Abstract

The Congested Clique model of distributed computing, which was introduced by Lotker, Patt-Shamir, Pavlov, and Peleg [SPAA’03, SICOMP’05] and was motivated as “a simple model for overlay networks”, has received extensive attention over the past few years. In this model, nodes of the system are connected as a clique and can communicate in synchronous rounds, where per round each node can send $O(\log n)$ bits to each other node, all the same time. The fact that this model allows each node to send and receive a linear number of messages at the same time seems to limit the relevance of the model for overlay networks. Towards addressing this issue, in this paper, we introduce the Node-Congested Clique as a general communication network model. Similarly to the Congested Clique model, the nodes are connected as a clique and messages are sent in synchronous communication rounds. However,
here, per round, every node can send and receive only $O(\log n)$ many messages of size $O(\log n)$.

To initiate research on our network model, we present distributed algorithms for the Minimum Spanning Tree, BFS Tree, Maximal Independent Set, Maximal Matching, and Coloring problem for an input graph $G = (V,E)$, where each clique node initially only knows a single node of $G$ and its incident edges. For the Minimum Spanning Tree problem, our runtime is polylogarithmic. In all other cases the runtime of our algorithms mainly depends on the arboricity $a$ of $G$, which is a constant for many important graph families such as planar graphs. At the core of these algorithms is a distributed algorithm that assigns directions to the edges of $G$ so that at the end, every node is incident to at most $O(a)$ outgoing edges.

1 Introduction

A standard assumption for network algorithms is that we have a fixed communication network whose topology can be modeled as an undirected or directed graph $G = (V,E)$. Many variants of message passing models have been studied for these communication networks. In the synchronous setting, in which message passing is performed in synchronous rounds of communication, the most prominent models are the LOCAL and CONGEST model [49]. In the LOCAL model, any finite amount of information can be sent along an edge in one round, while in the CONGEST model every node can only send a message consisting of $O(\log n)$ bits along each of its incident edges per round, where $n$ is the total number of nodes in the system. Many fundamental results have been shown for these models that significantly advanced our understanding of the complexity of various computational problems. However, as these models focus on edge capacities, their usefulness is limited when it comes to the study of overlay networks.

Nowadays, most of the distributed systems and applications do not have a dedicated communication infrastructure, but instead share a common physical network with many others. The logical network formed on top of this infrastructure is called an overlay network. For these systems, the amount of information that a node can send out in a single round does not scale linearly with the number of its incident edges. Instead, it rather depends on the bandwidth of the connection of the node to the communication infrastructure as a whole. For these networks, it is therefore more reasonable to impose a bound on the amount of information that a node can send and receive in one round, rather than imposing a bound on the amount of information that can be sent along each of its incident edges. Also, the topology of the overlay
network may change over time, and these changes are usually under the
cut of the distributed application. To capture these aspects, we propose
to study the so-called **Node-Congested Clique**. The Node-Congested Clique
model is inspired in part by the Congested Clique model introduced first by
Lotker, Patt-Shamir, Pavlov, and Peleg [44], which has received significant
attention recently [7,9,10,12,20,23,25,27,30,33,34,42,44].

Similarly to the Congested Clique model, the nodes of the Node-Congested
Clique are interconnected by a complete graph. However, in the Node-
Congested Clique every node can only send and receive at most \( O(\log n) \)
messages consisting of \( O(\log n) \) bits in each round. This limitation is added
precisely to address the issue explained above. It particularly rules out the
possibility that the model allows one node to be in contact with up to \( \Theta(n) \)
other nodes at the same time; a property of the Congested Clique that seems
to severely limit its practicality. We comment that the bound of \( O(\log n) \)
messages per node per round is a natural choice: it ensures a high scalability
while avoiding the design of unnecessarily complicated algorithms, which
would be the case for a bound of \( o(\log n) \).

Compared to traditional overlay network research, the Node-Congested
Clique model has the advantage that it abstracts away the issue of designing
and maintaining a suitable overlay network, for which many solutions have
already been found in recent years. Nevertheless, it is closely related to
overlay networks: every overlay network algorithm can be simulated in the
Node-Congested Clique without any overhead, and the Node-Congested
Clique can be simulated with just \( O(\log n) \) overhead in the CRCW PRAM
model, which in turn can be simulated with only \( O(\log n) \) overhead by
a network of constant degree [51]. The Congested Clique model and its
broadcast variant, on the other hand, are far more powerful (and arguably
beyond what is possible in overlay networks): Whereas in the Congested
Clique a total of \( \Theta(n^2) \) bits can be transmitted in each round, in the Node-
Congested Clique only \( \Theta(n) \) bits may be sent. For example, the gossip
problem — i.e., delivering one message from each node to each other node
— can be solved in a single round in the Congested Clique, whereas the
problem requires at least \( \Omega(n/\log n) \) rounds in the Node-Congested Clique
model. Even the simple broadcast problem — i.e., delivering one message
from one node to all nodes — already takes time \( \Omega(\log n/\log \log n) \) in the
Node-Congested Clique.

We also note that the results for graph problems in the Node-Congested
Clique turn out to be useful for some other theoretical models as well:
they are relevant for **hybrid networks** [24] and also the **k-machine model of
processing large scale graphs** [32].
Hybrid networks is a concept that has just recently been considered in theory (e.g., [24]). In a hybrid network, nodes have different communication modes: We are given a network of cheap links of arbitrary topology that is not under the control of the nodes and may potentially be changing over time. In addition to that, the nodes have the ability to build arbitrary overlay networks of costly links that are fully under the control of the nodes. Cell phones, for example, can communicate in an ad-hoc fashion via their WiFi interfaces, which is for free but only has a limited range, and whose connections may change as people move. Moreover, they may use their cellular infrastructure, which comes at a price, but remains fully under their control. Although in the idealized setting this overlay network may form a clique, to save costs, the nodes might want to exchange only a small amount of messages of small size in each communication round. This can be captured by the Node-Congested Clique. The network of cheap links, on the other hand, can be seen as the input graph for the Node-Congested Clique for which the nodes want to solve a graph problem of interest.

Another interesting application of the Node-Congested Clique is the recently introduced $k$-machine model [32], which was designed for the study of data center level distributed algorithms for large scale graph problems. Here, a data center with $k$ servers is modeled as $k$ machines that are fully interconnected and capable of executing synchronous message passing algorithms. A standard approach for the $k$-machine model is to partition the input graph in a fair way so that each machine stores a set of nodes of the input graph with their incident edges. It is quite natural to simulate algorithms designed for the Node-Congested Clique model in the $k$-machine model. Precisely, any algorithm that requires $T$ rounds in the Node-Congested Clique model can be simulated to take at most time $\tilde{O}(nT/k^2)$. The details of this simulation can be found in the Appendix. To illustrate the usefulness of this simulation, we remark that the running time of the fast minimum spanning tree algorithm provided by Pandurangan et al. [48] can be obtained simply by converting the algorithm we provide in this work to the $k$-machine model.

As we will demonstrate in this paper, many graph problems can be solved efficiently in the Node-Congested Clique, which shows that many interesting problems can be solved efficiently in distributed systems based on an overlay network over a shared infrastructure as well as hybrid networks and server systems.
1.1 Model and Problem Statement

In the Node-Congested Clique model we consider a set $V$ of $n$ computation entities that we model as nodes of a graph. Each node has a unique identifier consisting of $O(\log n)$ bits and the nodes know the identifiers of all nodes such that, on a logical level, they form a complete graph. Note that since every node knows the identifier of every other node, the nodes also know the total number of nodes $n$. Without loss of generality we assume that the identifiers are from the set $\{0, 1, \ldots, n-1\}$.

The network operates in a synchronous manner with time measured in rounds. In every round, each node can perform an arbitrary amount of local computation and send distinct messages consisting of $O(\log n)$ bits to up to $O(\log n)$ other nodes. The messages are received at the beginning of the next round. A node can receive up to $O(\log n)$ messages. If more messages are sent to a node, it receives an arbitrary subset of $O(\log n)$ messages. Additional messages are simply dropped by the network.

Let $G$ be an undirected graph $G = (V, E)$ with an arbitrary edge set, but the same node set as the Node-Congested Clique. We aim to solve graph problems on $G$ in the Node-Congested Clique model. At the beginning, each node locally knows which identifiers correspond to its neighbors in $G$, but has no further knowledge about the graph.

1.2 Related Work

The Congested Clique model has already been studied extensively in the past years. Problems studied in prior work include routing and sorting [42], minimum spanning trees [23, 25, 30, 34, 41], subgraph detection [9, 12], shortest path problems [7, 9], local problems [10, 26, 27], and problems related to matrix multiplication [9, 18]. Some of the upper bounds are astonishingly small, such as the constant-time upper bound for routing and sorting and for the computation of a minimum spanning tree, demonstrating the power of the Congested Clique model.

While almost no non-trivial lower bounds exist for the Congested Clique model (due to their connection to circuit complexity [13]), various lower bounds have already been shown for CONGEST model [15, 17, 39, 40, 46, 50, 52]. As pointed out in [35], the reductions used in these lower bounds usually boil down to constructing graphs with bottlenecks, that is, graphs where large amounts of information have to be transmitted over a small cut. As this is not the case for the Node-Congested Clique, the lower bounds are of limited use here. Therefore, it remains interesting to determine upper and
lower bounds for the Node-Congested Clique.

Hybrid networks have only recently been studied in theory. An example is the hybrid network model proposed in [24], which allows the design of much faster distributed algorithms for graph problems than with a classical communication network. Also the problem of finding short routing paths with the help of a hybrid network approach has been considered recently [29]. A priori, these papers do not assume that the nodes are completely interconnected, so extra measures have to be taken to build up appropriate overlays. Abstracting from that problem, the Node-Congested Clique allows one to focus on how to efficiently exchange information in order to solve the given problems.

The graph problems considered in this paper have already been extensively studied in many different models. In the CONGEST model, for example, a breadth-first search can trivially be performed in time $O(D)$. There exists an abundance of algorithms to solve the maximal independent set, the maximal matching, and the coloring problem in the CONGEST model (see, e.g., [5] for a comprehensive overview). Computing a minimum spanning tree has also been well studied in that model, see for example [15, 16, 50]. Whereas the running times of the above-mentioned algorithms depend on $D$ and additional polylogarithmic factors, there have only recently been proposed algorithm to solve such problems more efficiently in graphs with small arboricity [2, 5, 36, 37]. Notably, Barenboim and Khazanov [6] show how to solve a variety of graph problems in the Congested Clique efficiently given such graphs, e.g., compute an $O(a)$-orientation in time $O(\log a)$, an MIS in time $O(\sqrt{a})$, and an $O(a)$-coloring in time $O(a^\epsilon)$. The algorithms make use of the Nash-Williams forests-decomposition technique [47], which is one of the key techniques used in our work.

1.3 Our Contribution

We present a set of basic communication primitives and then show how they can be applied to solve certain graph problems (see Table 1 for an overview). Note that for many important graph families such as planar graphs, our algorithms have polylogarithmic runtime (except when depending on the diameter $D$).

