BORDISM INVARIANTS OF THE MAPPING CLASS GROUP

AARON HEAP

ABSTRACT. We define new bordism and spin bordism invariants of certain subgroups of the mapping class group of a surface. In particular, they are invariants of the Johnson filtration of the mapping class group. The second and third terms of this filtration are the well-known Torelli group and Johnson subgroup, respectively. We introduce a new representation in terms of spin bordism, and we prove that this single representation contains all of the information given by the Johnson homomorphism, the Birman-Craggs homomorphism, and the Morita homomorphism.

1. INTRODUCTION

Let $\Sigma_{g,1}$ be a compact, oriented surface of genus $g$ with one boundary component. Let $\Gamma_{g,1}$ be the mapping class group of $\Sigma_{g,1}$. That is, $\Gamma_{g,1}$ is the group of isotopy classes of orientation-preserving homeomorphisms of $\Sigma_{g,1}$ which fix the boundary. The study of mapping class groups has important applications in many different areas of topology, differential geometry, and algebraic geometry. Here we are particularly interested in $\Gamma_{g,1}$ within the area of 3-manifold topology.

The mapping class group $\Gamma_{g,1}$ acts naturally by automorphisms on the fundamental group $F = \pi_1(\Sigma_{g,1})$, which is a free group of rank $2g$. Then we have the induced representation $\rho_{g,1} : \Gamma_{g,1} \rightarrow \text{Aut}(F)$, and this representation is known classically to be injective. Let $\{F_k\}_{k \geq 1}$ be the lower central series of $F$. That is, $F_1 = F$ and the rest of the terms are defined inductively by $F_{k+1} = [F_k, F_1]$ for any $k \geq 1$. Then $\Gamma_{g,1}$ acts naturally on the nilpotent quotients $F/F_k$, providing a series of representations

$$\rho_k : \Gamma_{g,1} \rightarrow \text{Aut} \left( \frac{F}{F_k} \right).$$

Note that $F/F_2$ is isomorphic to the first homology group $H_1 = H_1(\Sigma_{g,1}; \mathbb{Z})$, and $\rho_2$ is the same as the classical representation $\Gamma_{g,1} \rightarrow \text{Sp}(2g; \mathbb{Z})$ of the mapping class group onto the Siegel modular group, which is the group of symplectic automorphisms of $H_1$ with respect to the skew-symmetric intersection pairing.

The generalized Johnson subgroup $J(k) \subseteq \Gamma_{g,1}$ is defined to be the kernel of $\rho_k$. That is, $J(k)$ is the subgroup of the mapping class group consisting of those homeomorphisms which induce the identity on $F/F_k$. The subgroup $J(2) = J_{g,1}$ is more commonly known as the Torelli group, and $J(3) = K_{g,1}$ is traditionally referred to as the Johnson subgroup. The Johnson subgroup was originally defined to be the subgroup of $\Gamma_{g,1}$ generated by all Dehn twists about separating simple closed curves on $\Sigma_{g,1}$. The fact that these two definitions of $K_{g,1}$ are equivalent was proved by D. Johnson in [16].

Date: November 10, 2004.
To get a better understanding of the structure of the subgroup \( \mathcal{J}(k) \), it is natural to seek abelian representations for it. That is, we would hope to understand \( \mathcal{J}(k) \) better by investigating abelian quotients of it. The first such quotient of the Torelli group \( \mathcal{J}(2) \) was given by a homomorphism due to D. Sullivan in [Su]. Johnson gave another homomorphism for \( \mathcal{J}(2) \), of which Sullivan’s is a quotient, in [J2]. He later generalized this homomorphism to \( \mathcal{J}(k) \) for all \( k \geq 2 \) in [J3], thus giving a family of homomorphisms

\[
\tau_k : \mathcal{J}(k) \to \text{Hom} \left( H_1, \frac{F_k}{F_{k+1}} \right),
\]

now known as the Johnson homomorphisms. In the case \( k = 2 \), the image of \( \tau_2 \) is known to be a submodule \( D_2(H_1) \) of \( \text{Hom}(H_1, F/F_2) = \text{Hom}(H_1, H_1) \). Moreover, the kernel of \( \tau_2 \) is known to be \( \mathcal{J}(3) \). In general, \( \ker \tau_k = \mathcal{J}(k+1) \). However, the image of \( \tau_k \) is not known for \( k \geq 3 \), and it is a fundamental problem in the study of the mapping class group to determine its image.

In [BC] J. Birman and R. Craggs produced a collection of abelian quotients of \( \mathcal{J}(2) \) given by homomorphisms onto \( \mathbb{Z}_2, \rho : \mathcal{J}(2) \to \mathbb{Z}_2 \). These are finite in number and unrelated to Johnson’s homomorphism. However, Johnson showed in [J6] that the Johnson homomorphism \( \tau_2 \) and the totality of these Birman-Craggs homomorphisms, together, completely determine the abelianization of the Torelli group \( \mathcal{J}(2) \) for \( g \geq 3 \). The abelianization of \( \mathcal{J}(k) \) is not known for \( k > 2 \).

In this paper we give new representations in terms of the 3-dimensional bordism groups \( \Omega_3(F/F_k) \) and \( \Omega_3^{pin}(F/F_k) \). The former is a faithful representation of the abelian quotient \( \mathcal{J}(k)/\mathcal{J}(2k-1) \), and the latter is a homomorphism which combines the Johnson and Birman-Craggs homomorphisms into a single homomorphism. See Sections [BC] and [J6] for specific details.

2. THE JOHNSON HOMOMORPHISM

2.1. Johnson’s Original Definition of \( \tau_k \). In this section we give a description of Johnson’s homomorphisms. Let \( \Sigma_{g,1} \) be a compact, oriented surface of genus \( g \) with one boundary component and with fundamental group \( F \). Let \( \{F_k\}_{k \geq 1} \) be the lower central series of \( F \). Let the generalized Johnson subgroup \( \mathcal{J}(k) \) be the subgroup of the mapping class group consisting of those homeomorphisms that induce the identity on \( F/F_k \).

Consider any \( f \in \mathcal{J}(k) \). Choose a representative \( \gamma \in \pi_1(\Sigma_{g,1}) = F \) for any given element \( [\gamma] \in H_1 = H_1(\Sigma_{g,1}; \mathbb{Z}) = F/F_2 \), and consider the element \( f_*(\gamma)\gamma^{-1} \) which belongs to \( F_k \) since \( f \in \mathcal{J}(k) \) implies \( f_* \) acts trivially on \( F/F_k \). Then let \( [f_*(\gamma)\gamma^{-1}] \in F_k/F_{k+1} \) denote the equivalence class of \( f_*(\gamma)\gamma^{-1} \) under the projection \( F_k \to F_k/F_{k+1} \). Then we define the Johnson homomorphisms

\[
\tau_k : \mathcal{J}(k) \to \text{Hom} \left( H_1, \frac{F_k}{F_{k+1}} \right)
\]

by letting \( \tau_k(f) \) be the homomorphism \( [\gamma] \to [f_*(\gamma)\gamma^{-1}] \). The skew-symmetric intersection pairing on \( H_1 \) defines a canonical isomorphism \( H_1 \cong \text{Hom}(H_1, \mathbb{Z}) \), and this induces an isomorphism

\[
\text{Hom} \left( H_1, \frac{F_k}{F_{k+1}} \right) \cong \text{Hom}(H_1, \mathbb{Z}) \otimes \frac{F_k}{F_{k+1}} \cong H_1 \otimes \frac{F_k}{F_{k+1}}.
\]
Thus we could also write

\[ \tau_k : J(k) \to H_1 \otimes \frac{F_k}{F_{k+1}}. \]

This is Johnson’s original definition [J3], but there are several equivalent definitions of his homomorphism. Also in [K3], one can see a definition in terms of the intersection ring of the mapping torus of \( f \). There is a definition of \( \tau_k \) in terms of the Magnus representation of the mapping class group \( \Gamma_{g,1} \) that may be found in [K] or [Mo].

The final definition we mention in this paper will be given in Section 2.3. It was stated by Johnson [J3] and verified by T. Kitano [Ki]. This definition gives a computable description of \( \tau_k \) in terms of Massey products of mapping tori.

We complete this section with a few well-known facts about the Johnson homomorphisms \( \tau_k \) and the subgroups \( J(k) \). It was shown by Morita in [Mo] that

\[ [J(k), J(l)] \subset J(k + l - 1). \]

In particular, the commutator subgroup \([J(k), J(k)]\) is a subgroup of \( J(2k - 1) \) for \( k \geq 2 \). As mentioned before, \( \ker \tau_k = J(k + 1) \). Then the image of \( \tau_k \) is isomorphic to the abelian quotient \( J(k)/J(k + 1) \). Thus the information provided by the \( k - 1 \) homomorphisms \( \tau_k, ..., \tau_{2k-2} \) can be combined to determine the abelian quotient \( J(k)/J(2k - 1) \). Unfortunately this only at most detects the free-abelian part of the abelianization \( J(k)/[J(k), J(k)] \cong H_1(J(k)) \). For example, the image of \( \tau_2 \) is given by \( J(2)/J(3) = T_{g,1}/K_{g,1} \), and \( J(2)/J(3) \otimes \mathbb{Q} \cong H_1(T_{g,1}; \mathbb{Q}) \), whereas the abelianization of the Torelli group \( H_1(T_{g,1}) \) has 2-torsion. We will discuss this 2-torsion in more detail in Section 6.

2.2. Massey Products. Let \((X, A)\) be a pair of topological spaces, and unless otherwise stated we assume that the coefficients for homology and cohomology groups are always the integers \( \mathbb{Z} \). In this section we will give the definition of the Massey product

\[ H^1(X, A) \otimes \cdots \otimes H^1(X, A) \to H^2(X, A) \]

since these are the only dimensions that we are interested in using, and we will give a few useful properties of which we wish to take advantage. The general definition is completely analogous except for various sign conventions, and we refer the reader to D. Kraines [K]. For a more complete description of this specific definition we are giving and for some useful examples, we refer you to R. Fenn’s book [F].

Massey products may be viewed as higher order analogues of cup products and are defined when certain cup products vanish. Let \( u_1, ..., u_n \in H^1(X, A) \) be cohomology classes with cocycle representatives \( a_1, ..., a_n \in C^1(X, A) \), respectively. A defining set for the Massey product \( \langle u_1, ..., u_n \rangle \) is a collection of cochains \( a = (a_{i,j}) \), \( 1 \leq i \leq j \leq n \) and \( (i, j) \neq (1, n) \), satisfying

(1) \( a_{i,i} = a_i \) for any \( i \in \{1, ..., n\} \),

(2) \( a_{i,j} \in C^1(X, A) \),

(3) \( \delta a_{i,j} = \sum_{r=1}^{j-1} a_{i,r} \cup a_{r+1,j} \).

For such a defining set \( a \) consider the cocycle \( u(a) \in C^2(X, A) \) given by

\[ u(a) = \sum_{r=1}^{n-1} a_{1,r} \cup a_{r+1,n}. \]
The Massey product \(\langle u_1, \ldots, u_n \rangle\) is defined if a defining set \(a\) exists, and it is defined to be the subset of \(H^2(X, A)\) consisting of the values \(u(a)\) of all such defining sets \(a\).

The length 1 Massey product \(\langle u_1 \rangle\) is simply defined to be \(u_1\), and its defining set is any cocycle representative of \(u_1\). The length 2 Massey product \(\langle u_1, u_2 \rangle\) is the cup product \(u_1 \cup u_2\). The triple Massey product \(\langle u_1, u_2, u_3 \rangle\) is defined only when \(\langle u_1, u_2 \rangle\) and \(\langle u_2, u_3 \rangle\) are zero. As you may notice from the definition, Massey products of length 3 or greater may not be uniquely defined but in fact may be a set of elements. However, if a sufficient number of smaller Massey products vanish, then \(\langle u_1, \ldots, u_n \rangle\) is uniquely defined. We have the following useful properties.

\(2.2.1\) Uniqueness. For \(n \geq 3\), the Massey product \(\langle u_1, \ldots, u_n \rangle\) is uniquely defined if all Massey products of length less than \(n\) are defined and vanish. (This hypothesis is stronger than necessary for uniqueness, but it is sufficient for our purposes.)

\(2.2.2\) Naturality. Let \((Y, B)\) be a pair of topological spaces, and consider a map of pairs \(f : (Y, B) \to (X, A)\). If \(\langle u_1, \ldots, u_n \rangle\) is defined then so is \(\langle f^*(u_1), \ldots, f^*(u_n) \rangle\), and \(f^* \langle u_1, \ldots, u_n \rangle \subset \langle f^*(u_1), \ldots, f^*(u_n) \rangle\). Furthermore, if \(f^*\) is an isomorphism, then equality holds.

2.3. Massey Product Description of \(\tau_k\). We are now prepared to describe Johnson’s homomorphisms \(\tau_k\) using Massey products of mapping tori. For a more complete description, see the work of Kitano \([K]\). As before, \(\Sigma_{g,1}\) is an oriented surface of genus \(g\) with one boundary component \(\partial \Sigma_{g,1}\). Consider any homeomorphism \(f \in \mathcal{F}(k)\), and let \(T_{f,1}\) denote the mapping torus of \(f\). That is, \(T_{f,1} = \Sigma_{g,1} \times [0, 1]\) with \(x \times \{0\}\) glued to \(f(x) \times \{1\}\). Note that the boundary \(\partial T_{f,1}\) is the torus \(\partial \Sigma_{g,1} \times S^1\). With the natural orientation on \([0, 1]\), we have a local orientation on \(T_{f,1}\) given by the product orientation. Moreover, since \(f \in \mathcal{F}(k)\) acts trivially on \(H_1 = H_1(\Sigma_{g,1})\) as long as \(k \geq 2\), the mapping torus \(T_{f,1}\) is an oriented homology \(\Sigma_{g,1} \times S^1\), but the Massey product structure may be different than that of \(\Sigma_{g,1} \times S^1\).

First, fix a basis \(\{\alpha_1, \ldots, \alpha_{2g}\}\) for the free group \(F = \pi_1(\Sigma_{g,1})\). Then if \(\gamma\) represents a generator of \(\pi_1(S^1)\), we get the following presentation of \(\pi_1(T_{f,1})\):

\[
\pi_1(T_{f,1}) = \langle \alpha_1, \ldots, \alpha_{2g}, \gamma | [\alpha_1, \gamma] f_*(\alpha_1)^{-1}, \ldots, [\alpha_{2g}, \gamma] f_*(\alpha_{2g})^{-1} \rangle.
\]

By denoting the homology classes of \(\alpha_i\) and \(\gamma\) by \(x_i\) and \(y\), respectively, we obtain a basis for \(H_1(T_{f,1})\):

\[
\{x_1, \ldots, x_{2g}, y\} \in H_1(T_{f,1}).
\]

Then since \(H^1(T_{f,1}) \cong \text{Hom}(H_1(T_{f,1}), \mathbb{Z})\), we have a dual basis for \(H^1(T_{f,1})\):

\[
\{x_1^*, \ldots, x_{2g}^*, y^*\} \in H^1(T_{f,1}).
\]

Let \(j : (T_{f,1}, \emptyset) \to (T_{f,1}, \partial T_{f,1})\) be the inclusion map. The long exact sequence of a pair shows \(j_* : H_1(T_{f,1}) \to H_1(T_{f,1}, \partial T_{f,1})\) has kernel generated by \(y\). So we have a basis for \(H_1(T_{f,1}, \partial T_{f,1})\):

\[
\{j_*(x_1), \ldots, j_*(x_{2g})\} \in H_1(T_{f,1}, \partial T_{f,1}).
\]

And this gives a corresponding basis for \(H_2(T_{f,1}) \cong H^1(T_{f,1}, \partial T_{f,1})\):

\[
\{X_1, \ldots, X_{2g}\} \in H_2(T_{f,1}).
\]

Let \(\varepsilon : \mathbb{Z}[F] \to \mathbb{Z}\) be the augmentation map and let

\[
\frac{\partial}{\partial \alpha_i} : \mathbb{Z}[F] \to \mathbb{Z}[F], \ 1 \leq i \leq 2g
\]
be the Fox’s free derivatives. Here \( \mathbb{Z}[F] \) is the integral group ring of the free group \( F \). Finally, let \( \mathfrak{X} \) denote the ring of formal power series in the noncommutative variables \( t_1, ..., t_{2g} \), and let \( \mathfrak{X}_k \) denote the submodule of \( \mathfrak{X} \) corresponding to the degree \( k \) part. One can show \( F_k/F_{k+1} \) is a submodule of \( \mathfrak{X}_k \), where the inclusion map is induced by

\[
F_k \ni \zeta \longmapsto \sum_{j_1, \ldots, j_k} \varepsilon \frac{\partial}{\partial \alpha_{j_1}} \cdots \frac{\partial}{\partial \alpha_{j_k}}(\zeta)t_{j_1} \cdots t_{j_k} \in \mathfrak{X}_k.
\]

Then we have the following theorem.

**Theorem 2.1 (Kitano).** There is a homomorphism \( \tau_k : \mathcal{J}(k) \rightarrow \text{Hom}(H_1, \mathfrak{X}_k) \) defined by letting \( \tau_k(f) \) be the homomorphism

\[
x_i \longmapsto \sum_{j_1, \ldots, j_k} \langle \langle x^*_{j_1}, \ldots, x^*_{j_k} \rangle, X_i \rangle t_{j_1} \cdots t_{j_k}
\]

where \( \langle \ , \ \rangle \) is the dual pairing of \( H^2(T_{f,1}) \) and \( H_2(T_{f,1}) \). Moreover, this homomorphism is the same as the Johnson homomorphism.

