An introduction to o-minimal structures *

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September 3, 1999

Abstract

The first papers on o-minimal structures appeared in the mid 1980s, since then the subject has grown into a wide ranging generalisation of semialgebraic, subanalytic and subpfaffian geometry. In these notes we will try to show that this is in fact the case by presenting several examples of o-minimal structures and by listing some geometric properties of sets and maps definable in o-minimal structures. We omit here any reference to the pure model theory of o-minimal structures and to the theory of groups and rings definable in o-minimal structures.

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*To appear in the proceedings of the TMR Junior and Summer School in Complex Dynamics, 2-11 September 1999, CMAF Lisboa, Portugal.
†Partially supported by JNICT grant PRAXIS XXI/BD/5915/95 and a TMR grant for the meeting.
1 Basic model theory

We start by recalling some basic notions of model theory (structures, expansions and reducts, definable sets, and elementary extensions, etc., - the reader already familiarised with these notions can skip this section) and we end the section by illustrating with examples these model theoretic notions.

Basic model theory. A structure \( \mathcal{N} \) consists of: (1) a non empty set \( N \); (2) a set of constants \((c_k^N)_{k \in K}\), where \( c_k^N \in N \); (3) a family of maps \((f_j^N)_{j \in J}\), where \( f_j^N \) is an \( n_j \)-ary map, \( f_j^N : N^{n_j} \rightarrow N \) and (4) a family \((R_i^N)_{i \in I}\) of relations that is, for each \( i \), \( R_i^N \) is a subset of \( N^{n_i} \) for some \( n_i \geq 1 \). We often use the following notation \( \mathcal{N} = (N, (c_k^N)_{k \in K}, (f_j^N)_{j \in J}, (R_i^N)_{i \in I}) \), and sometimes we omit the superscripts. The language \( \mathcal{L} \) associated to a structure \( \mathcal{N} \) consists of: (1) For each constant \( c_k^N \), a constant symbol \( c_k \); (2) For each map \( f_j^N \) a function symbol, \( f_j \) of arity \( n_j \) and (3) For each relation \( R_i^N \), a relation symbol, \( R_i \) of arity \( n_i \). We also include in \( \mathcal{L} \) a countable set of variables \((x_q)_{q \in Q}\) and use the notation \( \mathcal{L} = \{(c_k)_{k \in K}, (f_j)_{j \in J}, (R_i)_{i \in I}, (x_q)_{q \in Q}\} \). If \( \mathcal{L} \) is the language associated with the structure \( \mathcal{N} \) we say that \( \mathcal{N} \) is an \( \mathcal{L} \)-structure. If \( \mathcal{L}' \subseteq \mathcal{L} \) are two languages and \( \mathcal{N}' \) and \( \mathcal{N} \) are respectively an \( \mathcal{L}' \)-structure and an \( \mathcal{L} \)-structure, such that \( \mathcal{N}' = \mathcal{N} \) then we say that \( \mathcal{N} \) is an expansion of \( \mathcal{N}' \) or that \( \mathcal{N}' \) is a reduct of \( \mathcal{N} \).

