The Thomas theorem and the Efimov States within a generalized Bohr model

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Abstract

Using a generalized Bohr model and the hyper-spherical formalism for a three-body system, we derive the universal energy functions. We apply our model to 4He atom where only the Coulomb potential is dominant and find that its binding energy is well reproduced (less than 5.5% difference). Later, we focus on the equal mass three-body systems and derive the Thomas theorem assuming a simple parameter fi and that its binding energy is well reproduced. Prediction for excited (Efimov) levels are also given and in particular we demonstrate that for some hyper-angles two equal minima appear which indicate a phase (shape) transition similar to the Landau’s theory of phase transition. We suggest that the observed excited levels in two different experiments for the triton nucleus are indeed Efimov levels and there may be more surprises.

1. Introduction

In 1935, Thomas [1] demonstrated that if a quantum two-body system has at least one loosely bound state then a system made of three particles becomes strongly bound. In particular he showed in the limit that the two-body interaction becomes a delta function (i.e., its range r → 0), that the corresponding three-body system is unbound from below, i.e., its binding energy becomes −∞. In 1970 Efimov [2], building up on the Thomas effect, showed that if the scattering length a → ∞ then the three-body system displays a geometrical series of excited levels roughly spaced in the interval ∂a and ∂a. Thus, if r → 0 and a → ∞, an infinite series of excited levels appears. In particular, the number of excited three-body levels, is roughly given by the simple relation [2]:

\[ N \sim \frac{\epsilon_0}{\pi} \ln(|a|/\eta_0), \]

where \( \epsilon_0 = 1.00624 \) is a universal constant. We stressed the fact above that a quantum system shows these properties, i.e., there is no classical analog to the Thomas and Efimov effects. Originally, these properties were discussed for nuclear systems in particular to derive the binding energy of triton (t) and 3He nuclei [1] as well as the first excited level of 12C, the Hoyle state, which was suggested to be an Efimov level [2], see [3] for more details. The deuterium nucleus is a loosely bound system, E_d = 2.225 MeV, and the nucleon-nucleon scattering length in the triplet state a_d = 5.42 fm while the effective range r_d = 1.76 fm [5]. These values are not very large and equation (1) would support at most one excited (Efimov) level for the three body system, while the ground state of deuterium should be of the order of \( \frac{\epsilon_0}{\pi} \approx 1.4 \text{ MeV} \), i.e., close to the experimental value. The three-body nuclei t and 3He are much more bound than the d-nucleus, E_t = 8.48 MeV and E_{3He} = 7.72 MeV in agreement to the Thomas theorem [1]. Until recently, no excited levels of these nuclei were known, but two independent experiments [4, 5] have found some anomalies in the energy spectrum of t nuclei, which have been interpreted as an excited level of t at about \( E_t^* \approx 7.0 \text{ MeV} \) in [5] or final state interaction in [4]. Of course, even if
this finding is confirmed as an excited level of \( t \), it does not imply to be an Efimov state (ES). Some definitions of the ES have been given in the literature \([3]\), but we will discuss below other possible conditions for the ES. The skepticism \([6]\) to find ES in nuclei is mainly due to the fact that the scattering lengths are not terribly large. But the derivation of the Thomas (TS) and Efimov states does not involve any particular physical system, just that the two-body interaction is short range. Thus we are dealing with universal properties and we can apply the same to other systems, for instance atoms. In this case we can have a large variety of natural scattering and effective ranges for Fermion and Boson systems. Furthermore, using magnets or other devices we can effectively change the scattering lengths to almost any value using for instance the Feshbach resonances \([3]\). In this framework, the TS and the ES have been experimentally demonstrated in recent years for a variety of atomic systems \([7–13]\) and most criticism has been put definitely to bed.

