THE TRILINEAR EMBEDDING THEOREM

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Abstract. Let $\sigma_i$, $i = 1, 2, 3$, denote positive Borel measures on $\mathbb{R}^n$, let $\mathcal{D}$ denote the usual collection of dyadic cubes in $\mathbb{R}^n$ and let $K : \mathcal{D} \to [0, \infty)$ be a map. In this paper we give a characterization of the trilinear embedding theorem. That is, we give a characterization of the inequality
\[
\sum_{Q \in \mathcal{D}} K(Q) \left| \prod_{i=1}^{3} f_i \right|_{d\sigma_i} \leq C \prod_{i=1}^{3} \|f_i\|_{L^{p_i}(d\sigma_i)},
\]
in terms of discrete Wolff’s potential and Sawyer’s checking condition, when $1 < p_1, p_2, p_3 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$.

1. Introduction

The purpose of this paper is to investigate the trilinear embedding theorem. We first fix some notations. We will denote by $\mathcal{D}$ the family of all dyadic cubes $Q = 2^{-k}(m + [0, 1)^n)$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^n$. Let $K : \mathcal{D} \to [0, \infty)$ be a map and let $\sigma_i$, $i = 1, 2, 3$, be positive Borel measures on $\mathbb{R}^n$. In this paper we give a necessary and sufficient condition for which the inequality
\[
(1.1) \quad \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{3} \left| \int_Q f_i \, d\sigma_i \right| \leq C \prod_{i=1}^{3} \|f_i\|_{L^{p_i}(d\sigma_i)},
\]
to hold when $1 < p_1, p_2, p_3 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$. By duality, (1.1) is equivalent to two-weight norm inequality for the bilinear positive operators
\[
\|T_K[f_1d\sigma_1, f_2d\sigma_2]\|_{L^{p'_3}(d\sigma_3)} \leq C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(d\sigma_i)},
\]
Here, for each $1 < p < \infty$, $p'$ denote the dual exponent of $p$, i.e., $p' = \frac{p}{p-1}$, and the bilinear positive operator $T_K[\sigma_1, \sigma_2]$ is given by
\[
T_K[f_1d\sigma_1, f_2d\sigma_2](x) := \sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{2} \left( \int_Q f_i \, d\sigma_i \right) 1_{Q}(x), \quad x \in \mathbb{R}^n,
\]
where $1_E$ stands for the characteristic function of the set $E$.

For the bilinear embedding theorem, in the case $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$, Sergei Treil gives a simple proof of the following.

Proposition 1.1 ([16, Theorem 2.1]). Let $K : \mathcal{D} \to [0, \infty)$ be a map and let $\sigma_i$, $i = 1, 2$, be positive Borel measures on $\mathbb{R}^n$. Let $1 < p_1, p_2 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} \geq 1$. The following statements are equivalent:

2010 Mathematics Subject Classification. 42B20, 42B35 (primary), 31C45, 46E35 (secondary).

Key words and phrases. discrete Wolff’s potential; bilinear positive dyadic operator; Sawyer’s checking condition; trilinear embedding theorem; two-weight trace inequality.

The author is supported by the FMS4 program at Graduate School of Mathematical Sciences, the University of Tokyo, and Grant-in-Aid for Scientific Research (C) (No. 23540187), the Japan Society for the Promotion of Science.

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(a) The following bilinear embedding theorem holds:

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{2} \left| \int_{Q} f_i \, d\sigma_i \right| \leq c_1 \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;$$

(b) For all $Q \in \mathcal{D}$,

$$\begin{align*}
\left( \int_{Q} \left( \sum_{Q' \subset Q} K(Q') \sigma_1(Q') 1_{Q'} \right)^{\frac{p_2}{\sigma_2}} \, d\sigma_2 \right)^{1/p_2} &\leq c_2 \sigma_1(Q)^{1/p_1} < \infty, \\
\left( \int_{Q} \left( \sum_{Q' \subset Q} K(Q') \sigma_2(Q') 1_{Q'} \right)^{\frac{p_1}{\sigma_1}} \, d\sigma_1 \right)^{1/p_1} &\leq c_2 \sigma_2(Q)^{1/p_2} < \infty.
\end{align*}$$

Moreover, the least possible $c_1$ and $c_2$ are equivalent.

