Geometric actions for three-dimensional gravity

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Abstract
The solution space of three-dimensional asymptotically anti-de Sitter or flat Einstein gravity is given by the coadjoint representation of two copies of the Virasoro group in the former and the centrally extended BMS\(_3\) group in the latter case. Dynamical actions that control these solution spaces are usually constructed by starting from the Chern–Simons formulation and imposing all boundary conditions. In this note, an alternative route is followed. We study in detail how to derive these actions from a group-theoretical viewpoint by constructing geometric actions for each of the coadjoint orbits, including the appropriate Hamiltonians. We briefly sketch relevant generalizations and potential applications beyond three-dimensional gravity.

Keywords: three-dimensional gravity, AdS/CFT correspondence, Chern–Simons theories

1. Introduction

Even though three-dimensional general relativity does not admit local degrees of freedom, there is both an infinite dimensional symmetry structure \cite{1} and rich dynamics in two dimensions \cite{2} when allowing for non-trivial boundary conditions. The standard ‘bulk’ approach to derive this dynamics starts from the Chern–Simons formulation \cite{3, 4}: when taking into account the boundary conditions and the associated surface terms \cite{5}, this leads one in a first step to a Wess–Zumino–Witten theory along the lines of \cite{6–8}, and in a second step to Liouville...
theory through a Hamiltonian reduction [9–11]. In this approach, the Hamiltonian of the dual theory is inherited from the surface term associated with time translations. Alternatively, the dual theory may be constructed through holographic renormalization in the context of the AdS/CFT correspondence [12].

In this paper, we follow a different method to construct dual two-dimensional action principles for the gauge fixed solution spaces of three-dimensional gravity. Indeed, both in the asymptotically AdS and flat cases, the solution space coincides with the centrally extended coadjoint representation, at fixed values of the central charges, of the asymptotic symmetry groups, viz. two copies of the Virasoro group [1, 12–17] and the centrally extended BMS3 group [18–26], respectively (see also [27–31] for recent related considerations). More precisely, the general solution to the Einstein equations with AdS asymptotics is given by

\[
d s^2 = \frac{l^2}{r^2} d\sigma^2 - (r dx^+ - \frac{8\pi G}{r} b^- d\sigma)(r dx^- - \frac{8\pi G}{r} b^+ d\sigma),
\]  

(1.1)

with \( x^\pm = \frac{1}{l} \pm \varphi \) and the arbitrary \( 2\pi \)-periodic functions \( b^\pm(x^\pm) \) transforming as

\[
\tilde{b}^\pm = (\partial_x f^\pm)^2 b^\pm \circ f^\pm - c^\pm S_x[f^\pm], \quad c^\pm = \frac{3l}{2G},
\]  

(1.2)

under the conformal transformations \( x^\pm \to f^\pm(x^\pm) \), \( f^\pm(x^\pm + 2\pi) = f^\pm(x^\pm) \pm 2\pi \) of the cylinder at infinity, with the Schwarzian derivative given by

\[
S_x[f] = \frac{1}{24\pi} \left[ \partial_x^2 (\log \partial_x f) - \frac{1}{2} (\partial_x (\log \partial_x f))^2 \right].
\]  

(1.3)

This coincides with the coadjoint representation of the two copies of the Virasoro group for fixed values of the central charges. For asymptotically flat spacetimes, one finds instead

\[
d s^2 = 2[8\pi G p d\sigma - dr + 8\pi G (j + u p') d\varphi] d\sigma + r^2 d\varphi^2,
\]  

(1.4)

where the arbitrary \( 2\pi \)-periodic functions \( p = p(\varphi), j = j(\varphi) \) transform as

\[
\tilde{p} = (f')^2 p \circ f - c_2 S_x[f], \quad c_2 = \frac{3}{G},
\]  

\[
\tilde{j} = (f')^2 [j + u p' + 2\alpha p - \frac{c_2}{24\pi} \alpha''] \circ f - c_1 S_x[f], \quad c_1 = 0,
\]  

(1.5)

under the BMS3 transformations \( (u, \varphi) \to (f' u + \alpha(f(\varphi)), f(\varphi)) \), with \( f(\varphi + 2\pi) = f(\varphi) + 2\pi \) and \( \alpha(\varphi + 2\pi) = \alpha(\varphi) \). In turn, this coincides with the coadjoint representation of the centrally extended BMS3 group for fixed values of the central charges.

As a consequence, the gravitational solution space admits a partition into coadjoint orbits. For any group \( G \), the individual orbits are homogeneous symplectic spaces \( G/\mathcal{H} \) (see e.g. [32–35] and original references therein). The aspect we will exploit here is that one can associate to each of these orbits, in a canonical way, geometric actions which admit \( G \) as a global and \( \mathcal{H} \) as a gauge symmetry group [36]. What is fixed in these actions is the kinetic term determined by the symplectic structure on the coadjoint orbit. If the aim is to construct \( G \)-invariant dynamical systems, one may choose a suitable Hamiltonian defined on the coadjoint orbit that respects \( G \)-invariance (see also e.g. [37, 38] for reviews).

When this method is applied to three-dimensional gravity, we will get finer actions than those of [2, 22, 39, 40], precisely adapted to the individual orbits. From the bulk point of view, they take additional information on non-trivial holonomies into account. In the anti-de Sitter case for example, one finds an intriguing connection between 3d and 2d gravity in the sense that the geometric actions for each copy of the Virasoro group differ from the action for
two-dimensional gravity by taking into account a more general covector, for which not only the central charge differs from zero \([11, 41–44]\). In the flat case, this approach allows us to construct novel BMS\(_3\) invariant actions (see also \([45]\)). Note that the Hamiltonian can be fixed by reverting to the bulk approach described above in order to identify a suitable generator.

Another interesting aspect of this approach to two-dimensional conformal or BMS\(_3\) invariant actions is that, exactly like in the case of loop groups and the associated Wess–Zumino–Witten theories, they can also be interpreted as one-dimensional particle/world-line actions associated to infinite-dimensional groups. The spatial dimension is hidden or emergent, depending on whether one uses a Fourier expansion for the Lie algebra generators and their duals with associated infinite mode sums or an inner product with an explicit integration over the circle. For instance, from the two-dimensional point of view, the deformation by a Wess–Zumino–Witten term involves a 3-cocycle of the Lie algebra of the group \(G\), whereas in the worldline approach, this deformation comes from a related 2-cocycle on the Lie algebra of the loop group of \(G\).

The structure of the paper is as follows: in the next section, we review the construction of geometric actions, with a special emphasis on how to include Hamiltonians that preserve \(G\)-invariance. We continue with a discussion of central extensions and the well-known examples of the Kac–Moody and Virasoro group relevant for three-dimensional asymptotically anti de-Sitter gravity. We then move to semi-direct product groups in order to be able to treat three-dimensional flat gravity. In this section, we provide novel iso\((2, 1)\) WZW models and BMS\(_3\) invariant field theories in two dimensions. In the last section, we relate our considerations to recent developments in the field and discuss future prospects, both for three and four dimensional gravity.

### 2. Review of geometric actions

#### 2.1. Kinetic term

The adjoint action of a Lie group \(G\) on its Lie algebra \(g\) is defined as the differential of the automorphism \(h \mapsto ghg^{-1}\) at the identity

\[
\text{Ad}_g X = \frac{d}{ds} \left( gh(s)g^{-1} \right) \bigg|_{s=0},
\]

where \(X = \frac{d h(s)}{ds} \bigg|_{s=0} \in g\). The coadjoint action of \(G\) on the dual space \(g^*\) of \(g\) is defined as

\[
\langle \text{Ad}^*_g b, X \rangle = \langle b, \text{Ad}_g X \rangle,
\]

where \(b \in g^*\) and \(\langle \cdot, \cdot \rangle\) is the pairing between \(g\) and \(g^*\). For a fixed element \(b_0\) of \(g^*\), this action generates a coadjoint orbit \(O_{b_0}\), the set of elements \(b \in g^*\) such that

\[
b = \text{Ad}^*_g b_0,
\]

which is a manifold isomorphic to \(G/\mathcal{H}_{b_0}\), with \(\mathcal{H}_{b_0}\) the isotropy group of \(b_0\) under the coadjoint action, i.e. the subgroup of elements \(h \in G\) satisfying \(\text{Ad}_h b_0 = b_0\).

Coadjoint orbits are particularly interesting as they are symplectic manifolds. The Kirillov-Kostant symplectic form is the pull-back to a coadjoint orbit \(O_{b_0}\) of the pre-symplectic form on \(G\) given by

\[
\Omega = \frac{1}{2} \langle b, \text{ad}_\theta b \rangle,
\]

where \(b = \text{Ad}^*_g b_0\) is a point on the orbit, \(\theta\) is the left invariant Maurer–Cartan form satisfying
\[ d\theta = -\frac{1}{2} \text{ad}_g \theta, \]  
\text{(2.5)}

and \( \text{ad} \) denotes the adjoint action of \( g \) on itself. As \( \Omega \) is closed, it is locally exact. In fact, 
\[ \Omega = da, \quad a = \langle b, \theta \rangle, \]  
\text{(2.6)}

and therefore, a geometric action \( I_G[g; b_0] \) can be defined on the phase space through 
\[ I_G[g; b_0] = \int_\gamma a, \]  
\text{(2.7)}

where \( \gamma \) is a path on the coadjoint orbit \( O_{b_0} \).

