Random Rank-Based, Hierarchical or Trivial: Which Dynamic Graph Algorithm Performs Best in Practice?

Monika Henzinger* Alexander Noe †

Abstract

Fully dynamic graph algorithms that achieve polylogarithmic or better time per operation use either a hierarchical graph decomposition or random-rank based approach. There are so far two graph properties for which efficient algorithms for both types of data structures exist, namely fully dynamic \((\Delta+1)\) coloring and fully dynamic maximal matching. In this paper we present an extensive experimental study of these two types of algorithms for these two problems together with very simple baseline algorithms to determine which of these algorithms are the fastest. Our results indicate that the data structures used by the different algorithms dominate their performance.

1 Introduction

Real-world graphs are huge and changing dynamically, creating the need for data structures that efficiently maintain properties of such graphs. Such data structures are called fully dynamic graph algorithms and usually they assume that the number of vertices is fixed and each update operation either inserts or deletes an edge. While there exist graph properties, such as single-source shortest paths distances or maximum flow values in weighted graphs, for which (under some popular complexity assumptions) no sublinear update time is possible [1, 13, 8], there also exists other graph properties where algorithms with polylogarithmic or even constant update times are known. Such graph properties are connectivity [15, 17], minimum spanning tree [17], and maximal independent set [3, 7], which all have polylogarithmic time per operation, and maximal matching [20], \((\Delta+1)\)-vertex coloring [14, 5], where \(\Delta\) is the maximum degree in the graph, and \((1+\epsilon)\)-approximate minimum spanning tree value [15], which all have constant update time. All (non-trivial) dynamic algorithms with polylogarithmic or faster update time used some variant of hierarchical decomposition of a graph into logarithmic “layers” [16, 17, 20, 5, 15] but in 2019 a new technique, based on giving either each vertex or each edge in the graph a random value of \([0,1]\), called rank, was introduced into the field independently in three papers [3, 7, 14].

As the asymptotic running times achieved by the two techniques are similar an obvious question is how they compare in empirical evaluation. The purpose of this paper is to answer exactly this question. As \((\Delta+1)\)-vertex coloring and maximal matching are the only graph properties for which efficient dynamic algorithms exist that use either one or the other technique we implemented the best dynamic algorithms for these problems, together with one or two trivial dynamic algorithms as a baseline. Let \(n\) denote the number of nodes in the graph, \(m\) its current number of edges, and \(\Delta\) its maximum degree over the whole sequence of operations. Specifically, we compared the following algorithms:

(a) For maximal matching we compared the random-rank based algorithm of Behnezhad et al. [3] which takes \(O(\log^4 n)\) amortized update time, the hierarchical algorithm of Solomon [20] which takes \(O(1)\) amortized update time, the 2-level version of the hierarchical algorithm of Baswana, Gupta and Sen [2] which takes \(O(\sqrt{n})\) amortized update time, our own variant of [3] with \(O(nm)\) worst-case update time, and a trivial dynamic matching algorithm with \(O(n)\) worst-case matching time.

(b) For \((\Delta+1)\)-vertex coloring we compared the random-rank based algorithm of Henzinger and Peng [14] that takes constant amortized time per operation, the hierarchical algorithm of Bhattacharya et al. [5], which also takes constant time per operation, and two simple trivial algorithms; the first of them takes \(O(\Delta)\) worst-case time per operation and the second one might not terminate at all in the worst-case. Note that all but the last algorithm use \(\Theta(n\Delta)\) space, which kept us evaluating them for values of \(\Delta\) close to \(n\).

Related work. Recently there has been a large body of work on the experimental evaluation of fully dynamic algorithms, see [11] for a survey. The work most closely related to our work is a recent experimental eval-
A matching $\mathcal{M}$ in a graph $G$ is a subset of the edge set $\mathcal{M} \subseteq E$ so that each vertex $v$ has only up to one incident edge $e \in \mathcal{M}$. A vertex that has an incident edge $e \in \mathcal{M}$ is matched, otherwise a vertex is unmatched. A matching is maximal, if no edge $e = (u,v) \in E$ with $e \notin \mathcal{M}$ can be added to the matching, as either $u$ or $v$ are already matched. For a edge $e = (u,v) \in \mathcal{M}$, $u$ and $v$ are the partner $P$ of the respective other vertex, $P(u) = v$ and $P(v) = u$. For a vertex $v \in V$ without an incident matching edge, $P(v) = \bot$.

For an undirected graph $G = (V,E)$, an integral parameter $\lambda > 0$ and a color palette $\mathcal{C} = \{1, \ldots, \lambda\}$, let a $\lambda$-coloring in $G$ be a function $\xi : V \rightarrow \mathcal{C}$ which assigns a color $\xi(v) \in \mathcal{C}$ to each vertex $v \in V$. A coloring is proper if no two neighboring vertices in $G$ have the same color in $\xi$.

3 Algorithms for the Fully Dynamic $(\Delta + 1)$ Coloring Problem

In this section we briefly outline the implemented algorithms for the fully dynamic $(\Delta + 1)$ coloring problem. We implement two simple algorithms as well as the recent constant time algorithms of Henzinger and Peng [14] and of Bhattacharya et al. [5]. The algorithm of Henzinger and Peng uses random vertex ranks while the algorithm of Bhattacharya et al. is based on a hierarchical partition of the vertex set. We briefly outline the central ideas of the algorithms, starting with the simple ones, for further details of the often lengthy algorithm descriptions in [14] [5] [2] [20] we refer the reader to the original works. Note that the algorithms are all randomized.

3.1 Recursive Fully-Dynamic $(\Delta + 1)$ Coloring. For a fully dynamic graph with maximum degree $\Delta$, we can give a trivial $(\Delta + 1)$-coloring algorithm $\texttt{RecursiveCol}$, which initially assigns a random color $\xi(v)$ to each vertex $v \in V$. Each vertex $v$ maintains its neighborhood $N(v)$ as a dynamic size array ($\texttt{std::vector}$). When deleting an edge from this dynamic size array, we swap it with the last array element and then remove it. As the edge set $E$ is empty initially, $\xi$ is a proper coloring. On insertion of an edge $e$, $\texttt{RecursiveCol}$ adds the newly incident vertices to the respective neighborhoods and checks whether their colors in $\xi$ are equal. If they are, we randomly choose a vertex $v$ incident to $e$, and recolor $v$ with a random color $\xi(v)$ from the color palette $\mathcal{C}$. We then check the neighborhood $N(v)$ and recursively recolor all neighboring vertices whose color is equal to $\xi(v)$. If the vertices have different colors, the coloring remains proper and no action is taken. On an edge deletion, the coloring also remains proper and thus we only update the affected neighborhood data struc-
tures. This algorithm is, thus, very fast on edge deletions and non-clashing insertions, but an edge insertion between two vertices of the same color can cause expensive recursive cascading effects.

3.2 Counting Fully-Dynamic $(\Delta + 1)$ Coloring.

Another simple approach tries to avoid recursive cascading by keeping for each vertex $v \in V$ and color $\xi$ a count of how many neighbors of $v$ have color $\xi$. In this CountCol approach, we create an array of size $\Delta + 1$ for each vertex $v$ and set all initial values set to 0. This array is used to count how many neighbors of $v$ have certain colors. We use the same neighborhood data structure as previously. On edge deletion and non-clashing edge insertion, CountCol additionally updates the color counting array. On insertion of edge $e$ with equal incident colors, we choose a random endpoint $v$ of $e$. We then draw new random colors for $v$ until we find a color that no neighbor of $v$ currently has. As the number of colors is larger than the maximum degree, there is at least one such color. We then recolor $v$ and update the color counting data structure.

3.3 Constant-Time Algorithm of Henzinger and Peng.

The algorithm of Henzinger and Peng [14] (RandRCol) takes constant expected time per update against an oblivious adversary and is based on random vertex ranks. Initially, each vertex $v \in V$ is assigned a random rank $r(v) \in [0, 1]$ and a random color $\xi(v) \in C$. For each vertex $v \in V$, the set of neighbors is partitioned into $H_v$, the neighbors with rank $> r(v)$ and $L_v$, the neighbors with rank $< r(v)$. We implement $L_v$ and $H_v$ as hash sets. The algorithm aims to maintain a proper $(\Delta + 1)$-coloring. Edge deletions or non-clashing edge insertions do not lead to a violation of a proper coloring, only the insertion of edge $e = (u,v)$ with $\xi(u) = \xi(v)$ triggers a recolor of vertex $v$, where $r(v) > r(u)$. RandRCol samples in a carefully designed procedure a random new color that is either not used by any neighbor of $v$ or by a single neighbor $w$ with $v \in L_w$, and assigns this color to $v$. If such a “conflicting” neighbor $w$ exists, it triggers a recoloring of $w$.

When a vertex $v \in V$ is recolored, HierCol picks a new color for $v$ with level $l(v)$ in the following way: if $v$ has fewer than $3^{l(v)+2}$ neighbors of lower or equal level, we set $l(v) = -1$ and pick the new color $\xi(v)$ from the set of colors that no neighbor of $v$ has. Otherwise, the level of $v$ is raised to the lowest level $l(v)$, so that $v$ has fewer than $3^{l(v)+2}$ neighbors of lower or equal level and then pick a random new color that is either not used by any neighbor of $v$ or by exactly one neighbor $w$, where the level of $w$ is lower than the level of $v$. If there is such a neighbor $w$, the recoloring algorithm is then called on $w$.

4 Algorithms for the Fully Dynamic Maximal Matching Problem

We now briefly outline the implemented algorithms for the fully dynamic maximal matching problem. We implement a trivial maximal matching algorithm, as well as the level set partitioning algorithms of Solomon [20] and Baswana et al. [2]: and the random rank algorithm of Behnezhad et al. [8]. For the algorithm of Baswana et al., we use an implementation of Henzinger et al. [12]. Additionally, we introduce a variation of the algorithm of Behnezhad et al. that does not guarantee a poly-logarithmic running time but performs much faster in practice.

