String theory predictions for future accelerators

E. Dudas\textsuperscript{a} and J. Mourad\textsuperscript{b}\textsuperscript{†}

\textsuperscript{a} LPT\textsuperscript{‡}, Bât. 210, Univ. de Paris-Sud, F-91405 Orsay, France
\textsuperscript{b} LPTM, Site Neuville III, Univ. de Cergy-Pontoise, Neuville sur Oise
F-95031 Cergy-Pontoise, France

January 31, 2022

Abstract

We consider, in a string theory framework, physical processes of phenomenological interest in models with a low string scale. The amplitudes we study involve tree-level virtual gravitational exchange, divergent in a field-theoretical treatment, and massive gravitons emission, which are the main signatures of this class of models. First, we discuss the regularization of summations appearing in virtual gravitational (closed string) Kaluza-Klein exchanges in Type I strings. We argue that a convenient manifestly ultraviolet convergent low energy limit of type I string theory is given by an effective field theory with an arbitrary cutoff \( \Lambda \) in the closed (gravitational) channel and a related cutoff \( M^2_s/\Lambda \) in the open (Yang-Mills) channel. We find the leading string corrections to the field theory results. Second, we calculate exactly string tree-level three and four-point amplitudes with gauge bosons and one massive graviton and examine string deviations from the field-theory result.

PACS: 11.10Kk, 11.15.-q, 11.25-w

Keywords: Type I String Theory, Kaluza-Klein compactifications.

\textsuperscript{*}e.mail: dudas@qcd.th.u-psud.fr
\textsuperscript{†}e.mail: mourad@qcd.th.u-psud.fr
\textsuperscript{‡}Unité mixte de recherche du CNRS (UMR 8627).
1 Introduction and Summary of the results.

Shortly after the birth of string theory as a theory of hadronic interactions with a mass scale of the order of nucleon masses, it was realized that string theory is actually the natural framework to quantize gravity [1]. For a long time, the phenomenologically most interesting theories were considered to be the heterotic strings, where the string scale is of the order of the Planck scale. This rendered string theory predictions not directly accessible to current or future accelerators, with the notable exception of some peculiar models [3]. Recent progress in the understanding of string dualities and D-branes [2] led to other string constructions [4], where the string scale can have values directly accessible in future accelerators.

Consequently, a lot of efforts were made in order to understand the main features of low-scale string theories, from the point of view of possible existence of submilimeter dimensions which can provide testable deviations from the Newton law [5], gauge coupling unification [6] and corresponding string embedding [7]. The main interest of these theories comes from their possible testability at the future colliders, through the direct production or indirect (virtual) effects of Kaluza-Klein states [8] in various cross-sections. This paper is devoted to the (Type I) string computations of the relevant amplitudes. For the convenience of the reader we provide in the following a brief summary of our results.

A subtle issue concerning the virtual effects of gravitational Kaluza-Klein particles is that for a number of compact dimensions \( d \geq 2 \) the corresponding field theory summations diverge in the ultraviolet (UV). Indeed, let us consider a four-fermion interaction of particles stuck on a D3 brane mediated by Kaluza-Klein gravitational excitations orthogonal to it. Then the amplitude of the process, depicted in Figure 1, reads

\[
A = \frac{1}{M_P^2} \sum_{m_i} \frac{1}{-s + \frac{m_i^2 + \cdots + m_2^2}{R_{\perp}^2}}, \tag{1.1}
\]

where for simplicity we considered equal radii denoted by \( R_{\perp} \) and \( s = -(p_1 + p_2)^2 \) is the
The traditional attitude to adopt in this case is to cut the sums for masses heavier than a cutoff $\Lambda \gg R_\perp^{-1}$, of the order of the fundamental scale $M_s$ in the string theory $[8]$. This can be implemented in a proper-time representation of the amplitude

$$A = \frac{1}{M_P^2} \sum m_i \int_{l/\Lambda^2}^{\infty} dl \, e^{-l^{(s+m_1^2+\cdots+m_2^2)} R_\perp} = \frac{1}{M_P^2} \int_{l/\Lambda^2}^{\infty} dl \, e^{sl} \theta_3(0, \frac{il}{\pi R_\perp^2}), \quad (1.2)$$

where $\theta_3(0, \tau) = \sum_k \exp(i\pi k^2 \tau)$ is one of the Jacobi functions. We shall be interested in the following in the region of the parameter space $-R_\perp^2 s \gg 1$, $R_\perp \Lambda \gg 1$ and $-s \ll \Lambda^2$ in which the available energy is smaller (but not far away) from the UV cutoff $\Lambda$ but much bigger than the (inverse) compact radius $R_\perp^{-1}$, of submilimeter size. In this case, the amplitude can be evaluated to give

$$A = \frac{\pi^d R_\perp^d}{M_P^2} \int_{l/\Lambda^2}^{\infty} dl \, e^{sl} \theta_3(0, \frac{i\pi R_\perp^2}{l}) \simeq \frac{2\pi^d}{d-2} \frac{R_\perp^d \Lambda^{d-2}}{M_P^2} = \frac{4\pi^d}{d-2} \frac{\alpha_G^2}{\Lambda^{d-2}} \frac{\Lambda^{d-2}}{M_s^{d+2}}, \quad (1.3)$$

where in the last step we used the relation $M_P^2 = (2/\alpha_G^2) R_\perp^d M_s^{2+d}$, valid for Type I strings, where $\alpha_G = g_Y^2/(4\pi)$ and $g_Y$ is the Yang-Mills coupling on our brane. The high sensitivity of the result on the cutoff asks for a more precise computation in a full Type I string context. This is one of the aims of this paper. In what follows we present qualitatively the results which we derive in Section 3.

The computation in the following is done for the $SO(32)$ Type I 10D superstring compactified down to 4D on a six-dimensional torus. However, as we shall argue later on, the

$^1$Within our conventions $s$ is negative in Euclidean space.
The string nonplanar amplitude.

result holds for a large class of orbifolds, including $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetric vacua [9]. The Type I string diagram which contains in the low-energy limit the gravitational exchange mentioned above is the nonplanar cylinder diagram depicted in Fig. 2, in which for simplicity we prefer to put gauge bosons instead of fermions in the external lines. This diagram has a twofold dual interpretation [10] a) tree-level exchange of closed-string states, if the time is chosen to run horizontally (see Fig. 3) b) one-loop diagram of open strings, if the time runs vertically in the diagram (see Fig. 4). In the two dual representations, the nonplanar amplitude reads symbolically

$$ A = \sum_n \int_0^\infty dl \sum_{n_i} A_2(l, n_1 \cdots n_d, n) $$

$$ = \sum_{k_1 \cdots k_4} \int_0^\infty d\tau_2 \tau_2^{d/2-2} \sum_{m_i} A_1(\tau_2, m_1 \cdots m_d, k_1 \cdots k_4), $$

(1.4)

where $l$ denotes the cylinder parameter in the tree-level channel and $\tau_2 = 1/l$ is the one-loop open string parameter. In the first representation, the amplitude is interpreted as tree-level exchange of closed-string particles of mass $(n_1^2 + \cdots n_d^2)R^2M_s^4 + nM_s^2$, where $n_1 \cdots n_d$ are winding quantum numbers and $n$ is the string oscillator number. In particular the $n = 0$ term reproduces the field-theory result (1.1) and therefore the full expression (1.4) is its string regularization. In the second representation, the amplitude is interpreted as a sum of box diagrams with particles of masses $(m_1^2 + \cdots m_d^2)/R^2 + k_iM_s^2$ ($i = 1 \cdots 4$) running in the four propagators of the diagram.

The UV limit ($l \to 0$) of the gravitational tree-level diagram is related to the IR limit.
\(\tau_2 \rightarrow \infty\) of the box diagram. In particular, in four dimensions when an IR regulator \(\mu\) is introduced in the box diagram, the divergence in the Kaluza-Klein (KK) summation in the gravitational-exchange diagram cancels out. The final result for the nonplanar cylinder amplitude in the low energy limit \(E/M_s << 1\) (\(E\) is a typical energy scale), which is one of the main results of this paper to be discussed in Section 3.5, is in four-dimensions (D=4)

\[
A = -\frac{1}{\pi M_p^2 s} + \frac{2g_Y^4 M_s^2}{\pi^2} \left[ \frac{1}{st} \ln \frac{-s}{4\mu^2} \ln \frac{-t}{4\mu^2} + \text{perms.} \right] \\
- \frac{g_Y^4 M_s^2}{3M_s^4} \left[ \ln \frac{s}{t} \ln \frac{st}{\mu^4} + \ln \frac{s}{u} \ln \frac{su}{\mu^4} \right] + \cdots,
\]

where \text{perms.} denotes two additional contributions coming from the permutations of \(s, t, u\) and \(\cdots\) denote terms of higher order in the low energy expansion. Notice in (1.5) the absence of the contact term (1.3) in the string result, which is replaced by the leading string correction, given by the second line in (1.3). The string correction in (1.5) is indeed of the same order of magnitude as (1.3) for \(\Lambda \sim M_s\), however it has an explicit energy dependence coming from the logarithmic terms.
In order to find the appropriate interpretation of (1.5) in terms of field-theory diagrams, it is convenient to separate the integration region in (1.4) into two parts, by introducing an arbitrary parameter $l_0$ and writing

$$A = \sum_n \int_{l_0}^{\infty} dt \sum_{n_i} A_2 + \sum_{k_1, \ldots, k_4} \int_{1/l_0}^{\infty} d\tau \tau^{d/2-2} \sum_m A_1.$$  \hspace{1cm} (1.6)

This has the effect of fixing an UV cutoff $\Lambda = M_s/\sqrt{l_0}$ in the tree-level exchange diagram, similar to the one introduced in (1.2), (1.3), and simultaneously of a related UV cutoff $\Lambda' = M_s\sqrt{l_0} = M_s^2/\Lambda$ in the one-loop box diagram described here by $A_1$. This “mixed” representation of the non-planar amplitude is depicted in Figure 5. By computing the low-energy limit of $A_1$ and $A_2$ we find in D=4

$$A_1 = \frac{2g_{YM}^4}{\pi^2} \left[ \frac{1}{st} \ln \frac{-s}{4\mu^2} \ln \frac{-t}{4\mu^2} + \text{perms.} \right] - \frac{g_{YM}^4}{3M_s^4} \left[ \ln \frac{s}{t} \ln \frac{st}{\mu^4} + \ln \frac{s}{u} \ln \frac{su}{\mu^4} + \frac{6}{l_0^2} \right] + \cdots$$

$$A_2 = -\frac{1}{\pi M_p^2 s} + \frac{2g_{YM}^4}{M_s^4} \left[ \frac{1}{l_0^2} + \cdots + O\left(\frac{s^2}{M_s^4}\right) \right].$$  \hspace{1cm} (1.7)

The $g_{YM}^4$ terms in $A_1$ describe a box diagram with four light particles (of mass $\mu$) circulating in the loop, while the $g_{YM}^4/M_s^4$ terms are the first string corrections coming from box diagrams with one massive particle (of mass $M_s$) and three light particles of mass $\mu$ in the loop. It contains also the $l_0$ dependent part of the box diagram with four light particles in the loop. The $1/M_s^4 l_0^2$ term in $A_2$ can be written as $\Lambda^4/M_s^8$ and reproduces therefore the field theory computation (1.3) in the case $d = 6$. However, as expected, a similar term with opposite sign appears in $A_1$ and the $l_0$ dependent terms cancel. In $A_2$ the first dots contain $l_0$ dependent terms which cancel with higher-order contributions in $A_1$ and the second dots denote higher-order contributions, while the $O(s^2/M_s^4)$ term is $l_0$ independent and is actually the first correction to the tree-level graviton exchange. We emphasize, however, that the only physically meaningful amplitude is the full expression (1.5) and the leading string correction is therefore the second line of (1.5), coming from box diagrams $A_1$ with one massive particle in the loop.

Strictly speaking, the result described above is valid for the toroidal compactification...
of the $SO(32)$ 10D Type I string. For a general $\mathcal{N} = 1$ supersymmetric 4D Type I vacuum the amplitude $A$ has contributions from sectors with various numbers of supersymmetries

$$A = A^{\mathcal{N}=4} + A^{\mathcal{N}=2} + A^{\mathcal{N}=1},$$

where the $\mathcal{N} = 4$ sector contains the six-dimensional compact KK summations, $\mathcal{N} = 2$ sectors contain two-dimensional compact KK summations and $\mathcal{N} = 1$ sectors contain no KK summations. From the tree-level ($A_2$) viewpoint, the $\mathcal{N} = 2$ sectors give logarithmic divergences which correspond in the one-loop box ($A_1$) picture to additional infrared divergences associated to wave-functions or vertex corrections, which were absent (by nonrenormalization theorems) for the $\mathcal{N} = 4$ theory. Similarly, $\mathcal{N} = 1$ sectors give no KK divergences. As the important (power-type) divergences come from the gravitational $\mathcal{N} = 4$ sector, the toroidally compactified Type I superstring contains therefore the relevant information for our purposes. Moreover, even if we place ourselves in the context of Type I superstring, the formalism we use can be easily adapted to a Type II string context and the associated D-branes. This can be done by exchanging some of the Neumann boundary conditions in the compactified Type I string with the appropriate Dirichlet ones for the D-branes [11, 12].
The basic results and conclusions of our paper can be easily seen to be unchanged.

