Gaussian Fluctuations and Moderate Deviations of Eigenvalues in Unitary Invariant Ensembles

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Abstract

We study the limiting behavior of the \( k \)-th eigenvalue \( x_k \) of unitary invariant ensembles with Freud-type and uniform convex potentials. As both \( k \) and \( n - k \) tend to infinity, we obtain Gaussian fluctuations for \( x_k \) in the bulk and soft edge cases, respectively. Multidimensional central limit theorems, as well as moderate deviations, are also proved. This work generalizes earlier results in the GUE and unitary invariant ensembles with monomial potentials of even degree. In particular, we obtain the precise asymptotics of corresponding Christoffel–Darboux kernels as well.

Keywords
Gaussian fluctuations · Moderate deviation principle · Riemann–Hilbert approach · Unitary invariant ensembles

Mathematics Subject Classification (2010) 60B20 · 60F05 · 60F10

1 Introduction and Main Results

We are concerned with the unitary invariant ensemble of \( n \times n \) Hermitian matrices \( \mathcal{H}_n \) with the probability distribution defined by

\[
\mathbb{P}_n(dH) = C_n e^{-n Tr V(H)} dH, \quad H \in \mathcal{H}_n,
\]

(1.1)

where \( C_n \) is a normalization constant, \( V(x) \) is an external potential, which is real analytic and satisfies \( V(x)/\log(x^2 + 1) \to \infty \), as \( |x| \to \infty \), and \( dH \) stands for the Lebesgue measure on the algebraically independent entries of \( H \), i.e., \( dH = \prod_{1 \leq i < j \leq n} d\text{Re}H_{ij}d\text{Im}H_{ij} \prod_{i=1}^n dH_{ii} \).
It is well known (cf. [5]) that the distribution (1.1) induces a probability density function of the corresponding $n$ ordered real eigenvalues \( \{x_i\}_{i=1}^n \), $x_1 < \cdots < x_n$, given by

\[
R_{n,n}(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \exp \left( -n \sum_{i=1}^n V(x_i) \right),
\]  

(1.2)

where $Z_n$ is a normalization. In particular, the quadratic potential (i.e., $V(x) = 2x^2$) corresponds to the classical Gaussian unitary ensemble (GUE).

Unitary invariant ensembles have been extensively studied in literature. The physical significance of probability distribution (1.2) can be interpreted as a Gibbs measure for $n$ identical charged particles in $\mathbb{R}$, at the inverse temperature $\beta = 2$, with a logarithmic interaction and with an external potential $V$. The generalization to general inverse temperatures $\beta > 0$ is known as beta ensembles or log-gases. See, e.g., [17,19].

One remarkable global property is that the 1-point correlation function of (1.2) converges weakly to an equilibrium measure, which is the well-known semicircle law in the GUE (see [5,21]). Moreover, the dynamical interpretations of (1.2) and equilibrium measure are related closely to the generalized Dyson Brownian motion and McKean–Vlasov equation, respectively. We refer, e.g., to [4,26] for the GUE case and the recent work [22] for beta ensembles.

The main interests of this paper are concerned with local fluctuations, as well as moderate deviations, of the $k$-th eigenvalue $x_k$ of a matrix taken randomly from a unitary ensemble, when both $k$ and $n-k$ tend to infinity. Local fluctuations of the $k$-th eigenvalue $x_k$, in a general context, turn out to be universal.

On the one hand, when $k$ or $n-k$ is fixed, these fluctuations obey the celebrated Tracy–Widom distribution. We refer to [32] for the GUE, [6,25] for unitary invariant ensembles with Freud-type potentials and analytic potentials, respectively. See also [1] for general beta ensembles and [27] for orthogonal ensembles.

On the other hand, when both $k$ and $n-k$ tend to infinity, the $k$-th eigenvalue $x_k$ is asymptotically normally distributed in both the bulk and edge cases. This result was first proved by Gustavsson [20] for the GUE and later extended to various other matrix models. In the bulk case, see [34] for unitary invariant ensembles with monomial potentials, and [23] for more general potentials including uniform convex potentials. In the edge case, we refer to [1] for general beta ensembles with potentials independent of $n$. See also [30] for complex covariance matrices, [31] for Wigner Hermitian matrices, and [24] for real symmetric Wigner matrices.

Moreover, it is well known (see, e.g., [12,14,35]) that moderate deviations are related closely to central limit theorems. Recently, a moderate deviation principle was obtained for general determinantal point processes in [15, Theorem 1.4]. This result implies the moderate deviations of $k$-th eigenvalue of Wigner matrix in the bulk and edge cases ([15,16]), which is indeed another motivation of the present work.

Here we are mainly concerned with unitary ensembles with Freud-type weights or uniform convex potentials, that is,
(i) Freud-type potential,

\[(V(x) :=) V_n(x) = \frac{1}{n} Q(c_n x + d_n), \tag{1.3}\]

where \(Q(x) = \sum_{k=0}^{2m} q_k x^k, m \in \mathbb{N}^+, q_{2m} = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(\frac{2m+1}{2})}, m \geq 1, c_n = \frac{1}{2}(\beta_n - \alpha_n),
\]
\[d_n = \frac{1}{2}(\beta_n + \alpha_n), \text{ and } \alpha_n, \beta_n \text{ are the n-th Mhaskar–Rakhmanov–Saff numbers (see Sect. 2 for details).} \]

(ii) Uniform convex potential,

\[\inf_{x \in \mathbb{R}} V''(x) \geq c > 0 \tag{1.4}\]

for some \(c > 0\).

We obtain the Gaussian fluctuations of \(k\)-th eigenvalue in both the bulk and edge cases. Multi-dimensional central limit theorems of eigenvalues are obtained as well. These generalize earlier results in the GUE [20] and unitary invariant ensembles with monomial potentials [34]. Furthermore, the moderate deviations of \(k\)-th eigenvalue are obtained in both the bulk and edge cases as well, thereby generalizing the results in the GUE [15,16] to unitary invariant ensembles with Freud-type and uniform convex potentials.

The proof is mainly based on the central limit theorems in [3,28,29] and the moderate deviation principle in [15]. In particular, we apply the Riemann–Hilbert approach, developed in [7,8], to obtain the precise asymptotics of corresponding Christoffel–Darboux kernels, which enable us to obtain the asymptotics of expectation and variance for the counting statistic for an interval and would be also of independent interest.

After this paper was finished, we learned about the works [1,23] where the Gaussian fluctuations are proved in the edge and bulk cases, respectively, for uniform convex potentials (and also other general potentials or matrix models) based on different approaches. We would like to refer to [1,23] for more details.

Below we formulate the main results of this paper. Recall that the equilibrium measure \(\mu_V\) is the unique minimizer of variational problem

\[\mu_V = \arg \min_{\mu \in \mathcal{M}_1(\mathbb{R})} I_V(\mu). \tag{1.5}\]

where \(\mathcal{M}_1(\mathbb{R}) = \{\mu : \int_{\mathbb{R}} d\mu = 1\}, I_V\) is the Voiculescu free entropy defined by

\[I_V(\mu) = \iint \log |s - t|^{-1} d\mu(s) d\mu(t) + \int V(t) d\mu(t). \tag{1.6}\]

For Freud-type and uniform convex potentials, it is known (see [8, (4.17)], [7, (1.4),(1.5)]) that \(\mu_V\) has the density function \(\rho_V\) supported on \([b, a]\). With suitable scaling, we may assume that \(b = -1\) and \(a = 1\) without loss of generality. The density function \(\rho_V\) can be also characterized by the Euler–Lagrange equations below (cf. [8, (4.18), (4.19)], [7, (1.10), (1.11)])
\[2 \int \log |x - s| \rho_V(s) ds - V(x) = l, \ x \in [-1, 1], \quad (1.7)\]

\[2 \int \log |x - s| \rho_V(s) ds - V(x) \leq l, \ x \in \mathbb{R}/[-1, 1]. \quad (1.8)\]

The Gaussian fluctuation results are formulated in Theorems 1.1 and 1.2. As in [20], we use the notation \( k(n) \sim n^\theta \) to mean that \( k(n) = h(n)n^\theta \), where \( h \) is any function satisfying \( h(n)n^{-\epsilon} \to 0 \) and \( h(n)n^\epsilon \to \infty \) as \( n \to \infty \) for all \( \epsilon > 0 \).

**Theorem 1.1** (Bulk case) Consider the unitary invariant ensemble (1.1) with the Freud-type and uniform convex potential as in (1.3) and (1.4), respectively.

(i). Let \( G(s) = \int_{-1}^1 \rho_V(x) dx, -1 \leq s \leq 1 \), and \( t = t(k, n) = G^{-1}(k/n) \), where \( k = k(n) \in [cn, (1 - c)n] \), \( c \in (0, 1/2) \). Set

\[X_n := \frac{x_n - t}{\sqrt{\log n} \rho_V(t)}. \quad (1.9)\]

Then, \( X_n \to N(0, 1) \) in distribution, as \( n \to \infty \).

(ii). Let \( \{x_i\}_{i=1}^m \) be eigenvalues such that \( 0 < k_i - k_{i+1} \sim n^{\theta_i}, 0 < \theta_i \leq 1 \), and \( k_i \in [c_i n, (1 - c_i)n], c_i \in (0, 1/2) \). Set \( s_i = s_i(k_i, n) = G^{-1}(k_i/n) \) and

\[X_{i,n} := \frac{x_i - s_i}{\sqrt{\log n} \rho_V(s_i)}, \quad 1 \leq i \leq m.\]

Then, for any \( \xi_i \in \mathbb{R}, 1 \leq i \leq m \), as \( n \to \infty \),

\[\mathbb{P}_n[X_{1,n} \leq \xi_1, \ldots, X_{m,n} \leq \xi_m] \to \Phi_\Lambda(\xi_1, \ldots, \xi_m).\]

Here \( \Phi_\Lambda \) is the \( m \)-dimensional Normal distribution function with mean zero and the correlation matrix \( \Lambda, \Lambda_{i,i} = 1, 1 \leq i \leq m, \) and \( \Lambda_{i,j} = 1 - \max_{1 \leq k < j} \theta_k, 1 \leq i < j \leq m \).

**Theorem 1.2** (Edge case) Consider the unitary invariant ensemble (1.1) with the Freud-type and uniform convex potential as in (1.3) and (1.4), respectively.

(i). Let \( k \) be such that \( k \to \infty \) and \( k/n \to 0 \), as \( n \to \infty \). Set

\[Y_n := \frac{3\sqrt{2} \pi a_1^{2}}{2} x_n - \frac{\left[1 - \left(\frac{k}{a_1 n}\right)^{\frac{2}{3}}\right]}{\sqrt{\log k} \frac{1}{n^{\frac{2}{3}} k^{\frac{1}{5}}}}, \quad (1.10)\]

where for the Freud-type potential \( a_1 = \frac{2\sqrt{2}}{3\pi} \sum_{i=0}^{m-1} \frac{A_{m-1-i}}{A_{m}} \) with \( A_j = \prod_{i=1}^{j} \frac{2i-1}{2i} \), \( A_0 = 1, 1 \leq j \leq m \), while for the uniform convex potential \( a_1 = \frac{2\sqrt{2}}{3\pi} h(1) \) with \( h \) as in Lemma 6.1.

Then, \( Y_n \to N(0, 1) \) in distribution.
(ii). Let \( \{x_{ki}\}_{i=1}^{m} \) be eigenvalues such that \( k_1 \sim n^{\gamma}, 0 < \gamma < 1, \) and \( 0 < k_{i+1} - k_i \sim n^{\theta_i}, 0 < \theta_i < \gamma. \) For the Freud-type potential, we assume additionally that \( \theta > \gamma - \frac{1}{2m}. \)

Set

\[
Y_{i,n} := \frac{3\sqrt{2\pi}a_i^2}{2} \frac{x_{i,n-k_i}}{\sqrt{\log k_i/n^2}} \left[ 1 - \left( \frac{k_i}{a_i n} \right)^{\frac{3}{2}} \right], \; 1 \leq i \leq m.
\]

Then, for any \( \xi_i \in \mathbb{R}, 1 \leq i \leq m, \) as \( n \to \infty, \)

\[
\mathbb{P}_n [Y_{1,n} \leq \xi_1, \ldots, Y_{m,n} \leq \xi_m] \to \Phi_\Lambda (\xi_1, \ldots, \xi_m),
\]

where \( \Lambda \) is as in Theorem 1.1, but with \( \Lambda_{i,j} = 1 - \gamma^{-1} \max_{1 \leq k < j} \theta_k, 1 \leq i < j \leq m. \)

Remark 1.3 For the Freud-type potential, the additional condition \( \theta > \gamma - \frac{1}{2m} \) arises in the delicate estimate of remaining term of \( \tilde{a}(t) \) in Proposition 4.3. For more details see the proof of (4.18). In particular, when \( \gamma \leq \frac{1}{2m}, \) one can take any \( 0 < \theta < \gamma. \)

Remark 1.4 Theorems 1.1 and 1.2 generalize the results in the GUE [20] and the unitary invariant ensembles with monomial potentials [34]. After this work was finished, we learned about the works [1,23] which also obtained the Gaussian fluctuations in the edge and bulk cases, respectively, for the uniform convex potentials and also for other general potentials or matrix models. The proof presented below is different, based on the Riemann–Hilbert approach, and also gives the precise asymptotics of corresponding Christoffel–Darboux kernels (see Sect. 3 and Lemma 6.3).

Regarding the moderate deviations, we recall that a sequence of probability measures \( \{\mu_n\} \subseteq \mathbb{M}_1(\mathbb{R}) \) is said to satisfy the large deviation principle with speed \( s_n \to \infty \) and good rate function \( I : \mathbb{R} \to [0, \infty], \) if the level sets \( \{x \in \mathbb{R} : I(x) \leq c\} \) are compact for all \( c \in [0, \infty) \) and if for all Borel set \( A \) of \( \mathbb{R}, \)

\[
-\inf_{x \in A^o} I(x) \leq \liminf_{n \to \infty} \frac{1}{s_n} \log \mu_n(A) \leq \limsup_{n \to \infty} \frac{1}{s_n} \log \mu_n(A) \leq -\inf_{x \in \overline{A}} I(x),
\]

where \( A^o \) and \( \overline{A} \) denote the interior and closure of \( A, \) respectively. In that case, we simply say that \( \{\mu_n\} \) satisfies the LDP\( (s_n, I). \) We also say that a family of real-valued random variables satisfies the LDP\( (s_n, I) \) if the family of their laws does. In particular, if the deviation scale of random variables is between that of the law of large number and that of the central limit theorem, this sequence of random variables is said to satisfy the moderate deviation principle.

We set \( \tilde{a}(t) := (1-t)^{-\frac{3}{2}} \int_1^t \rho_V(x) \, dx \) for \( t \in [1 - \delta, 1] \) with \( \delta > 0 \) small enough. In particular, \( \tilde{a}(t) = a_1 (1 + o(1)), \) where \( a_1 \) is the constant as in Theorem 1.2.

Theorem 1.5 Consider the unitary invariant ensemble (1.1) with the Freud-type and uniform convex potential as in (1.3) and (1.4), respectively.
(i) (Bulk case) Let \( k = k(n) \in [cn, (1 - c)n] \) with \( c \in (0, 1/2) \) and \( t = t(k, n) = G^{-1}(k/n) \), where \( G \) is as in Theorem 1.1. Let \( X_n \) be as in (1.9).

Then, for any sequence \( \{\gamma_n\} \) such that \( 1 \ll \gamma_n \ll \sqrt{\log n} \), \( \{\gamma_n^{-1}X_n\} \) satisfies the LDP \((\gamma_n^2, x^2/2)\).

(ii) (Edge case) Let \( k \) be such that \( k \to \infty \) and \( k/n \to 0 \) as \( n \to \infty \). Set

\[
Y_n := \frac{3\sqrt{2\pi} (\tilde{a}(t))^{\frac{3}{2}}}{2} x_{n-k} - \frac{1 - \left( \frac{k}{\tilde{a}(t)n} \right)^{\frac{3}{2}}}{\frac{\sqrt{\log k}}{n^{\frac{3}{2}} k^{\frac{3}{2}}}} \gamma_n \xi.
\]

where \( t \) is the unique real number such that \( t = 1 - \left( \frac{k}{\tilde{a}(t)n} \right)^{\frac{3}{2}} + \frac{\sqrt{2}}{3\pi (\tilde{a}(t))^{\frac{3}{2}}} \frac{\sqrt{\log k}}{n^{\frac{3}{2}} k^{\frac{3}{2}}} \gamma_n \xi \).

