Application of exterior calculus to waveguides

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Exterior calculus is a powerful tool to search for solutions to the electromagnetic field equations, whose strength can be better appreciated when applied to work out non-trivial configurations. Here we show how to exploit this machinery to obtain the electromagnetic TM and TE modes in hollow cylindrical waveguides. The proper use of exterior calculus and Lorentz boosts will straightforwardly lead to such solutions and the respective power transmitted along the waveguide.

I. INTRODUCTION

Waveguides are an excellent arena to practice relativity and calculus with differential forms, since these tools greatly help the understanding of electromagnetic problems and their solutions. Although most of textbooks teaching exterior calculus to physicists display some basic applications to electromagnetism, any of them stresses the power of this language by applying it to solve Maxwell equations in waveguides and cavities. This is rather disappointing, because the skill in applying both tools—relativity and exterior calculus—allows a clean presentation of the subject and an easy way for working out the field configurations, so highlighting the power of the language not only as a theoretical weapon but as a practical tool to solve problems of technical relevance. On the contrary, normally the vector language is used to display the field configurations in waveguides (however see Ref. 5), which is hardly a natural language for electromagnetism. Because of this reason, the vector approach tends to be rather tedious. Frequently, the lack of a proper geometric language reduces the explanation to a simple case like the waveguide of rectangular or circular section (however see Ref. 6). We will work out the problem of propagating waves in hollow cylindrical waveguides of arbitrary section by employing exterior calculus. We will start with TM (TE) (non-propagating) stationary modes. We will turn these stationary solutions into propagating solutions by means of a Lorentz boost along the waveguide. Finally we will compute the transmitted power and emphasize its relativistic relation with the energy per unit of length in the waveguide. The calculus involving exterior derivatives, Hodge dualities and the generalized Stokes theorem will provide a straightforward way for building the solutions, since all the vector equations of the usual approach condense in just one equation in geometric language, so showing how powerful this machinery is.

II. A BRIEF REVIEW OF EXTERIOR CALCULUS

Exterior calculus is the natural language for electrodynamics. But not only electrodynamics greatly benefit from the compactness and simplicity of this geometric language. Also the developments in Hamiltonian mechanics, thermodynamics, Yang-Mills fields, geometric (Berry) phases in quantum mechanics, topological quantum fields as the Chern-Simons theory, gravity, symplectic geometry, connections in fiber bundles, etc. gain in clarity and depth when expressed through the tools of exterior calculus. There is a huge list of textbooks to learn exterior calculus. The reader is referred to Ref. 1–4 for a first approach to the subject. Here we will briefly review the main operational features of the exterior derivative $d$ and the wedge product $\wedge$ between differential forms.

Any linear combination of coordinate differentials at each point of space is a 1-form field (whatever the coordinates are, Cartesian or not; even if the geometry is non-Euclidean). For instance, we may call $\eta$ the 1-form

$$\eta = 3x^2y^7 \, dx + 5y \, dz$$

(1)

If the space is 3-dimensional and $(x, y, z)$ are the chosen coordinates, we say that $\{dx, dy, dz\}$ is a coordinate basis for 1-forms. The components of the above defined 1-form $\eta$ in this basis are $\eta_x = 3x^2y^7$, $\eta_y = 0$, $\eta_z = 5y$. Generically, a 1-form in a $n$-dimensional space is

$$\alpha = \alpha_\mu \, dx^\mu$$

(2)

(Einstein convention is used). The superindex in $dx^\mu$ labels the $n$ 1-forms of the coordinate basis, whereas the subindex in $\alpha_\mu$ labels the components of the 1-form $\alpha$, which are functions of the coordinates.
1-forms can be introduced in a geometric way as linear real valued functions on the tangent vector space to a manifold: they are covectors or covariant vectors. However, here we are not interested in the action of forms on vectors. Instead we will operate within the set of \( p \)-forms, which are defined as totally antisymmetric covariant tensors of \( p \) indexes (so \( p \leq n \)). \( p \)-forms can be obtained from (antisymmetrized) wedge tensor product \( \wedge \) of 1-forms. For instance, the wedge product between \( \eta \) and the 1-form \( \xi = z \, dx + 2 \, dy \) is the 2-form

\[
\omega = \eta \wedge \xi = 6 \, x^2 \, y^7 \, dx \wedge dy - 5 \, y \, z \, dx \wedge dz - 10 \, y \, dy \wedge dz
\]

Notice the absence of \( dx \). There are \( \binom{n}{2} \) linearly independent 2-forms \( dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \) (\( \otimes \) is the tensor product), which span the coordinate basis of 2-forms. Any 2-form can be written as

\[
\alpha = \frac{1}{2!} \alpha_{\mu\nu} \, dx^\mu \wedge dx^\nu
\]

where \( \alpha_{\mu\nu} = -\alpha_{\nu\mu} \). In the former example it is \( \omega_{xy} = 6 \, x^2 \, y^7 \), \( \omega_{yz} = 0 \), \( \omega_{xz} = -5 \, y \, z \). The factor \( 1/2! \) in Eq. (1) takes into account the fact that each independent element of the basis appears \( 2! \) times in the sum over \( \mu \) and \( \nu \).

