New Sharp Bounds for the Modified Bessel Function of the First Kind and Toader-Qi Mean †

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Abstract: Let $I_v(x)$ be the modified Bessel function of the first kind of order $v$. We prove the double inequality $\sqrt{\frac{\sinh t}{t}} \cosh^{1/q}(qt) < I_0(t) < \sqrt{\frac{\sinh t}{t}} \cosh^{1/p}(pt)$ holds for $t > 0$ if and only if $p \geq \frac{2}{3}$ and $q \leq (\ln 2)/\ln \pi$. The corresponding inequalities for means improve already known results.

Keywords: modified Bessel function of the first kind; hyperbolic function; mean; inequality

MSC: 39B62; 33B10

1. Introduction

The modified Bessel function of the first kind of order $v$, denoted by $I_v(x)$, is a particular solution of the second-order differential equation ([1], p. 77)

$$x^2 y''(x) + xy'(x) - \left(x^2 + v^2\right) y(x) = 0,$$

which can be represented explicitly by the infinite series as

$$I_v(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+v}}{n! \Gamma(v + n + 1)}, \quad x \in \mathbb{R}, \quad v \in \mathbb{R} \setminus \{-1, -2, \ldots\},$$

where $\Gamma(x)$ is the gamma function [2–4]. There are many properties of $I_v(x)$, see for example, [5–11].

In this paper, we are interested in a special case of $I_v(x)$, that is, $I_0(x)$, which is related to Toader-Qi mean of positive numbers $a$ and $b$ defined by

$$TQ(a,b) = \frac{2}{\pi} \int_{0}^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta = \sqrt{ab} I_0 \left(\ln \sqrt{\frac{a}{b}}\right)$$

(see [12–14]), where and in what follows $a, b > 0$ with $a \neq b$. It is undoubted that Toader-Qi mean $TQ(a,b)$ is a new newcomer. Recall that some classical means including the arithmetic mean, geometric mean, logarithmic mean, exponential mean and power mean of order $p$ defined by

$$A \equiv A(a,b) = \frac{a+b}{2}, \quad G \equiv G(a,b) = \sqrt{ab},$$

$$L \equiv L(a,b) = \frac{a-b}{\ln a - \ln b}, \quad I \equiv I(a,b) = e^{-\left(\frac{b}{a^2}\right)^{1/(b-a)}}$$

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\[A_p \equiv A_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}\text{ if } p \neq 0 \text{ and } A_0 \equiv A_0(a, b) = \sqrt{ab},\]

respectively. Clearly, \(A(a, b) = A_1(a, b)\) and \(G(a, b) = A_0(a, b)\). It is known that \(p \mapsto A_p(a, b)\) is increasing on \(\mathbb{R}\). A simple relation among these elementary means is the following inequalities:

\[G < L < A_{1/3} < \frac{A + 2G}{3} < A_{1/2} < \frac{2A + G}{3} < A_{2/3} < I < A_{\ln 2} < A_1\] (3)

(see [15–21]). Another interesting relation proven in [22] is that:

\[\sqrt{AG} < \sqrt{IL} < \frac{L + I}{2} < \frac{A + G}{2}.\] (4)

Let \(b > a > 0\) and \(t = \ln \sqrt{a/b}\). Then those means mentioned above can be represented in terms of hyperbolic functions:

\[L(a, b) = \frac{\sinh t}{t}, \quad I(a, b) = \exp \left(\frac{t}{\tanh t} - 1\right),\]
\[TQ(a, b) = I_0(t), \quad A_p(a, b) = \cosh^{1/p}(pt) \text{ for } p \neq 0.\]

Correspondingly, the inequalities mentioned above are equivalent to

\[1 < \frac{\sinh t}{t} < \cosh^{3} \left(\frac{t}{3}\right) < \cosh^{2} \left(\frac{t}{2}\right) < \cosh^{3/2} \left(\frac{2t}{3}\right) < \exp \left(\frac{t}{\tanh t} - 1\right) < \cosh^{1/\ln 2} (t \ln 2),\]

\[\sqrt{\cosh t} < \frac{\sinh t}{t} \exp \left(\frac{t}{\tanh t} - 1\right) < \frac{1}{2} \left[\frac{\sinh t}{t} + \exp \left(\frac{t}{\tanh t} - 1\right)\right] < \cosh t + 1.\]

for \(t > 0\).