Proposition 1.1 was first proved for $p_1 = p_2 = 2$ in [9] by the Bellman function method. Later in [10], this was proved in full generality. The checking condition in Proposition 1.1 is called “the Sawyer type checking condition”, since this was first introduced by Eric T. Sawyer in [10].

To describe the case $\frac{1}{p_1} + \frac{1}{p_2} < 1$, we need discrete Wolff’s potential.

Let $\mu$ and $\nu$ be positive Borel measures on $\mathbb{R}^n$ and let $K : \mathcal{D} \to [0, \infty)$ be a map. We will denote by $K_{\mu}(Q)(x)$ the function

$$K_{\mu}(Q)(x) := \frac{1}{\mu(Q)} \sum_{Q' \subset Q} K(Q') \mu(Q') 1_{Q'}(x), \quad x \in Q \in \mathcal{D},$$

and $K_{\mu}(Q)(x) = 0$ when $\mu(Q) = 0$. For $p > 1$, the discrete Wolff’s potential $W_{K,\mu}^{p}[\nu](x)$ of the measure $\nu$ is defined by

$$W_{K,\mu}^{p}[\nu](x) := \sum_{Q \in \mathcal{D}} K(Q) \mu(Q) \left( \int_{Q} K_{\mu}(Q)(y) \, d\nu(y) \right)^{p-1} 1_{Q}(x), \quad x \in \mathbb{R}^n.$$ 

The author proves the following, which describes the case $\frac{1}{p_1} + \frac{1}{p_2} < 1$.

**Proposition 1.2** ([13] Theorem 1.3). Let $K : \mathcal{D} \to [0, \infty)$ be a map and let $\sigma_i$, $i = 1, 2$, be positive Borel measures on $\mathbb{R}^n$. Let $1 < p_1, p_2 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} < 1$. The following statements are equivalent:

(a) The following bilinear embedding theorem holds:

$$\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{2} \left| \int_{Q} f_i \, d\sigma_i \right| \leq c_1 \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;$$

(b) For $\frac{1}{p_1} + \frac{1}{p_2} = 1$,

$$\begin{align*}
\|W_{K,\mu}^{p_2}[\sigma_1]^{\frac{1}{p_2}}\|_{L^r(d\sigma_1)} &\leq c_2 < \infty, \\
\|W_{K,\mu}^{p_1}[\sigma_2]^{\frac{1}{p_1}}\|_{L^r(d\sigma_2)} &\leq c_2 < \infty.
\end{align*}$$

Moreover, the least possible $c_1$ and $c_2$ are equivalent.

In his survey of the $A_2$ theorem [5], Tuomas P. Hytönen introduces another proof of Proposition 1.1 which uses the “parallel corona” decomposition from the recent work of Lacey, Sawyer, Shen and Uriarte-Tuero [7] on the two-weight boundedness of the Hilbert transform. In this paper, following Hytönen’s arguments in [5] and applying Propositions 1.1 and 1.2 we shall establish the following theorem (Theorem 1.3).
Let $\mathcal{I}$ be the set of all permutations of $(1, 2, 3)$, i.e.,
$$
\mathcal{I} := \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}.
$$
Let $\mu$ be a positive borel measure on $\mathbb{R}^n$ and let $K : \mathcal{D} \to [0, \infty)$ be a map. For $Q \in \mathcal{D}$, we will denote by $K(Q, \mu)$ the map
$$
K(Q, \mu)(Q') := \begin{cases}
K(Q')\mu(Q'), & q' \in \mathcal{D}, Q' \subset Q, \\
0, & \text{otherwise}.
\end{cases}
$$

**Theorem 1.3.** Let $K : \mathcal{D} \to [0, \infty)$ be a map and let $\sigma_i$, $i = 1, 2, 3$, be positive Borel measures on $\mathbb{R}^n$. Let $1 < p_1, p_2, p_3 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$. The following statements are equivalent:

(a) The following trilinear embedding theorem holds:
$$
\sum_{Q \in \mathcal{D}} K(Q) \prod_{i=1}^{3} \left| \int_{Q} f_i \, d\sigma_i \right| \leq c_1 \prod_{i=1}^{3} \|f_i\|_{L^{p_i}(d\sigma_i)} < \infty;
$$