Then, for any sequence \( \{\gamma_n\} \) such that \( 1 \ll \gamma_n \ll \sqrt{\log k} \), \( \{\gamma_n^{-1}Y_n\} \) satisfies the LDP \((\gamma_n^2, x^2/2)\).

Remark 1.6 The existence and uniqueness of \( t \) in Theorem 1.5 can be proved by using contraction mapping arguments on \([1 - \delta, 1]\) with \( \delta > 0 \) small enough. When \( k \sim n^\gamma \) and \( 0 < \gamma \leq \min\left(\frac{2}{5}, \frac{1}{m}\right) \), we can replace \( \tilde{a}(t) \) in (1.11) by the constant \( a_1 \) as in Theorem 1.2. See also Remark 4.4.

Remark 1.7 Theorem 1.5 is motivated by the works [15,16], where the moderate deviation principle of eigenvalues of Wigner matrices (including the GUE) was proved in the bulk and edge cases.

By virtue of the determinantal structure of unitary invariant ensembles, the proof of Theorems 1.1, 1.2 and 1.5 is mainly based on the central limit theorems in [3,28,29] and the moderate deviation principle in [15], which in turn rely on the asymptotical estimates of expectation and variance for the counting statistic for an interval. Such estimates actually can be derived by the analysis of corresponding Christoffel–Darboux kernels.

Unlike in [20,34], it is technically more involved to obtain the asymptotics of Christoffel–Darboux kernels \( \mathcal{K}_n(x, y) \) for the unitary invariant ensembles considered here, mainly due to the complicated formulations of equilibrium density functions (see (2.3) and (6.1)). As a matter of fact, when deriving the estimates of expectation, we have to obtain the asymptotics of \( \mathcal{K}_n(x, x) \) in the whole real line, not just in the interior of support \((-1, 1)\), because of the lack of symmetry \( \mathcal{K}_n(x, x) = \mathcal{K}_n(-x, -x) \). Moreover, for the estimates of variance, the straightforward computations as in [20, Lemma 2.3] and [34, Lemma 4.1] are no longer applicable here to obtain the asymptotics of kernels \( \mathcal{K}_n(x, y) \), \( x \neq y \).

The key idea to overcome these difficulties is to reformulate the Christoffel–Darboux kernels in terms of the solutions of Riemann–Hilbert problems (see (3.5) and (6.17)).

In [6–8] the steepest descent method, introduced by Deift and Zhou in [11], has been developed to obtain the asymptotics of solutions of Riemann–Hilbert problems and has been applied successfully to prove universality for a variety of statistical quantities arising in unitary invariant ensembles.
In view of the key identities (3.5) and (6.17), we employ here the Riemann–Hilbert approach to obtain the crucial asymptotic estimates of Christoffel–Darboux kernels, which indeed constitute the main part of present work and would be also of independent interests. Once these estimates obtained, the Gaussian fluctuations and moderate deviations can be proved by using similar arguments as in [20,34] and [15,16], respectively.

We would also like to mention that the growing variance statistics may be also derived by adjusting the arguments in [2].

The remainder of this article is organized as follows. Sections 2–5 are devoted to unitary invariant ensembles with Freud-type potentials. First, in Sect. 2 we briefly review the Riemann–Hilbert approach developed in [8], and then in Sect. 3 we prove the key asymptotic estimates of Christoffel–Darboux kernels. Section 4 mainly contains the proof of Gaussian fluctuations in Theorems 1.1 and 1.2. The precise asymptotics of expectations and variances are also given. Section 5 includes the proof of moderate deviations in Theorem 1.5. In Sect. 6 we treat unitary invariant ensembles with uniform convex potentials. For simplicity of exposition, some technical details are postponed to the “Appendix”.

**Notation** Throughout this article, \( \#I \) denotes the number of eigenvalues in the interval \( I \subseteq \mathbb{R} \). For two sequence of real numbers \( f_n \) and \( g_n \), \( n \geq 1 \), \( f_n = O(g_n) \) means that \( |f_n/g_n| \) stays bounded, and \( f_n \ll g_n \) means \( \lim_n f_n/g_n = 0 \). The notations \( C \) and \( c \) denote constants which may change from one line to another.

**2 Riemann–Hilbert Approach**

We start with the Freud-type potential (1.3). Let \( Q(x) = \sum_{j=0}^{2m} q_j x^j \), \( q_{2m} = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(2m+1)} \), \( m \geq 1 \). Define the n-th Mhasker–Rakhmanov–Saff numbers \( \alpha_n, \beta_n \) by

\[
\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(t - \alpha_n)}{\sqrt{(\beta_n - t)(t - \alpha_n)}} \, dt = n, \quad \frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(\beta_n - t)}{\sqrt{(\beta_n - t)(t - \alpha_n)}} \, dt = -n. \tag{2.1}
\]

It follows from [8, Proposition 5.2] that \( \alpha_n \) and \( \beta_n \) exist for \( n \) large enough and can be expressed in a power series in \( n^{-\frac{1}{2m}} \). Set

\[
(V(x) :=) V_n(x) = \frac{1}{n} Q(c_n x + d_n)
\]

with \( c_n = \frac{1}{2}(\beta_n - \alpha_n), \ d_n = \frac{1}{2}(\beta_n + \alpha_n) \). We have that ([8, (5.17), (5.18)]), \( V_n = \sum_{k=0}^{2m} v_{n,k} x^k \in \mathbb{P}_{2m}^+ \), where

\[
v_{n,2m} = \frac{1}{m A_m} + O\left(n^{-\frac{1}{m}}\right), \quad v_{n,k} = O\left(n^{\frac{k}{2m}-1}\right), \quad 0 \leq k \leq 2m - 1, \tag{2.2}
\]

and \( A_m = \prod_{j=1}^{m} \frac{2j-1}{2j} \).
**Remark 2.1** When \( Q \) is the monomial polynomial of even degree as in [34], we have that \( \beta_n = -\alpha_n = n^{1/m} \). Hence, \( c_n = n^{1/m}, d_n = 0 \) and \( V_n \equiv Q \).

We have the following formula for the equilibrium density function.

**Theorem 2.2** ([8, Proposition 5.3]) There exists \( N > 0 \), such that for all \( n \geq N \),

\[
\rho_{V_n}(x) = \frac{1}{2\pi} \sqrt{1 - x^2} h_n(x) \chi_{[-1,1]}(x),
\]

where

\[
h_n(x) = \sum_{k=0}^{2m-2} h_{n,k} x^k, \quad h_{n,k} = \sum_{j=0}^{\left\lfloor \frac{2m-2-k}{2} \right\rfloor} A_j(k + 2 + 2j) v_{n,k+2+2j}.
\]

Furthermore, for some \( h_0 > 0 \), \( h_n(x) > h_0 \) for all \( n \geq N \) and \( x \in \mathbb{R} \).

**Lemma 2.3** Let \( \delta \in (0, 1) \) and \( N \) be as in Theorem 2.2. Then \( 1/\rho_{V_n} \) and \( |\rho_{V_n}'| \) are uniformly bounded for all \( n \geq N \) and \( x \in [-1+\delta, 1-\delta] \).

(See the “Appendix” for the proof.)

Below we assume that \( n \) is large enough such that Theorem 2.2 holds. Set

\[
F_n(x) := \left| \int_x^1 \frac{1}{2\pi} \sqrt{1 - y^2} h_n(y) dy \right|, \quad \tilde{F}_n(x) := \left| \int_{-1}^x \frac{1}{2\pi} \sqrt{1 - y^2} h_n(y) dy \right|.
\]

Then, \( F_n(x) = \int_x^1 \rho_{V_n}(y) dy, x \in (-1, 1) \), and \( F_n(x) = \tilde{F}_n(-x) \) if \( Q(x) = Q(-x) \).

The \( j \)-th orthogonal polynomials \( p_j(x) \) and the Christoffel–Darboux kernels \( K_j(x, y) \) with respect to the weight \( e^{-Q(x)} \) are defined by

\[
p_j(x) = \gamma_j x^j + \cdots, \quad \gamma_j > 0, \quad j \geq 0,
\]

\[
\int p_i(x) p_j(x) e^{-Q(x)} dx = \delta_{ij}, \quad i, j \geq 0,
\]

\[
K_j(x, y) = \sum_{i=0}^{j-1} p_i(x) p_i(y) e^{-\frac{Q(x)+Q(y)}{2}}, \quad j \geq 1.
\]

For the scaled weight \( e^{-nV_n(x)} (= e^{-Q(c_n x+d_n)}) \), we define the \( j \)-th orthogonal polynomials \( p_j(x; n) \) and the corresponding kernels \( \tilde{K}_j(x, y) \) similarly as follows

\[
p_j(x; n) = \gamma_j^{(n)} \pi_j(x; n),
\]

\[
\int p_i(x; n) p_j(x; n) e^{-nV_n(x)} dx = \delta_{ij}, \quad i, j \geq 0,
\]

\[
\tilde{K}_j(x, y) = \sum_{i=0}^{j-1} p_i(x; n) p_i(y; n) e^{-n \frac{V_n(x)+V_n(y)}{2}}, \quad j \geq 1,
\]
where $\gamma_j^{(n)} > 0$ and $\pi_j(x; n)$ are monic polynomials. It is straightforward to verify

$$p_i(x; n) = \sqrt{c_n} p_i(c_n x + d_n), \quad (2.9)$$

$$\gamma_i^{(n)} = c_n \gamma_i, \quad i \geq 0, \quad (2.10)$$

$$\mathcal{K}_n(x, y) = c_n K_n(c_n x + d_n, c_n y + d_n). \quad (2.11)$$

Below we recall the Riemann–Hilbert problem and the steepest descent method, which was introduced by Deift and Zhou in [11] and later developed in [7–10] to analyze the asymptotics of the solutions of Riemann–Hilbert problem.

Let $U : \mathbb{C}/\mathbb{R} \to \mathbb{C}^{2 \times 2}$ be an analytic matrix-valued function, which solves the Riemann–Hilbert problem,

$$U_+(s) = U_-(s) \begin{pmatrix} 1 & e^{-n V_n(s)} \\ 0 & 1 \end{pmatrix}, \quad s \in \mathbb{R},$$

$$U(z) \begin{pmatrix} z^{-n} \\ z^n \end{pmatrix} = I + O\left(\frac{1}{|z|}\right), \quad \text{as } |z| \to \infty.$$  

The fundamental relation between the solutions of Riemann–Hilbert problem and the orthogonal polynomials, observed by Fokas, Its and Kitaev [18], is that

$$U_{11}(z) = \frac{1}{\gamma_n^{(n)}} p_n(z; n), \quad U_{21}(z) = -2\pi i \gamma_n^{(n)} p_n(z; n). \quad (2.12)$$

Set

$$g_n(z) := \int_{-1}^{1} \psi_n(t) \log(z - t) dt, \quad z \in \mathbb{C}/(-\infty, 1],$$

where

$$\psi_n(z) = \frac{1}{2\pi} (1 - z)^{\frac{1}{2}} (1 + z)^{\frac{1}{2}} h_n(z), \quad z \in \mathbb{C}/((-\infty, -1] \cup [1, \infty)) \quad (2.13)$$

with the analytic branch chosen by $\arg(1 - x) = \arg(1 + x) = 0, \quad x \in (-1, 1)$. Let

$$\xi_n(z) := -2\pi i \int_{1}^{z} \psi_n(y) dy, \quad z \in \mathbb{C}/(-\infty, -1] \cup [1, \infty).$$

We have that ([8, (8.29)])

$$g_n(z) = \frac{1}{2} (V_n(z) + l_n + \xi_n(z)), \quad z \in \mathbb{C}^+. \quad (2.14)$$

where $l_n$ is same as $l$ in (1.7) with $V_n$ replacing $V$ there.
Using the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we set

$$
T(z) := e^{-\frac{ln}{2} \sigma_3} U(z) e^{-\frac{ln}{2} (\xi_n(z) - \frac{ln}{2}) \sigma_3}, \quad z \in \mathbb{C}/\mathbb{R}.
$$

(2.15)

and

$$
S(z) :=
\begin{cases}
T(z), & \text{outside the lens-shaped region;} \\
T(z) \begin{pmatrix} 1 & 0 \\ -e^{-ln \xi_n} & 1 \end{pmatrix}, & \text{in the upper lens region;} \\
T(z) \begin{pmatrix} 1 & 0 \\ e^{ln \xi_n} & 1 \end{pmatrix}, & \text{in the lower lens region},
\end{cases}
$$

(2.16)

with the lens regions as in [8, fig. 6.1].

Next, we recall the delicate paramatrices $P_n$ in the small balls $U_{\pm 1}$ centered on $\pm 1$ with the radius $\delta$ sufficiently small, respectively. Define the functions $f_n$ and $\tilde{f}_n$ in $U_1$ and $U_{-1}$, respectively, by

$$
(-f_n(z))^3 = -\frac{3\pi}{2} \int_1^z \psi_n(y) dy, \quad z \in U_1/(1, \infty),
$$

(2.17)

$$
(\tilde{f}_n(z))^3 = \frac{3\pi}{2} \int_{-1}^z \psi_n(y) dy, \quad z \in U_{-1}/(-\infty, -1).
$$

(2.18)

We have that (see (7.14), (7.21), (7.38), (7.36) and (7.37) in [8]),

$$
\frac{2}{3} (f_n(z))^3 = n \varphi_n(z), \quad \text{or,} \quad f_n(z) = n^{\frac{2}{3}} (z - 1)(\tilde{\varphi}_n(z))^3,
$$

(2.19)

$$
\frac{2}{3} (-\tilde{f}_n(z))^3 = n \tilde{\varphi}_n(z), \quad \text{or,} \quad \tilde{f}_n(z) = n^{\frac{2}{3}} (z + 1)(\tilde{\varphi}_n(z))^3,
$$

(2.20)

where

$$
\varphi_n(z) = \begin{cases}
-\frac{i}{2} \xi_n(z) = \pi i \int_1^z \psi_n(y) dy, & z \in \mathbb{C}^+; \\
\frac{i}{2} \xi_n(z) = -\pi i \int_1^z \psi_n(y) dy, & z \in \mathbb{C}^-;
\end{cases}
$$

(2.21)

$$
\tilde{\varphi}_n(z) = \begin{cases}
\varphi_n(z) + \pi i = \pi i \int_1^z \psi_n(y) dy, & z \in \mathbb{C}^+; \\
\varphi_n(z) - \pi i = -\pi i \int_{-1}^z \psi_n(y) dy, & z \in \mathbb{C}^-;
\end{cases}
$$

(2.22)

and $\tilde{\varphi}_n, \tilde{\varphi}_n$ are analytic functions in $U_1$ and $U_{-1}$, respectively.

The paramatrices $P_n$ in $U_{\pm 1}$ are defined as follows.

(i). In the region $U_1/f_n^{-1}(\gamma_\sigma)$ with the contour $\gamma_\sigma$ as in [8, fig. 7.1], set

$$
P_n := E_n \Psi^\sigma (f_n) e^{ln \varphi_n \sigma_3},
$$

(2.23)
where \( E_n = \sqrt{\frac{\pi}{2}} e^{\frac{\pi i}{6}} \left( \frac{1}{i} - 1 \right) \left( \frac{H_n}{H_n^{-1}} \right) \), \( H_n = f_n a^{-1} \), and

\[
\Psi^\sigma(z) = \begin{cases} 
AI(z) e^{-\frac{\pi i}{6} \sigma_3}, & z \in I : 0 < \arg z < \frac{2\pi}{3}; \\
AI(z) e^{-\frac{\pi i}{6} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\
-1 & 1 \end{array} \right), & z \in II : \frac{2\pi}{3} < \arg z < \pi; \\
\tilde{AI}(z) e^{-\frac{\pi i}{6} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\
1 & 1 \end{array} \right), & z \in III : -\pi < \arg z < -\frac{2\pi}{3}; \\
\tilde{AI}(z) e^{-\frac{\pi i}{6} \sigma_3}, & z \in IV : -\frac{2\pi}{3} < \arg z < 0.
\end{cases}
\]

(2.24)

Here, \( AI(z) \) and \( \tilde{AI}(z) \) denote \( \left( \begin{array}{cc} AI(z) & Ai(\omega^2 z) \\
Ai'(z) & \omega^2 Ai'(\omega^2 z) \end{array} \right), \left( \begin{array}{cc} AI(z) & -\omega^2 Ai(\omega z) \\
Ai'(z) & -Ai'(\omega z) \end{array} \right) \), respectively, \( \omega = e^{\frac{2\pi i}{3}} \), and \( Ai \) is the Airy function, uniquely determined by the equation \( Ai''(z) = zAi(z) \) with \( \lim_{x \to \infty} \sqrt{4\pi x^2 e^\frac{\pi i}{6}} Ai(x) = 1 \).

