ARCHIMEDEAN-TYPE FORCE IN A COSMIC DARK FLUID:
I. EXACT SOLUTIONS FOR THE LATE-TIME ACCELERATED EXPANSION

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We establish a new self-consistent model in order to explain from a unified viewpoint two key features of the cosmological evolution: the inflation in the early Universe and the late-time accelerated expansion. The key element of this new model is the Archimedean-type coupling of the dark matter with dark energy, which form the so-called cosmic dark fluid. We suppose that dark matter particles immersed into the dark energy reservoir are affected by the force proportional to the four-gradient of the dark energy pressure. The Archimedean-type coupling is shown to play a role of effective energy-momentum redistributor between the dark matter and the dark energy components of the dark fluid, thus providing the Universe evolution to be a quasiperiodic and/or multistage process. In the first part of the work we discuss a theoretical base and new exact solutions of the model master equations. Special attention is focused on the exact solutions, for which the scale factor is presented by the anti-Gaussian function: these solutions describe the late-time acceleration and are characterized by a nonsingular behavior in the early Universe. The second part contains qualitative and numerical analysis of the master equations; we focus there on the solutions describing a multi-inflationary Universe.

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I. INTRODUCTION

The concepts of dark energy (DE) and dark matter (DM) (see, e.g., [13] and [18] for review and references) are the basic elements of modern cosmology and astrophysics. These elements were introduced into the scientific lexicon in two different ways: dark energy is considered to be a reason for the late-time accelerated expansion of the Universe [7, 8], while dark matter is usually associated with the explanation of the flat velocity curves of the spiral galaxies rotation [9, 10]. Nevertheless, there exists a tendency to consider dark energy and dark matter as two manifestations of one unified dark fluid (see, e.g., [11–16]). The models of interaction between these two constituents of the dark fluid, as well as the models of interactions of dark energy and/or dark matter with the standard (baryon) matter, are the subjects of wide discussion. The total contribution of dark energy and dark matter into the energy balance of the Universe is estimated to be about 95%. Thus, the coupling between these two constituents of the dark fluid seems to be the most important element in the list of cosmic medium interactions, and the dark fluid can be considered as a thermodynamic reservoir for baryon matter.

The most developed model of the coupling between DM and DE constituents of the dark fluid is based on the two-fluid representation of the cosmic medium (see, e.g., [17, 22]). In this approach the interaction terms $\pm Q$ appear in the right-hand sides of separate balance equations for the DE and DM with opposite signs and disappear in a sum, when one deals with the total balance equation. The modeling of the coupling term $Q$ has in most cases a phenomenological character and is based on the ansatz that $Q$ is a function (e.g., linear or power-law) of the energy densities of the DE and DM, of the Hubble function, $H$, etc. We formulate the theory of interaction between DE and DM using the relativistic hydrodynamics for dark energy and relativistic kinetics for dark matter. We suggest that the DE acts on the DM particles by means of some effective force, and the corresponding model force four-vector is introduced into the kinetic equation. The backreaction of the DM on the DE is described using a force-type term in the hydrodynamic equations. The total system (DM plus DE) is considered to be conservative. The concept of the Archimedean-type force can be naturally generalized for the description of the DE action on the baryon matter; however, now we restrict ourselves by the model of the DE and DM interaction.

We discussed the structure and properties of various effective forces, which appeared in the cosmological contexts, in the papers [23–28], the relativistic generalizations of the Stokes force, Langevin force, antifriction and tidal forces being investigated in detail. Concerning the force acting on the DM particles from the DE we would like to introduce the so-called Archimedean-type force. This choice can be motivated as follows. First, the DE pressure is assumed to be of the same order as the DE energy density (73% of the Universe energy density), thus, the Archimedean effect on the dark matter could be significant. Third, assuming that the DE pressure can be negative, we obtain that such a force can be attractive in contrast to explosive classical Archimedean force. Admitting that the DE pressure changes the sign in the course of the Uni-
verse evolution, one can describe a multistage (or even quasiperiodic) character of the cosmological expansion, for which epochs of deceleration are changed by epochs of acceleration and vice versa. (The interest in models of this type was renewed by the paper [29].)

We show that, in principle, the Archimedean-type force can effectively redistribute 95% of the Universe’s energy between the DE and DM constituents, thus guiding the time evolution of the cosmic medium as a whole. We divided the work into two parts: the first one contains pure analytical results and exact solutions of the model; in the second part we focus on the numerical and qualitative analysis of the model.

The first part of the work is organized as follows. In Sec. II we derive the master equations of the model with an Archimedean-type force. In particular, in Sec. II.B based on the kinetic approach we introduce the Archimedean-type force, obtain basic integrals of motion, construct the distribution functions and calculate their macroscopic moments as functions of the DE pressure. In Sec. II.C we formulate the balance equation for the dark fluid. In Sec. II.D we discuss the extended (inhomogeneous) equation of state for the dark energy and introduce the key equation for the DE pressure evolution. Sec. III contains discussions about two classes of exact solutions. In Sec. III.A we consider a special (constant) exact solution for the case when the guiding parameter of the model \( \sigma \) is not equal to its critical value, i.e. \( \sigma \neq -1 \), and we analyze the problem of asymptotic stability of this solution. In Sec. III.B we focus on the special case \( \sigma = -1 \) and obtain exact solutions of the anti-Gaussian type for the following submodels: (i) the massless DM, (ii) the cold dark matter, and (iii) the submodel with the DE domination. In Sec. IV we discuss obtained analytical results.

II. MASTER EQUATIONS

We consider a cosmic medium, which consists of two interacting components. One of them, the dark matter, can be described in the framework of general relativistic kinetic theory [30, 31]. The second component, dark energy, is considered as a perfect fluid with inhomogeneous equation of state. These two components interact gravitationally, i.e., both of them contribute the energy and momentum terms into the total stress-energy tensor of the system as a whole, indicated as the dark fluid. In addition, we assume that DE and DM interact by means of force of the Archimedean type. Mathematically, this model can be described as follows.

A. Equations for gravity field

We consider the spatially homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model with the metric

\[
\text{d}s^2 = a^2(t)(\text{d}x^1)^2 + (\text{d}x^2)^2 + (\text{d}x^3)^2.
\]

Here and below \( c = 1 \). The Einstein equations

\[
R_{ik} - \frac{1}{2}Rg_{ik} = \kappa T_{ik}^{\text{(total)}} + \Lambda g_{ik},
\]

can be reduced to the well-known system

\[
\frac{\dot{a}}{a} = -\frac{4\pi G}{3}[E + 3P] + \frac{\Lambda}{3}, \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}E + \frac{\Lambda}{3},
\]

where the dot denotes the derivative with respect to time, \( E(t) \) is the total energy of the system as a whole, and \( P \) is the corresponding pressure. Since the cosmological constant \( \Lambda \) is frequently interpreted in terms of vacuum energy and can be considered as a dark energy candidate, we assume the following decompositions of the total energy and pressure

\[
E + \frac{\Lambda}{8\pi G} = \rho + E, \quad P - \frac{\Lambda}{8\pi G} = \Pi + P.
\]

In such decomposition the functions \( \rho(t) \) and \( \Pi(t) \) describe the dark energy, while the functions \( E(t) \) and \( P(t) \) describe the dark matter; thus, the term \( \frac{\Lambda}{8\pi G} \) is considered to be incorporated into the DE energy density \( \rho \) and DE pressure \( \Pi \). Now the master equations for gravity field are

\[
\frac{\dot{a}}{a} = -\frac{4\pi G}{3}[(\rho + E) + 3(\Pi + P)],
\]

\[
H^2 = \frac{8\pi G}{3}(\rho + E),
\]

where, as usual, \( H(t) \equiv \frac{\dot{a}}{a} \) is the Hubble function.

