A structural geometrical analysis of weakly infeasible SDPs

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Abstract

In this article, we present a geometric theoretical analysis of semidefinite feasibility problems (SDFPs). This is done by decomposing a SDFP into smaller problems, in a way that preserves most feasibility properties of the original problem. With this technique, we develop a detailed analysis of weakly infeasible SDFPs to understand clearly and systematically how weak infeasibility arises in semidefinite programming. In particular, we show that for a weakly infeasible problem over $n \times n$ matrices, at most $n-1$ directions are required to approach the positive semidefinite cone. We also present a discussion on feasibility certificates for SDFPs and related complexity results.

1 Introduction.

In this paper, we deal with the following semidefinite feasibility problem

$$\max 0 \quad \text{s.t.} \quad x \in (L+c) \cap K_n, \tag{1}$$

where $L \subseteq \mathbb{S}_n$ is a vector subspace and $c \in \mathbb{S}_n$. By $\mathbb{S}_n$ we denote the linear space of $n \times n$ real symmetric matrices and $K_n \subseteq \mathbb{S}_n$ denotes the cone of $n \times n$ positive semidefinite matrices. We denote the problem (1) by $(K_n, L, c)$.

It is known that every instance of a semidefinite program falls into one of the following four statuses:

- **Strongly feasible**: $(L+c) \cap \text{int}(K_n) \neq \emptyset$, where $\text{int}(K_n)$ denotes the interior of $K_n$.
- **Weakly feasible**: $(L+c) \cap \text{int}(K_n) = \emptyset$, but $(L+c) \cap K_n \neq \emptyset$.
- **Weakly infeasible**: $(L+c) \cap K_n = \emptyset$ and $\text{dist}(K, L+c) = 0$.
- **Strongly infeasible**: $(L+c) \cap K_n = \emptyset$ and $\text{dist}(K, L+c) > 0$.

Among the four feasibility statuses, all but weak infeasibility afford simple finite certificates: an interior-feasible solution, a pair consisting of a feasible solution and a vector which is normal to a separating hyperplane, and a dual improving direction for strong feasibility, weak feasibility and strong infeasibility respectively. The last one is sometimes called a Farkas-type certificate, and plays an important role in optimization theory. However, it is not evident whether weak infeasibility affords such a finite certificate.

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By “finite certificate” we mean a finite sequence in some finite dimensional vector space. In this paper, we focus on the structural analysis of weak infeasibility in semidefinite programming and develop a procedure which distinguishes the four statuses. We also obtain a finite certificate for weak infeasibility. But we emphasize that the main feature of our approach is concreteness in analyzing weak infeasibility.

In view of finite certificates, we mention that it is possible to obtain a finite and polynomially bounded certificate of weak infeasibility by using Ramana’s extended Lagrangian dual [16]. This result is based on the fact that

\[(K_n, L, c) \text{ is weakly infeasible if and only if it is infeasible and not strongly infeasible}\]

as we will discuss in Section 2. Ramana developed a generalized Farkas’ Lemma for SDP which holds without any assumptions. Since infeasibility and not strong infeasibility have finite certificates, the same is true for weak infeasibility. As this argument has an existential flavour, it is not so clear the implications for the structure of the problem. In this paper, we study weak infeasibility in semidefinite programming from a more constructive point of view to answer, for instance, the following basic question:

**Given a weakly infeasible SDFP, how can we generate a sequence \( \{u^{(i)}| u^{(i)} \in L + c, i = 1, \ldots \infty\} \) such that \( \lim_{i \rightarrow \infty} \text{dist}(u^{(i)}, K_n) = 0 \)?**

Due to the fact that the distance between \( K_n \) and \( L + c \) is zero, we readily see that there exists a nonzero element \( a \) in \( K_n \cap L \). However, it is not clear how \( a \) is related to the weak infeasibility of \((K_n, L, c)\). Since the problem is infeasible, \( \text{dist}(ta + b, K_n) > 0 \) for any \( t > 0 \) and \( b \in L + c \). It would be natural to ask what to happen as \( t \) goes to infinity. Can \( \lim_{i \rightarrow \infty} \text{dist}(ta + b, K_n) = \infty \) or a finite nonzero value, or zero? If we cannot find any \( b \in L + c \) such that \( \lim_{i \rightarrow \infty} \text{dist}(ta + b, K_n) = 0 \) holds, how \( a \) can be used to construct points close to the cone?

