Furstenberg’s Times 2, Times 3 Conjecture (a Short Survey)

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Abstract
The following is a concise exposition on the conjecture and three of its proofs for the case of positive entropy, by D. Rudolph [22], by B. Host [14] and by W. Parry [21]. A simpler theorem of R. Lyons [19] - preceding them - is also presented and proved. This is a survey, no new results are introduced.

1 Introduction
Throughout this exposition, $\mathbb{T}$ will denote for us $\mathbb{R}/\mathbb{Z}$. Let $p, q > 1$ be multiplicatively independent integers, i.e. they are not both powers of the same integer. H. Furstenberg proved in [10] (1967) the following theorem:

**Theorem 1.1:** If $F \subseteq \mathbb{T}$ is infinite, closed and invariant under multiplication by both $p$ and $q$ then $F = \mathbb{T}$.

An equivalent formulation of the theorem is that the orbit of every irrational point under the semi-group $\langle p, q \rangle$ is dense. Furstenberg also raised the following conjecture which is the measurable analog of Theorem 1.1:

**Conjecture 1.2 (Furstenberg’s Times 2, Times 3 Conjecture):** If $\mu \in M(\mathbb{T})$ (the probability measures on $(\mathbb{T}, \mathcal{B}_\mathbb{T})$) is an atomless invariant measure under multiplication by both $p$ and $q$ then $\mu = \lambda$ (Lebesgue measure).

(Atomic ergodic measures which are likewise invariant of course do exist - those are the uniform measures on the orbits of the action of the multiplicative semi-group $\langle p, q \rangle$ on any rational number
by deploying ergodic decomposition one sees that requiring \( \mu \) to be also ergodic with respect to the action of the multiplicative semi-group \( \langle p, q \rangle \) results in an equivalent conjecture. It is tempting to think the conjecture immediately implies Theorem 1.1, by taking an ergodic invariant measure supported on the orbit closure of an irrational point. However, in order to prove that there exists such a measure that is not supported on a finite set of rationals, it seems one must argue in a similar fashion as the proof of Theorem 1.1 itself.

Contemporary knowledge about the conjecture’s validness is more or less summarized by Rudolph’s Theorem [22].

**Theorem 1.3 (Rudolph’s Theorem):** Let \( p, q > 1 \) be relatively prime integers. If \( \mu \in M(T) \) is an invariant measure under multiplication by both \( p \) and \( q \), ergodic under the action of the multiplicative semi-group \( \langle p, q \rangle \), and, moreover, there exists \( r \in \langle p, q \rangle \) with \( h_{\mu}(r) > 0 \) then \( \mu = \lambda \).

We shall see (Proposition 3.1) that there exists such an \( r \) if and only if multiplication by every element of \( \langle p, q \rangle \) has positive entropy. Therefore one could have equally required \( h_{\mu}(p) > 0 \) instead.

Actually, A. Johnson, Rudolph’s student, improved Rudolph’s proof to apply for multiplicatively independent integers \( p, q \) [15] (this full version of the theorem is called Rudolph-Johnson Theorem), but since Lyon’s and Host’s ideas presented here apply only to relatively prime integers \( p, q \) we stated the theorem as we did for our convenience. (Notice that by passing to the invertible extension, Rudolph’s theorem for the relatively prime integers \( p, q > 1 \) implies its validity also for \( p^{m_1}q^{n_1}, p^{m_2}q^{n_2} \) where \( m_1, m_2, n_1, n_2 \in \mathbb{N} \) and \( m_1n_2 - n_1m_2 \neq 0 \).)

Not much is known about the case of zero entropy - in [9], the existence of such an invariant measure without atoms is shown to be equivalent to the existence of two partitions that satisfy some condition that is formulated using only the Lebesgue measure of the circle. Additional known facts about the conjecture (and Theorem 1.1) that are not included in this exposition include its generalizations to higher dimensional tori and other connected compact Abelian groups [1] [2] [6].
some quantitative estimates \cite{3}, treatments of related conjectures and other proofs of Rudolph’s Theorem that exist such as that of Feldman \cite{17} (a weaker result than Rudolph-Johnson Theorem) and that of Hochman and Shmerkin \cite{12, 13} (a proof of Rudolph-Johnson Theorem - it uses ergodic theoretical methods associated with fractal geometry originating from \cite{11}). We shall also not elaborate on the importance of the subject, but only mention here that apart from raising a very natural question, the conjecture also serves as the leading toy example for the more general phenomenon of measure rigidity of higher rank hyperbolic actions (for which contemporary understanding is in a similar situation \cite{5, 16, 17, 18}).

Apart from the application of Prop 3.1 in section 4, the following sections are independent from one another.

I wish to thank my doctoral advisors Prof. H. Furstenberg and Prof. T. Meyerovitch for having the patience to hear my presentation of most of this survey and for some valuable remarks.

\section{Lyon’s Theorem}

The first significant advancement towards a proof of the Times 2, Times 3 conjecture was made by R. Lyons in \cite{19} (1988).

**Definition:** A measure preserving system $(X, T, \mu)$ is $K$-mixing (or $T$-exact) if for every $g \in L^2(X, \mu)$ the limit $\langle f \circ T^n, g \rangle - \langle f, 1 \rangle \langle 1, g \rangle \to 0$ as $n \to \infty$ exists uniformly for $\|f\|_2 \leq 1$.

**Theorem 2.1:** Let $p, q > 1$ be relatively prime integers. If $\delta_0 \neq \mu \in M(T)$ is an invariant measure under multiplication by both $p$ and $q$ and $K$-mixing for the times $p$ transformation then $\mu = \lambda$.

So let $p, q > 1$ be relatively prime integers. Lyons proves two elementary number theoretical lemmas.

**Lemma 2.2:** There exist $A, d, L \in \mathbb{N}$ such that $q^A = dp^L + 1$ and $L \geq 2$, gcd $(d, p) = 1$. 

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Lemma 2.3: Assuming the minimal $A$ for which $d, L$ exist satisfying Lemma 2.2, then for all $l \geq L$ the order of $q$ modulo $p^l$ is $p^l - L A$.

So $q^x$ attains $p^l - L A$ values Modulo $p^l$, and this implies a conclusion.

Conclusion 2.4: $q^x \equiv b \pmod{p^l}$ has a solution if and only if $q^x \equiv b \pmod{p^L}$ does.

The proof of Theorem 2.1 proceeds as follows. From Conclusion 2.4 we know that there exists a strictly increasing sequence $n_j$ for $j \geq L$ such that $q^{n_j} \equiv p^j + 1 \pmod{p^{2j}}$ (since $q^x \equiv p^j + 1 \pmod{p^L}$ has a solution for every such $j$), i.e. $q^{n_j} = d_j p^{2j} + p^j + 1$ for some integer $d_j$.

For $0 \neq m \in \mathbb{Z}$, we need to prove that $\hat{\mu}(m) = 0$. Writing

\[
\hat{\mu}(m) = \hat{\mu}(mq^{n_j}) = \hat{\mu}(m (d_j p^{2j} + p^j + 1)) = \hat{\mu}((md_j p^j + m) p^j + m),
\]

and substituting twice two characters for the functions $f, g$ in the definition of $K$-mixing, we obtain:

\[
\hat{\mu}(m) = \hat{\mu}(m) \lim_{j \to \infty} \hat{\mu}(md_j p^j + m) = \hat{\mu}(m)^2 \lim_{j \to \infty} \hat{\mu}(md_j).
\]

Assuming to the contrary that $\hat{\mu}(m) \neq 0$, the conclusion $|\hat{\mu}(m)| = 1$ is forced upon us. This means $\int_T e^{2m \pi x i} \, d\mu(x)$ has no cancellations whatsoever, i.e. $\mu$ is supported on finitely many points. This is a contradiction since then our system is not even mixing.