B. Kinetic equation for relativistic DM particles

Let us consider the general relativistic kinetic equations

\[
\frac{p_i}{m_{(a)}} \left( \frac{\partial}{\partial x^i} - \Gamma^k_{il} p^l \frac{\partial}{\partial p^k} \right) f_{(a)} + \frac{\partial}{\partial p^i} \left[ F^i_{(a)} f_{(a)} \right] = 0
\]

for the distribution functions \( f_{(a)} \), describing the evolution of the DM particles with the masses \( m_{(a)} \). It is worth mentioning that the DM can consist of a few sorts of particles (massless and massive), thus, we use the index \( a \) to distinguish them. The kinetic equations (7) are of the collisionless type, i.e., we neglect direct interparticle collisions, but consider the force four-vector \( F^i_{(a)} \) to guide the DM particle dynamics. Two subsets of characteristic equations

\[
\frac{dp^k}{ds} + \Gamma^k_{jl} p^l \frac{dx^j}{ds} = F^k_{(a)}; \quad \frac{dx^i}{ds} = \frac{p^i}{m_{(a)}},
\]
show that the DM particle motion is not geodesic because of the force $F_{(a)}^\alpha$, which appears as a result of the DE action. The third subset of characteristic equations

$$\frac{df_{(a)}}{ds} = - f_{(a)} \left( \frac{\partial F_k^{(a)}}{\partial p^k} \right)$$  \hspace{1cm} (9)$$
demonstrates the important role of the divergence $\frac{\partial F_k^{(a)}}{\partial p^k}$ in the evolution of the distribution functions.

1. Archimedean-type force

In classical physics the term “Archimedean force” appears when one deals with a body immersed into a nonhomogeneous liquid. This three-dimensional force $F_{(Arch)}$ is proportional to the spatial gradient of the Pascal pressure $P_{(Pascal)}$

$$F_{(Arch)} = -V_0 \nabla P_{(Pascal)}.$$  \hspace{1cm} (10)

Clearly, in the Archimedean historical experiment the gradient of the pressure was equal to the product of the water mass density $\rho_{(water)}$ and the free-fall acceleration $g$ in Syracuse. The coefficient $V_0$ relates to a body volume and can be represented as a quotient $\frac{M_{(body)}}{\rho_{(body)}}$, where $M_{(body)}$ is a mass of the body and $\rho_{(body)}$ is its mass density. The minus sign corresponds to the fact that the Archimedean force is the buoyancy (expulsive) one.

This classical force can be generalized for the case of relativistic DM particle immersed into the DE reservoir. The force four-vector $F_{(a)}^\alpha$ can be considered as a generalization. First of all, it is a force proportional again to the gradient of pressure $\Pi$, but now we deal with a four-gradient instead of gradient three-vector in classical physics, and $\Pi$ is the DE pressure instead of the Pascal one. Second, the projector in the square brackets guarantees that the force four-vector is orthogonal to the particle momentum four-vector, i.e., $F_{(a)}^\alpha p_{(a)} = 0$. The last property provides the existence of the first integral of motion $p^k \dot{p}_k = m_{(a)}^2 = \text{const}$, which guarantees the particle mass conservation. New constants $V_{(a)}$ have the dimensionality of inverse energy density (remember that $c=1$) and describe some new effective constants of interaction. The sign in front of these constants is positive, and this is in agreement with the definition of $\Pi$ and the accepted signature $(+---)$ of the metric $\eta$. This force does not belong to the class of gyroscopic forces, for which $\frac{\partial F_k^{(a)}}{\partial p^k} = 0$. Indeed, simple calculations give

$$\frac{\partial F_k^{(a)}}{\partial p^k} = -3 m_{(a)} V_{(a)} \frac{p^k \nabla k \Pi}{p^l p_l} = -3 V_{(a)} \frac{p^0 \Pi}{m_{(a)}} \neq \text{0},$$  \hspace{1cm} (12)

where $p^0$ is the DM particle energy

$$p^0 = p_0 = \sqrt{m_{(a)}^2 - p^\alpha p_\alpha}.$$  \hspace{1cm} (13)

(Greek indices run from 1 to 3). Thus, the Archimedean-type force $\Pi$ is divergence-free, if and only if the DE pressure $\Pi$ is constant.

2. Integrals of motion

Using the force four-vector given by (11) one can extract the following self-closed subsystem from the characteristic equations (8):

$$\frac{dp_\alpha}{dt} = -V_{(a)} \Pi, \quad \frac{dt}{ds} = \frac{p^0}{m_{(a)}}.$$  \hspace{1cm} (14)

These equations yield immediately three first integrals (for the covariant components of the particle momentum)

$$p_\alpha (t) = C_\alpha \exp \left\{ V_{(a)} [\Pi(t_0) - \Pi(t)] \right\}, \quad p_\alpha (t_0) = C_\alpha.$$  \hspace{1cm} (15)

Thus, any deviation of the DE pressure $\Pi(t)$ from the initial value $\Pi(t_0)$ generates particle acceleration/deceleration due to the Archimedean-type interaction. The energy of the DM particle of the sort (a)

$$p^0 (t) = m_{(a)} \sqrt{1 + q^2},$$  \hspace{1cm} (16)

deviates from its initial value

$$p^0 (t_0) = m_{(a)} \sqrt{1 + q^2},$$  \hspace{1cm} (17)

where

$$q^2 = \frac{1}{m_{(a)}^2 a^2 (t_0)} \left[ C_1^2 + C_2^2 + C_3^2 \right],$$  \hspace{1cm} (18)

due to the following effects. The first effect is the standard energy decreasing due to the Universe’s expansion; it is described by the term $a^2 (t_0)/a^2 (t)$ and tends to make the matter effectively nonrelativistic at $t \to \infty$. A new effect caused by the nonstationarity of the dark energy is described in [10] by the exponential term. Let us emphasize, that this effect decreases the particle energy, if the term $V_{(a)} [\Pi(t) - \Pi(t_0)]$ is positive, and increases it in the opposite case. Let us consider only one simple example, when $V_{(a)}$ is positive, the initial value $\Pi(t_0)$ is vanishing: one can see, that the DM particle energy grows due to the DE pressure counteracts the effective cooling of the dark matter in the course of expansion. In other words, even if a particle has small (but not equal identically to zero) initial kinetic energy, it can become ultrarelativistic due to the Archimedean-type interaction with the dark energy.
3. Macroscopic moments of the distribution function