We will show that \( a \) alone is not enough to generate such a sequence, but \((n - 1)\) directions including \( a \) are sufficient (with an appropriate choice of \( b \)), whenever the problem is weakly infeasible. In other words, if \((K_n, L, c)\) is weakly infeasible then there exists a \((n - 1)\) dimensional affine subspace \( F \subseteq L + c \) such that \( F \cap K_n = \emptyset \) but \( \text{dist}(F, K_n) = 0 \). This result is a bit surprising to us, because, in general, if \( K \) is a closed convex cone and \((K, L, c)\) is weakly infeasible, then the number of directions necessary to approach the cone could be as large as the dimension of \( L \), which could be up to \( \left( \frac{n(n+1)}{2} \right) - 1 \) in our context.

The proof is done by constructing a set of directions in \( L \) which we call hyper feasible partition. These direction are obtained recursively starting from an nonzero element in \( K_n \cap L \). An important feature of this set is that, even though each direction is not necessarily positive (semi)definite, we can always find a positive linear combination which is almost positive semidefinite (the minimum eigenvalue can be made to be arbitrarily close to zero). The introduction of hyper feasible partitions is another main contribution of this paper and they provide a new insight in the analysis of ill-conditioned semidefinite programs.

One possible application of our results is as follows. Consider the following SDP

\[
\max \langle b, x \rangle \quad \text{s.t. } x \in (L + c) \cap K_n, \quad (P)
\]

and suppose that the optimal value \( b^* \) is finite but not attained. The set \( \{x \in L + c \mid \langle b, x \rangle = b^*\} \) is non-empty and is also an affine space. Denoting by \( \tilde{L} \) the underlying vector space and letting \( \tilde{c} \) be any point which belongs to the affine space, we have that \((K_n, \tilde{L}, \tilde{c})\) is weakly infeasible. Indeed such problems arise in many applications in semidefinite programming including control theory and polynomial optimization [20].

The main tool we use is a simple decomposition result (Theorem 5), which implies that some semidefinite feasibility problems (SDFPs) can be decomposed into smaller subproblems in a way that the feasibility properties are mostly preserved. We also discuss two procedures for analyzing feasibility problems, a forward procedure (FP) and a backward procedure (BP). In particular, BP can distinguish the 4 different feasibility statuses in a systematic manner.

We review related previous works. The existence of weak infeasibility/feasibility and finite duality gap is one of the main difficulties in semidefinite programming. These situations may occur in the absence of interior-feasible solutions to the primal and/or dual. Two possible techniques to recover interior-feasibility by reducing the feasible region of the problem or by expanding the feasible region of its dual counter-part
are the facial reduction algorithm (FRA) and the conic expansion approach (CEA), respectively. FRA was developed by Borwein and Wolkowicz [3] for problems more general than conic programming, whereas CEA was developed by Luo, Sturm and Zhang [14] for conic programming.

In the earlier stages of research of semidefinite programming, Ramana [16] developed an extended Lagrange-Slater dual (ELSD) that has no duality gap. ELSD has the remarkable feature that the size of the extended problem is bounded by a polynomial in terms of the size of the original problem. In [17], Ramana, Tunçel and Wolkowicz demonstrated that ELSD can be interpreted as a facial reduction problem, however, we should note that in the original FRA, the size of the problem is not polynomially bounded, see also [10]. In [12], Polik and Terlaky provided strong duals for conic programming over symmetric cones. Recently, Klep and Schweighofer developed another dual based on real algebraic geometry where the strong duality holds without any constraint qualification [5]. Like ELSD, their dual is just represented in terms of the data of original problem and the size of the dual is bounded by a polynomial in terms of the size of the original problem. Complexity of SDFP is yet a subtle issue. This topic was studied extensively by Porkolab and Khachiyan [13].