In particular, for finite dimensional matrix groups, a local solution to (2.5) is simply 
\[ \theta = g^{-1}dg \] and the pre-symplectic potential becomes 
\[ a = \langle b_0, dgg^{-1} \rangle. \] The first order Euler–Lagrange equations of motion are equivalent to setting to zero the one-forms 
\[ i_{V} \Omega = 0, \] 
\text{(2.9)}

where \( V \) is the vector field associated to \( \frac{dx}{dt} \) and \( t \) parametrizes the path \( \gamma \). Note also that one may (trivially) write this action in Wess–Zumino–Witten form, 
\[ I_G[g; b_0] = \int_{\Sigma} \Omega. \]  
\text{(2.8)}

The assumptions here are that fields are extended to \( \Sigma \) and that \( \gamma \) is part of the boundary of \( \Sigma \), with suitable conditions on fields and their derivatives such that all other boundary terms vanish.

Consider the vector fields associated to one parameter families of right translations generated by \( X \), \[ \frac{dx}{dt} = \frac{d}{dt}(gh_{R}(s)) \big|_{s=0} \] with \( \frac{d}{dt}h_{R}(s) \big|_{s=0} = X \). These vector fields are the left invariant vector fields that reduce to \( X \) at the identity. For all \( X \in \mathfrak{g} \) that are constant along the path, they define global symmetries of \( I_G[g; b_0] \). This follows from 
\[ i_{\xi} \Omega = dQ_X, \quad Q_X = -\langle b, X \rangle, \]  
\text{(2.9)}

and \( \mathcal{L}_{\xi}a = i_{\xi} \Omega + d\langle b_0, \xi \rangle \). The associated Noether charge is \( Q_X \).

Elements \( \epsilon \) of \( h_{b_0} \), the ‘little’ algebra associated to \( H_{b_0} \), are defined by \( \text{ad}_g \epsilon = 0 \). Let \( \nu_{\epsilon} \) be the vector fields associated to one parameter families of left translations by elements \( h_{L}(s) \) of \( H_{b_0} \) and generated by \( \epsilon \), \( \nu_{\epsilon} = \frac{d}{dt}(h_{L}(s)g) \big|_{s=0} \) with \( \frac{d}{dt}h_{L}(s) \big|_{s=0} = \epsilon \). These vector fields are the right invariant vector fields that reduce to \( \epsilon \) at the identity. Let us now assume furthermore that \( \epsilon = \epsilon(t) \) depends \emph{a priori} arbitrarily on \( t \). It then follows from 
\[ \mathcal{L}_{\nu_{\epsilon}}a = i_{\nu_{\epsilon}} \Omega + d\langle b_0, \epsilon \rangle \] and 
\[ i_{\nu_{\epsilon}} \Omega = 0, \]  
\text{(2.10)}

that these transformations define gauge symmetries. More precisely, the action \( I_G[g; b_0] \) is gauge invariant provided that \( \epsilon(t) \) vanishes at the end points of \( \gamma \). Note that \( Q_X \) is gauge invariant since 
\[ \mathcal{L}_{\nu_{\epsilon}}Q_X = 0, \]  
\text{(2.11)}

by using (2.9) and (2.10). Note also that the Noether charges form a representation under the action of the global symmetries, 
\[ \mathcal{L}_{\nu_{X}}Q_{X} = Q_{[X,X_{\gamma}]} \]  
\text{(2.12)}
2.2. Symmetric Hamiltonians and deformations

In the spirit of effective field theories, once one has constructed a kinetic term that admits $G$ as global and $H$ as local symmetry groups, one now should add the most general Hamiltonian that is compatible with both of these groups. For instance, the local group is preserved by the addition of a Hamiltonian if its Poisson bracket weakly commutes with the first class constraints encoding the gauge symmetries. We will not do a systematic analysis of this question in the current paper but restrict ourselves to simple constructions that naturally occur in the applications below, and can be traced back for instance to the choice of suitable boundary conditions and boundary terms in the gravitational context. Deformations of the kinetic term are also allowed as long as they are compatible with all symmetries. Again, we are not performing a complete analysis.

(i) One possibility that preserves all global symmetries is to consider an extended action, where one of the Noether charges plays the role of the Hamiltonian, $H_X = Q_X$, so that the first order action becomes

$$I_G[g; b_0, H_X] = \int_\gamma (a - H_X dt),$$

for some constant $X \in g$. The equations of motion are then equivalent to the vanishing of

$$-iv \Omega - dH_X = -(iv + i_\xi) \Omega.$$ 

By construction, this action is gauge invariant under the same assumptions as before. It is also invariant under the global transformations associated to $v^\xi_X$. Indeed, by using (2.12), one finds that this is the case for instance when $X'(t)$ depends explicitly on time with an evolution determined by

$$\frac{dX'}{dt} = \text{ad}_X X'.$$ 

(ii) Another possibility uses an invariant symmetric tensor on $g^\ast$. Let $e_a$ denote the elements of a basis of $g$ and $e^a$ the elements of the dual basis. The Noether charges associated to $e_a$ are $Q_a = -\langle b, e_a \rangle$. If $k^{a_1 \cdots a_m}$ denote the components of the tensor, the Hamiltonian can be chosen to be,

$$H_k = \frac{1}{m!} k^{a_1 \cdots a_m} Q_{a_1} \cdots Q_{a_m}.$$ 

From (2.10) and (2.12), it follows that Lie derivatives with respect to $v^\xi$ and $v^\xi_X$ annihilate $H_k$. Thus, geometric actions supplemented by $H_k$ also preserve gauge and global symmetries, in this case with time independent $X'$. A Hamiltonian quadratic in the Noether charges may be constructed for instance for semi-simple Lie algebras by using the inverse of the Killing form.

(iii) Another deformation of $I_G[g, b_0]$ in (2.7) that changes the kinetic term is given by

$$I_G[g; b_0, c] = I_G[g, b_0] + \int_\Sigma \Omega_\omega, \quad \Omega_\omega = -\frac{1}{2} \epsilon^\omega (\text{Ad}_\theta, \text{Ad}_\theta)$$

with $\omega$ a Lie algebra 2-cocyle. Such a deformation is trivial in the sense that it can be absorbed by a redefinition of $b_0$ if $\omega$ is a coboundary. Non-trivial deformations are thus characterized by $[\omega] \in H^2(g, \mathbb{R})$. 


These deformations modify the gauge symmetries: the requirement \( i_{\hat{\epsilon}} \hat{\Omega} = 0 \) where \( \hat{\Omega} = \Omega + \Omega_\omega \) now restricts \( \epsilon \) to solve
\[
\text{ad}^* \epsilon b_0 - cs(\epsilon) = 0,
\]
with
\[
\langle s(X), Y \rangle = -\omega(X, Y).
\]
(2.18)

Concerning global symmetries, the cocycle condition for \( \omega \) implies that \( d i^*_R X \Omega_\omega = 0 \). When taking into account that \( i^*_R X \Omega_\omega = -c \omega(\text{Ad}_g X, \text{Ad}_g X) = c \langle s(X), X \rangle \), it follows that, locally, there exists \( S(g) \) such that \( i^*_R X \Omega_\omega = cd(\langle S(g), X \rangle) \). Hence, global symmetries are preserved by this deformation provided \( S(g) \) exists globally. This is the case for instance when \( H^1(G) = 0 \) or, as we will see in the next section, when \( \omega \) originates from a group 2-cocycle in \( G \). The associated Noether charges are
\[
\hat{Q}_X = -(b - cs(g), X).\]
(2.20)

They form a centrally extended representation of the symmetry algebra,
\[
L_{\epsilon^*_R} \hat{Q}_X = \hat{Q}_{[X_1, X_2]} + c \omega(X_1, X_2),
\]
and are gauge invariant under the modified gauge transformations, \( L_{\epsilon^*_L} \hat{Q}_X = 0 \).

When including a Hamiltonian \( \hat{H}_X = \hat{Q}_X \), one now finds that all global symmetries generated by \( \hat{\epsilon} \), with time evolution determined by (2.15) survive if in addition \( c \omega(X, X') = 0 \). When \( c \neq 0 \), this is a strong condition on allowed Hamiltonians respecting \( G \)-invariance.

For a Hamiltonian of the form \( \hat{H}_k = \frac{1}{m!} k^{a_1 \cdots a_m} \hat{Q}_{a_1} \cdots \hat{Q}_{a_m} \), one has
\[
L_{\epsilon^*_R} \hat{H}_k = \frac{1}{(m - 1)!} \epsilon_{a_1 \cdots a_m} X^{b_1} \hat{Q}_{a_1} \cdots \hat{Q}_{a_m} X^{b_2} \cdots X^{b_m} \omega_b,
\]
(2.22)

where \( \omega(X, Y) = \omega_{ab} X^a Y^b \). If we restrict ourselves to field independent Lie algebra elements \( X' \), invariance will hold in the quadratic case, \( m = 2 \), for instance when
\[
\frac{dX}{dt} = c X^b \omega_{b} X^a.
\]
(2.23)

These deformations will be systematically discussed in the next section from the viewpoint of centrally extended groups.

