since we always assume that \( l \geq 6 \), by Corollary 4.7 we have \( \eta \in C^{2,\alpha}_{\nu} \), and thus in particular

\[
|\eta| = O(\varrho^\nu) \quad \text{and} \quad |\nabla \eta| = O(\varrho^{\nu-1}).
\]

(137)

Note that we know nothing about \( |\nabla^k y| \) for \( k > 2 \). Now because \( \nu < 0 \) in the AC case with \( \varrho \to \infty \) on the end, and likewise because \( \nu > 0 \) in the CS case with \( \varrho \to 0 \) on each end, in either case we find that \( \varrho^\nu \), and thus \( \eta \), is bounded on \( M \).

To prove that \( Q_\nu(\eta) \in \Omega_{l,\nu}^3 \) we need to show that

\[
\nabla^j R \in L_{0,\nu-j}^2 \quad \text{for } 0 \leq j \leq l.
\]

By the chain rule, we have

\[
|\nabla^j R| \leq C_j \sum_{a+b \leq j} |(\nabla_x)^a (\partial_y)^b R(x, y(x))| \cdot \left( \sum_{m_1, \ldots, m_b \geq 1} \left( \prod_{i=1}^b |\nabla^{m_i} y(x)| \right) \right)
\]

(138)

for some positive constant \( C_j \) that is purely combinatorial and depends only on \( j \). To show that \( \nabla^j R \) is in \( L_{0,\nu-j}^2 \), we need to prove that the integral

\[
\int_M |\varrho^{2j-2\nu} \nabla^j R|^2 \varrho^{-7} \text{vol}_M
\]

is finite. From the inequality (138), it suffices to prove that each of the integrals

\[
\int_M \varrho^{2j-2\nu} |(\nabla_x)^a (\partial_y)^b R(x, y(x))|^2 \left( \prod_{i=1}^b |\nabla^{m_i} y(x)|^2 \right) \varrho^{-7} \text{vol}_M
\]

(139)

is finite, where \( a, b \geq 0, m_1, \ldots, m_b \geq 1, a+b \leq j \) and \( a+m_1+\cdots+m_b = j \).

Consider first the case \( b = 0 \). In this case, \( a = j \) and the product is empty. Hence, from (136) and the fact that \( |y| = |\eta| \) is bounded on \( M \), we have \( |(\nabla_x)^a R| \leq C \varrho^{-j} |y| \leq C \varrho^{-j} |\eta| \). Hence the integral in (139) is bounded above by

\[
C \int_M \varrho^{2j-2\nu} \varrho^{-2j} |\eta|^2 \varrho^{-7} \text{vol}_M = C \int_M \varrho^{-2\nu} |\eta|^2 \varrho^{-7} \text{vol}_M
\]

which is finite since \( \eta \in L_{l,\nu}^2 \subseteq L_{0,\nu}^2 \).

Next we consider the case \( b = 1 \). This time, \( m_1 \geq 1 \) and \( a+m_1 = j \). Thus, from (136) and (137), we have \( |(\nabla_x)^a (\partial_y) R| \leq C \varrho^{-a} |y| \leq C \varrho^{-a+\nu} \). Hence the integral in (139) is bounded above by

\[
C \int_M \varrho^{2j-2\nu} \varrho^{-2a+2\nu} |\nabla^{m_1} \eta|^2 \varrho^{-7} \text{vol}_M = C \int_M \varrho^{2m_1} |\nabla^{m_1} \eta|^2 \varrho^{-7} \text{vol}_M
\]

(140)

However, since \( \eta \in L_{l,\nu}^2 \), we have \( \nabla^{m_1} \eta \in L_{l+m_1,\nu-m_1}^2 \subseteq L_{0,\nu-m_1}^2 \) and therefore the integral

\[
\int_M \varrho^{-2\nu+2m_1} |\nabla^{m_1} \eta|^2 \varrho^{-7} \text{vol}_M
\]

(141)
is finite. But in either the AC case or the CS case, the function \( \varrho^{−2\nu} \to \infty \) at the ends, so outside of a compact set the integrand of (141) dominates the integrand of (140). Hence the integrals in (139) with \( b = 1 \) are indeed finite.

Finally we consider the general case of \( b \geq 2 \). Now from (136) we have \( |(\nabla x)^a(\partial y)^bR| \leq C\varrho^{−a} \). Hence the integral in (139) is bounded above by

\[
C \int_M \varrho^{2j−2\nu} \varrho^{−2a} \left( \prod_{i=1}^b |\nabla^{m_i} \eta|^2 \right) \varrho^{−7} \text{vol}_M.
\]

For \( i = 1, \ldots, b \), define \( q_i = \frac{l−a}{m_i} \). Since \( b \geq 2 \) and \( a + m_1 + \cdots + m_b = j \), we have \( q_i > 1 \), and also \( \sum_{i=1}^b \frac{1}{q_i} = 1 \). Observe also that the integrand of (142) can be written as \( \prod_{i=1}^b s_i \), where

\[
s_i = \varrho^{2m_i−2q_i |\nabla^{m_i} \eta|^2} \varrho^{−\frac{2}{q_i}}.
\]

Now by Hölder’s inequality, we have \( \int_M (\prod_{i=1}^b s_i)\text{vol}_M = \| \prod_{i=1}^b s_i \|_1 \leq \prod_{i=1}^b \| s_i \|_{q_i} \). Thus we can finish the proof if we can show the finiteness of the integrals

\[
\| s_i \|_{q_i}^2 = \int_M s_i^{q_i} \text{vol}_M = \int_M \varrho^{2m_i−2q_i |\nabla^{m_i} \eta|^2} \varrho^{−7} \text{vol}_M.
\]

We claim that the above integral is indeed finite, by the Sobolev embedding Theorem 4.6. To see this, let \( p = 2 \), and let \( q = 2q_i > 2 \) since \( q_i > 1 \). Let \( m = m_i \). We have \( l \geq m \) since \( m_i \leq j \leq l \). Furthermore, the last remaining inequality we need to use the embedding theorem is

\[
l−\frac{7}{2} \geq m−\frac{7}{q} = m_i−\frac{7}{2q_i} = m_i \left( 1−\frac{7}{2(j−a)} \right),
\]

which is easy to verify from \( 2 \leq j − a \leq l \) and \( 0 < \frac{m_i}{2} \leq 1 \). Thus Theorem 4.6 tells us that \( L^2_{l,\nu} \subseteq L^2_{m_i,\nu} \), and therefore

\[
\int_M \varrho^{2m_i−2q_i |\nabla^{m_i} \eta|^2} \varrho^{−7} \text{vol}_M = \int_M \varrho^{2\nu(1−q_i)} \varrho^{2m_i−2\nu |\nabla^{m_i} \eta|^2} \varrho^{−7} \text{vol}_M
\]

is finite. But now, just as in the \( b = 1 \) case, since \( q_i > 1 \), the integrand of (144) dominates the integrand of (143) outside of a compact set, and hence the proof is complete.

We now consider the linearization \( DF|_0 \) of the map \( F \) defined in equation (134).

**Lemma 5.13.** The linearization \( DF|_0 \) of \( F \) at the origin is the map

\[
DF|_0 : \Omega^3_{l,\nu} \to \Omega^3_{l−1,\nu−1} \quad \dot{\eta} \mapsto (d + d^*)\dot{\eta}.
\]

**Proof.** This follows immediately from the definition of \( Q_\varphi \) as the quadratic approximation of the nonlinear map \( \Theta \) in Lemma 2.8.

From Lemma 5.13, we see that \( DF|_0 \) always maps onto the space

\[
\mathcal{Y}_0 = (d + d^*)(\Omega^3_{l,\nu})
\]

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introduced in equation (75). In order to be able to apply the Banach space implicit function theorem to \( F \), we would need to know that \( F \) maps into \( \mathcal{Y}_0 \) and that \( \mathcal{Y}_0 \) is a Banach space. If we could show this, we could redefine the codomain of the map \( F \) to be \( \mathcal{Y}_0 \). Surjectivity would then be immediate and we would be able to apply the Banach space implicit function Theorem 5.11. However, the problem is that \( d^*(\Omega^3_{l,\nu}) \) is in general not contained in \( (d + d^*)(\Omega^3_{l,\nu}) \), so the nonlinear term in (134) does not actually lie in \( \mathcal{Y}_0 \). To simplify notation, define the space \( \mathcal{Y} = d(\Omega^3_{k,\nu}) + d^*(\Omega^3_{l,\nu}) \).

We showed in Lemma 4.38 and Definition 4.39 that \( d^*(\Omega^3_{l,\nu}) \subseteq \mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{O}_\nu \). Thus, we need to “correct” the map \( F \) to a map \( \hat{F} : \mathcal{U} \oplus \mathcal{O}_\nu \to \mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{O}_\nu \) by the rule

\[
\hat{F}(\eta, \xi) = (d + d^*)\eta - d^*(Q\varphi(\eta)) + \xi.
\]

Lemma 5.14. For generic rates \( \nu \), the space \( \mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{O}_\nu \) is a Banach space, and the map \( \hat{F} \) defined in equation (147) actually maps into this space.

Proof. The first statement is precisely Lemma 4.40. Since \( \mathcal{Y}_0 = (d + d^*)(\Omega^3_{l,\nu}) \), to show that the map \( \hat{F} \) of equation (147) maps into \( \mathcal{Y} \), we need only show that \( d^*(Q\varphi(\eta)) \) lies in \( \mathcal{Y} \). Recall that \( \mathcal{Y} = d(\Omega^3_{l,\nu}) + d^*(\Omega^3_{l,\nu}) \). We showed in Lemma 5.12 that the 3-form \( \chi = \ast(Q\varphi(\eta)) \) lies in \( \Omega^3_{l,\nu} \). The result is now immediate.