If \( \alpha \) and \( \beta \) are 1-forms on a 3-dimensional manifold, then the components of the product \( \alpha \wedge \beta \) will look as the Cartesian components of the vector product in Euclidean space:

\[
\alpha \wedge \beta = (\alpha_x \beta_y - \alpha_y \beta_x) \, dx \wedge dy - (\alpha_x \beta_z - \alpha_z \beta_x) \, dz \wedge dx + (\alpha_y \beta_z - \alpha_z \beta_y) \, dy \wedge dz
\]

Any \( p \)-form \( \alpha \) and \( q \)-form \( \beta \) satisfy

\[
\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha
\]

Thus 1-forms anti-commute but 2-forms commute, etc.

The exterior derivative \( d \) is a nilpotent operator (\( d^2 \equiv 0 \)). If \( d \) acts on a function \( f \) (0-form) the result is the 1-form \( df = (\partial f/\partial x^\mu) \, dx^\mu \). In general, if \( d \) acts on a \( p \)-form \( \alpha \) then the result is a \((p+1)\)-form \( d\alpha \). Since \( d(dx^\mu) \equiv 0 \), \( d\alpha \) is obtained by merely differentiating its components as exterior derivatives of functions:

\[
d\alpha = d \left( \frac{1}{p!} \alpha_{\lambda\mu\ldots} \right) \wedge dx^\lambda \wedge dx^\mu \wedge dx^\nu \ldots
\]

For instance

\[
d\eta = -21 \, x^2 \, y^6 \, dx \wedge dy + 5 \, dy \wedge dz
\]

If \( \alpha \) is a \( p \)-form then

\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta
\]

A given \( p \)-form \( \alpha \) is called closed if \( d\alpha = 0 \); it is called exact if it can be written as the exterior derivative of a \((p-1)\)-form. Each exact form is closed; the inverse proposition is locally true, but its global validity depends on the topology of the space (see Poincaré’s lemma).

Finally, we will introduce the Hodge star operator \( * \). This operator changes \( p \)-forms into \((n-p)\)-forms, and involves the components \( g_{\mu\nu} \) of the metric tensor present in the space-time interval. If \( \alpha \) is a \( p \)-form whose components are \( \alpha_{\mu_1 \ldots \mu_p} \) then

\[
* \alpha_{\mu_{p+1} \ldots \mu_n} = \sqrt{|\det(g_{\mu\nu})|} \, \varepsilon_{\mu_{p+1} \ldots \mu_n \mu_1 \ldots \mu_p} \, \alpha^{\mu_1 \ldots \mu_p}
\]

where the indexes are raised with the inverse metric tensor \( g^{\mu\nu} \); \( \alpha^{\mu_1 \ldots \mu_p} = g^{\mu_1 \nu_1} \ldots g^{\mu_p \nu_p} \alpha_{\nu_1 \ldots \nu_p} \). \( \varepsilon \) is the Levi-Civita symbol, which takes the value 1 \((-1)\) for even (odd) permutations of its indexes, and it is zero if there are indexes of equal value. The successive application of the Hodge star operator on a \( p \)-form \( \alpha \) is

\[
* * \alpha = (-1)^{p(n-p)+(n-\sigma)/2} \alpha
\]

where \( \sigma \) is the signature of the metric tensor (the difference between the numbers of positive and negative eigenvalues of the metric tensor).
III. POTENTIAL AND FIELD AS DIFFERENTIAL FORMS ON A MANIFOLD

The electromagnetic field is an exact 2-form \( F = dA \), where the 1-form \( A \) is the potential. Given a set of non-necessarily Cartesian coordinates \( x, y, z \) together with the time \( t \), one can express these forms in the coordinate basis \( \{ dt, dx, dy, dz \} \):

\[
A = A_\nu \, dx^\nu
\]  
(12)

If \( x, y, z \) are Cartesian coordinates, then the components \( A_\nu \) will coincide with the scalar and vector potential: \( A_\phi = (-\phi, \mathbf{A}) \) (SI units) or \( A_\nu = (-c \phi, \mathbf{A}) \) (Gaussian units). Therefore

\[
F = dA = dA_\nu \wedge dx^\nu = \partial_\mu A_\nu \, dx^\mu \wedge dx^\nu
\]  
(13)