3 Rudolph’s Theorem

Motivated by the implicit role positive entropy plays in Lyons’ result\(^1\), D. Rudolph published his theorem (Theorem 1.3) two years later in 1990\[22\]. In this section we present his proof (with very

\(^1\)The $K$-mixing hypothesis in Theorem 2.1, implies triviality of the Pinsker $\sigma$-algebra of $\sigma_p$, the multiplication by $p$ transformation. Indeed, If a set $C$ belongs to the Pinsker $\sigma$-algebra of $\sigma_p$, then - modulo $\mu$ - it is a tail event of the partition $\left\{\left[\frac{0}{p^n}, \frac{1}{p^n}\right)\right\}_{j=0}^{p^n-1}$. This means that for every $n > 0$ there exists a Borel measurable set $C_n$ such that $C = \sigma_p^{-n}(C_n)$. Taking $f_n = 1_{C_n}$ and $g = 1_C$ and using $K$-mixing we deduce $\mu(C)$ equals either 0 or 1.
We begin with a fundamental observation by Rudolph.

**Proposition 3.1**: Let $a, b \in \mathbb{N}$ and $\sigma_a$, $\sigma_b$ the maps of multiplication of $\mathbb{T}$ by each of them respectively. If $\mu \in M (\mathbb{T})$ is an invariant measure under $\sigma_a$ and $\sigma_b$, then $\log b \cdot h_\mu (\sigma_a) = \log a \cdot h_\mu (\sigma_b)$.

**Proof**: Assume $a, b > 1$ for otherwise there is nothing to prove. Denote by $\xi$ the partition $\{ \frac{i + \frac{1}{a}}{a} b \}^{a-1}_{j=0}$. The lengths of the intervals composing the partition $\vee_{i=0}^{l-1} \sigma_a^{-i} (\xi)$ is composed of $p$, and of $\vee_{i=0}^{m-1} \sigma_b^{-i} (\xi)$ is between $\frac{1}{a^b}$ and $\frac{1}{a}$, and hence none of which can intersect more than $b + 1$ of the intervals composing $\vee_{i=0}^{l-1} \sigma_a^{-i} (\xi)$. Thus

$$H_\mu (\vee_{i=0}^{l-1} \sigma_a^{-i} (\xi)) \leq H_\mu (\vee_{i=0}^{l-1} \sigma_a^{-i} (\xi) \vee \vee_{i=0}^{m(l)-1} \sigma_b^{-i} (\xi))$$

$$= H_\mu (\vee_{i=0}^{m(l)-1} \sigma_b^{-i} (\xi)) + H_\mu (\vee_{i=0}^{l-1} \sigma_a^{-i} (\xi) \mid \vee_{i=0}^{m(l)-1} \sigma_b^{-i} (\xi)) \leq H_\mu (\vee_{i=0}^{m(l)-1} \sigma_b^{-i} (\xi)) + \log (b + 1).$$

By Dividing both sides by $l$ we get

$$\frac{1}{l} H_\mu (\vee_{i=0}^{l-1} \sigma_a^{-i} (\xi)) \leq \frac{m(l)}{m(l)} H_\mu (\vee_{i=0}^{m(l)-1} \sigma_b^{-i} (\xi)) + \frac{1}{l} \log (b + 1),$$

and by letting $l \to \infty$ we conclude $\log b \cdot h_\mu (\sigma_a) \leq \log a \cdot h_\mu (\sigma_b)$. The reverse inequality is obtained similarly. ■

An immediate conclusion is that if $\mu \in M (\mathbb{T})$ is invariant under a semi-group of natural numbers different than 1, than the entropy is either positive for all its elements or zero for all its elements.

Now let $p, q, \mu$ be as in Rudolph’s Theorem (Theorem 1.3), $T_0 : \mathbb{T} \to \mathbb{T}$ the multiplication by $p$ and $S_0 : \mathbb{T} \to \mathbb{T}$ by $q$.

### 3.1 Preparations

We construct a subshift of finite type (SFT) $X$ for $\mathbb{Z}^2$ with the alphabet $\Lambda = \{0, 1, 2, \ldots, pq - 1\}$. Denote by $T$ the action of $(1, 0)$ and by $S$ that of $(0, 1)$. The rule of the subshift allows $i \in \Lambda$ to horizontally precede $j \in \Lambda$ if $\left( \frac{i + \frac{1}{pq}, i + \frac{1}{pq} \right) \cap T^{-1}_0 \left( \left( \frac{i}{pq}, \frac{i}{pq} \right) \right) \neq \emptyset$ (The pre-image on the right is composed of $p$ intervals of length $\frac{1}{pq}$), and likewise $i \in \Lambda$ to vertically precede $j \in \Lambda$ if
The purpose of this subsection is to clear the ground and understand properties of $X$ and its exact relation to our original system on the circle. The strategy of the proof is introduced only in the next subsection.

Associating to this the horizontal adjacency matrix $M_T$ and the vertical one $M_S$ we deduce that $M_T M_S = M_S M_T$ is just the $pq \times pq$ matrix with all entries equal to 1 (this is because the diagonal of the subshift is the full one-dimensional shift $\Lambda$). Notice then while we have complete freedom to choose the values of a positively infinite diagonal ray (of $(1,1)$ differences), it then determines all values of its corresponding quadrant.

Define $\varphi : X \to T$ by $\varphi(x) = \sum_{n=0}^{\infty} x(n,n)_{(pq)}^{n+1}$. It is equivariant and, on the $(\mathbb{N} \cup \{0\})^2$-restricted version of $X$ (taking only the part of $X$ which is in the non-negative quadrant) $\varphi$ is almost one-to-one. The bad set (which has two-point fibers) is $V = \left\{ \left( \frac{t}{pq} \right) : 0 \leq t, n \in \mathbb{Z} \right\}$. In particular, this implies that, in $X \setminus \varphi^{-1}(V)$, any positively infinite horizontal ray (of $(1,0)$ differences) determines everything above it, and any positively infinite vertical ray (of $(0,1)$ differences) determines everything to its right (one can verify that these statements are in fact true for all $X$- but we shall not make use of this fact).

Assuming $\mu$ is not supported on $V$ (that is $\mu \neq \delta_0$) we can lift it up to a unique $(T,S)$–ergodic measure on $X \setminus \varphi^{-1}(V)$ which we will denote by $\tilde{\mu}$. Denote by $\alpha$ the partition of $X$ by the value at the coordinate $(0,0)$.

**Lemma 3.2:** $h_{\tilde{\mu}}(T \mid A) = h_{\tilde{\mu}}(T, \alpha \mid A)$, $h_{\tilde{\mu}}(S \mid A) = h_{\tilde{\mu}}(S, \alpha \mid A)$ where $A$ is any sub-$\sigma$-algebra invariant under both $T$ and $S$ (i.e. $S^{-1}A = T^{-1}A = A$).

**Proof:** Just notice that $S^m \overset{\infty}{\underset{i=-\infty}{\vee}} T^{-i}(\alpha) \not\rightarrow$ the $\sigma$-algebra of $X$, so by Kolmogorov-Sinai Theorem $h_{\tilde{\mu}}(T \mid A) = \lim_{m \to \infty} h_{\tilde{\mu}}(T^m \alpha \mid A) = h_{\tilde{\mu}}(T, \alpha \mid A)$. The proof that $h_{\tilde{\mu}}(S \mid A) = h_{\tilde{\mu}}(S, \alpha \mid A)$ is similar. ■

The fact that $\gcd(p,q) = 1$ is taken advantage of only through the following observation and in the proof of Lemma 3.6.
Lemma 3.3: For any values \( x(n,k) \) on all \( n \geq 1, k \geq 0 \) together with \( x(0,m_0) \) for some \( m_0 \geq 0 \) all satisfying the SFT rule with their neighbors among those coordinates, there exists \( x_1, x_2, \ldots, x_k \) in \( X \) with those values and they are all equal also on \( (0,0) \) for all \( k \geq 0 \). Similarly, for any values \( x(n,k) \) on all \( n \geq 0, k \geq 1 \) together with \( x(m_0,0) \) for some \( m_0 \geq 0 \) all satisfying the SFT rule with their neighbors among those coordinates, there exists \( x_1, x_2, \ldots, x_k \) in \( X \) with those values and they are all equal also on \( (n,0) \) for all \( n \geq 0 \).