Let us return to Toader-Qi mean. In 2015, Qi, Shi, Liu and Yang [13] proved that the inequalities

\[L(a, b) < TQ(a, b) < \frac{A(a, b) + G(a, b)}{2} < \frac{2A(a, b) + G(a, b)}{3} < I(a, b)\] (5)

hold. Yang and Chu (Theorem 3.3 of [23]) established a series of sharp inequalities for \(TQ(a, b)\) and \(I_0(t)\), for example, the inequalities

\[\sqrt{\frac{\sinh (2t)}{\pi t}} < I_0(t) < \sqrt{\frac{\sinh (2t)}{2t}},\] (6)

\[\sqrt{\left(\frac{2}{\pi} \cosh t + 1 - \frac{2}{\pi}\right) t} \sinh t < I_0(t) < \sqrt{\left(\lambda_0 \cosh t + 1 - \lambda_0\right) t} \sinh t,\] (7)

\[\left(\frac{\sinh t}{t}\right)^{3/4} (\cosh t)^{1/4} < I_0(t) < \frac{3 \sinh t}{4t} + \frac{1}{4} \cosh t,\] (8)

hold for \(t > 0\) with \(\lambda_0 = 0.6766...\). Inspired by the inequalities (3) and (4), Yang and Chu conjectured further that the inequality

\[TQ(a, b) < \sqrt{L(a, b) I(a, b)}\] (9)
holds, which was proven in Theorem 3.1 of [24] by Yang, Chu and Song. In fact, they proved the following double inequality
\[ \sqrt{\frac{e}{\pi}} \sqrt{L(a,b)I(a,b)} < TQ(a,b) < \sqrt{L(a,b)I(a,b)} \] (10)
holds with the best coefficients \( \sqrt{e/\pi} = 0.930... \) and 1. More inequalities for \( TQ(a,b) \) can be seen in [25,26].

Motivated by the inequalities (9) and \( A_{2/3} < I \) listed in (3), the aim of this paper is to find the best constants \( p \) and \( q \) such that double inequality
\[ \sqrt{L(a,b)A_q(a,b)} < TQ(a,b) < \sqrt{L(a,b)A_p(a,b)} \] (11)
holds, or equivalently,
\[ \sqrt{\frac{\sinh t}{t}} \cosh^{1/q}(qt) < I_0(t) < \sqrt{\frac{\sinh t}{t}} \cosh^{1/p}(pt) \] (12)
for \( t > 0 \). Our main results are as follows.

**Theorem 1.** The function
\[ F(t) = \frac{tI_0(t)^2}{\cosh^{3/2}(2t/3) \sinh t} \]
is strictly decreasing from \((0, \infty)\) onto \( (\sqrt{8}/\pi, 1) \). Therefore, the double inequality
\[ \frac{2^{3/4}}{\sqrt{\pi}} \sqrt{\frac{\sinh t}{t}} \cosh^{3/2} \left( \frac{2t}{3} \right) < I_0(t) < \sqrt{\frac{\sinh t}{t}} \cosh^{3/2} \left( \frac{2t}{3} \right) \]
holds for \( t > 0 \), or equivalently,
\[ \frac{2^{3/4}}{\sqrt{\pi}} \sqrt{L(a,b)A_{2/3}(a,b)} < TQ(a,b) < \sqrt{L(a,b)A_{2/3}(a,b)} \] (13)
holds, where the coefficients \( 2^{3/4}/\sqrt{\pi} = 0.94885... \) and 1 are the best.

**Theorem 2.** The double inequality (12) holds for \( t > 0 \), or equivalently, (11) holds for \( a, b > 0 \) with \( a \neq b \), if and only if \( p \geq 2/3 \) and \( q \leq p_0 = (\ln 2)/\ln \pi = 0.605... \).

2. Tools And Lemmas

To prove our results, we need two tools. The first tool was due to Biernacki and Krzyz [27], which play an important role in dealing with the monotonicity of the ratio of power series.

**Lemma 1** ([27]). Let \( A(t) = \sum_{k=0}^\infty a_k t^k \) and \( B(t) = \sum_{k=0}^\infty b_k t^k \) be two real power series converging on \((-r, r)\) \((r > 0)\) with \( b_k > 0 \) for all \( k \). If the sequence \( \{a_k/b_k\} \) is increasing (decreasing) for all \( k \), then the function \( t \mapsto A(t)/B(t) \) is also increasing (decreasing) on \((0, r)\).

**Remark 1.** Recently, another monotonicity rule in the case when the sequence \( \{a_k/b_k\}_{k \geq 0} \) is piecewise monotonic was presented in Theorem 1 of [28], which is now applied preliminarily, see for example, [29–32].

The second tool is the so-called “L’Hospital Monotone Rule” (or, for short, LMR), which is very effective in studying the monotonicity of ratios of two functions.
Lemma 2 ([33], Theorem 2). Let $-\infty < a < b < \infty$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on $(a, b)$, with $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. Assume that $g'(x) \neq 0$ for each $x$ in $(a, b)$. If $f'/g'$ is increasing (decreasing) on $(a, b)$ then so is $f/g$.

The following two lemmas will be used to prove Proposition 1.

