A LOCAL LANGLANDS PARAMETERIZATION FOR
GENERIC SUPERCUSPIDAL REPRESENTATIONS OF
\( p \)-ADIC \( G_2 \)

WITH APPENDIX BY GORDAN SAVIN

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Introduction

The purpose of this article is to construct a Langlands parameterization
of supercuspidal representations of \( G_2 \) over a \( p \)-adic field. More precisely,
for any finite extension \( K/\mathbb{Q}_p \) we will construct a bijection
\[
\mathcal{L}_g : \mathfrak{A}_0^0(G_2, K) \to \mathfrak{G}^0(G_2, K)
\]
from the set of generic supercuspidal representations of \( G_2(K) \) to the set
of irreducible continuous homomorphisms \( \rho : W_K \to G_2(\mathbb{C}) \) with \( W_K \) the
Weil group of \( K \) (more precisely, between sets of equivalence classes). The
construction of the map is simply a matter of assembling arguments that
are already in the literature; the article [KLS10] effectively contains the
construction, although it doesn’t specifically point out the application to
supercuspidal representations. The proof of surjectivity is an application
of a recent result of Hundley and Liu [HL], which allows us to carry out
a strategy, based on automorphy lifting theorems, that was initially devel-
oped in [BHKT] as an application of Vincent Lafforgue’s global parameter-
ization of automorphic representations over function fields. The proof of
injectivity also uses global arithmetic methods, including automorphy lift-
ing theorems and the Ramanujan conjecture for self-dual, regular algebraic

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automorphic representations of $GL(n)$, alongside known results on liftings (especially $SWe$, $X$).

The parameterization is constructed in two steps. First, following [GRS97, GS04, SWe], among other references, we use the exceptional dual reductive pair $(G_2, PGSp(6))$ in $E_7$ to define local and global correspondences from representations of $G_2$ to representations of $PGSp(6)$. We then lift to $Sp(6)$ and use functorial transfer, as in [CKPS, A], to obtain an automorphic representation of $GL(7)$. Using the local Langlands correspondence for $GL(n)$, we can thus obtain a parameterization of supercuspidal representations of $G_2$ by Galois parameters with values in $GL(7)$. We use a global argument and Chebotarev density (following [Ch]) to show that the parameter takes values in the image of $G_2$ under its 7-dimensional irreducible representation $r_7$.

The proof of surjectivity is arithmetic. For the moment, let $K$ be a $p$-adic field and let $\rho$ be a continuous homomorphism

$$\rho : W_K \to G_2(\mathbb{C}).$$

We assume $\rho$ is irreducible: that its image is contained in no proper parabolic subgroup. Since the image is finite, we may replace the coefficient field $\mathbb{C}$ by a sufficiently large finite field $k$ of characteristic $\ell \neq p$. Following Moret-Bailly we show first that $K$ may be viewed as the completion at a $p$-adic place $v$ of a totally real field $F$, and that $\rho$ can be extended to a surjective homomorphism $Gal(F/F) \to G_2(k)$ that is odd, in an appropriate sense. We then use the lifting method in [KW] to lift $\rho$ to a homomorphism $\tilde{\rho} : Gal(F/F) \to G_2(W(k))$ in such a way that $r_7 \circ \tilde{\rho}$ is geometric, in the sense of Fontaine–Mazur, and Hodge–Tate regular.

Now we can apply automorphy lifting theorems, as in [BGGT], to show that $r_7 \circ \tilde{\rho}$ is potentially automorphic – that its restrictions to appropriate totally real Galois extensions $F'/F$ are attached to a cuspidal cohomological self-dual automorphic representation $\pi'$ of $GL(7, A_{F'})$. Choosing $F'$ carefully, we can then descend $\pi'$ to an automorphic representation $\pi''$ of $GL(7, A_{F''})$ over the fixed field $F''$ of a decomposition group $Gal(F'/F) \subset Gal(F''/F)$. At this point we apply the result of Hundley and Liu to show that $\pi''$ is in the image of the functorial transfer from $G_2(A_{F''})$ to $GL(7, A_{F''})$ of an automorphic representation $\Pi$ of $G_2(A_{F''})$, and we conclude by observing that the local component $\Pi_v$ is supercuspidal and has parameter $\rho$. As a bonus, the construction of [HL] provides a globally generic $\Pi$, so we see that $\rho$ is the parameter of a generic supercuspidal representation.

There has been a good deal of work on the local representation theory of as well as the automorphic theory of $G_2$. Notably, the articles [GS04], [SW07], and $SWe$ come very close to establishing a complete local Langlands correspondence for $G_2$ and to relate the correspondence to the exceptional theta correspondence used here; the article [HL] comes very close to characterizing the image of functoriality from $G_2$ to $GL(7)$. The purpose of this article is
not to replace the articles just cited – indeed, the results of these articles are used crucially in the proof of our main theorem – but rather to illustrate the possibility of applying a combination of arithmetic and automorphic methods to the local correspondence.

Acknowledgements. As mentioned above, this paper implements a strategy that was developed in our joint paper with Gebhard Böckle, and we are grateful to him for many discussions. Thanks are due to Joseph Hundley and Baiying Liu for bringing to our attention their recent result on descent for $G_2$, on which our argument crucially depends. We also thank Wee Teck Gan, Dihua Jiang, Aaron Pollack, and Gordan Savin for help with references. We thank Savin for agreeing to write the appendix that proves a global genericity result that allows us to define the local parameterization unambiguously.

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Notation. If $K$ is a perfect field, we will write $\Gamma_K$ for its Galois group relative to a fixed algebraic closure. When $K$ is a number field, we will fix an algebraic closure $\overline{K}/K$, algebraic closures $\overline{K}_v/K_v$ for each place $v$ of $K$, and embeddings $\overline{K} \to \overline{K}_v$ extending $K \to K_v$. These choices determine embeddings $\Gamma_{K_v} \to \Gamma_K$ for each place $v$ of $K$. If $v$ is a finite place, then $I_{K_v} \subset \Gamma_{K_v}$ is the decomposition group.

We write $\epsilon : \Gamma_K \to \mathbb{Z}_\ell^\times$ for the $\ell$-adic cyclotomic character. By abuse of notation, we also write $\epsilon$ for the pushforward of this character to the group of units of any $\mathbb{Z}_\ell$-algebra.

If $K$ is a totally real number field, $n$ is an odd integer, and $\pi$ is a cuspidal, regular algebraic automorphic representation of $GL(n, \mathbb{A}_K)$ which is self-dual, in the sense that $\pi \cong \pi^\vee$, then for any isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ there is an associated semi-simple $\ell$-adic Galois representation $r_{\iota}(\pi) : \Gamma_F \to GL(n, \overline{\mathbb{Q}}_\ell)$. This is characterized, up to isomorphism, by its compatibility with the local Langlands correspondence for $GL(n)$ at finite places. More precisely, if $v \nmid \ell$ is any finite place of $K$ then there is an isomorphism

$$WD(r_{\iota}(\pi))^{F-ss} \cong l^{-1} \text{rec}_{F_v}(\pi_v).$$

There is an isomorphism $r_{\iota}(\pi)^\vee \cong r_{\iota}(\pi)$. We note that this differs by a Tate twist from the normalization of $r_{\iota}(\pi)$ used in e.g. [BGGT], in which $\text{rec}_{F_v}(\pi_v)$ is replaced by $\text{rec}_{F_v}^T(\pi_v) = \text{rec}_{F_v}(\pi_v) \cdot |\iota(1-n)|/2$. This normalization, which only makes sense when $n$ is odd, suits our purposes here since we want representations $\pi$ which arise as functorial lifts from $G_2$ to give rise to Galois representations which are pure of weight 0.
1. Galois Parameterization of $G_2$

Let $G_2$ be the split group of that type over $\mathbb{Z}$. Let $K$ be a local field of characteristic 0 and let $\mathcal{A}_g(G_2, K)$ denote the set of equivalence classes of generic irreducible admissible representations of $G_2(K)$ over $\mathbb{C}$. If $K$ is non-archimedean, we let $\mathcal{A}_g^0(G_2, K) \subset \mathcal{A}_g(G_2, K)$ denote the subset of supercuspidal representations. Let $\mathcal{G}(G_2, K)$ denote the set of $G_2(\mathbb{C})$-conjugacy classes of $G_2$-completely reducible parameters

$$\rho : \mathcal{W}_K \to G_2(\mathbb{C}),$$

and let $\mathcal{G}^0(G_2, K) \subset \mathcal{G}(G_2, K)$ denote the subset of classes of $G_2$-irreducible parameters. We define $\mathcal{G}(GL(7), K)$ and $\mathcal{G}^0(GL(7), K)$ similarly.

The aim of [1] is to collect lifting results scattered in the literature (among them [GS04, GJ, GRS97, GS98, GW, HPS, HL, Li99, MS, SWe, ?]) to construct a map

$$(1) \quad \mathcal{L}_g : \mathcal{A}_g^0(G_2, K) \to \mathcal{G}^0(G_2, K).$$

We begin by constructing a map

$$(2) \quad \mathcal{L}_g' : \mathcal{A}_g(G_2, K) \to \mathcal{G}(GL(7), K)$$

using purely local means. Conjugacy results for $G_2$ (see [Gri95]) imply that the map

$$(3) \quad r_{7,*} : \mathcal{G}(G_2, K) \to \mathcal{G}(GL(7), K)$$

determined by the standard representation $r_7$ of $G_2$ is injective, so the main problem is to show that $\mathcal{L}_g'(\mathcal{A}_g^0(G_2, K))$ lies in the image of $r_{7,*}$. This we achieve using a global argument.

We note that if $K$ is a local field of positive characteristic, Genestier and Lafforgue construct the analogue of the map $\mathcal{L}_g'$ in [GLa] without reference to the theta lift, and at the same time show that its image is contained in the image of $r_{7,*}$.

1.1. Local generic theta lift. Let $K$ be a local field of characteristic 0, and let $E_i$ ($i = 6, 7$) denote the split adjoint reductive group of that type. We set $H_i = PGL(3)$ (if $i = 6$) and $H_i = PSp(6)$ (if $i = 7$). Let $\theta_{i, K}$ be the minimal representation of $E_i(K)$, which we consider by restriction to be a representation of $G_2(K) \times H_i(K)$, $i = 6, 7$. When $K$ is archimedean we work with the Harish-Chandra module of $\theta_{i, K}$ relative to a choice of maximal compact subgroup of $E_i(K)$ that contains a chosen product of maximal compact subgroups of $G_2(K)$ and $H_i(K)$. Let $\pi$ be an irreducible admissible representation of $G_2(K)$, and let $\theta_{i, K,[\pi]}$ denote the maximal quotient of $\theta_{i, K}$ which is isotypic for $\pi$ as representation of $G_2(K)$. Then we write

$$(4) \quad \theta_{i, K,[\pi]} = \pi \otimes \Theta_i(\pi)$$

where $\Theta_i(\pi)$ is a smooth representation of $H_i(K)$.

Proposition 1.2. (i) Suppose that $K$ is non-archimedean and that $\pi$ is generic. Then $\Theta_7(\pi)$ admits a unique generic subquotient.
(ii) Suppose that $K = \mathbb{R}$ and $\pi$ is a discrete series representation. If $\Theta_7(\pi)$ admits a generic constituent, then it is uniquely determined by $\pi$, up to isomorphism.

