ISOMORPHIC CLASSIFICATION OF $\ast$-ALGEBRAS OF LOG-INTEGRABLE FUNCTIONS

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Abstract. Using the notion of passport of a normed Boolean algebra, necessary and sufficient conditions for a $\ast$-isomorphism of $\ast$-algebras of log-integrable measurable functions are found.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure a measure space, and let $\mathcal{L}_{\text{log}}(\Omega, \mathcal{A}, \mu)$ be the symmetric function space consisting of complex valued measurable functions $f$ on $(\Omega, \mathcal{A}, \mu)$ such that $\int_{\Omega} \log(1 + |f|) \, d\nu < \infty$. It is known that $\mathcal{L}_{\text{log}}(\Omega, \mathcal{A}, \mu)$ is a $\ast$-subalgebra in the $\ast$-algebra $L^0(\Omega, \mathcal{A}, \mu)$ of complex valued measurable functions on $(\Omega, \mathcal{A}, \mu)$ (functions equal $\mu$-almost everywhere are identified) [3]. In addition, the space $\mathcal{L}_{\text{log}}(\Omega, \mathcal{A}, \mu)$ is a non-locally-convex Hausdorff topological $\ast$-algebra with respect to the $F$-norm

$$\|f\|_{\log} = \int_{\Omega} \log(1 + |f|) \, d\mu.$$ 

In the case where the measure space $(\Omega, \mathcal{A}, \mu)$ is the unit circle in the complex plane endowed with Lebesgue measure, the boundary values of Nevanlinna functions belong to the symmetric function space $\mathcal{L}_{\text{log}}(\Omega, \mathcal{A}, \mu)$, and the map assigning a Nevanlinna function its boundary values yields an injective and continuous algebraic homomorphism from the Nevanlinna class to $\mathcal{L}_{\text{log}}(\Omega, \mathcal{A}, \mu)$. Since the Nevanlinna class is not well behaved under the usual metric, it is natural to study its topological properties with respect to the $F$-norm $\| \cdot \|_{\log}$ (see [3]).

Taking into account various applications of $\ast$-algebras $\mathcal{L}_{\text{log}}(\Omega, \mathcal{A}, \mu)$ in the theory of functions of a complex variable, it is important to describe these algebras associated with different measures up to $\ast$-isomorphisms. This paper is devoted to solving this problem. Utilizing the notion of passport of a normed Boolean algebra, we give necessary and sufficient conditions for a $\ast$-isomorphism of $\ast$-algebras of log-integrable measurable functions. The proof uses the method of papers [1], [2], which give a description of the $\ast$-isomorphisms of Arens algebras of measurable functions.

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2. Preliminaries

Let \((\Omega, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space, and let \(L_0(\Omega, \mathcal{A}, \mu)\) be the \(*\)-algebra of complex valued measurable functions on \((\Omega, \mathcal{A}, \mu)\) (functions equal \(\mu\)-almost everywhere are identified). Following \[3\], we consider the \(*\)-subalgebra 
\[ L_{\log}(\Omega, \mathcal{A}, \mu) = \{ f \in L_0(\Omega, \mathcal{A}, \mu) : \int_{\Omega} \log(1 + |f|)d\mu < +\infty \} \]

of the \(*\)-algebra \(L_0(\Omega, \mathcal{A}, \mu)\). Since for any \(a, b > 0, \ a \neq 1, \ b \neq 1\), there are constants \(0 < d_1 < d_2\) such that 
\[ d_1 \log a c \leq \log b c \leq d_2 \log a c \text{ for all } c > 0, \]

it follows that the definition of \(L_{\log}(\Omega, \mathcal{A}, \mu)\) does not depend on the choice of base of the logarithm.