Let \( \mathcal{L} \) be a language and \( \mathcal{N} \) an \( \mathcal{L} \)-structure. We are going to define inductively the set of \( \mathcal{L} \)-formulas and satisfaction of an \( \mathcal{L} \)-formula \( \phi \) in the \( \mathcal{L} \)-structure \( \mathcal{N} \), in order to define the \( \mathcal{N} \)-definable sets. The set of \( \mathcal{L} \)-terms is generated inductively by the following rules: (i) every variable is an \( \mathcal{L} \)-term, (ii) every constant of \( \mathcal{L} \) is an \( \mathcal{L} \)-term and (iii) if \( f \) is in \( \mathcal{L} \) is an \( n \)-ary function, and \( t_1, \ldots, t_n \) are \( \mathcal{L} \)-terms, then \( f(t_1, \ldots, t_n) \) is an \( \mathcal{L} \)-term. An atomic \( \mathcal{L} \)-formula is an expression of the form: \( t_1 = f_2 \) or \( R(t_1, \ldots, t_n) \) where \( R \) is an \( n \)-ary relation in \( \mathcal{L} \) and \( t_1, \ldots, t_n \) are \( \mathcal{L} \)-terms. We sometimes write \( R(t_1(x_1, \ldots, x_k), \ldots, t_n(x_1, \ldots, x_k)) \) if we want to explicitly show the variables occurring in the atomic \( \mathcal{L} \)-formula. Given a tuple \( a \in N^k \), we say that \( a \) satisfies \( R(t_1(x), \ldots, t_n(x)) \) in \( \mathcal{N} \) where \( x = (x_1, \ldots, x_k) \) if \( R^N(t_1(a), \ldots, t_n(a)) \) holds. We denote this by \( \mathcal{N} \models R(t_1(a), \ldots, t_n(a)) \).

We say that \( S \subseteq N^k \) is an atomic \( \mathcal{N} \)-definable subset (defined over \( A \subseteq N \) if there is \( b \in A^m \) such that \( S = \{a \in N^k : \mathcal{N} \models \phi(a, b)\} \) for some atomic \( \mathcal{L} \)-formula \( \phi(x, y) \) with \( x = (x_1, \ldots, x_k) \) and \( y = (x_{k+1}, \ldots, x_{k+m}) \). We now generate the \( \mathcal{L} \)-formulas (resp., the \( \mathcal{N} \)-definable sets) from the atomic \( \mathcal{L} \)-formulas (resp., atomic \( \mathcal{N} \)-definable sets) using the following operators: \( \land \) (and) which corresponds to intersection, \( \lor \) (or) which corresponds to union, \( \neg \) (not) which corresponds to complementation, \( \exists \) (there exists) corresponding to projection and \( \forall \) (for all) which
corresponds to inverse image under projections. The construction is as follows: (i) all atomic $L$-formulas are $L$-formulas; (ii) if $\phi_1(x)$ and $\phi_2(x)$ are $L$-formulas, then $(\phi_1 \land \phi_2)(x)$ and $(\phi_1 \lor \phi_2)(x)$ are $L$-formulas, and for $a \in N^k$, $N \models (\phi_1 \land \phi_2)(a)$ iff $N \models \phi_1(a)$ and $N \models \phi_2(a)$, we also have the obvious clause for $\forall$; (iii) if $\phi(x)$ is an $L$-formula, $\neg \phi(x)$ is an $L$-formula, and for $a \in N^k$, $N \models \neg \phi(a)$ iff $\phi(a)$ does not hold in $N$ (this is denote by $N \not\models \phi(a)$); (iv) if $\phi(x, x_{k+1})$ is an $L$-formula, then $\exists x_{k+1} \phi(x, x_{k+1})$ is an $L$-formula, and for $a \in N^k$, $N \models \exists x_{k+1} \phi(a, x_k)$ iff there exists $b \in N$ such that $N \models \phi(a, b)$; (v) the obvious clauses for $\forall$. The $L$-formulas constructed using only (i), (ii) and (iii) are called quantifier-free $L$-formulas. Two $L$-formulas $\phi_1(x)$ and $\phi_2(x)$ are equivalent in $N$ if for all $a \in N^k$, $N \models \phi_1(a)$ iff $N \models \phi_2(a)$. We say that $N$ has quantifier elimination if every $L$-formula is equivalent in $N$ to a quantifier-free $L$-formula; we say that $N$ is model complete if every $L$-formula is equivalent in $N$ to an existential $L$-formula i.e., an $L$-formula of the form $\exists y \phi(x, y)$.

