ON PRIMITIVE DIVISORS OF \( n^2 + b \)

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ABSTRACT. We study primitive divisors of terms of the sequence
\( P_n = n^2 + b \), for a fixed integer \( b \) which is not a negative square. It
seems likely that the number of terms with a primitive divisor has
a natural density. This seems to be a difficult problem. We survey
some results about divisors of this sequence as well as provide upper
and lower growth estimates for the number of terms which have a
primitive divisor.

1. Primitive prime divisors

Given \( b \), an integer which is not a negative square, consider the integer sequence with \( n \)th term \( P_n = n^2 + b \). It seems likely [3] that infinitely many of the terms are prime but a proof seems elusive. Perhaps this mirrors the status of the Mersenne Prime Conjecture, which predicts that the sequence with \( n \)th term \( M_n = 2^n - 1 \) contains infinitely many prime terms. At least with the Mersenne sequence, an old result shows that primes are produced in a less restrictive sense.

Definition 1.1. Let \((A_n)\) denote a sequence with integer terms. We say an integer \( d > 1 \) is a primitive divisor of \( A_n \) if

1. \( d \mid A_n \)
2. \( \gcd(d, A_m) = 1 \) for all non-zero terms \( A_m \) with \( m < n \).

In 1886 Bang [2] showed that if \( a \) is any fixed integer with \( a > 1 \) then the sequence with \( n \)th term \( a^n - 1 \) has a primitive divisor for any index \( n > 6 \). This is remarkable because the number 6 is uniform across all \( a \) and it is small. Before we say any more about polynomials, a short survey follows indicating the incredible influence of Bang’s Theorem.

1.1. Primitive divisor theorems. In 1892 Zsigmondy obtained the generalization that for any choice of \( a \) and \( b \) with \( a > b > 0 \), the term \( a^n - b^n \) has a primitive divisor for any index \( n > 6 \). This lovely result was re-discovered several times in the early 20th century and it has turned out to be quite applicable. See [25] and the references

1991 Mathematics Subject Classification. 11A41, 11B32, 11N36.
Key words and phrases. prime, primitive divisor, quadratic polynomial.
therein where applications to Group Theory are discussed. For example, the order of the group $GL_n(\mathbb{F}_q)$ has a primitive divisor for all large $n$. Thus Sylow’s Theorem can be invoked to deduce information about the structure of the group.

The next major theoretical advance was made by Carmichael. Let $u$ and $v$ denote conjugate quadratic integers; in other words, zeros of a monic irreducible polynomial with integer coefficients. Consider the integer Lucas sequence defined by

$$U_n = \frac{(u^n - v^n)}{(u - v)}.$$

The Fibonacci sequence $(F_n)$ arises from the roots of the polynomial $x^2 - x - 1$. Carmichael [5] showed that if $u$ and $v$ are real then $U_n$ has a primitive divisor for $n > 12$. This is a sharp result because $F_{12}$ does not have a primitive divisor. Less is currently known about the corresponding Lehmer-Pierce sequence

$$V_n = (u^n - 1)(v^n - 1).$$

Kalman Győry pointed out to the first author that if $uv = 1$ then $V_n$ has a primitive divisor for all $n$ beyond some (actually uniform) bound; on the other hand, if $uv = -1$ then $V_{2k}$ does not have a primitive divisor if $k$ is odd, because $V_{2k} = -V_k^2$. This second observation is actually quite germane to this paper; see Theorem 1.2. In fact the set of terms with a primitive divisor has natural density equal to $\frac{3}{4}$ (cf. Conjecture 1.5). At the conference, Richard Pinch remarked that certain Lehmer-Pierce sequences count orders of groups: this time the groups are $E(\mathbb{F}_{p^n})$, where $E$ denotes an elliptic curve.

Bilu, Hanrot and Voutier [4] used powerful methods from Diophantine analysis to prove, in the general case, that $U_n$ has a primitive divisor for any $n > 30$. Again this is a sharp result as the sequence generated by the polynomial $x^2 - x + 2$ illustrates. Finally, Silverman [28] obtained a primitive divisor theorem for Elliptic Divisibility Sequences and a uniform version appears in [12] for a certain class of sequences.