For each \(f \in L_{\log}(\Omega, \mathcal{A}, \mu)\) we put 
\[ \|f\|_{\log} = \int_{\Omega} \log(1 + |f|)d\mu. \]

According to \[3, \text{Lemma 2.1}\], the function 
\[ \| \cdot \|_{\log} : L_{\log}(\Omega, \mathcal{A}, \mu) \to [0, \infty) \]

is an \(F\)-norm, that is, given \(f, g \in L_{\log}(\Omega, \mathcal{A}, \mu)\),

(i) \(\|f\|_{\log} > 0\) if \(f \neq 0\);
(ii) \(\|\alpha f\|_{\log} \leq \|f\|_{\log}\) if \(|\alpha| \leq 1\);
(iii) \(\lim_{\alpha \to 0} \|\alpha f\|_{\log} = 0\);
(iv) \(\|f + g\|_{\log} \leq \|f\|_{\log} + \|g\|_{\log}\).

Besides, \(L_{\log}(\Omega, \mathcal{A}, \mu)\) is a complete topological \(*\)-algebra with respect to the topology generated by the metric \(\rho(f, g) = \|f - g\|_{\log}\); see \[3, \text{Corollary 2.7}\].

Let \(\mu\) and \(\nu\) be \(\sigma\)-finite measures on a measure space \((\Omega, \mathcal{A})\) such that \(\mu \sim \nu\), that is,
\[ \mu(A) = 0 \iff \nu(A) = 0, \ A \in \mathcal{A}. \]

In this case,
\[ L_0(\Omega, \mathcal{A}, \mu) = L_0(\Omega, \mathcal{A}, \nu) := L_0(\Omega), \ L_\infty(\Omega, \mathcal{A}, \mu) = L_\infty(\Omega, \mathcal{A}, \nu) := L_\infty(\Omega). \]

We denote by \(\frac{d\nu}{d\mu}\) the Radon-Nikodym derivative of the measure \(\nu\) with respect to the measure \(\mu\). It is known that \(\frac{d\nu}{d\mu} \in L_0(\Omega, \mathcal{A}, \mu)\) and 
\[ f \in L_1(\Omega, \mathcal{A}, \nu) \iff f \cdot \frac{d\nu}{d\mu} \in L_1(\Omega, \mathcal{A}, \mu); \]
in addition,
\[ \int_{\Omega} f d\nu = \int_{\Omega} (f \cdot \frac{d\nu}{d\mu})d\mu. \]

Since \(\mu \sim \nu\), it follows that there exists an inverse function \((\frac{d\nu}{d\mu})^{-1} = \frac{d\mu}{d\nu}\).

**Proposition 2.1.** The following conditions are equivalent:

(i) \(\frac{d\nu}{d\mu} \in L_\infty(\Omega)\);
(ii) \(L_{\log}(\Omega, \mathcal{A}, \mu) \subset L_{\log}(\Omega, \mathcal{A}, \nu)\).

**Proof.** (i) \(\Rightarrow\) (ii): If \(\frac{d\nu}{d\mu} \in L_\infty(\Omega)\) and \(f \in L_{\log}(\Omega, \mathcal{A}, \mu)\), then 
\[ \int_{\Omega} \log(1 + |f|)d\nu = \int_{\Omega} \log(1 + |f|) \frac{d\nu}{d\mu}d\mu \]
\[ \leq \|\frac{d\nu}{d\mu}\|_\infty \int_{\Omega} \log(1 + |f|)d\mu < \infty. \]
Consequently, \( f \in \mathcal{L}_\log(\Omega, \mathcal{A}, \nu) \), hence \( \mathcal{L}_\log(\Omega, \mathcal{A}, \mu) \subset \mathcal{L}_\log(\Omega, \mathcal{A}, \nu) \).

