Kostant’s method for constructing Lie supergroups in the complex analytic category and applications

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Abstract

Integrable invariant almost complex structures are introduced in the setting of Lie-Hopf superalgebras. This leads to a convenient description of the subsheaf of holomorphic superfunctions in the sheaf of smooth superfunctions yielding an identification of complex Lie supergroups and complex Lie-Hopf superalgebras. Applications to the construction of a universal complexification of a real Lie supergroup and, in the case of type I supergroups, to the existence of a local product decomposition defined by conjugation at a generic point of a maximal torus are given. Deducing a Helgason type formula on the radial part of the Laplacian in the super setting we are able to give a very comprehensive approach to Berezin’s formula on radial parts of Laplace-Casimir operators on special type I Lie supergroups.

In contrast to the classical setting where a Lie group associated to a Lie algebra can be easily constructed, the question how to find a Lie supergroup with given Lie superalgebra is much more subtle. There are basically two well-known constructions: one by Berezin and Leites (see [1]) in the complex analytic category using Grassmann analytical continuation and one by Kostant in the smooth category using the notion of Lie-Hopf superalgebras (see [8]). Our goal is to combine the second approach with the notion an almost complex structure in order to construct complex Lie supergroups using the language of Lie-Hopf superalgebras. As we hope we have shown in the applications in this paper, this approach has substantial technical advantages. In particular we show how this method applies to a local theory of radial operators which yields a formula for the radial part of the Laplacian. This is in the spirit of classical results by Helgason (see [3]), but proved by different methods. Such a formula yields a local version of a global result on radial parts of Laplace-Casimir operators on special type I Lie supergroups (see [1]). Technical details on the results presented in this paper can be found in the authors dissertation [7]. Let us now outline our approach.

In the first two sections we briefly present the results by McHugh and Vaintrob (see [10] and [14]) on integrability of almost complex structures on real supermanifolds and by Kostant (see [8]) on the construction via Lie-Hopf superalgebras of real Lie supergroups associated to Harish-Chandra superpairs.

The main result is proved in the third section. There we give conditions for an almost complex structure on a real Lie supergroup to define the structure of a complex Lie supergroup. As a remark, this is consistent with the known equivalence of the categories of complex Lie

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supergroups and complex Harish-Chandra superpairs proved by Vishnyakova in \cite{15}. As a first small application, in the forth section we construct a universal complexification of a real Lie supergroup generalizing Hochschild’s classical result (see \cite{5} chap.XVII).

For later analysis in the fifth section we prove a local product decomposition of type I Lie supergroups defined by conjugation at a generic point of a maximal torus. Starting with an analysis of the radial part of the Laplacian in special geometrical settings in section six we deduce a Helgason type formula. Using this and the local product decomposition, we prove the desired structure theorem for radial parts of Laplace-Casimir operators, i.e. biinvariant differential operators, on type I Lie supergroups. A global formula for these objects is given in \cite{1}. The local formulation presented and proved here is needed for applications in \cite{2} and \cite{6} (see also \cite{4}) and is consistent with the global theorem. Most of the techniques used here are applicable in settings of greater generality and, above all, all of them easily comprehensible.

We use the following notation: a supermanifold is a pair \(\mathcal{M} = (M, \mathcal{A}_M)\) where \(M\) designates the underlying manifold and \(\mathcal{A}_M\) the sheaf of superfunctions. We denote its dimension by a pair \((n_M| m_M)\). A morphism of supermanifolds is denoted by \(m' = (m, m^\#)\) or \(\psi = (\psi, \psi^\#)\) with underlying morphism of manifolds \(m\), respectively \(\psi\), and pull-back of superfunctions \(m^\#\), respectively \(\psi^\#\).

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1 Integrability of almost complex structures

The Newlander-Nirenberg theorem (see \cite{11}) gives a condition for an almost complex structure \(J\) on an even-dimensional real manifold \(M\) to define a complex atlas. McHugh and Vaintrob proved an analogue version for supermanifolds (see \cite{10} and \cite{14}). For later application we recall these results with some additional brief remarks.

Let \(\mathcal{M} = (M, \mathcal{A}_M)\) be a real supermanifold of dimension \((2n|2m)\) and \(\text{Der}(\mathcal{A}_M)\) be the sheaf of left-\(\mathcal{A}_M\)-supermodules of superderivations on \(\mathcal{A}_M\).

**Definition 1.** An (even) automorphism of sheaves of left-\(\mathcal{A}_M\)-supermodules

\[ J : \text{Der}(\mathcal{A}_M) \to \text{Der}(\mathcal{A}_M) \]

with \(J^2 = -\text{Id}\) is called an almost complex structure on \(\mathcal{M}\).

Note that on a complex supermanifold \(\mathcal{M}\) multiplication by \(i\) on the sheaf of superderivations yields, as in the non-graded case, an almost complex structure on the associated real supermanifold \(\tilde{\mathcal{M}}\). Here \(\tilde{\mathcal{M}}\) is obtained from \(\mathcal{M}\) by the natural embedding of a complex superdomain \((U, \mathcal{O}_U \otimes \Lambda \mathbb{C}^m)\) for \(U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}\) into the real superdomain \((U, C^\infty_U \otimes \Lambda \mathbb{R}^{2m})\). For the inclusion \(\Lambda \mathbb{C}^m \subset \Lambda \mathbb{R}^{2m}\) both Grassmann algebras are regarded as algebras of alternating forms and holomorphic morphisms of superdomains are continued compatibly with complex conjugation.

Parallel to the classical notion of differential forms, set

\[
\mathcal{E}^0_{K, \mathcal{M}} := \mathcal{A}_{K, \mathcal{M}} := \mathcal{A}_M \otimes \mathbb{K}, \quad \mathcal{E}^1_{K, \mathcal{M}} := \text{Hom}_{\mathcal{A}_M}(\text{Der}(\mathcal{A}_{K, \mathcal{M}}), \mathcal{A}_{K, \mathcal{M}})
\]
as well as $\mathcal{E}_{k,M}^k := \Lambda^k \mathcal{E}_{k,M}^1$, where $\Lambda^k$ denotes the $k$-th super exterior algebra of a super vector space, and $\mathcal{E}_{k,M} := \bigoplus_k \mathcal{E}_{k,M}^k$. Here always $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The deRham operator $d$ on $\mathcal{E}_{p,M}$ can be continued $\mathbb{C}$-linearly to $\mathcal{E}_{c,M}$. Note that $J$ induces a decomposition into $(\pm i)$-eigenspaces

$$Der(\mathcal{A}_{c,M}) = Der^{(1,0)}(\mathcal{A}_{c,M}) \oplus Der^{(0,1)}(\mathcal{A}_{c,M})$$

while the dualization $J^*$ of $J$ defines a decomposition $\mathcal{E}_{c,M}^1 = \mathcal{E}_{c,M}^{(1,0)} \oplus \mathcal{E}_{c,M}^{(0,1)}$ which leads to

$$\mathcal{E}_{c,M}^k = \bigoplus_{p+q=k} \mathcal{E}_{c,M}^{(p,q)}$$

with $\mathcal{E}_{c,M}^{(p,q)} := \Lambda^p \mathcal{E}_{c,M}^{(1,0)} \wedge \Lambda^q \mathcal{E}_{c,M}^{(0,1)}$.

The following is a graded version of the Newlander-Nirenberg theorem (see [10] and [14]).

**Theorem 1** (McHugh,Vaintrob). For a given almost complex structure $J$ on a real supermanifold $M$ with dimension $(2n|2m)$ there exists a complex structure on $M$ associated to $J$ if and only if

$$d \omega = 0 \mod I^{(1,0)}(U) \quad \forall \omega \in \mathcal{E}_{c,M}^{(1,0)}(U), \quad U \subset M \text{ open},$$

(1)

where $I^{(1,0)}(U)$ is the ideal generated by $\mathcal{E}_{c,M}^{(1,0)}(U)$ in $\mathcal{E}_{c,M}(U)$. If (1) is satisfied, then $J$ is called integrable.

Parallel to the classical case (see [9, chap.IX.2]) one finds equivalent formulations of condition (1).

**Lemma 1.** Condition (1) is equivalent to

$$[Z,W] \in Der^{(1,0)}(\mathcal{A}_{c,M})(U)$$

for all $Z,W \in Der^{(1,0)}(\mathcal{A}_{c,M})(U)$ and $U$ an open subset of $M$. It is also equivalent to the vanishing of the Nijenhuis tensor,

$$[X,Y] + J([JX,Y] + [X,JY]) - [JX,JY] = 0,$$

(2)

for all $X,Y \in Der(\mathcal{A}_{c,M})(U)$.

According to the decomposition $\mathcal{E}^{1}_{c,M} = \mathcal{E}^{(1,0)}_{c,M} \oplus \mathcal{E}^{(0,1)}_{c,M}$ define the projections

$$pr^{(1,0)} : \mathcal{E}^{1}_{c,M} \to \mathcal{E}^{(1,0)}_{c,M} \quad \text{and} \quad pr^{(0,1)} : \mathcal{E}^{1}_{c,M} \to \mathcal{E}^{(0,1)}_{c,M}$$

and the operators

$$\partial := pr^{(1,0)} \circ d : \mathcal{A}_{c,M} \to \mathcal{E}^{(1,0)}_{c,M} \quad \text{and} \quad \bar{\partial} := pr^{(0,1)} \circ d : \mathcal{A}_{c,M} \to \mathcal{E}^{(0,1)}_{c,M}.$$

**Definition 2.** The sheaf of $J$-holomorphic superfunctions is defined by

$$\mathcal{O}_{M,J}(U) := \text{Ker}(\bar{\partial} U : \mathcal{A}_{c,M}(U) \to \mathcal{E}^{(0,1)}_{c,M}(U))$$

for open subsets $U \subset M$.

