Mixing and equilibration: Protagonists in the scene of nonextensive statistical mechanics

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Abstract

After a brief review of the present status of nonextensive statistical mechanics, we present a conjectural scenario where mixing (characterized by the entropic index $q_{mix} \leq 1$) and equilibration (characterized by the entropic index $q_{eq} \geq 1$) play central and inter-related roles, and appear to determine \textit{a priori} the values of the relevant indices of the formalism. Boltzmann-Gibbs statistical mechanics is recovered as the $q_{mix} = q_{eq} = 1$ particular case.

Human knowledge progresses along very many paths, and very rarely these paths follow a systematic and logical ordering. What we can presently witness about the formalism frequently referred to as \textit{nonextensive statistical mechanics} (first proposed in 1988 \cite{1} and further implemented in \cite{2,3}; for reviews, see \cite{4,5}) is by no means exception. Since it is the purpose of the present lines to start with a review, let us present it in an order which makes some epistemological sense, although it does not necessarily follow the chronology of the events. We will successively comment on (i) mathematical properties, which make the formalism to appear just as \textit{applied mathematics}; (ii) the successful confrontations with

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experimental (as well as computational) results, which we believe raises the formalism to the status of theoretical physics, since it has a compromise with phenomena indeed occurring in nature; and (iii) the lines along which the calculation of the relevant entropic indices \( q_{\text{mix}} \) and \( q_{\text{eq}} \) appears as possible, so that the formalism becomes a closed theory (in other words, a complete theory; see [7]), where everything can in principle be calculated a priori once the dynamics of the system is fully characterized.

**Applied mathematics**

The axiomatic starting point is the proposal of a possible generalization of Boltzmann-Gibbs (BG) statistical mechanics by postulating the following entropic form

\[
S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} = \langle \ln_q \frac{1}{p_i} \rangle \left( \sum_{i=1}^{W} p_i = 1 \right),
\]

where we are using \( k_B = 1 \) (without loss of generality), \( q \in \mathbb{R} \), \( \langle \cdots \rangle = \sum_{i=1}^{W} p_i (\cdots) \) and \( \ln_q x \equiv \frac{x^{1-q}-1}{1-q} \) (with \( \ln_1 x = \ln x \) [consistently, its inverse function is \( e_q^x \equiv [1+(1-q)x]^{1/(1-q)} \), with \( e_1^x = e^x \)]. Clearly, for \( q = 1 \), we recover the usual expression \( S_1 = -\sum_{i=1}^{W} p_i \ln p_i \), from now on referred to, for simplicity, as the BG entropy. Also, if \( A \) and \( B \) are two probabilistically independent systems (i.e., \( p_{ij}^{A+B} = p_i^A p_j^B \)), \( S_q \) satisfies the following pseudoadditivity property: \( S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \), hence \( q = 1, > 1, < 1 \) respectively correspond to extensive, subextensive and superextensive cases.

For an isolated system (microcanonical ensemble), optimization of \( S_q \) yields equiprobability, i.e., \( p_i = 1/W \ (\forall i) \), hence

\[
S_q = \ln_q W. \tag{2}
\]

Before going on, it is important to stress that the pseudoadditivity property mentioned above does not exclude that, for some classes of interdependency between \( A \) and \( B \), a special value \( q^* \) could exist such that extensivity could be recovered in the sense that \( S_{q^*}(A+B) = S_{q^*}(A) + S_{q^*}(B) \). If so, we could say that the adequacy of the entropic form enables the preservation of the traditional property of extensivity for the entropy. As mathematical illustrations of this fact, let us consider two possible cases at equiprobability. First, if
W \sim a\mu^N \text{ (with } a > 0, \mu > 1 \text{ and } N \equiv \text{number of elements } \to \infty), \text{ then } S_1 = \ln W \sim (\ln \mu)N \propto N, \text{ hence } q^* = 1. \text{ Second, if } W \sim bN^\nu \text{ (with } b > 0, \nu > 0 \text{ and } N \to \infty), \text{ then, for } q < 1, S_q = \frac{W^{1-q}-1}{1-q} \propto \frac{b^{1-q}N^{\nu(1-q)}}{1-q}, \text{ consequently the choice } q^* = 1 - 1/\nu \text{ implies } S_q^* \propto N, \text{ once again recovering the traditional proportionality between entropy and } N, \text{ when } N >> 1. 

