1. Two preliminary algorithms. In this Section, we provide algorithms introduced by Dor and Tarsi [3], and Chickering [1, 2] respectively. These results are necessary to implement our proposed approach technically.

Some definitions and notation are introduced first. A directed edge of a DAG is compulsory if it occurs in the corresponding completed PDAG, otherwise, the directed edge is reversible and the corresponding parents are reversible parents. Recall $N_x$ be the set of all neighbors of $x$, $\Pi_x$ is the set of all parent of $x$, $N_{xy} = N_x \cap N_y$ and $\Omega_{x,y} = \Pi_x \cap N_y$ and the concept of “strongly protected” is presented in Definition 1 in the paper [4].
Algorithm 3 generates a consistent extension of a PDAG [3]. Algorithm 4 creates the corresponding completed PDAG of a DAG [1]. They are used to implement Chickering’s approach.

**Algorithm 3:** (Dor and Tarsi [3]) Generate a consistent extension of a PDAG

**Input:** A PDAG $P$ that admits a consistent extension

**Output:** A DAG $D$ that is a consistent extension of $P$.

1. Let $D := P$;
2. while $P$ is not empty do

   3. Select a vertex $x$ in $P$ such that (1) $x$ has no outgoing edges and (2) if $N_x$ is not empty, then every vertex in $N_x$ is adjacent to all vertices in $N_x \cup \Pi_x$. */ Dor and Tarsi [3] show that a vertex $x$ with these properties is guaranteed to exist if $P$ admits a consistent extension. */

   4. Let all undirected edges adjacent to $x$ be directed toward $x$ in $D$

   5. Remove $x$ and all incident edges from $P$.

6. return $D$

2. **Additional examples, experiment and algorithms.** This section include three parts: (1) some examples to illuminate the methods proposed in the paper [4], (2) an experiment about v-structures, and (3) three algorithms to test the conditions $iu_3$, $id_3$ and $dd_2$ in Algorithm 1.1 only based on $e_i$ in an efficient manner.

2.1. **Examples.** Four examples are presented to illustrate operators, the generation of a resulting completed PDAG of an operator, the conditions of a perfect operator set, and the process of constructing a perfect operator set.

**Example 1.** This example illustrates six operators on a completed PDAG $C$ and their corresponding modified graphs. Figure 7 displays six operators: InsertU $x \rightarrow z$, DeleteU $y \rightarrow z$, InsertD $x \rightarrow v$, DeleteD $z \rightarrow v$, MakeV $z \rightarrow y \leftarrow u$, and RemoveV $z \rightarrow v \leftarrow u$. After inserting an undirected edge $x \rightarrow z$ into the initial graph $C$, we get a modified graph denoted as $P_1$ in Figure 7. By applying the other five operators to $C$ in Figure 7 respectively, we can obtain other five corresponding modified graphs $P_2, P_3, P_4, P_5$, and $P_6$. Here the operator “MakeV $z \rightarrow y \leftarrow u$” modifies $z \rightarrow y \leftarrow u$ to $z \rightarrow y \leftarrow u$ and the operator “Remove $z \rightarrow v \leftarrow u$” modifies $z \rightarrow v \leftarrow u$ to $z \rightarrow v \leftarrow u$. Notice that a modified graph might not be a PDAG though all modified graphs in this example are PDAGs.

In the above example, we see that the modified graph of an operator, denoted by $P$, might be a PDAG, but might not be a completed PDAG.
Algorithm 4: (Chickering [1]) Create the completed PDAG of a DAG

Input: $D$, a DAG
Output: The completed PDAG $C$ of DAG $D$.

1. Perform a topological sort on the vertices in $D$ such that for any pair of vertices $x$ and $y$ in $D$, $x$ must precede $y$ if $x$ is an ancestor of $y$;
2. Sort the edges first in ascending order for incident vertices and then in descending order for outgoing vertices; Label every edge in $D$ as unknown;
3. while there are edges labeled unknown in $D$ do
   4. Let $x \rightarrow y$ be the lowest ordered edge that is labeled unknown
   5. for every edge $w \rightarrow x$ labeled compelled do
      6. if $w$ is not a parent of $y$ then
         7. $x \rightarrow y$ and every edge incident into $y$ with compelled
         Goto 3
      else
         9. Label $w \rightarrow y$ with compelled
      10. if there exists an edge $z \rightarrow y$ such that $z = x$ and $z$ is not a parent of $x$ then
         11. Label $x \rightarrow y$ and all unknown edges incident into $y$ with compelled
      else
         14. Label $x \rightarrow y$ and all unknown edges incident into $y$ with reversible
   15. Let $C = D$ and undirect all edges labeled "reversible" in $C$.
4. return completed PDAG $C$

For example, the modified graphs $P_4$, and $P_6$ in Figure 7 are not completed PDAGs because the directed edge $y \rightarrow v$ is not strongly protected.