Proof. The first part is [GS04, Corollary 20]. This is based on Proposition 19 of [GS04], whose proof there is sketched. The proposition is restated, with a complete proof, as Theorem A.3 of Savin’s appendix. The second follows from the fact that $\Theta_7(\pi)$ has regular integral infinitesimal character determined by that of $\pi$, hence any generic constituent of $\Theta_7(\pi)$ must be the unique discrete series representation with that infinitesimal character. We describe this in more detail below.

When $\Theta_7(\pi)$ has a unique generic subquotient, we denote it by $\theta_7(\pi)$.

We can specify $\theta_7(\pi)$ precisely in two important special cases. First, the unramified, non-archimedean case:

**Proposition 1.3 ([GJ, Theorem 3.5], [SW07, Theorem 1.1]).** Suppose that $K$ is non-archimedean and that $\pi$ is an unramified, generic representation of $G_2(K)$. Then $\theta_7(\pi)$ is the unramified representation of $PSp(6, K)$ determined by the embedding $G_2 \to Spin(7)$ of $L$-groups.

Proof. The article [GJ] determines the Satake parameter of the theta lift $\Theta'(\pi)$ of $\pi$ to $GSp(6, F)$ by comparing the local Euler factors for the 8-dimensional Spin representation of the Langlands dual group $GSpin(7)$ of $GSp(6)$ (see the displayed formula at the bottom of p. 42 of [GJ]). The genericity of $\Theta'(\pi)$ is proved in the course of this comparison. The representation $\Theta'(\pi)$ is pulled back from the representation $\Theta_7(\pi)$ of $PSp(6, K)$, which implies that the Satake parameter of $\Theta'(\pi)$ lies in the subgroup $Spin(7)$ of $GSpin(7)$ and has the indicated form. See also the proof of [KLS10, Proposition 5.2].

Next, the real, discrete series case. We first recall an important fact. Let $G$ be a reductive group over $\mathbb{R}$ such that $G(\mathbb{R})$ admits discrete series representations. Let $Z(\mathfrak{g})$ denote the center of the enveloping algebra $U(\mathfrak{g})$ of the complexified Lie algebra of $G$. If $\pi$ is an irreducible representation of $G(\mathbb{R})$, we let $\xi_\pi : Z(\mathfrak{g}) \to \mathbb{C}$ denote its infinitesimal character. We recall

**Fact 1.4.** If $W$ is an irreducible (algebraic) representation of $G$, there is a unique (up to infinitesimal equivalence) generic representation $\pi(W)$ of $G(\mathbb{R})$ such that $\xi_W = \xi_\pi(W)$. Moreover, $\pi(W)$ is discrete series.

We have been unable to find an explicit statement of this well-known fact, but it can be derived from Theorem B of Kostant’s paper [Kos] on Whittaker vectors, Theorem 6.2 of Vogan’s paper [V], and the classification of cohomological representations.

We now fix normalizations, first for $G_2$, and then for $PSp(6)$. Let $\omega_1$ denote the highest weight of the irreducible 7-dimensional representation of $G_2$, and let $\omega_2$ be the other fundamental weight. Given non-negative integers $a, b$ we let $W(a, b)$ denote the irreducible representation of $G_2$ with
highest weight $a\omega_1 + b\omega_2$, and write $\pi_{a,b}$ for $\pi(W(a,b))$, a discrete series representation of $G_2(\mathbb{R})$.

We denote characters $Z(\mathfrak{sp}(6)) \to \mathbb{C}$ in the standard way by triples $(\alpha,\beta,\gamma)$ of integers with $\alpha > \beta > \gamma > 0$, and let $W(\alpha,\beta,\gamma)$ be the irreducible algebraic representation of $PSp(6)$ with the corresponding infinitesimal character.

**Theorem 1.5.** Let $a, b$ be non-negative integers. Then

$$\theta_7(\pi_{a,b}) = \pi(W(a + 2b + 3, a + b + 2, b + 1)).$$

**Proof.** See [HPS, Theorem 5.4]. In more detail, let us use the superscript (?) to denote the compact form of a real reductive group. The article [HPS] treats (among others) the exceptional theta correspondence $(G_2(\mathbb{R}), PSp(6, \mathbb{R})^c)$, [GAV] treats (among others) the correspondence $(G_2(\mathbb{R})^c, PSp(6, \mathbb{R}))$, and [Li97] treats the split case $(G_2(\mathbb{R}), PSp(6, \mathbb{R}))$ but only computes the correspondence for quaternionic discrete series of $G_2(\mathbb{R})$. However, as observed in [Li97, p. 204], the correspondence of infinitesimal characters is independent of real forms. By Fact 1.4 this suffices to identify $\theta_7(\pi_{a,b})$. The determination of the correspondence for generic discrete series is completed in [Li99] (see Table 1 on p. 375 of that paper). □

1.6. **Global generic theta lift.** Now let $F$ be a totally real number field. When $G = E_i$, we let $\mathcal{A}(G)$ denote the space of automorphic forms on $G(F)\backslash G(\mathbb{A}_F)$, and we let $\theta_i := \theta_G \subset \mathcal{A}(G)$ denote the minimal automorphic representation, as described in the article [GJ] (see also [GRS97]). Let $\pi$ be a cuspidal automorphic representation of $G_2(\mathbb{A}_F)$. We define $\Theta_i(\pi)$ and $\Theta_i(\pi)$ to be the spaces of automorphic forms on $H_7 = PSp(6, \mathbb{A}_F)$ and $H_6 = PGL(3, \mathbb{A}_F)$, respectively, defined to be the span of the functions $\Theta_i(f_\theta, \varphi)$, as $f_\theta \in \Theta_G$ and $\varphi$ runs through the automorphic forms in the space of the contragredient $\pi^\vee$ of $\pi$, and where

$$\Theta_i(f_\theta, \varphi)(h) = \int_{[G_2]} f_\theta(g, h) \varphi(g) dg, \ h \in H_i(\mathbb{A}_F).$$

Here the notation $\int_{[G_2]}$ is the standard abbreviation of $\int_{G_2(F)\backslash G_2(\mathbb{A})}$, and $(g, h)$ is a variable element of $G_2(\mathbb{A}_F) \times H_i(\mathbb{A}_F) \subset E_i(\mathbb{A}_F)$. In contrast to [GRS97], we let $\varphi$ to belong to $\pi^\vee$ (or equivalently to the complex conjugate of $\pi$) to guarantee compatibility with the local correspondence defined below.

**Theorem 1.7.** Let $\pi$ be a cuspidal automorphic representation of $G_2(\mathbb{A}_F)$. Then:

(i) Let $\Pi$ be an irreducible subquotient of $\Theta_i(\pi)$. Then for each place $v$ of $F$, $\Pi_v$ is an irreducible subquotient of $\Theta_i(\pi_v)$.

(ii) Suppose that $\pi$ is globally generic and that $\pi_\infty$ is a discrete series representation. Then $\Theta_i(\pi_\infty) = 0$, and $\Theta_i(\pi)$ is cuspidal and globally generic. In particular, it is non-zero.
1.10. **Definition of $L'$ and $L$.** Let $K$ be a non-archimedean local field of characteristic 0. We can now define the promised map $L'_g : \mathcal{A}_g(G_2, K) \to \mathcal{G}(GL(7, K))$. If $\pi \in \mathcal{A}_g(G_2, K)$, then we define $L'_g(\pi) = \text{rec}_K(\Psi(\pi))^{\ss}$, where $\sigma$ is the unique irreducible constituent of $\theta_\tau(\pi)|_{Sp(6, K)}$ which is generic (with respect to our fixed choice of Whittaker datum). This is independent of the choice of Whittaker datum on $Sp(6)$. The next step is to show that when $\pi$ is supercuspidal, $L'_g(\pi)$ can be conjugated to take values in $G_2(\mathbb{C})$. We will establish this using a global argument.
Proposition 1.11. Let $\pi \in A^0_g(G_2, K)$. Then $L'_g(\pi)$ lies in the image of $r_{7,*}$.

Proof. Fix a totally real number field $F$ with a finite place $v$ such that $F_v \sim K$. Let $\alpha_1, \ldots, \alpha_r$ be the real places of $F$. For each $i = 1, \ldots, r$, we choose a generic integrable discrete series representation $\pi_i$ of the (split) group $G_2(\mathbb{R})$. Fix a finite place $w \neq v$ of $F$ of residue characteristic different from 2 and a generic supercuspidal representation $\pi_w$ of $G_2(F_w)$ with the property that $\theta_7(\pi_w)$ is a supercuspidal representation of $PSp(6, F_w)$. (Such representations are constructed explicitly in the proof of Theorem 5.2 of [KLS10], at least when $F_w = \mathbb{Q}_p$, which we assume for convenience of reference.) By [KLS08, Theorem 4.5] (see also [V84, Theorem 2.2]), we can find a cuspidal, globally generic automorphic representation $\Pi$ of $G_2(A_F)$ such that $\Pi_v \cong \pi_v$, $\Pi_w \cong \pi_w$, and $\Pi_{\alpha_i} \cong \pi_i$ for each $i$.

Then $\theta_7(\Pi)$ is globally generic and cohomological, and its local component at $w$ is supercuspidal, by construction. Applying Theorem 1.9 to the restriction of $\theta_7(\Pi)$ to $Sp(6, A_F)$, we can find cuspidal, self-dual automorphic representations $\Psi_1, \ldots, \Psi_r$ of $GL(7, A_F)$ such that $\Psi(\theta_7(\Pi)) = \Psi_1 \oplus \cdots \oplus \Psi_r$ is cohomological. In particular, each $\Psi_i$ is cohomological up to twist. Fixing an isomorphism $\iota : \overline{\mathbb{Q}}_l \rightarrow \mathbb{C}$, we get a compatible family $r_i(\Psi_i)$ of $n_i$-dimensional representations with values in $GL(n_i, \overline{\mathbb{Q}}_l)$. We get a representation

$$r_i(\Pi) := r_i(\Psi(\theta_7(\Pi))) : \Gamma_F \rightarrow GL(7, \overline{\mathbb{Q}}_l).$$

In fact, [Ch. Theorem 6.4] shows that the representation $r_i(\Pi)$ is conjugate to a representation contained in $r_7(G_2(\overline{\mathbb{Q}}_l))$, and such a representation is unique up to conjugation in $G_2(\overline{\mathbb{Q}}_l)$ (noting that the cuspidality of $\Psi$ plays no role in the proof of that theorem). Our proof is now complete: we have $r_i(\Pi)^{ss}_{W_F} \cong r_i(\Psi(\theta_7(\pi)))^{ss} \cong \iota^{-1}L'_g(\pi)$, by definition, and this shows that $L'_g(\pi)$ is conjugate to a representation valued in $G_2(\mathbb{C})$. $\square$

Proposition 1.12. Let $\pi \in A^0_g(G_2, K)$, and let $\rho : W_K \rightarrow G_2(\mathbb{C})$ be the unique parameter, up to $G_2(\mathbb{C})$-conjugacy, with $r_7 \circ \rho = L'_g(\pi)$. Then $\rho$ is irreducible.

Proof. We split up into cases according to the dichotomy described in [SWc]. If $\theta_7(\pi)$ is supercuspidal then by [CKPS] Theorem 7.2 there are self-dual, non-isomorphic supercuspidal representations $\psi_1, \ldots, \psi_r$ of $GL(n_i, K)$ with $\sum_{i=1}^r n_i = 7$ and $r_7 \circ \rho$ conjugate in $GL(7, \mathbb{C})$ to $\bigoplus_{i=1}^r \text{rec}_K(\psi_i)$. There is a unique way to conjugate such a parameter into $SO(7, \mathbb{C})$, and it is irreducible there (indeed, its centralizer is finite). It follows that $\rho$ must also be irreducible in this case.

