A long pseudo-comparison of premice in $L[x]$

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Abstract

We describe an obstacle to the analysis of $\text{HOD}^{L[x]}$ as a core model: Assuming sufficient large cardinals, for a Turing cone of reals $x$ there are premice $M,N$ in $\text{HC}^{L[x]}$ such that the pseudo-comparison of $L[M]$ with $L[N]$ succeeds, is computed in $L[x]$, and lasts through $\omega_1^{L[x]}$ stages. Moreover, we can take $M = M_1|((\delta^+)^{M_1}$ where $M_1$ is the minimal iterable proper class inner model with a Woodin cardinal, and $\delta$ is that Woodin. We can take $N$ such that $L[N]$ is $M_1$-like and short-tree-iterable.

1 Introduction

A central program in descriptive inner model theory is the analysis of $\text{HOD}^W$, for transitive models $W$ satisfying $\text{ZF} + \text{AD}^+$; see [6], [5], [7], [4]. For the models $W$ for which it has been successful, the analysis yields a wealth of information regarding $\text{HOD}^W$ (including that it is fine structural and satisfies $\text{GCH}$), and in turn about $W$.

Assume that there are $\omega$ many Woodin cardinals with a measurable above. A primary example of the previous paragraph is the analysis of $\text{HOD}^{L(\mathbb{R})}$. Work of Steel and Woodin showed that $\text{HOD}^{L(\mathbb{R})}$ is an iterate of $M_\omega$ augmented with a fragment of its iteration strategy (where $M_n$ is the minimal iterable proper class inner model with $n$ Woodin cardinals). The addition of the iteration strategy does not add reals, and so the OD of reals are just $\mathbb{R} \cap M_\omega$. The latter has an analogue for $L[x]$, which has been known for some time: for a cone of reals $x$, the OD of reals are just $\mathbb{R} \cap M_1$. Given this, and further analogies between $L(\mathbb{R})$ and $L[x]$ and their respective HODs, it is natural to ask whether there the full $\text{HOD}^{L[x]}$ is an iterate of $M_1$, adjoined with a fragment of its iteration strategy. Woodin has
conjectured that this is so for a cone of reals \( x \); for a precise statement see [2, 8.23]. Woodin has proved approximations to this conjecture. He analyzed \( \text{HOD}^{L[x,G]} \), for a cone of reals \( x \), and \( G \subseteq \text{Coll}(\omega, < \kappa) \) a generic filter over \( L[x] \), where \( \kappa \) is the least inaccessible of \( L[x] \); see [2, 8.21] and [7]. However, the conjecture regarding \( \text{HOD}^{L[x]} \) is still open.

In this note, we describe a significant obstacle to the analysis of \( \text{HOD}^{L[x]} \).

Before proceeding, we give a brief summary of some relevant definitions and facts. We assume familiarity with the fundamentals of inner model theory; see [6], [3]. One does not really need to know the analysis of \( \text{HOD}^{L[x,G]} \), but familiarity does help in terms of motivation; the system \( F \) described below relates to that analysis. We do rely on some smaller facts from [7, §3].

Let us give some terminology, and recall some facts from [7]. We say that a premouse \( N \) is pre-\( \text{M}_1 \)-like iff \( N \) is proper class, 1-small, and has a (unique) Woodin cardinal, denoted \( \delta^N \). (The notion \( \text{M}_1 \)-like of [7] is stronger; it has some iterability built in.) Let \( P,Q \) be pre-\( \text{M}_1 \)-like. Given a normal iteration tree \( T \) on \( P \), \( T \) is maximal iff \( \text{lh}(T) \) is a limit and \( L[M(T)] \) has no \( Q \)-structure for \( M(T) \) (so \( L[M(T)] \) is pre-\( \text{M}_1 \)-like with Woodin \( \delta(T) \)). A premouse \( R \) is a (non-dropping) pseudo-normal iterate of \( P \) iff there is a normal tree \( T \) on \( P \) such that either \( T \) has successor length and \( R = M^T_\infty \), the last model of \( T \) (and \( [0, \infty)_T \) does not drop), or \( T \) is maximal and \( R = L[M(T)] \). A pseudo-comparison of \( (P,Q) \) is a pair \( (T,U) \) of normal iteration trees formed according to the usual rules of comparison, such that either \( (T,U) \) is a successful comparison, or either \( T \) or \( U \) is maximal. A \((z-)\)pseudo-genericity iteration is defined similarly, formed according to the rules for genericity iterations making a real \( z \) generic for Woodin’s extender algebra. We say that \( P \) is normally short-tree-iterable iff for every normal, non-maximal iteration tree \( T \) on \( P \) of limit length, there is a \( T \)-cofinal wellfounded branch through \( T \), and every putative normal tree \( T \) on \( P \) of length \( \alpha + 2 \) has wellfounded last model (that is, we never encounter an illfounded model at a successor stage). If \( P|\delta_P \in \text{HC}^{L[x]} \), then normal short-tree-iterability is absolute between \( L[x] \) and \( V \). If \( P,Q \) are normally short-tree-iterable then there is a pseudo-comparison \( (T,U) \) of \( (P,Q) \), and if \( T \) has a last model then \( [0, \infty)_T \) does not drop, and likewise for \( U \).

