THE CO-DIVERGENCE OF VECTOR VALUED CURRENTS

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Abstract. In the context of stress theory of the mechanics of continuous media, a generalization of the boundary operator for de Rham currents—the co-divergence operator—is introduced. While the boundary operator of de Rham’s theory applies to real valued currents, the co-divergence operator acts on vector valued currents, i.e., functionals dual to differential forms valued in a vector bundle. From the point of view of continuum mechanics, the framework presented here allows for the formulation of the principal notions of continuum mechanics on a manifold that does not have a Riemannian metric or a connection while at the same time allowing irregular bodies and velocity fields.

1. Introduction. In de Rham’s current theory [1] the boundary $\partial T$ of a current $T$, where $T$ is regarded as a continuous linear functional on the space $C_0^\infty \left( \Lambda^k T^* S \right)$ of compactly supported smooth $k$-forms on an $n$-dimensional manifold $S$, is defined as the dual of the exterior derivative operator so that $\partial T(\omega) = T(d\omega)$. Let $\pi : W \rightarrow S$ be a vector bundle over $S$. A vector valued $k$-form is a section of the vector bundle $\Lambda^k(T^*S,W)$ whose fiber $\Lambda^k(T^*S,W)_x$ at $x \in S$ is the vector space of alternating multi-linear mappings $(T_x S)^k \rightarrow W_x$. A vector valued current can be defined by analogy as a continuous linear functional on the space $C_0^\infty \left( \Lambda^k (T^*S,W) \right)$. Unlike the case of real valued forms, the exterior derivative of a vector valued form is not defined for a general vector bundle (see, for example, Palais [6, p. 10]). As a result, the boundary of a vector current is not defined.

In this article, we construct a generalization of the boundary operator that applies to vector valued $n$-currents. The construction may be described roughly as follows. A real valued locally integrable function $\varphi$ defined on an $n$-dimensional submanifold with boundary $B \subset S$, regarded as a 0-form, induces an $n$-current $\tilde{\varphi}$ by

$$\tilde{\varphi}(\theta) = \int_B \varphi \theta$$  \hspace{1cm} (1.1)

for any compactly supported $n$-form $\theta$. (This current is denoted as $\varphi B$ in [9].) Thus, the boundary $\partial \tilde{\varphi}$ is the $(n-1)$-current defined by

$$\partial \tilde{\varphi}(\omega) = \tilde{\varphi}(d\omega) = \int_B \varphi d\omega$$  \hspace{1cm} (1.2)

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for any compactly supported \((n-1)\)-form \(\omega\). In the suggested construction, a vector field \(w\) over the restriction of \(W\) to \(B\) will be the analog of the function \(\varphi\). Thus, in order to have an analogous structure so that a generalized boundary operator \(\partial\) is well defined, one has to specify an object \(S\) and a differential operator \(\text{div}\) setting

\[
\partial\hat{w}(S) = \int_B w \cdot \text{div} S
\]

(1.3)

for some paring \(w \cdot \text{div} S\). An indication as to the nature of the object \(S\) is suggested by the fact that in the real-valued case, an \((n-1)\)-form \(\omega\) induces a section \(\hat{\omega}\) of \(L(T^*S, \bigwedge^n T^*S)\) by \(\hat{\omega}(\psi) = \psi \wedge \omega\). Thus, \(\omega\) induces the linear map

\[
\hat{\omega} \circ d : C^\infty(S) \to C^\infty(\bigwedge^n T^*S) \quad \text{by} \quad \phi \mapsto \hat{\omega}(d\phi) = d\phi \wedge \omega.
\]

(1.4)

Since the exterior derivative of a vector field is not well-defined, we may generalize it to be the first jet of the vector field \(w\). Thus, in analogy with \(\hat{\omega}\) we set \(S\) to be a section of the bundle of linear mappings \(L(J^1(W), \bigwedge^n T^*S)\) so that for a section \(w\) one has the linear map \(C^\infty(W) \to C^\infty(\bigwedge^n T^*S)\) by \(w \mapsto S(j^1(w))\). Next, for the integral above to be well defined, the object \(\text{div} S\) should be a section of \(L(W, \bigwedge^n T^*S)\). Indeed, we use such a divergence differential operator

\[
\text{div} : C^\infty_0 \left( \bigwedge^n (T^*S, (J^1(W))^*) \right) \to C^\infty_0 \left( \bigwedge^n (T^*S, W^*) \right),
\]

(1.5)

which is a generalization of the traditional divergence operator (see [8]).

The mechanical interpretation of a section \(S\) of \(L(J^1(W), \bigwedge^n T^*S)\) is a variational stress field in the sense that for a vector field \(w\), regarded as a generalized velocity field, \(S(j^1(w))\) is the power expended by the stress on the “derivative” of the velocity field.