If $\theta_7(\pi)$ is not supercuspidal, then [SWc] Proposition 3.6 shows that $\theta_7(\pi)$ is a subquotient of an unnormalized induction $\text{Ind}_{Q_3}^{PSp(6)}(\rho \otimes |\det|$, where $\rho$ is a supercuspidal representation of $PGL(3, K)$. By [CKPS] Proposition 7.4, $r_7 \circ \rho$ is conjugate in $GL(7, \mathbb{C})$ to $\text{rec}_K(\rho) \oplus \mathbb{C} \oplus \text{rec}_K(\rho)^{\vee}$. Since $\text{rec}_K(\rho)$ is irreducible and 3-dimensional, [SWc] Proposition 1.5 shows that the image
of this representation is not contained in the image of any Levi subgroup of $G_2$, so once again $\rho$ must be irreducible.

We may therefore complete our definition of $L_g$ as follows: if $\pi \in A^0_\rho(G_2, K)$ then $L_g(\pi)$ is the unique parameter, up to conjugacy, with $r_7 \circ L_g(\pi) = L'_g(\pi)$. We record the following useful property of $L_g$.

**Proposition 1.13.** Let $\pi \in A^0_\rho(G_2, K)$ and let $R : G_2 \to SO(7)$ be the unique non-trivial homomorphism. Then $\theta_7(\pi)$ is supercuspidal if and only if $R \circ \theta_7(\pi)$ is $SO(7)$-irreducible.

**Proof.** This is a corollary of the proof of Proposition 1.12.

2. **Globalization of local $G_2$ parameters**

Let $G = G_2$ be a split reductive group over $\mathbb{Z}$ of that type, let $\mathfrak{g}$ denote its Lie algebra, and let $r_7 : G \to GL(7)$ denote the standard $7$-dimensional representation. We fix a split maximal torus and Borel subgroup $T \subset B \subset G$. We may assume that $r_7(T)$ is diagonal and $r_7(B)$ is contained in the upper-triangular Borel subgroup of $GL(7)$. Let $\Delta \subset \Phi = \Phi(G, T)$ denote the corresponding root basis, and $\Phi = \Phi^+ \sqcup \Phi^-$ the sets of positive and negative roots. We label $\Delta = \{\alpha_1, \alpha_2\}$ so that the fundamental weight $\omega_1$ is the highest weight of $r_7$. Let $\tilde{\omega}_1, \tilde{\omega}_2 \in X_*(T)$ denote the corresponding fundamental coweights, and let $\delta = \tilde{\omega}_1 + \tilde{\omega}_2$.

Let $K$ be finite extension of $\mathbb{Q}_p$, and let $k$ be a finite field of characteristic $\ell \neq p$. We suppose given a continuous representation $\overline{\rho} : \Gamma_K \to G(k)$ such that $\overline{\rho}(I_K)$ has order prime to $\ell$.

**Theorem 2.1.** There exists a totally real field $F$ and a continuous representation $\overline{\sigma} : \Gamma_F \to G(k)$ with the following properties:

(i) $\overline{\sigma}((\Gamma_F)) = G(k)$.

(ii) $\ell$ splits in $F$. For each place $v|\ell$ of $F$, $\overline{\sigma}|_{\Gamma_{F_v}}$ is $G(k)$-conjugate to $\tilde{\delta}^2 \circ \epsilon$.

(iii) For each place $v|p$ of $F$, there is an isomorphism $F_v \cong K$ such that $\overline{\sigma}|_{\Gamma_{F_v}}$ is $G(k)$-conjugate to $\overline{\rho}$.

(iv) If $v$ is a finite place of $F$ such that $v | \ell p$, then $\overline{\sigma}|_{\Gamma_{F_v}}$ is unramified.

(v) For each place $v|\infty$ of $F$ (which determines a complex conjugation $c_0 \in \Gamma_{F_v}$), $\overline{\sigma}(c_0)$ is $G(k)$-conjugate to $\delta(-1)$.

**Proof.** By weak approximation, we can find a totally real number field $E/\mathbb{Q}$ in which $\ell$ splits, and such that for each $v|p$ there is an isomorphism $E_v \cong K$. The existence of an extension $F/E$ and a homomorphism $\overline{\sigma} : \Gamma_F \to G(k)$ with the claimed properties then follows from the main theorem of [M90].

**Theorem 2.2.** Let $\overline{\sigma} : \Gamma_F \to G(k)$ be a representation satisfying the conclusion of Theorem 2.1 and suppose that $\ell > 28$. Then there exists a finite extension $E/\text{Frac} \mathcal{W}(k)$ with ring of integers $\mathcal{O}$ and a lift

$$\sigma : \Gamma_F \to G(\mathcal{O})$$
of $\pi$ with the following properties:

(i) $\sigma(\Gamma_F)$ contains a conjugate of $G(\mathbb{Z}_\ell)$.
(ii) For each place $v|\ell$ of $F$, there exists $g \in G(O)$ such that $g\sigma|_{\Gamma_{F_v}} g^{-1}$ takes values in $B(O)$, and the projection of $g\sigma|_{\Gamma_{F_v}} g^{-1}$ to $T(O)$ equals $\tilde{\delta}^2 \circ \epsilon$. Consequently, $r_7 \circ \sigma|_{\Gamma_{F_v}}$ is crystalline ordinary of Hodge–Tate weights $\{6, 4, 2, 0, -2, -4, -6\}$ (with respect to any embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_\ell$).
(iii) For each place $v|p$ of $F$, reduction modulo $m_O$ induces an isomorphism $\sigma(I_{F_v}) \sim \overline{\sigma}(I_{F_v})$.
(iv) For each finite place $v \nmid \ell p$ of $F$, $\sigma|_{\Gamma_{F_v}}$ is unramified.
(v) For each place $v \nmid \infty$ of $F$, $\sigma(c_v)$ is $G(O)$-conjugate to $\delta(-1)$.

Proof. We first remark that our hypotheses imply that $G(k)$ is its own derived group, hence $\overline{\pi}(\Gamma_{F(\sigma)}) = G(k)$, and $r_7 \circ \pi$ is absolutely irreducible (see [St]). This will be useful in applying the results of [BG] cited below. We next observe that [BHKT, Proposition 6.7] shows that any lift $\sigma$ of $\pi$ has the property that $G(O)$-conjugates of $\sigma(\Gamma_F)$ contains a conjugate of $G(\mathbb{Z}_\ell)$.

To construct a lift, we use the Khare–Wintenberger method (cf. [KW, Theorem 3.7]). Let $C$ denote the category of complete local Noetherian $W(k)$-algebras with residue field $k$, and let $\text{Def}_G : C \to \text{Sets}$ denote the functor which assigns to any $A \in C$ the set of $G(A)$-equivariant classes of homomorphisms $\sigma_A : \Gamma_F \to G(A)$ satisfying the following conditions:

- For each place $v|\ell$ of $F$, there exists $g \in G(A)$ such that $g\sigma_A|_{\Gamma_{F_v}} g^{-1}$ takes values in $B(A)$, and the projection of $g\sigma_A|_{\Gamma_{F_v}} g^{-1}$ to $T(A)$ equals $\tilde{\delta}^2 \circ \epsilon$.
- For each place $v|p$ of $F$, reduction modulo $m_A$ induces an isomorphism $\sigma_A(I_{F_v}) \sim \overline{\sigma}(I_{F_v})$.
- For each finite place $v \nmid \ell p$ of $F$, $\sigma_A|_{\Gamma_{F_v}}$ is unramified.

Then ([Pat16, Proposition 9.2]) $\text{Def}_G$ is represented by an object $R_G \in C$, and there exists an integer $g \geq 0$ such that $R_G$ can be expressed as a quotient of $W(k)[[X_1, \ldots, X_g]]$ by $g$ relations. (Note that the local conditions are liftable local deformation conditions, in the sense of [Pat16, Proposition 6.7] and [Pat16, §4.4]. The conditions $\ell > 28$ and $\ell$ split in $F$ imply that conditions (REG) and (REG*) of [Pat16, §4.1] hold.) To apply the Khare–Wintenberger method, we must show that $R_G$ is a finite $W(k)$-algebra. This will imply that $R_G$ is a finite flat complete intersection $W(k)$-algebra, and therefore that there exists a finite extension $W(k) \to O$ and a homomorphism $R_G \to O$, corresponding to a lift $\sigma : \Gamma_F \to G(O)$ with the desired properties.

To prove the finiteness of $R_G$, we must compare it with another deformation ring. Let $E/F$ be a totally imaginary quadratic extension in which the places of $F$ above $\ell$ and $p$ split; then $E$ is a CM field. Let $\overline{\pi} = r_7 \circ \overline{\sigma}|_{\Gamma_E}$. Then $\overline{\pi}$ is absolutely irreducible and $\overline{\pi} \cong \overline{\pi}^c$ (where $c \in \text{Gal}(E/F)$ is the
non-trivial element). Let $\mathcal{G}_\tau = (GL(7) \times GL(1)) \times \{\pm 1\}$ denote the group scheme defined in [BGGT §1.1]. As described there, $\overline{\tau}$ extends to a homomorphism $\overline{\tau} : \Gamma_F \to \mathcal{G}_\tau(k)$ such that $\nu \circ \overline{\tau} = \delta_{E/F}$ and $\overline{\tau}(\Gamma_E) \subset GL(7, k)$. (We write $\delta_{E/F} : \text{Gal}(E/F) \to \{\pm 1\}$ for the unique non-trivial character.)

Let $S$ denote the set of places of $F$ dividing $\ell p$. Fix for each $v \in S$ a place $\overline{v}$ of $E$ lying above $v$. Say that a finite totally real extension $F'/F$ is good if $\overline{\tau}(\Gamma_{F'}) = \overline{\tau}(\Gamma_F)$ and $\zeta_\ell \not\in \mathcal{E}'(F')$. If $F'$ is a good extension then we write $S_{F'}$ for the set of places of $F'$ lying above a place of $S$, $E' = EF'$, and $\overline{S}_{F'}$ for the set of places of $E'$ lying above a place $\overline{v}$ (some $v \in S$). If $v \in S_{F'}$ then we write $\overline{v}$ for the unique place of $\overline{S}_{F'}$ lying above $v$. We then write $\text{Def}_{F'} : \mathcal{C} \to \text{Sets}$ for the functor which assigns to any $A \in \mathcal{C}$ the set of ker($GL(7, A) \to GL(7, k)$)-conjugacy classes of homomorphisms $\tau_A : \Gamma_E \to \mathcal{G}_\tau(A)$ satisfying the following conditions:

- For each place $v|\ell$ of $F'$, $\tau_A|_{\Gamma_{E'_v}}$ defines an $A$-point of the ‘cr-ord’ lifting ring described in [BGGT §1.4], with sets
  \[ H_i = \{6, 4, 2, 0, -2, -4, -6\} \]
  of Hodge–Tate numbers ($i : E'_v \to \overline{Q}_\ell$ any embedding).
- For each place $v|p$ of $F'$, reduction modulo $\ell$ induces an isomorphism $\tau_A(I_{E'_v}) \to \overline{\tau}(I_{E'_v})$.
- For each finite place $v \nmid \ell p$ of $F'$, $\tau_A|_{\Gamma_{F'_v}}$ is unramified.
- $\nu \circ \tau_A = \delta_{E'/F'}$.