Since \( dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \), then the (antisymmetric) components of \( F \) are \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). For Cartesian coordinates, it is \( F_{it} = E_i \) in SI units (or \( eE_i \) in Gaussian units), \( F_{yz} = B_z \), \( F_{zx} = B_y \), \( F_{xy} = B_z \):

\[
F = E \wedge dt + B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy
\]  
(14)

where \( E = E_i \, dx^i \) (SI units) is a 1-form. As Eq. (14) shows, in the 3-dimensional Euclidean space the wedge product between 1-forms works as the vector product, since \( dy \wedge dz \) is the basis element for the \( x \)-component of the pseudo-vector \( \mathbf{B} \), and so on (think the wedge product as a vector product; see Eq. (5)).

Since \( F = dA \) is exact and \( d \) is nilpotent, then it results the identity

\[
d \wedge F = 0
\]  
(15)

which amounts those Maxwell equations used to define the potentials: \( \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t, \nabla \cdot \mathbf{B} = 0 \).

The dynamical sourceless Maxwell equations, \( \nabla \times \mathbf{B} = -c^{-2} \partial^2 \mathbf{E} / \partial t^2, \nabla \cdot \mathbf{E} = 0 \), come from varying the electromagnetic action \( S[A] = - (4\pi \mu_0 c)^{-1} \int *F \wedge F \). Since \( F \) is a 2-form in a manifold of dimension \( n = 4 \) (the spacetime), then \( *F \) is a 2-form too. \( *F \wedge F \) is a 4-form (a volume in spacetime). The resulting Euler- Lagrange equations are

\[
d \ast F = 0
\]  
(16)

IV. CYLINDRICAL WAVEGUIDES

We are going to solve the sourceless Maxwell equations for the electromagnetic field in hollow cylindrical waveguides. So, let us call \( z \) the Cartesian coordinate along the waveguide, and \( x, y \) two properly chosen coordinates spanning the waveguide section: Cartesian coordinates for rectangular section, polar coordinates for circular section, etc. We will begin by working out stationary waves. Afterwards we will introduce the propagation along the waveguide by means of a Lorentz boost in the \( z \)-direction. Therefore we will start by proposing a solution independent of \( z \).

A. Stationary TM modes

Let us consider a monochromatic potential having just a component along the waveguide:

\[
A = e^{i\Omega t} \, \psi(x, y) \, dz
\]  
(17)

Function \( \psi \) is defined in the waveguide section and has units of magnetic field times length. Thus the electromagnetic field is

\[
F = dA = e^{i\Omega t} \, (i\Omega \, \rho \, dt + \, d\psi) \wedge dz
\]
\[
= e^{i\Omega t} \, (i\Omega \, \rho \, dt \wedge dz + \partial_z \psi \, dx \wedge dz + \partial_x \psi \, dy \wedge dz)
\]  
(18)

The first term is an electric field along \( z \) and the other ones make up a magnetic field orthogonal to the \( z \)-axis, so lying on the waveguide section. Therefore the proposed solution is a TM mode. For the moment this solution does not propagate along the waveguide since the components do not depend on \( z \) (this feature will be introduced later). Field (18) is just a stationary wave bouncing between the boundaries. Function \( \psi \) is subjected to fulfill perfect conductor boundary conditions: the tangential electric field and the normal magnetic field must vanish on the boundary. Then \( \psi \) must be zero on the boundary to cancel out the (pure) tangential electric field:

\[
\psi|_{\text{boundary}} = 0
\]  
(19)
(Dirichlet condition for the potential $\psi(x, y)$). Since $\psi$ is constant on the boundary then the 1-form $d\psi = (\partial \psi / \partial x^\mu) \, dx^\mu$ is normal to the boundary; so the magnetic part $d\psi \wedge dz$ is tangential to the boundary and perpendicular to the $z$-axis (just think the product as a vector product). Thus, the boundary condition for the electric field guarantees the boundary condition for the magnetic field too.

The 2-form $*F$ involves the metric tensor. If orthogonal coordinates are used in the waveguide section, then the space-time interval will read

$$ds^2 = -c^2 dt^2 + g_{xx} \, dx^2 + g_{yy} \, dy^2 + dz^2$$  \hspace{1cm} (20)

where the metric $\text{diag}(g_{xx}, g_{yy})$ in the section depends on the choice of coordinates $x, y$. Then the determinant of the metric and the inverse metric tensor are

$$|\det(g_{\mu \nu})| = c^2 \, g_{xx} \, g_{yy}, \quad g^{\mu \nu} = \text{diag}(-c^{-2}, 1 / g_{xx}, 1 / g_{yy}, 1)$$  \hspace{1cm} (21)

To write $*F$ we just need a few results:

