NON-SIGNALLING BOXES AND BOHRIFICATION

JAN GUTT AND MAREK KUŚ

1. Introduction

1.1. The premise of this note is the following observation: the formalism of Bohrification, as developed by Heunen et al. [7], is a natural place for the interpretation of general non-signalling theories. The former, i.e. Bohrification, is an embodiment of Bohr’s idea that the account of all evidence concerning quantum phenomena, despite their non-classical character, must be expressed in classical terms [2]. The latter, i.e. non-signalling theories, are extensions of the notion of a probability theory of a physical system, allowing correlations forbidden in a quantum mechanical model (cf. [4]). We show that the condition of non-signalling allows one to present these extended probability theories in the same terms as ‘Bohrified’ quantum systems.

In standard quantum theory [1] the logic of a system described by a Hilbert space $\mathcal{H}$ is represented by the orthomodular lattice of closed subspaces in $\mathcal{H}$. The involution sending a subspace to its orthogonal complement represents logical negation, satisfying the law of excluded middle: measuring the spin of an electron will yield either ‘up’ or ‘down’, tertium non datur. On the other hand, the lattice is non-distributive: $x$-spin up does not imply $x$-spin up and $z$-spin up or $x$-spin up and $z$-spin down (incompatibility of the two measurements is reflected by non-distributivity of the sub-lattice they ‘generate’, just as by non-commutativity of the corresponding sub-algebra of operators). Having the lattice stand for the logic of the system, one derives its probability theory where states assign ‘probabilities’ to elements of the lattice, respecting the underlying structure (order and complementation). These states turn out to coincide with the usual density matrices by a celebrated theorem of Gleason (as long as $\dim \mathcal{H} \geq 3$, [5]).

One of the original purposes of the Bohrification programme has been to find an alternative logical foundation for the orthodox quantum theory, replacing the non-distributive orthomodular lattices with the distributive logic of a point-free space. From the classical perspective, distributivity comes at the price of the absence of the law of excluded middle; from the intuitionistic (or constructive) viewpoint this is however a feature rather than a flaw, and a characteristic of the true logic of physical observation (see [11]).

On the other hand, non-signalling theories embodied in so-called ‘box worlds’ are an attempt at extending quantum mechanics as a theory of probability, reaching beyond the well-known bounds imposed on correlations by the orthodox theory. We shall see that not only are such hypothetical ‘box worlds’ amenable to the process of Bohrification, but furthermore they acquire a natural logical structure, compatible with their probabilistic content. In particular, probability valuations on the logic thus associated with a box world are naturally identified with the standard ‘non-signalling box states’ (Theorem [11]). Moreover, one finds that a more general class
of non-signalling theories may be identified with their ‘Bohrified’ models, resulting in a common representation encompassing box worlds, orthodox quantum systems and a potentially interesting in-between (cf. [6]).

1.2. Bohrification. In certain sense, the goal of Bohrification is to find a well-behaved phase space for a quantum system. This is achieved by constructing its logical avatar: a frame, i.e. a complete distributive lattice where finite meets distribute over arbitrary joins. The lattice of open subsets of a given topological space provides an example of a frame, and in a leap of abstraction one may view any frame as a ‘pointless topology’ – i.e., a virtual space examined only through its collection of open subsets and the lattice operations on them. This does seem to correspond to the way a physicist observes the phase space of a system – and thus to the actual logic of such observations. To emphasise this interpretation, one defines the category of locales as the opposite of the category of frames (recall that the topology functor from topological spaces to frames is contravariant). The reference to open subspaces reflects the intuitionism of the logic: negation corresponds to taking the interior of the complement, whence the disjunction of a proposition and its negation need not be true.

However, we have not yet revealed a crucial technical aspect. Actually, the new intuitionistic logic of the quantum system is realised not as a frame (or any partially ordered set), but rather as a frame object in a suitable topos, intrinsically associated with the system under consideration. Thus, the meta-logic describing the logic of the system is the internal logic of a topos. For instance, probability valuations on the frame object are viewed as morphisms into a real numbers object (however, since the internal logic is intuitionistic, different constructions of the real numbers that would classically yield the same set, may lead to non-isomorphic objects when interpreted internally: e.g., one-sided Dedekind cuts are in general not equivalent to two-sided ones). Dually, the frame object is seen as an internal locale: the ‘phase space object’. Since the topos arising in Bohrification is simply that of sheaves on some base locale, one may represent the phase space externally by a locale (now in the category of sets) over the base locale – even more tangibly, these external locales are in fact topological spaces.

The original construction of Heunen et al. [7] is set in the framework of $C^*$-algebras, applicable to field-theoretic or statistical-mechanical systems with infinitely many degrees of freedom. Recovering the ‘correct’ topological structure of the phase space is then a delicate matter. Given a system described by a $C^*$-algebra $\mathcal{A}$, the authors consider the set $\mathcal{C}$ of abelian $C^*$-subalgebras of $\mathcal{A}$, partially ordered by inclusion. Viewing $\mathcal{C}$ as a category (with arrows expressing the order relation), we have a tautological functor $\mathcal{A}$ from $\mathcal{C}$ to commutative $C^*$-algebras, sending $A \in \mathcal{C}$ to $A$ itself. We may then view $\mathcal{A}$ as an object of the presheaf topos $\text{Set}^{\mathcal{C}}$ carrying the structure of an internal commutative $C^*$-algebra (the presheaf topos has functors $\mathcal{C} \to \text{Set}$ as objects, and natural transformations as morphisms; equivalently, its objects are $\mathcal{C}$-shaped diagrams of sets). Now, an internal Gelfand duality is invoked to produce an internal locale $X$ such that $C(X, \mathcal{C}) \cong \mathcal{A}$ (where $\mathcal{C}$ is a suitably defined internal locale of complex numbers, and $C(\cdot, \cdot)$ is the object of continuous maps between a pair of internal locales). This $X$ is the internal phase space of the system, and the corresponding internal frame is its logic.

For a finite-level system described by a finite-dimensional Hilbert space $\mathcal{H}$ the construction of the internal frame is much more direct (see [3]). Indeed, setting
\( \mathcal{A} = B(\mathcal{H}) \), i.e. linear operators on \( \mathcal{H} \), and proceeding as before, one may consider the internal Boolean algebra \( P(\mathcal{A}) \) of projections: it is the functor sending \( A \in \mathcal{C} \) to the Boolean algebra \( P(A) \) of projections in \( A \). This is not a frame yet, as \( P(\mathcal{A}) \) is not complete (even though its values are finite Boolean algebras, thus complete, \( P(\mathcal{A}) \) itself is only so-called \( K \)-finite, which does not suffice to establish completeness internally). One thus applies the operation of internal ideal completion, obtaining a frame \( \text{Idl} P(\mathcal{A}) \supset P(\mathcal{A}) \) defined by a suitable universal property.

The internal phase space is the corresponding locale (in fact, the passage from the internal Boolean algebra \( P(\mathcal{A}) \) to the latter locale is an internal version of the Stone spectrum). An important result, relying on Gleason’s theorem \[5\], is that internal probability valuations on \( \text{Idl} P(\mathcal{A}) \) are in a natural one-to-one correspondence with density matrices on \( \mathcal{H} \) for \( \text{dim} \mathcal{H} \geq 3 \).

Of course, the above description may be restated without any reference to \( B(\mathcal{H}) \). Namely, one lets \( \mathcal{C} \) be set of orthogonal decompositions of \( \mathcal{H} \), partially ordered by refinement, and then considers the functor \( L : \mathcal{C} \to \text{Bool} \) sending a given orthogonal decomposition \( \mathcal{H} = \bigoplus_{i \in I} V_i \) to \( 2^I \); a refinement of \( (V_i)_{i \in I} \) to \( (V_j)_{j \in J} \) is sent to the homomorphism \( 2^I \to 2^J \) induced by the inclusion map \( J \to I \). Now, \( L \) is isomorphic to the previous Boolean algebra \( P(\mathcal{A}) \) and its internal ideal completion \( \text{Idl} L \) corresponds to the internal phase space. The key insight we take from this discussion is: (1) \( \mathcal{C} \) is the set of compatible measurement contexts, partially ordered by refinement of available information; (2) the Boolean algebras describing the logic of measurements in each context give rise to an internal Boolean algebra in \( \text{Set} \); (3) the internal ideal completion of the latter corresponds to the internal phase space of the general system subject to our measurement arrangement; (4) the states of the system are identified with internal probability valuations.

1.3. Box worlds. Shifting the view away from logic, one recognizes the probabilistic essence of quantum mechanics as a theory prescribing, for a given collection of admissible measurements, the convex set of states assigning a ‘probability’ to each measurement and each outcome. A partial ‘conjunction’ allows for performing several compatible measurements simultaneously, and combining their outcomes. The classical scenario illustrating these notions involves two parties in causally separated laboratories, each equipped with a pair of binary measurements to be performed on a shared distributed system. In course of the experiment, each party subjects the system to one of the measurements at its disposal, and records the outcome. Assuming the initial state of the system may be consistently reproduced, the parties repeat the experiment and eventually communicate to assemble a list of ‘empirical probabilities’ for each combined outcome. The assumption that the parties be causally separated guarantees that the measurement chosen by the first party is always compatible with that chosen by the second party.

The non-local correlations between the results of measurements in the two laboratories distinguish a non-classical theory (more precisely, non-classical states) from a classical one. One expression detecting non-locality forms the CHSH inequality:

\[
\langle a_1 b_1 \rangle + \langle a_1 b_2 \rangle + \langle a_2 b_1 \rangle - \langle a_2 b_2 \rangle \leq 2.
\]

Here we encode the binary measurements in terms of \( \pm 1 \)-valued observables: \( a_1, a_2 \) in the first laboratory, \( b_1, b_2 \) in the second one; \( \langle \cdot \rangle \) denotes the expectation in a
given state $P$:

$$\langle a_i b_j \rangle = \sum_{\alpha,\beta = \pm 1} P(a_i = \alpha \land b_j = \beta)\alpha\beta.$$ 

If the inequality is satisfied, the state is consistent with a classical theory; otherwise, it describes a genuinely non-classical situation. Interestingly, the states described by standard quantum mechanics do not saturate the obvious algebraic upper bound for the CHSH expression, i.e. 4: instead, they satisfy an upper bound of $2\sqrt{2}$ \[4\].