4.1 Trivial Fully Dynamic Maximal Matching.

A matching is maximal if for each edge $e = (u,v) \in E$ at least one of the vertices $u$ and $v$ is matched. Initially $E$ is empty and therefore the empty matching is maximal. In the trivial fully dynamic maximal matching algorithm TrivialMatch, each vertex $v \in V$ maintains its neighborhood $N(v)$ as a dynamic size array (std::vector). For the insertion of edge $e = (u,v)$, we check whether $u$ and $v$ are both unmatched and if they are, we add $e$ to the matching. Otherwise we leave the matching unaltered, as the new edge already has an incident edge in the matching. On deletion of a matching edge $e = (u,v)$, we have to remove $e$ from the matching (as it no longer exists) and, thus, edges incident to $u$ or $v$ can be uncovered. In this case TrivialMatch iterates over the neighborhoods of the newly unmatched vertices $u$ and $v$ and checks whether new matching partners can be found.

4.2 Algorithms of Solomon and Baswana et al.

The algorithm of Baswana et al. [2] (Hier1Match) and later the algorithm of Solomon [20] (Hier2Match) takes $O(\log n)$, resp. $O(1)$ expected time per update against an oblivious adversary and both partition the vertex set into a hierarchy of $\log n$ levels, where intuitively a vertex rises in levels if it has many neighbors in lower levels. A vertex $v$ is in level $l(v) = -1$, if it
is unmatched and in a level \( l(v) \geq 0 \) if it is matched. Each vertex maintains a set of neighbors \( O_v \) of neighbors of lower level as well as a set of higher-level neighbors \( L_v[l] \) for each level \( l(v) \leq l < \log n \). For insertion of edge \( e = (u,v) \), the algorithms first add the edge to the respective data structures and check whether both \( u \) and \( v \) are unmatched (i.e. are in level \(-1\)) and if they both are, they add \( e \) to the matching and elevate both incident vertices to level 0. If at least one of \( u \) and \( v \) are already matched, no further action is taken. For deletion of edge \( e = (u,v) \), the algorithms remove the edge from the respective data structures. If \( e \) is not a matching edge, the vertex levels remain constant, otherwise \( u \) and \( v \) are now unmatched vertices and the algorithms aim to find new matching partners: a vertex \( v \) with few vertices in \( O_v \), checks whether there exists an unmatched \( u \in O_v \), and if it exists, adds \((v,w)\) to the matching and sets the levels of \( v \) and \( w \) to 0. If no such \( w \) exists, \( v \) remains unmatched. For a vertex \( v \) with many neighbors in \( O_v \), they instead pick a random (potentially matched) neighbor \( w \in O_v \) and adds \((v,w)\) to the matching. If a matching edge \((w,w')\) exists, it is removed from the matching and the algorithms use the same algorithm to try and find a new matching partner for the now unmatched vertex \( w' \).

The algorithm of Baswana et al. \cite{behr2018} explicitly maintains the vertex levels and achieves an amortized runtime of \( O(n \log n) \), the algorithm of Solomon \cite{solomon2020} achieved constant running time by lazily maintaining the vertex levels and therefore expensive waves of vertex level updates. We implement the neighborhood sets \( O_v \) and \( L_v[l] \) as hash sets.

4.3 Algorithm of Behnezhad et al. Behnezhad et al. \cite{behnezhad2017} give a random rank based algorithm (RandR1Match) for the fully dynamic maximal independent set problem and extend that algorithm to the fully dynamic maximal matching problem. The resulting algorithm takes \( O(\log^2 \Delta \log^2 n) \) expected time per update against an oblivious adversary and assigns during preprocessing a random rank to each edge, which induces a random order on the edges. It then maintains the lexicographically first maximal matching over this order of the edges.

A matching \( M \) is the lexicographically first maximal matching (LFMM) of a graph \( G = (V,E) \) with a ranking \( \pi : E \to [0,1] \) over the edges if it is equal to the matching created by the following greedy algorithm \cite{bible} that finds a maximal matching. Initially, all edges are white. We iteratively pick the white edge \( e \) with minimum rank \( \pi(e) \), add it to the matching, and then make \( e \) and all its incident white edges black. We call this matching \( LFMM(G, \pi) \). In this greedy algorithm, every edge is either added to the matching or turned black by an incident edge of smaller rank \( \pi \). In the following we say an edge \( e_1 \in M \) covers an edge \( e_2 \notin M \), if they are incident and \( \pi(e_1) < \pi(e_2) \). An edge \( e \notin M \) that does not have any matching edge which covers \( e \) is called uncovered.

The algorithm of Behnezhad et al. \cite{behnezhad2017} draws for each inserted edge \( e \) a random rank \( \pi(e) \) in \([0,1] \). For each edge \( e \) it also maintains its eliminator rank \( k(e) \leq \pi(e) \), where \( k(e) \) is the rank of the edge incident to \( e \) that is in the matching. For a matching edge \( e \in M \), \( r(e) = k(e) \), otherwise \( k(e) < \pi(e) \). Each vertex \( v \) maintains a self-balancing binary search tree of its incident edges ordered by their eliminator rank. This is necessary as it needs to be able to find all incident edges below a given eliminator rank. Each vertex also maintains its vertex rank \( k(v) \) which is equal to the edge rank of the incident matching edge if one exists and \( \infty \) otherwise. For the insertion of edge \( e = (u,v) \), we add \( e \) to the respective data structures and check whether the edge should be part of the LFMM (i.e. \( \pi(e) < k(u) \) and \( \pi(e) < k(v) \)).

If it is, we add \( e \) to the matching and remove the matching edges incident to \( u \) and \( v \). The algorithm then updates the eliminator ranks of all affected edges in the search tree data structures and finds new partners for the newly unmatched vertices. For the deletion of edge \( e = (u,v) \), the algorithm removes \( e \) from the respective data structures and if \( e \) was in the LFMM, checks all incident edges \( e' \) with \( k(e') \geq \pi(e) \) to find new matching partners for \( u \) and \( v \). Similar to the edge insertion, the algorithm updates the search tree data structure and finds new matching partners for all vertices that become unmatched over the course of the algorithm. The algorithm has amortized expected \( O(\log^4 n) \) update time.

4.4 New Random-Rank Algorithm for Maximal Matching. The algorithm of Behnezhad et al. \cite{behnezhad2017} performs many updates in search tree data structures. As these updates are very expensive in practice, we give an alternative random-rank algorithm (RandR2Match) based on their idea that does not need to perform these expensive updates. This new algorithm also maintains the lexicographically first maximal matching on a dynamic graph and is much faster in practice despite having a worse asymptotic running time bound. In the following we give a detailed description of this algorithm.

The data structures used in RandR2Match are closely related to the algorithm of Behnezhad et al. \cite{behnezhad2017}. For each inserted edge \( e \), RandR2Match draws a random edge rank \( \pi(e) \in [0,1] \). Each vertex maintains \( k(v) \), the rank of the incident matching edge if it exists - if \( v \) is unmatched, \( k(v) = \infty \), the set of neighbors \( N(v) \) and
Algorithm 1 findNewPartners(): Find new partners for unmatched vertices, cover all uncovered edges

Input: $S \leftarrow$ vertex priority queue

1: while $S$ is not empty do
2: $v, r_v \leftarrow$ remove lowest rank vertex in $S$
3: for $w \in N(v)$ with $r_w < \pi(v, w) < k(v)$ do
4: if $\pi(v, w) < k(w)$ then $\triangleright$ Add $(v, w)$ to $M$
5: if $k(w) < \infty$ then
6: $x \leftarrow P(w)$
7: $S$.insert$(x, k(w))$
8: $P(x) \leftarrow -1$
9: $k(x) \leftarrow \infty$
10: end if
11: if $k(v) < \infty$ then
12: $x \leftarrow P(v)$
13: $S$.insert$(x, k(v))$
14: $P(x) \leftarrow -1$
15: $k(x) \leftarrow \infty$
16: end if
17: $P(v) \leftarrow w$
18: $P(w) \leftarrow v$
19: $k(v) \leftarrow r_v$
20: $k(w) \leftarrow r_v$
21: return
22: end if
23: end for
24: end while

the matched neighbor $P(v)$, for an unmatched vertex $v$, $P(v) = -1$. In contrast to the algorithm of Behnezhad et al., the neighbors are ordered by their ranks and not by their eliminator rank and, thus, do not need to be updated when the matching changes. We next give a detailed description of the algorithm.

Algorithm description. Similar to the algorithm of Behnezhad et al. [3], we aim to maintain the lexicographically first maximal matching in $G$ according to a random edge ranking $\pi$. The set of vertices is static, the edge set is initially empty. On edge insertion, we call Algorithm 2 (pseudocode in Appendix B) and on edge deletion, we call Algorithm 3 (pseudocode in Appendix B), which each use Algorithm 1 (findNewPartners()) to find new partners for vertices whose previous partner was matched with a new matching partner in order to maintain the invariant that all non-matching edges are covered.

On insertion of edge $e = (u, v)$, Algorithm 2 first draws a random rank $\pi(e)$ and then inserts the edge in the appropriate data structures. We make sure $\pi(e)$ is unique by re-drawing the random rank if another edge already has equal rank. If $\pi(e)$ is smaller than the minimum of $k(v)$ and $k(u)$, $e$ is added into the matching. If $u$ or $v$ had previous matching partners, they are removed from the matching and added into a priority queue $S$ with priority equal to the rank of their previous matching edge $k(u)$ resp. $k(v)$. We then call Algorithm 1 to make sure all edges are covered again, i.e., all vertices in $S$ are processed.