The canonical restriction \( H^*(T_{f,1}, \partial T_{f,1}) \rightarrow H^*(T_{f,1}) \) leads to the following theorem that gives a relation between the algebraic structure of the mapping class group \( \Gamma_{g,1} \) and the topological structure of the mapping torus \( T_{f,1} \).

**Theorem 2.2 (Kitano).** For any \( f \in \Gamma_{g,1} \), \( f \in \mathcal{J}(k+1) \) if and only if all Massey products of length \( m \) of

\[
H^1(T_{f,1}, \partial T_{f,1}) \otimes \cdots \otimes H^1(T_{f,1}, \partial T_{f,1}) \rightarrow H^2(T_{f,1}, \partial T_{f,1}) \rightarrow H^2(T_{f,1})
\]

vanish for any \( m \) with \( 1 < m \leq k \).

### 2.4. Morita’s Refinement of \( \tau_k \)

In this section we point out the work of Morita in [Mo], where Johnson’s homomorphism \( \tau_k \) was refined so as to narrow the range of \( \tau_k \) to a submodule \( D_k(H_1) \) of \( H_1 \otimes F_k/F_{k+1} \). This enhancement is obtained via a homomorphism

\[
\tilde{\tau}_k : \mathcal{J}(k) \rightarrow H_3\left( \frac{F}{F_k} \right)
\]

defined below. Recall that the homology of a group \( G \) is \( H_i(G) \equiv H_i(K(G,1), \mathbb{Z}) \), where \( K(G,1) \) is an Eilenberg-MacLane space. (We determine the kernel of Morita’s refinement in Corollary 5.10 below.)

Let \( \zeta \in \pi_1(\Sigma_{g,1}) = F \) represent the homotopy class of a simple closed curve on \( \Sigma_{g,1} \) parallel to the boundary \( \partial \Sigma_{g,1} \). Now we choose a 2-chain \( \sigma \in C_2(F) \) such that \( \partial \sigma = -\zeta \). Since any \( f \in \Gamma_{g,1} \) is required by definition to fix the boundary, we have \( \partial(\sigma - f#(\sigma)) = -\zeta - (-\zeta) = 0 \). Thus \( \sigma - f#(\sigma) \) is a 2-cycle. Because \( H_2(F) \) is trivial, there is a 3-chain \( c_f \in C_3(F) \) such that \( \partial c_f = \sigma - f#(\sigma) \). Note that, essentially, this is just a mapping cylinder construction. Let \( \hat{c}_f \) denote the image of \( c_f \) in \( C_3(F/F_k) \). If \( f \in \mathcal{J}(k) \) then \( f# \) acts as the identity on \( F/F_k \). Thus we have \( \partial \hat{c}_f = \sigma - f#(\sigma) = \hat{\sigma} - f#(\hat{\sigma}) = 0 \), and \( \hat{c}_f \) is a 3-cycle. Finally define \( \hat{[c_f]} \in H_3(F/F_k) \) to be the corresponding homology class, and we define Morita’s homomorphism \( \hat{\tau}_k : \mathcal{J}(k) \rightarrow H_3(F/F_k) \) to be \( \hat{\tau}_k(f) = \hat{[c_f]} \). It is shown in [Mo] that the homology class \( \hat{[c_f]} \) does not depend on the choices that were made, and we refer you there for the details.
Now consider the extension

\[ 0 \to \frac{F_k}{F_{k+1}} \to \frac{F}{F_{k+1}} \to \frac{F}{F_k} \to 1, \]

and let \( \{ E^r_{p,q} \} \) be the Hochschild-Serre spectral sequence for the homology of this sequence. In particular, we have

\[ E^2_{p,q} = H_p \left( \frac{F}{F_k} ; H_q \left( \frac{F_k}{F_{k+1}} \right) \right). \]

Then we have the differential

\[ d^2 : E^2_{3,0} = H_3 \left( \frac{F}{F_k} \right) \to E^2_{1,1} = H_1 \left( \frac{F}{F_k} ; H_1 \left( \frac{F_k}{F_{k+1}} \right) \right) \cong H_1 \otimes \frac{F_k}{F_{k+1}}. \]

Finally, the refinement of Johnson’s homomorphism is given by the following theorem.

**Theorem 2.3** (Morita). The composition \( d^2 \circ \tilde{\tau}_k \) coincides with Johnson’s homomorphism \( \tau_k \) so that the following diagram commutes.

\[ \begin{array}{ccc}
J(k) & \xrightarrow{d^2} & H_1 \otimes \frac{F_k}{F_{k+1}} \\
\uparrow{\tilde{\tau}_k} & & \\
H_3 \left( \frac{F}{F_k} \right) & \xrightarrow{\gamma} & H_1 \otimes \frac{F_k}{F_{k+1}} \\
\end{array} \]

**Theorem 2.4** (Morita). Let \( D_k(H_1) \) be the submodule of \( H_1 \otimes F_k/F_{k+1} \) defined to be the kernel of the natural surjection

\[ H_1 \otimes \frac{F_k}{F_{k+1}} \to \frac{F_{k+1}}{F_{k+2}} \]

given by the Lie bracket map \((w, \xi) \mapsto [w, \xi] \). Then the image of the Johnson homomorphism \( \tau_k : J(k) \to H_1 \otimes F_k/F_{k+1} \) is contained in \( D_k(H_1) \) so that we can write \( \tau_k : J(k) \to D_k(H_1) \).

A short remark about this theorem is perhaps in order. It is known that the image of \( \tau_2 \) is exactly equal to \( D_2(H_1) \), and the image of \( \tau_3 \) is a submodule of \( D_3(H_1) \) of index a power of 2. Thus \( \text{Im} \tau_3 \) and \( D_3(H_1) \) have the same rank. However, for \( k \geq 4 \), \( k \) even, the rank of \( \text{Im} \tau_k \) is smaller than the rank of \( D_k(H_1) \). Please see [Mo] for more details.

### 3. Birman-Craggs Homomorphism

As mentioned at the end of Section 2.1, the Johnson homomorphism \( \tau_2 \) only detects the free abelian part of the abelianization of the Torelli group \( J(2) \), and some 2-torsion remains undetected. In this section we will say a word about this 2-torsion. In [BC] Birman and Craggs defined a (finite) collection of abelian quotients of \( J(2) \) given by homomorphisms onto \( \mathbb{Z}_2 \). Here we will give a description of these homomorphisms that is due to Johnson [J]. This somewhat more tractable description is different than (yet equivalent to) Birman and Craggs’ original definition, and it enabled Johnson to give the number of distinct Birman-Craggs homomorphisms.
Consider the surface $\Sigma_{g,1}$, and let $f \in \mathcal{J}(2)$. The definition of $\Gamma_{g,1}$ requires that $f$ be the identity on $\partial \Sigma_{g,1}$. Thus $f$ can easily be extended to a homeomorphism of the closed surface $\Sigma_g$. Let $h : \Sigma_g \rightarrow S^3$ be a Heegaard embedding of $\Sigma_g$ into the 3-sphere $S^3$, i.e. $\Sigma_g$ bounds handlebodies on both sides in $S^3$. Now cut $S^3$ open along $h(\Sigma_g)$ and reglue the two pieces using $f \in \mathcal{J}(2)$. The resulting manifold $S^3_{h,f}$ is a homology $S^3$, and its Rochlin invariant $\mu(S^3_{h,f}) \in \mathbb{Z}_2$ is defined.

In general, any closed, connected 3-manifold $M$, together with a fixed trivialization of its tangent bundle over the 2-skeleton, is the boundary of a 4-manifold $W$ whose tangent bundle can be trivialized in a compatible fashion. If $s$ denotes the choice of stable trivialization of the tangent bundle of $M$ over the 2-skeleton, then the Rochlin invariant $\mu(M,s) \in \mathbb{Z}_{16}$ is defined to be the signature $\sigma(W)$ reduced modulo 16. If $M$ happens to be a homology $S^3$ then $s$ is unique and $\sigma(W)$ is divisible by 8. Thus $\mu(S^3_{h,f}) = \mu(S^3_{h,f},s) = \sigma(W)$ can be considered an element of $\mathbb{Z}_2$.

For a fixed Heegaard embedding $h : \Sigma_g \rightarrow S^3$, the Birman-Craggs homomorphism $\rho_h : \mathcal{J}(2) \rightarrow \mathbb{Z}_2$ is defined by $\rho_h(f) = \mu(S^3_{h,f})$.

By relating the Birman-Craggs homomorphisms to a $\mathbb{Z}_2$-quadratic form, Johnson was able to show the dependence of $\rho_h$ on the embedding $h : \Sigma_g \rightarrow S^3$. The $\mathbb{Z}_2$-quadratic form $q : H_1(\Sigma_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ is defined as follows. Let $(\cdot, \cdot)$ be the Seifert linking form on $H_1(\Sigma_g; \mathbb{Z}_2)$ induced by $h : \Sigma_g \rightarrow S^3$ defined by letting $(x,y)$ be the linking number (modulo 2) of $h(x)$ and $h(y)^{\perp}$ in $S^3$, where $h(y)^{\perp}$ is the positive push-off of $h(y)$ in the normal direction determined by the orientations of $h(\Sigma_g)$ and $S^3$. Define $q(x) = (x,x)$, then it is a $\mathbb{Z}_2$-quadratic form on $H_1(\Sigma_g; \mathbb{Z}_2)$ induced by the embedding $h$. Because it is a quadratic form, $q$ satisfies $q(x+y) = q(x) + q(y) + x \cdot y$, where $x \cdot y$ is the intersection pairing of $H_1(\Sigma_g; \mathbb{Z}_2)$. Let $\{x_i, y_i\}, 1 \leq i \leq g$, denote the standard basis for $H_1(\Sigma_g; \mathbb{Z}_2)$, and the Arf invariant of $\Sigma_g$ with respect to $q$ is defined to be

$$\text{Arf}(\Sigma_g, q) = \sum_{i=1}^{g} q(x_i)q(y_i) \pmod{2}. $$

Johnson’s main results from [11] are as follows. Suppose $h_1, h_2 : \Sigma_g \rightarrow S^3$ are both Heegaard embeddings of the surface $\Sigma_g$.

**Theorem 3.1** (Johnson). The embeddings $h_1$ and $h_2$ induce the same mod 2 self-linking form if and only if the Birman-Craggs homomorphisms $\rho_{h_1}$ and $\rho_{h_2}$ are equal.

Therefore the homomorphism $\rho_h : \mathcal{J}(2) \rightarrow \mathbb{Z}_2$ only depends on the quadratic form $q$ induced by $h$, and we replace the notation $\rho_h$ with $\rho_q$ to emphasize this fact. Moreover, the $\mathbb{Z}_2$-quadratic forms $q$ which are induced by a Heegaard embedding $h$ are exactly those that satisfy $\text{Arf}(\Sigma_g, q) = 0$. Thus we are able to enumerate $\{\rho_q\}$.

**Corollary 3.2** (Johnson). There are precisely $2^{g-1}(2^g + 1)$ distinct Birman-Craggs homomorphisms $\rho_q : \mathcal{J}(2) \rightarrow \mathbb{Z}_2$.

Johnson also provided a means of computing $\rho_q$ in terms of the Arf invariant.

1. If $f \in \mathcal{J}(2)$ is a Dehn twist about a bounding simple closed curve $C$, then

$$\rho_q(f) = \text{Arf}(\Sigma', q|_{\Sigma'}),$$

where $\Sigma'$ is a subsurface of $\Sigma_g$ bounded by $C$. 
(2) If \( f \in J(2) \) is a composition of Dehn twists about cobounding curves \( C_1 \) and \( C_2 \), then

\[
\rho_q(f) = \begin{cases} 
0 & \text{if } q(C_1) = q(C_2) = 1 \\
\operatorname{Arf}(\Sigma', q|_{\Sigma'}) & \text{if } q(C_1) = q(C_2) = 0
\end{cases}
\]

where \( \Sigma' \) is a subsurface of \( \Sigma_g \) cobounded by \( C_1 \) and \( C_2 \).

For genus \( g = 2 \) surfaces, the Torelli group \( J(2) \) is generated by the collection of all Dehn twists about bounding simple closed curves. For genus \( g \geq 3 \), \( J(2) \) is generated by the collection of all Dehn twists about genus 1 cobounding pairs of simple closed curves, i.e. pairs of non-bounding, disjoint, homologous simple closed curves that together bound a genus 1 subsurface. Thus the list above is sufficient for computing \( \rho_q(f) \) for any \( f \in J(2) \).

4. Abelianization of the Torelli Group

We are now prepared to say something about the abelianization

\[ H_1(J(2); \mathbb{Z}) \cong \frac{J(2)}{[J(2), J(2)]} \]

of the Torelli group \( J(2) \). In fact, the main result of Johnson in [16] is that the Johnson homomorphism \( \tau_2 : J(2) \to D_2(H_1) \) and the totality of the Birman-Craggs homomorphisms \( \rho_q : J(2) \to \mathbb{Z}_2 \) completely determine \( H_1(J(2); \mathbb{Z}) \).

On the one hand, we have the composition

\[
\frac{J(2)}{[J(2), J(2)]} \to \frac{J(2)}{J(3)} \cong D_2(H_1)
\]

where the first map is the projection given by the fact that \([J(2), J(2)] \subset J(3)\) and the second map is given by \( \tau_2 \). After we tensor with the rationals \( \mathbb{Q} \), Johnson shows that we obtain an isomorphism

\[
\frac{J(2)}{[J(2), J(2)]} \otimes \mathbb{Q} \cong \frac{J(2)}{J(3)} \otimes \mathbb{Q} \cong D_2(H_1) \otimes \mathbb{Q}.
\]

Thus we have \( H_1(J(2); \mathbb{Q}) \cong J(2)/J(3) \otimes \mathbb{Q} \).

On the other hand, consider the totality of the Birman-Craggs homomorphisms \( \{\rho_q\} \), and let

\[ C = \bigcap_q \ker \rho_q \]

be the common kernel of all \( \rho_q \) for all \( q \) which satisfy \( \operatorname{Arf}(\Sigma_g, q) = 0 \). Also let \( J(2)^2 \) represent the subgroup generated by all squares in \( J(2) \), and let \( O_q \) be the subgroup of the mapping class group \( \Gamma_{g,1} \) which acts trivially on \( H_1(\Sigma_{g,1}; \mathbb{Z}_2) \). That is, \( O_q \) consists of those homeomorphisms which preserve the quadratic form \( q \). Then, by using the theory of Boolean quadratic and cubic forms, Johnson showed that

\[ C = J(2)^2 = [O_q, J(2)]. \]

Finally he showed that the commutator subgroup of \( J(2) \) is given by

\[ [J(2), J(2)] = C \cap \ker \tau_2 = C \cap J(3). \]

Thus we can completely determine \( H_1(J(2); \mathbb{Z}) \cong J(2)/[J(2), J(2)] \) from the homomorphisms \( \{\tau_2, \rho_q\} \).
5. A Bordism Representation of the Mapping Class Group

5.1. The Bordism Group $\Omega_3(X, A)$. Let $(X, A)$ be a pair of topological spaces $A \subseteq X$. The 3-dimension oriented relative bordism group $\Omega_3(X, A)$ is defined to be the set of bordism classes of triples $(M, \partial M, \phi)$ consisting of a compact, oriented 3-manifold $M$ with boundary $\partial M$ and a continuous map $\phi: (M, \partial M) \rightarrow (X, A)$. The triples $(M_0, \partial M_0, \phi_0)$ and $(M_1, \partial M_1, \phi_1)$ are equivalent, or bordant over $(X, A)$, if there exists a triple $(W, \partial W, \Phi)$ consisting of a compact, oriented 4-manifold $W$ with boundary $\partial W = (M_0 \cup -M_1) \cup \partial M$ and a continuous map $\Phi : (W, \partial W) \rightarrow (X, A)$ satisfying $\Phi|_{M_i} = \phi_i$ and $\Phi(M) \subset A$. We also require that $\partial M = \partial M_0 \cup -\partial M_1$ so that $\partial W$ is a closed 3-manifold.

![Figure 5.1. A relative bordism over $(X, A)$](image)

A triple $(M, \partial M, \phi)$ is said to be null-bordant (or trivial) over $(X, A)$ if it bounds $(W, \partial W, \Phi)$, that is, if it is bordant to the empty set $\emptyset$. The set $\Omega_3(X, A)$ forms a group with the operation of disjoint union and identity element $\emptyset$. In the case that $A = \emptyset$, we may write $\Omega_3(X) = \Omega_3(X, \emptyset)$ and restrict our definition to pairs $(M, \phi) = (M, \emptyset, \phi)$ of closed, oriented 3-manifolds.

5.2. A Bordism Invariant of $\mathcal{J}(k)$. The purpose of this section is to analyze $\mathcal{J}(k)$ from the point of view of bordism theory. Let $F = \pi_1(\Sigma_{g,1})$ as before, and consider the pair $(K(F/F_k, 1), \zeta)$, where $K(F/F_k, 1)$ is an Eilenberg-MacLane space and $\zeta \subset K(F/F_k, 1)$ is an $S^1$ corresponding to the image of $\partial \Sigma_{g,1}$ under a continuous map $\Sigma_{g,1} \rightarrow K(F/F_k, 1)$ induced by the canonical projection $F \rightarrow F/F_k$.