(b) For any $(a, b, c) \in \mathcal{I}$, if $\frac{1}{p_a} + \frac{1}{p_b} \geq 1$, then we have, for all $Q \in \mathcal{D}$,
$$
\left\{ \left( \int_{Q} \left( \sum_{Q' \subset Q} K(Q, \sigma_i)(Q') \sigma_a(Q') \right)^{p'_a} \right)^{1/p'_a} \right\}^{1/p_a} \leq c_2 \sigma_a(Q)^{1/p_a} \sigma_c(Q)^{1/p_c} < \infty,
$$
if $\frac{1}{p_a} + \frac{1}{p_b} < 1$, then we have, for all $Q \in \mathcal{D}$ and for $\frac{1}{p_a} + \frac{1}{p_b} = 1$,
$$
\left\{ \left( \int_{Q} \left( \sum_{Q' \subset Q} K(Q, \sigma_i)(Q') \sigma_a(Q') \right)^{p'_a} \right)^{1/p'_a} \right\}^{1/p_a} \leq c_2 \sigma_a(Q)^{1/p_a} \sigma_c(Q)^{1/p_c} < \infty.
$$

Moreover, the least possible $c_1$ and $c_2$ are equivalent.

In [3], Kangwei Li and Wenchang Sun establish the corresponding results of Theorem 1.3 for the bilinear fractional integrals in the case
$$
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} \geq 1.
$$
They also treat the weak-type estimates. For the works using Wolff’s potential, we refer the readers to [1] [2] [3] [4] [12] [14] [15].

**Remark 1.4.** To describe the case $1 < p_1, p_2, p_3 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$, probably, we need Wolff’s potential of two-measures. But, we can not find it until now.

The letter $C$ will be used for constants that may change from one occurrence to another.

2. Proof of Theorem 1.3

In what follows we shall prove Theorem 1.3. Let us start by proving that (a) implies (b). But, this is a direct consequence of Propositions 1.1 and 1.2. So, we concentrate on proving that (b) implies (a). We follow the arguments due to T. Hytönen in [5] with some necessary modifications. We will use $\int_{Q} f \, d\mu$ to denote the integral average $\frac{1}{\mu(Q)} \int_{Q} f \, d\mu$.

Let $Q_0 \in \mathcal{D}$ be taken large enough and be fixed. We shall estimate the quantity
$$
(2.1) \quad \sum_{Q \subset Q_0} K(Q) \prod_{i=1}^{3} \left( \int_{Q} f_i \, d\sigma_i \right),
$$
where \( f_i \in L^p_i(\sigma_i) \) is nonnegative and is supported in \( Q_0 \).

We define the collection of principal cubes \( \mathcal{F}_i \) for the pair \((f_i, \sigma_i), i = 1, 2, 3\). Namely,

\[
\mathcal{F}_i := \bigcup_{k=0}^{\infty} \mathcal{F}^k_i,
\]

where \( \mathcal{F}^0_i := \{Q_0\} \),

\[
\mathcal{F}^{k+1}_i := \bigcup_{F \in \mathcal{F}^k_i} ch_{\mathcal{F}_i}(F)
\]

and \( ch_{\mathcal{F}_i}(F) \) is defined by the set of all “maximal” dyadic cubes \( Q \subset F \) such that

\[
\int_Q f_i \, d\sigma_i > 2 \int_F f_i \, d\sigma_i.
\]

Observe that

\[
\sum_{F' \in ch_{\mathcal{F}_i}(F)} \sigma_i(F') \leq \left(2 \int_F f_i \, d\sigma_i\right)^{-1} \int_{F'} f_i \, d\sigma_i
\]

\[
\leq \left(2 \int_F f_i \, d\sigma_i\right)^{-1} \int_F f_i \, d\sigma_i = \frac{\sigma_i(F)}{2},
\]

which implies

\[
(2.2) \quad \sigma_i(E_{\mathcal{F}_i}(F)) := \sigma_i\left(F \setminus \bigcup_{F' \in ch_{\mathcal{F}_i}(F)} F'\right) \geq \frac{\sigma_i(F)}{2},
\]

where the sets \( E_{\mathcal{F}_i}(F), F \in \mathcal{F}_i \), are pairwise disjoint.

