Quantum-gravity effects for excited states of inflationary perturbations

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We generalize former findings regarding quantum-gravitational corrections arising from a canonical quantization of a perturbed FLRW universe during inflation by considering an initial state for the scalar and tensor perturbations that generalizes the adiabatic vacuum state and allows us to consider the scenario that the perturbation modes start their evolution in an excited state. Our result shows that the quantum-gravitationally corrected power spectra get modified by pre-factors including the excitation numbers.

I. INTRODUCTION

Looking for potentially observable effects of a candidate theory of quantum gravity is one of the crucial tasks necessary to eventually decide how to properly quantize gravity. Given that the standard paradigm of the beginning of the evolution of our universe with an inflationary phase is one of the physical scenarios with the highest energies available to be tested by measuring the anisotropies of the cosmic microwave background (CMB), many studies have focused on finding quantum-gravity corrections for inflationary perturbations that might lead to a measurable deviation of the CMB anisotropy spectrum; see, for example, [1–7].

In the context of a canonical quantization of gravity that leads to the Wheeler–DeWitt equation – which can be considered as one of the most conservative approaches to quantum general relativity [8] – one can calculate such corrections by applying a semiclassical approximation technique to the Wheeler–DeWitt equation of a perturbed Friedmann–Lemaître–Robertson–Walker (FLRW) universe, which effectively leads to a description of the quantized perturbations in the context of quantum field theory in curved spacetime with corrections arising from the fact that also the background geometry is quantized. The resulting quantum-gravitationally corrected Schrödinger equation was first derived in [9] and applied to a simplified inflationary model with perturbations in [2]. In [10, 11] this study was extended to gauge-invariant scalar and tensor perturbations and this is the formalism that we will base the present study on.

In the previous works, the initial state for the quantized perturbations was taken to be the adiabatic vacuum state, as it is usually done in the context of inflation; this is because of the assumption that the initial state should correspond to the vacuum in Minkowski space. However, given our very limited understanding of the physics present at the onset of inflation, this might not be the case.

The scenario that cosmological perturbations start their evolution in an excited state has thus already been discussed with various motivations in the literature ranging from general considerations [12] to the study of non-gaussianities [13, 14]. An early discussion of initial excited states in the Schrödinger picture and its semiclassical limit can be found in [15].

In this article, we intend to develop the tools needed to eventually answer the questions whether the inclusion of a quantized background and thus the emergence of quantum-gravity corrections can shed some light into the problem of choosing the initial state of inflationary perturbations and, in particular, whether quantum-gravitational effects might actually lead to a preference of initial excited states.

For this purpose, we will work with the invariants of a time-dependent harmonic oscillator [15, 16]; see also sections 8 and 20 of [20]. The advantage of this method is significant in that it provides a specific criterion for considering not only the vacuum state but also the excited states of the perturbation modes when applied to the study presented in [10, 11]. One can then impose an excited state as the initial state of the perturbation modes and analyze the residual effect that it would leave in the power spectrum of the CMB.

We have organized the article as follows. In Section II, we review the method of the invariants of a time-dependent harmonic oscillator leading us to the ansatz for the wave function that we then use in Section III to compute the power spectra of the uncorrected and quantum-gravitationally corrected perturbations for an initial excited eigenstate, generalizing the work in [10, 11]. In Section IV we consider a more general initial state given as a linear combination of eigenstates. We wind up the article with our conclusions in Section V.

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II. THE INVARIANT VACUUM

Given a harmonic oscillator with time dependent mass and frequency, \( m(t) \) and \( \omega(t) \) respectively, which obeys a wave equation

\[
y'' + \frac{m'}{m} y' + \omega^2 y = 0 \tag{1}
\]

with position variable \( y \) and the derivative of a generic time variable \( t \) marked by a prime, it is always possible to define a vacuum state of the quantum description that is stable along the entire evolution of the harmonic oscillator, that is, it represents the ground state of the harmonic oscillator, that is, it represents the ground state of the Wigner function. This invariant vacuum state is then given by the state that is annihilated by the number operator \( \hat{b}^\dagger \) of an invariant representation of the Hamiltonian \( H \). This operator, along with its creation counterpart \( \hat{b} \), can be defined as

\[
\hat{b} = \sqrt{\frac{m}{2\sigma}} \hat{y}, \quad \hat{b}^\dagger = \sqrt{\frac{m}{2\sigma}} \hat{y}^*.
\]

where \( \sigma \) is defined as the modulus of the solution to the classical evolution equation \( \dot{y} = \alpha y \), which can be written using a real phase \( \tau \) as

\[
y(t) \equiv \sigma(t)e^{i\tau(t)}.
\]

By replacing this polar decomposition into \( \hat{b} \), it can be easily shown that the invariant representation satisfies the auxiliary equation

\[
\dot{\sigma}^2 + \frac{m}{m'} \sigma' + \omega^2 \sigma = \alpha^2 \frac{m^2}{\sigma^2} \sigma', \tag{6}
\]

with an arbitrary real constant \( \alpha \) that is related to the normalization of the modes, while the phase is written as a quadrature,

\[
\tau(t) = \int_0^t \frac{\alpha}{m(t)\sigma^2(t)} \, dt. \tag{7}
\]