(ii). In the region \( U_{-1} / f_{n}^{-1}(\gamma_\sigma) \) with the contour \( \gamma_\sigma \) as in [8, fig. 7.3], set

\[
P_n := \tilde{E}_n \tilde{\Psi}^\sigma(f_n) e^{i\tilde{\phi}_n \sigma_3}
\]

(2.25)

with \( \tilde{\Psi}^\sigma(z) = \sigma_3 \Psi^\sigma(-z) \sigma_3, \tilde{E}_n = \sqrt{\frac{\pi}{2}} e^{\frac{\pi i}{6}} \left( \frac{1}{i} - 1 \right) \left( \frac{\tilde{H}_n}{\tilde{H}_n^{-1}} \right), \tilde{H}_n = (-f_n)^{\frac{1}{4}} a. \)

Finally, set

\[
R := \begin{cases} 
SP_n^{-1}, & \text{for } z \in U_1 \cup U_{-1}; \\
SN^{-1}, & \text{otherwise},
\end{cases}
\]

(2.26)

where

\[
N = \frac{1}{2} \left( \begin{array}{cc} a + a^{-1} & i(a^{-1} - a) \\
i(a - a^{-1}) & a + a^{-1} \end{array} \right),
\]

(2.27)

and

\[
a(z) = \left( \frac{z - 1}{z + 1} \right)^{\frac{1}{4}}, \quad z \in \mathbb{C}/[-1, 1]
\]

(2.28)

with the analytic branch chosen by \( \arg(x - 1) = \arg(x + 1) = 0 \), for \( x > 1 \). We have the asymptotic expansions of \( R \) below (see [8, (7.64)], [6, (3.6), (3.7)]),

\[
R(z) = I + \frac{1}{n} \sum_{k=0}^{\infty} r_k(z)n^{-\frac{k}{2m}}, \quad \frac{d}{dz} R(z) = \frac{1}{n} \sum_{k=0}^{\infty} \frac{d}{dz} r_k(z)n^{-\frac{k}{2m}},
\]

(2.29)
where $r_k(z), \frac{d}{dz}r_k(z), 0 \leq k < \infty$, are bounded functions and analytic in the complement of set $\partial U_1 \cup \partial U_{-1}$, and these expansions are uniform for $z \in \mathbb{C} \setminus \hat{\Sigma}_R$ with $\hat{\Sigma}_R$ as in [8, fig. 7.6].

**Remark 2.4** If $z = x \in \mathbb{R}$, we take the limiting expressions as $z$ is approaching from the upper half-plane. Thus, if $x > 1$, $\psi_n(x)$ means $\lim_{\epsilon \to 0^+} \psi_n(x + i\epsilon)$.

### 3 Asymptotics of Christoffel–Darboux Kernels

This section is mainly devoted to the asymptotics of Christoffel–Darboux kernels corresponding to Freud-type potentials.

**Lemma 3.1** Take any sufficiently small $\delta > 0$, we have

(i). For $x \in (-1 + \delta, 1 - \delta),$

$$\mathcal{K}_n(x, x) = n\rho V_n(x) + \mathcal{O}(1).$$

(ii). For $x \in (1 - \delta, 1 + \delta),$

$$\mathcal{K}_n(x, x) = \left[ \frac{1}{4} \frac{f_n'(x)}{f_n(x)} - \frac{a'(x)}{a(x)} \right] 2Ai(f_n(x))Ai'(f_n(x)) + f_n'(x) \left[ (Ai')^2(f_n(x)) - f_n(x)Ai^2(f_n(x)) \right] + \mathcal{O}(n^{-\frac{5}{6}}).$$

(iii). For $x \in (-1 - \delta, -1 + \delta),$

$$\mathcal{K}_n(x, x) = -\left[ \frac{1}{4} \frac{\tilde{f}_n'(x)}{\tilde{f}_n(x)} + \frac{a'(x)}{a(x)} \right] 2Ai(-\tilde{f}_n(x))Ai'(-\tilde{f}_n(x)) + \tilde{f}_n'(x) \left[ (Ai')^2(-\tilde{f}_n(x)) + \tilde{f}_n(x)Ai^2(-\tilde{f}_n(x)) \right] + \mathcal{O}(n^{-\frac{5}{6}}).$$

(iv). For $x \in \mathbb{R}/(-1 - \delta, 1 + \delta),$

$$\mathcal{K}_n(x, x) = \frac{1}{4\pi} \frac{1}{(x - 1)(x + 1)} e^{-2n\varphi_n(x)} + \mathcal{O}(n^{-1}).$$

**Proof** First by the Christoffel–Darboux formula (cf. [5, (3.48)]) and (2.12),

$$2\pi i(x - y)\mathcal{K}_n(x, y) = (1, 0)U(x)^T U(y)^{-T} (0, 1)^T e^{-n \frac{|V_n(x) + V_n(y)|}{2}}.$$
(i). For \( x, y \in (-1 + \delta, 1 - \delta) \), by (2.15) and (2.16),
\[
U = e^{\frac{n}{2} I_n \sigma_3 S} \begin{pmatrix} 1 & 0 \\ e^{-n \xi_n} & 1 \end{pmatrix} e^{n \left(g_{n} - \frac{\pi}{2}\right) \sigma_3}.
\]

Then, by (3.5), (2.14) and (2.21), direct computations show that
\[
2\pi i (x - y) \mathcal{K}_n(x, y) = (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)}) S(x)^T S(y)^{-T} (-e^{n\varphi_n(y)}, e^{-n\varphi_n(y)})^T.
\]

In order to obtain the leading term of the right-hand side above, we note that
\[
S^T(x) = S^T(y) + (x - y) \Delta_S(x, y),
\]
where
\[
\Delta_S(x, y) = \int_0^1 (S^T)'(y + t(x - y)) dt.
\]

This yields that
\[
S^T(x) S^{-T}(y) = Id + (x - y) \Delta_S(x, y) S^{-T}(y).
\]

Then, plugging (3.9) into (3.7), since \( \varphi_n(x) = -\pi i F_n(x), x \in (-1, 1) \), we obtain
\[
2\pi i (x - y) \mathcal{K}_n(x, y) = -2i \sin[n\pi (F_n(x) - F_n(y))] + (x - y) I_1(x, y),
\]
where \( I_1(x, y) := (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)}) [\Delta_S(x, y) S^{-T}(y)] (-e^{n\varphi_n(y)}, e^{-n\varphi_n(y)})^T. \)

Thus, taking \( y = x \) we get
\[
2\pi i \mathcal{K}_n(x, x) = 2\pi i n \rho \varphi_n(x) + (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)}) \left[(S^T)'(x) S^{-T}(x)\right] (-e^{n\varphi_n(x)}, e^{-n\varphi_n(x)})^T.
\]

In view of (2.26)–(2.29), \( S(x) \) and \( S'(x) \) are uniformly bounded for \( x \in [-1 + \delta, 1 - \delta] \), and hence, (3.1) follows.

(ii). For \( x, y \in (1 - \delta, 1) \), or \( x, y \in (1, 1 + \delta) \), similar calculations show that
\[
2\pi i (x - y) \mathcal{K}_n(x, y) = e^{-\frac{n\pi i}{2}} (1, 0) [AI(f_n(x))]^T E_n^T(x) R^T(x) \cdot R^{-T}(y) E_n^{-T}(y) [AI(f_n(y))]^{-T} (0, 1)^T.
\]

(see the “Appendix” for the proof.)

Regarding the leading term of the right-hand side above, using (3.9) with \( S \) replaced by \( R \), we obtain
\[
2\pi i (x - y) \mathcal{K}_n(x, y)
\]
\[
= e^{-\frac{n\pi i}{2}} (1, 0) [AI(f_n(x))]^T E_n^T(x) E_n^{-T}(y) [AI(f_n(y))]^{-T} (0, 1)^T + (x - y) e^{-\frac{n\pi i}{2}} I_2(x, y),
\]
\[
(3.12)
\]
where

\[
I_2(x, y) := (1, 0)[A I(f_n(x))]^T E_n^T(x) \\
\cdot \Delta_R(x, y) R_n^T(y) E_n^{-T}(y)[A I(f_n(y))]^{-T}(0, 1)^T.
\]

Then, using the expressions of \( AI, E_n \) and the asymptotics (2.29), we have that \( I_2(x, y) \) is of order \( n^{-\frac{5}{6}} \) and

\[
2\pi i(x - y) K_n(x, y) = (-2\pi i) \left[ -\text{Ai}^2(f_n(x)) + \frac{\text{Ai}'(f_n(x))}{\text{Ai}(f_n(x))} f_n^2(x) a(x) \right] + O((y - x)^2) + (x - y) O \left( n^{-\frac{5}{6}} \right).
\]

The proof is postponed to the “Appendix”.

Hence, taking the Taylor expansion and using \( \text{Ai}''(x) = x \text{Ai}(x) \) we obtain

\[
2\pi i(x - y) K_n(x, y) = -2\pi i(y - x) \left[ \frac{1}{4} f_n'(x) \right] 2\text{Ai}(f_n(x)) \text{Ai}'(f_n(x)) \]

\[
- f_n(x) f_n'(x) \text{Ai}^2(f_n(x)) + f_n'(x) (\text{Ai}')^2(f_n(x)) \right] + O((y - x)^2) + (x - y) O \left( n^{-\frac{5}{6}} \right),
\]

which implies (3.2).

(iii). For \( x, y \in (-1 - \delta, -1), \text{ or } x, y \in (-1, -1 + \delta) \), the proofs are similar to those in the previous case. First we compute that

\[
2\pi i(x - y) K_n(x, y) = (-1)e^{-\frac{\pi i}{2}} (1, 0) [\widetilde{A I}(-\widetilde{f}_n(x))]^T \sigma_3 \widetilde{E}_n^T(x) R^T(x) \\
\cdot R_n^T(y) \widetilde{E}_n^{-T}(y) \sigma_3^{-1} [\widetilde{A I}(-\widetilde{f}_n(y))]^{-T}(0, 1)^T.
\]

(See the “Appendix” for the proof.)

Then, using (3.9) with \( S \) replaced by \( R \) we have

\[
2\pi i(x - y) K_n(x, y) = (-1)e^{-\frac{\pi i}{2}} (1, 0) [\widetilde{A I}(-\widetilde{f}_n(x))]^T \sigma_3 \widetilde{E}_n^T(x) \widetilde{E}_n^{-T}(y) \sigma_3^{-1} [\widetilde{A I}(-\widetilde{f}_n(y))]^{-T}(0, 1)^T \\
- e^{-\frac{\pi i}{2}} (x - y) I_3(x, y),
\]

\( \Theta \) Springer
where

\[
I_3(x, y) := (1, 0)[\widetilde{AI}(\tilde{f}_n(x))]^T \sigma_3 \widetilde{E}_n^2(x) \\
\cdot \Delta_R(x, y) R^{-T}(y) \widetilde{E}_n^2(y) \sigma_3^{-1}[\widetilde{AI}(\tilde{f}_n(y))]^{-T}(0, 1)^T.
\]

(3.17)

Using the expressions of \(\widetilde{AI}, \tilde{f}_n\) and arguing as in the case (\(ii\)), we get that

\[
2\pi i(x - y) \mathcal{K}_n(x, y) = (-2\pi i) \left[ A_i(-\tilde{f}_n(x)) A_i'(-\tilde{f}_n(y)) \frac{\tilde{f}_n(x)^{\frac{1}{2}} a(x)}{\tilde{f}_n(y)^{\frac{1}{2}} a(y)} \\
- A_i'(-\tilde{f}_n(x)) A_i(-\tilde{f}_n(y)) \frac{\tilde{f}_n(y)^{\frac{1}{2}} a(y)}{\tilde{f}_n(x)^{\frac{1}{2}} a(x)} \right] + (x - y) O \left( n^{-\frac{5}{6}} \right).
\]

(3.18)

Consequently, the Taylor expansion yields that

\[
2\pi i(x - y) \mathcal{K}_n(x, y) = -2\pi i(y - x) \left\{ - \left[ \frac{1}{4} \tilde{f}_n^{-1}(x) \tilde{f}_n'(x) + \frac{a'(x)}{a(x)} \right] 2 A_i(-\tilde{f}_n(x)) A_i'(-\tilde{f}_n(x)) \\
+ \tilde{f}_n'(x) \left[ \tilde{f}_n(x) A_i^2(-\tilde{f}_n(x)) + (A_i')^2(-\tilde{f}_n(x)) \right] \right\} \\
+ O(y - x)^2 + (x - y) O \left( n^{-\frac{5}{6}} \right),
\]

which implies (3.3).

\(iv\). For \(x, y \in \mathbb{R}/(-1 - \delta, 1 + \delta)\), by (2.15) and (2.16),

\[
U = e^{n \frac{\ln \delta}{2}} \sigma_3 S e^{n (g_n - \frac{\ln n}{2})} \sigma_3,
\]

which along with (3.5) and (2.14) implies that

\[
2\pi i(x - y) \mathcal{K}_n(x, y) = e^{-n(\varphi_n(x) + \varphi_n(y))} (1, 0) S^T(x) S^{-T}(y)(0, 1)^T.
\]

Then, using \(S = R N\) and applying (3.9) twice with \(S\) replaced by \(R\) and \(N\), respectively, we obtain

\[
2\pi i(x - y) \mathcal{K}_n(x, y) = (x - y) e^{-n(\varphi_n(x) + \varphi_n(y))} (1, 0) \Delta_N(x, y) N^{-T}(y)(0, 1)^T \\
+ (x - y) I_4(x, y),
\]

(3.19)

where

\[
I_4(x, y) := e^{-n(\varphi_n(x) + \varphi_n(y))} (1, 0) N^T(x) \Delta_R(x, y) R^{-T}(y) N^{-T}(y)(0, 1)^T.
\]
Taking into account (2.29) and that \( N(x) \) and \( e^{-n\varphi_n(x)} \) are bounded for \( x \in \mathbb{R}/(-1 - \delta, 1 + \delta) \), we obtain that \( I_4(x, y) = \mathcal{O}(n^{-1}) \). Hence,

\[
\mathcal{K}'(x, y) = \frac{1}{2\pi i} e^{-n(\varphi_n(x) + \varphi_n(y))} (1, 0) \Delta_N(x, y) N^{-T}(y)(0, 1)^T + \mathcal{O}(n^{-1}),
\]

and

\[
\mathcal{K}(x, x) = \frac{1}{2\pi i} e^{-2n\varphi_n(x)} (1, 0) (N^T)'(x) N^{-T}(x)(0, 1)^T + \mathcal{O}(n^{-1}),
\]

which consequently yields (3.4) by (2.27) and (2.28). The proof is complete. \( \square \)

Lemmas 3.2 and 3.3 are concerned with the asymptotics of kernels \( \mathcal{K}_n(x, y) \), \( x \neq y \), in the bulk and edge cases, respectively.

**Lemma 3.2** (i). Let \( t \in (-1, 1) \) and set \( \Gamma^1 := \{(x, y) : t \leq x \leq t + \frac{1 - t}{\log n}, t - \frac{1 + t}{\log n} \leq y \leq t - \frac{1}{n}\} \). Then, for \( (x, y) \in \Gamma^1 \),

\[
\mathcal{K}_n(x, y) = \frac{\sin[\pi n(F_n(y) - F_n(x))] + \mathcal{O}(1)}{\pi(x - y)}, \tag{3.20}
\]

where \( F_n(x) \) is defined as in (2.5).