Solving Eq. (9) taking into account (12) and (15) we obtain

\[ f_{(a)} (t, p_\alpha) = f_{(a)}^0 (q^2) \exp \left\{ 3V_{(a)} \left[ \Pi (t) - \Pi (t_0) \right] \right\}, \tag{19} \]

where \( f_{(a)}^0 (q^2) \) is an initial distribution function of the DM particles of the sort \((a)\). Using (16) and (15) one can reduce the DM particle stress-energy tensor (the macroscopic moment of the second order)

\[ T_{ik}^{(DM)} (t) = \sum_{(a)} \int \frac{dp_1 dp_2 dp_3}{\sqrt{-g} \, p_0} f_{(a)} (t, p_{\alpha}) \, p_i p_k \] \tag{20}

to the pair of basic integrals for the energy \(E\) and pressure \(P\), respectively:

\[ E(x) = \sum_{(a)} \frac{4 \pi m_{(a)}^4}{x^3} \int_0^\infty q^2 dq f_{(a)}^0 (q^2) \sqrt{1 + q^2 F_{(a)} (x)}, \tag{21} \]

\[ P(x) = \sum_{(a)} \frac{4 \pi m_{(a)}^4 F_{(a)} (x)}{3 x^3} \int_0^\infty \frac{q^4 dq f_{(a)}^0 (q^2)}{\sqrt{1 + q^2 F_{(a)} (x)}}. \tag{22} \]

Here for the sake of convenience we have introduced a new dimensionless variable \(x\) and an auxiliary function \(F_{(a)} (x)\) defined as follows

\[ x = \frac{a(t)}{a(t_0)} \cdot \frac{F_{(a)} (x)}{x^2} = \frac{1}{x^2} \exp \left\{ 2V_{(a)} \left[ \Pi (1) - \Pi (x) \right] \right\}. \tag{23} \]

The initial moment \(t=t_0\) corresponds to the value \(x=1\) \((F_{(a)}(1)=1)\). The DE pressure can be also rewritten in these terms as

\[ \Pi (x) = \Pi (1) - \frac{1}{2 V_{(a)}} \log \left[ x^2 F_{(a)} (x) \right]. \tag{24} \]

The most conventional model of matter deals with the relativistic Maxwell - Boltzmann functions \[31\]

\[ f_{(a)}^0 (q^2) = \frac{N_{(a)} \lambda_{(a)}}{4 \pi m_{(a)}^3 K_2 (\lambda_{(a)})} e^{-\lambda_{(a)} \sqrt{1 + q^2}}, \tag{25} \]

where \(N_{(a)}\) is the particle number density, \(\lambda_{(a)} = \frac{m_{(a)}}{k_{(B)} T_{(a)}}\), and \(K_2 (\lambda_{(a)})\) is the modified Bessel function \[31\], defined as

\[ K_\nu (\lambda_{(a)}) \equiv \int_0^\infty dz \cosh \nu z \cdot \exp \left[ -\lambda_{(a)} \cosh z \right], \tag{26} \]

\(k_{(B)}\) is the Boltzmann constant. We assume that there is no thermodynamic equilibrium between different sorts of DM particles, thus, generally, the temperatures \(T_{(a)}\) do not coincide and are marked by the index of the sort.

In this model the DM energy and pressure have, respectively, the form

\[ E(x) = \sum_{(a)} \frac{E_{(a)}}{x^3} \int_0^\infty q^2 dq \sqrt{1 + q^2 F_{(a)} (x)} \, e^{-\lambda_{(a)} \sqrt{1 + q^2}}, \tag{27} \]

\[ P(x) = \sum_{(a)} \frac{E_{(a)}}{3 x^3} \int_0^\infty \frac{F_{(a)} (x) q^4 dq}{\sqrt{1 + q^2 F_{(a)} (x)}} \, e^{-\lambda_{(a)} \sqrt{1 + q^2}}, \tag{28} \]

where the constant \(E_{(a)}\) is given by the formula

\[ E_{(a)} = \frac{N_{(a)} m_{(a)} \lambda_{(a)}}{K_2 (\lambda_{(a)})}. \tag{29} \]

The formulas for the Fermi-Dirac and Bose-Einstein functions can be obtained analogously, and we do not write them here.

4. Massless particles \(m_{(a)} = 0\)

The dark matter can consist of massless and massive particles. When we deal with massless particles, i.e., \(m_{(a)}=0\), e.g., for the index \((a)=(0)\), we have to replace formally the quantity \(p^2/m_{(a)}\) by \(k^2\) with \(k_0 k_z=0\). Respectively, the formula \[29\] has to be modified as follows:

\[ E_{(0)} \rightarrow \frac{4 \pi \rho |k| \, T_{(0)} |x|}{h^3} \int_0^\infty \frac{q^3 dq}{e^q \pm 1}. \tag{30} \]

Here \(h\) is the Planck constant, \(T_{(0)}\) is an initial temperature of the massless particles, and \(\dot{\nu}\) is a degeneracy factor. Clearly, this massless constituent of the DM is described by the ultrarelativistic equation of state

\[ E_{(0)} (x) = 3 P_{(0)} (x) = E_{(0)} \sqrt{F_{(0)} (x) / x^3}. \tag{31} \]

When the Archimedean-type force is absent, one obtains that \(F_{(0)} (x)=x^{-2}\) and \(E_{(0)} (x) \propto x^{-4}\), as it should be.

C. Balance equations

1. Energy balance for interacting DE and DM

We consider the model in which dark matter and dark energy form a coupled conserved system. The total stress-energy tensor of DM and DE is divergence-free

\[ T_{(total)}^{ik} = T_{(DM)}^{ik} + T_{(DE)}^{ik}, \quad \nabla_k T_{(total)}^{ik} = 0. \tag{32} \]

We suppose that both cosmic substrates, DM and DE, have the same macroscopic velocity four-vector, \(U^i=\delta_0^i\), thus, due to the FLRW space-time symmetry only one equation among \[32\] is nontrivial:

\[ \dot{\rho} + \dot{E} + 3 H (\rho + E + \Pi + P) = 0. \tag{33} \]

It is the direct differential consequence of Eqs. \[13\] and \[14\].
Balance equations for DM can be obtained by integration of the kinetic equation (7) (see [30, 31] for details). The first-order macroscopic moment $N^k$ is defined as
\begin{equation}
N^k = \sum_{(a)} N^k_{(a)} = \sum_{(a)} \int dP f_{(a)} p^k \tag{34}
\end{equation}
and describes the total DM particle number vector. Using the kinetic equation (7) the four-divergence of this vector can be easily calculated:
\begin{equation}
\nabla_k N^k = \nabla_k \sum_{(a)} \int dP p^k f_{(a)} =
= - \sum_{(a)} m_{(a)} \int dP \frac{\partial}{\partial p^k} \left[ f_{(a)} \mathcal{F}^k_{(a)} \right] = 0 . \tag{35}
\end{equation}
This means that the DM particle number is conserved. Taking into account the symmetry of the model one can write
\begin{equation}
N^k = \delta^k_0 N(t_0) \left[ \frac{a(t_0)}{a(t)} \right]^3 . \tag{36}
\end{equation}
The second-order macroscopic moment, the DM stress-energy tensor $T^{(DM)}_{ik}$, satisfies the following equation
\begin{equation}
\nabla_k T^{(DM)}_{ik} = \nabla_k \sum_{(a)} T^{(a)}_{ik} - \sum_{(a)} m_{(a)} \int dP f_{(a)} \mathcal{F}^i_{(a)} =
= \nabla_k \Pi \sum_{(a)} m_{(a)}^2 V_{(a)} \int dP f_{(a)} \left[ g^{ik} \frac{p^i p^k}{p^m p_m} \right] =
= \delta^k_0 3 \Pi \sum_{(a)} V_{(a)} P_{(a)} . \tag{37}
\end{equation}
Only one equation among (37) is nontrivial:
\begin{equation}
\dot{E} + 3H(E + P) = - \mathcal{Q} . \tag{38}
\end{equation}
The source term in the right-hand side of this equation has the form
\begin{equation}
\mathcal{Q} = 3 \Pi \sum_{(a)} V_{(a)} P_{(a)} . \tag{39}
\end{equation}
The source $\mathcal{Q}$ in the DM energy balance equation vanishes, when the DE pressure $\Pi$ is constant and the Archimedean-type force disappears.

