Induced Domain Adaptation

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Abstract

We formulate the problem of induced domain adaptation (IDA) when the underlying distribution/domain shift is introduced by the model being deployed. Our formulation is motivated by applications where the deployed machine learning models interact with human agents, and will ultimately face responsive and interactive data distributions. We formalize the discussions of the transferability of learning in our IDA setting by studying how the model trained on the available source distribution (data) would translate to the performance on the induced domain. We provide both upper bounds for the performance gap due to the induced domain shift, as well as lower bound for the trade-offs a classifier has to suffer on either the source training distribution or the induced target distribution. We provide further instantiated analysis for two popular domain adaptation settings with covariate shift and label shift. We highlight some key properties of IDA, as well as computational and learning challenges.

1 Introduction

We formulate the problem of domain adaptation when the underlying distribution shift is induced by the model being deployed. We refer to this problem as induced domain adaptation. The major difference between our setting and the classical domain adaptation problem is that the source-to-target adaptation is no longer determined independently from the model that we are interested in obtaining from training. Our formulation is motivated by settings where the deployed machine learning models interact with human agents, and will ultimately face data distributions that reflect how human agents respond to the models. For instance, when a model is used to approve loan applications, candidates may adapt their features based on the model specification in order to maximize their chances of approval; thus the loan decision classifier observes a data distribution caused by its own deployment. See Figure 1 for a demonstration.

| FEATURE NAME   | CLASSIFIER WEIGHT | ORIGINAL VALUE | VALUE AFTER ADAPTATION |
|----------------|-------------------|----------------|------------------------|
| Income         | 2                 | $6,000         | $6,000                 |
| Education Level| 3                 | College        | College                |
| Debt           | -10               | $40,000        | $20,000                |
| Savings        | 5                 | $20,000        | 0                      |

Figure 1: An example of an agent who originally has both savings and debt, observes that the classifier penalizes debt (weight -10) more than it rewards savings (weight +5), and concludes that their most efficient adaptation is to use their savings to pay down their debt.

In this paper, we formally examine how training on the available source distribution $D_S$ translates to performance on the induced domain $D(h)$, which is a function of the model $h$ being deployed. A key concept in our IDA setting is the induced risk, defined as the error or risk a model incurs on the
We study several fundamental questions about this quantity:

\( \text{Induced Risk} : \quad \text{Err}_{\mathcal{D}(h)}(h) := \mathbb{P}_{\mathcal{D}(h)}(h(X) \neq Y) \)  \hfill (1)

We study several fundamental questions about this quantity:

- **Source risk \( \Rightarrow \) Induced risk** For a given model \( h \), how much different is \( \text{Err}_{\mathcal{D}(h)}(h) \), the error on the distribution induced by \( h \), from \( \text{Err}_{\mathcal{D}_s}(h) := \mathbb{P}_{\mathcal{D}_s}(h(X) \neq Y) \), the error on the source distribution?

- **Induced risk \( \Rightarrow \) Minimum induced risk** How much higher is \( \text{Err}_{\mathcal{D}(h)}(h) \), the error on the induced distribution, than \( \min_{h'} \text{Err}_{\mathcal{D}(h')}(h') \), the minimum achievable induced error?

- **Induced risk of source optimal \( \Rightarrow \) Minimum induced risk** Of particular interest, and as a special case of the above, how does \( \text{Err}_{\mathcal{D}(h^*_S)}(h^*_S) \), the induced error of the optimal model trained on the source distribution \( h^*_S := \min_h \text{Err}_{\mathcal{D}_S}(h) \), compare to \( \min_h \text{Err}_{\mathcal{D}(h)}(h) \)?

- **Lower bound for learning tradeoffs** What is the inevitable error a model has to suffer on either the source distribution \( \text{Err}_{\mathcal{D}_S}(h) \) or its induced distribution \( \text{Err}_{\mathcal{D}(h)}(h) \)?

For the first three questions, we prove upper bounds on the additional error incurred when a model trained on a source distribution is transferred over to its induced domain. We also provide lower bounds for the trade-offs a classifier has to suffer on either the source training distribution or the induced target distribution. We provide further instantiated analysis for two popular domain adaptation settings with covariate shift \( [27, 31, 28, 29, 34] \) and label shift \( [21, 14] \).

In addition to the above learning transferability study, we also investigate additional challenges that arise when minimizing the induced risk, including:

- **Convexity** When the original loss is convex, does the induced risk remain convex and therefore easy to optimize?

- **Learning challenges** What kinds of knowledge are required to minimize induced risk, in addition to the ones required in the classical domain adaptation setting?

- **Improving distribution** Does minimizing the induced risk lead to an “improved” distribution?

All omitted proofs can be found in the Appendix (supplementary materials).

### 1.1 Related works

Most relevant to us are three topics: the celebrated literature on domain adaptation \([16, 4, 29, 35, 17, 33]\), strategic classification \([15, 6, 9, 10, 5, 25, 19]\), and a recently proposed notion of performative prediction \([26, 24]\). The literature on domain adaptation began with a series of possibility \([4, 7]\) and impossibility results \([8]\). Later works looked into specific domain adaptation models, such as covariate shift \([27, 31, 28, 29, 34, 32]\) and label shift \([21, 14]\). A commonly established solution is to perform reweighted training on the source data, and robust and efficient solutions have been developed to estimate the weights accurately \([29, 34, 32, 21, 14]\).

Our work is also relevant to the literature on strategic classification. Hardt et al. \([15]\) pioneered the formalization of strategic behavior in classification based on a sequential two-player game between agents and classifiers. Subsequently, Chen et al. \([6]\) addressed the question of repeatedly learning linear classifiers against agents who are strategically trying to game the deployed classifiers. Most of the existing literature focuses on finding the optimal classifier by assuming fully rational agents (and by characterizing the equilibrium response). In contrast, we do not make these assumptions and primarily study the transferability when only having knowledge of the source training data.

Perdomo et al. \([26]\) advocate minimizing the error \( \mathbb{E}_{z \sim \mathcal{D}(\theta)}[\ell(\theta; z)] \), which they call the performative prediction risk: in above \( \theta \) is the model parameter to be optimized. This falls into the same category as induced risk, but the two concepts differ due to different requirements of knowing the distribution of a \( \mathcal{D}(\theta) \). Again, our focus is on the transferability when learning on the source training data. In addition, we study specific domain adaptation setting which generally do not assume the knowledge of \( \mathcal{D}(\theta) \) (and particularly we will not assume the knowledge of the supervision/label information on the transferred domain).
2 Formulation

Suppose we are learning a parametric model $h \in \mathcal{H}$ for a binary classification problem. Its training dataset $S := \{x_i, y_i\}_{i=1}^N$ is drawn from a source distribution $D_S$, where $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, +1\}$. However, $h$ will then be deployed in a setting where the samples come from a test or target distribution $D_T$ that can differ substantially from $D_S$. Therefore instead of minimizing the prediction error on the source distribution $\text{Err}_{D_S}(h) := \mathbb{P}_{D_S}(h(X) \neq Y)$, the goal is to find $h^*$ that minimizes $\text{Err}_{D_T}(h) := \mathbb{P}_{D_T}(h(X) \neq Y)$. This is often referred to as the domain adaptation problem, where typically, the transition from $D_S$ to $D_T$ is assumed to be independent of the model $h$ being deployed.

In this paper, we consider a setting in which the distribution shift depends on $h$, or is thought of as being induced by $h$. We will use $D(h)$ to denote the target distribution induced by $h$:

$$D_S \rightarrow \text{encounters model } h \rightarrow D(h)$$

Strictly speaking, the induced distribution is a function of both $D_S$ and $h$ and should be better denoted by $D_S(h)$. To ease the notation, we will stick with $D(h)$, but we shall keep in mind of its dependency of $D_S$. For now, we do not restrict the dependency of $D(h)$ of $D$ and $h$, but later in Section 4 and 5 we will further instantiate $D(h)$ under specific domain adaptation settings.