(ii). Let \( \delta > 0 \) be sufficiently small. Then, for \( x, y \in [-1 + \delta, 1 - \delta] \),

\[
\mathcal{K}_n^2(x, y) = \mathcal{O}\left(\frac{1}{(x - y)^2}\right). \tag{3.21}
\]

(iii). Let \( t \in (-1, 1) \). Then, for \( (x, y) \in \{(x, y) : t \leq x \leq t + \frac{1}{n}, t - \frac{1}{n} \leq y \leq t\} \),

\[
\mathcal{K}_n(x, y) = \mathcal{O}(n). \tag{3.22}
\]

**Proof** By (3.10), we have

\[
2\pi i(x - y) \mathcal{K}_n(x, y) = -2i \sin[n\pi(F_n(x) - F_n(y))] + \mathcal{O}(|x - y|), \tag{3.23}
\]

which immediately implies (3.20) and (3.21). As regards (iii), by (3.23),

\[
\mathcal{K}_n(x, y) = (-1)^{\sin[n\pi(F_n(x) - F_n(y))]}} + \mathcal{O}(1)
\]

\[
= (-1)^{\sin[n\pi(F_n(x) - F_n(y))]n(F_n(x) - F_n(y))} + \mathcal{O}(1).
\]

Since \( \sup_{x \in \mathbb{R}} |\sin x| = \mathcal{O}(1) \), and by Lemma 2.3, for some \( \xi \in (x, y) \),

\[
\frac{n(F_n(x) - F_n(y))}{x - y} = \frac{nF_n'(\xi)(x - y)}{x - y} = -n\rho_{\nu_n}(\xi) = \mathcal{O}(n),
\]

we obtain (3.22) and finish the proof. \( \square \)
Lemma 3.3 For $x \in (1 - \delta, 1 + \delta)$, $y \in (1 - \delta, 1)$ with $\delta > 0$ sufficient small,

\[
\mathcal{K}_n(x, y) = \frac{1}{x - y} \left[ Ai(f_n(x))Ai'(f_n(y)) \frac{f_n^4(x)}{f_n^4(y)} \frac{a(y)}{a(x)} - Ai'(f_n(x))Ai(f_n(y)) \frac{f_n^4(y)}{f_n^4(x)} \frac{a(x)}{a(y)} \right] + \mathcal{O}(n^{-\frac{3}{2}}). \tag{3.24}
\]

Proof In view of (3.14), we only need to prove (3.24) for $x \in (1, 1 + \delta)$ and $y \in (1 - \delta, 1)$. For $x \in (1, 1 + \delta)$, by (2.15), (2.16), (2.23) and (2.26),

\[
U(x) = e^{n \frac{ln}{2} \sigma_3} S(x) e^n (g_n - \frac{ln}{2}) \sigma_3, \tag{3.25}
\]

where $S(x) = R(x) En(x) [\text{AI}(f_n(x))] e^{-\frac{n}{6} \sigma_3} e^{n \rho_n \sigma_3}$. Moreover, for $y \in (1 - \delta, 1)$,

\[
U(y) = e^{n \frac{ln}{2} \sigma_3} S(y) \begin{pmatrix} 1 & 0 \\ -n \xi_n & 1 \end{pmatrix} e^n (g_n - \frac{ln}{2}) \sigma_3 \tag{3.26}
\]

with $S(y) = R(y) En(y) [\text{AI}(f_n(y))] e^{-\frac{n}{6} \sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{n \rho_n \sigma_3}$.

Then, plugging (3.25) and (3.26) into (3.5) we obtain

\[
2\pi i (x - y) \mathcal{K}_n(x, y) = e^{-\frac{n}{3} (1, 0) [\text{AI}(f_n(x))]^T E^T_n(x) R^T(x)} \cdot R^{-T}(y) En^{-T}(y) [\text{AI}(f_n(y))]^{-T} (0, 1)^T, \tag{3.27}
\]

which has the same expressions as in (3.11).

Thus, similar arguments there yield (3.24) for $x \in (1, 1 + \delta)$, $y \in (1 - \delta, 1)$. \hfill \Box

We conclude this section with the estimates of orthogonal polynomials.

Lemma 3.4 There exists a $\delta_0 > 0$, such that for all $0 < \delta \leq \delta_0$, we have

(i). For $x \in \mathbb{R}/(-1 - \delta, 1 + \delta)$,

\[
p_n(x; n)e^{-\frac{n}{2}V_n(x)} \leq C \left[ e^{-n\pi F_n(x)} \chi_{(1+\delta, \infty)}(x) + e^{-n\pi F_n(x)} \chi_{(-\infty, -1-\delta)}(x) \right].
\]

(ii). For $x \in (-1 - \delta, 1 + \delta)$,

\[
p_n(x; n)e^{-\frac{n}{2}V_n(x)} \leq C \left[ 1 + \frac{1}{|1 - x|^2} \chi_{(1-\delta, 1+\delta)}(x) + \frac{1}{|1 + x|^2} \chi_{(-1-\delta, -1+\delta)}(x) \right].
\]

Here $C$ is independent of $n$, and $\chi_I$ means the characteristic function of $I \subseteq \mathbb{R}$.

Proof This follows from the Plancherel–Rotach-type asymptotics of $p_n(x; n)$ in [8] and the asymptotics of Airy functions ([30, (2.60), (2.61), (3.6), (3.7)]). \hfill \Box
4 Gaussian Fluctuations

In this section we prove Theorems 1.1 and 1.2 for Freud-type potentials.

4.1 Bulk Case

We start with the asymptotics of the expectation and variance in Propositions 4.1 and 4.2, respectively.

Proposition 4.1 Let \( t = t(k, n) \) be as in Theorem 1.1. Fix \( \xi \in \mathbb{R} \) and set \( a_n := \frac{\sqrt{\log n}}{2\pi \sqrt{2} \rho V_n(t)} \), \( t_n := t + a_n \xi \) and \( I_n := [t_n, \infty) \). Then,

\[
\mathbb{E}_n(\#I_n) = n - k - \frac{\sqrt{\log n}}{\sqrt{2\pi^2}} \xi + O(1).
\] (4.1)

Proof First, since \(|t(k, n)| = |G^{-1}(k/n)| < 1\) for \( n \) large enough, Lemma 2.3 implies that \( 1/\rho V_n(t) \) are uniformly bounded and \( a_n = O(\sqrt{\log n/n}) \).

Using the estimates of \( \mathcal{H}_n(x, x) \) in Lemma 3.1 \((i)\) and \((iv)\), we have

\[
\mathbb{E}_n(\#I_n) = \int_{t_n}^{\infty} \mathcal{H}_n(x, x)dx = \int_{t_n}^{1-\delta} n\rho V_n(x)dx + \int_{1-\delta}^{1+\delta} \mathcal{H}_n(x, x)dx + O(1).
\] (4.2)

Moreover, using Lemma 3.1 \((ii)\) and arguing as in the proof of [30, Lemma 2] (see also the proof of [33, (2.2.4)]), we have

\[
\int_{1-\delta}^{1+\delta} \mathcal{H}_n(x, x)dx = \int_{1-\delta}^{1} n\rho V_n(x)dx + O(1).
\] (4.3)

Therefore, plugging (4.3) into (4.2) and using the Taylor expansion we obtain

\[
\mathbb{E}_n(\#I_n) = \int_{t_n}^{1} n\rho V_n(x)dx + O(1)
\]

\[
= n - n \int_{-1}^{t} \rho V_n(x)dx - n \int_{t}^{t+a_n\xi} \rho V_n(x)dx + O(1)
\]

\[
= n - k - n \left[ \rho V_n(t) a_n \xi + \frac{1}{2} \rho' V_n(\eta)(a_n \xi)^2 \right] + O(1)
\] (4.4)

with \( \eta \in (t, t + a_n \xi) \). Using Lemma 2.3 we obtain (4.1) and finish the proof.

Proposition 4.2 Let \( \{t_i\}_{i=1}^{\infty} \) be a sequence such that \( \sup_n |t_n| < 1 \). Set \( I_n := [t_n, \infty) \), \( n \in \mathbb{N} \). Then,

\[
\text{Var}_n(\#I_n) = \frac{1}{2\pi^2} \log n + O(\log \log n).
\] (4.5)
Proof The arguments are similar to those in the proof of [34, Lemma 3.2] (see also [20, Lemma 2.3], [33, Proposition 2.8]), so we give a sketch of it below.

First, we have
\[ \text{Var}_n(\#I_n) = \int_{\Omega_n} \mathcal{H}_n^2(x, y) \, dx \, dy \]
\[ = \int_{\Gamma} \mathcal{H}_n^2(x, y) \, dx \, dy + \int_{\Omega_n / \Gamma} \mathcal{H}_n^2(x, y) \, dx \, dy, \tag{4.6} \]
where \( \Omega_n = \{(x, y) : t_n \leq x < \infty, -\infty < y \leq t_n\} \) and \( \Gamma = \{(x, y) : t_n \leq x \leq 1 - \delta, -1 + \delta \leq y \leq t_n\} \).

By virtue of the asymptotic estimates of \( \mathcal{H}_n(x, y) \) in Lemma 3.2, we have
\[ \int_{\Gamma} \mathcal{H}_n^2(x, y) \, dx \, dy = \frac{1}{2\pi^2} \log n + O(\log \log n), \tag{4.7} \]
where the leading term comes from the integration on \( \Gamma^1 \) as in Lemma 3.2 (i).

Regarding the remaining region \( \Omega_n / \Gamma \), we have \( x - y \geq 2 - 2\delta > 0, (x, y) \in \Omega_n / \Gamma \). Moreover, by (2.10) and the asymptotic of \( \gamma_n \) in [8, (2.11)], we have \( \frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2} + O\left(\frac{1}{n^2}\right) \), which by the Christoffel–Darboux identity ([5, 3.48]) implies
\[ \mathcal{H}_n^2(x, y) \leq C[(p_n(x; n)p_n(y; n))^2 + (p_n(y; n)p_n(x; n))^2]e^{-n(V_n(x)+V_n(y))}. \]

Hence, in view of the estimates of orthogonal polynomials in Lemma 3.4 we obtain
\[ \int_{\Omega_n / \Gamma} \mathcal{H}_n^2(x, y) \, dx \, dy = O(1). \tag{4.8} \]

Therefore, plugging (4.7) and (4.8) into (4.6) we prove (4.5). \qed

Proof of Theorem 1.1 By virtue of Propositions 4.1 and 4.2, we can prove Theorem 1.1 by using similar arguments as in the proof of [20, Theorems 1.1 and 1.3] or [33, Theorems 2.9 and 2.10].

(i). Take \( t, \xi, a_n \) and \( I_n \) as in Proposition 4.1. Propositions 4.1 and 4.2 yield
\[ P_n \left( \frac{X_k - t}{a_n} < \xi \right) = P_n(\#I_n \leq n - k) = P_n \left( \frac{\#I_n - \mathbb{E}_n \#I_n}{\sqrt{\text{Var}_n(\#I_n)}} \leq \xi + o(1) \right), \tag{4.9} \]
which implies the assertion by the Costin–Lebowitz–Soshnikov theorem (see [28, p.497–498]).

(ii). The proof is based on the Soshnikov central limit theorem in [29, p.174]. The computations are straightforward but quite complicated. As in the proof of Proposition
4.2, the keypoint to calculate the correlation coefficients $\Lambda_{i,j}$ is that similarly to (4.7), for any given subset $\Lambda \subseteq \Omega_n$ with $\Omega_n$ as in the proof of Proposition 4.2,

$$\int_{\Lambda} K_{n}^{2}(x,y) \, dx \, dy = \int_{\Lambda \cap \tilde{\Gamma}_{1}} \frac{1}{2\pi^2 (x-y)^2} \, dx \, dy + \mathcal{O}(\log \log n),$$

where $\tilde{\Gamma}_{1} = \{(x,y) : t \leq x \leq t + \frac{1}{\log n}, t - \frac{1}{\log n} \leq y \leq t - \frac{1}{n}\}$. For simplicity of exposition, we refer to [20] and [33, Subsection 2.2.2] for more details. □

### 4.2 Edge Case

We start with Propositions 4.3 and 4.5 concerning the estimates of expectation and variance, respectively.

**Proposition 4.3** Set $I := [t, \infty)$ with $t \to 1^-$. Let $\tilde{a}(t) := (1 - t)^{-3/2} \int_{t}^{1} \rho V_n(x) \, dx$ and $a_1$ be as in Theorem 1.2. Then, we have

$$\mathbb{E}_n(\#I) = \tilde{a}(t)n(1 - t)^{3/2} + \mathcal{O}(1), \quad (4.10)$$

and

$$\tilde{a}(t) = a_1 + \eta(t) + \mathcal{O}\left(n^{-\frac{1}{2m}}\right), \quad (4.11)$$

where $\eta(t) = -(1 - t)^{-3/2} \int_{t}^{1} \tilde{h}'_n(\xi_x)(1 - x)^{3/2} \, dx$ for some $\xi_x \in (x, 1)$. In particular, for any $t_1, t_2 \in [1 - \delta, 1]$ with $\delta \in (0, 1)$ fixed, $|\eta(t_1) - \eta(t_2)| = \mathcal{O}(|t_1 - t_2|)$.

**Remark 4.4** Let $t$ be the real number such that $t = 1 - \left(\frac{k}{\tilde{a}(t)n}\right)^{\frac{2}{3}} + \frac{\sqrt{2}}{3\pi \tilde{a}(t)} \frac{\sqrt{\log k}}{\frac{5}{2m} t^{\frac{1}{2}}}$, where $k \sim n^\gamma$, $0 < \gamma < 1$. Then, $\eta(t) = \mathcal{O}(1 - t) = \mathcal{O}(n^{2/3} - \frac{1}{2})$. It follows that for $0 < \gamma \leq \min\left\{\frac{2}{5}, \frac{1}{2m}\right\}$, we have $\mathbb{E}_n(\#I) = a_1 n(1 - t)^{3/2} + \mathcal{O}(1)$.

**Proof** Similarly to (4.4), we have

$$\mathbb{E}_n(\#I) = \int_{t}^{1} n \rho V_n(x) \, dx + \mathcal{O}(1).$$

Since $t \to 1^-$, using the definition of $\tilde{a}(t)$ we obtain (4.10).

Moreover, set $\tilde{h}_n = \frac{1}{2\pi} \sqrt{1 + x h_n(x)}$, where $h_n$ is as in (2.3). Then, $\rho V_n(x) = \sqrt{1 - x \tilde{h}_n(x)}$. By Taylor’s expansion, mean value theorem and (7.2),
\[
\int_t^1 \rho_{V_n}(x) \, dx = \int_t^1 \sqrt{1-x} \left( \tilde{h}_n(1) + \tilde{h}'_n(\xi)(x-1) \right) \, dx \\
= \int_t^1 \sqrt{1-x} \left( \frac{\sqrt{2}}{\pi} \sum_{k=0}^{m-1} \frac{A_{m-1-k}}{A_m} + O \left( n^{-\frac{1}{2m}} \right) - \tilde{h}'_n(\xi)(1-x) \right) \, dx \\
= (1-t)^\frac{3}{2} \left( a_1 + O \left( n^{-\frac{1}{2m}} \right) + \eta(t) \right),
\]

where \( \xi \in (x,1) \), thereby implying (4.11). The last estimate follows from the uniform boundedness of differential \( \eta' \) implied by (7.3).

\[ \square \]

**Proposition 4.5** Let \( t \) be such that \( t \to 1^- \) and \( n(1-t)^\frac{3}{2} \to \infty \), and set \( I := [t, \infty) \).

Then,

\[
\text{Var}_n(\#I) = \frac{1}{2\pi^2} \log \left[ n(1-t)^\frac{3}{2} \right] (1 + o(1)). \tag{4.12}
\]

**Proof** By virtue of Lemma 3.3, the estimate (4.12) can be proved by using similar arguments as in [30, Lemma 4] and [33, Proposition 2.12].

In fact, similarly to (4.6), we have

\[
\text{Var}_n(\#I) = \iint_{\tilde{\Omega}_n} \mathcal{K}_n^2(x,y) \, dx \, dy + \iint_{\tilde{\Omega}_n/\tilde{\Gamma}} \mathcal{K}_n^2(x,y) \, dx \, dy, \tag{4.13}
\]

where \( \tilde{\Omega}_n = \{(x,y) : t \leq x < \infty, -\infty < y \leq t\} \), \( \tilde{\Gamma} = \{(x,y) : t \leq x \leq t+\frac{1-t}{r_n}, t-\frac{1-t}{r_n} \leq y \leq t-\epsilon\} \) with \( \epsilon = (n\sqrt{1-t})^{-1} \) and \( r_n^{-1} = \max\{\sqrt{1-t}, (\log[n(1-t)^\frac{3}{2}])^{-1}\} \).

Proceeding as in the proof of [30, (3.68)], we get from (3.24) that

\[
\iint_{\tilde{\Gamma}} \mathcal{K}_n^2(x,y) \, dx \, dy = \frac{1}{2\pi^2} \log \left[ n(1-t)^\frac{3}{2} \right] + O(\log r_n), \tag{4.14}
\]

which gives the leading term in (4.12).