The second aim of our paper is to calculate the tree-level string amplitudes with two and three gauge bosons and one winding graviton emission. For theories with low string scale and (sub)millimeter dimensions, this type of processes is one of the best signals for future accelerators and was computed in field theory in [8]. A full string formula is needed, however, for energies close to the string scale where string effects are important. We start by computing the two gauge bosons – one winding (KK mode \(m\) after T-dualities) graviton amplitude. The resulting expression has poles and zeroes for discrete values of energies, to be explained in Section 4. We then compute the technically more difficult and phenomenologically more interesting amplitude for three gauge bosons and one massive graviton. We study the deviations from the field-theory result and show that they are of order \(m^4/(R_\perp M_s)^4\). The full amplitude has an interesting structure of poles and zeroes and allows, as explained in Section 4, to define an off-shell form factor. By combining the results of Sections 3 and 4, the effective vertex of two gauge bosons (one of which can be off-shell) of momenta \(p_1, p_2\) and an off-shell graviton of momentum \(p\) (see Figure 6) can be written as

\[
\frac{1}{M_P\sqrt{\pi}} 2^{\frac{-p^2}{M_s^2}} \frac{\Gamma(-p^2/2M_s^2 + 1/2)}{\Gamma(-p_1^2/2M_s^2 + 1)} .
\]

From this we can deduce a form factor characterizing heavy graviton emission \((p^2 >> M_s^2)\)

\[
g(p^2) \sim 2\sqrt{\frac{2M_s^2}{\pi p^2}} (\tan \frac{\pi p^2}{M_s^2}) e^{-\frac{M_s^2}{M_s^2 \ln^2}},
\]

where for an on-shell graviton \(p^2\) is equal to the KK graviton mass \(p^2 = m^2/R_\perp^2\).

The plan of the paper is as follows. In Section 2 we review the mass scales and coupling constants in Type I string compactified on torii. Section 3 is devoted to the study of the virtual gravitational exchange. As explained above, this amounts in a Type I context to a one-loop nonplanar cylinder diagram described in Section 3.1. Sections 3.2 and 3.3 give two dual field-theoretical interpretations of the amplitude as one-loop box diagrams with open modes circulating in the loop and tree-level (winding) gravitational exchange,
respectively. A representation of the amplitude suitable for the low-energy manifestly UV convergent expansion is provided in Section 3.4 and applied to the compactified Type I string in Section 3.5, where the first string corrections to the field theory amplitude are computed. In Section 4 we consider the tree-level (disk) one-graviton emission amplitudes with two gauge bosons in Section 4.1 and three gauge bosons in Section 4.2. Finally, Appendix A contains definitions and some properties of Jacobi theta functions, Appendix B calculations of 4D box diagrams and Appendix C some details on the disk tree-level amplitudes of Section 4.2.

\section{Coupling constants}

Consider the type I superstring compactified to $D = 10-d$ dimensions on a torus $T^d$ with (equal for simplicity) radii $R$. The D-dimensional Planck mass and Yang-Mills coupling constant are given in terms of the string scale $M_s$ and the string coupling constant $g_s$ by

$$M_P^{8-d} = \frac{R^d M_s^8}{g_s^2}, \quad g_{YM}^{-2} = \frac{R^d M_s^6}{g_s}.$$  \hspace{1cm} \text{(2.1)}

Eliminating the radius $R$ in the above two relations for $D = 4$ we get

$$\lambda \equiv \frac{M_P^2}{M_s^2} = \frac{1}{g_s g_{YM}^2},$$ \hspace{1cm} \text{(2.2)}

which shows that the ratio $M_P/M_s$ can be very large if the string coupling constant $g_s$ is very small. The radius $R$ can be determined in terms of $M_s$ and the string and Yang-Mills coupling constants as

$$(RM_s)^6 = \frac{g_s}{g_{YM}^2} = \frac{1}{\lambda g_{YM}^4}. \hspace{1cm} \text{(2.3)}$$

So if the string scale is much lower than the four dimensional Planck scale, that is $\lambda \gg 1$, then the radius $R$ is very small compared to the string length $RM_s \ll 1$.

The equivalent T-dual description is given by a type II theory on $T''^6$ with 32 D3-branes and 64 orientifold planes. The radius of $T''$ is given by $R_{T''} = (RM_s)^{-1}$, so it is very large.
compared to the string length. The T-dual string coupling constant is given by

\[ g''^2 = g^2 \left( \frac{R_1}{R} \right)^6 = g_{YM}^4. \]  

(2.4)

Let \( E \) be the order of magnitude energy in a physical process. We shall mainly be interested in the low energy regime where \( E/M_s \ll 1 \). Moreover we shall suppose that \( \lambda^{-1} \ll E/M_s \), which is compatible with a low string scale. The low energy limit of type I superstrings was considered by Green, Schwarz and Brink \[13\] in the regime \( E/M_s \ll \lambda^{-1} \) with \( \lambda \) fixed, which corresponds to the gravitational decoupling limit \( M_P \to \infty \). It was shown there that this limit is given by the \( \mathcal{N} = 4 \) super Yang-Mills finite theory in four-dimensions. In the following, Section 3.1, we look for the leading stringy and KK corrections to the four-point amplitude described in the Introduction for values of parameters mentioned above and by keeping a finite value for \( M_P \).

### 3 Virtual gravitational exchange amplitude

#### 3.1 One loop type I amplitudes

The one loop type I amplitudes for the scattering of four external massless gauge bosons of momenta \( p_i \), polarisation \( \varepsilon_i^\mu \), and Chan-Paton factors \( \lambda_i \) are of the form

\[
A_\alpha(p, \varepsilon, \lambda) = \delta(\sum p_i) G_\alpha K_{\mu_1 \ldots \mu_4} \varepsilon_1^{\mu_1} \ldots \varepsilon_4^{\mu_4} A_\alpha(s, t, u), 
\]

(3.1)

where the index \( \alpha = 1, 2, 3 \) labels the three diagrams that contribute to the one loop level, the planar cylinder, nonplanar cylinder we are interested in and the Möbius amplitude. For the non planar cylinder with two vertex operators at each boundary, the corresponding group theory factor \( G_\alpha \) is

\[
G = tr(\lambda_1 \lambda_2) tr(\lambda_3 \lambda_4). 
\]

(3.2)

The kinematical factor \( K \) is a polynomial in the external momenta and is given by

\[
K_{\mu_1 \ldots \mu_4} = -(st\eta_{13}\eta_{24} + su\eta_{14}\eta_{23} + tu\eta_{12}\eta_{34}) + s(p_1^4 p_2^2 \eta_{24} + p_2^3 p_1^1 \eta_{13} + p_3^3 p_4^1 \eta_{23} + p_4^3 p_3^1 \eta_{14})
\]
\[ + t(p_2^4 p_4^3 \eta_{13} + p_3^4 p_1^3 \eta_{24} + p_1^4 p_2^3 \eta_{34} + p_2^1 p_2^1 \eta_{21}) + u(p_2^2 p_3^3 \eta_{23} + p_3^2 p_2^1 \eta_{14} + p_1^3 p_2^1 \eta_{34} + p_2^3 p_1^1 \eta_{12}) \], \quad (3.3)

where the upper index labels the external particles and the lower index \(i\) is an abbreviation of the Lorentz index \(\mu_i\). We have also used the Mandelstam variables

\[ s = -(p_1 + p_2)^2, \quad t = -(p_1 + p_4)^2, \quad u = -(p_1 + p_3)^2, \quad (3.4) \]

that verify \(s + t + u = 0\).

The amplitudes \(A_\alpha\) can be written as integrals over the modular parameter \(\tau_2\) of the corresponding surface and the positions \(w_i\) of the vertex operators

\[ A_\alpha = \frac{g_s^2}{M_s^{10}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{R_\alpha} dw \prod_{i>j} \exp \left( \frac{p_i \cdot p_j}{M_s^2} G(w_i - w_j) \right), \quad (3.5) \]

where \(G_\alpha\) is the Green function on the corresponding surface. It can be expressed with the aid of the Green function on the torus

\[ G(z, \tau) = -\ln \left| \frac{\vartheta_1(z, \tau)}{\vartheta_1(0, \tau)} \right|^2 + \frac{2\pi}{\tau_2} (\text{Im}(z))^2. \quad (3.6) \]

The definitions and some useful properties of Jacobi modular functions \(\theta_i(z, \tau)\) are given in Appendix A. From (A.3) we get

\[ G \left( z/(c\tau + d), (a\tau + b)/(c\tau + d) \right) = G(z, \tau) + \ln |c\tau + d|, \quad (3.7) \]

where \(a, b, c\) and \(d\) are elements of an SL(2,\(\mathbb{Z}\)) matrix.

The Green function on the cylinder is conveniently obtained from that of its covering torus. Let the torus with modular parameter \(\tau = \tau_1 + i\tau_2\) be parametrised by the complex coordinate \(w\) with \(w = w + 1 = w + \tau\) and let \(w = x + \tau \nu, x\) and \(\nu\) being two real 1-periodic coordinates. Then the cylinder is obtained by setting \(\tau_1 = 0\) and orbifolding with \(w = -\bar{w}\), the two boundaries being at \(x = \text{Re}(w) = 0, 1/2\). The parameter \(\tau_2\) represents the length of the circles at the boundaries. In the amplitudes (3.5), the region of integration over the positions \(w\) is given by \(\nu_i < \nu_{i+1}\) whenever the two vertex operators are on the same
boundary, the value of $\nu_1$ being fixed to 1 and the coordinate $x$ is fixed for the cylinder at 0 or 1/2. It will be convenient to use the notations:

$$\Psi(\nu, \tau_2) = \exp\left(\frac{1}{2} G(i\tau_2 \nu, i\tau_2)\right), \quad \Psi^T(\nu, \tau_2) = \exp\left(\frac{1}{2} G\left(\frac{1}{2} + i\tau_2 \nu, i\tau_2\right)\right). \quad (3.8)$$

With these notations the amplitude $A$ can be cast in the form (where $\psi_{12}$ stands for $\Psi(\nu_2 - \nu_1, i\tau_2)$ and so on)

$$A = \frac{g_s^2}{M_{s}^{10}} \int_{0}^{\infty} \frac{d\tau_2}{\tau_2^2} \int_{0}^{1} d\nu_2 \int_{0}^{\nu_2} d\nu_1 \int_{0}^{1} d\nu_3 \left(\frac{\Psi_{13}^{T} \Psi_{24}^{T}}{\Psi_{12} \Psi_{34}}\right)^{s/M_2^2} \left(\frac{\Psi_{13}^{T} \Psi_{24}^{T}}{\Psi_{14}^{T} \Psi_{23}^{T}}\right)^{t/M_2^2} F_6(\tau_2, R_i), \quad (3.9)$$

where

$$F_d(\tau_2, R_i) = \frac{M_{s}^{d}(2\tau_2)^{d/2}}{(RM_s)^{2d}} \sum_{m_i} \exp\left(-2\pi \tau_2 \alpha^d \sum_i \frac{m_i^2}{R^2}\right) = \frac{M_{s}^{d}(2\tau_2)^{d/2}}{(RM_s)^{2d}} \vartheta_3^d \left(0, \frac{2i\tau_2}{(RM_s)^{2}}\right) \quad (3.10)$$

is a factor coming from the toroidal compactification on torii with radii (taken equal for simplicity) $R$.