If $M$ is a complex supermanifold with induced almost complex structure $J$ on $\tilde{M}$, then a direct calculation in local coordinates shows that the subsheaf of $J$-holomorphic superfunctions $\mathcal{O}_{\tilde{M},J}$ in $\mathcal{A}_{c,\tilde{M}}$ is isomorphic to $\mathcal{A}_M$ as expected. Finally note that for $f \in \mathcal{A}_{c,M}(U)$ the condition $\bar{\partial} U f = 0$ is equivalent to $X(f) = 0$ for all $X \in Der^{(0,1)}(\mathcal{A}_{c,M})(U)$.
2 Construction of real Lie supergroups

A real Lie supergroup is uniquely determined by the Hopf superalgebra structure of its set of local differential operators. Here we first briefly discuss the needed algebraic objects as they are presented in [13]. Then we present Kostant’s construction of a real Lie supergroup in [8] which we will transfer to the complex setting later.

2.1 Lie-Hopf superalgebras

A superbialgebra is a superalgebra \((B, m, u)\) with multiplication \(m\) and unit \(u\) which is at the same time a supercoalgebra \((B, \Delta, \epsilon)\) with comultiplication \(\Delta\) and counit \(\epsilon\) such that both structures supercommute.

**Definition 3.** A linear map \(s : B \to B\) on a superbialgebra \((B, m, u, \Delta, \epsilon)\) satisfying

\[
m \circ (Id \otimes s) \circ \Delta = m \circ (s \otimes Id) \circ \Delta = u \circ \epsilon.
\]

is called an antipode. A Hopf superalgebra \(H\) is a superbialgebra \((H, m, u, \Delta, \epsilon)\) with antipode. If its supercoalgebra structure is super co-commutative, then \(H\) is called super co-commutative. A morphism of Hopf superalgebras is a morphism of bialgebras compatible with the antipodes.

An element \(h\) of a Hopf superalgebra \(H\) is called group-like if \(\Delta(h) = h \otimes h\) and primitive if \(\Delta(h) = h \otimes 1 + 1 \otimes h\). A morphism of Hopf superalgebras always maps group-like elements to group-like elements and primitive elements to primitive elements. The group-like elements, which are contained in \(H_0\) since \(\Delta\) is even, define a group with multiplication \(m\) and inverse \(s\). The primitive elements define a Lie superalgebra with bracket

\[
[X, Y] = m(X \otimes Y) - (-1)^{|X||Y|}m(Y \otimes X).
\]

There are two important basic examples of super co-commutative Hopf superalgebras. The first is the universal enveloping algebra \(E(\mathfrak{g})\) of a Lie superalgebra \(\mathfrak{g}\) equipped with comultiplication \(\Delta(X) = X \otimes 1 + 1 \otimes X\) for \(X \in \mathfrak{g}\). A second example is the group algebra \(K(G)\) of a Lie group \(G\) with comultiplication \(\Delta(g) = g \otimes g\) for \(g \in G\). In our case \(K\) is either \(\mathbb{R}\) or \(\mathbb{C}\).

These objects can be combined to the Lie-Hopf superalgebra of a Harish-Chandra superpair. For this recall that a Harish-Chandra superpair \((G, \mathfrak{g})\) is a finite dimensional Lie superalgebra \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) together with a Lie group \(G\) such that \(\mathfrak{g}_0 = Lie(G)\) and such that there exists an integrated adjoint representation denoted by \(Ad : G \to Aut(\mathfrak{g})\).

The *Lie-Hopf superalgebra* of a Harish-Chandra superpair \((G, \mathfrak{g})\) is the super vector space \(K(G) \otimes E(\mathfrak{g})\) with the \(\mathbb{Z}_2\)-grading given by the \(\mathbb{Z}_2\)-grading of \(\mathfrak{g}\) together with an algebra structure: the multiplication on the underlying super vector space \(K(G) \otimes E(\mathfrak{g})\) is given by

\[
(x_1 \# y_1)(x_2 \# y_2) = (x_1 \cdot x_2) \# (Ad(x_2^{-1})(y_1) \cdot y_2)
\]

for \(x_1, x_2 \in K(G)\) and \(y_1, y_2 \in E(\mathfrak{g})\). Here \(\cdot\) denotes the multiplication in \(K(G)\), respectively \(E(\mathfrak{g})\). One also refers to this object as the smash product and denotes it by \(K(G) \# E(\mathfrak{g})\). If \(Ad\) is replaced by a different representation \(\pi\) one usually writes \(K(G) \#_{\pi} E(\mathfrak{g})\).

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1Sweedler discusses the non-graded setting which in most cases can be directly transferred to the graded setting as Kostant indicates.

2Following the notation in [13] “unit” designates the neutral element and not a unit in a ring.
Remark 1. Note that $\mathbb{K}(G)\#E(\mathfrak{g})$ with the comultiplication defined as above and the antipode given by

$$s(g\#X) = -g^{-1}\#Ad(g)(X) \quad \text{for } g \in G \text{ and } X \in \mathfrak{g}. \quad (3)$$

is a super co-commutative Hopf superalgebra. A morphism of Lie-Hopf superalgebras is therefore defined to be a morphism of super co-commutative Hopf superalgebras.

2.2 Superfunctions on real Lie supergroups

For a real supermanifold $M = (M, \mathcal{A}_M(a))$ define the sheaf

$$\mathcal{A}_M^*(U) := \left\{ \varphi \in \text{Hom}_{R-\text{vect}}(\mathcal{A}_M(U), \mathbb{R}) \mid \exists \text{ ideal } I \subset \mathcal{A}_M(U) \text{ with } \text{codim}(I) < \infty, I \subset \ker(\varphi) \right\}$$

for open sets $U \subset M$. Set

$$\mathcal{A}_M^* := \mathcal{A}_M^*(M) = \bigoplus_{p \in M} \mathcal{A}_{M,p}^*$$

where $\mathcal{A}_{M,p}^*$ denotes the elements with support at $p$. Kostant proves the following fact (see [8, §2.11]).

Lemma 2. The space of global sections $\mathcal{A}_M^*$ carries the structure of a super co-commutative supercoalgebra induced by the structure of a sheaf of supercommutative superalgebras on $\mathcal{A}_M$.

Now let $\mathcal{G} = (G, \mathcal{A}_G)$ be a real Lie supergroup in the sense of the category theoretical approach to Lie supergroups, i.e., a group object in the category of supermanifolds. Denote its Lie superalgebra by $\mathfrak{g}$ and the morphisms of supermanifolds for multiplication, inverse and identity by $m_{\mathcal{G}} : G \times G \to G$ and $s_{\mathcal{G}} : G \to G$ and $u_{\mathcal{G}} : pt \to G$. The pull-back $m_{\mathcal{G}}^\#$ of the group multiplication and the evaluation $\delta_e$ of the numerical part of a superfunction at the identity $e \in G$ define the structure of a supercoalgebra on $\mathcal{A}_G^*(G)$. Dualizing this structure $\mathcal{A}_G^*$ obtains the structure of a superalgebra with multiplication $m_{\mathcal{G}}^\#$ and unit $u_{\mathcal{G}}^\#$. Together with the antipode induced by the push-forward $s_{\mathcal{G}}^\#$, these maps define the structure of a super co-commutative Hopf superalgebra on $\mathcal{A}_G^*$. The isomorphy of a finitely generated pointed super co-commutative Hopf superalgebra and the Lie-Hopf subsuperalgebra generated by its group-like and primitive elements (see [13, chap.13]) implies the following basic fact.

Lemma 3. For a real Lie supergroup $\mathcal{G}$ with Lie superalgebra $\mathfrak{g}$ there is an isomorphy of Hopf superalgebras $\mathcal{A}_G^* \cong \mathbb{R}(G)\#E(\mathfrak{g})$.

A question which arises is whether there is a unique real Lie supergroup for any real Harish-Chandra superpair $(G, \mathfrak{g})$. The following answers this in the positive (see [8, §3.7]).

Proposition 1. Let $(G, \mathfrak{g})$ be a real Harish-Chandra superpair. The real supermanifold denoted by $\mathcal{G} := (G, \mathcal{A}_G)$ with sheaf of superfunctions

$$\mathcal{A}_G(U) = \left\{ \Phi \in \text{Hom}_{R-\text{vect}}(\mathbb{R}(U)\#E(\mathfrak{g}), \mathbb{R}) \mid \begin{array}{c} U \to \mathbb{R}, \ g \mapsto \Phi(g\#Z) \end{array} \in C_G^\infty(U) \ \forall Z \in E(\mathfrak{g}) \right\}$$

and projection of superfunctions onto their numerical part given by

$$\text{pr}_{C_G^\infty}(U) : \mathcal{A}_G(U) \longrightarrow C_G^\infty(U), \quad \Phi \longmapsto (g \mapsto \Phi(g\#1)),$$
for open subsets $U \subset G$ has the structure of a real Lie supergroup with $A^*_G = \mathbb{R}(G)^{#}E(\mathfrak{g})$. The pull-backs for multiplication and inverse of the Lie supergroup structure are naturally induced by multiplication and the antipode in $\mathbb{R}(G)^{#}E(\mathfrak{g})$.

The construction is canonical in the sense that the morphisms of real Lie supergroups are up to isomorphy in 1:1-correspondence to the morphisms of the associated real Lie-Hopf superalgebras.

For later application we note the representation of an element $A^{*}_G(U)$ as a differential operator on a superfunction $\Phi \in A^*_G(U) \subset \text{Hom}_{\mathbb{R}-\mathrm{vect}}(\mathbb{R}(U)^{#}E(\mathfrak{g}), \mathbb{R})$.

**Lemma 4.** An element $g \in G$ embedded into $A^*_G$ by $g^#1$ is identified with numerical point-evaluation of superfunctions by $\Phi \mapsto \text{pr}_{\mathbb{C}^{\infty}}(\Phi)(g)$. An element $Z \in \mathfrak{g}$ embedded into $A^*_G$ by $e^#Z$ is identified with a left-invariant superderivation on $A_G(U)$ by the map

$$Z^L.\Phi = (w \mapsto (-1)^{|Z|} \Phi(w \cdot (e^#Z))).$$

### 3 Almost complex structures on real Lie supergroups

If an almost complex structure on a real Lie supergroup is to define a complex Lie supergroup, then of course it must be integrable. However, integrability only guarantees that the underlying supermanifold is complex, not that multiplication and inverse are holomorphic mappings. Here we determine explicit conditions for the compatibility of the almost complex structure and the supergroup structure. As a consequence we obtain a complex version of the correspondence of Lie supergroups and Harish-Chandra superpairs as it is proved for the real case by Kostant’s construction above and in the complex case by Vishnyakova in [15].

