SCATTERING PROBLEMS FOR PERTURBATIONS OF THE
MULTIDIMENSIONAL BIHARMONIC OPERATOR

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Abstract. Some scattering problems for the multidimensional biharmonic operator are studied. The operator is perturbed by first and zero order perturbations, which maybe complex-valued and singular. We show that the solutions to direct scattering problem satisfy a Lippmann-Schwinger equation, and that this integral equation has a unique solution in the weighted Sobolev space $H^{2-\delta}$. The main result of this paper is the proof of Saito’s formula, which can be used to prove a uniqueness theorem for the inverse scattering problem. The proof of Saito’s formula is based on norm estimates for the resolvent of the direct operator in $H^{1-\delta}$.

1. Introduction. We consider the following $n$-dimensional ($n \geq 2$) biharmonic operator
\begin{equation}
H_4 u = \Delta^2 u + \vec{q} \cdot \nabla u + Vu,
\end{equation}
where $\Delta$ is the Laplacian and $\cdot$ denotes the dot-product $x \cdot y = \sum_{j=1}^{n} x_j y_j$ for $x,y \in \mathbb{C}^n$. The bi-Laplacian is perturbed by first and zero order perturbations, vector-valued function $\vec{q}$ and a scalar function $V$, that may be complex-valued.

The motivation to study operators of order 4 appears for example in the study of elasticity and the theory of vibrations of beams. As a concrete example, the (linear) beam equation [9]
\[ \partial_t^2 U + \Delta^2 U + mU = 0 \]
der under time-harmonic assumption $U(x,t) = u(x)e^{-i\omega t}$ results in the equation
\[ \Delta^2 u + mu = \omega^2 u. \]
For scattering in the nonlinear case, see e.g. [20] and the references therein. Other examples of biharmonic problems include hinged plate configurations, described by equations of form
\[
\begin{cases}
\Delta^2 u = f & \text{in } \Omega \\
u = \Delta u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

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where \( u \) satisfies Navier boundary conditions [9] and scattering by grating stacks, described by the equation \( \Delta^2 u - \beta^4 u = 0 \) [17].

In terms of inverse problems for bi- and poly-harmonic operators we mention some solutions to inverse boundary value problems, see e.g., [14] and also [3, 4]. In these papers the aim is to recover the coefficients \( \vec{q} \) and \( V \) in operator \( \Delta^m + \vec{q} \cdot \nabla + V \), \( m \geq 2 \) in some bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 3 \). The main idea is to define the so-called Dirichlet-to-Neumann map and then to recover the coefficients \( \vec{q} \) and \( V \) from it. In [22] it was proved that the Dirichlet-to-Neumann map uniquely corresponds to \( \vec{q} \) and \( V \) and also to a second order perturbation \( F \). The motivation for the approach in the above texts is based on the fundamental papers [24, 25].

The present work is concerned with the following scattering problem for \( H_4 \) given by

\[
\begin{cases}
H_4 u = k^4 u, & k > 0, \quad u = u_0 + u_{sc}, \quad u_0(x, k, \theta) = e^{ik(x, \theta)}, \\
\frac{\partial}{\partial n} f - ik f = 0 \left( |x|^{-\frac{n-2}{2}} \right), & |x| \to \infty, \text{ for both } f = u_{sc} \text{ and } f = \Delta u_{sc},
\end{cases}
\tag{2}
\]

where the scattered wave \( u_{sc} \) should be from some suitable function space. Here \( \theta \in \mathbb{S}^{n-1} \) is the angle of incident wave and \((, ,)\) denotes the usual real inner product. The second line in (2) is interpreted as an analogue of Sommerfeld’s radiation condition at infinity for thisbiharmonic operator. To the best of our knowledge this radiation condition has not appeared in the literature before. We use it to reduce (2) to a related integral equation.

The authors were originally motivated to start studying scattering for fourth order operators by the article of Aktosun and Papanicolaou [2], where the time-evolution of several scattering coefficients for the 1D biharmonic operator was studied. In terms of inverse scattering problems for fourth order operator we mention K. Iwasaki’s results [12, 13]. Iwasaki studied the scattering problem in one-dimension and considered the inverse problem as a Riemann-Hilbert boundary value problem with respect to the wavenumber \( k \) in the complex cone \( \text{arg}([0, \pi/4]) \setminus \{0\} \). Given certain reflection and connection coefficients he then showed that it is possible to uniquely recover the potentials \( \vec{q} \) and \( V \) of the scattering operator. In abstract setting, Kuroda [15] has studied scattering for zero-order perturbations for self-adjoint operators. However, we are not aware of other results regarding inverse scattering for higher order operators even with just zero-order perturbations.

Let us now turn to (2) and define \( H_0 := \Delta^2 - k^4 \). Due to the equality \( H_0 = (-\Delta - k^2)(-\Delta + k^2) \) the operator \( H_4 \) inherits some Schrödinger-like properties which allows us to conclude that a fundamental solution to \( H_0 \) is given by

\[
G_k^+(|x|) = \frac{i}{8k^2} \left( \frac{|k|}{2\pi|x|} \right)^{\frac{n-2}{2}} \left( H_{\frac{n-2}{2}}^{(1)}(|k||x|) + \frac{2i}{\pi} K_{\frac{n-2}{2}}(|k||x|) \right).
\]

Here \( H_{\frac{n-2}{2}}^{(1)} \) and \( K_{\frac{n-2}{2}} \) are the Hankel function of the first kind and Macdonald’s function of orders \( \frac{n-2}{2} \). The function \( G_k^+ \) is also the kernel of the integral operator \( (\Delta^2 - k^4 - i0)^{-1} \). By applying the fundamental solution to (1) we obtain a Lippmann-Schwinger integral equation

\[
(3) \quad u(x, k, \theta) = e^{ik(x, \theta)} - \int_{\mathbb{R}^n} G_k^+(|x-y|) [\vec{q}(y) \cdot \nabla u(y, k, \theta) + V(y) u(y, k, \theta)] \, dy.
\]

Since \( u_0 = e^{ik(\theta,x)} \) is just a bounded function it is more convenient to study the equivalent integral equation for the scattered wave, namely...
\[ u_{sc} = -\int_{\mathbb{R}^n} G_k^+ (|x - y|) \left[ \tilde{q}(y) \cdot \nabla (u_0 + u_{sc}) + V(y)(u_0 + u_{sc}) \right] dy \]

where \( \tilde{u}_0 = L_k u_0 \).

The aim of the present text is to study the classical scattering theory for (1) and in particular to prove an analogue of so-called Saito’s formula. The approach is similar to scattering theory of Schrödinger operators see e.g., books by Cakoni and Colton [6] and Eskin [7] and to scattering theory of operators with constant coefficients, see the classic book of Hörmander [11]. We start by showing that a solution to the scattering problem (2) indeed also satisfies equation (3). This translates the study of the scattering problem to the somewhat simpler study of an integral equation for which we can then show the unique solvability by extending some well-known estimates for the Schrödinger operator to biharmonic case. This solution admits the asymptotic representation

\[ u(x, k, \theta) = e^{ik(x, \theta)} - C_n \frac{k^n}{|x|^{n+2}} e^{ik|x|} - A(k, \theta, \theta') + o \left( \frac{1}{|x|^n} \right), \quad |x| \to \infty, \]

where \( \theta' \in S^{n-1} \) is the angle of measurement and the function

\[ A(k, \theta, \theta') = \int_{\mathbb{R}^n} e^{-ik(\theta', y)} \left[ \tilde{q} \cdot \nabla u + Vu \right] dy, \]

is called the scattering amplitude. From the point of view of inverse problems we regard this scattering amplitude as the relevant scattering data. For our purposes we require the scattering amplitude to be known for all possible incident and measurement angles and all arbitrarily high frequencies \( (k > 0 \text{ large}) \). Then Saito’s formula is given by the following

**Theorem 1.1 (Saito’s formula).** Assume that \( \tilde{q} \in W^{1, 2\delta}(\mathbb{R}^n) \) and \( V \in L^{2\delta}(\mathbb{R}^n) \), where \( 2\delta > n - \frac{n}{p} \) and \( n < p \leq \infty \). Then the limit

\[ \lim_{k \to \infty} k^{n-1} \int_{S^{n-1} \times S^{n-1}} e^{-ik(\theta - \theta', x)} A(k, \theta, \theta') d\theta d\theta' = 2^n \pi^{n-1} \int_{\mathbb{R}^n} |y|^{-n} dy \]

holds uniformly in \( x \). Here \( \beta := -\frac{1}{2} \nabla \cdot \tilde{q} + V \).

To the best of our knowledge a formula of this type first appeared in [21] for the Schrödinger operator with real short-range potential \( |V(x)| \leq C(1 + |x|)^{-\nu} \), \( \mu > 1 \) in \( \mathbb{R}^3 \). This formula has since then been generalized to Schrödinger operators in any dimensions with more general singular potentials and in some cases to nonlinear coefficients, see e.g., [18, 10, 23]. The significance of Saito’s formula for inverse problems is apparent from its corollaries.

**Corollary 1 (Uniqueness).** Let \( q_1^1, V_1 \) and \( q_2^2, V_2 \) be as in Theorem 1.1. If the corresponding scattering amplitudes for these coefficients coincide for some sequence \( k_j \to \infty \) then the corresponding coefficients \( \beta_1 \) and \( \beta_2 \) are equal in the sense of distributions.

**Corollary 2 (Representation formula).** Under the same assumptions as in Theorem 1.1 we have

\[ \beta(x) = \frac{\Gamma \left( \frac{n-1}{2n+1} \right)}{2^{n+1} \pi^{\frac{3n}{2}}} \lim_{k \to \infty} k^{n-1} \int_{S^{n-1} \times S^{n-1}} A(k, \theta, \theta') |\theta - \theta'| e^{-ik(\theta - \theta', x)} d\theta d\theta' \]
in the sense of tempered distributions.

This paper is organized as follows. In Section 2 we fix some notations and recall asymptotic formulas for Hankel and Macdonald functions. Then in Section 3 we show that a solution to (2) satisfies (3). In Section 4 we prove the existence and the uniqueness of the solution to (3). Several estimates for the solution and its related operators are given. Section 5 concerns the asymptotic behaviour of the solution $u$. In particular, we define the scattering amplitude which is regarded as the scattering data for the inverse problem. Finally, in Section 6 we give a proof for Saito’s formula.