* $(dt \wedge dz) = -c^{-1} \, \sqrt{g_{xx} g_{yy}} \, dx \wedge dy$,
* $(dz \wedge dx) = -c \, \sqrt{g_{xx} g_{yy}} \, g^{xx} \, dt \wedge dy$,
* $(dy \wedge dz) = c \, \sqrt{g_{xx} g_{yy}} \, g^{yy} \, dt \wedge dx$  \hspace{1cm} (22)

Thus,

$$*F = c \, \sqrt{g_{xx} g_{yy}} \, e^{i \Omega t} \left\{ -i c^{-2} \Omega \psi \, dx \wedge dy + \left( g^{xx} \partial_x \psi \, dy - g^{yy} \partial_y \psi \, dx \right) \wedge dt \right\}$$  \hspace{1cm} (23)

$$d * F = c \, \sqrt{g_{xx} g_{yy}} \, e^{i \Omega t} \left( c^{-2} \Omega^2 \psi + (2) \Delta \psi \right) \, dt \wedge dx \wedge dy$$  \hspace{1cm} (24)

Eq. (16) implies that function $\psi$ must be an eigenfunction $\psi_{mn}$ of the two-dimensional Laplacian operator in the waveguide section, $-c^{-2} \Omega^2_{mn}$ being its respective eigenvalue:

$$(2) \Delta \psi_{mn} = \frac{1}{\sqrt{g_{xx} g_{yy}}} \left[ \partial_x \left( \sqrt{g_{xx} g_{yy}} \, g^{xx} \partial_x \psi_{mn} \right) + \partial_y \left( \sqrt{g_{xx} g_{yy}} \, g^{yy} \partial_y \psi_{mn} \right) \right] = \frac{\Omega^2_{mn}}{c^2} \psi_{mn}$$  \hspace{1cm} (25)

Notice that Eq. (23) contains the two-dimensional 1-form $(2) (\ast d\psi)$ (the superscript “(2)” means that the Hodge star is applied in a $n = 2$ submanifold; see Appendix). As can be seen, the Laplacian in the waveguide section is $(2) \Delta = (2) (-\ast d \ast d)$. Its eigenfunctions $\psi_{mn}$ satisfying proper boundary conditions are identified by two discrete indexes $m, n$. Table I summarizes the results for monochromatic TM stationary modes. Table II shows the solutions of Eqs. (10), (25) and their eigenvalues $\Omega$ (allowed frequencies for stationary TM modes) for typical waveguide sections.

### Table I: TM stationary modes.

| $\Omega_{mn}$ | $\psi_{mn}$ |
|--------------|-------------|
| $c \, \sqrt{g_{xx} g_{yy}} \, e^{i \Omega t}$ | $i \, \Omega_{mn} \, \psi_{mn} \, dt + \, d\psi_{mn} \wedge dz$ |
| $c \, \sqrt{g_{xx} g_{yy}} \, e^{-i \Omega t}$ | $-i \, c^{-1} \Omega_{mn} \, \sqrt{g_{xx} g_{yy}} \, \psi_{mn} \, dx \wedge dy + \, c \, \ast d \psi_{mn} \wedge dt$ |

$\ast \psi$ and $(2) (\ast d\psi)$ are defined in the waveguide section:

$$\ast \psi_{mn} (x, y) = -c^{-2} \Omega^2_{mn} \psi_{mn} (x, y)$$  \hspace{1cm} $\psi_{mn} |_{\text{boundary}} = 0$

metric in the section:  \hspace{1cm} $d\ell^2 = g_{xx} (x, y) \, dx^2 + g_{yy} (x, y) \, dy^2$

### B. Stationary TE modes

Remarkably $F$ and $*F$ are on an equal footing in Eqs. (15) and (16). This circumstance will allow to built the stationary TE modes by exchanging their roles. In Eq. (14), this exchange amounts to the interchanging of $E$ and
TABLE II: TM ψ’s and Ω’s for rectangular and circular sections.

| Section       | ψ                  | Ω                  |
|---------------|--------------------|--------------------|
| Rectangular   | $\psi_{mn}(x,y) = A_{mn} \sin \left( \frac{m\pi}{a}x \right) \sin \left( \frac{n\pi}{b}y \right)$ | $c^{-2} \Omega_{mn}^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2$ |
| Circular      | $\psi_{mn}(r,\varphi) = J_m(x_{mn} r/R) (A_{mn} \cos(m\varphi) + B_{mn} \sin(m\varphi))$ | $\Omega_{mn} = \frac{x_{mn}}{R}$ |

B. So we define

$$F^{TE} = *F^{TM}$$  \hspace{1cm} (26)