\((ii) \Rightarrow (i)\): Let \( \mathcal{L}_\log(\Omega, \mathcal{A}, \mu) \subset \mathcal{L}_\log(\Omega, \mathcal{A}, \nu) \) and suppose that \( \frac{d\nu}{d\mu} \notin \mathcal{L}_\infty(\Omega) \). In this case there exists a sequence of positive integers \( n_k \uparrow \infty \) such that \( \mu(A_{n_k}) > 0 \), where \( A_{n_k} = \{ \omega \in \Omega : n_k \leq \frac{d\mu}{d\nu}(\omega) < n_k + 1 \} \), for all natural \( k \). Consider a step measure function \( g = \sum_{k=1}^{\infty} \frac{1}{\mu(A_{n_k})} \chi_{A_{n_k}}, \) where \( \chi_A \) is the characteristic function of a set \( A \in \mathcal{A} \), that is, \( \chi_A(\omega) = 1 \) for \( \omega \in A \) and \( \chi_A(\omega) = 0 \), if \( \omega \notin A \). Putting \( f = e^g - 1 \), we obtain that

\[
\int_{\Omega} \ln(1 + |f|)d\mu = \sum_{k=1}^{\infty} \frac{\mu(E_{n_k})}{k^2 \mu(E_{n_k})} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,
\]

At the same time,

\[
\int_{\Omega} \log(1 + |f|)d\nu = \int_{\Omega} \log(1 + |f|) \frac{d\nu}{d\mu} d\mu \geq \sum_{k=1}^{\infty} \frac{n_k \cdot \mu(E_{n_k})}{k^2 \mu(E_{n_k})} = \sum_{k=1}^{\infty} \frac{n_k}{k^2} \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty.
\]

It follows from (1) and (2) that \( f \in \mathcal{L}_\log(\Omega, \mathcal{A}, \mu) \) and \( f \notin \mathcal{L}_\log(\Omega, \mathcal{A}, \mu) \), that is, \( \mathcal{L}_\log(\Omega, \mathcal{A}, \mu) \) is not a subset of \( \mathcal{L}_\log(\Omega, \nu) \) which contradicts the assumption. Consequently, \( \frac{d\nu}{d\mu} \notin \mathcal{L}_\infty(\Omega) \).

Directly from Proposition 2.1 we obtain the following.

**Corollary 2.1.** The following conditions are equivalent:

(i) \( \frac{d\nu}{d\mu} \in \mathcal{L}_\infty(\Omega) \) and \( \frac{d\nu}{d\mu} \in \mathcal{L}_\infty(\Omega) \);

(ii) \( \mathcal{L}_\log(\Omega, \mathcal{A}, \mu) \subset \mathcal{L}_\log(\Omega, \mathcal{A}, \nu) \).

Let \( \nabla_\mu \) be the complete Boolean algebra of equivalence classes \( e = [A] \) of \( \mu \)-almost everywhere equal sets. It is known that \( \hat{\mu}(e) = \mu(A) \) is a strictly positive countably additive \( \sigma \)-finite measure on \( \nabla_\mu \). Since \( \mu \sim \nu \), it follows that \( \nabla_\mu = \nabla_\nu := \nabla \). In what follows, the measure \( \hat{\mu} \) will be denoted by \( \mu \), and \( \mathcal{L}_0(\Omega) (\mathcal{L}_\infty(\Omega)) \) by \( \mathcal{L}_0(\nabla) \) (respectively, \( \mathcal{L}_\infty(\nabla) \)).

Let \( \varphi : \nabla \rightarrow \nabla \) be an arbitrary automorphism of the Boolean algebra \( \nabla \). It is clear that \( \lambda(e) = \mu(\varphi(e)) \), \( e \in \nabla_\mu \), is a strictly positive countably additive \( \sigma \)-finite measure on the Boolean algebra \( \nabla \). Denote by \( \Phi \) a \( * \)-isomorphism of the \( * \)-algebra \( \mathcal{L}_0(\nabla) \) such that \( \varphi(e) = \Phi(e) \) for all \( e \in \nabla \). The restriction of \( \Phi \) on the \( C^* \)-algebra \( \mathcal{L}_\infty(\nabla) \) is a \( * \)-isomorphism of \( \mathcal{L}_\infty(\nabla) \).