1.2. Primitive divisors of $n^2 + b$.

**Theorem 1.2.** Infinitely many terms of the sequence $n^2 + b$ do not have a primitive divisor.

The proof of Theorem 1.2 follows very easily from a result of Schinzel [26] and will be discussed shortly. Schinzel’s proof manufactures a very thin set of terms with no primitive divisor. Dartyge [8] has improved Schinzel’s result for $n^2 + 1$ (and in principle the method works for $n^2 + b$ also). The aim of this paper to obtain a better grasp on the
set of terms with no primitive divisor. We will also consider whether the set of indices \( n \) for which \( P_n \) has a primitive divisor has a natural density. Apparently this lies quite deep.

For other interesting approaches to the study of divisors of quadratic integral polynomials; consult [8], [10], [11], [18], [19], [23], [24] and [30]. For higher order polynomials there is also the paper [9].

1.3. The greatest prime factor. Let \( P^+(m) \) denote the greatest prime factor of the integer \( m > 1 \). There is a wealth of literature about \( P^+(n^2 + b) \) concerned with the fact that \( P^+(n^2 + b) \to \infty \) as \( n \to \infty \), see [27, Chapter 7]. In a slightly different direction, Luca [22] has recently revived an old method of Lehmer’s [21] to show that, given \( B \), the set of indices for which \( P^+(n^2 + 1) < B \) is efficiently computable. Carmichael’s result mentioned earlier for Lucas sequences plays a key role. He illustrates his method by showing that when \( B = 101, n \leq 24208144 \).

The following is an easy proposition, see [6] or [14], which relates \( P^+(n^2 + b) \) to the existence of a primitive divisor.

**Proposition 1.3.** For all \( n > |b| \), the term \( P_n = n^2 + b \) has a primitive divisor if and only if \( P^+(n^2 + b) > 2n \). For all \( n > |b| \), if \( P_n \) has a primitive divisor then that primitive divisor is a prime and it is unique.

**Proof of Theorem 1.2.** Results of Schinzel [26, Th. 13] show that for any \( \alpha > 0 \), \( P^+(n^2 + b) \) is bounded above by \( n^\alpha \) for infinitely many \( n \). Taking \( \alpha = \frac{1}{2} \), Proposition 1.3 shows that \( P_n = n^2 + b \) fails to have a primitive divisor infinitely often. \( \square \)

Given \( x > 1 \), Schinzel’s method constructs fewer than \( \log x \) terms \( P_n \) with \( n < x \) having no primitive divisor. For \( \alpha > \frac{149}{179} \), Dartyge [8] showed that
\[
| \{ n \leq x : P^+(n^2 + 1) < x^\alpha \} | \gg x.
\]
It should be noted that the implied constant is very small, involving, as it does, a term \( 2^{-\delta - 2} \) where \( \delta \) “est extrêmement petit” [8, p.3 line 10]. In this paper we prove the following, which provides good upper and lower estimates for the number of terms with a primitive divisor.

**Theorem 1.4.** Supposing \( -b \) is not an integer square, define
\[
\rho_b(x) = \left| \{ n \leq x : n^2 + b \text{ has a primitive divisor} \} \right|.
\]
For all sufficiently large \( x \) we have
\[
0.5324 < \frac{\rho_b(x)}{x} < 0.905.
\]
1.4. **Natural density.** Integers \( m \) with the property \( P^+(m) > 2\sqrt{m} \) were studied by Chowla and Todd [6]. They proved that the set of these numbers has natural density \( \log 2 \). Perhaps this suggests the following:

**Conjecture 1.5.** If \(-b\) is not an integer square then \( \rho_b(x) \sim x \log 2 \).

With the availability and power of modern computers, one would usually resort to some computational evidence in support of such a conjecture. The authors of [14] looked for such evidence. Whilst they found nothing to clearly contradict the conjecture, neither did they find overwhelming evidence to support it. The problem is that the convergence to the natural density is very slow.