We will apply Eq. (11) to obtain $*F^{TE} = * * F^{TM}$. In the space-time it is $n = 4$ and $\sigma = 2$ (see Eq. (20)); so $** = (-1)^{p+1}$. Thus

$$* F^{TE} = - F^{TM}$$  \hspace{1cm} (27)

The field (26) can be ascribed to the potential

$$A^{TE} = \frac{e^{i\Omega t} c}{i\Omega} \sqrt{g_{xx} g_{yy}} (g^{yy} \partial_y \psi dx - g^{xx} \partial_x \psi dy) = \frac{e^{i\Omega t} c}{i\Omega} (2(*d\psi))$$  \hspace{1cm} (28)

In fact, by differentiating the middle expression in Eq. (28) one recognizes the appearance of the Laplacian defined in Eq. (25):

$$F^{TE} = dA^{TE} = e^{i\Omega t} c \sqrt{g_{xx} g_{yy}} \left\{ i \Omega \frac{(2\Delta \psi dx \wedge dy + (g^{xx} \partial_x \psi dy - g^{yy} \partial_y \psi dx) \wedge dt)}{i\Omega} \right\}$$

$$= e^{i\Omega t} c \sqrt{g_{xx} g_{yy}} \left\{ i \Omega \frac{(2\Delta \psi dx \wedge dy + (2(*d\psi)) \wedge dt)}{(2(*d\psi)) \wedge dt} \right\}$$  \hspace{1cm} (29)

This result coincides with field (28) when the pair $\Omega, \psi$ is chosen among the solutions $\Omega_{mn}, \psi_{mn}$ of the eigenvalue equation (25). Although $F^{TE}$ solves both Eqs. (15) and (16), the boundary condition should be consistently changed: the electric field should be normal to the boundary to accomplish the perfect conductor boundary condition. If $n = n_x dx + n_y dy$ is a 1-form normal to the boundary, and $(2(*d\psi))$ in Eq. (29) is a 1-form proportional to the electric field in the waveguide section, then one demands

$$(2(*d\psi)) \wedge n |_{\text{boundary}} = 0$$  \hspace{1cm} (30)

(just think the product as a vector product). When written in vector language this requirement means

$$n \cdot \nabla \psi |_{\text{boundary}} = 0$$  \hspace{1cm} (31)

(Neumann boundary condition for the potential $\psi(x,y)$). Table III summarizes the monochromatic stationary TE modes. Table IV shows the functions $\psi$ for typical waveguide sections.

V. PROPAGATING MODES

The solutions studied in Section IV do not propagate energy along the waveguide. Since the Poynting vector is proportional to $E \times B$, then there should exist $E$ and $B$ in the waveguide section to have energy propagating along the waveguide. However, field $F^{TM}$ in Eq. (18) has only a magnetic field in the waveguide section, and field $F^{TM}$ in Eq. (29) has only an electric field in the section. Thus, the Poynting vector in solutions (18), (29) is orthogonal to the
TABLE III: TE stationary modes.

\[ F^{TE} = e^{i\Omega_{mn} t} \left( -ie^{-1} \Omega_{mn} \sqrt{g_{xx}g_{yy}} \psi_{mn} \ dx \wedge dy + c^{(2)}(\ast d\psi_{mn}) \wedge dt \right) \]

\[ \ast F^{TE} = -e^{i\Omega_{mn} t} \left( i\Omega_{mn} \psi_{mn} \ dt + d\psi_{mn} \right) \wedge dz \]

\[ \psi \text{ and } (2)\ast d\psi \text{ are defined in the waveguide section:} \]

\[ \Delta \psi_{mn}(x,y) = -c^{-2} \Omega^2_{mn} \psi_{mn}(x,y) \]

\[ n \cdot \nabla \psi |_{\text{boundary}} = 0 \]

metric in the section: \[ d\ell^2 = g_{xx}(x,y) \, dx^2 + g_{yy}(x,y) \, dy^2 \]

TABLE IV: TE \( \psi \)'s and \( \Omega \)'s for rectangular and circular sections.

| Section               | Eq. Details                                                                 |
|-----------------------|-----------------------------------------------------------------------------|
| Rectangular section   | \( \psi_{mn}(x,y) = A_{mn} \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \) \[ c^{-2}\Omega^2_{mn} = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \] |
| Circular section      | \( \psi_{mn}(r,\varphi) = J_m(y_{mn} r / R) \left( A_{mn} \cos(m\varphi) + B_{mn} \sin(m\varphi) \right) \) \[ \Omega_{mn} = \frac{y_{mn}}{R} \] \( y_{mn} \text{ are zeros of the derivatives of Bessel functions } J_m \) |