Since

\[
\int_{\Omega} \sum_{i=1}^{n} c_i e_i d\lambda = \sum_{k=i}^{n} c_i \lambda(e_i) = \sum_{k=i}^{n} c_i \mu(\varphi(e_i)) = \int_{\Omega} \Phi(\sum_{i=1}^{n} c_i e_i) \, d\mu
\]

for all \( e_i \in \nabla_\mu, \mu(e_i) < \infty, \, e_i e_j = 0, \, i \neq j, \, i, j = 1, ..., n \), it follows that

\[
\int_{\Omega} f d\lambda = \int_{\Omega} \Phi(f) d\mu, \quad \int_{\Omega} \Phi^{-1}(g) d\lambda = \int_{\Omega} g d\mu
\]

for all \( f \in \mathcal{L}_1(\nabla, \lambda) \) and \( g \in \mathcal{L}_1(\nabla, \mu) \). This means that \( \Phi(\mathcal{L}_1(\nabla, \lambda)) = \mathcal{L}_1(\nabla, \mu) \).

Let \( \mu(\nu) \) be a strong positive countably additive \( \sigma \)-finite measure on a complete Boolean algebra \( \nabla_1 \) (respectively, \( \nabla_2 \)), let \( \varphi : \nabla_1 \rightarrow \nabla_2 \) be an isomorphism, and let \( \Phi : \mathcal{L}_0(\nabla_1) \rightarrow \mathcal{L}_0(\nabla_2) \) be a \( * \)-isomorphism such that \( \varphi(e) = \Phi(e) \) for all \( e \in \nabla_1 \). It
is clear that \( \lambda(\varphi(e)) = \mu(e) \), \( e \in \nabla \), is a strong positive countably additive \( \sigma \)-finite measure on the Boolean algebra \( \nabla_2 \).

**Proposition 2.2.** \( \Phi(\log(1 + |f|)) = \log(1 + \Phi(|f|)) \) for all \( f \in \mathcal{L}_0(\nabla_1) \) and \( \Phi((\mathcal{L}_\log(\nabla_1, \mu)) = \mathcal{L}_\log(\nabla_2, \lambda) \).

**Proof.** The restriction \( \Psi \) of the \( * \)-isomorphism \( \Phi \) on the \( C^* \)-algebra \( \mathcal{L}_\infty(\nabla_1) \) is a \( * \)-isomorphism from \( C^* \)-algebra \( \mathcal{L}_\infty(\nabla_1) \) onto \( C^* \)-algebra \( \mathcal{L}_\infty(\nabla_2) \). Then we have

\[
\Psi(u \circ |g|) = u \circ \Psi(|g|)
\]

for any \( g \in \mathcal{L}_\infty(\nabla_1) \) and every continuous function \( u : [0, +\infty) \rightarrow \mathbb{R} \). The function \( u(t) = \log(1 + t) \) is continuous on the interval \([0, +\infty)\). Therefore

\[
\Phi(\log(1 + |g|)) = \Psi(\log(1 + |g|)) = \log(1 + \Phi(|g|))
\]

for all \( g \in \mathcal{L}_\infty(\nabla_1) \). If \( f \in \mathcal{L}_0(\nabla_1) \), then setting \( g_n = |f| : \chi_{\{|f| \leq n\}} \), we obtain \( g_n \in \mathcal{L}_\infty(\nabla_1) \), \( n \in \mathbb{N}, 0 \leq g_n \uparrow |f| \) and \( \log(1 + g_n) \uparrow \log(1 + |f|) \). Since the isomorphism \( \Phi : \mathcal{L}_0(\nabla_1) \rightarrow \mathcal{L}_0(\nabla_2) \) is order preserving, we have

\[
\Phi(g_n) \uparrow \Phi(|f|), \quad \log(1 + \Phi(g_n)) \uparrow \log(1 + \Phi(|f|)),
\]

and

\[
\log(1 + \Phi(g_n)) = \Phi(\log(1 + g_n)) \uparrow \Phi(\log(1 + |f|)).
\]

hence

\[
\Phi(\log(1 + |f|)) = \log(1 + \Phi(|f|))
\]

for all \( f \in \mathcal{L}_0(\nabla_1) \).