Let us begin with a real Harish-Chandra superpair $(G, \mathfrak{g})$ with associated real Lie supergroup $G = (G, A_G)$ of dimension $(2n|2m)$. Note that the three defining morphisms (multiplication, inverse and identity) of a Lie supergroup can be continued $\mathbb{C}$-linearly to the sheaf $A^{\mathbb{C}}_G$. As a consequence of Proposition[1] for an open subset $U \subset G$ we can write

$$A^{\mathbb{C}}_{G}(U) = \left\{ \Phi \in \text{Hom}_{\mathbb{R}-\mathrm{vect}}(\mathbb{R}(U)^{#}E(\mathfrak{g}), \mathbb{C}) \mid \begin{aligned}
(U \to \mathbb{C}, \ g \mapsto \Phi(g^#Z)) &\in C^{\infty}_{\mathbb{C},G}(U) \ \forall \ Z \in E(\mathfrak{g})
\end{aligned} \right\}$$

where $C^{\infty}_{\mathbb{C},G}$ denotes the sheaf of complex valued smooth functions on $G$.

For an almost complex structure $J$ on $G$ we start our analysis with a description of the $J$-holomorphic superfunctions. Regarding $Z \in \mathfrak{g}$ as a left-invariant superderivation denoted by $Z^L$, the expression $J(Z^L)$ is well-defined but not necessarily left-invariant. For each $g \in G$ there is exactly one $Z_g \in \mathfrak{g}$ with $g^#Z_g = \delta_g \circ J(Z^L)$. This defines a map

$$R_J : G \rightarrow \text{Aut}_{\mathbb{R}-\mathrm{vect}}(\mathfrak{g}), \ g \mapsto (Z \mapsto Z_g).$$

Note that the set of all left-invariant superderivations on the sheaf $A^{\mathbb{C}}_{G}$ is induced by the complex super vector space $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$. Later on we will often use the $\mathbb{C}$-linear continuations.
of $R_J(g)$ and $\text{Ad}(g)$ for a $g \in G$ regarding $\mathfrak{g}$ as being canonically embedded in $\mathfrak{g}_C$. A left-invariant superderivation induced by $Z + iZ' \in \mathfrak{g}_C$ is represented on superfunctions by

$$(Z + iZ').\Phi := (g\#X \mapsto \Phi((g\#X) \cdot (e\#Z)) + i\Phi((g\#X) \cdot (e\#Z')))$$

for $\Phi \in \mathcal{A}_G(U)$, open $U \subset G$ and $g\#X \in \mathbb{R}(G)\#E(\mathfrak{g})$.

The following gives a description of the $J$-holomorphic superfunctions.

**Lemma 5.** The sheaf of $J$-holomorphic superfunctions is given by

$$\mathcal{O}_{G,J}(U) = \left\{ \Phi \in \mathcal{A}_{G,J}(U) \ | \ \Phi((g\#X) \cdot (e\#Z)) = 0 \quad \forall \ X \in E(\mathfrak{g}), \ g \in G, \ Z \in \mathfrak{g}_C \quad \text{with} \quad R_J(g)(Z) = -iZ \right\}$$

for open subsets $U \subset G$.

**Proof.** The conditions on $g$ and $Z$ stated in the Lemma are equivalent to the condition $\chi.\Phi = 0$ \quad $\forall \ \chi \in \text{Der}^{(0,1)}(\mathcal{A}_{C,M})(U)$. \hfill \Box

Our goal is to understand which conditions $J$ must satisfy so that $\mathcal{G}_{C,J} = (G, \mathcal{O}_{G,J})$ carries the structure of a complex Lie supergroup. Of course by definition $J$ should be integrable, i.e., it should satisfy the equivalent conditions of Theorem $\text{[1]}$ and the multiplication and inverse morphisms should stabilize the sheaf $\mathcal{O}_{G,J}$. Let us analyze in detail what this means.

The condition that multiplication $m^\#: \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ stabilizes $\mathcal{O}_{G,J}$ is equivalent to the property $m^\#(\mathcal{O}_{G,J}) \subset \mathcal{O}_{G,J} \otimes \mathcal{O}_{G,J}$. Now if $\Phi \in \mathcal{O}_{G,J}$, then

$$m^\#(\Phi) : (\mathbb{R}(G)\#E(\mathfrak{g})) \otimes (\mathbb{R}(G)\#E(\mathfrak{g})) \longrightarrow \mathbb{C} \quad (h_1 \#X) \otimes (h_2 \#Y) \mapsto \Phi((h_1 \#X) \cdot (h_2 \#Y))$$

for $h_1, h_2 \in G$ and $X, Y \in E(\mathfrak{g})$ linearly continued. Hence $m^\#(\Phi) \in \mathcal{O}_{G,J} \otimes \mathcal{O}_{G,J}$ is equivalent to

$$\Phi((g\#X) \cdot (e\#Z) \cdot (h\#Y)) = 0 \quad \forall \ X, Y \in E(\mathfrak{g}), \ g, h \in G, \ Z \in \mathfrak{g}_C \quad \text{with} \quad R_J(g)(Z) = -iZ \quad \text{and} \quad R_J(h)(Z) = -iZ \quad \text{and}$$

$$\Phi((g\#X) \cdot (h\#Y) \cdot (e\#Z)) = 0 \quad \forall \ X, Y \in E(\mathfrak{g}), \ g, h \in G, \ Z \in \mathfrak{g}_C \quad \text{with} \quad R_J(g)(Z) = -iZ \quad \text{and} \quad R_J(h)(Z) = -iZ$$

For the following we note that for arbitrary $g \in G$ and $Z \in \mathfrak{g}_C$ satisfying the inequality $R_J(g)(Z) \neq -i \cdot Z$ there is locally a $J$-holomorphic superfunction $\Phi$ with $\Phi(g\#Z) \neq 0$.

**Lemma 6.** If $m^\#_G$ stabilizes $\mathcal{O}_{G,J}$, then $J$ maps left-invariant superderivations to left-invariant superderivations. The same is true for right-invariant superderivations.

**Proof.** From $[6]$, choosing $X = 1$, it follows that

$$\exists g' \in G \quad \text{with} \quad R_J(g')(Z) = -iZ \quad \Leftrightarrow \quad R_J(g')(Z) = -iZ \quad \forall g' \in G .$$
Anti-holomorphic superfunctions are obtained from holomorphic superfunctions in $\mathcal{A}_{C,G}$ by composition with complex conjugation. They are also preserved by $m'_g$ under the condition of the lemma and we obtain from (5) and (6) with $i$ replaced by $-i$

$$\exists g' \in G \text{ with } R_J(g')(Z) = i Z \iff R_J(g')(Z) = i Z \ \forall g' \in G.$$ 

Thus $J$ acts on left-invariant superderivations. From (4), now choosing $Y = 1$, we obtain:

$$\exists g' \in G \text{ with } R_J(g')(Z) = i Z \iff R_J(g')(Ad(g'')(Z)) = i Ad(g'')(Z) \ \forall g', g'' \in G.$$ 

From the parallel result concerning anti-holomorphic superfunctions it follows that $J$ commutes with the adjoint action on left-invariant superderivations. Since right-invariant superderivations are of the form $g \mapsto (e\#X) \cdot (g\#1) = g\#Ad(g^{-1})(X)$ the lemma is proved. $\square$

Defining $J_g : g \rightarrow g$ by $J_g := R_J(e)$ and then continuing it $C$-linearly to $g_C$ we have the decomposition $g_C = g_C^{(1,0)} \oplus g_C^{(0,1)}$ into $(\pm i)$-eigenspaces.

**Lemma 7.** If $m'_g$ stabilizes $\mathcal{O}_{G,J}$, then $[g_C^{(1,0)}, g_C^{(0,1)}] = 0$. Furthermore the Lie superbracket is $J$-linear.

**Proof.** For $Y \in g_C^{(1,0)}$ we apply equation (5). It follows that

$$(e\#Z) \cdot (h\#Y) = h\#(Ad(h^{-1})(Z)Y) = (-1)^{|Z||Y|}h\#(YAd(h^{-1})(Z)) + h\#[Ad(h^{-1})(Z),Y].$$

Since $Ad$ commutes with $J$ on left-invariant superderivations (see the proof of Lemma 6), a superfunction $\Phi$ holomorphic with respect to $J$ vanishes on the first summand. It vanishes on the second summand if and only if $[g_C^{(1,0)}, g_C^{(0,1)}] \subset g_C^{(0,1)}$. The analogous argument for anti-holomorphic superfunctions yields the inclusion $[g_C^{(1,0)}, g_C^{(0,1)}] \subset g_C^{(1,0)}$. Hence $[g_C^{(1,0)}, g_C^{(0,1)}] = 0$. The $J$-linearity of the Lie superbracket follows immediately from Lemma 6. $\square$

These results lead to the main theorem of this section.

**Theorem 2.** A real Lie supergroup $G = (G,A_G)$ of dimension $(2n|2m)$ with almost complex structure $J$ defines the structure of a complex Lie supergroup on $G_{C,J}$ if and only if $J$ is defined on left- and right-invariant superderivations and the Lie superbracket is $J$-linear.

**Proof.** Due to Lemma 6 and 7 there is only one direction to prove. Since $J$ is defined on left-invariant superderivations and the Lie superbracket is $J$-linear, the Nijenhuis tensor in (2) vanishes. Thus $J$ is integrable. Furthermore, the equality $[g_C^{(1,0)}, g_C^{(0,1)}] = 0$ follows from

$$i[g_C^{(1,0)}, g_C^{(0,1)}] = [J(g_C^{(1,0)}), g_C^{(0,1)}] = [g_C^{(1,0)}, J(g_C^{(0,1)})] = -i[g_C^{(1,0)}, g_C^{(0,1)}].$$

Since $J$ is defined on left- and right-invariant superderivations, the inverse arguments of the proofs of Lemmas 6 and 7 imply that $J$ commutes with $Ad$ on left-invariant superderivations and equations (3) and (6) are fulfilled. The inverse $s'_g : G \rightarrow G$ automatically preserves $\mathcal{O}_{G,J}$ as can be seen from the explicit formula (3). $\square$

As a consequence of Theorem 2 every real Harish-Chandra superpair $(G,g)$ of dimension $(2n|2m)$ together with an (even) isomorphism of super vector spaces $J_g : g \rightarrow g$ satisfying $J_g^2 = -Id$ and $[J_gX,Y] = J_g[X,Y] = [X,J_gY]$ defines a complex Lie supergroup and vice versa. Such a triple $(G,g,J_g)$ corresponds to a complex Harish-Chandra superpair.
Remark 2. This yields an isomorphy of categories which was first proved in [15]: Complex Lie supergroups naturally correspond to complex Harish-Chandra superpairs. The morphisms of complex Lie supergroups are up to isomorphy in 1:1-correspondence with the morphisms of the associated complex Lie-Hopf superalgebras.