For a system in thermal equilibrium with a thermostat (canonical ensemble), we must optimize $S_q$ with a further restriction, namely \[3\]

\[
\sum_{i=1}^{W} P_i \epsilon_i = U_q,
\]

where the escort distribution $P_i \equiv p_i^q / \sum_{j=1}^{W} p_j^q$, $\{\epsilon_i\}$ are the eigenvalues of the corresponding Hamiltonian (with the associated boundary conditions), and $U_q$ is a fixed finite value for the generalized internal energy. The optimizing distribution is then given by

\[
p_i = \frac{e^{-\epsilon_i - U_q}/T_q}}{\sum_{j=1}^{W} e^{-\epsilon_j - U_q}/T_q^q},
\]

where $T_q \equiv T \sum_{i=1}^{W} p_i^q$, $1/T \equiv \beta$ being the Lagrange parameter associated with constraint (3). We can verify that, for all values of $q$, $p_i$ is invariant under shifts of the zero of the spectrum of energies $\{\epsilon_i\}$. Also, for $q = 1$, we recover the usual BG equilibrium distribution $p_i = e^{-\epsilon_i/T} / \sum_{j=1}^{W} e^{-\epsilon_j/T}$. Eq. (4) can be rewritten in another form, namely

\[
p_i = \frac{e^{-\epsilon_i/T'_q}}{\sum_{j=1}^{W} e^{-\epsilon_j/T'_q}},
\]

with $T'_q \equiv T_q + (1 - q)U_q$. For fixed and large values of $T'_q$, we have $p_i \sim (1 - \epsilon_i/T'_q)/[\sum_{j=1}^{W} (1 - \epsilon_j/T'_q)]$ for all values of $q$. In other words, all nonextensive equilibrium thermostatistics share, at large temperatures, a common distribution which can be equivalently considered to be the BG one \[5\].

These are the essential steps. From these, many other have been developed for arbitrary $q$, such as the $H$-theorem, Ehrenfest theorem, Bogolyubov inequality, factorization of the likelihood function, Onsager’s reciprocity theorem, Kramers and Wannier relation, Pesin
theorem (conjecture), fluctuation-dissipation theorem, and others. Also, a variety of traditional statistical-mechanical techniques for treating many-body systems is now available for arbitrary $q$, such as Green functions, variational method, perturbative methods, path integrals, Lie-Trotter formula, simulated annealing, and others. For review of all these, see [4–6] and references therein.

Finally, in Table 1, we present the most relevant historical steps of the foundations of BG statistical mechanics, and their generalizations for arbitrary $q$. In 1860, Maxwell presented its celebrated Gaussian distribution of velocities [11]. In 1872, Boltzmann [12] arrived, through the molecular chaos hypothesis (stosszahlansatz), to the celebrated exponential weight as the stationary state of his partial derivative kinetic equation for distributions in the so called $\mu$-space (projection, on the one-particle phase space, of the states of all particles). In 1902, Gibbs [13] presented how, within a variational principle using entropy as the relevant functional, the exponential weight can be reobtained, this time in a more general framework, namely in the so called $\Gamma$-space (phase space of all particles). Gibbs equilibrium distribution was later reobtained through a variety of other manners, namely by Darwin and Fowler in 1922 [14] using a steepest descent argument, by Khinchin in 1949 [15] using the law of large numbers, by Balian and Balazs in 1987 [16] and by Kubo et al in 1988 [17], performing countings in the microcanonical ensemble (isolated system). In parallel with these developments, Shannon in 1948 [18] and Khinchin in 1953 [19] established necessary and sufficient conditions for the entropic functional to be $-\sum_{i=1}^{\mathcal{W}} p_i \ln p_i$. All these arguments have been generalized for arbitrary $q$. Gibbs path was followed in 1988 and thereafter [1–3], the Darwin-Fowler, Khinchin and Balian-Balazs paths were followed in 2000 by Abe and Rajagopal in [20], [21] and [22] respectively, and the Kubo path was followed by Abe and Rajagopal in 2001 [23]. Boltzmann path was followed in 2001 by Lima, Silva and Plastino [24] as well as by Kaniadakis in [25]. Shannon and Khinchin paths for the necessary and sufficient for the entropic form were respectively followed by Santos in 1997 [26] and by Abe in 2000 [27]. All these generalizations, without exception, proved to be consistent among them and consistent with Eqs. (1–4).
At this point it seems appropriate to comment that, as a whole, the formalism exhibits a remarkable mathematical “texture”, which quite naturally extends to arbitrary values of \( q \) the properties that are since long known for \( q = 1 \).

