A Generalized Prony Method for Filter Recovery in Evolutionary System via Spatiotemporal Trade Off

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Abstract

We consider the problem of spatiotemporal sampling in an evolutionary process \( x^{(n)} = A^nx \) where an unknown linear operator \( A \) driving an unknown initial state \( x \) is to be recovered from a combined set of coarse spatial samples \( \{ x|_{\Omega_0}, x^{(1)}|_{\Omega_1}, \cdots, x^{(N)}|_{\Omega_N} \} \). In this paper, we will study the case of infinite dimensional spatially invariant evolutionary process, where the unknown initial signals \( x \) are modeled as \( \ell^2(\mathbb{Z}) \) and \( A \) is an unknown spatial convolution operator given by a filter \( a \in \ell^1(\mathbb{Z}) \) so that \( Ax = a \ast x \). We show that \( \{ x|_{\Omega_0}, x^{(1)}|_{\Omega_1}, \cdots, x^{(N)}|_{\Omega_N} : N \geq 2m - 1, \Omega_m = m\mathbb{Z} \} \) contains enough information to recover the Fourier spectrum of a typical low pass filter \( a \), if the initial signal \( x \) is from a dense subset of \( \ell^2(\mathbb{Z}) \). The idea is based on a nonlinear, generalized Prony method similar to [1]. We provide an algorithm for the case when both \( a \) and \( x \) are compactly supported around the center. Finally, we perform the accuracy analysis based on the spectral properties of the operator \( A \) and initial state \( x \), and verify them by several numerical experiments.

Keywords: Distributed sampling, reconstruction, channel estimation.

1 Introduction

1.1 The dynamical sampling problem

In situations of practical interest, physical systems evolve in time under the action of well studied operators such as diffusion processes. Sampling of such an evolving system is done by sensors or measurement devices that are placed at various locations and can be activated at different times. For practical reasons, we aim to reconstruct any states in the evolutionary process using as few sensors as possible, but allow one to take samples at different time levels.
This setting has not been studied within the classical approach in sampling theory, where the samples are taken simultaneously at only one time level, see [2, 3, 22, 32, 11, 12, 10, 27, 8]. Dynamical sampling is a newly proposed sampling framework. It involves studying the time-space patterns formed by the locations of the measurement devices and the times of their activation. Mathematically speaking, suppose \( x \) is an initial distribution that is evolving in time satisfying the evolution rule:

\[
x_t = A_t x
\]

where \( \{A_t\}_{t \in [0, \infty)} \) is a family of evolution operators satisfying the condition \( A_0 = I \). Dynamical sampling asks the question: when do coarse samplings taken at varying times \( \{x|\Omega_0, (A_{t_1} x)|\Omega_1, \ldots, (A_{t_N} x)|\Omega_N \} \) contain the same information as a finer sampling taken at the earliest time? One goal of dynamical sampling is to find all spatiotemporal sampling sets \( (\chi, \tau) = \{\Omega_t, t \in \tau\} \) such that the certain classes of signals \( x \) can be recovered from the spatiotemporal samples \( x_t(\Omega_t), t \in \tau \). In the above cases, the evolution operators are assumed to be known. It has been well-studied in the context of various evolutionary systems in a very general setting, see [4, 16, 6, 18, 20].

Another important problem arises when the evolution operators are themselves unknown or partially known. In this case, we are interested in finding all spatiotemporal sampling sets and certain classes of evolution operators so that the family \( \{A_t\}_{t \in [0, \infty)} \) or their spectrum can be identified. We call such a problem the unsupervised system identification problem in dynamical sampling.

### 1.2 Problem Statement

We are going to introduce the notion of infinite dimensional spatially invariant evolutionary system and uniform spatiotemporal sampling problem. Let \( x \in \ell^2(\mathbb{Z}) \) be an unknown initial spatial signal and the evolution operator \( A \) be given by an unknown convolution filter \( a \in \ell^1(\mathbb{Z}) \) such that \( Ax = a \ast x \). At time \( t = n \in \mathbb{N} \), the signal \( x \) evolves to be \( x_n = A^n x = a^n \ast x \), where \( a^n = a \ast a \ast \cdots \ast a \). We call this evolutionary system spatially invariant. In this paper, we are interested in the recovery of the unknown filter \( a \) that drives the evolutionary process. Without loss of generality, assume \( m \) is a positive odd integer \( (m > 1) \) and denote by \( S_m : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) the sampling operator on \( \Omega_m = m\mathbb{Z} \), i.e., \( (S_m x)(k) = x(mk) \). At time level \( t = l \), we have partial observations

\[
y_l = S_m(a^l \ast x).
\]
The special case we are going to consider can be stated as follows:

Under what conditions on $a, m, N$ and $x$, can $a$ be recovered from the spatiotemporal samples $\{y_l : l = 0, \cdots, N - 1\}$, or equivalently, from $\{x|_{\Omega_m}, (a \ast x)|_{\Omega_m}, \cdots, (a^{N-1} \ast x)|_{\Omega_m}\}$?

