DUALITY STRUCTURES AND DISCRETE CONFORMAL VARIATIONS OF PIECEWISE CONSTANT CURVATURE SURFACES

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Abstract. A piecewise constant curvature manifold is a triangulated manifold that is assigned a geometry by specifying lengths of edges and stipulating that for a chosen background geometry (Euclidean, hyperbolic, or spherical), each simplex has an isometric embedding into the background geometry with the chosen edge lengths. Additional structure is defined either by giving a geometric structure to the Poincaré dual of the triangulation or by assigning a discrete metric, a way of assigning length to oriented edges. This notion leads to a notion of discrete conformal structure, generalizing the discrete conformal structures based on circle packings and their generalizations studied by Thurston and others. We define and analyze conformal variations of piecewise constant curvature 2-manifolds, giving particular attention to the variation of angles. We give formulas for the derivatives of angles in each background geometry, which yield formulas for the derivatives of curvatures. Our formulas allow us to identify particular curvature functionals associated with conformal variations. Finally, we provide a complete classification of discrete conformal structures in each of the background geometries.

1. Introduction

A triangulation of a manifold can be given a geometric structure by assigning compatible geometric structures to its component simplices. One of the easiest ways of doing this is to assign constant curvature geometries to the simplices, as these simplices are uniquely determined by their edge lengths. Such a structure gives a finitely parametrized set of geometric structures on a closed manifold.

In Thurston’s formulation of the discrete Riemann mapping problem (see [43]) as well as in applied methods such as discrete exterior calculus (see, e.g., [16], [15]), it is important to not only have a piecewise constant curvature metric assigned to simplices, but also to give a structure to the Poincaré dual of the triangulation. Such structures arise naturally as incircle duals in Thurston’s formulation of circle packings and as circumcentric duals in discrete exterior calculus. For piecewise Euclidean surfaces and 3-manifolds, in [22] and [24] the first author gives an axiomatic treatment of geometric duality structures that have orthogonal intersections with the primal simplices, and also relates these to discrete conformal variations.

The goal of the present work is to make precise the parametrization of duality structures by partial edge lengths (giving a discrete analogue of a Riemannian metric), define the general form of discrete conformal structures based on an axiomatic
development related to conformal variation of angle, and derive a local classification of such structures. The relationship between duality structures and discrete metrics requires some understanding of possible geometric centers for triangles, leading to the definition of the span of a triangle as the space of possible geometric centers. The axiomatic development of conformal structure follows that in [24] for piecewise Euclidean surfaces, while the construction in piecewise hyperbolic and spherical surfaces is new. The general formulas for angle and curvature variation of piecewise hyperbolic and spherical surfaces is new (however, see the parallel work in [48]), generalizing circle packings and other discrete conformal structures previously studied by many authors (see Section 1.3 for details). The local classification of discrete conformal structures, giving explicit formulas for the structures, is new for each geometry including Euclidean.

We will begin by making these geometric structures precise, and then give precise statements of the main results.

1.1. Geometric structures on triangulations. In this section, we make precise some geometric structures.

**Definition 1.** A triangulated manifold $(M, T)$ is a topological manifold $M$ together with a triangulation $T$ of $M$. A (triangulated) piecewise constant curvature manifold $(M, T, \ell)$ with background geometry $G$ is a triangulated manifold $(M, T)$ together with a function $\ell$ on the edges of the triangulation such that each simplex can be embedded in $G$, a space of constant curvature, as a (nondegenerate) simplex with edge lengths determined by $\ell$.

When the background geometry is Euclidean ($G = \mathbb{E}$), hyperbolic ($G = \mathbb{H}$), or spherical ($G = \mathbb{S}$), we call such a manifold piecewise flat, piecewise hyperbolic, or piecewise spherical, respectively.

When the background geometry is clear from context, we may omit it. Note that part of the definition is that the simplices are nondegenerate; this places inequality restrictions on the possible edge lengths. For instance, in Euclidean background the restrictions can be derived from Cayley-Menger determinants.

We will use $V = V(T)$ to denote the vertices in triangulation $T$ and label them with numbers or letter such as $i \in V$. We will use $E = E(T)$ to denote edges and label them as a set of vertices $\{i, j\} \in E$, although most of this work could allow multiple edges between the same vertices or edges between the same vertex. We will use $E_+ = E_+(T)$ to denote oriented edges and label them with ordered pairs $(i, j) \in E_+$. Triangles will be denoted as a set of vertices, such as $\{i, j, k\}$. In a piecewise constant curvature manifold, the angle at vertex $i$ in a triangle $\{i, j, k\}$ will be denoted $\gamma_i$. The set of real valued functions on $V$ or $E_+$ will be denoted by $V^*$ and $E_+^*$, respectively. We will use $\sigma < \tau$ to mean that $\sigma$ is a subsimplex of $\tau$.

1.1.1. Duality structures. The idea of a duality structure is that, in addition to the metric structure of a piecewise constant curvature manifold, we can put a geometric structure on the Poincaré dual cell complex by introducing geometric centers for pieces of the dual complex. Motivated by the Euclidean background case, we see that these geometric centers do not have to be constrained to the simplex, but its affine span. In the more general constant curvature case, we will need an analogue of the affine span that defines the space of possible simplex centers.
Since a piecewise constant curvature manifold is subdivided by simplices that can be embedded into the space $G$, each simplex $\sigma^k$ has a span defined as follows. First we need to define the underlying space of the span in each geometry.

**Definition 2.** Given a constant curvature geometry $G$, we define $\hat{G}$ as follows:

- If $G = \mathbb{E}^n$ then we take $\hat{G}$ to be the underlying space $\mathbb{R}^n$.
- If $G = \mathbb{H}^n$ then we take $\hat{G}$ to be the entire space of the Klein model, also described as the extended hyperbolic plane in [9]. Note that in this case, $\mathbb{H}^n \subset \hat{G}^n$.
- If $G = S^n$ then we take $\hat{G}$ to be the quotient $\mathbb{RP}^n$ of the sphere.

We note the following easy facts about $\hat{G}$.

- In each case, the isometry group of $G$ acts on $\hat{G}$.
- In each case, there is a notion of orthogonality between two vectors, induced from the Euclidean dot product in the cases of Euclidean space and the sphere, and the Lorentzian bilinear product using the hyperboloid model of hyperbolic space and projecting to the Klein model space.
- In each case, any two points in $\hat{G}$ can be connected by a line.

In what follows, we will assume that any simplex modeled on geometry $G$ can be isometrically embedded into $\hat{G}$, and that embedding is unique up to isometry of $G$. Note that, in the case of spherical geometry, the fact that a simplex embeds is a restriction on how big it can be. We are now ready to define the span.

**Definition 3.** Given a simplex $\sigma^k$ and an isometric embedding $\phi : \sigma^k \rightarrow \hat{G}$, the span of $\sigma^k$ under $\phi$, denoted $S_\phi \sigma^k$, is the set

$$S_\phi \sigma^k = \bigcup_{p,q \in \phi(\sigma^k)} L_{p,q}$$

where $L_{p,q} \subset \hat{G}$ is the line through the points $p$ and $q$.

The span of $\sigma^k$, denoted $S\sigma^k$, is the quotient space obtained from the disjoint union $\bigcup_{\phi} S_\phi \sigma^k$ by identifying each pair of summands $S_\phi \sigma^k$ and $S_\rho \sigma^k$ by an isometry of $\hat{G}$ that agrees with $\rho \circ \phi^{-1}$.

We remark that our definition is analogous to the definition of affine span in polytope theory (c.f., [34]). In both definitions, the span is viewed as a (geodesic) hyperplane tangent to the simplex/polytope-face as it sits in the ambient geometry.

The span has the property that for any points $x \in \sigma \subset S\sigma$ and $y \in S\sigma$, there is a unique line between $x$ and $y$ in $S\sigma \cong \hat{G}$. The span also has the property that if $\sigma < \sigma'$ then there is a natural way in which $S\sigma \subset S\sigma'$.

**Definition 4.** Suppose $(M,T,\ell)$ is a piecewise constant curvature manifold with background geometry $G$.

A duality structure for $(M,T)$ is a choice of one point $C[\sigma] \in S\sigma$ from each simplex $\sigma^k$ of $T$, subject to:

If $\sigma' < \sigma^k$ then for any simplex $\tau = \{C[\sigma^k], C[\sigma^{k+1}], \ldots, C[\sigma^\ell]\}$, we have that $S\tau$ is orthogonal to $S\sigma'$ intersecting only at $C[\sigma']$.

We say a duality structure is proper if it has Euclidean or spherical background or has hyperbolic background and the center of each edge is in $\mathbb{H}$. 
Notice that in the case of spherical background, the centers lie in $\mathbb{RP}^n$ and so correspond to two points in $S^n$. We will often consider the span as $S^n$ with pairs of points instead of $\mathbb{RP}^n$. Proper duality structures are ones such that edge centers are determined by signed distances from the vertices, as determined by the partial edge lengths in the next section.