The scalar of the DM entropy production
\begin{equation}
\sigma_{(DM)} = \nabla_k S^k = -k_B \nabla_k \sum_{(a)} \int dP p^k f_{(a)} \left[ \log h^3 f_{(a)} - 1 \right] \tag{40}
\end{equation}
is also proportional to the four-gradient of the DE pressure
\begin{equation}
\sigma_{(DM)} = k_B \sum_{(a)} m_{(a)} \int dP f_{(a)} \frac{\partial \mathcal{F}^k_{(a)}}{\partial p^k} =
= -3k_B \nabla_k \Pi \sum_{(a)} V_{(a)} N^k_{(a)} =
= -3k_B \Pi \left[ \frac{a(t_0)}{a(t)} \right]^3 \sum_{(a)} V_{(a)} N_{(a)}(t_0) . \tag{41}
\end{equation}
When the coefficient $\Re \equiv \sum_{(a)} V_{(a)} N_{(a)}(t_0)$ is positive and $\dot{\Pi} > 0$, the entropy production scalar $\sigma_{(DM)}$ is negative, i.e., the DE provides organization processes in the DM system by the Archimedean-type force; when $\dot{\Pi} < 0$ the Archimedean-type force produces a chaotization in the DM system.

3. Balance equations for dark energy

The combination of Eqs. (33) and (38) gives the following balance equation for the DE state functions $\rho$ and $\Pi$:
\begin{equation}
\dot{\rho} + 3H(\rho + \Pi) = \mathcal{Q} . \tag{42}
\end{equation}
The functions $P_{(a)}(t)$ depend on $\Pi(t)$ according to Eqs. (28) and (29) for the massive particles, and according to Eq. (31) for the massless ones. One can emphasize that the macroscopic balance equations (38), (39) and (42) look like the balance equations in the well-known two-fluid models [17–21]; the difference is that now the source term $\mathcal{Q}$ (40) is not modeled phenomenologically, but is directly calculated on the basis of kinetic approach.

D. Dark energy dynamics

1. Dark energy equation of state accounting for retardation of response

To describe the dark energy fluid we use the linear equation of state
\begin{equation}
\rho(t) = \rho_0 + \sigma \Pi + \frac{\xi}{H(t)} \dot{\Pi} . \tag{43}
\end{equation}
It belongs to the class of the so-called inhomogeneous equations of state, which is intensely discussed in the literature (see, e.g., [32–37]). When $\sigma = 0$ and $\xi = 0$, Eq. (43) introduces the model in which the dark energy relates to the cosmological constant $\Lambda$ [see, e.g., (4)]. Alternatively, $\rho_0$ can be introduced by analogy with the
so-called bag constant appearing in the theory of quark-gluon plasma \[38\]. When \(\rho_0=0\) and \(\xi=0\), Eq. (43) gives the well-known linear relation \(\Pi=\rho\phi\) with \(\phi=\frac{1}{2}\). Since the proportionality coefficient \(w\) depends on the choice of the epoch in the Universe’s evolution, many authors consider it as a function of cosmological time, i.e., \(w=w(t)\), thus introducing the nonstationary equation of state (see, e.g., \[32\,37\]). We follow another version of nonstationary equation of state, for which \(w\) and \(\sigma\) remain constant, but the retardation of response is taken into account by inserting the term containing the first derivative of the pressure \(\Pi\). An equivalent scheme is widely used in the extended thermodynamics and rheology [see, e.g., \[32\]], in which the extended constitutive equation for the thermodynamically coupled variables \(X\) and \(Y\) has the form
\[
\tau X + X = w Y .
\]
(44)

Here \(\tau\) is a relaxation time, a new coupling parameter of the model. In the cosmological context \(\tau\) is generally the function of time, \(\tau(t)\). We assume that \(\tau(t) = \frac{\xi}{\sigma H(t)}\), i.e., this relaxation time can be measured in natural cosmological scale \[40\]. Our ansatz here is that the dimensionless parameter \(\xi\) is constant.

2. Key equation of evolution of the DE pressure

When the quantities \(\rho(t)\) and \(\Pi(t)\) depend on time through the scale factor \(a(t)\) only, i.e., \(\rho=\rho(a(t)), \Pi=\Pi(a(t))\), the so-called \(x\)-representation is convenient, based on the following relations
\[
\frac{d}{dt} = xH(x) \frac{d}{dx}, \quad t-t_0 = \int_{1}^{t(t_0)} \frac{dx}{xH(x)} .
\]
(45)

In these terms the balance equation \[12\] takes the following form (the prime denotes the derivative with respect to \(x\))
\[
x\rho'(x) + 3(\rho + \Pi) =
- \sum_{(a)} E_{(a)} \frac{[x^2 F_{(a)}(x)]'}{2x^4} \int_{0}^{\infty} q^4 dq e^{-\lambda_{(a)} \sqrt{1+q^2}}.
\]
(46)

Using (43) one can transform this equation into the equation for the DE pressure only, yielding
\[
\xi x^2 \Pi''(x) + x \Pi'(x) (4\xi + \sigma) + 3(1+\sigma)\Pi + 3\rho_0 = \mathcal{J}(x),
\]
(47)

where the source \(\mathcal{J}(x)=\mathcal{J}(x, \Pi=\Pi(1), \Pi')\) is defined as follows:
\[
\mathcal{J}(x) = - \sum_{(a)} E_{(a)} \frac{[x^2 F_{(a)}(x)]'}{2x^4} \int_{0}^{\infty} q^4 dq e^{-\lambda_{(a)} \sqrt{1+q^2}}.
\]
(48)

The quantity \(\mathcal{J}(x)\) vanishes, when all the Archimedean parameters vanish, i.e., \(\lambda_{(a)} = 0\). Below we will address Eq. (47) as the key equation. It is a differential equation of the second-order linear in the derivatives but nonlinear in the unknown function \(\Pi(x)\). There are two important particular cases, when the source term \(\mathcal{J}(x)\) can be written in an explicit form; let us consider them in more detail.

