ON PROPER AND EXTERIOR SEQUENTIALITY

L. ESPAÑOL, J.M. GARCÍA-CALCINES AND M.C. MÍNGUEZ.

Abstract. In this article a sequential theory in the category of spaces and proper maps is described and developed. As a natural extension a sequential theory for exterior spaces and maps is obtained.

1. Introduction

Many mathematicians have been interested in studying relationships between the topology of a space and its convergent sequences. Among these we can mention M. Fréchet [10], P. Urysohn [19], A. Arhangel’ski [1] or J. Kisynski [17]. In the sixties, R. M. Dudley [5] suggested to re-examine topology from a sequential point of view; and S. P. Franklin in 1965 [9] arrived at the satisfactory notion of sequential space. Sequential spaces are the most general class of spaces for which sequences suffice to determine the topology. This class of spaces is so large that it includes the most important and useful examples of topological spaces, such as CW-complexes, metric spaces or topological manifolds. They also form a coreflective subcategory Seq of the category Top of topological spaces, and have good categorical properties, including being complete, cocomplete and cartesian closed. Sequential space theory not only interacts in general topology and analysis, but also in topos theory, as it was shown in [15] by P.T. Johnstone. In his work P.T. Johnstone presented the category of sequential spaces as a certain subcategory of a topos of sheaves, where the embedding preserves some useful colimits and exponentials.

It is natural to ask for a sequential theory in the category of spaces and proper maps. Continuing Dudley’s program, R. Brown studied in 1973 the sequential versions of proper maps and of one-point compactification [4]. In that paper he gives sufficient conditions on any space $X$ to ensure that its Alexandroff compactification $X^+$ is sequential. He also asks for more general conditions for $X^+$ to be sequential. However, until now, it seems that nobody has noticed the lack of an analogue notion of sequential space in the proper scope. In this article we present a proper sequential theory for spaces and solve the questions posed by R. Brown. In order to do this we firstly use a slightly different notion of sequentially proper map. This notion turns out to be equivalent to that given by R. Brown when the spaces have some natural sequential properties. Then we introduce what we call $\omega$-sequential spaces, in which the closed compact subsets are completely determined by the proper sequences (and the convergent ones). This new class of spaces plays in
the category of spaces and proper maps \( P \) the same role played in \( \text{Top} \) by the sequential spaces. We also show that the \( \omega \)-sequential spaces verify the expected natural properties. Perhaps one of the most significant properties is that given any map \( f : X \to Y \) between topological spaces, where \( X \) is \( \omega \)-sequential, then \( f \) is proper if and only if \( f \) preserves convergent sequences and proper sequences (see Proposition 3.6 (ii) for more details). We also check that the \( \omega \)-sequential spaces contain enough important examples, where the CW-complexes, the metric spaces and the topological manifolds are included.

As it is known, \( P \) does not have good properties as far as limits and colimits is concerned so many topological constructions are not possible in the proper category. Subsequently, several classical sequential results cannot be transferred to the proper category. A way to solve this problem is to consider a greater category having better properties and then define a convenient notion of sequential object. For instance, we could use the Edwards-Hastings embedding \( [7] \) of the proper category of locally compact \( \sigma \)-compact Hausdorff spaces into the category of pro-spaces. From our point of view this embedding has important disadvantages. On the one hand, it is necessary to consider strong restrictions of the proper category. On the other hand many constructions give rise to pro-spaces that cannot be interpreted as regular spaces. The category of exterior spaces (see [11][12]) is a good possibility. Broadly speaking, an exterior space is a topological space with a ‘neighborhood system at infinity’ which we call externology, while an exterior map is a continuous map which is ‘continuous at infinity’. This category not only contains in its totality the proper category but also does not lose the geometric notion. Furthermore, it is complete and cocomplete. Simplicity in their description and the similarity with the classical limit and colimit constructions turn the exterior spaces to a useful and powerful tool for the study of non-compact spaces.

Taking into account the above discussion, the last part of this paper is devoted to extend our proper sequential theory to the category \( E \) of exterior spaces. Such extension begins establishing the definition of \( e \)-sequential exterior space in such a way that an \( \omega \)-sequential space is a particular case. We will analyze the new resulting category verifying that it has analogous properties to those of the classic sequential spaces. In particular the category \( E_{\text{seq}} \) of \( e \)-sequential exterior spaces is a coreflective subcategory of \( E \), a property which is not inherited in the proper case. Finally, and similarly as done in the classical topological case by P.T. Johnstone, we also prove that the category \( E_{\text{seq}} \) is a full subcategory of a topos of sheaves.

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2. Preliminary definitions and properties

First of all, we will establish the more relevant notions and properties, as well as their corresponding notation that will be used throughout this work.

Recall that a proper map is a continuous map \( f : X \to Y \) such that \( f^{-1}(K) \) is closed compact, for every closed compact subset \( K \subset Y \). Taking the filter of open subsets in a space \((X, \tau_X)\)

\[
\varepsilon_{cc}(X) = \{ U \in \tau_X; U^c \text{ is compact}\},
\]

where \( U^c \) denotes the complement of \( U \) in \( X \), then it is easy to check that \( f \) is proper if and only if it is continuous and \( f^{-1}(V) \in \varepsilon_{cc}(X), \forall V \in \varepsilon_{cc}(Y) \).
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We will denote by $\mathbf{P}$ the category of spaces and proper maps. If we consider for any space $X$ its Alexandrov compactification as a based space $(X^+, \infty)$ then we have a functor $(-)^+ : \mathbf{P} \to \mathbf{Top}^\infty$. Here $\mathbf{Top}^\infty$ denotes the subcategory of $\mathbf{Top}^*$ (the category of all based spaces and maps) whose objects are based spaces $(X, x_0)$ such that $\{x_0\}$ is closed in $X$, and the morphisms are based maps $f : (X, x_0) \to (Y, y_0)$ verifying $f^{-1}(\{y_0\}) = \{x_0\}$. On the set of natural numbers $\mathbb{N}$ we consider the discrete topology, so $\mathbb{N}^+$ represents the sequence of natural numbers with its limit $\{\infty\}$. On any space $X$ we have sequences as (continuous) maps $s : \mathbb{N} \to X$, $s(n) = x_n$. The set of all sequences in $X$ will be denoted by $X^\mathbb{N}$. On the other hand a continuous map $s : \mathbb{N}^+ \to X$ is a sequence $s$ and a limit point $s(\infty)$. A sequence $s : \mathbb{N} \to X$ is said to be convergent if it factorizes through a continuous map $\mathbb{N}^+ \to X$. The set of all convergent sequences of a space $X$ will be denoted by $\Sigma_c(X) \subset X^\mathbb{N}$, and we will write $\Sigma_c^+(X) = \mathbf{Top}(\mathbb{N}^+, X)$. We are interested in the monoid $E$ of all monotone and injective sequences $u : \mathbb{N} \to \mathbb{N}$, because each composition $s \circ u$ is a subsequence of $s$, for all $u \in \mathbb{N}$. Note that $E \subset \mathbf{P}[\mathbb{N}, \mathbb{N}]$. It is clear that for any space $X$, $\Sigma_c(X)$ is an $E$-set with the (right) action given by composition. The same is true for $X^\mathbb{N}$, and in order to ‘measure the convergency’ of an arbitrary sequence $s : \mathbb{N} \to X$ we can use the ideal