We further define the stopping parents, for \( Q \in \mathcal{D} \),

\[
\begin{cases}
\pi_{\mathcal{F}_i}(Q) := \min\{F \supset Q : F \in \mathcal{F}_i\}, \\
\pi(Q) := (\pi_{\mathcal{F}_1}(Q), \pi_{\mathcal{F}_2}(Q), \pi_{\mathcal{F}_3}(Q)).
\end{cases}
\]

Then we can rewrite the series in \( (2.1) \) as follows:

\[
(2.3) \quad \sum_{Q \subset Q_0} = \sum_{(F_i) \in \mathcal{F}_i} \sum_{Q : \pi(Q) = (F_i)}.
\]

We notice the elementary fact that, if \( P, R \in \mathcal{D} \), then \( P \cap R \in \{P, R, \emptyset\} \). This fact implies, if \( \pi(Q) = (F_i) \), then

\[
Q \subset F_a \subset F_b \subset F_c \quad \text{for some} \quad (a, b, c) \in \mathcal{I}.
\]

Thus, by symmetry of the problem in \( (2.3) \), we shall concentrate ourselves on the estimate

\[
(i) := \sum_{H \in \mathcal{F}_3} \sum_{(F_i) \in \mathcal{F}_i, F \subset G \subset H} \sum_{Q : \pi(Q) = (F_i, F)} K(Q) \prod_{i=1}^{3} \left(\int_Q f_i \, d\sigma_i\right).
\]
It follows that, for $H \in F_3$,

$$
\sum_{F \subseteq G \subseteq H} \sum_{Q: \pi(Q) = (F,G,H)} K(Q) \prod_{i=1}^{3} \left( \int_{Q} f_i \, d\sigma_i \right)
\leq 2 \int_{H} f_3 \, d\sigma_3 \sum_{F \subseteq G \subseteq H} \sum_{Q: \pi(Q) = (F,G,H)} K(Q) \prod_{i=1}^{3} \left( \int_{Q} f_i \, d\sigma_i \right)
= 2 \int_{H} f_3 \, d\sigma_3 \sum_{F \subseteq G \subseteq H} \sum_{Q: \pi(Q) = (F,G,H)} K(H, \sigma_3)(Q) \prod_{i=1}^{2} \left( \int_{Q} f_i \, d\sigma_i \right).
$$

We need two observations. Suppose that $\pi(Q) = (F,G,H)$ and $F \subseteq G \subseteq H$. Let $i = 1, 2$. If $H' \in ch_{F_3}(H)$ satisfies $H' \subseteq Q$, then, by definition of $\pi_{F_3}$, we must have

$$
(2.4) \quad \pi_{F_3}(\pi_{F_i}(H')) = H.
$$

By this observation, we define

$$
ch_{F_3}^i(H) := \{ H' \in ch_{F_3}(H) : \pi_{F_3}(\pi_{F_i}(H')) = H \}.
$$

We further observe that, when $H' \in ch_{F_3}^i(H)$, we can regard $f_i$ as a constant on $H'$ in the above integrals, that is, we can replace $f_i$ by $f_i^H$ in the above integrals, where

$$
f_i^H := f_i 1_{E_{F_3}(H)} + \sum_{H' \in ch_{F_3}^i(H)} \int_{H'} f_i \, d\sigma_1 1_{H'}.
$$

A little thought confirms that, by the assumption (b) and Propositions 1.1 and 1.2,

$$
\sum_{F \subseteq G \subseteq H} \sum_{Q: \pi(Q) = (F,G,H)} K(H, \sigma_3)(Q) \prod_{i=1}^{2} \left( \int_{Q} f_i^H \, d\sigma_i \right)
\leq C c_2 \sigma_3(H)^{1/p_3} \prod_{i=1}^{2} \| f_i^H \|_{L^{p_i}(d\sigma_i)}.
$$

Thus, we obtain

$$
(i) \leq C c_2 \sum_{H \in F_3} \prod_{i=1}^{2} \| f_i^H \|_{L^{p_i}(d\sigma_i)} \int_{H} f_3 \, d\sigma_3 \sigma_3(H)^{1/p_3}.
$$