Waki and Muramatsu [21] considered a FRA for conic programming and showed that FRA can be regarded as a dual version of CEA. See an excellent review by Pataki [10] for FRA, where he points out the relation between facial reduction and extended duals. Pataki also found that all ill-conditioned semidefinite programs can be reduced to a common $2 \times 2$ semidefinite program [11]. Finally, we mention that Waki showed that weakly infeasible instances can be obtained from semidefinite relaxation of polynomial optimization problems [20].

The problem of weak infeasibility is closely related to closedness of the image of $K_n$ by a certain linear map. A comprehensive treatment of the subject was given by Pataki [9]. We will discuss the connection between Pataki’s results and weak infeasibility in Section 2.

This paper is organized as follows. In Section 2, we discuss certificates for the different feasibility statuses and point the connections to previous works. In Section 3 we present Theorem 5 and discuss how certain SDFPs can be broken in smaller problems. We also prove the bound $n - 1$ for the number of the directions needed to approach $K_n$. In Section 4, a procedure to distinguish between the 4 different feasibility statuses is given. Section 5 summarizes this work.

2 Characterization of different feasibility statuses

In this section, we review the characterization of different feasibility statuses of semidefinite programs with emphasis on weak infeasibility.

2.1 Certificates and NP class in the Blum-Shub-Smale model

Our main interest is on finite certificates and computational complexity. The model of computation we use is the Blum-Shub-Smale model (BSS model) [1] of real computation. The main aspects are that we do not care about the bit length of a real number, we can evaluate any rational function over $\mathbb{R}$ and the machine can deviate the flow of execution by evaluating a linear inequality. “Finite” in this context means that the certificates are composed of a finite number of vectors contained in some finite dimensional vector space. The length of the certificate is then the total number of coordinates among all the vectors it contains. It is also required that a verifier procedure exists. Such a procedure receives as input the problem and the certificate and attest that the certificate is indeed valid in a finite amount of time. If a decision problem admits a finite certificate with a verification procedure such that the length of the former and time complexity of the latter are polynomials in terms of the size of the problem then it is in NP under the BSS model. The main decision problem we are interested in is: given $(K_n, L, c)$, what is its feasibility status?\footnote{Strictly speaking, a decision problem should have “yes” or “no” as answers, but in our case the possible answers are strong/weakly feasible, strong/weakly infeasible. We could have broken down the decision problem in 4 different decision problems having “yes” or “no” as answers. We did not do so, because we wanted to treat them in a unified manner in our procedure BP.}
Suppose we have an algorithm for a decision problem which employs oracles for some problems, for instance, returning a feasible solution to a SDP. (This is a typical situation in the literature when talking about regularization procedures.) In such a situation, if we want to show that the decision problem is in \( \text{NP} \), we can do the following. First, we prove that correctness of the output of each oracle can be verified in polynomial time (with respect the size of the problem). Then, we evaluate the time complexity of the algorithm assuming that the cost for each call of the oracle is one. If the running time of the algorithm is bounded by a polynomial in the size of the problem, then the decision problem is in \( \text{NP} \). The set of outputs given by the oracles can be used as a certificate and the algorithm itself acts as a verifier procedure.

Throughout this paper, we assume that \( L \) is represented as the set of solution of the system of linear equations, where the coefficients and left hand side is explicitly given. We also note that checking positive semidefiniteness of a symmetric matrix can be done in polynomial time by using a variant of \( LDL^T \) decomposition.

### 2.2 Characterization of feasibility statuses

We start with the following proposition which characterizes strong feasibility, weak feasibility and strong infeasibility.