$$\langle s \in \Sigma_c(X) \rangle = \{u \in E; s \circ u \in \Sigma_c(X)\}$$

of $E$, which is equal to $E$ if and only if $s$ is convergent. We can also consider those sequences without convergent subsequences, that is, sequences $\mathcal{s}$ such that $\langle s \in \Sigma_c(X) \rangle = \emptyset$. We will denote by $\neg \Sigma_c(X) \subset X^\mathbb{N}$ the set of all such sequences, where the symbol $\neg$ is used because it corresponds to the negation in the Heyting algebra $\mathbf{H}$ of all $E$-subsets of $X^\mathbb{N}$. In this way $\neg \Sigma_c(X)$ is the biggest $E$-subset $E$ of $X^\mathbb{N}$ such that $\Sigma_c(X) \cap E = \emptyset$.

Topological notions are defined in terms of open sets, and a notion is called sequential or sequentially defined when it is defined in terms of sequences. There are the typical sequential spaces, those which can be characterized by their convergent sequences. A different kind of sequentiality is that of sequential bornological spaces, which can be characterized by their bounded sequences. Now we are interested in the sequentiality with respect to proper sequences. Proper maps are continuous maps, so proper sequentiality shall be related to the sequentiality by convergent sequences. We will assume that the reader is familiarized to sequential spaces. Nevertheless, although the reader may also refers to S. P. Franklin’s article [9] we will recall here some basic notions.

Given a space $X = (X, \tau)$, a subset $U \subset X$ is said to be sequentially open if any sequence $s : \mathbb{N} \to X$ that has a limit point $x \in U$ is eventually in $U$. This can be stated in the form

$$s \to x \in U \quad \text{implies} \quad s \propto U,$$

where $s \propto U$ means that $s^{-1}(U)$ is cofinite. The family of all sequentially open subsets in $X$ is a topology $\tau_{seq}$. A space $X$ is said to be sequential if the open subsets agree with the sequentially open subsets, that is, $\tau_{seq} = \tau$. A subset $C \subset X$ is sequentially closed if its complement in $X$ is sequentially open, that is,

$$s \to x, \quad s \propto C \quad \text{implies} \quad x \in C.$$

A map $f : X \to Y$ is sequentially continuous if it preserves convergent sequences and limits, that is, $f \circ s \in \Sigma_c^+(Y)$, for all $s \in \Sigma_c^+(X)$; in order to express this condition we will write $f \circ \Sigma_c^+(X) \subset \Sigma_c^+(Y)$. Every continuous map is sequentially continuous.
closed, so

\[\Sigma_{p} \subseteq K\]

Then sequentially compact by Theorem 2.3, so there is a convergent subsequence of

Note that if

Remark 2.5.

Theorem 2.3.

In order to prove that

Proof. In order to prove that \(\neg \Sigma_{c}(X) \subseteq \Sigma_{p}(X)\), suppose a non-proper sequence \(s : \mathbb{N} \to X\), that is, a sequence such that \(s^{-1}(K)\) is finite for every closed compact subset \(K \subseteq X\); in other words, \(s \nless U\) for all \(U \in \varepsilon_{cc}(X)\). Denoting by \(\Sigma_{p}(X)\) the set of the proper sequences in \(X\), the following result shows how different are proper sequences and convergent sequences in \(S_{2}\)-spaces.

Theorem 2.4. Let \(X\) be an \(S_{2}\)-space. Then \(\Sigma_{p}(X) = \neg \Sigma_{c}(X)\).

Remark 2.5. Note that if \(X\) is Hausdorff then the set \(K\) in the above proof is also closed, so \(\Sigma_{p}(X) \subseteq \neg \Sigma_{c}(X)\) by the proof of Theorem 2.4.

Remark 2.1. It is also important to remark that every sequentially Hausdorff space is \(T_{1}\) (so \(T_{0}\)) by [20] Th.1.

The next definition will be crucial for our purposes.

Definition 2.2. We say that a space \(X\) is \(S_{2}\) (or \(X\) is an \(S_{2}\)-space) when it is sequential and sequentially Hausdorff.

Recall also that a space is sequentially compact if any sequence has a convergent subsequence. Considering the \(\mathcal{E}\)-subsets of \(X^{\mathbb{N}}\), this means that \(\langle s \in \Sigma_{c}(X) \rangle \neq \emptyset\) for any sequence \(s\), that is, \(\neg \Sigma_{c}(X) = \emptyset\). The following theorem, due to I. Gotchev and H. Minchev [14], characterizes sequential compactness and will be important for the next result. Here, a sequentially open cover of \(X\) means a cover whose elements are sequentially open sets.

Theorem 2.3. For a \(T_{0}\) topological space \(X\) the following conditions are equivalent:

(i) \(X\) is a sequentially compact space.

(ii) Every countable sequentially open cover of \(X\) has a finite subcover.

A proper sequence in \(X\) is just a proper map \(s : \mathbb{N} \to X\), that is, a sequence such that \(s^{-1}(K)\) is finite for every closed compact subset \(K \subseteq X\); in other words, \(s \nless U\) for all \(U \in \varepsilon_{cc}(X)\). Denoting by \(\Sigma_{p}(X)\) the set of the proper sequences in \(X\), the following result shows how different are proper sequences and convergent sequences in \(S_{2}\)-spaces.

Theorem 2.4. Let \(X\) be an \(S_{2}\)-space. Then \(\Sigma_{p}(X) = \neg \Sigma_{c}(X)\).

Remark 2.5. Note that if \(X\) is Hausdorff then the set \(K\) in the above proof is also closed, so \(\Sigma_{p}(X) \subseteq \neg \Sigma_{c}(X)\) by the proof of Theorem 2.4.
In general spaces, proper sequences in $X$ are convergent sequences in its Alexandroff compactification $X^+$ with limit the based point $\infty$. We have the following statement, which is easy to prove.

**Proposition 2.6.** The functor $(-)^+ : \mathbf{P} \to \mathbf{Top}^\infty$ is full and faithful, and it induces an equivalence

$$\mathbf{P}_{lcH} \simeq \mathbf{Top}^\infty_{cH}$$

between the full subcategory $\mathbf{P}_{lcH}$ of $\mathbf{P}$ whose objects are locally compact Hausdorff spaces and the full subcategory $\mathbf{Top}^\infty_{cH}$ of $\mathbf{Top}^\infty$ whose objects are compact Hausdorff spaces.

The quasi-inverse of $(-)^+$ is obtained as follows: Given a based space $(X, x_0)$ we take $\bar{X} = X - \{x_0\}$ equipped with the relative topology $\tau_{\bar{X}} = \{A - \{x_0\} | A \in \tau_X\}$.