One may ponder whether this is indeed the bound chosen by nature. Seeking an axiomatics for general ‘probabilities’ of the form $P(a_i = \alpha \land b_j = \beta)$, one would certainly like to ensure that the one-party expectation $\langle a_i \rangle$, or the probability $P(a_i = \alpha)$, may be consistently inferred as a marginal. That is, that

$$P(a_i = \alpha \land b_1 = 1) + P(a_i = \alpha \land b_1 = -1) = P(a_i = \alpha \land b_2 = 1) + P(a_i = \alpha \land b_2 = -1)$$

for all $i = 1, 2$ and $\alpha = \pm 1$: a property referred to as non-signalling (for otherwise the first laboratory would obtain information on the choice of a measurement at the second laboratory, violating causality). Popescu and Rohrlich \[10\] provided the first example of a non-signalling state $P$ for which the CHSH expression achieves its algebraic maximum of 4. Their construction, referred to as the PR box, has developed into the study of ‘non-signalling boxes’ or ‘box worlds’, modelling super-quantum correlations.

1.4. Let us attempt a preliminary ‘Bohrification’ of the above box-world. Following the ‘key insight’ of \[12\] we first identify the partially ordered set $C$ of compatible measurement contexts. As indicated in the description of the measurement protocol, each party selects one measurement out of two, whence we obtain four contexts represented by subsets

$$\{a_1, b_1\}, \{a_1, b_2\}, \{a_2, b_1\}, \{a_2, b_2\} \subset \{a_1, a_2, b_1, b_2\}.$$ 

Furthermore, since we have assumed that expectations of the form $\langle a_i \rangle, \langle b_j \rangle$ may be consistently obtained as marginals, we may add another four contexts

$$\{a_1\}, \{a_2\}, \{b_1\}, \{b_2\} \subset \{a_1, a_2, b_1, b_2\}.$$ 

These might be alternatively interpreted as corresponding to a situation where either (1) only one party performs a measurement, or (2) both parties perform measurements, but one of them discards the result. Finally, we may for completeness add the empty context $\emptyset$, corresponding to either no measurements being taken, or all results being discarded. It follows that the partially ordered set $C$ of measurement contexts is a subset of the power-set $2^{\{a_1, a_2, b_1, b_2\}}$ consisting of subsets whose intersection with $\{a_1, a_2\}$ and $\{b_1, b_2\}$ is of cardinality at most one. In particular, viewed as a category, $C$ is a sub-category of Set.

We now need to construct a functor assigning to each context $S \in C$ a Boolean algebra representing the measurement logic in that context. Since $S$ is a set of compatible binary measurements, it is immediate that the set of possible outcomes if $2^S$. The logic of measurements is the power-set of the set of outcomes, in our case $2^S$. Equivalently, it is simply the free Boolean algebra on $S$. Hence, the desired functor $L : C \to \text{Bool}$ is simply a restriction of the free functor $Set \to \text{Bool}$ to $C \subseteq Set$ (in particular, for $S \leq S'$ in $C$ we obtain a homomorphism $L(S \to S') : LS \to LS'$ of measurement logics, induced by the restriction map $2^S \to 2^{S'}$ on
measurement outcomes). Viewed as an object of $\mathbf{Set}^{\mathcal{C}}$, the functor $L$ is an internal Boolean algebra.

We may also describe the internal frame $\text{Idl} \, L$ corresponding to the internal phase space. For now, it will be enough to look at its global sections, or alternatively at its value at $\emptyset \in \mathcal{C}$. This is the set of sub-functors $I \subseteq L$ such that for each $S \in \mathcal{C}$, the subset $I(S) \subseteq L(S)$ is an order ideal. Since each $L(S)$ is finite, $I(S)$ is necessarily principal, i.e. a lower set $\downarrow x$ for some $x \in L(S)$. Thus, we may identify $(\text{Idl} \, L)(\emptyset)$ with the set of maps $\xi : \mathcal{C} \to \coprod_{S \in \mathcal{C}} L(S)$ such that $\xi(S) \subseteq L(S)$ for all $S \in \mathcal{C}$ and $L(S \to S')\xi(S) \subseteq \xi(S')$ whenever $S \subseteq S'$ in $\mathcal{C}$. Alternatively, $(\text{Idl} \, L)(\emptyset)$ is the set of all upper subsets of $X = \coprod_{S \in \mathcal{C}} 2^S$. Here $X$ is viewed as the set of measurement outcomes, fibred over the set of measurement contexts $\mathcal{C}$, and partially ordered by refinement ($f' : S' \to 2$ refines $f : S \to 2$ if $S \subseteq S'$ and $f'|S = f$). We may thus finally say that $(\text{Idl} \, L)(\emptyset)$ is the frame $\mathcal{O}(X)$ of open subsets in $X$ equipped with the Alexandrov topology. Accordingly, $X$ is the external phase space of our box world (this description of the external phase space is taken from Heunen et al., [7]).

2. Categorical preliminaries

2.1. The logical structure of $L$, $\text{Idl} \, L$ and lastly $X$ may be interesting, but it may only be justified by its compatibility with the probabilistic content of the box world. We will show in the next section that probability valuations on $\text{Idl} \, L$ (or on $L$) are indeed in one-to-one correspondence with non-signalling box-world states. For that purpose, we will need to make these notions more precise.

2.2. Recall that a category $\mathcal{E}$ (for convenience assumed to be locally small) is a topos if it has finite limits and colimits, exponential objects, and a sub-object classifier. Note that the existence of finite limits and colimits implies the existence of a terminal object (denoted $1_\mathcal{E}$) and initial object (denoted $0_\mathcal{E}$). The existence of exponential objects means that for each object $X$ of $\mathcal{E}$, the functor $X \times - : \mathcal{E} \to \mathcal{E}$ has a right adjoint $-X : \mathcal{E} \to \mathcal{E}$. A sub-object classifier is an object $\Omega$ representing the functor $\text{Sub} : \mathcal{E}^{\text{op}} \to \mathbf{Set}$ of sub-objects. That is, for each object $X$ of $\mathcal{E}$ there is a natural bijection between $\text{Hom}_\mathcal{E}(X, \Omega)$ and the set of equivalence classes of monomorphisms into $X$. The sub-object classifier is unique up to unique isomorphism.

The basic example of a topos is of course the category $\mathbf{Set}$ of sets, with the two-point set $2$ as a sub-object classifier. The topoi that arise in the context of Bohrification are still of a very simple kind. Namely, given a partially ordered set $\mathcal{C}$, we identify it with a category (its objects are the elements of $\mathcal{C}$, and its hom-sets are either empty or singletons, reflecting the order relation). Then, we form the Kripke topos $\mathbf{Set}^{\mathcal{C}}$: its objects are functors $\mathcal{C} \to \mathbf{Set}$, and its morphisms are natural transformations (equivalently, the objects may be viewed as $\mathcal{C}$-shaped diagrams of sets). One generically interprets $\mathcal{C}$ as a set of ‘contexts’ ordered by ‘refinement’; then, an object of the corresponding Kripke topos is a ‘context-dependent set’, transforming as the context is being refined. Limits and colimits in $\mathbf{Set}^{\mathcal{C}}$ are computed ‘point-wise’, i.e. independently for each context. The terminal (resp. initial) object is a functor sending each context to a singleton (resp. empty set). A sub-object of a functor $F : \mathcal{C} \to \mathbf{Set}$ is just a sub-functor $F'$, i.e. a subset $F'(c) \subseteq F(c)$ for each $c \in \mathcal{C}$ such that $F'(c)$ maps to $F'(c')$ whenever $c \leq c'$ in $\mathcal{C}$. It is then easy to see that a sub-object classifier is given by the functor $\Omega : \mathcal{C} \to \mathbf{Set}$ sending $c \in \mathcal{C}$ to the set of upper subsets contained in $\uparrow c$. 
2.3. Given a pair of topoi $\mathcal{E}$, $\mathcal{E}'$, a geometric morphism $f : \mathcal{E} \to \mathcal{E}'$ is a pair of functors $f_*, f^* : \mathcal{E} \leftrightarrow \mathcal{E}' : f^*$ such that $f^*$ is left adjoint to $f_*$ and preserves finite limits (the notation is of course reminiscent of the direct and inverse image functors for sheaves). Such $f$ is furthermore essential if $f^*$ possesses a further left adjoint, denoted $f_!$. Every topos admits at most a unique (up to equivalence) geometric morphism to $\textbf{Set}$.

Geometric morphisms of Kripke topos arise from isotone maps on the underlying posets. Indeed, given an isotone map $f : \mathcal{E} \to \mathcal{E}'$ (i.e. a functor when posets are viewed as categories), we define a geometric morphism $f : \textbf{Set}^{\mathcal{E}} \to \textbf{Set}^{\mathcal{E}'}$ where, for functors $F : \mathcal{E} \to \textbf{Set}$ and $F' : \mathcal{E}' \to \textbf{Set}$, we have $f^* F' = F' \circ f$ and

$$ (f_! F)(c') = \lim_{f(c)=c'} F(c) $$

for all $c' \in \mathcal{E}'$ (the action of $f_* F$ on arrows of $\mathcal{E}$ is easy to deduce). In fact, $f$ is essential with

$$ (f_! F)(c') = \mathrm{colim}_{f(c)=c'} F(c). $$

In particular, identifying $\textbf{Set}$ with the Kripke topos over the one-element poset $\ast$, we always have the essential geometric morphism $f : \textbf{Set}^{\mathcal{E}} \to \textbf{Set}$ induced by the unique map $f : \mathcal{E} \to \ast$. Its components $f_*, f_! : \textbf{Set}^{\mathcal{E}} \to \textbf{Set}$ compute, respectively, limits and colimits of functors, while $f^* : \textbf{Set} \to \textbf{Set}^{\mathcal{E}}$ gives the constant functor on a given set.