Example 2. This example illustrates Chickering’s approach to obtain the resulting completed PDAG of a valid operator from its modified graph. Consider the initial completed PDAG $C$ and the operator “Remove $z \rightarrow v \leftarrow u$” in Figure 7. We illustrate in Figure 8 the steps of Chickering’s approach that generates the resulting completed PDAG $C_1$ by applying “Remove $z \rightarrow v \leftarrow u$” to $C$. The first step (step 1) extends the modified graph (a PDAG $P_6$) to a consistent extension ($D_6$) via Algorithm 3. The second step (step 2) constructs the resulting completed PDAG $C_1$ of the operator “Remove $z \rightarrow v \leftarrow u$” from the DAG $D_6$ via Algorithm 4.

Example 3. This example illustrates that $O$ in Equation (3.3) will not be reversible if condition $iu_3$ or $dd_2$ is not contained in Definition 9. Consider the operator set $O$ defined in Equation (3.3) for $S_5$ and the completed PDAG $C \in S_5$ in Figure 9. We have that operator InsertU $z \rightarrow u$ and DeleteD $z \rightarrow v$ are valid. As shown in Figure 9, InsertU $z \rightarrow u$ transfers $C$ to the completed PDAG $C_1$ and DeleteD $z \rightarrow v$ transfers $C$ to the completed PDAG $C_2$. However, deleting $z \rightarrow u$ from $C_1$ will result in an undirected PDAG distinct
Fig 7. Examples of six operators of PDAG $C$. $P_1$ to $P_6$ are the modified graphs of six operators.

from $C$ and InsertD $z \rightarrow v$ is not valid for $C_2$. As a consequence, if $O$ contains InsertU $z - u$ and DeleteD $z \rightarrow v$, it will be not reversible. According to Definition 9, these two operators do not appear in $O_C$ because they do not satisfy the conditions $iu_3$ and $dd_2$ respectively.

**Example 4.** This toy example is given to show how to construct a concrete perfect set of operators following Definition 9 in the paper [4]. Consider the completed PDAG $C$ in Example 3. Here we introduce the procedure to determine $InsertU_C$. All possible operators of inserting an undirected edge to $C$ include: “InsertU $x - z$”, “InsertU $x - u$”, “InsertU $x - v$” and “InsertU $z - u$”. The operator “InsertU $x - v$” is not valid according to Lemma 3 in the paper [4] since $\Pi(x) \neq \Pi(v)$. The operator “InsertU $z - u$” is valid; however, condition $iu_3$ does not hold. According to Definition 9 in the paper [4], we have that only “InsertU $x - z$” and “InsertU $x - u$” are in $InsertU_C$. Thus $InsertU_C = \{x - z, x - u\}$, where “$x - z$” denotes “InsertU $x - z$” in the set. Table 1 lists the six sets of operators on $C$.

2.2. Experiment about v-structures. Below, we present the experiment result in Figure 10 about the numbers of v-structures of completed PDAGs in $\mathcal{S}_p^p$. 
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Fig 8. Example for constructing the unique resulting completed PDAG of a valid operator. An operator “Remove $z \rightarrow v \leftarrow u$” in Figure 7 is applied to the initial completed PDAG $C$ and finally results in the resulting completed PDAG $C_1$.

Fig 9. Example: Two valid operators bring about irreversibility. It shows valid conditions are not sufficient for perfect operator set.

For $S_{p}^{1.5p}$ in the main window, the medians of the four distributions are 108, 220, 557 and 1110 for $p$=100, 200, 500, and 1000 respectively. Figure 10 shows that the numbers of $v$-structures are much less than $(p^2)$ for most completed PDAGs in $S_p^{1.5p}$ when $r$ is set to 1.2, 1.5 or 3. This result is useful to analyze the time complexities of Algorithm 1 and Algorithm 1.1 in Section 3.2.2 of the paper [4].