The reason for this might best be explained as follows. Chowla and Todd’s proof uses Mertens’ Theorem about the asymptotic formula for the sum of inverse primes:

\[
\sum_{p<x} \frac{1}{p} = \log \log x + C + O \left( \frac{1}{\log x} \right).
\]

The main term of this formula grows very slowly and the error term shrinks very slowly as well. Perhaps, somehow, this lies behind the extremely slow convergence to the natural density of terms with primitive divisor, as in Conjecture 1.5. In addition, the arithmetical nature of the sequence \( n^2 + b \) plays a significant rôle when discussing its very large prime divisors (see (11) below) and this will affect what happens for ‘small’ \( x \). Our paper concludes with an explanation as to why we are not holding our breath about a proof of Conjecture 1.5.

## 2. Simple bounds

The article [14] gives some simple estimates for \( \rho_b(x) \) which are sketched below. These are recalled here as a way in to the harder methods. The first bound in (1) counts indices which produce no primitive divisor. It is much better than the bound obtained from [26] but the set of indices still has density zero and perhaps indicates the limit of elementary methods. The second bound in (1) is very easy but already gives a good estimate for the density of terms with a primitive divisor if it exists.

**Theorem 2.1.** For all sufficiently large \( x \),

\[
\frac{x}{\log x} \ll x - \rho_b(x) \quad \text{and} \quad \frac{1}{2}x - \rho_b(x) \ll \frac{x}{\log x}.
\]

(1)

The proofs use little apart from well-known estimates for sums over primes, which can be found in the book of Apostol [11]. Both begin with an old idea of Chebychev which is used frequently as the starting point
of investigating the greatest prime factor of certain sequences (see [20, Chapter 2] for example).

Apart from a finite number of primes, any prime \( p \) that divides \( n^2 + b \) has the property that \( -b \) is a quadratic residue modulo \( p \). Let \( \mathcal{R} \) denote the set of odd primes for which \( -b \) is a quadratic residue; notice that \( \mathcal{R} \) comprises the intersection of a finite union of arithmetic progressions with the set of primes. Write

\[
Q_x = \prod_{n=1}^{x} |P_n|
\]

and denote by \( \omega(Q_x) \) the number of prime divisors of \( Q_x \). By Proposition 1.3 it is sufficient to bound \( \omega(Q_x) \) because, with finitely many exceptions, a primitive divisor is unique.

2.1. Sketch proof of Theorem 2.1. Define

\[
\mathcal{S} = \{ p \in \mathcal{R} : p | Q_x, p < 2x \} \quad \text{and} \quad \mathcal{S}' = \{ p \in \mathcal{R} : p | Q_x, p \geq 2x \}.
\]

Let \( s = |\mathcal{S}| \) and \( s' = |\mathcal{S}'| \). We seek bounds for \( s + s' \). By Dirichlet’s Theorem on primes in arithmetic progression it is sufficient to estimate \( s' \).

Following Chebychev’s method, use Stirling’s Formula to obtain

\[
\sum_{p | Q_x} e_p \log p = \log Q_x = 2x \log x + O(x) \quad (2)
\]

where the left-hand side corresponds to the prime decomposition of \( Q_x \), for positive integers \( e_p \). The sum on the left-hand side of (2) decomposes according to the definitions of \( \mathcal{S} \) and \( \mathcal{S}' \) to give

\[
\sum_{p \in \mathcal{S}} e_p \log p + \sum_{p \in \mathcal{S}'} \log p = \log Q_x, \quad (3)
\]

noting that \( e_p = 1 \) whenever \( p \geq 2x \). It is easy to show that

\[
\sum_{p \in \mathcal{S}} e_p \log p = x \log x + O(x). \quad (4)
\]

Combining (2), (3) and (4) gives

\[
x \log x + O(x) = \sum_{p \in \mathcal{S}'} \log p. \quad (5)
\]

The right hand side is bounded above by \( s' \log(x^2 + 1) \) yielding a lower bound for \( s' \).
The second bound in (1) arises similarly using a finer partition of the set $S'$

$$T = \{ p \in \mathbb{R} : p|Q_x, 2x < p < Kx \};$$
$$U = \{ p \in \mathbb{R} : p|Q_x, Kx < p \},$$

for $K > 2$. Write $t = |T|$ and $u = |U|$ then we seek an upper bound for expression $t+u$. Using the definitions of $T$ and $U$ as well as equation (4) shows that

$$\sum_{p \in T} \log p + \sum_{p \in U} \log p = x \log x + O(x).$$

The extra leverage comes because the left-hand side is greater than

$$t \log x + u \log(Kx).$$

Now $K$ can be chosen judiciously to beat the other $O$-constants. An upper bound for $t+u$ follows easily and hence the second bound in (1).

