UNFOLDING OF THE UNRAMIFIED IRREGULAR SINGULAR GENERALIZED ISOMONODROMIC DEFORMATION

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Abstract. We introduce an unfolded moduli space of connections, which is an algebraic relative moduli space of connections on complex smooth projective curves, whose generic fiber is a moduli space of regular singular connections and whose special fiber is a moduli space of unramified irregular singular connections. On the moduli space of unramified irregular singular connections, there is a subbundle of the tangent bundle defining the generalized isomonodromic deformation produced by the Jimbo-Miwa-Ueno theory. On an analytic open subset of the unfolded moduli space of connections, we construct a non-canonical lift of this subbundle, which we call an unfolding of the unramified irregular singular generalized isomonodromic deformation. Our construction of an unfolding of the unramified irregular singular generalized isomonodromic deformation is not compatible with the asymptotic property in the unfolding theory established by Hurtubise, Lambert and Rousseau which gives unfolded Stokes matrices for an unfolded linear differential equation in a general framework.

Introduction

The intention of this paper is to produce a tool toward understanding the confluence phenomena connecting the regular singular isomonodromic deformation and the irregular singular generalized isomonodromic deformation. In the case of connections on $\mathbb{P}^1$, the regular singular isomonodromic deformation is the Schlesinger equation and the irregular singular generalized isomonodromic deformation is the Jimbo-Miwa-Ueno equation which is completely given in [15], [16], [17]. The most fundamental example of the confluence phenomena will be the confluence of the classical hypergeometric functions, though their isomonodromic deformations may not be mentioned because of the rigidity. There are extended results in [21], [35], [18] and [19]. The next important example of the confluence phenomena will be the degeneration of Painlevé equations, where the irregular singular generalized isomonodromic deformation arises when we take a limit of the regular singular isomonodromic deformation. In [20], a generalization of this machinery to a general Schlesinger equation is given. There is an approach via monodromy manifolds in [26] to the confluence of Painlevé equations. An origin of confluence problems is given by Ramis in [29] and unfolding of Stokes data is one of the important problems. Unfolding of Stokes matrices using an analysis of flows is given in [7] and a general framework of unfolded Stokes data of an unfolded linear differential equation is established by Hurtubise, Lambert and Rousseau in [9] and [10]. In [22], confluence of unfolded Stokes data in rank two case is given. One of the key ideas in [9] and [10] is to adopt fundamental solutions with an asymptotic property, which is estimated by a flow of the vector field $v_\epsilon = p_\epsilon(x) \frac{\partial}{\partial x}$, where $p_\epsilon(x) = 0$ is a local unfolding equation. They construct unfolded Stokes matrices of a linear differential equation on $\mathbb{P}^1$ via connecting fundamental solutions with an asymptotic property around points in the unfolding divisor and that around $\infty$. In order to reconstruct an unfolded linear differential equation, they consider another regular singular point, whose monodromy reflects the analytic continuation along the ‘inner side’ of the unfolded divisor. In [10], they introduce a delicate condition called the ‘compatibility condition’ in order that the corresponding linear differential equation is a well-defined analytic family.

In this paper, we adopt a moduli theoretic approach to the construction of an unfolding or a perturbation of the generalized isomonodromic deformation.

The Schlesinger type equation, or the regular singular isomonodromic deformation is defined on a family of moduli spaces of regular singular connections on smooth projective curves. In order to get a good moduli space, we consider a parabolic structure to the given connection and the moduli space is constructed in [27], [1], [11] and [12], which is a smooth and quasi-projective moduli space. The algebraic moduli construction is basically given by modifying the standard method by Simpson in [33], [34] or by Nitsure in [28]. In [11] and [12], we formulate the regular singular isomonodromic deformation and prove the geometric Painlevé property of the isomonodromic deformation using the properness of the Riemann-Hilbert morphism. In

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satisfies $C(\text{data of local exponents}).$ For example, the regular singular case, unfolding problem of the moduli space of irregular singular connections does not completely determined by the given connection. Let us recall that the essential number of independent variables of the regular singular isomonodromic deformation is $3g - 3 + \deg D$, where $D$ is the divisor consisting of all the regular singular points and $g$ is the genus of base curves.

Moduli space of unramified irregular singular connections is constructed in [3] analytically and in [14] algebraically. The moduli theoretic description of the irregular singular generalized isomonodromic deformation is given in [12, Proposition 8.1] on the locus where the parabolic structure is not completely determined by the given connection. Let us recall that the essential number of independent variables of the regular singular isomonodromic deformation is $3g - 3 + \deg D$, where $D$ is the divisor consisting of all the regular singular points and $g$ is the genus of base curves.

In this paper, we adopt another method of parameterizing the local exponents. If we fix distinct complex numbers $\mu_1, \ldots, \mu_r$ and if we take generic unramified local exponents $\nu_1 \frac{dz}{z^m}, \ldots, \nu_r \frac{dz}{z^m}$ at a singular point $p$, then we can observe that there is a polynomial $\nu(T) \in \mathbb{C}[z]/(z^m)[T]$ satisfying $\nu_k = \nu(\mu_k)$ for any $k$. So we can regard $(\nu(T), \mu_1, \ldots, \mu_r)$ as a data of local exponents. We can see that a connection $\nabla$ on a vector bundle $E$ has the local exponents $\nu_1 \frac{dz}{z^m}, \ldots, \nu_r \frac{dz}{z^m}$ at $p$ if and only if there is an endomorphism $N \in \text{End}(E|_{mp})$ whose eigenvalues are $\mu_1, \ldots, \mu_r$ and $\nu(N) \frac{dz}{z^m} = \nabla|_{mp}$.

In the construction of the unfolded moduli space of connections, we consider a smooth projective curve $C$ of genus $g$ and a divisor $D = D^{(1)} \cup \cdots \cup D^{(n)}$ on $C$ locally given by the equation $D^{(i)} = \{z^m - \epsilon^{m_i} = 0\}$. Then we regard a polynomial $\nu^{(i)}(T) \in \mathcal{O}_{D^{(i)}}[T]$ and distinct complex numbers $\mu_1^{(i)}, \ldots, \mu_r^{(i)} \in \mathbb{C}$ as a data of local exponents. For $\mu = (\mu^{(i)}_k)$, we introduce the concept of $(\nu, \mu)$-connections $(E, \nabla, \{N^{(i)}\})$ in Definition 2.3 where $E$ is an algebraic vector bundle on a base projective algebraic curve $C$, $\nabla$ is a connection on $E$ admitting poles along the divisor $D = D^{(1)} + \cdots + D^{(n)}$ and $N^{(i)} \in \text{End}(E|_{D^{(i)}})$ satisfies $\nabla|_{D^{(i)}} = \nu(N^{(i)}) \frac{dz}{z^m} - \epsilon^{m_i}$ and $\varphi^{(i)}_{\mu}(N^{(i)}) = 0$, where $\varphi^{(i)}_{\mu}(T) = (T - \mu_1^{(i)})(T - \mu_2^{(i)}) \cdots (T - \mu_r^{(i)})$.

In subsection 5.1, we define the relative moduli space $M^0_{\mathbb{C}, D}(\tilde{\nu}, \lambda)$ of $\alpha$-stable $(\nu, \mu)$-connections whose existence is provided by Theorem 2.11. Here we define $T_{\mu, \lambda} \rightarrow \Delta_{\epsilon, 0}$ in subsection 2.1 on which there are a full family of pointed curves $(C, t_1, \ldots, t_n)$, perturbations $D^{(i)}$ of the divisor $m_i t_i$ given by the equation $z^m - \epsilon^{m_i} = 0$ and a full family of exponents $\nu$. The fiber of the moduli space $M^0_{\mathbb{C}, D}(\tilde{\nu}, \lambda)$ over $\epsilon \neq 0$ is a moduli space of regular singular connections and the fiber over $\epsilon = 0$ is a moduli space of generic unramified irregular singular connections. We can give an expression of the relative symplectic form in Proposition 2.26 on the algebraic splitting $\Psi_0 : (\pi_{T_{\mu, \lambda}, \mathbb{C}, 0})^* T_{T_{\mu, \lambda}, \mathbb{C}, 0} \rightarrow T_{M^0_{\mathbb{C}, D}(\tilde{\nu}, \lambda), \mathbb{C}, 0}$ of the surjection $d\pi_{T_{\mu, \lambda}, \mathbb{C}, 0} : T_{M^0_{\mathbb{C}, D}(\tilde{\nu}, \lambda)} \rightarrow (\pi_{T_{\mu, \lambda}, \mathbb{C}, 0})^* T_{T_{\mu, \lambda}, \mathbb{C}, 0}$, where $T_{T_{\mu, \lambda}, \mathbb{C}, 0}$ and $T_{M^0_{\mathbb{C}, D}(\tilde{\nu}, \lambda), \mathbb{C}, 0}$ are the tangent bundles of $T_{\mu, \lambda}$ and $M^0_{\mathbb{C}, D}(\tilde{\nu}, \lambda)$, respectively. The splitting $\Psi_0$ is the irregular singular generalized isomonodromic deformation arising from the theory by Jimbo, Miwa and Ueno in [15], which is extended in [6]. The idea of the construction of $\Psi_0$ is to construct a horizontal lift of the universal relative connection, which is a first order infinitesimal extension of the relative connection with an integrability condition. We notice here that the complete description of the Jimbo-Miwa-Ueno equation in [15] says that the essential number of independent variables of the unramified irregular singular generalized isomonodromic deformation is $3g - 3 + \sum_{i=1}^n (r(m_i - 1) + 1)$.

One of the reasons of the difficulty in the confluence problem will be that the number $3g - 3 + \deg D$ of independent variables of the regular singular isomonodromic deformation is much smaller than the number $3g - 3 + \sum_{i=1}^n (r(m_i - 1) + 1)$ of independent variables of the irregular singular generalized isomonodromic deformation. Here we have $\deg D = \sum_{i=1}^n m_i$, because the divisors are connected by a flat family. In this paper, we try to extend the splitting $\Psi_0$ locally to the unfolded moduli space $M^0_{\mathbb{C}, D}(\tilde{\nu}, \lambda)$ via regarding $T_{\mu, \lambda}$ as the space of independent variables. The main theorem of this paper is the following:
Theorem 0.1. For a general point \( x \in M^\alpha_{C,D}(\tilde{\nu}, \lambda) \) satisfying Assumption 5.6 in subsection 5.3, there exist an analytic open neighborhood \( M^\circ \subset M^\alpha_{C,D}(\tilde{\nu}, \lambda) \) of \( x \) whose image in \( T_{\mu,\lambda} \) is denoted by \( T^\circ \), blocks of local horizontal lifts \( (\nabla_{\bar{P}^1 \times M^\circ[\tilde{\nu}],\nu_{i,j}^{(i)})} \) defined in Definition 5.7 and a holomorphic splitting

\[
\Psi: (\pi T^\circ)^* T_{M^\circ/\Delta_{\alpha}} \longrightarrow T_{M^\circ/\Delta_{\alpha}}
\]

of the canonical surjection of the tangent bundles \( T_{M^\circ/\Delta_{\alpha}} \) coincides with \( \Psi\big|_{M^\circ_{C,D}(\tilde{\nu}, \lambda)=0 \cap M^\circ} \) such that the restriction \( \Psi\big|_{M^\circ_{C,D}(\tilde{\nu}, \lambda)\cap M^\circ} \) is isomorphic to the cotangent space of \( M^\circ_{C,D}(\tilde{\nu}, \lambda) \) and a holomorphic splitting

\[
\Psi: (\pi T^\circ)^* T_{M^\circ/\Delta_{\alpha}} \longrightarrow T_{M^\circ/\Delta_{\alpha}}
\]

The main idea of the construction of \( \Psi \) in Theorem 0.1 is to extend a local restriction of the universal family of connections to a family of connections on \( \mathbb{P}^1 \) admitting regular singularity at \( \infty \). We can construct, using the arguments in section 4, a data \( (\tilde{\Xi}^{(i)}_l(z)) \) adjusting the residue part and an integrable connection \( \nabla_{\bar{P}^1 \times M^\circ[\tilde{\nu}],\nu_{i,j}^{(i)}} \) which is an extension of the restricted universal family of connections. In section 5, we glue local integrable connections obtained by restrictions of \( \nabla_{\bar{P}^1 \times M^\circ[\tilde{\nu}],\nu_{i,j}^{(i)}} \) and get an unfolding in Theorem 0.1.

In our unfolded generalized isomonodromic deformation determined by \( \Psi \), the monodromy along a loop surrounding \( \infty \) is the unfolding divisor \( D^{(i)} \) is preserved constant, but the local monodromy around each regular singular point in \( D^{(i)} \) is not preserved constant, because the local exponents are not constant. So our unfolded generalized isomonodromic deformation does not mean the usual regular singular isomonodromic deformation. We notice that the splitting \( \Psi \) in the theorem is not canonical because it is essentially determined by the blocks of local horizontal lifts \( (\nabla_{\bar{P}^1 \times M^\circ[\tilde{\nu}],\nu_{i,j}^{(i)})} \) constructed in subsection 4.2 which depend on the data \( (\tilde{\Xi}^{(i)}_l(z)) \) adjusting the residue part. So we cannot expect the splitting \( \Psi \) to be defined globally on \( M_{C,D}(\tilde{\nu}, \lambda) \). Moreover, we cannot expect the integrability condition for the subbundle im \( \Psi \subset T_{M^\circ/\Delta_{\alpha}} \).

The author’s hope was to construct the unfolding \( \Psi \) via adopting the asymptotic arguments in the unfolding theory established by Hurtubise, Lambert and Rousseau in a series of papers [23, 24, 9, 10]. Unfortunately we cannot achieve in such an easy way, because we do not know that the unfolded Stokes matrices defined in [10] are constant for our generalized isomonodromic deformation \( \Psi \). This is another reason why the splitting \( \Psi \) cannot be extended globally.

At the present, the framework of this paper is tentative because the moduli space \( M_{C,D}(\tilde{\nu}, \lambda) \) does not seem to be enough for the description of the unfolded generalized isomonodromic deformation. The author’s hope is to find a good replacement of the moduli space which describes our splitting \( \Psi \) adequately.

1. AN OBSERVATION FROM LINEAR ALGEBRA ON A \( GL_r(\mathbb{C}) \) ADJOINT ORBIT

In this section, we give a small remark on an adjoint orbit of \( GL_r(\mathbb{C}) \) on \( \mathfrak{gl}_r(\mathbb{C}) \). From the idea of the observation in this section, we will get in section 2 a convenient parametrization of the local exponents of connections. Furthermore, we will get a pertinent expression of the relative symplectic form on an unfolded moduli space of connections on smooth projective curves in section 2.

1.1. Factorization of a linear endomorphism whose minimal polynomial is of maximal degree.

Let \( V \) be a vector space over \( \mathbb{C} \) of dimension \( r \) and \( \mu_1, \ldots, \mu_r \in \mathbb{C} \) be mutually distinct complex numbers. If we consider the subvariety

\[
C(\mu_1, \ldots, \mu_r) := \{ f : V \longrightarrow V : \text{linear map with the eigenvalues } \mu_1, \ldots, \mu_r \}
\]

of the affine space \( \text{Hom}_\mathbb{C}(V, V) \), then \( C(\mu_1, \ldots, \mu_r) \) is isomorphic to the \( GL_r(\mathbb{C}) \)-adjoint orbit of the diagonal matrix

\[
\begin{pmatrix}
\mu_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \mu_r
\end{pmatrix}
\]

So \( C(\mu_1, \ldots, \mu_r) \) has a symplectic structure given by the Kirillov-Kostant symplectic form. Indeed there is a canonical morphism from \( C(\mu_1, \ldots, \mu_r) \) to the complete flag variety \( F(V) \) by sending each \( f \) to the flag of \( V \) induced by the eigen space decomposition of \( f \). The fiber is isomorphic to the set of upper triangular nilpotent matrices which is also isomorphic to the cotangent space of \( F(V) \). So \( C(\mu_1, \ldots, \mu_r) \) is locally isomorphic over \( F(V) \) to the cotangent bundle over \( F(V) \) and the symplectic structure from the cotangent bundle coincides with the Kirillov-Kostant symplectic form. In subsection 1.2 we give another expression
of the symplectic form on the adjoint orbit $C(\mu_1, \ldots, \mu_r)$. For the construction of the symplectic form, we extend to a slightly more general setting.

Let $\varphi(T) \in \mathbb{C}[T]$ be a monic polynomial of degree $r$ and $V$ be a vector space over $\mathbb{C}$ of dimension $r$. We put

$$C_{\varphi(T)} := \{ f : V \to V \mid f \text{ is a linear map whose minimal polynomial is } \varphi(T) \}.$$ 

Recall that $\varphi(T)$ is a minimal polynomial of $f : V \to V$ if and only if $\varphi(f) = 0$ and the induced map

$$\mathbb{C}[T]/(\varphi(T)) \ni P(T) \mapsto P(f) \in \text{End}(V)$$

is injective.

**Proposition 1.1.** For each $f \in C_{\varphi(T)}$, there are an isomorphism $\theta : V^\vee \cong V$ and a linear map $\kappa : V \to V^\vee$ satisfying $f = \theta \circ \kappa$, $\text{^t}\theta = \theta$ and $\text{^t}\kappa = \kappa$. Here $V^\vee$ is the dual vector space of $V$, $\text{^t}\theta : V^\vee \to (V^\vee)^\vee = V$ is the dual of $\theta$ and $\text{^t}\kappa : V = (V^\vee)^\vee \to V^\vee$ is the dual of $\kappa$.

**Proof.** The ring homomorphism $\mathbb{C}[T] \ni P(T) \mapsto P(f) \in \text{End}(V)$ induces a $\mathbb{C}[T]$-module structure on $V$. By an elementary theory of linear algebra, there is an isomorphism $V \cong \mathbb{C}[T]/(\varphi(T))$, of $\mathbb{C}[T]$-modules, because the minimal polynomial $\varphi(T)$ of $f$ has degree $r = \dim V$. Since the minimal polynomial of $\text{^t}f$ coincides with $\varphi(T)$, there is an isomorphism $V^\vee \cong \mathbb{C}[T]/(\varphi(T))$ of $\mathbb{C}[T]$-modules. So we can take an isomorphism

$$\theta : V^\vee \cong V$$

of $\mathbb{C}[T]$-modules. If we put

$$\kappa := \theta^{-1} \circ f : V \to V^\vee,$$

then $\kappa$ becomes a homomorphism of $\mathbb{C}[T]$-modules and $f = \theta \circ \kappa$. We take a generator $v^* \in V^\vee$ of $V^\vee$ as a $\mathbb{C}[T]$-module. Then $v := \theta(v^*) \in V$ is a generator of $V$ as a $\mathbb{C}[T]$-module. Take any $w_1^*, w_2^* \in V^\vee$. Then we can write $w_1^* = P_1(\text{^t}f)v^*$ and $w_2^* = P_2(\text{^t}f)v^*$ for certain polynomials $P_1(T), P_2(T) \in \mathbb{C}[T]$. For the dual pairing $\langle \cdot, \cdot \rangle : V^\vee \times V \to \mathbb{C}$, we have

$$\langle \text{^t}\theta(w_1^*), w_2^* \rangle = \langle w_1^* \circ \theta, w_2^* \rangle = \langle w_1^* , \theta(w_2^*) \rangle = (P_1(\text{^t}f)v^*, \theta(P_2(\text{^t}f)v^*)) = (v^* \circ P_1(f), P_2(f)(\theta(v^*))) = (v^*, P_1(f)P_2(f)(\theta(v^*))) = (v^*, P_2(f)P_1(f)(\theta(v^*))) = (P_2(\text{^t}f)v^*, \theta(P_1(\text{^t}f)v^*)) = \langle w_2^*, \theta(w_1^*) \rangle.$$

So we have $\text{^t}\theta(w_1^*) = \theta(w_1^*)$ and $\text{^t}\theta = \theta$.

Take any $w_1, w_2 \in V$. Then there are polynomials $P_1(T), P_2(T) \in \mathbb{C}[T]$ satisfying $w_1 = P_1(f)v$ and $w_2 = P_2(f)v$. We have

$$\langle \text{^t}\kappa(w_1), w_2 \rangle = \langle \kappa(w_2), w_1 \rangle = \langle \kappa(P_2(f)v), P_1(f)v \rangle = \langle \theta^{-1}fP_2(f)v, P_1(f)v \rangle = \langle \theta^{-1}(fP_2(f))v, P_1(f)v \rangle = \langle \theta^{-1}(v), fP_2(f)P_1(f)v \rangle = \langle \theta^{-1}(v), fP_1(f)P_2(f)v \rangle = \langle \kappa(P_1(f)v), P_2(f)v \rangle = \langle \kappa(w_1), w_2 \rangle.$$ 

So we have $\text{^t}\kappa(w_1) = \kappa(w_1)$ and $\text{^t}\kappa = \kappa$ holds. \hfill \qed

**Proposition 1.2.** For $f \in C_{\varphi(T)}$, assume that $\theta_1, \theta_2 : V^\vee \cong V$ are isomorphisms and $\kappa_1, \kappa_2 : V \to V^\vee$ are linear maps satisfying $f = \theta_1 \circ \kappa_1 = \theta_2 \circ \kappa_2$, $\text{^t}\theta_1 = \theta_1$, $\text{^t}\theta_2 = \theta_2$, $\text{^t}\kappa_1 = \kappa_1$ and $\text{^t}\kappa_2 = \kappa_2$. Then there exists $P(T) \in (\mathbb{C}[T]/(\varphi(T)))^\times$ satisfying $\theta_2 = \theta_1 \circ P(\text{^t}f)$ and $\kappa_2 = (P(\text{^t}f))^{-1} \circ \kappa_1$. 
Proof. Put $\sigma := \theta^{-1}_1 \circ \theta_2 : V^\vee \to V^\vee$. Then $t f \circ \sigma = t \kappa_1 \circ t \theta_1 \circ \theta^{-1}_2 = \kappa_1 \circ \theta_1 \circ \theta^{-1}_2 = \kappa_1 \circ \theta_2$ and $\sigma \circ t f = \theta^{-1}_1 \circ \theta_2 \circ t \kappa_2 \circ \theta_2 = \theta^{-1}_1 \circ \theta_2 \circ \kappa_2 \circ \theta_2 = \theta^{-1}_1 \circ \theta_2 = \theta^{-1}_1 \circ \theta_1 \circ \kappa_1 \circ \theta_2 = \kappa_1 \circ \theta_2$. So $\sigma \circ t f = t f \circ \sigma$ and $\sigma : V^\vee \to V^\vee$ becomes a $C[T]$-isomorphism. Since $C[T]/(\varphi(T)) \cong \text{Hom}_{C[T]}(V^\vee, V^\vee)$, there exists $P(T) \in (C[T]/(\varphi(T)))^\times$ satisfying $P(t f) = \sigma = \theta^{-1}_1 \circ \theta_2$. So we have $\theta_1 \circ P(t f) = \theta_2, \kappa_1 = \theta^{-1}_1 \circ f = \theta^{-1}_1 \circ \theta_2 \circ \kappa_2 = \sigma \circ \kappa_2$ and $\kappa_2 = \sigma^{-1} \circ \kappa = P(t f)^{-1} \circ \kappa_1$. \hfill $\square$

1.2. An expression of the symplectic form on a $GL_r(C)$ adjoint orbit. Let the notations $V, \varphi(T), r$ and $C_{\varphi(T)}$ be as in subsection 1.1. We set

$$S(V^\vee, V) = \{ \theta \in \text{Hom}_C(V^\vee, V) \mid t \theta = \theta \}$$

and

$$S(V^\vee, V^\vee) = \{ \kappa \in \text{Hom}_C(V^\vee, V^\vee) \mid t \kappa = \kappa \}$$

Then there is an action of the commutative algebraic group $(C[T]/(\varphi(T)))^\times$ on $S$ defined by

$$P(T) \cdot (\theta, \kappa) = (\theta \circ P(\kappa \circ \theta), P(\kappa \circ \theta)^{-1} \circ \kappa).$$

for $P(T) \in (C[T]/(\varphi(T)))^\times$. We can see by Proposition 1.1 and Proposition 1.2 that the quotient of $S$ by the action of $(C[T]/(\varphi(T)))^\times$ is isomorphic to $C_{\varphi(T)}$:

$$S/(C[T]/(\varphi(T)))^\times \cong C_{\varphi(T)}.$$ 

We describe the tangent space of $C_{\varphi(T)}$ at $f = \theta \circ \kappa$ via this isomorphism. Let us consider the complex

$$C[T]/(\varphi(T)) \xrightarrow{\partial^0} S(V^\vee, V) \oplus S(V, V^\vee) \xrightarrow{d^1} (C[T]/(\varphi(T)))^\vee$$

defined by

$$\partial^0(P(T)) = (\theta \circ P(t f), -P(t f) \circ \kappa) \quad (P(T) \in C[T]/(\varphi(T)))$$

$$d^1(\tau, \xi) : C[T]/(\varphi(T)) \ni P(T) \mapsto \text{Tr}(P(f) \circ (\theta \circ \xi + \tau \circ \kappa)) \in C \quad ((\tau, \xi) \in S(V^\vee, V) \oplus S(V, V^\vee)).$$

Proposition 1.3. The tangent space $T_S(\theta, \kappa)$ of $S$ at $(\theta, \kappa)$ is isomorphic to $\ker d^1$.

Before proving the proposition, we prove the following lemma.

Lemma 1.4. For $f \in C_{\varphi(T)}$, the sequence

$$0 \to C[T]/(\varphi(T)) \xrightarrow{\iota_f} \text{End}_C(V) \xrightarrow{\text{ad}(f)} \text{End}_C(V) \xrightarrow{\pi_f} (C[T]/(\varphi(T)))^\vee \to 0$$

is exact, where $\iota_f$ is defined by $\iota_f(P(T)) = P(f)$ and $\pi_f$ is the dual of $\iota_f$.

Proof of Lemma 1.4. The map $\iota_f$ is injective since $f$ belongs to $C_{\varphi(T)}$. Since the minimal polynomial of $f$ is of degree $r = \dim V$, the linear map

$$\text{ad}(f) : \text{End}_C(V) \ni g \mapsto f \circ g - g \circ f \in \text{End}_C(V)$$

satisfies $\ker \text{ad}(f) = C[f] = \text{im} \iota_f$. In particular, we have rank $\text{ad}(f) = r^2 - r$. The map $\pi_f$ is given by

$$\pi_f(g)(P(T)) = \text{Tr}(g \circ P(f))$$

for $g \in \text{End}_C(V)$ and $P(T) \in C[T]/(\varphi(T))$. So we have

$$\pi_f(\text{ad}(f)(g))(P(T)) = \text{Tr}((g \circ f - f \circ g)(P(f))$$

$$= \text{Tr}(P(f) \circ g - g \circ f \circ P(f)) = \text{Tr}(f \circ P(f) \circ g) - \text{Tr}(f \circ P(f) \circ g) = 0$$

for $g \in \text{End}_C(V)$ and $P(T) \in C[T]/(\varphi(T))$, which means $\pi_f \circ \text{ad}(f) = 0$. So we have

$$\text{im} \text{ad}(f) = \ker \pi_f = \{ g \in \text{End}_C(V) \mid \text{Tr}(f^i \circ g) = 0 \text{ for } i = 0, 1, \ldots, r - 1 \},$$

because the right hand side is of dimension $r^2 - r$. Thus we have proved the lemma. \hfill $\square$
Proof of Proposition 1.3: If we take \((\tau, \xi) \in \ker d^1\), we have \(\pi f(\theta \circ \xi + \tau \circ \kappa) = d^1(\tau, \xi) = 0\). By Lemma 1.4, there is \(g \in \operatorname{End}(V)\) satisfying \(\theta \circ \xi + \tau \circ \kappa = f \circ g - g \circ f\). We write \(\varphi(T) = b_r T^r + b_{r-1} T^{r-1} + \cdots + b_1 T + b_0\) with \(b_r = 1\). Then the \(\mathbb{C}[t]/(t^2)\)-valued point \((\theta + \tau t, \kappa + \xi t)\) of \(S(V^\vee, V) \times S(V, V^\vee)\) satisfies

\[
\varphi((\theta + \tau t) \circ (\kappa + \xi t)) = \varphi(f + (f \circ g - g \circ f)t) = \sum_{i=0}^r b_i (f + (f \circ g - g \circ f)t)^i
\]

\[
= \sum_{i=0}^r b_i \left( f^i + \sum_{j=0}^{i-1} f^j (f \circ g - g \circ f) f^{i-j-1} t \right) = \sum_{i=0}^r b_i (f^i + (f^i \circ g - g \circ f^i)t)
\]

\[
= \varphi(f) + (\varphi(f) \circ g - g \circ \varphi(f))t = 0.
\]

So \((\theta + \tau t, \kappa + \xi t)\) gives a tangent vector of \(S\) at \((\theta, \kappa)\).

Conversely take a tangent vector of \(S\) and let \((\theta + \tau t, \kappa + \xi t)\) be the corresponding \(\mathbb{C}[t]/(t^2)\)-valued point of \(S\). Then we have \(\varphi((\theta + \tau t) \circ (\kappa + \xi t)) = 0\) and

\[(2) \quad \mathbb{C}[t]/(t^2)[T]/(\varphi(T)) \ni P(T) \mapsto P(\theta + \tau t) \circ (\kappa + \xi t) \in \operatorname{End}_{\mathbb{C}[t]/(t^2)}(V \otimes \mathbb{C}[t]/(t^2))
\]

is injective, whose cokernel is flat over \(\mathbb{C}[t]/(t^2)\). Recall that there is an isomorphism \(\sigma: \mathbb{C}[T]/(\varphi(T)) \cong V\). So we can take a generator \(\sigma = \sigma(1)\) of \(V\) as a \(\mathbb{C}[T]\)-module. If we take a lift \(\tilde{v} \in V \otimes \mathbb{C}[t]/(t^2)\) of \(v\), then \(\tilde{v}\) becomes a generator of \(V \otimes \mathbb{C}[t]/(t^2)\) as a \(\mathbb{C}[t]/(t^2)[T]\)-module with respect to the action of \(\mathbb{C}[t]/(t^2)[T]\) induced by the ring homomorphism \(\tilde{\sigma}\).

So we have an isomorphism

\[
\tilde{\sigma}: \mathbb{C}[t]/(t^2)[T]/(\varphi(T)) \cong V \otimes \mathbb{C}[t]/(t^2)
\]

satisfying \(\tilde{\sigma}(1) = \tilde{v}\). If we denote by \(\id\) the identity map, \(\sigma \otimes \id: \mathbb{C}[T]/(\varphi(T)) \otimes \mathbb{C}[t]/(t^2) \cong V \otimes \mathbb{C}[t]/(t^2)\) is another \(\mathbb{C}[t]/(t^2)[T]\)-isomorphism with respect to the action of \(\mathbb{C}[t]/(t^2)[T]\) on \(V \otimes \mathbb{C}[t]/(t^2)\) via the ring homomorphism

\[
\mathbb{C}[t]/(t^2)[T] \ni P(T) \mapsto P(\theta \circ \kappa \otimes \id) \in \operatorname{End}_{\mathbb{C}[t]/(t^2)}(V \otimes \mathbb{C}[t]/(t^2))
\]

Composing \(\tilde{\sigma}^{-1}\) with \(\sigma \otimes \id\), we obtain a \(\mathbb{C}[t]/(t^2)\)-automorphism of \(V \otimes \mathbb{C}[t]/(t^2)\) of the form \(\id + Q\tilde{t}\) with \(Q \in \operatorname{End}_{\mathbb{C}[t]}(V)\) which makes the diagram

\[
\begin{array}{ccc}
V \otimes \mathbb{C}[t]/(t^2) & \xrightarrow{(\theta + \tau t) \circ (\kappa + \xi t)} & V \otimes \mathbb{C}[t]/(t^2) \\
\id + Q\tilde{t} & \downarrow & \id + Q\tilde{t} \\
V \otimes \mathbb{C}[t]/(t^2) & \xrightarrow{\theta \circ \kappa \otimes \id} & V \otimes \mathbb{C}[t]/(t^2)
\end{array}
\]

commutative. Then we have

\[
(\theta \circ \xi + \tau \circ \kappa)\tilde{t} = (\theta + \tau t) \circ (\kappa + \xi t) = \theta \circ \kappa = (\id - Q\tilde{t}) \circ (\theta \circ \kappa) = (\id + Q\tilde{t}) - (\theta \circ \kappa) = (f \circ Q - Q \circ f)\tilde{t}
\]

and

\[
\operatorname{Tr}(f^i \circ (\theta \circ \xi + \tau \circ \kappa)) = \operatorname{Tr}(f^i \circ (f \circ Q - Q \circ f)) = \operatorname{Tr}(f^{i+1} \circ Q - Q \circ f^{i+1}) = 0
\]

for any \(i \geq 0\). Thus we have \((\tau, \xi) \in \ker d^1\). By the correspondence \((\tau, \xi) \mapsto (\theta + \tau t, \kappa + \xi t)\), we get the isomorphism from \(\ker d^1\) to the tangent space of \(S\) at \((\theta, \kappa)\).

We can see that \(\operatorname{im}(d^0)\) coincides with the tangent space of the \((\mathbb{C}[T]/(\varphi(T))^{\times})\)-orbit of \((\theta, \kappa)\) in \(S\). So the tangent space of \(C_{\varphi(T)} = S/(\mathbb{C}[T]/(\varphi(T))^{\times})\) at \(f = \theta \circ \kappa\) is isomorphic to \(T_{S}(\theta, \kappa)/\operatorname{im}d^0\) which is the first cohomology of the complex \((1)\):

\[
T_{C_{\varphi(T)}}(f) \cong H^1\left( \mathbb{C}[T]/(\varphi(T)) \xrightarrow{\delta} S(V^\vee, V) \oplus S(V, V^\vee) \xrightarrow{d^1} (\mathbb{C}[T]/(\varphi(T))^{\times}) \right).
\]

We define a pairing

\[
\omega_{C_{\varphi(T)}}: T_{C_{\varphi(T)}}(f) \times T_{C_{\varphi(T)}}(f) \rightarrow \mathbb{C}
\]

by

\[(3) \quad \omega_{C_{\varphi(T)}}([[(\tau, \xi)], [(\tau', \xi')]]) = \frac{1}{2} \operatorname{Tr}(\tau \circ \xi' - \tau' \circ \xi).
\]

If \(\left[ [\tau, \xi] \right] = 0\), then we can write \(\tau = \theta \circ P(1)f\) and \(\xi = -P(1)f \circ \kappa\). So we have

\[
\operatorname{Tr}(\tau \circ \xi' - \tau' \circ \xi) = \operatorname{Tr}(\theta \circ P(1)f \circ \xi' + \xi' \circ P(1)f \circ \kappa) = \operatorname{Tr}(P(f) \circ (\theta \circ \xi' + \tau' \circ \kappa)) = 0.
\]

Similarly we can see that \(\operatorname{Tr}(\tau \circ \xi' - \tau' \circ \xi) = 0\) if \(\left[ [\tau', \xi'] \right] = 0\). Thus the pairing \((3)\) is well-defined. On the other hand, there is a well-known symplectic form so called the Kirillov-Kostant form. For two
tangent vectors \([[(\tau, \xi)], [(\tau', \xi')]] \in T_{C_{e(f)}}(f)\) of \(C_{e(f)}\) at \(f = \theta \circ \kappa\), we can see by Lemma 1.4 that there exist \(g, g' \in \text{Hom}(V, V)\) satisfying \(f \circ g - g \circ f = \theta \circ \xi + \tau \circ \kappa\) and \(f \circ g' - g' \circ f = \theta \circ \xi' + \tau' \circ \kappa\). The Kirillov-Kostant symplectic form \(\omega_{\text{K-K}}\) is defined in [21] page 5, Definition 1] by
\[
\omega_{\text{K-K}}([(\tau, \xi)], [(\tau', \xi')]) = \text{Tr}(f \circ ([g, g'])).
\]

**Proposition 1.5.** The pairing \(\omega_{C_{e(f)}}\) defined in [3] coincides with the Kirillov-Kostant symplectic form \(\omega_{\text{K-K}}\) on the adjoint orbit \(C_{e(f)}\).

**Proof.** Take any member \((a, b) \in S(V^\vee, V) \oplus S(V, V^\vee)\) satisfying \(\theta \circ b + a \circ \kappa = 0\). Then we have
\[
(\theta + a \bar{f}) \circ (\kappa + b \bar{f}) = \theta \circ \kappa = f \in \text{End}_{C_{e(f)}}(V \otimes_C \mathbb{C}[t]/(t^2)),
\]
from which we can see
\[
(\kappa + b \bar{f}) \circ (\theta + a \bar{f}) = \bar{f} ((\kappa + b \bar{f}) \circ (\theta + a \bar{f})) = \bar{f} = \kappa \circ \theta.
\]
So we have
\[
(\text{id} + \theta^{-1} a \bar{f}) \circ \bar{f} = \theta^{-1} \circ (\theta + a \bar{f}) \circ (\kappa + b \bar{f}) \circ (\theta + a \bar{f}) = \text{id} \circ \kappa \circ \theta \circ \kappa \circ \theta^{-1} \circ a \bar{f} \circ \text{id} = \bar{f} \circ \text{id} + \theta^{-1} \circ a \bar{f} \circ \text{id}.
\]
Then we have \(\theta^{-1} \circ a \in \text{End}_{C_{e(f)}}(V^\vee) \cong \mathbb{C}[t]/(\phi(t))\) and there exists \(P \in \mathbb{C}[t]/(\phi(t))\) satisfying \(\theta^{-1} a = P(\bar{f})\). So we have \(a = \theta \circ P(\bar{f})\) and \(b = -\theta^{-1} \circ a \circ \kappa = -P(\bar{f} \circ \kappa)\), which means that \((a, b) \in \text{dm}(\theta)\). Thus we have proved
\[
\text{im}(\text{dm}(\theta)) = \ker \left(S(V^\vee, V) \oplus S(V, V^\vee) \ni (a, b) \mapsto \theta \circ b + a \circ \kappa = \text{Hom}(V, V)\right).
\]

Take two tangent vectors \([(\tau, \xi)], [(\tau', \xi')]) \in T_{C_{e(f)}}(f)\) of \(C_{e(f)}\) at \(f = \theta \circ \kappa\). Since \((\tau, \xi), (\tau', \xi') \in \text{ker} \text{d}^f\), we can see from Lemma 1.4 that there exist \(g, g' \in \text{Hom}(V, V)\) satisfying \(f \circ g - g \circ f = \theta \circ \xi + \tau \circ \kappa\) and \(f \circ g' - g' \circ f = \theta \circ \xi' + \tau' \circ \kappa\). Note that we have
\[
\theta \circ (\kappa \circ g + \tau \circ g') + (- \theta \circ \kappa \circ g - \theta \circ \xi' - \tau' \circ \kappa) = f \circ g - g \circ f = \theta \circ \xi + \tau \circ \kappa.
\]
By the equality [3], we have \([[(\tau, \xi)], [(\tau', \xi')]] = \left[\left(- \theta \circ \kappa \circ g + \theta \circ \xi' + \tau' \circ \kappa, \kappa \circ g + \tau \circ g'\right)\right]\) in \(T_{C_{e(f)}}(f)\) and we may assume that \(\tau' = -\theta \circ \kappa \circ \xi' + \tau \circ \xi' + \theta \circ \kappa \circ g'\) and \(\xi' = \kappa \circ g' + \theta \circ \xi'\). We have
\[
\omega_{\text{K-K}}([(\tau, \xi)], [(\tau', \xi')]) = \text{Tr}(f \circ ([g, g'])),
\]
\[
= \text{Tr}(f \circ (g \circ g' - g' \circ g)),
\]
\[
= \text{Tr}(f \circ (g \circ g' - f \circ g' - f \circ g)),
\]
\[
= \text{Tr}((\theta \circ \xi + \tau \circ \kappa) \circ g' + \text{Tr}(g \circ (f \circ g') - (f \circ g') \circ g)),
\]
\[
= \text{Tr}((\theta \circ \xi + \tau \circ \kappa) \circ g' + \text{Tr}(\theta \circ \xi \circ \kappa \circ g' + \text{Tr}(\tau \circ \kappa \circ g')),
\]
\[
= \frac{1}{2} \left(\text{Tr}(g' \circ \theta \circ \xi) + \text{Tr}(\xi \circ \theta \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g' + \text{Tr}(\tau \circ \xi \circ \kappa g'))\right).
\]

**Claim 1.6.** \(\text{Tr}(u \circ v) = \text{Tr}(v \circ u)\) for any \(u \in \text{Hom}(V, V^\vee)\) and any \(v \in \text{Hom}(V^\vee, V)\).

Using the above claim, we have \(\text{Tr}(\xi' \circ \kappa \circ g' \circ \tau = \kappa \circ g' \circ \tau \circ \xi' = \text{Tr}(\tau \circ g' \circ \xi'))\) and \(\text{Tr}(\xi' \circ \kappa \circ g' \circ \tau = \kappa \circ g' \circ \tau \circ \xi')\). So we have
\[
\omega_{\text{K-K}}([(\tau, \xi)], [(\tau', \xi')]) = \frac{1}{2} \left(\text{Tr}(g' \circ \theta \circ \xi) + \text{Tr}(\xi \circ \theta \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g')\right)
\]
\[
= \frac{1}{2} \left(\text{Tr}(g' \circ \theta \circ \xi) + \text{Tr}(\xi \circ \theta \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g')\right)
\]
\[
= \frac{1}{2} \left(\text{Tr}(\tau \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g')\right) = \frac{1}{2} \left(\text{Tr}(\tau \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g')\right)
\]
\[
= \frac{1}{2} \left(\text{Tr}(\tau \circ \kappa \circ g') + \text{Tr}(\tau \circ \kappa \circ g')\right) = \omega_{C_{e(f)}}([(\tau, \xi)], [(\tau', \xi')]).
\]

For the proof of Claim 1.6, we take a basis \(e_1, \ldots, e_r\) of \(V\) and its dual basis \(e_1^*, \ldots, e_r^*\) of \(V^\vee\). If write \(u(e_j) = \sum_{i=1}^r a_{ij} e_i^*\) and \(v(e_i) = \sum_{k=1}^r b_{ki} e_k\), then we have
\[
\text{Tr}(u \circ v) = \text{Tr} \left( \sum_{i=1}^r \sum_{k=1}^r a_{ik} b_{ki} e_i^* \otimes e_l \right) = \sum_{k=1}^r a_{ik} b_{ki}
\]
\[
\text{Tr}(v \circ u) = \text{Tr} \left( \sum_{j=1}^r \sum_{i=1}^r a_{ij} b_{ki} e_k \otimes e_j^* \right) = \sum_{i=1}^r a_{ik} b_{ki}
\]
So we have \(\text{Tr}(u \circ v) = \text{Tr}(v \circ u)\) and Claim 1.6 follows. Thus we have proved \(\omega_{\text{K-K}} = \omega_{C_{e(f)}}\).
2. ALGEBRAIC CONSTRUCTION OF AN UNFOLDING OF THE MODULI SPACE OF UNRAMIFIED IRREGULAR SINGULAR CONNECTIONS

2.1. Regular singular and unramified irregular singular connections as \((\nu, \mu)\)-connections. Let \(C\) be a complex smooth projective irreducible curve of genus \(g\). We take an effective divisor \(D \subset C\), which has a decomposition \(D = D^{(1)} + D^{(2)} + \cdots + D^{(m)} = D^{(1)} \sqcup \cdots \sqcup D^{(m)}\), where each \(D^{(i)}\) is an effective divisor of degree \(m_i\) and \(D^{(i)} \cap D^{(j)} = \emptyset\) for \(i \neq j\). We write \(D^{(i)} = p_1^{(i)} + p_2^{(i)} + \cdots + p_{m_i}^{(i)}\) for \(1 \leq i \leq n\), where each \(p_j^{(i)}\) is a reduced point in \(C\) and it may be possible that \(p_j^{(i)} = p_j^{(j')}\) for \(j \neq j'\).

Using the Chinese remainder theorem

\[
\mathcal{O}_{2D^{(i)}} \cong \prod_{p \in D^{(i)}} \mathcal{O}_{2D^{(i)}, p},
\]

we can choose \(z^{(i)} \in \mathcal{O}_{2D^{(i)}}\) satisfying

\[
\bar{z}^{(i)}(p_j^{(i)}) \neq z^{(i)}(p_j^{(i)}) \quad \text{for} \quad p_j^{(i)} \neq p_j^{(i)} \quad \text{and} \quad d\bar{z}^{(i)}|_{p_j^{(i)}} \neq 0 \in \Omega^1_{C, p_j^{(i)}} \quad \text{for} \quad j = 1, \ldots, m_i.
\]

We write \(z^{(i)} = z^{(i)}(p_j^{(i)})\), where \(z^{(i)}(p_j^{(i)}) \in \mathbb{C}\) is the value of \(z^{(i)}\) at \(p_j^{(i)}\). We take local lifts \(z^{(i)} \in \mathcal{O}_{\mathbb{C}}\) of \(z^{(i)}\), put \(z^{(i)} := \bar{z}^{(i)} - z^{(i)}(p_j^{(i)})\) and define

\[
\frac{d\bar{z}^{(i)}}{(z_1^{(i)})^{l_1}(z_2^{(i)})^{l_2} \cdots (z_{m_i}^{(i)})^{l_{m_i}}} = \sum_{p \in D^{(i)}} \sum_{1 \leq j \leq m_i} \frac{a_p^{(i)} d\bar{z}^{(i)}}{(z^{(i)} - z^{(i)}(p))^{l_j}}
\]

with \(a_p^{(i)} \in \mathbb{C}\). Since \(a_p^{(i)}\) is determined by

\[
a_p^{(i)} = \lim_{z^{(i)} \to p} \frac{1}{d(z^{(i)})^{m_p^{(i)} - j}} \frac{d^{m_p^{(i)} - j}}{(z^{(i)} - z^{(i)}(p))^{m_p^{(i)} - j}} \frac{(z^{(i)} - z^{(i)}(p))^{m_p^{(i)}}}{(z_1^{(i)})^{l_1}(z_2^{(i)})^{l_2} \cdots (z_{m_i}^{(i)})^{l_{m_i}}},
\]

we can see that \(a_p^{(i)}\) is independent of the choice of the lift \(z^{(i)}\) of \(z^{(i)}\). Then we define

\[
\text{res}_p \left( \frac{d\bar{z}^{(i)}}{(z^{(i)})^{l_1}(z_2^{(i)})^{l_2} \cdots (z_{m_i}^{(i)})^{l_{m_i}}} \right) = a_p^{(i)}.
\]

Lemma 2.1. If \(l_1, \ldots, l_{m_i}\) are integers satisfying \(0 \leq l_1, \ldots, l_{m_i} \leq 1\) and \(l_1 + \cdots + l_{m_i} \geq 2\), the equality

\[
\sum_{p \in D^{(i)}} \text{res}_p \left( \frac{d\bar{z}^{(i)}}{(z_1^{(i)})^{l_1}(z_2^{(i)})^{l_2} \cdots (z_{m_i}^{(i)})^{l_{m_i}}} \right) = 0
\]

holds.

Proof. It is sufficient to prove the equality for the case \(l_1 = l_2 = \cdots = l_{m_i} = 1\). Since the equality which we want is a formal equality determined by \(\mathbb{C}\), it is sufficient to prove the equality

\[
\sum_{p \in \{p_1, \ldots, p_m\}} \text{res}_p \left( \frac{dz}{(z - p_1)(z - p_2) \cdots (z - p_m)} \right) = 0
\]

when \(z\) is a coordinate of the complex plane \(\mathbb{C}\), \(m \geq 2\) and \(p_1, \ldots, p_m \in \mathbb{C}\) may not be distinct. If we take a circle \(\gamma\) in \(\mathbb{C}\) which is a boundary of a large disk containing all the points \(p_1, \ldots, p_m\) within, then we have

\[
\sum_{p \in \{p_1, \ldots, p_m\}} \text{res}_p \left( \frac{dz}{(z - p_1)(z - p_2) \cdots (z - p_m)} \right) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{(z - p_1)(z - p_2) \cdots (z - p_m)}
\]

\[
= - \text{res}_{z = \infty} \left( \frac{dz}{(z - p_1)(z - p_2) \cdots (z - p_m)} \right) = 0
\]

because \(m \geq 2\). Thus the equality \([7]\) holds. \(\square\)
We take \( \mu = (\mu_1^{(i)}|_{\Omega_{D^{(i)}}}^{(i)})_{1 \leq i \leq r} \in H^0(D^{(i)}, \mathcal{O}^{nr}_{D^{(i)}}) \) such that \( \mu_1^{(i)}|_{p}, \ldots, \mu_r^{(i)}|_{p} \) are mutually distinct at any point \( p \in D^{(i)} \). Then we define a polynomial \( \varphi_\mu^{(i)}(T) \in H^0(D^{(i)}, \mathcal{O}_{D^{(i)}})[T] \) by setting
\[
\varphi_\mu^{(i)}(T) := \prod_{k=1}^r (T - \mu_k^{(i)}).
\]
We fix a tuple of complex numbers \( \lambda = (\lambda_k^{(i)})_{1 \leq k \leq r} \in \mathbb{C}^{nr} \) satisfying \( \sum_{i=1}^n \sum_{k=1}^r \lambda_k^{(i)} \in \mathbb{Z} \) and put
\[
a := -\sum_{i=1}^n \sum_{k=1}^r \lambda_k^{(i)}.
\]
For each \( i \), we take a polynomial \( \nu^{(i)}(T) = c_0^{(i)} + c_1^{(i)}T + \cdots + c_{r-1}^{(i)}T^{r-1} \in H^0(D^{(i)}, \mathcal{O}_{D^{(i)}})[T] \) such that the expression
\[
\nu^{(i)}(\mu_k^{(i)}) = \sum_{0 \leq i_1, \ldots, i_{m_i} \leq 1} \sum_{0 \leq i_1 + \cdots + i_{m_i} \leq m_i} a_{k,i_1,\ldots,i_{m_i}} (\lambda_1^{(i)} \lambda_2^{(i)} \cdots (\bar{z}_1^{(i)} \bar{z}_2^{(i)} \cdots (\bar{z}_{m_i}^{(i)})^{l_{m_i}})
\]
with \( a_{k,i_1,\ldots,i_{m_i}} \in \mathbb{C} \) satisfies the equality
\[
\lambda_k^{(i)} = (a_{k,0,1,\ldots,1} + a_{k,0,1,\ldots,1} + \cdots + a_{k,1,0,\ldots,0})
\]
for any \( i, k \). We can see by Lemma 2.1 that
\[
\sum_{p \in D^{(i)}} \text{res}_p \left( \frac{\nu^{(i)}(\mu_k^{(i)})}{\bar{z}_1^{(i)} \cdots \bar{z}_{m_i}^{(i)}} \right) = \sum_{0 \leq i_1, \ldots, i_{m_i} \leq 1} \sum_{0 \leq i_1 + \cdots + i_{m_i} \leq m_i} a_{k,i_1,\ldots,i_{m_i}} \sum_{p \in D^{(i)}} \text{res}_p \left( \frac{\bar{z}_1^{(i)} \cdots \bar{z}_{m_i}^{(i)}}{\bar{z}_1^{(i)} \cdots \bar{z}_{m_i}^{(i)}} \right)
\]
\[
= \sum_{i=1}^{m_i} a_{k,1,\ldots,1} = \sum_{i=1}^{m_i} a_{k,1,\ldots,1} + \cdots + a_{k,1,\ldots,1}.
\]
So (8) means the equality
\[
\lambda_k^{(i)} = \sum_{p \in D^{(i)}} \text{res}_p \left( \frac{\nu^{(i)}(\mu_k^{(i)})}{\bar{z}_1^{(i)} \cdots \bar{z}_{m_i}^{(i)}} \right)
\]
where \( \sum_{p \in D^{(i)}} \) runs over the set theoretical points \( p \) of \( D^{(i)} \).

We assume the following assumption on \( \nu = (\nu^{(i)}(T))_{1 \leq i \leq n} \):

**Assumption 2.2.** For each \( i \), \( \nu^{(i)}(\mu_1^{(i)})|_p, \ldots, \nu^{(i)}(\mu_r^{(i)})|_p \) are mutually distinct at any point \( p \in D^{(i)} \).

**Definition 2.3.** We say that a tuple \( (E, \nabla, \{N^{(i)}\}_{1 \leq i \leq n}) \) is a \( (\nu, \mu) \)-connection on \( (C, D) \) if

1. \( E \) is an algebraic vector bundle on \( C \) of rank \( r \) and degree \( a \),
2. \( \nabla : E \rightarrow E \otimes \Omega^1_C(D) \) is an algebraic connection on \( E \) admitting poles along \( D \),
3. \( N^{(i)} : E|_{D^{(i)}} \rightarrow E|_{D^{(i)}} \) is an \( \mathcal{O}_{D^{(i)}} \)-homomorphism satisfying \( \varphi_\mu^{(i)}(N^{(i)}) = 0 \), the homomorphism
\[
\mathcal{O}_{D^{(i)}}[T]/(\varphi_\mu^{(i)}(T)) \ni P(T) \mapsto P(N^{(i)}) \in \text{End}(E|_{D^{(i)}})
\]
is injective and \( \nu^{(i)}(N^{(i)}) \frac{dz^{(i)}}{z_1^{(i)} z_2^{(i)} \cdots z_{m_i}^{(i)}} = \nabla|_{D^{(i)}} \) for \( 1 \leq i \leq n \).

**Remark 2.4.** The injectivity of the homomorphism [10] in Definition 2.3 implies that \( \mathcal{O}_{D^{(i)}}[T]/(\varphi_\mu^{(i)}(T)) \) becomes an \( \mathcal{O}_{D^{(i)}} \)-subbundle of \( \text{End}(E|_{D^{(i)}}) \).

**Proposition 2.5.** Assume that \( D \) is a reduced divisor on \( C \). In other words, we assume that \( p_j^{(i)} \neq p_j^{(i')} \) for \( j \neq j' \). Then giving a \( (\nu, \mu) \)-connection on \( (C, D) \) is equivalent to giving a regular singular connection \( (E, \nabla) \) on \( C \) admitting poles along \( D \) whose residue \( \text{res}_{p_j^{(i)}}(\nabla) \) at \( p_j^{(i)} \) has the distinct eigenvalues
\[
\left\{ \nu^{(i)}(\mu_k^{(i)})|_{p_j^{(i)}} \text{res}_{p_j^{(i)}} \left( \frac{dz^{(i)}}{z_1^{(i)} z_2^{(i)} \cdots z_{m_i}^{(i)}} \right) \right\}_{1 \leq k \leq r}.
\]
Proof. Let \((E, \nabla, \{N(i)\})\) be a \((\nu, \mu)\)-connection on \((C, D)\). The restriction \(N^{(i)}|_{p_j} : E|_{p_j} \rightarrow E|_{p_j}\) of \(N^{(i)}\) to the fiber \(E|_{p_j}\) of \(E\) at \(p_j\) satisfies \(\prod_{k=1}^{d} (N^{(i)}|_{p_j} - \mu^{(i)} \nu_{p_j} \id_{E|_{p_j}}) = 0\), because \(\phi\mu^{\nu}(N^{(i)}) = 0\). From the injectivity of the homomorphism \([10]\) in Definition 2.3, the induced homomorphism
\[
\mathbb{C}[\nabla]/(\phi^{(i)}(\nabla)) \supset \mathbb{P}(\nabla) \rightarrow P(N^{(i)}|_{p_j}) \in \text{End}(E|_{p_j})
\]
is injective. So \(N^{(i)}|_{p_j}\) has the distinct eigenvalues \(\mu^{(i)}|_{p_j}, \ldots, \mu^{(i)}|_{p_j}\). By Assumption 2.2, the linear endomorphism on \(E|_{p_j}\)
\[
\nu^{(i)}(N^{(i)}|_{p_j}) = c_0^{(i)}|_{p_j} \id_{E|_{p_j}} + c_1^{(i)}|_{p_j} N^{(i)}|_{p_j} + \cdots + c_r^{(i)}|_{p_j} (N^{(i)}|_{p_j})^{m_{i-r}} : E|_{p_j} \rightarrow E|_{p_j}
\]
has the distinct eigenvalues \(\nu^{(i)}(\mu^{(i)}|_{p_j}), \ldots, \nu^{(i)}(\mu^{(i)}|_{p_j})\). Since \(\nu^{(i)}(N^{(i)}) = \frac{dz(i)}{z_1(i) \cdots z_m(i)} = \nabla|_{D^{(i)}}\), the residue homomorphism \(\text{res}_{p_j}(\nabla) : E|_{p_j} \rightarrow E|_{p_j}\) has the eigenvalues
\[
\nu^{(i)}(\mu^{(i)}|_{p_j}) \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right) \quad 1 \leq k \leq r.
\]
Conversely let \(E\) be a vector bundle on \(C\) of rank \(r\) and \(\nabla : E \rightarrow E \otimes \Omega^1_C(D)\) be a connection whose residue \(\text{res}_{p_j}(\nabla)\) at \(p_j\) has the distinct eigenvalues \(\nu^{(i)}(\mu^{(i)}|_{p_j}) \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right) \quad 1 \leq k \leq r\).

