The solution of the Painleve equations as special functions of catastrophes, defined by a rejection in these equations of terms with derivative

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February, 4, 1999

Abstract

The relation between the Painleve equations and the algebraic equations with the catastrophe theory point of view are considered. The asymptotic solutions with respect to the small parameter of the Painleve equations different types are discussed. The qualitative analysis of the relation between algebraic and fast oscillating solutions is done for Painleve-2 as an example.

1. From six equations the Painleve \((a, b, c, d - \text{constants})\)

\[
P1: \quad w_{xx} = 6w^2 + x,
\]

\[
P2: \quad w_{xx} = 2w^3 + xv + a,
\]

\[
P3: \quad w_{xx} = \frac{w^2}{w} - \frac{w_x}{x} + \frac{(aw^2 + b)}{x} + cw^3 + \frac{d}{w},
\]

\[
P4: \quad w_{xx} = \frac{w^2}{2w} + \frac{3w^3}{2} + 4xw^2 + 2(x^2 - a)w + \frac{8b}{w},
\]

\[
P5: \quad w_{xx} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w_x^2 - \frac{w_x}{x} + \frac{2}{x^2}(w - 1)^2\left(\frac{aw + b}{w} + \frac{cw}{x} + \frac{dw(w + 1)}{w - 1}\right),
\]

\[
P6: \quad w_{xx} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-x}\right)w_x^2 - \left(\frac{a}{x} + \frac{1}{x-1} + 1 \frac{w-x}{1}\right)w_x + \frac{w(w - 1)(w - x)}{x^2(x - 1)^2}(a + bx + \frac{c}{(w-1)^2(x-1)} + \frac{dx(x - 1)}{(w-x)^2}),
\]

first five equations can be obtained from sixth using following passages to the limit [1]:

1
substitution in $P_6$ $1 + cx$ instead of $x$, $d/e^2$ instead of $d$, $ce - de^2$ instead of $c$ with $\epsilon$ tending to zero gives the equation $P_5$;

in the equation $P_5$ a substitution $1 + \epsilon w$ instead of $w$, $-b/e^2$ instead of $b$, $b/e^2 + a/e$ instead of $a$, $c/e$ instead of $c$, $d/e$ instead of $d$ and the passage to the limit $\epsilon \rightarrow 0$ gives the equation $P_3$;

in the same fifth equation the Painleve we shall substitute $\epsilon w \sqrt{2}$ instead of $w$, $1 + \epsilon \sqrt{2}$ instead of $w$, $1/(2e^4)$ instead of $a$, $-1/e^4$ instead of $c$, $-1/(2e^4)$ $d/e$ instead of $d$. In this case in a limit there will be an equation $P_3$;

in turn, substitution in equation $P_3$ $1 + \epsilon x$ instead of $x$, $1 + 2\epsilon w$ instead of $w$, $1/(4\epsilon^6)$ instead of $c$, $-1/(2\epsilon^6)$ instead of $a$, $1/(2\epsilon^6) + 2b/e^3$ instead of $b$ gives in a limit the equation $P_2$;

the equation $P_2$ can be obtained by a passage to the limit as well from the equation $P_4$ using a substitution $\epsilon x 2^{-1/3} - 1/e^3$ instead of $x$, $2^{2/3}\epsilon w + 1/e^3$ instead of $w$, $-1/(2\epsilon^6) - a$ instead of $a$, $-1/(2\epsilon^{12})$ instead of $b$;

and, at last, $P_1$ it turns out as a limit from the equation $P_2$ at a substitution $\epsilon^2 x - 6/e^{10}$ instead of $x$, $\epsilon w + 1/e^5$ instead of $w$, $4/e^{15}$ instead of $a$.

Thus, the Painleve equations form the following hierarchy:

$$P_4 \leftarrow P_6 \rightarrow P_5 \rightarrow P_2 \rightarrow P_1 \leftarrow P_3$$

The hierarchies

$$H_4 \leftarrow H_6 \rightarrow H_5 \rightarrow H_2 \rightarrow H_1 \leftarrow H_3 \leftarrow H_4,$$

$$L_6 \rightarrow L_5 \rightarrow L_2 \rightarrow L_1 \leftarrow L_3$$

form also Hamiltonians $H_1 - H_6$ of the Painleve equations and the linear ordinary differential equations of the second order $L_1 - L_6$ on an auxiliary parameter $\lambda$ found in [1], [2] which permit to investigate asymptotic behavior of the equations $PJ$ with the help of monodromy preserving method.