In [7], the authors present Cauchy’s flux theory from the point of view of Whitney’s geometric integration theory, [9]. Whitney’s integration theory is formulated in the setting of a Euclidean space. Hence, in [7], the analysis is restricted essentially to \(n\)-tuples of scalar valued fluxes in a Euclidean space. An alternative approach to geometric integration theory as in [2, 4] is based on the theory of de Rham currents. Although the theory of flat chains in [2, 4] is set in a Euclidean space, it may be generalized to manifolds without a metric in the scalar case (see [3]). In this article, adopting the approach of de Rham, we present some necessary components for vector valued geometric integration theory on a manifold in the context of stress theory in continuum mechanics.

2. Notation and preliminaries. We consider a smooth vector bundle \(\pi : W \to S\) and the space \(C^\infty_0(W)\) containing the smooth sections of \(W\) having compact supports in \(S\). Let \(\{(U_k, \varphi_k, \Phi_k)\}\) be a vector bundle atlas so that for each \(k\), \(U_k \subset S\) is open and \(\Phi_k : \pi^{-1}(U_k) \to \varphi_k(U) \times V\). Thus, if the dimension of \(S\) is \(n\), \(\varphi_k(U_k)\) is open in \(\mathbb{R}^n\) and \(\Phi_k|_{\pi^{-1}(x)} : \pi^{-1}(x) = W_x \to V\), is a linear isomorphism onto the typical fiber \(V\) for every \(x \in U_k\). A section \(w\) of \(W\) is therefore represented locally by \(\pi_2 \circ \Phi_k \circ w|_{U_k} \circ \varphi_k^{-1} : \varphi_k(U_k) \to V\), where \(\pi_2\) denotes the projection on the second component of the Cartesian product. We will also denote by \(\{c_\alpha\},\ \alpha = 1, \ldots, m\), a given basis in \(V\) and \(w^\alpha\) will denote the components of the local representative relative to this basis.

Vector valued \(r\)-forms are sections of the vector bundle \(\bigwedge^r(T^*S, W)\) whose fiber at \(x \in S\) contains \(r\)-alternating multi-linear mappings defined on \(T_xS\) valued in \(W_x\).
Let $L(W, \bigwedge^r T^* S)$ be the vector bundle whose typical fiber at $x \in S$ is the vector space of linear mappings $W_x \to \bigwedge^r T^*_x S$ and let $\omega \in \bigwedge^r(T^* S, W^*)$ be an alternating multi-linear map into the dual bundle $W^*$. One can associate with $\omega$ an element $\omega^T \in L(W, \bigwedge^r T^* S)$ by

$$\omega^T(w)(v^1, \ldots, v^r) = \omega(v^1, \ldots, v^r)(w).$$  \hfill (2.1)

One can easily verify that this relation induces a natural isomorphism

$$\bigwedge^r(T^* S, W^*) \cong L(W, \bigwedge^r T^* S)$$  \hfill (2.2)

that we will often use below.

Just like the case of real valued smooth forms on $S$, a locally convex topology may be defined on the vector space $C_0^\infty \left( \bigwedge^r(T^* S, W) \right)$ using local representatives. A vector valued $r$-current on the manifold $S$ is defined as a continuous linear functional on the space of compactly supported vector valued forms defined on $S$.

2.1. The velocity field on a body as a vector valued current. We interpret the base manifold $S$ as the physical space where a certain extensive property $P$ is distributed. Typically, the extensive property will be the mass distribution in space. A body force field is modeled by an $n$-form $\beta$ valued in the dual $W^*$ of the vector bundle $\pi : W \to S$. Thus, $\beta$ is a section of $\bigwedge^n(T^* S, W^*)$ so that for each $x \in S$, $\beta(x)$ is an alternating $n$-multi-linear mapping $(T_x S)^n \to W^*_x$. It is assumed that $\beta$ is a $C^\infty$-section having a compact support in $S$.

Using integration of $n$-forms on $S$, a body force field $\beta$ has a natural bi-linear pairing with sections of $W$ defined by

$$\langle w, \beta \rangle = \int_S \beta^T(w).$$  \hfill (2.3)

This pairing is interpreted mechanically as follows. The section $w$ is interpreted as the flux of the extensive property $P$ so that $\beta$ is interpreted as the force density per unit of the property $P$. In particular, $w$ may be interpreted as a velocity field so that $\beta$ is the body force density in space. Thus, $\langle w, \beta \rangle$ is interpreted as the power expended by the body force for the given flux distribution.

If the local representatives of a section $w$ of $W$ in any vector bundle atlas are locally integrable, they will be locally integrable in any other atlas and $w$ is considered as a locally integrable section. The collection of locally integrable sections of $W$ where two sections are identified if they are equal almost everywhere will be denoted as $L^1(W)$. Thus, $w \in L^1(W)$ induces an $n$-current $\tilde{w}$ of order zero such that $\tilde{w}(\beta) = \langle w, \beta \rangle$.

Let $B$ be a compact $n$-dimensional submanifold with boundary of $S$ and let $\iota_B : B \to S$ be the natural inclusion. A section $w$ of the pullback bundle $\iota_{B*}^* \pi : \iota_{B*}^* W \to B$ induces a vector-valued $n$-current $\tilde{w}$ by

$$\tilde{w}(\beta) = \int_B \beta^T(w).$$

We will interpret $B$ as a body in continuum mechanics and the vector field $w$ is interpreted as a flux field on $B$ of the property $P$ that makes up the body.