**Theoretical physics**

Let us now argue in the sense of transforming the above mathematical formalism into theoretical physics by comparing theoretical with experimental results. We must however warn the reader that many of the available applications of nonextensive statistical mechanics concern open systems (stationary states of open systems), and not only (meta)equilibrium states of the time-independent Hamiltonian systems addressed in Eqs. (3–5). To be more precise, all thermodynamical equilibrium states are stationary solutions of some family of partial derivative equations (e.g., of the Boltzmann kinetic equation, for classical systems), but the opposite is not true. There are stationary states which are not tractable in thermo-statistical terms, this is to say susceptible of being founded in geometrical aspects of some phase space (or of some Riemann subspace in that phase space) or analogous spaces such as the Hilbert or Fock ones. To illustrate what we mean about geometrical founding, it is certainly instructive to recall that it is the symmetrization or the antisymmetrization of the corresponding wave functions that determines the transmutation of Maxwell-Boltzmann statistics into Bose-Einstein and Fermi-Dirac ones, consistently leading to entropic functionals which differ from the classical one. The point is that if the dynamics of the system is completely known, its possible stationary states always are in principle calculable, but for some classes of such systems it is not necessary to follow its dynamics: we can directly, geometrically, calculate the stationary state, which can then be considered as optimizing some entropic functional. Statistical mechanics focuses on such states. The difficulty is of course to know *a priori* what specific statistics is to be applied to a given system. We come back onto this point later on. But at the present stage, let us emphasize that in our understanding statistical mechanics emerges if and only if dynamics can in some sense be replaced by geometry. Unfortunately it is by no means clear when this is possible, but we shall refer to it as geometrizable dynamics. As an attempt to clarify these arguments, we
schematically display in Figs. 1 and 2 respectively the traditional and the present views on the connections between dynamics and statistics (see also Cohen’s contribution to the present proceedings). In Fig. 1 it is stressed that, in the traditional view, an unique type of thermal equilibrium exists and this is the BG one (either in the micro-canonical, the canonical or the various grand-canonical forms). We believe that statistical mechanics is wider than that, as indicated in Fig. 2. Every time that dynamics (of a finite or infinite system) can be automatically taken into account by geometrical considerations, theory of probabilities can, for a variety of purposes, efficiently replace the knowledge of the time evolution of the mechanical system. Within classical BG statistical mechanics, dynamics of vast classes of isolated systems (typically involving short-range interactions) can be replaced by the hypothesis of equiprobability in the occupation of the accessible phase space together with a connection of macroscopic entropy with the relevant phase space volume. We are of course referring to the Boltzmann principle \( S = k \ln W \) (see also Gross’ contribution to the present proceedings). It is however physically appealing to think that more subtle situations can also be handled within statistical-mechanical methods. Such could be the case when, in addition to the knowledge of the volume \( W \), we need to characterize a physical, dynamical bias: this is the role of \( q \) within nonextensive statistical mechanics. The just mentioned Boltzmann principle would have to be generalized in such cases by \( S_q = k \ln W^{1-q} - 1 /[1 - q] \). Of course, there is no reason for thinking that such geometrization of dynamics could not be in principle done for other, possibly more complex, systems, outside of the \( q \)-statistics focused here. Within this scenario, nonextensive statistical mechanics appears to be just the first non-Boltzmannian thermostatistical formalism; many others are in principle thinkable, corresponding to various manners for replacing (specific classes of anomalous) dynamics by geometry. In our present formalism it is clear that \( W \) is a geometrical concept, but the reader might be puzzled by the fact that we are including \( q \) in the same category. This point will become transparent soon, when we shall illustrate how multifractal geometrical considerations enable us, at least for some simple specific cases, to uniquely determine \( q \) \textit{a priori} from the mechanical characterization of the system (and not only from fitting procedures,
as frequently done, *faute de mieux*, in the experimental literature).

To make connection with nature, it is mandatory to mention that the present formalism has been successfully applied to a considerable variety of systems, such as Lévy [28] and correlated [29] anomalous diffusions, peculiar velocities in spiral galaxies [30], turbulence in electron plasma [31], fully developed turbulence [32–35], citations of scientific papers [36], linguistics [37], reassociation in folded proteins [38], quantum entanglement [39,40], electron-positron annihilation [41], quark-gluon plasma [42], cosmology [43,44], hadronic scattering [45], motion of *Hydra viridissima* [46], low-dimensional maps [47,48], inertial classical planar rotators ferromagnetically coupled at long distances [49,50], among others. To be more precise, this formalism addresses systems which, in one way or another, exhibit some relevant multifractal structure. This can occur through a variety of physical mechanisms, such as spatial and/or temporal long-range interactions, mesoscopic dissipation, multifractal boundary conditions, quantum entanglement, and others.

This is a good point for warning the reader about the fact that stretched exponentials and $q$-exponentials can be numerically very close to each other for intermediate values of the abscissa (see Fig. 3), although they are definitively very different both close to the origin and approaching infinity. It is important to realize these features in order to really appreciate the strength of experimental and computational evidences favoring one or the other functional form whenever fittings are involved. The present formalism naturally leads to $q$-exponentials, rather than to stretched exponentials (frequently used for fittings during the last 10–15 years). But only fittings that are satisfactory over relatively large physical ranges can be acceptable in order to distinguish between these two functional forms. For example, in the case of *Hydra viridissima* just mentioned [46], both a $q$-exponential and a stretched exponential fit well the experimental distribution of velocities if only relatively small velocities are taken into account. In this particular case, it is because large velocities were experimentally measured as well, that it became possible to dismiss the stretched exponential function, and retain the $q$-exponential one.

Closed theory
We shall from now on note \( q_{eq} \) the value of \( q \) characterizing the equilibrium distribution (or the stationary distribution if the system is open); typically \( q_{eq} \geq 1 \) (see, for instance, [34,35,41,44,46,50]). Let us also address a different quantity, noted \( q_{mix} \), related to the mixing properties of the system; typically \( q_{mix} \leq 1 \) (see, for instance, [33,47]). The relationship between these two different values of \( q \) is under intensive study nowadays (see [51]).