2.3. Three Algorithms to check $iu_3$, $id_3$ and $dd_2$ in Algorithm 1.1. The conditions $iu_3$, $id_3$ and $dd_2$ in the fourth group depend on both $e_t$ and the resulting completed PDAGs of the operators. Intuitively, checking these three conditions requires that we obtain the corresponding resulting completed PDAGs. We know that the time complexity of getting a resulting completed PDAG of $e_t$ is $O(p_{n_{e_t}})$ [2, 3], where $n_{e_t}$ is the number of edges in $e_t$.
Table 1: The six sets of operators of \( C \). These operators are perfect.

\[ \begin{align*}
q & \quad q \\
q & \quad q \\
r & \quad x \\
y & \quad z \\
u & \quad v \\
v & \quad u \\
\end{align*} \]

\[ \begin{align*}
\text{InsertU} & : C = \{ x - u, x - u \} \\
\text{DeleteU} & : C = \{ y - z, y - u \} \\
\text{InsertD} & : C = \{ x \rightarrow u \} \\
\text{DeleteD} & : C = \{ y \rightarrow v \} \\
\text{MakeV} & : C = \{ x - y - z, x - y - u, z - y - u \} \\
\text{RemoveV} & : C = \{ u \rightarrow v, z \rightarrow v \} \\
\end{align*} \]

Avoid generating resulting completed PDAG, we provide three algorithms to check \( iu_3, id_3 \), and \( dd_2 \) respectively.

In these three algorithms, we use the concept of strongly protected edges, defined in Definition 2. Let \( \Delta_v \) contain all vertices adjacent to \( v \). To check whether a directed edge \( v \rightarrow u \) is strongly protected or not in a graph \( G \), from Definition 2, we need to check whether one of the four configurations in Figure 1 occurs in \( G \). This can be implemented by local search in \( \Delta_v \) and \( \Delta_u \). We know that when a PDAG is sparse, in general, these sets are small, so it is very efficient to check whether an edge is “strongly protected”.

We are now ready to provide Algorithm 1.1.1, Algorithm 1.1.2, and Algorithm 1.1.3 to check \( iu_3, id_3 \), and \( dd_2 \) only based on \( e_t \), respectively. In these three algorithms, we just need to check whether a few directed edges are strongly protected or not in \( P_{t+1} \), which has only one or a few edges different from \( e_t \). We prove in Theorem 2 that these three algorithms are equivalent to checking conditions \( iu_3, id_3 \), and \( dd_2 \), respectively.

---

**Fig 10.** The distributions of the numbers of v-structures of completed PDAGs in \( S_{rp}^p \). The red lines in the boxes indicate the medians.
Algorithm 1.1.1: Check the condition $iu_3$ in Definition 9

Input: a completed PDAG $e_t$ and a valid operator on it: InsertU $x - y$.
Output: True or False

1. Insert $x - y$ to $e_t$, get the modified PDAG denoted as $P_{t+1}$.
2. for each common child $u$ of $x$ and $y$ in $P_{t+1}$ do
   3. if either $x \rightarrow u$ or $y \rightarrow u$ is not strongly protected in $P_{t+1}$ then
      4. return False
3. return True ($iu_3$ holds for InsertU $x - y$)

Algorithm 1.1.2: Check the condition $id_3$ in Definition 9

Input: a completed PDAG $e_t$ and a valid operator: InsertD $x \rightarrow y$.
Output: True or False

1. Insert $x \rightarrow y$ to $e_t$, get a PDAG, denoted as $P_o$.
2. for each undirected edge $u - y$ in $P_o$, where $u$ is not adjacent to $x$ do
   3. update $P_o$ by orienting $u - y$ to $y \rightarrow u$.
4. for each edge $v \rightarrow y$ in $P_o$ do
   5. if $v \rightarrow y$ is not strongly protected in $P_o$ then
      6. update $P_o$ by changing $v \rightarrow y$ to $v - y$,
7. Set $P_{t+1} = P_o$
8. for each common child $u$ of $x$ and $y$ in $P_{t+1}$ do
   9. if $y \rightarrow u$ is not strongly protected in $P_{t+1}$ then
      10. return False
11. return True ($id_3$ holds for InsertD $x \rightarrow y$)

Theorem 2 (Correctness of Algorithms 1.1.1, 1.1.2 and 1.1.3). Let $e_t$ be a completed PDAG. We have the following results.