Corollary 5.15. The linearization \( D\hat{F}|_0 \) of \( \hat{F} \) at the origin is the map

\[
D\hat{F}|_0 : \Omega^3_{l,\nu} \oplus \mathcal{O}_\nu \to \mathcal{Y}
(\hat{\eta}, \hat{\xi}) \mapsto (d + d^*)\hat{\eta} + \hat{\xi}.
\]

and \( D\hat{F}|_0 \) is surjective onto \( \mathcal{Y} \).

Proof. The first statement follows from Lemma 5.13 and equation (147), while the second statement is immediate from Definition 4.39.

Consider the Banach space \( \mathcal{X} = \Omega^3_{l,\nu} \oplus \mathcal{O}_\nu \). For generic rates \( \nu \), we have shown that the differential \( D\hat{F}|_0 \) maps surjectively from \( \mathcal{X} \) onto \( \mathcal{Y} \). It is also clear that \( \hat{F} \) is a \( C^\infty \) map from \( \tilde{U} = \mathcal{U} \times \mathcal{O}_\nu \) to \( \mathcal{Y} \). Finally, note from Corollary 5.15 and Definition 4.39 that

\[
\mathcal{K} = \ker D\hat{F}|_0 = \ker DF|_0 \oplus \{0\} = \mathcal{H}^3_{\nu},
\]

and we have \( \mathcal{X} = \mathcal{K} \oplus \mathcal{Z} \) for a closed subspace \( \mathcal{Z} \) by Propositions 4.29 and 4.31. Thus, we can apply Theorem 5.11 to conclude that there exist open sets \( \mathcal{V} \subseteq \mathcal{K} \) and \( \mathcal{W} \subseteq \mathcal{Z} \), both containing 0, with \( \mathcal{V} \times \mathcal{W} \subseteq \tilde{U} = \mathcal{U} \times \mathcal{O}_\nu \), and a \( C^\infty \)-map \( G : \mathcal{V} \to \mathcal{W} \) such that

\[
\hat{F}^{-1}(0) \cap (\mathcal{V} \times \mathcal{W}) = \{(x, G(x)); x \in \mathcal{V}\}
\]

in \( \mathcal{X} = \mathcal{K} \oplus \mathcal{Z} \). We have therefore established the following result.

Corollary 5.16. The set \( \hat{F}^{-1}(0) \) is a smooth manifold, diffeomorphic to an open neighbourhood \( \mathcal{V} \) of the origin in \( \mathcal{K} = \mathcal{H}^3_{\nu} \), and hence with dimension \( \dim \hat{F}^{-1}(0) = \dim \mathcal{H}^3_{\nu} \).
Notice that \( Z = Z_0 \oplus O_\nu \) for some closed subspace \( Z_0 \) of \( \Omega^3_{l,\nu} \). Thus the projection map \( \pi_o : W \to O_\nu \) is well defined and smooth. It is also clear that \( F^{-1}(0) \) can be identified homeomorphically with the subset \( (\pi_o \circ G)^{-1}(0) \) of \( V \subseteq \Omega^3_\nu \). Hence we have shown the following.

**Corollary 5.17.** The composition \( \Psi_\nu = \pi_o \circ G \) is a smooth map \( \Psi_\nu : V \to O_\nu \) from the open subset \( V \) of the finite-dimensional vector space \( \Omega^3_\nu \) to the finite-dimensional vector space \( O_\nu \), whose zero set \( \Psi_\nu^{-1}(0) \) is homeomorphic to \( F^{-1}(0) \).

By combining Corollary 5.17 with Corollary 5.7 and Theorem 5.8 we have the following result.

**Theorem 5.18.** The reduced moduli space \( \mathcal{M}_\nu \) is locally homeomorphic to the zero set \( \Psi_\nu^{-1}(0) \) of a smooth map \( \Psi_\nu \) from an open subset \( V \) of a finite-dimensional vector space \( H^3_\nu \) to a finite-dimensional vector space \( O_\nu \). In particular, in the AC case when \( \nu \in (-4, -\frac{3}{2}) \), we have \( O_\nu = \{0\} \) and \( \mathcal{M}_\nu = \mathcal{M}_\nu \), and thus in this case we conclude that the actual moduli space \( \mathcal{M}_\nu \) is a smooth manifold of dimension \( \dim H^3_\nu \).

**Remark 5.19.** We could conclude that \( \mathcal{M}_\nu \) is a smooth manifold if we knew that the map \( \Psi_\nu \) was the zero map. However, this will not be true in general.

**5.2.4 Step 4: Computing the virtual dimension of the moduli space**

In this section we compute the expected (virtual) dimension of the reduced moduli space \( \mathcal{M}_\nu \), including an exact result for the dimension in the unobstructed case. Explicitly, the integer \( v\cdot\dim \mathcal{M}_\nu \) would be the dimension of \( \mathcal{M}_\nu \) if it were a smooth manifold. From Corollary 5.17, this virtual dimension is

\[
v\cdot\dim \mathcal{M}_\nu = \dim \mathcal{H}^3_\nu - \dim O_\nu.
\]

From equations (75), (146), and Definition 4.39 this virtual dimension is precisely the index of the map \( D_{l,\nu} : \Omega^3_{l,\nu} \to \mathcal{Y} \) defined in equation (71). Thus, we have

\[
v\cdot\dim \mathcal{M}_\nu = \text{ind}(D_{l,\nu}).
\]

To compute \( v\cdot\dim \mathcal{M}_\nu \), we need to use our special index change Theorem 4.50 for the operator \( D_{l,\nu} \). The first step is to compute the index of \( D_{l,\lambda} \) exactly in some special cases. Recall from Corollary 4.52 and Proposition 4.61 we have, for \( \varepsilon > 0 \) sufficiently small,

\[
\begin{align*}
\dim \mathcal{H}^3_{-3+\varepsilon} &= b^3_{cs} + \dim(\text{im } \mathcal{Y}^3) \\
\dim \mathcal{H}^3_{-\lambda} &= b^3_{cs}, \quad \lambda \in (-4, -3) \\
\dim \mathcal{H}^3_{-4-\varepsilon} &= b^3_{cs} - \dim(\text{im } \mathcal{Y}^4)
\end{align*}
\]

\[
\begin{align*}
\dim \mathcal{H}^3_{-3+\varepsilon} &= b^3 - \dim(\text{im } \mathcal{Y}^3) \\
\dim \mathcal{H}^3_{-\lambda} &= b^3, \quad \lambda \in (-4, -3) \\
\dim \mathcal{H}^3_{-4-\varepsilon} &= b^3 + \dim(\text{im } \mathcal{Y}^4)
\end{align*}
\]

(AC),

(152)

where \( b^3 = \dim H^3(M, \mathbb{R}) \) and \( b^3_{cs} = \dim H^3_{cs}(M) \) are the ordinary and compactly supported third Betti numbers of \( M \), respectively. The above equations say that the rates \( \lambda = -3 \) and \( \lambda = -4 \) contribute to changes in the kernel of \( D \), not the cokernel. This fact also follows from Theorem 4.45 as we stated during the proof of Theorem 4.49, but the above equations tell us exactly how the kernel (and thus the index) changes at these rates.
Proposition 5.20. The index of $\mathcal{O}_{t,\lambda}$ is purely topological for a certain range of rates, as follows.

\[
\begin{align*}
\text{ind}(\mathcal{O}_{t,-3+\epsilon}) &= b_{cs}^3 + \dim(\text{im } \mathcal{T}^3) \quad (AC), \\
\text{ind}(\mathcal{O}_{t,\lambda}) &= b_{cs}^3, \lambda \in (-4, -3) \quad (AC), \\
\text{ind}(\mathcal{O}_{t,-4-\epsilon}) &= b_{cs}^3 - \dim(\text{im } \mathcal{T}^4) \\
\text{ind}(\mathcal{O}_{t,\lambda}) &= b_{cs}^3, \lambda \in (-4, -3) \quad (CS), \\
\text{ind}(\mathcal{O}_{t,-4-\epsilon}) &= b_{cs}^3 + \dim(\text{im } \mathcal{T}^4) \quad (CS),
\end{align*}
\]

Proof. By Proposition 4.17 and Proposition 4.41 (b), the space $\text{coker}(\mathcal{O}_{t,\lambda}) = \{0\}$ if $-6 - \lambda < -\frac{5}{2}$ (AC) or if $-6 - \lambda > -\frac{3}{2}$ (CS). This corresponds to $\lambda > -\frac{7}{2}$ (AC) or $\lambda < -\frac{7}{2}$ (CS). Moreover, we know that the index cannot change at all in the interval $(-4, -3)$ since by Corollary 4.27 there are no critical rates for $\mathcal{O}$ in this interval. Finally, the cokernel does not change at the rates $-4$ and $-3$, and the change in the kernel at those rates is given by $\{152\}$. When these facts are all combined we obtain the statements above. $\square$

Corollary 5.21. For generic rates $\nu$, the virtual dimension $\text{v-dim } \hat{\mathcal{M}}_{\nu}$ of the (reduced) moduli space $\mathcal{M}_{\nu}$ or the actual dimension $\dim \mathcal{M}_{\nu}$ of the full moduli space is as follows.