Lemma 3 ([23], Lemma 2.8). We have

$$I_0(t)^2 = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!} t^{2n}. \tag{14}$$

Lemma 4 ([34], Problems 85, 94). The two given sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ satisfy the conditions

$$b_n > 0; \sum_{n=0}^{\infty} b_n t^n \text{ converges for all values of } t; \lim_{n \to \infty} \frac{a_n}{b_n} = s.$$

Then $\sum_{n=0}^{\infty} a_n t^n$ converges too for all values of $t$ and in addition

$$\lim_{t \to \infty} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s.$$

3. Three Propositions

The proofs of Theorems 1 and 2 rely on the following propositions.

Proposition 1. Let

$$f_0(t) = \theta^2 \frac{\cosh t + \sinh t}{3} + (1 - \theta) \left(1 + \frac{1}{2} t^2 + \frac{229}{6720} t^4\right), \tag{15}$$

where $\theta = 11,009/10,449$. The function

$$F_0(t) = \frac{I_0(t)^2}{f_0(t)}$$

is strictly decreasing from $(0, \infty)$ onto $(3/((\theta \pi), 1)$.

Proof. Expanding in power series yields

$$f_0(t) = \theta \frac{\sinh 2t + \sinh t}{3t} + (1 - \theta) \left(1 + \frac{1}{2} t^2 + \frac{229}{6720} t^4\right)$$

$$= \theta \sum_{n=0}^{\infty} \frac{2(2n+1) + 1}{3(2n+1)!} t^{2n} + (1 - \theta) \left(1 + \frac{1}{2} t^2 + \frac{229}{6720} t^4\right)$$

$$= 1 + \frac{1}{2} t^2 + \frac{387\theta + 229}{6720} t^4 + \sum_{n=3}^{\infty} \frac{\theta (2n+1) + 1}{3(2n+1)!} t^{2n} : = \sum_{n=0}^{\infty} v_n t^{2n},$$

where $v_0 = 1$, $v_1 = 1/2$,

$$v_2 = \frac{387\theta + 229}{6720} \quad \text{and} \quad v_n = \frac{\theta (2n+1) + 1}{3(2n+1)!} \quad \text{for } n \geq 3.$$

By Lemma 3, we see that

$$I_0(t)^2 = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!} t^{2n} : = \sum_{n=0}^{\infty} u_n t^{2n}. \quad \text{and} \quad u_n = \frac{\theta (2n+1) + 1}{3(2n+1)!}.$$
Direct calculations give
\[
\frac{u_0}{v_0} = \frac{u_1}{v_1} = 1, \quad \frac{u_2}{v_2} = \frac{630}{387\theta + 229}
\]
\[
\frac{u_n}{v_n} = \frac{(2n)!}{2^{2n}n!^4} \sqrt{\frac{\theta (2^{2n+1} + 1)}{3(2n)!}} \quad \text{for } n \geq 3,
\]
then
\[
\frac{u_1}{v_1} \frac{u_0}{v_0} = 0, \quad \frac{u_2}{v_2} - \frac{u_1}{v_1} = \frac{387\theta - 401}{387\theta + 229} < 0,
\]
\[
\frac{u_3}{v_3} - \frac{u_2}{v_2} = \frac{-35}{172} < 0,
\]
\[
\frac{u_{n+1}}{v_{n+1}} \frac{u_n}{v_n} - 1 = -\frac{2^{2n+1} - (3n^2 + 6n + 2)}{(n+1)^2 (2^{2n+3} + 1)} < 0 \quad \text{for } n \geq 3,
\]
where the last inequality holds due to
\[
2^{2n+1} - (3n^2 + 6n + 2) > 1 + (2n + 1) \frac{(2n+1)(2n)}{2!} + \frac{(2n+1)(2n)(2n-1)}{3!}
\]
\[
- (3n^2 + 6n + 2) = \frac{1}{3} n (4n + 5) (n - 2) > 0 \quad \text{for } n \geq 3.
\]
This shows that the sequence \( \frac{u_n}{v_n} \) is strictly decreasing, so is \( (t_0 / f_0 (t) \) on \( (0, \infty) \) by Lemma 1. It is easy to check that
\[
\lim_{t \to 0} \frac{f_0 (t)}{f_0 (t)} = \lim_{t \to 0} \frac{\theta}{\sinh t} = 1 \quad \text{and} \quad \lim_{t \to \infty} \frac{f_0 (t)}{f_0 (t)} = \lim_{t \to \infty} \frac{u_n}{v_n} = \frac{3}{\pi \theta},
\]
where the second limit holds due to Lemma 4, thereby completing the proof. \( \square \)

**Proposition 2.** Let \( f_0 (t) \) be defined by (15). The function
\[
F_1 (t) = \frac{tf_0 (t)}{\cosh^{3/2} (2t/3) \sinh t}
\]
is strictly decreasing from \( 0, \infty \) onto \( \left( \sqrt{8\theta} / 3, 1 \right) \), where \( \theta = 11,009 / 10,449. \)