A subset $D \subseteq N^k$ is an $N$-definable subset (defined over $A \subseteq N$) if there is an $L$-formula $\phi(x, y)$ with $x = (x_1, \ldots, x_k)$ and $y = (x_{k+1}, \ldots, x_{k+m})$ and some $b \in A^m$ such that $D = \{a \in N^k : N \models \phi(a, b)\}$. The “$N$-constructible sets” are those $N$-definable sets determined by some quantifier-free $L$-formula, they are finite boolean combination of atomic $N$-definable sets. If $N$ has quantifier elimination then every $N$-definable set is an $N$-constructible set and if $N$ is model complete then every $N$-definable set is a projection of an $N$-constructible set. If $A \subseteq N^k$ and $B \subseteq N^m$ are $N$-definable sets (over $A \subseteq N$), a function $f : A \rightarrow B$ is $N$-definable (over $A$) if its graph is an $N$-definable set (over $A$). More generally, a structure $M = (M, (c^M_k)_{k \in K}, (f^M_j)_{j \in J}, (R^M_i)_{i \in I})$ is $N$-definable (over $A$) if: (i) $M \subseteq N^l$ is $N$-definable (over $A$); (ii) for each $k \in K$ there is a point $m_k \in M$ corresponding to $c^M_k$; (iii) for each $j \in J$ the function $f^M_j : M^{n_j} \rightarrow M$ is $N$-definable (over $A$) and (iv) for each $i \in I$ the relation $R^M_i \subseteq M^{m_i}$ is $N$-definable (over $A$). Note that, in this case every $M$-definable set is also an $N$-definable set.

Given two $L$-structures $N$ and $M$, a map $h : N \rightarrow M$ (which determines in the obvious way a map $h : N^k \rightarrow M^k$) is a homomorphism if: (i) for every constant $c$ in $L$, $h(c^N) = c^M$; (ii) for every $n$-ary function $f$ in $L$, for every $a \in N^n$, $h(f^N(a)) = f^M(h(a))$ and (iii) for every $n$-ary relation $R$ in $L$, for every $a \in N^n$, if $R^N(a)$ then $R^M(h(a))$. An injective homomorphism $h : N \rightarrow M$ is an embedding if for every $n$-ary relation $R$ in $L$, for every $a \in N^n$, $R^N(a)$ if and only if $R^M(a)$. An isomorphism is a bijective embedding. We say that $N$ is an $L$-substructure of $M$, denoted by $N \subseteq M$, if $N \subseteq M$ and the inclusion map is an embedding. Let $N \subseteq M$. We say that $M$ is an elementary extension of $N$ (or that $N$ is an elementary substructure of $M$), denoted by $N \preceq M$, if for every $L$-formula $\phi(x)$, for all $a \in N^k$, we have $N \models \phi(a)$ iff $M \models \phi(a)$. This is equivalent (by
Tarski-Vaught test) to saying that for every non empty $\mathbf{M}$-definable set $E \subseteq M^l$, defined with parameters from $N$, $E(N) := E \cap N^l$ ("the set of $N$-points of $E$") is a non empty $\mathbf{N}$-definable set. Clearly, if $S \subseteq N^l$ is an $\mathbf{N}$-definable set defined with parameters from $N$ and $\mathbf{N} \preceq \mathbf{M}$, then the $\mathbf{L}$-formula which determines $S$ determines an $\mathbf{M}$-definable set $S(M) \subseteq M^l$ ("the $M$-points of $S$"). The theory $Th(\mathbf{N})$ of an $\mathbf{L}$-structure $\mathbf{N}$ is the collection of all $\mathbf{L}$-sentences (i.e., $\mathbf{L}$-formulas without free variables) $\sigma$ such that $\mathbf{N} \models \sigma$. $\mathbf{N}$ is elementarily equivalent to $\mathbf{M}$, denoted $\mathbf{N} \equiv \mathbf{M}$ iff $Th(\mathbf{N}) = Th(\mathbf{M})$. Clearly, if $\mathbf{N} \preceq \mathbf{M}$ then $\mathbf{N} \equiv \mathbf{M}$. Note also that if $\mathbf{N}$ has quantifier elimination (resp., is model complete) and $\mathbf{N} \equiv \mathbf{M}$ then $\mathbf{M}$ has quantifier elimination (resp., is model complete).