In the present work we call attention to natural and usefulness relation of the Painleve equations with a hierarchy

$$P_{4_0} \leftarrow P_{6_0} \rightarrow P_{5_0} \rightarrow P_{2_0} \rightarrow P_{1_0} \leftarrow P_{3_0}.$$
The following algebraic equations were obtained from the equations $P1 - P6$ by a neglect by terms containing derivative

\begin{align*}
P_{10} & : \quad 6w^2 + x = 0, \\
P_{20} & : \quad 2w^3 + xw + a = 0, \\
P_{30} & : \quad (aw^2 + b)/x + cw^3 + d/w = 0, \\
P_{40} & : \quad 3w^3/2 + 4xw^2 + 2(x^2 - a)u^2 + b/w = 0, \\
P_{50} & : \quad (w - 1)^2(aw + b/w)/x^2 + cw/x + 3w(w + 1)/(w - 1) = 0, \\
P_{60} & : \quad w(w - 1)(w - x)(a + bx/w^2 + c(x - 1)/(w - 1)^2 + \\
& \quad \frac{dx(x - 1)}{(w - x)^2})x^2(x - 1)^2 = 0.
\end{align*}

2. It is well known, that it is possible to consider the Painleve transcendents as nonlinear analogs of special functions, allowing integrated representations of the Fourier type. Arising in applications at exposition of various fast transients, transcendents of the Painleve allow, due to a possibility of application to they of the monodromy preserving method, similarly to the special functions, effectively to solve problems of exposition of asymptotics at various values of argument $x[7], [8].$

In view of this analogy the problem of exposition of uniform asymptotics for the Painleve transcendents is actual (that is, about them asymptotics also at $x^2 + a^2 + b^2 + c^2 + d^2 \to \infty$), effectively solved in case of special functions, supposing integrated representation. (The first step in this direction was actually made recently in works by A.A. Kapaev[9], [10], devoted to an asymptotics of a common solution of the equation $p2$ at $\Re(a) \to \infty$ for any value of $x$. In spite of their importance, we shall underline at once, that in these works the speech nevertheless goes about exposition pointwise, instead of uniform on $x$ asymptotics.)

First of all just in view of this problem alongside with a hierarchy of equations the Painleve, it is reasonable to separate reviewing of the entered above appropriate hierarchy of the algebraic equations $P_{10} - P_{60}$. Their solutions define the most simplest asymptotics of the Painleve transcendents. (We shall mark also, that until now in most of all mathematical physics applications arose the Painleve transcendents only extremely in relation just with such asymptotics.)

All terms of this hierarchy of the algebraic equations can be consecutively obtained from each other with the help of the same replacements, that were described in section 1 for terms of a hierarchy ordinary differential equations $P1 - P6$.

Thus a quadratic equation $P_{10}$ is the canonical equation of a fold type, defining in a terminology of the theory of catastrophes [5], arises from each of the stayed algebraic equations of a hierarchy in an exactitude at those values $x$ and parameters $a, b, c, d$ (forming in aggregate gang controlling of parameters of catastrophes circumscribed by the equations $P_{J0}$), at which happens confluence of two roots of the appropriate equation $P_{J0}$ (in a situation of ”the general provisions” multiple roots are absent). The solution $P_{10}$ describes processes these confluence, being not limited immediately by moment of degeneration (to
which in the equation $P_{10}$ only a value $x = 0$, but includes exposition of the appropriate reorganizations, when "... the parameter, varying, passes through a degenerate value ".

Similarly, the cubic equation $P_{20}$, defining canonical equation of the cusp catastrophe, arises from each of higher one to ratio to it of the algebraic equations $P_{30} - P_{60}$ hierarchies when three various roots stick together.

The equations $P_{30}$ and $P_{40}$ turn out from the equations $P_{50} - P_{60}$ when four various radicals stick together.

Extreme the left arrow of a hierarchy $P_{10} - P_{60}$ corresponds the confluence of five solutions of an equation $P_{60}$, and from it in an outcome of a passage to the limit there is an equation $P_{50}$.

