Best approximation of functions in generalized Hölder class

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Abstract
Here, for the first time, error estimation of the functions \( g \in H_z^w \) and \( \tilde{g} \in H_z^w \) classes using \( TC_1 \) method of F.S. (Fourier Series) and C.F.S. (Conjugate Fourier Series), respectively, are determined. The results of (Dhakal in Int. Math. Forum 5(35):1729–1735, 2010; Dhakal in Int. J. Eng. Technol. 2(3):1–15, 2013; Kushwaha and Dhakal in Nepal J. Sci. Technol. 14(2):117–122, 2013) become the particular cases of our Theorem 2.1. Some important corollaries are also deduced from our main theorems.

MSC: 41A10; 41A25; 42B05; 42A50; 40G05; 40C05

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1 Introduction
Several results on the error estimation of a function \( g \) in Lipschitz and Hölder classes by a trigonometric polynomial using different single and product means have been obtained by the researchers like [1–11], and [12].

Our motivation for this work is to consider a more advanced class of functions that can provide best approximation by a trigonometric polynomial of degree not more than \( r \). Therefore, in this work, we generalize the results of Kushwaha and Dhakal [3] and Dhakal [1, 2]. In fact, we obtain the results on the error estimation for the function \( f \in H_z^w (z \geq 1) \) by \( T.C^1 \) method by F. S. Thus, the results of Kushwaha and Dhakal [3] and Dhakal [1, 2] become the particulars cases of our Theorem 2.1.

We also obtain the results on the error estimation of the function \( \tilde{g} \in H_z^w (z \geq 1) \) by \( T.C^1 \) method of C. F. S.

Let \( T = (a_{r,m}) \) be an infinite triangular matrix satisfying the conditions of regularity [13], i.e.,

\[
\sum_{m=0}^{r} a_{r,m} = 1 \quad \text{as} \quad r \to \infty,
\]

\[
a_{r,m} = 0 \quad \text{for} \quad m > r,
\]

\[
\sum_{m=0}^{r} |a_{r,m}| \leq M, \quad \text{a finite constant.}
\]
The sequence-to-sequence transformation

\[ t^T_r := \sum_{m=0}^{r} a_{r,m} s_m = \sum_{m=0}^{r} a_{r,m-r} s_{r-m} \]  

(2)

defines the sequence \( t^T_r \) of triangular matrix means of the sequence \( \{s_r\} \) generated by the sequence of coefficients \( (a_{r,m}) \).

If \( t^T_r \to s \) as \( r \to \infty \), then the infinite series \( \sum_{r=0}^{\infty} h_r \) or the sequence \( \{s_r\} \) is summable to \( s \) by a triangular matrix \( (T\text{-method}) \) [14].

Let

\[ C^1_r = \frac{s_0 + s_1 + \cdots + s_r}{r+1} \]

\[ = \frac{1}{r+1} \sum_{m=0}^{r} s_m \to s \text{ as } r \to \infty. \]  

(3)

If \( C^1_r \to s \) as \( r \to \infty \), then the infinite series \( \sum_{r=0}^{\infty} h_r \) is summable to \( s \) by \( C^1 \) means [14].

The \( TC^1 \) means \( (T\text{-means of } C^1 \text{ means}) \) is given by

\[ t^{T.C^1}_r := \sum_{m=0}^{r} a_{r,m} C^1_m \]

\[ = \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} s_m. \]  

(4)

If \( t^{T.C^1}_r \to s \) as \( r \to \infty \), then the series \( \sum_{r=0}^{\infty} h_r \) or the sequence \( \{s_r\} \) is summable to \( s \) by \( T.C^1 \) means.

The regularity of \( T \) and \( C^1 \) methods implies the regularity of \( T.C^1 \) method.

**Remark 1 (Example)** Consider an infinite series

\[ 1 + \sum_{n=1}^{\infty} (-1)^n 2n. \]  

(5)

The \( n \)th partial sum of (5) is given by

\[ s_n = \begin{cases} 
  n + 1, & \text{n is even}, \\
  0, & \text{n is odd} \end{cases} \]

and so

\[ C^1_n = \begin{cases} 
  1, & \text{n is even}, \\
  0, & \text{n is odd}. \end{cases} \]

Therefore, series (5) is not summable by \( (C,1) \) means.

If we take \( a_{n,k} = \frac{1}{n^k} \), then series (5) is also not summable by \( T \) means. But series (5) is summable by \( T.C^1 \) means. So, the product means is more powerful than the individual means.
Remark 2. \( TC^1 \) means reduces to

(i) \((H, \frac{1}{r\pi})C^1 \) or \( H.C^1 \) means if \( a_{r,m} = \frac{1}{r\pi(\log(r\pi))^l} \);

(ii) \((N, p)_r C^1 \) or \( NpC^1 \) means if \( a_{r,m} = \frac{p_m}{P_r} \), where \( P_r = \sum_{m=0}^{r} p_m \neq 0 \);

(iii) \((N, p, q)(C, 1) \) or \( Np,qC^1 \) means if \( a_{r,m} = \frac{p_m q_{r-m}}{R_r} \), where \( R_r = \sum_{m=0}^{r} p_m q_{r-m} \);

(iv) \((\tilde{N}, p, q)(C, 1) \) or \( \tilde{N}pC^1 \) means if \( a_{r,m} = \frac{p_m}{R_r} \).