### 3. Geometric actions for centrally extended groups

The procedure outlined at the beginning of section 2 can be straightforwardly generalized for infinite dimensional groups and central extensions thereof. In applications to three-dimensional gravity, the asymptotic symmetry algebras of the theory is usually infinite dimensional, with central extensions in the representation through surface charges [1, 19, 46, 47]. From the boundary point of view, one should thus study geometric actions associated to centrally extended groups.
### 3.1. Central extensions

A central extension of a group $G$ is a direct product $\hat{G} = G \times \mathbb{R}$, whose elements are pairs $(g, m)$ with group operation $(g_1, m_1)(g_2, m_2) = (g_1g_2, m_1 + m_2 + \Xi(g_1, g_2))$, where $\Xi : G \times G \to \mathbb{R}$ is a 2-cocycle on $G$ that satisfies

\[
\Xi(g_1g_2, g_3) + \Xi(g_1, g_2) = \Xi(g_1, g_2g_3) + \Xi(g_2, g_3),
\]

which we assume for simplicity to satisfy $\Xi(e, g) = 0 = \Xi(g, e)$. Two such central extensions denoted by $\Xi$ and $\Xi'$ are isomorphic iff

\[
\Xi'(g_1, g_2) = \Xi(g_1, g_2) + \mu(g_1) + \mu(g_2) - \mu(g_1g_2),
\]

where $\mu : G \to \mathbb{R}$. Denoting the elements of the corresponding centrally extended Lie algebra $\hat{g} = g \oplus \mathbb{R}$ by $(X, n)$, the adjoint representation of $G$ can be written as

\[
\text{Ad}_{(g, n)}(X, n) = (\text{Ad}_g X, n - \langle S(g), X \rangle),
\]

where $S : G \to \text{g}^*$ is the Souriau cocycle on $G$ defined by

\[
\langle S(g), X \rangle = - \frac{d}{ds} \left[ \Xi(g, h(s)g^{-1}) + \Xi(h(s), g^{-1}) \right] \bigg|_{s=0},
\]

with differential at the identity given by

\[
\frac{d}{ds} S(h(s))|_{s=0} = s(X).
\]

Due to (3.1), it satisfies the 1-cocycle condition

\[
S(g_1g_2) = \text{Ad}_{g_1}^* S(g_1) + S(g_2).
\]

The adjoint action in $\hat{g}$ is given by

\[
\text{ad}^*_{(X, n)}(Y, k) = (\text{ad}_X Y, \omega(X, Y)),
\]

where (2.19) has been taken into account and where $s$ and $\text{ad}_X$ are the differentials of $S$ and $\text{Ad}_X$ at the identity respectively. Note that $s$ is entirely determined by the Lie algebra cocycle $[\omega] \in H^2(\text{g}, \mathbb{R})$ associated to $\Xi$ according to equation (2.19).

Elements in $\text{g}^*$ are denoted by pairs $(b, c)$ where the dual element $c$ to the central extension of $\text{g}$ is the central charge. The pairing between $\text{g}$ and its dual space $\text{g}^*$ is defined by

\[
\langle (b, c), (X, n) \rangle = \langle b, X \rangle + cn,
\]

the coadjoint action is given by

\[
\text{Ad}^*_{(b, c)}(X, n) = (\text{Ad}_b^* X - cS(g)^{-1}b, c),
\]

while its associated action in $\text{g}^*$ reads

\[
\text{ad}^*_{(b, c)}(X, n) = (\text{ad}_b^* X + cs(X), 0).
\]

The extended Maurer–Cartan one-form is denoted by $(\theta, \theta_\Xi)$. The additive piece is

\[
\theta_\Xi = dm + [\delta_2 \Xi(g_1, g_2)] \big|_{g_1 = g^{-1}, g_2 = e, \delta g_1 = \delta g_2 = 0},
\]

where $\delta_2$ denotes an infinitesimal variation of $g_2$. Equation (2.5) is supplemented by

\[
d\theta_\Xi = \frac{1}{2} \langle s(\theta), \theta \rangle.
\]
Differentiating \((3.6)\) with \(g_1 = g\) and \(g_2 = h(s)\) gives at \(s = 0\),
\[
dS(g) = -\text{ad}^*_Y S(g) + s(\theta).
\] (3.13)
By using \((3.6)\) applied to \(gh(s)g^{-1}\) and differentiating at \(s = 0\), with \(X\) as in \((2.1)\), one also gets
\[
\text{ad}^*_Y S(g) = -\text{Ad}^*_{Y^{-1}} S(\text{Ad}_Y X) + s(Y),
\] (3.14)
where \(Y = \text{Ad}_{Y^{-1}} X\). Combining \((3.13)\) with \((3.14)\) yields
\[
dS(g) = \text{Ad}^*_{Y^{-1}} S(\text{Ad}_Y \theta).
\] (3.15)

**Remark.**

(i) Suppose in particular that \(H^1(g, \mathbb{R}) = 0\). It can then be shown that the Souriau map \((2.19)\) on the level of the Lie algebra, \(s : H^2(g, \mathbb{R}) \to H^1(g, \mathfrak{g}^*)\), \([\omega] \mapsto [s]\) is an isomorphism. If furthermore \(H^2(g, \mathbb{R})\) is of dimension 1 and the Lie group \(G\) is connected, \((3.14)\) determines \(S(g)\) uniquely from \(\omega\) (see e.g. [37] and original references therein), without the need for an explicit expression for \(\Xi\).

(ii) In the case of a centrally extended group, one can parametrize the elements of a coadjoint orbit by \((b, c) = \text{Ad}^*_{(g;m)}^{-1}(b_0, c)\). For later use, note that, if \(c \neq 0\) and
\[
b_0/c = -S(\Upsilon),
\] (3.16)
for some group element \(\Upsilon\), it follows from \((3.6)\) that the coadjoint orbit generated from \((b_0, c)\) can also be generated from \((0, c)\) provided one changes \(S(g)\) to \(S(\Upsilon g)\) in the coadjoint action,
\[
\text{Ad}^*_{(g;m)}^{-1}(b_0, c) = (-c S(\Upsilon g), c).
\] (3.17)

### 3.2. Geometric actions for central extensions

The pre-symplectic potential for centrally extended groups is \(a = \langle (b, c), (\theta, \theta \Xi) \rangle\) and the kinetic term of the geometric action associated to a coadjoint orbit \(O_{(b_0, c)}\) can be written as
\[
I_G[g; m; b_0, c] = I_G[g; b_0, c] + c \int (-\langle S(g), \theta \rangle + \theta \Xi).
\] (3.18)
Using the relations of the previous section, the pre-symplectic 2-form \(\tilde{\Omega}\) for centrally extended groups can then be worked out to be \(\tilde{\Omega} = \Omega + \Omega_\Xi\).

As compared to the analysis at the end of the previous section, the Lie algebra associated to the centrally extended group has an additional dimension consisting of vectors of the form \((0, n)\). The associated left invariant vector fields \(\nu^B\) are global symmetries that generate constant shifts of \(m\). They are all trivial however since these vectors belong to the extended little algebra. This can also be seen from the fact that
\[
I_G[g; m; b_0, c] = I_G[g; 0; b_0, c],
\] (3.19)
since the dependence on \(m\) is only through a total time derivative that can be omitted. In the following we will simplify the notation and use \(I_G[g; b_0, c]\). The additional Noether charges are trivial constants. More generally, the Noether charges can be choosen as
\[
Q_{X,n} = -\iota_X (\text{Ad}^*_{(g;m)}^{-1}(b_0, c), (\theta, \theta \Xi)) = \hat{Q}_X - cn,
\] (3.20)
and now form an ordinary representation of the centrally extended symmetry algebra,
\[ L_{\psi(x_1, \epsilon_1)}^x Q(x_1, \epsilon_1) = Q[(x_1, \epsilon_1), (x_2, \epsilon_2)]. \]  
(3.21)

For orbits generated by \((b_0, c)\) with \(c \neq 0\) and where (3.16) holds, it follows from (3.17) and the left invariance of the Maurer–Cartan form that \(I_G[g; b_0]\) can be absorbed into the term proportional to the central charge \(c\), using a new a group element \(u = \Upsilon g\),
\[ I_G^g [g; b_0, c] = c \int (- \langle S(u), \theta \rangle + \theta_\Xi) = I_G^g [u; 0, c] \]  
(3.22)

and analogously, the charges (2.20) can be written as
\[ \hat{Q}_X = \langle cS(u), X \rangle. \]  
(3.23)

This allows one to absorb the term proportional to the orbit representative \(b_0\) also in geometric actions deformed by a Hamiltonian and to study the geometric actions corresponding to various coadjoint orbits in a unified fashion.

### 3.3. Examples

As a preparation for the cases of direct interest below, we briefly revisit in this subsection the well-known geometric actions for semi-simple loop groups \(G\) and for the Virasoro group, first derived in [11, 41]. More details can be found for instance in [38, 48].