In order to fix the rest of the integration constants, it is usual practice to look for an asymptotic limit where the mass and frequency of the harmonic oscillator are constants, given by \( m_0 \) and \( \omega_0 \), respectively. In this limit, the invariant representation has to become the customary representation of the harmonic oscillator with constant mass and frequency. This is accomplished by choosing the modes to have a constant amplitude,

\[
\sigma \to \sqrt{\frac{\alpha}{m_0\omega_0}}, \quad \sigma' \to 0, \tag{8}
\]

in that limit. Note that the value of the amplitude has been chosen such that \( \sigma'' \) is vanishing, as can be seen from its equation of motion \( \ddot{\sigma} \). This makes the representation to be unchanged as long as the mass and frequency keep their constant values. These conditions translate to the following asymptotic form for the classical solution,

\[
y \to \sqrt{\frac{\alpha}{m_0\omega_0}} e^{\omega_0 \eta}, \quad |y| \to 0. \tag{9}
\]

At this point, the normalization \( \alpha \) is the only constant to be fixed and, for that, one usually considers the Wronskian defined using the complex conjugate \( \overline{y} \) as

\[
W[\overline{y}, y] = \overline{y}y' - \overline{y}y = \frac{2i\alpha}{m}, \tag{10}
\]

which is a constant of motion. We note that the constant \( \alpha \) has dimension of an action. Since we use here units in which \( \hbar = 1 \) (in addition to \( c = 1 \), \( \alpha \) is dimensionless and we can set for convenience \( \alpha = 1 \). (Alternatively, one could re-scale \( \alpha \to \alpha \hbar \).)

The invariant representation has the important property

\[
\frac{d}{dt}(\hat{b}^\dagger \hat{b}) = \frac{\partial(\hat{b}^\dagger \hat{b})}{\partial t} + i [H, \hat{b}^\dagger \hat{b}] = 0, \tag{11}
\]

such that

\[
\hat{N}(t)|N\rangle_I \equiv \hat{b}^\dagger \hat{b}|N\rangle_I = N|N\rangle_I, \tag{12}
\]

with \( N \) being a constant integer number. It means that once the harmonic oscillator is in a number state it remains in the same state along the entire evolution. In particular, once it is in the invariant vacuum state it remains in the vacuum state for all time.

As usual, the solutions of the Schrödinger equation \( \psi \) correspond to the eigenstates of the number operator \( \hat{N} \); see, for instance, Refs. \[21\] \[22\]. They are given by the following wave function \( \psi \):

\[
\psi_N(y, t) \equiv \langle y|N \rangle = \frac{1}{\sqrt{\sigma(t)}} \exp \left( \frac{i m}{2} \sigma'(t) y^2 \right) \varphi_N(y, t), \tag{13}
\]

\[1\] Or more generally in a superposition of number states.
where $\varphi_N$ is the customary wave function of the $N$th eigenstate of the harmonic oscillator with unit mass and frequency, that is,

$$
\varphi_N(y, t) = \frac{e^{-i(N + \frac{1}{2})\tau(t)}}{\sqrt{2^{N}N!\pi^{1/4}}} H_N \left( \frac{y}{\sigma(t)} \right),
$$

(14)

with $H_N$ being the Hermite polynomial of degree $N$, and $\tau(t)$ is the phase of the solution to the classical equation defined in [14].

In particular, the wave function of the vacuum state ($N = 0$) for the harmonic oscillator with time dependent frequency and mass reads

$$
\psi_{0\text{eig}}(y, t) = \frac{e^{-\frac{1}{2}\tau(t)}}{\sqrt{\sigma(t)\pi^{1/4}}} \exp \left[ -\frac{1}{2} \left( \frac{1}{\sigma^2(t)} - i m \frac{\sigma'(t)}{\sigma(t)} \right) y^2 \right],
$$

which is a Gaussian wave function with variance $\sigma^2(t)$. (The above constant $\alpha$, which was set equal to one, can be recovered by the substitution $\sigma \rightarrow \sigma / \sqrt{\alpha}$.)

### III. POWER SPECTRA FOR AN INITIAL EXCITED EIGENSTATE

In this section, we will use the method explained above in order to obtain the power spectra corresponding to both scalar and tensorial perturbations, which can be related to the anisotropies seen in the cosmic microwave background. We will do so by first following the standard approximation given by quantum field theory on classical background spacetimes and then by including quantum-gravity effects coming from the quantization of that background spacetime. In both cases, the equation to be solved will correspond to a Schrödinger equation of a harmonic oscillator with time-dependent frequency and unit mass.

