CONTACT ANGLE FOR IMMERSED SURFACES IN $S^{2n+1}$

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Abstract. In this paper we introduce the notion of contact angle. We deduce formulas for Laplacian and Gaussian curvature of a minimal surface in $S^{2n+1}$ and give a characterization of the generalized Clifford Torus as the only non-legendrian minimal surface in $S^5$ with constant Contact and Kähler angles.

1. Introduction

The Kähler angle was introduced by Chern and Wolfson in [2] and [10] and it is the most fundamental invariant for minimal surfaces in complex manifolds. Using the technique of moving frames Wolfson obtained equations for the Laplacian and Gaussian curvature for an immersed minimal surface in $\mathbb{C}P^n$.

Later, Kenmotsu in [6], Ohnita in [8] and Ogata in [9] classified minimal surfaces with constant Gaussian curvature and constant Kähler angle.

A few years ago, Zhenqui in [11] gave a counterexample for the Bolton’s conjecture, in [1], that says that a minimal immersion (non-holomorphic, non anti-holomorphic, non totally real) of two-sphere in $\mathbb{C}P^n$ with constant Kähler angle have constant Gaussian curvature.

In [7] we introduced the contact angle and it has can be consider a new geometric invariant in order to investigate immersed surfaces in $S^3$. Geometrically, the contact angle is the complementary angle between the contact distribution and the tangent space of the surface. Also in [7], we deduced Gaussian curvature and Laplacian formulae for an immersed minimal surface in $S^3$ and gave a characterization of the Clifford Torus as the only minimal surface in $S^3$ with constant Contact angle.

In this work, a corresponding statement to Bolton’s conjecture (see [1]) for an odd dimmensional sphere has been proved. We deduce Gaussian curvature and Laplacian formulae for an immersed minimal surface in $S^{2n+1}$:

$$K = -\sec^2 \beta |\nabla \beta|^2 - \tan \beta \Delta \beta - 2 \cos \alpha (1 + 2 \tan^2 \beta) / \beta_1$$
$$+ 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos^2 \alpha$$
\[
\tan \beta \Delta \beta = -1 - \tan^2 \beta (|\nabla \beta|^2 + 4 \cos \alpha \beta_1) + 2 \tan \beta \sin \alpha \alpha_1 \\
- \cos^2 \alpha (4 \tan^2 \beta - 1) + (1 + \sin^2 \beta) \left( \frac{d\alpha}{2} + w_1^2 \right) \\
+ \sum_{\lambda=3}^{2n} (h_1^\lambda h_2^\lambda - (h_1^\lambda)^2)
\]

Using Gaussian curvature, we characterize the Clifford Torus in \( S^5 \):

**Theorem 1**  The Clifford Torus is the only non-legendrian minimal surface in \( S^5 \) with constant Contact and Kähler angles.

When the Kähler angle is null, we have an interesting characterization of the Clifford Torus without suppose that the Contact angle is constant:

**Theorem 2**  The Clifford Torus is the only non-legendrian minimal surface in \( S^5 \) with Contact angle \( 0 \leq \beta < \frac{\pi}{2} \) and null Kähler angle.

2. Preliminaries

Consider in \( C^{n+1} \) the Hermitian product:

\[
(z, w) = \sum_{j=0}^{n} z^j \bar{w}^j
\]

The inner product is given by:

\[
\langle z, w \rangle = \text{Re}(z, w)
\]

The unitary sphere

\( S^{2n+1} = \{ z \in C^{n+1} | (z, z) = 1 \} \)

The Reeb field in \( S^{2n+1} \) is given by:

\( \xi(z) = iz \)

The usual contact distribution in \( S^{2n+1} \) is orthogonal to \( \xi \):

\( \Delta = \{ v \in T_z S^{2n+1} | \langle \xi, v \rangle = 0 \} \)

\( \Delta \) is invariant by the complex structure of \( C^{n+1} \).

A unitary frame \( (f_0, f_1, ..., f_n) \) of \( C^{n+1} \) induces an adapted frame of \( S^{2n+1} \), where \( f_0 = z \) is the position vector in \( S^{2n+1} \) and \( (f_1, ..., f_n) \) is the complex basis of \( \Delta \).