In [1], Aldroubi and Kristal consider the recovery of an unknown $d \times d$ matrix $B$ and an unknown initial state $x \in \ell^2(\mathbb{Z}_d)$ from coarse spatial samples of its successive states $\{B^k x, k = 0, 1, \cdots\}$. Given an initial sampling set $\Omega \subset \mathbb{Z}_d = \{1, 2, \cdots, d\}$, they employ techniques related to Krylov subspace methods to show how large $l_i$ should be to recover all the eigenvalues of $B$ that can possibly be recovered from spatiotemporal samples $\{B^k x(i) : i \in \Omega, k = 0, 1, \cdots, l_i - 1\}$. Our setup is very similar to the special case of regular invariant dynamical sampling problem in [1]. In this special case, they employ a generalization of the well known Prony method that uses these regular undersampled spatiotemporal data first for the recovery of the Fourier spectrum of the correlating filter. Since the filter is a typical low pass filter with the symmetry and monotonicity condition, it is completely determined by its Fourier spectrum. By using techniques developed in [20], one can recover the initial state. In this paper, we will generalize this idea and address the infinite dimensional analog of this special case. In recent years, the Prony method or its generalized form has been successfully applied to different inverse problems. In [7], Peter and Plonka use a generalized prony method to reconstruct the sparse sums of the eigenfunctions of some known linear operators. Our generalization shares some similar spirits with it, but deals with a fundamentally different problem.

The remainder of the paper is organized as follows: In section 2, we discuss the noise free case, and propose a generalized prony method to show that we can reconstruct a typical low pass filter $a$ via the spatiotemporal samples $\{y_l\}_{l=1}^N$, provided $N \geq 2m - 1$. In section 3, we provide an accuracy analysis of the algorithm derived from the generalized prony method. The estimation results are formulated in the rigid $\ell^\infty$ norm. In section 4, we do several numerical stimulations to verify some estimation results. Finally, we summarize the work in section 5.

1.3 Notations

Let us introduce some relevant notations.

Definition 1. Let $M = (m_{ij})$ be an $n \times n$ matrix, the infinity norm of $M$, 

is defined by
\[ ||M||_\infty = \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} |m_{ij}| \right). \]

For a vector \( z = (z_i) \in \mathbb{C}^n \), we define the infinity norm \( ||z||_\infty = \max_{i=1,\cdots,n} |z_i| \). It is easy to see that
\[ ||M||_\infty = \max_{z \in \mathbb{C}^n, ||z||_\infty = 1} ||Mz||_\infty. \]

We use \( z^T \) and \( M^T \) to denote their transpose.

**Definition 2.** Let \( M = (m_{ij}) \) be an \( n \times n \) matrix. The minimal annihilating polynomial of \( M \), denoted by \( p_M \), is the monic polynomial of smallest degree among all the polynomials \( p \) such that \( p(M) = 0 \). We will denote the degree of \( p_M \) by \( r_M \).

**Definition 3.** Let \( w_1, w_2, \cdots, w_n \) be \( n \) distinct complex numbers, denote \( w = (w_1, \cdots, w_n) \), the \( n \times n \) Vandermonde matrix generated by \( w \)'s is defined by
\[
V_n(w) = \begin{pmatrix}
1 & w_1 & \cdots & w_1^{n-1} \\
1 & w_2 & \cdots & w_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & w_n & \cdots & w_n^{n-1}
\end{pmatrix}.
\]

**Definition 4.** For a sequence \( c = (c_n)_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z}) \) or \( \ell^2(\mathbb{Z}) \), we define its Fourier transformation to be the function on the Torus \( \mathbb{T} = [0, 1) \)
\[
\hat{c}(\xi) = \sum_{n \in \mathbb{Z}} c_n e^{-2\pi in\xi}, \xi \in \mathbb{T}.
\]

## 2 Noise-free recovery

We consider the recovery of a frequently encountered case in applications when the filter \( a \in \ell^1(\mathbb{Z}) \) is a typical low pass filter so that \( \hat{a}(\xi) \) is real, symmetric and strictly decreasing on \([0, \frac{1}{2}]\). An example of such a typical low pass filter is shown in Figure 1. The symmetry reflects the fact that there is often no preferential direction for physical kernels and monotonicity is a reflection of energy dissipation. Without loss of generality, we also assume \( a \) is a normalized filter, i.e., \( |\hat{a}(\xi)| \leq 1, \hat{a}(0) = 1 \). Let \( \mu \) denote the lebesgue measure on \( \mathbb{T} \), and \( X \) be a subclass of \( \ell^2(\mathbb{Z}) \) defined by
\[
X = \{x \in \ell^2(\mathbb{Z}) : \mu(\{\xi \in \mathbb{T} : \hat{x}(\xi) = 0\}) = 0\}. \]
Clearly, $X$ is a dense class of $\ell^2(\mathbb{Z})$ under the norm topology. In noise free scenario, our first result shows that we can recover $a$ provided that our initial state $x \in X$.

**Theorem 1.** Let $x \in X$ be the initial state and the evolution operator $A$ be a convolution operator given by $a \in \ell^1(\mathbb{Z})$ so that $\hat{a}(\xi)$ is real, symmetric, and strictly decreasing on $[0, \frac{1}{2}]$. Then $a$ can be recovered from measurements $\{y_l\}_{l=0}^{2m-1}$ defined in (1).

