Waves along fractal coastlines: From fractal arithmetic to wave equations

Marek Czachor
Katedra Fizyki Teoretycznej i Informatyki Kwantowej, Politechnika Gdańska, 80-233 Gdańsk, Poland
Centrum Leo Apostel (CLEA), Vrije Universiteit Brussel, 1050 Brussels, Belgium
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Beginning with addition and multiplication intrinsic to a Koch-type curve we formulate and solve wave equation describing wave propagation along a fractal coastline. As opposed to examples known from the literature, we do not replace the fractal by the continuum in which it is embedded. This seems to be the first example of a truly intrinsic description of wave propagation along a fractal curve.

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Introduction— The issue of waves propagating in fractal media, or interacting with fractal boundary conditions, is not new and is related to various phenomena occurring in natural systems. In a ‘fractal drum’ of Berry [1–3], and diffusion through its boundary [4–6], a partial differential equation is solved with a Koch snowflake boundary condition. The differential equations occurring in the problem are the usual ones, but what is unusual is the boundary which is treated by prefractal-to-fractal limits. A program of fractal continua [7–12], in principle applicable to gels, polymers, biological materials, rocks, soils, or carbonate reservoirs, is based on the idea of mapping a mechanical problem in a fractal into a problem in space in which the fractal is embedded. So, in this respect the approach cannot be regarded as an intrinsically fractal analysis. Yet another formulation of fractal continua begins with Sikorski’s differential spaces [13–14]. Here a smooth function with fractal domain embedded in $\mathbb{R}^n$ is, by definition, a function which is smooth in the ordinary $\mathbb{R}^n$ sense. A wave propagating along the Koch curve would be just any wave propagating on the plane, but restricted to the curve. The differential space approach seems appropriate for problems such as deformation or stiffness of a Koch-curve beam, since analysis of the beam from the point of view of its embedding in plane is then justified. However, the Sikorski-type gradients span, at any point of the curve, a two-dimensional space, so it is debatable to what extent the formalism is really intrinsic to the curve. Rather, it is yet another formulation of fractal continuum, and in such a context it is typically mentioned [15]. In the approach of Stillinger [16] one begins with integration in noninteger dimension and then guesses the form of an appropriate Laplacian. Taking into account Svozil’s generalization [17] of the Stillinger formalism, Palmer and Stavrinou arrived at Euler-Lagrange equations in noninteger-dimensional space [18]. Still, the resulting differential operators are defined in terms of the usual derivatives in $\mathbb{R}^n$, and thus are closer in spirit to fractal continuum than to genuine fractal analysis. The formalism developed by Freiberg [19–21] leads in a natural way to strings with fractal mass distributions, but it is unclear how to deal with sets whose fractal dimensions are higher than 1, such as those of a Koch-type variety. Finally, within the fully intrinsic formalism of fractal analysis [22,24] some results are known for wave equations describing waves propagating in fractal subsets of an interval [25,26], but the unsolved problem of a wave equation on the Koch curve is regarded as hopelessly difficult [27].

The goal of this Letter is to show that things get simplified if one begins with fractal calculus constructed in terms of ‘fractal arithmetic’. In particular, wave equations on Koch-type coastlines are easy to formulate and solve. A philosophy behind the arithmetic approach can be summarized as follows: first learn how to add, subtract, multiply, and divide in your fractal, and only then proceed with the calculus.

The calculus we employ was introduced in [28] and further developed in [29–31], with some fractal applications in mind (for applications beyond fractals see [32–34]). The main idea is based on the general fact that sets whose cardinality is continuum can be bijectively mapped onto $\mathbb{R}$. For a given set there are infinitely many such bijections. However, whichever bijection $f$ we choose, it turns our set into a field, and then $f$ itself becomes a field isomorphism. ‘Arithmetic’ operations in $\mathbb{R}$ induce, by means of $f$, appropriate ‘plus’ and ‘times’ in the set in question. And this is enough to build a calculus with fractal or even more general applications.