3. Two explicit examples

(i) Massless dark matter

When one deals with massless particles [enumerated, e.g., by the index \(0\)], the source term \(\mathcal{J}(x)\) takes explicit form
\[
\mathcal{J}(0)(x) = E_{(0)} \frac{\mathcal{V}_0(x)}{x^3} \exp \{\mathcal{V}_0[\Pi(1)-\Pi(x)]\}
\]
(49)

with \(E_{(0)}\) given by \[30\]. When DM particles are massive but effectively ultrarelativistic \([q^2 F_{(a)}(x) >> 1\) and \(q^2 >> 1\)], the source term \(\mathcal{J}(x)\) has the same form \[19\].

(ii) Cold dark matter

When one deals with the models using the concept of cold dark matter, one assumes that the corresponding particles are effectively nonrelativistic, i.e., \(q^2 F_{(a)} << 1\) and \(q^2 << 1\). The corresponding key equation contains the source term
\[
\mathcal{J}(C)(x) = \frac{3N(C)T(C)\mathcal{V}_C(x)}{x^4} e^{2\mathcal{V}_C[\Pi(1)-\Pi(x)]}.
\]
(50)

Ultrarelativistic and nonrelativistic models can be studied analytically and qualitatively. When the parameters \(\lambda_{(a)}\) are arbitrary, one needs numerical analysis for the key equation.

III. TWO EXAMPLES OF EXACT SOLUTIONS

A. Constant solution to the key equation

The behavior of the function \(\Pi(x)\), the solution to the Eq. (47), essentially depends on the initial data \(\Pi(1)\), \(\Pi'(1)\), on the values of the parameters \(\lambda_{(a)}, \sigma, \rho_0, \xi\) and \(\mathcal{V}_a\). When \(\sigma \neq -1\), the key equation admits a special constant exact solution
\[
\Pi(x) = \Pi(1) = - \frac{\rho_0}{1 + \sigma} = \Pi(\infty).
\]
(51)

For this solution the energy density \(\rho(x)\) is also constant
\[
\rho(x) = \rho(1) = -\Pi(x) = - \frac{\rho_0}{1 + \sigma} = -\Pi(\infty),
\]
(52)

and the parameters \(\xi, \lambda_{(a)}\) and \(\mathcal{V}_{(a)}\) are arbitrary. For this special constant solution the Archimedean-type force
vanishes (or more precisely becomes hidden), and the cosmological model turns into the FLRW-type model with dark matter and nonvanishing cosmological constant $\Lambda$. For this model we should write

$$\Pi(x) = -\rho(x) = -\frac{\Lambda}{8\pi G},$$

and assume that

$$\rho_0 = \Lambda \left(1 + \frac{\sigma}{8\pi G}\right),$$

taking into account the solution \((51)\). This constant solution also exists when the Archimedean-type force is absent, i.e., when $\mathcal{V}(\rho)=0$ and $\mathcal{F}(x)=0$. In addition, this solution is an asymptotic limit for a family of integral curves at $x \to \infty$.

A question arises: for what values of the parameters $\sigma$, $\rho_0$, $\xi$ and $\mathcal{V}(\rho)$ is this special constant solution asymptotically stable? In other words, when does the deviation from the asymptotic value tend to zero, $Z(x \to \infty) \to 0$, where

$$\Pi(x) = -\frac{\rho_0}{1+\sigma} + Z(x), \quad Z(1)=0, \quad Z'(1)=\Pi'(1) \neq 0. \quad (55)$$

In order to answer this question let us analyze Eq. \((47)\) linearized with respect to $Z$ [see \((53)\)] at $x \to \infty$. In this case the leading-order term in the decomposition \((55)\) satisfies the equation

$$\xi^2 Z''(x) + xZ'(x) (4\xi + \sigma) + 3(1+\sigma)Z = 0. \quad (56)$$

This equation is clearly the well-known Euler equation. First, it can be obtained as an exact solution for the case $\mathcal{V}(\rho)=0$; second, this equation describes integral curves slightly deviating from the constant solution \((51)\) at $x \to \infty$. The characteristic polynomial of the Euler equation \((56)\) has two roots

$$s_{1,2} = \frac{1}{2\xi} \left[ -(\sigma + 3\xi) \pm \sqrt{(\sigma - 3\xi)^2 - 12\xi} \right], \quad (57)$$

which can be real or complex depending on the values of the parameters $\xi$ and $\sigma$. One can distinguish three subcases.

1. **Two different real roots \([\sigma - 3\xi]^2 > 12\xi\]**

When the discriminant in \((57)\) is positive, one obtains

$$Z(x) = \frac{\Pi'(1)}{2\Gamma} x^{-\gamma} (x^\Gamma - x^{-\Gamma}), \quad (58)$$

where

$$\gamma \equiv \frac{\sigma + 3\xi}{2\xi}, \quad \Gamma \equiv \frac{1}{2\xi} \sqrt{(\sigma - 3\xi)^2 - 12\xi}. \quad (59)$$

The solution \((58)\) tends to zero asymptotically at $x \to \infty$, when $\Gamma < \gamma$; it is possible, first, when $\sigma > 3\xi + 2\sqrt{3\xi}$ (for arbitrary $\xi$), second, when $-1 < \sigma < 3\xi - 2\sqrt{3\xi}$ for $\xi > \frac{1}{3}$.

2. **Double real roots \([\sigma - 3\xi]^2 = 12\xi\]**

In this case the solution for $Z(x)$ is

$$Z(x) = \frac{\Pi'(1)}{\beta} x^{-\gamma} \sin (\beta \log x), \quad (60)$$

where $\gamma=3+\sqrt{3}$, if $\sigma=3\xi+2\sqrt{3\xi}$, and $\gamma=3-\sqrt{3}$, if $\sigma=3\xi-2\sqrt{3\xi}$. The first solution relates to the asymptotically vanishing $Z(x \to \infty)$ for all $\xi$, the parameter $\sigma$ being positive. The second solution tends to zero asymptotically, when $\xi > \frac{1}{3}$, and the parameter $\sigma$ satisfies the inequality $\sigma > -1$. Finally, when $\xi=\frac{1}{3}$ and $\sigma=-1$, the solution is logarithmically unstable.

3. **Complex roots \([\sigma - 3\xi]^2 < 12\xi\]**

For the complex roots the solution is quasiperiodic

$$Z(x) = \frac{\Pi'(1)}{\beta} x^{-\gamma} \log (\gamma \log x), \quad (61)$$

where

$$\gamma \equiv \frac{\sigma + 3\xi}{2\xi}, \quad \beta \equiv \frac{1}{2\xi} \sqrt{12\xi - (\sigma - 3\xi)^2}. \quad (62)$$

This solution remains quasiperiodic and asymptotically small, when $\gamma > 0$; it is possible in the following two cases:

(i) when $0 < \xi < \frac{1}{3}$ and $-3\xi < \sigma < 3\xi + 2\sqrt{3\xi}$;

(ii) when $\xi \geq \frac{1}{3}$ and $3\xi - 2\sqrt{3\xi} \leq \sigma \leq 3\xi + 2\sqrt{3\xi}$.

A special case, when the roots are purely imaginary, relates to the condition $\gamma = 0$ or equivalently $\sigma = -3\xi$. This special case is realized, when $\xi < \frac{1}{3}$.