In general, we will denote the center of edge $\{i,j\}$ by $c_{ij}$ and the center of triangle $\{i,j,k\}$ by $c_{ijk}$. These centers determine edge heights.

**Definition 5.** Given a proper duality structure on a triangle $\{i,j,k\}$, each edge $\{i,j\}$ has a corresponding edge height $h_{ij}$ determined by one of the following:

- If the center $c_{ijk}$ is in the same half plane determined by the span $S\{i,j\}$ as the simplex $\{i,j,k\}$ is, $h_{ij}$ is the distance between $c_{ij}$ and $c_{ijk}$.
- If the center $c_{ijk}$ is not in the same half plane determined by the span $S\{i,j\}$ as the simplex $\{i,j,k\}$ is, $h_{ij}$ is the negative of the distance between $c_{ij}$ and $c_{ijk}$.
- If the center $c_{ijk}$ is in $\mathbb{H}$ but not in $\mathbb{H}$, then the height is the distance from $c_{ij}$ to $c_{ijk}$ (see Section 3) with the same sign convention.

1.1.2. **Discrete metric structure.** The definition of duality structure requires choosing centers. For a more explicit parametrization, we will try to adjust the metric structure $\ell$ in some way to ensure a duality structure. This is the role of metrics and pre-metrics.

The notion of a pre-metric is to reassign parts of the length function to the vertices. This is motivated partly by the definition of Riemannian metrics as tensor valued functions of the points of a manifold.

**Definition 6.** Let $(M,T)$ be a triangulated manifold. A pre-metric is an element $d \in E_{+}(T)^*$ such that $(M,T,\ell)$ is a piecewise constant curvature manifold with background geometry $\mathcal{G}$ for the assignment $\ell_{ij} = d_{ij} + d_{ji}$ for every edge $\{i,j\}$.

The $d_{ij}$ are sometimes called partial edge lengths, since one considers the edge $\{i,j\}$ divided into two partial edges of length $d_{ij}$ and $d_{ji}$. If the partial edge lengths are nonnegative, there is a point on the edge that is distance $d_{ij}$ from vertex $i$ and distance $d_{ji}$ from vertex $j$, and this point is called the edge center. Note that if one of the partial edge lengths is negative, there is an interpretation in terms of signed distance, and there is still a center, this time on the span of the edge.

We would like to restrict pre-metrics to those that generate geometries on the Poincaré dual structure such that dual and primal cells intersect orthogonally. If one considers the point $c_{ij}$ on the span of an edge $\{i,j\}$ that is distance $d_{ij}$ from vertex $i$ and $d_{ji}$ from vertex $j$ (distance can be considered with sign so one partial edge length can be negative), a center is determined. One can consider the plane orthogonal to the span $S\{i,j\}$ through $c_{ij}$, and use the intersections of these planes to construct more centers (e.g., if the planes of the three edges of a triangle intersect at a point then we use that point as the center of the triangle). This construction is explained in detail for Euclidean background in [22]. We wish to characterize which conditions on the pre-metrics guarantee that these centers exist and give a duality structure. We call these metrics, and the actual motivations for the following definitions are characterization theorems given later. The main advantage of metrics over duality structures is that the metrics entirely parametrize the geometry, and so the space of metrics is relatively easy to describe.
Definition 7. A **discrete metric**, or **metric**, on \((M, T)\) with background geometry \(G\) is a pre-metric \(d\) such that for every triangle \(\{i, j, k\}\) in \(T\),

\[
\begin{align*}
    d_{ij}^2 + d_{jk}^2 + d_{ki}^2 &= d_{ji}^2 + d_{kj}^2 + d_{ik}^2 & \text{if } G = E, \\
    \cosh(d_{ij}) \cosh(d_{jk}) \cosh(d_{ki}) &= \cosh(d_{ji}) \cosh(d_{kj}) \cosh(d_{ik}) & \text{if } G = H, \\
    \cos(d_{ij}) \cos(d_{jk}) \cos(d_{ki}) &= \cos(d_{ji}) \cos(d_{kj}) \cos(d_{ik}) & \text{if } G = S. 
\end{align*}
\]

A piecewise constant curvature, metrized manifold \((M, T, d)\) with background geometry \(G\) is a triangulated manifold \((M, T)\) together with a metric \(d\). We denote the space of all metrics with background geometry \(G\) on a given triangulated manifold \((M, T)\) by \(\text{met}_G(M, T)\).

Note that the space of metrics \(\text{met}_G(M, T)\) on a finite triangulation is determined as a subset of \(\mathbb{R}^{|E_+|}\) by a number of equalities of the form above (one for each triangle) and a number of inequalities (to ensure the simplices are nondegenerate).

### 1.1.3. Discrete conformal structure

A discrete conformal structure is a particular way of determining the metric from information assigned to points (vertices). It is partly motivated by this characterization of conformal change of a Riemannian metric, and also by Thurston’s formulation of conformal circle packing structure. A general formulation for Euclidean background is described in [24], and there are a number of formulations of specific cases of analogous structures in hyperbolic and spherical backgrounds (see Section 1.3).

Based on Propositions [1][6] and [10] if we suppose that the pre-metric is determined by weights on the vertex endpoints, there is a restriction that ensures that the resulting pre-metric is actually a discrete metric, i.e., it determines a duality structure. In addition, we want conformal structures to have nice formulas for angle variations. This motivates the following definition.

**Definition 8.** A discrete conformal structure \(C(M, T, U)\) on a triangulated manifold \((M, T)\) with background geometry \(G\) on an open set \(U \subset V(T)^*\) is a smooth map

\[
C(M, T, U) : U \to \text{met}_G(M, T)
\]

such that if \(d = C(M, T, U)[f]\) then for each \((i, j) \in E_+(T)\) and \(k \in V(T)\),

\[
\begin{align*}
    \frac{\partial \ell_{ij}}{\partial f_i} &= d_{ij} & \text{if } G = E, \\
    \frac{\partial \ell_{ij}}{\partial f_i} &= \tanh d_{ij} & \text{if } G = H, \\
    \frac{\partial \ell_{ij}}{\partial f_i} &= \tan d_{ij} & \text{if } G = S, \\
\end{align*}
\]

and

\[
\frac{\partial d_{ij}}{\partial f_k} = 0
\]

if \(k \neq i\) and \(k \neq j\).

A conformal variation of a metric \(d = C(M, T, U)[f]\) is the change of the metric in the conformal class as \(f\) changes, and is determined by derivatives such as \(\partial d_{ij}/\partial f_i\).
We have chosen the parameter $f$ so that the variation formulas above are as simple as possible. However, we will sometimes choose to parametrize the structures differently (see Theorem 3). Also note that with conformal variations, the choice of the set $U$ is not particularly important; we only need the existence of a neighborhood around any point in $U$.

1.2. Main theorems. In this paper, we study the relationships between duality structures, discrete metrics, and conformal variations. The main new contributions are the following: (1) a characterization of duality structures in hyperbolic and spherical backgrounds, generalizing the notion of length structures arising from circles with given radii and inversive distances, (2) calculation of the conformal variation of angles in a triangle for hyperbolic and spherical backgrounds together with determining a functional making the curvature variational, and (3) a classification theorem for discrete conformal variations of Euclidean, hyperbolic, and spherical triangles, including the formulation of the notion of discrete conformal variations from basic principles.

1.2.1. Equivalence of duality and metric structures. The following theorem characterizes duality structures on surfaces in each of the constant curvature backgrounds.

**Theorem 1.** Let $(M, T, \ell)$ be a piecewise constant curvature 2-manifold. There is a one-to-one correspondence between proper duality structures on $(M, T, \ell)$ and discrete metric structures on $(M, T, \ell)$.

This theorem follows from Propositions 1, 6, and 10.

1.2.2. Discrete conformal variations of angle. The following theorem gives the variation of angle formulas. The Euclidean result is in [24], and the hyperbolic and spherical results are new (compare [48]).

**Theorem 2.** For any conformal variation of a metric $d = C(M, T, U)[f]$ with background geometry $G$ of a surface $M^2$, we have for any edge $\{i, j\}$ the following formulas.