Since the diagonal matrix
\[
R = \begin{pmatrix}
\nu^{(i)}(\mu^{(i)}|_{p_j}) \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \nu^{(i)}(\mu^{(i)}|_{p_j}) \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right)
\end{pmatrix}
\]
has the distinct eigenvalues and commutes with the diagonal matrix \(N = \begin{pmatrix} \mu^{(i)}|_{p_j} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \mu^{(i)}|_{p_j} \end{pmatrix}\), the matrix \(N\) can be written as a polynomial \(\psi^{(i)}(R)\) in \(R\) with coefficients in \(\mathbb{C}\), that is, \(N = \psi^{(i)}(R)\). Consider the linear map
\[
\psi^{(i)}(\text{res}_{p_j}(\nabla)) : E|_{p_j} \rightarrow E|_{p_j}.
\]
By the Chinese remainder theorem \(\mathcal{O}_{D^{(i)}} \xrightarrow{\sim} \bigoplus_{j=1}^{m_i} \mathcal{O}_{p_j}\), we have an isomorphism
\[
\text{Hom}_{\mathcal{O}_{D^{(i)}}}(E|_{D^{(i)}}, E|_{D^{(i)}}) \xrightarrow{\sim} \bigoplus_{j=1}^{m_i} \text{Hom}_{\mathcal{O}_{p_j}}(E|_{p_j}, E|_{p_j}).
\]
So there is an endomorphism \(N^{(i)} : E|_{D^{(i)}} \rightarrow E|_{D^{(i)}}\) satisfying \(N^{(i)}|_{p_j} = \psi^{(i)}(\text{res}_{p_j}(\nabla))\) for \(1 \leq j \leq m_i\).

Since
\[
\nu^{(i)}(N)|_{p_j} \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right) = \nu^{(i)}(\psi^{(i)}(R)) \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right),
\]
we can see
\[
\text{res}_{p_j}(\nabla) = \nu^{(i)} \left( \psi^{(i)}(\text{res}_{p_j}(\nabla)) \right) \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right) = \nu^{(i)}(N)|_{p_j} \text{ res}_{p_j} \left( \frac{dz(i)}{z_1(i) \cdots z_m(i)} \right),
\]
which implies $\nu^{(i)}(N^{(i)}) \frac{d\overline{z}^{(i)}}{z^{(i)}_1 z^{(i)}_2 \cdots z^{(i)}_{m^{(i)}}} = \nabla|_{D^{(i)}}$. From the definition, each $N^{(i)}|_{p^{(i)}_j}$ has the distinct eigenvalues $\mu^{(i)}_1|_{p^{(i)}_j}, \ldots, \mu^{(i)}_r|_{p^{(i)}_j}$ and so the identity $\varphi^{(i)}_\mu(N^{(i)}) = 0$ follows. Thus $(E, \nabla, \{N^{(i)}\})$ becomes a $(\nu, \mu)$-connection.

The following definition of unramified irregular singular parabolic connection is given in [14]. Here we restrict to the case of generic exponents and a notation of suffix is slightly changed.

**Definition 2.6.** Let $t_1, \ldots, t_n \in C$ be distinct points and $m_1, \ldots, m_n$ be integers satisfying $m_i > 1$ for any $i$. Take a generator $z_i \in \mathfrak{m}_i$ of the maximal ideal $\mathfrak{m}_i$ of $\mathcal{O}_{C, t_i}$. Assume that $\nu^{(i)}_1, \ldots, \nu^{(i)}_{r} \in \mathcal{O}_{m_i, t_i}$ satisfy $\nu^{(i)}_k|_{t_i} \neq \nu^{(i)}_{k'}|_{t_i}$ for $k \neq k'$. Then $(E, \nabla, \{l^{(i)}_k\})$ is said to be an unramified irregular singular parabolic connection with the exponents $\nu^{(i)}_1 \frac{dz_{i}}{z_{i}^{m_i}}, \ldots, \nu^{(i)}_{r} \frac{dz_{i}}{z_{i}^{m_i}}$, at $t_i$ if $E$ is an algebraic vector bundle on $C, \nabla : E \rightarrow E \otimes \Omega^1_C(\sum_{i=1}^n m_i t_i)$ is an algebraic connection, $E|_{m_i t_i} = l^{(i)}_1 \supset \ldots \supset l^{(i)}_r \supset \ldots \supset 0$ is a filtration satisfying $\nabla|_{m_i t_i} - \nu^{(i)}_k \frac{dz_i}{z_i^{m_i}}|_{l^{(i)}_k} \supset \ldots \supset 0$ is a filtration for any $k$. Then we can see as in the proof of [14] Proposition 2.3 that there is a decomposition

$$E|_{m_i t_i} = \bigoplus_{k=1}^r \ker \left( \nabla|_{m_i t_i} - \nu^{(i)}_k \frac{dz_i}{z_i^{m_i}} \right)$$

which induces the filtration $l^{(i)}_k$ and the diagonal representation matrix of $\nabla|_{m_i t_i}$

$$
\begin{pmatrix}
\nu^{(i)}_1 \frac{dz_i}{z_i^{m_i}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \nu^{(i)}_r \frac{dz_i}{z_i^{m_i}}
\end{pmatrix}
$$

with respect to a basis of $E|_{m_i t_i}$ obtained from the decomposition [14].

**Proposition 2.8.** Under Assumption 2.2 suppose that each $D^{(i)}$ is a multiple divisor of degree $m_i$ for $1 \leq i \leq n$. In other words, we assume that $p^{(i)}_j = p^{(i)}_{j'}$ for any $j, j'$ and $D^{(i)} = m_i p^{(i)}_1$. Then giving a $(\nu, \mu)$-connection on $(C, D)$ is equivalent to giving an unramified irregular singular parabolic connection $(E, \nabla, \{l^{(i)}_k\})$ on $(C, D)$ with the exponents $\nu^{(i)}_k|_{t_i} \neq \nu^{(i)}_{k'}|_{t_i}$ for $k \neq k'$. Then we can see as in the proof of [14] Proposition 2.3 that there is a decomposition

$$E|_{m_i t_i} = \bigoplus_{k=1}^r \ker \left( \nabla|_{m_i t_i} - \nu^{(i)}_k \frac{dz_i}{z_i^{m_i}} \right)$$

which induces the filtration $l^{(i)}_k$ and the diagonal representation matrix of $\nabla|_{m_i t_i}$

$$
\begin{pmatrix}
\nu^{(i)}_1 \frac{dz_i}{z_i^{m_i}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \nu^{(i)}_r \frac{dz_i}{z_i^{m_i}}
\end{pmatrix}
$$

with respect to a basis of $E|_{m_i t_i}$ obtained from the decomposition [14].

**Proposition 2.8.** Under Assumption 2.2 suppose that each $D^{(i)}$ is a multiple divisor of degree $m_i$ for $1 \leq i \leq n$. In other words, we assume that $p^{(i)}_j = p^{(i)}_{j'}$ for any $j, j'$ and $D^{(i)} = m_i p^{(i)}_1$. Then giving a $(\nu, \mu)$-connection on $(C, D)$ is equivalent to giving an unramified irregular singular parabolic connection $(E, \nabla, \{l^{(i)}_k\})$ on $(C, D)$ with the exponents $\nu^{(i)}_k|_{t_i} \neq \nu^{(i)}_{k'}|_{t_i}$ for $k \neq k'$. Then we can see as in the proof of [14] Proposition 2.3 that there is a decomposition

$$E|_{m_i t_i} = \bigoplus_{k=1}^r \ker \left( \nabla|_{m_i t_i} - \nu^{(i)}_k \frac{dz_i}{z_i^{m_i}} \right)$$

which induces the filtration $l^{(i)}_k$ and the diagonal representation matrix of $\nabla|_{m_i t_i}$

$$
\begin{pmatrix}
\nu^{(i)}_1 \frac{dz_i}{z_i^{m_i}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \nu^{(i)}_r \frac{dz_i}{z_i^{m_i}}
\end{pmatrix}
$$

with respect to a basis of $E|_{m_i t_i}$ obtained from the decomposition [14].
is an injection to the eigen subspace of $E|_{p_i^{(i)}}$ with respect to the eigenvalue $\mu_k^{(i)}|_{p_i^{(i)}}$ of $N^{(i)}|_{p_i^{(i)}}$. Therefore we can see that

$$\prod_{k' \neq k} (N^{(i)} - \mu_k^{(i)}): \text{coker}(N^{(i)} - \mu_k^{(i)}) \to E|_{D^{(i)}}$$

is also injective and its cokernel is a free $\mathcal{O}_{D^{(i)}}$-module of rank $r-1$. So

$$\text{coker}(N^{(i)} - \mu_k^{(i)}) \cong \ker(N^{(i)} - \mu_k^{(i)}) \subset E|_{D^{(i)}}$$

is a rank one subbundle of $E|_{D^{(i)}}$ and we have a decomposition

(12)

$$E|_{D^{(i)}} = \bigoplus_{k=1}^r \ker(N^{(i)} - \mu_k^{(i)}).$$

By the equality $\nu^{(i)}(N^{(i)}) \frac{dz_i}{\bar{z}_1^{(i)} \bar{z}_2^{(i)} \cdots \bar{z}_m^{(i)}} = \nabla|_{D^{(i)}}$, we can see that the representation matrix of $\nabla|_{D^{(i)}}$ with respect to a basis giving the direct sum decomposition (12) of $E|_{D^{(i)}}$ is

$$
\begin{pmatrix}
\nu^{(i)}(\mu_1^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_1}} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \nu^{(i)}(\mu_r^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_r}}
\end{pmatrix}.
$$

If we choose the parabolic structure $\{l_k^{(i)}\}$ compatible with the decomposition (12), then $(E, \nabla, \{l_k^{(i)}\})$ becomes an unramified irregular singular parabolic connection with the exponents $\left\{ \nu^{(i)}(\mu_k^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}} \right\}_{1 \leq k \leq r}$ at $p_i^{(i)}$ for $1 \leq i \leq n$.

Conversely, let $(E, \nabla, \{l_k^{(i)}\})$ be an unramified irregular singular parabolic connection with the exponents $\left\{ \nu^{(i)}(\mu_k^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}} \right\}_{1 \leq k \leq r}$ at $p_i^{(i)}$. Since $\nu^{(i)}(\mu_1^{(i)})|_{p_i^{(i)}}, \ldots, \nu^{(i)}(\mu_r^{(i)})|_{p_i^{(i)}}$ are mutually distinct, we have a decomposition

$$E|_{D^{(i)}} = \bigoplus_{k=1}^r \ker \left( \nabla|_{D^{(i)}} - \nu^{(i)}(\mu_k^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}} \right)$$

as in Remark 2.7 which is compatible with $\{l_k^{(i)}\}$. If we define a homomorphism $N^{(i)}: E|_{D^{(i)}} \to E|_{D^{(i)}}$ by setting

$$N^{(i)}|_{\ker \left( \nabla|_{D^{(i)}} - \nu^{(i)}(\mu_k^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}} \right)} = \mu_k^{(i)} - \text{id}|_{\ker \left( \nabla|_{D^{(i)}} - \nu^{(i)}(\mu_k^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}} \right)}$$

for each $k$, then $N^{(i)}$ satisfies $\varphi^{(i)}(N^{(i)}) = 0$ and $\nabla|_{D^{(i)}} = \nu^{(i)}(N^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}}$. Since $N^{(i)}|_{p_i^{(i)}}$ has the distinct eigenvalues $\mu_1^{(i)}|_{p_i^{(i)}}, \ldots, \mu_r^{(i)}|_{p_i^{(i)}}$, the homomorphism

$$\mathcal{O}_{D^{(i)}}[T]/(\varphi^{(i)}(T)) \ni P(T) \mapsto P(N^{(i)}) \in \text{End}(E|_{D^{(i)}})$$

is injective, because of the injectivity of its restriction to the reduced point $p_i^{(i)}$ of $D^{(i)}$. So $(E, \nabla, \{N^{(i)}\})$ becomes a $(\nu, \mu)$-connection.

Now we come back to the general setting in Definition 2.3 and define a stability for a $(\nu, \mu)$-connection $(E, \nabla, \{N^{(i)}\})$ which is necessary for the construction of the moduli space. By Assumption 2.2 there is a unique filtration

(13)

$$E|_{D^{(i)}} = f_1^{(i)} \supset f_2^{(i)} \supset \cdots \supset f_r^{(i)} \supset f_{r+1}^{(i)} = 0$$

such that $f_k^{(i)} / f_{k+1}^{(i)} \cong \mathcal{O}_{D^{(i)}}$, $\left( \nabla|_{D^{(i)}} - \nu^{(i)}(\mu_k^{(i)}) \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}} \right) f_k^{(i)} \subset f_{k+1}^{(i)} \frac{dz_i}{(\bar{z}_1^{(i)})^{m_i}}$, and $(N^{(i)} - \mu_k^{(i)} \text{id})(f_k^{(i)}) \subset f_{k+1}^{(i)}$ for any $i, k$. \qed
We take a tuple of positive rational numbers \( \alpha = (\alpha_k^{(i)})_{1 \leq i \leq n} \), satisfying \( 0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1 \) for any \( i \) and \( \alpha_k^{(i)} \neq \alpha_{k'}^{(i)} \) for \( (i, k) \neq (i', k') \). The following definition in fact depends on the ordering of \( \mu_1^{(i)}, \ldots, \mu_r^{(i)} \).

**Definition 2.9.** A \((\nu, \mu)\)-connection \((E, \nabla, \{N^{(i)}\})\) on \((C, D)\) is \( \alpha \)-stable (resp. \( \alpha \)-semistable) if the inequality
\[
\text{deg } E + \sum_{i=1}^{n} \sum_{k=1}^{r} \alpha_k^{(i)} \text{length}((F|_{D^{(i)}} \cap l_k^{(i)})/(F|_{D^{(i)}} \cap l_{k+1}^{(i)})) < \text{rank } F
\]
(resp. \( \leq \))
holds for any subbundle \( 0 \neq F \subset E \) satisfying \( \nabla(F) \subset F \otimes \Omega^1_C(D) \), where \( \{l_k^{(i)}\} \) is the filtration of \( E|_{D^{(i)}} \) determined by \( \nabla|_{D^{(i)}} \).

2.2. **Relative moduli space of \((\hat{\nu}, \hat{\mu})\)-connections.** Let \( S \) be an irreducible algebraic variety over \( \text{Spec } \mathbb{C} \) and let \( \mathcal{C} \rightarrow S \) be a smooth projective morphism whose geometric fibers are smooth projective irreducible curves of genus \( g \). Assume that \( \mathcal{D} \) is an effective Cartier divisor on \( \mathcal{C} \) flat over \( S \), which has a decomposition
\[
\mathcal{D} = \mathcal{D}^{(1)} + \cdots + \mathcal{D}^{(n)} = \mathcal{D}^{(1)} \sqcup \cdots \sqcup \mathcal{D}^{(n)},
\]
where \( \mathcal{D}^{(i)} \) is an effective Cartier divisor on \( \mathcal{C} \) flat over \( S \), which also has a decomposition
\[
\mathcal{D}^{(i)} = D_1^{(i)} + D_2^{(i)} + \cdots + D_{m_i}^{(i)},
\]
such that the composition \( D_j^{(i)} \hookrightarrow \mathcal{C} \rightarrow S \) is isomorphic. Here we assume that \( \mathcal{D}^{(i)} \cap D^{(i')} = \emptyset \) for \( i \neq i' \) and \( (D_j^{(i)})_s \cap (D_{j'}^{(i)})_s = \emptyset \) for \( j \neq j' \) if \( (D_j^{(i)})_s, (D_{j'}^{(i)})_s \) are generic fibers but \( D_j^{(i)} \) and \( D_{j'}^{(i)} \) may intersect.

Assume that we can take a section \( \tilde{z}^{(i)} \in \mathcal{O}_{2\mathcal{D}^{(i)}} \) such that \( \tilde{z}^{(i)} - \tilde{z}^{(i)}(D_j^{(i)}) = 0 \) is a defining equation of \( D_j^{(i)} \) in \( 2\mathcal{D}^{(i)} \) and that \( \tilde{z}^{(i)}|_p \) gives a local basis of \( \Omega^1_{\mathcal{C}/S} \otimes \mathcal{O}_{\mathcal{D}^{(i)}}|_p \) for any point \( p \in \mathcal{D}^{(i)} \), where \( \tilde{z}^{(i)}(D_j^{(i)}) \in \mathcal{O}_S \) corresponds to \( \tilde{z}^{(i)}|_{D_j^{(i)}} \) via the isomorphism \( \mathcal{D}_j^{(i)} \approx S \). We denote \( \tilde{z}^{(i)} - \tilde{z}^{(i)}(D_j^{(i)}) \in \mathcal{O}_{2\mathcal{D}^{(i)}} \) by \( \tilde{z}_j^{(i)} \). Then we can define
\[
\frac{dz^{(i)}}{z_1^{(i)} \cdots z_{m_i}^{(i)}} \in \Omega^1_{\mathcal{C}/S}(\mathcal{D}^{(i)})|_{\mathcal{D}^{(i)}}
\]
similarly to \([5]\) which is a local basis of \( \Omega^1_{\mathcal{C}/S}(\mathcal{D}^{(i)})|_{\mathcal{D}^{(i)}} \).

We fix \( \hat{\mu} = (\hat{\mu}_k^{(i)})_{1 \leq i \leq r} \in H^0(\mathcal{D}^{(i)}, \mathcal{O}_{\mathcal{D}^{(i)}}^{nr}) \) such that \( \hat{\mu}_k^{(i)}|_p, \ldots, \hat{\mu}_r^{(i)}|_p \in \mathbb{C} \) are mutually distinct at any point \( p \in \mathcal{D}^{(i)} \). Then we define a tuple \( \varphi_{\hat{\mu}} = (\varphi_{\hat{\mu}}^{(i)}(T))_{1 \leq i \leq n} \) of polynomials by
\[
\varphi_{\hat{\mu}}^{(i)}(T) = \prod_{k=1}^{r} \frac{T - \tilde{\mu}_k^{(i)}}{T - \hat{\mu}_k^{(i)}} \in H^0(\mathcal{D}^{(i)}, \mathcal{O}_{\mathcal{D}^{(i)}})[T].
\]
Assume that \( a \in \mathbb{Z} \) and \( \lambda = (\lambda_k^{(i)}) \in H^0(S, \mathcal{O}_S)^{nr} \) satisfying
\[
a + \sum_{i=1}^{n} \sum_{k=1}^{r} \lambda_k^{(i)} = 0
\]
are given. We also take a tuple \( \hat{\nu} = (\hat{\nu}^{(i)}(T))_{1 \leq i \leq n} \) of polynomials
\[
\hat{\nu}^{(i)}(T) = c_0^{(i)} + c_1^{(i)} T + \cdots + c_{r-1}^{(i)} T^{r-1} \in H^0(\mathcal{D}^{(i)}, \mathcal{O}_{\mathcal{D}^{(i)}})[T]
\]
such that the expression
\[
\varphi_{\hat{\mu}}^{(i)}(\mu_k^{(i)}) = \sum_{0 \leq l_1 \ldots l_{m_i} \leq 1} a_{k,l_1 \ldots l_{m_i}}^{(i)} \prod_{i=1}^{l_1} (\tilde{z}_1^{(i)})^{l_1} \cdots \prod_{i=1}^{l_{m_i}} (\tilde{z}_{m_i}^{(i)})^{l_{m_i}}
\]
with \( a_{k,l_1 \ldots l_{m_i}}^{(i)} \in H^0(S, \mathcal{O}_S) \) satisfies the equality
\[
\tilde{\lambda}_k^{(i)} = a_{k,0,1,\ldots,1}^{(i)} + a_{k,1,0,1,\ldots,1}^{(i)} + \cdots + a_{k,1,\ldots,1}^{(i)}
\]
for any \( i, k \). Furthermore, we assume that \( \hat{\nu}^{(i)}(\mu_1^{(i)}|_p, \ldots, \hat{\nu}^{(i)}(\mu_r^{(i)})|_p \) are mutually distinct for each \( i \) and \( p \in \mathcal{D}^{(i)} \).
Before the definition of moduli functor, we mention a convention of notation used in this paper. For a noetherian scheme $S'$ with a morphism $S' \to S$, we denote $\mathcal{C} \times_S S'$ by $\mathcal{C}_{S'}$ and denote $\mathcal{D} \times_S S'$ by $\mathcal{D}_{S'}$ and so on. For a coherent sheaf $E$ on $\mathcal{C}$, we denote the pull-back of $E$ under the morphism $\mathcal{C} \times_S S' \to \mathcal{C}$ by $E_{S'}$ and so on.

**Definition 2.10.** We define a contravariant functor $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}}): (\text{Sch}/S)^{\circ} \to (\text{Sets})$ from the category $(\text{Sch}/S)$ of noetherian schemes over $S$ to the category $(\text{Sets})$ of sets by setting

\[
\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})(S') = \left\{ (E, \nabla, \{N^{(i)}\}_{1 \leq i \leq n}) \mid (E, \nabla, \{N^{(i)}\}) \text{ satisfies the following (a), (b), (c), (d)} \right\} / \sim,
\]

for a noetherian scheme $S'$ over $S$, where

(a) $E$ is a vector bundle on $\mathcal{C}_{S'}$ of rank $r$ and $\deg(E|_{\mathcal{C}_{S'}}) = a$ for any geometric point $s$ of $S$,

(b) $\nabla: E \to E \otimes \Omega^1_{\mathcal{C}_{S'}/S'}(\mathcal{D}_{S'})$ is an $S'$-relative connection, in other words, $\nabla(fa) = a \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathcal{C}_{S'}}$ and $a \in E$,

(c) $N^{(i)}: E|_{\mathcal{D}_{S'}} \to \mathcal{O}_{\mathcal{D}_{S'}}$ is an $\mathcal{O}_{\mathcal{D}_{S'}}$-homomorphism satisfying $\varphi^{(i)}(N^{(i)}) = 0$, the homomorphism $\mathcal{O}_{\mathcal{D}_{S'}}/\mathcal{O}_{\mathcal{D}_{S'}} \otimes \mathcal{O}_{\mathcal{D}_{S'}}(T)/\mathcal{O}_{\mathcal{D}_{S'}} = \mathcal{O}_{\mathcal{D}_{S'}}$ for $1 \leq i \leq n$ and

(d) $(E|_{\mathcal{C}_{S'}}, \nabla|_{\mathcal{C}_{S'}}, \{N^{(i)}|_{\mathcal{D}_{S'}}\})$ is $\alpha$-stable for any geometric point $s$ of $S'$.

Here $(E, \nabla, \{N^{(i)}\}) \sim (E', \nabla', \{N^{(i)}\})$ if there are a line bundle $\mathcal{L}$ on $S'$ and an isomorphism $\sigma: E \to E' \otimes \mathcal{L}$ satisfying $(\text{id} \otimes \sigma) \circ \nabla = \nabla' \otimes \sigma$ and $\sigma|_{\mathcal{D}_{S'}} \circ N^{(i)} = (N^{(i)} \otimes \text{id}) \circ \sigma|_{\mathcal{D}_{S'}}$ for any $i$.

**Theorem 2.11.** There exists a coarse moduli scheme $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})$ of $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})$. The structure morphism $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}}) \to S$ is a smooth and quasi-projective morphism whose non-empty fiber is of dimension $2r^2(g-1) + 2 + r(r-1)\sum m_i$. Moreover, there is a relative symplectic form on $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})$ over $S$.

We call $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})$ in Theorem 2.11 the relative moduli space of $\alpha$-stable $(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})$ connections on $(\mathcal{C}, \mathcal{D})$ over $S$. First we give a proof of the existence of the moduli space $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})$. We define a moduli functor $\mathcal{M}: (\text{Sch}/S)^{\circ} \to (\text{Sets})$ by

\[
\mathcal{M}(S') = \left\{ (E, \nabla, \{t^{(i)}_k\}) \mid (E, \nabla, \{t^{(i)}_k\}) \text{ satisfies the following (i), (ii), (iii), (iv)} \right\} / \sim
\]

for a noetherian scheme $S'$ over $S$, where

(i) $E$ is a vector bundle on $\mathcal{C} \times_S S'$ of rank $r$ and $\deg(E|_{\mathcal{C}_s}) = a$ for any geometric point $s$ of $S'$,

(ii) $\nabla: E \to E \otimes \Omega^1_{\mathcal{C}_{S'}/S'}(\mathcal{D}_{S'})$ is a relative connection,

(iii) $E|_{\mathcal{D}_{S'}} = \mathcal{O}_{\mathcal{D}_{S'}}(t^{(i)}_0) \supset \mathcal{O}_{\mathcal{D}_{S'}}(t^{(i)}_1) \supset \cdots \supset \mathcal{O}_{\mathcal{D}_{S'}}(t^{(i)}_{r-1}) \supset \mathcal{O}_{\mathcal{D}_{S'}}(t^{(i)}_r) = 0$ is a filtration by coherent $\mathcal{O}_{\mathcal{D}_{S'}}$-submodules such that each $t^{(i)}_k/t^{(i)}_{k+1}$ is flat over $S'$ and $\text{length}(t^{(i)}_k/t^{(i)}_{k+1})|_{\mathcal{D}_{S'}} = m_i$ for any $s \in S'$,

(iv) for any geometric point $s$ of $S'$, the fiber $(E, \nabla, \{t^{(i)}_k\})|_{\mathcal{C}_s}$ satisfying the stability condition

\[
\frac{-\deg E|_{\mathcal{D}_{S'}} + \sum_{i=1}^n \sum_{k=1}^r \alpha^{(i)}_k \text{length}(t^{(i)}_k|_{\mathcal{D}_{S'}}/t^{(i)}_{k+1}|_{\mathcal{D}_{S'}})}{\text{rank } E} < \frac{-\deg E + \sum_{i=1}^n \sum_{k=1}^r \alpha^{(i)}_k \text{length}(t^{(i)}_k|_{\mathcal{D}_{S'}}/t^{(i)}_{k+1}|_{\mathcal{D}_{S'}})}{\text{rank } F}
\]

for any subbundle $0 \neq F \subseteq E|_{\mathcal{C}_s}$ satisfying $\nabla|_{\mathcal{C}_s}(F) \subseteq F \otimes \Omega^1_{\mathcal{C}_s}(\mathcal{D}_s)$.

Here $(E, \nabla, \{t^{(i)}_k\}) \sim (E', \nabla', \{t^{(i)}_k\})$ if there are a line bundle $\mathcal{L}$ on $S'$ and an isomorphism $(E, \nabla, \{t^{(i)}_k\}) \to (E', \nabla', \{t^{(i)}_k\}) \otimes_{\mathcal{O}_{\mathcal{D}_S}} \mathcal{L}$. Note that the parabolic structure $\{t^{(i)}_k\}$ in (iii) has no relationship with the connection $\nabla$ in (ii). The following lemma is already used in [11], [12] and [14].

**Lemma 2.12.** There exists a coarse moduli scheme $\mathcal{M}$ of $\mathcal{M}$. $M$ is quasi-projective over $S$ and represents the étale sheafification of the moduli functor $\mathcal{M}$.

**Proof.** By [11 Theorem 5.1], there exists a relative coarse moduli scheme $\mathcal{M}^\alpha_{\mathcal{C}, \mathcal{D}}(\mathbf{\bar{\nu}}, \mathbf{\bar{\mu}})$ over $S$ of parabolic $\mathbf{\bar{\nu}}$-triples $(E_1, E_2, \phi, \nabla, \{t^{(i)}_k\})$, where $E_1$ and $E_2$ are algebraic vector bundles of rank $r$ on a fiber of $\mathcal{C}$ over $S$, $\phi: E_1 \to E_2$ is an $\mathcal{O}_{\mathcal{C}}$-homomorphism, $\nabla: E_1 \to E_2 \otimes \Omega^1_{\mathcal{C}/S}(\mathcal{D})$ satisfies $\nabla(fa) = \text{...}$
\( \phi(a) \otimes df + f \nabla(a) \) for \( f \in \mathcal{O}_C, \ a \in E_1, \ E_1|_{D^{(i)}} = i_0^{(i)} \supset i_1^{(i)} \supset \cdots \supset i_m^{(i)} = 0 \) is a filtration satisfying length(\( i_0^{(i)}/i_{k+1}^{(i)} \)) = m_i and \( (E_1, E_2, \phi, \nabla, \{ i_k^{(i)} \}) \) satisfies a stability condition with respect to \( (\alpha', \beta, \gamma) \). Furthermore, \( M^{\omega, \beta, \gamma}_{C/S}(r, a, \{ m_i \}) \) is quasi-projective over \( S \). The detail is written in [11] section 5. If we denote the moduli functor corresponding to \( M^{\omega, \beta, \gamma}_{C/S}(r, a, \{ m_i \}) \) by \( M^{\omega, \beta, \gamma}_{C/S}(r, a, \{ m_i \}) \) and choose an appropriate stability parameter \( (\alpha', \beta, \gamma) \) by a similar argument to that in [11] section 5, then we can define a morphism of functors

\[
\mathcal{M} \longrightarrow M^{\omega, \beta, \gamma}_{C/S}(r, a, \{ m_i \})
\]

given by \( (E, \nabla, \{ i_k^{(i)} \}) \rightarrow (E, E, \text{id}_E, \nabla, \{ i_k^{(i)} \}) \) which is represented by an open immersion. So there is a Zariski open subset \( M \subset M^{\omega, \beta, \gamma}_{C/S}(r, a, \{ m_i \}) \) satisfying

\[
\mathcal{M} \times_M M^{\omega, \beta, \gamma}_{C/S}(r, a, \{ m_i \}) \cong M^{\omega, \beta, \gamma}_{C/S}(r, a, \{ m_i \}) \]

Then \( M \) represents the étale sheafification of \( \mathcal{M} \) and becomes a coarse moduli scheme of \( \mathcal{M} \).

**Proof of the existence of \( M^{\omega, \beta, \gamma}_{C,D}(\tilde{\nu}, \tilde{\mu}) \).**

For some quasi-finite étale covering \( \tilde{M} \rightarrow M \), there is a universal family \( (\tilde{E}, \tilde{\nabla}, \{ i_k^{(i)} \}) \) on \( C \times S \tilde{M} \). Let \( Y \) be the maximal locally closed subscheme of \( \tilde{M} \) such that \( (i_k^{(i)})_Y \) is a locally free \( \mathcal{O}_{D^{(i)}} \)-module of rank one for \( i = 1, \ldots, n \) and \( (i_k^{(i)})_Y \subset (i_{k+1}^{(i)})_Y \otimes \Omega^1_{\mathcal{C}_Y/Y}(D_Y) \) for \( 1 \leq k \leq r \). We set

\[
P := \prod_{i=1}^n \text{Spec} \mathcal{O}_Y \left( \text{Hom}(\tilde{E}|_{D^{(i)}}, \tilde{E}|_{D^{(i)}})^Y \right)
\]
and take universal families \( \tilde{\nu}^{(i)}: \tilde{E}|_{D^{(i)}_p} \rightarrow \tilde{E}|_{D^{(i)}_p} \) for \( i = 1, \ldots, n \), where \( S_Y \left( \text{Hom}(\tilde{E}|_{D^{(i)}}, \tilde{E}|_{D^{(i)}})^Y \right) \) denotes the symmetric algebra of \( \text{Hom}(\tilde{E}|_{D^{(i)}}, \tilde{E}|_{D^{(i)}})^Y \) over \( Y \). Let \( Z \) be the maximal locally closed subscheme of \( P \) satisfying \( \varphi(\tilde{\nu}^{(i)}|_Z) = 0 \) in \( \text{End}(\tilde{E}|_{D^{(i)}_p}) \), \( \tilde{\nu}^{(i)}(\tilde{N}^{(i)}) \left( \frac{dz^{(i)}}{z_1^{(i)} z_2^{(i)} \cdots z_m^{(i)}} \right) \bigg|_{D^{(i)}_z} = \tilde{\nabla}|_{D^{(i)}_z} \) and

\[
\mathcal{O}_{D^{(i)}_p}[T]/(\varphi^{(i)}(T)) \ni P(T) \mapsto P((\tilde{N}^{(i)})_p) \in \text{End}(\tilde{E}|_{D^{(i)}_p})
\]
is injective for any \( \mathbb{C} \)-valued point \( p \) of \( Z \). By construction, we can easily see that \( Z \) descends to a quasi-projective scheme \( M^{\omega, \beta, \gamma}_{C,D}(\tilde{\nu}, \tilde{\mu}) \) over \( M \), which is the desired moduli space.

The proof of Theorem 2.11 will be completed at the end of subsection 2.7.

### 2.3. **Factorized \((\nu, \mu)\)-connection.**

For the rest of the proof of Theorem 2.11 we need to describe the tangent space of the moduli space. We will describe the tangent space and give a symplectic structure via the idea in section 1.2. So we introduce the notion of factorized \((\nu, \mu)\)-connection which comes from the idea of factorization of a linear map in subsection 1.1.

Let \( C, D, D^{(i)}_p, \mu, \varphi, \nu, \tilde{\nu}, \tilde{\mu}, \tilde{\nu}^{(i)}, \nu^{(i)}, \tilde{\nu}^{(i)} \) be as in Definition 2.3. The following notion of factorized connection is useful for describing the deformation theory of \((\nu, \mu)\)-connections and the relative symplectic form on the moduli space.

**Definition 2.13.** We say that a tuple \((E, \nabla, \{ \theta^{(i)}, \kappa^{(i)} \})\) is a factorized \((\nu, \mu)\)-connection if

1. \( E \) is an algebraic vector bundle on \( C \) of rank \( r \) and degree \( a \),
2. \( \nabla: E \rightarrow E \otimes \Omega^1_C(D) \) is an algebraic connection admitting poles along \( D \),
3. \( \theta^{(i)}: E^{(i)}|_{D^{(i)}} \rightarrow E|_{D^{(i)}} \) is an \( D^{(i)} \)-isomorphism satisfying \( t^{(i)} \theta^{(i)} = \theta^{(i)} \),
4. \( \kappa^{(i)}: E|_{D^{(i)}} \rightarrow E^{(i)}|_{D^{(i)}} \) is an \( D^{(i)} \)-homomorphism satisfying \( t^{(i)} \kappa^{(i)} = \kappa^{(i)} \),
5. the composition \( N^{(i)} := \theta^{(i)} \circ \kappa^{(i)}: E|_{D^{(i)}} \rightarrow E|_{D^{(i)}} \) satisfies \( \nu^{(i)}(N^{(i)}) \frac{dz^{(i)}}{z_1^{(i)} z_2^{(i)} \cdots z_m^{(i)}} = \nabla|_{D^{(i)}} \),

\[ \varphi^{(i)}(N^{(i)}) = 0 \] and the injectivity of the ring homomorphism

\[ \mathcal{O}_{D^{(i)}}[T]/(\varphi^{(i)}(T)) \ni P(T) \mapsto P(N^{(i)}) \in \text{End}_{D^{(i)}}(E|_{D^{(i)}}). \]
Two factorized \((\nu, \mu)\)-connections \((E, \nabla, \{\theta^{(i)}, \kappa^{(i)}\})\) and \((E', \nabla', \{\theta'^{(i)}, \kappa'^{(i)}\})\) are isomorphic if there is an isomorphism \(\sigma: E \xrightarrow{\sim} E'\) of algebraic vector bundles such that \((\sigma \otimes 1) \circ \nabla = \nabla' \circ \sigma\), and the diagrams

\[
\begin{align*}
E|_{D^{(i)}} & \xrightarrow{\kappa^{(i)}} E'|_{D^{(i)}} \\
\sigma|_{D^{(i)}} & \cong \\
E'|_{D^{(i)}} & \xrightarrow{\kappa'^{(i)}} E|_{D^{(i)}}
\end{align*}
\]

\[
\begin{align*}
E|_{D^{(i)}} & \xrightarrow{\theta^{(i)}} E|_{D^{(i)}} \\
\sigma|_{D^{(i)}} & \cong \\
E|_{D^{(i)}} & \xrightarrow{\theta'^{(i)}} E'|_{D^{(i)}}
\end{align*}
\]

are commutative for some \(\overline{P}^{(i)}(T) \in \left(\mathcal{O}_{D^{(i)}}[T]/(\varphi^{(i)}_{\mu}(T))\right)^{\times}\).

**Proposition 2.14.** The correspondence \((E, \nabla, \{\theta^{(i)}, \kappa^{(i)}\}) \mapsto (E, \nabla, \{\theta'^{(i)}, \kappa'^{(i)}\})\) gives a bijective correspondence between the isomorphism classes of factorized \((\nu, \mu)\)-connections and the isomorphism classes of \((\nu, \mu)\)-connections on \((C, D)\).

**Proof.** We will give the inverse correspondence. Let \((E, \nabla, \{N^{(i)}\})\) be a \((\nu, \mu)\)-connection on \((C, D)\). We can define an \(\mathcal{O}_{D^{(i)}}[T]\)-module structure on \(E|_{D^{(i)}}\) by

\[
\mathcal{O}_{D^{(i)}}[T] \times E|_{D^{(i)}} \ni (P(T), v) \mapsto P(N^{(i)})v \in E|_{D^{(i)}}.
\]

We also define an \(\mathcal{O}_{D^{(i)}}[T]\)-module structure on \(E'|_{D^{(i)}}\) by

\[
\mathcal{O}_{D^{(i)}}[T] \times E'|_{D^{(i)}} \ni (P(T), v) \mapsto P(tN^{(i)})v \in E'|_{D^{(i)}}.
\]

For any point \(x \in D^{(i)}\), the homomorphism \(\mathbb{C}[T]/(\varphi^{(i)}_{\mu}(T)) \ni \overline{P}(T) \mapsto \overline{P}(N^{(i)}x) \in \text{End}_{\mathbb{C}}(E|_{x})\) is injective by Remark 2.4. So the minimal polynomial of the endomorphism \(\overline{N}^{(i)}|_{x}\) on the vector space \(E|_{x}\) is \(\varphi^{(i)}_{\mu}|_{x}(T)\) whose degree is \(r = \dim_{\mathbb{C}} E|_{x}\). Thus an elementary theory of linear algebra implies that there is an element \(v_{x} \in E|_{x}\) such that the homomorphism \(\mathbb{C}[T]/(\varphi^{(i)}_{\mu}(T)) \ni \overline{P}(T) \mapsto \overline{P}(N^{(i)}v_{x}) \in E|_{x}\) is an isomorphism of \(\mathbb{C}[T]\)-modules. If we take an element \(v \in E|_{D^{(i)}}\) such that \(v|_{x} = v_{x}\) for any \(x \in D^{(i)}\), then the homomorphism

\[
\mathcal{O}_{D^{(i)}}[T]/(\varphi^{(i)}_{\mu}(T)) \ni \overline{P}(T) \mapsto \overline{P}(N^{(i)}v) \in E|_{D^{(i)}}
\]

is an isomorphism of \(\mathcal{O}_{D^{(i)}}[T]\)-modules. Similarly \(E'|_{D^{(i)}}\) is isomorphic to \(\mathcal{O}_{D^{(i)}}[T]/(\varphi^{(i)}_{\mu}(T))\) as an \(\mathcal{O}_{D^{(i)}}[T]\)-module. So we can take an \(\mathcal{O}_{D^{(i)}}[T]\)-isomorphism \(\theta^{(i)}: E|_{D^{(i)}} \xrightarrow{\sim} E'|_{D^{(i)}}\), which makes the diagram

\[
\begin{align*}
E|_{D^{(i)}} & \xrightarrow{\theta^{(i)}} E'|_{D^{(i)}} \\
\overline{E} & \xrightarrow{\overline{\theta}^{(i)}} \overline{E}'
\end{align*}
\]

commutative. If we define

\[
\kappa^{(i)} := (\theta^{(i)})^{-1} \circ N^{(i)}: E|_{D^{(i)}} \rightarrow E'|_{D^{(i)}},
\]

then \(\kappa^{(i)}\) also becomes a homomorphism of \(\mathcal{O}_{D^{(i)}}[T]\)-modules. By definition, we have \(\theta^{(i)} \circ \kappa^{(i)} = N^{(i)}\) and we can verify the equalities \(t^{\theta^{(i)}} = \theta^{(i)}\) and \(t^{\kappa^{(i)}} = \kappa^{(i)}\) in the same way as Proposition 1.1. We can see by the same argument as Proposition 1.2 that the ambiguity of the choice of \(\theta^{(i)}\) is just a composition with the automorphism of \(E|_{D^{(i)}}\) of the form \(P(tN^{(i)})\) for some \(P(T) \in \mathbb{C}[T]\). Thus we can define a correspondence \((E, \nabla, \{N^{(i)}\}) \mapsto (E, \nabla, \{\theta^{(i)}, \kappa^{(i)}\})\) which is the desired inverse correspondence by its construction. \(\square\)

We extend the above proposition to a relative setting over a noetherian local scheme, that is, a scheme isomorphic to Spec \(A\) for some noetherian local ring \(A\). Let \(C, D, \mathcal{D}^{(i)}, \mathcal{D}^{(j)}, \nu, \mu, \varphi^{(i)}_{\mu}, \overline{\varphi}^{(i)}_{\mu}, \overline{S}^{(i)}\) and \(\overline{z}^{(i)}\) be as in subsection 2.2. Assume that \(S' := \text{Spec } A'\) is a noetherian local scheme with a morphism \(S' \rightarrow S\). We say that \((E, \nabla, \{N^{(i)}\})\) is a flat family of \((\nu_{S'}, \mu_{S'})\)-connections on \((C_{S'}, D_{S'})\) over \(S'\) if \(E\) is a vector bundle on \(C_{S'}\) of rank \(r\), \(\nabla: E \rightarrow E \otimes \Omega^{1}_{C_{S'}/S'}(D_{S'})\) is an \(S'\)-relative connection and \(N^{(i)}: E|_{D_{S'}^{(i)}} \rightarrow E|_{D_{S'}^{(i)}}\) is an \(\mathcal{O}_{D_{S'}^{(i)}}\)-homomorphism such that \(\varphi^{(i)}_{\mu}(N^{(i)}) = 0, \overline{\varphi}^{(i)}_{\mu}(N^{(i)}) = \overline{\nabla}|_{\mathcal{D}^{(i)}_{S'}}\) and the homomorphism

\[
\mathcal{O}_{D_{S'}^{(i)}}[T]/(\varphi^{(i)}_{\mu}(T)) \ni \overline{P}(T) \mapsto \overline{P}(N^{(i)}) \in \text{End}(E|_{D_{S'}^{(i)}})
\]

is an injection whose cokernel is flat over \(S'\). Similarly we say that \((E, \nabla, \{\theta^{(i)}, \kappa^{(i)}\})\) is a flat family of factorized \((\nu_{S'}, \mu_{S'})\)-connections on \((C_{S'}, D_{S'})\) over \(S'\) if \(E\) is a vector bundle on \(C_{S'}\) of rank \(r\),
\[ \nabla : E \to E \otimes \Omega^1_{\nu_s'/S'}(D_{s'}) \text{ is an } S'\text{-relative connection, } \theta^{(i)} : E^{(i)}|_{D^\nu_s'} \to E|_{D^\nu_s'} \text{ is an isomorphism,} \\
ue^{(i)}(\nu) \circ \kappa^{(i)}(\nu) = \nabla|_{D^\nu_s'} \text{ and the homomorphism} \\
\nu^{(i)}(\theta^{(i)} \circ \kappa^{(i)}) = \nabla|_{D^\nu_s'} \text{ is a homomorphism such that } t^{(i)} \theta^{(i)} = \theta^{(i)}, t^{(i)} \kappa^{(i)} = \kappa^{(i)}, \varphi^{(i)}(\theta^{(i)} \circ \kappa^{(i)}) = 0, \]

is an injection whose cokernel is flat over \( S' \).

**Proposition 2.15.** Let \( \mathcal{C}, \mathcal{D}, D^{(i)}, \mathcal{E}^{(i)}, \tilde{\nu}_s, \tilde{\mu}_s, \tilde{\rho}^{(i)}_s, \tilde{\varsigma}^{(i)}_s \) and \( \tilde{\varsigma}^{(i)}_s \) be as in subsection 2.2 and let \( S' \) be a noetherian local scheme with a morphism \( S' \to S \). Then the correspondence

\[ (E, \nabla, \{ \theta^{(i)}(\nu), \kappa^{(i)}(\nu) \}) \to (E, \nabla, \{ \theta^{(i)} \circ \kappa^{(i)} \}) \]

gives a bijective correspondence between the flat families of factorized \((\tilde{\nu}_s', \tilde{\mu}_s')\)-connections on \((\mathcal{C}_s', \mathcal{D}_s')\) over \( S' \) and the flat families of \((\tilde{\nu}_s', \tilde{\mu}_s')\)-connections on \((\mathcal{C}_s', \mathcal{D}_s')\) over \( S' \).

**Proof.** The proof is exactly the same as that of Proposition 2.14. \( \Box \)

2.4. **Tangent space of the moduli space of \((\tilde{\nu}, \tilde{\mu})\)-connections.** We use the same notations as in subsection 2.2. We take a \( \mathcal{C} \)-valued point \( x \) of \( M_{\mathcal{C}, \mathcal{D}}(\tilde{\nu}, \tilde{\mu}) \) over a \( \mathcal{C} \)-valued point \( s \) of \( S \). Let \((E, \nabla, \{ \theta^{(i)}(\nu), \kappa^{(i)}(\nu) \})\) be the \((\nu, \mu)\)-connection on the fiber \((\mathcal{C}_s, \mathcal{D}_s)\) corresponding to \( x \), where we put \((\nu, \mu) := (\tilde{\nu}_s, \tilde{\mu}_s)\). By Proposition 2.14, we can take a factorized \((\nu, \mu)\)-connection \((E, \nabla, \{ \theta^{(i)}(\nu), \kappa^{(i)}(\nu) \})\) corresponding to \((E, \nabla, \{ \theta^{(i)}(\nu), \kappa^{(i)}(\nu) \})\).

We will consider the deformation theory of \((E, \nabla, \{ \theta^{(i)}(\nu), \kappa^{(i)}(\nu) \})\).

Recall that \( \tilde{\nu}^{(i)}(T) \) is given by

\[ \tilde{\nu}^{(i)}(T) = \sum_{j=0}^{r-1} c_j^{(i)} T^j \in H^0(D^{(i)}, \mathcal{O}_{D^{(i)}})[T]. \]

We define homomorphisms

\[ \sigma^{(i)\nu}_\theta : \text{End}(E|_{D^{\nu}_s}) \otimes \mathcal{O}_{D^{\nu}_s}[T]/(\varphi^{(i)}(\mu)(T)) \to \text{Hom}(E^{(i)}|_{D^{\nu}_s}, E|_{D^{\nu}_s}) \]
\[ \sigma^{(i)\nu}_\kappa : \text{End}(E|_{D^{\nu}_s}) \otimes \mathcal{O}_{D^{\nu}_s}[T]/(\varphi^{(i)}(\mu)(T)) \to \text{Hom}(E^{(i)}|_{D^{\nu}_s}, E^{(i)}|_{D^{\nu}_s}) \]
\[ \delta^{(i)\nu}_{\nu, N^{(i)}} : \text{End}(E|_{D^{\nu}_s}) \to \text{End}(E|_{D^{\nu}_s}) \otimes \Omega^1_{\mathcal{C}_s}(D_s) \]

by setting

\[ \sigma^{(i)\nu}_\theta(u, \tilde{P}(T)) = -u \circ \theta^{(i)} - \theta^{(i)} \circ t u + \theta^{(i)} \circ P( t N^{(i)}) \]
\[ \sigma^{(i)\nu}_\kappa(u, \tilde{P}(T)) = \kappa^{(i)} \circ u + t u \circ \kappa^{(i)} - P( t N^{(i)}) \circ \kappa^{(i)} \]
\[ \delta^{(i)\nu}_{\nu, N^{(i)}}(u) = \sum_{j=1}^{r-1} \sum_{i=1}^j c_j^{(i)} (N^{(i)})^{j-l} \circ u \circ (N^{(i)})^{l-1} \frac{d \tilde{\varsigma}^{(i)}}{\tilde{z}_1^{(i)} \cdots \tilde{z}_{m_i}^{(i)}} \]

for \( u \in \text{End}(E|_{D^{\nu}_s}) \) and \( \tilde{P}(T) \in \mathcal{O}_{D^{\nu}_s}[T]/(\varphi^{(i)}(\mu)(T)) \). For each fixed \( u \in \text{End}(E|_{D^{\nu}_s}) \), we define a homomorphism \( \Theta^{(i)\nu}_u : \mathcal{O}_{D^{\nu}_s}[T]/(\varphi^{(i)}(\mu)(T)) \to \Omega^1_{\mathcal{C}_s}(D_s)|_{D^{\nu}_s} \) by setting

\[ \Theta^{(i)\nu}_u(\tilde{P}(T)) = \text{Tr}(P(N^{(i)}) \circ u) \frac{d \tilde{\varsigma}^{(i)}}{\tilde{z}_1^{(i)} \cdots \tilde{z}_{m_i}^{(i)}} \]

for \( \tilde{P}(T) \in \mathcal{O}_{D^{\nu}_s}[T]/(\varphi^{(i)}(\mu)(T)) \). We put

\[ G^0 := \text{End}(E), \quad G^1 := \text{End}(E) \otimes \Omega^1_{\mathcal{C}_s}(D_s), \quad G^1 := \bigoplus_{i=1}^n \text{Hom}(E|_{D^{\nu}_s}, E|_{D^{\nu}_s} \otimes \Omega^1_{\mathcal{C}_s}(D_s^{\nu})). \]
Furthermore we put

\[
S(E|_{D_s}, E|_{D_s}) = \left\{ (\tau^{(i)}) \in \bigoplus_{i=1}^n \text{Hom}(E|_{D^{(i)}}, E|_{D^{(i)}}) \left| t^{(i)} = \tau^{(i)} \text{ for any } i \right. \right\}
\]

\[
S(E|_{D_s}, E|_{D_s}) = \left\{ (\xi^{(i)}) \in \bigoplus_{i=1}^n \text{Hom}(E|_{D^{(i)}}, E|_{D^{(i)}}) \left| t^{(i)} = \xi^{(i)} \text{ for any } i \right. \right\}
\]

and

\[
Z^0 := \bigoplus_{i=1}^n O_{D^{(i)}}[T]/(\varphi_{\mu}^{(i)}(T)), \quad Z^1 := \bigoplus_{i=1}^n \text{Hom}_{O_{D^{(i)}}} \left( O_{D^{(i)}}[T]/(\varphi_{\mu}^{(i)}(T)), \Omega^1_{D^{(i)}} \right). \]

We define sheaves \( \mathcal{F}^0, \mathcal{F}^1, \mathcal{F}^2 \) on \( C_s \) by

\[
\mathcal{F}^0 := \mathcal{G}^0 \oplus Z^0,
\]

\[
\mathcal{F}^1 := \mathcal{G}^1 \oplus S(E|_{D_s}, E|_{D_s}) \oplus S(E|_{D_s}, E|_{D_s}),
\]

\[
\mathcal{F}^2 := \mathcal{G}^1 \oplus Z^1
\]

and define homomorphisms \( d^0 : \mathcal{F}^0 \rightarrow \mathcal{F}^1, \ d^1 : \mathcal{F}^1 \rightarrow \mathcal{F}^2 \) by

\[
d^0(u, (P(i)(T))) = \left( \nabla \circ u - u \circ \nabla, \left( \sigma^{(i)}_{\theta(i)}(u|_{D^{(i)}}, P(i)(T)) \right), \left( \sigma^{(i)}_{\kappa(i)}(u|_{D^{(i)}}, P(i)(T)) \right) \right)
\]

\[
d^1(v, (\tau^{(i)}), (\xi^{(i)})) = \left( \left( v|_{D^{(i)}} - \delta^{(i)}_{\nu, N(i)}(\tau^{(i)} \circ \kappa^{(i)} + \theta^{(i)} \circ \xi^{(i)}) \right), \left( \Theta^{(i)}_{(\tau^{(i)} \circ \kappa^{(i)}) + \theta^{(i)} \circ \xi^{(i)}} \right) \right)
\]

**Lemma 2.16.** Under the above notation, \( d^1 \circ d^0 = 0 \).

**Proof.** Take \( (u, (P(i)(T))) \in \mathcal{F}^0 = \mathcal{G}^0 \oplus Z^0 \). Note that

\[
\sigma^{(i)}_{\theta(i)}(u|_{D^{(i)}}, (P(i)(T))) \circ \kappa^{(i)} + \theta^{(i)} \circ \sigma^{(i)}_{\kappa(i)}(u|_{D^{(i)}}, (P(i)(T)))
\]

\[
= \left( -u|_{D^{(i)}} \circ \theta^{(i)} - \theta^{(i)} \circ t^{(i)} u|_{D^{(i)}} + \theta^{(i)} \circ P(i)(tN^{(i)}) \circ \kappa^{(i)} \right.
\]

\[\quad + \theta^{(i)} \circ \left( \kappa^{(i)} \circ u|_{D^{(i)}} + t^{(i)} u|_{D^{(i)}} \circ \kappa^{(i)} - P(i)(tN^{(i)}) \circ \kappa^{(i)} \right)
\]

\[= \theta^{(i)} \circ \kappa^{(i)} \circ u|_{D^{(i)}} - u|_{D^{(i)}} \circ \theta^{(i)} \circ \kappa^{(i)}
\]

\[= N^{(i)} \circ u|_{D^{(i)}} - u|_{D^{(i)}} \circ N^{(i)}.
\]

So the first component of \( d^1(d^0(u, (P(i)(T)))) \) is

\[
\left( \left( \nabla \circ u - u \circ \nabla \right)|_{D^{(i)}} - \delta^{(i)}_{\nu, N(i)}(\sigma^{(i)}_{\theta(i)}(u|_{D^{(i)}}, (P(i)(T))) \circ \kappa^{(i)} + \theta^{(i)} \circ \sigma^{(i)}_{\kappa(i)}(u|_{D^{(i)}}, (P(i)(T)))) \right)
\]

\[
= \left( \left( \nabla \circ u - u \circ \nabla \right)|_{D^{(i)}} - \delta^{(i)}_{\nu, N(i)}(N^{(i)} \circ u|_{D^{(i)}} - u|_{D^{(i)}} \circ N^{(i)}) \right)
\]

\[
= \left( \left( \nabla \circ u - u \circ \nabla \right)|_{D^{(i)}} - \sum_{j=1}^{r-1} \sum_{i=1}^j c^{(i)}_{j} (N^{(i)})^{j-1} \circ (N^{(i)} \circ u|_{D^{(i)}} - u|_{D^{(i)}} \circ N^{(i)}) \circ (N^{(i)})^{j-1} \frac{d\zeta^{(i)}}{\zeta_1 \cdots \zeta^{(i)} m_i} \right)
\]

\[
= \left( \left( \nabla \circ u - u \circ \nabla \right)|_{D^{(i)}} - \sum_{j=0}^{r-1} \sum_{i=0}^j c^{(i)}_{j} (N^{(i)})^{j} \circ u|_{D^{(i)}} - \sum_{j=0}^{r-1} c^{(i)}_{j} u|_{D^{(i)}} \circ (N^{(i)})^{j} \frac{d\zeta^{(i)}}{\zeta_1 \cdots \zeta^{(i)} m_i} \right)
\]

\[= \left( \left( \nabla \circ u - u \circ \nabla \right)|_{D^{(i)}} - (\nabla|_{D^{(i)}} \circ u|_{D^{(i)}} - u|_{D^{(i)}} \circ \nabla|_{D^{(i)}}) \right)
\]

\[= 0.
\]

The second component of \( d^1(d^0(u, (P(i)(T)))) \) is

\[
\left( \Theta^{(i)}_{\sigma^{(i)}_{\theta(i)}(u|_{D^{(i)}}, (P(i)(T))) \circ \kappa^{(i)} + \theta^{(i)} \circ \sigma^{(i)}_{\kappa(i)}(u|_{D^{(i)}}, (P(i)(T)))} \right) = \left( \Theta^{(i)}_{N^{(i)} \circ u|_{D^{(i)}} - u|_{D^{(i)}} \circ N^{(i)}} \right)
\]
which is zero because
\[
\Theta_{N(n)}(Q(T)) = \text{Tr} \left( Q(N(n)) \circ N(n) \circ u |_{D_A} - Q(N(n)) \circ u |_{D_A} \circ N(n) \right) \frac{dz}{z_1^m} \ldots \frac{dz}{z_m^m},
\]

\[
= \left( \text{Tr} \left( Q(N(n)) \circ N(n) \circ u |_{D_A} \right) - \text{Tr} \left( N(n) \circ Q(N(n)) \circ u |_{D_A} \right) \right) \frac{dz}{z_1^m} \ldots \frac{dz}{z_m^m} = 0
\]

for any \( Q(T) \in \mathcal{O}_{D_A}[T]/(\varphi_\mu(T)) \). Thus we have proved \( d^1(d^0(u,(P^n(T)))) = 0 \). \( \square \)

By Lemma 2.16 \( \mathcal{F}^* \) becomes a complex. Note that there is an exact commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & 0 & \to & G^0 & \oplus & Z^0 & \to & 0 \\
\downarrow & & \downarrow & & d^0 & & \downarrow & & \\
0 & \to & G^1 & \oplus & S(E|_{D_A},E|_{D_A}) & \to & G^1 & \oplus & Z^1 & \to & 0 \\
\downarrow & & \downarrow & & d^1 & & \downarrow & & \\
0 & \to & 0 & \to & G^1 & \oplus & Z^1 & \to & 0 & \to & 0.
\end{array}
\]

If we denote by \( \mathcal{F}^*_0 \) the complex \( G^0 \oplus Z^0 \to S(E|_{D_A},E|_{D_A}) \) concentrated in degree 0 and 1 and if we denote by \( \mathcal{F}^*_1 \) the complex \( G^1 \oplus S(E|_{D_A},E|_{D_A}) \to G^1 \oplus Z^1 \) concentrated in degree 0 and 1, then the above commutative diagram is a short exact sequence of complexes

\[
(19) \quad 0 \to \mathcal{F}^*_1[-1] \to \mathcal{F}^* \to \mathcal{F}^*_0 \to 0
\]

which induces a long exact sequence of hyper cohomologies:

\[
(20) \quad 0 \to H^0(\mathcal{F}^*) \to H^0(\mathcal{F}^*_1) \to H^1(\mathcal{F}^*_1) \to H^1(\mathcal{F}^*_0) \to H^2(\mathcal{F}^*) \to 0.
\]

Proposition 2.17. Let \( A \) be an artinian local ring over \( S \) with the maximal ideal \( m \) satisfying \( A/m = \mathbb{C} \) and let \( I \) be an ideal of \( A \) satisfying \( mI = 0 \). Assume that there exists a flat family \((E',\nabla',\{N^{(i)}\}) \in \mathcal{M}_{C,D}(\tilde{\nu},\tilde{\mu})(A)\) of (\( \tilde{\nu}, \tilde{\mu} \))-connections over \( A \) such that \((E',\nabla',\{N^{(i)}\}) \otimes A/m \cong (E,\nabla,\{N^{(i)}\}) \). Consider the restriction map

\[
\rho_{A/I}: \mathcal{M}_{C,D}(\tilde{\nu},\tilde{\mu})(A) \ni (\tilde{E},\tilde{\nabla},\{\tilde{N}^{(i)}\}) \to (\tilde{E},\tilde{\nabla},\{\tilde{N}^{(i)}\}) \otimes A/I \in \mathcal{M}_{C,D}(\tilde{\nu},\tilde{\mu})(A/I).
\]

Then there exists a bijective correspondence \( \rho_{A/I}^{-1}((E',\nabla',\{N^{(i)}\}) \otimes A/I) \cong H^1(\mathcal{F}^*) \otimes \mathbb{C} I \).