2. Preliminaries. We use the following notations throughout the text. The letter $C$ will be used to denote a generic constant $C > 0$ which may have different values from step to step. The symbol $p'$ denotes the Hölder conjugate of $p$, i.e. $1/p' + 1/p = 1$. The weighted Lebesgue spaces $L^p_\delta$ are defined by the norm

$$
\|f\|_{L^p_\delta(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} (1 + |x|)^\delta |f(x)|^p \, dx \right)^{\frac{1}{p}}.
$$

The Sobolev spaces are defined by $W^{1,p}_\delta(\mathbb{R}^n) := \{ f \in L^p_\delta(\mathbb{R}^n) \mid \nabla f \in L^p_\delta(\mathbb{R}^n) \}$. We say that $\vec{q} \in W^{1,p}_\delta(\mathbb{R}^n)$, if each scalar-valued component of $\vec{q}$ belongs to $W^{1,p}_\delta(\mathbb{R}^n)$ in the above sense. As usual, in the case of $L^2_\delta$-based Sobolev space, we write $W^{s,2}_\delta(\mathbb{R}^n) =: H^s_\delta(\mathbb{R}^n)$.

Later we will need some knowledge about the behaviour of the function $G^+_k$. We start by recalling that the asymptotic behaviour of Hankel functions is

$$
H^{(1)}_{\nu}(x) = \begin{cases} 
O(|x|^{-\nu}), & \nu > 0, \\
O(\log (2x)), & \nu = 0
\end{cases} \quad \text{as} \quad x \to 0+ \text{ (see e.g., [26, 16])}.
$$

The Macdonald function $K_{\nu}$ has the same behaviour at $x \to 0+$. Moreover, for all $\nu \geq 0$

$$
H^{(1)}_{\nu}(x) = \frac{2}{\pi x} e^{i \left( x^{-\frac{\nu}{2}} - \frac{1}{4} \pi \right)} + O(x^{-\frac{3}{2}}), \quad x \to +\infty.
$$

Correspondingly, Macdonald’s function has the asymptotic representation

$$
K_{\nu}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} + O\left(\frac{e^{-x}}{x^{\frac{3}{2}}}\right), \quad x \to +\infty.
$$

The explicit form of the fundamental solution $G^+_k$ allows us to conclude that for $k > 0$ the function $G^+_k$ behaves like

$$
G^+_k(|x|) = \begin{cases} 
O\left(\frac{1}{x^{\frac{1}{2}}} (1 + |\log (k|x|)|)\right), & n = 2, \\
O\left(\frac{1}{k^{n-1}|x|^{n-2}}\right), & n \geq 3
\end{cases}
$$

as $k|x| \to 0$ and

$$
G^+_k(|x|) = O\left(\frac{k^{n-3}}{|x|^{\frac{n-1}{2}}}\right), \quad k|x| \to +\infty.
$$

On the other hand, the relations

$$
\frac{d}{dz} z^{-\nu} H^{(1)}_{\nu}(z) = -z^{-\nu} H^{(1)}_{\nu+1}(z), \quad \frac{d}{dz} z^{-\nu} K_{\nu}(z) = -z^{-\nu} K_{\nu+1}(z)
$$
Lemma 3.1. Let \( C \) as Cakoni and Colton \([6]\) and Eskin \([7]\). The proof is classical and uses similar techniques.

3. From scattering problem to Lippmann-Schwinger equation. This section is devoted to showing that a solution to the scattering problem (2) must satisfy the Lippmann-Schwinger equation. The proof is classical and uses similar techniques as Cakoni and Colton \([6]\) and Eskin \([7]\).

Lemma 3.2. Let \( \tilde{q} \in W_{1,23}^1(\mathbb{R}^n) \) and \( V \in L_{23}^0(\mathbb{R}^n) \), where \( 2\delta > n - \frac{2}{p} \) and \( n < p \leq \infty \). If \( f \in H_{1,4}^1(\mathbb{R}^n) \) then \( \tilde{q} \cdot \nabla f + V f \in L_2^1(\mathbb{R}^n) \).

Proof. By Hölder’s inequality we obtain

\[
\|V f\|_{L_2^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |x|)^{2\delta}|V(x)|^2|f(x)|^2 dx \\
\leq \left( \int_{\mathbb{R}^n} (1 + |x|)^{4\delta r}|V(x)|^{2r} dx \right)^{\frac{1}{r}} \left( \int_{\mathbb{R}^n} (1 + |x|)^{-2\delta r'}|f(x)|^{2r'} dx \right)^{\frac{1}{r'}}.
\]

We choose \( 2r = p \) which implies \( 2r' = \frac{2p}{p-2} \). By the Sobolev embedding theorem we have the continuous embedding

\( f \in H_{1,4}^1(\mathbb{R}^n) \subset L_{2,\delta}^{2p}(\mathbb{R}^n) \),

when \( p > n \), with the norm estimate

\[
\|f\|_{L_{2,\delta}^{2p}(\mathbb{R}^n)} \leq c_0 \|f\|_{H_{1,4}^1(\mathbb{R}^n)}
\]

for some constant \( c_0 > 0 \) depending only on \( p \) and \( n \). Since \( W_{1,23}^1(\mathbb{R}^n) \subset L_{23}^\infty(\mathbb{R}^n) \) we conclude that

\[
\|\tilde{q} \cdot \nabla f + V f\|_{L_2^1(\mathbb{R}^n)} \leq \|\tilde{q}\|_{L_{2,\delta}^\infty(\mathbb{R}^n)} \|\nabla f\|_{L_{2,\delta}^1(\mathbb{R}^n)} + c_0 \|V\|_{L_{23}^\infty(\mathbb{R}^n)} \|f\|_{H_{1,4}^1(\mathbb{R}^n)} < \infty.
\]

Lemma 3.3. Let \( \tilde{q} \in W_{p,23}^1(\mathbb{R}^n) \) and \( V \in L_{23}^p(\mathbb{R}^n) \), where \( 2\delta > n - \frac{2}{p} \) and \( n < p \leq \infty \). If \( u = u_0 + u_{sc}, \ u_{sc} \in H_{loc}^4(\mathbb{R}^n) \cap H_{\delta,4}^2(\mathbb{R}^n) \), solves (2) then

\[
\lim_{R \to \infty} \int_{|y|=R} (|\Delta u_{sc}|^2 + |u_{sc}|^2) d\sigma(y) \leq C
\]

for some constant \( C > 0 \) and for fixed \( k > 0 \).
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Bedenotes the open ball of radius \( R \leq \infty \) then implies the claim. If
\[
\begin{aligned}
&2k \int_{|y|=R} (|\Delta u_{sc}|^2 + k^4 |u_{sc}|^2) d\sigma(y) \\
&= \int_{|y|=R} (\Delta u_{sc}(ik\Delta u_{sc}) - \Delta u_{sc}(ik\Delta u_{sc}) + k^4 \nabla u_{sc}(iku_{sc}) - k^4 u_{sc}(iku_{sc})) d\sigma(y) \\
&= \int_{|y|=R} (\Delta u_{sc} \frac{\partial}{\partial n} \Delta u_{sc} - \Delta u_{sc} \frac{\partial}{\partial n} \Delta u_{sc} + k^4 \nabla u_{sc} \frac{\partial}{\partial n} u_{sc} - k^4 u_{sc} \frac{\partial}{\partial n} u_{sc}) d\sigma(y) \\
&+ o(R^{-\frac{n-1}{2}}) \int_{|y|=R} (\Delta u_{sc} + \Delta u_{sc} + k^4 u_{sc} + k^4 u_{sc}) d\sigma(y).
\end{aligned}
\]

Note that \( \Delta^2 u_{sc} - k^4 u_{sc} = -\vec{q} \cdot \nabla u - Vu \). Therefore
\[
\begin{aligned}
&2k \int_{|y|=R} (|\Delta u_{sc}|^2 + k^4 |u_{sc}|^2) d\sigma(y) \\
&= \int_{|y| \leq R} (\Delta u_{sc}(\vec{q} \cdot \nabla u - Vu) - \Delta u_{sc}(\vec{q} \cdot \nabla u - Vu)) d\sigma(y) \\
&+ o(R^{-\frac{n-1}{2}}) \int_{|y|=R} (\Delta u_{sc} + \Delta u_{sc} + k^4 u_{sc} + k^4 u_{sc}) d\sigma(y).
\end{aligned}
\]

Using the Cauchy-Schwartz inequality in both terms gives
\[
\begin{aligned}
\int_{|y|=R} (|\Delta u_{sc}|^2 + |u_{sc}|^2) d\sigma(y) &\leq C \|u_{sc}\|_{H^2(R^n)} \|\vec{q} \cdot \nabla u + Vu\|_{L^2(R^n)} \\
&+ o(1) \left( \int_{|y|=R} (|\Delta u_{sc}|^2 + |u_{sc}|^2) d\sigma(y) \right)^{\frac{1}{2}},
\end{aligned}
\]

where the first term on the right-hand side is finite according to Lemma 3.1. This
inequality then implies the claim. \( \square \)

**Theorem 3.3.** Let \( \vec{q} \in W^1_{p,2d}(R^n) \) and \( V \in L_{2d}^p(R^n) \) where \( 2d > n - \frac{n}{p} \) and \( n < p \leq \infty \). If \( u = u_0 + u_{sc} \), \( u_{sc} \in H^4_{loc}(R^n) \cap H^2_{2d}(R^n) \), solves (2) then it also solves (3).