Algorithm 1 takes vertex $v$ of $S$ with lowest priority, called $r_v$, and checks whether there exists an incident edge $e = (v, w)$ with priority $r_v < \pi(e) < k(v)$ and $\pi(e) < k(w)$ that is added to LFMM. Note that $r_v$ is the rank of the last edge with which $v$ was matched at the time that it was added to $S$. If $v$ and $w$ are currently matched, i.e. $P(v) \neq -1$, we add $P(v)$ to $S$ with priority $\pi((v, P(v)))$, as all edges incident to $v$ with priority $> \pi((v, P(v)))$ are now potentially uncovered. Note that even though vertex $v$ was unmatched when it was added to priority queue $S$, it is possible that an edge incident to $v$ was added to the matching in the meantime and, thus, $v$ is currently matched. We repeat this process until priority queue $S$ is empty.

On deletion of an edge $e = (u, v)$, we delete it from the neighborhoods of $u$ and $v$ and check whether it is a matching edge. If it is, we set both $u$ and $v$ to unmatched and add them to $S$ with a starting rank of $\pi(e)$ to check every edge with a rank of $> \pi(e)$. Similar to the insertion algorithm, we then call Algorithm 1.

In the following we show that this algorithm terminates and correctly maintains a LFMM on edge insertion and deletion. We first prove Lemma 4.1, which says that the LFMM is the only matching, in which each non-matching edge is covered by an incident matching edge of smaller rank.

Lemma 4.1. A matching $\mathcal{M}$ is LFMM($G, \pi$) if and only if for each edge $e = (u, v) \in \mathcal{E}$ either $e \in \mathcal{M}$ or $\exists w \in N(u) \cup N(v)$ s.t. $(u, w) \in \mathcal{M}$ with $\pi(u, w) < \pi(e)$ or $(v, w) \in \mathcal{M}$ with $\pi(v, w) < \pi(e)$. (Proof in Appendix A)

Claim 1. In Algorithm 1 when vertex $v$ is removed from $S$ with rank $r_v$, then any new vertex $w$ that is added to $S$ during this iteration of Algorithm 1 is added with rank $> r_v$. (Proof in Appendix A)

Claim 2. Algorithm 1 checks vertices in non-decreasing priority order. (Proof in Appendix A)

Claim 3. Algorithm 1 terminates after at most $m$ iterations of the while-loop. (Proof in Appendix A)

Each iteration takes time $O(n)$ leading to a $O(nm)$ worst-case time per operation. To show the correctness of the algorithm we will show that before each call to Algorithm 1 the following two invariants hold:

1. At least one endpoint $v$ of each uncovered edge $e$ is in $S$ (or currently processed in Algorithm 1) and $\pi(e) > r_v$,

2. Every vertex $v$ in $S$ has an edge in $\mathcal{M}$ with rank $\pi$ equal to $\pi(v)$ and has at least one other non-matching edge with rank $\pi(e)$ smaller than $\pi(v)$.

Proof of Claim 1. The proof of Claim 2 is similar.

Proof of Claim 3. Consider any iteration of Algorithm 1.

Case 1. The while-loop is not executed. Then $S$ is empty and so is $\mathcal{M}$.

Case 2. The while-loop is executed. Then Algorithm 1 finds all unmatched vertices.

Case 3. Algorithm 1 terminates. Then $S$ is empty and so is $\mathcal{M}$.

Each call to Algorithm 1 updates at most $\frac{m}{n}$ vertices and in each iteration at least one vertex is added to $S$. Thus, the algorithm executes at most $m$ iterations.
I2 All edges of rank at most the lowest rank in \(S\) are covered.

First, however, we show the following lemma:

**Lemma 4.2.** If I1 and I2 hold at the beginning of Algorithm 1 then at its termination the matching is a lexicographically first maximal matching.

**Proof.** Assume I1 and I2 hold when Algorithm 1 is called. The algorithm removes repeatedly the lowest priority vertex \(v\), say with priority \(r_v\) from \(S\) and check in increasing rank order (starting from \(r_v\)) whether the edges incident to \(v\) are covered. All uncovered edges incident to \(v\) must have rank > \(r_v\) by our invariant. If an edge \(e = (v,w)\) with \(\pi(e) > r_v\) is not covered when it is checked, we add \(e\) to the matching, which immediately covers all remaining uncovered edges incident \(v\). Thus, all edges incident to \(v\) are now covered. Then we evict the current matching edges incident to \(v\) and \(w\) from \(M\). Note that both must have rank > \(\pi(e)\), but both are now covered. As their partners might now have incident uncovered edges of rank > \(\pi(x,P(x))\), \(\pi(e)\) and > \(\pi(y,P(y))\), \(\pi(e)\), we add \(P(x)\) and \(P(y)\) to the priority queue \(S\) with priority \(\pi(x,P(x)), \pi(y,P(y))\). This maintains Invariant I1. It also maintain Invariant I2 as (a) all new potentially uncovered edges are incident to either \(P(x)\) or \(P(y)\) and have rank > \(\min\{\pi(x,P(x)),\pi(y,P(y))\}\), which is larger than the minimum rank in \(S\), (b) all edges incident to \(v\) are now covered, and (c) by the Invariant 1 for all other uncovered edges \(e'\) there exists an endpoint \(u\) in \(S\) with \(\pi(e') > r_u\). As the queue \(S\) is guaranteed to be empty after the execution of Algorithm 1 (Claim 2), it follows that all edges are covered, which implies by Lemma 4.1 that the matching is a lexicographic first maximal matching.

**Lemma 4.3.** Given the matching \(M = LFMM(G,\pi)\) and the insertion of an edge \(e = (u,v)\), Algorithm 2 maintains the lexicographically first maximum matching \(LFMM(G + e,\pi)\). *(Proof in Appendix A)*

**Lemma 4.4.** Given the matching \(M = LFMM(G,\pi)\) and the deletion of an edge \(e = (u,v)\), Algorithm 2 maintains the lexicographically first maximum matching \(LFMM(G - e,\pi)\). *(Proof in Appendix A)*

5 Experiments and Results

5.1 Experimental Setup and Methodology. We implemented the algorithms using C++-20 and compiled all codes using g++-11.1.0 with full optimization (-O3). Our experiments were conducted on a machine with two Intel Xeon E5-2643 v4 with 3.4GHz with 6 CPU cores each and 1.5 TB RAM in total. We performed five repetitions per instance and report average running time. In this section we first describe the experimental methodology. Afterwards, we evaluate different algorithmic choices in our algorithm and then we compare our algorithm to the state of the art. When we report a mean result we give the geometric mean as problems differ strongly in result and time. All codes are sequential.

5.1.1 Instances. For our experiments we used both both real-world and generated instances. For generated instances, we use the KaGen graph generator [9], which generates a wide variety of static graph families. We use a family of random Erdős-Rényi graphs and a family of random hyperbolic geometric graphs to create our fully dynamic instances.

An Erdős-Rényi graph (ER graph) \(G(n, p)\) is a random graph with \(n\) vertices where each two vertices are connected by an edge with probability \(p\), independently from the other edges in the graph. The graph has on average \(\binom{n}{2} \cdot p\) edges and a Poisson degree distribution.

Random Hyperbolic Geometric Graphs (RHG graphs) [13] are random graphs that replicate many features of social graphs [6]: the degree distribution follows a power law, they have small diameter and often exhibit a community structure. These graphs are geometric graphs on a disk in hyperbolic space, in which nodes that are close to each other (in hyperbolic space) are connected by an edge. In these graphs, the nodes that are close to the center of the disk have a very high degree, less central vertices usually have a much lower degree.

We generated static ER and RHG graphs with varying number of vertices and edges and used different techniques to create fully dynamic instances from the static graphs. We explain the dynamization methods when we report about the respective experiments. On the ER graphs, the maximum degree \(\Delta\) is generally 20–40% larger than the average degree, on the RHG graphs, \(\Delta\) is generally a constant factor of \(n\), i.e. there are vertices (close to the center of the hyperbolic plane) with a very high degree.

Additionally, we used as data sets six real-world Wikipedia update sequences that were obtained from the Koblenz Network Collection KONECT [19]. The graphs are directed graphs and given as a sequence of individual edge insertions and deletions and model the link structure between Wikipedia articles. While the instances are given as directed graphs, we interpret them as undirected graphs, i.e. while there exists either an edge from \(u\) to \(v\) or from \(v\) to \(u\), our graph instance has an undirected edge \((u, v)\). The instances are initially empty and have between 100K and 2.2M vertices and
between 1.6M and 86M edge updates. As these graphs have very high maximum degree \( \Delta \) and, thus, our implementations of the dynamic coloring algorithms \( \text{RandRCol} \), \( \text{CountCol} \) and \( \text{HierCol} \) require space that is quadratic in \( n \), we only run experiments on them for the fully dynamic matching problem.

5.2 Fully Dynamic \((\Delta + 1)\) Coloring. We implemented all algorithms described above, namely the random rank algorithm of Henzinger and Peng \cite{HP15} (\( \text{RandRCol} \)), the hierarchical algorithm of Bhattacharya et al. \cite{BBPP13} (\( \text{HierCol} \)), as well as two trivial algorithms (\( \text{CountCol} \) and \( \text{RecurseCol} \)) for the problem.

The non-trivial algorithms \( \text{RandRCol} \) and \( \text{HierCol} \) use Google Dense Hash Set \cite{GHH10} to implement the neighborhood data structures efficiently, while the trivial algorithms \( \text{CountCol} \) and \( \text{RecurseCol} \) use dynamic size arrays (\texttt{std::vector}). For the trivial algorithms we can use dynamic size arrays, as updates generally affect all neighbors of a vertex and so we need to iterate over the whole neighborhood of a vertex. In \( \text{RandRCol} \) and \( \text{HierCol} \) there are more neighborhood queries (i.e. is vertex \( u \) a neighbor of vertex \( v \)?) which can be answered in expected constant time using a hash table.