For the scalar and tensor perturbations of the FLRW spacetime that we are considering, we use the perturbation variables $v_k$ that correspond to the Mukhanov–Sasaki variables [23] in the scalar sector and to the two different independent components of the symmetric tensor that describes the gravitational waves in the tensorial sector. The exact definitions can be found, for example, in [10, 11]. The power spectrum $P_v$ of the variable $v_k$ can then be defined using the wave function [13] for a number eigenstate $N_k$, where $N_k$ is an integer, with $v_k$ taking the role of the position variable and the conformal time $\eta$ as the time variable, as follows (see e.g. [24]):

$$
\frac{2\pi^2}{k^3} P_v(k) \delta(k - p) = \langle \hat{v}_k \hat{v}_p \rangle
$$

$$
= \int \prod_q dq \, \overline{\psi_{0\text{eig}}(v_q, \eta)} \hat{v}_k \psi_{0\text{eig}}(v_q, \eta).
$$

(16)

Evaluating the integral above then leads to

$$
P_v(k) = \frac{k^3}{2\pi^2} \frac{2N_k + 1}{2} \sigma^2.
$$

(17)

The quantity $\sigma(k, \eta)$ is the modulus of the mode variable $y(k, \eta)$ that obeys an equation analogous to (1).

The real observable power spectrum of the scalar sector corresponds to the spectrum of the comoving curvature perturbation $\zeta$, which is related to the Mukhanov–Sasaki variable as follows,

$$
\zeta_k = \sqrt{\frac{4\pi G}{\alpha}} \frac{v_k}{a}.
$$

(18)

The power spectrum of the scalar sector is thus given by

$$
P_{\zeta} = \frac{4\pi G}{a^2} \frac{k^3}{2\pi^2} \frac{2N_k + 1}{2} \sigma^2.
$$

(19)

Concerning the tensorial sector, the power spectrum is given by

$$
P_{T\theta} = \frac{64\pi G}{a^2} \frac{k^3}{2\pi^2} \frac{2N_k + 1}{2} \sigma^2.
$$

(20)

In both cases, $\sigma(k, \eta)$ should be obtained by solving its equation of motion with its corresponding frequency.

#### A. Uncorrected perturbation modes

Following [10, 11], the wave function of the uncorrected perturbation modes, $\psi_k^{(0)}$, is the solution of the Schrödinger equation

$$
\mathcal{H}_k \psi_k^{(0)} = i \frac{\partial}{\partial \eta} \psi_k^{(0)},
$$

(21)

with

$$
\mathcal{H}_k = -\frac{1}{2} \frac{\partial^2}{\partial \eta^2} + \frac{1}{2} \omega^2(k, \eta) v_k^2.
$$

(22)

It corresponds to the Hamiltonian of a harmonic oscillator with constant mass, $m = 1$, and frequency given by

$$
\omega_S^2(k, \eta) := k^2 - \frac{z''}{z}, \quad \omega_T^2(k, \eta) := k^2 - \frac{a''}{a},
$$

(23)

for the scalar modes, $\omega_S$, and for the tensorial modes, $\omega_T$. The prime now stands for the derivative with respect to the conformal time $\eta$, while $a = a(\eta)$ is the scale factor and $z := a' / a$, the inflaton field and $H := a'/a$ (see Refs. [10, 11] for the details).

For simplicity, in this paper we will just work in the de Sitter limit with a constant Hubble factor $H$. In that case both scalar and tensorial frequencies coincide and take the following form:

$$
\omega_0^2(k, \eta) = k^2 - 2 \frac{2}{\eta^2}.
$$

(24)

2 We will use the subindex 0 to refer to objects related to the uncorrected wave function $\psi_k^{(0)}$, like $\omega_0, \sigma_0$ and $y_0$, whereas the subindex 1 will stand for objects associated with the corrected wave function $\Psi_k^{(1)}$ that will be introduced in the next subsection.
In order to obtain the power spectrum, one assumes that the wave function is in a given eigenstate $N_k$ and sets \( \psi_k^{(0)}(v_k, \eta) \equiv \psi_{N_k}^{\text{eig}}(v_k, \eta) \) using (13). Then one just needs to solve the equation for the mode variable \( y_0(k, \eta) \) and insert its modulus \( \sigma(k, \eta) \) into (19) and (20). The equation for the modes \( y_0 \) takes the form
\[
y_0'' + \omega_0^2 y_0 = 0,
\]
and can be solved straightforwardly by
\[
y_0 = c_1 \left( \sin \frac{\xi}{k} - \cos \frac{\xi}{k} \right) + c_2 \left( \cos \frac{\xi}{k} + \sin \frac{\xi}{k} \right),
\]
where, for convenience, we have defined the new dimensionless time variable \( \xi := -k\eta \) and have introduced two integration constants \( c_1 \) and \( c_2 \).

As commented in the previous section, in order to fix the integration constants, one should find a limit where the frequency tends to a certain constant. In inflationary dynamics, this happens at the beginning of inflation \( \eta \to -\infty \) (or equivalently \( \xi \to \infty \)). In that limit, we have \( \omega_0 \to k \), so one gets the modes corresponding to Minkowski spacetime. Therefore, we will apply the conditions
\[
y_0 \to e^{-i\xi} \sqrt{k}, \quad |y_0'| \to 0.
\]
With these conditions, and the normalization of the Wronskian (10), one gets the solution
\[
y_0 = \frac{\xi - i}{\sqrt{k}\xi} e^{-i\xi},
\]
whose modulus is given by
\[
\sigma_0 = \left( \frac{\xi^2 + 1}{k^2 \xi^2} \right)^{1/2}.
\]
Inserting this result into the power spectra (19) and (20) and taking into account that the scale factor behaves as \( a = -1/(H\eta) = k/(H\xi) \), one gets the usual scale-invariant form for the power spectra for super-Hubble scales (\( \xi \to 0 \)),
\[
S_{P_{N_k}^{(0)}}(k) = (2N_k + 1) \frac{G H^2}{\pi \epsilon},
\]
\[
T_{P_{N_k}^{(0)}}(k) = (2N_k + 1) \frac{16G H^2}{\pi}.
\]
Note that in these objects we have included the superindex \( (0) \) to explicitly mark that they correspond to the uncorrected case. In the limit \( N_k \to 0 \) we recover the results of Ref. 10 for the power spectra corresponding to the ground state.

