NEW SIMPLE METHOD OF EXPANSION OF ITERATED ITO STOCHASTIC INTEGRALS OF MULTIPlicity 2 BASED ON EXPANSION OF THE BROWNIAN MOTION USING LEGENDRE POLYNOMIALS AND TRIGONOMETRIC FUNCTIONS

DMITRIY F. KUZNETSOV

ABSTRACT. The article is devoted to the expansion of iterated Ito stochastic integrals of second multiplicity based on expansion of the Brownian motion (standard Wiener process) using complete orthonormal systems of functions in the space $L_2([t,T])$. The cases of Legendre polynomials and trigonometric functions are considered in details. We obtained a new representation of the Levy stochastic area based on the Legendre polynomials. This representation was first derived in the author’s work [1] (1997). In this article, we obtain the mentioned representation by a simpler method compared to [1] (1997). Also, we get the polynomial representation of the Levy stochastic area using the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. The polynomial representation of the Levy stochastic area has more simple form in comparison with the classical trigonometric representation of the Levy stochastic area. The convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) as well as the convergence with probability 1 for approximations of the Levy stochastic area are proved. The results of the article can be applied to the numerical solution of Ito stochastic differential equations as well as to the numerical approximation of mild solution for non-commutative semilinear stochastic partial differential equations.

CONTENTS

1. Introduction .......................................................... 2
2. Method of Expansion of Iterated Ito stochastic integrals Based on Generalized Multiple Fourier Series .................. 3
3. New Representation of the Levy Stochastic Area Based on the Legendre Polynomials .................................. 10
4. The Classical Representation of the Levy Stochastic Area .......................................................... 11
5. New Simple Method for Obtainment of Representation of the Levy Stochastic Area .................................. 13
6. Convergence in the Mean of Degree $2n$ and With Probability 1 .......................................................... 17

References .................................................................. 19

Mathematics Subject Classification: 60H05, 60H10, 42B05.

Keywords: Iterated Ito stochastic integral, Generalized multiple Fourier series, Multiple Fourier–Legendre series, Levy stochastic area, Mean square-convergence, Milstein method, Ito stochastic differential equation, Approximation, Expansion.
1. Introduction

Let \((\Omega, F, P)\) be a complete probability space, let \(\{F_t, t \in [0, T]\}\) be a nondecreasing right-continuous family of \(\sigma\)-algebras of \(F\), and let \(w_t\) be a standard \(m\)-dimensional Wiener stochastic process, which is \(F_t\)-measurable for any \(t \in [0, T]\). We assume that the components \(w_t^{(i)} (i = 1, \ldots, m)\) of this process are independent. Consider an Ito stochastic differential equation in the integral form

\[
x_t = x_0 + \int_0^t a(x_\tau, \tau) d\tau + \int_0^t B(x_\tau, \tau) dW_\tau, \quad x_0 = x(0, \omega), \quad \omega \in \Omega.
\]

Here \(x_t\) is some \(n\)-dimensional stochastic process satisfying the equation (1). The nonrandom functions \(a : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n\), \(B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}\) guarantee the existence and uniqueness up to stochastic equivalence of a solution of the equation (1) [2]. The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let \(x_0\) be an \(n\)-dimensional random variable, which is \(F_0\)-measurable and \(M\{|x_0|^2\} < \infty\) (\(M\) denotes a mathematical expectation). We assume that \(x_0\) and \(w_t - w_0\) are independent when \(t > 0\).

One of the effective approaches to the numerical integration of Ito stochastic differential equations is an approach based on the Taylor–Ito expansion [3]–[5]. The most important feature of the Taylor–Ito expansion is a presence in this expansion of the so-called iterated Ito stochastic integrals, which play the key role for solving the problem of numerical integration of Ito stochastic differential equations and have the following form

\[
J[\psi^{(k)}]_{T, t} = \int_t^T \psi_1(t_1) \cdots \int_t^{t_1} \psi_2(t_2) \cdots \int_t^{t_2} \psi_k(t_k) \, dW^{(i_1)}_{t_1} \cdots dW^{(i_k)}_{t_k} \quad (i_1, \ldots, i_k = 0, 1, \ldots, m),
\]

where \(\psi_1(\tau), \ldots, \psi_k(\tau)\) are nonrandom functions on \([t, T]\), \(w^{(i)}(i = 1, \ldots, m)\) are independent standard Wiener processes, and \(w^{(0)}(\tau) = \tau\).

In this article, we pay a special attention to the case \(k = 2, i_1, i_2 = 1, \ldots, m, \psi_1(\tau), \psi_2(\tau) \equiv 1\). This case corresponds to the so-called Milstein method [4], [5] for the numerical integration of Ito stochastic differential equations. It is well known that the Milstein method has the order 1.0 of strong convergence under the specific conditions [4], [5].

The Milstein method has the following form [4], [5]

\[
y_{p+1} = y_p + \sum_{i_1=1}^m B_{i_1} f^{(i_1)}_{\tau_{p+1}, \tau_p} + \Delta a + \sum_{i_1, i_2=1}^m G_{i_1} B_{i_2} f^{(i_1, i_2)}_{\tau_{p+1}, \tau_p},
\]

where \(\Delta = T/N\) \((N > 1)\) is a constant (for simplicity) step of integration, \(\tau_p = p\Delta\) \((p = 0, 1, \ldots, N)\),

\[
G_i = \sum_{j=1}^n B_{ij}(x, t) \frac{\partial}{\partial x_j} \quad (i = 1, \ldots, m),
\]

\(B_i\) is the \(i\)th column of the matrix function \(B\) and \(B_{ij}\) is the \(ij\)th element of the matrix function \(B\), \(a_i\) is the \(i\)th element of the vector function \(a\), and \(x_i\) is the \(i\)th element of the column \(x\), the columns \(B_{i1}, a, G_i, B_{i2}\) are calculated in the point \((y_p, p)\).
NEW SIMPLE METHOD OF EXPANSION OF ITERATED ITO STOCHASTIC INTEGRALS

Consider the iterated Ito stochastic integrals (2) and define the following function on the hypercube $[t, T]^k$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\ 0, & \text{otherwise} \end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2,$$

and $K(t_1) \equiv \psi_1(t_1)$ for $t_1 \in [t, T]$. Here we suppose that $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$.

Assume that $\{\phi_j(x)\}_{j=0}^\infty$ is a complete orthonormal system of functions in the space $L_2([t, T])$. The function $K(t_1, \ldots, t_k)$ belongs to the space $L_2([t, T]^k)$. At this situation it is well known that the generalized multiple Fourier series of $K(t_1, \ldots, t_k) \in L_2([t, T]^k)$ is converging to $K(t_1, \ldots, t_k)$ in the hypercube $[t, T]^k$ in the mean-square sense, i.e.
\[
\lim_{p_1, \ldots, p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right\|_{L_2([t,T]^k)} = 0,
\]

where
\[
C_{j_k \ldots j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k
\]
is the Fourier coefficient and
\[
\|f\|_{L_2([t,T]^k)} = \left( \int_{[t,T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}
\]

Consider the partition \( \{\tau_j\}_{j=0}^N \) of \([t, T]\) such that
\[
t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \text{ if } N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]