**Proof.** Fix a point \( x \in R^n \) and choose \( R > 0 \) so that \( x \in B_R(0) \), where \( B_R(0) \) denotes the open ball of radius \( R \) centered at origin. Let \( \varepsilon > 0 \) be such that \( B_{\varepsilon}(x) \subset B_R(0) \) and denote \( \Omega_{R,\varepsilon} := B_R(0) \setminus B_{\varepsilon}(x) \). Then by Lemma A.1 we calculate
\[
\begin{aligned}
&\int_{\Omega_{R,\varepsilon}} (u_{sc}(y)(\Delta^2_y - k^4)G^+_k(|x-y|) - G^+_k(|x-y|)(\Delta^2 - k^4)u_{sc}(y)) dy \\
&= \int_{\Omega_{R,\varepsilon}} (u_{sc}(y)\Delta^2_y G^+_k(|x-y|) - G^+_k(|x-y|)\Delta^2 u_{sc}(y)) dy \\
&= \int_{\partial \Omega_{R,\varepsilon}} \left[ u_{sc} \frac{\partial}{\partial n} \Delta_y G^+_k(|x-y|) + \Delta u_{sc} \frac{\partial}{\partial n} G^+_k(|x-y|) \right] d\sigma(y).
\end{aligned}
\]
Lemma 4.1. The operator 
\[ -G_k^+ (|x - y|) \frac{\partial }{\partial n} \Delta u_{sc} - \Delta_y G_k^+ (|x - y|) \frac{\partial }{\partial n} u_{sc} \] 
\[ = \int_{\partial \Omega_{R,s}} \left[ u_{sc} \left( \frac{\partial }{\partial n} - ik \right) \Delta_y G_k^+ (|x - y|) + \Delta u_{sc} \left( \frac{\partial }{\partial n} - ik \right) G_k^+ (|x - y|) 
- G_k^+ (|x - y|) \left( \frac{\partial }{\partial n} - ik \right) \Delta u_{sc} - \Delta_y G_k^+ (|x - y|) \left( \frac{\partial }{\partial n} - ik \right) u_{sc} \right] d\sigma(y). \]

Letting \( \varepsilon \to 0+ \) we obtain the equality
\[ u_{sc}(x) = - \int_{\Omega_R} G_k^+ (|x - y|)(\vec{q} \cdot \nabla u + Vu)dy + \int_{\partial \Omega_R} \left[ u_{sc} \left( \frac{\partial }{\partial n} - ik \right) \Delta_y G_k^+ (|x - y|) + \Delta u_{sc} \left( \frac{\partial }{\partial n} - ik \right) G_k^+ (|x - y|) 
- G_k^+ (|x - y|) \left( \frac{\partial }{\partial n} - ik \right) \Delta u_{sc} - \Delta_y G_k^+ (|x - y|) \left( \frac{\partial }{\partial n} - ik \right) u_{sc} \right] d\sigma(y). \]

Note that for fixed \( x \in \mathbb{R}^n \) and large \( R > 0 \) we have \( G_k^+ (|x - y|) = O \left( R^{-\frac{n+1}{2}} \right) \)
(same for \( \Delta G_k^+ \)). Therefore, letting \( R \to +\infty \) we see that the latter integral tends to zero. This follows from Lemma 3.2 and the fact that all \( u_{sc}, \Delta u_{sc}, G_k^+ \) and \( \Delta G_k^+ \) satisfy the radiation condition.

Corollary 3. If \( v \in H^4_{loc}(\mathbb{R}^n) \) satisfies the radiation condition at infinity and fulfils the homogeneous equation \( \Delta^2 v - k^4 v = 0 \), then \( v = 0 \).

4. Solution to Lippmann-Schwinger equation. Next we show that when \( k > 0 \) the integral equation (4) can be solved by iterations defined by \( u^{(j)}_{sc} = L_k^j \tilde{u}_0 \) for \( j = 0, 1, \ldots \). We denote by \( \widehat{G}^+_k \) the convolution operator with kernel \( G_k^+ \). Our first lemma gives the required mapping properties for this integral operator. Similar estimates for Schrödinger operator are familiar in the literature and can be met for example in [7] and [19]. The first lemma is analogous to the limiting absorption principle.

Lemma 4.1. The operator \( \widehat{G}^+_k \) maps from \( L^2_\delta(\mathbb{R}^n) \) to \( H^2_{\delta,j}(\mathbb{R}^n) \) with the estimates
\[ \| \widehat{G}^+_k f \|_{H^2_{\delta,j}(\mathbb{R}^n)} \leq C_0 \frac{k^3}{k^{3-j}} \| f \|_{L^2_{\delta}(\mathbb{R}^n)}, \quad j = 0, 1, 2 \]
when \( k > 1 \) and \( \delta > \frac{1}{2} \) for \( n \geq 2 \). Here the constant \( C_0 \) only depends on \( n \) and \( \delta \).

Proof. The claim follows from Agmon’s estimate [1, Appendix A, Remark 2] for \( p = 2, \delta > \frac{1}{2} \) and \( k > 1 \) given by
\[ \sum_{|\alpha| \leq 4} k^{3-|\alpha|} \| D^\alpha f \|_{L^2_{\delta,j}(\mathbb{R}^n)} \leq C_0 \| (\Delta^2 - k^4) f \|_{L^2_{\delta}(\mathbb{R}^n)} \]
for any \( f \in H^4(\mathbb{R}^n) \), where the constant \( C_0 > 0 \) only depends on \( n \) and \( \delta \). We use this result to the opposite direction and take only the terms where \( |\alpha| \) equals 0,1 or 2 to get the claim.

We now proceed to prove norm estimates for the operator \( L_k \).

Theorem 4.2. Let \( \vec{q} \in W^{1,2\delta}(\mathbb{R}^n) \) and \( V \in L^p_{2\delta}(\mathbb{R}^n) \) where \( 2\delta > n - \frac{n}{p} \) and \( n < p \leq \infty \). Then the following properties are satisfied.
1. The function $\tilde{u}_0$ belongs to $H^2_\delta(\mathbb{R}^n)$ with the estimates
   $$\|\tilde{u}_0\|_{H^j_\delta(\mathbb{R}^n)} \leq \frac{C_1}{k^{2-j}}, \quad j = 0, 1, 2$$

2. The operator $L_k : H^1_\delta(\mathbb{R}^n) \to H^2_\delta(\mathbb{R}^n)$ is bounded and satisfies the norm estimates
   $$\|L_k f\|_{H^j_\delta(\mathbb{R}^n)} \leq \frac{C_2}{k^{2-j}} \|f\|_{H^j_\delta(\mathbb{R}^n)}, \quad j = 0, 1, 2$$
   for $k > 1$ and for some $C_1, C_2 > 0$ independent of $k$.

**Proof.** In view of Lemma 4.1 we consider only the case $j = 0$ and the estimates for derivatives follow. We apply Lemma 4.1 to estimate
   $$\|\tilde{u}_0\|_{L^2_\delta(\mathbb{R}^n)} = \|\tilde{G}^n_k(\tilde{q} \cdot \nabla u_0 + Vu_0)\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C_0}{k^2} \|\tilde{q} \cdot \nabla u_0 + Vu_0\|_{L^2_\delta(\mathbb{R}^n)}$$
   $$\leq \frac{C_0}{k^3} \left( k\|\tilde{q}\|_{L^2_\delta(\mathbb{R}^n)} + \|V\|_{L^2_\delta(\mathbb{R}^n)} \right) \leq \frac{C_1}{k^3},$$
   where the constant $C_0 > 0$ comes from Lemma 4.1 and $C_1 := C_0(\|\tilde{q}\|_{L^2_\delta(\mathbb{R}^n)} + \|V\|_{L^2_\delta(\mathbb{R}^n)})$. Next, suppose that $f \in H^2_\delta(\mathbb{R}^n)$. By Lemmata 3.1 and 4.1 we see that
   $$\|L_k f\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C_0}{k^3} \left( \|\tilde{q}\|_{L^2_\delta(\mathbb{R}^n)} \|\nabla f\|_{L^2_\delta(\mathbb{R}^n)} + c_0 \|V\|_{L^2_\delta(\mathbb{R}^n)} \|f\|_{H^2_\delta(\mathbb{R}^n)} \right)$$
   $$\leq \frac{C_2}{k^3} \|f\|_{H^2_\delta(\mathbb{R}^n)},$$
   where $C_2 := C_0(\|\tilde{q}\|_{L^2_\delta(\mathbb{R}^n)} + c_0 \|V\|_{L^2_\delta(\mathbb{R}^n)})$. The proof is then finished.

**Corollary 4.** Let $\tilde{q} \in W^1_{p,2\delta}(\mathbb{R}^n)$ and $V \in L^p_{2\delta}(\mathbb{R}^n)$ where $n < p \leq \infty$ and $2\delta > n - \frac{n}{p}$. Then there exists a constant $k_0 > 1$ such that the function $u_{sc}(x, k, \theta)$ defined by the series
   $$u_{sc}(x, k, \theta) = \sum_{j=0}^{\infty} L^j_k \tilde{u}_0(x, k, \theta)$$
   solves integral equation (4) uniquely in $H^2_\delta(\mathbb{R}^n)$, when $k \geq k_0$. Moreover,
   $$\|u_{sc}\|_{H^1_\delta(\mathbb{R}^n)} \leq \frac{C}{k},$$
   when $k \geq k_0$ is large enough.

**Proof.** We show that
   $$\|L^j_k \tilde{u}_0\|_{H^j_\delta(\mathbb{R}^n)} \leq \frac{C_1(C_2)^j}{k^{2j+1}}, \quad j = 0, 1, \ldots$$
   By Theorem 4.2 the claim holds for $j = 0$. Suppose then that the claim is proved for some $j > 0$. By induction hypothesis
   $$\|L^{j+1}_k \tilde{u}_0\|_{H^{j+1}_\delta(\mathbb{R}^n)} \leq \frac{C_2}{k^2} \|L^j_k \tilde{u}_0\|_{H^j_\delta(\mathbb{R}^n)} \leq \frac{C_2}{k^2} \cdot \frac{C_1(C_2)^j}{k^{2j+1}} = \frac{C_1(C_2)^{j+1}}{k^{2(j+1)+1}}$$
   so the claim holds for all $j = 0, 1, \ldots$. We may now choose any $k_0 > \max\{1, \sqrt{C_2}\}$ to conclude that the series (6) converges in $H^1_\delta(\mathbb{R}^n)$. Because the operator $L_k$ is linear and maps continuously in $H^1_\delta(\mathbb{R}^n)$ the series solves (4). Similar calculation shows that $u_{sc} \in H^2_\delta(\mathbb{R}^n)$. 