### B. Corrected perturbation modes

The consideration of a non-vacuum state for the initial state of the perturbations of the scalar field or the spacetime would have two consequences in the next-order corrected wave functions \( \psi_k^{(1)} \) computed in [11, 12]. One is the modification that the choice of a non-vacuum state introduces in the computation of the corrected frequency \( \tilde{\omega} \) (see below). The other is that one can also apply the method of the invariants of the harmonic oscillator to obtain the excited states of the corrected wave functions, \( \psi_k^{(1)}(v_k, \eta) \equiv \psi_{N_k}^{\text{eig}}(v_k, \eta) \) using (13), and check their influence in the generated power spectrum.

The Schrödinger equation for the perturbation modes corrected by the quantum-gravitational effects of the Wheeler–DeWitt equation is given by [11, 12]
\[
i \frac{\partial}{\partial \eta} \psi_k^{(1)} = H_k \psi_k^{(1)}
\]
\[
- \psi_k^{(1)} \frac{1}{2 m_p^2} \mathbb{R} \left\{ \frac{1}{V} \left[ \frac{H_k}{V} \psi_k^{(0)} + i \frac{\partial}{\partial \eta} \left( \frac{H_k}{V} \psi_k^{(0)} \right) \right] \right\}.
\]
Here we use \( m_p \) to denote a rescaled Planck mass \( m_p := 3/(4\pi G) \) (recall \( h = 1 = c \)) and \( V \) is a minisuperspace potential that in de Sitter space takes the form
\[
V(\eta) = \frac{1}{H^2 \eta^4}.
\]
In order to avoid unitarity violations, and following the considerations presented in [11, 12], we have taken the real part of the term between see Refs. 23, 24. In order to compute that term, we will use the ansatz \( \psi_k^{(0)}(v_k, \eta) = \psi_{N_k}^{\text{eig}}(v_k, \eta) \) and perform a power expansion in \( v_k \) of the result, dropping terms of order \( v_k^4 \) and higher.

Note that this power expansion does not lead to any odd power because of the parity of the eigenfunctions \( \psi_{N_k}^{\text{eig}} \).

In this way, the above equation is rewritten as follows,
\[
i \frac{\partial}{\partial \eta} \psi_k^{(1)} = H_k \psi_k^{(1)} + \left( f_0 + v_k^2 f_2 \right) \psi_k^{(1)},
\]
where \( f_0 \) and \( f_2 \) are real functions of time that depend on \( \sigma_0, \omega_0, V \) and the number of the state \( N_k \), but not on \( v_k \). In fact, these functions differ whether \( N_k \) is an even or odd number. Explicitly, they are given by:
\[
f_0 = \frac{1}{8 m_p^2 V} \left[ \frac{3 \sigma_0^2 + 2 \omega_0^2 - 3 + 4 N_k(N_k + 1) + 2 \sigma_0 V'}{\sigma_0^3} \right],
\]
\[
f_2 = \frac{2 N_k + 1}{4 m_p^2 V \sigma_0^2} \left[ 3 \left( 1 - 3 \sigma_0^2 \omega_0^2 - \sigma_0^4 \omega_0^2 \right) - 2 \sigma_0^3 \sigma_0 V' \right],
\]
for even \( N_k \), and
\[
f_0 = \frac{1}{8 m_p^2 V} \left[ \frac{15 \sigma_0^2 + 6 \omega_0^2 - 7 + 4 N_k(N_k + 1)}{\sigma_0^4} \right],
\]
\[
f_2 = \frac{2 N_k + 1}{12 m_p^2 V \sigma_0^2} \left[ 5 \left( 1 - 3 \sigma_0^2 \omega_0^2 - \sigma_0^4 \omega_0^2 \right) - 3 \sigma_0^3 \sigma_0 V' \right],
\]
for odd \( N_k \).