Let \( L^2[0, 2\pi] = \{ g : [0, 2\pi] \rightarrow \mathbb{R} : \int_0^{2\pi} |g(x)|^2 \, dx < \infty, z \geq 1 \} \) be the space of functions \( (2\pi \text{- periodic and integrable}) \). We define the norm \( \| \cdot \|_z \) by

\[
\left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(x)|^z \, dx \right\}^{\frac{1}{z}}, \quad z \geq 1.
\]

As defined in ["14"], \( w : [0, 2\pi] \rightarrow \mathbb{R} \) is an arbitrary function with \( w(l) > 0 \) for \( 0 < l \leq 2\pi \) and \( \lim_{l \to 0^+} w(l) = w(0) = 0 \). Now we define

\[
H_z^{(w)} = \left\{ g \in L^2[0, 2\pi] : \sup_{l \neq 0} \frac{\|g(\cdot + l) - g(\cdot)\|_z}{w(l)} < \infty, z \geq 1 \right\}
\]

and

\[
\| \cdot \|_z^{(w)} = \| g \|_z^{(w)} = \| g \|_z + \sup_{l \neq 0} \frac{\|g(\cdot + l) - g(\cdot)\|_z}{w(l)}; \quad z \geq 1.
\]

Note 1. \( w(l) \) and \( \nu(l) \) denote “Zygmund moduli of continuity [14].”

If we consider \( \frac{w(l)}{\nu(l)} \) as positive and non-decreasing,

\[
\| g \|_z^{(w)} \leq \max \left( 1, \frac{w(2\pi)}{\nu(2\pi)} \right) \| g \|_z^{(w)} < \infty.
\]

Thus,

\[
H_z^{(w)} \subset H_z^{(\nu)} \subset L^2; \quad z \geq 1.
\]

Remark 3

(i) If \( w(l) = l^p \) in \( H_z^{(w)}, H_z^{(\nu)} \) implies \( H_a \) class.

(ii) If \( w(l) = l^p \) in \( H_z^{(w)}, H_z^{(\nu)} \) implies \( H_{a, z} \) class.

(iii) If \( z \to \infty \) in \( H_z^{(w)} \), \( H_z^{(\nu)} \) implies \( H^{(\nu)} \) class and \( H_{a, z} \) class implies \( H_a \) class.

Remark 4. We are not representing here the F. S. and C. F. S. as these trigonometric series are well known and the detailed work on these series can be found in [14].

We denote the rth partial sum of the F. S. as

\[
s_r(g;x) - g(x) = \frac{1}{2\pi} \int_0^\pi \phi_s(l) \frac{\sin(r + \frac{1}{2})l}{\sin \frac{l}{2}} \, dl.
\]

The rth partial sum of C. F. S. is defined as

\[
s_r(\tilde{g};x) - \tilde{g}(x) = \frac{1}{2\pi} \int_0^\pi \psi_s(l) \frac{\cos(r + \frac{1}{2})l}{\sin \frac{l}{2}} \, dl,
\]
where

\[ \tilde{g} = -\frac{1}{2\pi} \int_{0}^{\pi} \psi_x(l) \cot \left(\frac{l}{2}\right) dl. \]

“The error estimation of function \(g\) is given by

\[ E_r(g) = \min \|g - t_r\|_z, \]

where \(t_r\) is a trigonometric polynomial of degree \(r\) [14].”

We write

\[ \phi_x(l) = \phi(x, l) = g(x + l) + g(x - l) - 2g(x), \]

\[ \psi_x(l) = \psi(x, l) = g(x + l) - g(x - l), \]

\[ \Delta p_m = p_m - p_{m+1}, \quad m \geq 0, \]

\[ H_r(l) = \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \frac{\sin(v + \frac{l}{2})}{\sin(\frac{l}{2})}, \]

\[ \tilde{H}_r(l) = \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \frac{\cos(v + \frac{l}{2})}{\sin(\frac{l}{2})}. \]

2 Main theorems

**Theorem 2.1** If \(g \in H^{(w)}_z\) class; \(z \geq 1\) and \(\frac{w(l)}{v(l)}\) are positive and non-decreasing, then the error estimation of \(g\) by \(TC^1\) means of F.S. is

\[ \|t_r^{TC^1} - g\|_z = O\left(\frac{1}{r+1} \int_{\frac{l}{2\pi}}^{\pi} \frac{w(l)}{P^2(v(l))} dl\right), \]

where \(T = (a_{r,m})\) is an infinite triangular matrix satisfying (1) and \(w, v\) are defined as in Note 1 provided

\[ \sum_{m=0}^{r-1} |\Delta a_{r,m}| = O\left(\frac{1}{r+1}\right) \quad \text{and} \quad (r+1)a_{r,r} = O(1). \] (6)

**Theorem 2.2** If \(\tilde{g} \in H^{(w)}_z\) class; \(z \geq 1\) and \(\frac{w(l)}{v(l)}\) are positive and non-decreasing, then the error estimation of \(\tilde{g}\) by \(TC^1\) means of C. F. S. is

\[ \|t_r^{TC^1} - \tilde{g}\|_z = O\left(\frac{\log(r+1) + 1}{r+1} \int_{\frac{l}{2\pi}}^{\pi} \frac{w(l)}{P^2(v(l))} dl\right), \]

where \(T = (a_{r,m})\) is an infinite triangular matrix satisfying (1), (6) and \(w, v\) are defined as in Note 1.