Structure equations of \( U(n+1) \), \( df_j = \psi^k_j f_k \), where \( \psi^k_j + \bar{\psi}_j^k = 0 \), can be interpreted as a mobile frame in \( S^{2n+1} \)

\[
dz = \psi^1_0 f_1 + ... + \psi^n_0 f_n + \theta \xi
\]
where $\theta = -i\psi_0^0$ is a real form.

$(\psi^0_j, \theta)$ constitute a coframe in $S^{2n+1}$ and unitary in $\Delta$.

Taking:

$$\psi^0_j = w^j + iw^{j+n} \quad (1)$$

we obtain a coframe $(w^\lambda)^{\lambda=1,...,2n+1}$.

$(\psi^k_j)$ forms satisfy structure equations of $\mathbb{U}(n)$:

$$d\psi^k_j + \psi^k_j \wedge \psi^r_j = 0 \quad (2)$$

Taking $j = 0$ in (1), we obtain that:

$$dw^j + w^j_k \wedge w^k_j = w^{j+n} \wedge w^{k+n} - w^{j+n} \wedge w^{2n+1} = 0$$
$$dw^{j+n} + w^{j+n}_k \wedge w^k_j = w^{j+n} \wedge w^{k+n} + w^{j+n} \wedge w^{2n+1} = 0$$
$$dw^{2n+1} - 2w^j \wedge w^{j+n} = 0$$

Therefore Riemannian connexion forms satisfy:

$$w^{j+n}_k = w^j_k$$
$$w^{j+n}_j = -w^{j+n}_k$$
$$w^{2n+1}_j = -w^{j+n}_j$$
$$w^{2n+1}_j = w^j_j$$

Therefore we conclude that curvature forms of $S^{2n+1}$ are:

$$\Omega^j_i = w^i \wedge w^j$$

Note that in this adapted frame connexion forms of maximum degree $w^{2n+1}_j$ and $w^{2n+1}_{j+n}$ are:

$$w^{2n+1}_{j+n} = w^{j+n}$$
$$w^{2n+1}_{j+n} = -w^j$$

3. CONTACT ANGLE FOR IMMERSED SURFACES IN $S^{2n+1}$

Consider $S$ an immersed surface in $S^{2n+1}$.

The **contact angle** $\beta$ is the complementary angle between the contact distribution $\Delta$ and the tangent space $TS$ of the surface.

Consider $(e_1, e_2)$ a local frame of $TS$, where $e_1 \in TS \cap \Delta$. Then:

$$\cos \beta = \langle \xi, e_2 \rangle$$

Let $v$ the projection of unitary vector field $e_2$ in $\Delta$:

$$e_2 = \sin \beta v + \cos \beta \xi$$
We define $\alpha$ as the angle given by:

$$\cos \alpha = \langle ie_1, v \rangle$$

The angle $\alpha$ is the corresponding to the Kähler angle and it was introduced by Chern and Wolfson in [2].

Consider $(f_j)$ an adapted frame to the distribution $\Delta$.

The $(f_j)$ is given by:

\[
\begin{align*}
    f_1 &= \frac{e_1 - i\nu}{2\cos \frac{\alpha}{2}} \\
    f_2 &= \frac{e_1 + i\nu}{2\sin \frac{\alpha}{2}} \\
    f_j &\in \Delta \\
    f_{n+j} &= if_j, \quad j = 1, \ldots, n \\
    f_{2n+1} &= \xi
\end{align*}
\]

Therefore $(e_1, e_2)$ can be written, as:

\[
\begin{align*}
    e_1 &= \cos \frac{\alpha}{2} f_1 + \sin \frac{\alpha}{2} f_2 \\
    e_2 &= \sin \beta (\cos \frac{\alpha}{2} f_{n-1} - \sin \frac{\alpha}{2} f_{n+2}) + \cos \beta \xi
\end{align*}
\]

Consider normal vector fields:

\[
\begin{align*}
    e_{n+1} &= \sin \frac{\alpha}{2} f_1 - \cos \frac{\alpha}{2} f_2 \\
    e_{n+2} &= \sin \frac{\alpha}{2} f_{n+1} + \cos \frac{\alpha}{2} f_{n+2} \\
    e_{2n+1} &= -\cos \beta (\cos \frac{\alpha}{2} f_{n+1} - \sin \frac{\alpha}{2} f_{n+2}) + \sin \beta \xi \\
    e_{\lambda} &= f_{\lambda}; \quad 3 \leq \lambda \leq n, \quad n + 3 \leq \lambda \leq 2n
\end{align*}
\]

where $(e_{\lambda})_{\lambda=1,...,2n+1}$ constitute a Darboux frame of $S$.