The condition of being Hausdorff cannot be removed in Proposition 2.6. The based space $(\mathbb{2}S, 0)$ is compact but it does not come from the Alexandroff compactification. Otherwise, $(1^+, \infty) \cong (\mathbb{2}S, 0)$ would be homeomorphic, where $1 = \{0\}$ is the one-point space. But this is impossible.

3. **On a proper notion of sequentiality: $\omega$-sequential spaces**

In this section we will give the notion of $\omega$-sequential space, which is the core of this work. These spaces play in the proper case an analogous role to that played by the sequential spaces since proper maps and sequentially proper maps between them agree (see (ii) in Proposition 3.6). We will also see that the class of $\omega$-sequential spaces contains, among others, the CW-complexes, the metric spaces and the topological manifolds.

Before going to our definition we will firstly deal with the proper sequentiality of maps.

3.1. **Sequentially proper maps.** Since a proper map is a continuous map with a condition on (closed) compact subsets, a sequentially proper map will be a sequentially continuous map with a condition on proper sequences.

**Definition 3.1.** Given spaces $X, Y$, a map $f : X \to Y$ is sequentially proper if it is sequentially continuous and it preserves proper sequences, that is,

1. $f \circ s \in \Sigma_+^c(Y)$, for all $s \in \Sigma_+^c(X)$; and
2. $f \circ s \in \Sigma_p(Y)$, for all $s \in \Sigma_p(X)$.

In other words, $f \circ \Sigma_+^c(X) \subseteq \Sigma_+^c(Y)$ and $f \circ \Sigma_p(X) \subseteq \Sigma_p(Y)$.

There is an almost obvious relationship between the sequentially proper maps and the sequentially continuous maps when we consider Alexandroff compactifications.

**Theorem 3.2.** Let $f : X \to Y$ be a map between spaces. Then $f$ is sequentially proper if and only if $f^+ : X^+ \to Y^+$ is sequentially continuous.

**Proof.** The fact that $f : X \to Y$ is sequentially continuous means that $f^+ : X^+ \to Y^+$ preserves convergent sequences with limit in $X$. Similarly, the fact that $f : X \to Y$ preserves proper sequences means that $f^+ : X^+ \to Y^+$ preserves convergent sequences with $\infty$ as limit, so the theorem follows. We leave the details to the reader. \qed
Now we compare our Definition 3.1 with a similar notion given by R. Brown [4]. We say that \( f : X \to Y \) is a sequentially proper map in the sense of Brown if it is sequentially continuous and \( f \times 1_Z : X \times Z \to Y \times Z \) is sequentially closed (that is, preserves sequentially closed subsets), for every space \( Z \). In order to avoid confusion with our definition we will say that a sequentially proper map in the sense of Brown is a B-proper map. The following theorem relates B-proper maps and sequentially proper maps.

**Theorem 3.3.** Let \( f : X \to Y \) be a function between \( S_2 \)-spaces. Then \( f \) is sequentially proper if and only if \( f \) is B-proper.

**Proof.** By Theorem 2.4 the following conditions are equivalent for \( S_2 \)-spaces:

(i) \( f \circ \Sigma_p(X) \subseteq \Sigma_p(Y) \)

(ii) \( f \circ -\Sigma_c(X) \subseteq -\Sigma_c(Y) \)

Suppose that \( f \) is sequentially continuous. Then (i) means that \( f \) is sequentially proper, and (ii) means that \( f \) is B-proper by [4, Th. 2.6]. □

**Remark 3.4.** Note that the statement ‘\( f \) B-proper implies \( f \) sequentially proper’ is true when \( X \) is just Hausdorff instead of being \( S_2 \), and the converse is also true when \( Y \) is just Hausdorff instead of being \( S_2 \).

It is clear that every proper map is sequentially proper. The following task is to find the class of spaces \( X, Y \) in which the proper maps \( f : X \to Y \) and the sequentially proper maps agree.

### 3.2. s-compact subsets and \( \omega \)-sequential spaces.

Now we give a sequential notion which is weaker than that of closed compact subset and well adapted to proper sequentiality. When these two families of subsets agree we have a \( \omega \)-sequential space.

**Definition 3.5.** Let \( X \) be a space.

(i) We say that \( C \subset X \) is s-compact if \( C \) is sequentially closed and every proper sequence is eventually in the complement of \( C \).

(ii) \( X \) is said to be \( \omega \)-sequential if it is a sequential space and the s-compact subsets agree with the closed compact subsets.

For any space \( X \) we introduce the family of sequentially open subsets

\[ \varepsilon_{sc}(X) = \{ U \subset X; U^c \text{ is s-compact} \}. \]

Note that in an \( \omega \)-sequential space \( X \), the family \( \varepsilon_{sc}(X) \) is a filter of open subsets. The latter notation was given to express the following immediate result. The proof is straightforward and left to the reader.

**Proposition 3.6.** Let \( f : X \to Y \) be a map between topological spaces. Then

(i) \( f : X \to Y \) is a sequentially proper map if and only if it is sequentially continuous and \( f^{-1}(V) \in \varepsilon_{sc}(X) \), for all \( V \in \varepsilon_{sc}(Y) \).

(ii) Suppose that \( X \) is \( \omega \)-sequential. Then \( f \) is proper if and only if \( f \) is sequentially proper.

Now we will give some interesting properties about \( \omega \)-sequential spaces. In order to do this we must give relationships between s-compact, sequentially compact and countably compact subsets. Under certain weak properties on the space \( X \) these subsets agree. Recall that a space is said to be **countably compact** when every countable open cover has a finite subcover.
Lemma 3.7. Let $X$ be a space and $C \subset X$. If $C$ is closed and $s$-compact then $C$ is countably compact. In particular, when $X$ is a sequential space, every $s$-compact subset is countably compact.

Proof. Suppose that an open cover $C = \bigcup_{n \in \mathbb{N}} U_n$ does not admit any finite subcover. Then for all $k \in \mathbb{N}$
\[ W_k = \bigcup_{i=1}^{k} U_i \not\subseteq C, \]
and the sequence of open subsets $W_1 \subset W_2 \subset \cdots \subset W_n \subset \cdots$ is such that for any $n$ there exists $k > n$ such that $W_n \not\subseteq W_k$. Set $n_1 = 1$ and pick a point $x_1 \in W_1$; next consider the smallest natural $n_2 > n_1$ such that $W_{n_1} \not\subseteq W_{n_2}$ and pick $x_2 \in W_{n_2} - W_{n_1}$. Thus, coming from an inductive process, we obtain a strictly increasing sequence of natural numbers $u \in \mathbb{E}$, defined as $u(k) = n_k$, and a sequence $s : \mathbb{N} \to X$, $s(k) = x_k$, such that $s(1) \in W_1$, $s(k+1) \in W_{u(k+1)} - W_{u(k)}$, $k \in \mathbb{N}$. This sequence is proper; indeed, if $L$ is any closed compact subset of $X$ then $K = L \cap C$ is a closed compact verifying that $s^{-1}(K) = s^{-1}(L)$. But $K \subset C = \bigcup_{k=1}^{\infty} W_k$ implies $K \subset \bigcup_{k=1}^{\infty} W_k \subseteq W_p$ for some $p$, so $s^{-1}(K) \subset s^{-1}(W_p) = \{n_1, n_2, \ldots, n_p\}$ is finite. Therefore, $s$ is proper and $s^{-1}(C)$ must be finite. But $s^{-1}(C) = \mathbb{N}$, which is a contradiction.