**Theorem 1 [7] (2006), [8]-[52].** Suppose that every \( \psi_l(\tau) \ (l = 1, \ldots, k) \) is a continuous nonrandom function on \([t, T]\) and \( \{\phi_j(x)\}_{j=0}^\infty \) is a complete orthonormal system of continuous functions in \(L_2([t,T])\). Then
\[
J[\psi^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} - \right.
\]

\[
- \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta w_{\tau_{l_1}}^{(i_1)} \cdots \phi_{j_k}(\tau_{l_k}) \Delta w_{\tau_{l_k}}^{(i_k)} \),
\]

where
\[
G_k = H_k \backslash L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1\},
\]

\[
L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1; l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},
\]
l.i.m. is a limit in the mean-square sense, \( i_1, \ldots, i_k = 0,1,\ldots,m \), \( C_{j_k \ldots j_1} \) is the Fourier coefficient \([5]\),
\[
\zeta_{j}^{(i)}(s) = \int_t^T \phi_{j}(s)dw_s^{(i)}
\]
are independent standard Gaussian random variables for various \( i \) or \( j \) (in the case when \( i \neq 0 \), \( \Delta w_{\tau_j}^{(i)} = w_{\tau_{j+1}}^{(i)} - w_{\tau_j}^{(i)} \) \((i = 0,1,\ldots,m)\), \( \{\tau_j\}_{j=0}^N \) is a partition of \([t,T]\), which satisfies the condition \([6]\).
In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for \( k = 1, \ldots, 5 \) \[\text{[7]-[52]}\]

\[
J[\psi^{(1)}]_{T,t} = \lim_{p_1 \to \infty} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(1)},
\]

\[
J[\psi^{(2)}]_{T,t} = \lim_{p_1, p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \left( \zeta_{j_1}^{(1)} \zeta_{j_2}^{(1)} - 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} \right);
\]

\[
J[\psi^{(3)}]_{T,t} = \lim_{p_1, \ldots, p_3 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_3j_2j_1} \left( \zeta_{j_1}^{(1)} \zeta_{j_2}^{(1)} \zeta_{j_3}^{(1)} - \right.
\]

\[
-1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} \zeta_{j_3}^{(1)} - 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_2=j_3\}} \zeta_{j_1}^{(1)} - 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} \zeta_{j_2}^{(1)} \right),
\]

\[
J[\psi^{(4)}]_{T,t} = \lim_{p_1, \ldots, p_4 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \sum_{j_4=0}^{p_4} C_{j_4j_3j_2j_1} \left( \prod_{l=1}^{4} \zeta_{j_l}^{(1)} - \right.
\]

\[
-1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} \zeta_{j_3}^{(1)} \zeta_{j_4}^{(1)} - 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} \zeta_{j_2}^{(1)} \zeta_{j_4}^{(1)} - \right.
\]

\[
-1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} \zeta_{j_1}^{(1)} \zeta_{j_3}^{(1)} - 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_2=j_3\}} \zeta_{j_1}^{(1)} \zeta_{j_4}^{(1)} + \right.
\]

\[
+1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} 1_{\{j_1=j_3\}} + \right.
\]

\[
+1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} 1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} \right),
\]

\[
J[\psi^{(5)}]_{T,t} = \lim_{p_1, \ldots, p_5 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \sum_{j_4=0}^{p_4} \sum_{j_5=0}^{p_5} C_{j_5j_4j_3j_2j_1} \left( \prod_{l=1}^{5} \zeta_{j_l}^{(1)} - \right.
\]

\[
-1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} \zeta_{j_3}^{(1)} \zeta_{j_4}^{(1)} \zeta_{j_5}^{(1)} - 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} \zeta_{j_2}^{(1)} \zeta_{j_4}^{(1)} \zeta_{j_5}^{(1)} - \right.
\]

\[
-1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} \zeta_{j_1}^{(1)} \zeta_{j_3}^{(1)} \zeta_{j_5}^{(1)} - 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_2=j_3\}} \zeta_{j_1}^{(1)} \zeta_{j_4}^{(1)} \zeta_{j_5}^{(1)} - \right.
\]

\[
-1_{\{i_2=i_5 \neq 0\}} 1_{\{j_2=j_5\}} \zeta_{j_1}^{(1)} \zeta_{j_3}^{(1)} \zeta_{j_4}^{(1)} - 1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} \zeta_{j_1}^{(1)} \zeta_{j_3}^{(1)} \zeta_{j_5}^{(1)} + \right.
\]

\[
+1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_3=i_4 \neq 0\}} 1_{\{j_3=j_4\}} \zeta_{j_5}^{(1)} + 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_3=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta_{j_4}^{(1)} + \right.
\]

\[
+1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_4=i_5 \neq 0\}} 1_{\{j_4=j_5\}} \zeta_{j_3}^{(1)} + \right.
\]

\[
+1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} 1_{\{i_2=i_5 \neq 0\}} 1_{\{j_2=j_5\}} \zeta_{j_4}^{(1)} + 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} 1_{\{i_4=i_5 \neq 0\}} 1_{\{j_4=j_5\}} \zeta_{j_2}^{(1)} + \right.
\]

\[
+1_{\{i_1=i_4 \neq 0\}} 1_{\{j_1=j_4\}} 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_2=j_3\}} \zeta_{j_5}^{(1)} + 1_{\{i_1=i_4 \neq 0\}} 1_{\{j_1=j_4\}} 1_{\{i_2=i_5 \neq 0\}} 1_{\{j_2=j_5\}} \zeta_{j_3}^{(1)} + \right.
\]

\[
+1_{\{i_1=i_4 \neq 0\}} 1_{\{j_1=j_4\}} 1_{\{i_3=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta_{j_2}^{(1)} + 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} 1_{\{i_2=i_5 \neq 0\}} 1_{\{j_2=j_5\}} \zeta_{j_4}^{(1)} +
\]
It was shown that Theorem 1 is valid for convergence w. p. 1 \([12] - [14], [32], [44]\) (the cases of Legendre polynomials and trigonometric functions) and for convergence in the mean of degree \(2n\) \((n \in \mathbb{N})\) \([12] - [14], [26], [30]\). Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in the space \(L_2([t, T])\) can also be applied in Theorem 1 \([7] - [21]\). The modification of Theorem 1 for complete orthonormal with weight \(r(x) \geq 0\) systems of functions in the space \(L_2([t, T])\) can be found in \([11], [14], [42]\). Recently, Theorem 1 and Theorem 2 (see below) has been applied to the expansion and mean-square approximation of iterated stochastic integrals with respect to the infinite-dimensional Q-Wiener process \([12] - [14]\) (Chapter 7), \([28], [29], [45] - [47]\). These results can be directly applied to construction of high-order strong numerical methods for non-commutative semi-linear stochastic partial differential equations with multiplicative trace class noise \([12] - [14]\) (Chapter 7), \([29], [47]\).

Note that we obtain the following useful possibilities of the approach based on Theorem 1.

1. There is the explicit formula \([5]\) for calculation of expansion coefficients of the iterated Ito stochastic integral \([2]\) with any fixed multiplicity \(k\).

2. We have new possibilities for exact calculation of the mean-square approximation error of iterated Ito stochastic integral \([2]\) \([9], [13], [22], [33]\) (also see Theorem 3 below).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space \(L_2([t, T])\), then we have new possibilities for approximation — we can use not only trigonometric functions as in \([3] - [5]\) but Legendre polynomials.

4. As it turned out \([7] - [13]\) it is more convenient to work with Legendre polynomials for approximation of the iterated Ito stochastic integrals \([2]\). Approximations based on the Legendre polynomials are much simpler than their analogues based on the trigonometric functions. Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in \([12] - [14], [26], [30]\).

5. The approach to expansion of iterated Ito and Stratonovich stochastic integrals based on the Karhunen–Loeve expansion of the Brownian bridge process \([4]\) (also see \([3], [5]\)) as well as the approach from \([53]\) lead to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1 and Theorem 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case \(\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1\); \(i_1, i_2, i_3 = 1, \ldots, m\)) of iterated stochastic integrals. Multiple series from Theorems 1, 2 (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as \(p_1, \ldots, p_k\)). For example, when \(p_1 = \ldots = p_k = p \rightarrow \infty\). For iterated series, the condition \(p_1 = \ldots = p_k = p \rightarrow \infty\) obviously does not guarantee the convergence of this series. However, the authors of the works \([3]\) (Sect. 5.8, pp. 202–204), \([54]\) (pp. 82-84), \([55]\) (pp. 438-439), \([56]\) (pp. 263-264) use the Wong–Zakai approximation \([57] - [59]\) (without rigorous proof) within the frames of the method of expansion of iterated stochastic integrals \([4] (1988)\) based on the series expansion of the Brownian bridge process (version of the so-called Karhunen-Loeve expansion). See discussions in \([12]\) (Sect. 2.16, 6.2), \([13]\) (Sect. 2.6.2, 6.2), \([14]\) (Sect. 2.6.2, 6.2), \([32]\) (Sect. 11), \([33]\) (Sect. 8), \([36]\) (Sect. 6) for detail.