It has been suggested\(^1\) that one might analyze \( \text{HOD}^{L[x]} \) using an \( \text{OD}^{L[x]} \) directed system \( F \) such that:

- the nodes of \( F \) are pairs \( (N, s) \) such that \( s \in \text{OR}^{< \omega} \) and \( N \) is a normally

\(^1\)For example, at the AIM Workshop on Descriptive inner model theory, June, 2014.
short-tree iterable, pre-$M_1$-like premouse with $N|\delta^N \in \text{HC}^{L[x]}$ and such that there is an $L[N]$-generic filter $G$ for $\text{Coll}(\omega, \delta^N)$ in $L[x]$.\[2\]

- for $(P,t), (Q,u) \in \mathcal{F}$, we have $(P,t) \leq_{\mathcal{F}} (Q,u)$ iff $t \subseteq u$ and $Q$ is a pseudo-iterate of $P$, and

- $(M_1, \emptyset) \in \mathcal{F}$.

There are also further conditions, regarding the sets $s$, strengthening the iterability requirements; these and other details regarding how the direct limit is formed from $\mathcal{F}$ are not relevant here.

The main difficulty in analyzing $\text{HOD}^{L[x]}$ in this manner is in arranging that $\mathcal{F}$ be directed. For this, it seems most obvious to try to arrange that $\mathcal{F}$ be closed under pseudo-comparison of pairs.

However, we show here that, given sufficient large cardinals, there is a cone of reals $x$ such that if $\mathcal{F}$ is as above, then $\mathcal{F}$ is not closed under pseudo-comparison. The proof proceeds by finding a node $(N, \emptyset) \in \mathcal{F}$ such that, letting $(T, U)$ be the pseudo-comparison of $(M_1, N)$, then $T, U$ are in fact pseudo-genericty iterations of $M_1, N$ respectively, making reals $y, z$ generic, where $\omega_1^{L[y]} = \omega_1^{L[z]} = \omega_1^{L[x]}$. Letting $W$ be the output of the pseudo-comparison, we have $W|\delta^W \in L[x]$, so $\omega_1^{W[z]} = \omega_1^{L[x]}$, which implies that $\delta^W = \omega_1^{L[x]}$, so $(W, \emptyset) \notin \mathcal{F}$. We now proceed to the details.

2 The comparison

For a formula $\varphi$ in the language of set theory (LST), $\zeta \in \text{OR}$, and $x \in \mathbb{R}$, let $A_{\varphi, \zeta}^x$ be the set of all $M \in \text{HC}^{L[x]}$ such that $L[x] \models \varphi(\zeta, M)$, and $L[M]$ is a normally short-tree-iterable pre-$M_1$-like premouse with $\delta^{L[M]} = \text{OR}^M$ and $M = L[M]|\delta^M$.

Note that $\varphi$ does not use $x$ as a parameter. So by absoluteness of normal short-tree-iterability (between $L[x]$ and $V$, for elements of $\text{HC}^{L[x]}$), $A_{\varphi, \zeta}^x$ is $\text{OD}^{L[x]}$. So $A_{\varphi, \zeta}^x$ is a collection of premice like those involved in the system $\mathcal{F}$ (restricted to their Woodins).

**Theorem.** Assume Turing determinacy and that $M_1^\#$ exists and is fully iterable. Then for a cone of reals $x$, for every formula $\varphi$ in the LST and

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\[2\]The point of $G$ is that we can then use Neeman’s genericity iterations, working inside $L[x]$. We cannot use Woodin’s, as closure under Woodin’s would produce premice with Woodin cardinal $\omega_1^{L[x]}$. 