3 Lemmas

**Lemma 3.1** Under condition (1), \(H_r(l) = O(r + 1)\) for \(0 < l < \frac{1}{r\pi}\).
Proof. For $0 < l < \frac{1}{r+1}$, $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$, $\sin(rl) \leq rl$.

$$H_r(l) = \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \frac{\sin(\nu + \frac{1}{2})l}{\sin(\frac{l}{2})},$$

$$|H_r(l)| \leq \frac{1}{2\pi} \times \pi \left| \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \sin \left( \nu + \frac{1}{2} \right) l \right|$$

$$\leq \frac{1}{2l} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} (2\nu + 1) \frac{l}{2}$$

$$= \frac{1}{4} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \sin(2\nu + 1)$$

$$= \frac{1}{4} \sum_{m=0}^{r} a_{r,m} (m+1)$$

$$= \frac{1}{2} \sum_{m=0}^{r} a_{r,m}$$

$$= O(r + 1).$$

Lemma 3.2 Under conditions (1) and (6), $H_r(l) = O\left(\frac{1}{(r+1)^2}\right)$ for $\frac{1}{r+1} \leq l \leq \pi$.

Proof. For $\frac{1}{r+1} \leq l \leq \pi$, $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$, $\sin^2 rl \leq 1$ and using Abel’s lemma, we have

$$H_r(l) = \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \sin(\nu + \frac{1}{2})l,$$

$$|H_r(l)| \leq \frac{1}{2\pi} \times \pi \left| \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \sin \left( \nu + \frac{1}{2} \right) l \right|$$

$$\leq \frac{1}{2l} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \sin^2(\nu + \frac{1}{2})l$$

$$\leq \frac{1}{2l} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sin^2(m+1) \frac{l}{2}$$
Lemma 3.3 Under condition (1), $\tilde{H}_r(l) = O(\frac{1}{l})$ for $0 < l < \frac{1}{r+1}$.

Proof For $0 < l \leq \frac{1}{r+1}$, using $\sin\left(\frac{l}{2}\right) \geq \frac{l}{\pi}$ and $|\cos l| \leq 1$, we obtain

$$\tilde{H}_r(l) = \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \frac{1}{\sin\left(\frac{l}{2}\right)}$$

$$|\tilde{H}_r(l)| \leq \frac{1}{2\pi} \times \frac{\pi}{l} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \cos\left(\frac{v + \frac{1}{2}}{l}\right)$$

$$\leq \frac{1}{2l} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{1}$$

$$\leq \frac{1}{2l} \sum_{m=0}^{r} a_{r,m}$$

$\therefore \tilde{H}_r(l) = O\left(\frac{1}{l}\right)$. \hfill \Box

Lemma 3.4 Under conditions (1) and (6), $\tilde{H}_r(l) = O\left(\frac{1}{(r+1)^2}\right)$ for $\frac{1}{r+1} \leq l \leq \pi$.

Proof For $\frac{1}{r+1} \leq l \leq \pi$, using $\sin\left(\frac{l}{2}\right) \geq \frac{l}{\pi}$, Abel’s lemma, and $\left|\sum_{m=0}^{r} \frac{\sin(m+1)l}{m+1}\right| \leq 1 + \frac{l}{2} \forall r$ and $l [15]$, we get

$$|\tilde{H}_r(l)|$$

$$\leq \frac{1}{2\pi} \times \frac{\pi}{l} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \cos\left(\frac{v + \frac{1}{2}}{l}\right)$$

$$\leq \frac{1}{2l} \left|\sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \left\{ 2 \sin\left(\frac{l}{2}\right) \cos\left(\frac{l}{2}\right) + 2 \sin\left(\frac{l}{2}\right) \cos\frac{3l}{2} + \cdots + 2 \sin\left(\frac{l}{2}\right) \cos\frac{(2m+1)l}{2} \right\} \right|$$

$$\leq \frac{1}{4l} \times \frac{\pi}{l} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \left| \sin l + \sin 2l - \sin l + \sin 3l - \sin 2l + \cdots \\
+ \sin (m+1)l - \sin ml \right|$$
\[\leq \frac{\pi}{4l^2} \left| \sum_{m=0}^{r} a_{r,m} \frac{\sin(m+1)l}{m+1} \right|\]
\[\leq \frac{\pi}{4l^2} \left| \sum_{m=0}^{r-1} (a_{r,m} - a_{r,m+1}) \sum_{\nu=0}^{m} \frac{\sin(\nu+1)l}{\nu+1} + a_{r,r} \sum_{m=0}^{r} \frac{\sin(m+1)l}{m+1} \right|\]
\[\leq \frac{\pi}{4l^2} \left| \sum_{m=0}^{r-1} |\Delta a_{r,m}| \left( \sum_{\nu=0}^{m} \frac{\sin(\nu+1)l}{\nu+1} + a_{r,r} \sum_{m=0}^{r} \frac{\sin(m+1)l}{m+1} \right) \right|\]
\[\leq \left[ \frac{1}{l^2} \left( \sum_{m=0}^{r-1} |\Delta a_{r,m}| + a_{r,r} \right) \right].\]
\[= \left[ \frac{1}{l^2} \left( O\left( \frac{1}{r+1} \right) + O\left( \frac{1}{r+1} \right) \right) \right]\]
\[= O\left( \frac{1}{(r+1)^2} \right). \quad \square\]

**Lemma 3.5** "([16], p. 93)" Let \( g \in H_{\pi}(w) \), then for \( 0 < l \leq \pi \):

(i) \( \| \phi(\cdot, l) \|_z = O(w(l)) \);

(ii) \( \| \phi(\cdot + y, l) - \phi(\cdot, l) \|_z = O(w(y)) \);

(iii) If \( w(l) \) and \( v(l) \) are defined as in Note 1, then \( \| \phi(\cdot + y, l) - \phi(\cdot, l) \|_z = O(v(y)) \).