**Proposition 1.** Let \( L \) be a subspace of \( S_n \) and \( c \in S_n \) then \((K_n, L, c)\) is:

1. **Strongly feasible**, if and only if there is \( x \in L + c \) such that \( x \) is positive definite.
2. **Weakly feasible** if and only if there is
   
   i. \( x \in L + c \) such that \( x \) is positive semidefinite
   
   ii. \( y \in L^\perp \cap K_n \) such that \( y \neq 0 \), \( \langle y, c \rangle = 0 \).
3. **Strongly infeasible** if and only if \( c \neq 0 \) and \((K_n, L_{SI}, c_{SI})\) is feasible, where \( L_{SI} = L^\perp \cap c^\perp \), and \( c_{SI} = -\frac{c}{\|c\|^2} \).

**Proof.** Item i is immediate. Items ii. and iii. follow easily from Theorem 11.3 and 11.4 of [18]. Item ii. correspond to the situation where \( K_n \) and \( L + c \) can be properly separated but still a feasible point exists and item iii. to the case where they can be strongly separated. Also, for a proof of iii. see, for instance, Lemma 5 of [15].

Proposition 1 already implies that deciding deciding strong feasibility, weak feasibility and strong infeasibility are in \( \text{NP} \), in the BSS model. In addition, \((K_n, L, c)\) is not strongly feasible if and only if item 2.ii holds (but not necessarily 2.i). This means that deciding strong feasibility lies in \( \text{coNP} \) as well\(^2\).

In Proposition 1, weak infeasibility is absent. When proving weak infeasibility, it is necessary to show that the distance between \( K_n \) and \( L + c \) is 0. The obvious way is to produce a sequence \( \{x_k\} \in L + c \) such that \( \lim_{k \to \infty} \text{dist}(x_k, K_n) = 0 \). In [14] it was shown that \((K_n, L, c)\) is weakly infeasible if and only if there is no dual improving direction and there is a dual improving sequence (see Lemma 6 and Table 1 in [14]). But this is not a finite certificate of weak infeasibility.

A finite certificate for weak infeasibility can be obtained by using Ramana’s results on an extended Lagrangian dual for semidefinite programming [16]. Ramana’s dual has a number of key properties: it is written explicitly in terms of problem data, it has no duality gap and the optimal value is always attained when finite. With his dual, it was possible to develop an exact Farkas-type lemma for semidefinite programming. In Theorem 19 of [16], he constructed another SDFP \( \mathcal{RD}(K_n, L, c) \) for which the following holds without any regularity conditions:

\[
(K_n, L, c) \text{ is feasible if and only if } \mathcal{RD}(K_n, L, c) \text{ is infeasible.}
\]

Furthermore, the size of \( \mathcal{RD}(K_n, L, c) \) is bounded by a polynomial that depends only on the size of the system \((K_n, L, c)\). Based on this strong result, we obtain a finite certificate of weak infeasibility as in the following proposition:

\(^2\)In this case, the decision problem has either “yes” or “no” as possible answers, so it makes sense to talk about \( \text{coNP} \).
Proposition 2. We have the following:

i. \((K_n, L, c)\) is weakly infeasible \(\iff\) \(c \neq 0\), \(RD(K_n, L, c)\) and \(RD(K_n, L_{SI}, c_{SI})\) are feasible.

ii. The problem of deciding whether a given \((K_n, L, c)\) is weakly infeasible is in \(NP \cap coNP\) in the BSS model.

Proof. A feasible solution to \(RD(K_n, L, c)\) attests the infeasibility of \((K_n, L, c)\). As \(RD(K_n, L, c)\) has polynomial size, it is possible to check that a point is indeed a solution to it in polynomial time.

Note that a problem is weakly infeasible if and only if it is infeasible and is not strongly infeasible. Due to Proposition 1, we have that \((K_n, L, c)\) is not strongly infeasible if and only if \(c = 0\) or \(c \neq 0\) and \(RD(K_n, L_{SI}, c_{SI})\) is feasible. Hence, feasible solutions to \(RD(K_n, L, c)\) and \(RD(K_n, L_{SI}, c_{SI})\) can be used together as a certificate for weak infeasibility. Such a certificate can be checked in polynomial time, hence the problem is in \(NP\).