The quantal and universal features of the TS and ES prompted us to investigate a simple geometric model to help our physical intuition and possibly explore new venues. In exchange we had to sacrifice some of the mathematical beauty, which has been derived in the past 80 years or so \([1–3, 8]\). At the heart of our approach is the Bohr model \([14]\), which gives a reasonable description of many systems ground and excited states. In order to greatly simplify the math and derive universal expressions, we have used hyper-angles to describe the three-body system \([3]\). There are advantages of this geometric model. First, the numerical calculation is very simple and no Schrödinger equation is needed to solve which generally is not a easy job. Second, the geometric configuration of the three-body system can be derived directly from the values of the hyper-angle. Within this framework, we first study the binding energy of \(^4\)He atom. Then, we focus on the equal mass system and derive the TS result and go beyond it by investigating how ‘loosely’ bound the two-body system needs to be before the three-body one becomes unbound. We later apply the model to some physical examples from atomic and nuclear physics. We show that by simply fitting the interaction strength to the two-body ground state (when known) gives a very good prediction for the three-body system (when known) or vice versa. Ground states, or TS, are found when the three hyper-angles are equal, which corresponds to an equilateral triangular geometric distribution. Excited levels are also found and the ES corresponds to one small hyper-angle (less than \( \approx \frac{\pi}{12} \)) and the other two approaching \( \frac{\pi}{3} \). Geometrically this corresponds to a monomer plus a dimer system \((1 + 2)\). This is, in our opinion, an experimental signature of an ES, like for instance the \(^{12}\)C Hoyle state (HS) which decays mostly into the \(^{8}\)Be(g.s) + α \([2, 15, 16]\) or \(^{4}\)He atoms \([9]\). Recent experimental results of the HS decay give an upper value for the ratio of the direct decay (DD) to sequential decay (SD) less than 0.043% \([17–25]\). A new upper limit \( \sim 0.019\% \) has been reported with the new high precision study in \([26]\). For an ES we expect the probability to decay into different channels to be exactly zero, thus the need for higher experimental precision \([15]\). For atomic systems, the situation is more advanced respect to nuclei for the reason discussed above and our definition of ES may fit well the experimental results \([3, 6, 7, 9]\). However, distinctions must be done when the scattering length is positive (which admits two-body bound states) or negative (no bound states, resonances at most in the two-body channel). Triton nuclei and \(^4\)He atoms are examples of the first case while \(^8\)Be and polarized triton \(^3\)H atoms might be examples of the unbound two-body systems. These examples involve Bosons and non-identical Ferrions, and in some cases the Coulomb interaction comes into play. This can be easily incorporated in the model and explicitly breaks the universality of the functions we derive below.

A novel result from this approach is the transition from the ground state (TS) to an excited (ES) level. For positive scattering lengths and comparable effective lengths we find a critical hyper-angle for which two energy minima exist for the trimer. A comparison to phase transitions from the ground state can be drawn and in particular we can write the Landau’s (free) energy for phase transitions in terms of one hyper-angle (which plays the role of the ‘temperature’) and the hyper-distance (which plays the role of the order parameter, for instance the density in a liquid-gas phase transition). In particular if the effective and scattering lengths are comparable, then the ‘phase transition’ is first order (similar to the isotropic-nematic transition \([27]\)). When the effective length is negligible respect to the scattering length, the two minima disappear and the ‘phase transition’ becomes second order. Large negative scattering lengths display bound states but for a region in hyper-angles (near \( \frac{\pi}{8} \approx \frac{\pi}{10} \)) unbound resonances appear which become bound states again (ES) for smaller hyper-angles. The concept of phase transition in this context might be associated to a shape transition or a transition from TS to ES. It is quite impressive to have so many features in a quantum system made of three particles only!