Regarding the remaining integration on \( \tilde{\Omega}_n/\tilde{\Gamma} \), using (3.24), Lemma 3.4 and Proposition 4.3 and arguing as in [30], we have that

\[
\iint_{\tilde{\Omega}_n/\tilde{\Gamma}} \mathcal{K}_n^2(x,y) \, dx \, dy = O(\log r_n). \tag{4.15}
\]

(See also the proof of [33, (2.3.15)] for details.)

Consequently, plugging (4.14) and (4.15) into (4.13) we obtain (4.12). \( \square \)

**Proof of Theorem 1.2** (i) Fix \( \xi \in \mathbb{R} \), let \( \tilde{a}(t) \) and \( t \) be as in Proposition 4.3 and Remark 4.4, respectively, and set \( I_n := [t, \infty) \), \( a_2 := (2\pi^2)^{-\frac{1}{2}} \). Note that \( t \to 1^- \), as

\[ \square \]
\( n \to \infty \). Moreover,

\[
1 - t = \left( \frac{k}{\tilde{a}(t)n} \right)^{\frac{3}{2}} \left( 1 - \frac{2a_2 \sqrt{\log k}}{3k} \xi \right),
\]

which implies that

\[
n(1 - t)^{\frac{3}{2}} = \frac{k}{\tilde{a}(t)} \left( 1 - \frac{2a_2 \sqrt{\log k}}{3k} \xi \right)^{\frac{3}{2}}
\]

\[
= \frac{k}{\tilde{a}(t)} \left( 1 - \frac{a_2 \sqrt{\log k}}{k} \xi + \mathcal{O} \left( \frac{\log k}{k^2} \right) \right) \to \infty.
\]

Then, define \( \tilde{Y}_n \) as in (1.11). Similarly to (4.9), by Propositions 4.3 and 4.5,

\[
\mathbb{P}_n(\tilde{Y}_n < \xi) = \mathbb{P}_n(\# I_n \leq k) = \mathbb{P}_n \left( \frac{\# I_n - \mathbb{E}_n \# I_n}{\text{Var}_n \# I_n} \leq \xi + o(1) \right),
\]

(4.16)

which along with the Costin–Lebowitz–Soshnikov theorem (cf. [28, p.497–498]) implies the asymptotic normality of \( \tilde{Y}_n \).

Therefore, taking into account \( \tilde{a}(t) \to a_1 \) as \( n \to \infty \), we see that for any \( \tau \in \mathbb{R} \),

\[
|\mathbb{E}_n e^{i\tau \tilde{Y}_n} - \mathbb{E}_n e^{i\tau \tilde{Y}_n}| \to 0 \quad \text{as} \quad n \to \infty,
\]

which yields that \( Y_n \) is also normally distributed in the limit.

(ii). The proof is similar but more involved. Let \( \tilde{s}_i \) be such that \( \tilde{s}_i = c_i + d_i \xi_i \), where

\[
c_i := 1 - \left( \frac{k_i}{\tilde{a}(s_i)n} \right)^{\frac{3}{2}} \quad \text{and} \quad d_i := \frac{2a_2 \sqrt{\log k}}{3 \tilde{a}(s_i)^{\frac{3}{2}}} \quad \text{with} \quad a_2 = (2\pi^2)^{-1/2}, \ 1 \leq i \leq m.
\]

Define \( \tilde{Y}_{i,n} \) as in (1.11) with \( \tilde{a}(s_i) \) replacing \( \tilde{a}(t) \), and set \( Z_i := \sum_{k=1}^{i} \# I_k \) (\( = \#(\tilde{s}_i, \infty) \)),

\[
1 \leq i \leq m, \quad \text{where} \quad I_1 := (\tilde{s}_1, \infty) \quad \text{and} \quad I_i := (\tilde{s}_i, \tilde{s}_{i-1}), \ 2 \leq i \leq m.
\]

Since \( \tilde{a}(s_i) \to a_1, \ 1 \leq i \leq m, \) we see that for any \( \{c_i\}_{i=1}^{m} \subseteq \mathbb{R} \) and \( \tau \in \mathbb{R} \),

\[
|\mathbb{E}_n e^{i\tau \sum_{i=1}^{m} c_i Y_{i,n}} - \mathbb{E}_n e^{i\tau \sum_{i=1}^{m} c_i \tilde{Y}_{i,n}}| \to 0, \quad \text{as} \quad n \to \infty.
\]

Hence, it is equivalent to prove the assertion for \( \{\tilde{Y}_{i,n}\}_{i=1}^{m} \). Moreover, similarly to (4.16), we have

\[
\mathbb{P}_n(\tilde{Y}_{1,n} < \xi_1, \ldots, \tilde{Y}_{m,n} < \xi_m) = \mathbb{P}_n(Z_1 \leq \xi_1 + o(1), \ldots, Z_m \leq \xi_m + o(1)).
\]

Thus, we can reduce the proof of \( \{\tilde{Y}_{i,n}\}_{i=1}^{m} \) to that of \( \{Z_i\}_{i=1}^{m} \).

Now, the asymptotic normality of \( \{Z_i\}_{i=1}^{m} \) can be proved by using Soshnikov’s central limit theorem (cf. [29, p.174]), and the correlations can be calculated by using similar arguments as in the proof of Proposition 4.5. The key fact is that for any given set \( \Lambda \) in the neighborhood of \( (t, t) \) with \( t \to 1^- \) and \( n(1 - t)^{\frac{3}{2}} \to \infty \),

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\[
\int_\Lambda \int_\Lambda \mathcal{K}_n^2(x, y) = \frac{1}{2 \pi} \int_\Lambda \int_\Lambda \frac{1}{(x - y)^2} \mathrm{d}x \mathrm{d}y + \mathcal{O}(\log r_n), \tag{4.17}
\]

where \( \widehat{\Gamma} \) and \( r_n \) are as in the proof of Proposition 4.5. For more details we refer to the proof of \([33, \text{Theorem 2.14}]\) where the constant \( \tilde{a} \) needs to be modified by the function \( \tilde{a}(\cdot) \) here, yet the arguments still go through since \( \tilde{a}(t) = a_1(1 + o(1)) \).

It should be mentioned that the additional condition \( \theta > \gamma - \frac{1}{2m} \) arises in the estimate below (see, e.g., \([33, \text{p.61}]\))

\[
\Delta \tilde{s}_{i,j} := \tilde{s}_i - \tilde{s}_j = \frac{2}{3a^2_1 n^{\frac{2}{3}} \left( 1 + \varepsilon_i \right)^{rac{2}{3}} \left( 1 + \varepsilon_j \right)^{rac{2}{3}} + \left( 1 + \varepsilon_i \right)^{rac{2}{3}} \left( 1 + \varepsilon_j \right)^{rac{2}{3}}} + \mathcal{O}\left( \frac{\sqrt{\log k}}{n^{\frac{2}{3}} + \gamma} \right).
\]

for which we have to take into account the delicate remaining order of \( \tilde{a} \) in (4.11). Precisely, we write \( \tilde{a}(\tilde{s}_i) = a_1(1 + \varepsilon_i) \). Then, by the definitions of \( \tilde{s}_i \) and \( \tilde{s}_j \),

\[
\Delta \tilde{s}_{i,j} = \frac{1}{3a^2_1 n^{\frac{2}{3}}} \left( \frac{2}{3} n^{\frac{2}{3}} \gamma + \mathcal{O}\left( n^{\frac{2}{3}} \gamma - \frac{1}{2m} \right) \right) (1 + o(1)) + o(\Delta \tilde{s}_{i,j}),
\]

which yields (4.18) under the condition \( \theta > \gamma - \frac{1}{2m} \). \( \square \)

**Remark 4.6** The reason we first treat \( \widehat{Y}_n \) (instead of \( Y_n \)) in the proof of Theorem 1.2 is that the identity (4.16) may be not valid for \( Y_n \) due to the remaining terms of \( \tilde{a}(t) \). Precisely, if we consider \( Y_n \) instead, then \( \mathbb{P}_n(Y_n < \xi) = \mathbb{P}_n\left( \frac{\#I_n - \mathbb{E}_n \#I_n}{\sqrt{\text{Var}_n \#I_n}} \leq \frac{k - \mathbb{E}_n \#I_n}{\sqrt{\text{Var}_n \#I_n}} \right) \),

where \( I_n = [t, \infty) \) with a different choice of \( t \) such that \( t = 1 - \left( \frac{k}{\mathbb{E}_n} \right)^{\frac{1}{2}} + \frac{\sqrt{\frac{\log k}{\mathbb{E}_n}}}{{\pi} a^2_1 n^{\frac{1}{2}} k^{\frac{1}{2}}} \xi \)

instead. However, Proposition 4.3 yields

\[
k - \mathbb{E}_n \#I_n = k - \frac{\tilde{a}(t)}{a_1} k \left( 1 - \frac{a_2 \sqrt{\log k}}{k} \frac{\xi}{\mathbb{E}_n} \right) + \mathcal{O}\left( \frac{\log k}{k^2} \right) = a_2 \sqrt{\log k} \left( 1 + o(1) \right) + \mathcal{O}(\log n) + \mathcal{O}(k \eta(t)) + \mathcal{O}(k^{n^{\frac{2}{3}} - \frac{1}{2m}}),
\]

which implies that the right-hand side above may exceed the order of \( \sqrt{\text{Var}_n(\#I_n)} \) (i.e., \( \sqrt{\log k} \)) if \( \gamma > \frac{2}{3} \) or \( \gamma > \frac{1}{2m} \).
5 Moderate Deviations

This section contains the proof of Theorem 1.5 for unitary invariant ensembles with Freud-type potentials as in (1.3).

Proof of Theorem 1.5 (i) Fix $\xi \in \mathbb{R}$. Let $t = t(k, n)$, $a_n$, $t_n$ and $I_n$ be as in Proposition 4.1. Since $1 \ll \gamma_n \ll \sqrt{\log n}$ and $|t(k, n)| < 1$ for $n$ large enough, we have that $a_n \gamma_n \xi = o(1)$ and $|t_n| < 1$ for large $n$. Thus, arguing as in (4.2)–(4.4), we have

$$
\mathbb{E}_n(\#I_n) = n - k - \frac{\sqrt{\log n}}{\sqrt{2\pi}} \gamma_n \xi + O(1). \tag{5.1}
$$

Moreover, by Proposition 4.2,

$$
\text{Var}_n(\#I_n) = \frac{1}{2\pi^2} \log n + O(\log \log n). \tag{5.2}
$$

Hence, we have

$$
\frac{n - k - \mathbb{E}_n(\#I_n)}{\gamma_n \sqrt{\text{Var}_n(\#I_n)}} = \xi + o(1). \tag{5.3}
$$

This yields that

$$
\mathbb{P}_n(\gamma_n^{-1} X_n < \xi) = \mathbb{P}_n(\#I_n \leq n - k) = \mathbb{P}_n\left( \frac{\#I_n - \mathbb{E}_n \#I_n}{\gamma_n \sqrt{\text{Var}_n(\#I_n)}} \leq \xi + o(1) \right), \tag{5.4}
$$

which implies by [15, Theorem 1.4] that for every $\xi < 0$,

$$
\lim_{n \to \infty} \gamma_n^{-2} \log \mathbb{P}_n(\gamma_n^{-1} X_n \leq \xi) = - \frac{\xi^2}{2}. \tag{5.5}
$$

Similarly, for every $\xi > 0$, by (5.1) and (5.2),

$$
\mathbb{P}_n(\gamma_n^{-1} X_n \geq \xi) = \mathbb{P}_n(\#I_n \geq n - k + 1) = \mathbb{P}_n\left( \frac{\#I_n - \mathbb{E}_n \#I_n}{\gamma_n \sqrt{\text{Var}_n(\#I_n)}} \geq \xi + o(1) \right),
$$

which implies by [15, Theorem 1.4] that

$$
\lim_{n \to \infty} \gamma_n^{-2} \log \mathbb{P}_n(\gamma_n^{-1} X_n \geq \xi) = - \frac{\xi^2}{2}. \tag{5.6}
$$

Now, as in the proof of [15, Theorem 2.1], we denote by $\mathcal{U}$ the set of all open intervals $(c, d)$, where $c, d \neq 0$ and at least one of the endpoints is finite. Define $\mathcal{L}_U := - \lim_{n \to \infty} \gamma_n^{-2} \log \mathbb{P}_n(\gamma_n^{-1} X_n \in U)$, $U \in \mathcal{U}$. By (5.5) and (5.6),
\[
\mathcal{L}_U = \begin{cases} 
d^{2}/2, & c < d < 0; \\
0, & c < 0 < d; \\
c^{2}/2, & 0 < c < d.
\end{cases}
\] (5.7)

Then, it follows from [13, Theorem 4.1.11] that \( \{\gamma_n^{-1} X_n\} \) satisfies a weak LDP with speed \( \gamma_n^2 \) and rate function \( I(x) := \sup_{U \in \mathcal{U}, x \in U} \mathcal{L}_U(x) = x^2/2 \).

Moreover, for any \( \alpha < \infty \), consider the compact set \( K_\alpha = [-c, c] \) with \( c = \sqrt{2\alpha} \).

By [13, Lemma 1.2.15] and (5.7),

\[
\lim_{n \to \infty} \gamma_n^{-2} \log \mathbb{P}_n(\gamma_n^{-1} X_n \notin K_\alpha) = \max \left\{ \lim_{n \to \infty} \gamma_n^{-2} \log \mathbb{P}_n(\gamma_n^{-1} X_n \in (-\infty, -c)), \lim_{n \to \infty} \gamma_n^{-2} \log \mathbb{P}_n(\gamma_n^{-1} X_n \in (c, \infty)) \right\} = -\frac{c^2}{2} = -\alpha,
\]

which implies the exponential tightness of \( \{\gamma_n^{-1} X_n\} \), thereby yielding that \( \{\gamma_n^{-1} X_n\} \) satisfies the LDP \((\gamma_n^2, x^2/2)\).

(ii). Let \( t \) be the real number such that \( t = 1 - (\frac{k}{\tilde{a}(t)n})^{\frac{3}{2}} + \frac{2a_2}{3(\tilde{a}(t))^{\frac{3}{2}}} \sqrt{\log k} \gamma_n \xi, \)

\( I_n := [t_n, \infty) \), where \( k, \tilde{a}(t) \) are as in Theorem 1.5, and \( a_2 := (2\pi^2)^{-\frac{1}{2}} \). Since \( k/n \to 0 \) and \( \gamma_n \ll \sqrt{\log k} \), we have that \( t \to 1^- \) and

\[
n(1 - t)^{\frac{3}{2}} = \frac{k}{\tilde{a}(t)} \left( 1 - \frac{a_2 \sqrt{\log k}}{k} \gamma_n \xi + O\left( \frac{\log k \gamma_n^2}{k^2} \right) \right) \to \infty.
\]

Then, using Proposition 4.3 we have

\[
\mathbb{E}_n(\#I_n) = k - a_2 \sqrt{\log k} \gamma_n \xi + O\left( \frac{\log k \gamma_n^2}{k} \right),
\] (5.8)

which implies that

\[
k - \mathbb{E}_n(\#I_n) = a_2 \sqrt{\log k} \gamma_n \xi (1 + o(1)).
\] (5.9)

Moreover, by Proposition 4.5,

\[
\sqrt{\text{Var}_n(\#I_n)} = a_2 \sqrt{\log k (1 + o(1))}.
\] (5.10)

Thus, we obtain from the estimates above that

\[
\frac{k - \mathbb{E}_n \#I_n}{\gamma_n \sqrt{\text{Var}_n(\#I_n)}} = \xi + o(1),
\]

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which yields that
\[
\mathbb{P}_n(\gamma_n^{-1} Y_n < \xi) = \mathbb{P}_n(\#I_n \leq k) = \mathbb{P}_n\left( \frac{\#I_n - \mathbb{E}_n \#I_n}{\gamma_n \sqrt{\text{Var}_n(\#I_n)}} \leq \xi + o(1) \right),
\]
and
\[
\mathbb{P}_n(\gamma_n^{-1} Y_n \geq \xi) = \mathbb{P}_n(\#I_n \geq k + 1) = \mathbb{P}_n\left( \frac{\#I_n - \mathbb{E}_n \#I_n}{\gamma_n \sqrt{\text{Var}_n(\#I_n)}} \geq \xi + o(1) \right).
\]

Using [15, Theorem 1.4] again, we obtain (5.5) and (5.6) with \(Y_n\) replacing \(X_n\).