The transformation of the torus Green function under the modular group suggests the possibility of using other modular parameters than $\tau_2$. Defining $l = 1/\tau_2$, the transformation (3.7) gives

$$G(z, i\tau_2) = G(z/(i\tau_2), i/\tau_2) - \ln(\tau_2), \quad (3.11)$$

which implies that

$$\Psi(\nu, \tau_2) = \sqrt{l} \exp\left(\frac{1}{2} G(\nu, il)\right) \equiv \tilde{\Psi}(\nu, l), \quad (3.12)$$

$$\Psi^T(\nu, \tau_2) = \sqrt{l} \exp\left(\frac{1}{2} G(\nu - \frac{l}{2}, il)\right) \equiv \tilde{\Psi}^T(\nu, l).$$

With the new modular parameter $l$ the amplitude can be cast in the following form

$$A = \frac{g_s^2}{M_{s}^{10}} \int_{0}^{\infty} dl \int_{0}^{1} d\nu_2 \int_{0}^{\nu_2} d\nu_1 \int_{0}^{1} d\nu_3 \left(\frac{\tilde{\Psi}_{13}^{T} \tilde{\Psi}_{24}^{T}}{\tilde{\Psi}_{12} \tilde{\Psi}_{34}}\right)^{s/M_2^2} \left(\frac{\tilde{\Psi}_{13}^{T} \tilde{\Psi}_{24}^{T}}{\tilde{\Psi}_{14}^{T} \tilde{\Psi}_{23}^{T}}\right)^{t/M_2^2} F_6. \quad (3.13)$$

By performing a Poisson transformation one can also write the Kaluza-Klein contributions as

$$F_d = \frac{M_{s}^{d}}{(RM_s)^{2d}} \vartheta_3^{d} \left(0, \frac{il(RM_s)^{2}}{2}\right). \quad (3.14)$$
3.2 The one loop amplitude as a sum of box diagrams

We consider here in more detail the nonplanar diagram in the representation given in (3.9). The relevant exponentials of the Green functions (3.8) are explicitly given by

\[ \Psi = ie^{-\pi \tau_2 \nu^2} \frac{\partial_1(-i \nu \tau_2, i \tau_2)}{\eta^3}, \quad \Psi^T = e^{-\pi \tau_2 \nu^2} \frac{\partial_2(-i \nu \tau_2, i \tau_2)}{\eta^3}. \]  

(3.15)

In order to obtain a field theory interpretation of this string diagram, it is convenient to first divide the region of integration over \( \nu_i \) into three disjoint regions with a given ordering

\[ R_1 : \nu_1 < \nu_2 < \nu_3 < 1, \quad R_2 : \nu_1 < \nu_3 < \nu_2 < 1, \quad R_3 : \nu_3 < \nu_1 < \nu_2 < 1, \quad (3.16) \]

then we can write

\[ A = A^{(1)}(s, t) + A^{(2)}(u, t) + A^{(3)}(u, s), \quad (3.17) \]

where

\[ A^{(i)}(s, t) = \frac{8g_M^4}{s t} \int_0^\infty d\tau_2 \int_{\eta_i > 0} d^4 \eta_\delta(1 - \sum_i \eta_i e^{\frac{2\pi \tau_2 s \eta_i + t \eta_i}{s t}} f_\eta(0, \frac{2i \tau_2}{(RM_s)^2}) R^{(i)}). \]  

(3.18)

The variables \( \eta_i \) in the region \( R_1 \) are given by

\[ \eta_1 = \nu_1, \quad \eta_i = \nu_i - \nu_{i-1}, \quad i = 2, 3, 4 \]

(3.19)

and similar expressions in the other regions. The factors \( R^{(i)} \) are given by

\[ R^{(1)} = \left( \frac{f^T(\eta_3 + \eta_2) f^T(\eta_3 + \eta_4)}{f^T(\eta_2) f^T(\eta_4)} \right)^{s/M_s^2} \left( \frac{f^T(\eta_2 + \eta_3) f^T(\eta_3 + \eta_4)}{f^T(\eta_1) f^T(\eta_3)} \right)^{t/M_s^2}, \]

\[ R^{(2)} = \left( \frac{f(\eta_3 + \eta_2) f(\eta_3 + \eta_4)}{f(\eta_2) f(\eta_4)} \right)^{s/M_s^2} \left( \frac{f(\eta_2 + \eta_3) f(\eta_3 + \eta_4)}{f(\eta_1) f(\eta_3)} \right)^{t/M_s^2}, \]

\[ R^{(3)} = \left( \frac{f^T(\eta_3 + \eta_2) f^T(\eta_3 + \eta_4)}{f^T(\eta_2) f^T(\eta_4)} \right)^{s/M_s^2} \left( \frac{f^T(\eta_2 + \eta_3) f^T(\eta_3 + \eta_4)}{f(\eta_1) f(\eta_3)} \right)^{t/M_s^2}, \]

(3.20)

where we introduced the convenient definitions

\[ f(\eta_i) = e^{-\pi \tau_2 (\eta_i - 1/6)} \frac{\partial_1(-i \eta_i \tau_2, i \tau_2)}{\eta}, \quad f^T(\eta_i) = e^{-\pi \tau_2 (\eta_i - 1/6)} \frac{\partial_2(-i \eta_i \tau_2, i \tau_2)}{\eta}. \]

(3.21)

\footnote{A similar parametrization for the four-point amplitude on the torus in the Type II and heterotic strings was considered in \cite{1}. There, the functions \( R^{(i)} \) are identical.}
such that \( f(f^T) \) can be expanded in powers of \( \exp(-2\pi \eta_1 \tau_2) \) (B.10,B.11). By using the explicit definitions given in Appendix A, it can be checked that \( f(1-\eta_1) = f(\eta_1) \) and \( f^T(1-\eta_1) = f^T(\eta_1) \), which was used in deriving (3.21). The field theory interpretation of \( A \) is clarified by the formal expansion of the factor \( R^{(i)} \) as a power series in \( e^{-2\pi \tau_2 \eta_j} \)

\[
R^{(i)} = \sum_{n_1, \ldots, n_4 \geq 0} p^{(i)}_{n_1, \ldots, n_4} \left( \frac{s}{M_s^2}, \frac{t}{M_s^2} \right) e^{-2\pi \tau_2 (n_1 \eta_1 + n_2 \eta_2 + n_3 \eta_3 + n_4 \eta_4)}, \quad (3.22)
\]

where \( p^{(i)}_{n_1, \ldots, n_4} \) are polynomials in \( s/M_s^2 \) and \( t/M_s^2 \), whose explicit expressions are not important in the following. Our definition is such that \( p^{(i)}_{0, \ldots, 0} (s/M_s^2, t/M_s^2) = 1 \).

By using the expansion (3.22), the amplitude (3.18) can be interpreted as a sum of an infinit set of box diagrams

\[
A^{(i)} = \frac{12g^4_{YM}}{\pi^4} \sum_{k_1, \ldots, k_6 = -\infty}^{+\infty} \sum_{n_1, \ldots, n_4 \geq 0} p^{(i)}_{n_1, \ldots, n_4} \left( \frac{s}{M_s^2}, \frac{t}{M_s^2} \right) B(s, t, \{ n_i M_s^2 + (k_i^2 + \ldots + k_6^2)/R^2 \}). \quad (3.23)
\]

Indeed, the Feynman representation of a box diagram in 4D with particles of masses \( m_i^2 \) in the loop reads

\[
B(s, t, \{ m_i^2 \}) = \int d^4k \prod_{i=1}^{4} \frac{1}{(k + p_i)^2 + m_i^2} = \frac{2\pi^4}{3M_s^4} \int d\tau_2 \tau_2 d^4\eta \delta(1 - \sum_i \eta_i) e^{\frac{2\pi \tau_2 (s \eta_1 \eta_3 + t \eta_2 \eta_4)}{M_s^2}} e^{-\frac{2\pi \tau_2}{M_s^2} \sum_i m_i^2 \eta_i}. \quad (3.24)
\]

Note that in (3.23), the particles circulating in the loop are open string oscillators and KK states.

For \( D \leq 4 \) the box diagram with massless particles in the loop, which is the leading contribution to the above amplitude, is IR divergent. Infrared divergences are as usual harmless and in order to obtain a finite intermediate result it suffices to add a small mass to the particles circulating in the loop. Since \( \sum_i \eta_i = 1 \), it can be seen from (3.24) that this is equivalent to the replacement

\[
R^{(i)} \to R^{(i)} e^{\frac{-2\pi \tau_2 m_i^2}{M_s^2}}, \quad (3.25)
\]
where $\mu$ is a small mass which, as usual, is replaced by the resolution over the energy of final particles in a given physical process.\(^3\)

In practice, the expansion (3.23) is not very useful. It does not correspond to an expansion in powers of $s/M_s^2$ and $t/M_s^2$ and merely gives an interpretation of the nonplanar amplitude as an infinite sum of box diagrams. The low energy limit is naturally given by the box diagram with massless particles circulating in the loop. However, this box diagram is UV divergent for spacetime dimension $D \geq 8$, whereas string theory is finite in the UV. This shows that this expansion is not manifestly UV finite. In fact for $D \geq 8$ the series is divergent term by term. This is to be contrasted with the expansion of the similar torus amplitude in heterotic and Type II strings, where the modular invariance provides an explicit UV cutoff [15]. Furthermore, for $D \leq 8$, even though the diagrams are finite in the UV, infinitely many terms contribute to a given order in $s/M_s^2$. In section 3.4 we show how it is possible to get a systematic low energy expansion of the amplitude, which is also manifestly UV finite. Before doing that, we will need however another interpretation of the nonplanar diagram.

### 3.3 The nonplanar amplitude as exchange of closed string modes

In this subsection we shall discuss a representation which is the string generalisation of the proper time parametrisation used in equation (1.2). In this representation we must express the amplitude (3.13) as a function of $l = \tau_2^{-1}$, with $l$ the modulus of the cylinder. The factor $F_6$ due to the Kaluza-Klein modes is given by (3.14), while the functions $\tilde{\Psi}, \tilde{\Psi}^T$ defined in (3.12) are given by

$$
\tilde{\Psi} = \frac{1}{l} \frac{\vartheta_1(\nu, il)}{\eta^3}, \quad \tilde{\Psi}^T = \frac{1}{l} \frac{\vartheta_4(\nu, il)}{\eta^3}.
$$

\(^3\)Another way of regularizing the IR divergence is to add a Wilson line in the Chan-Paton sector.
In order to analyse this amplitude it is convenient to define

\[
\left( \frac{\bar{\psi}_{13}^T \bar{\psi}_{24}^{1/2}}{\bar{\psi}_{14}^T \bar{\psi}_{34}^{1/2}} \right)^{s/M_s^2} \left( \frac{\bar{\psi}_{13}^T \bar{\psi}_{24}^{1/2}}{\bar{\psi}_{14}^T \bar{\psi}_{34}^{1/2}} \right)^{t/M_s^2} = e^{\pi t s/2M_s^2} \left[ 4 \sin(\nu_2 - \nu_1) \sin(\pi(1 - \nu_3)) \right]^{-s/M_s^2} \tilde{R} \left( \frac{s}{M_s^2}, \frac{t}{M_s^2}, \nu_i \right),
\]

where

\[
\tilde{R} \equiv \left( \frac{\bar{f}_{13} \bar{f}^{1/2}_{24}}{\bar{f}_{12} \bar{f}^{1/2}_{34}} \right)^{s/M_s^2} \left( \frac{\bar{f}_{13} \bar{f}^{1/2}_{24}}{\bar{f}_{14} \bar{f}^{1/2}_{34}} \right)^{t/M_s^2}
\]

and

\[
f (\nu, l) = e^{\pi l/6} \frac{\vartheta_1(\nu, il)}{2 \sin \pi \nu} \frac{\eta}{\vartheta_4(\nu, il)}, \quad \bar{f}^T (\nu, l) = e^{-\pi l/12} \frac{\vartheta_4(\nu, il)}{\eta}.
\]

Similarly to \( f \) and \( f^T \) in (3.21), these functions can be expanded in positive powers of \( e^{-2\pi l} \).

Therefore it is possible to cast \( \tilde{R} \) in the form

\[
\tilde{R} = 1 + \sum_{n=1}^{\infty} \tilde{p}_n(s/M_s^2, t/M_s^2, \nu)e^{-2\pi nl},
\]

where \( \tilde{p}_n \) is a polynomial in \( s/M_s^2, t/M_s^2 \) and \( e^{2i\pi \nu} \), whose exact expression is not important here. Assembling the different terms and by finally using (2.1), the amplitude can be cast in the suggestive form

\[
A = \frac{1}{M_p M_s^2} \sum_{n=0}^{\infty} c_n \int_0^\infty dl \sum_{n_1, \ldots, n_6} e^{-\pi l/2} \left[ -s/M_s^2 + (n_1^2 + \ldots + n_6^2)(RM_s)^2 + 4n \right],
\]

where we defined

\[
c_n = \int d\nu_1 d\nu_2 d\nu_3 \tilde{p}_n(s/M_s^2, t/M_s^2, \nu) [4 \sin(\nu_2 - \nu_1) \sin(\pi(1 - \nu_3))]^{-s/M_s^2},
\]

for \( n \geq 0 \). The field theory result (1.1) is obtained by truncating in (3.31) the massive string oscillators and taking the low energy limit in \( c_0 \), which gives 1/2. We therefore keep only the winding modes, T-dual to the KK states appearing in (1.1).