Let $(w^j), (\theta^j)$ respective coframes of $(f_j)$ and $(e_j)$.

When we restrict to $S$, we obtain:

\[
\begin{align*}
    w^1 &= \cos \frac{\lambda}{2} \theta^1 \\
    w^2 &= \sin \frac{\lambda}{2} \theta^1 \\
    w^{n+1} &= \cos \frac{\lambda}{2} \sin \beta \theta^2 \\
    w^{n+2} &= -\sin \frac{\lambda}{2} \sin \beta \theta^2 \\
    w^{2n+1} &= \cos \beta \theta^2 \\
    w^{\lambda} &= 0 \quad 3 \leq \lambda \leq n, \quad n + 3 \leq \lambda \leq 2n
\end{align*}
\]

4. EQUATIONS FOR CURVATURE AND LAPLACIAN OF A MINIMAL SURFACE IN $S^{2n+1}$

Consider $D$ the covariant derivative of Riemannian metric in $S^{2n+1}$.

Then:

$$Df_j = w^k_j f_k$$

where $(w^k_j)$ satisfy (3).

Consider connexion forms $(\theta^j_k)$ associate to the Darboux frame $(e_j)$. Then:

$$De_j = \theta^k_j e_k$$
Second fundamental forms in this frame are given by:

\[
\begin{align*}
II_1^{n+1} &= \theta_1^{n+1} + \theta_2^{n+1} \\
II_2^{n+1} &= \theta_1^{n+1} + \theta_2^{n+1} \\
II_1^{2n+1} &= \theta_1^{2n+1} + \theta_2^{2n+1} \\
II_2^{2n+1} &= \theta_1^{2n+1} + \theta_2^{2n+1} \\
II_1^{\lambda} &= \theta_1^{\lambda} + \theta_2^{\lambda}
\end{align*}
\]  

Using formulae (4) and (5), we deduce:

\[
\begin{align*}
\theta_1^{n+1} &= \frac{d\alpha}{2} + w_1^2 \\
\theta_2^{n+1} &= \sin \beta \left( \frac{\sin \alpha}{2} w_1^{n+1} - \frac{\sin^2 \alpha}{2} w_1^{n+2} - \frac{\cos^2 \alpha}{2} w_2^{n+1} + \frac{\sin \alpha}{2} w_2^{n+2} \right) \\
\theta_1^{n+2} &= \frac{\sin \alpha}{2} w_1^{n+1} + \frac{\sin^2 \alpha}{2} w_1^{n+2} + \frac{\cos^2 \alpha}{2} w_2^{n+1} + \frac{\sin \alpha}{2} w_2^{n+2} \\
\theta_2^{n+2} &= \sin \beta \left( \frac{d\alpha}{2} - w_1^{n+2} \right) \\
\theta_1^{2n+1} &= -\cos \beta \left( \cos \frac{\sin \alpha}{2} w_1^{n+1} + \frac{\sin \alpha}{2} w_1^{n+2} - \frac{\sin \alpha}{2} w_2^{n+1} - \frac{\sin \alpha}{2} w_2^{n+2} \right) \\
&\quad - \cos \alpha \sin^2 \left( \frac{\beta}{2} \right) \theta^2 \\
\theta_2^{2n+1} &= d\beta + \cos \alpha \theta^1
\end{align*}
\]

Minimality and symmetry conditions are:

\[
\begin{align*}
\theta_1^\lambda \wedge \theta^1 + \theta_2^\lambda \wedge \theta^2 &= 0 \\
\theta_1^\lambda \wedge \theta^1 - \theta_2^\lambda \wedge \theta^2 &= 0
\end{align*}
\]  

From (7), we obtain:

\[
\begin{align*}
\theta_1^{n+1} &= \frac{d\alpha}{2} + w_1^2 \\
\theta_2^{n+1} &= \frac{\cos \beta}{2} + w_1^2 \circ J \\
\theta_1^{n+2} &= \sin \beta \left( -\frac{d\alpha}{2} \circ J - w_1 \circ J \right) \\
\theta_2^{n+2} &= \sin \beta \left( \frac{d\alpha}{2} + w_1^2 \right) \\
\theta_1^{2n+1} &= -d\beta \circ J + \cos \alpha \theta^2 \\
\theta_2^{2n+1} &= d\beta + \cos \alpha \theta^1
\end{align*}
\]

where the complex structure $J$ of $S$ is given by $Je_1 = e_2$ and $Je_2 = -e_1$.