Therefore, using similar arguments as those below (5.6) we obtain that \(\{\gamma_n^{-1} Y_n\}\) satisfies the LDP \((\gamma_n^2, x^2/2)\). The proof of Theorem 1.5 is complete. \(\square\)

### 6 Uniform Convex Potential

This section is devoted to unitary invariant ensembles with uniform convex potentials. Since the arguments are similar to those in the case of Freud-type potentials, we mainly show the estimates of Christoffel–Darboux kernels and orthogonal polynomials. Some technical details are postponed to the “Appendix”.

**Lemma 6.1** Consider the uniform convex potential as in (1.4). We have
\[
\rho_V(x) = \frac{1}{2\pi} \sqrt{1 - x^2} h(x) \chi_{[-1,1]}(x),
\]
where \(h(x)\) is an analytic function satisfying for some \(c > 0\), \(h(x) \geq c, \forall x \in \mathbb{R}\).

(See the “Appendix” for the proof.) As a consequence, we have

**Lemma 6.2** Consider the uniform convex potential as in (1.4). Let \(\delta \in (0, 1)\). Then, \(\rho_V^{-1}\) and \(|\rho_V'|\) are bounded on \([-1 + \delta, 1 - \delta]\).

Set
\[
F(x) := \left| \int_x^1 \frac{1}{2\pi} \sqrt{1 - y^2} h(y) dy \right|, \quad \tilde{F}(x) := \left| \int_{-1}^x \frac{1}{2\pi} \sqrt{1 - y^2} h(y) dy \right|.
\]

We also keep the notations \(p_j(x; n), \gamma_j^{(n)}\) and \(\mathcal{K}_n(x, y)\) for the orthogonal polynomials, leading coefficients and reproducing kernels with respect to \(e^{-n V(x)}\).

Below we recall the Riemann–Hilbert approach developed in [7]. Let \(Y(z)\) be an analytic \(2 \times 2\) matrix-valued function, solving the Riemann–Hilbert problem
\[
Y_+(z) = Y_-(z) \begin{pmatrix} 1 & e^{-n V(z)} \\ 0 & 1 \end{pmatrix}, \text{ for } z \in \mathbb{R}
\]
\[
Y(z) \begin{pmatrix} z^{-n} \\ z^n \end{pmatrix} = I + O\left( \frac{1}{|z|} \right), \text{ as } |z| \to \infty.
\]

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Similarly to (2.12), we have

$$Y_{11}(z) = \frac{1}{\gamma(n)} p_n(z; n), \quad Y_{21}(z) = -2\pi i \gamma(n) p_{n-1}(z; n),$$

(6.3)

Set

$$g(z) := \int \log(z - s) \psi(s) ds, \quad z \in \mathbb{C}/(-\infty, 1),$$

(6.4)

where \( \psi(z) = \frac{1}{2\pi i} R^{\frac{1}{2}}(z) h(z), z \in \mathbb{C}/[-1, 1], R^{\frac{1}{2}}(z) = (z + 1)^{\frac{1}{2}} (z - 1)^{\frac{1}{2}}, \) which is analytic in \( \mathbb{C}/[-1, 1] \) and satisfies \( \sqrt{R(z)} \sim z \) as \( z \to \infty. \) Set

$$G(z) := - \int_1^z (s - 1)^{\frac{1}{2}} (s + 1)^{\frac{1}{2}} h(s) ds, \quad z \in \mathbb{C}.$$  

(6.5)

We have

$$g = \frac{1}{2} (V + l + G),$$

(6.6)

where \( l \) is as in (1.7). Let

$$M := e^{-\frac{nl}{2} \sigma_3} Y e^{-\frac{1}{2} (g - \frac{i}{2}) \sigma_3},$$

(6.7)

and

$$M^{(1)}(z) := \begin{cases} 
M(z), & \text{outside the lens-shaped region;} \\
M(z) \begin{pmatrix} 1 & 0 \\
0 & -e^{-nG} \\
-e^{-nG} & 1 \\
e^{nG} & 1 
\end{pmatrix}, & \text{in the upper lens region;} \\
M(z) \begin{pmatrix} 1 & 0 \\
0 & e^{nG} \\
e^{nG} & 1 \\
1 & 1 
\end{pmatrix}, & \text{in the lower lens region},
\end{cases}$$

(6.8)

where the lens are as in Sect. 2.

Below are the paramatrices \( M_p \) in small neighborhoods \( U_{\pm 1} \) of endpoints \( \pm 1. \) (i). For \( z \in U_1, \) we set (cf. [7, (4.76)])

$$M_p := B(z) P(\Phi(z)),$$

(6.9)

Here, \( B(z) = \frac{1}{\sqrt{2i}} N \begin{pmatrix} i & -i \\
i & 1 \end{pmatrix} (\Phi(z))^{\sigma_3} \) with \( N \) as in (2.27),
We have similar asymptotic expansions of $R$ where $\tilde{R}$

\[
P(z) = \begin{cases} 
\sqrt{2\pi}e^{-\frac{z^4}{4}}A(z)e^{\left(\frac{z^3}{3} - \frac{z}{6}\right)\sigma_3}, & \text{for } z \in I; \\
\sqrt{2\pi}e^{-\frac{z^4}{4}}A(z)e^{\left(\frac{z^3}{3} - \frac{z}{6}\right)\sigma_3}\left(1 \ 0\right), & \text{for } z \in II; \\
\sqrt{2\pi}e^{-\frac{z^4}{4}}\tilde{A}(z)e^{\left(\frac{z^3}{3} - \frac{z}{6}\right)\sigma_3}\left(1 \ 0\right), & \text{for } z \in III; \\
\sqrt{2\pi}e^{-\frac{z^4}{4}}\tilde{A}(z)e^{\left(\frac{z^3}{3} - \frac{z}{6}\right)\sigma_3}, & \text{for } z \in IV,
\end{cases}
\]

the regions $I$–$IV$ and the matrices $A$, $\tilde{A}$ are as in Sect. 2, and $\Phi_1(z) = \left(\frac{3n}{4}\right)^2(-G)^{\frac{\bar{s}}{2}} = \left(\frac{3n}{4}\right)^2\left(\int_1^\infty R^2(s)h(s)ds\right)^{\frac{3}{2}}$.

(ii). For $z \in U_{-1}$, we set (cf. [7, (4.92)])

\[
M_p := \tilde{B}(z)P(\Phi_{-1}(z))\sigma_3,
\]

where $\tilde{B}(z) = \frac{1}{\sqrt{2\pi}}N\sigma_3\left(\begin{array}{c} i \\ -1 \\ 1 \end{array}\right)(\Phi_{-1})^{\frac{\bar{s}}{2}}$, $\Phi_{-1}(z) = \left(\frac{3n}{4}\right)^2\left(-\int_1^\infty R^2(s)h(s)ds\right)^{\frac{3}{2}}$.

Finally, set (cf. [7, (p. 1377)])

\[
R := \begin{cases} 
M^{(1)}M_p^{-1}, & \text{for } z \in U_1 \cup U_{-1}; \\
M^{(1)}N^{-1}, & \text{otherwise}.
\end{cases}
\]

We have similar asymptotic expansions of $R$ as in (2.29).

Similarly to Lemma 3.1, we have the crucial asymptotic estimates of $\mathcal{K}(x, x)$.

**Lemma 6.3** Take any sufficiently small $\delta > 0$, we have

(i). For $x \in (-1 + \delta, 1 - \delta)$,

\[
\mathcal{K}(x, x) = n\rho_V(x) + O(1).
\]

(ii). For $x \in (1 - \delta, 1 + \delta)$,

\[
\mathcal{K}(x, x) = \left[\frac{1}{4}\Phi_1(x) - a'(x)\right]2\Phi_1(x)\Phi_1'(x) + \Phi_1(x)\left[(ai')^2(\Phi_1(x)) - \Phi_1(x)ai^2(\Phi_1(x))\right] + O\left(n^{-\frac{5}{6}}\right).
\]

(iii). For $x \in (-1 - \delta, -1 + \delta)$,

\[
\mathcal{K}(x, x) = -\left[\frac{1}{4}\Phi_{-1}(x) + a'(x)\right]2\Phi_{-1}(x)\Phi_{-1}'(x) - \Phi_{-1}(x)\left[(ai'^2(\Phi_{-1}(x)) - \Phi_{-1}(x)ai^2(\Phi_{-1}(x))\right] + O\left(n^{-\frac{5}{6}}\right).
\]
(iv). For $x \in \mathbb{R}/(-1 - \delta, 1 + \delta)$,
\[
\mathcal{X}_n(x, x) = \frac{1}{4\pi} \frac{1}{(x - 1)(x + 1)} e^{nG(x)} + O(n^{-1}).
\] (6.16)

**Proof** First note that similarly to (3.5),
\[
2\pi i(x - y)\mathcal{X}_n(x, y) = (1, 0)Y(x)T Y(y)^{-T} (0, 1)^T e^{-n \frac{\nu(x) + \nu(y)}{2}}.
\] (6.17)

(i). The estimate (6.13) follows from the same calculations as in Lemma 3.1 (i), with $U, T, S, \xi_n$ replaced by $Y, M, M^{(1)}$ and $G$, respectively.

(ii). First, for $x \in (1 - \delta, 1)$, we note that
\[
N \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = i \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \begin{pmatrix} a^{-1} \\ a \end{pmatrix}.
\]
and $\frac{2}{3} \Phi_1^3 = -\frac{nG}{2}$. Then, it follows from (6.9) and (6.10) with $z \in II$ that
\[
M_p = \widehat{E_n} [AI(\Phi_1)] e^{-\frac{\pi i}{6} \sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{-\frac{nG}{2} \sigma_3},
\] (6.18)

where $\widehat{E_n} = \sqrt{\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \begin{pmatrix} a^{-1}(\Phi_1)^{\frac{1}{2}} \\ 0 \\ 0 \\ a(\Phi_1)^{-\frac{1}{2}} \end{pmatrix}$. By (6.12),
\[
M^{(1)} = RE_n [AI(\Phi_1)] e^{-\frac{\pi i}{6} \sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{-\frac{nG}{2} \sigma_3}.
\] (6.19)

Similarly, for $x \in (1, 1 + \delta)$, using (6.10) with $z \in I$ we have
\[
M_p = \widehat{E_n} [AI(\Phi_1)] e^{-\frac{\pi i}{6} \sigma_3} e^{-\frac{nG}{2} \sigma_3},
\] (6.20)
which implies that
\[
M^{(1)} = RE_n [AI(\Phi_1)] e^{-\frac{\pi i}{6} \sigma_3} e^{-\frac{nG}{2} \sigma_3}.
\] (6.21)

Then, comparing (6.19) and (6.21) with (7.5) and (7.7), we see that the expressions of $M^{(1)}$ and $S$ are similar on $(1 - \delta, 1 + \delta)$, with the only difference that $\widehat{E_n}, \Phi_1, G$ above are replaced by $E_n, f_n$ and $-2\varphi_n$, respectively. Thus, using the same arguments as in the proof of Lemma 3.1 (ii) we obtain (6.14).

(iii). First, for $x \in (-1, -1 + \delta)$, note that
\[
N \sigma_3 \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = i \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \sigma_3,
\]
and \( \frac{2}{3} (\Phi - 1)^{\frac{3}{2}} = n(-\frac{G}{2} + \pi i) \). Using (6.11) and (6.10) with \( z \in III \), we have

\[
M_p = \hat{E}_n \sigma_3 \left[ \tilde{A} I (\Phi - 1) \right] e^{-\frac{\pi i}{6} \sigma_3 \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array} \right) \sigma_3 e^n (\frac{-G}{2} + \pi i) \sigma_3},
\]

where \( \hat{E}_n = \sqrt{\pi} e^{\frac{\pi i}{6}} \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right) \left( \begin{array}{cc} a(\Phi - 1)^{\frac{1}{2}} & 0 \\ 0 & a^{-1}(\Phi - 1)^{-\frac{1}{2}} \end{array} \right) \). Hence, by (6.12),

\[
M^{(1)} = R \hat{E}_n \sigma_3 \left[ \tilde{A} I (\Phi - 1) \right] e^{-\frac{\pi i}{6} \sigma_3 \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array} \right) \sigma_3 e^n (\frac{-G}{2} + \pi i) \sigma_3}.
\]

Similarly, for \( x \in (-1 - \delta, -1) \), by (6.10) with \( z \in IV \),

\[
M_p = \hat{E}_n \sigma_3 \left[ \tilde{A} I (\Phi - 1) \right] e^{-\frac{\pi i}{6} \sigma_3 \sigma_3 e^n (\frac{-G}{2} + \pi i) \sigma_3},
\]

which yields that

\[
M^{(1)} = R \hat{E}_n \sigma_3 \left[ \tilde{A} I (\Phi - 1) \right] e^{-\frac{\pi i}{6} \sigma_3 \sigma_3 e^n (\frac{-G}{2} + \pi i) \sigma_3}.
\]

Thus, comparing (6.23) and (6.25) with (7.9) and (7.11), we see that \( M^{(1)} \) and \( S \) enjoy the same expressions on \((-1 - \delta, -1 + \delta)\), with the only difference that \( \hat{E}_n, \Phi - 1, -\frac{G}{2} + \pi i \) here are replaced by \( \tilde{E}_n, -\tilde{f}_n \) and \( \tilde{\varphi}_n \), respectively. Hence, similar arguments as in the proof of Lemma 3.1 (iii) yield (6.15).

(i v). The proof is the same as that in Lemma 3.1 (i v). \( \Box \)

Arguing as in the proof of Lemmas 3.2 and 3.3, we have also the asymptotics of \( K_n(x, y) \) as in (3.20), (3.21), (3.22) and (3.24) with \( F \) and \( \Phi_1 \) replacing \( F_n \) and \( f_n \), respectively.

Below are the asymptotics of orthogonal polynomials, which would be also of independent interest. The proof is postponed to the “Appendix”.

Lemma 6.4 (i). For \( x > 1 + \delta \),

\[
\begin{align*}
p_n(x; n) e^{-\frac{\pi}{8} V(x)} &= \frac{1}{\sqrt{4\pi}} e^{-\pi n F(x)} \left[ \frac{x + 1}{x - 1} \right]^{\frac{1}{4}} + \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{4}} + O(n^{-1}) \right], \\
p_{n-1}(x; n) e^{-\frac{\pi}{8} V(x)} &= \frac{1}{\sqrt{4\pi}} e^{-\pi n F(x)} \left[ \frac{x + 1}{x - 1} \right]^{\frac{1}{4}} - \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{4}} + O(n^{-1}) \right],
\end{align*}
\]

where \( F(x) \) is defined as in (6.2).

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For \( x \in (-\infty, -1 - \delta) \),

\[
p_n(x; n) e^{-\frac{2n}{2}V(x)} = (-1)^n \frac{1}{\sqrt{4\pi}} e^{-n\pi \tilde{F}(x)} \left[ \left( \frac{x + 1}{x - 1} \right)^{\frac{1}{4}} + \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{4}} + \mathcal{O}(n^{-1}) \right],
\]

(6.28)

\[
p_{n-1}(x; n) e^{-\frac{2n}{2}V(x)} = (-1)^n \frac{1}{\sqrt{4\pi}} e^{-n\pi \tilde{F}(x)} \left[ \left( \frac{x + 1}{x - 1} \right)^{\frac{1}{4}} - \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{4}} + \mathcal{O}(n^{-1}) \right].
\]

(6.29)

where \( \tilde{F}(x) \) is defined as in (6.2).