Since $1 < p_1, p_2, p_3 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$, we can select the auxiliary parameters $s_i$, $i = 1, 2$, that satisfy

$$
\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1 \text{ and } 1 < p_i \leq s_i < \infty.
$$
It follows from Hölder’s inequality with exponent $s_1$, $s_2$ and $p_3$ that

\[(i) \leq C_{C_2} \prod_{i=1}^{2} \left( \sum_{H \in \mathcal{F}_3} \| f_i^H \|^s_{L^{p_i}(d\sigma_i)} \right)^{1/s_i} \times \left( \sum_{H \in \mathcal{F}_3} \left( \int_H f_3^H d\sigma_3 \right)^{p_3} \sigma_3(H) \right)^{1/p_3} \]

\[\leq C_{C_2} \prod_{i=1}^{2} \left( \sum_{H \in \mathcal{F}_3} \| f_i^H \|^p_{L^{p_i}(d\sigma_i)} \right)^{1/p_i} \times \left( \sum_{H \in \mathcal{F}_3} \left( \int_H f_3^H d\sigma_3 \right)^{p_3} \sigma_3(H) \right)^{1/p_3} \]

\[=: C_{C_2}(i_1) \times (i_2) \times (i_3),\]

where we have used $\| \cdot \|_{L^{p_i}} \geq \| \cdot \|_{L^{s_i}}$.

For $(i_3)$, using $\sigma_3(H) \leq 2\sigma_3(E_{\mathcal{F}_3}(H))$ (see (2.2)), the fact that

\[\int_H f_3^H d\sigma_3 \leq \inf_{y \in H} M_{\sigma_3} f_3(y)\]

and the disjointness of the sets $E_{\mathcal{F}_3}(H)$, we have

\[(i_3) \leq C \left( \sum_{H \in \mathcal{F}_3} \int_{E_{\mathcal{F}_3}(H)} (M_{\sigma_3} f_3)^{p_3} d\sigma_3 \right)^{1/p_3} \]

\[\leq C \left( \int_{Q_0} (M_{\sigma_3} f_3)^{p_3} d\sigma_3 \right)^{1/p_3} \leq C \| f_3 \|_{L^{p_3}(d\sigma_3)}.\]

Here, $M_{\sigma_3}$ is the dyadic Hardy-Littlewood maximal operator and we have used the $L^{p_3}(d\sigma_3)$-boundedness of $M_{\sigma_3}$.

It remains to estimate $(i_1)$. ($(i_2)$ can be estimated by the same manner.) It follows that

\[(i_1)^{p_1} = \sum_{H \in \mathcal{F}_3} \int_{E_{\mathcal{F}_3}(H)} f_1^{p_1} d\sigma_1 + \sum_{H \in \mathcal{F}_3} \sum_{H' \in \mathcal{F}_{\lambda^2}(H)} \left( \int_H f_1 d\sigma_1 \right)^{p_1} \sigma_1(H').\]

By the pairwise disjointness of the sets $E_{\mathcal{F}_3}(H)$, it is immediate that

\[\sum_{H \in \mathcal{F}_3} \int_{E_{\mathcal{F}_3}(H)} f_1^{p_1} d\sigma_1 \leq \| f_1 \|_{L^{p_1}(d\sigma_1)}^{p_1}.\]
For the remaining double sum, we use the definition of $\text{ch}^1 F_3 (H)$ (see (2.4)) to reorganize:

$$
\sum_{H \in F_3} \sum_{H' \in \text{ch}^1 F_3 (H)} \left( \int_{H'} f_1 \, d\sigma_1 \right)^{p_1} \sigma_1 (H')
$$

$$
= \sum_{H \in F_3} \sum_{F \in F_1} \sum_{\pi_{\text{F}_3} (F) = H} \sum_{\pi_{\text{F}_3} (H') = F} \left( \int_{H'} f_1 \, d\sigma_1 \right)^{p_1} \sigma_1 (H')
$$

$$
\leq \sum_{H \in F_3} \sum_{F \in F_1} \left( \frac{2}{F} \int_{F} f_1 \, d\sigma_1 \right)^{p_1} \sigma_1 (F)
$$

$$
\leq \sum_{F \in F_1} \left( \frac{2}{F} \int_{F} f_1 \, d\sigma_1 \right)^{p_1} \sigma_1 (F)
$$

$$
\leq C \|
M_{\sigma_1} f_1 \|_{L^{p_1} (d\sigma_1)}^{p_1} \leq C \|
M_{\sigma_1} f_1 \|_{L^{p_1} (d\sigma_1)}^{p_1}.
$$

Altogether, we obtain

$$(i) \leq C \sigma_2 \prod_{i=1}^{3} \|
M_{\sigma_1} f_i \|_{L^{p_i} (d\sigma_i)}^{p_i}.
$$

This yields the theorem.

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