We define the induced risk of a classifier $h$ as the 0-1 error on the distribution $h$ induces:

$$\text{(Induced risk)} \quad \text{Err}_{D(h)}(h) := \mathbb{P}_{D(h)}(h(X) \neq Y)$$

Denote by $h_T^* := \text{argmin}_{h \in \mathcal{H}} \text{Err}_{D_T(h)}(h)$ the classifier with minimum induced risk. More generally, when the loss may not be the 0-1 loss, we define the induced $\ell$-risk as

$$\text{(Induced } \ell\text{-risk)} \quad \text{Err}_{\ell, D(h)}(h) := \text{argmin}_{h \in \mathcal{H}} \mathbb{E}_{z \sim D(h)}[\ell(h(z))]$$

The induced risks will be the primary quantities that we are interested in minimizing. The following additional notation will also be helpful:

- Distributions of $Y$ on a distribution $D$: $D_Y := \mathbb{P}_D(Y = y)$, and in particular $D_Y(h) := \mathbb{P}_{D(h)}(Y = y)$.
- Distribution of $h$ on a distribution $D$: $D_h := \mathbb{P}_D(h(X) = y)$, and in particular $D_h(h) := \mathbb{P}_{D(h)}(h(X) = y)$.
- Marginal distribution of $X$ for a distribution $D$: $D_X := \mathbb{P}_D(X = x)$, and in particular $D_X(h) := \mathbb{P}_{D(h)}(X = x)$.
- Total variation distance defined between $D$ and $D'$:

$$d_{TV}(D, D') := \sup_O |\mathbb{P}_D(O) - \mathbb{P}_{D'}(O)|.$$  

3 Transferability of learning to induced domains

In this section, we first provide upper bounds for the transfer error of a classifier $h$ (that is, the difference between $\text{Err}_{D(h)}(h)$ and $\text{Err}_{D_S}(h)$), as well as between $\text{Err}_{D(h)}(h)$ and $\text{Err}_{D(h^*_T)}(h_T^*)$. We then provide lower bounds for $\max\{\text{Err}_{D_S}(h), \text{Err}_{D(h)}(h)\}$; that is, the minimum error a model $h$ must incur on either the source distribution $D_S$ or the induced distribution $D(h)$.

3.1 Upper bound

We first investigate upper bounds for the transfer errors. We begin by showing generic upper bounds, but later in Section 4 and 5, we strengthen the bound for specific domain adaptation settings.

We begin with answering a central question in domain adaptation:

How does a model $h$ trained on its training dataset fare on the induced distribution $D(h)$?

To that end, define the minimum and maximum combined error of two distributions $D$ and $D'$ as:

$$\lambda_{D \rightarrow D'} := \min_{h' \in \mathcal{H}} \text{Err}_{D'}(h') + \text{Err}_D(h'), \quad \Lambda_{D \rightarrow D'} := \max_{h' \in \mathcal{H}} \text{Err}_{D'}(h') + \text{Err}_D(h')$$

$^1$The “:=” defines the RHS as the probability measure function for the LHS.

$^2$For continuous $X$, the probability measure shall be read as the density function.
The above theorem informs us that the induced transfer error is bounded by the “average” achievable error on both distributions. Reflecting on the difference between the bounds of Theorem 1 and Theorem 2, we see that the primary change is replacing the minimum achievable error by the average of \( \lambda \) and \( \Lambda \).

### 3.2 Lower bound

Now we provide a lower bound on the induced transfer error. We particularly want to show that at least one of the two errors \( \text{Err}_{D_S}(h) \), \( \text{Err}_{D_{h^*}}(h) \) must be lower-bounded by a certain quantity.

**Theorem 3 (Lower bound for learning tradeoffs).** Any model must incur the following error on either the source or induced distribution:

\[
\max \{ \text{Err}_{D_S}(h), \text{Err}_{D_{h^*}}(h) \} \geq \frac{d_{TV}(D_{Y|S}, D_{Y}(h)) - d_{TV}(D_{h|S}, D_{h}(h))}{2}.
\]

The proof leverages the triangle inequality of \( d_{TV} \). This bound is dependent on \( h \); however, by the data processing inequality of \( d_{TV} \) (and \( f \)-divergence functions in general) [20], we have

\[
d_{TV}(D_{h|S}, D_{h}(h)) \leq d_{TV}(D_{X|S}, D_{X}(h))
\]

Applying this to Theorem 3 gives the following model-independent bound:

**Corollary 4.** For any model \( h \),

\[
\max \{ \text{Err}_{D_S}(h), \text{Err}_{D_{h^*}}(h) \} \geq \frac{d_{TV}(D_{Y|S}, D_{Y}(h)) - d_{TV}(D_{X|S}, D_{X}(h))}{2}
\]

A couple of remarks:

- Without further assumptions, it is unclear if \( d_{TV}(D_{Y|S}, D_{Y}(h)) - d_{TV}(D_{h|S}, D_{h}(h)) \geq 0; \) if not, this bound is vacuous. Later, after introducing specific domain adaptation settings, we will revisit this bound.
- When \( d_{TV}(D_{Y|S}, D_{Y}(h)) - d_{TV}(D_{X|S}, D_{X}(h)) > 0 \), we know there is a positive tradeoff between a model’s achievable training error on the source distribution and the induced distribution.
4 Covariate shift

In this section, we focus on a particular domain adaptation setting known as covariate shift, in which the distribution of features changes, but the distribution of labels conditioned on features does not:

\[ P_{D(h)}(Y = y|X = x) = P_{D_S}(Y = y|X = x), \quad P_{D(h)}(X = x) \neq P_{D_S}(X = x) \] (5)

Thus with covariate shift, we have

\[ P_{D(h)}(X = x, Y = y) = P_{D(h)}(Y = y|X = x) \cdot P_{D(h)}(X = x) = P_{D_S}(Y = y|X = x) \cdot P_{D(h)}(X = x) \]

Let \( \omega_x(h) := \frac{P_{D(h)}(X=x)}{P_{D_S}(X=x)} \) be the importance weight at \( x \), which characterizes the amount of adaptation induced by \( h \) at instance \( x \). Then for any loss function \( \ell \) we have

\[
\mathbb{E}_{D(h)}[\ell(h; X, Y)] = \int P_{D(h)}(X = x, Y = y)\ell(h; x, y) \, dx \, dy \\
= \int P_{D_S}(Y = y|X = x) \cdot P_{D(h)}(X = x)\ell(h; x, y) \, dx \, dy \\
= \int P_{D_S}(Y = y|X = x) \cdot P_{D_S}(X = x) \cdot \frac{P_{D(h)}(X = x)}{P_{D_S}(X = x)} \cdot \ell(h; x, y) \, dx \, dy \\
= \int P_{D_S}(Y = y|X = x) \cdot P_{D_S}(X = x) \cdot \omega_x(h) \cdot \ell(h; x, y) \, dx \, dy \\
= \mathbb{E}_{D_S}[\omega_x(h) \cdot \ell(h; x, y)]
\]

The above derivation was not new and offered the basis for performing importance reweighting when learning under covariate shift [29]. The particular form informs us that \( \omega_x(h) \) controls the generation of \( D(h) \) and encodes its dependency of both \( D_S \) and \( h \), and is critical for deriving our results below.

4.1 Upper bound

We now derive an upper bound for transferability under covariate shift. We will focus particularly on the optimal model trained on the source data \( S \), which we denote as \( h_S^* := \arg\min_{h \in H} \text{Err}_S(h) \). Recall that the classifier with minimum induced risk is denoted as \( h_T^* := \arg\min_{h \in H} \text{Err}_{D(h)}(h) \).

We can upper bound the difference between \( h_S^* \) and \( h_T^* \) as follows:

**Theorem 5** (Suboptimality of \( h_S^* \)). Let \( X \) be distributed according to \( D_S \). We have:

\[
\text{Err}_{D(h_S)}(h_S^*) - \text{Err}_{D(h_T^*)}(h_T^*) \leq \sqrt{\text{Err}_{D_S}(h_T^*)} \cdot \left( \sqrt{\text{Var}(\omega_X(h_S^*))} + \sqrt{\text{Var}(\omega_X(h_T^*))} \right)
\]

We leave the proof to the Appendix. This result can be interpreted as follows: \( h_T^* \) incurs an irreducible amount of error on the source dataset, represented by \( \sqrt{\text{Err}_{D_S}(h_T^*)} \). Moreover, the difference in error between \( h_S^* \) and \( h_T^* \) is at its maximum when the two classifiers induce adaptations in “opposite” directions; this is represented by the sum of the standard deviations of their importance weights, \( \sqrt{\text{Var}(\omega_X(h_S^*))} + \sqrt{\text{Var}(\omega_X(h_T^*))} \).

