Quantum entanglement entropy and classical mutual information in long-range harmonic oscillators

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We study different aspects of quantum von Neumann and Rényi entanglement entropy of one dimensional long-range harmonic oscillators that can be described by well-defined non-local field theories. We show that the entanglement entropy of one interval with respect to the rest changes logarithmically with the number of oscillators inside the subsystem. This is true also in the presence of different boundary conditions. We show that the coefficients of the logarithms coming from different boundary conditions can be reduced to just two different universal coefficients. We also study the effect of the mass and temperature on the entanglement entropy of the system in different situations. The universality of our results is also confirmed by changing different parameters in the coupled harmonic oscillators. We also show that more general interactions coming from general singular Toeplitz matrices can be decomposed to our long-range harmonic oscillators. Despite the long-range nature of the couplings we show that the area law is valid in two dimensions and the universal logarithmic terms appear if we consider subregions with sharp corners. Finally we study analytically different aspects of the mutual information such as its logarithmic dependence to the subsystem, effect of mass and influence of the boundary. We also generalize our results in this case to general singular Toeplitz matrices and higher dimensions.

I. INTRODUCTION

Quantum entanglement entropy as an interesting quantity in many body systems has been studied in many different locally interacting systems by using different techniques, see the reviews\textsuperscript{1–6} and references therein. Among the most important results (which are related to this work) one can list the classical result of Bombelli, et.al\textsuperscript{7} which they compute the entanglement entropy of free field theory by using the discrete version of the field theory which is simply coupled harmonic oscillators. The result was rediscovered in\textsuperscript{8} and used to introduce the area law. In\textsuperscript{9} the result was generalized to the Rényi entropy and the validity of the replica trick is checked. This method is also used to study free fermionic systems in a series of papers by Peschel and collaborators in\textsuperscript{6,10}. The techniques used in these works are applicable in any dimension. In two dimensions for the short-range interacting systems one can also derive exact formulas for the entanglement entropy using Euclidean methods\textsuperscript{11}. In the special cases when one have integrable models one can use the form factor techniques and calculate the entanglement entropy\textsuperscript{11}. Finally at the quantum critical point when we have conformal field theory many explicit results are known, see\textsuperscript{4} and references therein.

Although in short-range interacting systems numerous results has been discovered in last ten years there are just few results concerning long-range interacting systems. The main difficulty is the lack of exact solution in most of this kind of systems. The entanglement entropy in Lipkin-Meshkov-Glick (LMG) model which in that all spins interact among themselves is studied numerically and analytically in\textsuperscript{12,13}. In\textsuperscript{14}, the static and the dynamical properties of the entanglement entropy is

Contents

I. Introduction 1

II. Definitions and settings 2

III. von Neumann and Rényi entanglement entropy 3
   A. Hamiltonian approach 3
   B. Numerical evaluation 6
   C. Finite-size effects 9
   D. Massive LRHO 11
   E. Finite temperature 11
   F. Universality 13
      1. Long-range HO in the presence of short-range HO 13
      2. Generalization to singular Toeplitz matrices 14
   G. Two dimensions: area law and logarithmic term for polygonal region 15

IV. Mutual information 15
   A. Finite systems 17
   B. Massive systems 18
   C. Generalized singular Toeplitz matrices 19
   D. Two dimensions: area law and logarithmic term for polygonal region 19

V. Conclusions and Discussions 19

Acknowledgments 20

A. Appendix: Fisher-Hartwig theorem 20

References 21
studied in a long-range Ising type model without an external magnetic field. In the same direction the entanglement entropy is also calculated numerically for the anti-ferromagnetic long-range Ising chain in\textsuperscript{15}. In an interesting work a logarithmically divergent geometric entropy is found in free fermions with long-range unshielded Coulomb interaction in\textsuperscript{16}. Plenio, et. al\textsuperscript{17-19}, see also\textsuperscript{20,21} studied the general properties of the entanglement entropy for coupled harmonic oscillators and found an interesting bounds for the entanglement entropy. Finally using the matrix product states it was argued in\textsuperscript{22} that for those long-range systems that one can not approximate the ground state of the model with the ground state of another short range model, we expect larger entanglement. One can find some other results concerning entanglement entropy in long-range systems in\textsuperscript{23}. Recently using the methods of\textsuperscript{24} we studied the entanglement entropy of block of long-range coupled harmonic oscillators\textsuperscript{25}. We showed that the entanglement of the gapless system is logarithmically dependent to the system size and we calculated the prefactor of the logarithm in different situations. The idea of studying this particular non-local system is manyfold: firstly the hamiltonian\textsuperscript{11} that we are going to study is a simple discretization of fractional laplacian and so it has a very clean continuum limit. This is useful because then we can claim that we are actually studying the entanglement entropy of a non-local field theory. This field theory is a well-known field theory which also appears in the study of long-range Ising model\textsuperscript{26} so in principle any analytical understanding of the entanglement entropy of long-range Ising model will be based on the system that we are studying. Having the above motivations in mind we extended our study in many different directions.

The organization of the paper is as follows: in section two we present the model and give the definitions of the quantities that we are going to study. In section three we study different aspects of von Neumann and Rényi entanglement entropy in long-range harmonic oscillators. We first summarize the main formulas that we need to calculate the entanglement entropy. Most of the formulas are in the discrete level but we also provide the eigenvalue problems in the continuum limit. Then we study the entanglement entropy numerically both at the purely discrete level and also at the level of discretization of the eigenvalue problem. This part of the paper is the extension of the work done in\textsuperscript{22}. Then we study the finite size effects in different kind of situations such as, periodic boundary conditions and Dirichlet boundary conditions. Then we compare the results with the massive coupled long-range oscillators. After that we study the effect of temperature on the entanglement entropy of our system. Our main result will be presented at the end of this section which concerns the universality of our results. In this section we will show that the results presented in the previous sections are robust against many small changes in the form of the interaction. We will also show that one can calculate the entanglement entropy of larger set of coupled oscillators, to be specific oscillators coupled with singular Toeplitz interactions, to the cases that we studied in previous sections. we will conclude this section with some comments about the entanglement entropy in higher dimensions especially in the presence of polygonal regions. Finally in section three we will study different aspects of the classical mutual information in long-range harmonic oscillators. We will presents two definitions and then using Fisher-Hartwig theorem we will show that in contrast to the von Neumann entanglement entropy one can actually analytically calculate these quantities. In this section we also address the finite size effects and also the massive case. The generalization to the singular Toeplitz matrices will be also discussed.

II. DEFINITIONS AND SETTINGS

We start by describing the coupled harmonic oscillators, the perhaps simplest lattice model available to the research where the hamiltonian is a quadratic form:

\[
\mathcal{H} = \frac{1}{2} \sum_{n=1}^{N} \pi_n^2 + \frac{1}{2} \sum_{n,n' = 1}^{N} \phi_n K_{nn'} \phi_{n'}.
\]

We would like to study coupled harmonic oscillators with long-range interaction. To define the $K$ matrix for the long-range harmonic oscillator problem one can use the fractional operator. In principle there are many ways to write a long-range $K$ matrix, however, we are interested in those that they have a very simple continuum counterpart. In principle in the continuum the fractional laplacian is usually defined by its Fourier transform $|q|^\alpha$ or $(q^2)^{\frac{\alpha}{2}}$ which $q^2$ is just the Fourier transform of a simple laplacian. Since the fourier transform of the discrete laplacian is $2 - 2 \cos q$ one may use some powers of this to define the discrete fractional laplacian. Then the elements of the matrix $K$, representing the discretized fractional Laplacian, are

\[
K_{l,m} = -\int_0^{2\pi} dq \frac{i q^{l-m}}{2\pi} \{ \frac{|2(1 - \cos q)|^{\frac{\alpha}{2}}}{\sin(\frac{\alpha}{2})} + M^\alpha \}
\]

\[
= \frac{\Gamma(-\frac{\alpha}{2} + n) \Gamma(\alpha + 1)}{\pi \Gamma(1 + \frac{\alpha}{2} + n)} \sin(\frac{\alpha}{2}) + M^\alpha \delta_{l,m}
\]

where $n = |l - m|$ and fractional order $\alpha > 0$. In the future $M$ will play the role of the mass of the fractional field theory. In the special case $\alpha = 2$ the $K$ matrix is equal to the simple laplacian. When $\alpha/2$ is an integer the elements $K(n) = (-1)^{n}C_{\alpha,\frac{\alpha}{2}+n}$ for $n \leq \alpha/2$ and $K(n) = 0$ for $n > \alpha/2$, where $C_{\alpha,\frac{\alpha}{2}+n}$ are binomial coefficients\textsuperscript{27}.

For sufficiently large one dimensional system, $K$ and the two point correlator matrices $K^{\pm 1/2}$ are Toeplitz matrices and all off-diagonal elements of them are identical. The elements of $K^{\pm 1/2}$ can be expressed as a Fourier series
The matrix $K^{-1/2}$ corresponds to the spatial correlation of an oscillator system $(\phi_t \phi_m)$, and for the system with periodic boundary condition one can find the spatial correlation length $\xi_s$ as

$$\xi_s^{-1} = -\lim_{n \to \infty} \frac{1}{n} \log |\langle \phi_t \phi_{-n} \rangle|$$

$$= -\lim_{n \to \infty} \frac{1}{n} \log |K^{-1/2}(n)|.$$  \hfill (4)

For the massless system $\xi_{s}^{-1} = \lim_{n \to \infty} \frac{1}{n} \log \left| \frac{\Gamma(n+\alpha/4)\Gamma(1-\alpha/4+n)}{\pi^{1-\alpha/2} n^{\alpha/2}} \sin(\frac{\alpha}{4} \pi) \right| = 0$ and for the massive case $\xi_{s}^{-1} \propto M$. We note that for $M = 0$, the correlation length $\xi_s$ is infinite and the system is gapless, and for non-zero value of $M$ the system is gapped.

The $K$ matrix in the continuum limit has the following form:

$$\frac{1}{2} \sum_{n,n'=1}^{N} \phi_n K_{nn'} \phi_{n'} \to \int \{ -\frac{1}{2} \phi(x) (-\nabla)^{\alpha/2} \phi(x) + \frac{1}{2} M^{\alpha} \phi^2(x) \} dx,$$ \hfill (5)

where $-(\nabla)^{\alpha/2}$ is defined by its Fourier transform $|q|^\alpha$.