#### 3.3.1. Kac–Moody groups.

#### 3.3.2. Loop groups and their extension.

Consider a finite dimensional simple and simply connected group \(G\). The Kac–Moody group \(\hat{L}G\) is given by the central extension of the loop group \(L\) of \(G\), whose elements are given by the continuous maps from the unit circle to \(G\)
\[ g : S^1 \rightarrow G, \quad \varphi \mapsto g(\varphi), \]  
(3.24)

with \(g(\varphi + 2\pi) = g(\varphi)\). In the same way, the loop algebra \(\hat{L}g\) corresponds to the algebra of continuous maps from \(S^1\) to \(\hat{g}\). The pairing between \(\hat{L}g\) and its dual \(\hat{L}g^*\) reads
\[ \langle b(\varphi), X(\varphi) \rangle = \int_0^{2\pi} d\varphi \, Tr [b(\varphi)X(\varphi)]. \]  
(3.25)

where \(Tr\) denotes the normalized Killing form. The central extension is determined by the 2-cocycle on the loop group
\[ \Xi (g_1, g_2) = \frac{1}{4\pi} \int_D Tr [g_1^{-1}dg_1dg_2g_2^{-1}], \]  
(3.26)

where \(d\) denotes the exterior derivative on the disk \(D\) whose boundary is \(S^1\). The 1-cocycle defining the adjoint action can then be obtained from (3.4),
\[ S(g) = \frac{1}{2\pi} g^{-1}\partial_\varphi g, \]  
(3.27)

with \(s(X) = \frac{1}{2\pi} \partial_\varphi X\). Equation (3.11) gives
\[ \theta_\Xi = dm(\varphi) + \frac{1}{4\pi} \left( \int_0^{2\pi} d\varphi Tr [g^{-1}\partial_\varphi gg^{-1}dg] + \int_D Tr [g^{-1}dgg^{-1}dg^{-1}dg] \right). \]  
(3.28)
3.3.3. Geometric actions. The geometric action (3.18) therefore turns out to be
\[ I_{\Sigma_G}[g; b_0, c] = \int_0^{\pi/2} d\varphi \text{Tr} \left[ b_0 dg g^{-1} - \frac{c}{4\pi} g^{-1} \partial \varphi g^{-1} \right] + c \Gamma, \]  
(3.29)
where
\[ \Gamma = \frac{1}{4\pi} \int_D \text{Tr} \left[ g^{-1} dgg^{-1} \right]. \]  
(3.30)
Using the notation \( d = dt \partial_t \), and defining a manifold \( M = \gamma \times D \) where \( t \) is the coordinate along \( \gamma \), the Wess–Zumino term \( \Gamma \) can be put into the standard form
\[ \Gamma = \frac{1}{12\pi} \int_M \text{Tr} \left[ \left( d^f g^{-1} \right)^3 \right], \]  
(3.31)
where \( d^f \) denotes the exterior derivative on the whole of \( M \), \( dt \wedge d\varphi \wedge dr \) is considered as orientation for the integration on \( M \), with boundary conditions such that the only contribution from \( \partial M \) arises from \( \gamma \times S^1 \).

According to (3.20), the Noether charges associated to the symmetries corresponding to right multiplication by group elements \( (g(\varphi), m(\varphi)) \) and generated by \( (X(\varphi), n(\varphi)) \) are
\[ Q_{\{X, n\}} = \int_0^{\pi/2} d\varphi \right( \text{Tr} [Q(\varphi)X(\varphi)] - cn(\varphi) \right), \quad Q(\varphi) = \frac{c}{2\pi} g^{-1} \partial \varphi g - g^{-1} b_0 g. \]  
(3.32)
In order to make contact with 3d gravity, we will choose the following bilinear combination of \( Q \) as a Hamiltonian,
\[ H_2 = \frac{\pi}{c} \int d\varphi \text{Tr} [Q^2]. \]  
(3.33)
Indeed, this Hamiltonian arises from the Chern–Simons formulation of AdS\(_3\) gravity when imposing Brown–Henneaux boundary conditions [2]. Under global symmetries generated by \( \psi^R \), it transforms as
\[ L^R_{\psi^R} H_2 = \int d\varphi \text{Tr} [Q\partial \varphi X^I]. \]  
(3.34)
Since under the same transformation \( \delta_{\psi^R} \Sigma_G = \int d\varphi \text{Tr} [Q\partial \varphi X^I] \), it follows that
\[ I_{\Sigma_G}[g; b_0, c, H_2] = I_{\Sigma_G}[g; b_0, c] - \int d\varphi H_2 \]  
(3.35)
is invariant under the global symmetries generated by \( X^I(t, \varphi) = X^I(t + \varphi) \).

3.3.4. Relation to chiral WZW theories. Defining now \( 2\partial_\varphi = \partial_t - \partial_\varphi \), we can write
\[ I_{\Sigma_G}[g; b_0, c, H_2] = -\int d\varphi \text{Tr} \left[ 2b_0 g^{-1} \partial \varphi g^{-1} \right] + I_{\text{WZW}}[g; c], \]  
(3.36)
where \( I_{\text{WZW}}[g; c] \) corresponds to the chiral WZW model
\[ I_{\text{WZW}}[g; c] = \frac{c}{2\pi} \int d\varphi \text{Tr} \left[ g^{-1} \partial_\varphi g^{-1} \partial_\varphi g^{-1} \right] + c \Gamma, \]  
(3.37)
after neglecting a time independent \( \text{Tr} b_0^2 \) in the integrand. In the particular case of a \( \varphi \) independent \( b_0 \) action (3.36) has been obtained in [8] after solving the constraints of Chern–Simons
theory based on a semisimple group $G$ on a spatial disk with a source. In the context of \textit{AdS$_3$ gravity} it has been used in [49] and more recently in [50].

The term proportional to $b_0$ can be absorbed into the chiral WZW model by considering a group element $\Upsilon = \Upsilon(\varphi)$ that solves equation (3.16), which in this case takes the form

$$\Upsilon^{-1} \partial_{\varphi} \Upsilon = -\frac{2\pi}{c} b_0.$$  (3.38)

The action (3.29) can then be written as a chiral WZW action for a non periodic field $u = \Upsilon g$, i.e.

$$I_{\text{WZW}}[u,c] = I_{\text{LG}}[g; b_0, c, H_2].$$  (3.39)

In this formulation, the dependence on the orbit representative $b_0$ is translated into a nontrivial periodicity of the field $u$,

$$u(\varphi + 2\pi) = M(b_0) u(\varphi), \quad M(b_0) = \exp \left[ -\frac{2\pi}{c} \oint d\varphi b_0 \right].$$  (3.40)

### 3.3.5. Virasoro group.

#### 3.3.6. Diffeomorphism group and its extension.

The Virasoro group is the central extension of $\text{Diff}(S^1)$, which in turn corresponds to the orientation-preserving diffeomorphism group of the circle with elements $f$ satisfying

$$f(\varphi + 2\pi) = f(\varphi) + 2\pi, \quad f' > 0.$$  (3.41)

The associated Lie algebra will be denoted by $\text{Vec}(S^1)$. Its elements are vector fields on the circle $X = \partial_{\varphi}$, while elements of the dual $\text{Vec}(S^1)^*$ are taken as quadratic differentials on $S^1$, $b = b(\varphi) (d\varphi)^2$. The natural pairing between $\text{Vec}(S^1)$ and its dual is

$$\langle b, X \rangle = \int_0^{2\pi} d\varphi \ b(\varphi) X(\varphi).$$  (3.42)

The adjoint and coadjoint actions of $\text{Diff}(S^1)$ are

$$\text{Ad}_{f^{-1}} X = \frac{1}{f'(\varphi)} X(f(\varphi)) \partial_{\varphi}, \quad \text{Ad}^*_{f^{-1}} b = f'(\varphi)^2 b(f(\varphi)) (d\varphi)^2.$$  (3.43)

The associated infinitesimal adjoint action is minus the Lie bracket for vector fields on $S^1$.

The 2-cocycle determining the Virasoro group is the Thurston–Bott cocycle

$$\Xi(f_1, f_2) = -\frac{1}{48\pi} \int_0^{2\pi} d\varphi \log(\partial_{\varphi} f_1 \circ f_2) \partial_{\varphi} (\log(\partial_{\varphi} f_2)).$$  (3.44)

One then finds the Schwarzian derivative (1.3) as the corresponding Souriau cocycle, with differential at the identity given by

$$s(X) = \frac{1}{24\pi} X'''(\varphi),$$  (3.45)

while the Maurer–Cartan form is

$$\langle \theta, \theta \Xi \rangle = \left( \frac{df}{f} \partial_{\varphi}, dm + \frac{1}{48\pi} \int_0^{2\pi} d\varphi \frac{df}{f} \left( \frac{f'''}{f'} \right) \right).$$  (3.46)
3.3.7 Geometric actions. The geometric action (3.18) for the Virasoro group is then found to be

\[
I_{\text{Diff}(S)} \left[ f; b_0, c \right] = \int d\varphi \, dt \left[ b_0(f)f'f' + \frac{c}{48\pi} f''f' \right].
\] (3.47)

The next step is to add a Hamiltonian preserving diffeomorphisms on the circle. Again, in order to make contact with three-dimensional gravity, we chose a Hamiltonian associated with a suitable vector field. From the discussion of section 2 after equation (2.21), it follows that we may choose \( X = -\partial_x \) with associated Noether charge

\[
H_1 = \int d\varphi \left[ b_0(f)f'2 \right] + \frac{c}{48\pi} \phi'2 f'^2 f'.
\] (3.48)