**Proof.** Let
\[
f_1 (t) = \ln F_1 (t) = \ln \left[ \frac{\theta}{3} \frac{2 \cosh t + 1 \sinh t}{t} + (1 - \theta) \left( 1 + \frac{2}{3t} + \frac{229}{6720} t^4 \right) \right] - \frac{3}{2} \ln \left( \cosh \frac{2t}{3} \right) - \frac{1}{t} \ln \sinh t.
\]
Differentiation yields
\[
f_1' (t) = -\frac{1}{6t \sinh t \cosh (2t/3)} \frac{f_2 (t)}{f_0 (t)}
\]
where
\[
f_2 (t) = t^5 f_{25} (t) + t^4 f_{24} (t) + t^3 f_{23} (t) + t^2 f_{22} (t) + tf_{21} (t) + f_{20} (t),
\]
(16)
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\[
f_{25}(t) = \frac{229}{1120} (1 - \theta) \left( \cosh \frac{2t}{3} \cosh t + 3 \sinh \frac{2t}{3} \sinh t \right),
\]
\[
f_{24}(t) = \frac{229}{244} (\theta - 1) \cosh \frac{2t}{3} \sinh t,
\]
\[
f_{23}(t) = \frac{3}{2} (1 - \theta) \left( 2 \cosh \frac{2t}{3} \cosh t + 3 \sinh \frac{2t}{3} \sinh t \right),
\]
\[
f_{22}(t) = 9 (\theta - 1) \cosh \frac{2t}{3} \sinh t,
\]
\[
f_{21}(t) = 3 (1 - \theta) \left( 2 \cosh \frac{2t}{3} \cosh t + 3 \sinh \frac{2t}{3} \sinh t \right),
\]
\[
f_{20}(t) = 6 (\theta - 1) \cosh \frac{2t}{3} \sinh t - 4 \theta \cosh \frac{2t}{3} \sinh^3 t
\]
\[
+ 3 \theta \sinh \frac{2t}{3} \sinh^2 t + 6 \theta \sinh \frac{2t}{3} \cosh t \sinh^2 t.
\]

Expanding in power series gives

\[
f_{25}(3s) = -\frac{229}{4480} (\theta - 1) (5 \cosh 5s - \cosh s) = -\frac{229}{4480} (\theta - 1) \sum_{n=2}^{\infty} \frac{5^{2n-3} - 1}{(2n-4)!} s^{2n-4},
\]
\[
f_{24}(3s) = \frac{229}{448} (\theta - 1) (\sinh 5s + \sinh s) = \frac{229}{448} (\theta - 1) \sum_{n=2}^{\infty} \frac{5^{2n-3} + 1}{(2n-3)!} s^{2n-3},
\]
\[
f_{23}(3s) = -\frac{3}{4} (\theta - 1) (5 \cosh 5s - \cosh s) = -\frac{3}{4} (\theta - 1) \sum_{n=1}^{\infty} \frac{5^{2n-1} - 1}{(2n-2)!} s^{2n-2},
\]
\[
f_{22}(3s) = \frac{9}{2} (\theta - 1) (\sinh 5s + \sinh s) = \frac{9}{2} (\theta - 1) \sum_{n=1}^{\infty} \frac{5^{2n-1} + 1}{(2n-1)!} s^{2n-1},
\]
\[
f_{21}(3s) = -\frac{3}{2} (\theta - 1) (5 \cosh 5s - \cosh s) = -\frac{3}{2} (\theta - 1) \sum_{n=0}^{\infty} \frac{5^{2n+1} - 1}{(2n)!} s^{2n},
\]
\[
f_{20}(3s) = \frac{1}{4} \theta \sinh 11s + \frac{3}{4} \theta \sinh 8s - \frac{5}{4} \theta \sinh 7s + \left( \frac{15}{4} \theta - 3 \right) \sinh 5s
\]
\[
- \frac{3}{4} \theta \sinh 4s - \frac{3}{2} \theta \sinh 2s + \left( \frac{21}{4} \theta - 3 \right) \sinh s
\]
\[
= \sum_{n=0}^{\infty} \left[ \frac{\theta}{4} \frac{11}{2}^{2n+1} + \frac{3 \theta}{4} \frac{8}{2}^{2n+1} - \frac{5 \theta}{4} \frac{5}{2}^{2n+1} + \left( \frac{15 \theta}{4} - 3 \right) \frac{3}{2} \frac{3}{2}^{2n+1} \right] \frac{s^{2n+1}}{(2n+1)!}.
\]