Our focus here is, as mentioned previously, to develop a scenario where \( q_{mix}, q_{eq} \) and the metaequilibrium states that frequently emerge for some nonextensive systems, play deeply entangled roles. The final outcome is to illustrate the lines along which dynamics can a priori determine the values of \( q \) to be used for specific physical models, so that the whole thermostatistical formalism becomes a closed and complete theory.

To start uncovering the scenario it is enough to consider systems whose phase space (noted \( x \)) is one-dimensional, e.g., one-dimensional maps such as the logistic one. If we note \( \Delta x(t) \) the discrepancy, as a function of time, of two trajectories initially discrepant of \( \Delta x(0) \), we can define the sensitivity to the initial conditions (or mixing function) \( \xi(t) \equiv \lim_{\Delta x(0) \to 0} \Delta x(t)/\Delta x(0) \). The most frequent case is that where \( \xi \) satisfies the following differential equation:

\[
\frac{d\xi}{dt} = \lambda_1 \xi , \tag{6}
\]

where \( \lambda_1 \) is the Lyapunov exponent (the subscript 1 will become clear soon). It follows that

\[
\xi = e^{\lambda_1 t} \tag{7}
\]

Positive and negative values for \( \lambda_1 \) respectively correspond to strong sensitivity and insensitivity to the initial conditions. What happens when \( \lambda_1 = 0 \)? The generic answer is that Eq. (6) is not applicable anymore, and we must take into account the next term, so we consider now

\[
\frac{d\xi}{dt} = \lambda_{q_{mix}} \xi^{q_{mix}} , \tag{8}
\]

where we focus on the case \( \lambda_{q_{mix}} > 0 \) and \( q_{mix} \leq 1 \). The solution now is
\[ \xi = \left[ 1 + (1 - q_{\text{mix}}) \lambda_{q_{\text{mix}}} t \right]^{1/(1-q_{\text{mix}})}, \] (9)

which reproduces solution (7) if \( q_{\text{mix}} = 1 \), and asymptotically behaves like \( t^{1/(1-q_{\text{mix}})} \) if \( q_{\text{mix}} < 1 \). The latter will be referred to as weak sensitivity to the initial conditions (or weak mixing). Let us finally consider the most general case along this line, namely

\[ \frac{d\xi}{dt} = \lambda_1 \xi + (\lambda_{q_{\text{mix}}} - \lambda_1) \xi^{q_{\text{mix}}}, \] (10)

whose solution is

\[ \xi = \left[ 1 - \frac{\lambda_{q_{\text{mix}}}}{\lambda_1} + \frac{\lambda_{q_{\text{mix}}}}{\lambda_1} e^{(1-q_{\text{mix}})\lambda_1 t} \right]^{1/q_{\text{mix}}}. \] (11)

This function is illustrated in Fig. 4 in such a way as to exhibit the crossover from the power-law regime at intermediate times to the exponential regime at long times, which occurs when \( 0 < \lambda_1 < q_{\text{mix}} \) and \( q_{\text{mix}} < 1 \). More precisely, for times \( t \) satisfying \( 0 \leq t < t^* \equiv 1/[(1 - q_{\text{mix}}) \lambda_{q_{\text{mix}}}] \), we have an integrable-like regime (with \( \xi \sim 1 + \lambda_{q_{\text{mix}}} t \)) (i.e., characterized by \( q = 0 \)), for times satisfying \( t^* < t < t^{**} \equiv 1/[(1 - q_{\text{mix}}) \lambda_1] \) we have a power-law regime (with \( \xi \sim [(1 - q_{\text{mix}}) \lambda_{q_{\text{mix}}}]^{1/(1-q_{\text{mix}})} \)), i.e., characterized by \( q = q_{\text{mix}} \), and finally for times satisfying \( t > t^{**} \) we have an exponential regime (with \( \xi \sim \frac{\lambda_{q_{\text{mix}}}}{\lambda_1} e^{\lambda_1 t} \), i.e., characterized by \( q = 1 \)). These facts lead to the following nonuniform convergence: \( \lim_{\lambda_1 \to 0} \lim_{t \to \infty} \frac{\ln \xi}{\ln t} = \lim_{t \to \infty} \frac{\ln \xi}{\ln t} = \lim_{t \to \infty} \frac{\ln \xi}{\ln t} = \lim_{t \to \infty} \frac{\ln \xi}{\ln t} = \lim_{t \to \infty} \frac{\ln \xi}{\ln t} = \lim_{t \to \infty} \frac{\ln \xi}{\ln t} = \frac{1}{1-q_{\text{mix}}} < \infty \). Analogously, using the \( q \)-logarithm function \( \ln_q x \), we have \( \lim_{\lambda_1 \to 0} \lim_{t \to \infty} \frac{\ln_q \xi}{\ln_q t} = \lim_{\lambda_1 \to 0} \lim_{t \to \infty} \frac{\ln_q \xi}{\ln_q t} = \lim_{\lambda_1 \to 0} \lim_{t \to \infty} \frac{\ln_q \xi}{\ln_q t} = \lim_{\lambda_1 \to 0} \lim_{t \to \infty} \frac{\ln_q \xi}{\ln_q t} = \lambda_{q_{\text{mix}}} < \infty \). In other words, for \( 1 << t << t^{**} \), we have \( \frac{\ln_q \xi}{\ln_q t} \simeq \lambda_{q_{\text{mix}}} \), whereas, for \( t > t^{**} \), we have \( \frac{\ln_q \xi}{\ln_q t} \simeq \lambda_1 \). We believe that these features constitute the basic scenario of validity of Boltzmann-Gibbs statistical mechanics versus validity of nonextensive statistical mechanics.