Although many of our algorithms rely on existing algorithms from literature, we point out that most of these algorithms cannot be executed in the Node-Congested Clique in a straight-forward fashion. The main reason for that is that high-degree nodes cannot efficiently communicate with all of their neighbors directly in our model, which imposes significant difficulties
Table 1: An overview of our results. We use $a$ for arboricity and $D$ to denote the diameter of the given graph.

to the application of the algorithms. To overcome these difficulties, we present a set of basic tools that still allow for efficient communication, and combine it with variations of well-known algorithms and novel techniques. Notably, we present an algorithm to compute an orientation of the graph, in which each edge gets assigned a direction, ensuring that the outdegree of any node is at most $O(a)$. The algorithm is later used to efficiently construct multicast trees to be used for communication between nodes. Achieving this is a highly nontrivial task in our model and requires a combination of techniques, ranging from aggregation and multicasting to shared randomness and coding techniques. We believe that many of the presented ideas might also be helpful for other applications in the Node-Congested Clique.

Although proving lower bounds for the presented problem seems to be a highly nontrivial task, we believe that many problems require a running time linear in the arboricity. For the MIS problem, for example, it seems that we need to communicate at least 1 bit of information about every edge (typically in order for a node of the edge to learn when the edge is removed from the graph because the other endpoint has joined the MIS). However, explicitly proving such a lower bound in this model seems to require more than our current techniques in proving multi-party communication complexity lower bounds.

2 Preliminaries

In this section, we first give some basic definitions, observations and tools used in throughout this paper.
2.1 Basic Definitions and Notation

Let $G = (V, E)$ be an undirected graph. The neighborhood of a node $u$ is defined as $N(u) = \{v \in V \mid \{u, v\} \in E\}$, and $d(u) = |N(u)|$ denotes its degree. With $\Delta = \max_{u \in V} (d(u))$ we denote the maximum degree of all nodes in $G$, and $\bar{d} = \sum_{u \in V} d(u)/n$ is the average degree of all nodes. The diameter $D$ of $G$ is the maximum length of all shortest paths in $G$.

The arboricity $a$ of $G$ is the minimum number of forests into which its edges can be partitioned. Since the edges of any graph with maximum degree $\Delta$ can be greedily assigned to $\Delta$ forests, $a \leq \Delta$. Furthermore, since the average degree of a forest is at most 2, and the edges of $G$ can be partitioned into $a$ forests, $\bar{d} \leq 2a$. Graphs of many important graph families have small arboricity although their maximum degree might be unbounded. For example, a tree obviously has arboricity 1. Nash-Williams \[47\] showed that the arboricity of a graph $G$ is given by $\max_{H \subseteq G} (m_H/(n_H - 1))$, where $H \subseteq G$ is a subgraph of $G$ with at least two nodes and $n_H$ and $m_H$ denote the number of nodes and edges of $H$, respectively. Therefore, any planar graph, which has at most $6n - 3$ edges, has arboricity at most 3. In fact, any graph with genus $g$, which is the minimum number of handles that must be added to the plane to embed the graph without any crossings, has arboricity $O(\sqrt{g})$ \[3\]. Furthermore, it is known that the family of graphs that exclude a fixed minor \[11\] and the family of graphs with bounded treewidth \[14\] have bounded arboricity.

An orientation of $G$ is an assignment of directions to each edge, i.e., for every $\{u, v\} \in E$ either $u \rightarrow v$ ($u$ is directed to $v$) or $v \rightarrow u$ ($v$ is directed to $u$). If $u \rightarrow v$, then $u$ is an in-neighbor of $v$ and $v$ is an out-neighbor of $u$. For $u \in V$ define $N_{in}(u) = \{v \in V \mid v \rightarrow u\}$ and $N_{out}(u) = \{v \in V \mid u \rightarrow v\}$. The indegree of a node $u$ is defined as $d_{in}(u) = |N_{in}(u)|$ and its outdegree is $d_{out}(u) = |N_{out}(u)|$. A $k$-orientation is an orientation with maximum outdegree $k$. For a graph with arboricity $a$, there always exists an $a$-orientation: we root each tree of every forest arbitrarily and direct every edge from child to parent node.

2.2 Probability Theory

Since the outcome of many of our algorithms heavily rely on randomization, we require their correctness and runtime bounds to hold with high probability (w.h.p.). We say an event holds with high probability if it holds with probability at least $1 - 1/n^c$ for any constant $c > 0$. To bound the probability of certain events, we use a generalization of the Chernoff bound in \[54\].

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Lemma 1. Let $X_1, \ldots, X_n$ be $k$-wise independent random variables with $X_i \in [0, b]$ and let $X = \sum_{i=1}^n X_i$. Then it holds for all $\delta \geq 1$, $\mu \geq E[X]$, and $k \geq \lceil \delta \mu \rceil$

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-\min[\delta^2, \delta] \cdot \mu / (3b)}.$$ 

2.3 The Butterfly Network

Our communication primitives rely on an emulation of the so-called butterfly network. In the following, we denote $[k] = \{0, \ldots, k - 1\}$. For $d \in \mathbb{N}$, the $d$-dimensional butterfly is a graph with node set $[d + 1] \times [2^d]$ and an edge set $E_1 \cup E_2$ with

$$E_1 = \{(i, \alpha), (i + 1, \alpha)\} \mid i \in [d], \alpha \in [2^d];$$
$$E_2 = \{(i, \alpha), (i + 1, \beta)\} \mid i \in [d], \alpha, \beta \in [2^d],$$

$\alpha$ and $\beta$ differ only at the $i$-th bit.

The node set $\{(i, j) \mid j \in [2^d]\}$ represents level $i$ of the butterfly, and node set $\{(i, j) \mid i \in [d + 1]\}$ represents column $j$ of the butterfly. An example of a butterfly can be found in Figure 1.

2.4 Aggregate Functions

To allow nodes to efficiently gather information sent to it by other nodes, our communication primitives make heavy use of aggregate functions. An aggregate function $f$ maps a multiset $S = \{x_1, \ldots, x_N\}$ of input values to some value $f(S)$. For some functions $f$ it might be hard to compute $f(S)$ in a distributed fashion, so we will focus on so-called distributive aggregate functions: An aggregate function $f$ is called distributive if there is an aggregate function $g$ such that for any multiset $S$ and any partition $S_1, \ldots, S_\ell$ of $S$, $f(S) = g(f(S_1), \ldots, f(S_\ell))$. Classical examples of distributive aggregate functions are MAX, MIN, and SUM.

2.5 The Random Rank Protocol

Finally, our communication primitives make use of a variant of the random rank protocol, which has its origin in a paper by Aleliunas [1] and Upfal [55], and can be found in a similar form as described below in Leighton’s book [40].

A path collection $P = \{p_1, \ldots, p_N\}$ is a leveled path collection if every node $v$ can be given a level $\ell(v) \in \mathbb{N}$ so that for every edge $(v, w)$ of a path in that collection, $\ell(w) = \ell(v) + 1$. The depth of $P$ is the minimum over all level
assignments of the maximum level of a node, and the congestion of \( P \) is the maximum number of paths in which an edge is contained. The random rank protocol routes packets along an arbitrary leveled path collection of size \( N \), one packet per path, with congestion \( C \) and depth \( D \) in \( O(C + D + \log N) \) steps, w.h.p.

The random rank protocol works as follows: At the beginning, every packet \( p \) gets assigned a random rank denoted by \( \text{rank}(p) \) that is stored in its routing information. We require \( \text{rank}(p) \) to be chosen uniformly and independently at random from some fixed range \([K]\) to be determined later. Additionally, each packet stores an identification number \( \text{id}(p) \) in its routing information that is different from all other identification numbers of the other packets. The random rank protocol uses the following contention resolution rule: If two or more packets contend to use the same link at the same time, then the one with minimal rank is chosen to be forwarded first. If two packets have the same rank, then the one with minimum identifier is preferred.

## 3 Communication Primitives

Our algorithms to solve graph problems make heavy use of a set of communication primitives, which are presented in this section. The general idea of the primitives is to distribute local communication load over all nodes of the network by using an emulation of the \( \lceil \log n \rceil \)-dimensional butterfly network: Every node \( u \) with \( \text{id}(u) \leq 2^d - 1 \) emulates all nodes of column \( \text{id}(u) \) of the butterfly. Since \( u \) knows the identifiers of all other nodes, it knows exactly which nodes emulate its neighbors in the butterfly network. Since the butterfly has degree 4, and every node in the Node-Congested Clique can send and receive \( O(\log n) \) messages in each round, a communication round in the butterfly can be simulated in a single round in the Node-Congested Clique. To distinguish the emulated butterfly nodes from the nodes of the Node-Congested Clique, in the following we refer to them as BF-nodes. Moreover, we sometimes refer to a BF-node \((i, j)\) as BF-node \( j \) of level \( i \).

The communication primitives presented in this section are specifically tailored for the algorithms in this paper. For a broader applicability, some of the primitives can be generalized to solve more complex problems efficiently, e.g., to allow nodes to be source of multiple multicasts, or target of multiple distinct multi-aggregations.
3.1 Aggregate-and-Broadcast Algorithm

The first and most basic communication primitive is the Aggregate-and-Broadcast Algorithm. Assume we are given a distributive aggregate function $f$ and a set $A \subseteq V$, where each member of $A$ stores exactly one input value. The goal is to aggregate all input values so that eventually every node in $V$ knows $f$(inputs of $A$). This problem can easily be solved using the emulated butterfly network. From a high-level perspective, the nodes first aggregate all data at a single node using a path system in the butterfly, and then distribute the result to all nodes using the same path system in the other direction.

More precisely, first, every node that stores an input value, but does not emulate a node of the butterfly (in which case the most significant bit of its identifier must be 1), sends it to the BF-node $j$ of level 0 such that $j$ equals the remaining bits of its identifier. Afterwards, every BF-node of level 0 stores at most two input values, i.e., its own value and at most one value of a node that does not emulate a node of the butterfly. Note, that for every BF-node of level 0 there is a unique path of length $d$ from that node to any BF-node of level $d$ in the butterfly. In the aggregation phase, we send all input values to BF-node 0 of level $d$, which in the following we refer to as the root of the butterfly, along that path system. Whenever two values $x, y$ reach the same BF-node $u$, $u$ only forwards $g(\{x, y\})$. Thereby, the root eventually computes the aggregate of all values. This value is finally broadcast to all BF-nodes of level 0 in the broadcast phase: Every BF-node of level $i$ that receives the value forwards it to all of its neighbors in level $i - 1$. Finally, every node that does not emulate a BF-node gets informed by the BF-node of level 0 whose identifier differs only in the most significant digit. Since $d = O(\log n)$, we have the following lemma.

Lemma 2. The Aggregate-and-Broadcast Algorithm takes time $O(\log n)$.