Then \( f_2(3s) \) defined by (16) can be written as

\[
f_2(3s) = 243s^3 f_{25}(3s) + 81s^4 f_{24}(3s) + 27s^3 f_{23}(3s) + 9s^2 f_{22}(3s) + 3sf_{21}(3s) + f_{20}(3s)
\]
\[
= 54s^3 + (\theta - 1) \sum_{n=2}^{\infty} a_n^{[1]} \frac{s^{2n+1}}{(2n+1)!} + \sum_{n=2}^{\infty} a_n^{[2]} s^{2n+1} + \frac{3 \theta}{4} \sum_{n=2}^{\infty} a_n^{[3]} s^{2n+1} \frac{s^{2n+1}}{(2n+1)!}.
\]
where

\[
a_n^{[1]} = -\frac{55,647 \, (2n + 1)! \, 5^{2n-3}}{4480 \, (2n - 4)!} + \frac{18,549 \, (2n + 1)! \, 5^{2n-3}}{448 \, (2n - 3)!} + \frac{81 \, (2n + 1)! \, 5^{2n-1}}{4 \, (2n - 2)!} + \frac{81 \, (2n + 1)! \, 5^{2n-1}}{2 \, (2n - 1)!} - \frac{9 \, (2n + 1)! \, 5^{2n+1}}{2 \, (2n)!} + \frac{3 \, 550 - 4 \, 5^{2n+1}}{4 \, \theta - 1} + \frac{1 \, \theta}{4 \, \theta - 1} \times 11^{2n+1},
\]

\[
a_n^{[2]} = \frac{55,647 \, \theta - 1}{4480 \, (2n - 4)!} + \frac{18,549 \, \theta - 1}{448 \, (2n - 3)!} + \frac{81 \, \theta - 1}{4 \, (2n - 2)!} + \frac{81 \, \theta - 1}{2 \, (2n - 1)!} + \frac{9 \, \theta - 1}{2 \, (2n)!} + \frac{1 \, 3 \, (7 \, \theta - 4)}{4 \, (2n + 1)!},
\]

\[
a_n^{[3]} = 8^{2n+1} - \frac{5}{3} \times 2^{2n+1} - 4^{2n+1} - 2^{2n+2}.
\]

It remains to prove \( a_n^{[i]} > 0 \) for \( i = 1, 2, 3 \) and \( n \geq 2 \). It is clear that \( a_n^{[2]} > 0 \) due to \( \theta = 11,009/10,449 > 1 \). For \( a_n^{[3]} \), it is easy to check that

\[
a_n^{[3]} - 49a_n^{[3]} = 12 \left( 10 \times 2^{4n} + 11 \times 2^{2n} + 15 \right) \times 2^{2n} > 0,
\]

which together with \( a_2^{[3]} = 11,005 > 0 \) yields \( a_n^{[3]} > 0 \) for all \( n \geq 2 \). For \( a_n^{[1]} \), since \((5 \theta - 4) > 5 \, (\theta - 1)\) and

\[
\frac{\theta}{\theta - 1} = \frac{11,009}{560} = 19.659... > 18,
\]

we have

\[
a_n^{[1]} > -\frac{55,647 \, (2n + 1)! \, 5^{2n-3}}{4480 \, (2n - 4)!} + \frac{18,549 \, (2n + 1)! \, 5^{2n-3}}{448 \, (2n - 3)!} - \frac{81 \, (2n + 1)! \, 5^{2n-1}}{4 \, (2n - 2)!} + \frac{81 \, (2n + 1)! \, 5^{2n-1}}{2 \, (2n - 1)!} - \frac{9 \, (2n + 1)! \, 5^{2n+1}}{2 \, (2n)!} + \frac{15}{4} \times 5^{2n+1} + \frac{9}{2} \times 11^{2n+1}
\]

\[
= \frac{9}{2} \times 11^{2n+1} - 3 \left( \frac{18,549}{28} \, n^5 - \frac{154,575}{56} \, n^4 + \frac{138,915}{16} \, n^3 - \frac{1,357,425}{224} \, n^2 + \frac{848,523}{224} \, n + \frac{3125}{4} \right) \times 5^{2n-4} := a_n^{[0]}.
\]

The sequence \( \{a_n^{[0]}\}_{n \geq 2} \) satisfies the recurrence relation

\[
a_n^{[0]} - 121a_n^{[0]} = \frac{148,392}{7} \, n^5 - \frac{463,725}{4} \, n^4 + \frac{2,202,435}{7} \, n^3 - \frac{36,754,425}{112} \, n^2 + \frac{3,895,809}{56} \, n - \frac{21,875}{2}.
\]
which can be written as
\[
\frac{148,392}{7} (n - 2)^5 + \frac{2,689,605}{28} (n - 2)^4 + \frac{1,645,965}{7} (n - 2)^3 + \frac{52,997,895}{112} (n - 2)^2 + \frac{29,042,799}{56} (n - 2) + \frac{312,145}{2} > 0
\]
for \( n \geq 2 \). This in combination with \( a_n^0 = 10,126,407/16 > 0 \) leads to \( a_n^0 > 0 \) for \( n \geq 2 \), and so is \( a_n^1 \).