As in the case of real Lie supergroups the superfunctions on a complex Lie supergroup can be constructed from a complex Lie-Hopf superalgebra in the following way.

Proposition 2. Let \( \mathcal{G} = (G, \mathcal{A}_G) \) denote a complex Lie supergroup and \( \mathbb{C}(G) \# E(\mathfrak{g}) \) the associated complex Lie-Hopf superalgebra. The sheaf of holomorphic superfunctions can be identified with

\[
\mathcal{A}_G(U) = \left\{ \Phi \in \text{Hom}_{\mathbb{C}-\text{eect}}(\mathbb{C}(U) \# E(\mathfrak{g}), \mathbb{C}) \mid \left( U \to \mathbb{C}, \; g \mapsto \Phi(g \# Z) \right) \in \mathcal{O}_G(U) \; \forall Z \in E(\mathfrak{g}) \right\}.
\]

Proof. Let \( \tilde{\mathcal{G}} = (\tilde{G}, \mathcal{A}_{\tilde{G}}) \) be the associated real supermanifold of the complex supermanifold \( \mathcal{G} \) and \( \tilde{\mathfrak{g}} \) be the associated real Lie superalgebra which is equipped with the induced complex structure \( J \). The morphisms of multiplication, inverse and identity can be continued from the sheaf of holomorphic superfunctions to the sheaf of smooth superfunctions via \( J \)-equivariance yielding a real Lie supergroup structure on \( \tilde{\mathcal{G}} \) with Lie superalgebra \( \tilde{\mathfrak{g}} \). Finally we use \( \text{pr}_{\mathcal{C}}(\mathcal{A}_G) = \mathcal{O}_G \subset \mathcal{C}_G^{\infty} \) and the identification (4). \( \square \)

From now on all almost complex structures on supermanifolds are supposed to be integrable and compatible with the group structure in the case of Lie supergroups.

4 Universal complexification of real Lie supergroups

A first application where Kostant’s construction generalized to complex Lie supergroups turns out to be very useful is the proof of the existence of a universal complexification for a real Lie supergroup. We give the construction here which is strongly reliant on the result in the non-graded case due to Hochschild [5].

A universal complexification of a real Lie group \( G \) is a complex Lie group \( G^C \) together with a morphism of real Lie groups \( \gamma : \mathbb{R} \to \mathbb{R}^C \) with the following property: For every morphism of real Lie groups \( \eta : \mathbb{R} \to \mathbb{H} \) into a complex Lie group \( \mathbb{H} \) there exists a unique morphism of complex Lie groups \( \eta^C : \mathbb{R}^C \to \mathbb{H} \) such that \( \eta^C \circ \gamma = \eta \). Hochschild proves the following theorem (see [5, chap.XVII.5]).

Theorem 3. For every real Lie group \( G \) there exists a universal complexification \( G^C \). The Lie algebra \( \text{Lie}(G^C) \) is isomorphic to \( (\text{Lie}(G) \otimes \mathbb{C})/\mathfrak{p}^C \) for an ideal \( \mathfrak{p} \) of \( \text{Lie}(G) \) and \( \mathfrak{p}^C := \mathfrak{p} \otimes \mathbb{C} \).

Note that the universal complexification is unique up to isomorphisms of complex Lie groups. Here is an example where \( \mathfrak{p} \) is nontrivial. Let \( \tilde{S} \) be the universal cover of \( S := \text{SL}_2(\mathbb{R}) \) with covering group \( \Gamma \cong \mathbb{Z} \). Let \( \rho : \Gamma \to \mathbb{S}^1 \) be a representation with dense image and embed \( \Gamma \) as a closed subgroup in \( \tilde{S} \times \mathbb{S}^1 \) by \( k \mapsto (k, \rho(k)) \). Define \( G = (\tilde{S} \times \mathbb{S}^1)/\Gamma \). Since \( \text{Lie}(G^C) = \text{Lie}(G) \), it follows that \( G^C = \mathbb{S}^1 \) as well.

We define the notion of a universal complexification \( \gamma^C : \mathcal{G} \to \mathcal{G}^C \) for a real Lie supergroup \( \mathcal{G} \) – whenever it exists – parallel to the definition for real Lie groups. As in the classical case
uniqueness of the universal complexification then follows from the definition. Let us now turn to the super setting.

Let $G = (G,A_G)$ be a real Lie supergroup with Lie superalgebra $g = g_0 \oplus g_1$ and universal complexification $\gamma : G \to G^C$. Furthermore, let $p$ be as in the above theorem. Our goal is to construct a unique universal complexification for $G$. As a first step, recall that the Lie-Hopf superalgebra of $G$ has the form $\mathbb{R}(G)\#E(g)$ which is isomorphic to $\mathbb{R}(G)\#(E(g_0) \otimes \Lambda(g_1))$ due to the Poincaré-Brinkhoff-Witt Theorem. By definition $G$ acts on $g_1$ by the adjoint action $\rho : G \to GL_C(g_1)$. Let $\bar{\rho} : G \to GL_C(g_1 \otimes \mathbb{C})$ denote the $\mathbb{C}$-linearly extended action. By Theorem 3 there is a unique morphism of complex Lie groups $\tilde{\rho}^C : G^C \to GL_C(g_1 \otimes \mathbb{C})$ such that $\tilde{\rho}^C \circ \gamma = \bar{\rho}$.

**Lemma 8.** The super vector space $(g_0 \otimes \mathbb{C})/p^C \oplus (g_1 \otimes \mathbb{C})$ is a Lie superalgebra with superbracket given by the Lie bracket on $(g_0 \otimes \mathbb{C})/p^C$ and the representation $\tilde{\rho}^C : (g_0 \otimes \mathbb{C})/p^C \to End(g_1 \otimes \mathbb{C})$ and the $\mathbb{C}$-linear continuation of the Lie superbracket on $g_1$ followed by projection onto the quotient $(g_0 \otimes \mathbb{C})/p^C$.

**Proof.** Since $p$ is in the kernel of $\gamma_*$, $p^C$ acts trivially on $g_1 \otimes \mathbb{C}$. Consequently $p$ is an ideal of $g$ and hence the object under consideration is the complexification of the quotient $g/p$. \(\square\)

As a result we obtain a candidate for a universal complexification.

**Proposition 3.** The super co-commutative Hopf superalgebra

$$\mathbb{C}(G^C)\pi(E((g_0 \otimes \mathbb{C})/p^C) \otimes \Lambda(g_1 \otimes \mathbb{C}))$$

with $\pi|_{g_0} : G^C \to End((g_0 \otimes \mathbb{C})/p^C)$ given by the adjoint action on $\mathbb{C}(G^C)\pi(E((g_0 \otimes \mathbb{C})/p^C)$ and $\pi|_{g_1} : G^C \to End(g_1 \otimes \mathbb{C})$ given by $\tilde{\rho}^C$ is a Lie-Hopf superalgebra.

**Proof.** By the above Lemma $(g_0 \otimes \mathbb{C})/p^C \oplus (g_1 \otimes \mathbb{C})$ is a Lie superalgebra. Furthermore, $(g_0 \otimes \mathbb{C})/p^C = Lie(G^C)$ and $\pi$ integrates the adjoint representation of $(g_0 \otimes \mathbb{C})/p^C$ on the super vector space $(g_0 \otimes \mathbb{C})/p^C \oplus (g_1 \otimes \mathbb{C})$. \(\square\)

We denote the complex Lie supergroup associated to this Lie-Hopf superalgebra by $G^C$ and define the morphism of real Lie-Hopf superalgebras $\gamma^* : G \to G^C$ by

$$\gamma_* : \mathbb{R}(G)\#(E(g_0) \otimes \Lambda(g_1)) \to \mathbb{C}(G^C)\pi(E((g_0 \otimes \mathbb{C})/p^C) \otimes \Lambda(g_1 \otimes \mathbb{C}))$$

with $\gamma_*(Z) = \gamma_*(Z)$ for $Z \in \mathbb{R}(G)\#E(g_0)$ and $\gamma_* (e\#X_1) = e\#X_1$ for $X_1 \in g_1$.

**Theorem 4.** The complex Lie supergroup $G^C$ together with the morphism of real Lie supergroups $\gamma^* : G \to G^C$ is a universal complexification of $G$.

**Proof.** Let $\mathbb{C}(H)\#(E(h_0) \otimes \Lambda h_1)$ be a Lie-Hopf superalgebra defining a complex Lie supergroup $H$ and $\eta' = (\eta, \eta^\#: G \to H$ a morphism of real Lie supergroups. Note that the corresponding map on the Lie group level $\eta : G \to H$ is uniquely continued by Theorem 3 to a morphism of complex Lie groups denoted by $\eta^C : \mathbb{C}(G^C)\pi(E((g_0 \otimes \mathbb{C})/p^C) \to \mathbb{C}(H)\#E(h_0)$. Continuing $\eta^{|g_1}$ $\mathbb{C}$-linearly to $g_1 \otimes \mathbb{C}$, we obtain a unique morphism $\eta^C = (\eta^C, \eta^C\#)$ of complex Lie supergroups with the property $\eta^C \circ \gamma^* = \eta'$. \(\square\)
5 Conjugation on complex quadratic type I Lie supergroups

In order to analyze the radial part of invariant differential operators for a special type of Lie supergroups, certain results on the structure of the action by conjugation are needed. With the tool developed in the third section of this article, i.e., the identification of a complex Lie supergroup with its complex Lie-Hopf superalgebra, we are able to prove these facts in the complex case.