The functor $\text{Def}_{F'}$ is represented by an object $R_{F'} \in \mathcal{C}$ (see [CHT Proposition 2.2.9]). The representation $r_7$ determines a natural map $R_{F'} \to R_G$. (The only point to check here is that if $\sigma_A : \Gamma_F \to G(A)$ arises from a homomorphism $R_G \to A$, then for any place $v|\ell$ of $F'$, $\tau_7 \circ \sigma_A|_{\Gamma_{E'_v}}$ defines a point of the ‘cr-ord’ lifting ring. This can be reduced to the universal case, in which case the key point is [G99 Lemma 3.3.2].) If $F = F'$, the map $R_F \to R_G$ is surjective (same proof as [BHKT Lemma 5.7]). For any $F'$, the map $R_{F'} \to R_F \to R_G$ therefore a finite algebra homomorphism [BGGT Lemma 1.2.3]. To finish the proof of the theorem, we therefore just need to show there exists a good extension $F'/F$ such that $R_{F'}$ is a finite $W(k)$-algebra.

By [BGGT Theorem 2.4.2], this will follow if we can find a good extension $F'/F$ and an isomorphism $\iota : \overline{Q}_\ell \to \mathbb{C}$ such that $\overline{\tau}|_{\Gamma_E}$ is the residual representation of an $\iota$-ordinary, cuspidal, regular algebraic, polarizable automorphic representation of $GL(7, \mathbb{A}_{E'})$. The existence of such a representation follows from [BGGT Proposition 3.3.1]. This completes the proof.

\[ \square \]

**Theorem 2.3.** Let $\sigma : \Gamma_F \to G(\mathcal{O})$ be as in Theorem 2.2. Let

$$\sigma_7 = \tau_7 \circ \sigma : \Gamma_F \to GL(7, \mathcal{O}) \hookrightarrow GL(7, \overline{Q}_\ell).$$
Then there exists a totally real Galois extension $F'/F$ and a cuspidal, cohomological automorphic representation $\Psi(\sigma)$ of $GL(7, A_{F'})$, and an isomorphism $\overline{Q}_\ell \to \mathbb{C}$ such that $r_\ell(\Psi(\sigma)) \cong \sigma_7|_{\Gamma_{F'}}$. Moreover, we can assume that $\Psi(\sigma)$ is everywhere unramified.

Proof. This from [BGGT] Corollary 4.5.2. We note that the statement of [BGGT] Corollary 4.5.2 does not include the assertion of disjointness from $E_{\text{avoid}}/F$. However, the analogous assertion is part of [BGGT] Theorem 4.5.1, to which that corollary is reduced after a quadratic base change, and this can be used to get the statement we need here. \qed

We now forget our existing assumptions and restate the above results in a form suitable for application in \[8\]

**Theorem 2.4.** Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $k$ be a finite field of characteristic $\ell \neq p$. Let $\rho : W_K \to G_2(W(k))$ be a continuous homomorphism, irreducible over $\overline{\mathbb{Q}}_\ell$. Then $\rho(I_K)$ is finite; we assume that $\ell > 28$ and that $\ell$ does not divide the order of $\rho(I_K)$. Then we can find the following data:

(i) A totally real number field $L$, together with a non-empty set $\Sigma$ of $p$-adic places of $L$ such that for each $v \in \Sigma$, $L_v \cong K$.

(ii) A finite extension $E/\text{Frac} W(k)$ with ring of integers $O$ and a continuous representation $\sigma : \Gamma_L \to G_2(O)$ of Zariski dense image such that for each place $v \in \Sigma$, $\sigma|_{W_{L_v}}$ and $\rho$ are $G_2(O)$-conjugate.

(iii) A cuspidal, tempered, regular algebraic, self-dual automorphic representation $\Psi(\sigma)$ of $GL(7, A_L)$, unramified outside $\Sigma$, and an isomorphism $\iota : \overline{Q}_\ell \to \mathbb{C}$ such that $r_\ell(\Psi(\sigma)) \cong r_\ell \circ \sigma$.

Proof. We begin with some remarks. First, $\rho$ has finite image, and so extends to a representation of $\Gamma_K$. Indeed, $\rho(I_K)$ is finite since $r_\ell \circ \rho$ is semisimple; the $\ell$-adic monodromy theorem implies the existence of a finite extension $K'/K$ such that $r_\ell \circ \rho(I_{K'})$ is semisimple and unipotent, hence trivial. Let $\phi \in W_K$ be a Frobenius lift. Since $\rho(I_K)$ is finite, some power $\rho(\phi)^N$ centralizes $\rho(W_K)$. Since $\rho(W_K)$ is irreducible, this forces $\rho(\phi)^N$ to lie in the centre of $G$, hence (since $G$ is adjoint) to be trivial. Thus $\rho(W_K)$ is finite.

Second, let $\overline{\rho} : W_K \to G_2(k)$ denote the reduction of $\rho$ modulo $\ell$. Then any minimally ramified lift of $\overline{\rho}$ to $G_2(O)$ is $G_2(O)$-conjugate to $\rho$. Indeed, [FKP] Lemma A.2] shows that $h^0(\Gamma_K, g_k) = 0$, hence (cf. [Pat16] Lemma 4.17]) that the tangent space to the minimal deformation functor is trivial. This implies that any minimally ramified lift is even ker($G_2(O) \to G_2(k)$)-conjugate to $\rho$.

By Theorem 2.3 we can find the following objects:

- A totally real field $F$, and a totally real Galois extension $F'/F$.
- A continuous representation $\sigma : \Gamma_F \to G_2(O)$ such that $\sigma|_{\Gamma_{F'}}$ has Zariski dense image, $\sigma$ is unramified outside $\ell p$, and for each place $v|\ell$ of $F$, $r_\ell \circ \sigma|_{\Gamma_F}$ is crystalline with Hodge-Tate numbers $\{6, 4, 2, 0, -2, -4, -6\}$ (with respect to any embedding of $F_v$ in $\overline{Q}_{\ell}$).
• For each place $v|p$ of $F$, an isomorphism $F_v \cong K$ such that $\sigma|_{\Gamma_{F_v}}$ and $\rho$ are $G_2(\mathcal{O})$-conjugate.

• A cuspidal, regular algebraic, self-dual automorphic representation $\Psi(\sigma)'$ of $GL(7, \mathcal{A}_F')$ which is everywhere unramified and an isomorphism $\iota : \mathbb{Q}_\ell \to \mathbb{C}$ such that $r_\iota(\Psi(\sigma)') \cong r_\iota \circ \sigma|_{\Gamma_{F'}}$.

Let $v'$ be a place of $F'$ above $F$, and let $D_{v'/v} \subset \text{Gal}(F'/F)$ be the decomposition group. We set $F'' = (F')^{D_{v'/v}}$, and $v'' = v'|_{F''}$. Then $F'' \cong K$ and $\sigma|_{\Gamma_{F''}}$ and $\rho$ are $G_2(\mathcal{O})$-conjugate. Moreover, $F'/F''$ is a soluble extension. By soluble descent (see e.g. [BGGT, Lemma 2.2.2]), there exists a cuspidal, regular algebraic, self-dual automorphic representation $\Psi(\sigma)''$ of $GL(7, \mathcal{A}_{F''})$ such that $r_\iota(\Psi(\sigma)') \cong r_\iota \circ \sigma|_{\Gamma_{F''}}$. Finally, choose a soluble totally real extension $L/F$ in which $v''$ splits and such that the base change $\Psi(\sigma)$ of $\Psi(\sigma)''$ to $L$ is unramified away from $v''$. The proof is complete on taking $\Sigma$ to be the set of places of $L$ lying above $v''$, and noting that [S, Corollary 1.3] shows that $\Psi(\sigma)$ is tempered. □

3. Automorphic descent from $GL(7)$ to $G_2$

The following theorem was recently proved by Hundley and Liu:

**Theorem 3.1.** Let $F$ be a number field, and let $\Psi$ be a cuspidal automorphic representation of $GL(7, \mathcal{A}_F)$. Suppose that the following conditions hold.

(i) For almost all places $v$ of $F$ at which $\Psi_v$ is unramified, the Satake parameter of the local component $\Psi_v$ is conjugate, in $GL(7, \mathbb{C})$, to an element of $r_7(G_2(\mathbb{C}))$.

(ii) The partial $L$-function $L^S(s, \Psi, \wedge^3)$ has a pole at $s = 1$, for some finite set $S$.

Then there exists a globally generic automorphic, cuspidal automorphic representation $\Pi$ of $G_2(\mathcal{A}_F)$ such that for all but finitely many places at which $\Pi_v$ is unramified, the Satake parameter of $\Psi_v$ is the image under $r_7$ of the Satake parameter of $\Pi_v$.

Moreover, if $v$ is a finite place at which $\Psi_v$ is both unramified and tempered, with Satake parameter conjugate to $r_7(G_2(\mathbb{C}))$, then $\Pi_v$ is unramified and the Satake parameter of $\Psi_v$ is the image under $r_7$ of the Satake parameter of $\Pi_v$.

**Proof.** The theorem up to “moreover” is contained in [HL, Theorem 6.1.17] [HL, Theorem 6.4.31] and [HL, Theorem 6.5.6] (giving genericity, cuspidality and weak lifting, respectively). If $v$ is any finite place at which $\Psi_v$ is unramified with Satake parameter conjugate to $r_7(G_2(\mathbb{C}))$, then $\Pi_v$ is the generic subquotient of the unramified principal series representation of $G_2$ with Satake parameter corresponding to that of $\Psi_v$. If $\Psi_v$ is tempered then this unramified principal series representation is also tempered and irreducible, implying the final claim. □
We remark that if $F$ is totally real and $\Psi$ is regular algebraic with infinitesimal character “integral of $G_2$ type”, then $\Pi_\infty$ must be discrete series. Indeed, the theta lift of $\Pi$ is an automorphic representation $\Psi'$ of $GL(7, A_F)$ such that the infinitesimal character of $\Psi'$ respects that of $\Pi$ (cf. the discussion in \textsection1). By strong multiplicity one, $\Psi = \Psi'$, and the infinitesimal character of $\Pi_\infty$ is integral. By Fact 1.4, $\Pi_\infty$ is discrete series.

**Corollary 3.2.** Let $L$ (a totally real number field), $\sigma : \Gamma_L \to G_2(\mathbb{Q}_\ell)$ (a continuous homomorphism) and $\Psi(\sigma)$ (a cuspidal tempered automorphic representation of $GL(7, A_L)$ with $r_7 \circ \sigma \cong r_7(\Psi(\sigma))$) be as in the statement of Theorem 2.4. Then there exists a globally generic, cuspidal automorphic representation of $\Gamma(\Pi(\sigma))$ of $G_2(\mathbb{A}_L)$ such that $\Pi(\sigma)_\infty$ is discrete series and for every place $v$ of $L$ at which $\Psi(\sigma)$ is unramified, $\Pi(\sigma)_v$ is unramified and the Satake parameter of $\Psi(\sigma)_v$ is the image in $GL(7, \mathbb{C})$ of the Satake parameter of $\Pi(\sigma)_v$.

**Proof.** Since the local parameter of $\Psi(\sigma)$ at every place at which $\sigma$ is unramified factors through $r_7(G_2)$, $\Psi(\sigma)$ satisfies hypothesis (i) of Theorem 3.1. If hypothesis (ii) is satisfied, then $\Psi(\sigma)$ is the functorial lift of some cuspidal $\Pi(\sigma)$. Finally, the temperedness of $\Psi(\sigma)$ will imply that $\Psi(\sigma)_v$ is the lift of $\Pi(\sigma)_v$ for all unramified places.