Therefore, \( f_1(t) > 0 \) for \( t > 0 \), so \( f_0(t) \) is strictly increasing on \((0, \infty)\). An easy computation yields
\[
\lim_{t \to 0} f_1(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} f_1(t) = \ln \frac{\sqrt{5} \theta}{3},
\]
which completes the proof. \( \square \)

Using Lemma 2 we can prove the following lemma, which will be use to prove Theorem 2.

**Proposition 3.** Let \( q \neq 0, 1/2, 1 \). The ratio
\[
t \mapsto \frac{\cosh^{1/4}(qt) - 1}{\cosh t - 1}
\]
is strictly increasing on \((0, \infty)\) if \( q \in (-\infty, 0) \cup (1/2, 1) \) and strictly decreasing on \((0, \infty)\) if \( q \in (0, 1/2) \cup (1, \infty) \). Consequently, the double inequality
\[
q \cosh t + 1 - q < \cosh^{1/4}(qt) < c_q \cosh t + 1 - c_q
\]
holds for \( t > 0 \) if \( q \in (-\infty, 0) \cup (1/2, 1) \), where the weights \( q \) and \( c_q = 2^{1-1/q} \) if \( q > 0 \) and \( c_q = 0 \) if \( q < 0 \) are the best possible. If \( q \in (0, 1/2) \cup (1, \infty) \), then the double inequality (17) is reversed.

**Proof.** Let
\[
g_1(t) = \cosh^{1/4}(qt) - 1 \quad \text{and} \quad g_2(t) = \cosh t - 1.
\]
Clearly, \( g_1(0^+) = g_2(0^+) = 0 \), and
\[
\lim_{t \to 0} g_1(t) = q \quad \text{and} \quad \lim_{t \to \infty} g_1(t) = c_q = \begin{cases} 2^{1-1/q} & \text{if } q > 0, \\ 0 & \text{if } q < 0. \end{cases}
\]

Differentiation yields
\[
\frac{g_1'(t)}{g_2'(t)} = \frac{\cosh^{1/4-1}(qt) \sinh (qt)}{\sinh t} = \frac{1}{2} \frac{(1 - 2q) t}{\cosh^{2-1/q}(qt)} \left( \frac{\sinh [(1 - 2q)t]}{\sinh t} \right) - \frac{\sinh [(1 - 2q)t]}{\sinh t}.
\]

Since the function \((\sinh x)/x\) is strictly increasing on \((0, \infty)\), we find that
\[
\begin{array}{ll}
\frac{g_1'(t)}{g_2'(t)} > 0 & \text{if } (|1 - 2q| - 1)(1 - 2q) > 0, \text{ i.e., } q \in (-\infty, 0) \cup \left(\frac{1}{2}, 1\right), \\
< 0 & \text{if } (|1 - 2q| - 1)(1 - 2q) < 0, \text{ i.e., } q \in (1, \infty) \cup \left(0, \frac{1}{2}\right).
\end{array}
\]

By Lemma 2, the desired monotonicity follows. The double inequality (17) and its reverse follow from the monotonicity of \( g_1(t)/g_2(t) \) on \((0, \infty)\). This completes the proof. \( \square \)
Remark 2. Taking \( q = p_0 = (\ln 2) / \ln \pi \) in the double inequality (17) we obtain the double inequality
\[
p_0 \cosh t + 1 - p_0 < \cosh^{1/p_0} (p_0 t) < \frac{2}{\pi} \cosh t + 1 - \frac{2}{\pi}
\]  
for \( t > 0 \).

Remark 3. The generalized Heronian mean \([35]\) is defined by
\[
H_w (a, b) = \frac{a + b + w\sqrt{ab}}{w + 2}.
\]

Let \( t = \ln \sqrt{a/b} \) with \( b > a > 0 \) and \( q = w/(w + 2) > 0 \). Then Proposition 3 give a best approximation for \( H_w (a, b) \) by power means:
\[
H_w (a, b) < A_{w/(w+2)} (a, b) \text{ if } w \in (2, \infty), \\
H_w (a, b) > A_{w/(w+2)} (a, b) \text{ if } q \in (0, 2).
\]

Our proof is clearly concise than Li, Long and Chu’s given in \([35]\).