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every $\zeta \in \text{OR}$, if $M_1|\delta^{M_1} \in A^x_{\varphi,\zeta}$ then there is $R \in A^x_{\varphi,\zeta}$ such that the pseudo-comparison of $M_1$ with $L[R]$ has length $\omega_1^{L[x]}$.

**Proof.** Suppose not. Then we may fix $\varphi$ such that for a cone of $x$, the theorem fails for $\varphi, x$. Fix $z$ in this cone with $z \geq_T M_1$. Let $W$ be the $z$-genericity iteration on $M_1$ (making $z$ generic for the extender algebra), and $Q = M_{1,\omega}$. By standard arguments (see [7]), $Q[z] = L[z]$,

$$\text{lh}(W) = \omega_1^{L[z]} + 1 = \delta^Q + 1,$$

$Q|\delta^Q = M(W|\delta^Q)$, and $T = \text{def} W|\delta^Q$ is the $z$-pseudo-genericity iteration, and $T \in L[z]$.

Let $\mathbb{B}$ be the extender algebra of $Q$ and let $\mathbb{P}$ be the finite support $\omega$-fold product of $\mathbb{B}$. For $p \in \mathbb{P}$ let $p_i$ be the $i^{th}$ component of $p$. Let $G \subseteq \mathbb{P}$ be $Q$-generic, with $z_0 = z$ where $x = \text{def} \langle z_i \rangle_{i < \omega}$ is the generic sequence of reals. Then

$$Q[G] = Q[x] = L[x]$$

and $x >_T z$. Let $\zeta \in \text{OR}$ witness the failure of the theorem with respect to $\varphi, x$. So $M_1|\delta^{M_1} \in A^x_{\varphi,\zeta}$.

By [1] Lemma 3.4 (essentially due to Hjorth), $\mathbb{P}$ is $\delta^Q$-cc in $Q$, so $\delta^Q \geq \omega_1^{L[z]}$, but $\delta^Q = \omega_1^{L[z]}$, so $\delta^Q = \omega_1^{L[z]}$. So it suffices to see that there is some $R \in A^x_{\varphi,\zeta}$ such that the pseudo-comparison of $M_1$ with $L[R]$ has length $\delta^Q$.

For $e \in \omega$ and $y \in \mathbb{R}$ let $\Phi^y_e : \omega \to \omega$ be the partial function coded by the $e^{th}$ Turing program using the oracle $y$. Let $e \in \omega$ be such that $\Phi^x_e$ is total and codes $M_1|\delta^{M_1}$. Let $\dot{x}$ be the $\mathbb{P}$-name for the $\mathbb{P}$-generic sequence of reals, and for $n < \omega$ let $\dot{z}_n$ be the $\mathbb{P}$-name for the $n^{th}$ real. Let $p \in G$ be such that $p \forces^{\mathbb{Q}}_\mathbb{P} \psi(\dot{z}_0)$, where $\psi(v)$ asserts “$\Phi^x_e$ is total and codes a premouse $R$ such that $R \in A^x_{\varphi,\zeta}$, and the $v$-pseudo-genericity iteration of $L[R]$ produces a maximal tree $U$ of length $\delta^Q$ with $M(U) = L[\dot{E}]|\delta^Q$”. In the notation of this formula,

$$p \forces^{\mathbb{Q}}_\mathbb{P} “R \notin \dot{V}”, \text{ because } p \forces^{\mathbb{Q}}_\mathbb{P} “E^U_0 \notin M(U)”. $$

By genericity, we may fix $q \in G$ such that $q \leq p$ and for some $m > 0$, $q_m = q_0$. Note that $q \forces^{\mathbb{Q}}_\mathbb{P} \psi(\dot{z}_m)$.

Let $\dot{R}_i$ be the $\mathbb{P}$-name for the premouse coded by $\Phi^y_e$ (or for $\emptyset$ if this does not code a premouse). Also let $\dot{z}'_0, \dot{z}'_1$ be the $\mathbb{B} \times \mathbb{B}$-names for the two $\mathbb{B} \times \mathbb{B}$-generic reals (in order), and let $\dot{R}'_i$ be the $\mathbb{B} \times \mathbb{B}$-name for the premouse coded by $\Phi^y_e$. 