To conclude, we can state that the solution $\Pi(x) = -\frac{\rho_0}{1+\sigma}$ is asymptotically stable, first, when $0 < \xi < \frac{1}{3}$ and $\sigma > -3\xi$, second, when $\xi \geq \frac{1}{3}$ and $\sigma > -1$. This statement is illustrated by the Fig.1.

Since the zone of stability relates to the case $\sigma > -1$, the asymptotic value of the DE pressure is negative and the dark energy is positive at $\rho_0 > 0$. In the asymptotic limit $x \to \infty$ the energy density of the dark matter asymptotically vanishes, $\mathcal{E}(\infty) = 0$, thus the Hubble function and the scale factor tend to the following values

$$H(\infty) = \sqrt{\frac{8\pi G \rho_0}{3(1+\sigma)}}, \quad a(t) \to a(t_0) e^{H(\infty)t}. \quad (63)$$

Such a behavior corresponds to the de Sitter law with positive acceleration parameter $-q \equiv \frac{\ddot{a}}{a^2} = 1$, thus, one can indicate this exact solution as a $\Lambda$-type solution of the model with Archimedean-type interaction between DM and DE.

When $\sigma < -1$, the constant solution \((51)\) is unstable for arbitrary $\xi$. The case $\sigma = -1$ is the special one, and we consider this case in detail in the next section.
FIG. 1. The stability domain of the special solution \( \rho = - \Pi = \frac{1}{\Pi(1)} \), presented on the plane of the parameters \( \xi \) and \( \sigma \). The deviation from this constant solution decreases according to the power law \( (58) \), first, in the zone situated higher than the upper branch of the parabola, \( \sigma = 3 \xi + 2 \sqrt{\xi} \) for arbitrary \( \xi \), second, in the zone between the straight line \( \sigma = -1 \) and the lower branch of the parabola, \( \sigma = 3 \xi - 2 \sqrt{\xi} \) for \( \xi > \frac{3}{4} \). In the zone inside the parabola higher than the straight line \( \sigma = -3 \xi \) the deviation from the constant solution behaves as a damped sinusoid \( (61) \). On the branches of parabola the decreasing is described by the product of a power-law function and logarithm \( (60) \).

B. Special model with \( \sigma = -1 \):

Exact solutions of the anti-Gaussian type

For this specific model the key equation \( (47) \) does not admit constant solutions if \( \rho_0 \neq 0 \). In order to analyze the new situation we consider, first, massless dark matter, then the model with cold dark matter, and discuss exact solutions of the key equation, which now become logarithmic, providing the scale factor to be of the anti-Gaussian type.

1. Special model for a massless dark matter

When \( \sigma = -1 \) the key equation of the second order for massless dark matter

\[
\xi x^2 \Pi''(x) + x \Pi'(x) (4 \xi - 1) - \Pi'(x) E(0) \frac{\mathcal{V}(0)}{x^2} e^{\mathcal{V}(0)(\Pi(1) - \Pi(x))} = 0
\]

can be reduced to the first-order equation

\[
\xi [4 + \mathcal{V}(0) W] \frac{dW}{dz} = W [3 \xi - 1 - E(0) \mathcal{V}(0) z] + 3 \rho_0 = 0 \quad (64)
\]

can be reduced to the first-order equation

\[
\xi [4 + \mathcal{V}(0) W] \frac{dW}{dz} = W [3 \xi - 1 - E(0) \mathcal{V}(0) z] + 3 \rho_0 = 0 \quad (65)
\]

based on the following definitions (see, e.g., \( [41] \)):

\[
y = \Pi(x) - \Pi(1), \quad x \frac{dy}{dx} = W, \quad z = x^{-4} e^{-\mathcal{V}(0) y} \quad (66)
\]

Eq. \( (65) \) is the Abel equation of the second kind \( [41] \). The special solution to this equation can be obtained if we put the coefficient \([4 + \mathcal{V}(0) W]\) in front of the derivative equal to zero. Two constants

\[
W = - \frac{4}{\mathcal{V}(0)}, \quad z = \frac{1}{E(0)} \left[ \frac{3 \xi - 1}{\mathcal{V}(0)} - \frac{3}{4} \rho_0 \right] \quad (67)
\]

satisfy the key equation, if

\[
y(x) = - \frac{4}{\mathcal{V}(0)} \log x, \quad z = 1, \quad E(0) = \left[ \frac{3 \xi - 1}{\mathcal{V}(0)} - \frac{3}{4} \rho_0 \right], \quad (68)
\]

Of course, this special solution exists when \( \mathcal{V}(0) \neq 0 \), i.e., when the Archimedean-type force is present.

2. Logarithmic solution for the DE pressure

When the parameters of the model are coupled by the last relation in \( (68) \), we obtain an exact solution

\[
\Pi(x) = \Pi(1) - \frac{4}{\mathcal{V}(0)} \log x, \quad (69)
\]

\[
\rho(x) = \rho(1) + \frac{4}{\mathcal{V}(0)} \log x, \quad (70)
\]

where the relation

\[
\Pi'(1) = - \frac{4}{\mathcal{V}(0)} = \frac{1}{\xi} [\rho(1) - \rho_0 + \Pi(1)] \quad (71)
\]

is used. The sum \( \Pi(x) + \rho(x) \) remains constant for all values of \( x \)

\[
\rho(x) + \Pi(x) = \rho(1) + \Pi(1) = \rho_0 - \frac{4 \xi}{\mathcal{V}(0)}. \quad (72)
\]

The DM energy density \( E(x) \) is also constant, i.e.,

\[
E(x) = const = E(0) = - \frac{3}{4} \rho_0 + \frac{3 \xi - 1}{\mathcal{V}(0)}, \quad (73)
\]

due to the specific structure of the corresponding Archimedean-type force. The Hubble function \( H(x) \) can be found from Eq. \( (65) \), which now takes the form

\[
H^2(x) = \frac{8 \pi G}{3} \left[ \rho(1) + E(0) + \frac{4}{\mathcal{V}(0)} \log x \right]. \quad (74)
\]

Since the right-hand side of this equation has to be non-negative for every \( x > 1 \), we suppose that in addition to \( \mathcal{V}(0) > 0 \) the following inequality holds

\[
\rho(1) + E(0) \geq 0 \Rightarrow \Pi(1) \leq \frac{1}{4} \rho_0 - \frac{1 + \xi}{\mathcal{V}(0)}. \quad (75)
\]

According to \( (15) \) and \( (74) \) the scale factor \( a(t) \) can be written in the form

\[
a(t) = a^* \exp \left\{ \frac{8 \pi G}{3 \mathcal{V}(0)} (t - t^*)^2 \right\}, \quad (76)
\]
where the parameters with asterisks are defined as follows
\[ a^* = a(t_0) \exp \left\{ -\frac{V(0)}{4} \left[ \rho(1) + E(0) \right] \right\}, \]
\[ t^* = t_0 - \frac{3}{2\pi G} \sqrt{\frac{3}{2\pi G}} \frac{\rho(1) + E(0)}{\rho(1) + E(0)}. \] (77)
Such a scale factor describes an anti-Gaussian expansion without initial singularity. The acceleration parameter \(-q(t)\) for the anti-Gaussian expansion is positive and exceeds one:
\[ -q(t) = \frac{\ddot{a}}{a^2} = 1 + \frac{3V(0)}{16\pi G(t-t^*)^2} \geq 1. \] (78)
Moreover, it is a regular function of time (since \( t > t^* \)), and tends to one, when \( t \to \infty \). Comparing the anti-Gaussian function \( \text{(76)} \) with the corresponding de Sitter type function
\[ a(t) = a(t^*) \exp \left\{ \sqrt{\frac{A}{3}}(t-t^*) \right\}, \] (79)
one can see that the former increases slowly at \( t \approx t^* \), but then grows much more quickly.