**Proof.** We are going to show that the regular subsampled data $\{y_l\}_{l=0}^{2m-1}$ contains enough information to recover the Fourier spectrum of $a$ on $T$ up to a measure zero set. As in [1], we prefer to look for the solution in the Fourier domain and rewrite the problem in the following way. First, for each $\xi \in T$, we define the matrix

$$A_m(\xi) = \frac{1}{m} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \hat{a}\left(\frac{\xi}{m}\right) & \hat{a}\left(\frac{\xi+1}{m}\right) & \cdots & \hat{a}\left(\frac{\xi+m-1}{m}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}^{m-1}\left(\frac{\xi}{m}\right) & \hat{a}^{m-1}\left(\frac{\xi+1}{m}\right) & \cdots & \hat{a}^{m-1}\left(\frac{\xi+m-1}{m}\right) \end{pmatrix}, \quad (3)$$

and

$$D_m(\xi) = \begin{pmatrix} \hat{a}\left(\frac{\xi}{m}\right) & 0 & \cdots & 0 \\ 0 & \hat{a}\left(\frac{\xi+1}{m}\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{a}\left(\frac{\xi+m-1}{m}\right) \end{pmatrix}, \quad B_{m,x}(\xi) = \begin{pmatrix} \hat{x}\left(\frac{\xi}{m}\right) & 0 & \cdots & 0 \\ 0 & \hat{x}\left(\frac{\xi+1}{m}\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{x}\left(\frac{\xi+m-1}{m}\right) \end{pmatrix}. \quad (4)$$

By our assumptions of $x$, there exists a measurable subset $E_0$ of $T$ with $\mu(E_0) = 1$, so that $B_{m,x}(\xi)$ is an invertible matrix for $\xi \in E_0$. Let $E = \{ \xi \in T : \hat{x}(\xi) \neq 0 \}$.
$E_0 - \{0, \frac{1}{2}\}$, we will show that, using the same idea with the Prony method, we can recover the diagonal entries of matrix $D_m(\xi)$ for $\xi \in E$. Define

$$H_m(\xi) = \begin{pmatrix} \hat{y}_0(\xi) & \hat{y}_1(\xi) & \cdots & \hat{y}_{m-1}(\xi) \\ \hat{y}_1(\xi) & \hat{y}_2(\xi) & \cdots & \hat{y}_m(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{m-1}(\xi) & \hat{y}_m(\xi) & \cdots & \hat{y}_{2m-2}(\xi) \end{pmatrix}, b_m(\xi) = \begin{pmatrix} \hat{y}_m(\xi) \\ \hat{y}_{m+1}(\xi) \\ \vdots \\ \hat{y}_{2m-1}(\xi) \end{pmatrix}. \ \ \ (5)$$

**Claim:** For each $\xi \in E$, the minimal annihilating polynomial of $D_m(\xi)$ is given by

$$p_{D_m(\xi)}[z] = m-1 \prod_{i=0}^{m-1} (z - \hat{a}(\frac{\xi+i}{m})) = z^m + q_{m-1}(\xi)z^{m-1} + \cdots + q_0(\xi),$$

and $q(\xi) = (q_0(\xi), \cdots, q_{m-1}(\xi))^T$ is the unique solution of the equation

$$H_m(\xi)q(\xi) = -b_m(\xi). \ \ \ (6)$$

Now we are going to prove this claim. Using the Poisson Summation Formula

$$\hat{S}_m x(\xi) = \frac{1}{m} \sum_{l=0}^{m-1} \hat{x}(\frac{\xi+l}{m}), \ \ \ (7)$$

and the convolution theorem

$$\hat{a} * x(\xi) = \hat{a}(\xi) \hat{x}(\xi), \ \ \ (8)$$

we can rewrite the $(l+1)$th column of $H_m(\xi)$ and $b_m(\xi)$ in the following way

$$\begin{pmatrix} \hat{y}_l(\xi) \\ \hat{y}_{l+1}(\xi) \\ \vdots \\ \hat{y}_{l+m-1}(\xi) \end{pmatrix} = A_m(\xi)(D_m(\xi))^l \begin{pmatrix} \hat{x}(\frac{\xi}{m}) \\ \hat{x}(\frac{\xi+1}{m}) \\ \vdots \\ \hat{x}(\frac{\xi+m-1}{m}) \end{pmatrix}, \ \ \ (9)$$

for $l = 0, 1, \cdots, m-1, m$. If $\xi \neq 0, \frac{1}{2}$, by the symmetry and monotonicity condition of $\hat{a}(\xi)$, $D_m(\xi)$ has $m$ distinct eigenvalues, and hence $r_{D_m(\xi)} = m,$
\[ p^{D_m}(\xi)[z] = \prod_{i=0}^{m-1} \left( z - \hat{a}\left(\frac{\xi+i}{m}\right) \right) = z^m + q_{m-1}(\xi)z^{m-1} + \cdots + q_0(\xi). \] By (9),

\[
H_m(\xi)q(\xi) = \sum_{l=0}^{m-1} A_m(\xi)q_l(\xi)(D_m(\xi))^l \begin{pmatrix} \hat{x}(\frac{\xi}{m}) \\ \hat{x}(\frac{\xi+1}{m}) \\ \vdots \\ \hat{x}(\frac{\xi+m-1}{m}) \end{pmatrix} 
= -A_m(\xi)(D_m(\xi))^m \begin{pmatrix} \hat{x}(\frac{\xi}{m}) \\ \hat{x}(\frac{\xi+1}{m}) \\ \vdots \\ \hat{x}(\frac{\xi+m-1}{m}) \end{pmatrix}
= -b_m(\xi).
\]

Let \( \{e_i : i = 1, \ldots, m\} \) be the canonical basis for \( \mathbb{C}^m \). For \( k = 1, \ldots, m \), a simple computation shows that

\[
B_{m,x}(\xi)A_m^T(\xi)e_k = (D_m(\xi))^{k-1} \begin{pmatrix} \hat{x}(\frac{\xi}{m}) \\ \hat{x}(\frac{\xi+1}{m}) \\ \vdots \\ \hat{x}(\frac{\xi+m-1}{m}) \end{pmatrix}.
\]

By (9) and (11),

\[
H_m(\xi)e_k = A_m(\xi)B_{m,x}(\xi)A_m(\xi)^T e_k, k = 1, \ldots, m.
\]

Hence

\[
H_m(\xi) = A_m(\xi)B_{m,x}(\xi)A_m(\xi)^T.
\]

Since \( A_m(\xi) \) and \( B_{m,x}(\xi) \) are invertible when \( \xi \in E \), it follows that \( H_m(\xi) \) is invertible. Therefore \( q(\xi) \) is the unique solution of (6). Now to determine \( D_m(\xi) \) amounts to finding the roots of \( p^{D_m}(\xi) \) and ordering them according to the monotonicity condition on \( \hat{a} \).