- **In Euclidean background,**
  
  \[
  \frac{\partial \gamma_i}{\partial f_j} = \frac{h_{ij}}{\ell_{ij}} \tag{1.7}
  \]
  
  \[
  \frac{\partial \gamma_i}{\partial f_i} = -\frac{h_{ij}}{\ell_{ij}} - \frac{h_{ik}}{\ell_{ik}}. \tag{1.8}
  \]

- **In hyperbolic background,**
  
  \[
  \frac{\partial \gamma_i}{\partial f_j} = \frac{1}{\cosh d_{ji}} \frac{\tanh \beta h_{ij}}{\sinh \ell_{ij}} \tag{1.9}
  \]
  
  \[
  \frac{\partial \gamma_i}{\partial f_i} = -\frac{1}{\cosh d_{ji}} \frac{\tanh \beta h_{ij}}{\tanh \ell_{ij}} - \frac{1}{\cosh d_{ki}} \frac{\tanh \beta h_{ik}}{\tan \ell_{ik}} \tag{1.10}
  \]
  
  where $\beta$ is 1 if $c_{ijk}$ is timelike and -1 if $c_{ijk}$ is spacelike.

- **In spherical background,**
  
  \[
  \frac{\partial \gamma_i}{\partial f_j} = \frac{1}{\cos d_{ji}} \frac{\tan h_{ij}}{\sin \ell_{ij}} \tag{1.11}
  \]
  
  \[
  \frac{\partial \gamma_i}{\partial f_i} = -\frac{1}{\cos d_{ji}} \frac{\tan h_{ij}}{\tan \ell_{ij}} - \frac{1}{\cos d_{ki}} \frac{\tan h_{ik}}{\tan \ell_{ik}}. \tag{1.12}
  \]
This theorem follows from Theorems 5, 7, and 9 together with Propositions 9 and 11.

It turns out that although the variables $f$ for the conformal variations are quite natural, a change of variables gives that the curvatures are the gradient of a functional, where the curvatures are defined as

$$K_i = 2\pi - \sum_{\{i,j,k\}} \gamma_i$$

for each vertex $i$, where the sum is over all triangles containing $i$.

**Theorem 3.** Consider a piecewise constant curvature, metrized 2-manifold $(M, T, d)$, where $d = d(f)$ is determined by a conformal structure. There is a change of variables $u = u(f)$ such that

$$\frac{\partial \gamma_i}{\partial u_j} = \frac{\partial \gamma_j}{\partial u_i}$$

and hence if we fix a $\bar{u}$ there is a functional

$$F = 2\pi \sum_{i \in V} u_i - \sum_{\{i,j,k\}} \int_{\bar{u}} (\gamma_i du_i + \gamma_j du_j + \gamma_k du_k)$$

with the property that

$$\frac{\partial F}{\partial u_i} = K_i.$$

Furthermore, if all $d_{ij} > 0$ and $h_{ij} > 0$ and then this function is strictly convex if $G = \mathbb{H}$ and weakly convex (strictly convex except for scaling) if $G = E$.

This theorem follows from Theorems 6, 8, and 10.

1.2.3. **Classification of discrete conformal structures.** The following theorems classify discrete conformal variations in each of the constant curvature backgrounds. The results are new for all background geometries.

**Theorem 4.** Let $C(M, T, U)$ be a discrete conformal class with background geometry $G$ on a surface $M$. Then there exist $\alpha \in \mathbb{R}^{|V|}$ and $\eta \in \mathbb{R}^{|E|}$ such that the conformal structure can be written as

$$d_{ij} = \frac{\alpha_i e^{2f_i} + \eta_{ij} e^{f_i + f_j}}{\ell_{ij}}$$

with

$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i + f_j}.$$

if $G = E$,

$$\tanh d_{ij} = \frac{\alpha_i e^{2f_i}}{\sinh \ell_{ij}} \sqrt{\frac{1 + \alpha_i e^{2f_i}}{1 + \alpha_j e^{2f_j}} + \frac{\eta_{ij} e^{f_i + f_j}}{\sinh \ell_{ij}}}$$

with

$$\cosh \ell_{ij} = \sqrt{(1 + \alpha_i e^{2f_i}) (1 + \alpha_j e^{2f_j})} + \eta_{ij} e^{f_i + f_j}.$$

if $G = \mathbb{H}$, or

$$\tan d_{ij} = \frac{\alpha_i e^{2f_i}}{\sin \ell_{ij}} \sqrt{\frac{1 - \alpha_j e^{2f_j}}{1 - \alpha_i e^{2f_i}} + \frac{\eta_{ij} e^{f_i + f_j}}{\sin \ell_{ij}}}$$

if $G = \mathbb{H}$.
with
\[
\cos \ell_{ij} = \sqrt{(1 - \alpha_i e^{2f_i}) (1 - \alpha_j e^{2f_j}) - \eta_{ij} e^{f_i + f_j}}.
\]

if \( G = S \).

This theorem is proven in each case in Sections 2.3, 5.2, and 6.

In light of Theorem 4 one can also calculate angle variations from Theorem 2 based on the conformal structures determined by \( \alpha \) and \( \eta \). These conformal structures are sometimes referred to as \( C_{\alpha,\eta} \) (see, e.g., [25]).

1.3. Comparison with previous formulations. In this section we briefly compare our parametrizations with other parametrizations of certain discrete conformal structures. The formulation in this paper unifies the previous work into a single formula for each background geometry and generalizes some of these. Independently, [48] derived a formula for the variation of angle that is essentially the same as ours, though we express it and prove it in a different way. We note that the Euclidean background case was treated in [24], which also describes the relationship of the general case to previous formulations.

The first formulation of the circle packing conformal structure (corresponding, in our notation, to \( \alpha_i = 1 \) and \( \eta_{ij} = 1 \) for all vertices and edges) is in Thurston’s work [44]. Many of the relevant calculations are followed through in [35], and the first variational formulation is due to Colin de Verdière in [12]. In each of these cases, the Euclidean and hyperbolic cases were treated, and the conformal structures were either circles with given intersection angles between 0 and \( \pi/2 \) (corresponding, in our notation, to \( \alpha_i = 1 \) and \( 0 \leq \eta_{ij} \leq 1 \) for all vertices and edges). Additional work was done by Chow-Luo in [10]. The case of circles with fixed inversive distances (corresponding, in our notation, to \( \alpha_i = 1 \) and \( |\eta_{ij}| \geq 1 \) for all vertices and edges) was introduced by Bowers and Stephenson [4] and the variational perspective was pursued by Guo in [26] (this was anticipated by Springborn’s work on volumes of hyperideal simplices in [41]).

The multiplicative conformal structure (corresponding, in our notation, to \( \alpha_i = 0 \) for all vertices) was apparently first suggested in [40], but most of the mathematical ideas arose in work of Luo [33] and Springborn-Schrader-Pinkall [42] in the Euclidean case. Generalizing to the hyperbolic case was not obvious, but work in this direction first appeared in work by Bobenko-Pinkall-Springborn [1]. It is notable that the proper parametrization variable is not clear in this case, and this issue is discussed in Section 5.3. The unified case for Euclidean background is given in [24] and the hyperbolic case was first described in this paper and independently in [48]. For more on some of these discrete conformal structures, see the books [43], [14], and [47].

Explicit calculation of the variation of angle coefficients in the Euclidean circle packing case is due to Z. He [28], and followed by the first author in [24]. The coefficients are closely related to the discrete Laplacians found in [15], [11], [30], [17], [3] [23], [16], [29], [35], [36], and many other places.

There are close connections between these variational viewpoints and hyperbolic volumes, as evidenced by work of Brägger [5], Rivin [39], Garret [20], Leibon [32], Bobenko-Springborn [2], Springborn [41], Springborn-Schrader-Pinkall [42], and Bobenko-Pinkall-Springborn [1], Fillastre-Izmestiev [19], and Zhang et. al. [48].

Some of this work was generalized to discrete conformal structures in three dimensions by Cooper-Rivin in [13] and the first author in [21] and [24]. While the
functionals whose variations lead to curvatures in two dimensions are possibly re-
lated to the log determinant of the Laplacian and surface entropy (see [32]), in three
dimensions the functional is related to Regge’s formulation of the Einstein-Hilbert
(total scalar curvature) functional. See, e.g., [38], [7], [27], [6], [30], [31].

2. Euclidean geometry

2.1. Duality structures on Euclidean triangles. Clearly, the choice of a pre-
metric with Euclidean background determines the geometry of each triangle \{i, j, k\}
and for any isometric embedding, specifies the triangle’s sides \{e_{ij}\} with lengths
\{\ell_{ij}\}. Through each finite edge \(e_{ij}\) of the triangle we have a unique line \(E_{ij}\),
considered in \(\mathbb{E}\).

Suppose we identify \(E_{ij}\) with the real number line such that \(v_i\) is at the origin
and \(v_j\) is on the positive \(x\) axis. Given these coordinates, we specify the edge
centers \(c_{ij} = c_{ji} = C(\{i, j\})\) to be the point \(d_{ij}\) on the line. Note that \(d_{ij}\) denotes
the distance between \(c_{ij}\) and \(v_j\), considered with a sign determined by which side
of \(v_j\) in \(E_{ij}\) contains \(c_{ij}\).