We are now in a position to introduce the entanglement entropy and it’s value in two dimensional CFT’s. Here, we shall only discuss the von Neumann and Rényi entanglement entropies. Nevertheless, there are many other measures that have been explored.

Consider a system with the density matrix of a pure state $\rho$, which is divided into two subsystems $A$ and $B$. Then the entanglement may be characterized by the properties of the reduced density matrix $\rho_A$ of the subsystem $A$. Density matrix $\rho_A$ is obtained by tracing out the remaining degrees of freedom $\rho_A = \text{tr}_B \rho$. The von Neumann entanglement entropy associated to the local density matrix $\rho_A$ reduced to a region $A$ of the space is

$$S(A) = -\text{tr}(\rho_A \log(\rho_A)) .$$ \hfill (6)

Another related measure to the local density matrix, is a family of functions called the Rényi entropies,

$$S_n(A) = \frac{1}{1-n} \log(\text{tr} \rho_A^n), \quad n \geq 0, \quad n \neq 1 .$$ \hfill (7)

The Rényi entropy $S_n$ has similar properties as the entanglement entropy $S$.

For general quantum field theories in $d$ spatial dimensions the entanglement entropy is always divergent in a continuum system and the coefficient of the leading divergence term is proportional to the area of the boundary of the subsystem $A$ and it is given by the simple formula

$$S(A) = g_d^{-1} \left( \frac{l}{\epsilon} \right)^{d-1} + \cdots + g_1 \left( \frac{l}{\epsilon} \right)^1 + g_0 \log(l/\epsilon) + S_0(A) ,$$ \hfill (8)

where $\{g_d,\ldots,g_1\}$ and $S_0$ are non-universal constants which depend on the system. The coefficient $g_0$ of the log term is expected to be universal and $l^d$ is the volume in $d$ dimensional space and $\epsilon$ is a short distance cutoff (or a lattice spacing). The simple area law, however, can not describe the scaling of the entanglement entropy in generic cases. Indeed the entanglement entropy of conformal field theory in one special dimension, scales logarithmically with respect to the size of the subsystem $A$. If the total system is infinitely long, it is given by the simple formula

$$S = \frac{c}{3} \log \frac{l}{\epsilon} ,$$ \hfill (9)

where $c$ is the central charge of the CFT. In $1+1$ dimensional conformal invariant systems the Rényi entropy follows

$$S_n = \frac{c}{6} (1 + \frac{1}{n}) \log \frac{l}{\epsilon} .$$ \hfill (10)

It is also worth mentioning that for a finite system of length $L$ with boundary, at zero temperature and one special dimension, divided into two pieces of lengths $l$ and $L-l$, the Rényi entropy obeys

$$S_n = \frac{c}{12} (1 + \frac{1}{n}) \log((L/\pi a) \sin(\pi l/L)) + c' .$$ \hfill (11)

The above formulas are a few among many others that are known for different cases in two dimensional CFT’s, see. In the next sections we will introduce many of them as the limiting behavior of our long-range harmonic oscillators.

In the next section we will review a method where one can use it to calculate $\rho_A$ and consequently $S$ and $S_n$ for generic quadratic bosonic systems. Then we will hire this technique to study our particular long-range system.

## III. VON NEUMANN AND RÉNYI ENTANGLEMENT ENTROPY

### A. Hamiltonian approach

A useful method to obtain entanglement entropy is introduced in\textsuperscript{2,5} and rediscovered in\textsuperscript{5} and generalized to Rényi entropy in\textsuperscript{5}. In this method one would like to measure the quantum entanglement entropy of the ground state of the free field $\{\phi\}$, generated by tracing over fields inside of the region of the boundary surface. To fix the
notation and for the later use we give here a brief summary of the work described in more detail in the Ref.\cite{26,27}.

The ground state wave functional is given by

$$\Psi_0(\{\phi\}) \propto (\det\Gamma)^{1/2} \exp\left\{-\sum_{n,n'=1}^N \phi_n \Gamma_{nn'} \phi_{n'}\right\}. \quad (12)$$

where \(\{\phi\}\) denotes the collection of all \(\phi\)'s, one for each oscillator and \(\Gamma = K^{1/2}\).

Now consider a subregion in the total space and split the field variables into inside \((\{\phi\}_A)\) and outside \((\{\phi\}_B)\) parts, then one can rewrite the ground state wave function as

$$\Psi_0 \propto \exp\{-\langle\{\phi\}_A,\{\phi\}_B\rangle\} \left(\begin{array}{cc} \Gamma_{AA} & \Gamma_{AB} \\ \Gamma_{BA} & \Gamma_{BB} \end{array}\right) \left(\begin{array}{c} \{\phi\}_A \\ \{\phi\}_B \end{array}\right), \quad (13)$$

where \(\Gamma_{\boxplus\oplus} = \{A,B\}\) and \(\oplus = \{A,B\}\) denotes the kernel matrix restricted to the inside or the outside.

For the fields \(\{\phi^1,2\}_A\) which are defined in the inside region, the ground state density matrix \(\rho_A(\{\phi^1\}_A,\{\phi^2\}_A)\), is given by

$$\rho_A(\{\phi^1\}_A;\{\phi^2\}_A) \propto (\det(\Gamma_{AA}))^{-1/2} \exp\left\{-\frac{1}{2} \times \right\}

\left(\begin{array}{c} \{\phi^1\}_A \\ \{\phi^2\}_A \end{array}\right) \left(\begin{array}{cc} A & 2B \\ 2B & A \end{array}\right) \left(\begin{array}{c} \{\phi^1\}_A \\ \{\phi^2\}_A \end{array}\right), \quad (14)$$

where

$$A = 2(\Gamma_{AA} - \frac{1}{2} \Gamma_{AB}(\Gamma_{BB})^{-1}\Gamma_{BA});$$

$$B = -\frac{1}{2} \Gamma_{AB}(\Gamma_{BB})^{-1}\Gamma_{BA}. \quad (15)$$

From now on one can follow two different methods to get the entanglement entropy: one is based on direct diagonalization of the above reduced density matrix and the other based on using replica trick. For later use we will summarize the results for both of them. Using appropriate transformations one can write the reduced density matrix as

$$\rho_A(\{\phi^1\}_A;\{\phi^2\}_A) = \prod_{n=1}^N \frac{1}{\sqrt{\pi}} \times \exp\left\{-\frac{1}{2} \left(\phi_n^1 \phi_n^1 + \phi_n^2 \phi_n^2\right) - \frac{1}{4} E(n) \phi_n^1 - \frac{1}{2} E(n) \phi_n^2\right\} \quad (16)$$

where \(E_i\)'s are the eigenvalues of the matrix \(A\) with the following simple form

$$\Lambda = -\left(\Gamma^{-1}\right)_{AB} \Gamma_{BA}. \quad (17)$$

The interesting point about the equation (15) is that it has the form of the reduced density matrix of two body harmonic oscillator. In other words for the ground state of coupled harmonic oscillator the problem of calculating the entanglement entropy can be reduced to the problem of calculating the entanglement entropy of two coupled harmonic oscillators. One can then show that the entropy can be expressed in terms of the eigenvalues \(E_i\) of \(A\) as:

$$S = \sum_i \left[ \log \frac{\sqrt{E_i}}{2} + \sqrt{1+E_i} \log \left( \frac{1}{\sqrt{E_i}} + \sqrt{1+1/E_i} \right) \right]. \quad (18)$$

It is worth mentioning that having larger coupling between two oscillators leads to larger \(E\) and consequently larger entanglement entropy.

The second method which is also useful to get the Rényi entropy is based on Replica trick. Using Eq. (14), and rescaling the reduced density matrix one can calculate \(\text{tr} \rho^\alpha \) and ultimatley the entropy\(^2\) as following sum

$$S = \lim_{n \to 1} \left( \frac{1}{1-n} \log(\text{tr} \rho^n) \right) = -\sum_{i=1}^N \left\{ \ln(1-\xi_i) + \frac{\xi_i}{1-\xi_i} \ln \xi_i \right\}, \quad (19)$$

where \(\xi_i\) is related to the eigenvalue of the matrix \(C = -2A^{-1}B\) by \(C_i = \frac{2E_i}{1+\xi_i} \).

It is also useful to consider the matrix \(\Lambda = (1-P)^{-1}P\) where \(P = \Gamma^{-1} A \Gamma_{AB}^{-1} \Gamma_{BA}\) which has also the simple form (17) and write Eq. (19) in terms of eigenvalues of the matrix \(\Lambda\) as (18). The eigenvalues \(E_i\) of the matrix \(\Lambda\) are positive and related to \(\xi_i\) by

$$\xi_i = \frac{\sqrt{1+E_i} - 1}{\sqrt{1+E_i} + 1}. \quad (20)$$

It is also straightforward to write the Rényi entropy \(S_n\) in term of \(\xi_i\) as:

$$S_n = \frac{1}{n-1} \sum_i \left( \log(1-\xi_i^n) - n \log(1-\xi_i) \right). \quad (21)$$

In order to compute the entanglement entropy obtained by tracing over the fields in the region \(B\) for a given problem, one should find the eigenvalues of the matrix \(\Lambda\). For a given hamiltonian \(H\) one can easily find the operators \(K\) and consequently \(\Gamma\) and \(\Gamma^{-1}\). In the continuum limit, the operator \(\Lambda\) is obtained after integration over the oscillators in the region \(B\) as

$$\Lambda(x,y) = -\int_B \int_B \partial z \Gamma^{-1}(x,z) \Gamma(z,y). \quad (22)$$

The eigenvalue problem to be solved is then

$$\int dy \Lambda(x,y) \psi(y) = E \psi(x), \quad (23)$$

where \(\psi(x)\) is an eigenfunction with eigenvalue \(E\).