By the definition of the \( * \)-algebra \( \mathcal{L}_\log(\nabla_1, \mu) \),

\[
f \in \mathcal{L}_\log(\nabla_1, \mu) \Leftrightarrow (f \in \mathcal{L}_0(\nabla_1) \text{ and } \log(1 + |f|) \in \mathcal{L}_1(\nabla_1, \mu)).
\]

Consequently, in view of

\[
\Phi(\mathcal{L}_1(\nabla_1, \mu)) = \mathcal{L}_1(\nabla_2, \lambda)
\]

and

\[
\Phi(\log(1 + |f|)) = \log(1 + \Phi(|f|)), \quad f \in \mathcal{L}_0(\nabla_1),
\]

we conclude that

\[
\log(1 + |\Phi(f)|) = \log(1 + \Phi(|f|)) \in \mathcal{L}_\log(\nabla_2, \lambda).
\]

Therefore \( \Phi(f) \in \mathcal{L}_\log(\nabla_2, \lambda) \) for all \( f \in \mathcal{L}_\log(\nabla_1, \mu) \). Similarly, using the inverse \( * \)-isomorphism \( \Phi^{-1} \), we see that \( \Phi^{-1}(h) \in \mathcal{L}_\log(\nabla_1, \mu) \) for all \( h \in \mathcal{L}_\log(\nabla_2, \lambda) \). Therefore \( \Phi(\mathcal{L}_\log(\nabla_1, \mu)) = \mathcal{L}_\log(\nabla_2, \lambda) \).

Let \( \mu (\nu) \) be a strong positive countably additive \( \sigma \)-finite measure on a complete Boolean algebra \( \nabla_1 \) (respectively, \( \nabla_2 \)). The measures \( \mu \) and \( \nu \) are called log-equivalent if there exists an isomorphism \( \varphi : \nabla_1 \rightarrow \nabla_2 \) such that \( \mathcal{L}_\log(\nabla_2, \nu) = \mathcal{L}_\log(\nabla_2, \mu \circ \varphi) \), where \( \mu \circ \varphi \) is a strictly positive countably additive \( \sigma \)-finite measure on the Boolean algebra \( \nabla_2 \). Therefore, by virtue of Corollary 2.1 condition

\[
\mathcal{L}_\log(\nabla, \nu) = \mathcal{L}_\log(\nabla, \mu \circ \varphi)
\]

is equivalent to the system \( \frac{d\lambda}{dx} \in \mathcal{L}_\infty(\nabla_2) \) and \( \frac{d\mu}{d\varphi} \in \mathcal{L}_\infty(\nabla_2) \).

**Theorem 2.1.** Let \( \mu (\nu) \) be a strong positive countably additive \( \sigma \)-finite measure on a complete Boolean algebra \( \nabla_1 \) (respectively, \( \nabla_2 \)). Then the algebras \( \mathcal{L}_\log(\nabla_1, \mu) \) and \( \mathcal{L}_\log(\nabla_2, \nu) \) are \( * \)-isomorphic if and only if the measures \( \mu \) and \( \nu \) are log-equivalent.
Proof. Let measures $\mu$ and $\nu$ be log-equivalent, that is, there exists an isomorphism $\varphi : \nabla_1 \to \nabla_2$ such that $L_{\log}(\nabla_2, \nu) = L_{\log}(\nabla_2, \mu \circ \varphi)$. Let $\Phi : L_0(\nabla_1) \to L_0(\nabla_2)$ be a $*$-isomorphism for which $\varphi(e) = \Phi(e)$ for all $e \in \nabla$. By Proposition 2.2 we have

$$\Phi(L_{\log}(\nabla_1, \mu)) = (L_{\log}(\nabla_2, \mu \circ \varphi)) = L_{\log}(\nabla_2, \nu).$$

This means that the algebras $L_{\log}(\nabla_1, \mu)$ and $L_{\log}(\nabla_2, \nu)$ are $*$-isomorphic.