\[
\begin{align*}
&+1_{\{i_1 = i_5 \neq 0\}} 1_{\{j_1 = j_5\}} 1_{\{i_2 = i_4 \neq 0\}} 1_{\{j_2 = j_4\}} \zeta_{j_1} \zeta_{j_2} \\
&+1_{\{i_2 = i_5 \neq 0\}} 1_{\{j_2 = j_5\}} 1_{\{i_3 = i_5 \neq 0\}} 1_{\{j_1 = j_5\}} \zeta_{j_1} \zeta_{j_2} \\
&+1_{\{i_3 = i_5 \neq 0\}} 1_{\{j_3 = j_5\}} 1_{\{i_5 = i_4 \neq 0\}} 1_{\{j_3 = j_4\}} \zeta_{j_1} \zeta_{j_2} + \\
&\text{where } 1_A \text{ is the indicator of the set } A.
\end{align*}
\]
Note that the correctness of the formulas [9]–[13] can be verified by the fact that if $i_1 = \ldots = i_5 = i = 1, \ldots, m$ and $\psi_1(s), \ldots, \psi_5(s) \equiv \psi(s)$ in [9]–[13], then we can obtain from [9]–[13] the following equalities

\[
J[\psi^{(1)}]_{T,t} = \frac{1}{1!} \delta_{T,t},
\]

\[
J[\psi^{(2)}]_{T,t} = \frac{1}{2!} \left( \delta_{T,t}^2 - \Delta_{T,t} \right),
\]

\[
J[\psi^{(3)}]_{T,t} = \frac{1}{3!} \left( \delta_{T,t}^3 - 3 \delta_{T,t} \Delta_{T,t} \right),
\]

\[
J[\psi^{(4)}]_{T,t} = \frac{1}{4!} \left( \delta_{T,t}^4 - 6 \delta_{T,t}^2 \Delta_{T,t} + 3 \Delta_{T,t}^2 \right),
\]

\[
J[\psi^{(5)}]_{T,t} = \frac{1}{5!} \left( \delta_{T,t}^5 - 10 \delta_{T,t}^3 \Delta_{T,t} + 15 \delta_{T,t}^2 \Delta_{T,t}^2 \right)
\]

w. p. 1, where

\[
\delta_{T,t} = \int_T^T \psi(s) d\mathbf{w}_k^{(i)}, \quad \Delta_{T,t} = \int_T^T \psi^2(s) ds.
\]

The above formulas can be independently obtained using the Ito formula and Hermite polynomials. Note that the cases $k = 2, 3$ and $p_1 = p_2 = p_3 = p$ are considered in detail in [8]–[21], [32].

Consider the generalization of formulas [9]–[13] for the case of an arbitrary multiplicity $k$ of the stochastic integral $J[\psi^{(k)}]_{T,t}$ as well as for the case of an arbitrary complete orthonormal systems of functions in the space $L_2([t,T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])$. In order to do this, let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of $r$ unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

\[
(\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}, \{q_1, \ldots, q_{k-2r}\}),
\]

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, braces mean an unordered set, and parentheses mean an ordered set.

We will say that (14) is a partition and consider the sum with respect to all possible partitions

\[
\sum_{(\{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}, \{q_1, \ldots, q_{k-2r}\}\})\ (g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}) \equiv \{1, 2, \ldots, k\}} a_{g_1g_2 \ldots g_{2r-1}g_{2r}q_1 \ldots q_{k-2r}},
\]

Below there are several examples of sums in the form (15)

\[
\sum_{(\{g_1, g_2\})\ (g_1, g_2) \equiv (1, 2)} a_{g_1g_2} = a_{12},
\]

\[
\sum_{(\{g_1, g_2\}, \{q_3, q_4\})\ (g_1, g_2, q_3, q_4) \equiv (1, 2, 3, 4)} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},
\]
\[
\sum_{((q_1,q_2) \cdot (q_1,q_2)) \cdot (q_1,q_2) = (1,2,3,4)} a_{q_1,q_2,q_1,q_2} = a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},
\]

\[
\sum_{((q_1,q_2) \cdot (q_1,q_2) \cdot (q_1,q_2)) \cdot (q_1,q_2,q_1,q_2) = (1,2,3,4,5)} a_{q_1,q_2,q_1,q_3} = a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{23,14,5} + a_{24,13,5} + a_{34,12,5} + a_{25,13,4} + a_{34,125} + a_{35,124} + a_{45,123},
\]

\[
\sum_{((q_1,q_2),(q_3,q_4),(q_1)) \cdot (q_1,q_2,q_3,q_4) = (1,2,3,4,5)} a_{q_1,q_2,q_3,q_4} = a_{12,34,5} + a_{13,24,5} + a_{14,23,5} + a_{23,14,5} + a_{24,13,5} + a_{34,12,5} + a_{12,54,3} + a_{15,24,3} + a_{14,25,3} + a_{15,34,2} + a_{13,54,2} + a_{14,53,2} + a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.
\]

Now we can generalize Theorem 1.

Theorem 2 [12] (Sect. 1.11), [32] (Sect. 15). Suppose that \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])\) and \(\phi_j(x)\) is an arbitrary complete orthonormal system of functions in the space \(L_2([t,T])\). Then the following expansion

\[
J[\psi^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1 = 0}^{p_1} \ldots \sum_{j_k = 0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^{k} \psi^{(i_l)}_{j_l} \right) + \sum_{r=1}^{k-2r} \left( \prod_{l=1}^{r} \psi^{(i_l)}_{j_l} \right)
\]

(16)

that converges in the mean-square sense is valid, where \([x]\) is an integer part of a real number \(x\); another notations are the same as in Theorem 1.

In particular from (16) for \(k = 5\) we obtain

\[
J[\psi^{(5)}]_{T,t} = \lim_{p_1, \ldots, p_5 \to \infty} \sum_{j_1 = 0}^{p_1} \ldots \sum_{j_5 = 0}^{p_5} C_{j_5 \ldots j_1} \left( \prod_{l=1}^{5} \psi^{(i_l)}_{j_l} \right) - \sum_{((q_1,q_2),(q_1,q_2,q_1)) \cdot (q_1,q_2,q_1,q_2,q_1) = (1,2,3,4,5)} 1_{i_{q_1} = i_{q_2} \neq 0} 1_{j_{q_1} = j_{q_2}} \prod_{l=1}^{3} \psi^{(i_l)}_{j_{q_l}} + \sum_{((q_1,q_2),(q_3,q_4),(q_1)) \cdot (q_1,q_2,q_3,q_4,q_1) = (1,2,3,4,5)} 1_{i_{q_1} = i_{q_2} \neq 0} 1_{j_{q_1} = j_{q_2}} 1_{i_{q_3} = i_{q_4} \neq 0} 1_{j_{q_3} = j_{q_4}} \psi^{(i_{q_1})}_{j_{q_1}}.
\]

The last equality obviously agrees with [13].

It should be noted that an analogue of Theorem 2 for multiple Ito stochastic integrals was considered in [60]. Note that we use another notations in comparison with [60]. Moreover, the proof of an
analogue of Theorem 2 from [60] is somewhat different from the proof given in [12] (Sect. 1.11), [32] (Sect. 1.12), [33] (Sect. 6).

Let us denote

\[ E_{k}^{p_{1}, \ldots, p_{k}} \overset{\text{def}}{=} M \left\{ \left( J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{p_{1}, \ldots, p_{k}} \right)^{2} \right\}, \quad E_{k}^{p} \overset{\text{def}}{=} E_{k}^{p_{1}, \ldots, p_{k}} \mid_{p_{1}=\ldots=p_{k}=p}, \]

\[ I_{k} \overset{\text{def}}{=} \|K\|_{L_{2}([t, T])}^{2} = \int_{[t, T]} K^{2}(t_{1}, \ldots, t_{k})dt_{1} \ldots dt_{k}, \]

where \( J[\psi^{(k)}]_{T,t}^{p_{1}, \ldots, p_{k}} \) is the expression on the right-hand side of (16) before passing to the limit \( l.i.m. \rightarrow \infty \), i.e.

\[ J[\psi^{(k)}]_{T,t}^{p_{1}, \ldots, p_{k}} = \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}} \left( \prod_{l=1}^{k} \sum_{s=1}^{\lfloor k/2 \rfloor - 1} (-1)^{s} \right) \times \]

\[ \times \sum_{\{(i_{1}, q_{1}), \ldots, (i_{k}, q_{k}) \} \in \{(1, \ldots, j_{1}), \ldots, \}, \ldots, \}} \prod_{s=1}^{r} 1\{i_{s_{2s-1}} = i_{s_{2s}} \neq 0\} 1\{j_{s_{2s-1}} = j_{s_{2s}} \} \prod_{l=1}^{k-2r} S_{i_{q_{l}}}^{(i_{q_{l}})} \). \]

In [12, 14, 32] it was shown that

\[ E_{k}^{p_{1}, \ldots, p_{k}} \leq k! \left( I_{k} - \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}}^{2} \right) \]

if \( i_{1}, \ldots, i_{k} = 1, \ldots, m \) and \( 0 < T - t < \infty \) or \( i_{1}, \ldots, i_{k} = 0, 1, \ldots, m \) and \( 0 < T - t < 1 \).