It is worth mentioning that for the general Hamiltonian Eq. (1), one can calculate the two point correlators \(X_A = \text{tr}(\rho_A \phi \phi)\) and \(P_A = \text{tr}(\rho_A \pi \pi)\) using the \(K\) matrix by

$$\frac{1}{2} K^{-1/2} = \begin{pmatrix} X_A & X_{AB} \\ X_{AB}^T & X_B \end{pmatrix}, \quad \frac{1}{2} K^{1/2} = \begin{pmatrix} P_A & P_{AB} \\ P_{AB} & P_B \end{pmatrix}. \quad (24)$$
Then one can define matrix $C = \sqrt{X_A P_A}$, which has the eigenvalues,
\begin{equation}
\nu_i = \coth(-\log((\sqrt{1+E_{i}^2} - 1)/\sqrt{1+E_{i}^2} + 1)) / 2, \quad (25)
\end{equation}
where $\nu_i$ are the eigenvalues of $C$. With respect to the new operators the entropy is given by
\begin{equation}
S = \text{tr} \left[ (C + \frac{1}{2}) \log(C + \frac{1}{2}) - (C - \frac{1}{2}) \log(C - \frac{1}{2}) \right] = \sum_{i=1}^l \left[ (\nu_i + \frac{1}{2}) \log(\nu_i + \frac{1}{2}) - (\nu_i - \frac{1}{2}) \log(\nu_i - \frac{1}{2}) \right]. \quad (26)
\end{equation}
We also have
\begin{equation}
S_n = \frac{1}{n-1} \text{tr} \left[ \log \left( (C + \frac{1}{2})^n - (C - \frac{1}{2})^n \right) \right] = \frac{1}{n-1} \sum_{i=1}^l \left[ (\nu_i + \frac{1}{2})^n - (\nu_i - \frac{1}{2})^n \right]. \quad (27)
\end{equation}
where $l$ is the size of the subsystem $A$. In this formulation we need only the correlators inside the region $A$, to calculate $S$ and $S_n$.

In order to clarify the hamiltonian approach in the continuum limit, we briefly review the procedure followed to find an approximate analytical solution for the harmonic oscillator system with short-range interaction. This method is introduced in order to determine $E$ and also $S$ by using eigenvalue problem Eq. (23). They considered a one dimensional coupled harmonic oscillator with mass $M$, confined to the region $-L < x < L$ and the subsystem is taken to be half of a finite system.

To calculate the eigenvalue $E$, for system of harmonic oscillators with short-range interactions, it is better to first consider a system with infinite size $L \to \infty$. At this limit the kernels $\Gamma^{\pm 1}$ needed to construct $E$ have the following forms:
\begin{equation}
\Gamma(x,y) = MK_1(M(x-y))/\pi(x-y), \quad \Gamma^{-1}(x,y) = K_0(M(x-y))/\pi. \quad (28)
\end{equation}
where $M$ is the mass term. In the $M \to 0$ limit it is easy to show that $\psi = \exp(\omega \ln x)$ is an eigensolution of the Eq. (23) with eigenvalue
\begin{equation}
E = \sinh^{-2}(\pi \omega). \quad (29)
\end{equation}
To discretize the spectrum and calculate the entropy, one needs to impose Dirichlet boundary conditions at some large $x = L$ and further Dirichlet condition at some small $x = \epsilon$. The eigenvalues and eigenvectors are then:
\begin{equation}
\psi(x) = \sin(\omega (L/\epsilon)) \ln(x/\epsilon)), \quad \omega(E) \ln(L/\epsilon) = \pi i. \quad (30)
\end{equation}

It is useful to note that the density of states per unit $\omega$ interval is constant. Now one can rewrite the continuum limit of the Rényi entropy Eq. (21) and the entanglement entropy as an integral over $\omega$,
\begin{equation}
S_n = \frac{\log L}{\pi(n-1)} \int_0^\infty d\omega (\log(1 - \xi^n) - n \log(1 - \xi)), \quad (31)
\end{equation}
where $\xi(\omega)$ is defined in the Eq. (20).

As discussed before, conformal invariance implies universal properties for the entanglement entropy. The entanglement entropy and also the Rényi entropy for these models, diverge logarithmically with the subsystem size with prefactors proportional to $c$ and $c_n$, respectively.

By using the Eq. (22) and also Eqs. (20) and (29) one can find the entanglement entropy $S$ for the harmonic oscillator problem, giving the result $S = \frac{1}{2} \log(L/\epsilon)$ which is consistent with $c = 1$. In addition using (31) one can also find the Rényi entropy $S_n = \frac{1}{n \pi} \ln(L/\epsilon)$ consistent with the CFT predictions.

Next we consider short-range harmonic oscillator with infinite size and sub-system with length $l$. This kind of configuration is completely different from the Ref.2. We should remember that the Eqs. (29) and (30) are no longer true in this configuration. We proceed with the Eq. (22). To evaluate this integral we must consider $B \in (-\infty < z < 0) \cup (l < z < \infty)$ as the complement of the sub-region $l$. The matrix $\Lambda$ becomes:
\begin{equation}
\Lambda(x, y) = -\frac{1}{\pi^2} \int_B \frac{\ln(x+z) - \ln(x-y)}{(z+y)^2} \frac{dz}{y}. \quad (33)
\end{equation}
Therefore, according to Eq. (23), the eigenvalues $E_i$ and the corresponding eigenfunctions $\psi_i(x)$ can be obtained by diagonalizing $\Lambda$ matrix. Unfortunately we were unable to find $E_i$ analytically. One can numerically evaluate $E_i$ and $\psi_i$ using direct diagonalization of the matrix $\Lambda$, then try to guess the formula for eigenvalues and eigenfunctions. We shall come to this problem in the next section by means of numerical calculations.

We are now ready to speak more about the LRHO with $\alpha < 2$. To determine $E$ and $\psi$ for LRHO, we calculated first the matrices $\Gamma = K^{1/2}$ and $\Gamma^{-1} = K^{-1/2}$. The continuum limit of the matrices $\Gamma$ and $\Gamma^{-1}$ has the following forms:
\begin{equation}
\Gamma^{\pm 1}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk |k|^\alpha + M^\alpha)^{\pm 1/2} e^{ik(x-y)} \times 
\end{equation}
\begin{equation}
= \frac{1}{2\Gamma[+\alpha/2]} \frac{1}{|r|^{1+\alpha/2}} \times H_{1,2}^{1,2} \left( |M|r|^\alpha \right) \left( (1,1)(\frac{1}{\alpha},1)(\frac{1}{\alpha},\frac{1}{\alpha}) (\frac{1}{\alpha},1)(\frac{1}{\alpha},\frac{1}{\alpha}) \right) \quad (34)
\end{equation}
\begin{equation}
= \frac{1}{2\Gamma[+\alpha/2]} \cos(\frac{1}{\alpha}) |r|^{1+\alpha/2} + O(M^\alpha), \quad (35)
\end{equation}
where $r = x - y$ and $H_{1,2}^{1,2}$ is the Fox H-Function. Then in a similar way as in Eq. (22), we found the matrix
Λ by multiplying Γ and Γ^{-1} in the complement region $B \in (-\infty < z < 0) \cup (l < z < \infty)$ as

$$\Lambda(x,y) = \mathfrak{A} \frac{2 \left( \left( \frac{z}{\pi} \right)^{\alpha/2} - \left( \frac{y}{\pi} \right)^{\alpha/2} \right)^{\alpha < 2}}{\alpha(x-y)}$$

(35)

where $\mathfrak{A} = \frac{1}{4\Gamma^{1-\alpha/2}[\Gamma(\alpha/2)]^2 \cos^{\alpha/2}(\pi/2)}$. Unfortunately, the exact solution of the eigenvalues and the corresponding eigenfunctions for Eq. (35) are not known and remain an open problem. It is nonetheless both possible and interesting to investigate the properties of the $E$ and $\psi$, for LRHO problem, numerically. In the next section we will discuss our numerical findings.

It is worth mentioning that, the Eq. (35) is only true for an infinitely large system compared to the sub-system size. However one can also study LRHO problem in the presence of boundary but since boundary of the finite system breaks the translational invariance, we have not been able to find $\Gamma \pm 1$ explicitly because we are not allowed to use Fourier transform for the finite systems. Therefore we studied this case just numerically, which we will present the results in the next sections.

As explained above, we can find the eigenvalues $E$ and the corresponding eigenvectors $\psi(x)$ for a given matrix $K$, and we can also study the scaling behaviors of the entanglement entropy $S$ and the Rényi entropy $S_\alpha$.

In the next section we will speak more about our results but here we will discuss about different configurations for the system and also subsystem that we have used in our study. In this work we consider five main kinds of configurations depicted in Fig. (1) for system and subsystem. In the massless case:

- **C$_1$**: System is very large and $A$ is a small sub-system with length $l$.
- **C$_2$**: System with periodic boundary condition has finite size $L$, and $A$ is a sub-system with length $l$.
- **C$_3$**: System with size $L$ has boundary and is divided to two adjacent parts. The first part is a sub-system with length $l < L$ and the second part is the complement with size $L - l$.
- **C$_4$**: System with size $2L$ has boundary and is divided to two adjacent equal intervals with length $l = L$ where one of them is the sub-system.

In the massive case:

- **C$_5$**: System is very large and $A$ is a sub-system with length $l$.

**B. Numerical evaluation**

We now numerically evaluate the von Neumann entanglement entropy $S$ and the Rényi entropy $S_\alpha$ for LRHO problem in different cases ($C_{i,i} = 1 \ldots 5$), by using Eqs. 15 and 21 or equivalently Eqs. 20 and 27 which was first studied in24. In this respect, we follow the method explained in the last section. We will measure the eigenvalues $E_i$ and the eigenfunctions $\psi_i(x)$ in Eq. 20 numerically and then we introduce an expression for $E$ and $\psi$, which matches to the numerical simulations. Our motivation to study these quantities with full detail is related to our interest in better understanding the operator (35) which its eigenvalues provide the entanglement entropy. We should here stress that we calculate the entanglement entropy using the numerical $\Lambda$ matrix and not by discretizing the operator (35). However we will confirm that these two operators are very close to each other if we consider large systems and consequently can approximate each other.

In order to calculate $E$ and $\psi$, we first need to construct the matrix $\Lambda$ for a given $K$ matrix. Numerically one can find the matrix $\Lambda \equiv -\Gamma^{-1}_+ \Gamma^{-1} \Gamma_+$ by multiplying $\Gamma^{-1}$ and $\Gamma$, where $\Gamma = K^{-1/2}$ and $\Gamma^{-1} = K^{-1/2}$. For example we applied this method to the LRHO with very large system size and small sub-region $l$. There is a very good agreement between numerical $\Lambda$ and the matrix $\Lambda(x,y)$ coming from Eqs. (33) and (35), when the distances are more than four lattice sizes.