Now, \(c = 0\), a solution to \((K_n, L_{SI}, c_{SI})\) or to \((K_n, L, c)\) can be used to certify that a system is not weakly infeasible. This shows that deciding weak infeasibility is indeed in \(coNP\). \(\Box\)

The important point in the argument above is having both a certificate of infeasibility for the original system and a certificate of infeasibility for the system \((K_n, L_{SI}, c_{SI})\). Any method of obtaining finite certificates of infeasibility can be used in place of \(RD\), as long as it takes polynomial time to verify them. See the comments after Theorem 3.5 in Sturm [19] and also Theorem 7.5.1 of [7] for another certificate of infeasibility.

Klep and Schweighofer [5] also developed certificates for infeasibility and a hierarchy of infeasibility in which 0-infeasibility corresponds to strong infeasibility and \(k\)-infeasibility to weak infeasibility, when \(k > 0\). Liu and Pataki [6] also introduced an infeasibility certificate for semidefinite programming. They defined what is a reformulation of a feasibility system and showed that \((K_n, L, c)\) is infeasible if and only if it admits a reformulation that converts the systems to a special format, see Theorem 1 therein.

We mention a few more related works on weak infeasibility. The feasibility problem \((K_n, L, c)\) is weakly infeasible if and only if \(c \in cl(K_n + L) \setminus (K_n + L)\), where \(cl\) denotes the closure operator. Hence, a necessary condition for weak infeasibility is that \(K_n + L\) fails to be closed. This problem is closely related to closedness of the image of \(K_n\) by a linear map which is the problem analyzed in detail by Pataki [9].

Theorem 1.1 in [9] provides a necessary and sufficient condition for the failure of closedness of \(K_n + L\). Pataki’s result implies that there is some \(c \in S_n\) such that \((K_n, L, c)\) is weakly infeasible if and only if \(L^\perp \cap (cl\ dir (x, K_n) \setminus dir (x, K_n)) \neq 0\), where \(x\) belongs to the relative interior of \(L \cap K_n\) and \(dir (x, K_n)\) is the cone of feasible directions at \(x\). This tells us whether \(K_n\) and \(L\) can accommodate a weakly infeasible problem. If it is indeed possible, Corollary 3.1 of [9] shows how to find an appropriate \(c\).

Bonnans and Shapiro [2] also discussed generation of weakly infeasible semidefinite programming problems. As a by-product of the proof of Proposition 2.193 therein, it is shown how to construct weakly infeasible problems.

In [11], Pataki introduced the notion of well-behaved system. \((K_n, L, c)\) is said to be well-behaved if for all \(b \in S_n\), the optimal value of \((P)\) and of its dual are the same and the dual is attained whenever it is finite. A SDP which is not well-behaved is said to be badly-behaved. Pataki showed that badly-behaved SDPs can be put into a special shape, see Theorem 6 in [11]. Then, a necessary condition for weak infeasibility is that the homogenized system \((K_n, \tilde{L}, 0)\) be badly-behaved, where \(\tilde{L}\) is spanned by \(L\) and \(c\). See the comments before Section 4 in [11].

3 A decomposition result.

In this section, we develop a key decomposition result. Given an SDFP, we show how to construct a smaller dimensional SDFP which preserves most of the feasibility properties.

3.1 Preliminaries

First we introduce the notation. If \(C, D\) are subsets of some real space, we write \(dist(C, D) = \inf\{\|x - y\| \mid x \in C, y \in D\}\), where \(\|\cdot\|\) is the Euclidean norm or the Frobenius norm, in the case of subsets of \(S_n\). By
int(\mathcal{C})$ and $\text{ri}(\mathcal{C})$ we denote the interior and the relative interior of $\mathcal{C}$, respectively. We use $I_n$ to denote the $n \times n$ identity matrix. Given $(K_n, L, c)$ and a matrix $A \in K_n \cap L$ with rank $k$, we will call $A$ a hyper feasible direction of rank $k$. We remark that when $(K_n, L, c)$ is feasible, $A$ is also a recession direction of the feasible region.