One may ask why we speak here of ‘arithmetic’ and not just ‘field’ structures. The first reason is pragmatic: In application to physics the notion of a ‘field’ (in the algebraic sense) could be easily confused with physical fields (say, electromagnetic). The resulting terminology would be misleading. The second reason is deeper. A non-Diophantine arithmetic of natural numbers was introduced by Burgin [35–37] and applied in particular to economics [38]. The phrase ‘non-Diophantine’ stresses the fact that the generalized arithmetic does not follow all the rules formalized by Diophantos of Alexandria, in exact analogy to the non-Euclidean geometry which violates some of the axioms of Euclid of Alexandria. Our generalized addition and multiplication, combined with the resulting calculus, can be regarded as an implementation and extension of the Burgin idea. An analogous generalized multiplication is also employed in the number-scaling formalism of Benioff [39,41]. Yet, Benioff’s $f$s are always linear whereas in typical examples a non-
Diophantine $f$ is nonlinear. The key element of our construction is a one-one map of the Koch curve onto $\mathbb{R}$. We first show how to construct Koch-type curves (with fractal dimensions $1 \leq D \leq 2$) that are automatically equipped with appropriate bijections. Then we explain how to differentiate and integrate functions with Koch-curve domains in a way that is intrinsic to the domain. The resulting derivatives are different from those discussed in fractal analysis and cannot be identified with differentiations based on Sierpinski differential spaces. Finally, the wave equation on a Koch curve is formulated and solved.

Koch curve supported on unit interval—For convenience we represent $\mathbb{R}^2$ by $\mathbb{C}$. Let us begin with the Koch curve $K_{[0,1]} \subset \mathbb{C}$, beginning at $0$ and ending at $1$ (Fig. 1). A point $z \in K_{[0,1]}$ can be parametrized by a real number in quaternary representation,

$$y = (0.q_1 \ldots q_j \ldots) \in [0,1] \quad (1)$$

where $q_k = 0, 1, 2, 3$. The parametrization is defined by a bijection $g : [0,1] \to K_{[0,1]}$, $z = g(y)$, constructed as follows. Consider $a = e^{i\alpha}$, $0 \leq \alpha \leq \pi/2$, $L = 1/(2 + 2 \cos \alpha)$, and

$$\hat{0}(z) = Lz, \quad \hat{1}(z) = L(1 + az), \quad \hat{2}(z) = L(1 + a + \bar{a}z), \quad \hat{3}(z) = L(1 + 2 \cos \alpha + z). \quad (2-5)$$

An $n$-digit point $z \in K_{[0,1]}$ corresponding to $y = (0.q_1 \ldots q_n)_{4}$, $q_n \neq 0$, is given by

$$\hat{q}_1 \circ \ldots \circ \hat{q}_n(0) = g(y) \quad (6)$$

(value at 0 of the composition of maps). If $y_n = (0.q_1 \ldots q_n)_{4}$ is a Cauchy sequence convergent to $y = \lim_{n \to \infty} y_n$, then $g(y) = \lim_{n \to \infty} g(y_n)$. Curves from Fig. 1 are the images $g([0,1])$ for various $\alpha$. $g$ is one-one, so it defines the inverse bijection $g^{-1} = f : K_{[0,1]} \to [0,1]$. For $\alpha = \pi/3$ we obtain the standard curve, generated by equilateral triangles. Similarity dimension of a curve generated by (2-6) is given by (Fig. 2)

$$D = \frac{\log 4}{\log(2 + 2 \cos \alpha)}. \quad (7)$$

There are many ways of extending the Koch curve from $K_{[0,1]}$ to $K_{\mathbb{R}}$. For example, let $K_{[k,k+1]}$ be the curve $K_{[0,1]}$ shifted according to $z \mapsto z + k$, $k \in \mathbb{Z}$. Then $K_{\mathbb{R}} = \cup_{k \in \mathbb{Z}} K_{[k,k+1]}$ is a periodic Koch curve, with the bijection $f : K_{\mathbb{R}} \to \mathbb{R}$ constructed from appropriately shifted maps $g$ defined above. Non-periodic but self-similar extensions can be obtained by shifts and rescalings. From our point of view the only condition we impose on $f$ is the continuity of $g = f^{-1}$ at $0$, i.e. $\lim_{y \to 0} g(y) = \lim_{y \to -\alpha} g(y) = g(0)$. We take $g(0) = 0$.