5.2.1 Random Update Sequence on Generated Graph. For this experiment, we generated ER and RHG graphs with varying number of vertices \( n \) and number of edges \( m \). For both we generated two sets of graphs, in set (1) the average density remains constant while the number of vertices changes; while in set (2) the number of vertices is constant while the average density changes. In set (1), we generate graphs with \( n = 2^{13}, 2^{14}, 2^{15}, 2^{16}, 2^{17}, 2^{18} \) and \( m = n \cdot 2^{64} \), whereas in set (2), the number of vertices is constant \( n = 2^{16} \), while \( m = n \cdot \{64, 128, 256, 512, 1024\} \). For each of these 22 graph instances, we generate random update sequences that start with an empty graph with \( n \) nodes and no edges and insert the whole edge set \( S \) in random order, interspersed with deletions of previously inserted edges. Let \( \rho \in [0, 1] \) be the deletion rate. The random update sequence is generated as follows: first we randomly shuffle the edges in \( S \), afterwards with probability \( \frac{1}{1+\rho} \) we insert the next edge in \( S \) into \( G \) or with probability \( \frac{\rho}{1+\rho} \) we remove a random edge (if one exists) from \( G \). This process is repeated until every edge in \( S \) is inserted, resulting in a total of \( |S| \) edge insertions and about \( \rho |S| \) edge deletions. For each graph, we create an update sequence each with \( \rho = \{0, 10\%, 25\%, 50\%, 75\%\} \), which results in a total of 220 graph instances. Note that \( \rho = 0 \) indicates that there are no deletions. Insertions are random and not dependent on earlier insertions. Recall that the algorithms have to recolor a vertex whenever we insert an edge \( e = (u,v) \) where \( \xi(u) = \xi(v) \), due to the random nature of the insertions and the fact that there are \( \Delta + 1 \) available colors this happens with a probability of roughly \( \frac{1}{\Delta} \) independent of the choice of the algorithm and the experiment shows average update times on random updates. As the RHG graphs have very high maximum degree \( \Delta \) and therefore data structures grow very large, we exclude the initialization time in this experiment; so we can report the update performance of the algorithms and not how much time is required to initialize large arrays and hash tables.

Figure 1 shows the results for all random insertion and deletion sequences. We can see that in these random instances, the simple \( \text{RecurseCol} \) algorithm is fastest for almost all instances, followed by \( \text{CountCol} \) and \( \text{RandRCol} \), which is on average about 30 - 50% faster than \( \text{HierCol} \). Interestingly, there is no major effect of density (i.e., set (1) vs. set (2)), graph type and deletion rate \( \rho \) on the relative performance of the algorithms, generally \( \text{RandRCol} \) performs slightly better than \( \text{CountCol} \) on denser instances with a higher deletion rate \( \rho \). For all algorithms we can see that the average time per operation increases slightly on larger instances, as the basic operations are slightly slower on larger arrays and hash tables.

For all algorithms, the majority of the running time is not spent in recoloring, but in the insertion and removal of vertices from neighborhood data structures and the use of different neighborhood data structures has a major impact on the performance: While finding a neighboring vertex can take up to \( O(\Delta) \) using arrays, the constant time operations such as insertion and deletions are significantly faster on a hash table array compared to dynamic arrays. Specifically, on a data structure benchmark on the same hardware inserting 1M (random integer) elements (into an empty resp. already filled with 10M elements data structures) into a hash table vs. a dynamic size array is a factor 10, resp. 13 faster, accessing all elements sequentially is
roughly a factor of 20 faster, and deleting 1M elements is roughly a factor of 100 faster.

One explanation for the superiority of RandRCol over HierCol is that RandRCol only updates neighborhood counts on the lower-rank side of a deleted edge, while HierCol counts the color of the incident vertex on both sides. Therefore, edge deletions are faster in RandRCol. Additionally, HierCol changes vertex levels in the hierarchy over the course of the algorithm, which induces additional data structure insertions and deletions. In this experiment, according to performance profiles HierCol spends 20 – 25% of the time spent in edge insertions in level changes, i.e. moving neighboring vertices to different data structures for the respective hierarchy levels. The vertex ranking of RandRCol remains constant over the course of the algorithm, so neighborhood data structures only need to be updated when edges are inserted or deleted.

Performance profiles also indicate that the more complex updating schemes of HierCol and RandRCol have a much larger impact on the running time compared to the simple iteration over neighboring vertices in the trivial algorithms. RandRCol spends about 25 – 30% of the running time of insertions in recoloring operations, HierCol about 35 – 40% of insertion time, whereas the trivial algorithms rarely exceed 10%. For all algorithms, the vast majority of the remaining running time is spent in adding and removing neighboring vertices from the basic neighborhood data structures. Note that these numbers are derived from sampling-based performance profiles and also vary depending on the instance. They should therefore be seen as estimates, however they clearly indicate that the simple recoloring scheme of trivial algorithm is significantly faster than more elaborate recoloring approaches.

5.2.2 Incremental Clashing Sequence. In this experiment we generate incremental (insertions-only) worst-case update sequences for both of the trivial algorithms CountCol and RecurseCol, by selecting two random vertices u and v with $\xi(u) = \xi(v) = \xi$ and then add an edge $e = (u, v)$ to the graph, forcing the algorithm to recolor one of the two. Note that this creates two different worst-case updates sequences as the two algorithms generate different colorings. Note that both RandRCol and HierCol achieve constant amortized worst-case running time against an oblivious adversary (that must generate all updates without knowing the colors assigned to the vertices) by making sure that the palette of new possible colors when recoloring has $\Theta(\Delta)$ colors and choosing a color from it randomly. However, their running time guarantees do not hold against an adaptive adversary as the one above.

For either algorithm we generated instances with $n = 2^{12}, 2^{13}, 2^{14}, 2^{15}$ and a maximum degree of 128, 256, 512, 1024, i.e. a total of 16 instances per algorithm. Figure 2 shows the clashing sequence for RecurseCol and Figure 3 shows the clashing sequence for CountCol. Note that in these figures, both the x- and the y-axis are logarithmic. We can see the expected result: the time per operation in the clashing algorithm increases quickly achieving the worst performance over all algorithms. Furthermore, the running time of CountCol increases on its worst-case sequence much faster (reaching over $10^{14}$ ns per operation after $10^6$ operations) than RecurseCol on its respective sequence.

The reason is that when recoloring, CountCol updates the neighborhood color counts of all neighbors of the recolored vertex, while RecurseCol recolors only one or few vertices and checks that no neighbors have the same color. While this also takes time linear to the vertex degree, it is much faster in practice, as there are no write operations for all neighbors of the recolored vertex. Note also that for reasonably large $n$ both RandRCol and HierCol perform better than the clashing algorithm and RandRCol always performs better than HierCol.

5.2.3 Vertices of Almost Equal Degree. In Sections 5.2.1 and 5.2.2 we can see that the simplicity of RecurseCol makes the algorithm very fast when there are enough free colors for many vertices and the recursive cascading does not recurse too deeply. However, the situation changes when the average vertex degree is very close to $\Delta$ and an average vertex does not have many free colors in its neighborhood. In this experiment, we first generate random initial instances $G_i$ with a maximum degree of $(\Delta - 1)$ by adding random edges between vertices of degree $< (\Delta - 1)$ until the average degree is very close to $(\Delta - 1)$. We then perform a random update sequence on top of this graph, where we
overlay this graph with a graph $G_o$ of maximum degree 1, which results in a graph $(G_i + G_o)$ of maximum degree $\Delta$. For each update, we draw a random vertex $v$ and check whether it has a neighbor $u$ in $G_o$. If it does, we remove $(v, u)$ from the graph, otherwise we draw random vertices until we find a vertex $u$ that does not have a neighbor in $G_o$ and $(v, u) \notin E(G_i)$, and add $(v, u)$ to the graph. Therefore, the average degree is very close to the maximum degree over the whole edge update sequence. We generate $G_i$ with $n = 2^{13}, 2^{14}, 2^{15}, 2^{16}$ and a maximum degree of 512, 1024, 2048, 4096 and perform $10^5$ random updates.

In Figure 4 we can see that RecurseCol performs worse compared to other algorithms when the average degree is very close to the maximum degree $\Delta$ and the recursion depth grows. On the instances, in which the average degree is $> 0.999\Delta$, the algorithm is significantly slower than CountCol, which does not experience slowdown when the average degree is close to the maximum degree. The two non-trivial algorithms also do not experience slowdown when the average degree is almost as large as the maximum degree.

We also generated graphs $G_i$ in which we made sure that every vertex has equal degree $(\Delta - 1)$. When performing this experiment on those graphs, RecurseCol often runs into non-ending recursive calls where the recursion depth keeps growing until the recursion stack overflows. Therefore, in Figure 4 we use graphs where the average degree is almost equal to the maximum degree.

5.3 Fully Dynamic Maximal Matching. For the fully dynamic maximal matching problem, we implemented the algorithms of Behnezhad et al. 8 (RandR1Match), the algorithm of Solomon 20 (Hier2Match), our random-rank algorithm described in detail in Section 4 (RandR2Match) and the trivial algorithm described in Section 4 (TrivialMatch), and we used an implementation of Henzinger et al. 12 of the algorithm of Baswana et al. 2 (Hier1Match).