For each edge \{i, j\}, there exists a unique line \(P_{ij}\) that passes through \(c_{ij}\) and is
orthogonal to \(E_{ij}\).

In [22] (Proposition 4), the first author presented a necessary and sufficient
condition on the partial edges to guarantee the three lines \{\(P_{ij}\)\} meet at a single
point:

**Proposition 1.** Suppose \{\(d_{ij}\)\} is a Euclidean pre-metric. Then the perpendiculars
\{\(P_{ij}\)\} meet at a single point if and only if

\[
d_{12}^2 + d_{23}^2 + d_{31}^2 = d_{21}^2 + d_{32}^2 + d_{13}^2.
\]

This motivates the Euclidean case of Definition 7 and proves the Euclidean case
of Theorem 1.

2.2. Conformal variation of angle. The conformal structure is defined in such
a way as to give the following variational formula.

**Theorem 5.** Given a conformal structure, we have

\[
\frac{\partial \gamma_i}{\partial f_j} = \frac{h_{ij}}{\ell_{ij}}
\]

if \(i \neq j\) and

\[
\frac{\partial \gamma_i}{\partial f_i} = -\frac{h_{ij}}{\ell_{ij}} - \frac{h_{ik}}{\ell_{ik}}.
\]

This theorem is proven in [22], generalizing the theorems in special cases given
in [28] and [21]. It follows easily (see, e.g., [24]) that the curvature is variational
with respect to a convex functional.

**Theorem 6.** The partial derivatives of the angles in a triangle are symmetric, i.e.,

\[
\frac{\partial \gamma_i}{\partial f_j} = \frac{\partial \gamma_j}{\partial f_i}
\]

and hence if we fix a \(f\) there is a functional

\[
F = 2\pi \sum_{i \in \mathcal{V}} f_i - \sum_{\{i, j, k\}} \int f \left( \gamma_i df_i + \gamma_j df_j + \gamma_k df_k \right)
\]
with the property that

\[ \frac{\partial F}{\partial f_i} = K_i. \]

Furthermore, if all \( d_{ij} > 0 \) and \( h_{ij} > 0 \) and then this function is weakly convex (strictly convex except for scaling).

2.3. Characterization of discrete conformal structures. In this section we prove the characterization theorem. Recall that the only assumptions are:

- The compatibility condition \( 1.1 \) for the triangle with vertices \( v_i, v_j, \) and \( v_k. \)
- The assumption that \( d_{ij} \) depends only on \( f_i \) and \( f_j. \)

Proof of the Euclidean case of Theorem 2. We first note that

\[ \frac{\partial^2}{\partial f_i \partial f_j} (d_{ij}^2 - d_{ji}^2) = 0 \]

since for any triangle with vertices \( v_i, v_j, v_k \) we have

\[ d_{ij}^2 - d_{ji}^2 = d_{ik}^2 + d_{kj}^2 - d_{jk}^2 - d_{ki}^2. \]

We can compute that

\[ \left( \frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j} \right) d_{ij} = \frac{\partial d_{ij}}{\partial f_i} + \frac{\partial d_{ij}}{\partial f_j} + \frac{\partial d_{ji}}{\partial f_i} + \frac{\partial d_{ji}}{\partial f_j} = d_{ij} \]

since

\[ \frac{\partial d_{ij}}{\partial f_i} = \frac{\partial^2 e_{ij}}{\partial f_i f_j} = \frac{\partial d_{ji}}{\partial f_i}. \]

It follows that

\[ \left( \frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j} \right) (d_{ij}^2 - d_{ji}^2) = 2 (d_{ij}^2 - d_{ji}^2) \]

and so it follows that

\[ \frac{\partial^2}{\partial^2 f_i} (d_{ij}^2 - d_{ji}^2) = 2 \frac{\partial}{\partial f_i} (d_{ij}^2 - d_{ji}^2) \]

and

\[ \frac{\partial^2}{\partial^2 f_j} (d_{ij}^2 - d_{ji}^2) = 2 \frac{\partial}{\partial f_j} (d_{ij}^2 - d_{ji}^2). \]

We can solve these equations, getting

\[ \frac{\partial}{\partial f_i} (d_{ij}^2 - d_{ji}^2) = 2a_{ij} e^{2f_i}, \]

\[ \frac{\partial}{\partial f_j} (d_{ij}^2 - d_{ji}^2) = -2a_{ji} e^{2f_j}, \]

for constants \( a_{ij} \) and \( a_{ji}. \) Hence

\[ d_{ij}^2 - d_{ji}^2 = a_{ij} e^{2f_i} - a_{ji} e^{2f_j}. \]
We can now use (2.2) to find that for a constant $\eta_{ij}$

$$\ell^2_{ij} = a_{ij} e^{2f_i} + a_{ji} e^{2f_j} + 2\eta_{ij} e^{f_i+f_j}.$$  

(2.3)

From this, we compute that

$$d_{ij} = \frac{\partial \ell_{ij}}{\partial f_i} = a_{ij} e^{2f_i} + \eta_{ij} e^{f_i} + \ell_{ij}.$$  

We note that in a triangle, since

$$d_{ij}^2 - d_{ji}^2 + d_{ik}^2 - d_{jk}^2 = d_{kj}^2 - d_{jk}^2$$  

and the right side is independent of $f_i$, differentiating with respect to $f_i$ gives

$$2 (a_{ij} - a_{ik}) e^{2f_i} = 0$$

and hence $a_{ij} = a_{ik}$ and $a$ is independent of the edge, only depending on the vertex, hence we rename $\alpha_i = a_{ij} = a_{ik}$.

To see that the $\alpha_i$ and $\eta_{ij}$ must be consistent across triangles, consider Equation 2.3 on both triangles and differentiate with respect to $f_i$ and $f_j$ to see that the $\eta_{ij}$ agree and then $f_i$ to see that the $\alpha_i$ agree.  

□

3. Basic calculations in hyperbolic geometry

Before we move to the hyperbolic versions of the previous work, we will review some techniques for computing in hyperbolic geometry. This section summarizes the elementary facts about the hyperbolic plane $\mathbb{H}$ that we will use in later calculations. All of the propositions in this section are discussed in Chapter 3 of [37]. See also [8]. For the reader’s convenience, we have included some, but not all, proofs.

We use the hyperboloid model of $\mathbb{H}$ for the majority of our calculations. In this model, the vector space $\mathbb{R}^3$ is equipped with a Lorentzian inner product $\ast$ given by $u \ast v := u^T J v$ where $J$ is the diagonal matrix with entries 1,1,-1. We define a “hyperbolic magnitude” $\|u\| := \sqrt{u \ast u};$ the only possible hyperbolic lengths are nonnegative scalar multiples of 1 and $i$. $\mathbb{H}$ corresponds to those vectors $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ satisfying $u \ast u = -1$ and $u_3 > 0$.

**Definition 9.** A vector $u \in \mathbb{R}^3$ is termed spacelike if $u \ast u > 0$, lightlike (or “on the light cone”) if $u \ast u = 0$, and timelike if $u \ast u < 0$.

The vector space structure on $(\mathbb{R}^3, \ast)$ gives us several ways to describe a geodesic in $\mathbb{H}$:

- As a nonempty intersection $\mathbb{H} \cap \text{Span}(p,q)$ for linearly independent $p, q \in \mathbb{R}^3$.
- As a nonempty intersection $\mathbb{H} \cap p^\perp$, where $p$ is a spacelike vector and $p^\perp := \{v \in \mathbb{R}^3 : p \ast v = 0\}$.
- As a path, parametrized by arclength, given by $\gamma(t) = \cosh(t) p + \sinh(t) v$. In this form, $p \in \mathbb{H}, v \in p^\perp$ with $v \ast v = 1$. Note $p$ and $v$ encode the position and direction of $\gamma$ at $t = 0$.

The second characterization becomes particularly useful when combined with the Lorentzian cross product, which is given by $p \otimes q := J(x \times y)$. Clearly, the Lorentzian cross product has two useful properties:

- $p \otimes q = 0$ if and only if $p$ and $q$ are linearly dependent.
- $p \otimes q$ is $\ast$-orthogonal to both $p$ and $q$.  

A consequence of the second observation is that given distinct points \( p, q \in \mathbb{H} \), one simple way to describe the geodesic through \( p \) and \( q \) is \((p \otimes q)^\perp\).