The invariance under diffeomorphisms \( \delta X = \partial_x f \) of (3.47) then survives the deformation \( H_1 dt \) if \( X' \) evolves according to (2.15). This becomes explicitly \( \partial_t X' = \partial_x f + \partial_x \phi \), and implies \( X' = X'(t + \phi) \). At this point, it is convenient to define \( \partial_x f = e^\phi \). This can be done because of (3.41). In that case, after adding the Hamiltonian \( H_1 \) to (3.47), we find

\[
I_{\text{Diff}(S)} \left[ f; b_0, c, H_1 \right] = 2 \int d\varphi \, dt \left[ b_0(f)f'2 \partial_\varphi f' - \frac{c}{48\pi} \phi'2 \partial_\varphi f' \right].
\] (3.49)

3.3.8. Relation to chiral bosons. One may again eliminate the term proportional to the representative \( b_0(\varphi) \) from the action by defining a new field with a suitable periodicity. Following section 3, this can be done by defining a new field \( F = \Upsilon \circ f \) where \( \Upsilon \) satisfies

\[
c S_\varphi [\Upsilon] = -b_0(\varphi).
\] (3.50)

The ansatz \( F = e^{\mu(f)} \) turns this equation into

\[
\frac{c}{48\pi} \left( \frac{d\mu}{df} \right)^2 - c S_f [\mu] = b_0(f).
\] (3.51)

In terms of the new field \( F \), the action (3.49) reduces to \( I_{\text{Diff}(S)} [(F,0); (0,c)] \), which can be rewritten as the action of a chiral boson \( \chi \)

\[
I_{\text{Diff}(S)} [\chi; c] = \frac{c}{24\pi} \int d\varphi \, d\varphi \chi \partial_\varphi \chi,
\] (3.52)

where \( \partial_\varphi F = e^\chi \). In this case, the field redefinition that relates (3.52) with (3.49) is

\[
\chi = \mu(f) + \phi + \log \left( \frac{d\mu}{df} \right).
\] (3.53)

As a side remark, note that (3.51) turns out to be Hill’s equation: defining \( \psi(f) = (\partial_\varphi f)^{1/2} \exp(-\chi/2) \), (3.51) becomes

\[
\left( -\frac{c}{12\pi} \partial_\varphi^2 + b_0(f) \right) \psi(f) = 0.
\] (3.54)

It is well-known that conjugacy classes of monodromy matrices associated to the Hill’s equation characterize Virasoro coadjoint orbits (see e.g. [51]).
4. Geometric action for semi-direct products

4.1. Semi-direct product groups

A semidirect product of a semi-simple Lie group $G$ and an abelian group $A$, under some representation $\sigma$ of $G$ on $A$,

$$S \sigma = G \ltimes_{\sigma} A,$$

(4.1)

is a group with elements of the form $(g, \alpha)$, where $g \in G$ and $\alpha \in A$. The group operation is given by $(g_1, \alpha_1) \cdot (g_2, \alpha_2) = (g_1 g_2, \alpha_1 + \sigma_{g_1} \alpha_2)$. As $A$ is abelian, its Lie algebra is isomorphic to itself and therefore the Lie algebra associated to $S \sigma$ is given by $s = \mathfrak{g} \to \mathfrak{a}$. Denoting the elements of $s$ by $(X, \alpha)$, and the elements of its dual space by $(j, p)$, the bilinear form on $s$ is

$$\langle (j, p), (X, \alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle_A,$$

(4.2)

where $(\cdot, \cdot)_A$ are the natural pairings in $\mathfrak{g}$ and $A$ respectively. The adjoint and coadjoint actions of $S$ on $s$ and $s^*$ [52] follow from (2.1) and (2.2),

$$\text{Ad}_{(g, \alpha)} (X, \beta) = (\text{Ad}_g X, \sigma_g \beta - \Sigma_{\text{Ad}_g \alpha}),$$

(4.3)

$$\text{Ad}_{(g, \alpha)}^* (j, p) = (\text{Ad}_g^* j + \sigma_g^* p \circ \alpha, \sigma_g^* p),$$

(4.4)

where $\Sigma$ is the infinitesimal form of $\sigma$, $p \circ \alpha$ is defined as

$$\langle p \circ \alpha, X \rangle = \langle p, \Sigma_X \alpha \rangle_A = - \langle \Sigma^*_X p, \alpha \rangle_A,$$

(4.5)

and $\sigma^*, \Sigma^*$ are the dual maps of $\sigma, \Sigma$ respectively (with respect to the pairing $(\cdot, \cdot)_A$). The commutation relations for $s$ are defined by the infinitesimal form of (4.3), i.e.

$$[[X, \alpha], [Y, \beta]] = \text{ad}_{(X, \alpha)}(Y, \beta) = (\text{ad}_X Y, \Sigma_X \beta - \Sigma_Y \alpha).$$

(4.6)

We will construct geometric actions for the group $S \sigma$ when $\sigma$ corresponds to the adjoint representation and $A$ is given by the Lie algebra of $G$ seen as an abelian vector space, which will be denoted by $\mathfrak{g}_{ab}$. Using (4.3), the adjoint action of $S_{\text{Ad}} = G \ltimes_{\text{Ad}} \mathfrak{g}_{ab}$ takes the form

$$\text{Ad}_{(g, \alpha)} (Y, \beta) = (\text{Ad}_g Y, \text{Ad}_g \beta - \text{ad}_{\text{Ad}_g} Y \alpha),$$

(4.7)

while its infinitesimal form becomes

$$\text{ad}_{(X, \alpha)} (Y, \beta) = (\text{ad}_X Y, \text{ad}_X \beta - \text{ad}_Y \alpha).$$

(4.8)

For the coadjoint action, (4.4) leads to [25, 26]

$$\text{Ad}_{(g, \alpha)}^* (j, p) = (\text{Ad}_g^* j - \text{Ad}_g^* \text{ad}_{\text{Ad}_g} p, \text{Ad}_g^* p).$$

(4.9)

The Maurer–Cartan one-form consists of a pair $(\theta, \theta_{\alpha})$, where $\theta$ is defined in (2.5) and

$$\theta_{\alpha} = \text{Ad}_g^{-1} \text{d} \alpha.$$

(4.10)

4.2. Geometric actions for $S_{\text{Ad}}$

The geometric action (2.7) for a semi-direct product with an adjoint action is given by

$$I_{S_{\text{Ad}}} [g; \alpha; p_0, j_0] = I_G [g; j_0] - I_G [g; \text{ad}_{\alpha} p_0].$$

(4.11)
In terms of \((h_L, \alpha_L)\) and \((h_R, \alpha_R)\), left and right actions in a semi-direct product group act as
\[
\begin{align*}
g \to h_L g, & \quad \alpha \to \alpha_L + \text{Ad}_{h_L} \alpha, \\
g \to g h_R, & \quad \alpha \to \alpha + \text{Ad}_{h_R} \alpha. 
\end{align*}
\] (4.11)

Geometric actions (4.10) are invariant under gauge and global transformations generated by vector fields \(v^R_{(x, \psi)}\) and \(v^L_{(c, \zeta)}\) respectively. They are given by
\[
\begin{align*}
v^R_{(x, \psi)} &= \frac{d}{ds}(gh_R(s), \alpha + \text{Ad}_{h_R}(s))|_{s=0}, X = \frac{d}{ds}h_R(s)|_{s=0}, \quad v = \frac{d}{ds}\alpha_R(s)|_{s=0}, \\
v^L_{(c, \zeta)} &= \frac{d}{ds}(h_L(s)g, \alpha_L(s) + \text{Ad}_{h_L}(s))|_{s=0}, \quad \epsilon = \frac{d}{ds}h_L(s)|_{s=0}, \quad \zeta = \frac{d}{ds}\alpha_L(s)|_{s=0}. 
\end{align*}
\] (4.12)

where \((\epsilon, \zeta)\) depends arbitrary on time \(t\) and belongs to the little algebra of the representatives \((p_0, j_0)\).

### 4.3. Geometric action for centrally extended \(\hat{S}_{\text{Ad}}\)

Let us consider now the centrally extended group \(\hat{S} = \hat{G} \rtimes \text{Ad} \hat{\mathfrak{g}}\), i.e. a central extension of a semi-direct product group under the adjoint action, whose elements will be denoted by \((g, m_1, \alpha, m_2)\). The elements of the algebra \(\hat{\mathfrak{g}}\) will be denoted by \((X, n_1, \alpha, n_2)\), while the elements of the dual by \((\mathbf{j}, c_1, p, c_2)\). The coadjoint action as well as for the geometric action are constructed using (4.8) and (4.10) and replacing
\[
\begin{align*}
j_0 &\to (j_0, c_1), \quad p_0 \to (p_0, c_2), \quad \text{Ad}^*_g \to \text{Ad}^*_{g(m_1)}, \quad \text{ad}^*_\alpha \to \text{ad}^*_{(\alpha, n_2)}, 
\end{align*}
\]
where \(\text{Ad}^*_{(g,m_1)}\) is defined in (3.9) and \(\text{ad}^*_{(\alpha,n_1)}\) corresponds to its infinitesimal form given by (3.10). In the same way, the one-form \((\theta, \bar{\theta}_a)\) introduced in the previous section must be replaced by \((\theta, \bar{\theta}_z, \bar{\theta}_{za}, \bar{\theta}_{a\varnothing})\) where \(\bar{\theta}_{a\varnothing} = \langle \mathcal{S}(g), \bar{\theta}_a \rangle\).