The hierarchical algorithms Hier1Match and Hier2Match use Google Dense Hash Set 10 for neighborhood data structures so that they can perform neighborhood queries in expected constant time, the trivial algorithm TrivialMatch uses dynamic size arrays (std::vector). As RandR1Match needs to keep all neighbors in order and often updates this order, it uses a self-balancing binary tree data structure (std::set). Compared to Google Dense Hash Set, this self-balancing binary tree is slower by a factor of 12, resp. 25 on insert of 1M random elements in an empty (or pre-filled with 10M elements) data structure, by a factor of 11 on sequential access and by a factor of 30, resp. 35 on deletion of 1M elements. RandR2Match has neither neighborhood queries nor needs to update the order of neighbors (as edge ranks remain constant). The fastest implementation of the neighborhood data structure therefore has for each vertex $v$ an unsorted dynamic size array of neighbors and a dynamic size array of deleted neighbors. When accessing a vertex $v$, we sort both arrays and remove the neighbors that were since removed; and then clear the array of deleted neighbors. While this incurs $O(\Delta \log \Delta)$ running time, it is faster than both balancing and non-balancing binary trees.

5.3.1 Random Update Sequence on Generated Graph. We performed experiments for the fully dynamic maximal matching problem using the same random update sequences that we used in Section 5.2.1 for the fully dynamic $(\Delta + 1)$ coloring problem.

In Figure 5 we can see the results for this experiment. We can clearly see that RandR1Match is more than an order slower than the other algorithms on most instances and also scales the worst. The algorithm performs especially poorly on instances with a high average degree. Most of the update time is spent on updating the neighborhood data structure $N(v)$, as whenever an edge is added or removed from the matching, the eliminator rank of all incident edges needs to be updated, which results in a large number of updates.
in a self-balancing binary search tree (implemented as std::set). Similar to the fully-dynamic \((\Delta + 1)\) coloring problem, we can see that the simple trivial algorithm TrivialMatch performs very well with random insertions and deletions. For the closely related algorithms Hier1Match and Hier2Match we can see that Hier2Match is faster all instances, with a speedup factor of up to 2.5, not significantly affected by graph type, density, deletion rate or number of vertices. On most instances, RandR2Match is 10 − 30% faster than Hier2Match, even though the algorithm does not have good theoretical performance guarantees. Looking at performance profiles we can see that Hier2Match spends up to 40% in hierarchy level updates, while the simpler two-level implementation of Hier1Match only spends 20 − 25% in level updates. This is not all to surprising, as the implementation in Hier2Match uses log \(n\) levels while the implementation in Hier1Match only uses two. For both algorithms this is the largest part of the time spent in updating the maximal matching, the rest of the running time of the algorithms is spent in adding and removing elements from neighborhood data structures.

For RandR1Match, the vast majority of running time is spent in updating the neighborhood binary trees. RandR2Match spends about a quarter of the total running time in Algorithm 1 which updates the matching and TrivialMatch spends about 10% of the running time in matching updates. The rest of the respective running times are spent in updating neighborhood data structures when inserting and removing edges.

Size of the maximal matchings We also evaluated the size of the maximal matchings found. The geometric mean of the matching size RandR1Match and RandR2Match about 0.3% lower than the geometric mean of the matching size of the other algorithms whose matching sizes are all very similar. This is not very surprising as the matchings of RandR1Match and RandR2Match are lexicographically first maximal matching, i.e., maximal matchings that must fulfill an additional requirement.

5.3.2 Generated Sliding Window Instances. In this experiment, we generate sliding window instances, in which we insert edges from generated ER and RHG graphs in random order and delete the oldest edge whenever the dynamic graph has \(\phi\) (see below) edges. We also give an additional parameter \(\eta \in [0, 1]\) that is used to generate harder instances for TrivialMatch. For each edge deletion, with probability \(\eta\) we delete the oldest edge that is part of the matching of TrivialMatch, with probability \(1 - \eta\) we delete the oldest edge in the sliding window. Similar to the coloring problems, an adversary that is oblivious to the random choices made by the algorithms cannot create such worst-case instances for the non-trivial algorithms. For either graph generator, we generate graphs with \(n = 2^{14}, 2^{15}, 2^{16}, 2^{17}, 2^{18}\) and \(m = 2^{25}\). For each graph, we generated sliding windows with window size \(\phi = 2^0, 2^1, 2^2, 2^3\) and \(\eta = 0, 25\%, 50\%, 75\%, 100\%\), which results in a total of 100 instances. Figure 6 shows box plots for these instances. In these box plots, the box extends from the lower to the upper quartile of the data, the line indicates the median. The whiskers extend to show the range, with outlier results denoted by black dots. The results with \(\eta = 0\) are similar to the results in Section 5.3.1 with increased parameter \(\eta\) TrivialMatch is much slower, as each deleted matching edge means that both adjacent vertices are now unmatched and need to check whether they have an incident edge to another unmatched vertex. However, even when \(\eta = 1\), TrivialMatch is still significantly faster than RandR1Match.

5.3.3 Real-world Graphs. Figure 7 shows results of the wikipedia edit instances described in Section 5.1.1. On these, the algorithms have similar performances to the instances in Section 5.3.1 TrivialMatch is the fastest algorithm, followed by RandR2Match and Hier2Match, then followed by RandR1Match. The algorithm RandR1Match is slower by one to two orders of magnitude.
6 Conclusion

We empirically evaluated a variety of algorithms for the fully dynamic ($\Delta + 1$) coloring problem and the fully dynamic maximal matching problem. For both problems the trivial algorithms performed best, except on specifically chosen worst-case input sequences. The reason is the use of different data structures for storing the neighborhood of each vertex and how they interact with the caching system. Dynamic algorithms with random rank also clearly outperformed algorithms using a hierarchical graph decompositions. The reason for this results is the frequent use of expensive data structures for updating the hierarchy. However, for maximal matching the random-rank based algorithm with best asymptotic performance that time $O(\log^4 n)$ and it was clearly outperformed by the hierarchical algorithms with $O(\log n)$ or constant expected update time. It would be very interesting to find a constant-time random rank algorithm and compare its performance empirically. We gave a random-rank based algorithm that outperforms the constant-time hierarchical algorithms, but its asymptotic bound is $O(nm)$.

To summarize this work shows that the extensive use of data structures that support random accesses significantly slow down the non-trivial dynamic algorithm in practice, pointing to a (known) weakness of the RAM model which need to be adapted to increase the impact of theoretical algorithms research. One such attempt is the external-memory model, but as it completely ignores computation time, it is also overlooking a large contribution to the running time of dynamic algorithms.

References

[1] A. Abboud and V. V. Williams. Popular conjectures imply strong lower bounds for dynamic problems. In Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on, pages 434–443. IEEE, 2014.

[2] S. Baswana, M. Gupta, and S. Sen. Fully dynamic maximal matching in $O(\log n)$ update time. SIAM Journal on Computing, 44(1):88–113, 2015.

[3] S. Behnezhad, M. Derakhshian, M. Hajiaghayi, C. Stein, and M. Sudan. Fully dynamic maximal independent set with polylogarithmic update time. In 2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS), pages 382–405. IEEE, 2019.

[4] S. Behnezhad, M. T. Hajiaghayi, and D. G. Harris. Exponentially faster massively parallel maximal matching. In 2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS), pages 1637–1649. IEEE, 2019.

[5] S. Bhattacharya, F. Grandoni, J. Kulkarni, Q. C. Liu, and S. Solomon. Fully dynamic $(\delta + 1)$-coloring in constant update time. arXiv preprint arXiv:1910.02063, 2019.

[6] D. Chakrabarti and C. Faloutsos. Graph mining: Laws, generators, and algorithms. ACM computing surveys (CSUR), 38(1):2-es, 2006.

[7] S. Chechik and T. Zhang. Fully dynamic maximal independent set in expected poly-log update time. In D. Zuckerman, editor, 60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019, pages 370–381. IEEE Computer Society, 2019.

[8] S. Dahlgaard. On the hardness of partially dynamic graph problems and connections to diameter. In I. Chatzigiannakis, M. Mitzenmacher, Y. Rabani, and D. Sangiorgi, editors, 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy, volume 55 of LIPIcs, pages 48:1–48:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.

[9] D. Funke, S. Lamm, U. Meyer, M. Penschuck, P. Sanders, C. Schulz, D. Strash, and M. von Loos. Communication-free massively distributed graph generation. Journal of Parallel and Distributed Computing, 131:200–217, 2019.

[10] Google. Google sparse hash. https://github.com/sparsehash/sparsehash Accessed: 2021-07-02.

[11] K. Hanauer, M. Henzinger, and C. Schulz. Recent advances in fully dynamic graph algorithms. CoRR, abs/2102.11169, 2021.

[12] M. Henzinger, S. Khan, R. Paul, C. Schulz, et al. Dynamic matching algorithms in practice. In 28th Annual European Symposium on Algorithms (ESA 2020). Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2020.

[13] M. Henzinger, S. Krinninger, D. Nanongkai, and T. Saranurak. Unifying and strengthening hardness for dynamic problems via the online matrix-vector multiplication conjecture. In Proc. of the forty-seventh annual ACM symposium on Theory of computing, pages 21–30, 2015.

[14] M. Henzinger and P. Peng. Constant-time dynamic $(\delta + 1)$-coloring. In 37th International Symposium
M. Henzinger and P. Peng. Constant-time dynamic weight approximation for minimum spanning forest. CoRR, abs/2011.00977, 2020.

M. R. Henzinger and V. King. Randomized fully dynamic graph algorithms with polylogarithmic time per operation. J. ACM, 46(4):502–516, 1999.

J. Holm, K. de Lichtenberg, and M. Thorup. Polylogarithmic deterministic fully-dynamic algorithms for connectivity, minimum spanning tree, 2-edge, and biconnectivity. In J. S. Vitter, editor, Proc. of the Thirtieth Annual ACM Symposium on the Theory of Computing, Dallas, Texas, USA, May 23-26, 1998, pages 79–89. ACM, 1998.