Proof. We can take an affine open covering \( C_A = \bigcup \alpha U_\alpha \) such that \#\{\alpha | D_A \cap U_\alpha \neq 0\} \leq 1 for any \( \alpha \) and \#\{\alpha | D_A \subset U_\alpha \} = 1 for any \( i \). We may assume that \( E'|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}^{\oplus r} \) for any \( \alpha \). Take any member \((\tilde{E},\tilde{\nabla},\{\tilde{N}^{(i)}\}) \in \rho_{A/I}^{-1}((E',\nabla',\{N^{(i)}\}) \otimes A/I)\). Let \((E',\nabla',\{\theta^{(i)}\},\kappa^{(i)})) \) and \((\tilde{E},\tilde{\nabla},\{\tilde{\theta}^{(i)},\tilde{\kappa}^{(i)}\})\) be the flat families of factorized \((\tilde{\nu},\tilde{\mu})\) \( A \)-connections on \( (C_A,D_A) \) over \( A \) corresponding to \((E',\nabla',\{N^{(i)}\})\) and \((\tilde{E},\tilde{\nabla},\{\tilde{N}^{(i)}\})\), respectively. We can take an isomorphism \( \sigma_\alpha: \tilde{E}|_{U_\alpha} \cong E'|_{U_\alpha} \) which is a lift of the given isomorphism \( \tilde{E} \otimes A/I|_{U_\alpha \otimes A/I} \cong E'|_{U_\alpha} \). Then we put

\[
u_{\alpha} := \sigma_\alpha \circ \nabla \circ \sigma^{-1}_\alpha - \nabla' \in G^1(U_\alpha) \otimes I
\]

and

\[
u_{\alpha} := \sigma_\alpha \circ \theta' \circ \sigma^{-1}_\alpha - \theta \in G^1(U_\alpha) \otimes I
\]

if \( D_A \subset U_\alpha \). Note that we have \(((\tau_{\alpha}^{(i)}),\xi_{\alpha}^{(i)}) \in \mathcal{M}(E'|_{D_A},E|_{D_A}) \oplus \mathcal{M}(E|_{D_A},E'|_{D_A})(U_\alpha) \otimes \mathbb{C} I \). We can easily check the equalities

\[
u_{\beta\gamma} - \nu_{\alpha\gamma} + \nu_{\alpha\beta} = 0, \quad \nabla \circ \nu_{\alpha\beta} - \nu_{\alpha\beta} \circ \nabla = \nu_\beta - \nu_\alpha.
\]
Since $r^{(i)}_\alpha \circ K^{(i)} + \theta^{(i)} \circ \xi^{(i)}_\alpha = (\sigma^{(i)}_\alpha |_{D^i} \circ \tilde{\theta}^{(i)}_\alpha \circ t^{(i)} \sigma |_{D^i} - \theta^{(i)}_\alpha) \circ t^{(i)} \sigma^{(i)}_\alpha^{-1} |_{D^i} \circ \tilde{K}^{(i)}_\alpha \circ \sigma^{(i)}_\alpha^{-1} |_{D^i} + \theta^{(i)}_\alpha \circ (t^{(i)} \sigma^{(i)}_\alpha^{-1} |_{D^i} \circ \tilde{K}^{(i)}_\alpha \circ \sigma^{(i)}_\alpha^{-1} |_{D^i} - K^{(i)}_\alpha))$

$= \sigma^{(i)}_\alpha |_{D^i} \circ \tilde{\theta}^{(i)}_\alpha \circ \tilde{K}^{(i)}_\alpha \circ \sigma^{(i)}_\alpha^{-1} |_{D^i} - \theta^{(i)} \circ K^{(i)}_\alpha$

$= \sigma^{(i)}_\alpha |_{D^i} \circ \tilde{N}^{(i)}_\alpha \circ \sigma^{(i)}_\alpha^{-1} |_{D^i} = N^{(i)}_\alpha$,

we have

$\delta^{(i)}_{\nu, N^{(i)}}(r^{(i)}_\alpha \circ K^{(i)} + \theta^{(i)} \circ \xi^{(i)}_\alpha) = \sum_{j=1}^{r-1} \sum_{j=1}^{j} c_j^{(i)} (\sigma^{(i)}_\alpha |_{D^i} \circ \tilde{N}^{(i)}_\alpha \circ \sigma^{(i)}_\alpha^{-1} |_{D^i})^{j-1} \circ (\sigma^{(i)}_\alpha |_{D^i} \circ \tilde{N}^{(i)}_\alpha \circ \sigma^{(i)}_\alpha^{-1} |_{D^i} - N^{(i)}_\alpha) \circ (N^{(i)}_\alpha)^{j-1} \frac{d\tilde{z}^{(i)}}{\tilde{z}^{(i)}_1 \cdots \tilde{z}^{(i)}_m}$

$= \sigma^{(i)}_\alpha |_{D^i} \circ \tilde{N}^{(i)}_\alpha \circ \sigma^{(i)}_\alpha^{-1} |_{D^i} - \nabla |_{D^i}$. 

So the first component of $d^1(v_\alpha, (r^{(i)}_\alpha), (\xi^{(i)}_\alpha))$ becomes

$v_\alpha |_{D^i} - \delta^{(i)}_{\nu, N^{(i)}}(r^{(i)}_\alpha \circ K^{(i)} + \theta^{(i)} \circ \xi^{(i)}_\alpha)$

$= (\sigma \circ \nabla \circ \sigma^{-1} - \nabla') |_{D^i} - (\sigma |_{D^i} \circ \nabla |_{D^i} \circ \sigma^{-1} |_{D^i} - \nabla |_{D^i}) = 0$.

On the other hand, $N^{(i)}_\alpha$ has a representation matrix

$$
\begin{pmatrix}
\mu_1^{(i)} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \mu_r^{(i)}
\end{pmatrix}
$$

with respect to a basis $e_1^{(i)}, \ldots, e_r^{(i)}$ of $E^{(i)} |_{D^i}$ and $\tilde{N}^{(i)}_\alpha$ has the same representation matrix with respect to a basis $\tilde{e}_1, \ldots, \tilde{e}_r$ of $\tilde{E}^{(i)} |_{D^i}$ from Definition 2.10 (c). Moreover, we may assume that $(e_1^{(i)}, \ldots, e_r^{(i)}) \otimes \tilde{A} / \tilde{A} = (\tilde{e}_1, \ldots, \tilde{e}_r) \otimes A / A$, because $\tilde{N}^{(i)}_\alpha \otimes A / A = N^{(i)}_\alpha \otimes A / A$. So there exists $g \in I \text{End}(E^{(i)} |_{D^i})$ satisfying $(\text{id} - g) \circ N^{(i)}_\alpha = (\text{id} + g) \circ \sigma |_{D^i} \circ \tilde{N}^{(i)}_\alpha \circ \sigma^{-1} |_{D^i}$. In other words, $\sigma |_{D^i} \circ \tilde{N}^{(i)}_\alpha \circ \tau^{(i)}_\alpha = N^{(i)}_\alpha = N^{(i)}_\alpha \circ g - g \circ N^{(i)}_\alpha$. So the second component of $d^1(v_\alpha, (r^{(i)}_\alpha), (\xi^{(i)}_\alpha))$ becomes

$\Theta^{(i)}_{(r^{(i)}_\alpha, \tau^{(i)}_\alpha, \xi^{(i)}_\alpha)} = \Theta^{(i)}_{(\sigma |_{D^i} \circ \tilde{N}^{(i)}_\alpha \circ \sigma^{-1} |_{D^i}, -N^{(i)}_\alpha)} = \Theta^{(i)}_{(N^{(i)}_\alpha \circ g - g \circ N^{(i)}_\alpha)} = 0$.

Thus the element

$\Phi(v) := \{\{u_{\alpha \beta}, 0\}, \{v_\alpha, (r^{(i)}_\alpha), (\xi^{(i)}_\alpha)\}) \in H^1(F^* \otimes I)$

can be defined.

Conversely assume that $u = \{\{u_{\alpha \beta}, 0\}, \{v_\alpha, (r^{(i)}_\alpha), (\xi^{(i)}_\alpha)\}) \in H^1(F^* \otimes I)$ is given. We put $E_\alpha := E^i |_{U_\alpha}$ and define a connection $\nabla_\alpha: E_\alpha \rightarrow E_\alpha \otimes \Omega^1_{C^*_\alpha / A}(D_\alpha)$ by $\nabla_\alpha = \nabla' + v_\alpha$. Furthermore, we put $\theta^{(i)}_\alpha := \theta^{(i)} + r^{(i)}_\alpha + \kappa^{(i)}_\alpha := n^{(i)} + \xi^{(i)}_\alpha$ if $D^{(i)}_\alpha \subset U_\alpha$. We define the isomorphism

$\varphi^{(i)}_\alpha = \text{id} + u_{\alpha \beta}: E_\alpha |_{U_{\alpha \beta}} \cong E_\beta |_{U_{\alpha \beta}}$.

Since $(\{u_{\alpha \beta}, 0\}, \{v_\alpha, (r^{(i)}_\alpha), (\xi^{(i)}_\alpha)\})$ satisfies the cocycle conditions $\nabla \circ u_{\alpha \beta} - u_{\alpha \beta} \circ \nabla = v_\beta - v_\alpha$ and $u_{\beta \alpha} - u_{\gamma \alpha} + u_{\gamma \beta} = 0$, we have the gluing condition

$\varphi^{(i)}_\gamma \circ \varphi^{(i)}_\alpha \circ \varphi^{(i)}_\beta = (\varphi^{(i)}_\alpha \circ \varphi^{(i)}_\beta) \circ \varphi^{(i)}_\gamma$,

So we can patch the local connections $(\{E_\alpha, \nabla_\alpha, (\theta^{(i)}_\alpha, \kappa^{(i)}_\alpha)\})$ together via $\{\varphi^{(i)}_\alpha\}$ and obtain a flat family $(\tilde{E}, \tilde{\nabla}, \{\tilde{\theta}^{(i)}_\alpha, \tilde{\kappa}^{(i)}_\alpha\})$ of factorized $\tilde{\mu} \otimes A$-connections over $A$, which we denote by $\Psi(w)$. By construction the correspondence $H^1(F^* \otimes I) \ni w \mapsto \Psi(w) \in \tilde{E}^{-1}((E', \nabla', \{N^{(i)}_\alpha\}) \otimes A / I)$ gives the inverse of $\Phi$. □

As a corollary of Proposition 2.17 we get the following.
Corollary 2.18. The relative tangent space of the moduli space $M_{C,D}(\nu, \mu)$ over $S$ at $(E, \nabla, \{N^{(i)}\}) \in M_{C,D}(\nu, \mu)$ is isomorphic to $H^1(F^\bullet)$.

2.5. Nondegenerate pairing on the cohomologies. We use the same notations as in subsection 2.4.

If we denote the complex

$$\mathcal{O}_C \xrightarrow{d} \Omega^1_{C/S}(D) \rightarrow \Omega^1_{C/S}(D)|_D,$$

by $L^\bullet$, then there is a canonical quasi-isomorphism $\Omega^\bullet_{C/S} \rightarrow L^\bullet$ and there is an isomorphism

$$H^2(L^\bullet_\omega) \cong H^2(\Omega^\bullet_\omega) \cong \mathbb{C},$$

where $L^\bullet := L^\bullet|_{C_s}$ is the restriction of the complex $L^\bullet$ to the fiber $C_s$. We consider the modified complex $\tilde{L}^\bullet_s : L^\bullet_s \xrightarrow{d^0} L^\bullet_s \oplus Z^1 \xrightarrow{d^1} L^\bullet_s \oplus Z^1,$

defined by

$$\tilde{d}^0(u) = (du, 0), \quad \tilde{d}^1(v, (Q^j)) = \left((v|_{D_s} - Q^j((\nu^{(i)})'(T))), (Q^j)\right),$$

where $(\nu^{(i)})'(T)$ is the derivative of the polynomial $\nu^{(i)}(T)$ in $T$. Then there is a canonical quasi-isomorphism $L^\bullet_s \rightarrow \tilde{L}^\bullet_s$.

We define a morphism of complexes $Tr : F^\bullet \rightarrow \tilde{L}^\bullet_s$ by

$$Tr^0\left(u, (P^{(i)}(T))\right) = Tr(u), \quad Tr^1\left(v, (\tau^{(i)}), (\xi^{(i)})\right) = \left(Tr(v), (\Theta_{\tau^{(i)}}, (\nu^{(i)}(T))\right),$$

Indeed we can check the following commutative diagram:

$$\begin{array}{ccc}
G^0 \oplus Z^0 & \xrightarrow{\tau} & G^1 \oplus S(E|_{D_s}^\oplus, E|_{D_s}^\oplus) \\
\downarrow & & \downarrow \\
\mathcal{O}_{C_s} & \xrightarrow{d} & \Omega^1_{C_s}(D_s) \oplus Z^1
\end{array}$$

For $((\tau^{(i)}), (\xi^{(i)})), ((\tau^{(i)}), (\xi^{(i)})) \in S(E|_{D_s}^\oplus, E|_{D_s}^\oplus) \otimes S(E|_{D_s}^\oplus, E|_{D_s}^\oplus)$, we define $\Xi^{\tau^{(i)}(\xi^{(i)})} \in \Omega^1_{C_s}(D_s)|_{D_s^{(i)}}$ by setting

$$\Xi^{\tau^{(i)}(\xi^{(i)})} = \frac{1}{2} \sum_{l=0}^{r-1} \sum_{j=0}^{r-1} c_j^{(i)} Tr \left(\tau^{(i)} \circ (\xi^{(i)})^l \circ (\xi^{(i)})^{j-l} \right) \frac{dz^{(i)}}{\bar{z}_1^{(i)}} \cdots \frac{dz^{(i)}}{\bar{z}_m^{(i)}} - \frac{1}{2} \sum_{l=0}^{r-1} \sum_{j=0}^{r-1} c_j^{(i)} Tr \left(\tau^{(i)} \circ (\xi^{(i)})^l \circ (\xi^{(i)})^{j-l} \right) \frac{dz^{(i)}}{\bar{z}_1^{(i)}} \cdots \frac{dz^{(i)}}{\bar{z}_m^{(i)}}.$$

Remark 2.19. In the extreme case when $\mu^{(i)}_k = \nu^{(i)}(\mu_k^{(i)})$ for any $k$, we have $c_1^{(i)} = 1$ and $c_j^{(i)} = 0$ for $j \neq 1$. So we have

$$\Xi^{\tau^{(i)}(\xi^{(i)})} = \frac{1}{2} Tr \left(\tau^{(i)} \circ (\xi^{(i)}) - (\xi^{(i)}) \circ (\tau^{(i)}) \right) \frac{dz^{(i)}}{\bar{z}_1^{(i)}} \cdots \frac{dz^{(i)}}{\bar{z}_m^{(i)}}$$

which is almost the same form as the expression in subsection 1.2 (3) of the Kirillov-Kostant form in Proposition 1.15.

We define a bilinear pairing

$$\omega_{(E, \nabla, \{N^{(i)}\})} : H^1(F^\bullet) \times H^1(F^\bullet) \rightarrow H^2(L^\bullet) \cong \mathbb{C}$$

on $H^1(F^\bullet)$ by setting

$$\omega_{(E, \nabla, \{N^{(i)}\})}(\{(u_{\alpha \beta}, 0)\}, \{(v_{\alpha}, (\tau^{(i)}_\alpha, \xi^{(i)}_\alpha))\}) = \left(\{\tau_{\alpha \beta} \circ u_{\alpha \beta} \circ \theta_\gamma\right), \{-\tau_{\alpha \beta} \circ u_{\alpha \beta} \circ \theta_\gamma\}, \left(\Xi^{(\tau^{(i)}_\alpha, (\xi^{(i)}_\alpha))}\right)) \in H^2(L^\bullet).$$

We will check that the cohomology class $\omega_{(E, \nabla, \{N^{(i)}\})}$ in $H^2(L^\bullet)$ is independent of the choice of the representatives $\{(u_{\alpha \beta}, 0)\}, \{(v_{\alpha}, (\tau^{(i)}_\alpha, \xi^{(i)}_\alpha))\}$ and $\{(u_{\alpha \beta}, 0)\}, \{(v_{\alpha}, (\tau^{(i)}_\alpha, \xi^{(i)}_\alpha))\}$, respectively. Indeed assume
that $\{(u_{\alpha\beta}, 0)\}, \{(u_{\alpha}, ((\tau_{\alpha}^{(i)}), (\xi_{\alpha}^{(i)})))\}] = 0$ in $H^{1}(F^{*})$. Then there is $\{u_{\alpha}, (P_{\alpha}^{(i)}(T))\} \in C^{0}(\{U_{\alpha}\}, \mathcal{G}^{0} + \mathcal{Z}^{0})$ satisfying

$$
\begin{align*}
\tau^{(i)}_{\alpha} &= \sigma_{\theta_{\alpha}^{(i)}}^{(-)}(u_{\alpha}|_{D^{i}(\alpha)}, \bar{P}(T)_{\alpha}) = -(u_{\alpha}|_{D^{i}(\alpha)} \circ \theta^{(i)} + \theta^{(i)} \circ t_{u_{\alpha}|_{D^{i}(\alpha)}} + \theta^{(i)} \circ P(\tau^{(i)} N^{i}))
\xi^{(i)}_{\alpha} &= \sigma_{\kappa_{\alpha}}^{(+)}(u_{\alpha}|_{D^{i}(\alpha)}, \bar{P}(T)) = \kappa^{(i)} \circ u_{\alpha}|_{D^{i}(\alpha)} + t_{u_{\alpha}|_{D^{i}(\alpha)}} \circ \kappa^{(i)} - P(\tau^{(i)} N^{i}) \circ \kappa^{(i)}.
\end{align*}
$$

So we can write

$$
\omega_{E, \nabla, (N^{i})} \left( \left[ \{(u_{\alpha\beta}, 0)\}, \{(u_{\alpha}, ((\tau_{\alpha}^{(i)}), (\xi_{\alpha}^{(i)})))\}] \right), \left[ \{(u_{\alpha\beta}', 0)\}, \{(u_{\alpha}', ((\tau_{\alpha}'^{(i)}), (\xi_{\alpha}'^{(i)})))\}] \right) = \left\{ \begin{array}{l}
\{ \text{Tr}((u_{\beta} - u_{\alpha}) \circ u_{\alpha}^{\beta_{\gamma}}) \}, - \{ \text{Tr}((u_{\beta} - u_{\alpha}) \circ v_{\beta}^{\gamma} - (\nabla \circ u_{\alpha} - u_{\alpha} \circ \nabla) \circ u_{\alpha}^{\beta_{\gamma}}) \}, \\
\left\{ \left( \sigma_{\kappa_{\alpha}}^{(+)}(u_{\alpha}|_{D^{i}(\alpha)}, \bar{P}(T)), \sigma_{\kappa_{\alpha}}^{(+)}(u_{\alpha}|_{D^{i}(\alpha)}, \bar{P}(T)) \right) \right\}.
\end{array} \right.
$$

If we put $c_{\alpha\beta} := \text{Tr}(u_{\alpha} \circ u_{\alpha}^{\beta_{\gamma}})$, then $\{c_{\alpha\beta}\} \in C^{1}(\{U_{\alpha}\}, L^{0})$ and

$$
\{ \text{Tr}((u_{\beta} - u_{\alpha}) \circ u_{\alpha}^{\beta_{\gamma}}) \} = \{ \text{Tr}(u_{\beta} \circ u_{\alpha}^{\beta_{\gamma}} - u_{\alpha} \circ (u_{\alpha}^{\beta_{\gamma}} - u_{\alpha}^{\beta_{\gamma}})) \} = \{ c_{\alpha\gamma} + c_{\alpha\beta} \}.
$$

If we put $b_{\alpha} := \text{Tr}(u_{\alpha} \circ v_{\alpha}^{\beta_{\gamma}})$, then $\{b_{\alpha}\} \in C^{0}(\{U_{\alpha}\}, L^{0})$ and we have

$$
\begin{align*}
d^{\star}_{E^{\star}}(\{c_{\alpha\beta}\}) &= \{ d \text{Tr}(u_{\alpha} \circ u_{\alpha}^{\beta_{\gamma}}) \} = \{ \text{Tr}(\nabla \circ u_{\alpha} \circ u_{\alpha}^{\beta_{\gamma}} - u_{\alpha} \circ u_{\alpha}^{\beta_{\gamma}} \circ \nabla) \} \\
&= \{ \text{Tr}((\nabla \circ u_{\alpha} - u_{\alpha} \circ \nabla) \circ u_{\alpha}^{\beta_{\gamma}} + u_{\alpha} \circ (\nabla \circ u_{\alpha}^{\beta_{\gamma}} \circ \nabla)) \} \\
&= \{ \text{Tr}((\nabla \circ u_{\alpha} - u_{\alpha} \circ \nabla) \circ u_{\alpha}^{\beta_{\gamma}} + u_{\alpha} \circ (v_{\beta_{\gamma}} - v_{\alpha}) \} \\
&= \{ \text{Tr}((\nabla \circ u_{\alpha} - u_{\alpha} \circ \nabla) \circ u_{\alpha}^{\beta_{\gamma}} + (u_{\alpha} - u_{\beta}) \circ v_{\beta_{\gamma}} + (u_{\beta} \circ v_{\beta_{\gamma}} - u_{\alpha} \circ v_{\alpha}) \} \\
&= - \{ \text{Tr}((u_{\beta} - u_{\alpha}) \circ v_{\beta_{\gamma}} - (\nabla \circ u_{\alpha} - u_{\alpha} \circ \nabla) \circ u_{\alpha}^{\beta_{\gamma}} \} + \{ b_{\beta} - b_{\alpha} \}.
\end{align*}
$$

Since $\text{Tr}((\tau^{(i)}_{\alpha} \circ \kappa^{(i)} + \theta^{(i)} \circ \xi^{(i)}_{\alpha}) \circ (N^{i})^{l} \circ P(\tau^{(i)}(N^{i})) \circ (N^{i})^{j-1-l} = 0$ follows from $\Theta_{\tau^{(i)}(N^{i}) + \theta^{(i)}(N^{i})} = 0,$

$$
\begin{align*}
\text{Tr}((\tau^{(i)}_{\alpha} \circ (\tau^{(i)}(N^{i}))^{l} \circ \sigma_{\kappa^{(i)}}^{(i)}(u_{\alpha}|_{D^{i}(\alpha)}, \bar{P}(T)) \circ (N^{i})^{j-1-l}) \\
- \text{Tr}(\sigma_{\theta_{\alpha}^{(i)}}^{(i)}(u_{\alpha}|_{D^{i}(\alpha)}, \bar{P}(T)) \circ (\tau^{(i)}(N^{i}))^{l} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{j-1-l}) \\
= \text{Tr}(\tau^{(i)}_{\alpha} \circ (\tau^{(i)}(N^{i}))^{l} \circ \kappa^{(i)} \circ u_{\alpha}|_{D^{i}(\alpha)} + t_{u_{\alpha}|_{D^{i}(\alpha)}} \circ \kappa^{(i)} - P(\tau^{(i)}(N^{i})) \circ \kappa^{(i)} \circ (N^{i})^{j-1-l}) \\
- \text{Tr}(u_{\alpha}|_{D^{i}(\alpha)} \circ \theta^{(i)} - \theta^{(i)} \circ t_{u_{\alpha}|_{D^{i}(\alpha)}} + \theta^{(i)} \circ P(\tau^{(i)}(N^{i}))) \circ (\tau^{(i)}(N^{i}))^{l} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{j-1-l}) \\
= \text{Tr}(\tau^{(i)}_{\alpha} \circ (\tau^{(i)}(N^{i}))^{l} \circ \kappa^{(i)} \circ u_{\alpha}|_{D^{i}(\alpha)} \circ (N^{i})^{j-1-l} + (\tau^{(i)}(N^{i}))^{j-1-l} \circ \kappa^{(i)} \circ u_{\alpha}|_{D^{i}(\alpha)} \circ (N^{i})^{l} \circ \tau^{(i)}_{\alpha} \circ (N^{i})^{j-1-l}) \\
+ \text{Tr}(u_{\alpha}|_{D^{i}(\alpha)} \circ \theta^{(i)} \circ (\tau^{(i)}(N^{i}))^{l} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{j-1-l} + (\tau^{(i)}(N^{i}))^{j-1-l} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{l} \circ u_{\alpha}|_{D^{i}(\alpha)} \circ \theta^{(i)} \circ (N^{i})^{j-1-l}) \\
- \text{Tr}(\tau^{(i)}_{\alpha} \circ \kappa^{(i)} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{j-1-l} \circ u_{\alpha}|_{D^{i}(\alpha)} \circ (N^{i})^{j-1-l} + u_{\alpha}|_{D^{i}(\alpha)} \circ (N^{i})^{j-1-l} \circ \theta^{(i)} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{j-1-l}) \\
+ \text{Tr}(u_{\alpha}|_{D^{i}(\alpha)} \circ (N^{i})^{j-1-l} \circ \theta^{(i)} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{j-1-l} + u_{\alpha}|_{D^{i}(\alpha)} \circ (N^{i})^{j-1-l} \circ \theta^{(i)} \circ \xi^{(i)}_{\alpha} \circ (N^{i})^{j-1-l}).
\end{align*}
$$
So we have
\[
\sum_{\gamma}^{(\sigma)} \sigma_{\gamma}^{(\pi)} \left( u_{\alpha_{\gamma}} \right)_{D_{\alpha}^{(\pi)}(T)} \sigma_{\gamma}^{(\pi)} \left( u_{\alpha_{\gamma}} \right)_{D_{\alpha}^{(\pi)}(T)}
\]
\[
= \frac{1}{2} \sum_{j=1}^{r-1} \sum_{i=0}^{j-1} c_{j}^{(i)} \operatorname{Tr} \left( \tau_{\alpha}^{(i)} \circ \left( t N^{(i)} \right)^{t} \circ \sigma_{\gamma}^{(\pi)} \left( u_{\alpha_{\gamma}} \right)_{D_{\alpha}^{(\pi)}(T)} \circ \left( N^{(i)} \right)^{j-1-t} \right)
- \sigma_{\theta}^{(\pi)} \left( u_{\alpha} \right)_{D_{\alpha}^{(\pi)}(T)} \circ \left( t N^{(i)} \right)^{t} \circ \xi_{\alpha}^{(\pi)} \circ \left( N^{(i)} \right)^{j-1-t} \right)
\]
\[
= \sum_{j=1}^{r-1} \sum_{i=0}^{j-1} c_{j}^{(i)} \operatorname{Tr} \left( u_{\alpha} \right)_{D_{\alpha}^{(\pi)}} \circ \left( N^{(i)} \right)^{t} \circ \left( t N^{(i)} \right)^{t} \circ \theta \circ \xi_{\alpha}^{(\pi)} \circ \left( N^{(i)} \right)^{j-1-t} \right)
\]
\[
= \operatorname{Tr} \left( u_{\alpha} \right)_{D_{\alpha}^{(\pi)}} \circ \left( \frac{d\xi^{(i)}}{z_{1}^{(i)} z_{2}^{(i)} \ldots z_{m_{i}^{(i)}}} \right)
\]
Since \( v_{\alpha}^{(i)} \in D_{\alpha}^{(\pi)} \) is determined by \( \mu_{N}^{(\gamma)}(\tau_{\alpha}^{(i)} \circ \kappa_{\gamma} + \theta \circ \xi_{\alpha}^{(i)}) \), we have
\[(26) \quad d_{Z}^{(i)} \left( \left( b_{\alpha} \right) \right) = \left\{ \left( \operatorname{Tr} \left( u_{\alpha} \circ v_{\alpha}^{(i)} \right) \right) \right\} = \left\{ \left( \operatorname{Tr} \left( u_{\alpha} \circ \nu \left( \tau_{\alpha}^{(i)} \circ \kappa_{\gamma} + \theta \circ \xi_{\alpha}^{(i)} \right) \right) \right\}
\]
The equalities (24), (25) and (26) mean that the cohomology class (23) is represented as the coboundary of \( \left( \{\xi_{\alpha}^{(i)} \}, \{b_{\alpha} \} \right) \in C^{0}(U_{1} \times C^{0}(L^{*}) \), which should be zero in \( H^{2}(L^{*}) \). Similarly (22) becomes zero when \( \left( \{u_{\alpha_{\gamma}}^{(i)} \}, \{v_{\alpha}^{(i)}, (\tau_{\alpha}^{(i)}), (\xi_{\alpha}^{(i)}) \} \right) = 0 \) in \( H^{1}(F^{\ast}) \). Thus we have proved that the bilinear pairing \( \omega_{\gamma} \) is well-defined.

**Lemma 2.20.** The bilinear pairing \( \omega_{\gamma} : H^{1}(F^{*}) \times H^{1}(F^{*}) \rightarrow H^{2}(L^{*}) \cong \mathbb{C} \) defined in (22) is a non-degenerate pairing.

**Proof.** Let \( \sigma : H^{1}(F^{*}) \rightarrow H^{1}(F)^{\gamma} \) be the morphism determined by the pairing \( \omega_{\gamma} \). We have to show that \( \sigma \) is an isomorphism. We can see that \( \sigma \) induces the following exact commutative diagram

\[
\begin{array}{cccccc}
H^{0}(F_{0}^{*}) & \longrightarrow & H^{0}(F_{0}^{*}) & \longrightarrow & H^{1}(F^{*}) & \longrightarrow & H^{1}(F_{0}^{*}) \\
\sigma_{1} \downarrow & & \sigma_{2} \downarrow & & \sigma \downarrow & & \sigma_{3} \downarrow \\
H^{1}(F_{1}^{*})^{\gamma} & \longrightarrow & H^{1}(F_{0}^{*})^{\gamma} & \longrightarrow & H^{0}(F^{*})^{\gamma} & \longrightarrow & H^{0}(F_{1}^{*})^{\gamma} \\
\end{array}
\]

Here \( \sigma_{2} : H^{0}(F_{1}^{*}) \rightarrow H^{1}(F_{0}^{*})^{\gamma} \) and \( \sigma_{3} : H^{1}(F_{1}^{*}) \rightarrow H^{0}(F_{0}^{*})^{\gamma} \) are given by the pairing

\[
\left( \{u_{\alpha}, (\xi_{\alpha}^{(i)})) \right), \{u_{\alpha}^{(i)}, (\tau_{\alpha}^{(i)}) \} \right) \rightarrow \left\{ \operatorname{Tr} (u_{\alpha} \circ u_{\alpha}^{(i)}) \right\} \right) \rightarrow \left\{ \left( \sum_{\gamma}^{(\sigma)} \sigma_{\gamma}^{(\pi)} \left( u_{\alpha_{\gamma}} \right)_{D_{\alpha}^{(\pi)}(T)} \right) \right\}
\]
and \( \sigma_{1} : H^{0}(F_{0}^{*}) \rightarrow H^{1}(F_{0}^{*})^{\gamma} \) and \( \sigma_{4} : H^{1}(F_{1}^{*}) \rightarrow H^{0}(F_{0}^{*})^{\gamma} \) are defined by the pairing

\[
\left( \{u_{\alpha}, (\xi_{\alpha}^{(i)})) \right), \{u_{\alpha}^{(i)}, (\tau_{\alpha}^{(i)}) \} \right) \rightarrow \left\{ - \operatorname{Tr} (u_{\alpha} \circ u_{\alpha}^{(i)}) \right\} \right) \rightarrow \left\{ \left( \sum_{\gamma}^{(\sigma)} \sigma_{\gamma}^{(\pi)} \left( u_{\alpha_{\gamma}} \right)_{D_{\alpha}^{(\pi)}(T)} \right) \right\}
\]
We denote the short exact sequence of complexes

\[
\begin{array}{cccccccc}
0 & \longrightarrow & G^{1} & \longrightarrow & G^{1} \oplus S(E|_{D_{\gamma}}, E_{D_{\gamma}}) & \longrightarrow & S(E|_{D_{\gamma}}, E_{D_{\gamma}}) & \longrightarrow & 0 \\
0 & \longrightarrow & G^{1} & \longrightarrow & G^{1} \oplus Z^{1} & \longrightarrow & Z^{1} & \longrightarrow & 0 \\
\end{array}
\]
simply by $0 \to [G^1 \to G^1] \to \mathcal{F}_1^* \to [S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1] \to 0$ and denote the short exact sequence of complexes

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Z^0 & \longrightarrow & \mathcal{G}^0 \oplus Z^0 & \longrightarrow & \mathcal{G}^0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) & \longrightarrow & S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

simply by $0 \to [Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s})] \to \mathcal{F}_0^* \to \mathcal{G}^0 \to 0$. These short exact sequences of complexes induce the exact commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\ker(G^1 \to G^1)) & \longrightarrow & \mathbf{H}^0(\mathcal{F}_1^*) & \longrightarrow & \ker(S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1) & \longrightarrow & H^1(\ker(G^1 \to G^1)) \\
\eta_1 \downarrow & & \sigma_2 \downarrow & & \eta_2 \downarrow & & \eta_3 \downarrow & & \\
0 & \longrightarrow & H^1(\mathcal{G}^0) & \longrightarrow & \mathbf{H}^1(\mathcal{F}_1^*) & \longrightarrow & \cok(Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s})) & \longrightarrow & H^0(\mathcal{G}^0)^\vee.
\end{array}
\]

Here $\eta_1$ and $\eta_3$ are induced by the trace pairing

\[
\mathcal{G}^0 \otimes \ker(G^1 \to G^1) \ni u \otimes v \mapsto \text{Tr}(u \otimes v) \in \Omega_{\mathcal{C}}^4
\]

and the isomorphism $H^1(\Omega_{\mathcal{C}}^4) \cong H^2(\mathcal{L}_s^*) \cong \mathbb{C}$. Since the above trace pairing induces the isomorphism $\ker(G^1 \to G^1) \cong (\mathcal{G}^0)^\vee \otimes \Omega_{\mathcal{C}}^4$, $\eta_1$, $\eta_3$ are the isomorphisms induced by this isomorphism and the Serre duality. The homomorphism $\eta_2$ is induced by the pairing

\[(27) \quad \ker(S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1) \times \cok(Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s})) \to \mathbf{H}^2(\mathcal{L}_s^*) \cong \mathbb{C}
\]

\[
((\xi^{(i)}), (\tau^{(i)})) \mapsto \left\{ \left( \Xi^{(0, \xi^{(i)})}_{(\tau^{(i)}, 0)} \right) \right\}.
\]

Note that $\left[ \left( \Xi^{(0, \xi^{(i)})}_{(\tau^{(i)}, 0)} \right) \right] \in \mathbf{H}^2(\mathcal{L}_s^*)$ corresponds to

\[
\frac{1}{2} \sum_{i=1}^n \text{res}_{p \in \mathcal{D}_s(i)} \left( \sum_{j=1}^{r-i-j} c_{ij} \text{Tr} \left( \tau^{(i)} \circ (N^{(i)})^{j-1} \circ (\xi^{(i)}) \circ (N^{(i)})^{j-1-l} \right) \right)
\]

via the isomorphism $\mathbf{H}^2(\mathcal{L}_s^*) \cong \mathbb{C}$. Let us consider the restriction to each $p \in \mathcal{D}_s$ of the pairing

\[(28) \quad \ker(S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1) \times \cok(Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s})) \to \mathcal{O}_{\mathcal{D}_s}
\]

\[
((\xi^{(i)}), (\tau^{(i)})) \mapsto \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{r-i-j} c_{ij} \text{Tr} \left( \tau^{(i)} \circ (N^{(i)})^{j-1} \circ (\xi^{(i)}) \circ (N^{(i)})^{j-1-l} \right).
\]

Assume that $((\xi^{(i)}) \in \ker(S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1)_p$ satisfies

\[
\sum_{i=1}^n \sum_{j=1}^{r-i-j} c_{ij} \text{Tr} \left( \tau^{(i)} \circ (N^{(i)})^{j-1} \circ (\xi^{(i)}) \circ (N^{(i)})^{j-1-l} \right) = 0
\]

for any $(\tau^{(i)}) \in \cok(Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s})_p$. Since the usual trace pairing is nondegenerate, we have $\sum_{j=1}^{r-i-j} c_{ij} \tau^{(i)} \circ (N^{(i)})^{j-1} \circ (\xi^{(i)}) \circ (N^{(i)})^{j-1-l} = 0$. Recall that $\Theta_{g \circ \xi^{(i)}}^{(i)} = 0$ by the choice of $(\xi^{(i)})$, which is equivalent to the existence of some $g \in \text{End}(E|_p)$ satisfying $\Theta_{g \circ \xi^{(i)}}^{(i)} = N^{(i)} \circ g - g \circ N^{(i)}$. So we have $\sum_{j=1}^{r-i-j} c_{ij} \tau^{(i)} \circ (N^{(i)})^{j-1} \circ g \circ (N^{(i)})^{j} = 0$, which means $\nu^{(i)}(N^{(i)} \circ g - g \circ N^{(i)})$. Since $\nu^{(i)}$ satisfies Assumption 2.2, we have $N^{(i)} \circ g = g \circ N^{(i)}$ and $\xi^{(i)} = 0$. Thus the pairing $(28)$ is nondegenerate because $\text{rank}_{\mathcal{O}_p} \ker(S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1) = \frac{r(2r-1)}{2} = \text{rank}_{\mathcal{O}_p} \cok(Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}))$. So the pairing $(28)$ becomes a nondegenerate pairing of vector spaces over $\mathbb{C}$ and $\eta_2$ becomes isomorphic. Thus the homomorphism $\sigma_2 : \mathbf{H}^0(\mathcal{F}_1^*) \cong \mathbf{H}^1(\mathcal{F}_1^*)$ becomes an isomorphism by the five lemma. The homomorphism $\sigma_3 : \mathbf{H}^1(\mathcal{F}_1^*) \mapsto \mathbf{H}^0(\mathcal{F}_1^*)$ is isomorphic because it is the dual of $\sigma_2$. On the other hand, we have the exact commutative diagram

\[
\begin{array}{ccccccccc}
\text{ker}(Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s})) & \longrightarrow & \mathbf{H}^0(\mathcal{F}_1^*) & \longrightarrow & H^0(G^0) & \longrightarrow & \cok(Z^0 \to S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s})) \\
\downarrow & & \sigma_1 \downarrow & & \eta_4 \downarrow & & ^\sim \eta_2 \downarrow \\
\cok(S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1)^\vee & \longrightarrow & \mathbf{H}^1(\mathcal{F}_1^*) & \longrightarrow & H^1(\ker(G^1 \to G^1))^\vee & \longrightarrow & \ker(S(E|_{\mathcal{D}_s}, E|_{\mathcal{D}_s}) \to Z^1)^\vee.
\end{array}
\]
Note that ker\((Z^0 \to S(E|_{D_1}, E|_{D_2}))\) = 0 and coker\((S(E|_{D_1}, E|_{D_2}) \to Z^1)\) = 0. The homomorphism \(\eta_4\) is isomorphic since it is induced by the isomorphism ker\((G^1 \to G^1)\) \(\otimes \Omega^1_{D_3} \cong G^0\) and the Serre duality. Thus the homomorphism \(\sigma_1\) is an isomorphism. The homomorphism \(\sigma_4: H^1(F^*_f) \to H^0(F^*_f)^\vee\) is isomorphic, because it is the dual of \(\sigma_1\).

From all the above arguments, the homomorphism \(\sigma: H^1(F^*_f) \to H^1(F^*_f)^\vee\) is isomorphic by the five lemma, because \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) are all isomorphic.

**Lemma 2.21.** \(H^2(\text{Tr}): H^2(F^*_f) \to H^2(\tilde{L}_f^*) \cong \mathbb{C}\) is an isomorphism.

**Proof.** From the proof of Lemma 2.20, the exact commutative diagram

\[
\begin{array}{cccc}
H^1(F^*_f) & \longrightarrow & H^1(F^*_f) & \longrightarrow & H^2(F^*_f) & \longrightarrow & 0 \\
\sigma_3 & \downarrow & \sigma_4 & \downarrow & \sigma_5 & \downarrow & \\
H^0(F^*_f)^\vee & \longrightarrow & H^0(F^*_f)^\vee & \longrightarrow & H^0(F^*_f)^\vee & \longrightarrow & 0
\end{array}
\]

is induced and \(\sigma_5: H^2(F^*_f) \to H^0(F^*_f)^\vee\) is an isomorphism because \(\sigma_3\) and \(\sigma_4\) are isomorphic. Note that \(H^0(F^*_f) = \mathbb{C}\) because \((E, \nabla, \{N^{(i)}\})\) is \(\alpha\)-stable whose endomorphisms are only scalar multiplications. We can see from the construction that the composition

\[
H^2(F^*_f) \xrightarrow{\sigma_5} H^0(F^*_f)^\vee \xrightarrow{\sim} H^0(\tilde{L}_f^*)^\vee \xrightarrow{\sim} H^2(\tilde{L}_f^*)
\]

coinsides with \(H^2(\text{Tr})\) and the result follows. \(\square\)

**Corollary 2.22.** The dimension of the relative tangent space of \(M^a_{\mathbb{C}, D}(\tilde{\nu}, \tilde{\mu})\) over \(S\) at \((E, \nabla, \{N^{(i)}\})\) is given by

\[
\dim H^1(F^*_f) = 2r^2(g - 1) + 2 + r(r - 1) \sum_{i=1}^n m_i.
\]

**Proof.** Since we will prove the smoothness of the moduli space \(M^a_{\mathbb{C}, D}(\tilde{\nu}, \tilde{\mu})\) over \(S\) in Proposition 2.25, we can deduce the corollary from [12 Theorem 2.1] and [14 Theorem 2.2]. We give here a direct proof using the proof of Lemma 2.20. Since \(H^0(F^*_f) \cong \mathbb{C}\) and \(H^2(F^*_f) \cong \mathbb{C}\), the exact sequence (20) becomes

\[
0 \longrightarrow \mathbb{C} \longrightarrow H^0(F^*_f) \longrightarrow H^0(F^*_f) \longrightarrow H^1(F^*_f) \longrightarrow H^1(F^*_f) \longrightarrow \mathbb{C} \longrightarrow 0.
\]

Since \(H^0(F^*_f) \cong H^0(F^*_f)^\vee\) and \(H^1(F^*_f) \cong H^1(F^*_f)^\vee\) by the proof of Lemma 2.20, we have

\[
\dim H^1(F^*_f) = \dim H^0(F^*_f)^\vee + \dim H^0(F^*_f) - \dim H^0(F^*_f) - \dim H^1(F^*_f) + \dim \mathbb{C} + \dim \mathbb{C}
\]

\[
= 2 \dim H^1(F^*_f) - 2H^0(F^*_f) + 2
\]

\[
= -2\chi(F^*_f) + 2
\]

Using the Riemann-Roch formula, we can see

\[
\chi(F^*_f) = \chi(G^0) + \text{length } Z^0 - \text{length } S(E|_{D_1}, E|_{D_2}))
\]

\[
= r^2(1 - g) + \sum_{i=1}^n rm_i - \sum_{i=1}^n \frac{r(r + 1)}{2} m_i.
\]

Substituting this in (29) we get the corollary. \(\square\)

### 2.6 Smoothness of the moduli space of \((\tilde{\nu}, \tilde{\mu})\)-connections

We use the same notations as in subsection 2.4 and subsection 2.5.

**Proposition 2.23.** Let \(A\) be an artinian local ring over \(S\) with the maximal ideal \(m\) and \(I\) be an ideal of \(A\) satisfying \(mI = 0\) and \(A/m = \mathbb{C}\). Let \((E', \nabla', \{N^{(i)}\})\) be a flat family of \((\tilde{\nu}, \tilde{\mu}) \otimes A/I\)-connections on \((C_{A/I}, D_{A/I})\) over \(A/I\) such that \((E', \nabla', \{N^{(i)}\}) \otimes A/m \cong (E, \nabla, \{N^{(i)}\})\). Then there is an obstruction class \(\alpha(E', \nabla', \{N^{(i)}\}) \in H^2(F^*_f) \otimes I\) whose vanishing is equivalent to the existence of a lift of \((E', \nabla', \{N^{(i)}\})\) to a flat family of \((\tilde{\nu}, \tilde{\mu}) \otimes A\)-connections on \((C_{A/I}, D_{A/I})\) over \(A/I\).

**Proof.** We can define the \(O_{D_{A/I}}\) \([T]\)-module structures on \(E'|_{D_{A/I}}\) and on \(E'^{\vee}|_{D_{A/I}}\) by \(N^{(i)}\) and \(N'^{\vee}\), respectively. Then we can take an \(O_{D_{A/I}}\) \([T]\)-isomorphism \(\theta^{(i)}: E'^{\vee}|_{D_{A/I}} \cong E'^{\vee}|_{D_{A/I}}\) which is a lift of \(\theta^{(i)}\). If we put \(\kappa^{(i)} := (\theta^{(i)})^{-1} \circ N^{(i)}: E'|_{D_{A/I}} \to E'^{\vee}|_{D_{A/I}}\), then \((E', \nabla', \{\theta^{(i)}, \kappa^{(i)}\})\) is a flat family of factorized \((\tilde{\nu}, \tilde{\mu}) \otimes A/I\)-connections on \((C_{A/I}, D_{A/I})\) over \(A/I\).
We can take an affine open covering \(C_A = \bigcup U_\alpha\) such that \(\mathbb{Z}\{i | D_A^{(i)} \cap U_\alpha \neq \emptyset\} \leq 1\) for any \(\alpha\) and \(\mathbb{Z}\{i | D_A^{(i)} \subset U_\alpha\} = 1\) for any \(i\). Furthermore, we may assume that \(E'|_{\mathbb{Z}\{U_\alpha \otimes A/I\}} \cong \mathbb{O}^{\oplus r}_{\mathbb{Z}\{U_\alpha \otimes A/I\}}\). Take a free \(\mathbb{O}_{U_\alpha}\)-module \(E_\alpha\) with an isomorphism \(\psi_\alpha: E_\alpha \otimes A/I \xrightarrow{\cong} E'|_{\mathbb{Z}\{U_\alpha \otimes A/I\}}\) and a lift \(\sigma_{\beta \alpha}: E_\alpha|_{U_\alpha \otimes A/I} \xrightarrow{\cong} E_\beta|_{U_\alpha \otimes A/I}\) of the composite \(\psi_\beta^{-1} \circ \psi_\alpha: E_\alpha|_{U_\alpha \otimes A/I} \xrightarrow{\psi_\beta^{-1}} E'|_{\mathbb{Z}\{U_\alpha \otimes A/I\}} \xrightarrow{\psi_\beta} E_\beta|_{U_\alpha \otimes A/I}\).

If we write \(\varphi^{(i)}_{\mu \otimes A}(T) = T^r + b_{r-1}T^{r-1} + \cdots + b_1T + b_0\) with \(b_i \in \mathbb{O}_{D_A^{(i)}}\) and define matrices \(N, \Phi_1, \Phi_2\) by
\[
N = \begin{pmatrix}
-b_{r-1} & 1 & 0 & \cdots & 0 \\
-b_{r-2} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-b_1 & 0 & \cdots & 0 & 1 \\
-b_0 & 0 & \cdots & \cdots & 0 \\
\end{pmatrix},
\]
\[
\Phi_1 = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & b_{r-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 1 & b_{r-1} & \cdots & b_2 \\
1 & b_{r-2} & \cdots & b_1 \\
\end{pmatrix},
\]
\[
\Phi_2 = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 1 & b_{r-1} & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & b_{r-1} & \cdots & b_3 & 0 \\
1 & b_{r-2} & \cdots & b_2 & 0 \\
0 & 0 & \cdots & 0 & 0 & -b_0 \\
\end{pmatrix},
\]
then \(t\Phi_1 = \Phi_1\), \(t\Phi_2 = \Phi_2\) and \(\Phi_1\) and \(\Phi_2\) is invertible. We can check \(N\Phi_1 = \Phi_2\), which is equivalent to \(N = \Phi_2\Phi_1^{-1}\). So there is a matrix factorization
\[
tN = \Phi_1^{-1}\Phi_2: (\mathbb{O}^{\oplus r}_{D_A^{(i)}}) \xrightarrow{\psi^{-1}} \mathbb{O}^{\oplus r}_{D_A^{(i)}}.
\]
After replacing the representative \(((\theta^{(i)}), (\kappa^{(i)})\)) by the action of an element of \((\mathbb{O}^{\oplus r}_{D_A^{(i)}}) \xrightarrow{\psi^{-1}} \mathbb{O}^{\oplus r}_{D_A^{(i)}}\), we may assume that there is an isomorphism \(g: \mathbb{O}^{\oplus r}_{D_A^{(i)}} \xrightarrow{\psi^{-1}} \mathbb{O}^{\oplus r}_{D_A^{(i)}}\) satisfying \(g\circ (\Phi_1^{-1}\otimes A/I)\circ t\).

If we put \(\theta^{(i)} = \gamma\), \(\kappa^{(i)} = \gamma\), \(\theta^{(i)}\), \(\kappa^{(i)}\) becomes a lift of \((\theta^{(i)}), (\kappa^{(i)})\) and \(N^{(i)}\) becomes a lift of \(N^{(i)}\). We can take an \(A\)-relative local connection \(\nabla\alpha: E_\alpha \xrightarrow{\psi_\alpha^{-1}} \nabla^{(i)}_{\mathbb{Z}\{\sigma_{\beta \alpha}\}}(D_A)\) satisfying \(\nabla^{(i)}(N^{(i)}(\tau^{(i)})) = \frac{d\tau^{(i)}_{\zeta_1}}{\zeta_1} \cdots \frac{d\tau^{(i)}_{\zeta_{m_i}}}{\zeta_{m_i}} = \nabla^{(i)}_{\mathbb{Z}\{\sigma_{\beta \alpha}\}}\) and \(\nabla_{\psi^{-1}} : N^{(i)}\).