The geometric action on a coadjoint orbit \(O_{(j_0, c_1, p_0, c_2)}\) is given by the centrally extended version of (4.10),
\[
I_{\hat{S}_{\text{Ad}}} [g, \alpha; p_0, j_0, c_1, c_2] = I_G [g; j_0, c_1] - I_G [g; \text{ad}^*_\alpha p_0 + c_2 s(\alpha)],
\] (4.13)
where \(I_G\) is given by (3.18).

As before, the orbit representatives \(j_0\) and \(p_0\) can be absorbed into the terms proportional to the central charges by defining suitable fields \(u = \Upsilon g\) and \(a = \eta + \text{Ad}_\Upsilon \alpha\). Using (3.14) allows one to write the action (4.13) in the form
\[
I_{\hat{S}_{\text{Ad}}} [g, \alpha; p_0, j_0, c_1, c_2] = I_G [g, \alpha; j_0, p_0] + c_1 \int \langle \text{Ad}^*_{\eta\Upsilon} S(\Upsilon), \theta \rangle \\
+ c_2 \int \langle \text{Ad}^*_{\eta\Upsilon} (\text{Ad}^*_\Upsilon^{-1} s(\eta) - \text{ad}^*_\alpha S(\Upsilon)), \theta \rangle + I_{\hat{S}_{\text{Ad}}} [u, a; 0, 0, c_1, c_2].
\] (4.14)

An inspection of the latter expression makes evident that, provided the pair \((\Upsilon, \eta)\) satisfies
\[
c_2 S(\Upsilon) = -p_0, \quad c_2 \text{Ad}^*_\Upsilon^{-1} s(\eta) = -j_0 + \frac{c_1}{c_2} p_0,
\] (4.15)
the geometric action reduces to
\[
I_{\hat{S}_{\text{Ad}}} [g, \alpha; p_0, j_0, c_1, c_2] = I_{\hat{S}_{\text{Ad}}} [u, a; 0, 0, c_1, c_2].
\] (4.16)
As we will see in the next examples, the latter equality will allow us to link geometric actions based on groups having a semi-direct product structure with actions appearing in the Hamiltonian reduction of 3d gravity in the case of vanishing cosmological constant.

### 4.4. Examples

#### 4.4.1. Loop group of $G \ltimes \mathfrak{g}$ and its extension.

Let us consider the group $\tilde{S}_\text{Ad} = \tilde{L}G \ltimes \tilde{L}G_{\text{ab}}$, where $G$ is a semi-simple Lie group. Its elements are of the form $(g(\varphi), m_1, \alpha(\varphi), m_2)$, where $g(\varphi)$ is a map of the form (3.24), $\alpha$ is an element the loop algebra $L\mathfrak{g}$ and $m_1, m_2$ correspond to the central extensions of $L\mathfrak{g}$ and $L\mathfrak{g}$ respectively. The geometric action can be obtained directly from (4.13) using the machinery developed in section 3.3.1,

$$I_{I_{\text{Ad}}^{\tilde{S}_\text{Ad}};\tilde{L}G_{\text{ab}}}[g, \alpha; p_0, j_0, c_1, c_2] = I_G[g; j_0, c_1] - \int d\varphi \text{Tr} \left[ \left( [\alpha, p_0] + \frac{c_2}{2\pi} \alpha' \right) g g^{-1} \right],$$

(4.17)

where $I_G$ is given by (3.29). From (4.11), this action is invariant under right multiplication of $g$, but also under the adjoint action on $\alpha$, i.e. under

$$\delta_{(X,\alpha)}(g, \alpha) = (gX(\varphi), g\alpha)(\varphi)g^{-1}.$$  

(4.18)

These global symmetries give rise to the following noether charges

$$J_X = \int_0^{2\pi} d\varphi \text{Tr}[Xg], \quad j = \frac{c_1}{2\pi} g^{-1} \partial_\varphi g - g^{-1} \left( j_0 - \frac{c_2}{2\pi} \partial_\varphi \alpha - [\alpha, p_0] \right) g,$$

$$P_\nu = \int_0^{2\pi} d\varphi \text{Tr}[\nu g], \quad p = \frac{c_2}{2\pi} g^{-1} \partial_\varphi g - g^{-1} p_0 g.$$  

(4.19)

A Hamiltonian motivated by asymptotically flat gravity in three dimensions is

$$H_2 = \frac{\pi}{c_2} \int d\varphi \text{Tr} \left[ p^2 \right].$$

(4.20)

Since $H_2$ transforms as (3.34) under the symmetries (4.18) and $\delta_{(X,\alpha)}I_{I_{\text{Ad}}^{\tilde{S}_\text{Ad}};\tilde{L}G_{\text{ab}}}$ is given by $\int d\varphi \text{Tr} \left[ \partial_\varphi \nu g + \partial_\varphi Xg \right]$, one concludes that in presence of the Hamiltonian, action

$$I_{I_{\text{Ad}}^{\tilde{S}_\text{Ad}};\tilde{L}G_{\text{ab}}}[g, \alpha; p_0, j_0, c_1, c_2] - \int dH_2$$

(4.21)

is invariant provided $X = X_0(\varphi), \nu = \nu_0(\varphi) + i\partial_\varphi X_0$.

#### 4.4.2. Relation to Flat WZW model.

The terms in (4.17) proportional to the orbit representatives $j_0$ and $p_0$ can be absorbed into the kinetic term of the flat WZW model by defining new fields $u = \eta \gamma$ and $a = \eta + \gamma \alpha \gamma^{-1}$ satisfying equations (4.15), which in this case take the form

$$-\frac{c_2}{2\pi} \gamma^{-1} \partial_\varphi \gamma = p_0, \quad -\frac{c_2}{2\pi} \gamma^{-1} \partial_\varphi \eta \gamma = j_0 - \frac{c_1}{c_2} p_0.$$  

(4.22)

After including the Hamiltonian, the geometric action (4.21) can be written in terms of the new fields $u$ and $a$ as
\[ I_{\tilde{G} \times \text{Adj} G}^{\alpha, \psi, \xi} [g, \alpha; p_0, j_0, c_1, c_2] = I_{\tilde{G}}^{\alpha} [u; 0, c_1] - \frac{c_2}{2\pi} \int d\varphi d\Omega \left[ i u u^{-1} a' - \frac{1}{2} (u^{-1} a')^2 \right]. \]

(4.23)

This corresponds to the flat WZW model obtained in [40, 53] in the context of asymptotically flat three-dimensional Einstein gravity. In this representation, the information on \((p_0, j_0)\) encoded in the periodicity of the fields \(u\) and \(a\),

\[ u(\varphi + 2\pi) = \mathcal{M}(p_0) u(\varphi), \quad a(\varphi + 2\pi) = \mathcal{M}(p_0) a(\varphi) \mathcal{M}^{-1}(p_0) + \mathcal{N}(j_0, p_0), \]

(4.24)

where \(\mathcal{M}(p_0)\) is given by (3.40) and \(\mathcal{N}(j_0, p_0) = -\frac{2\pi}{c_2} \int \mathcal{T} \left( j_0 - \frac{c_2}{2\pi} p_0 \right) \mathcal{T}^{-1}. \)

4.4.3. \(\tilde{\text{BMS}}_3\) group. The \(\tilde{\text{BMS}}_3\) group is the semidirect product of the Virasoro group and its algebra (seen as an abelian vector space) under the adjoint action

\[ \tilde{\text{BMS}}_3 = \text{Diff} (S^1) \ltimes \text{Vec} (S^1). \]

Its elements are pairs \((f, \alpha)\), where \(f\) is a diffeomorphism of the circle (3.41) and \(\alpha\) satisfies \(\alpha(\varphi + 2\pi) = \alpha(\varphi)\).

Therefore, the corresponding geometric action has the form (4.13) where the coadjoint action is the one of the Virasoro group (3.43) and \(S\) is the Schwarzian derivative (1.3). The resulting action is

\[ I_{\text{BMS}_3} [f, \alpha; p_0, j_0, c_1, c_2] = I_{\text{Diff}(S^1)} [f; j_0, c_1] + \int d\varphi \left[ f' d\varphi \left( p'_0 \alpha + 2p_0 \alpha' - \frac{c_2}{24\pi} \alpha''' \right) \right] f. \]

(4.26)

Defining \(e^\phi = f', \xi = \alpha' (f)\), the geometric action on an orbit of the \(\tilde{\text{BMS}}_3\) group takes the form

\[ I_{\text{BMS}_3} [f, \alpha; p_0, j_0, c_1, c_2] = I_{\text{Diff}(S^1)} [f; j_0, c_1] + \int d\varphi d\tau \left[ f' f \left( p'_0 \alpha + 2p_0 \alpha' \right) \circ f + \frac{c_2}{24\pi} \phi'^2 \right]. \]

(4.27)

From (4.11), we can infer that the global transformations laws of the fields are

\[ \delta_{(\mathcal{X}, v)} (f, \alpha(f)) = (\mathcal{X}(\varphi) \partial_x f, v(\varphi) \partial_x f). \]

(4.28)

We will choose the Hamiltonian as the charge associated to rigid translation\(^5\), \((X, v) = (0, -\partial_x)\)

\[ H = \int_0^{2\pi} d\varphi \left[ f'^2 p_0(f) + \frac{c_2}{48\pi} f''^2 \right]. \]

(4.29)

As in the previous examples, this choice is inspired by three-dimensional Einstein gravity without cosmological constant. Let us now consider the geometric action (4.27) deformed by the Hamiltonian (4.29). Symmetry (4.28) will be preserved in this new action provided

\(^5\) Note that when representing \(\text{bms}_3\) as vector fields on \(\mathcal{M}^+\), the generator \((0, -\partial_x)\) becomes the usual retarded time translation generator \(\partial_x\).
the suitable extension of (2.15) is satisfied, i.e. \( \partial_t(X', \nu') = \text{ad}_{(0,-\partial_\nu)}(X', \nu') \), which gives \( X' = X_0(\varphi) \) and \( \nu' = \nu_0(\varphi) + \partial_{\nu} X_0 \).