### 2. The model

The generalized total energy of the N-body system is (in the center of mass frame)

\[
E = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i<j=1}^{N} V(r_{ij}) = \sum_{i<j=1}^{N} \frac{m_i + m_j}{M} \frac{p_{ij}^2}{2\mu_{ij}} + \sum_{i<j=1}^{N} V(r_{ij}).
\]
The reduced mass $\mu_{ij}$ is

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j},$$

and the relative momentum is

$$p_{ij} = \mu_{ij}(r_i - r_j) = \frac{m_i p_i - m_j p_j}{m_i + m_j}.$$  

The total mass of the system is

$$M = \sum_{i=1}^{N} m_i,$$

and the relative distance between particle $i$ and $j$ is

$$r_{ij} = r_i - r_j.$$  

It is easy to verify that equation (2) recovers the well known formula for the two-body system when $N = 2$. In order to obtain the total energy of the N-body system, we need to know the relative momentum $p_{ij}$ and the potential $V(r_{ij})$ between the particle $i$ and $j$. We also need to know the mass of each particle in the system.

In our consideration, we adopt the Bohr Model. We write $p_{ij} r_{ij} = \hbar$, and we will restrict our considerations to $n = 1$, i.e., the system’s ground state, unless otherwise noted. Thus, the total energy of the N-body system given by equation (2) becomes only a function of $r_{ij}$ and we can reconstruct the configuration of the system with those $r_{ij}$. If someone is only interested about the three-body system, e.g. the $^4$He atom, it is more convenient to use the hyper-angles and hyper-radius coordinates to describe the configuration of the system [3]. For a generalized three-body system, we can introduce the three-body reduced mass

$$m_{123} = \frac{m_1 m_2 m_3}{m_1 m_2 + m_2 m_3 + m_3 m_1},$$

the hyper-radius

$$R^2 = \frac{m_1 m_2 r_{12}^2 + m_2 m_3 r_{23}^2 + m_3 m_1 r_{31}^2}{m_1 m_2 + m_2 m_3 + m_3 m_1},$$

and the hyper-angle

$$\eta_{ij} = \left(\frac{m_1 m_2 m_3}{m_i m_{123} M}\right)^{1/2} R \sin \alpha_k,$$

where $m_{ij} = \mu_{ij}$. Thus, we have

$$R^2 \sin^2 \alpha_k = \frac{m_0 m_{123} M}{m_1 m_2 m_3} r_{ij}^2 = \frac{M}{(m_i + m_j) m_1 m_2 + m_2 m_3 + m_3 m_1}.$$  

Comparing with equation (8), one can obtain the identity

$$\frac{m_2 + m_3}{M} \sin^2 \alpha_3 + \frac{m_3 + m_1}{M} \sin^2 \alpha_2 + \frac{m_1 + m_2}{M} \sin^2 \alpha_1 = 1.$$

This means the three hyper-angles are correlated and the correlation depends on the particle masses.

In order to test our model, we plot the total energy of $^4$He atom versus the hyper-radius for different hyper-angles and our semiclassical approach is in good agreement with the experimental data shown in figure 2. As one can see, our results are different from the ones obtained.
from the quantal approach, but the deviation becomes less when $n$ increases as expected since screening is less important. Since one electron remains in its lowest level and the other is excited for very large $n$, the $^4\text{He}$ atom becomes hydrogen-like system where the Bohr model works perfectly. These results encouraged us to investigate other interesting systems, i.e. the TS and ES in atomic system and nuclear system, with short range interactions only.

3. The Thomas theorem and beyond

For simplicity, we will only consider equal masses system in the following. We make a further assumption and write the short-range two-body potential as:

$$V(r_{ij}) = -c \frac{\hbar^2}{ma^2} \exp[-(r_{ij} - r_0)^2/a^2].$$

(12)

Where $c$ is the (only) fitting parameter. Using this form (or any other short range potential form which can be written as $V(r_{ij}, r_0, a)$) of the two-body potential and the Heisenberg relation above, we can write the energy of a two-body system in a-dimensional form:

$$E_2 = \frac{\hbar^2}{mr_{ij}^2} + V(r_{ij}) = \frac{\hbar^2}{ma^2} \left[ \frac{1}{x^2} - c e^{-(x-x_0)^2} \right].$$