The derivative of the solutions for the given hierarchy of the algebraic equations will be tend to infinity in points, in which they have the multiple roots. Therefore it becomes untrue a neglect terms with derivative in small neighborhoods of such points at exposition the Painleve transcendents assigned in a principal order to the appropriate solutions $P_{J0}$. Just therefore in small neighborhoods of such points there is essential a hierarchy, circumscribed in section 1, of degenerations the Painleve transcendents.

It is naturally to consider this hierarchy as a hierarchy of nonlinear special functions of catastrophes (SFC) of a hierarchy, circumscribed by the equations $P_{J0}$.

Note 1. Similarly to the linear analogs (solutions of such equations, as, for example, equation Bessel, Weber - Hermit, Whittaker and hypergeometric equation of the Gauss [11]) all solutions of the Painleve equations simultaneously are solutions of difference equations by a variable $a, b, c, d$ [12] on all from their included in right hand side. Per last years the subjects devoted to these equations in a combination to their continual limits, being lower terms of a hierarchy the Painleve transcendents has become popular. The popularity of this subjects was stimulated in the large degree by series of works of a beginning of the 90-th years under the quantum theory of a gravitation [13] - [15]. We mark here, that similarly to degenerations of a hierarchy of the Painleve equations, all these continual limits are carried out also during such small modification controlling parameters of catastrophes, circumscribed by the algebraic equations $P_{J0}$, at which their solutions pass through singularities.

3. It is traditional [16], the Painleve equations are considered as nonlinear analogs known of the linear theory (SFC), whose integrated representations are similar to a behaviour in neighborhoods of singular points $\lambda$ of solutions of simple equations of the monodromy preserving method.

The treatment, offered in the previous section, the Painleve transcendents as is essential nonlinear special functions is based on essence other principle ascending to variant [17] of the approach to nonlinear SFC, which initially being as solutions of the partial differential equations, (such, as the Korteweg-de Vries, Burgers and Nonlinear Schrodinger equations). The traditional treatment of the Painleve transcendents as analogs of the Fourier integrals was used in offered earlier [18], [19] variant of the approach to such SFC.

It is curious, that in all considered until now examples this variant and
variant offered in [17] "commuted" in the sense that proceeding from different principles, invariable associated with the same algebraic equation, defining the appropriate catastrophe.

Thus, for example, that fact, that associated with the canonical equation of catastrophe of cusp

\[ x - tv + v^3 = 0 \]  \hspace{1cm} (1)

known special Gurevich-Pitaevskii solution (GP) [20], [21] of the Korteweg-de Vries equation

\[ v_t + vv_x + v_{xxx} = 0 \]  \hspace{1cm} (2)

is simultaneously solution of the ordinary differential equation

\[ v_{xxxx} + 5vv_{xx}/3 + 5v^2_x/6 + 5(x - tv + v^3)/18 = 0, \]

initially appeared in [22] proceeding from that circumstance, that the GP solution [20] is analog of special function for the cusp catastrophe:

\[ J = \int_R \lambda \exp(-2i(x\lambda + 4t\lambda^3 + 3456\lambda^7/35))d\lambda \]

\( J \) satisfies of a linear part of (3), and its asymptotics at \( x^2 + t^2 \to \infty \) following to a method of a stationary phase [25], is defined in terms of solutions of (1).

In [17] was specified, that the validity for a special GP solution of the given equation could be deduced from that fact, that at a rejection in it the derivative there is in exactitude a solution of the cusp equation (1), which defines [20] an asymptotics on infinity of a solution GP outside of area of its fast oscillations.

Let’s mark, that similar "commutation" of two treatments of the Painleve equations as nonlinear SFC has a place of the equations \( P1 \) and \( P2 \). Really, according to traditional treatment the equation \( P1 \) is nonlinear analog of the Fourier integral

\[ \int_R \exp(ix\lambda + i\lambda^5)d\lambda, \]

and equation \( P2 \) is the analog of an integral

\[ \int_R \lambda^a \exp(-ix\lambda + i\lambda^4)d\lambda. \]

As on a method of a stationary phase an asymptotics of first of them at \( x \to \infty \) can be expressed in terms of a solution for the equation \( P1_0 \), and the asymptotics of second integral at \( x^2 + a^2 \to \infty \) in terms of solutions \( P2_0 \), at both treatments the first Painleve is considered in quality SFC such as canonical fold catastrophe, and second such as cusp catastrophe.