2.4. A partially ordered object in $\mathcal{E}$ is a pair $(X, \leq_X)$, where $X$ is an object together with a sub-object $(\leq_X) \subset X \times X$ such that (1) the intersection of $(\leq_X)$ with its ‘transpose’ is the diagonal $\Delta_X \subset X \times X$, and (2) $p_{13} : (\leq_X) \times X (\leq_X) \to X \times X$ factors through $(\leq_X)$, where the fibre product is with respect to the right projection from the left factor and left projection from the right factor. These two conditions are equivalent to requiring that the representable functor $\text{Hom}(-, X) : \mathcal{E}^{\mathrm{op}} \to \textbf{Set}$ factor through the category $\textbf{Ord}$ of partially ordered sets, where the order relation on $\text{Hom}(Y, X)$ is the preimage of $\text{Hom}(Y, \leq_X)$ under the map $\text{Hom}(Y, X) \times \text{Hom}(Y, X) \to \text{Hom}(Y, X \times X)$. A morphism between $X \to Y$ between partially ordered objects is isotone if it maps $\leq_X$ into $\leq_Y$. We let $\textbf{Ord}_{\mathcal{E}}$ denote the category of partially ordered objects in $\mathcal{E}$ and isotone morphisms. The power-object functor $\Omega^- : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ factors naturally through $\textbf{Ord}_{\mathcal{E}}$, where the ordering on $\Omega^X$ is given by factorisation, i.e. inclusion, of sub-objects of $X$. In particular, $\Omega$ itself is naturally a partially ordered object. In a Kripke topos $\textbf{Set}^{\mathcal{E}}$, partially ordered objects are simply functors $\mathcal{E} \to \textbf{Ord}$.

The object $(X, \leq_X)$ of $\textbf{Ord}_{\mathcal{E}}$ is an internal distributive lattice if the representable functor $\text{Hom}(-, X) : \mathcal{E}^{\mathrm{op}} \to \textbf{Ord}$ factors through the forgetful functor from the category $\mathbf{dLat}$ of bounded distributive lattices. If that is the case, there exist morphisms $\wedge_X, \vee_X : X \times X \to X$ such that for each $Y$, the induced maps $\text{Hom}(Y, X) \times \text{Hom}(Y, X) \to \text{Hom}(Y, X)$ are the meet and join in $\text{Hom}(Y, X)$. There are also morphisms $0_X, 1_X : 1_{\mathcal{E}} \to X$ such that for each $Y$, the induced elements in $\text{Hom}(Y, X)$ are the bottom and top. An isotone morphism $\phi : X \to Y$ between distributive lattices is a distributive lattice homomorphism if $\vee_Y \circ (\phi \times \phi) = \phi \circ \wedge_X$ and likewise for $\land$, and if $\phi(0_X) = 0_Y$ and likewise for 1 (equivalently if $\text{Hom}(Z, \phi)$ is a distributive lattice homomorphism for each $Z$). The category of internal distributive lattices and their homomorphisms in $\mathcal{E}$ is denoted $\mathbf{dLat}_{\mathcal{E}}$. It is a sub-category of $\textbf{Ord}_{\mathcal{E}}$. Again, the power-object functor factors naturally through $\mathbf{dLat}_{\mathcal{E}}$ and
in particular $\Omega$ is itself an internal distributive lattice. In a Kripke topos $\text{Set}^\mathcal{E}$, internal distributive lattices are simply functors $\mathcal{E} \to \text{dLat}$.

Finally, $(X, \leq_X)$ is an internal Boolean algebra if $\text{Hom}(-, X) : \mathcal{E}^{\text{op}} \to \text{Ord}$ factors through the category $\text{Bool}$ of Boolean algebras. That is, if it is an internal distributive lattice together with an antitone negation/complementation morphism $\neg_X : X \to X$ such that $\forall_X \circ (\text{id}_X \times \neg_X) \circ \Delta_X$ factors through $1_X$, and an analogous composite with $\wedge_X$ factors through $0_X$. Internal Boolean algebras form a full subcategory $\text{Bool}_\mathcal{E}$ of $\text{dLat}_\mathcal{E}$. In a Kripke topos $\text{Set}^\mathcal{E}$ these are simply functors $\mathcal{E} \to \text{Bool}$.

2.5. As completeness is a higher-order notion, the definition of internal frames (as well as internal dcpo) is somewhat more involved. We use monadicity to keep the discussion explicit. Recall that a monad on a category $\mathcal{K}$ is a triple $T = (T, \mu, \eta)$ where $T$ is an endofunctor of $\mathcal{K}$ together with natural transformations $\mu : T^2 \to T$ (‘multiplication’), $\eta : \text{id} \to T$ (‘unit’) such that $\mu \circ T\eta = \text{id}$ and $\mu \circ T\mu = \mu \circ \mu T$. A $T$-algebra is a pair $(X, \xi)$ where $X$ is an object, and $\xi : T^X \to X$ a morphism (‘structure map’) such that $\xi \circ T\xi = \text{id}_X$ and $\xi \circ \mu = \xi \circ T\xi$. A homomorphism of $T$-algebras $(X, \xi) \to (Y, \eta)$ is a morphism $\phi : X \to Y$ such that $\eta \circ T\phi = \phi \circ \xi$. The category of $T$-algebras and homomorphisms is the Eilenberg-Moore category of $\mathcal{K}$, denoted $\mathcal{K}^T$. Note that for any $X$ in $\mathcal{K}$, there is a natural $T$-algebra structure on $TX$ given by $\eta_X$. This is the ‘free’ $T$-algebra ‘generated’ by $X$: indeed, the functor $\mathcal{K} \to \mathcal{K}^T$ sending $X$ to $(TX, \mu_X)$ is left adjoint to the obvious forgetful functor. Finally, given a monad $T$ on $\mathcal{K}$ and a monad $T'$ on another category $\mathcal{K}'$, a morphism of monads from $T$ to $T'$ is a functor $\Phi : \mathcal{K} \to \mathcal{K}'$ together with a natural transformation $\phi : T'\Phi \to T$ compatible with the monad structures in a rather obvious sense. The pair $(\Phi, \phi)$ induces a functor $\mathcal{K}^T \to \mathcal{K}'^{T'}$ of Eilenberg-Moore algebras, where given a $T$-algebra $(X, \xi)$ in $\mathcal{K}$, the object $\Phi X$ becomes a $T'$-algebra via the composite $\Phi\xi \circ \phi_X : T'\Phi X \to \Phi X$.

Consider the internal ideal completion functor $\text{Idl} : \text{Ord}_\mathcal{E} \to \text{Ord}_\mathcal{E}$. Its action on objects sends $X$ to the sub-object of $\Omega^X$ parameterising order ideals: i.e., a morphism $\phi : T \to \Omega^X$ factors through $\text{Idl} X$ if the corresponding sub-object $\Phi \subset T \times X$ satisfies:

1. (lower) $\text{id}_T \times \text{pr}_1 : (\leq_X) \times_X \Phi \to T \times X$ ( fibre product using $\text{pr}_2$) factors through $\Phi$.

2. (directed) for any $\alpha : T' \to T$ and $x_1, x_2 : T' \to \Phi$ over $T$, there is an epi $\beta : T'' \to T'$ and $x : T'' \to \Phi$ such that $\beta^* x_1, \beta^* x_2 \leq x$.

The partial order on $\text{Idl} X$ is induced from $\Omega^X$. The action on morphisms sends an isotope $\phi : X \to Y$ to the restriction of the direct image morphism $\phi_* : \Omega^X \to \Omega^Y$ to $\text{Idl} X \to \text{Idl} Y$. There is a natural transformation $\eta : \text{id} \to \text{Idl}$ such that $\eta_X : X \to \text{Idl} X \subset \Omega^X$ is adjoint to the characteristic morphism $X \times X \to \Omega$ classifying the transpose of $(\leq_X)$. (For $\mathcal{E} = \text{Set}$, this map sends $x \in X$ to the principal ideal $\downarrow x$.) We also have a natural transformation $\mu : \text{Idl}^2 \to \text{Idl}$ such that $\mu_X : \text{Idl}^2 X \to \text{Idl} X$ is a restriction of the ‘union’ morphism $\Omega^{\Omega^X} \to \Omega^X$.

One checks that (1) $\text{Idl} = (\text{Idl}, \mu, \eta)$ is a monad on $\text{Ord}_\mathcal{E}$, (2) if a partially ordered object $X$ is an internal distributive lattice, then so is $\text{Idl} X$, and furthermore $\mu_X$ and $\eta_X$ are homomorphisms of distributive lattices, so that $(\text{Idl}, \mu, \eta)$ defines a monad $\text{Idl}^\mathcal{E}$ on $\text{dLat}_\mathcal{E}$. Their Eilenberg-Moore categories are, respectively, the internal dcpo and internal frames: $\text{Ord}^\mathcal{E} = \text{Dcpo}_\mathcal{E}$ and $\text{dLat}^\mathcal{E} = \text{Frm}_\mathcal{E}$. (For
2.6. Let \( \mathcal{C} = \text{Set} \), these are the usual categories of dcpo, resp. frames, with Scott-continuous maps as morphisms.

In a Kripke topos \( \text{Set}^\mathcal{E} \), the internal ideal completion \( \text{Idl} \mathcal{S} \) of a functor \( \mathcal{S} : \mathcal{E} \to \text{Ord} \) sends \( c \in \mathcal{E} \) to the poset of sub-functors \( J \subset \mathcal{S}|_{\mathcal{C}} \) such that \( J(c') \) is an order ideal in \( \mathcal{S}(c') \) for all \( c' \geq c \) in \( \mathcal{E} \). In particular, for each \( c \in \mathcal{E} \) and each \( s \in \mathcal{S}(c) \) we may consider the sub-functor \( \downarrow s \subset \mathcal{S} \) sending \( c' \in \mathcal{E} \) to \( \downarrow \mathcal{S}(c \to c') s \) if \( c' \geq c \) and to \( \{0\} \) otherwise. This yields an isotone map \( \mathcal{S}(c) \to f_* \text{Idl} \mathcal{S} \) where \( f : \text{Set}^\mathcal{E} \to \text{Set} \) is the usual geometric morphism.