If we put \(u_{\alpha \beta} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \sigma_{\alpha \beta} - \text{id}_{E_\alpha}) \circ \psi_\alpha^{-1}, \quad v_{\alpha \beta} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \nabla \circ \sigma_{\alpha \beta} - \nabla_{\alpha}) \circ \psi_\alpha^{-1}, \quad u_{\beta \alpha} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \nabla \circ \sigma_{\alpha \beta} - \nabla_{\alpha}) \circ \psi_\alpha^{-1}, \quad v_{\beta \alpha} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \nabla \circ \sigma_{\alpha \beta} - \nabla_{\alpha}) \circ \psi_\alpha^{-1}, \quad u_{\beta \gamma} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \sigma_{\alpha \beta} - \text{id}_{E_\alpha})\),

we have
\[
v_{\beta \gamma} - v_{\alpha \gamma} + u_{\alpha \beta} = \nabla^{(i)}(u_{\alpha \beta} - u_{\alpha \beta} - u_{\alpha \beta} - u_{\alpha \beta} - u_{\alpha \beta} = 0
\]
and we can define an element
\[
o(E', \nabla', \{N^{(i)}\}) := ((u_{\alpha \beta}, 0), (u_{\alpha \beta}, (0, 0)), (0, 0)) \in \mathbb{H}^2(F^*) \otimes I.
\]
Assume that \(o(E', \nabla', \{N^{(i)}\}) = 0\). Then there are
\[
\{a_{\beta \gamma}\} \in \mathbb{Z}\{U_\alpha\}, \mathbb{Z}^0\),
\[
\{b_{\alpha \tau}(\tau^{(i)}), (\xi^{(i)})\} \in \mathbb{Z}\{U_\alpha\}, \mathbb{Z}^0(\{U_\alpha\}, \mathbb{C}^0(S(E'|_{D_A}, E|_{D_A}) \oplus S(E|_{D_A}, E|_{D_A})), \mathbb{C}^0(\{U_\alpha\}, \mathbb{C}^0(S(E'|_{D_A}, E|_{D_A}) \oplus S(E|_{D_A}, E|_{D_A})))
\]
satisfying
\[
u_{\beta \alpha} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \sigma_{\alpha \beta} - \text{id}_{E_\alpha}) \circ \psi_\alpha^{-1}, \quad u_{\beta \gamma} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \sigma_{\alpha \beta} - \text{id}_{E_\alpha}) \circ \psi_\alpha^{-1}, \quad v_{\alpha \beta} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \nabla \circ \sigma_{\alpha \beta} - \nabla_{\alpha}) \circ \psi_\alpha^{-1}, \quad v_{\beta \alpha} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \nabla \circ \sigma_{\alpha \beta} - \nabla_{\alpha}) \circ \psi_\alpha^{-1}, \quad u_{\beta \gamma} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \sigma_{\alpha \beta} - \text{id}_{E_\alpha})\).
\[
\Theta_{\psi^{-1}_{\sigma^{(i)}}(\sigma^{(i)}_{\alpha \beta} \circ \sigma^{(i)}_{\alpha \beta} - \text{id}_{E_\alpha})} = 0.
\]
If we put \(\theta^{(i)} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \sigma_{\alpha \beta} - \text{id}_{E_\alpha}) \circ \psi_\alpha^{-1}, \quad v_{\alpha \beta} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \nabla \circ \sigma_{\alpha \beta} - \nabla_{\alpha}) \circ \psi_\alpha^{-1}, \quad v_{\beta \alpha} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \nabla \circ \sigma_{\alpha \beta} - \nabla_{\alpha}) \circ \psi_\alpha^{-1}, \quad u_{\beta \gamma} = \psi_\alpha \circ (\sigma^{-1}_{\alpha \beta} \circ \sigma_{\alpha \beta} - \text{id}_{E_\alpha})\),

then \(\psi^{-1}_{\sigma^{(i)}}(\sigma^{(i)}_{\alpha \beta} \circ \sigma^{(i)}_{\alpha \beta} - \text{id}_{E_\alpha}) = 0\), because there is \(g_{\alpha} \in \text{End}(E|_{D_A^{(i)}}) \otimes I\).
satisfying $N^{(i)} \circ g^{(i)}_a - g^{(i)}_a \circ N^{(i)} = \tau^{(i)}_a \circ \kappa^{(i)} + \theta^{(i)} \circ \xi^{(i)}_a$ from the condition $\Theta^{(i)}_{\kappa^{(i)} + \theta^{(i)} \circ \xi^{(i)}_a} = 0$. We define a connection $\tilde{\nabla}_\alpha$ on $E_{\alpha}$ by $\tilde{\nabla}_\alpha := \nabla_\alpha + \psi_{a, \alpha}^{-1} \circ b_a \circ \psi_a$. Then we have

$$
\tilde{\nabla}_\alpha|_{D^*_A} = \nabla_\alpha|_{D^*_A} + (\psi_{a, \alpha}^{-1} \circ b_a \circ \psi_a)|_{D^*_A} = \tilde{\nabla}^{(i)}(N^{(i)}) \frac{d\tilde{z}^{(i)}}{\tilde{z}_1^{(i)} \cdots \tilde{z}_m^{(i)}} + \tilde{\theta}^{(i)}_{\nu, N^{(i)}(N^{(i)} - N^{(i)})} = \tilde{\nabla}^{(i)}(N^{(i)}) \frac{d\tilde{z}^{(i)}}{\tilde{z}_1^{(i)} \cdots \tilde{z}_m^{(i)}} + \tilde{\theta}^{(i)}_{\nu, N^{(i)}(N^{(i)} - N^{(i)})}.
$$

If we put $\tilde{\sigma}_{\beta \alpha} := \sigma_{\beta \alpha} \circ (id - \psi_{a, \alpha}^{-1} \circ a_{\alpha \beta} \circ \psi_a)$, then

$$
(\tilde{\sigma}_{\gamma \beta})^{-1} \circ \tilde{\sigma}_{\gamma \beta} \circ \tilde{\sigma}_{\beta \alpha} = (id + \psi_{a, \alpha}^{-1} \circ a_{\alpha \beta} \circ \psi_a) \circ \sigma_{\gamma \beta} \circ \sigma_{\beta \alpha} \circ (id - \psi_{a, \alpha}^{-1} \circ a_{\alpha \beta} \circ \psi_a)
$$

because $\sigma_{\beta \alpha} \otimes A/I = id$. We also have

$$
\tilde{\nabla}^{(i)} = \nabla_\alpha + \psi_{a, \alpha}^{-1} \circ b_a \circ \psi_a.
$$

Thus we can patch $(E_{\alpha}, \tilde{\nabla}_\alpha, \{\tilde{\theta}^{(i)}_a, \tilde{\kappa}^{(i)}_a\})$ together via the gluing isomorphisms $\{\tilde{\sigma}_{\beta \alpha}\}$ and obtain a flat family $(E, \nabla, \{\theta^{(i)}_a, \kappa^{(i)}_a\})$ of factorized $(\bar{v}, \bar{\mu}) \otimes A$-connections over $A$ which is a lift of $(E', \nabla', \{\theta^{(i)}_a, \kappa^{(i)}_a\})$. Conversely, we can immediately see that $o(E', \nabla', \{\kappa^{(i)}_a\}) = 0$ if there is a lift of $(E', \nabla', \{\theta^{(i)}_a, \kappa^{(i)}_a\})$ over $A$, which corresponds to a lift of $(E', \nabla', \{\kappa^{(i)}_a\})$ over $A$. Thus the proposition is proved.

**Lemma 2.24.** The isomorphism $H^2(\text{Tr}): H^2(F^* \otimes I) \cong H^2(L^* \otimes I)$ in Lemma 2.21 sends the obstruction class $o(E', \nabla', \{\kappa^{(i)}_a\})$ defined in the proof of Proposition 2.23 to an element of $H^2(L^* \otimes I)$ whose vanishing is equivalent to the existence of an extension of $(\det(E', \nabla'))$ to a pair $(L, \nabla_L)$ of a line bundle $L$ on $C \times \text{Spec} A$, a connection $\nabla_L: L \rightarrow L \otimes \Omega^1_{C/A}(D_A)$ satisfying $(L, \nabla_L) \otimes A/I \cong \det(E', \nabla')$, and $\nabla_L|_{D^*_A} = \sum_{k=1}^r \tilde{\mu}^{(i)}(k^{(i)}_a) A$.

**Proof.** Take the same affine open covering $\{U_{\alpha}\}$ of $C_A$ and the lifts $(E_{\alpha}, \nabla_\alpha)$ of $(E', \nabla')|_{U_{\alpha} \times \text{Spec} A/I}$ as in the proof of Proposition 2.23. Then $det(E_{\alpha}, \nabla_\alpha)$ is a lift of $det(E', \nabla')|_{U_{\alpha} \times \text{Spec} A/I}$ and the class

$$
o(det(E', \nabla')) := \left[\left\{\det(\nu_{\alpha}) \circ (\det(\sigma^{-1}_{\gamma \beta} \circ \sigma_{\beta \alpha}) - id_{det E_{\alpha}}) \circ det(\psi_{a, \alpha}^{-1})\right\}\right] \in H^2(F^* \otimes I)
$$

is nothing but the obstruction for the existence of a lift $(L, \nabla_L)$ of $(E', \nabla')$ over $A$ satisfying $\nabla_L|_{D^*_A} = \sum_{k=1}^r \tilde{\mu}^{(i)}(k^{(i)}_a) A$. Here $\nabla: E_{\alpha} \rightarrow E_{\alpha} \otimes \Omega^1_{C/A}(D_A)$ is the $A$-relative connection on $det(E_{\alpha})$ induced from $\nabla_\alpha$, which is defined by

$$
(det(\nabla_\alpha))(v_1 \wedge v_2 \wedge \cdots \wedge v_r) = \nabla_\alpha(v_1) \wedge v_2 \wedge \cdots \wedge v_r + \cdots + v_1 \wedge \cdots \wedge v_{r-1} \wedge \nabla_\alpha(v_r)
$$
for $v_1, \ldots, v_r \in E_\alpha$. For the notations $\{u_{\alpha\beta}\}, \{v_{\alpha\beta}\}$ in the proof of Proposition 2.23 we have
\[
\text{Tr}(u_{\alpha\beta}) = \det(\psi_\alpha) \circ (\det(\sigma_{\alpha}^{-1} \circ \sigma_{\beta} \circ \sigma_{\alpha}) - \text{id}_{E_\alpha}) \circ \det(\psi_\alpha^{-1})
\]
\[
\text{Tr}(v_{\alpha\beta}) = \det(\psi_\alpha) \circ (\det(\sigma_{\alpha}^{-1} \circ \det(\nabla) \circ \det(\sigma_{\beta} \circ \det(\nabla_\alpha)) \circ \det(\psi_\alpha^{-1})
\]
So $o(\det(E', \nabla'))$ is nothing but the image of the obstruction class $o(E', \nabla', \{t^{(i)}_j, \{N^{(i)}_j\}\}) \in H^2(F^* \otimes I)$ under the isomorphism $H^2(Tr): H^2(F^* \otimes I) \sim H^2(L^* \otimes I)$.

\[\square\]

**Proposition 2.25.** The moduli space $M^\alpha_{C,D}(\tilde{\nu}, \tilde{\mu})$ is smooth over $S$.

**Proof.** Consider the $S$-relative moduli space $M_{C,D}(\text{Tr}(\tilde{\nu}), \text{Tr}(\tilde{\mu}))$ whose $S'$-valued points are the pairs $(L, \nabla_L)$ of a line bundle $L$ on $C$, and a relative connection $\nabla_L: L \rightarrow L \otimes \Omega^1_{C/S'}(\mathcal{D}_S)$ satisfying $\nabla_L|_{\mathcal{D}'_{S'}} = \sum_{k=1}^r \tilde{\nu}(i)(\mu_k)^{S'}$. Then $M_{C,D}(\text{Tr}(\tilde{\nu}), \text{Tr}(\tilde{\mu}))$ is an affine space bundle over the Jacobian variety of $C$ over $S$ whose fiber is isomorphic to $H^0(\Omega^1_{C/S})$. So we can prove by the same method as in the proof of Theorem 2.1 that $M_{C,D}(\text{Tr}(\tilde{\nu}), \text{Tr}(\tilde{\mu}))$ is smooth over $S$ and the obstruction class $o(\det(E', \nabla'))$ should vanish. Thus the obstruction class $o(E', \nabla', \{N^{(i)}_j\})$ also vanishes by Lemma 2.24 and the moduli space $M^\alpha_{C,D}(\tilde{\nu}, \tilde{\mu})$ is smooth over $S$.

\[\square\]

2.7. Relative symplectic form on the moduli space.

**Proposition 2.26.** There exists an $S$-relative symplectic form $\omega \in H^0(M^\alpha_{C,D}(\tilde{\nu}, \tilde{\mu}), \Omega^2_{M^\alpha_{C,D}(\tilde{\nu}, \tilde{\mu})}$ on the moduli space $M^\alpha_{C,D}(\tilde{\nu}, \tilde{\mu})$.

**Proof.** For some quasi-finite étale covering $\tilde{M} \rightarrow M^\alpha_{C,D}(\tilde{\nu}, \tilde{\mu})$, there is a universal flat family of $(\tilde{\nu}, \tilde{\mu})$-connections $(\tilde{E}, \nabla, \{\tilde{N}^{(i)}(i)\})$ on $C \times \tilde{M}$ over $M$. Replacing $\tilde{M}$ by a refinement, there is a corresponding flat family $(\tilde{E}, \nabla, \{\tilde{N}^{(1)}(i)\})$ of factorized $(\tilde{\nu}, \tilde{\mu})$-connections on $C \times \tilde{M}$ over $\tilde{M}$. We define homomorphisms
\[
\sigma^{(i)}_{\theta(i)}: \text{End}(E|_{\mathcal{D}_{\tilde{M}}}) \otimes \Omega^1_{C \times \tilde{M}/\tilde{M}}(\mathcal{D}_{\tilde{M}}) \rightarrow \text{Hom}(E|_{\mathcal{D}_{\tilde{M}}}, E|_{\mathcal{D}_{\tilde{M}}})
\]
\[
\sigma^{(i)+}_{\kappa^{(i)}}: \text{End}(E|_{\mathcal{D}_{\tilde{M}}}) \otimes \Omega^1_{C \times \tilde{M}/\tilde{M}}(\mathcal{D}_{\tilde{M}}) \rightarrow \text{Hom}(E|_{\mathcal{D}_{\tilde{M}}}, E|_{\mathcal{D}_{\tilde{M}}})
\]
\[
\delta^{(i)}_{\nu^{(i)}, \sigma^{(i)}, \kappa^{(i)}}: \text{End}(E|_{\mathcal{D}_{\tilde{M}}}) \rightarrow \text{End}(E|_{\mathcal{D}_{\tilde{M}}}) \otimes \Omega^1_{C \times \tilde{M}/\tilde{M}}(\mathcal{D}_{\tilde{M}})
\]
by the same formulas as in subsection 2.4. For each $u \in \text{End}(E|_{\mathcal{D}_{\tilde{M}}})$, we define a homomorphism
\[
\theta_u^{(i)}: \text{End}(E|_{\mathcal{D}_{\tilde{M}}}) \rightarrow \Omega^1_{C \times \tilde{M}/\tilde{M}}(\mathcal{D}_{\tilde{M}})
\]
by the same formula as subsection 2.4. We put
\[
\tilde{\mathcal{G}}^0 := \text{End}(\tilde{E}), \quad \tilde{\mathcal{G}}^1 := \text{End}(\tilde{E}) \otimes \Omega^1_{C \times \tilde{M}/\tilde{M}}(\mathcal{D}_{\tilde{M}}), \quad \tilde{\mathcal{G}}^1 := \tilde{\mathcal{G}}^1|_{\mathcal{D}_{\tilde{M}}},
\]
\[
S(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}}) := \left\{ (\tau^{(i)}) \in \bigoplus_{i=1}^n \text{Hom}(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}}) \mid \tau^{(i)} = \tau^{(i)} \text{ for any } i \right\},
\]
\[
S(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}}) := \left\{ (\xi^{(i)}) \in \bigoplus_{i=1}^n \text{Hom}(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}}) \mid \xi^{(i)} = \xi^{(i)} \text{ for any } i \right\},
\]
\[
\tilde{Z}^0 := \bigoplus_{i=1}^n \text{Hom}(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}})/\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \quad \tilde{Z}^1 := \bigoplus_{i=1}^n \text{Hom}(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}}), \quad \Omega^1_{C \times \tilde{M}/\tilde{M}}(\mathcal{D}_{\tilde{M}})|_{\mathcal{D}_{\tilde{M}}},
\]
We define a complex $\tilde{F} := [\tilde{F}^0 \rightarrow \tilde{F}^1 \rightarrow \tilde{F}^2]$ in the same way as subsection 2.4.
\[
\tilde{F}^0 = \tilde{\mathcal{G}}^0, \quad \tilde{F}^1 = \tilde{\mathcal{G}}^1 \otimes S(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}}) \otimes S(\tilde{E}|_{\mathcal{D}_{\tilde{M}}}, \tilde{E}|_{\mathcal{D}_{\tilde{M}}}), \quad \tilde{F}^2 = \tilde{\mathcal{G}}^1 \otimes \tilde{Z}^1
\]
\[
d^0(u, (P^{(i)}(T))) = (\nabla u - \nabla u, \sigma^{(i)}_{\theta^{(i)}}(u|_{\mathcal{D}_{\tilde{M}}}, P^{(i)}(T)))
\]
\[
d^1(v, (\tau^{(i)}), (\xi^{(i)})) = \left( u|_{\mathcal{D}_{\tilde{M}}}, -\delta^{(i)}_{u^{(i)}, \sigma^{(i)}, \kappa^{(i)}}(\tau^{(i)} \circ \kappa^{(i)} + \theta^{(i)} \circ \xi^{(i)}) \right), \left( \Theta^{(i)}(\tau^{(i)} \circ \kappa^{(i)} + \theta^{(i)} \circ \xi^{(i)}) \right)
\]
Then we can see by the same proof as Proposition 2.17 that the relative tangent bundle $T_{\tilde{M}/S}$ of $\tilde{M}$ over $S$ is isomorphic to $R^1(p_{\tilde{M}})_{*(\tilde{F}^*)}$, where $p_{\tilde{M}}: C \times \tilde{M} \rightarrow \tilde{M}$ is the structure morphism. We define
\[(\Xi_{(\tau^{(i)}, \xi^{(i)})}, (\tau^{(i)}, \xi^{(i)})) \in \Omega^1_{\tilde{M}, \tilde{M}}(\mathcal{D}_M)|_{\mathcal{D}_M} \text{ for } ((\tau^{(i)}, (\xi^{(i)})), ((\tau^{(i)}, (\xi^{(i)}))) \in S(\tilde{E}|_{\tilde{M}}, \tilde{E}|_{\mathcal{D}_M}) \oplus S(\tilde{E}|_{\mathcal{D}_M}, \tilde{E}|_{\mathcal{D}_M}) \text{ in the same way as } [21] \text{ in subsection 2.5. We take an affine open covering } \{U_\alpha\} \text{ of } \mathcal{C} \text{ and define a pairing}
\]
\[\omega_{\tilde{M}} : \mathbf{R}^1(p_{\tilde{M}})_*(\tilde{F}^\bullet) \times \mathbf{R}^1(p_{\tilde{M}})_*(\tilde{F}^\bullet) \to \mathbf{R}^2(p_{\tilde{M}})_*(\mathcal{L}^\bullet_M) \cong \mathcal{O}_{\tilde{M}}\]
by
\[\omega_{\tilde{M}} \left[ \left\{ (u_{\alpha \beta}, v_\alpha, ((\tau_{\alpha}^{(i)}, (\xi_{\alpha}^{(i)}))), ((u_{\alpha \beta}^{(i)}, (v_\alpha^{(i)}), (\tau_{\alpha}^{(i)}, (\xi_{\alpha}^{(i)})))) \right\}, \left\{ (v_{\alpha \beta}, v_\alpha, ((\tau_{\alpha}^{(i)}, (\xi_{\alpha}^{(i)}))), ((v_{\alpha \beta}^{(i)}, (v_\alpha^{(i)}), (\tau_{\alpha}^{(i)}, (\xi_{\alpha}^{(i)})))) \right\} \right]
\]
using the Čech cohomology with respect to the covering \( \{U_\alpha \times S \tilde{M}\} \). Then the restriction \( \omega_{\tilde{M}}|_x \) at a point \( x \) of \( \tilde{M} \) whose image in \( M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}) \) corresponds to \((E, \nabla, \{l^{(i)}\})\) is nothing but the pairing \( \omega_{(E, \nabla, \{l^{(i)}\})} \) in Lemma 2.20 which is nondegenerate. We can easily see that \( \omega_{\tilde{M}} \) descends to a pairing
\[\omega_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})} : T_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})/S} \times T_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})/S} \to \mathcal{O}_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})}\]
which is nondegenerate. If we take a tangent vector \( v \in T_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})/S}(x) \) at a point \( x \in M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}) \) corresponding to a \((\tilde{\nu}, \tilde{\mu})\)-connection \((E, \nabla, \{l^{(i)}\})\), \( v \) corresponds to a \( C[t]/(t^2) \)-valued point \((E', \nabla', \{l'^{(i)}\})\) of \( M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}) \) which is a lift of \((E, \nabla, \{l^{(i)}\})\). Then we can check that \( \omega_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})}(v, v) \) coincides with the image by \( \text{Tr}: \mathbf{H}^2(F^\bullet) \to \mathbf{H}^2(C^\bullet) \) of the obstruction class \( o(E', \nabla', \{l'^{(i)}\}) \) for the lifting of \((E', \nabla', \{l'^{(i)}\})\) to a \( C[t]/(t^3) \)-valued point of \( M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}) \) which is given in Proposition 2.25. Since \( M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}) \) is smooth over \( S \) by Proposition 2.25 we have \( \omega_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})}(v, v) = 0 \). So the pairing \( \omega_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})} \) is skew-symmetric and define a relative 2-form \( \omega_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})} \in \Omega^2(M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}))/S \).

A generic geometric fiber \( M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})_s \) over \( S \) is the moduli space of regular singular connections on \( \mathcal{C}_s \) along the reduced divisor \( \mathcal{D}_s \). If we put \( \tilde{M}_s := \tilde{M} \times M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})_s \), there is a universal parabolic structure \( \tilde{E}_{\tilde{M}_s}|_{(\tilde{D}_s)^{(i)}_{\tilde{M}_s}} = \tilde{E}_0^{(i)} \cup \cdots \cup \tilde{E}_r^{(i)} \) determined by \( \nabla_{\tilde{M}_s} \). If we put
\[\tilde{F}^{\alpha}_{\text{par}} := \left\{ u \in \tilde{G}M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}) \mid u|_{(\tilde{D}_s)^{(i)}_{\tilde{M}_s}} \tilde{E}_j^{(i)} \subset \tilde{E}_j^{(i)} \text{ for any } i, j, k \right\}
\[\tilde{F}^{\alpha}_{\text{par}} := \left\{ v \in \tilde{G}M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu}) \mid v|_{(\tilde{D}_s)^{(i)}_{\tilde{M}_s}} \tilde{E}_j^{(i)} \subset \tilde{E}_j^{(i)} \text{ for any } i, j, k \right\}
\[\nabla_{\text{par}} : \tilde{F}^{\alpha}_{\text{par}} \ni u \mapsto \nabla u - u \nabla = \tilde{F}^\alpha_{\text{par}} \ni \tilde{F}^\alpha_{\text{par}} \ni \tilde{G}M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})_s \text{ induce a morphism } \tilde{F}^\alpha_{\text{par}} \to \tilde{F}^\alpha_{\text{par}} \text{ of complexes which induces an isomorphism}
\]
\[\mathbf{R}^1(\pi_{\tilde{M}_s})_*(\tilde{F}^\alpha_{\text{par}}) \cong \mathbf{R}^1(\pi_{\tilde{M}_s})_*(\tilde{F}^\alpha_{\text{par}})\]
because they are both isomorphic to the tangent bundle of \( \tilde{M}_s \). A symplectic form \( \omega_{\tilde{M}_s} \) on \( \tilde{M}_s \) is defined in [12] Proposition 7.2. which satisfies \( dw_{\tilde{M}_s} = 0 \) by [12] Proposition 7.3. By construction, we can see that \( \omega_{\tilde{M}_s} = \omega_{\tilde{M}}|_{\tilde{M}_s} \). So we have \( d\omega_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})}|_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})_s} = 0 \), which implies that \( \omega_{M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})} \) is relatively \( d \)-closed on \( M^\alpha_{\tilde{C}, D}(\tilde{\nu}, \tilde{\mu})_s \) over \( S \).

Eventually Theorem 2.11 follows from Corollary 2.22 Proposition 2.25 and Proposition 2.26.

3. Fundamental solution of an unfolded linear differential equation with an asymptotic property

In this section, we introduce the existence theorem of fundamental solutions with an asymptotic property of an unfolded linear differential equation, which is one of the main tools in the unfolding theory of linear differential equations established by Hurtubise, Lambert and Rousseau in [9] and [10]. Unfortunately, the unfolded generalized isomonodromic deformation in Theorem 0.1 is not compatible with the asymptotic property given in the unfolding theory in [9], [10]. However, it will be worth pointing out what is the difficulty in adopting the asymptotic property in [9], [10] to our moduli theoretic setting constructed in section 2. Since the unfolding theory in [9], [10] are written in a very general setting and hard to follow all of them, we restrict to the easy case when the unfolding of the singular divisor is given by the equation \( z^m - e^m = 0 \).
3.1. Flows for an asymptotic estimate. Let $\Delta = \{z \in \mathbb{C} | |z| < 1\}$ be a unit disk in the complex plane $\mathbb{C}$. For an integer $m$ with $m \geq 2$, we put $\zeta_m := \exp\left(\frac{2\pi \sqrt{-1}}{m}\right)$. Then we have $z^m - \epsilon^m = (z - \epsilon \zeta_m)(z - \epsilon \zeta_m^2) \cdots (z - \epsilon \zeta_m^m)$ for $z, \epsilon \in \Delta$. We set

$$D := \{(z, \epsilon) \in \Delta \times \Delta \mid z^m - \epsilon^m = 0\}. $$

Note that there is an equality

$$\frac{1}{z^m - \epsilon^m} = \frac{1}{(z - \epsilon \zeta_m)(z - \epsilon \zeta_m^2) \cdots (z - \epsilon \zeta_m^m)} = \sum_{j=1}^{m} \frac{1}{\prod_{j \neq i} \epsilon (\zeta_m^j - \zeta_m^i)} \frac{dz}{z - \epsilon \zeta_m^i}$$

for $(z, \epsilon) \in (\Delta \times \Delta) \setminus D$. By Lemma 2.1 we have

$$\sum_{i=1}^{m} \frac{1}{\prod_{j \neq i} (\epsilon \zeta_m^i - \epsilon \zeta_m^j)} = \sum_{i=1}^{m} \text{res}_{z=\epsilon \zeta_m^i} \left( \frac{dz}{(z - \epsilon \zeta_m^i)(z - \epsilon \zeta_m^2) \cdots (z - \epsilon \zeta_m^m)} \right) = 0$$

for $\epsilon \neq 0$, since $m \geq 2$.

For a fixed $\theta \in \mathbb{R}$, we consider a holomorphic differential equation

$$\frac{dz}{dt} = e^{\sqrt{-1}\theta}(z^m - \epsilon^m) = e^{\sqrt{-1}\theta}(z - \epsilon \zeta_m)(z - \epsilon \zeta_m^2) \cdots (z - \epsilon \zeta_m^m).$$

Under the above equation, we can regard $\tau$ as a multi-valued function in $z \in (\Delta \times \Delta) \setminus D$. We substitute into $\tau \in \mathbb{C}$ a real variable $t \in \mathbb{R}$ and consider the restricted differential equation

$$\frac{dz}{dt} = e^{\sqrt{-1}\theta}(z^m - \epsilon^m) = e^{\sqrt{-1}\theta}(z - \epsilon \zeta_m)(z - \epsilon \zeta_m^2) \cdots (z - \epsilon \zeta_m^m).$$

Note that giving a solution $z(t) = x(t) + \sqrt{-1}y(t)$ of the differential equation (31) is equivalent to giving a flow of the vector field

$$v_{\epsilon, \theta} = \text{Re} \left( e^{\sqrt{-1}\theta}(z^m - \epsilon^m) \right) \frac{\partial}{\partial x} + \text{Im} \left( e^{\sqrt{-1}\theta}(z^m - \epsilon^m) \right) \frac{\partial}{\partial y}.$$

For the investigation of the flow of the vector field $v_{\epsilon, \theta}$, we consider the surjective morphism

$$\varpi: \Delta \times [0, 1) \times S^1 \rightarrow \Delta \times \Delta$$

defined by

$$\varpi(z, s, e^{\sqrt{-1}\psi}) = (z, se^{\sqrt{-1}\psi})$$

and we call $\varpi$ a polar blow up of $\Delta \times \Delta$ along $\Delta \times \{0\}$. Here we denote $\{t \in \mathbb{R} \mid a \leq t < b\}$ by $[a, b)$ for real numbers $a, b$ satisfying $a < b$.

We consider the following proposition which treats an easy restricted case of the analysis of flows in a series of papers [23, 24, 9, 10]. We give here just an elementary proof in an easy restricted case for the purpose of the author’s understanding. So it may seem trivial for experts.

**Proposition 3.1.** There is an open neighborhood $U$ of $\{0\} \times \{0\} \times S^1$ in $\Delta \times [0, 1) \times S^1$ and an open covering

$$U \setminus (U \cap \varpi^{-1}(D)) = \bigcup_{j=1}^{m} \bigcup_{0 \leq \psi_0 \leq 2\pi \xi = 1, 2} W^{(j)}_{\psi_0, \xi}$$

such that any flow of the vector field

$$v_{\epsilon, \theta_{\psi_0, \xi}}^{(j)} = \text{Re} \left( e^{\sqrt{-1}\theta_{\psi_0, \xi}^{(j)}(z^m - \epsilon^m)} \right) \frac{\partial}{\partial x} + \text{Im} \left( e^{\sqrt{-1}\theta_{\psi_0, \xi}^{(j)}(z^m - \epsilon^m)} \right) \frac{\partial}{\partial y}$$

starting at a point of $W^{(j)}_{\psi_0, \xi}$ converges to a point in $\varpi^{-1}(D)$, where $\theta_{\psi_0, \xi}^{(j)}$ is determined by $j, \psi_0, \xi$.

**Proof.** We take a point $(z_0, s_0, e^{\sqrt{-1}\psi_0}) \in (\Delta \setminus \{0\}) \times [0, \frac{1}{2}) \times S^1$ satisfying $0 < |z_0| < \frac{1}{4}$. We can choose an integer $j$ with $1 \leq j \leq m$ satisfying

$$-\frac{\pi}{m} \leq \arg(z_0) - \psi_0 - \frac{2j\pi}{m} \leq \frac{\pi}{m}.$$  

We divide into two cases:

$$0 \leq \arg(z_0) - \psi_0 - \frac{2j\pi}{m} \leq \frac{\pi}{m}, \quad -\frac{\pi}{m} \leq \arg(z_0) - \psi_0 - \frac{2j\pi}{m} < 0.$$
Case 1. $0 \leq \arg(z_0) - \frac{2j\pi}{m} - \psi_0 \leq \frac{\pi}{m}$.

In this case we choose small $\delta > 0$ satisfying $\delta < \frac{\pi}{24m}$ and put

$$\theta^{(j)}_{\psi_0,1} := -\frac{2j(m-1)\pi}{m} - (m-1)\psi_0 + \pi + \delta.$$  

We simply denote $\theta^{(j)}_{\psi_0,1}$ by $\theta$ in the following. So $\theta$ is given by

$$\frac{\theta - \pi}{m-1} = -\frac{2j\pi}{m} - \psi_0 + \frac{\delta}{m-1}.$$  

Note that we have

$$\frac{\delta}{m-1} \leq \arg(z_0) + \frac{\theta - \pi}{m-1} \leq \frac{\pi}{m} + \frac{\delta}{m-1}, \quad \psi_0 + \frac{2j\pi}{m} + \frac{\theta - \pi}{m-1} = \frac{\delta}{m-1}.$$  

If we replace $\delta > 0$ sufficiently smaller, we may assume that the two segments

$$l_1 = \left\{ z \in \mathbb{C} \mid \arg(e^{\sqrt{-1}\frac{\pi m}{m-1}} - e^{\sqrt{-1}\frac{\pi}{m-1}z}) = \frac{(2m+1)\pi}{m-1}, \quad |z| < 1, \quad \text{Re}(z) > 0 \right\}$$

$$l_2 = \left\{ z \in \mathbb{C} \mid \arg(z) + \frac{\theta - \pi}{m-1} = \frac{\pi}{m} + \frac{2\delta}{m-1}, \quad |z| < 1, \quad \text{Re}(z) > 0 \right\}$$

intersects at a point $s_1 e^{\sqrt{-1}(\frac{\pi}{m} - \frac{\theta + \psi_0 + \pi}{m-1})}$ satisfying $\frac{1}{4} < s_1 < 1$. Then we put

$$P^{(j)}_{\psi_0,1} = \left\{ (z, s, e^{\sqrt{-1}\psi}) \in \Delta \times \left[ 0, \frac{1}{3} \right] \times S^1 \mid \begin{array}{c}
-\frac{3\delta}{2m-2} < \psi + \frac{2j\pi}{m} + \frac{\theta - \pi}{m-1} - \frac{3\delta}{2m-2}, \quad z \neq 0 \\
\frac{2m+1}{m-1} < \arg(e^{\sqrt{-1}\frac{\pi m}{m-1}} - e^{\sqrt{-1}\frac{\pi}{m-1}z}) < \frac{\pi}{2m-2} + \frac{3\delta}{2m-2}
\end{array}
\right\}.$$  

A figure of the region $\left\{ \hat{z} = e^{\sqrt{-1}\frac{\pi m}{m-1}z} \mid (z,s,e^{\sqrt{-1}\psi}) \in P^{(j)}_{\psi_0,1} \cap (\Delta \times \{(s,e^{\sqrt{-1}\psi})\}) \right\}$ looks like [figure 1].

Since $\arg(e^{\sqrt{-1}\frac{\theta + \psi_0 + \pi}{m-1}z}) = \frac{\theta - \pi}{m-1} + \theta + m\psi = m\theta + \theta + m\psi + \pi$, we have

$$\pi - \frac{3m\delta}{2m-2} < \arg(e^{\sqrt{-1}\frac{\theta + \psi_0 + \pi}{m-1}z}) < \pi + \frac{3m\delta}{2m-2}$$

if $-\frac{3\delta}{2m-2} < \psi + \frac{2j\pi}{m} + \frac{\theta - \pi}{m-1} < \frac{3\delta}{2m-2}$. So we can take $\eta > 0$ depending on $m,j,\theta,\delta$ such that

$$-\frac{2m\delta}{m-1} < \arg(e^{\sqrt{-1}\frac{\theta + \psi_0 + \pi}{m-1}z}) < \frac{2m\delta}{m-1}$$

holds for any $w \in \Delta$ satisfying $|w| \leq \eta$ and when $-\frac{3\delta}{2m-2} < \psi + \frac{2j\pi}{m} + \frac{\theta - \pi}{m-1} < \frac{3\delta}{2m-2}$. We put

$$Q^{(j)}_{\psi_0,1} := \left\{ (z, s, e^{\sqrt{-1}\psi}) \in \Delta \times \left[ 0, \frac{1}{3} \right] \times S^1 \mid \begin{array}{c}
eq 0, \quad \frac{6m}{m} < \arg(e^{\sqrt{-1}\frac{\theta + \psi_0 + \pi}{m-1}z} - \eta se^{\sqrt{-1}\pi}) < \frac{6m}{m} \\
-\frac{3\delta}{2m-2} < \psi + \frac{2j\pi}{m} + \frac{\theta - \pi}{m-1} - \frac{3\delta}{2m-2}, \quad \text{Re}(z) < 1
\end{array}
\right\}.$$  

and set

$$R^{(j)}_{\psi_0,1} := P^{(j)}_{\psi_0,1} \cup Q^{(j)}_{\psi_0,1}.$$  

We may assume $\eta < \frac{1}{4}$ and then the segment

$$l_3' = \left\{ z \in \mathbb{C} \mid \arg(e^{\sqrt{-1}\frac{\pi m}{m-1}z - \eta se^{\sqrt{-1}\pi}}) = \frac{\pi}{6m}, \quad -\eta < \text{Re}(z) < 1 \right\}$$

intersects with the segment $l_2$ at a point $s_2 e^{\sqrt{-1}(\frac{\pi}{m} - \frac{\theta + \psi_0 + \pi}{m-1})}$ satisfying $0 < s_2 < \frac{1}{4} < s_1$ if $s > 0$. A figure of the region $\left\{ \hat{z} = e^{\sqrt{-1}\frac{\pi m}{m-1}z} \mid (z, s, e^{\sqrt{-1}\psi}) \in R^{(j)}_{\psi_0,1} \cap (\Delta \times \{(s,e^{\sqrt{-1}\psi})\}) \right\}$ looks like [figure 2].
$e^{\sqrt{-1}(\frac{\pi}{m} + 2\delta_m - 1)}$
In the case of \( \epsilon = se^{\sqrt{-1}\psi} = 0 \), we can see \( Q^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(0, e^{\sqrt{-1}\psi})\}) = \emptyset \) by the definition of \( Q^{(j)}_{\psi_{0,1}} \), from which we have
\[
R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(0, e^{\sqrt{-1}\psi})\}) = P^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(0, e^{\sqrt{-1}\psi})\}).
\]
In any case, \((z_0, s_0, e^{\sqrt{-1}\psi})\) lies in \(R^{(j)}_{\psi_{0,1}}\) and
\[
R^{(j)}_{\psi_{0,1}} \cap \varpi^{-1}(D) = \left\{(e^{\zeta_j m}, s, e^{\sqrt{-1}\psi}) \in R^{(j)}_{\psi_{0,1}} \mid \epsilon = se^{\sqrt{-1}\psi}\right\}.
\]
Consider the differential equation
\[
\frac{dz(t)}{dt} = e^{\sqrt{-1}\theta}(z(t)^m - e^m) = e^{\sqrt{-1}\theta}(z(t) - e^\zeta_m)(z(t) - e^{\zeta_m^2}) \cdots (z(t) - e^{\zeta_m^m})
\]
with respect to a real time variable \(t\) and the initial point \(z(0) \in R^{(j)}_{\psi_{0,1}} \cap \varpi^{-1}(D) \cap R^{(j)}_{\psi_{0}}\). The solution of the above differential equation is equivalent to the flow of the vector field
\[
v_{\epsilon, \theta} = \text{Re} \left( e^{\sqrt{-1}\theta}(z^m - e^m) \right) \frac{\partial}{\partial x} + \text{Im} \left( e^{\sqrt{-1}\theta}(z^m - e^m) \right) \frac{\partial}{\partial y},
\]
starting at a point in \(R^{(j)}_{\psi_{0,1}} \cap \varpi^{-1}(D) \cap R^{(j)}_{\psi_{0,1}}\). Notice that the direction of the vector \(v_{\epsilon, \theta}\) is given by \(\arg \left( e^{\sqrt{-1}\theta}(z(t)^m - e^m) \right)\). We investigate the direction of the vector \(v_{\epsilon, \theta}\) at each boundary point of the fiber \(R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(s, e^{\sqrt{-1}\psi})\})\) of \(R^{(j)}_{\psi_{0,1}}\) over \((s, e^{\sqrt{-1}\psi}) \in [0, \frac{1}{2}) \times S^1\).

First take a boundary point \((z, s, e^{\sqrt{-1}\psi})\) of \(R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(s, e^{\sqrt{-1}\psi})\})\) satisfying \(\arg(z) + \frac{\theta - \pi}{m - 1} = \frac{\pi}{m} + \frac{2\delta}{m - 1}\). Then we have
\[
\arg \left( e^{\sqrt{-1}\frac{\theta - \pi}{m - 1}}e^{\sqrt{-1}\theta}z^m \right) = \frac{\theta - \pi}{m - 1} + \theta + m\arg(z) = \frac{m(\theta - \pi)}{m - 1} + m\arg(z) + \pi = 2\pi + \frac{2m\delta}{m - 1}.
\]
Combined with the inequality \([35]\), we have
\[
-\frac{3m\delta}{2m - 2} < \arg \left( e^{\sqrt{-1}\frac{\theta - \pi}{m - 1}}e^{\sqrt{-1}\theta}(z^m - (e^{\zeta_m})^m) \right) < \frac{2m\delta}{m - 1} < \frac{\pi}{m} + \frac{2\delta}{m - 1},
\]
from which we can see that the vector \(v_{\epsilon, \theta}\) faces toward the interior of the region \(R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(s, e^{\sqrt{-1}\psi})\})\).

Secondly take a boundary point \((z, s, e^{\sqrt{-1}\psi})\) of \(R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(s, e^{\sqrt{-1}\psi})\})\) satisfying \(\arg(z) + \frac{\theta - \pi}{m - 1} = -\frac{\pi}{3m}\). Then we have
\[
\arg \left( e^{\sqrt{-1}\frac{\theta - \pi}{3m}}e^{\sqrt{-1}\theta}z^m \right) = \frac{\theta - \pi}{m - 1} + \theta + m\arg(z) = \frac{m(\theta - \pi)}{m - 1} + m\arg(z) + \pi = \frac{2\pi}{3}.
\]
Combined with \([35]\), we have
\[
-\frac{\pi}{3m} < -\frac{3m\delta}{2m - 2} < \arg \left( e^{\sqrt{-1}\frac{\theta - \pi}{3m}}e^{\sqrt{-1}\theta}(z^m - (e^{\zeta_m})^m) \right) < \frac{2\pi}{3}.
\]
So the vector \(v_{\epsilon, \theta}\) faces toward the interior of the region \(R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(s, e^{\sqrt{-1}\psi})\})\).

Thirdly we take a boundary point \((z, s, e^{\sqrt{-1}\psi})\) of \(R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(s, e^{\sqrt{-1}\psi})\})\) which satisfies the equality
\[
\arg \left( e^{\sqrt{-1}\frac{\pi}{3m}} - e^{\sqrt{-1}\frac{\pi}{3m} + z} \right) = \frac{(2m + 1)\delta}{m - 1},
\]
which means that \(z\) lies on the segment \(l_1\). Since \(\frac{\pi}{3} + \pi \leq \arg \left( e^{\sqrt{-1}\frac{\pi}{3m}}e^{\sqrt{-1}\theta}z^m \right) \leq 2\pi + \frac{2m\delta}{m - 1}\), we can see by the inequality \([35]\) that the inequality
\[
-\frac{2\pi}{3} \leq \arg \left( e^{\sqrt{-1}\frac{\pi}{3m}}e^{\sqrt{-1}\theta}(z^m - (e^{\zeta_m})^m) \right) \leq \frac{2m\delta}{m - 1} < \frac{(2m + 1)\delta}{m - 1}
\]
holds. So the vector \(v_{\epsilon, \theta}\) faces toward the interior of the region \(R^{(j)}_{\psi_{0,1}} \cap (\Delta \times \{(s, e^{\sqrt{-1}\psi})\})\) at this point. A picture of the direction of the vector \(v_{\epsilon, \theta}\) is [figure 3]. Fourthly we take a boundary point \((z, s, e^{\sqrt{-1}\psi})\)
of $R_{\psi_0,1}^{(j)} \cap \left( \Delta \times \{ (s, e^{\sqrt{-1} \psi}) \} \right)$ satisfying $|z| = 1$. Note that we have $-\frac{\pi}{3m} \leq \arg(z) + \frac{\theta - \pi}{m - 1} \leq \frac{\pi}{3m}$. If $\frac{\pi}{6m} \leq \arg(z) + \frac{\theta - \pi}{m - 1} \leq \frac{\pi}{3m}$, then

$$\frac{7\pi}{6} \leq \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} z^m \right) = \frac{\theta - \pi}{m - 1} + \theta + m \arg(z) = \frac{m(\theta - \pi)}{m - 1} + m \arg(z) + \pi \leq \frac{4\pi}{3}.$$ 

Since $|e^m| \leq s < \frac{1}{3} = \frac{1}{3} |z|^m$ and $\frac{5\pi}{6} \leq \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} e^m \right) \leq \frac{7\pi}{6}$ by (35), we have a rough estimate

$$\frac{7\pi}{6} \leq \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} (z^m - (e^{\epsilon_0})^m) \right) < \frac{3\pi}{2}.$$ 

So the vector $v_{\epsilon, \theta}$ faces toward the interior of the region $R_{\psi_0,1}^{(j)} \cap \left( \Delta \times \{ (s, e^{\sqrt{-1} \psi}) \} \right)$ at this point. If $-\frac{\pi}{3m} \leq \arg(z) + \frac{\theta - \pi}{m - 1} \leq -\frac{\pi}{6m}$, then we have

$$\frac{2\pi}{3} \leq \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} z^m \right) = \frac{m(\theta - \pi)}{m - 1} + m \arg(z) + \pi \leq \frac{5\pi}{6}$$

and we have, from (35) and $|e^m| < \frac{1}{3} = \frac{1}{3} |z|^m$, a rough estimate

$$\frac{\pi}{2} < \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} (z^m - e^m) \right) = \frac{m(\theta - \pi)}{m - 1} + m \arg(z) + \pi \leq \frac{5\pi}{6}.$$ 

So the vector $v_{\epsilon, \theta}$ faces toward the interior of the region $R_{\psi_0,1}^{(j)} \cap \left( \Delta \times \{ (s, e^{\sqrt{-1} \psi}) \} \right)$ at this point. If $-\frac{\pi}{6m} \leq \arg(z) + \frac{\theta - \pi}{m - 1} \leq \frac{\pi}{6m}$, then we have $\frac{5\pi}{6} \leq \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} z^m \right) \leq \frac{7\pi}{6}$, from which we obtain a rough estimate

$$\frac{2\pi}{3} < \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} (z^m - e^m) \right) < \frac{4\pi}{3}$$

using (35) and $|e^m| < \frac{1}{3} = \frac{1}{3} |z|$. So $v_{\epsilon, \theta}$ faces toward the interior of the region $R_{\psi_0}^{(j)} \cap \left( \Delta \times \{ (s, e^{\sqrt{-1} \psi}) \} \right)$.}

Finally we take a boundary point $(z, s, e^{\sqrt{-1} \psi})$ of $R_{\psi_0,1}^{(j)} \cap \left( \Delta \times \{ (s, e^{\sqrt{-1} \psi}) \} \right)$ satisfying $(z, s, e^{\sqrt{-1} \psi}) \in Q_{\psi_0,1}^{(j)}$ and $\arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} z - \eta e^{\sqrt{-1} \psi} \right) = \pm \frac{\pi}{6m}$. Then we have $|z| \leq s\eta$ and

$$-\frac{\pi}{6m} < -\frac{2m \delta}{m - 1} < \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m - 1}} e^{\sqrt{-1} \theta} (z^m - (s e^{\sqrt{-1} \psi})^m) \right) < \frac{2m \delta}{m - 1} < \frac{\pi}{6m}$$

because of the inequality (36) and the assumption $0 < \delta < \frac{\pi}{24m}$. Thus the vector $v_{\epsilon, \theta}$ faces toward the interior of the region $R_{\psi_0,1}^{(j)} \cap \left( \Delta \times \{ (s, e^{\sqrt{-1} \psi}) \} \right)$ at this point. A picture of the direction of $v_{\epsilon, \theta}$ is [figure 4].
\[
e \sqrt{-1} \left( \pi m + 2 \delta m - 1 \right) e^{\sqrt{-1} \theta - \pi m - 1} l_1
\]

\[
e \sqrt{-1} \left( \pi m + 2 \delta m - 1 \right) e^{\sqrt{-1} \theta - \pi m - 1} l_2
\]

\[
e \sqrt{-1} \left( \pi m + 2 \delta m - 1 \right) R^{(j)}_{\psi_0,1}
\]

\[
e \sqrt{-1} \left( \pi m + 2 \delta m - 1 \right) \eta_{se} e^{\sqrt{-1} \pi m + \frac{\pi}{m}}
\]
From all the above arguments, we can see that the flows of the vector field $v_{r,\theta}$ stay inside the region $R^{(j)}_{\psi_0,1} \setminus (\omega^{-1}(D) \cap R^{(j)}_{\psi_0,1})$. Take a flow $\{(z(t), (s, e^{\sqrt{-1} \psi})) | t \geq 0\}$ inside $R^{(j)}_{\psi_0,1} \setminus (\omega^{-1}(D) \cap R^{(j)}_{\psi_0,1})$. If we set

$$R' := \left\{ (z, s, e^{\sqrt{-1} \psi}) \in \Delta \times \left[ 0, \frac{1}{3} \right] \times S^1 \mid -\frac{2\delta}{m+2} < \theta - \pi < \frac{3\delta}{m+2}, \quad \frac{\pi}{3} \leq \arg(z) + \frac{\theta - \pi}{m-1} < \frac{\pi}{m} + \frac{2\delta}{m-1} \right\},$$

then we have $R' \subset R^{(j)}_{\psi_0,1}$ and we can see by the argument similar to the former analysis on the direction of $v_{r,\theta}$ that flows of $v_{r,\theta}$ starting at points in $R' \setminus (\omega^{-1}(D) \cap R')$ stay inside $R' \setminus (\omega^{-1}(D) \cap R')$. Take any point $(z, s, e^{\sqrt{-1} \psi}) \in R^{(j)}_{\psi_0,1} \setminus R'$. If $z \neq 0$, then we have either $(z, s, e^{\sqrt{-1} \psi}) \in Q_{\psi_0,1}^{(j)}$ or $\frac{\pi}{3} < \arg(z) + \frac{\theta - \pi}{m-1} < \frac{\pi}{m} + \frac{2\delta}{m-1}$. So we have either $|z| < \eta_5$ or

$$4\pi \leq \arg \left( e^{\sqrt{-1} \frac{\theta - \pi}{m+1} e^{\sqrt{-1} \psi} z} \right) < 2\pi + \frac{2m\delta}{m-1}. \tag{37}$$

Combined with (35), we have $e^{\sqrt{-1} \psi}(z_m - e^m) \neq 0$. If $z = 0$, then $s > 0$ and we have $e^{\sqrt{-1} \psi}(z_m - e^m) \neq 0$ again. So $v_{r,\theta}$ does not vanish on $R^{(j)}_{\psi_0,1} \setminus R'$ and there is no limit point limit as $t \to \infty$ of $(z(t), s, e^{\sqrt{-1} \psi})$. Since the inequality (37) holds as long as $(z, s, e^{\sqrt{-1} \psi})$ lies in $R^{(j)}_{\psi_0,1} \setminus R'$, flows of $v_{r,\theta}$ do not stay inside $R^{(j)}_{\psi_0,1} \setminus R'$ and there exists $t_0 > 0$ such that $(z(t_0), s, e^{\sqrt{-1} \psi})$ is contained in the region $R' \setminus (\omega^{-1}(D) \cap R')$. If $(z(t), s, e^{\sqrt{-1} \psi}) \in R' \setminus (\omega^{-1}(D) \cap R')$, then we have

$$-\frac{(m-1)\pi}{3m} \leq \arg \left( \sum_{j=0}^{m-1} \left( z(t)e^{\sqrt{-1} \frac{\theta - \pi}{m+1} \psi} \right)^{m-1-j} \left( e^{\sqrt{-1} \frac{\theta - \pi}{m+1} \psi} \epsilon_{\psi_0}^{(j)} \right)^j \right) \leq \frac{(m-1)\pi}{3m}. \tag{38}$$

By the calculation

$$\frac{d}{dt} \left( \frac{1}{|z(t) - \epsilon_{\psi_0}^{(j)}|^{2m}} \right) = \frac{d}{dt} \left( \frac{1}{ (z(t) - \epsilon_{\psi_0}^{(j)})^m } \right) + \frac{1}{ (z(t) - \epsilon_{\psi_0}^{(j)})^m } \frac{d}{dt} \left( \frac{1}{ (z(t) - \epsilon_{\psi_0}^{(j)})^m } \right),$$

$$= \frac{1}{ (z(t) - \epsilon_{\psi_0}^{(j)})^m } \frac{dz(t)}{dt} \left( \frac{1}{ (z(t) - \epsilon_{\psi_0}^{(j)})^m } - \frac{m e^{\sqrt{-1} \psi}(z(t)m - \epsilon_{\psi_0}^{(j)}m) \frac{dz(t)}{dt} }{ (z(t) - \epsilon_{\psi_0}^{(j)})^{m+1} } \right),$$

$$= \frac{2m}{|z(t) - \epsilon_{\psi_0}^{(j)}|^{2m}} \text{Re} \left( e^{\sqrt{-1} \psi} \frac{z(t)m - \epsilon_{\psi_0}^{(j)}m}{z(t) - \epsilon_{\psi_0}^{(j)}} \right),$$

we can see

$$\frac{d}{dt} \left( \frac{1}{|z(t) - \epsilon_{\psi_0}^{(j)}|^{2m}} \right) = \frac{2m \text{Re} \left( e^{\sqrt{-1} \psi}(z(t)m - \epsilon_{\psi_0}^{(j)}m + \epsilon_{\psi_0}^{(j)}m - \epsilon_{\psi_0}^{(j)}m)^{-2} \cdot (\epsilon_{\psi_0}^{(j)}m - \epsilon_{\psi_0}^{(j)}m)^{-1} \right)}{|z(t) - \epsilon_{\psi_0}^{(j)}|^{2m}},$$

$$= \frac{2m}{|z(t) - \epsilon_{\psi_0}^{(j)}|^{2m}} \text{Re} \left( \sum_{j=0}^{m-1} \left( z(t)e^{\sqrt{-1} \frac{\theta - \pi}{m+1} \psi} \right)^{m-1-j} \left( e^{\sqrt{-1} \frac{\theta - \pi}{m+1} \psi} \epsilon_{\psi_0}^{(j)} \right)^j \right),$$

$$\geq \frac{2m}{|z(t) - \epsilon_{\psi_0}^{(j)}|^{2m}} \max \{|z(t)|, |\epsilon_j|\} m^{-1} \cos \left( \frac{(m-1)\pi}{3m} \right),$$

$$\geq \frac{2m}{|z(t) - \epsilon_{\psi_0}^{(j)}|^{2m}} \left( \frac{|z(t) - \epsilon_{\psi_0}^{(j)}|}{2} \right)^{m-1} \frac{1}{2},$$

$$\geq \frac{m}{2^{m-1}|z(t) - \epsilon_{\psi_0}^{(j)}|^{m+1}} \geq \frac{m}{4^m} > 0.$$
So we have \( \frac{1}{|z(t) - \epsilon \zeta_m|^2m} \geq \frac{m}{4^m} - C \) for some constant \( C > 0 \). Thus we have

\[
\lim_{t \to \infty} z(t) = \epsilon \zeta_m.
\]

and the flow of \( \nu_{\theta, \epsilon} \) starting at any point of \( R_{\psi_0,1}^{(j)} \setminus (\varpi^{-1}(D) \cap R_{\psi_0,1}^{(j)}) \) converges to \( (\epsilon \zeta_m, s, e^{\sqrt{-1} \psi}) \in \varpi^{-1}(D) \).

Case 2. \( -\frac{\pi}{m} \leq \arg(z_0) - \frac{2j\pi}{m} - \psi_0 < 0 \).