It remains to verify that (ii) is satisfied. Now $L^S(s, \Psi(\sigma), \wedge^3) = L^S(s, \wedge^3 \circ \sigma_7)$ if $S$ contains all ramified places. Moreover, since $\sigma_7 = r_7 \circ \sigma$ factors through $r_7(G_2)$, it is well known that

$$\wedge^3 \circ \sigma_7 \sim \sigma_7 \oplus \text{Sym}^2 \circ \sigma_7.$$  

Thus

$$L^S(s, \Psi(\sigma), \wedge^3) = L^S(s, \Psi(\sigma)) \cdot L^S(s, \Psi(\sigma), \text{Sym}^2).$$

The first factor does not vanish at $s = 1$ by the theorem of Jacquet–Shalika and Shahidi, whereas the second factor has a simple pole at $s = 1$ because $\Psi(\sigma)$ is a self-contragredient representation of an odd general linear group (cf. the discussion of \cite[p. 139]{BG}).

**Theorem 3.3.** Let $K$ be a $p$-adic local field. Then the map

$$L_g : A_0^0(G_2, K) \to G_0^0(G_2, K)$$

is surjective.

**Proof.** Let $\rho : W_K \to G_2(\mathbb{C})$ be an irreducible representation. We will construct $\pi \in A_0^0(G_2, K)$ such that $L_g(\pi) = \rho$. Fix a prime $\ell > 28$ which does not divide the (finite) order of $\rho(W_K)$, and let $\iota : \mathbb{Q}_\ell \to \mathbb{C}$ be an isomorphism. We may assume that $\iota^{-1}\rho$ takes values in $G_2(\mathbb{Z}_\ell)$. We take the totally real number field $L$ and homomorphism $\sigma : \Gamma_L \to G_2(\mathbb{O})$ as in Theorem 2.4 and apply Corollary 3.2. Thus there is a non-empty set $\Sigma$ of $p$-adic places of $L$, and for each $v \in \Sigma$ an isomorphism $K \cong \mathbb{C}$ such that $\iota^{-1}\rho, \sigma|_{\Gamma_L}$ are $G_2(\mathbb{O})$-conjugate. Fix a choice of $v \in \Sigma$, and let $\pi = \Pi(\sigma)_v$. We must show that $\pi$ is supercuspidal and that $L_g(\pi) = \rho$. 


We first note that $\tau_1(\Pi(\sigma))$ (defined as in the proof of Proposition 1.11) is $G_2(\mathbb{Q}_p)$-conjugate to $\sigma$; indeed, this can be checked at unramified places, so follows from [Gri95]. We therefore just need to check that $\pi$ is supercuspidal. If $\theta_7(\pi)$ is supercuspidal, then Proposition 3.4 of [SWe] shows that $\pi$ is also supercuspidal.

We can therefore assume that $\theta_7(\pi)$ is not supercuspidal; thus its Jacquet module $J_P(\theta_7(\pi)) \neq 0$ for some maximal parabolic subgroup $P \subset PSp(6)$. It follows from [SWe, Proposition 1.1] and [CKPS, Proposition 7.5] that $P$ must be the Siegel parabolic subgroup, with Levi quotient $GL(3)$. Since $\theta_7(\pi)$ is a subquotient of $\theta_7, K$, $\pi$ is a subquotient of $J_P(\theta_7, K)$, hence of $\theta_6, K$, by [MS, Theorem 5.3] (which computes $J_P(\theta_7, K)$). By construction, $J_P(\theta_7(\pi))$ is a non-zero direct sum of supercuspidal representations of $GL(3, K)$, so [SWe, Proposition 3.5]) shows that $\pi$ must be supercuspidal.

\section{Injectivity}

Let $K$ be a $p$-adic local field. The final step in our argument is to prove that the parameterization

$$L_g : \mathcal{A}_g^0(G_2, K) \to \mathcal{G}^0(G_2, K)$$

is injective (hence bijective). We recall that if $\pi \in \mathcal{A}_g^0(G_2, K)$, then $\theta_7(\pi)$ is a representation of $PSp(6, K)$. Its restriction to $Sp(6, K)$ contains a unique generic constituent, which lifts to $GL(7, K)$ using the results of [CKPS] or [A], and $L_g(\pi)$ is defined to be the unique $G_2$-valued parameter which is conjugate in $GL(7, \mathbb{C})$ to the Galois parameter of the lift to $GL(7, K)$.

Arthur shows in [A] that the lifting from generic representations of $Sp(6, K)$ to representations of $GL(7, K)$ is injective. On the other hand, Savin–Weissman study the injectivity of the lift to $PSp(6, K)$ (see [SWe, Theorem 4.7]). The problematic step for us is therefore restriction from $PSp(6, K)$ to $Sp(6, K)$. Fortunately this has been analyzed by Xu [X], and we will be able to prove injectivity by combining these results with a global argument.

**Lemma 4.1.** Let $\pi, \pi' \in \mathcal{A}_g^0(G_2, K)$, and suppose that $L_g(\pi) = L_g(\pi')$. Then there exists a quadratic character $\omega : PSp(6, K) \to \mathbb{C}^\times$ such that $\theta_7(\pi) \cong \theta_7(\pi') \otimes \omega$.

**Proof.** Let $\sigma$ be the unique generic constituent of the restriction of $\theta_7(\pi)$ to $Sp(6, K)$, and define $\sigma'$ similarly. Then [A, Theorem 1.5.1] shows that $\sigma = \sigma'$. Then [X, Lemma 2.17] shows the existence of a character $\omega$ with the claimed property. \qed

**Corollary 4.2.** Let $\rho \in \mathcal{G}^0(G_2, K)$. Then the following are equivalent:

(i) $L_g^{-1}(\rho)$ is a singleton.

(ii) There exists $\pi \in \mathcal{A}_g^0(G_2, K)$ with the following property: for any non-trivial character $\omega : PSp(6, K) \to \mathbb{C}^\times$, either (a) $\theta_7(\pi) \cong$
\(\theta_\tau(\pi) \otimes \omega\) or (b) \(\theta_\tau(\pi) \otimes \omega\) is not of the form \(\theta_\tau(\pi')\) for any \(\pi' \in \mathcal{A}_g^0(G_2, K)\).

**Proof.** Suppose that (i) holds, and choose any \(\pi\) with \(L_\rho(\pi) = \rho\). If \(\pi' \in \mathcal{A}_g^0(G_2, K)\) and \(\theta_\tau(\pi) \otimes \omega \cong \theta_\tau(\pi')\) then (as \(L_\rho\) is injective) we have \(\pi = \pi'\), hence \(\theta_\tau(\pi) \cong \theta_\tau(\pi) \otimes \omega\).

Suppose instead that (ii) holds, and let \(\pi', \pi'' \in \mathcal{A}_g^0(G_2, K)\) be representations with \(L_\rho(\pi') = L_\rho(\pi'') = \rho\). Let \(\pi \in \mathcal{A}_g^0(G_2, K)\) be the given representation with \(L_\rho(\pi) = \rho\). Thus there exist characters \(\omega', \omega''\) such that \(\theta_\tau(\pi') \cong \theta_\tau(\pi) \otimes \omega'\), \(\theta_\tau(\pi'') \cong \theta_\tau(\pi) \otimes \omega''\). Consideration of possibilities (a) and (b) shows that \(\theta_\tau(\pi) = \theta_\tau(\pi')\). By symmetry, we also have \(\theta_\tau(\pi) = \theta_\tau(\pi'')\), hence \(\theta_\tau(\pi') = \theta_\tau(\pi'')\), hence \(\pi' = \pi''\) by [SWc, Theorem 4.7]. \(\square\)

We first dispense with the simplest case.

**Proposition 4.3.** Let \(\pi, \pi' \in \mathcal{A}_g^0(G_2, K)\), and suppose that \(L_\rho(\pi) = L_\rho(\pi')\). Suppose further that \(\theta_\tau(\pi)\) is not supercuspidal. Then \(\pi = \pi'\).

**Proof.** Our hypotheses imply that \(\theta_\tau(\pi')\) is also not supercuspidal. By [SWc, Theorem 3.9] there exist supercuspidal representations \(\tau, \tau'\) of \(GL(3, K)\) of trivial central character such that \(\theta_\tau(\pi)\) is the generic subquotient of \(i_{Q_2}^{PSp(6)}\tau\), and similarly for \(\theta_\tau(\pi')\) (and we use normalized induction). The existence of an isomorphism \(\theta_\tau(\pi) \otimes \omega \cong \theta_\tau(\pi')\) implies that \(\tau \otimes \omega, \tau'\) are conjugate under the stabilizer in the Weyl group of \(PSp(6)\) of the Levi subgroup of \(Q_3\) (uniqueness of supercuspidal support). This in turn implies that \(\tau \otimes \omega\) is isomorphic to one of \(\tau'\) or \((\tau')^\vee\). This is a contradiction, because the central character of \(\tau \otimes \omega\) is non-trivial. \(\square\)

**Lemma 4.4.** To show that \(L_\rho\) is injective, it is enough to prove the following statement:

- Let \(\rho \in \mathcal{G}_c^0(G_2, K)\) be a parameter which remains irreducible in \(SO(7)\). Then there exists a representation \(\pi \in \mathcal{A}_g^0(G_2, K)\) such that \(L_\rho(\pi) = \rho\) and for any character \(\omega : PSp(6, K) \to \mathbb{C}^\times\) such that \(\theta_\tau(\pi) \not\cong \theta_\tau(\pi) \otimes \omega\), the Shahidi \(L\)-function \(L(\theta_\tau(\pi) \otimes \omega, Spin, s)\) is holomorphic at \(s = 0\).

The Shahidi \(L\)-function is the one defined in [SWc, §5.1] using the realisation of \(GSp(6)\) as a Levi subgroup of \(F_4\).

**Proof.** By Proposition 4.3, it’s enough to show that the condition (ii) of Corollary 4.2 holds for parameters \(\rho\) which are irreducible in \(SO(7)\). Then [SWc, Theorem 5.10] and [SWc, Proposition 4.6] together show that the holomorphy of \(L(\theta(\pi) \otimes \omega, Spin, s)\) at \(s = 0\) implies that \(\theta_\tau(\pi) \otimes \omega\) does not have the form \(\theta_\tau(\pi')\) for any \(\pi' \in \mathcal{A}_g^0(G_2, K)\). \(\square\)

**Proposition 4.5.** To show that \(L_\rho\) is injective, it is enough construct for each \(\rho \in \mathcal{G}_c^0(G_2, K)\) which remains irreducible in \(SO(7)\) the following data:
(i) A totally real field $F$, together with a non-empty set $\Sigma$ of $p$-adic places of $F$ and for each $v \in \Sigma$ an isomorphism $F_v \cong K$.

(ii) A cuspidal, globally generic automorphic representation $\pi$ of $G_2(A_F)$ which is unramified outside $\Sigma$ and discrete series at infinity.

(iii) For each character $\omega : \text{PSp}(6, F) \to \mathbb{C}^\times$, a character $\Omega : \text{PSp}(6, F) \backslash \text{PSp}(6, A_F) \to \mathbb{C}^\times$ which is unramified away from $\Sigma$ and satisfies $\Omega|_{\text{PSp}(6,F_v)} = \omega$ for each $v \in \Sigma$.

with the following property:

(iv) Let $\Psi$ be the lift of $\pi$ to $\text{GL}(7, A_F)$. Then $\Psi$ is cuspidal and there exists a prime $\ell \neq p$ and an isomorphism $\iota : \mathbb{Q}_\ell \to \mathbb{C}$ such that $r_\iota(\Psi)|_{W_{F_v}} \cong \iota^{-1} r_\ell \circ \rho$.