Moreover, in [12, 14, 32] the following estimate is obtained

\[ E_{k}^{p_{1}, \ldots, p_{k}} \leq (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)! \times \]

\[ \times \left( I_{k} - \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}}^{2} \right)^{n}, \]

where \( n \in \mathbb{N} \).

The value \( E_{k}^{p} \) can be calculated exactly.

**Theorem 3** [12] (Sect. 1.12), [33] (Sect. 6). Suppose that \( \{\psi_{j}(x)\}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_{2}([t, T]) \) and \( \psi_{1}(\tau), \ldots, \psi_{k}(\tau) \in L_{2}([t, T]) \). Then

\[ E_{k}^{p} = I_{k} - \sum_{j_{1}, \ldots, j_{k}=0}^{p} C_{j_{k} \ldots j_{1}} M \left\{ J[\psi^{(k)}]_{T,t} \sum_{(j_{1}, \ldots, j_{k})}^{T} \phi_{j_{k}}(t_{k}) \ldots \phi_{j_{1}}(t_{1})d\tau_{1}^{i_{1}} \ldots d\tau_{k}^{i_{k}} \right\}, \]

where \( i_{1}, \ldots, i_{k} = 1, \ldots, m \); the expression

\[ \ldots \]
\[
\sum_{(j_1,\ldots,j_k)}
\]
means the sum with respect to all possible permutations \((j_1,\ldots,j_k)\). At the same time if \(j_r\) swapped with \(j_q\) in the permutation \((j_1,\ldots,j_k)\), then \(i_r\) swapped with \(i_q\) in the permutation \((i_1,\ldots,i_k)\); another notations are the same as in Theorems 1, 2.

Note that
\[
M \left\{ J[\psi^{(k)}]_{T,t} \int_I \phi_{j_k}(t_k) \ldots \int_I \phi_{j_1}(t_1) df(t_1)^{i_1} \ldots df(t_k)^{i_k} \right\} = C_{j_k\ldots j_1}.
\]

Then from Theorem 3 for pairwise different \(i_1,\ldots,i_k\) and for \(i_1 = \ldots = i_k\) we obtain
\[
E_{k}^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1},
\]

\[
E_{k}^p = I_k - \sum_{j_1,\ldots,j_k=0}^{p} C_{j_k\ldots j_1} \left( \sum_{(j_1,\ldots,j_k)} C_{j_k\ldots j_1} \right).
\]

3. NEW REPRESENTATION OF THE LEVY STOCHASTIC AREA BASED ON THE LEGENDRE POLYNOMIALS

Let us consider (10) for the case \(i_1 \neq i_2\). \(\psi_1(s), \psi_2(s) \equiv 1\). At that we suppose that \(\{\phi_j(x)\}_{j=0}^{\infty}\) is the complete orthonormal system of Legendre polynomials in the space \(L_2([t,T])\). Then
\[
I_{T,t}^{(i_1,i_2)} = \frac{T-t}{2} \left( \zeta_{0}^{(i_1)} \zeta_{0}^{(i_2)} + \sum_{i=1}^{\infty} \frac{1}{\sqrt{2i^2 - 1}} \left( \zeta_{i}^{(i_1)} \zeta_{i}^{(i_2)} - \zeta_{i-1}^{(i_1)} \zeta_{i-1}^{(i_2)} \right) \right),
\]

where
\[
I_{T,t}^{(i_1,i_2)} = \int_I \int_I dw_s^{(i_1)} dw_s^{(i_2)} \quad (i_1, i_2 = 1, \ldots, m),
\]

\(\zeta_{i}^{(i)}\) are independent standard Gaussian random variables (for various \(i\) or \(j\)), which have the following form
\[
\zeta_{i}^{(i)} = \int_I \phi_i(s) dw_s^{(i)},
\]

where
\[
\phi_i(s) = \sqrt{\frac{2i+1}{T-t}} P_i \left( \left( s - t - \frac{T-t}{2} \right) \frac{2}{T-t} \right), \quad i = 0, 1, 2, \ldots,
\]

and \(P_i(x) (i = 0, 1, 2, \ldots)\) is the Legendre polynomial.

Note that the representation (10) was first obtained in the author’s works [1] (1997), [61] (1998).
From (19) we obtain

\[ T - t \sum_{i=1}^{\infty} \frac{1}{\sqrt{4i^2 - 1}} \left( \zeta_i^{(i)} \zeta_{i-1}^{(i)} - \zeta_i^{(i)} \zeta_{i-1}^{(i)} \right) = \frac{1}{2} \left( I_{T, t}^{(i_1, i_2)} - I_{T, t}^{(i_2, i_1)} \right). \]

Then, a new representation of the Levy stochastic area based on the Legendre polynomials has the following form

\[ A_{T, t}^{(i_1, i_2)} = \frac{T - t}{2} \sum_{i=1}^{\infty} \frac{1}{\sqrt{4i^2 - 1}} \left( \zeta_i^{(i)} \zeta_{i-1}^{(i)} - \zeta_i^{(i)} \zeta_{i-1}^{(i)} \right). \]

4. The Classical Representation of the Levy Stochastic Area

Let us consider (10) for the case \( i_1 \neq i_2, \) \( \psi_1(s), \psi_2(s) \equiv 1. \) At that we suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is the complete orthonormal system of trigonometric functions in \( L_2([t, T]). \) Then

\[ I_{T, t}^{(i_1, i_2)} = \frac{1}{2} (T - t) \left( \zeta_0^{(i_1)} \zeta_0^{(i_2)} + \frac{1}{\pi} \sum_{r=1}^{\infty} \frac{1}{r} \left( \zeta_2^{(i_1)} \zeta_{2r-1}^{(i_2)} - \zeta_0^{(i_1)} \zeta_0^{(i_2)} \right) + \sqrt{2} \left( \zeta_2^{(i_1)} \zeta_0^{(i_2)} - \zeta_0^{(i_1)} \zeta_2^{(i_2)} \right) \right), \]

where we use the same notations as in (19), but \( \phi_j(s) \) has the following form

\[ \phi_j(s) = \frac{1}{\sqrt{T - t}} \begin{cases} 1, & \text{if } j = 0 \\ \sqrt{2} \sin(2\pi r(s - t)/(T - t)), & \text{if } j = 2r - 1, \ r = 1, 2, \ldots \\ \sqrt{2} \cos(2\pi r(s - t)/(T - t)), & \text{if } j = 2r \end{cases} \]

From (23) we obtain

\[ \frac{T - t}{2\pi} \sum_{r=1}^{\infty} \frac{1}{r} \left( \zeta_2^{(i_1)} \zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)} \zeta_2^{(i_2)} + \sqrt{2} \left( \zeta_2^{(i_1)} \zeta_0^{(i_2)} - \zeta_0^{(i_1)} \zeta_2^{(i_2)} \right) \right) = \frac{1}{2} \left( I_{T, t}^{(i_1, i_2)} - I_{T, t}^{(i_2, i_1)} \right). \]