waveguide axis; so solutions (18), (29) are stationary waves where the energy bounces between the boundaries but does not propagate along the waveguide. Moreover, it is easy to prove that the time averaged Poynting vector vanishes in this case. This means that the fields (18), (29) are displayed in their proper frame. The stationary solutions (18), (29) can be transformed into solutions that propagates energy along the waveguide by performing a Lorentz boost in the \( z \)-direction. To do this we use

\[ t = \gamma(V)(t' - Vc^{-2}z'), \quad dt = \gamma(V)(dt' - Vc^{-2}dz'), \quad \gamma(V) = (1 - V^2c^{-2})^{-1/2} \]

where \( \gamma(V) = (1 - V^2c^{-2})^{-1/2} \). Thus

\[ dt \wedge dz = \gamma(V)^2(1 - V^2c^{-2}) \ dt' \wedge dz' = dt' \wedge dz' \]

A. TM modes

Eq. (33) means that the longitudinal electric field remains invariant in Eq. (18). On the contrary the transverse term \( d\psi(x,y) \wedge dz \) changes to

\[ d\psi(x,y) \wedge dz = \gamma(V) \ d\psi(x,y) \wedge (dz' - V dt') \]

which amounts not only a change of the magnetic field in the waveguide section but the appearance of a transverse electric field. Of course this is nothing but the usual rules to transform electric and magnetic fields (see for instance Ref. [16]). However the geometric language shows it in a quite elegant way.
B. TE modes

In this case the Eq. (33) means that the longitudinal magnetic field remains invariant in Eq. (29). Instead, the Lorentz boost changes the electric transverse sector of $F_{TE}$ to

$$c (2) (\ast d\psi_{mn}) \wedge dt = c \gamma(V) (2) (\ast d\psi_{mn}) \wedge (dt' - Vc^{-2}dz')$$

$$= c \gamma(V) \sqrt{g_{xx} g_{yy}} (g^{yy} \partial_{y} \psi \, dx - g^{xx} \partial_{x} \psi \, dy) \wedge (dt' - Vc^{-2}dz')$$

Therefore, not only the electric transverse field is changed by the boost, but a magnetic field appears in the waveguide section in the new frame.

As a conclusion of this Section, in any frame differing from the proper frame where the solutions (18), (29) were built, the propagating TM and TE modes display both electric and magnetic fields in the waveguide section and so they propagate energy along the waveguide.

VI. TRANSMITTED POWER

As already stated, the existence of both magnetic and electric transverse fields in the new frame produces a Poynting vector along the waveguide. Thus, in the new frame there is energy propagating in the waveguide. Velocity $V$ is the velocity relative to the proper frame. In this sense $V$ has the right to be called energy velocity, since no energy propagates along the waveguide in the original proper frame (there is just an energy flux orthogonal to the waveguide axis whose time averaging vanishes).

The time averaged energy flux along the waveguide, in the frame moving with velocity $V$ relative to the proper frame, results from the $t'z'$ component of the electromagnetic energy-momentum tensor $T_{\mu\nu}$

$$\mu_{\alpha} T_{\mu\nu} = F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho}$$

(35)

Only real fields should be considered; so one has to average products of trigonometric functions. It is well known that $<\sin^2(\Omega t)> = \frac{1}{2} = <\cos^2(\Omega t)>, <\sin(\Omega t)\cos(\Omega t)> = 0$. Thus one obtains

$$\mu_{\alpha} <T_{t'z'}> = g^{xx} <F_{t'x} F_{z'x}> + g^{yy} <F_{t'y} F_{z'y}> = -\frac{1}{2} \gamma(V)^2 V |\nabla \psi|^2$$

(36)

(we remark that the Lorentz boost does not change the components of the metric tensor). The result (36) is shared by TM and TE modes (although it is harder to get it for TE modes).

$$c^2 T_{t'z'} = -T_{t'z'}$$ is the energy per unit of time and area going through the waveguide section. The transmitted power results from integrating this quantity in the section $S$. Since $\psi$ vanishes on the boundary of the section (its normal derivative vanishes at the boundary for TE modes), then the Green’s first identity implies that

$$\int \nabla \psi \cdot \nabla \psi \, dS = -\int \psi (2) \Delta \psi \, dS$$

(37)

(both for TM and for TE modes). Thus one uses Eq. (25) to obtain

$$\int |\nabla \psi|^2 \, dS = \int c^{-2} \Omega^2 \psi^2 \, dS$$

(38)

Therefore the transmitted power is

$$P_{mn} = \int c^2 T_{t'z'} \, dS = \frac{\gamma(V)^2 V \Omega_{mn}}{2 \mu_{\alpha} c^2} \int \psi_{mn}^2 \, dS$$

(39)

