Orbital angular momentum of the down converted photons

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We calculate the relative amplitude of orbital angular momentum (OAM) entangled photon pairs from the spontaneous parametric down conversion. The results show that the amplitude depends on both the two Laguerre indices \( l, p \). We also discuss the influences of the mostly used holograms and mono-mode fibers for mode analysis. We conclude that only a few dimensions can be explored from the infinite OAM modes of the down-converted photon pairs.

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I. INTRODUCTION

Quantum entanglement is a very important property of quantum mechanics. It is the foundation of quantum teleportation [1], quantum computation [2–4], quantum cryptography [5], superdense coding [6], etc. Up to now most of the theoretical discussions and experiments are focused on quantum states belonging to two-dimensional states, or qubits [7–10]. In recent years, the interest in multi-dimensional states, or qudits, is steadily growing in not only polarization, or spin angular momentum, but also OAM [22,24]. This provides a promise to explore OAM of the down-converted photons from SPDC and also OAM [22,24]. This provides a promise to explore OAM of the down-converted photons from SPDC and also OAM [22,24].

II. SPONTANEOUS PARAMETRIC DOWN CONVERSION AND OAM

It is well known that photons can carry both spin angular momentum and OAM [26]. Spin angular momentum is associated with polarization and OAM with the azimuthal phase of the electric field. The normalized LG mode is given in cylindrical coordinates by

\[
LG^l_p(\rho, \varphi, z) = D^l_p \frac{1}{\sqrt{\pi l! \omega^2}} \frac{\sqrt{2^p \rho}}{\omega} L_p^{l+1}(\frac{2\rho^2}{\omega^2}) \exp(-\rho^2/\omega^2) \exp(-i2p + (l+1)\varphi) \exp(-il\varphi),
\]

where \( L_p^m(x) \) are the associated Laguerre polynomials,

\[
L_p^m(x) = \sum_{m=0}^{p} (-1)^m \frac{(\frac{l}{p} + p)!}{(p-m)!((l+m)!m!)} x^m,
\]

and the standard definitions for Gaussian beam parameters are used:

- \( \omega(z) = \omega_0 \sqrt{1 + (z/z_R)^2} \) : spot size,
- \( R(z) = z(1 + (z_R/z)^2) \) : radius of wavefront curvature,
- \( \psi(z) = \arctan(z/z_R) \) : Gouy phase,
- \( z_R = \frac{k \omega_0^2}{2} \) : Rayleigh range,

\( \omega_0 \) is the beam width at the beam waist, the index \( l \) is referred to as the winding number, and \( p + 1 \) is the number of radial nodes. If the mode function is a pure LG mode with winding number \( l \), then every photon of this beam carries an OAM of \( lh \). This corresponds to an eigenstate of the OAM operator with eigenvalue \( lh \) [26]. If the mode function is not a pure LG mode, the state is...
a superposition state, with the weights dictated by the contribution of the \( l \)th harmonic modes.

At the beam waist \((z = 0)\), the LG mode can be written as

\[
LG^l_p(\rho, \varphi) = \sqrt{\frac{2p!}{\pi(|l| + p)!}} \frac{1}{\omega_0} \left( \frac{\sqrt{2} \rho}{\omega_0} \right)^{|l| + |p|} \frac{2^p \rho^2}{\omega_0^2} \exp(-\rho^2/\omega_0^2) \exp(-il\varphi),
\]

\( (3) \)

In the following calculation, we use this equation because in the experiment we always manipulate the light at its beam waist.

In the SPDC process, a thin quadratic nonlinear crystal is illuminated by a laser pump beam propagating in the \( z \) direction, with wave number \( k_p \) and waist \( \omega_0 \). The generated two-photon quantum state is given by [20]

\[
|\Psi\rangle = \sum_{l_1, p_1} \sum_{l_2, p_2} C_{l_1, l_2, p_1, p_2} |l_1, p_1; l_2, p_2\rangle,
\]

\( (4) \)

where \((l_1, p_1)\) corresponds to the mode of the signal beam and \((l_2, p_2)\) the mode of the idler beam. The probability amplitude \( C_{l_1, l_2, p_1, p_2} \) is given as [16,17,19,20]

\[
C_{l_1, l_2, p_1, p_2} \sim \int d\rho \Phi(r_\perp)[LG^l_{p_1}(r_\perp)]^* [LG^{l_2}_{p_2}(r_\perp)]^*,
\]

\( (5) \)

where \( r_\perp \) is the radial coordinate in the transverse \( X - Y \) plane. \( \Phi(r_\perp) \) is the spiral distribution of the pump beam at the input face of the crystal, \( LG^l_{p_1}(r_\perp) \) is the spiral distribution of the LG mode beam at the same plane.