**Lemma 3.6** Let \( \tilde{g} \in H_{\pi}(w) \), then for \( 0 < l \leq \pi \):

(i) \( \| \psi(\cdot, l) \|_z = O(w(l)) \);

(ii) \( \| \psi(\cdot + y, l) - \psi(\cdot, l) \|_z = O(w(y)) \);

(iii) If \( w(l) \) and \( v(l) \) are defined as in Note 1, then \( \| \psi(\cdot + y, l) - \psi(\cdot, l) \|_z = O(v(y)) \).

**Proof** This lemma can be proved along the same lines as the proof of Lemma 3.5(iii). \( \square\)

### 4 Proof of the main theorems

#### 4.1 Proof of Theorem 2.1

**Proof** Following Titchmarsh [17], \( s_r(g;x) \) of F. S. is given by

\[ s_r(g;x) - g(x) = \frac{1}{2\pi} \int_0^\pi \phi_x(l) \frac{\sin(m+\frac{1}{2})l}{\sin(\frac{l}{2})} dl. \]

Now, denoting \( T_rC^1 \) transform of \( s_r(g;x) \) by \( t_rT_rC^1 \),

\[ t_rT_rC^1(x) - g(x) = \sum_{m=0}^r a_{r,m} (C^1_{m}(x) - g(x)) \]
\[ = \sum_{m=0}^r a_{r,m} \left( \frac{1}{m+1} \sum_{\nu=0}^m s_r(g;x) - g(x) \right) \]
\[ = \sum_{m=0}^r a_{r,m} \left( \frac{1}{2\pi} \int_0^\pi \phi_x(l) \left( \frac{1}{2\pi} \sum_{m=0}^r a_{r,m} \frac{1}{m+1} \sum_{\nu=0}^m \frac{\sin(\nu+\frac{1}{2})l}{\sin(\frac{l}{2})} \right) dl \right). \]
\[ t_rT_rC^1(x) - g(x) = \int_0^\pi \phi_x(l) H_r(l) dl. \quad (7) \]
Let
\[ R_r(x) = t_r^{T,C^1}(x) - g(x) = \int_0^\pi \phi_2(l)H_r(l) dl. \] (8)

Then
\[ R_r(x + y) - R_r(x) = \int_0^\pi (\phi(x + y, l) - \phi(x, l))H_r(l) dl. \]

"Using generalized Minkowski's inequality Chui [18]," we get
\[ \|R_r(\cdot + y) - R_r(\cdot)\|_z \leq \int_0^\pi \|(\cdot + y, l) - (\cdot, l)\|_z H_r(l) dt \]
\[ = \left( \int_0^{\frac{\pi}{r+1}} + \int_{\frac{\pi}{r+1}}^\pi \right) \|(\cdot + y, l) - (\cdot, l)\|_z H_r(l) dl \]
\[ = I_1 + I_2. \] (9)

Using Lemmas 3.1 and 3.5(iii), we have
\[ I_1 = \int_0^{\frac{\pi}{r+1}} \|\phi(\cdot + y, l) - \phi(\cdot, l)\|_z H_r(l) dl \]
\[ = O(r + 1)\left( v(|y|) \int_0^{\frac{\pi}{r+1}} \frac{w(l)}{v(l)} dl \right) \]
\[ = O\left( v(|y|) \frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})} \right). \] (10)

Also, using Lemmas 3.2 and 3.5(iii), we get
\[ I_2 = \int_{\frac{\pi}{r+1}}^\pi \|\phi(\cdot + y, l) - \phi(\cdot, l)\|_z H_r(l) dl \]
\[ = O\left( \frac{1}{r+1} \int_{\frac{\pi}{r+1}}^\pi v(|y|) \frac{w(l)}{P v(l)} dl \right). \] (11)

By (9), (10), and (11), we have
\[ \sup_{\gamma \neq 0} \frac{\|R_r(\cdot + y) - R_r(\cdot)\|_z}{v(|y|)} = O\left( \frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})} \right) + O\left( \frac{1}{r+1} \int_{\frac{\pi}{r+1}}^\pi \frac{w(l)}{P v(l)} dl \right). \] (12)

Again applying Minkowski's inequality, Lemma 3.1, Lemma 3.2, and \( \|\phi(\cdot, l)\|_z = O(w(l)) \), we obtain
\[ \|R_r(\cdot)\|_z = \|t_r^{T,C^1} - g\|_z \]
\[ \leq \left( \int_0^{\frac{\pi}{r+1}} + \int_{\frac{\pi}{r+1}}^\pi \right) \|\phi(\cdot, l)\|_z H_r(l) dl \]
\[ = O\left( (r + 1) \int_{0}^{\pi} w(l) \, dl \right) + O\left( \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2} \, dl \right) \]
\[ = O\left( w \left( \frac{1}{r + 1} \right) \right) + O\left( \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2} \, dl \right). \quad (13) \]

Now, we have
\[ \| R_r(\cdot) \|_z^v = \| R_r(\cdot) \|_z + \sup_{y \neq 0} \frac{\| R_r(\cdot + y) - R_r(\cdot) \|_z}{v(|y|)}. \quad (14) \]

Using (12) and (13), we get
\[ \| R_r(\cdot) \|_z^v = O\left( w \left( \frac{1}{r + 1} \right) \right) + O\left( \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2 v(l)} \, dl \right). \quad (15) \]