We obtain the following useful result.

Proposition 3.8. Let $X$ be an $S_2$-space and $C \subset X$. The following statements are equivalent:

(i) $C$ is $s$-compact.
(ii) $C$ is countably compact.
(iii) $C$ is sequentially compact.

Proof. By Theorem 2.3 (ii) and (iii) are equivalent; and by the above lemma (i) implies (ii). Now we prove (iii) implies (i). Let $C \subset X$ be a sequentially compact subset. Then $C$ is sequentially closed since $X$ is sequentially Hausdorff. On the other hand consider $s : \mathbb{N} \to X$ any proper sequence. If $s^{-1}(C)$ were infinite, then it would exist a subsequence $s \circ u$, $u \in \mathbb{E}$ in $C$ and therefore a convergent subsequence $s \circ u \circ v$, $v \in \mathbb{E}$, which is not possible by Theorem 2.4.

3.3. Brown’s questions. Given an open $U$ in a space $X$, we consider the set $\Sigma_U(X)$ of sequences $s : \mathbb{N} \to X$ such that $s \propto U$. It is clear that for any space $X$ with topology $\tau$, the family
\[ \varepsilon = \{ U \in \tau; -\Sigma_{\tau}(X) \subseteq \Sigma_U(X) \} \]
is a filter of open subsets. Now we consider the set $X \cup \{\infty\}$ equipped with the topology $\tau^\wedge = \tau \cup \varepsilon_\infty$, where $\varepsilon_\infty = \{ U \cup \{\infty\}; U \in \varepsilon \}$. Thus we get a space $X^\wedge$, which is the one-point sequential compactification defined by R. Brown [4]. If $X$ is already sequentially compact, then $\tau^\wedge$ is the coproduct topology. Brown proves that: (1) $X^\wedge = X^+$ if both $X$ and $X^+$ are $S_2$, and (2) $X^+$ is sequential if $X$ is first countable and a countable union of closed compact subsets $K_i$ such that every compact subset is contained in some $K_i$. Finally, he poses this problem: ‘find more general conditions for $X^+$ to be sequential’.

Now we will give a satisfactory solution to this problem and improve these earlier results. Namely, we will prove that the statement ‘$X^+$ is sequential’ is equivalent to
‘X is ω-sequential’. First, we will see that, under the not very restrictive condition of being $S_2$, the ω-sequential condition is equivalent to $X^\omega = X^\omega$.

**Theorem 3.9.** An $S_2$-space $X$ is ω-sequential if and only if $X^\omega = X^\omega$.

**Proof.** By Theorem 2.4, $\Sigma(X) = \neg \Sigma_c(X)$. Hence, the s-compact subsets agree with the closed subsets $C$ such that $\neg \Sigma_c(X) \subset \Sigma_U(X)$, where $U$ is the complement of $C$ in $X$. When $X$ is ω-sequential these subsets clearly agree with the closed compact subsets of $X$, so $X^\omega = X^\omega$. Conversely, when $X^\omega = X^\omega$ the s-compact subsets agree with the closed compact subsets, that is, $X$ is ω-sequential. □

And now we give our characterization.

**Theorem 3.10.** A space $X$ is ω-sequential if and only if $X^\omega$ is sequential.

**Proof.** Suppose that $X$ is ω-sequential and let $V \subset X^\omega$ be any sequentially open subset. Then we must prove that $V$ is an open subset of $X^\omega$. It is clear that $U = V - \{\infty\} \subset X$ is sequentially open in $X$ and therefore $U$ is an open subset in $X$. If $\infty \notin V$ then $V = U$ is also open in $X^\omega$. Now assume that $\infty \in V$; taking into account that a sequence in $X$ is proper if and only if it converges to $\infty$ in $X^\omega$ we have that $\Sigma_p(X) \subset \Sigma_U(X)$. Hence, the complement $C$ of $U$ in $X$ is s-compact. By the hypothesis on $X$, $C$ is closed compact, so $V \subset X^\omega$ is open.

Conversely, suppose that $X^\omega$ is a sequential space. Since $X$ is open in $X^\omega$ then $X$ is also a sequential space. Now, we will check that every s-compact subset $C$ is closed compact. If $U$ is the complement of $C$ in $X$ and $V = U \cup \{\infty\}$, we must prove that $V$ is open, that is, $V$ is a sequentially open subset of $X^\omega$. Consider a sequence $s : \mathbb{N} \to X^\omega$ such that $s \to x \in V$; then we have $s \propto V$. Indeed, if $x \neq \infty$ or $s$ is eventually constant at $\infty$, then the condition $s \propto V$ is clear. Otherwise, the complement $A$ of $s^{-1}(\infty)$ in $\mathbb{N}$ is infinite, and we can take the map $u \in E$ enumerating $A$, so $t = s \circ u$ is a subsequence of $s$ contained in $V$. Moreover $t$ is proper, so $t \propto U$ because $C$ is s-compact. This fact implies that $s \propto V$. □

**Remark 3.11.** As a consequence of the above theorem, if $\mathbf{Pseq}_{lcH}$ denotes the full subcategory of $\mathbf{P}_{lcH}$ whose objects are ω-sequential spaces and $\mathbf{Seq}_{lcH}$ the full subcategory of $\mathbf{Top}_{lcH}$ whose objects are sequential spaces we have, by Proposition 2.6 an equivalence of categories

$$(-)^+: \mathbf{Pseq}_{lcH} \xrightarrow{\sim} \mathbf{Seq}_{lcH}.$$

### 3.4. Examples of ω-sequential spaces

We finish this section giving some examples of ω-sequential spaces. For the topological notions and relations involved in the next proposition see for instance [6].

**Proposition 3.12.** The following spaces are ω-sequential:

(i) Sequential and paracompact spaces.

(ii) Sequential Lindelöf spaces.

(iii) Hausdorff and second countable spaces.

**Proof.** (i) Consider $C$ any s-compact subspace of $X$. Since $X$ is sequential $C$ is closed, so it is also paracompact. On the other hand $C$ is countably compact by Lemma 5.7. But every paracompact countably compact space is compact.

(ii) Similarly, if $C$ is any s-compact subset, we have that $C$ is closed and therefore Lindelöf. Being $C$ countably compact and Lindelöf, $C$ is compact.
(iii) Every second countable Hausdorff space is Lindelöf and first countable (in particular sequential). Hence we may apply (ii).

From the above proposition it follows that CW-complexes, metric spaces, and usual topological manifolds are \(\omega\)-sequential spaces. We can find other examples in the spaces \(X\) considered by R. Brown (first countable spaces which are countable union of closed compact subsets \(K_i\) such that every compact is contained in some \(K_i\)). In this case \(X^+\) is first countable and we may apply Theorem 3.10. The spaces studied by R. Brown are \(\sigma\)-compact but CW-complexes are examples of \(\omega\)-sequential spaces which are not necessarily \(\sigma\)-compact.