4.4.4. Relation to chiral BMS\(_3\) theory. Defining new fields \( F = \Upsilon \circ f \) and \( a = \eta + \text{Ad}_\Upsilon \alpha \) satisfying (4.15), which in this case takes the form

\[
c_2 S_\varphi \left[ \Upsilon \right] = -p_0, \quad -\frac{c_2}{24\pi} \Upsilon'''' \eta''(\Upsilon) = j_0 = \frac{c_1}{c_2} p_0, \tag{4.30}
\]

the terms in (4.27) proportional to the orbit representatives \( j_0 \) and \( p_0 \) can be absorbed in this field redefinition. Including \( H \) in (4.27), the geometric action for the BMS\(_3\) group can be written as

\[
I_{\text{BMS}_3}[f, \alpha; p_0, j_0, c_1, c_2, H] = I_{\text{Diff}(S^3)}[\chi; 0, c_1] + \frac{c_2}{24\pi} \int d\varphi dt \left( \chi' \zeta' - \frac{1}{2} \chi'^2 \right), \tag{4.31}
\]

where \( I_{\text{Diff}(S^3)}[\chi; 0] \) is given in (3.52), \( \chi = \log(\partial_\varphi F) \) and \( \zeta = a'(F) \). This is the chiral BMS\(_3\) model constructed as the classical dual of three-dimensional asymptotically flat gravity [40, 53] (see also [54] for a higher spin extension).

Note that, as in the Virasoro case, the first relation of (4.30) produces the Hill’s equation for the variable \( \psi(f) \) defined in (3.54) with \((p_0, c_2)\) playing the role of \((b_0, c)\). In the same way, the second equation of (4.30) controls the orbits associated to the pair \((j_0, c_1)\).

5. Discussion and perspectives

We have studied geometric actions for various groups arising in three-dimensional gravity. In a first stage, we have analyzed geometric actions for loop groups reobtaining (3.29), and constructing (4.17) for the semi-direct product case. Introducing monodromies in the groups elements, these actions can be written as chiral WZW models (4.23) and (3.37). When considering Chern–Simons theories on manifolds with non contractible cycles, it has been shown in [8] that the term proportional to \( b_0 \) in (3.29) arises in the associated WZW theory. Thus, one should expect these models to originate from the Chern–Simons formulation of gravity after solving the constraints inside the action once holonomies are properly taken into account. More precisely, this would be the case when adopting the new boundary conditions that have been proposed recently in the context of three dimensional gravity [55, 56] and that give rise to loop groups as asymptotic symmetry groups: for AdS\(_3\) gravity the charge algebra yields two copies of \( \text{SL}(2, \mathbb{R}) \), while in the case of vanishing cosmological constant, it is the centrally extended Poincaré loop group \( \text{SL}(2, \mathbb{R}) \rtimes \text{sl}(2, \mathbb{R}) \) that appears. In this sense, actions (3.29) and (4.17) are the \((1+1)\)-models representing these boundary degrees of freedom.

In the case of Brown–Henneaux boundary conditions, the asymptotic symmetry group is given by two copies of the diffeomorphism group of the circle and the dual dynamics is controlled by the difference of two \( \text{Diff}(S^1) \) invariant actions (3.49). For asymptotically flat spacetimes, similar boundary conditions lead to a boundary dynamics that is controlled by the BMS\(_3\) invariant model (4.26). In both cases, once we remove the representative term, we obtain the difference of two chiral bosons (3.52) for AdS\(_3\) boundary conditions, and (4.31) in the case of flat geometries. These results are consistent with the earlier derivations [39, 40]. Note however that the periodicity of the chiral fields is determined by the value of the representative(s).
The results of this paper can readily be generalized to other situations that arise in the context of three-dimensional gravity: one could for instance use the general formula (4.13) to work out geometric actions associated to other groups with semi-direct product structure like Warped Virasoro [57] or extensions of BMS$_3$ with spin-one generators [58].

Geometric actions can be used to compute one-loop partition functions associated to three-dimensional gravity. It has been shown in [59] for the Virasoro case and for constant representatives that the partition function leads to characters associated with highest weight representations. The key ingredient in this derivation was provided by a transformation that removes the representative from the action (see also [60]). In this paper, we have found a generalization of that transformation for any centrally extended group. Thus, we hope to prove more general connections between characters and partition functions by using relations (3.16) and (3.22).

Another interesting direction corresponds to exploring the connection established between geometric actions and Berry phases [31]. More precisely, it would be interesting to understand the physical content of these phases in the case of systems with BMS$_3$ symmetry or other groups with a semi-direct product structure.

Coadjoint orbits have also appeared recently in the study of the SYK model (see e.g. [61] for a review). In [62], the Hamiltonian associated to a rigid rotation (3.48) in the case where $b_0 = -\frac{c}{3\pi}$ is taken as the Euclidean action of the model. In this regard, it would be interesting to understand whether other conserved charges for the Virasoro group, for BMS$_3$ symmetry, or for the Poincaré loop group, could play a similar role.

A most relevant extension of the considerations here consists in modifying from the very beginning the set-up of section 2 by letting $b_0 = b_0(t)$ be a dynamical variable. This means that one no longer considers the dynamics on a fixed coadjoint orbit, but rather a suitable collection of orbits and the associated dynamics. In practice, this can be done for instance by introducing an additional vector $a_0 = a_0(t) \in g$. One may then choose the extended presymplectic potential

$$a^E = a + \langle b_0, da_0 \rangle, \quad (5.1)$$

and the extended kinetic term

$$I_{[g, h_0, a_0]}^E = \int_{\gamma} a^E. \quad (5.2)$$

The associated pre-symplectic 2-form is

$$\Omega^E = \Omega + \langle dh_0, \text{Ad}_g \theta \rangle + \langle b_0, da_0 \rangle. \quad (5.3)$$

The extended equations of motion are equivalent to

$$iV\Omega + \langle \dot{h}_0, \text{Ad}_g iV \theta \rangle = 0, \quad \dot{a}_0 = -\text{Ad}_g iV \theta, \quad \dot{b}_0 = 0. \quad (5.4)$$

Hence, when choosing the integration constants $b_0(t) = \bar{b}_0$ to coincide with the constant values of section 2, the dynamics of the group variables $g$ is unchanged. The additional integration constants $\bar{a}_0$ are controlled by additional global symmetries that correspond to constant shifts of $a_0$.

In particular, in the case of centrally extended groups, we get the kinetic term

$$I_{[(g, m), (h_0, c), (a_0, d)]}. \quad (5.5)$$

In this case, the group element $m$ no longer drops out of the problem and the associated global symmetry becomes relevant, and so does the quadratic term $b_0^2$ in section 3.3.1. Both the orbit
representative $b_0$ and the central charge $c$ are now dynamical variables. For the orbit representative, this has been analyzed from the geometric actions point of view in the context of ‘model spaces’ in [59, 63] and from the bulk of viewpoint in [64]. For the central extension, it seems that such a generalization has not yet been considered in the context of geometric actions, whereas from the bulk viewpoint it has recently been discussed in [65].

Finally, we will discuss elsewhere the implications of the extended set-up for three-dimensional gravity. In particular, we will study in more detail (i) the bulk duals of the geometric actions obtained here in terms of suitable choices of boundary conditions and by properly taking into account the holonomies and the dynamics of the associated particles, (ii) the interpretation in terms of Goldstone bosons and the connection to non-linear realizations. Most importantly, since the present framework is entirely group-theoretical, there are \textit{a priori} no obstructions to constructing dynamical actions appropriate to BMS symmetry in four dimensions.