However for the remaining Painleve equations the fact of similar "commutation" is not traced, probably, truth, only at present. But, in any case, in an association from a context of a problem, in which there can be that or other Painleve equation, it is useful to mean both treatments.

4. That fact, that in process of the onfluence of the roots of a hierarchy of the algebraic equations \( PJ_0 \) at exposition of asymptotics of the equations \( PJ \)
is necessary to use the solutions of the Painleve equations $PK$ with numbers $K < J$ does not solve completely problem on exposition of a behaviour of these solutions of the equations $PJ$ in small neighborhoods of values controlling parameters of catastrophes $PJ_0$, at which happens the onfluence of the various roots of this equation. The additional analysis is necessary here. It is clear even on an example of an outcome of degeneration of the second Painleve equation to the first Painleve equation:

While the solutions of the second Painleve equation can not have poles higher than first order, all real solutions of the equation $P1$ at real $x$ have an infinite set of second order poles.

The similar position has a place and for degenerations others the Painleve equations. (It is also for degenerations of difference equations, with which simultaneously satisfy appropriate the Painleve equations.)

Besides the analysis about influence of derivative asymptotics is necessary and after completion of reorganization of solutions of the equations $PJ_0$ during a modification controlling parameters $x, a, b, c, d$, during which these solutions of the algebraic equations pass through singularities.

In the stayed sections of the paper, we shall not concern a problem on an improvement a behaviour of solutions of the equations $PJ$ in neighborhoods of poles of solutions of the lowest terms of a hierarchy of the Painleve equations, by which they are reduced during passages to the limit, circumscribed in section 1. Up to the present moment it remains not investigated.

Let’s mark only that circumstance, that the special solutions of first Painleve equations having at $x \to \infty$ asymptotic expansions as ascending power series should arise similarly at exposition of asymptotics of solutions for a wide class of the nonlinear equations (as ordinary, and in partial derivatives) with small parameters at derivative.

In particular, these solutions arise at exposition formal asymptotics of a solution for singular - perturbed equation

$$i\epsilon \Psi_t + (|\Psi|^2 - t)\Psi = 1 \quad (3)$$

in a small neighborhood of a point $t = T = 3/2^{1/3}$, which at $t > T$ has asymptotic expansion as a series

$$\Psi_0(t) + \epsilon \Psi_1(t) + \epsilon^2 \Psi_2(t) + \ldots,$$

in which the principal term $\Psi_0 = u(t)$ is one of solutions of a cubic equation

$$u^3 - tu = 1,$$

having a singularity in a critical point $t = T$. The problem that happens, for example, with an asymptotics of a solution at passage through a critical point, which first of two authors of the given paper was set of problem, and it have reduced at the end to this preprint.

4. The rest of the given work is devoted, in basic, research of connection between nonoscillatory and fast oscillating by asymptotics on a small parameter
ε to the real solution of the equation:

$$\varepsilon^2 \partial_t^2 u + 2u^3 - tu = 1,$$

(4)

by obvious replacement reducible to only imaginary (at real x) solutions of the equation $P2$. Asymptotic solutions of this equation as a series

$$u = u_0 + \varepsilon^2 u_1 + \varepsilon^4 u_2 + ...,$$

(5)

where the principal term $u_0$ is the solution of a cubic equation

$$2u^3 - tu = 1,$$

(6)

This asymptotics becomes unsuitable at passage through points of a singularity of these solutions (after their passage $u_0$ ceases to be clean by real function).

It is necessary to take into consideration the influence of the derivative for exposition of a principal term of an asymptotics. It is naturally for the considered problem to expect, that there will be fast oscillating asymptotic solutions (WKB-solutions, averaged on Kuzmak-Witham of a solution) of the equation (4), which principal term was actually constructed by Kuzmak in him widely to known work [23] (construction of full asymptotic expansion and its justification is present in [24]).

The results of the numerical account, reduced in section 5, confirm this supposition for one of three solutions of a cubic equation (6).

We will be interested by a problem on connection of the asymptotic solution of the equation (4), which main term is in the root of a cubic equation, and the oscillating asymptotic solutions for the equation Painleve-2 is constructed by Kuzmak [23], which, as we shall assume, are served for explanation of the results for this numerical account.