The following two facts (the Löwenheim-Skolem theorems) are fundamental theorems of basic model theory: (1) Let $\mathbf{L}$ be a language, $\mathbf{N}$ an $\mathbf{L}$-structure and $X \subseteq N$. Then for every cardinal $\kappa$ such that $|X| + |L| \leq \kappa \leq |N|$, $\mathbf{N}$ has an elementary substructure $\mathbf{M}$ such that $X \subseteq M$ and $|M| = \kappa$; (2) Let $\mathbf{L}$ be a language, let $\mathbf{N}$ be an infinite $\mathbf{L}$-structure. Then for any cardinal $\kappa > |N|$, $\mathbf{N}$ has an elementary extension of cardinality $\kappa$.

A set of $\mathbf{L}$-sentences $\Sigma$ is consistent if there is an $\mathbf{L}$-structure $\mathbf{N}$ such that for all $\sigma \in \Sigma$ we have $\mathbf{N} \models \sigma$. In this case we say that $\mathbf{N}$ is a model of $\Sigma$, denoted $\mathbf{N} \models \Sigma$. An $\mathbf{L}$-theory (resp., a complete $\mathbf{L}$-theory) is a consistent set of $\mathbf{L}$-sentences (resp., a maximal consistent set of $\mathbf{L}$-sentences). An $\mathbf{L}$-theory $T$ is axiomatizable if there is a set of $\mathbf{L}$-sentences $\Sigma$ (the set of axioms) such that for every $\mathbf{L}$-structure $\mathbf{N}$, $\mathbf{N} \models T$ iff $\mathbf{N} \models \Sigma$. The Compactness theorem says that: if $\Sigma$ is a set of $\mathbf{L}$-sentences then $\Sigma$ is consistent iff every finite subset of $\Sigma$ is consistent. Moreover, if $\mathbf{N}$ is an $\mathbf{L}$-structure and $F$ is a family of $\mathbf{N}$-definable subsets of $N^k$ with finite intersection property in $N^k$ then there is an elementary extension $\mathbf{M}$ of $\mathbf{N}$ such that $F$ has non empty intersection in $M^k$.

**Examples.** Let $\mathbf{L}_{\text{rings}} := \{0,1,+,-,\cdot\}$ be the language of rings. Then $\mathbb{C} := (\mathbb{C},0,1,+,-,\cdot)$ is an $\mathbf{L}_{\text{rings}}$-structure. The atomic sets in $\mathbb{C}$ are exactly the Zariski closed sets, and by Chevalley’s theorem, the $\mathbb{C}$-definable sets are the constructible sets (i.e., boolean combinations of Zariski closed sets) which means that $\mathbb{C}$ has quantifier elimination. The theory $Th(\mathbb{C})$ of $\mathbb{C}$ is the theory $ACF_0$ of algebraically closed fields of characteristic zero which is axiomatised by the usual axioms for fields of characteristic zero, together with $\forall x_1 \ldots \forall x_l \exists y(y^l + x_1 y^{l-1} + \cdots + x_{l-1} y + x_l = 0)$, for each positive integer $l$. Another model of $ACF_0$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The compactness theorem shows that $ACF_0$ has model of any transcendence degree and the Löwenheim-Skolem theorem shows that $ACF_0$ has models of any infinite cardinality. Two model of $ACF_0$ are isomorphic iff their transcendence base over $\mathbb{Q}$ has the same cardinality, in particular for an
uncountable cardinal $\kappa$, up to isomorphism there is only one model of $ACF_0$ of cardinality $\kappa$. (And there are $2^{\aleph_0}$ countable models). The models of $ACF_0$ are called algebraically closed fields (of characteristic zero) and are examples of strongly minimal structures i.e., structures $N$ such that any $N$-definable subset of $N$ is either finite or co-finite. Other examples of such structures are a nonempty set and a vector space over a division ring.