We denote the bordism group over $(K(F/F_k, 1), \zeta)$ by $\Omega_3(F/F_k, \zeta)$. Moreover, we have an isomorphism $j_* : \Omega_3(F/F_k) \rightarrow \Omega_3(F/F_k, \zeta)$ induced by the inclusion map $j : (K(F/F_k, 1), \emptyset) \rightarrow (K(F/F_k, 1), \zeta)$. We will make use of both of these groups in what follows, but our main focus will be on the group $\Omega_3(F/F_k)$.

Below, in Theorem 5.2 we define a homomorphism $\sigma_k : \mathcal{J}(k) \rightarrow \Omega_3(F/F_k)$ whose kernel is $\ker \sigma_k = \mathcal{J}(2k - 1)$. Thus the image of $\sigma_k$ is $\mathcal{J}(k)/\mathcal{J}(2k - 1)$. We already saw that the image of the Johnson homomorphism $\tau_k$ is $\mathcal{J}(k)/\mathcal{J}(k + 1)$. However, since we know $[\mathcal{J}(k), \mathcal{J}(k)] \subset \mathcal{J}(2k - 1) \subset \mathcal{J}(k + 1)$, the image of this new homomorphism $\sigma_k$ is, in general, much closer to the abelianization of $\mathcal{J}(k)$.

Consider a surface homeomorphism $f \in \mathcal{J}(k)$ for some $k \geq 2$. As before let $T_{f,1}$ be the mapping torus of $f$, i.e. $\Sigma_{g,1} \times [0, 1]$ with $x \times \{0\}$ glued to $f(x) \times \{1\}$. The boundary $\partial T_{f,1}$ of $T_{f,1}$ is the torus $\partial \Sigma_{g,1} \times S^1$, and the mapping torus $T_{f,1}$ is
an (oriented) homology \( \Sigma_{g,1} \times S^1 \). Fixing a basis \( \{ \alpha_1, \ldots, \alpha_{2g} \} \) for the free group \( F = \pi_1(\Sigma_{g,1}) \) gives a presentation of \( \pi_1(T_{f,1}) \):

\[
\pi_1(T_{f,1}) = \langle \alpha_1, \ldots, \alpha_{2g}, \gamma \mid [\alpha_1, \gamma] \phi_* (\alpha_1) \alpha_1^{-1}, \ldots, [\alpha_{2g}, \gamma] \phi_* (\alpha_{2g}) \alpha_{2g}^{-1} \rangle
\]

where \( \gamma \) represents a generator of \( \pi_1(S^1) \). We now wish to obtain a closed 3-manifold from \( T_{f,1} \) by filling in its boundary. Let \( T_f^\gamma = T_{f,1} \) be the result of performing a Dehn filling along a curve on \( \partial T_{f,1} \) represented by the homotopy class \( \gamma \). That is, \( T_f^\gamma \) is obtained by filling in the torus \( \partial T_{f,1} \cong \partial \Sigma_{g,1} \times S^1 \) with the solid torus \( \partial \Sigma_{g,1} \times D^2 \). Then we also have a presentation for \( \pi_1(T_f^\gamma) \):

\[
\pi_1(T_f^\gamma) = \langle \alpha_1, \ldots, \alpha_{2g} \mid \phi_* (\alpha_1) \alpha_1^{-1}, \ldots, \phi_* (\alpha_{2g}) \alpha_{2g}^{-1} \rangle
\]

Note that if \( \phi \) is isotopic to the identity, then \( T_f^\gamma \) is homeomorphic to the connected sum of \( 2g \) \((S^1 \times S^2)\)'s.

Now for all \( m \leq k \) we can define \( \phi_{f,m} : (T_{f,1}, \partial T_{f,1}) \to (K(F/F_m, 1), \zeta) \) to be a continuous map induced by the canonical epimorphism

\[
\pi_1(T_{f,1}) \twoheadrightarrow \frac{\pi_1(T_{f,1})}{\langle \gamma, (\pi_1(T_{f,1}))_m \rangle} \cong \frac{F}{F_m}
\]

where the isomorphism requires the fact that \( f \in J(k) \subseteq J(m) \) (see Lemma 5.1 below.) Also, since we kill the homotopy class \( \gamma \) in our construction of \( T_f^\gamma \), the map \( \phi_{f,m} \) extends to a continuous map \( \phi_f^\gamma : T_f^\gamma \to K(F/F_m, 1) \), and \( \phi_f^\gamma \) induces the canonical epimorphism

\[
\pi_1(T_f^\gamma) \twoheadrightarrow \frac{\pi_1(T_f^\gamma)}{(\pi_1(T_f^\gamma))_m} \cong \frac{F}{F_m}
\]

Moreover, we have the following lemma.

**Lemma 5.1.** The following are equivalent:

(a) \( f \in J(m) \),

(b) \( \frac{\pi_1(T_f^\gamma)}{(\pi_1(T_f^\gamma))_m} \cong \frac{F}{F_m} \) and \( \frac{\pi_1(T_{f,1})}{\langle \gamma, (\pi_1(T_{f,1}))_m \rangle} \cong \frac{F}{F_m} \), and

(c) the continuous maps \( \phi_{f,m} \) and \( \phi_f^\gamma \) exist as defined.

**Proof.** This is an obvious fact, but we wish to emphasize it because of the important role it will play later.

(a) \( \iff \) (b). If \( f \in J(m) \) then the relations \( [\alpha_i, \gamma] \phi_* (\alpha_i) \alpha_i^{-1} \) in \( \pi_1(T_{f,1}) \) become trivial modulo \( \langle \gamma, (\pi_1(T_{f,1}))_m \rangle \) since \( \phi_* \) acts as the identity on \( F/F_m \), and we clearly have a homomorphism (in fact, an isomorphism.) On the other hand, no such homomorphism exists if \( f \notin J(m) \) because the relations \( [\alpha_i, \gamma] \phi_* (\alpha_i) \alpha_i^{-1} \equiv \phi_* (\alpha_i) \alpha_i^{-1} \mod \gamma \) are certainly not trivial modulo \( (\pi_1(T_{f,1}))_m \).

(b) \( \iff \) (c). It is a well-known property of Eilenberg-MacLane spaces that continuous maps into them are in one-to-one correspondence with homomorphisms into their fundamental group. (See [5], Theorem V.4.3.) Thus \( \phi_{f,m} \) (and similarly for \( \phi_f^\gamma \)) is defined if and only if the homomorphism

\[
\pi_1(T_{f,1}) \twoheadrightarrow \frac{\pi_1(T_{f,1})}{\langle \gamma, (\pi_1(T_{f,1}))_m \rangle} \cong \frac{F}{F_m}
\]

exists. \( \square \)
Let us now consider the pair \( (T_f^γ, φ_f^γ) \) ∈ \( Ω_3(F/F_k) \). We introduce a new homomorphism giving a representation of \( J(k) \) which is very geometric in nature.

**Theorem 5.2.** The map

\[
σ_k : J(k) \to Ω_3 \left( \frac{F}{F_k} \right)
\]

defined by \( σ_k(f) = (T_f^γ, φ_f^γ) \) is a well-defined homomorphism.

We point out that one can similarly define a homomorphism into the relative bordism group \( J(k) \to Ω_3(F/F_k, ζ) \) which sends a mapping class \( f \in J(k) \) to \( (T_{f,1}, ∂T_{f,1}, φ_{f,k}) \). However, we will mainly focus on the homomorphism given in Theorem 5.2.

**Proof.** Consider two homeomorphisms \( f, g \in J(k) \) for the oriented surface \( Σ_1 \) with one boundary component. If \( f \) and \( g \) are isotopic, i.e. they represent the same mapping class, then of course \( T_f^γ \) and \( T_g^γ \) are homeomorphic and \( (T_f^γ, φ_f^γ) \) and \( (T_g^γ, φ_g^γ) \) are bordant. Thus \( σ_k \) is certainly well-defined.

To show \( σ_k \) is indeed a homomorphism we need to show that \( (T_f^γ, φ_f^γ) \Pi (T_g^γ, φ_g^γ) \) is bordant to \( (T_{fg}^γ, φ_{fg,k}) \) in \( Ω_3(F/F_k) \) for any mapping classes \( f, g \in J(k) \). To do so, we simply construct a bordism, i.e. we build a 4-manifold \( W \) and continuous map \( Φ : W \to K(F/F_k, 1) \) with boundary given by

\[
(∂W, Φ|_{∂W}) = \left[ (T_f^γ, φ_f^γ) \Pi (T_g^γ, φ_g^γ) \right] ∪ \left[ - (T_{fg}^γ, φ_{fg,k}) \right]
\]

We begin by first constructing a 4-manifold between the mapping tori \( T_{f,1} \Pi T_{g,1} \) and \( T_{fg,1} \). Recall that

\[
T_{f,1} = \frac{Σ_1 \times [0, 1]}{(x, 0) \sim (f(x), 1)}.
\]

We may also consider \( T_{fg,1} \) in pieces as depicted in Figure 5.2.

![Figure 5.2. T_{fg,1} considered in pieces](image)

That is,

\[
T_{fg,1} = \frac{Σ_1 \times [0, 1]}{(x, 0) \sim (f(g(x)), 1)} \cong \frac{(Σ_1 \times [0, \frac{1}{2}]) \cup (Σ_1 \times [\frac{1}{2}, 1])}{(x, 0) \sim (f(x), 1), (x, \frac{1}{2}) \sim (g(x), \frac{1}{2})}
\]

We can assume there is a product neighborhood of \( Σ_1 \times \{\frac{1}{2}\} \) in \( T_{f,1}, \) i.e. a cylinder \( (Σ_1 \times \{\frac{1}{2}\}) \times [-ε, ε] \). Let \( V = (T_{f,1} \Pi T_{g,1}) \times [0, 1] \). Then \( V \) has boundary

\[
∂V = (T_{f,1} \Pi T_{g,1}) \times \{0\} \cup -(T_{f,1} \Pi T_{g,1}) \times \{1\} \cup (∂T_{f,1} \Pi ∂T_{g,1}) \times [0, 1].
\]
Now consider the piece \((T_{f,1} \amalg T_{g,1}) \times \{1\}\) of \(\partial V\) and attach a 4-dimensional “strip” \(\Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]\) to \((T_{f,1} \amalg T_{g,1}) \times \{1\}\) by gluing \(\Sigma_1 \times [-\varepsilon, \varepsilon] \times \{-\delta\}\) to the neighborhood \((\Sigma_1 \times \{1\}) \times [-\varepsilon, \varepsilon]\) in \(T_{f,1}\) and gluing \(\Sigma_1 \times [-\varepsilon, \varepsilon] \times \{\delta\}\) to the neighborhood \((\Sigma_1 \times \{1\}) \times [-\varepsilon, \varepsilon]\) in \(T_{g,1}\). Let \(V'\) be the result of this gluing, then

\[
\partial V' = ((T_{f,1} \amalg T_{g,1}) \times \{0\}) \cup (-T_{fog,1}) \times \{1\}) \cup ((\partial T_{f,1} \amalg \partial T_{g,1}) \times [0, 1]) \cup (\partial \Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]).
\]

We now fill in the boundary component \((\partial T_{f,1} \amalg \partial T_{g,1}) \times [0, 1]\) with

\[(*) \quad ((\partial \Sigma_1 \times D^2) \amalg (\partial \Sigma_1 \times D^2)) \times [0, 1]\]

to obtain a new 4-manifold \(W\). At one end, this has the effect of filling in the boundary of \((T_{f,1} \amalg T_{g,1}) \times \{0\}\), thus creating \(T^\gamma_{f,g} \amalg T^\gamma_{f,g}\) with \((\partial \Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta])\) above, and the filling by \((*)\) has the effect of filling in the rest of the boundary of \(T_{fog,1} \times \{1\}\) Thus we have actually created \(T^\gamma_{fog,1}\). Therefore we have created a 4-manifold \(W\) with boundary \(\partial W = (T^\gamma_{f} \amalg T^\gamma_{g}) \amalg -T^\gamma_{fog}\). Also, the continuous map \(\phi_{f,k} \amalg \phi_{g,k}\) clearly extends over \(V = (T_{f,1} \amalg T_{g,1}) \times [0, 1]\). It is also easy to see that it extends over \(V'\) as well since \(\Sigma_1 \times [-\varepsilon, \varepsilon] \times [-\delta, \delta]\) deformation retracts to \(\Sigma_1\). Finally it extends to a continuous map \(\Phi : W \rightarrow K(F/F_k, 1)\) in a similar way that \(\phi_{f,k}\) extends to \(\phi^\gamma_{f,k}\). Therefore \(T^\gamma_{f,g} \amalg \phi^\gamma_{f,g}\) is bordant to \(T^\gamma_{fog,1}\) in \(\Omega_3(F/F_k)\) and we have completed the proof of Theorem 5.2.

Notice that if a surface homeomorphism \(f\) is isotopic to the identity then its mapping class is in \(\mathcal{J}(k)\) for all \(k\), and \((T_{f,1}, \partial T_{f,1}, \phi_{f,k}) = (T_{id,1}, \partial T_{id,1}, \phi_{id,k})\) and \((T^\gamma_{f}, \phi^\gamma_{f,k}) = (T^\gamma_{id}, \phi^\gamma_{id,k})\) are null-bordant in \(\Omega_3(F/F_k)\) and \(\Omega_3(F/F_k)\), respectively, since they each bound \(\Sigma_{g,1} \times D^2\) and the respective maps clearly extend.
The homomorphism $\phi$ is defined as the induced map on $\ker (\gamma)$, which is nullbordant.

Proof. Suppose we have $(T_f, \partial T_f, 1^k, k)$ is bordant to $(T_g, \partial T_g, 1^k, k)$ in $\Omega_3(F/F_k)$ for all $k$. This is equivalent to having $T_f = T_g$ and $\partial T_f = \partial T_g$. However, we showed in the proof of Theorem 5.3 that this is equivalent to $T_{f,g}$ being nullbordant.

We prove the theorem for the pair $(T_f, 1^k, k)$ and the proof for the triple $(T_f, 1^k, 1^k, k)$ is completely analogous. Suppose $f \in \mathcal{J}(m)$, then for $l \leq m$ let $\pi_{m,l} : K(F/F_m, 1) \rightarrow K(F/F_l, 1)$ be the projection map such that $\phi_{f,l} = \pi_{m,l} \circ \phi_{f,m}$.

$(\Leftarrow)$. Let us first suppose that $f \in \mathcal{J}(2k-1)$. Then the pair $(T_f, 1^k, 2k-1)$ is defined and is an element of $\Omega_3(F/F_{2k-1})$. The following lemma is due to K. Igusa and K. Orr (10), Theorem 6.7.

**Lemma 5.6 (Igusa-Orr).** Let $(\pi_{m,k})_*$ be the induced map on $H_3$ and consider $x \in H_3(F/F_m)$. Then $x \in \ker (\pi_{m,k})_*$ if and only if $x \in \text{Image}(\pi_{2k-1,m})_*$ for $k \leq m \leq 2k-1$. In particular, the homomorphism $(\pi_{2k-1,1})_* : H_3(F/F_{2k-1}) \rightarrow H_3(F/F_k)$ is trivial.

We have the following corollary.

**Corollary 5.7.** The homomorphism

$$(\pi_{2k-1,1})_* : \Omega_3 \left( \frac{F}{F_{2k-1}} \right) \rightarrow \Omega_3 \left( \frac{F}{F_k} \right)$$

is trivial. Moreover, a bordism class is in $\ker (\pi_{m,k})_*$ if and only if it lies in the image of $(\pi_{2k-1,m})_*$ for $k \leq m \leq 2k-1$. The equivalence of (c) is proved similarly. 

$\square$

The following lemma is due to K. Igusa and K. Orr (10), Theorem 6.7.
Proof. In general, $\Omega_n(X,A)$ is the $n$-dimensional bordism group, and it is an extraordinary homology theory. Using the Atiyah-Hirzebruch spectral sequence, (see G. Whitehead [WR] for details,) one can express $\Omega_n(X,A)$ in terms of ordinary homology with coefficient group $\Omega_q$, where $\Omega_q = \Omega_q(\cdot)$ is the bordism group of a single point. In particular, $E^2_{p,q} = H_p(X,A;\Omega_q)$ and the boundary operator is $\partial^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$, and $\Omega_n(X,A)$ is built using $H_p(X,A;\Omega_q)$ with $p + q = n$. Now $\Omega_0 \cong \mathbb{Z}$ and $\Omega_1$, $\Omega_2$, and $\Omega_3$ are all trivial. So in the case $n = 3$ we have $\Omega_3(X,A) \cong H_3(X,A;\Omega_0) \cong H_3(X,A)$. In fact, the isomorphism is given by $(M,\partial M,\phi) \mapsto \phi_*([M,\partial M])$ where $[M,\partial M]$ denotes the fundamental class in $H_3(M,\partial M)$. Of course it follows directly that $\Omega_3(F/F_k) \cong H_3(F/F_k)$ (and $\Omega_3(F/F_k,\zeta) \cong H_3(F/F_k,\zeta)$) and we have the following commutative diagram:

\[
\begin{array}{ccc}
H_3\left(\frac{F}{F_{2k-1}}\right) & \xrightarrow{(\pi_{2k-1,m})_*} & H_3\left(\frac{F}{F_m}\right) \\
\cong & & \cong \\
\Omega_3\left(\frac{F}{F_{2k-1}}\right) & \xrightarrow{(\pi_{2k-1,m})_*} & \Omega_3\left(\frac{F}{F_m}\right)
\end{array}
\]

Since the map $(\pi_{2k-1,m})_*$ on $H_3$ is the zero-homomorphism, the conclusion of the first part of the corollary is proved. The proof of the latter part is also immediate.