Let $x$ be a $n \times n$ matrix, and $0 \leq k \leq n$. We denote by $\pi_k(x)$ the upper left $k \times k$ principal submatrix of $x$. For instance, if

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix},$$

then,

$$\pi_2(x) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$ 

We define the subproblem $\pi_k(K_n, L, c)$ of $(K_n, L, c)$ to be

$$\text{find } u \in \pi_k(L + c), \quad u \geq 0.$$ 

In other words, it is the feasibility problem $(\pi_k(K_n), \pi_k(L), \pi_k(c))$. We denote by $\pi_k(x)$, the lower right $(n - k) \times (n - k)$ principal submatrix. In the example above, we have $\pi_2(x) = 6$. In a similar manner, we write $\pi_k(K_n, L, c)$ for the feasibility problem $(\pi_k(K_n), \pi_k(L), \pi_k(c))$. We remark that $\pi_n(x) = \pi_0(x) = x$ and we define $\pi_0(x) = \pi_n(x) = 0$.

The proposition below summarizes the properties of the Schur Complement. For proofs, see Theorem 7.7.6 of [4].

**Proposition 3** (Schur Complement). Suppose $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ is a symmetric matrix divided in blocks in a way that $A$ is positive definite, then:

- $M$ is positive definite if and only if $C - B^T A^{-1} B$ is.
- $M$ is positive semidefinite if and only if $C - B^T A^{-1} B$ is.

The properties of a semidefinite program are not changed when a congruence transformation is applied, i.e, for any non-singular matrix $P$, we have that $(K_n, L, c)$ and $(K_n, PLP^T, PcP^T)$ have the same feasibility properties, where $PLP^T = \{PLP^T \mid l \in L\}$.

### 3.2 The main result

It will be convenient for now to collapse weak feasibility and weak infeasibility into a single status. We say that $(K_n, L, c)$ is in weak status if it is either weakly feasible or weakly infeasible. We start with the following basic observation. The proof is left to the readers.

**Proposition 4.** If $(K_n, L, c)$ is weakly infeasible, there exists a nonzero vector in $K_n \cap L$.

Now we present a key result in our paper. The following theorem says that if $(K_n, L, c)$ has a hyper feasible direction, then, we can construct another SDFP of smaller size whose feasibility status is almost identical to the original problem.

**Theorem 5.** Let $(K_n, L, c)$ be a SDFP, and consider a subproblem $\pi_k(K_n, L, c)$ for some $k > 0$. If the subproblem $\pi_k(K_n, L, c)$ admits an interior hyper feasible direction (i.e, $\text{int} \pi_k(K_n) \cap \pi_k(L) \neq \emptyset$) then:

1. $(K_n, L, c)$ is strongly feasible if and only if $\pi_k(K_n, L, c)$ is.
2. $(K_n, L, c)$ is strongly infeasible if and only if $\pi_k(K_n, L, c)$ is.
3. $(K_n, L, c)$ is in weak status if and only if $\pi_k(K_n, L, c)$ is.
Proof. Due to the assumption, there exists a $n \times n$ matrix
\[
x = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}
\]
where $A$ is a $k \times k$ positive definite matrix.

We now prove items 1 and 2. Item 3 will follow by elimination.

(1) $\Rightarrow$ If $y \in L + c$ is positive definite, all its principal submatrices are also positive definite. Therefore, $\pi_k(y)$ is positive definite.