(13)

where $x = r_{ij}/a$ and $x_0 = r_0/a$. The coefficient before the square parenthesis determines the systems’ units. For a three-body system, the hyper-angles $\alpha_k$ and hyper-radius $R$ coordinates become [3]:

$$R^2 = \frac{1}{3} (r_{12}^2 + r_{23}^2 + r_{31}^2),$$

(14)

and

$$r_{ij} = \sqrt{2} R \sin \alpha_k.$$
The energy of the three-body system is:

\[
E_3 = \frac{1}{1.5m} \times \left[ \frac{\hbar^2}{\left( \sqrt{2} R \sin \alpha_3 \right)^2} + \frac{\hbar^2}{\left( \sqrt{2} R \sin \alpha_1 \right)^2} + \frac{\hbar^2}{\left( \sqrt{2} R \sin \alpha_2 \right)^2} \right] \\
+ V(\sqrt{2} R \sin \alpha_3) + V(\sqrt{2} R \sin \alpha_1) + V(\sqrt{2} R \sin \alpha_2) = \frac{\hbar^2}{ma^2} \\
\times \left\{ \left[ \frac{1}{(\sqrt{2} \sin \alpha_3)^2} + \frac{1}{(\sqrt{2} \sin \alpha_1)^2} + \frac{1}{(\sqrt{2} \sin \alpha_2)^2} \right] \right\} 1.5x^2 \\
- ce^{-x(\sqrt{2} \sin \alpha_3-x_1)^2} - ce^{-x(\sqrt{2} \sin \alpha_1-x_1)^2} - ce^{-x(\sqrt{2} \sin \alpha_2-x_1)^2} \right\},
\]

where \(x = R/a, \alpha_1 \) is in \([0, \pi/2]\), \(\frac{1}{2} \pi - \alpha_1 < \alpha_2 < \frac{1}{2} \pi \) and \(\frac{1}{6} \pi - \alpha_1 \) and the identity of the relation of the three hyper-angles reduces to

\[
\sin \alpha_1^2 + \sin \alpha_2^2 + \sin \alpha_3^2 = \frac{3}{2}.
\]

Similar to equation (15) the dimensions are contained in the term before the curly brackets. Equations (13) and (16) can be reduced to a one-dimensional form by defining the universal energies \(\varepsilon_2 = \frac{E_3}{\hbar^2/ma^2}\) and \(\varepsilon_3 = \frac{E_3}{\hbar^2/ma^2}\).

Some notable cases: three equal hyper-angles \(\left(=\frac{\pi}{3}\right)\) geometrically correspond to an equilateral triangle and it is a ground state (TS); one of the hyper-angles equals \(\frac{\pi}{2}\) (the other two hyper-angles can be derived from the constraints above) results in a linear configuration (similar to the \(^4\)He atom ground state discussed above). Another extreme case is when one of the hyper-angles approaches zero resulting in the monomer + dimer configuration (which we indicate as an ES, i.e., the ES, i.e., the energy has a finite, while the other hyper-angles \(\alpha_{j,k} \rightarrow \frac{\pi}{2}\)). This is probably the largest limitation of the model. In this limit we expect the series of ES to appear, see equation (1), but we have not been able to derive any relation to recover them.