Proof: We only prove the first statement (the proof of the second one is similar). \( x(0,m_0) \) determines \( x(0,m_0 + 1) \) since \( M_T M_S \) has all its entries equal to 1. It then suffices to prove that \( x(0,0) \) is determined when \( m_0 = 1 \).

\[
T_0^{-1}(\varphi(T(x))) = \left\{ \frac{x(T(x)) + i}{p} : 0 < i < p \right\}
\]

and thus \( S_0(T_0^{-1}(\varphi(T(x)))) = \left\{ \frac{x(T(x) + ia)}{p} : 0 < i < p \right\} \).

Since \( q \) is invertible in the ring \( \mathbb{Z}/p\mathbb{Z} \) there exists a one-to-one correspondence between allowed entries for \( x(0,0) \) and allowed entries for \( x(0,1) \). Thus the choice of the former determines the latter. ■

Lemma 3.4: For \( \tilde{\mu} \)-a.e. \( x \) the events \( \{x' \in X : x'(-1,0) = k_1\} \) and \( \{x' \in X : x'(0,-1) = k_2\} \) for \( 0 \leq k_1, k_2 < pq \) are independent when conditioned on the first quadrant

\[
(\text{i.e. the quadrant } \bigvee_{i=0}^{\infty} \bigvee_{j=0}^{\infty} T^{-i}S^{-j}(\alpha) = \bigvee_{i=0}^{\infty} T^{-i}S^{-i}(\alpha)).
\]

Proof: It suffices to prove that

\[
H_{\tilde{\mu}} \left( T^{-1}(\alpha) \vee S^{-1}(\alpha) \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = H_{\tilde{\mu}} \left( T^{-1}(\alpha) \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) + H_{\tilde{\mu}} \left( S^{-1}(\alpha) \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right).
\]

Since a positive diagonal ray determines a quadrant and also by applying Lemma 3.3:

\[
\alpha \vee \left( \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = T^{-1}(\alpha) \vee S^{-1}(\alpha) \vee \left( \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right). \text{ Hence}
\]

\[
H_{\tilde{\mu}} \left( \alpha \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = H_{\tilde{\mu}} \left( T^{-1}\alpha \vee S^{-1}\alpha \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right)
\]

\[
\leq H_{\tilde{\mu}} \left( T^{-1}(\alpha) \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) + H_{\tilde{\mu}} \left( S^{-1}(\alpha) \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = h_{\tilde{\mu}}(T) + h_{\tilde{\mu}}(S) \text{ (the last equality is by Lemma 3.2).}
\]

So we want to show the inequality is in fact an equality, namely that

\[
H_{\tilde{\mu}} \left( \alpha \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = h_{\tilde{\mu}}(T) + h_{\tilde{\mu}}(S). \text{ This is an immediate consequence of Prop. 3.1 (because}
\]

\[
H_{\tilde{\mu}} \left( \alpha \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = h_{\tilde{\mu}}(TS), \text{ but let us also present a computation that does not make use of it.}
\]

\[
H_{\tilde{\mu}} \left( \alpha \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = H_{\tilde{\mu}} \left( \alpha \vee \left( \bigvee_{i=1}^{\infty} T^{-i}(\alpha) \right) \mid \bigvee_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) \text{ since the diagonal ray}
\]

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\[ \lim_{i=0}^{\infty} T^{-i}S^{-i}(\alpha) = \alpha \lor \left( \lim_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) \] determines the corresponding quadrant which includes \[ \lim_{i=1}^{\infty} T^{-i}\alpha. \] But

\[ H_{\tilde{\mu}} \left( \alpha \lor \left( \lim_{i=1}^{\infty} T^{-i}(\alpha) \right) \right) = H_{\tilde{\mu}} \left( \alpha \lor \left( \lim_{i=1}^{\infty} \lim_{j=0}^{\infty} T^{-i}S^{-j}(\alpha) \right) \right) + H_{\tilde{\mu}} \left( T^{-1}(\alpha) \mid \lim_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) \]

(because \( T^{-1}(\alpha) \lor \left( \lim_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) = \lim_{i=1}^{\infty} \lim_{j=0}^{\infty} T^{-i}S^{-j}(\alpha) \)) and

\[ H_{\tilde{\mu}} \left( \alpha \mid \lim_{i=1}^{\infty} \lim_{j=0}^{\infty} T^{-i}S^{-j}(\alpha) \right) + H_{\tilde{\mu}} \left( T^{-1}(\alpha) \mid \lim_{i=1}^{\infty} T^{-i}S^{-i}(\alpha) \right) \]

\[ = H_{\tilde{\mu}} \left( \alpha \mid \lim_{i=1}^{\infty} T^{-i}(\alpha) \right) + H_{\tilde{\mu}} \left( \alpha \mid \lim_{i=1}^{\infty} S^{-i}(\alpha) \right) = h_{\tilde{\mu}}(T) + h_{\tilde{\mu}}(S). \]

**Lemma 3.5:** If \( \mathcal{A} \) is a sub-\( \sigma \)-algebra invariant under both \( T \) and \( S \) (i.e. \( T^{-1}\mathcal{A} = S^{-1}\mathcal{A} = \mathcal{A} \))

then

\[ h_{\tilde{\mu}}(T; \mathcal{A}) = \frac{\log p}{\log q} h_{\tilde{\mu}}(S; \mathcal{A}) \] - where \( h_{\tilde{\mu}}(S; \mathcal{A}) \) denotes the entropy of the action of \( S \) on \((X, \mathcal{A}, \tilde{\mu})\).

**Proof:** By lifting Prop. 3.1 from the circle to \( X \) (through \( \varphi \)) one concludes that \( h_{\tilde{\mu}}(T) = \frac{\log p}{\log q} h_{\tilde{\mu}}(S) \), and by a similar proof to Prop. 3.1 that \( h_{\tilde{\mu}}(T \mid \mathcal{A}) = \frac{\log p}{\log q} h_{\tilde{\mu}}(S \mid \mathcal{A}) \). The Abramov-Rokhlin formula yields the desired result. ■

### 3.2 The Heart of the Proof

Supposing \( \mu \) is not Lebesgue measure \( \lambda \), we want to show that \( h_{\tilde{\mu}}(T) = 0 \). The strategy of the proof is based on the insight that it is enough to find a sub-\( \sigma \)-algebra \( \mathcal{A} \) invariant under both \( T \) and \( S \) (i.e. \( T^{-1}\mathcal{A} = S^{-1}\mathcal{A} = \mathcal{A} \)) such that (i) \( h_{\tilde{\mu}}(S; \mathcal{A}) = 0 \) and (ii) \( \alpha \subseteq \mathcal{A} \lor \left( \lim_{i=1}^{\infty} T^{-i}(\alpha) \right) \). This is because then \( h_{\tilde{\mu}}(T) = h_{\tilde{\mu}}(T \mid \mathcal{A}) + h_{\tilde{\mu}}(T; \mathcal{A}) = h_{\tilde{\mu}}(T; \mathcal{A}) \) (the first equality is by the Rokhlin-Abramov formula, and the second by Lemma 3.2 and (ii)). But \( h_{\tilde{\mu}}(T; \mathcal{A}) = \frac{\log p}{\log q} h_{\tilde{\mu}}(S; \mathcal{A}) = 0 \) by lemma 3.5. The rest of this subsection is dedicated to finding such an \( \mathcal{A} \) and proving that it satisfies those two desired properties.

For this purpose, given any fixed \( n \in \mathbb{N} \) we define a function \( \nu_n(\cdot) : X \to M(T) \) for \( \tilde{\mu} \)-a.e. \( x \) by the formula

\[ \nu_n(x) = \sum_{t=0}^{p^n-1} a_n^{(t)}(x) \cdot \delta_{\frac{t}{p^n}} \] where
\( a_n^{(t)}(x) = \mu_{x}^{\circ T^{-i \alpha}} \left( \left\{ x' \in X : x'(-j,0) = T^n \circ \varphi^{-1} \left( \varphi \left( T^{-n}(x) \right) + \frac{1}{p^n} \right)(-j,0) \text{ for all } 1 \leq j \leq n \right\} \right) \)

\( \mu_{x}^{\circ T^{-i \alpha}} \) is the conditional measure at \( x \) with respect to \( \mu_{x}^{\circ T^{-i \alpha}} \). \( \nu_n \) is measurable according to the weak-* topology on its range (equivalently, all \( a_n^{(t)}(x) \) are real measurable functions).