Our problem is from a two-parameter set of fast oscillating asymptotic solutions to choose a solution, in which passes an asymptotic solution (5) after passage $t$ through a critical value $t^* = t_0$.

Let’s underline, that we consider this problem in the first turn as a model example for study of the connection of various types of asymptotic solutions of a wide class of the singular-perturbed nonlinear differential equations (and not only ordinary). The choice of this equation as a model is explained by its simple form, and also analytically any solution at real $t$ is not used, that allows to hope on applicability of similar reasonings and at the analysis of unintegrable problems.

Note 2. Besides the results [3],[10] give uniform asymptotics in small ranges of an explanatory variable $x - x_0 \ll \alpha^{-1/3 - \delta}$, for anyone as much as small $\delta > 0$ or, accordingly $t - t_0 \ll \varepsilon^{-1 - \delta}$ (here $x_0 = const$). In our work the uniform asymptotics on $\varepsilon$ are constructed in the large area (order $O(1)$) of the explanatory variable $t$ in natural terms for a considered problem of stretchings and replacements. It is clear, that only use of similar stretchings and replacements can allow to receive the satisfactory answers in unintegrable problems.
But as one more argument for the benefit of our choice, certainly, the connection with explained above in sections 1-3 has served also. The equation (4) by the stretchings $X = \varepsilon^{2/3}t$, $W = \varepsilon^{1/3}u$, $\alpha = \varepsilon^{-1}$ is reduced to the equation

$$y_{xx} - xy + 2y^3 = \kappa.$$  \hfill (7)

It is one of the forms of the second Painlevé equation (replacement $Y = iw$, $k = ia$, reduce it to canonical form $P^2$). Outside of direct $K = 0$ the problem on an asymptotics at $x^2 + k^2 \to \infty$ of solutions of this equation is reduced to study an asymptotics on $\varepsilon$ uniform on $t$ of the solution (4).

5. The asymptotic solution (3) is defined completely by a choice of the roots of the cubic equation for the principal term of the asymptotics. There is a point $t_*$, more to the right of which the cubic equation has three material roots $u_{00}(t) > u_{01}(t) > u_{02}(t)$. In a point $t = t_*$ the roots $u_{01}(t)$ and $u_{02}(t)$ stick together. The value $T_*$ and magnitude $u_* = u_{01}(t_*) = u_{02}(t_*)$ are defined from the equations:

$$2u_*^3 - t_* u_* = t, \quad 6u_*^2 - t_* = 0.$$  

Thus asymptotic solution (3) with a principal term $u(t)_0 = u_{00}(t), i = 1, 2$ is suitable at $t > t_*$. In the neighborhood of the point $t_*$ such expansion (3) loses the asymptotic character because of growth of the correction terms. If as a principal term to take $u_{00}(t)$, then the asymptotic solution (3) will be suitable at any value $t$. (This reasoning is confirmed in the following paragraph by results of numerical experiments).

The choice of a considered below principal term of a solution (3) is connected also to the problem on its stability in relation to small perturbations. The solutions with principal terms $u_0(t)$ or $u_2(t)$ are stable. It follows from the analysis of the equation (4), linearized on the principal term of the formal asymptotics:

$$\varepsilon^2 \partial^2_t v + (6 u^2(t) - t)v = 0.$$  

Therefore for the analysis of connection between an asymptotic solution (3) of a dispersionless limit and oscillating asymptotic solutions at $t > t_*$ we shall choose an asymptotic solution in the form (3) where the principal term is $\hat{u}(t) = u_{02}(t)$. At first, - it is stable, secondly, - it is not suitable at $t < t_*$. More to the left of the point $t_*$ we shall construct fast oscillating asymptotic solution, which is in turn unsuitable at $t > t_*$ and its principal term continuously passes in a leading term asymptotic solution (3).