- In the asymptotically conical (AC) case, we have

\[
\text{v-dim } \hat{\mathcal{M}}_{\nu} = b_{cs}^3 - \dim(\text{im } \mathcal{T}^4) - \sum_{\lambda \in \mathcal{D}_{\mathcal{C}} \cap (-\nu, -4)} \dim \mathcal{K}(\lambda), \quad \nu < -4;
\]
\[
\dim \mathcal{M}_{\nu} = b_{cs}^3, \quad \nu \in (-4, -3);
\]
\[
\dim \mathcal{M}_{\nu} = b_{cs}^3 + \dim(\text{im } \mathcal{T}^4) + \sum_{\lambda \in \mathcal{D}_{\mathcal{C}} \cap (-3, \nu)} \dim \mathcal{K}(\lambda), \quad \nu \in (-3, -\frac{5}{2}).
\]

Note that when $\nu \in (-4, -\frac{5}{2})$, the deformation problem is unobstructed and moreover $\hat{\mathcal{M}}_{\nu} = \mathcal{M}_{\nu}$, so the moduli space $\mathcal{M}_{\nu}$ is a smooth manifold, and the virtual dimension is the actual dimension of $\mathcal{M}_{\nu}$.

- In the conically singular (CS) case, we have

\[
\text{v-dim } \hat{\mathcal{M}}_{\nu} = \dim(\text{im } H_{cs}^3 \to H^3)) - \sum_{i=1}^{n} \sum_{\lambda \in \mathcal{D}_{\mathcal{C}_i} \cap (-3, \nu_i)} \dim \mathcal{K}(\lambda),
\]

where each $\nu_i > 0$.

Note that in general there are both topological and analytic contributions to the virtual dimension.

Proof. This is immediate from Proposition 5.20 and equation \{151\}, along with Theorem 4.50. In the CS case we have also used equation \{155\}. $\square$

Example 5.22. Let us apply Theorem 5.1 and Corollary 5.21 to the three known examples of AC $G_2$ manifolds, the Bryant–Salamon manifolds of Example 3.17. The manifolds $\Lambda^2(S^4)$ and $\Lambda^2(\mathbb{CP}^2)$ are of rate $\nu = -4$ and the manifold $S(S^3)$ is of rate $\nu = -3$. Since $-4$ and $-3$ are excluded in Theorem 5.1 we can only describe their deformations as AC $G_2$ manifolds of rate $\nu + \epsilon$. For the first two examples, we find that the moduli space $\mathcal{M}_{-4+\epsilon}$ is a smooth manifold of dimension $b_{cs}^3 = b^3 = 1$. (Recall by Remark 5.24 that the dimension has to be at least one.)
For $N = \mathcal{S}(S^3)$, we find that the moduli space $\mathcal{M}_{-3+\varepsilon}$ is a smooth manifold of dimension $b_2^3 + \text{dim}(\text{im} \, T^3) = b^3 + \text{dim}(\text{im} \, T^3) = 0 + \text{dim}(\text{im} \, T^3)$. A simple diagram chase in \cite{04} using the facts that $H^3(\Sigma) = \mathbb{R}^2$, $H^3_{\text{ker}}(N) = \mathbb{R}$, and $H^3(N) = \{0\}$ gives $\text{dim}(\text{im} \, T^3) = 1$.

That is, the Bryant–Salamon manifolds are \textit{locally rigid} as $\text{AC G}_2$ manifolds of rate $\nu + \varepsilon$, modulo the scalings which are always present. In Section 6.2 we show how to extend this result to establish that the Bryant–Salamon manifolds are in fact locally rigid as $\text{AC G}_2$ manifolds of rate $-\varepsilon$ for any small $\varepsilon > 0$.

In the remainder of this section we make some brief remarks about interpreting these dimension formulae.

In the CS case, what we have done is to describe a subspace $\tilde{M}_\nu$ of the moduli space, which are those deformations parameterized by $\eta \in \mathcal{G}_{t,\nu}$ satisfying $d(\Theta(\varphi + \eta)) = 0$. It is not difficult to set things up to describe the whole moduli space. In this case the PDE which generalizes equation (129) would be given by

$$(d + d^*)\eta = d^* (\ast L_\varphi(\zeta)) + d^* (\ast Q_\varphi(\eta + \zeta)), $$

which by the same arguments as in Theorem 5.8 gives $\pi_\tau(d^*\eta) = 0$. For each $\zeta$ this is an elliptic equation on $\eta$ and hence elliptic regularity implies that $\eta$ is smooth. We now have a map $\tilde{F} : \Omega^3_{\nu} \to (\Omega^3_{\nu}), d^* (\Omega^3_{\nu})$. The linearization is $d\tilde{F}|_0 : (\eta, \zeta) \to (d + d^*)\eta - d^* (\ast L_\varphi(\zeta))$, which has a better chance of being surjective, but in this case we still probably need an obstruction space $\tilde{O}$, which will be a subspace of the obstruction space we have defined in the case of the reduced moduli space. The implicit function theorem then tells us that the moduli space is a smooth manifold with dimension equal to $\text{dim ker}(d\tilde{F}|_0) - \text{dim} \tilde{O}$. We would then have to try to determine this dimension. The authors decided not to pursue the technical issues involved in such an analysis.

However, we \textit{can understand} what some of the extra deformations mean in the CS case, as follows. Consider the $i$th end of CS $G_2$ manifold $M$ and let $X_i$ be the dilation vector field $\frac{d}{dr}$ on the $i$th asymptotic cone $C_i$ pulled back to $M$ and extended smoothly so that it vanishes on the other ends. Hence $\exp(x_i) \in D_1$ is a diffeomorphism of rate 1 on the ends. Now consider $d(X_i \varphi)$. Since $\varphi$ is asymptotic to $\varphi_{C_i}$ and $\varphi_{C_i}$ is dilation equivariant we see that $d(X_i \varphi) - 3\varphi$ decays on the $i$th end with rate $\nu$ and vanishes on the other ends. We reiterate here that, since $X_i$ is of rate 1 on the ends, $d(X_i \varphi) - 3\varphi$ should \textit{a priori} be of rate 0 on the ends, but in fact it has \textit{faster} decay, being of rate $\nu > 0$ on the ends. Thus $\xi_i = d(X_i \varphi) - 3\varphi \in C_i$, but $\xi_i$ is transverse to $(\ker \pi_{\tau} d^*)_\nu$, because it has a component which is not in $(\ker \pi_{\tau} d^*)_\nu$. Therefore the 3-form $\xi_i$ in fact corresponds to a nonzero element in $(\mathcal{E}_\varphi)_L,\nu$. This means that if we consider an extension $\tilde{M}_\nu$ of the reduced moduli space $\tilde{M}_\nu$, where we count such diffeomorphisms as additional nontrivial deformations, we find that the virtual dimension $v\text{-dim} \tilde{M}_\nu$ of this extension is

$$v\text{-dim} \tilde{M}_\nu = \text{dim}(\text{im} (H^3_{cs} \to H^3)) - \sum_{i=1}^{n} \sum_{L \in \mathcal{D}_{C_i}} \text{dim} (K(\lambda)_{\mathcal{D}_{C_i}} + n),$$

where $n$ is the number of conical singularities. Each such additional infinitesimal deformation corresponds to a reparametrization of the conical model for the singularity, given by rescaling the nearly Kähler structure on the link $\Sigma_i$ or equivalently rescaling the $r_i$ coordinate near the singular point.

\textbf{Remark 5.23.} This is essentially what we do in Section 6.3 for CS $G_2$ conifolds whose asymptotic links come from the three known Gray manifolds.

\textbf{Remark 5.24.} We note that in the AC case as well, the deformation that corresponds to rescaling at infinity, which is generated by the dilation vector field, is always included in our actual moduli
space $\mathcal{M}_\nu$. Note that in particular this observation, together with our dimension formula from Corollary 5.21 implies topological restrictions on AC $G_2$ manifolds.

6 Applications and open problems

In this section we present several applications of our results, and discuss some open problems. In Section 6.1 we discuss some aspects of the spectrum of the Laplacian on Gray manifolds. These are used in the next three sections. In Section 6.2 we first extend our results on AC deformation theory from $\nu < -\frac{5}{2}$ all the way up to $\nu < 0$ under suitable hypotheses, and use this to establish the local rigidity (modulo trivial scalings) of the Bryant–Salamon manifolds as AC $G_2$ manifolds of rate $\nu < 0$. In Section 6.3 we establish that under certain conditions, an AC $G_2$ manifold must be of cohomogeneity one, and this implies that at least two of the Bryant–Salamon manifolds are unique as AC $G_2$ manifolds with given asymptotic cone. In Section 6.4 we investigate when the CS moduli space is smooth and unobstructed. In Section 6.5 we relate our main theorem to the desingularization theorem of [20], providing evidence that CS $G_2$ manifolds likely make up a large part of the “boundary” of the moduli space of smooth compact $G_2$ manifolds. In Section 6.6 we show that a gauge-fixing condition that is needed in [20] can always be achieved. Finally we conclude in Section 6.7 with some open problems for the future.

6.1 The spectrum of the Laplacian on Gray manifolds

We first need to discuss some results about the spectrum of the Laplacian on 2-forms, for compact Gray manifolds, that is, for 6-dimensional strictly nearly Kähler manifolds. These results are required for the applications in Sections 6.2, 6.3 and 6.4. The current section has a different flavour from the rest of the paper. Readers who are only interested in the use of these results for the applications can just note Proposition 6.1 and Proposition 6.2.

Slightly similar calculations can be found (at least implicitly if not explicitly) in [37]. Note that in [37], the form $\Psi^+ + i \Psi^-$ corresponds to our $-\Omega = -\text{Re}(\Omega) - i \text{Im}(\Omega)$.