The image of $(T^*_{f^*},\phi_{j^*F,2k-1})$ under $(\pi_{2k-1,m})_* : \Omega_3(F/F_{2k-1}) \to \Omega_3(F/F_k)$ is

$(\pi_{2k-1,m})_*(T^*_{f^*},\phi_{j^*F,2k-1}) = (T^*_{f^*},\pi_{2k-1,m} \circ \phi_{j^*F,2k-1}) = (T^*_{f^*},\phi_{j^*F,k})$,

and Corollary 5.4 tells us that this image is trivial in $\Omega_3(F/F_k)$. Thus the condition $f \in J(2k - 1)$ is certainly sufficient.

( $\implies$ ). The proof of the necessity of $f \in J(2k - 1)$ is much more subtle. If we assume that $(T^*_{f^*},\phi_{j^*F,k})$ is trivial in $\Omega_3(F/F_k)$, then Corollary 5.4 tells us that there is a pair $(M,\phi) \in \Omega_3(F/F_{2k-1})$ that gets sent to $(T^*_{f^*},\phi_{j^*F,k})$, but we do not know anything more than that. We want to show that $\phi_{j^*F,2k-1}$ is defined, and by Lemma 5.1 we may achieve the desired conclusion $f \in J(2k - 1)$.

Lemma 5.8 (Cochran-Gerges-Orr). Let $M$ be any oriented manifold such that $\pi_1(M) = G$, and suppose $F$ is a free group. Then for $k > 1$, $G/G_k \cong F/F_k$ if and only if $H_1(M)$ is torsion-free and all Massey products for $H^1(F/F_k)$ of length less than $k$ vanish. Under the latter conditions, any isomorphism $G/G_{k-1} \cong F/F_{k-1}$ extends to $G/G_k \cong F/F_k$.

Proof. If $G/G_k \cong F/F_k$ for $k > 1$, there is a continuous map $\phi : M \to K(F/F_k,1)$ that induces an isomorphism $\phi^* : H^1(F/F_k) \to H^1(M)$ and $H_1(M)$ is clearly torsion-free. In [OR] (Lemma 16) it is shown that Massey products for $H^1(F/F_k)$ of length less than $k$ vanish and length $k$ Massey products generate $H^2(F/F_k)$. Consider $x_i \in H^1(F/F_k)$, then $\langle x_1,\ldots,x_n \rangle = 0$ for all $n < k$. Also, the naturality of Massey products (see property (2.2.2)) tells us that $\phi^* \langle x_1,\ldots,x_n \rangle \subset \langle \phi^*x_1,\ldots,\phi^*x_n \rangle$. Thus for $n < k$ we certainly have $0 \in \langle \phi^*x_1,\ldots,\phi^*x_n \rangle$. However, the uniqueness of Massey products given in property (2.2.1) tells us that
the first nonzero Massey product is uniquely defined, and we conclude that $0 = \langle \phi^* x_1, \ldots, \phi^* x_n \rangle$ for $n < k$. Therefore, since $\phi^*$ is an isomorphism, all Massey products for $H^1(M)$ of length less than $k$ are zero.

On the other hand, if $H_1(M)$ is torsion-free and all Massey products for $H^1(M)$ of length less than $k$ vanish, then we easily see that $H_1(M) \cong G_2 \cong F/F_2$. Now assume by induction that $G/G_{k-1} \cong F/F_{k-1}$, and let $\psi : F \to G$ be a homomorphism that induces this isomorphism. We will extend this isomorphism to $G/G_k \cong F/F_k$. It is sufficient to show that $G_{k-1}/G_k \cong F_{k-1}/F_k$. We have the following commutative diagram

$$0 \longrightarrow H_2 \left( \frac{F}{F_{k-1}} \right) \cong \frac{F_{k-1}}{F_k} \longrightarrow 0$$

$$\xymatrix{ H_2(G) \ar[r]^{\pi_*} \ar[d]_{\psi_*} & H_2 \left( \frac{G}{G_{k-1}} \right) \ar[d]_{\psi_*} \ar[r] & \frac{G_{k-1}}{G_k} \ar[r] \ar[d]_{\psi_*} & 0 }$$

in which the horizontal maps are exact sequences. The fact that the sequences are exact is a result of J. Stallings [St]. This diagram shows us that it is sufficient to show that $\pi_* : H_2(G) \to H_2(G/G_{k-1})$ is trivial. However, since $H_2(M) \to H_2(G)$ is onto, we need only show that $\pi_* : H_2(M) \to H_2(G/G_{k-1})$ is trivial. As mentioned above, length $k - 1$ Massey products $(x_1, \ldots, x_{k-1})$ generate $H^2(G/G_{k-1}) \cong H^2(F/F_{k-1})$. Then $\pi^* \langle x_1, \ldots, x_{k-1} \rangle = \langle \pi^* x_1, \ldots, \pi^* x_{k-1} \rangle = 0$ since length $k - 1$ Massey products vanish for $M$. Therefore $\pi^*$ and $\pi_*$ are trivial homomorphisms, and the conclusion follows.

A slightly more general version of the following lemma is proved in [CGO] (Theorem 4.2), and we include a proof here for your convenience.

**Lemma 5.9** (Cochran-Gerges-Orr). Suppose $M_0$ and $M_1$ are closed, oriented 3-manifolds with $\pi_1(M_0) = G_0$ and $\pi_1(M_1) = G_1$. Further suppose that there is an epimorphism $\psi : G_1 \to G_0/(G_0)_k$, and then let $\phi_0 : M_0 \to K(G_0/(G_0)_k, 1)$ and $\phi_1 : M_1 \to K(G_0/(G_0)_k, 1)$ be continuous maps so that $(\phi_1)_* = \psi$ and $(M_0, \phi_0) = (M_1, \phi_1)$ in $\Omega_3(G_0/(G_0)_k)$. Then $(M_0, \phi_0)$ and $(M_1, \phi_1)$ are bordant over $K(G_0/(G_0)_k, 1)$ via a 4-manifold with only 2-handles (rel $M_0$) whose attaching circles lie in $(G_0)_k$.

**Proof.** Since $(M_0, \phi_0)$ and $(M_1, \phi_1)$ are bordant, we know there exists a compact, oriented 4-manifold $W$ and a continuous map $\Phi : W \to K(G_0/(G_0)_k, 1)$ such that $\partial(W, \Phi) = (M_0, \phi_0) \cup (\partial M_1, \phi_1)$. $\Phi$ is already a surjection on $\pi_1$, and we can make it an injection by performing surgery on loops in $W$. Thus we may assume $\Phi$ is an isomorphism. Now we choose a handlebody structure for $W$ relative to $M_0$ with no 0-handles or 4-handles. We then get rid of the 1-handles by trading them for 2-handles, i.e., we perform a surgery along a loop $c$ passing over the 1-handles in the interior of $W$. In a similar manner, we can get rid of the 3-handles by thinking of them as 1-handles relative to $M_1$. Let $V$ be the result of this handle swapping. We want to make sure $\Phi$ extends to $V$, so because $\Phi_*$ is an isomorphism it is necessary to make sure $c$ was null-homotopic in $W$ since it is null-homotopic in $V$. However, since $(\phi_1)_*$ is surjective and $c$ is in the interior of $W$, we can alter $c$ by a loop in $M_0$ so that the altered $c$ is null-homotopic in $W$. Thus we may assume that the 2-handles are attached along loops $c$ in $(G_0)_k$. \qed
Lemma 5.10. Let \( M_i \) and \( G_i \) \((i = 0, 1)\) be as in Lemma \ref{lem:massey_products}. For some free group \( F \) suppose that \( \phi_0 : M_0 \to K(F/F_k, 1) \) and \( \phi_1 : M_1 \to K(F/F_k, 1) \) are continuous maps such that \( \phi_0 \) induces an isomorphism \( G_0/(G_0)_k \cong F/F_k \) and \( \phi_1 \) extends to a continuous map \( \phi_1 : M_1 \to K(F/F_k, 1) \) inducing \( G_1/(G_1)_{k+1} \cong F/F_{k+1} \). If \( (M_0, \phi_0) \) is bordant to \( (M_1, \phi_1) \) in \( \Omega_3(F/F_k) \), then \( \phi_0 \) also extends so that it induces \( G_0/(G_0)_{k+1} \cong F/F_{k+1} \).

Proof. Lemma \ref{lem:massey_products} tells us there exists a bordism \( (W, \Phi) \) between \( (M_0, \phi_0) \) and \( (M_1, \phi_1) \) over \( K(F/F_k, 1) \) such that \( W \) contains only 2-handles with attaching circles in \( F_k \) and \( \pi_1(W) \cong F/F_k \). Let \( j_i : M_i \to W \) be inclusion maps so that \( \Phi \circ j_i = \phi_i, i = 0, 1 \).

Consider any collection \( \{x_1, \ldots, x_k\} \in H^1(M_0) \) of cohomology classes. Then choose \( y_i \in H^1(F/F_k) \) so that \( \phi^*_0(y_i) = x_i \). Since \( \pi_1(W) \cong F/F_k \) and \( G_0/(G_0)_k \cong F/F_k \), Lemma \ref{lem:massey_products} says that Massey products of length less than \( k \) vanish. Thus each of the following Massey products are uniquely defined:

\[
\langle x_1, \ldots, x_k \rangle = \langle \phi^*_0(y_1), \ldots, \phi^*_0(y_k) \rangle = j^*_0 \langle \Phi^*(y_1), \ldots, \Phi^*(y_k) \rangle.
\]

If we can actually show that these Massey products vanish then we can use Lemma \ref{lem:massey_products} to show that \( \phi_0 \) also extends so as to induce \( G_0/(G_0)_{k+1} \cong F/F_{k+1} \), thus completing the proof. We will show \( \langle \Phi^*(y_1), \ldots, \Phi^*(y_k) \rangle = 0 \). Since \( G_1/(G_1)_{k+1} \cong F/F_{k+1} \), Lemma \ref{lem:massey_products} says Massey products for \( H^1(M_1) \) of length less than \( k + 1 \) vanish. In particular, length \( k \) Massey products are zero, thus

\[
j^*_1 \langle \Phi^*(y_1), \ldots, \Phi^*(y_k) \rangle = \langle \phi^*_1(y_1), \ldots, \phi^*_1(y_k) \rangle = 0.
\]

Now consider the following short exact sequence

\[
0 \to H_2(M_1) \to H_2(W) \to H_2(W, M_1) \to 0.
\]

Since we can view \( W \) as \( M_1 \times [0, 1] \) with 2-handles attached along circles in \( F_k \), we see that \( H_2(W, M_1) \) is a free abelian group generated by the cores of the 2-handles (rel \( M_1 \)). Thus this sequence splits and we can write \( H_2(W) \cong H_2(M_1) \oplus H_2(W, M_1) \). Because the attaching circles of the 2-handles lie in \( F_k \), the images of the generators of the latter summand are clearly spheres in \( K(F/F_k, 1) \). But since \( K(F/F_k, 1) \) has trivial higher homotopy groups, they must vanish in \( H_3(F/F_k) \). Then by considering the dual splitting \( H^2(W) \cong H^2(M_1) \oplus H^2(W, M_1) \) we know that the image of \( H^2(F/F_k) \) must be contained in the summand \( H^2(M_1) \) of \( H^2(W) \). Therefore \( j^*_1 : H^2(W) \to H^2(M_1) \) must be injective on the image of \( H^2(F/F_k) \), and we are able to conclude that \( \langle \Phi^*(y_1), \ldots, \Phi^*(y_k) \rangle = 0 \). \( \square \)

Consider the following result of V. Turaev \cite{Turaev}.

Lemma 5.11 (Turaev). Let \( G \) be a finitely generated nilpotent group of nilpotency class at most \( k \geq 1 \), and let \( \alpha \in H_3(G) \). Then there exists a closed, connected, oriented 3-manifold \( M \) and a continuous map \( \psi : M \to K(G, 1) \) such that \( \psi_*(\langle M \rangle) = \alpha \) and such that \( \psi \) induces an isomorphism \( \pi_1(M)/(\pi_1(M))_k \cong G \) if and only if
(a) the homomorphism $\text{Hom}(H^2(G)) \to \text{Hom}(H_1(G))$ defined by sending $x$ to $x \cap \alpha$ is an isomorphism, and
(b) for any $h \in H_2(G)$, there exists $y \in H^1(G)$ such that
$$h - (y \cap \alpha) \in \ker \left( H_2(G) \to H_2 \left( \frac{G}{G_{k-1}} \right) \right).$$

**Corollary 5.12.** For any bordism class $\alpha \in \Omega_3(F/F_k)$ there exists a closed, connected, oriented 3-manifold $M$ and a continuous map $\psi : M \to K(F/F_k, 1)$ such that $(M, \psi) = \alpha$ in $\Omega_3(F/F_k)$ and $\psi$ induces an isomorphism $\pi_1(M)/(\pi_1(M))_k \cong F/F_k$.

**Proof.** We simply use the fact proved earlier that $\Omega_3(F/F_k) \cong H_3(F/F_k)$ and apply the lemma in the case that $G \cong F/F_k$. The group $F/F_k$ is nilpotent with nilpotency $k - 1$. The groups $H^2(F/F_k)$ and $H_1(F/F_k)$ are each torsion-free. Thus condition (a) of Lemma 5.11 is satisfied trivially. Using Stallings’ exact sequence given in [St], we have the following commutative diagram
\[
\begin{array}{c}
H_2(F) = 0 \xrightarrow{\sim} H_2 \left( \frac{F}{F_k} \right) \xrightarrow{\sim} \frac{F_k}{F_{k+1}} \xrightarrow{0 - \text{map}} 0 \\
H_2(F) = 0 \xrightarrow{\sim} H_2 \left( \frac{F}{F_{k-1}} \right) \xrightarrow{\sim} \frac{F_{k-1}}{F_k} \xrightarrow{0} 0
\end{array}
\]
which shows us that the map $H_2(F/F_k) \to H_2(F/F_{k-1})$ is the zero homomorphism. Thus condition (b) of Lemma 5.11 is also satisfied trivially.

**Lemma 5.13.** Let $M$ be any closed, oriented 3-manifold with $\pi_1(M) = G$, and suppose there is a continuous map $\phi_k : M \to K(F/F_k, 1)$ inducing an isomorphism $G/G_k \cong F/F_k$ for some free group $F$. For $m \geq k$, $(M, \phi_k)$ is in the image of $(\pi_m, k) : \Omega_3(F/F_m) \to \Omega_3(F/F_k)$ if and only if the isomorphism $G/G_k \cong F/F_k$ can be extended to an isomorphism $G/G_m \cong F/F_m$ induced by a continuous map $\phi_m : M \to K(F/F_m, 1)$ such that $(\pi_m, k)(M, \phi_m) = (M, \phi_k)$.

**Proof.** Suppose $(M, \phi_k) = (\pi_m, k)(\alpha)$, for some $\alpha \in \Omega_3(F/F_m)$. By Corollary 5.12 there exists a closed, connected, oriented 3-manifold $M'$ and a continuous map $\psi : M' \to K(F/F_m, 1)$ that induces an isomorphism $\pi_1(M')/(\pi_1(M'))_m \cong F/F_m$ such that $(M', \psi) = \alpha$ in $\Omega_3(F/F_m)$. Therefore we have $(M, \phi_k) = (\pi_m, k)(\alpha) = (\pi_m, k)(M', \psi) = (M', \pi_m, k \circ \psi)$. Thus $(M, \phi_k)$ and $(M', \pi_m, k \circ \psi)$ are bordant in $\Omega_3(F/F_k)$. In the case $m = k + 1$, Lemma 5.11 gives the desired result. The case $m > k + 1$ is achieved via induction. The converse is clear.

We are now ready to continue our proof of Theorem 5.3. First, we are assuming that $\phi_{f,k}$ exists, so Lemma 5.11 tells us that at the very least $f \in \mathcal{F}(k)$. We also assume that $(T_f^\gamma, \phi_{f,k})$ is trivial in $\Omega_3(F/F_k)$. In particular, $(T_f^\gamma, \phi_{f,k}) = (T_{id}^\gamma, \phi_{id,k})$ in $\Omega_3(F/F_k)$. Also, we have
$$\frac{\pi_1(T_{id}^\gamma)}{(\pi_1(T_{id}))_m} \cong \frac{F}{F_m},$$
for all $m$, and
\[
\frac{\pi_1(T_f^\gamma)}{\pi_1(T_f^\gamma)} \cong \frac{F}{F_m}, \text{ for all } m \leq k.
\]

Then by Lemma \ref{lem:extension}, we can extend the latter isomorphism to
\[
\frac{\pi_1(T_f^\gamma)}{\pi_1(T_f^\gamma)} \cong \frac{F}{F_{k+1}}.
\]

By Lemma \ref{lem:existence} we are able to conclude that \( f \in J(k+1) \) and that the continuous map \( \phi_{f,k+1}^\gamma \) exists, allowing us to consider \( (T_f^\gamma, \phi_{f,k+1}^\gamma) \in \Omega_3(F/F_{k+1}) \). Moreover, since we are assuming that \( (T_f^\gamma, \phi_{f,k}^\gamma) \) is trivial in \( \Omega_3(F/F_k) \), we have
\[
(T_f^\gamma, \phi_{f,k+1}^\gamma) \in \text{ker} \left( \Omega_3 \left( \frac{F}{F_{k+1}} \right)^{\pi_{k+1,k}}, \Omega_3 \left( \frac{F}{F_k} \right) \right),
\]
and by Corollary \ref{cor:kernel}
\[
(T_f^\gamma, \phi_{f,k+1}^\gamma) \in \text{Image} \left( \Omega_3 \left( \frac{F}{F_{2k-1}} \right)^{\pi_{2k-1,k+1}}, \Omega_3 \left( \frac{F}{F_{k+1}} \right) \right).
\]

Thus Lemma \ref{lem:extension} implies that the isomorphism
\[
\frac{\pi_1(T_f^\gamma)}{\pi_1(T_f^\gamma)} \cong \frac{F}{F_{k+1}}
\]
extends to an isomorphism
\[
\frac{\pi_1(T_f^\gamma)}{\pi_1(T_f^\gamma)} \cong \frac{F}{F_{2k-1}}.
\]

Therefore, by Lemma \ref{lem:extension}, we are able to conclude that \( f \in J(2k-1) \). This completes the proof of Theorem \ref{thm:extension}.