The integral in (3.31) is simply the proper time representation of a Feynman propagator with the mass

\[
m^2 = M_s^2 [(n_1^2 + \ldots + n_6^2)(RM_s)^2 + 4n].
\]

This fact reflects a familiar result: the one loop open string amplitude can be seen as a tree diagram in the closed string channel [10] where the masses of the closed string
particles are given by (3.33), the integer $2n$ being the closed string oscillator level and $n_1, \ldots, n_6$ the winding numbers. Note however that the expansions we performed and therefore a truncation for some value on $n$ are valid for large $l$. Similarly to the case of the representation of the non planar amplitude as a sum of box diagrams, this representation is not manifestly UV convergent. In fact the sum over the winding modes behaves as $l^{-d/2}$ for small $l$, so that the integral diverges for $d \geq 2$ and in particular in the present case $d = 6$ we obtain a quartic divergence.

### 3.4 Type I ultraviolet regularisation of 10D field theory

In the two preceding sections we have given two representations of the non planar amplitude. Both of them were not manifestly UV convergent and did not allow a systematic (in $s/M_s^2$ and $t/M_s^2$) low energy expansion. Here we combine both of them in a new representation which is free of these two drawbacks.

The two dual expressions (3.9) and (3.13) are typically of the form

$$I = \int_0^\infty dx \ h(x)[g(x)]^\epsilon, \quad (3.34)$$

and we are interested in the small $\epsilon$ expansion of the amplitude. In the easiest case where $g$ is strictly positive and bounded from above, the expansion is given by expanding the integrand, that is

$$I = \int_0^\infty dx \ h(x) + \epsilon \int_0^\infty dx \ h(x) \ln (g(x)) + \ldots. \quad (3.35)$$

A less trivial case is when $g$ vanishes somewhere between 0 and $\infty$ and possibly at the boundaries, in which case one cannot perform an expansion of the above form. Suppose however that $g$ can be put in the form $g = g_1(x)g_2(x)$ where $g_2$ is strictly positive and bounded. Then one can expand $I$ as

$$I = \int_0^\infty dx \ h[g_1]^\epsilon + \epsilon \int_0^\infty dx \ h[g_1]^\epsilon \ln(g_2) + \ldots, \quad (3.36)$$
which can be useful when $g_1$ is much simpler than $g$.

Now we come back to the string amplitude, which in the open string representation has the form

$$A = \int_0^\infty d\tau_2 \int d^4\eta \delta(1 - \sum_i \eta_i) e^{\frac{2\pi \tau_2}{M_s^2}(\eta_1\eta_3 + t_0\eta_4)} I_s/M_s^2 J^{t_1}/M_s^2,$$  

(3.37)

where both $I$ and $J$ are bounded and nonvanishing at $\tau_2 = \infty$. However at $\tau_2 = 0$ they vanish and furthermore it is not possible to factorise a finite number of vanishing terms. The closed string representation of the amplitude has the same problem: it is possible to isolate the dangerous piece at $l = \infty$ but not at $l = 0$. In fact the closed string representation can be put in the form

$$A = \int_0^\infty dl \ e^{\frac{2\pi l s}{M_s^2}} \int d^3\nu \ [4 \sin \pi (\nu_2 - \nu_1) \sin \pi (1 - \nu_3)]^{-s/M_s^2} \tilde{I}^{s/M_s^2} \tilde{J}^{t/M_s^2},$$  

(3.38)

where $\tilde{I}$ and $\tilde{J}$ are bounded and nonzero at $\infty$ but not at $l = 0$. Note that the problematic region of each representation corresponds to the nice region of the other representation. This suggests a solution which consists in using a mixed representation of the amplitude: choose a finite nonvanishing $l_0$ and write the 10D amplitude as

$$A = g^2_s \int_0^\infty \frac{d\tau_2}{\tau_2} \int_0^{v_2} dv_2 \int_0^{v_1} dv_1 \int_0^{v_0} dv_0 \ \left( \frac{\tilde{\Psi}_{13}^T \tilde{\Psi}_{24}^T}{\tilde{\Psi}_{12} \tilde{\Psi}_{34}} \right)^{s/M_s^2} \left( \frac{\tilde{\Psi}_{13}^T \tilde{\Psi}_{24}^T}{\tilde{\Psi}_{14} \tilde{\Psi}_{23}} \right)^{t/M_s^2}$$

$$+ \ g^2_s \int_0^\infty dl \int_0^{v_2} dv_2 \int_0^{v_1} dv_1 \int_0^{v_0} dv_0 \ \left( \frac{\tilde{\Psi}_{13}^T \tilde{\Psi}_{24}^T}{\tilde{\Psi}_{12} \tilde{\Psi}_{34}} \right)^{s/M_s^2} \left( \frac{\tilde{\Psi}_{13}^T \tilde{\Psi}_{24}^T}{\tilde{\Psi}_{14} \tilde{\Psi}_{23}} \right)^{t/M_s^2}$$

$$\equiv A_1(l_0) + A_2(l_0).$$  

(3.39)

Now in each integral we can use the factorisation described above. Note that even if the full amplitude is independent on $l_0$, each part of it clearly does depend. In fact $l_0$ plays the rôle of an UV cutoff $\Lambda_c$ for the closed string exchange and $l_0^{-1}$ plays the rôle of an UV cutoff $\Lambda_o$ in the one loop box diagrams. The two cutoffs are given by

$$\Lambda_c = \frac{M_s}{\sqrt{l_0}}, \quad \Lambda_o = M_s \sqrt{l_0}$$  

(3.40)

and are clearly inversely proportional to each other:

$$\Lambda_o \Lambda_c = M_s^2.$$  

(3.41)
This mixed representation is thus manifestly UV convergent. The low energy expansion is also manifestly finite term by term.

Let’s elaborate more on this 10D example and obtain the finite result for low energy limit of the amplitude. Consider first the $A_1(l_0)$ part

$$A_1(l_0) = \frac{g_s^2}{M_s^{10}} \int_{1/l_0}^{\infty} \frac{d\tau_2}{\tau_2^2} \int d^4\eta \delta\left(1 - \sum_i \eta_i\right) e^{\frac{2\pi\tau_2}{M_s^2}(s\eta_1\eta_3 + t\eta_2\eta_4)} \left(1 + \frac{s}{M_s^2} \ln I + \frac{t}{M_s^2} \ln J + \ldots\right). \quad (3.42)$$

Let us first neglect the terms multiplying $s/M_s^2$ and $t/M_s^2$, as well as higher order terms and integrate over $\tau_2$ to obtain

$$A_1(l_0) = \frac{g_s^2}{M_s^{10}} l_0 \int d^4\eta \delta\left(1 - \sum_i \eta_i\right) E_2\left(-\frac{2\pi}{l_0 M_s^2}(s\eta_1\eta_3 + t\eta_2\eta_4)\right) + \ldots \quad (3.43)$$

where

$$E_m(z) = \int_{1}^{\infty} \frac{dx}{x^m} e^{-zx} \quad (3.44)$$

For small $z$ we have

$$E_2(z) = 1 + z \ln z - (1 - \gamma)z + \ldots \quad (3.45)$$

so to the next to leading order in $s/M_s^2$ we have

$$A_1^{(1)}(l_0) = \frac{l_0 g_s^2}{6 M_s^{10}} - \frac{2\pi g_s^2}{M_s^{12}} \int d^4\eta \delta\left(1 - \sum_i \eta_i\right)(s\eta_1\eta_3 + t\eta_2\eta_4) \ln \frac{-\left(s\eta_1\eta_3 + t\eta_2\eta_4\right)}{M_s^2} + \ldots \quad (3.46)$$

if we neglect higher order terms. Note that the leading term depends on $l_0$ and becomes infinite in the $l_0 \to \infty$ limit, which signals that the 10D Yang-Mills box diagram is UV divergent. The $l_0$ dependence must of course cancel in the full amplitude. In order to check it explicitly let us consider the second part of the amplitude

$$A_2(l_0) = \frac{g_s^2}{M_s^{10}} \int_{l_0}^{\infty} d\nu e^{\pi s/2M_s^2} \int d^3\nu\left[4 \sin \pi(\nu_2 - \nu_1) \sin \pi(1 - \nu_3)\right]^{-s/M_s^2} \left\{1 + \frac{s}{M_s^2} \ln \tilde{I} + \frac{t}{M_s^2} \ln \tilde{J} + \ldots\right\}. \quad (3.47)$$

The $s/M_s^2$ and $t/M_s^2$ terms in the brackets, representing oscillator contributions, give a vanishing contribution to the first order in $s/M_s^2$. In fact

$$\int d^3\nu \ln \tilde{I} = \int d^3\nu \ln \tilde{J} = 0 \quad (3.48)$$
as can be easily verified upon expanding the logarithms in powers of $e^{2\pi i \nu}$. So to the first
order in $s/M_s^2$ or, more generally, if we neglect string oscillator exchanges in $A_2$, we can
replace the terms in the curly brackets by 1. The integral over $\nu$ can then be performed
and gives

$$
\int d^3 \nu [\sin \pi (\nu_2 - \nu_1) \sin \pi (1 - \nu_3)]^{-s/M_s^2} = \frac{1}{2\pi} \left( \frac{\Gamma (-s/2M_s^2 + 1/2) \sqrt{\pi}}{\Gamma (-s/2M_s^2 + 1)} \right)^2 .
$$

(3.49)

This result, due to the massless graviton tree-level exchange in 10D, presents a perfect
square structure which allows the identification of the tree-level (disk) form factor $g$ between
two (on-shell) gauge bosons and one (off-shell) massless graviton of momentum squared $s$
to be

$$
g(s) = \frac{1}{\sqrt{\pi}} 2^{-s/3M_s^2} \frac{\Gamma (-s/2M_s^2 + 1/2)}{\Gamma (-s/2M_s^2 + 1)} .
$$

(3.50)

The presence of poles (and also zeroes) in this form factor is interpreted as due to a tree-
level mixing between the massless graviton and open string singlets, present at odd mass
levels in the $SO(32)$ Type I superstring. The result (3.50) will be rederived and generalized
for off-shell gauge bosons and compactified 4D theory in Section 4.

To the first order in $s/M_s^2$, (3.49) becomes $1/2 + s/M_s^2 \ln 2 + \ldots$. When combined
with $4^{-s/M_s^2}$, the first order terms in $A_2$ in $s/M_s^2$ cancel and we are left with

$$
A_2(l_0) = -\frac{g_s^2}{sM_s^8} (e^{s l_0/2M_s^2} + O((s^2/M_s^4))) = -\frac{1}{\pi s M_p^2} - \frac{l_0 g_s^2}{2 M_s^6} + \cdots ,
$$

(3.51)

where in identifying the graviton pole in the last equality we used the first relation in (2.1)
for 10D (d=0). To order $(s g_s^2/M_s^4)^2 \ln(s/M_s^2)$, the amplitude is then given by

$$
A = A_1^{(1)} (l_0) (s,t) + A_1^{(1)} (l_0) (u,t) + A_1^{(1)} (l_0) (u,s) + A_2 (l_0) .
$$

(3.52)

Notice first of all that up to this order the terms dependent on $l_0$ cancel in the sum
$A_1(l_0) + A_2(l_0)$, as it should. The first term in the r.h.s. of $A_2$ represents the exchange of
the graviton multiplet between the two gauge bosons. One may be tempted to interpret $A_2$
as the gravitational contribution to the amplitude. This is unambiguous as long as we are
considering the leading contributions in $A_1$ and $A_2$. However, if we consider higher order terms the distinction between $A_1$ and $A_2$ looses its relevance. In other words the gauge and gravitational contributions are mixed and only their sum in meaningful. Notice also the absence of terms of order 0 in $s/M_s^2$ in $A$. This is a direct verification of the absence of one loop $(trF^2)^2$ terms in the effective action of the type I superstring, a result which was important in checking the type I/heterotic duality [16].

Let us consider the effect of including higher order terms. The dominant contribution of $\Delta A_1$ is of order $sg_s^2/M_s^{12}$. However this contribution is $l_0$ dependent and thus must cancel with a similar contribution from $A_2$. The first $l_0$ independent contribution to $\Delta A_1$ is of order $(g_s^2 s^2/M_s^{14}) \ln(-s/M_s^2)$, which can be easily checked explicitly.

Notice that in this way, one sees clearly how type I strings regularise the UV divergent field theory. In field theory language, this is equivalent to introducing an arbitrary UV cutoff in the divergent (box) diagram and using a related cutoff in the graviton exchange one. The product of the two is $M_s^2$ and the sum of the two diagrams is cutoff-independent. One may interpret the result as a regularisation of the Yang-Mills theory by gravity. Notice that this works in a subtle way because the gravity diagram here is UV finite. The regularisation is possible because the cutoff used on the gravitational side is inversely proportional to the one used for the Yang-Mills diagram.