The Hausdorff, locally compact and \(\sigma\)-compact spaces considered by Edwards and Hastings in [7] are also \(\omega\)-sequential if they are also first countable.

Now we give an example of sequential space which is not \(\omega\)-sequential. Consider \(X = [0, \Omega)\) the ordinal space [6], where \(\Omega\) is the first non countable ordinal. Then \(X\) is sequentially compact so \(X^\wedge\) is the coproduct of \(X\) with \(\{\infty\}\). Then \(X^\wedge \neq X^+\), since \(X\) is not compact. By Theorem 3.9 we conclude that \(X\) is not \(\omega\)-sequential.

4. Sequentiality in exterior spaces

As we have commented in the introduction, the category \(P\) of spaces and proper maps does not have good categorical properties so several classical sequential results cannot be translated to \(P\). Recall the embedding \((-)^+ : P \hookrightarrow \text{Top}^\omega\) given in Proposition 2.6. Now we will define a category equivalent to \(\text{Top}^\omega\), which is the category \(E\) of exterior spaces and maps. This category, which was firstly presented in [11] and [12], contains the proper category and is complete and cocomplete. It is for this reason that we complete our analysis considering the category of exterior spaces.

4.1. Exterior spaces. Now we will provide some necessary background about exterior spaces. For a detailed and ampler vision of this topic, [11] and [12] can be consulted.

An externology in a space \((X, \tau)\) is a nonempty subfamily \(\varepsilon \subset \tau\) which is a filter of open subsets. This means that \(\varepsilon\) is closed under finite intersections and \(U \in \tau\), \(E \in \varepsilon\), \(U \supset E\) implies \(U \in \varepsilon\). Note that \(\varepsilon\) is a filter in the lattice \(\tau\), not a filter on the set \(X\) as in set theory. So in our algebraic sense, \(\tau\) is the maximal filter of open subsets, and \(\{X\}\) is the minimal filter of open subsets (see [16]). An exterior space \((X, \varepsilon \subset \tau)\) consists of a topological space together with an externology. The elements of \(\varepsilon\) are called exterior-open subsets or, in short, e-open subsets.

Given an space \(X = (X, \tau)\), we may consider the discrete exterior space \(X_d = (X, \varepsilon = \tau)\), and the indiscrete exterior space \(X_i = (X, \{X\} \subset \tau)\). An exterior space \((X, \varepsilon \subset \tau)\) is discrete if and only the empty set is contained in the filter \(\varepsilon\).

A function between exterior spaces, \(f : (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')\), is said to be an exterior map if it is continuous and

\[
 f^{-1}(E) \in \varepsilon, \forall E \in \varepsilon'.
\]

The category of exterior spaces and exterior maps will be denoted by \(E\).
The externology $\varepsilon_{cc}$ constituted by the complements of the closed compact subsets (see Section 2) is called the \textit{cocompact externology}. The exterior space

$$X_{cc} = (X, \varepsilon_{cc} \subset \tau)$$

is a topological space enriched with a system of open neighborhoods at infinity. It is clear that we have a full and faithful functor

$$(-)_{cc} : \mathcal{P} \to \mathcal{E}$$

The category $\mathcal{E}$ has better properties than $\mathcal{P}$. In particular it has limits and colimits given respectively by final and initial structures with respect to the forgetful functor $O : \mathcal{E} \to \text{Top}$, which has two adjoints: $(-)_{d} \dashv O \dashv (-)_{i} : \text{Top} \to \mathcal{E}$.

\textbf{Remark 4.1.} We also note that $\emptyset (\emptyset_{d} = \emptyset_{i})$ is the initial object in $\mathcal{E}$, and $\mathbf{1}_{i}$ is the final object. It is clear that $\mathbf{1}_{i}$ and $\mathbf{1}_{d}$ are non-isomorphic exterior spaces. Every exterior map $\mathbf{1}_{d} \to X$ can be considered as an element of $X$, but an exterior map $\mathbf{1}_{i} \to X$ is an element which belongs to all the open subsets of the filter.

Let us call limit of $(X, \varepsilon \subset \tau)$ the intersection $\ell(X)$ of all open subsets of the filter. Elements of $\ell(X)$ are limit points of the exterior space. For every subset $A \subset X$, the space $X$ endowed with the filter $\mathcal{U}(A) = \{U \in \tau; A \subset U\}$ is an exterior space such that $\ell(X) = A$. It is clear that $\ell(X_{d}) = \emptyset$, but also $\ell(N) = \emptyset$, where $N$ is the discrete space with the cofinite externology $\varepsilon_{cc}$. For every exterior map $f : X \to Y$ we have $f(\ell(X)) \subset \ell(Y)$. In particular, every exterior map $X_{i} \to Y$ must have its image contained in $\ell(Y)$. Thus $(-)_{i} \dashv \ell : \mathcal{E} \to \text{Top}$, where we consider $\ell(Y)$ as an indiscrete space.

If $X$ is an exterior space, an \textit{exterior sequence} is an exterior map $N \to X$, where $N$ is the discrete space with the externology $\varepsilon_{cc}$, that is, the cofinite externology. In other words, a sequence $s : N \to X$ is exterior if

$$s \propto U, \forall U \in \varepsilon.$$ 

The set of all exterior sequences $N \to X$ will be denoted by $\Sigma_{e}(X)$. In particular, $\mathcal{E}$ is a submonoid of $\Sigma_{e}(N) = \Sigma_{e}(\mathbb{N})$, so $\Sigma_{e}(X)$ is an $\mathcal{E}$-subsets of $X^{N}$. We also note that Theorem 2.4 is a very particular case, because there are exterior spaces satisfying $\Sigma_{e}(X) \subset \Sigma_{e}(\mathbb{N})$. Take for instance the filter of open subsets $\mathcal{U}(\{x\})$, $x \in X$.

\subsection*{4.2. Sequentially exterior maps, and e-sequential exterior spaces.} Now we analyze the sequentiality of exterior spaces and maps, extending the sequentiality of spaces and proper maps. The next definition and theorem are analogous to Definition 3.1 and Theorem 3.2 respectively. Now we formulate them in the general exterior scope.

\textbf{Definition 4.2.} A map between exterior spaces $f : X \to Y$ is said to be sequentially exterior or e-sequential if it is sequentially continuous and it preserves exterior sequences, that is,

1. $f \circ s \in \Sigma_{e}^{+}(Y),$ for all $s \in \Sigma_{e}^{+}(X)$; and
2. $f \circ s \in \Sigma_{e}(Y),$ for all $s \in \Sigma_{e}(X)$.

In other words $f \circ \Sigma_{e}^{+}(X) \subset \Sigma_{e}^{+}(Y)$ and $f \circ \Sigma_{e}(X) \subset \Sigma_{e}(Y)$.