**Lemma 3.6:** \( a_n^{(qt \ mod \ p^n)}(S(x)) = a_n^{(t)}(x) \) for \( \mu \)-a.e. \( x \).

**Proof:** We will explain only the case of \( n = 1 \) as the general case follows by a similar inductive argument. So why does \( a_1^{(qt \ mod \ p)}(S(x)) = a_1^{(t)}(x) \)?

\[
\begin{align*}
\mu_{x}^{\circ S^{-1T^{-i \alpha}}(x)} & \left( \left\{ x' \in X : x'(-1,0) = T \circ \varphi^{-1} \left( \varphi \left( S^{-1T^{-i \alpha}}(x) \right) + \frac{1}{p} \right)(-1,0) \right\} \right) \\
= \mu_{x}^{\circ S^{-1T^{-i \alpha}}(x)} & \left( \left\{ x' \in X : x'(-1,0) = T \circ \varphi^{-1} \left( \varphi \left( T^{-1}(x) \right) + \frac{1}{p} \right) \right\} \right) \\
= \mu_{x}^{\circ S^{-1T^{-i \alpha}}(x)} & \left( \left\{ x' \in X : x'(-1,0) = T \circ \varphi^{-1} \left( \varphi \left( T^{-1}(x) \right) + \frac{1}{p} \right) \right\} \right)
\end{align*}
\]

where in the last equality we use Lemma 3.4. Since \( \alpha \vee \left( \bigvee_{i=0}^{\infty} S^{-1T^{-i \alpha}}(x) \right) = \bigvee_{i=0}^{\infty} T^{-i \alpha} \), this is indeed \( a_1^{(t)}(x) \).

We are now ready to define our invariant \( \sigma \)-algebra \( A \) as the minimal \( T \)-invariant (i.e. \( T^{-1}A = A \)) sub-\( \sigma \)-algebra on \( X \) for which all the functions \( \nu_n \) are measurable. By Lemma 3.6 it is also \( S \) invariant (i.e. \( S^{-1}A = A \)). Notice that \( A_k \not\supset A \) where - for every \( k \geq 0 \) - \( A_k \) is the minimal sub-\( \sigma \)-algebra for which the function \( \nu_{2k+1} \circ T^k \) is measurable.

**Lemma 3.7:** There is \( j_k \) such that \( S^{j_k}(A) = A \) for all \( A \in A_k \).

**Proof:** Take \( j_k \) to be the order of \( q \) in \((\mathbb{Z}/p^{2k+1}\mathbb{Z})^X \). By Lemma 3.6, \( a_{2k+1}^{(qt \ mod \ p^{2k+1})}(S^{j_k}T^k(x)) = a_{2k+1}^{(t)}(T^k(x)) \), and so \( a_{2k+1}^{(t \ mod \ p^{2k+1})}(S^{j_k}T^k(x)) = a_{2k+1}^{(t)}(T^k(x)) \) which means the function \( \nu_{2k+1} \circ T^k \) stays invariant under the action of \( S^{j_k} \).

We can now conclude desired property (i) of \( A \).

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\(^2\)Using also the trivial fact that if events \( A_1, A_2, A_3, B_1, B_2, B_3 \) in a probability space satisfy \( \mathbb{P}(A_2 \mid A_3) = \mathbb{P}(B_2 \mid B_3) \) and \( \mathbb{P}(A_1 \mid A_2 \cap A_3) = \mathbb{P}(B_1 \mid B_2 \cap B_3) \) then \( \mathbb{P}(A_1 \cap A_2 \mid A_3) = \mathbb{P}(B_1 \cap B_2 \mid B_3) \).
Corollary 3.8: Each ergodic component with respect to the action of $S$ on the factor corresponding to $\mathcal{A}$ has a rational pure point spectrum. In particular, $h_{\bar{\mu}}(S; \mathcal{A}) = 0$.

Proof: Let $\tilde{\eta}$ be such an ergodic component, then each $S$-factor corresponding to $\mathcal{A}_k$ is just a periodic measure on a finite number of atoms and our $S$-system with $\tilde{\eta}$ is isomorphic to the inverse limit of these $S$-factors. In particular, our $S$-system with $\tilde{\eta}$ is isomorphic to a Kronecker system (an ergodic rotation of a compact group) and hence has 0 entropy. ■

If $x, y \in X$ agree on the non-negative horizontal axis then $\nu_n(x)$ is just a translation of $\nu_n(y)$ by $\varphi(T^{-n}(x)) - \varphi(T^{-n}(y))$. Let us call a point $x \in X$ symmetric if there exists $y \in X$ agreeing with it on the non-negative horizontal axis but disagreeing with it on a coordinate $(-i_0, 0)$ for some $i_0 > 0$, and $\nu_n(T^m(x)) = \nu_n(T^m(y))$ for all $m \geq 0$ and $n \in \mathbb{N}$. This implies that $\nu_n(T^m(x))$ is invariant under a translation by $\varphi(T^{-n+m}(x)) - \varphi(T^{-n+m}(y))$.

Next, we want to prove that $x$ is $\bar{\mu}$-a.s. not symmetric (Prop. 3.11).

Lemma 3.9: The set of symmetric points is $T$ and $S$ invariant (and hence of measure either 0 or 1).

Proof: $T$ invariance is obvious. For $S$ invariance, let $x, y$ be a pair of corresponding points as in the definition of a symmetric point. We claim that $S(x)$ and $S(y)$ also form such a pair. Lemma 3.3 implies that if $i_0 = \min \{i : x(-i, 0) \neq y(-i, 0)\}$ then $S(x)(-i_0, 0) \neq S(y)(-i_0, 0)$. So it suffices to prove that $\nu_n(T^m(S(x))) = \nu_n(T^m(S(y)))$, and this will follow if we prove $a_n^{(qt \mod p^n)}(T^mS(x)) = a_n^{(qt \mod p^n)}(T^mS(y))$ since $q$ is invertible in the ring $\mathbb{Z}/p^n\mathbb{Z}$ as

$$\gcd(p, q) = 1.$$ 

Indeed, $a_n^{(qt \mod p^n)}(T^mS(x)) = a_n^{(t)}(T^m(x)) = a_n^{(t)}(T^m(y)) = a_n^{(qt \mod p^n)}(T^mS(y))$ - this follows from Lemma 3.6 together with the fact that $T^m$ is a symmetry of the action of $S$. ■

Lemma 3.10: Given $r \in \mathbb{N}$, and integers $-r < c_0, \ldots, c_n < r$ with $c_n \neq 0$, if $\sum_{i=0}^{n} \frac{c_i}{r^i}$ is equal to

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3This is the only place in the proof of Theorem 1.3 where the fact $\gcd(p, q) = 1$ is used not through Lemma 3.3.
such an \( x \in \mathbb{N} \) in least terms then \( w \geq 2^n \).

**Proof:** By induction on \( n \). The case of \( n = 1 \) is clear. Continuing , if \( r \frac{w}{w} = c_0 + \sum_{i=0}^{n-1} \frac{c_{i+1}}{r} \) equals \( \frac{w}{w} \) in least terms then \( w \) is a non-trivial multiple of \( w' \) since all prime divisors of \( w \) divide \( r \). Hence \( w \geq 2w' \geq 2^n \) - where the last inequality is by the induction hypothesis. \( \blacksquare \)

**Proposition 3.11:** \( \hat{\mu} \)-a.e. \( x \in X \) is not symmetric.

**Proof:** By Lemma 3.9 and the ergodicity of \( \hat{\mu} \) to the \( \mathbb{Z}^2 \)-action, if the proposition is false then \( \hat{\mu} \)-a.e. \( x \in X \) is symmetric. We prove that this implies that \( \mu \) is Lebesgue measure \( \lambda \) (in contradiction to our assumption in the beginning of this subsection).