In order to derive the Thomas theorem we set \(r_0 = 0\) and interpret \(a\) as the potential range \([1]\). In the limit \(a \rightarrow 0\) the Gaussian potential in equation (12) becomes a \(\delta\)-function and the term \(\frac{\hbar^2}{ma^2} \rightarrow \infty\), thus in order to recover the theorem we need to show that \(\varepsilon_3\) has a negative minimum and the minimum of \(\varepsilon_2\) equal to zero. There are many other methods to demonstrate the Thomas theorem within our framework and our ‘short cut’ here will be useful for the remaining of the paper. Thus we require that \(\varepsilon_2 = \varepsilon_3 = 0\), i.e., the dimer has a minimum with zero energy. With these conditions we can determine the parameter \(c\) in equation (13) and derive the trimer energy from equation (16). Using symmetry considerations, it is clear that the minimum energy of the three-body system is recovered when the three hyper-angles are equal (TS) and this can be easily demonstrated analytically. In figure 3 we plot \(\varepsilon_2\) (top) and \(\varepsilon_3\) (bottom) versus \(x\). The dimer is ‘loosely’ bound while the trimer becomes strongly bound and taking \(a \rightarrow 0\) results in the diverging \(\varepsilon_3\). We can study under which conditions of the parameter \(c\) gives a bound trimer. In figure 3 we have plotted the cases where the interaction strength is obtained imposing \(\varepsilon_2'' = \varepsilon_3'' = 0\), i.e., the energy has a (flex)-dashed line, \(\varepsilon_2'' = \varepsilon_3'' = 0\)-dashed line and \(\varepsilon_2'' = \varepsilon_3'' = 0\)-dashed-dotted line. Thus the trimer is loosely bound when the two-body channel displays a flex in energy. A weak two-body attraction still results in a resonance in the trimer.

4. Atoms and nuclei with short range forces

We now discuss some physical systems starting with the \(^4\)He atomic case. This system has been studied at length both experimentally and theoretically [3]. The experimental scattering length \(a = 197.335 \times a_0\) is the Bohr radius but the effective range is not known. Theoretical models, which reproduce the large scattering length \([3, 30–36]\), give \(r_i = 13.85a_0\) and such value depends somehow on the chosen interaction. From equation (1) we expect to have at least one ES. The binding energy of the dimer is \(E_{2} = 1.15 m\) kK [3, 29], which we can use to fit the parameter \(c\). Detailed theoretical models give two bound states for the trimer. For instance Motovilov et al [30] give \(E_{3} = 125.8\) mK and \(E_{3} = 2.28\) mK, the highest binding has been confirmed experimentally [37]. There is a large consensus that \(E_{4}\) is an ES [3] and a recent experimental confirmation presented in [9]. We can build an analogy with the \(^{12}\)C nucleus, its binding energy is \(E_{4} = (−) 92.162\) MeV while the Hoyle excited state is at \(E_{4} = (7.68 − 7.65) \times 12 = 0.36\) MeV. The large difference between the ground state and the excited one is due to the large binding energy of the \(\alpha\) particle. In fact, because of the large binding, the first particle that can be emitted (not a \(g\) from the excited \(^{12}\)C is an \(\alpha\)-particle leaving a \(^{8}\)Be that subsequently decays into \(^{2}\)Be since its g.s. is unbound. The Hoyle state decays into a monomer + dimer (which later decays again) thus the suggestion that the Hoyle state is an ES [2]. On the other
hand, the ground state of $^{12}$C cannot be explained in terms of $\alpha$-clusters but rather of single nucleons degrees of freedom [38] and this fact is at the root of its large binding energy thus it is not a TS. Given this preamble, in figure 4 we plot the $E_2$ (panel (a)), $E_3$ (panel (b), $r_s = 0$) and $E_3$ (panel (c), $r_s = 13.85a_0$) versus $x$ for the $^4$He atoms. We have fitted the potential parameter to $E_2$ and considered two cases with and without effective range correction. The two limits are very similar since $r_s \ll a$. In the panel (b), we display $E_3$ for $r_s = 0$ and we have changed the scattering length within the experimental errors to see the changes in the ground energy (shaded region), corresponding to three equal hyper-angles. Similar results are obtained for finite effective ranges as expected. As we see from the figure the TS is relatively unbound (6 mK) respect to the theoretical $E_3^0 = 125.8$ mK, while the ES which can be obtained in the limit $\alpha \rightarrow 0$ results in $E_3^0 = 1.98^{+0.2}_{-0.06}$ mK, very close to other models [3]. Thus, in analogy to the $^{13}$C comparison above, we conclude that the TS does not occur because of the dominance of other degrees of freedom not included in our simple model. Other hyper-angle values, for instance corresponding to the linear configuration, are not the true g.s. (the one with the lowest energy) which is given by the equal sides configuration. If an external field, for instance strong magnets, could influence the interaction and break the symmetry, then one would be able to see other geometrical configurations.