### 3.5 Type I ultraviolet regularisation of winding modes in 4D

In 4D the box diagrams are UV convergent but the closed string exchange is UV divergent. Introducing the parameter $l_0$ and using a mixed representation we get a manifestly UV finite form of the amplitude. The IR divergence of the box diagram is eliminated by adding a small mass $\mu$ to the open string modes or, in a SUSY manner, by adding a Wilson line in the open sector.

The amplitudes can be expanded in $RM_s$ as well as in $s/M_s^2$ and $t/M_s^2$. Since we are
interested in the small \( RM_s \) limit we shall keep the leading contribution in \( RM_s \) and the next to leading contribution in \( s/M_s^2 \). The leading contribution in \( (RM_s) \) in \( A_1 \) is obtained by neglecting all the Kaluza-Klein modes whose relative contributions are of order \( (RM_s)^4 \). In \( A_2 \) the sum over the winding modes is replaced by the zero mode and an integral over non-zero modes, that is

\[
\sum_{n_1, \ldots, n_6} e^{-\pi l[-s/2M_s^2+(n_1^2+\ldots+n_6^2)(RM_s)^2/2+2n]} = e^{\pi ls/2M_s^2} \left( 1 + \frac{8}{(RM_s)^6l^3} \right) + \ldots \quad (3.53)
\]

Note that strictly speaking one should neglect the zero mode contribution, however this is the only term that diverges in the \( s/M_s^2 \to 0 \) limit. The \( A_2(l_0) \) contribution becomes

\[
A_2(l_0) = \frac{1}{M_s^2M_P^2} \int_{l_0}^{\infty} dl e^{\pi ls/2M_s^2} \left( 1 + \frac{8}{(RM_s)^6l^3} \right) \int d^8\nu [4 \sin \pi (\nu_2-\nu_1) \sin \pi (1-\nu_3)]^{-s/M_s^2} \left\{ 1 + \frac{s}{M_s^2} \ln \tilde{I} + \frac{t}{M_s^2} \ln \tilde{J} + \ldots \right\} . \quad (3.54)
\]

As in the previous Section, eq. (3.47), if we neglect string oscillator contributions coming from \( \tilde{I}, \tilde{J}, \) the \( \nu \) integral gives an effective form factor (3.50), which is seen now to be the same for all winding states, result which will be rederived and shown to be true even for off-shell gauge bosons in Section 4.

As in the 10D case the dominant contribution in \( A_2 \) comes from

\[
A_2^{(0)}(l_0) = \frac{1}{2M_s^2M_P^2} \int_{l_0}^{\infty} dl e^{\pi los/2M_s^2} \left( 1 + \frac{8}{(RM_s)^6l^3} \right) , \quad (3.55)
\]

where the factor \( 1/2 \) comes from the integration over \( \nu \). This contribution can be expressed with the aid of the \( E_3 \) function (3.44) as

\[
A_2^{(0)}(l_0) = -\frac{1}{\pi s M_P^2} e^{\pi los/2M_s^2} + \frac{4}{l_0^2 M_P^6 R^6 M_s^2} E_3\left(-\frac{\pi l_0 s}{2M_s^2}\right) . \quad (3.56)
\]

The small \( s/M_s^2 \) limit (3.56) is obtained with the aid of the developpement of \( E_3 \)

\[
E_3(z) = \frac{1}{2} \left( 1 - 2z - z^2 \ln z + \left( \frac{3}{2} - \gamma \right) z^2 + \ldots \right) , \quad (3.57)
\]

where \( \gamma \) is the Euler constant. Therefore we find, by using again (2.1)

\[
A_2^{(0)}(l_0) = -\frac{1}{\pi s M_P^2} + \frac{2 g_Y^4}{M_s^4} \left( \frac{1}{l_0^2} + \frac{\pi s}{l_0 M_s^2} - \frac{\pi^2 s^2}{4 M_s^4} \ln\left(-\frac{\pi l_0 s}{2M_s^2}\right) + \left( \frac{3}{2} - \gamma \right) \frac{\pi^2 s^2}{4 M_s^4} + \ldots \right) . \quad (3.58)
\]
It is transparent in (3.58), because of the factor $g_{YM}^4$, that the leading corrections to the
graviton exchange diagram are actually mostly related to the one-loop box diagram and
not to massive tree-level exchanges. However, as already emphasized, the real physical
quantity is the sum of $A_1$ and $A_2$.

We now turn to the $A_1(l_0)$ contribution, where we rely heavily on technical results
derived in Appendix B. The dominant contribution is obtained from

$$A_1^{(i,0)}(s, t) = \frac{8g_{YM}^4}{M_s^4} \int_{1/l_0}^{\infty} d\tau_2 \tau_2 \int_{\eta_i > 0} d^4\eta \delta(1 - \sum_i \eta_i) e^{\frac{2\pi\tau_2(s \eta_3 + t \eta_4)}{M_s^2}} e^{-\frac{2\pi\tau_2\mu^2}{M_s^2}}. \quad (3.59)$$

Since the box diagram is UV convergent, we can safely write the above integral as

$$A_1^{(i,0)}(s, t) = \int_0^{\infty} ... - \int_0^{1/l_0} ... \quad (3.60)$$

The first term is obtained from the box diagram calculated in the Appendix B

$$\int_0^{\infty} ... = \frac{2g_{YM}^4}{\pi^2} \frac{1}{st} \ln \frac{-s}{4\mu^2} \ln \frac{-t}{4\mu^2} \quad (3.61)$$

and the second is given by the following expansion in $s/M_s^2$ and $t/M_s^2$

$$\int_0^{1/l_0} ... = \frac{8g_{YM}^4}{M_s^4} \int_0^{1/l_0} d\tau_2 \tau_2 \int_{\eta_i > 0} d^4\eta \delta(1 - \sum_i \eta_i)[1 + \frac{2\pi\tau_2}{M_s^2}(s \eta_3 + t \eta_4) + ...]. \quad (3.62)$$

These integrals can be easily evaluated and yield

$$\int_0^{1/l_0} ... = \frac{8g_{YM}^4}{M_s^4} \left[ \frac{1}{12l_0^2} + \frac{\pi(s + t)}{180l_0^3M_s^2} \right] + ... \quad (3.63)$$

Note that the second term when considering the sum $A^{(1)} + A^{(2)} + A^{(3)}$ gives a vanishing
contribution due to $s + t + u = 0$. The next contribution to $A_1$ comes from box diagrams
with a massive string mode in one propagator, the other three propagators containing light
particles (of mass $\mu$). As before, since the diagrams are UV convergent, in order to get
the $l_0$ independent terms it suffices to calculate the corresponding box diagram. These
diagrams are calculated in Appendix B. The $l_0$ independent terms in the sum of the three
terms are

$$\Delta A_1 = -\frac{g_{YM}^4}{3M_s^4} \left( \frac{\ln s}{t} \frac{\ln st}{\mu^4} + \frac{\ln s}{u} \frac{\ln su}{\mu^4} \right). \quad (3.64)$$
Notice that the terms in \((\ln \mu^2)^2\) have cancelled in the sum of the three terms in \((\text{B.9})\). Collecting terms of lowest order in \(s/M_s^2\) the contribution of \(A_1\) reads

\[
A^{(1,0)}(s,t) + A^{(1,0)}(u,t) + A^{(1,0)}(u,s) + \Delta A_1 =
\]

\[
\frac{2g_{YM}^4}{\pi^2} \left[ \frac{1}{st} \ln \frac{-s}{4\mu^2} \ln \frac{-t}{4\mu^2} + \text{perms.} \right] - \frac{g_{YM}^4}{3M_s^4} \ln \frac{s}{t} \ln \frac{st}{\mu^4} + \ln \frac{s}{u} \ln \frac{su}{\mu^4} + \frac{6}{l_0^2} + \cdots, (3.65)
\]

the result announced in \((1.7)\). The \(l_0\) dependent terms to this order are \((2g_{YM}^4/l_0^2M_s^4)\) which exactly cancels the first \(l_0\) dependent term in \(A_2\) as it should.

The next corrections in \(s/M_s^2\) to \(A_1\) come from terms of the type \((\ln \eta_2)^2\) and from terms of the type \(\ln \eta_2 + \eta_3\), which represent a sum of box diagrams with two massive modes circulating in the loop. By using the result \((\text{B.8})\) in Appendix B, it can be shown that the corresponding corrections to \((3.65)\) are of the order \(g_{YM}^4 sM_s^{-6} \ln M_s^2/\mu^2\).

4 Couplings of brane states to bulk states

Another interesting computation, closely related to the tree-level exchange of virtual closed string states is the tree-level (disk) coupling between two open (brane) states and one closed (bulk) winding excitation of mass \(w^2 \equiv n^2R^2M_s^4\), where \(n^2 = n_1^2 + \cdots n_6^2\). We show here, in agreement with the results obtained in Sections 3.4 and 3.5, that all winding modes couple the same way to the gauge bosons with a form factor written in \((3.50)\). Recently this issue was investigated in an effective theory context \([17]\) and an exponential supression in the winding (KK after T-duality) modes was found, interpreted there as the brane thickness. In a field theory context, the result depends not only on the fundamental mass scale but also on other (dimensionless) parameters. The full result has a nonperturbative origin from string theory viewpoint. In the perturbative string framework we discuss here, the result depends only on the string scale \(M_s\). The form factor that we found in \((3.50)\) depends actually on the energy squared of the graviton and not on its mass, a difference which is important for off-shell calculations. Moreover, its presence is actually, as discussed
in detail in previous Sections, not directly related to the regularization of winding (KK) virtual summations.

4.1 Two gauge bosons - one winding graviton amplitude

We consider for illustration the case of the open bosonic string. A similar computation in the superstring case was performed in [11] and we compare here their result with ours in order to understand the role played by supersymmetry in these computations. We compute the correlation function of two gauge boson vertex operators $V_i^{\mu_i}$ with one one

bulk graviton vertex operator $V_3^{\mu_3 \nu_4}$ on the disk represented here as the upper half complex plane $z, \text{Im}z > 0$. We use the doubling trick to represent the antiholomorphic piece of the graviton vertex operator as an holomorphic operator at the point $z' = \bar{z}$. Then the vertex operators are of the form

$$V_i^{\mu_i} = g_s^{1/2} \lambda^a \partial X^{\mu_i}(y_i)e^{2ip_i \cdot X(y_i)} : ,$$

$$V_3^{\mu_3 \nu_4} = V_3^{\mu_3}(z)V_3^{\nu_4}(\bar{z}) = g_s : \partial X^{\mu_3}(z)e^{ip_3 \cdot X(z) + i\omega Y(z)} : \partial X^{\nu_4}(\bar{z})e^{ip_3 \cdot X(\bar{z})} - i\omega Y(\bar{z}) : , \quad (4.1)$$

where $X$ are spacetime coordinates, $Y$ compact coordinates and $\lambda^a$ Chan-Paton factors gauge bosons. The gauge vector vertex operators are inserted on boundary points $y_1, y_2$ and the graviton vertex operator on a bulk point $z$. The Green functions on the disk we need in the computation are

$$<X^\mu(z_1)X^\nu(z_2)> = -\frac{1}{2M_s^2} \eta^{\mu\nu} \ln(z_1 - z_2) , \quad <X^\mu(z_1)X^\nu(\bar{z}_2)> = -\frac{1}{2M_s^2} \eta^{\mu\nu} \ln(z_1 - \bar{z}_2) , \quad (4.2)$$

where $\eta^{\mu\nu}$ is the Minkowski metric. The disk has three conformal Killing vectors which allow to fix three parameters in the positions of the vertex operators. We choose to fix the position of the graviton and the position of the second gauge boson. Introducing polarization vectors $\epsilon_i$ for gauge bosons and $\epsilon_3$ for the graviton, the amplitude to consider
is then

$$A_{ab} = g_s^{-1} N_3 \text{tr}(\lambda_a \lambda_b) \int_{-\infty}^{\infty} dy < c(y_2) c(z) c(\bar{z}) > < \epsilon_1 V_1(y_1) \epsilon_2 V_2(y_2) \epsilon_3 V_3(z) \epsilon_3 V_3(\bar{z}) > ,$$

(4.3)

where $< c(y_2) c(z) c(\bar{z}) > = |(y_2 - z)(y_2 - \bar{z})(\bar{z} - z)|$ is the factor coming from fixing the three positions on the disk, the factor $g_s^{-1}$ comes from the topological factor of the disk and $N_3$ is a normalization constant to be fixed by factorization later on. The conditions of transverse polarizations are $p_1 \epsilon_1 = p_2 \epsilon_2 = 0$, $p_3 \epsilon_3 = 0$ and the graviton polarization is also constrained by eliminating the scalar component $\epsilon_3^2 = 0$. Is to be understood that in the final result, $\epsilon_3^\mu \epsilon_3^\nu$ is to be replaced by the symmetric polarization tensor of the graviton, $\epsilon_3^{\mu \nu}$.

