GLOBAL SOLUTION AND DECAY RATE FOR A REDUCED GRAVITY TWO AND A HALF LAYER MODEL

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ABSTRACT. In this paper we investigate the reduced gravity two and a half model in oceanic fluid dynamics. In a finite domain (for the initial-boundary value problem), we obtain time-independent estimates, which allow us to show the existence and uniqueness of regular solutions as well as the decay rate estimates. A collection of the decay rate estimates for $h_i - \tilde{h}_i$ (with $\tilde{h}_i$ being the stationary layer thickness) and $u_i (i = 1, 2)$ in $L^2(\Omega)$-norm as well as $H^1(\Omega)$-norm as time $t \to \infty$ are established.

1. Introduction. In this paper, we consider a reduced gravity two and a half layer model in one dimension, which can be stated as follows [13]

$$
\begin{align*}
&h_{1,t} + (h_1 u_1)_x = 0, \\
&h_{2,t} + (h_2 u_2)_x = 0, \\
&(h_1 u_1)_t + (h_1 u_1^2)_x - (h_1 u_1 h_2)_x + (g_1 + g_2) h_1 h_1 x + g_2 h_1 h_2_x = 0, \\
&(h_2 u_2)_t + (h_2 u_2^2)_x - (h_2 u_2 h_1)_x + g_2 h_2 h_1_x + g_2 h_2 h_2_x = 0,
\end{align*}
$$

for $(x, t) \in \Omega \times (0, \infty)$, where $\Omega = (0, \ell)$. The unknowns are $h_i(x, t)$ and $u_i(x, t) (i = 1, 2)$, denoting the upper and middle layer thickness and horizontal velocities, $g_1$ and $g_2$ are positive constants, $v_1, v_2$ are constants denoting the coefficient of lateral friction.

As stated in [6], system (1.1) with two active layers is a useful model of the stratified upper ocean overlying a nearly stationary and nearly unstratified abyss. As is known to all, the movement of the sea water is the core of ocean environment. Hence, there are more and more people try their best to investigate the ocean environment. It is obvious that the simplest way to stimulate the ocean circulation is to assume the ocean is homogeneous in density. Such the model has no vertical structure (i.e. homogeneous model). In fact, there is a prominent main thermocline in the oceans, so a nature way of stimulating the ocean circulation is to treat the main thermocline as a step function in density. Consequently, the stratification in

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the ocean is simplified as a two layer fluid. Under the circumstances, fluid above the main thermocline moves faster than that below the main thermocline. As a good approximation, one can assume that fluid in the lower layer is near stagnant. For more information about this model, see for instance \cite{26} and references cited therein.

Let’s simply summarize results about the reduced gravity two and a half layer model. For one dimension, Guo, Li and Yao \cite{13} proved the existence of global weak solutions in bounded domain $\Omega = [0,1]$ and periodic domain $\Omega = T^1$. For two dimension, Duan and Zhou \cite{6} proved the stability of weak solutions in periodic domain $\Omega = T^2$. Cui, Yao and Yao \cite{5} showed the global existence of weak solutions in two-dimensional exterior spatial domain, when the initial data are large and spherically symmetric.

Recently, by constructing suitable approximate solutions and using the method of weak convergence, Yao, Li and Wang \cite{30} obtained the global existence of weak solutions in two-dimensional exterior spatial domain, when the initial data are large and spherically symmetric.

In this paper, without the loss of generality, we set $\nu_1 = \nu_2 = 1$ in the sequel. Then system \eqref{1.1} can be rewritten in the following form

\begin{equation}
\begin{cases}
    h_{1,t} + (h_1 u_1)_x = 0, \\
    h_{2,t} + (h_2 u_2)_x = 0, \\
    (h_1 u_1)_t + (h_1 u_1^2)_x - (h_1 u_1 x)_x + (g_1 + g_2) h_1 h_1 x + g_2 h_1 h_2 x = 0, \\
    (h_2 u_2)_t + (h_2 u_2^2)_x - (h_2 u_2 x)_x + g_2 h_2 h_1 x + g_2 h_2 h_2 x = 0,
\end{cases}
\tag{1.2}
\end{equation}

for $(x, t) \in \Omega \times (0,\infty)$. The initial and boundary data of the system are given as,

\begin{align}
    h_i(x,t)|_{t=0} &= h_{i,0} > 0, & x &\in \Omega, \\
    u_i(0,t) &= u_i(\ell,t) = 0, & t &\geq 0,
\end{align}

where $\Omega$ denotes the spatial domain $(0,\ell)$.

When we take $h_1 \equiv 0$ or $h_2 \equiv 0$, then the model \eqref{1.1} is the compressible Navier-Stokes equation as follows

\begin{equation}
\begin{cases}
    \rho_t + (\rho u)_x = 0, \\
    (\rho u)_t + (\rho u^2)_x + (\rho \gamma)_x = \nu(\rho u)_x,
\end{cases}
\tag{1.5}
\end{equation}

where the viscosity coefficient is taken as $\mu(\rho) = \nu \rho$, and the pressure is $P(\rho) = \rho^2 (\gamma = 2)$. The multi-dimensional form is

\begin{equation}
\begin{cases}
    \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\
    \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho \gamma = \text{div}(\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)) + \nu(\lambda \text{div} \mathbf{u}),
\end{cases}
\tag{1.6}
\end{equation}

where the viscosity coefficient $\mu$ and $\lambda$ satisfy $\lambda(\rho) = 2(\mu'(\rho) - \mu(\rho))$, if $\mu(\rho) = \nu \rho$, then $\lambda(\rho) = 0$.

Now we review some previous works about the compressible Navier-Stokes system. For one dimensional compressible Navier-Stokes \eqref{1.6} with $\mu(\rho) = \rho^\theta$ and $\lambda(\rho) = 0$, the well-posedness of the solutions are investigated by a number of mathematicians, see for instance \cite{9, 15, 21, 28, 31, 32, 33} and references therein. However, there are few results about multi-dimensional problems. The first multi-dimensional result is due to Bresch, Desjardins and Lin \cite{2}, they showed the $L^1$ stability of weak solutions for the Korteweg’s system (with the Korteweg stress tensor $\kappa \rho \nabla \Delta \rho$) and their result was later improved by Bresch and Desjardins \cite{1} to include the case of vanishing capillarity ($k = 0$), but with an additional quadratic friction term $r \rho |\mathbf{u}| \mathbf{u}$. In fact, an interesting new entropy estimate is established in \cite{1, 2, 3}, which provided some high regularity of density. Guo, Li, Jiu and Xin
showed how to construct approximate smooth solutions and obtain the global existence of weak solution to the Dirichlet and free boundary problem via the BD entropy to the system (1.6) with spherical symmetric initial data. Recently, Vasseur and Yu have proved in [27] that there exists a global weak solution to the compressible Navier-Stokes systems (1.6) in two or three dimensional periodic domains by constructing some smooth multipliers allowing to derive the Mellet-Vasseur type inequality [22] for the weak solutions. Almost at the same time, Li and Xin gave an another approach in [20]. What’s more, for the well-posedness of Vaigant-Kazhikhov model((1.6) with \( \mu = \text{constant} \) and \( \lambda = \rho^2 \)), see [16, 17, 14, 25].

For the multi-dimensional Navier-Stokes system (1.6), the global existence and optimal time-decay rates have been obtained in [7, 10, 19] and references therein. The point-wise estimates of compressible Navier-Stokes system in multi-dimensions were considered in [18, 29].

On the other hand, there are numerous papers devoted to the analysis of the asymptotic behavior as \( t \to \infty \) of the solution as well as other properties for the initial boundary value problem. For more information about this problem, see for instance [23, 24] and references cited therein. In this paper, with time \( t \to \infty \), we consider the related stationary problem for \( u_i = 0 \) with \( \tilde{h}_i (i = 1, 2) \) being the stationary layer thickness, and the form is as follows

\[
\begin{aligned}
( g_1 + g_2 ) \tilde{h}_1 \tilde{h}_1, x + g_2 \tilde{h}_1 \tilde{h}_2, x &= 0, \\
g_2 \tilde{h}_2 \tilde{h}_1, x + g_2 \tilde{h}_2 \tilde{h}_2, x &= 0,
\end{aligned}
\]

and satisfy the mass conservation law

\[
\int_0^\ell \tilde{h}_i dx = \int_0^\ell h_i dx = \int_0^\ell h_{i,0} dx > 0, \quad (i = 1, 2).
\]

As far as we know, for the decay rate estimates of the reduced gravity two and a half layer model, there are few results so far. We are particularly interested in the decay rate estimates of \( h_i - \tilde{h}_i \) \((i = 1, 2)\) with \( \tilde{h}_i \) being the stationary thickness) and the velocity \( u_i \) in \( L^2(\Omega) \)-norm and \( H^1(\Omega) \)-norm as time \( t \to \infty \) in a bounded domain. For the sake of obtaining the basic energy equality, we introduce an auxiliary function \( Q_i \) which is designed to fit the model problem. It takes the following form

\[
Q_i(h_i, \tilde{h}_i) = h_i \int_{\tilde{h}_i}^{h_i} \frac{s^2 - \tilde{h}_i^2}{s^2} ds, \quad (i = 1, 2),
\]

where the positive constants \( \tilde{h}_i (i = 1, 2) \) are the stationary solution of the (1.7)-(1.8).

Then we introduce the following two qualities for sake of characterizing the initial state

\[
E_0 = \frac{1}{2} \int_0^\ell (h_{1,0} u_{1,0}^2 + h_{2,0} u_{2,0}^2) dx + \int_0^\ell Q_1(h_1, \tilde{h}_1) dx + \int_0^\ell Q_2(h_2, \tilde{h}_2) dx,
\]

\[
E_1 = 3E_0 + 4 \int_0^\ell \left( (\sqrt{h_{1,0}})^2 + (\sqrt{h_{2,0}})^2 \right) dx.
\]

Appropriate smallness conditions on \( E_0 \) and \( E_1 \) will play a key role in obtaining the result as summed up by Theorem 2.1.

More precisely, for the initial-boundary value problem (1.2)-(1.4), we will assume that the initial layer thickness \( h_{1,0} \) and \( h_{2,0} \) have positive lower and upper bounds and there is a key estimate that is the so-called BD entropy estimate developed by
Bresch and Desjardins for Navier-Stokes equations, please refer to [1, 2, 3] for details. Furthermore, combining the BD entropy estimate with the smallness assumption on the initial data represented by $E_0$ and $E_1$, we can obtain the desired uniform lower and upper bounds for $h_1$ and $h_2$ as expressed by Theorem 2.1. For the stationary problem (1.7)-(1.8), it is easy to see that the constant $\hat{h}_i > 0$ satisfying the mass conservation law (1.8) is the solution of (1.7). In order to obtain the decay rate estimates of $h_i - \hat{h}_i$ (with $\hat{h}_i$ being the stationary thickness) and $u_i$ in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm as time $t \to \infty$, we need more regularities of the solutions independent of time. Then a series of lemmas on the energy estimates are necessarily given; Among which, the property of positive definite quadratic form is applied in the estimates of $I_7$ and $J_5$, since $g_1 > 0$, $g_2 > 0$, see (4.4) and (4.19). In the meantime, those lemmas are conducive to making the strong generalized solution satisfy (2.2) with the help of smallness assumptions.