We now turn to the cases where the scattering length is negative. One example is given by the polarized triton ($^3$H) atoms whose (triplet) scattering length $a_0 = -82.1a_0$, while the effective length is not known and models give $r_\alpha = 13.7a_0$ [3]. Since the scattering length is negative, there should be no dimer bound state, while the experimental value of the trimer $E_3 = 4.59$ mK [3, 39]. Since we cannot fit the $c$-value to $E_2$, we decided to fit $E_3$ instead assuming that its experimental value corresponds to the TS. In the panels (d)–(f) of figure 4 we report the results obtained under this (strong) assumption. The cases with zero (panel (e)) and finite (panel (f)) effective range are given as well. Since we fix the parameter to $E_3$ the two results are not much different while some difference can be noticed in $E_3$, which now becomes a model prediction for the dimer suggesting a resonant state. If our assumption that the TS is the experimental $E_3$ then we expect an ES at $E_3^0 = 1.53$ mK. On the other hand, if the TS is hindered by other degrees of freedom, in analogy with the $^4$He and $^{12}$C above, then we should fit the parameter $c$ to the ES. In the panels (g)–(i) of figure 4, we plot the results where we fit the parameter $c$ assuming that the ES energy is equal to the experimental $E_3$. As we see from the figure, the TS becomes largely bound as well as the dimer state in contrast with the negative value of the scattering length. Thus our approach suggests

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Universal dimer (top) and trimer energy (bottom) versus scaled hyper-radius $x = r_j/a$ for two-body system and $= R/a$ for three-body system, see text. The potential strength $c$ is obtained imposing $\epsilon_1 = \epsilon_1' = 0$ ($c = 2.718$, full-black line), $\epsilon_1' = \epsilon_1'' = 0$, ($c = 1.847$, long-dashed red line), $\epsilon_1'' = \epsilon_1''' = 0$, ($c = 1.331$, short-dashed blue line), $\epsilon_1''' = \epsilon_1'''' = 0$, ($c = 0.988$, dotted-dashed purple line). Thus bound or resonance states in the trimers disappear when $c < 1$.}
\end{figure}
that there might be an ES around 1.53 mK. Our ansatz can be tested experimentally, but what we want to point out as a consequence of negative scattering lengths is the appearance of unbound levels when $a_{\text{pp}} \approx \frac{\pi}{10}$, figure 4.

For t nuclei all the physical quantities needed to test our model are well known and reported in the introduction. A complication might be due to the fact that the interaction between protons and neutrons and neutrons-neutrons with different spins might not be the same. We will assume the same potential for all non-identical Fermions (the two neutrons must have different spins because of the Pauli blocking). The small difference between our results and experiments will be discussed in more detail in a following paper. In figure 5, we plot the d and t energies versus $x$ with and without effective range correction. The parameter $c$ was fitted to the binding energy of the deuterium. We see a very good agreement with $r_s$ given by the experimental value. Since we do not expect the quarks degrees of freedom to play an important role, we conclude that the trimer ground state is indeed a TS. Small hyper-angles give an ES at about 7 MeV excitation energy in agreement with the experimental results [4, 5], which still await further experimental confirmation.

It should be pointed out that our approach is not only applicable to the light equal masses three-body system, i.e., 4He and triton atoms as well as triton nuclei, but also to heavier atoms or nuclei. The procedure is the same as discussed above. Our approach can also be applied to the unequal masses three-body system, i.e., 4He atom (see the discussion in section 2), 19B nuclei discussed in [40] and $^4$Be nuclei discussed in [41].