First we show that if $G$ is a type I Lie supergroup, then in a neighborhood of a generic point $p$ of a maximal torus $G$ is locally a product $\hat{A}_p \times \hat{B}_p$ of complex subsupermanifolds where $\hat{A}_p$ is an open neighborhood of the given point in the torus and $\hat{B}_p$ contains the identity and follows the directions of the root spaces. The local isomorphism of supermanifolds is given by the $G$-action by conjugation.

Secondly, under the additional assumption of the existence of an invariant non-degenerate even supersymmetric bilinear form, we briefly tend to the superdeterminant of the Jacobian of the local isomorphism from the product structure $A \times B$ into the complex Lie supergroup $G$. This globally defined meromorphic function $\gamma$ plays a central role in Berezin’s analysis of the radial part of Laplace-Casimir operators (see [1, chap.II.3.8]). This will become clear in section six.

5.1 Local product structure

Let $g = g_0 \oplus g_1$ be a complex Lie superalgebra of dimension $(t + r|s)$ with even Cartan algebra $\mathfrak{h}$ of dimension $t$ and set of even and odd roots $R = R_0 \cup R_1$ defining 1-dimensional even and odd root spaces. Note that we count a root twice if it appears as an even and an odd root. Such a Lie superalgebra is called a Lie superalgebra of type I. Two well-known Lie superalgebras of this type are the orthosymplectic and the special linear ones. Now let $G = (G, A_G)$ be a complex Lie supergroup with Lie superalgebra $g$ of type I and morphisms of supermanifolds $m^*_G : G \times G \to G$, $s^*_G : G \to G$ and $w^*_G : pt \to G$ for multiplication, inverse and identity. We decompose the Lie superalgebra into root spaces $g = \mathfrak{h} \oplus \bigoplus_{\epsilon \in R} g_\epsilon$, and denote conjugation on Lie group level by

$$\psi : G \times G \to G, \quad (g_1, g_2) \mapsto g_2 \cdot g_1 \cdot g_2^{-1}.$$ \hspace{1cm} (7)

The morphisms of multiplication $m^*_G$ and inverse $s^*_G$ on $G$ induce multiplication and antipode in the associated Lie-Hopf superalgebra $\mathbb{C}(G)\#E(g)$. Hence conjugation is identified with conjugation in $\mathbb{C}(G)\#E(g)$ with respect to $m^*_{G*}$ and $s^*_{G*}$. Thus conjugating on the Lie supergroup level a first argument by a second one is given by

$$m^*_{G*} \circ (m^*_{G*} \otimes s^*_{G*}) \circ (T \otimes \text{Id}) \circ (\text{Id} \otimes \Delta),$$

where $T : \otimes^2 E(G, g) \to \otimes^2 E(G, g)$ is given by

$$T(g_1 \# Z_1 \otimes g_2 \# Z_2) = (-1)^{|Z_1||Z_2|} g_2 \# Z_2 \otimes g_1 \# Z_1$$

for homogeneous elements and $\Delta$ is the comultiplication in $\mathbb{C}(G)\#E(g)$.
Lemma 9. Conjugation on the Lie supergroup level

\[ \psi^* = (\psi, \psi^\#) : G \times G \to G \]

is given by \( \psi \) in (7) and pull-back \( \psi^\# : \psi^*O_G(U) \to O_G \times G(U) \) by

\[ \psi^\#(a\#X_a, b\#X_b) = \psi(a, b)\#Ad(b)(X_aX_b - (-1)^{|X_a||X_b|}Ad(a^{-1})(X_b)X_a) \]

for homogeneous \( a\#X_a, b\#X_b \in A_G^\#. \) Furthermore,

\[ \psi^\#(a\#X_a, b\#1) = \psi(a, b)\#Ad(b)(X_a) \]
\[ \psi^\#(a\#1, b\#X_b) = \psi(a, b)\#Ad(b)(X_b - Ad(a^{-1})(X_b)) \text{ and} \]
\[ \psi^\#(a\#1, b\#1) = \psi(a, b)\#1. \]

Proof. On one hand, for \( g_1\#Z_1 \) and \( g_2\#Z_2 \) with \( Z_2 \in g, \) we use

\( (T \otimes Id) (g_1\#Z_1 \otimes \Delta(g_2\#Z_2)) = (T \otimes Id) (g_1\#Z_1 \otimes g_2\#(Z_2 \otimes 1 + 1 \otimes Z_2)) \)
\( = (-1)^{|Z_1||Z_2|}g_1\#Z_2 \otimes g_1\#Z_1 \otimes g_2\#1 + g_2\#1 \otimes g_1\#Z_1 \otimes g_2\#Z_2 \) \hfill (8)

On the other hand, for homogeneous \( \tilde{g}_1\#\tilde{Z}_1, \tilde{g}_2\#\tilde{Z}_2 \) and \( \tilde{g}_3\#\tilde{Z}_3 \) with \( \tilde{Z}_3 \in \mathbb{C} \cup g \) we have

\[ m^\#_G \left( m^\#_G (\tilde{g}_2\#\tilde{Z}_2 \otimes \tilde{g}_1\#\tilde{Z}_1) \otimes s^\#_G (\tilde{g}_3\#\tilde{Z}_3) \right) = (-1)^km^\#_G \left( (\tilde{g}_2\#1)\#(Ad(\tilde{g}_1^{-1})(\tilde{Z}_2)\tilde{Z}_1) \otimes (\tilde{g}_3\#1)\#Ad(\tilde{g}_3)(\tilde{Z}_3) \right) \]
\( = (-1)^k(\tilde{g}_2\#1)\#Ad(\tilde{g}_1^{-1})(\tilde{Z}_2)\tilde{Z}_1\tilde{Z}_3 \) \hfill (9)

with \( k = 0 \) if \( \tilde{Z}_3 \in \mathbb{C} \) and \( k = 1 \) if \( \tilde{Z}_3 \in g. \) Here we have applied the explicit form of the antipode in (3). Combining (8) and (9) we obtain the lemma. \( \square \)

Aiming at an explicit computation of the restriction of \( \psi^* \) to a suitable product region, we fix and analyze two subsupermanifolds \( A \) and \( B \) of \( G \) according to the decomposition

\( g = h \oplus \bigoplus_{e \in R} \mathfrak{g}_e. \) So let \( A \) denote the complex subsupermanifold of \( G \) with underlying manifold \( A = \exp(h) \) and the restricted sheaf of numerical holomorphic functions \( A_A = O_G|_A. \) Secondly let \( B \subset \exp(\bigoplus_{e \in R} \mathfrak{g}_e) \) be the subset of elements satisfying

\[ b \in B \iff b\#X_a \text{ is transversal to } \exp(\bigoplus_{e \in R} \mathfrak{g}_e) \text{ for all } X_a \in h. \]

Due to the local biholomorphic of the exponential mapping near the identity the open set \( B \) contains the identity \( e \in G. \) Define \( B \) to be the complex subsupermanifold of \( G \) with underlying manifold \( B \) equipped with the sheaf \( A_B \) of those superfunctions on \( G \) near \( B \) which are annihilated by all \( g\#X_a \) for \( X_a \in S(h) \subset E(g). \) Note that by definition of \( B \) these differential operators are always transversal to \( B. \)

The supermanifold \( A \) is a Lie subsupergroup of \( G \) where the embedding is given by the natural embedding of Lie-Hopf superalgebras \( C(A)\#S(h) \subset C(G)\#E(g). \) Furthermore \( B \) is by definition a subsupermanifold of \( G. \) Thus we obtain the structure of a product of supermanifolds \( A \times B \) as a subsupermanifold of the product of Lie supergroups \( G \times G. \)

We now turn to the proof of the expected decomposition for Lie supergroups. For this denote the connected component of the identity in a Lie group \( G \) by \( G_0. \)
Finally we release \( X \) all since \( \text{Ad} \).

Now fix additionally satisfying \( S \) is less than the rank of \( \text{End}(\mathfrak{g}) \).

Therefore we consider the map \( \text{Ad} \) isomorphic to \( \hat{g}_p \) via the restriction of \( \psi^\ast \) in Lemma 3 denoted by

\[
\psi_p^\ast = (\psi_p, \psi_p^\#) : \hat{A}_p \times \hat{B}_p \to \hat{G}_p.
\]

Proof. It is necessary to show that \( \psi_p^\# \) is bijective on primitives. For this we first observe that from Lemma 3 it follows that

\[
\psi_{(a_e),e}^\#(\bigoplus_{e \in R} \mathfrak{g}_e) \subset \bigoplus_{e \in R} \mathfrak{g}_e \quad \text{and} \quad \psi_{(a_e),e}^\#(\mathfrak{h}) \subset \mathfrak{h}.
\]

Thus we can analyze the question of bijectivity separately on the components of \( \mathfrak{g} \) near \( A \times \{e\} \). We do this by first setting \( X_a = 1 \) and analyzing bijectivity on the argument \( X_b \in \bigoplus_{e \in R} \mathfrak{g}_e \).

Therefore we consider the map \( G \to \text{End}(\mathfrak{g}) \) given by \( g \mapsto \text{Id} - \text{Ad}(g) \) and assume that the rank of \( S(g) \) given by

\[
S : G \to \text{Hom}(\mathfrak{g}, \mathfrak{g}), \quad S(g) := (\text{Id} - \text{Ad}(g))|_{\bigoplus_{e \in R} \mathfrak{g}_e} : \bigoplus_{e \in R} \mathfrak{g}_e \to \mathfrak{g}
\]

is less than \( r + s \) for all \( g \in G \). Then there is a vector \( X_g \in \bigoplus_{e \in R} \mathfrak{g}_e \setminus \{0\} \) for each \( g \in G \) satisfying \( S(g)(X_g) = 0 \) which yields \( \text{Ad}(g)(X_g) = X_g \).

Now fix additionally \( g = a^{-1} \) for an \( a = \exp(-Y) \in A \) satisfying \( |\epsilon(Y)| \in (0, 2\pi) \) for all \( \epsilon \in R \).