Note Actually $K$ can be taken as large as $\log x$ which yields

$$\frac{x \log \log x}{\log x} < x - \rho_b(x)$$

for all large $x$. But this still fails to produce a positive density set.

3. Better bounding

It is the aim of this section to prove Theorem 1.4. Take $b = 1$ for simplicity, so we can drop the subscript $b$ on $\rho$; as with [10] the arguments in [20] can be used to generalise to $b \neq 1$. We then prove the following.

**Theorem 3.1.** For all sufficiently large $x$ we have

$$0.5324 < \frac{\rho(x)}{x} < 0.905.$$

Let

$$N_x(p) = \sum_{x \leq n < 2x \atop p | n^2 + 1} 1.$$  

The previous section shows it is sufficient to estimate

$$\sum_{p \geq 2x} N_x(p).$$

Re-casting (5) using this definition:

$$\sum_{p \geq 2x} N_x(p) \log p = x \log x + O(x). \quad (6)$$
The extreme cases arise if most of the contribution to this sum comes from \( p \) around \( 2x \) in size, or around \( 4x^2 \) in size. In the former case the bound \( \log p \geq \log x \) gives the trivial bound
\[
\sum_{p \geq 2x} N_x(p) < x,
\]
which is weaker than the first bound of the last section. On the other hand, \( \log p \leq 2 \log x + O(1) \) gives
\[
\sum_{p \geq 2x} N_x(p) > \frac{1}{2} x + o(x),
\]
which is essentially the second bound of the last section.

We could obtain improved results if we had better information about the following expression:
\[
V_x(v) = \sum_{v < p \leq ev} N_x(p).
\]
It is a good exercise to show that
\[
V_x(v) \sim \frac{x}{\log v}
\]
implies the conjecture. Unfortunately, the asymptotic formula \((9)\) is not expected to be true for very large \( v \), in view of the arithmetic nature of \( n^2 + 1 \) (see below). However, it is expected that \((9)\) will be true for \( v < x^{2-\epsilon} \) for any \( \epsilon > 0 \) and this suffices to prove the conjecture.

#### 3.1. A better upper bound for \( \rho(x) \)

We begin by modifying the definitions to allow us to use the Deshouillers-Iwaniec method in [10]. To be precise we must use smooth functions in order to apply the mean-value estimates in [10] for Kloosterman sums. Let \( \epsilon, \eta \) be two small positive quantities. Let \( b(u) \) be a function satisfying \( b(u) \in [0,1] \) for all \( u \in \mathbb{R} \), with
\[
b(u) = \begin{cases} 
1 & \text{if } (1 + \epsilon)x \leq u \leq (2 - \epsilon)x \\
0 & \text{if } u \leq x \text{ or } u \geq 2x,
\end{cases}
\]
\[
\frac{d^r b(u)}{du^r} \ll_{r,\epsilon} u^{-r} \text{ for all } r \in \mathbb{N}.
\]
We redefine \( N_x(p) \) to be
\[
N_x(p) = \sum_{x \leq n < 2x \atop p \mid n^2 + 1} b(n).
\]
An upper bound for this summed over \( p \) will give us an upper bound for the original problem, since the two quantities will differ by at most \( \frac{3}{2} \epsilon x \).

Now write
\[
X = \int_x^{2x} b(u) \, du, \quad |A_d| = \sum_{n^2 + 1 \equiv 0 \pmod{d}} b(n).
\]

By the working on [10, p.2] we can modify the Chebychev argument to give
\[
\sum_p |A_p| \log p = 2X \log x + O(x).
\]

Also, as shown in [10], we have
\[
\sum_{p \leq x} |A_p| = X \log x + O(x).
\]

Let
\[
P_x = \max_{|A_p| \neq 0} p = x^\sigma \quad \text{say}.
\]

We therefore have
\[
\sum_{x \leq p \leq P_x} |A_p| \log p = X \log x + O(x).
\]

Deshouillers and Iwaniec then estimate this sum as
\[
\sum_{1 \leq j \leq J} S(X, V_j) + O(x),
\]
where \( V_j = 2^j x \) and
\[
S(x, V_j) = \sum_{V_j \leq p \leq 4V_j} C_j(p) \log p.
\]