Let’s reduce outcomes of a numerical solution by a Runge-Kutta method of a Cauchy problem for (4) with initial data -, appropriate to the three roots of the equation (4).
From the figure 1 it is obvious, that after passage through a point $t_*$ on the right to the left the solution becomes fast oscillating. The process of origin of these oscillations represented on figure 1 is pertinent, using analogies from the theory of bifurcations [3], to name to a rigid regime of establishment of oscillations. As it is visible from this figure at $a \to -\infty$, these the amplitude of these oscillations aspires to zero. In a combination to results of numerical experiment reduced on Figure 1 these circumstances allows to put forward a hypothesis that in a neighborhood direct $k = 0$ in an asymptotics of a solution of the equation (7) $Y$ at $x^2 + k^2 \to \infty$ qualitative reorganization (which, again using arising analogies to a standard terminology of the theory of bifurcations also happens, it is natural to name as a soft regime of origin of oscillations).

Unfortunately, even in case of a validity of this hypothesis, solution of the important problem of deriving uniform asymptotics $y$ at $T \to -\infty$ on the present moment is not obtained. We shall be forced to be limited to exposition of this reorganization at a level of presentation in spirit [9], [10] asymptotics $y$ at $t \to -\infty$ for anyone fixed $k$:

At $k < 0$ it carries an algebraic character and easily it turns out by approaching replacements from a series (5):

At $k = 0$ the principal term of an asymptotics $y$ is obtained in recent work [26] (exposition after that full asymptotic expansion turns out with the help of standard reasonings)

$$Y = d(-t)^{-1/4} \sin(2/3(-t)^{3/2}) - 3d^2 ln(-t)/4 + \beta,$$

with some constants $d, \beta$, which are calculated in [26];
And at \( k > 0 \) the answer is described with the help by an fast oscillating asymptotic solution from units 7 of the present work. (From results of unit 7 follows, that the considered there asymptotic solution aspires to zero at \( t \to \infty \).

6. The special character of direct \( k = 0 \) for considered by us SFC of cusp, which in this case is described with the help of the solutions of the cubic equation

\[
2y^3 - xy - k = 0, \tag{8}
\]

carries a not casual character. It forms so-called Maxwell stratum of cusp catastrophes. (From a point of view of the catastrophes theory the Maxwell stratum what is selected because on it the various radicals (3) define the same value of a primitive \( y^4/2 - xy^2/2 - ky \) it was specified of the left part of this cubic equation.) It was specified already in [22], the special character of the Maxwell stratum one can find in the asymptotic expansions of a lot various linear and nonlinear SFC associated with catastrophe of cusp.

We would like to pay attention here for the following:

In one of papers V.R.Kudashev and second of the authors of the present paper preparing in the present moment for the publication, it is marked, that for a some nonlinear SFC (equations, being simultaneously solutions of Burgers and Nonlinear Schrodinger equations) the Maxwell stratum is selected not only at a level of asymptotic expansions:

for these SFC on it the "increased" integrability has a place. The ordinary differential equations, which (alongside with initial integrable partial equations) the data nonlinear special functions satisfy, because of their parity, or the oddness, suppose on the Maxwell stratum lowering of the order.

There is, a property of the "increased" integrability on the Maxwell stratum is not limited at all to cases SFC, which are initially solutions of integrable equations and follows from their parity or oddness not necessarily. Really, ordinary differential equation (3) is reduced to the equation

\[
iQz + (|Q|^2 - z)Q = a,
\]

which at \( a = 0 \) is integrable in quadratures.

One more similar example give SFC of the cusp catastrophe being a solution of the Abel equation:

\[
Qz = Q^3 - zQ + a.
\]

7. For clearing up of a qualitative character of a behaviour of a (4) solution, interesting for us, after passage through \( t_* \) the following reception is applicable. By "freezing" a value of factor \( t \) in the equation (3) in some point \( t = T \), we shall consider equation

\[
\varepsilon^2 \partial_t^2 V + 2V^3 - TV = 1.
\]

Integrating it one, we shall receive a relation:

\[
-\varepsilon^2 (\partial_t V)^2 = V^4 - TV^2 - 2V - E.
\]
In an association from a value of the constant $T$ the potential has one of the curve:

Fig. 2

At $T > t_*$ the points $V_0$ and $V_2$ are the point of a stable equilibrium, and $V_1$ is the point of a labile equilibrium. It is easy to see, that at $t = T$ the value of a principal term of an asymptotic solution (5) (the function $u_2(t)$) coincides with $V_2$. At $T = t_*$ the points $V_1$ and $V_2$ stick together and $u_*$ is the point of the labile equilibrium. In particular, in this point the value of a parameter $E = E_*$ is easy for calculating for an equilibrium condition:

$$E_* = u_*^4 - t_* u_*^2 - 2 u_* .$$

The sequence the picture at $T > t_*$, $T = t_*$, $T < t_*$ gives the qualitative answer to the problem: that happens to a true solution of the equation (4) more to the left of a point $t_*$. The solution carries an oscillatory character. The value of a parameter $E$ - analog an energy - in a point $t_*$ is equal to $E_*$. 