Then, the representation of the Levy stochastic area based on the trigonometric functions has the following form

\[ A_{T, t}^{(i_1, i_2)} = \frac{T - t}{2\pi} \sum_{r=1}^{\infty} \frac{1}{r} \left( \zeta_2^{(i_1)} \zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)} \zeta_2^{(i_2)} + \sqrt{2} \left( \zeta_2^{(i_1)} \zeta_0^{(i_2)} - \zeta_0^{(i_1)} \zeta_2^{(i_2)} \right) \right). \]
As we mentioned above, Milstein G.N. proposed \cite{4} the method of expansion of iterated Ito stochastic integrals of multiplicity 2 based on the trigonometric Fourier expansion of the following Brownian bridge process

\[ w_t - \frac{t}{\Delta} w_\Delta, \quad t \in [0, \Delta], \quad \Delta > 0, \]

where \( w_t \) is a standard multidimensional Wiener process with independent components \( w^{(i)}_t, \quad i = 1, \ldots, m. \)

The trigonometric Fourier expansion of the Brownian bridge process (version of the so-called Karunen–Loeve expansion) has the form \cite{4}

\[
(26) \quad w^{(i)}_t - \frac{t}{\Delta} w^{(i)}_\Delta = \frac{1}{2} a^{(i)}_{i,0} + \sum_{r=1}^{\infty} \left( a^{(i)}_{i,r} \cos \frac{2\pi rt}{\Delta} + b^{(i)}_{i,r} \sin \frac{2\pi rt}{\Delta} \right),
\]

where

\[
a^{(i)}_{i,r} = \frac{2}{\Delta} \int_0^\Delta \left( w^{(i)}_s - \frac{s}{\Delta} w^{(i)}_\Delta \right) \cos \frac{2\pi rs}{\Delta} ds,
\]

\[
b^{(i)}_{i,r} = \frac{2}{\Delta} \int_0^\Delta \left( w^{(i)}_s - \frac{s}{\Delta} w^{(i)}_\Delta \right) \sin \frac{2\pi rs}{\Delta} ds,
\]

\( r = 0, 1, \ldots, \quad i = 1, \ldots, m. \)

It is easy to demonstrate \cite{4} that the random variables \( a^{(i)}_{i,r}, b^{(i)}_{i,r} \) are Gaussian ones and they satisfy the following relations

\[
M \{a^{(i)}_{i,r} b^{(i)}_{i,k} \} = M \{a^{(i)}_{i,r} a^{(i)}_{i,k} \} = 0,
\]

\[
M \{a^{(i)}_{i,r} a^{(i)}_{i,k} \} = M \{b^{(i)}_{i,r} b^{(i)}_{i,k} \} = 0,
\]

\[
M \{a^{(i)}_{i,r} a^{(i)}_{i,k} \} = M \{b^{(i)}_{i,r} b^{(i)}_{i,k} \} = 0,
\]

\[
M \left\{ a^{(i)}_{i,r} \right\} = \frac{\Delta}{2\pi^2 r^2}.
\]

where \( i, \quad i_1, i_2 = 1, \ldots, m, \quad r \neq k, \quad i_1 \neq i_2. \)

According to \cite{26}, we have

\[
(27) \quad w^{(i)}_t = w^{(i)}_\Delta \frac{t}{\Delta} + \frac{1}{2} a^{(i)}_{i,0} + \sum_{r=1}^{\infty} \left( a^{(i)}_{i,r} \cos \frac{2\pi rt}{\Delta} + b^{(i)}_{i,r} \sin \frac{2\pi rt}{\Delta} \right),
\]

where the series converges in the mean-square sense.

The expansion \cite{23} has been obtained in \cite{4} using \cite{27}. 
5. New Simple Method for Obtainment of Representation of the Levy Stochastic Area

It is well known that the idea of representing of the Wiener process as a functional series with random coefficients using the complete orthonormal system of trigonometric functions in $L_2([0,T])$ goes back to the works of Wiener [62] (1924) and Levy [63] (1951). The specified series was used in [62] and [63] for construction of the Brownian motion process (Wiener process). A little later, Ito and McKean in [64] (1965) used for this purpose the complete orthonormal system of Haar functions in $L_2([0,T])$.

Let $w_\tau$, $\tau \in [0,T]$ be an $m$-dimensional standard Wiener process with independent components $w^{(i)}_\tau$ $(i = 1, \ldots, m)$. We have

$$w^{(i)}_s - w^{(i)}_t = \int_t^s d w^{(i)}_\tau = \int_t^s 1_{(\tau < s)} d w^{(i)}_\tau,$$

where

$$1_{(\tau < s)} = \begin{cases} 1, & \tau < s \\ 0, & \text{otherwise} \end{cases}, \quad \tau, s \in [t, T], \quad 0 \leq t < T.$$

Consider the Fourier expansion of $1_{(\tau < s)}$ at the interval $[t, T]$ (see, for example, [65])

$$1_{(\tau < s)} = \sum_{j=0}^\infty \int_t^s 1_{(\tau < s)} \phi_j(\tau) d\tau \cdot \phi_j(\tau) = \sum_{j=0}^\infty \int_t^s \phi_j(\tau) d\tau \cdot \phi_j(\tau),$$

where $\{\phi_j(\tau)\}_{j=0}^\infty$ is a complete orthonormal system of functions in the space $L_2([t, T])$ and the series on the right-hand side of (28) converges in the mean-square sense, i.e.

$$\int_t^T \left( 1_{(\tau < s)} - \sum_{j=0}^q \int_t^s \phi_j(\tau) d\tau \cdot \phi_j(\tau) \right)^2 d\tau \to 0 \quad \text{if} \quad q \to \infty.$$

Let $(w^{(i)}_s - w^{(i)}_t)^{(q)}$ be the mean-square approximation of the process $w^{(i)}_s - w^{(i)}_t$, which has the following form

$$(29) \quad (w^{(i)}_s - w^{(i)}_t)^{(q)} = \int_t^T \left( \sum_{j=0}^q \int_t^s \phi_j(\tau) d\tau \cdot \phi_j(\tau) \right) d w^{(i)}_\tau = \sum_{j=0}^q \int_t^s \phi_j(\tau) d\tau \cdot \int_t^T \phi_j(\tau) d w^{(i)}_\tau.$$

Moreover,

$$M \left\{ \left( w^{(i)}_s - w^{(i)}_t - (w^{(i)}_s - w^{(i)}_t)^{(q)} \right)^2 \right\} =$$
\[ = M \left\{ \left( \int_T^t \left( 1_{\tau<s} - \sum_{j=0}^{q} \int_t^s \phi_j(\tau) \cdot \phi_j(\tau) \right) \, dw_\tau^{(i)} \right)^2 \right\} = \]

\[ = \int_T^t \left( 1_{\tau<s} - \sum_{j=0}^{q} \int_t^s \phi_j(\tau) \cdot \phi_j(\tau) \right)^2 \, d\tau \rightarrow 0 \quad \text{if} \quad q \rightarrow \infty. \]

In [53] it was proposed to use the expansion similar to (29) for construction of expansion of the iterated Ito stochastic integral \((20)\) of multiplicity 2. At that, to obtain the mentioned expansion of \((20)\), the truncated expansions \((29)\) of components of the Wiener process \(w_s\) have been iteratively substituted in the single integrals [53]. This procedure leads to the calculation of coefficients of the double Fourier series, which is a time-consuming task for not too complex problem of expansion of the iterated Ito stochastic integral \((20)\).

In contrast to [53] we substitute the truncated expansion \((29)\) only one time and only into the innermost integral in \((20)\). This procedure leads to the simple calculation of the coefficients

\[ \int_t^s \phi_j(\tau) \, d\tau \quad (j = 0, 1, 2, \ldots) \]

of the usual (not double) Fourier series.

Moreover, we use the Legendre polynomials for construction of the expansion of \((20)\). For the first time the Legendre polynomials have been applied in the framework of the mentioned problem in the author’s papers [1] (1997), [61] (1998), [66] (2000), [67] (2001) (also see [7]-[52], [68], [69]). At the same time in the papers of other author’s these polynomials have not been considered as the basis functions for construction of expansions of iterated Ito and Stratonovich stochastic integrals.