5.3. Relating \( \sigma_k \) to the Johnson Homomorphism. The goal of this section is to describe how the homomorphism \( \sigma_k : J(k) \to \Omega_3(F/F_k) \) relates to Johnson’s homomorphism \( \tau_k : J(k) \to D_k(H_1) \subset \text{Hom}(H_1, F_k/F_{k+1}) \). It turns out that \( \tau_k \) factors through \( \Omega_3(F/F_k) \). To see this, we will use Kitano’s definition of \( \tau_k \) in terms of Massey products, which we reviewed in Section \ref{sec:massey}.

Let \( \mathfrak{X} \) denote the ring of formal power series in the noncommutative variables \( t_1, \ldots, t_{2g} \), and let \( \mathfrak{X}_k \) denote the submodule of \( \mathfrak{X} \) corresponding to the degree \( k \) part. Because \( F_k/F_{k+1} \) is a submodule of \( \mathfrak{X}_k \), we can consider the homomorphism
\[
\tau_k : J(k) \to \text{Hom}(H_1, \mathfrak{X}_k)
\]
defined in Theorem \ref{thm:kitano}. Recall from Section \ref{sec:massey} that we are considering the following dual bases:
\[
\{x_1, \ldots, x_{2g}, y\} \in H_1(T_f, 1),
\]
\[
\{x_1^*, \ldots, x_{2g}^*, y^*\} \in H^1(T_f, 1), \text{ and}
\]
\[
\{X_1, \ldots, X_{2g}\} \in H_2(T_f, 1).
\]
Define $\Psi' : \Omega_3(F/F_k, \zeta) \to \text{Hom}(H_1, X_k)$ to be the map that sends the bordism class $(T_{f,1}, \partial T_{f,1}, \phi_{f,k})$ to the homomorphism

$$x_i \mapsto \sum_{j_1, \ldots, j_k} \langle (x^*_{j_1}, \ldots, x^*_{j_k}), X_i \rangle t_{j_1} \cdots t_{j_k}.$$  

Let $i_* : \Omega_3(F/F_k) \to \Omega_3(F/F_k, \zeta)$ be the homomorphism induced by inclusion which sends $(T^*, \phi^*_{f,k})$ to $(T_{f,1}, \partial T_{f,1}, \phi_{f,k})$. Then we define the homomorphism $\Psi : \Omega_3(F/F_k) \to \text{Hom}(H_1, X_k)$ to be the composition $\Psi = \Psi' \circ i_*$.  

**Theorem 5.14.** The map $\Psi$ is a well-defined homomorphism. Moreover, the composition $\Psi \circ \sigma_k$ corresponds to the Johnson homomorphism $\tau_k$ so that we have the following commutative diagram.

$$\begin{array}{ccc}
\Omega_3(F/F_k) & \xrightarrow{\Phi} & \text{Hom}(H_1, X_k) \\
\delta \downarrow & & \Psi \downarrow \tau_k \\
J(k) & \xrightarrow{\sigma_k} & \text{Hom}(H_1, X_k)
\end{array}$$

**Proof.** We only need to show that $\Psi' : \Omega_3(F/F_k, \zeta) \to \text{Hom}(H_1, X_k)$ is a well-defined homomorphism, and the rest of the theorem clearly follows. We will need the following lemma.

**Lemma 5.15.** Suppose $(M_0, \phi_0)$ and $(M_1, \phi_1)$ are closed, oriented 3-manifolds with $\pi_1(M_i) = G_i$ and continuous maps $\phi_i : M_i \to K(G_0/(G_0)_k, 1)$. Further suppose $\phi_1$ induces an isomorphism $G_1/(G_1)_k \cong G_0/(G_0)_k$. If $(M_0, \phi_0)$ is bordant to $(M_1, \phi_1)$ in $\Omega_3(G_0/(G_0)_k)$ and all Massey products for $H^1(M_0)$ of length less than $k$ vanish, then $\phi = (\phi_0)_*^{-1} \circ (\phi_1)_* : H_1(M_1) \to H_1(M_0)$ is an isomorphism such that for $x_i \in H^1(M_0)$, $E_i \in H_2(M_0)$ Poincaré dual to $x_i$, and $F_i \in H_2(M_1)$ Poincaré dual to $\phi^* (x_i)$ in $H^1(M_1)$ we have

$$\langle \langle x_{j_1}, \ldots, x_{j_k} \rangle, E_i \rangle = \langle (\phi^* (x_{j_1}), \ldots, \phi^* (x_{j_k})), F_i \rangle$$

where $\langle , \rangle$ is the dual pairing of $H^2(M_i)$ and $H_2(M_i)$.

**Proof.** Since $(M_0, \phi_0)$ is bordant to $(M_1, \phi_1)$ in $\Omega_3(G_0/(G_0)_k)$, we must also have $(\phi_0)_* ([M_0]) = (\phi_1)_* ([M_1])$ in $H_3(G_0/(G_0)_k)$ where $[M_i]$ is the fundamental class in $H_3(M_i)$. The bordism $(W, \Phi)$ between $(M_0, \phi_0)$ and $(M_1, \phi_1)$ can be chosen so that $\Phi$ induces an isomorphism $\pi_1(W) \cong G_0/(G_0)_k$ and the inclusion maps $j_i : M_i \to W$ induce isomorphisms $G_i/(G_i)_k \cong \pi_1(W)/\pi_1(W)_k$.

W. Dwyer proves in [Dw] (Corollary 2.5) that for cohomology classes $\alpha_i \in H^1(W)$ we have $\langle \alpha_1, \ldots, \alpha_m \rangle = 0$ if and only if $j_0^* \langle \alpha_1, \ldots, \alpha_m \rangle = 0$ for $m < k$. However, by the naturality of Massey products given in property (2), we know that $j_0^* \langle \alpha_1, \ldots, \alpha_m \rangle \subset \langle j_0^* (\alpha_1), \ldots, j_0^* (\alpha_m) \rangle$, and the latter is 0 since Massey products of length less than $k$ vanish for $H^1(M_0)$. Thus $\langle \alpha_1, \ldots, \alpha_m \rangle = 0$ for all $\alpha_i \in H^1(W)$. Moreover, $j_1^* : H^1(W) \to H^1(M_1)$ is an isomorphism. Then for any $y_i \in H^1(M_1)$ there exists an $\alpha_i \in H^1(W)$ such that $j_1^* (\alpha_i) = y_i$. Thus for $m < k$ we have

$$\langle y_1, \ldots, y_m \rangle = \langle j_1^* (\alpha_1), \ldots, j_1^* (\alpha_m) \rangle = j_1^* (\alpha_1, \ldots, \alpha_m) = 0$$
where the second equality follows from naturality. So then we have shown that all Massey products of length less than $k$ vanish also for $H^3(W)$ and $H^1(M_1)$. Thus Massey products for $H^1(M_0)$, $H^1(M_1)$, and $H^1(W)$ of length $k$ are uniquely defined.

Consider $x_i \in H^1(M_0)$ with Poincaré dual $\xi_i \in H_2(M_0)$. Let $\mathcal{F}_i \in H_2(M_1)$ be Poincaré dual to $\phi^*(x_i) \in H^1(M_1)$, where $\phi$ is the isomorphism given by the composition $\phi = (\phi_0)^{-1} \circ (\phi_1)_*: H_1(M_1) \to H_1(M_0)$. Then we have

$$\begin{align*}
(\phi_0)_*(\mathcal{E}_i) &= (\phi_0)_*(x_i \cap [M_0]) \\
&= (\phi_0^*)^{-1} (x_i) \cap (\phi_0)_*([M_0]) \\
&= \left( (\phi_1^*)^{-1} \circ \phi^* \right) (x_i) \cap (\phi_1)_*([M_1]) \\
&= (\phi_1)_*(\phi^*(x_i) \cap [M_1]) \\
&= (\phi_1)_*(\mathcal{F}_i),
\end{align*}$$

where the second and fourth equalities follow from the naturality of cap products.

Now choose $\beta_i \in H^1(G_0/(G_0)_k)$ such that $\phi^*_0(\beta_i) = x_i$. Then

$$\begin{align*}
\langle (x_{j_1}, \ldots, x_{j_k}), \mathcal{E}_i \rangle &= \langle (\phi_0^*(\beta_{j_1}), \ldots, \phi_0^*(\beta_{j_k})), \mathcal{E}_i \rangle \\
&= \langle (\beta_{j_1}, \ldots, \beta_{j_k}), (\phi_0)_*(\mathcal{E}_i) \rangle \\
&= \langle (\beta_{j_1}, \ldots, \beta_{j_k}), (\phi_1)_*(\mathcal{F}_i) \rangle \\
&= \langle (\phi_1^*(\beta_{j_1}), \ldots, \phi_1^*(\beta_{j_k})), \mathcal{F}_i \rangle \\
&= \langle (\phi_1^* \phi_0^*)(\beta_{j_1}, \ldots, (\phi^* \phi_0^*)(\beta_{j_k})), \mathcal{F}_i \rangle \\
&= \langle (\phi^*(x_{j_1}), \ldots, \phi^*(x_{j_k})), \mathcal{F}_i \rangle.
\end{align*}$$

This completes the proof of the lemma. \qed

Consider the mapping classes $f, h \in J(k)$. We have the dual bases mentioned above for specific homology and cohomology groups of $T_{f,1}$. Consider the following dual bases defined in the same manner for $T_{h,1}$:

$$\begin{align*}
\{w_1, \ldots, w_{2g}, z\} &\in H_1(T_{h,1}), \\
\{w_1^*, \ldots, w_{2g}^*, z^*\} &\in H^1(T_{h,1}), \text{ and} \\
\{W_1, \ldots, W_{2g}\} &\in H_2(T_{h,1}).
\end{align*}$$

Recall that $T_f^2$ was constructed from $T_{f,1}$ by filling the boundary $\partial T_{f,1} = \partial \Sigma_{g,1} \times S^1$ with a solid torus $\partial \Sigma_{g,1} \times D^2$. Let $\psi_f : T_{f,1} \to T_f^2$ be the inclusion map, and then we have a basis

$$\{a_1^*, \ldots, a_2^g\} \in H^1(T_f^2)$$

where $a_i^* = ((\psi_f)_*(x_i))^*$. Since $x_i^*$ is the Hom dual of $x_i$, by definition we have that the dual pairing is $\langle x_i^*, x_j \rangle = \delta_{ij}$. Similarly $a_i^*$ is the Hom dual of $((\psi_f)_*(x_i))$, and thus $\langle \psi_f^*(a_i^*), x_j \rangle = \langle a_i^*, (\psi_f)_*(x_i) \rangle = \delta_{ij}$. Note that this implies that $\psi_f^*(a_i^*) = x_i^*$. Letting $A_i$ denote the Poincaré dual of $a_i^*$ gives a basis for $H_2(T_f^2)$:

$$\{A_1, \ldots, A_{2g}\} \in H_2(T_f^2).$$
By carefully examining the following commutative diagram, we see \((\psi_f)_* (X_i) = A_i\).

\[
\begin{array}{cccc}
H_1(T_{f,1}, \partial T_{f,1}) & \cong & \text{Hom dual} & H^1(T_{f,1}, \partial T_{f,1}) \cap [T_{f,1}, \partial T_{f,1}] \cong H_2(T_{f,1}) \\
\downarrow j_* & & \downarrow j^* & \\
H_1(T_f) & \cong & \text{Hom dual} & H^1(T_f) \cap [T_f, \partial T_f] \\
(\psi_f)_* & & \psi_f^* & \\
H_1(T_f^\gamma) & \cong & \text{Hom dual} & H^1(T_f^\gamma) \cap [T_f^\gamma] \\
& & \psi_f^* & \\
& & \text{Hom dual} & H_2(T_f^\gamma)
\end{array}
\]

Finally, let \(\bar{A}_i \in H_2(T_f^\gamma)\) denote the Poincaré dual to \(\phi^*(a_i^*)\), where \(\phi\) is the isomorphism guaranteed by the following corollary. Then for \(T_{h,1}\) we similarly have \(\psi_f^* (\phi^*(a_i^*)) = w_i^*\) and \((\psi_h)_* (W_i) = \bar{A}_i\). We have the following immediate corollary to Lemma 5.15.

**Corollary 5.16.** If \(\left(T_f^\gamma, \phi_f^\gamma, k\right) = \left(T_f, \phi, k\right)\) in \(\Omega_3(F/F_k)\), then the isomorphism \(\phi = (\phi_0)_* \circ (\phi_1)_* : H_1(T_f^\gamma) \to H_1(T_f)\) satisfies

\[
\left\langle \langle a_{j_1}, \ldots, a_{j_k} \rangle, A_i \right\rangle = \left\langle \langle \phi^*(a_{j_1}^*), \ldots, \phi^*(a_{j_k}^*) \rangle, \bar{A}_i \right\rangle
\]

where \(\bar{A}_i\) is Poincaré dual to \(\phi^*(a_i^*)\).

**Lemma 5.17.** If \((T_{f,1}, \partial T_{f,1}, \phi_{f,k}) = (T_{h,1}, \partial T_{h,1}, \phi_{h,k})\) in \(\Omega_3(F/F_k, \zeta)\), then

\[
\left\langle \langle x_{j_1}^*, \ldots, x_{j_k}^* \rangle, X_i \right\rangle = \left\langle \langle w_{j_1}^*, \ldots, w_{j_k}^* \rangle, W_i \right\rangle.
\]

**Proof.** Since \(f, h \in J(k)\), Theorem 2.2 says that the Massey products of length less than \(k\) for \((T_{f,1}, \partial T_{f,1})\) and \((T_{h,1}, \partial T_{h,1})\) must vanish. Thus \(\langle x_{j_1}^*, \ldots, x_{j_k}^* \rangle\) and \(\langle w_{j_1}^*, \ldots, w_{j_k}^* \rangle\) are uniquely defined. By Corollary 5.15 we know that \(\left(T_f^\gamma, \phi_f^\gamma, k\right) = \left(T_f, \phi, k\right)\) in \(\Omega_3(F/F_k)\). So we let \(\phi\) be the isomorphism guaranteed by Corollary 5.15. Then we have

\[
\left\langle \langle x_{j_1}^*, \ldots, x_{j_k}^* \rangle, X_i \right\rangle = \left\langle \langle \psi_f^* (a_{j_1}^*), \ldots, \psi_f^* (a_{j_k}^*) \rangle, X_i \right\rangle = \left\langle \langle a_{j_1}, \ldots, a_{j_k} \rangle, (\psi_f)_* (X_i) \right\rangle = \left\langle \langle a_{j_1}^*, \ldots, a_{j_k}^* \rangle, A_i \right\rangle = \left\langle \langle \phi^*(a_{j_1}^*), \ldots, \phi^*(a_{j_k}^*) \rangle, \bar{A}_i \right\rangle.
\]

This proves that \(\Psi': \Omega_3(F/F_k, \zeta) \to \text{Hom}(H_1, X_k)\) is a well-defined homomorphism and completes the proof of Theorem 5.14. \(\square\)
5.4. Relating $\sigma_k$ to Morita’s Homomorphism. As we have already seen in the proof of Corollary 5.17, there is an isomorphism $\Phi : \Omega_3(F/F_k) \rightarrow H_3(F/F_k)$ given by $(M, \phi) \mapsto \phi_k([M])$, where $[M]$ is the fundamental class in $H_3(M)$. Because of this, one may guess that there is a relationship between $\sigma_k : \mathcal{J}(k) \rightarrow \Omega_3(F/F_k)$ and Morita’s refinement $\tilde{\tau}_k : \mathcal{J}(k) \rightarrow H_3(F/F_k)$ discussed in Section 2.4. This assumption turns out to be correct, and the two homomorphisms are in fact equivalent. However, $\sigma_k$ gives a representation that is much more geometric, and as we will see in Section 5.4, $\sigma_k$ leads to interesting questions that $\tilde{\tau}_k$ does not.