In the remainder of this paper, we will extensively use the above algorithm to achieve synchronization: Assume that the nodes execute some distributed algorithm that finishes in different rounds at the nodes. In order to start a follow-up algorithm at the same round, the nodes can make use of a slight modification of the Aggregate-and-Broadcast algorithm: Every node delays its participation in the aggregation phase until it has finished the current algorithm. Once it has finished, it sends a token to its corresponding BF-node at level 0. Once a BF-node at level 0 has received a token from each node of the Node-Congested Clique associated with it, it sends a token in the direction of the root. Similarly, once a BF-node at level $i > 0$ has received
tokens from both incoming edges, it sends a token in the direction of the root. Thus, once the root has received tokens from both incoming edges, it knows that all nodes have finished the current algorithm. The broadcast phase will then allow all nodes to start the follow-up algorithm at the same round. It is easy to see that the synchronization just produces an overhead of $O(\log n)$ rounds.

Finally, we also use the algorithm to achieve shared randomness. Although in the remainder of this paper we assume that all hash functions behave like perfect random functions, it can be shown that it suffices to use $\Theta(\log n)$-wise independent hash functions (see, e.g., [8] and the references therein). By partitioning events in a suitable way, and using Lemma [1], we can show that all of our results still hold with high probability. To agree on such hash functions, the root only needs to broadcast $\Theta(\log^2 n)$ random bits to all nodes in the broadcast phase, which can be done by broadcasting $\Theta(\log n)$ messages consisting of $\log n$ bits in a pipelined fashion. That is, whenever a BB-node has sent out a message, it already receives the next message in the subsequent round and forwards it immediately. Thereby, all messages are broadcast in time $O(\log n)$.

### 3.2 Aggregation Algorithm

Next, we introduce the *Aggregation Algorithm*, which allows a node to efficiently communicate with all nodes in its (possibly very large) neighborhood, under the requirement that messages sent to the same node can be combined. As in the previous algorithm, the main idea is to distribute the communication load among all nodes of the network using the simulated butterfly. More formally, we are given a distributive aggregate function $f$ and a set of aggregation groups $A = \{A_1, \ldots, A_N\}$, $A_i \subseteq V$, $i \in \{1, \ldots, N\}$ with targets $t_1, \ldots, t_N \in V$, where each node holds exactly one input value $s_{u,i}$ for each aggregation group $A_i$ of which it is a member, i.e., $u \in A_i$. Note that a node may be member or target of multiple aggregation groups.

The goal is to aggregate these input values so that eventually $t_i$ knows $f(s_{u,i} \mid u \in A_i)$ for all $i$. We define $L = \sum_{i=1}^{N} |A_i|$ to be the global load and $\ell = \max_{u \in V} \{|i \in \{1, \ldots, N\} \mid u \in A_i \text{ or } u = t_i\}$ to be the local load of the aggregation problem, and we assume that the nodes know an upper bound $\hat{\ell}$ on $\ell$.

We divide the execution of the algorithm into three phases, the *Preprocessing Phase*, the *Combining Phase*, and the *Postprocessing Phase*. An
Figure 1: An execution of the Aggregation Algorithm on the 3-dimensional butterfly (white nodes). The nodes on top are members aggregation groups $A_1$ (black), $A_2$ (gray), or both $A_1$ and $A_2$ (black/gray). The bold (dashed) lines represent the paths of the packets corresponding to $A_1$ and $A_2$, respectively. If a line is both bold and dashed, then packets of both aggregation groups contend to use the same edge in the same round.

Example of the execution of the algorithm can be found in Figure 1. First, in the **Preprocessing Phase**, all input values are sent in batches of size $\lceil \log n \rceil$ to BF-nodes of level 0 chosen uniformly at random. More specifically, every node $u \in V$ transforms each input value $s_{u,i}$ for all $A_i$ of which $u$ is a member into a packet of the form $(i, s_{u,i})$, and enumerates all of its packets arbitrarily from 1 to $k \leq \ell$ as $p_1, \ldots, p_k$. Then, for each $j \in \{1, \ldots, \lceil k / \log n \rceil\}$, $u$ sends out packets $p(j-1)\lceil \log n \rceil + 1, \ldots, p(\min\{j\lceil \log n \rceil, k\})$ in communication round $j$ to BF-nodes chosen uniformly and independently at random among all BF-nodes of level 0. To achieve synchronization after this phase, the nodes perform the Aggregate-and-Broadcast algorithm.

In the **Combining Phase**, the input values of each aggregation group $A_i$ are aggregated to a node $h(i)$ chosen uniformly and independently at random from the BF-nodes of level $d$ using a (pseudo-)random hash-function $h$. This is achieved by using a variant of the random rank protocol: Each packet $p = (i, s_{u,i})$ stored at some BF-node of level 0 gets assigned a rank $rank(p) = \rho(i)$ using some (pseudo-)random hash function $\rho: \{1, \ldots, N\} \rightarrow [K]$ that is
known to all nodes. Then, all packets belonging to aggregation group \( A_i \) are routed towards their target \( h(i) \) along the unique paths on the butterfly, and using the following rules:

1. Whenever a BF-node stores multiple packets belonging to the same aggregation group \( A_i \), it combines them into a single packet of rank \( \rho(i) \), combining their values using the given aggregate function.

2. Whenever multiple packets from different aggregation groups contend to use the same edge in the same round, the one with smallest rank wins (preferring the one with smallest aggregation group identifier in case of a tie), and all others get delayed.

Note that a packet can never get delayed by a packet belonging to the same aggregation group. Clearly, in each round at most one packet is sent along each edge of the butterfly, and eventually all (combined) packets have reached their targets.

In order to determine whether the combining phase has finished, every BF-node of level 0 sends out a token to all neighbors at level 1 once it has sent out all packets. Correspondingly, every BF-node at level \( i > 0 \) that has sent out all packets and has received tokens from both neighbors at level \( i - 1 \) sends a token to both its neighbors at level \( i + 1 \). By performing the Aggregate-and-Broadcast Algorithm to determine whether all BF-nodes of level \( d \) have received two tokens, the nodes eventually detect that the combining phase has finished.

Finally, in the Postprocessing Phase the BF-nodes of level \( d \) send their packets to the corresponding targets in rounds that are randomly chosen from \( \{1, \ldots, s\} \), where \( s = \lfloor \ell / \log n \rfloor \). More specifically, for each packet \( p \) stored at some node \( u \), which contains the result \( f(\{s_{u,i} \mid u \in A_i\}) \) for some aggregation group \( A_i \), \( u \) selects a round \( r \in \{1, \ldots, s\} \) uniformly and independently at random and sends \( p \) to \( t_i \) in round \( r \). Again, the end of the phase is determined by using the Aggregate-and-Broadcast Algorithm.

We now turn to the analysis of the algorithm.

**Lemma 3.** The Preprocessing Phase takes time \( O(\ell / \log n) \). Moreover, in each round every node sends and receives at most \( O(\log n) \) packets, w.h.p.

**Proof.** The runtime and the bound on the number of packets sent out in each round are obvious. Hence, it remains to bound the number of packets that are received in each round.

Fix any BF-node \( u \) of level 0 and round \( t \in \{1, \ldots, \lfloor \ell / \log n \rfloor \} \). Altogether, at most \( n \lfloor \log n \rfloor \) packets are sent out in round \( t \), which we denote by
For each $p_i$, let the binary random variable $X_i$ be 1 if and only if $p_i$ is sent to BF-node $u$ in round $t$. Furthermore, let $X = \sum_{i=1}^{k} X_i$. Certainly, $E[X_i] = \Pr[X_i = 1] = 1/2^d$ and therefore, $E[X] \leq (n/\log n)/2^d \leq 2\log n + 1$. Since the packets choose their destinations uniformly and independently at random, it follows from Lemma 1 that $X = O(\log n)$, w.h.p.

In order to bound the runtime of the Combining Phase, we first analyze our variant of the random rank protocol in a general setting: Given a leveled path collection $P$ of size $n$ in which packets belonging to the same aggregation group have the same destination, let the \textit{congestion} $C$ of $P$ be defined as the maximum number of aggregation groups that have packets that want to cross the same edge, and let the \textit{degree} $d$ of $P$ be defined as the maximum number of edges in $E(P)$ leading to the same node, where $E(P)$ is the set of all edges used by the paths in $P$. The proof of the following general theorem can be found in the Appendix.

\textbf{Theorem 1.} For any leveled path collection $P$ of size $n$ with congestion $C$, depth $D$, and degree $d$, the routing strategy used in the Combining Phase with parameter $K \geq 8C$ needs at most $O(C + D\log d + \log n)$ steps, w.h.p., to finish routing in $P$.

Using Theorem 1, we are now able to bound the runtime of the Combining Phase by determining the parameters of the underlying routing problem.

\textbf{Lemma 4.} The Combining Phase takes time $O(L/n + \log n)$, w.h.p.

\textit{Proof.} The depth of the butterfly is $O(\log n)$ and its degree is 4. Furthermore, the size of the routing problem is $L$. Therefore, it only remains to show that the congestion of the routing problem is $O(L/n + \log n)$, w.h.p.

Consider some fixed edge $e$ from level $i$ to $i+1$ in the butterfly. For any $A \in A$ let the binary random variable $X_A$ be 1 if and only if there is at least one packet from $A$ crossing $e$. Clearly, there are $2^i \cdot 2^{d-i-1} = 2^d/2$ source-destination pairs, where the source is in level 0 while the destination is in level $d$, whose unique shortest path passes through $e$. If the source of every packet is chosen uniformly and independently at random among all BF-nodes of level 0 and the destinations of the aggregation groups are chosen uniformly and independently at random from all BF-nodes of level $d$, then the probability for an individual packet to pass through $e$ is $(2^d/2)/(2^d) = 1/(2^{d+1})$. Hence, $E[X_A] = \Pr[X_A = 1] \leq |A|/2^{d+1}$. Let $X = \sum_{A \in A} X_A$. Then

$$E[X] = \sum_{A \in A} E[X_A] \leq \frac{\sum_{A \in A} |A|}{2^{d+1}} = \frac{L}{2^{d+1}} \leq \frac{L}{n}.$$
Since the $X_A$’s are independent, it follows from the Chernoff bounds that $X = O(L/n + \log n)$, w.h.p. \hfill \qed

Using Chernoff bounds and the fact that every node at level $d$ of the butterfly is target of at most $O(\hat{\ell} + \log n)$ aggregation groups, w.h.p., the following result can be shown similarly to Lemma 3.

**Lemma 5.** The Postprocessing Phase takes time $O(\hat{\ell}/\log n)$, w.h.p. Moreover, in each round every node sends and receives at most $O(\log n)$ packets, w.h.p.

We conclude the following theorem.