The paper is organized as follows. In section 2, we summarize our main results which include the global existence of strong solution satisfying (2.4) to the initial-boundary problem of (1.2) and a collection of the decay rate estimates for $h_i - \hat{h}_i$ and $u_i$ in $L^2(\Omega)$-norm and $H^1(\Omega)$-norm. In section 3, we establish some essential energy estimates independently in time and prove Theorem 2.1. There are utilized essentially in section 3 to obtain the decay rate estimates and we give the elaborate detail proofs of Theorem 2.2 in section 4.

We should mention that the methods introduced by Evje-Wen-Yao in [8] and Stráskraba-Zlotnik in [23, 24] will play a crucial role in our proof here.

Notation. $K^{(i)}, \tilde{K}_i, K_i (i = 1, 2, \cdots)$ are arbitrarily positive constants, which can also depend on the initial data, but independent of the time. $L^p$ denotes the Lebesgue spaces $L^p(\Omega)$ with the usual norm $\| \cdot \|_{L^p}$ $(0 \leq p \leq \infty)$, and $H^n$ denotes the Sobolev space $H^n(\Omega)$ with the usual norm $\| \cdot \|_{H^n}$ $(n \in N)$ and $L^2 = H^0$. For two quantities $a$ and $b$, $a \sim b$ means $\frac{1}{C} |a| \leq |b| \leq C |a|$ for a generic constant $C$.

2. Main results. The followings are the main results. Firstly, the unique global solution to the problem (1.2)-(1.4) is stated as follows.

Theorem 2.1. (Existence) Assume that $(h_{1,0}, h_{2,0}) \in H^1(\Omega)$, $(u_{1,0}, u_{2,0}) \in H^1_0(\Omega)$, and for any given positive constants $h_1, \tilde{h}_2, h_1, \tilde{h}_1, h_2$ and $\tilde{h}_2$, where $\tilde{h}_1 \in (0, \tilde{h}_1), \tilde{h}_2 \in (0, \tilde{h}_2)$ such that

$$\begin{align*}
\begin{cases}
\inf h_{1,0}(x) \leq \sup h_{1,0}(x) \leq \tilde{h}_1,
\inf h_{2,0}(x) \leq \sup h_{2,0}(x) \leq \tilde{h}_2.
\end{cases}
\end{align*}$$

Then there exists a positive constants $\alpha$, which is only dependent on initial data and other known constants, such that the initial boundary problem (1.2)-(1.4) admits a unique global solution $(h_1, h_2, u_1, u_2)$ satisfying

$$\begin{align*}
(h_1, h_2) \in C([0, \infty); H^1(\Omega)), & \quad (h_{1,t}, h_{2,t}) \in C([0, \infty); L^2(\Omega)),
(u_1, u_2) \in C([0, \infty); H^1_0(\Omega)) \cap L^2([0, \infty); H^2(\Omega)),
(u_{1,t}, u_{2,t}) \in L^2([0, \infty); L^2(\Omega)),
\end{align*}$$

provided that

$$\begin{align*}
E_2 = E_1 + \frac{1}{|\Omega|} \int_{\Omega} |h_{1,0} - \tilde{h}_1| dx + \frac{1}{|\Omega|} \int_{\Omega} |h_{2,0} - \tilde{h}_2| dx \leq \alpha,
\end{align*}$$

where $E_1$ and $E_2$ are defined in Theorem 2.1.
where the positive constant $\alpha$ is given by (3.35).

What’s more, $h_i (i = 1, 2)$ have uniform upper bound and lower bound, that is

\[
\underline{h_i} \leq h_i \leq \overline{h_i}, \quad (x, t) \in \Omega \times [0, \infty),
\]

\[
\underline{h_i} \leq h_i \leq \overline{h_i}, \quad (x, t) \in \Omega \times [0, \infty),
\]

and the solution satisfies the following estimates

\[
\|(h_1, h_2)\|_{H^1} + \|(h_{1,t}, h_{2,t})\|_{L^2} + \|(u_1, u_2)\|_{H^1} \leq K(1),
\]

where

\[
\overline{h_i} \in (\overline{h}, \infty), \underline{h_i} \in (0, \underline{h}), \overline{h_2} \in (\overline{h}, \infty), h_2 \in (0, h_2).
\]

**Remark 2.1.** As the time tends to infinity ($t \to \infty$), we consider the stationary problem (1.7)-(1.8) (the velocity $u_{i,\infty}=0$, the thickness $h_{i,\infty} = \overline{h_i}$). In particular, the positive constants $\overline{h}_i (i = 1, 2)$ are the solution of (1.7)-(1.8). It is obvious that there are the positive constant $K(2)$, $K(3)$ such that $0 < K(2) \leq \overline{h_i} \leq K(3)$ as well as $\overline{h}_{i,x} = 0, \overline{h}_{i,t} = 0$.

Then we give the main theorem about the decay rate estimates.

**Theorem 2.2.** (Decay estimates) Let the hypotheses of Theorem 2.1 be satisfied, the following decay rate estimates holds

\[
\|h_1 - \bar{h}_1\|_{L^2} + \|u_1\|_{L^2} + \|h_2 - \bar{h}_2\|_{L^2} + \|u_2\|_{L^2} \\
\leq e^{-\frac{K_{48}}{\sqrt{K_8}}t} \max \left\{ (\|h_{1,0} - \bar{h}_1\|_{L^2} + \|u_{1,0}\|_{L^2} + \|h_{2,0} - \bar{h}_2\|_{L^2} + \|u_{2,0}\|_{L^2}), \sqrt{2K_8} \right\},
\]

\[
\|h_1 - \bar{h}_1\|_{H^1} + \|u_1\|_{H^1} + \|h_2 - \bar{h}_2\|_{H^1} + \|u_2\|_{H^1} \\
\leq e^{-\frac{K_{18}}{\sqrt{K_{17}}}t} \max \left\{ (\|h_{1,0} - \bar{h}_1\|_{H^1} + \|u_{1,0}\|_{H^1} + \|h_{2,0} - \bar{h}_2\|_{H^1} + \|u_{2,0}\|_{H^1}), \sqrt{2K_{17}} \right\},
\]

and

\[
\|h_1 - \bar{h}_1\|_{H^1} + \|u_1\|_{H^1} + \|h_2 - \bar{h}_2\|_{H^1} + \|u_2\|_{H^1} \\
\leq e^{-\frac{K_{48}}{\sqrt{K_{25}}t}} \max \left\{ (\|h_{1,0} - \bar{h}_1\|_{H^1} + \|u_{1,0}\|_{H^1} + \|h_{2,0} - \bar{h}_2\|_{H^1} + \|u_{2,0}\|_{H^1}), \sqrt{2K_{25}} \right\}.
\]

**Remark 2.2.** In this paper, we are particularly interested in obtaining the decay rate estimates of the reduced gravity two and a half layer model. In order to study this, we need more regularities of the solutions, such as the uniform with respect to the time estimates for $h_{i,x}$ and $u_{i,x}(i = 1, 2)$ in $L^2(0, t)$-norm, which usually requires that $h_i (i = 1, 2)$ have the uniform positive bounds. To achieve this, for the initial boundary value problem, we assume that the initial data $h_i|_{t=0} (i = 1, 2)$ are positive and then get the uniform positive lower bounds for both $h_1$ and $h_2$ with the help of smallness assumptions. On the other hand, the viscosity coefficients $\mu_1 = \nu_1 h_1$ and $\mu_2 = \nu_2 h_2$ make it difficult to get the positive lower bounds when the initial data can be large. So the smallness assumptions of initial data in (2.3) is necessary in Theorem 2.2.
A priori estimates and global existence and uniqueness of problem (1.2)-(1.4). 3.1. Global existence. In a finite domain \(\Omega = (0, \ell)\), we make use of the classical method, which is the local existence in combination with global a priori estimate in time, to prove the global existence. The local existence of the solutions can be done by using the classical iteration arguments, see for instance [4]. For the sake of simplicity, we omit it. It is essential for us to get some a priori estimates globally in time.

We start with the following basic energy estimates for \((h_1, h_2, u_1, u_2)\).

**Lemma 3.1.** Let the hypotheses of Theorem 2.1 be satisfied, the following equality holds for any \(t \in [0, T]\)

\[
\frac{1}{2} \int_0^\ell (h_1 u_1^2 + h_2 u_2^2) dx + \frac{g_1 + g_2}{2} \int_0^\ell Q_1 dx + \frac{g_2}{2} \int_0^\ell Q_2 dx + g_2 \int_0^\ell h_1 h_2 dx + \int_0^\ell \left( h_1 u_1^2 + h_2 u_2^2 \right) dx + g_1 + g_2 \int_0^\ell Q_{1,0} dx
\]

\[
\frac{1}{2} \int_0^\ell (h_{1,0} u_{1,0}^2 + h_{2,0} u_{2,0}^2) dx + g_1 + g_2 \int_0^\ell Q_{2,0} dx + g_2 \int_0^\ell h_{1,0} h_{2,0} dx = E_0.
\]

**Proof.** Taking \(L^2[0, \ell]\) inner product of (1.2) \(_3\) with \(u_1\), using (1.2) \(_1\) and integration by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^\ell h_1 u_1^2 dx + \int_0^\ell h_1 u_{1,x}^2 dx = -(g_1 + g_2) \int_0^\ell h_1 h_{1,x} u_1 dx - g_2 \int_0^\ell h_1 h_{2,x} u_1 dx.
\]

(3.2)

Similarly, taking \(L^2[0, \ell]\) inner product of (1.2) \(_4\) with \(u_2\), using (1.2) \(_2\) and integration by parts, we get

\[
\frac{1}{2} \frac{d}{dt} \int_0^\ell h_2 u_2^2 dx + \int_0^\ell h_2 u_{2,x}^2 dx = -g_2 \int_0^\ell h_2 h_{1,x} u_2 dx - g_2 \int_0^\ell h_2 h_{2,x} u_2 dx.
\]

(3.3)

Then adding (3.2) and (3.3) up, we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^\ell (h_1 u_1^2 + h_2 u_2^2) dx + \int_0^\ell (h_1 u_{1,x}^2 + h_2 u_{2,x}^2) dx
\]

\[
= -(g_1 + g_2) \int_0^\ell h_1 h_{1,x} u_1 dx - g_2 \int_0^\ell h_2 h_{1,x} u_2 dx - g_2 \int_0^\ell h_1 h_{2,x} u_1 dx
\]

\[
- g_2 \int_0^\ell h_2 h_{2,x} u_2 dx := I_1.
\]

