CONIC KÄHLER-EINSTEIN METRICS ALONG SIMPLE NORMAL CROSSING DIVISORS ON FANO MANIFOLDS

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Abstract. We prove that on one Kähler-Einstein Fano manifold without holomorphic vector fields, there exists a unique conical Kähler-Einstein metric along a simple normal crossing divisor with admissible prescribed cone angles. We also establish a curvature estimate for conic metrics along a simple normal crossing divisor which generalizes Li-Rubinstein’s estimate and derive high order estimates from this estimate.

1. Introduction

Conic Kähler metrics are very useful in the study of Kähler geometry. Recently there are a lot of works on this topic [1] [4] [5] [15] [16] [17] [18] [20] [23] [24] [30] [36] [38] [39]. It played a crucial role in the solution to Yau-Tian-Donaldson Conjecture [35] [7] [8] [9] on Fano manifolds. In this solution one important part is the analysis of the continuity path of the cone angle, which has been studied in [15]. The idea is to investigate the behavior of the solutions to conical Kähler-Einstein metrics as the angle tends to 1. Then the stability condition guarantees that the solution could be extended, which generates a smooth Kähler-Einstein metric. On the other hand, we can consider the existence of conical Kähler-Einstein metrics assuming the existence of smooth Kähler-Einstein metrics, which could be studied along the continuity path of decreasing cone angles. Related works in this direction could be found in [20] [30] [38] in case of one smooth divisor. In [30], Song-Wang also considered the case of simple normal crossing divisors on toric Fano manifolds. In this paper, we consider the general situation of simple normal crossing divisors and our main result is as below, which could be thought as the generalization of Theorem 1.1 of [20]:

**Theorem 1.1.** Given a Fano manifold \((M, \omega_0)\) without holomorphic vector fields where \([\omega_0] = c_1(M)\), which admits a Kähler-Einstein metric \(\omega_{KE}\) satisfying \(\text{Ric}(\omega_{KE}) = \omega_{KE}\).

For any simple normal crossing divisor \(D = \sum_{r=1}^{m} D_r\) and a sequence of positive rational numbers \(\lambda_1, \cdots, \lambda_m\) satisfying

\[
\sum_{r=1}^{m} c_r[D_r] = c_1(M),
\]

(1.1)

if for all \(r = 1, \cdots, m\) it holds that \(c_r \leq 1\) and

\[
\lambda_r := \inf\{\lambda > 0 | \lambda K_M^{-1} - [D_r] > 0\} \geq \frac{n}{n + 1}
\]
in case that \( c_r = 1 \), then for \( \mu, \beta_1, \cdots, \beta_m \in (0, 1) \) which satisfy
\[
1 - \beta_r = (1 - \mu)c_r, \tag{1.2}
\]
there exists a unique \( C^{2, \alpha, \beta} \) conical Kähler-Einstein metric in \( c_1(M) \) with cone angle \( 2\pi \beta_r \) along each irreducible divisor \( D_r \) for \( r = 1, \cdots, m \) for some \( \alpha \in (0, 1) \) depending on \( B := (\beta_1, \cdots, \beta_m) \).

We recall that a Kähler \((1, 1)\)-current \( \omega \) is a conic metric along the simple normal divisor \( D = \sum_{r=1}^m \) with cone angle \( \beta_r \) along each irreducible divisor \( D_r \) if it is smooth outside the divisor \( D \) and equivalent to the standard conic metric
\[
\omega_{\text{cone}} = \sqrt{-1} \left( \sum_{r=1}^m \frac{dz_r \wedge d\bar{z}_r}{|z_r|^{2(1-\beta_r)}} + \sum_{r=m+1}^n dz_r \wedge d\bar{z}_r \right) \tag{1.3}
\]
The study of such metrics is related to klt singularities in algebraic geometry. However in this general setting even the linear theory developed in [15] [18] is unclear to hold so the continuity method and a prior \( C^0 \)-estimate in [18] for smooth divisor case cannot be adapted directly. Alternately, we will make use the approximation method developed in [35] [24] [38], to construct smooth solutions to the complex Monge-Ampere equations which approximate the conical Monge-Ampere equation associated to the conical Kähler-Einstein metric along a simple normal crossing divisor. To obtain the solutions, we need to derive the properness of perturbed energy functionals, e.g., Ding energy or Mabuchi energy from the existence of smooth Kähler-Einstein metric. We also need to establish a uniform \( C^0 \)-estimate for the approximating solutions and then deduce a unique weak solution to conical Monge-Ampere equation by Berndtsson’s uniqueness theorem [3].

However, the approximation method can only guarantee the existence of weak solution to conical Monge-Ampere equation. To establish the regularity of the solution we first need to establish the Laplacian estimate of the solution. In case of one irreducible divisor [18] this estimate comes from an application of Chern-Lu’s Inequality [22] which requires a bisectional upper bound estimate of the background conic metric by Li-Rubinstein (see the appendix of [18] or C. Li’s thesis [19]). In simple normal crossing case the curvature estimate is much more complicated. In [5] [16] they construct an approximating sequence of background conic metric. As their metrics do not have a uniformly curvature bound from any side they need more complicated calculations to derive the Laplacian estimate. Alternately in [12] Datar-Song gave a simple Laplacian estimate depending on Li-Rubinstein’s curvature estimate. In this paper, we generalize Li-Rubinstein’s curvature estimate to simple normal crossing case:

**Theorem 1.2.** For a Kähler manifold \( M \) with a simple normal crossing divisor consisting of \( m \) irreducible divisors \( D_r, r = 1, \cdots, m \) on \( M \), suppose we have a conic metric
\[
\omega = \omega_0 + \sum_{r=1}^m \sqrt{-1} \partial \bar{\partial} ||S_r||^{2\beta_r}
\]
with cone angle $2\pi \beta_r$ along each irreducible divisor $D_r$. Then we have two cases:

1. for either all cone angles $\beta_r \leq \frac{1}{2}$ or $D$ is composed by irreducible divisors free of triple singularities, the bisectional curvature of $\omega$ is uniformly bounded from above on $M \setminus D$.

2. for $D$ containing higher multiple singularities, there exists a smooth potential function $\varphi_0$ such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_0$ is a conic Kähler metric which is equivalent to $\omega$ and its bisectional curvature is uniformly bounded from above on $M \setminus D$.

**Remark 1.3.** By Y. Rubinstein and the author’s computation, until now we can only obtain the upper bisectional curvature bound for the case that either all cone angles $\beta_i \leq \frac{1}{2}$ or $D$ is composed by irreducible divisors free of triple singularity, which is just the first part of 1.2. The problem for higher multiple singularities is that near such singularities, the diagonal terms of the background metric $\omega_0$ cannot supply enough control over its non-diagonal terms. However, if we add a smooth function $\varphi_0$ to enlarge the diagonal terms then such control can be achieved. In this sense the two conclusions could be combined as we can set $\varphi_0 = 0$ in the first case. Although this setting changes the original background metric, as the modified metric is still equivalent to the original one, the Laplacian estimate can still be established by Chern-Lu’s Inequality.

Finally, the $C^{2,\alpha,\beta}$-estimate could be derived by Tian’s beautiful estimate in [36]. The simple normal crossing case could also be derived by some generalization and the readers can see [26] which established a parabolic estimate but the elliptic version can be derived similarly and even more easily.

**Acknowledgment.** First of all the author wants to thank his Ph.D thesis advisor Professor Gang Tian for suggesting this problem to him and a lot of guidance and encouragement. He also wants to thank Professor Xiaohua Zhu for a lot of discussions on conic metric problems. And he also wants to thank Professor Jingyi Chen, Chi Li, Rafe Mazzeo, Yanir Rubinstein, Xiaowei Wang, Yuan Yuan and Zhenlei Zhang for their interest in this work and useful advice. Finally this material is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while the author is in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2016 semester.

2. Basic setting, energy functionals and log $\alpha$-invariants

As [35], first we need to derive a complex Monge-Ampere equation from the equation (6.1) and conical Kähler-Einstein condition. We know that a simple normal crossing divisor $D = \sum_{r=1}^{m} D_r$ is consisting of $m$ irreducible divisors which can be written as $D_r = \{z_r = 0\}$ locally. Denote $S_r$ as the defining holomorphic section of $D_r$ and $\| \cdot \|_r$ as the Hermitian metric defined on the holomorphic line bundle $[D_r]$. By (6.1) (1.2) we have that

$$\sum_{r=1}^{m} (1 - \beta_r) R(\| \cdot \|_r) = (1 - \mu)(\omega_0 + \sqrt{-1}\partial\bar{\partial}h),$$

(2.1)
where $R(||\cdot||_r)$ is the curvature of $||\cdot||_r$ and $h$ is a smooth function. Suppose $\omega_c = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ is the required conical Kähler-Einstein metric, then by equations above it satisfies

$$Ric(\omega_c) = \mu \omega_c + \sum_{r=1}^{m} 2\pi(1 - \beta_r)[D_r]$$

(2.2)

If the background metric $\omega_0$ satisfies

$$Ric(\omega_0) = \omega_0 + \sqrt{-1}\partial\bar{\partial}h_0,$$

where $h$ is a smooth function satisfies $\int_M (e^h - 1)\omega^n_0 = 0$, then by these equations above we could derive a conical Monge-Ampere equation for $\varphi$:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_0 - (1-\mu)h - \mu\varphi - \sum_{r=1}^{m}(1-\beta_r)\log||S_t||^2 + a_\mu \omega_0^n},$$

(2.3)

where $a_\mu$ is a constant satisfying

$$\int_M (e^{h_0 - (1-\mu)h - \sum_{r=1}^{m}(1-\beta_r)\log||S_t||^2 + a_\mu} - 1)\omega^n_0 = 0.$$

Note that in [18] [20] they applied continuity method to solve such conical Monge-Ampere equation. Actually their works highly rely on the linear theory with respect to the standard conic metric along one smooth divisor. For simple normal crossing divisors such linear theory is not known yet. Alternately, we try to establish the existence of the solution by approximating method in [33] [24] [38]. Basically, we can perturb the equation (2.3) to the following smooth equation:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{h_0 - (1-\mu)h - \mu\varphi - \sum_{r=1}^{m}(1-\beta_r)\log||S_t||^2 + a_\mu,\epsilon \omega_0^n} + a_\mu,\epsilon \omega_0^n,$$

(2.4)

where $a_{\mu,\epsilon}$ is a constant satisfying

$$\int_M (e^{h_0 - (1-\mu)h - \sum_{r=1}^{m}(1-\beta_r)\log||S_t||^2 + a_\mu,\epsilon} - 1)\omega^n_0 = 0.$$

Our strategy is to solve this approximating equation and try to obtain the compactness of the approximating solutions. To achieve this goal, as [33] [35] we will introduce corresponding energy functionals, analyze the properness, and finally establish the uniform $C^0$-estimate.