We conclude that in the general case, $n$-currents $T_{(\beta)}$ of order zero over $S$ valued in the vector bundle $\pi : W \to S$ may serve to model generalized velocity fields or generalized flux fields over irregular bodies.
2.2. A velocity field on the boundary of a body as a vector valued current.
Consider the boundary $\partial B$ of a body $B$ and let $\iota_\partial : \partial B \to B$ be the natural embedding of the boundary in $B$. Letting $\iota_\partial^*W^*$ denote the pullbak of the vector bundle $W^* \to B$ onto $\partial B$ by $\iota_\partial$, a surface field force $\tau$ is a smooth section of $\bigwedge^{n-1}(T^*\partial B, \iota_\partial^*W^*)$. For a given surface force field $\tau$, the section $\tau^T$ may be paired with local sections of $\iota_\partial^*W$. If $C$ is an $(n-1)$-dimensional submanifold with boundary of $\partial B$ and $v$ is a section of $\iota_C^* \circ \iota_{\partial B}^*W$ we may write

$$\langle v, \tau \rangle = \int_C \tau^T(v). \quad (2.4)$$

Thus, the section $v$ induces an $(n-1)$-current $\tilde{v}$ on $\partial B$ valued in $\iota_{\partial B}^*W$ whose action is defined as $\tilde{v}(\tau) = \langle v, \tau \rangle$. For a given body $B$, local velocity fields or flux fields on $\partial B$ are therefore modeled as W-valued $(n-1)$-currents $T_{\langle \tau \rangle}$ on $\partial B$.

2.3. The Cauchy-stress as a vector valued form. An $(n-1)$-form $\sigma$ on $\mathcal{S}$ valued in $W^*$ induces surface forces on the boundaries of bodies. If we have a section $\sigma \in C_0^\infty \left( \bigwedge^{n-1}(T^*\mathcal{S}, W^*) \right)$, a body $B$, and $\iota_{\partial B} : \partial B \to \mathcal{S}$ is the natural embedding, then, $\iota_{\partial B}^*(\sigma)$ is a smooth $W^*$-valued $(n-1)$-form on $\partial B$—a surface force. Here, $\iota_{\partial B}^*$ is the pullback of forms induced by the tangent mapping $T_{\iota_{\partial B}} : T(\partial B) \to T\mathcal{S}$, which restricts the form $\sigma$ to vectors tangent to $\partial B$. Thus, sections of $\bigwedge^{n-1}(T^*\mathcal{S}, W^*)$ model the Cauchy stress fields and the relation $\tau = \iota_{\partial B}^*(\sigma)$ is a generalization of the Cauchy formula.

In the sequel, we will often write $\sigma(w)$ for the $(n-1)$-form $\sigma^T(w)$. Given an $(n-1)$-dimensional submanifold $C$ of $\mathcal{S}$, a section $v$ of $\iota_C^*W$ induces a vector valued $(n-1)$-current $\tilde{v}$ on $\mathcal{S}$ by

$$\tilde{v}(\sigma) = \int_C \sigma(v). \quad (2.5)$$

Thus, one may regard $(n-1)$-currents $T_{\langle \tau \rangle}$ on $\mathcal{S}$ valued in $W$ as velocity fields defined on generalized hyper-surfaces in $\mathcal{S}$.

2.4. The variational-stress as a vector valued form. For a vector bundle $W$, we use the notation $J^r(W) \to W$ for the $r$-th jet bundle (see [5]). We recall that for $x \in \mathcal{S}$, $J^1(W)_x$ is isomorphic with $W_x \times L(T_x\mathcal{S}, W_x)$. The jet extension mapping $j^1 : C^\infty(W) \to C^\infty(J^1(W))$ assigns a section $j^1(w)$ of the jet bundle to a vector field $w$. If $w$ is represented locally by the components $(w^\alpha)$, then, $j^1(w)$ is represented locally by the components $(w^\alpha, w^\gamma_\alpha)$, $\alpha, \gamma = 1, \ldots, m$, $p = 1, \ldots, n$.

A variational stress is a compactly supported smooth n-form valued in the dual of the first jet bundle of the vector bundle $W$. Thus, a variational stress $\mathcal{S}$ is an element of $C_0^\infty \left( \bigwedge^n(T^*\mathcal{S}, (J^1(W))^*) \right)$.

Let $B$ be a body and $A$ a smooth section of the pullback $\iota_B^*J^1(W) = J^1(\iota_B^*W)$, then, there is a natural pairing

$$\langle A, \mathcal{S} \rangle = \int_B \mathcal{S}^T(A). \quad (2.6)$$

Another natural action on variational stress fields may be defined as follows. For a given body $B$ and a differentiable section $w$ of $\iota_B^*W$, the jet of $w$, $j^1(w) : B \to
J^1(\iota^*_B W), may be paired with a variation stress field S as before, i.e.,

\[
\langle j^1(w), S \rangle = \int_B S^T(j^1(w)) = \langle w, j^1(S) \rangle.
\] (2.7)

It is noted that in the pairing \(\langle w, j^1(S) \rangle\), \(j^1(S)\) cannot be regarded as a body force because its pairing with a vector field \(w\) is not of order zero, rather, it is a linear differential operator.