(ii). For \( x \in (-1 + \delta, 1 - \delta) \),

\[
p_n(x; n) e^{-\frac{2n}{2}V(x)} = \sqrt{\frac{2}{\pi}} \frac{1}{(1-x)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \left\{ \cos \left[ \frac{1}{2} \arcsin x - \pi n F(x) \right] + \mathcal{O}(n^{-1}) \right\},
\]

(6.30)

\[
p_{n-1}(x; n) e^{-\frac{2n}{2}V(x)} = \sqrt{\frac{2}{\pi}} \frac{1}{(1-x)^{\frac{1}{4}}(1+x)^{\frac{1}{4}}} \left\{ \sin \left[ \frac{1}{2} \arcsin x + \pi n F(x) \right] + \mathcal{O}(n^{-1}) \right\}.
\]

(6.31)

(iii). For \( x \in (1 - \delta, 1 + \delta) \),

\[
p_n(x; n) e^{-\frac{2n}{2}V(x)} = -\Phi_1^\frac{1}{2} Ai(\Phi_1)(1 + \mathcal{O}(n^{-1})) - \Phi_1^{-\frac{1}{2}} Ai'(\Phi_1)(1 + \mathcal{O}(n^{-1})),
\]

(6.32)

\[
p_{n-1}(x; n) e^{-\frac{2n}{2}V(x)} = a^{-1} \Phi_1^\frac{1}{2} Ai(\Phi_1)(1 + \mathcal{O}(n^{-1})) + a \Phi_1^{-\frac{1}{2}} Ai'(\Phi_1)(1 + \mathcal{O}(n^{-1})),
\]

(6.33)

and for \( x \in (-1 - \delta, -1 + \delta) \),

\[
p_n(x; n) e^{-\frac{2n}{2}V(x)}
\]

\[
= (-1)^n \left\{ a \Phi_1^{-\frac{1}{2}} Ai(\Phi_1)(1 + \mathcal{O}(n^{-1})) - a^{-1} \Phi_1^{-\frac{1}{2}} Ai'(\Phi_1)(1 + \mathcal{O}(n^{-1})) \right\}.
\]

(6.34)

\[
p_{n-1}(x; n) e^{-\frac{2n}{2}V(x)}
\]

\[
= (-1)^{n+1} \left\{ a \Phi_1^\frac{1}{2} Ai(\Phi_1)(1 + \mathcal{O}(n^{-1})) + a^{-1} \Phi_1^\frac{1}{2} Ai'(\Phi_1)(1 + \mathcal{O}(n^{-1})) \right\}.
\]

(6.35)

As a consequence of Lemma 6.4 and the asymptotics of Airy functions, we also have similar estimates of orthogonal polynomials as in Lemma 3.4.

Now, by virtue of the asymptotics of Christoffel–Darboux kernels and orthogonal polynomials, Theorems 1.1, 1.2 and 1.5 in the case of uniform convex potentials can
be proved by using similar arguments as in Sects. 4 and 5. Note that the condition \( \theta > \gamma - 1/(2m) \) is not needed here, since we do not have the addition order \( O(n^{-1/(2m)}) \) in (4.11). For simplicity of exposition, the details are omitted.

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**Appendix**

**Proof of Lemma 2.3** First, using Theorem 2.2, we have for all \( n \geq N \) and \( x \in [-1 + \delta, 1 - \delta] \),

\[
\rho_V^{-1}(x) = 2\pi \frac{1}{\sqrt{1 - x^2}} \frac{1}{h_n(x)} < \frac{2\pi}{h_0} \frac{1}{\sqrt{1 - (1 - \delta)^2}} < \infty.
\]

As regards \( \rho'_V \), we have

\[
2\pi |\rho'_V| \leq \frac{|x|}{\sqrt{1 - x^2}} |h_n(x)| + \sqrt{1 - x^2} |h'_n(x)|. \tag{7.1}
\]

Note that by (2.2) and (2.4),

\[
h_n(x) = \sum_{k=0}^{m-1} 2^k \frac{A_{m-k-1}}{A_m} x^{2k} + O\left(n^{-\frac{1}{2m}}\right), \tag{7.2}
\]

\[
h'_n(x) = \sum_{k=0}^{m-1} 4^k \frac{A_{m-k-1}}{A_m} x^{2k-1} + O\left(n^{-\frac{1}{2m}}\right), \tag{7.3}
\]

which implies that \( |h_n(x)| \) and \( |h'_n(x)| \) are uniformly bounded for all \( n \geq N \) and \( x \in [-1 + \delta, 1 - \delta] \), thereby completing the proof. \( \square \)

**Proof of (3.11)** First consider \( x, y \in (1 - \delta, 1) \). As in the case where \( x, y \in (-1 + \delta, 1 - \delta) \), (3.7) still holds, i.e.,

\[
2\pi i (x - y) \mathcal{K}(x, y) = (e^{-n\psi_n(x)}, e^{n\psi_n(x)}) S(x)^T S(y)^{-T} (-e^{n\psi_n(y)}, e^{-n\psi_n(y)})^T. \tag{7.4}
\]

Since for \( x \in (1 - \delta, 1) \), \( f_n(x + i\epsilon) \) lies in the region II in (2.24), taking \( \epsilon \to 0 \) we obtain

\[
\Psi^\sigma (f_n(x)) = [AI(f_n(x))] e^{-\frac{\pi i}{6} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right),
\]

which along with (2.26) and (2.23) yields that

\[
S = RE_n [AI(f_n)] e^{-\frac{\pi i}{6} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right) e^{n\psi_n \sigma_3}. \tag{7.5}
\]
Consequently, plugging (7.5) into (7.4), we obtain (3.11) for \( x, y \in (1 - \delta, 1) \).

Regarding the case where \( x, y \in (1, 1 + \delta) \), by (2.14), (2.15), (2.16) and (3.5),

\[
U = e^{\frac{\pi}{6} \sigma_3} S e^n \left( g_n - \frac{f_n}{2} \right) \sigma_3,
\]

and

\[
2\pi i (x - y) \mathcal{X}_n(x, y) = (e^{-n\varphi_n(x)}, 0) S^T(x) S^{-T}(y)(0, e^{-n\varphi_n(y)})^T.
\]  

(7.6)

Since for \( x \in (1, 1 + \delta) \), \( f_n(x + i\epsilon) \) is in the region \( I \) in (2.24), by (2.26) and (2.23),

\[
S = RE_n[AI(f_n)] e^{-\frac{n\pi}{6} \sigma_3} e^{n\varphi_n \sigma_3}.
\]

(7.7)

Hence, combining (7.6) and (7.7) we get (3.11) for \( x, y \in (1, 1 + \delta) \), thereby completing the proof of (3.11). \( \square \)

**Proof of (3.14)** We first show that \( I_2(x, y) = \mathcal{O}(n^{-\frac{2}{3}}) \). Indeed, by (3.13), expressions of \( AI, E_n \) and that \( \det[AI(z)] = -\frac{1}{2\pi i} e^{-\frac{n\pi}{6}} \) (see [6, p. 890]),

\[
I_2(x, y) = -2\pi i e^{\frac{n\pi}{6}} (H_n(x) Ai(f_n(x)), H_n^{-1}(x) Ai'(f_n(x))) \left( \begin{array}{cc} 1 & -1 \\ -i & -i \end{array} \right)^T
\]

\[
\cdot \Delta_R(x, y) R^{-T}(y) \left( \begin{array}{cc} 1 & -1 \\ -i & -i \end{array} \right)^{-T}
\]

\[
\times (-H_n^{-1}(y) Ai'(f_n(y)), H_n(y) Ai(f_n(y)))^T.
\]

Note that since \( H_n = f_n^{\frac{1}{4}} a^{-1} \), by (2.19),

\[
H_n = n^{\frac{1}{6}} (x - 1)^{\frac{1}{2}} \left( \frac{x}{x + 1} \right)^{-\frac{1}{4}} = n^{\frac{1}{6}} (x + 1)^{\frac{1}{2}} \left( \hat{\varphi}_n \right)^{\frac{1}{2}}. \]

Moreover, for \( x \in \mathbb{R} \), \( |Ai(x)| = \mathcal{O}(1) \) and \( |Ai'(f_n(x))| = \mathcal{O}(1) \), and by (2.29), \( \Delta_R(x, y) R^{-T}(y) = \mathcal{O}(n^{-1}) \). Thus, we conclude that \( I_2(x, y) \) is of order \( \mathcal{O}(n^{-\frac{1}{3}}) \). It remains to check the first term on the right-hand side of (3.12). To this end, it follows from (3.12) and the computations as above that

\[
e^{-\frac{n\pi}{6}} (1, 0) [AI(f_n(x))]^T E_n^T(x) E_n^{-T}(y) [AI(f_n(y))]^{-T} (0, 1)^T
\]

\[
= (-2\pi i) \left[ -Ai(f_n(x)) Ai'(f_n(y)) \frac{f_n^{\frac{1}{4}}(x)}{f_n^{\frac{1}{4}}(y)} \frac{a(y)}{a(x)} + Ai'(f_n(x)) Ai(f_n(y)) \frac{f_n^{\frac{1}{4}}(y)}{f_n^{\frac{1}{4}}(x)} \frac{a(x)}{a(y)} \right].
\]

which yields the first term on the right-hand side of (3.14). \( \square \)
**Proof of (3.15)** The proofs are similar to those of (3.11). First consider \( x, y \in (-1, -1 + \delta) \). As in the case where \( x, y \in (1 - \delta, 1) \), we have

\[
2\pi i (x - y) \mathcal{K}_n(x, y) = (e^{-n\varphi_n(x)}, e^{n\varphi_n(x)}) S(x)^T S(y)^{-T} (-e^{n\varphi_n(y)}, e^{-n\varphi_n(y)})^T.
\]

(7.8)

Since for \( x \in (-1, -1 + \delta) \), \(-\tilde{f}_n(x + i \epsilon)\) lies in the region \( III \) in (2.24), letting \( \epsilon \to 0 \) we have

\[
\mathcal{P} \mathcal{K}^\sigma (-\tilde{f}_n(x)) = [\tilde{A} I (-\tilde{f}_n(x))] e^{-\frac{\pi i}{6} \sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

which along with (2.26) and (2.25) yields

\[
S = R \tilde{E}_n \sigma_3 [\tilde{A} I (-\tilde{f}_n)] e^{-\frac{\pi i}{6} \sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_3 e^{n\mathcal{P} \mathcal{K}^\sigma}.
\]

(7.9)

Thus, plugging (7.9) into (7.8), since \( \mathcal{P} \mathcal{K}^\sigma = \varphi_n(z) + \pi i, z \in \mathbb{C}^+ \), we get (3.15).

Regarding the case where \( x, y \in (-1 - \delta, -1) \). As in the case where \( x, y \in (1, 1 + \delta) \) in the proof of (3.11), we have

\[
2\pi i (x - y) \mathcal{K}_n(x, y) = (e^{-n\varphi_n(x)}, 0) S^T(x) S^{-T}(y)(0, e^{-n\varphi_n(y)})^T.
\]

(7.10)

Since for \( x \in (-1 - \delta, -1) \), \(-\tilde{f}_n(x + i \epsilon)\) is in the region \( IV \) in (2.24), taking \( \epsilon \to 0 \), we obtain from (2.25) and (2.26) that

\[
S = R \tilde{E}_n \sigma_3 [\tilde{A} I (-\tilde{f}_n)] e^{-\frac{\pi i}{6} \sigma_3} \sigma_3 e^{n\mathcal{P} \mathcal{K}^\sigma}.
\]

(7.11)

Therefore, plugging (7.11) into (7.10) yields (3.15).

**□**

**Proof of Lemma 6.1** Define the Hilbert transform \( \mathcal{H} \) and the Borel transform \( \mathcal{B} \) by

\[
\mathcal{H} \rho_V(x) = \frac{1}{\pi} P.V. \int \frac{\rho_V(y)}{x - y} dy,
\]

\[
\mathcal{B} \rho_V(z) = \frac{1}{\pi i} \int \frac{\rho_V(s)}{s - z} ds, \quad z \in \mathbb{C}/\mathbb{R}.
\]

In view of [7, (3.10), (3.12)], for \( x \in \mathbb{R} \) we have

\[
(\mathcal{B} \rho_V)_\pm(x) = \pm \rho_V(x) + i \mathcal{H} \rho_V(x) = \pm \rho_V(x) - \frac{1}{2\pi i} V'(x).
\]

Moreover, by virtue of [7, (3.17), (3.18)],

\[
\mathcal{B} \rho_V(z) = -\frac{1}{2\pi i} V'(z) - \frac{\sqrt{R(z)}}{4\pi^2} \oint_{\Gamma_z} \frac{V'(s)}{\sqrt{R(s)}} ds - z,
\]

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where \( \sqrt{R(z)} \) is as in Sect. 6, and \( \Gamma_z \) is a counterclockwise contour with \( z \) and \([-1, 1]\) in its interior. Note that due to the analytic branch of \( \sqrt{R(z)} \), we have

\[
(\sqrt{R(x)})_+ = i\sqrt{(x+1)(1-x)} = -(\sqrt{R(x)})_-. \tag{7.12}
\]

Hence, it follows that for \( x \in (-1, 1) \),

\[
\rho_V(x) = \frac{1}{2\pi} \sqrt{(1-x)(x+1)} \left( \frac{1}{2\pi i} \oint_{\Gamma_z} \frac{V'(s)}{\sqrt{R(s)} s-z} \, ds \right)_+,
\]

and so

\[
h(z) = \frac{1}{2\pi i} \oint_{\Gamma_z} \frac{V'(s)}{\sqrt{R(s)} s-z} \, ds,
\]

which is an analytic function.

It remains to prove that for some \( c > 0 \), \( h(x) \geq c \forall x \in \mathbb{R} \). We first claim that

\[
\frac{1}{2\pi i} \oint_{\Gamma_z} \frac{1}{\sqrt{R(s)} s-z} \, ds = 0 \tag{7.14}
\]

To this end, we have for \( r \) large enough that

\[
\frac{1}{2\pi i} \oint_{\Gamma_z} \frac{1}{\sqrt{R(s)} s-z} \, ds = \frac{1}{2\pi i} \oint_{|s|=r} \frac{1}{\sqrt{(s+1)(s-1)} s-z} \, ds. \tag{7.15}
\]

Using Taylor’s extension, we see that

\[
\frac{1}{\sqrt{S^2-1}} \frac{1}{s-z} = \left( A_0 \frac{1}{s} + A_1 \frac{1}{s^3} + A_2 \frac{1}{s^5} + \cdots \right) \left( \frac{1}{s} + \frac{z}{s^3} + \frac{z^2}{s^5} + \cdots \right),
\]

which by Cauchy’s theorem implies that the right-hand side of (7.15) equals the coefficient of \( \frac{1}{s} \) which is exactly zero, thereby yielding (7.14), as claimed.

Now, it follows from (7.13) and (7.14) that

\[
h(z) = \frac{1}{2\pi i} \oint_{\Gamma_z} \frac{V'(s) - V'(z)}{s-z} \frac{1}{\sqrt{R(s)} s-z} \, ds = \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{V'(s) - V'(z)}{s-z} \frac{1}{\sqrt{R(s)} s-z} \, ds,
\]

where \( \Gamma_1 \) is a counterclockwise contour with \([-1, 1]\), but not \( z \), in the interior.
Therefore, in view of (7.16) we have that for \( x \in (-1, 1) \),

\[
h(x) = \lim_{z \in \mathbb{C}^+ \to x} h(z) = \frac{1}{\pi i} \int_{-1}^{1} \frac{V'(s) - V'(x)}{s - x} \frac{1}{(\sqrt{R(s)})_+} ds = \frac{1}{\pi} \int_{-1}^{1} \frac{V'(s) - V'(x)}{s - x} \frac{1}{\sqrt{(s + 1)(1 - s)}} ds,
\]

which implies by the mean value theorem and the uniform convexity of \( V \) that

\[
h(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{V''(\xi)}{\sqrt{(s + 1)(1 - s)}} ds \geq c \frac{1}{\pi} \int_{-1}^{1} \frac{ds}{\sqrt{(s + 1)(1 - s)}} = c > 0.
\]

where \( \xi \in (-1, 1) \). The proof is complete. \( \square \)

Before proving Lemma 6.4, we recall that

**Theorem 7.1** ([7, (1.62)–(1.64)]) Consider the uniform convex potential as in (1.4). For the leading coefficients of orthogonal polynomials, we have

\[
(\gamma_n(n))^{-2} = e^{-nl} \left[ \frac{1}{4\pi} + \mathcal{O}(n^{-1}) \right],
\]

\[
\gamma_{n-1}(n) = \frac{1}{2} + \mathcal{O}(n^{-1/2}).
\]

**Theorem 7.2** ([7, Theorems 1.1–1.3]) Consider the uniform convex potential as in (1.4). For the monic polynomials, we have

(i). For \( x \in \mathbb{R}/(-1 - \delta, 1 + \delta) \),

\[
\pi_n(x; n) = e^{n g(x)} \left( M_1(x) + \mathcal{O}(n^{-1}) \right),
\]

\[
-2\pi i (\gamma_{n-1}(n))^2 \pi_{n-1}(x; n) = e^{ng(x) - l} \left( M_2(x) + \mathcal{O}(n^{-1}) \right),
\]

where \( M_1 = \frac{a + a^{-1}}{2} \), \( M_2 = \frac{a^{-1} - a}{2l} \) with \( a \) as in (2.28) and \( l \) is as in (1.7).