Let \( f : \mathcal{E} \to \mathcal{E}' \) be a geometric morphism. We lift both \( f^* \) and \( f_* \) to an adjoint pair of functors \( f_* : \text{Ord}_\mathcal{E} \Rightarrow \text{Ord}_{\mathcal{E}'} : f^* \), using the fact that both preserve finite limits. Given \( (X, \leq_X) \) in \( \text{Ord}_\mathcal{E} \), we let \( f_* \) send it to \( (f_*X, f_* (\leq_X)) \). Given \( (X', \leq_{X'}) \) in \( \text{Ord}_{\mathcal{E}'} \), we let \( f^* \) send it to \( (f^*X, f^* (\leq_{X'})) \). Furthermore, these functors restrict to functors between the sub-categories of internal distributive lattices and internal Boolean algebras.

The situation is more complicated for internal frames and internal dcpos. Let us first observe that the canonical morphism \( \Omega^E_{\mathcal{E},X} \to f_* \Omega^E_{\mathcal{E}',X} \) induces a natural transformation \( \text{Idl}_{\mathcal{E}}, f_* \to f_* \text{Idl}_{\mathcal{E}'} \) of functors \( \text{Ord}_{\mathcal{E}} \to \text{Ord}_{\mathcal{E}'} \), inducing a morphism of monads from \( \text{Ord}_{\mathcal{E}} \) to \( \text{Ord}_{\mathcal{E}'} \). It follows that, given an \( \text{Idl}_{\mathcal{E}} \)-algebra \( X \) in \( \text{Ord}_{\mathcal{E}} \), we may turn \( f_*X \) into an \( \text{Idl}_{\mathcal{E}'} \)-algebra in \( \text{Ord}_{\mathcal{E}'} \) with a composite structure

\[
\text{Idl}_{\mathcal{E}'}, f_*X \to f_* \text{Idl}_{\mathcal{E}} X \to f_*X.
\]

An analogous construction equips the direct image of an \( \text{Idl}_{\mathcal{E}'} \)-algebra in \( \text{dLat}_{\mathcal{E}} \) with the structure of an \( \text{Idl}_{\mathcal{E}''} \)-algebra in \( \text{dLat}_{\mathcal{E}''} \). Hence, the direct image functor lifts to \( f_* : \text{Dcpo}_{\mathcal{E}} \to \text{Dcpo}_{\mathcal{E}'} \), and \( f_* : \text{Frm}_{\mathcal{E}} \to \text{Frm}_{\mathcal{E}'} \). The inverse image functor does not have this property. However, there does exist a left adjoint to the direct image functor on dcpos, resp. frames, denoted \( f^! : \text{Dcpo}_{\mathcal{E}'} \to \text{Dcpo}_{\mathcal{E}} \), resp. \( f^! : \text{Frm}_{\mathcal{E}'} \to \text{Frm}_{\mathcal{E}} \).

2.7. A natural numbers object in a general topos \( \mathcal{E} \) is an object \( \mathbb{N}_\mathcal{E} \) equipped with morphisms \( 0 : 1_\mathcal{E} \to \mathbb{N}_\mathcal{E} \) (‘zero’) and \( S : \mathbb{N}_\mathcal{E} \to \mathbb{N}_\mathcal{E} \) (‘successor map’), satisfying a universal property: for every \( X \) equipped with morphisms \( x : 1_\mathcal{E} \to X \) and \( \varphi : X \to X \) there is a unique morphism \( \Phi : \mathbb{N}_\mathcal{E} \to X \) such that \( \Phi \circ 0 = x \) and \( \Phi \circ S = \varphi \circ \Phi \). The morphism \( \Phi \) describes the sequence of images of \( x \) under subsequent iterates of \( \varphi \). In particular, iterating the composition-with-\( S \) as an endomorphism of \( \mathbb{N}_\mathcal{E}^{\mathbb{N}_\mathcal{E}} \) gives rise to an addition \( \mathbb{N}_\mathcal{E} \times \mathbb{N}_\mathcal{E} \to \mathbb{N}_\mathcal{E} \). This makes \( \mathbb{N}_\mathcal{E} \) (if it exists) into a monoid object, and we may proceed to construct the integers \( \mathbb{Z}_\mathcal{E} \) as a quotient of \( \mathbb{N}_\mathcal{E} \times \mathbb{N}_\mathcal{E} \) by the diagonal action of \( \mathbb{N}_\mathcal{E} \). The sub-object \( 0 : 1_\mathcal{E} \to \mathbb{Z}_\mathcal{E} \) admits a complement \( \mathbb{Z}_\mathcal{E}^\times \); the latter is a monoid object for ‘multiplication’ and we obtain the rationals \( \mathbb{Q}_\mathcal{E} \) as a quotient of \( \mathbb{Z}_\mathcal{E} \times \mathbb{Z}_\mathcal{E}^\times \) by the diagonal action of \( \mathbb{Z}_\mathcal{E}^\times \). If \( \mathcal{E} \) admits a geometric morphism \( f : \mathcal{E} \to \text{Set} \), we have simply \( \mathbb{N}_\mathcal{E} \cong f^* \mathbb{N} \), \( \mathbb{Z}_\mathcal{E} \cong f^* \mathbb{Z} \) and \( \mathbb{Q}_\mathcal{E} \cong f^* \mathbb{Q} \). In particular, in a Kripke topos \( \text{Set}^\mathcal{E} \) the natural numbers, integers and rationals are simply the corresponding constant functors.

The construction of the reals is more involved. Observe first that the objects of natural numbers, integers and rationals are naturally partially ordered (and their inclusions are isotone). We may consider the lower reals \( \mathbb{R}_{\ell,\mathcal{E}} \) as a sub-object of \( \Omega^{\mathbb{R}_\mathcal{E}} \) such that a morphism \( \varphi : T \to \Omega^{\mathbb{R}_\mathcal{E}} \) factors through \( \mathbb{R}_{\ell,\mathcal{E}} \) if and only if the corresponding sub-object \( \Phi \subset T \times \mathbb{Q}_\mathcal{E} \) satisfies:
(1) (lower) $\text{id}_T \times \text{pr}_1 : (\leq_{Q_\mathcal{E}}) \times_{Q_\mathcal{E}} \Phi \to T \times Q_\mathcal{E}$ (fibre product using $\text{pr}_2$) factors through $\Phi$,
(2) (rounded) for each $q' : T' \to \Phi$ there is an epimorphism $\beta : T'' \to T'$ and morphism $q'' : T'' \to \Phi$ such that $\beta^* q' \leq q''$ and the equaliser of these two is $0_\mathcal{E}$,
(3) (epic) the projection $\Phi \to T$ is epic,
(4) (bounded) there is an epi $\alpha : T'' \to T$ and a morphism $b : T' \to Q_\mathcal{E}$ such that $\alpha^* \Phi \leq b$ in the obvious sense.

Without the last two conditions, we would obtain the extended lower reals. Finally, the Dedekind reals $R_{d,\mathcal{E}}$ may be constructed as a sub-object of $R_{t,\mathcal{E}} \times Q_\mathcal{E}$ parameterising pairs consisting of a lower real and a complement. More precisely, a morphism $\varphi : T \to R_{t,\mathcal{E}} \times Q_\mathcal{E}$ is a Dedekind real if its two components $\varphi_1 : T \to R_{t,\mathcal{E}} \subset Q_\mathcal{E}$ and $\varphi_2 : T \to Q_\mathcal{E}$ define complementary sub-objects of $T \times Q_\mathcal{E}$. The Dedekind reals are an honest ring object in $\mathcal{E}$; the lower reals are an additive monoid object, and carry an action of the multiplicative monoid object of positive lower reals. They are also naturally partially ordered in a compatible way.

These constructions have a simple interpretation in a Kripke topos $\text{Set}^\mathcal{E}$. The lower reals here are ‘context-dependent’ inhabited, bounded, rounded lower subsets of $\mathbb{Q}$ that ‘grow’ as the context is being refined: that is, the functor $R_{t,\mathcal{E}}$ maps $c \in \mathcal{E}$ to the set of isotone maps from $\uparrow c$ to the (usual) reals. On the other hand, the Dedekind reals are the constant functor $f^* R$ (for the geometric morphism $f : \text{Set}^\mathcal{E} \to \text{Set}$). Indeed, a Dedekind real consists of a lower real and a complement – the latter is also a ‘context-dependent’ subset that ‘grows’, so that in effect both halves of the Dedekind cut are forced to remain (locally) constant.

3. BOX PRESENTATIONS AND FRAMED TOPOI

3.1. Box worlds. A box world is an abstraction of an experiment in which a single system is shared among a number of parties. Each party has a collection of yes–no ‘questions’, i.e. binary measurements, it may subject the system to. It is assumed that during a single run of the experiment, the system had been prepared in a certain state, and each party chooses and asks at most one ‘question’, and that the order in which the parties perform their measurements is irrelevant (the latter is usually expressed by saying that the parties are causally separated). This procedure is then repeated, with the system consistently being prepared in the same state prior to observation. Each party records its choice of a question and the answer obtained, and eventually the records are compiled into a list of pairs, each consisting of a sequence of questions (‘context’) and a sequence of answers (‘outcome’). Given sufficiently many records, and a fixed context, empirical probabilities are assigned to the possible outcomes.

We formalise this situation as a map $S \to I$ of finite sets, alternatively viewed as a family $(S_i)_{i \in I}$ of fibres. The set $I$ indexes the parties, while $S_i$ is the collection of questions available to the $i$-th party. A context is then a partial section, i.e. a subset $c \subset S$ such that the projection $c \to I$ is injective. The set of contexts $\mathcal{C}_{S/I}$ is then a subset of the power-set of $S$, and thus inherits a partial order. We may also view $\mathcal{C}_{S/I}$ as a sub-category of $\text{Set}$. In particular, restricting the free Boolean algebra functor $\text{Set} \to \text{Bool}$ to $\mathcal{C}_{S/I}$ yields the functor $L_{S/I} : \mathcal{C}_{S/I} \to \text{Bool}$. 
The latter is an internal Boolean algebra in the Kripke topos
\[ \mathcal{E}_{S/I} = \text{Set}^{\mathcal{E}_{S/I}}. \]
Finally, applying the internal ideal completion viewed as a free frame functor \( \text{Idl} : \text{dLat}^{\mathcal{E}_{S/I}} \to \text{Frm}_{\mathcal{E}_{S/I}} \) yields an internal frame
\[ F_{S/I} = \text{Idl} L_{S/I}. \]

3.2. **Functoriality.** We have thus associated with \( S/I \) the pair \((\mathcal{E}_{S/I}, F_{S/I})\) of a topos and an internal frame. We shall now make it functorial.