(i) Let InsertU $x - y$ be any valid operator of $e_t$, then condition $iu_3$ holds for the operator InsertU $x - y$ if and only if the output of Algorithm 1.1.1 is True.

(ii) Let InsertD $x \rightarrow y$ be any valid operator of $e_t$, then condition $id_3$ holds for the operator InsertD $x \rightarrow y$ if and only if the output of Algorithm 1.1.2 is True.

(iii) Let DeleteD $x \rightarrow y$ be any valid operator of $e_t$, then condition $dd_2$ holds for the operator DeleteD $x \rightarrow y$ if and only if the output of Algorithm 1.1.3 is True.

In Theorem 2, we show that an algorithm (Algorithm 1.1.1, Algorithm 1.1.2, or Algorithm 1.1.3) returns True for an operator if and only if the
Algorithm 1.1.3: Check the condition \( dd_2 \) in Definition 9

\begin{verbatim}
1. Input: a completed PDAG \( e_t \) and a valid operator \( \text{DeleteD} \ x \rightarrow y \)
2. Output: True or False
3. 1 Delete \( x \rightarrow y \) from \( e_t \), get a PDAG, denoted as \( P_{t+1} \);
4. for each parent \( v \) of \( y \) in \( P_{t+1} \) do
5. if \( v \rightarrow y \) is not strongly protected in \( P_{t+1} \) then
6. return False
7. return True (\( dd_2 \) holds for \( \text{DeleteD} \ x \rightarrow y \))
\end{verbatim}

corresponding condition (\( iu_3 \), \( id_3 \) or \( dd_2 \)) holds for the operator. Theorem 2 says that we do not have to examine the resulting completed PDAG to check conditions \( iu_3 \), \( id_3 \) and \( dd_2 \), which saves much computation time.

3. Proofs. We will provide a proof of Theorem 2 in Subsection 3.1 below. Notice that we present Theorem 2 in Subsection 2.3 to show the correctness of Algorithm 1.1.1, Algorithm 1.1.2 and Algorithm 1.1.3.

3.1. Proof of Theorem 2 introduced in Subsection 2.3. To prove Theorem 2, we need the following lemmas that have been introduced in the paper [4].

**Lemma 6.** For any operator \( o \in O_C \) denoted by “\( \text{InsertD} \ x \rightarrow y \)”, the operator “\( \text{DeleteD} \ x \rightarrow y \)” is the reversible operator of \( o \).

**Lemma 12.** Let graph \( C \) be a completed PDAG, \( \{w,v,u\} \) be three vertices that are adjacent each other in \( C \). If there are two undirected edges in \( \{w,v,u\} \), then the third edge is also undirected.

**Lemma 14.** Let \( C \) be any completed PDAG, and let \( P \) denote the PDAG that results from adding a new edge between \( x \) and \( y \). For any edge \( v \rightarrow u \) in \( C \) that does not occur in the resulting completed PDAG extended from \( P \), there is a directed path of length zero or more from both \( x \) and \( y \) to \( u \) in \( C \).

**Lemma 15.** Let \( \text{InsertU}_C \) and \( \text{DeleteU}_C \) be the operator sets defined in Definition 9 in the paper [4]. For any \( o \) in \( \text{InsertU}_C \) or in \( \text{DeleteU}_C \), where \( P' \) is the modified graph of \( o \) that is obtained by applying \( o \) to \( C \), we have that \( P' \) is a completed PDAG.

**Lemma 17.** If the graph \( P_1 \) obtained by deleting \( a \rightarrow b \) from a completed PDAG \( C \) can be extended to a new completed PDAG, \( C_1 \), then we have that for any directed edge \( x \rightarrow y \) in \( C \), if \( y \) is not \( b \) or a descendent of \( b \), then \( x \rightarrow y \) occurs in \( C_1 \).
There are three statements in Theorem 2; we prove them one by one below.