In the sequel, we will use \( d_H(u, v) \) to denote the hyperbolic distance between two timelike points, and \( d_H(u, v^\perp) \) to denote the hyperbolic distance between a timelike point and a geodesic in hyperbolic space determined as the orthogonal complement of a spacelike point. When \( u, v \in \mathbb{R}^3 \) satisfy \(|u \ast u| = |v \ast v| = 1\), we have the following interpretations of the quantity \( u \ast v \):

- If \( u \) and \( v \) are both timelike, then \( u \ast v = -\cosh(d_H(u, v)) \).
- If \( u \) is timelike and \( v \) is spacelike, then \( u \ast v = \pm \sinh(d_H(u, v^\perp)) \) and the sign depends upon which of the halfspaces bounded by \( v^\perp \) contains \( u \).
- If \( u \) and \( v \) are both spacelike and \( u^\perp \) and \( v^\perp \) intersect in angle \( \alpha \) within \( \mathbb{H} \), \( u \ast v = \cos(\alpha) \).

Notice that the last item implies that for spacelike \( u \) and \( v \), \( u^\perp \) and \( v^\perp \) meet at a right angle if and only if \( u \ast v = 0 \).

The following identities simplify calculations that involve Lorentzian cross products. Suppose \( x, y, z, w \in \mathbb{R}^3 \):

\[
\begin{align*}
(3.1) & \quad x \otimes y = -y \otimes x, \\
(3.2) & \quad (x \otimes y) \ast z = \det(x, y, z), \\
(3.3) & \quad x \otimes (y \otimes z) = (x \ast y)z - (z \ast x)y, \\
(3.4) & \quad (x \otimes y) \ast (z \otimes w) = \begin{vmatrix} x \ast w & x \ast z \\ y \ast w & y \ast z \end{vmatrix}.
\end{align*}
\]

We have already seen that several different kinds of data can be used to specify a geodesic on \( \mathbb{H} \). This allows us to extend our understanding of where geodesics intersect.

**Definition 10.** Given a geodesics \( \gamma \) on \( \mathbb{H} \), we will identify \( \gamma \) with the unique 2-dimensional subspace \( P_{\gamma} \) of \( \mathbb{R}^3 \) such that \( P_{\gamma} \cap \mathbb{H} \) is the image of \( \gamma \).

Given geodesics \( \gamma, \omega \) on \( \mathbb{H} \), we define their intersection to be their intersection as subspaces of \( \mathbb{R}^3 \), namely \( P_{\gamma} \cap P_{\omega} \).

Readers familiar with the Klein model of \( \mathbb{H} \) (the central projection of \( \mathbb{H} \) onto the plane \( z = 1 \)) should note that this definition is simply a linear-algebraic way of formulating the notion of intersecting 1-hyperplanes in the Klein model.

Introducing a broader notion of intersection allows us to generalize familiar equations (like the law of cosines) and express them in terms of linear algebra. Understanding how to interpret the Lorentzian inner product is key to relating these different formulas. Often, the linear algebraic interpretation allows us to efficiently treat several seemingly different cases at once.

Recall the definition of a triangle (see Section 3.5 in [37]), which allows some of the vertices to be timelike, lightlike, or spacelike. We will concentrate on triangles with at least two timelike vertices.

**Proposition 2.** Suppose \( x \in \mathbb{H} \) and \( y, z \in \mathbb{R}^3 \) are either timelike or spacelike. Then

\[
(z \otimes x) \ast (x \otimes y) = -\|z \otimes x\| \cdot \|x \otimes y\| \cos(\alpha),
\]

where \( \alpha \) is the angle at \( x \) in the (clockwise oriented) triangle \( \{x, y, z\} \).
Proposition 3 (The Generalized Law of Cosines). Suppose \( x, y, z \in \mathbb{R}^3 \), with \( \|x\| = \|z\| = 1 \) and \( \|y\| = 1 \) or \( i \), are the vertices of a triangle in \( \mathbb{H} \), with angle \( \alpha \) at \( x \). Then
\[
z \ast y + (z \ast x)(x \ast y) = \|z \otimes x\|\|x \otimes y\| \cos(\alpha).
\]

Proof. Assume, without loss of generality, that \( x, y, z \) label the vertices of the triangle in clockwise order. Equation [3.4] implies
\[
-(z \otimes x) \ast (x \otimes y) = (z \ast y) + (z \ast x)(x \ast y).
\]
Now apply Proposition 2 to obtain the desired equality. \( \square \)

By setting \( \alpha = \pi/2 \), we obtain a generalized version of the Pythagorean theorem:

Corollary 1 (The Generalized Pythagorean Theorem). Suppose \( x, y, z \) are the vertices of a right triangle, with the right angle at \( x \). Then:
\[
-(z \ast y) = (z \ast x)(x \ast y).
\]

We will require formulas for performing trigonometry in a hyperbolic right triangle where one of the vertices (not the one adjacent to the right angle) may be spacelike or timelike. Suppose we have a right triangle labeled like the one in Figure 1.

![Figure 1. Two (Generalized) Right Triangles in the Klein Model](image)

Proposition 4. Given a triangle labeled as in Figure 7 we have:
\[
\cos(\alpha) = \frac{\tanh(B)}{\tanh(C)}, \quad \sin(\alpha) = \frac{\sinh(A)}{\sinh(C)}, \quad \tan(\alpha) = \frac{\tanh(A)}{\sinh(B)}
\]
if \( b \) is timelike and
\[
\cos(\alpha) = \tanh(B) \tanh(C), \quad \sin(\alpha) = \frac{\cosh(A)}{\cosh(C)}, \quad \tan(\alpha) = \frac{1}{\sinh(B) \tanh(A)}
\]
if \( b \) is spacelike.

Deriving these formulas is an easy application of the generalized Pythagorean theorem and the generalized law of cosines.

The next corollary generalizes the familiar formula for the cosine of an angle in a hyperbolic right triangle.

Corollary 2. Suppose \( x, y, z \) are the vertices of a right triangle (with the right angle at \( z \)) satisfying the assumptions of Proposition 3. Then
\[
\cos(\alpha) = -\frac{x \ast y}{\|x \otimes y\| \tanh(d_{\mathbb{H}}(z, x))}.
\]
Proof. The right angle at \( z \) means that:

\[
0 = (y \otimes z) \ast (z \otimes x) = -(y \ast x) - (y \ast z)(z \ast x)
\]

and so

\[
z \ast y = \frac{y \ast x}{z \ast x}.
\]

Substituting this into the equation we obtain from the Law of Cosines, we learn:

\[
\|z \otimes x\|\|x \otimes y\| \cos(\alpha) = z \ast y + (z \ast x)(x \ast y)
\]

\[
= \frac{y \ast x}{z \ast x} + (z \ast x)(x \ast y)
\]

\[
= (x \ast y) \frac{(z \ast x)^2 - 1}{z \ast x}
\]

\[
= (x \ast y) \frac{\sinh^2(d_{\mathbb{H}}(z, x))}{\cosh(d_{\mathbb{H}}(z, x))}.
\]

Using Equation \([3.4]\) it is easy to check \( \|z \otimes x\| = \sinh(d_{\mathbb{H}}(z, x)) \). Hence:

\[
\cos(\alpha) = \frac{x \ast y}{\|x \otimes y\|} \tanh(d_{\mathbb{H}}(z, x)).
\]

\[\Box\]

Because the Lorentzian inner product is nondegenerate, we have a well defined notion of \( \ast \)-orthogonality and may apply the Gram-Schmidt procedure to obtain a basis of mutually \( \ast \)-orthogonal vectors. This procedure can be used to parametrize a geodesic given in the form \( \mathbb{H} \cap \text{Span}(p, q) \) by arclength.

**Proposition 5.** Suppose \( p \in \mathbb{H} \), and \( q \in \mathbb{R}^3 \). Then the geodesic \( \mathbb{H} \cap \text{Span}(p, q) \) may be parametrized by arclength as:

\[
\gamma(t) = \cosh(t)p + \sinh(t)\frac{q + (p \ast q)p}{\sqrt{q \ast q + (p \ast q)^2}}.
\]

**Proof.** The geodesic in question can be parametrized by arclength as \( \gamma(t) = \cosh(t)p + \sinh(t)v \) for some spacelike \( v \) with \( v \ast v = 1 \); we simply need to use the Gram-Schmidt procedure to guarantee that \( \text{Span}(p, v) = \text{Span}(p, q) \) and \( v \in p^\perp \).

So consider the vector \( q + (p \ast q)p \). Notice \( -(p \ast q)p \) is the \( \ast \)-projection of \( q \) onto the subspace spanned by \( p \), and

\[
p \ast (q + (p \ast q)p) = p \ast q - p \ast q = 0.
\]

To find \( v \), we only need to rescale this projection. Since

\[
(q + (p \ast q)p) \ast (q + (p \ast q)p) = q \ast q + 2(p \ast q)^2 + (p \ast q)^2(p \ast p)
\]

\[
= q \ast q + (p \ast q)^2
\]

the appropriate \( v \) is

\[
v = \frac{q + (p \ast q)p}{\sqrt{q \ast q + (p \ast q)^2}}.
\]

\[\Box\]
4. Duality structures on hyperbolic triangles

We interpret a piecewise hyperbolic pre-metric as subdividing each edge \( \{i, j\} \) of length \( \ell_{ij} \) into two portions of length \( d_{ij} \) and \( d_{ji} \), that are assigned to the vertices \( i \) and \( j \) respectively.