**Definition 1.** A framed topos is a pair \((\mathcal{E}, F)\) where \( \mathcal{E} \) is a topos and \( F \) is an internal frame in \( \mathcal{E} \). A homomorphism of framed topoi from \((\mathcal{E}, F)\) to \((\mathcal{E}', F')\) is a geometric morphism \( f : \mathcal{E} \to \mathcal{E}' \) together with a homomorphism \( f^\sharp F' \to F \) of internal frames in \( \mathcal{E} \). The category of framed topoi is denoted \( \text{FrmTop} \).

**Definition 2.** The category \( \text{Box} \) of box presentations is a sub-category of the arrow category \( \text{fSet}^\rightarrow \) with maps \( S \to I \) of finite sets as objects, and commutative diagrams
\[
\begin{array}{ccc}
S' & \longrightarrow & S \\
\downarrow & & \downarrow \\
I' & \longrightarrow & I
\end{array}
\]
such that \( S'_i \to S \) is injective for all \( i' \in I' \) as morphisms.

**Proposition 1.** The assignment sending \( S/I \) to \((\mathcal{E}_{S/I}, F_{S/I})\) is the object part of a functor \( \text{Box}^\rightarrow \to \text{FrmTop} \).

**Proof.** Suppose given a morphism in \( \text{Box} \), i.e. a commutative diagram
\[
\begin{array}{ccc}
S' & \overset{\varphi}{\longrightarrow} & S \\
\downarrow & & \downarrow \\
I' & \overset{\varphi}{\longrightarrow} & I
\end{array}
\]
of finite sets, such that \( S'_i \to S \) is injective for all \( i' \in I \). We obtain an isotone map \( \varphi : \mathcal{E}_{S/I} \to \mathcal{E}_{S'/I'} \) as a restriction of the inverse image map \( 2^{S'} \to 2^S \) on power-sets. In fact, viewing \( \varphi \) as a functor between sub-categories of \( \text{Set} \), we also have a natural transformation \( \tilde{\varphi} : \varphi^* \iota' \to \iota \) between the embedding functors \( \iota : \mathcal{E}_{S/I} \to \text{Set} \) and \( \iota' : \mathcal{E}_{S'/I'} \to \text{Set} \). Letting \( B : \text{Set} \to \text{Bool} \) be the free Boolean algebra functor, we then have a natural transformation \( B\tilde{\varphi} : \varphi^* B\iota' \to B\iota \) of functors \( \mathcal{E}_{S/I} \to \text{Bool} \). That is nothing but a homomorphism \( \phi^* L_{S'/I'} \to L_{S/I} \) of internal Boolean algebras in \( \text{Set}^{\mathcal{E}_{S/I}} \), where we use \( \phi : \text{Set}^{\mathcal{E}_{S/I}} \to \text{Set}^{\mathcal{E}_{S'/I'}} \) to denote the geometric morphism induced by the map on posets. Passing to the adjoint \( L_{S'/I'} \to \phi_* L_{S/I} \), applying \( \text{Idl}' \) and using the natural transformation \( \text{Idl}'_{\phi_* \mathcal{E}_{S/I}} \phi_* \to \phi_* \text{Idl}'_{\mathcal{E}_{S/I}} \) we have the composite
\[
F_{S'/I'} = \text{Idl}' L_{S'/I'} \to \text{Idl}' \phi_* L_{S/I} \to \phi_* \text{Idl}' L_{S/I} = \phi_* F_{S/I}.
\]
It is a homomorphism of internal frames in \( \mathcal{E}_{S'/I'} \) by construction of the \( \text{Idl}' \)-algebra structure on \( \phi_* F_{S/I} \). By adjunction, we obtain an internal frame homomorphism \( \phi^* F_{S'/I'} \to F_{S/I} \) in \( \mathcal{E}_{S/I} \). Hence, \( (\phi, \phi^*) \) is a morphism in \( \text{FrmTop} \) between \((\text{Set}^{\mathcal{E}_{S/I}}, F_{S/I})\) and \((\text{Set}^{\mathcal{E}_{S'/I'}}, F_{S'/I'})\): the value of the desired functor on the morphism \( (\varphi, \varphi^*) \) in \( \text{Box} \) between \( S/I \) and \( S'/I' \). A careful inspection of all the
3.3. **Products.** Observe that the category $\mathbf{Box}$ has finite coproducts: indeed, $\emptyset \to \emptyset$ is an initial object, while the coproduct of $S/I$ and $T/J$ is $(S \sqcup T)/(I \sqcup J)$. On the other hand, the category $\mathbf{FrmTop}$ does not have all finite products, and in particular it lacks a final object. It is convenient to restrict to its full subcategory $\mathbf{bFrmTop}$ of bounded framed topoi, consisting of those $(\mathcal{E}, F)$ for which there exists a geometric morphism $f : \mathcal{E} \to \mathbf{Set}$ and an object $B$ in $\mathcal{E}$ such that every object of $\mathcal{E}$ is a sub-quotient of $B \times f^* A$ for some set $A$. Now, $\mathbf{bFrmTop}$ does have finite products: indeed, $(\mathbf{Set}, 2)$ is the final object, while the product of $(\mathcal{E}, E)$ and $(\mathcal{F}, F)$ is $(\mathcal{E} \times \mathcal{F}, p^1_1 E \otimes p^2_2 F)$ where

$$\mathcal{E} \xrightarrow{p^1_1} \mathcal{E} \times \mathcal{F} \xrightarrow{p^1_2} \mathcal{F}$$

is the product in the category of bounded topoi and geometric morphisms, while $\otimes$ is the coproduct in $\mathbf{Frm}_{\mathcal{E} \times \mathcal{F}}$. It is easy to compute in our case.

**Lemma 1.** Let $\mathcal{E}, \mathcal{F}$ be bounded topoi, and let $L$ be an internal distributive lattice in $\mathcal{E}$, and $M$ an internal distributive lattice in $\mathcal{F}$. Then the product of $(\mathcal{E}, \text{Idl}^L L)$ and $(\mathcal{F}, \text{Idl}^M M)$ in $\mathbf{bFrmTop}$ is naturally isomorphic to $(\mathcal{E} \times \mathcal{F}, \text{Idl}^L (p^1_1 L \otimes p^2_2 M))$ where $\otimes$ is the coproduct in $\mathbf{dLat}_{\mathcal{E} \times \mathcal{F}}$, together with projections

$$(\mathcal{E}, \text{Idl}^L L) \xleftarrow{(p^1_1, p^1_2)} (\mathcal{E} \times \mathcal{F}, \text{Idl}^L (p^1_1 L \otimes p^2_2 M)) \xrightarrow{(p^2_1, p^2_2)} (\mathcal{F}, \text{Idl}^M M)$$

in $\mathbf{bFrmTop}$ in which the frame homomorphisms

$$p^1_1 \text{Idl}^L L \to \text{Idl}^L (p^1_1 L \otimes p^2_2 M) \xleftarrow{p^2_1} p^2_2 \text{Idl}^M M$$

are induced, using adjunctions, by distributive lattice homomorphisms

$$p^1_1 L \to p^1_1 L \otimes p^2_2 M \xleftarrow{} p^2_2 M$$

arising from the universal property of the coproduct.

**Proof.** Let $(\mathcal{G}, F)$ be an object of $\mathbf{bFrmTop}$ together with morphisms

$$(\mathcal{E}, \text{Idl}^L L) \xleftarrow{(s^t, s^t)} (\mathcal{G}, F) \xrightarrow{(t^t, t)} (\mathcal{F}, \text{Idl}^M M).$$

The geometric morphisms $s$ and $t$ factor uniquely through $\langle s,t \rangle : \mathcal{G} \to \mathcal{E} \times \mathcal{F}$. The homomorphisms of internal frames

$$p^L \text{Idl}^L L \to F \xleftarrow{q^L \text{Idl}^M M}$$

correspond by adjunction to

$$\text{Idl}^L L \to s_* F, \quad t_* F \leftarrow \text{Idl}^M M$$

and thus to

$$L \to s_* F, \quad t_* F \leftarrow M$$

in $\mathbf{dLat}_{\mathcal{E}}$ and $\mathbf{dLat}_{\mathcal{F}}$. These pull back to

$$p^1_1 L \to p^1_1 p_{1*} \langle s,t \rangle_* F, \quad p^2_2 p_{2*} \langle s,t \rangle_* F \leftarrow p^2_2 M$$

in $\mathbf{dLat}_{\mathcal{G}}$. Composing with the counits $p^1_{1*} \to \text{id}$ and $p^2_2 p_{2*} \to \text{id}$ we do finally obtain a morphism

$$p^1_1 L \otimes p^2_2 M \to \langle s,t \rangle_* F.$$
in \text{dLat}_\varphi$, extending by universality to a frame homomorphism from \(\text{Idl}'(p_1^*L \otimes p_2^*M)\) and giving by adjunction a morphism

\[
\langle s, t \rangle^\varphi \text{Idl}'(p_1^*L \otimes p_2^*M) \rightarrow F
\]

in \text{Frm}_\varphi. We have thus produced a morphism

\[
\langle (s, s^2), (t, t^5) \rangle : (\varphi, F) \rightarrow (\varphi \times \varphi', \text{Idl}'(p_1^*L \otimes q_2^*M))
\]

in \text{bFrmTop}. By construction, its composites with \((p_1, p_1^*)\) and \((p_2, p_2^*)\) give \((s, s^2)\) and \((t, t^5)\). Uniqueness follows from the universal property of the coproduct \(\otimes\) in \text{dLat}_\varphi.