Recall in [10] [11] [33], we have the following energy functionals for $\varphi \in PSH(M, \omega_0)$:

**Definition 2.1.**

(1) $J_{\omega_0}(\varphi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \sqrt{-1}\partial\bar{\partial}\varphi \wedge \partial\bar{\partial}\varphi \wedge \omega_0^{n-i-1}$

(2) $I_{\omega_0}(\varphi) = \frac{1}{V} \int_M \varphi(\omega_0^n - \omega_\varphi^n)$,

where $V = \int_M \omega_0^n$, $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$. 

There are some nice properties of those functionals, see [34] for more details. Then we can introduce the twisted Ding functional and the twisted Mabuchi functional (see [18] [20] [24]), which are the Lagrangians of the conical Monge-Ampere equation (2.3). For simplicity here we set

\[ H_{0,\mu} = h_0 - (1 - \mu)h - \sum_{r=1}^{m}(1 - \beta_r) \log ||S_r||^2 + a_\mu, \]

and we can choose a family \( \varphi_t \) connected 0 and \( \varphi \in PSH(M, \omega_0) \):

**Definition 2.2.** (1) We define twisted Ding functional as

\[
F_{\omega_0,\mu}(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi \omega_0^n - \frac{1}{\mu} \log \left( \frac{1}{V} \int_M e^{H_{0,\mu} - \mu \varphi} \omega_0^n \right),
\]

(2) we define twisted Mabuchi functional as

\[
\nu_{\omega_0,\mu}(\varphi) = -\frac{n}{V} \int_0^1 \int_M \varphi_t (\text{Ric}(\omega_\varphi) - \mu \omega_\varphi - \sum_{r=1}^{m} 2\pi (1 - \beta_r) [D_r]) \wedge \omega^{n-1}_\varphi dt
\]

\[
= \frac{1}{V} \int_M \log \omega_\varphi^n \omega_0^n + \frac{1}{V} \int_M H_{0,\mu}(\omega_0^n - \omega_\varphi^n) - \mu (I_{\omega_0}(\varphi) - J_{\omega_0}(\varphi))
\]

\[
= \frac{1}{V} \int_M \log \omega_\varphi^n \omega_0^n + \frac{1}{V} \int_M H_{0,\mu}(\omega_0^n - \omega_\varphi^n) + \mu (F_{\omega_0}^0(\varphi) + \frac{1}{V} \int_M \varphi \omega_\varphi^n),
\]

where

\[
F_{\omega_0}^0(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi \omega_0^n.
\]

In [33], Tian pointed out that the \( C^0 \)-estimate for the Kähler-Einstein problem could be established if the Ding functional or the Mabuchi functional is proper. Similarly, to establish the uniform \( C^0 \)-estimates for the approximating equation (2.4), we need to check the properness of the corresponding Ding functionals or Mabuchi functionals, which could be derived from the properness of corresponding twisted functionals. First, we recall the definition of the properness:

**Definition 2.3.** Suppose the twisted Ding functional \( F_{\omega_0,\mu}(\varphi) \) (twisted Mabuchi functional \( \nu_{\omega_0,\mu}(\varphi) \)) is bounded from below, i.e. \( F_{\omega_0,\mu}(\varphi) \geq -C_{\omega_0} \) \( \nu_{\omega_0,\mu}(\varphi) \geq -C_{\omega_0} \), we say it is proper on \( PSH(M, \omega_0) \), if there exists an increasing function \( f : [-C_{\omega_0}, \infty) \rightarrow \mathbb{R} \), and \( \lim_{t \rightarrow \infty} f(t) = \infty \), such that for any \( \varphi \in PSH(M, \omega_0) \), we have

\[
F_{\omega_0,\mu}(\varphi) \geq f(J_{\omega_0}(\varphi)) \quad (\nu_{\omega_0,\mu}(\varphi) \geq f(J_{\omega_0}(\varphi))).
\]

There are a lot of properties of these functionals, for more details, see [34] [20]. The main result of this section is the properness of the twisted Ding functional assuming the conditions in Theorem 1.1.
Theorem 2.4. Assume the conditions of Theorem 1.1, then \( F_{\omega, \mu}(\varphi) \) is proper with respect to \( J_{\omega_0}(\varphi) \), more precisely, there exists an \( \eta > 0 \) and a constant \( C_\eta \) such that
\[
F_{\omega, \mu}(\varphi) \geq \eta J_{\omega_0}(\varphi) - C_\eta.
\] (2.6)

To prove this theorem, we will adapt the interpolation method using in [20] [30]. First, due to the existence of the Kähler-Einstein metric \( \omega_{KE} \) and no holomorphic vector field condition, we derive that
\[
F_{\omega_0}(\varphi) \geq \eta' J_{\omega_0}(\varphi) - C_{\eta'}
\]
for some positive constants \( \eta', C_{\eta'} \) by [33], where
\[
F_{\omega_0}(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi \omega_0^n - \log \left( \frac{1}{V} \int_M e^{h_0 - h + a \omega_0^n} \right)
\]
is the original Ding functional corresponding to smooth Monge-Ampere equation. To apply the interpolation, we also need to find a group of small data \( \mu, \beta_1, \cdots, \beta_m \) such that the corresponding properness holds. That could be done by generalizing Berman’s log \( \alpha \)-invariant estimate [1], which is the generalization of Tian’s \( \alpha \)-invariant [31]. First we introduce the definition of log \( \alpha \)-invariant:

**Definition 2.5.** Fix a smooth volume form \( vol \), for any Kähler class \( [\omega] \), we define log \( \alpha \)-invariant as below:
\[
\alpha([\omega], D, B) = \sup\{ \alpha > 0 : \exists C_\alpha < \infty \text{ s.t. } \frac{1}{V} \int_M e^{\alpha(\sup \varphi - \varphi)} vol \prod_{r=1}^m ||S_r||^{2(1-\beta_r)} \leq C_\alpha \}
\]
for any \( \phi \in PSH(M, \omega_0) \).

For the estimate of the log \( \alpha \)-invariant in case of simple normal crossing divisors, we have the following lemma:

**Lemma 2.6.** For data \( \mu, \beta_1, \cdots, \lambda_r \) where \( 1 \leq r \leq n \) in the main theorem, we have
\[
\alpha([\omega], D, B) \geq \min\{\lambda_1 \beta_1, \cdots, \lambda_m \beta_m, \alpha(K^{-1}_M), \alpha(K^{-1}_M|_{\cap_r D_r})\}. \tag{2.7}
\]

**Proof.** Similar to Berman’s estimate [1], by Demailly’s work [14], it suffices to prove the integrability in the definition for the functions \( \frac{\log ||\sigma_k||^2}{k} \) for each positive integer \( k \) where \( \sigma_k \in H^0(kK^{-1}_M) \). For each \( \sigma_k \) there exists nonnegative integers \( l_1, \cdots, l_m \) such that
\[
\sigma_k = \prod_{r=1}^l s_r^{\otimes l_r} \otimes \sigma',
\]
where \( \sigma' \) does not vanish identically on each irreducible component \( D_r \). For \( t > 0 \) we have that
\[
e^{-t \frac{\log ||\sigma_k||^2}{k}} \prod_{r=1}^m ||S_r||^{2(1-\beta_r)} = e^{-t \frac{\log ||\sigma'||^2}{k}} \prod_{r=1}^m ||S_r||^{2(1-\beta_r) + \frac{\delta}{t}}. \tag{2.8}
\]
In case that \( \sigma' \equiv 1 \), by the definition of \( \lambda_r \), we have \( \frac{t}{k} \geq \frac{1}{\lambda_r} \) thus for any \( \delta > 0 \) if \( t < \lambda_r \beta_r - \delta \), the integral of (2.8) over \( M \) is finite. On the other hand, as \( \sigma' \) does not vanish identically
on each irreducible component $D_r$, the zero set of $\sigma'$ have at least complex codimension 1 on each $D_r$. Meanwhile $\deg \sigma' \leq k \deg K^{-1}_M$, we have that $\frac{\log ||\sigma'||^2}{k} \in PSH(M, \omega_0)$.

