Abstract. We analyze a new framework for expressing finite element methods on arbitrarily many intersecting meshes: multimesh finite element methods. The multimesh finite element method, first presented in [10], enables the use of separate meshes to discretize parts of a computational domain that are naturally separate: such as the components of an engine, the domains of a multiphysics problem, or solid bodies interacting under the influence of forces from surrounding fluids or other physical fields. In the present paper, we prove optimal order \textit{a priori} error estimate as well as optimal order estimates of the condition number.

Key words. FEM, unfitted mesh, non-matching mesh, multimesh, CutFEM, Nitsche

AMS subject classifications. 65N30, 65N85, 65Y99, 68U20

1. Introduction. The multimesh finite element method presented in [10] extends the finite element method to arbitrarily many overlapping and intersecting meshes. This is of great value for problems that are naturally formulated on domains composed of parts, such as complex domains composed of simpler parts that may be more easily meshed than their composition. This is of particular importance when the parts are moving, either relative to each other or relative to a fixed background mesh, as part of a time-dependent simulation or optimization problem. Figure 1.1 provides some illustrative examples.

The multimesh finite element method is based on Nitsche’s method [14], which was initially presented as a method for weakly imposing Dirichlet boundary conditions and later extended to form the basis of discontinuous Galerkin methods [1]. Nitsche’s method is also the basis for CutFEM, the finite element method on cut meshes. The cut finite element was originally proposed in [8, 9] and has since been applied to a range of problems [7, 6, 2, 13]. For an overview, see [5]. See also [12, 11] where related formulations for the Stokes problem are presented and analyzed.

In the remainder of this paper, we analyze the multimesh finite element method for the Poisson problem for an arbitrary number of intersecting meshes.

2. Notation. We first review the notation for domains, interfaces, meshes, overlaps, function spaces and norms used to formulate and analyze the multimesh finite element method. For a more detailed exposition, we refer to [10].
Fig. 1.1. (Top left) The flow around a propeller may be computed by immersing a mesh of the propeller into a fixed background mesh. (Top right) The geometry of a composite object may be discretized by superimposing meshes of each component. (Bottom) The interaction of a set of solid bodies may be simulated using individual meshes that move and intersect freely relative to each other and a fixed background mesh.

### Notation for domains

Let $\Omega = \hat{\Omega}_0 \subset \mathbb{R}^d$, $d = 2, 3$, be a domain with polygonal boundary (the background domain).

Let $\hat{\Omega}_i \subset \hat{\Omega}_0$, $i = 1, \ldots, N$ be the so-called predomains with polygonal boundaries (see Figure 2.1).

Let $\Omega_i = \hat{\Omega}_i \setminus \bigcup_{j=i+1}^{N} \hat{\Omega}_j$, $i = 0, \ldots, N$ be a partition of $\Omega$ (see Figure 2.2).

**Remark 2.1.** To simplify the presentation, the domains $\Omega_1, \ldots, \Omega_N$ are not allowed to intersect the boundary of $\Omega$.

### Notation for interfaces

Let the interface $\Gamma_i$ be defined by $\Gamma_i = \partial \hat{\Omega}_i \setminus \bigcup_{j=i+1}^{N} \hat{\Omega}_j$, $i = 1, \ldots, N - 1$ (see Figure 2.3a).

Let $\Gamma_{ij} = \Gamma_i \cap \Omega_j$, $i > j$ be a partition of $\Gamma_i$ (see Figure 2.3b).
Fig. 2.1. (a) Three polygonal predomains. (b) The predomains are placed on top of each other in an ordering such that \( \hat{\Omega}_0 \) is placed lowest, \( \hat{\Omega}_1 \) is in the middle and \( \hat{\Omega}_2 \) is on top.

Fig. 2.2. Partition of \( \Omega = \hat{\Omega}_0 \cup \hat{\Omega}_1 \cup \hat{\Omega}_2 \). Note that \( \hat{\Omega}_2 = \tilde{\Omega}_2 \).

**Notation for meshes**

Let \( \tilde{\mathcal{K}}_{h,i} \) be a quasi-uniform [3] premesh on \( \hat{\Omega}_i \) with mesh parameter \( h_i = \max_{K \in \tilde{\mathcal{K}}_{h,i}} \text{diam}(K) \), \( i = 0, \ldots, N \) (see Figure 2.4a).

Let \( \mathcal{K}_{h,i} = \{ K \in \tilde{\mathcal{K}}_{h,i} : K \cap \hat{\Omega}_i \neq \emptyset \} \), \( i = 0, \ldots, N \) be the active meshes (see Figure 2.4b).

The multimesh is formed by the active meshes placed in the given ordering (see Figure 2.5b).

Let \( \Omega_{h,i} = \bigcup_{K \in \mathcal{K}_{h,i}} K \), \( i = 0, \ldots, N - 1 \) be the active domains.
Fig. 2.3. (a) The two interfaces of the domains in Figure 2.1: \( \Gamma_1 = \partial \hat{\Omega}_1 \setminus \hat{\Omega}_2 \) (dashed line) and \( \Gamma_2 = \partial \hat{\Omega}_2 \) (filled line). Note that \( \Gamma_1 \) is not a closed curve. (b) Partition of \( \Gamma_2 = \Gamma_{20} \cup \Gamma_{21} \).

Fig. 2.4. (a) The three premeshes. (b) The corresponding active meshes (cf. Figure 2.1).

### Notation for overlaps

Let \( \mathcal{O}_i \) denote the overlap defined by \( \mathcal{O}_i = \Omega_{h,i} \setminus \Omega_i \), \( i = 0, \ldots, N - 1 \).

Let \( \mathcal{O}_{ij} = \mathcal{O}_i \cap \Omega_j = \Omega_{h,i} \cap \Omega_j \), \( i < j \) be a partition of \( \mathcal{O}_i \).

For \( i < j \), let

\[
\delta_{ij} = \begin{cases} 
1, & \mathcal{O}_{ij} \neq \emptyset, \\
0, & \text{otherwise},
\end{cases}
\]

be a function indicating which overlaps are non-empty. For ease of notation, we further let \( \delta_{ii} = 1 \) for \( i = 0, \ldots, N \).