(1) $\Leftarrow$ Suppose that $y \in L + c$ is such that $\pi_k(y) \in \text{int} K_{n-k}$. Then, we may write $y = \begin{pmatrix} F & E \\ E^T & G \end{pmatrix}$, where $G$ is $(n-k) \times (n-k)$ and positive definite. For large and positive $\alpha$, $F + \alpha A$ is positive definite and the Schur complement of $y + x\alpha$ is $G - E^T(F + \alpha A)^{-1}E$. Since $G$ is positive definite, it is clear that, increasing $\alpha$ if necessary, the Schur complement is also positive definite. For such an $\alpha$, $y + x\alpha \in (L + c) \cap \text{int} K_n$.

(2) $\Rightarrow$ Suppose $(K_n, L, c)$ strongly infeasible. Then there exists $s \in K_n$ such that $s \in L^\perp$ and $\langle s, c \rangle = -1$. As $x \in L$, we have $s \in K_n \cap \{x\}^\perp$. This means that $s$ can be written as $\begin{pmatrix} 0 & 0 \end{pmatrix}$, where $D$ belongs to $K_{n-k}$. It follows that $\pi_k(s) \in \pi_k(L)^\perp$ and $\langle \pi_k(s), \pi_k(c) \rangle = -1$. By item iii. of Proposition 1, $\pi_k(K_n, L, c)$ is strongly infeasible.

(2) $\Leftarrow$. Now, suppose $\pi_k(K_n, L, c)$ is strongly infeasible. Note that $\pi_k$ is a non-expansive map, i.e., $\|\pi_k(y) - \pi_k(z)\| \leq \|y - z\|$ holds. In particular, if $\inf_{y \in L + c, z \in K_n} \|\pi_k(y) - \pi_k(z)\| > 0$, then the same is true for $\inf_{y \in L + c, z \in K_n} \|y - z\|$. \hfill $\square$

3.3 Forward Procedure

Assume that $(K_n, L, c)$ admits a hyper feasible direction $\tilde{A}_1$ of rank $k_1$. Theorem 5 might not be directly applicable but after appropriate congruence transformation by a nonsingular matrix $P_1$, we have that $(K_n, P_1^TLP_1, P_1^TcP_1)$ admits a hyper feasible direction of the form
\[
A_1 = \begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & 0 \end{pmatrix} = P_1^T\tilde{A}_1P_1,
\]
where $\tilde{A}_1$ is a $k_1 \times k_1$ positive definite matrix. The feasibility status of $(K_n, L, c)$ and
\[
(K_{n-k_1}, \pi_{k_1}(P_1^TLP_1), \pi_{k_1}(P_1^TcP_1))
\]
are mostly the same in the sense that items 1 - 3 of Theorem 5 hold.

Now, suppose that $(K_{n-k_1}, \pi_{k_1}(P_1^TLP_1), \pi_{k_1}(P_1^TcP_1))$ admits a hyper feasible direction $\tilde{A}_2$ of rank $k_2$. Then, after appropriate congruence transformation by $\tilde{P}_2$, we obtain that
\[
(K_{n-k_1}, \tilde{P}_2^T\tilde{\pi}_{k_1}(P_1^TLP_1)\tilde{P}_2, \tilde{P}_2^T\tilde{\pi}_{k_1}(P_1^TcP_1)\tilde{P}_2)
\]
admits a hyper feasible direction of the form
\[
\begin{pmatrix} \tilde{A}_2 & 0 \\ 0 & 0 \end{pmatrix},
\]
where $\tilde{A}_2$ is $k_2 \times k_2$ positive definite matrix.