Similar to Berman’s estimate, by Ohsawa-Takegoshi extension theorem we have that for each $\delta' > 0$ and a small neighborhood $U$ of the divisor $D = \sum_{r=1}^m D_r$ it holds that

$$\int_U e^{-t \frac{\log ||\sigma'||^2}{k}} \leq C' \int_{\gamma, D} e^{-t \frac{\log ||\sigma'||^2}{k}}.$$  

Thus whenever $t \leq \min\{\lambda_1\beta_1, \ldots, \lambda_m\beta_m, \alpha(K^{-1}_M | r, D_r)\} - \delta$ the integral over $U$ is finite. On $M \setminus U$ the denominator is uniformly bounded so whenever

$$t \leq \min\{\lambda_1\beta_1, \ldots, \lambda_m\beta_m, \alpha(K^{-1}_M), \alpha(K^{-1}_M | r, D_r)\} - \delta$$

the integral is finite. Combine these cases, the lemma is concluded.  

Given an $\alpha > 0$ such that

$$\frac{1}{V} \int_M e^{\frac{\alpha (\sup \varphi - \varphi) + h_0(1-\mu) + e^{-\lambda_0}}{\omega_0} \omega_0^n} \leq C_\alpha,$$

we could derive that

$$\log C_\alpha \geq \log \left(\frac{1}{V} \int_M e^{\frac{\alpha (\sup \varphi - \varphi) + h_0(1-\mu) + e^{-\lambda_0}}{\omega_0} \omega_0^n}\right)$$

$$\geq \frac{\alpha}{V} \int_M (\sup \varphi - \varphi) \omega_0^n + \frac{1}{V} \int_M H_{0, \mu} \omega_0^n - \frac{1}{V} \int_M \log \frac{\omega_0^n}{\omega_0^n} \omega_0^n$$

$$\geq \alpha I_{\omega_0}(\varphi) + \frac{1}{V} \int_M H_{0, \mu} \omega_0^n - \frac{1}{V} \int_M \log \frac{\omega_0^n}{\omega_0^n} \omega_0^n,$$

then by the expression of the twisted Mabuchi functional, we have that

$$\nu_{\omega_0, \mu}(\varphi) = \frac{1}{V} \int_M \log \frac{\omega_0^n}{\omega_0^n} \omega_0^n + \frac{1}{V} \int_M H_{0, \mu} (\omega_0^n - \omega_0^n) - \mu (I_{\omega_0}(\varphi) - J_{\omega_0}(\varphi))$$

$$\geq \alpha I_{\omega_0}(\varphi) - \mu (I_{\omega_0}(\varphi) - J_{\omega_0}(\varphi)) + C'$$

$$\geq (\alpha - \frac{n}{n+1} \mu) I_{\omega_0}(\varphi) + C'.$$

Thus if we want to obtain the properness of the twisted Mabuchi functional we only need to make $\alpha - \frac{n}{n+1} \mu > 0$. By Lemma 2.6 we want to find $\mu > 0$ such that

$$\min\{\lambda_1\beta_1, \ldots, \lambda_m\beta_m, \alpha(K^{-1}_M), \alpha(K^{-1}_M | r, D_r)\} > \frac{n}{n+1} \mu.$$  

It is easy to see if $\mu$ is small enough we have that $\alpha(K^{-1}_M), \alpha(K^{-1}_M | r, D_r) > \frac{n}{n+1} \mu$. It remains to prove that for sufficiently small $\mu > 0$ and all $r$, it holds that $\lambda_r \beta_r > \frac{n}{n+1} \mu$, which is equivalent to

$$0 < \lambda_r (1 - (1 - \mu)c_r) - \frac{n}{n+1} \mu = \lambda_r (1 - c_r) + (\lambda_r c_r - \frac{n}{n+1}) \mu,$$
Proposition 2.7. In case that $c_r < 1$ or $c_r = 1$ with $\lambda_r > \frac{n}{n+1}$, for sufficiently small $\mu > 0$ the corresponding twisted Mabuchi functional $\nu_{\omega_0, \mu}(\varphi)$ is proper.

Recall that in [19] [24] the properness of twisted Mabuchi functional implies the properness of the corresponding twisted Ding functional, which implies the properness of the twisted Ding functional when $\mu > 0$ sufficiently small. To derive the properness of all $\mu \in (0, 1)$ we need such a lemma from [20] [24]:

Lemma 2.8. Suppose $0 < \mu_0 < \mu_1$, write $\mu = (1 - t)\mu_0 + t\mu_1$ where $0 \leq t \leq 1$, we have

$$\mu F_{\omega_0, \mu}(\varphi) \geq (1 - t)\mu_0 F_{\omega_0, \mu}(\varphi) + t\mu_1 F_{\omega_0, \mu}(\varphi).$$

The proof is an easy application of the concavity of logarithmic functions. By this lemma, we obtain the properness of the twisted Ding functional for all $\mu \in (0, 1)$. The Theorem 2.4 is followed.

3. Approximation procedure and uniform $C^0$-estimate

In the last section, we obtained the properness of the twisted Ding functional for any data $\mu, \beta_1, \cdots, \beta_m$ satisfying the condition of Theorem 1.1. Now we expect to establish the existence of corresponding conical Kähler-Einstein metric. Recall that in smooth case [33] and one smooth divisor case [18] [20] [30], they applied continuity method. One key ingredient is the linear theory of Laplacian in ordinary and conical along one smooth divisor case. As such linear theory is still unknown in simple normal crossing divisor case, in [2] they used pluripotential approach to obtain the existence of weak solution. Now we expect to use approximation approach to establish the existence and regularity of such solutions. Recall that we consider the solution to the equation (2.4). We want to establish uniform estimates for such solutions and prove that the solutions converge to the solution to the original conical Monge-Ampere equation (2.3). Actually this approximation approach was first used by Tian [35] to approximate conical Kähler-Einstein metric and later in [24] it was used to approximate conic metrics with lower Ricci curvature bound. As (2.4) is an ordinary smooth Monge-Ampere equation, we could apply classical continuity method to solve this equation. More precisely, we consider the following equation:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_{\epsilon,t})^n = e^{h_0 - (1 - \mu)h - t\varphi_{\epsilon,t} - \sum_{r=1}^{m}(1 - \beta_r)\log(||S_r||^2 + \epsilon) + a_{\mu, \epsilon}\omega_0^n,}$$

for $t \in [0, \mu]$. Suppose $E \in [0, \mu]$ is the solvable set for $t$. First, when $t = 0$ by Yau’s solution to Calabi conjecture [40], it is solvable so $0 \in E$. Next, we know the linear operator at some $t \in E$ is $\Delta_t + t$, where $\Delta_t$ is the Laplacian with respect to the metric
\( \omega_{\epsilon,t} := \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon,t} \). We can compute the Ricci curvature of \( \omega_{\epsilon,t} \):

\[
\text{Ric}(\omega_{\epsilon,t}) = \omega_0 + (1 - \mu) \sqrt{-1} \partial \bar{\partial} h + t \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon,t} + \sum_{r=1}^{m} (1 - \beta_r) \sqrt{-1} \partial \bar{\partial} \log(||S_r||^2 + \epsilon) + R(|| \cdot ||_r)
\]

\[
= \mu \omega_0 + t \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon,t} + \sum_{r=1}^{m} (1 - \beta_r) (\sqrt{-1} \partial \bar{\partial} \log(||S_r||^2 + \epsilon) + R(|| \cdot ||_r))
\]

\[
= t \omega_{\epsilon,t} + (\mu - t) \omega_0 + \sum_{r=1}^{m} (1 - \beta_r) (\epsilon R(|| \cdot ||_r) ||S_r||^2 + \epsilon) + \frac{\epsilon \sqrt{-1} DS_r \wedge DS_r}{(||S_r||^2 + \epsilon)^2}
\]

\[
> t \omega_{\epsilon,t}.
\]

So by Bochner’s formula, we know that the linear operator \( \Delta_t + t \) is invertible thus \( E \) is open.

The most difficult part is the closeness, i.e., the uniform \( C^0 \)-estimate for \( \varphi_{\epsilon,t} \) for each \( t > 0 \). To establish such estimate, we will prove the properness of corresponding Ding functionals and obtain a uniform upper bound for them. Thus we can give a uniform bound for \( J_{\omega_0}(\varphi_{\epsilon,t}) \) which implies the uniform \( C^0 \)-bound for \( \varphi_{\epsilon,t} \).