In this case, we take \( \delta > 0 \) satisfying \( \delta < \frac{\pi}{24m} \) and put

\[
\theta := \frac{2j(m-1)\pi}{m} - (m-1)\psi_0 + \pi - \delta.
\]

Then we have

\[
-\frac{\pi}{m} - \frac{\delta}{m-1} \leq \arg(z_0) + \frac{\theta - \pi}{m-1} \leq -\frac{\delta}{m-1}.
\]

We take \( \frac{1}{4} < s_1 < 1 \) and \( \eta > 0 \) similarly to Case 1 and put

\[
P_{\psi_0,2}^{(j)} = \left\{ (z, (s, e^{\sqrt{-1} \psi})) \in \Delta \times \left[ 0, \frac{1}{3} \right] \times S^1 \left| \begin{array}{l}
\frac{3\delta}{2m} < \psi + \frac{2j\pi}{m} + \frac{\theta - \pi}{m-1} < \frac{3\delta}{2m} \quad (m \neq 0) \\
\frac{2m-2}{3m} \leq \arg \left( e^{\sqrt{-1} \frac{m-1}{m} \psi} - e^{\sqrt{-1} \frac{s-1}{m} \psi} \right) < \frac{2m-2}{3m} \\
\theta = 0 \\
\frac{\pi}{m} - \frac{\delta}{m-1} < \arg(z) + \frac{\theta - \pi}{m-1} \leq \frac{\pi}{m} - \frac{\delta}{m-1} \quad \text{and} \quad \delta \neq 0
\end{array} \right. \right\}.
\]

\[
Q_{\psi_0,2}^{(j)} := \left\{ (z, (s, e^{\sqrt{-1} \psi})) \in \Delta \times \left[ 0, \frac{1}{3} \right] \times S^1 \left| \begin{array}{l}
\frac{3\delta}{2m} < \psi + \frac{2j\pi}{m} + \frac{\theta - \pi}{m-1} < \frac{3\delta}{2m} \quad (m \neq 0) \\
\frac{2m-2}{3m} \leq \arg \left( e^{\sqrt{-1} \frac{m-1}{m} \psi} - e^{\sqrt{-1} \frac{s-1}{m} \psi} \right) < \frac{2m-2}{3m} \\
\theta = 0 \\
\frac{6m}{\pi} < \arg \left( e^{\sqrt{-1} \frac{m-1}{m} \psi} - e^{\sqrt{-1} \frac{s-1}{m} \psi} \right) < \frac{6m}{2m-2} \\
\frac{2m-2}{3m} < \psi + \frac{2k\pi}{m} + \frac{\theta - \pi}{m-1} < \frac{2m-2}{3m} \quad \text{and} \quad \frac{\pi}{m} \leq \arg(z) + \frac{\theta - \pi}{m-1} \leq \frac{2\pi}{m} - \frac{2\delta}{m} \quad \text{for} \quad z \neq 0
\end{array} \right. \right\}.
\]

\[
R_{\psi_0,2}^{(j)} := P_{\psi_0,2}^{(j)} \cup Q_{\psi_0,2}^{(j)}.
\]

By the similar argument to Case 1, we can see that \( (z_0, s_0, e^{\sqrt{-1} \psi_0}) \in R_{\psi_0,2}^{(j)} \) and the flow \( (z(t), s, e^{\sqrt{-1} \psi})_{t \geq 0} \) of \( \nu_{\epsilon, \theta} \) starting at a point in \( R_{\psi_0,2}^{(j)} \setminus (\varpi^{-1}(D) \cap R_{\psi_0,2}^{(j)}) \) satisfies

\[
\lim_{t \to \infty} z(t) = \epsilon \zeta_m.
\]

If we put

\[
U := \{(0) \times \{(0) \times S^1 \} \cup \bigcup_{\psi_0,2} R_{\psi_0,2}^{(j)}
\]

then we can see by the construction of \( R_{\psi_0,2}^{(j)} \) that \( \{z \in \Delta \mid |z| < \frac{1}{4} \} \times \left[ 0, \frac{1}{3} \right] \times S^1 \) is contained in \( U \). So we can write

\[
U = \left( \left\{ z \in \Delta \mid |z| < \frac{1}{4} \right\} \times \left[ 0, \frac{1}{3} \right] \times S^1 \right) \cup \bigcup_{\psi_0,2} R_{\psi_0,2}^{(j)}
\]

and we can see that \( U \) is an open neighborhood of \( \{(0) \times \{(0) \times S^1 \} \cup \bigcup_{\psi_0,2} R_{\psi_0,2}^{(j)} \) in \( \Delta \times \left[ 0, \frac{1}{3} \right] \times S^1 \). If we put

\[
W_{\psi_0,2}^{(j)} := R_{\psi_0,2}^{(j)} \setminus (\varpi^{-1}(D) \cap R_{\psi_0,2}^{(j)})
\]

then we have an open covering

\[
U \setminus (U \cap \varpi^{-1}(D)) = \bigcup_{\psi_0,2} W_{\psi_0,2}^{(j)}
\]

This covering satisfies the statement of the proposition. \( \square \)

3.2. Fundamental solution with an asymptotic property. We use the same notations as in subsection 3.1. Take a point \( p_0 \in W_{\psi_0,2}^{(j)} \) and consider the holomorphic solution \( (z(\tau), s, e^{\sqrt{-1} \psi}) \) of the differential equation

\[
\frac{dz(\tau)}{d\tau} = e^{\sqrt{-1} \theta}(z(\tau)^m - e^m)
\]

satisfying \( (z(0), s, e^{\sqrt{-1} \psi}) = p_0 \), where \( \epsilon = se^{\sqrt{-1} \psi} \) and \( \theta = \theta_{\psi_0,2}^{(j)} \). If we take \( t_1, u_1 \in \mathbb{R} \) and if we fix \( t_1 + \sqrt{-1} u_1 \) constant, \( (z(t_1 + t_1 + \sqrt{-1} u_1), s, e^{\sqrt{-1} \psi})_{t \geq 0} \) coincides with the flow \( (z(t_1 + \sqrt{-1} u_1)(t), s, e^{\sqrt{-1} \psi}) \) of \( \nu_{\epsilon, \theta} \) satisfying \( z(t_1 + \sqrt{-1} u_1) = 0 \) at \( z(t_1 + \sqrt{-1} u_1) = 0 \). So we can extend the solution \( (z(\tau), s, e^{\sqrt{-1} \psi}) \) by an analytic
continuation to a holomorphic function in \( \tau \) on an open neighborhood of \( \mathbb{R}_{\geq 0} \) whose image by \( z(\tau) \) is an open neighborhood of the flow of \( v_{\epsilon, \theta} \) starting at the point \( p_0 \). Note that we have

\[
\lim_{t \to \infty} z(t + \sqrt{-1}u_1) = \epsilon C^j_m
\]

and \( z_{\epsilon, \theta}(t) = z(t + t_1 + \sqrt{-1}u_1) = z_{\epsilon, \theta}(t + t_1) \).

The following theorem is a weak unfolded analogue of the existence theorem of fundamental solutions with an asymptotic property [39, Theorem 12.1] in the irregular singular case. It is an easy restricted case of a more general theorem in [9] and [10], which is one of the main tools in the unfolding theory by Hurtubise, Lambert and Rousseau.

**Theorem 3.2** ([9, Theorem 5.3], [10, Theorem 2.5]). Consider the linear differential equation

\[
\begin{pmatrix}
\frac{df_1}{dz} \\
\vdots \\
\frac{df_r}{dz}
\end{pmatrix} = A(z, \epsilon, w)
\begin{pmatrix}
f_1 \\
\vdots \\
f_r
\end{pmatrix}
\]

on the polydisk \( \Delta \times \Delta \times \Delta^s \), where \( A(z, \epsilon, w) \) is an \( r \times r \) matrix of holomorphic functions in \( (z, \epsilon, w) = (z, \epsilon, w_1, \ldots, w_s) \in \Delta \times \Delta \times \Delta^s \) such that

\[
A(z, \epsilon, w) - \begin{pmatrix}
\nu_1(z, \epsilon, w) & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \nu_r(z, \epsilon, w)
\end{pmatrix} \in (z^m - \epsilon^m)M_r(C^0_{\Delta \times \Delta \times \Delta^s}),
\]

where \( \nu_1(z, \epsilon, w), \ldots, \nu_r(z, \epsilon, w) \) are polynomials in \( z \) whose coefficients are holomorphic functions in \( \epsilon, w \) and \( \nu_1(\epsilon^j_m, \epsilon, w), \ldots, \nu_r(\epsilon^j_m, \epsilon, w) \) are mutually distinct for any fixed \( j, \epsilon \) and \( w \). Then for a certain choice of the open covering \( \{W_{\psi, \xi}^{(j)}\} \) of \( U \) \( \setminus (\varpi^{-1}(D) \cap U) \) in Proposition 3.1, there are an open covering

\[
W_{\psi, \xi}^{(j)} \times \Delta^s = \bigcup_{p \in W_{\psi, \xi}^{(j)}} S_{\psi, \xi, p}^{(j)},
\]

and a matrix \( Y_\theta(z) = (y_1^\theta(z), \ldots, y_r^\theta(z)) \) of solutions on \( S_\theta := S_{\psi, \xi, p}^{(j)} \) of the differential equation (44), that is,

\[
\frac{dY_\theta(z)}{dz} = A(z, \epsilon, w)\frac{z^m - \epsilon^m}{z^m - \epsilon^m}Y_\theta(z)
\]

such that for the solution \( z(\tau) \) of the holomorphic differential equation [39] with the initial value \( z(0) = p \in S_{\psi, \xi, p}^{(j)}, \) the limit

\[
\lim_{t \to \infty} Y_\theta(z(t + u)) \exp\left(-\begin{pmatrix}
\int_0^t \nu_1(z(t + u))e^{\sqrt{-1}\theta dt} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \int_0^t \nu_r(z(t + u))e^{\sqrt{-1}\theta dt}
\end{pmatrix}\right) = C_u^\theta(s, e^{\sqrt{-1}\psi}, w)
\]

along the flow \( (z(t + u))_{t \geq 0} \) exists and the limit \( C_u^\theta(se^{\sqrt{-1}\psi}, w) \) is a diagonal matrix of functions continuous in \( s, e^{\sqrt{-1}\psi}, w, t_1, t_u \) and holomorphic in \( w \) and \( \psi \neq 0 \).

**Proof.** For the solution \( z(\tau) \) of the differential equation (39) with an initial value \( (z(0), s, e^{\sqrt{-1}\psi}) = p \) in \( W_{\psi, \xi}^{(j)} \), we consider \( z(t + u) \) for \( u \in \mathbb{C} \) with \( |u| \ll 1 \). If we write \( \epsilon := se^{\sqrt{-1}\psi} \), the restriction of the differential equation (40) to the flow \( z(t + u) \) of \( v_{\epsilon, \theta} \) becomes

\[
\begin{pmatrix}
\frac{df_1(z(t + u), \epsilon, w)}{dt} \\
\vdots \\
\frac{df_r(z(t + u), \epsilon, w)}{dt}
\end{pmatrix} = e^{\sqrt{-1}\theta}A(z(t + u), \epsilon, w)
\begin{pmatrix}
f_1(z(t + u), \epsilon, w) \\
\vdots \\
f_r(z(t + u), \epsilon, w)
\end{pmatrix}.
\]
Since the flow \((z(t + u), s, e^{\sqrt{-1} \psi}, w)\) is contained in \(W_{\psi_0, \xi}^{(j)} \times \Delta^4\), we have \(\lim_{t \to \infty} z(t + u) = \epsilon \zeta_m^j\) and
\[
\lim_{t \to \infty} e^{\sqrt{-1} \theta} A(z(t + u), \epsilon, w) = e^{\sqrt{-1} \theta} A(\epsilon \zeta_m^j, \epsilon, w)
\]
\[
= \left(\begin{array}{cccc}
(\epsilon^{\sqrt{-1} \theta} \nu_1(\epsilon \zeta_m^j, \epsilon, w)) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & (\epsilon^{\sqrt{-1} \theta} \nu_r(\epsilon \zeta_m^j, \epsilon, w))
\end{array}\right).
\]

We may assume by a suitable choice of \(\delta > 0\) for defining \(\theta = \theta_{\psi_0, \xi}^{(j)}\) in (34) and (38), that the real parts \(\text{Re} \left( e^{\sqrt{-1} \theta} \nu_1(\epsilon \zeta_m^j, \epsilon, w) \right), \ldots, \text{Re} \left( e^{\sqrt{-1} \theta} \nu_r(\epsilon \zeta_m^j, \epsilon, w) \right)\) of the eigenvalues of the matrix \(e^{\sqrt{-1} \theta} A(\epsilon \zeta_m^j, \epsilon, w)\) are mutually distinct. Moreover we may assume by replacing the order of a holomorphic frame that
\[
\text{Re} \left( e^{\sqrt{-1} \theta} \nu_1(\epsilon \zeta_m^j, \epsilon, w) \right) < \cdots < \text{Re} \left( e^{\sqrt{-1} \theta} \nu_r(\epsilon \zeta_m^j, \epsilon, w) \right)
\]
holds. As in the proof of Proposition 3.1 we have
\[
-\frac{(m - 1)\pi}{3m} \leq \text{arg} \left( \left( e^{\sqrt{-1} \frac{m - 1}{m} z(t + u)} \right)^{m - 1} \right) \leq \frac{(m - 1)\pi}{3m}
\]
for sufficiently large \(t > 0\). So we have
\[
\frac{d}{dt} |z(t + u)^m - \epsilon|^m = \frac{1}{2(|z(t + u)^m - \epsilon|^m)^{1/2}} \frac{d}{dt} (z(t + u)^m - \epsilon)|z(t + u)^m - \epsilon|^m
\]
\[
= 2 \text{Re} \left( m z(t + u)^{m - 1} \nu'(t + u)(z(t + u)^m - \epsilon) \right)
\]
\[
= \text{Re} \left( -m \left( e^{\sqrt{-1} \frac{m - 1}{m} z(t + u)} \right)^{m - 1} \right) |z(t + u)^m - \epsilon|^m
\]
\[
\leq -m \cos \left( \frac{(m - 1)\pi}{3\pi} \right) |z(t + u)^m - \epsilon|^m
\]
\[
\leq -m \left| z(t + u)^m - \epsilon \right| ^{m-1} |z(t + u)^m - \epsilon|^m
\]
for sufficiently large \(t > 0\), from which we have
\[
\frac{d}{dt} \left( |z(t + u)^m - \epsilon|^m \right) \geq \frac{m - 1}{m} |z(t + u)^m - \epsilon|^m
\]
\[
\geq \frac{m - 1}{2}.
\]
So there exists a constant \(C > 0\) such that
\[
|z(t + u)^m - \epsilon|^m \geq \frac{m - 1}{2} t - C
\]
holds for sufficiently large \(t > 0\). If we write \(\nu_k = \sum_{i=0}^q b_i(\epsilon, w) z^i\), we have
\[
\frac{d}{dt} e^{\sqrt{-1} \theta} \nu_k(z(t + u), \epsilon, w) = e^{\sqrt{-1} \theta} \sum_{i=0}^q i b_i(\epsilon, w) z(t + u)^i e^{\sqrt{-1} \theta} (z(t + u)^m - \epsilon).
\]
So there is a constant \(C' > 0\) satisfying \(\frac{d}{dt} e^{\sqrt{-1} \theta} \nu_j(z(t + u), \epsilon, w) \leq C' |z(t + u)^m - \epsilon|^m\) and
\[
\int_{a_0}^{\infty} \frac{d}{dt} e^{\sqrt{-1} \theta} \nu_k(z(t + u), \epsilon, w) \, dt \leq C' \int_{a_0}^{\infty} |z(t + u)^m - \epsilon|^m \, dt
\]
\[
\leq C' \int_{a_0}^{\infty} \left( \frac{m - 1}{2} t - C \right)^{-\frac{1}{m-1}} \, dt < \infty
\]
for a reference point \( a_0 \in \mathbb{R}_{>0} \). Similarly we have
\[
\int_{a_0}^{\infty} \left\| A(z(t+u), \epsilon, w) - \begin{pmatrix} \nu_1(z(t+u), \epsilon, w) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_r(z(t+u), \epsilon, w) \end{pmatrix} \right\| dt < \infty
\]
because the absolute values of the entries of the matrix
\[
A(z(t+u), \epsilon, w) - \begin{pmatrix} \nu_1(z(t+u), \epsilon, w) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_r(z(t+u), \epsilon, w) \end{pmatrix}
\]
are bounded by \( C'' |z(t+u)^m - \epsilon^m| \) for some constant \( C'' > 0 \). Thus, by the theorem of Levinson ([25 Theorem 1]), there are \( t_0 > 0 \) and a matrix
\[
Y^u(t, s, e^{\sqrt{-1}\theta}, w) = \left( y_1^u(t, s, e^{\sqrt{-1}\theta}, w), \ldots, y_r^u(t, s, e^{\sqrt{-1}\theta}, w) \right)
\]
of solutions \( y_1^u(t, s, e^{\sqrt{-1}\theta}, w), \ldots, y_r^u(t, s, e^{\sqrt{-1}\theta}, w) \) of the differential equation
\[
\frac{dy(t)}{dt} = e^{\sqrt{-1}\theta} A(z(t+u), \epsilon, w) y(t)
\]
defined for \( t > t_0 - b \) for some \( b > 0 \), which satisfies
\[
\lim_{t \to \infty} Y^u(t, s, e^{\sqrt{-1}\theta}, w) \exp \left( - \begin{pmatrix} \int_{t_0}^{t} \nu_1(z(t+u)) e^{\sqrt{-1}\theta} dt \\ \vdots \\ 0 \\ \int_{t_0}^{t} \nu_r(z(t+u)) e^{\sqrt{-1}\theta} dt \end{pmatrix} \right) = C_u(s, e^{\sqrt{-1}\theta}, w) = \begin{pmatrix} c_1(u) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_r(u) \end{pmatrix}
\]
with \( C_u(\epsilon, w) \) constant in \( z \) satisfying
\[
A(\epsilon \zeta_m^j, \epsilon, w) C_u(s, e^{\sqrt{-1}\psi}, w) = C_u(s, e^{\sqrt{-1}\psi}, w) \begin{pmatrix} \nu_1(\epsilon \zeta_m^j, \epsilon, w) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_r(\epsilon \zeta_m^j, \epsilon, w) \end{pmatrix}.
\]
Notice that \( y_k^u(t, s, e^{\sqrt{-1}\theta}, w) \) is constructed in [25] by applying an infinite sum and integrations of the form \( \int_{a_1}^{t} \) or \( \int_{t}^{\infty} \) to given functions in \( t, s, e^{\sqrt{-1}\psi}, w, u \) constructed from \( A(z, \epsilon, w) \). So we can see by their construction in [25] that the solutions \( y_k^u(t, s, e^{\sqrt{-1}\theta}, w) \) are functions continuous in \( s, e^{\sqrt{-1}\theta}, w, u \) and holomorphic in \( w, u \) and \( \epsilon \neq 0 \). Furthermore, \( C_u(s, e^{\sqrt{-1}\theta}, w) \) is a matrix of functions continuous in \( s, e^{\sqrt{-1}\theta}, w, u \) and holomorphic in \( w, u \) and \( \epsilon \neq 0 \). Since \( A(\epsilon \zeta_m^j, \epsilon, w) \) is a diagonal matrix with the distinct eigenvalues by the assumption, \( C_u(\epsilon, w) \) becomes a diagonal matrix.

By the fundamental theorem of ordinary linear differential equations, there exists a fundamental solution
\[
Y_\vartheta(z, s, e^{\sqrt{-1}\psi}, w) = \left( y_1^\vartheta(z, s, e^{\sqrt{-1}\psi}, w), \ldots, y_r^\vartheta(z, s, e^{\sqrt{-1}\psi}, w) \right)
\]
of the differential equation ([10], that is to say,
\[
\frac{dY_\vartheta}{dz} = A(z) \frac{z^m - s^me^{\sqrt{-1}\psi} \vartheta}{z^m - s^me^{\sqrt{-1}\psi} \vartheta} Y_\vartheta
\]
in a neighborhood of \( (z(t_0), s, e^{\sqrt{-1}\psi}, w) \) which satisfies the initial condition
\[
Y_\vartheta(z(t_0), s, e^{\sqrt{-1}\psi}, w) = \vartheta^\circ(t_0, s, e^{\sqrt{-1}\psi}, w).
\]
Here the suffix \( \vartheta \) means the data \( p, j, \psi_0, \xi \). Since the solutions of the linear differential equation ([10]) form a local system on \( U \setminus (D \cap U) \), we can extend \( Y_\vartheta(z) \) to a matrix of holomorphic functions in a neighborhood of \( \{ z(t) | t \geq t_0 \} \) by an analytic continuation. We fix \( w_0 \in \mathbb{C} \) close to the origin 0. Since both \( Y_\vartheta(z(t+u), s, e^{\sqrt{-1}\psi}, w) \) and \( Y_\psi(t, s, e^{\sqrt{-1}\psi}, w) \) satisfy the same linear differential equation
\[
\frac{dY}{dt} = e^{\sqrt{-1}\theta} A(z(t+u)) Y,
\]
there is a matrix $P(u)$ of functions continuous in $s, e^{\sqrt{-1}\psi}, w, u$ and holomorphic in $w$ and $u$ satisfying

$$Y_\theta(z(t+u), s, e^{\sqrt{-1}\psi}, w) = Y^u(t, s, e^{\sqrt{-1}\psi}, w) \ P(u)$$

for $t$ close to $t_0$. We put $\Lambda_k(t, u) := \exp \left( \int_{t_0}^{t} \nu_k(z(t+u)) e^{\sqrt{-1}\theta} dt \right)$. By (41), $\lim_{t \to \infty} \Lambda_k(t)^{-1} \Lambda_{k'}(t)$ is convergent if $k < k'$. If $u \in \mathbb{R}$ is a real number, we can see by the property (48) for $u = 0$ that

$$\lim_{t \to \infty} Y_\theta(z(t+u)) \ \exp \begin{pmatrix} -\int_{t_0}^{t} \nu_1(z(t+u)) e^{\sqrt{-1}\theta} dt & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\int_{t_0}^{t} \nu_r(z(t+u)) e^{\sqrt{-1}\theta} dt \end{pmatrix} = \lim_{t \to \infty} Y_\theta(z(t+u)) \ \exp \begin{pmatrix} -\int_{t_0}^{t+u} \nu_1(z(t')) e^{\sqrt{-1}\theta} dt' & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\int_{t_0}^{t+u} \nu_r(z(t')) e^{\sqrt{-1}\theta} dt' \end{pmatrix} = C_u(s, e^{\sqrt{-1}\psi}, w)$$

is convergent and its limit is a diagonal matrix. If we put

$$Y^u(t) = (y_1^u(t), \ldots, y_r^u(t)), \quad P(u) = \begin{pmatrix} p_{1,1}(u) & \cdots & p_{1,r}(u) \\ \vdots & \ddots & \vdots \\ p_{r,1}(u) & \cdots & p_{r,r}(u) \end{pmatrix},$$

then, for $u \in \mathbb{R},$

$$Y_\theta(z(t+u)) \ \begin{pmatrix} \Lambda_1(t)^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_r(t)^{-1} \end{pmatrix} = Y^u(t) \ P(u) \ \begin{pmatrix} \Lambda_1(t)^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_r(t)^{-1} \end{pmatrix}$$

$$= (y_1^u(t), \ldots, y_r^u(t)) \ \begin{pmatrix} p_{1,1}(u) & \cdots & p_{1,r}(u) \\ \vdots & \ddots & \vdots \\ p_{r,1}(u) & \cdots & p_{r,r}(u) \end{pmatrix} \ \begin{pmatrix} \Lambda_1(t)^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_r(t)^{-1} \end{pmatrix}$$

$$= \left( \sum_{k=1}^{r} p_{k,1}(u) \Lambda_1(t)^{-1} y_k^u(t), \ldots, \sum_{k=1}^{r} p_{k,r}(u) \Lambda_r(t)^{-1} y_k^u(t) \right)$$

is bounded when $t \to \infty$. Note that

$$\Lambda_i(t)^{-1} y_k^u(t) = (\Lambda_i(t)^{-1} \Lambda_k(t)) \ \left( \Lambda_k(t)^{-1} y_k^u(t) \right)$$

is divergent for $i < k$ when $t \to \infty$, because $\lim_{t \to \infty} \Lambda_i(t)^{-1} \Lambda_k(t)$ is divergent and $\lim_{t \to \infty} \Lambda_k(t)^{-1} y_k^u(t) = c_k(u) e_k \neq 0$. So we should have $p_{k,l}(u) = 0$ for $k > l$ and $u \in \mathbb{R}$ with $|u| < 1$. Since $p_{k,l}(u)$ is holomorphic in $u$, we have $p_{k,l}(u) = 0$ for $u \in \mathbb{C}$ with $|u| < 1$. In other words, $P(u)$ is an upper triangular matrix of holomorphic functions in $u$. Then we have, for $u \in \mathbb{C}$ with $|u| < 1$, that

$$\lim_{t \to \infty} Y_\theta(z(t+u)) \ \exp \begin{pmatrix} -\int_{t_0}^{t} \nu_1(z(s+u)) e^{\sqrt{-1}\theta} ds & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\int_{t_0}^{t} \nu_r(z(s+u)) e^{\sqrt{-1}\theta} ds \end{pmatrix} = \lim_{t \to \infty} \begin{pmatrix} p_{1,1}(u) & \cdots & p_{1,r}(u) \\ \vdots & \ddots & \vdots \\ p_{r,1}(u) & \cdots & p_{r,r}(u) \end{pmatrix} \ \begin{pmatrix} \Lambda_1(t)^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_r(t)^{-1} \end{pmatrix}$$

converges to a diagonal matrix $C_u^\theta(s, e^{\sqrt{-1}\psi}, w)$.
4. Construction of a Local Horizontal Lift

In this section, we construct an integrable connection which is a first order infinitesimal extension of a given local relative connection. We call this extension a local horizontal lift, or a block of local horizontal lifts in section 5, which is a key part in the construction of an unfolding of the unramified irregular singular generalized isomonodromic deformation. A basic idea in this section is to extend a local connection to a global connection on \( \mathbb{P}^1 \) with regular singularity at \( \infty \). Unfortunately, our construction of a local horizontal lift is not canonical but it is systematically determined. So it enables us to construct a non-canonical global horizontal lift in section 6 which induces an unfolded generalized isomonodromic deformation.

4.1. Extension of a Local Connection to a Global Connection on \( \mathbb{P}^1 \). Consider the divisor

\[
D := \{(z, \epsilon, w) \in \Delta \times \Delta^s \ | \ z^m - \epsilon^m = 0\}
\]

on the polydisk \( \Delta \times \Delta^s \), where \( \Delta = \{z \in \mathbb{C} \mid |z| < 1\} \). If we put

\[
D_j := \{(z, \epsilon, w) \in \Delta \times \Delta^s \mid z - \epsilon^j \zeta_m = 0\}
\]

for \( j = 1, \ldots, m \) with \( \zeta_m = \exp(\frac{2\pi i}{m}) \), then we can write

\[
D = D_1 + \cdots + D_m
\]

as an effective divisor on \( \Delta \times \Delta^s \). We consider a family of intervals

\[
\Gamma_{\Delta, j} = \{(s\zeta_m \epsilon, \epsilon, w) \in \Delta \times \Delta^s \mid 0 \leq s \leq 1\}
\]

which join the origin 0 and \( \zeta_m \epsilon \) and consider their union

\[
\Gamma_\Delta := \bigcup_{j=1}^m \Gamma_{\Delta, j}.
\]

We consider the embedding \( \Delta \times \Delta \times \Delta^s \in \mathbb{P}^1 \times \Delta \times \Delta^s = \mathbb{P}^1_{\Delta \times \Delta^s} \) and regard \( D \) as an effective divisor on \( \mathbb{P}^1 \times \Delta \times \Delta^s \).

We prepare a notation of diagonal matrix.

**Notation 4.1.** We denote the diagonal matrix whose \((k, k)\) entry is \(a_k\) by \( \text{Diag}(a_k) \):

\[
\text{Diag}(a_k) = \begin{pmatrix}
a_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & a_r
\end{pmatrix}
\]

Take mutually distinct complex numbers \(\mu_1, \ldots, \mu_r\) and a polynomial \(\nu(T) \in \mathcal{O}_D[T]\) given by

\[
\nu(T) = \sum_{i=0}^{r-1} \left( \sum_{j=0}^{m-1} c_{i,j} T^j \right) T^i
\]

with \(c_{i,j} \in \mathcal{O}_{\Delta \times \Delta^s}\) such that \(\nu(\mu_1)|_{p}, \ldots, \nu(\mu_r)|_{p}\) are distinct complex numbers at any point \(p \in D\).

We denote the closed interval \(\{t \in \mathbb{R} \mid 0 \leq t \leq 1\}\) by \([0, 1]\). We take a continuous map

\[
\tilde{\gamma} : [0, 1] \times \Delta \times \Delta^s \rightarrow \Delta \times \Delta \times \Delta^s
\]

and an open subset \(W \subset \Delta \times \Delta \times \Delta^s\) such that each fiber \(W_b\) over \(b \in \Delta \times \Delta^s\) is a disk containing \(D_b\) and that the boundary \(\partial W_b\) coincides with the image \(\tilde{\gamma}([0, 1] \times \{b\})\).

Let

\[
\nabla_{\Delta} : \mathcal{O}^{\oplus r}_{\Delta \times \Delta^s} \ni \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \mapsto \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + A(z, \epsilon, w) \begin{pmatrix} dz \\ z^m - \epsilon^m \end{pmatrix}
\]

be a relative connection on \(\Delta \times \Delta \times \Delta^s\) over \(\Delta \times \Delta^s\) satisfying

\[
A(z, \epsilon, w)|_{D} = \text{Diag}(\nu(\mu_k))|_D = \begin{pmatrix}
\nu(\mu_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \nu(\mu_r)
\end{pmatrix}
\]

be a relative connection on \(\Delta \times \Delta \times \Delta^s\) over \(\Delta \times \Delta^s\) satisfying

\[
A(z, \epsilon, w)|_{D} = \text{Diag}(\nu(\mu_k))|_D = \begin{pmatrix}
\nu(\mu_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \nu(\mu_r)
\end{pmatrix}
\]

For each point \(b \in \Delta \times \Delta^s\), we consider the restriction \(\nabla_{\Delta_b} := \nabla_{\Delta}|_{\Delta \times \{b\}}\) and its associated connection

\[
\nabla_{\Delta_b}^t : \mathcal{E}nd(\mathcal{O}^{\oplus r}_{\Delta \times \{b\}}) \ni u \mapsto \nabla_{\Delta_b} \circ u - u \circ \nabla_{\Delta_b} \in \mathcal{E}nd(\mathcal{O}^{\oplus r}_{\Delta \times \{b\}}) \otimes \Omega^1_{\Delta \times \{b\}}(D_b).
\]
We assume the following condition for $\nabla_\Delta$:

**Assumption 4.2.**

(i) the monodromy of $\nabla_\Delta$ along $\tilde{\gamma}_b$ has a diagonal representation matrix of holomorphic functions over $\Delta \times \Delta^*$ with $r$ distinct eigenvalues for any $b \in \Delta \times \Delta^*$ and

(ii) $H^0(\Delta \times \{b\}, \ker(\nabla^1_\Delta)) = \mathbb{C}$ for each $b \in \Delta \times \Delta^*$.

**Proposition 4.3.** There exist an open neighborhood $\mathcal{V}$ of $(0, 0)$ in $\Delta \times \Delta^*$ and a relative connection

$$
\nabla^1: (O^h_{\mathbb{P}^1 \times \mathcal{V}})^{\oplus r} \to (O^h_{\mathbb{P}^1 \times \mathcal{V}})^{\oplus r} \otimes \Omega^1_{\mathbb{P}^1 \times \mathcal{V}/\mathbb{P}^1 \cap (\Delta \times \mathcal{V}) \cup \{\infty\} \times \mathcal{V}})^{\oplus r}
$$

on $\mathbb{P}^1 \times \mathcal{V}$ over $\mathcal{V}$ admitting poles along $(D \cap (\Delta \times \mathcal{V})) \cup \{\infty\} \times \mathcal{V}$ such that the restriction $\nabla^1|_{\Delta \times \mathcal{V}}$ is isomorphic to the restriction $\nabla_\Delta|_{\Delta \times \mathcal{V}}$ of $\nabla_\Delta$ in $\partial \mathbb{D}$.

**Proof.** Let $\text{Mon}_r(\nabla_\Delta)$ be the monodromy matrix of $\nabla_\Delta$ along $\tilde{\gamma}$ with respect to a local basis of $\ker \nabla_\Delta$. We can take a contractible open subset $\mathcal{W} \subset \Delta \times \Delta^*$ with $\mathcal{W} \subset \subset \mathbb{C}$ such that the fiber $\mathcal{W}_b$ is a closed disk for each $b \in \Delta \times \Delta^*$ and that the fundamental group $\pi_1((\Delta \times \Delta^*) \setminus \mathcal{W}_b, *)$ is isomorphic to $\mathbb{Z}$ which is generated by $\tilde{\gamma}$. We can take a regular singular relative connection

$$
\nabla_{\infty}: O^h_{\mathbb{P}^1 \times \Delta \times \Delta^* \setminus \mathcal{W}} \to O^h_{\mathbb{P}^1 \times \Delta \times \Delta^* \setminus \mathcal{W}} \otimes \Omega^1_{\mathbb{P}^1 \times \Delta \times \Delta^* \setminus \mathcal{W}/\Delta \times \Delta^*}(\{\infty\} \times \Delta \times \Delta^*)
$$

such that the monodromy of $\nabla_{\infty}$ along $\tilde{\gamma}$ is given by $\text{Mon}_r(\nabla_\Delta)$ and that the set of eigenvalues of $\text{Res}_{\infty}(b') \left( \nabla_{\infty}|_{\mathbb{P}^1 \times \{b'\} \cap \mathcal{W} \cap \{\infty\} \times \mathcal{V}} \right)$ is contained in $\{z \in \mathbb{C} | 0 \leq \text{Re}(z) < 1 \}$ for any $b' \in \Delta \times \Delta^*$. Note that $((O^h_{\Delta \times \Delta^* \setminus \mathcal{W}})^{\oplus r}, \nabla_\Delta|_{\Delta \times \Delta^* \setminus \mathcal{W}})$ and $((O^h_{\Delta \times \Delta^* \setminus \mathcal{W}})^{\oplus r}, \nabla_{\infty}|_{\Delta \times \Delta^* \setminus \mathcal{W}})$ are isomorphic, because their corresponding representations of the fundamental group $\pi_1((\Delta \times \Delta^*) \setminus \mathcal{W}, \star) \cong \mathbb{Z}$ are given by the same monodromy matrix $\text{Mon}_r(\nabla_\Delta)$. So we can patch $\nabla_{\infty}$, $\nabla_\Delta|_{\Delta \times \Delta^*}$ and obtain a global relative connection

$$
\nabla_0: E_0 \to E_0 \otimes \Omega^1_{\mathbb{P}^1 \times \Delta \times \Delta^* \setminus \mathcal{W}/\mathcal{W}}(D \cup \{\infty\} \times \Delta \times \Delta^*)
$$

on $\mathbb{P}^1 \times \Delta \times \Delta^*$ over $\Delta \times \Delta^*$. We can write

$$
E_0|_{\mathbb{P}^1 \times \{(0, 0)\}} \cong \bigoplus_{k=1}^r O_{\mathbb{P}^1}(a_k)
$$

with $a_1 \geq a_2 \geq \cdots \geq a_r$. Assume that $a_1 > a_r$. For some choice of $k$, the projection

$$
\psi'_k: E_0 \to \ker(\nabla_0|_{\{\infty\} \times \Delta \times \Delta^*} - \nu(\mu_k) \frac{dz}{z^m - \epsilon^m}|_{\{\infty\} \times \Delta \times \Delta^*})
$$

$$
\to \ker(\nabla_0|_{\{\infty\} \times \Delta \times \Delta^*} - \nu(\mu_k) \frac{dz}{z^m - \epsilon^m}|_{\{\infty\} \times \Delta \times \Delta^*})
$$

satisfies $\psi'_0|_{\{\infty, (0, 0)\} \cap (\{\infty\} \times \mathcal{V})} = \ker(\nabla_0|_{\{\infty, (0, 0)\} \cap \mathcal{V}} - \nu(\mu_k) \frac{dz}{z^m - \epsilon^m}|_{\{\infty, (0, 0)\} \cap \mathcal{V}}) \cong O_{\{\infty, (0, 0)\}}$. Then there is an open neighborhood $\mathcal{V}$ of $(0, 0)$ in $\Delta \times \Delta^*$ such that

$$
\psi_0 := \psi'_0|_{\mathbb{P}^1 \times \mathcal{V}: E_0|_{\mathbb{P}^1 \times \mathcal{V}} \to \ker(\nabla_0|_{\{\infty\} \times \mathcal{V}} - \nu(\mu_k) \frac{dz}{z^m - \epsilon^m}|_{\{\infty\} \times \mathcal{V}})
$$

is surjective. If we put $\left( E_1, \nabla_1 \right) := (\ker \psi_0, \nabla_0|_{\ker \psi_0})$, then $\nabla_1$ is a relative connection on $\mathbb{P}^1 \times \mathcal{V}$ over $\mathcal{V}$ admitting poles along $(D \cap (\Delta \times \mathcal{V})) \cup \{\infty\} \times \mathcal{V}$ and we have

$$
E_1|_{\mathbb{P}^1 \times \{(0, 0)\}} \cong O_{\mathbb{P}^1}(a_1 - 1) \oplus \bigoplus_{k=2}^r O_{\mathbb{P}^1}(a_k).
$$

Similarly we can choose a surjection $\psi_1: E_1 \to O_{\{\infty\} \times \mathcal{V}}$ after shrinking $\mathcal{V}$ such that $\ker \psi_1$ is preserved by $\nabla_1$ and that $\psi_1(O_{\{\infty\}}) = O_{\{\infty\} \times \mathcal{V}}$ for $\tilde{a}_1 := \max\{a_1 - 1, a_2\}$. Then we put $\left( E_2, \nabla_2 \right) := (\ker \psi_1, \nabla_1|_{\ker \psi_1})$. Repeating this procedure, we finally obtain $\left( E_N, \nabla_N \right)$ such that $E_N|_{\mathbb{P}^1 \times \mathcal{V}} \cong O_{\mathbb{P}^1}(N_0)^{\oplus r}$. So the connection $\nabla_N \otimes O(-N_0)$ satisfies the condition of the proposition. 

\qed
4.2. The construction of a local horizontal lift. We use the same notations as in subsection 4.1. We consider the non-reduced analytic space \( \mathbb{P}^1 \times \Delta \times \Delta^* \times \text{Spec} \mathbb{C}[h]/(h^2) \). For an analytic open subset \( U \subset \mathbb{P}^1 \times \Delta \times \Delta^* \), we denote by \( U[h] \) the analytic open subspace of \( \mathbb{P}^1 \times \Delta \times \Delta^* \times \text{Spec} \mathbb{C}[h]/(h^2) \) whose underlying set of points coincides with \( U \). In this subsection, we will construct an extension of the relative connection \( \nabla^{\mathbb{P}^1_1} \) constructed in Proposition 4.3 to an integrable connection on \( \mathbb{P}^1 \times \mathcal{V}[h] \) over \( \mathcal{V} \). This produces a block of local horizontal lifts defined in Definition 5.7 which is a key concept in the construction of a global horizontal lift in subsection 5.3.

Recall that the sheaf of holomorphic differential forms \( (\Omega^1_{(\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h]/\Delta \times \Delta^*})^{\text{hol}} \) on \( (\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h] \) is given by

\[
(\Omega^1_{(\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h]/\Delta \times \Delta^*})^{\text{hol}} = I^{\text{hol}}_{\Delta_1}(\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h]/\Delta \times \Delta^* / (I^{\text{hol}}_{\Delta_1}((\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h]/\Delta \times \Delta^*)^2,
\]

where \( I^{\text{hol}}_{\Delta_1}((\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h]/\Delta \times \Delta^*) \) is the ideal sheaf of \( \mathcal{O}^{\text{hol}}_{(\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h]/\Delta \times \Delta^*} \) which defines the diagonal

\[
(\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h] \rightarrow (\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h] \times \Delta \times \Delta^* \rightarrow (\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h].
\]

Let \( t_{(\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h]} : (\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h] \rightarrow (\mathbb{P}^1_1 \times \Delta^* \times \Gamma_\Delta)[h] \) be the inclusion. We put \( \mathcal{V}[h] := \mathcal{V} \times \text{Spec} \mathbb{C}[h]/(h^2) \). We denote \( D \times \Delta \times \Delta^* \mathcal{V}, \Gamma \times \Delta \times \Delta^* \mathcal{V} \) by \( D_\mathcal{V}, \Gamma_\mathcal{V} \), respectively and denote \( D \times \Delta \times \Delta^* \mathcal{V}[h] \) by \( D_\mathcal{V}[h] \). We first construct an extension of the relative connection \( \nabla^{\mathbb{P}^1_1} \) to a relative connection on \( \mathbb{P}^1 \times \mathcal{V}[h] \) over \( \mathcal{V}[h] \). We need the following lemma:

**Lemma 4.4.** Let \( A_1, \ldots, A_m \) be elements of \( \text{End}_\mathbb{C}(\mathcal{C}^r) \) satisfying

\[
\bigcap_{j=1}^m \ker \text{ad}(A_j) = \mathbb{C} \cdot \text{id},
\]

where \( \text{ad}(A_j) : \text{End}_\mathbb{C}(\mathcal{C}^r) \ni X \mapsto A_j X - X A_j \in \text{End}_\mathbb{C}(\mathcal{C}^r) \) is the adjoint map. Then we have

\[
\sum_{j=1}^m \ker \text{ad}(A_j) = \ker \left( \text{End}_\mathbb{C}(\mathcal{C}^r) \xrightarrow{\text{Tr}} \mathbb{C} \right).
\]

**Proof.** In general we have \( \text{ad}(A_j) = -\text{ad}(A_j) \), because

\[
\text{Tr}(\text{ad}(A_j)(X) \cdot B) = \text{Tr}(X \cdot \text{ad}(A_j)(B)) = \text{Tr}(X \cdot (A_j B - BA_j)) = \text{Tr}((X A_j - A_j X)B + A_j XB - XBA_j) = \text{Tr}((X A_j - A_j X)B) + \text{Tr}(A_j XB) - \text{Tr}(XBA_j) = \text{Tr}(-\text{ad}(A_j)(X) \cdot B)
\]

for any \( X, B \in \text{End}_\mathbb{C}(\mathcal{C}^r) \). So there are exact sequences

\[
0 \longrightarrow \ker \text{ad}(A_j) \longrightarrow \text{End}_\mathbb{C}(\mathcal{C}^r) \xrightarrow{\text{ad}(A_j)} \text{End}_\mathbb{C}(\mathcal{C}^r) \longrightarrow (\ker \text{ad}(A_j))^\vee \longrightarrow 0.
\]

for \( j = 1, \ldots, m \). Since \( \text{End}_\mathbb{C}(\mathcal{C}^r) \xrightarrow{\pi} \text{End}_\mathbb{C}(\mathcal{C}^r)/\sum_{j=1}^m \ker \text{ad}(A_j) = \ker \text{ad}(A_j) \) is the largest quotient vector space satisfying \( \pi \circ \text{ad}(A_j) = 0 \) for \( j = 1, \ldots, m \), its dual is given by

\[
\left( \text{End}_\mathbb{C}(\mathcal{C}^r)/\sum_{j=1}^m \ker \text{ad}(A_j) \right)^\vee = \bigcap_{j=1}^m \ker \text{ad}(A_j) = \bigcap_{j=1}^m \ker \text{ad}(A_j) = \mathbb{C} \cdot \text{id} \subset \text{End}_\mathbb{C}(\mathcal{C}^r).
\]

Taking the dual again, we obtain

\[
\text{End}_\mathbb{C}(\mathcal{C}^r)/\sum_{j=1}^m \ker \text{ad}(A_j) = (\mathbb{C} \cdot \text{id})^\vee = \text{End}_\mathbb{C}(\mathcal{C}^r)/\ker \left( \text{End}_\mathbb{C}(\mathcal{C}^r) \xrightarrow{\text{Tr}} \mathbb{C} \right).
\]

Thus we have \( \sum_{j=1}^m \ker \text{ad}(A_j) = \ker \left( \text{End}_\mathbb{C}(\mathcal{C}^r) \xrightarrow{\text{Tr}} \mathbb{C} \right). \)

For the relative connection

\[
\nabla^{\mathbb{P}^1_1} : (\mathcal{O}_{\mathbb{P}^1_1 \times \mathcal{V}}^{\text{hol}})^{\otimes r} \longrightarrow (\mathcal{O}_{\mathbb{P}^1_1 \times \mathcal{V}}^{\text{hol}})^{\otimes r} \otimes \Omega^{1_1}_{\mathbb{P}^1_1 \times \mathcal{V}/\mathcal{V}} (D_\mathcal{V} \cup (\{\infty\} \times \mathcal{V}))^{\text{hol}}
\]
constructed in Proposition 4.3, let \( A_\infty(z, \epsilon) \frac{dz}{z^m - \epsilon^m} \) be the connection matrix of \( \nabla^{p_1} \). Since \( \nabla^{p_1} \) is regular singular at \( z = \infty \), we can write

\[
A_\infty(z, \epsilon) = A_{\infty, 0}(\epsilon) + A_{\infty, 1}(\epsilon) z + \cdots + A_{\infty, m-1}(\epsilon) z^{m-1}
\]

with matrices \( A_{\infty, 0}(\epsilon), \ldots, A_{\infty, m-1}(\epsilon) \) of holomorphic functions in \((\epsilon, w) \in \mathcal{V}\). From the proof of Proposition 4.3, there exists an invertible matrix \( P(z, \epsilon) \) of holomorphic functions on a neighborhood of \( D_{\epsilon} \) such that

\[
\left( P(z, \epsilon)^{-1} dP(z, \epsilon) + P(z, \epsilon)^{-1} \nabla A_\infty(z, \epsilon) \frac{dz}{z^m - \epsilon^m} P(z, \epsilon) \right) \bigg|_{D_{\epsilon}} = \text{Diag}(\nu(\mu_k)) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}}.
\]

Here recall that \( \nu(T) \) is given in (44). Since \( \nu(\mu_1), \ldots, \nu(\mu_r) \) are distinct for any \( p \in D_{\epsilon} \), there is a polynomial \( \tilde{\psi}(T) = \tilde{a}_{r-1} T^{r-1} + \cdots + \tilde{a}_1 T + \tilde{a}_0 \in \mathcal{O}^{\text{hol}}_{D_{\epsilon}}[T] \) satisfying

\[
\tilde{\psi} \left( \text{Diag}(\nu(\mu_k)) \right) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}} = \text{Diag}(\mu_k) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}}.
\]

After shrinking \( \mathcal{V} \), we can take lifts \( a_0(z, \epsilon), a_1(z, \epsilon), \ldots, a_{r-1}(z, \epsilon) \in \mathcal{O}^{\text{hol}}_{\mathcal{V}}[z] \) of \( \tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{r-1} \) and put

\[
\psi(T) := a_{r-1}(z, \epsilon) T^{r-1} + \cdots + a_1(z, \epsilon) T + a_0(z, \epsilon) \in \mathcal{O}[z][T] = \mathcal{O}[z][\nabla].
\]

Here we may assume that \( a_0(z, \epsilon), \ldots, a_{r-1}(z, \epsilon) \) are polynomials in \( z \) of degree less than \( m \). Then \( \psi(A_\infty(z, \epsilon)) \) is a matrix of polynomials in \( z \) and we have

\[
P(z, \epsilon)^{-1} \psi(A_\infty(z, \epsilon)) P(z, \epsilon) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}} = \text{Diag}(\mu_k) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}}.
\]

For \( l = 0, 1, \ldots, r - 1 \) and for \( j' = 0, 1, \ldots, m - 2 \), we have

\[
\text{res}_{z=\infty} \left( \text{Tr} \left( \psi(A_\infty(z, \epsilon))^l \frac{z^{j'} dz}{z^m - \epsilon^m} \right) \right) = -\sum_{j=1}^{m} \text{res}_{z=\epsilon \zeta_k} \left( \text{Tr} \left( \psi(A_\infty(z, \epsilon))^l \frac{z^{j'} dz}{z^m - \epsilon^m} \right) \right)
\]

\[
= -\sum_{j=1}^{m} \text{res}_{z=\epsilon \zeta_k} \left( \text{Tr} \left( P(z, \epsilon)^{-1} \psi(A_\infty(z, \epsilon))^l P(z, \epsilon) \frac{z^{j'} dz}{z^m - \epsilon^m} \right) \right)
\]

\[
= -\sum_{j=1}^{m} \text{res}_{z=\epsilon \zeta_k} \left( \text{Tr} \left( \psi \left( \text{Diag}(\nu(\mu_k)) \right) \frac{z^{j'} dz}{z^m - \epsilon^m} \right) \right)
\]

\[
= \text{res}_{z=\infty} \left( \text{Tr} \left( \text{Diag}(\mu_k) \frac{z^{j'} dz}{z^m - \epsilon^m} \right) \right) = 0.
\]

We can write

\[
\psi(A_\infty(z, \epsilon))^l = \sum_{q=0}^{Q} C_q^{(l)}(\epsilon) z^q
\]

for matrices \( C_q^{(l)}(\epsilon) \) constant in \( z \). We define

\[
\Xi_{l, j}(z, \epsilon) := \sum_{j'=0}^{r-1-m} \sum_{0 \leq p + j' = j \leq Q} c_{pm} z^{j'} C_{pm+j'-j}^{(l)}(\epsilon)
\]

for \( j = 0, 1, \ldots, m - 1 \) and \( l = 0, 1, \ldots, r - 1 \). In other words, \( \Xi_{l, j}(z, \epsilon) \) is obtained from \( z^{j'} \psi(A_\infty(z, \epsilon))^l \) by substituting \( \epsilon^m \) in \( z^m \). Then we have

\[
A_\infty(z, \epsilon) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}} = \nu(\text{Diag}(\mu_k)) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}}
\]

\[
= \sum_{l=0}^{r-1} \sum_{j=0}^{m-1} c_{l,j} z^{j} \psi(A_\infty(z, \epsilon))^l \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}} = \sum_{l=0}^{r-1} \sum_{j=0}^{m-1} c_{l,j} \Xi_{l, j}(z, \epsilon) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\epsilon}},
\]

from which we have

\[
A_\infty(z, \epsilon) = \sum_{l=0}^{r-1} \sum_{j=0}^{m-1} c_{l,j} \Xi_{l, j}(z, \epsilon).
\]
Note that we have

\[
\text{res}_{z=\infty} \left( \text{Tr} \left( \Sigma_{l,j}(z, \varepsilon) \frac{dz}{z^m - \varepsilon^m} \right) \right) = - \text{Tr} \left( \sum_{0 \leq p_m + m-1-j \leq Q} \varepsilon^{p_m} c_{p_m+m-1-j}^{(l)}(\varepsilon) \right) = \text{res}_{z=\infty} \left( \text{Tr} \left( z^j \psi(A_{\infty}(z, \varepsilon))^j \frac{dz}{z^m - \varepsilon^m} \right) \right) = 0
\]

for \( j = 0, 1, \ldots, m - 2 \).

We put \( \mathcal{V}_m := \mathcal{V} \times \Delta \times \Delta^* (\text{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^m) \times \Delta^*) \) and \( \mathcal{V}_m[\tilde{h}] := \mathcal{V}_m \times \text{Spec} \mathbb{C}[\tilde{h}]/(\tilde{h}^2) \). Then the restriction

\[
\nabla^{\text{flat}}_{\Delta \times \mathcal{V}_m[\tilde{h}],\mathcal{V}_m} : (\mathcal{O}^{\text{hol}}_{\Delta \times \mathcal{V}_m[\tilde{h}]} \otimes \Omega^1_{\Delta \times \mathcal{V}_m[\tilde{h}]/\mathcal{V}_m}) \rightarrow \left( (\mathcal{O}^{\text{hol}}_{\Delta \times \mathcal{V}_m[\tilde{h}]} \otimes \Omega^1_{\Delta \times \mathcal{V}_m[\tilde{h}]/\mathcal{V}_m}) \right)^{\otimes r} \]

becomes an irregular singular connection, where \( A_{\infty}(z, \varepsilon) \) is the restriction of \( A_{\infty}(z, \varepsilon) \) to \( \mathbb{P}^1 \times \mathcal{V}_m \). If we put

\[
B_{0,l,j}(z) := P(z, \varepsilon) \text{Diag} \left( \left( \mu_k^j z^j \text{d}z \right)^{\otimes r} \right) P(z, \varepsilon)^{-1}
\]

for \( j = 0, 1, \ldots, m - 2 \), then \( B_{0,l,j}(z) \) becomes a matrix of single valued meromorphic forms whose pole order at \( z = 0 \) is at most \( m - 1 \), because \( \mu_k^j z^j \text{d}z \) has no residue part. If we put

\[
A_{\infty}(z, \varepsilon) \left( z, \varepsilon \right) \left( \nabla_{\Delta \times \mathcal{V}_m,\mathcal{V}_m} \right)^{\otimes r} = \text{Diag} \left( \mu_k^j z^j \text{d}z \right)^{\otimes r} \left( \nabla_{\Delta \times \mathcal{V}_m,\mathcal{V}_m} \right)^{\otimes r}
\]

then we can see that \( A_{\infty}(z, \varepsilon) \left( z, \varepsilon \right) \left( \nabla_{\Delta \times \mathcal{V}_m,\mathcal{V}_m} \right)^{\otimes r} = \text{Diag} \left( \mu_k^j z^j \text{d}z \right)^{\otimes r} \left( \nabla_{\Delta \times \mathcal{V}_m,\mathcal{V}_m} \right)^{\otimes r} \).

Let us consider the integrable connection

\[
\nabla^{\text{flat}}_{\Delta \times \mathcal{V}_m[\tilde{h}],\mathcal{V}_m} : (\mathcal{O}^{\text{hol}}_{\Delta \times \mathcal{V}_m[\tilde{h}]} \otimes \Omega^1_{\Delta \times \mathcal{V}_m[\tilde{h}]}) \rightarrow \left( (\mathcal{O}^{\text{hol}}_{\Delta \times \mathcal{V}_m[\tilde{h}]} \otimes \Omega^1_{\Delta \times \mathcal{V}_m[\tilde{h}]}) \right)^{\otimes r} \]

If we choose a fundamental solution \( Y_{0,\infty}(z) \) of \( \nabla^{\text{flat}}_{\Delta \times \mathcal{V}_m[\tilde{h}],\mathcal{V}_m} \), then we have

\[
\frac{\partial}{\partial z} \left( Y_{0,\infty} - \tilde{h} B_{0,l,j}(z) Y_{0,\infty} \right) \left( \nabla_{\Delta \times \mathcal{V}_m[\tilde{h}],\mathcal{V}_m} \right)^{\otimes r} = - A_{\infty}(z, \varepsilon) \left( \nabla_{\Delta \times \mathcal{V}_m[\tilde{h}],\mathcal{V}_m} \right)^{\otimes r}
\]

Thus

\[
\tilde{Y}_{0,\infty}(z, \tilde{h}) := Y_{0,\infty}(z) - \tilde{h} B_{0,l,j}(z) Y_{0,\infty}(z)
\]

is a fundamental solution of \( \nabla^{\text{flat}}_{\Delta \times \mathcal{V}_m[\tilde{h}],\mathcal{V}_m} \). If denote the monodromy matrix of \( Y_{0,\infty}(z) \) along \( \tilde{\gamma} \) by \( \text{Mon}_{\tilde{\gamma}} \), then \( \tilde{Y}_{0,\infty}(z, \tilde{h}) = Y_{0,\infty}(z) - \tilde{h} B_{0,l,j}(z) Y_{0,\infty}(z) \) has the monodromy matrix \( \text{Mon}_{\tilde{\gamma}} \) along \( \tilde{\gamma} \), because \( B_{0,l,j}(z) \) is single valued. By the similar method to that in the proof of Proposition 4.3, we can construct a global connection

\[
\nabla^{\text{flat}}_{\Delta \times \mathcal{V}_m[\tilde{h}],\mathcal{V}_m} : (\mathcal{O}^{\text{hol}}_{\Delta \times \mathcal{V}_m[\tilde{h}]} \otimes \Omega^1_{\Delta \times \mathcal{V}_m[\tilde{h}]}) \rightarrow \left( (\mathcal{O}^{\text{hol}}_{\Delta \times \mathcal{V}_m[\tilde{h}]} \otimes \Omega^1_{\Delta \times \mathcal{V}_m[\tilde{h}]}) \right)^{\otimes r} \]

satisfying

\[
\text{res}_{z=\infty} \left( \tilde{A}_{e_m,\tilde{h},l,j}(z) \frac{dz}{z^m} \right) = 0
\]
such that the restriction of $\nabla_{\mathbb{P}^1 \times \mathcal{V}_m}[\tilde{\mathfrak{h}}, \nu_{i,j}]$ to $\mathbb{P}^1 \times \mathcal{V}_m$ coincides with the restriction $\nabla^{\Omega_l}_{\mathbb{P}^1 \times \mathcal{V}_m}$ of $\nabla^{\Omega_l}$ to $\mathbb{P}^1 \times \mathcal{V}_m$ and that the restriction of $\nabla^{\Omega_l}_{\mathbb{P}^1 \times \mathcal{V}_m}[\tilde{\mathfrak{h}}, \nu_{i,j}]$ to $\Delta \times \mathcal{V}_m[\tilde{\mathfrak{h}}]$ is isomorphic to the irregular singular relative connection induced by $\nabla^{\Omega_l}_{\Delta \times \mathcal{V}_m}[\tilde{\mathfrak{h}}, \nu_{i,j}]$. By construction, there is a convergent power series

$$\sum_{l'=0}^{\infty} R_{0,j,l'}^{(l)} z^{l'}$$

such that

$$(A_{\infty}(z, \epsilon) + \tilde{\mathfrak{h}} A_{\epsilon, m, \tilde{\mathfrak{h}}, \nu_{i,j}}(z)) \frac{dz}{z^m} = \tilde{\mathfrak{h}} \sum_{l'=1}^{\infty} l' R_{0,j,l'}^{(l)} z^{l'-1} dz + \left(1 - \tilde{\mathfrak{h}} \sum_{l'=0}^{\infty} R_{0,j,l'}^{(l)} z^{l'}\right) \left(A_{\infty}(z, \epsilon) + \tilde{\mathfrak{h}} A_{\epsilon, m, \tilde{\mathfrak{h}}, \nu_{i,j}}(z)\right) \frac{dz}{z^m} \left(1 + \tilde{\mathfrak{h}} \sum_{l'=0}^{\infty} R_{0,j,l'}^{(l)} z^{l'}\right),$$

which implies

$$\Xi_{i,j}(z, \epsilon)|_{D_{\nu_{i,j}} = \psi(A_{\infty}(z, \epsilon))^{1/2} |_{D_{\nu_{i,j}}} = \text{Diag}(\nu_z z^j) |_{D_{\nu_{i,j}}} = A_{\epsilon, m, \tilde{\mathfrak{h}}, \nu_{i,j}}(z)|_{D_{\nu_{i,j}}} = \left(\tilde{A}_{\epsilon, m, \tilde{\mathfrak{h}}, \nu_{i,j}}(z) + \sum_{j'=0}^{m-1} \sum_{l'=0}^{l'-1} \left[A_{\infty, j', l'}(\tilde{\mathfrak{h}}), R_{0,j,l'}^{(l)}\right] z^{l'}\right)|_{D_{\nu_{i,j}}}. $$

So we have

$$(49) \quad \Xi_{i,j}(z, \epsilon) = \tilde{A}_{\epsilon, m, \tilde{\mathfrak{h}}, \nu_{i,j}}(z) + \sum_{j'=0}^{m-1} \sum_{l'=0}^{l'-1} \left[A_{\infty, j', l'}(\tilde{\mathfrak{h}}), R_{0,j,l'}^{(l)}\right] z^{l'}.$$ 

If we put

$$B_{0,i,j}(z) := B_{0,i,j}(z) - \sum_{l'=0}^{\infty} R_{0,j,l'}^{(l)} z^{l'},$$

then the connection on $(\mathcal{O}^{(\Omega_l)}_{\Delta \times \mathcal{V}_m}[\mathfrak{h}])^{\oplus r}$ given by the connection matrix

$$(A_{\infty}(z, \epsilon) + \tilde{\mathfrak{h}} A_{\epsilon, m, \tilde{\mathfrak{h}}, \nu_{i,j}}(z)) \frac{dz}{z^m} + B_{0,i,j}(z) d\tilde{\mathfrak{h}}$$

is isomorphic to $\nabla^{\Omega_l}_{\Delta \times \mathcal{V}_m}[\mathfrak{h}, \nu_{i,j}]$ and satisfies the integrability condition.