Conversely, suppose that the algebras $L_{\log}(\nabla_1, \mu)$ and $L_{\log}(\nabla_2, \nu)$ are $*$-isomorphic, that is, there exists a $*$-isomorphism

$$\Psi : L_{\log}(\nabla_1, \mu) \to L_{\log}(\nabla_2, \nu).$$

Let $\{e_n\}_{n=1}^\infty$ be a partition of a unity of the Boolean algebra $\nabla_1$ such that

$$\mu(e_n) < \infty \quad \text{and} \quad \nu(\Psi(e_n)) < \infty \quad \text{for all} \quad n.$$ 

It is clear that $\Psi : L_{\log}(e_n \nabla_1, \mu) \to L_{\log}(\Psi(e_n) \nabla_2, \nu)$ is a $*$-isomorphism. Since $e_n \nabla_1 \subset L_{\log}(e_n \nabla_1, \mu)$ and $\Psi(e_n) \nabla_2 \subset L_{\log}(\Psi(e_n) \nabla_2, \nu)$, it follows that the restriction $\varphi_n$ of that $*$-isomorphism $\Psi : L_{\log}(e_n \nabla_1, \mu) \to L_{\log}(\Psi(e_n) \nabla_2, \nu)$ on the Boolean algebra $e_n \nabla_1$ is an isomorphism from $e_n \nabla_1$ onto $\Psi(e_n) \nabla_2$. Define the map $\varphi : \nabla_1 \to \nabla_2$ by the formula

$$\varphi(e) = \sup_{n \geq 1} \varphi_n(e \cdot e_n), \quad e \in \nabla_1.$$ 

It is clear that $\varphi$ is an isomorphism from $\nabla_1$ onto $\nabla_2$ and $\Psi(e) = \varphi(e)$ for all $e \in \nabla_1$. This means that $\mu$ and $\nu$ are log-equivalent.

Let $\nabla$ be an arbitrary complete Boolean algebra, $\nabla_e = \{g \in \nabla : g \leq e\}$, where $0 \neq e \in \nabla$. Denote by $\tau(\nabla_e)$ the minimum cardinality of a set that is dense in $\nabla_e$ with respect to the order topology ($(\sigma)$-topology). An infinite Boolean algebra $\nabla$ is said to be homogeneous if $\tau(\nabla_e) = \tau(\nabla_g)$ for any nonzero $e, g \in \nabla$. The cardinality of $\tau(\nabla)$ is called the weight of the homogeneous Boolean algebra $\nabla$ (see, for example, [4 Chapter VII]).

Let $1_{\nabla}$ be the unity in a Boolean algebra $\nabla$. It is known that any infinite Boolean algebra $(\nabla, \mu)$ with $\mu(1_{\nabla}) < \infty$ is a direct product of homogeneous Boolean algebras $\nabla_{e_n}, e_n \cdot e_m = 0, \ n \neq m$, $\tau_n = \tau(\nabla_{e_n}) < \tau_{n+1}$ ([3 Chapter VII, §2, Theorem 3]). Set $\mu_n = \mu(e_n)$. The matrix

$$\begin{pmatrix}
\tau_1 & \tau_2 & \cdots \\
\mu_1 & \mu_2 & \cdots 
\end{pmatrix}$$

is called the passport of the Boolean algebra $(\nabla, \mu)$.

The following theorem gives a classification of Boolean algebras with finite measure ([4 Chapter VII, §2, Theorem 5]).