**Theorem 4** [12]-[14], [68], [69]. Let \(\phi_j(\tau) \quad (j = 0, 1, \ldots)\) be an arbitrary complete orthonormal system of functions in the space \(L_2([t, T])\). Let

\[ \int_T^t \left( w_s^{(i_1)} - w_t^{(i_1)} \right)^{(q)} \, dw_s^{(i_2)} = \sum_{j=0}^{q} \int_T^t \phi_j(\tau) \, dw_t^{(i_1)} \int_T^t \int_t^s \phi_j(\tau) \, d\tau \, dw_s^{(i_2)} \]

be the approximation of the iterated Ito stochastic integral

\[ \int_T^t \int_t^s d\tau \, dw_s^{(i_1)} \, dw_s^{(i_2)} \quad (i_1 \neq i_2), \]

where \(i_1, i_2 = 1, \ldots, m\). Then

\[ \int_T^t \int_t^s d\tau \, dw_s^{(i_1)} \, dw_s^{(i_2)} = \lim_{q \rightarrow \infty} \int_T^t \left( w_s^{(i_1)} - w_t^{(i_1)} \right)^{(q)} \, dw_s^{(i_2)} = \]

\[ = \lim_{q \rightarrow \infty} \sum_{j=0}^{q} \int_T^t \phi_j(\tau) \, dw_t^{(i_1)} \int_T^t \int_t^s \phi_j(\tau) \, d\tau \, dw_s^{(i_2)}, \]
where \( i_1 \neq i_2 \) \((i_1, i_2 = 1, \ldots, m)\).

**Proof.** Using standard properties of the Ito stochastic integral as well as (30) and the property of orthonormality of the functions \( \phi_j(\tau) \) \((j = 0, 1, \ldots)\) at the interval \([t, T]\), we obtain

\[
M \left\{ \left( \int_t^s dw_s^{(i_1)} dw_s^{(i_2)} - \int_t^s \left( w_s^{(i_1)} - w_t^{(i_1)} \right)^{(q)} \right) dw_s^{(i_2)} \right\}^2 =
\]

\[
= \int_t^T M \left\{ \left( w_s^{(i_1)} - w_t^{(i_1)} - \left( w_s^{(i_1)} - w_t^{(i_1)} \right)^{(q)} \right)^2 \right\} ds =
\]

\[
= \int_t^T \int_t^T \left( \sum_{j=0}^q \phi_j(\tau) \int_t^s \phi_j(\tau) d\tau \right)^2 d\tau ds =
\]

\[
= \int_t^T \left( s - t - \sum_{j=0}^q \left( \int_t^s \phi_j(\tau) d\tau \right)^2 \right) ds.
\]

(32)

Applying the continuity of the functions \( u_q(s) \) (see below), the nondecreasing property of the functional sequence

\[
u_q(s) = \sum_{j=0}^q \left( \int_t^s \phi_j(\tau) d\tau \right)^2,
\]

and the continuity of the limit function \( u(s) = s - t \) according to Dini’s Theorem, we have the uniform convergence \( u_q(s) \) to \( u(s) \) at the interval \([t, T]\).

Then from this fact as well as from (32) we obtain

\[
\int_t^T \int_t^s dw_s^{(i_1)} dw_s^{(i_2)} = \text{l.i.m.} \int_t^T \left( w_s^{(i_1)} - w_t^{(i_1)} \right)^{(q)} dw_s^{(i_2)}.
\]

(33)

Theorem 4 is proved.

Let \( \{\phi_j(\tau)\}_{j=0}^\infty \) be the complete orthonormal system of Legendre polynomials in the space \( L_2([t, T]) \), which has the form (21). Then

\[
\int_t^s \phi_j(\tau) d\tau = \frac{T - t}{2} \left( \frac{\phi_{j+1}(s)}{\sqrt{(2j + 1)(2j + 3)}} - \frac{\phi_{j-1}(s)}{\sqrt{4j^2 - 1}} \right) \quad \text{for} \quad j \geq 1.
\]

(34)

Denote (see Theorem 1)

\[
\zeta_j^{(i)} = \int_t^T \phi_j(\tau) d\mathbf{w}_\tau^{(i)} \quad (i = 1, \ldots, m).
\]
From (31) and (34) we get

\[
\int_t^T \left( w^{(i_1)}_s - w^{(i_1)}_t \right) \left( q \right) \, dw^{(i_2)}_s \, d\tau = \frac{1}{\sqrt{T-t}} \zeta^{(i_1)}_{s_0} \int_t^T (s-t) w^{(i_2)}_s + \frac{T-t}{2} q \sum_{j=1}^q \zeta^{(i_1)}_j \left( \frac{1}{\sqrt{(2j+1)(2j+3)}} \zeta^{(i_2)}_{j+1} - \frac{1}{\sqrt{4j^2-1}} \zeta^{(i_2)}_{j-1} \right) =
\]

\[
= \frac{T-t}{2} \zeta^{(i_1)}_{s_0} \left( \zeta^{(i_2)}_{s_0} + \frac{1}{\sqrt{3}} \zeta^{(i_2)}_{s_1} \right) + \frac{T-t}{2} \sum_{j=1}^q \zeta^{(i_1)}_j \left( \frac{1}{\sqrt{(2j+1)(2j+3)}} \zeta^{(i_2)}_{j+1} - \frac{1}{\sqrt{4j^2-1}} \zeta^{(i_2)}_{j-1} \right) =
\]

\[
= \frac{T-t}{2} \left( \zeta^{(i_1)}_0 \zeta^{(i_2)}_{s_0} + \sum_{j=1}^q \frac{1}{\sqrt{4j^2-1}} \left( \zeta^{(i_2)}_{j+1} s_{j+1} - \zeta^{(i_2)}_{j} s_{j-1} \right) \right) +
\]

\[
\frac{T-t}{2} \frac{1}{\sqrt{(2q+1)(2q+3)}} \zeta^{(i_1)} q \zeta^{(i_2)}_{q+1}.
\]

(35)

Then from (33) and (35) we obtain

\[
\int_t^s \int_t^T \left( w^{(i_1)} \right) \left( q \right) \, dw^{(i_2)}_s \, d\tau = \int_t^T \left( w^{(i_1)} - w^{(i_1)}_t \right) \left( q \right) \, dw^{(i_2)}_s =
\]

\[
= \frac{T-t}{2} \left( \zeta^{(i_1)}_0 \zeta^{(i_2)}_{s_0} + \sum_{j=1}^\infty \frac{1}{\sqrt{4j^2-1}} \left( \zeta^{(i_2)}_{j+1} s_{j+1} - \zeta^{(i_2)}_{j} s_{j-1} \right) \right).
\]

(36)

From (33) it follows that the equality (22) is fulfilled. It is not difficult to see that the relation (25) can also be obtained using the approach from this section.

Let \( \{ \phi_j(\tau) \}_{j=0}^\infty \) be the complete orthonormal system of trigonometric functions in the space \( L_2([t, T]) \), which has the form (24).

We have

\[
\int_t^s \phi_j(\tau) d\tau = \frac{T-t}{2\pi r} \begin{cases} 
\phi_{2r-1}(s), & j = 2r \\
\sqrt{2} \phi_0(s) - \phi_{2r}(s), & j = 2r - 1
\end{cases},
\]

(37)

where \( j \geq 1 \) and \( r = 1, 2, \ldots \).

From (31) and (37) we obtain
\[
\int_{t}^{T} \left( w_s^{(i_1)} - w_t^{(i_1)} \right) (q) \, dw_s^{(i_2)} = \frac{1}{\sqrt{T-t}} c^{(i_1)} \int_{t}^{T} (s-t) w_s^{(i_2)} +
\]
\[
\frac{T-t}{2} \sum_{r=1}^{q} \frac{1}{\pi r} \left( \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} - \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} + \sqrt{2} \zeta_{0}^{(i_2)} \zeta_{2r}^{(i_1)} \right)
\]
\[
= \frac{1}{\sqrt{T-t}} c^{(i_1)} \frac{(T-t)^{3/2}}{2} \left( \zeta_0^{(i_2)} - \frac{\sqrt{2}}{\pi} \sum_{r=1}^{\infty} \frac{1}{r^{2r-1}} \right) +
\]
\[
\frac{T-t}{2} \sum_{r=1}^{q} \frac{1}{\pi r} \left( \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} - \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} + \sqrt{2} \zeta_{0}^{(i_2)} \zeta_{2r}^{(i_1)} \right)
\]
\[
= \frac{1}{2} (T-t) \left( \zeta_0^{(i_1)} \zeta_0^{(i_2)} + \frac{1}{\pi} \sum_{r=1}^{q} \frac{1}{r} \left( \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} - \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} + \sqrt{2} \zeta_{0}^{(i_2)} \zeta_{2r}^{(i_1)} \right) \right) +
\]
\[
- \frac{T-t}{\pi \sqrt{2}} \zeta_0^{(i_1)} \sum_{r=q+1}^{\infty} \frac{1}{r} \zeta_{2r-1}^{(i_2)}.
\]

From (38) and (39) we obviously get (23).