\begin{proof}

Note first that the final object \(\emptyset/\emptyset\) of \text{Box}^{\text{op}} is sent to the final object \((\text{Set}, 2)\) of \text{bFrmTop}. Suppose now given \(S/I\) and \(T/J\), and let \(U = S \sqcup T, K = I \sqcup J\). A context for \(U/K\) is a subset of \(S \sqcup T\) such that its intersection with \(S\) is a context for \(S/I\) and its intersection with \(T\) is a context for \(T/J\); hence

\[
\mathcal{C}_{U/K} \simeq \mathcal{C}_{S/I} \times \mathcal{C}_{T/J}
\]

as posets. Let

\[
\mathcal{C}_{S/I} \xrightarrow{s} \mathcal{C}_{U/K} \xrightarrow{t} \mathcal{C}_{T/J}
\]

be the two projections, and denote by \(t_S : \mathcal{C}_{S/I} \rightarrow \text{Set}, t_T : \mathcal{C}_{T/J} \rightarrow \text{Set}, t_U : \mathcal{C}_{U/K} \rightarrow \text{Set}\) the canonical embeddings as sub-categories of \text{Set}. Observe that

\[
t_U \simeq s^*t_S \sqcup t^*t_T \quad \text{in } \mathcal{C}_{U/K}
\]

so that

\[
L_{U/K} = B_{t_U} \simeq s^*B_{t_S} \otimes t^*B_{t_T} \simeq s^*L_{S/I} \otimes t^*L_{T/J},
\]

where \(\otimes\) is the coproduct in \(\text{Bool}^{\mathcal{C}_{U/K}}\), given point-wise by the tensor product of Boolean algebras in \text{Set}, and thus coinciding with the coproduct in \text{dLat}^{\mathcal{C}_{U/K}}. Hence, by Lemma \ref{lem:coproduct} \((\mathcal{C}_{U/K}, \mathcal{F}_{U/K})\) is the product of \((\mathcal{C}_{S/I}, F_{S/I})\) and \((\mathcal{C}_{T/J}, F_{T/J})\).

\end{proof}

### 3.4. Example

The \text{gbit} is a box world presented by the unique map \(2 \rightarrow 1\). The Popescu-Rohrlich box world is then presented by the coproduct of a pair of gbits in \text{Box}, i.e. a map \(2 \sqcup 2 \rightarrow 1 \sqcup 1\). Accordingly, the framed topos associated to the P-R box world is the product of two copies of the framed topos \((\mathcal{C}_{2/1}, F_{2/1})\) associated with a gbit. For a general object \((\varphi, F)\) of \text{bFrmTop}, it is interesting to consider \text{epimorphisms} to the product of \(n\) copies of \((\mathcal{C}_{2/1}, F_{2/1})\): the supremum of \(n\) for which such an epimorphism exists is a certain measure of \text{information capacity} of a hypothetical system described by \((\varphi, F)\). This in particular applies to \((\mathcal{C}_{S/I}, F_{S/I})\) for a given \(S/I\) in \text{Box}.

### 3.5. Remark

Given a box presentation \(S/I\) and an object \((\varphi, F)\) of \text{FrmTop}, to give a morphism \((\varphi, F) \rightarrow (\mathcal{C}_{S/I}, F_{S/I})\) is the same as to give a geometric morphism \(\phi : \varphi \rightarrow \mathcal{C}_{S/I}\) together with a homomorphism \(\varphi : L_{S/I} \rightarrow \phi_*F\) of \text{internal distributive lattices} in \(\mathcal{C}_{S/I}\). Furthermore, \(\varphi\) factors through a homomorphism of internal Boolean algebras into the internal sub-lattice of ‘complemented elements’ in \(\phi_*F\). It thus seems somewhat spurious to consider internal frames rather than simply internal Boolean algebras (or even distributive lattices); this will be seen even more strongly when we characterise states or probability valuations on \(F_{S/I}\).
However, frames are a natural setting for the discussion of phase spaces. In any case, \( L_{S/I} \) may be recovered from \( F_{S/I} \) as, again, the sub-lattice of ‘complemented elements’.

4. States

4.1. Box world states. Consider a box world presented by a map \( S \to I \) of finite sets. According to the interpretation as a series of ‘simultaneous’ observations performed by parties indexed by \( I \), the most general form of an ‘empirical’ probability distribution is a map, assigning a certain probability distribution to the possible outcomes in each context. The condition of non-signalling requires that the distribution in a ‘partial’ context (where some parties do not perform a measurement) arise consistently as a marginal of the distribution in a ‘total’ context refining the partial one.

This motivates the following definition. Recall first that for an object \( S/I \) of Box, we consider the elements of \( \mathcal{C}_{S/I} \) as subsets of \( S \). Denote by \( \mathcal{C}^*_m \subset \mathcal{C}_{S/I} \) the set of maximal elements.

Definition 3. A box world state on \( S/I \) is a map

\[
p : \bigoplus_{c \in \mathcal{C}^*_m} 2^c \to [0, 1]
\]

such that

1. (normalisation) \( \sum_{x \in 2^c} p(x) = 1 \) for each \( c \in \mathcal{C}^*_m \),
2. (non-signalling) for each \( c \in \mathcal{C}_{S/I} \) and \( x \in 2^c \) the value

\[
p(x) := \sum_{y \in 2^d, y|_c = x} p(y)
\]

does not depend on the choice of \( d \in \mathcal{C}^*_m \), \( c \leq d \).

Recall that embedding \( \mathcal{C}_{S/I} \to \text{Set} \), composing with the free Boolean algebra functor and the forgetful functor gives the object \( L_{S/I} \) in \( \text{Set} = \text{Set}^\delta \). We may view it as a \( \mathcal{C}_{S/I} \)-shaped diagram in \( \text{Set} \), and take its colimit \( \text{colim} L_{S/I} \) (this is not a colimit in \( \text{Bool} \)). Observe that for each \( c \in \mathcal{C}_{S/I} \) there are natural maps

\[2^c \overset{\delta}\longrightarrow 2^{2^c} \cong L_{S/I}(c) \overset{i_c}\longrightarrow \text{colim} L_{S/I}\]

where \( \delta \) takes \( x \in 2^c \) to its indicator function \( \delta_c(x) : 2^c \to 2 \). The following will soon be useful.

Lemma 2. The map

\[
\bigoplus_{c \in \mathcal{C}^*_m} 2^c \longrightarrow \text{colim} L_{S/I}
\]

induces a one-to-one correspondence between box-world states on \( S/I \) and maps

\[
\rho : \text{colim} L_{S/I} \to [0, 1]
\]

such that \( \rho \circ i_c \) is a probability valuation on the Boolean algebra \( L_{S/I}(c) \) for each \( c \in \mathcal{C}^*_m \).
Proof. Note first that for each finite set $A$, the set of probability valuations on the power-set Boolean algebra $2^A$ is in natural bijection with the set of functions $q : A \to [0, 1]$ such that $\sum_{a \in A} q(a) = 1$.

Suppose now $p$ is a box world state on $S/I$. It follows that $p$ induces a probability valuation $p_c$ on $L_{S/I}(c)$ for each $c \in \mathcal{C}_{S/I}^m$. Furthermore, by the non-signalling condition, $p$ may be consistently extended to $2^c$ for each non-maximal $c'$, and thus defines a probability valuation $p_{c'}$ on $L_{S/I}(c')$. By construction, $p_{c'} = p_c \circ L_{S/I}(c' \to c)$ whenever $c' \leq c$ so that $\prod_c p_c : \prod_c L_{S/I}(c) \to [0, 1]$ descends to a unique map $\rho : \colim L_{S/I} \to [0, 1]$ such that $\rho \circ \iota_c = p_c$ for all $c \in \mathcal{C}_{S/I}$.

In fact, since for every $c' \in \mathcal{C}_{S/I}$ there exists $c \in \mathcal{C}_{S/I}^m$ with $c' \leq c$, it follows that $\rho$ is the unique $[0, 1]$-valued map from $\colim L_{S/I}$ such that $\rho \circ \iota_c = p_c$ for all $c \in \mathcal{C}_{S/I}$.

Conversely, it is immediate that given a map $\rho : \colim L_{S/I} \to [0, 1]$ such that $\rho \circ \iota_c = p_c$ for all $c \in \mathcal{C}_{S/I}$, it induces a probability valuation for all $c \in \mathcal{C}_{S/I}$, and pulling back along $\iota \circ \delta$, one obtains a box-world state $p : \prod_{c \in \mathcal{C}_{S/I}^m} 2^c \to [0, 1]$. Applying the construction of the previous paragraph to $p$ gives $\rho$, whence the one-to-one correspondence. \hfill $\square$

4.2. Internal states. We shall now recall the definition of probability valuations on internal distributive lattices and frames in a topos $E$. We require that $E$ possess a natural numbers object $\mathbb{N}_E$, so that we may construct the lower reals object $\mathbb{R}_{\ell,E}$. Recall that the latter is an additive monoid and a partially ordered object. We denote by $[0,1]_{\ell,E}$ the sub-object of $\mathbb{R}_{\ell,E}$ such that a morphism $r : T \to [0,1]_{\ell,E}$ factors through $[0,1]_{\ell,E}$ if and only if $0_T \leq r \leq 1_T$ (where $0_T, 1_T : T \to \mathbb{R}_{\ell,E}$ are the obvious `constant’ morphisms). Now, $[0,1]_{\ell,E}$ is an internal dcpo, i.e. an $\Idl_E$-algebra.

**Definition 4.** Let $L$ an internal distributive lattice in $E$. An **probability valuation** on $L$ as a distributive lattice is an isotope morphism $P : L \to [0,1]_{\ell,E}$ satisfying the conditions of:

1. (normalisation) $P0_L = 0_{[0,1]}$ and $P1_L = 1_{[0,1]}$;
2. (modularity) the diagram

   \[
   \begin{array}{ccc}
   L \times L & \xrightarrow{P \times P} & [0,1]^2_{\ell,E} \\
   \downarrow \langle \vee, \wedge \rangle & & \downarrow \\
   L \times L & \xrightarrow{P \times P} & [0,1]^2_{\ell,E}
   \end{array}
   \]

   commutes.