D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Boguná. Hyperbolic geometry of complex networks. Physical Review E, 82(3):036106, 2010.

J. Kunegis. Konect: the koblenz network collection. In Proceedings of the 22nd international conference on World Wide Web, pages 1343–1350, 2013.

S. Solomon. Fully dynamic maximal matching in constant update time. In 2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), pages 325–334. IEEE, 2016.
Appendices

Appendix A Additional Proofs

Lemma 4.1. A matching \( \mathcal{M} \) is LFMM\((G, \pi)\) if and only if for each edge \( e = (u, v) \in E \) either \( e \in \mathcal{M} \) or \( \exists w \in N(u) \cup N(v) \) s.t. \( (u, w) \in \mathcal{M} \) with \( \pi(u, w) < \pi(e) \) or \( (v, w) \in \mathcal{M} \) with \( \pi(v, w) < \pi(e) \).

Proof. In the construction algorithm of LFMM\((G, \pi)\), every edge is either added to the matching or covered by an incident edge. For all edges \( e \notin \mathcal{M} \), there therefore exists an edge of smaller rank that is incident to \( e \).

To prove the other direction, let \( \mathcal{M} \) be the LFMM\((G, \pi)\) and \( \mathcal{M}_o \) be a matching with \( \mathcal{M} \neq \mathcal{M}_o \), i.e., they differ in at least one edge. As \( \mathcal{M} \) is a maximal matching, \( \mathcal{M} \not\subseteq \mathcal{M}_o \), i.e., it is not fully encompassed in matching \( \mathcal{M}_o \). Thus, there exists an edge \( e \in E \) that is in one of the matchings but not the other. Let \( e_f \in E \) be the lowest rank edge that is in one matching but not the other. For every edge \( e_i \in E \) with \( \pi(e_i) < \pi(e_f) \), \( e_i \in \mathcal{M}_o \) if and only if \( e_i \in \mathcal{M} \). We now look at both cases where \( e_f \in \mathcal{M} \) and \( e_f \in \mathcal{M}_o \) and show that \( \mathcal{M}_o \) is not an LFMM for \( G \) with ranking \( \pi \).

Case 1: \( e_f \in \mathcal{M}_o \), \( e_f \notin \mathcal{M} \). As \( e_f \notin \mathcal{M} \), it must have been covered by an incident edge of smaller rank. Thus there exists an edge \( e_x \in \mathcal{M} \) incident to \( e_f \) with \( \pi(e_x) < \pi(e_f) \) that covers \( e_f \). Due to the assumption that \( e_f \) is the lowest rank edge in which the matchings differ, \( e_x \in \mathcal{M}_o \) and \( \mathcal{M}_o \) is not a matching.

Case 2: \( e_f \notin \mathcal{M}_o \), \( e_f \in \mathcal{M} \). As \( e_f \in \mathcal{M} \), we know that no incident edge of rank \( < e_f \) covers \( e_f \). As the matchings do not differ in any edge of rank \( < e_f \), this is also true for \( \mathcal{M}_o \). Therefore \( e_f \notin \mathcal{M}_o \) but no incident matching edge with smaller rank exists. This contradicts the assumption that every non-matching edge is covered.

This shows that any matching that differs from \( \mathcal{M} \) in at least one edge is not LFMM\((G, \pi)\).

Claim 1. In Algorithm 7, when vertex \( v \) is removed from \( S \) with rank \( r_v \), then any new vertex \( w \) that is added to \( S \) during this iteration of Algorithm 7 is added with rank \( > r_v \).

Proof. In Algorithm 1, we try to find a incident matching edge for vertex \( v \). We iterate over all incident edges \( e = (v, w) \) with \( \pi(e) > r_v \). The edge \( e \) is added to the matching, if \( \pi(e) < k(v) \) and \( \pi(e) < k(w) \) and we add the current matching partners of \( v \) and \( w \) to \( S \) (if they exist). \( P(v) \) (and \( P(w) \) analogously) is added to \( S \) with priority \( k(v) \), which is \( > \pi(v) > r_v \).

Claim 2. Algorithm 7 checks vertices in non-decreasing priority order.

Proof. Let \( v \) and \( r_v \) be the vertex and its priority that is currently checked in the loop of Algorithm 1. In order to show that all vertices checked at a later point of the algorithm have priority \( \geq r_v \), we need to show the following: (1) All vertices that are currently in \( S \) have priority \( \geq r_v \) and (2) all vertices that are added to \( S \) at any later point have priority \( \geq r_v \). (1) follows from the fact that \( S \) is a priority queue and returns the vertex with smallest priority, (2) follows directly from Claim 1.

Lemma 4.3. Given the matching \( \mathcal{M} = LFMM(G, \pi) \) and the insertion of an edge \( e = (u, v) \), Algorithm 2 maintains the lexicographically first maximum matching LFMM\((G + e, \pi)\).

Proof. Algorithm 2 adds \( e \) to the data structures and checks whether it is uncovered, i.e., has to be inserted into matching \( \mathcal{M} \). In this case, we need to evict the current matching edges incident to \( u \) and \( v \) from \( \mathcal{M} \) (if they exist). All edges incident to \( u \) and \( v \) are still covered, as the ranking of their incident matching edge only decreased, but their previous matching partners \( P(u) \) and \( P(v) \) are now unmatched. Let \( e_x = (u, P(u)) \) be an evicted matching edge (analogous for \( v \) and \( P(v) \)). Note that \( e_x \) is covered by \( e \). However, all edges \( e_2 \) incident to \( P(u) \) with \( e_2 > e_x \) were covered by \( e_x \) and are now potentially uncovered. Edges \( e_2 \) incident to \( P(u) \) with \( e_2 < e_x \) were previously already not covered by \( e_x \) and therefore are guaranteed to remain covered even though \( e_x \) was removed from the matching. As no other edges were changed, we know that all edges not incident to \( P(u) \) and \( P(v) \) and all edges of rank \( \leq \min(\pi(u, P(u)), \pi(v, P(v))) \) are covered. We add the unmatched partner vertices \( P(u) \) and \( P(v) \) (if they exist) to priority queue \( S \). The invariants I1 and I2 currently hold as only edges incident to \( P(u) \) and \( P(v) \) with rank greater than their original matching edge are potentially uncovered.

Then Algorithm 2 is called and the correctness follows from Lemma 1.2.

Lemma 4.4. Given the matching \( \mathcal{M} = LFMM(G, \pi) \) and the deletion of an edge \( e = (u, v) \), Algorithm 3 maintains the lexicographically first maximum matching LFMM\((G - e, \pi)\).

Proof. In Algorithm 3 we delete an edge \( e \) and check whether it is in the matching \( \mathcal{M} \). If it is not, no edges were covered by \( e \), thus every edge in \( G \) is still covered after removal of \( e \), and \( \mathcal{M} \) is still the LFMM\((G - e, \pi)\) due to Lemma 4.1. If \( e \) is a matching edge, we remove it from \( \mathcal{M} \) and add \( u \) and \( v \) to \( S \), as all incident edges with rank \( > \pi(e) \) are now potentially
uncovered. Analogously to the insertion in Lemma 4.3, all potentially uncovered edges are incident to \( u \) or \( v \) and have rank \( > \pi(e) \). Thus Invariants I1 and I2 hold. Then Algorithm 1 is called and the correctness follows from Lemma 4.2.

Claim 3. Algorithm 1 terminates after at most \( m \) iterations.

Proof. In order to prove that the update operations terminate, we need to look at how vertices are added to the priority queue \( S \), as the algorithm terminates as soon as \( S \) is empty. A vertex \( v \) is added with priority \( r_v \) exactly when an incident edge \( e \) with rank \( r_v \) is removed from the matching \( \mathcal{M} \). We process the vertices in \( S \) in increasing priority order, thus, according to Claim 2 when \( v \) with priority \( r_v \) is at the front of \( S \), all edges with rank \( < r_v \) are settled, i.e. they will not be added to or removed from \( \mathcal{M} \) at a later point of the update operation. In Algorithm 1 we try to find a new matching edge \( e' \) and check all incident edges with rank \( > r_v \). Thus, as the edge ranks are all unique, \( e' \) is now settled as well and will not be updated at a later point. As each iteration of the while-loop in Algorithm 1 settles at least one edge, the algorithm terminates after at most \( m \) iterations.