(ii). For \( x \in (-1 + \delta, 1 - \delta) \),

\[
\pi_n(x; n) = 2e^{\frac{2}{\pi}(V(x) + l)} \left[ Re(M_1 e^{i\pi n F(x)}) + \mathcal{O}(n^{-1}) \right],
\]

\[
-2\pi i (\gamma_{n-1}(n))^2 \pi_{n-1}(x; n) = 2e^{\frac{2}{\pi}(V(x) - l)} \left[ Im(M_2 e^{i\pi n F(x)}) + \mathcal{O}(n^{-1}) \right],
\]

where \( F \) is as in (6.2).
(iii). For \( x \in (1 - \delta, 1) \cup (-1, -1 + \delta), \)

\[
\pi_n(x; n) = \left( e^{\frac{n}{2} \sigma_3} (I + \mathcal{O}(n^{-1})) M_p e^{n g(x) - \frac{i}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ e^{nV} & 1 \end{pmatrix} \right)^{11},
\]

(7.21)

\[-2\pi i (\gamma_{n-1}^{(n)})^2 \pi_{n-1}(x; n) = \left( e^{\frac{n}{2} \sigma_3} (I + \mathcal{O}(n^{-1})) M_p e^{n g(x) - \frac{i}{2} \sigma_3} \begin{pmatrix} 1 & 0 \\ e^{nV} & 1 \end{pmatrix} \right)^{21},
\]

(7.22)

and for \( x \in (1, 1 + \delta) \cup (-1 - \delta, -1), \)

\[
\pi_n(x; n) = \left( (I + \mathcal{O}(n^{-1})) M_p \right)^{11} e^{ng(x)},
\]

(7.23)

\[-2\pi i (\gamma_{n-1}^{(n)})^2 \pi_{n-1}(x; n) = \left( (I + \mathcal{O}(n^{-1})) M_p \right)^{21} e^{ng(x) - l},
\]

(7.24)

**Proof of Lemma 6.4** We first note that by the analytic branch of \( R^{1/2}(z) \),

\[
(z - 1)^{\frac{1}{2}} = i(1 - z)^{\frac{1}{2}}, \quad (z + 1)^{\frac{1}{2}} = i(-1 - z)^{\frac{1}{2}}, \quad z \in \mathbb{C}_+.
\]

Then, using (6.5) we have

\[
G(x) = -\int_1^x \sqrt{(s - 1)(s + 1)} h(s) ds = -2\pi F(x), \quad x > 1,
\]

(7.25)

\[
G(x) = 2\pi i F(x), \quad x \in (-1, 1),
\]

(7.26)

\[
G(x) = -2\pi \tilde{F}(x) + 2\pi i, \quad x < -1,
\]

(7.27)

where \( F(x) \) and \( \tilde{F}(x) \) are as in (6.2).

(i). For \( x \in \mathbb{R}/(-1 - \delta, 1 + \delta), \) by (7.17) and (6.6),

\[
\pi_n(x; n) = e^{\frac{n}{2} (V + G)} \left\{ \frac{1}{2} \left[ \left( \frac{x + 1}{x - 1} \right)^{\frac{1}{2}} + \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{2}} \right] + \mathcal{O}(n^{-1}) \right\}.
\]

Then, by Theorem 7.1,

\[
p_n(x; n) = \gamma_n^{(n)} \pi_n(x; n) = \frac{1}{\sqrt{\pi}} e^{\frac{n}{2} (V + G)} \left\{ \frac{1}{2} \left[ \left( \frac{x + 1}{x - 1} \right)^{\frac{1}{2}} + \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{2}} \right] + \mathcal{O}(n^{-1}) \right\}.
\]

Thus, using (7.25) and (7.27) for \( x > 1 + \delta \) and \( x < -1 - \delta \), respectively, we obtain (6.26) and (6.28).
Similarly, by Theorem 7.1, (7.18) and (6.6),

\[
p_{n-1}(x; n) e^{-\frac{n^2}{4} V} = \frac{1}{\sqrt{4\pi}} e^{\frac{nG}{2}} \left\{ \left[ \left( \frac{x + 1}{x - 1} \right)^{\frac{1}{2}} - \left( \frac{x - 1}{x + 1} \right)^{\frac{1}{2}} \right] + O(n^{-1}) \right\},
\]

which yields (6.27) and (6.29) by (7.25) and (7.27), respectively.

(ii). By the definitions of \( M_1 \) and \( M_2 \), we have that (cf. [8, (8.33), (8.34)])

\[
M_1 = \frac{a + a^{-1}}{2} = \frac{\sqrt{2}}{2} \frac{1}{(1 - x)^{\frac{1}{2}}(1 + x)^{\frac{1}{2}}} e^{-\frac{i}{2} \arcsin x}
\]

\[
M_2 = \frac{a^{-1} - a}{2i} = -\frac{\sqrt{2}}{2} \frac{1}{(1 - x)^{\frac{1}{2}}(1 + x)^{\frac{1}{2}}} e^{\frac{i}{2} \arcsin x}.
\]

Then, in view of Theorems 7.1 and 7.2 (ii), we obtain (6.30) and (6.31).

(iii). We consider four cases (iii.1) – (iii.4).

(iii.1). For \( x \in (1 - \delta, 1) \), by (6.18) and (6.6),

\[
\left. e^{\frac{nG}{2} \sigma_3 M_p e^{\left( g - \frac{1}{2} \right) \sigma_3}} \begin{pmatrix} 1 & 0 \\ e^{nV} & 1 \end{pmatrix} \right|^1_{11} = \left. e^{\frac{nG}{2} \sigma_3 \hat{E}_n [AI(\Phi_1)]} e^{-\frac{nG}{2} \sigma_3 e^{\frac{n}{2} \sigma_3} e^{\frac{n}{2} \sigma_3} V} \right|^{11}_{11}
\]

Then, since \( \hat{E}_n = \sqrt{\pi} e^{\frac{ni}{2}} \begin{pmatrix} a^{-1} \Phi_1^{\frac{1}{2}} & -a \Phi_1^{-\frac{1}{2}} \\ -ia^{-1} \Phi_1^{\frac{1}{2}} & -ia \Phi_1^{-\frac{1}{2}} \end{pmatrix} \), direct calculations show that

\[
\left. \left( e^{\frac{nG}{2} \sigma_3 M_p e^{\left( g - \frac{1}{2} \right) \sigma_3}} \begin{pmatrix} 1 & 0 \\ e^{nV} & 1 \end{pmatrix} \right) \right|^1_{11} = \sqrt{\pi} e^{\frac{nG}{2} (V + 1)} \begin{pmatrix} a^{-1} \Phi_1^{\frac{1}{2}} Ai(\Phi_1) - a \Phi_1^{-\frac{1}{2}} Ai'(\Phi_1) \end{pmatrix}.
\] (7.28)

Similarly,

\[
\left. \left( e^{\frac{nG}{2} \sigma_3 M_p e^{\left( g - \frac{1}{2} \right) \sigma_3}} \begin{pmatrix} 1 & 0 \\ e^{nV} & 1 \end{pmatrix} \right) \right|_{21} = (-i) \sqrt{\pi} e^{\frac{nG}{2} (V - 1)} \begin{pmatrix} a^{-1} \Phi_1^{\frac{1}{2}} Ai(\Phi_1) + a \Phi_1^{-\frac{1}{2}} Ai'(\Phi_1) \end{pmatrix}.
\] (7.29)

Plugging these into (7.21) and (7.22) and using Theorem 7.1, we obtain (6.32) and (6.33).

(iii.2). For \( x \in (1, 1 + \delta) \), using (6.20) we note that in (7.23) and (7.24), \((M_p)_{11} e^{ng}\) and \((M_p)_{21} e^{ng/2}\) have the same formulations as (7.28) and (7.29). Thus, arguing as above we obtain (6.32) and (6.33).
For $x \in (-1, -1 + \delta)$, it follows from (6.22) that

$$e^{\frac{n}{2} \sigma_3 M_p e^{n(g - \frac{1}{2}) \sigma_3} \left( \begin{array}{c} 1 \\ e^{nV} \\ 1 \end{array} \right)} = (-1)^n e^{\frac{n}{2} \sigma_3 \tilde{E}_n \sigma_3} [\widetilde{A} I(\Phi_1)] e^{-\frac{n}{2} \sigma_3 e^{nV} \sigma_3}. $$

Then, since

$$\tilde{E}_n = \sqrt{\pi} e^{-\frac{n}{2}} \left( \begin{array}{ccc} a \Phi_{\frac{-1}{4}} & a^{-1} \Phi_{\frac{-1}{4}} \\ ia \Phi_{\frac{-1}{4}} & -ia^{-1} \Phi_{\frac{-1}{4}} \end{array} \right),$$

we have

$$\left( e^{\frac{n}{2} \sigma_3 M_p e^{n(g - \frac{1}{2}) \sigma_3} \left( \begin{array}{c} 1 \\ e^{nV} \\ 1 \end{array} \right)} \right)_{11} = (-1)^n \sqrt{\pi} e^{-\frac{n(Y+i)}{2}} \left[ a \Phi_{\frac{-1}{4}} Ai(\Phi_1) - a^{-1} \Phi_{\frac{-1}{4}} Ai'(\Phi_1) \right].$$

Similarly,

$$\left( e^{\frac{n}{2} \sigma_3 M_p e^{n(g - \frac{1}{2}) \sigma_3} \left( \begin{array}{c} 1 \\ e^{nV} \\ 1 \end{array} \right)} \right)_{21} = (-1)^{n+1} (-i) \sqrt{\pi} e^{-\frac{n(Y-i)}{2}} \left[ a \Phi_{\frac{-1}{4}} Ai(\Phi_1) + a^{-1} \Phi_{\frac{-1}{4}} Ai'(\Phi_1) \right].$$

Thus, (6.34) and (6.35) follow from (7.21), (7.22) and Theorem 7.1.

For $x \in (-1 - \delta, -1)$, by (6.24) we note that $(M_p)_{11} e^{n g}$ and $(M_p)_{11} e^{n(g - l)}$ in (7.23) and (7.24) have the same formulations as (7.30) and (7.31), which consequently implies (6.34) and (6.35). The proof of Lemma 6.4 is complete.

References

1. Bourgade, P., Erdös, L., Yau, H.T.: Edge universality of beta ensembles. Commun. Math. Phys. 332(1), 261–353 (2014)
2. Breuer, J., Duits, M.: The Nevai condition and a local law of large numbers for orthogonal polynomial ensembles. Adv. Math. 265, 441–484 (2014)
3. Costin, O., Lebowitz, J.L.: Gaussian fluctuation in random matrices. Phys. Rev. Lett. 75(1), 69–72 (1995)
4. Chan, T.: The Wigner semi-circle law and eigenvalues of matrix-valued diffusions. Probab. Theory Relat. Fields 93(2), 249–272 (1992)
5. Deift, P.: Orthogonal polynomials and random matrices: a Riemann–Hilbert approach, Courant Lecture Notes in Mathematics 3. Courant Institute of Mathematical Science, New York (1999)
6. Deift, P., Gioev, D.: Universality at the edge of the spectrum for unitary, orthogonal, and symplectic ensembles of random matrices. Commun. Pure Appl. Math. 60(6), 867–910 (2007)
7. Deift, P., Kriecherbauer, T., McLaughlin, K.T.-R., Venakides, S., Zhou, X.: Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. Commun. Pure Appl. Math. 52(11), 1335–1425 (1999)
8. Deift, P., Kriecherbauer, T., McLaughlin, K.T.-R., Venakides, S., Zhou, X.: Strong asymptotics of orthogonal polynomials with respect to exponential weights. Commun. Pure Appl. Math. 52(12), 1491–1552 (1999)
9. Deift, P., Kriecherbauer, T., McLaughlin, K.T.-R., Venakides, S., Zhou, X.: A Riemann–Hilbert approach to asymptotic questions for orthogonal polynomials. J. Comput. Appl. Math. 133(1–2), 47–63 (2001)
10. Deift, P., Venakides, S., Zhou, X.: New results in small dispersion KdV by an extension of the steepest descent method for Riemann–Hilbert problems. Int. Math. Res. Not. 6, 286–299 (1997)
11. Deift, P., Zhou, X.: A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation. Ann. Math. 137(2), 295–368 (1993)
12. Dembo, A., Guionnet, A., Zeitouni, O.: Moderate deviations for the spectral measure of certain random matrices. Ann. Inst. H. Poincaré Probab. Stat. 39(6), 1013–1042 (2003)
13. Dembo, A., Zeitouni, O.: Large deviations techniques and applications. Corrected reprint of the second (1998) edition. Stochastic Modelling and Applied Probability, 38. Springer-Verlag, Berlin (2010). xvi+396 pp
14. Döring, H., Eichelsbacher, P.: Moderate deviations via cumulants. J. Theor. Probab. 26(2), 360–385 (2013)
15. Döring, H., Eichelsbacher, P.: Moderate deviations for the eigenvalue counting function of Wigner matrices. ALEA Lat. Am. J. Probab. Math. Stat. 10(1), 27–44 (2013)
16. Döring, H., Eichelsbacher, P.: Edge fluctuations of eigenvalues of Wigner matrices. High dimensional probability VI, pp. 261–275, Progr. Probab., 66, Birkhäuser/Springer, Basel (2013)
17. Erdős, L.: Universality for Random Matrices and Log-Gases. Current Developments in Mathematics, pp. 59–132. International Press, Somerville (2013)
18. Fokas, A.S., Its, A.R., Kitaev, A.V.: Discrete Painlevé equations and their appearance in quantum gravity. Commun. Math. Phys. 142(2), 313–344 (1991)
19. Forrester, P.J.: Log-Gases and Random Matrices. London Mathematical Society Monographs Series, 34. Princeton University Press, Princeton (2010)
20. Gustavsson, J.: Gaussian fluctuations of eigenvalues in the GUE. Ann. Inst. H. Poincaré Probab. Stat. 41(2), 151–178 (2005)
21. Johansson, K.: On fluctuations of eigenvalues of random Hermitian matrices. Duke Math. J. 91(1), 151–204 (1998)
22. Li, S.Z., Li, X.D., Xie, Y.X.: Generalized Dyson Brownian motion, McKean–Vlasov equation and eigenvalues of random matrices (2013). arXiv:1303.1240v1
23. Lubinsky, D.S.: Gaussian fluctuations of eigenvalues of random Hermitian matrices associated with fixed and varying weights. Random Matrices Theory Appl. 5(3), 1650009, 31 (2016)
24. O’Rourke, S.: Gaussian fluctuations of eigenvalues in Wigner random matrices. J. Stat. Phys. 138(6), 1045–1066 (2010)
25. Pastur, L., Shcherbina, M.: On the edge universality of the local eigenvalue statistics of matrix models. Mat. Fiz. Anal. Geom. 10(3), 335–365 (2003)
26. Rogers, L.C.G., Shi, Z.: Interacting Brownian particles and the Wigner law. Probab. Theory Relat. Fields 95(4), 555–570 (1993)
27. Shcherbina, M.: Edge universality for orthogonal ensembles of random matrices. J. Stat. Phys. 136(1), 35–50 (2009)
28. Soshnikov, A.: Gaussian fluctuation for the number of particles in Airy, Bessel, sine, and other determinantal random point fields. J. Stat. Phys. 100(3–4), 491–522 (2000)
29. Soshnikov, A.: Gaussian limit for determinantal random point fields. Ann. Probab. 30(1), 171–187 (2002)
30. Su, Z.G.: Gaussian fluctuations in complex sample covariance matrices. Electron. J. Probab. 11(48), 1284–1320 (2006)
31. Tao, T., Vu, V.: Random matrices: universality of local eigenvalue statistics. Acta Math. 206(1), 127–204 (2011)
32. Tracy, C., Widom, H.: Level-spacing distributions and the Airy kernel. Commun. Math. Phys. 159(1), 151–174 (1994)
33. Zhang, D.: Random matrices, stochastic nonlinear Schrödinger equations (in Chinese). Ph.D. thesis, Chinese Academy of Sciences (2014)
34. Zhang, D.: Gaussian fluctuations of eigenvalues in log-gas ensemble: bulk case I. Acta Math. Sin. (Engl. Ser.) 31(9), 1487–1500 (2015)
35. Zhang, D.: Tridiagonal random matrix: Gaussian fluctuations and deviations. J. Theor. Probab. 30(3), 1076–1103 (2017)

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