Here the infinitely differentiable functions \( C_j(u) \in [0, 1] \) are supported in \([V_j, 4V_j]\), with
\[
\sum_{1 \leq j \leq J} C_j(u) = \begin{cases} 
1 & \text{if } 2x < u \leq P_x \\
0 & \text{if } u < x \text{ or } u > P_x.
\end{cases}
\]

After several transformations and an application of the Rosser-Iwaniec sieve in tandem with their own sophisticated mean-value estimate for averages of Kloostermann sums, they prove that
\[
S(x, V_j) \leq \frac{2}{\log D_j} \int C_j(u) \frac{\log u}{u} \, du \left(1 + O\left(\frac{1}{\log D}\right)\right), \quad (10)
\]
Here $D_j = x^{1-\eta} V_j^{-\frac{1}{2}}$. From this they deduce that $\sigma$ is not less than the solution to
\[ 2 - \sigma - 2 \log(2 - \sigma) = \frac{\delta}{4}. \]
That is, $\sigma = 1.202468 \ldots$

Now, the worst case scenario for the upper bound (7) is if (10) holds for each $V_j$. This gives
\[
\sum_{p \geq 2x} N_x(p) \leq \sum_{1 \leq j \leq J} S(X, V_j)(\log V_j)^{-1} + O((\log x)^{-1})
\]
\[
= x (1 + O((\log x)^{-1}) \int_1^\sigma \frac{2}{1 - t/2} dt
\]
\[
= (2\sigma - \frac{3}{2})x (1 + O((\log x)^{-1})) < 0.905x.
\]

3.2. A better lower bound for $\rho(x)$. Now we need to show that not all the contribution comes from primes near $x^2$. This is a relatively simple application of an upper bound sieve to the set
\[
\{ m : m\ell = n^2 + b, x < n \leq 2x \} \text{ for } \ell \text{ in some range.}
\]
In this case we can apply the sieve with distribution level
\[
D_\ell = \frac{x}{\ell(\log x)^A}
\]
for some $A$, by an elementary argument: this corresponds to $V_j/x$ in the last section. Of course, this is why the elementary argument is no good for $V_j$ near $x$ in size. The crossover point between the two methods is at $V_j = x^4$, but we can get nowhere near this value for the problem discussed in [10]. For $\ell = 1$ the problem is the well-known one of representing almost-primes by values of $n^2 + 1$ and giving an upper bound for the number of prime values of this polynomial. By \[16, \text{Theorem } 5.3\] (or see \[15, \text{p.66}\]) we have
\[
\sum_{x \leq n \leq 2x \atop n^2 + 1 = p} 1 \leq \frac{2x}{\log x} \prod_p \left( 1 - \frac{\chi(p)}{p - 1} \right) \left( 1 + O\left( \frac{\log \log 3x}{\log x} \right) \right). \tag{11}
\]
Here $\chi(n)$ is the non-trivial character (mod 4). Note the important product over primes above which encodes arithmetical information relevant to the polynomial $n^2 + 1$. This did not arise in the previous section since summing over a sufficiently long range for $\ell$ smooths out this factor (compare \[10, \text{§8}\]). It is expected that (11) holds with equality if the factor 2 is replaced by $\frac{1}{2}$ on the right-hand side, see \[17, 8\].
We obtain our desired bound by first considering

\[ W(L, x) = \sum_{L \leq \ell \leq eL} \sum_{x \leq n \leq 2x} \sum_{n^2 + 1 = \ell p} 1. \]

Write \( \omega(d) \) for the number of solutions to \( n^2 + 1 \equiv 0 \, (\text{mod } d) \) and let \( \{\lambda_d\}_{d \leq D} \) be the Rosser upper bound sieve of level \( D_L = x(L(\log x)^A)^{-1} \) as described in [10, §4] and explicitly constructed in [15, Chapter 4]. We then have

\[ W(L, x) \leq \sum_{L \leq \ell \leq eL} \sum_{d \leq D_L} \lambda_d \sum_{x \leq n \leq 2x} \sum_{n^2 + 1 \equiv 0 \, (\text{mod } d\ell)} 1 \]

\[ = \sum_{L \leq \ell \leq eL} \sum_{d \leq D_L} \lambda_d \omega(d\ell) \left( \frac{x}{d\ell} + O(1) \right) \]

\[ = x \sum_{L \leq \ell \leq eL} \sum_{d \leq D_L} \lambda_d \frac{\omega(d\ell)}{d\ell} + O \left( LD_L(\log x)^2 \right). \]