**Theorem 2.** The Aggregation Algorithm takes time $O(L/n + \hat{\ell}/\log n + \log n)$, w.h.p.

### 3.3 Multicast Tree Setup Algorithm

Some of our algorithms rely on a structure of precomputed multicast trees, which enables the nodes to multicast a message to their neighbors. Assume we are given a set of multicast groups $\mathcal{A} = \{A_1, \ldots, A_N\}$, $A_i \subseteq V$ with sources $s_1, \ldots, s_N \in V$ such that each node is source of at most one multicast group. The goal of the Multicast Tree Setup Algorithm is to set up a multicast tree $T_i$ in the butterfly for each $i \in \{1, \ldots, N\}$ with root $h(i)$, which is a node uniformly and independently chosen among the BF-nodes of level $d$, and a unique and randomly chosen leaf $l(i,u)$ in level 0 for each $u \in A_i$. Let $L = \sum_{i=1}^{N} |A_i|$, $\ell = \max_{u \in V} \{\sum_{i \in \{1, \ldots, N\}} |A_i| \mid u \in A_i\}$ and define the congestion of the multicast trees to be the maximum number of trees that share the same BF-node.

The algorithm shares many similarities with the Aggregation Algorithm; in fact, the multicast trees stem from the paths taken by packets from random BF-nodes of level 0 to random targets in level $d$, which we route using the very same routing strategy. First, every node $u$ injects an (empty) packet $(i,u)$ for each $i$ such that $u \in A_i$ into a BF-node $l(i,u)$ of level 0 chosen uniformly and independently at random (we say, $u$ joins the multicast group $A_i$). As before, packets are sent in batches of size $[\log n]$. Then, for all $i$, all packets of $A_i$ are aggregated at $h(i)$ using the same routing strategy as in the Aggregation Algorithm and an arbitrary aggregate function. Alongside the algorithm’s execution, every BF-node $u$ records for every $i \in \{1, \ldots, N\}$ all edges along which packets from group $A_i$ arrived at $v$ during the routing towards $h(i)$, and declares them as edges of $T_i$. Again, the intermediate
steps are synchronized using the Aggregate-and-Broadcast Algorithm, and the final termination is determined using a token passing strategy.

The following theorem follows from the analysis of the Aggregation Algorithm.

**Theorem 3.** The Multicast Tree Setup Algorithm computes multicast trees in time $O(L/n + \ell/\log n + \log n)$, w.h.p. The resulting multicast trees have congestion $O(L/n + \log n)$, w.h.p.

### 3.4 Multicast Algorithm

If we have constructed multicast trees for a set of multicast groups $\mathcal{A}$ as defined before, then, naturally, these can be used to let each source $s_i$ of a multicast group $A_i$ multicast a message $p_i$ to all members of its group by using the Multicast Algorithm. As before, let $C$ be the congestion of the multicast trees and $\ell = \max_{v \in V} |\{i \in \{1, \ldots, N\} \mid v \in A_i\}|$ and assume that the nodes know an upper bound $\hat{\ell}$ on $\ell$.

The algorithm shares many similarities to the Aggregation Algorithm. First, every source $s_i$ directly sends $p_i$ to $h(i)$. Then, in the Spreading Phase, $h(i)$ sends $p_i$ to all $l(i, u)$ for all $i$ and $u \in A_i$. This is done by using the multicast trees and our variant of the random rank routing protocol of the Combining Phase in "reverse order": First, each packet $p_i$ is assigned a $\text{rank}(p_i) = \rho(i)$. Whenever a multicast packet $p_i$ of some aggregation group $A_i$ is stored by an inner node of $T_i$, i.e., by some BF-node $u$ of level $j \in \{1, \ldots, d\}$, then a copy of $p_i$ is sent over each outgoing edge of $u$ in $T_i$, i.e., towards one or both of $u$’s neighbors in level $j - 1$. If two packets from different multicast groups contend to use the same edge at the same time, the one with largest rank is sent (preferring the one with largest multicast group identifier in case of a tie), and the others get delayed. Once there are no packets in transit anymore, which is determined by using the token passing strategy of the Aggregation Algorithm from level $d$ in the direction of level 0, all leaves of the multicast trees have received their multicast packet. Finally, every leaf node $l(i, u)$ sends $p_i$ to $u$ in a round randomly chosen from $\{1, \ldots, \lceil \hat{\ell}/\log n \rceil\}$.

The following theorem follows from discussion of the previous sections.

**Theorem 4.** The Multicast Algorithm takes time $O(C + \hat{\ell}/\log n + \log n)$, w.h.p.
3.5 Multi-Aggregation Algorithm

Finally, we want to allow a node $u$ to first multicast a message to all of its neighbors, and then aggregate all messages destined at $u$. More formally, we are given a set of multicast groups $\mathcal{A} = \{A_1, \ldots, A_N\}$, $A_i \subseteq V$ with sources $s_1, \ldots, s_N \in V$ such that every source $s_i$ stores a multicast packet $p_i$, and every node is source of at most one multicast group. We again assume that multicast trees for the multicast groups with congestion $C$ have already been set up. The goal of the Multi-Aggregation Algorithm is to let every node $u \in V$ receive $f(\{p_i \mid u \in A_i\})$ for a given distributive aggregate function $f$.

The algorithm essentially first performs a multicast, then maps each multicast packet to a new aggregation group corresponding to its target, and finally aggregates the packets to their targets. More precisely, first every node $s_i$ sends its multicast packet to $h(i)$. Then, by using the same strategy as in the Multicast Algorithm, we let each $l(i, u)$ receive $p_i$ for all $i$ and $u \in A_i$. Every node $l(i, u)$ then maps $p_i$ to a packet $(\id(u), p_i)$ for all $i$ and $u \in A_i$. We randomly distribute the resulting packets by letting each BF-node send out its packets, one after the other, to BF-nodes of level 0 chosen uniformly and independently at random. By using the same strategy as in the Aggregation Algorithm, we then aggregate all packets $(\id(u), p_i)$ for all $i$ to $h(\id(u))$, and finally send the result $f(\{p_i \mid u \in A_i\})$ from $h(\id(u))$ to $u$.

The following theorem follows from discussion of the previous sections and from the fact that the mapping takes time $O(C)$.

**Theorem 5.** The Multi-Aggregation Algorithm takes time $O(C + \log n)$, w.h.p.

4 Computing an $O(a)$-Orientation

In this section we present the Orientation Algorithm, which computes an $O(a)$-orientation of $G$. More specifically, the goal is to let every node learn a direction of all of its incident edges in $G$. We first describe the general approach from a high level, before we provide the details of its realization in our model. In the subsequent section, we will use the algorithm to efficiently construct multicast trees that connect each node with all of its neighbors.

4.1 High-Level Algorithm

The Orientation Algorithm essentially constructs a Nash-Williams forests-decomposition [47] using the approach of [3]. From a high-level perspective,
the algorithm repeatedly identifies low-degree nodes and removes them from the graph until the graph is empty. Whenever a node leaves, all of its adjacent edges are directed away from it. More precisely, the algorithm proceeds in phases $1, \ldots, t$. Let $d_i(u)$ be the number of incident edges of a node $u$ that have not yet been assigned a direction at the beginning of phase $i$. Define $\overline{d_i}$ to be the average degree of all nodes $u$ with $d_i(u) > 0$. In phase $i$, a node $u$ is called inactive if $d_i(u) = 0$, active if $d_i(u) \leq 2\overline{d_i}$, and waiting if $d_i(u) > 2\overline{d_i}$. In each phase, an edge $\{u, v\}$ gets directed from $u$ to $v$, if $u$ is active and $v$ is waiting, or if both nodes are active and $\text{id}(u) < \text{id}(v)$. Thereby, each node is waiting until it becomes active in some phase, and remains inactive for all subsequent phases. This results in a partition of the nodes into levels $L_1, \ldots, L_t$, where level $i$ is the set $L_i$ of active nodes in phase $i$.

**Lemma 6.** In every phase, at least half of all nodes that are not yet inactive become inactive.

**Proof.** Note that a node $u$ becomes inactive in phase $i$ if it is active in that phase, i.e., if $d_i(u) \leq 2\overline{d_i}$. Let $A_i$ be the set of nodes that are not inactive at the beginning of some phase $i$, and assume to the contrary that more than $|A_i|/2$ nodes have a degree greater than $2\overline{d_i}$. Then we arrive at a contradiction since

$$\sum_{v \in A_i} d(v) > (|A_i|/2) \cdot 2\overline{d_i} = \sum_{v \in A_i} d(v).$$

The above lemma immediately implies that $O(\log n)$ phases suffice until all edges are directed. Furthermore, note that $\overline{d_i} \leq 2a$, since any subgraph of $G$ can be partitioned into $a$ forests and the average degree of a forest is at most 2. Since in the worst case all incident edges of an active node are directed away from it, this implies that the high-level algorithm constructs an $O(a)$-orientation.

### 4.2 Identification Algorithm

Before we present how a phase of the high-level algorithm is executed in our model, we present the **Identification Algorithm**, which will later be used as a subroutine. The algorithm solves the following problem. We are given a set $L \subseteq V$ of learning nodes and a set $P \subseteq V$ of playing nodes. Every playing node knows a subset of its neighbors that are potentially learning, i.e., it knows that none of the other neighbors are learning. The goal is to let every learning node determine which of its neighbors are playing.
In this subsection, we represent each edge \(\{u, v\}\) by two directed edges \((u, v)\) and \((v, u)\). We assume that all nodes know \(s\) (pseudo-)random hash functions \(h_1, \ldots, h_s : E \rightarrow [q]\) for some parameters \(s\) and \(q\). The hash functions are used to map every directed edge to \(s\) trials. We say an edge \(e\) participates in trial \(i\) if \(h_j(e) = i\) for some \(j\).

Let \(u \in L\). We refer to an edge \((u, v)\) as a red edge of \(u\), if \(v\) is not playing, and a blue edge of \(u\), if \(v\) is playing. We identify each edge \((u, v)\) by the identifiers of its endpoints, i.e., \(id(u, v) = id(u) \circ id(v)\). Let \(X(i)\) be the XOR of the identifiers of all edges \((u, v)\) that participate in trial \(i\), and \(X'(i)\) be the XOR of the identifiers of all blue edges \((u, v)\) that participate in trial \(i\). Furthermore, let \(x(i)\) be the total number of edges adjacent to \(u\) that participate in trial \(i\), and let \(x'(i)\) be the number of blue edges that participate in trial \(i\).

Our idea is to let \(u\) use these values to identify all of its red edges; then it can conclude which of its neighbors must be playing. Before describing this, we explain how the values are determined. Clearly, the values \(X(i)\) and \(x(i)\) can be computed by \(u\) by itself for all \(i\). The other values are more difficult to obtain as \(u\) does not know which of its edges are blue. To compute these values, we use the Aggregation Algorithm: Each playing node \(v\) is in aggregation group \(A_{id(w)}\) for every potentially learning neighbor \(w\) and every trial \(i\) such that \((w, v)\) participates in trial \(i\). The input of \(v\) for the group \(A_{id(w)}\) is \((id(w, v), 1)\), where the first coordinate is used to let \(w\) compute \(X'(i)\), and the second coordinate is used to compute \(x'(i)\). Correspondingly, the aggregate function \(f\) combines two inputs corresponding to the same aggregation group by taking the XOR of the first coordinate and the sum of the second coordinate. Thereby, \(u\) eventually receives both \(X'(i)\) and \(x'(i)\).