We begin by showing that if \( x \in X \) is a symmetric point then \( \nu_n (x) \xrightarrow[n \to \infty]{} \lambda \) (weakly). So given such an \( x \), there exists \( x \neq y \in X \) agreeing with it on the non-negative horizontal axis such that the measure \( \nu_n (x) \) is invariant under translation by

\[
\varphi (T^{-n}x) - \varphi (T^{-n}y) = \sum_{i=1}^{n} \frac{y(i-n-i) - x(i-n-i)}{(pq)^{n-i+1}}.
\]

Let \( i_0 = \min \{ i : x(-i,0) \neq y(-i,0) \} \), and assume \( n \geq i_0 \), then by Lemma 3.7, \( \varphi (T^{-n} (x)) - \varphi (T^{-n} (y)) \) is a fraction with denominator \( \geq 2^{n-i_0+1} \) (in its least terms representation). So the group of translations under which \( \nu_n (x) \) is invariant is of order at least \( 2^{n-i_0+1} \) and thus contains an element \( \frac{1}{2^{n-i_0+1}} \). This implies that

\[
\int_T f(s) d(\nu_n (x)) (s) \xrightarrow[n \to \infty]{} \int_T f(s) d\lambda (s) \text{ for every } f \in C(T).
\]

\[
\mu = \varphi_* \left( \int_T \hat{\mu} d\hat{\mu} (x) \right) = \int_T \varphi_* \left( \hat{\mu} \right) d\hat{\mu} (x) = \int_T (\varphi (T^{-n} (x)) + \nu_n (x)) d\mu (x),
\]

but \( \varphi (T^{-n} (x)) + \nu_n (x) \xrightarrow[n \to \infty]{} \lambda \) (weakly) and so \( \mu = \lambda \). \( \blacksquare \)

The following lemma is desired property (ii) of \( \mathcal{A} \) and thus concludes the proof of Theorem 1.3.

**Lemma 3.12:** \( T(\alpha) \subseteq_{mod \hat{\mu}} \mathcal{A} \lor \left( \bigvee_{i=0}^{\infty} T^{-i} (\alpha) \right) \).

**Proof:** The statement will follow if we prove that for \( \hat{\mu} \)-a.e. \( x \), its atom with respect to

\( \mathcal{A} \lor \left( \bigvee_{i=0}^{\infty} T^{-i} (\alpha) \right) \) is contained in its atom with respect to \( T(\alpha) \). But if \( x \) is not such and belongs to the full measure set on which the functions \( \nu_n \circ T^m \) are defined, it implies that there exists \( x \neq y \in X \) as in the definition of a symmetric point, i.e. \( x \) is symmetric, and by Lemma 3.11

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we are done. ■

Remark: Rather disappointingly, a corollary from Theorem 1.3 is that $\mathcal{A} \equiv \{X, \emptyset\}$. This is because $h_\mu(T, \alpha) = 0$ implies that all the functions $\nu_\alpha$ are $\mu$-a.s. constant.

4 Parry’s Proof of Rudolph’s Theorem

This section is based on a proof of W. Parry to Rudolph’s Theorem [21] (with some adjustments). The proof has similarities to Rudolph’s original proof (presented in section 3), but is still different. Among the more superficial differences between the two is that it is not formulated in the language of symbolic dynamics. The only result from other sections we shall need here is Prop. 3.1.

Lemma 4.1: Assume $T$ is a surjective measure preserving transformation on a Borel probability space $(X, \mathcal{B}, \mu)$ (by Borel we mean that $X$ is a subset of a compact metric space and $\mathcal{B}$ its Borel $\sigma$-algebra), and that there exists a partition $\xi$ for which $\bigvee_{i=0}^\infty T^{-i} \xi \equiv \mu \mathcal{B}$. Then every $T$-invariant sub-$\sigma$-algebra $\mathcal{A}$ (i.e. $T^{-1}\mathcal{A} = \mathcal{A}$) is contained in the Pinsker $\sigma$-algebra of the system.

Proof: Note that

$$h_\mu(T | \mathcal{A}) = H_\mu \left( \xi | \bigvee_{i=1}^\infty T^{-i} \xi \vee \mathcal{A} \right) = H_\mu \left( \xi | T^{-1} \mathcal{B} \vee T^{-1} \mathcal{A} \right) = H_\mu \left( \xi | T^{-1} \mathcal{B} \right) = h_\mu(T).$$

The result follows by passing to the invertible extension (that is the reason for the surjectivity requirement), and applying the Abramov-Rokhlin formula. ■

4.1 Invariance of Conditional Informations for Certain Commuting Maps

Let $(X, \mathcal{B})$ be a Standard Borel space (i.e. $X$ is a subset of a compact metric space and $\mathcal{B}$ its Borel $\sigma$-algebra), and $S : X \rightarrow X$ be a measurable map that preserves a probability measure $\mu$.

Any function $w \in L^1(X, \mu)$ induces a measure $\nu_w$ defined by setting $d\nu_w = w \, d\mu$. $\nu_w$ can be pushed-forward through $S$ to obtain a new measure on $X$ which also is absolutely continuous with
respect to $\mu$, and let us denote its Radon-Nikodym derivative by $L_{S\mathcal{W}}$. One may verify that

$$L_{S\mathcal{W}}(Sx) = \mathbb{E}_\mu (w \mid S^{-1}\mathcal{B})(x) = \int w(y) \, d\mu_x^{S^{-1}\mathcal{B}},$$

where $\mu_x^{S^{-1}\mathcal{B}}$ is the conditional measure at $x$ with respect to $S^{-1}\mathcal{B}$ (notice that hence, in particular, $L_S (w \circ S) = w$). $L_S : L^1(X, \mu) \to L^1(X, \mu)$ is called the transfer operator of $S$.

Thus, if $S$ is countable-to-one and we define $f(x) = I_\mu (\mathcal{B} \mid S^{-1}\mathcal{B})(x)$ (the conditional information function), then $L_{S\mathcal{W}}(x) = \sum_{y \in S^{-1}x} e^{-f(y)}w(y)$.

Assuming this and throwing into the game another countable-to-one measurable map $T : X \to X$ that preserves $\mu$ and commutes with $S$ we define $g(x) = I_\mu (\mathcal{B} \mid T^{-1}\mathcal{B})(x)$, and so

$$L_{T\mathcal{W}}(x) = \sum_{y \in T^{-1}x} e^{-g(y)}w(y).$$

The commutation relation implies the important identity

(*) $f(Tx) - f(x) = g(Sx) - g(x)$, proved by exponentiating both sides of the equation

$$-f(x) - g(Sx) = -g(x) - f(Tx),$$

which then reads

$$\mu_x^{S^{-1}\mathcal{B}}(\{x\}) \cdot \mu_x^{T^{-1}\mathcal{B}}(\{Sx\}) = \mu_x^{T^{-1}\mathcal{B}}(\{x\}) \cdot \mu_x^{S^{-1}\mathcal{B}}(\{Tx\}),$$

i.e.

$$\mu_x^{(TS)^{-1}\mathcal{B}}(\{x\}) = \mu_x^{(ST)^{-1}\mathcal{B}}(\{x\}).$$

We now assume further that for $\mu$-almost-every $x$ the map $T : S^{-1}x \to S^{-1}Tx$ is bijective, and show that this implies that $f \circ T = f$ and $g \circ S = g$. $L_{S\mathcal{E}^g}(x) = \sum_{y \in S^{-1}x} e^{-f(y)+g(y)}$ and by identity ($*$) this is equal to

$$\sum_{y \in S^{-1}x} e^{-f(Ty)+g(Sy)} = e^{g(x)} \sum_{y \in S^{-1}x} e^{-f(Ty)} = e^{g(x)} \cdot L_S 1 = e^{g(x)}.$$

So $\mathbb{E}_\mu (e^g \mid S^{-1}\mathcal{B}) = e^{\mathcal{E}^gS}$. Together with the fact that $e^g \in L^\infty(X, \mu) \subseteq L^2(X, \mu)$ we deduce that $\|\mathbb{E}_\mu (e^g \mid S^{-1}\mathcal{B})\|_2 = \|e^g\|_2$ ($S$ is measure preserving). But on $L^2(X, \mu)$ conditional expectation is an orthogonal projection and hence $e^{\mathcal{E}^gS} = \mathbb{E}_\mu (e^g \mid S^{-1}\mathcal{B}) = e^g$. Hence $g \circ S = g$ and by ($*$) also $f \circ T = f$. 

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4.2 The Proof

Sticking to Parry’s original notation, let \( p, q > 1 \) be relatively prime integers and \( S, T : \mathbb{T} \rightarrow \mathbb{T} \) be the multiplications by \( p, q \) respectively.