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References

[1] Brown J D and Henneaux M 1986 Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity \textit{Commun. Math. Phys.} \textbf{104} 207–26
[2] Coussaert O, Henneaux M and van Driel P 1995 The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant \textit{Class. Quantum Grav.} \textbf{12} 2961–6
[3] Achucarro A and Townsend P 1986 A Chern–Simons action for three-dimensional anti-de Sitter supergravity theories \textit{Phys. Lett.} B \textbf{180} 89
[4] Witten E 1988 (2 + 1)D gravity as an exactly soluble system \textit{Nucl. Phys.} B \textbf{311} 46
[5] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of general relativity \textit{Ann. Phys.} \textbf{88} 286
[6] Witten E 1989 Quantum field theory and the Jones polynomial \textit{Commun. Math. Phys.} \textbf{121} 351
[7] Moore G W and Seiberg N 1989 Taming the conformal zoo \textit{Phys. Lett.} B \textbf{220} 422
[8] Elitzur S, Moore G W, Schwimmer A and Seiberg N 1989 Remarks on the canonical quantization of the Chern–Simons–Witten theory \textit{Nucl. Phys.} B \textbf{326} 108
[9] Forgacs P, Wipf A, Balog J, Feher L and O’Raifeartaigh L 1989 Liouville and Toda theories as conformally reduced WZNW theories \textit{Phys. Lett.} B \textbf{227} 214
[10] Bershadsky M and Ooguri H 1989 Hidden SL(n) symmetry in conformal field theories \textit{Commun. Math. Phys.} \textbf{126} 49
[11] Alekseev A and Shatashvili S L 1989 Path integral quantization of the coadjoint orbits of the virasoro group and 2D gravity \textit{Nucl. Phys.} B \textbf{323} 719
[12] Skenderis K and Solodukhin S N 2000 Quantum effective action from the AdS/CFT correspondence \textit{Phys. Lett.} B \textbf{472} 316–22
[13] Banados M 1998 Three-dimensional quantum geometry and black holes (arXiv:9901148 [hep-th])
[14] Navarro-Salas J and Navarro P 1999 Virasoro orbits, AdS(3) quantum gravity and entropy J. High Energy Phys. JHEP05(1999)009
[15] Nakatsu T, Umetsu H and Yokoi N 1999 Three-dimensional black holes and Liouville field theory Prog. Theor. Phys. 102 867–96
[16] Garbarz A and Leston M 2014 Classification of boundary gravitons in AdS3 gravity J. High Energy Phys. JHEP05(2014)141
[17] Barnich G and Oblak B 2014 Holographic positive energy theorems in three-dimensional gravity Class. Quantum Grav. 31 152001
[18] Ashtekar A, Bicak J and Schmidt B G 1997 Asymptotic structure of symmetry reduced general relativity Phys. Rev. D 55 669–86
[19] Barnich G and Compere G 2007 Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions Class. Quantum Grav. 24 F15
[20] Barnich G and Troessaert C 2010 Aspects of the BMS/CFT correspondence J. High Energy Phys. JHEP05(2010)062
[21] Barnich G and Troessaert C 2010 Supertranslations call for superrotations PoS CNCFG2010 010
[22] Barnich G, Gomberoff A and Gonzalez H A 2013 Three-dimensional Bondi–Metzner–Sachs invariant two-dimensional field theories as the flat limit of Liouville theory Phys. Rev. D 87 124032
[23] Duval C, Gibbons G W and Horvathy P A 2014 Conformal carroll groups and BMS symmetry Class. Quantum Grav. 31 092001
[24] Duval C, Gibbons G W and Horvathy P A 2014 Conformal carroll groups J. Phys. A: Math. Theor. 47 335204
[25] Barnich G and Oblak B 2014 Notes on the BMS group in three-dimensional: I. Induced representations J. High Energy Phys. JHEP06(2014)129
[26] Barnich G and Oblak B 2015 Notes on the BMS group in three-dimensional: II. Coadjoint representation J. High Energy Phys. JHEP03(2015)033
[27] Raeymaekers J 2015 Quantization of conical spaces in three-dimensional gravity J. High Energy Phys. JHEP03(2015)060
[28] Garbarz A and Leston M 2016 Quantization of BMS3 orbits: a perturbative approach Nucl. Phys. B 906 133–46
[29] Campoleoni A, Gonzalez H A, Oblak B and Riegler M 2016 BMS modules in three-dimensional Int. J. Mod. Phys. A 31 1650068
[30] Oblak B 2017 BMS Particles in Three Dimensions (Springer Theses) (Cham: Springer) (https://doi.org/10.1007/978-3-319-61878-4)
[31] Oblak B 2017 Berry phases on Virasoro orbits J. High Energy Phys. JHEP1710(2017)114
[32] Kostant B 1970 Quantization and unitary representations J. High Energy Phys. 31 152001
[33] Souriau J-M 1969 Structure des Systèmes Dynamiques (Paris: Dunod)
[34] Kirillov A A 1970 Elements of the Theory of Representations (Paris: Dunod)
[35] Kirillov A A 2004 Lectures on the Orbit Method (Providence, RI: American Mathematical Society)
[36] Alekseev A, Faddeev L D and Shatashvili S L 1988 Quantization of symplectic orbits of compact Lie groups by means of the functional integral J. Geom. Phys. 5 391–406
[37] Guieu L and Roger C 2007 L’Algèbre et le Groupe de Virasoro (Montréal: Les Publications CRM)
[38] Khesin B and Wendt R 2009 The Geometry of Infinite-Dimensional Groups (Berlin: Springer)
[39] Henneaux M, Miao E and Schwimmer A 2000 Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity Ann. Phys. 282 31–66
[40] Barnich G and Gonzalez H A 2013 Dual dynamics of three-dimensional asymptotically flat Einstein gravity at null infinity J. High Energy Phys. JHEP05(2013)016
[41] Rai B and Rodgers V G J 1990 From coadjoint orbits to scale invariant WZNW type actions and 2D quantum gravity action Nucl. Phys. B 341 119–33
[42] Delius G W, van Nieuwenhuizen P and Rodgers V 1990 The method of coadjoint orbits: an algorithm for the construction of invariant actions Int. J. Mod. Phys. A 05 3943–83
[43] Arutyun H, Nissimov E, Pacheva S and Zimerman A H 1990 Symplectic actions on coadjoint orbits Phys. Lett. B 240 127–32
[44] Nissimov E and Pacheva S 2001 Gauging of geometric actions and integrable hierarchies of KP type Int. J. Mod. Phys. A 16 2311–64
[45] Salgado-Rebolledo P 2015 Symplectic structure of constrained systems: gribov ambiguity and classical duals for three-dimensional gravity PhD Thesis Universidad de Concepción U. & Université libre de Bruxelles
21

[46] Balachandran A, Bimonte G, Gupta K and Stern A 1992 Conformal edge currents in Chern–Simons theories Int. J. Mod. Phys. A 7 4655–70

[47] Banados M 1996 Global charges in Chern–Simons field theory and the (2 + 1) black hole Phys. Rev. D 52 5816

Barnich G and Compère G 2007 Class. Quantum Grav. 24 3139

[48] Pressley A and Segal G 1986 Loop Groups (Oxford: Oxford University Press)

[49] Troost J and Tsuchiya A 2003 Three-dimensional black hole entropy J. High Energy Phys. JHEP06(2003)029

[50] Kim J and Porrati M 2015 On a canonical quantization of three-dimensional anti de Sitter pure gravity J. High Energy Phys. JHEP10(2015)096

[51] Balog J, Feher L and Palla L 1998 Coadjoint orbits of the Virasoro algebra and the global Liouville equation Int. J. Mod. Phys. A 13 315–62

[52] Baguis P 1998 Semidirect products and the Pukanszky condition J. Geom. Phys. 25 245–70

[53] Barnich G, Donnay L, Matalich J and Troncoso R 2017 Super-BMS3 invariant boundary theory from three-dimensional flat supergravity J. High Energy Phys. JHEP01(2017)029

[54] Gonzalez H A and Pino M 2014 Boundary dynamics of asymptotically flat three-dimensional gravity coupled to higher spin fields J. High Energy Phys. JHEP05(2014)127

[55] Grumiller D and Riegler M 2016 Most general AdS3 boundary conditions J. High Energy Phys. JHEP10(2016)023

[56] Grumiller D, Merbis W and Riegler M 2017 Most general flat space boundary conditions in three-dimensional Einstein gravity 2017 Class. Quantum Grav. 34 184001

[57] Compère G, Song W and Strominger A 2013 New boundary conditions for AdS3 J. High Energy Phys. JHEP05(2013)152

[58] Detournay S and Riegler M 2017 Enhanced asymptotic symmetry algebra of 2 + 1 dimensional flat space Phys. Rev. D 95 046008

[59] Alekseev A and Shatashvili S L 1990 From geometric quantization to conformal field theory Commun. Math. Phys. 128 197–212

[60] Afshar H, Grumiller D, Sheikh-Jabbari M M and Yavartanoo H 2017 Horizon fluff, semi-classical black hole microstates—log-corrections to BTZ entropy and black hole/particle correspondence J. High Energy Phys. JHEP08(2017)087

[61] Maldacena J and Stanford D 2016 Remarks on the Sachdev–Ye–Kitaev model Phys. Rev. D 94 106002

[62] Stanford D and Witten E 2017 Fermionic localization of the Schwarzian theory J. High Energy Phys. JHEP1710(2017)008

[63] La H, Nelson P and Schwarz A S 1990 Virasoro model space Commun. Math. Phys. 134 539–54

[64] Compère G, Mao P, Seraj A and Sheikh-Jabbari M M 2016 Symplectic and Killing symmetries of AdS3 gravity: holographic versus boundary gravitons J. High Energy Phys. JHEP01(2016)080

[65] Bunster C and Pérez A 2015 Superselection rule for the cosmological constant in three-dimensional spacetime Phys. Rev. D 91 024029