(3.4)

For the sake of handling \(I_1\), we introduce new energy functions

\[
Q_i(h_i, \bar{h}_i) = h_i \int_{\bar{h}_i}^{h_i} \frac{s^2 - \bar{h}_i^2}{s^2} ds, \quad (i = 1, 2),
\]

(3.5)

where the positive constants \(\bar{h}_i(i = 1, 2)\) are the stationary solution of (1.7)-(1.8).
Differentiating \( Q_1 \) with respect to \( t \), using (1.2)_1 and (1.2)_2, we obtain
\[
Q_{1,t}(h_1, \tilde{h}_1) = h_{1,t} \int_{h_1}^{h_1} \frac{s^2 - \tilde{h}_1^2}{s^2} ds + h_{1,h_{1,t}} \frac{h_1^2 - \tilde{h}_1^2}{h_1^2} \\
= -[h_{1,u_1} \int_{h_1}^{h_1} \frac{s^2 - \tilde{h}_1^2}{s^2} ds]_x + u_{1,h_{1,x}} \frac{h_1^2 - \tilde{h}_1^2}{h_1} + h_{1,t} \frac{h_1^2 - \tilde{h}_1^2}{h_1} \\
= -[h_{1,u_1} \int_{h_1}^{h_1} \frac{s^2 - \tilde{h}_1^2}{s^2} ds]_x - u_{1,x}(h_1^2 - \tilde{h}_1^2).
\]

Combining with the boundary condition, we deduce
\[
- \frac{d}{dt} \int_0^\ell Q_1(h_1, \tilde{h}_1) dx = \int_0^\ell u_{1,x}(h_1^2 - \tilde{h}_1^2) dx = -\int_0^\ell u_1(h_1^2)_x dx. \tag{3.6}
\]

Similarly, we get
\[
- \frac{d}{dt} \int_0^\ell Q_2(h_2, \tilde{h}_2) dx = \int_0^\ell u_{2,x}(h_2^2 - \tilde{h}_2^2) dx = -\int_0^\ell u_2(h_2^2)_x dx. \tag{3.7}
\]

For the last two terms on the right hand side of (3.4), using integration by parts, we have
\[
- g_2 \int_0^\ell (h_1 u_1) h_{2,x} dx - g_2 \int_0^\ell (h_2 u_2) h_{1,x} dx \\
= g_2 \int_0^\ell h_2 (h_1 u_1)_x dx + g_2 \int_0^\ell h_1 (h_2 u_2)_x dx \\
= -g_2 \int_0^\ell h_2 h_{1,t} dx - g_2 \int_0^\ell h_1 h_{2,t} dx \\
= -g_2 \frac{d}{dt} \int_0^\ell (h_1 h_2) dx,
\]

where we have used (1.2)_1 and (1.2)_2.

Substituting (3.6), (3.7) and (3.8) into (3.4), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_0^\ell (h_1 u_1^2 + h_2 u_2^2) dx + \frac{g_1 + g_2}{2} \frac{d}{dt} \int_0^\ell Q_1 dx + \frac{g_2}{2} \frac{d}{dt} \int_0^\ell Q_2 dx \\
+ g_2 \frac{d}{dt} \int_0^\ell h_1 h_{2,t} dx + \int_0^\ell h_1 u_{1,x}^2 dx + \int_0^\ell h_2 u_{2,x}^2 dx = 0. \tag{3.9}
\]

Integrating (3.9) with respect to \( \tau \) over \((0, t)\), we can finish the proof of Lemma 3.1.

Next, we get the BD type of entropy estimate that is about the derivatives of \( h_i (i = 1, 2) \), which are essential to provide more available assistance on higher regularity of the upper and middle layer thickness.
Lemma 3.2. Let the hypotheses of Theorem 2.1 be satisfied, the following inequality holds for any \( t \in [0, T] \)

\[
\frac{1}{2} \left[ \int_0^t \left( h_1(u_1 + \frac{h_1}{h_1}) + 2h_2(u_2 + \frac{h_2}{h_2}) \right) dx + \frac{g_1 + g_2}{2} \int_0^t Q_1 dx + \frac{g_2}{2} \int_0^t Q_2 dx \right] + g_2 \int_0^t h_1h_2 dx + g_1 \int_0^t \int_0^x h_1^2 dxd\tau + g_2 \int_0^t \int_0^x (h_1 + h_2)^2 dxd\tau \\
\leq 2E_0 + 4 \int_0^t [(\sqrt{h_1})^2 + (\sqrt{h_2})^2] dx.
\]

(3.10)

Proof. Differentiating (1.2)_1 with respect to \( x \), we obtain

\( h_{1,x} (h_1 u_1)_x + (h_1 u_1)_x = 0. \) (3.11)

Summing (3.11) and (1.2)_3 up, yields

\[
[h_1(u_1 + \frac{h_1}{h_1})]_x + [h_1 u_1(u_1 + \frac{h_1}{h_1})]_x = -(g_1 + g_2)h_1 h_{1,x} + g_1 h_1 h_{2,x}.
\]

Similarly, we have

\[
[h_2(u_2 + \frac{h_2}{h_2})]_x + [h_2 u_2(u_2 + \frac{h_2}{h_2})]_x = -(g_2 h_2 h_{1,x} + g_2 h_2 h_{2,x}].
\]

(3.13)

Taking \( L^2[0,\ell] \) inner product of (3.12) and (3.13) with \( u_1 + \frac{h_1}{h_1} \) and \( u_2 + \frac{h_2}{h_2} \), respectively, using (1.2)_1, (1.2)_2 and adding them up, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \int_0^t \left( h_1(u_1 + \frac{h_1}{h_1})^2 + h_2(u_2 + \frac{h_2}{h_2})^2 \right) dx \right] = - \int_0^t [(g_1 + g_2)h_1 h_{1,x} + g_2 h_1 h_{2,x}] (u_1 + \frac{h_1}{h_1}) dx \\
- \int_0^t (g_2 h_2 h_{1,x} + g_2 h_2 h_{2,x}) (u_2 + \frac{h_2}{h_2}) dx := I_2.
\]

(3.14)

Using (1.2)_1, (1.2)_2, (3.4) and (3.9), \( I_2 \) can be rewritten as follows

\[
I_2 = I_1 - [(g_1 + g_2) \int_0^t h_1 h_{1,x} (u_1 + \frac{h_1}{h_1}) dx + g_2 \int_0^t h_1 h_{2,x} (u_1 + \frac{h_1}{h_1}) dx] \\
+ g_2 \int_0^t h_2 h_{1,x} (u_1 + \frac{h_1}{h_1}) dx + g_2 \int_0^t h_2 h_{2,x} (u_1 + \frac{h_1}{h_1}) dx] \\
= I_1 - [(g_1 + g_2) \int_0^t h_1^2 dx + g_2 \int_0^t h_2^2 dx] \\
- [(g_1 + g_2) \int_0^t h_1 h_{1,x} dx + g_2 \int_0^t h_1 h_{2,x} dx + g_2 \int_0^t h_2^2 dx] \\
= I_1 - \frac{g_1 + g_2}{2} \int_0^t \frac{d}{dt} Q_1 dx - \frac{g_2}{2} \int_0^t \frac{d}{dt} Q_2 dx - g_2 \int_0^t \frac{d}{dt} h_1 h_2 dx \\
- g_2 \int_0^t h_1^2 dx + g_2 \int_0^t (h_1 + h_2)^2 dx]
\]

(3.15)
Substituting the above equality (3.15) into (3.14), we have

\[
\frac{1}{2} \int_0^t \left[ h_1(u_1 + \frac{h_{1,x}}{h_1})^2 + h_2(u_2 + \frac{h_{2,x}}{h_2})^2 \right] dx + \frac{g_1 + g_2}{2} \int_0^t Q_1 dx + \frac{9g_2}{2} \int_0^t Q_2 dx
\]

\[
+ g_2 \int_0^t \left( (h_{1,x} + h_{2,x})^2 \right) dx = 0.
\]

Integrating (3.16) over \((0, t)\), we get

\[
\frac{1}{2} \int_0^t \left[ h_1(u_1 + \frac{h_{1,x}}{h_1})^2 + h_2(u_2 + \frac{h_{2,x}}{h_2})^2 \right] dx + \frac{g_1 + g_2}{2} \int_0^t Q_1 dx + \frac{9g_2}{2} \int_0^t Q_2 dx
\]

\[
+ g_2 \int_0^t \left( (h_{1,x} + h_{2,x})^2 \right) dx = 0.
\]

By Lemma 3.1 and 3.2, we can obtain the following important corollary.

**Corollary 3.1.** Let the hypotheses of Theorem 2.1 be satisfied, the following inequality holds

\[
\int_0^t (|\sqrt{h_{1,x}}|^2 + |\sqrt{h_{2,x}}|^2) dx \leq E_1.
\]

**Proof.** By the definition of \(E_0\) and \(E_1\), from (3.17), we obtain

\[
\frac{1}{2} \int_0^t \left[ h_1(u_1 + \frac{h_{1,x}}{h_1})^2 + h_2(u_2 + \frac{h_{2,x}}{h_2})^2 \right] dx + \frac{g_1 + g_2}{2} \int_0^t Q_1 dx
\]

\[
+ \frac{9g_2}{2} \int_0^t Q_2 dx + g_2 \int_0^t \left( (h_{1,x} + h_{2,x})^2 \right) dx = E_1 - E_0 (\geq 0).
\]

Using inequality \((a + b)^2 \geq \frac{1}{2}(a^2 + b^2)\), the above inequality (3.19) can be rewritten as follows

\[
\frac{1}{4} \int_0^t \left[ h_1 \left( \frac{h_{1,x}}{h_1} \right)^2 + h_2 \left( \frac{h_{2,x}}{h_2} \right)^2 \right] dx - \frac{1}{2} \int_0^t (h_{1,u_1}^2 + h_{2,u_2}^2) dx + \frac{9g_2}{2} \int_0^t Q_1 dx
\]

\[
+ \frac{9g_2}{2} \int_0^t Q_2 dx + g_2 \int_0^t \left( (h_{1,x} + h_{2,x})^2 \right) dx = E_1 - E_0.
\]
From (3.1), we know
\[
\frac{1}{2} \int_0^\ell (h_1 u_1^2 + h_2 u_2^2) dx = E_0 - \left[ \frac{g_1 + g_2}{2} \int_0^\ell Q_1 dx + \frac{g_2}{2} \int_0^\ell Q_2 dx \right. \\
+ g_2 \int_0^\ell h_1 h_2 dx + \int_0^\ell \int_0^\ell (h_1 u_{1,x}^2 + h_2 u_{2,x}^2) dx d\tau \left. \right].
\] (3.21)

Substituting (3.21) into (3.20), we obtain
\[
\int_0^\ell [(\sqrt{h_1})^2 + (\sqrt{h_2})^2] dx = \frac{1}{4} \int_0^\ell \left[ h_1 \left( \frac{h_{1,x}}{h_1} \right)^2 + h_2 \left( \frac{h_{2,x}}{h_2} \right)^2 \right] dx \\
\leq E_1 - 2 \left[ \frac{g_1 + g_2}{2} \int_0^\ell Q_1 dx + \frac{g_2}{2} \int_0^\ell Q_2 dx + g_2 \int_0^\ell h_1 h_2 dx \right. \\
- \int_0^\ell \int_0^\ell (h_1 u_{1,x}^2 + h_2 u_{2,x}^2) dx d\tau - g_1 \int_0^\ell \int_0^\ell h_{1,x}^2 dx d\tau \\
- g_2 \int_0^\ell \int_0^\ell (h_{1,x} + h_{2,x})^2 dx d\tau \leq E_1.
\] (3.22)

where
\[
Q_i(h_i, \tilde{h}_i) = h_i \int_{\tilde{h}_i}^{h_i} \frac{s^2 - \tilde{h}_i^2}{s^2} ds = (h_i - \tilde{h}_i)^2 \geq 0, \quad (i = 1, 2),
\]
where the positive constants \( h_i (i = 1, 2) \) are the stationary solution of (1.7)-(1.8).
\[
\square
\]

The following argument is mainly about the upper and lower bounds of \( h_1 \) and \( h_2 \).