If $(X, \varepsilon_{X} \subset \tau_{X})$ is an exterior space and $\infty$ is a point which does not belong to $X$ then we consider the based space $X^{\infty} = X \cup \{\infty\}$ with base point $\infty$, and topology $\tau^{\infty} = \tau_{X} \cup \{E \cup \{\infty\} : E \in \varepsilon_{X}\}$. In this way the canonical inclusion
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\( X \hookrightarrow X^\infty \) is a homeomorphism onto its image. Notice that Brown’s construction \( X^\wedge \) in Section 3 is a particular case of \( X^\infty \).

If \( f : X \to X' \) is an exterior map, we define \( f^\infty : X^\infty \to X'^\infty \) by \( f^\infty(x) = f(x) \) when \( x \in X \) and \( f^\infty(\infty) = \infty \). Thus we obtain a functor

\[ (-)^\infty : E \to \text{Top}^\infty, \]

which is an equivalence of categories. Indeed, an easy verification shows that it is full and faithful. On the other hand, the quasi-inverse is described as follows: Let \( (X, x_0) \) be an object in \( \text{Top}^\infty \); then consider \( \bar{X} = X - \{x_0\} \) equipped with

\[ \tau_{\bar{X}} = \{A - \{x_0\} : A \in \tau_X \} \subset \tau_X, \quad \varepsilon_{\bar{X}} = \{A - \{x_0\} : A \in \tau_X, x_0 \notin A \}. \]

Then \( (\bar{X}^\infty, \infty) \cong (X, x_0) \) in \( \text{Top}^\infty \) by means of

\[ \alpha : (\bar{X}^\infty, \infty) \to (X, x_0), \quad \alpha(x) = \begin{cases} x, & \text{if } x \in \bar{X} \\ \infty, & \text{if } x = x_0 \end{cases} \]

As a consequence, the category \( \text{Top}^\infty \) is complete and cocomplete. The initial object is \((1, 0)\) and the final object is \((2_S, 0)\). It is also clear that we have a commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{(-)^e} & E \\
\downarrow \cong & & \downarrow \cong \\
\text{Top}^\infty & \xrightarrow{(-)^{\infty}} & \text{Top}^\infty
\end{array}
\]

The next result gives us a relationship between the sequentially exterior maps and the sequential continuous maps when we consider the functor \((-)^\infty\). We leave the proof to the reader, since it is analogous to that given in the previous section.

**Theorem 4.3.** Let \( f : X \to Y \) be a map between exterior spaces. Then \( f \) is e-sequential if and only if \( f^\infty : X^\infty \to Y^\infty \) is sequentially continuous.

As in the proper case we want to find a suitable class of exterior spaces in which exterior maps and sequentially exterior maps agree. Now we will give a notion which corresponds in the proper case to the complements of s-compacts. In fact, Definition 3.5 and Proposition 3.6 have a natural extension to the exterior setting.

**Definition 4.4.** Let \( X \) be an exterior space.

(i) We say that \( E \subset X \) is a sequentially e-open subset if it is a sequentially open subset and every exterior sequence is eventually in \( E \).

(ii) \( X \) is said to be e-sequential if it is a sequential space and every sequentially e-open subset is e-open.

Given a space \( X \), we have the family of subsets

\[ \varepsilon_{\text{seq}}(X) = \{U \subset X ; U \text{ is sequentially e-open} \}. \]

Obviously, when \( X \) is an e-sequential space then \( \varepsilon_{\text{seq}}(X) \) is a filter of open subsets.

**Remark 4.5.** Let us recall that an exterior set (see [13]) is an exterior space with discrete topology. The category \( \text{ESet} \) of exterior sets is a full subcategory of \( E \). Given an exterior set \( (X, \varepsilon) \), the final filter on \( X \) with respect to \( \Sigma_\varepsilon(X) \) is

\[ \bar{\varepsilon} = \{U \subset X ; s \propto U, \forall s \in \Sigma_\varepsilon(X) \}. \]

It is clear that \( \varepsilon \subset \bar{\varepsilon} \), and, following Definition 3.4, an exterior set is e-sequential if \( \varepsilon = \bar{\varepsilon} \). This particular definition was used in [8].
We obtain the following immediate result:

**Proposition 4.6.** Let \( f : X \to Y \) be a map between exterior spaces. Then

(i) \( f : X \to Y \) is e-sequential if and only if it is sequentially continuous and 
\( f^{-1}(V) \in \varepsilon_{\text{seq}}(X), \forall V \in \varepsilon_{\text{seq}}(Y). \)

(ii) Suppose that \( X \) is e-sequential. Then \( f \) is exterior if and only if \( f \) is e-sequential.

The next result is completely analogous to that of Theorem 3.10 so its proof is omitted and left to the reader.

**Theorem 4.7.** An exterior space \( X \) is e-sequential if and only if \( X^{\infty} \) is a sequential space.

**Remark 4.8.** If \( \text{Seq}^{\infty} \) denotes the full subcategory of \( \text{Top}^{\infty} \) whose objects are sequential spaces, and \( \text{Eseq} \) the full subcategory of \( \mathcal{E} \) whose objects are the e-sequential exterior spaces, then we have an equivalence of categories

\[ (-)^{\infty} : \text{Eseq} \xrightarrow{\cong} \text{Seq}^{\infty}. \]

As a new example of exterior spaces we consider the exterior version of first countable spaces, which are sequential. After a convenient definition, properties of exterior spaces become properties of spaces.

**Definition 4.9.** Let \( X \) be an exterior space.

(i) An exterior base in \( X \) is a nonempty collection of e-open subsets \( \beta \subset \varepsilon \) such that for every e-open \( E \) one can find \( B \in \beta \) such that \( B \subset E \).

(ii) \( X \) is said to be e-first countable (or first countable at infinity) if it is first countable and it has a countable exterior base.

**Proposition 4.10.**

(i) An exterior space \( X \) is e-first countable if and only if \( X^{\infty} \) is a first countable space.

(ii) Every e-first countable exterior space \( X \) is e-sequential.

**Remark 4.11.** Proposition 4.10(ii) follows immediately from Theorem 4.7 and Proposition 4.10(i), but it is also easy to prove it directly. In general, these exterior spaces are simpler to handle. They are the equivalent, in the category of pro-spaces, to the towers of spaces (see for example [7]). In addition, they play an important role in the theory of sequential homology, which is defined in [12]. As examples of this nature we can mention the ones of the form \( X_{cc} \), where \( X \) is a first countable, \( \sigma \)-compact, locally compact Hausdorff space.