We set

\[
H_{0,\mu,\epsilon} = h_0 - (1 - \mu) h - \sum_{r=1}^{m} (1 - \beta_r) \log(||S_r||^2 + \epsilon) + a_{\mu,\epsilon},
\]

then by the choice of \( a_{\mu,\epsilon} \) we know that \( \int_M (e^{H_{0,\mu,\epsilon}} - 1) \omega_0^m = 0 \). Now we define the approximating Ding functional as following:

\[
F_{\mu,\epsilon}(\varphi) := J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi \omega_0^m - \frac{1}{\mu} \log \left( \frac{1}{V} \int_M e^{H_{0,\mu,\epsilon}} - \mu \varphi \omega_0^m \right),
\] (3.2)

which is the Lagrangian of the approximating equation (2.4). Recall the computation in [33], suppose \( \varphi_{\epsilon,t} \) solves the equation (3.1), take the derivative of both sides of (3.1) we have that

\[
\Delta_{\epsilon,t} \varphi_{\epsilon,t} = -\varphi_{\epsilon,t} - t \dot{\varphi}_{\epsilon,t},
\]

where \( \Delta_{\epsilon,t} \) is the Laplacian with respect to \( \omega_{\epsilon,t} := \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon,t} \). Consequently,

\[
\frac{d}{dt} (t(J_{\omega_0}(\varphi_{\epsilon,t}) - \frac{1}{V} \int_M \varphi_{\epsilon,t}(\omega_0^m)) = J_{\omega_0}(\varphi_{\epsilon,t}) - \frac{1}{V} \int_M \varphi_{\epsilon,t}(\omega_0^m - t \int_M \varphi_{\epsilon,t}(\omega_0^m)
\]

\[
= J_{\omega_0}(\varphi_{\epsilon,t}) - \frac{1}{V} \int_M \varphi_{\epsilon,t}(\omega_0^m - \omega_{\epsilon,t}^m)
\]

\[
= -(I_{\omega_0}(\varphi_{\epsilon,t}) - J_{\omega_0}(\varphi_{\epsilon,t})) \leq 0.
\]
Integrate from 0 to \( t \) and make use of the concavity of logarithmic function we deduce that

\[
F_{\mu,\epsilon}(\varphi_{\epsilon,t}) \leq -\frac{1}{\mu} \log \left( \frac{1}{V} \int_M e^{H_{0,\mu,\epsilon} - \mu \varphi_{\epsilon,t}\omega_0^n} \right)
= -\frac{1}{\mu} \log \left( \frac{1}{V} \int_M e^{H_{0,\mu,\epsilon} - t \varphi_{\epsilon,t} - (\mu - t) \varphi_{\epsilon,t}\omega_0^n} \right)
\leq -\frac{\mu - t}{\mu} \int_M \varphi_{\epsilon,t}\omega_{\epsilon,t}^n.
\]

As for each \( t \geq t_0 > 0 \), it holds that \( \text{Ric}(\omega_{\epsilon,t}) > t\omega_{\epsilon,t} \geq t_0\omega_{\epsilon,t} \) and the volume is fixed, we could have a uniform control of the Sobolev constant and the first eigenvalue. Thus it follows from standard Moser’s iteration that

\[
-\inf_M \varphi_{\epsilon,t} \leq C(C' + \frac{1}{V} \int_M \varphi_{\epsilon,t}\omega_{\epsilon,t}^n),
\]

where all constants depends on \( t_0 \). As \( \inf_M \varphi_{\epsilon,t} \) by the normalization condition we obtain that

\[
F_{\mu,\epsilon}(\varphi_{\epsilon,t}) \leq C. \tag{3.3}
\]

As \( n + \Delta_0 \varphi_{\epsilon,t} = tr\omega_0\omega_{\epsilon,t} > 0 \), by standard Green formula it holds that

\[
\sup_M \varphi_{\epsilon,t} \leq c + \frac{1}{V} \int_M \varphi_{\epsilon,t}\omega_{\epsilon,t}^n,
\]

Combine these two estimates for \( \varphi_{\epsilon,t} \) we have that

\[
\text{osc}_M \varphi_{\epsilon,t} = \sup_M \varphi_{\epsilon,t} - \inf_M \varphi_{\epsilon,t} \leq C(1 + I_{\omega_0}(\varphi_{\epsilon,t})) \leq (n + 1)C(1 + J_{\omega_0}(\varphi_{\epsilon,t})). \tag{3.4}
\]

Now considering that

\[
H_{0,\mu,\epsilon} = h_0 - (1 - \mu)h - \sum_{r=1}^m (1 - \beta_r) \log(||S_r||_r^2 + \epsilon) + a_{\mu,\epsilon}
\leq h_0 - (1 - \mu)h - \sum_{r=1}^m (1 - \beta_r) \log ||S_r||_r^2 + a_{\mu,\epsilon} = H_{0,\mu} - a_{\mu} + a_{\mu,\epsilon},
\]

as \( a_{\mu,\epsilon} \) and \( a_{\mu} \) are uniformly bounded, it is easy to obtain the properness of the approximating Ding functional from Theorem 2.4:

\[
F_{\mu,\epsilon}(\varphi) \geq \eta J_{\omega_0}(\varphi) - C'_\eta. \tag{3.5}
\]

Combine (3.3) (3.4) (3.5) and note that by normalization condition

\[
\sup_M \varphi_{\epsilon,t} \geq 0, \quad \inf_M \varphi_{\epsilon,t} \leq 0,
\]

we obtain that \( |\varphi_{\epsilon,t}| \leq C(t_0) \) for \( t \geq t_0 > 0 \). Thus the solvable set \( E \) is close for each \( \epsilon > 0 \). To summarize, we have that
Theorem 3.1. For each $\epsilon > 0$ there exists a unique smooth solution $\varphi_\epsilon$ to (2.4) such that $\omega_\epsilon = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon$ have uniform lower Ricci curvature bound $\mu$. Moreover, $|\varphi_\epsilon| \leq C$ which is independent of $\epsilon$.

4. Convergence and high order estimates

As we have established a uniform $C^0$-estimate for the approximating solutions $\varphi_\epsilon$ for any $\epsilon > 0$, in this section we want to prove that there exists one subsequence converging to a weak solution of the conical Monge-Ampere equation (2.3) and such weak solution is unique. Then we establish high order estimates for the weak solution, which completes the proof of Theorem 1.1.

First, it is easy to see that there exists a subsequence of $\varphi_\epsilon$ which converge to a function $\varphi \in PSH(M, \omega_0) \cap L^\infty$ such that $\varphi$ is a weak solution to the equation (2.3) in distribution sense. By Berndtsson’s uniqueness theorem [3], the weak solution to (2.3) is unique. Moreover, by Chern-Lu’s Inequality [22], as $Ric(\omega_\epsilon) \geq \mu \omega_\epsilon$, we have that

$$\Delta_\epsilon \log tr_{\omega} \omega_0 \geq -atr_{\omega} \omega_0,$$

where $\Delta_\epsilon$ is the Laplacian of $\omega_\epsilon$ and $a$ is the upper bound of the bisectional curvature of $\omega_0$. Then put

$$u = \log tr_{\omega} \omega_0 - (a + 1) \varphi_\epsilon$$

then by the inequality above we obtain that

$$\Delta_\epsilon u \leq e^{u-(a+1)c} - n(a+1),$$

which implies $u \leq C$ by maximal principle, so we have $c_1 \omega_0 \leq \omega_\epsilon$. By the equation (2.3), we obtain that

$$c_1 \omega_0 \leq \omega_\epsilon \leq \frac{c_2 \omega_0}{\prod_{r=1}^{m} (||S_r||^2 + \epsilon)^{(1-\beta_r)}}.$$

By standard high order estimate, we have that

$$||\varphi_\epsilon||_{C^l(K)} \leq C(l, K)$$

uniformly on each compact set $K \in M \setminus D$, which implies

Proposition 4.1. $\omega_\epsilon$ converges to $\omega_\varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi$ in current sense, and the convergence is $C^\infty$ on each compact set $K \in M \setminus D$. Moreover $\omega_\varphi$ is smooth outside the divisor $D$.

Assuming the existence of $L^\infty$ weak solution to (2.3), the next step is to establish the Laplacian estimate, i.e., the equivalence between the metric $\omega_\varphi$ and the standard conic metric $\omega_{cone}$. In one irreducible divisor case, Brendle proved the $C^3$-estimates in [4] using Calabi’s estimate when $\beta < \frac{1}{2}$. For any $\beta < 1$, in [18] the Laplacian estimate follows from Chern-Lu’s Inequality based on Li-Rubinstein’s upper bisectional curvature bound estimate for basic type conic metrics. In simple normal crossing divisor case, one way is to set up a new approximating approach [5] [16]. However as the approximating conic metrics have no uniform control of bisectional curvature from either above or below, they need to
establish a much more complicated Laplacian estimate based on Paun’s trick. Meanwhile, in \cite{[12]} Datar-Song gave a simpler estimate based on Li-Rubinstein’s estimate. Their trick is that for each $r$ keep the divisor $D_r$ component and regularize other components. Then compare each approximating solution metric with standard conic metrics along $D_r$ with cone angle $2\pi\beta_r$ by Li-Rubinstein’s curvature estimate and finally the Laplacian estimate follows. In Fano case to construct approximating solutions both of their works need Demailly’s regularization process \cite{[13]}.

In this section, we will make use of the new curvature estimate Theorem 1.2 to derive the Laplacian estimate directly analogous to \cite{[18]}, without approximation and Demailly’s regularization. The proof of Theorem 1.2 will be left to the end of this paper. Assuming this theorem, we denote $\omega_{bg} := \omega_0 + \epsilon \sum_{r=1}^{m} \sqrt{-1}\partial\bar{\partial} \left| S_r \right|^{2/\beta_r} + \sqrt{-1}\partial\bar{\partial} \varphi_0$

as the background conic metric whose bisectional curvature has an upper bound $\Lambda$ on $M \setminus D$, where $\epsilon$ is small and $\varphi_0 \in PSH(M, \omega_0) \cap C^\infty(M)$. Actually it is equivalent to the standard conic metric $\omega_{\text{cone}}$ defined in \cite{[13]}. Similar to \cite{[29]} \cite{[12]}, for any $\delta > 0$, put

$$Q := \log \left( \prod_{r=1}^{m} \left| S_r \right|^{2\delta} tr_{\omega_\varphi} \omega_{bg} \right) - A(\varphi - \varphi_0 - \epsilon \sum_{r=1}^{m} \left| S_r \right|^{2/\beta_r}).$$

On $M \setminus D$, as $\omega_\varphi$ has positive Ricci curvature and the bisectional curvature of $\omega_{bg}$ is less than $\Lambda$, using Chern-Lu’s Inequality we immediately have that $\Delta Q \geq (A - \Lambda) tr_{\omega_\varphi} \omega_{bg} + \delta \sum_{r=1}^{m} tr_{\omega_\varphi} R(|| \cdot ||_r) - An.$

Take $A = \Lambda + 1$ by the setting we know that the maximal of $Q$ on $M$ is attained at $p \in M \setminus D$, where

$$tr_{\omega_\varphi} \omega_{bg}(p) \leq -\delta \sum_{r=1}^{m} tr_{\omega_\varphi} R(|| \cdot ||_r) + (\Lambda + 1)n \leq (\Lambda + 1)n,$$

by maximal principle. As $\varphi, \varphi_0$ are uniformly bounded on $M$ we conclude that it holds that $tr_{\omega_\varphi} \omega_{bg} \leq \frac{C_0}{\prod_{r=1}^{m} \left| S_r \right|^2}$. Let $\delta$ tend to 0 we prove that $\omega_\varphi \geq C_1 \omega_{bg}$.