**Lemma 3.3.** There exists a positive constant \( \alpha \) depending only on the initial data and other known constants, such that for any \( t \in [0, T] \)
\[
\bar{h}_1 \leq h_1 \leq \tilde{h}_1, \quad (x, t) \in [0, \ell] \times [0, T],
\] (3.23)
\[
\bar{h}_2 \leq h_2 \leq \tilde{h}_2, \quad (x, t) \in [0, \ell] \times [0, T],
\] (3.24)
provided that \( E_2 \leq \alpha \), where \( \bar{h}_1 \in (0, \bar{h}_1) \subseteq (0, \tilde{h}_1), \bar{h}_2 \in (0, \bar{h}_2) \subseteq (0, \tilde{h}_2), \) and \( \bar{h}_1 \in (\tilde{h}_1, \infty), \bar{h}_2 \in (\tilde{h}_2, \infty) \).

**Proof.** From Corollary 3.1, using the mass conservation law
\[
\int_0^\ell h_i dx = \int_0^\ell h_{i,0} dx, \quad (i = 1, 2),
\]
we deduce
\[
|h_1(x, t) - \tilde{h}_1| = |h_1(x, t) - \tilde{h}_1 - \frac{1}{\ell} \int_0^\ell (h_1 - \tilde{h}_1) dx + \frac{1}{\ell} \int_0^\ell (h_1 - \tilde{h}_1) dx| \\
= |h_1(x, t) - \frac{1}{\ell} \int_0^\ell h_1 dx + \frac{1}{\ell} \int_0^\ell (h_{1,0} - \tilde{h}_1) dx| \\
= \left| \frac{1}{\ell} \int_0^\ell \int_y^x h_{1,\xi} d\xi dy + \frac{1}{\ell} \int_0^\ell (h_{1,0} - \tilde{h}_1) dx \right|.
\]
This combined with Hölder inequality, using the mass conservation law and (1.2)_1, we obtain

\[
|h_1(x, t) - \hat{h}_1| \leq 2 \int_0^\ell \sqrt{h_1} |(\sqrt{h_1})_x| dx + \frac{1}{\ell} \int_0^\ell |h_{1,0} - \hat{h}_1| dx
\]

\[
\leq 2 \left( \int_0^\ell h_1 dx \right)^{\frac{1}{2}} E_1^{\frac{1}{2}} + \frac{1}{\ell} \int_0^\ell |h_{1,0} - \hat{h}_1| dx
\]

(3.25)

Let \( \alpha_1 > 0 \) sufficiently small such that

\[
2 \alpha_1^\frac{1}{2} \left( \int_0^\ell h_{1,0} dx \right)^{\frac{1}{2}} + \alpha_1 \leq \overline{h}_1 - \hat{h}_1.
\]

(3.26)

If we choose \( E_2 \leq \alpha_1 \), from (3.25), we get

\[
h_1 \leq 2 \alpha_1^\frac{1}{2} \left( \int_0^\ell h_{1,0} dx \right)^{\frac{1}{2}} + \alpha_1 + \hat{h}_1 \leq \overline{h}_1.
\]

(3.27)

Let \( \alpha_2 > 0 \) also sufficiently small such that

\[
2 \alpha_2^\frac{1}{2} \left( \int_0^\ell h_{1,0} dx \right)^{\frac{1}{2}} + \alpha_2 \leq \hat{h}_1 - h_1.
\]

(3.28)

If we choose \( E_2 \leq \alpha_2 \), from (3.25), we have

\[
h_1 \geq \hat{h}_1 - 2 \alpha_2^\frac{1}{2} \left( \int_0^\ell h_{1,0} dx \right)^{\frac{1}{2}} - \alpha_2 \geq \overline{h}_1.
\]

(3.29)

Using the similar arguments as that in (3.25), we obtain

\[
|h_2(x, t) - \hat{h}_2| \leq 2 \int_0^\ell \sqrt{h_2} |(\sqrt{h_2})_x| dx + \frac{1}{\ell} \int_0^\ell |h_{2,0} - \hat{h}_2| dx
\]

\[
\leq 2 \left( \int_0^\ell h_{2,0} dx \right)^{\frac{1}{2}} E_1^{\frac{1}{2}} + \frac{1}{\ell} \int_0^\ell |h_{2,0} - \hat{h}_2| dx
\]

(3.30)

Let \( \alpha_3 > 0 \) also sufficiently small such that

\[
2 \alpha_3^\frac{1}{2} \left( \int_0^\ell h_{2,0} dx \right)^{\frac{1}{2}} + \alpha_3 \leq \overline{h}_2 - \hat{h}_2.
\]

(3.31)

If we choose \( E_2 \leq \alpha_3 \), we obtain from (3.30)

\[
h_2 \leq 2 \alpha_3^\frac{1}{2} \left( \int_0^\ell h_{2,0} dx \right)^{\frac{1}{2}} + \alpha_3 + \hat{h}_2 \leq \overline{h}_2.
\]

(3.32)

Similarly, if we choose \( E_2 \leq \alpha_4 \), where \( \alpha_4 \) also sufficiently small such that

\[
2 \alpha_4^\frac{1}{2} \left( \int_0^\ell h_{2,0} dx \right)^{\frac{1}{2}} + \alpha_4 \leq \overline{h}_2 - h_2,
\]

(3.33)

then

\[
h_2 \geq \hat{h}_2 - 2 \alpha_4^\frac{1}{2} \left( \int_0^\ell h_{2,0} dx \right)^{\frac{1}{2}} - \alpha_4 \geq \overline{h}_2.
\]

(3.34)
Lemma 3.4. Let the hypotheses of Theorem 2.1 be satisfied, the following inequality holds
\[
\int_0^\ell (u_{1,x}^2 + u_{2,x}^2) dx + \int_0^T \int_0^\ell (u_{1,t}^2 + u_{2,t}^2) dt \leq K(4).
\]  
(3.36)

Proof. Taking $L^2[0, \ell]$ inner product of (1.2) with $u_{2,t}$, using integration by parts, Cauchy inequality, (1.2) and (3.24), we have
\[
= \frac{1}{2} \int_0^\ell h_{2,x} u_{2,x}^2 dx - \int_0^\ell h_{2} u_{2} u_{2,x} u_{2,t} dx - |g_2| \int_0^\ell h_{2,1,x} u_{2,t} dx + g_2 \int_0^\ell h_{2,2,x} u_{2,t} dx
\]
\[
\leq - \frac{1}{2} \int_0^\ell h_{2,x} u_{2,x}^2 dx - \frac{1}{2} \int_0^\ell h_{2} u_{2} u_{2,x}^2 dx + \frac{1}{4} \int_0^\ell h_{2} u_{2,t}^2 dx
\]
\[
+ K(5) \int_0^\ell h_{2,x} u_{2,x}^2 dx + K(6) \int_0^\ell (h_{1,x} + h_{2,x})^2 dx
\]
\[
\leq K(7) \|h_{2,x}\|_L^\infty \int_0^\ell h_{2,x} u_{2,x}^2 dx + \frac{1}{2} \int_0^\ell \left( \frac{1}{h_2} \right) u_2 (h_2 u_{2,x})^2 dx + \frac{1}{4} \int_0^\ell h_{2} u_{2,t}^2 dx
\]
\[
+ K(5) \int_0^\ell h_{2,x} u_{2,x}^2 dx + K(6) \int_0^\ell (h_{1,x} + h_{2,x})^2 dx.
\]

Using integration by parts, (1.2), (3.24) and Cauchy inequality, we obtain
\[
= \frac{1}{2} \int_0^\ell h_{2,x} u_{2,x}^2 dx - \frac{1}{2} \int_0^\ell \frac{1}{h_2} u_2 (h_2 u_{2,x})^2 dx
\]
\[
+ g_2 h_{2,x} dx + K(8) \|u_2\|_L^2 \int_0^\ell h_{2,x} u_{2,x}^2 dx + K(6) \int_0^\ell (h_{1,x} + h_{2,x})^2 dx
\]
\[
\leq K(9) \|h_{2,x}\|_L^\infty \int_0^\ell h_{2,x} u_{2,x}^2 dx - \int_0^\ell u_2 u_{2,x} (h_2 u_{2,t} + h_2 u_{2,x} + g_2 h_{2,1,x}
\]
\[
+ g_2 h_{2,x} dx + K(8) \|u_2\|_L^2 \int_0^\ell h_{2,x} u_{2,x}^2 dx + K(6) \int_0^\ell (h_{1,x} + h_{2,x})^2 dx
\]
\[
\leq K(10) \|h_{2,x}\|_L^\infty \int_0^\ell h_{2,x} u_{2,x}^2 dx + \frac{1}{4} \int_0^\ell h_{2} u_{2,t}^2 dx
\]
\[
+ K(11) \|u_2\|_L^2 \int_0^\ell h_{2,x} u_{2,x}^2 dx + K(12) \int_0^\ell (h_{1,x} + h_{2,x})^2 dx.
\]
Similarly, we can get

\begin{align*}
\|h_2u_{2,x}\|_{L^\infty} &\leq K^{(13)}(\|h_2u_{2,x}\|_{L^2} + \|(h_2u_{2,x})_x\|_{L^2}) \\
&= K^{(13)}(\|h_2u_{2,x}\|_{L^2} + \|h_2u_{2,t} + h_2u_{2,x} + g_2h_{2,1,x} + g_2h_{2,2,x}\|_{L^2}) \\
&\leq K^{(14)}[(1 + \|u_2\|_{L^\infty})\sqrt{h_2u_{2,x}}]\|_{L^2} \\
&\quad + \|h_2u_{2,t}\|_{L^2} + \|g_2h_{2,1,x} + g_2h_{2,2,x}\|_{L^2},
\end{align*}