Let $(\Sigma, J, \omega, \Omega)$ be a Gray manifold, so in particular from (23) we have

$$d\omega = -3 \text{Re}(\Omega) \quad \text{and} \quad d\text{Im}(\Omega) = 2 \omega^2.$$  \hspace{1cm} (154)

From Lemma 4.36 we find that the closed and coclosed 3-forms $\gamma = r^\lambda (r^3 \alpha_3 + r^2 dr \wedge \alpha_2)$ on the cone, homogeneous of order $\lambda$, correspond to coclosed (in fact coexact, if $\lambda \neq -4$) 2-forms $\alpha_2$ on the link that satisfy $\Delta_\Sigma \alpha_2 = (\lambda + 3)(\lambda + 4) \alpha_2$. From Corollary 5.21 we are particularly interested in such forms for $\lambda \in (-3, 0]$, in which case $0 < (\lambda + 3)(\lambda + 4) \leq 12$, with equality if and only if $\lambda = 0$. This motivates us to study on $\Sigma$ the pair of equations

$$\Delta_\Sigma \xi = \mu \xi \quad \text{and} \quad d^*_\Sigma \xi = 0$$  \hspace{1cm} (155)

for $\mu \in (0, 12]$ and $\xi$ a section of $\Lambda^2(T^*\Sigma)$.

We can decompose the bundle of real 2-forms on $\Sigma$ into $\text{SU}(3)$ representations as follows:

$$\Lambda^2(T^*\Sigma) = \Lambda^{(2,0)+(0,2)}(T^*\Sigma) \oplus \Lambda^{(1,1)}_0(T^*\Sigma) \oplus \mathbb{R}(\omega),$$

where

$$T\Sigma \cong \Lambda^{(2,0)+(0,2)}T^*\Sigma \quad \text{via} \quad X \mapsto X \downarrow \text{Re}(\Omega).$$
We may then write any section $\xi$ of $\Lambda^2 T^* \Sigma$ as
\[
\xi = X \nabla \text{Re}(\Omega) + f\omega + \gamma
\]  
(156)
for a vector field $X$, a smooth function $f$, and $\gamma$ a section of $\Lambda^{(1,1)}_0 T^* \Sigma$. Following [37], define the Hermitian connection $\nabla$ by $\nabla_X = \nabla_X - \frac{1}{2} A_X$, where $A_X = J(\nabla_X J)$, and define $\Delta$, the Hermitian Laplace operator, explicitly via the formula $\Delta = \nabla^* \nabla + q(\hat{R})$ where $\hat{R}$ is the curvature tensor of $\nabla$, and $q(\hat{R})$ is the associated curvature operator.

**Proposition 6.1.** Let $\eta = r^{3}\eta d\xi + r^{2}dr \wedge (\lambda + 3)\xi$ be a closed and coclosed 3-form on the cone, homogeneous of order $\lambda$. We have seen that $\xi$ satisfies (155). Then $\eta = d(r^{\lambda+3}\xi)$, and the 2-form $\beta = r^{\lambda+3}\xi$ is homogeneous of order $\lambda + 1$ and satisfies $\Delta_\Sigma \beta = 0$ and $d^*_\Sigma \beta = 0$.

Moreover, write $\xi$ in the form (156), and hence $\xi$ satisfies (155), where $\mu = (\lambda + 3)(\lambda + 4)$ for some $\lambda \in (-3, 0]$. Then:

- if $\lambda \leq -1$, we have $f = 0$ and $X = 0$ and $\gamma$ satisfies $d^*_\Sigma \gamma = 0$ and $\Delta \gamma = \mu \gamma$;
- if $\lambda \in (-1, 0)$, then suppose $\xi_1 = X \nabla \text{Re}(\Omega) + f\omega + \gamma_1$ and $\xi_2 = X \nabla \text{Re}(\Omega) + f\omega + \gamma_2$ are two solutions of (155) with the same $X$ and $f$. Then $\gamma = \gamma_1 - \gamma_2$ satisfies $d^*_\Sigma \gamma = 0$ and $\Delta \gamma = \mu \gamma$.

**Proof.** Because $\eta$ is closed and coclosed on the cone, it is harmonic. Since $\eta = d\beta$, it follows that the 2-form $\beta = r^{\lambda+3}\xi$ is a harmonic 2-form on the cone, homogeneous of order $\lambda + 1$. In particular we have $\Delta_\Sigma \beta = 0$. Moreover, by equation (25) we find that $d^*_\Sigma \beta = r^{\lambda+1} d^*_\Sigma \xi = 0$.

By Remark 2.4, we have $\beta_\gamma = * (Y \wedge \psi)$ for some harmonic 1-form $Y$ on the cone. An easy computation gives
\[
Y = r^{\lambda+1}( -f dr + \frac{2}{3} X).
\]  
(157)
By Proposition 3.6, we know that $Y = 0$ if $\lambda + 1 \in [-5, 0]$, hence if $\lambda \in [-6, -1]$. Thus $f = 0$ and $X = 0$ if $\lambda \in (-3, -1]$. Thus in this case we have $\xi = \gamma$, and hence $d^*_\Sigma \gamma = 0$. The fundamental formula [37] equation (17)] states that for a section $\gamma$ of $\Lambda^{(1,1)}_0 T^* \Sigma$, we have
\[
(\Delta_\Sigma - \Delta_\gamma) \gamma = -(Jd^*_\Sigma \gamma \nabla \text{Re}(\Omega)).
\]  
(158)
Hence we conclude that $\Delta_\gamma = \Delta_\Sigma \gamma = \mu \gamma$ and this completes the proof for $\lambda \in (-3, -1]$. The case $\lambda \in (-1, 0)$ is immediate from (158) and the linearity of $\Delta_\Sigma$ and $d^*_\Sigma$.

Therefore, an important equation to study is $\Delta_\gamma = \mu \gamma$, with $d^*_\Sigma \gamma = 0$, where $\gamma \in \Lambda^{(1,1)}_0 T^* \Sigma$ and $\mu \in (0, 12)$. This is the equation we proceed to consider. Specifically, we need to observe that the calculations in [37] enable us to determine the eigenvalues (and their multiplicities) in $(0, 12)$ for $\Delta$ acting on coclosed forms in $\Lambda^{(1,1)}_0 (T^* \Sigma)$ for the three known Gray manifolds $\Sigma = \mathbb{CP}^3$, $S^3 \times S^3$, and $\text{SU}(3)/T^2$.

**Proposition 6.2.** Let $\Sigma$ be one of the three known Gray manifolds: $\mathbb{CP}^3$, $S^3 \times S^3$, or $\text{SU}(3)/T^2$. There are no nontrivial coclosed primitive $(1, 1)$-forms which are eigenforms of the Hermitian Laplace operator $\Delta$ with eigenvalue in $(0, 12)$. Moreover, in the first two cases, we can also exclude the eigenvalue 12. For the flag manifold $\text{SU}(3)/T^2$, we get an 8-dimensional space of such forms with eigenvalue 12. These forms correspond to infinitesimal deformations of the nearly Kähler structure, for which it is still not known whether or not they can be integrated to actual deformations.
Proof. Case 1: Let us start with the case of \( \mathbb{CP}^3 \), which follows from the work in \textit{[37 §5.5]}. Let \( E \) denote the usual representation of \( SU(2) \) on \( \mathbb{C}^2 \) and let \( C_i \) be the \( U(1) \) representation on \( \mathbb{C} \) given by multiplication by \( z^i \) for \( z \) in the unit circle in \( \mathbb{C} \). Let \( E_{k,l} = \text{Sym}^k(E) \otimes C_l \) for \( k \geq 1 \) and \( l \in \mathbb{Z} \), and \( k \equiv l \mod 2 \), which are the irreducible representations of \( U(2) \). From \textit{[37 Lemma 5.8]} and the discussion before it, we have decompositions

\[
T^*\mathbb{CP}^3 \cong E_{0,-2} \oplus E_{1,1} \oplus E_{0,2} \oplus E_{1,-1} \quad \text{and} \quad \Lambda^{(1,1)}_0(T^*\mathbb{CP}^3) \cong E_{0,0} \oplus E_{1,3} \oplus E_{1,-3} \oplus E_{2,0}.
\]

If \( V_{a,b} \) is an irreducible \( \text{SO}(5) \) representation with highest weight \( (a,b) \) for \( a \geq b \geq 0 \), it corresponds to a possible eigenspace of \( \Delta \) on \( \Lambda_0^{(1,1)}(T^*\mathbb{CP}^3) \) with eigenvalue \( 2(a(a+3) + b(b+1)) \) if there is a homomorphism from \( V_{a,b} \) to \( \Lambda_0^{(1,1)}(T^*\mathbb{CP}^3) \). So the only possible positive eigenvalue less than 12 is 8 for \( (a,b) = (1,0) \). However, since \( V_{1,0} \cong T^*(\mathbb{CP}^3) \) we see that there are no such homomorphisms from \( V_{1,0} \) to \( \Lambda_0^{(1,1)}(T^*\mathbb{CP}^3) \), and thus the lowest possible positive eigenvalue is 12. Moreover, the multiplicity of the possible eigenvalue 12 on coclosed forms in \( \Lambda_0^{(1,1)}(T^*\mathbb{CP}^3) \) is shown to be 0 in \textit{[37 Theorem 5.10]}, so there are no solutions of (155) for \( \mu \in (0,12] \) on \( \mathbb{CP}^3 \) other than \( \xi = \omega \).