By the conical Monge–Ampere equation (2.3) it is easy to see that $\omega_\varphi^n$ is equivalent to $\omega_{bg}^n$ so

$$\omega_\varphi \leq C_2 \omega_{bg}.$$

Thus we have proved that

Proposition 4.2. There exist $C_1, C_2 > 0$ such that $C_1 \omega_{cone} \leq \omega_\varphi \leq C_2 \omega_{cone}$. 

Finally, to finish the proof of Theorem 1.1 we need to establish a $C^{2,\alpha,B}$ estimate for the solution $\varphi$, i.e., to show $\sqrt{-1}\partial\bar{\partial}\varphi$ is $C^\alpha$ with respect to the standard conic metric $\omega_{\text{cone}}$. By [4] in one smooth divisor case if $\beta < \frac{1}{2}$ we can even find that $||\varphi||C^3$ is bounded using Calabi’s 3rd order estimate, because in this case the bisectional curvature is uniformly bounded outside the divisor. Actually this result could be generalized to simple normal crossing divisors whose cone angles are all smaller than $\frac{1}{2}$ as the bisectional curvature is also uniformly bounded outside the divisor by the proof of Theorem 1.2 in the next section. However, in case that $\beta \geq \frac{1}{2}$ one can only have $C^\alpha$-bound for $\sqrt{-1}\partial\bar{\partial}\varphi$ where $\alpha \in (0, \min\{\frac{1}{\beta} - 1, 1\})$. Based on Tian’s 3rd order estimate [36], we could establish similar estimates in general simple normal crossing case. We refer [26] for the generalization to simple normal crossing divisor case when some angle $\beta_r > \frac{1}{2}$. The corresponding conical Kähler-Einstein case should be simpler than conical Kähler-Ricci flow case in [26]:

**Proposition 4.3.** If the solution $\varphi$ of conical Monge-Ampere equation (2.3) satisfies that $C_1\omega_{\text{cone}} \leq \omega \leq C_2\omega_{\text{cone}}$, then for any point $p \in M$, $r < 1$ and $\alpha \in (0, \min\{1, \frac{1}{\beta_1} - 1, \cdots, \frac{1}{\beta_m} - 1\})$ there exists a constant $C_\alpha > 0$ such that

$$\int_{B_r(p)} |\partial\bar{\partial}\varphi|^2 \omega_{\text{cone}}^n \leq C_\alpha r^{2n-2+2\alpha}.$$  

Moreover, $\sqrt{-1}\partial\bar{\partial}\varphi$ is $C^\alpha$-bounded.

Now the proof of Theorem 1.1 is complete.

5. **Proof of Theorem 1.2**

In this section we supply a detailed proof for Theorem 1.2 which is purely technical. We will compute along Li-Rubinstein’s approach in the appendix of [18]. First, as [18], we denote $\hat{g}, g$ as the Kähler metrics associated to $\omega_0, \omega$. Without loss of confusion we temporarily still use $\omega_0$ to represent the whole background metric even if the precise form is $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ in the second case of the main theorem. Here we only deal with the most complicated case that the point we consider is near the intersection of all divisors. For other cases the estimates will be similar but easier. Now we extend Lemma A.2 in [18] which came from [37] originally to choose appropriate local holomorphic frames and coordinate system near the intersection (actually by the private communication Y. Rubinstein also gave the same extension), which we will use in the following computation:

**Lemma 5.1.** There exists $\epsilon_0 > 0$ such that if $0 \leq \text{dist}_\hat{g}(p, D_r) \leq \epsilon_0$ for all $r = 1, \cdots, m$, then we can choose local holomorphic frames $e_r$ of each holomorphic line bundle $[D_r]$ and local holomorphic coordinates $(z_1, \cdots, z_n)$ valid in a neighborhood of $p$, such that (i) $S_r = z_re_r$, and $a_r := ||e_r||^2$ satisfies $a_r(p) = 1$, $\partial a_r(p) = 0$, $\frac{\partial^2 a_r(p)}{\partial z_i \partial \bar{z}_j} = 0$, and (ii) $\hat{g}_{ij}(p) = 0$ for $i \leq m, j > m$, and $\hat{g}_{i,j,k}(p) = \frac{\partial}{\partial z_k} \omega_0(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})|_p = 0$, whenever $j > m$. 

Proof. Actually this proof can be modified from the proof of Lemma A.2 in [18]. Fix any point \( q \) on the intersection of \( D_1, \ldots, D_m \), and choose local holomorphic frames \( e'_1, \ldots, e'_m \) and holomorphic coordinates \((w_1, \ldots, w_n)\) in \( B_\overline{g}(q, \epsilon(q))\) for some sufficient small \( \epsilon(q) \). Let \( S_r = f'_s e'_r \) with \( f'_s \) holomorphic functions and \(||e'_r||^2 = c_r||\). Let \( e_r = F_r e'_r \) for some nonvanishing holomorphic functions \( F_r \) to be determined later. Then we have \( a_r = ||F_r e'_r||^2_r = |F_r|^2 c_r \). Now fix any point \( p \in B_\overline{g}(q, \epsilon(q)) \). In order for \( a_r \) to satisfy condition (i) with respect to coordinates \((w_1, \ldots, w_n)\) at point \( p \), we choose \( F_r \) such that \( F_r(p) = c_r(p)^{-\frac{1}{2}} \) and

\[
\begin{align*}
\partial_{w_i} F_r(p) &= -c_r^{-1} F_r(p) \partial_{w_i} c_r(p) = -c_r^{-\frac{3}{2}} \partial_{w_i} c_r(p) \\
\partial_{w_i} \partial_{w_j} F_r(p) &= -c_r^{-1} (F_r \partial_{w_i} \partial_{w_j} c_r + \partial_{w_i} c_r \partial_{w_j} F_r + \partial_{w_i} F_r \partial_{w_j} c_r)(p) \\
&= -c_r^{-\frac{3}{2}} \partial_{w_i} \partial_{w_j} c_r(p) + 2c_r^{-\frac{5}{2}} \partial_{w_i} c_r \partial_{w_j} c_r(p).
\end{align*}
\]

Since \( c_r = ||e'_r||^2 \) is nonzero as \( \epsilon(q) \) is small, which implies \(|w - w(p)|\) is small, we can assume \( F_r \neq 0 \) in \( B_\overline{g}(q, \epsilon(q)) \). Now \( S_r = f'_s e'_r = f_r e_r \) with \( f_r = f'_s F_r^{-1} \) holomorphic functions. As \( D_r = \{z_r = 0\} \) are smooth divisors, we can assume that \( \partial_{w_r} f_r(q) \neq 0 \), but \( \partial_{w_r} f_r(q) = 0 \) for \( s \neq r \) among \( 1, \ldots, m \), and choose small \( \epsilon(q) \), we can assume that \( \partial_{w_r} f_r(\cdot) \neq 0 \) but \( \partial_{w_r} f_r(q) \) sufficiently small for \( s \neq r \) among \( 1, \ldots, m \), in \( B_\overline{g}(q, \epsilon(q)) \). Then by the inverse function theorem,

\[
z_1 = f_1(w_1, \ldots, w_n), \ldots, z_m = f_m(w_1, \ldots, w_n), z_{m+1} = w_{m+1}, \ldots, z_n = w_n
\]

are holomorphic coordinates in \( B_\overline{g}(q, \epsilon(q)/2) \) and now \( S_r = f_r w e_r = z_r e_r \). By the chain rule, (i) holds locally in the new coordinates. By covering argument (i) holds globally.