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Our assumption \( k \geq k_0 > \sqrt{C_2} \) means that \( \frac{C_2}{k^2} \leq \frac{C_2}{k_0^2} < 1 \) and therefore \( \tilde{u} = \tilde{v} \) in \( H^2_{-\delta}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \).

With these results the limiting absorption principle can be extended for the full operator \( H_4 \).

**Corollary 5.** Let \( \bar{q} \in W^{1,1}_{p,2\delta}(\mathbb{R}^n) \) and \( V \in L^p_{2\delta}(\mathbb{R}^n) \) with \( n < p \leq \infty \) and \( 2\delta > n - \frac{n}{p} \). Then the operator

\[
\hat{G}_p := \lim_{\varepsilon \to 0^+} (H_4 - k^4 - i\varepsilon)^{-1}
\]

exists in the uniform operator topology from \( L^2_\delta(\mathbb{R}^n) \) to \( H^1_{-\delta}(\mathbb{R}^n) \) with the norm estimates

\[
\|\hat{G}_p f\|_{H^1_{-\delta}(\mathbb{R}^n)} \leq \frac{C}{k^{3-j}} \|f\|_{L^2_\delta(\mathbb{R}^n)}, \quad j = 0, 1,
\]

for \( k > 0 \) large enough.

**Proof.** Consider the equation \( \hat{G}_p f = \hat{G}^+_k f - \hat{G}^+_k (\bar{q} \cdot \nabla + V) \hat{G}_p f \) in \( H^1_{-\delta}(\mathbb{R}^n) \). By (formally) applying this equation repeatedly we obtain

\[
\hat{G}_p f = \hat{G}^+_k f + \sum_{j=1}^{\infty} \left(-\hat{G}^+_k (\bar{q} \cdot \nabla + V)\right)^j \hat{G}^+_k f = \hat{G}^+_k f + \sum_{j=1}^{\infty} L^j_k \hat{G}^+_k f.
\]

The convergence of this series in \( H^1_{-\delta} \) follows from the estimates of Theorem 4.2 and the norm estimates follow from those of \( \hat{G}^+_k \), so there exists a unique solution. Then it is straightforward to confirm that \( \hat{G}_p \) satisfies \( (H_4 - k^4) \hat{G}_p = I \).

**Remark 2.** The previous proof yields a useful identity

\[
u_{sc} = -\hat{G}^+_k (\bar{q} \cdot \nabla u + Vu) = -\hat{G}_p (\bar{q} \cdot \nabla u_0 + Vu_0).
\]

**Proposition 1.** Suppose \( n = 2 \) or \( 3 \) and let \( \bar{q} \in W^{1,1}_{p,2\delta}(\mathbb{R}^n) \) and \( V \in L^p_{2\delta}(\mathbb{R}^n) \), with \( 2\delta > n - \frac{n}{p} \) and \( n < p \leq \infty \). Then the operator \( L_k \) maps \( W^{1}_{\infty}(\mathbb{R}^n) \) to \( W^{1}_{\infty}(\mathbb{R}^n) \) with the estimates

\[
\|L_k f\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{k^3} \|f\|_{W^{1}_{\infty}(\mathbb{R}^3)} \text{ and } \|
abla L_k f\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{k} \|f\|_{W^{1}_{\infty}(\mathbb{R}^n)}
\]

for \( k \geq 1 \).

**Proof.** We start from the case \( n = 3 \) and choose some \( f \in W^{1}_{\infty}(\mathbb{R}^3) \). Then a direct estimation by using Remark 1 yields

\[
|L_k f(x)| \leq \frac{1}{4\pi k^2} \int_{\mathbb{R}^3} \frac{|\bar{q}(y)|}{|x-y|} \|f\|_{L^\infty(\mathbb{R}^3)} + \frac{1}{4\pi k^2} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} \|f\|_{L^\infty(\mathbb{R}^3)}.
\]
For the gradient of $L_k f$ we have
\[
\frac{\partial}{\partial x_i} L_k f(x) = -\int_{\mathbb{R}^3} \frac{x_i - y_i}{8\pi k^2 |x - y|^3} \left( (ik|x - y| - 1)e^{ik|x - y|} + (k|x - y| + 1)e^{-ik|x - y|} \right) [\vec{q}(y) \cdot \nabla f(y) + V(y) f(y)] \, dy,
\]
so that
\[
|\nabla L_k f(x)| \leq \frac{\sqrt{3}}{4\pi k} \int_{\mathbb{R}^3} \frac{|\vec{q}(y)|}{|x - y|} \|\nabla f\|_{L^\infty(\mathbb{R}^3)} + \frac{\sqrt{3}}{4\pi k} \int_{\mathbb{R}^3} |V(y)| \|f\|_{L^\infty(\mathbb{R}^3)}
\]
\[
+ \frac{\sqrt{3}}{4\pi k^2} \int_{\mathbb{R}^3} \frac{|\vec{q}(y)|}{|x - y|^2} \|\nabla f\|_{L^\infty(\mathbb{R}^3)} + \frac{\sqrt{3}}{4\pi k^2} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|^2} \|f\|_{L^\infty(\mathbb{R}^3)}.
\]
These integrals converge uniformly in $x$ when $n < p \leq \infty$, which yields the claim in $\mathbb{R}^3$.

Consider next the case $n = 2$. Suppose that $f \in W^1_\infty(\mathbb{R}^2)$. Then by using the asymptotic representation obtained in Section 2 we get
\[
|L_k f(x)| \leq \frac{C}{k^2} \int_{k|x - y| < 1} (1 + |\log(k|x - y|)|) |\vec{q} \cdot \nabla f + V f| \, dy
\]
\[
+ \frac{C}{k^2} \int_{k|x - y| > 1} \frac{|\vec{q} \cdot \nabla f + V f|}{\sqrt{k|x - y|}} \, dy =: I_1 + I_2.
\]
Since $\vec{q}, V \in L^1(\mathbb{R}^2)$ then $I_2$ satisfies the claim. In the first integral we may use Hölder’s inequality to get
\[
I_1 \leq \frac{C}{k^2} \left( \|\vec{q}\|_{L^p(\mathbb{R}^2)} \|\nabla f\|_{L^\infty(\mathbb{R}^2)} + \|V\|_{L^p(\mathbb{R}^2)} \|f\|_{L^\infty(\mathbb{R}^2)} \right)
\]
\[
\times \left( \int_{k|x - y| < 1} (1 + |\log(k|x - y|)|)^{p'} \, dy \right)^\frac{1}{p'}
\]
\[
= \frac{C}{k^{2 + \frac{p'}{2}}} \|f\|_{W^1_\infty(\mathbb{R}^2)} \left( \int_{|z| < 1} (1 + |\log(|z|)|)^{p'} \, dz \right)^\frac{1}{p'}.
\]
The above integral converges when $1 \leq p' < \infty$, that is, $1 < p \leq \infty$, and therefore the claim holds also for $I_1$. The gradient of $L_k f$ can be estimated straight-forwardly as
\[
|\nabla L_k f(x)| \leq \frac{C}{k^2} \int_{k|x - y| < 1} \frac{|\vec{q} \cdot \nabla f + V f|}{|x - y|} \, dy
\]
\[
+ \frac{C}{k} \int_{k|x - y| > 1} \frac{|\vec{q} \cdot \nabla f + V f|}{\sqrt{k|x - y|}} \, dy =: J_1 + J_2.
\]
Again, $J_2$ clearly satisfies the claimed estimate. In the term $J_1$ Hölder’s inequality yields
\[
J_1 \leq \frac{C}{k^2} \|f\|_{W^1_\infty(\mathbb{R}^2)} \left( \int_{k|x - y| < 1} \frac{1}{|x - y|^{p'}} \, dy \right)^\frac{1}{p'}
\]
\[
= \frac{C}{k^{1 + \frac{p'}{2}}} \|f\|_{W^1_\infty(\mathbb{R}^2)} \left( \int_{|z| < 1} \frac{1}{|z|^{p'}} \, dz \right)^\frac{1}{p'}.
\]
This integral converges when $1 \leq p' < 2$, or $2 < p \leq \infty$, concluding the proof.
Corollary 6. Let \( n = 2 \) or \( 3 \) and \( \vec{q} \in W^{1,2}_{p,2\delta}(\mathbb{R}^n) \), \( V \in L^p_{2\delta}(\mathbb{R}^n) \) with \( 2\delta > n + \frac{n}{p} \) and \( n < p \leq \infty \). Then there exists \( k'_0 > 0 \) such that the unique solution \( u \) to (3) is bounded uniformly in \( k \geq k'_0 \).

Proof. The proof is essentially the same as the proof of Corollary 4, but uses the estimates of Proposition 1. \( \square \)

5. Asymptotics for Lippmann-Schwinger equation. Let us first prove an elementary fact about an integral that appears in the sequel.

Lemma 5.1. Let \( \psi \in L^1(\mathbb{R}^n) \) and \( 0 < a < 1 \). Then
\[
e^{-k|x|} \int_{|y| \leq |x|^a} e^{k(|\theta'|,y)} \psi(y)dy = o(1), \quad |x| \to +\infty
\]
for all \( \theta' \in \mathbb{S}^{n-1} \) and fixed \( k > 0 \).

Proof. For all \( \theta' \in \mathbb{S}^{n-1} \) and \( y \in \mathbb{R}^n \) it holds that \( (\theta', y) \leq |(\theta', y)| \leq |y| \). Therefore \( e^{k(|\theta'|,y)} \leq e^{k|y|} \). Then we estimate the integral
\[
\left| e^{-k|x|} \int_{|y| \leq |x|^a} e^{k(|\theta'|,y)} \psi(y)dy \right| \leq e^{-k|x|} \int_{|y| \leq |x|^a} e^{k|y|} \psi(y)dy \\
\leq e^{-k|x|} e^{k|x|^a} \int_{|y| \leq |x|^a} \psi(y)dy \\
\leq e^{-k|x|} e^{k|x|^a} \| \psi \|_{L^1(\mathbb{R}^n)}.
\]
Since \( 0 < a < 1 \) then \( e^{-k|x|} e^{k|x|^a} = e^{-k(|x|-|x|^a)} = o(1), \quad |x| \to +\infty \).