Case 2: Next we consider the case of \( S^3 \times S^3 \) as in \textit{[37 §5.4]}. If \( E \) is as above, then by \textit{[37 Lemma 5.5]},

\[
\Lambda_0^{(1,1)}(T^*(S^3 \times S^3)) \cong \text{Sym}^2(E) \oplus \text{Sym}^3(E).
\]

Then the irreducible representation of \( SU(2) \times SU(2) \times SU(2) \) given by

\[
V_{a,b,c} = \text{Sym}^a(E) \otimes \text{Sym}^b(E) \otimes \text{Sym}^c(E)
\]

for \( a, b, c \geq 0 \) corresponds to a possible eigenspace of \( \Delta \) with eigenvalue \( \frac{3}{2}(a(a+2) + b(b+2) + c(c+2)) \) if there is a homomorphism from \( V_{a,b,c} \) to \( \Lambda_0^{(1,1)}(T^*(S^3 \times S^3)) \). The only possible positive eigenvalues less than 12 are \( \frac{9}{2} \) and 9 for \( (a,b,c) = (1,0,0) \) and \( (a,b,c) = (1,1,0) \), respectively, up to permutation. There are no homomorphisms from \( V_{1,0,0} = \text{Sym}^1(E) \to \text{Sym}^2(E) \oplus \text{Sym}^3(E) \), but there is one from \( V_{1,1,0} = \text{Sym}^1(E) \otimes \text{Sym}^1(E) = \text{Sym}^0(E) \oplus \text{Sym}^2(E) \to \text{Sym}^2(E) \oplus \text{Sym}^4(E) \). However, the fact that \( \text{Sym}^0(E) \) is a factor in \( V_{1,1,0} \) means that \( V_{1,1,0} \) also corresponds to an eigenspace for \( \Delta \) on functions with eigenvalue 9, and such functions, by \textit{[37 Proposition 4.11]}, define elements of \( \Lambda_0^{(1,1)}(T^*(S^3 \times S^3)) \) that are eigenforms of \( \Delta \) with eigenvalue 12 but which are \emph{not} coclosed, as the eigenvalue is not 6. We hence deduce that 9 does not arise as an eigenvalue for \( \Delta \) acting on coclosed forms in \( \Lambda_0^{(1,1)}(T^*(S^3 \times S^3)) \). Moreover, the multiplicity of the possible eigenvalue 12 on coclosed forms in \( \Lambda_0^{(1,1)}(T^*(S^3 \times S^3)) \) is shown to be 0 in \textit{[37 Corollary 5.7]}, so again there are no solutions of (155) for \( \mu \in (0,12] \) on \( S^3 \times S^3 \) other than \( \xi = \omega \).

Case 3: Finally, we consider the case of \( SU(3)/T^2 \) as in \textit{[37 §5.6]}. Let \( E \) denote the usual representation of \( SU(3) \) on \( \mathbb{C}^3 \) and let

\[
V_{k,l} = \ker(\text{Sym}^k(E) \otimes \text{Sym}^l(E) \to \text{Sym}^{k-1}(E) \otimes \text{Sym}^{l-1}(E)),
\]

where the map is the contraction map. If \( \xi_i \) for \( i = 1, 2, 3 \) is the standard basis on \( \mathbb{R}^3 \) then the weights of \( V_{k,l} \) are

\[
(a-a')\xi_1 + (b-b')\xi_2 + (c-c')\xi_3
\]

where \( k = a + b + c \), \( l = a' + b' + c' \) and \( a, b, c, a', b', c' \geq 0 \). The representation \( V_{k,l} \) corresponds to a possible eigenvalue of \( 2(k(k+2) + l(l+2)) \) of \( \Delta \). So we need \( (k,l) = (1,0) \) or \( (k,l) = (0,1) \)

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for a positive eigenvalue less than 12. Now by [37, Corollary 5.11] the possible weights of the $T^2$ representation on $\Lambda_0^{1,1}(T^*(\mathbf{SU}(3)/T^2))$ are 0 and $\pm 3\varepsilon_i$ for $i = 1, 2, 3$. However, for eigenvalues less than 12 we cannot achieve weights $\pm 3\varepsilon_i$ since $k, l \in \{0, 1\}$ and we cannot achieve weight 0 since $k \neq l$. Thus there are no eigenvalues in $(0, 12)$. Moreover, by the work in [37, §5.6 & §6], the space of solutions to (155) for $\mu \in (0, 12)$ is zero unless $\mu = 12$, in which case the solutions are of the form $\xi = \omega + \gamma$, where the $\gamma$'s lie in a space of dimension 8, isomorphic to $\mathfrak{su}(3)$.

6.2 Extension of the AC results from $\nu < -\frac{5}{2}$ to $\nu < 0$

Recall the statement of Theorem 5.3 in the AC case. The decomposition

$$C_{l, \nu} = T_\nu(D_{l+1, \nu+1} \cdot \varphi) + (G_\nu)_{l, \nu}.$$ 

works all the way up to $\nu < 0$. We only needed the assumption that $\nu \leq -1$ to establish that the sum was direct. This means that the solutions to (128) include all gauge-fixed torsion-free $G_2$ structures, for any $\nu < 0$, but the solution set of (128) could be strictly bigger if $\nu \in (-1, 0)$. Since Theorem 5.8 says that the solutions to (129) include the solutions to (128) for any $\nu < 0$, we conclude that the gauge-fixed torsion-free $G_2$ structures of rate $\nu$ are a subset of the solutions to (129), for any $\nu < 0$.

Let $\nu \in (-3, 0)$. By Corollary 5.16 since the obstruction space $O_\nu = \{0\}$ for $\nu > -4$, we know that the space of solutions to (129) is locally diffeomorphic to $H^0_{\nu}$. Now suppose that the interval $(-3, 0)$ contains no critical rates for $D^3_{\nu}$ on the cone. This means that $H^0_{\nu} = H^3_{\nu, \nu+3\varepsilon}$ for any $\nu \in (-3, 0)$. Since solutions of (128) are a subset of solutions of (129) and solutions to (128) can only increase or stay the same as we increase $\nu$, we find that solutions to (128) must stay the same for all $\nu \in (-3, 0)$.

More generally, suppose that $\lambda$ is a critical rate of $O_C$ in the interval $(-3, 0)$. By Proposition 6.1 if $\lambda \in (-3, -1]$, then this critical rate corresponds to the existence of a solution of

$$\Delta \gamma = (\lambda + 3)(\lambda + 4)\gamma, \quad d_\gamma^* \gamma = 0, \quad \gamma \text{ is of type } \Lambda_0^{1,1}. \tag{159}$$

If, on the other hand, we have $\lambda \in (-1, 0)$, then by Proposition 6.1 such a critical rate corresponds to a homogeneous of order $\lambda$ closed and coclosed 3-form $\eta$ on the cone of the form $\eta = d\beta$ with $\Delta c_\beta = 0$ and $d_\gamma^* \beta = 0$. Then $\pi_7 d_\gamma^* d\beta = \pi_7 \Delta c_\beta = \Delta c_\pi_7 \beta = 0$, and so we have a critical rate for $\pi_7 d^*d\eta$. By Proposition 2.7 and Lemma 4.76, crossing this critical rate must add to the kernel, so Proposition 4.21 applied to the elliptic operator $P = \pi d^*d$ tells us that this critical rate corresponds to the existence of a 2-form $\tilde{\beta}_7 \in (\Omega^2)_{1+1, \lambda+1}$ in $(\ker \pi d^* d)_{1+1}$, asymptotic at infinity to $\beta_7$. Hence, $d\tilde{\beta}_7$ lies in the intersection $T_\varphi(D_{l+1, \lambda+1} \cdot \varphi) \cap (G_\varphi)_{l, \lambda}$ and thus is tangent to the orbit of the diffeomorphism action, not to our slice. Therefore, the only contributions to increasing the dimension of the moduli space from critical rates in $(-1, 0)$ will come from elements of the form $\tilde{\eta} - d\tilde{\beta}_7$, where $\tilde{\eta}$ is asymptotic to $\eta$ at infinity. Again by Proposition 6.1 this corresponds precisely to the existence of solutions of (159) for $\lambda \in (-1, 0)$.

Putting together all of the above discussion, we have established the following result.

**Proposition 6.3.** Let $(M, \varphi)$ be an AC $G_2$ manifold of rate $\nu \in (-3, 0)$. Suppose that there are no solutions to the system of equations (159) for any $\lambda \in (-\frac{5}{2} - \varepsilon, 0)$. Then the moduli space $\mathcal{M}_\nu$ of AC deformations of $M$, asymptotic to the same cone $C$ at the same rate $\nu$, is a smooth manifold of dimension $\dim H^3_{-\frac{5}{2} - \varepsilon}$. 

Thus, we have shown that our deformation theory of AC $G_2$ manifolds can be extended from $\nu < -\frac{5}{2}$ to any noncritical rate $\nu < 0$ if there are no solutions to (159) on the link of the asymptotic cone for any $\lambda \in \left(-\frac{5}{2} - \varepsilon, 0\right)$.

**Corollary 6.4.** The Bryant–Salamon $G_2$ conifolds are locally rigid, modulo scalings, as AC $G_2$ manifolds with the same asymptotic cones, up to any rate $\nu < 0$.

**Proof.** Proposition 6.2 says that for the three Bryant–Salamon manifolds, there are no solutions to (159) for any $\lambda \in (-3, 0)$. The conclusion now follows for $S^3$ by Example 5.22 and Proposition 6.3. For the cases $\Lambda^2(S^4)$ and $\Lambda^2(\mathbb{C}P^2)$, we have to also use the fact that there are no new deformations as we cross $\lambda = -3$, arising from the term $\dim(\text{im } \Upsilon^3)$ in Corollary 5.21. But this is immediate since $H^3(\Sigma) = \{0\}$ for both these manifolds, and thus $\Upsilon^3 = 0$. 

### 6.3 Cohomogeneity of AC $G_2$ manifolds

In this section we apply our deformation theory together with the spectral theory results of Section 6.1 to obtain a strong result about the cohomogeneity of AC $G_2$ manifolds under certain conditions.

**Remark 6.5.** The ideas in this section were suggested to the authors by one of the anonymous referees of a first draft of this article. We thank the referee for this suggestion.

In the following, we choose $\varepsilon > 0$ sufficiently small so that $(-3, -3 + \varepsilon)$ contains no rates for homogeneous closed and coclosed 3-forms on the given asymptotic cone.