Let \( \mu \in M(\mathbb{T}) \) be invariant under both \( S \) and \( T \) and ergodic under their joint action. We assume further it is of positive entropy with respect to the action of each and need to prove that it is Lebesgue measure.

The fact that \( p \) and \( q \) are relatively prime assures that \( T : S^{-1}x \rightarrow S^{-1}Tx \) for every \( x \in \mathbb{T} \). Hence all conclusions of section 4.1 apply in this case. In particular, \( f \circ T = f \).

For \( \mu \)-almost-every \( x \in \mathbb{T} \) and \( n \in \mathbb{N} \) we define \( d(x, n) \in M(\mathbb{T}) \) supported on \( 0, \frac{1}{p^n}, \frac{2}{p^n}, \ldots, \frac{p^n-1}{p^n} \)

where the mass of each \( \frac{i}{p^n} \) is \( \mu S^{-n}B \left( \left\{ x + \frac{i}{p^n} \right\} \right) \).

Notice that hence \( d(x, n) \) determines \( d(x, n-1) \). Also, if we define

\[
 f^n = f + f \circ S + \cdots + f \circ S^{n-1}
\]

then \( \mu_x S^{-n}B \left( \left\{ x + \frac{i}{p^n} \right\} \right) = f^n \left( x + \frac{i}{p^n} \right) \).

We denote by \( \mathcal{H}_n \subseteq \mathcal{B} \) the smallest \( \sigma \)-algebra on \( X \) for which the function \( d(\cdot, n) : \mathbb{T} \rightarrow M(\mathbb{T}) \) is measurable (so \( \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \ldots \)). Since \( f \circ T = f \) and \( S, T \) commute then \( f^n \circ T = f^n \).

Applying this we note that

\[
 \mu_x S^{-n}B \left( \left\{ x + \frac{i}{p^n} \right\} \right) = f^n \left( x + \frac{i}{p^n} \right) = f^n \left( T \left( x + \frac{i}{p^n} \right) \right) = f^n \left( T x + \frac{qi}{p^n} \right) = \mu_{T x} S^{-n}B \left( \left\{ T x + \frac{qi}{p^n} \right\} \right),
\]

thus \( T^{-1} \mathcal{H}_n = \mathcal{H}_n \). Defining \( \mathcal{H} = \bigvee_{n=1}^{\infty} \mathcal{H}_n \) we obtain that \( \mathcal{H} \) is \( T \)-invariant (i.e. \( T^{-1} \mathcal{H} = \mathcal{H} \)). By Lemma 4.1, this means that \( \mathcal{H} \) is contained in the Pinsker \( \sigma \)-algebra of \( T \).

**Lemma 4.2:** The Pinsker \( \sigma \)-algebras of \( S \) and \( T \) are equal.

**Proof:** Denote by \( \tilde{\mu} \) the measure in the invertible extension of our system relative to the joint action of both \( S \) and \( T \), each of the corresponding maps there \( \tilde{S}, \tilde{T} \) is an automorphism of the system.
defined by the other and hence the Pinsker $\sigma$-algebra of each of the maps there is invariant relative to the other. Denote these Pinsker $\sigma$-algebras by $\mathcal{P}_A, \mathcal{P}_T$. We claim $\mathcal{P}_T = \mathcal{P}_A$. By a similar proof to Prop. 3.1 $h_\mu \left( \hat{T} \mid \mathcal{P}_T \right) = \frac{\log E}{\log q} h_\mu \left( \hat{S} \mid \mathcal{P}_T \right)$ but $h_\mu \left( \hat{T} \mid \mathcal{P}_T \right) = h_\mu \left( \hat{S} \right)$ (the latter equality is implied by Prop. 3.1) and hence $h_\mu \left( \hat{S} \mid \mathcal{P}_T \right) = h_\mu \left( \hat{S} \right)$. By the Abaramov-Rokhlin formula $\mathcal{P}_T \subseteq \mathcal{P}_A$. In the same manner one shows the reverse containment.

Now, if a set $A$ belongs to the Pinsker $\sigma$-algebra down at the circle of $T$ (resp. $S$) then its inverse image up at the joint action invertible extension belongs to the Pinsker $\sigma$-algebra of $T$ (resp. $S$) there and hence of $S$ (resp. $T$) there, and this means that $A$ belongs to the Pinsker $\sigma$-algebra of $S$ (resp. $T$) down at the circle. $\blacksquare$

Hence, by Lemma 4.2, $\mathcal{H} \subseteq \bigcap_{n=1}^\infty S^{-n}B$ (modulo $\mu$). This last containment is the key to the proof.

There exists a set $N \subseteq \mathbb{T}$ of measure 0 such that the conditional measures $\mu^{S^{-n}B}_x$ are defined on $\mathbb{T} \setminus N$ for all $n \in \mathbb{N}$, and $\mu^{S^{-n}B}_x \left( [x]_{S^{-n}B} \setminus N \right) = 1$, $[x]_{S^{-n}B} \setminus N \subseteq [x]_{\mathcal{H}}$ for all $x \in \mathbb{T} \setminus N$ (where the square brackets denote an atom of the $\sigma$-algebra that appears in the subscript) - here we use the fact $\mathcal{H} \subseteq \bigcap_{n=1}^\infty S^{-n}B$.

Given any $x \in \mathbb{T}$ and any $g \in [x]_{S^{-1}B} \setminus N$ the equality $d(y, n) = d(y + g, n)$ holds for any $g$ which is a difference of two elements of the set $[x]_{S^{-n}B} \setminus N$. Therefore for every $x \in \mathbb{T}$ and $n \in \mathbb{N}$ there exists a subgroup $G^n_x$ of $\left\{ 0, \frac{1}{p^n}, \frac{2}{p^n}, \ldots, \frac{p^n-1}{p^n} \right\}$ such that $[x]_{S^{-n}B} \setminus N$ is a coset of $G^n_x$. Thus $\mu^{S^{-n}B}_x$ is the uniform measure on $[x]_{S^{-n}B} \setminus N$ (each element with probability $e^{-f^n(x)}$).

$h_\mu (S) > 0$ implies that the set $E \subseteq \mathbb{T} \setminus N$ composed of the points $x$ such that $|[x]_{S^{-1}B} \setminus N| > 1$ is of positive measure, so for them $G^n_x \neq \{0\}$ for every $n$. But $X \setminus E \subseteq T^{-1} (X \setminus E)$ (modulo $\mu$) and, since the joint action of $S$ and $T$ is ergodic, the Pointwise Ergodic Theorem implies that $S^n x \in E$ infinitely often for $x \in \mathbb{T}$ almost-surely, and thus $f^n(x) \xrightarrow{n \to \infty} 0$ almost-surely. But $\frac{1}{f^n(x)} = |G^n_x|$ and thus $|G^n_x| \xrightarrow{n \to \infty} \infty$ almost-surely.

So $\mathbb{E}_\mu \left( e^{2\pi k x} \mid S^{-n}B \right) (x) = \sum_{g \in G^n_x} \frac{e^{2\pi k (x+g)}}{|G^n_x|}$ equals 0 for $n$ sufficiently large and this guarantees $\mathbb{E}_\mu \left( e^{2\pi k x} \right) = 0$ for every integer $k \neq 0$. This means $\mu$ is Lebesgue measure.
5 Host’s Theorem

The main source for the discussion of Host’s Theorem presented here is [20] by D. Meiri (it is both a mathematical and a linguistical strengthening of B. Host’s original paper which is in French [14] - for predecessors see [8, 23]).

Definition: For an integer $p > 1$, a sequence $c_k \in \mathbb{N}$ is called a $p$-Host sequence if for every $\mu \in M(\mathbb{T})$ invariant under multiplication by $p$, ergodic and with entropy $h > 0$, the sequence $c_k x$ equidistributes $\mu$-a.s..

Note that the $\mu$-a.s. equidistribution of $c_k x$ implies that every $f \in C(\mathbb{T})$ satisfies

$$\frac{1}{N} \sum_{k=0}^{N-1} f(c_k x) = \int f(x) d\mu$$

(where $\lambda$ is Lebesgue measure), i.e. the sequence $c_k \mu$ equidistributes.