Combining the generalized Koch curves we can construct a curve which is in a one-one relation with $\mathbb{R}$, with explicitly given bijection $f$, and whose fractal dimensions vary from segment to segment in a prescribed way. This type of generalization may be useful for applications involving realistic coastlines, whose fractal dimensions coincide with the data described by the Richardson law [12]. In what follows we will concentrate on the simple case $\alpha = \pi/3$, $L = 1/3$, of the standard Koch curve.

Calculus on the Koch curve—Consider two sets, $\mathcal{X}$ and $\mathcal{Y}$ say, equipped with bijections $f_X : \mathcal{X} \to \mathbb{R}$ and $f_Y : \mathcal{Y} \to \mathbb{R}$, and arithmetics $\{\oplus_Y, \otimes_Y : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}\}$, $\{\oplus_X, \otimes_X : \mathcal{X} \times \mathcal{X} \to \mathcal{X}\}$, defined by $f_X$ and $f_Y$. The two arithmetics are defined in a non-Diophantine way [30,58]

$$x \oplus_X y = f_X^{-1}(f_X(x) + f_Y(y)), \quad (8)$$

$$x \otimes_X y = f_X^{-1}(f_X(x) - f_Y(y)), \quad (9)$$

$$x \oplus_Y y = f_Y^{-1}(f_X(x) + f_Y(y)), \quad (10)$$

$$x \otimes_Y y = f_Y^{-1}(f_X(x)/f_Y(y)), \quad (11)$$

(and analogously for $\mathcal{Y}$).

For a function $A : \mathcal{X} \to \mathcal{Y}$ we define

$$\frac{DA(x)}{dx} = \lim_{h \to 0} \left( A(x \oplus_X f_X^{-1}(h)) \otimes_Y A(x) \right) \otimes_Y f_Y^{-1}(h)$$

$$= f_Y^{-1} \left( \frac{d}{df_X(x)} f_Y \circ A \circ f_X^{-1} [f_X(x)] \right), \quad (12)$$

\[ \text{FIG. 1: Koch curves and their generator parametrized by } \alpha \text{ and corresponding to (2)-(6): } \alpha = \pi/2.5 \text{ (orange), } \alpha = \pi/3 \text{ (black), } \alpha = \pi/4 \text{ (blue), } \alpha = \pi/6 \text{ (red).} \]

\[ \text{FIG. 2: Similarity dimension } D \text{ and the length } L \text{ of the generator from Fig. 1 as functions of } \alpha. \text{ The horizontal lines show the values for the standard } \pi/3 \text{ Koch curve.} \]
where \( f_x^{-1} : \mathbb{R} \to \mathbb{X} \) and \( f_y^{-1} : \mathbb{R} \to \mathbb{Y} \) are bijections continuous at \( 0 \in \mathbb{R} \). Note that \( f_x^{-1}(0) = 0_x \) and \( f_y^{-1}(0) = 0_y \) are the neutral elements of addition with respect to \( \oplus_\mathbb{X} \) and \( \oplus_\mathbb{Y} \), respectively. The derivative is linear with respect to \( \oplus_\mathbb{Y} \), satisfies the Leibnitz rule for \( \oplus_\mathbb{Y} \), and fulfills an appropriate chain rule for composition of functions. The formula,

\[
\frac{DA(x)}{dx} = f_y^{-1} \circ \hat{A} \circ f_x(x)
\]  

(13)

where \( \hat{A}(x) = \lim_{h \to 0} \frac{(\hat{A}(x + h) - \hat{A}(x))/h}{h} \), shows that a change of bijection implies a ‘covariant’ modification of the derivative. Note the peculiarity of the derivative defined in terms of generalized arithmetic: the bijections do not get differentiated. A change of variable is not just a change of variables or a gauge transformation.

The integral is defined in a way guaranteeing the fundamental laws of calculus, relating derivatives and integrals:

\[
\int_Y^X A(x) dx = f_y^{-1} \left( \int_Y^{f_x(X)} f_y \circ A \circ f_x^{-1}(x) dx \right)
\]  

(14)

where \( f \circ a(x)dx \) is the usual (say, Lebesgue) integral of a function \( a : \mathbb{R} \to \mathbb{R} \). One proves that

\[
\frac{D}{DX} \int_Y^X A(x) dx = A(X),
\]  

(15)

\[
\int_Y^X \frac{DA(x)}{Dx} dx = A(X) \oplus_\mathbb{Y} A(Y).
\]  

(16)

In the case of interest \( \mathbb{X} = K_\mathbb{R}, \mathbb{Y} = \mathbb{R} \), and \( f_y(x) = x \).