Let $L_{ord} := \{0, 1, +, -, \cdot, <\}$ be the language of ordered rings. Then $\mathbb{R} := (\mathbb{R}, 0, 1, +, -, \cdot, <)$ is an $L_{ord}$-structure. Tarski [1] showed that $\mathbb{R}$ has quantifier elimination and so the $\mathbb{R}$-definable subsets of $\mathbb{R}^l$ are boolean combinations of sets of the form $\{a \in \mathbb{R}^l : f(a) = 0\}$ and $\{a \in \mathbb{R}^l : g(a) > 0\}$ where $f, g \in \mathbb{R}[x_1, \ldots, x_l]$. These sets are called semi-algebraic. The theory $Th(\mathbb{R})$ of $\mathbb{R}$ is the theory $RCF$ of real closed fields which is axiomatised by the usual axioms for ordered fields together with (i) $\forall x_1 \ldots \forall x_l(x_1^2 + \cdots + x_l^2 \neq 1)$ (for each positive integer $l$), (ii) $\forall x \exists y(x = y^2 \lor -x = y^2)$ and (iii) $\forall x_1 \ldots \forall x_l \exists y(y^l + x_1y^{l-1} + \cdots + x_{l-1}y + x_l = 0)$ (for all odd $l$). Another model of $RCF$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$. Again, the Löwenheim-Skolem theorem shows that $RCF$ has models of any infinite cardinality, but unlike $ACF$, for any infinite cardinal $\kappa$, up to isomorphism there are $2^\kappa$ model of $RCF$ of cardinality $\kappa$. The models of $RCF$ are called real closed fields and they are examples of o-minimal structures (see the definition below).

2 O-minimal structures

O-minimal structures. An o-minimal structure is an expansion $N = (N, <, \ldots)$ of a linearly ordered nonempty set $(N, <)$, such that every $N$-definable subset of $N$ is a finite union of points and intervals with endpoints in $N \cup \{-\infty, +\infty\}$.

Note the following important results: let $N$ be an o-minimal structures then: (0) every $N$-definable structure $M$ which is an expansion of a linearly ordered nonempty set $(M, <_M)$ is also o-minimal; (1) [KPS] if $M$ is a structure (in the language of $N$) such that $N \equiv M$ then $M$ is also o-minimal; (2) [PS] for every $A \subseteq N$ there is a prime model of $Th(N)$ over $A$ (or simply prime model if $A$ is empty) i.e., there is an o-minimal structure $P$ such that $A \subseteq P$, $P \equiv N$, $P$ is unique up isomorphism over $A$ and for all $M \equiv N$ with $A \subseteq M$, there is an elementary embedding $P \rightarrow M$ which is the identity over $A$. In particular by (2), if $S$ is an $N$-definable set over $P$ and $M \equiv N$ with $A \subseteq M$, then $S$ determines an $M$-definable set $S(M)$. Let $N$ be an o-minimal structure in the language $L$. For every $\kappa > \max\{\aleph_0, |L|\}$ there up to isomorphism $2^\kappa$ o-minimal structures $M$ such that $|M| = \kappa$ and $M \equiv N$ (see [Sh]), and if $L$ is countable then up to isomorphism there are either $2^{\aleph_0}$ or $6^{3^{\aleph_0}}$ countable o-minimal structures $M$ such that $M \equiv N$.
Let \( N \) be an o-minimal structure. We now list some geometric properties of \( N \)-definable sets and \( N \)-definable maps, most of these can be found in book [D2] but the proof appear elsewhere. Note that because of the presence of an ordering in \( N \), each \( N^m \) has a natural topology and if \( N \) is an expansion of ordered ring it makes sense to talk about differentiability (but in general, it does not make sense to talk about integrability). Two of the most powerful results are the \( C^p \)-cell decomposition theorem for \( N \)-definable sets and \( N \)-definable maps (where \( p = 0 \) if \( N \) is not an expansion of an ordered ring) and the monotonicity theorem for \( N \)-definable one variable functions. The cell decomposition theorem has several consequences: (1) it is used to define the notion of o-minimal dimension and o-minimal Euler characteristic for \( N \)-definable sets, these notions are well behaved under the usual set theoretic operations on \( N \)-definable sets, are invariant under \( N \)-definable bijections and given an \( N \)-definable family of \( N \)-definable sets, the set of parameters whose fibre in the family has a fixed dimension (resp., Euler characteristic) is also an \( N \)-definable set; (2) it shows that every \( N \)-definable set has only finitely many \( N \)-definably connected components, and given an \( N \)-definable family of \( N \)-definable sets there is a uniform bound on the number of \( N \)-definably connected components of the fibres in the family.