Theorem 5.1 (Host’s Theorem): Let $p, q > 1$ be relatively prime integers, then $q^k$ is a $p$-Host sequence.

Before presenting the proof, let us see why Host’s Theorem implies Rudolph’s Theorem (and thus serves as an independent proof of the latter). Given an atomless $\mu \in M(\mathbb{T})$ invariant and ergodic with respect to the action of the multiplicative semi-group $\langle p, q \rangle$ and $h_\mu(p) > 0$, consider the ergodic decomposition of $\mu$ with respect to the times $p$ transformation: $\mu = \int_T \mu_\varepsilon^p d\mu(x)$ ($\mu_\varepsilon^p$ is the conditional measure at $x$ with respect to $\varepsilon$ - the $\sigma$-algebra of invariant sets). By averaging with respect to iterates of the times $q$ transformation we obtain $\mu = \frac{1}{N} \sum_{k=0}^{N-1} q^k \mu = \int_T \frac{1}{N} \sum_{k=0}^{N-1} q^k \mu_\varepsilon^p d\mu(x)$. Now $h_\mu(p) = \int_T h_{\mu_\varepsilon^p}(p) d\mu(x)$ and thus there exists $A \subseteq \mathbb{T}$ with $\mu(A) > 0$ for which $h_{\mu_\varepsilon^p}(p) > 0$ for every $x \in A$. By Host’s Theorem $\int_A \frac{1}{N} \sum_{k=0}^{N-1} q^k \mu_\varepsilon^p d\mu(x)$ converges to $\mu(A) \lambda$, and so $\mu \gg \lambda$ (if $\mu(B) = 0$ then $\mu_\varepsilon^p(B) = 0$ $\mu$-a.s.), but $\mu$ is ergodic with respect to the semi-group action and hence $\mu = \lambda$.

We need one number theoretical fact which is an immediate conclusion from Lemma 2.3 (this is the only place in this section where the comprimality of $p, q$ is exploited).

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4 The stronger version - for $p, q$ multiplicatively independent, was proved a few years ago by Hochman and Shmerkin [12].

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Proposition 5.2: For every $0 \neq a \in \mathbb{Z}$ there exists some $M > 0$ for which

$$\# \{ k : 0 \leq k < p^n, \; aq^k \equiv t \pmod{p^n} \} < M \text{ for all integers } n > 0 \text{ and } t.$$ 

We now turn to prove Host’s Theorem. Let $\mu$ be a measure such as in the definition of a $p$-Host sequence. In order to prove the theorem it is enough to show that $g_N(x) = \frac{1}{N} \sum_{k=0}^{N} e(aq^k x)$ \(\xrightarrow{N \to \infty} 0\) \(\mu\)-a.s. for every $0 \neq a \in \mathbb{Z}$, where $e(x) = e^{2\pi i x}$ for $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. $\int |g_N(x)|^2 \, d\mu$ is bounded by 1, but this does not solve the problem. As we shall now see, Host observed that if one considers a sum of translates $\omega_{n(N)} = \sum_{j=0}^{p^m(N)-1} \delta_{j} * \mu$ instead of $\mu$, the integral of $|g_N(x)|^2$ by this measure (which is of total mass $p^n$) is surprisingly still bounded uniformly in $N$ by a constant due to cancelations.

We choose $n(N)$ to be the natural number satisfying $p^{n-1} \leq N < p^n$. Evaluating the expression

$$\int |g_N(x)|^2 \, d\omega_n = \int \sum_{j=0}^{p^m(N)-1} |g_N(x + \frac{j}{p^n})|^2 \, d\mu = \frac{1}{N^2} \sum_{k,l=0}^{N-1} \sum_{j=0}^{p^m(N)-1} e \left( (aq^k - aq^l) \frac{j}{p^n} \right) \cdot \left| \int e \left( (a (q^k - q^l) x) \right) \, d\mu \right| \leq$$

$$\leq \frac{1}{N^2} \sum_{k,l=0}^{N-1} \left| \sum_{j=0}^{p^m(N)-1} e \left( (aq^k - aq^l) \frac{j}{p^n} \right) \right| \cdot \left| \int e \left( (a (q^k - q^l) x) \right) \, d\mu \right| \leq$$

$$\leq \frac{1}{N^2} \sum_{k,l=0}^{N-1} \left| \sum_{j=0}^{p^m(N)-1} e \left( (aq^k - aq^l) \frac{j}{p^n} \right) \right|.$$

The summation over $j$ vanishes if $aq^k \not\equiv aq^l \pmod{p^n}$ and is $p^n$ otherwise, hence

$$\int |g_N(x)|^2 \, d\omega_n \leq \frac{p^n}{N^2} \cdot \# \left\{ (k,l) \in \{0,\ldots,N-1\}^2 : \; aq^k \equiv aq^l \pmod{p^n} \right\}$$

$$= \frac{p^n}{N^2} \sum_{l=0}^{p^m(N)-1} \left( \# \left\{ 0 \leq k \leq N-1 : \; aq^k \equiv t \pmod{p^n} \right\} \right)^2.$$ 

Proposition 5.2 implies that $\int |g_N(x)|^2 \, d\omega_n \leq \frac{p^2 M^2}{N^2} \leq \frac{N^2 p^2 M^2}{N^2} = p^2 M^2$ for all $N$.

On the other side $\int \frac{dg_N}{dx_n} \cdot |g_N(x)|^2 \, d\mu = \int \frac{dg_N}{dx_n} \cdot |g_N(x)|^2 \, d\omega_n \cdot (x) \, d\mu \leq \int |g_N(x)|^2 \, d\omega_n$, so we conclude $\int \frac{|g_N(x)|^2}{\frac{dg_N}{dx_n} (x)} \, d\mu \leq p^2 M \left( \frac{d\mu}{d\omega_n} (x) > 0 \right) \mu$-a.s. hence, in particular, the integrand is well-defined.

This already looks interesting.

Let us now investigate the Radon-Nikodym derivative $\frac{d\mu}{d\omega_n}$. Denote by $\alpha$ the partition $\left\{ \left[ \frac{t-1}{p}, \frac{t+1}{p} \right) \right\}_{j=0}^{p^n-1}$. 
and by $\alpha_k \alpha^{k+1}$ denote $\sigma_p^{-k}(\alpha) \cap \ldots \cap \sigma_p^{-(k+1)}(\alpha)$, where $\sigma_p$ is the transformation of multiplication by $p$.

**Proposition 5.3:** $-\log \frac{d\mu}{d\omega_n} = I_\mu (\alpha_0^{n-1}|\alpha_0^n)$ (the conditional information function) $\mu$-a.s..

**Proof:** It is sufficient to prove that $\mathbb{E}_\mu (f|\alpha_0^n) (x) = \sum_{j=0}^{n-1} f \left( x + \frac{j}{p^n} \right) \frac{d\mu}{d\omega_n} \left( x + \frac{j}{p^n} \right)$ $\mu$-a.s. for any $f \in L^1 (\mathbb{T}, \mu)$ (then for every element $S$ of the partition $\alpha_0^n$ one takes $f = 1_S$). The function on the right hand side is indeed $\alpha_0^n$-measurable, and for every set $A \in \alpha_0^n$

$$\int_A f(x) \, d\mu(x) = \sum_{j=0}^{n-1} \int_A f \left( x + \frac{j}{p^n} \right) \frac{d\mu}{d\omega_n} \left( x + \frac{j}{p^n} \right) \, d\mu(x).$$

The following proposition is reminiscent of the Shannon-McMillan-Breiman Theorem (although it is much easier to prove).