**Definition 11.** Given a pre-metric \( d \) and an isometric embedding of a simplex \( \{i, j\} \) into \( \mathbb{H} \):

- The vertices \( p_i, p_j \in \mathbb{H} \) of \( \{i, j\} \) are the images of \( i \) and \( j \) under the embedding.
- The edge center \( c_{ij} \) induced by \( d \) is the unique point along the line \( E_{ij} \) through \( p_i \) and \( p_j \) such that \( c_{ij} \) is (signed) distance \( d_{ij} \) from \( p_i \) and \( d_{ji} \) from \( p_j \).
- The edge perpendicular \( P_{ij} \) is the line through \( c_{ij} \) that is orthogonal to \( E_{ij} \).

Unlike in the Euclidean setting, it is possible that the geodesics \( P_{ij} \) and \( P_{jk} \) do not intersect within \( \mathbb{H} \). However, these two 1-hyperplanes can be understood as intersecting in the more general sense of Definition 10, namely the two-dimensional subspaces of \((\mathbb{R}^3, \ast)\) associated to \( P_{ij} \) and \( P_{jk} \) intersect in a one-dimensional subspace. One can then ask for necessary and sufficient conditions on the pre-metric that guarantee that for each simplex \( \{i, j, k\} \)

\[
P_{ij} \cap P_{jk} = P_{jk} \cap P_{ki} = P_{ki} \cap P_{ij}
\]

(4.1)

or, colloquially, the three perpendiculars of \( \{i, j, k\} \) intersect in a single point (this point is in the span of \( \{i, j, k\} \)). This condition can also be interpreted in the Klein model of hyperbolic space as the condition that the three lines representing the geodesics intersect at the same point in the plane of the Klein model.

**Proposition 6.** Suppose \( d \) is a piecewise hyperbolic pre-metric. Equation (4.1) holds if and only if the following compatibility equation

\[
(p_i * c_{ij})(p_j * c_{jk})(p_k * c_{ki}) = (p_i * c_{ki})(p_j * c_{ij})(p_k * c_{jk})
\]

(4.2)

is satisfied for every simplex \( \{i, j, k\} \).

Since the vectors \( p_i \) and \( c_{ij} \) are timelike of length -1, Equation (4.2) has the following equivalent formulation:

\[
cosh(d_{ij}) \cosh(d_{jk}) \cosh(d_{ki}) = \cosh(d_{ji}) \cosh(d_{kj}) \cosh(d_{ik})
\]

**Proof.** To simplify our notation, we shall consider a single 2-simplex \( \{1, 2, 3\} \). The vertices of the embedded 2-simplex are linearly independent vectors \( p_1, p_2, p_3 \in \mathbb{H} \).

Consider that if \( c \) is a point on the perpendicular \( P_{ij} \), then \( P_{ij} = (c \otimes c_{ij})^\perp \). Likewise the span of edge \( e_{ij} \) is given by \( (p_i \otimes p_j)^\perp \). Since \( c_{ij} \) belongs to both \( P_{ij} \) and \( e_{ij} \), the fact that \( P_{ij} \) and \( e_{ij} \) are perpendicular is equivalent to the equation:

\[
(c \otimes c_{ij}) \ast (p_i \otimes p_j) = 0.
\]

Identities 3.1 3.3 imply this is equivalent to the equation:

\[
c \ast ((c_{ij} \ast p_i)p_j - (c_{ij} \ast p_j)p_i) = 0.
\]
Hence, Equation 4.1 holds for simplex \{1, 2, 3\} if and only if there is a nontrivial solution \(c\) to the system:

\[
\begin{align*}
    c * ((c_{12} * p_1)p_2 - (c_{12} * p_2)p_1) &= 0 \\
    c * ((c_{23} * p_2)p_1 - (c_{23} * p_3)p_2) &= 0 \\
    c * ((c_{31} * p_3)p_1 - (c_{31} * p_1)p_3) &= 0 
\end{align*}
\]

This system can be reformulated as a matrix equation

\[
\begin{bmatrix}
    ((c_{12} * p_1)p_2 - (c_{12} * p_2)p_1)^T \\
    ((c_{23} * p_2)p_3 - (c_{23} * p_3)p_2)^T \\
    ((c_{31} * p_3)p_1 - (c_{31} * p_1)p_3)^T 
\end{bmatrix} \cdot J \cdot c = 0
\]

that has a nontrivial solution if and only if the determinant of the first matrix is zero.

Expanding that determinant and canceling the (nonzero) factors of \(\det(p_1, p_2, p_3)\) that arise yields Equation 4.2. The last statement follows easily.

This proposition motivates the hyperbolic case of Definition 7.

**Remark 1.** One can gain insight into how the Euclidean and hyperbolic compatibility conditions are related by comparing Equation 1.1 and Equation 1.2 for small \(d_{ij}\) in the same way one compares the Euclidean Pythagorean Theorem with the hyperbolic version, \(\cosh(c) = \cosh(a)\cosh(b)\).

5. **Conformal variations of hyperbolic triangles**

Various formulations of conformal variations of hyperbolic triangulations of surfaces have been studied in [44, 33, 12, 10, 41, 26, 1, 48]. We present a unified approach from the perspective of the metric triangulations as defined above.

5.1. **Motivation and variation formula.** Suppose we wanted to generate a metric from weights assigned to vertices, so that \(d_{ij} = d_{ij}(f_i, f_j)\) for some function \(f\) on the vertices. If this our starting point for conformal structure, in order to compute conformal variations, we will consider what happens to the metric on a triangle \{1, 2, 3\} when the conformal parameter \(f_3\) changes but the other two do not, i.e., \(\delta f_1 = \delta f_2 = 0\). We will call this a \(f_3\)-conformal variation in this section.

The next two propositions analyze the configuration shown in Figure 2. We assume throughout that \(v_1, v_2, v_3\) are linearly independent in \(\mathbb{R}^3\), with \(v_i \ast v_i = -1\).

**Proposition 7.** Under an \(f_3\)-conformal variation:

\[
\begin{align*}
    v_1 \ast \delta v_3 &= -\sinh \ell_{13} \frac{\partial \ell_{13}}{\partial f_3} \delta f_3 \\
    v_2 \ast \delta v_3 &= -\sinh \ell_{23} \frac{\partial \ell_{23}}{\partial f_3} \delta f_3 \\
    v_3 \ast \delta v_3 &= 0 
\end{align*}
\]

**Proof.** Bilinearity of the Lorentzian inner product implies:

\[
\delta(v_1 \ast v_3) = v_1 \ast \delta v_3.
\]

However, since \(v_1 \ast v_3 = -\cosh(\ell_{13})\), we can also write:

\[
\delta(v_1 \ast v_3) = -\sinh \ell_{13} \frac{\partial \ell_{13}}{\partial f_3} \delta f_3.
\]
Hence
\[ v_1 \ast \delta v_3 = -\sinh \ell_{13} \frac{\partial \ell_{13}}{\partial f_3} \delta f_3. \]
We get the formula for \( v_2 \ast \delta v_3 \) similarly. Finally, since \( v_3 \ast v_3 = -1 \):
\[ 0 = \delta (v_3 \ast v_3) = 2 v_3 \ast \delta v_3. \]

The following proposition makes precise what we mean by the colloquial statement that conformal variations give good angle variations.

**Proposition 8.** Let \( c_{123} \) denote the center of the triangle specified by the vertices \( v_i \) and the (compatible) partial edge lengths \( d_{ij} \). Suppose further that the edge centers on edges \( \{1,3\} \) and \( \{2,3\} \) are timelike. Then under a \( f_3 \)-conformal variation, the points \( v'_3, v_3 \), and \( c_{123} \) lie on a line in \( \mathbb{H} \) if and only if
\[ \frac{\partial \ell_{ij}}{\partial f_i} = (\tanh d_{ij}) F(f_i), \]
for some function \( F(f_i) \).