Let \( N_{\mathcal{O}} = \max(\max_i \sum_j \delta_{ij}, \max_j \sum_i \delta_{ij}) \) be the maximum number of overlaps. Note that \( N_{\mathcal{O}} \) is bounded by \( N \) but is usually much smaller.
Fig. 2.5. (a) Given three ordered triangles $K_0$, $K_1$ and $K_2$, the overlaps are $O_{01}$ in green, $O_{02}$ in red and $O_{12}$ in blue. (b) The multimesh of the domains in Figure 2.1b consists of the active meshes in Figure 2.4b.

Notation for function spaces

Let $V_{h,i}$ be a continuous piecewise polynomial finite element space on $K_{h,i}$. Let the multimesh finite element space $V_h$ be defined by $V_h = \bigoplus_{i=0}^{N} V_{h,i}$. An element $v \in V_h$ is a tuple $(v_0,...,v_N)$. The inclusion $V_h \hookrightarrow L^2(\Omega)$ is defined by $v(x) = v_i(x)$ for $x \in \Omega_i$.

Notation for norms

Let $c > 0$ and $C > 0$ be constants. The inequality $x \leq Cy$ is denoted by $x \lesssim y$. The equivalence $cx \leq y \leq Cx$ is denoted by $x \sim y$. Let $W_p^s(X)$ denote the standard Sobolev spaces on $X$ with norm denoted by $\| \cdot \|_{W_p^s(X)}$ and semi-norm $| \cdot |_{W_p^s(X)}$. The special case $p = 2$ is denoted by $H^s(X)$ and the space with $p = 2$ and zero trace is denoted by $H^s_0(X)$ (see also e.g. [3, 4]). The Euclidean norm on $\mathbb{R}^N$ is denoted by $| \cdot |_N$. The corresponding inner products are labeled accordingly. The same notation is used for the Lebesgue measure and absolute value. It will be clear from the argument which is used.

We shall make use of the following custom norms. We first let $\| \cdot \|_{s_h}$ denote the semi-norm defined by

\[
\|v\|_{s_h}^2 = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \|\nabla v\|_{O_{ij}}^2,
\]

and we let $\| \cdot \|_h$ denote the norm

\[
\|v\|_h^2 = \sum_{i=0}^{N} \|v_i\|_{\Omega_{h,i}}^2.
\]

Note the relation $s_h(v,v) = \beta_1 \|v\|_{s_h}^2$, between the stabilization term $s_h$ and the norm $\| \cdot \|_{s_h}$ (see Section 3). Also note that for the norm $\| \cdot \|_h$, the domain of integration extends to each active domain $\Omega_{h,i}$, meaning that each overlap will be counted (at least) twice.
Let $||| \cdot |||_h$ denote the energy norm defined by
(2.4)
$$
|||v|||_h^2 = \sum_{i=0}^{N} |||\nabla v_i|||_{H_i}^2 + ||v|||_{S_h}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} (h_i |||\nabla v_i|||_{H_{i,j}}^2 + h_j |||\nabla v_j|||_{H_{i,j}}^2) + \sum_{i=1}^{N} \sum_{j=0}^{i-1} (h_i^{-1} + h_j^{-1}) |||v|||_{H_{i,j}}^2.
$$
The numbering of the terms will be used to alleviate the analysis of the proposed method.

3. Finite element method. As a model problem we consider the Poisson problem
\begin{align}
(3.1a) & \quad -\Delta u = f \quad \text{in } \Omega, \\
(3.1b) & \quad u = 0 \quad \text{on } \partial \Omega,
\end{align}
where $\Omega \subset \mathbb{R}^d$ is a polygonal domain. The multimesh finite element method for (3.1) is to find $u_h \in V_h$ such that
(3.2)
$$
A_h(u_h, v) = l_h(v) \quad \forall v \in V_h,
$$
where
(3.3)
$$
A_h(v, w) = a_h(v, w) + s_h(v, w),
$$
(3.4)
$$
a_h(v, w) = \sum_{i=0}^{N} (\nabla v_i, \nabla w_i)_{\Omega_i} - \sum_{i=1}^{N} \sum_{j=0}^{i-1} (\langle n_i \cdot \nabla v_i, [w] \rangle_{\Gamma_{i,j}} + \langle [v], n_i \cdot \nabla w \rangle_{\Gamma_{i,j}}) + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \beta_0 (h_i^{-1} + h_j^{-1}) ([v], [w])_{\Gamma_{i,j}},
$$
(3.5)
$$
s_h(v, w) = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \beta_1 ([\nabla v], [\nabla w])_{\Omega_{i,j}},
$$
(3.6)
$$
l_h(v) = \sum_{i=0}^{N} (f, v_i)_{\Omega_i}.
$$
Here, $\beta_0 > 0$ is the Nitsche (interior) penalty parameter and $\beta_1 > 0$ is a stabilization parameter. The jump and average operators on $V_h$ are defined by
(3.7)
$$
[v] = v_i - v_j, \quad i < j,
$$
(3.8)
$$
\langle n_i \cdot \nabla v \rangle = (n_i \cdot \nabla v_i + n_i \cdot \nabla v_j)/2,
$$
where \( v_i \) and \( v_j \) are the finite element solutions on represented on the active meshes \( K_{h,i} \) and \( K_{h,j} \).

4. Consistency, Galerkin orthogonality and continuity. We may easily establish the consistency, Galerkin orthogonality and continuity of the form \( A_h \).

**Proposition 4.1 (Consistency).** The form \( A_h \) is consistent; that is,

\[
A_h(u, v) = l_h(v) \quad \forall v \in V_h,
\]

where \( u \in H^1_0(\Omega) \) is the weak solution of (3.1).

**Proof.** The result follows from integration by parts and noting that the jump terms vanish for the exact solution \( u \). \( \square \)

**Proposition 4.2 (Galerkin orthogonality).** The form \( A_h \) satisfies the Galerkin orthogonality; that is,

\[
A_h(u - u_h, v) = 0 \quad \forall v \in V_h,
\]

where \( u \in H^1_0(\Omega) \) is the weak solution of (3.1) and \( u_h \in V_h \) is a solution of (3.2).

**Proof.** The result follows directly from Proposition 4.1 and (3.2). \( \square \)

**Proposition 4.3 (Continuity).** The form \( A_h \) is continuous; that is,

\[
A_h(v, w) \lesssim \|v\|_h \|w\|_h \quad \forall v, w \in H^{3/2}(\Omega) \cap V_h.
\]

**Proof.** The result follows by repeated use of the Cauchy–Schwarz inequality. \( \square \)

**Remark 4.4.** Note that we have continuity not only for the functions in \( V_h \) but also for functions in \( H^{3/2}(\Omega) \). This is used in the a priori estimates in Section 5.2.