(3.38)

and

\|u_2\|_{L^\infty} \leq K^{(15)}(\|u_2\|_{L^2} + \|u_{2,x}\|_{L^2}) \leq K^{(16)}(1 + \sqrt{h_2u_{2,x}}). \tag{3.39}

Substituting (3.38) and (3.39) into (3.37), and using Young inequality, we obtain

\begin{align*}
\frac{1}{2} \frac{d}{dt}\int_0^\ell h_2u_{2,x}^2 \, dx + \frac{1}{4} \int_0^\ell h_2u_{2,t}^2 \, dx \\
&\leq K^{(17)} \left( \int_0^\ell h_2u_{2,x}^2 \, dx \right)^2 + K^{(18)} \left[ \int_0^\ell (h_{1,x} + h_{2,x})^2 \, dx + \int_0^\ell h_2u_{2,x}^2 \, dx \right]. \tag{3.40}
\end{align*}

Then by (3.23), (3.24), Lemma 3.1, Lemma 3.2 and the Gronwall inequality, we deduce

\begin{align*}
\int_0^\ell h_2u_{2,x}^2 \, dx &\leq \text{exp} \left( 2K^{(17)} \int_0^T \int_0^\ell h_2u_{2,x}^2 \, dx \, dt \right) \left\{ \int_0^\ell h_{2,0}u_{2,0,x}^2 \, dx \\
&\quad + 2K^{(18)} \int_0^T \left[ \int_0^\ell h_2u_{2,x}^2 \, dx + \int_0^\ell (h_{1,x} + h_{2,x})^2 \, dx \right] \, dt \right\} \leq K^{(19)},
\end{align*}

(3.41)

where we have used by (3.10)

\begin{align*}
\int_0^T \int_0^\ell (h_{1,x} + h_{2,x})^2 \, dx \, dt \leq K^{(20)}.
\end{align*}

By (3.40) and (3.41), we obtain

\begin{align*}
\int_0^\ell h_2u_{2,x}^2 \, dx + \int_0^T \int_0^\ell h_2u_{2,t}^2 \, dx \, dt \leq K^{(21)}.
\end{align*}

This combined with (3.24) concludes that

\begin{align*}
\int_0^\ell u_{2,x}^2 \, dx + \int_0^T \int_0^\ell u_{2,t}^2 \, dx \, dt \leq K^{(4)} \frac{1}{2}.
\end{align*}

(3.43)

Similarly, we can get

\begin{align*}
\int_0^\ell u_{1,x}^2 \, dx + \int_0^T \int_0^\ell u_{1,t}^2 \, dx \, dt \leq K^{(4)} \frac{1}{2}.
\end{align*}

(3.44)

\begin{flushright}
\Box
\end{flushright}

**Lemma 3.5.** Let the hypotheses of Theorem 2.1 be satisfied, the following inequality holds

\begin{align*}
\|(u_1, u_2)\|_{L^\infty} \leq K^{(22)},
\end{align*}

(3.45)

and

\begin{align*}
\int_0^\ell (h_{1,t}^2 + h_{2,t}^2) \, dx \leq K^{(23)}.
\end{align*}

(3.46)
Proof. Using the Sobolev imbedding inequality $W^{1,2}([0, \ell]) \hookrightarrow L^\infty([0, \ell])$, and (3.24), we get
\[
\|u_2\|_{L^\infty} \leq K^{(15)}(\|u_2\|_{L^2} + \|u_{2,x}\|_{L^2}) \leq K^{(16)}(1 + \|\sqrt{h_2u_{2,x}}\|_{L^1}).
\]
Then by virtue of Lemma 3.4 and (3.24), we obtain
\[
\int_0^\ell h_2^2 u_{2,x}^2 \, dx \leq K^{(24)},
\]
therefore
\[
\|u_2\|_{L^\infty} \leq K^{(22)}.
\]
Similarly, we conclude
\[
\|u_1\|_{L^\infty} \leq K^{(22)}.
\]
Then, using (1.2), (3.24) and Corollary 3.1 and Lemma 3.4, we deduce
\[
\int_0^\ell h_1^2 \, dx = \int_0^\ell (h_1u_1)^2 \, dx = \int_0^\ell (h_{1,x}u_1 + h_1u_{1,x})^2 \, dx \leq K^{(25)} \int_0^\ell [(h_{1,x}u_1^2) + (h_1^2u_{1,x}^2)] \, dx \leq \frac{K^{(23)}}{2}.
\]
Similarly,
\[
\int_0^\ell h_1^2 \, dx \leq \frac{K^{(23)}}{2}.
\]

Lemma 3.6. Let the hypotheses of Theorem 2.1 be satisfied, the following inequality holds
\[
\int_0^T \int_0^\ell (u_{1,xx}^2 + u_{2,xx}^2) \, dx \, dt \leq K^{(26)}, \tag{3.47}
\]
\[
\int_0^T (\|u_{1,x}\|_{L^\infty}^2 + \|u_{2,x}\|_{L^\infty}^2) \, dt \leq K^{(27)}. \tag{3.48}
\]

Proof. Combining with (1.2) and (1.2), we can obtain
\[
h_2u_{2,xx} = h_2u_{2,t} + h_2u_{2,x} + g_2h_2h_{1,x} + g_2h_2u_{2,x} - h_{2,xx}u_{2,xx}. \tag{3.49}
\]
Using (3.24), (3.38), (3.45) and Lemma 3.1, Lemma 3.2, Lemma 3.4, we have
\[
\int_0^T \int_0^\ell u_{2,xx}^2 \, dx \, dt \leq K^{(28)} \left[ \int_0^T \int_0^\ell u_{2,t}^2 \, dx \, dt + \int_0^T \left( \|u_2\|_{L^\infty}^2 \int_0^\ell u_{2,x}^2 \, dx \right) \, dt \right]
+ \int_0^T \int_0^\ell (h_{1,x} + h_{2,x})^2 \, dx \, dt + \int_0^T \left( \|u_{2,x}\|_{L^\infty}^2 \int_0^\ell h_{2,x}^2 \, dx \right) \, dt
\leq K^{(29)} \left[ 1 + \int_0^T (\|u_{2,x}\|_{L^\infty}^2 \int_0^\ell h_{2,x}^2 \, dx) \, dt \right]
\leq K^{(30)} \left[ 1 + \int_0^T \|u_{2,x}\|_{L^\infty}^2 \, dt \right]
\leq K^{(31)} \left( 1 + \int_0^T \int_0^\ell u_{2,x}^2 \, dx \, dt \right) \leq \frac{K^{(26)}}{2}.
\]
By the Sobolev imbedding inequality $W^{1,2}([0, \ell]) \hookrightarrow L^\infty([0, \ell])$, then using (3.24), (3.36), (3.38), (3.42), (3.45) and Lemma 3.1, we get
\begin{align*}
\int_0^T \|h_2 u_{2,x}\|_{L^\infty}^2 dt \leq K^{(32)} \left[ \int_0^T (1 + \|u_2\|_{L^\infty}^2) \|\sqrt{h_2 u_{2,x}}\|_{L^2}^2 dt \right. \\
+ \left. \int_0^T \|h_2 u_{2,t}\|_{L^2}^2 dt \right] \leq K^{(33)}.
\end{align*}
(3.50)

This combined with (3.24) concludes that
\begin{align*}
\int_0^T \|u_{2,x}\|_{L^\infty}^2 dt \leq \frac{K^{(27)}}{2}.
\end{align*}

Similarly, we have
\begin{align*}
\int_0^\ell \int_0^T u_{1,x}^2 dx dt \leq \frac{K^{(26)}}{2},
\end{align*}
and
\begin{align*}
\int_0^T \|u_{1,x}\|_{L^\infty}^2 dt \leq \frac{K^{(27)}}{2}.
\end{align*}

Conclusion. Based on the a priori estimates globally in time as obtained in Section 3.1, it can be concluded that the maximal existence time of the solution $T^* = \infty$. Thus, the proof of the global existence part of Theorem 2.1 is complete.

3.2. Uniqueness. For the uniqueness of the solutions as stated in Theorem 2.1, it can be proved by a similar way as that in [8] and references cited therein. For the brevity, we omit it.

4. Decay rate estimates. Now, we are in position to obtain the global decay rate estimates, we start from an estimate in $L^2(\Omega)$-norm. Then, we are going to deduce global decay rate estimates in $H^1(\Omega)$-norm for the layer thickness $h_1$ and velocity $u_i$ $(i = 1, 2)$. Here, we should remark that the methods in [23, 24] give us some ideas.

Proof of Theorem 2.2
(1). Combining with (1.2)2 and (1.2)4, yields
\begin{align*}
h_2 u_{2,t} + h_2 u_2 u_{2,x} + g_2 h_2 h_{1,x} + g_2 h_2 h_{2,x} = (h_2 u_{2,x})_x.
\end{align*}
Dividing the above equality by $h_2$, we can obtain
\begin{align*}
u_{2,t} + u_2 u_{2,x} + g_2 (h_{1,x} + h_{2,x}) - \frac{(h_2 u_{2,x})_x}{h_2} = 0.
\end{align*}
(4.1)
Motivated by [23], define the operator of integration
\begin{align*}
\int_0^\ell (h_2 - \bar{h}_2)(\xi) d\xi := I(h_2 - \bar{h}_2).
\end{align*}
Multiplying (4.1) by $-I(h_2 - \bar{h}_2)$ and integrating over $[0, \ell]$. Then using the formula $u_{2,t} I(h_2 - \bar{h}_2) = [u_2 I(h_2 - \bar{h}_2)]_t + h_2 u_2^2$ and the mass conservation law $\int_0^\ell \bar{h}_i dx =...
Similarly, with functionals $\nu$ given by

\[
\nu = \left( \frac{1}{2} \int_0^\ell h_1 u_1^2 dx + \frac{1}{2} \int_0^\ell h_2 u_2^2 dx \right) + \left( \frac{g_1 + g_2}{2} \int_0^\ell Q_1 dx + \frac{g_2}{2} \int_0^\ell Q_2 dx \right)
\]