This concludes the proof. \( \square \)

Next we study the asymptotic behaviour of the solution \( u \) obtained in the previous section.

Theorem 5.2. Let \( \vec{q} \in W^{1,2}_{p,2\delta}(\mathbb{R}^n) \) and \( V \in L^p_{2\delta}(\mathbb{R}^n) \) with \( 2\delta > n - \frac{n}{p} \) and \( n < p \leq \infty \). Then for fixed \( k > 0 \) the solution \( u(x, k, \theta) \) to (3) admits the representation
\[
u(x, k, \theta) = e^{ik(x, \theta)} - C_n k^\frac{n-2}{2} e^{k|x|} |x|^{n-4} A(k, \theta, \theta') + o \left( \frac{1}{|x|^{n-2}} \right), \quad |x| \to \infty,
\]
if \( n = 2 \) or \( 3 \). The same representation holds for any \( n \geq 4 \) if we assume in addition that \( \vec{q} \) and \( V \) are compactly supported. The constant \( C_n \) is given by
\[
C_n := \frac{ie^{\frac{i(n+2)}{2} \pi}}{4(2\pi)^{\frac{n+4}{2}}}
\]
The function \( A(k, \theta', \theta) \) is called the scattering amplitude and is defined by
\[
A(k, \theta, \theta') = \int_{\mathbb{R}^n} e^{-ik(|\theta'|, y)} \left[ \vec{q} \cdot \nabla u + Vu \right] dy,
\]
where \( \theta' = \frac{\vec{r}}{|\vec{r}|} \) is the direction of observation.
Proof. We begin the proof along the lines of [10, Lemma 3.1]. Let $0 < a < 1$ be a parameter, so that we can decide its value to suit our interests later. Since

$$u_{sc}(x, k, \theta) = -\int_{R^n} G_k^+ (|x - y|) [q \cdot \nabla u + Vu] \, dy$$

we can divide the region of integration in two parts as $|y| \leq |x|^a$ and $|y| > |x|^a$. In the former case we apply the Maclaurin series $(1 + \omega)^s = 1 + s\omega + O(\omega^2)$ to see that

$$|x - y| = |x|^2 - 2(x, y) + |y|^2|^{1/2}$$

$$= |x| \left[ 1 - \frac{(x, y)}{|x|^2} + \frac{|y|^2}{2|x|^2} + O \left( \left\{ -\frac{2(x, y)}{|x|^2} + \frac{|y|^2}{|x|^2} \right\}^2 \right) \right]$$

$$= |x| - (\theta', y) + \frac{|y|^2}{2|x|} + O(|x|^{2a-1})$$

Similarly

$$|x - y|^{-\frac{a}{2-1}} = |x|^2 - 2(x, y) + |y|^2|^{-\frac{a}{2-1}}$$

$$= |x|^{-\frac{a}{2-1}} \left[ 1 + \frac{n-1}{2} \frac{(x, y)}{|x|^2} - \frac{n-1}{4} \frac{|y|^2}{|x|^2} + O \left( \left\{ -\frac{2(x, y)}{|x|^2} + \frac{|y|^2}{|x|^2} \right\}^2 \right) \right]$$

$$= |x|^{-\frac{a}{2-1}} \left[ 1 + O \left( |x|^{a-1} \right) + O \left( |x|^{2a-2} \right) \right]$$

and $|x - y|^{-1} = |x|^{-1} \left[ 1 + O(|x|^{a-1}) \right]$. Then, for fixed $k > 0$ and $|y| \leq |x|^a$ we take $a < \frac{1}{2}$. Because $|y| \leq |x|^a \leq |x| \rightarrow +\infty$ then we may assume that $k|x - y| > 1$. This allows us to use the large argument asymptotic behaviour of Hankel and Macdonald functions and the Maclaurin expansion $e^{\omega} = 1 + O(\omega)$ to compute

$$G_k^+ (|x - y|) = \frac{i}{8k^2} \left( \frac{k}{2\pi|x - y|} \right)^{-\frac{a}{2-1}} \left( H_{\frac{a}{2}} (k|x - y|) + \frac{2i}{\pi} K_{\frac{a}{2}} (k|x - y|) \right)$$

$$= \frac{i}{8k^2} \left( \frac{k}{2\pi|x - y|} \right)^{-\frac{a}{2-1}} \left( \sqrt{\frac{2}{\pi k|x - y|}} \epsilon^{2(k|x - y| - \frac{a}{2-1}) - \frac{a}{2-1} x} \right)$$

$$+ C \sqrt{\frac{\pi}{2k|x - y|}} e^{-k|x - y|} + O \left( |x - y|^{-\frac{a}{2-1}} \right)$$

$$= C_n k^{-\frac{a}{2-1}} |x|^{-\frac{a}{2-1}} \left[ 1 + O(|x|^{a-1}) \right] \left( e^{ik|x|} e^{-ik(\theta', y)} e^{-ikO(|x|^{2a-1})} \right.$$

$$+ C e^{-k|x|} e^{k(\theta', y)} e^{kO(|x|^{2a-1})} + O(|x - y|^{-1}) \right)$$

$$= C_n k^{-\frac{a}{2-1}} |x|^{-\frac{a}{2-1}} \left( e^{ik|x|} e^{-ik(\theta', y)} + C e^{-k|x|} e^{k(\theta', y)} \right) + O(|x|^{2a-1 - \frac{a}{2-1}}).$$

Let us apply these estimates to $u_{sc}$. We have

$$u_{sc}(x, k, \theta) = -\int_{|y| \leq |x|^a} G_k^+ (|x - y|) [q \cdot \nabla u + Vu] \, dy$$
Now integral \( I_1 \) equals

\[
I_1 = -C_n \frac{k^{\frac{n-2}{2}} e^{ik|x|}}{|x|^\frac{n-2}{2}} \int_{\mathbb{R}^n} e^{-ik(|\theta'|, y)} [\bar{q} \cdot \nabla u + Vu] \, dy \\
+ C_n \frac{k^{\frac{n-2}{2}} e^{ik|x|}}{|x|^\frac{n-2}{2}} \int_{|y|>|x|^\alpha} e^{-ik(|\theta'|, y)} [\bar{q} \cdot \nabla u + Vu] \, dy \\
+ C_n \frac{k^{\frac{n-2}{2}} e^{-k|x|}}{|x|^\frac{n-2}{2}} \int_{|y|>|x|^\alpha} e^{k(|\theta'|, y)} [\bar{q} \cdot \nabla u + Vu] \, dy
\]

As \( |x| \to \infty \) the second integral above is \( o\left(|x|^{-\frac{n-1}{2}}\right) \), since \( \bar{q} \cdot \nabla u \) and \( Vu \) are integrable. On the other hand, by Lemma 5.1 the third term in this expression is also \( o\left(|x|^{-\frac{n-1}{2}}\right) \).

Next we consider \( I_2 \). We first split the region of integration as

\[
|I_2| \leq \int_{|y|>|x|^\alpha} |G_k^+ (|x-y|) [\bar{q} \cdot \nabla u + Vu]| \, dy \\
= \int_{|x|^\alpha < |y| < |x|^\alpha} |G_k^+ (|x-y|)||\bar{q} \cdot \nabla u + Vu| \, dy \\
+ \int_{|y| \geq \frac{|x|^\alpha}{2}} |G_k^+ (|x-y|)||\bar{q} \cdot \nabla u + Vu| \, dy : =: J_1 + J_2.
\]

In the case \( J_1 \) we have \( |x-y| \geq |x| - |y| > \frac{|x|}{2} \). Thus as \( |x| \to +\infty \) by the asymptotic of \( G_k^+ \) we have

\[
J_1 \leq \frac{C}{|x|^\frac{n-1}{2}} \int_{|x|^\alpha < |y| < \frac{|x|^\alpha}{2}} |\bar{q} \cdot \nabla u + Vu| \, dy = o\left(|x|^{-\frac{n-1}{2}}\right).
\]

Note that so far we have only used the fact that \( \bar{q} \in W_{1,2}^1(\mathbb{R}^n) \) and \( V \in L_2^p(\mathbb{R}^n) \), \( n \geq 2 \). If these coefficients are compactly supported, then choosing large enough \( |x| \) immediately yields \( J_2 = 0 \), giving the claimed asymptotic representation. On the other hand, even if the coefficients are not compactly supported, in dimensions \( n = 2, 3 \) we know that the solution \( u \) to (3) is in \( W_{2,1}^1(\mathbb{R}^n) \).

We study first the case \( n = 3 \). By Remark 1 we have for a fixed \( k > 0 \) that

\[
J_2 \leq C \int_{|y| \geq \frac{|x|}{2}} \frac{|\bar{q} + |V|}{|x-y|} \, dy \\
\leq C \left( \int_{|y| \geq \frac{|x|}{2}} \frac{(1 + |y|^{-2\delta p'})}{|x-y|^{-p'}} \, dy \right)^{1/p'} \left( \|\bar{q}\|_{L_2^p(\mathbb{R}^3)} + \|V\|_{L_2^p(\mathbb{R}^3)} \right)^{1/p'} \\
\leq C \left( \int_{|z| \geq \frac{1}{2}} \frac{(1 + |z|^2)^{-2\delta p'}}{|z|^{-p'} |x/y|^{-p'} |z|^3} \, dz \right)^{1/p'} \\
\leq C |x|^{\frac{3}{p'} - 2\delta - 1} \left( \int_{|z| \geq \frac{1}{2}} \frac{|z|^{-2\delta p'}}{|x/y|^{-p'} - z^{-p'}} \, dy \right)^{1/p'}.
\]
where we did the change of variables \( y = z|x| \). Dividing the region of integration to 
\( 2 > |z| > \frac{1}{2} \) and \( |z| > 2 \) we get
\[
\int_{2>|z|>\frac{1}{2}} \frac{|z|^{-2\delta p'}}{|(x/|x|) - z|^{p'}} \, dz \leq C \int_{2>|z|>\frac{1}{2}} \frac{1}{|(x/|x|) - z|^{p'}} \, dz < \infty,
\]
because \( p' < n/2 \). In the region \( |z| > 2 \) we have \( |(x/|x|) - z| \geq |z| - 1 > |z|/2 \) and therefore
\[
\int_{|z|>2} \frac{|z|^{-2\delta p'}}{|(x/|x|) - z|^{p'}} \, dz \leq C \int_{|z|>2} \frac{1}{|z|^{2\delta p' + p'}} \, dz < \infty.
\]
Recalling that \( \frac{2}{p'} - 2\delta < 0 \) we conclude that \( J'_2 \) is \( o \left( \frac{1}{|z|} \right) \).

