A Note on Fermion and Gauge Couplings in Field Theory Models for Tachyon Condensation

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Abstract

We study soliton solutions in supersymmetric scalar field theory with a class of potentials. We study both bosonic and fermionic zero-modes around the soliton solution. We study two possible couplings of gauge fields to these models. While the Born-Infeld like coupling has one normalizable mode (the zero mode), the other kind of coupling has no normalizable modes. We show that quantum mechanical problem which determines the spectrum of fluctuation modes of the scalar, fermion and the gauge field is identical. We also show that only the lowest lying mode, i.e., the zero mode, is normalizable and the rest of the spectrum is continuous.

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1 Introduction

The study of field theory models of tachyon dynamics has proved useful in understanding the dynamics and stability of non-BPS D-branes, in the light of the Sen conjectures [1–4] on tachyon condensation in string theory [5–12]. Zwiebach [13] and then Minahan and Zwiebach [14] have studied some solvable toy models containing one scalar field representing the tachyon, with Lagrangian densities containing at most two derivatives of the field. Their models support lump solutions, whose fluctuation spectra contain an unstable (tachyonic) mode. Minahan and Zwiebach [15] have also studied a two-derivative model for tachyon dynamics in superstring theory. They have shown that this model has a stable kink solution.

In a recent paper [16], we generalized the model of Minahan and Zwiebach, where we considered potentials of the form

\[ V(\phi) = Aq^n \phi^2 (-\ln \phi)^n, \quad n > 1, \]

where \( \phi(x) \) is the scalar field, \( A \) and \( q \) are arbitrary parameters and \( n \) is a positive integer. These potentials are not bounded below for odd values of \( n \). However, the corresponding field theory was found to possess a stable kink solution. In this paper, we will first look at models with even \( n \). They have potentials (1) which are bounded below and, not surprisingly, have stable kink solutions. We will show that these models can be supersymmetrized by introducing a fermionic field as a supersymmetric partner of the original scalar field. The derivation of bosonic fluctuation modes of the kink, which was studied in [16] for \( n \) odd, is equally valid for \( n \) even. The functional form of the fermionic zero mode turns out to be similar.

We then introduce a gauge field into the model. This is equally applicable for both even and odd values of \( n \). Minahan and Zwiebach have shown that there exists more than one way of coupling the gauge field. We will show that only the Born-Infeld like coupling has a normalizable zero mode. We also show that the ground state wavefunction of the underlying quantum mechanical problem is the only normalizable wavefunction. The rest of the spectrum is continuous. Since the fluctuation spectrum of bosons as well as fermions is governed by the same quantum mechanical problem, we find that fluctuations of all these fields, for any \( n \geq 2 \), contain only one normalizable mode.

2 The Supersymmetric Soliton

The field theory model considered in [16] has the Lagrangian density

\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \]

where \( \phi(x) \) is the scalar field, \( A \) and \( q \) are arbitrary parameters and \( n \) is a positive integer.
where $V(\phi)$ is given by Eq. (1). For any positive integer $n$ this model has a soliton solution of the form

$$\bar{\phi}(x) = \begin{cases} 
\exp(e^{-\alpha_2 x}) & \text{for } n = 2 \\
\exp(-\alpha_n x^2 - \frac{1}{n-2}) & \text{for } n > 2
\end{cases},$$

(3)

where

$$\alpha_n = \left[\left(\frac{n}{2} - 1\right)^2 2Aq^n\right]^{-1/(n-2)}.$$

(4)

While the potential (1) is not bounded below for odd $n$, for even $n$, it is. The generic form of the potential for even $n$ is shown in Fig 1. In either case we have a local minimum at $\phi = 0$ and a maximum at $\phi = \exp(-n/2)$. The value of the potential at the maximum is

$$V_{\max} = Aq^n \frac{n^n}{2^n} e^{-n}$$

(5)

At $\phi = 1$, $n - 1$ derivatives of the potential vanish. For $n$ odd, this is a point of inflection and the potential becomes negative for $\phi > 1$. On the other hand, for $n$ even, this is another minimum of the potential which is degenerate to the one at $\phi = 0$. Due to the point of inflection at $\phi = 1$ in the odd $n$ case, the kink solution turns out to be stable. The functional form of the kink solution as well as the
quantum mechanical problem for the fluctuation spectrum is the same for the even $n$ case. As mentioned earlier, the potential is bounded for even $n$ and it is possible to write down a supersymmetric extension to the corresponding bosonic field theory model.