3. Stability analysis of the anti-Gaussian solution

The differential equation \( \text{(65)} \) can be evidently reduced to the autonomous dynamic system
\[ \frac{dz}{dt} = \xi z[4 + V(0)W], \] (80)
\[ \frac{dW}{dt} = W \left[ 3\xi - 1 - E(0) \right] + 3\rho_0. \] (81)

The exact constant solution \( W_* = -\frac{4}{V(0)}, \quad z^* = 1, \) \( \text{(82)} \) valid at the condition \( \text{(68)} \), describes the stationary (singular) point of this dynamic system. In order to determine the type of this singular point let us consider the linearized system
\[ \frac{d\xi}{dt} = \xi V(0)\omega, \quad \frac{d\omega}{dt} = \frac{3}{4}\rho_0 V(0) + 4E(0)\xi, \] (83)
where
\[ W \to W_* + \omega, \quad z \to z^* + \xi. \] (84)
Since we consider the case, when \( \xi > 0, \quad E(0) > 0, \quad V(0) > 0 \), the roots of the corresponding characteristic equation are real
\[ s_{1,2} = \frac{3}{8} \rho_0 V(0) \left[ 1 \pm \sqrt{1 + \frac{256\xi E(0)}{9V(0)\rho_0^5}} \right]. \] (85)
The product of the roots \( s_1 \cdot s_2 = -4\xi E(0) V(0) < 0 \) is negative, thus, the singular point is the saddle one, and the anti-Gaussian solution is unstable.

4. Cold dark matter

The key equation \( \text{(47)} \) with the nonrelativistic source \( \text{(50)} \) can also be reduced to the first-order equation
\[ \xi z[5+2V(C)W] \frac{dW}{dz} = W \left[ 3\xi - 1 - 3N(C)T(C)\right] + 3\rho_0 \] (86)
based on the following definitions:
\[ y = \Pi(x) - \Pi(1), \quad \frac{dy}{dx} = W, \quad \xi = x^{-5} e^{-2V(C)y}. \] (87)
The corresponding exact special solution is
\[ \Pi(x) = \Pi(1) - \frac{5}{2V(C)} \log x, \] (88)

when
\[ \frac{6}{5}\rho_0 + 3N(C)T(C) = \frac{1}{V(C)}(3\xi - 1). \] (89)
The corresponding value of the DE energy density is
\[ \rho(x) = \rho(1) + \frac{5}{2V(C)} \log x, \] (90)
so, that the sum of \( \Pi(x) \) and \( \rho(x) \) again remains constant for all values of \( x \)
\[ \rho(x) + \Pi(x) = \rho(1) + \Pi(1) = \rho_0 - \frac{5\xi}{2V(C)}. \] (91)
The DM energy-density function \( E(x) \) is now a decreasing function of \( x \) (not a constant contrary to the massless case)
\[ E(x) = \frac{m(C)N(C)}{x^3}, \] (92)
and the Hubble function \( H(x) \) can be now found from the equation
\[ H^2(x) = \frac{8\pi G}{3} \left[ \rho(1) + \frac{5}{2V(C)} \log x + \frac{m(C)N(C)}{x^3} \right]. \] (93)
In the asymptotic regime \( x \to \infty \) the logarithmic term in \( \text{(33)} \) dominates, thus, we obtain again the anti-Gaussian solution
\[ a(t) = a^{**} \exp \left\{ \frac{20\pi G}{3V(C)}(t-t^{**})^2 \right\}, \] (94)
where the parameters with double asterisks are defined as follows
\[ a^{**} = a(t_0) \exp \left\{ -\frac{\rho(1) V(C)}{5} \right\}, \]
\[ t^{**} = t_0 - \frac{V(C)}{10} \sqrt{\frac{3\rho(1)}{2\pi G}}. \] (95)
Again it will be an accelerated expansion of the Universe with \( -q > 1 \), and this solution is also unstable.
5. Model with DE domination

In order to complete the analysis of the model with \( \sigma = -1 \), let us consider the case, when the dark matter is absent \((E_{(a)} = 0)\). In fact such a model can be considered as the approximate one, since at \( x \rightarrow \infty \) the DM contribution to the total energy decreases as \( x^{-3} \). The key equation for the DE pressure reduces now to the inhomogeneous Euler equation

\[
\xi x^2 \Pi''(x) + x \Pi'(x) (4\xi - 1) + 3\rho_0 = 0. \tag{96}
\]

The characteristic equation for the corresponding homogeneous Euler equation

\[
s(s - \nu) = 0, \tag{97}
\]

with \( \nu \equiv \frac{1-3\xi}{3} \), gives double roots \( s = 0 \), when \( \xi = 1/3 \), thus, let us consider two different cases.

(i) Special case: \( \sigma = -1, \xi \neq \frac{1}{3} \).

Exact solution to \((96)\) contains in this case a sum of logarithmic and power-law terms

\[
\Pi(x) = \Pi(1) + \frac{3\rho_0 \log x}{1-3\xi} + \left(\frac{x''}{x''(1)} - 1\right) \left(\rho(1) + \Pi(1) + \frac{\rho_0}{3\xi - 1}\right),
\]

\[
\rho(x) = \rho(1) - 3\xi \left[\Pi(x) - \Pi(1)\right] - 3\rho_0 \log x. \tag{98}
\]

Asymptotic behavior of the DE pressure and DE energy density at \( 3\xi > 1 \) for arbitrary initial data are the following

\[
\Pi(x \rightarrow \infty) \rightarrow -\frac{3\rho_0}{3\xi - 1} \log x,
\]

\[
\rho(x \rightarrow \infty) \rightarrow \frac{3\rho_0}{3\xi - 1} \log x. \tag{99}
\]

In the asymptotic regime we deal again with the anti-Gaussian expansion

\[
a(t \rightarrow \infty) \propto \exp \left\{ \frac{2\pi G\rho_0}{(3\xi - 1)^2} t^2 \right\}, \tag{100}
\]

when \( \xi > \frac{1}{3} \). We would like to mention that when the initial data \( \Pi(1), \rho(1) \) and \( \Pi'(1) \) take special values

\[
\Pi(1) = -\frac{\rho_0}{3\xi - 1}, \quad \rho(1) = 0, \quad \Pi'(1) = -\frac{3\rho_0}{3\xi - 1}, \tag{101}
\]

the power-law terms disappear from the exact formulas

\[
\Pi(x) = \frac{\rho_0}{1-3\xi} (1 + 3 \log x), \quad \rho(x) = \frac{3\rho_0}{3\xi - 1} \log x, \tag{102}
\]

and the anti-Gaussian solution

\[
a(t) = a(t_0) \exp \left\{ \frac{2\pi G\rho_0}{(3\xi - 1)} (t - t_0)^2 \right\}, \tag{103}
\]

become not only asymptotic, but the exact one. When \( \nu > 0 \), then \( \rho(x) \propto x^\nu, \quad H \propto x \nu \) and \( a(t \rightarrow \infty) \rightarrow 0 \). Thus, the Universe does not expand, when \( \xi < \frac{1}{3} \).