4. Proofs of Theorem 1 and 2

We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1. We have
\[
F (t) = \frac{t I_0 (t)^2}{\cosh^{3/2} (2t/3) \sinh t} = \frac{I_0 (t)^2}{f_0 (t)} \times \frac{tf_0 (t)}{\cosh^{3/2} (2t/3) \sinh t} = F_0 (t) \times F_1 (t).
\]

As shown in Propositions 1 and 2, the functions \( F_0 (t) \) and \( F_1 (t) \) are both strictly positive and decreasing on \((0, \infty)\), so is \( F (t) \). And, we easily obtain
\[
\lim_{t \to 0} F (t) = \lim_{t \to 0} F_0 (t) \times \lim_{t \to 0} F_1 (t) = 1, \\
\lim_{t \to \infty} F (t) = \lim_{t \to \infty} F_0 (t) \times \lim_{t \to \infty} F_1 (t) = \frac{3}{\sqrt{8}} \frac{\sqrt{3} \theta}{3} = \frac{\sqrt{8}}{\pi}.
\]

Using the monotonicity of \( F (t) \), the desired double inequality follows. This completes the proof. \( \square \)

Proof of Theorem 2. (i) The necessary condition for the right hand side inequality of (12) to hold follows from the limit relation
\[
\lim_{t \to 0} \frac{I_0 (t)^2 - \cosh^{1/p} (pt) (\sinh t) / t}{t^2} = -\frac{1}{6} (3p - 2) \leq 0.
\]

The sufficiency follow from Theorem 1 and the increasing property of \( p \mapsto \cosh^{1/p} (pt) \) on \( \mathbb{R} \).

(ii) The necessary condition for the left hand side inequality of (12) to hold follows from the limit relation
\[
\lim_{t \to \infty} \frac{\cosh (qt) \frac{1/2}{t} (\sinh t) / t}{I_0 (t)^2} \leq 1.
\]

Since \( I_0 (t) \sim e^t / \sqrt{2 \pi t} \) as \( t \to \infty \) (see \([36]\), 9.7.1) and
\[
\cosh^{1/q} (qt) \frac{\sinh t}{t} \leq \frac{\sinh t}{t} \sim \frac{e^t}{2t} \text{ if } q \leq 0, \\
\cosh^{1/q} (qt) \frac{\sinh t}{t} = e^t \left( \frac{1 + e^{-2qt}}{2} \right)^{1/q} \sim \frac{e^t}{2^{1/q} \sqrt{2t}},
\]

\[\text{Proof of Theorem 2. (ii).)}\]
we have
\[
\lim_{t \to \infty} \frac{\cosh^{1/q} (qt) (\sinh t) / t}{I_0 (t)^2} = \left\{ \begin{array}{ll}
0 & \text{if } q \leq 0, \\
\frac{\pi}{2^{1/4}} & \text{if } q > 0.
\end{array} \right.
\]

Therefore, the necessary condition is that \( \pi/2^{1/4} \leq 1 \) if \( q > 0 \) and \( q \leq 0 \), that is, \( q \leq (\ln 2) / \ln \pi = p_0 \).

By the increasing property of \( q \mapsto \cosh^{1/q} (qt) \), to prove the sufficiency, it suffices to prove the
left hand side inequality of (12) holds when \( q = p_0 \). From the first inequality of (7) and the second inequality of (18) it follows that
\[
I_0 (t) > \sqrt{\frac{\sinh t}{t}} \left( \frac{2}{\pi} \cosh t + 1 - \frac{2}{\pi^2} \right) > \sqrt{\frac{\sinh t}{t}} \cosh^{1/p_0} (p_0 t)
\]
for \( t > 0 \), which proves the sufficiency, and the proof is completed. \( \square \)

5. Concluding Remarks

In this paper, we obtained the best constants \( p \) and \( q \) such that the double inequality (12) holds for \( t > 0 \), or equivalently, (11) holds for \( a, b > 0 \) with \( a \neq b \). This improved the result in [24]. We close the paper by giving two remarks on our results.

Remark 4. It was shown in ([20], 5.25) that
\[
A_{2/3} (a, b) < I (a, b) < \frac{2\sqrt{2}}{e} A_{2/3} (a, b).
\]
Then the double inequality (11) can be extended as
\[
\sqrt{\frac{e}{\pi}} \sqrt{L (a, b) I (a, b)} < \frac{2^{3/4}}{\sqrt{\pi}} \sqrt{L (a, b) A_{2/3} (a, b)} < TQ (a, b) \\
< \sqrt{L (a, b) A_{2/3} (a, b)} < \sqrt{L (a, b) I (a, b)}.
\]