2.1 Mass of the Soliton

The mass of the soliton can be calculated from the following expression for the total energy of the field configuration $\bar{\phi}(x)$:

$$E_{\text{sol}} = \int_0^\infty dx \left[\frac{1}{2}(\partial_x \bar{\phi})^2 + V(\bar{\phi})\right]$$

Using the equation of motion $\partial_x \bar{\phi} = \sqrt{2}V(\bar{\phi})$ and substituting from Eq. (3) for $\bar{\phi}(x)$,

$$E_{\text{sol}} = \sqrt{2Aq^{n/2}} \Gamma(n/2 + 1).$$

The ratio of the soliton mass to the tension of the original brane, which is same as the value of the potential at the maximum, is

$$\frac{E_{\text{sol}}}{V_{\text{max}}} = \frac{e^n \Gamma(n/2 + 1)}{2\sqrt{2Aq^{n/2}n^n}}$$

It is straightforward to see that this result agrees with that obtained by Minahan and Zwiebach for $n = 1$, viz. $e/\sqrt{2\pi}$ (with $A = 1/4$ and $q = 2$). For other $n$, the ratio is considerably smaller than 1.

2.2 Coupling to Fermions

In this section, we will study a supersymmetric model with the bosonic potential given in Eq. (1) for $n$ even. Consider a Lagrangian density with a fermion field $\Psi(x)$ coupled to the scalar field $\phi$,

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \gamma^2(\phi) + \bar{\Psi} i \not{\partial} \Psi + \bar{\Psi} \Psi \frac{d\mathcal{V}(\phi)}{d\phi},$$

where

$$\mathcal{V}(\phi) = \sqrt{2V(\phi)} = \sqrt{2Aq^{n/2}} \phi (-\ln \phi)^{n/2}.$$
A classical solution to these equations of motion is obtained by setting the fermion field $\Psi$ to zero and then solving the pure bosonic problem. The equation of motion thus obtained was solved in [16] to get the soliton solution $\vec{\phi}(x)$ of Eq. (3). Within the bosonic sector, the small fluctuation spectrum around the soliton solution was studied in [16]. In particular, the bosonic zero mode was calculated exactly:

$$\psi_0(x) = x^{-n/(n-2)} \exp \left( -\alpha_n x^{-2/(n-2)} \right).$$

(13)

Here we will study the fermionic zero mode in the background of the soliton (3). This is obtained by solving for the zero mode $\Psi^{(0)}$ of the spinor equation with a potential $U'(\vec{\phi})$. Since the equation is linear in $\Psi$, the zero mode will essentially satisfy the same equation of motion. The two components $\Psi^{(0)}_{1,2}(x)$ of the zero mode satisfy the equations

$$\mp \partial_x \Psi^{(0)}_{1,2} + V'(\vec{\phi})\Psi^{(0)}_{1,2} = 0,$$

(14)

the general solutions to which are

$$\Psi^{(0)}_{1,2}(x) = x^{\mp \frac{n}{(n-2)}} \exp \left( \mp \alpha_n x^{-\frac{2}{n-2}} \right).$$

(15)

Of these two solutions, only $\Psi^{(0)}_1$, corresponding to the ‘−’ sign, is well behaved when $x \to 0$ and $x \to \infty$ as can be checked easily. So the normalizable fermionic zero mode is given by

$$\Psi^{(0)}_1(x) = \begin{cases} 
\exp \left[ e^{-\alpha_2 x} - \alpha_2 x \right] & n = 2 \\
\exp \left[ -\alpha_n x^{-\frac{2}{n-2}} \right] & n > 2 
\end{cases},$$

$$\Psi^{(0)}_2(x) = 0.$$  \hspace{1cm} (16)

Thus we can see that the soliton breaks half the supersymmetry of the original model. Notice, the wavefunction $\Psi^{(0)}_1(x)$ is identical to the bosonic zero mode (13). This is expected since $\Psi$ is a superpartner of $\phi$. Hence, the quantum mechanical problem governing the two is identical. We will encounter this quantum mechanical problem again when we discuss gauge field couplings.