5. Coercivity. To prove that the form \( A_h \) is coercive, we will make use of the following lemma.

**Lemma 5.1.** For all \( v \in V_h \), we have

\[
\|\nabla v\|^2_K \lesssim N_0 \sum_{i=0}^N \|\nabla v_i\|^2_{K \cap \Omega_i} + \|v\|^2_{h}.
\]

**Proof.** Take an element \( K \in \mathcal{K}_{h,i} \) and observe that \( K = \bigcup_{j=i}^N K \cap \Omega_j \). It follows that

\[
\|\nabla v_i\|^2_K = \|\nabla v_i\|^2_{K \cap \Omega_i} + \sum_{j=i+1}^N \|\nabla v_i\|^2_{K \cap \Omega_j}
\]

\[
\leq \|\nabla v_i\|^2_{K \cap \Omega_i} + 2 \sum_{j=i+1}^N \|\nabla (v_i - v_j)\|^2_{K \cap \Omega_j} + \|\nabla v_j\|^2_{K \cap \Omega_j}
\]

\[
\leq 2 \sum_{j=i}^N \|\nabla v_j\|^2_{K \cap \Omega_j} + 2 \sum_{j=i+1}^N \|\nabla (v_i - v_j)\|^2_{K \cap \Omega_j}.
\]

Here we have made use of the inequality \( a^2 \leq 2(a-b)^2 + 2b^2 \), which follows by Young’s inequality \( 2ab \leq a^2 + b^2 \) applied to \( a^2 = (a - b)^2 + b^2 = (a - b)^2 + b^2 + 2(a - b)b \).
Summing over all elements \( K \in K_{h,i} \) we have

\[
\|\nabla v_i\|_{\Omega_{h,i}}^2 \leq 2 \sum_{j=i}^{N} \|\nabla v_j\|_{\Omega_{h,i} \cap \Omega_j}^2 + 2 \sum_{j=i+1}^{N} \|\nabla (v_i - v_j)\|_{\Omega_{h,i} \cap \Omega_j}^2
\]

where we have used \( \Omega_{h,i} \cap \Omega_j = \mathcal{O}_{ij} \subseteq \Omega_j \) for \( i < j \). Note that the second sum is empty for \( i = N \).

Summing over all domains, we obtain by (2.1)

\[
\|\nabla v\|_{h}^2 \leq 2 \sum_{i=0}^{N} \sum_{j=i+1}^{N} \|\nabla v_j\|_{\Omega_{ij}}^2 + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \|\nabla (v_i - v_j)\|_{\mathcal{O}_{ij}}^2
\]

which proves the estimate.

Remark 5.2. There is a dependence on the maximum number of overlaps \( N_{\mathcal{O}} \) in Lemma 5.1. In practice, \( N_{\mathcal{O}} \) is of moderate size and this dependence is not an issue.

The interpolation error estimates and condition number estimates (shown below) have a different kind of dependence.

Remark 5.3. Using an inverse bound of the form (see e.g. [3])

\[
\|v\|_{H^l(K)} \lesssim h^{m-l} |v|_{H^m(K)} \quad m, l \in \mathbb{Z}^+, \quad m \leq l,
\]

one can show that the stabilization term \( s_h(v,w) \) may alternatively be formulated as

\[
s_h(v,w) = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \beta_1 h^{-2}([v],[w])_{\mathcal{O}_{ij}}.
\]

Using Lemma 5.1, we may now proceed to prove the coercivity of the bilinear form.

Proposition 5.4 (Coercivity). The form \( A_h \) is coercive. More precisely, for \( \beta_0 \) and \( \beta_1 \) large enough, we have

\[
\|v\|_{h}^2 \lesssim A_h(v,v) \quad \forall v \in V_h.
\]

Proof. We first note that

\[
A_h(v,v) \geq \sum_{i=0}^{N} \|\nabla v_i\|_{\Omega_i}^2 + \sum_{i=1}^{N-1} \sum_{j=0}^{i} \beta_0 (h_i^{-1} + h_j^{-1}) \|v\|_{\Gamma_{ij}}^2 + \beta_1 s_h(v,v) - \sum_{i=1}^{N-1} \sum_{j=0}^{i-1} 2 \|\langle n_i \cdot \nabla v, [v] \rangle_{\Gamma_{ij}} \|.
\]
Now, for \( l = i \) or \( l = j \), let

\[
\mathcal{K}_{h,t}(\Gamma_{ij}) = \{ K \in \mathcal{K}_{h,t} : K \cap \Gamma_{ij} \neq \emptyset \}
\]
denote the set of elements in \( \mathcal{K}_{h,t} \) which intersect \( \Gamma_{ij} \). Using an inverse estimate (see [9]), we have

\[
h \|
abla v_l \|_{\mathcal{K} \cap \Gamma_{ij}}^2 \lesssim \|
abla v_l \|_{K}^2,
\]
where the constant is independent of the position of \( \Gamma_{ij} \). It follows that

\[
\sum_{j=0}^{i-1} h \|
abla v_l \|_{\Gamma_{ij}}^2 = \sum_{j=0}^{i-1} \delta_{ji} h \|
abla v_l \|_{\Gamma_{ij}}^2 \lesssim \sum_{j=0}^{i-1} \delta_{ji} \|
abla v_l \|_{\mathcal{K}_{h,t}(\Gamma_{ij})}^2 \leq \sum_{j=0}^{i-1} \delta_{ji} \|
abla v_l \|_{\Omega_{h,t}}^2,
\]
where we have noted that \( \Gamma_{ij} \) (the part of \( \Gamma_i \) bordering to \( \Omega_j \) for \( j < i \)) is empty if the overlap \( \mathcal{O}_{ji} \) (the part of \( \Omega_{h,j} \) intersected by \( \Gamma_i \) for \( j < i \)) is empty, as indicated by \( \delta_{ji} \).