4.2 Lower bound

Now we further extend the lower bound

\[
\max\{\text{Err}_{D_S}(h), \text{Err}_{D(h)}(h)\} \geq \frac{d_{TV}(D_Y|S, D_Y(h)) - d_{TV}(D_h|S, D_h(h))}{2}
\]

under covariate shift. Denote \( X_+(h) = \{x : \omega_x(h) \geq 1\} \) and \( X_-(h) = \{x : \omega_x(h) < 1\} \). First we observe that

\[
\int_{X_+(h)} P_{D_S}(X = x)(1 - \omega_x(h)) \, dx + \int_{X_-(h)} P_{D_S}(X = x)(1 - \omega_x(h)) \, dx = 0
\]

This is simply because of \( \int_x P_{D_S}(X = x) \cdot \omega_x(h) \, dx = \int_x P_{D(h)}(X = x) \, dx = 1 \). We make the following assumptions:
We thus have
\[ \int_{X_+(h)} \mathbb{P}_{D_S}(Y = +1, X = x)(1 - \omega_x(h))dx \geq \int_{X_-(h)} \mathbb{P}_{D_S}(Y = +1, X = x)(1 - \omega_x(h))dx \]

Assumption 6 (increased \( \omega_x(h) \) value points are more likely to have \( Y = +1 \)).

Assumption 7 (increased \( \omega_x(h) \) value points are more likely to be classified as \( +1 \)).

Assumption 8. \( \mathbb{P}_{D_S}(Y = +1|X = x) - \mathbb{P}_{D_S}(h(x) = +1|X = x) \) and \( \omega_x(h) \) is positively correlated, which means:

\[ \text{Cov}(\mathbb{P}_{D_S}(Y = +1|X = x) - \mathbb{P}_{D_S}(h(x) = +1|X = x), \omega_x(h)) > 0 \]

The above assumption states that for a deterministic classifier \( h \), within all \( h(X) = +1 \) or \( h(X) = -1 \), a higher \( \mathbb{P}_{D_S}(Y = +1|X = x) \) associates with a higher \( \omega_x(h) \). We next show under these assumptions, our previously provided lower bound is indeed strictly positive:

**Theorem 9.** With Assumption 6-8 the following lower bound is strictly positive for covariate shift.

\[ \max\{\text{Err}_{D_S}(h), \text{Err}_{D(h)}(h)\} \geq \frac{d_{TV}(D_{Y|S}, D_Y(h)) - d_{TV}(D_h|S, D_h(h))}{2} > 0 \]

5 **Label shift**

Now we consider another popular domain adaptation setting known as *label shift*, in which the distribution of labels changes, but not the distribution of features conditioned on the label:

\[ \mathbb{P}_{D(h)}(X = x|Y = y) = \mathbb{P}_{D_S}(X = x|Y = y), \quad \mathbb{P}_{D(h)}(Y = y) \neq \mathbb{P}_{D_S}(Y = y) \tag{6} \]

We thus have

\[ \mathbb{P}_{D(h)}(X = x, Y = y) = \mathbb{P}_{D(h)}(X = x|Y = y) \cdot \mathbb{P}_{D(h)}(Y = y) = \mathbb{P}_{D_S}(X = x|Y = y) \cdot \mathbb{P}_{D(h)}(Y = y) \]

In the case of binary classification, let \( \omega(h) := \mathbb{P}_{D(h)}(Y = +1), \) and \( \mathbb{P}_{D(h)}(Y = -1) = 1 - \omega(h) \). Again, \( \omega(h) \) encodes the induced adaptation from \( D_S \) and \( h \). Then we have for any proper loss function \( \ell \):

\[ \mathbb{E}_{D(h)}[\ell(h; X, Y)] = \omega(h) \cdot \mathbb{E}_{D(h)}[\ell(h; X, Y)|Y = +1] + (1 - \omega(h)) \cdot \mathbb{E}_{D(h)}[\ell(h; X, Y)|Y = -1] \]

\[ = \omega(h) \cdot \mathbb{E}_{D_S}[\ell(h; X, Y)|Y = +1] + (1 - \omega(h)) \cdot \mathbb{E}_{D_S}[\ell(h; X, Y)|Y = -1] \]

We will adopt the following shorthands:

\[ \text{Err}_+(h) := \mathbb{E}_{D_S}[\ell(h; X, Y)|Y = +1], \quad \text{Err}_-(h) := \mathbb{E}_{D_S}[\ell(h; X, Y)|Y = -1] \tag{7} \]

Note that \( \text{Err}_+(h), \text{Err}_-(h) \) are both defined on the conditional source distribution, which is invariant under the label shift assumption.

5.1 **Upper bound**

We again upper bound the transferability of \( h^*_S \) under label shift. Denote by \( D_+ \) the positive label distribution \( \mathbb{P}_{D_S}(X = x|Y = +1) \) and \( D_- \) the negative label distribution \( \mathbb{P}_{D_S}(X = x|Y = -1) \). Let \( p := \mathbb{P}_{D_S}(Y = +1) \).

**Theorem 10.** Under label shift, the difference between \( \text{Err}_{D(h^*_S)}(h^*_S) \) and \( \text{Err}_{D(h^*_T)}(h^*_T) \) bounds as:

\[ \text{Err}_{D(h^*_S)}(h^*_S) - \text{Err}_{D(h^*_T)}(h^*_T) \leq |\omega(h^*_S) - \omega(h^*_T)| \]

\[ + (1 + p) \cdot (d_{TV}(D_+(h^*_S), D_+(h^*_T)) + d_{TV}(D_-(h^*_S), D_-(h^*_T))) \tag{8} \]
The proof is involved but intuitive. Our main idea is to first relate $\text{Err}_{D(h^*_S)}(h^*_S), \text{Err}_{D(h^*_T)}(h^*_T)$ to $\text{Err}_{D_S}(h^*_S), \text{Err}_{D_S}(h^*_T)$, for instance:

$$\text{Err}_{D(h^*_S)}(h^*_S) = p \cdot \text{Err}_+(h^*_S) + (1 - p) \cdot \text{Err}_-(h^*_S) + (\omega(h^*_S) - p)\left[\text{Err}_+(h^*_S) - \text{Err}_-(h^*_S)\right]$$

Then we will be able to use the optimality of $h^*_S$ on $D_S$. $(\omega(h^*_S) - p)\left[\text{Err}_+(h^*_S) - \text{Err}_-(h^*_S)\right]$ will be handled separately.

The above upper bound consists of two components. The first quantity captures the difference between the two induced distributions $D(h^*_S)$ and $D(h^*_T)$. The second quantity characterizes the difference between the two classifiers $h^*_S, h^*_T$ on the source distribution.

### 5.2 Lower bound

Now we discuss lower bounds. Denote by $\text{TPR}_S(h)$ and $\text{FPR}_S(h)$ the true positive and false positive rates of $h$ on the source distribution $D_S$. We prove the following:

**Theorem 11.** Under label shift, any model $h$ must incur the following error on either the source or induced distribution:

$$\max\{\text{Err}_{D_S}(h), \text{Err}_{D(h)}(h)\} \geq \frac{|p - \omega(h)| \cdot (1 - |\text{TPR}_S(h) - \text{FPR}_S(h)|)}{2}$$

The proof extends the bound of Theorem 3 by further explicating each of $d_{\text{TV}}(D_Y|S, D_Y(h)), d_{\text{TV}}(D(h)|S, D(h))$ under the assumption of label shift.

Since $|\text{TPR}_S(h) - \text{FPR}_S(h)| < 0$ unless we have a trivial classifier that has either $\text{TPR}_S(h) = 1, \text{FPR}_S(h) = 0$ or $\text{TPR}_S(h) = 0, \text{FPR}_S(h) = 1$, the lower bound is strictly positive. Taking a closer look, the lower bound is determined linearly by how much the label distribution shifts: $p - \omega(h)$. Then the difference is further determined by the performance of $h$ on the source distribution through $1 - |\text{TPR}_S(h) - \text{FPR}_S(h)|$. For instance, when $\text{TPR}_S(h) > \text{FPR}_S(h)$, the quality becomes $\text{FNR}_S(h) + \text{FPR}_S(h)$, that is the more error $h$ makes, the larger the lower bound will be.