A section \(A\) of the jet bundle is represented locally in the form \((A^\alpha, A^i_\alpha)\), a variational stress \(S\) is represented locally in the form \((R_{\alpha_1...n}, S_{\gamma_1...n}^i)\) and the single component of the \(n\)-form \(S^T(A)\) is represented by \(\sum_\alpha R_{\alpha_1...n} A^\alpha + \sum_{\alpha,i} S_{\gamma_1...n}^i A^\gamma_i\). For \(A = j^1(w)\), the representative of \(S^T(j^1(w))\) is (comma denoting partial differentiation)

\[
\sum_\alpha R_{\alpha_1...n} w^\alpha + \sum_{\alpha,i} S_{\gamma_1...n}^i w^\alpha_{,i} = \sum_\alpha (R_{\alpha_1...n} - S_{\alpha_1...n,i}^i) w^\alpha + \sum_{\alpha,i} (S_{\gamma_1...n}^i w^\alpha_{,i}). \tag{2.8}
\]

It will be shown below that the terms \(\sum_\alpha S_{\gamma_1...n}^i w^\alpha\) induce the components of an \((n-1)\)-form whose exterior derivative is represented by \(\sum_{\alpha,i} (S_{\gamma_1...n}^i w^\alpha_{,i})\), so that \(\sum_\alpha (R_{\alpha_1...n} - S_{\alpha_1...n,i}^i) w^\alpha\) is the single component of an \(n\)-form.

We conclude that \(n\)-currents \(T(S)\) on \(S\) valued in \(J^1(W)\) satisfying some compatibility conditions may serve as generalized jets of vector fields. However, since \(j^1\) is not an exterior differentiation, \(j^1\) cannot be regarded as a boundary operator for currents. Furthermore, \(j^1\) is not a differential operator so that \(j^1\) cannot be regarded as the dual of a differential operator, a variation of the notion of a boundary of a current.

### 2.5. The Cauchy-stress induced by a variational stress.

Let \(S\) be a variational stress field. We show that \(S\) induces a Cauchy stress. Specifically, we construct a surjective vector bundle morphism

\[
p_\sigma : \bigwedge^n (T^*S, J^1(W)^*) \rightarrow \bigwedge^{n-1} (T^*S, W^*) \tag{2.9}
\]

as follows.

Using the linear jet projection mapping \(\pi_0^1 : J^1(W) \rightarrow W\), consider the vertical sub-bundle \(V J^1(W) = \text{Kernel} \pi_0^1\). An element of \(V J^1(W)\) is represented locally in the form \((x^i, 0, A^\alpha_0)\) and it is noted that there is a natural isomorphism \(V J^1(W) \cong L(TS, W)\). We use \(\iota_V : V J^1(W) \hookrightarrow J^1(W)\) to denote the inclusion of the vertical sub-bundle. Then, the dual vector bundle morphism \(\iota_V^* : J^1(W)^* \rightarrow V J^1(W)^*\) is a projection represented locally in the form \((x^i, s_\alpha, b^\alpha_i) \mapsto (x^i, s_\alpha^i)\)—the restriction of \(S\) to vertical elements of the jet bundle. Thus, for \(S \in C_0^\infty \left( \bigwedge^n (T^*S, (J^1(W)^*)) \right)\),

\[
\iota_V^* \circ S \in C_0^\infty \left( \bigwedge^n (T^*S, V J^1(W)^*) \right) \cong C_0^\infty \left( \bigwedge^n (T^*S, L(W, TS)) \right). \tag{2.10}
\]