For \( l = i \), we thus obtain the estimate

\[
\sum_{i=1}^{N} \sum_{j=0}^{i-1} h \|
abla v_i \|_{\Gamma_{ij}}^2 \lesssim \sum_{i=1}^{N} \sum_{j=0}^{i-1} \delta_{ji} \|
abla v_i \|_{\Omega_{h,i}}^2 \leq \sum_{i=1}^{N} \left( \sum_{j=0}^{N} \delta_{ji} \right) \|
abla v_i \|_{\Omega_{h,i}}^2,
\]

\[
\leq N_{\mathcal{O}} \sum_{i=0}^{N} \|
abla v_i \|_{\Omega_{h,i}}^2 = N_{\mathcal{O}} \|
abla v \|_{h}^2,
\]

while for \( l = j \), we obtain the similar estimate

\[
\sum_{i=1}^{N} \sum_{j=0}^{i-1} h \|
abla v_j \|_{\Gamma_{ij}}^2 \lesssim \sum_{i=1}^{N} \sum_{j=0}^{i-1} \delta_{ji} \|
abla v_j \|_{\Omega_{h,j}}^2 \leq \sum_{j=0}^{N} \left( \sum_{i=1}^{N} \delta_{ji} \right) \|
abla v_j \|_{\Omega_{h,j}}^2
\]

\[
= N_{\mathcal{O}} \sum_{j=0}^{N} \|
abla v_j \|_{\Omega_{h,j}}^2 = N_{\mathcal{O}} \|
abla v \|_{h}^2.
\]

We next use the Cauchy–Schwarz inequality with weight \( ch \), Young’s inequality \( 2ab \leq a^2 + b^2 \), and the quasiconformality of the multimesh to obtain

\[
\star = \sum_{i=1}^{N} \sum_{j=0}^{i-1} 2|\langle n_i \cdot \nabla v \rangle, [v] \rangle_{\Gamma_{ij}}| \leq \sum_{i=1}^{N} \sum_{j=0}^{i-1} 2c h^{1/2} h^{1/2} \|\langle n_i \cdot \nabla v \rangle \|_{\Gamma_{ij}} \|v\|_{\Gamma_{ij}} \epsilon^{-1/2} h^{-1/2} \|v\|_{\Gamma_{ij}}
\]

\[
\leq \sum_{i=1}^{N} \sum_{j=0}^{i-1} c h \|\langle n_i \cdot \nabla v \rangle \|_{\Gamma_{ij}}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \epsilon^{-1} h^{-1} \|v\|_{\Gamma_{ij}}^2
\]

\[
\leq \sum_{i=1}^{N} \sum_{j=0}^{i-1} \epsilon (h_i \|\nabla v_i \|_{\Gamma_{ij}}^2 + h_j \|\nabla v_j \|_{\Gamma_{ij}}^2) + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \epsilon^{-1} (h_i^{-1} + h_j^{-1}) \|v\|_{\Gamma_{ij}}^2
\]

\[
\lesssim \epsilon N_{\mathcal{O}} \|\nabla v \|_{h}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \epsilon^{-1} (h_i^{-1} + h_j^{-1}) \|v\|_{\Gamma_{ij}}^2.
\]
By Lemma 5.1, we may now estimate the $\| \cdot \|_h$ norm in terms of the $\| \cdot \|_\Omega$ norm and the $\| \cdot \|_{s_h}$ norm to obtain

$$\star \lesssim \epsilon N^2 \sum_{i=0}^N \| \nabla v_i \|_{\Omega_i}^2 + \epsilon N \| v \|_{s_h}^2 + \epsilon^{-1} \sum_{i=1}^{N-1} \sum_{j=0}^{i-1} (h_i^{-1} + h_j^{-1}) \| [v] \|_{\Gamma_{ij}}^2.$$  

(5.25)

Combining (5.13) and (5.25), we find that

$$A_h(v, v) \geq \sum_{i=0}^N (1 - \epsilon CN^2) \| \nabla v_i \|_{\Omega_i}^2 + (\beta_1 - \epsilon CN) \| v \|_{s_h}^2$$

$$+ \sum_{i=1}^{N-1} \sum_{j=0}^{i-1} (\beta_0 - \epsilon^{-1} C)(h_i^{-1} + h_j^{-1}) \| [v] \|_{\Gamma_{ij}}^2,$$

(5.26)

and thus, by choosing $\epsilon$ small enough and then $\beta_0, \beta_1$ large enough,

$$A_h(v, v) \gtrsim \sum_{i=0}^N \| \nabla v_i \|_{\Omega_i}^2 + \| v \|_{s_h}^2 + \sum_{i=1}^{N-1} \sum_{j=0}^{i-1} (h_i^{-1} + h_j^{-1}) \| [v] \|_{\Gamma_{ij}}^2.$$  

(5.27)

The coercivity now follows by noting that term III in (2.4) may be controlled by terms I and II as above in the estimate of $\star$. □

**Remark 5.5.** By continuity (4.3), coercivity (5.12) and a continuous $l_h(v)$, there exists a unique solution to (3.2) by the Lax–Milgram theorem (see e.g. [3]).

**5.1. Interpolation error estimate.** To construct an interpolation operator into $V_h$, we pick a standard interpolation operator into $V_{h,i}$,

$$\pi_{h,i} : L^2(\Omega_{h,i}) \longrightarrow V_{h,i}, \quad i = 0, 1, \ldots, N,$$

(5.28)

where $\pi_{h,i}$ satisfies the standard interpolation error estimate (see e.g. [3])

$$\| v - \pi_{h,i} v \|_{H^m(K)} \lesssim h^{k+1-m} | v|_{H^{k+1}(N_h(K))}.$$  

(5.29)

Here, $N_h(K)$ denotes the set of elements that share a vertex with $K$. We may then define the interpolation operator into $V_h$ by

$$\pi_h : L^2(\Omega) \ni v \longmapsto \bigoplus_{i=0}^N \pi_{h,i}(v|_{\Omega_{h,i}}) \in V_h.$$  

(5.30)

To prove an interpolation error estimate for $\pi_h$, we let $U_\delta(\Gamma_{ij})$ denote the half-tubular neighborhood defined by

$$U_\delta(\Gamma_{ij}) = \bigcup_{x \in \Gamma_{ij}} B_\delta(x),$$

(5.31)

where $B_\delta(x)$ is the ball of radius $\delta$ centered at $x$; see Figure 5.1. In addition, let

$$U_\delta(\Gamma) = \bigcup_{i,j} U_\delta(\Gamma_{ij}).$$  

(5.32)
Proposition 5.6 (Interpolation error estimate). The interpolation operator \( \pi_h \) satisfies the interpolation error estimate

\[
||| v - \pi_h v |||_h \lesssim \sqrt{1 + hN h^k |v|_{H^{k+1}(\Omega) \cap W^{k+1}_\infty(\mathcal{U}_h(\Gamma))}},
\]

where \( |v|_{H^{k+1}(\Omega) \cap W^{k+1}_\infty(\mathcal{U}_h(\Gamma))} = \max(|v|_{H^{k+1}(\Omega)}, |v|_{W^{k+1}_\infty(\mathcal{U}_h(\Gamma))}). \)