The weights of the quantum superposition are given by \( A_{l_1, l_2, p_1, p_2}^l \sim |C_{l_1, l_2, p_1, p_2}|^2 \). It is the ideal joint detection probability for finding one photon in the signal mode \((l_1, p_1)\) and one photon in the idler mode \((l_2, p_2)\).

To consider the case that the pump beam is in a pure LG mode \( LG^{l_0}_0 \) with \( p_0 = 0 \). The LG^l_0 mode light at \( z = 0 \) can be written as

\[
LG^l_0(\rho, \varphi) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega_0} \left( \frac{\sqrt{2} \rho}{\omega_0} \right)^{|l|} \exp(-\rho^2/\omega_0^2) \exp(-il\varphi).
\]

\( (6) \)

Substitute \( LG^l_0(\rho, \varphi) \) for \( \Phi(r_\perp) \) into Eq. (5), and use the OAM conservation law in SPDC [16,22]:

\[
l_1 + l_2 = l_0,
\]

\( (7) \)

where \( l_1, l_2, l_0 \) are the winding numbers of signal beam, idler beam and pump beam respectively, we can achieve the probability amplitude

\[
C_{l_1, l_2, p_1, p_2} \sim \sum_{m=0}^{p_1} \sum_{n=0}^{p_2} \frac{2}{3} \left( \frac{2m + 2n + l_1 + l_2}{2} \right) (-1)^{m+n} \frac{p_1!p_2!(l_1 + n)!((l_2 + n)!}{(l_1 + l_2 + n)!m!} \omega_0^2 \exp(-\rho^2/\omega_0^2) \exp(-il\varphi),
\]

\( (8) \)

In the case \( l_0 = 0 \), or the input beam is Gaussian mode light, Eq. (8) can be simplified as \((l > 0)\)

\[
C_{p_1, p_2}^{l, -l} \sim \sum_{m=0}^{p_1} \sum_{n=0}^{p_2} \frac{2}{3} \left( \frac{2m + 2n + l_1 + l_2}{2} \right) (-1)^{m+n} \omega_0^2 \exp(-\rho^2/\omega_0^2) \exp(-il\varphi),
\]

\( (9) \)

It can be easily proved that \( C_{p_1, p_2}^{l, -l} = C_{p_2, p_1}^{-l, l} = C_{p_1, p_2}^{l, -l} = C_{p_2, p_1}^{-l, l} \).

Table 1 gives the relative value for \( p_1, p_2, 0, 1, 2 \) and \( l = 0, 1, 2 \). We can also illustrate the dependence of the relative probability amplitude \( C_{p_1, p_2}^{l, -l} \) on \( p_1, p_2, l \), and Fig. 1 \((l = 0 \text{ and } p_1, p_2 = 0, 1, 1, 2, 3, 4)\) and Fig. 2 \((p_1 = p_2 = 0 \text{ and } l = 0, 1, 2, 3, 4)\).

From the above table and figures, we can see that the probability amplitude decreases very rapidly with the growing of \( p_1, p_2 \) and \( l \). We then just consider the cases with \( p_1, p_2 = 0, 1, 2 \) and \( l = 0, 1, 2 \) when \( p_0 = 0 \). In papers [22,24], they also just consider the cases with \( l = 0, 1, 2 \).

If additionally assume \( p_1 = p_2 = 0 \) as in the previous works [22,24,25,20], we can obtain:

\[
C_{0,0}^{l, -l} \sim \left( \frac{2}{3} \right)^l.
\]

\( (10) \)

This result can also be achieved from the Eq. (14) of the paper [20], when the condition \( l_1 = -l_2 = l \) is assumed.