To obtain a better understanding of the long-range harmonic oscillator problem, we studied first the eigenvalues $E_i$ and the eigenfunctions $\psi_i$ of the short-range harmonic oscillator. We considered a system with size $2L$ and the subsystem is taken to be half of the system size ($C_4$). Then using the numerical methods, we diagonalized the matrix $\Lambda$ to find $E_i$ and $\psi_i$. In Fig. 2 we sketched logarithm of the eigenvalues $E_i$ with respect to $\omega$($E_i$). As can be seen, the result obtained from Eq. 24 has similar asymptotic behavior as numerical simulations. In addition the eigenvectors $\psi(x)$ for the first and second largest eigenvalues, $E_1$ and $E_2$ verify the behavior predicted in Eq. 30. We have also calculated the prefactor $c$ numerically and our result is consistent.
with the theoretical prediction. The numerical results of entanglement entropy for LRHO ($\alpha < 2$) for the systems with boundary e.g., $\mathcal{C}_3$ and $\mathcal{C}_4$, are summarized in the next sections.

Next we discuss the case, where the subsystem is very small with length $l$ and the system is very large ($\mathcal{C}_1$). For this configuration, as a first step, we have studied the properties of $E_i$ and $\psi_i$ for harmonic oscillator problem with short-range interaction by a direct diagonalization of the matrix $\Lambda$. Numerical results are shown in Fig. 3. It is interesting to note that, when we choose $\omega(E_i) = \pi i/2(\log(l) + \zeta)$ ($\zeta = 1.3$), apart from a constant, which it appears ubiquitously in this kind of studies, the behavior of the eigenvalues $E_i$ are in very good agreement with $E(\omega) = 1/\sinh^2(\pi \omega)$ (see Fig. 4). Let us remark that $\omega(E)$ for the configuration $\mathcal{C}_3$, differs from Ref. 2 by a factor two and a constant $\zeta$. We studied the scaling of $S$ versus the logarithm of the sub-system size, $\log l$, and compared with Eq. (10). Our result agrees with $c = 1$.

The next step is to analyze the eigenvalues $E$ of the Eq. (35) for LRHO with $\alpha < 2$. As we remarked before, if we consider very small sub-region of LRHO with $\alpha = 2$ and very large system size ($\mathcal{C}_1$), we expect $E_i \sim \sinh^{-2}(\pi \omega_i)$ and $\omega(E_i) = \pi i/2(\log(l) + \zeta)$. For other values of $\alpha$, the eigenvalues behavior can be seen in the Fig. 1, where we compared $\log(E)$ vs. $\omega$ for various $\alpha$‘s. Let us first address the behavior of small eigenvalues $E_i$ (large $i$, i.e. $i > 3$). Our results show that the small eigenvalues are independent of $\alpha$ and $\log(E_i)$ is linearly dependent to $\omega_i$ by scaling factor $-2\pi$. Then one can get the asymptotic behavior $E_i \propto e^{-2\pi \omega_i}$ for $i > 3$ and from our previous knowledge about $E_i$ for $\alpha = 2$, one can conjecture the simple behavior $E_i \sim \sinh^{-2}(\pi \omega_i)$. In our numerical simulations we used $\omega_i = \pi i/2(\log(l) + \zeta)$, where $\zeta$ is a $\alpha$ dependent parameter ($\zeta \in [1.0, 2.0]$), to get the best fit to numerical data. We may use this behavior and guess the asymptotic expression for the eigenvalue $E$ as

$$E(\omega) = \frac{a(\alpha)}{\sinh^2(\pi \omega) + b(\alpha)}.$$  \hspace{1cm} (36)

The best fit parameters to our numerical data were $a(\alpha) = \frac{3}{2}\sin^2(\frac{\alpha}{2})$ and $b(\alpha) = 0.12\alpha + 0.19\alpha^2 - 0.20\alpha^3 + 0.04\alpha^4$. The value of $b(\alpha)$ is zero at $\alpha = 0$ and $\alpha = 2$ and it has a maximum at $\alpha = 1$.

Next, we studied the eigenvector $\psi_i(x)$ of the matrix $\Lambda$ for LRHO numerically. By diagonalizing $\Lambda$ we can also find the eigenfunctions $\psi_i(x)$ are symmetric around $x = l/2$ for odd $i$, and antisymmetric for even $i$. We found that the best fit to the eigenvectors $\psi_i(x)$ is

$$\psi_i(x) = \frac{1}{\mathcal{N}} \left\{ (x^{\omega_i+b} + x^{-\omega_i+b}) - (-1)^i (l-x)^{\omega_i+b} + (l-x)^{-\omega_i+b} \right\};$$  \hspace{1cm} (37)

where $\mathcal{N}$ is the normalization coefficient and $b \in [-1,0]$.
The eigenfunction $\psi$ for LRHO ($\alpha E$) is the free parameter to get the best fit to the numerical $\psi$ Fig. (5) the behavior of the $\psi$ Eq. (23) for LRHO, in order to find the eigenfunction as a function of $x$. Possible to find $S$ and also $S_n$ using Eq. (26) and Eq. (27), respectively.

Finally, we discuss the goodness of Eq. (36). For arbitrary values of the long-range interaction $\alpha$, the von Neumann entanglement entropy $S$ and the Rényi entropy $S_n$ can in practice be obtained by (i) evaluating Eq. (2) numerically for system with total size $L$ and compute $X_A$ and $P_A$ from $K$ matrix (see Eq. (24)), (ii) diagonalizing $C$ to obtain $\nu_i$, and (iii) evaluating (29) and (27), where $l$ is the number of lattice sites in the subsystem $A$.

We observe that, in the LRHO problem the entanglement entropy and the Rényi entropy increase logarithmically with the sub-system size as

$$S \sim \frac{\tilde{c}(\alpha)}{3} \log l,$$

and

$$S_n \sim \frac{\tilde{c}_n(\alpha)}{3} \log l,$$

respectively. By studying the scaling behavior of $S$ and also $S_n$, vs. $\log l$, one can find the scaling parameters $\tilde{c}(\alpha)$ and $\tilde{c}_n(\alpha)$. We display the resulting quantities for different values of $\alpha$ and $n$, in Fig. (6).

For arbitrary values of $\alpha$ and $n$, according to Eqs. (33) and (32) and also Eq. (29), one can find the prefactors $\tilde{c}(\alpha)$ and $\tilde{c}_n(\alpha)$. We have depicted the results coming from these formulas in Fig. (6), and we found perfect agreement between our results, confirming the validity of the Eq. (36). There are some comments in order: the fact that the coefficient of the logarithm is an increasing function of $\alpha$ is somehow counter intuitive because we know that for bigger $\alpha$’s the interaction get weaker by the distance faster than the smaller $\alpha$’s. There are some ways to roughly understand this result: from mathematical point of view one might argue that the entanglement entropy is actually related to the eigenvalues of the matrix $A$ and those eigenvalues are smaller when we take smaller $\alpha$’s. This can be seen easily by looking to the equation (35). These eigenvalues are also the parameters that appear after mapping the many body harmonic oscillator to the two body case in equation (14). Stronger couplings between two oscillators leads to bigger $E$ and consequently bigger entanglement among them. The fact that after diagonalization we have smaller $E_i$’s for smaller $\alpha$’s shows that although the interactions between oscillators far from each other is much stronger for smaller $\alpha$’s, that still does not guaranty bigger entanglement entropy. One might understand this phenomena as follows: based on the equation (2) in the range $0 < \alpha < 2$ one can see that $K(1)$, which is related to the nearest neighbor interaction, is an increasing function with respect to $\alpha$ but $K(n)$ with $n > 1$ first increases with $\alpha$ and then decreases. It seems like the value of $E_i$ is mostly dependent on the value of the nearest neighbor interaction and follows the same trend. So although in some range of $\alpha$’s the next nearest neighbor interaction for bigger $\alpha$ is smaller the entanglement after considering the nearest

![Graph](image-url)

Fig. 5: (Color online) Top: The eigenvalues log($E_i$) versus $\omega_i$ for LRHO ($\alpha = 1$) with the configuration $C_1$. The blue stars correspond to $E = \frac{a(1)}{\sinh(\pi \omega_1) + i b(1)}$, where $\omega = n \pi/2(\log(l) + \zeta)$ ($\zeta = 1.26$) and $a(1) = 0.25$ and also $b(1) = 0.14$. Middle: The eigenfunction $\psi_1(x)$ corresponds to the first eigenvalue $E_1$. Bottom: The eigenfunction $\psi_2(x)$ corresponds to $E_2$. Solid red lines correspond to normalized form of equation (37) ($b = -0.26, -0.34$ for $n = 1, 2$ respectively).
In conformal invariant systems neighbor interaction is bigger. This also explain qualitatively why we get an increasing function of $a(\alpha)$ in the equation \((40)\). This reasoning is consistent with the area law observation in the massive case and also higher dimensions that we are going to discuss later.

We now turn to determine the behavior of $\tilde{c}_n(\alpha)$ with respect to $n$. Interestingly, we find that the best fit to $\tilde{c}_n$ is

$$\tilde{c}_n(\alpha) = \frac{\tilde{c}(\alpha)}{2}(A(\alpha) + B(\alpha) \frac{1}{n}), \quad (40)$$

The coefficients $A(\alpha)$ and $B(\alpha)$ are functions of $\alpha$ (see Fig. \(\bullet\)), which indicates that LRHO is not conformally invariant(notice that by definition $A(\alpha) + B(\alpha) = 2$). In conformal invariant systems $\tilde{c}_n = \frac{\tilde{c}(\alpha)}{2}(1 + \frac{1}{n})$, where $c$ is the central charge of the system. At this point it is worth mentioning that one can also calculate single copy entanglement introduced in\(^{29}\). Since this quantity is equivalent to the Rényi entropy with $n \to \infty$, see\(^{30}\) we get simply the result $S_{\infty} = \frac{\tilde{c}(\alpha)}{6} A(\alpha)$ which shows that in this case in contrast to the short-range case the single copy entanglement is not just half of the von Neumann entanglement entropy.

FIG. 6: (Color online) Top: The prefactor $\tilde{c}(\alpha)$ for discrete system with size $L = 6000$ with the configuration $\mathcal{C}_4$. The prefactor is measured using the scaling relation $S$ with $\log l$ in the region $0 < l < L/100$. The red line represents the same quantity coming from the continuum limit approximation. Bottom: $\tilde{c}_n(\alpha)$ vs. $n$ for different $\alpha$’s (from top to bottom: $\alpha = 2.0, 1.8, 1.6, 1.4$). The red lines come from the continuum limit approximation. Inset: $A$ and $B$ coefficients vs. $\alpha$.