This belief is supported by at least three examples, namely the standard map (see [52] and references therein), another, billiard-inspired, two-dimensional conservative map [53], and the system of \( N \) classical inertial planar rotators coupled all with all through long-range interactions (so called \( \alpha \)-XY model, which for \( \alpha = 0 \) reproduces the HMF model; see [49,50] and references therein).
The standard map is a conservative map with a two-dimensional phase space (the lowest dimension at which a map can be conservative). It includes a nonlinear term introduced by a constant $a$. In the limit $a = 0$ the system is integrable, and for $a \neq 0$ it is chaotic (i.e., it has over entire regions of the phase space two nonzero Lyapunov exponents of opposite sign and equal in absolute value). The positive Lyapunov exponent $\lambda_1$ is a monotonic function of $a$ which vanishes for $a = 0$. In the limit $0 < a \ll 1$, this map precisely exhibits [52] the scenario described above with $q_{\text{mix}} \simeq 0.3$ as studied through the time evolution of the entropic form $S_q$. Indeed, for times below a crossover time (which appears to diverge when $a$ approaches zero), $S_{0.3}$ increases linearly with time (see [52] for details), whereas for times above that crossover time, it is $S_1$ which linearly increases with time. The same behavior is observed for the above mentioned billiard-like map, but with $q_{\text{mix}} \simeq 0.5$.

Let us now turn onto our third example. The system of rotators mentioned above has been studied numerically and presents, in the microcanonical ensemble (i.e., isolated) a second order phase transition at some total energy (conveniently scaled with $N$). This system presents anomalies both above and below that critical energy. Above that point, the entire Lyapunov spectrum collapses to zero when $N \to \infty$. This is to say, $1/N$ plays a role analogous to $a$ in the map just described. Below that critical point, the system exhibits at least two (probably only two) basins of attraction with respect to the initial conditions at which the system is dynamically started. There is a basin of attraction which exhibits an equilibrium at the temperature that the recipe of BG statistical mechanics provides, and whose distribution of velocities precisely is the Maxwellian one. But if we start from the other basin of attraction, the system equilibrates at a finite temperature different (below) from that indicated by BG statistics, and its distribution of velocities exhibit unusual long tails. After some long time the system crosses over essentially to the BG equilibrium state. The duration of this anomalous metastable state diverges with $N$, in a manner which, once again, strongly reminds the scenario we advanced in the present paper. Indeed, if we consider the $\lim_{N \to \infty} \lim_{t \to \infty}$ case, the equilibrium appears to be correctly described within BG statistical mechanics, but if we consider the $\lim_{t \to \infty} \lim_{N \to \infty}$ case, this is definitively not true, and
some other thermostatistical description becomes necessary, apparently the nonextensive one. To make the scenario stronger, it can be verified that, during this metastable state, the Lyapunov spectrum also vanishes in the limit $N \to \infty$.

We have sketched above what happens with mixing. What can we say about equilibration and $q_{eq}$? In other words, how physical quantities relax onto the corresponding equilibrium values? Let us refer once again to the logistic map as an illustration. In the region where strong chaos exists (hence above the chaos threshold), for fixed $W$ and using an ensemble where all initial conditions belong to a single among the $W$ windows of the partition, $S_1(t)$ exhibits a linear increase with time and then a saturation at $S_1(\infty)$. Moreover we verify that $\sigma_1 \equiv |1 - \frac{S_1(t)}{S_1(\infty)}| \sim e^{-t/\tau_1}$, $\tau_1$ being of the order of $1/\lambda_1$ (consistently with Krylov’s ideas more than half a century ago [54]). In other words, $\sigma_1$ essentially satisfies $\frac{d\sigma_1}{dt} = -\sigma_1/\tau_1$.

However, at the chaos threshold, $\tau_1$ diverges, and the time evolution of $\sigma_{q_{mix}} \equiv |1 - \frac{S_{q_{mix}}(t)}{S_{q_{mix}}(\infty)}|$ is essentially given [51] by $\frac{d\sigma_{q_{mix}}}{dt} = -(\sigma_{q_{mix}})^{q_{eq}}/\tau_{q_{eq}}$, with $q_{eq} > 1$ and $\tau_{q_{eq}}$ hopefully of the order of $1/\lambda_{q_{mix}}$. The solution of this differential equation is given by $\sigma_{q_{mix}} = 1/[1 + (q_{eq} - 1) t/\tau_{q_{eq}}]^{1/(q_{eq}-1)}$, which reproduces $e^{-t/\tau_1}$ for $q_{eq} = 1$. Furthermore, $q_{eq}$ depends on $W$, and, in the $W \to \infty$ limit (infinitely fine graining), we observe (within some degree of accuracy) [51] the following finite size scaling

$$q_{eq}(\infty) - q_{eq}(W) \propto \frac{1}{W^{q_{mix}}},$$

with $q_{eq}(\infty) > 1$. This fascinating relation has up to now been verified only for the $z$-logistic maps. For these maps and all values of $z$ that have been checked, the values for $q_{eq}(\infty)$ precisely coincide – supreme suggestion of correctness of the present conjectural scenario! –, with the values numerically obtained in [48], where the initial conditions were spread uniformly all over the entire accessible phase space (in a typically Gibbsian manner).