**Proof of (i) of Theorem 2**

(If)

Figure 1 shows the four cases that ensure that an edge is strongly protected. We first show that for any edge $x \rightarrow u$ (or $y \rightarrow u$), where $u$ is a common child of $x$ and $y$, if $x \rightarrow u$ is strongly protected in $P_{t+1}$ by configuration (a), (b), or (d) in Figure 1 (replace $v \rightarrow u$ by $x \rightarrow u$), it is also directed in $e_{t+1}$.

Case (1), (2) and (3) in Figure 11 show the sub-structures of $P_{t+1}$ in which $x \rightarrow u$ is protected by case (a), (b) and (d) in Figure 1 respectively, where $P_{t+1}$ is the modified graph obtained by inserting $x \rightarrow y$ into $e_t$.

If $x \rightarrow u$ is protected in $P_{t+1}$ like case (1) in Figure 11, $w \rightarrow x \rightarrow u$ occurs and $w$ and $u$ are not adjacent in $P_{t+1}$. If $w \rightarrow x$ is undirected in $e_{t+1}$, from Lemma 14, there exists a directed path from $y$ to $x$. Any parent of $x$ that is in this path must not be a parent of $y$; otherwise, there exists a directed cycle from $y$ to $y$ in $e_t$. Hence we have that the parent sets of $y$ and $x$ are not equal. This is a contradiction of the condition $\Pi_x = \Pi_y$ in Lemma 3 in the paper [4]. We have that $w \rightarrow x$ and $x \rightarrow u$ occur in $e_{t+1}$.

If $x \rightarrow u$ is protected in $P_{t+1}$ by $v$-structure $x \rightarrow u \leftarrow w$, like case (2) in Figure 11, clearly, the $v$-structure also occurs in $e_{t+1}$, so $x \rightarrow u$ occurs in $e_{t+1}$.

If $x \rightarrow u$ is protected in $P_1$ like case (3) in Figure 11, we have that the $v$-structure $w \rightarrow u \leftarrow w_1$ also occurs in $e_{t+1}$. If either $x \rightarrow u$ or $u \rightarrow x$ is in $e_{t+1}$, we have that $w_1 \rightarrow x$ and $w \rightarrow x$ are both in $e_{t+1}$ and the $v$-structure $w_1 \rightarrow x \leftarrow w$ occurs. Hence we have have that $x \rightarrow u$ occur in $e_{t+1}$.

---

**Fig 11.** strongly protected in $P_{t+1}$
Now we show that if \( x \to u \) is protected in \( P_{t+1} \) like (c) in Figure 1, it is also protected in \( e_{t+1} \). For any \( u_1 \) in \( x \to u_1 \to u \), there are only two cases: \( u_1 \) and \( y \) are adjacent or nonadjacent.

When \( u_1 \) and \( y \) are not adjacent, like (4) in Figure 11, there is a \( v \)-structure \( u_1 \to u \leftarrow y \in P_{t+1} \). Then \( u_1 \to u \) occurs in \( e_{t+1} \). If \( x - u \) occurs in \( e_{t+1} \), by Lemma 12, the edge between \( x \) and \( u_1 \) must be directed and oriented as \( u_1 \to x \) in \( e_{t+1} \). This is impossible, because there exists some extension of \( P_{t+1} \) that has an edge oriented as \( x \to u_1 \). Thus, \( x \to u \) occurs in \( e_{t+1} \).

When \( u_1 \) and \( y \) are adjacent, we have that \( u_1 \to y \) and \( u_1 - y \) do not occur in \( P_{t+1} \) since \( P_x = P_y \) must hold in \( e_t \) for the validity of the operator InsertU \( x - y \). Hence we have that \( y \to u_1 \) occurs in \( P_{t+1} \) and \( x \to u \) is strongly protected like case (5) in Figure 11. We consider two cases: \( x \to u_1 \) occurs or does not occur in \( e_{t+1} \).

Assume \( x \to u_1 \) occurs in \( e_{t+1} \). If \( u_1 \to u \) occurs in \( e_{t+1} \), clearly, \( x \to u \) must occur in \( e_{t+1} \) because there is a partially directed path \( x \to u_1 \to u \) in \( e_{t+1} \). If \( u_1 \to u \) is undirected in \( e_{t+1} \), from Lemma 12, \( x \to u \) must occur in \( e_{t+1} \).