The object \(\iota_V^* \circ S\) is the symbol of the linear differential operator \(S\) as defined in [5].
Next, it is observed that
\[ \bigwedge^n (T^* S, L(W, TS)) \cong L(L(W, TS)^*, \bigwedge^n T^* S), \]
\[ \cong \bigwedge^n T^* S \otimes L(W, TS), \]
\[ \cong \bigwedge^n T^* S \otimes TS \otimes W^*, \]
so that any element in \( \bigwedge^n (T^* S, L(W, TS)) \) may be represented locally in the form \( \sum_\alpha \theta \otimes v_\alpha \otimes \varphi^\alpha \) for an \( n \)-form \( \theta \) and pairs \( v_\alpha, \varphi^\alpha \) of elements of \( T_x S \) and \( W^*_x \), respectively. We can use the contraction of the second and first factors in the product to obtain \( \sum_\alpha (v_\alpha \downarrow \theta) \otimes \varphi^\alpha \). Thus, we have a canonical mapping
\[ C : \bigwedge^n (T^* S, L(W, TS)) \rightarrow \bigwedge^{n-1} T^* S \otimes W^* \]
\[ \cong L(W, \bigwedge^{n-1} T^* S) \cong \bigwedge^n (T^* S, W^*). \quad (2.11) \]
Let \( \{ e_\alpha \} \) be a local basis of \( W \) with dual basis \( \{ e^\alpha \} \), then, an element of the vector bundle \( \bigwedge^n (T^* S, L(W, TS)) \) is represented in the form
\[ \sum_{i,\alpha} s_{\alpha_1 \ldots n} \frac{\partial}{\partial x^i} \otimes e^\alpha \otimes dx^1 \wedge \cdots \wedge dx^n \]
and an element of \( \bigwedge^{n-1} (T^* S, W^*) \) is of the form
\[ \sum_{p,\alpha} \sigma_{\alpha_1 \ldots \hat{p} \ldots n} e^\alpha \otimes dx^1 \wedge \cdots \wedge \hat{dx^p} \wedge \cdots \wedge dx^n, \]
where a superimposed “hat” indicates the omission of the corresponding term. The mapping \( C \) is represented locally by
\[ \sum_{p,\alpha} s_{\alpha_1 \ldots n} \frac{\partial}{\partial x^p} \otimes e^\alpha \otimes dx^1 \wedge \cdots \wedge dx^n \rightarrow \sum_{p,\alpha} s_{\alpha_1 \ldots n} e^\alpha \otimes \frac{\partial}{\partial x^p} \big( dx^1 \wedge \cdots \wedge dx^n \big), \]
\[ = \sum_{p,\alpha} (-1)^{p-1} s_{\alpha_1 \ldots \hat{n}} e^\alpha \otimes \big( dx^1 \wedge \cdots \wedge \hat{dx^p} \wedge \cdots \wedge dx^n \big), \quad (2.14) \]
so that if \( C(s) = \sigma \), then, \( \sigma_{\alpha_1 \ldots \hat{p} \ldots n} \).\( = (-1)^{p-1} s_{\alpha_1 \ldots n}^p \).

Now, it is possible to consider the composition
\[ p_\alpha = C \circ \iota^*_\alpha : \bigwedge^n (T^* S, J^1(W)^*) \rightarrow \bigwedge^{n-1} (T^* S, W^*) \]
\[ \quad \rightarrow \bigwedge^n (T^* S, W^*) \]
so that \( \sigma = p_\alpha \circ S \) is indeed a Cauchy stress associated with the variational stress \( S \).

In order to simplify the notation, we use the same notation for the element of \( L(W, \bigwedge^{n-1} T^* S) \) induced by \( \sigma = p_\alpha \circ S \) (rather than \( (p_\alpha \circ S)^T \)). Thus, for a variational stress field \( S \) and a vector field \( w \), we write \( d(p_\alpha(S)(w)) \) for the corresponding \( n \)-form obtained by exterior differentiation.
2.6. The divergence of a variational stress. We now define the divergence, an invariant linear differential operator

\[
\text{div} : C^\infty_0 \left( \bigwedge^n (T^* S, (J^1(W))^*) \right) \rightarrow C^\infty_0 \left( \bigwedge^n (T^* S, W^*) \right).
\]  

(2.16)

The divergence operator \text{div} is a first order differential operator in the sense of Palais [5], i.e., it is induced by a vector bundle morphism

\[
J^1 \left( \bigwedge^n (T^* S, (J^1(W))^*) \right) \rightarrow \bigwedge^n (T^* S, W^*).
\]

(2.17)

Using the isomorphism, \( \bigwedge^n (T^* S, W^*) \cong L(W, \bigwedge^n T^* S) \) and \( \bigwedge^{n-1} (T^* S, W^*) \cong L(W, \bigwedge^{n-1} T^* S) \) it is defined invariantly by the relation

\[
\text{div} S(w) = d \left( p_\sigma(S)(w) \right) - S\left(j^1(w)\right),
\]

(2.18)

for every differentiable vector field \( w \). To present the local representatives of \( \text{div} S \) we first note that if \( \sigma = p_\sigma(S) \), then \( d\left(\sigma(w)\right) \) is represented locally by

\[
\sum_{\alpha,p} d(\sigma_{\alpha_1...\hat{\alpha}_p...\alpha_n} w^\alpha) \wedge dx^1 \wedge \cdots \wedge \hat{dx}^p \wedge \cdots \wedge dx^n
\]

\[
= \sum_{\alpha,p} (\sigma_{\alpha_1...\hat{\alpha}_p...\alpha_n} w^\alpha)_p dx^p \wedge dx^1 \wedge \cdots \wedge \hat{dx}^p \wedge \cdots \wedge dx^n,
\]

\[
= \sum_{\alpha,p} (\sigma_{\alpha_1...\hat{\alpha}_p...\alpha_n} w^\alpha)_p (-1)^{p-1} dx^1 \wedge \cdots \wedge dx^n,
\]

(2.19)

\[
= \sum_{\alpha,p} (S^p_{\alpha_1...\alpha_n} w^\alpha)_p dx^1 \wedge \cdots \wedge dx^n.
\]

The local expression for \( \text{div} S(w) \) is therefore

\[
\sum_{\alpha,p} (S^p_{\alpha_1...\alpha_n} - R_{\alpha_1...\alpha_n}) w^\alpha dx^1 \wedge \cdots \wedge dx^n,
\]

(2.20)

so that \( \text{div} S \) is represented locally by

\[
\sum_{\alpha,p} (S^p_{\alpha_1...\alpha_n} - R_{\alpha_1...\alpha_n}) e^\alpha \otimes dx^1 \wedge \cdots \wedge dx^n.
\]

(2.21)

It is noted that in the case where \( R_{\alpha_1...\alpha_n} = 0 \) locally, the expression for the divergence reduces to the traditional expression for the divergence of a tensor field in a Euclidean space.