Gauss equation is:

\[
d\theta_1^1 + \theta_1^2 \wedge \theta_2^k = \theta_1 \wedge \theta^2
\]

We define:

\[
\theta_j^k = h_{jk}^\lambda \theta^k
\]

Then using (8), (9) and (10), we obtain that:

\[
K = 1 - |\nabla(\beta)|^2 - 2\beta \cos \alpha - \cos^2 \alpha \\
-(1 + \sin^2 \beta) \left( \frac{|\nabla \alpha|^2}{4} + w_1^2 (e_1) + w_2^2 (e_2) + |\nabla w_1|^2 + \sum_{\lambda=3}^{2n} (h_{11}^\lambda h_{12}^\lambda - (h_{12}^\lambda)^2) \right)
\]
Or, to be equivalent to:

\[
K = 1 - |\nabla \beta + \cos \alpha e_1|^2 - (1 + \sin^2 \beta) \left| \frac{d\alpha}{2} + w_1^2 \right|^2 \\
+ \sum_{\lambda=3}^{2n} (h_{11}^\lambda h_{22}^\lambda - (h_{12}^\lambda)^2)
\]  

(12)

Taking the differential of \((\theta^1, \theta^2)\), we have:

\[
d\theta^1 = - \sin \beta [\cos^2 \frac{\alpha}{2} w_1^{n+1} + \frac{\sin \alpha}{2} (w_2^{n+1} - w_1^{n+2}) \\
- \sin^2 \frac{\alpha}{2} w_2^{n+2}] \wedge \theta^2 = 0 \\
d\theta^2 = \sin \beta [\cos^2 \frac{\alpha}{2} w_1^{n+1} + \frac{\sin \alpha}{2} (w_2^{n+1} - w_1^{n+2}) - \sin^2 \frac{\alpha}{2} w_2^{n+2} \\
- \cos \beta \cos \alpha \theta^2] \wedge \theta^1 = 0
\]

Therefore:

\[
\theta^1_2 = - \sin \beta [\cos^2 \frac{\alpha}{2} w_1^{n+1} + \frac{\sin \alpha}{2} (w_2^{n+1} - w_1^{n+2}) - \sin^2 \frac{\alpha}{2} w_2^{n+2} - \cos \beta \cos \alpha \theta^2]
\]

Using the symmetry and minimality and the complex structure of \(S\), we obtain that:

\[
\theta^1_2 = \tan \beta (d\beta \circ J - 2 \cos \alpha \theta^2)
\]  

(13)

Computing the differential of (13), we can get:

\[
d\theta_2^1 = (- \sec^2 \beta |\nabla \beta|^2 - \tan(\beta) \Delta(\beta) - 2 \cos \alpha \beta_1 (1 + 2 \tan^2 \beta) \\
+ 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos \alpha \theta^1 \wedge \theta^2)
\]

(14)

where \(\Delta \beta = tr \nabla d\beta\).

Therefore the Gaussian curvature is:

\[
K = - \sec^2 \beta |\nabla \beta|^2 - \tan \beta \Delta(\beta) - 2 \cos \alpha \beta_1 (1 + 2 \tan^2 \beta) + \\
+ 2 \tan \beta \sin \alpha \alpha_1 - 4 \tan^2 \beta \cos \alpha \theta^1 \wedge \theta^2
\]

(15)

Jointing (11) and (15), we obtain a new Laplacian equation:

\[
\tan \beta \Delta(\beta) = -1 - \tan^2 \beta (|\nabla \beta|^2 + 4 \cos \alpha \beta_1) + 2 \tan \beta \sin \alpha \alpha_1 \\
- \cos^2 \alpha (4 \tan^2 \beta - 1) + (1 + \sin^2 \beta) \left| \frac{d\alpha}{2} + w_1^2 \right|^2 \\
+ \sum_{\lambda=3}^{2n} (h_{11}^\lambda h_{22}^\lambda - (h_{12}^\lambda)^2)
\]

(16)

5. Minimal Surfaces in \(S^5\)

In this section, we compute Contact angle for examples of minimal torus given by Kenmotsu, in [5].
5.1. Contact Angle of Legendrian Minimal Torus.