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We may fix $r \leq q$, $r \in G$, such that

$$r \not\models_P Q \ “\hat{R}_0 \neq \hat{R}_m”. \quad (1)$$

For otherwise there is $r \leq q$, $r \in G$, such that $r \not\models_P Q \ “\hat{R}_0 = \hat{R}_m”$. But since

$$M_1 |\delta^{M_1} = \hat{R}_0 \notin Q,$$

there are $s, t \in B$, $s, t \leq r_0$, such that

$$(s, t) \not\models_{B \times B} Q \ “\hat{R}_0 \neq \hat{R}_1”.$$

Therefore there are $u, v \in B$, with $u \leq r_0$ and $v \leq r_m$, such that

$$(u, v) \not\models_{B \times B} Q \ “\hat{R}_0 \neq \hat{R}_1”.$$

Let $w \leq r$ be the condition with $w_i = r_i$ for $i \neq 0, m$, and $w_0 = u$ and $w_m = v$. Then

$$w \not\models_P Q \ “\hat{R}_0 \neq \hat{R}_m”, \text{ a contradiction.}$$

So letting $R = \hat{R}^G_m$, we have $R \neq M_1 |\delta^{M_1}$ and $R \in A^\tau_{\varphi, \zeta}$ and $Q |\delta^Q = M(\mathcal{U})$, where $\mathcal{U}$ is the $z^G_m$-pseudo-genericy iteration of $L[R]$, and $\text{lh}(\mathcal{U}) = \delta^Q$. We defined $\mathcal{T}$ earlier. Let $\mathcal{T}^*, \mathcal{U}^*$ be the padded trees equivalent to $\mathcal{T}, \mathcal{U}$, such that for each $\alpha$, either $E^T_\alpha \neq \emptyset$ or $E^U_\alpha \neq \emptyset$, and if $E^T_\alpha \neq \emptyset \neq E^U_\alpha$ then $\text{lh}(E^T_\alpha) = \text{lh}(E^U_\alpha)$. Let $(\mathcal{T}', \mathcal{U}')$ be the pseudo-comparison of $(M_1, L[R])$.

We claim that $(\mathcal{T}', \mathcal{U}') = (\mathcal{T}^*, \mathcal{U}^*)$; this completes the proof. For this, we prove by induction on $\alpha$ that

$$(\mathcal{T}', \mathcal{U}') \upharpoonright (\alpha + 1) = (\mathcal{T}^*, \mathcal{U}^*) \upharpoonright (\alpha + 1).$$

This is immediate if $\alpha$ is a limit, so suppose it holds for $\alpha = \beta$; we prove it for $\alpha = \beta + 1$. Let $\lambda = \text{lh}(E^T_\beta)$ or $\lambda = \text{lh}(E^U_\beta)$, whichever is defined. Because $M(\mathcal{T}^*) = Q |\delta^Q = M(\mathcal{U}^*)$, the least disagreement between $M^T_\beta$ and $M^U_\beta$ has index $\geq \lambda$, so we just need to see that $E^T_\beta \neq E^U_\beta$.

So suppose that $E^T_\beta = E^U_\beta$. In particular, both are non-empty. Then there is $s \in G$ such that $s \leq r$ (see line (1)) and $s \not\models_P \sigma$ where $\sigma$ asserts “For $i = 0, m$, let $\mathcal{T}_i$ be the $\hat{z}_i$-pseudo-genericy iteration of $L[\hat{R}_i]$. Then $\mathcal{T}_0$ and $\mathcal{T}_m$ use identical non-empty extenders $E$ of index $\hat{\lambda}$.” Because

$$s \not\models_P \psi(\hat{z}_0) \ & \ \psi(\hat{z}_m),$$
also \( s \Vdash^Q \sigma' \), where \( \sigma' \) asserts “Letting \( E \) be as above, \( E \subseteq L[\mathcal{E}]|\check{\lambda} \), but \( E \notin V'' \); here \( E^{\beta'} \notin Q \) because \( \lambda \) is a cardinal of \( Q \). But since \( T_i^{\infty} \) is computed in \( Q[z_i^{G_i}] \) (for \( i = 0, m \)) we can argue as before (as in the proof of the existence of \( r \) as in line (1)) to reach a contradiction. \( \square \)

A slightly simpler argument, using \( \mathbb{B} \times \mathbb{B} \) instead of \( \mathbb{P} \), proves the weakening of the theorem given by dropping the parameter \( \zeta \).

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