If $L$ is an internal frame, $P$ is a probability valuation on $L$ as a frame if additionally it satisfies

3. (continuity) the diagram

   \[
   \begin{array}{ccc}
   \Idl L & \xrightarrow{\Idl P} & \Idl [0,1]_{\ell,E} \\
   \downarrow & & \downarrow \\
   L & \xrightarrow{P} & [0,1]_{\ell,E}
   \end{array}
   \]

   commutes, i.e. $P$ is a morphism of internal dcpo.
Note that $P$ is a morphism in $\text{Ord}_\mathcal{E}$, so that $\text{Idl} P$ is a well-defined. The vertical arrows in the last diagram are the structure morphisms for $\text{Idl}_\mathcal{E}$-algebras (recall that a frame is in particular a dcpo).

**Lemma 3.** Let $L$ be an internal distributive lattice in $\mathcal{E}$. Then the natural morphism $\eta_L : L \to \text{Idl}' L$ induces a one-to-one correspondence between probability valuations on $L$ (as a distributive lattice) and probability valuations on $\text{Idl}' L$ (as a frame).

**Proof.** By adjunction, $\eta_L$ induces a bijection

$$\text{Hom}_{\text{Dcpo}}(\text{Idl} L, [0, 1]_{\ell, \mathcal{E}}) \simeq \text{Hom}_{\text{Ord}}(L, [0, 1]_{\ell, \mathcal{E}})$$

where we recall that $\text{Idl}' L$ as an internal dcpo is the same as $\text{Idl} L$. Hence an isotone morphism $\text{Idl}' L \to [0, 1]_{\ell, \mathcal{E}}$ satisfying conditions (1), (2), (3) of the Definition is the same as an isotone morphism $L \to [0, 1]_{\ell, \mathcal{E}}$ satisfying conditions (1), (2). □

**Lemma 4.** Let $L$ be an internal Boolean algebra in $\mathcal{E}$. Then every probability valuation $P : L \to [0, 1]_{\ell, \mathcal{E}} \subset R_\mathcal{E}$ factors through the Dedekind reals $R_{d, \mathcal{E}}$.

**Proof.** Let $\nu : R_{d, \mathcal{E}} \to R^2_{\ell, \mathcal{E}}$ be the morphism taking a Dedekind cut to its lower and negative upper part. We then have a pullback diagram

Now, consider the diagram

$$L \xrightarrow{(\text{id}, \neg)} L \times L \xrightarrow{\text{id} \land \land \lor} L \times L \xrightarrow{P \times (P^{-1})} R^2_{\ell, \mathcal{E}} \xrightarrow{+} R_{\ell, \mathcal{E}}$$

where the two composites coincide by modularity of $P$, and are in fact equal to the composite

$$L \to 1_{\mathcal{E}} \xrightarrow{0_{\ell}} R_{\ell, \mathcal{E}}.$$ It follows that the top composite $L \to R^2_{\ell, \mathcal{E}}$ factors through $\nu$. Composing with the first projection $R^2_{\ell, \mathcal{E}} \to R_{\ell, \mathcal{E}}$, we have that $P : L \to R_\mathcal{E}$ factors through the ‘lower part’ map $R_{d, \mathcal{E}} \to R_{\ell, \mathcal{E}}$. □

**4.3. Main result.**

**Lemma 5.** Let $L$ be an internal Boolean algebra in a Kripke topos $\mathcal{E} = \text{Set}_\mathcal{E}$, and $f : \mathcal{E} \to \text{Set}$ an essential geometric morphism. Then for each probability valuation $P : L \to [0, 1]_{\ell, \mathcal{E}}$ there is a map $\bar{P} : f^* L \to [0, 1]$ such that the diagram

commutes.
Proof. By Lemma 4, $P$ factors through $\mathbb{R}_{d,\varepsilon}$. Since $\mathcal{E}$ is a Kripke topos, $\mathbb{R}_{d,\varepsilon}$ is isomorphic to $f^*\mathbb{R}$. By adjunction $f_! \dashv f^*$ we have $\bar{P} : f_! L \to \mathbb{R}$ such that $f^* \bar{P}$ composed with the unit $L \to f^* f_! L$ gives $P$. Finally, since $P$ factors through $[0, 1]_{d,\varepsilon}$ and thus through $[0, 1]_{d,\varepsilon} \simeq f^*[0, 1]$, it follows that $\bar{P}$ factors through $[0, 1]$.

**Theorem 1.** Let $S/I$ be an object of $\text{Box}$. The are natural one-to-one correspondences between:

1. probability valuations on $F_{S/I}$ as an internal frame in $\mathcal{E}_{S/I}$,
2. probability valuations on $L_{S/I}$ as an internal distributive lattice in $\mathcal{E}_{S/I}$,
3. box world states on $S/I$.

Proof. The bijection between (1) and (2) follows from Lemma 3. Since $L_{S/I}$ is Boolean, probability valuations on $L_{S/I}$ as an internal distributive lattice are in bijection with maps $\bar{P} : f_! L_{S/I} \to [0, 1]$ such that composing $f^* \bar{P}$ with $L \to f^* f_! L$ is a probability valuation (Lemma 4). Since $f_! L_{S/I} \simeq \text{colim} L_{S/I}$, we then have a further bijection with maps $\rho : \text{colim} L_{S/I} \to [0, 1]$ such that the lift

$$\hat{\rho} : \bigsqcup_{c \in \mathcal{E}_{S/I}} L_{S/I} \to [0, 1]$$

defines a probability valuation $L_{S/I} \to f^*[0, 1]$. The latter condition is equivalent to requiring that the restriction of $\hat{\rho}$ to $L_{S/I}(c)$ be a probability valuation for each $c \in \mathcal{E}_{S/I}$. In fact, since for each $\ell' \in \mathcal{E}_{S/I}$ there is a $c \in \mathcal{E}_{S/I}$ with $\ell' \leq c$, the above requirement need only be stated for maximal contexts. Hence, by Lemma 2 we have a bijection between (2) and (3).

$\square$

5. Phase space

5.1. Frames and locales. Internal locales in $\mathcal{E}$ are just internal frames, with the direction of morphism reversed: $\text{Loc}_\mathcal{E} = \text{Frm}_\mathcal{E}^{\text{op}}$. Given a geometric morphism $f : \mathcal{E} \to \mathcal{E}'$, the adjunction $f^* \dashv f_*$ may be viewed as a pair of functors between $\text{Loc}_\mathcal{E}$ and $\text{Loc}_{\mathcal{E}'}$. We use $\mathcal{O}$ to denote the tautological contravariant functor from frames to locales. Now, given a locale $X$ in $\text{Set}$, the frame $\mathcal{O}(X)$ viewed as a category comes with a Grothendieck topology where a sieve $(u_i \to u)_{i \in I}$ is covering whenever $u = \bigvee_{i \in I} u_i$. Thus $\mathcal{O}(X)$ becomes a site, and its topos of sheaves is denoted $\text{Sh}(X)$. The functor $\text{Sh}$ from locales to toposi is full and faithful, and its essential image is the category of localic toposi. These are in particular bounded, with a geometric morphism $\text{Sh}(X) \to \text{Set}$ induced by the unique frame homomorphism $2 \to \mathcal{O}(X)$. We observe that if $M$ is a topological space, its topology is a frame and thus we may consider the corresponding locale: this gives a functor from topological spaces to locales, admitting a right adjoint. Stone duality restricts it to an equivalence between the category of sober topological spaces with the category of spatial locales in $\text{Loc}$. In any case, the usual topos of sheaves on a topological space $M$ is canonically equivalent to the topos of sheaves on the corresponding locale.

We let $\text{FrmTop} \subset \text{bf FrmTop}$ denote the full subcategory of localic framed topoi. In particular, for $S/I$ in $\text{Box}$ we have $\mathcal{E}_{S/I} \simeq \text{Sh}(\mathcal{E}_{S/I})$ where we view $\mathcal{E}_{S/I}$ as a sober topological space using the Alexandrov topology. Thus the functor $\text{Box}^{\text{op}} \to \text{FrmTop}$ factors through $\text{FrmTop}$.

Note that for each $(\text{Sh}(X), F)$ in $\text{FrmTop}$ the internal frame $F$ may be viewed as an internal locale. The direct image functor $\text{Loc}_{\text{Sh}(X)} \to \text{Loc}$ sends the final internal locale, corresponding to the initial frame $\Omega_{\text{Sh}(X)}$, to $X$ itself, and thus
factors through the slice category $\text{Loc}/X$. A fundamental result in topos theory [8, Thm. C1.6.3] is that this gives an equivalence of categories $\text{Loc}_{\text{Sh}(X)} \simeq \text{Loc}/X$. Following the definitions, we thus obtain an equivalence

$$\text{fFrmTop} \simeq \text{Loc}^\to$$

between localic framed topoi and the arrow category of locales in $\text{Set}$. Composing with the functor from box world presentations gives the external phase space functor

$$\mathcal{X} : \text{Box}^{\text{op}} \to \text{Loc}^\to.$$ 

It turns out that the latter factors through the arrow sub-category of spatial locales, and thus through the category of homeomorphisms of sober topological spaces. We will give its explicit description.