### 6. Convergence in the Mean of Degree 2n and With Probability 1

Let us denote

\[
A_{T,t}^{(i_1i_2)q} = \frac{T-t}{2} \sum_{i=1}^{q} \frac{1}{\sqrt{4i^2-1}} \left( \zeta_{4i-1}^{(i_1)} \zeta_{4i}^{(i_2)} - \zeta_{i}^{(i_1)} \zeta_{i}^{(i_2)} \right),
\]

\[
\hat{A}_{T,t}^{(i_1i_2)q} = \frac{T-t}{2\pi} \sum_{r=1}^{q} \frac{1}{r} \left( \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} - \zeta_{2r}^{(i_1)} \zeta_{2r-1}^{(i_2)} + \sqrt{2} \zeta_{0}^{(i_2)} \zeta_{2r}^{(i_1)} \right).
\]

Then, from (39) we get

\[
J_{T,t}^{(i_1i_2)q} = \frac{T-t}{2} \zeta_0^{(i_1)} \zeta_0^{(i_2)} + A_{T,t}^{(i_1i_2)q},
\]

\[
\hat{J}_{T,t}^{(i_1i_2)q} = \frac{T-t}{2} \zeta_0^{(i_1)} \zeta_0^{(i_2)} + \hat{A}_{T,t}^{(i_1i_2)q}.
\]

It is not difficult to demonstrate [4] that from (23) we can get an another representation for the Levy stochastic area.
\[
\frac{T - t}{2\pi} \left( \sum_{r=1}^{q} \frac{1}{r} \left( \zeta_{2r-1}^{(i_1)} - \zeta_{2r-1}^{(i_2)} \right) \right) + \sqrt{2} \left( \zeta_{2r-1}^{(i_1)} - \zeta_{2r-1}^{(i_2)} \right) + \sqrt{2} \left( \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right)^{1/2} \left( \zeta_{q}^{(i_1)} - \zeta_{q}^{(i_2)} \right),
\]

where

\[
\zeta_{q}^{(i)} = \left( \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right)^{-1/2} \sum_{r=q+1}^{\infty} \frac{1}{r} \zeta_{2r-1}^{(i)},
\]

and \( \zeta_{0}^{(i)}, \zeta_{2r}^{(i)}, \zeta_{2r-1}^{(i)}, \zeta_{q}^{(i)} (r = 1, \ldots, q, i = 1, \ldots, m) \) are independent standard Gaussian random variables.

From (39) and (40) we obtain

\[
\mathbb{M} \left\{ \left( A_{T,t}^{(i_1 i_2)} - A_{T,t}^{(i_1 i_2)q} \right)^2 \right\} = \frac{(T-t)^2}{2} \left( \frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2 - 1} \right) =
\]

\[
= \frac{(T-t)^2}{2} \sum_{i=q+1}^{\infty} \frac{1}{4i^2 - 1} \leq \frac{(T-t)^2}{2} \int_{q+1}^{\infty} \frac{1}{4x^2 - 1} \, dx =
\]

\[
= -\frac{(T-t)^2}{8} \ln \left| 1 - \frac{2}{2q+1} \right| \leq C_1 \frac{(T-t)^2}{q},
\]

\[
\mathbb{M} \left\{ \left( \hat{A}_{T,t}^{(i_1 i_2)} - \hat{A}_{T,t}^{(i_1 i_2)q} \right)^2 \right\} = \frac{3(T-t)^2}{2\pi^2} \left( \frac{\pi^2}{6} - \sum_{r=1}^{q} \frac{1}{r^2} \right) =
\]

\[
= \frac{3(T-t)^2}{2\pi^2} \sum_{r=q+1}^{\infty} \frac{1}{r^2} \leq \frac{3(T-t)^2}{2\pi^2} \int_{q+1}^{\infty} \frac{dx}{x^2} =
\]

\[
= \frac{3(T-t)^2}{2\pi^2 q} \leq C_2 \frac{(T-t)^2}{q},
\]

where constants \( C_1, C_2 \) does not depend on \( q \).

For the case \( k = 2, i_1 \neq i_2, \) and \( \psi_1(s), \psi_2(s) \equiv 1 \) from (17) we obtain

\[
\mathbb{M} \left\{ \left( I_{T,t}^{(i_1 i_2)} - I_{T,t}^{(i_1 i_2)q} \right)^{2n} \right\} \leq C_{n,2} \left( \frac{(T-t)^2}{2} \left( \frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2 - 1} \right) \right)^n \rightarrow 0 \quad \text{if} \quad q \rightarrow \infty,
\]
where \(C_{n,k} = \left(\frac{k!}{2n}\right) \cdot \left(\frac{n(2n - 1)}{n!}\right)\) and \(I_{T,t}^{(i_1 i_2)}\) has the form (41) in the inequality (45), and \(I_{T,t}^{(i_1 i_2)}\) has the form (42) in the inequality (46).

From (43)–(46) we get

\[
\mathbb{M}\left\{\left(\frac{A_{T,t}^{(i_1 i_2)} - A_{T,t}^{(i_1 i_2)}}{q}\right)^2\right\} \to 0 \quad \text{if} \quad q \to \infty,
\]

\[
\mathbb{M}\left\{\left(\frac{\hat{A}_{T,t}^{(i_1 i_2)} - \hat{A}_{T,t}^{(i_1 i_2)}}{q}\right)^2\right\} \to 0 \quad \text{if} \quad q \to \infty.
\]

Let us address now to the convergence w. p. 1 for \(A_{T,t}^{(i_1 i_2)}\). First, note the well known fact.

**Lemma 1.** If for the sequence of random variables \(\xi_q\) and for some \(\alpha > 0\) the number series

\[
\sum_{q=1}^{\infty} \mathbb{M}\left\{|\xi_q|^\alpha\right\}
\]

converges, then the sequence \(\xi_q\) converges to zero w. p. 1.

From (43) and (45) \((n=2)\) we obtain

\[
\mathbb{M}\left\{\left(\frac{I_{T,t}^{(i_1 i_2)} - I_{T,t}^{(i_1 i_2)}}{q}\right)^4\right\} = \mathbb{M}\left\{\left(A_{T,t}^{(i_1 i_2)} - A_{T,t}^{(i_1 i_2)}\right)^4\right\} \leq \frac{K}{q^2},
\]

where constant \(K\) does not depend on \(q\).

Since the series

\[
\sum_{q=1}^{\infty} \frac{K}{q^2}
\]

converges, then according to Lemma 1 we obtain that \(A_{T,t}^{(i_1 i_2)} - A_{T,t}^{(i_1 i_2)}\) → 0 if \(q \to \infty\) w. p. 1. Then \(A_{T,t}^{(i_1 i_2)}\) → \(A_{T,t}^{(i_1 i_2)}\) if \(q \to \infty\) w. p. 1.

In addition, using (44) and (46) \((n=2)\), we get \(\hat{A}_{T,t}^{(i_1 i_2)}\) → \(\hat{A}_{T,t}^{(i_1 i_2)}\) if \(q \to \infty\) w. p. 1.

**References**

[1] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal “Differential Equations and Control Processes” ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: [http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html](http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html)

[2] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982.

[3] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995. 632 pp.

[4] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988. 225 pp.
[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004. 616 pp.

[6] Watanabe S. Levy's stochastic area formula and Brownian motion on compact Lie groups. In: Ikeda N., Watanabe S., Fukushima M., Kunita H. (Eds.) Itô's Stochastic Calculus and Probability Theory. Springer, Tokyo, 1996, pp. 401–412.