Then we obtain

\[
\frac{d}{dt} \nu + \omega = 0, \quad (4.3)
\]

with functionals $\nu$ and $\omega$ given by

\[
\nu = \left( \frac{1}{2} \int_0^\ell h_1 u_1^2 dx + \frac{1}{2} \int_0^\ell h_2 u_2^2 dx \right) + \left( \frac{g_1 + g_2}{2} \int_0^\ell Q_1 dx + \frac{g_2}{2} \int_0^\ell Q_2 dx \right)
\]

\[
+ \left( g_2 \int_0^\ell h_1 h_2 dx \right) + \left( -\varepsilon \int_0^\ell u_1 I(h_1 - \bar{h_1}) dx - \varepsilon \int_0^\ell u_2 I(h_2 - \bar{h_2}) dx \right)
\]

\[
:= I_1 + I_2 + I_3 + I_4,
\]
\[ \omega_c = \left( \int_0^\ell h_1 u_{1,x}^2 \, dx + \int_0^\ell h_2 u_{2,x}^2 \, dx \right) \\
+ \left[ \frac{-\varepsilon}{2} \int_0^\ell u_1^2 (h_1 + \bar{h}_1) \, dx - \frac{\varepsilon}{2} \int_0^\ell u_2^2 (h_2 + \bar{h}_2) \, dx \right] \\
+ \left[ \varepsilon g_2 \int_0^\ell (h_1 + h_2) (h_1 - \bar{h}_1) \, dx + \varepsilon (g_1 + g_2) \int_0^\ell h_1 (h_1 - \bar{h}_1) \, dx \right] \\
+ \left[ \varepsilon g_2 \int_0^\ell h_2 (h_2 - \bar{h}_1) \, dx \right] \\
+ \left[ -\varepsilon \int_0^\ell \frac{1}{h_1} h_1 u_{1,x} (h_1 - \bar{h}_1) \, dx - \varepsilon \int_0^\ell \frac{1}{h_1} I(h_1 - \bar{h}_1) h_1 u_{1,x} \, dx \right] \\
- \varepsilon \int_0^\ell \frac{1}{h_2} h_2 u_{2,x} (h_2 - \bar{h}_2) \, dx - \varepsilon \int_0^\ell \frac{1}{h_2} I(h_2 - \bar{h}_2) h_2 u_{2,x} \, dx \right] \\
:= I_5 + I_6 + I_7 + I_8. \\
\]

Clearly,

\[ \nu_c(t) = \nu_c[h_1(t, \cdot), h_2(t, \cdot), u_1(t, \cdot), u_2(t, \cdot)], \]

\[ \omega_c(t) = \omega_c[h_1(t, \cdot), h_2(t, \cdot), u_1(t, \cdot), u_2(t, \cdot)]. \]

Now we can estimate \( I_i \) \((i = 1, 2, \ldots, 8)\) as follows.

For \( I_2 \), we can obtain

\[ \hat{K}_1 (||h_1 - \bar{h}_1||^2_{L^2} + ||h_2 - \bar{h}_2||^2_{L^2}) \]

\[ \leq \frac{g_1 + g_2}{2} \int_0^\ell Q_1 \, dx + \frac{g_2}{2} \int_0^\ell Q_2 \, dx \]

\[ \leq K_1 (||h_1 - \bar{h}_1||^2_{L^2} + ||h_2 - \bar{h}_2||^2_{L^2}), \]

where

\[ Q_i = h_i \int_{h_i}^{\bar{h}_i} \frac{s^2 - \bar{h}_i^2}{s^2} \, ds = (h_i - \bar{h}_i)^2 \geq 0, \quad (i = 1, 2), \]

where the positive constants \( \bar{h}_i(i = 1, 2) \) are the stationary solution of the (1.7)-(1.8).

For \( I_3 \), by using (3.23) and (3.24), we can obtain the following estimate

\[ 0 < \int_0^\ell h_1 h_2 \, dx \leq K_2. \]

For \( I_4 \), by using (3.23), (3.24) and Young inequality, we deduce

\[ \varepsilon \left| \int_0^\ell u_1 I(h_1 - \bar{h}_1) \, dx + \int_0^\ell u_2 I(h_2 - \bar{h}_2) \, dx \right| \]

\[ \leq \varepsilon \hat{K}_2 (||h_1 u_1^2||_{L^1} + ||h_2 u_2^2||_{L^1}) + (||h_1 - \bar{h}_1||^2_{L^2} + ||h_2 - \bar{h}_2||^2_{L^2}). \]

For \( I_6 \), by using (3.23), (3.24) and the Poincaré inequality, we get

\[ \frac{\varepsilon}{2} \left| \int_0^\ell u_1^2 (h_1 + \bar{h}_1) \, dx + \int_0^\ell u_2^2 (h_2 + \bar{h}_2) \, dx \right| \]

\[ \leq \varepsilon \hat{K}_3 ||u_1||^2_{L^2} + \varepsilon \hat{K}_4 ||u_2||^2_{L^2} \leq \varepsilon K_3 K_4 (||u_{1,x}||^2_{L^2} + ||u_{2,x}||^2_{L^2}), \]

where

\[ K_3 := \max \{ \bar{h}_1, \bar{h}_2 \}, \]
\[ ||u_1||_{L^2}^2 + ||u_2||_{L^2}^2 \leq K_4(||u_{1,x}||_{L^2}^2 + ||u_{2,x}||_{L^2}^2). \]

By using the mass conservation law, from (1.8), we have

\[ \int_0^t h_idx = \int_0^t \tilde{h}_idx. \]

It is easy to see that

\[ \int_0^t (h_i - \tilde{h}_i)dx = 0, \quad (i = 1, 2). \]

Then, \( I_7 \) can be rewritten as follows

\[
\begin{align*}
\varepsilon g_2 \int_0^t & (h_1 + h_2)(h_2 - \tilde{h}_2)dx + \varepsilon (g_1 + g_2) \int_0^t h_1(h_1 - \tilde{h}_1)dx + \varepsilon g_2 \int_0^t h_2(h_1 - \tilde{h}_1)dx \\
= & \varepsilon g_2 \int_0^t (h_1 - \tilde{h}_1)(h_2 - \tilde{h}_2)dx + \varepsilon g_2 \tilde{h}_1 \int_0^t (h_2 - \tilde{h}_2)dx + \varepsilon g_2 \int_0^t (h_2 - \tilde{h}_2)^2dx \\
+ & \varepsilon g_2 \tilde{h}_2 \int_0^t (h_2 - \tilde{h}_2)dx + \varepsilon (g_1 + g_2) \int_0^t (h_1 - \tilde{h}_1)^2dx + \varepsilon (g_1 + g_2) \int_0^t h_1(h_1 - \tilde{h}_1)dx \\
+ & \varepsilon g_2 \int_0^t (h_2 - \tilde{h}_2)(h_1 - \tilde{h}_1)dx + \varepsilon g_2 h_2 \int_0^t (h_1 - \tilde{h}_1)dx \\
= & 2\varepsilon g_2 \int_0^t (h_1 - \tilde{h}_1)(h_2 - \tilde{h}_2)dx + \varepsilon g_2 \int_0^t (h_2 - \tilde{h}_2)^2dx + \varepsilon (g_1 + g_2) \int_0^t (h_1 - \tilde{h}_1)^2dx \\
\approx & \varepsilon \int_0^t (h_1 - \tilde{h}_1)^2dx + \varepsilon \int_0^t (h_2 - \tilde{h}_2)^2dx,
\end{align*}
\]

(4.4)

where \( \tilde{h}_i(i = 1, 2) \) are the positive constants, therefore we can get

\[
\begin{align*}
\frac{1}{K_5} & (||h_1 - \tilde{h}_1||_{L^2}^2 + ||h_2 - \tilde{h}_2||_{L^2}^2) \\
\leq & \varepsilon g_2 \int_0^t (h_1 + h_2)(h_2 - \tilde{h}_2)dx + \varepsilon (g_1 + g_2) \int_0^t h_1(h_1 - \tilde{h}_1)dx \\
+ & \varepsilon g_2 \int_0^t h_2(h_1 - \tilde{h}_1)dx \\
\leq & \varepsilon K_5(||h_1 - \tilde{h}_1||_{L^2}^2 + ||h_2 - \tilde{h}_2||_{L^2}^2),
\end{align*}
\]

where we assume \( K_5 > 1 \).

For \( I_8 \), by using Young inequality with \( \varepsilon \), Lemma 3.2, (3.23) and (3.24), we have

\[
\begin{align*}
\varepsilon & \left| \int_0^t u_{1,x} \left[ h_1 \frac{1}{h_1} (h_1 - \tilde{h}_1) + h_1 \left( \frac{1}{h_1} \right)_x I(h_1 - \tilde{h}_1) \right] dx \\
+ & \int_0^t u_{2,x} \left[ h_2 \frac{1}{h_2} (h_2 - \tilde{h}_2) + h_2 \left( \frac{1}{h_2} \right)_x I(h_2 - \tilde{h}_2) \right] dx \right| \\
\leq & \frac{1}{4} (||u_{1,x}||_{L^2}^2 + ||u_{2,x}||_{L^2}^2) + \varepsilon^2 K_6 (||h_1 - \tilde{h}_1||_{L^2}^2 + ||h_2 - \tilde{h}_2||_{L^2}^2).
\end{align*}
\]

Therefore taking \( 0 < \varepsilon \leq \varepsilon_1 := \frac{1}{K_5} \), with \( K_7 \) large enough, then (4.3) can be rewritten as follows

\[
\frac{d}{dt} \nu_{\varepsilon_1} + \omega_{\varepsilon_1} = 0.
\]