For each $u \in \bigcap_{j=0}^{m-1} \ker(\text{ad}(A_{\infty, j}(\epsilon)))$, we have $u \cdot A_{\infty}(z, \epsilon) \frac{dz}{z^m - \epsilon^m} - A_{\infty}(z, \epsilon) \frac{dz}{z^m - \epsilon^m} u = 0$. So $u|_{\Delta \times \{b\}}$ is a section of $\ker(\nabla_{\Delta}^l)$ on $\Delta \times \{b\}$ for each $b \in \mathcal{V}$, which is a scalar endomorphism by Assumption 4.2 (ii). Then we have $u \in \mathcal{O}_\mathcal{V} \cdot \text{id}$ and

$$(50) \quad \bigcap_{j=0}^{m-1} \ker(\text{ad}(A_{\infty, j}(\epsilon))) = \mathcal{O}_\mathcal{V} \cdot \text{id}$$

follows. So we can see

$$\sum_{j=0}^{m-1} \text{im}(\text{ad}(A_{\infty, j}(\epsilon))) = \text{ker} \left(\text{End}_{\mathcal{O}_\mathcal{V}}(\mathcal{O}_{\mathcal{V}}^{\oplus r}) \xrightarrow{\text{Tr}} \mathcal{O}_\mathcal{V}\right),$$

because the equality for the restriction to each $b' \in \mathcal{V}$ holds by Lemma 4.4. Then, after shrinking $\mathcal{V}$, there are matrices $R_{0,j,0}^{(l)}(\epsilon), \ldots, R_{j,m-1}^{(l)}(\epsilon)$ constant in $z$ such that the $z^{m-1}$-coefficient of $\Xi_{i,j}(z, \epsilon)$ given in (46) is expressed by

$$\sum_{0 \leq pm + m' - 1 - j \leq Q} e^{pm + m' - 1 - j}(\epsilon) = \sum_{l'=0}^{m-1} \left[A_{\infty, m-l'-1}(\epsilon), R_{j,l'}^{(l)}(\epsilon)\right].$$
because of (47). Here we may assume $R_{j,t}^{(l)}(\epsilon)|_{\epsilon m = 0} = R_{0,j,t'}^{(l)}$ by using (49). For $l = 0, 1, \ldots, r - 1$ and for $j = 0, 1, \ldots, m - 2$, we define

$$
\tilde{\Xi}_{l,j}(z, \epsilon) := \Xi_{l,j}(z, \epsilon) - \sum_{q=0}^{m-1} \sum_{0 < t' \leq m-1-q} \left[ A_{\infty,q}(\epsilon), R_{j,t'}^{(l)}(\epsilon) \right] z^{q+t'} - \sum_{q=0}^{m-1} \sum_{m-1 < t' \leq m-1} \left[ A_{\infty,q}(\epsilon), R_{j,t'}^{(l)}(\epsilon) \right] \epsilon^{m-q+t'-m}.
$$

Then we have

$$\text{res}_{z=\infty} \left( \frac{d z}{z^m - \epsilon^m} \frac{dz}{z^m - \epsilon^m} \right) = 0$$

for $j = 0, 1, \ldots, m - 2$, $l = 0, 1, \ldots, r - 1$ and

$$\text{res}_{z=\infty} \left( \frac{dz}{z^m - \epsilon^m} \right) = P(z, \epsilon) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_V} - \frac{dz}{z^m - \epsilon^m} \bigg|_{D_V}.$$

Let

$$\nabla_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi}}}} : (C^0_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi}}}})^{\otimes r} \rightarrow (C^0_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi}}}})^{\otimes r} \oplus \Omega^1_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi}}}} \left( D_{\psi^1} \cup (\infty \times \psi_{\varphi^1}) \right)^{\otimes m}$$

be the relative connection defined by

$$\nabla_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi}}}} \left( \begin{array}{c} f_1 \\ \vdots \\ f_r \end{array} \right) = \left( \begin{array}{c} df_1 \\ \vdots \\ df_r \end{array} \right) + \left( A_{\infty}(z) + \epsilon \tilde{\Xi}_{l,j}(z) \right) \frac{dz}{z^m - \epsilon^m} \left( \begin{array}{c} f_1 \\ \vdots \\ f_r \end{array} \right).$$

Note that $\text{res}_{z=\infty}(\nabla_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi_{\varphi^1 \times \psi}}}}) = 0$. By construction, there is an invertible matrix $\tilde{P}(z, \tilde{h})$ such that

$$\left( \tilde{P}(z, \tilde{h})^{-1} d \tilde{P}(z, \tilde{h}) + \tilde{P}(z, \tilde{h})^{-1} \left( A_{\infty}(z) + \epsilon \tilde{\Xi}_{l,j}(z) \right) \frac{dz}{z^m - \epsilon^m} \tilde{P}(z, \tilde{h}) \right) \bigg|_{D_{\psi^1}} = \text{Diag}(\nu(\mu_k + \epsilon \mu_k \bar{z}^l)) \frac{dz}{z^m - \epsilon^m} \bigg|_{D_{\psi^1}}.$$
So we can write

\[
C(\tilde{\epsilon}, \tilde{h}) = \sum_{l=0}^{r-1} b_l(\tilde{\epsilon}, \tilde{h})\text{Mon}_\infty(\tilde{\epsilon})^l,
\]

because \(\text{Mon}_\infty(\tilde{\epsilon})\) has the \(r\) distinct eigenvalues at each \(b \in \mathcal{V}_\infty\). Shrinking \(\mathcal{V}\), we can take lifts \(b_l(\epsilon, \tilde{h})\) of \(b_l(\tilde{\epsilon}, \tilde{h})\) as holomorphic functions in \(\epsilon\). If we replace \(\tilde{Y}_\infty(z, \epsilon, \tilde{h})\) by \(\hat{Y}_\infty(z, \epsilon, \tilde{h}) = \sum_{l=0}^{r-1} b_l(\tilde{\epsilon}, \tilde{h})\text{Mon}_\infty(\tilde{\epsilon})^l\), then the restriction of \(\hat{Y}_\infty(z, \epsilon, \tilde{h})\) to \((U_\infty \times \mathcal{V}_\infty[\tilde{h}]) \setminus (\Gamma_\infty \times \mathcal{V}_\infty[\tilde{h}])\) coincides with \(Y_\infty(z, \epsilon) - \hat{h}B_{0,l,j}(z)Y_\infty(z, \epsilon)\).

If we define

\[
B_{l,j}(z, \epsilon, \tilde{h}) := -\frac{\partial \hat{Y}_\infty(z, \epsilon, \tilde{h})}{\partial \tilde{h}}Y_\infty(z, \epsilon, \tilde{h})^{-1},
\]

we have \(B_{l,j}(z, \epsilon, 0) = B_{0,l,j}(z)\). By an analytic continuation, we can regard \(B_{l,j}(z, \epsilon)\) as a matrix of single valued holomorphic functions on \((\mathbb{P}^1 \times \mathcal{V}) \setminus \Gamma_\mathcal{V}\). Let us consider the connection

\[
\nabla_{\mathbb{P}^1 \times \mathcal{V}[\tilde{h}], v_{l,j}}^{flat} : (\Omega^\text{hol}_{\mathbb{P}^1 \times \mathcal{V}[\tilde{h}]} \otimes \Omega^1_{\mathcal{V}[\tilde{h}]}/\mathcal{V}(\mathbb{P}^1 \times \mathcal{V}[\tilde{h}])^\text{hol}) = \left( f_1, \ldots, f_r \right) \mapsto \left( df_1, \ldots, df_r \right) + \left( (A_\infty(z) + \tilde{h}B_{l,j}(z, \epsilon)) \frac{dz}{z_m - \epsilon} + B_{l,j}(z, \epsilon) \tilde{h} d\tilde{h} \right)
\]

where \(t_{\mathcal{V}[\tilde{h}]} : (\mathbb{P}^1 \times \mathcal{V} \setminus \Gamma_\mathcal{V})[\tilde{h}] \to (\mathbb{P}^1 \times \mathcal{V}[\tilde{h}])\) is the canonical inclusion.

**Proposition 4.5.** The connection \(\nabla_{\mathbb{P}^1 \times \mathcal{V}[\tilde{h}], v_{l,j}}^{flat}\) defined in [53] is an integrable connection.

**Proof.** The curvature form of \(\nabla_{\mathbb{P}^1 \times \mathcal{V}[\tilde{h}], v_{l,j}}^{flat}\) becomes

\[
d \left( A_\infty(z) + \tilde{h}B_{l,j}(z) \frac{dz}{z_m - \epsilon} + B_{l,j}(z) d\tilde{h} \right) + \left[ \left( A_\infty(z) + \tilde{h}B_{l,j}(z) \frac{dz}{z_m - \epsilon} + B_{l,j}(z) d\tilde{h} \right) \left( A_\infty(z) + \tilde{h}B_{l,j}(z) \frac{dz}{z_m - \epsilon} + B_{l,j}(z) d\tilde{h} \right) \right] = -\tilde{\Xi}_{l,j}(z) \frac{dz}{z_m - \epsilon} + \partial B_{l,j}(z) \frac{dz}{z_m - \epsilon} + \left( \frac{\partial A_\infty(z)}{\partial \tilde{h}} + \tilde{h} \frac{\partial B_{l,j}(z)}{\partial \tilde{h}} \right) \frac{dz}{z_m - \epsilon}
\]

So \(\nabla_{\mathbb{P}^1 \times \mathcal{V}[\tilde{h}], v_{l,j}}^{flat}\) is an integrable connection. \(\square\)

4.3 Comparison with the asymptotic property in the unfolding theory by Hurtubise, Lambert and Rousseau. In the unfolding theory by Hurtubise, Lambert and Rousseau in [9, 10], unfolded Stokes matrices for unfolded linear differential equations are defined. So our integrable connection \(\nabla_{\mathbb{P}^1 \times \mathcal{V}[\tilde{h}], v_{l,j}}^{flat}\)
constructed in Proposition 4.1 induces unfolded Stokes matrices but we cannot expect that these matrices are constant in \( \hat{h} \). Although we cannot produce any positive result on the asymptotic property concerned with the integrable connection \( \nabla^{\text{flat}}_{\mathfrak{p}^1 \times \mathfrak{r}[\hat{h},\psi,\zeta]} \) defined in (4.4), it will be worth pointing out what is the difficulty.

We use the same notation as in subsection 4.1 and in subsection 4.2. We consider the multivalued function
\[
\tau_\epsilon(z) := \int \frac{dz}{z^m - \epsilon^m}
\]
which is single valued on \( \mathbb{P}^1_{\Delta \times \Delta^*} \setminus \Gamma_\Delta \). Under a suitable choice of path integral, we may assume that \( \tau_\epsilon(z) \) does not vanish on \( \Gamma_\Delta \setminus (\Gamma_\Delta \cap D) \). Let
\[
\varpi: [0,1) \times S^1 \rightarrow \Delta
\]
\[
(s, e^{\sqrt{-1}t}) \mapsto se^{\sqrt{-1}t}
\]
be the polar blow up. We can regard \( \Delta \times [0,1] \times S^1 \times \Delta^* \subset \mathbb{C} \times [0,1] \times S^1 \times \Delta^* \subset \mathbb{P}^1 \times [0,1] \times S^1 \times \Delta^* \) and an open covering
\[
U \setminus ((id \times \varpi \times id)^{-1}(D) \cap U) = \bigcup_{\psi_0, \xi} \bigcup_{j=1}^{2} W_{\psi_0, \xi}^{(j)}
\]
such that any flow of the vector field
\[
v_{\epsilon, \theta} = \Re \left( e^{\sqrt{-1}\theta}(z^m - \epsilon^m) \right) \frac{\partial}{\partial x} + \Im \left( e^{\sqrt{-1}\theta}(z^m - \epsilon^m) \right) \frac{\partial}{\partial y}
\]
starting at a point in \( W_{\psi_0, \xi}^{(j)} \) has an accumulation point in \((id \times \varpi^{-1} \times id)^{-1}(D) \cap U\), where \( x = \Re(z) \), \( y = \Im(z) \). Here \( \theta = \theta_{\psi_0, \xi}^{(j)} \in \mathbb{R} \) is determined by \( j, \psi_0, \xi \) as in the proof of Proposition 3.1.

We take an open covering
\[
(\varpi \times id_{\Delta^*})^{-1}(\mathcal{V}) = \bigcup_{b \in (\varpi \times id_{\Delta^*})^{-1}(\mathcal{V})} \tilde{\mathcal{Y}}_b
\]
by small contractible open subsets \( \mathcal{V}^\prime \) of \((\varpi \times id_{\Delta^*})^{-1}(\mathcal{V})\). By Theorem 3.2 we can see that there are an open covering
\[
(\Delta \times \mathcal{V}^\prime \setminus \mathcal{Y}_{\psi_0, \xi}) \cap W_{\psi_0, \xi}^{(j)} = \bigcup_{p \in W_{\psi_0, \xi}^{(j)}} S_{\psi_0, \xi, \varrho}^{(j)} = \bigcup_{p \in S_{\partial}} S_{\partial}
\]
with \( \varrho = (j, \psi_0, \xi, p) \) and a matrix
\[
Y_{\partial}(z, s, e^{\sqrt{-1}\theta}, w, h) = \left( \begin{array}{c} \tilde{y}_1^{\varrho}(z, s, e^{\sqrt{-1}\theta}, w, h), \ldots, \tilde{y}_r^{\varrho}(z, s, e^{\sqrt{-1}\theta}, w, h) \end{array} \right)
\]
of functions on \( S_{\partial} \times \Delta_\delta \) for some \( \delta > 0 \), satisfying
\[
\frac{d\tilde{Y}_{\partial}(z, s, e^{\sqrt{-1}\theta}, w, h)}{dz} = -A_\infty(z, \epsilon, w) + h\tilde{z}_0^{(j)}(z, \epsilon, w) \tilde{Y}_{\partial}(z, s, e^{\sqrt{-1}\theta}, w, h),
\]
such that the limit
\[
\lim_{t \rightarrow \infty} \tilde{P}(z_\partial(t), \hat{h}) \tilde{Y}_{\partial}(z_\partial(t), \hat{h}) \text{ Diag}(\exp(\int_{t_0}^{t} (\nu_{(\mu, \kappa)}(z_\partial(t)) + h\rho_{\hat{h}}\mu_\zeta(t)\bar{z}_0(t)e^{\sqrt{-1}\theta}dt))) = I_r
\]
is the identity matrix, where \( z_\partial(t) \) is a flow of \( v_{\epsilon, \theta} \) in \( S_{\partial} = S_{\psi_0, \xi, \varrho}^{(j)} \) and \( \theta = \theta_{\psi_0, \xi}^{(j)} \) is determined from \( \vartheta = (j, \psi_0, \xi, p) \). We denote the restriction of \( \tilde{Y}_{\partial}(z, s, e^{\sqrt{-1}\theta}, w, h) \) to \( S_{\partial}[\hat{h}] \) by \( \tilde{Y}_{\partial}(z, \hat{h}) \) and denote the restriction of \( \tilde{Y}_{\partial}(z, s, e^{\sqrt{-1}\theta}, h) \) to \( S_{\partial} \setminus \{0\} \) by \( Y_{\partial}(z, \hat{h}) \). By (56), we have
\[
\lim_{t \rightarrow \infty} \tilde{P}(z_\partial(t), \hat{h}) \tilde{Y}_{\partial}(z_\partial(t), \hat{h}) \text{ Diag}(\exp(\int_{t_0}^{t} (\nu_{(\mu, \kappa)}(z_\partial(t)) + h\rho_{\hat{h}}\mu_\zeta(t)\bar{z}_0(t)e^{\sqrt{-1}\theta}dt))) = I_r
\]
from which \( \tilde{Y}_{\partial}(z, \hat{h}) \text{ Diag}(\exp(\int_{t_0}^{t} (\nu_{(\mu, \kappa)}(z_\partial(t)) + h\rho_{\hat{h}}\mu_\zeta(t)\bar{z}_0(t)e^{\sqrt{-1}\theta}dt))) = I_r \) is bounded on \( S_{\partial}[\hat{h}] \) and in particular \( Y_{\partial}(z, \hat{h}) \text{ Diag}(\exp(\int_{t_0}^{t} (\nu_{(\mu, \kappa)}(z_\partial(t)) + h\rho_{\hat{h}}\mu_\zeta(t)\bar{z}_0(t)e^{\sqrt{-1}\theta}dt))) = I_r \) is bounded on \( S_{\partial} \).

Recall that we can write \( Y_\infty(z) = (y_1^{\infty}(z), \ldots, y_r^{\infty}(z)) \) for \( y_\infty^{\infty}(z) := \hat{y}_k(z, 0) \).

We take a family of loops \( \gamma: [0,1] \times \mathcal{V}^\prime \rightarrow (\Delta \times \mathcal{V}^\prime) \setminus \Gamma_\mathcal{V'} \) satisfying \( \gamma(0, w) = \gamma(1, w), p_2(\gamma(t, w)) = w \) and that \( \gamma(\bullet, w): [0,1] \rightarrow \Delta \times \{w\} \) is homotopic to \( \hat{\gamma}(\bullet, w) \) for any \( w \in \mathcal{V} \). From the analysis of flows in Proposition 3.1 we may assume that there are points \( 0 = t_1 < t_2 < \cdots < t_f < 1 \) such that \( t_i \in S_{\partial} \),
lim_{t \to \infty} z_\delta(t) = \epsilon_{\ell m}^j$ and that either $j_{i+1} = j_i + 1$ or $j_{i+1} = j_i$ with $\epsilon_{\ell m}^j \in S_{\delta_i} \cap S_{\delta_{i+1}}$ holds. Here in the case of $\epsilon_{\ell m}^j \in S_{\delta_i} \cap S_{\delta_{i+1}}$, we can further assume that a flow $z_\delta(t)$ lie in $S_{\delta_i} \cap S_{\delta_{i+1}}$ which is accumulated to $\epsilon_{\ell m}^j$ and a flow $z_{\delta_{i+1}}(t)$ lie in $S_{\delta_i} \cap S_{\delta_{i+1}}$ which is accumulated to $\epsilon_{\ell m}^j$.

**Lemma 4.6.** Assume that flows $z_\delta(t)$ (resp. $z_{\delta'}$) of $v_{z, h}$ (resp. $v_{z, h'}$) in $S_{\delta}$ (resp. $S_{\delta'}$) lie in $S_{\delta} \cap S_{\delta'}$ for $\delta, \delta'$ and that $\lim_{t \to \infty} z_{\delta}(t) = \lim_{t \to \infty} z_{\delta'}(t) = \epsilon_{\ell m}^j \in S_{\delta} \cap S_{\delta'}$. We take a permutation $\sigma$ of $\{1, \ldots, r\}$ satisfying

$$\text{Re}(e^{\sqrt{-1} \theta}(\mu_{\sigma(1)})(\epsilon_{\ell m}^j)) > \cdots > \text{Re}(e^{\sqrt{-1} \theta}(\mu_{\sigma(r)})(\epsilon_{\ell m}^j)),$$

Assume that

$$\tilde{Y}_{\delta'}(z, \bar{h}) = \tilde{Y}_{\delta}(z, \bar{h})C_{\delta, \delta'}(\bar{h})$$

holds under an analytic continuation along a path in $S_{\delta} \cup S_{\delta'}$. Then

$$(e_{\sigma(1)}, \ldots, e_{\sigma(r)})^{-1}C_{\delta, \delta'}(\bar{h})(e_{\sigma(1)}, \ldots, e_{\sigma(r)})$$

is an upper triangular matrix.

**Proof.** We put

$$\Lambda_k(z, \bar{h}) := \exp \left( \int (\nu(\mu_k) + \bar{h}\mu_k^1 z^q) \frac{dz}{z^m - e^{m}} \right).$$

If $k < k'$, then $\Lambda_k(z, \bar{h})^{-1}\Lambda_{k'}(z, \bar{h})$ tends to 0 when $z$ tends to $\epsilon_{\ell m}^j$. Note that

$$(\text{Diag}((\Lambda_k(z, \bar{h})))^{-1}C_{\delta, \delta'}(\bar{h})\text{Diag}((\Lambda_k(z, \bar{h}))) = (\tilde{Y}_{\delta}(z, \bar{h})\text{Diag}((\Lambda_k(z, \bar{h}))))^{-1}\tilde{Y}_{\delta'}(z, \bar{h})\text{Diag}((\Lambda_k(z, \bar{h})))$$

tends to a matrix of bounded functions when $z$ tends to $\epsilon_{\ell m}^j$ on $S_{\delta} \cap S_{\delta'}$.

If we put

$$C'(\bar{h}) := (e_{\sigma(1)}, \ldots, e_{\sigma(r)})^{-1}C_{\delta, \delta'}(\bar{h})(e_{\sigma(1)}, \ldots, e_{\sigma(r)}) = \begin{pmatrix} c_{1,1}(\bar{h}) & \cdots & c_{1,r}(\bar{h}) \\ \vdots & \ddots & \vdots \\ c_{r,1}(\bar{h}) & \cdots & c_{r,r}(\bar{h}) \end{pmatrix},$$

then we have

$$(e_{\sigma(1)}, \ldots, e_{\sigma(r)})^{-1}(\text{Diag}((\Lambda_k(z, \bar{h})))^{-1}C_{\delta, \delta'}(\bar{h})\text{Diag}((\Lambda_k(z, \bar{h}))) (e_{\sigma(1)}, \ldots, e_{\sigma(r)})$$

$$= \begin{pmatrix} \Lambda_{\sigma(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_{\sigma(r)} \end{pmatrix}^{-1} \begin{pmatrix} c_{1,1}(\bar{h}) & \cdots & c_{1,r}(\bar{h}) \\ \vdots & \ddots & \vdots \\ c_{r,1}(\bar{h}) & \cdots & c_{r,r}(\bar{h}) \end{pmatrix} \begin{pmatrix} \Lambda_{\sigma(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_{\sigma(r)} \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda_{\sigma(1)}(z, \bar{h}) \Lambda_{\sigma(1)}(z, \bar{h})^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Lambda_{\sigma(r)}(z, \bar{h}) \Lambda_{\sigma(r)}(z, \bar{h})^{-1} \end{pmatrix}.$$

Since $\Lambda_{\sigma(k)}(z, \bar{h})^{-1}\Lambda_{\sigma(k')}(z, \bar{h})$ is divergent for $k > k'$, we should have $c_{k', k}(\bar{h}) = 0$ for $k' > k$.

By an analytic continuation we can write

$$\tilde{Y}_{\delta_i}(z, \bar{h}) = \tilde{Y}_{\infty}(z, \bar{h})C_{\infty, \delta_i}(\bar{h})$$

from which we have

$$\tilde{Y}_{\theta_{i+1}}(z, \bar{h}) = \tilde{Y}_{\theta_i}(z, \bar{h})C_{\infty, \theta_i}(\bar{h})^{-1}C_{\infty, \theta_{i+1}}(\bar{h}).$$

If $j_i = j_{i+1}$, then $(e_{\sigma(1)}, \ldots, e_{\sigma(r)})^{-1}C_{\infty, \theta_i}(\bar{h})^{-1}C_{\infty, \theta_{i+1}}(\bar{h})(e_{\sigma(1)}, \ldots, e_{\sigma(r)})$ is an upper triangular matrix for a permutation $\sigma$ by Lemma 4.6. The matrix $C_{\infty, \theta_i}(\bar{h})$ is analogous to an unfolded Stokes matrix given in [7] but we cannot say from its construction that it is constant in $\bar{h}$.

We remark that the restriction $\tau_{n}(z)^{-1}B_{0, i, q}(z) |_{m = 0} = -(m - 1)^{-1}B_{0, i, q}(z)$ to the irregular singular locus $m = 0$ is bounded around $z = 0$ by its construction. We can see that

$$B_{0, i, q}(z) = -\frac{\partial \tilde{Y}_{\infty}(z, \bar{h})}{\partial \bar{h}} \tilde{Y}_{\infty}(z)^{-1} Y_{\theta_i}(z)^{-1 - \frac{\partial Y_{\theta_i}(z, \bar{h})}{\partial \bar{h}} Y_{\theta_i}(z)^{-1} + Y_{\theta_i}(z)^{-1}C_{\infty, \theta_i}(\bar{h})^{-1} \frac{\partial C_{\infty, \theta_i}(\bar{h})}{\partial \bar{h}} Y_{\theta_i}(z)^{-1}.}$$
By the following proposition, we can say that $\tau_{e}(z)^{-1} \frac{\partial \tilde{Y}_{\psi}(z, \tilde{h})}{\partial h} Y_{\psi}(z)^{-1}$ is bounded on $S_{\psi}$. However, $\tau_{e}(z)^{-1} Y_{\psi}(z)^{-1} \frac{\partial C_{S_{\psi}, \psi}(\bar{h})}{\partial h} C_{S_{\psi}, \psi}(0)^{-1} Y_{\psi}(z)^{-1}$ is not bounded unless

$$(e_{\sigma(1)}, \ldots, e_{\sigma(r)})^{-1} C_{S_{\psi}, \psi}(0)^{-1} \frac{\partial C_{S_{\psi}, \psi}(\bar{h})}{\partial h} (e_{\sigma(1)}, \ldots, e_{\sigma(r)})$$

is an upper triangular matrix. So we can not say the boundedness of $\tau_{e}(z)^{-1} B_{\gamma}(z)$ on $S_{\gamma}$. This is one of the reasons why we cannot get a canonical global horizontal lift in section 5.

**Proposition 4.7.** $\tau_{e}(z)^{-1} \frac{\partial}{\partial h} \tilde{Y}_{\psi}(z, \tilde{h}) Y_{\psi}(z)^{-1}$ is bounded on $S_{\psi}$.

**Proof.** Since the limit in (57) is uniform in $\tilde{h}$, we can see that

$$T_{\phi}(z, \tilde{h}) := \tilde{Y}_{\phi}(z, \tilde{h}) \text{Diag} \left( \exp \left( \int \nu(\mu_{k}) + \overline{h} \mu_{k}^{*} \right) \frac{d\tilde{z}}{z_{m}} \right)$$

and its partial derivative in $\tilde{h}$ is bounded on $\Delta \times \mathbb{V}[\tilde{h}]$. So

$$\frac{\partial T_{\phi}(z, \tilde{h})}{\partial \tilde{h}} = \frac{\partial \tilde{Y}_{\phi}(z, \tilde{h})}{\partial \tilde{h}} \text{Diag} \left( \exp \left( \int \nu(\mu_{k}) + \overline{h} \mu_{k}^{*} \right) \frac{d\tilde{z}}{z_{m}} \right) + \tilde{Y}_{\phi}(z) \frac{\partial}{\partial \tilde{h}} \text{Diag} \left( \exp \left( \int \nu(\mu_{k}) + \overline{h} \mu_{k}^{*} \right) \frac{d\tilde{z}}{z_{m}} \right)$$

is bounded on $S_{\psi}$. So it is sufficient to show that $\tau_{e}(z)^{-1} \text{Diag} (f_{\mu_{k}^{*}} z^{q} \frac{d\tilde{z}}{z_{m}})$ is bounded.

If $\epsilon = 0$,

$$\left| \tau_{e}(z)^{-1} \int \mu_{k}^{*} q \frac{d\tilde{z}}{z_{m}} \right| = \left| \left(1 - \frac{m - 1}{(m - 1)z_{m - 1}} \right)^{-1} \int \mu_{k}^{*} q \frac{d\tilde{z}}{z_{m - 1}} \right|$$

$$= \left| (m - 1)z_{m - 1} \left( \frac{-\mu_{k}^{*}}{(m - 1)z_{m - 1}} + (\text{constant}) \right) \right|$$

$$\leq (m - 1) |\mu_{k}^{*}| |\tilde{z}|^{q} \left( m - q - 1 \right) + (\text{constant})$$

is bounded on each $S_{\psi} \cap (\Delta \times \mathbb{V}^{-1}(0) \times \Delta^{*})$.

If $\epsilon \neq 0$, we can write

$$\mu_{k}^{*} q \frac{d\tilde{z}}{z_{m} - \epsilon_{m}} = \sum_{j=1}^{m} a_{j}^{k} \left( z - \epsilon_{m}^{j} \right) \frac{d\tilde{z}}{z_{m}}$$

for $0 \leq q \leq m - 2$. Then

$$\left| \tau_{e}(z)^{-1} \int \mu_{k}^{*} q \frac{d\tilde{z}}{z_{m} - \epsilon_{m}} \right| = \left| \sum_{j=1}^{m} \frac{\log(z - \epsilon_{m}^{j})}{\epsilon_{m}} \prod_{j' \neq j} \left( \zeta_{j'} - \zeta_{j}^{m} \right) \right|^{-1} \int_{z_{0}}^{z} \sum_{j=1}^{m} a_{j}^{k} \frac{d\tilde{z}}{z - \zeta_{j}^{m} \epsilon}$$

$$= \left| \sum_{j=1}^{m} \frac{\log(z - \epsilon_{m}^{j})}{\epsilon_{m}} \prod_{j' \neq j} \left( \zeta_{j'} - \zeta_{j}^{m} \right) \right|^{-1} \left( \sum_{j=1}^{m} a_{j}^{k} \log(z - \zeta_{m}^{j} \epsilon) + (\text{constant}) \right)$$

$$\leq \sum_{j=1}^{m} \sum_{j' \neq j} \frac{\log(z - \epsilon_{m}^{j})}{\epsilon_{m}} \left( \zeta_{j'} - \zeta_{j}^{m} \right)^{-1} \left| a_{j}^{k} \right| \frac{\log(z - \zeta_{m}^{j} \epsilon)}{|\epsilon^{m-1}|} + (\text{constant})$$

is bounded on each $S_{\psi} \cap \{ \epsilon \neq 0 \}$.

Thus $\tau_{e}(z)^{-1} \text{Diag}(f_{\mu_{k}^{*}} z^{q} \frac{d\tilde{z}}{z_{m}})$ is bounded on $S_{\psi}$ and the proposition follows. \qed

In a precise setting in the paper [10] by Hurtubise and Rousseau, they consider a linear differential equation on $\mathbb{P}^{1}$ with poles along the unfolding divisor and two regular singular points $\infty^{H,R}$, $R^{H,R}$. So we should associate a relative connection

$$\nabla_{\mathbb{P}^{1} \times \mathbb{V}[\tilde{h}], \nu_{r,q}} : \left( \mathcal{O}^{\text{hol}}_{\mathbb{P}^{1} \times \mathbb{V}[\tilde{h}]} \right)^{\nu_{r}} \longrightarrow \left( \mathcal{O}^{\text{hol}}_{\mathbb{P}^{1} \times \mathbb{V}[\tilde{h}]} \right)^{\nu_{r}} \otimes \Omega_{\mathbb{P}^{1} \times \mathbb{V}[\tilde{h}] / \mathbb{V}[\tilde{h}]} \left( D_{\mathbb{V}[\tilde{h}]} \cup \left( \{ \infty^{H,R}, R^{H,R} \} \times \mathbb{V}[\tilde{h}] \right) \right)^{\text{hol}}.$$
such that $\nabla^{p_{i}^{3}}_{X \times Y, m_{i}, v_{i}} \mid_{\Delta \times Y, t} \cong \nabla^{p_{1}^{3}}_{X \times Y, m_{i}, v_{i}} \mid_{\Delta \times Y, t}$. In other words, we decompose the monodromy of $\nabla^{p_{1}^{3}}_{X \times Y, m_{i}, v_{i}}$ along $\Delta \times Y, t$ to the composition of the monodromy of $\nabla^{p_{i}^{3}}_{X, \bar{h}, v_{q}}$ around $\infty^{H-R}$ and that around a point $P_{H-R}$ other than $\infty^{H-R}$. The monodromy of $\nabla'$ around $R^{H-R}$ reflects the analytic continuation of fundamental solutions of $\nabla^{p_{i}^{3}}_{X \times Y, m_{i}, v_{i}}$ along the `inner side' of the unfolded divisor $D_{Y, t}$. We can take a fundamental solution $Y'_{\infty^{H-R}}(z, \bar{h})$ of $\nabla^{p_{1}^{3}}_{X \times Y, m_{i}, v_{i}}$ near $\infty^{H-R} \times Y, t$. Then we can write

$$Y_{\infty^{H-R}}(z, \bar{h}) = Q(z, \bar{h})Y'_{\infty^{H-R}}(z, \bar{h})C_{\infty^{H-R}, \bar{t}}(\bar{h})$$

for an invertible matrix $Q(z, \bar{h})$ giving the isomorphism $\nabla^{p_{1}^{3}}_{X \times Y, m_{i}, v_{i}} \mid_{\Delta \times Y, t} \cong \nabla^{p_{i}^{3}}_{X \times Y, m_{i}, v_{i}} \mid_{\Delta \times Y, t}$. Here the matrix $C_{\infty^{H-R}, \bar{t}}(\bar{h})$ is a more close analogue of an unfolded Stokes matrix in [10]. Though there is an ambiguity of the choice of $C_{\infty^{H-R}, \bar{t}}(\bar{h})$ coming from the choices of $\nabla^{p_{i}^{3}}_{X \times Y, m_{i}, v_{i}}$ and $Y^{p_{i}^{3}}_{\infty^{H-R}}(z, \bar{h})$, we cannot say from its construction that $C_{\infty^{H-R}, \bar{t}}(\bar{h})$ is constant in $\bar{h}$, because we do not know the compatibility of the asymptotic properties between $Y_{\bar{t}}(z, \bar{h})$ and $Y_{\bar{t}+1}(z, \bar{h})$ when $j_{i} \neq j_{i+1}$.

5. Construction of an Unramified Generalized Isomonodromic Deformation

5.1. Setting of the moduli space for an unfolded generalized isomonodromic deformation. In this subsection, we introduce the moduli theoretic setting for describing an unfolding of the unramified irregular singular generalized isomonodromic deformation, which basically comes from [15]. We consider unramified irregular singular connections $\nabla: E \rightarrow E \otimes \Omega_{C}^{1}(m_{1}t_{1} + \cdots + m_{n}t_{n})$ and we take a certain étale covering $U \rightarrow M_{g,n}^{reg}$ of the moduli stack $M_{g,n}^{reg}$ of $n$-pointed smooth projective curves of genus $g$ with a universal family $(C, \bar{t}_{1}, \ldots, \bar{t}_{n})$ over $U$. Then

$$(\Omega_{C/U}^{1}(m_{1}t_{1} + \cdots + m_{n}t_{n})/\Omega_{C/U}^{1}(\bar{t}_{1} + \cdots + \bar{t}_{n}))^{r}$$

becomes the space of independent variables of the generalized isomonodromic deformation of $(E, \nabla)$. We will give a certain perturbation of this space.

First we construct a smooth covering $H \rightarrow M_{g,n}^{reg}$ of the moduli stack of $n$-pointed smooth projective curves of genus $g$ as follows. If $g = 0$, we put $H := \text{Spec} \mathbb{C}$, $Z := \mathbb{P}^{1}$ and regard $Z$ as a curve over $H$. If $g = 1$, we put $H := \{D \in |\Omega_{\mathbb{P}^{n}}| \mid D \text{ is a smooth cubic curve}\}$ and we set $Z \subset \mathbb{P}^{2} \times H$ as the universal family of smooth cubic curves. Assume that $g \geq 2$. Then we fix $l \geq 3$ and put $N := h^{0}(C, \omega_{C}^{l}) - 1$ for a smooth projective irreducible curve $C$ of genus $g$, where $\omega_{C}$ is the canonical bundle of $C$. We consider the locally closed subscheme $H \subset \text{Hilb}_{N}$ of the Hilbert scheme which parametrizes the closed subvarieties $C \subset \mathbb{P}^{N}$ isomorphic to the $l$-th canonical embeddings $C \hookrightarrow \mathbb{P}(H^{0}(C, \omega_{C}^{l}))$ of smooth projective curves $C$ of genus $g$. Let $Z \subset \mathbb{P}^{N} \times H$ be the universal family. For any case $g \geq 0$, we define a Zariski open subset

$$H := \left\{ (p_{i}) \in \prod_{i=1}^{n} Z \mid p_{i} \neq p_{i}' \text{ for } i \neq i' \right\}$$

of the fiber product $\prod_{i=1}^{n} Z$ of $n$ copies of $Z$ over $H$. Similarly we define a Zariski open subset

$$\mathcal{P} := \left\{ (p_{i}), (p_{j}^{(i)}) \in \prod_{i=1}^{n} Z \times H \prod_{i=1}^{n} m_{i} \mid p_{i} \neq p_{i}' \neq p_{j}^{(i)} \neq p_{j}^{(i)'} \text{ for } i \neq i' \right\}$$

of the fiber product $\prod_{i=1}^{n} Z \times H \prod_{i=1}^{n} m_{i}$ of $n + \sum_{i=1}^{n} m_{i}$ copies of $Z$ over $H$. Then there is a canonical projection

$$\pi_{\mathcal{P}, H}: \mathcal{P} \rightarrow H$$

defined by $\pi_{\mathcal{P}, H}((p_{i}), (p_{j}^{(i)})) = (p_{i})$ and there is a section

$$\tau_{H, \mathcal{P}}: H \rightarrow \mathcal{P}$$

defined by $\tau((p_{i})) = ((p_{i}), (p_{i}))$.

We put $C := Z \times H$ and $C_{\mathcal{P}} := Z \times H \mathcal{P}$. Then there are universal sections $\sigma_{i}: \mathcal{P} \rightarrow C$ and $\sigma_{j}^{(i)}: \mathcal{P} \rightarrow C_{\mathcal{P}}$ defined by $\sigma_{i}((p_{i}), (p_{j}^{(i)})) = (p_{i}, (p_{j}^{(i)}))$, $\sigma_{j}^{(i)}((p_{i}), (p_{j}^{(i)})) = (p_{j}^{(i)}, (p_{j}^{(i)}))$ which satisfy $\sigma_{i}(\mathcal{P}) \cap \sigma_{j}^{(i)}(\mathcal{P}) = \emptyset$, $\sigma_{i}(\mathcal{P}) \cap \sigma_{j}^{(i)'}(\mathcal{P}) = \emptyset$ and $\sigma_{j}^{(i)}(\mathcal{P}) \cap \sigma_{j}^{(i)'}(\mathcal{P}) = \emptyset$ for $i \neq i'$ and any $j, j'$. We define divisors $D_{i},$
$D_j$, $D_i$, and $D$ on $C_P$ by putting $D_i := \sigma_i(P)$, $D_i^{(i)} := \sigma_i^{(i)}(P)$, $D_i^{(i)} := \sum_{j=1}^m D_j^{(i)}$ and $D := \sum_{i=1}^n D_i^{(i)}$.

We consider the closed subvariety $\tau_{H,P}(H) \subset P$ which can be written

$$\tau_{H,P}(H) = \left\{ ((p_i), (p_j^{(i)})) \in P \mid p_i = p_j^{(i)} \text{ for any } i, j \right\}.$$

It was necessary to set the differential form \([14]\) in subsection 2.2 for the formulation of the moduli space of $(\nu, \mu)$-connections. For its construction, we use the following lemma.

**Lemma 5.1.** Let $f : X \to S$ be a smooth morphism of algebraic schemes over $\text{Spec} \mathbb{C}$ such that all the geometric fibers of $X$ over $S$ are smooth curves. Assume that $X \to S$ has a section $\sigma : S \to X$. Consider the diagonals

$$\Delta_{1,2} = \{(x, y, z) \in X \times_S X \times_S X \mid x = y\},$$

$$\Delta_{1,3} = \{(x, y, z) \in X \times_S X \times_S X \mid x = z\},$$

$$\Delta_{2,3} = \{(x, y, z) \in X \times_S X \times_S X \mid y = z\}.$$

We denote the ideal sheaf of $\mathcal{O}_{X \times_S X \times_S X}$ defining $\Delta_{i,j}$ by $I_{\Delta_{i,j}}$. Then for each closed point $p \in \sigma(S) \subset X$, there exists an affine open neighborhood $W$ of $(p, p, p)$ in $X \times_S X \times_S X$ such that the ideal $I_{\Delta_{1,2}}|_W$ is generated by a section $z_{1,2} \in H^0(W, I_{\Delta_{1,2}}|_W)$, the ideal $I_{\Delta_{1,3}}|_W$ is generated by a section $z_{1,3} \in H^0(W, I_{\Delta_{1,3}}|_W)$, the ideal $I_{\Delta_{2,3}}|_W$ is generated by $z_{1,2} - z_{1,3}$ and that $z_{1,2} - z_{1,3} \in p_2^{-1}(\mathcal{O}_U)$ for some open neighborhood $V$ of $(p, p)$ in $X \times_S X$.

**Proof.** If we put $s = f(p)$, the stalk of $I_{\sigma(S)} \otimes \mathcal{O}_{X_s} = I_{\sigma(S)}|_{X_s}$ at $p$ is a principal ideal of $\mathcal{O}_{X_s,P}$. So there is an affine open neighborhood $U$ of $p$ in $X$ and a section $z \in H^0(U, I_{\sigma(S)}|_U)$ such that $z|_U$ is a generator of $I_{\sigma(S)}|_U$. By Nakayama’s lemma, $z$ becomes a generator of $I_{\sigma(S)}|_U$ after shrinking $U$ if necessary. Since

$$z \otimes 1 \otimes 1 - 1 \otimes z \otimes 1 = dz \otimes 1 \in I_{\Delta_{1,2}}|_{U \times_S U \times_S U} = \Omega^1_{U/S} \otimes \mathcal{O}_U$$

is a generator after shrinking $U$, Nakayama’s lemma implies that $z_{1,2} := z \otimes 1 \otimes 1 - 1 \otimes z \otimes 1$ becomes a generator of $I_{\Delta_{1,2}}|_W$ for some affine open neighborhood $W$ of $(p, p, p)$ in $X \times_S X \times_S X$. If we put $z_{1,3} := z \otimes 1 \otimes 1 - 1 \otimes 1 \otimes z$, then $z_{1,3}$ similarly becomes a generator of $I_{\Delta_{1,3}}|_W$ after shrinking $W$ again. Since

$$z_{1,2} - z_{1,3} = (z \otimes 1 \otimes 1 - 1 \otimes z \otimes 1) - (z \otimes 1 \otimes 1 - 1 \otimes 1 \otimes z) = 1 \otimes (1 \otimes z - z \otimes 1) \in p_2^{-1}(\mathcal{O}_{U \times_S U}),$$

and $1 \otimes (1 \otimes z - z \otimes 1)$ becomes a generator of $I_{\Delta_{2,3}}$ after shrinking $W$, the lemma is proved. \;

In the above lemma, we may further assume that $p_2^{-1}(V) \cap \Delta_{1,2} \subset W$ and $p_2^{-1}(V) \cap \Delta_{1,3} \subset W$.

For each point $h_0 \in H$, we consider the fiber $C_{h_0}$ of $C_P$ over $\tau_{H,P}(h_0)$. If we put $p_0 := \sigma_1(\tau_{H,P}(h_0))$, then, by Lemma 5.1, there is an affine open neighborhood $W$ of $p_0$ in $C_P$ and sections $z^{(i)}, z^{(i)} \in H^0(W, \mathcal{O}_W)$ such that $z^{(i)} = 0$ is a defining equation of $D_i \cap W$, $z^{(i)} = 0$ is a defining equation of $D_i^{(i)} \cap W$ for each $j$ and $z^{(i)} - z^{(i)} \in \mathcal{O}_P$ for any $i, j$. So we can take an affine open neighborhood $\mathcal{P}'$ of $p_0$ in $\mathcal{P}$ and an affine open covering $\{U_a\}$ of $\mathcal{P} \times P'$ such that $\{a \mid D_i^{(i)} \times P' \subset U_a\} = \{a \mid D_i \times P' \subset U_a\}$ consists of a single element $\alpha_i$ for each $i$, $z^{(i)}(D_i \times P') \cap U_a \neq \emptyset \leq 1$ and $z^{(i)}(D_i^{(i)} \times P') \cap U_a \neq \emptyset \leq 1$ for each $\alpha_i$, $(D_i)_{P'}$ coincides with the zero scheme of $z^{(i)} \in H^0(U_a \otimes \mathcal{O}_{U_a}, (D_i)_{P'})$ coincides with the zero scheme of $z^{(i)} \in H^0(U_a \otimes \mathcal{O}_{U_a}, z^{(i)} - z^{(i)} \in \mathcal{O}_P$, and $(z^{(i)} - z^{(i)})|_{\tau_{H,P}(h) \times P'} = 0$ for any $i, j$. We denote the image of $z^{(i)}$ and $z^{(i)}$ in $\mathcal{O}_{2D_{i}^{(i)} \times P'}$ by $z^{(i)}$ and $z^{(i)}$, respectively. We put

$$\zeta_{m_i} := \exp \left( \frac{2\pi \sqrt{-1}}{m_i} \right)$$

and consider the locus

$$B := \left\{ h \in \mathcal{P}' \mid (z^{(i)} - z^{(i)})|_{h} = \zeta_{m_i}(z^{(i)} - z^{(i)})|_{h} \text{ for any } i, j \right\}$$

$$\text{and} (z^{(i)} - z^{(i)})|_{h} = (z^{(i)} - z^{(i)})|_{h} \text{ for any } i, i'$$
which is a smooth subvariety of $\mathcal{P}'$. Note that we have $z_{j}^{(i)} - z_{j}^{(i)} \in H^0(\mathcal{O}_{\mathcal{P}'})$ from the choice of $\mathcal{P}'$. If we put $\epsilon(h) := (z_{m}^{(i)} - z_{j}^{(i)})(h)$ for $h \in \mathcal{B}$, then $\epsilon : \mathcal{B} \rightarrow \mathbb{A}^1 = \mathbb{C}$ is an algebraic function. There is a diagram

$$
\begin{array}{ccc}
\mathcal{C}_{\mathcal{B}} & \longrightarrow & \mathcal{B} \\
\downarrow & & \downarrow \epsilon \\
\mathcal{H} & \longrightarrow & \mathbb{A}^1 = \mathbb{C}
\end{array}
$$

and we have $z_{j}^{(i)} = z_{j}^{(i)} + c_{m}^{(i)} \epsilon$ on $\mathcal{U}_{\mathcal{B}} \times \mathcal{B} \subset \mathcal{C}_{\mathcal{B}}$.

Let $(w_1, \ldots, w_s)$ be a holomorphic coordinate system in a neighborhood of $h_0$ in $\mathcal{H}$. Then we can see that $(z_{j}^{(i)}, \epsilon, w_1, \ldots, w_s)$ becomes a holomorphic coordinate system in a neighborhood of $\sigma_{i}(\tau_{\mathcal{H},\mathcal{P}}(h_0))$ in $\mathcal{U}_{\mathcal{B}} \times \mathcal{B}$. So we can take a disk $\Delta_{\epsilon_0} = \{ z \in \mathbb{C} \mid |z| < \epsilon_0 \}$ for small $\epsilon_0 > 0$, an analytic open neighborhood $\mathcal{B}'$ of $\tau_{\mathcal{H},\mathcal{P}}(h_0)$ in $\epsilon^{-1}(\Delta_{\epsilon_0}) \subset \mathcal{B}$ and an analytic open neighborhood $U_i \subset \mathcal{U}_{\mathcal{B}} \times \mathcal{B}'$ of $\sigma_{i}(\tau_{\mathcal{H},\mathcal{P}}(h_0))$ containing $\mathcal{D}_{i} \times \mathcal{B}'$ such that $U_i \cap U_{i'} = \emptyset$ for $i \neq i'$ and

$$(58) \quad U_i \frac{(z_{j}^{(i)}, \epsilon, w_1, \ldots, w_s)}{\sim} \Delta_{a} \times \Delta_{\epsilon_0} \times \Delta_{r}^{s}$$

becomes biholomorphic for any $i$, where $a, r > 0$, $\Delta_{a} = \{ z \in \mathbb{C} \mid |z| < a \}$ and $\Delta_{r} = \Delta_{r} \times \cdots \times \Delta_{r}$ with $\Delta_{r} = \{ z \in \mathbb{C} \mid |z| < r \}$. We define a subset $\Gamma_{j, i, k}^{(i)}$ of the fiber $\mathcal{C}_b$ of $\mathcal{C} \times \mathcal{B}'$ over $b \in \mathcal{B}'$ by setting

$$
\Gamma_{j, i, k}^{(i)} := \bigcup_{0 \leq s \leq 1} \left\{ x \in \mathcal{C}_b \cap U_i \left| (z_{j}^{(i)} + s c_{j,m}^{(i)} \epsilon)(x) = 0 \right. \right\} .
$$

Then $\Gamma_{j, i, k}^{(i)}$ becomes a simple path in $\mathcal{C}_b$ joining the two points $(\mathcal{D}_i)_b$ and $(\mathcal{D}_{j}^{(i)})_b$ for $\epsilon \in \Delta_{\epsilon_0} \setminus \{ 0 \}$ because of the bijectivity of (58). If we set

$$
\Gamma_{j}^{(i)} := \bigcup_{b \in \mathcal{B}'} \Gamma_{j, i, k}^{(i)}, \quad \Gamma := \bigcup_{i,j} \Gamma_{j}^{(i)},
$$

then $\Gamma_{j}^{(i)}$ and $\Gamma$ are closed subsets of $\mathcal{C} \times \mathcal{B}'$ with respect to the analytic topology.

We fix distinct complex numbers $\mu_{1}^{(i)}, \ldots, \mu_{r}^{(i)} \in \mathbb{C}$ for $i = 1, \ldots, n$ and write $\mu = (\mu_{k}^{(i)})_{1 \leq k \leq r}$. Then we put

$$
\varphi_{\mu}^{(i)}(T) := (T - \mu_{1}^{(i)})(T - \mu_{2}^{(i)}) \cdots (T - \mu_{r}^{(i)}) \in \mathbb{C}[T].
$$

We take an integer $a \in \mathbb{Z}$ and a tuple of complex numbers $\lambda = (\lambda_{k}^{(i)}) \in \mathbb{C}^{nr}$ satisfying

(i) $a + \sum_{k=1}^{r} \sum_{i=1}^{n} \lambda_{k}^{(i)} = 0,$

(ii) $\lambda_{k}^{(i)} - \lambda_{k'}^{(i)} \notin \mathbb{Z}$ for $k \neq k'.$

We define an algebraic variety $\mathcal{T}_{\mu, \lambda}$ over $\mathcal{B}$ whose set of $S$-valued points is given by

$$
\mathcal{T}_{\mu, \lambda}(S) := \left\{ (\mu^{(i)}(T))_{1 \leq i \leq n} \left| \mu^{(i)}(T) = \sum_{l=0}^{r-1-m} c^{(i)}_{l,m-1} (z^{(i)} \epsilon)^{l} T^{l} \text{ with } c_{l,m}^{(i)} \in H^0(\mathcal{O}_{S}) \right. \right\}
$$

satisfying the following (a) and (b):

(a) $\lambda_{k}^{(i)} = \sum_{l=0}^{r-1-m} c^{(i)}_{l,m-1} (\mu_{k}^{(i)})^{l}$ for each $i, k$

(b) $\mu^{(i)}(\mu_{k}^{(i)})_{p} = \mu^{(i)}(\mu_{k}^{(i)})_{p}$ for $k \neq k'$, $1 \leq i \leq n$ and any $p \in \mathcal{D}_{S}^{(i)}$.