**Theorem 2.2.** Let $\mu (\nu)$ be a probability measure on infinite complete Boolean algebra $\nabla_1$ (respectively, $\nabla_2$). Let $\begin{pmatrix}
\tau_1^{(1)} & \tau_2^{(1)} & \cdots \\
\mu_1 & \mu_2 & \cdots 
\end{pmatrix}$ be the passport of the Boolean algebra $(\nabla_1, \mu)$, and let $\begin{pmatrix}
\tau_1^{(2)} & \tau_2^{(2)} & \cdots \\
\nu_1 & \nu_2 & \cdots 
\end{pmatrix}$ be the passport of the Boolean algebra $(\nabla_2, \nu)$. The following conditions are equivalent:

(i) There exists an isomorphism $\varphi : \nabla_1 \to \nabla_2$ such that $\mu(e) = \nu(\varphi(e))$ for all $e \in \nabla_1$;

(ii) $\tau_n^{(1)} = \tau_n^{(2)}$ and $\mu_n = \nu_n$ for all $n$. 

In addition, the Boolean algebras $\nabla_1$ and $\nabla_2$ are isomorphic if and only if the upper rows of their passports coincide.

Now we are ready to give a criterion for a $*$-isomorphism between $*$-algebras $L_{\log}(\nabla, \mu)$ of log-integrable measurable functions associated with finite measure spaces. Let $\mu$ and $\nu$ be finite measures on complete Boolean algebras $\nabla_1$ and $\nabla_2$, respectively. Since $L_{\log}(\nabla, \mu) = L_{\log}(\nabla, \frac{\mu}{\mu(1_{\nabla_1})})$, we can assume without loss of generality that $\mu(1_{\nabla_1}) = 1 = \nu(1_{\nabla_2})$.

**Theorem 2.3.** Let $\mu$ ($\nu$) be a probability measure on an infinite complete Boolean algebra $\nabla_1$ (respectively, $\nabla_2$), and let

$$(\tau_1^{(1)} \tau_2^{(1)} \cdots \mu_1 \mu_2 \cdots), \left(\tau_1^{(2)} \tau_2^{(2)} \cdots \nu_1 \nu_2 \cdots\right)$$

be the passports of $(\nabla_1, \mu)$ and $(\nabla_2, \nu)$.

The following conditions are equivalent:

(i) The $*$-algebras $L_{\log}(\nabla_1, \mu)$ and $L_{\log}(\nabla_2, \nu)$ are $*$-isomorphic;

(ii) The upper rows of the passports of $(\nabla_1, \mu)$ and $(\nabla_2, \nu)$ coincide and the sequences $\left(\frac{\mu_n}{\nu_n}\right)$ and $\left(\frac{\tau_n^{(2)}}{\nu_n}\right)$ are bounded.

**Proof.** (i) $\Rightarrow$ (ii): Let $\Psi : L_{\log}(\nabla_1, \mu) \to L_{\log}(\nabla_2, \nu)$ be a $*$-isomorphism. Then the restriction $\varphi$ of $\Psi$ on the Boolean algebra $\nabla_1$ is an isomorphism from $\nabla_1$ onto $\nabla_2$, hence, by Theorem 2.2, the upper rows of passports of $(\nabla_1, \mu)$ and $(\nabla_2, \nu)$ coincide.

Let $\{e_n\}_{n=1}^{\infty}$ be a partition of a unity $1_{\nabla_1}$ of the Boolean algebra $\nabla_1$ such that $e_n\nabla_1$ is a homogeneous Boolean algebra, $\tau_n^{(1)} = \tau(e_n\nabla_1) < \tau_n^{(1)}$ and $\mu_n = \mu(e_n)$ for each $n$. Set $q_n = \varphi(e_n)$. It is clear that $\{q_n\}_{n=1}^{\infty}$ is a partition of the unity $1_{\nabla_2}$ of the Boolean algebra $\nabla_2$ such that $q_n\nabla_2$ is a homogeneous Boolean algebra, $\tau_n^{(2)} = \tau(q_n\nabla_2) < \tau_n^{(2)}$ and $\nu_n = \nu(q_n)$ for each $n$.