(4.5)
Combining with the estimates of $I_i (i = 1, 2, \cdots, 8)$ above, we have
\[
K_{8}^{-1} \left( |h_1 - \tilde{h}_1|_{L^2}^2 + |u_1|_{L^2}^2 + |h_2 - \tilde{h}_2|_{L^2}^2 + |u_2|_{L^2}^2 \right) \leq \nu_{\varepsilon_1}
\leq K_{8} \left( |h_1 - \tilde{h}_1|_{L^2}^2 + |u_1|_{L^2}^2 + |h_2 - \tilde{h}_2|_{L^2}^2 + |u_2|_{L^2}^2 + 1 \right),
\]
where we assume $K_{8} > 1$. And,
\[
\frac{1}{2} K^{-1}_{5} \varepsilon_{1} \left( |h_1 - \tilde{h}_1|_{L^2}^2 + |h_2 - \tilde{h}_2|_{L^2}^2 \right) + \frac{1}{2} \tilde{K}_{5} \left( |u_1|_{L^2}^2 + |u_2|_{L^2}^2 \right) \leq \omega_{\varepsilon_1},
\]
where
\[
\tilde{K}_{5} := \min \{ h_1, h_2 \}.
\]
Then by using (4.6), (4.7) together with
\[
K_{4}^{-1} \left( |u_1|_{L^2}^2 + |u_2|_{L^2}^2 \right) \leq \left( |u_1|_{L^2}^2 + |u_2|_{L^2}^2 \right),
\]
we obtain inequality as follows
\[
\omega_{\varepsilon_1} \geq K_{9} \left( \frac{1}{K_{8}} \nu_{\varepsilon_1} - 1 \right) = K_{9} \nu_{\varepsilon_1} - K_{9},
\]
where
\[
K_{9} := \min \left\{ \frac{1}{2 \tilde{K}_{5} K_{7}}, \frac{\tilde{K}_{5}}{2 K_{4}} \right\}.
\]
Then (4.5) can be rewritten as follows
\[
\frac{d}{dt} \nu_{\varepsilon_1} + K_{9} \nu_{\varepsilon_1} \leq K_{9}.
\]
Using the Gronwall inequality, we get
\[
\nu_{\varepsilon_1} \leq e^{-\frac{K_{9} t}{K_{8}}} \left( \nu_{\varepsilon_1}(0) + \int_{0}^{t} K_{9} ds \right) = e^{-\frac{K_{9} t}{K_{8}}} \nu_{\varepsilon_1}(0) + K_{9} e^{-\frac{K_{9} t}{K_{8}}} t
\leq e^{-\frac{K_{9} t}{K_{8}}} \nu_{\varepsilon_1}(0) + K_{9} 2 K_{8} e^{-\frac{K_{9} t}{K_{8}} t} \leq e^{-\frac{K_{9} t}{K_{8}} t} \max \{ \nu_{\varepsilon_1}(0), 2 K_{8} \}.
\]
Hence, we get
\[
\left( |h_1 - \tilde{h}_1|_{L^2} + |u_1|_{L^2} + |h_2 - \tilde{h}_2|_{L^2} + |u_2|_{L^2} \right)
\leq e^{-\frac{K_{9} t}{K_{8}} t} \max \left\{ \left( |h_{1,0} - \tilde{h}_{1,0}|_{L^2} + |u_{1,0}|_{L^2} + |h_{2,0} - \tilde{h}_{2,0}|_{L^2} + |u_{2,0}|_{L^2} \right), \sqrt{2 K_{8}} \right\}. \tag{4.8}
\]
(2). Differentiating (1.2)\_1 with respect to $x$, we have
\[
h_{1,xt} + (h_{1,x} u_{1})_{x} + (h_{1} u_{1,x})_{x} = 0. \tag{4.9}
\]
Then dividing (4.9) by $h_1$ and using (1.2)\_1, we obtain
\[
\frac{1}{h_1} h_{1,xt} - \frac{h_{1,x}}{h_1} \left[ h_{1,t} + (h_{1} u_{1})_{x} \right] + \frac{1}{h_1} \left( h_{1,x} u_{1} \right)_{x} + \frac{1}{h_1} \left( h_{1} u_{1,x} \right)_{x}
= \left( \frac{1}{h_1} h_{1,x} \right)_{t} + u_{1} \left( \frac{1}{h_1} h_{1,x} \right)_{x} + \frac{1}{h_1} \left( h_{1} u_{1,x} \right)_{x} = 0. \tag{4.10}
\]
Combining with (1.2)\_1 and (1.2)\_3, we get
\[
h_{1} u_{1,t} + h_{1} u_{1,x} - (h_{1} u_{1,x})_{x} + (g_1 + g_2) h_{1} h_{1,x} + g_2 h_{1} h_{2,x} = 0. \tag{4.11}
\]
Multiplying (4.10) by $h_1$, then substituting it into (4.11), we have

\[
\begin{align*}
  & h_1 u_{1,t} + h_1 u_1 u_{1,x} + h_1 \left( \frac{1}{h_1} h_{1,x} \right)_t + h_1 u_1 \left( \frac{1}{h_1} h_{1,x} \right)_x \\
  & \quad + (g_1 + g_2) h_1 h_{1,x} + g_2 h_1 h_{2,x} = 0. 
\end{align*}
\]

(4.12)

Define

\[
d[h_i] := \frac{h_{i,x}}{h_i},
\]

\[
D_t \omega_i := \omega_{i,t} + u_i \omega_{i,x}, \quad (i = 1, 2).
\]

By virtue of $d[\tilde{h}_1] = 0 (i = 1, 2)$, (4.12) can be rewritten as

\[
h_1 D_t (u_1 + d[h_1] - d[\tilde{h}_1]) + (g_1 + g_2) h_1 h_{1,x} + g_2 h_1 h_{2,x} = 0,
\]

(4.13)

Similarly, we obtain

\[
h_2 D_t (u_2 + d[h_2] - d[\tilde{h}_2]) + g_2 h_2 h_{2,x} + g_2 h_2 h_{1,x} = 0.
\]

(4.14)

Taking $L^2[0, \ell]$ inner product of (4.13) with the function $(u_1 + d[h_1] - d[\tilde{h}_1])$, and using the fact $\tilde{h}_{1,t} = 0$ and $\tilde{h}_{1,x} = 0$ ($\tilde{h}_1$ is the constant), we obtain

\[
\begin{align*}
  & \frac{1}{2} \frac{d}{dt} \int_0^\ell \left[ h_1 (u_1 + d[h_1] - d[\tilde{h}_1])^2 \right] dx + \frac{g_1 + g_2}{2} \int_0^\ell \left( h_1^2 \right)_x \\
  & \quad - \left( \tilde{h}_1^2 \right)_x (u_1 + d[h_1] - d[\tilde{h}_1]) dx \\
  & \quad + g_2 \int_0^\ell h_1 u_1 h_{1,x} dx + g_2 \int_0^\ell h_1 h_{2,x} (d[h_1] - d[\tilde{h}_1]) dx = 0.
\end{align*}
\]

Similarly, taking $L^2[0, \ell]$ inner product of (4.14) with the function $(u_2 + d[h_2] - d[\tilde{h}_2])$, and using the fact $\tilde{h}_{2,t} = 0$ and $\tilde{h}_{2,x} = 0$, we deduce

\[
\begin{align*}
  & \frac{1}{2} \frac{d}{dt} \int_0^\ell \left[ h_2 (u_2 + d[h_2] - d[\tilde{h}_2])^2 \right] dx + \frac{g_2}{2} \int_0^\ell \left( h_2^2 \right)_x - \left( \tilde{h}_2^2 \right)_x (u_2 + d[h_2] - d[\tilde{h}_2]) dx \\
  & \quad + g_2 \int_0^\ell h_2 u_2 h_{1,x} dx + g_2 \int_0^\ell h_2 h_{1,x} (d[h_2] - d[\tilde{h}_2]) dx = 0.
\end{align*}
\]
By using integration by parts, using (1.2)1 and (1.2)2, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^\ell [h_1(u_1 + d[h_1] - d[\bar{h}_1])^2] dx + \frac{1}{2} \frac{d}{dt} \int_0^\ell [h_2(u_2 + d[h_2] - d[\bar{h}_2])^2] dx
\]

\[+ (g_1 + g_2) \int_0^\ell h_1 u_1 h_{1,x} dx + (g_1 + g_2) \int_0^\ell h_1 h_{1,x} (d[h_1] - d[\bar{h}_1]) dx + g_2 \int_0^\ell h_1 u_1 h_{2,x} dx
\]

\[+ g_2 \int_0^\ell h_1 h_{2,x} (d[h_1] - d[\bar{h}_1]) dx + g_2 \int_0^\ell h_2 u_2 h_{2,x} dx + g_2 \int_0^\ell h_2 h_{2,x} (d[h_2] - d[\bar{h}_2]) dx
\]

\[+ g_2 \int_0^\ell h_2 u_2 h_{1,x} dx + g_2 \int_0^\ell h_2 h_{1,x} (d[h_2] - d[\bar{h}_2]) dx = 0. \tag{4.15}\]

Now adding (4.5) to (4.15) multiplied by a parameter \( \varepsilon > 0 \), we obtain

\[
\frac{d}{dt} \tilde{\nu}_\varepsilon + \tilde{\omega}_\varepsilon = 0, \tag{4.16}
\]

with functionals \( \tilde{\nu}_\varepsilon \) and \( \tilde{\omega}_\varepsilon \) given by

\[
\tilde{\nu}_\varepsilon = \nu_\varepsilon + \frac{\varepsilon}{2} \left[ \int_0^\ell h_1(u_1 + d[h_1] - d[\bar{h}_1])^2 dx + \int_0^\ell h_2(u_2 + d[h_2] - d[\bar{h}_2])^2 dx \right]
\]

\[+ \varepsilon \left( \frac{g_1 + g_2}{2} \int_0^\ell h_1^2 dx + g_2 \int_0^\ell h_1 h_{2,x} dx + \frac{g_2}{2} \int_0^\ell h_2^2 dx \right) := J_1 + J_2 + J_3,
\]

\[
\tilde{\omega}_\varepsilon = \omega_\varepsilon + \varepsilon \left[ (g_1 + g_2) \int_0^\ell h_{1,x} (d[h_1] - d[\bar{h}_1]) dx + g_2 \int_0^\ell h_{1,x} (d[h_2] - d[\bar{h}_2]) dx
\]

\[+ g_2 \int_0^\ell h_{2,x} (d[h_2] - d[\bar{h}_2]) dx + g_2 \int_0^\ell h_{2,x} (d[h_2] - d[\bar{h}_2]) dx \right] := J_4 + J_5.
\]

Using \( \tilde{h}_{i,x} = 0 \) (i = 1, 2) and Lemma 3.2, we have

\[
d[h_1] - d[\bar{h}_1] + d[h_2] - d[\bar{h}_2] = \frac{1}{h_1} (h_{1,x} - \bar{h}_{1,x}) + \frac{1}{h_1} (\bar{h}_1 - \bar{h}_1) \tag{4.17}
\]

\[+ \frac{1}{h_2} (h_{2,x} - \bar{h}_{2,x}) + \frac{1}{h_2} (\bar{h}_2 - h_{2,x}) \]

\[+ \frac{1}{h_2} (h_{2,x} - \bar{h}_{2,x}) + \frac{1}{h_2} (\bar{h}_2 - h_{2,x}) \]

\[+ \frac{1}{h_2} (h_{2,x} - \bar{h}_{2,x}) + \frac{1}{h_2} (\bar{h}_2 - h_{2,x}), \tag{4.18}
\]

and

\[
||d[h_1] - d[\bar{h}_1]||_{L^2} + ||d[h_2] - d[\bar{h}_2]||_{L^2} \leq K_10(||h_1 - \bar{h}_1||_{H^1} + ||h_2 - \bar{h}_2||_{H^1}). \tag{4.18}
\]
For $J_2$, using the formula $(a + b)^2 \geq \frac{1}{2}a^2 - b^2$, (3.23), (3.24), (4.17), (4.18) and $d[\tilde{h}_1] = 0$, we get
\begin{align*}
\varepsilon \frac{g_1 + g_2}{2} \int_0^\ell h_1^2 dx + \varepsilon g_2 \int_0^\ell h_2 dx & \leq K_3 (||h_1||_2^2 + ||h_2||_2^2), \\
K_3 := \max \{\overline{h_1}, \overline{h_2}\}.
\end{align*}