A local version of the o-minimal analog of Zilber’s conjecture holds [PeS]: if \( a \in N \) then the structure induced by \( N \) on an open interval containing \( a \) is either trivial, or the structure of an open interval in an ordered vector space over some ordered division ring or an o-minimal expansion of an ordered ring. If \( N \) expands an ordered group, then \( N \) is either linear, eventually linear, or linearly bounded (several useful characterisations of these three cases are given in [Li], [E] and [MS]). When \( N \) expands an ordered ring then \( N \) is either power bounded (polynomially bounded if \( N \) is Archimedean) or is exponential (see [Mi1]) moreover, several geometric properties from semialgebraic and subanalytic geometry also hold for \( N \)-definable sets and \( N \)-definable maps: we have (1) an \( N \)-definable curve selection theorem (this in fact holds in the more general case where \( N \) has \( N \)-definable Skolem functions e.g., \( N \) expands an ordered group); (2) an \( N \)-definable triangulation theorem; (3) an \( N \)-definable trivialization theorem; (4) and finally (see [DM2]) an \( N \)-definable analog of the uniform bounds on growths theorem, the \( C^p \)-multiplier theorem, the generalised Łojasiewicz inequality, the \( C^p \) zero set theorem, the \( C^p \) Whitney stratification theorem, etc.

**Examples.** As examples of o-minimal structures apart from the trivial ones such as: (1) dense linearly ordered nonempty sets without endpoints -conversely by
if $N = (N, <, \ldots)$ is an o-minimal structure, then $(N, <)$ is elementarily equivalent to an ordered set of the form $C_1 + \cdots + C_m$ where: (i) $C_i$ is elementarily equivalent to one of the following ordered sets: a finite ordered set, $\omega$, $\omega^*$, $\omega + \omega^*$, $\omega^* + \omega$, $\mathbb{Q}$ and (ii) if $C_i$ does not have a last element, then $C_{i+1}$ has a first element, also by results from $\text{PiS1}$ and $\text{PiS3}$ one usually assumes, without loss of generality, that $(N, <)$ is a dense linearly ordered set without end points;

(2) ordered divisible abelian groups, in fact also ordered vector spaces over a division ring (semilinear geometry) - and conversely by $\text{PiS1}$ if $N$ is an o-minimal expansion of an ordered group $(N, 0, +, <)$ then $(N, 0, +, <)$ is an ordered divisible abelian group, in fact it is also an ordered vector spaces over the ordered division ring $\Lambda(N)$ of all $N$-definable endomorphisms of $(N, 0, +, <)$ - we also have the following examples which are of special interest to geometers:

(3) $\mathbb{R} := (\mathbb{R}, 0, 1, +, \cdot, <)$ (semialgebraic geometry, by Tarski-Seidenberg theorem $\text{[1]}$ this structure has quantifier elimination and therefore every $\mathbb{R}$-definable set is a semialgebraic set and o-minimality follows from this, in fact any real closed field is o-minimal i.e., any ordered ring $\mathbb{R} := (\mathbb{R}, 0, 1, +, \cdot, <)$ such that $\mathbb{R} \cong \mathbb{R}$, for example the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$ is a real closed field - and conversely by $\text{PiS1}$ if $N$ is an o-minimal expansion of an ordered ring $(N, 0, 1, +, \cdot, <)$ then $(N, 0, 1, +, \cdot, <)$ is a real closed field;