For a given variational stress, set \( \beta = -\text{div} S \). Then, the definition of the divergence implies that

\[
- \int_B \beta^T(w) = \int_B \text{div} S(w) = \int_{\partial B} p_\sigma(S)(w) - \int_{\partial B} S(j^1(w))
\]

(2.22)

which may be rearranged as

\[
\int_B \beta^T(w) + \int_{\partial B} p_\sigma(S)(w) = \int_B S(j^1(w)),
\]

(2.23)

a generalization of the principle of virtual work in continuum mechanics.
3. The case where the vector bundle is trivial. In order to demonstrate the notions defined above, we present them in the special case where the vector bundle is trivial. Assuming a basis is given for the typical fiber, we consider the case where \( W = S \times \mathbb{R}^m \). This is the case where the manifold \( S \) is interpreted as the material universe manifold and the typical fiber of the vector bundle is the tangent space (with an assumed basis) to an affine space modeling the physical space.

In the case of a trivial bundle, a body force is of the form \( \beta = (\beta_1, \ldots, \beta_m) \), where \( \beta_\alpha \in \wedge^n T^* S \) and a generalized velocity field is an \( m \)-tuple of \( n \)-currents of order zero \( T_{(\beta)} = (T_{(\beta)}^1, \ldots, T_{(\beta)}^m) \) so that \( T_{(\beta)}(\beta) = \sum_{\alpha=1}^m T_{(\beta)}^\alpha(\beta_\alpha) \). Let \( B \) be a body and regarding the body as a real valued \( n \)-current, consider the case where \( T_{(\beta)} = \tilde{\omega}^\alpha = w^\alpha B \) for an \( m \)-tuple of functions \( (w^1, \ldots, w^m) \) defined on \( B \). Thus, the action of the current \( \tilde{\omega} = (\tilde{\omega}^1, \ldots, \tilde{\omega}^m) \), induced by the vector field \( (w^1, \ldots, w^m) \), is \( \tilde{\omega}(\beta) = \sum B w^\alpha \beta_\alpha \). It is noted that we will naturally treat the functions \( w^\alpha \) as 0-forms on \( B \).

For a given body \( B \), a surface force is represented by an \( m \)-tuple of \( (n - 1) \)-forms \( \tau = (\tau_1, \ldots, \tau_m) \) on \( \partial B \). A generalized velocity field on the boundary is an \( m \)-tuple of \( (n - 1) \)-currents of order zero \( T_{(\tau)} = (T_{(\tau)}^1, \ldots, T_{(\tau)}^m) \) so that \( T_{(\tau)}(\tau) = \sum_{\alpha=1}^m T_{(\tau)}^\alpha(\tau_\alpha) \). Let \( C \) be an \( (n - 1) \)-dimensional submanifold of \( \partial B \) and regarding \( C \) as a scalar, \( (n - 1) \)-current, set \( T_{(\tau)}^\alpha = \tilde{\upsilon}^\alpha = v^\alpha C \) for an \( m \)-tuple of functions \( u = (u^1, \ldots, u^m) \) defined on \( C \). The action of the induced current \( \tilde{\upsilon} \) is given by \( \tilde{\upsilon}(\tau) = \sum C \upsilon^\alpha \tau_\alpha \).

A Cauchy stress is an \( m \)-tuple \( \sigma = (\sigma_1, \ldots, \sigma_m) \) of \( (n - 1) \)-forms over \( S \). A generalized surface velocity field is an \( m \)-tuple \( T_{(\sigma)} = (T_{(\sigma)}^1, \ldots, T_{(\sigma)}^m) \) of \( (n - 1) \)-currents so that \( T_{(\sigma)}(\sigma) = \sum_{\alpha=1}^m T_{(\sigma)}^\alpha(\sigma_\alpha) \). Let \( C \) be an \( (n - 1) \)-dimensional submanifold of \( S \) regarded as an \( (n - 1) \)-scalar current and consider the case where \( T_{(\sigma)}^\alpha = \tilde{\upsilon}^\alpha = v^\alpha C \) for an \( m \)-tuple of functions defined on \( C \). The action of the induced current \( \tilde{\upsilon} \) is given by \( \tilde{\upsilon}(\sigma) = \sum C \upsilon^\alpha \sigma_\alpha \).