**Theorem 5.18.** The homomorphism $\sigma_k : \mathcal{J}(k) \rightarrow \Omega_3(F/F_k)$ coincides with the Morita refinement of the Johnson homomorphism so that we have a commutative diagram.

\[
\begin{array}{ccc}
\Omega_3 \left( \frac{F}{F_k} \right) & \xrightarrow{\delta^k} & \Phi \\
\downarrow \Phi & & \downarrow \Phi \\
\mathcal{J}(k) & \xrightarrow{\tilde{\tau}_k} & H_3 \left( \frac{F}{F_k} \right)
\end{array}
\]

**Corollary 5.19.** The kernel of Morita’s refinement $\tilde{\tau}_k$ is $\mathcal{J}(2k - 1)$.

**Proof.** This is an immediate consequence of Theorem 5.18 and Corollary 5.14. □

**Proof of Theorem 5.18.** Consider a genus $g$ surface with one boundary component $\Sigma = \Sigma_{g,1}$ and $f \in \mathcal{J}(k)$. Let $r : \Sigma \times [0, 1] \rightarrow \Sigma$ be a retraction, $\psi : \Sigma \rightarrow K(F/F_k, 1)$ be a continuous map that induces the canonical epimorphism $F \rightarrow F/F_k$, and $i : K(F/F_k, 1) \rightarrow (K(F/F_k, 1), \zeta)$ be the inclusion map. Also let $G : \Sigma \times [0, 1] \rightarrow (T_{f,1}, \partial T_{f,1})$ be the composition of the “gluing map” $\Sigma \times [0, 1] \rightarrow T_{f,1}$ and the inclusion $T_{f,1} \rightarrow (T_{f,1}, \partial T_{f,1})$. Recall that the maps $\phi_{f,k}$ and $\phi_{f,k}^\gamma$ defined at the beginning of Section 5.2 are defined only up to homotopy. We choose them so that the following diagram commutes.

\[
\begin{array}{ccc}
\Sigma \times [0, 1] & \xrightarrow{r} & \Sigma \\
\downarrow G & & \downarrow \psi \\
(T_{f,1}, \partial T_{f,1}) & \xrightarrow{\phi_{f,k}^\gamma} & K(F/F_k, 1) \\
\downarrow \phi_{f,k} & & \downarrow i \\
(K(F/F_k, 1), \zeta)
\end{array}
\]

That is, we have $\phi_{f,k} \circ G = i \circ \psi \circ r$.

Consider the fundamental class $[T_{f,1}, \partial T_{f,1}] \in H_3(T_{f,1}, \partial T_{f,1})$, and suppose that $(t_f, \partial t_f) \in C_3(T_{f,1}, \partial T_{f,1})$ is a corresponding relative 3-cycle. Now we choose a 2-chain $\sigma \in C_2(\Sigma \times [0, 1])$ so that $\partial \sigma$ is in the homotopy class of a simple closed curve on $\Sigma \times \{0\}$ parallel to the boundary $\partial \Sigma \times \{0\}$. Let $\sigma$ also denote $r_{\bar{\gamma}}(\sigma) \in C_2(\Sigma)$, and choose a 3-chain $\rho \in C_3(\Sigma \times [0, 1])$ so that $G_{\bar{\gamma}}(\rho) = (t_f, \partial t_f)$ and $\partial \rho = \sigma - f\#(\sigma) + (\partial \sigma \times [0, 1])$. 

Consider the restriction \( r|_{\partial \Sigma \times [0,1]} \). Then \( r\#(\partial \sigma \times [0,1]) = \varepsilon \in C_2(\partial \Sigma) \), and
\[
\partial r\#(\rho) = r\#(\partial(\rho)) \\
= r\#(\rho - f\#(\sigma) + (\partial \sigma \times [0,1])) \\
= r\#(\sigma - f\#(\sigma)) + r\#(\partial \sigma \times [0,1]) \\
= \sigma - f\#(\sigma) + \varepsilon
\]
If \( f \) is the identity on the boundary, we must have \( \partial \sigma - f\#(\partial \sigma) = 0 \), and therefore \( 0 = \partial(\partial r\#(\rho)) = \partial(\partial \sigma - f\#(\sigma) + \varepsilon) = \partial \sigma - f\#(\partial \sigma) + \partial \varepsilon = \partial \varepsilon \). Since \( H_2(\partial \Sigma) \) is trivial, there must be a 3-chain \( \eta \in C_3(\partial \Sigma) \) such that \( \partial \eta = \varepsilon \). Let \( j : \Sigma \to \Sigma \times [0,1] \) be the inclusion map, and consider \( j\#(\eta) \in C_3(\partial \Sigma \times [0,1]) \to C_3(\Sigma \times [0,1]) \). Define \( c_f \in C_3(\Sigma) \) to be
\[
c_f = r\#(\rho - j\#(\eta)) \\
= r\#(\rho) - r\#j\#(\eta) \\
= r\#(\rho) - \eta.
\]
Then \( \partial c_f = \partial r\#(\rho) - \partial \eta = (\sigma - f\#(\sigma) + \varepsilon) - \varepsilon = \sigma - f\#(\sigma) \).

Also, since \( j\#(\eta) \in C_3(\Sigma \times [0,1]) \) is carried by the subcomplex \( \partial \Sigma \times [0,1] \), \( G\#(j\#(\eta)) \) must be carried by \( \partial T_{f,1} \). Thus \( G\#(j\#(\eta)) = 0 \), and
\[
G\#(\rho - j\#(\eta)) = G\#(\rho) - G\#(j\#(\eta)) = (t_f, \partial t_f).
\]
Let \( \tilde{c}_f = \psi\#(c_f) \in C_3(F/F_k) \). Then \( \tilde{c}_f \) is a 3-cycle since \( f \in \mathcal{J}(k) \) induces the identity on \( F/F_k \). Let \( \tilde{c}_f \in H_3(F/F_k) \) denote the corresponding homology class, and
\[
i_*([\tilde{c}_f]) = [i_*([\tilde{c}_f])] \\
= [(i \circ \psi)_\#(c_f)] \\
= [(i \circ \psi \circ r)_\#(\rho - j\#(\eta))] \\
= ([\phi_{f,k} \circ G\#(\rho - j\#(\eta))] \\
= ([\phi_{f,k}]_\#(t_f, \partial t_f)) \\
= ([\phi_{f,k}]_\#([T_{f,1}, \partial T_{f,1}]))
\]
On the other hand, we also have \( i_*((\phi_{f,k})_\#([T_f])) = (\phi_{f,k})_\#([T_{f,1}, \partial T_{f,1}]) \), and since \( i_* : H_3(F/F_k) \to H_3(F/F_k, \zeta) \) is an isomorphism, we must have \( [\tilde{c}_f] = ([\phi_{f,k}]_\#([T_f])) \).

Finally, notice that our choices of \( \sigma \in C_2(\Sigma) \) and \( c_f \in C_3(\Sigma) \) certainly qualify as choices for \( \sigma \in C_2(F) \) and \( c_f \in C_3(F) \), respectively, in the construction of Morita’s homomorphism in Section 24. Thus we have \( (\Phi \circ \sigma_k)(f) = \Phi(T_f, \phi_{f,k}) = (\phi_{f,k})_\#([T_f]) = [\tilde{c}_f] = \tilde{\tau}_k(f) \), and the theorem is proved. \( \square \)

Now that we see that \( \sigma_k : \mathcal{J}(k) \to \Omega_3(F/F_k) \) and Morita’s homomorphism are indeed equivalent, we can describe in a different way how \( \sigma_k \) relates to Johnson’s homomorphism \( \tau_k : \mathcal{J}(k) \to H_1 \otimes F_k/F_{k+1} \). Recall the differential
\[
d^2 : H_3 \left( \frac{F}{F_k} \right) \to H_1 \otimes \frac{F_k}{F_{k+1}}
\]
discussed in Section 2.4. Then $\tau_k$ factors through $\Omega_3(F/F_k)$ so that the following diagram commutes.

\[
\begin{array}{ccc}
\sigma & \Omega_3 \left( \frac{F}{F_k} \right) & \delta \psi \\
J(k) & H_1 \otimes \frac{F_k}{F_{k+1}} & d^2 \circ \Phi
\end{array}
\]

6. A Spin Bordism Representation of the Mapping Class Group

We introduced in Section 5 a new representation $\sigma_k : J(k) \to \Omega_3(F/F_k)$ which we then showed was equivalent to Morita’s homomorphism $\tilde{\tau}_k : J(k) \to H_5(F/F_k)$. Because of the range of the latter homomorphism, it may seem preferable to those who have a firm understanding of homology. However, $\sigma_k$ has its advantages. First, it simply has a more geometric nature to it. Second, and perhaps most importantly, it naturally leads to an interesting question that $\tilde{\tau}_k$ does not. What happens when we add more structure to the bordism group? More specifically, what is the result of replacing the bordism group $\Omega_5(F/F_k)$ with the spin bordism group $\Omega_5^{spin}(F/F_k)$?

6.1. A Spin Bordism Invariant of $J(k)$. Recall that a spin structure can be thought of as a trivialization of the stable tangent bundle restricted to the 2-skeleton, and every oriented 3-manifold has a spin structure. Since a spin structure on a manifold induces a spin structure on its boundary, we can define the 3-dimensional spin bordism group $\Omega_3^{spin}(X)$ in exactly the same way as the oriented bordism group $\Omega_3(X)$ with the additional requirement that spin structures on spin bordant 3-manifolds must extend to a spin structure on the 4-dimensional bordism between them. That is, elements of $\Omega_3^{spin}(X)$ are equivalence classes of triples $(M, \phi, \sigma)$ consisting of a closed, spin 3-manifold $M$ with spin structure $\sigma$ and a continuous map $\phi : M \to X$. We say two elements $(M_0, \phi_0, \sigma_0)$ and $(M_1, \phi_1, \sigma_1)$ are equivalent, or spin bordant over $X$, if there is a triple $(W, \Phi, \sigma)$ consisting of a compact, spin 4-manifold $(W, \sigma)$ with boundary $\partial(W, \sigma) = (M_0, \sigma_0) - (M_1, \sigma_1)$ and a continuous map $\Phi : W \to X$ satisfying $\Phi|_{M_i} = \phi_i$.

Further recall that the spin structures for a spin manifold $M$ are enumerated by $H^1(M; \mathbb{Z}_2)$. Thus, for example, the number of possible spin structures on an oriented surface $\Sigma_{g,1}$ of genus $g$ with one boundary component is $|H^1(\Sigma_{g,1}; \mathbb{Z}_2)| = 2^{2g}$. If we fix a spin structure on $\Sigma_{g,1}$, then we can extend it to the product $\Sigma_{g,1} \times [0,1]$. Now consider the mapping class $f \in J(k)$ for $\Sigma_{g,1}$. For $k \geq 2$, $f$ acts trivially on $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$ and on the set of spin structures. Thus the spin structure on $\Sigma_{g,1} \times [0,1]$ can be extended to the mapping torus $T_{f,1}$. The number of possible spin structures for $T_{f,1}$ is $|H^1(T_{f,1}; \mathbb{Z}_2)| = 2^{2g+1}$, where the extra factor of 2 corresponds to the extra generator $\gamma \in \pi_1(T_{f,1})$. Remember that we construct $T_{f,1}$ from $T_{f}^j$ by performing a Dehn filling along $\gamma$, i.e. filling the boundary $\partial T_{f,1} = \partial \Sigma_{g,1} \times S^1$ with $\partial \Sigma_{g,1} \times D^2$. Then, as long as we choose the spin structure for $\gamma$ which extends over a disk, we can extend the spin structure on $T_{f,1}$ to a spin structure $\sigma$ on $T_{f,1}^j$. Again, the number of possible spin structures for $T_{f,1}^j$
is \( H^1(T^*_j; \mathbb{Z}_2) = \mathbb{Z}_2^{2g} \), and these exactly correspond to the spin structures on \( \Sigma_{g,1} \). Let \( \phi^*_{j,k} : T^*_j \rightarrow K(F/F_k, 1) \) be as before.

**Theorem 6.1.** Let \( \Sigma_{g,1} \) be a surface of genus \( g \) with one boundary component and a fixed spin structure. Let \( \sigma \) denote the corresponding spin structure on \( T^*_j \) for all \( f \in \mathcal{J}(k) \), \( k \geq 2 \). Then there is a finite family of well-defined homomorphisms

\[
\eta_{\sigma,k} : \mathcal{J}(k) \rightarrow \Omega_{3,\text{spin}}^3 \left( \frac{F}{F_k} \right)
\]

defined by \( \eta_{\sigma,k}(f) = (T^*_j, \phi^*_{j,k}, \sigma) \).

**Proof.** This follows directly from the proof that \( \sigma_k : \mathcal{J}(k) \rightarrow \Omega_3(F/F_k) \) is a well-defined homomorphism (see Theorem 5.2) since the spin structure on \( \Sigma_{g,1} \) naturally extends over the product \( \left( T^*_j \right) \times \mathbb{I} \) and the spin structure on \( \Sigma_{g,1} \) naturally extends over the product \( \Sigma_{g,1} \times (\varepsilon, \varepsilon) \times (\delta, \delta) \).

First, we point out that if we compose this homomorphism with a “forgetful” map which ignores the spin structure then we obtain our original homomorphism \( \gamma \). Second, recall the proof of Corollary 5.14 where we pointed out that, by using the Atiyah-Hirzebruch spectral sequence, one could build the \( n \)-dimensional bordism group \( \Omega_n(X, A) \) using \( H_p(X, A; \Omega_q) \) with \( p + q = n \) as building blocks. In the same way, the \( n \)-dimensional spin bordism group \( \Omega_{n,\text{spin}}^3(X, A) \) is constructed out of \( H_p(X, A; \Omega_q^{\text{spin}}) \) with \( p + q = n \), where \( \Omega_q^{\text{spin}} = \Omega_q^{\text{spin}}(\cdot) \) is the spin bordism group of a single point. Unlike the previous case, all but one of these coefficient groups are nontrivial for \( n = 3 \). In particular, since \( \Omega_0^{\text{spin}} \cong \mathbb{Z} \), \( \Omega_1^{\text{spin}} \cong \Omega_2^{\text{pin}} \cong \mathbb{Z}_2 \), and \( \Omega_3^{\text{spin}} \cong \{ e \} \), we have that \( \Omega_{3,\text{spin}}^3(F/F_k) \) is built out of

\[
\begin{align*}
H_3(F/F_k; \Omega_0^{\text{spin}}) &\cong H_3(F/F_k) \cong \Omega_3(F/F_k), \\
H_2(F/F_k; \Omega_1^{\text{spin}}) &\cong H_2(F/F_k) \otimes \mathbb{Z}_2 \cong F_k/F_{k+1} \otimes \mathbb{Z}_2, \\
H_1(F/F_k; \Omega_2^{\text{spin}}) &\cong H_1(F/F_k) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2^{2g}, \text{ and} \\
H_0(F/F_k; \Omega_3^{\text{spin}}) &\cong 0.
\end{align*}
\]

And so at the very least we see that there is potential for \( \eta_{\sigma,k} \) to give much more information about the structure of the group \( \mathcal{J}(k) \).

### 6.2. A Closer Look at \( \eta_{\sigma,2} \)

We will now investigate the specific case when \( k = 2 \) and see what information \( \eta_{\sigma,2} : \mathcal{J}(2) \rightarrow \Omega_{3,\text{spin}}^3(F/F_2) \) gives us about the Torelli group \( \mathcal{J}(2) \). We have already seen that the original Johnson homomorphism \( \tau_2 \) factors through \( \Omega_{3,\text{spin}}^3(F/F_2) \) (see Theorem 6.4). In this section we will see that, in fact, the Birman-Craggs homomorphisms \( \{ \rho_q \} \) also factor through \( \Omega_{3,\text{spin}}^3(F/F_2) \). Therefore, this new homomorphism \( \eta_{\sigma,2} \) combines the Johnson homomorphism and Birman-Craggs homomorphism into a single one.

Consider any mapping class \( f \in \mathcal{J}(2) \) and fix a spin structure on \( \Sigma_{g,1} \). Let \( \sigma \) be the corresponding spin structure on \( T^*_j \). Finally let \( \phi^*_{j} = \phi^*_{j,2} : T^*_j \rightarrow K(F/F_2, 1) \) be a continuous map which induces the canonical epimorphism \( \pi_1(T^*_j) \rightarrow F/F_2 \cong \mathbb{Z}_2^{2g} \).

Then the image under \( \eta_{\sigma,2} \) of \( f \) is \( (T^*_j, \phi^*_{j}, \sigma) \).

The group \( [T^*_j, S^1] \) of homotopy classes of maps \( T^*_j \rightarrow S^1 \) is in one-to-one correspondence with \( \text{Hom}(\pi_1(T^*_j), \mathbb{Z}) \). In fact, there is an isomorphism \( [T^*_j, S^1] \cong \)
Let \( \alpha \in H^1(T^*_f;\mathbb{Z}) \) be a primitive cohomology class, then there is a continuous map \( \psi_\alpha : T^*_f \rightarrow S^1 \) corresponding to \( \alpha \). There is a connected surface \( S \) embedded in \( T^*_f \) which represents a class in \( H_2(T^*_f) \) Poincaré dual to \( \alpha \), and this surface \( S \) represents the same homology class in \( H_2(T^*_f) \) as \( \psi_\alpha^{-1}(p) \) does, where \( p \in S^1 \) is a regular value of \( \psi_\alpha \). (If \( p \in S^1 \) is a regular value of \( \psi_\alpha \), then \( \psi_\alpha^{-1}(p) \) is an embedded, codimension 1 submanifold of \( T^*_f \). That is, \( \psi_\alpha^{-1}(p) \) is an embedded surface in \( T^*_f \).)

\[
\text{Figure 6.1. Embedding of } S \text{ into } T^*_{f,1} \hookrightarrow T^*_f \text{ and the map } \psi_\alpha.
\]

Let \( \pi_\alpha : K(F/F_2,1) \rightarrow S^1 \) be a continuous map such that \( \psi_\alpha \) is homotopic to \( \pi_\alpha \circ \phi^*_f \), and let \( (\pi_\alpha)_* : \Omega^{}_{3}^{\text{spin}}(F/F_2) \rightarrow \Omega^{}_{3}^{\text{spin}}(S^1) \) denote the induced bordism homomorphism. Then we can define a homomorphism

\[
\omega_{\sigma,\alpha} = (\pi_\alpha)_* \circ \eta_{3,2} : J(2) \rightarrow \Omega^{}_{3}^{\text{spin}}(S^1)
\]

by sending \( f \in J(2) \) to the bordism class \( \left(T^*_f, \psi_\alpha, \sigma \right) \in \Omega^{}_{3}^{\text{spin}}(S^1) \). Again, using the Atiyah-Hirzebruch spectral sequence, we see that \( \Omega^{}_{3}^{\text{spin}}(S^1) \cong \Omega^{}_{2}^{\text{spin}} \cong \mathbb{Z}_2 \). The specific isomorphism is given by \( (M, \phi, \sigma) \rightarrow (\phi^{-1}(p), \sigma|_{\phi^{-1}(p)}) \), where \( p \in S^1 \) is a regular value of \( \phi \). We can see by this isomorphism that the spin structure on \( T^*_f \) restricts to a spin structure on \( S = \psi_\alpha^{-1}(p) \).