At this point, it is useful to redefine the wave function as
\[
\Psi_k^{(1)} := e^{-i \int^\eta f_0(\tilde{\eta}) d\tilde{\eta}} \psi_k^{(1)},
\]
which just differs from the original wave function by a phase and thus does not change the expression for the power spectrum. This new wave function obeys the Schrödinger equation of a time-dependent harmonic oscillator,

$$i \frac{\partial \Psi_k^{(1)}}{\partial \eta} = -\frac{1}{2} \frac{\partial^2 \Psi_k^{(1)}}{\partial \epsilon_k^2} + \frac{1}{2} \omega_1^2 \epsilon_k^2 \Psi_k^{(1)}, \tag{36}$$

with the modified frequency $\omega_1(k, \eta)$ given by

$$\omega_1^2 := \omega_0^2 + 2f_2. \tag{37}$$

By inserting the form of $\omega_0$ \textsuperscript{24} and $V$ \textsuperscript{33} for de Sitter spacetime and the solution for $\sigma_0$ \textsuperscript{26} in the expressions above, one gets the following form of the modified frequency:

$$\omega_1^2 = k^2 - 2k^2 + \frac{(2N_k + 1) - H^2}{m_p^2} \omega_0^2, \tag{38}$$

where $\omega_1$ has a different expression depending on whether $N_k$ is an even or odd number:

$$\omega_1^2 = \begin{cases} \frac{1}{2} \frac{(7^2-11)\xi^4}{(7^2+1)\xi^4} & \text{for } N_k \text{ even,} \\ \frac{1}{2} \frac{(7^2-13)\xi^4}{(7^2+1)\xi^4} & \text{for } N_k \text{ odd.} \end{cases} \tag{39}$$

It is interesting to note that the form of the corrected frequency is quite similar for both even and odd cases. They both have a global factor $(2N_k + 1)/k$ multiplying the correction terms of the frequencies, such that the dependence on $N_k$ and $k$ of the correction to the power spectra will be the same for both even and odd $N_k$. In addition, the corrective term is given by the ratio between two polynomials in the dimensionless time $\xi$. These polynomials are of the same order in both cases, but just with slightly different coefficients. This implies that the asymptotic limits of the frequency will be quite similar in both cases.

The next step will be to solve equation \textsuperscript{30}. This can be done following the steps of the previous section by assuming that $\Psi_k^{(1)}$ takes a form

$$\Psi_k^{(1)}(v_k, \eta) = \psi_{M_k}^{(1)}(v_k, \eta) \tag{40}$$

based on \textsuperscript{13}. Note that the excitation number $M_k$ used here does not necessarily need to be the same as the one used when inserting $\psi_k^{(0)}$ into \textsuperscript{32}, which is why we have added a tilde. Nevertheless, since we have $\psi_1 = \psi_0$ up to a certain order in $m_p$, it seems reasonable to assume that they be in the same number eigenstates $N_k$; for this reason, we set in the following $M_k = N_k$.

In this way, one just needs to solve the equation for the corresponding mode variable $y_1(k, \eta)$,

$$y_1'' + \omega_1^2 y_1 = 0, \tag{41}$$

with the following conditions at the beginning of inflation $\eta \rightarrow -\infty$,

$$y_1 \rightarrow \frac{e^{i \omega_1(\infty) \eta}}{\sqrt{\omega_1(\infty)}}, \quad |y_1|' \rightarrow 0. \tag{42}$$

where $\omega_1(\infty)$ is the value of $\omega_1$ in that limit, and the normalization $W[y_1, y_1'] = 2i$. The modulus $\sigma_1 := |y_1|$ should then simply be plugged into the form of the power spectra \textsuperscript{19, 20}.

Equation \textsuperscript{41} turns out to be quite difficult to be solved analytically. Therefore, we will perform the following change of variable,

$$y_1 \equiv y_0 + \frac{2N_k + 1}{k^{3/2}} \frac{H^2}{m_p^2} y_1, \tag{43}$$

where $y_0$ is the solution to the uncorrected equation \textsuperscript{25} and is explicitly given by \textsuperscript{25}. Dropping terms of the order $(H/m_p)^4$, the equation for $\tilde{y}_1$, written in terms of $\xi$, is given by

$$\frac{d^2 \tilde{y}_1}{d \xi^2} + \frac{\omega_0^2}{k^2} \tilde{y}_1 = -\sqrt{k} y_0 \tilde{\omega}_1^2, \tag{44}$$

which is the equation of the uncorrected harmonic oscillator with an inhomogeneous term. The general solution is then given by the solution of the homogeneous part \textsuperscript{26} plus some particular solution of the inhomogeneous equation, which can be systematically obtained. We will do so in the following for both even and odd cases. In addition, all the $k$-dependence of this equation is encoded in $\xi$. Note that the explicit $k$ that are written in that expression are absorbed by the dependence on $k$ of $\omega_0^2$ and $y_0$. The modulus of the mode function $y_1$ is then given by

$$\sigma_1^2 = \sigma_0^2 + \frac{2N_k + 1}{k^4} \frac{H^2}{m_p^2} \tilde{\sigma}_1^2, \tag{45}$$

where we have defined

$$\tilde{\sigma}_1^2 := 2\sqrt{k} \text{Re} \left( y_0 \tilde{y}_1 \right). \tag{46}$$

This last $\sqrt{k}$ has been inserted to compensate the $k$-dependence of $y_0$ and make, in this way, $\tilde{\sigma}_1^2$ independent of $k$. The initial conditions for $\tilde{y}_1$ can be obtained by expanding in a power series of $(H/m_p)^2$ the above conditions for $y_1$, and they are given in a compact way in terms of this last object as follows,

$$\tilde{\sigma}_1^2 \rightarrow -\frac{\tilde{\omega}_1(\infty)^2}{2}, \quad (\tilde{\sigma}_1^2)' \rightarrow 0, \tag{47}$$