Proof. Recall $\theta_7(\pi)$ is the globally generic, cuspidal lift of $\pi$ to $\text{PSp}(6, A_F)$, and $\Psi$ is the lift of the globally generic constituent of $\pi|_{\text{PSp}(6,A_F)}$ to $\text{GL}(7, A_F)$. Note that $\Psi$ is tempered (by [S, Corollary 1.3]), hence so are $\pi$ and $\theta_7(\pi)$. Fix $v_0 \in \Sigma$, and let $\omega : \text{PSp}(6, F_{v_0}) \to \mathbb{C}^\times$ be a character such that $\theta_7(\pi_{v_0}) \neq \theta_7(\pi_{v_0}) \otimes \omega_{v_0}$. We must show that the Shahidi $L$-function $L(\theta_7(\pi_{v_0}) \otimes \omega_{v_0}, \text{Spin}, s)$ is holomorphic at $s = 0$ (as then the criterion of Lemma 4.4 will be satisfied with $L_\rho(\pi_{v_0}) = \rho$).

Arguing as in the proof of [SWc, Proposition 5.9], we have an identity

\[ (8) \quad \gamma(\theta_7(\pi)_\infty \otimes \Omega_\infty, s) \prod_{v \in \Sigma} \gamma(\theta_7(\pi_v) \otimes \omega, s) = \frac{L^\Sigma(\theta_7(\pi) \otimes \Omega, \text{Spin}, 1 - s)}{L^\Sigma(\theta_7(\pi) \otimes \Omega, \text{Spin}, s)}, \]

where the $\gamma$-factors are those of Shahidi and the zeroes and poles of $\gamma(\theta_7(\pi)_\infty \otimes \Omega_\infty, s)$ lie on finitely many lines parallel to the real axis. The proof of [SWc, Theorem 5.10] shows that it will be enough for us to check that $\gamma_{v_0}(\theta_7(\pi)_{v_0} \otimes \omega, s)$ has finitely many zeroes of the form $s = 2\pi ik/\log q_v$ ($k \in \mathbb{Z}$). However, the factors $\gamma_v(\theta_7(\pi_v) \otimes \omega, s)$ for $v \in \Sigma$ are holomorphic on the line $\text{Re } s = 0$ (for example, by [Sh, Proposition 7.3]) so it will suffice to show that the right-hand side of (8) has only finitely many zeroes of the form $s = 2\pi ik/\log q_{v_0}$.

Using the computation of unramified $L$-functions, the right-hand side of (8) equals the quotient

\[ \frac{L^\Sigma(\Psi \otimes \Omega, 1 - s)L^\Sigma(\Omega, 1 - s)}{L^\Sigma(\Psi \otimes \Omega, s)L^\Sigma(\Omega, s)} \]

of standard $L$-functions. Using the functional equation for these standard $L$-functions, we find that this equals

\[ \frac{L_\Sigma(\Psi \otimes \Omega, 1 - s)L_\Sigma(\Omega, 1 - s)}{L_\Sigma(\Psi \otimes \Omega, s)L_\Sigma(\Omega, s)}, \]

up to product with a meromorphic function with all of its zeroes and poles on the real axis. Since $\omega$ is non-trivial the factor $L_\Sigma(\Omega, 1 - s)/L_\Sigma(\Omega, s)$ is holomorphic and non-vanishing on the line $s = 0$. By purity, $L_\Sigma(\Psi \otimes \Omega, 1 - s)$
is holomorphic and non-vanishing on the line $s = 0$. We are therefore reduced to showing that

$$L_{\Sigma}(\Psi \otimes \Omega, s) = L_{\nu_0}((r_7 \circ \rho) \otimes \omega, s)_{|\Sigma}$$

does not have poles at infinitely many points of the form $s = 2\pi ik/\log q_v \ (k \in \mathbb{Z})$. Equivalently, $(r_7 \circ \rho) \otimes \omega$ does not contain the trivial representation of $W_K$.

Now we make use of [X, Corollary 4.2]. Writing $R : G_2 \rightarrow Spin(7)$ for the natural homomorphism, it states that $\theta_7(\pi_{\nu_0}) \cong \theta_7(\pi_{\nu_0}) \otimes \omega$ if and only if $R \circ \rho$, $(R \circ \rho) \otimes \omega$ are $Spin(7)$-conjugate. We are assuming that this is not the case. Lemma 4.6 below implies that $(r_7 \circ \rho) \otimes \omega$ does not contain the trivial representation; and this completes the proof of the proposition. □

**Lemma 4.6.** Let $\Gamma$ be a group, and let $\rho : \Gamma \rightarrow G_2(\mathbb{C})$ be a completely reducible representation and $\omega : \Gamma \rightarrow \{\pm 1\}$ a non-trivial character such that $R \circ \rho$ and $(R \circ \rho) \otimes \omega$ are not $Spin(7)$-conjugate. Then $(r_7 \circ \rho) \otimes \omega$ does not contain the trivial representation.

**Proof.** Let $R' : Spin(7) \rightarrow GL(8)$ be the spin representation, $R'' : Spin(7) \rightarrow GL(7)$ the vector representation. Then $R' \circ R = r_7 \oplus \mathbb{C}$ and $R'' \circ R = r_7$. In particular, $R' \circ ((R \circ \rho) \otimes \omega) = (r_7 \circ \rho) \otimes \omega \oplus \omega$. We see that $(r_7 \circ \rho) \otimes \omega$ contains the trivial representation if and only if $(R \circ \rho) \otimes \omega$ fixes a non-zero vector in the spin representation.

Since the stabilizers of non-zero vectors in the spin representation are exactly the $Spin(7)$-conjugates of $R(G_2)$, we see that $(r_7 \circ \rho) \otimes \omega$ contains the trivial representation if and only if there exists $g \in Spin(7, \mathbb{C})$ such that $Ad(g) \circ ((R \circ \rho) \otimes \omega)$ is valued in $R(G_2(\mathbb{C}))$.

Suppose for contradiction that this is case, and let $\rho' : \Gamma \rightarrow G_2(\mathbb{C})$ be the homomorphism such that $R \circ \rho' = Ad(g) \circ ((R \circ \rho) \otimes \omega)$. Then $r_7 \circ \rho' = Ad(R''(g)) \circ r_7 \circ \rho$. By [Gr95], this implies that $\rho, \rho'$ are themselves $G_2$-conjugate, hence that $R \circ \rho$ and $R \circ \rho'$ are $R(G_2)$-conjugate, hence that $R \circ \rho$ and $(R \circ \rho) \otimes \omega$ are $Spin(7)$-conjugate. This contradicts our hypothesis. □

To complete the proof of injectivity, it remains to construct data as in Proposition 4.5. Given an irreducible representation $\rho : W_K \rightarrow G_2(\mathbb{C})$ which remains irreducible in $SO(7)$, Theorem 2.3 implies the existence of the following data:

(i) A totally real field $F$, together with a non-empty set $\Sigma$ of $p$-adic places of $F$ and for each $v \in \Sigma$ an isomorphism $F_v \cong K$.

(ii) A prime $\ell \neq p$, an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$, and a homomorphism $\sigma : \Gamma_F \rightarrow G_2(\overline{\mathbb{Q}}_\ell)$ of Zariski dense image such that for each $v \in \Sigma$, $\sigma|_{W_{F_v}}$ is conjugate to $\iota^{-1}\rho$.

(iii) A cuspidal, regular algebraic, self-dual automorphic representation $\Psi$ of $GL(7, \mathbb{A}_F)$ such that $r_7(\Psi) \cong r_7\circ\sigma$ and $\Psi$ is unramified outside $\Sigma$.
We choose for each character \( \omega : \text{PSp}(6, K) \to \mathbb{C}^\times \) a globalization \( \Omega \) such that for each \( v \in \Sigma \), the restriction of \( \Omega \) to \( \text{PSp}(6, F_v) \) equals \( \omega \). After making a quadratic base change, split at \( \Sigma \), we can assume moreover that each character \( \Omega \) is unramified away from \( \Sigma \).

Applying Theorem 3.1, we obtain a cuspidal, globally generic representation \( \pi \) of \( G_2(A_F) \), unramified outside \( \Sigma \) and discrete series at infinity. We have now constructed all of the required data.

5. Final remarks

It is possible to use the same strategy of passing to \( GL(7) \), combined with properties of \( L \)-functions, to show that no pure generic supercuspidal representation of \( G_2 \) is incorrigible. But this can also be derived from the dichotomy of Savin and Weissman [SWe] and any proof using \( L \)-functions ultimately reduces to the dichotomy property.

Appendix A. Genericity of a lift by Gordan Savin

Let \( F \) be a global field and \( \mathbb{A} \) its ring of adèles. The goal of the appendix is to show that any generic cuspidal automorphic form on \( G_2(\mathbb{A}) \) lifts to a generic automorphic form on \( \text{PGSp}_6(\mathbb{A}) \).

A.1. Octonions and \( G_2 \). We follow the exposition [SW15] and work over \( \mathbb{Q} \). Let \( H \) be the algebra of Hamilton quaternions, with the usual basis \( \{1, i, j, k\} \). The split octonion algebra over \( \mathbb{Q} \) is \( O = H \oplus H \) with multiplication

\[
(a, b) \cdot (c, d) = (ac + db, \bar{ad} + cb).
\]

If \( x = (a, b) \), let \( \bar{x} = (\bar{a}, -b) \). Then \( x \mapsto \bar{x} \) is a linear anti-involution of \( O \), defining norm and trace maps

\[
N : O \to F, \quad x \mapsto x\bar{x} = \bar{x}x, \quad \text{Tr} : O \to F, \quad x \mapsto x + \bar{x}
\]
satisfying

\[
N(x \cdot y) = N(x)N(y), \quad \text{Tr}(x \cdot y) = \text{Tr}(y \cdot x), \quad \text{Tr}(x \cdot (y \cdot z)) = \text{Tr}((x \cdot y) \cdot z).
\]

Let \( l = (0, 1) \), so \( O \) has a basis \( \{1, i, j, k, l, li, lj, lk\} \). The following basis is particularly useful.

\[
\begin{align*}
s_1 &= \tfrac{1}{2}(i + li), & s_2 &= \tfrac{1}{2}(j + lj), & s_3 &= \tfrac{1}{2}(k + lk), & s_4 &= \tfrac{1}{2}(1 + l), \\
t_1 &= \tfrac{1}{2}(i - li), & t_2 &= \tfrac{1}{2}(j - lj), & t_3 &= \tfrac{1}{2}(k - lk), & t_4 &= \tfrac{1}{2}(1 - l).
\end{align*}
\]

The multiplication table for this basis is given in Table 1.