**Theorem 6.2.** The fixed spin structure \( \sigma \) on \( \Sigma_{g,1} \) has a canonically associated quadratic form \( q : H_1(\Sigma_{g,1};\mathbb{Z}) \rightarrow \mathbb{Z}_2 \). If \( \text{Arf}(\Sigma_{g,1}, q) = 0 \), there is a primitive cohomology class \( \alpha \in H^1(T^*_f;\mathbb{Z}) \) such that the homomorphism \( \omega_{\sigma,\alpha} : J(2) \rightarrow \Omega^{}_{3}^{\text{spin}}(S^1) \) is equivalent to the Birman-Craggs homomorphism \( \rho_q : J(2) \rightarrow \mathbb{Z}_2 \).

We note that the hypothesis \( \text{Arf}(\Sigma_{g,1}, q) = 0 \) is necessary for the Birman-Craggs homomorphism \( \rho_q : J(2) \rightarrow \mathbb{Z}_2 \) to be defined. See Section 3 for details.

We have a surface \( S = \psi_\alpha^{-1}(p) \) embedded in \( T^*_f \). To determine whether the image of \( f \) under the homomorphism \( \omega_{\sigma,\alpha} : J(2) \rightarrow \Omega^{}_{3}^{\text{spin}}(S^1) \) is trivial or not, we simply need to determine \( (S, \sigma|_S) \in \Omega^{}_{2}^{\text{spin}} \). However, this is just the well-known Arf invariant of \( S \) with respect to \( \sigma|_S \). We defined the Arf invariant \( \text{Arf}(\Sigma, q) \) for
a closed surface $\Sigma$ and $\mathbb{Z}_2$-quadratic form $q : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$ in Section 8. For the spin structure $\sigma|_S$ on $S$ let $q_\sigma : H_1(S; \mathbb{Z}_2) \to \mathbb{Z}_2$ be the corresponding $\mathbb{Z}_2$-quadratic form. Namely, $q_\sigma$ is defined to be the quadratic form given by $q_\sigma(x) = 0$ if $\sigma|_x$ is the spin structure that extends over a disk and $q_\sigma(x) = 1$ if $\sigma|_x$ does not extend over a disk. It is the work of Johnson in [J1] that tells us this quadratic form is equivalent to the quadratic form discussed in Section 8. Then we have

$$\text{Arf}(S, q_\sigma) = \text{Arf}(S, \sigma|_S) = (S, \sigma|_S) \in \Omega_2^{\text{spin}}.$$ 

We will also need a more general definition of the Arf invariant which includes surfaces with boundary. The definition is the same except for a small change to the $\mathbb{Z}_2$-quadratic form $q$. In particular we have a $\mathbb{Z}_2$-quadratic form

$$q : \frac{H_1(\Sigma; \mathbb{Z}_2)}{i_* (H_1(\partial \Sigma; \mathbb{Z}_2))} \to \mathbb{Z}_2$$

where $i_*$ is induced by inclusion $i : \partial \Sigma \to \Sigma$. Then for a symplectic basis $\{x_i, y_i\}$ of the quotient $H_1(\Sigma; \mathbb{Z}_2)/i_* (H_1(\partial \Sigma; \mathbb{Z}_2))$, the Arf invariant of $\Sigma$ with respect to $q$ is defined to be

$$\text{Arf}(\Sigma, q) = \sum_{i=1}^{g} q(x_i)q(y_i) \pmod{2}.$$ 

Notice that if the surface $\Sigma$ happens to be embedded in $S^3$ then this definition is the same as the definition of the Arf invariant $\text{Arf}(L)$ of an oriented link $L$ in $S^3$ with components $\{L_i\}$ and satisfying the property that the linking number is $\text{lk}(L_i, L - L_i) \equiv 0 \pmod{2}$. The surface $\Sigma$ would be a Seifert surface for the link, and $q$ would be the mod 2 Seifert self-linking form on $H_1(\Sigma; \mathbb{Z}_2)/i_* (H_1(\partial \Sigma; \mathbb{Z}_2))$, where the self-linking is computed with respect to a push-off in a direction normal to the surface. See the W. Lickorish text [Li] for more details.

Now consider the surface $S = \psi_\alpha^{-1}(p)$ embedded in $T_f^\gamma$, and suppose that $S$ has genus $k$. There exists a symplectic basis $\{x_i, y_i\}$, $1 \leq i \leq k$, of $H_1(S; \mathbb{Z}_2)$ such that $x_k$ is homologous to the homology class $[\gamma]$ corresponding to $\gamma$ in $T_f^\gamma$ and $y_k$ is homologous to the homology class of $\beta = S \cap \Sigma_{g,1} \subset T_f^\gamma$. But $\gamma$ was required to have the spin structure that extends over a disk (so the spin structure on $T_f,1$ may be extended to a spin structure on $T_f^\gamma$). Thus $q_\sigma(x_k) = q_\sigma([\gamma]) = 0$, and

$$\text{Arf}(S, q_\sigma) = \sum_{i=1}^{k} q_\sigma(x_i)q_\sigma(y_i) = \sum_{i=1}^{k-1} q_\sigma(x_i)q_\sigma(y_i).$$

If we cut $S$ open along a simple closed curve parallel to $\beta = S \cap \Sigma_{g,1}$ then the result deformation retracts to a surface $S'$ with boundary $\partial S' = \beta \coprod f(\beta)$ and such that $H_1(S'; \mathbb{Z}_2)$ has symplectic basis $\{x_i, y_i\}$, $1 \leq i \leq k - 1$. (See Figure 6.2.) If we let $q'_\sigma : H_1(S'; \mathbb{Z}_2)/i_* (H_1(\partial S'; \mathbb{Z}_2)) \to \mathbb{Z}_2$ be the induced $\mathbb{Z}_2$-quadratic form, then

$$\text{Arf}(S, q_\sigma) = \sum_{i=1}^{k-1} q_\sigma(x_i)q_\sigma(y_i) = \sum_{i=1}^{k-1} q'_\sigma(x_i)q'_\sigma(y_i) = \text{Arf}(S', q'_\sigma).$$

According to Johnson in [J1], the quadratic form $q$ (in the statement of Theorem 6.2) corresponds to a Heegaard embedding of $\Sigma_{g,1}$ into $S^3$. Thus we get an induced embedding of $\Sigma_{g,1} \times [0, 1]$, and thus of $S'$, into $S^3$, and the quadratic form $q'_\sigma$ is precisely the same as the mod 2 Seifert self-linking form. Thus we see that to
calculate \( \text{Arf}(S, q) \), we really only need to calculate the Arf invariant of the link \( \{ \beta, f(\beta) \} \) with Seifert surface \( S' \).

![Image of a cut-open surface](image)

**Figure 6.2.** \( S \) cut open along \( \beta \) to obtain \( S' \)

Since there is an isomorphism \( H^1(T_j^g; \mathbb{Z}) \cong H^1(\Sigma_{g,1}; \mathbb{Z}) \), \( \alpha \in H^1(T_j^g; \mathbb{Z}) \) has a corresponding class in \( H^1(\Sigma_{g,1}; \mathbb{Z}) \) which we will also call \( \alpha \). The homology class of \( \beta = S \cap \Sigma_{g,1} \) in \( H_1(T_j^g) \) also has a corresponding class \( [\beta] \) in \( H_1(\Sigma_{g,1}) \). Since the homology class of \( S \) is Poincaré dual to \( \alpha \in H^1(T_j^g; \mathbb{Z}) \), \([\beta]\) \in H_1(\Sigma_{g,1}) \) must be Poincaré dual to \( \alpha \in H^1(\Sigma_{g,1}; \mathbb{Z}) \).

**Proof of Theorem 6.2.** We have a fixed spin structure \( \sigma \) on \( \Sigma_{g,1} \). Let \( q \) be the associated \( \mathbb{Z}_2 \)-quadratic form. Recall from Section 3 that the hypothesis \( \text{Arf}(\Sigma_{g,1}, q) = 0 \) was necessary for the Birman-Craggs homomorphism \( \rho_q : J(2) \to \mathbb{Z}_2 \) to be defined. We have already seen that the spin structure on \( \Sigma_{g,1} \) induces a spin structure on \( T_j^g \) which we will also denote by \( \sigma \) and which in turn induces a spin structure \( \sigma|_S \) on the surface \( S \) defined above. To prove the theorem, we need to find a primitive cohomology class \( \alpha \in H^1(T_j^g; \mathbb{Z}) \) such that \( \omega_{\sigma, \alpha}(f) = \rho_q(f) \). To accomplish this we need to find a surface \( S \) that represents a homology class Poincaré dual to \( \alpha \) and such that \( \text{Arf}(S, q) = \text{Arf}(S', q') = \rho_q(f) \). To do so, we will construct a simple closed curve \( \beta \) on \( \Sigma_{g,1} \) and calculate the Arf invariant \( \text{Arf}(\beta, f(\beta)) \) with Seifert surface \( S' \) in \( \Sigma_{g,1} \times [0, 1] \to S^3 \).

Recall that for genus \( g = 2 \) surfaces, the Torelli group \( J(2) \) is generated by the collection of all Dehn twists about bounding simple closed curves, and for genus \( g \geq 3 \), \( J(2) \) is generated by the collection of all Dehn twists about genus 1 cobounding pairs of simple closed curves, i.e. pairs of non-bounding, disjoint, homologous simple closed curves that together bound a genus 1 subsurface. Thus it is sufficient to prove the claim for such elements of \( J(2) \).

First assume that \( g = 2 \) and \( C \) is a genus 1 bounding simple closed curve on \( \Sigma_{2,1} \). Let \( f \) be a Dehn twist about \( C \). Then \( C \) splits \( \Sigma_{2,1} \) into two genus 1 surfaces \( \Sigma_a \) and \( \Sigma_b \). Let \( \{x_a, y_a\} \) and \( \{x_b, y_b\} \) be symplectic bases of \( H_1(\Sigma_a)/i_*(H_1(\partial \Sigma_a)) \) and \( H_1(\Sigma_b)/i_*(H_1(\partial \Sigma_b)) \), respectively. Then we have two cases:

(i) \( \rho_q(f) = \text{Arf}(\Sigma_a, q|_{\Sigma_a}) = \text{Arf}(\Sigma_b, q|_{\Sigma_b}) = 1 \)
\( \iff \) \( q(x_a) = q(y_a) = q(x_b) = q(y_b) = 1 \), or

(ii) \( \rho_q(f) = \text{Arf}(\Sigma_a, q|_{\Sigma_a}) = \text{Arf}(\Sigma_b, q|_{\Sigma_b}) = 0 \)
\( \iff \) at least one of \( \{q(x_a), q(y_a)\} \) and one of \( \{q(x_b), q(y_b)\} \) are 0.

Without loss of generality, let us assume in case (ii) that \( q(x_a) = q(x_b) = 0 \). Then in either case we have \( \rho_q(f) = q(x_a) \). Let \( \beta \) be a simple closed curve on \( \Sigma_{2,1} \to T_j^g \)
which intersects $C$ exactly twice and such that $[\beta] \in H_1(\Sigma_{g,1})$ is homologous to $x_a + x_b$. Then we also have the simple closed curve $f(\beta)$ on $f(\Sigma_{g,1}) \to T_f^g$. Near $C$ the picture will always be as in Figure 6.3 and we choose $S'$ to be this particular surface pictured in Figure 6.3 with boundary $\partial S' = \beta II - f(\beta)$. This surface $S'$ has

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig63}
\caption{Surface $S'$ in $T_f^g$ with boundary $\beta II - f(\beta)$ (for $g = 2$)}
\end{figure}

spin structure $\sigma|_{S'}$ and a corresponding quadratic form
\[ q'_\sigma : H_1(S';\mathbb{Z}_2)/i_*(H_1(\partial S';\mathbb{Z}_2)) \to \mathbb{Z}_2 \]
given by the mod 2 self-linking form. Notice that $\{x_a, [C]\}$ is a symplectic basis for the quotient $H_1(S';\mathbb{Z}_2)/i_*(H_1(\partial S';\mathbb{Z}_2))$. Then we have
\[ \omega_{\sigma,\alpha}(f) = \text{Arf}(S', q'_\sigma) \overset{def.}{=} \text{Arf}(\beta, f(\beta)) = q'_\sigma(x_a)q'_\sigma([C]). \]

Note that, while $C$ is a product of commutators on $\Sigma_{g,1}$, it is not a product of commutators on $S'$. But it is easy to see from Figure 6.3 that $q'_\sigma([C]) = \text{lk}(C, C^+) \equiv 1$ modulo 2. It is also clear that $q'_\sigma(x_a) = q(x_a)$. Thus
\[ \omega_{\sigma,\alpha}(f) = q'_\sigma(x_a)q'_\sigma([C]) = q(x_a) = \text{Arf}(\Sigma_a, q|_{\Sigma_a}) = \rho_q(f). \]

Now assume that $g \geq 3$ and $C_1$ and $C_2$ are genus 1 cobounding pairs of simple closed curves on $\Sigma_{g,1}$. Let $f$ be a composition of Dehn twists about $C_1$ and $C_2$. Then $C_1$ and $C_2$ cobound a genus 1 subsurface $\Sigma'$. Let $\{x, y\}$ be a symplectic basis of $H_1(\Sigma')/i_*(H_1(\partial \Sigma'))$. There are two cases:

1. $q(C_1) = q(C_2) = 1$ and
2. $q(C_1) = q(C_2) = 0$.

For case (1), we simply let $\beta$ be a simple closed curve on $\Sigma_{g,1} \to T_f^g$ which does not intersect $C_1$ or $C_2$. Then $f$ will not affect $\beta$, and we can choose $S'$ to be a straight cylinder between $\beta$ and $f(\beta)$ so that $H_1(S';\mathbb{Z}_2)/i_*(H_1(\partial S';\mathbb{Z}_2))$ is trivial. Thus $\omega_{\sigma,\alpha}(f) = \text{Arf}(S', q'_\sigma) = 0$. We also know from the end of Section 3 that in this case $\rho_q(f) = 0$.

For case (2), we have two subcases:

(i) $\rho_q(f) = \text{Arf}(\Sigma', q|_{\Sigma'}) = 1$ \iff $q(x) = q(y) = 1$, or
(ii) $\rho_q(f) = \text{Arf}(\Sigma', q|_{\Sigma'}) = 0$ \iff at least one of $\{q(x), q(y)\}$ is 0.
Again without loss of generality, let us assume in case (ii) that \( q(x) = 0 \). In both cases let \( \beta \) be a simple closed curve on \( \Sigma_{g,1} \to T^2_f \) which intersects each of \( C_1 \) and \( C_2 \) exactly once and such that \( [\beta] \in H_1(\Sigma_{g,1}) \) is homologous to \( x + x' \), where \( x' \) is any nontrivial homology class in \( H_1(\Sigma_{g,1} - \Sigma') \). Then we also have the simple closed curve \( f(\beta) \) on \( f(\Sigma_{g,1}) \to T^2_f \). Near \( C_1 \) and \( C_2 \) the picture will always be as in Figure 6.4 and we choose \( S' \) to be this particular surface pictured in Figure 6.4 with boundary \( \partial S' = \beta \Pi - f(\beta) \). Again this surface \( S' \) has spin structure \( \sigma|S' \)

and a corresponding quadratic form \( q'_\sigma : H_1(S'\setminus f(\beta)/\partial S';\Z) \to \Z_2 \) by the mod 2 self-linking form. Let \( y' \) be any homology class such that \( \{x, y'\} \) is a symplectic basis for \( H_1(S'\setminus f(\beta)/\partial S';\Z) \). Then we have

\[
\omega_{\sigma,\alpha}(f) = \text{Arf}(S', q'_\sigma) \overset{\text{def.}}{=} \text{Arf}(\beta, f(\beta)) = q'_\sigma(x)q'_\sigma(y').
\]

Notice that \( \{x, y'\} \) is also a basis for \( H_1(\Sigma';\Z)/\imath_*(H_1(\partial\Sigma';\Z)) \) and that \( q'_\alpha(x) = q(x) \) and \( q'_\alpha(y') = q(y') \). Thus we see that

\[
\omega_{\sigma,\alpha}(f) = q'_\alpha(x)q'_\alpha(y') = q(x)q(y') = \text{Arf}(\Sigma', q|\Sigma') = \rho_\eta(f).
\]

This completes the proof of Theorem 6.2. \( \square \)

As a result of this theorem and Theorem 5.14 we see that \( \eta_{\sigma,2} \) contains the necessary information for determining both the Johnson homomorphism \( \tau_2 \) and the Birman-Craggs homomorphism \( \rho_\eta \). Recall from Section 4 that the abelianization \( H_1(\mathcal{J}(2);\Z) \cong \mathcal{J}(2)/[\mathcal{J}(2), \mathcal{J}(2)] \) of the Torelli group is completely determined by \( \tau_2 \) and \( \rho_\eta \) (over all possible \( q \)) since the commutator subgroup of the Torelli group is given by the kernels of these homomorphisms. Namely, we have \([\mathcal{J}(2), \mathcal{J}(2)] = \mathcal{C} \cap \mathcal{J}(3)\), where \( \mathcal{C} = \bigcap_q \ker \rho_q \). Suppose we take a mapping class \( f \in \ker \eta_{\sigma,2} \). Certainly it is true that \( f \in \ker \rho_\eta \cap \mathcal{J}(3) \) since \( \tau_2 \) and \( \rho_\eta \) factor through \( \eta_{\sigma,2} \). Moreover, if

\[
\mathcal{D} = \bigcap_\sigma \ker \eta_{\sigma,2}
\]

is the common kernel over all possible spin structures, then \( \mathcal{D} \subset \mathcal{C} \cap \mathcal{J}(3) \). Of course it would be nice to know if the converse is also true.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure64.png}
\caption{Surface \( S' \) in \( T^2_f \) with boundary \( \beta \Pi - f(\beta) \) (for \( g \geq 3 \))}
\end{figure}
Problem 6.3. What is $\ker \eta_{\sigma,2}$? Is $\ker \eta_{\sigma,2} = \ker \rho_2 \cap J(3)$?