### 3 Coupling Gauge Fields

Minahan and Zwiebach [17] have considered two possible ways of introducing gauge fields in these kinds of models. We consider both of them below and study fluctuations of the gauge field about the brane (soliton).

We first consider the Born-Infeld action for a D-p-brane coupled to a gauge field $A_\mu$:

$$S_{BI} = -\mathcal{T} \int dt d^{p+1} x \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})},$$

(17)
where $V(T)$ is the potential for the tachyon field $T(x)$. We are studying tachyon condensation only along one direction, say $x$ (all other $p$ directions will be referred to as $y$), and its potential is derived from our field theory potential of Eq. (1). Consider a field redefinition (as in [16] following [10])

$$\phi = e^{-T} \Rightarrow V(T) = A q^n T^n e^{-2T}. \quad (18)$$

Since we wish to study small fluctuations of the gauge field around the soliton solution, we can make the approximation

$$\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu})} \approx 1 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \cdots, \quad (19)$$

and retain terms up to quadratic order. The action involving the gauge and tachyon fields is then given by

$$S = -\frac{1}{2} \int dt d^p x \ e^{-2T} \left[ \partial_{\mu} T \partial^{\mu} T + A q^n T^n \left( 1 + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right]. \quad (20)$$

We choose the axial gauge condition, i.e. $A_x = 0$. This is equivalent to setting $F_{x\bar{\mu}} = \partial_x A_{\bar{\mu}}$, where $\bar{\mu}$ represents the coordinates other than $x$. To calculate the spectrum of gauge fluctuations about the kink, we look at the gauge part of the action with $T(x) = \tilde{T}(x) = -\ln \phi$. At this point, it is useful to make a field redefinition for the gauge fields:

$$B_{\bar{\mu}} \equiv \sqrt{A q^n / 2 e^{-\tilde{T}} T^n / A}, \quad (21)$$

$$F_{\mu\bar{\nu}} F^{\mu\bar{\nu}} = \partial_{\mu} B_{\bar{\nu}} - \partial_{\bar{\nu}} B_{\mu}. \quad (22)$$

With this field redefinition and substitution of $T(x) = \tilde{T}(x)$ in the action for the gauge field gives

$$S(\tilde{T}, B(A)) = -\int dt d^p y dt \left[ \frac{1}{4} F_{\mu\bar{\nu}} F^{\mu\bar{\nu}} + B_{\bar{\mu}} O(x) B^\bar{\mu} \right] \quad (23)$$

where $O(x)$ is the Schrödinger operator governing gauge field fluctuations in the soliton background. It can be written in terms of $\tilde{T}$ as

$$O(x) = -\frac{\partial^2}{\partial x^2} + U(x)$$

$$= -\frac{\partial^2}{\partial x^2} + \tilde{T}^n \left( \frac{n}{2} \tilde{T}^{-1} - 1 \right) + (\tilde{T}^n)^2 \left( \frac{n}{2} \tilde{T}^{-1} - 1 \right)^2 - \frac{n}{2} (\tilde{T}^n)^2. \quad (24)$$

It is clear from the field redefinition (18) that

$$\tilde{T}(x) = \alpha_n x^{-\frac{2}{n-2}}. \quad (25)$$
Substituting this in (24), we find that the potential $U(x)$ in the Schrödinger operator is the same as that for the scalar fluctuations about the kink solution obtained in [16]:

$$U(x) = \frac{2}{(n-2)^2} \left[ 2\alpha_n \left( \frac{1}{x^2} \right)^{n/(n-2)} - 3n\alpha_n \left( \frac{1}{x^2} \right)^{(n-1)/(n-2)} + n(n-1) \frac{1}{x^2} \right]. \quad (26)$$