On the other hand if \( \mathcal{E}_{\text{efc}} \) denotes the full subcategory of \( \mathcal{E} \) whose objects are e-first countable exterior spaces, and \( \text{Top}^{\infty}_{\text{fc}} \) denotes the full subcategory of \( \text{Top}^{\infty} \) whose objects are first countable spaces, then we have an equivalence

\[ (-)^{\infty} : \mathcal{E}_{\text{efc}} \xrightarrow{\cong} \text{Top}^{\infty}_{\text{fc}}. \]

Naturally, \( F \subset X \) is a sequentially e-closed subset if its complement in \( X \) is sequentially e-open. Every e-open (e-closed) subset is sequentially e-open (e-closed) subset. Recall that a sequential space \( X \) has the final topology with respect to its continuous maps \( \mathbb{N}^{+} \to X \). In the same way, it is clear that an e-sequential space \( X \) has the final externology with respect to its continuous maps \( \mathbb{N}^{+} \to X \) and its exterior sequences \( \mathbb{N} \to X \).
Many properties of the e-sequential exterior spaces are analogous to the properties of sequential spaces, and they are proved in a similar way. For instance, $\text{Eseq}$ is a coreflective subcategory of $\text{E}$. Recall that given an exterior space $X = (X, \varepsilon \subset \tau)$, we can consider $\tau_{\text{seq}}$ and $\varepsilon_{\text{seq}} = \{U \subset X; U$ is sequentially e-open} $\subset \tau_{\text{seq}}$. Then we have an e-sequential exterior space $\sigma X = (X, \varepsilon_{\text{seq}} \subset \tau_{\text{seq}})$. With $\sigma(f) = f$ we complete the definition of the coreflector functor $\sigma : \text{E} \rightarrow \text{Eseq}$.

The subcategory $\text{Eseq}$ has limits and colimits. The colimits are taken in $\text{E}$ and for limits in $\text{Eseq}$ we take limits in $\text{E}$ and then we apply the coreflector functor. The counit of the coreflection is the identity $\sigma X \rightarrow X$, and an exterior space is e-sequential if and only if the counit is an isomorphism. Hence:

**Proposition 4.12.** Let $X$ be an exterior space. The following statements are equivalent:

1. $X$ is e-sequential.
2. For all map $f : X \rightarrow Y$ with $Y$ exterior space, $f$ is exterior if and only if $f$ is e-sequential.

**Remark 4.13.** The property of being e-sequential is not hereditary since being sequential is not. Nevertheless, it is straightforward to check that in an e-sequential exterior space the e-open and e-closed subsets are e-sequential exterior spaces.

On the other hand the image of an e-sequential exterior space is not necessarily e-sequential. Take for instance an identity map $\text{id}_X : (X, \{X\} \subset \tau_d) \rightarrow (X, \{X\} \subset \tau)$ where $\tau_d$ is the discrete topology and $\tau$ is any non-sequential topology. However, an exterior quotient (that is, a quotient space with the corresponding final externology) of any e-sequential exterior space is an e-sequential exterior space.

If $X$ is any space, an immediate verification shows that $X$ is $\omega$-sequential if and only if $X_{\text{cc}}$ is e-sequential. Hence we have a commutative diagram

$$
\begin{array}{ccc}
\text{Eseq} & \hookrightarrow & \text{E} \\
(-)_{\text{cc}} & \downarrow & \downarrow (-)_{\text{cc}} \\
\text{Pseq} & \hookrightarrow & \text{P}
\end{array}
$$

where $\text{Pseq}$ denotes the full subcategory of $\text{P}$ whose objects are $\omega$-sequential spaces. Of course, there are e-sequential exterior spaces which do not come from any $\omega$-sequential space; for instance, a first countable space $X$, provided with the indiscrete externology.

Finally, we give an example of space $X$ such that $\sigma(X_{\text{cc}})$ does not belongs to $\text{Pseq}$. Take the exterior space $X_{\text{cc}}$, where $X = [0, \Omega)$ is the ordinal space considered at the end of Section 3. It is clear that $\sigma(X_{\text{cc}})$ is equal to $X$ as topological spaces, because $X$ is sequential. On the other hand $\Sigma_p(X) = \emptyset$, so $\sigma(X_{\text{cc}})$ has the discrete externology. Hence $\sigma(X_{\text{cc}})$ is not $X_{\text{cc}}$ because $X$ is not compact.

**4.3. Sequential exterior spaces as sheaves.** We finish this paper showing a new extension of results from topological spaces to exterior spaces. Namely we will prove that the category $\text{Eseq}$ is a full subcategory of certain topos of sheaves. Subsequently, since $(-)_{\text{cc}} : \text{Pseq} \hookrightarrow \text{Eseq}$ is a full embedding, $\text{Pseq}$ will also be a full subcategory of a topos of sheaves.
Let \( \mathbb{C} \) be the subcategory of \( \mathbb{E} \) defined by the objects \( 1 \equiv 1_d, \mathbb{N}^+ \equiv \mathbb{N}^+_d, \mathbb{N} \equiv \mathbb{N}_{cc} \) and the following morphisms:

\[
\begin{array}{c}
\mathbb{N}^+ \ni u \in \mathbb{M}^+ \\
\downarrow \\
\mathbb{N} \ni c_n \in \mathbb{C} \\
\downarrow \\
\mathbb{N}^+ \ni n \in \mathbb{M}
\end{array}
\]

where \( C \) denotes the set of all constant maps \( c_n = n \circ ! : \mathbb{N}^+ \to \mathbb{N} \) (we shall also use \( C \) for the constant maps \( \mathbb{N}^+ \to \mathbb{N}^+ \)), and the monoids on the objects \( \mathbb{N}^+, \mathbb{N} \) are respectively \( \mathbb{M}^+ = \Sigma^+_{\mathbb{N}^+} \) and \( \mathbb{M} = \Sigma_{\mathbb{N}}(\mathbb{N}) \).

Let us denote \( \mathbb{C}\text{-Set} \) the category of presheaves over \( \mathbb{C} \) or \( \mathbb{C} \)-sets. Then, we may see each \( \mathbb{C} \)-set as a commutative diagram

\[
\begin{array}{c}
\mathbb{X}_c \ni cte \ni \mathbb{X}_e \\
\downarrow cte \downarrow \\
\mathbb{X} \ni ev_n \\
\downarrow \\
\mathbb{X}_c \ni ev_n, ev_\infty
\end{array}
\]

where \( X_c \) is an \( M^+ \)-set, \( X_e \) an \( M \)-set, and the compositions with the formal constant and evaluation maps are clear: \( c_n = cte \circ ev_n, ev_n(s \circ u) = ev_{n(u)}(s) \), etcetera. Briefly, we shall denote a \( \mathbb{C} \)-set \( P \) as a triple

\[
P = (X, X_c, X_e)
\]

where the morphisms can be forgotten in the notation. Given another \( \mathbb{C} \)-set, \( Q = (Y, Y_c, Y_e) \), a \( \mathbb{C} \)-map \( \phi : P \to Q \) is given by a triple \( \phi = (f, f_c, f_e) \), where \( f : X \to Y \) is a map, \( f_c : X_c \to Y_c \) an \( M^+ \)-map (equivariant), \( f_e : X_e \to Y_e \) an \( M \)-map, and the compositions with constant and evaluation maps are satisfied.

It is clear that a functor \( \Sigma : \mathbb{E} \to \mathbb{C}\text{-Set} \) is defined by \( \Sigma(X) = \mathbb{E}(-, X) \). Briefly,

\[
\Sigma(X) = (X, \Sigma^+_c(X), \Sigma_c(X))
\]

and if \( f : X \to Y \) is an exterior map, then the natural transformation \( \Sigma(f) : \Sigma(X) \to \Sigma(Y) \) is formed by \( f, f_c = f \circ (-) : \Sigma^+_c(X) \to \Sigma^+_c(Y), \) and \( f_e = f \circ (-) : \Sigma_c(X) \to \Sigma_c(Y) \). Note that the functor \( \Sigma^+_c(X) \) was used in [15] to embed \( \text{Seq} \) in the topos \( M^+\text{-Set} \) of \( M^+ \)-sets, and \( \Sigma_e : \mathbb{E} \to \mathbb{M}\text{-Set} \) is a construction used in [13].