Problem 6.4. Is it true that $D = C \cap J(3) = [J(2), J(2)]$?

6.3. Analysis of $\eta_{\sigma,k}$. In this section we shift our focus to the general homomorphism $\eta_{\sigma,k} : J(k) \to \Omega_3^{spin}(F/F_k)$ for arbitrary values of $k$. We already know that $\ker \eta_{\sigma,k} \subset J(2k - 1) = \ker \sigma_k$ since the oriented bordism homomorphism $\sigma_k : J(k) \to \Omega_3(F/F_k)$ factors through $\Omega_3^{spin}(F/F_k)$. However, the additional structure on the bordism given by the spin structures should refine the kernel of $\eta_{\sigma,k}$.

Problem 6.5. What is the kernel of $\eta_{\sigma,k}$?

Problem 6.6. Does $\eta_{\sigma,k}$ give a faithful representation of the abelianization of $J(k)$? In other words, is $\text{Im} \eta_{\sigma,k} \equiv J(k)/[J(k), J(k)]$?

A sufficient condition for $f \in \ker \eta_{\sigma,k}$ is given in the following theorem, but it is most likely not necessary. Consider the entire collection $\{\omega_{\sigma,\alpha}\}$ of the homomorphisms $\omega_{\sigma,\alpha} : J(2) \to \Omega_3^{spin}(S^1)$ defined in Section 6.2 and let

$$B = \bigcap_{\alpha} \ker \omega_{\sigma,\alpha}$$

be the common kernel of all $\omega_{\sigma,\alpha}$ for all $\alpha \in H^1(T_1'; \mathbb{Z})$.

Theorem 6.7. If $f \in B \cap J(2k + 1)$, then $f \in \ker \eta_{\sigma,k}$.

Note that the hypothesis requires $f \in J(2k + 1)$, not just $f \in J(2k - 1)$. The purpose of this will be revealed in the proof of the theorem, but it is probably not necessary. However, as stated above, it is certainly necessary that $f \in J(2k - 1)$.

Before we give the proof of this theorem, let us first set up some necessary notation. For a more complete discussion, we refer the reader to Whitehead’s book [WH]. We will be using the Atiyah-Hirzebruch spectral sequence. In particular, let

$$(*) \quad J_{p,q}^m = \text{Image} \left( (i_{p,q})_* : \tilde{\Omega}_{p,q}^{spin} \left( \frac{F}{F_m} \right)^{(p)} \to \tilde{\Omega}_{p,q}^{spin} \left( \frac{F}{F_m} \right) \right).$$

Here $(F/F_m)^{(p)}$ denotes the $p$-skeleton of $K(F/F_m, 1)$, $(i_{p,q})_*$ is induced by the inclusion map $(F/F_m)^{(p)} \hookrightarrow K(F/F_m, 1)$, and $\Omega_n^{spin}(F/F_m)$ denotes the reduced spin bordism group defined by

$$\Omega_n^{spin} \left( \frac{F}{F_m} \right) \cong \Omega_n^{spin} \oplus \tilde{\Omega}_n^{spin} \left( \frac{F}{F_m} \right).$$

Note that if $(M, \phi, \sigma) \in J_{p,q}^m$ then for $l \leq m$ the triple $(M, \pi_{m,l} \circ \phi, \sigma)$ is in $J_{p,q}^l$, where $\pi_{m,l} : K(F/F_m, 1) \to K(F/F_l, 1)$ is the projection map. Let

$$E_2^{p,q} \cong H_p(F/F_m; \Omega_q^{spin}),$$

and the boundary operator is

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p-2,q+1}.$$

The groups $E_2^{p,q}$ may be thought of as the building blocks for $\tilde{\Omega}_n^{spin}(F/F_m)$ with $p + q = n$. In actuality, the building blocks are the groups $E_\infty^{p,q} = \lim E_r^{p,q}$, where
for $r \geq 3$

$$E_{p,q}^r = \frac{\ker d_{p,q}^{r-1}}{\text{Im} d_{p+r-1,q-r+2}} \text{ and } d_{p,q}^{r-1} : E_{p,q}^{r-1} \to E_{p-r+1,q+r-2}^r.$$ 

We also have an isomorphism

$$E_{p,q}^{\infty} \cong J_{p,q}^m / J_{p-1,q+1}^m.$$ 

Since $\Omega_3^{\text{spin}} = 0$, we then have $\Omega_3^{\text{spin}}(F/F_m) \cong \Omega_3^{\text{spin}}(F/F_m)$ and

$$\Omega_3^{\text{spin}}(F/F_m) = J_{3,0}^m \supset J_{2,1}^m \supset J_{1,2}^m \supset J_{0,3}^m = 0.$$ 

Then one can show that the relevant $E_{p,q}^\infty$ are as follows.

- $E_{3,0}^3 = E_{3,0}^3 = \ker d_{3,0}^2 \subset H_3(F/F_m) \cong H_3(F/F_m)$
- $E_{2,1}^3 = E_{2,1}^3 = \text{coker } d_{2,0}^2 \cong H_2(F/F_m; \Omega_1^{\text{spin}}) / \text{Im } d_{2,0}^2$
- $E_{1,2}^3 = E_{1,2}^3 = \text{coker } d_{1,0}^2 \cong H_1(F/F_m; \Omega_2^{\text{spin}}) / \text{Im } d_{1,1}^2$
- $E_{0,3}^{\infty} = 0$

We can now begin our proof of Theorem 6.7.

**Proof of Theorem 6.7.** We assume that $f \in B \cap J(2k + 1)$, and we want to show that $(T_f^g, \phi_{f,k}, \sigma) = 0$ in $\Omega_3^{\text{spin}}(F/F_k)$. Since $\Omega_3^{\text{spin}}(F/F_m) = J_{3,0}^m$, we may perturb any $(M, \phi, \sigma) \in \Omega_3^{\text{spin}}(F/F_m)$ to ensure that $\phi(M)$ is contained in the 3-skeleton $(F/F_m)_3$ of $K(F/F_m, 1)$. That is, by the definition of $J_{3,0}^m$ given in (4) we can choose $\phi'$ homotopic to $\phi$ so that $(M, \phi', \sigma) \in \Omega_3^{\text{spin}}((F/F_m)_3)$ and $J_{3,0}^m(M, \phi', \sigma) = (M, \phi, \sigma)$ in $\Omega_3^{\text{spin}}(F/F_m)$.

Since $f \in J(2k + 1) \subset J(k + 1)$, Lemma 6.8 says $\phi_{f,k+1}^g$ exists. We start with $(T_f^g, \phi_{f,k+1}^g, \sigma) \in J_{3,0}^{k+1}$. Then Theorem 5.8 says that the pair $(T_f^g, \phi_{f,k+1}^g) = 0$ in $\Omega_3(F/F_{k+1})$. Thus $(\phi_{f,k+1}^g, (\phi_{f,k+1}^g)) = 0$ in $H_3(F/F_{k+1}) \cong E_{3,0}^{k+1}$, and we therefore know from (5) that $(T_f^g, \phi_{f,k+1}^g, \sigma)$ must be in $J_{2,1}^{k+1}$. Thus by (5) there exists a triple $(M', \phi', \sigma') \in \Omega_3^{\text{spin}}((F/F_{k+1})_3)$ such that $(i_{2,1})_*(M', \phi', \sigma') = (T_f^g, \phi_{f,k+1}^g, \sigma)$ in $\Omega_3^{\text{spin}}(F/F_{k+1})$ as indicated in the following diagram.

$$\Omega_3^{\text{spin}}((F/F_{k+1})_3) \quad (M', \phi', \sigma')$$

$$\Omega_3^{\text{spin}}(F/F_{2k+1}) \quad \Omega_3^{\text{spin}}(F/F_{k+1})$$

$$\xrightarrow{(T_f^g, \phi_{f,2k+1}^g, \sigma)} \xrightarrow{(T_f^g, \phi_{f,k+1}^g, \sigma)}$$

**Lemma 6.8.** The homomorphism $(\pi_{k+1})_*: J_{2,1}^{k+1} / J_{1,2}^{k+1} \rightarrow J_{2,1}^k / J_{1,2}^k$ is trivial.
Proof. By \( \pi \) we have \( J_{k+1}^j / J_{k+1}^j \cong E_{2,1}^\infty \cong H_2(F/F_{k+1};\Omega_1^{\text{spin}}) / \text{Im} \, d_{2,0}^2 \). Similarly, we have \( J_{k+1}^j / J_{k+1}^j \cong H_2(F/F_k;\Omega_1^{\text{spin}}) / \text{Im} \, d_{2,0}^2 \). So this homomorphism is equivalent to
\[
\frac{H_2(F/F_{k+1};\Omega_1^{\text{spin}})}{\text{Im} \, d_{2,0}^2} \rightarrow \frac{H_2(F/F_k;\Omega_1^{\text{spin}})}{\text{Im} \, d_{2,0}^2}.
\]
In the proof of Corollary 5.12 we showed that \( H_2(F/F_{k+1}) \rightarrow H_2(F/F_k) \) is the zero map. Thus \( H_2(F/F_{k+1};\Omega_1^{\text{spin}}) \rightarrow H_2(F/F_k;\Omega_1^{\text{spin}}) \) is also trivial, and the conclusion follows. \( \square \)

Consider the image of \( (T_f^j, \phi_f, \sigma) \) in \( \Omega_3^{\text{spin}}(F/F_k) \) under the homomorphism \( (\pi_{k+1,k})_* : \Omega_3^{\text{spin}}(F/F_{k+1}) \rightarrow \Omega_3^{\text{spin}}(F/F_k) \). This image is of course \( (T_f^j, \phi_f, \sigma) \), and since \( (T_f^j, \phi_f, \sigma) \) is a triple \( (M'', \phi'', \sigma'') \), Lemma 6.8 tells us \( (T_f^j, \phi_f, \sigma) \in J_{k+1}^j \). By \( \pi \) there is a triple \( (M'', \phi'', \sigma'') \) in \( \Omega_3^{\text{spin}}(F/F_k) \) such that \( (i_{1,2})_* (M'', \phi'', \sigma'') = (T_f^j, \phi_f, \sigma) \) in \( \Omega_3^{\text{spin}}(F/F_k) \) as indicated in the following diagram.

\[
\begin{array}{ccc}
\Omega_3^{\text{spin}}((F/F_{k+1})^{(2)}) & \rightarrow & \Omega_3^{\text{spin}}(F/F_{k+1}) \\
(M', \phi', \sigma') & \downarrow & \Omega_3^{\text{spin}}(F/F_k) \\
\Omega_3^{\text{spin}}(F/F_{k+1}) & \rightarrow & \Omega_3^{\text{spin}}(F/F_k) \\
(T_f^j, \phi_f, \sigma) & \downarrow & (T_f^j, \phi_f, \sigma) \\
(T_f^j, \phi_f, \sigma) & \rightarrow & (T_f^j, \phi_f, \sigma) \\
\end{array}
\]

Now we use the fact that \( f \in B \). Recall \( \omega_{\sigma, \alpha} = (\pi_{\alpha})_* \circ \eta_{\sigma, 2} : \mathcal{J}(2) \rightarrow \Omega_3^{\text{spin}}(S^1) \) and \( (\pi_{\alpha})_* : \Omega_3^{\text{spin}}(F/F_k) \rightarrow \Omega_3^{\text{spin}}(S^1) \). Since the 1-skeleton \( (F/F_k)^{(1)} \) is homotopy equivalent to the wedge of 2g circles, we have
\[
\Omega_3^{\text{spin}}((F/F_k)^{(1)}) \cong \Omega_3^{\text{spin}}(S^1 \vee \cdots \vee S^1) \cong 2g \circ\Omega_3^{\text{spin}}(S^1)
\]
and the following commutative diagram.
\[
\begin{array}{ccc}
2g \circ\Omega_3^{\text{spin}}(S^1) & \rightarrow & \Omega_3^{\text{spin}}(S^1) \\
(\mathcal{J}(k) \eta_{\sigma, k})_* & \downarrow & (\pi_{\alpha})_* \\
\mathcal{J}(k) & \rightarrow & \Omega_3^{\text{spin}}(F/F_k) \rightarrow \Omega_3^{\text{spin}}(F/F_k)
\end{array}
\]
There is a basis of $H^1(T^g_f; \mathbb{Z})$ such that for each basis element $\alpha_i$, the range of the homomorphism $\omega_{\sigma, \alpha_i}$ corresponds to a summand of $\Omega^{spin}_3((F/F_k)^{(1)}) = \mathbb{Z}^{2g} \oplus \Omega^{spin}_3(S^1)$. Since $f \in B$, $(M'', \phi'', \sigma'') \in \Omega^{spin}_3((F/F_k)^{(1)})$ must be trivial in each summand of $\mathbb{Z}^{2g} \oplus \Omega^{spin}_3(S^1)$, and thus it must be trivial in $\Omega^{spin}_3((F/F_k)^{(1)})$. Therefore $0 = (i_{1,2})_{*} (M'', \phi'', \sigma'') = \left( T^g_f, \phi^g_f, \sigma \right)$ in $\Omega^{spin}_3(F/F_k)$.

□

References

[BC] J. Birman and R. Craggs, The $\mu$-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold, Trans. of the AMS 237 (1978) 283-309.

[CGO] T. Cochran, A. Gerges, and K. Orr, Dehn surgery equivalence relations on 3-manifolds, Math. Proc. Camb. Phil. Soc. 131 (2001) 97-127.

[Dw] W. Dwyer, Homology, Massey products, and maps between groups, J. Pure and Applied Algebra 6 (1975) 177-190.

[Fe] R. Fenn, Techniques of Geometric Topology, London Mathematical Society Lecture Notes Series 57, Cambridge University Press (1983).

[IO] K. Igusa and K. Orr, Links, pictures and the homology of nilpotent groups, Topology 40, Issue 6 (2001) 1125-1166.

[J1] D. Johnson, Quadratic forms and the Birman-Craggs homomorphisms, Trans. of the AMS 261, no. 1 (1980) 235-254.

[J2] An abelian quotient of the mapping class group $T_g$, Math. Annalen 249 (1980) 225-242.

[J3] A survey of the Torelli group, Contemporary Math. 20 (1983) 163-179.

[J4] The structure of the Torelli group I: A finite set of generators for $T_g$, Annals of Math. 118 (1983) 423-442.

[J5] The structure of the Torelli group II: A characterization of the group generated by twists on bounding curves, Topology 24, Issue 2 (1985) 113-126.

[J6] The structure of the Torelli group III: The abelianization of $T_g$, Topology 24, Issue 2 (1985) 127-144.

[Ki] T. Kitano, Johnson’s homomorphisms of subgroups of the mapping class group, the Magnus expansion and Massey higher products of mapping tori, Topology and its Appl. 69 (1996) 165-172.

[Kr] D. Kraines, Massey Higher Products, Trans. of the AMS 124 (1966) 431-449.

[Li] W.B.R. Lickorish, An Introduction to Knot Theory, Springer-Verlag (1997).

[Mo] S. Morita, Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. Journal 70 (1993) 699-726.

[Or] K. Orr, Link concordance invariants and Massey products, Topology 30, Issue 4 (1991) 699-710.

[St] J. Stallings, Homology and central series of groups, J. Algebra 2 (1965) 170-181.

[Su] D. Sullivan, On the intersection ring of compact 3-manifolds, Topology 14 (1975) 275-277.

[Tu] V. Turaev, Nilpotent homotopy types of closed 3-manifolds, Proceedings of Topology Conference in Leningrad, Lecture Notes in Mathematics 1060 (1982), Springer-Verlag, 355-366.

[Wh] G. W. Whitehead, Elements of Homotopy Theory, Springer-Verlag (1978).