In summary, for each \( \xi \in E \), we can uniquely determine \( \{\hat{a}(\frac{\xi+i}{m}) : i = 0, \ldots, m-1\} \). Note \( \mu(E) = 1 \), and hence we can recover the Fourier spectrum of \( a \) up to a measure zero set. The conclusion is followed by applying the inverse Fourier transformation on \( \hat{a}(\xi) \).

Theorem 1 addresses the infinite dimensional analog of Theorem 4.1 in [1]. Once \( a \) is recovered, we can recover \( x \) using techniques developed in [6]. If the shape information of \( a \) is not a priori knowledge, with minor modifications of the above proof, one can show the recovery of the range of \( \hat{a} \) on a measurable subset of \( \mathbb{T} \), where the measure of this subset is 1.
Algorithm 2.1. Input: Choose a recovery of the Fourier spectrum of this case, the proof of Theorem 1 essentially provides an algorithm for the 

Definition 5. Let $a = (a(n))_{n \in \mathbb{Z}}$, the support set of $a$ is defined by $\text{Supp}(a) = \{k \in \mathbb{Z} : a(k) \neq 0\}$. If $\text{Supp}(a)$ is a finite set, $a$ is said to be compactly supported. In particular, $a$ is said to be compactly supported around the center with radius $r$, if there exists a positive integer $r$ so that $\text{Supp}(a) \subset \{-r, -r+1, \ldots, r\}$.

If $a$ is compactly supported around the center with radius $r$, we can immediately get the following:

Corollary 1. In addition to the assumptions of Theorem 1 if $a$ is compactly supported around the center with radius $r$, it is enough to recover $\{\hat{a}(\eta_i) : i = 1, \ldots, r\}$ at $r$ distinct locations via equation (6).

Proof. Under these assumptions, we know

$$\hat{a}(\xi) = a(0) + \sum_{k=1}^{r} a(k) \cos(2\pi k \xi).$$

(14)

By solving (6), suppose $\{\hat{a}(\eta_i) : i = 1, \ldots, r, \eta_i \neq \eta_j \text{ if } i \neq j\}$ is recovered, we set up the following linear equation

$$
\begin{pmatrix}
1 & \cos(2\pi \eta_1) & \cdots & \cos(2r\pi \eta_1) \\
1 & \cos(2\pi \eta_2) & \cdots & \cos(2r\pi \eta_2) \\
\vdots & \vdots & \ddots & \vdots \\
1 & \cos(2\pi \eta_r) & \cdots & \cos(2r\pi \eta_r)
\end{pmatrix}
\begin{pmatrix}
a(0) \\
a(1) \\
\vdots \\
a(r)
\end{pmatrix}
= 
\begin{pmatrix}
\hat{a}(\eta_1) \\
\hat{a}(\eta_2) \\
\vdots \\
\hat{a}(\eta_r)
\end{pmatrix}.
$$

(15)

Note that $\{1, \cos(2\pi \eta), \ldots, \cos(2r\pi \eta)\}$ is a Chebyshev system on $[0, 1]$(see [35]), and hence (15) has a unique solution, which finishes the proof.

The proof of Theorem 1 does not give a practical method in general, since it involves computing the Fourier transformation of infinite sequences and solving the roots of uncountably many polynomials. However, if we know that $\text{Supp}(a)$ and $\text{Supp}(x)$ are contained in $\{-r, -r+1, \ldots, r\}$ for some $r \in \mathbb{N}^+$ as a priori, then $\{y_l\}_{l=0}^{2m-1}$ consists of sequences supported in $\{-2mr, \ldots, 2mr\}$. We are able to compute $\{\hat{y}_l(\xi)\}_{l=0}^{2m-1}$ for any $\xi \in \mathbb{T}$. In this case, the proof of Theorem 1 essentially provides an algorithm for the recovery of the Fourier spectrum of $a$. We summarize it as follows:

Algorithm 2.1. Input: Choose $\xi \in \mathbb{T} - \{0, \frac{1}{2}\}$.

1. From the measurements $\{y_l\}_{l=0}^{2m-1}$, compute $\{\hat{y}_l(\xi)\}_{l=0}^{2m-1}$. Form the Hankel matrix $H_m(\xi)$ and $b_m(\xi)$ as in (5). Test the rank of $H_m(\xi)$, if $\text{rank}(H_m(\xi)) = m$, solve $\mathbf{q}(\xi)$ via the following equation:

$$H_m(\xi)\mathbf{q}(\xi) = -b_m(\xi).$$
2. For \( q(\xi) = (q_0(\xi), \cdots, q_m(\xi))^T \), form the Prony polynomial \( p_{Dm}(\xi)[z] = z^m + \sum_{l=0}^{m-1} q_l(\xi)z^l \). Find the roots of \( p_{Dm}(\xi)[z] \) and order them to get \( \{\hat{a}(\xi) : i = 0, \cdots, m-1\} \).