For (ii), we denote by \( w^1, \ldots, w^n \) the coordinates obtained in (i). First we can make a coordinate change such that at the point \( p \), \( \partial / \partial z_{j+m} \) perpendicular to \( \partial / \partial z_j \). Let

\[
\tilde{z}^k = w^k - w^k(p) + \frac{1}{2} b^k_{st} (w^s - w^s(p))(w^t - w^t(p)),
\]

with \( b^k_{st} = b^k_{ts} \), define a new coordinate system. Then we have that

\[
\begin{align*}
\omega_0(\partial / \partial w^1, \partial / \partial w^2) &= \omega_0(\partial / \partial \tilde{z}^1, \partial / \partial \tilde{z}^2) + \tilde{g}^{ij} b^i_{st} (w^s - w^s(p)) + \hat{g} b^i_{sj} (w^s - w^s(p)) \\
&+ O(\sum_{r=1}^n |w^r - w^r(p)|^2),
\end{align*}
\]

\[
d_{ijk} := \left. \frac{\partial}{\partial w^k} \omega_0(\partial / \partial w^1, \partial / \partial w^2) \right|_p = \frac{\partial}{\partial \tilde{z}^k} \omega_0(\partial / \partial \tilde{z}^1, \partial / \partial \tilde{z}^2) + \tilde{g}_{ij}(p) b^i_{jk} =: e_{ijk} + \hat{g}_{ij}(p) b^i_{jk}.
\]

Let \( \tilde{g}_{rs} := \hat{g}_{rs} \), for \( r, s > m \), and denote the inverse of the \((n-m) \times (n-m)\) matrix \([\tilde{g}^{rs}] \) by \([\hat{g}^{rs}] \). Let \( b^r_{ik} = 0 \) for \( r = 1, \ldots, m \). Then for each \( j > m \), the equations can be rewritten as \( d_{ijk} - \sum_{t>m} \hat{g}_{ij}(p) b^i_{tk} = e_{ijk} \). Hence if we define \( b^i_{tj} = \sum_{j>m} \hat{g}^{ij} d_{ijk} \) for each \( t > m \), we will
have that \(e_{ijk} = 0\) for each \(j > m\). Finally, we set \(z^r = \tilde{z}^r + w^r(p)\) for each \(r = 1, \ldots, n\), then these coordinates satisfy conditions (i) and (ii). \(\square\)

Using Lemma 5.1 we can compute the metric tensors and their derivatives as following:

\[
g_{ij} = \hat{g}_{ij} + \sum_{r=1}^{m} \left( \beta_r ||S_r||_r^{2\beta_r} (\log ||S_r||_r^2)_i \right)_j
\]

\[
= \hat{g}_{ij} + \sum_{r=1}^{m} \left( \beta_r \frac{||S_r||_r^{2\beta_r} (||S_r||_r^2)_i}{||S_r||_r^{4-2\beta_r}} - \beta_r ||S_r||_r^{2\beta_r} \Theta_{r,i} \right)_j,
\]

where the last equality comes from Poincare-Lelong equation and \(\Theta_r = -\sqrt{-1} \partial \bar{\partial} \log a_r\) represents the curvature form of the line bundle \([D_r]\). Now for the first order derivatives, we have

\[
g_{ij,k} = \hat{g}_{ij,k} + \sum_{r=1}^{m} \frac{\beta_r^2 (\beta_r - 2) (||S_r||_r^2)_i (||S_r||_r^2)_j (||S_r||_r^2)_k}{||S_r||_r^{2(3-\beta_r)}}
\]

\[
+ \sum_{r=1}^{m} \beta_r \frac{||S_r||_r^{2\beta_r} (||S_r||_r^2)_i (||S_r||_r^2)_j (||S_r||_r^2)_k}{||S_r||_r^{2(2-\beta_r)}}
\]

\[
- \sum_{r=1}^{m} \left( \beta_r \frac{||S_r||_r^{2\beta_r} (||S_r||_r^2)_k}{||S_r||_r^{2(1-\beta_r)}} \Theta_{r,i} + \beta_r ||S_r||_r^{2\beta_r} \Theta_{r,i,j,k} \right),
\]

Note that \(||S_r||_r^2 = a_r |z_r|^2\), using (i) of Lemma 5.1 we can obtain the second order derivatives at \(p\):

\[
g_{ij,kl}(p) = \hat{g}_{ij,kl} + \sum_{r=1}^{m} \frac{\beta_r^2 (\beta_r - 1)^2}{|z_r|^{2(2-\beta_r)}} \delta_{r,i} \delta_{r,j} \delta_{r,k} \delta_{r,l}
\]

\[
+ \sum_{r=1}^{m} \frac{\beta_r^3}{|z_r|^{2(1-\beta_r)}} (a_{r,ij} \delta_{r,k} \delta_{r,l} + a_{r,ik} \delta_{r,j} \delta_{r,l} + a_{r,kj} \delta_{r,i} \delta_{r,l} + a_{r,kl} \delta_{r,i} \delta_{r,j})
\]

\[
+ \sum_{r=1}^{m} \beta_r |z_r|^{2\beta_r} \left( (a_{r,ikl} \delta_{r,j} + a_{r,ijkl} \delta_{r,l}) z_r + (a_{r,ijk} \delta_{r,k} + a_{r,ijkl} \delta_{r,l}) |z_r|^2 \right)
\]

\[
+ \sum_{r=1}^{m} \beta_r |z_r|^{2\beta_r} (a_{r,ij} a_{r,kl} + a_{r,il} a_{r,jk}) |z_r|^2 - \sum_{r=1}^{m} \beta_r \Theta_{r,i,j,kl}.
\]
Meanwhile we can simplify the expressions of metric tensors and first order derivatives at point \( p \):

\[
\begin{align*}
g_{ij}(p) &= \hat{g}_{ij} + \sum_{r=1}^{m} \left( \frac{\beta_r^2 \delta_{ri} \delta_{rj}}{|z_r|^{2(1-\beta_r)}} + \beta_r^2 |z_r|^{2\beta_r} a_{r,ij} \right) \\
g_{ij,k}(p) &= \hat{g}_{ij,k} + \sum_{r=1}^{m} \beta_r^2 (\beta_r - 1) \frac{\varepsilon_r \delta_{ri} \delta_{rj} \delta_{rk}}{|z_r|^{2(1-\beta_r)}} \varepsilon_r \delta_{ri} \delta_{rj} \delta_{rk} \\
&+ \sum_{r=1}^{m} \frac{\beta_r^2 \varepsilon_r}{|z_r|^{2(1-\beta_r)}} (a_{r,ij} \delta_{rk} + a_{r,kj} \delta_{ri}) + \sum_{r=1}^{m} \beta_r^2 |z_r|^{2\beta_r} a_{r,ijk}.
\end{align*}
\]

For \( 1 \leq r, s \leq m, r \neq s \) and \( m + 1 \leq t, t' \leq n \), we can easily have

\[
g^{rt}(p) = O(z_r^{2(1-\beta_r)}), \quad g^{rt'} = O(1). \tag{5.1}
\]

We need the following lemma to give a more precise estimate for \( g^{rs}(p)(1 \leq r, s \leq m) \):

**Lemma 5.2.**

\[
\begin{align*}
g^{rt}(p) &= \frac{\beta_r^{-2} |z_r|^{2(1-\beta_r)}}{1 + c_r(p) \frac{|z_r|^{2(1-\beta_r)}}{|z_s|^{2(1-\beta_s)}}} + O(|z_r|^{4(1-\beta_r)} \sum_{s=1}^{m} (|z_s|^{2(1-\beta_s)} + |z_s|^{2\beta_s})), \tag{5.2} \\
g^{rs}(p) &= \beta_r^{-2} \beta_s^{-2} |z_r|^{2(1-\beta_r)} |z_s|^{2(1-\beta_s)} ((-1)^{r+s} g_{st}^{rs} + o(1)) \quad (r \neq s), \tag{5.3}
\end{align*}
\]

where \( c_r(p) := \frac{\beta_r^{-2} \det(\hat{g}_{ij})_{i,j=r,m+1,\ldots,n}}{\det(\hat{g}_{ij})_{i,j=m+1,\ldots,n}}(p) \) and \( 0 < C_1 < c_r(p) < C_2 \) for all \( p \in M \) and \( r = 1, \ldots, m \).

**Proof.** Actually we only need to prove the result for \( r = 1, s = 2 \). For simplicity we denote

\[
a_{ij} := \hat{g}_{ij} + \sum_{r=1}^{m} \beta^2 |S_r|^{2\beta_r} a_{r,ij}, \quad b_r = \frac{\beta^2_r}{|z_r|^{2(1-\beta_r)}},
\]

then we know that \( g_{ij} = a_{ij} + b_{ij} \delta_{ij} \) where \( i \leq m \). By determinant rule we know that \( g^{ij}(p) = \frac{C_{ij}}{\det g} \) where \( G_{11} \) represents the cofactor of \( g_{11} \). Let us compute \( G_{11} \) and \( \det g \) respectively. We can denote \( A_{r_1,r_2,\ldots,r_k,R} \) as the \((k+n-m)\)-th minor \( \det(a_{ipq})_{r_1,\ldots,r_k,m+1,\ldots,n} \), where \( 1 \leq r_1 < \cdots < r_k \leq m \). Now make use of determinant rules, we can have such
decomposition of $\det g$:

\[
\begin{vmatrix}
    b_1 & 0 & \cdots & 0 \\
    a_{21} & a_{22} + b_2 & \cdots & a_{2n} \\
    a_{n1} & \cdots & a_{n\bar{n}} \\
  \end{vmatrix}
+ \begin{vmatrix}
    a_{1\bar{1}} & a_{1\bar{2}} & \cdots & a_{1n} \\
    a_{2\bar{1}} & a_{2\bar{2}} + b_2 & \cdots & a_{2n} \\
    a_{n\bar{1}} & \cdots & a_{n\bar{n}} \\
  \end{vmatrix}
\]
\[
= \begin{vmatrix}
    b_1 & 0 & \cdots & 0 \\
    0 & b_2 & \cdots & 0 \\
    a_{n1} & \cdots & a_{n\bar{n}} \\
  \end{vmatrix}
+ \begin{vmatrix}
    a_{1\bar{1}} & a_{1\bar{2}} & \cdots & a_{1n} \\
    a_{2\bar{1}} & a_{2\bar{2}} & \cdots & a_{2n} \\
    a_{n\bar{1}} & \cdots & a_{n\bar{n}} \\
  \end{vmatrix}
\]
\[
+ \begin{vmatrix}
    b_1 & 0 & \cdots & 0 \\
    0 & b_2 & \cdots & 0 \\
    a_{n1} & \cdots & a_{n\bar{n}} \\
  \end{vmatrix}
= \cdots + \cdots
\]

\[
\begin{vmatrix}
    b_1 & 0 & \cdots & 0 & \cdots & 0 \\
    0 & b_2 & \cdots & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & b_{m-1} & 0 & \cdots & 0 \\
    a_{m,\bar{1}} & \cdots & a_{m\bar{n}} \\
    a_{n1} & \cdots & a_{n\bar{n}} \\
  \end{vmatrix}
\]
\[
+ \cdots + \begin{vmatrix}
    a_{1\bar{1}} & a_{1\bar{2}} & \cdots & a_{1n} \\
    a_{m,\bar{1}} & \cdots & a_{m\bar{n}} \\
    a_{n\bar{1}} & \cdots & a_{n\bar{n}} \\
  \end{vmatrix}
\]
\[
= b_1 b_2 \cdots b_m A_R + b_1 b_2 \cdots b_m \sum_{r=1}^{m} \frac{A_{r,R}}{b_r} + b_1 b_2 \cdots b_m \sum_{1 \leq r < s}^{m} \frac{A_{r,s,R}}{b_r b_s} + \cdots + \det(a_{ij}).
\]
Similarly, we have that
\[ G_{11} = b_2b_3 \cdots b_m \left( A_R + \sum_{r=2}^{m} \frac{A_{r,R}}{b_r} + \sum_{2 \leq r < s} \frac{A_{r,s,R}}{b_r b_s} \right) + \cdots + \det(a_{ij})_{2 \leq i,j \leq m}, \]
\[ G_{12} = -a_2 b_3 \cdots b_m A_R + \cdots. \]