We are unfortunately not in position to rigorously prove the validity of the above relations nor discuss the detailed hypothesis they must involve. It has however been possible to give some physical consistency to the conjectural scenario by analyzing three different phenomena, namely related to electron-positron annihilation [11], to fully developed turbulence
and to the Henon-Heiles Hamiltonian \[44\].

The distributions of transverse momenta of hadronic jets produced by electron-positron annihilation experiments at CERN have been discussed by Bediaga et al in \[41\]. The values for \( q_{eq} \) they obtained depend on the collision center-of-mass energy \( E \), which plays a role similar to \( W \) since the graining is finer for increasingly high energies. The data in their Table 1 can be organized as indicated in the present Fig. 5, with \( q_{eq}(\infty) \approx 1.30 \), and the role played by the exponent 1/2 is that of \( q_{mix} \) in Eq. (12).

The distributions of velocity differences in fully developed turbulence in Couette-Taylor experiments have been discussed by Beck et al in \[35\] for four typical values of the Reynolds number \( Re \) and millions of experimental distances \( r \) (in units of the Kolmogorov length \( \eta \)). The data they present in their Fig. 3 can be organized as shown in the present Fig. 6, with \( q_{eq}(\infty) \approx 1.45 \), and the role played by the exponent 0.37 is that of \( q_{mix} \) in Eq. (12). Incidentally, 0.37 is the value used by the Arimitsu’s \[33\] to produce (in one of their calculations) such distributions of velocity differences. It is worthy mentioning also that the value 1.45 is very close to \( 3/2 \) recently advanced by Beck \[55\] for Lagrangian turbulence.

Saddle-point dynamics of the Henon-Heiles system have been discussed by Soares et al \[44\]. In their Table 1 they show the numerical values obtained for \( \gamma \equiv 1/(q_{eq} - 1) \) as a function of the control parameter \( k \), known to play a role analogous to a Reynolds number, hence \( k \) characterizes \( W \) and the \( k \rightarrow \infty \) limit corresponds to infinitely fine graining. The data presented in that Table can be organized as shown in the present Fig. 7, thus giving support to relation (12), with \( q_{eq}(\infty) \approx 2.81 \), and the role played by the exponent 0.35 is that of \( q_{mix} \).

Summarizing, the basic picture which emerges from all the above considerations is as follows. If the system is strongly chaotic (in the sense that its spectrum of Lyapunov exponents includes positive values, i.e., exponential mixing), then the measure of ignorance (entropy) to be used is, as well known, \( S_1 \), from which the entire Boltzmann-Gibbs statistical mechanics can be derived. If, however, the system is only weakly chaotic (zero Lyapunov exponent spectrum, and power-law mixing), then several indications exist which suggest
that we should instead use $S_{q_{\text{mix}}}$ in what concerns a finite entropy production, and $S_{q_{\text{eq}}}$ to discuss the corresponding equilibrium thermodynamics, both $q_{\text{mix}}$ and $q_{\text{eq}}$ being uniquely determined by the dynamics of the specific physical system (to be more precise, the graining degree, characterized here by $W$, also enters in the determination of $q_{\text{eq}}(W)$; it is the value $q_{\text{eq}}(\infty)$ the one which only depends on the fundamental dynamics). As we see, in this scene there are two protagonists, namely mixing and equilibration. Although deeply entangled, they are different concepts, typically represented by two different values of $q$, one of them ($q_{\text{mix}}$) not above unity, and the other one ($q_{\text{eq}}$) not below unity. These two values of $q$ merge on a single one ($q = 1$) for Boltzmann-Gibbs thermostatistics (thus transforming power laws in exponentials). It is certainly allowed to think that this is perhaps at the origin of not few of the warm controversies in our community about the foundations of statistical mechanics!

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APPENDIX

This Appendix focuses on an interesting point that was raised during the meeting by V.V. Uchaikin, and reports A.R. Plastino’ s related remarks.