But till now, this assumption has not been proven either in theoretical discussions [16,20] or in experiments [22,24]. We will discuss the two cases separately in Section 5. Before proceed, we analyze the two main experiment elements, computer generated holograms and mono-mode fibre, which will unavoidably affect the detected relative probability amplitude.

### III. Computer generated holograms and the mode analysis

In most of the recent experiments [21–25], the authors always use computer generated holograms to transform Gaussian mode light into other LG modes light, or change the winding number of LG mode light. It is a kind of transmission holograms with the transmittance function:

\[
T(\rho, \varphi) = \exp(\left( \frac{\delta}{2\pi} \right)^i \left\langle \frac{\pi}{2} \right\rangle \mod(l \varphi - \frac{2\pi}{\Lambda} \rho \cos \varphi, 2\pi)),
\]

\( (11) \)

where \( \delta \) is the amplitude of the phase modulation, \( \Lambda = \frac{2\pi}{\Delta \ell_{in}} \) is the period of the grating at a large distance away from the fork, \( k_x \) is the \( x \) component of the simplest reference beam’s wave vector. Corresponding to the diffraction order \( m \), the hologram can change the winding number of the input beam by \( \Delta l_m = ml \). The diffraction efficiency depends on the phase modulation \( \delta \). When \( \delta = 2\pi \), almost 100% of the incident intensity is diffracted into the first-order.
However, even the input beam is a pure LG mode light, the output beam after the hologram is not a pure LG mode light. The output light will be the superposition of the various LG modes with the same \( l \) and different \( p \) [27]. In addition, the beam waists of the input and output beam will affect the weights of different components of the output beam. Assume the input beam and the output beam have the same waists \( \omega_0 \), thus the complex expansion coefficients of the decomposition of the \( n \)th diffraction order can be calculated as:

\[
a_{pl} = \int \int (LG_{p1}^l(\rho, \varphi) \exp(-im\frac{2\pi}{\lambda} r \cos \varphi))^* \times T(\rho, \varphi) E_{in}(\rho, \varphi) \rho d\rho d\varphi,
\]

where \( E_{in}(\rho, \varphi) = LG_{p1}^l(\rho, \varphi) \). Consider the first-order diffraction, or \( m = 1 \), Eq. (12) can be then rewritten as

\[
a_{l_1',l_1'}^{l_1,l_1} = \int \int (LG_{p1}^{l_1}(\rho, \varphi))^* \exp(-i\Delta l \varphi) LG_{p1}^{l_1}(\rho, \varphi) \rho d\rho d\varphi,
\]

where \( \Delta l = l_1 - l_1' \) is the winding number changed by the hologram. The relative weight of the output modes is given by

\[
p_{p_1,p_1'}^{l_1,l_1'} = \left| a_{l_1,l_1'} \right|^2.
\]

As the mono-mode fibres can only detect the photons with \( l = 0 \), we consider the case that the output light is in the modes with \( l_1' = 0 \). Then \( p^{l_1,l_1'} \) can be simplified as \( P_{p_1,p_1'}^{\Delta l} \), and \( P_{p_1,p_1'}^{\Delta l} = P_{p_1-p_1',p_1'}^{\Delta l} \). In most of the experiments [22–25], only the holograms of \( \Delta l = 1 \) or 2 are employed. Table 2 and Table 3 give the relative weight of different modes after the computer generated hologram with \( \Delta l = 1 \) and 2.

From Table 2 and Table 3, we can see most of the input mode \( LG_{p}^{\Delta l} \) is converted into \( LG_{p}^{0} \) and \( LG_{p+1}^{0} \). Thereby, we only consider the case that the output light is in the modes with \( p' = 0, 1, 2, 3 \).