This result can be written in terms of the frequency $\omega$ and the wavenumber $k_{z'}$. The transformation of the coordinate $t$ implies that the wave phase becomes

$$\Omega t = \Omega \gamma(V)(t' - Vc^{-2}z')$$

(40)

Then

$$\omega = \gamma(V)\Omega, \quad k_{z'} = \gamma(V)\Omega V c^{-2}$$

(41)
which leads to the dispersion relation

\[ \omega = \sqrt{c^2 k_{z'}^2 + \Omega^2} \] (42)

(then \(\Omega_{mn}\) is the cut-off frequency for each mode). Thus the energy velocity \(V\) is written in terms of \(\omega\) and \(k_{z'}\) as

\[ V = \frac{c^2 k_{z'}}{\omega} = \frac{\partial \omega}{\partial k_{z'}} < c \] (43)

As could be expected, the energy velocity coincides with the group velocity \(\partial\omega/\partial k_{z'}\).

Notice that \(\omega k_{z'}\) equals the expression contained in the transmitted power. So

\[ P_{mn} = \frac{\omega_{mn} k_{z'}}{2\mu_o} \int \psi_{mn}^2 \, dS \] (44)

If the waveguide is filled with an homogeneous linear medium, then the Eq. (44) for the transmitted power can be used in the frame where the medium is at rest by replacing \(\mu_o\) with the permeability \(\mu\) (the constitutive relations are only valid in the media proper frames).

To end the study of the energy transmission let us compute the time-averaged energy density \(\omega k_{z'}\) as the momentum per unit of length equaled to the energy per unit of length over the momentum. Therefore, since the energy-momentum tensor is symmetric, the former expression can also be read as the momentum per unit of length equaled to the energy per unit of length over \(c^2\) times the energy velocity (the usual relativistic relation for massive particles!). Even though the electromagnetic field is massless its energy in the waveguide behaves like the one of a massive field, as a consequence of the boundary conditions imposed by the waveguide. The same feature emerges in theories with compactified dimensions, which impose periodic boundary conditions to the (otherwise massless) fields. 

The invariant \(< F_{\lambda\rho} F^{\lambda\rho} >\) can be computed with the components of the stationary field (it is invariant under a Lorentz boost!). The field of stationary TM modes has only three independent components. The real fields are

\[ F_{tz} = -\Omega_{mn} \psi_{mn} \sin(\Omega_{mn} t) \]
\[ F_{xz} = \partial_x \psi_{mn} \cos(\Omega_{mn} t) \]
\[ F_{yz} = \partial_y \psi_{mn} \cos(\Omega_{mn} t) \] (46)

Therefore, the time-averaged scalar invariant \(< F_{\lambda\rho} F^{\lambda\rho} >\) is

\[ < F_{\lambda\rho} F^{\lambda\rho} > = 2 < F_{tz} F^{tz} + F_{xz} F^{xz} + F_{yz} F^{yz} > \]
\[ = -c^{-2} \Omega^2_{mn} \psi_{mn}^2 + g^{xx} (\partial_x \psi_{mn})^2 + g^{yy} (\partial_y \psi_{mn})^2 = -c^{-2} \Omega^2_{mn} \psi_{mn}^2 + |\nabla \psi_{mn}|^2 \] (47)

Then

\[ \mu_o < T^{z'z'} > = \frac{1}{2} \gamma(V)^2 V^2 |\nabla \psi|^2 + \frac{1}{4} \Omega^2 \psi^2 - \frac{1}{4} \Omega^2 \psi^2 + \frac{c^2}{4} |\nabla \psi|^2 = \frac{1}{4} \Omega^2 \psi^2 + \frac{c^2}{4} \frac{1 + \frac{\gamma^2}{c^2}}{c^2} |\nabla \psi|^2 \] (48)

By performing the integral in the waveguide section we obtain the energy per unit of length:

\[ U_{mn} = \frac{c^2 T^{z'z'}}{2\mu_o c^2} \int \left\{ \Omega^2_{mn} \psi_{mn}^2 + c^2 \frac{1 + \frac{\gamma^2}{c^2}}{1 - \frac{\gamma^2}{c^2}} |\nabla \psi_{mn}|^2 \right\} \, dS = \frac{\gamma(V)^2 \Omega^2_{mn}}{2\mu_o c^2} \int \psi_{mn}^2 \, dS \] (49)

where we have used the result (48). By comparing with Eq. (44) one gets

\[ P_{mn} = V U_{mn} \] (50)