Let \( R \subset O \) be the \( \mathbb{Z} \)-lattice spanned by \( s_i \) and \( t_i \). It follows, from the multiplication table, that \( R \) is an order. It is maximal since the determinant of the trace pairing \( \text{Tr}(x \cdot y) \) on \( R \) is 1. Let \( O^0 \) be the subspace of trace zero elements. For every subspace \( V \subset O^0 \), let \( V^\Delta \) be the subspace of all \( x \in O^0 \) such that \( x \cdot y = 0 \) for all \( y \in V \). A subspace \( V \subset O^0 \) on which multiplication is trivial is at most 2-dimensional. (We call such a subspace a null space or a null subspace.) Indeed, let \( \{i, j, k\} = \{1, 2, 3\} \). Then from
the multiplication table we see that $\langle s_i \rangle^\Delta = \langle s_i, t_j, t_k \rangle$, and the null spaces of $O_0$ which contain $s_i$ are all of the form $\langle s_i, at_j + bt_k \rangle$ for fixed $a, b \in \mathbb{Q}$. Since $G_2$, the group of automorphisms of $O$, acts transitively on (nonzero) elements of trace zero and norm zero, this phenomenon is generic.

The group $G_2$ has two conjugacy classes of maximal parabolic subgroups, and they can be described as the stabilizers of null subspaces in $O_0$. Let $V_1 \subset V_2$ be 1 and 2-dimensional null subspaces. Let $V_3 = V_1^\perp \subset V_2$. Let $Q_1 = L_1 U_1$ and $Q_2 = L_2 U_2$ be the stabilizers of $V_1$ and $V_2$, respectively. The Levi factors $L_1$ and $L_2$ are isomorphic to $\text{GL}(V_3/V_1)$ and $\text{GL}(V_2)$, respectively. The Borel subgroup $Q_0 = L_0 U_0 = Q_1 \cap Q_2$ stabilizes the full flag

$$V_1 \subset V_2 \subset V_3 \subset V_3^\perp \subset V_2^\perp \subset V_1^\perp$$

in $O_0$, where $V^\perp$ stands for the orthogonal complement of $V$ with respect to the trace pairing.

**A.2. Albert algebra and $E_7$.** This an exceptional 27-dimensional Jordan algebra $J$ over $\mathbb{Q}$. It is the set of matrices

$$A = \begin{pmatrix} \gamma & x & \bar{y} \\ \bar{x} & \beta & z \\ y & \bar{z} & \alpha \end{pmatrix}$$

where $\alpha, \beta, \gamma, x, y, z \in \mathbb{O}$. We have a cubic form (the determinant) on $J$

$$\det A = \alpha \beta \gamma - \alpha \mathbb{N}(x) - \beta \mathbb{N}(y) - \gamma \mathbb{N}(z) + \text{Tr}(xyz).$$

The group of isogenies of the cubic form is a reductive group of type $E_6$. Its orbits on $J$ are classified by the rank of the matrix $A$. If $A \neq 0$ then $A$ has rank one if $A^2 = \text{Tr}(A) \cdot A$. Explicitly, this means that the entries of $A$ satisfy the equalities

$$\mathbb{N}(x) = \beta \gamma, \mathbb{N}(y) = \gamma \alpha, \mathbb{N}(z) = \alpha \beta, \alpha x = \bar{z} y, \beta \bar{y} = x z, \gamma \bar{z} = y x.$$

Let $G$ be the split, adjoint group of type $E_7$. This group can be constructed from $J$ by the Koecher-Tits construction, see Section 3 in [KS15]. In

|   | $s_1$ | $s_2$ | $s_3$ | $t_1$ | $t_2$ | $t_3$ | $s_4$ | $t_4$ |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| $s_1$ | 0    | $-t_3$ | $t_2$ | $s_4$ | 0    | 0    | 0    | $s_1$ |
| $s_2$ | $t_3$ | 0    | $-t_1$ | 0    | $s_4$ | 0    | 0    | $s_2$ |
| $s_3$ | $-t_2$ | $t_1$ | 0    | 0    | 0    | $s_4$ | 0    | $s_3$ |
| $t_1$ | $t_4$ | 0    | 0    | 0    | $s_3$ | $-s_2$ | $t_1$ | 0    |
| $t_2$ | 0    | $t_4$ | 0    | $-s_3$ | 0    | $s_1$ | $t_2$ | 0    |
| $t_3$ | 0    | 0    | $t_4$ | $s_2$ | $-s_1$ | 0    | $t_3$ | 0    |
| $s_4$ | $s_1$ | $s_2$ | $s_3$ | 0    | 0    | 0    | $s_4$ | 0    |
| $t_4$ | 0    | 0    | 0    | $t_1$ | $t_2$ | $t_3$ | 0    | $t_4$ |

**Table 1. Multiplication Table for Octonions**
particular, $G$ has a pair of opposite maximal parabolic subgroups $P = MN$ and $P = M\bar{N}$, $N \cong J$ and $\bar{N} \cong \bar{J}$, such that the conjugation action of $M$ on $N$ (this action is faithful since $G$ is adjoint) gives an isomorphism of $M$ and the group of isogenies of the cubic form on $J$. The action of $M$ on $J$ resulting from the isomorphism $\bar{N} \cong J$ is dual to the action arising from $N$. Observe that $G_2$ acts naturally on $J$ (by acting on off-diagonal entries). This gives an embedding of $G_2$ into $M$. Let $M_3$, $N_3$ and $\bar{N}_3$ be the centralizers of $G_2$ in $M$ and $N$, respectively. It is clear that $N_3 \cong J_3$ and $\bar{N}_3 \cong \bar{J}_3$, where $J_3$ is the Jordan algebra of symmetric $3 \times 3$-matrices. The group $M_3$ is isomorphic to $GL_3$. This isomorphism is realized by observing that $GL_3$ acts on $J$ by isogenies

$$A \mapsto \det(g)^{-1}gAg^t$$

for all $g \in GL_3$. Thus the centralizer of $G_2$ in $G$ is $PGSp_6$ with $P_3 = M_3N_3$ and $P_3 = M_3\bar{N}_3$ a pair of opposite maximal parabolic subgroups.

**A.3. Minimal representation of $G$.** Let $F$ be a local field of characteristic 0. In this section, $J$, $G$ etc stand for their sets of $F$-points. Fix $\psi : F \to \mathbb{C}^\times$, a non-trivial unitary additive character. Every $A \in J$ defines a character $\psi_A$ of $J$

$$\psi_A(B) = \psi(\text{Tr}(A \circ B)) = \psi_B(A)$$

for all $B \in J$, where $A \circ B$ denotes the Jordan multiplication. We view $\psi$ a character of $N$, every unitary character of $N$ is equal to $\psi_A$ for some $A$. Let $\Omega \subseteq J$ be the set of rank one elements in $J$. We view $\Omega \subseteq \bar{N}$, where $\bar{N}$ is the opposite of $N$. A unitary model of the minimal representation is $L^2(\Omega)$. Here only the action of the maximal parabolic $P = MN$ is obvious: Let $n \in N$ correspond to $B \in J$ via the isomorphism $N \cong J$. Then, for $f \in L^2(\Omega)$,

$$\pi(n)f(A) = \psi_B(A) \cdot f(A).$$

Any $m \in M$ acts on $\bar{N}$ by conjugation, and therefore on $\Omega \subset J$ via the identification $\bar{N} \cong J$. Then, for $f \in L^2(\Omega)$,

$$\pi(m)f(A) = \chi(m)f(m^{-1}A)$$

where $\chi$ is a positive character of $M$ that we shall not need. We have the following, see Propositions 7.2 and 8.3 in [KS15]:

**Theorem A.4.** Let $\Pi$ be the subspace of $G$-smooth vectors in the minimal representation. Then $C_c^\infty(\Omega) \subseteq \Pi \subseteq C^\infty(\Omega)$. If $A \in J$, non-zero, then any continuous functional $\ell$ on $\Pi$ such that $\ell(\pi(n)f) = \psi_A(n) \cdot \ell(f)$ for all $n \in N$ and $f \in \Pi$ is equal to a multiple of the delta functional

$$f \mapsto f(A).$$

In particular, $\ell = 0$ if $A$ is not rank one. If $F$ is $p$-adic, we moreover have an exact sequence of $P$-modules

$$0 \to C_c^\infty(\Omega) \to \Pi \to \Pi_N \to 0.$$
The representation $\Pi$ is spherical, and we describe a spherical vector in the $p$-adic case. Let $O$ be the ring of integers in $F$, $\varpi \in O$ a uniformizing element and $q$ the order of the residual field. The maximal order $\mathcal{R}$ in $\mathbb{O}$ defines an integral structure on $J$, let $J(O)$ be the lattice of $O$-points in $J$. The greatest common divisor of entries of $A \in J(O)$, is the largest power $\varpi^n$ dividing $A$ i.e. such that $A/\varpi^n$ is in $J(O)$. We have the following Theorem 6.1 in [SW07]:

**Theorem A.5.** Assume $F$ is a $p$-adic field. Assume the conductor of $\psi$ is $O$. Then the spherical vector in $\Pi$ is a function $f \circ u \in C^\infty(\Omega)$ supported in $J(O)$. Its value at $A \in \Omega$ depends on the gcd of entries of $A$. More precisely, if the gcd of $A$ is $\varpi^n$, and $q$ is the order of the residual field, then

$$f^n(A) = 1 + q^3 + \ldots + q^{3(n-1)}.$$ 

**A.6. Local non-vanishing.** In this section $F$ is a $p$-adic field. Let $U_3 \subset M_3$ be the maximal of unipotent subgroup that, via the isomorphism $M_3 \cong \text{GL}_3$, is the group of matrices

$$u = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Then $U = U_3N_3$ is a maximal unipotent subgroup of $\text{PGSp}_6$. We define a Whittaker character $\psi_U$ on $U$ as follows. For every $u \in U_3$, written as above, $\psi_U(u) = \psi(a + b)$. For every $u \in N_3$, which we identify with $B \in J_3 \cong N_3$, $\psi_U(u) = \psi_E(B)$ where

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

The following result was stated in Proposition 17 of [GS04] without a proof. We give details following Theorem 7.1 in [G99], where a version of this result was proved for groups over finite fields.

**Theorem A.7.** Let $\Pi$ be the minimal representation of $G$, $U$ the maximal unipotent subgroup of $\text{PGSp}_6$ and $\psi_U$ the Whittaker character of $U$. Then there exists a maximal unipotent subgroup $U_0$ of $G_2$ and a Whittaker character $\psi_0$ of $U_0$ such that

$$\Pi_{U,\psi_U} \cong \text{ind}_{U_0}^{G_2} \psi_0.$$ 

**Proof.** We shall compute $(U, \psi_U)$-coinvariants in two steps, $(N_3, \psi_U)$-coinvariants, followed by $(U_3, \psi_U)$-coinvariants. From Theorem A.4 it follows that

$$\Pi_{N_3,\psi_U} \cong C_0^\infty(\Omega)_{N_3,\psi_U} \cong C_0^\infty(\omega)$$ 

where $\omega$ is the set of all rank one elements $A$ such that $\psi_A = \psi_E$ on $N_3 \cong J_3$. This works out to all

$$A = \begin{pmatrix} 0 & x & -y \\ -x & 0 & z \\ y & -z & 1 \end{pmatrix} \in J$$
where \( x, y, z \in \mathbb{O} \) satisfying
\[
\text{Tr}(x) = \text{Tr}(y) = \text{Tr}(z) = 0, \\
x^2 = y^2 = z^2 = 0, \\
\begin{align*}
xz &= yz = 0, \\
uyz &= x.
\end{align*}
\]

It is now useful to write \( \omega = \omega' \cup \omega'' \) where \( \omega' \) is the open subset given by \( x \neq 0 \). Observe that \( G_2 \) acts transitively on \( \omega' \). Indeed, \( x, y, z \) can be \( G_2 \)-conjugated into the elements \( s_1, t_2, t_3 \). This also implies that \( x, y, z \) are linearly independent. The group \( U_3 \) acts on \( \omega \), and hence on the triples \((x, y, z)\) of off-diagonal terms and explicitly this action is
\[
u^{-1} \cdot (x, y, z) = (x, y + bx, z + ay + cx).
\]