[7] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-227 Available at: http://www.sde-kuznetsov.spb.ru/06.pdf (ISBN 5-7422-1191-0)

[8] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 2nd Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2011, 284 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-233 Available at: http://www.sde-kuznetsov.spb.ru/11a.pdf (ISBN 978-5-7422-3162-2)

[9] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: Fourier-Legendre and trigonometric expansions, approximations, formulas. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2017), A.1–A.385. DOI: http://doi.org/10.18720/SPBPU/2/s17-3 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.1/article.2.1.html

[10] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Programs on MATLAB, 5th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2017), A.1-A.1000. DOI: http://doi.org/10.18720/SPBPU/2/s17-4 Available at: http://diffjournal.spbu.ru/EN/numbers/2017.2/article.2.1.html

[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MATLAB Programs, 6th Edition. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2018), A.1-A.1073. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.4/article.2.1.html

[12] Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Ito SDEs and Semilinear SPDEs, arXiv:2003.14184 [math.PR], 2020, 869 pp. [In English].

[13] Kuznetsov D.F. Strong Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), A.1-A.606. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.4/article.1.8.html

[14] Kuznetsov D.F. Mean-Square Approximation of Iterated Itô and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2011, XXX+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-230 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[15] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Matlab programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-228 Available at: http://www.sde-kuznetsov.spb.ru/07b.pdf (ISBN 5-7422-1394-8)

[16] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Matlab programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-229 Available at: http://www.sde-kuznetsov.spb.ru/07a.pdf (ISBN 5-7422-1439-1)

[17] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Matlab programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/09.pdf (ISBN 978-5-7422-2132-6)

[18] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With Matlab programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-231 Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2148-8)

[19] Kuznetsov D.F. Multiple stochastic Ito and Stratonovich integrals and multiple Fourier series. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2010), A.1-A.257. DOI: http://doi.org/10.18720/SPBPU/2/s17-7 Available at: http://diffjournal.spbu.ru/EN/numbers/2010.3/article.2.1.html

[20] Kuznetsov D.F. Strong Approximation of Multiple Ito and Stratonovich Stochastic Integrals: Multiple Fourier Series Approach. 1st Edition. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2021, 250 pp. DOI: http://doi.org/10.18720/SPBPU/2/s17-292 Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

[21] Kuznetsov D.F. Multiple Ito and Stratonovich stochastic integrals: approximations, properties, formulas. [In English]. Polytechnical University Publishing House, Saint-Petersburg, 2013, 382 pp.
Kuznetsov D.F. Development and application of the Fourier method for the numerical solution of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 58, 7 (2018), 1058-1070. DOI: http://doi.org/10.1134/S096554410800906

Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 1.5 and 2.0 orders of strong convergence [In English]. Automation and Remote Control, 79, 7 (2018), 1240-1254. DOI: http://doi.org/10.1134/S0005117918070060

Kuznetsov D.F. To numerical modeling with strong order 1.0, 1.5, and 2.0 of convergence for multidimensional dynamical systems with random disturbances. [arXiv:1802.00888] [math.PR], 2018, 28 pp. [In English].

Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

Kuznetsov D.F. Expansion of multiple iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers2018.1/article.1.1.html

Kuznetsov D.F. Application of the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [arXiv:1905.03724] [math.GM], 2019, 41 pp. [In English].

Kuznetsov D.F. Application of the method of approximation of iterated stochastic Itô integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3, (2019), 18-62. Available at: http://diffjournal.spbu.ru/EN/numbers2019.3/article.1.2.html

Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Itô stochastic differential equations. [arXiv:1901.02345] [math.GM], 2019, 40 pp. [In English]

Kuznetsov D.F. Expansion of multple iterated Stratonovich stochastic integrals of second multiplicity, based on double Fourier-Legendre series summarized by Pringsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers2018.1/article.1.1.html

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 1 to 6 from the Taylor-Itô and Taylor-Stratonovich expansions using Legendre polynomials. [arXiv:1801.00231] [math.PR]. 2017, 105 pp. [In English].

Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [arXiv:1801.03195] [math.PR]. 2022, 104 pp. [In English].

Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Itô stochastic integrals based on generalized multiple Fourier series. [arXiv:1801.01079] [math.PR]. 2018, 67 pp. [In English].

Kuznetsov D.F. Mean-square approximation of iterated Itô and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Itô and Taylor-Stratonovich expansions using Legendre polynomials. [arXiv:1801.00231] [math.PR]. 2017, 105 pp. [In English].

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [arXiv:1801.03195] [math.PR]. 2022, 104 pp. [In English].

Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 5 and beyond. [arXiv:1712.09516] [math.PR]. 2022, 174 pp. [In English].

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 3 based on generalized multiple Fourier series converging in the mean: general case of series summation. [arXiv:1801.01564] [math.PR]. 2018, 64 pp. [In English].

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [arXiv:1801.01962] [math.PR]. 2018, 40 pp. [In English].

Kuznetsov D.F. Development and application of the Fourier method to the mean-square approximation of iterated Itô and Stratonovich stochastic integrals. [arXiv:1712.09095] [math.PR]. 2017, 46 pp. [In English].

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [arXiv:1801.00754] [math.PR]. 2018, 78 pp. [In English].

Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Itô stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series. [arXiv:1807.02190] [math.PR]. 2018, 42 pp. [In English].
Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. arXiv:1801.06501 [math.PR]. 2018, 37 pp. [In English].

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth multiplicity based on generalized multiple Fourier series. arXiv:1802.06643 [math.PR]. 2018, 94 pp. [In English].

Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.2/article.1.6.html

Kuznetsov D.F. Application of multiple Fourier-Legendre series to implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. arXiv:1912.02613 [math.PR]. 2019, 32 pp. [In English].

Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry (AMMAI-2020). MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20c.pdf

Kuznetsov D.F. Application of multiple Fourier-Legendre series to strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear stochastic partial differential equations. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2020), 129-162. Available at: http://diffjournal.spbu.ru/EN/numbers/2020.3/article.1.6.html

Kuznetsov D.F. Strong approximation of iterated Ito and Stratonovich stochastic integrals. Abstracts of talks given at the 4th International Conference on Stochastic Methods (Divnomorskoe, Russia, June 2-9, 2019), Theory of Probability and its Applications, 65, 1 (2020), 141-142. DOI: http://doi.org/10.1137/S0040585X19T898578

Kuznetsov D.F. Explicit one-step numerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. [In English]. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542520030100

Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Ito SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated Ito stochastic integrals from strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Ito SDEs with non-commutative noise based on the unified Taylor-Ito and Taylor-Stratonovich Expansions and multiple Fourier–Legendre series. arXiv:2009.14011 [math.PR]. 2020, 342 pp. [In English].

Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107.

Kloeden P.E., Platen E., Schurz H. Numerical solution of SDE through computer experiments. Berlin: Springer, 1994, 292 pp.

Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stoch. Anal. Appl., 10, 4 (1992), 431-441.

Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.

Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.

Rybakov K.A. Orthogonal expansion of multiple Ito stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

Kuznetsov D.F. Problems of the numerical analysis of Ito stochastic differential equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html
[62] Wiener N. Un problème de probabilités dénombrables. Bulletin de la Société Mathématique de France. 52 (1924), 569-578.

[63] Lévy P. Wiener’s random function and other Laplacian random functions. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. 1951, 171-187.

[64] Ito K., McKea H. Diffusion processes and their sample paths. Springer-Verlag, Berlin-Heidelberg-New York, 1965, 395 p.

[65] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Institute of Technology, 2006, 225 p.

[66] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: [http://doi.org/10.1615/JAutomatInfScien.v32.i12.80]

[67] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: [http://www.sde-kuznetsov.spb.ru/01b.pdf]

[68] Kuznetsov D.F. Approximation of iterated Ito stochastic integrals of the second multiplicity based on the Wiener process expansion using Legendre polynomials and trigonometric functions. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4, (2019), 32-52. Available at: [http://diffjournal.spbu.ru/EN/numbers/2019.4/article.1.2.html]

[69] Kuznetsov D.F. New representation of the Levy stochastic area based on Legendre polynomials. [In English]. arXiv:1807.00409v1 [math.PR]. 2018, 10 pp.

Dmitriy Feliksovich Kuznetsov
Peter the Great Saint-Petersburg Polytechnic University,
Polytechnicheskaya ul., 29,
195251, Saint-Petersburg, Russia
Email address: sde_kuznetsov@inbox.ru