Output: \( \{\hat{a}(\xi) : i = 0, \cdots, m-1\} \).

Remark 1. Note in this case, \( H_m(\xi) \) is not invertible only at finitely many locations of \( \mathbb{T} \), \( H_m(\xi) \) is invertible with probability 1.

Let \( \{\xi_i : i = 1, \cdots, K\} \) be a subset of \( \mathbb{T} \) satisfying the condition \(|\xi_i - \xi_j| \neq \frac{k}{m} \) for \( k = 0, \cdots, m-1 \), and \( Km > r \). Assume we have recovered \( \{\hat{a}(\xi) : i = 0, \cdots, m-1, j = 1, \cdots, K\} \) via Algorithm 2.1 by Corollary 1, we can completely determine \( a \).

3 Accuracy Analysis

In previous sections, we show that if we are able to compute the spectral data \( \{\hat{y}_l(\xi)\}_{l=0}^{2m-1} \) at \( \xi \), then we can recover the Fourier spectrum \( \{\hat{a}(\xi) : i = 0, \cdots, m-1\} \) by Algorithm 2.1. However, a critical issue still remains. We need to analyze the accuracy of the solution achieved by our algorithm. The motivation to study the accuracy comes from two aspects. Firstly, for the case when one or both of \( a \) and \( x \) are not compactly supported, although we only have access to a finite section of each exact measurement \( y_l \in \ell^2(\mathbb{Z}) \) in practice, we may have a good approximation \( \tilde{y}_l \) of \( y_l \), so that \( ||\hat{y}_l(\xi) - \tilde{y}_l(\xi)||_{\infty} \leq \epsilon_l \ll 1 \). Consequently, we can employ Algorithm 2.1 to compute an approximation of the Fourier spectrum of \( a \). A natural question to ask is how large the error will be between the approximate solutions and the actual solutions. Secondly, for the case when both \( x \) and \( a \) are compactly supported, what if we have noise in the process of computing \( \{\hat{y}_l(\xi)\}_{l=0}^{2m-1} \)?

We can summarize our accuracy analysis problem in the following:

Assume the measurements are given by \( \{\tilde{y}_l\}_{l=0}^{2m-1} \) compared to (1) so that \( ||\hat{y}_l(\xi) - \tilde{y}_l(\xi)||_{\infty} \leq \epsilon_l \) for all \( \xi \in \mathbb{T} \). Given an estimation for \( \epsilon = \max_l|\epsilon_l| \), how large can the error be in the worst case for the reconstructed parameters in Step I and Step II of Algorithm 2.1 in terms of \( \epsilon \), and the true parameters.

Our accuracy analysis will consist of two steps. Suppose our measurements are perturbed from \( \{y_l\}_{l=0}^{2m-1} \) to \( \{\tilde{y}_l\}_{l=0}^{2m-1} \). For any \( \xi \), we first measure
the perturbation of $\mathbf{q}(\xi)$ in terms of $\ell^\infty$ norm. This step is linear and standard. Then we measure the perturbation of the roots. It is well known that the roots of a polynomial are continuously dependent on the small change of its lower degree coefficients. Hence, for a small perturbation, although the roots of the perturbed polynomial $\tilde{p}^{D_m}(\xi)$ may not be real, we can order them according to their modulus and have a one to one correspondence with the roots of $p^{D_m}(\xi)$. Before presenting our main results in this section, let us introduce some useful notations and terminologies.

**Definition 6.** Let $\xi \in \mathbb{T} - \{0, \frac{1}{2}\}$, consider the set $\{\hat{a}(\frac{\xi+i}{m}) : i = 0, \cdots, m-1\}$ that consists of $m$ distinct nodes.

1. For $0 \leq k \leq m - 1$, the separation between $\hat{a}(\frac{\xi+k}{m})$ with other $m - 1$ nodes is measured by
   $$\delta_k(\xi) = \frac{1}{\prod_{0 \leq j \leq |m-1| \neq k} |\hat{a}(\frac{\xi+j}{m}) - \hat{a}(\frac{\xi+k}{m})|}$$

2. For $0 \leq k \leq m$, the $k$th elementary symmetric function generated by the $m$ nodes is denoted by
   $$\sigma_k(\xi) = \sum_{0 \leq j_1 < \cdots < j_k \leq m-1} \hat{a}(\frac{\xi+j_1}{m}) \hat{a}(\frac{\xi+j_2}{m}) \cdots \hat{a}(\frac{\xi+j_k}{m})$$

For $0 \leq k, i \leq m-1$, the $k$th elementary symmetric function generated by $m-1$ nodes with $\hat{a}(\frac{\xi+i}{m})$ missing is denoted by $\sigma_k(i)(\xi)$.