**Proposition 5.4:** $-\frac{1}{n} \log \frac{d\mu}{d\omega_n} \xrightarrow{n \to \infty} h$ $\mu$-a.s..

**Proof:** The proof is a simple implementation of the Pointwise Ergodic Theorem:

$$-\frac{1}{n} \log \frac{d\mu}{d\omega_n} = \frac{1}{n} I_\mu (\alpha_0^{n-1}|\alpha_0^n) = \frac{1}{n} (I_\mu (\alpha|\alpha_0^n) + I_\mu (\alpha|\alpha_1^n) + \ldots + I_\mu (\alpha|\alpha_{n-1}^n))$$

$$= \frac{1}{n} (I_\mu (\alpha|\alpha_1^n) + I_\mu (\alpha|\alpha_2^n) \circ \sigma_p + \ldots + I_\mu (\alpha|\alpha_{n-1}^n) \circ \sigma_{p^{-1}}).$$

Summing up what we know up to now: (1) $\left( \frac{d\mu}{d\omega_n} \right) \xrightarrow{n \to \infty} e^{-h}$ $\mu$-a.s. and (2) $\int \frac{|g(x)|^2}{e^{\frac{N}{n} \cdot \frac{d\mu}{d\omega_n}(x)}} \, d\mu$ is bounded as a sequence in $N$ ($n$ is determined by the condition $p^n - 1 \leq N < p^n$). Given any $C > 1$, (2) implies that

$$\int \sum_{N=0}^{\infty} \frac{|g(x)|^2}{e^{\frac{N}{n} \cdot \frac{d\mu}{d\omega_n}(x)}} \, d\mu = \sum_{N=0}^{\infty} \int \frac{|g(x)|^2}{e^{\frac{N}{n} \cdot \frac{d\mu}{d\omega_n}(x)}} \, d\mu < \infty$$

and so $\frac{|g(x)|^2}{e^{\frac{N}{n} \cdot \frac{d\mu}{d\omega_n}(x)}} \xrightarrow{N \to \infty} 0 \mu$-a.s. - where here we define $n$ by $p^n - 1 \leq |C^N| < p^n$. But (1) implies that for $\mu$-almost every $x$ the inequality $\frac{d\mu}{d\omega_n}(x) \leq e^{-\frac{nh}{2}}(x)$ is satisfied for $n$ large enough, and thus for $\mu$-almost every $x$ the expression $e^{\frac{nh}{2}} \frac{d\mu}{d\omega_n}(x)$ is bounded. This means $|g(x)|^2 \xrightarrow{N \to \infty} 0$ for $\mu$-almost every $x$. It is only in these last steps that we use the positivity of $h$.

I learned from E. Lindenstrauss the following simple and useful lemma from which one can easily
deduce that $|g_N(x)| \xrightarrow{N \to \infty} 0$ $\mu$-a.s. and this finishes the proof of Host’s Theorem.

**Lemma 5.5:** Given any real number $C > 1$, if $\frac{1}{[C^n]} \sum_{k=0}^{[C^n]-1} e(x_k) \xrightarrow{n \to \infty} 0$ for a sequence $x_k \in \mathbb{T}$ then $\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{k=0}^{N-1} e(x_k) \right| \leq 1 - \frac{1}{C}$.

**Proof:** Let $N$ be a natural number, and take $n$ for which $\lfloor C^n \rfloor - 1 \leq N < \lfloor C^n \rfloor$.

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$$\left| \frac{1}{[C^n]} \sum_{k=0}^{[C^n]-1} e(x_k) - \frac{1}{N} \sum_{k=0}^{N-1} e(x_k) \right| = \left| \frac{1}{[C^n]} \sum_{k=0}^{[C^n]-1} e(x_k) - \frac{1}{N} \sum_{k=0}^{[C^n]-1} e(x_k) + \left( \frac{1}{[C^n]} - \frac{1}{N} \right) \sum_{k=[C^n]-1}^{N-1} e(x_k) + \frac{1}{N} \sum_{k=N}^{[C^n]-1} e(x_k) \right|.$$

Each of the first two summands converges to 0 when $N \to \infty$. The sum of the two others is less in absolute value than

$$\left( \frac{1}{[C^n]} - \frac{1}{N} \right) \left( \lfloor C^n \rfloor - [C^n]-1 \right) + \frac{[C^n]-1}{[C^n]} \leq \left( \frac{[C^n]-1}{[C^n]} - \frac{1}{C} \right) \left( \frac{[C^n]-1}{[C^n]} - 1 \right) + 1 - \frac{[C^n]-1}{[C^n]}$$

$$= 2 - 2 \frac{[C^n]-1}{[C^n]} - \frac{[C^n]}{C[C^n]-1} + \frac{1}{C} \text{ which converges to } 1 - \frac{1}{C}. \blacksquare$$

**References**

[1] D. Berend. *Multi-Invariant Sets on Compact Abelian Groups*. Trans. Amer. Math. Soc. 286 (1984), pages 505–535.

[2] D. Berend. *Multi-Invariant Sets on Tori*. Trans. Amer. Math. Soc. 280 (1983), pages 509–532.

[3] J. Bourgain, E. Lindenstrauss, P. Michel, A. Venkatesh. *Some Effective Results for $\times a, \times b$*. Ergodic Theory and Dynamical Systems 29 (2009), pages 1705–1722.

[4] T. Downarowicz, D. Huczek. *Empirical Approach to the $\times 2, \times 3$ Conjecture*. Experimental Mathematics - Taylor & Francis (2019).

[5] M. Einsiedler, A. Katok, E. Lindenstrauss. *Invariant Measures and the Set of Exceptions to Littlewood’s Conjecture*. Ann. of Math. (2), 164(2) (2006), pages 513–560.
[6] M. Einsiedler, E. Lindenstrauss. *Rigidity Properties of $\mathbb{Z}^d$-actions on Tori and Solenoids*. Electron. Res. Announc. Amer. Math. Soc. 9 (2003), pages 99-110.

[7] J. Feldman. *A Generalization of a Result of R. Lyons About Measures on [0, 1)*. Israel Journal of Mathematics 81 (1993), pages 281-287.

[8] J. Feldman, M. Smorodinsky. *Normal Numbers from Independent Processes*. Ergodic Theory and Dynamical Systems 12 (1992), pages 707-712.

[9] S. Friedland, B. Weiss. *Generalized Interval Exchanges and the 2–3 Conjecture*. Open Mathematics 3 no. 3 (2005), pages 412-429.

[10] H. Furstenberg, Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation, Mathematical Systems Theory 1 (1967), pages 1-49.

[11] H. Furstenberg. *Intersections of Cantor Sets and Transversality of Semigroups*. Problems in Analysis (Sympos. Salomon Bochner), Princeton Univ. Press (1970), pages 41-59.

[12] M. Hochman, P. Shmerkin. *Equidistribution from Fractals*. Inventiones Mathematicae 202 (2015), pages 427-479.

[13] M. Hochman, P. Shmerkin. *Local Entropy Averages and Projections of Fractal Measures*. Ann. of Math. (2), 175(3) (2012), pages 1001–1059.

[14] B. Host. *Nombres Normaux, Entropie, Translations*. Israel Journal of Mathematics 91 (1995), pages 419-428.

[15] A. Johnson. *Measures on the Circle Invariant under Multiplication by a Nonlacunary Subgroup of the Integers*. Israel Journal of Mathematics 77 (1992), pages 211-240.

[16] B. Kalinin, A. Katok. *Invariant Measures for Actions of Higher Rank Abelian Groups*. Proceedings of Symposia in Pure Mathematics, vol. 69 (2001), pages 593-638.

[17] B. Kalinin, A. Katok, F. Rodriguez Hertz. *Nonuniform Measure Rigidity*. Annals of mathematics (2011), pages 361-400.

[18] A. Katok, R. Spatzier. *Invariant Measures for Higher-Rank Hyperbolic Abelian Actions*. Ergodic Theory and Dynamical Systems 16, no. 4 (1996), pages 751-778.

[19] R. Lyons. *On Measures Simultaneously 2- and 3-Invariant*. Israel Journal of Mathematics 61 (1998), pages 219-224.
[20] D. Meiri. *Entropy and Uniform Distributions of Orbits in $T^d$*. Israel Journal of Mathematics 105 (1998), pages 155-183.

[21] W. Parry. *Squaring and Cubing the Circle - Rudolph’s Theorem*. Ergodic Theory of $\mathbb{Z}^d$ Actions, Cambridge Univ. Press (1996), pages 177–183.

[22] D. J. Rudolph. $\times 2$ and $\times 3$ Invariant Measures and Entropy. Ergodic Theory and Dynamical Systems 10 (1990), 395-406.

[23] W. Schmidt. *On Normal Numbers*. Pacific J. Math. 10 (1960), pages 661-672.

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