Appendix B Additional Pseudocode and Tables

Algorithm 2 Edge Insertion

Input: \( e = (u, v) \) ← edge to insert
1. \( \pi(e) \leftarrow \text{rand}(0, 1) \)
2. \( N(v).\text{insert}(\pi(e), u) \)
3. \( N(u).\text{insert}(\pi(e), v) \)
4. \( S \leftarrow \text{empty priority queue} \)
5. if \( \pi(e) < \min(k(u), k(v)) \) then
6. \( p_v \leftarrow P(v) \)
7. if \( p_v \neq -1 \) then
8. \( \text{S.insert}(p_v, k(p_v)) \)
9. \( k(p_v) \leftarrow \infty \)
10. \( P(p_v) \leftarrow -1 \)
11. end if
12. \( p_u \leftarrow P(u) \)
13. if \( p_u \neq -1 \) then
14. \( \text{S.insert}(p_u, k(p_u)) \)
15. \( k(p_u) \leftarrow \infty \)
16. \( P(p_u) \leftarrow -1 \)
17. end if
18. \( k(u) \leftarrow \pi(e) \)
19. \( k(v) \leftarrow \pi(e) \)
20. \( P(u) \leftarrow v \)
21. \( P(v) \leftarrow u \)
22. end if
23. findNewPartners(S)

Algorithm 3 Edge Deletion

Input: \( e = (u, v) \) ← edge to remove
1. \( N(v).\text{delete}(u) \)
2. \( N(u).\text{delete}(v) \)
3. \( S \leftarrow \text{empty priority queue} \)
4. if \( P(v) = u \) then
5. \( \text{S.insert}(u, \pi(e)) \)
6. \( \text{S.insert}(v, \pi(e)) \)
7. \( k(v) \leftarrow \infty \)
8. \( k(u) \leftarrow \infty \)
9. \( P(v) \leftarrow -1 \)
10. \( P(u) \leftarrow -1 \)
11. end if
12. findNewPartners(S)
| $\rho$ | HierCol | RecurseCol | RandRCol | CountCol |
|-------|---------|------------|-----------|----------|
| 0.0   | 1385.65 (9.39) | 147.58 (1.00) | 838.87 (5.68) | 440.00 (2.98) |
| 0.1   | 1366.87 (9.94) | 230.04 (1.00) | 775.95 (3.37) | 548.09 (2.38) |
| 0.25  | 1314.98 (4.42) | 297.32 (1.00) | 780.06 (2.62) | 623.54 (2.10) |
| 0.5   | 1208.33 (3.83) | 315.64 (1.00) | 682.45 (2.16) | 634.96 (2.01) |
| 0.75  | 1106.47 (4.40) | 251.55 (1.00) | 615.94 (2.45) | 528.27 (2.10) |

Table 1: Average time in $\text{ns}$ per operation (and slowdown to fastest), Section 5.2.1, for all different values of $\rho$. 

| Graph | HierCol | RecurseCol | RandRCol | CountCol |
|-------|---------|------------|-----------|----------|
| ER $n = 2^{14}, m = n \cdot 2^8$ | 612.57 (5.61) | 109.16 (1.00) | 317.38 (2.91) | 197.19 (1.81) |
| ER $n = 2^{14}, m = n \cdot 2^8$ | 733.53 (5.52) | 132.92 (1.00) | 373.52 (2.81) | 240.90 (1.81) |
| ER $n = 2^{15}, m = n \cdot 2^8$ | 974.99 (6.12) | 159.26 (1.00) | 500.24 (3.14) | 312.55 (1.96) |
| ER $n = 2^{16}, m = n \cdot 2^6$ | 1165.88 (7.47) | 156.14 (1.00) | 679.39 (4.35) | 400.40 (2.56) |
| ER $n = 2^{16}, m = n \cdot 2^7$ | 1252.77 (7.11) | 176.28 (1.00) | 708.24 (4.02) | 433.68 (2.46) |
| ER $n = 2^{16}, m = n \cdot 2^8$ | 1371.70 (6.46) | 212.18 (1.00) | 745.25 (3.51) | 505.44 (2.38) |
| ER $n = 2^{16}, m = n \cdot 2^9$ | 1479.21 (5.57) | 265.45 (1.00) | 800.28 (3.01) | 584.39 (2.20) |
| ER $n = 2^{16}, m = n \cdot 2^{10}$ | 1645.04 (4.53) | 362.78 (1.00) | 886.99 (2.45) | 750.66 (2.07) |
| ER $n = 2^{17}, m = n \cdot 2^8$ | 1742.95 (6.08) | 286.83 (1.00) | 1001.00 (3.49) | 701.61 (2.45) |
| ER $n = 2^{18}, m = n \cdot 2^8$ | 2111.15 (5.53) | 381.66 (1.00) | 1260.63 (3.30) | 935.03 (2.45) |
| RHG $n = 2^{13}, m = n \cdot 2^8$ | 506.94 (4.40) | 115.24 (1.00) | 297.43 (2.58) | 223.28 (1.94) |
| RHG $n = 2^{14}, m = n \cdot 2^8$ | 673.94 (4.65) | 144.82 (1.00) | 384.58 (2.66) | 286.35 (1.98) |
| RHG $n = 2^{15}, m = n \cdot 2^8$ | 985.15 (5.18) | 190.01 (1.00) | 515.13 (2.71) | 421.89 (2.22) |
| RHG $n = 2^{16}, m = n \cdot 2^6$ | 1158.60 (7.20) | 160.91 (1.00) | 618.36 (3.84) | 465.90 (2.90) |
| RHG $n = 2^{16}, m = n \cdot 2^7$ | 1250.88 (5.92) | 211.21 (1.00) | 674.32 (3.19) | 545.16 (2.58) |
| RHG $n = 2^{16}, m = n \cdot 2^8$ | 1278.62 (4.85) | 263.53 (1.00) | 680.36 (2.58) | 607.43 (2.30) |
| RHG $n = 2^{16}, m = n \cdot 2^9$ | 1408.32 (4.12) | 341.94 (1.00) | 727.64 (2.13) | 704.60 (2.07) |
| RHG $n = 2^{16}, m = n \cdot 2^{10}$ | 1520.41 (3.29) | 462.34 (1.00) | 824.23 (1.78) | 863.87 (1.87) |
| RHG $n = 2^{17}, m = n \cdot 2^8$ | 1636.89 (4.66) | 351.57 (1.00) | 905.24 (2.57) | 817.54 (2.33) |
| RHG $n = 2^{18}, m = n \cdot 2^8$ | 2019.66 (4.17) | 484.80 (1.00) | 1858.53 (3.83) | 1101.60 (2.27) |

Table 2: Average time in $\text{ns}$ per operation (and slowdown to fastest), Section 5.2.1, for different Erdős-Renyi and random hyperbolic geometric graph families.
| \(n = 2^{12}, m = n \cdot 2^7\) | HierCol | RecurseCol | RandRCol | CountCol |
|---|---|---|---|---|
| 384.92 (4.25) | 200.51 (2.21) | 237.26 (2.62) | 90.66 (1.00) |
| \(n = 2^{12}, m = n \cdot 2^8\) | 484.63 (4.05) | 308.30 (2.57) | 280.07 (2.34) | 119.73 (1.00) |
| \(n = 2^{12}, m = n \cdot 2^9\) | 632.33 (4.28) | 773.94 (5.24) | 320.88 (2.17) | 147.61 (1.00) |
| \(n = 2^{12}, m = n \cdot 2^{10}\) | 712.27 (4.21) | 1959.38 (11.57) | 360.09 (2.13) | 169.29 (1.00) |
| \(n = 2^{13}, m = n \cdot 2^7\) | 497.63 (3.89) | 226.20 (1.77) | 302.26 (2.36) | 128.02 (1.00) |
| \(n = 2^{13}, m = n \cdot 2^8\) | 583.08 (3.63) | 455.31 (2.83) | 345.00 (2.15) | 160.83 (1.00) |
| \(n = 2^{13}, m = n \cdot 2^9\) | 723.19 (4.09) | 1049.81 (5.94) | 373.34 (2.11) | 176.64 (1.00) |
| \(n = 2^{13}, m = n \cdot 2^{10}\) | 811.83 (4.28) | 2297.92 (12.12) | 407.48 (2.15) | 189.58 (1.00) |
| \(n = 2^{14}, m = n \cdot 2^7\) | 659.06 (3.68) | 328.14 (1.83) | 395.15 (2.21) | 179.02 (1.00) |
| \(n = 2^{14}, m = n \cdot 2^8\) | 770.18 (3.99) | 608.97 (3.16) | 424.16 (2.20) | 192.90 (1.00) |
| \(n = 2^{14}, m = n \cdot 2^9\) | 868.08 (4.38) | 1144.72 (5.77) | 436.99 (2.20) | 198.23 (1.00) |
| \(n = 2^{14}, m = n \cdot 2^{10}\) | 955.15 (4.68) | 2539.51 (12.43) | 464.66 (2.27) | 204.27 (1.00) |
| \(n = 2^{15}, m = n \cdot 2^7\) | 872.28 (3.61) | 425.36 (1.76) | 539.78 (2.23) | 241.87 (1.00) |
| \(n = 2^{15}, m = n \cdot 2^8\) | 965.36 (3.83) | 780.35 (3.10) | 565.28 (2.24) | 251.98 (1.00) |
| \(n = 2^{15}, m = n \cdot 2^9\) | 1107.30 (4.31) | 1491.58 (5.81) | 574.00 (2.24) | 256.68 (1.00) |
| \(n = 2^{15}, m = n \cdot 2^{10}\) | 1223.99 (4.69) | 2911.95 (11.15) | 640.29 (2.45) | 261.22 (1.00) |