**Proposition 1.** Let the perturbed measurements $\{\tilde{y}_i\}_{i=0}^{2m-1}$ be given with an error satisfying $||\tilde{y}_i(\xi) - \hat{y}_i(\xi)||_\infty \leq \epsilon, \forall i$. Let $\tilde{H}_m(\xi)$ and $\tilde{b}_m(\xi)$ be formed by $\{\tilde{y}_i(\xi)\}_{i=0}^{2m-1}$ in the same way as in (5). Assume $H_m(\xi)$ is invertible and $\epsilon$ is sufficient small so that $\tilde{H}_m(\xi)$ is also invertible. Denote by $\tilde{q}(\xi)$ the solution of $\tilde{H}_m(\xi)\tilde{q}(\xi) = -\tilde{b}_m(\xi)$. Form the Prony polynomial $\tilde{p}^{D_m}(\xi)$ using $\tilde{q}(\xi)$ and let $\{\hat{a}(\frac{\xi+i}{m}) : i = 0, \cdots, m-1\}$ be its roots, then we have the following estimates as $\epsilon \to 0$,

$$||\mathbf{q}(\xi) - \tilde{q}(\xi)||_\infty \leq ||H^{-1}_m(\xi)||_\infty (1 + m\beta_1(\xi))\epsilon + O(\epsilon^2),$$

where $\beta_1(\xi) = \max_{k=1,\cdots,m} |\sigma_k(\xi)|$. As a result, we achieve the following first order estimation

$$|\hat{a}(\frac{\xi+i}{m}) - \hat{a}(\frac{\xi+i}{m})| \leq C_1(\xi)(1 + m\beta_1(\xi))||H^{-1}_m(\xi)||_\infty \epsilon + O(\epsilon^2),$$
where \( C_i(\xi) = \delta_i(\xi) \cdot (\sum_{k=0}^{m-1} |\hat{a}_k(\frac{\xi+i}{m})|) \).

**Proof.** Note that equation (18) is perturbed to be

\[
\tilde{H}_m(\xi)\hat{q}(\xi) = -\tilde{b}_m(\xi).
\]

By our assumptions, we have

\[
||\Delta H_m(\xi)||_{\infty} = ||\tilde{H}_m(\xi) - H_m(\xi)||_{\infty} \leq m\epsilon, \quad (19)
\]

\[
||\Delta b_m(\xi)||_{\infty} = ||\tilde{b}_m(\xi) - b_m(\xi)||_{\infty} \leq \epsilon. \quad (20)
\]

Define \( \Delta q(\xi) = \tilde{q}(\xi) - q(\xi) \), by simple computation,

\[
\Delta q(\xi) = H_m^{-1}(\xi)(I + H_m^{-1}(\xi)\Delta H_m(\xi))^{-1}\Delta b_m(\xi) - \Delta H_m(\xi)q(\xi). \quad (21)
\]

Hence if \( \epsilon \to 0 \), we obtain

\[
\Delta q(\xi) = H_m^{-1}(\xi)(\Delta b_m(\xi) - \Delta H_m(\xi)q(\xi)) + O(\epsilon^2). \quad (22)
\]

Now we can easily get an estimation of \( \epsilon^\infty \) norm of \( \Delta q(\xi) \)

\[
||\Delta q(\xi)||_{\infty} \leq ||H_m^{-1}(\xi)||_{\infty}(1 + m||q(\xi)||_{\infty})\epsilon + O(\epsilon^2). \quad (23)
\]

Since \( \{\hat{a}(\frac{\xi+i}{m}) : i = 0, \ldots, m - 1\} \) is the root of \( p_m^{Dm}(\xi) \), using Vieta’s Formulas (see [30]), we know

\[
||q(\xi)||_{\infty} = \max_{1 \leq k \leq m} |\sigma_k(\xi)|.
\]

Let \( (\Delta p(\xi))[z] \) be the polynomial of degree less than or equal to \( m-1 \) defined by the vector \( \Delta q(\xi) \). Using Proposition V.1 in [29], and denote by \( (p_m^{Dm}(\xi))' \) the derivative function of \( p_m^{Dm}(\xi) \), for \( 0 \leq i \leq m-1 \), we conclude

\[
|\hat{a}(\frac{\xi+i}{m}) - \hat{a}(\frac{\xi+i}{m})| = \frac{|\Delta p(\xi)[\hat{a}(\frac{\xi+i}{m})]|}{(p_m^{Dm}(\xi))'[\hat{a}(\frac{\xi+i}{m})]} + O(\epsilon^2)
\]

\[
\leq \frac{||\Delta q(\xi)||_{\infty} (\sum_{k=0}^{m-1} |\hat{a}_k(\frac{\xi+i}{m})|)}{\prod_{0 \leq j \leq m-1} |\hat{a}(\frac{\xi+j}{m}) - \hat{a}(\frac{\xi+i}{m})|} + O(\epsilon^2)
\]

\[
\leq C_i(\xi)||H_m^{-1}(\xi)||_{\infty}(1 + m \max_{1 \leq k \leq m} |\sigma_k(\xi)|)\epsilon + O(\epsilon^2), \quad (24)
\]

where \( C_i(\xi) = \delta_i(\xi) (\sum_{k=0}^{m-1} |\hat{a}_k(\frac{\xi+i}{m})|) \).
Therefore it is important to understand the relation between the behavior of \(\|H_m^{-1}(\xi)\|_\infty\) and our system parameters, i.e., \(m\), \(a\) and \(x\). Next, we are going to estimate \(\|H_m^{-1}(\xi)\|_\infty\) and reveal their connection with the spectral properties of \(a\) and \(x\).