By the monotonicity of \( v(l) \), \( w(l) = \frac{w_0(l)}{v(l)} \leq v(\pi) \frac{w_0(l)}{v(\pi)} \) for \( 0 < l \leq \pi \), we get
\[ \| R_r(\cdot) \|_z^v = O\left( \frac{w(\frac{1}{r + 1})}{v(\frac{1}{r + 1})} \right) + O\left( \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2 v(l)} \, dl \right). \quad (16) \]

Since \( w \) and \( v \) are moduli of continuity such that \( \frac{w_0}{v(l)} \) is positive and non-decreasing, therefore
\[ \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2 v(l)} \, dl \geq \frac{w(\frac{1}{r + 1})}{v(\frac{1}{r + 1})} \left( \frac{1}{r + 1} \right) \int_{\pi/2}^{\pi} \frac{1}{l^2} \, dl \geq \frac{w(\frac{1}{r + 1})}{2v(\frac{1}{r + 1})}. \]

Then
\[ \frac{w(\frac{1}{r + 1})}{v(\frac{1}{r + 1})} = O\left( \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2 v(l)} \, dl \right). \quad (17) \]

From (16) and (17), we get
\[ \| R_r(\cdot) \|_z^{(v)} = O\left( \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2 v(l)} \, dl \right), \]
\[ \| t_r^{T C_l} - g \|_z^{(v)} = O\left( \frac{1}{r + 1} \int_{\pi/2}^{\pi} \frac{w(l)}{l^2 v(l)} \, dl \right). \quad (18) \]

4.2 Proof of Theorem 2.2

Proof. The integral representation of \( s_r(\tilde{g}; x) \) is given by
\[ s_r(\tilde{g}; x) - \tilde{g}(x) = \frac{1}{2\pi} \int_{0}^{\pi} \psi_x(l) \frac{\cos(r + \frac{1}{2})l}{\sin(\frac{l}{2})} \, dl. \]
Now, denoting $T.C^1$ transform of $s_r(\tilde{g}; \cdot)$ by $\tilde{t}_r^{T.C^1}$, we get

$$\tilde{t}_r^{T.C^1}(x) - \tilde{g}(x) = \sum_{m=0}^{r} a_{r,m} \left( C^1_m(x) - \tilde{g}(x) \right)$$

$$= \sum_{m=0}^{r} a_{r,m} \left( \frac{1}{m+1} \sum_{v=0}^{m} s_r(\tilde{g}; x) - \tilde{g}(x) \right)$$

$$= \int_0^\pi \psi_x(l) \left( \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \cos(v + \frac{1}{2}) \right) \sin^2 \left( \frac{l}{2} \right) dl,$$

$$\tilde{t}_r^{T.C^1}(x) - \tilde{g}(x) = \int_0^\pi \psi_x(l) \tilde{H}_r(l) dl.$$ 

Let

$$\tilde{R}_r(x) = \tilde{t}_r^{T.C^1}(x) - \tilde{g}(x) = \int_0^\pi \psi_x(l) \tilde{H}_r(l) dl.$$ 

Then

$$\tilde{R}_r(x + y) - \tilde{R}_r(x) = \int_0^\pi \{ \psi_x(x + y, l) - \psi_x(x, l) \} \tilde{H}_r(l) dl.$$ 

Using "generalized Minkowski’s inequality Chui [18]," we get

$$\| \tilde{R}_r(\cdot + y) - \tilde{R}_r(\cdot) \|_z \leq \int_0^\pi \| \psi_x(\cdot + y, l) \|_z \tilde{H}_r(l) dl$$

$$= \left( \int_0^{\pi/2} + \int_{\pi/2}^\pi \right) \| \psi(\cdot + y, l) - \psi(\cdot, l) \|_z \tilde{R}_r(l) dl$$

$$= I_1 + I_2. \quad (19)$$

Using Lemmas 3.3 and 3.6(iii), we have

$$I_1 = \int_0^{\pi/2} \| \psi(\cdot + y, l) - \psi(\cdot, l) \|_z \tilde{H}_r(l) dl$$

$$= O\left( \nu(|y|) \frac{w(1)}{\nu^{1/2}} \int_0^{\pi/2} \frac{1}{l} \, dl \right)$$

$$= O\left( \nu(|y|) \frac{w(1)}{\nu^{1/2}} \log(r + 1) \right). \quad (20)$$

Again using Lemmas 3.4 and 3.6(iii), we have

$$I_2 = \int_{\pi/2}^\pi \| \psi(\cdot + y, l) - \psi(\cdot, l) \|_z \tilde{H}_r(l) dl$$

$$= O\left( \frac{1}{r + 1} \int_{\pi/2}^\pi \nu(|y|) \frac{w(l)}{\nu^{1/2} l} \, dl \right). \quad (21)$$
Using (19), (20), and (21), we have
\[
\sup_{y \neq 0} \frac{\| \tilde{R}_r(\cdot + y) - \tilde{R}_r(\cdot) \|_z}{\nu(|y|)} = O\left( \frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)} \log(r+1) \right) + O\left( \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l} dl \right). \tag{22}
\]

Again applying Minkowski's inequality, Lemma 3.3, Lemma 3.4, and \( \| \psi(\cdot, l) \|_z = O(w(l)) \), we have
\[
\| \tilde{R}_r(\cdot) \|_z = \| \tilde{R}_r(\cdot) \|_z + \sup_{y \neq 0} \frac{\| \tilde{R}_r(\cdot + y) - \tilde{R}_r(\cdot) \|_z}{\nu(|y|)}.
\]