With \( X_g = \sum_{e \in R} a_e \cdot X_e \in \bigoplus_{e \in R} \mathfrak{g}_e \setminus \{0\} \) we obtain

\[
\text{Ad}(g)(X_g) = X_g \quad \Rightarrow \quad \exp(ad(Y))(X_g) = X_g \quad \Rightarrow \quad \sum_{e \in R} a_e \cdot (1 - e^{\epsilon(Y)}) \cdot X_e = 0
\]

which is a contradiction to the assumption that the rank of \( S(g) \) is smaller than \( r + s \) for all \( g \in G \). Since \( S \) is holomorphic, the rank of \( S(g) \) is equal to \( r + s \) for \( g \) in an open dense subset of \( G_0 \). Due to the fact that \( \text{Ad}(b) \) is bijective, the rank of \( \psi_{(a,b),e}^\#|_{\bigoplus \mathfrak{g}_e} \) is maximal near \( \hat{A} \times \{e\} \) for an open dense subset \( \hat{A} \subset A \).

Finally we release \( X_a \in \mathfrak{h} \) and fix \( X_b = 1 \) obtaining that the rank of \( \psi_{(a,b),e}^\#|_{\mathfrak{h}} \) is constantly \( t \) since \( \text{Ad}(b) \) is bijective. So altogether the rank of \( \psi_{(a,b),e}^\# \) is maximal near \( \hat{A} \times \{e\} \).

5.2 The Jacobian of conjugation

We start by fixing some additional structure. Let \( G = (G, \mathcal{A}_G) \) be a Lie supergroup with Lie superalgebra \( \mathfrak{g} \) and assume additionally that there is an invariant, supersymmetric, non-degenerate, even bilinear form \( b \) on \( \mathfrak{g} \). We call such a Lie superalgebra with fixed form \( b \) a quadratic Lie superalgebra and \( b \) a quadratic form. Note that \( b \) induces an invariant bilinear form \( b_{\mathcal{M}} \) on the sheaf \( \text{Der}(\mathcal{A}_G) \).

If \( \mathcal{M} \) and \( \mathcal{N} \) are complex supermanifolds of dimension \( (n,2m) \) with fixed supersymmetric, non-degenerate, even bilinear forms \( b_{\mathcal{M}} \) and \( b_{\mathcal{N}} \), then the superdeterminant of the Jacobian of an isomorphism \( \varphi : \mathcal{M} \to \mathcal{N} \) can be defined up to a sign in the following way: First choose local orthosymplectic frames for the sheaves of superderivations on the domain of definition.
M and the image space N. These are frames such that the bilinear form is given by the block-
diagonal matrix consisting of I_n := diag(1, ..., 1) in the first diagonal block and \((0 -I_m 0)\) in
the second. Secondly determine the linearization matrix for the isomorphism \(\varphi^*\) according to
these frames. Change of orthosymplectic frames has at most the effect of changing the sign
of the superdeterminant of this matrix.

Now let \(\mathcal{G} = (G, \mathcal{A}_g)\) be a complex Lie supergroup with quadratic type I Lie superalgebra
\(\mathfrak{g}\). Let \(\mathcal{A}\) and \(\mathcal{B}\) be as in the previous subsection and define quadratic forms on both by
the restriction of the quadratic form on \(\mathcal{G}\). Furthermore let \(\psi^*_p\) be as in Proposition 4 We are
interested in information about \(\gamma = sdet(Jac(\psi^*_p))\) and find at first:

**Lemma 10.** In the notation of Proposition 4 the superfunction \(\gamma = sdet(Jac(\psi^*_p))\) is annihi-
lated by all superderivatives along \(\hat{\mathcal{B}}_p\). As a consequence \(\gamma\) may be identified with a numerical
function on \(\hat{\mathcal{A}}_p\).

**Proof.** The explicit form of \(\psi^*_p\) given in Lemma 9 shows that \(X_\beta.\gamma \equiv 0\) for all \(X_\beta \in \mathfrak{g}_\beta, \beta \in R_1\). Due to the
\(Ad\)-invariance of the quadratic form the superfunction \(\gamma\) also satisfies \(X_\alpha.\gamma \equiv 0\) for all \(X_\alpha \in \mathfrak{g}_\alpha, \alpha \in R_0\). Thus \(\gamma\) can be identified with a (super-)function on the
non-graded manifold \(\hat{\mathcal{A}}\).

A formula for \(\gamma\) in the non-graded setting is determined and used by Helgason in [3, chap.I.5.2].
Berezin deduces it regarding the density of integrals using Grassmann analytical continuation in
[1, chap.II.2.9]. A direct proof can be found in the authors dissertation [7]. There the
superdeterminant of the Jacobian is calculated with respect to specific orthosymplectic frames.

**Proposition 5.** In the situation of Proposition 4 with the additional assump-
tion of the existence of an invariant quadratic form, the superdeterminant of the Jacobian of \(\psi^*_p\) is

\[
\gamma(\exp(Y)) = 2^{\left|R_0\right| - \left|R_1\right|} \cdot i^{\left|R_0\right|} \cdot \prod_{\alpha \in R_0^+} \sinh^2(\alpha(Y)/2) \prod_{\beta \in R_1^+} \sinh^2(\beta(Y)/2)
\]

for \(\exp(Y) \in \hat{\mathcal{A}}_p\) (10)

where \(R^+ = R_0^+ \cup R_1^+\) is the set of positive roots in \(R\).

Finally we state a remark:

**Remark 3.** Note that the function \(\gamma\) is defined globally on \(\hat{\mathcal{A}}\) in Proposition 4 Hence Propo-
sition 4 yields a better description of this set:

\[
\hat{\mathcal{A}} = (A \cap \text{Dom}(\gamma)) \setminus \{\gamma = 0\} = A \setminus \bigcup_{\epsilon \in R^+} \exp(\text{Ker}(\epsilon))
\]

6 Radial parts of Laplace-Casimir operators

In preparation for the analysis of Laplace-Casimir operators we define a second order Lapla-
cian \(L_M\) on a split complex supermanifold \(M\) with supermetric and analyze its restriction to
a subsupermanifold which is embedded respecting some geometric restrictions. Here we can
prove a Helgason type formula for the radial part of \(L_M\) (see [3]) which yields, when applied
in the setting of the last section, a local formula for radial parts of Laplace-Casimir operators.
In the second subsection we prove in a main step that the decomposition in Proposition \ref{prop:decomposition} satisfies the geometric conditions in which the Helgason type formula holds. Then the last part of the proof of the Berezin type formula in the local setting is parallel to his global proof in \cite{Berezin}.

6.1 A Helgason type formula for the Laplacian

In contrast to the classical setting it is not quite clear how a Laplacian can be defined on a supermanifold with supermetric. We aim at a form for the Laplacian \(L_M = \ast_M d_M \ast_M d_M\) for appropriate Hodge and de Rham operators, having to decide first which is the correct super vector bundle where these operators act. Here we will restrict to the case of a real or complex split supermanifold \(M = (M, \mathcal{A}_M)\) of dimension \((n_M|m_M)\) with fixed splitting, i.e. an isomorphism of sheaves from the sheaf of section in the full exterior power of the associated vector bundle (see \cite{Witten}) to the sheaf of superfunctions.

First we discuss the operators \(d_M\) and \(\ast_M\). Due to the fixed splitting we obtain a well defined decomposition of the tangent sheaf \(T_M = T_0^r M \oplus T_1^r M\), where for local homogeneous coordinates on \(U \subset M\) the super vector space \(T_i^r M\) is the left-\(A_M\)-supermodule generated by the superderivations along the even, respectively odd coordinates. Regarding the induced decomposition \(T^*M = T_0^r M \oplus T_1^r M\) set

\[
\tilde{E}_r^r := \Lambda^r(T_0^r M \oplus \Pi(T_1^r M)) \quad \text{and} \quad \tilde{E}_M := \bigoplus_r \tilde{E}_r^r.
\]

(11)

Here \(\Pi\) denotes the interchange of parity. Note that the sheaves in (11) are locally free sheaves of left-\(A_M\)-supermodules. We call sections in \(\tilde{E}_r^r\) **differential forms of degree** \(r\). In contrast to the sheaf \(E_r^r\) defined before these forms are really dual objects to super vector fields. So this is the right object to define the \(d_M\)-operator parallel to the classical situation as an exterior derivative. Note furthermore that \(\tilde{E}_r^{n_M + m_M}\) is a 1-dimensional left-\(A_M\)-supermodule.

A global section defining everywhere a local frame is called a **volume form**.

Now assume in addition that \(M\) is equipped with an even, supersymmetric, non-degenerate bilinear form \(b_M\) on the tangent sheaf. This fixes a Berezinian for given homogeneous coordinates \((x_i, \xi_j)\) by

\[
\omega^\phi_{(x_i, \xi_j)} := \sqrt{\text{sdet}_{(x_i, \xi_j)}(b_M)} \cdot dx_1 \wedge \ldots \wedge dx_{n_M} \otimes \frac{\partial}{\partial x_1} \circ \cdots \circ \frac{\partial}{\partial x_{n_M}}
\]

whenever the square root is well defined. Let us assume this now and call \((M, b_M)\) a quadratic supermanifold. The dualization defined by \(b_M\) on the tangent sheaf allows us to transform the Berezinian form into a volume form which we will denote \(\omega_{(x_i, \xi_j)}\). This object comes from integration theory, so here is the right setting to define the \(\ast_M\)-operator by contraction of \(\omega_{(x_i, \xi_j)}\) with the dualization of a differential form.

Secondly we geometrically further restrict the underlying situation where we wish to discuss radial operators, already having the situation in section five in mind: Let \((M, b_M)\) be a quadratic supermanifold. Assume further that there are fixed quadratic subsupermanifolds \((N, b_N)\) of dimension \((n_N|m_N)\) and \((B, b_B)\) of \((M, b_M)\) and an isomorphism of supermanifolds \(\psi^* : N \times B \to M\) such that the induced decomposition \(TM \cong TN \oplus TB\) is orthogonal.
Here we call a superfunction on $\mathcal{M}$ a **radial superfunction** if it is annihilated by all super differential operators along $\mathcal{B}$. This yields a bijection between radial superfunctions on $\mathcal{M}$ and superfunctions on $\mathcal{N}$. A differential operator on $\mathcal{M}$ is called a **radial differential operator** if it maps radial superfunctions to radial superfunctions and the **radial part** of such an operator $D$ is defined to be the differential operator on $\mathcal{N}$ by

$$\Delta(D) := R_{\mathcal{N}}^\omega \circ D \circ F_{\mathcal{M}}^\omega,$$

where $R_{\mathcal{N}}^\omega$ and $F_{\mathcal{M}}^\omega$ are the well defined restriction, respectively extension of (radial) superfunctions.