$\tilde{\omega}_1(\infty)$ being the value of the function $\tilde{\omega}_1$ at $\xi \rightarrow \infty$. Note that these conditions are also $k$-independent, since $\tilde{\omega}_1$ is $k$-independent, so all $k$-dependence is explicitly displayed. Finally, the normalization of the Wronskian \textsuperscript{11} implies that

$$\text{Im} (y_0 \tilde{y}_1 + y_0' \tilde{y}_1) = 0. \tag{48}$$

As a side remark, we note that these conditions fix completely $\tilde{\sigma}_1^2$, but not $\tilde{y}_1$, which still has some freedom. In particular, given a $\tilde{y}_1$ that obeys the conditions above, one could add to it a term of the form $i\gamma y_0$ with any real
constant \( \gamma \), and this new \( \tilde{y}_1 \) would still obey the above conditions. This does not affect the form of \( \hat{\sigma}_1^2 \), since its contribution to \( \sigma_1 \) is of order \((\hat{H}/m_p)^4\), which we are neglecting. Therefore, this freedom is just an artifact of the level of approximation we are using.

From these results, we can already display the functional form of the corrected power spectra for both scalar and tensor perturbations,

\[
P^{(1)}_{N_k}(k) = P^{(0)}_{N_k}(k) \left(1 + \frac{2N_k + 1}{k^3} \frac{H^2}{m_p^2} \beta_{N_k}\right),
\]

where \( \beta_{N_k} \) is a number given by

\[
\beta_{N_k} = \lim_{\xi \to \infty} \xi^2 \tilde{\sigma}_1^2.
\]

In fact, the only dependence of this number will be on \( N_k \), whether it is even or odd. The calculation to obtain its explicit form for both cases is presented in the Appendix, and the result reads

\[
\beta_{N_k} \approx \begin{cases} 
0.9876 & \text{for } N_k \text{ even}, \\
0.104 & \text{for } N_k \text{ odd}.
\end{cases}
\]

As can be seen from (11), we recover again the \( 1/k^3 \) dependence of the corrected power spectra as in \([2, 10, 11]\) and for even \( N_k \), the number \( \beta_{N_k} \) corresponds to the value computed in \([10]\). In addition, the relative correction term appears multiplied by a factor \( (2N_k + 1) \), in addition to another \( (2N_k + 1) \) global factor in front of the power spectrum that arises from the ansatz \([10]\) for the corrected wave function \( \Psi^{(1)} \).

Our final result for the quantum-gravitationally corrected scalar and tensor power spectra in de Sitter space using excited initial states thus reads:

\[
S^p_{N_k}(k) = \frac{G H^2}{\pi \epsilon} (2N_k + 1) \left(1 + \frac{2N_k + 1}{k^3} \frac{H^2}{m_p^2} \beta_{N_k}\right),
\]

\[
T^p_{N_k}(k) = \frac{16G H^2}{\pi} (2N_k + 1) \left(1 + \frac{2N_k + 1}{k^3} \frac{H^2}{m_p^2} \beta_{N_k}\right).
\]

Again, the results of \([10]\) for the case of an adiabatic ground state are recovered in the limit \( N_k \to 0 \).

We have not considered here the issue of the quantum-to-classical transition for the primordial modes. This is of importance, because these modes are of fundamental quantum nature, whereas the observed CMB anisotropies can be described by classical stochastic quantities. The mechanism for the quantum-to-classical transition is environmental decoherence. For the uncorrected wave functions, decoherence has been addressed for the case of an initial ground state as well as initial excited states \([27]\). Using the explicit expressions for the wave functions presented in \([10]\), it was found that decoherence for number eigenstates is stronger by a factor \( 2N_k + 1 \); see Eq. (84) in \([27]\). The quantum-to-classical transition thus proceeds faster than for the case of the ground state.

### IV. Power Spectra for a Superposition of States

In the previous section, we have computed the power spectra by considering an initial eigenstate for the uncorrected \( \psi^{(0)} \) and corrected wavefunctions \( \Psi^{(1)} \). In order to generalize this result, in this section we will instead assume a linear combination of those eigenstates, that is, \( \psi = \sum_{N_k} C_{N_k} \psi^{(0)}_{N_k} \). In particular, with this choice of wave function, the form of the power spectrum \([17]\) will change and take the following form:

\[
P_v = \frac{k^3}{2\pi^2} \sum_{N_k, M_k} C_{N_k} \overline{C_{M_k}} \langle \psi^{(0)}_{N_k} | v_k \bar{v}_k | \psi^{(0)}_{M_k} \rangle
\]

\[
= \frac{k^3 \sigma^2}{4\pi^2} \sum_{N_k} \left(2(N_k + 1) |C_{N_k}|^2 + 2 \text{Re}(C_{N_k} \overline{C_{N_k+2}} \sqrt{(N_k+2)(N_k+1)})\right).
\]

Note that the change only affects the global factor \((2N_k + 1) \) in \([17]\), which is now replaced by the summatory in the equation above, but the power spectrum is still proportional to \( \sigma^2 \). Concerning the power spectra for scalar and tensorial modes \([19-20]\), they will also change in the same way, by just replacing the factor \((2N_k + 1) \) by the sum above.