Using (22) and (23), we get
\[
\| \tilde{R}_r(\cdot) \|_z^{(v)} = O\left( \left( \log(r+1)w\left(\frac{1}{r+1}\right) \right) \right) + O\left( \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l} dl \right)
\]
\[
+ O\left( \frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)} \log(r+1) \right) + O\left( \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l} dl \right).
\]

By the monotonicity of \( v(l) \), we have \( w(l) = \frac{w(l)}{v(l)} v(l) \leq v(\pi) \frac{w(l)}{v(\pi)}, 0 < l \leq \pi \), we get
\[
\| \tilde{R}_r(\cdot) \|_z^{(v)} = O\left( \frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)} \log(r+1) \right) + O\left( \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l} dl \right). \tag{24}
\]

Using the fact that \( \frac{w(l)}{v(l)} \) is positive and non-decreasing, we have
\[
\frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l} dl \geq \frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)} \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{1}{l} dl
\]
\[
\geq \frac{w\left(\frac{1}{r+1}\right)}{2v\left(\frac{1}{r+1}\right)}.
\]

Then
\[
\frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)} = O\left( \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l} dl \right). \tag{25}
\]
From (24) and (25), we get
\[
\|\tilde{T}_{C^1}\|_{r}^{(\nu)} = O\left(\frac{\log(r+1)}{r+1} \int_{\frac{1}{r+1}}^{r} \frac{w(l)}{l} \, dl\right) + O\left(\frac{1}{r+1} \int_{\frac{1}{r+1}}^{r} \frac{w(l)}{l^2} \, dl\right).
\]
\[
\therefore \|\tilde{T}_{C^1} - \tilde{g}\|_{r}^{(\nu)} = O\left(\frac{\log(r+1) + 1}{r+1} \int_{\frac{1}{r+1}}^{r} \frac{w(l)}{l^2} \, dl\right).
\]
\[
(26)
\]

5 Corollary

Corollary 5.1 Let \(0 \leq \beta < \alpha \leq 1\) and \(\tilde{g} \in H_{(\nu)_z}; \). Then
\[
\|\tilde{T}_{C^1} - \tilde{g}\|_{(\nu)_z} = \begin{cases} O[(\log(r+1)e)(r+1)^{\beta-\alpha}] & \text{if } 0 \leq \beta < \alpha < 1, \\ O\left[\frac{\log(r+1)e}{r+1}\right] & \text{if } \beta = 0, \alpha = 1. \end{cases}
\]

Proof Putting \(w(l) = l^r, \nu(l) = l^\alpha, 0 \leq \beta < \alpha \leq 1\) in (26)
\[
\|\tilde{T}_{C^1} - \tilde{g}\|_{(\nu)_z} = O\left[\frac{\log(r+1)e}{r+1}\right] \int_{\frac{1}{r+1}}^{r} l^{r-\beta-2} \, dl
\]
\[
\Rightarrow \|\tilde{T}_{C^1} - \tilde{g}\|_{(\nu)_z} = \begin{cases} O\left[\frac{\log(r+1)e}{r+1}\right] \int_{\frac{1}{r+1}}^{r} l^{r-\beta-2} \, dl & \text{if } 0 \leq \beta < \alpha < 1, \\ O\left[\frac{\log(r+1)e}{r+1}\right] \int_{\frac{1}{r+1}}^{r} l^{\alpha-1} \, dl & \text{if } \beta = 0, \alpha = 1, \end{cases}
\]
\[
\therefore \|\tilde{T}_{C^1} - \tilde{g}\|_{(\nu)_z} = \begin{cases} O[(\log(r+1)e)(r+1)^{\beta-\alpha}] & \text{if } 0 \leq \beta < \alpha < 1, \\ O\left[\frac{\log(r+1)e}{r+1}\right] & \text{if } \beta = 0, \alpha = 1. \end{cases}
\]

Corollary 5.2 Let \(0 \leq \beta < \alpha \leq 1\), \(a, b \in \mathbb{R}\) and suppose \(w(l) = \frac{e}{(\log l)^{\alpha}}, \nu(l) = \frac{e}{(\log l)^{\beta}}, 0 < l \leq \pi, \tilde{g} \in H_{(\nu)_z}; , \). Then
\[
\|\tilde{T}_{C^1} - \tilde{g}\|_{z}^{(\nu)} = \begin{cases} O\left[\frac{\log(r+1)e}{(\log l)^{\alpha}}\right] & \text{if } \alpha = \beta \text{ and } a - b \geq -1, \\ O\left[\frac{\log(r+1)e}{(\log l)^{\beta}}\right] & \text{if } \alpha = \beta \text{ and } a - b = -1. \end{cases}
\]

Proof We have
\[
\|\tilde{T}_{C^1} - \tilde{g}\|_{z}^{(\nu)} = O\left(\frac{\log(r+1)e}{r+1} \int_{\frac{1}{r+1}}^{r} l^{a} \left(\log \frac{1}{l}\right)^{b-a} \, dl\right)
\]
\[
= O\left(\frac{\log(r+1)e}{r+1} \int_{\frac{1}{r+1}}^{r} l^{a-\beta-2} \left(\log \frac{1}{l}\right)^{b-a} \, dl\right)
\]
\[
\therefore \|\tilde{T}_{C^1} - \tilde{g}\|_{z}^{(\nu)} = \begin{cases} O\left[\frac{\log(r+1)e}{(\log l)^{\alpha}}\right] & \text{if } \alpha = \beta \text{ and } a - b \geq -1, \\ O\left[\frac{\log(r+1)e}{(\log l)^{\beta}}\right] & \text{if } \alpha = \beta \text{ and } a - b = -1. \end{cases}
\]