8. Let’s pass to a construction of an asymptotics of this oscillating solution. Following [25] we shall search for it as:

$$u(t, \varepsilon) = 0 U_{0} (t_1, t, \varepsilon) + \varepsilon \frac{1}{0} U_{1} (t_1, t, \varepsilon) + \ldots . \quad (9)$$

As argument $t_1$ we shall use the expression $S(t)/\varepsilon$, where $S(t)$ is unknown function. The equations for the definition of an association from a parameter $t_1$ of first two terms of an asymptotics look like:

$$(S')^2 \partial_{t_1}^2 U_{0} + 2 U_{0}^{3/2} - U_{0} t = 1,$$

$$(S')^2 \partial_{t_1}^2 U_{1} + (6 U_{0}^2 - t) \frac{1}{U_{1}} = -2 S' \partial_{t_1}^2 U_{0} - S'' \partial_{t_1}^0 U_{0} .$$

Integrate once on $t_1$ the equation for $U_{0}$ in an outcome we get:

$$(S')^2 (\partial_{t_1} U_{0})^2 = - \frac{1}{U_{1}}^{4} + t \frac{1}{U_{1}}^{2} + 2 \frac{1}{U_{1}} + E(t), \quad (10)$$
where $E(t)$ is "the constant of integration". In [23] it is shown, that the condition of periodicity on a parameter $t_1$ for the function $U$ reduces in the equation for function $S(t)$:

$$S' \int_{0}^{T} \left[ \partial_{t_1} U(t_1, t) \right]^2 dt_1 = c_0.$$  

Here $T$ is period of the oscillations, $c_0$ - constant. By taking in account the explicit expression for the derivative on $t_1$ this formula can be copied in a little bit other form:

$$2 \int_{\alpha(t)}^{\beta(t)} \frac{dx}{\sqrt{-x^4 + tx^2 + 2x + E(t)}} = c_0,$$

(11)

where $\alpha(t)$ and $\beta(t)$ are the solutions of the equation $-x^4 + tx^2 + 2x + E(t) = 0$. For the present the function $E(t)$ is not defined here. Its connection with a phase of the fast oscillations $S(t)$ is given by the formula [25]:

$$T = \sqrt{2S'} \int_{\beta(t)}^{\alpha(t)} \frac{dx}{\sqrt{-x^4 + tx^2 + 2x + E(t)}}.$$

(12)

**Note 3.** A.N. Belogrudov has specified to the authors, that the integral in the left part (11) is generalized hypergeometric function satisfying to a set of equations in partial derivatives on parameters $\alpha$ and $\beta$ [27].

The equations (10) - (12) define to within some constant $c_0$ the principal term of the asymptotics (9). Let’s remark, that we build the asymptotic solution of the equation (4) at $t < t^*$. Thus the polynomial of the fourth degree on $U$ in a right hand side of the equation (10) can have no more two various real roots $\alpha(t)$ and $\beta(t)$. Hence, this polynomial can be look as:

$$F(x, t) = (\alpha(t) - x)(x - \beta(t))\left((x - m(t))^2 + n^2(t)\right).$$

In the point $t = t^*$ the curve on the figure 2 has a point of inflection. The degeneration of an elliptic integral at $t = t^*$ corresponds to a case $M(t^*) = \beta(t^*) = u^*$ and $n(t^*) = 0$, when one of the roots of a polynomial corresponds to the value of the polynomial in a point of inflection. For this case it is easy to calculate the constant in the right hand side of the equation (11): $C_0 = \pi$ and value of the parameter $E(t^*) = E_*$. Express gratitude L.A. Kalyakin, S.G. Glebov and A.N. Belogrudov for discussions and also V.E. Adler for help in realization of the numerical calculations.

The work was maintained RFBR 97-01-00459, 96-01-00382 and Fund of support of scientific schools 96-15-96241.

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