The mass-shell conditions are $p_1^2 = p_2^2 = 0$, $p_3^2 = -w^2$ and the kinematics of the process is described by

$$s = -(p_1 + p_2)^2 = w^2 , \quad p_1 p_2 = -\frac{1}{2} w^2 , \quad p_1 p_3 = \frac{1}{2} w^2 , \quad p_2 p_3 = \frac{1}{2} w^2 .$$

(4.4)

The gauge choice we make in the following is $y_1 = y$, $y_2 = 0$ and $z = i$. Then, by a straightforward computation of (4.3) and by using the formula

$$B(a, b) = \int_{-\infty}^{\infty} y^{2a-1}(1 + y^2)^{-(a+b)} = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)} ,$$

(4.5)

where $\Gamma(a)$ is the Euler function, we find the final result for the amplitude

$$A_{ab} = \frac{4}{\sqrt{\pi} M_P} 2^{-\frac{s^2}{M_s^2}} \frac{w^2}{M_s^2} \{[(2 \epsilon_1 \epsilon_2)(p_1 \epsilon_3) - \epsilon_1 \epsilon_3][\epsilon_2 \epsilon_3] + 8(\epsilon_1 p_2)(\epsilon_2 p_1)(\epsilon_3 p_1)^2] B\left(-\frac{s^2}{2M_s^2} + 3, \frac{1}{2}\right)$$

$$+ [2 \epsilon_1 \epsilon_2 (p_2 \epsilon_3)^2 + (\epsilon_1 \epsilon_3)(\epsilon_2 \epsilon_3) + 4(p_1 \epsilon_2)(p_2 \epsilon_3)(\epsilon_1 \epsilon_3) +$$

$$4(\epsilon_1 p_2)(\epsilon_2 \epsilon_3)(p_1 \epsilon_3) + 8(\epsilon_1 p_2)(\epsilon_2 p_1)(\epsilon_3 p_2)^2] B\left(-\frac{s^2}{2M_s^2} + \frac{3}{2}, \frac{1}{2}\right)\} \delta_{ab} ,$$

(4.6)

where $N_3$ was determined by factorization of the one-loop four-point amplitude of Section 3.5. Notice that the amplitude has a sequence of poles for $s = w^2 = (2n - 1) M_s^2$, with $n = 1 \cdots \infty$ a positive integer, and zeroes for $s = 2(n + 1) M_s^2$. The poles can be interpreted...
as due to massive open string states at odd levels coupled to the gauge fields and to the massive graviton through a tree-level diagram, as in Figure 6. In order for this to be possible, these states must be gauge singlets. Indeed, in the toroidal compactification we are considering, the gauge group is orthogonal ($SO(2^{13})$ for the bosonic string) and the spectrum contains adjoint (antisymmetric) representations at even mass levels and symmetric representations at odd mass levels. The symmetric representations however are reducible and contain the singlets which produce the poles. Note that, even if the spectrum at odd mass levels starts with a tachyonic state, this does not couple and therefore produce no pole in the amplitude. The particular case $w^2 = 0$ of the amplitude is in agreement with the field-theoretical computation of the three-point amplitude computed from the interaction Lagrangian

$$\mathcal{L} = \frac{1}{M_P} h_{\mu \nu} T^{\mu \nu} + \cdots = \frac{1}{M_P} h_{\mu \nu} \text{tr} \left( F^{\mu \rho} F_{\rho \nu} - \frac{1}{4} \eta^{\mu \nu} F_{\rho \sigma} F^{\rho \sigma} + \cdots \right), \quad (4.7)$$

where $T^{\mu \nu}$ is the energy-momentum tensor of the gauge-fields and $h_{\mu \nu}$ represents the graviton. More precisely, an explicit 3-point computation from (4.7) reproduces exactly all terms in (4.6) except the terms quartic in momenta. Up to these terms actually the result is exactly the same as in the superstring case$^4$ $^{11}$ and therefore the conclusions we present below are largely independent on supersymmetry$^5$. In particular, we find here again the selection rule which make the amplitude vanish for $s = 2(n + 1)M_s^2$. The quartic terms, absent in the superstring case, are to be interpreted as arising from the higher-derivative term in the Lagrangian $F^{\mu \nu} F^{\sigma \nu} R_{\mu \sigma \nu}$, with $R_{\mu \sigma \nu}$ the gravitational Riemann tensor.

The computation presented is on-shell $s = w^2$. We are now interested in the large $s = w^2$ behaviour of the above amplitude (for values which avoid the poles and zeros we just discussed), which on-shell is equivalent of considering couplings to very massive winding

---

$^4$This can be shown by using the identity $2^{2z-1} \Gamma(z) \Gamma(z + 1/2) = \sqrt{\pi} \Gamma(2z)$.

$^5$This allowed us to determine the normalisation constant $N_3$ in (4.3) by factorizing the one-loop amplitude of Section 3.5.
Figure 6: The two gauge bosons – one graviton vertex, where the intermediate states are open string singlets.

By using the asymptotic expansion (valid for $s >> M_s^2$)

$$B(-\frac{s}{2M_s^2} + \frac{3}{2}, \frac{1}{2}) \simeq \sqrt{\frac{2\pi M_s^2}{s}} \tan \frac{\pi s}{2M_s^2} ,$$  

we find the (on-shell) effective coupling of gauge fields to massive winding (or Kaluza-Klein in the T-dual picture where the gauge field is stuck on a D3 brane orthogonal to the compact space)

$$g_n = \frac{1}{\sqrt{\pi}} 2^{\frac{w^2}{M_s^2}} \frac{\Gamma(-w^2/2M_s^2 + 1/2)}{\Gamma(-w^2/2M_s^2 + 1)} \sim 2 \sqrt{\frac{2M_s^2}{\pi w^2}} e^{-\frac{w^2}{M_s^2} \ln 2} ,$$  

where in the last formula we took the heavy mass limit $w^2 >> M_s^2$. So, modulo the power in front of it, we find an exponential suppression of states heavier than a cutoff mass $\Lambda^2 = M_s^2/\ln 2$. However, we emphasize again that this interpretation is valid for masses not very close to poles and zeros of the full expression \((4.6)\), where the interpretation is completely different. In addition, as shown in \((3.50)\), for off-shell gravitons the winding mass $w$ is actually replaced by the (squared) momentum of the graviton, $k^2$.

### 4.2 Three gauge bosons - one winding graviton amplitude

This (tree-level) amplitude is of direct interest for accelerator searches and was calculated at the effective-field theory level in \([8]\). It is however important to have a full string expression in order to control the string corrections for energies close enough to the string scale $M_s$. 

$$\sqrt{\frac{2\pi M_s^2}{s}} \tan \frac{\pi s}{2M_s^2} .$$
Moreover, this computation allows an off-shell continuation of the form factors (3.50), (4.9) for one of the two gauge bosons.

The amplitude involves the correlation function of three gauge vertex operators of polarisations $\epsilon_i$ and momenta $p_i \ (i = 1, 2, 3)$ and of a massive winding-type graviton of polarisation $\epsilon_4$ and momentum $p_4$. The three conformal Killing vectors allow us to fix the positions of the gauge vertex operators on the boundary of the disk $y_1 = 0, y_2 = 1$ and $y_3 = \infty$. Then the position of the graviton is unfixed on the disk, represented as usual as the upper half plane. Some details about this computation in type I superstring are displayed in the Appendix C. We choose to use the 0-picture vertex operators for the gauge bosons and the (-2) picture vertex on the disk (corresponding to the (-1,-1) picture on the sphere) for the graviton vertex operator [18]. Therefore we have

$$V_i^{\mu} = g_s^{1/2} \lambda^{a_i} : (i \partial X^\mu(y_i) + \frac{2p_i}{M_i^2} \psi_i \psi_i(y_i)) e^{2i p_i X(y_i)} : , \quad (4.10)$$

for the gauge bosons and

$$V^{\mu\nu} = V^\mu(z) V^\nu(\bar{z}) = g_s : e^{-\phi(z)} \psi^{\mu}(z) e^{i p_4 X(z) + i w Y(z)} \psi_i^{\nu}(\bar{z}) e^{i p_4 X(\bar{z}) - i w Y(\bar{z})} : \quad (4.11)$$

for the graviton, where $\psi^{\mu}$ and $\phi$ are the world-sheet fermions and the bosonised ghosts, respectively. The amplitude of interest reads

$$A_4 = g_s^{-1} N_1 tr(\lambda_{a_1} \lambda_{a_2} \lambda_{a_3}) \int_{\mathbb{C}^+} d^2 z < c(y_1)c(y_2)c(y_3) > \prod_{i=1}^3 \epsilon_i V_i(y_i) \epsilon_4 V(z) \epsilon_4 V(\bar{z}) > +1 \leftrightarrow 2 , \quad (4.12)$$

where we introduced the normalization constant, to be fixed by unitarity. By using the Mandelstam variables (3.4), the kinematics of the amplitude is summarized by the equations

$$s = -2p_1.p_2 = -2p_3p_4 + w^2 , \quad t = -2p_2.p_3 = -2p_1.p_4 + w^2 ,$$

$$u = -2p_1.p_3 = -2p_2.p_4 + w^2 , \quad s + t + u = w^2 . \quad (4.13)$$

The details of the calculation are rather long and some of the steps are sketched in Appendix
The final result can be put in the form

\[
A_4 = \frac{g_{YM}}{\sqrt{\pi} M_P} 2 \frac{w^2}{M_s^2} \text{tr}([\lambda^{a_1}, \lambda^{a_2}]\lambda^{a_3}) K \frac{\Gamma\left(\frac{-w^2}{2M_s^2} + \frac{1}{2}\right) \Gamma\left(-\frac{s}{2M_s^2} + 1\right) \Gamma\left(-\frac{t}{2M_s^2} + 1\right) \Gamma\left(-\frac{u}{2M_s^2} + 1\right)}{\Gamma\left(\frac{4-w^2}{2M_s^2} + 1\right) \Gamma\left(\frac{4-s^2}{2M_s^2} + 1\right) \Gamma\left(\frac{4-t^2}{2M_s^2} + 1\right) \Gamma\left(\frac{4-u^2}{2M_s^2} + 1\right)}, \tag{4.14}
\]

where \(K\) is a kinematical factor displayed in Appendix C and \(N_4\) was determined by unitarity from the three gauge bosons amplitude and two gauge bosons – one graviton amplitude. In order to make connection with the field-theory result, we notice that we can actually rewrite (4.14) in the form

\[
A_4 = 1 \frac{1}{\sqrt{\pi}} 2 \frac{w^2}{M_s^2} \frac{\Gamma\left(\frac{-w^2}{2M_s^2} + \frac{1}{2}\right) \Gamma\left(-\frac{s}{2M_s^2} + 1\right) \Gamma\left(-\frac{t}{2M_s^2} + 1\right) \Gamma\left(-\frac{u}{2M_s^2} + 1\right)}{\Gamma\left(\frac{4-w^2}{2M_s^2} + 1\right) \Gamma\left(\frac{4-s^2}{2M_s^2} + 1\right) \Gamma\left(\frac{4-t^2}{2M_s^2} + 1\right) \Gamma\left(\frac{4-u^2}{2M_s^2} + 1\right)} A_{\text{FT}}^4, \tag{4.15}
\]

where \(A_{\text{FT}}^4\) turns out to be exactly the field-theory amplitude \([8]\). The full string result (4.15) obviously reduces to the field theory result in the low energy \(s,t,u << M_s^2\) and low graviton mass \(w^2 << M_s^2\) limit. The analytic structure of the string amplitude shows the presence of poles for \(s,t,u = (2n - 2)M_s^2\) for \(n\) a positive integer, corresponding to tree-level open modes exchanges in the \(s\), \(t\) and \(u\) channel, respectively. Moreover, we find, like in the previous Section, poles for graviton winding masses \(w^2 = (2n - 1)M_s^2\), interpreted as a tree-level mixing between the graviton and the gauge singlets present at the odd open string levels, which couples afterwards to the gauge fields. We find also interesting zeroes of the amplitudes for very heavy gravitons \(w^2 = s + 2nM_s^2\) or similar equations obtained by the replacement \(s \rightarrow t, u\), giving interesting selection rules. By using (4.15) we are now able to extend the form factor (3.50) to the case where one of the gauge bosons is off-shell

\[
g(p_1, p_2, p) = \frac{1}{M_P \sqrt{\pi}} 2 \frac{w^2}{M_s^2} \frac{\Gamma\left(-\frac{p^2}{2M_s^2} + 1/2\right)}{\Gamma\left(-\frac{p_1 p_2}{2M_s^2} + 1\right)}, \tag{4.16}
\]

the result displayed in (1.9). An important question is certainly the string deviations in (4.15) from the field theory result \(A_{\text{FT}}^4\). The energy corresponding to the first string

\[\text{We added also in Appendix C the similar but much simpler computation of an amplitude in the bosonic string of three open string tachyons and open winding closed string tachyon, which has similar analyticity properties to the supersymmetric amplitude we discuss here.}\]
resonance is, in the s-channel, $s = 2M_s^2$ and similarly for $t$ and $u$, meaning that field theory computations certainly break down for energies above. For energies well below this value $s, t, u, w^2 \ll M_s^2$, it is easy to find the corrections to the field-theory computation by performing a power-series expansion in (4.18). The first corrections turn out to be of the form

$$A_4 = (1 + \frac{\zeta(2)}{4} \frac{w^4}{M_s^4} + \frac{\zeta(3)}{4} \frac{stu + w^6}{M_s^6} + \cdots)A_4^{FT}$$