### 6 Computational challenges

The literature of domain adaptation has provided us solutions to minimize the risk on the target distribution via a nicely developed $s$[29, 28, 27]. This allows us to extend the solutions to minimize the induced risk too. Nonetheless we will highlight additional computational challenges. We focus on the covariate shift setting. The scenario for label shift is similar. For covariate shift, recall that earlier we derived the following fact:

**(Importance Reweighting):** $E_{D(h)}[\ell(h; X, Y)] = E_{D}[\omega_x(h) \cdot \ell(h; x, y)]$

This formula informs us that a promising solution that uses $\omega_x(h)$ to perform reweighted ERM. Of course, the primary challenge that stands in the way is how do we know $\omega_x(h)$. There are different methods proposed in the literature to estimate $\omega_x(h)$ when one has access to $D(h)$ [24, 23, 12]. How any of the specific techniques work in our induced domain adaptation setting will be left for a more thorough future study, and in Section 6.2 we will also highlight the additional challenges in our IDA setting. In this section, we focus on explaining the computational challenges even when such knowledge of $\omega_x(h)$ can be obtained for each model $h$ being considered during training.

#### 6.1 Non-convexity

Though $\omega_x(h), \ell(h; x, y)$ might both be convex with respect to (the output of) the classifier $h$, their product is not necessarily convex. Consider the following 1-dimensional example:

**Example 1** $(\omega_x(h) \cdot \ell(h; x, y)$ is generally non-convex). Let $\mathcal{X} = (0, 1]$. Let the true label of each $x \in \mathcal{X}$ be $y(x) = 1$ if $x \geq \frac{1}{2}$, Let $\ell(h; x, y) = \frac{1}{2}(h(x) - y)^2$, and let $h(x) = x$ (simple linear model). Notice that $\ell$ is convex in $h$. Let $D$ be the uniform distribution, whose density function is $f_D = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & otherwise \end{cases}$. Notice that if the training data is drawn from $D$, then $h$ is the linear
Then we discuss the challenges in performing induced domain adaptation. In the standard domain adaptation settings, one often assumes the access to a sample set of $X$, which already poses challenges when there is no access to label $Y$ after the adaptation. Nonetheless, the literature has observed a fruitful development of solutions [29] [34] [12].

One might think the above idea can be applied to our IDA setting rather straightforwardly by assuming observing samples from $\mathcal{D}(h)$, the induced distribution under each model $h$ during the training. However, we often do not know precisely how the distribution would shift under a model $h$ until we deploy it. This is particularly true when the distribution shifts are caused by human responding to a model. Therefore, the ability to “predict” accurately how samples “react” to $h$ plays a very important role [30]. Indeed, the strategic classification literature enables this capability by assuming full rational human agents. For a more general setting, building robust domain adaptation tools that are resistant to the above “prediction error” is also going to be a crucial criterion.

6.2 Challenges due to the lack of access to data

We discuss the challenges in performing induced domain adaptation. In the standard domain adaptation settings, one often assumes the access to a sample set of $X$, which already poses challenges when there is no access to label $Y$ after the adaptation. Nonetheless, the literature has observed a fruitful development of solutions [29] [34] [12].

One might think the above idea can be applied to our IDA setting rather straightforwardly by assuming observing samples from $\mathcal{D}(h)$, the induced distribution under each model $h$ during the training. However, we often do not know precisely how the distribution would shift under a model $h$ until we deploy it. This is particularly true when the distribution shifts are caused by human responding to a model. Therefore, the ability to “predict” accurately how samples “react” to $h$ plays a very important role [30]. Indeed, the strategic classification literature enables this capability by assuming full rational human agents. For a more general setting, building robust domain adaptation tools that are resistant to the above “prediction error” is also going to be a crucial criterion.

7 Improving distribution

A key aspect of our study is distribution shift. Now we ask whether we are inducing better distributions under our induced domain adaptation framework. There are potentially different ways to define what we mean by a “better distribution”. We instantiate our discussion in the context of loan approval and would call a distribution $\mathcal{D}$ with higher qualification rate $\mathbb{P}_D(Y = +1)$ a better one. Naturally, for the discussion of $\mathbb{P}_D(Y = +1)$, we will focus on label shift setting and will evaluate the induced +1 label distribution $\omega(h) = \mathbb{P}_{\mathcal{D}(h)}(Y = +1)$. We first observe the following for the label shift setting:

**Theorem 13.** For two models $h, h'$ with the same training accuracy $Err_+(h), Err_-(h)$, if $Err_+(h) < Err_-(h)$, the model with higher $\omega(h)$ leads to lower induced risk $Err_{\mathcal{D}(h)}(h)$.

The above result is arguably a bit contrived since we effectively forced the classifiers to perform better on the $X | Y = +1$ distribution, but it reveals a fact that indeed the goal of inducing a better $\omega(h)$ can align with the goal of minimizing the induced risk. We are interested in finding what loss functions $\ell$, when used in induced risk, can incentivize a better $\omega(h)$ with less assumptions.
Theorem 14. where $t$ is the threshold.

In our experiments, $h_T^*$ is trained on the source data directly, $h_T^*$ is trained by optimizing $\omega_x(h) \cdot \ell(h; x, y)$ on the source data. Results are summarized in Table 8.

- We do observe positive gaps of $\text{Err}_{D(h_T^*)} - \text{Err}_{D(h_T^*)}$, indicating suboptimality of training on the source distribution. The gaps are well bounded by the upper bound (UB).

- Our lower bounds (LB) do return meaningful positive gaps except for Synthetic Case 2. Because these lower bounds should hold for any models, and we do observe in our settings that the errors for both domains can be small, the lower bounds are close to 0.

Concluding remarks In this paper, we introduce the problem of induced domain adaptation, which differs from the classical domain adaptation literature in that the adaptation is induced by the output model from the training. We presented a sequence of transferrability results, both for general adaptation setting and for specific ones (covariate shift and label shift). Our paper ends with

| Dataset                | $\text{Err}_{D(h_T^*)}$ | $\text{Err}_{D(h_T^*)} - \text{Err}_{D(h_T^*)}$ | UB (Thm. 5) | LB (Thm. 9) |
|------------------------|-------------------------|-----------------------------------------------|-------------|-------------|
| Synthetic Dataset 1    | 0.0717                  | 0.0643                                        | 0.0755      | 0.0058      |
| Synthetic Dataset 2    | 0.1070                  | 0.0134                                        | 0.3560      | -0.0431     |
| Heart                  | 0.0885                  | 0.0016                                        | 0.2247      | 0.0040      |
| Diabetes               | 0.2122                  | 0.0122                                        | 0.4100      | 0.0258      |

Table 1: Experiment results for synthetic and UCI datasets.

Next we show that a recently proposed loss function called **peer loss** [22] induces higher improvement while used in minimizing the induced risk, so that induced risk minimization aligns with incentivizing improvement. Our inspiration of looking into peer loss is due to the similarity between the label shift problem and the learning with noisy label literature, as well as the nice knowledge-free property of peer loss when dealing with label shifts. Peer loss is defined in the following way:

- For each $(x, y)$ we aim to evaluate, randomly draw two other sample indices $p_1, p_2 \in [n]$.
- Pairing $x_{p_1}$ with $y_{p_2}$, the peer loss is defined as $\ell_{PL}(h(x), y) := \ell(h(x), y) - \ell(h(x_{p_1}), y_{p_2})$.

We propose a corrected version of $\ell_{PL}$ that accounts for label shift:

$$
\ell_{PL}^t(h(x), y) := \ell(h(x), y) - \ell(h(x_{p_1}), y_{p_2}) + 2\omega(h)(1 - \omega(h))
$$

We show that under specific conditions, minimizing $\ell_{PL}^t(h(x), y)$ balances improving $\omega(h)$ and the error rate on the source data distribution. Denote by $\omega^-(h) := \mathbb{P}_{D(h)}(Y = +1 | Y = -1)$ the improvement fraction. Suppose that $\omega^t(h) := \mathbb{P}_{D(h)}(Y = +1 | Y = -1) = 0$, so that a qualified agent will not update to become unqualified. Then the adapted label distribution is $\omega(h) = p + (1 - p)\omega^-(h)$. Now we prove the following:

**Theorem 14.** When $\ell = 0$-1 loss, $\mathbb{E}_{D(h)}[\ell_{PL}^t(h(X), Y])] = (1 - (\omega^-(h))^2) \cdot (\text{Err}_r(h) + \text{Err}_l(h))$. In other words, for any two models $h, h'$ with the same training accuracy $\text{Err}_{D(h)}(h)$, the model with higher improvement fraction $\omega^-(h)$ has lower induced risk $\ell^t$ on $D(h)$.