For $J_3$, using (3.23), (3.24) and the estimates of $I_3$, we have
\begin{align*}
0 < \varepsilon \frac{g_1 + g_2}{2} \int_0^\ell h_1^2 dx + \varepsilon g_2 \int_0^\ell h_1 h_2 dx + \varepsilon \frac{g_2}{2} \int_0^\ell h_2^2 dx \leq K_4.
\end{align*}

Using $d[\tilde{h}_1] = 0$, $J_5$ can be rewritten as follows
\begin{align*}
\varepsilon (g_1 + g_2) \int_0^\ell h_1 h_1(x) (d[h_1] - d[\tilde{h}_1]) dx + \varepsilon g_2 \int_0^\ell h_1 h_2(x) (d[h_1] - d[\tilde{h}_1]) dx \\
+ \varepsilon g_2 \int_0^\ell h_2 h_2(x) (d[h_2] - d[\tilde{h}_2]) dx + \varepsilon g_2 \int_0^\ell h_2 h_1(x) (d[h_2] - d[\tilde{h}_2]) dx \\
= \varepsilon (g_1 + g_2) \int_0^\ell h_1^2 dx + 2\varepsilon g_2 \int_0^\ell h_1 h_2 dx + \varepsilon g_2 \int_0^\ell h_2^2 dx, (4.19)
\end{align*}

therefore we can obtain
\begin{align*}
\varepsilon (g_1 + g_2) \int_0^\ell h_1 h_1(x) (d[h_1] - d[\tilde{h}_1]) dx + \varepsilon g_2 \int_0^\ell h_1 h_2(x) (d[h_1] - d[\tilde{h}_1]) dx \\
+ \varepsilon g_2 \int_0^\ell h_2 h_2(x) (d[h_2] - d[\tilde{h}_2]) dx + \varepsilon g_2 \int_0^\ell h_2 h_1(x) (d[h_2] - d[\tilde{h}_2]) dx \\
\geq \varepsilon K_{15} (||h_1||_2^2 + ||h_2||_2^2).
\end{align*}

Using estimates of $J_i (i = 1, 2, \cdots, 5)$, (4.6), (4.7), (4.18) and we choose $0 < \varepsilon \leq \frac{1}{K_{16}}$ small enough and $\varepsilon_2 < \varepsilon_1$, we get
\begin{align*}
\frac{1}{K_{17}} (||h_1 - \tilde{h}_1||_2^2 + ||h_2 - \tilde{h}_2||_2^2) \leq \overline{\nu}_\varepsilon, (4.20)
\end{align*}

where we assume $K_{17} > 1$.
\begin{align*}
K_{18} (||h_1 - \tilde{h}_1||_2^2 + ||h_2 - \tilde{h}_2||_2^2) + (||h_1||_2^2 + ||h_2||_2^2) \leq \overline{\omega}_\varepsilon. (4.21)
\end{align*}

By simple and direct computation, from (4.20) and (4.21), we obtain
\begin{align*}
\overline{\omega}_\varepsilon \geq K_{18} (\frac{1}{K_{17}} - 1) = \frac{K_{18}}{K_{17}} \overline{\nu}_\varepsilon - K_{18}.
\end{align*}
Then we can get the following inequality

\[ \frac{d}{dt}\tilde{\nu}_{\varepsilon_2} + K_{18}\frac{\tilde{\nu}_{\varepsilon_2}}{K_{17}} \leq K_{18}. \]  

(4.22)

Using the Gronwall inequality, we get

\[ \tilde{\nu}_{\varepsilon_2} \leq e^{-\frac{K_{18}}{K_{17}}t}\left(\tilde{\nu}_{\varepsilon_2}(0) + \int_0^t K_{18}e^{-\frac{K_{18}}{K_{17}}s}ds\right) \]
\[ \leq e^{-\frac{K_{18}}{K_{17}}t}\tilde{\nu}_{\varepsilon_2}(0) + 2K_{17}e^{-\frac{K_{18}}{K_{17}}t} \]
\[ \leq e^{-\frac{K_{18}}{K_{17}}t}\max\{\tilde{\nu}_{\varepsilon_2}(0), 2K_{17}\}. \]

Hence, we have

\[ ||h_1 - \tilde{h}_1||_{H^1} + ||u_1||_{L^2} + ||h_2 - \tilde{h}_2||_{H^2} + ||u_2||_{L^2} \leq e^{-\frac{K_{18}}{K_{17}}t}\max\{(||h_{1,0} - \tilde{h}_1||_{H^1} + ||u_{1,0}||_{L^2} + ||h_{2,0} - \tilde{h}_2||_{H^1} + ||u_{2,0}||_{L^2}), \sqrt{2K_{17}}\}. \]

(4.23)

(3). It follows from (1.2)4 and (1.2)2, we obtain

\[ h_2u_{2,1} + h_2u_2u_{2,x} + g_2h_2h_{1,x} + g_2h_2h_{2,x} - h_{2,xx}u_2 - h_2u_{2,xx} = 0. \]

Dividing the above equality by \( h_2 \), we get

\[ u_{2,t} - u_{2,xx} + (u - \frac{1}{h_2}h_{2,x})u_{2,x} = -g_2h_{1,xx} - g_2h_{2,xx}. \]

(4.24)

Similarly,

\[ u_{1,t} - u_{1,xx} + (u - \frac{1}{h_1}h_{1,x})u_{1,x} = -(g_1 + g_2)h_{1,xx} - g_2h_{2,xx}. \]

(4.25)

By Lemma 3.2 and Lemma 3.3, it is easy to see that

\[ ||u_i - \hat{h}_{i,x}||_{L^2} \leq K_{19}, \quad (i = 1, 2). \]

(4.26)

Using the Sobolev inequality, for any \( \varepsilon > 0 \),

\[ ||u_{i,x}||_{L^\infty} \leq \varepsilon||u_{i,x}||_{L^2} + K_{20}\varepsilon^{-\frac{3}{2}}||u_i||_{L^2}, \quad (i = 1, 2). \]

(4.27)

Taking \( L^2[0, t] \) inner product of (4.24) and (4.25) with \( u_{2,xx} \) and \( u_{1,xx} \), respectively, using (4.26), (4.27) and adding the resulting inequalities, we obtain

\[ \frac{1}{2}\frac{d}{dt}\int_0^t (u_{1,xx}^2 + u_{2,xx}^2)dx + \frac{1}{2}\int_0^t (u_{1,xx}^2 + u_{2,xx}^2)dx \leq K_{21}(||u_1||_{L^2}^2 + ||u_2||_{L^2}^2 + ||h_1 - \tilde{h}_1||_{H^1}^2 + ||h_2 - \tilde{h}_2||_{H^1}^2) := R. \]

(4.28)

Then (4.28) can be rewritten as follows

\[ \frac{d}{dt}\int_0^t (u_{1,xx}^2 + u_{2,xx}^2)dx + \int_0^t (u_{1,xx}^2 + u_{2,xx}^2)dx \leq K_{22}R. \]

(4.29)

Then adding (4.5) and (4.15) multiplied by a parameter \( \varepsilon_2 > 0 \) to (4.29) multiplied by a parameter \( \varepsilon_3 \), then we take \( 0 < \varepsilon \leq \varepsilon_3 := \frac{1}{K_{23}} \) small enough and \( \varepsilon_3 < \varepsilon_2 \), we obtain

\[ \frac{d}{dt}\tilde{\nu}_{\varepsilon_3} + \tilde{\omega}_{\varepsilon_3} \leq 0, \]

(4.30)
with functionals $\tilde{\nu}_{\varepsilon_3}$ and $\tilde{\omega}_{\varepsilon_3}$ given by
\[
\tilde{\nu}_{\varepsilon_3} = \tilde{\nu}_{\varepsilon_2} + \varepsilon_3 \int_0^t (u_{1,x}^2 + u_{2,x}^2) dx,
\]
\[
\tilde{\omega}_{\varepsilon_3} = \tilde{\omega}_{\varepsilon_2} + \varepsilon_3 \int_0^t (u_{1,xx}^2 + u_{2,xx}^2) dx - \varepsilon_3 K_{22} R.
\]
It is easy to see that
\[
\frac{1}{K_{25}} (||h_1 - \tilde{h}_1||^2_{H^1} + ||u_1||^2_{H^1} + ||h_2 - \tilde{h}_2||^2_{H^1} + ||u_2||^2_{H^1}) \leq \tilde{\nu}_{\varepsilon_3}
\]
\[
\leq K_{25} (||h_1 - \tilde{h}_1||^2_{H^1} + ||u_1||^2_{H^1} + ||h_2 - \tilde{h}_2||^2_{H^1} + ||u_2||^2_{H^1} + 1),
\]
where we assume $K_{25} > 1$.

By virtue of the inequality
\[
K_{26}^{-1} (||u_{1,x}||^2_{L^2} + ||u_{2,x}||^2_{L^2}) \leq (||u_{1,x}||^2_{L^2} + ||u_{2,x}||^2_{L^2}),
\]
then we get
\[
K_{26} (||h_1 - \tilde{h}_1||^2_{H^1} + ||h_2 - \tilde{h}_2||^2_{H^1} + (||u_1||^2_{H^1} + ||u_2||^2_{H^1})) \leq \tilde{\omega}_{\varepsilon_3}.
\]
By simple and direct computation, from (4.31) and (4.32), we deduce
\[
\tilde{\omega}_{\varepsilon_3} \geq K_{26} (\frac{1}{K_{25}} \tilde{\nu}_{\varepsilon_3} - 1) = \frac{K_{26}}{K_{25}} \tilde{\nu}_{\varepsilon_3} - K_{26}.
\]
Then (4.30) can be rewritten the following inequality
\[
\frac{d}{dt} \tilde{\nu}_{\varepsilon_3} + \frac{K_{26}}{K_{25}} \tilde{\nu}_{\varepsilon_3} \leq K_{26}.
\]
Using the Gronwall inequality, we get
\[
\tilde{\nu}_{\varepsilon_3} \leq e^{-\frac{K_{26}}{K_{25}} t} \left( \tilde{\nu}_{\varepsilon_3}(0) + \int_0^t K_{26} ds \right)
\]
\[
\leq e^{-\frac{K_{26}}{K_{25}} t} \tilde{\nu}_{\varepsilon_3}(0) + 2K_{26} e^{-\frac{K_{26}}{K_{25}} t} \max \{ \tilde{\nu}_{\varepsilon_3}(0), 2K_{25} \}.
\]
Hence, we obtain
\[
||h_1 - \tilde{h}_1||_{H^1} + ||u_1||_{H^1} + ||h_2 - \tilde{h}_2||_{H^1} + ||u_2||_{H^1}
\]
\[
\leq e^{-\frac{K_{26}}{K_{25}} t} \max \left\{ (||h_{1,0} - \tilde{h}_{1,0}||_{H^1} + ||u_{1,0}||_{H^1} + ||u_{2,0}||_{H^1}) , \sqrt{2K_{25}} \right\}.
\]

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A REDUCED GRAVITY TWO AND A HALF LAYER MODEL

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