In the case \( n = 2 \) we use the asymptotic behaviour of \( G^+_k \). We consider the regions \( k|x - y| < 1 \) and \( k|x - y| > 1 \) separately, with the respective integrals denoted by \( J''_2 \) and \( J''_{2'} \). In the first case
\[
J''_2 \leq C \int_{|y| \geq \frac{|x|}{2} \atop k|x - y| < 1} \frac{|y| + |V|}{|x - y|^{\frac{2}{2}}} \, dy
\]
\[
\leq C \left( \int_{|y| \geq \frac{|x|}{2} \atop k|x - y| < 1} (1 + |k|x - y|))^{p'} (1 + |y|)^{-2\delta p'} \, dy \right)^{\frac{1}{p'}}
\]
\[
\leq C|x|^{-2\delta} \left( \int_{|y| < 1} (1 + |z|)^{p'} \, dz \right)^{\frac{1}{p'}} = o(|z|^{-1}).
\]
In the second case \( J''_{2'} \) we pick some \( \sigma > 0 \) such that \( 0 < 2\delta p' - 2 < \sigma < \frac{2}{p'} + 2\delta p' - 2 \). Then
\[
J''_{2'} \leq C \int_{|y| \geq \frac{|x|}{2} \atop k|x - y| > 1} \frac{|y| + |V|}{|x - y|^{\frac{2}{2}}} \, dy
\]
\[
\leq C \left( \int_{|y| \geq \frac{|x|}{2} \atop k|x - y| > 1} \frac{(1 + |y|)^{-2\delta p'}}{|x - y|^{\frac{2}{2}}} \, dy \right)^{\frac{1}{p'}}
\]
\[
\leq C|x|^{-2\delta} \left( \int_{|y| \geq \frac{|x|}{2} \atop k|x - y| > 1} \frac{|y|^{-2\delta p' + \sigma}}{|x - y|^{\frac{2}{2}}} \, dy \right)^{\frac{1}{p'}}.
\]
By changing the variables \( y/|x| = z \) in the above integral we get
\[
\left( \int_{|y| \geq \frac{|x|}{2} \atop k|x - y| > 1} \frac{|y|^{-2\delta p' + \sigma}}{|x - y|^{\frac{2}{2}}} \, dy \right)^{\frac{1}{p'}}
\]
\[
= |x|^{-\frac{1}{2} - 2\delta + \frac{\sigma}{p''} + \frac{2}{p'}} \left( \int_{|z| \geq \frac{1}{2} \atop k|x - |x|| > 1} \frac{|z|^{-2\delta p' + \sigma}}{|(x/|x|) - z|^{p'}} \, dz \right)^{\frac{1}{p'}}.
\]
If the above integral converges, we have that \( J''_{2'} = o \left( |z|^{-\frac{1}{2}} \right) \), since \( -2\delta + \frac{\sigma}{p''} + \frac{2}{p'} < 0 \).
And indeed, the integral converges because

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Remark 3. In similar spirit as in the proof of Theorem 5.2 one can confirm that $u_{sc}$ and $\Delta u_{sc}$ satisfy the radiation condition at infinity at least if $\vec{q}$ and $V$ are smooth and compactly supported. For such coefficients, in light of Section 3 we can then conclude that the scattering problem (2) and Lippmann-Schwinger equation (3) are equivalent.

6. Proof of Theorem 1.1. Note that (unlike the proof for the asymptotic formula of $u$) this theorem does not require us to assume that $\vec{q}$ and $V$ have compact supports.

Proof. Let us denote

$$I := k^{n-1} \int_{\mathbb{R}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta - \theta', x)} A(k, \theta, \theta') d\theta d\theta'.$$

Then, because $u = u_0 + u_{sc}$, we can write

$$I = k^{n-1} \int_{\mathbb{R}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} [ik \cdot \vec{q} + V] dy d\theta d\theta' + k^{n-1} \int_{\mathbb{R}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} [\vec{q} \cdot \nabla u_{sc} + Vu_{sc}] dy d\theta d\theta' =: I_1 + I_2.$$

Consider $I_1$ first. We can further split $I_1$ in two terms as

$$I_1 = k^{n-1} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} ik \cdot \vec{q}(y) dy d\theta d\theta' + k^{n-1} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} V(y) dy d\theta d\theta' =: I_1^{(1)} + I_1^{(2)}.$$

In $I_1^{(1)}$ we can first split the integral into two parts and then integrate the first one by parts and do a change of variables $\theta = -\gamma'$ and $\theta' = -\gamma$ in the second term as follows

$$I_1^{(1)} = k^{n-1} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} ik(\theta - \theta') \cdot \vec{q}(y) dy d\theta d\theta' + k^{n-1} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} ik \theta' \cdot \vec{q}(y) dy d\theta d\theta'.$$
\[
= -k^{n-1} \int_{S^{n-1} \times S^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} \nabla \cdot \vec{q}(y) dy d\theta d\theta' \\
- k^{n-1} \int_{S^{n-1} \times S^{n-1}} e^{-ik(-\gamma' + \gamma, x)} \int_{\mathbb{R}^n} e^{-ik(-\gamma + \gamma', y)} i k \gamma \cdot \vec{q}(y) dy d\gamma',
\]

Here \( \nabla \cdot \vec{q} \) denotes the divergence of \( \vec{q} \). The integration by parts is valid, since \( \vec{q} \in W^1_{p,2}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap W^1_p(\mathbb{R}^n) \subset C(\mathbb{R}^n) \) when \( n < p \leq \infty \) (see Theorem A.2).

But now the last integral is equal to \(-I_1^{(1)}\), so that
\[
I_1^{(1)} = -k^{n-1} \int_{S^{n-1} \times S^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} \frac{1}{2} \nabla \cdot \vec{q}(y) dy d\theta d\theta'.
\]

and
\[
I_1 = k^{n-1} \int_{S^{n-1} \times S^{n-1}} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^n} e^{-ik(\theta' - \theta, y)} \left(-\frac{1}{2} \nabla \cdot \vec{q}(y) + V(y)\right) dy d\theta d\theta'.
\]

An integral of this form was studied in [10] and we carry out the analogous analysis here. It is known that
\[
\int_{S^{n-1}} e^{ik(\theta, x - y)} d\theta = (2\pi)^\frac{n}{2} \frac{J_{\frac{n-2}{2}}(k|x - y|)}{(k|x - y|)^{\frac{n-2}{2}}},
\]

and one proof for this fact can be found for example from [10, Lemma 3.6]. Here \( J_\nu(x) \) is the Bessel function of first kind and of order \( \nu \). From [16] we find that
\[
J_\nu(x) = \begin{cases} 
O(x^{\nu}), & x \to 0, \\
\sqrt{\frac{2}{\pi x}} \cos(x - \frac{1}{2} \nu \pi - \frac{1}{4} \pi) + O(x^{-\frac{4}{2}}), & x \to \infty.
\end{cases}
\]

We split the region of integration as
\[
I_1 = (2\pi)^n k^{n-1} \int_{|x - y| < 1} \frac{J_{\frac{n-2}{2}}^2(k|x - y|)}{(k|x - y|)^{n-2}} \beta(y) dy + (2\pi)^n k^{n-1} \int_{|x - y| > 1} \frac{J_{\frac{n-2}{2}}^2(k|x - y|)}{(k|x - y|)^{n-2}} \beta(y) dy =: I_1' + I_1''.
\]

The term \( I_1' \to 0 \) as \( k \to +\infty \). Indeed,
\[
|I_1'| \leq C k^{n-1} \int_{|x - y| < 1} \frac{(k|x - y|)^{n-2}}{(k|x - y|)^{n-2}} |\beta(y)| dy \leq k^{n-1-\frac{n}{p}} \|\beta\|_{L^p(\mathbb{R}^n)} \left( \int_{|z| < 1} \frac{1}{y^p} \right),
\]

where \( n - 1 - \frac{n}{p} = \frac{n}{p} - 1 < 0 \) when \( p > n \). The term \( I_1'' \) converges to the claimed limit of this theorem as \( k \to +\infty \). To see this we utilize the large argument behaviour of Bessel function to get
\[
I_1'' = (2\pi)^n k \int_{k|x - y| > 1} \beta(y) \left( \frac{2}{\pi k|x - y|} \cos \left( k|x - y| - \frac{n - 1}{4} \pi \right) \right)
\]