Consider the torus in $S^5$ defined by:

$$T^3 = \{(z_1, z_2, z_3) \in C^3 | z_1 \bar{z}_1 = \frac{1}{3}, z_2 \bar{z}_2 = \frac{1}{3}, z_3 \bar{z}_3 = \frac{1}{3}\}$$

We consider the immersion:

$$f(u_1, u_2) = \frac{\sqrt{3}}{3}(e^{iu_1}, e^{iu_2}, e^{-i(u_1 + u_2)})$$

Then the line vector fields are:

$$e_1 = \frac{\sqrt{2}}{2}(e^{iu_1}, 0, -e^{-i(u_1 + u_2)})$$

$$e_2 = \frac{\sqrt{2}}{\sqrt{3}}(\frac{1}{2}e^{iu_1}, e^{iu_2}, -\frac{1}{2}e^{-i(u_1 + u_2)})$$

The normal vector field is:

$$\xi = \frac{i\sqrt{3}}{3}(e^{iu_1}, e^{iu_2}, e^{-i(u_1 + u_2)})$$

The Contact angle is:

$$\cos \beta = \langle e_2, \xi \rangle = 0$$

Therefore,

$$\beta = \frac{\pi}{2}$$

5.2. Contact Angle of Generalized Clifford Torus.

Consider the torus in $S^5$ defined by:

$$T^3 = \{(z_1, z_2, z_3) \in C^3 | z_1 \bar{z}_1 = \frac{1}{3}, z_2 \bar{z}_2 = \frac{1}{3}, z_3 \bar{z}_3 = \frac{1}{3}\}$$

We consider the immersion:

$$f(u_1, u_2) = \frac{\sqrt{3}}{3}(e^{iu_1}, e^{iu_2}, e^{i(u_2 - u_1)})$$

Then the line vector fields are:

$$e_1 = \frac{i\sqrt{2}}{2}(e^{iu_1}, 0, -e^{i(u_2 - u_1)})$$

$$e_2 = \frac{i}{\sqrt{3}}(\frac{1}{2}e^{iu_1}, e^{iu_2}, \frac{1}{2}e^{i(u_2 - u_1)})$$

The normal vector field is:

$$\xi = \frac{i\sqrt{3}}{3}(e^{iu_1}, e^{iu_2}, e^{i(u_2 - u_1)})$$

The Contact angle is:

$$\cos \beta = \langle e_2, \xi \rangle = \frac{2\sqrt{3}}{3}$$
Therefore,

\[ \beta = \arccos \left( \frac{2\sqrt{2}}{3} \right) \]

5.3. Contact Angle of Clifford Torus.

Consider the torus in \( S^5 \) defined by:

\[ T^2 = \{(z_1, z_2, 0) \in C^3 | z_1 \bar{z}_1 = \frac{1}{2}, z_2 \bar{z}_2 = \frac{1}{2} \} \]

We consider the immersion:

\[ f(u_1, u_2) = \frac{\sqrt{2}}{2} (e^{iu_1}, e^{iu_2}, 0) \]

Then the line vector fields are:

\[ e_1 = \frac{i}{2} (e^{iu_1}, -e^{iu_2}, 0) \]
\[ e_2 = \frac{i}{2} (e^{iu_1}, e^{iu_2}, 0) \]

The normal vector field is:

\[ \xi = \frac{i\sqrt{2}}{2} (e^{iu_1}, e^{iu_2}, 0) \]

The Contact angle is:

\[ \cos \beta = \langle e_2, \xi \rangle = 1 \]

Therefore,

\[ \beta = 0 \]

6. Main Results

6.1. Proof of the Theorem 1.

Suppose that \( \alpha \) and \( \beta \) are constants, then equation (15) reduces to:

\[ K = -4 \tan^2 \beta \cos^2 \alpha \] (17)

As the line field determined by \( e_1 \) is globally defined, we have therefore that \( S \) is parallelizable and by equation (17) Gaussian curvature is constant. Therefore, using Gauss-Bonnet theorem, we have that Gaussian curvature of \( S \) is null everywhere.

Using the work of Kenmotsu, in [4], we have that \( S \) is the generalized Clifford Torus in \( S^5 \).

q.e.d.
6.2. Proof of the Theorem 2.