(4) $\mathbb{R}_{an} := (\mathbb{R}, 0, 1, +, \cdot, <, (f)_{f\in an})$ where $an$ is the collection of all functions which are the restriction to $[-1, 1]^n$ (for some $n$) of analytic functions on some open neighbourhood of $[-1, 1]^n$ (by results of Gabrielov $\text{[3]}$, Lojasiewicz $\text{[4]}$ and Bierstone and Milman $\text{[BM]}$ the $\mathbb{R}_{an}$-definable sets are exactly the subanalytic sets in the projective spaces we therefore get (global) subanalytic geometry, this structure is model complete and o-minimal as remarked by van den Dries $\text{[D2]}$; $\mathbb{R}_{an}$ has quantifier elimination after adding the function $1/x$ (with $1/0 = 0$) to its language, an axiomatization of its theory is given in $\text{[DMM1]}$. Some model complete reducts of $\mathbb{R}_{an}$ expanding $\mathbb{R}$ were constructed: the expansions of $\mathbb{R}$ by restricted elementary functions $\text{[D3]}$ and expansions of $\mathbb{R}$ by abelian and elliptic functions. By $\text{[Mi2]}$, the expansion of $\mathbb{R}_{an}$ by power functions has quantifier elimination, is o-minimal and has an explicit axiomatization;

(5) Wilkie $\text{[W1]}$ uses model theory, valuation theory and results by Khovanskii $\text{[4]}$ to show that $\mathbb{R}_{exp} := (\mathbb{R}, 0, 1, +, \cdot, <, exp)$ and the expansion of $\mathbb{R}$ by restricted Pfaffian functions are model complete and o-minimal; By $\text{[Mi2]}$, the expansion of $\mathbb{R}$ by restricted Pfaffian functions, power functions and a constant symbol for each exponent is model complete and o-minimal. A quantifier elimination result and a (non trivial) axiomatisation of the expansion of $\mathbb{R}$ by the restricted Pfaffian functions (and also by power functions and a constant symbol for each exponent) is not known. Ressayre gives an axiomatisation of the theory of $\mathbb{R}_{exp}$, as for quantifier
elimination, van den Dries [D2] adapts an old result of Osgood to show that an expansion of $\mathbb{R}$ by a family of total real analytic functions admits elimination of quantifiers iff each such function is semialgebraic. Macintyre and Wilkie [MW] show that if the Schanuel conjecture holds then $\text{Th} (\mathbb{R}_{\exp})$ is decidable;

(6) Wilkie’s method and Khovanskii result are refined in [DM1] to show that the structure $\mathbb{R}_{\text{an,exp}} := (\mathbb{R}, 0, 1, +, \cdot, <, (f)_{f \in \text{an,exp}})$ is model complete and o-minimal, van den Dries, Macintyre and Marker [DMM1], inspired by work of Ressayre (a preliminary version of [Re]) give a different proof of this fact and in [DMM2] explicit nonstandard models of $\mathbb{R}_{\text{an,exp}}$ are constructed leading to a solution of a conjecture posed by Hardy and to some non definability results: the following functions are not $\mathbb{R}_{\text{an,exp}}$-definable: $\Gamma |_{(0, +\infty)}$ (the gamma function), the error function, the logarithmic integral and $\zeta |_{(1, +\infty)}$ (the Riemann zeta function).