### IV. MONO-MODE FIBRE AND DETECTION EFFICIENCY OF THE OAM MODES

It is known that only one mode of light can transmit in the mono-mode fibre: \( HE_{11} \) mode. And in practical calculation, we always use Gaussian mode to replace the \( HE_{11} \) mode. The Gaussian mode is

\[
E(\rho) = E(0) \exp(-\rho^2/\omega^2),
\]

where \( d = 2\omega \) is the Mode Field Diameter(MFD) of the fibre, \( E(0) \) is amplitude of field at the fibre center. For \( LG_{p}^{0} \) mode light, the detection efficiency is given as

\[
Q_{l,p} = \frac{\iint (LG_0^p)^* E(\rho) \rho d\rho d\varphi \rho d\rho d\varphi}{\iint (LG_0^p)^* LG_0^p \rho d\rho d\varphi \rho d\rho d\varphi \rho d\rho d\varphi}.
\]

Obviously if \( l \neq 0 \), then \( Q_{l,p} = 0 \). For the case \( l = 0 \), Eq. (16) can be simplified as:

\[
Q_p = \frac{\iint (LG_0^0)^* E(\rho) \rho d\rho d\varphi \rho d\rho d\varphi \rho d\rho d\varphi}{\iint (LG_0^p)^* LG_0^p \rho d\rho d\varphi \rho d\rho d\varphi \rho d\rho d\varphi}.
\]

To calculate the relative joint detection probability of the down-converted photons from SPDC, we only need the relative detection efficiency of the \( LG_0^0, LG_1^0, LG_2^0 \) and \( LG_3^0 \), but not their absolute efficiency. Assume the waist size of the input beam is adjusted equal to \( d/2 \). Then when the detection area is much more larger than the cross-section of the input light, only \( LG_0^0 \) mode light can be detected. But in practice, the detection area is determined by the fibre diameter. To simplify calculation, we further assume that the detection area is a round area with diameter equal to the MFD. Thus the integral for \( p \) is from 0 to \( \omega \). With these assumptions, the relative efficiencies for \( p = 0, 1, 2, 3 \) can be written as

\[
Q_0 : Q_1 : Q_2 : Q_3 = 1 : 0.263 : 0 : 0.036
\]

### V. RELATIVE JOINT DETECTION PROBABILITIES OF OAM ENTANGLED PHOTONS FROM SPDC

With the above discussions about the influence of computer generated holograms and mono-mode fibres, we now consider the joint detection probability for OAM entangled photons generated from an experimental set-up similar to the work [22]. The joint detection probability can be written as

\[
R_l = \sum_{p_1=0}^{2} \sum_{p_2=0}^{2} (\sum_{p_1'=0}^{1} \sum_{p_2'=0}^{1} (P_{p_1'p_1,p_2'p_2}^{l} Q_{p_1} Q_{p_2})).
\]

When \( l = 0 \), \( R_0 \) gives the joint detection probability that there is no hologram in both the signal and idler beam. And for the case \( l \neq 0 \), \( R_l \) represents the joint detection probability with one \( \Delta l = -l \) hologram in the signal beam and one \( \Delta l = l \) hologram in the idler beam. Obviously, \( R_l = R_{-l} \).

Substitute the values of \( C, P \) and \( Q \) calculated in the above sections to Eq. (19), we can get the relative joint detection probability of the three cases \( l = 0, 1, 2 \) as

\[
R_0 : R_1 : R_2 = 1 : 0.346 : 0.101.
\]

If we also assume that \( p_1 = p_2 = 0 \) for the SPDC process as the recent papers [22,24,25,20], the joint detection probability can thus be written as

\[
R_l = (\sum_{p_1'=0}^{1} \sum_{p_2'=0}^{1} (P_{p_1'p_1,p_2'p_2}^{l} Q_{p_1} Q_{p_2})).
\]
And the relation for \( l = 0, 1, 2 \) becomes:

\[
R_0 : R_1 : R_2 = 1 : 0.311 : 0.079.
\] (22)

The difference between Eq. (20) and (22) is caused by the additional assumption that Laguerre index \( p_1 = p_2 = 0 \). But this assumption has not been proven either in theoretical discussions [16,20] or in experiments [22,24]. From the above calculations, we can see that this assumption will cause non-trivial influence to the relative joint detection probability. Our results put forward a feasible method to verify the assumption.