**Theorem 2.** Assume \(H_m(\xi)\) is invertible, we have the lower bound estimation

\[
\|H_m^{-1}(\xi)\|_\infty \geq m^2 \max_{i=0, \ldots, m-1} \frac{\beta_2(i, \xi) \delta_i(\xi)}{|\hat{x}(\frac{i+1}{m})|},
\]

where \(\beta_2(i, \xi) = \max_{k=0, \ldots, m-1} |\sigma_k^{(i)}(\xi)|\), and the upper bound estimation

\[
\|H_m^{-1}(\xi)\|_\infty \leq m^2 \max_{i=0, \ldots, m-1} \frac{\delta_i(\xi) \prod_{0 \leq j \leq m-1} (1 + |\hat{\alpha}(\frac{j+1}{m})|)^2}{|\hat{x}(\frac{i+1}{m})|}.
\]

**Proof.** Firstly, we prove the lower bound for \(\|H_m^{-1}(\xi)\|_\infty\). Note \((A_m^{-1})^T(\xi) = (b_{ki})_{1 \leq k, i \leq m}\) is the scaling inverse of a standard Vandermonde matrix \(A_m^{-1}(\xi)\), by the inverse formula for a standard Vandermonde matrix,

\[
b_{ki} = (-1)^{m-k} m \sigma_{m-k}^{(i-1)}(\xi) \delta_{i-1}(\xi).
\]

Let \(\{e_i\}_{i=1}^m\) be the standard basis for \(\mathbb{C}^m\) and \(w_i(\xi) = A_m(\xi)e_i\) for \(i = 1, \ldots, m\). Since \(|\hat{\alpha}(\xi)| \leq 1\), we conclude that \(\|w_i\|_\infty = \frac{1}{m}\).

\[
\|H_m^{-1}(\xi)\|_\infty \geq m \cdot \max_{i=1, \ldots, m} \|H_m^{-1}(\xi)w_i(\xi)\|_\infty
\]

\[
\geq \max_{i=1, \ldots, m} \frac{m}{|\hat{x}(\frac{i+1}{m})|} \|(A_m^{-1})^T(\xi)e_i\|_\infty
\]

\[
= m^2 \max_{i=0, \ldots, m-1} \frac{\beta_2(i, \xi) \delta_i(\xi)}{|\hat{x}(\frac{i+1}{m})|}.
\]

On the other hand, using (13) and the norm estimation for the inverse of a Vandermonde matrix in [36], we show that

\[
\|H_m^{-1}(\xi)\|_\infty \leq \|A_m^{-1}(\xi)\|_\infty \|(A_m^{-1})^T(\xi)\|_\infty \|B_m(\xi)\|_\infty
\]

\[
\leq m^2 \max_{i=0, \ldots, m-1} \frac{\delta_i(\xi) \prod_{j \neq i} (1 + |\hat{\alpha}(\frac{j+1}{m})|)^2}{|\hat{x}(\frac{i+1}{m})|}.
\]
As an application of Theorem 2, the following corollary gives us an idea about the dependence of $||H_m^{-1}(\xi)||_\infty$ on $m$.

**Corollary 2.** If $|\hat{x}(\xi)| \leq M$ for every $\xi \in \mathbb{T}$, then $||H_m^{-1}(\xi)||_\infty \geq O(m2^m)$. Therefore, $||H_m^{-1}(\xi)||_\infty \to \infty$ as $m \to \infty$.

**Proof.** We show this by proving $m^2 \max_{i=0, \ldots, m-1} \delta_i(\xi) \geq O(m2^m)$. Note $\beta_2(i, x) \geq |\sigma_0^{(i)}(\xi)| = 1$. By (27),

$$||H_m^{-1}(\xi)||_\infty \geq m^2 \cdot \frac{\max_{i=0, \ldots, m-1} \delta_i(\xi)}{M} = O(m2^m),$$

the conclusion follows. Let $c(\xi) = \max_{i=0, \ldots, m-1} \delta_i(\xi), \eta(\xi) = (\hat{a}(\xi/m), \ldots, \hat{a}(\xi + (m-1)/m))$. Note

$$\frac{1}{c(\xi)^m} \leq \prod_{i=0}^{m-1} \frac{1}{\delta_i(\xi)} = \prod_{0 \leq i < j \leq m-1} |\hat{a}(\xi + i/m) - \hat{a}(\xi + j/m)|^2 = |\det(V_m(\eta(\xi)))|^2.$$

Since every entry of $\eta(\xi)$ is contained in $[-1, 1]$, the Chebyshev points on $[-1, 1]$ maximize the determinant of Vandermonde matrix, see [17]. Therefore, by the formula for the determinant of a Vandermonde matrix on the Chebyshev points in [34], we get

$$|\det(V_m(\eta(\xi)))|^2 \leq \frac{m^m}{2^{(m-1)^2}}.$$ 

By (30),

$$c(\xi) \geq \frac{2^{(m-1)^2}}{m}, m^2 c(\xi) \geq O(m2^m).$$

Hence by (29)

$$||H_m^{-1}(\xi)||_\infty \to \infty, m \to \infty.$$

**Remark 2.** From our proof, we also see that $||H_m^{-1}(\xi)||_\infty$ grows at least geometrically when $m$ increases.

By Proposition 1 and Theorem 2, our results suggest that
1. For $0 \leq k \leq m-1$, the accuracy of recovering the node $\hat{a}(\xi \pm k)$ not only depends on its separation with other nodes $\delta_k(\xi)$ (see Definition 6), but also depends on the global minimal separation $\delta(\xi) = \max_{k=0,\ldots,m-1} \delta_k(\xi)$ among the nodes. Fix $m, x$, our estimations (24) and (28) suggest that error $|\Delta_k(\xi)| = |\hat{a}(\xi \pm k) - \hat{a}(\xi \pm \frac{k}{m})|$ in the worst possible case could be proportional to $\delta_k(\xi)\delta^2(\xi)$. Our numerical experiment confirms this is sharp, see Figure 2(b).

2. The accuracy of recovering all nodes is inversely proportional to the lowest magnitude of $\{\hat{x}(\xi \pm i) : i = 0, \ldots, m-1\}$.