**Wave equation on Koch curves** First of all, let us assume we discuss a real-valued field, whose evolution on the Koch curve \( \mathbb{X} = K_\mathbb{R} \) is described with respect to a ‘normal’ non-fractal time \( t \). The field is then represented by \( \mathbb{R} \times K_\mathbb{R} \to \phi_t(x) \in \mathbb{R} \), with \( x \in \mathbb{X} \). Since \( \mathbb{Y} = \mathbb{R} \) we take \( f_y = \text{id}_\mathbb{R} \). The wave equation is

\[
1 \frac{d^2}{dt^2} \phi_t(x) - \frac{D^2}{Dx^2} \phi_t(x) = 0,
\]  

(17)

where

\[
\frac{d}{dt} \phi_t(x) = \lim_{h \to 0} \left( \phi_{t+h}(x) - \phi_t(x) \right)/h,
\]  

(18)

\[
\frac{D}{DX} \phi_t(x) = \lim_{h \to 0} \left( \phi_t(x \oplus_x f_x^{-1}(h)) - \phi_t(x) \right)/h.
\]  

(19)

We search solutions in the form (here \( y = ct \))

\[
\phi_t(x) = A(x, y) + B(x, y),
\]  

(20)

where

\[
\left( \frac{d}{dy} - \frac{D}{DX} \right) A(x, y) = \left( \frac{d}{dy} + \frac{D}{DX} \right) B(x, y) \equiv 0,
\]  

(21)

suggesting simply

\[
A(x, y) = a(f_x(x) + y),
\]  

(22)

\[
B(x, y) = b(f_x(x) - y),
\]  

(23)

for some twice differentiable \( a, b : \mathbb{R} \to \mathbb{R} \).

One similarly verifies that \( d/dy \) and \( D/Dx \) commute, and

\[
\frac{D}{DX} B(x, y) \equiv - \frac{d}{dy} B(x, y).
\]  

(25)

Fig. 3 shows the dynamics of \( \phi_t(x) \) with \( a = 0 \). The energy of the wave is given by

\[
E = \frac{1}{2} \int_{t=1-(-\infty)}^{t=1+\infty} \left( \left| \frac{d\phi_t(x)}{dt} \right|^2 + \left| \frac{D\phi_t(x)}{DX} \right|^2 \right) Dx,
\]  

(26)

where the integral is defined by (14).

Let us explicitly check the time independence of \( E \) for the particular case of \( \phi_t(x) = a(f_x(x) + ct) \). Let \( a'(x) = da(x)/dx \). Then,

\[
E = \int_{-\infty}^{\infty} |a'(f_x \circ f_x^{-1}(x) + ct)|^2 dx
\]  

(27)

\[
= \int_{-\infty}^{\infty} |a'(x)|^2 dx
\]  

(28)

is independent of time, as it should be.
To conclude, we have obtained a wave that propagates along a Koch-type curve. The wave possesses finite conserved energy and satisfies the usual wave equation, defined with respect to appropriately defined derivatives. The derivatives are not the ones we know from the standard mathematical literature of the subject, but are very naturally defined and easy to work with. We have not proved that the solution we have found is a general one, but it is very unlikely that it is not. The velocity of the wave is intriguing. On the one hand, it is described by the parameter \( c \) in the wave equation. On the other hand, however, the length of any piece of a fractal coast is infinite and yet the wave moves from point to point in a finite time, and with speed that looks finite and natural. This is possible since the fractal sum \( z = x \oplus_{Y} y \) of two points in a Koch curve is uniquely defined in spite of the apparently ‘infinite’ distances between \( x, y, z \) and the origin 0. Another interesting aspect of the resulting motion is the lack of difficulties with combining non-fractal time with fractal space. The corresponding Lorentz transformations in 1+1 dimensional Minkowski-Koch space-time can be formulated in analogy to those introduced in [28, 29, 32], so space-time symmetries are here implicitly included as well. Fractal arithmetic automatically takes the infinities inherent in the length of the curve. It would not be very surprising if our fractal calculus found applications also in other branches of physics, where finite physical results are buried in apparently infinite theoretical predictions.

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