If \( w = (w^1, \ldots, w^m) \) is a section of \( W \), then \( \sigma^T(w) = \sum \upsilon^\alpha \sigma_\alpha \), an \( (n - 1) \)-form on \( S \). Regarding \( w^\alpha \) as zero forms, we note that

\[
d(\sigma^T(w)) = \sum \alpha (dw^\alpha \wedge \sigma_\alpha + w^\alpha d\sigma_\alpha). \tag{3.1}\]

The jet of a vector field \( w = (w^1, \ldots, w^m) \) is represented by the collection \( (w^\alpha, dw^\alpha) \), \( \alpha, \gamma = 1, \ldots, m \), where we regard each component \( w^\alpha \) as a 0-form and \( dw^\alpha \) is the exterior derivative of a generic component, a 1-form. Thus, the fiber \( J^1(W)_x \) of the jet bundle may be identified with \( \mathbb{R}^m \times (T_x S)^m \) so that \( J^1(W) \cong W \times (T^* S)^m \). It follows that a fiber at \( x \in S \) of the dual of the jet bundle may be identified with \( \mathbb{R}^m \times (T_x S)^m \) and for the dual of the jet bundle one has \( J^1(W)^* \cong W^* \times (TS)^m \).

A variational stress \( S \), an \( n \)-form valued in \( J^1(W)^* \), is represented by a collection \( (R_\alpha, S_\gamma) \), \( \alpha, \gamma = 1, \ldots, m \), where the \( R_\alpha \) are \( n \)-forms over \( S \) and a generic \( S_\gamma \) is an \( n \)-form valued in \( T \mathcal{S} \). The pairing of a section \( A \) of the jet bundle, given in terms of the vector field \( (v^1, \ldots, v^m) \) and the collection of 1-forms \( (A^1, \ldots, A^m) \), is given by

\[
(A, S) = \int_S \sum \alpha R_\alpha v^\alpha + \sum \gamma S_\gamma^T(A^\gamma), \tag{3.2}\]

where \( S_\gamma^T(A^\gamma) \) is the \( n \)-form given by \( S_\gamma^T(A^\gamma)(u_1, \ldots, u_n) = A^\gamma(S_\gamma(u_1, \ldots, u_n)) \).
A generalized section of the jet bundle $T_{(S)}$ is given in terms of a collection of $n$-currents $(T^\alpha_{(R)}, T^\gamma_{(S)})$ so that $T^\alpha_{(R)}$ are real valued, $T^\gamma_{(S)}$ are valued in $T^*S$ and

$$T_{(S)}(S) = \sum_\alpha T^\alpha_{(R)}(R_\alpha) + \sum_\gamma T^\gamma_{(S)}(S_\gamma). \quad (3.3)$$

If $B$ is a body regarded as an $n$-current on $S$, a collection $A = (v^\alpha, A^\beta)$ as above induces a generalized section $\tilde{A}$ of the jet bundle by

$$\tilde{A}(S) = \sum_\alpha \int_B v^\alpha R_\alpha + \sum_\gamma \int_B S^T_\gamma (A^\gamma). \quad (3.4)$$

For the case where $A = j^1(w)$ for a differentiable vector field $w$ given in terms of its components, the induced current is

$$\tilde{j}^1(w)(S) = \sum_\alpha \int_B w^\alpha R_\alpha + \sum_\gamma \int_B S^T_\gamma (dw^\gamma). \quad (3.5)$$

It is noted that unlike the general situation, in the case of a trivial vector bundle, the components $(S_\alpha)$ and $(dw^\alpha)$ may be extracted from the variational stress and jet object, respectively.

It follows from the general local expression, $p_\sigma(S)_{\alpha_1...\beta...n} = (-1)^{p-1} S^p_{\alpha_1...n}$, that

$$\sum_{p,r} w^\alpha_r dx^r \wedge (p_\sigma(S)_{\alpha_1...\beta...n}) dx^1 \wedge \cdot \cdot \cdot \wedge dx^n$$

$$= \sum_{p,r} (-1)^{p-1} S^p_{\alpha_1...n,p} dx^r \wedge dx^1 \wedge \cdot \cdot \cdot \wedge dx^n,$$

$$= \sum_p S^p_{\alpha_1...n,p} dx^1 \wedge \cdot \cdot \cdot \wedge dx^n, \quad (3.6)$$

which we will denote by div $(S_\alpha)$.

From

$$\sum_{p,r} w^\alpha_r dx^r \wedge (p_\sigma(S)_{\alpha_1...\beta...n}) dx^1 \wedge \cdot \cdot \cdot \wedge dx^n$$

$$= \sum_p w^\alpha_r (-1)^{p-1} p_\sigma(S_{\alpha_1...\beta...n}) dx^1 \wedge \cdot \cdot \cdot \wedge dx^n,$$

$$= \sum_p w^\alpha_r (S_{\alpha_1...n,p}) dx^1 \wedge \cdot \cdot \cdot \wedge dx^n,$$

we conclude that

$$d(w^\alpha) \wedge p_\sigma(S_\alpha) = S^T_\alpha (d(w^\alpha)) \quad (3.7)$$

which could be used define $p_\sigma$. 