Now we can have such estimate:
\[ \frac{G_{11}}{\det g} = b_1^{-1} \left[ 1 + \frac{1}{b_1} \sum_{r=2}^{m} \frac{A_{r,R}}{A_R b_r} + \sum_{1 \leq r < s} \frac{A_{r,s,R}}{A_R b_r b_s} + \cdots \right] = \frac{b_1^{-1}}{B}, \]
where
\[ B = (1 + \sum_{r=1}^{m} \frac{A_{r,R}}{A_R b_r} + \sum_{1 \leq r < s} \frac{A_{r,s,R}}{A_R b_r b_s} + \cdots) \left[ 1 - \left( \sum_{r=2}^{m} \frac{A_{r,R}}{A_R b_r} + \sum_{2 \leq r < s} \frac{A_{r,s,R}}{A_R b_r b_s} + \cdots \right) \right] \]
\[ + \left( \sum_{r=2}^{m} \frac{A_{r,R}}{A_R b_r} + \sum_{2 \leq r < s} \frac{A_{r,s,R}}{A_R b_r b_s} + \cdots \right)^2 + \cdots \]
\[ = 1 + \frac{A_{1,R}}{A_R b_1} + \frac{1}{A_R b_1} \sum_{r=2}^{m} \frac{1}{b_r} (A_{1,r,R} - \frac{A_{1,R} A_{r,R}}{A_R}) + O(\frac{1}{b_1} \sum_{r=2}^{m} \frac{1}{b_r^2}). \]

Note that
\[ a_{ij} := \hat{g}_{ij}(p) + \sum_{s=1}^{m} \beta_s^2 |z_s|^{2\beta_s} a_{s,ij} = \hat{g}_{ij}(p) + O(\sum_{s=1}^{m} |z_s|^{2\beta_s}), \]
(5.2) follows. (5.3) follows from the computation of $G_{12}$ more easily.

Take two unit vectors $\eta = \eta^i \frac{\partial}{\partial x_i}, \nu = \nu^i \frac{\partial}{\partial x_i} \in T_p^{\perp} M$, so that $g(\eta, \eta)|_p = g(\nu, \nu)|_p = 1$. Then from the expression of $g_{ij}$ we have
\[ \eta^r, \nu^r = O(|z_r|^{1-\beta_r}), \eta^t, \nu^t = O(1) \text{ for } r = 1, \cdots, m, \ t = m + 1, \cdots, n. \]
(5.4)

By the definition of bisectional curvature, we set
\[ \text{Bisec}_\omega(\eta, \nu) = R(\eta, \nu, \nu, \nu) = R_{ijkl} \eta^i \eta^j \nu^k \nu^l = \sum_{i,j,k,l} (\Lambda_{ijkl} + \Pi_{ijkl}), \]
with $\Lambda_{ijkl} := -g_{ij,kl} \eta^i \eta^j \nu^k \nu^l$, and $\Pi_{ijkl} := g^{pq} g_{iq,kl} g_{pj,ij} \eta^i \eta^j \nu^k \nu^l$ (no summations on $i, j, k, l$).

By (5.1)–(5.4) we have $|\Lambda_{ijkl}| \leq C$ except for the terms $\sum_{r=1}^{m} \Lambda_{rrrr}$, hence
\[ \sum_{i,j,k,l} \Lambda_{ijkl}(p) = O(1) - \sum_{r=1}^{m} \frac{\beta_r^2 (\beta_r - 1)^2}{|z_r|^{2(2-\beta_r)}} |\eta^r|^2 |\nu^r|^2. \]
(5.5)

Now we can deal with the first case in Theorem 1.2. For this case we have the following lemma:
Lemma 5.3. In case that either no three irreducible divisors intersect or all angles $\beta_i \leq \frac{1}{2}$, there exists a uniform constant $C > 0$ such that for every $p \in M$,

$$\sum_{i,j,k,l} \Pi_{ijkl}(p) \leq C + \sum_{r=1}^{m} \frac{\beta_r^2 (\beta_r - 1)}{|z_r|^{2(2-\beta_r)}} |\eta^r|^2 |\nu^r|^2.$$  \hspace{1cm} (5.6)

Proof. By Brendle’s computation in [4], we can easily bound all the terms if $\beta_i \leq \frac{1}{2}$ for all $i$. Now we consider the general case. As lemma A.3 in [18], we define a bilinear Hermitian form of two tensors $a = [a_{i\ell k}], b = [b_{j\mu l}] \in (\mathbb{C}^n)^3$ satisfying $a_{i\ell k} = a_{k\ell i}$, $b_{j\mu l} = b_{l\mu j}$ by setting

$$\langle [a_{i\ell k}], [b_{j\mu l}] \rangle := \sum_{i,j,k,l,p,q} g^{pq}(\eta^i g_{i\ell k} \nu^k)(\eta^p g_{p\mu l} \nu^l).$$

Obviously it is a nonnegative bilinear form. We denote by $|| \cdot ||$ the associated norm. Then $\sum_{i,j,k,l} \Pi_{ijkl} = ||[a_{i\ell k}]||^2$. We write

$$g_{i\ell,j,k} = A_{i\ell k} + B_{i\ell k} + D_{i\ell k} + E_{i\ell k}$$

with

$$A_{i\ell k} := \hat{g}_{i\ell,k}, \quad B_{i\ell k} := \sum_{r=1}^{m} \frac{\beta_r^2 |z_r|^{2\beta_r} a_{r,i\ell k}}{|z_r|^{2(1-\beta_r)}},$$

$$D_{i\ell k} := \sum_{r=1}^{m} \frac{\beta_r^2 |z_r|^{2(1-\beta_r)}}{|z_r|^{2(2-\beta_r)}} \left( a_{r,i\ell} \delta_{rk} + a_{r,k\ell} \delta_{ri} \right),$$

$$E_{i\ell k} := \sum_{r=1}^{m} \frac{\beta_r^2 (\beta_r - 1)}{|z_r|^{2(2-\beta_r)}} \bar{z}_r \delta_{ri} \delta_{j\ell} \delta_{rk}.$$  

Denote $A := [A_{i\ell k}]$ and similarly $B, D, E$. Using (5.1), we can bound $||A + B + D||^2$ easily. For the crossing terms of $A, B, D$ and $E$, we have that

$$2Re(A, E) = 2Re \left( \sum_{i,j,k,r} g^{rj} \hat{g}_{i\ell,k} \overline{E_{r\delta r}} \right)$$

$$\leq C \sum_{i,j,k} |\hat{g}_{i\ell,j,k}|^2 + \delta \sum_{r} |z_r|^{2(1-\beta_r)}||E_{r\delta r}||^2 \leq C + \delta \sum_{r} |z_r|^{4(1-\beta_r)} \hat{g}_{r\delta r}(p) \beta_r^{-2} |E_{r\delta r}|^2,$$

where $\delta$ can be chosen small enough. By the same argument, we also have that

$$2Re(B + D, E) \leq C + \delta \sum_{r} |z_r|^{4(1-\beta_r)} \hat{g}_{r\delta r}(p) \beta_r^{-2} |E_{r\delta r}|^2.$$  

Now let us consider $||E||^2$ : In case that at most two divisors intersect transversely, we can take $m = 2$. As there exists a uniform constant $c_0 < 1$ such that $|\hat{g}_{12}|^2(p) \leq c_0^2 \hat{g}_{11}(p) \hat{g}_{22}(p)$,
By (5.2) and (5.3), we have that

\[
||E||^2(p) = \sum_{r=1}^{2} g^{rr} |E_{rr}|^2 + \sum_{r\neq s}^{2} g^{rs} E_{rr} E_{ss}
\]

\[
\leq \sum_{r=1}^{2} \frac{\beta^{-2} |z_r|^{2(1-\beta)}}{1 + c_r(p) |z_r|^{2(1-\beta)}} |E_{rr}|^2 + \sum_{r=1}^{2} c_0 \hat{g}_{rr}(p) \beta^{-4} |z_r|^{4(1-\beta)} |E_{rr}|^2
\]

\[
= \sum_{r=1}^{2} \beta^{-2} |z_r|^{2(1-\beta)} \left( \frac{1 + c_0 \hat{g}_{rr}(p) \beta^{-2} |z_r|^{2(1-\beta)}}{1 + c_r(p) |z_r|^{2(1-\beta)}} + o(1) \right) |E_{rr}|^2.
\]

Add these estimates together when \( m = 2 \) or all cone angles \( \beta_r \in (0, \frac{1}{2}) \), we obtain that

\[
\sum_{i,j,k,l} \Pi_{ijkl}(p) \leq C + \sum_{r=1}^{m} \beta^{-2} |z_r|^{2(1-\beta)} \left( \frac{1 + (c_0 + 2\delta) \hat{g}_{rr}(p) \beta^{-2} |z_r|^{2(1-\beta)}}{1 + c_r(p) |z_r|^{2(1-\beta)}} + o(1) \right) |E_{rr}|^2
\]

As in our coordinate system \( c_r(p) = \beta^{-2} \frac{4 \delta \beta}{\lambda p} = \beta^{-2} \hat{g}_{rr}(p) \), by choosing \( \delta > 0 \) such that \( 2\delta + c_0 < 1 \), we obtain the lemma.