8 Experimental results

Now we present a set of toy empirical examples to highlight some of our findings. Due to space limit, more details can be found in the Appendix. **Data** We performed experiments both on synthetic data as well as the UCI datasets [11] Heart and Diabetes. For the synthetic data, the goal is to predict a label $y \in \{0, 1\}$ using 1-dimensional feature $x \in \mathbb{R}$. We assume the label $y \sim \text{Bernoulli}(\alpha)$, and that $x|y \sim \mathcal{N}((\mu_y, \sigma^2)$ (normal distribution). We train a linear threshold classifier $h(x) = 1[x \geq t_h]$, where $t_h$ is the threshold. **Adaptation function** We consider a covariate shift setting, and set $\omega_x(h)$ accordingly. We assume the importance weight term $\omega_x(h)$ is of the following form:

$$
\omega_x(h) \propto \begin{cases} 
1 + c_1 \cdot \sqrt{x - t_h}, & \text{if } x \geq t_h \\
1 - c_2 \cdot \sqrt{t_h - x}, & \text{if } x \leq t_h 
\end{cases}
$$

where $c_1, c_2$ are parameters that can be tuned. For the UCI dataset, we adopt $\omega_x(h)$ of similar flavor.

In our experiments, $h_T^*$ is trained on the source data directly, $h_T^*$ is trained by optimizing $\omega_x(h) \cdot \ell(h; x, y)$ on the source data. Results are summarized in Table 8.

- We do observe positive gaps of $\text{Err}_{D(h_T^*)} - \text{Err}_{D(h_T^*)}$, indicating suboptimality of training on the source distribution. The gaps are well bounded by the upper bound (UB).

- Our lower bounds (LB) do return meaningful positive gaps except for Synthetic Case 2. Because these lower bounds should hold for any models, and we do observe in our settings that the errors for both domains can be small, the lower bounds are close to 0.

The parameters are $\alpha = 0.15, \mu_0 = 0, \mu_1 = 1, \sigma = 0.5, C_1 = 1.2, C_2 = 0.8, N = 300$.

The parameters are $\alpha = 0.7, \mu_0 = 0, \mu_1 = 1, \sigma = 0.45, C_1 = 1.2, C_2 = 0.8, N = 500$.
discussing the challenges in performing induced domain adaptation, including the convexity when performing importance reweighting, as well as the challenges of not observing induced samples.

Unawareness of the potential distribution shift might lead to unintended consequence when training a machine learning model. One goal of this paper is to raise awareness of this issue for a safe deployment of machine learning methods in high-stake societal applications. We believe this is a promising research direction for the machine learning community, both as an unaddressed technical problem and a stepstone for putting human in the center when training a machine learning model.

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Appendix

The Appendix is organized as follows:

- Section A includes omitted proofs for theoretical conclusions in the main paper.
- Section B presents the missing experimental details.

A Proofs

A.1 Proof of Theorem 1

Proof. We first establish two lemmas that will be helpful for bounding the errors of a pair of classifiers. Both are standard results from the domain adaptation literature [4].

Lemma 15. For any hypotheses \( h, h' \in \mathcal{H} \) and distributions \( \mathcal{D}, \mathcal{D}' \),

\[
|\text{Err}_\mathcal{D}(h, h') - \text{Err}_{\mathcal{D}'}(h, h')| \leq \frac{d_{\mathcal{H} \times \mathcal{H}}(\mathcal{D}, \mathcal{D}')}{2}.
\]

Proof. Define the cross-prediction disagreement between two classifiers \( h, h' \) on a distribution \( \mathcal{D} \) as \( \text{Err}_\mathcal{D}(h, h') := \mathbb{P}_\mathcal{D}(h(X) \neq h'(X)) \). By the definition of the \( \mathcal{H} \)-divergence,

\[
d_{\mathcal{H} \times \mathcal{H}}(\mathcal{D}, \mathcal{D}') = 2 \sup_{h, h' \in \mathcal{H}} |\mathbb{P}_\mathcal{D}(h(X) \neq h'(X)) - \mathbb{P}_{\mathcal{D}'}(h(X) \neq h'(X))| \]

\[
= 2 \sup_{h, h' \in \mathcal{H}} |\text{Err}_\mathcal{D}(h, h') - \text{Err}_{\mathcal{D}'}(h, h')| \]

\[
\geq 2 |\text{Err}_\mathcal{D}(h, h') - \text{Err}_{\mathcal{D}'}(h, h')|.
\]

Another helpful lemma for us is the well-known fact that the 0-1 error obeys the triangle inequality (see, e.g., [2]):

Lemma 16. For any distribution \( \mathcal{D} \) over instances and any labeling functions \( f_1, f_2, \) and \( f_3 \), we have \( \text{Err}_\mathcal{D}(f_1, f_2) \leq \text{Err}_\mathcal{D}(f_1, f_3) + \text{Err}_\mathcal{D}(f_2, f_3) \).

Denote by \( \tilde{h}^* \) the ideal joint hypothesis, which minimizes the combined error:

\[
\tilde{h}^* := \arg\min_{h' \in \mathcal{H}} \text{Err}_{\mathcal{D}(h)}(h') + \text{Err}_{\mathcal{D}_S}(h')
\]

We have:

\[
\text{Err}_{\mathcal{D}(h)}(h) \leq \text{Err}_{\mathcal{D}(h)}(\tilde{h}^*) + \text{Err}_{\mathcal{D}(h)}(h, \tilde{h}^*) \quad \text{(Lemma 16)}
\]

\[
\leq \text{Err}_{\mathcal{D}(h)}(\tilde{h}^*) + \text{Err}_{\mathcal{D}_S}(h, \tilde{h}^*) + |\text{Err}_{\mathcal{D}(h)}(h, \tilde{h}^*) - \text{Err}_{\mathcal{D}_S}(h, \tilde{h}^*)|
\]

\[
\leq \text{Err}_{\mathcal{D}(h)}(\tilde{h}^*) + \text{Err}_{\mathcal{D}_S}(h) + \text{Err}_{\mathcal{D}_S}(\tilde{h}^*) + \frac{1}{2} d_{\mathcal{H} \times \mathcal{H}}(\mathcal{D}_S, \mathcal{D}(h)) \quad \text{(Lemma 15)}
\]

\[
= \text{Err}_{\mathcal{D}_S}(h) + \lambda_{\mathcal{D}_S \rightarrow \mathcal{D}(h)} + \frac{1}{2} d_{\mathcal{H} \times \mathcal{H}}(\mathcal{D}_S, \mathcal{D}(h)). \quad \text{(Definition of } \tilde{h}^*)
\]

A.2 Proof of Theorem 2

Proof. Invoking Theorem 1 and replacing \( h \) with \( h^*_T \) and \( S \) with \( \mathcal{D}(h^*_T) \), we have

\[
\text{Err}_{\mathcal{D}(h)}(h^*_T) \leq \text{Err}_{\mathcal{D}(h^*_T)}(h^*_T) + \lambda_{\mathcal{D}(h^*_T) \rightarrow \mathcal{D}(h)} + \frac{1}{2} d_{\mathcal{H} \times \mathcal{H}}(\mathcal{D}(h^*_T), \mathcal{D}(h)) \quad (11)
\]
Now observe that
\[
\begin{align*}
\text{Err}_{D(h)}(h) & \leq \text{Err}_{D(h)}(h^*_T) + \text{Err}_{D(h)}(h, h^*_T) \\
& \leq \text{Err}_{D(h)}(h^*_T) + \text{Err}_{D(h^*_T)}(h, h^*_T) + \left| \text{Err}_{D(h)}(h, h^*_T) - \text{Err}_{D(h^*_T)}(h, h^*_T) \right| \\
& \leq \text{Err}_{D(h)}(h^*_T) + \text{Err}_{D(h^*_T)}(h, h^*_T) + \frac{1}{2} d_{H \times H}(D(h^*_T), D(h)) \quad \text{(by Lemma 15)} \\
& \leq \text{Err}_{D(h^*_T)}(h^*_T) + \text{Err}_{D(h^*_T)}(h) + \text{Err}_{D(h^*_T)}(h^*_T) + \frac{1}{2} d_{H \times H}(D(h^*_T), D(h)) \quad \text{(by Lemma 16)} \\
& \leq \text{Err}_{D(h^*_T)}(h^*_T) + \lambda_{D(h) \rightarrow D(h^*_T)} + \frac{1}{2} d_{H \times H}(D(h^*_T), D(h)) \\
& + \text{Err}_{D(h^*_T)}(h) + \text{Err}_{D(h^*_T)}(h^*_T) + \frac{1}{2} d_{H \times H}(D(h^*_T), D(h)) \\
\end{align*}
\]
Adding \( \text{Err}_{D(h)}(h) \) to both sides and rearranging terms yields
\[
2\text{Err}_{D(h)}(h) - 2\text{Err}_{D(h^*_T)}(h^*_T) \leq \text{Err}_{D(h)}(h) + \text{Err}_{D(h^*_T)}(h) + \lambda_{D(h) \rightarrow D(h^*_T)} + d_{H \times H}(D(h^*_T), D(h))
\]
Dividing both sides by 2 completes the proof.