Here the first integral gives the claimed limit as $k \to +\infty$, since
\[ \int_{\mathbb{R}^n} \frac{|\beta(y)|}{|x - y|^{n-1}} dy < \infty \]
uniformly in $x$ when $p > n$. The second integral above gives
\[
\int_{k|x - y| > 1} \frac{\beta(y)}{|x - y|^{n-1}} \cos \left( \frac{2k|x - y| - \frac{n-1}{2}\pi}{\beta(y)} \right) dy \\
= \int_{\mathbb{R}^n} \frac{\beta(y)}{|x - y|^{n-1}} \cos \left( \frac{2k|x - y| - \frac{n-1}{2}\pi}{\beta(y)} \right) dy \\
- \int_{|x - y| < \frac{1}{k}} \frac{\beta(y)}{|x - y|^{n-1}} \cos \left( \frac{2k|x - y| - \frac{n-1}{2}\pi}{\beta(y)} \right) dy =: I_1^* + I_1^{**}.
\]
The latter integral $I_1^{**} \to 0$ as $k \to +\infty$, while the former integral requires little more effort. We do the change of variables $y = x - z$ and then swap to polar coordinates to obtain
\[ I_1^* = \int_0^\infty \int_{S^{n-1}} \beta(x - r\gamma) \cos(2kr - \frac{n-1}{2}\pi) d\gamma dr. \]
Define
\[ f_x(r) := \int_{S^{n-1}} \beta(x - r\gamma) d\gamma. \]
Now it suffices to conclude that $f_x \in L^1(0, \infty)$ uniformly in $x$ and then apply the Riemann-Lebesgue lemma to conclude that $I_1^* \to 0$ as $k \to +\infty$. We calculate the $L^1$-norm of $f_x$ to see that
\[ \int_0^\infty |f_x(r)| dr = \int_0^\infty \int_{S^{n-1}} |\beta(x - r\gamma)| d\gamma dr \\
= \int_{\mathbb{R}^n} \frac{|\beta(y)|}{|x - y|^{n-1}} dy < \infty \]
uniformly in $x$. To finish the consideration of $I_1^{**}$ we only need to show that
\[ \frac{1}{k} \int_{k|x - y| > 1} \frac{\beta(y)}{|x - y|^{n-1}} dy \to 0. \]
as $k \to +\infty$. Let $\varepsilon > 0$ be a parameter whose value will be chosen later. Estimate
\[
\frac{1}{k} \int_{|k(x-y)|>1} \frac{|\beta(y)|}{|x-y|^n} \, dy \leq k^{-1} \int_{|k(x-y)|>1} \frac{|\beta(y)|}{|x-y|^{n-\varepsilon}} \, dy
\]
\[
\leq k^{-1} \left( \int_{|z|>\frac{1}{k}} \frac{1}{|z|^{(n-\varepsilon)p'}} \, dz \right) \left\| \beta \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{p'}}
\]
\[
= Ck^{-1} \left( \int_{\frac{1}{k}}^{\infty} r^{-(n-\varepsilon)p'+n-1} \, dr \right) \cdot
\]
\[
= Ck^{-1} \left( -\left. r^{-(n-\varepsilon)p'+n} \right|_{r=\frac{1}{k}}^{\infty} \right) = Ck^{\frac{n}{p'}}^{-1},
\]
where we required $\varepsilon < \frac{n}{p'}$. This estimate shows that all of the terms of $I''_1$ tend to 0 as $k \to \infty$.

Next we confirm that the term $I_2 \to 0$ as $k \to \infty$. To achieve this we calculate the integrals with respect to $\theta$ and $\theta'$ as in [18] and use the mapping properties of $G_p$. First we separate the integral into two parts with respect to $\vec{q}$ and $V$

\[
I_2 = k^{n-1} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta'-\theta',x)} \int_{\mathbb{R}^n} e^{-ik(\theta',y)} (\vec{q} \cdot \nabla u_{sc} + V u_{sc}) \, dy \, d\theta \, d\theta'
\]
\[
= I^{(1)}_2 + I^{(2)}_2.
\]

Recall from Remark 2 that $u_{sc} = -\widehat{G}_p(\vec{q} \cdot \nabla u_0 + V u_0)$. This allows us to write
\[
I^{(2)}_2 = -k^{n-1} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} e^{-ik(\theta'-\theta',x)} \int_{\mathbb{R}^n} e^{-ik(\theta',y)} V(y)
\times \widehat{G}_p(ik \vec{q} \cdot \vec{e} e^{ik(\theta, \cdot)} + V e^{ik(\theta, \cdot)}) (y) \, dy \, d\theta \, d\theta'
\]
\[
= -k^{n-1} \int_{\mathbb{R}^n} \frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{\frac{n-2}{2}}} V(y) \widehat{G}_p \left( \int_{\mathbb{S}^{n-1}} ik \vec{q} \cdot \vec{e} e^{-ik(\theta,x-z)} \, d\theta \right) \, dy
\]
\[
- k^{n-1} \int_{\mathbb{R}^n} \frac{J_{\frac{n-2}{2}}(k|x-y|)}{(k|x-y|)^{\frac{n-2}{2}}} V(y) \widehat{G}_p \left( \int_{\mathbb{S}^{n-1}} V e^{-ik(\theta,x-z)} \, d\theta \right) \, dy
\]
\[
= I^{(2)*}_2 + I^{(2)**}_2.
\]

From [26] we find that
\[
\frac{\partial}{\partial x}(x^{-\nu} J_{\nu}(x)) = -x^{-\nu} J_{\nu+1}(x)
\]
and therefore in $I^{(2)*}_2$ we have
\[
\int_{\mathbb{S}^{n-1}} \vec{q}(z) \cdot ik e^{-ik(x-z,\theta)} \, d\theta = \vec{q}(z) \cdot \nabla z \int_{\mathbb{S}^{n-1}} e^{-ik(x-z,\theta)} \, d\theta
\]
\[
= -k(2\pi)^{\frac{n}{2}} \vec{q}(z) \cdot \frac{x-z \cdot J_{\frac{n}{2}}(k|x-z|)}{|x-z| (k|x-z|)^{\frac{n-2}{2}}}.
\]

Next, we may write $\vec{q} = (\vec{q}/|\vec{q}|)|\vec{q}|^{\frac{1}{2}}|\vec{q}|^{\frac{1}{2}} =: q_{\frac{1}{2}} |\vec{q}|^{\frac{1}{2}}$. Now we define operator $\widehat{K}_1$ as the integral operator with the kernel $|V(y)|^{\frac{1}{2}} \widehat{G}_p(y - \cdot) q_{\frac{1}{2}} (\cdot)$. Actually, $\widehat{K}_1 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ uniformly in $x$ with the same norm estimates as $\widehat{G}_p$. Indeed, let $\vec{f} \in L^2(\mathbb{R}^n)$. Then, since $\vec{q} \in W^{1,p}_{p,23} (\mathbb{R}^n)$, we have that
\[
\|q_{\frac{1}{2}} \cdot \vec{f}\|_{L^2(\mathbb{R}^n)} \leq \|q\|_{L^2_{p,23}(\mathbb{R}^n)}^\frac{1}{2} \|\vec{f}\|_{L^2(\mathbb{R}^n)}.
\]
Due to the Sobolev embedding theorem we have the continuous embedding

\[ H^1_{-\delta}(\mathbb{R}^n) \subset L^{\frac{2n}{n-\delta}}(\mathbb{R}^n). \]

Thus, taking a function \( g \in H^1_{-\delta}(\mathbb{R}^n) \) and applying Hölder’s inequality gives

\[ \| V^{\frac{1}{2}} g \|_{L^2(\mathbb{R}^n)} \leq \| V^{\frac{1}{2}} \|_{L^2 \times (\mathbb{R}^n)}^{\frac{n}{2n}} \| g \|_{L^2(\mathbb{R}^n)} \leq C \| V^{\frac{1}{2}} \|_{L^2 \times (\mathbb{R}^n)}^{\frac{n}{2n}} \| g \|_{H^1_{-\delta}(\mathbb{R}^n)}. \]

These estimates and the mapping property of \( \hat{G}_p \) finally yield

\[ \| V^{\frac{1}{2}} \hat{G}_p(q \frac{1}{2} f) \|_{L^2(\mathbb{R}^n)} \leq \| V^{\frac{1}{2}} \|_{L^2 \times (\mathbb{R}^n)}^{\frac{n}{2n}} \| \hat{G}_p(q \frac{1}{2} f) \|_{H^1_{-\delta}(\mathbb{R}^n)} \]

\[ \leq \frac{C}{k^{\frac{1}{2}}} \| V^{\frac{1}{2}} \|_{L^2 \times (\mathbb{R}^n)}^{\frac{n}{2n}} \| q \frac{1}{2} f \|_{L^2(\mathbb{R}^n)} \]

\[ \leq \frac{C}{k^{\frac{1}{2}}} \| V^{\frac{1}{2}} \|_{L^2 \times (\mathbb{R}^n)}^{\frac{n}{2n}} \| \hat{q} \|_{L^2(\mathbb{R}^n)} \| f \|_{L^2(\mathbb{R}^n)}. \]

Next we show that the function from (7) satisfies

\[ \left\| \hat{q} \left( \frac{1}{2} J_n(k|x - |) \right) \right\|_{L^2(\mathbb{R}^n)} \leq C k^{\frac{1-n}{2}}. \]

This is a straightforward calculation by using the asymptotic behaviour of the Bessel function. We estimate

\[ \int_{\mathbb{R}^n} |\hat{q}(y)| \left| J_n \left( \frac{k}{2} |x - y| \right) \right|^2 dy \leq C \int_{|x-y|<1} |\hat{q}(y)| \left( \frac{k}{2} |x - y| \right)^n \frac{1}{n-2} dy \]

\[ + C \int_{|x-y|>1} |\hat{q}(y)| \left( \frac{k}{2} |x - y| \right)^{n-1} dy \]

\[ \leq \frac{C}{k^n} \int_{\mathbb{R}^n} |\hat{q}(y)| \frac{1}{|x-y|^{n-1}} dy \leq C k^{1-n}. \]

In the same manner

\[ \left\| V^{\frac{1}{2}} \left( \frac{1}{2} J_n \left( \frac{k|x-|}{|x-|} \right) \right) \right\|_{L^2(\mathbb{R}^n)} \leq C k^{\frac{1-n}{2}}. \]

To finish the consideration of the term \( I_2^{(2)*} \) we integrate with respect to \( \theta' \) to obtain

\[ J_1 \leq \frac{k^{-n}}{n-2} \int_{\mathbb{R}^n} \left| V(y) \right|^{\frac{1}{2}} \left| \frac{1}{2} J_n \left( \frac{k}{2} |x - y| \right) \right| \left| \hat{G}_1 \left( \frac{k}{2} |y - | \right) \right| dy \]

\[ \leq C k^{n+\frac{1-n}{2} - 2} \left( \int_{\mathbb{R}^n} \left| V(y) \right|^{\frac{1}{2}} \left| \frac{1}{2} J_n \left( \frac{k}{2} |x - y| \right) \right| |\hat{G}_1| \left( \frac{k}{2} |y - | \right) dy \right)^\frac{1}{2} \]

\[ \leq C k^{n+\frac{1-n}{2} - 2} k^{\frac{1-n}{4}} = C k^{-\frac{n}{4}}. \]

This means that the term \( I_2^{(2)*} \to 0 \) as \( k \to \infty \).