**Proposition 6.6.** Let $(M, \varphi)$ be an AC $G_2$ manifold with rate $\nu = -3 + \varepsilon$. The map from the moduli space $\mathcal{M}_\nu$ to $H^3(M) \times H^4(M)$ given by

$$D_{\nu+1} \cdot \tilde{\varphi} \mapsto (\tilde{\varphi}, [\Theta(\tilde{\varphi})])$$

is an immersion.

**Proof.** The map is well-defined because we are considering diffeomorphisms isotopic to the identity, so choosing different elements of the orbit will not change the cohomology classes. Hence it suffices to show that its derivative at the orbit of any point $\tilde{\varphi}$ is injective. Since the argument is identical (modulo cumbersome notation) at any point, we show the case $\tilde{\varphi} = \varphi$. The tangent space to the moduli space at the orbit of $\varphi$ is $\mathcal{H}_\nu^3$ hence the derivative is

$$\eta \mapsto ([\eta], [L_\varphi(\eta)]),$$

which maps $\eta \in \mathcal{H}_\nu^3$ to $H^3(M) \times H^4(M)$.

Since $d\eta = d^*\eta = 0$ we know that $\Delta \eta = 0$. Hence $\pi_1(\Delta \eta) = 0$ and $\pi_7(\Delta \eta) = 0$ and thus $\pi_1(\eta)$ and $\pi_7(\eta)$ are harmonic by the torsion-freeness of $\varphi$. Arguing exactly as in the proof of Theorem 5.8 we find that $\pi_1(\eta) = 0$ and $\pi_7(\eta) = 0$. Therefore

$$L_\varphi(\eta) = \ast_\varphi \left( \frac{4}{3} \pi_1(\eta) + \pi_7(\eta) - 2\pi_2(\eta) \right) = \ast_\varphi \left( \frac{7}{3} \pi_1(\eta) + 2\pi_7(\eta) - \eta \right) = *_\varphi \eta.$$ 

Thus we need to show that the map

$$\eta \mapsto ([\eta], [-*_\varphi \eta])$$

(160)

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from $\mathcal{H}^3$ to $H^3(M) \times H^4(M)$ is injective. Suppose that $\eta$ maps to $(0,0)$ under this map. Since the only exceptional rate of $d+d^*$ in $[-\frac{2}{3}, \nu]$ is $-3$ we know by Lemma 4.26 that we can write $\eta = \eta_+ + \eta_-$ where $\eta_- \in \mathcal{H}^3$ and $\eta_+$ is asymptotic to a closed and coclosed 3-form $\gamma$ on the asymptotic cone $C$ of rate $-3$, and $\eta_+$ is nonzero if and only if $\gamma$ is nonzero. Moreover, from equation (25) we know that $\gamma$ is independent of the radial direction on $C$ as it is a closed and coclosed 3-form on the link $\Sigma$ of $C$.

Now, $\mathcal{H}^3 = \mathcal{H}^3_{L^2}$ is the space of $L^2$ closed and coclosed 3-forms and so by Proposition 4.51 is isomorphic to $H^3_0(M)$. Hence $[\eta_-]$ lies in the image of $H^3_0(M)$ in $H^3(M)$ under inclusion. Therefore, from the long exact sequence (94), we find that under the natural map $\Upsilon: H^3(M) \to H^3(\Sigma)$ we have $\Upsilon^3(\eta_-) = 0$. Furthermore, $\Upsilon^3([\eta_+]) = \gamma$, since $\eta_+$ is asymptotic to $\gamma$. As we are assuming that $[\eta] = 0$ we find that $0 = \Upsilon^3([\eta]) = \Upsilon^3([\eta_+] + [\eta_-]) = \gamma$ which implies that $\gamma = 0$. We find therefore that $\eta_+ = 0$ and hence $\eta = \eta_-$. Again from Proposition 4.51 we know that $\mathcal{H}^3_{L^2}$ is isomorphic to $H^3(M)$, which means that $[\ast \varphi \eta_-] = 0$ in $H^4(M)$ if and only if $[\ast \varphi \eta_-] = 0$. We conclude that $\eta = \eta_- = 0$ and the thus the map (160) is indeed injective.

**Remark 6.7.** A similar immersion (but with more structure) exists for the moduli space of smooth compact $G_2$ manifolds. See [13, Theorem 10.4.5] for details.

We now apply Proposition 6.6 to “lift” an automorphism of the link to an “almost-automorphism” of the $AC G_2$ manifold, as follows.

**Proposition 6.8.** Let $(M, \varphi_M)$ be an $AC G_2$ manifold with rate $\nu = -3+\varepsilon$, where $M$ has asymptotic cone $C$ with link $\Sigma$. Suppose that there are no solutions to (159) for any $\lambda \in (-3, \nu)$. Let $F_\Sigma$ be a diffeomorphism of $\Sigma$ isotopic to and sufficiently close to the identity which preserves the nearly Kähler structure on $\Sigma$. Then there exists a diffeomorphism $F_M$ of $M$ preserving $\varphi_M$ which is asymptotic to $F_\Sigma$ with rate $\nu + 1$.

**Proof.** By Proposition 6.6 we know that $\mathcal{M}_\nu$ is smooth and equal to $\mathcal{M}_{-3+\varepsilon}$, so we can apply Proposition 6.6 to $\mathcal{M}_\nu$. By hypothesis, we can write $F_\Sigma = \exp(X_\Sigma)$ for a Killing field $X_\Sigma$ on $\Sigma$. The Killing field $X_\Sigma$ naturally defines a Killing field $X_C$ on the cone $C$. Define a smooth increasing cutoff function $\rho: [0, \infty) \to [0, 1]$ such that $\rho(r) = 0$ for $r \in (0, R)$ and $\rho(r) = 1$ for $r \geq R + 1$. Using the notation of Definition 3.15 we can then define a vector field $X$ on $M$ such that $h_\ast(\rho X_C) = X$ on $h((R, \infty) \times \Sigma) = M \setminus L$ and which vanishes on the compact subset $L$ of $M$. Finally, we let $F = \exp(X)$. Now, since $\varphi_M$ is asymptotic to $\varphi_C$ and $F$ is asymptotic to $F_C = \exp(X_C)$, we see that $F^\ast \varphi_M$ is asymptotic to $F_C^\ast \varphi_C$ with rate $\nu$. Moreover, since $F$ is isotopic to the identity we see that $[F^\ast \varphi_M] = [\varphi_M]$ and $[\Theta(F^\ast \varphi_M)] = [\Theta(\varphi_M)]$. Using Proposition 6.6 we find that the orbits of $F^\ast \varphi_M$ and $\varphi_M$ under $D_{\nu+1}$ in $\mathcal{M}_\nu$ are equal. Thus there exists $F_\rho \in D_{\nu+1}$ such that $F_\rho^\ast (F^\ast \varphi_M) = \varphi_M$. Since $F_\rho$ is asymptotic to the identity and $F$ is asymptotic to $F_\Sigma$ we can set $F_M = F \circ F_\rho$. □

**Remark 6.9.** A similar but slightly weaker uniqueness result for $AC$ Calabi-Yau manifolds was obtained by Conlon–Hein in [10]. They show that if a biholomorphism of an $AC$ Calabi-Yau manifold $M$ is asymptotic to an isometry of the cone, then it must be an isometry of $M$.

**Corollary 6.10.** Let $(M, \varphi_M)$ be an $AC G_2$ manifold with rate $\nu < 0$ such that the link $\Sigma$ of the asymptotic cone is one of the three possible homogeneous Gray manifolds, namely $\mathbb{C}P^3$, $SU(3)/T^3$, or $S^3 \times S^3$. Then $(M, \varphi_M)$ has cohomogeneity one.
In particular, the Bryant–Salamon $G_2$ manifolds $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ are the unique AC $G_2$ manifolds of rate $\nu < 0$ asymptotic to the cones on $\mathbb{CP}^3$ and SU(3)/T$^2$, respectively.

Proof. It follows from Proposition 6.2 that there are no solutions to (159) for any $\lambda \in (-3, 0)$ for these links. Hence Proposition 6.8 applies to $(M, \varphi_m)$. Consequently, any automorphism of the homogeneous nearly Kähler manifold can be extended to an automorphism of the AC $G_2$ manifold, so $M$ must have cohomogeneity one.

The cohomogeneity one $G_2$ manifolds where the action is by a simple group are classified in [9, Theorem 9.3]. The uniqueness of $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ amongst such cohomogeneity one $G_2$ manifolds follows.

Remark 6.11. It is likely that the spinor bundle $\mathcal{S}(S^3)$ of $S^3$ is also unique in the same sense, but this is not considered in [9] as they only study the case where the cohomogeneity one action is by a simple group.

6.4 Smoothness of the CS moduli space for certain cones

In this section we establish that the extended reduced CS moduli space $\tilde{\mathcal{M}}_\nu$ is in fact smooth if the singularities are all modeled on $G_2$ cones satisfying certain conditions. This includes two of the known $G_2$ cones, and may include the third as well.