Corollary 5.3 If \(\alpha, \nu = \frac{1}{(r+1)(\log r+1)}\), then \(T.C^1\) means reduces to \((H, \frac{1}{r+1})(C, 1)\) means and error estimation of a function \(g \in H_{(\nu)_z}; \) by \((H, \frac{1}{r+1})(C, 1)\) means of \(F.S.\) is
\[
\|\tilde{T}_{C^1} - \tilde{g}\|_{z}^{(\nu)} = O\left(\frac{1}{r+1} \int_{\frac{1}{r+1}}^{r} \frac{w(l)}{l^{2}} \, dl\right).
\]
Corollary 5.4 If \( a_{r,m} = \frac{p_{r-m}}{r_r} \), then \( T.C^1 \) means reduces to \( N_p.C^1 \) and the error estimation of \( g \in H^{(w)}_r \) by \( N_p.C^1 \) means of F. S. is

\[
\left\| L_{N_p.C^1} - g \right\|_z^{(v)} = O\left( \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{F^2(v(l))} \, dl \right).
\]

Corollary 5.5 If \( a_{r,m} = \frac{p_{r-m}}{R_r} \), then \( T.C^1 \) means reduces to \( N_{p,q}.C^1 \) and the error estimation of \( g \in H^{(w)}_r \) by \( N_{p,q}.C^1 \) means of F. S. is

\[
\left\| L_{N_{p,q}.C^1} - g \right\|_z^{(v)} = O\left( \frac{1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{F^2(v(l))} \, dl \right).
\]

Corollary 5.6 If \( a_{r,m} = \frac{1}{(r-m+1)\log(r+1)} \), then \( T.C^1 \) means reduces to \( (H, \frac{1}{r+1})(C,1) \) means and the error estimation of a function \( \tilde{g} \in H^{(w)}_r \) by \( (H, \frac{1}{r+1})(C,1) \) means of C. F. S. is

\[
\left\| L_{H^1} - \tilde{g} \right\|_z^{(v)} = O\left( \frac{\log(r+1) + 1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{F^2(v(l))} \, dl \right).
\]

Corollary 5.7 If \( a_{r,m} = \frac{p_{r-m}}{r_r} \), then \( T.C^1 \) means reduces to \( N_p.C^1 \) and the error estimation of \( \tilde{f} \in H^{(w)}_r \) by \( N_p.C^1 \) means of C. F. S. is

\[
\left\| L_{N_p.C^1} - \tilde{f} \right\|_z^{(v)} = O\left( \frac{\log(r+1) + 1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{F^2(v(l))} \, dl \right).
\]

Corollary 5.8 If \( a_{r,m} = \frac{p_{r-m}}{R_r} \), then \( T.C^1 \) means reduces to \( N_{p,q}.C^1 \) and the error estimation of \( \tilde{f} \in H^{(w)}_r \) by \( N_{p,q}.C^1 \) means of C. F. S. is

\[
\left\| L_{N_{p,q}.C^1} - \tilde{f} \right\|_z^{(v)} = O\left( \frac{\log(r+1) + 1}{r+1} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{F^2(v(l))} \, dl \right).
\]

Remark 5
(i) If \( z \rightarrow \infty \) in \( H^{(w)}_z \) class, then \( H^{(w)}_z \) class reduces to \( H^{(w)} \) class. Also putting \( w(l) = l^\mu \) and \( v(l) = l^\nu \) in our Theorem 2.1, \( H^{(w)} \) class reduces to \( H_a \) class; then, by putting \( \beta = 0 \) in \( H_a \) class, \( H_{a,z} \) class reduces to \( \text{Lip} \alpha \) class.
(ii) In our Theorem 2.1, by putting \( w(l) = l^\mu \), \( v(l) = l^\nu \) in \( H^{(w)}_z \) class, \( H^{(w)}_z \) class reduces to \( H_{a,z} \); then, by putting \( \beta = 0 \) in \( H_{a,z} \) class, \( H_{a,z} \) class reduces to \( \text{Lip}(\alpha, z) \) class.

6. Particular cases
6.1. Using Remark 4(i), our Theorem 2.1 becomes a particular case of Dhakal [1].
6.2. Using Remark 4(ii) and putting \( a_{r,m} = \frac{p_{r-m}}{R_r} \), where \( R_r = \sum_{m=0}^{\nu} p_m q_{r-m} \) in our of Theorem 2.1, our result of Theorem 2.1 becomes a particular case of the main theorem of Kushwaha and Dhakal [3].
6.3. Using Remark 4(i) and putting \( a_{r,m} = \frac{p_{r-m}}{R_r} \), where \( R_r = \sum_{m=0}^{\nu} p_m q_{r-m} \) in our Theorem 2.1, our Theorem 2.1 becomes a particular case of the main theorem of Dhakal [2].
7 Conclusion
Approximation by trigonometric polynomials is at the heart of approximation theory. Much of the advances in the theory of trigonometric approximation are due to the periodicity of the functions. The study of error approximation of periodic functions in Lipschitz and Hölder classes has been of great interest among the researchers [1–11], and [12] in recent past. The trigonometric Fourier approximation (TFA) is of great importance due to its wide applications in different branches of engineering such as electronics and communication engineering, electrical and electronics engineering, computer science engineering, etc. Several elegant results on TFA can be found in a monograph [14].