Here we intend to regard $(\mu_{k}^{(i)})_{0 \leq i \leq r-1, 0 \leq j \leq m}$ with $\mu_{k}^{(i)} \in H^0(\mathcal{O}_{S})$ as a precise data denoted by $(\mu^{(i)}(T))$. We take a universal family

$$
\tilde{\mu}^{(i)}(T) = \sum_{l=0}^{r-1-m} c_{l,m}^{(i)} (z^{(i)} \epsilon)^{l} T^{l}
$$
with $c^{(i)}_{i,j} \in H^0(\mathcal{O}_{\mathcal{M},\lambda})$ and write $\tilde{\nu} := (\tilde{\nu}^{(i)}(T))$. If we denote by $\tilde{\nu}^{(i)}_s, (c^{(i)}_{i,j})_s$ the restrictions of $\tilde{\nu}^{(i)}, c^{(i)}_{i,j}$ to $s \in \mathcal{T}_{\mu,\lambda}$, respectively, we can see by Lemma 2.1 that

$$
\sum_{p \in \mathcal{D}'^{(i)}_s} \text{res}_p \left( \tilde{\nu}^{(i)}_s(\mu_k^{(i)}) \frac{dz^{(i)}}{(z^{(i)})^{m_i} - \epsilon^{m_i}} \right) = \sum_{l=0}^{r-1} \sum_{m_i=1}^{m_i-1} (c^{(i)}_{i,j})_s(\mu_k^{(i)})^l \sum_{p \in \mathcal{D}'^{(i)}_s} \text{res}_{z^{(i)}} = p \left( \frac{(z^{(i)})^l dz^{(i)}}{(z^{(i)})^{m_i} - \epsilon^{m_i}} \right)
$$

$$
= - \sum_{l=0}^{r-1} \sum_{m_i=1}^{m_i-1} (c^{(i)}_{i,j})_s(\mu_k^{(i)})^l \text{res}_{z^{(i)}} = \infty \left( \frac{(z^{(i)})^l dz^{(i)}}{(z^{(i)})^{m_i} - \epsilon^{m_i}} \right)
$$

$$
= \sum_{l=0}^{r-1} (c^{(i)}_{i,m_i-1})_s(\mu_k^{(i)})^l.
$$

So the equality (a) in the definition of $\mathcal{T}_{\mu,\lambda}$ means the equality

$$
\lambda^{(i)}_k = \sum_{p \in \mathcal{D}'^{(i)}_s} \text{res}_p \left( \tilde{\nu}^{(i)}_s(\mu_k^{(i)}) \frac{dz^{(i)}}{(z^{(i)})^{m_i} - \epsilon^{m_i}} \right)
$$

where $\sum_{p \in \mathcal{D}'^{(i)}_s}$ runs over the set theoretical points $p$ of $\mathcal{D}'^{(i)}_s$. For each point $p = \epsilon c_{i,j} \in \mathcal{D}'^{(i)}_s$, we have

$$
\tilde{\nu}^{(i)}_s(\mu_k^{(i)})|_p = \sum_{l=0}^{r-1} \sum_{m_i=1}^{m_i-1} (c^{(i)}_{i,j})_s(\epsilon(s)c^{(i)}_{i,m_i})^l(\mu_k^{(i)})^l
$$

for $1 \leq k \leq r$. The condition (b) in the definition of $\mathcal{T}_{\mu,\lambda}$ is that

$$
\sum_{l=0}^{r-1} \sum_{m_i=1}^{m_i-1} (c^{(i)}_{i,j})_s(\epsilon(s)c^{(i)}_{i,m_i})^l(\mu_k^{(i)})^l \neq \sum_{l=0}^{r-1} \sum_{m_i=1}^{m_i-1} (c^{(i)}_{i,j})_s(\epsilon(s)c^{(i)}_{i,m_i})^l(\mu_{k'}^{(i)})^l
$$

for $k \neq k'$, when $\epsilon(s) \neq 0$ and that

$$
\sum_{l=0}^{r-1} (c^{(i)}_{i,j})_s(\mu_k)^l \neq \sum_{l=0}^{r-1} (c^{(i)}_{i,j})_s(\mu_{k'})^l
$$

for $k \neq k'$ when $\epsilon(s) = 0$.

By Theorem 2.11 there is a relative moduli space

$$
\pi_{\mathcal{T}_{\mu,\lambda}}: M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu) \rightarrow \mathcal{T}_{\mu,\lambda}
$$

of $(\tilde{\nu}, \mu)$-connections over $\mathcal{T}_{\mu,\lambda}$. Note that the morphism $\pi_{\mathcal{T}_{\mu,\lambda}}$ in (60) is an algebraic smooth morphism of quasi-projective schemes. We consider the pull-back diagram

$$
\begin{array}{ccc}
M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu) \times_B B' & \longrightarrow & M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu) \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B
\end{array}
$$

where the horizontal arrows are open immersions as analytic spaces.

5.2. Unramified irregular singular generalized isomonodromic deformation. The unramified irregular singular generalized isomonodromic deformation is the well-known theory by Jimbo, Miwa and Ueno, which is completely given in [15], [16], [17] with explicit calculations using formal solutions based on the Malgrange-Sibuya theorem (2, Theorem 4.5.1). We recall here a moduli theoretic construction of the unramified irregular singular generalized isomonodromic deformation given in [14], which is valid in a higher genus case.

Recall that there are compositions of morphisms $M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu) \rightarrow \mathcal{T}_{\mu,\lambda} \rightarrow B -\epsilon\rightarrow \Delta_{\epsilon_0}$. We consider the fibers

$$
B_{\epsilon=0} := B \times \Delta_{\epsilon_0} \{0\}, \quad \mathcal{T}_{\mu,\lambda,\epsilon=0} := \mathcal{T}_{\mu,\lambda} \times_B B_{\epsilon=0}, \quad M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu)_{\epsilon=0} := M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu) \times_B B_{\epsilon=0}
$$

over $\epsilon = 0 \in \Delta_{\epsilon_0}$. Then $\pi_{\mathcal{T}_{\mu,\lambda,\epsilon=0}}: M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu)_{\epsilon=0} \rightarrow \mathcal{T}_{\mu,\lambda,\epsilon=0}$ is the relative moduli space of unramified irregular singular connections. In our moduli theoretic setting, the unramified irregular singular generalized isomonodromic deformation is given in [14] Theorem 6.2] as an algebraic splitting

$$
\Psi_0: \pi_{\mathcal{T}_{\mu,\lambda,\epsilon=0}}^* \mathcal{T}_{\mu,\lambda,\epsilon=0} \rightarrow T_{M_{\mathcal{C},\mathcal{D}}^{\alpha}(\tilde{\nu}, \mu)_{\epsilon=0}}
$$
of the canonical surjection $T_{\mathcal{M}_0}[\tilde{\nu}, \mu] = 0 \rightarrow (\pi_{\mathcal{M}_0, x = 0})^* T_{\mathcal{M}_0, x = 0}$. Here we use the symbol $\Psi_0$ instead of the symbol $D$ used in [14], for the purpose of avoiding confusion with the divisor of singularity of the connection.

Let us recall the construction of $\Psi_0$. For each Zariski open subset $T_0 \subset T_{\mathcal{M}_0, x = 0}$ and for each vector field $v \in H^0(T_0, T_{\mathcal{M}_0, x = 0})$, let $T_0[v] = T_0 \times \text{Spec} \mathbb{C}[h]/(h^2) \xrightarrow{\iota} T_0$ be the corresponding morphism satisfying $I_v \otimes \mathbb{C}[h]/(h) = \text{id}_{T_0}$. If we put

$$\nu^{(i)}_{0, \text{hor}}(T) := \sum_{i=0}^{n} \sum_{j=0}^{(i)} (I_v^* e_i^{(j)} T_0 - \tilde{h} v ((e_i^{(j)} T_0)) (z)^i)\, T^j,$$

then we have $I_v^* (\tilde{\nu}^{(i)}(T)) = \nu^{(i)}_{0, \text{hor}}(T) + \tilde{h} \nu^{(i)}_{0, \text{hor}}(T)$ and $\nu^{(i)}_{0, \text{hor}}(T)$ is the pullback of $\tilde{\nu}^{(i)}(T)$ via the trivial projection $T_0[v] = T_0 \times \text{Spec} \mathbb{C}[h]/(h^2) \rightarrow T_0 \rightarrow T_{\mathcal{M}_0, x = 0}$. We consider the fiber product $C_{T_0[v]} = C_{T_0} \times_{T_0} T_0[v]$ with respect to $I_v, T_0[v] \rightarrow T_0$ and the trivial projection $C_{T_0} \rightarrow T_0$. We denote the pullback of $z^{(i)}$ under the morphism $C_{T_0[v]} \rightarrow C_{T_0} \times_{T_0} T_0[v]$ by $\tilde{z}^{(i)}$.

For some étale surjective morphism $\tilde{M} \rightarrow M_{0, D}(\tilde{\nu}, \mu)$, there is a universal family $(\tilde{E}, \tilde{\nabla}, \{\tilde{N}^{(i)}\})$ on $\tilde{C}_M$. We put $M'_0 := \tilde{M} \times_{\tilde{M}_0} T_0$, $\tilde{M}'[v] := \tilde{M} \times_{\tilde{M}_0} T_0[v]$ and denote the restriction of $(\tilde{E}, \tilde{\nabla}, \{\tilde{N}^{(i)}\})$ to $\tilde{C}_{M'_0}$ by $(\tilde{E}_{M'_0}, \tilde{\nabla}_{M'_0}, \{\tilde{N}_{M'_0}^{(i)}\})$. In the following definition, $C_{M'_0[v]}$ means the fiber product $C_{T_0[v]} \times_{T_0} T_0[v]$ with respect to the universal family $\tilde{M}'[v] \rightarrow T_0$ and the composition $\tilde{M}'[v] \rightarrow T_0[u] \rightarrow T_0[u] \rightarrow T_0$. On the other hand, relative differentials in $\Omega^1_{C_{M'_0[v]}/M'_0}$ are with respect to the composition $C_{M'_0[v]} \rightarrow M'_0[v] = \tilde{M}'_0 \times \text{Spec} \mathbb{C}[h]/(h^2) \rightarrow \tilde{M}'_0$ of the trivial projections.

**Definition 5.2.** $(\tilde{E}_0, \tilde{\nabla}_0, \{\tilde{N}^{(i)}_{0,v}\})$ is a horizontal lift of $(\tilde{E}_{M'_0}, \tilde{\nabla}_{M'_0}, \{\tilde{N}^{(i)}_{M'_0}\})$ with respect to $v$ if

1. $\tilde{E}_0$ is an algebraic vector bundle on $C_{M'_0[v]}$ of rank $r$;
2. $\nabla_0 : \tilde{E}_0 \rightarrow \tilde{E}_0 \otimes \Omega^1_{C_{M'_0[v]}/M'_0}(\mathcal{D}_{M'_0[v]})$ is a morphism of sheaves satisfying $\nabla_0(f a) = a \otimes df + f \nabla_0(a)$ for $f \in \mathcal{O}_{M'_0[v]}$ and $a \in \tilde{E}_0$;
3. $\nabla_0$ is integrable in the sense that the restriction of $\nabla_0$ to any open set $U[v] \subset C_{M'_0[v]} \setminus \mathcal{D}_{M'_0[v]}$ satisfying $\nabla_0|_{U[v]} \cong (\mathcal{O}_{U[v]} \otimes \Omega^1_{C_{M'_0[v]}/M'_0}(\mathcal{D}_{M'_0[v]}))^{\otimes r}$ is expressed by

$$(\mathcal{O}_{U[v]} \otimes \Omega^1_{C_{M'_0[v]}/M'_0}(\mathcal{D}_{M'_0[v]}))^{\otimes r} \cong \left(\begin{array}{c} f_1 \\ \vdots \\ f_r \end{array}\right) \rightarrow \left(\begin{array}{c} df_1 \\ \vdots \\ df_r \end{array}\right) + \left(\begin{array}{c} \hat{A}d\tilde{z} + B\hat{d}\tilde{h} \\ \vdots \\ \hat{A}d\tilde{z} + B\hat{d}\tilde{h} \end{array}\right) \left(\begin{array}{c} f_1 \\ \vdots \\ f_r \end{array}\right) \in \left(\mathcal{O}_{U[v]} \otimes \Omega^1_{C_{M'_0[v]}/M'_0}(\mathcal{D}_{M'_0[v]})\right)^{\otimes r}$$

satisfying $d (\hat{A}d\tilde{z} + B\hat{d}\tilde{h}) + \left(\hat{A}d\tilde{z} + B\hat{d}\tilde{h}\right) = 0$ in $\Omega^2_{C_{M'_0[v]}/M'_0}(\mathcal{D}_{M'_0[v]})$;
4. $\nabla_0^{(i)} : \tilde{E}_0|_{\mathcal{D}_{M'_0[v]}^{(i)}} \rightarrow \tilde{E}_0|_{\mathcal{D}_{M'_0[v]}^{(i)}}$ is an endomorphism satisfying $\nabla_0^{(i)}(\lambda_{0,v}^{(i)}) = 0$;
5. the relative connection $\nabla^{(i)}_0$ defined by the composition

$$\nabla^{(i)}_0 : \tilde{E}_0 \xrightarrow{\nabla_0} \tilde{E}_0 \otimes \Omega^1_{C_{M'_0[v]}/M'_0}(\mathcal{D}_{M'_0[v]}) \rightarrow \tilde{E}_0 \otimes \Omega^1_{C_{M'_0[v]}/M'_0}(\mathcal{D}_{M'_0[v]})$$

satisfies

$$(\nu^{(i)}_{0, \text{hor}} + \tilde{h} \nu^{(i)}_{0, v})(\tilde{N}^{(i)}_{0,v}) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^m_i} = \nabla^{(i)}_0 $$

for any $i$ and
6. $(\tilde{E}_0, \tilde{\nabla}_0, \{\tilde{N}^{(i)}_{0,v}\}) \cong (\tilde{E}_{M'_0}, \tilde{\nabla}_{M'_0}, \{\tilde{N}^{(i)}_{M'_0}\})$.

The following proposition is essentially given in the proof of [14] Theorem 6.2 and we omit its proof here.

**Proposition 5.3.** There exists a unique horizontal lift $(\tilde{E}_0, \tilde{\nabla}_0, \{\tilde{N}^{(i)}_{0,v}\})$ of $(\tilde{E}_{M'_0}, \tilde{\nabla}_{M'_0}, \{\tilde{N}^{(i)}_{M'_0}\})$ with respect to $v$. 

For each vector field \( v \in H^0(\mathcal{T}'_0, T_{\mathcal{M}'_0, \lambda, x=0}|_{\mathcal{T}'_0}) \), the horizontal lift of \( (\hat{E}_{\mathcal{M}'_0}^i, \hat{\nabla}_{\mathcal{M}'_0}^i, \{\hat{N}_{\mathcal{M}'_0}^i\}) \) with respect to \( v \) induces a relative connection \( (\mathcal{E}_0^i, \mathcal{\nabla}_0^i, \{\mathcal{N}_0^i\}) \) which gives a morphism \( \mathcal{M}'_0[v] \to \mathcal{M}'_0(\hat{\nu}, \mu) \) making the diagram

\[
\begin{array}{ccc}
\mathcal{M}'_0[v] & \to & \mathcal{M}'_0(\hat{\nu}, \mu) \\
\downarrow & & \downarrow \\
\mathcal{T}'_0[v] & \xrightarrow{\mathcal{T}'_0[v]} & \mathcal{T}'_0 \to \mathcal{T}_{\mathcal{M}'_0, \lambda, x=0}
\end{array}
\]

commutative. This morphism corresponds to a section of \( T_{\mathcal{M}'_0(\hat{\nu}, \mu), x=0} \otimes \mathcal{O}_{\mathcal{M}'_0} \) over \( \mathcal{M}'_0 \) which descends to a vector field \( \Phi_0(v) \in H^0(\pi_{\mathcal{M}'_0, \lambda, x=0}^{-1}(\mathcal{T}'_0), T_{\mathcal{M}'_0(\hat{\nu}, \mu), x=0} \otimes \pi_{\mathcal{M}'_0, \lambda, x=0}^{-1}(\mathcal{T}'_0)) \). We can show that the correspondence

\[
\mathcal{T}_{\mathcal{M}'_0, \lambda, x=0} \ni v \mapsto \Phi_0(v) \in (\pi_{\mathcal{M}'_0, \lambda, x=0})^* T_{\mathcal{M}'_0(\hat{\nu}, \mu), x=0}
\]

is an \( \mathcal{O}_{\mathcal{M}'_0, \lambda, x=0} \)-homomorphism. We omit its proof because it is as that as Proposition 5.13 which is given later. So \( \Phi_0 \) is equivalent to the morphism

\[
\Psi_0 : (\pi_{\mathcal{M}'_0, \lambda, x=0})^* T_{\mathcal{M}'_0, \lambda, x=0} \to T_{\mathcal{M}'_0(\hat{\nu}, \mu), x=0}.
\]

We devote the rest of this subsection to the proof of the integrability of the subbundle \( \text{im} \Psi_0 \subset T_{\mathcal{M}'_0(\hat{\nu}, \mu), x=0} \). The rest of the irregular singular generalized isomonodromic deformation in the zero genus case is proved by Jimbo, Miwa and Ueno in [13] Theorem 4.2, which is extended by Bremer and Sage in [6] Theorem 5.1. Although it is almost a consequence of the Malgrange-Sibuya isomorphism [2] Theorem 4.5.1 in a general case, it will be worth giving a proof of the integrability of \( \Psi_0 \), because the situation in an unfolded case is different.

For the proof of the integrability condition of \( \Psi_0 \), we extend the definition of horizontal lift given in Definition 5.2. We consider a morphism

\[
u : \mathcal{T}'_0[u] := \mathcal{T}'_0 \times \text{Spec} \mathbb{C}[h_1, h_2]/(h_1^2, h_2^2) \to \mathcal{T}'_0 \subset \mathcal{T}_{\mathcal{M}'_0, \lambda, x=0}
\]

satisfying \( u \otimes \mathbb{C}[h_1, h_2]/(h_1^2, h_2^2) = \text{id}_{\mathcal{T}_0} \) and write

\[
u^* \tilde{\nu}^{(i)}(T) = \nu^{(i)}_{\text{hor}}(T) + \nu_1^{(i)}(T)h_1 + \nu_2^{(i)}(T)h_2 + \nu_1^{(i)}(T)h_1h_2
\]

where \( \nu^{(i)}_{\text{hor}}(T) \) is the pullback of \( \tilde{\nu}^{(i)}(T) \) by the composition \( \mathcal{T}'_0 \times \text{Spec} \mathbb{C}[h_1, h_2]/(h_1^2, h_2^2) \to \mathcal{T}'_0 \to \mathcal{T}_{\mathcal{M}'_0, \lambda} \) of the trivial projection and the inclusion and \( \nu_1^{(i)}(T), \nu_2^{(i)}(T) \) are pullbacks of polynomials in \( \mathcal{O}_{\mathcal{D}'_0/T} \) via the trivial projection \( \mathcal{T}'_0 \times \text{Spec} \mathbb{C}[h_1, h_2]/(h_1^2, h_2^2) \to \mathcal{T}'_0 \).

We consider the fiber product \( \mathcal{M}_0'[u] := \mathcal{M}_0'[\mathcal{T}'_0[u]] \) with respect to the canonical morphism \( \mathcal{M}_0' \to \mathcal{T}'_0 \) and \( \mathcal{T}'_0[u] \to \mathcal{T}'_0 \). We can extend the notion of horizontal lift given in Definition 5.2 to the morphism \( u : \mathcal{T}'_0 \times \text{Spec} \mathbb{C}[h_1, h_2]/(h_1^2, h_2^2) \to \mathcal{T}_0 \).

We say that a tuple \( (\mathcal{E}_0^i, \mathcal{\nabla}_0^i, \{\mathcal{N}_0^i\}) \) is a horizontal lift of \( (\hat{E}_{\mathcal{M}_0}', \hat{\nabla}_{\mathcal{M}_0}', \{\hat{N}_{\mathcal{M}_0}'\}) \) with respect to \( u \) if \( \mathcal{E}_0^i \) is a locally free sheaf on \( C_{\mathcal{M}_0'[u]}, \mathcal{\nabla}_0^i \to \mathcal{E}_0^u \otimes \Omega^1_{\mathcal{M}_0'[u]/\mathcal{M}_0[u]}(\mathcal{D}_{\mathcal{M}_0'[u]}) \) is an integrable connection and \( \mathcal{N}_0^i|_{\mathcal{D}_0[u]} \to \mathcal{E}_0^u|_{\mathcal{D}_0[u]} \) is an endomorphism such that the conditions (3), (4), (5) and (6) of Definition 5.2 hold after replacing \( v \) by \( u \). Then we have the following:

Lemma 5.4. There exists a unique horizontal lift \( (\mathcal{E}_0^i, \mathcal{\nabla}_0^i, \{\mathcal{N}_0^i\}) \) of \( (\hat{E}_{\mathcal{M}_0}', \hat{\nabla}_{\mathcal{M}_0}', \{\hat{N}_{\mathcal{M}_0}'\}) \) with respect to \( u \).

Proof. We consider the restriction of \( \hat{\nabla}_{\mathcal{M}_0}' \) to an affine open neighborhood \( U_i \) of \( \mathcal{D}_0 \) such that \( \hat{E}_{\mathcal{M}_0}'|_{U_i} \cong \mathcal{O}_{U_i}^{\text{gr}} \). It can be written

\[
\hat{\nabla}_{\mathcal{M}_0}'|_{U_i} : \mathcal{O}_{U_i}^{\text{gr}} \ni \begin{pmatrix} f_1 \\ f_r \end{pmatrix} \mapsto \begin{pmatrix} df_1 \\ df_r \end{pmatrix} + A(z^{(i)}) \frac{dz^{(i)}}{(z^{(i)})^m} \begin{pmatrix} f_1 \\ f_r \end{pmatrix}
\]

Here we may assume that

\[
A(z^{(i)})|_{\mathcal{D}_0} = \text{Diag}(\mu_{z^{(i)}})|_{\mathcal{D}_0}.
\]
We can take a lift $A(\tilde{z}^{(i)})$ of $A(z^{(i)})$ as a matrix of algebraic functions on $U^{(i)}_{\bar{M}_0[u]}$, satisfying
\begin{equation}
\frac{\partial A(\tilde{z}^{(i)})}{\partial h_1} = \frac{\partial A(\tilde{z}^{(i)})}{\partial h_2} = 0.
\end{equation}
Indeed for an arbitrary lift $\tilde{A}(\tilde{z}^{(i)})$ of $A(z^{(i)})$, we can write $d\tilde{A} = A_0 d\tilde{z}^{(i)} + A_1d\bar{h}_1 + A_2d\bar{h}_2$ with respect to the identification
\[\Omega^1_{U^{(i)}_{\bar{M}_0[u]}/\bar{M}_0} = \mathcal{O}_{U^{(i)}_{\bar{M}_0[u]}} d\tilde{z}^{(i)} \oplus \mathcal{O}_{U^{(i)}_{\bar{M}_0[u]}} d\bar{h}_1 \oplus \mathcal{O}_{U^{(i)}_{\bar{M}_0[u]}} d\bar{h}_2.\]
Here relative differential forms in $\Omega^1_{U^{(i)}_{\bar{M}_0[u]}/\bar{M}_0}$ are with respect to the composition of the trivial projections $U^{(i)}_{\bar{M}_0[u]} \rightarrow \bar{M}_0[u] = \bar{M}_0 \times \text{Spec} \mathbb{C}[h_1, h_2]/(h_1^2, h_2^2) \rightarrow \bar{M}_0$. Then the replacement
\[A(\tilde{z}^{(i)}) := \tilde{A} - \bar{h}_1 A_1 - \bar{h}_2 A_2 + \bar{h}_1 \bar{h}_2 \frac{\partial A_1}{\partial \bar{h}_2}\]
satisfies the condition (61) because of the equalities
\[\frac{\partial A_1}{\partial \bar{h}_2} = \frac{\partial^2 A}{\partial \bar{h}_1 \partial \bar{h}_2} = \frac{\partial A_2}{\partial \bar{h}_1}.
\]
We put
\[B_1(\tilde{z}^{(i)}) := \text{Diag} (\nu_1^{(i)}(\mu_k) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}), \quad B_2(\tilde{z}^{(i)}) := \text{Diag} (\nu_2^{(i)}(\mu_k) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}, \quad B_{1,2}(\tilde{z}^{(i)}) := \text{Diag} (\nu_1^{(i)}(\mu_k) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}, \nu_2^{(i)}(\mu_k) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}).
\]
Note that $(\tilde{z}^{(i)})^{m_i-1} B_1(\tilde{z}^{(i)})$, $(\tilde{z}^{(i)})^{m_i-1} B_2(\tilde{z}^{(i)})$ and $(\tilde{z}^{(i)})^{m_i-1} B_{1,2}(\tilde{z}^{(i)})$ are matrices of polynomials in $\tilde{z}^{(i)}$, because $\nu_1^{(i)}(\mu_k)$ and $\nu_2^{(i)}(\mu_k)$ have no residue part. If we define
\[C_1(\tilde{z}^{(i)}) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} := dB_1(\tilde{z}^{(i)}) + [A(\tilde{z}^{(i)}), B_1(\tilde{z}^{(i)})] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}},
\]
\[C_2(\tilde{z}^{(i)}) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} := dB_2(\tilde{z}^{(i)}) + [A(\tilde{z}^{(i)}), B_2(\tilde{z}^{(i)})] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}},
\]
we have $C_1(\tilde{z}^{(i)}) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} = \text{Diag}\left(\nu_1^{(i)}(\mu_k)\right) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}$ and $C_2(\tilde{z}^{(i)}) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} = \text{Diag}\left(\nu_2^{(i)}(\mu_k)\right) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}$. Since $B_1(\tilde{z}^{(i)})$, $B_2(\tilde{z}^{(i)})$, $dB_1(\tilde{z}^{(i)})$ and $dB_2(\tilde{z}^{(i)})$ commute to each other, we have
\[\left[ C_1(\tilde{z}^{(i)}), B_2(\tilde{z}^{(i)}) \right] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} = \left[ dB_1(\tilde{z}^{(i)}) + [A(\tilde{z}^{(i)}), B_1(\tilde{z}^{(i)})] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}, B_2(\tilde{z}^{(i)}) \right] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}},
\]
\[= [[A(\tilde{z}^{(i)}), B_1(\tilde{z}^{(i)})], B_2(\tilde{z}^{(i)})] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}},
\]
\[= [[A(\tilde{z}^{(i)}), B_2(\tilde{z}^{(i)})], B_1(\tilde{z}^{(i)})] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}},
\]
\[= [C_2(\tilde{z}^{(i)}), B_1(\tilde{z}^{(i)})] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}.
\]
If we put
\[C(\tilde{z}^{(i)}) := [C_1(\tilde{z}^{(i)}), B_2(\tilde{z}^{(i)})] = [C_2(\tilde{z}^{(i)}), B_1(\tilde{z}^{(i)})],
\]
then we can see that $C(\tilde{z}^{(i)})$ is a matrix of algebraic functions on $U^{(i)}_{\bar{M}_0[u]}$ such that $C(\tilde{z}^{(i)}) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} = 0$. We can check that the connection matrix
\[\eta = (A + \bar{h}_1 C_1 + \bar{h}_2 C_2 + \bar{h}_1 \bar{h}_2 C) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} + B_1 d\bar{h}_1 + B_2 d\bar{h}_2
\]
satisfies the integrability condition:

\[ d\eta + [\eta, \eta] = (C_1 + \tilde{h}_2 C) d\bar{h}_1 \wedge \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} + (C_2 + \tilde{h}_1 C) d\bar{h}_2 \wedge \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} + dB_1 \wedge d\bar{h}_1 + dB_2 \wedge d\bar{h}_2 \]

\[+ \left( [A, B_1] + \tilde{h}_2 [C, B_1] \right) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \wedge d\bar{h}_1 + \left( [A, B_2] + \tilde{h}_1 [C, B_2] \right) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \wedge d\bar{h}_2\]

\[= \left( dB_1 + (-C_1 + [A, B_1]) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \right) \wedge d\bar{h}_1 + \left( dB_2 + (-C_2 + [A, B_2]) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \right) \wedge d\bar{h}_2\]

\[= 0.\]

If we put

\[ C_{1,2}(\tilde{z}^{(i)}) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} := dB_{1,2}(\tilde{z}^{(i)}) + [A(\tilde{z}^{(i)}), B_{1,2}(\tilde{z}^{(i)})] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}}, \]

then the connection matrix

\[ \tilde{\eta} := \eta + \tilde{h}_1 \tilde{h}_2 C_{1,2} \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} + \tilde{h}_2 B_{1,2}(\tilde{z}^{(i)}) d\bar{h}_1 + \tilde{h}_1 B_{1,2}(\tilde{z}^{(i)}) d\bar{h}_2 \]

satisfies the integrability condition

\[ d\tilde{\eta} + [\tilde{\eta}, \tilde{\eta}] = d\eta + [\eta, \eta] + \tilde{h}_2 C_{1,2} d\bar{h}_1 \wedge \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} + \tilde{h}_1 C_{1,2} d\bar{h}_2 \wedge \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} + \tilde{h}_2 dB_{1,2} \wedge d\bar{h}_1 \]

\[+ \tilde{h}_1 dB_{1,2} \wedge d\bar{h}_2 + \tilde{h}_2 [A, B_{1,2}] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \wedge d\bar{h}_1 + \tilde{h}_1 [A, B_{1,2}] \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \wedge d\bar{h}_2 \]

\[= \tilde{h}_2 \left( dB_{1,2} + (-C_{1,2} + [A, B_{1,2}]) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \right) \wedge d\bar{h}_1 \]

\[+ \tilde{h}_1 \left( dB_{1,2} + (-C_{1,2} + [A, B_{1,2}]) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \right) \wedge d\bar{h}_2 \]

\[= 0.\]

Then the connection

\[ \nabla^\mu_u : \mathcal{O}^{\bar{\mathfrak{g}}} U_{U^{(i)}_\mu} \to \mathcal{O}^{\bar{\mathfrak{g}}} U_{U^{(i)}_\mu} \otimes \mathcal{O}^{\mathfrak{g}} U_{U^{(i)}_\mu} / \mathfrak{g} \mathcal{O}^{\mathfrak{g}} U_{U^{(i)}_\mu} \]

given by the connection matrix

\[ \tilde{\eta} = \left( A(\tilde{z}^{(i)}) + \tilde{h}_1 C_1(\tilde{z}^{(i)}) + \tilde{h}_2 C_2(\tilde{z}^{(i)}) + \tilde{h}_1 \tilde{h}_2 (C(\tilde{z}^{(i)}) + C_{1,2}(\tilde{z}^{(i)})) \right) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})^{m_i}} \]

\[+ \left( B_1(\tilde{z}^{(i)}) + \tilde{h}_2 B_{1,2}(\tilde{z}^{(i)}) \right) d\bar{h}_1 + \left( B_2(\tilde{z}^{(i)}) + \tilde{h}_1 B_{1,2}(\tilde{z}^{(i)}) \right) d\bar{h}_2 \]

becomes an integrable connection. If we put \( N^{(i)}_{U^{(i)}_\mu} := \text{Diag}_{\mu} \), then \( (\mathcal{O}^{\bar{\mathfrak{g}}} U_{U^{(i)}_\mu}, \nabla^\mu_u, N^{(i)}_{U^{(i)}_\mu}) \) is a local horizontal lift of \( (E_{\mathfrak{g}^{\mu}}, \nabla^\mu_{\mathfrak{g}^{\mu}}, \{ \tilde{N}^{\mu(\mu)}_{\mathfrak{g}^{\mu}} \})_{U^{(i)}_\mu} \).

Assume that \( (\mathcal{O}^{\bar{\mathfrak{g}}} U_{U^{(i)}_\mu}, \nabla^\mu_u, N') \) is another local horizontal lift given by a connection matrix

\[ (A(\tilde{z}^{(i)}) + \tilde{h}_1 C'_1(\tilde{z}^{(i)}) + \tilde{h}_2 C'_2(\tilde{z}^{(i)}) + \tilde{h}_1 \tilde{h}_2 C'_{1,2}(\tilde{z}^{(i)})) \frac{d\tilde{z}^{(i)}}{\tilde{z}^{m_i}} \]

\[+ B'_1(\tilde{z}^{(i)}) d\bar{h}_1 + B'_2(\tilde{z}^{(i)}) d\bar{h}_2 + B'_{1,2}(\tilde{z}) \tilde{h}_2 d\bar{h}_1 + B'_{2,1}(\tilde{z}) \tilde{h}_1 d\bar{h}_2. \]
We want to construct an isomorphism between $\nabla_u \|_{U'(i)}$ and $\nabla'$. Since $C'_1(\tilde{z}'(i)) \|_{D'_{\tilde{M}'_0}[u]}$, $C'_2(\tilde{z}'(i)) \|_{D'_{\tilde{M}'_0}[u]}$ and $C'_{1,2}(\tilde{z}'(i)) \|_{D'_{\tilde{M}'_0}[u]}$ are diagonal matrices by the assumption, the integrability condition

$$
- ((C'_1 + \tilde{h}_2 C'_{1,2}) d\tilde{h}_1 + (C'_2 + \tilde{h}_1 C'_{1,2}) d\tilde{h}_2) \land \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''} \\
= (dB'_1(\tilde{z}'(i)) + \tilde{h}_2 dB'_{1,2}(\tilde{z}'(i))) \land d\tilde{h}_1 + (dB'_2(\tilde{z}'(i)) + \tilde{h}_1 dB'_{2,1}(\tilde{z}'(i))) \land d\tilde{h}_2 \\
+ (B'_{2,1}(\tilde{z}'(i)) - B'_{1,2}(\tilde{z}'(i)) + [B'_1(\tilde{z}'(i)), B'_2(\tilde{z}'(i))] d\tilde{h}_1 \land d\tilde{h}_2 \\
+ ([A(\tilde{z}'(i)), B'_1(\tilde{z}'(i)) + \tilde{h}_2 B'_{1,2}(\tilde{z}'(i))] + \tilde{h}_2 [C'_2, B'_2(\tilde{z}'(i))]) \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''} \land d\tilde{h}_1 \\
+ ([A(\tilde{z}'(i)), B'_2(\tilde{z}'(i)) + \tilde{h}_1 B'_{2,1}(\tilde{z}'(i))] + \tilde{h}_1 [C'_1, B'_2(\tilde{z}'(i))]) \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''} \land d\tilde{h}_2
$$

implies that $dB'_1(\tilde{z}'(i)) \|_{D'_{\tilde{M}'_0}[u]} = \text{Diag}(v'_1(\mu_k) \cdot \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''}) \|_{D'_{\tilde{M}'_0}[u]}$ and $dB'_2(\tilde{z}'(i)) \|_{D'_{\tilde{M}'_0}[u]} = \text{Diag}(v'_2(\mu_k) \cdot \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''}) \|_{D'_{\tilde{M}'_0}[u]}$. Then $B'_1(\tilde{z}'(i)) - B'_1(\tilde{z}'(i))$, $B'_2(\tilde{z}'(i)) - B'_2(\tilde{z}'(i))$ are matrices of algebraic functions on $U'(i)[u]$ and applying the transform $(I_r + \tilde{h}_1 B'_1(\tilde{z}'(i)) - B'_1(\tilde{z}'(i)) + \tilde{h}_2 B'_2(\tilde{z}'(i)) - B'_2(\tilde{z}'(i)))$ to $\nabla'$, we may assume that $B'_1 = B_1$, $B'_2 = B_2$ and consequently, $C'_1 = dB_1 + [A, B_1] \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''} = C_1$ and $C'_2 = dB_2 + [A, B_2] \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''} = C_2$.

Since $[B_1, B_2] = 0$, we have $B'_{1,2} = B'_{2,1}$ and $C'_{1,2} = dB'_{1,2} + ([A, B]_{1,2} + [C, B_1]) \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''}$. implies that $dB'_{1,2} \|_{D'_{\tilde{M}'_0}[u]} = \text{Diag}(v'_{1,2}(\mu_k) \cdot \frac{d\tilde{z}'(i)}{(\tilde{z}'(i))''})$. So we can see that $B_1, B_2$ is a matrix of regular functions on $U'(i)[u]$ and the transform $I_r + \tilde{h}_1 B_2(\tilde{z}'(i)) - B'_2(\tilde{z}'(i))$ gives an isomorphism between $(O^{\tilde{r}}_{U'(i)[u]}, \nabla'_U \|_{U'(i)}, \nabla'_U \|_{U'(i), u})$ and $(O^{\tilde{r}}_{U'(i)[u]}, \nabla', N'_U)$. We can see that such an isomorphism is unique because it is determined by the coefficients of $d\tilde{h}_1$ and $d\tilde{h}_2$.

If an open subset $U \subset C_{\tilde{M}'_0}$ is disjoint from $D_{\tilde{M}'_0}$, then we can easily give a local horizontal lift of $(\tilde{E}_{\tilde{H}'_0}, \tilde{\nabla}_{\tilde{H}'_0}, \{\tilde{N}'_{H'_0}(i)\})_U$. In that case $\{\tilde{N}'_{H'_0}(i)\}_U$ is nothing. Patching local horizontal lifts altogether, we obtain a unique horizontal lift $(\epsilon''_U, \nabla''_U, \{N''_{H''_0, u}(i)\})$ of $(\tilde{E}_{\tilde{H}'_0}, \tilde{\nabla}_{\tilde{H}'_0}, \{\tilde{N}'_{H'_0}(i)\})$ with respect to $u$. 

\[ \square \]

**Theorem 5.5.** The subbundle $\Psi_0((\pi_{\mu, \lambda, v, 0})^*T_{\mu, \lambda, v, 0}) \subset T_{C_{\tilde{H}'_0}}(\pi_{\mu, \lambda, v, 0})$ satisfies the integrability condition

$$[\Psi_0((\pi_{\mu, \lambda, v, 0})^*T_{\mu, \lambda, v, 0}), \Psi_0((\pi_{\mu, \lambda, v, 0})^*T_{\mu, \lambda, v, 0})] \subset \Psi_0((\pi_{\mu, \lambda, v, 0})^*T_{\mu, \lambda, v, 0}).$$

**Proof.** Take a Zariski open set $T' \subset T_{\mu, \lambda, v, 0}$ and vector fields $v_1, v_2 \in H^0(T', T'_n)$. We will prove the equality

$$[\Psi_0((\pi_{\mu, \lambda, v, 0})^*T_{\mu, \lambda, v, 0}), \Psi_0((\pi_{\mu, \lambda, v, 0})^*T_{\mu, \lambda, v, 0})] = \Psi_0([v_1, v_2])$$

from which the theorem follows immediately. Let $T'_0 \times C[h_1, h_2]/(h_1^2, h_2^2) \xrightarrow{\tilde{I}_{v_1}} T'_0 \times \text{Spec } C[h_1, h_2]/(h_1^2, h_2^2)$ be the morphism over Spec $C[h_1, h_2]/(h_1^2, h_2^2)$ corresponding to the ring homomorphism

$$\tilde{I}_{v_1} : O_{T'_0[h_1, h_2]/(h_1^2, h_2^2)} \ni f + f_1 \tilde{h}_1 + f_2 \tilde{h}_2 + f_{1,2} \tilde{h}_1 \tilde{h}_2 \nrightarrow (f + f_1 v_1(f)) \tilde{h}_1 + f_2 \tilde{h}_2 + (f_{1,2} + v_1(f_2)) \tilde{h}_1 \tilde{h}_2 \in O_{T'_0[h_1, h_2]/(h_1^2, h_2^2)}$$

and let $T'_0 \times C[h_1, h_2]/(h_1^2, h_2^2) \xrightarrow{\tilde{I}_{v_2}} T'_0 \times \text{Spec } C[h_1, h_2]/(h_1^2, h_2^2)$ be the morphism corresponding to the ring homomorphism

$$\tilde{I}_{v_2} : O_{T'_0[h_1, h_2]/(h_1^2, h_2^2)} \ni f + f_1 \tilde{h}_1 + f_2 \tilde{h}_2 + f_{1,2} \tilde{h}_1 \tilde{h}_2 \nrightarrow f + f_1 \tilde{h}_1 + (f_2 + v_2(f)) \tilde{h}_2 + (f_{1,2} + v_2(f_1)) \tilde{h}_1 \tilde{h}_2 \in O_{T'_0[h_1, h_2]/(h_1^2, h_2^2)}.$$

The proof is completed.
By the calculation

\[
\begin{align*}
&f + f_1 h_1 + f_2 h_2 + f_1.2 h_1 h_2 \tilde{I}_{f_2}^* f + f_1 h_1 + (f_2 + v_2(f))h_2 + (f_1.2 + v_2(f_1))h_1 h_2 \\
&\tilde{I}_{v_1}^* f + (f_1 + v_1(f))h_1 + (f_2 + v_2(f))h_2 + (f_1.2 + v_1(f_2) + v_1 v_2(f))h_1 h_2 \\
&\tilde{I}_{v_2}^* f + (f_1 + v_1(f))h_1 + f_2 h_2 + (f_1.2 + v_1(f_2) + v_1 v_2(f) - v_2 v_1(f))h_1 h_2 \\
&\tilde{I}_{v_1}^* f + f_1 h_1 + f_2 h_2 + (f_1.2 + (v_1 v_2 - v_2 v_1)(f))h_1 h_2,
\end{align*}
\]

we can see that the composition \( \tilde{I}_{v_1}^* \tilde{I}_{v_2}^* \tilde{I}_{v_1}^* \tilde{I}_{v_2}^{*} \) is given by

\[
\tilde{I}_{v_1}^* \tilde{I}_{v_2}^* \tilde{I}_{v_1}^* \tilde{I}_{v_2}^{*} : \mathcal{O}_{T_0} \to \mathcal{O}_{T_0}^* \to \mathcal{O}_{T_0}^* \to \mathcal{O}_{T_0}^* \to \mathcal{O}_{T_0}^* \to \mathcal{O}_{T_0}^*.
\]

(63)

The vector field \( \Phi_0(v_1) \) corresponds to a morphism \( \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1]/(h_2^2) \to \tilde{M}_0' \). This morphism together with the second projection \( \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1]/(h_2^2) \to \text{Spec} \mathbb{C}[h_1]/(h_2^2) \) gives a morphism

\[
(64) \quad \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1]/(h_2^2) \to \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1]/(h_2^2)
\]

over \( \text{Spec} \mathbb{C}[h_1]/(h_2^2) \). Let

\[
(65) \quad \tilde{I}_{\Phi_0(v_1)} : \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2) \to \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2)
\]

be the base change of \[(64)\] under the projection \( \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2) \to \text{Spec} \mathbb{C}[h_1]/(h_2^2) \). Similarly we can define a morphism

\[
(66) \quad \tilde{I}_{\Phi_0(v_2)} : \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2) \to \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2)
\]

from the morphism \( \tilde{M}_0' \times \text{Spec} \mathbb{C}[h_2]/(h_2^2) \to \tilde{M}_0' \) corresponding to \( \Phi_0(v_2) \). We can see by a similar calculation to that of \[(63)\] that the composition \( \tilde{I}_{\Phi_0(v_1)} \tilde{I}_{\Phi_0(v_2)} \tilde{I}_{\Phi_0(v_1)} \tilde{I}_{\Phi_0(v_2)} \) corresponds to the ring homomorphism

\[
(67) \quad \tilde{I}_{\Phi_0(v_1)} \tilde{I}_{\Phi_0(v_2)} \tilde{I}_{\Phi_0(v_1)} \tilde{I}_{\Phi_0(v_2)} : \mathcal{O}_{\tilde{M}_0'} \to \mathcal{O}_{\tilde{M}_0'} \to \mathcal{O}_{\tilde{M}_0'} \to \mathcal{O}_{\tilde{M}_0'}
\]

\[
\to f + f_1 h_1 + f_2 h_2 + (f_1.2 + (\Phi_0(v_1)\Phi_0(v_2) - \Phi_0(v_2)\Phi_0(v_1)(f))h_1 h_2 \in \mathcal{O}_{\tilde{M}_0'}[h_1,h_2]/(h_1^2,h_2^2).
\]

Let \( \pi_{T_0} : T_0 \times \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2) \to T_0 \) be the first projection. By Lemma \[5.4\] there exists a unique horizontal lift \((E_0\tilde{M}_0', \nabla_{\tilde{M}_0'}, \{N^{(i)}_{\tilde{M}_0'}\})\) with respect to the composition \(\pi_{T_0} \circ \tilde{I}_{v_1} \circ T_0 \times \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2) \to T_0'. \) Then we can see that

\[
(id \times \tilde{I}_{\Phi_0(v_1)})^* (id \times \tilde{I}_{\Phi_0(v_2)})^* (id \times \tilde{I}_{\Phi_0(v_1)})^* (E_0\pi_{T_0} \circ \tilde{I}_{v_2} \circ \tilde{I}_{v_1} \circ I_{v_2} \circ I_{v_1} = I_{[v_1,v_2]} \circ \rho),
\]

be the morphism whose corresponding ring homomorphism \(\rho^* : \mathcal{O}_{T_0'}[h_1,h_2]/(h_1^2,h_2^2) \to \mathcal{O}_{T_0'}[h_1,h_2]/(h_1^2,h_2^2)\) is given by \(\rho^*(f + gh) = f + gh h_2\) for \(f, g \in \mathcal{O}_{T_0'}\). Then we have

\[
\pi_{T_0} \circ \tilde{I}_{v_2} \circ \tilde{I}_{v_1} \circ \tilde{I}_{v_2} \circ \tilde{I}_{v_1} = I_{[v_1,v_2]} \circ \rho,
\]

where \(I_{[v_1,v_2]} : T_0' \times \text{Spec} \mathbb{C}[h_1,h_2]/(h_1^2,h_2^2) \to T_0'\) means the morphism corresponding to the commutator vector field \([v_1,v_2] = v_1 v_2 - v_2 v_1\). If we denote by \((E_0[v_1,v_2], \nabla_{[v_1,v_2]}, \{N^{(i)}_{[v_1,v_2]}\})\) the horizontal lift of \((E_0\tilde{M}_0', \nabla_{\tilde{M}_0'}, \{N^{(i)}_{\tilde{M}_0'}\})\), in the sense of Proposition \[5.3\] with respect to the the commutator vector field \([v_1,v_2] \in H^0(T_0',T_{T_0'})\), we can see that \((id \times \rho)^* (E_0[v_1,v_2], \nabla_{[v_1,v_2]}, \{N^{(i)}_{[v_1,v_2]}\})\) is also a horizontal lift of \((E_0\tilde{M}_0', \nabla_{\tilde{M}_0'}, \{N^{(i)}_{\tilde{M}_0'}\})\), in the sense
of Lemma \[\text{[5.4]}\] with respect to \(I_{[v_1,v_2]}\) we have an isomorphism
\[
(id \times \tilde{I}_{\Phi_0(v_1)})^*(id \times \tilde{I}_{\Phi_0(v_2)})^*(id \times \tilde{I}_{\Phi_0(v_1)})^* \left( \pi_{\tau_0}^{-1} \circ \tilde{I}_{\tau_2}, \nabla_0^{-1} \circ \tilde{I}_{\tau_2}, \{N_0^{(i)}(\tau_2)\} \right)
\]
\[
\cong (id \times \rho)^*(\mathcal{E}^{[v_1,v_2]}_{\tau_0}, \mathcal{V}^{[v_1,v_2]}_{\tau_0}, \{\Lambda^{(i)}_{0,\tau_0} \circ \tilde{I}_{\tau_2}\}).
\]

By the uniqueness of the horizontal lift in Lemma \[\text{[5.4]}\] we have the isomorphism
\[
(id \times \tilde{I}_{\Phi_0(v_1)})^*(id \times \tilde{I}_{\Phi_0(v_2)})^*(id \times \tilde{I}_{\Phi_0(v_1)})^* \left( \pi_{\tau_0}^{-1} \circ \tilde{I}_{\tau_2}, \nabla_0^{-1} \circ \tilde{I}_{\tau_2}, \{N_0^{(i)}(\tau_2)\} \right)
\]
\[
\cong (id \times \rho)^*(\mathcal{E}^{[v_1,v_2]}_{\tau_0}, \mathcal{V}^{[v_1,v_2]}_{\tau_0}, \{\Lambda^{(i)}_{0,\tau_0} \circ \tilde{I}_{\tau_2}\}).
\]

The flat family \((id \times \tilde{I}_{\Phi_0(v_1)})^*(id \times \tilde{I}_{\Phi_0(v_2)})^*(id \times \tilde{I}_{\Phi_0(v_1)})^* \left( \pi_{\tau_0}^{-1} \circ \tilde{I}_{\tau_2}, \nabla_0^{-1} \circ \tilde{I}_{\tau_2}, \{N_0^{(i)}(\tau_2)\} \right)\) associated to \(68\) corresponds to the composition
\[
\pi_{\tilde{M}_0} \circ \tilde{I}_{\Phi_0(v_2)} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_2)} \circ \tilde{I}_{\Phi_0(v_1)} : \tilde{M}_0 \times \text{Spec } \mathbb{C}[h_1,h_2]/(h_1^2, h_2^2) \longrightarrow \tilde{M}_0,
\]
where \(\pi_{\tilde{M}_0} : \tilde{M}_0 \times \text{Spec } \mathbb{C}[h_1,h_2]/(h_1^2, h_2^2) \longrightarrow \tilde{M}_0\) is the first projection. The same associated flat family \((id \times \rho)^*(\mathcal{E}^{[v_1,v_2]}_{\tau_0}, \mathcal{V}^{[v_1,v_2]}_{\tau_0}, \{\Lambda^{(i)}_{0,\tau_0} \circ \tilde{I}_{\tau_2}\})\) induced by \(68\) corresponds to the composition
\[
\pi_{\tilde{M}_0} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_2)} \circ \tilde{I}_{\Phi_0(v_2)} : \tilde{M}_0 \times \text{Spec } \mathbb{C}[h_1,h_2]/(h_1^2, h_2^2) \longrightarrow \tilde{M}_0.
\]
Thus we have \(\pi_{\tilde{M}_0} \circ \tilde{I}_{\Phi_0(v_2)} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_2)} \circ \tilde{I}_{\Phi_0(v_1)} = \pi_{\tilde{M}_0} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_2)} \circ \tilde{I}_{\Phi_0(v_2)}\) is given by the ring homomorphism
\[
O_{\tilde{M}_0} \ni f \mapsto f + (\Phi_0(v_1) \Phi_0(v_2) - \Phi_0(v_2) \Phi_0(v_1)) \tilde{h}_1 \tilde{h}_2 \in O_{\tilde{M}_0}[h_1,h_2]/(h_1^2, h_2^2).
\]

On the other hand, the morphism \(\pi_{\tilde{M}_0} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_1)} \circ \tilde{I}_{\Phi_0(v_2)} \circ \tilde{I}_{\Phi_0(v_2)}\) is given by the ring homomorphism
\[
O_{\tilde{M}_0} \ni f \mapsto f + \Phi_0(v_1 v_2 - v_2 v_1) \tilde{h}_1 \tilde{h}_2 \in O_{\tilde{M}_0}[h_1,h_2]/(h_1^2, h_2^2).
\]
Hence we have \(\Phi_0(v_1) \Phi_0(v_2) - \Phi_0(v_2) \Phi_0(v_1) = \Phi_0(v_1 v_2 - v_2 v_1)\), which is nothing but the equation \(62\) and the proposition is proved. \(\square\)

5.3. Global horizontal lift in the unfolded case. In this subsection, we give a construction of a global horizontal lift via patching local horizontal lifts given in Proposition \[\text{[4.3]}\] The consequent global horizontal lift given in Proposition \[\text{[5.10]}\] produces a proof of Theorem \[\text{[0.1]}\]

Take a point \(x \in M^0_{C,D}(\nu,\mu)_{v_0} \times_{B^s} B^s\) which corresponds to a \((\nu,\mu)\)-connection \((E,\nabla,\{N^{(i)}\})\). Recall that we are given an analytic open subset \(U_i \subset C_{B^s}\) with a biholomorphic map \(U_i \xrightarrow{\sim} \Delta_a \times \Delta_a \times \Delta_a\) in subsection \[\text{[5.1]}\] We take a loop \(\tilde{\gamma}_{x} \in (U_i)_{x} \subset C_{x}\) which is a boundary of a disk containing \(D^0_{\nu,\mu}\). We consider the morphism
\[
\nabla^1 : \text{End}(E) \supseteq u \mapsto \nabla \circ u - u \circ \nabla \in \text{End}(E) \otimes \Omega^1_{C_{x}}(D_{x})
\]
and assume the following:

Assumption 5.6. (1) The monodromy of \(\nabla_x : E_x \longrightarrow E_x \otimes \Omega^1_{C_{x}}(D_{x})\) along \(\tilde{\gamma}_{x}\) has the \(r\) distinct eigenvalues and

(2) \(H^{0}(\text{End}(E)_{x}, \ker \nabla^1|_{(U_i)_{x}}) = \mathbb{C}\).

There is an \(\acute{\text{e}}\text{tale}\) morphism \(\tilde{M} \longrightarrow M^0_{C,D}(\nu,\mu)\) whose image contains \(x\) such that there is a universal family \((\tilde{E}, \tilde{\nabla}, \{\tilde{N}^{(i)}\})\) on \((C,D)_{\tilde{M}}\) over \(\tilde{M}\). We can take an analytic open neighborhood \(M^0_{\nu,\mu}\) of \(x\) in \(M^0_{C,D}(\nu,\mu) \times_{B^s} B^s\) with a factorization \(M^0 \longrightarrow \tilde{M} \longrightarrow M^0_{C,D}(\nu,\mu)\). We denote by \((\tilde{E}^{(i)}_{M^0}, \tilde{\nabla}^{(i)}_{M^0}, \{\tilde{N}^{(i)}_{M^0}\})\) the pullback of \((\tilde{E}, \tilde{\nabla}, \{\tilde{N}^{(i)}\})\) to \((C,D)_{M^0}\).

In the following, we successively replace \(M^0\) by its shrink till Definition \[\text{[5.7]}\] We may assume that \((U_i)_{M^0} \xrightarrow{\sim} \Delta_a \times \Delta_a\) is an isomorphism if \(M^0\) by its shrink. We denote the image of \(M^0\) under the morphism \(M^0_{C,D}(\nu,\mu) \times_{B^s} B^s \longrightarrow \tilde{T}_{\mu,\lambda} \times_{B^s} B^s\) by \(T^0\), which is an analytic open subset of \(\tilde{T}_{\mu,\lambda} \times_{B^s} B^s\).