(4.17)

which, after T-duality in order to make connection with the notation in the Introduction, becomes

$$A_4 = (1 + \frac{\pi^2}{24} \frac{m^4}{(R_M M_s)^4} + \cdots)A_4^{FT}.$$  

(4.18)

This result can be interpreted as a modification of the effective coupling of massive graviton to matter

$$\frac{1}{M_P} \to \frac{1}{M_P} \left(1 + \frac{\zeta(2)}{4} \frac{w^4}{M_s^4}\right).$$

(4.19)

Notice that the first correction to the amplitude with a massless graviton (of fixed energy) is of order $E^6/M_s^6$, so the deviation from the field theoretical result is first expected to be seen from massive gravitons.

An experimentally more useful way to define deviations from the field theory result is in the integrated cross-section $\sigma$, obtained by summing over all graviton masses, up to the available energy $E$

$$\sigma = \sum_{m_1 \cdots m_6=0}^{R_M E} |A_4|^2, \quad \sigma^{FT} = \sum_{m_1 \cdots m_6=0}^{R_M E} |A_4^{FT}|^2,$$

(4.20)

where $\sigma^{FT}$ is the corresponding field theory value. Surprisingly enough, terms of order $E^2$ (or $m^2$ in (4.18)) are absent in (4.20) and therefore at low energies the string corrections are smaller than expected, of order

$$\frac{\sigma - \sigma^{FT}}{\sigma^{FT}} \sim \frac{E^4}{M_s^4}.$$  

(4.21)

However, as mentioned above, strong deviations appear close to the value $E^2 = 2M_s^2$, where the first string resonance appear and the field theory approach breaks down.
A  Jacobi functions and their properties

For the reader’s convenience we collect in this Appendix the definitions, transformation properties and some identities among the modular functions that are used in the text. The Dedekind function is defined by the usual product formula (with \( q = e^{2\pi i \tau} \))

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]

(A.1)

whereas the Jacobi \( \vartheta \)-functions with general characteristic and arguments are

\[
\vartheta_{[\alpha \beta]}(z, \tau) = \sum_{n \in \mathbb{Z}} e^{i \pi \tau (n-\alpha)^2} e^{2 \pi i (z-\beta)(n-\alpha)}.
\]

(A.2)

We give also the product formulae for the four special \( \vartheta \)-functions

\[
\begin{align*}
\vartheta_1(z, \tau) &\equiv \vartheta_{[1/2 1/2]}(z, \tau) = 2 q^{1/8} \sin \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2\pi i z})(1 - q^n e^{-2\pi i z}), \\
\vartheta_2(z, \tau) &\equiv \vartheta_{[1/2 0]}(z, \tau) = 2 q^{1/8} \cos \pi z \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n e^{2\pi i z})(1 + q^n e^{-2\pi i z}), \\
\vartheta_3(z, \tau) &\equiv \vartheta_{[0 0]}(z, \tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2} e^{2\pi i z})(1 + q^{n-1/2} e^{-2\pi i z}), \\
\vartheta_4(z, \tau) &\equiv \vartheta_{[0 1/2]}(z, \tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2} e^{2\pi i z})(1 - q^{n-1/2} e^{-2\pi i z}).
\end{align*}
\]

(A.3)

The modular properties of these functions are described by

\[
\begin{align*}
\eta(\tau + 1) &= e^{i\pi/12} \eta(\tau), & \vartheta_{[\alpha \beta]}(z, \tau + 1) &= e^{-i\pi \alpha(\alpha-1)} \vartheta_{[\alpha \alpha + \beta - 1/2]}(z, \tau), \\
\eta(-1/\tau) &= \sqrt{-i \tau} \eta(\tau), & \vartheta_{[\alpha \beta]}(z, -1/\tau) &= \sqrt{-i \tau} e^{2i\pi \alpha \beta + i\pi z^2/\tau} \vartheta_{[\beta - \alpha]}(z, \tau).
\end{align*}
\]

(A.4)

We consider the box amplitude \( B(s, t, m_1, \ldots, m_4) \) in four dimensions (see Figure 7) in the euclidean formulation

\[
\frac{M_s^{-4} \pi^4}{3} \int d\tau_2 d\tau_3 \eta^4 \delta(1 - \sum_i \eta_i) e^{2\pi \tau_2 (\eta_1 \eta_3 + \eta_2 \eta_4)} e^{-\frac{2\pi \tau_3}{M_s} \sum_i m_i^2 \eta_i} \equiv \frac{\pi^2}{6} B'.
\]

(B.1)
Figure 7: The box diagram with particles of masses $m_i$ in the loop

It is possible to perform the integral over $\tau_2$ as well as the integration over two $\eta$ variables and obtain

$$B' = \int_0^1 d\eta_1 \int_0^{1-\eta_1} d\eta_2 \frac{1 - \eta_1 - \eta_2}{[(1-\eta_1-\eta_2)(m_3^2 - t\eta_2) + \eta_1 m_1^2 + \eta_2 m_2^2][(1-\eta_1-\eta_2)(m_3^2 - s\eta_1) + \eta_1 m_1^2 + \eta_2 m_2^2]}. \quad (B.2)$$

For equal masses the expression simplifies to

$$B' = \int_0^1 d\eta_1 \int_0^{1-\eta_1} d\eta_2 \frac{1 - \eta_1 - \eta_2}{m^2 - t\eta_2(1-\eta_1-\eta_2)[m^2 - s\eta_1(1-\eta_1-\eta_2)]}. \quad (B.3)$$

If $-s << m^2$ and $-t << m^2$ we get the dominant contribution $B' = 1/6m^4 + \ldots$. In the opposite limit when $m^2 << -s$ and $m^2 << -t$, it is convenient to change variables to $\alpha = \eta_2(1-\eta_1-\eta_2)$ and $\beta = \eta_1(1-\eta_1-\eta_2)$, so that the integral becomes

$$B' = 2 \int_0^{1/4} d\alpha \int_0^{1/4-\alpha} d\beta \frac{1}{\sqrt{1 - 4(\alpha + \beta)}} \frac{1}{m^2 - t\alpha} \frac{1}{m^2 - s\beta}. \quad (B.4)$$

In the limit where $m$ is very small one can neglect the first factor in the integral and we get

$$B' = \frac{1}{st} \ln \frac{-s}{4m^2} \ln \frac{-t}{4m^2} + \ldots. \quad (B.5)$$

Another case encountered is the one where $m_4 = M$ is large and the other masses are equal ($\mu$) and small. The integral is approximated by

$$\frac{1}{M^2} \int_0^1 d\eta_1 \int_0^{1-\eta_1} d\eta_2 \frac{1}{\mu^2 - s\eta_1\eta_2} = \frac{-1}{M^2s} \int_0^1 d\eta \frac{1}{\eta} \ln[1 - \frac{s\eta_1(1-\eta)}{\mu^2}], \quad (B.6)$$
which is approximately equal to
\[-\frac{1}{2sM^2}\ln^2 \frac{s}{\mu^2}.
\] (B.7)

The case where \(m_2\) is very large and the other masses small gives the same answer and the one where \(m_1\) or \(m_3\) are very large is obtained by changing \(s\) into \(t\). Another case of interest corresponds to \(m_4 = m_3 = M\) and \(m_1 = m_2 = \mu\). The box diagram to leading order is given in this case by
\[B' = \frac{1}{M^4} \int_0^1 d\eta_1 \int_0^{1-\eta_1} d\eta_2 \frac{1}{1 - (\eta_1 + \eta_2)(1 + \mu^2/M^2)} = \frac{1}{M^4} \ln \frac{M^2}{\mu^2}.
\] (B.8)

We are now in a position to calculate the leading correction, \(\Delta A_1\), to the amplitude \(A_1\) discussed in Section 3.5. It is the sum of the three terms
\[A_1^{(1,1)}(l_0)(s, t) = \frac{8g_4^2 M}{M^4} \int_{1/0}^\infty d\tau_2 d\tau_2 \int_{\eta > 0} d^4 \eta \delta(1 - \sum_i \eta_i) e^{\frac{2\pi\eta^2}{M^2}(s\eta_1 \eta_2 + t\eta_1 \eta_2)} e^{-\frac{2\pi\tau^2 \mu_2^2}{M^2}} \left\{ -(s/M^2)(\ln f(\eta_2) + \ln f(\eta_4)) - (t/M^2)(\ln f^T(\eta_1) + \ln f^T(\eta_3)) \right\},
\]
\[A_1^{(2,1)}(l_0)(u, t) = \frac{8g_4^2 M}{M^4} \int_{1/0}^\infty d\tau_2 d\tau_2 \int_{\eta > 0} d^4 \eta \delta(1 - \sum_i \eta_i) e^{\frac{2\pi\eta^2}{M^2}(u\eta_1 \eta_2 + s\eta_1 \eta_2)} e^{-\frac{2\pi\tau^2 \mu_2^2}{M^2}} \left\{ -(u/M^2)(\ln f^T(\eta_2) + \ln f^T(\eta_4)) - (t/M^2)(\ln f^T(\eta_1) + \ln f^T(\eta_3)) \right\},
\]
\[A_1^{(3,1)}(l_0)(u, s) = \frac{8g_4^2 M}{M^4} \int_{1/0}^\infty d\tau_2 d\tau_2 \int_{\eta > 0} d^4 \eta \delta(1 - \sum_i \eta_i) e^{\frac{2\pi\eta^2}{M^2}(u\eta_1 \eta_2 + s\eta_1 \eta_2)} e^{-\frac{2\pi\tau^2 \mu_2^2}{M^2}} \left\{ -(u/M^2)(\ln f^T(\eta_2) + \ln f^T(\eta_4)) - (s/M^2)(\ln f(\eta_1) + \ln f(\eta_3)) \right\}.\] (B.9)

Recall that
\[f(\eta) = (1 - e^{-2\pi\eta \tau_2}) \prod_{n=1}^\infty (1 - e^{-2\pi(n-\eta) \tau_2})(1 - e^{-2\pi(n+\eta) \tau_2})\]
\[f^T(\eta) = (1 + e^{-2\pi\eta \tau_2}) \prod_{n=1}^\infty (1 + e^{-2\pi(n-\eta) \tau_2})(1 + e^{-2\pi(n+\eta) \tau_2})\] (B.10) (B.11)

The development of \(\ln f(\eta)\) as
\[\ln f(\eta) = - \sum_{m=1}^\infty \frac{e^{-2\pi m \eta \tau_2}}{m} + \ldots\] (B.12)

and a similar expansion of \(\ln f^T\) shows that every term in the above sum represents a box diagram with a massive string mode in one leg and the other three particles of mass \(\mu\),
while the dots represent box diagrams with more than one leg having a massive string mode and are thus of higher order. As explained in section 3.5, since the diagrams are UV convergent, in order to get the $l_0$ independent terms it suffices to calculate the corresponding box diagram. From (B.7) we obtain the leading contribution to $A_1^{(1,1)}$ as
\[
A_1^{(1,1)}(s, t) = -\frac{g_4^4 YM^3}{3 M_s^4} \left( \ln^2 \frac{s}{\mu^2} - \frac{1}{2} \ln^2 \frac{t}{\mu^2} \right),
\]
$A_1^{(2,1)}(u, t) = \frac{g_4^4 YM^3}{3 M_s^4} \left( \frac{1}{2} \ln^2 \frac{t}{\mu^2} + \frac{1}{2} \ln^2 \frac{u}{\mu^2} \right)$,
$A_1^{(3,1)}(u, s) = -\frac{g_4^4 YM^3}{3 M_s^4} \left( \ln^2 \frac{s}{\mu^2} - \frac{1}{2} \ln^2 \frac{u}{\mu^2} \right),$
(B.13)
where we have used $\sum \frac{1}{m^2} = \frac{\pi^2}{6}$. The sum of the three terms in (B.13) gives
\[
\Delta A_1 = \frac{g_4^4 YM^3}{3 M_s^2} \left( \ln \frac{s}{t} \ln \frac{st}{\mu^2} + \ln \frac{s}{u} \ln \frac{su}{\mu^2} \right),
\]
(B.14)
which is the result used in the text (3.65). Notice that the terms in $\ln^2 \mu^2$ have cancelled in $\Delta A_1$.