We saw in [16] that this defines a supersymmetric (SUSY) quantum mechanics problem [18] with superpotential given by

$$W(x) = \frac{1}{n-2} \left[ n - \frac{2\alpha_n}{x^{n/(n-2)}} \right], \quad (27)$$

whose (massless) ground state can be calculated exactly. The higher excited states can be calculated algebraically using SUSY techniques if one can solve the Ricatti equation

$$W'^2 + W' = \tilde{W}'^2 - \tilde{W}' + R \quad (28)$$

where $\tilde{W}$ is another superpotential and $R$ is some constant [18]. In our case, we find $\tilde{W} = -W$ and $R = 0$, which implies that the ground state of the problem with potential $\tilde{U} = \tilde{W}'^2 - \tilde{W}'$ is not normalizable. Thus it supports no bound state, and the entire spectrum is continuous. Using methods of SUSY quantum mechanics we can relate the spectrum of $U$ with that of $\tilde{U}$. The continuous spectrum of $\tilde{U}$ implies that for the potential $U$ of Eq. (26), there exists only one bound state and that is the zero-energy ground state calculated in [16]. This fact is also clear from the shape of the potential (see Fig.(2)), which approaches zero for large $x$. Using SUSY, we also conclude that there is only one normalizable fermionic mode.
A simpler gauge coupling than the Born-Infeld kind has also been considered in [17]. The action looks like

\[ S' = -\frac{1}{4} \int dt d^{p+1}x \ e^{-2\bar{T}} F_{\mu\nu} F^{\mu\nu}. \]

(29)

Once again choosing the axial gauge and using the following field redefinition

\[ B_{\bar{\mu}} = e^{-\bar{T}} A_{\bar{\mu}}, \]

(30)

we find that the gauge fluctuations are governed by a Schrödinger operator

\[ \mathcal{O}(x) = -\frac{\partial^2}{\partial x^2} + (\bar{T}')^2 - 2\bar{T}'' \]

(31)

The zero energy ground state eigenfunction of this operator can be expressed in terms of \( \bar{T} \). Substituting the explicit form of \( \bar{T} \) from Eq. (25), it is evident that this eigenfunction is not normalizable. Thus we find that gauge coupling of this type does not lead to normalizable gauge fluctuations about the soliton. Apart from the numerical factors the profile of the gauge zero mode is identical to that of the kink, which means the zero mode fluctuations grow as we go away from the core of the soliton. This feature is contrary to that of the D-branes where one expects the gauge zero mode to be localized near the core of the soliton. Thus this coupling is less suitable compared to the Born-Infeld coupling.

4 Conclusions

This paper is a continuation along the lines of investigation started in [16] where we studied soliton solutions for a class of scalar field theory models with potentials that are not bounded below. They correspond to odd values of a parameter \( n \). In this paper we study the same for potentials that are bounded below, corresponding to even values of \( n \).

We calculate the mass of the soliton and compare it to the tension of the original D-brane. This calculation is valid for both odd and even values of \( n \). Our formula reproduces the result of Minahan and Zwiebach [14] for \( n = 1 \). As \( n \) increases, the mass decreases and the ratio of mass to D-brane tension becomes considerably lesser than 1.

The models with \( n \) even can be supersymmetrized, and accordingly, we add a fermionic field to the model and study its fluctuations about the soliton. We find one normalizable zero mode and show that it breaks half the supersymmetry. The wavefunction for this mode is identical to that of the scalar zero mode.

We then study two types of gauge couplings following the proposals of Minahan and Zwiebach [17]. This analysis is also valid for any value of \( n \), even or odd. We find that for a Born-Infeld type of coupling, the gauge fluctuations have only one
normalizable mode, *viz.* the zero mode. In fact, the quantum mechanical problem governing the gauge fluctuations in this case turns out to be the same as that for the scalar as well as the fermionic fluctuations about the soliton. Thus, the spectrum of fluctuations in all these fields have only one normalizable mode, which has zero energy. The rest of the fluctuation modes have continuous spectrum.

We show that the other kind of gauge coupling has no normalizable fluctuations about the soliton. The zero mode, which is also not normalizable, grows away from the core of the soliton. This is contrary to what we know about the D-branes and tachyon condensation. Even within field theory, growth of gauge fluctuations away from the core of the soliton is counterintuitive.

Comparing contrasting features of these two possible gauge couplings, especially the behaviour of zero modes, we conclude that the Born-Infeld type gauge coupling is more appropriate.

With the Born-Infeld type coupling of the gauge field to the scalar field and consequently to the soliton, it is then straightforward to obtain lower dimensional branes as well as intersecting brane configurations, using techniques studied in [19], i.e., by turning on an electromagnetic field in different directions.

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