Proof. We first let \( \eta = v - \pi_h v \) and recall the numbering of the terms in the definition of the energy norm \( ||| \cdot |||_h \) (2.4). Starting with term I, we have

\[
I(\eta) = \sum_{i=0}^{N} \| \nabla \eta_i \|_{\Omega_i}^2 \leq \sum_{i=0}^{N} \| \nabla \eta_i \|_{\Omega_{h,i}}^2.
\]

For term II, we have, since \( \cup_{j=0}^{N} \mathcal{O}_{ij} \subseteq \Omega_{h,i} \) and \( \mathcal{O}_{ij} \subseteq \mathcal{U}_h(\Gamma_{ij}) \cap \Omega_{h,i} \) with \( \delta \sim h_i \),

\[
II(\eta) \lesssim \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} (\| \nabla \eta_i \|_{\Omega_{h,i}}^2 + \| \nabla \eta_j \|_{\Omega_{h,j}}^2) \leq \sum_{i=0}^{N-1} \| \nabla \eta_i \|_{\Omega_{h,i}}^2 + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \delta_{ij} \| \nabla \eta_j \|_{\mathcal{U}_h(\Gamma_{ij}) \cap \Omega_{h,i}}^2.
\]

For term III, recall the inverse estimate (5.16) and note that \( \mathcal{K}_{h,j}(\Gamma_{ij}) \subseteq \mathcal{U}_h(\Gamma_{ij}) \) with \( \delta \sim h_j \). Thus,

\[
III(\eta) = \sum_{i=1}^{N} \sum_{j=0}^{i-1} (h_i \| \nabla \eta_i \|_{\Gamma_{ij}}^2 + h_j \| \nabla \eta_j \|_{\Gamma_{ij}}^2) \lesssim N \sum_{i=1}^{N} \| \nabla \eta_i \|_{\Omega_{h,i}}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \delta_{ij} \| \nabla \eta_j \|_{\mathcal{U}_h(\Gamma_{ij})}^2.
\]

For term IV, we first handle the jump term as in II and then proceed as for III,

\[
IV(\eta) \leq \sum_{i=1}^{N} \sum_{j=0}^{i-1} (h_i^{-1} + h_j^{-1})(\| \eta_i \|_{\Gamma_{ij}}^2 + \| \eta_j \|_{\Gamma_{ij}}^2) \lesssim N \sum_{i=1}^{N} h_i^{-2} \| \eta_i \|_{\Omega_{h,i}}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \delta_{ij} h_j^{-2} \| \eta_j \|_{\mathcal{U}_h(\Gamma_{ij})}^2.
\]
Now, note that since \( \Omega_{h,i} \subseteq \Omega_i \cup_{j=1}^{N} U_\delta(\Gamma_{ij}) \), we have

\[
\sum_{i=0}^{N} \|v_i\|_{\Omega_{h,i}}^2 \lesssim \sum_{i=0}^{N} \|v_i\|_{\Omega_i}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \delta_{ij} \|v_i\|_{U_\delta(\Gamma_{ij})}^2.
\]

(5.41)

Therefore, there are only two terms in \( I - IV \) that need to be estimated. First

\[
\sum_{i=0}^{N} \|v_i\|_{\Omega_{h,i}}^2 \lesssim \sum_{i=0}^{N} \|v_i\|_{\Omega_i}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} \delta_{ij} \|v_i\|_{U_\delta(\Gamma_{ij})}^2.
\]

(5.42)

which follows immediately by (5.29). Second, we make use of the disjoint partition of \( \Gamma_i \) and noting that \( |U_\delta(\Gamma_i)| \lesssim h |\Gamma_i| \) to obtain

\[
\sum_{i=1}^{N} \sum_{j=0}^{i-1} h_i^{2(m-1)} |\eta_i|^2_{H^{m}(U_\delta(\Gamma_{ij}))} \lesssim h^{2k} \|v\|_{H^{k+1}(\Omega)}^2.
\]

(5.43)

Due to the maximum norm, this estimate also holds with \( \eta_i \) replaced by \( \eta_j \), and the desired estimate holds.

\[
\sum_{i=1}^{N} \sum_{j=0}^{i-1} h_i^{2(m-1)} |\eta_i|^2_{H^{m}(U_\delta(\Gamma_{ij}))} \lesssim h^{2k} \|v\|_{H^{k+1}(\Omega)}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} h_i^{2(k+1-m)} |v_i|^2_{H^{k+1}(U_\delta(\Gamma_{ij}))}.
\]

(5.44)

\[
\lesssim h^{2k} \sum_{i=1}^{N} \sum_{j=0}^{i-1} |v_i|^2_{H^{k+1}(U_\delta(\Gamma_{ij}))}.
\]

(5.45)

\[
= h^{2k} \sum_{i=1}^{N} |v_i|^2_{H^{k+1}(U_\delta(\Gamma_i))}.
\]

(5.46)

\[
\lesssim h^{2k} \sum_{i=1}^{N} h_i |\Gamma_i| \|v_i\|^2_{W^{k+1}_\infty(U_\delta(\Gamma_i))}.
\]

(5.47)

5.2. A priori error estimates. We may now prove the following optimal order a priori error estimates. The estimates are supported by the numerical results presented in Figure 5.2. For details on these results, we refer to the accompanying paper [10].