In case (5), we have that \( u_1 \) is also a common child of \( x \) and \( y \), so, \( x \to u_1 \) will also be strongly protected in \( P_{t+1} \) from the condition \( \text{iu}_3 \). Now, consider \( x \to u_1 \); if it is protected in \( P_{t+1} \) like any of case (1), (2), (3), or (4), then, by our proof, \( x \to u_1 \) occurs in \( e_{t+1} \). Thus, \( x \to u \) must occur in \( e_{t+1} \). If \( x \to u \) is protected in \( P_{t+1} \) like case (5), we can find another vertex \( u_2 \) that is a common child of \( x \) and \( y \) like case (6). From the proof above, we know if \( x \to u_2 \) occurs in \( e_{t+1} \), \( x \to u_1 \) and \( x \to u \) also occur in \( e_{t+1} \). Since the graph has finite vertices, we can find a common child of \( x \) and \( u_1 \), say \( u_k \), such that \( x \to u_k \) is protected in \( P_{t+1} \) like one of cases (1), (2), (3) or (4). Thus, \( x \to u_k \) occurs in \( e_{t+1} \), implying that \( x \to u_{k-1} \) occurs in \( P_{t+1} \), so, finally, \( x \to u \) occurs in \( P_{t+1} \).

(Only if) From Lemma 15, we have that the modified graph \( P_{t+1} \) is also the resulting completed PDAG \( e_{t+1} \). Hence, all directed edges in \( e_{t+1} \) are strongly protected in \( P_{t+1} \), so the Algorithm 1.1.1 will return True.

□

Proof of (ii) of Theorem 2

To prove (ii) of Theorem 2, we need following lemma.

**Lemma 21.** Let \( e_t \) be a completed PDAG, \( P_{t+1} \) be the PDAG obtained in Algorithm 1.1.2 with input of a valid operator InsertD \( x \to y \), and \( e_{t+1} \) be the resulting completed PDAG extended from \( P_{t+1} \). We have:

1. If \( u \) is not a common child of \( x \) and \( y \), then all directed edges \( y \to u \) in \( P_{t+1} \) are also in \( e_{t+1} \).
2. All directed edges $v \to y$ in $\mathcal{P}_{t+1}$ are also in $e_{t+1}$;

Proof. (1)

If $u$ is not a common child of $x$ and $y$, and $y \to u$ occurs in $\mathcal{P}_{t+1}$, we have that there is a structure like $x \to y \to u$ in $\mathcal{P}_{t+1}$. Because $x \to y$ occurs in $e_{t+1}$, $y \to u$ must be in $e_{t+1}$ too.

(2)

From Algorithm 1.1.2, all directed edges $v \to y$ are strongly protected in $\mathcal{P}_{t+1}$. When $v$ is not adjacent to $x$ in $e_{t+1}$, $v \to y \leftarrow x$ is a $v$-structure, so $v \to y$ occurs in $e_{t+1}$. When $v$ is adjacent to $x$, we show below that if $v \to y$ is strongly protected like one of four cases in Figure 12, it is also strongly protected in $e_{t+1}$.

![Figure 12](image)

In case (1) of Figure 12, because there is no path from $y$ to $v$, we have that $w \to v$ occurs in $e_{t+1}$ from Lemma 14. Hence we have that $v \to y$ occurs in $e_{t+1}$.

In case (2), there is a $v$-structure $w \to y \leftarrow v$ in $\mathcal{P}_{t+1}$. So, $v \to y$ occurs in $e_{t+1}$.

In case (3), because there is no path from $y$ to $u$, we have that $v \to u$ occurs in $e_{t+1}$ according to Lemma 14. If $u \to y$ occurs in $e_{t+1}$, $v \to y$ occurs in $e_{t+1}$. If $u \to y$ become $u - y$ in $e_{t+1}$, $v \to y$ must also be in in $e_{t+1}$ from Lemma 12.

From the proof of (i) of Theorem 2, we also have that $v \to y$ must be in $e_{t+1}$ when case (4) occurs in $\mathcal{P}_{t+1}$.

Notice that the above proof also holds when we replace $x - v$ by a directed edge or add an edge between $x$ and $w$( or $u$). Hence we have that $v \to y$ in $\mathcal{P}_{t+1}$ also occurs in $e_{t+1}$.

We now give a proof for (ii) of Theorem 2.