We begin by analyzing the effect of critical rates in $(-3, 0]$ in the CS case. If $\lambda$ is a critical rate of $\mathcal{O}_{C_i}$ in the interval $(-3, 0)$, then by Proposition 6.1, for $\lambda \in (-3, -1]$ such a critical rate corresponds to the existence of a solution on the link $\Sigma_i$ of

$$\bar{\Delta}_i \gamma = (\lambda + 3)(\lambda + 4) \gamma, \quad d^* \gamma = 0, \quad \gamma \text{ is of type } \Lambda^{(1,1)}_0. \quad (161)$$

If, on the other hand, we have $\lambda \in (-1, 0]$, then by Proposition 6.1 such a critical rate corresponds to a homogeneous of order $\lambda$ closed and coclosed 3-form $\eta$ on one of the cones $C_i$ of the form $\eta = d\beta$ with $\Delta_{C_i} \beta = 0$ and $d^* \beta = 0$. Then $\pi_\tau d^* \eta = \pi_\tau \Delta_{C_i} \beta = \Delta_{C_i} \pi_\tau \beta = 0$, and so we have a critical rate for $\pi_\tau d^* d$. By Proposition 2.7 and Lemma 4.76 crossing this critical rate must subtract from the cokernel. Thus, from the relation between the cokernel of the map $(\pi_\tau d^* d)_{|\lambda=1}$ and the finite-dimensional space $(E_x)_{|\lambda}$, as explained in the proof of the $L^2$ version of Theorem 5.3, such a critical rate must correspond to a nonzero infinitesimal deformation in $(E_x)_{|\lambda}$, and those are precisely the deformations we had to ignore in order to obtain meaningful results for a reduced moduli space $\tilde{\mathcal{M}}_\nu$.

Therefore, the only contributions to decreasing the dimension of the reduced moduli space from critical rates in $(-1, 0]$ will come from critical rates for which $\beta_\tau = 0$. By Proposition 6.1, these correspond precisely to the existence of solutions of (161) for $\lambda \in (-1, 0]$.

Hence, the critical rates in $(-3, 0]$ which would act to subtract from the dimension of the reduced moduli space $\mathcal{M}_\nu$ in the CS case correspond precisely to those for which there exist solutions of (161) for some $i$. Moreover, by the work of Moroianu–Nagy–Semmelmann [36, Theorem 4.1], the solutions of (161) when $\lambda = 0$ correspond to infinitesimal deformations of the nearly Kähler structure on the link $\Sigma_i$.

This motivates the following definition.

Definition 6.12. Let $M$ be a CS $G_2$ manifold. We say that $M$ has good singularities if for each link $\Sigma_i$ of the corresponding asymptotic cone $C_i$, equation (161) has no nontrivial solutions for $\lambda \in (-3, 0]$. 

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Theorem 6.13. Let $M$ be a CS $G_2$ manifold with good singularities, and choose $\nu > 0$ close enough to zero so that there are no critical rates between 0 and $\nu$. Then the extension $\mathcal{M}'_\nu$ of the reduced moduli space of deformations of $M$ as a CS $G_2$ manifold, defined just before equation (153), is smooth with dimension

$$\dim \mathcal{M}'_\nu = \dim(\text{im}(H^3_{cs} \to H^3)).$$

Proof. First, we show that the extension of the reduced moduli space is smooth because the obstruction space vanishes. From the hypotheses about good singularities, we know that for any $\lambda < 0$, we have $\text{coker} \, \mathcal{D}_{l,\lambda} = \text{coker} \, \mathcal{D}_{l,-3+\varepsilon}$. As we cross the rate $\lambda = -3$, the argument at the end of the first paragraph of the proof of Theorem 4.49 says that $\text{coker} \, \mathcal{D}_{l,\lambda}$ still does not change. Corollary 4.47 then allows us to conclude that $\text{coker} \, \mathcal{D}_{l,\lambda} = \text{coker} \, \mathcal{D}_{l,-4+\varepsilon}$ for any $\lambda < 0$. Now Remark 4.42 can be applied in the CS case, with all inequalities reversed, to conclude that $\text{coker} \, \mathcal{D}_{l,-4+\varepsilon} = \{0\}$, since $-4+\varepsilon < -\frac{7}{2}$.

The virtual dimension formula now follows from Corollary 5.21, the discussion preceding equation (153), and Section 6.1. The point is that each $K(\lambda)_{\mathcal{D}_C} = \{0\}$ for $\lambda \in (-3,0)$ and each $K(0)_{\mathcal{D}_C}$ is one-dimensional, so the contributions of $-n$ and $+n$ cancel each other.

Corollary 6.14. Let $M$ be a CS $G_2$ manifold of rate $\nu = 0+\varepsilon$, all of whose conical singularities are modeled on $G_2$ cones whose links are either $\mathbb{CP}^3$ or $S^3 \times S^3$. Then the extension $\mathcal{M}'_\nu$ of the reduced moduli space of CS deformations of $M$ with rate $\nu$ is a smooth manifold.

Proof. This follows from Theorem 6.13 and Proposition 6.2.

6.5 Relation with the moduli of resolved CS $G_2$ manifolds

In this section we relate our results to the resolution of singularities construction of [20]. Our observations provide evidence that CS $G_2$ manifolds likely arise as the “most common” form of singular object in any attempt to compactify the moduli space of compact smooth $G_2$ manifolds.

Let $M$ be a CS $G_2$ manifold with one conical singularity and let $N$ be an AC $G_2$ manifold asymptotic to the same $G_2$ cone, with link $\Sigma$ at infinity, as $M$ has at its singularity. The main result of [20] says that, if a particular necessary topological condition [20, Theorem 3.8] is satisfied, then one can desingularize $M$ by gluing in $N$ to obtain a compact smooth $G_2$ manifold, which we will denote by $X$. When $M$ has a single conical singularity, the topological condition can be expressed using the maps $\Upsilon^k$ of Definition 4.54 as follows:

$$\Upsilon^k_N(\varphi_N) \in \text{im}(\Upsilon^k_M), \quad \Upsilon^k_N(\psi_N) \in \text{im}(\Upsilon^k_M).$$

Remark 6.15. In [20] Definition 2.40, the elements $\Upsilon^k_N(\varphi_N)$ and $\Upsilon^k_N(\psi_N)$ are denoted by $\Phi(N)$ and $\Psi(N)$, respectively.

Since $X = M \cup N$, and since $M \cap N$ is homotopy equivalent to $\Sigma$, the Mayer–Vietoris long exact sequence gives

$$\cdots \to H^k(X) \to H^k(M) \oplus H^k(N) \xrightarrow{\delta^k} H^k(\Sigma) \to H^{k+1}(X) \to \cdots$$

(163)

where the map $\delta^k : H^k(M) \oplus H^k(N) \to H^k(\Sigma)$ is given by $\delta^k(a) = \Upsilon^k_M(a) - \Upsilon^k_N(a)$, and the $\Upsilon^k$ maps are those of Definition 4.54.
Lemma 6.16. Let $M$, $N$, $X$, and $\Sigma$ be as above. The following equation holds.

\[ b^k(X) = b^k(M) - \dim(\text{im} \, \Upsilon_M^k) \]
\[ + b^k_\cs(N) - \dim(\text{im} \, \Upsilon_N^k) + \dim((\text{im} \, \Upsilon_M^k \cap (\text{im} \, \Upsilon_N^k)) \]
\[ + \dim((\text{im} \, \Upsilon_M^k) \cap (\text{im} \, \Upsilon_N^k)). \]  

(164)

Proof. We will use the shorthand notation $H^k_M$ for $H^k(A)$. By the rank-nullity theorem we have

\[ b^k(X) = \dim(\text{ker}(H^k_X \to H^k_M \oplus H^k_N)) + \dim(\text{im}(H^k_X \to H^k_M \oplus H^k_N)). \]  

(165)

The exactness of (163) gives

\[ \dim(\text{im}(H^k_X \to H^k_M \oplus H^k_N)) = \dim(\ker \delta^k), \]  

(166)

but since $\delta^k = \Upsilon_M^k - \Upsilon_N^k$, it is easy to check that

\[ \dim(\ker \delta^k) = \dim(\ker \Upsilon_M^k) + \dim(\ker \Upsilon_N^k) + \dim((\text{im} \, \Upsilon_M^k) \cap (\text{im} \, \Upsilon_N^k)). \]  

(167)

Substituting (166) into (165) gives

\[ b^k(X) = \dim(\ker(H^k_X \to H^k_M \oplus H^k_N)) \]
\[ + \dim(\ker \Upsilon_M^k) + \dim(\ker \Upsilon_N^k) + \dim((\text{im} \, \Upsilon_M^k) \cap (\text{im} \, \Upsilon_N^k)). \]  

(168)

Again using the exactness of (163), we find

\[ \dim(\ker(H^k_X \to H^k_M \oplus H^k_N)) = \dim(\text{im}(H^{k-1}_X \to H^k_X)) \]
\[ = b^{k-1}(\Sigma) - \dim(\ker \Upsilon_{\Sigma}^{k-1}) \]
\[ = b^{k-1}(\Sigma) - \dim(\ker \delta^{k-1}) \]
\[ = \dim(\text{coker} \delta^{k-1}). \]

But the dimension of this cokernel is equal to the dimension of the kernel of the formal adjoint, which is a map $(\delta^{k-1})^* : H^{6-(k-1)}(\Sigma) \to H_{\cs}^{7-(k-1)}(M) \oplus H_{\cs}^{7-(k-1)}(N)$. It can be easily checked from the definitions of all the maps involved that $(\delta^{k-1})^*(a) = (\partial M^{7-k}(a), \partial N^{7-k}(a))$, where the maps $\partial^k_M : H^k(\Sigma) \to (H_{\cs}^{k+1})(A)$ are the connecting homomorphisms in the long exact sequence (94). Thus we have

\[ \dim(\ker(H^k_X \to H^k_M \oplus H^k_N)) = \dim(\ker(\delta^{k-1})^*) \]
\[ = \dim((\ker \partial^k_M \cap (\ker \partial^k_N)) \]
\[ = \dim((\text{im} \, \Upsilon_{M}^{7-k}) \cap (\text{im} \, \Upsilon_{N}^{7-k})). \]  

(169)

where in the last step we have used the exactness of (94) for both $M$ and $N$.