For triple or higher multiple singularities, i.e. \( m > 2 \), generally we do not have that the inequality

\[
\begin{bmatrix}
0 & -\hat{g}_{12} & \cdots & (-1)^{m+1} \hat{g}_{1m} \\
-\hat{g}_{21} & 0 & \cdots & (-1)^{m+2} \hat{g}_{2m} \\
& \ddots & \ddots & \vdots \\
(-1)^{m+1} \hat{g}_{m1} & \cdots & 0 & \hat{g}_{mm}
\end{bmatrix}
\leq
\begin{bmatrix}
\hat{g}_{m\bar{m}} & 0 & \cdots & 0 \\
0 & \hat{g}_{m\bar{m}} & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{g}_{m\bar{m}}
\end{bmatrix}
\]

That is why we cannot control the crossing terms in \( ||E||^2 \) by its diagonal terms so well as \( m = 2 \). Fortunately this observation implies that we can increase the diagonal terms of \( \hat{g} \) such that this matrix inequality holds for the modified metric. In applications to geometric problems we only change the metric in the same cohomology class. One natural idea is to consider the new background metric with the following form

\[
\omega' = \omega_0 + \sqrt{-1} \partial \bar{\partial} \sum_{i=1}^{m} \varphi_r(\|S_r\|^2),
\]

where \( \varphi_r(\|S_r\|^2) \) behaves as \( \lambda_r \|S_r\|^2 \) for some suitable \( \lambda_r > 0 \) near \( D_r = (S_r = 0) \) and tends to 0 far away such that it preserves the positivity of the new background metric. This can be done by suitable cutoff argument. To see how much this modification changes the bisectional curvature we first compute the corresponding derivatives of the new background metric tensors (we denote \( \hat{g}' \) as the metric tensors of \( \omega'_0 \)) as in the previous
section:
\[
\hat{g}'_{ij} = \hat{g}_{ij} + \sum_{r=1}^{m} \varphi'_{r}(||S_r||_{p}^2) (a_{r,i} \delta_{jr} + a_{r,j} \delta_{ir} z_r + a_{r,ij} |z_r|^2) \\
+ \sum_{r=1}^{m} \varphi''_{r}(||S_r||_{p}^2) (a_{r,i} \delta_{jr} \hat{z}_r + a_{r,j} \delta_{ir} \hat{z}_r + a_{r,ij} |\hat{z}_r|^2).
\]
At the chosen point \(p\), by our assumption that \(\varphi_{r}(t) = \lambda_{r} t\) when \(t\) is small and the chosen coordinate system, we get that
\[
\hat{g}'_{ij}(p) = \hat{g}_{ij}(p) + \sum_{r=1}^{m} \lambda_{r} (\delta_{ir} \delta_{jr} + O(F_{r})),
\]
Meanwhile by the order of the error terms we can easily construct cut-off functions so that after modification the new metric is uniformly equivalent to the original metric. And similarly, we can also get that
\[
\hat{g}'_{ij,k}(p) = \hat{g}_{ij,k}(p) + O(1), \quad \hat{g}'_{ij,kl}(p) = \hat{g}_{ij,kl}(p) + O(1).
\]
Using these estimates in the computation of the previous section, we find that almost any estimates do not change except for the estimate of \(||E||^2\), due to the change of the background metric tensors. Let us rewrite this formula for the modified metric:
\[
||E||^2(p) = \sum_{r=1}^{m} g^{rr} |E_{rrr}|^2 + \sum_{r \neq s}^{m} g^{rs} E_{rr} \overline{E_{rss}}
\leq \sum_{r=1}^{m} \beta_{r}^{-2} |z_r|^{2(1-\beta_{r})} |E_{rrr}|^2
+ \beta_{r}^{-2} \beta_{s}^{-2} |z_{r}|^{2(1-\beta_{r})} |z_{s}|^{2(1-\beta_{s})} (-1)^{r+s} (g'_{s\hat{s}} + o(1)) E_{rrr} \overline{E_{rss}}.
\]
As in the lemma [5,3] we need to control the second term by the diagonal terms. Now we could choose \(\lambda_{r}\) large enough, \(r = 1, \cdots, m\) so that for some constant \(c'_{0} < 1\) the following inequality holds for all points lying in some tubular neighborhood of the simple normal crossing divisor \(D\):
\[
\begin{bmatrix}
0 & -\hat{g}'_{12} & \cdots & (-1)^{m+1} \hat{g}'_{1m} \\
-\hat{g}'_{21} & 0 & \cdots & (-1)^{m+2} \hat{g}'_{2m} \\
& \ddots & \ddots & \ddots \\
& & (-1)^{m+1} \hat{g}'_{m1} & \cdots & 0
\end{bmatrix}
< c'_{0}
\begin{bmatrix}
\hat{g}'_{11} & 0 & \cdots & 0 \\
0 & \hat{g}'_{22} & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \hat{g}'_{mm}
\end{bmatrix}
\]
Then the corresponding estimate will follow as above and the following proposition holds:

**Proposition 5.4.** In general situations (triple or higher multiple singularities with arbitrary cone angles), there exist a smooth function \(\varphi_{0} = \sum_{i=1}^{m} \varphi_{r}(||S_{r}||_{p}^2)\) where \(\varphi_{r}(t) = \lambda_{r} t\) for some enough large constants \(\lambda_{r}, r = 1, \cdots, m\) near 0 and vanish when \(t\) is larger, such
that there exists a uniform constant $C > 0$ such that for every $p \in M$, and new metric $\omega'_c = \omega_c + \sqrt{-1}\partial\bar{\partial}\varphi$,

$$\sum_{i,j,k,l} \Pi_{ijkl}(p) \leq C + \sum_{r=1}^{m} \frac{\beta_r^2(\beta_r - 1)^2}{|z_r|^{2(2-\beta_r)}} |\eta_r|^2 |\nu_r|^2.$$  \hspace{1cm} (5.8)

Combine Lemma 5.3 and this proposition, the proof of Theorem 1.2 is finished.

6. Further discussions

In Theorem 1.1 we assume that $M$ does not have any holomorphic vector fields, which guarantees the properness of the smooth Ding functional or Mabuchi functional by [33]. However, in case of simple normal crossing divisor with $m \geq 2$, there is no holomorphic vector fields tangential to all the components of the divisor. In this sense, the assumption of nonexistence of holomorphic vector fields is not necessary in the following theorem which is modified from Theorem 1.1:

\textbf{Theorem 6.1.} Given a Fano manifold $(M, \omega_0)$ where $[\omega_0] = c_1(M)$, for any simple normal crossing divisor $D = \sum_{r=1}^{m}$ and a sequence of positive rational numbers $\lambda_1, \cdots, \lambda_m$ satisfying

$$\sum_{r=1}^{m} c_r [D_r] = c_1(M), \hspace{1cm} (6.1)$$

if for all $r = 1, \cdots, m$ it holds that $c_r \leq 1$ and

$$\lambda_r := \inf \{ \lambda > 0 | \lambda K^{-1}_M - [D_r] > 0 \} \geq \frac{n}{n+1}$$

in case that $c_r = 1$, then for $\mu > 0$ small enough, and $\beta_1, \cdots, \beta_m \in (0, 1)$ satisfies

$$1 - \beta_r = (1 - \mu)c_r,$$

there exists a unique $C^{2,\alpha}\beta$ conical Kähler-Einstein metric in $c_1(M)$ with cone angle $2\pi \beta_r$ along each irreducible divisor $D_r$ for $r = 1, \cdots, m$. Moreover in case that $M$ admits a conical Kähler-Einstein metric $\omega_{\mu_0} \in c_1(M)$ such that

$$\text{Ric}(\omega_{\mu_0}) = \mu_0 \omega_{\mu_0} + \sum_{r=1}^{m} 2\pi (1 - \beta_{r,0})[D_r]$$

then for $\mu \in (0, \mu_0)$, the above existence result still hold.

Note that the properness of the twisted Ding functional corresponding to $\mu_0$ could be derived from a modification of [38] and we want to thank Professor X. H. Zhu for pointing out this to us.

Another interesting problem is the requirement of the coefficients $c_r, \lambda_r$. In one smooth divisor case, by [20] [30] such similar requirement is necessary due to the obstruction of log-K stability and the example of $\mathbb{P}^2$ with a quadratic curve shows this point. But in the case of simple normal crossing divisor with $m \geq 2$, such obstruction will not exist.
due to the nonexistence of holomorphic vector fields tangential all the components and we do not know whether our theorem could be extended or not due to another unknown obstruction. We hope to investigate and analyze more general cases in the future.

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