We now show how \(u\) can identify its red edges using the aggregated information. First, it determines a trial \(i\) for which \(x(i) = x'(i) + 1\). Since neighbors that are not playing did not participate in the aggregation, in this case there is exactly one red edge \((u, v)\) such that \(id(u, v)\) is included in \(X(i)\) but not in \(X'(i)\). Therefore, \(id(u, v)\) can be retrieved by taking the XOR of both values. Having identified \(id(u, v)\), \(u\) determines all trials in which \((u, v)\) participates using the common hash functions and “removes” \(id(u, v)\) from \(X(i)\) by again computing the XOR of both. It then decreases \(x(i)\) by 1 and repeats the above algorithm until no further edge can be identified. If \(u\) always finds a trial \(i\) for which \(x(i) = x'(i) + 1\), then it eventually has identified all red edges. Clearly, all the remaining neighbors must be playing.

**Lemma 7.** Let \(u \in L\) and assume that \(u\) is incident to at most \(p\) red edges.
Let \( s \) be the number of hash functions, and \( q \) be the number of trials.

\[
\Pr[u \text{ fails to identify at least } k \text{ red edges}] \leq 2 \left( \frac{2sk}{q} \right)^{(s-2)k/2}
\]

for \( q \geq 4sp \) and \( s \geq 3 \).

**Proof.** \( u \) fails to identify at least \( k \) red edges if at some iteration of the above process there are \( j \geq k \) edges left such that all edges participate only in trials in which at least two of the \( j \) edges participate. Here, the \( j \) edges participate in at most \( \lfloor s \cdot j/2 \rfloor \) many different trials, since otherwise there must be a trial in which only one edge participates. Therefore, the probability for that event is

\[
\Pr \leq \sum_{j=k}^{p} \binom{p}{j} \left( \frac{q}{s} \right)^{sj/2} \left( \frac{s_j/2}{q} \right)^{sj}
\]

\[
\leq \sum_{j=k}^{p} \left( \frac{ep}{j} \right)^{j} \left( \frac{2eq}{s} \right)^{s_j/2} \left( \frac{s_j/2}{2q} \right)^{sj}
\]

\[
= \sum_{j=k}^{p} \left[ \left( \frac{ep}{j} \right) \cdot \left( \frac{2eq}{s} \right) \cdot \left( \frac{s_j/2}{2q} \right)^{s_j} \right]^{s/2-1} j
\]

\[
= \sum_{j=k}^{p} \left[ \frac{e^2ps}{2q} \right]^{(s/2-1)} j
\]

\[
\leq \sum_{j=k}^{p} \frac{2sj}{q} \left( \frac{s-j}{s} \right)^{s/2} \leq 2 \left( \frac{2sk}{q} \right)^{(s-2)k/2},
\]

where the last inequality holds because

\[
\left( \frac{2s(j + 1)}{q} \right)^{(s-2)(j+1)/2} \leq 1/2 \left( \frac{2sj}{q} \right)^{(s-2)j/2}.
\]

4.3 Details of the Algorithm

Next, we show how the Identification Algorithm can be used to efficiently realize a phase of the high-level algorithm. In our algorithm every node learns the direction of all its adjacent edges in the phase in which it is active; however, its neighbors might learn their direction only in subsequent phases. Each phase is divided into three stages: In Stage 1, every node determines whether it is active in this phase. In Stage 2, every active node learns which
of its neighbors are inactive. Finally, in Stage 3 every active node learns which of its remaining neighbors, which must be either active or waiting, are active. From this information, and since every node knows the identifiers of all of its neighbors, every active node concludes the direction of each of its incident edges. In the following we describe the three stages of a phase \( i \) in detail.

**Stage 1: Determine Active Nodes.** We assume that all nodes start the stage in the same round. First, every node \( u \) that is not inactive needs to compute \( d_i(u) \) to determine whether it remains waiting or becomes active in this phase. This value can easily be computed using the Aggregation Algorithm: Every inactive node \( v \), which already knows the orientation of each of its incident edges, is a member of every aggregation group \( A_{\text{id}(w)} \) such that \( v \rightarrow w \). As the input value of each node we choose 1, and the aggregate function \( f \) is the sum. By performing the Aggregation Algorithm, \( u \) determines the number of inactive neighbors, and, by subtracting the value from \( d(u) \), computes \( d_i(u) \). Afterwards, the nodes use the Aggregate-and-Broadcast Algorithm to compute \( d_i \) and to achieve synchronization.

**Stage 2: Identify Inactive Neighbors.** The goal of this stage is to let every active node learn which of its neighbors are inactive. The stage is divided into two steps: In the first step, a large fraction of active nodes succeeds in the identification of inactive neighbors. The purpose of the second step is to take care of the nodes that were unsuccessful in the first step, i.e., that only identify some, but not all, of their incident red edges. In both steps we use the Identification Algorithm described in the previous section, and carefully choose the parameters to minimize the overall runtime.

At the beginning of the first step, the nodes perform the Aggregate-and-Broadcast Algorithm to compute \( d_i^* = \max_{u \in L_i} (d_i(u)) \). Let \( d^* = \max_{j \leq i} d_i^* \), which is a value known to all nodes, and note that \( d^* = O(a) \). Then, the nodes perform the Identification Algorithm, where the active nodes are learning and the inactive nodes are playing. As the parameters of the process we choose \( s = c \) and \( q = 4cd^* \log n \) for some constant \( c \geq 4 \). Hence, the endpoints of the red edges learned by the active nodes must either be active or waiting.

The goal of the second step is to let all nodes that have been unsuccessful in the first step identify its remaining red edges. Let \( U = \{ u \in V \mid u \text{ is unsuccessful} \} \). To minimize the runtime of the algorithm, we divide \( U \) into sets of high-degree nodes \( U_{\text{high}} = \{ u \in U \mid (d(u) - d_i(u)) > n/\log n \} \) and
of low-degree nodes $U_{\text{low}} = \{u \in U \mid (d(u) - d_i(u)) \leq n/\log n\}$ and consider the nodes of each set separately. First, the nodes of $U_{\text{high}}$ broadcast their identifiers by using a variant of the Aggregate-and-Broadcast Algorithm: Using the path system of the butterfly, every node $u \in U_{\text{high}}$ sends its identifier to the butterfly’s root; however, messages are not combined. Instead, whenever multiple identifiers contend to use the same edge in the same round, the smallest identifier is sent first. After the root has received all identifiers, it broadcasts them in a pipelined fashion. For every node $u \in A := \{u \in V \mid u \text{ is active or waiting}\}$ define $R_u = U_{\text{high}} \cap N(u)$, i.e., $(v, u)$ is a red edge of $v$ for all $v \in R_u$. By using the Aggregate-and-Broadcast Algorithm, the nodes first compute $r = \max_{a \in A} |R_a|$. Let $u \in A$. For each $v \in R_u$, $u$ chooses a round from $\{1, \ldots, \max\{r, d^*_i\}\}$ uniformly and independently at random and sends its own identifier to $v$ in that round. Afterwards, every high-degree node can identify all of its red edges.

To let the low-degree nodes identify their red edges, we again use the Identification Algorithm. First, in order to narrow down its set of potentially learning neighbors, every inactive node determines which of its neighbors are unsuccessful low-degree nodes. Therefore, we let every inactive node $u$ join multicast group $A_{\text{id}(v)}$ for all $u \rightarrow v$ such that $v$ is not inactive. Note that every inactive node knows both the directions of all of its incident edges, and whether the other endpoint of each edge is inactive or not, so it can inject the required packets according to the algorithm. Every node $v \in U_{\text{low}}$ then informs its inactive neighbors by using the Multicast Algorithm. Since every node is member of at most $d^*$ multicast groups, which is a value known to all nodes, the nodes know an upper bound on $\ell$ as required by the algorithm. Having narrowed down the set of learning nodes and the sets of potentially learning neighbors to the unsuccessful ones only, the Identification Algorithm is performed once again. As the parameters of the algorithm we choose $s = c \log n$ and $q = 4c \log^2 n$ for some positive constant $c$.

**Stage 3: Identify Active Neighbors.** Finally, every active node has to learn which of the endpoints of its red edges are active. In the following, let $\text{id}(e) = \text{id}(u) \circ \text{id}(v)$ be the identifier of an edge given by its endpoints $u$ and $v$ such that $\text{id}(u) < \text{id}(v)$. The nodes use two (pseudo-)random hash-function $h, r$, where $h$ maps the identifier of an edge $e$ to a node $h(\text{id}(e)) \in V$ uniformly and independently at random, and $r$ maps its identifier to a round $r(\text{id}(e)) \in \{1, \ldots, d^*_i\}$ uniformly and independently at random. Every active node $u$ sends an edge-message containing $\text{id}(e)$ to $h(\text{id}(e))$ in round $r(\text{id}(e))$ for every incident edge $e$ leading to an active or waiting node. Using this
strategy, two adjacent active nodes $u, v$ send an edge-message containing $\text{id} \{u, v\}$ to the same node in the same round. Whenever a node receives two edge-messages with the same edge identifier, it immediately responds to the corresponding nodes, which thereby learn that both endpoints are active.

### 4.4 Analysis

We present the analysis of the Orientation Algorithm in three parts: first, we show the correctness of the algorithm, then analyze its runtime, and finally show that every node receives at most $O(\log n)$ messages in each round. It is easy to see that the algorithm correctly mimics the high-level algorithm if every active node is able to identify all of its red edges in each phase, which is shown in the following two lemmas.

**Lemma 8.** In the first step, every active node fails to identify at most $\log n$ red edges, w.h.p.

**Proof.** Note that every active node can only be adjacent to at most $p \leq d^*$ active or waiting nodes, i.e., it is incident to at most $p$ red edges. Therefore, by Lemma 7 the probability that an active node $u$ fails to identify at least $\log n$ red edges is

$$2 \left( \frac{2c \log n}{4cd^* \log n} \right)^{(c-2) \log n/2} \leq \frac{1}{2^{(c/2-1) \log n - 1}} = \frac{1}{n^{c/2-2}}.$$

Taking the union bound over all nodes implies the lemma.

**Lemma 9.** After the second step, every active node has identified all of its red edges, w.h.p.

**Proof.** If $u \in U_{\text{high}}$, then after having received the identifiers of all neighbors that are active or waiting, $u$ immediately knows its red edges. Now let $u \in U_{\text{low}}$. Since by Lemma 8 $u$ has at most $p \leq \log n$ remaining red edges, by Lemma 7 we have that the probability that $u$ fails to identify at most one of its remaining red edges is at most

$$2 \left( \frac{2c \log n}{4c \log^2 n} \right)^{c \log n/2 - 2} \leq \frac{1}{2^{c \log n/2 - 1}} = \frac{1}{n^{c/2-1}}.$$

Taking the union bound over all nodes implies the lemma.