(iI)

We need to consider four cases in Figure 1 in which $y \to u$ is strongly protected in $\mathcal{P}_{t+1}$. Similar to the proof of (i) of Theorem 2, we first prove that the theorem holds in the first three cases in Figure 1, which correspond
to the cases (1)', (2)' and (3)' shown in Figure 13. Notice that the following proof holds for any configuration of the edge between $x$ and $w$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig13}
\caption{Five cases in which $x \to u$ or $y \to u$ is strongly protected.}
\end{figure}

Consider the case (1)' in Figure 13. From Lemma 21, $w \to y$ occurs in $e_{t+1}$. We have that $x \to u$ must occur in $e_{t+1}$.

Because there is a v-structure $w \to u \leftarrow y$ in case (2)', we have that $w \to u \leftarrow y$ also occurs in $e_{t+1}$. After implementing Algorithm 1.1.2, if case (3)' occurs in $P_{t+1}$, we have that $y \to w$ is not strongly protected in $P_{t+1}$ and the edge between $y$ and $w$ have opposite directions in different consistent extensions of $P_{t+1}$. Hence $y \to w$ occurs in $e_{t+1}$. Similarly, $y \to w_1$ also occurs in $e_{t+1}$. Moreover, the v-structure $w \to u \leftarrow w_1$ occurs in $e_{t+1}$. We have that $y \to u$ is strongly protected and occurs in $e_{t+1}$.

We now just need to show that a directed edge $y \to u$ that is strongly protected in $P_{t+1}$ like case (4)' ($x$ and $u_1$ are nonadjacent) or (4)'' ($x$ and $u_1$ are adjacent) in Figure 13 is also directed in $e_{t+1}$.

In case (4)', from delete Lemma 21, $y \to u_1$ occurs in $e_{t+1}$. Moreover, $x \to u \leftarrow u_1$ is a v-structure, so $u_1 \to u$ also occurs in $e_{t+1}$. So we have $y \to u$ must occur in $e_{t+1}$.

In case (4)'', we have that $u_1$ is also a common child of $x$ and $y$; hence, $y \to u_1$ will also be strongly protected in $P_{t+1}$ from the condition of this Theorem. Consider $y \to u_1$; if it is protected in $P_{t+1}$ like at least one case other than (4)'', from our proof, $y \to u_1$ is also compelled in $e_{t+1}$, so $y \to u$ must be compelled in $e_{t+1}$. If $y \to u_1$ is protected in $P_{t+1}$ like case (4)'', we can find another vertex $u_2$ that is a common child of $y$ and $x$; from the proof above, we know if $y \to u_2$ is directed in $e_{t+1}$, $y \to u_1$ and $y \to u$ are directed too. Since the graph has finite vertices, we can find a common child of $x$ and $y$, say $u_k$, such that $u_k$ is protected in $P_{t+1}$ like at least one case other than
(4)”. It is compelled in $e_{t+1}$, so we can get $y \to u_{k-1}$ is compelled in $P_{t+1}$, so, finally, $y \to u$ is also compelled in $P_{t+1}$. We have that $y \to u$ must occur in $e_{t+1}$ and id$_3$ holds.

(Only if) Let $u$ be a common child of $x$ and $y$ in $e_t$. If condition id$_3$ holds for a valid operator InsertD $x \to y$, we have that $y \to u$ in $e_t$ occurs in $e_{t+1}$ and is strongly protected in $e_{t+1}$. We need to show that $y \to u$ must be strongly protected in $P_{t+1}$, obtained in Algorithm 1.1.2. From the proof of this statement above, we know we just need to consider the five configurations in which $y \to u$ is strongly protected in $e_{t+1}$ in Figure 13.

We know that v-structures in $e_{t+1}$ occur in $P_{t+1}$ therefore, the v-structure in the cases (2)$'$, (3)$'$ and (4)$'$ in $e_{t+1}$ must occur in $P_{t+1}$ too.

For case (2)$'$, $y \to u$ is also strongly protected in $P_{t+1}$, since the v-structure $y \to u \leftarrow w$ occurs in $P_{t+1}$.

For case (3)$'$, we have that (1) the v-structure $w_1 \to u \leftarrow w$ occurs in $P_{t+1}$; (2) $e_{t+1}$ and $P_{t+1}$ have the same set of v-structures. Hence the v-structure $w_1 \to y \leftarrow w$ does not occur in $P_{t+1}$. We have that $y \to u$ is also strongly protected in $P_{t+1}$ for any configuration of edges between $w_1$, $y$ and $w$.