By Proposition 2.2 for a probability measure $\lambda(\varphi(e)) = \mu(e)$, $e \in \nabla_1$, on $\nabla_2$ we have

$$L_{\log}(\nabla_2, \nu) = \Psi(L_{\log}(\nabla_1, \mu)) = L_{\log}(\nabla_2, \lambda).$$

Using Corollary 2.1 we see that

$$\frac{dv}{d\lambda} \in L_{\infty}(\nabla_2) \text{ and } \frac{d\lambda}{dv} \in L_{\infty}(\nabla_2).$$

Consequently,

$$\nu_n = \nu(q_n) = \int_{q_n} \frac{dv}{d\lambda} d\lambda \leq \|\frac{dv}{d\lambda}\|_{\infty}(q_n) = \|\frac{dv}{d\lambda}\|_{\infty}(\mu(e_n))$$

for all $n$. Therefore, the sequence $\left(\frac{\mu_n}{\nu_n}\right)$ is bounded. That the sequence $\left(\frac{\tau_n^{(2)}}{\nu_n}\right)$ is also bounded is shown similarly.

(ii) $\Rightarrow$ (i): Let the upper rows of the passports of $(\nabla_1, \mu)$ and $(\nabla_2, \nu)$ coincide and the sequences $\left(\frac{\mu_n}{\nu_n}\right)$ and $\left(\frac{\tau_n^{(2)}}{\nu_n}\right)$ are bounded. By Theorem 2.2 there exists an isomorphism $\varphi : \nabla_1 \to \nabla_2$. Let $\{e_n\}_{n=1}^{\infty}$ and $q_n = \varphi(e_n)$ be as in the proof of the implication (i) $\Rightarrow$ (ii). Consider the probability measure $\gamma(q) = \sum_{n=1}^{\infty} \mu_n \nu_n^{-1} \nu(q_n q)$, $q \in \nabla_2$, on $\nabla_2$. Since the passports of the Boolean algebras $(\nabla_1, \mu)$ and $(\nabla_2, \gamma)$ coincide, it follows by Theorem 2.2 that there exists an isomorphism $\psi : \nabla_1 \to \nabla_2$ such that $\mu(e) = \gamma(\psi(e))$ for all $e \in \nabla_1$. 


Let now $\Psi : L_0(\nabla_1, \mu) \to L_0(\nabla_2, \gamma)$ be a $*$-isomorphism such that $\psi(e) = \Psi(e)$ for all $e \in \nabla_1$. By Proposition 2.21 $\Psi(L_{\log}(\nabla_1, \mu)) = L_{\log}(\nabla_2, \gamma)$. Since $\gamma(q) = \sum_{n=1}^{\infty} \mu_n \nu_n^{-1} \nu(q_n q), q \in \nabla_2$, it follows that

$$\frac{d\gamma}{d\nu} = \sum_{n=1}^{\infty} \mu_n \nu_n^{-1} q_n \in L_\infty(\nabla_2).$$

On the other hand, if $q \in q_n \nabla_2$, then $\nu(q) = \nu(q_n q) = \nu_n \mu_n^{-1} \gamma(q)$, that is,

$$\nu(q) = \sum_{n=1}^{\infty} \nu_n \mu_n^{-1} \gamma(q_n q), q \in \nabla_2.$$

Consequently,

$$\frac{d\nu}{d\gamma} = \sum_{n=1}^{\infty} \nu_n \mu_n^{-1} q_n \in L_\infty(\nabla_2).$$

Therefore, by Corollary 2.1 we have

$L_{\log}(\nabla_2, \nu) = L_{\log}(\nabla_2, \gamma) = \Psi(L_{\log}(\nabla_1, \mu)).$

\[ \square \]

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