A.3 Proof of Theorem 3

Proof. Using the triangle inequality of \( d_{TV} \), we have
\[
d_{TV}(D_{Y|S}, D_Y(h)) \leq d_{TV}(D_{Y|S}, D_{h|S}) + d_{TV}(D_{h|S}, D_h(h)) + d_{TV}(D_h(h), D_Y(h)) \quad (12)
\]
and by the definition of \( d_{TV} \), the divergence term \( d_{TV}(D_{Y|S}, D_Y(h)) \) becomes
\[
d_{TV}(D_{Y|S}, D_{h|S}) = |\mathbb{P}_{D_Y}(Y = +1) - \mathbb{P}_{D_{h}}(h(x) = +1)| \\
= \frac{\mathbb{E}_{D_{h}}[Y] + 1 - \mathbb{E}_{D_{h}}[h(X)] + 1}{2} \\
= \frac{\mathbb{E}_{D_{h}}[Y] - \mathbb{E}_{D_{h}}[h(X)]}{2} \\
\leq \frac{1}{2} \mathbb{E}_{D_{h}}[|Y - h(X)|] \\
= \text{Err}_{D_{h}}(h)
\]
Similarly, we have
\[
d_{TV}(D_{h}(h), D_Y(h)) \leq \text{Err}_{D(h)}(h)
\]
As a result, we have
\[
\text{Err}_{D_{h}}(h) + \text{Err}_{D(h)}(h) \geq d_{TV}(D_{Y|S}, D_{h|S}) + d_{TV}(D_h(h), D_Y(h)) \\
\geq d_{TV}(D_{Y|S}, D_Y(h)) - d_{TV}(D_{h|S}, D_h(h)) \quad \text{(by (12))}
\]
which implies
\[
\max\{\text{Err}_{D_{h}}(h), \text{Err}_{D(h)}(h)\} \geq \frac{d_{TV}(D_{Y|S}, D_Y(h)) - d_{TV}(D_{h|S}, D_h(h))}{2} .
\]

A.4 Proof of Theorem 5

Proof. We start from the error induced by \( h^*_S \). Let the average importance weight induced by \( h^*_S \) be \( \bar{\omega}(h^*_S) = \mathbb{E}_{D_S}[\omega_x(h^*_S)] \); we add and subtract this from the error:
\[
\begin{align*}
\text{Err}_{D(h^*_S)}(h^*_S) &= \mathbb{E}_{D_S}[\omega_x(h^*_S) \cdot 1(h^*_S(x) \neq y)] \\
&= \mathbb{E}_{D_S}[\omega(h^*_S) \cdot 1(h^*_S(x) \neq y)] + \mathbb{E}_{D_S}[(\omega_x(h^*_S) - \bar{\omega}(h^*_S)) \cdot 1(h^*_S(x) \neq y)]
\end{align*}
\]
In fact, \( \bar{\omega}(h_S^*) = 1 \), since

\[
\omega(h_S^*) = \mathbb{E}_{D_S}[\omega_x(h_S^*)] = \int \omega_x(h_S^*)\mathbb{P}_{D_S}(X = x)dx
\]

\[
= \int \frac{\mathbb{P}_{D(h)}(X = x)}{\mathbb{P}_{D_S}(X = x)}\mathbb{P}_{D_S}(X = x)dx = \mathbb{P}_{D(h)}(X = x)dx = 1
\]

Now consider any other classifier \( h \). We have

\[
\text{Err}_{D(h)}(h_S^*) = \mathbb{E}_{D_S}[\mathbb{I}(h(x) \neq y)] + \mathbb{E}_{D_S}[(\omega_x(h_S^*) - \bar{\omega}(h_S^*)) \cdot \mathbb{I}(h(x) \neq y)]
\]

\[
\leq \mathbb{E}_{D_S}[\mathbb{I}(h(x) \neq y)] + \mathbb{E}_{D_S}[(\omega_x(h_S^*) - \bar{\omega}(h_S^*)) \cdot \mathbb{I}(h(x) \neq y)]
\]

(by optimality of \( h_S^* \) on \( D_S \))

\[
= \mathbb{E}_{D_S}[\mathbb{I}(h(x) \neq y)] + \mathbb{E}_{D_S}[(\omega_x(h_S^*) - \bar{\omega}(h_S^*)) \cdot \mathbb{I}(h(x) \neq y)]
\]

(multiply by \( \bar{\omega}(h_S^*) = 1 \))

\[
= \mathbb{E}_{D_S}[\omega_x(h) \cdot \mathbb{I}(h(x) \neq y)] + \mathbb{E}_{D_S}[(\bar{\omega}(h) - \omega_x(h)) \cdot \mathbb{I}(h(x) \neq y)]
\]

(adding and subtracting \( \bar{\omega}(h_S^*) \))

\[
= \text{Err}_{D(h)}(h) + \text{Cov}(\omega_x(h_S^*), \mathbb{I}(h(x) \neq y)) - \text{Cov}(\omega_x(h), \mathbb{I}(h(x) \neq y))
\]

Moving the error terms to one side, we have

\[
\text{Err}_{D(h)}(h_S^*) - \text{Err}_{D(h)}(h) 
\]

\[
\leq \text{Cov}(\omega_x(h_S^*), \mathbb{I}(h(x) \neq y)) - \text{Cov}(\omega_x(h), \mathbb{I}(h(x) \neq y))
\]

\[
\leq \sqrt{\text{Var}(\omega_x(h_S^*)) \cdot \text{Var}(\mathbb{I}(h(x) \neq y))} 
\]

\[\text{Cov}(X, Y) \leq \sqrt{\text{Var}(X) \cdot \text{Var}(Y)}\]

\[
+ \sqrt{\text{Var}(\omega_x(h)) \cdot \text{Var}(\mathbb{I}(h(x) \neq y))}
\]

\[
= \sqrt{\text{Var}(\omega_x(h_S^*)) \cdot \text{Err}_{D_S}(h_S^*)(1 - \text{Err}_{D_S}(h_S^*)) + \text{Var}(\omega_x(h)) \cdot \text{Err}_{D_S}(h)\text{Err}_{D_S}(h)}
\]

\[
\leq \sqrt{\text{Var}(\omega_x(h_S^*)) \cdot \text{Err}_{D_S}(h_S^*) + \text{Var}(\omega_x(h)) \cdot \text{Err}_{D_S}(h)}
\]

\[1 - \text{Err}_{D_S}(h) \leq 1\]

\[
\leq \sqrt{\text{Err}_{D_S}(h) \cdot \left(\sqrt{\text{Var}(\omega_x(h_S^*))} + \sqrt{\text{Var}(\omega_x(h))}\right)}
\]

Since this holds for any \( h \), it certainly holds for \( h = h_T \).

\[\square\]

A.5 Proof of Theorem 9

Proof. Notice that in the setting of binary classification, we can write the total variation distance between \( D_{Y|S} \) and \( D_Y(h) \) as the difference between the probability of \( Y = +1 \) and the probability of \( Y = -1 \):

\[
d_{TV}(D_{Y|S}, D_Y(h))
\]

\[
= |\mathbb{P}_{D_S}(Y = +1) - \mathbb{P}_{D(h)}(Y = +1)|
\]

\[
= \left| \int \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x)dx - \int \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x)\omega_x(h)dx \right|
\]

\[
= \left| \int \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (1 - \omega_x(h))dx \right|
\]

Similarly we have

\[
d_{TV}(D_{h|S}, D_h(h)) = \left| \int \mathbb{P}_{D_S}(h(x) = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (1 - \omega_x(h))dx \right|
\]
We can further expand the total variation distance between $D_{Y|S}$ and $D_Y(h)$ as follows:

$$d_{TV}(D_{Y|S}, D_Y(h)) = \left| \int \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (1 - \omega_x(h)) \, dx \right|$$

$$= \left| \int_{X_+(h)} \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (1 - \omega_x(h)) \, dx \right| + \left| \int_{X_-(h)} \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (1 - \omega_x(h)) \, dx \right|$$

$$= -\int_{X_+(h)} \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (1 - \omega_x(h)) \, dx$$

$$- \int_{X_-(h)} \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (1 - \omega_x(h)) \, dx$$