Table 3: Average time in ns per operation (and slowdown to fastest), Section 5.2.2 Clashing Sequence for RecurseCol

| \(n = 2^{12}, m = n \cdot 2^7\) | HierCol | RecurseCol | RandRCol | CountCol |
|---|---|---|---|---|
| 438.18 (6.87) | 63.76 (1.00) | 250.12 (3.92) | 412.22 (6.47) |
| \(n = 2^{12}, m = n \cdot 2^8\) | 536.57 (7.88) | 68.07 (1.00) | 291.50 (4.28) | 1902.24 (16.05) |
| \(n = 2^{12}, m = n \cdot 2^9\) | 639.16 (8.21) | 77.83 (1.00) | 341.07 (4.38) | 4693.29 (60.30) |
| \(n = 2^{12}, m = n \cdot 2^{10}\) | 782.89 (8.57) | 91.38 (1.00) | 360.64 (3.95) | 17327.34 (189.62) |
| \(n = 2^{13}, m = n \cdot 2^7\) | 543.10 (6.98) | 77.81 (1.00) | 311.17 (4.00) | 601.15 (7.73) |
| \(n = 2^{13}, m = n \cdot 2^8\) | 632.55 (7.29) | 86.83 (1.00) | 351.94 (4.05) | 2148.88 (24.75) |
| \(n = 2^{13}, m = n \cdot 2^9\) | 751.01 (7.66) | 98.02 (1.00) | 464.69 (4.74) | 3897.90 (80.57) |
| \(n = 2^{13}, m = n \cdot 2^{10}\) | 844.29 (8.16) | 103.42 (1.00) | 406.32 (3.93) | 22293.45 (221.72) |
| \(n = 2^{14}, m = n \cdot 2^7\) | 675.75 (6.88) | 98.18 (1.00) | 387.18 (3.94) | 137.84 (11.59) |
| \(n = 2^{14}, m = n \cdot 2^8\) | 769.93 (7.29) | 105.62 (1.00) | 435.88 (4.13) | 3706.01 (35.09) |
| \(n = 2^{14}, m = n \cdot 2^9\) | 903.79 (8.30) | 108.85 (1.00) | 448.75 (4.12) | 10364.31 (95.21) |
| \(n = 2^{14}, m = n \cdot 2^{10}\) | 959.41 (8.69) | 110.40 (1.00) | 462.33 (4.19) | 27090.15 (245.39) |
| \(n = 2^{15}, m = n \cdot 2^7\) | 901.02 (7.66) | 117.55 (1.00) | 556.56 (4.73) | 1978.43 (16.83) |
| \(n = 2^{15}, m = n \cdot 2^8\) | 1016.79 (8.68) | 117.11 (1.00) | 577.63 (4.93) | 5462.71 (46.65) |
| \(n = 2^{15}, m = n \cdot 2^9\) | 1141.97 (9.71) | 117.58 (1.00) | 598.01 (5.09) | 14135.98 (120.22) |
| \(n = 2^{15}, m = n \cdot 2^{10}\) | 1262.60 (10.75) | 117.48 (1.00) | 644.49 (5.49) | 34874.17 (296.85) |

Table 4: Average time in ns per operation (and slowdown to fastest), Section 5.2.2 Clashing Sequence for CountCol
Table 5: Average time in ns per operation (and slowdown to fastest), Section 5.2.3

| Graph          | RandR1Match      | Hier1Match     | RandR2Match      | Hier2Match      | TrivialMatch     |
|----------------|------------------|----------------|------------------|-----------------|-----------------|
| n = 2^{13}, m = n · 2^8 | 68 232.92 (254.55) | 1010.38 (3.77) | 555.35 (2.08) | 417.68 (1.56) | 268.06 (1.00) |
| n = 2^{14}, m = n · 2^8 | 65 916.21 (254.60) | 1005.68 (3.88) | 563.03 (2.17) | 620.09 (2.40) | 258.90 (1.00) |
| n = 2^{15}, m = n · 2^8 | 56 402.74 (219.98) | 1011.97 (3.95) | 548.94 (2.14) | 633.58 (2.47) | 256.39 (1.00) |
| n = 2^{16}, m = n · 2^8 | 38 825.19 (158.09) | 1002.12 (4.08) | 526.47 (2.14) | 658.69 (2.68) | 245.59 (1.00) |
| n = 2^{17}, m = n · 2^8 | 20 261.76 (94.55) | 911.05 (4.25) | 483.74 (2.26) | 667.51 (3.11) | 214.29 (1.00) |

Table 6: Average time in ns per operation (and slowdown to fastest), Section 5.3.1 for all different values of \( \rho \).

| Graph | RandR1Match | Hier1Match | RandR2Match | Hier2Match | TrivialMatch |
|-------|-------------|------------|-------------|------------|--------------|
| ER n = 2^{13}, m = n · 2^8 | 37 194.02 (261.91) | 535.18 (3.77) | 341.80 (2.41) | 323.63 (2.28) | 142.01 (1.00) |
| ER n = 2^{14}, m = n · 2^8 | 39 242.31 (244.56) | 639.29 (3.98) | 390.64 (2.43) | 374.46 (3.23) | 160.46 (1.00) |
| ER n = 2^{15}, m = n · 2^8 | 40 378.22 (224.28) | 806.29 (4.48) | 480.36 (2.67) | 500.07 (2.78) | 180.03 (1.00) |
| ER n = 2^{16}, m = n · 2^8 | 12 425.32 (55.74) | 974.62 (4.37) | 553.41 (2.48) | 612.15 (2.75) | 222.90 (1.00) |
| ER n = 2^{17}, m = n · 2^8 | 21 547.75 (103.62) | 967.58 (4.65) | 536.05 (2.58) | 624.13 (3.00) | 207.94 (1.00) |
| ER n = 2^{18}, m = n · 2^8 | 44 139.36 (197.16) | 1088.99 (4.86) | 582.90 (2.60) | 648.29 (2.90) | 223.88 (1.00) |
| ER n = 2^{19}, m = n · 2^8 | 106 605.75 (451.90) | 1162.39 (4.75) | 649.76 (2.75) | 690.57 (3.93) | 235.91 (1.00) |
| ER n = 2^{20}, m = n · 2^8 | 237 704.22 (926.70) | 1097.79 (4.28) | 702.56 (2.74) | 744.72 (2.90) | 256.51 (1.00) |
| ER n = 2^{21}, m = n · 2^8 | 51 162.74 (162.03) | 1280.94 (4.06) | 671.22 (2.13) | 849.03 (2.69) | 315.75 (1.00) |
| ER n = 2^{22}, m = n · 2^8 | 54 897.60 (132.92) | 1529.76 (3.70) | 765.26 (1.85) | 1022.91 (2.48) | 413.00 (1.00) |
| RHG n = 2^{13}, m = n · 2^8 | 20 293.78 (141.88) | 567.05 (3.96) | 301.04 (2.10) | 314.12 (2.20) | 143.03 (1.00) |
| RHG n = 2^{14}, m = n · 2^8 | 22 767.57 (129.68) | 680.43 (3.88) | 374.21 (2.13) | 390.13 (2.22) | 175.57 (1.00) |
| RHG n = 2^{15}, m = n · 2^8 | 24 662.86 (113.24) | 838.95 (3.85) | 440.53 (2.02) | 480.77 (2.21) | 217.79 (1.00) |
| RHG n = 2^{16}, m = n · 2^8 | 10 728.51 (51.36) | 858.08 (4.11) | 443.96 (2.13) | 484.52 (3.22) | 208.88 (1.00) |
| RHG n = 2^{17}, m = n · 2^8 | 18 692.50 (79.10) | 938.83 (3.97) | 489.00 (2.07) | 529.89 (2.24) | 236.32 (1.00) |
| RHG n = 2^{18}, m = n · 2^8 | 28 621.40 (112.17) | 966.38 (3.79) | 496.04 (1.94) | 558.27 (2.19) | 255.15 (1.00) |
| RHG n = 2^{19}, m = n · 2^8 | 49 899.54 (169.68) | 1054.92 (3.59) | 568.25 (1.93) | 621.73 (2.11) | 294.09 (1.00) |
| RHG n = 2^{20}, m = n · 2^8 | 96 823.97 (291.86) | 1165.62 (3.51) | 634.83 (1.91) | 685.68 (2.07) | 331.74 (1.00) |
| RHG n = 2^{21}, m = n · 2^8 | 33 715.72 (100.35) | 1225.00 (3.65) | 608.60 (1.81) | 705.11 (2.10) | 335.98 (1.00) |
| RHG n = 2^{22}, m = n · 2^8 | 47 052.31 (113.11) | 1428.94 (3.43) | 691.69 (1.66) | 829.97 (2.00) | 416.00 (1.00) |

Table 7: Average time in ns per operation (and slowdown to fastest), Section 5.3.1 for different Erdős-Rényi and random hyperbolic geometric graph families
| $\eta$ | RandR1Match | Hier1Match | RandR2Match | Hier2Match | TrivialMatch |
|-------|-------------|------------|-------------|------------|--------------|
| 0     | 1 053.71 (174.69) | 25.33 (4.20) | 13.07 (2.17) | 25.29 (4.19) | 6.03 (1.00) |
| 0.25  | 1 057.44 (80.77)  | 25.40 (1.94) | 13.09 (1.00) | 24.03 (1.84) | 14.05 (1.07) |
| 0.5   | 1 063.82 (80.23)  | 25.59 (1.93) | 13.26 (1.00) | 22.40 (1.69) | 22.17 (1.67) |
| 0.75  | 1 093.46 (79.42)  | 26.28 (1.91) | 13.77 (1.00) | 20.71 (1.50) | 30.55 (2.22) |
| 1     | 1 133.05 (77.56)  | 27.08 (1.85) | 14.61 (1.00) | 18.97 (1.30) | 39.32 (2.69) |

Table 8: Average time in $\text{ns}$ per operation (and slowdown to fastest), Section 5.3.2

| Graph | RandR1Match | Hier1Match | RandR2Match | Hier2Match | TrivialMatch |
|-------|-------------|------------|-------------|------------|--------------|
| dewiki| 33 801.67 (239.28) | 499.93 (3.54) | 328.57 (2.33) | 334.01 (2.36) | 141.26 (1.00) |
| frwiki| 71 420.54 (515.89) | 547.24 (3.95) | 320.29 (2.31) | 391.16 (2.83) | 138.44 (1.00) |
| itwiki| 73 972.12 (476.47) | 530.66 (3.42) | 337.78 (2.18) | 398.15 (2.56) | 155.25 (1.00) |
| nlwiki| 21 734.20 (146.27) | 516.84 (3.48) | 321.31 (2.16) | 398.45 (2.68) | 148.59 (1.00) |
| plwiki| 50 364.98 (349.47) | 489.49 (3.40) | 303.27 (2.10) | 349.70 (2.43) | 144.12 (1.00) |
| simplewiki| 6 025.97 (70.93) | 316.30 (3.72) | 197.13 (2.32) | 288.12 (3.39) | 84.96 (1.00) |

Table 9: Average time in $\text{ns}$ per operation (and slowdown to fastest), Section 5.3.3