**Proof.** Without loss of generality, assume \( c_{123} \ast c_{123} = \pm 1 \). Consider the geodesic through \( v_3 \) and \( c_{123} \). As a set, this geodesic can be described by \( \mathbb{H} \cap \text{Span}(v_3, c_{123}) \), a characterization we will use to parametrize the geodesic by arclength as
\[ u = h_{13} \cosh t + \sinh t \delta v_3 \]
for some \( u \in v_3^\perp \cong T_{v_3} \mathbb{H} \). Specifically, Proposition 5 implies:
\[ u = \frac{c_{123} + (v_3 \ast c_{123}) v_3}{\sqrt{c_{123} \ast c_{123} + (v_3 \ast c_{123})^2}}. \]

The points \( v'_3, v_3 \), and \( c_{123} \) lie on a geodesic if and only if \( u \) and \( \delta v_3 \) are collinear. The three numbers \( \{\delta v_3 \ast v_i\}_{i=1}^3 \) completely characterize the vector \( \delta v_3 \in v_3^\perp \cong T_{v_3} \mathbb{H} \). Hence, \( u \) and \( \delta v_3 \) are collinear if and only if there exists \( \lambda \in \mathbb{R} \) such that \( u \ast v_i = \lambda \delta v_3 \ast v_i \) for \( i = 1, 2, 3 \). We already know \( v_3 \ast u = v_3 \ast \delta v_3 = 0 \), so only \( v_1 \ast u \) and \( v_2 \ast u \) require consideration.
Consider our equation for $u$. The scalar in the denominator will appear in both $v_1 \ast u$ and $v_2 \ast u$. To simplify our notation, we will write $\lambda_3 := (c_{123} \ast c_{123} + (v_3 \ast c_{123})^2)^{-1/2}$. Now

$$v_1 \ast u = \lambda_3 (c_{123} \ast v_1 + (v_3 \ast c_{123})(v_3 \ast v_1))$$

and we can apply the Generalized Law of Cosines (Proposition 3) to the triangle $\{v_1, v_3, c_{123}\}$ in order to rewrite this equation as

$$v_1 \ast u = \lambda_3 \|v_1 \otimes v_3\| \|v_3 \otimes c_{123}\| \cos(\alpha) = \lambda_3 \sinh(\ell_{13}) \|v_3 \otimes c_{123}\| \cos(\alpha).$$

Next consider the right triangle with vertices $\{c_{123}, c_{13}, v_3\}$. By Corollary 2, we have

$$\|v_3 \otimes c_{123}\| \cos(\alpha) = -(v_3 \ast c_{123}) \sinh(\ell_{13}) \tanh(d_{31}).$$

A final substitution into our equation for $v_1 \ast u$ implies

$$v_1 \ast u = -(\lambda_3 \cdot v_3 \ast c_{123}) \sinh(\ell_{13}) \tanh(d_{31}).$$

A similar argument for $v_2$ yields

$$v_2 \ast u = -(\lambda_3 \cdot v_3 \ast c_{123}) \sinh(\ell_{23}) \tanh(d_{32}).$$

From Proposition 7, we know that for $k = 1, 2$

$$v_k \ast \delta v_3 = -\sinh(\ell_{k3}) \frac{\partial f_{k3}}{\partial f_3} \delta f_3.$$

Comparing these two equations, we see that there exists $\lambda \in \mathbb{R}$ so that $v_k \ast u = \lambda v_k \ast \delta v_3$ if and only if there exists a smooth function $F(f_3)$ for which $\frac{\partial f_{k3}}{\partial f_3} = \tanh(d_{3k}) F(f_3)$. □

Proposition 8 motivates the hyperbolic case of Definition 8, where we have chosen to simplify to parameters that make $F$ equal to the constant function 1.

We will now study how the angles change under a conformal variation. First we see the following.

**Theorem 7.** Given a conformal structure, then for any simplex $\{i, j, k\}$

\[
\begin{align*}
\frac{\partial \gamma_i}{\partial f_j} &= \frac{1}{\cosh d_{ij}} \frac{\tanh^\beta h_{ij}}{\sinh \ell_{ij}}, \\
\frac{\partial \gamma_i}{\partial f_i} &= -\frac{\partial A_{ijk}}{\partial f_i} - \frac{\partial \gamma_j}{\partial f_i} - \frac{\partial \gamma_k}{\partial f_i}
\end{align*}
\]

where $\beta$ is 1 if $c_{ijk}$ is timelike and -1 if $c_{ijk}$ is spacelike.

**Proof.** For simplicity, we shall consider the problem for a single simplex $\{1, 2, 3\}$ labeled as in Figure 2 with $i = 1, j = 3$. We will address the case where $c_{123}$ is timelike: the case where $c_{123}$ is spacelike is similar. Once (5.1) is proven, (5.2) follows immediately because of the area formula for a hyperbolic triangle:

$$A_{123} = \pi - \gamma_1 - \gamma_2 - \gamma_3.$$

Because the variation is conformal, $\delta \ell_{13} = \tanh(d_{31}) \delta f_3$. Using the formula for a segment of a circle in the hyperbolic plane, we have $\omega = \delta \gamma_1 \sinh(\ell_{13})$. 


By Proposition 8, under a conformal variation \( v_3, v'_3 \) and \( c_{123} \) are collinear. Consequently, the angle adjacent to \( v_3 \) in the triangle with side lengths \( \omega, \delta l_{13} \) and \( \delta v_3 \) is \( \pi/2 - \alpha \). This, together with the formulas in Proposition 4, allows us to write:

\[
\tan(\alpha) = \frac{\tanh h_{13}}{\sinh d_{31}},
\]

\[
\cot(\alpha) = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\tanh \delta l_{13}}{\sinh(\delta \gamma_1 \sinh l_{13})}
\]

Using the Taylor series for sinh and tanh, we have:

\[
\frac{\tanh h_{13}}{\sinh d_{31}} = \frac{\delta \gamma_1 \sinh l_{13} + O(\delta \gamma_1^2)}{\frac{1}{\sinh d_{31}} + O(\delta l_{13}^2)}
\]

and hence,

\[
\frac{\delta \gamma_1}{\delta f_3} = \frac{1}{\cosh d_{31}} \frac{\tanh h_{13}}{\sinh l_{13}} \left(\frac{1 + O(\delta f_3^2)}{1 + O(\delta \gamma_1^2)}\right).
\]

We can also compute the variation of area explicitly.

**Proposition 9.** Given a conformal structure, then for any simplex \( \{i,j,k\} \) with area \( A_{ijk} \),

\[
\frac{\partial A_{ijk}}{\partial f_k} = \frac{\partial \gamma_i}{\partial f_k} (\cosh \ell_{ik} - 1) + 2 \frac{\partial \gamma_j}{\partial f_k} (\cosh \ell_{jk} - 1).
\]

In particular, if the derivatives \( \partial \gamma_i/\partial f_k \) are positive whenever \( k \neq i \), then the derivative of the area is positive.

**Proof.** This follows from the formula for the area of a sector of circle as a function of the radius for a hyperbolic surface, since in Figure 2 we find that the area of each of the small triangles is higher order, leaving only the areas of the skinny triangles in the picture. 

5.2. **Characterization of discrete conformal structures.** The proof of the hyperbolic case of Theorem 4 is similar to the proof of the Euclidean case, though the calculation is a bit harder in hyperbolic background.

**Proof of the hyperbolic case of Theorem 4.** We first note the following:

\[
\frac{\partial}{\partial f_i} \cosh \ell_{ij} = \cosh \ell_{ij} - \frac{\cosh d_{ij}}{\cosh d_{ij}^2},
\]

\[
\frac{\partial}{\partial f_j} \cosh \ell_{ij} = \cosh \ell_{ij} - \frac{\cosh d_{ij}}{\cosh d_{ij}^2}.
\]

A straightforward calculation gives that

\[
\left(\frac{\cosh^2 d_{ij}}{\cosh d_{ij}^2} \frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j}\right) \log \frac{\cosh^2 d_{ij}}{\cosh^2 d_{ij}^2} = 2 \left(\frac{\cosh^2 d_{ij}}{\cosh^2 d_{ij}^2} - 1\right)
\]

or if \( H = \log \cosh^2 d_{ij} \), then

\[
\left(e^H \frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j}\right) H = 2 \left(e^H - 1\right).
\]
Since
\[ \frac{\partial^2 H}{\partial f_i \partial f_j} = 0 \]
it follows that
\[ e^H \frac{\partial^2 H}{\partial f_i^2} + e^H \left( \frac{\partial H}{\partial f_i} \right)^2 = 2 e^H \frac{\partial H}{\partial f_i} \]
and
\[ e^{-H} \frac{\partial^2 H}{\partial f_j^2} - e^{-H} \left( \frac{\partial H}{\partial f_j} \right)^2 = 2 e^{-H} \frac{\partial H}{\partial f_j}. \]

One can then easily solve this ODE to obtain that:
\[ \frac{\partial H}{\partial f_i} = 2 a_{ij} e^{2f_i} \]
for some constant \( a_{ij} \) and
\[ \frac{\partial H}{\partial f_j} = -2 a_{ji} e^{2f_j} \]
for some constant \( a_{ji} \). It follows that
\[ \cosh^2 d_{ij} = D \left( 1 + a_{ij} e^{2f_i} \right) \]
\[ \cosh^2 d_{ji} = D \left( 1 + a_{ji} e^{2f_j} \right) \]
for some constant \( D \). We can now use Equation 5.4 to see that
\[ \cosh \ell_{ij} - \frac{\partial}{\partial f_i} \cosh \ell_{ij} = \frac{1}{D} \left( \frac{1 + a_{ij} e^{2f_i}}{1 + a_{ji} e^{2f_j}} \right)^{-1/2} \]
\[ \cosh \ell_{ij} - \frac{\partial}{\partial f_j} \cosh \ell_{ij} = D \left( \frac{1 + a_{ij} e^{2f_i}}{1 + a_{ji} e^{2f_j}} \right)^{1/2} \]
and so we find that \( D = 1 \) and
\[ \cosh \ell_{ij} = \sqrt{(1 + a_{ij} e^{2f_i}) (1 + a_{ji} e^{2f_j}) + \eta_{ij} e^{f_i + f_j}} \]
for some constant \( \eta_{ij} \).