5.2. The external phase space. Consider a box world presented by $S \to I$. Recall once again that $\mathcal{C}_{S/I}$ is a subset of the power-set $2^S$, ordered by inclusion. The functor $L_{S/I}$ takes a context $c \subseteq S$ to the free Boolean algebra on $c$, and may be viewed as a functor into the category $\text{fBool}$ of finite Boolean algebras. Now, we have an adjunction

$$2^- : \text{Set} \rightleftarrows \text{fBool}^{\text{op}} : \text{Sp}$$

where the spectrum functor is right-adjoint to the powerset functor. Thus, the composite

$$\text{Sp} \circ L_{S/I} : \mathcal{C}_{S/I}^{\text{op}} \to \text{Set}$$

is a contravariant set-valued functor, i.e. a presheaf on $\mathcal{C}_{S/I}$. We construct the category of elements

$$X_{S/I} = \int \text{Sp} \circ L_{S/I},$$

fibred over $\mathcal{C}_{S/I}$. Its set of objects is the disjoint union $\bigsqcup_{c \in \mathcal{C}_{S/I}} \text{Sp} L_{S/I}(c)$, while the hom-set $\text{Hom}_{X_{S/I}}(x, x')$ is a singleton if $x$ is the image of $x'$ under $\text{Sp} L_{S/I}(c' \to c)$, or empty otherwise. Thus, $X_{S/I}$ is in fact a poset, with an isotone projection to $\mathcal{C}_{S/I}$. We may give a very straightforward description as follows: just as $\mathcal{C}_{S/I}$ is the poset of contexts, $X_{S/I}$ is the poset of outcomes, ordered by refinement. That is:

$$X_{S/I} \simeq \{(c, x) \mid c \in \mathcal{C}_{S/I}, \ x : c \to 2\}$$

where $(c', x') \leq (c, x)$ if and only if $c' \leq c$ and $x|_{c'} = x'$.

**Lemma 6.** The external phase space of $S/I$ is isomorphic, as an object of $\text{Loc}^\to$, to the continuous map of Alexandrov spaces $X_{S/I} \to \mathcal{C}_{S/I}$.

**Proof.** By construction, the external phase space is the homeomorphism of locales corresponding to the frame homomorphism $f_* \Omega_{S/I} \to f_* F_{S/I}$ where $f : \mathcal{C}_{S/I} \to \text{Set}$ is the unique geometric morphism, and $\Omega_{S/I}$ is the sub-object classifier in $\mathcal{C}_{S/I}$. Since the poset $\mathcal{C}_{S/I}$ admits a lower bound $\emptyset$ (the empty context), we have that

$$f_* F = \lim F = F(\emptyset)$$

for each functor $F : \mathcal{C} \to \text{Set}$. We already know that $f_* \Omega_{S/I} = \Omega_{S/I}(\emptyset)$ is the frame $\mathcal{O}(\mathcal{C}_{S/I})$ for the Alexandrov topology: indeed, it is the lattice of subfunctors of $1_{\mathcal{C}_{S/I}}$, coinciding with the lattice of upper sets in $\mathcal{C}_{S/I}$. On the other hand, $f_* F_{S/I} = (\text{Idl} L_{S/I})(\emptyset)$ is the lattice of subfunctors $J \subseteq L_{S/I}$ such that $J(c) \subseteq L_{S/I}(c)$ is an ideal for all $c \in \mathcal{C}_{S/I}$. Since $L_{S/I}(c)$ is finite, every ideal is principal, and thus $f_* F_{S/I}$ is naturally identified with the poset of ‘sections’ $u : \mathcal{C}_{S/I} \to \bigsqcup L_{S/I}(c)$.
such that $u(c) \geq L_{S/I}(c \to c)u(c')$ whenever $c' \leq c$. With this latter description, the homomorphism $\Theta(\mathcal{E}_{S/I}) \to f_*F_{S/I}$ sends an upper set $U \subset \mathcal{E}_{S/I}$ to a section $u$ such that $u(c) = 1$ if $c \in U$, or $u(c) = 0$ otherwise. Now, note that $L_{S/I}(c)$ is precisely the powerset of the fibre of $X_{S/I}$ over $c$, i.e. $2^{\text{Sp}L_{S/I}(c)}$. Thus, a section $u$ in $f_*F_{S/I}$ is the same as a subset $G \subset X_{S/I}$ such that the preimage of the fibre $G_c \subset \text{Sp}L_{S/I}(c')$ under $\text{Sp}L_{S/I}(c) = \mathcal{E}_{S/I}(c \to c)$ is contained in $G_c \subset \text{Sp}L_{S/I}(c)$. Hence, $f_*F_{S/I}$ is canonically identified with the lattice of upper sets in $X_{S/I}$, i.e. the frame $\Theta(X_{S/I})$ for the Alexandrov topology. Furthermore, the homomorphism $\Theta(\mathcal{E}_{S/I}) \to f_*F_{S/I} \simeq \Theta(X_{S/I})$ now takes an upper set $U \subset \mathcal{E}_{S/I}$ to its preimage in $X_{S/I}$ under the natural projection, whence the corresponding morphism of locales — i.e., of Alexandrov spaces — is the projection again. □

Recall that given a morphism $S'/I' \to S/I$ in $\text{Box}$, with underlying map $\varphi : S' \to S$, the corresponding map on contexts $\mathcal{E}_{S/I} \to \mathcal{E}_{S'/I'}$ is given by the pullback map $\varphi^* : 2^S \to 2^{S'}$. Now, given an outcome $x : c \to 2$ with $c \in S$, we obtain $\varphi^* x : \varphi^* c \to 2$, an outcome in the context $\varphi^* c$. It thus follows that the map on contexts lifts to a map on the external phase spaces $X_{S/I} \to X_{S'/I'}$:

\[
\begin{array}{ccc}
X_{S/I} & \xrightarrow{\varphi^*} & X_{S'/I'} \\
\downarrow & & \downarrow \\
\mathcal{E}_{S/I} & \xrightarrow{\varphi^*} & \mathcal{E}_{S'/I'}
\end{array}
\]

All arrows in the above diagram are isotone, and thus may be viewed as continuous maps between Alexandrov spaces. Identifying the latter with spatial locales, we obtain a functor

$$\mathcal{X} : \text{Box}^{\text{op}} \to \text{Loc}.$$

Lemma 7. $\mathcal{X}$ is naturally isomorphic to $\mathcal{X}$.

Proof. The components $\lambda_{S/I} : \mathcal{X}(S/I) \to \mathcal{X}(S/I)$ have already been constructed in the proof of Lemma 6 as diagrams

\[
\begin{array}{ccc}
f_*\Omega_{S/I} & \xrightarrow{\simeq} & \Theta(\mathcal{E}_{S/I}) \\
\downarrow & & \downarrow \\
f_*F_{S/I} & \xrightarrow{\simeq} & \Theta(X_{S/I})
\end{array}
\]

of frame homomorphisms, where $f : \mathcal{E}_{S/I} \to \text{Set}$ is the geometric morphism. Checking that they combine into a natural transformation is straightforward, if somewhat tedious, unwinding of the constructions performed in the proof of Proposition 1 and comparison with the definition of $\mathcal{X}$. □

5.3. Examples. The external phase space of the gbit is a five-point set $X_{S/1} = \{*, a, a', b, b'\}$ where $*$ is the unique closed point, and the $\{a\}, \{a'\}, \{b\}, \{b'\}$ form a basis for the topology. More precisely, $\mathcal{X}(2/1)$ is the projection $X_{2/1} \to \mathcal{E}_{2/1}$ with fibres $\{*, \{a\}, \{b\}, \{a', b', b'\}\}$. Note that since $\text{IFrmTop} \to \text{Loc}^{\text{op}}$ is an equivalence, and $\text{Box}^{\text{op}} \to \text{IFrmTop}$ preserves finite product, it follows that so does $\mathcal{X}$. Thus, the external phase space of the PR box world is the product $X_{2/1} \times X_{2/1} \to \mathcal{E}_{2/1} \times \mathcal{E}_{2/1}$. This may of course be seen on a completely elementary level, viewing the external
phase space as a fibration of ‘outcomes’ over ‘contexts’ (with topologies consisting of subsets stable under refinement).

6. Discussion

6.1. States and morphisms. The assignment of the set of box world states to a presentation $S/I$ in Box gives rise to a functor $\text{Box}^{\text{op}} \to \text{Conv}$ into the category of convex spaces. It would be desirable to extend such functoriality to more general framed topoi. A possible version of this may be achieved if one works with the locale of probability valuations, rather than a set. Given a bounded topos $\mathcal{E}$, Vickers [12] defines a valuation monad $V_\mathcal{E}$ on $\text{Loc}_\mathcal{E}$. It sends a locale corresponding to a frame $F$ to the locale whose global points are precisely the probability valuations on $F$. Furthermore, $V_\mathcal{E}$-algebras are the internal convex locales, so that the locale of probability valuations is tautologically convex. The most important aspect of his construction is however geometricity, giving rise to a natural isomorphism $f^\sharp V_{\mathcal{E}} \rightarrow V_{\mathcal{E}'} f^\sharp$ for a geometric morphism $f : \mathcal{E} \rightarrow \mathcal{E}'$. It then follows that sending $(\mathcal{E}, F)$ to $(\mathcal{E}, V_\mathcal{E} F)$ induces a functor $V : \text{bFrmTop} \rightarrow \text{bFrmTop}$.

6.2. Channels. The most general notion of a channel between a pair of box worlds is a map on their state spaces, preserving the convex structure. For classical systems, this is equivalent to giving a probability kernel, or stochastic map, between their phase spaces. In particular, a channel from a system to itself is simply a Markov kernel, or a stochastic self-map. Given a bounded framed topos $(\mathcal{E}, F)$ one might by analogy consider homomorphisms $V_\mathcal{E} F \rightarrow F'$ for a geometric morphism $f : \mathcal{E} \rightarrow \mathcal{E}'$. It then follows that sending $(\mathcal{E}, F)$ to $(\mathcal{E}, V_\mathcal{E} F)$ induces a functor $V : \text{bFrmTop} \rightarrow \text{bFrmTop}$.

6.3. General non-signalling theories. Our framework suggests a moderate generalisation of non-signalling box worlds, where the system is specified by a partially ordered set $\mathcal{C}$ of contexts, and a functor $L : \mathcal{C} \rightarrow \mathbf{fBool}$ of measurement logics (or, equivalently, the ‘spectral presheaf’ $\text{Sp} L : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$). A convenient assumption on $\mathcal{C}$ is that its chains be finite. This representation is applicable to orthodox quantum systems (with a finite-dimensional Hilbert space), quantum logics modelled by orthomodular lattices (see [9] for the general notion of Bohrification of models of an algebraic theory with respect to a sub-theory), and possibly more exotic structures such as the logic of properties induced by a convex state space à la Mielnik [9].

7. Acknowledgements

The authors acknowledge partial support of the John Templeton Foundation grant no. 43174.

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