To bound the runtime of the complete algorithm, we now prove that each stage takes time $O(a + \log n)$, w.h.p.
Lemma 10. Stage 1 takes time $O(a + \log n)$, w.h.p.

Proof. In the execution of the Aggregation Algorithm, every inactive node is member of at most $O(a)$ aggregation groups and every active node is target of at most one aggregation, i.e., $L = O(na)$ and $\ell = O(a)$. Note that the nodes do not need to explicitly know an upper bound on $\ell$, as every node is target of at most one aggregation group and its result can be sent to it immediately. The lemma follows from Theorem 2.

For the runtime of Stage 2 we need the following two lemmas.

Lemma 11. $|U_{\text{high}}| = O(a + \log n)$, w.h.p.

Proof. Let $A = \{u \in L_i \mid (d(u) - d_i(u)) > n/\log n\}$. Note that since $d \leq 2a$, we have that $\sum_{u \in V} d(u) \leq 2an$, and therefore $|A| \leq 2a \log n$. For $u \in A$ let $X_u$ be the binary random variable that is 1, if $u$ is unsuccessful in the first step, and 0, otherwise. By Lemma 7 and since $c \geq 4$, we have

$$\Pr[X_u = 1] \leq \frac{2}{(2d^* \log n)^{c/2-1}} \leq \frac{1}{\log n}.$$ 

Let $X = \sum_{u \in A} X_u$. The expected value of $X$ is therefore $E[X] \leq 2a \log n / \log n = 2a =: \mu$. Let $\delta = \min\{\alpha \log n / \mu, 1\}$ for a constant $\alpha$, then by using the Chernoff bounds we have that

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-\alpha \log n / 3} \leq \frac{1}{n^{1/2}},$$

and thus $X = O(a + \log n)$.

Lemma 12. $\sum_{u \in U_{\text{low}}} (d(u) - d_i(u)) = O(an/\log n + n)$, w.h.p.

Proof. Let $A = \{u \in L_i \mid (d(u) - d_i(u)) > n/\log n\}$. For a node $u \in A$, let $X_u$ be the random variable that is $d_u$, if $u$ is unsuccessful in the first step, and 0, otherwise. By Lemma 7 and since $c \geq 4$, we have

$$\Pr[X_u = 1] \leq \frac{2}{(2d^* \log n)^{c/2-1}} \leq \frac{1}{\log n}.$$ 

for some constant $\alpha = c/2 - 1$. Let $A$ be the set of active nodes. Then $X = \sum_{u \in A} X_u$ is a sum of independent random variables with expected value $E[X] \leq \sum_{u \in A} d(u) / \log n \leq an / \log n =: \mu$. Note that $d(u) \leq n / \log n$ for all $u \in A$. Therefore, we can use the general Chernoff bound with $\delta = \min(\alpha n / \mu, 1)$ for some constant $\alpha$, and get

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-\alpha n \log n / (n3)} \leq \frac{1}{n^{45}}.$$ 

Therefore, we have that $X = O(an/\log n + n)$, w.h.p.
We are now ready to bound the runtime of Stage 2.

**Lemma 13.** Stage 2 takes time $O(a + \log n)$, w.h.p.

*Proof.* The computation of $d^*$ at the beginning of the first step takes time $O(\log n)$. In the first execution of the Identification Algorithm, every active node $u$ is target of aggregation group $A_{\text{id}(u)}$ for every trial $i$, and every inactive neighbor $v$ of $u$ is member of all aggregation groups $A_{\text{id}(u)}$ such that $(u, v)$ participates in trial $i$. Therefore, every active node is target of at most $4c d^* \log n$ and every inactive node is a member of at most $c d^*$ aggregation groups. Since both values are known to every node, the nodes know an upper bound $\hat{\ell} = 4c d^* \log n$ on $\ell$. Since every inactive node is a member of at most $c d^*$ aggregation groups, the global load $L$ is bounded by $ncd^*$. By Theorem 2, the Aggregation Algorithm takes time $O(ncd^* n + 4c d^* \log n \log n + \log n) = O(a + \log n)$, w.h.p., to solve the problem.

Now consider the second step. By Lemma 11, $|U_{\text{high}}| = O(a + \log n)$, w.h.p., and therefore all identifiers of high-degree nodes can be broadcasted in time $O(a + \log n)$. Informing each node in $U_{\text{high}}$ about its red edges takes an additional $O(a + \log n)$ rounds, as $\tau = O(a + \log n)$ and $d^*_i = O(a)$.

The multicast trees to handle low-degree nodes are constructed in time $O(a + \log n)$, as every inactive node joins at most $d^*$ multicast groups, and have congestion $O(a + \log n)$, w.h.p. Correspondingly, the multicast can be performed in time $O(a + \log n)$, w.h.p.

We now bound the runtime of the final execution of the Identification Algorithm. Every inactive node is a member of at most $O(a \log n)$ aggregation groups, w.h.p., and every node is a target of at most $4c \log^2 n$ aggregation groups. By Lemma 12, $\sum_{u \in U_{\text{low}}}(d(u) - d_i(u)) = O(an / \log n + n)$, w.h.p. As this is also a bound on the number of edges that participate in any trial, and each edge participates in $c \log n$ trials, the global load $L$ is bounded by $O(an + n \log n)$. Therefore, by Theorem 2, the Aggregation Algorithm takes time $O(a + \log n)$, w.h.p.

The lemma below immediately follows from the fact that $d^*_i = O(a)$.

**Lemma 14.** Stage 3 takes time $O(a + \log n)$, w.h.p.

Finally, it remains to show that no node receives too many messages.

**Lemma 15.** In each round of the Orientation Algorithm, every node sends and receives at most $O(\log n)$ messages, w.h.p.
Proof. By the discussion of Section 3, the executions of the Aggregation, Multicast Tree Setup, and Multicast Algorithm ensure that every node receives only $O(\log n)$ messages in each round. It remains to show the claim for the second step of Stage 2, where high-degree nodes broadcast their identifiers and receive their red edges, and for Stage 3, where active nodes learn which of their red edges lead to other active nodes.

For the first part, note that after all high-degree nodes have broadcasted their identifiers, every active or waiting node sends out $O(\log n)$ messages containing its identifier in every round, w.h.p., which can easily be shown using Chernoff bounds. Second, as every high-degree node receives at most $d^*_i$ identifiers, it also follows from the Chernoff bound that every such node receives at most $O(\log n)$ messages.

Now consider Stage 3 of the algorithm. Again, by using the Chernoff bound, it can easily be shown that no node sends out more than $O(\log n)$ edge-messages in any round. Therefore, every node only receives $O(\log n)$ response messages in every round. It remains to show that every node receives at most $O(\log n)$ edge-messages in every round, from which it follows that it only sends out $O(\log n)$ response messages in every round. Let $A = \{\{u, v\} \mid u \text{ or } v \text{ is active}\}$ and note that $|A| \leq nd^*_i$. Fix a node $u \in V$ and a round $i \in \{1, \ldots, d^*_i\}$ and let $X_e$ be the binary random variable that is 1 if and only if $h(\text{id}(e)) = u$ and $r(\text{id}(e)) = i$ for $e \in A$. Then $\Pr[X_e = 1] = 1/(nd^*_i)$. $X = \sum_{e \in A} X_e$ has expected value $E[X] \leq 1$. Using the Chernoff bound we get that $X = O(\log n)$, w.h.p., which implies that $u$ receives at most $O(\log n)$ edge-messages in round $i$. The claim follows by taking the union bound over all nodes and rounds.

Since no node is active anymore after $O(\log n)$ phases, and by applying the union bound over all phases, we conclude the following theorem.

Theorem 6. The Orientation Algorithm computes an $O(a)$-orientation in time $O((a + \log n) \log n)$, w.h.p.

5 Graph Problems

In this section we present algorithms to efficiently solve some graph problems in our model using the techniques presented in the previous sections. Except for the algorithm to compute a minimum spanning tree, the presented algorithms rely on a structure of precomputed multicast trees. More specifically, for every node $u \in V$ we construct a multicast tree $T_{\text{id}(u)}$ for the multicast
group $A_{\text{id}(u)} = N(u)$. Since such trees enable the nodes to send messages to all of their neighbors, in the following we refer to them as broadcast trees.

The broadcast trees are constructed by using a variant of the Multicast Tree Setup Algorithm. Instead of letting each node $u$ inject packets $(\text{id}(v), u)$ for all of its neighbors $v \in N(u)$, which would imply a runtime linear in the maximum degree, we first construct an $O(a)$-orientation of the edges as shown in the previous section and let $u$ only inject packets $(\text{id}(v), u)$ for neighbors $v$ such that $u \rightarrow v$. Additionally, it includes each such $v$ into its own multicast group by injecting $(\text{id}(u), v)$. Thereby, we obtain the following result.

**Lemma 16.** Setting up the broadcast trees takes time $O(a + \log n)$, w.h.p. The congestion of the broadcast trees is $O(a + \log n)$, w.h.p.

The following theorem establishes one of the key techniques used by most of the algorithms presented in this section.

**Theorem 7.** Let $S \subseteq V$. Using the broadcast trees, the Multi-Aggregation Algorithm solves any multi-aggregation problem with multicast groups $A_{\text{id}(u)} = N(u)$ and $s_{\text{id}(u)} = u$ for all $u \in S$ in time $O(\sum_{u \in S} d(u)/n + \log n)$, w.h.p.

### 5.1 Breadth-First Search Trees

As a first example, we show how to compute Breadth-First Search (BFS) Trees: Let $s$ be some node and let $\delta(u)$ be the length of a shortest path from $s$ to $u$ in $G$. Furthermore, let $\pi(u)$ be the immediate predecessor of $u$ on a shortest path from $s$ to $u$ that has smallest identifier. The goal is to let each node $u \in V$ eventually store $\delta(u)$ and $\pi(u)$. The problem can be solved by the following simple algorithm, which proceeds in phases. In Phase 1, only $s$ is active, and in Phase $i > 1$, all nodes that have received an identifier in Phase $i - 1$ for the first time are active. In each phase, every active node sends its identifier to all of its neighbors using the broadcast trees and the Multi-Aggregation Algorithm. By choosing $f$ as the minimum function, every neighbor thereby receives the minimum identifier of all active nodes. Furthermore, in every Phase $i > 1$, every active node $u$ sets $\delta(u) = i - 1$ and $\pi(u)$ to the node whose identifier it has received in the previous phase. Clearly, after at most $D + 1$ phases all nodes have been reached.