Once again we find the velocity \(V\) in the expected role of the energy velocity: in this case as the quotient of transmitted power and energy per unit of length as is usually defined. Notice that \(T^{z'z'}\) is the density of the \(z'\)-component of the momentum. Therefore, since the energy-momentum tensor is symmetric, the former expression can also be read as the momentum per unit of length equaled to the energy per unit of length over \(c^2\) times the energy velocity (the usual relativistic relation for massive particles!). Even though the electromagnetic field is massless its energy in the waveguide behaves like the one of a massive field, as a consequence of the boundary conditions imposed by the waveguide. The same feature emerges in theories with compactified dimensions, which impose periodic boundary conditions to the (otherwise massless) fields.
VII. APPENDIX

Let \( \phi, \psi \) be two differentiable functions (0-forms). Then one has the following identity among volumes (\( n \)-forms),

\[
d\phi \wedge \ast d\psi + \phi \, d \ast d\psi = d(\phi \ast d\psi)
\] (51)

which can be integrated to become

\[
\int_S (d\phi \wedge \ast d\psi + \phi \, d \ast d\psi) = \int_{\partial S} \phi \ast d\psi
\] (52)

(Stokes theorem\(^3\) has been used on the right side). The \( n \)-form \( d \ast d\psi \) is connected with \( \Delta \psi = - \ast d \ast d\psi \). According to Eq. (11) it is \( \ast \Delta \psi = - d \ast d\psi \) whenever \( \psi \) is a 0-form and the space has signature \( \sigma = n \). Thus we get the Green’s first identity

\[
\int_S (d\phi \wedge \ast d\psi - \phi \ast \Delta \psi) = \int_{\partial S} \phi \ast d\psi
\] (53)

To prove Eq. (38) we will apply Green’s first identity to the case \( \phi = \psi \) in the waveguide section, which is a 2-dimensional manifold with coordinates \( x, y \) and signature \( \sigma = 2 \). For any differentiable function (0-form) \( \psi \) one has

\[
\begin{align*}
d\psi &= \partial_x \psi \, dx + \partial_y \psi \, dy \\
\ast d\psi &= \sqrt{g_{xx}g_{yy}} \left( g^{yy} \partial_y \psi \, dx - g^{xx} \partial_x \psi \, dy \right) \\
d\psi \wedge \ast d\psi &= -\sqrt{g_{xx}g_{yy}} \left( g^{xx} \left( \partial_x \psi \right)^2 + g^{yy} \left( \partial_y \psi \right)^2 \right) \, dx \wedge dy
\end{align*}
\] (54)

As stated, the 2-form \( d\psi \wedge \ast d\psi \) is a volume in the 2-dimensional waveguide section \( S \). It contains the surface element \( dS = \sqrt{g_{xx}g_{yy}} \, dx \, dy \). Notice that the integral we are interested in Eq. (38) is

\[
\int_S |\nabla \psi|^2 \, dS = -\int_S \psi \ast \Delta \psi
\] (55)

So, we can compute it by using Eq. (53) for \( \phi = \psi \). Since we are interested in functions \( \psi \) that vanish on the boundary (TM modes), or their normal derivatives vanish on the boundary (TE modes) (i.e., \( \ast d\psi \) restricted to the boundary is null; see Eq. (30)), then the right side of Eq. (53) is zero in these cases. Besides, it is \( \ast 1 = \sqrt{g_{xx}g_{yy}} \, dx \wedge dy \), which will be used to compute the Hodge star of the 0-form \( \Delta \psi \). Moreover, we are working with functions accomplishing the eigenvalue equation \( \Delta \psi = -\Omega^2 c^{-2} \psi \); then it is \( \ast \Delta \psi = -\Omega^2 c^{-2} \, \psi \sqrt{g_{xx}g_{yy}} \, dx \wedge dy \). Thus the result is

\[
\int_S |\nabla \psi|^2 \, dS = -\int_S \psi \ast d\psi = -\int_S \psi \ast \Delta \psi = \int_S c^{-2} \Omega^2 \psi^2 \sqrt{g_{xx}g_{yy}} \, dx \wedge dy = \int c^{-2} \Omega^2 \psi^2 \, dS
\] (56)

It should be emphasized that expressions like the ones in Eq. (54) only depend on the normalized basis of forms and vectors:

\[
\begin{align*}
e^x &= \sqrt{g_{xx}} \, dx, & e^y &= \sqrt{g_{yy}} \, dy \\
e_x &= \sqrt{g^{xx}} \frac{\partial}{\partial x}, & e_y &= \sqrt{g^{yy}} \frac{\partial}{\partial y}
\end{align*}
\] (57) (58)

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Green’s first identity can be stated as \( \int_S (\phi \Delta \psi + \nabla \phi \cdot \nabla \psi) \, dS = \int_{\partial S} \phi \psi \frac{\partial \phi}{\partial n} \, dl \), where \( dl \) is the length element on the boundary \( \partial S \) and \( \partial / \partial n \) is the normal derivative. See the Appendix for a proof using exterior calculus.