Theorem 5.8 (A priori error estimates). The finite element solution \( u_h \) of (3.2) satisfies the following a priori error estimates:

\[
\|u - u_h\|_{H^{k+1}(\Omega) \cap W^{k+1}_\infty(U_\delta(\Gamma))} \lesssim \sqrt{1 + hN} \|u|_{H^{k+1}(\Omega) \cap W^{k+1}_\infty(U_\delta(\Gamma))},
\]

(5.48)

\[
\|u - u_h\|_{H^{k+1}(\Omega) \cap W^{k+1}_\infty(U_\delta(\Gamma))} \lesssim (1 + hN) \|u|_{H^{k+1}(\Omega) \cap W^{k+1}_\infty(U_\delta(\Gamma))}.
\]

(5.49)

Proof. The proof of (5.48) follows the standard procedure of splitting the error and using the energy norm interpolation error estimate from Proposition 5.6.

\[
\|u - u_h\|_{h} \leq \|u - \pi_h u\|_{h} + \|\pi_h u - u_h\|_{h}
\]

(5.50)

\[
\lesssim \sqrt{1 + hN} \|u|_{H^{k+1}(\Omega) \cap W^{k+1}_\infty(U_\delta(\Gamma))} + \|\pi_h u - u_h\|_{h}.
\]

(5.51)
To estimate the second term on the right-hand side of (5.51), we use the coercivity (Proposition 5.4), Galerkin orthogonality (Proposition 4.2) and continuity (Proposition 4.3) of $A_h$ to obtain

\begin{align}
\|\pi_h u - u_h\|_h^2 & \lesssim A_h(\pi_h u - u_h, \pi_h u - u_h) \\
& = A_h(\pi_h u - u, \pi_h u - u_h) \\
& \lesssim \|\pi_h u - u\|_h \|\pi_h u - u_h\|_h.
\end{align}

(5.52) \hspace{1cm} (5.53) \hspace{1cm} (5.54)

It follows that

\begin{align}
\|\pi_h u - u_h\|_h & \lesssim \|u - \pi_h u\|_h \\
& \lesssim \sqrt{1 + hN} \|u\|_{H^k(U_h, \Gamma)}.
\end{align}

(5.55)

where we have again used the interpolation error estimate of Proposition 5.6. Combining (5.51) and (5.55) now yields (5.48).

To prove (5.49), we use a standard duality argument (see e.g. [3]). Let $\phi \in V = H^{3/2}(\Omega)$ be the solution to the dual problem

\begin{equation}
A_h(v, \phi) = (v, \psi)_\Omega \quad \forall v \in V.
\end{equation}

(5.56)

We now take $v = e = u - u_h$ and use the Galerkin orthogonality, continuity and interpolation error estimate to obtain

\begin{align}
(e, \psi)_\Omega &= A_h(e, \phi) \\
& = A_h(e, \phi - \pi_h \phi) \\
& \leq \|e\|_h \|\phi - \pi_h \phi\|_h \\
& \lesssim \|e\|_h \sqrt{1 + hN} \|\phi\|_{W^{k+1}(U_h, \Gamma)} \\
& \lesssim \|e\|_h \sqrt{1 + hN} \|\psi\|_{\Omega},
\end{align}

(5.57) \hspace{1cm} (5.58) \hspace{1cm} (5.59) \hspace{1cm} (5.60) \hspace{1cm} (5.61)

where in the last step we have used the standard elliptic regularity estimate (see e.g. [3]). Note that we have continuity (5.59) also for functions in $H^{3/2}(\Omega)$ as noted in Proposition 4.3. The desired estimate (5.49) now follows from (5.61) by (5.48) and taking $\psi = e$. \hfill \square

6. Condition number estimate. To prove a bound on the condition number, we first introduce some notation and definitions. Let $\{\phi_{i,j}\}_{j=1}^{M_i}$ be the finite element basis of $V_{h,i}$. We then have the expansion

\begin{equation}
v_i = \sum_{j=1}^{M_i} \hat{v}_{i,j} \phi_{i,j},
\end{equation}

(6.1)

for each part $v_i$ of a multimesh function $v = (v_0, \ldots, v_N)$. Collecting all expansion coefficients for the $1 + N$ parts into a vector $\hat{v}$ of dimension $M = \sum_{i=0}^{N} M_i$, the total stiffness matrix $\hat{A}$ for the multimesh system is defined by

\begin{equation}
(\hat{A}\hat{v}, \hat{w})_M = A_h(v, w) \quad \forall v, w \in V_h,
\end{equation}

(6.2)

with condition number

\begin{equation}
\kappa(\hat{A}) = |\hat{A}|_M |\hat{A}^{-1}|_M.
\end{equation}

(6.3)

To derive an estimate of $\kappa(\hat{A})$ we make use of the following Lemmas.
Fig. 5.2. Rate of convergence in the $L^2(\Omega)$ (top) and $H^1_0(\Omega)$ (bottom) norms for $p = 1$ (blue), $p = 2$ (red), $p = 3$ (yellow) and $p = 4$ (purple). For each $p$, the convergence rate is shown for $N = 1, 2, 4, 6, 16, 32$ meshes (six lines) and the errors for $N = 0$ (the standard single mesh discretization) are marked with \times and dashed lines.

**Lemma 6.1 (Inverse inequality).** It holds that

\[
\|v\|_{h}^2 \lesssim (1 + N \Theta) h^{-2} \|v\|_h^2 \quad \forall v \in V_h.
\]

*Proof.* Recall the definition of the energy norm (2.4). We first note that

\[
I = \sum_{i=0}^{N} \|\nabla v_i\|_{\Omega_h}^2 \leq \sum_{i=0}^{N} \|\nabla v_i\|_{\Omega_h,i}^2.
\]

For term $II$, we have by recalling (5.36)

\[
II \lesssim \sum_{i=0}^{N-1} \|\nabla v_i\|_{\Omega_h,i}^2 + \sum_{i=0}^{N-1} \sum_{j=i+1}^{N} \delta_{ij} \|\nabla v_j\|_{\Omega_{h(\Gamma_{ij})\cup \Omega_j}}^2.
\]

\[
\lesssim N \Theta \sum_{i=0}^{N} \|\nabla v_i\|_{\Omega_{h,i}}^2.
\]

Term $III$ may be estimated similarly to obtain

\[
III \lesssim N \Theta \sum_{i=1}^{N} \|\nabla v_i\|_{\Omega_{h,i}}^2.
\]
For term IV, we have by recalling (5.40)

\begin{align}
IV & \lesssim N_O \sum_{i=0}^{N} h_{i}^{-2} \| v_i \|^2_{O_h} + \sum_{i=1}^{N} \sum_{j=0}^{i-1} h_j^{-1} \| v_j \|^2_{U_\delta(\Gamma_{i,j})} \\
& \lesssim N_O \sum_{i=0}^{N} h_{i}^{-2} \| v_i \|^2_{O_h}.
\end{align}

(6.9) (6.10)

The desired estimate now follows using the standard inverse inequality (5.10).