**Theorem 8.** The algorithm computes a BFS Tree in time $O((a + D + \log n) \log n)$, w.h.p.

**Proof.** By Lemma 16, the broadcast trees are constructed in time $O((a + \log n) \log n)$, w.h.p. Let $S_i$ be the set of nodes active in Phase $i$. By
Theorem 7, the Multi-Aggregation Algorithm takes time $O(\sum_{u \in S_i} d(u)/n + \log n)$, w.h.p. We conclude a total runtime of

\[
O \left( (a + \log n) \log n + \sum_{i=1}^{D+1} \left( \sum_{u \in S_i} d(u)/n + \log n \right) \right)
\]

\[
= O \left( (a + \log n) \log n + \sum_{u \in V} d(u)/n + (D + 1) \log n \right)
\]

\[
= O((a + D + \log n) \log n), \text{ w.h.p.}
\]

5.2 Maximal Independent Set

In this section we show how to compute a maximal independent set (MIS): A set $U \subseteq V$ is an MIS if (1) it is an independent set, i.e., no two nodes of $U$ are adjacent in $G$, and (2) there is no set $U' \subseteq U$ such that $U \subset U'$. To solve the problem, we propose the algorithm of Mètivier et al [45], which works as follows. First, all nodes are active and no node is in the MIS. The algorithm proceeds in phases, where in each phase every active node $u$ first chooses a random number $r(u) \in [0, 1]$ and broadcasts the value to all of its neighbors. $u$ then joins the MIS (and becomes inactive) if $r(u)$ is smaller than the minimum of all received values. If so, it broadcasts a message to all of its neighbors, instructing them to become inactive.

Note that we can easily perform a phase of the algorithm using two executions of the Multi-Aggregation Algorithm, the first to let every node aggregate the minimum of all values chosen by its neighbors, and the second to let every node that is not in the MIS determine whether it is adjacent to another node that is not in the MIS. This information is then used to determine whether the nodes have reached an MIS using the Aggregate-and-Broadcast Algorithm. Since by [45] $O(\log n)$ phases suffice, and each phase can be performed in time $O(a + \log n)$ by [7] we conclude the following theorem.

**Theorem 9.** The algorithm computes an MIS in time $O((a + \log n) \log n)$, w.h.p.

5.3 Maximal Matching

Similar to an MIS, a maximal matching $M \subseteq E$ is defined as a maximal set of independent (i.e., node-disjoint) edges. To compute a maximal matching, we propose to use the algorithm of Israeli and Itai [28], which works as follows.
Initially, no node is matched. The algorithm proceeds in phases, where in each phase every unmatched node $u$ performs the following procedure. First, it chooses an edge to an unmatched neighbor uniformly at random. If $u$ itself has been chosen by multiple neighbors, it accepts only one choice arbitrarily and informs the respective node. The outcome is a collection of paths and cycles. Each node of a path or cycle finally chooses one of its at most two neighbors. If thereby two adjacent nodes choose the same edge, the edge joins the matching and the two nodes become matched. Afterwards, all matched nodes and their incident edges are removed from the graph.

The algorithm lends itself to be realized using our communication primitives. First, we let every unmatched node randomly pick one of its unmatched neighbors by performing the Multi-Aggregation Algorithm with a slight modification. Here, every node $u$ that is still unmatched sends out a packet $p_{\text{id}(u)}$. Recall, that after $p_{\text{id}(u)}$ has reached BF-node $l(\text{id}(u), v)$ for all $v \in N(u)$ in the execution of the Multi-Aggregation Algorithm, it is mapped to a new packet $(\text{id}(v), p_{\text{id}(u)})$. Here, we additionally let $l(\text{id}(u), v)$ choose a value $r \in [0, 1]$ uniformly at random, and annotate $(\text{id}(v), p_{\text{id}(u)})$ by $r$. Whenever thereafter two packets of the same aggregation group are combined, the packet annotated by the minimum value remains. Thereby, every node that still has an unmatched neighbor receives the identifier of a node chosen uniformly and independently at random among its unmatched neighbors.

 Afterwards, every node that has been chosen by multiple neighbors has to choose one of them arbitrarily. This can be done by performing the Aggregation Algorithm, in which we let every node $u$ aggregate the minimum of the identifiers of all nodes by which it has been chosen in the previous step. In the resulting collection of paths and cycles, neighbors can directly send messages to each other to determine which edges join the matching. Finally, the nodes have to determine whether the matching is maximal, which can be done as described in the previous section. Using Corollary 3.5 of [28] and the Chernoff bounds, it can be shown that $O(\log n)$ phases suffice. Therefore, we conclude the following theorem.

**Theorem 10.** The algorithm computes a maximal matching in time $O((a + \log n) \log n)$, w.h.p.

5.4 $O(a)$-Coloring

The goal of this section is to compute an $O(a)$-coloring, in which every node has to choose one of $O(a)$ colors such that no color is chosen by two adjacent nodes. Following the idea of Barenboim and Elkin [3], we consider
the partition of nodes into levels $L_1, \ldots, L_t$ and color the nodes of each level separately. Recall that after the algorithm to compute the $O(a)$-orientation, every node knows its own level. Furthermore, for all $i$ every node $u \in L_i$ knows which of its neighbors are in lower levels $L_1, \ldots, L_{i-1}$, the same level $L_i$, and higher levels $L_{i+1}, \ldots, L_t$, since it knows which of its neighbors were inactive, active, or waiting in phase $i$. First, the nodes use the Aggregate-and-Broadcast Algorithm to compute $\hat{a} = \max_{u \in V} \{\max(d_{L_i}(u), d_{\text{out}}(u))\} = O(a)$, where $d_{L_i}(u)$ is the number of neighbors of $u$ that are in the same level as $u$. Furthermore, the nodes set up multicast trees for multicast groups $A_{\text{id}(u)} = N_{\text{in}}(u)$ with source $s_{\text{id}(u)} = u$ for all $u \in V$.

Afterwards, the algorithm proceeds in phases $1, \ldots, t$, where in each phase $i$ the nodes of level $L_{t-i+1}$ get colored. Throughout the algorithm’s execution, every node $u$ maintains a color palette $C(u)$ initially set to $[2(1 + \varepsilon)\hat{a}]$ for some constant $\varepsilon > 0$. After each phase, the color palette of every remaining uncolored node has been narrowed down to all colors that have not yet been chosen by its neighbors. Since every $u \in L_{t-i+1}$ has at most $\hat{a}$ neighbors in higher levels, $C(u)$ still consists of at least $(1 + \varepsilon)\hat{a}$ colors at the beginning of phase $i$.

In phase $i$ of the algorithm, the nodes of level $L_{t-i+1}$ essentially perform the Color-Random Algorithm of Kothapalli et al. [38]. First, every node $u \in L_{t-i+1}$ chooses a color $c_u$ from its color palette uniformly at random. Then, it informs its in-neighbors about its choice by performing the Multicast Algorithm using the precomputed multicast trees and $\hat{a}$ as an upper bound on $\ell$. Thereby, $u$ receives the colors chosen by its out-neighbors of the same level. If $u$ does not receive its own color $c_u$, it permanently chooses $c_u$. In that case, it first informs all of its in-neighbors about its permanent choice by again performing the Multicast Algorithm. Afterwards, it informs all of its out-neighbors by performing the Aggregation Algorithm. Here, $u$ is a member of aggregation groups $A_{\text{id}(u)c_{uv}}$ for all $v \in N_{\text{out}}$ and target of aggregation groups $A_{\text{id}(u)i}$ for all $i \in [2(1 + \varepsilon)\hat{a}]$. Note that every node is a member of at most $\hat{a}$ and a target of at most $2(1 + \varepsilon)\hat{a}$ aggregation groups. Afterwards, all nodes (including nodes of lower levels) remove all colors permanently chosen by neighbors from their palettes.

The above procedure is repeated until all nodes of level $L_{t-i+1}$ have permanently chosen a color, which is determined by performing the Aggregate-and-Broadcast Algorithm after each repetition. Then, if $i > 1$, the nodes start the next phase, and terminate, otherwise. The following theorem can be shown using the fact that $O(\sqrt{\log n})$ repetitions during a phase suffice to color all nodes of the corresponding level, w.h.p., which follows from the discussion in Section 4 of [38]. Furthermore, each repetition takes time
$O(a + \log n)$, w.h.p.

**Theorem 11.** The algorithm computes an $O(a)$-coloring in time $O((a + \log n) \log^{3/2} n)$, w.h.p.

### 5.5 Minimum Spanning Tree

Finally, we describe an algorithm that computes a minimum spanning tree (MST) in time $O(\log^4 n)$. More specifically, for every edge, one of its endpoints eventually knows whether the edge is in the MST or not. We assume that each edge in the input graph has an integral weight in $\{1, 2, \ldots, W\}$ for some positive integer $W = \text{poly}(n)$.

**High-Level Description.** From a high level, our algorithm mimics Boruvka’s algorithm with Heads/Tails clustering, which works as follows. Start with every node as its own component. For $O(\log n)$ iterations, every component $C$ (1) finds its lightest, i.e., minimum-weight, edge out of the component that connects to the other components, (2) flips a Heads/Tails coin, and (3) learns the coin flip of the component $C'$ on the other side of the lightest edge. If $C$ flips Tails and $C'$ flips Heads, then the edge connecting $C$ to $C'$ is added to the MST, and thus effectively component $C$ merges with component $C'$ (and whatever other components that are merging with $C'$ simultaneously).

It is well known that, w.h.p., all nodes get merged into one component within $O(\log n)$ iterations and the added edges form an MST (see, e.g., [21,22]).

**Details of the Algorithm.** Over the course of the algorithm, each component $C \subseteq V$ maintains a leader node $l(C) \in C$ whose identifier is known to every node in the component. Furthermore, we maintain a multicast tree in each component $C$ in which the source is $l(C)$, and the multicast group is the set $A = C \setminus \{l(C)\}$. In each round of Boruvka’s algorithm with the partition of $V$ into components $\{C_1, \ldots, C_N\}$, every leader $l(C_i)$ flips Heads/Tails and multicasts the result to all nodes in its component by using the Multicast Algorithm. Since the components are disjoint, and therefore the multicast trees have congestion $O(\log n)$ by Theorem 3, the Multicast Algorithm takes time $O(\log n)$, w.h.p., by Theorem 4.

For each component $C_i$, the leader then learns the lightest edge to a neighbor in $V \setminus C_i$ in time $O(\log^2 n \log W)$. This is a highly nontrivial task which we address later. Afterwards, the leader multicasts the lightest edge $\{u_i, v_i\} \in (C_i \times (V \setminus C_i)) \cap E$ to every node in its component, which can again be done in time $O(\log n)$ by Theorem 4. For each component $C_i$ that
flips Tails, the node $u_i \in C_i$ incident to the lightest outgoing edge $\{u_i, v_i\}$ joins the multicast group $A_{id(v_i)}$ with source $v_i$. Setting up the multicast trees with congestion $O(\log n)$ takes time $O(\log n)$. By using the Multicast Algorithm, the endpoints of all lightest edges learn the result of the coin flip and the identifier of their adjacent component. If for the edge $\{u_i, v_i\}$ the component $C_j$ of $v_i$ has flipped Heads, then $u_i$ sends the identifier of $C_j$ to its leader, which in turn informs all nodes of $C_i$. Note that thereby only $u_i$ learns that $\{u_i, v_i\}$ is an edge of the MST. Finally, the multicast trees of the resulting components are rebuilt by letting the nodes join corresponding multicast groups, which can again be done in time $O(\log n)$.