The compatibility condition (1.2) implies that \( \log \frac{\cosh d_{ij}}{\cosh d_{ji}} + \log \frac{\cosh d_{ik}}{\cosh d_{ki}} \) is independent of \( f_i \) and so we can use Equation 5.6 to see that \( a_{ij} = a_{ik} \) and so we can define \( \alpha_i = a_{ij} = a_{ik} \).

It follows that
\[ \tanh d_{ij} = \frac{1}{\sinh \ell_{ij}} \frac{\partial}{\partial f_i} \cosh \ell_{ij} \]
\[ = \frac{\alpha_i e^{2f_i}}{\sinh \ell_{ij}} \sqrt{\frac{1 + \alpha_j e^{2f_j}}{1 + \alpha_i e^{2f_i}} + \frac{\eta_{ij} e^{f_i + f_j}}{\sinh \ell_{ij}}} \]

Finally, we can use Equation 5.6 again to write \( 2 \log \frac{\cosh d_{ij}}{\cosh d_{ji}} \) in terms of the coefficients determined in the two triangles adjacent to edge \( \{i, j\} \) and differentiate to see that the \( \alpha_i \) derived in each triangle must be equal. It then follows from Equation 5.7 that the \( \eta_{ij} \) derived in each triangle must be equal as well.
5.3. Variational formulation for curvature. While the formula (1.9) is not symmetric in \( i \) and \( j \), we can reparametrize to get a symmetric variation formula. Notice that Equation 5.6 (recall that we proved \( D = 1 \)) implies that
\[
\sqrt{1 + \alpha_i e^{2f_i}} \cosh d_{ij} = \sqrt{1 + \alpha_j e^{2f_j}} \cosh d_{ji}.
\]
If we take new coordinates \( u_i = u_i(f_i) \) such that
\[
\frac{\partial f_i}{\partial u_i} = \sqrt{1 + \alpha_i e^{2f_i}},
\]
then we have the symmetry
\[
\frac{\partial \gamma_i}{\partial u_j} = \frac{\partial \gamma_j}{\partial u_i}.
\]

**Remark 2.** The function \( u_i(f_i) \) can be computed explicitly. It is not hard to see that if \( \alpha_i = 0 \) then \( u_i = f_i \) and if not then
\[
u_i = \frac{1}{2} \log \frac{\sqrt{1 + \alpha_i e^{2f_i}} - 1}{\sqrt{1 + \alpha_i e^{2f_i}} + 1}.
\]
If \( \alpha_i < 0 \) then this is
\[-\tanh u_i = \sqrt{1 + \alpha_i e^{2f_i}}\]
and if \( \alpha_i > 0 \) then this is
\[-\coth u_i = \sqrt{1 + \alpha_i e^{2f_i}}.
\]
Compare to the formulations in [26], [1], and [48].

It then follows that for a triangle \( t = \{1, 2, 3\} \) the following form is closed:
\[
(5.8) \quad \omega_t = \sum_{i=1}^{3} \gamma_i du_i.
\]
We can now integrate to get a function on the whole triangulation, where we fix some \( \bar{u} \):
\[
(5.9) \quad F(u) = 2\pi \sum_i u_i - \sum_t \int_{\bar{u}}^{u} \omega_t.
\]

**Theorem 8.** The function \( F \) has the property that
\[
\frac{\partial F}{\partial u_i} = K_i.
\]
Furthermore, if all \( d_{ij} > 0 \) and \( h_{ij} > 0 \) then this function is strictly convex.

**Proof.** The first statement follows from the definition. The second follows from the facts that in a triangle \( \{1, 2, 3\} \),
\[
\frac{\partial \gamma_i}{\partial u_j} \geq 0
\]
\[
\frac{\partial \gamma_i}{\partial u_i} > \frac{\partial \gamma_i}{\partial u_j} + \frac{\partial \gamma_i}{\partial u_k}
\]
for \( \{i, j, k\} = \{1, 2, 3\} \) since
\[
\frac{\partial A_{123}}{\partial f_i} > 0
\]
by Proposition [9]. It follows that the matrix of partial derivatives is diagonally dominant.

6. Spherical Geometry

The arguments presented in the case of hyperbolic background geometry can be adjusted for the case of spherical background geometry. Essentially, this occurs because in the hyperbolic case we are studying properties of the Lorentzian inner product $\ast$, while in spherical geometry we study analogous properties of the Euclidean inner product. Because the definitions and arguments in the spherical case are so similar to those of previous sections, we will only state the main results in the spherical case.

To work in the spherical case, we work with the usual dot product $\cdot$ on $\mathbb{R}^3$. Geodesics on the sphere correspond to planes in $\mathbb{R}^3$ and so given a triangle $\{i,j,k\}$ in the sphere and a pre-metric, a given embedding induces planes $P_{ij}$, etc. through edge centers and the condition for inducing a duality structure is

$$P_{ij} \cap P_{jk} = P_{jk} \cap P_{ki} = P_{ki} \cap P_{ij} \tag{6.1}$$

As in the hyperbolic case, this corresponds to a compatibility condition on the partial edge lengths.

**Proposition 10.** Suppose $d$ is a piecewise spherical pre-metric. Equation (6.1) holds if and only if the following compatibility equation

$$(p_i \cdot c_{ij})(p_j \cdot c_{jk})(p_k \cdot c_{ki}) = (p_i \cdot c_{ki})(p_j \cdot c_{ij})(p_k \cdot c_{jk}) \tag{6.2}$$

is satisfied for every simplex $\{i,j,k\}$. Equation (6.2) has the following equivalent formulation:

$$\cos(d_{ij}) \cos(d_{jk}) \cos(d_{ki}) = \cos(d_{ji}) \cos(d_{kj}) \cos(d_{ik}).$$

We can also look at discrete conformal structures. The angle variation theorem takes the following form.

**Theorem 9.** Given a conformal structure, then for any simplex $\{i,j,k\}$:

$$\frac{\partial \gamma_i}{\partial f_j} = 1 \tan h_{ij} \cos d_{ij} \sin \ell_{ij},$$

$$\frac{\partial \gamma_i}{\partial f_i} = \frac{\partial A_{ijk}}{\partial f_i} - \frac{\partial \gamma_j}{\partial f_i} - \frac{\partial \gamma_k}{\partial f_i}. \tag{6.4}$$

Note that although the heights $h_{ij}$ require choosing one of the two possible centers, the term $\tan h_{ij}$ does not depend on this choice, since choosing the other center leads to heights $h_{ij}' = -(\pi - h_{ij})$ and so $\tan h_{ij}' = \tan h_{ij}$.

We can also compute the variation of area explicitly.

**Proposition 11.** Given a spherical conformal structure, then for any simplex $\{i,j,k\}$ with area $A_{ijk}$, we have

$$\frac{\partial A_{ijk}}{\partial f_k} = \frac{\partial \gamma_i}{\partial f_k} (1 - \cos \ell_{ik}) + \frac{\partial \gamma_j}{\partial f_k} (1 - \cos \ell_{jk}).$$

Using this theorem and the definition of a spherical conformal structure, one can derive the spherical case of Theorem 4. As in the hyperbolic case, it is desirable...
to change from the variables \( f_i \) to variables \( u_i = u_i(f_i) \), so that one can recognize that \( \partial \gamma_i / \partial u_j = \partial \gamma_j / \partial u_i \). The variables \( u_i \) are given by

\[
\frac{\partial f_i}{\partial u_i} = \sqrt{1 - \alpha_i e^{2 f_i}}.
\]

Finally, we may define closed forms \( \omega_i \) and a function \( F \) as in Equations 5.8 and 5.9. We have the following analog to Theorem 8.

**Theorem 10.** The function \( F \) has the property that

\[
\frac{\partial F}{\partial u_i} = K_i.
\]

Notice that we do not have a corresponding notion of convexity for this functional, as we do in the cases of Euclidean and hyperbolic backgrounds.

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