Finally, since in the uncorrected case the computation of \( \sigma_0 \) does not depend on the number state \( N_k \), one obtains the same result \([20]\). Nonetheless, regardless of the computation of the corrected \( \sigma_1 \), the corrected Schrödinger equation is nonlinear in \( \psi^{(0)} \). Therefore, if one assumes a linear combination of eigenstates for \( \psi^{(0)} = \sum_{N_k} C_{N_k} \psi^{(0)}_{N_k} \), different modes will couple and the corrected frequency \( \omega_1 \) will have an intrinsic dependency on the different numbers \( N_k \) and coefficients \( C_{N_k} \). Although lengthy, the computation is straightforward for a finite number of modes, but obtaining the result for the generic (in principle infinite) linear combination does not seem feasible.

As an example, we include here the form of the corrected frequency for the particular case \( \psi^{(0)} = C_1 \psi_1^{(0)} + C_3 \psi_3^{(0)} \) for the de Sitter spacetime:

\[
\omega_1^2 = k^2 - \frac{2k^2}{\xi^2} + \frac{H^2 \xi^4}{24k m_p^4} \left(\xi^2 + 1\right)^3 \left(2C_1^2 - 2\sqrt{6}C_1 C_3 + 3C_3^2\right)^2 \times
\]

\[
\left[12C_1^4 (7\xi^2 - 13) - 16\sqrt{6}C_1 C_3 (5\xi^6 + 14\xi^2 - 26) + 60C_2 C_3^2 (8\xi^6 + 21\xi^2 - 39) - 24\sqrt{6}C_1 C_3^2 (5\xi^6 + 21\xi^2 - 39) + 63C_3^4 (7\xi^2 - 13)\right].
\]

With this corrected frequency at hand, one can just apply the same method of resolution explained above to compute \( \sigma_1 \).
V. CONCLUSIONS

In this article, we have computed the power spectra of scalar and tensor perturbations in a de Sitter universe using excited initial states that arise from the invariants of a time-dependent harmonic oscillator in the context of a full quantization of a perturbed FLRW spacetime based on canonical quantum gravity with the Wheeler–DeWitt equation. The quantization not only of the perturbations, but also of the background spacetime, which leads to a Schrödinger equation with quantum-gravitational correction terms, modifies the power spectra with a specific $1/k^3$ correction. In the limit of the ground state, the former results of Ref. [10] for the corrected power spectra are recovered.

Given that such excited states also lead to higher-order correlations contributing to the bispectrum of the perturbations, a further project would be to study non-Gaussianities arising from quantum-gravity corrections in this context. Apart from that, the study of more general states, in particular, coherent states would also be of interest. By considering a wide class of initial states, one can obtain different results for the power spectra, which would lead to different observational features. Whether one can encounter a situation for which the tiny quantum-gravitational correction terms are really measurable, is a big and important open issue.

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Appendix A: Computation of $\beta_{N_k}$

1. $N_k$ even

For the case $N_k$ even, the equation for $\tilde{y}_1$ is,

$$\frac{d^2\tilde{y}_1}{d\xi^2} + \left(1 - \frac{2}{\xi^2}\right)\tilde{y}_1 = \frac{e^{-i\xi}}{8\xi(\xi + i)} \left[2i\xi^3 + \xi^2 - 14i\left(\xi^2 + 1\right)\arctan\xi + 8i\xi + 1\right] \left[3e^4\text{Ci}(2i - 2\xi) - 7\text{Ci}(2\xi + 2i) + 16\text{Ei}(2 - 2\xi) + 3ie^4\text{Si}(2i - 2\xi) + 7i\text{Si}(2\xi + 2i)\right],$$

and the initial conditions,

$$\tilde{\sigma}_1^2 \rightarrow -\frac{1}{4}, \quad (\tilde{\sigma}_1^2)' \rightarrow 0.$$  

By solving this system one gets the following form for the mode,

$$\tilde{y}_1 = \frac{25 - 3e^4}{8e^2\xi} \left(\xi \sin\xi + \cos\xi\right) - \frac{e^{-i\xi}}{8\xi(\xi + i)} \left[2i\xi^3 + \xi^2 - 14i\left(\xi^2 + 1\right)\arctan\xi + 8i\xi + 1\right] \left[3e^4\text{Ci}(2i - 2\xi) - 7\text{Ci}(2\xi + 2i) + 16\text{Ei}(2 - 2\xi) + 3ie^4\text{Si}(2i - 2\xi) + 7i\text{Si}(2\xi + 2i)\right].$$  