If we consider the Yoneda embedding \( y : \mathbb{C} \hookrightarrow \mathbb{C}\text{-Set} \) and the representable \( \mathbb{C} \)-sets we get \( y(1) = \Sigma(1_d) = (1, 1, \emptyset), \) \( y(\mathbb{N}^+) = \Sigma(\mathbb{N}^+_d) = (\mathbb{N}^+, M^+, \emptyset), \) and \( y(\mathbb{N}) = \Sigma(\mathbb{N}) = (\mathbb{N}, C, M) \). That is, the Yoneda functor is the restriction of \( \Sigma \) to \( \mathbb{C} \). For
instance, the diagram of $y(\mathbb{N})$ is, with $\mathbb{N} \cong C$:

![Diagram](image)

We point out that since $E$ has colimits and $\Sigma(X)(A) = E(A, X)$ for any object $A$ of $\mathbb{C}$, then $\Sigma$ has a left adjoint by the universal property of the categories of presheaves. Moreover all the three objects of $\mathbb{C}$ are e-sequential, hence the last remark is also true when $\Sigma$ is defined on $\text{Eseq}$.

Actually, the functor $\Sigma$ has a better behavior when its domain is restricted to $\text{Eseq}$. In fact, we have the following result:

**Proposition 4.14.** The functor $\Sigma : \text{Eseq} \to \text{CSet}$ is full, faithful and injective on objects.

**Proof.** It is clear that $\Sigma$ is faithful, and it is injective on objects because $\text{Eseq}$ is a coreflective subcategory of $E$. Finally, given a natural transformation $(f, f_c, f_e) : \Sigma(X) \to \Sigma(Y)$, it is easy to verify by naturality that $f_c = f \circ (-)$; and the same is true in the exterior component $f_e$. Hence a categorical reading of Proposition 4.6(ii) shows that $\Sigma$ is full. $\square$

From now on we consider e-sequential spaces as $\mathbb{C}$-sets by the full embedding $\Sigma : \text{Eseq} \hookrightarrow \text{CSet}$. Thus an e-sequential space is completely determined by its points, convergent sequences and exterior sequences, together with the maps between them given by the notion of $\mathbb{C}$-set. Our last goal is to determine a smaller topos of sheaves $E \hookrightarrow \text{CSet}$ such that $\Sigma : \text{Eseq} \hookrightarrow E$.

Recall from [15] (with a different formulation based on monoids) that given a topological space $X$, the $M^+$-set $\Sigma^e(X)$ is a sheaf when we consider on the monoid $M^+$ the Grothendieck topology $\mathbb{J}_e$ formed by the ideals $I$ such that:

1. $C \subseteq I$,
2. $\forall u \in E, \exists v \in E : u \circ v \in I$,

where $C$ is the set of all constant maps in $M^+$, and $E$ is the submonoid of $M^+$ constituted by all the monotone and injective maps $u : \mathbb{N}^+ \to \mathbb{N}^+$ (hence $u^{-1}(\infty) = \{\infty\}$). We can translate this scheme to the monoid $M$, which has not constants. By deleting $u(\infty) = \infty$, we can see $E$ as a submonoid of $M$ and define a family $\mathbb{J}_e$ of ideals of $M$ by means of the condition (ii) above. Then $\mathbb{J}_e$ is a Grothendieck topology on $M$ $\mathbb{S}$ (actually the double negation topology). The proof of the following result is long but straightforward.

**Lemma 4.15.** For any object $A$ of $\mathbb{C}$ we define a collection $\mathbb{J}(A)$ of families of morphisms of $\mathbb{C}$ with codomain $A$ as follows:

1. The unique family in $\mathbb{J}(1)$ is all the morphisms with codomain $1$.
2. A family in $\mathbb{J}(\mathbb{N}^+)$, denoted $\hat{I}$, is formed by all the morphisms $1 \to \mathbb{N}^+$ and the morphisms $\mathbb{N}^+ \to \mathbb{N}^+$ of an ideal $I \in \mathbb{J}_e$. 
(iii) A family in \( \mathcal{J}(\mathbb{N}) \), denoted \( \hat{I} \), is formed by all the morphisms \( 1 \to \mathbb{N} \), all the morphisms \( \mathbb{N}^+ \to \mathbb{N} \), and the morphisms \( \mathbb{N} \to \mathbb{N} \) of an ideal \( I \in \mathcal{J}_e \).

Then \( \mathcal{J} \) is a Grothendieck topology on \( \mathbb{C} \).

We denote the subtopos of sheaves \( \mathcal{E} = \text{sh}(\mathbb{C}, \mathcal{J}) \hookrightarrow \mathbb{C}\text{Set} \).

**Theorem 4.16.** The embedding \( \Sigma : \text{Eseq} \hookrightarrow \mathcal{E} \) holds.

**Proof.** By Proposition 4.14 it suffices to prove that \( \Sigma(X) \) is a sheaf for any e-sequential space \( X \). This should be done on the three objects \( 1, \mathbb{N}^+, \mathbb{N} \). The case \( 1 \) is obvious and for the case \( \mathbb{N}^+ \) we refer the reader to [15]. The case \( \mathbb{N} \) is analogous to the last one. We consider \( \hat{I} \) as a presheaf of \( y(\mathbb{N}) \). Given a natural transformation \( \theta : \hat{I} \to \Sigma(X) \) we must find a unique \( s \in \Sigma_c(X) \) such that \( \theta \) is the restriction of \( y(s) = (s, s \circ (-), s \circ (-)) \). But \( \theta \) is of the form \( \theta = (s, s \circ (-), H) \), where \( s : \mathbb{N} \to X \) and \( H : \hat{I} \to \Sigma_c(X) \) is an \( M \)-equivariant map; and for any \( n \in \mathbb{N} \), \( ev_n \circ H = s \circ ev_n \), that is, \( H(g) = s \circ g \) for any \( g \in I \). Since \( I \in \mathcal{J}_e \) and each \( H(g) \) is exterior [8] the latter condition means that \( s \) is exterior. \( \square \)

**Remark 4.17.** Note that the sequentiality is not needed for the sheaf condition, but only to apply Proposition 4.14.

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L. ESPAÑOL GONZÁLEZ; M.C. MÍNGUEZ HERRERO
DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN
UNIVERSIDAD DE LA RIOJA
26004 LOGROÑO
E-mail address: luis.espanol@dmc.unirioja.es
E-mail address: carmen.minguez@dmc.unirioja.es

J.M. GARCÍA CALCINES
DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL
UNIVERSIDAD DE LA LAGUNA
38271 LA LAGUNA.
E-mail address: jmgarcal@ull.es