Hence \( U_3 \) acts freely on \( \omega' \), by the linear independence of \( x, y, z \), but with large stabilizers on \( \omega'' \), assuring that
\[
\Pi_{U, \psi} \simeq \text{ind}^{G_2}_{U_0} (\psi_0).
\]

A.8. **Global non-vanishing.** Assume now that \( F \) is a global field, and let \( \mathbb{A} \) be the ring of adèles over \( F \). Let \( \Pi = \otimes \Pi_v \) be the restricted tensor product of minimal representations over all local places \( v \) of \( F \). Every element in \( \Pi \) is a finite linear combination of pure tensors \( f = \otimes f_v \), where \( f_v = f_v^0 \) for almost all places \( v \). There is a unique, up to a non-zero scalar, embedding \( \theta : \Pi \to \mathbb{A} (G(F) \backslash G(\mathbb{A})) \) of \( \Pi \) into the space of automorphic functions of uniform moderate growth. We Fourier expand \( \theta(f) \) along \( N \). More precisely, fix a non-trivial character
ψ : \mathbb{A}/F \to \mathbb{C}^\times. Then, as in the local case, any \( A \in J(F) \) defines a character \( \psi_A \) of \( N(F) \backslash N(\mathbb{A}) \) by the isomorphism \( N(\mathbb{A}) \cong J(\mathbb{A}) \). Let

\[
\theta(f)_A(g) = \int_{N(F) \backslash N(\mathbb{A})} \theta(f)(ng)\tilde{\psi}_A(n) \, dn.
\]

We have a Fourier expansion

\[
\theta(f)(g) = \theta(f)_0(g) + \sum_{A \in \Omega(F)} \theta(f)_A(g)
\]

supported on the set of rank one elements. Let \( A \in \Omega(F) \). Observe that \( f \mapsto \theta(f)_A \) is a continuous, \((N(\mathbb{A}), \psi_A)\)-equivariant functional on \( \Pi \). By uniqueness of local functionals, Theorem A.4, there exists a non-zero scalar \( c_A \) such that \( \theta_A(f) = c_A \cdot f(A) \), for all \( f \in \Pi \). Hence

\[
\theta(f)_A(g) = c_A \cdot (\pi(g)f)(A)
\]

for all \( f \in \Pi \) and all \( g \in G(\mathbb{A}) \), where \( \pi \) denotes the action of \( g \in G(\mathbb{A}) \) on \( f \in \Pi \). This formula is particularly useful if \( g \in G(\mathbb{A}) \). Then \( (\pi(g)f)(A) = f(g^{-1}A) \), where \( g^{-1}A \) is the result of the natural action of \( g^{-1} \) on the off-diagonal entries of \( A \).

For every cusp form \( h \in \mathcal{A}(G_2(F) \backslash G_2(\mathbb{A})) \) consider the function \( \Theta(f, h) \) on \( \text{PGSp}_6(\mathbb{A}) \) defined by

\[
\Theta(f, h)(g_1) = \int_{G_2(F) \backslash G_2(\mathbb{A})} \theta(f)(g_1g)h(g) \, dg.
\]

The function \( \theta(f) \) is of moderate growth on \( G \), hence it is also on \( G_2(\mathbb{A}) \times \text{PGSp}_6(\mathbb{A}) \). In particular, the integral converges absolutely, since \( h \) is a cusp form, and the output \( \Theta(f, h) \) is a function of uniform moderate growth on \( \text{PGSp}_6(\mathbb{A}) \). Let \( U_0 \) is the maximal unipotent subgroup stabilizing the partial flag \( \langle s_1 \rangle \subset \langle s_1, t_2 \rangle \subset \langle s_1, t_2, t_3 \rangle \). Let \( U'_0 \subset U_0 \) be the stabilizer of the triple \( \langle s_1, t_2, t_3 \rangle \). Then \( U_0/U'_0 \) is isomorphic to \( U_3 \) and, via this isomorphism, the Whittaker character \( \psi_U \) of \( U(F) \backslash U(\mathbb{A}) \) transfers to a Whittaker character \( \tilde{\psi}_0 \) of \( U_0(F) \backslash U_0(\mathbb{A}) \) as in the local case. Let

\[
h_{U_0, \psi_0}(g) = \int_{U_0(F) \backslash U_0(\mathbb{A})} h(ug)\tilde{\psi}_0(u) \, du.
\]

**Theorem A.9.** Let \( \psi_U \) be the Whittaker character of \( U(F) \backslash U(\mathbb{A}) \). If \( h_{U_0, \psi_0} \neq 0 \) then, for some choice of \( f \in \Pi \),

\[
\Theta(f, h)_{U, \psi_U}(1) := \int_{U(F) \backslash U(\mathbb{A})} \Theta(f, h)(u)\tilde{\psi}_U(u) \, du \neq 0.
\]

**Proof.** The first part of the proof involves using the Fourier expansion of \( \theta(f) \) and unfolding the integral. This follows closely the proof of the local Theorem A.7 and we shall be brief. Firstly we integrate over \( N_3(F) \backslash N_3(\mathbb{A}) \).
This reduces the Fourier sum to over the subset $\omega(F)$

$$
\Theta(f, h)_{N_3, \psi_U}(1) = \int_{G_2(F) \backslash G_2(\mathbb{A})} \sum_{A \in \omega(F)} \theta(f)_A(g) h(g) \, dg.
$$

Write $\omega = \omega' \cup \omega''$ as in the local case. We can ignore the sum over $\omega''$ since it will vanish after integrating over $U_3(F) \backslash U_3(\mathbb{A})$. On the other hand, $\omega'(F)$ is the $G_2(F)$-orbit of $A_0$ corresponding to the triple $(x, y, z) = (s_1, t_2, t_3)$ with the stabilizer $U_0'(F)$. Then

$$
\int_{G_2(F) \backslash G_2(\mathbb{A})} \sum_{A \in \omega'(F)} \theta(f)_A(g) h(g) \, dg = c_{A_0} \int_{U_0'(F) \backslash G_2(\mathbb{A})} f(g^{-1}A_0) h(g) \, dg.
$$

We integrate over $U_0'(F) \backslash U_0'(\mathbb{A})$, and use that the function $g \mapsto f(g^{-1}A_0)$ is left $U_0'(\mathbb{A})$-invariant, hence

$$
c_{A_0} \int_{U_0'(F) \backslash G_2(\mathbb{A})} f(g^{-1}A_0) h(g) \, dg = c_{A_0} \int_{U_0'(\mathbb{A}) \backslash G_2(\mathbb{A})} f(g^{-1}A_0) h_{U_0'}(g) \, dg
$$

where $h_{U_0'}$ is the constant term of $h$ along $U_0'$. Finally, we integrate over $U_3(F) \backslash U_3(\mathbb{A})$ against the character $\tilde{\psi}_U$. Using the isomorphism $U_3 \cong U_0/U_0'$, we obtain

$$
\Theta(f, h)_{U, \tilde{\psi}_U}(1) = c_{A_0} \int_{U_0'(\mathbb{A}) \backslash G_2(\mathbb{A})} \tilde{f}(g) h_{U_0'}(g) \, dg
$$

where $\tilde{f}$ is the product of

$$
\tilde{f}_u(g) = \int_{U_0'(F_v) \backslash U_0(F_v)} f_v((ug)^{-1}A_0) \psi_0(u) \, du.
$$

Observe that $\tilde{f}$ is a Whittaker function i.e. $\tilde{f}(ug) = \tilde{\psi}_0(u) \tilde{f}(g)$, for all $u \in U_0(\mathbb{A})$. Hence

$$
\Theta(f, h)_{U, \tilde{\psi}_U}(1) = c_{A_0} \int_{U_0(\mathbb{A}) \backslash G_2(\mathbb{A})} \tilde{f}(g) h_{U_0, \psi_0}(g) \, dg.
$$

The next step is to show that this integral reduces to a finite number of places $S$, as in Section 7 of [GS03]. Assume that $S$ contains all archimedean places and, if $v \notin S$, then

- $f_v$ is the spherical vector $f_v^0$,
- the cusp form $h$ is $G_2(O_v)$-invariant,
- the conductor of $\psi$ restricted to $F_v$ is $O_v$,
- $l_vt_2 = \lambda_2 \cdot t_2$, $l_vt_3 = \lambda_3 \cdot t_3$, $l_vs_1 = (\lambda_2 \lambda_3) \cdot s_1$,

for some non-zero scalars $\lambda_2$ and $\lambda_3$, where the last identity follows from $s_1 = t_2t_3$. 

Let $B_0$ be the Borel subgroup containing $U_0$. We fix a Levi factor $L_0$ (a torus) so that is stabilizes the lines through $s_1, t_2, t_3$. Thus, if $l_v \in L_0(F_v)$ then

$$
\begin{align*}
 l_vt_2 &= \lambda_2 \cdot t_2, \\
 l_vt_3 &= \lambda_3 \cdot t_3, \\
 l_vs_1 &= (\lambda_2 \lambda_3) \cdot s_1,
\end{align*}
$$

for some non-zero scalars $\lambda_2$ and $\lambda_3$, where the last identity follows from $s_1 = t_2t_3$. 

Lemma A.10. If $v \notin S$ then the Whittaker function $\tilde{f}_v^o$ is supported on $U_0(F_v)G_2(O_v)$ and $\tilde{f}_v^o(1) = 1$.

Proof. Let $\text{ord}_v$ denote the valuation on $F_v^\times$. Since $\tilde{f}_v^o$ is a spherical Whittaker function, it is determined by its restriction to $L_0(F_v)$ and there it is supported on the cone consisting of $l_v$ such that $\text{ord}_v(\alpha_1(l_v)) \geq 0$ and $\text{ord}_v(\alpha_2(l_v)) \geq 0$ where $\alpha_1$ and $\alpha_2$ are the simple roots. In terms of the explicit action of $l_v$ on $s_1, t_2, t_3$ described above, these inequalities are $\text{ord}_v(\lambda_2) \geq \text{ord}_v(\lambda_3) \geq 0$.

On the other hand, since $f_v^o$ is supported on the lattice $J(O_v)$, it is easily seen that, for any $u_v \in U_0(F_v)$, the function $l_v \mapsto f_v^o((u_v l_v)^{-1} A_0)$ is supported on the cone $\text{ord}_v(\lambda_2), \text{ord}_v(\lambda_3) \leq 0$.

This completes the support part of the statement. Using again that $f_v^o$ is supported on the lattice $J(O_v)$, the function $u_v \mapsto f_v^o(u_v^{-1} A_0)$, is supported on $U_0^o(F_v)U_0(O_v)$ and this implies that $\tilde{f}_v^o(1) = 1$, since the character $\psi_0$ is trivial on $U_0(O_v)$.

Let $A_S$ be the product of $F_v$ over all $v \in S$. The previous lemma implies that

$$\Theta(f, h)_{U, \psi_0}(1) = c_{A_0} \int_{U_0(A_S) \backslash G_2(A_S)} \tilde{f}_S(g) h_{U_0, \psi_0}(g) \, dg.$$ 

Since, by Theorem A.4, $f_v$ can be any smooth, compactly supported function on $\Omega(F_v)$, for every $v \in S$, the integral can be arranged to be non-zero. This completes the proof of the theorem.

Remark: Note that $\Theta(f, h)$ is unramified for all $v \notin S$.

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