Equation (3.1) may now be rewritten as
\[ d(p_\alpha(S)(w)) = \sum_\alpha (S^T_\alpha(dw^\alpha) + w^\alpha dp_\alpha(S_\alpha)) = \sum_\alpha (S^T_\alpha(dw^\alpha) + \text{div}(S_\alpha(w^\alpha))). \]  
\[ (3.9) \]

Finally, the definition of the divergence of the stress may be written for the case of trivial vector bundle as
\[ \text{div} S(w) = \sum_\alpha (\text{div} S_\alpha)(w^\alpha) = \sum_\alpha (\text{div}(S_\alpha) - R_\alpha)(w^\alpha). \]

We conclude that \((\text{div} S)_\alpha = \text{div}(S_\alpha) - R_\alpha.\)

4. The boundary and co-divergence of a vector field.

4.1. The boundary of a vector field in the case of a trivial bundle. The fact that a Cauchy stress is represented by \((n-1)\)-forms, enables one to define the boundary of an \(\mathbb{R}^n\)-valued \(n\)-currents by
\[ \partial T(S)(\sigma) = \sum_\alpha T^\alpha_{(\beta)}(d\sigma_\alpha). \]  
\[ (4.1) \]

In case \(T_{(\beta)} = 0\), a current induced by vector field \(w = (w_1, \ldots, w_m)\) defined on a body \(B\), one has
\[ \partial \tilde{w}(\sigma) = \int_B \sum_\alpha w^\alpha d\sigma_\alpha, \]
\[ = \int_B \sum_\alpha (d(w^\alpha \sigma_\alpha) - dw^\alpha \wedge \sigma_\alpha), \]
\[ = \int_{\partial B} \sum_\alpha w^\alpha \sigma_\alpha - \int_B \sum_\alpha dw^\alpha \wedge \sigma_\alpha, \]
\[ = \sum_\alpha \left( \tilde{w}^\alpha |_{\partial B}(\sigma_\alpha) - \tilde{dw}^\alpha(\sigma_\alpha) \right), \]
so that (cf. \[9, p. 282, Equation (11)])
\[ \partial \tilde{w}^\alpha = \tilde{w}^\alpha |_{\partial B} - \tilde{dw}^\alpha. \]  
\[ (4.2) \]

4.2. The co-divergence in the general case. On the basis of the divergence differential operator one can now define the co-divergence \(\theta\) of a vector field as a mapping that takes a vector valued \(n\)-current \(T_{(S)}\) on \(S\) and gives a \(J^1(W)^*\)-valued \(n\)-current by
\[ \theta T_{(S)}(S) = T_{(S)}(\text{div} S), \]  
\[ (4.3) \]

for any smooth compactly supported \(J^1(W)^*\)-valued \(n\)-form \(S\).

Consider the case where the current \(T_{(S)}\) is induced by a vector field \(w\) on a body \(B\). It is observed that for a vector field \(w\) which is not differentiable, the terms \(S(j^1(w))\) and \(d(p_\alpha(S)(w))\) in the definition of the divergence in Equation (2.18) cannot be represented separately as actions of generalized vector fields on
smooth variational stress fields, hence, the significance of the combination of the terms in the definition of the divergence, utilized in the definition of $\partial$. In the case where the vector field $w$ is differentiable, Equation (2.23) gives
\[ \partial \tilde{w}(S) = \tilde{w}\big|_{\partial B}(p_\sigma(S)) - j^1(\tilde{w})(S), \quad \text{or} \quad \partial \tilde{w} = \tilde{w}\big|_{\partial B} \circ p_\sigma - j^1(\tilde{w}) \] (4.4)
in analogy with (4.2).

4.3. The co-divergence for the case of a trivial bundle. For the case of a trivial bundle $W$, let a variational stress $S$ be represented by $(R_\alpha, S_\gamma)$, where $R_\alpha$ are $n$-forms and $S_\gamma$ are $T^*S$ valued $n$-forms on $S$. The dual of the divergence, the co-divergence, of $\text{div}(S_\alpha)$ will be denoted by $\delta$, so that for a real valued $n$-current $T$, $\delta T$ is the $T^*S$ valued $n$-current
\[ \delta T(S_\alpha) = T(\text{div}(S_\alpha)) \] (4.5)
for any $T^*S$-valued $n$-form $S_\alpha$. Thus,
\[ \delta T(S_\alpha) = T(dp_\sigma(S_\alpha)), \]
\[ = \partial T(p_\sigma(S_\alpha)), \]
and one may write
\[ \delta T = \partial T \circ p_\sigma = p_\sigma^*(\partial T). \] (4.6)

For the co-divergence $\partial T_{(S)}$, one has
\[ \partial T_{(S)}(S) = T_{(S)}(\text{div} S), \]
\[ = \sum_\alpha T_{(S)}^\alpha(\text{div} S_\alpha - R_\alpha), \]
so
\[ \partial T_{(S)}(S) = \sum_\alpha \left( \delta T_{(S)}(S_\alpha) - T_{(S)}^\alpha(R_\alpha) \right). \] (4.7)

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