In the next subsection, we will report the results of LRHO in the case of a system which has a finite size and also we will report the effect of boundary on the entanglement entropy.

C. Finite-size effects

Until now to avoid any finite size effect, we concentrated on very large system size $L \to \infty$ and small subsystem size $l$ (configuration $\mathcal{C}_4$). As mentioned previously, the entanglement entropy $S$ and the Rényi entropy $S_n$, scale logarithmically with the size of the subregion $l$ ($l \ll L$). However, from the numerical computation of $\tilde{c}_n(\alpha)$, we argued that the LRHO is not conformally invariant except at $\alpha = 2$.

We shall now present a computation of the entanglement entropy for systems with finite size. Conformal field theory (CFT) predicts\(^{31}\) following formulas for the Rényi entropy and the von Neumann entropy of conformally invariant systems with periodic BC’s:

$$S^{CFT}(L, l) = \frac{c}{3} \log \left[ \frac{L}{\pi \sin \left( \frac{\pi l}{L} \right)} \right] + c', \quad (41)$$

$$S_n^{CFT}(L, l) = \frac{c}{6} \frac{1}{n} \log \left[ \frac{L}{\pi \sin \left( \frac{\pi l}{L} \right)} \right] + c'_n, \quad (42)$$

where $c$ is the central charge and $c'$ and $c'_n$ are non-universal constants. Note that Eqs. \((41)\) and \((42)\) are symmetric under $l \to L - l$, and they are maximal when $l = L/2$. For infinite system size $L \to \infty$ and also the finite one with the condition $l \ll L$ the entanglement entropy scales like Eq. \((41)\). Notice that the Eqs. \((41)\) and \((42)\) are only true for conformally invariant systems and we expect different function in our system.

Here we will discuss the effect of boundary on the entanglement and Rényi entropies of the LRHO problem. We are interested to study the case, which we take a finite
system with half of it as the sub-system. We considered a system with total size $2L$, and the sub-system size $L$ ($\mathcal{C}_4$). The important subtlety here is the definition of the $K$ matrix. Since we have a finite system the fractional laplacian cannot be easily defined by its Fourier transform (for more details see\textsuperscript{26}). One way to define the fractional laplacian is based on non-local integrals in bounded domain\textsuperscript{23}. Although this approach is precise it is difficult to use it in discrete level for numerical evaluations. We will follow the simpler path the so called absorbing boundary condition considered in\textsuperscript{26}.

The main difference between $K$ matrix in the finite system with boundary and the infinite one defined in\textsuperscript{26} is that, the $K$ matrix for the system with boundary is defined by throwing away the elements of the infinite matrix which are in the outside of the system.

Let us now consider the $\Lambda$ matrix and its eigenvalues $E$ for the configuration $\mathcal{C}_4$. For the short range interaction problem ($\alpha = 2.0$) the eigenvalues are described by $E = \sinh^{-2}(\pi \omega)$ with $\omega = n\pi / \log(L)$ (see Eq. (29)). Our calculations for other cases $\alpha < 2$ show that the small eigenvalues are independent of $\alpha$ (see Fig. 7). We found that $E = a(\alpha)/(\sinh^2(\pi \omega) + b(\alpha))$ (see also Eq. (61)) is a good approximation for the eigenvalues of $\Lambda$ with $a(\alpha) = \frac{3}{2} \sin^2(\frac{\pi \alpha}{n})$ and $b(\alpha) = 0.32a - 0.0832 - 0.16a^2 + 0.06a^4$ as the best numerical fit parameters to our data. The parameter $b(\alpha)$ for the configuration $\mathcal{C}_4$ differs from the same quantity for the configuration $\mathcal{C}_3$ except at $\alpha = 2$.

Numerical measurement shows that the entanglement entropy $S$ and the Rényi entropy $S_n$ follow $S \sim \tilde{c}_4^F(\alpha) \log L$ and $S_n \sim \tilde{c}_4^F(\alpha)/n \log L$ respectively, where the indices 4 indicates the case that we study. In Fig. 8 we report the numerically calculated values $\tilde{c}_4^F(\alpha)$ and $\tilde{c}_4^F(\alpha)$ for several values of $\alpha$ and $n$.

These prefactors are generally different from $\tilde{c}(\alpha)$ and $\tilde{c}_n(\alpha)$ except at the point $\alpha = 2$. Finally we found that, $\tilde{c}_4^F(\alpha) = \tilde{c}_3^F(\alpha)(B^F(\alpha)/n)$, is the best fit to $\tilde{c}_4^F(\alpha)$ with respect to $n$ (notice that by definition $A^F(\alpha) + B^F(\alpha) = 2$). The coefficients $A^F$ and also $B^F$ are functions of $\alpha$ (see Fig. 5).

One can do the same calculations also for the configurations $\mathcal{C}_2$ (in this case we considered $l = L/2$ and $S \sim \tilde{c}_F(\alpha) \log L$) and $\mathcal{C}_3$ (where we take $S \sim \tilde{c}_F(\alpha) \log L$). In Fig. 9 we sketched $\tilde{c}_F(\alpha)$. It is clear that the results for different configurations $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are similar. In other words

$$\tilde{c}_4^F(\alpha) = \tilde{c}_3^F(\alpha) = \tilde{c}_2^F(\alpha) = \tilde{c}_F(\alpha)$$

In the next section we will discuss this similarity and we will show that these results are also the same as the massive systems. In case $\mathcal{C}_3$ to have an idea about the function which controls the finite size effect we first realized that one can fit the data to the following function

$$S = \frac{\tilde{c}_3^F(\alpha)}{6} \log(L f_{3\alpha}(\frac{l}{L})),$$

where $f_{\alpha}(x \to 0) \sim x$ and $f_{\alpha}(\frac{L}{2}) \sim 1$. One can determine the function $f_{\alpha}$ by using the formula

$$f_{3\alpha}(\frac{l}{L}) = e^{\frac{\alpha}{2}(S_{n}(l)-S_{n}(\frac{L}{2})).}$$

As one can see in Fig 10 the function is smoothly $\alpha$ dependent. At the same Fig 10 we also depicted the
calculate the entropy $S$ with periodic boundary condition $c\bar{\kappa}$, it seems that except at $\alpha = 2$ the form of the functions are different in two different configurations.

D. Massive LRHO

As noted before, the entanglement entropy $S$ and the Rényi entropy $S_n$ in massless LRHO (for all configurations $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$), increase logarithmically with the sub-system size. We also calculated the prefactors of the logarithms, $\bar{c}(\alpha)$ and $\bar{c}_n(\alpha)$ for the case $\mathcal{C}_1$ and $\bar{c}^F(\alpha)$ and $\bar{c}^F_n(\alpha)$ for other cases, as a function of the long-range parameter $\alpha$ and $n$. Here we are interested in characterizing the massive long-range interacting harmonic oscillators.

First we consider a finite interval of length $l$ in a massive system (configuration $\mathcal{C}_2$). Following an argument given by Cardy-Calabrese, the entanglement entropy for such a system gets saturated by a mass scale and increases logarithmically with $S = -\kappa \frac{c}{2} \log M$, where $c$ is the central charge of the system and it is equal to one for short-range harmonic oscillators and $M$ is the mass of the system. The prefactor $\kappa$ is the number of boundary points between subsystem $A$ and its complement with $\kappa = 1$ for system with boundary and $\kappa = 2$ for system with periodic boundary conditions.

We now consider the LRHO problem, Eq. 2 with mass $M > 0$. As discussed before, we are again going to calculate the entropy $S$ numerically. The results, clearly show that $S$ saturates in the $l \rightarrow \infty$ and the entropy $S$ changes logarithmically with respect to the mass

$$S = -\frac{c^0(\alpha)}{3} \log M .$$

Using such scaling form, we have obtained the prefactor $c^0(\alpha)$, as illustrated in Fig. 11, as a function of $\alpha$. Surprisingly we found that the prefactor is the same as the prefactor of the system with periodic boundary condition when we take half of the system (see Fig. 11). We also considered massive system with boundary and numerical results perfectly agree with $S = -\frac{c^0(\alpha)}{3} \log M$.

Next, we turn to speak more about the Rényi entropy $S_n$ for the massive case. Our analysis show that $S_n$ has also logarithmic scaling with the subsystem size and we measured the prefactor $c^F(\alpha)$. Interestingly we found that this exponent is the same as $c^F_n(\alpha)$. This is shown in Fig. 11. To have an understanding of this equality we note that we generated the $K$ matrix for the system with boundary ($M = 0$) by throwing away those elements of the infinite system which are not inside the corresponding finite system. In this case the summation of the every row of the $K$ matrix is non-zero. This corresponds to an effective mass in the system and the system will be gapped. This effective mass is equivalent to the correlation length $\xi = \frac{1}{m^{\alpha/2}}$. Therefore, this argument hints that the results of massive LRHO should be similar to the massless one when the system has boundary.

E. Finite temperature

In this section we present numerical results for the coupled harmonic oscillators with long range interaction in thermal states. Consider the Hamiltonian $\mathcal{H}$ at some temperature $T > 0$. The Gibbs state corresponding to
this temperature, associated with the canonical ensemble, is given by

\[ \rho(\beta) = \exp(-\beta H) / \text{tr}[\exp(-\beta H)], \]  

(47)

where \( \beta = 1/T \). Similar to the zero temperature case, one can obtain the covariance matrix \( C(\beta) \) and also two point correlators \( P(\beta) \) and \( X(\beta) \), of the state \( \rho(\beta) \) in the basis in which the Hamiltonian matrix is diagonal. These matrices are given by:

\[ P(\beta) = \frac{1}{2}K^{1/2}W(T), \quad X(\beta) = \frac{1}{2}K^{-1/2}W(T), \]  

(48)

and

\[ C^2(\beta) = \frac{1}{4}(K^{-1/2}W(T)) \oplus (K^{1/2}W(T)), \]  

(49)

where \( W(T) := I + 2(\exp(K^{1/2}/T) - I)^{-1} \). It is worth mentioning that the entropy of the subsystem with length \( l \) at temperature \( T \) for CFT is given by the formula:

\[ S = \frac{c}{3} \log \left( \frac{\beta}{\pi \sinh \frac{\beta l}{\beta}} \right) + c_1. \]  

(50)

As expected, in the zero and high temperature limits, the von Neumann entropy reduces to \( S = \frac{c}{4} \log l + c' \) and \( S = \frac{c}{3} l + c'_1 \), respectively. In the high-temperature limit, the von Neumann entropy has an extensive form and it reduces to the standard CFT and agrees with the Gibbs entropy of an isolated system of length \( l \).