**Lemma 6.2 (Poincaré inequality).** It holds that

\begin{equation}
\| v \|^2_h \lesssim (1 + h^{2/d} N_O) \| v \|^2_h \quad \forall v \in V_h.
\end{equation}

(6.11)

**Proof.** First note that by a Taylor expansion argument and Lemma 5.1, we have

\begin{equation}
\| v \|^2_h \lesssim \sum_{i=0}^{N} (\| v_i \|^2_{\Omega_i} + h_i^2 \| \nabla v_i \|^2_{\Omega_{h,i}})
\end{equation}

(6.12)

\begin{equation}
\lesssim \sum_{i=0}^{N} \| v_i \|^2_{\Omega_i} + h^2 \left( N_O \sum_{i=0}^{N} \| \nabla v_i \|^2_{\Omega_i} + \| v \|^2_{h_i} \right).
\end{equation}

(6.13)

To control the first term on the right-hand side in (6.13), let \( \phi \in H^2(\Omega) \) be the solution to the dual problem

\begin{align}
-\Delta \phi &= \psi \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \partial \Omega,
\end{align}

(6.14) (6.15)

where \( \psi \in L^2(\Omega) \). Multiplying the dual problem with \( v \in V_h \) and integrating by parts, we obtain using the Cauchy-Schwartz inequality

\begin{equation}
\sum_{i=0}^{N} (v_i, \psi )_{\Omega_i} = \sum_{i=0}^{N} (v_i, -\Delta \phi )_{\Omega_i}
\end{equation}

(6.16)

\begin{equation}
= \sum_{i=0}^{N} (\nabla v_i, \nabla \phi )_{\Omega_i} - \sum_{i=1}^{N} \sum_{j=0}^{i-1} ( [v_i, n_i \cdot \nabla \phi ]_{\Gamma_{i,j}}
\end{equation}

(6.17)

\begin{equation}
\leq \sum_{i=0}^{N} \| \nabla v_i \|^2_{\Omega_i} \| \nabla \phi \|^2_{\Omega_i} + \sum_{i=1}^{N} \sum_{j=0}^{i-1} h^{-1/2} \| [v] \|_{\Gamma_{i,j}} h^{1/2} \| \nabla \phi \|_{\Gamma_{i,j}}
\end{equation}

(6.18)

\begin{equation}
\lesssim \left( \sum_{i=0}^{N} \| \nabla v_i \|^2_{\Omega_i} + \sum_{i=1}^{N} \sum_{j=0}^{i-1} h^{-1} \| [v] \|^2_{\Gamma_{i,j}} \right)^{1/2}
\end{equation}

\begin{equation}
\times \left( \| \nabla \phi \|^2_{\Omega} + \sum_{i=1}^{N} \sum_{j=0}^{i-1} h \| \nabla \phi \|^2_{\Gamma_{i,j}} \right)^{1/2}.
\end{equation}

(6.19)

Now we continue with the second factor in (6.19). Using the trace inequality

\begin{equation}
\| v \|^2_{\gamma \cap K} \lesssim h^{-1} \| v \|^2_K + h \| \nabla v \|^2_K,
\end{equation}

(6.20)
with constant independent of the position of an interface $\gamma$ (see [9]), we have

$$\sum_{i=1}^{N} \sum_{j=0}^{N-1} h_i \| \nabla \phi \|^2_{K_{h,i}(\Gamma_{ij})} \lesssim \sum_{i=1}^{N} \sum_{j=0}^{N-1} \delta_{ji} \left( \| \nabla \phi \|^2_{K_{h,i}(\Gamma_{ij})} + h_i^2 \| \nabla^2 \phi \|^2_{K_{h,i}(\Gamma_{ij})} \right).$$

By the construction of $U_\delta(\Gamma_{ij})$, see (5.31), we have $K_{h,i}(\Gamma_{ij}) \subseteq U_\delta(\Gamma_{ij})$ with $\delta \sim h_i$. Furthermore, by the Hölder inequality [4] with coefficients $r, s$ such that $1/r + 1/s = 1$ we have

$$\| \nabla \phi \|^2_{K_{h,i}(\Gamma_{ij})} \lesssim \| \nabla \phi \|^2_{L^2(\Gamma_{ij})}$$

$$= \| 1 \cdot | \nabla \phi|^2 \|_{L^2(U_\delta(\Gamma_{ij}))}$$

$$\leq \| 1 \|_{L^r(U_\delta(\Gamma_{ij}))} \| | \nabla \phi|^2 \|_{L^r(U_\delta(\Gamma_{ij}))}$$

$$= \| U_\delta(\Gamma_{ij}) \|^{1/s}_{L^r(U_\delta(\Gamma_{ij}))} \| | \nabla \phi|^2 \|_{L^r(U_\delta(\Gamma_{ij}))}$$

$$\lesssim h_i^{1/s} | \Gamma_{ij}|^{1/s} \| | \nabla \phi|^2 \|_{L^r(U_\delta(\Gamma_{ij}))}$$

$$\lesssim h_i^{1-2/p} \| | \nabla \phi|^2 \|_{L^p(U_\delta(\Gamma_{ij}))}$$

$$\lesssim h_i^{1-2/p} \| | \nabla \phi|^2 \|_{W^2_p(U_\delta(\Gamma_{ij}))}$$

with $p = 2r$ and $1/s = 1 - 1/r = 1 - 2/p$.

To determine $p$ in (6.28), we use the Sobolev embedding $W^d_2(\Omega) \subseteq W^k_2(\Omega)$ [4] with $k = 1, l = 2$ and $q = 2$. This is motivated by the fact that due to elliptic regularity and $\psi \in L^2(\Omega)$, we have $\phi \in H^2(\Omega)$. Since the embedding holds for $1/p - k/d = 1/q - l/d$ [4], we obtain $p = 2d/(d - 2)$, where $p = \infty$ for $d = 2$. Thus

$$h_i^{1-2/p} \| | \nabla \phi|^2 \|_{W^2_p(U_\delta(\Gamma_{ij}))} \lesssim h_i^{2/d} \| \phi \|^2_{W^2_{2d/(d-2)}(U_\delta(\Gamma_{ij}))} \lesssim h_i^{2/d} \| \phi \|^2_{H^2(U_\delta(\Gamma_{ij}))}.$$  