Remark 5. As a computable bound, the upper bound \( \sqrt{t^{-1} \sinh t \cosh^{3/2} (2t/3)} \) for \( I_0 (t) \) is superior to those given (6) and (8). In fact, we have
\[
I_0 (t) < \sqrt{\frac{\sinh t}{t} \cosh^{3/2} \left( \frac{2t}{3} \right)} < \sqrt{\frac{\sinh t}{t} \cosh t} = \sqrt{\frac{\sinh (2t)}{2t}} \quad (19)
\]
and
\[
I_0 (t) < \sqrt{\frac{\sinh t}{t} \cosh^{3/2} \left( 2t/3 \right)} < \frac{3}{4} \frac{\sinh t}{t} + \frac{1}{4} \cosh t \quad (20)
\]
for \( t > 0 \). The inequalities (19) are clear, and we have to check (20). Let
\[
h (t) = \ln \left[ \frac{\sinh (3t/2)}{3t/2} \cosh^{3/2} (t) \right] - \ln \left[ \frac{3 \sinh (3t/2)}{4 - 3t/2} + \frac{1}{4} \cosh (3t/2) \right] \cdot
\]
Differentiation yields
\[
h' (t) = \frac{1}{6} t \left[ \frac{h_1 (t)}{\sinh (3t) + t \cosh (3t) \cosh (2t) \cosh (3t/2)} \right]'
\]
where
\[ h_1 (t) = t^2 \left( 3 \cosh 2t \cosh^2 3t + 3 \sinh 2t \cosh 3t \sinh 3t - 6 \cosh 2t \sinh^2 3t \right) \]
\[ + \left( 3 \sinh 2t \sinh^2 3t - 4 \cosh 2t \cosh 3t \sinh 3t \right) t + \cosh 2t \sinh^2 3t. \]

Using “product into sum” formulas for hyperbolic functions and expanding in power series give
\[ h_1 (t) = \frac{9}{2} t^2 \cosh 2t - \frac{3}{2} t^2 \sinh 4t - \frac{3}{2} t \sinh 2t - \frac{7}{4} t \sinh 4t - \frac{1}{4} t \sinh 8t \]
\[ + \frac{1}{4} \cosh 4t - \frac{1}{2} \cosh 2t + \frac{1}{4} \cosh 8t. \]
\[ h_1 (t) = \frac{9}{2} \sum_{n=1}^{\infty} \frac{2n-2}{(2n-2)!} t^{2n} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{4n-2}{(2n-2)!} t^{2n} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{2n-1}{(2n-1)!} t^{2n} - \frac{7}{4} \sum_{n=1}^{\infty} \frac{4n-1}{(2n-1)!} t^{2n} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{8n-1}{(2n)!} t^{2n} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{4n}{(2n)!} t^{2n} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{2n}{(2n)!} t^{2n} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{8n}{(2n)!} t^{2n} - \frac{1}{16} \sum_{n=1}^{\infty} \frac{b_n (2t)^{2n}}{(2n)!}, \]

where
\[ b_n = (n - 4) 4^{2n} + \left( 6n^2 + 11n - 4 \right) 2^{2n} - 4 \left( 18n^2 - 15n - 2 \right). \]

Since \( b_1 = b_2 = 0 \), \( b_3 = 756 \) and for \( n \geq 4 \),
\[ b_n \geq \left( 6n^2 + 11n - 4 \right) 2^n - 4 \left( 18n^2 - 15n - 2 \right) = 4 \left( 366n^2 + 719n - 254 \right) > 0, \]

we have \( h_1 (t) < 0 \) for \( t > 0 \), so is \( h' (t) \). This leads to \( h (t) < \lim_{t \to 0} h (t) = 0 \), which proves the second inequality of (20) holds for \( t > 0 \).

**Remark 6.** Due to
\[ L (a, b) = \frac{a - b}{\ln a - \ln b} = \int_0^1 a^s b^{1-s} ds, \]
\[ TQ (a, b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta = \frac{1}{\pi} \int_0^1 a^{s} b^{1-s} (s (1-s))^{-1/2} ds, \]
the referee introduces a new family of means \( L_\alpha (a, b) \) defined for \( \alpha > 0 \) by
\[ L_\alpha (a, b) = \frac{\Gamma (2\alpha)}{\Gamma (\alpha)^2} \int_0^1 a^s b^{1-s} (s (1-s))^{\alpha-1} ds. \]

The referee also gives an interesting relation between this new mean and the modified Bessel functions of the first kind:
\[ \frac{L_\alpha (a, b)}{\sqrt{ab}} = \Gamma \left( \alpha + \frac{1}{2} \right) \left( \frac{t}{2} \right)^{1/2-\alpha} L_{\alpha-1/2} (t), \quad t = \ln \sqrt{\frac{a}{b}}. \]

It is easy to check that
\[ \lim_{\alpha \to 0} L_\alpha (a, b) = \frac{a + b}{2}. \]
\[ L_\alpha(a, b) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} \int_0^1 a^s b^{1-s} (s (1-s))^{\alpha-1} ds < \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} \int_0^1 (sa + (1-s)b) (s (1-s))^{\alpha-1} ds = \frac{a+b}{2}. \]

However, more problems remain to be researched on this new family of means, for example: (i) checking the monotonicity of this mean with respect to the parameter \( \alpha \); (ii) finding the lower and upper bounds for this mean in terms of elementary means; (iii) comparing this new mean with others.

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