(ii) Special case: \( \sigma = -1, \xi = \frac{1}{3} \).

The root \( s = 0 \) of characteristic Eq.\((97)\) is now double, so, the exact solution contains the logarithmic terms in square,

\[
\Pi(x) = \Pi(1) + 3(\rho(1) + \Pi(1) - \rho_0) \log x - \frac{9}{2} \rho_0 \log^2 x, \tag{104}
\]

\[
\rho(x) = \rho(1) - 3(\rho(1) + \Pi(1)) \log x + \frac{9}{2} \rho_0 \log^2 x, \tag{105}
\]

the sum of these functions being linear in logarithm,

\[
\Pi(x) + \rho(x) = \Pi(1) + \rho(1) - 3\rho_0 \log x. \tag{106}
\]

The scale factor evolves now superexponentially. For instance, when the initial data satisfy the condition \( \rho(1) + \Pi(1) = 0 \), the scale factor has the form

\[
a(t) = a(t_0) e^{H(t_0)(t - t_0)}, \quad H(t_0) = \sqrt{\frac{8\pi G\rho(1)}{3}}, \tag{108}
\]

while asymptotically at \( t \rightarrow \infty \) the scale factor grows as

\[
a(t) \rightarrow a(t_0) \exp \left\{ \sqrt{\frac{2\rho(1)}{9\rho_0}} e^{\frac{12\pi G\rho_0(t - t_0)}} \right\}. \tag{109}
\]

For the Universe expansion described by the law \( H(t) \), the Hubble function is monotonic

\[
H(t) = \sqrt{\frac{8\pi G\rho(1)}{3}} \cosh \left( \sqrt{12\pi G\rho_0(t - t_0)} \right), \tag{110}
\]

and increases infinitely. The acceleration parameter

\[
-q(t) = 1 + \sqrt{\frac{9\rho_0}{2\rho(1)}} \sinh^2 \left( \sqrt{\frac{12\pi G\rho_0(t - t_0)}}{2\sqrt{12\pi G\rho_0}} \right), \tag{111}
\]

starts with \(-q(t_0) = 1\), reaches the maximum \(-q_{(\text{max})} = 1 + \sqrt{\frac{9\rho_0}{2\rho(1)}} \) at \( t = t_0 + \frac{\log(1 + \sqrt{2})}{\sqrt{12\pi G\rho_0}} \), and tends asymptotically to \(-q(\infty) = 1\).

6. Remark on the coupling of the baryon matter to DE

The baryon matter can be naturally included into the scheme of Archimedean-type coupling: for this purpose one can add to the sum the terms, which correspond to the standard particles. If one assumes that the baryon matter is not affected by the Archimedean-type coupling, one can put the corresponding coupling constants \( V_{(a)} \) equal to zero; nevertheless, the contribution of the baryon matter to the total stress-energy tensor now will be taken into account in the modified formula \([20]\).
IV. DISCUSSION

The study of the cosmological model, into which the Archimedean-type interaction between dark energy and dark matter is introduced, shows that the roles of DM and DE in the energy balance of the Universe can be revised. According to the obtained formula (21), the contribution of the DM particles, \( E_{\text{D}}(a(t)) \), into the total energy density depends on the state of DE pressure \( \Pi(a(t)) \) through the functions \( F_{\text{D}}(a) \) (see (23)). In the models without Archimedean-type force the energy of the DM particle decreases effectively because of the factor \( a^{-2}(t) \); in other words, all the particles, both nonrelativistic and ultrarelativistic at the initial moment \( t_0 \), inevitably become (effectively) nonrelativistic in the process of Universe expansion. When the Archimedean-type force acts on the DM particles, the particle energy (16) becomes much more complicated function of cosmological time due to the exponential dependence on the DE pressure. This means, in particular, that, nonrelativistic particles can become (effectively) ultrarelativistic due to the Archimedean-type force action, thus the corresponding contribution of cold dark matter into the total energy can be reestimated taking into account the sign, the value and DE in the energy balance of the Universe can be redistributed be-

In the first part of the work we focused on exact solutions of this new model. The first exact solution is the constant one, \( \Pi(x)=-\rho(x)=-\frac{c_0}{a^3} \), and relates to the case \( \sigma \neq -1 \). Since the DE pressure for this exact solution is constant, the Archimedean-type force becomes hidden, and we obtain the standard cosmology with \( \Lambda \) term. This solution is asymptotically stable, when the guiding parameters of the model, \( \xi \) and \( \sigma \), satisfy the inequalities \( 0<\xi<\frac{1}{3} \) and \( \sigma>-3\xi \), or \( \xi>\frac{1}{3} \) and \( \sigma>-1 \). When \( \sigma<-1 \), the solution is asymptotically unstable for arbitrary parameter \( \xi \).

Exact solutions of the second class, with \( \sigma=-1 \), are much more sophisticated, since the DE pressure is described by the logarithmic function of the ratio \( a(t)/a(t_0) \). The corresponding scale factor \( a(t) \) is given by the anti-Gaussian function (70), it has no singular points in the early Universe, and describes late-time accelerated expansion with the acceleration parameter \(-q>1\); this acceleration parameter is bigger than for the de Sitter model. Exact solutions of the anti-Gaussian type happen to be typical for different physical situations: for massless and massive nonrelativistic DM, for the case with DE domination, etc. This solution is unstable.

There are two specific values of the guiding parameters of the model: \( \sigma=-1 \) and \( \xi=\frac{1}{3} \). The first one, \( \sigma=-1 \), can be associated with the so-called phantom divider \( w(t)=-1=\frac{1}{\sigma} \). The second value, \( \xi=\frac{1}{3} \), can be denoted as a resonance value of the relaxation time parameter. Indeed, according to the formula (13) the function \( C(t) \) plays a role of the relaxation time for the DE pressure \( \Pi(t) \), say, \( \tau_1 \). When \( \xi=\frac{1}{3} \), one obtains that \( \tau_1=\frac{3H(t)}{\Theta(t)}=\frac{1}{\Theta(t)} \), where \( \Theta(t)=3H(t)=\nabla U_k \) is the expansion parameter. Thus, the characteristic time of expansion \( \frac{1}{\Theta(t)} \) coincides with the relaxation time parameter for the DE pressure, introducing some specific resonance condition. When \( \sigma=-1 \) and \( \xi=\frac{1}{3} \) simultaneously, there exists superexponential solution of the model, described by the scale factor (107).

A physical origin of the Archimedean-type force is not yet clear; nevertheless, this force seems to be very interesting from the viewpoints of a new model of interaction and a new scheme of redistribution of the cosmic energy between the interacting DE and DM constituents. The presented model is self-consistent, simple from the point of view of analysis and very promising. We keep in mind the story of the Chaplygin gas model (42), starting from
a classical analogy a new evolutionary model has been elaborated and applied to cosmology, although physical meaning of the Chaplygin scheme of interaction is under discussion till now.

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