Let us first remind, along the lines of [56] (see also [37]), the $q = 1$ case. Let us assume that we have a continuous probability distribution $p(x)$ (with $\int dx \; p(x) = 1$, $x$ being a one-dimensional real variable) and, using the entropic functional $S_1[p(x)] = - \int dx \; p(x) \ln p(x)$, we wish to consider its discretization. In other words, we consider $p_i = p(x_i)\Delta$, where $\Delta$ represents a graining for $x$, and $i = 1, 2, ..., W$. It follows that, in the limit $\Delta \to 0,$
\[ S_1[p(x)] \sim -\sum_{i=1}^W p(x_i) \ln \frac{p(x_i)}{\Delta} = -\sum_{i=1}^W p(x_i) \ln p(x_i) + \ln \Delta, \text{ hence,} \]

\[ S_1[p(x)] \sim S_1(\{p(x_i)\}) + \ln \Delta. \tag{13} \]

We see that \( \Delta \to 0 \) leads to a divergence. This divergence is not surprising and corresponds to the fact that it is necessary an infinite number of yes/no answers to resolve the uncertainty associated with a continuous distribution. In practice, it is chosen \( \Delta = 1 \), therefore the continuous and discrete versions of the entropy provide the same result. The choice \( \Delta = 1 \) corresponds, when we consider the passage from quantum to classical statistical mechanics, to the measure of phase space in units of \( \hbar \) for every pair of conjugate mechanical variables.

Let us now address this point for arbitrary \( q \). Using the entropic functional \( S_q[p(x)] \equiv \frac{1-\int dx [p(x)]^q}{q-1} \) we straightforwardly obtain

\[ S_q[p(x)] \sim \Delta^{1-q} S_q(\{p(x_i)\}) + \ln_q \Delta, \tag{14} \]

where we have used Eq. (1). Of course, this equation recovers Eq. (13) for \( q = 1 \). We see that, on top of an extra additive term, like in the \( q = 1 \) case, we have here an extra multiplicative term. Nevertheless, like in the \( q = 1 \) case, the choice \( \Delta = 1 \) makes the continuous and discrete versions of the nonextensive entropy to coincide.
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[8] If we reintroduce in the formalism the Boltzmann constant $k_B$ with its nominal value in standard units, at the place of $(1-q)/T_q'$ appears $(1-q)/k_BT_q'$ (or, more precisely, $(1-q)/kT_q'$ with $k = f(q)k_B$, where $f(q)$ is some dimensionless function such that $f(1) = 1$). Therefore, the confluence onto BG statistics emerges for asymptotically large values of $(1-q)/k_BT_q'$, or equivalently in the limit $1/k_B \to 0$. This limit consistently re-emerges in the pseudoadditivity property of the entropic form $S_q$, which in usual units reads $S_q(A + B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)/k_B$. Once again, in the limit $1/k_B \to 0$, the BG entropy additivity is satisfied. All this makes one to think about some analogies concerning mechanics. Indeed, in the limits $h \to 0$, $1/c \to 0$ and
$G \to 0$, Newtonian mechanics emerges from quantum mechanics, special and general relativity. From this standpoint, we certainly agree with Cohen-Tannoudji’s understanding \[9\] that the basic universal constants of the physical world (or at least of our contemporary understanding of it) are four in number ($c$, $h$, $G$ and $k_B$, referred to as Einstein, Planck, Newton and Boltzmann constants respectively), and not only three ($c$, $h$ and $G$) as sometimes claimed. The subtle interplay of the universal constants can be illustrated, for instance, on Bose-Einstein and Fermi-Dirac statistics: They both converge onto Maxwell-Boltzmann statistics in the limit $h/k_B \to 0$, which can be indistinctively realized through $h \to 0$ or $1/k_B \to 0$. Along related lines, see also \[10\].

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TABLES

ENTROPIC FORM AND EQUILIBRIUM STATISTICS: FOUNDATIONS

| Distribution of velocities at equilibrium | BG | $q \neq 1$ |
|------------------------------------------|----|------------|
| Maxwell 1860                             | R.S. Mendes and C. Tsallis |
|                                          | Phys Lett A 285, 273 (2001) |
| Boltzmann 1872                           | J.A.S. Lima, R. Silva and A.R. Plastino |
|                                          | Phys Rev Lett 86, 2938 (2001) |
|                                          | G. Kaniadakis |
|                                          | Physica A 296, 405 (2001) |

| Molecular chaos hypothesis (Stosszahlansatz) | Optimization of entropy with constraints |
|---------------------------------------------|------------------------------------------|
| Boltzmann 1872                              | C. Tsallis |
|                                             | J Stat Phys 52, 479 (1988) |
|                                             | E.M.F. Curado and C. Tsallis |
|                                             | J Phys A 24, L69 (1991) |
|                                             | C. Tsallis, R.S. Mendes and A.R. Plastino |
|                                             | Physica A 261, 534 (1998) |

| Steepest descent                            | S. Abe and A.K. Rajagopal |
|---------------------------------------------|----------------------------|
| Darwin-Fowler 1922                          | J Phys A 33, 8733 (2000) |

| Conditions of uniqueness of the entropy     | S. Abe and A.K. Rajagopal |
|---------------------------------------------|----------------------------|
| Shannon 1948                                | J Math Phys 38, 4104 (1997) |

| Law of large numbers                        | S. Abe and A.K. Rajagopal |
|---------------------------------------------|----------------------------|
| Khinchin 1949                               | Europhys Lett 52, 610 (2000) |

| Compact conditions of uniqueness of the entropy | S. Abe |
|-----------------------------------------------|-------|
| Khinchin 1953                                 | Phys Lett A 271, 74 (2000) |

| Counting in the microcanonical ensemble | S. Abe and A.K. Rajagopal |
|----------------------------------------|----------------------------|
| Balian-Balazs 1987                     | Phys Lett A 272, 341 (2000) |
| Kubo et al 1988                         | Europhys Lett 55, 6 (2001) |