(by Assumption 6)

$$= \int_{X_+(h)} \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (\omega_x(h) - 1) \, dx$$

$$+ \int_{X_-(h)} \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (\omega_x(h) - 1) \, dx$$

(by (13))

Similarly, by assumption 7 and equation (14), we have

$$d_{TV}(D_{h|S}, D_h(h)) = \int \mathbb{P}_{D_S}(h(x) = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (\omega_x(h) - 1) \, dx$$

Thus we can bound the difference between $d_{TV}(D_{Y|S}, D_Y(h))$ and $d_{TV}(D_{h|S}, D_h(h))$ as follows:

$$d_{TV}(D_{Y|S}, D_Y(h)) - d_{TV}(D_{h|S}, D_h(h))$$

$$= \int \mathbb{P}_{D_S}(Y = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (\omega_x(h) - 1) \, dx$$

$$- \int \mathbb{P}_{D}(h(x) = +1|X = x)\mathbb{P}_{D_S}(X = x) \cdot (\omega_x(h) - 1) \, dx$$

$$= \int [\mathbb{P}_{D_S}(Y = +1|X = x) - \mathbb{P}_{D_S}(h(x) = +1|X = x)]\mathbb{P}_{D_S}(X = x) \cdot (\omega_x(h) - 1) \, dx$$

$$= \mathbb{E}_{D_S}[(\mathbb{P}_{D_S}(Y = +1|X = x) - \mathbb{P}_{D_S}(h(x) = +1|X = x))(\omega_x(h) - 1)]$$

(by Assumption 8)

$$> \mathbb{E}_{D_S}[(\mathbb{P}_{D_S}(Y = +1|X = x) - \mathbb{P}_{D_S}(h(x) = +1|X = x))(\omega_x(h) - 1)]$$

$$= 0$$

Combining the above with Theorem 3 we have

$$\max\{\text{Err}_{D_S}(h), \text{Err}_{D_h(h)}\} \geq \frac{d_{TV}(D_{Y|S}, D_Y(h)) - d_{TV}(D_{h|S}, D_h(h))}{2} > 0$$

\(\square\)

A.6 Proof of Theorem 10

Proof. Defining $p := \mathbb{P}_{D_S}(Y = +1)$, we have

$$\text{Err}_{D_h(h^*)}(h^*_S) = \omega(h^*_S) \cdot \text{Err}_+(h^*_S) + (1 - \omega(h^*_S)) \cdot \text{Err}_-(h^*_S)$$

(by definitions of $\omega(h^*_S), \text{Err}_+(h^*_S),$ and $\text{Err}_-(h^*_S)$)

$$= p \cdot \text{Err}_+(h^*_S) + (1 - p) \cdot \text{Err}_-(h^*_S) + (\omega(h^*_S) - p) |\text{Err}_+(h^*_S) - \text{Err}_-(h^*_S)|$$

\(\text{(15)}\)
We can expand (I) as follows:

\[
p \cdot \text{Err}_+(h_S^+) \leq p \cdot \text{Err}_+(h_T^+) + (1 - p) \cdot \text{Err}_-(h_T^+)
\]

(by optimality of \(h_T^+\) on \(D_S\))

\[
= \omega(h_T^+) \cdot \text{Err}_+(h_T^+) + (1 - \omega(h_T^+)) \cdot \text{Err}_-(h_T^+) + (p - \omega(h_T^+)) \cdot [\text{Err}_+(h_T^+) - \text{Err}_-(h_T^+)]
\]

Adding these two equations yields

\[
\text{Err}_{D(h_S^+)}(h_T^+) - \text{Err}_{D(h_S^+)}(h_T^+) \leq (\omega(h_S^+) - p)[\text{Err}_+(h_S^+) - \text{Err}_-(h_S^+)] + (p - \omega(h_T^+)) \cdot [\text{Err}_+(h_T^+) - \text{Err}_-(h_T^+)]
\]

Notice that

\[
0.5(\text{Err}_+(h) - \text{Err}_-(h)) = 0.5 \cdot 1 - 0.5 \cdot \mathbb{P}(h(X) = +1|Y = +1) - 0.5 \cdot \mathbb{P}(h(X) = +1|Y = -1)
\]

\[
= 0.5 - \mathbb{P}_{D_n}(h(X) = +1)
\]

where \(D_n\) is a distribution with uniform prior. Then

\[
(\omega(h_S^+) - p)[\text{Err}_+(h_S^+) - \text{Err}_-(h_S^+)] = 2(\omega(h_S^+) - p) \cdot (0.5 - \mathbb{P}_{D_n}(h(X) = +1))
\]

\[
(p - \omega(h_T^+))\text{[Err}_+(h_T^+) - \text{Err}_-(h_T^+)] = 2(p - \omega(h_T^+)) \cdot (0.5 - \mathbb{P}_{D_n}(h(X) = +1))
\]

Adding together these two equations yields

\[
(\omega(h_S^+) - p)[\text{Err}_+(h_S^+) - \text{Err}_-(h_S^+)] + (p - \omega(h_T^+)) \cdot [\text{Err}_+(h_T^+) - \text{Err}_-(h_T^+)]
\]

\[
= 2(\omega(h_S^+) - p) \cdot (0.5 - \mathbb{P}_{D_n}(h(X) = +1)) + 2(p - \omega(h_T^+)) \cdot (0.5 - \mathbb{P}_{D_n}(h(X) = +1))
\]

\[
= 2(\omega(h_S^+) - \omega(h_T^+)) - 2(\omega(h_S^+) - \omega(h_T^+))\mathbb{P}_{D_n}(h(X) = +1) + 2(\omega(h_T^+) - \omega(h_T^+))\mathbb{P}_{D_n}(h(X) = +1)
\]

\[
+ 2p \cdot (\mathbb{P}_{D_n}(h_S^+(X) = +1) - \mathbb{P}_{D_n}(h_T^+(X) = +1))
\]

\[
\leq |\omega(h_S^+) - \omega(h_T^+)| \cdot (1 + 2|\mathbb{P}_{D_n}(h_S^+(X) = +1) - \mathbb{P}_{D_n}(h_T^+(X) = +1)|)
\]

\[
+ 2p \cdot |\mathbb{P}_{D_n}(h_S^+(X) = +1) - \mathbb{P}_{D_n}(h_T^+(X) = +1)|
\]

(16)

Meanwhile,

\[
|\mathbb{P}_{D_n}(h_S^+(X) = +1) - \mathbb{P}_{D_n}(h_T^+(X) = +1)|
\]

\[
\leq 0.5 \cdot |\mathbb{P}_{D|Y=+1}(h_S^+(X) = +1) - \mathbb{P}_{D|Y=+1}(h_T^+(X) = +1)|
\]

\[
+ 0.5 \cdot |\mathbb{P}_{D|Y=-1}(h_S^+(X) = +1) - \mathbb{P}_{D|Y=-1}(h_T^+(X) = +1)|
\]

\[
= 0.5 (d_{TV}(D_+(h_S^+), D_+(h_T^+)) + d_{TV}(D_-(h_S^+), D_-(h_T^+))
\]

(17)

Combining (16) and (17) gives

\[
|\omega(h_S^+) - \omega(h_T^+)| \cdot (1 + 2 \cdot [\mathbb{P}_{D_n}(h_S^+(X) = +1) - \mathbb{P}_{D_n}(h_T^+(X) = +1)])
\]

\[
+ 2p \cdot [\mathbb{P}_{D_n}(h_S^+(X) = +1) - \mathbb{P}_{D_n}(h_T^+(X) = +1)]
\]

\[
\leq |\omega(h_S^+) - \omega(h_T^+)| \cdot (1 + d_{TV}(D_+(h_S^+), D_+(h_T^+)) + d_{TV}(D_-(h_S^+), D_-(h_T^+))
\]

\[
+ p \cdot (d_{TV}(D_+(h_S^+), D_+(h_T^+)) + d_{TV}(D_-(h_S^+), D_-(h_T^+))
\]

\[
+ |\omega(h_S^+) - \omega(h_T^+)| \cdot (1 + p \cdot (d_{TV}(D_+(h_S^+), D_+(h_T^+)) + d_{TV}(D_-(h_S^+), D_-(h_T^+))
\]