For case (4)$'$, from Algorithm 1.1.2, $y \to u_1$ occurs in $P_{t+1}$. Hence $y \to u$ is strongly protected in $P_{t+1}$.

Because the valid operator “Insert $x \to y$” satisfies condition id$_3$, from Lemma 6, we have that the operator “Delete $x \to y$”, when applied to $e_{t+1}$, results in $e_t$. From the condition dd$_2$, any directed edge $v \to y$ in $e_{t+1}$ also occurs in $e_t$. For case (1)$'$, we have that $v \to y \to u$ is strongly protected in $P_{t+1}$.

Consider the case (4)$''$, we have that v-structures $x \to u \to y$ and $x \to u_1 \to y$ occur in $e_t$ since $e_t$ is the resulting completed PDAG of the operator “Delete $x \to y$” from $e_{t+1}$. According to Algorithm 1.1.2, $x \to y$, $x \to u \to y$ and $x \to u_1 \to y$ occur in $P_{t+1}$. We have that $u \to u_1$ does not occur in $e_t$, otherwise $u \to u_1$ occurs in at least one consistent extension of $P_{t+1}$ and consequently $u_1 \to u$ does not occur in $e_{t+1}$. To prove that $y \to u$ is strongly protected in $e_{t+1}$, we need to show that $u_1 \to u$ occurs in $e_t$. Equivalently, we show $u_1 \to u$ does not occur in $e_t$. If $u_1 \to u$ occurs in a chain component denoted by $\tau$ in $e_t$, we have that neither $x$ nor $y$ are in $\tau$. The undirected edges adjacent to $x$ or $y$ are in chain components different from $\tau$. Hence id$_3$ holds for the operator “Insert $x \to y$”, and all parents of $\tau$ occur in $e_{t+1}$ too. We have that $u_1 \to u$ occurs in $e_{t+1}$ too. It’s a contradiction that $u_1 \to y$ occurs in $e_{t+1}$. □

Proof of (iii) of Theorem 2
(If)
Since Algorithm 1.1.3 returns True, all directed edges like $v \rightarrow y$ are strongly protected in $\mathcal{P}_{t+1}$. Consider the four configurations in which $v \rightarrow y$ is strongly protected in $\mathcal{P}_{t+1}$ in Figure 14. Notice that $\mathcal{P}_{t+1}$ is obtained by deleting $x \rightarrow y$ from completed PDAG $\mathcal{C}$, by Lemma 17, all directed edges with no vertices being descendants of $y$ (excluding $y$) in $\mathcal{P}_{t+1}$ will occur in $e_t$.

Hence, we have the edges $w \rightarrow v$ in case (1), and $v \rightarrow w$ in case (3) will remain in $e_{t+1}$. We have $v \rightarrow y$ in case (1) and case (3) must occur in $e_{t+1}$. Because v-structures in case (2) and case (4) will also remain in $e_{t+1}$, $v \rightarrow y$ in case (2) and case (4) must occur in $e_{t+1}$ too.

$$(1) : v \quad y(2) : v \quad y(3) : v \quad y(4) : v \quad y, (w \neq w_1)$$

Fig 14. Four configurations of $v \rightarrow y$ being strongly protected.

(Only if) If condition $dd_2$ holds for a valid operator $\text{DeleteD} \ x \rightarrow y$, all edges like $v \rightarrow y \ (v \neq x)$ in $e_t$ will occur in $e_{t+1}$. $v \rightarrow y$ must be strongly protected in $e_{t+1}$. Consider the four configurations in which $v \rightarrow y$ is strongly protected in $e_{t+1}$ as Figure 14. We know that v-structures in $e_{t+1}$ must occur in $e_t$; consequently, all directed edges in $e_{t+1}$ must occur in $e_t$; they also occur in $\mathcal{P}_{t+1}$. From Lemma 17, $w \rightarrow v \rightarrow w_1$ in case (4) in Figure 14 must be in $\mathcal{P}_{t+1}$, so an edge $v \rightarrow y$ that is strongly protected in $e_{t+1}$ is also strongly protected in $\mathcal{P}_{t+1}$.

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