3. Increasing $m$ may result in amplifying the error caused by the noise significantly. Since by the proof of Corollary 2, $\|H^{-1}_m\|_\infty$ grows at least geometrically when $m$ increases and a small error $\epsilon$ could be at least amplified to $O(m^{2m}\epsilon)$. Thus, When m increses, our solutions become less robust to noise.

4 Numerical Experiment

In this section, we provide some simple numerical stimulations to verify some theoretical accuracy estimations in section 3.

4.1 Experiment Setup

Suppose our filter $a$ is around the center of radius 3. For example, $\hat{a}(\xi) = 0.1 + 0.8\cos(2\pi \xi) + 0.1\cos(4\pi \xi)$, $x$ is dirac at the center so that $\hat{x}(\xi) = 1$, and $m = 3$.

1. Choose $\xi_1, \xi_2, \ldots, \xi_d$, and caculate $\hat{y}_l(\xi_i)$ and the perturbed $\tilde{\hat{y}}_l(\xi_i) = \hat{y}_l(\xi_i) + \epsilon_l$ for $l = 0, \ldots, 5$, where $y_l$ is defined as in (1) and $\epsilon_l \ll 1$.

2. Use Algorithm 2.1 to calculate the roots of $p_{Dm}(\xi)$ and the perturbed roots of $\tilde{p}_{Dm}(\xi)$ respectively, then compute $|\Delta_k(\xi)| = |\hat{a}(\xi \pm k) - \hat{a}(\xi \pm \frac{k}{m})|$.

4.2 Experiment Results

1. Sharpness of estimation (17) and (26). Fix $m$ and $x$, our estimation (17) and (26) suggest that error $|\Delta_k(\xi)| = |\hat{a}(\xi \pm k) - \hat{a}(\xi \pm \frac{k}{m})|$ in the worst possible case could be proportional to $\delta_k(\xi)\delta^2(\xi)$. In
this experiment, we choose six points $0 < \xi_1 < \cdots < \xi_6 < \frac{1}{2}$ such that $\delta_0(\xi_i)\delta(\xi_i)$ grows geometrically at rate $10^3$, and set the noise level $\epsilon \sim 10^{-14}$. In Figure 2(b), we plot the value of $\Delta_0(\xi_i) = |\hat{a}(\xi_i^3) - \hat{a}(\xi_i^4)|$ for $i = 1, \cdots , 6$. We can see that $\Delta_0(\xi_i)$ grows approximately proportionally to the growth of $\delta_0(\xi)\delta^2(\xi)$, which verified the sharpness of estimation (17) and (26).

2. Dependence of $\Delta_k(\xi)$ on the measurement error $\epsilon$. In this experiment, we fix some $0 < \xi < \frac{1}{2}$, we only change the magnitude of the error $\epsilon = \max_{l=0,\cdots,5} \epsilon_l$ such that $\epsilon$ grows geometrically and keep other parameters unchanged. We plot the value of $\Delta_k(\xi) = |\hat{a}(\xi^3) - \hat{a}(\xi^3)|$ for different noise levels. The results are presented in Figure 2(c), we show that $|\Delta_k(\xi)| \sim O(\epsilon)$ for $k = 0, 1, 2$. 

Figure 2: Experiment Results
3. The infinity norm of $H^{-1}_m(\xi)$. In this experiment, we choose $m = 2, 3, \cdots, 6$ and $\xi = 0.3$. We compute and plot the value of $\|H^{-1}_m(\xi)\|_\infty$ for different $m$. The results are presented in Figure 2(a). It is shown that $\|H^{-1}_m(\xi)\|_\infty$ grows geometrically.

5 Conclusion

In this paper, we investigate the conditions under which we can recover a typical low pass convolution filter $a \in \ell^1(\mathbb{Z})$ and a vector $x \in \ell^2(\mathbb{Z})$ from the combined regular subsampled version of the vector $x, \cdots, A^{N-1}x$ defined in (1), where $A = a \ast x$. A generalized Prony method is proposed to show that $\{x|\Omega_m, x^{(1)}|\Omega_m, \cdots, x^{(N)}|\Omega_m: N \geq 2m - 1, \Omega_m = m\mathbb{Z}\}$ contains enough information to recover $a$ almost surely. Our accuracy estimates are formulated in very simple geometric terms involving Fourier spectral function of $a$, $x$ and $m$, shedding some light on the structure of the problem. Our results suggest that when the generalized Prony method is used, the parameters of the problem are coupled to each other, in the sense that the accuracy of recovering the nodes $\{\hat{a}(\xi + i\frac{m}{m}): i = 0, \cdots, m - 1\}$ depends on the values of all the parameters at once. This unfavorable behavior is reflected by our numerical experiments in section 4. For example, the accuracy of recovering the node $\hat{a}(\xi + k\frac{m}{m})$ not only depends on its separation with other nodes $\delta_k(\xi)$ (see Definition 6), but also depends on the minimum separation $\delta(\xi) = \max_{k=0,\cdots,m-1}\delta_k(\xi)$ among the nodes. The classical Prony method performs poorly when noisy sampled data are given. In our case, we have similar issues, since our Hankel matrix $H_m(\xi)$ is ill conditioned. In practice, we can employ denoising techniques to process sampled data such as Cadzow denoising algorithm to make the method more robust to noise. However, we believe that a full answer to our somewhat rigid $\ell^\infty$ formulation of the accuracy problem may contribute to the understanding of limitations of using Prony type methods in spatiotemporal sampling.

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