5. Link to Landau’s theory of phase transitions

Another interesting feature arises and can be clearly seen in the figure 5. When the hyper-angle approaches a critical value $\alpha_{\text{cr}} = \frac{\pi}{12.228}$ two equal minima appear. We interpret such critical value as the border between TS, i.e., larger hyper-angles result in one absolute minimum, and the ES, i.e., smaller $\alpha_i$ result in an excited level which scales as the product $x\alpha_i$. The critical $\alpha_i$ appears when the $r_i$ is close to $a$, compare figures 4 and 5, and its
shape resembles an isotropic-nematic first order phase transition. The question arises: when the $r_3$ is negligible does the phase transition become second order?

Following Landau’s theory of phase transitions we assume that the (free) energy has a singular part, which can be written as:

$$\varepsilon_3 = \varepsilon^*_3 + d(\alpha_i - \alpha^*_i) \left( \frac{x - x^*}{2} \right)^2 + e \left( \frac{x - x^*}{3} \right)^3 + f \left( \frac{x - x^*}{4} \right)^4 + \cdots,$$

(18)

where we have neglected any external field [42]. $d$, $e$ and $f$ are fitting parameters. We have assumed that $\alpha_i$ and $x$ play the roles of the control and order parameters respectively. In figure 6 (top panels) we plot the a-dimensional energy opportunely shifted versus $x - x^*$ for cases where the ratio $\frac{\alpha_i}{x^*}$ is not small, see equation (18). In the right panel we have plotted the results using the Landau’s (free) energy, equation (18), for an opportune choice of the fitting parameters.

The odd order term in equation (18) is crucial and breaks the symmetry $(x - x^*) \rightarrow -(x - x^*)$ similar to a nematic first order phase transition [27]. The phase transition is due to a change of shape, in fact for $\alpha_i = \alpha^*_i$ the two minima corresponding to two different distances between the monomer and dimer as given by $x_{\pm} = \sqrt{2} x \sin(\alpha_i)$, it also signals the point where we have a transition from the TS (dominated by the absolute minimum for equal energies) and the ES where the energy scales with $x \sin(\alpha_i)$. Taking the ratio $\frac{\alpha_i}{x^*} \rightarrow 0$, the two minima disappear (for $\frac{\alpha_i}{x^*} = 0.12$) and now we can reproduce the reduced energy versus $x$ with the Landau’s form with $\varepsilon = 0$ and $\alpha^*_i = \alpha^*_i$, equation (18).
6. Summary

In conclusion, in this work we have discussed a simple model based on the Bohr atom and hyper-coordinates to study the transition from Thomas states to Efimov states. The model can be applied to the equal masses and unequal masses system, but limited to three-body system. We first recovered the Thomas theorem. We have shown the t nuclei have both the TS (the ground state) and ES (around 7 MeV excitation energy) if existing experiments are confirmed, while in $^4$He and $^3$H atoms (as well as $^{12}$C nucleus) the TS might disappear because other degrees of freedom which can not be taken into account in our model (for instance nucleons versus $\alpha$-clusters in $^{12}$C) become dominant giving rise to a deeper minimum. We feel confident that the transparent features discussed here will be of great help in guiding future experimental and theoretical investigations.

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Figure 6. (Top panels) A-dimensional energy opportunely shifted versus $x - x^*$ for cases where the ratio $\frac{a}{\alpha}$ is not small (left) and the results using the Landau’s (free) energy, equation (18) for an opportune choice of the fitting parameters (right); (Bottom panels) Same as the top panels, for case where the ratio $\frac{a}{\alpha} = 0$ (left) and $\varepsilon = 0$ and $\alpha_r = \alpha_\lambda$ in Landau’s (free) energy, equation (18).
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