Finally, we substitute (169) into (168) and use (96) for $M$ and (97) for $N$ to obtain (164). \qed

Specializing Lemma 6.16 to $k = 3$ gives

\[ b^3(X) = b^3(M) - \dim(\text{im} \, \Upsilon_M^3) \]
\[ + b^3_\cs(N) - \dim(\text{im} \, \Upsilon_N^3) + \dim((\text{im} \, \Upsilon_M^3) \cap (\text{im} \, \Upsilon_N^3)) \]
\[ + \dim((\text{im} \, \Upsilon_M^3) \cap (\text{im} \, \Upsilon_N^3)). \]  

(170)
Notice that the left hand side gives the dimension of the moduli space of deformations of the smooth, compact $G_2$ manifold $X$. Also, by Theorem 6.13 and equation (95) the first two terms on the right hand side give the dimension of the extension $\mathcal{M}_0'$ of the reduced moduli space of deformations of the $CS$ $G_2$ manifold $M$, in the cases when Theorem 6.13 applies. In particular, by Corollary 6.14 this is true if there is one conical singularity whose link is either $\mathbb{C}P^3$ or $S^3 \times S^3$.

Let us consider two particular cases.

Case 1. Suppose $b^3(N) = b^3_{cs}(N) = 1$, $b^3(N) = b^3_{cs}(N) = 0$, and $b^3(\Sigma) = 0$. Further assume that $\Upsilon^3_N(\psi_x) \neq 0$ in $H^3(\Sigma)$. [In particular, these assumptions all hold for the Bryant–Salamon manifolds $\Lambda^2(S^3)$ and $\Lambda^2(\mathbb{C}P^2)$, with links $\Sigma = \mathbb{C}P^3$ and $\Sigma = SU(3)/T^2$, respectively.]

With these assumptions, some simple diagram chasing using the exact sequence (94) gives

$$\Upsilon^3_M = 0, \quad \Upsilon^3_N = 0, \quad \ker(\Upsilon^3_N) = \{0\}, \quad \dim(\text{im}(\Upsilon^3_N)) = 1.$$  

The condition (162) in this case thus becomes $\text{im} \Upsilon^1_N = (\text{im} \Upsilon^1_N) \cap (\text{im} \Upsilon^3_M)$, and (170) therefore becomes $b^3(X) = b^3(M) + 1$.

Case 2. Suppose $b^3(N) = b^3_{cs}(N) = 0$, $b^3(N) = b^3_{cs}(N) = 1$, and $b^3(\Sigma) = 0$. Further assume that $\Upsilon^3_N(\psi_x) \neq 0$ in $H^3(\Sigma)$. [In particular, these assumptions all hold for the Bryant–Salamon manifold $S(S^2)$, with link $\Sigma = S^3 \times S^3$.]

As before, diagram chasing using (94) in this case yields

$$\Upsilon^4_M = 0, \quad \Upsilon^4_N = 0, \quad \ker(\Upsilon^4_N) = \{0\}, \quad \dim(\text{im}(\Upsilon^4_N)) = 1.$$  

The condition (162) in this case therefore becomes $\text{im} \Upsilon^1_N = (\text{im} \Upsilon^1_N) \cap (\text{im} \Upsilon^3_M)$, and (170) thus becomes $b^3(X) = b^3(M) = b^3(M) - \dim(\text{im}(\Upsilon^3_M)) + 1$.

In summary, in both cases (which include all the known examples of $AC$ $G_2$ manifolds), we find that the dimension of the moduli space of glued compact $G_2$ manifolds that are constructed in [20] is exactly one dimension higher than the “extended reduced moduli space” $\mathcal{M}_0'$ of the $CS$ $G_2$ manifold which has been resolved, so we can view the $CS$ moduli space almost literally as “the boundary” of the moduli space of compact $G_2$ manifolds, at least locally.

### 6.6 Existence of a gauge-fixing diffeomorphism

In [20], a gauge-fixing condition was defined for $AC$ $G_2$ manifolds that is slightly different from our Definition 4.65. Specifically, an $AC$ $G_2$ manifold $(M, \varphi_M)$ comes equipped with a choice of diffeomorphism $h : (R, \infty) \times \Sigma \to N \setminus L$ for some compact subset $L \subset M$ and some $R > 0$, where $\Sigma$ is the link of the asymptotic $G_2$ cone. In [20] Definition 3.3], the diffeomorphism $h$ is said to satisfy the gauge-fixing condition if $h^* \varphi_M - \varphi_C$ lies in $\Omega^3_{27}$ with respect to the $G_2$ structure $\varphi_C$ on the cone.

In [20], it was promised that the present paper would give a proof that such a diffeomorphism $h$ can always be chosen to satisfy the gauge-fixing condition. In fact, it turns out that it is easier to establish a gauge-fixing condition in which the roles of $M$ and $C$ are reversed. Specifically, let us make the following alternative definition of gauge-fixing.

**Definition 6.17.** Consider an $AC$ $G_2$ manifold $(M, \varphi_M)$, which comes equipped with a choice of diffeomorphism $h : (R, \infty) \times \Sigma \to N \setminus L$ for some compact subset $L \subset M$ and some $R > 0$, where $\Sigma$ is the link of the asymptotic $G_2$ cone. The diffeomorphism $h$ is said to satisfy the gauge-fixing condition if $\varphi_M - (h^{-1})^* \varphi_C$ lies in $\Omega^3_{27}$ with respect to the $G_2$ structure $\varphi_M$ on $M$.

One can show quite easily that [20] Theorem 3.6], the result which needed the gauge-fixing condition, is still true with the version of gauge-fixing given by Definition 6.17 above. More precisely,
the statements and proofs of [20] Lemma 6.1] and [20] Corollary 6.2] can be modified to work with this version of gauge-fixing. The fact that such a gauge-fixing always exists is a consequence of our slice theorem, Theorem 6.6 for AC $G_2$ conifolds. The statement of the existence of a good gauge is as follows.

**Proposition 6.18.** Let $(M, \varphi_M)$ be an AC $G_2$ manifold, with diffeomorphism $h : (R, \infty) \times \Sigma \to M \setminus L$ as given in Definition [3.1]. Then, after possibly making $R$ larger, there exists some diffeomorphism $\tilde{h} : (R, \infty) \times \Sigma \to M \setminus L$ for some compact subset $\tilde{L}$ of $M$ such that $\varphi_M - (\tilde{h})^* \varphi_C$ is in $\Omega^3_R$ with respect to $\varphi_M$.

**Proof.** The 3-form $\varphi_M$ is a torsion-free $G_2$ structure on the open subset $M_R = h((R, \infty) \times \Sigma)$ of $M$, and the AC condition says that $\varphi_M$ is close to the torsion-free $G_2$ structure $(\tilde{h})^* \varphi_C$ on $M_R$. By the slice theorem for AC $G_2$ conifolds, after possibly making $R$ larger to make $(\tilde{h})^* \varphi_C$ sufficiently close to $\varphi_M$, there exists a diffeomorphism $f : M \to M$ such that $\pi_7 d^* ((f^* (\tilde{h})^* \varphi_C) = 0$, where $d^*$ and $\pi_7$ are taken with respect to $\varphi_M$. Define $\tilde{h} = f^{-1} \circ h$, which is a diffeomorphism from $(R, \infty) \times \Sigma$ to $f^{-1}(M \setminus L) = M \setminus f^{-1}(L)$. Let $\tilde{L} = f^{-1}(L)$.

Since $\tilde{h}^{-1} = h^{-1} \circ f$, we have that $\pi_7 d^* (\varphi_M - (\tilde{h})^* \varphi_C) = 0$. Now write the 3-form $\zeta = \varphi_M - (\tilde{h})^* \varphi_C$ in the form (107). By the arguments in the proof of Lemma 4.64, the function $f$ and the 1-form $X$ are harmonic, of order $O(r^\nu)$ as $r \to \infty$. Since $\nu < 0$, by Proposition 3.6 we can conclude that both $f$ and $X$ are of order $O(r^{-5+\epsilon})$, and thus by Lemma 4.64 both $f = 0$ and $X = 0$. So $\zeta \in \Omega^3_R$ with respect to $\varphi_M$, which is what we wanted to show. 

### 6.7 Open problems

There remain several interesting and important open problems for future study.

- We need to find more examples, especially with little or no symmetry, of Gray manifolds (compact strictly nearly Kähler 6-manifolds). This would provide new examples of $G_2$ cones, and hopefully one could construct new AC $G_2$ manifolds with these asymptotic cones.

- The work of Moroianu–Nagy–Semmelmann [36] describes in detail the *infinitesimal* deformations of Gray manifolds. It is still an open problem to understand the integrability of such infinitesimal deformations to actual deformations. Understanding this would allow us to consider more general deformations of $G_2$ conifolds where we allow the asymptotic cones to also deform.

- A related question is to better understand the spectrum of the Laplacian on 2-forms for Gray manifolds. Some work on this already appears in Moroianu–Nagy–Semmelmann [36] and Moroianu–Semmelmann [37]. But a more thorough understanding would allow us to conclude whether the results in Sections 6.2 and 6.4 are more general or are particular to $G_2$ conifolds whose links are the known Gray manifolds.

- One can define a *stability index* for $G_2$ cones in a similar way to the stability index for special Lagrangian cones [15] or for coassociative cones [28]. Results about the spectrum of the Laplacian for Gray manifolds would also tell us something about the stability index of $G_2$ cones. Knowledge of the stability index tells us more about when the CS deformation theory is unobstructed.
• We need to construct the first examples of CS $G_2$ manifolds. As mentioned earlier, the approach in [16] of constructing compact smooth $G_2$ manifolds may possibly be generalizable to construct CS $G_2$ manifolds.

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