**Finding the Lightest Edge.** To find the lightest edge of a component, we “sketch” its incident edges. Our algorithm follows the procedure $\text{FindMin}$ of \cite{31}, with the “broadcast-and-echo” subroutine replaced by executions of the Multicast and the Aggregation Algorithm. We highlight the main steps of $\text{FindMin}$, and refer the reader to \cite{31} for the details and proof. All multicasts and aggregations are from/to the leader node of each component to/from the entire component, and since the components are disjoint, both steps can be performed in time $O(\log n)$, w.h.p.

Initially, we bidirect each edge into two arcs, in opposite directions, and let the identifier $id(u,v) = id(u) \circ id(v)$. We will apply binary search to the weights of edges so that we can find the lightest outgoing edge. Every iteration has a current range $[L,R] \subseteq [1,W]$ such that the lightest edge out has weight in that range. To compute the next range, the algorithm determines whether there is an edge out of $[L,M]$, where $M := \lfloor (L + R) / 2 \rfloor$. If so, the new range becomes $[L,M]$; otherwise, the new range is $[M+1,R]$.

The remaining task is to solve the following subproblem: given a range $[a,b]$, determine whether there exists an outgoing edge with weight in $[a,b]$.

To sketch their incident edges, the nodes use a (pseudo-)random hash function $h$ that maps each edge identifier to $\{0,1\}$. For a node $u$, define

$$h^+(u) := \sum_{v \in N(u) : w(u,v) \in [a,b]} h(id(u,v)) \mod 2,$$

and

$$h^-(u) := \sum_{v \in N(u) : w(u,v) \in [a,b]} h(id(v,u)) \mod 2,$$

\footnote{The algorithm $\text{FindMin}$ of \cite{31} actually uses a “$\Theta(\log n)$-ary” search instead of binary search, but we replace it with binary search here for simplicity of explanation.}
and for component $C \subseteq V$, define $h^\uparrow(C) := \sum_{u \in C} h^\uparrow(u)$ and $h^\downarrow(C)$ similarly. Observe that the unordered sets $\{\text{id}(u,v) : u \in C, v \in N(u), w(u,v) \in [a,b]\}$ and $\{\text{id}(v,u) : u \in C, v \in N(u), w(u,v) \in [a,b]\}$ are the same if and only if component $C$ does not have an outgoing edge with weight in the range $[a,b]$. Also, the hash function $h$ satisfies the property that, if two sets $S_1, S_2$ of integers are not equal, then the values of $\sum_{x \in S_1} h(x) \mod 2$ and $\sum_{x \in S_2} h(x) \mod 2$ are not equal with constant probability. To compute the values of $h^\uparrow(C)$ and $h^\downarrow(C)$, each node $u \in C$ computes $h^\uparrow(u)$ and $h^\downarrow(u)$, and an aggregation problem to the leader node is performed on each component $C$ with addition mod 2 as the aggregate function. We can repeat this procedure $O(\log n)$ times so that w.h.p., there is no outgoing edge out of $C$ with weight in $[a,b]$ if and only if $h^\uparrow(C)$ and $h^\downarrow(C)$ are equal in every trial.

The running time analysis from [31], modified to count the number of “broadcast-and-echo” subroutines, can be rewritten as follows.

Lemma 17 ([31], Lemma 2). In $O(\log W \log n)$ phases of performing the Multicast and the Aggregation Algorithm, the leader node can learn the lightest edge out of each component, w.h.p.

Since each phase can be performed in time $O(\log n)$, we conclude the following theorem.

Theorem 12. The algorithm computes an MST in time $O(\log^4 n)$, w.h.p.

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A Simulations in the $k$-Machine Model

In this section we consider the simulation of an algorithm for the Node-Congested Clique in the $k$-machine model. For the Congested Clique model, Klauck et al. [32] provide a conversion theorem that states the following.

**Theorem 13** (Theorem 4.1 in [32]). *Any algorithm $A^C$ in the Congested Clique model that executes in $T^C$ rounds and passes at most $M^C$ messages over the course of the algorithm’s execution can be simulated in the $k$-machine model so that it requires at most $\tilde{O}(M^C/k^2 + T^C \Delta'/k)$ rounds. Here, $\Delta'$ is the communication degree complexity and refers to the maximum number of messages sent by any node at any round.*

The simulation alluded to in Theorem 13 is quite straightforward. Each node from the Congested Clique model is placed randomly on one of the $k$ machines in the $k$-machine model. Under this random vertex partitioning scheme, each machine will get at most $\tilde{O}(n/k)$ nodes from the Congested Clique model. So it is natural for the messages sent by each node $u$ in the Congested Clique model to be simulated by the machine that holds $u$.

The following conversion result suited for the Node-Congested Clique model follows as a corollary when we notice that the number of messages per round is at most $\tilde{O}(n)$ and, furthermore, $\Delta'$ under the Node-Congested Clique model is at most $O(\log n)$.

**Corollary 1.** *Any algorithm $A^{NCC}$ in the Node-Congested Clique model that executes in $T^{NCC}$ rounds can be simulated in the $k$-machine model so that it requires at most $\tilde{O}(nT^{NCC}/k^2)$ rounds.*

B Proof of Theorem 1

We closely follow the analysis of the random rank protocol in [53] and extend it with ideas from [41] so that the analysis covers the case that packets can be combined. In order to bound the runtime, we will use the following delay sequence argument.

Consider the runtime of the routing strategy to be at least $T \geq D + s$. We want to show that it is very improbable that $s$ is large. For this we need to find a structure that witnesses a large $s$. This structure should become more and more unlikely to exist the larger $s$ becomes.

Let $p_1$ be a packet that arrived at its destination $v_1$ in step $T$, and let $A_1$ be the aggregation group of $p_1$. We follow the path of $p_1$ (or one of its predecessors, if $p_1$ is the result of the combination of two packets at some
point) backwards until we reach a link $e_1$, where it was delayed the last time. Let us denote the length of the path from the destination of $p_1$ to $e_1$ (inclusive) by $\ell_1$, and the packet that delayed $p_1$ by $p_2$. Let $A_2$ be the aggregation group of $p_2$. From $e_1$ we follow the path of $p_2$ (or one of its predecessors) backwards until we reach a link $e_2$ where $p_2$ was delayed the last time, by a packet $p_3$ from some aggregation group $A_3$. Let us denote the length of the path from $e_1$ (exclusive) to $e_2$ (inclusive) by $\ell_2$. We repeat this construction until we arrive at a packet $p_{s+1}$ from some aggregation group $A_{s+1}$ that prevented the packet $p_s$ at edge $e_s$ from moving forward.

Altogether it holds for all $i \in \{1, \ldots, s\}$: a packet from aggregation group $A_i + 1$ leaves the buffer of $e_i$ at time step $T - \sum_{j=1}^{i-1} (\ell_j + 1) + 1$, and prevents at that time step a packet from aggregation group $A_i$ from moving forward.

The path from $e_s$ to $v_1$ recorded by this process in reverse order is called delay path. It consists of $s$ contiguous parts of routing paths of length $\ell_1, \ldots, \ell_s \geq 0$ with $\sum_{i=1}^{s} \ell_i \leq D$. Because of the contention resolution rule it holds $\rho(i) \geq \rho(i+1)$ for all $i \in \{1, \ldots, s\}$. A structure that contains all these features is defined as follows.

**Definition 1 (s-delay sequence).** An $s$-delay sequence consists of

- $s$ not necessarily different delay links $e_1, \ldots, e_s$;
- $s + 1$ delay groups $a_1, \ldots, a_{s+1}$ such that the path of a packet from $a_i$ traverses $e_i$ and $e_{i-1}$ in that order for all $i \in \{2, \ldots, s\}$, the path of $p_1$ contains $e_1$, and the path of $p_{s+1}$ contains $e_s$;
- $s$ integers $\ell_1, \ldots, \ell_s \geq 0$ such that $\ell_1$ is the number of links on the path of $p_1$ from $e_1$ (inclusive) to its destination, and for all $i \in \{2, \ldots, s\}$, $\ell_i$ is the number of links on the path of $p_i$ from $e_i$ (inclusive) to $e_{i-1}$ (exclusive), and $\sum_{i=1}^{s} \ell_i \leq D$; and
- $s + 1$ integers $r_1, \ldots, r_{s+1}$ with $0 \leq r_{s+1} \leq \ldots \leq r_1 < K$.

A delay sequence is called active if for all $i \in \{1, \ldots, s+1\}$ we have $\rho(a_i) = r_i$.

Our observations above yield the following lemma.

**Lemma 18.** Any choice of the ranks that yields a routing time of $T \geq D + s$ steps implies an active $s$-delay sequence.

**Lemma 19.** The number of different $s$-delay sequences is at most

$$n \cdot d^s \cdot C^s \cdot \binom{D + s}{s} \cdot \binom{s + K}{s + 1}.$$
Proof. There are at most \( \binom{D+s}{s} \) possibilities to choose the \( \ell_i \)'s such that \( \sum_{i=1}^{s} \ell_i \leq D \). Furthermore, there are at most \( n \) choices for \( v_1 \), which will also fixed \( a_1 \). Once \( v_1 \) and \( \ell_1 \) is fixed, there are at most \( d^{\ell_1} \) choices for \( e_1 \). Once \( e_1 \) is fixed, there are at most \( d^{\ell_2} \) choices for \( e_3 \), and so on. So altogether, there are at most \( d^D \) possibilities for \( e_1, \ldots, e_s \). Since the congestion at every edge is at most \( C \), there are at most \( C^s \) possibilities for each \( e_i \) to pick \( a_{i+1} \), so altogether, there are at most \( C^n \) possibilities to select \( a_2, \ldots, a_{s+1} \). Finally, there are at most \( \binom{s+K}{s+1} \) ways to select the \( r_i \) such that \( 0 \leq r_{s+1} \leq \ldots \leq r_1 < K \).

Note that we assumed that there is a unique, total ordering on the ranks of the aggregation groups once \( \rho \) is fixed. Hence, every aggregation group can only occur once in an \( s \)-delay sequence. Since \( \rho \) is assumed to be a random hash function, the probability that an \( s \)-delay sequence is active is \( 1/K^{s+1} \). Thus,

\[
\text{Pr}[\text{The protocol needs at least } D + s \text{ steps}] \leq \frac{1}{K^{s+1}}
\]

If we set \( K \geq 8C \) and \( s = K + D(\log d + 1) + (\alpha + 1) \log n \), where \( \alpha > 0 \) is an arbitrary constant, then

\[
\text{Pr}[\text{The algorithm needs at least } D + s \text{ steps}] \leq n \cdot 2^{2s+D(\log d+1)+K} \cdot 2^{-3s} = n \cdot 2^{-s+D(\log d+1)+K} = \frac{1}{n^\alpha}
\]

which concludes the proof of Theorem 1.