As in previous cases, our aim is to study the properties of the von Neumann entropy of system of harmonic oscillators with long range interaction at finite temperature. For simplicity we focus on high temperature limit. In order to measure the von Neumann entropy, we needed to restrict the system size to the finite values with total size \( L \), and subsystem size \( 1 \ll l \ll L \), to avoid finite size problem. In order to calculate the von Neumann in this state, we need to consider the covariance matrix Eq. 49 associated with the reduced state of an interval with length \( l \). Thus, we calculated \( C(\beta) \) at some particular values \( T = 1/\beta \) and then performed the diagonalization of the covariance matrix to find \( S(T,l) \).

As shown in Fig. 12, \( S(T,l) \) for various values of \( T \) and \( \alpha \), in the high-temperature limit is a linear function of \( l \), so in this case one has

\[ S \sim \lambda T^\xi \]  

(51)

where \( \lambda \) and \( \xi \) are functions of \( \alpha \). The log-log plot \( S/l \) with respect to \( T \) is shown in the Fig. 13. The scaling parameter \( \xi \) and the prefactor \( \lambda \) are shown in Fig. 13. The scaling exponent \( \xi \) and the quantity \( 2/\alpha \) are the same, and one can nicely interpolates \( \xi = 2/\alpha \). This
Returning to our LRHO problem we can conclude that the requirement of dimensionlessness and extensivity is not surprising because it is well known that in the long-range systems the dynamical exponent is $z = \frac{2}{3}$ and this exponent controls the relative scaling of time and space leading to the invariant form $I T^{1/3}$. In general the thermal entropy for the theories with the dynamical exponent $z \neq 1$ scale as $I T^{1/2}$ which it follows from the requirement of dimensionlessness and extensivity. Returning to our LRHO problem we can conclude that the entropy in high temperature limit should follow the simple form $S \propto I T^{2/\alpha}$.

It is interesting to note that the Rényi entropy $S_n$ for finite temperature LRHO, in the high-temperature limit is

$$S_n \sim \lambda_n I T^\xi. \quad (52)$$

In Fig. 14 we show the prefactor $\lambda_n$ as a function of $n$ for several $\alpha$’s. It is worth mentioning that all the curves have similar behavior at large $n$ ($\lambda_\infty = \pi/6$).

### F. Universality

In the previous sections we studied a particular case of long-range harmonic oscillator which leads to a well-defined continuum limit field theory. This is a hint to believe that probably the results that we found are robust and valid for larger set of harmonic oscillators. In this section we would like to address this question by first studying long-range harmonic oscillator in the presence of short-range harmonic oscillator and then by investigating larger set of interactions which can be decomposed to our studied systems.

1. **Long-range HO in the presence of short-range HO**

So far, we have only considered the harmonic oscillator systems with the long-range interaction. In the last section, we studied LRHO by means of eigenvalue problem and we computed the eigenvalue $E$ and the eigenfunction $\psi$ numerically. In a similar way we will try to do the same calculation for harmonic oscillator systems with long-range plus short-range interaction. Then, we will study the logarithmic scaling of the entanglement entropy $S$ and also Rényi entropy $S_n$ for this model. Finally, we are going to analyze the scaling coefficient $\tilde{c}(\alpha)$ and $\tilde{c}_n(\alpha)$ as functions of $\alpha$.

Consider the hamiltonian Eq. (1), with long-range plus short-range interaction:

$$K = K_{LR} + K_{SR}. \quad (53)$$

where $K_{LR}$ is again defined as Eq. (2), and $K_{SR}$ is just a simple laplacian. We have only considered the massless system with $M = 0$ but one can also generalize them to $m \neq 0$.

We have carried out simulations for $0 < \alpha < 4$. For each value of $\alpha$, we have determined the matrix $K$ for the large enough system size with $L = 6000$ with the sub-system size less than $L/100$. The entanglement entropy grows logarithmically with the sub-system size as $S = \tilde{c}(\alpha) \log(l)$, where $\tilde{c}(\alpha)$ is a function of $\alpha$. We have depicted $\tilde{c}(\alpha)$ versus $\alpha$ in Fig. 15, where the solid line comes from LRHO case. It is also interesting to note the similarity of $\tilde{c}(\alpha)$ in the range $\alpha < 2$ with the results of harmonic oscillator with pure long-range interaction and also $\alpha \geq 2$ with the result of HO with pure short-range interaction (see Fig. 14). The entanglement entropy of harmonic oscillator system with long range plus short range interaction with the exponent $\alpha < 2$ ($\alpha \geq 2$) is the same as harmonic oscillator system with pure long-range (short-range) interaction. This might look not surprising because we know that from the renormalization group point of view the short-range interaction is irrelevant as far as $\alpha < 2$. In our numerical calculation the reason of discrepancy in the region $2 < \alpha \leq 2.5$ is unclear to us.

We also calculated the Rényi entropy $S_n$ for coupled harmonic oscillators with long-range plus short range couplings. To get $S_n$ numerically, we used Eq. (21). We found that for $l \ll L$ the Rényi entropy also logarithmi-
a large discrepancy in the numerical results. The prefactor \( \tilde{c}(\alpha) \) is measured using the scaling relation \( S \) with \( \log l \) in the region \( 0 < l < L/100 \) for the system size \( L = 6000 \).

We now consider the entanglement entropy of a system of long-range harmonic oscillator with range \( |m| \sim \alpha^2 \) for \( \alpha > 1 \) and \( \alpha = 0 \) for \( \alpha < 1 \). The results are shown in the Fig. 17 and Fig 18. The best fit is \( \tilde{c}(\alpha) \sim 2 \) for several \( \alpha \)'s and one can simply show that for \( \alpha > 1 \) the system is gapped and otherwise it is gapless.

The above Toeplitz matrices are usually called singular Toeplitz matrices. For our purpose we need to also consider some particular restrictions on \( q_r \) to have real interactions between harmonic oscillators. From now on we will consider just those \( q_r \)'s that \( e^{i qr} \)'s are either real or the complex conjugate of each other for every \( \alpha_r \). Harmonic oscillators with the above interactions are critical and one can simply show that \( \xi^{-1} = 0 \). The above interactions are natural generalizations of the ones that we considered in previous sections. One way to see this is by considering \( 2m - 2 \cos q \) instead of \( 2 - \cos q \) in the equation (2). For \( m = \cos q_r \) one can simply show that \( |2m - 2 \cos q| = (2 - 2 \cos (q + q_r))^{\frac{1}{2}}(2 - 2 \cos (q - q_r))^{\frac{1}{2}} \) which is in the form of (54). It is worth mentioning that for \( m > 1 \) the system is gapped and otherwise it is gapless.

Using the techniques of the previous sections one can calculate easily the entanglement entropy of these harmonic oscillators. The entanglement entropy changes logarithmically with the subsystem size and the prefactor of the logarithm is a function which is independent of \( b(q) \) and \( q_r \) but it is strongly dependent on the \( \alpha_r \)'s. To show that the results are \( b(q) \) independent we took \( b(q) = 2g - 2 \cos q \) with \( g > 1 \) for several \( g \) for \( R = 1 \). The results are shown in the Fig 17 and Fig 18 where one can see that the prefactor of the logarithm is the same in all the different cases. We conjecture that the prefactor of the logarithm is independent of the form of the function \( b(q) \). Next we calculated the prefactor of the logarithm for different values of \( \alpha_r \) and \( q_r \). The results are shown in the table II.

It is easy to see that firstly the results are independent of \( q_r \)'s and secondly one can get the results of the last three rows by just summing the results of the first four rows. Based on the results of the table one can conjecture that for the interaction (53) the following result is valid:

\[
K_{l,m} = -\int_0^{2\pi} \frac{dq}{2\pi} e^{i(ql-m)} b(q) \prod_{r=1}^{R} u(\alpha_r, q - q_r),
\]

(54)

where \( b(q) : S^1 \rightarrow \mathbb{C} \) is a smooth non-vanishing function with zero winding number and

\[
u(\alpha, q) = (2 - 2 \cos q)^{\frac{1}{2}}.
\]

(55)
1. the prefactor is independent of the presence of boundary for non-trivial $b(q)$ function. For $g > 2$ the prefactor is independent of the $b(q)$ function.

2. FIG. 18: (Color online) The prefactor of logarithm in the presence of boundary for non-trivial $b(q)$ function. For $g > 2$ the prefactor is independent of the $b(q)$ function.

3. FIG. 19: (Color online) $s(\theta, \alpha)$ as a function of $\alpha$ for subsystems with different vertex angles $\theta$. Inset: the nonuniversal coefficient of the area term $a_1$ with respect to different values of $\alpha$.

4. The results are depicted in Fig. (19). Using the equation (57) then we found $s(\theta, \alpha)$ for different values of $\theta$ such as: $\theta = \frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ and different values of $\alpha$. The results are depicted in Fig. (19). We also showed that the coefficients $s(\theta, \alpha)$ are independent of the orientation of the subregion with respect to the symmetry axes of the lattice. One can summarize this section as follows: the entanglement entropy of a polygonal region for long-range harmonic oscillators follows the same formula as the short-range one but with different coefficients. We also confirmed that the same kind of behavior is also valid for Rényi entropy.

5. IV. MUTUAL INFORMATION

In the previous sections we studied the von Neumann entropy $S$ and the Rényi entropy $S_n$ for long range harmonic oscillators with different configurations of systems and subsystems. It is also of considerable interest to quantify the Shannon’s classical mutual information for system of harmonic oscillators with short range and long range interactions. The Shannon information for spin systems were first studied in and much more investigated in for different quantum systems. Here we focus to the definitions given in and.