Cf. [4] regarding the last inequality for $d = 2, 3$. Returning to (6.21) we have, using a standard duality argument (see e.g. [3]), elliptic regularity, a stability estimate and the Poincaré equality, that

$$\| \nabla \phi \|^2_{\Omega_i} + \sum_{i=1}^{N} \sum_{j=0}^{N-1} \delta_{ji} \left( \| \nabla \phi \|^2_{K_{h,i}(\Gamma_{ij})} + h_i^2 \| \nabla^2 \phi \|^2_{K_{h,i}(\Gamma_{ij})} \right) \lesssim (1 + h_2^{2/d} N_\mathcal{O}) \| \psi \|^2.$$  

The bound on $\sum_{i} (v_i, \psi)_{\Omega_i}$ from (6.19) with $\psi = v$ now reads

$$\sum_{i=0}^{N} \| v_i \|^2_{\Omega_i} \lesssim \left( \sum_{i=0}^{N} \| \nabla v_i \|^2_{\Omega_i} + \sum_{i=1}^{N} \sum_{j=0}^{N-1} h^{-1}_i \| v \|^2_{\Gamma_{ij}} \right)^{1/2} \sqrt{1 + h_2^{2/d} N_\mathcal{O}} \| v \|,$$

from which follows that

$$\sum_{i} \| v_i \|^2_{\Omega_i} \lesssim \left( \sum_{i=0}^{N} \| \nabla v_i \|^2_{\Omega_i} + \sum_{i=1}^{N} \sum_{j=0}^{N-1} h^{-1}_i \| v \|^2_{\Gamma_{ij}} \right)(1 + h_2^{2/d} N_\mathcal{O}).$$
To conclude, recall (6.13) and insert (6.32) to obtain the desired estimate

\[
\|v\|_h^2 \lesssim \left( \sum_{i=0}^{N} \|\nabla v_i\|_{\Omega_h}^2 + \sum_{i=1}^{N} \sum_{j=0}^{i-1} h^{-1} \|v\|_{\Gamma_{ij}}^2 \right) (1 + h^{2/d} N_O) \\
+ h^2 \left( N_O \sum_{i=0}^{N} \|\nabla v_i\|_{\Omega_h}^2 + \|v\|_{s_h}^2 \right) \\
\lesssim \left( 1 + h^{2/d} N_O \right) \|v\|_h^2.
\]

(6.34)

**Theorem 6.3 (Condition number estimate).** It holds that

\[
\kappa(\hat{A}) \lesssim (1 + h^{2/d} N_O)(1 + N_O) h^{-d}.
\]

**Proof.** Since \( K_{h,i} \) is conforming and quasi-uniform we have the equivalence

\[
\|v_i\|_{\Omega_h,i}^2 \sim h^d |\hat{v}_i|_{M,i}^2 \quad \forall v_i \in V_{h,i};
\]

see e.g. [3]). It follows that

\[
\|v\|_h^2 = \sum_{i=0}^{N} \|v_i\|_{\Omega_h,i}^2 \sim \sum_{i=0}^{N} h^d |\hat{v}_i|_{M,i}^2 \sim h^d |\hat{v}|_M^2.
\]

(6.37)

Recall the definition of the matrix norm

\[
|\hat{A}|_M = \sup_{\hat{v} \neq 0} \frac{|\hat{A}\hat{v}|_M}{|\hat{v}|_M}.
\]

(6.38)

To estimate \( |\hat{A}|_M \), we use the definition of the stiffness matrix (6.2), the inverse inequality (6.4) and the equivalence (6.37) to obtain

\[
|\hat{A}\hat{v}|_M = \sup_{\hat{w} \neq 0} \frac{(\hat{A}\hat{v}, \hat{w})_M}{|\hat{w}|_M} \\
= \sup_{\hat{w} \neq 0} \frac{A_h(v, w)}{|\hat{w}|_M} \\
\lesssim \sup_{\hat{w} \neq 0} \frac{\|v\|_h \|w\|_h}{|\hat{w}|_M} \\
\lesssim \sup_{\hat{w} \neq 0} \frac{(1 + N_O) h^{-2} \|v\|_h \|w\|_h}{|\hat{w}|_M} \\
\sim \sup_{\hat{w} \neq 0} \frac{(1 + N_O) h^{-2} h^d |\hat{v}|_M |\hat{w}|_M}{|\hat{w}|_M} \\
\lesssim (1 + N_O) h^{d-2} |\hat{v}|_M.
\]

(6.42)

(6.43)

Dividing by \( |\hat{v}| \) and using the definition of the matrix norm (6.38) yields

\[
|\hat{A}|_M \lesssim (1 + N_O) h^{d-2}.
\]

(6.45)
To estimate $|\tilde{A}^{-1}|_M$, we proceed similarly, and additionally use the Poincaré inequality (6.11) and the coercivity of the bilinear form (5.12) to obtain

(6.46) $h^{d|\tilde{v}|^2_M} \sim ||v||^2_h$
(6.47) $\lesssim (1 + h^{2/d}N_{\Omega})||v||^2_h$
(6.48) $\lesssim (1 + h^{2/d}N_{\Omega})A_h(v, v)$
(6.49) $= (1 + h^{2/d}N_{\Omega})(\tilde{A}\tilde{v}, \tilde{v})_M$
(6.50) $\leq (1 + h^{2/d}N_{\Omega})|\tilde{A}\tilde{v}|_M|\tilde{v}|_M$.

The inequality thus reads

(6.51) $h^{d|\tilde{v}|^2_M} \lesssim (1 + h^{2/d}N_{\Omega})|\tilde{A}\tilde{v}|_M$.

Setting $\tilde{v} = \tilde{A}^{-1}\tilde{w}$ yields

(6.52) $h^{d|\tilde{A}^{-1}\tilde{w}|^2_M} \lesssim (1 + h^{2/d}N_{\Omega})|\tilde{w}|_M$.

Dividing by $|\tilde{w}|_M$ and using the definition of the matrix norm (6.38) now gives

(6.53) $|\tilde{A}^{-1}|_M \lesssim (1 + h^{2/d}N_{\Omega})(1 + N_{\Omega})h^{-d}$.

By using (6.45) and (6.53) in the definition of the condition number (6.3), we obtain the desired estimate (6.35).

The estimate for the condition number is supported by the numerical results presented in Figure 6.1. The details on this example is found in [10].

7. Conclusions and Future Work. We have analyzed a general framework for discretization of partial differential equations posed on a domain defined by an arbitrary number of intersecting meshes. The framework was analyzed in the context of the Poisson problem. In the accompanying paper [10], numerical results are presented in support of the theoretical analysis presented in the current paper.
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