When the surface is immersed in $S^5$, equation (12) is:

$$K = 1 - |\nabla \beta + \cos \alpha e_1|^2 - (1 + \sin^2 \beta)\frac{d\alpha}{2} + w_1^2$$

(18)

In $S^5$ equation (16) is given by:

$$\tan \beta \Delta(\beta) = -1 - \tan^2 \beta(\|\nabla \beta\|^2 + 4 \cos \alpha \beta) + 2 \tan \beta \sin \alpha \alpha_1$$

$$- \cos^2 \alpha(4 \tan^2 \beta - 1) + (1 + \sin^2 \beta)\frac{d\alpha}{2} + w_1^2$$

(19)

Suppose that $\alpha = 0$, then the adapted frame $(f_i)$ is given by:

$$f_1 = e_1$$
$$f_3 = ie_1$$
$$(f_2, f_4) \in \Delta$$
$$f_5 = \xi$$

Therefore $(e_1, e_2)$ can be written, as:

$$e_1 = f_1$$
$$e_2 = \sin \beta f_3 + \cos \beta \xi$$

Normal vector fields are:

$$e_3 = f_2$$
$$e_4 = f_4$$
$$e_5 = -\cos \beta f_3 + \sin \beta \xi$$

(20)

Restrict to $S$, we obtain:

$$w^1 = \theta^1$$
$$w^2 = 0$$
$$w^3 = \sin \beta \theta^2$$
$$w^4 = 0$$
$$w^5 = \cos \beta \theta^2$$

(21)

Taking the differential of $e_1$, we obtain:

$$De_1 = Df_1$$
$$= w_1^2 f_2 + w_1^3 f_3 + w_1^4 f_4 + w_1^5 f_5$$
$$= (\sin \beta w_1^3 + \sin 2\beta \theta^2)e_2 - w_1^2 e_3 + w_1^4 e_4 + (- \cos \beta w_1^3 + \sin^2 \beta \theta^2)e_5$$

(22)

Therefore:

$$\theta_1^2 = \sin \beta w_1^3 + \frac{\sin 2\beta \theta^2}{2}$$
$$\theta_4 = -w_1^2$$
$$\theta_4^2 = w_1^4$$
$$\theta_5 = -\cos \beta w_1^3 + \sin^2 \beta \theta^2$$

(23)
Taking the differential of $e_2$, we obtain:

$$
\begin{align*}
\text{De}_2 &= \cos \beta d\beta f_3 - \sin \beta d\beta f_5 + \sin \beta Df_3 + \cos \beta Df_5 \\
&= w_1^2 f_2 + w_2^3 f_3 + w_4^4 f_4 + w_5^5 f_5 \\
&= (\sin \beta w_3^1 - \frac{\sin 2\beta}{2} \theta^2)e_1 - \sin \beta w_7^1 e_3 + \sin \beta w_7^1 e_4 - (d\beta + \theta^1) e_5
\end{align*}
$$

Therefore:

$$
\begin{align*}
\theta_2^1 &= \sin \beta w_3^1 - \frac{\sin 2\beta}{2} \theta^2 \\
\theta_3^2 &= -\sin \beta w_4^1 \\
\theta_4^3 &= \sin \beta w_4^1 \\
\theta_5^5 &= -(d\beta + \theta^1)
\end{align*}
$$

Equivalently to minimallity and symmetry conditions, we have:

$$
\begin{align*}
\theta_3^1 \circ J &= \theta_3^3 \\
\theta_1^1 \circ J &= \theta_4^2
\end{align*}
$$

Substituting (23) and (25) at the equations (26), we obtain:

$$
\begin{align*}
w_1^2 \circ J &= \sin \beta w_4^1 \\
w_1^1 \circ J &= \sin \beta w_4^1
\end{align*}
$$

Therefore:

$$
(1 - \sin^2 \beta)w_1^2 \circ J = 0
$$

The surface $S$ is non-legendrian, we conclude that $w_1^2 = 0$. Then the equation (19) reduces to:

$$
\Delta(\beta) = -\tan(\beta)|\nabla \beta + 2e_1|^2
$$

Suppose that $0 \leq \beta < \frac{\pi}{2}$, then using Hopf’s lemma we obtain that $\beta$ is constant and using equation (18), we obtain that $K=0$, and therefore $S$ is the Clifford torus.

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