Consider a chain of $L$ harmonic oscillators described by canonical variables $(\phi_i, \pi_i)$, $i = 1, 2, \ldots, L$, and the system is divided in to two parts $A$ and $B$ with $l$ and $L - l$ oscillators, respectively. The oscillators are coupled by a quadratic hamiltonian Eq. (1). Let us now consider $\Phi_A = (\phi_1, \phi_2, \ldots, \phi_l)$ and $\Phi_B = (\phi_{l+1}, \phi_{l+2}, \ldots, \phi_L)$ the

6. TABLE I: Numerical values of the prefactor $\tilde{c}$ for different values of $\alpha, \beta$, and $q_i$.

| $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\tilde{c}/2$ |
|-------|-------|-------|-------|-------|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0.75  | 0.33  | 0.33  | 0.33  | 0.33  | 0.33  | 0.33(0.01)  | 0.33(0.01)  | 0.33(0.01)  | 0.33(0.01)  | 0.33(0.01)  | 0.33(0.01)  | 0.33(0.01)  |

7. for the prefactor of the logarithm:

$$
\tilde{c}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{R-1}, \alpha_{R-1}) = \\
\tilde{c}(\alpha_1, \alpha_1, \ldots, 0) + \tilde{c}(0, 0, \alpha_2, \alpha_2, \ldots, 0) + \ldots
$$

8. In other words one can get the prefactor of the logarithm in the model (14) by just having the same quantities for the long-range harmonic oscillator that we have discussed in the previous sections.

9. G. Two dimensions: area law and logarithmic term for polygonal region

It was shown in\cite{32} that the area law is valid for short-range harmonic oscillator if we consider a sphere like region in higher dimensions. The coefficient of the area term is a non-universal number. For example, if we take an square like subregion then the coefficient of the area term will be dependent to the orientation of the polygon with respect to the symmetry axes of the lattice. Later it was shown in\cite{36} that if we consider a region with sharp corners then in the entanglement entropy there will be also some extra logarithmic terms with universal coefficients. In other words one can write the entanglement entropy of a polygon as

$$
S(\theta) = a_0 + a_1 L + a_{-1} L^{-1} + a_{-2} L^{-2} - s(\theta, \alpha) \log L
$$

where $L$ is the size of the system and $\theta$ is the vertex angle of the polygon. Following the same procedure as previous sections we first found the entanglement entropy of the square like regions for different values of $\alpha$’s and confirmed that the leading term is the area law. The coefficient $a_1$ was an increasing function of $\alpha$ (See Fig. (19)). Using the equation (57) then we found $s(\theta, \alpha)$ for different values of $\theta$ such as: $\theta = \frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ and different values of $\alpha$. The results are depicted in Fig. (19). We also showed that the coefficients $s(\theta, \alpha)$ are independent of the orientation of the subregion with respect to the symmetry axes of the lattice. One can summarize this section as follows: the entanglement entropy of a polygonal region for long-range harmonic oscillators follows the same formula as the short-range one but with different coefficients. We also confirmed that the same kind of behavior is also valid for Rényi entropy.
position vectors of the subsystems $A$ and $B$ and $\Pi_{A,B}$ the respective momentum vectors. The classical mutual information can be defined as:

$$I(A, b) = S_A + S_B - S_{(A+B)} ,$$

(58)

where $S$ is the Shannon’s classical entropy. There are in fact two different definitions to evaluate Shannon’s mutual information. The difference comes from the source of probabilities that we use to define the entropy. In the first case we use the ground state of the quantum system as the source of the probabilities for appearing different configurations and in the second case it will be just the Gibbs distribution. The first definition which has recently found many interesting applications in the study of spin chains can be defined in the context of harmonic oscillators as follows: The Shannon’s classical mutual information $I(A : B)$ between two regions $A$ and $B$ is

$$I_1(A : B) = \int d^N \Phi p(\Phi_A, \Phi_B) \ln \frac{p(\Phi_A, \Phi_B)}{p(\Phi_A)p(\Phi_B)}$$

(59)

where $p(\Phi_A, \Phi_B) = |\Psi_0|^2$ is the total and $p(\Phi_A) = \int [\prod_{m \in (A)} d\phi_m] |\Psi_0|^2$ and $p(\Phi_B) = \int [\prod_{m \in (B)} d\phi_m] |\Psi_0|^2$ are the reduced probability densities in position space $(\Psi_0$ is the ground state wave function i.e. Eq. (12)).

The mutual information $I(A : B)$ or $I(A : B)$ quantifies how correlated two parts are when the system is in the ground state and for harmonic oscillators has the following simple form:

$$I_1(A : B) = \frac{1}{2} \ln \left( \frac{\det 2X_A}{\det 2X_B} \right) \frac{\det K^{-1/2}}{\det K^{-1/2}}$$

$$= \frac{1}{2} \ln \left( \frac{\det 2P_A}{\det 2P_B} \right) \frac{\det K^{1/2}}{\det K^{1/2}}$$

$$= \frac{1}{2} \ln(\det 4X_A P_A) = \sum_{i=1}^{l} \ln 2\nu_i ,$$

(60)

where $X_A$ and $P_A$ are $l$ dimensional matrices describing correlations within a compact block of $l$ oscillators (subsystem $A$) and $\nu_i$ is the eigenvalue of the matrix $C = \sqrt{X_A P_A}$ and $X_B$ and $P_B$ are $(L-l) \times (L-l)$ matrices describing correlations within subsystem $B$, and the matrices $X_{AB}$ and $P_{AB}$ describe the correlations between them (see Eq. (24)). It is worth mentioning that the mutual information $I_1$ is the lower bound to the quantum entanglement entropy. Note that Shannon’s mutual information $I_1$ (see Eq. (60)) is equal to the Rényi entropy $S_n$ (see Eq. (29)) when $n = 242$.

According to Eqs. (2) and (6), $K$ and $K^{1/2}$ matrices, for a translational invariant system, are Toeplitz matrices. Therefore, to compute the Shannon’s classical mutual information Eq. (59), we need to compute the Toeplitz determinants. As shown by Fisher-Hartwig and proved later by Widom, for translational invariant system, are Toeplitz matrices. Therefore, to compute the Shannon’s classical mutual information Eq. (59), we need to compute the Toeplitz determinants. As shown by Fisher-Hartwig and proved later by Widom, the asymptotic behavior of the Toeplitz determinants $\det(P_A)$ for the massless system i.e. Eq. (8), with subsystem size $l > 1$ is

$$\det P_A \propto l^{42/16} ,$$

(61)

It is also possible to apply the Fisher-Hartwig theorem to $X_A$ when $\alpha < 2$. Then one can find the power law behavior

$$\det X_A \propto l^{42/16} .$$

(62)

We have numerically calculated $X_A$ for $\alpha = 2$, and found an agreement with the Eq. (62).

The Eqs. (61) and (62), provide an explicit way to find the logarithmic behavior of the mutual information $I_1$ in terms of the system size. In the case where the system is very large and the subsystem has small size $l$, the mutual information $I_1$ can be obtained

$$I_1 = \frac{\alpha^2}{16} \ln l + c_0 .$$

(63)

Numerical simulation results (see Fig. (20)), in a wide range of $\alpha$, are in good agreement with the Eq. (63),

FIG. 20: (Color online) The prefactor of the logarithmic scaling of the mutual information $I_1$ for LRHO with configuration $\xi_1$.

FIG. 21: (Color online) The prefactor of the logarithmic scaling of the mutual information $I_2$ for LRHO with configuration $\xi_1$. 
but when $1.5 < \alpha < 2$ we observe small discrepancy in the numerical results. The reason of this discrepancy is unclear to us.

Here we also focus on the other definition considered by Cramer et al. to evaluate Shannon’s mutual information. They determined the classical Shannon entropy of the total lattice $S_{(A+B)}$, as well as the entropy $S_A$ and $S_B$ determined by the reduced densities describing the two regions $A$ and $B$, respectively. The classical Shannon entropy for harmonic oscillator at finite temperature $T = 1/\beta$ is

$$S_{(\oplus)} = -\frac{1}{2} \ln \det (K|_{\oplus})^{-1} + v(\oplus) \ln \frac{2\pi}{\beta} + v(\oplus),$$

(64)

where $\oplus \in \{A, B, (A + B)\}$ and $K|_{\oplus}$ denotes the $K$ matrix associated with the corresponding region $\oplus$ and $v$ is the size of the region. Then for a hamiltonian of the form Eq. 1, one can compute the Shannon’s mutual information by the following formula:

$$I_2 = \frac{1}{2} \ln \frac{\det K|_A \det K|_{\oplus}}{\det K} = \frac{1}{2} \ln (\det K|_A K^{-1}|_A),$$

(65)

where $K^{-1}|_A$ denotes the $K^{-1}$ matrix associated with the interior region $A$. It is worth mentioning that the mutual information $I_2$ is independent of temperature.

Using Fisher-Hartwig theorem one can get the asymptotic behavior of the Toeplitz determinants $\det (K|_A)$ for the massless system i.e. Eq. (22) as

$$\det K|_A \propto l^{\alpha^2/4}.$$  

(66)

We will now discuss our numerical calculations. First suppose very large system and very small subsystem size (configuration $\mathcal{C}_4$). In order to compute the mutual information we have numerically calculated the $K|_A$ and $K^{-1}|_A$ matrices. Then we calculated the eigenvalues of the matrix product $K|_A K^{-1}|_A$ and we measured the mutual information $I_2$ by the Eq. 45. Our results show the mutual information for LRHO increases logarithmically with the subsystem size as

$$I_2 = \frac{\alpha^2}{8} \ln l + c_0.$$  

(67)

We then measured the prefactor of the logarithm and our results are shown in the Fig. 21).

As we shall discuss in the next sections, it is easy to extend our numerical computation to general configurations of systems and sub-systems i.e. the configurations $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$.

### A. Finite systems

Here we focus on the effect of the finite size system on the mutual information. Hence we shall first consider the mutual information $I_1$. Consider the case when the system has size $L$ and the subsystem has size $l = L/2$ (configuration $\mathcal{C}_2$ and $\mathcal{C}_3$). The mutual information $I_1$ for systems with size $L$ and subsystem size $l = L/2$ with periodic boundary condition (configuration $\mathcal{C}_2$) follows:

$$I_1 = \frac{\alpha^2}{16} \ln L + c_0,$$

(68)

where $\alpha$ is the scaling exponent for the LRHO and $L$ is the size of the system and $c_0$ is the non universal constant.

Then consider the case with configuration $\mathcal{C}_4$. In this case the mutual information $I_1$ follows

$$I_1 = \frac{\alpha^2}{32} \ln L + c_0.$$  

(69)

The numerical results of the prefactor of the logarithmic scaling Eqs. 65 and 69 for various $\alpha$’s are displayed in Fig. 22). The agreement between the theoretical results given by Eqs. 65 and 69 and the numerical results is fairly good.