Next we consider the integral \( I_2^{(2)**} \). The method is essentially the same as in case of \( I_2^{(2)*} \). By computing the integral with respect to \( \theta \) we get

\[ I_2^{(2)**} = -k^{n-1} \int_{\mathbb{R}^n} J_n \left( \frac{k}{2} |x - y| \right) \frac{1}{n-2} V(y) \hat{G}_1 \left( \frac{k}{2} |y - | \right) dy. \]
Here it appears to be simpler to conclude directly that
\[ \left\| \frac{J_{n-2} (k|x - \cdot|)}{(k|x - \cdot|)^{n-2}} \right\|_{L^2_2 (\mathbb{R}^n)} \leq C k^{\frac{2-n}{2}} \]
and to not factorize the integrals into new operators. We estimate the square of the above norm by
\[
\int_{\mathbb{R}^n} (1 + |z|)^{2\delta} |V(z)|^2 \frac{J_{n-2}^2 (k|x - z|)}{(k|x - z|)^{n-2}} \, dz \leq C \int_{k|x - z| < 1} (1 + |z|)^{2\delta} |V(z)|^2 \, dz \\
+ C \int_{k|x - z| > 1} (1 + |z|)^{2\delta} \frac{|V(z)|^2}{(k|x - z|)^{n-1}} \, dz \\
\leq C k^{2-n} \int_{\mathbb{R}^n} (1 + |z|)^{2\delta} \frac{|V(z)|^2}{|x - z|^{n-2}} \, dz.
\]
Dividing the region of integration to \(|x - z| < 1\) and \(|x - z| > 1\) we see that
\[
\int_{\mathbb{R}^n} (1 + |z|)^{2\delta} \frac{|V(z)|^2}{|x - z|^{n-2}} \, dz \leq \int_{|x - z| < 1} (1 + |z|)^{2\delta} \frac{|V(z)|^2}{|x - z|^{n-2}} \, dz \\
+ \int_{|x - z| > 1} (1 + |z|)^{2\delta} |V(z)|^2 \, dz \\
\leq \left( \int_{|x - z| < 1} \frac{1}{|x - z|^{n-2}} \, dz \right)^{\frac{n-2}{p}} \|V\|_{L^p_2 (\mathbb{R}^n)} \\
+ \|V\|_{L_{\infty}^2 (\mathbb{R}^n)}.
\]
The above integral converges uniformly in \(x\), when \(n < p \leq \infty\). Thus, since \(\tilde{G}_p : L^2_2 (\mathbb{R}^n) \rightarrow H^1_\delta (\mathbb{R}^n)\), we have
\[
\|I_2^{(2)\ast}\| \\
\leq C k^{n-1} \left( \int_{\mathbb{R}^n} |V(y)| \frac{J_{n-2}^2 (k|x - y|)}{(k|x - y|)^{n-2}} \, dy \right)^{\frac{1}{2}} \left\| |V|^\frac{1}{2} \tilde{G}_p \left( \frac{J_{n-2} (k|x - \cdot|)}{(k|x - \cdot|)^{n-2}} \right) \right\|_{L^2_2 (\mathbb{R}^n)} \\
\leq C k^{n-1} \left( \frac{1}{p} + \frac{n-2}{p} - 2 \right) \leq C k^{-\frac{2}{n}} \rightarrow 0
\]
as \(k \rightarrow +\infty\).

The term \(I_2^{(1)}\) can be considered analogously to \(I_2^{(2)}\). Compared to the term \(I_2^{(2)}\), where we utilized the fact that \(\tilde{G}_p\) maps \(L^2_2\) to \(H^1_\delta\), in term \(I_2^{(1)}\) we can use the previous calculations and the fact that \(\nabla \tilde{G}_p\) maps \(L^2_2\) to \(L^2_2\) with the same estimates as earlier. Now for \(g \in L^2_2 (\mathbb{R}^n)\) we have that
\[
\|\hat{q} \frac{1}{2} g\|_{L^2_2 (\mathbb{R}^n)} \leq \|\hat{q}\|_{L^{2\delta}_2 (\mathbb{R}^n)} \|g\|_{L^2_2 (\mathbb{R}^n)}.
\]
Replacing the first \(|V|^\frac{1}{2}\) by \(|\hat{q}|^\frac{1}{2}\) in the previous calculations and using the above norm estimate we may conclude that \(I_2^{(1)}\) satisfies the same estimate in \(k\) as \(I_2^{(2)}\). Therefore, \(I_2 \rightarrow 0\) as \(k \rightarrow +\infty\) and the theorem is proved. \(\square\)

The proofs of Corollaries 1 and 2 are well-known in literature and can be found for example in [18] or [10] (case \(n = 3\).
Conclusion. A direct scattering problem for a first-order perturbation of the biharmonic operator was studied. It was shown that a solution to scattering problem with certain radiation conditions satisfies the Lippmann-Schwinger equation. This integral equation has a unique solution in the weighted Sobolev space $H^1_{−\delta}$ in any dimension and in particular this solution is bounded if the dimension is 2 or 3. The proofs of these results are based on a Green-type formula for bi-Laplacian and Agmon’s estimate [1]. The asymptotic behaviour of the solution for fixed $k > 0$ as $|x| \to +\infty$ was studied and a formula for the scattering amplitude was obtained.

The main result of this paper, Saito’s formula, was proved under quite general assumptions on the coefficients. More precisely, the formula for scattering amplitude was proved under the assumption that the coefficients of the direct operator are compactly supported, but this assumption was not necessary for the proof of Saito’s formula. The proof itself was based on explicit calculations starting from the formula for scattering amplitude and utilizing the properties of certain resolvent operator related to the direct operator. Some consequences of Saito’s formula were discussed. Namely, the scattering amplitude uniquely determines a combination of the coefficients for the direct problem and in turn gives a uniqueness result for the inverse problem. A representation formula for this combination of the coefficients was given.

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Appendix A. A Green-type formula and an embedding theorem.

Lemma A.1. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial \Omega$. If $f, g \in H^4(\Omega)$ then the following equality holds

$$
\int_{\Omega} (f \Delta^2 g - g \Delta^2 f) \, dx = \int_{\partial \Omega} \left( f \frac{\partial}{\partial n} \Delta g + \Delta f \frac{\partial}{\partial n} g - g \frac{\partial}{\partial n} \Delta f - \Delta g \frac{\partial}{\partial n} f \right) \, d\sigma(x).
$$

Proof. Let $f$ and $g$ be $C^4$-functions on $\Omega$ and define a $C^1$-vector field $\vec{F}$ by the formula

$$
\vec{F} = \Delta f \nabla g + f \nabla (\Delta g) - \Delta g \nabla f - g \nabla (\Delta f).
$$

Then the divergence theorem (see e.g., [8]) gives the claimed formula. To extend the result for functions from $H^4(\Omega)$ we note that left-hand side defines a bounded bilinear form. Similarly, for smooth boundary $\partial \Omega$ the trace operator is bounded from $H^3(\Omega)$ to $H^3(\partial U)$ so the right-hand side is also bounded. Since $C^4$-functions are dense in $H^4$ the continuity and a density argument show that the claim holds also for functions from $H^4(\Omega)$.

Theorem A.2. If $n < p < \infty$, then $W^1_p(\mathbb{R}^n) \subset \hat{C}(\mathbb{R}^n)$.

Proof. We start from the known [5] embedding $W^1_p(\mathbb{R}^n) \subset C^{1-\frac{n}{p}}(\mathbb{R}^n)$ for $n < p < \infty$, where $C^{\alpha}(\mathbb{R}^n)$ is the Hölder space for $0 < \alpha < 1$. It is also known that functions in the space $W^1_p(\mathbb{R}^n)$ are bounded when $p > n$. It remains to verify that the functions in $W^1_p(\mathbb{R}^n)$ vanish at infinity.
Let $f \in W^1_p(\mathbb{R}^n)$. If $a \neq b$ then by the mean value theorem we have the equality
\[ |a^p - b^p| = p|\xi|^{p-1}|a - b| \]
for some $\xi \in ]a, b[$. Thus for the function $|f|$ we find that
\[ \|f(x)|^p - |f(y)|^p\| \leq p\|f\|_{L^\infty(\mathbb{R}^n)}^{p-1}\|f(x) - f(y)\| \leq C|x - y|^{1-\frac{p}{n}} \quad \text{a.e.,} \]
and therefore $|f|^p$ can be chosen to be uniformly (even $(1-n/p)$-Hölder) continuous.

Next, assume in contrary that $|f|^p$ does not vanish at infinity. It means that there exists a constant $c > 0$ and points $x_j \in \mathbb{R}^n$ so that $|f(x_j)|^p \geq c$. We may pick a sequence of these points with the property that for each $m \in \mathbb{N}$ we have $|x_m| \geq m$. Since $|f|^p$ is uniformly continuous, there exists $\delta > 0$ (that only depends on $c$) such that $|f(x)|^p \geq \frac{c}{2}$ when $|x - x_m| < \delta$. This is a contradiction since $|f|^p$ is integrable, but $|x_m| \to \infty$ and
\[
\int_{|x-x_m|<\delta}|f(x)|^p\,dz \geq c\frac{|B_\delta(x_m)|}{2},
\]
where $|B_\delta(x_m)|$ denotes the volume of a ball of radius $\delta$ centered at $x_m$.

Finally, if $f$ did not vanish at infinity then we would find constant $\tilde{c} > 0$ and points $\tilde{x}_m \in \mathbb{R}^n$ so that $|f(\tilde{x}_m)| \geq \tilde{c}$. But this means that $|f(\tilde{x}_m)|^p \geq \tilde{c}^p$, which is a contradiction, as we just saw. \hfill $\square$

**Remark 4.** In dimension $n = 1$ it is possible to prove straightforwardly that $W^1_1(\mathbb{R}) \subset C(\mathbb{R})$. Moreover, Theorem A.2 does not hold for $p = \infty$, since for example all constant functions belong to $W^1_\infty(\mathbb{R}^n)$, but the only constant function vanishing at infinity is the constant $C \equiv 0$.

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