Then the inclusion \(T^0 \xrightarrow{\sim} \tilde{T}_{\mu,\lambda} \times_{B^s} B^s \longrightarrow \tilde{T}_{\mu,\lambda}\) corresponds to a tuple of polynomials \(\nu = (\nu^{(i)}(T))_{1 \leq i \leq n}\) given by
\[
\nu^{(i)}(T) = \sum_{i=0}^{r-1} \sum_{j=0}^{m_{i}-1} c^{(i)}_{i,j} (z^{(i)})^{j} T^{i}
\]
with \(c^{(i)}_{i,j} \in H^{0}(T^0, \mathcal{O}_{T^0}^{B^s})\) satisfying (a) and (b) of the definition of \(T_{\mu,\lambda}\).
We apply the process in subsection 4.2 to the restricted relative connection \( \tilde{E}^{\text{hol}}_{M^\circ} \), \( \tilde{N}^{\text{hol}}_{M^{\circ}} \), \( \{ (U_i)_{M^\circ} \} \). Using Proposition 4.3, there is an isomorphism \( \vartheta^{(i)} : \tilde{E}^{\text{hol}}_{M^\circ} \mid (U_i)_{M^\circ} \xrightarrow{\sim} (\mathcal{O}^{\text{hol}}_{(U_i)_{M^\circ}}) \oplus \mathcal{R} \) after shrinking \( M^\circ \) such that the connection \( (\vartheta^{(i)} \otimes \text{id}) \circ \tilde{\nabla}^{\text{hol}}_{M^\circ} \mid (U_i)_{M^\circ} \circ (\vartheta^{(i)})^{-1} \) is canonically extended to a global relative connection

\[
\nabla^{(i), \text{proj}}_{M^\circ} : (\mathcal{O}^{\text{hol}}_{P^1 \times M^\circ}) \oplus \Omega_{P^1 \times M^\circ/M^\circ}^1 (D/M^\circ \cup \{ \infty \} \times M^\circ) \to \text{hol},
\]

where we are assuming the identification \( (U_i)_{M^\circ} = \Delta \times M^\circ \hookrightarrow P^1 \times M^\circ \). Let

\[
A^{(i)}(z^{(i)}, \epsilon) = \sum_{j=0}^{m_i-1} A_{i,j}^{(i)}(\epsilon)(z^{(i)})^j \frac{dz^{(i)}}{(z^{(i)})^{m_i} - \epsilon m_i^n}
\]

be the connection matrix of \( \nabla^{(i), \text{proj}}_{M^\circ} \). By Assumption 5.6, we can see, after shrinking \( M^\circ \), that

\[
\bigcap_{j=0}^{m_i-1} \ker \left( \text{ad}(A_{i,j}^{(i)}(\epsilon)) \right) = \mathcal{O}^{\text{hol}}_{M^\circ}
\]

in the same way as in 50 in subsection 4.2. As in the argument in subsection 4.2 producing (60), we can take matrices \( \Xi^{(i)}_{i,j} \) of polynomials in \( z^{(i)} \) of degree less than \( m_i \) satisfying

\[
A^{(i)}(z^{(i)}, \epsilon) = \sum_{j=0}^{m_i-1} c^{(i)}_{i,j} \Xi^{(i)}_{i,j}(z^{(i)})
\]

and

\[
z^{i} \vartheta^{(i)} \circ \left( \tilde{N}^{(i), \text{hol}} \right) \circ (\vartheta^{(i)})^{-1} \mid D_{M^\circ}^{(i)} = \Xi^{(i)}_{i,j} \mid D_{M^\circ}^{(i)}.
\]

By the same method as in 51, we can take, using Lemma 4.4, matrices \( R^{(i), \text{proj}}_{i,j} \) with coefficients in \( \mathcal{O}^{\text{hol}}_{M^\circ} \) after shrinking \( M^\circ \) such that

\[
\Xi^{(i)}_{i,j}(z^{(i)}) = \Xi^{(i)}_{i,j}(z^{(i)}) - \sum_{q=0}^{m_i-1} \sum_{0 \leq l' \leq m_i-1-q} \left[ A_{q}^{(i)}(\epsilon), R^{(i), \text{proj}}_{i,j,l'} \right] (z^{(i)})^{q+l'}
\]

and

\[
\Xi^{(i)}_{i,j}(z^{(i)}) \mid D_{M^\circ}^{(i)} = \Xi^{(i)}_{i,j}(z^{(i)}) \mid D_{M^\circ}^{(i)} - \left[ A^{(i)}(z^{(i)}), \sum_{l'=0}^{m_i-1} R^{(i), \text{proj}}_{i,j,l'}(z^{(i)})^{l'} \right] \mid D_{M^\circ}^{(i)}.
\]

In the same way as in subsection 4.2 defining (53), we take a fundamental solution \( \tilde{Y}_{\infty}(z^{(i)}, \tilde{h}) \) of the relative connection

\[
\nabla_{P^1 \times M^{\circ} \backslash \{ \infty \}}^{\text{proj}} : (\mathcal{O}^{\text{hol}}_{P^1 \times M^{\circ} \backslash \{ \infty \}}) \oplus \Omega_{P^1 \times M^{\circ} \backslash \{ \infty \}}^1 (D^{(i)}/M^{\circ} \backslash \{ \infty \} \times M^{\circ}) \to \text{hol},
\]

where we write \( M^{\circ} \backslash \tilde{h} := M^{\circ} \times \text{Spec} \mathbb{C}[\tilde{h}]/(\tilde{h}^2) \). Then we define

\[
B_{i,j}^{(i)}(z^{(i)}) = \frac{\partial \tilde{Y}_{\infty}(z^{(i)}, \tilde{h})}{\partial \tilde{h}} \tilde{Y}_{\infty}(z^{(i)}, 0)^{-1}
\]

as in (53). If we denote by \( t_{M^\circ \backslash \tilde{h}} : (P^1 \times M^{\circ} \backslash \{ \infty \}) \Rightarrow P^1 \times M^{\circ} \backslash \tilde{h} \) the canonical inclusion, then the connection

\[
\nabla_{P^1 \times M^{\circ} \backslash \{ \infty \}}^{\text{proj}} : (\mathcal{O}^{\text{hol}}_{P^1 \times M^{\circ} \backslash \{ \infty \}}) \oplus (t_{M^\circ \backslash \tilde{h}} \mathcal{O}_{P^1 \times M^{\circ} \backslash \{ \infty \}}) \oplus \Omega_{P^1 \times M^{\circ} \backslash \{ \infty \}}^1 (D^{(i)}/M^{\circ} \backslash \{ \infty \} \times M^{\circ}) \to \text{hol}
\]
given by the connection matrix

\[ (A(z^{(i)}, \epsilon) + \tilde{h} z^{(i)} \partial_i \tilde{z}^{(i)}) \frac{dz^{(i)}}{(z^{(i)})^{m_i - e_{m_i}}} + B_{i,j}^{(i)}(z^{(i)}) d\bar{h} \]

satisfies the integrability condition by Proposition 4.2. Furthermore, we may assume as in the construction of (54) in subsection 4.2 that the restriction of \( \nabla_{\mathcal{P}^1 \times M^\nu \mid \mathcal{H}^{(i)} \times \Sigma_{\mathcal{U}} \mathbb{C} \langle \epsilon \rangle / (\mathbb{C} \times \mathcal{H}^{(i)}) \) coincides with the horizontal lift giving the unramified irregular singular generalized isomonodromic deformation.

**Definition 5.7.** We call the collection \( \{ \nabla_{\mathcal{P}^1 \times M^\nu \mid \mathcal{H}^{(i)} \times \Sigma_{\mathcal{U}} \mathbb{C} \langle \epsilon \rangle / (\mathbb{C} \times \mathcal{H}^{(i)}) \} \) of integrable connections determined by \( \{ \tilde{z}^{(i)}_{i,j}(z^{(i)}) \} \), \( \{ B_{i,j}^{(i)}(z^{(i)}) \} \) in \( \{ \mathcal{H}^{(i)} \} \) with the above property a block of local horizontal lifts.

Take an analytic open subset \( \mathcal{T}' \subset \mathcal{T}^\circ \subset \mathcal{T}_{\mu, \lambda} \times_{\lambda_{\mathcal{B}}} \mathcal{B}' \) and a \( \Delta_{\mathcal{U}} \)-relative holomorphic vector field \( v \in H^0(\mathcal{T}', T_{\mathcal{T}' / \mathcal{U}}) \) on \( \mathcal{T}' \). Then \( v \) corresponds to an analytic morphism

\[ I_v : \mathcal{T}' \times \text{Spec} \mathbb{C}[h]/(h^2) \to \mathcal{T}' \to \mathcal{T}_{\mu, \lambda} \times_{\lambda_{\mathcal{B}}} \mathcal{B}' \]

over \( \Delta_{\mathcal{U}} \) satisfying \( I_v |_{\mathcal{T}' \times \text{Spec} \mathbb{C}[h]/(h^2)} = \text{id}_{\mathcal{T}'} \). We put \( \mathcal{T}'[v] := \mathcal{T}' \times \text{Spec} \mathbb{C}[h]/(h^2) \) which is regarded as an analytic space over \( \mathcal{T}' \) via \( I_v \) and consider the fiber product

\[ \mathcal{T}' \times \text{Spec} \mathbb{C}[h]/(h^2) \]

of \( \mathcal{T}_{\mathcal{T}'[v]} := \mathcal{T}_v \times \mathcal{T}' \) and \( \mathcal{T}' \times \text{Spec} \mathbb{C}[h]/(h^2) \) \( I_v \to \mathcal{T}' \). The morphism \( I_v \) corresponds to an analytic morphism

\[ I_{\frac{1}{2}v} : \mathcal{T}' \times \text{Spec} \mathbb{C}[h]/(h^2) \to \mathcal{B}' \]

over \( \Delta_{\mathcal{U}} \) and a tuple of polynomials

\[ (72) \]

\[ \nu_{\text{hor}} + \tilde{h} \nu_v = (\nu_{\text{hor}}^{(i)}(T) + \tilde{h} \nu_v^{(i)}(T)) \]

where \( \nu_{\text{hor}}^{(i)}(T) \in \mathcal{O}_{\mathcal{T}'[v]}[h]/(h^2) [T] \) and \( \nu_v^{(i)}(T) \in \mathcal{O}_{\mathcal{T}'[v]}[T] \) are given by

\[ \nu_{\text{hor}}^{(i)}(T) = \sum_{i=0}^{r-1} \sum_{j=0}^{m_i - 1} (i^r T_{c_{i,j}} - \tilde{h} \nu_v^{(i)}(\tilde{z}^{(i)}) \partial_i \tilde{z}^{(i)}) T_j \]

\[ \nu_v^{(i)}(T) = \sum_{i=0}^{r-1} \sum_{j=0}^{m_i - 1} v(c_{i,j}^{(i)}(\tilde{z}^{(i)}) \partial_i \tilde{z}^{(i)}) T_j. \]

Here \( \tilde{z}^{(i)} \) is the pull-back of \( z^{(i)} \) under the morphism \( \mathcal{T}_{\mathcal{T}'[v]} \) \( \xrightarrow{id \times I_v} \mathcal{T}_{\mathcal{T}'[v]} \to \mathcal{B}' \) and \( \nu_{\text{hor}}^{(i)}(T) + \tilde{h} \nu_v^{(i)}(T) \in \mathcal{O}_{\mathcal{T}'[v]}[h]/(h^2) [T] \) should satisfy (a) in the definition of \( \mathcal{T}_{\mu, \lambda} \). For an analytic open subset \( U \subset \mathcal{T}_{\mathcal{T}'[v]} \), we denote by \( U[v] \) the open subspace of \( \mathcal{T}_{\mathcal{T}'[v]} \) whose underlying set of points is \( U \).

We consider the sheaf of differential forms \( \left( \Omega^1_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}'[v]} \right)^{\text{hor}} \) with respect to the composite of the trivial projections

\[ \mathcal{T}_{\mathcal{T}'[v]} = \mathcal{C} \times \mathcal{T}' / \text{Spec} \mathbb{C}[h]/(h^2) \to \mathcal{T}' / \text{Spec} \mathbb{C}[h]/(h^2) \to \mathcal{T}' \]

which is different from the structure of \( \mathcal{T}_{\mathcal{T}'[v]} \) over \( \mathcal{T}' \) coming from the fiber product structure. Note that \( \left( \Omega^1_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}'[v]} \right)^{\text{hor}} \) is locally generated by \( d\tilde{z}^{(i)} \) and \( d\bar{h} \), where \( \tilde{z} \) is the pullback of a uniformizing parameter \( z \) of \( \mathcal{T}_{\mathcal{T}'[v]} \) via the first projection \( \mathcal{T}_{\mathcal{T}'[v]} \times \mathcal{T}'[v] \to \mathcal{T}' \). Let

\[ \iota_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]} : (\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]) \to \mathcal{T}_{\mathcal{T}'[v]} \]

be the inclusion morphism. We denote by \( \iota_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]} : (\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]) \to \mathcal{T}_{\mathcal{T}'[v]} \) its restriction to the underlying sets of points.

**Definition 5.8.** We define the \( \mathcal{O}^{\text{hor}}_{\mathcal{T}_{\mathcal{T}'[v]}[v]} \)-subsheaf \( \Omega^1_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]} \) of \( (\iota_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v])^*_v \left( \Sigma_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]} \right)^{\text{hor}} \) by the condition that \( \Omega^1_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]} \) is locally generated by \( d\tilde{z}^{(i)} \) and \( (\iota_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v])^*_v \left( \Sigma_{\mathcal{T}_{\mathcal{T}'[v]} / \mathcal{T}_{\mathcal{T}'[v]}[v]} \right)^{\text{hor}} d\bar{h} \) around points in \( \Gamma^{(i)}_{\mathcal{T}_{\mathcal{T}'[v]}[v]} \).
and locally generated by $dz$ and $dh$ around points in $(C_T \setminus \Gamma_T)[v]$ where $z$ is a local holomorphic coordinate of $C_T \setminus \Gamma_T$. We denote by $\Omega_{C,T,v}^2$ the canonical image of $\Omega_{C,T,v}^1 \wedge \Omega_{C,T,v}^1$ in $(\iota_{(C_T \setminus \Gamma_T)[v]} \ast (\Omega_{C,T,v}^2)_{(C_T \setminus \Gamma_T)[v]/T})^{\text{hol}}$.

We put $M' := M^o \times_{\Gamma} T'$ and consider the analytic space $M'[v] := M' \times \text{Spec} \mathbb{C}[h]/(h^2)$ with the structure morphisms

$$M'[v] := M' \times \text{Spec} \mathbb{C}[h]/(h^2) \longrightarrow T' \times \text{Spec} \mathbb{C}[h]/(h^2) \xrightarrow{\iota_v} T'.$$

We denote the base change of $C \times_T T'$, $D \times_T T'$ and $\mathcal{D}^{(i)} \times_T T'$ via $M'[v] \rightarrow T'$ by $\mathcal{C}_M[v]$, $\mathcal{D}_M[v]$ and $\mathcal{D}_M^{(i)}[v]$, respectively. We denote the pullback of a local holomorphic coordinate $z$ of $C_{T'}$ under the morphism $C_{M'[v]} \rightarrow C_{T'}$ by $\tilde{z}$.

Let us consider the analytic open subspace $(U_i)_{M'[v]} \subset \mathcal{C}_M[v] = C_T \times_T (M' \times \text{Spec} \mathbb{C}[h]/(h^2))$. We have an analytic isomorphism

$$(U_i)_{M'[v]} \cong \Delta_a \times M'[v] = \Delta_a \times M' \times \text{Spec} \mathbb{C}[h]/(h^2)$$

whose structure morphism over $T_{\mu, \lambda}$ is given by

$$\Delta_a \times M' \times \text{Spec} \mathbb{C}[h]/(h^2) \longrightarrow M' \times \text{Spec} \mathbb{C}[h]/(h^2) \longrightarrow T' \times \text{Spec} \mathbb{C}[h]/(h^2) \xrightarrow{\iota_v} T' \rightarrow T_{\mu, \lambda}.$$

We remark that elements in $\Omega^i_{C,T,v} \otimes_{\mathcal{O}_{M'[v]}} \mathcal{O}_{M'[v]} \subset (\iota_{(C_M \setminus \Gamma_M)[v]} \ast (\Omega^1_{(C_M \setminus \Gamma_M)[v]/M})^{\text{hol}}$ are relative differentials with respect to the morphism

$$\mathcal{C}_{M'[v]} = C_T \times_T (M' \times \text{Spec} \mathbb{C}[h]/(h^2)) \longrightarrow M' \times \text{Spec} \mathbb{C}[h]/(h^2) \longrightarrow M',$$

where the arrows are the trivial projections. The restriction of the above morphism to $(U_i)_{M'[v]}$ is just the trivial projection $(U_i)_{M'[v]} \cong \Delta_a \times M' \times \text{Spec} \mathbb{C}[h]/(h^2) \longrightarrow M'$. The corresponding inclusion $\mathcal{O}_{M'} \rightarrow \mathcal{O}_{(U_i)_{M'[v]}}$ induces the ring homomorphism

$$\mathcal{O}_{M'}^{\text{hol}}[\tilde{z}^{(i)}] \longrightarrow \mathcal{O}_{(U_i)_{M'[v]}}^{\text{hol}}$$

from the polynomial ring. We denote the image under this ring homomorphism of a matrix $A(z^{(i)})$ of polynomials with coefficients in $\mathcal{O}_{M'}$ by $A(\tilde{z}^{(i)})$.

We denote the restriction of $(\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}_{M'}^{(i), \text{hol}}\})$ to $\mathcal{C}_{M'}$ by $(\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}_{M'}^{(i), \text{hol}}\})$.

**Definition 5.9.** We say that a tuple $(\nabla^v, \nabla^v_t, \{N_v^{(i)}\})$ is a horizontal lift of $(\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}_{M'}^{(i), \text{hol}}\})$ with respect to $v \in H^0(T', T_{\mu, \lambda}/\Delta_a)$ and with respect to blocks of horizontal lifts $(\nabla^v, \nabla^v_t, \{N_v^{(i)}\})$ if

1. $\nabla^v$ is a rank $r$ holomorphic vector bundle on $\mathcal{C}_M[v]$;
2. $\nabla^v : \mathcal{E}^v \longrightarrow \mathcal{E}^v \otimes_{\mathcal{O}_{M'[v]}} \mathcal{O}_{C,T,v}^1$ is a morphism of sheaves satisfying $\nabla^v(f \partial) = a \otimes df + f \nabla^v(a)$ for $f \in \mathcal{O}_{M'}^{\text{hol}}$, and $a \in \mathcal{E}^v$.
3. $\nabla^v$ is integrable in the sense that the restriction of $\nabla^v$ to any open set $U[v] \subset (\mathcal{C}_M \setminus \Gamma_M)[v]$ is expressed by

   $$\mathcal{E}^v|_{U[v]} \cong \left(\mathcal{O}_{U[v]}^{\text{hol}}\right)^{\otimes r} \ni \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \mapsto \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \begin{pmatrix} A \tilde{d} \tilde{z} + B \tilde{d} \tilde{h} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \in \left(\mathcal{O}_{U[v]}^{\text{hol}}\right)^{\otimes r} \otimes_{\mathcal{O}_{C,T,v}} \mathcal{O}_{C,T,v}^1$$

   satisfies

   $$d \begin{pmatrix} A \tilde{d} \tilde{z} + B \tilde{d} \tilde{h} \end{pmatrix} + \begin{pmatrix} A \tilde{d} \tilde{z} + B \tilde{d} \tilde{h} \end{pmatrix} \wedge \begin{pmatrix} A \tilde{d} \tilde{z} + B \tilde{d} \tilde{h} \end{pmatrix} = 0$$

   in $\mathcal{E} \text{nd}(\mathcal{O}_{U[v]}^{\text{hol}})^{\otimes r} \otimes_{\mathcal{O}_{C,T,v}} \mathcal{O}_{C,T,v}^2$.
4. $\nabla^v_0 : \mathcal{E}^v|_{\mathcal{D}_M^1[v]} \longrightarrow \mathcal{E}^v|_{\mathcal{D}_M^1[v]}$ is an endomorphism satisfying $\nabla^v_0(N_v^{(i)}) = 0$,
5. the relative connection $\nabla^v_t$ defined by the composition

   $$\nabla^v_t : \mathcal{E}^v \longrightarrow \mathcal{E}^v \otimes \mathcal{O}_{C,T,v}^1 \longrightarrow \mathcal{E}^v \otimes \mathcal{O}_{C,M[v]/M'}[v](\mathcal{D}_M[v])^{\text{hol}}$$

   satisfies

   $$\left(\iota_{\mu,T,v} + \hat{h}_v^{(i)}(\tilde{z}^{(i)}) \epsilon_{M'}^m - \epsilon_{M'}^m \right) = \nabla^v_0\left|_{\mathcal{O}_{C,M[v]/M'}[v]} \right.$$ for any $i$.
6. $\left(\mathcal{E}^v, \nabla^v_t, \{N_v^{(i)}\} \right) \otimes \mathcal{O}_{M'[v]}^{\text{hol}} / \mathcal{H} \mathcal{O}_{M'[v]}^{\text{hol}} \cong \left(\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}_{M'}^{(i), \text{hol}}\} \right)$. 


(7) there is an isomorphism \( \theta^{i,v} : \mathcal{E}^v|_{(U_i,v)} \xrightarrow{\sim} (\mathcal{O}_{(U_i,v)}^{\text{hol}})^{\oplus r} \) which is a lift of the restriction \( \theta^{i}|_{(U_i,v)} \) of the given isomorphism \( \theta^{i} : \mathcal{E}^v|_{(U_i,v)} \xrightarrow{\sim} (\mathcal{O}_{(U_i,v)}^{\text{hol}})^{\oplus r} \) such that the consequent connection matrix of \( (\theta^{i,v} \otimes \text{id}) \circ \nabla^v \circ (\theta^{i,v})^{-1} \) is given by

\[
\left( A^{(i)}(z^{(i)}, \epsilon) + \tilde{h} \sum_{l=0}^{r-1} \sum_{j=0}^{m_i-1} v(c^{(i)}_{l,j} \tilde{z}^{(i)}_{l,j}(z^{(i)})) \frac{dz^{(i)}}{(z^{(i)})_{m_i} - \epsilon m_i} + \sum_{l=0}^{r-1} \sum_{j=0}^{m_i-2} v(c^{(i)}_{l,j} \tilde{B}^{(i)}_{l,j}(z^{(i)})) \right) \frac{d\bar{h}}{\bar{h}}
\]

and that \( (\theta^{i,v} \otimes \text{id}) \circ \mathcal{N}^{(i)} \circ (\theta^{i,v})^{-1} = \left( \tilde{z}^{(i)}_{l,j}(z^{(i)}) + \tilde{h} v(c^{(i)}_{l,j} \tilde{z}^{(i)}_{l,j}(z^{(i)})) \right)_{|\mathcal{D}^{(i)}}. \)

The following proposition on the existence of a global horizontal lift is a key process in the construction of an unfolded generalized isomonodromic deformation.

**Proposition 5.10.** For any \( \Delta_{\alpha_0} \)-relative holomorphic vector field \( v \in H^0(T', T'^{\text{hol}}_{T'/\Delta_{\alpha_0}}) \), there exists a unique horizontal lift \( (\mathcal{E}^v, \nabla^v, \{ \mathcal{N}^{(i)} \}) \) of \( (\mathcal{E}_{\text{hol}}^{\text{flat}}_{T', T'^{\text{hol}}_{T'/\Delta_{\alpha_0}}}, \nabla_{T', T'^{\text{hol}}_{T'/\Delta_{\alpha_0}}}, \{ \mathcal{N}^{(i)}_{T'^{\text{hol}}_{T'/\Delta_{\alpha_0}}} \}) \) with respect to \( v \) and with respect to the blocks of local horizontal lifts \( (\nabla_{\text{flat}}^{T'^{\text{flat}}_{T'/\Delta_{\alpha_0}}} |_{\mathbb{P}^{1} \times M'} : \mathcal{O}_{\mathbb{P}^{1} \times M'}^{\text{hol}})^{\oplus r} \rightarrow (\mathcal{O}_{\mathbb{P}^{1} \times M'}^{\text{hol}})^{\oplus r} \otimes \Omega^1_{\mathbb{P}^{1} \times M'/M'}(\mathcal{D}_{T'} \cup \{ \infty \} \times M'). \)

**Proof.** We can take an analytically open covering \( \{ U_{\beta} \} \) of \( M' \) which is a refinement of \( \{ U_{\alpha} \times_\alpha M' \} \) such that \( U_{\beta} \) is contractible and \( \bar{E}_{\text{hol}}^{\text{flat}} / U_{\beta} \cong (\mathcal{O}_{(U_i,v)}^{\text{hol}})^{\oplus r} \) for any \( \beta \). Moreover, we may assume that \( U_{\beta} \cap \mathcal{T}'_{M'} = \emptyset \) unless \( U_{\beta} = (U_i,v) \). Recall that \( (\theta^{i,v} \otimes \text{id}) \circ \nabla_{T'^{\text{flat}}_{T'/\Delta_{\alpha_0}}} |_{(U_i,v)} \circ (\theta^{i,v})^{-1} \) is canonically extended to a global connection

\[
\nabla^{(i,v)} |_{\mathbb{P}^{1} \times M'} : \mathcal{O}_{\mathbb{P}^{1} \times M'}^{\text{hol}} |_{\mathbb{P}^{1} \times M'}^{\text{hol}} \rightarrow (\mathcal{O}_{\mathbb{P}^{1} \times M'}^{\text{hol}})^{\oplus r} \otimes \Omega^1_{\mathbb{P}^{1} \times M'/M'}(\mathcal{D}_{T'} \cup \{ \infty \} \times M'). \)

given by the connection matrix

\[
A^{(i)}(z^{(i)}, \epsilon) = \frac{dz^{(i)}}{(z^{(i)})_{m_i} - \epsilon m_i}.
\]

Here we use the identification \( (U_i,M') = \Delta_{\alpha} \times M' \rightarrow \mathbb{P}^{1} \times M' \). As in Definition 5.7, there is a block \( (\nabla^{\text{flat}}_{\mathbb{P}^{1} \times M'/[h],v} |_{\mathbb{P}^{1} \times M'/[h],v}^{(i)}) \) of local horizontal lifts given by \( (\tilde{z}^{(i)}_{l,j}(z^{(i)})) \) and \( (B^{(i)}_{l,j}(z^{(i)})) \). We put

\[
A_v^{(i)}(z^{(i)}) := \sum_{l=0}^{r-2} \sum_{j=0}^{m_i-1} v(c^{(i)}_{l,j} \tilde{z}^{(i)}_{l,j}(z^{(i)}))
\]

and denote by \( \iota_{M'/[h]} : \mathbb{P}^{1} \times M'/M' |_{[h]} \rightarrow \mathbb{P}^{1} \times M'/[h] \) the inclusion morphism. Consider the connection

\[
\nabla^{\text{flat}}_{\mathbb{P}^{1} \times M'/[h],v} : \mathcal{O}_{\mathbb{P}^{1} \times M'/[h],v}^{\text{hol}} \rightarrow (\mathcal{O}_{\mathbb{P}^{1} \times M'/[h],v}^{\text{hol}})^{\oplus r} \otimes (\iota_{M'/[h],v})_{*} \Omega^1_{\mathbb{P}^{1} \times M'/M'}(\mathcal{D}_{T'} \cup \{ \infty \} \times M').
\]

determined by the connection matrix

\[
\left( A^{(i)}(\tilde{z}^{(i)}, \epsilon) + \tilde{h} A_v^{(i)}(\tilde{z}^{(i)}) \right) \frac{d\tilde{z}^{(i)}}{(\tilde{z}^{(i)})_{m_i} - \epsilon m_i} + B^{(i)}(z^{(i)}) d\bar{h}.
\]

Then we can see by the same calculation as Proposition 4.5 that \( \nabla^{\text{flat}}_{\mathbb{P}^{1} \times M'/[h],v} \) is an integrable connection. If we put \( \mathcal{N}^{(i)}_v := (\tilde{z}^{(i)}_{l,j}(z^{(i)}) + \tilde{h} v(c^{(i)}_{l,j} \tilde{z}^{(i)}_{l,j}(z^{(i)})) |_{\mathcal{D}_{T'}^{(i)}} \), then \( (\mathcal{O}_{(U_i,v)}^{\text{hol}})^{\oplus r} \mathcal{N}^{(i)}_v |_{\mathcal{D}_{T'}^{(i)}} \) gives a local horizontal lift of \( (\mathcal{E}_{\text{hol}}^{\text{flat}}_{T', T'^{\text{hol}}_{T'/\Delta_{\alpha_0}}}, \nabla_{T'^{\text{hol}}_{T'/\Delta_{\alpha_0}}}, \{ \mathcal{N}^{(i)}_{T'^{\text{hol}}_{T'/\Delta_{\alpha_0}}} \}) |_{(U_i,v)} \) with respect to \( v \).

Assume that \( U_{\beta} \cap \mathcal{D}^{(i)}_{M'} = \emptyset \) for any \( i \). Then the connection \( \mathcal{N}^{\text{hol}}_{M'_{U_{\beta}}} \) is given by a connection matrix \( A(z) dz \), for some local holomorphic coordinate \( z \) of \( \mathcal{C}_{T'} \) over \( T' \). We can take a matrix \( \tilde{A}(\tilde{z}) \) with entries in \( \mathcal{O}_{U_{\beta}[v]}^{\text{hol}} \) which is a lift of \( A(z) \), where \( \tilde{z} \) is the pullback of \( z \) under the morphism \( \mathcal{C}_{T'}[v] \xrightarrow{\text{id} \times \tilde{h}} \mathcal{C}_{T'} \). We can write

\[
d\tilde{A}(\tilde{z}) = C(\tilde{z})d\tilde{z} + B(z) d\bar{h}.
\]
If we put $A'(z) := \hat{A}(z) - \bar{h}B(z)$, then we have $dA'(z) \in M_r(\Omega^1_{U^h})d\bar{z}$ and

$$\nabla^\alpha_{\beta} : (\Omega^1_{U^h})^{\oplus r} \to (\Omega^1_{U^h})^{\oplus r} \otimes \Omega^1_{C^\tau_{r},v}$$

$$(f_1) \mapsto (df_1) \quad \vdots \quad + A'(z)d\bar{z} \quad \vdots \quad (f_r)$$

becomes a flat connection. So $((\Omega^1_{U^h})^{\oplus r}, \nabla^\alpha_{\beta})$ becomes a local horizontal lift of $\left(\hat{E}^h_{M'}, \hat{\nabla}^h_{M'}, \{\hat{\mathcal{N}}^i_{M'}\} \right)_{U^h}$, where $\{\hat{\mathcal{N}}^i_{M'}\}_{U^h}$ is nothing in this case.

From the above arguments, we obtain a local horizontal lift $(\mathcal{E}^u_{\beta}, \nabla^u_{\beta}, \{\mathcal{N}^i_{\beta}\})$ of $(\hat{E}^h_{M'}, \hat{\nabla}^h_{M'}, \{\hat{\mathcal{N}}^i_{M'}\} \{\alpha^i_{M'}\})_{U^h}$ for each piece $U_\beta$ of the covering $C_{M'} = \bigcup_\beta U_\beta$. If $U_\beta \neq U_{\beta'}$, then $\Gamma_{M'} \cap U_\beta \cap U_{\beta'} = \emptyset$ by the assumption. Assume that a connection matrix of $\nabla^u_{\beta}$ is given by

$$\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \mapsto \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \begin{pmatrix} \hat{A}_{\beta}(z)d\bar{z} + B_{\beta}(z)d\bar{h} \end{pmatrix}$$

where the integrability condition

$$-\frac{\partial \hat{A}_{\beta}(z)}{\partial \bar{h}}d\bar{z} \wedge d\bar{h} + d\bar{h}B_{\beta}(z) \wedge d\bar{h} + (\hat{A}_{\beta}(z)B_{\beta}(z) - B_{\beta}(z)\hat{A}_{\beta}(z))d\bar{z} \wedge d\bar{h} = 0$$

is satisfied and so for $\nabla^u_{\beta}$. There is an invertible matrix $P_{\beta,\beta'}(z)$ of holomorphic functions on $U_{\beta \cap U_{\beta'}}$ satisfying

$$P_{\beta,\beta'}(z)^{-1}dP_{\beta,\beta'}(z) + P_{\beta,\beta'}(z)^{-1}\hat{A}_{\beta}(z)dzP_{\beta,\beta'}(z) = \hat{A}_{\beta'}(z)dz$$

coming from the isomorphism $(\mathcal{E}^u_{\beta}, \nabla^u_{\beta})_{U_{\beta\beta'}} \cong (\hat{E}^h_{M'}, \hat{\nabla}^h_{M'})_{U_{\beta\beta'}} \cong (\mathcal{E}^u_{\beta'}, \nabla^u_{\beta'})$ of holomorphic functions on $U_{\beta\beta'}[\bar{h}]$ which is a lift of $P_{\beta,\beta'}(z)$. If we put

$$\hat{A}_{\beta'}(z)dz + B_{\beta'}(z)d\bar{h} := \hat{P}_{\beta,\beta'}(z, \bar{h})^{-1}dP_{\beta,\beta'}(z, \bar{h}) \hat{P}_{\beta,\beta'}(z, \bar{h})^{-1}(\hat{A}_{\beta}(z)d\bar{z} + B_{\beta}(z)d\bar{h})\hat{P}_{\beta,\beta'}(z, \bar{h}),$$

then we can write $A_{\beta'}(z) = \hat{A}_{\beta'}(z) + \bar{h}C_{\beta'}(z)$. If we put $Q_{\beta,\beta'}(z) := B_{\beta'}(z) - B_{\beta}(z)$, then $Q_{\beta,\beta'}(z)$ is holomorphic on $U_{\beta \cap U_{\beta'}} = (U_{\beta \cap U_{\beta'}} \setminus (\Gamma_{M'} \cap U_\beta \cap U_{\bar{\beta}}))$ and we have

$$\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} \mapsto \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix} + \begin{pmatrix} \hat{A}_{\beta}(z)d\bar{z} + B_{\beta}(z)d\bar{h} \end{pmatrix}$$

Thus the composition of $P_{\beta,\beta'}(z, \bar{h})$ with $I_r + \bar{h}Q_{\beta,\beta'}(z)$ gives an isomorphism between $(\mathcal{E}^u_{\beta}, \nabla^u_{\beta})_{U_{\beta\beta'}}[\bar{h}]$ and $(\mathcal{E}^u_{\beta'}, \nabla^u_{\beta'})_{U_{\beta\beta'}}[\bar{h}]$ whose restriction to $U_{\beta\beta'} = U_{\beta\beta'}[\bar{h}] \cap [\bar{h}]$ is the identity. By construction, we can see that this isomorphism is unique, because it is essentially determined by the $d\bar{h}$-coefficients. So we can patch $(\mathcal{E}^u_{\beta}, \nabla^u_{\beta}, \{\mathcal{N}^i_{\beta}\})$ together and obtain a global horizontal lift $(\mathcal{E}^u, \nabla^u, \{\mathcal{N}^i_{\beta}\})$ of $(\hat{E}^h_{M'}, \hat{\nabla}^h_{M'}, \{\hat{\mathcal{N}}^i_{M'}\})$ with respect to $v$ and with respect to the blocks $(\nabla^u_{\beta\beta'}[\bar{h}])$ of local horizontal lifts. Since the local horizontal lift is unique up to a unique isomorphism, we can see that a global horizontal lift $(\mathcal{E}^u, \nabla^u, \{\mathcal{N}^i_{\beta}\})$ is unique up to an isomorphism.

For a vector field $v \in H^0(T', \hat{T}^h_{M', \lambda \times B'})$ over an analytic open subset $T' \subset T \subset T \times B'$, we have by Proposition 3.10 a unique horizontal lift $(\mathcal{E}^u, \nabla^u, \{\mathcal{N}^i_{\beta}\})$ of the restriction $(\hat{E}^h_{M'}, \hat{\nabla}^h_{M'}, \{\hat{\mathcal{N}}^i_{M'}\})$. 

of the universal family to \( C \times H M' \) with respect to \( v \) and with respect to the blocks \( (\nabla_{\text{flat}}^{\text{flat}} P^2 \times M^{\tau}[h], v_i_{\tau}^{(i)}) \) of local horizontal lifts. Let

\[
\nabla^{v \circ}: E^{v \circ} \otimes \Omega^1_{C'_{\tau},v} \longrightarrow E^{v \circ} \otimes \Omega^1_{C'_{\tau},v/M'[v]}(\mathcal{D}_{M'[v]})
\]

be the relative connection induced by \( \nabla^{v \circ} \). Then \( (E^{v \circ}, \nabla^{v \circ}, \{\Lambda_{C'}^{(i)}\}) \) becomes a holomorphic flat family of \((\nu, \mu)\)-connections on \( C'_{\tau}[v] \) over \( M'[v] \), which determines a morphism \( M'[v] \longrightarrow M_{C,D}^{\nu,\mu}(\nu, \mu) \times_{\tau_{\nu, \lambda}} T' \)
making the diagram

\[
\begin{array}{c}
M'[v] \longrightarrow M_{C,D}^{\nu,\mu}(\nu, \mu) \times_{\tau_{\nu, \lambda}} T' \\
\downarrow \quad \downarrow \\
T'[v] \quad \longrightarrow T'
\end{array}
\]

commutative. This morphism corresponds to a vector field \( \Phi(v) \in H^0((\pi^{\circ})^{-1}(T'), T'^{\text{hol}}_{M'/\Delta_{\nu}}(\pi^{\circ})^{-1}(T')) \), where \( \pi^{\circ}: M' \longrightarrow T' \) is the projection morphism. We can see \( d\pi^{\circ}(\Phi(v)) = v \) by the commutative diagram (74), where \( d\pi^{\circ}: \pi^{\circ}_{T'^{\text{hol}}_{M'}/\Delta_{\nu}} \longrightarrow T'^{\text{hol}}_{T'/\Delta_{\nu}} \) is the differential of \( \pi^{\circ} \). Thus we have defined a map

\[
\Phi: T'^{\text{hol}}_{T'/\Delta_{\nu}} \ni v \mapsto \Phi(v) \in (\pi^{\circ})_{T'^{\text{hol}}_{M'/\Delta_{\nu}}}
\]

(75)

In the rest of this subsection, we will prove that the correspondence [75] defined above is an \( \mathcal{O}_{C'}^{\text{hol}} \)-homomorphism. In order to prove it, we extend the notion of horizontal lift. Let \( C[I] = C \oplus I \) be a finite dimensional local algebra over \( C \) with the maximal ideal \( I \) satisfying \( I^2 = 0 \). For a morphism \( u: T' \times \text{Spec} \, C[I] \longrightarrow T' \) over \( \Delta_{\nu} \) satisfying \( u|\Gamma' \times \text{Spec} \, C[I]/I = \text{id}_{T'} \), we write \( T'[u] := T' \times \text{Spec} \, C[I] \) which is endowed with the structure morphism \( u: T'[u] \longrightarrow T' \). We endow the fiber product \( C'_{T'[u]} := C \times_{H} T' \times \text{Spec} \, C[I] \) with the structure

\[
C'_{T'[u]} = C \times_{H} T' \times \text{Spec} \, C[I] \longrightarrow T' \times \text{Spec} \, C[I] \overset{u}{\longrightarrow} T'.
\]

For an analytic open subset \( U \subset C_T' \), we denote by \( U[u] \) the open subspace of \( C'_{T'[u]} \) whose underlying set of points is \( U \).

We consider the sheaf of differential forms \( (\Omega^1_{(C'_{T'} \backslash \Gamma_{T'})[u]/T'})^{\text{hol}} \) with respect to the composite of the trivial projections

\[
C'_{T'[u]} = C \times T' \times \text{Spec} \, C[I] \longrightarrow T' \times \text{Spec} \, C[I] \longrightarrow T'
\]

which is different from the structure of \( C'_{T'[u]} \) over \( T' \) coming from the fiber product structure. We can consider the quotient sheaf

\[
(\Omega^1_{(C'_{T'} \backslash \Gamma_{T'})[u]/T'})^{\text{hol}} / (I\Omega^1_{(C'_{T'} \backslash \Gamma_{T'})[u]/T'})
\]

and define a subsheaf \( \Omega^1_{C'_{T',u}} \) of \( (\Omega^1_{(C'_{T'} \backslash \Gamma_{T'})[u]/T'})^{\text{hol}} / (I\Omega^1_{(C'_{T'} \backslash \Gamma_{T'})[u]/T'}) \) locally generated by

\[
\left\{ \frac{d\bar{z}^{(i)}}{(\bar{z}^{(i)})^{n_i} - \epsilon_{m_i}} \right\} \cup \sum_{q=1}^{K} (\kappa_{C'_{T'} \backslash \Gamma_{T'}})^{\text{hol}}_{\text{C'_{T',u}}} (\bar{h}^{\tilde{q}}_{i}) (\bar{h}^{\tilde{q}}_{i}) \mathrm{d}h^{\tilde{q}}_{i}
\]

around points \( p \in (\Gamma^{(i)})_{T'[u]} \) and locally generated by \( \{d\bar{z} \cup \{d\bar{h} : \bar{h} \in I\} \) around points \( p \in (C'_{T'} \backslash \Gamma_{T'})[u] \).

Here \( h_1, \ldots, h_\kappa \) is a basis of \( I \) and \( z \) is a local holomorphic coordinate of \( C'_{T'} \backslash \Gamma_{T'} \) over \( T' \). We denote the image of \( \Omega^1_{C'_{T',u}} \wedge \Omega^1_{C'_{T',u}} \) in \( (\Omega^2_{(C'_{T'} \backslash \Gamma_{T'})[u]/T'})^{\text{hol}} / (I\Omega^1_{(C'_{T'} \backslash \Gamma_{T'})[u]/T'}) \) by \( \Omega^2_{C'_{T',u}} \).

For each \( i = 1, \ldots, n \), we consider the sheaf of differential forms \( \Omega^1_{U^{(i)}_{M'[u]}/M'} \) with respect to

\[
(U^{(i)}_{M'[u]} \hookrightarrow C \times M' \times \text{Spec} \, C[I] \longrightarrow M' \times \text{Spec} \, C[I] \longrightarrow M',
\]

where the last two arrows are the trivial projections. From the above projection, a ring homomorphism from the polynomial ring

\[
\mathcal{O}_{M'}[z^{(i)}] \longrightarrow \mathcal{O}_{U^{(i)}_{M'[u]}}^{\text{hol}}
\]

is induced. We denote the image of a matrix \( A(z^{(i)}) \) of polynomials in \( z^{(i)} \) with coefficients in \( \mathcal{O}_{M'} \) under this ring homomorphism by \( A(z^{(i)}) \).

Note that we can write

\[
u^{(i)}(T) = \nu_{\text{hor}}^{(i)}(T) + \sum_{q=1}^{s} \bar{h}^{\tilde{q}}_{i} v^{(i)}_{u, q}(T)
\]
Lemma 5.12. \[ \nu_{\text{hor}}^{(i)}(T) = \sum_{l=0}^{r-1} \sum_{j=0}^{m_{l}-1} c_{\text{hor},l,j}^{(i)}(\bar{z}^{(i)})^{j}T^{l} \]
\[ \nu_{\text{u,q}}^{(i)}(T) = \sum_{l=0}^{r-1} c_{\text{u,q},l,j}^{(i)}(\bar{z}^{(i)})^{j}T^{l}, \]
where \( c_{\text{hor},l,j}^{(i)} \) and \( c_{\text{u,q},l,j}^{(i)} \) are pullbacks of \( c_{l,j}^{(i)} \) and \( \bar{c}_{l,j}^{(i)} \in \mathcal{O}_{M'}^{\text{hol}} \) under the composition of the trivial projections \((U_{i})_{M'[u]} \to M'[u] \to M'\).

Definition 5.11. Under the above notation, we say that a tuple \((\mathcal{E}^{u}, \nabla^{u}, \{N_{u}^{(i)}\})\) is a horizontal lift of \((\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}_{M'}^{(i)}\})\) with respect to \(u\) and with respect to blocks of local horizontal lifts \( (\nabla_{\text{flat}}^{\text{hor}}_{\mathbb{P}^{1} \times M'[\bar{h}],v_{i,j}^{(i)}}) \) if
\begin{enumerate}
\item \( \mathcal{E}^{u} \) is a rank \( r \) holomorphic vector bundle on \( \mathcal{C}_{\text{M'[u]}} \),
\item \( \nabla^{u} : \mathcal{E}^{u} \to \mathcal{E}^{u} \otimes \mathcal{O}_{\tilde{C}_{\text{M'[u]}}}^{\text{hol}} \Omega^{1}_{\tilde{C}_{\text{M'[u]}}} \) is a morphism of sheaves satisfying \( \nabla^{u}(fa) = a \otimes df + f\nabla^{u}(a) \) for \( f \in \mathcal{O}^{\text{hol}}_{\tilde{C}_{\text{M'[u]}}} \) and \( a \in \mathcal{E}^{u} \),
\item \( \nabla^{u} \) is integrable in the sense that for each local expression
\[
\left( \begin{array}{c}
    f_{1} \\
    \vdots \\
    f_{r}
\end{array} \right) \mapsto \left( \begin{array}{c}
    df_{1} \\
    \vdots \\
    df_{r}
\end{array} \right) + \left( \begin{array}{c}
    A \bar{d}z + \sum_{l=1}^{\kappa} B_{l} d\bar{h}_{l} \\
    \vdots \\
    \end{array} \right) = 0
\]
of \( \nabla^{u} \) on \( \mathcal{E}^{u}|_{U[u]} = \mathcal{O}_{U[u]}^{\text{hor}} \) for an open subset \( U[u] \subset (\mathcal{C}_{\text{M'[u]}} \setminus \Gamma_{\text{M'[u]}})[u] \), the equality \( d\left( A \bar{d}z + \sum_{l=1}^{\kappa} B_{l} d\bar{h}_{l} \right) + \left( A \bar{d}z + \sum_{l=1}^{\kappa} B_{l} d\bar{h}_{l} \right) \wedge \left( A \bar{d}z + \sum_{l=1}^{\kappa} B_{l} d\bar{h}_{l} \right) = 0 \) holds in \( \Omega^{2}_{\tilde{C}_{\text{M'[u]}}} \), where \( \{\bar{h}_{1}, \ldots, \bar{h}_{\kappa}\} \) is a basis of \( I \) over \( \mathbb{C} \),
\item \( \mathcal{N}_{u}^{(i)} : \mathcal{E}^{u}|_{\tilde{D}_{\text{M'[u]}}} \to \mathcal{E}^{u}|_{\tilde{D}_{\text{M'[u]}}} \) is an endomorphism satisfying \( \varphi_{i}^{(i)}(\mathcal{N}_{u}^{(i)}) = 0 \),
\item the relative connection \( \nabla^{u} \) defined by the composition
\[
\nabla^{u} : \mathcal{E}^{u} \to \mathcal{E}^{u} \otimes \Omega^{1}_{\tilde{C}_{\text{M'[u]}}} \to \mathcal{E}^{u} \otimes \Omega^{1}_{\tilde{C}_{\text{M'[u]}}} \]
satisfies
\[
(u^{*}L^{(i)})\left( \mathcal{N}_{u}^{(i)} \right) \frac{d(\bar{z}^{(i)})}{(\bar{z}^{(i)})^{m_{i}} - \epsilon^{m_{i}}} = \nabla^{u}|_{\tilde{D}_{\text{M'[u]}}}^{(i)}
\]
for any \( i \),
\item \( (\mathcal{E}^{u}, \nabla^{u}, \{N_{u}^{(i)}\}) \otimes \mathcal{O}^{\text{hol}}_{\tilde{C}_{\text{M'[u]}}}/I\mathcal{O}^{\text{hol}}_{\tilde{C}_{\text{M'[u]}}} \cong (\tilde{E}^{\text{hor}}_{M'}, \tilde{\nabla}^{\text{hor}}_{M'}, \{\tilde{N}_{M'}^{(i)}\}) \),
\item there is an isomorphism \( \theta^{(i),u} : \mathcal{E}^{u}|_{(U_{i})_{M'[u]}} \to (\mathcal{O}^{\text{hol}}_{(U_{i})_{M'[u]}})^{\text{hor}} \) which is a lift of the given isomorphism \( \theta^{(i),u}|_{(U_{i})_{M'}} : \tilde{E}|_{(U_{i})_{M'}} \to (\mathcal{O}^{\text{hol}}_{(U_{i})_{M'}})^{\text{hor}} \) such that the connection matrix of \( (\theta^{(i),u} \otimes \text{id}) \circ \nabla^{u} \circ (\theta^{(i),u})^{-1} \) is given by
\[
\left( A^{(i)}(\bar{z}^{(i)}, \epsilon) + \sum_{q=1}^{\kappa} \bar{h}_{q} \sum_{l=0}^{r-1} \sum_{j=0}^{m_{l}-1} c_{u,q,l,j}^{(i)} \bar{z}^{(i)} \right) \frac{d(\bar{z}^{(i)})}{(\bar{z}^{(i)})^{m_{i}} - \epsilon^{m_{i}}} + \sum_{q=1}^{\kappa} \sum_{l=1}^{r-1} \sum_{j=0}^{m_{l}-1} c_{u,q,l,j}^{(i)} \bar{h}_{q} d\bar{h}_{q}
\]
and that
\[
(\theta^{(i),u} \otimes \text{id}) \circ \mathcal{N}_{u}^{(i)} \circ (\theta^{(i),u})^{-1} = \left( \sum_{q=1}^{s} \bar{h}_{q} c_{u,q,l,j}^{(i)} \bar{z}^{(i)} \right) \Big|_{\tilde{D}_{\text{M'[u]}}}^{(i)}
\]
\end{enumerate}

Lemma 5.12. There exists a unique horizontal lift \((\mathcal{E}^{u}, \nabla^{u}, \{N_{u}^{(i)}\})\) of \((\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}_{M'}^{(i)}\})\) with respect to \(u\) and with respect to blocks of local horizontal lifts \( (\nabla_{\text{flat}}^{\text{hor}}_{\mathbb{P}^{1} \times M'[\bar{h}],v_{i,j}^{(i)}}) \).

Proof. The proof of this lemma is the same as that of Proposition 5.10 and we omit the detail.
We take the same open covering \(\{U_\beta\}\) as in the proof of Proposition 5.10. We consider the connection \(\nabla_{flat}^{hol}_{P^1 \times M'}[h,u]\) on \((\mathcal{O}_{P^1 \times M'}^{hol})^{gr}\) given by the connection matrix

\[
A(i)(z(i), \epsilon) + \sum_{q=1}^k \bar{h}_q \sum_{l=0}^{r-1} \sum_{j=0}^{m_i-1} c_{u,q,l,j}^{i} \Xi_{l,j}(z(i)) \frac{dz^{(i)}}{(z^{(i)})^{m_i - \epsilon m_i}} + \sum_{q=1}^k \sum_{l=0}^{r-1} \sum_{j=0}^{m_i-1} c_{u,q,l,j}^{i} \bar{P}_{l,j}^{(i)}(z^{(i)}) d\bar{h}_q
\]

with respect to \(u\). Together with the endomorphism \(\mathcal{N}^{(i)}_u\) given by

\[
\mathcal{N}^{(i)}_u = \left( \Xi_{l,j}^{(i)}(z^{(i)}) + \sum_{q=1}^k \bar{h}_q c_{u,q,l,j}^{i} \Xi_{l,j}^{(i)}(z^{(i)}) \right) \bigg|_{\mathcal{D}_{l,j}^{(i)}[u]},
\]

we can get a local horizontal lift \(\left((\mathcal{O}_{P^1 \times M'}^{hol})^{gr}, \nabla_{flat}^{hol}_{P^1 \times M'}[h,u]|_{(U_\beta)_M[u]} \mathcal{N}^{(i)}_u\right)\). Patching the local horizontal lifts altogether, we obtain a unique horizontal lift in the same way as Proposition 5.10.

**Proposition 5.13.** The morphism

\[
T^{hol}_{T'/T}/\Delta_0 \ni v \mapsto \Phi(v) \in \left(\pi', T^{hol}_{M'/\Delta_0}\right)
\]

defined in (73) is an \(\mathcal{O}^{hol}_{T'/T}\)-homomorphism.

**Proof.** Take an open subset \(T' \subset T^0\) and holomorphic vector fields \(v_1, v_2 \in H^0(T', T^{hol}_{T'/T}/\Delta_0)\). Let

\[
u : T' \times \text{Spec } \mathbb{C}[h_1, h_2]/(h_1^2, h_1 h_2, h_2^2) \rightarrow T'
\]

be the morphism such that the restriction \(u|_{T' \times \text{Spec } \mathbb{C}[h]/(h^2)}\) corresponds to \(v_1\) for \(i = 1, 2\). Applying Lemma 5.12 to \(\mathbb{C}[I] = \mathbb{C}[h_1, h_2]/(h_1^2, h_1 h_2, h_2^2)\), we can take a horizontal lift \(\{(E_u, \nabla_u, \{\mathcal{N}^{(i)}_u\})\}\) of \((\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}^{(i)}_{M'}\})\) with respect to \(u\) and with respect to the blocks \((\nabla_{flat}^{hol}_{P^1 \times M'}[h,u]_{i}^{(i)})\) of local horizontal lifts. We can see by construction that the restriction \(\{(E_u, \nabla_u, \{\mathcal{N}^{(i)}_u\})\}|_{M' \times \text{Spec } \mathbb{C}[h]/(h^2)}\) coincides with the horizontal lift \(\{(E_{v_1}, \nabla_{v_1}, \{\mathcal{N}^{(i)}_{v_1}\})\}\) of \((\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}^{(i)}_{M'}\})\) with respect to \(v_1\). So the morphism

\[
M' \times \text{Spec } \mathbb{C}[h_1, h_2]/(h_1^2, h_1 h_2, h_2^2) \rightarrow M_{\Delta_0}^{\mathcal{O}_{T'/T}}(\tilde{\mathcal{D}}, \mu) \times_B \mathcal{B}'
\]

determined by the flat family \(\{(E_u, \nabla_u, \{\mathcal{N}^{(i)}_u\})\}\) coincides with the one given by the pair \((\Phi(v_1), \Phi(v_2))\) of vector fields, where \(\nabla_u^{hol} : E_u \rightarrow E_u \otimes \Omega_{\text{Spec } \mathbb{C}[I]/(I)}^{1}\) is the relative connection induced by \(\nabla_u\). From the definition of the addition of vector fields, the restriction \((\Phi(v_1), \Phi(v_2))|_{M' \times \text{Spec } \mathbb{C}[h_1, h_2]/(h_1 h_2)}\) to the diagonal coincides with \(\Phi(v_1) + \Phi(v_2)\). On the other hand, we can see by the construction that the restriction \(\{(E_u, \nabla_u, \{\mathcal{N}^{(i)}_u\})\}|_{M' \times \text{Spec } \mathbb{C}[h_1, h_2]/(h_1 h_2)}\) is a horizontal lift of \((\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}^{(i)}_{M'}\})\) with respect to \(v_1 + v_2\) and with respect to the blocks of local horizontal lifts \((\nabla_{flat}^{hol}_{P^1 \times M'}[h,v_1]_{i})\) in the sense of Proposition 5.10. So we have \(\Phi(v_1 + v_2) = \Phi(v_1) + \Phi(v_2)\).

Take a holomorphic function \(f \in H^0(T', \mathcal{O}^{hol}_{T'})\) and a holomorphic vector field \(v \in H^0(T', T^{hol}_{T'/T}/\Delta_0)\). Let

\[
\sigma_f : T' \times \text{Spec } \mathbb{C}[h]/(h^2) \rightarrow T' \times \text{Spec } \mathbb{C}[h]/(h^2)
\]

be the morphism corresponding to the ring homomorphism \(\mathcal{O}^{hol}_{T'/T}[h]/(h^2) \ni a + bh \mapsto a + b f h \in \mathcal{O}^{hol}_{T'/T}[h]/(h^2)\) and let

\[
\text{id} \times \sigma_f : M' \times T' \times \text{Spec } \mathbb{C}[h]/(h^2) \rightarrow M' \times T' \times \text{Spec } \mathbb{C}[h]/(h^2)
\]

be its base change. If \(\{(E^v, \nabla^v, \{N^{(i)}_v\})\}\) is a horizontal lift of \((\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}^{(i)}_{M'}\})\) with respect to \(v\) and with respect to the blocks of local horizontal lifts \((\nabla_{flat}^{hol}_{P^1 \times M'}[h,v]_{i}^{(i)})\), then we can see by the construction that the pull back \((1 \times \sigma_f)^*(\{(E^v, \nabla^v, \{N^{(i)}_v\})\}\) is a horizontal lift of \((\tilde{E}_{M'}, \tilde{\nabla}_{M'}, \{\tilde{N}^{(i)}_{M'}\})\) with respect to \(f v\) and with respect to the blocks of local horizontal lifts \((\nabla_{flat}^{hol}_{P^1 \times M'}[h,v]_{i}^{(i)})\). By the definition of \(\mathcal{O}^{hol}_{T'/T}\)-module structure on the tangent bundle, we can see that the pull-back \((\text{id} \times \sigma_f)^* \nabla^v, (\text{id} \times \sigma_f)^* N^{(i)}_v\) of the flat family \(\{(E^v, \nabla^v, \{N^{(i)}_v\})\}\) corresponds to \(f \Phi(v)\). So we have \(\Phi(fv) = f \Phi(v)\). Hence we have proved that \(\Phi\) is an \(\mathcal{O}^{hol}_{T'/T}\)-homomorphism. 

\(\square\)
By the adjoint bijection
\[ (\pi_\circ)^* \rightarrowhol_{\mathcal{M}^v_\circ} (\pi_\circ)^* \rightarrowhol_{\mathcal{M}^v_\circ} = \text{Hom}_{\mathcal{O}hol_{\mathcal{M}^v_\circ}} \left( \rightarrowhol_{\mathcal{T}_{\mathcal{M}^v_\circ}} (\pi_\circ)^* \rightarrowhol_{\mathcal{M}^v_\circ} \right), \]
the \( \mathcal{O}hol_{\mathcal{M}^v_\circ} \)-homomorphism \( \Phi: \rightarrowhol_{\mathcal{T}_{\mathcal{M}^v_\circ}} (\pi_\circ)^* \rightarrowhol_{\mathcal{M}^v_\circ} \) given in (75) corresponds to an \( \mathcal{O}hol_{\mathcal{M}^v_\circ} \)-homomorphism \( \Psi: (\pi_\circ)^* \rightarrowhol_{\mathcal{T}_{\mathcal{M}^v_\circ}} \rightarrowhol_{\mathcal{M}^v_\circ} \). Since \( \Phi \) satisfies \( d\pi_\circ \circ \Phi(v) = v \) for vector fields \( v \in \rightarrowhol_{\mathcal{T}_{\mathcal{M}^v_\circ}} \), the homomorphism \( \Psi \) is a splitting of the surjection \( \rightarrowhol_{\mathcal{M}^v_\circ} \rightarrowhol_{\mathcal{M}^v_\circ} \). Furthermore we can see \( \Psi|_{\mathcal{M}^v_{\alpha,\varphi,\lambda}} = \Psi_0|_{\mathcal{M}^v_{\alpha,\varphi,\lambda}} \) canonically induced by the smooth morphism \( \pi_\circ: \mathcal{M}^v \rightarrow \mathcal{T} \). Thus we have proved Theorem 0.1.

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