In this paper, we, for the first time, obtain the best approximation of the functions \( g \) and \( \tilde{g} \) in a generalized Hölder class \( H_r^{(w)} (r \geq 1) \) using Matrix-C \( C^1 \) (T.C) method of F. S. and C. F. S. respectively. Since, in view of Remark 2, the product summability means \( H.C^1 \), \( N_p C^1 \), \( N_p q C^1 \), and \( \tilde{N}_p C^1 \) are the particular cases of Matrix-C \( C^1 \) method, so our results also hold for these methods, which are represented in a form of corollaries. In view of Remark 1, it has been shown that (TC \( C^1 \) ) method is more powerful than the individual \( T \) and \( C^1 \) method. Moreover, in view of Remark 5, some previous results (see Sect. 6) become the particular cases of our Theorem 2.1. We also deduce a corollary for the \( H_{\alpha r} \) class \( (r \geq 1) \).

Some other studies regarding the modulus of continuity (smoothness) of functions using more generalized functional spaces may be addressed as a future work.

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Authors’ contributions
HK framed the problems. HK and MH carried out the results and wrote the manuscript. All the authors read and approved the final manuscript.

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References
1. Dhakal, B.P.: Approximation of functions belonging to Lip \( \alpha \) class by matrix-Cesàro summability method. Int. Math. Forum 5(35), 1729–1735 (2010)
2. Dhakal, B.P.: Approximation of a function \( f \) belonging to Lip class by \( (N_{p,q})C^1 \) means of its Fourier series. Int. J. Eng. Technol. 2(9), 1–15 (2013)
3. Kushwaha, J.K., Dhakal, B.P.: Approximation of a function belonging to Lip(\( \alpha \),r) class by \( N_{p,q} C^1 \) summability method of its Fourier series. Nepal J. Sci. Technol. 14(2), 117–122 (2013)
4. Khatri, K.: Degree of approximation of functions \( f \in H_v \) class by the \( (N_{p,E}) \) means in the Hölder metric. Int. J. Math. Math. Sci. 2014, Article ID 837408 (2014)
5. Khatri, K., Mishra, V.N.: Degree of approximation by the \( T.E^1 \) means of conjugate series of Fourier series in the Hölder metric. Iran. J. Sci. Technol., Trans. A Sci. (2017). https://doi.org/10.1007/s40995-017-0272-3
6. Mishra, L.N.: On existence and behaviour of solutions to some nonlinear integral equations with applications. Ph.D. thesis, National Institute of Technology, Silchar 788 010, Assam, India (2017)
7. Mishra, V.N., Khatri, K., Mishra, L.N., Deepmala: Trigonometric approximation of periodic signals belonging to generalized weighted Lipschitz \( W (L_\infty, \xi (t)) (r \geq 1) \)-class by Norlund–Euler \( (N_{p,q})E^1 \) operator of conjugate series of its Fourier series. J. Class. Anal. 5(2), 91–105 (2014). https://doi.org/10.7155/jca-05-08
8. Mishra, L.N., Mishra, V.N., Khatri, K., Deepmala: On the trigonometric approximation of signals belonging to
generalized weighted Lipschitz $W^{\frac{r}{r}}(\xi_{r}) (r \geq 1)$-class by matrix $C^1_{N_p}$ operator of conjugate series of its Fourier
series. Appl. Math. Comput. 237, 252–263 (2014)
9. Mishra, V.N., Khatri, K., Mishra, L.N.: Product $(N,p_n)C_1$ summability of a sequence of Fourier coefficients. Math. Sci. 6, 38
(2012)
10. Mishra, V.N., Mishra, L.N.: Trigonometric approximation of signals (functions) in $L_\varphi^p$ ($\varphi \geq 1$)-norm. Int. J. Contemp. Math.
Sci. 7(19), 909–918 (2012)
11. Mittal, M.L., Rhoades, B.E., Mishra, V.N., Singh, U.: Using infinite matrices to approximate functions of class $Lip(\alpha, p)$
using trigonometric polynomials. J. Math. Anal. Appl. 326, 667–676 (2007)
12. Tiwari, K.S., Barwal, C.S.: The degree of approximation of functions in the Hölder metric by triangular matrix method
of Fourier series. Int. J. Pure Appl. Math. 76(2), 227–232 (2012)
13. Toeplitz, O.: Über eine allgemeine lineare Mittelbildung. J. Reine Angew. Math. 113, 113–119 (1912)
14. Zygmund, A.: Trigonometric Series, 3rd edn. Cambridge University Press, Cambridge (2002)
15. Gronwall, T.H.: Über die Gibbssche Erscheinung und die trigonometrische summation
$\sin x + (1/2)\sin 2x + (1/3)\sin 3x + \cdots + (1/n)\sin nx$. Math. Ann. 72, 228–243 (1912)
16. Lal, S., Mishra, A.: The method of summation $(E, 1)(N, p_n)$ and trigonometric approximation of function in generalized
Hölder metric. J. Indian Math. Soc. 80(1–3), 87–98 (2013)
17. Titchmarsh, E.C.: The Theory of Functions. Oxford University Press, London (1939)
18. Chui, C.K.: An Introduction to Wavelets: Wavelet Analysis and Applications, vol. 1. Academic Press, San Diego (1992)