For our analysis of radial parts of radial differential operators with respect to $\mathcal{N}$ we add two further modifications: First we demand that $\gamma := \text{sdet}(\text{Jac}(\psi')) \in A_{\mathcal{N} \times \mathcal{B}}$ is annihilated by all differential operators along $\mathcal{B}$ hence it is the pull-back of a radial superfunction on $\mathcal{M}$. Secondly let $\iota : \mathcal{N} \to \mathcal{M}$ be the embedding of the subsupermanifold. We demand that $\mathcal{N}$ is embedded compatible with the product structure $\iota \circ \psi = \text{Id}_{\mathcal{N}}$.

**Definition 4.** We call such a setting $(\mathcal{M}, b_{\mathcal{M}}, \mathcal{N}, b_{\mathcal{N}}, \mathcal{B}, b_{\mathcal{B}}, \psi')$ geometrically compatible. It is called geometrically compatible with fixed splitting if $\mathcal{N}$ is a supermanifold with fixed splitting and $\mathcal{B}$ and $\mathcal{N}$ are subsupermanifolds with compatible fixed splitting.

In a third step we determine the radial part of the Laplacian in a geometrically compatible setting with fixed splitting. For this we need two technical lemmas on the two operators which build the Laplacian on $\mathcal{M}$ in order to show on the one hand that $L_{\mathcal{M}}$ is radial with respect to $\psi'$ and $\mathcal{N}$ and on the other hand to calculate its radial part.

We denote the induced Berezinian forms by $\omega_{\mathcal{M}}, \omega_{\mathcal{N}}$ and $\omega_{\mathcal{B}}$ and the associated volume forms by $\omega_{\mathcal{N}}$, $\omega_{\mathcal{B}}$ and $\omega_{\mathcal{N}}$. For the de Rham operator we have the following result.

**Lemma 11.** In a geometrically compatible setting with fixed splitting we have the equalities $d_{\mathcal{M}}(F_{\mathcal{N}}^M f_N) = d_N(f_N)$ for a function $f \in A_N$ and furthermore $d_{\mathcal{M}}(\eta_N) = d_N(\eta_N)$ for $\eta_N \in E_{\mathcal{N}}^{n_N+m_N-1}$. In both cases we have used the embedding $\mathcal{E}_{\mathcal{N}} \subset \mathcal{E}_{\mathcal{M}}$ by continuation on $TB$ by zero.

**Proof.** The first equality follows from $(d_{\mathcal{M}} \circ F_{\mathcal{N}}^M)(f_N)(\chi) = 0$ for a superderivation $\chi$ with values in $TB$. This holds since $F_{\mathcal{N}}^M f_N$ is a radial function. The second equality follows from an analogue argument.

Now let us look more closely at the behavior of the Hodge operators in such a situation. For this we set $\omega := \gamma \cdot \omega_B$.

**Lemma 12.** In a geometrically compatible setting with fixed splitting it is $*_{\mathcal{M}}(\eta_a) = *_{\mathcal{N}}(\eta_a) \cdot \omega$ for a differential 1-form $\eta_a$ on $\mathcal{M}$ which is zero on $TB$ and $*_{\mathcal{N}}(\eta_b) = (R_{\mathcal{N}}^\omega \circ *_{\mathcal{M}})(\eta_b \cdot \omega)$ for a volume form $\eta_b$ on $\mathcal{N}$.

**Proof.** We have $\gamma \cdot \psi^{-1} *_{\mathcal{N}}(\omega_{\mathcal{N}}^\omega \cdot \omega_{\mathcal{B}}) = \omega_{\mathcal{M}}^\omega$, which transfers to the associated volume forms since $(\mathcal{N}, b_{\mathcal{N}})$ and $(\mathcal{B}, b_{\mathcal{B}})$ are quadratic submanifolds of $(\mathcal{M}, b_{\mathcal{M}})$ and since the splitting $T_{\mathcal{M}} \simeq T_{\mathcal{N}} \oplus TB$ is orthogonal. With $pr_{\mathcal{N}}^\omega \circ \psi^{-1} \circ \iota = \text{Id}_{\mathcal{N}}$ we can identify $\psi^* \omega_{\mathcal{N}} = \omega_{\mathcal{N}}$ and follow $\iota_{\mathcal{N}}^{-1}(\psi^{-1} *_{\mathcal{B}}(\psi^{-1} *_{\mathcal{N}} \omega_{\mathcal{B}})) = \gamma^{-2}$ using $\iota_{\mathcal{N}} \omega_{\mathcal{M}} \omega_{\mathcal{N}} \equiv 1$. The proof is then completed by a direct calculation.

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*aWe omit the symbols $\psi^*_{\mathcal{N}}$ as well as $pr_{\mathcal{N}}^\#$, in order to keep the calculation readable.
Hence we obtain the following central proposition

**Proposition 6.** In a geometrically compatible setting with fixed splitting the Laplacian $L_{M}$ is a radial differential operator with respect to $\psi'$ and $N$.

**Proof.** For a function $f_{N} \in A_{N}$ it follows from Lemmas 11 and 12 that

$$\ast_{M} d_{M} F_{N}^{M} f_{N} = \ast_{N}(d_{N} f_{N}) \cdot \omega . \quad (12)$$

Now $d_{M}(\omega_{B}) = 0$ since $\omega_{B}$ is a volume form on $B$ and hence

$$d_{M} \omega = \gamma^{-1} \cdot d_{N} \gamma \cdot \omega . \quad (13)$$

So $d_{M} \ast_{M} d_{M} F_{N}^{M} f_{N}$ is of the form $\eta \cdot \omega$ for a highest degree differential form $\eta$ on $N$. Applying $\ast_{M}$ and Lemma 12 together with the fact that $\gamma$ comes from a radial superfunction on $M$ we obtain the proposition.

Thus the radial part of the Laplacian in a geometrically compatible setting with fixed splitting is defined and we obtain a generalization of theorem 3.7 in [3, chap.II.3] first to the graded setting and secondly to the complex case without the necessity of a group action.

**Proposition 7.** In a geometrically compatible setting with fixed splitting the radial part of the Laplacian $L_{M}$ on $(M, b_{M})$ with respect to $\psi'$ and $N$ is given by

$$\Delta(L_{M}) = \gamma^{-\frac{1}{2}} \cdot L_{N} \circ \gamma^{\frac{1}{2}} - \gamma^{-\frac{1}{2}} \cdot L_{N}(\gamma^{\frac{1}{2}}) , \quad (14)$$

where $\gamma = sdet(Jac(\psi'))$ is seen as a superfunction on $N$.

**Proof.** We calculate the radial part of $L_{M} = \ast_{M} \circ d_{M} \circ \ast_{M} \circ d_{M}$ applying $L_{M}$ to the radial continuation of a function $f_{N} \in A_{N}$. Using (12) we have

$$d_{M} \ast_{M} d_{M} F_{N}^{M} f_{N} = d_{M}(\ast_{M} d_{N} f_{N}) \cdot \omega$$

$$+ (-1)^{n_{N}+m_{N} - 1} \ast_{N}(d_{N} f_{N}) \cdot d_{M} \omega$$

which together with (13) and Lemma 11 and 12 implies

$$\Delta(L_{M}) f_{N} = L_{N}(f_{N})$$

$$+ (-1)^{n_{N}+m_{N} - 1} \gamma^{-1} \cdot \ast_{N}(\ast_{N} d_{N}(f_{N}) \cdot d_{N}(\gamma)) . \quad (15)$$

In addition, for a unit $g \in A_{N,0}$ we observe that

$$L_{N}(g \cdot f_{N}) = (\ast_{N} \circ d_{N} \circ \ast_{N})(d_{N}(f_{N}) \cdot g + f_{N} \cdot d_{N} g)$$

$$= L_{N}(f_{N}) \cdot g + L_{N}(g) \cdot f_{N} + \ast_{N}(d_{N}(g) \cdot \ast_{N} d_{N}(f_{N}))$$

$$+ \ast_{N}(d_{N}(f_{N}) \cdot \ast_{N} d_{N}(g)) .$$

A calculation in local coordinates shows that the last two terms coincide. Consequently we have

$$L_{N}(f_{N}) = g^{-1} \cdot L_{N}(g \cdot f_{N}) = g^{-1} \cdot L_{N}(g) \cdot f_{N}$$

$$- 2 \cdot g^{-1} \cdot \ast_{N}(d_{N}(g) \cdot \ast_{N} d_{N}(f_{N})) .$$

Now we use

$$\ast_{N}(d_{N}(g) \cdot \ast_{N} d_{N}(f_{N})) = (-1)^{n_{N}+m_{N} - 1} \ast_{N}(\ast_{N} d_{N}(g) \cdot d_{N} f_{N}) ,$$

set $g = \gamma^{\frac{1}{2}}$ and apply equation (15). The lemma follows from $d_{N} \gamma = d_{N}(\gamma^{2}) = 2g \cdot d_{N} g$. □

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The square root is well defined according to the definition of a quadratic supermanifold above.
6.2 Berezin’s formula for radial parts

First we will prove that the situation in section five satisfies the definition of a geometrically compatible setting with fixed splitting. Then we prove that the Laplacian on \((G, b_G)\) is a biinvariant second order differential operator. Finally we use the formula in Proposition 7 to proof a local version of a result by Berezin.

The first step is contained in the following Lemma:

**Lemma 13.** Using the notation in Proposition 7 the septuple

\[(M, b_M, N, b_N, \mathcal{B}, b_B, \psi^*) = (\hat{G}_p, b_G, \hat{A}_p, b_A, \hat{B}_p, b_B, \psi_p^*)\]

for \(p \in \hat{A}\) together with the splitting induced by the canonical splitting on \(G\) by Kostant’s construction defines a geometrically compatible setting with fixed splitting.