C The type I disk amplitude

In order to compute the amplitude (4.12) depicted in Figure 8, it is convenient first to write the vertex operators with the aid of Grassmann variables $\theta_i$ and $\phi_i$ as
\[
\epsilon_i V_i = \int \theta_i \phi_i e^{2ip_i X(y_i) + i \theta_i \phi_i \epsilon_i \partial X_i - \frac{1}{2} p_i \theta_i \phi_i} + \frac{1}{2} \phi_i \epsilon_i \theta_i ,
\]
\[
\epsilon_4 V_4(z) = \int \phi_4 e^{ip_4 X(z) + i w Y(z) + \phi_4 \epsilon_4 \psi(z)} ,
\]
(C.1)
\[
\epsilon_4 V_4(\bar{z}) = \int \bar{\phi}_4 e^{ip_4 X(\bar{z}) - i w Y(\bar{z}) + \bar{\phi}_4 \epsilon_4 \psi(z)} .
\]
The correlation functions are then easily calculated with the aid of
\[
< \psi^\mu(z_1) \psi^\nu(z_2) > = \eta^{\mu\nu} \frac{1}{z_1 - z_2} ,
\]
(C.2)
\[
< c(y_1) c(y_2) c(y_3) > = (y_1 - y_2)(y_1 - y_3)(y_2 - y_3) , \quad < e^{-\phi(z)} e^{-\phi(\bar{z})} > \approx \frac{1}{\bar{z} - z} .
\]
Then we fix the positions $y_i$ to 0, 1 and $\infty$ and calculate the integral over the Grassmann variables. The resulting amplitude involve integrals of the form
\[
I_n(\alpha, \beta, \gamma) = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \ x^n y^\alpha (x^2 + y^2)^\beta \left( (1 - x)^2 + y^2 \right)^\gamma ,
\]
(C.3)
Figure 8: The disk amplitude with three open string particles and one closed string particle.

where \( n = 0, 1 \) and \( \alpha, \beta \) and \( \gamma \) are real. These integrals can be calculated using the standard tricks to yield

\[
I_0(\alpha, \beta, \gamma) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma(-\beta)\Gamma(-\gamma)} \Gamma\left(-\beta-\frac{\alpha}{2}-1\right) B\left(\gamma+\frac{\alpha}{2}+1, \beta+\frac{\alpha}{2}+1\right),
\]

\[
I_1(\alpha, \beta, \gamma) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma(-\beta)\Gamma(-\gamma)} \Gamma\left(-\beta-\frac{\alpha}{2}-1\right) B\left(\gamma+\frac{\alpha}{2}+1, \beta+\frac{\alpha}{2}+2\right). \tag{C.4}
\]

With the aid of these integrals the amplitude can be calculated and after some arrangements and summing the two cyclically inequivalent permutations of the open states it can be put in the form \((4.14)\) with the kinematical factor \( K \) given by

\[
K = s(u+t)\left(\epsilon_1, \epsilon_3, \epsilon_2, p_3, \epsilon_4, p_1, \epsilon_4, p_2 - \epsilon_1, p_3, \epsilon_2, \epsilon_3, \epsilon_4, p_1, \epsilon_4, p_2 - \epsilon_1, \epsilon_4, \epsilon_2, \epsilon_3, \epsilon_4, p_2, p_1 \right.
\]

\[
+ \epsilon_1, \epsilon_4, \epsilon_2, \epsilon_3, \epsilon_4, p_1, p_3, \epsilon_1, \epsilon_4, \epsilon_2, \epsilon_4, \epsilon_3, \epsilon_4, p_1, p_2, p_3 - \epsilon_1, \epsilon_3, \epsilon_2, \epsilon_4, \epsilon_3, \epsilon_4, p_1, p_2, p_3
\]

\[
- \epsilon_1, \epsilon_4, \epsilon_2, p_3, \epsilon_3, p_1, \epsilon_4, p_2 + \epsilon_1, p_3, \epsilon_2, \epsilon_4, \epsilon_3, p_2, p_4
\]

\[
+ \frac{1}{2} (st\epsilon_3, p_1 - su\epsilon_3, p_2)\left(\epsilon_1, \epsilon_2, \epsilon_3, p_1, \epsilon_4, p_2 - \epsilon_1, \epsilon_4, \epsilon_2, \epsilon_3, p_1, \epsilon_4, p_2 - \epsilon_1, p_2, \epsilon_2, \epsilon_4, \epsilon_3, p_1, p_2, p_1, \epsilon_4, p_2
\]

\[
- \epsilon_1, p_2, \epsilon_2, \epsilon_4, \epsilon_3, p_1 \right) + (1, 2, 3, 4) \rightarrow (3, 1, 2, 4) + (1, 2, 3, 4) \rightarrow (2, 3, 1, 4). \tag{C.5}
\]

For pedagogical reasons, we present here also a similar but much simpler computation in the bosonic string, the four-point function of three open string tachyons and one winding state closed string tachyon. The vertex operators \( V_i \) for open tachyons and \( V_4 \) for the winding state closed tachyon in this case are

\[
V_i = g_s^{1/2} \lambda^{\alpha_i} \cdot e^{2p_i X(y_i)} ;
\]
\[ V_4 = g_s : e^{ip_3 X(z) + iwY(\bar{z})} :: e^{ip_3 X(\bar{z}) - iwY(z)} : \] \hspace{1cm} (C.6)

By using unitarity arguments as in Section 4 in order to fix the overall normalization constant, the amplitude to compute becomes therefore

\[ A = \frac{gYM}{\pi M_P} \text{tr}(\lambda_{a_1} \lambda_{a_2} \lambda_{a_3}) \int_{C^+} d^2 z |z|^{\frac{2p_4}{M_s^2}} |1 - z|^{\frac{2p_2 p_4}{M_s^2}} |z - \bar{z}|^{\frac{p_4^2 - w^2}{2M_s^2} + 2} + 1 \leftrightarrow 2, \] \hspace{1cm} (C.7)

where the complex integral is in the upper half complex plane. The kinematics of the process is described by

\[
\begin{align*}
p_1^2 &= p_2^2 = p_3^2 = M_s^2, & p_4^2 + w^2 &= 4M_s^2, \\
s &= M_s^2 - w^2 + 2(p_1 + p_2)p_4, & t &= -5M_s^2 + w^2 - 2p_2 p_4, \\
u &= -5M_s^2 + w^2 - 2p_1 p_4, & s + t + u &= -5M_s^2 + w^2.
\end{align*}
\] \hspace{1cm} (C.8)

The simplest way to compute the amplitude is to use equalities of the type

\[ \left( \frac{1}{z\bar{z}} \right)^a = \frac{1}{\Gamma(a)} \int_0^\infty dt t^{a-1} e^{-tz}, \] \hspace{1cm} (C.9)

which was also used in order to obtain (C.4) from (C.3). The final result is

\[ A = \frac{gYM}{\sqrt{\pi} M_P} 2^{\frac{5}{2} + 1} \text{tr}(\lambda^{a_1} \lambda^{a_2} \lambda^{a_3}) \frac{\Gamma\left(-\frac{w^2}{2M_s^2} + \frac{3}{2}\right) \Gamma\left(-\frac{s}{2M_s^2} - \frac{1}{2}\right) \Gamma\left(-\frac{t}{2M_s^2} - \frac{1}{2}\right) \Gamma\left(-\frac{u}{2M_s^2} - \frac{1}{2}\right)}{\Gamma\left(-\frac{s-w^2}{2M_s^2} + \frac{5}{2}\right) \Gamma\left(-\frac{t-w^2}{2M_s^2} + \frac{5}{2}\right) \Gamma\left(-\frac{u-w^2}{2M_s^2} + \frac{5}{2}\right)}. \] \hspace{1cm} (C.10)
References

[1] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Vol. I,II, Cambridge University Press, 1987.

[2] J. Polchinski, *String Theory*, Vol. I,II, Cambridge University Press, 1998.

[3] I. Antoniadis, *Phys. Lett.* B246 (1990) 377; I. Antoniadis and K. Benakli, *Phys. Lett.* B326 (1994) 69; I. Antoniadis, K. Benakli and M. Quiros, *Phys. Lett.* B331 (1994) 313.

[4] E. Witten, *Nucl. Phys.* B471 (1996) 135, J.D. Lykken, *Phys. Rev.* D54 (1996) 3693.

[5] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* B429 (1998) 263, hep-ph/9807344.

[6] K.R. Dienes, E. Dudas and T. Gherghetta, *Phys. Lett.* B436 (1998) 55, *Nucl. Phys.* B537 (1999) 47, hep-ph/9807522; C. Bachas, hep-th/9807415; D. Ghilencea and G.G. Ross, *Phys. Lett.* B442 (1998) 165; Z. Kakushadze, hep-th/9811193; A. Delgado and M. Quirós, hep-ph/9903400; Z. Kakushadze and T.R. Taylor, hep-th/9905137; I. Antoniadis, C. Bachas and E. Dudas, hep-th/9906039.

[7] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* B436 (1998) 263, G. Shiu and S.-H.H. Tye, *Phys. Rev.* D58 (1998) 106007; K. Benakli, *Phys. Rev.* D60 (1999) 104002; C. Burgess, L.E. Ibáñez and F. Quevedo, hep-ph/9810335; I. Antoniadis and C. Bachas, hep-th/9812093; I. Antoniadis and B. Pioline, *Nucl. Phys.* B550 (1999) 41.

[8] G.F. Giudice, R. Rattazzi and J.D. Wells, *Nucl. Phys.* B544 (1999) 3; E.A. Mirabelli, M. Perelstein and M.E. Peskin, *Phys. Rev. Lett.* 82 (1999) 2236; T. Han, J.D. Lykken and R.J. Zhang, *Phys. Rev.* D59 (1999) 105006; J.L. Hewett, *Phys. Rev. Lett.* 82 (1999) 4765; T.G. Rizzo, *Phys. Rev.* D59 (1999) 115010, *Phys. Rev.* D60 (1999)
075001; P. Mathews, S. Raychaudhuri and K. Sridhar, Phys. Lett. B450 (1999) 343 and hep-ph/9904232, M. Besancon, hep-ph/9909364.

[9] A. Sagnotti, hep-th/9302099; C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Ya.S. Stanev, Phys. Lett. B387 (1996) 743; G. Zwart, Nucl. Phys. B526 (1998) 378; Z. Kakushadze, G. Shiu and S.H.H. Tye, Nucl. Phys. B533 (1998) 25; G. Aldazabal, A. Font, L.E. Ibañez and G. Violero, Nucl. Phys. B536 (1998) 29.

[10] E. Cremmer and J. Scherk, Nucl. Phys. B50 (1972) 222; L. Clavelli and J.A. Shapiro, Nucl. Phys. B57 (1973) 490; C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, Nucl. Phys. B293 (1987) 83, Nucl. Phys. B308 (1988) 221; J. Polchinski and Y. Cai, Nucl. Phys. B296 (1988) 91.

[11] A. Hashimoto and I.R. Klebanov, Phys. Lett. B381 (1996) 437, Nucl. Phys. Proc. Suppl. B55 1997 118.

[12] M.R. Garousi and R.C. Myers, Nucl. Phys. B475 (1996) 193.

[13] M.B. Green, J.H. Schwarz and L. Brink, Nucl. Phys. B198 (1982) 474.

[14] M.B. Green and J.H. Schwarz, Nucl. Phys. B198 (1982) 441.

[15] E. D’Hoker and D.H. Phong, Nucl. Phys. B440 (1995) 24.

[16] A.A Tseytlin, Phys. Lett. B367 (1996) 84, Nucl. Phys. B467 (1996) 383; C. Bachas and E. Kiritsis, Nucl. Phys. Proc. Suppl. B55 (1997) 194.

[17] M. Bando, T. Kugo, T. Nagoshi and K. Yoshioka, hep-ph/9906549, J. Hisano and N. Okada, hep-ph/9909553.

[18] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.