TABLE I. Historical steps of the foundations of BG statistical mechanics (both equilibrium distribution and entropic form), and their $q \neq 1$ counterparts. Gibbs 1902 and Kubo et al 1988 refer not to the dates when the original works were essentially done, but rather to the books where they are reproduced.
FIG. 1. Traditional view (schematic) of the place of statistical mechanics for classical, quantum or relativistic dynamical systems with a finite or infinite number \( N \) of particles. Thermal equilibrium only occurs for conservative systems and is necessarily of the BG class, independently of the ordering of limits such as the \( t \to \infty \) and the \( N \to \infty \) ones. This is certainly the case of short-range interactions which present no delicate singularity at say the origin. The micro-canonical ensemble can be seen as the particular case of the canonical one when the temperature diverges; the canonical one can in turn be seen as the particular case of the grand-canonical ensemble when all chemical potentials vanish. KSLPK stands for Kolmogorov-Sinai-Lyapunov-Pesin-Krylov, thus meaning the region where, for the particular case of classical dynamical systems, the Kolmogorov-Sinai entropy is positive, the Lyapunov spectrum includes a positive branch, the Pesin identity is nontrivial, i.e. connecting nonvanishing quantities, and Krylov’s emphasis on exponential mixing being essential in the foundations of BG statistical mechanics. The Sinai-Ruelle-Bowen (SRB) distributions are stationary states which can be both at or out from thermal equilibrium; the BG equilibrium distributions are particular cases.
FIG. 2. Present view (schematical) of the place of statistical mechanics for classical, quantum or relativistic dynamical systems with a finite or infinite number \( N \) of particles. The traditional ensembles are enlarged in the sense that \( q \) can differ from unity. Distributions similar to those occurring at thermal \( q \)-equilibrium, or metaequilibrium (in the sense of metastability), can occur even for dissipative systems. Ordering of limits such as the \( t \to \infty \) and the \( N \to \infty \) ones can be very relevant. For example, for conservative many-body systems including long-range interactions, the \( \lim_{t \to \infty} \lim_{N \to \infty} \) ordering, which is the physical one, is non-Boltzmannian, whereas the \( \lim_{N \to \infty} \lim_{t \to \infty} \), physically unobservable, corresponds to BG statistics. The \( q \)-region includes the 1-region, which precisely is the KSLPK region of Fig. 1.
FIG. 3. Comparison between $q$- and stretched exponentials. (a) The circles have been calculated with a stretched exponential, and have been fitted with a $q$-exponential. (b) The circles have been calculated with a $q$-exponential, and have been fitted with a stretched exponential. The numerical discrepancies emerge only for $x << 1$ and for $x >> 1$. 

$$y_s = e^{-x^{3/4}}$$

$$y_q = A / [1+(q-1)\beta x]^{1/(q-1)}$$

$A = 0.8544$

$\beta = 0.8520$

$q = 1.1627$

$$y_q = 1 / [1 + 0.2x]^{5}$$

$y_s = B e^{-\gamma x}$

$B = 1.2243$

$\gamma = 1.1638$

$\alpha = 0.70196$
FIG. 4. Time evolution of the sensitivity to the initial conditions $\xi$. The crossover occurring in the limit $\lambda_1 \to 0$ becomes apparent: the smaller $\lambda_1$ is compared with $\lambda_{q_{\text{mix}}}$, the larger is the domain of validity of the power law $\xi \propto t^{1/(1-q_{\text{mix}})}$. The crossover time $t^*$ does not depend on $\lambda_1$; the crossover time $t^{**}$ depends on $\lambda_1$ and diverges when $\lambda_1$ vanishes.
FIG. 5. Values of $q_{eq}$ obtained by Bediaga et al through the analysis of distributions of hadronic transverse momenta in electron-positron experiments. We have chosen the abscissa in such a way as to produce a linear form. $R^2$ is the square linear correlation factor.
FIG. 6. Values of $q_{eq}$ obtained by Beck et al through the analysis of distributions of velocity differences in Couette-Taylor experiments. We have scaled $\ln(r/\eta)$ with $(\ln Re)^{7/4}$ in order to produce data collapse for different values of $Re$. Also, we have chosen the exponent 0.37 in such a way as to produce a linear form for intermediate distances. The error bar on this exponent is of the order of 0.1.
FIG. 7. Values of $q_{eq}$ obtained by Soares et al through the analysis of distributions emerging in the analysis of saddle point dynamics of the Henon-Heiles system. We have chosen the abscissa in such a way as to produce a linear form. $R^2$ is the square linear correlation factor. For the value $\lambda = 1/3$ see [44].