By [DMM1], the expansion $\mathbb{R}_{\text{an,exp,log}}$ of $\mathbb{R}_{\text{an,exp}}$ has quantifier elimination and both have an explicit axiomatisation. A geometric proof of o-minimality and model completeness of $\mathbb{R}_{\text{an,exp}}$ has been given recently by Lion and Rolin [LR1];

(7) Denef and van den Dries [DD] give a proof of model completeness and o-minimality of $\mathbb{R}_{\text{an}}$ using more explicitly the Weirstrass preparation theorem, this is then generalised to establish the model completeness and o-minimality of $\mathbb{R}_{\text{an}}^{*}$, the expansion of $\mathbb{R}$ by (restricted convergent) generalized power series ([DS1]) and of $\mathbb{R}_{G}$ the expansion of $\mathbb{R}$ by (a variant of) Tougeron’s class of Gevrey functions ([DS2]). In [DS2] the expansions $\mathbb{R}_{\text{an}}^{*,exp}$ and $\mathbb{R}_{G,exp}$ of $\mathbb{R}_{\text{an}}^{*}$ and $\mathbb{R}_{G}$ by $exp$ are shown to be o-minimal and model complete, in particular $\zeta(1, +\infty)$ is definable in $\mathbb{R}_{\text{an}}^{*,exp}$ and $\Gamma |_{[0, +\infty)}$ is definable in $\mathbb{R}_{G,exp}$ since $\zeta(-log(x)) = \sum_{n=1}^{\infty} x^{-ln}$ is definable in $\mathbb{R}_{\text{an}}^{*}$ and $\Gamma |_{[0, +\infty)}$ is definable in $\mathbb{R}_{G,exp}$ since $log\Gamma(x) = (x - \frac{1}{2})logx - x + \frac{1}{2} log(2\pi) + \phi(x)$ where $\phi$ is definable in $\mathbb{R}_{G}$; Quantifier elimination results and (non trivial) axiomatisations for $\mathbb{R}_{\text{an}}^{*}$ and $\mathbb{R}_{G}$ are not known. But by [DS2], if $\tilde{\mathbb{R}}$ is a polynomially bounded o-minimal expansion of $\mathbb{R}$ such that $exp |_{[0, 1]}$ is $\tilde{\mathbb{R}}$-definable, then the expansion of $\tilde{\mathbb{R}}$ by $exp$ (resp., $exp$ and $log$) is model complete (resp., has quantifier elimination) and is o-minimal, moreover they have explicit axiomatisations - the axiomatisation of $\text{Th}(\tilde{\mathbb{R}})$ plus Ressayre axioms for $exp$ (resp., $exp$ and $log$)- and they are exponentially bounded.

(8) Finally, building on work of Charbonnel, Wilkie [W2] gives necessary and sufficient conditions for an expansion of $\mathbb{R}$ by total $C^\infty$ functions to be o-minimal, in particular o-minimality of the expansion of $\mathbb{R}$ by total $C^\infty$ Pfaffian functions is established, a geometric treatment of Wilkie’s result in the subanalytic context is given by Lion and Rolin [LR2] using Moussu and Roche’s [MR] notion of Rolle leafs and the Khovanskii-Rolle theorem, this is later generalised [S] to show that any o-minimal expansion $\tilde{\mathbb{R}}$ of $\mathbb{R}$ has an o-minimal Pfaffian closure $\mathcal{P}(\tilde{\mathbb{R}})$ i.e., the Rolle leafs of 1-forms with $C^1$ coefficients definable in $\mathcal{P}(\tilde{\mathbb{R}})$ are already definable.
in $\mathcal{P}(\mathbb{R})$. In [LS] the prove of the existence of the o-minimal Pfaffian closure $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ is refined to show that if $\mathbb{R}$ has analytic cell decomposition (resp., is exponentially bounded) then $\mathcal{P}(\mathbb{R})$ has analytic cell decomposition (resp., is exponentially bounded). It's not known if the model completeness of $\mathbb{R}$ implies that of $\mathcal{P}(\mathbb{R})$, but a relative model completeness result is proved in [LS].

All the examples of o-minimal expansions of $\mathbb{R}$ mentioned above have analytic cell decomposition and those which do not have $\text{exp}$ on their language are polynomially bounded (the other ones are exponentially bounded). There is no known example of an exponential o-minimal expansion of $\mathbb{R}$ which is not exponentially bounded.

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