2. $N_k$ odd

For the case $N_k$ odd, the equation for $\tilde{y}_1$ is,

$$\frac{d^2\tilde{y}_1}{d\xi^2} + \left(1 - \frac{2}{\xi^2}\right)\tilde{y}_1 = \frac{e^{-i\xi}}{8\xi(\xi + i)} \left[2i\xi^3 + \xi^2 - 14i\left(\xi^2 + 1\right)\arctan\xi + 8i\xi + 1\right] \left[3e^4\text{Ci}(2i - 2\xi) - 7\text{Ci}(2\xi + 2i) + 16\text{Ei}(2 - 2\xi) + 3ie^4\text{Si}(2i - 2\xi) + 7i\text{Si}(2\xi + 2i)\right],$$

and the initial conditions,

$$\tilde{\sigma}_1^2 \rightarrow -\frac{7}{12}, \quad (\tilde{\sigma}_1^2)' \rightarrow 0.$$  

By solving this system one gets the following form for the mode,

$$\tilde{y}_1 = \frac{63 - 5e^4}{24e^2\xi} \left(\xi \sin\xi + \cos\xi\right) - \frac{e^{-i\xi}}{24\xi(\xi + i)} \left[14i\xi^3 + 7\xi^2 - 34i\left(\xi^2 + 1\right)\arctan\xi + 24i\xi + 7\right] \left[5e^4\text{Ci}(2i - 2\xi) - 33\text{Ci}(2\xi + 2i) + 48\text{Ei}(2 - 2i\xi) + 5ie^4\text{Si}(2i - 2\xi) + 33i\text{Si}(2\xi + 2i)\right].$$

Replacing this functional form in the definition of $\tilde{\sigma}_1$ and computing the limit, one gets the correction number:

$$\beta_{N_k} = \frac{1}{12e^2} \left[5e^4\text{Ei}(-2) + 15\text{Ei}(2) - 7e^2\right] \approx 0.104.$$

(A8)
[1] D. Brizuela and M. Krämer, in: *Experimental Search for Quantum Gravity*, edited by S. Hossenfelder (FIAS Interdisciplinary Science Series, Springer, 2018), pp. 9–14.

[2] C. Kiefer and M. Krämer, Phys. Rev. Lett. 108, 021301 (2012); D. Bini, G. Esposito, C. Kiefer, M. Krämer, and F. Pessina, Phys. Rev. D 87, 104008 (2013).

[3] A. Y. Kamenshchik, A. Tronconi, and G. Venturi, Phys. Lett. B 726, 518 (2013); Phys. Lett. B 734, 72 (2014); J. Cosmol. Astropart. Phys. 04 (2015) 046; Phys. Rev. D 94, 123524 (2016).

[4] A. Ashtekar and A. Barrau, Class. Quantum Grav. 32, 234001 (2015).

[5] L. Castelló Gomar, D. Martín de Blas, G. A. Mena Marugán and J. Olmedo, Phys. Rev. D 96, 103528 (2017).

[6] J. Morais, M. Bouhmadi-López, M. Krämer, and S. Robles-Pérez, Eur. Phys. J. C 78, 240 (2018); M. Bouhmadi-López, M. Krämer, J. Morais, and S. Robles-Pérez, J. Cosmol. Astropart. Phys. 02 (2019) 057.

[7] D. Brizuela and U. Muniain, arXiv:1901.08391 [gr-qc] (2019).

[8] C. Kiefer, *Quantum Gravity*, International Series of Monographs on Physics 155, third edition (Oxford University Press, Oxford, UK, 2012).

[9] C. Kiefer and T. P. Singh, Phys. Rev. D 44, 1067 (1991).

[10] D. Brizuela, C. Kiefer, and M. Krämer, Phys. Rev. D 93, 104035 (2016).

[11] D. Brizuela, C. Kiefer, and M. Krämer, Phys. Rev. D 94, 123527 (2016).

[12] C. Armendariz-Picon, J. Cosmol. Astropart. Phys. 02 (2007) 031.

[13] I. Aguilló and L. Parker, Phys. Rev. D 83, 063526 (2011).

[14] J. Ganc, Phys. Rev. D 84, 063514 (2011).

[15] A. Aravind, D. Lorshbough, S. Paban, J. High Energy Phys. 07 (2013) 076.

[16] J. Lesgourgues, D. Polarski, and A. A. Starobinsky, Nucl. Phys. B 497, 479 (1997).

[17] S. Robles-Pérez, Phys. Lett. B 774, 608 (2017).

[18] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).

[19] C. M. A. Dantas, I. A. Pedrosa and B. Baseia, Phys. Rev. A 45, 1320 (1992).

[20] W. Dittrich and M. Reuter, *Classical and Quantum Dynamics*, third edition (Springer, Berlin, 2001).

[21] P. G. L. Leach, J. Phys. A 16, 3261 (1983).

[22] H. Kanasugui and H. Okada, Prog. Theor. Phys. 93, 949 (1995).

[23] V. F. Mukhanov, Sov. Phys. JETP 68, 1297 (1988); M. Sasaki, Prog. Theor. Phys. 76, 1036 (1986).

[24] J. Martin, V. Vennin, and P. Peter, Phys. Rev. D 86, 103524 (2012).

[25] C. Kiefer and D. Wichmann, Gen. Relativ. Gravit. 50, 66 (2018).

[26] H. Kanasugui, arXiv:1901.07104 [gr-qc] (2019).

[27] C. Kiefer and D. Polarski, Ann. Phys. (Berlin) 7, 137 (1998).