Now we are interested to find the mutual information $I_2$ for systems with finite size. First consider configuration $\mathcal{C}_4$, when the subsystem has size $1 < l < L/2$ and the system has periodic boundary condition. Finite size effects bend down the mutual information when the size of the sub-system approaches half of the system size.

Recall from Eq. 66) that $\det K|_A \propto l^{\alpha^2/4}$ and $\det K|_B \propto (L - l)^{\alpha^2/4}$ for a subsystem of length $l$ in a finite system of length $L$ with periodic boundary condition. It is then natural to expect that the mutual information $I_2$ (see Eq. 45) for systems with finite size, obeys the following formula:

$$I_2 = \frac{\alpha^2}{8} \ln (l(L - l)) + c_0'.$$  

(70)

We notice that the logarithmic scaling Eq. 67) can be recovered from Eq. 70 for $l \ll L$. The numerical computation of the mutual information $I_2$ in this case can
be easily achieved using equation (65). The results are shown in Fig. (23), obtaining excellent agreement between the numerical data and the Eq. (70). The mutual information $I_2$ for a system with periodic boundary condition and subsystem with size $l = L/2$ changes logarithmically as

$$I_2 = \frac{\alpha^2}{4} \ln L + c_0^' . \quad (71)$$

The numerical results are shown in Fig. (23). We obtain good agreement with the available theoretical prediction. We also studied $I_2$ (see Eq. (65)) for the long range harmonic oscillator with configuration $C_3$. Here, we examine the behavior of the mutual information $I_2$ for harmonic oscillator with long-range interaction, when the system has boundary. The boundary breaks translational symmetry and it is not therefore possible to use the method followed in $C_2$. However, it is possible to find the $K$ matrix and then we can use numerical diagonalization of the matrix $K|A K^−1|A$ to find the eigenvalues and calculate the mutual information $I_2$. In this case we observe that

$$I_2 = \frac{\alpha^2}{8} \ln(l(L - l)) - \frac{\alpha^2}{8} \ln(L) + c_0^' , \quad (72)$$

where in our numerical simulations we found good agreement with our prediction (see Fig. (24)). It is interesting to note that the mutual information $I_2$ for LRHO with configuration $C_4$ grows logarithmically with $L$:

$$I_2 = \frac{\alpha^2}{8} \ln L + c_0 , \quad (73)$$

where this simple behavior is expected from the Eq. (72) when $l = L/2$. In Fig. (24) we show our numerical results for the prefactor of the logarithmic term of the mutual information.

**B. Massive systems**

In this subsection we consider massive LRHO with $M > 0$ (configuration $C_5$). For the massive case, we study the behavior of the mutual information $I_1$ and $I_2$ numerically. It is interesting to note that the mutual information $I_1$ and $I_2$ increase logarithmically with the mass and obey the following formula:

$$I_1 = -\frac{\alpha^2}{16} \ln M , \quad I_2 = -\frac{\alpha^2}{4} \ln M . \quad (74)$$

In Fig. (25) we report the results of the simulation of the mutual information $I_1$ and $I_2$ for the massive LRHO, where we calculated the prefactor of the logarithm which is in good agreement with the Eq. (74).

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**FIG. 23:** (Color online) Top: Mutual information for LRHO ($\alpha = 1.0$) with the configuration $C_2$. The solid line represents $I_2 = \frac{1}{2} \ln(l(L - l)) + c_0^'$. Bottom: The prefactor of the logarithmic scaling of the mutual information $I_2$ for LRHO for a system with periodic boundary condition and configuration $C_2$ when $l = L/2$.

**FIG. 24:** (Color online) Top: Mutual information for LRHO ($\alpha = 1.0$) with the configuration $C_3$. The solid line represents $I_2 = \frac{1}{2} \ln(l(L - l)) - \frac{1}{2} \ln(L) + c_0^'$. Bottom: The prefactor of the logarithmic scaling of the mutual information $I_2$ for LRHO with configuration $C_4$. 

**FIG. 25:** (Color online) Top: Mutual information for LRHO ($\alpha = 1.0$) with the configuration $C_5$. The solid line represents $I_2 = \frac{1}{2} \ln(l(L - l)) - \frac{1}{2} \ln(L) + c_0^'$. Bottom: The prefactor of the logarithmic scaling of the mutual information $I_2$ for LRHO with configuration $C_5$. 

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C. Generalized singular Toeplitz matrices

In this subsection we generalize our results to the general Toeplitz matrices that we have studied in the section [III.F.2]. For $I_1$ the discussion follows the argument given in [22] which is based on the Fisher-Hartwig theorem. It is very simple to see that since $P_A$ and $X_A$ are Toeplitz matrices for the $\alpha_r < 2$ one can simply get the following results for the prefactor of the logarithm of the mutual information $c_{I_1}$ of the subsystem

$$
\tilde{c}_{I_1}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_R) = \\
\tilde{c}_{I_1}(\alpha_1, \ldots, 0) + \tilde{c}_{I_1}(0, 0, \alpha_2, \alpha_3, \ldots, 0) + \ldots \\
+ \tilde{c}_{I_1}(0, 0, \ldots, \alpha_{R-1}, \alpha_{R-1}).
$$

(75)

Similar result has been already announced in [22] for the mutual information of a periodic system with half of the system as the subsystem for $\alpha$ = even. We numerically checked the above result in table [III] It is worth mentioning that the prefactors are independent of $b(q)$ and $q_r$.

It is not difficult to show that the same formula is also valid for the cases with boundary.

Finally we should mention that the above argument works perfectly also for $I_2$. The results of the numerical calculations are shown in the table [III] for a large system with the small subsystem $A$.

D. Two dimensions: area law and logarithmic term for polygonal region

In this section we study the behavior of the Shannon mutual information in two dimensions. Since $I_1$ is nothing except the $n = 2$ R"enyi entanglement entropy one just expect that the equation (57) be valid also for this case. In this section we mostly concentrate on $I_2$. We first confirmed that the area law is valid also in this case for different values of $\alpha = 2, 1.5$ and 1. Then we checked the effect of sharp corner as we did in the case of entanglement entropy. The best fit for the data is

$$
I_2(\theta) = b_0 + b_1 L + b_{-1} L^{-1} + b_{-2} L^{-2} - i(\theta, \alpha) \log L(76)
$$

where $\theta$ as before is the vertex angle. The coefficient of the area term is a nonuniversal quantity and increases with the $\alpha$, see Fig. [26]. Similar to what we had in the case of the entanglement entropy the coefficient of the logarithm is a universal function and increases with $\alpha$ and decreases with $\theta$, see Fig. [26]. It is worth mentioning that we also calculated the same quantity for $I_1$ and we found that $i(\theta, \alpha)$ is $\frac{1}{2}$ of the result for $I_2$.

V. CONCLUSIONS AND DISCUSSIONS

In this paper we studied quantum entanglement entropy of coupled long-range harmonic oscillators. We...
showed that the von Neumann and Rényi entanglement entropy of a subsystem of an infinite system changes logarithmically with the subsystem size which the prefactor is dependent to the fractional power of the interaction \( \alpha \). We also studied the same quantities in the presence of different kinds of boundary conditions and found that the entanglement entropy changes logarithmically with the subsystem size but with a prefactor which is different from the case without a boundary. The prefactor is interestingly the same as the prefactor coming from the massive case. Having the above results we concluded that there are just two universal prefactors in our system. Later we extended our results to the finite temperature case and found \( T^\frac{\theta}{4} \) dependence of the entanglement entropy to the temperature. Our main result was studying the universality of our results by changing the interactions. For example we considered long-range HO plus short-range HO and found that the short-range interaction does not have any effect as far as \( \alpha < 2 \). For \( \alpha > 2 \) the result is the same as the short-range interaction. We also showed that one can change some other parameters in the interaction and get always the same results. We generalized our findings by studying general singular Toeplitz like couplings which in this case we showed that one can calculate the entanglement entropy by just having the results for the simple cases that we have studied. Although in this case we were able to prove the result for the \( n = 2 \) Rényi case, proving it for the von Neumann entanglement entropy is far from obvious. We also generalized our findings to two dimensional cases and showed that despite the long-range nature of the couplings the area law is valid in this case. In addition we showed that universal logarithmic terms will appear if we consider regions with sharp corner in our system. Finally we also studied mutual shannon entropy in our system. We used two definitions; one coming from purely classical considerations and the other comes from using the ground state of the quantum system as the source of probabilities. We showed that the latter case is actually equal to the \( n = 2 \) Rényi entanglement entropy and one can calculate many things analytically by using Fisher-Hartwig theorem for Toeplitz matrices. We also provided many simple exact results by using the same method. The generalization to the singular Toeplitz matrices is immediate in these two cases and one can prove that the decomposition mentioned in the case of von Neumann entropy is valid also in this case. There are many other directions that one can extend our work among the immediate ones one can call the study of our system in the presence of the quantum quench, the other direction is studying the entanglement entropy of excited states. Another important study can be investigating the entanglement entropy of long-range Ising model in the mean field regime where one can relate it to the field theory that we have studied in this paper. We hope to be able to come back to some of these questions in future.

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### Appendix A: Appendix: Fisher-Hartwig theorem

The Fisher-Hartwig conjecture which is proved later by Widom is about the asymptotic behavior of the determinants of a certain class of Toeplitz matrices. The singular Toeplitz matrices have the following form

\[ K_{l,m} = -\int_0^{2\pi} dq \frac{dq}{2\pi} e^{iq(l-m-b(q)\phi(q-q_r)}, \tag{A1} \]

where \( b(q) : S^1 \rightarrow \mathbb{C} \) is a smooth non-vanishing function with zero winding number and

\[ \phi(q) = \prod_{r=1}^{R} u(\alpha_r, q) t(\beta_r, q), \tag{A2} \]

\[ u(\alpha, q) = (2 - 2\cos q)^\frac{\alpha}{2}, \quad \text{Re} \alpha > -1 \tag{A3} \]

\[ t(\beta, q) = \exp[-i\beta(\pi - q)], \quad 0 < q < 2\pi. \tag{A4} \]

Fisher and Hartwig conjectured that the determinant of the matrix \( K \) follows

\[ D_n[K] \sim E G^n(b)n^{\sum_r(\frac{\beta_r^2}{2} - \beta_r^2)} \tag{A5} \]

where \( E \) is a constant and \( G(b) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log b(q) dq\right) \). In our study we took the cases with \( \beta_r = 0 \), however, we believe that generalizations to \( \beta \neq 0 \) are straightforward.
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