**Proof.** First all three supermanifolds are quadratic, due to the fact that \(b_G\) is invariant. From Lemma 10 we already know that \(\gamma\) is radial. Furthermore, we have \(Ad(a^{-1})(X) \in \bigoplus_{c \in R} \mathfrak{g}_c\) if \(X \in \bigoplus_{c \in R} \mathfrak{g}_c\). For \(Y \in \mathfrak{h}\), \(X_c \in \mathfrak{g}_c\) and \(Y' \in \mathfrak{h}\) with \(\epsilon(Y') \neq 0\) we have

\[\epsilon(Y')b(Y, X_c) = b(Y, [Y', X_c]) = -b([Y', Y], X_c) = 0.\]

So \(\mathfrak{h}\) is orthogonal to each root space and the decomposition \(T \hat{G}_p \cong T \hat{A}_p \oplus T \hat{B}_p\) is orthogonal. By direct calculation \(pr_{\hat{N}} \circ \psi^{-1} \circ i^* = Id_{\hat{N}}\) follows from Lemma 9.

Next we turn our attention to invariant operators: **Laplace-Casimir operators** on a quadratic Lie superalgebra \((\mathfrak{g}, b)\) are the biinvarinat continuations of elements in the center of the universal enveloping algebra \(E(\mathfrak{g})\) called **Casimir elements**. For a homogeneous basis \(V_i\) of \(\mathfrak{g}\) there is a Casimir element of order two which can be defined as follows (see e.g. \([6]\)):

\[C_2(V_1, \cdots, V_n) := \sum_{1 \leq i, k \leq n} b(\theta(V_i), \theta(V_k))V_i \cdot V_k, \quad (16)\]

where \(\theta: \mathfrak{g} \rightarrow \mathfrak{g}\) maps \(V_i\) to the \(i\)-th element of the dual basis of the basis \(\{b(V_j, \cdot)\}_j\) of \(\mathfrak{g}^*\). The invariant continuation of \(C_2(V_1, \cdots, V_n)\) turns out to be exactly the Laplacian of \((G, b_G)\) where \(b_G\) is the invariant continuation of \(b\):

**Lemma 14.** The Laplacian \(L_G\) equals the Laplace-Casimir operator \(\hat{C}_2\) defined by the element \(C_2 = C_2(V_1, \cdots, V_n)\) being thus independent of the homogeneous basis \(\{V_i\}\) of \(\mathfrak{g}\).

**Proof.** We will use

\[d_G = \sum_i \tilde{V}_i \cdot d v_i, \quad (\tau)^{-1}(d v_i) = \overline{\theta}(V_i) \quad \text{and} \quad \iota_X \omega_G = c \cdot \sum_j \sigma_b(\tilde{V}_j, \tilde{X}) \prod_{k \neq j} d v_k\]

for \(X \in \mathfrak{g}\), where \(\sim\) denotes left-invariant continuation. For \(L_G = \ast_G d_G \ast_G d_G\) and \(f \in A_G\) we
have:

\[
L_G(f) = \ast_G d_G \ast_G \left( \sum_i \tilde{V}_i(f) \, dv_i \right) \\
= \ast_G d_G \left( \sum_i \sum_j (-1)^j \cdot c \cdot \sigma_b(\tilde{V}_j, \theta(\tilde{V}_i)) \tilde{V}_i(f) \prod_{k, k \neq j} dv_k \right) \\
= \ast_G \left( \sum_i \sum_j c \cdot \sigma_b(V_j, \theta(V_i)) \tilde{V}_j \tilde{V}_i(f) \prod_k dv_k \right) = \sum_{i,j} b(\theta(V_j), \theta(V_i)) \tilde{V}_j \tilde{V}_i(f)
\]

Now we argue in a similar way as Berezin: We aim at information about the radial part of Laplace-Casimir operators of arbitrary order. So let \( G \) be a complex Lie Supergroup of type I with quadratic Lie superalgebra \((g, b)\) as discussed in the previous subsection and quadratic superform \( b_G \) induced by \( b \). Assume from now on that \( \gamma_1^2 \) in Proposition 5 seen as a function on \( A \) is an eigenfunction of \( L_A \).

Using (14) we can write in the geometrically compatible setting with fixed splitting of Proposition 7

\[
\Delta(L_{\tilde{G}}^p) = j^{-1} \cdot P_{L_{\tilde{G}}^p} \circ j
\]

for a polynomial with constant coefficients \( P_{L_{\tilde{G}}^p} \) on \( \mathfrak{h} \) and \( j := \gamma_1^2 \). Our first goal is a similar global expression for higher order Laplace-Casimir Operators.

Let \( C \in Z(E(g)) \) be a Casimir element and \( \tilde{C} \) be the associated Laplace-Casimir operator on \( G \). We can state now a local form of theorem 3.2 in [1, chap.II.3.8]:

**Theorem 5.** Let \( G \) be a type I Lie supergroup with quadratic Lie superalgebra \((g, b)\) such that \( \gamma \) is an eigenfunction of the Laplacian on the subset \( \tilde{A} \) of a fixed maximal torus. The differential operator \( \Delta(C) \) associated to a Laplace-Casimir operator defined by a Casimir element \( C \in Z(E(g)) \) and acting on superfunctions on \( \tilde{A}_p \) in the notion of Proposition 7 is of the form

\[
\Delta(C) = j^{-1} \cdot P_C \circ j
\]

for a polynomial with constant coefficients \( P_C \in \mathbb{C}[\mathfrak{h}] \).

**Proof.** The mapping on the radial part

\[
\Delta_p : \{ \tilde{C} \in \text{Diff}_{\tilde{G}} | C \in Z(E(g)) \} \to \text{Diff}_{\tilde{A}_p}
\]

is a homomorphism of superalgebras, so for \( C_1, C_2 \in Z(E(g)) \) it follows that

\[
\Delta_p(C_1) \circ \Delta_p(C_2) = \Delta_p(C_1 \cdot C_2) = \Delta_p(C_2 \cdot C_1) = \Delta_p(C_2) \circ \Delta_p(C_1).
\]

The radial part of a Laplace-Casimir operator \( \Delta_p(C) \) acts on functions in \( A_{\tilde{A}_p} \). So the operator \( \tilde{\Delta}_p(C) := j \Delta_p(C) \circ j^{-1} \) acts on the sheaf of functions \( \{ f \cdot j | f \in A_{\tilde{A}_p} \} \). Let \( \{ Y_i \} \)

---

^Note that all these conditions are satisfied by two important examples of quadratic Lie superalgebras/groups – the orthosymplectic and the special linear case – as it is shown explicitly in [1, chap.II.4.1-6].
denote an orthogonal basis of \( h \) and the symbols \( \tilde{Y}_i \) replace the induced left invariant vector fields on \( G \). This way we have an expression \( \tilde{\Delta}_p(C) = \sum g_I \tilde{Y}_i \) for holomorphic functions \( g_I \) on \( \hat{A}_p \) and restricted fields \( \tilde{Y}_i \) on \( G \). With (19) and (17) we have

\[
\tilde{\Delta}_p(C) \tilde{\Delta}_p(L_{\hat{A}_p}) = \tilde{\Delta}_p(L_{\hat{A}_p}) \tilde{\Delta}_p(C) \Rightarrow \tilde{\Delta}_p(C) L_{\hat{A}_p} \tilde{\Delta}_p(C)
\]

and obtain with \( L_{\hat{A}_p} = \sum_i b(Y_i, Y_i) \frac{\partial^2}{\partial Y_i^2} \) from Lemma 14 the equations

\[
\sum_i b(Y_i, Y_i) \frac{\partial^2 g_I}{\partial Y_i^2} = -2 \sum_i b(Y_i, Y_i) \frac{\partial g_I}{\partial Y_i} \frac{\partial}{\partial Y_i} \quad \forall k \in \mathbb{N} \text{ and } j = 1, \ldots, \dim(a_0).
\]

So all partial derivatives of the \( g_I \) vanish identically and hence \( \Delta_p(C) \) is of the form \( j^{-1} \cdot P_C \circ j \) for a polynomial in \( h \) of constant coefficients.

\[\text{Remark 4. As a consequence of Leibniz rule the highest order term } P_C \text{ equals the highest order term of the orthosymplectic projection of } C \text{ to } S(h) \subset E(g).\]

We finish with a note on Berezin’s global formula. Since the operator \( \tilde{C} \) is defined not only on \( \hat{G}_p \) but also on \( G \) the question arises whether \( \Delta(C) \) can be regarded as an operator on \( A \). Therefore we consider the global morphism \( \varphi : A \times B \rightarrow G \) in Lemma 9. Chevalley’s restriction theorem 6 describes the sheaf of superfunctions \( f \in A \) which are mapped by the pullback of the projection \( pr_A : A \times B \rightarrow A \) to a superfunction \( pr_A^\#(f) \in A \times B \) which is also in the image of the pullback of \( \varphi^* \). We call this sheaf on \( A \) the sheaf of extendable superfunctions \( \tilde{A}_A \subset A_A \). The associated function \( F_A^G(f) \in A_G \) satisfying \( \varphi^*(F_A^G(f)) = pr_A^\#(f) \) is called the extension of \( f \). By definition an element \( f \in \tilde{A}_A \) can be extended to a unique superfunction \( F_A^G f \in A_G \) with the property:

\[
(F_A^G f)|_{\hat{G}_p} = F_{\hat{A}_p}^G (f|_{\hat{A}_p})
\]

Furthermore \( \tilde{C}(F_A^G f) \in A_G \) is again the extension of a superfunction on \( A \). This leads to the result in [1] chap.II.3.8:

\[\text{Corollary 1. Altogether we can regard } \Delta(C) \text{ as an operator } \Delta(C) : \tilde{A}_A \rightarrow \tilde{A}_A, \quad \Delta(C) := R_A^G \circ \tilde{C} \circ F_A^G of the form (18).\]

\[\text{Remark 6. The graded version of this theorem stated by Sergeev can be found in [12].}\]
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