A.7 Proof of Theorem

We will make use of the following fact:

**Lemma 17.** Under label shift, \(TPR_S(h) = TPR_{h}(h)\) and \(FPR_S(h) = FPR_{h}(h)\).
Proof. We have

\[ TPR_h(h) = \mathbb{P}_{D(h)}(h(X) = +1 | Y = +1) \]
\[ = \int \mathbb{P}_{D(h)}(h(X) = +1, X = x | Y = +1) dx \]
\[ = \int \mathbb{P}_{D(h)}(h(X) = +1 | X = x, Y = +1) \mathbb{P}_{D(h)}(X = x | Y = +1) dx \]
\[ = \int \mathbbm{1}(h(x) = +1) \mathbb{P}_{D(h)}(X = x | Y = +1) dx \]
\[ = \int \mathbb{P}_{D_S}(h(X) = +1 | X = x, Y = +1) \mathbb{P}_{D_S}(X = x | Y = +1) dx \]
\[ = TPR_S(h) \]

The argument for \( TPR_h(h) = TPR_S(h) \) is analogous. \( \square \)

Now we proceed to prove the theorem.

Proof of Theorem[17] In section 3.2 we showed a general lower bound on the maximum of \( \text{Err}_{D_S}(h) \) and \( \text{Err}_{D(h)}(h) \):

\[
\max \{ \text{Err}_{D_S}(h), \text{Err}_{D(h)}(h) \} \geq \frac{d_{TV}(D_{Y|S}, D_Y(h)) - d_{TV}(D_{h|S}, D_h(h))}{2}
\]

In the case of label shift, and by the definitions of \( p \) and \( \omega(h) \),

\[
d_{TV}(D_{Y|S}, D_Y(h)) = |\mathbb{P}_{D_S}(Y = +1) - \mathbb{P}_{D(h)}(Y = +1)| = |p - \omega(h)| \tag{18}
\]

In addition, we have

\[
D_{h|S} = \mathbb{P}_S(h(X) = +1) = p \cdot TPR_S(h) + (1 - p) \cdot FPR_S(h) \tag{19}
\]

Similarly

\[
D_h(h) = \mathbb{P}_{D(h)}(h(X) = +1)
\]
\[ = \omega(h) \cdot TPR_h(h) + (1 - \omega(h)) \cdot FPR_h(h) \]
\[ = \omega(h) \cdot TPR_S(h) + (1 - \omega(h)) \cdot FPR_S(h) \] (by Lemma[17]) \tag{20}

Therefore

\[
d_{TV}(D_{h|S}, D_h(h)) = |\mathbb{P}_{D_S}(h(X) = +1) - \mathbb{P}_{D(h)}(h(X) = +1)|
\]
\[ = |(p - \omega(h)) \cdot TPR_S(h) + (\omega(h) - p) \cdot FPR_S(h)| \] (By \(20\) and \(19\)) \tag{21}

which yields:

\[
d_{TV}(D_{Y|S}, D_Y(h)) - d_{TV}(D_{h|S}, D_h(h)) = |p - \omega(h)||(1 - TPR_S(h) - FPR_S(h))| \]

(by \(18\) and \(21\))

completing the proof. \( \square \)

A.8 Proof of Proposition[12]

Proof. Let us use the shorthand \( \omega(h) := \omega_x(h) \) and \( \ell(h) := \ell(h; x, y) \). To show that \( \omega(h) \cdot \ell(h) \) is convex, it suffices to show that for any \( \alpha \in [0, 1] \) and any two hypotheses \( h, h' \) we have

\[
\omega(\alpha \cdot h + (1 - \alpha) \cdot h') \cdot \ell(\alpha \cdot h + (1 - \alpha) \cdot h') \leq \alpha \cdot \omega(h) \cdot \ell(h) + (1 - \alpha) \cdot \omega(h') \cdot \ell(h')
\]

By the convexity of \( \omega \),

\[
\omega(\alpha \cdot h + (1 - \alpha) \cdot h') \leq \alpha \cdot \omega(h) + (1 - \alpha) \cdot \omega(h')
\]
and by the convexity of $\ell$,

$$\ell(\alpha \cdot h + (1 - \alpha) \cdot h') \leq \alpha \cdot \ell(h) + (1 - \alpha) \cdot \ell(h')$$

Therefore it suffices to show that

$$[\alpha \cdot \omega(h) + (1 - \alpha) \cdot \omega(h')] \cdot [\alpha \cdot \ell(h) + (1 - \alpha) \cdot \ell(h')] - \alpha \cdot \omega(h) \cdot \ell(h) + (1 - \alpha) \cdot \omega(h') \cdot \ell(h') \leq 0$$

$$\Leftrightarrow \alpha(\alpha - 1) \cdot \omega(h) \ell(h) - \alpha(\alpha - 1) \cdot [\omega(h) \ell(h') + \omega(h') \ell(h)] + \alpha(\alpha - 1) \cdot \omega(h') \ell(h') \leq 0$$

$$\Leftrightarrow [\omega(h) - \omega(h')] \cdot [\ell(h) - \ell(h')] \leq 0$$

$$\Leftrightarrow [\omega(h) - \omega(h')] \cdot [\ell(h) - \ell(h')] \geq 0$$

By Assumption (9), the left-hand side is indeed non-negative, which proves the claim.

A.9 Proof of Theorem 13

Proof. This is simply because

$$\omega(h) \cdot \text{Err}_+(h) + (1 - \omega(h)) \cdot \text{Err}_-(h) = \omega(h) \cdot (\text{Err}_+(h) - \text{Err}_-(h)) + \text{Err}_-(h)$$

Then clearly, when $\text{Err}_+(h) < \text{Err}_-(h)$, a model that induces higher $\omega(h)$ leads to a smaller loss.

A.10 Proof of Theorem 14

Proof.

$$\mathbb{P}_D(h(X) \neq Y) - \mathbb{P}_D(h(X) = \ Y) + 2\omega(h)(1 - \omega(h))$$

$$= \omega(h)\text{Err}_+(h) + (1 - \omega(h))\text{Err}_-(h) - \mathbb{P}_D(h(X) = +1)\omega(h) - \mathbb{P}_D(h(X) = -1)(1 - \omega(h))$$

$$= \omega(h)\text{Err}_+(h) + (1 - \omega(h))\text{Err}_-(h) - (\omega(h)\text{Err}_+(h) + \text{Err}_+(h) + (1 - \omega(h))(1 - \text{Err}_-(h)))\omega(h)$$

$$- (\omega(h)(1 - \text{Err}_+(h)) + (1 - \omega(h))\text{Err}_-(h))(1 - \omega(h)) + 2\omega(h)(1 - \omega(h))$$

$$= 4\omega(h)(1 - \omega(h)) (\text{Err}_+(h) + \text{Err}_-(h) - 0.5) + 2\omega(h)(1 - \omega(h))$$

$$= 4\omega(h)(1 - \omega(h)) (\text{Err}_+(h) + \text{Err}_-(h))$$

Noticing that

$$4\omega(h)(1 - \omega(h)) = 1 - (\omega^-(h))^2$$

(22)

completes the proof. A higher $\omega^-(h)$ leads to a smaller $4\omega(h)(1 - \omega(h))$.

B Missing experiment details

B.1 Implementation details on UCI datasets

The optimization In our experiments, we adopt cross-entropy $\ell(h; x, y)$ as the loss function for data sample $(x, y)$. $h^*_S$ is trained on the source data (training data) directly. $h^*_T$ is trained by optimizing $\omega_x(h) \cdot \ell(h, x, y)$. Adaption function $\omega_x(h)$ For UCI datasets, we adopt the adaption function $\omega_x(h)$ as:

$$\omega_x(h) = \lambda \cdot (h(x) - y)^2 + 20x^2 + 5.$$ 

$\lambda = -5, -0.5$ for Heart and Diabetes datasets respectively. In the training of each batch, $\omega_x(h)$ is subtracted by the minimum value in this batch and then proceed with a normalization before feeding $\omega_x(h)$ into the optimization of $h^*_T$. Hyper-parameter settings We adopted logistic regression for classification tasks on two UCI datasets, trained for 1000 episodes with batch-size 64 and Adam [13] optimizer. The learning rate for Heart is 0.07, and 0.1 for Diabetes.