In the above we have noted that \( \omega(d\ell) \leq \tau(d)\tau(\ell) \) and used the well-known average value of the divisor function \( \tau(n) \) to give a bound for the error term. We then use a similar analysis to that in [10, §8] to produce the ‘main term’. We give all the details that differ from [10] here for completeness.

Firstly write

\[ \sum_{L \leq \ell \leq eL} \sum_{d \leq D_L} \lambda_d \frac{\omega(d\ell)}{d\ell} = \sum_{d \leq D_L} \lambda_d \frac{\omega(d)}{d} J(d, L). \]

Now put

\[ L(s, d) = \sum_{m=1}^{\infty} \frac{\omega(dm)}{\omega(d)m^s}. \]

Note by [10] Lemma 4] that

\[ L(s, d) = \frac{\zeta(s)L(s, \chi)}{\zeta(2s)} \prod_{p \mid d} \left( 1 + \frac{1}{p^s} \right)^{-1}. \]

Using Perron’s formula ([29 Theorem 3.12]) with \( T = x, c = (\log x)^{-1} \) we have

\[ J(d, L) = \frac{1}{2\pi i} \int_{c-ix}^{c+ix} L(s + 1, d) \frac{(eL)^s - L^s}{s} ds + O \left( x^{-\frac{1}{4}} \right). \]

The final term is negligible. (Actually, this has been estimated very crudely, in reality it is \( O(x^{c-1}) \)). Now take the contour of integration
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back to $\text{Re } s = -\frac{1}{2}$. The pole at $s = 0$ gives a term

$$\frac{L(1, \chi)}{\zeta(2)} \prod_{p | d} \left( 1 + \frac{1}{p} \right)^{-1}.$$  

The pair of integrals on $\text{Im } s = \pm \epsilon$ give a negligible contribution ($O(x^{\epsilon - \frac{T}{2}})$), using

$$\max(|\zeta(s)|, |L(s, \chi)|) \ll T^{\frac{1}{2}} \text{ for } 1 \leq |\text{Im } s| \leq T, \text{Re } s \geq \frac{1}{2}.$$  

The integral on the new contour can be estimated using:

$$\int_{-T}^{T} \frac{|\zeta(\frac{1}{2} + it)|^2}{1 + |t|} dt \ll (\log T)^2,$$

with the same bound applying when $\zeta(s)$ is replaced by $L(s, \chi)$, together with ([29, p.135])

$$\frac{1}{\zeta(1 + it)} \ll \log T \text{ for } |t| \leq T;$$

and

$$\left| \prod_{p | d} \left( 1 + \frac{1}{p^{s+1}} \right)^{-1} \right| \leq \prod_{p | d} \left( 1 - \frac{1}{p^{\frac{1}{2}}} \right)^{-1} < \tau(d).$$

This gives a bound for the integral which is

$$\ll \frac{\tau(d)(\log x)^3}{L^\frac{1}{2}}.$$  

Thus

$$\sum_{L \leq \ell \leq eL} \sum_{d \leq DL} \lambda_d \omega(d\ell) \frac{d\ell}{d} = \sum_{d \leq DL} \lambda_d \omega'(d) \frac{1}{d} + O \left( \sum_{d \leq DL} \frac{\omega(d)\tau(d)(\log x)^3}{L^\frac{1}{2}d} \right).$$

Here

$$\omega'(d) = \omega(d) \prod_{p | d} \left( 1 + \frac{1}{p} \right)^{-1}.$$  

The rest of the working follows mutatis mutandis from [10, p.10]. Hence

$$x \sum_{L \leq \ell \leq eL} \sum_{d \leq DL} \lambda_d \omega(d\ell) \frac{d\ell}{d} = \frac{2x}{\log D_L} \left( 1 + O \left( \frac{1}{\log D_L} \right) \right) + O \left( \frac{x(\log x)^7}{L^\frac{1}{2}} \right).$$