If we rule out the influence of the holograms and mono-mode fibres, and make the assumption of \( p_1 = p_2 = 0 \), the relationship for the relative joint detection probability of the cases \( l = 0, 1, 2 \) becomes

\[
R_0 : R_1 : R_2 = 1 : 0.444 : 0.198.
\] (23)

Compare Eq. (22) with Eq. (23), we can see that the experimental elements can apparently influence the joint detection probability. In the experiment by Vaziri and co-workers [24], they found that the state of the OAM entangled photons from SPDC was given by

\[
\psi = 0.65 |0,0\rangle + 0.60 |1,-1\rangle + 0.47 |-1,1\rangle.
\] (24)

From this equation we can get the relationship for the relative joint detection probabilities of the cases \( l = 0, 1, -1 \) as

\[
R_0 : R_1 : R_{-1} = 1 : 0.852 : 0.523.
\] (25)

Including the influence of computer generated holograms and mono-mode fibres, and loosening the assumption of \( p_1 = p_2 = 0 \), we expect this relation be

\[
R_0 : R_1 : R_{-1} = 1 : 0.346 : 0.346.
\] (26)

The reason for the difference between Eq. (25) and (26) might be as follows: in experiment, the diffraction efficiency of the hologram can not be 100% . Generally, different holograms have different diffraction efficiencies. The waists of the input beam and the output beam of holograms will also affect the final experiment detection probabilities. In addition, the fibre diameter and MFD of the practical mono-mode fibre will also affect the detection efficiencies of different modes light. Evidently, the detection efficiency of avalanche detectors has little effect on the relative joint detection probabilities. Thus in the practical experiment, the \( P \) and \( Q \) values have to be adjusted according to the particular conditions.

VI. CONCLUSION

In conclusion, we have calculated the probability amplitudes of different LG modes of the down converted photons. Our results show that the relative amplitude decreases almost exponentially with growing of OAM. We also discussed the impact of the previous assumption for \( p \) on the joint detection probability. In addition, we analyzed the influences of the experiment elements. We concluded that only a few dimensions can be explored from the infinite OAM modes of the down-converted photon pairs. The experiment verification of the present theory is straightforward and is currently in progress in our laboratory.

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Table 1. The relative probability amplitude $C_{l,p1,p2}^{l,0}$ of the down converted photons from SPDC. We let $C_{0,0}^{0,0}$ be unity.

Fig. 1. The relative probability amplitude $C$ for $l = 0$ and $p1, p2 = 0, 1, 2, 3, 4$. We let $C_{0,0}^{0,0}$ be unity.

Fig. 2. The relative probability amplitude $C$ for $p1 = p2 = 0$ and $l = 0, 1, 2, 3, 4$. We let $C_{0,0}^{0,0}$ be unity.

Table 2. The relative weight of different modes after the computer generated hologram with $\Delta l = 1$.

Table 3. The relative weight of different modes after the computer generated hologram with $\Delta l = 2$. 

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| \( C^{l,-l}_{p_1, p_2} \) | \( l \) | \( 0 \) | \( 1 \) | \( 2 \) |
|---|---|---|---|---|
| (0,0) | | 1 | 0.6667 | 0.4444 |
| (0,1) | 0.3333 | 0.3143 | 0.2566 |
| (0,2) | 0.1111 | 0.1283 | 0.1210 |
| (1,1) | 0.5556 | 0.4444 | 0.3457 |
| (1,2) | 0.3333 | 0.3024 | 0.2561 |
| (2,2) | 0.4074 | 0.3539 | 0.2963 |

Table 1.
| $p_{1}^{1}$ | $p_{1}$  |
|----------|---------|
| $p_{1}$  | 0       | 1       | 2       | 3       |
| 0        | 0.7854  | 0.1963  | 0.0123  | 0.0031  |
| 1        | 0.0982  | 0.6136  | 0.2592  | 0.0188  |
| 2        | 0.0368  | 0.0828  | 0.5528  | 0.2912  |

Table 2.
| $p^2_{n,n}$ | $p_i$ |     |     |     |
|------------|------|-----|-----|-----|
|            | 0    | 1   | 2   | 3   |
| $p_1$      |      |     |     |     |
| 0          | 0.5000 | 0.5000 | 0 | 0 |
| 1          | 0.1667 | 0.1667 | 0.6667 | 0 |
| 2          | 0.0833 | 0.0833 | 0.0833 | 0.7500 |

Table 3.