The reader can thus see that the extra error term $O(x(\log x)^7 L^{-\frac{1}{2}})$ (the log power could be reduced here by more careful working) corresponds to the averaging over $\ell$ smoothing out the influence of the product in ([11]), and this must dominate the main term for small $L$ since the ‘main
term’ will be incorrect in this case. Assuming that $D_L = xL^{-1}(\log x)^{-4}$ and $x^\frac{3}{4} > L > (\log x)^{18}$ we obtain

$$W(N, L) \leq \frac{2x}{\log D_L} + O\left(\frac{x}{(\log x)^2}\right). \quad (12)$$

For $L \leq (\log x)^{18}$ we can establish a slightly cruder upper bound as follows. For each value of $\ell$ we do not sieve by primes dividing $\ell$. This makes the $\lambda_d$ depend on $\ell$, but we have $\lambda_d = 0$ if $(d, \ell) > 1$. Hence we can write $\omega(d\ell) = \omega(d)\omega(\ell)$. Following the analysis above, the remainder term remains $O(LD_L(\log x)^2)$. The ‘main term’ for the upper bound is now

$$\frac{2x}{\log D_L} \sum_{L \leq \ell \leq eL} \frac{\omega(\ell)}{\phi(\ell)} \prod_{p \mid \ell} \left(1 - \frac{\chi(p)}{p-1}\right) \leq \frac{Kx}{\log D_L}$$

for some absolute constant $K$. The contribution from the terms with $L \leq (\log x)^{18}$ is thus $\ll (\log \log x)(\log x)^{-1}$ times the total contribution for larger $L$. These terms may therefore be neglected asymptotically.

Now the worst case scenario for (7) has equality in (12) for

$$x^{2-\theta} \geq L \geq (\log x)^{18}, \text{ where } \int_{\theta}^{2} \frac{2t}{t-1} \, dt = 1.$$ 

In other words, $\theta$ is the solution to

$$2(2 - \theta) - 2\log(\theta - 1) = 1.$$ 

This is the limit of the sequence

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2} \left(\frac{3}{2} + a_n - \log(a_n - 1)\right) \quad (n \geq 1),$$

quickly giving the value $1.766249 \ldots$. We then calculate

$$\int_{\theta}^{2} \frac{2}{t-1} \, dt = 2\theta - 3 > 0.5324 \ldots$$

4. SOME IMPLICATIONS OF CONJECTURE 1.5

The following argument shows that we do not expect Conjecture 1.5 to be settled in the near future. In the previous section we have used the tools that have been developed for the investigation of the greatest prime factor of $n^2 + 1$ to obtain (rather weak) approximations to the conjecture. Now we assume the conjecture and demonstrate that it would lead to a phenomenal improvement for the greatest prime factor problem.
The conjecture leads to
\[ \sum_{p \geq x} N_x(p) \sim x \log 2. \]
By the Chebychev argument (6), on average in these sums,
\[ \frac{\log p}{\log x} \sim \frac{1}{\log 2} = \sigma \text{ (say)} = 1.4416 \ldots \]
Hence the greatest prime factor of \( n^2 + 1 \) infinitely often exceeds \( n^\sigma \).
This more than doubles the improvement of Deshouillers-Iwaniec over the trivial estimate! However, we can do still better using the elementary bound from the last section. The worst case scenario now has all the contribution to the left hand side of (6) coming from \( p \) close to \( x^\sigma \).
Since the bounds of the last section must hold (and they are better than the Deshouillers-Iwaniec estimates in this region), this corresponds to finding \( \alpha < \sigma < \beta \) with
\[ \int_\alpha^\beta \frac{2}{t-1} dt = \log 2, \quad \int_\alpha^\beta \frac{2t}{t-1} dt = 1. \]
A little bit of manipulation gives the solution to be
\[ \beta = 1 + \frac{1 - \log 2}{2 - \sqrt{2}} = 1.52383 \ldots \] (13)
This gives the following result.

**Theorem 4.1.** If Conjecture 1.5 is true, then infinitely often the greatest prime factor of \( n^2 + 1 \) exceeds \( n^\beta \) where \( \beta \) is given by (13).

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