Now, the feasibility status of $(K_{n-k_1}, \pi_{k_1}(P_1^TLP_1), \pi_{k_1}(P_1^TcP_1))$ and
\[
(K_{n-k_1-k_2}, \pi_{k_2}(\tilde{P}_2^T\tilde{\pi}_{k_1}(P_1^TLP_1)\tilde{P}_2), \pi_{k_2}(\tilde{P}_2^T\tilde{\pi}_{k_1}(P_1^TLP_1)\tilde{P}_2))
\]
are mostly the same. Note that instead of applying a congruence transformation by $\tilde{P}_2$ to $(K_{n-k_1}, \pi_{k_1}(P_1^TLP_1), \pi_{k_1}(P_1^TcP_1))$, we can apply a congruence transformation by
\[
P_2 = \begin{pmatrix} I_{k_1} & 0 \\ 0 & \tilde{P}_2 \end{pmatrix}
\]
to the original problem \((K_n, P_1^TLP_1, P_1^TcP_1)\), i.e., we consider
\[
(K_n, P_2^T P_1^TLP_1P_2, P_2^T P_1^TcP_1P_2)
\]
Then the subproblem defined by the \((n - k_1) \times (n - k_1)\) lower right block matrix is precisely
\[
(K_{n-k_1}, \tilde{P}_2^T \bar{\pi}_{k_1}(P_1^TLP_1) \tilde{P}_2, \tilde{P}_2^T \bar{\pi}_{k_1}(P_1^TcP_1) \tilde{P}_2),
\]
and we may pick \(A_2 \in P_2^T P_1^T LP_1 P_2\) such that
\[
\bar{\pi}_{k_1 + k_2}(A_2) = \begin{pmatrix}
\tilde{A}_2 & 0 \\
0 & 0
\end{pmatrix}.
\]
Note that \(A_2\) has the following shape
\[
A_2 = \begin{pmatrix}
* & * & * \\
* & \tilde{A}_2 & 0 \\
* & 0 & 0
\end{pmatrix}.
\]
Generalizing the process outlined above, we obtain the following procedure, which we call “forward procedure”. The set of matrices \(\{A_1, \ldots, A_n\}\) obtained in this way will be called a hyper feasible partition. After each application of Theorem 5, the size of the matrices is reduced at least by one. This means that after at most \(n\) iterations, a subproblem with no nonzero hyper feasible directions is found. At this point, no further directions can be added and we will say that the partition is maximal.

We note that the problem of checking whether a SDFP \((\tilde{K}_n, \tilde{L}, \tilde{c})\) has a nonzero hyper-feasible direction lies in \(\text{NP} \cap \text{coNP}\), in the real computation model. In fact, by Gordan’s Theorem, \((\tilde{K}_n, \tilde{L}, \tilde{c})\) does not have a nonzero hyper-feasible direction if and only if \((K_n, \tilde{L}^\perp, 0)\) is strongly feasible.

[Procedure FP]

Input: \((K_n, L, c)\)

Output: a non-singular matrix \(P\), a sequence \(k_1, \ldots, k_m\) and a maximal hyper feasible partition \(\{A_1, \ldots, A_m\}\) contained in \(P^TLP\). The \(A_i\) are such that \(A_1 = \begin{pmatrix} \hat{A}_1 & 0 \\ 0 & 0 \end{pmatrix} \), \(A_2 = \begin{pmatrix} * & * & * \\ * & \hat{A}_2 & 0 \\ * & 0 & 0 \end{pmatrix} \), \(A_3 = \begin{pmatrix} * & * & * & * \\ * & * & A_3 & 0 \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} \)

and so forth, where \(\hat{A}_i\) is positive definite and lies in \(K_{k_i}\) for every \(i\).

1. Set \(i := 1, \tilde{L} := L, \tilde{c} := c K := K_n, P := I_n\).
2. Find (i) \(\tilde{A}_i \in \tilde{L} \cap K, \text{tr}(\tilde{A}_i) = 1\) or (ii) \(\tilde{B} \in \tilde{L}^\perp \cap \text{int}K, \text{tr}(\tilde{B}) = 1\). (Exactly one of (i) and (ii) is solvable.) If (ii) is solvable, then stop. (No nonzero hyper-feasible direction exists.)
3. Compute a non-singular \(\tilde{P}\) such that,
\[
\tilde{P}^T \tilde{A}_i \tilde{P} = \begin{pmatrix}
\hat{A}_i & 0 \\
0 & 0
\end{pmatrix}
\]

where \(\hat{A}_i\) is a positive definite matrix. Let \(k_i := \text{rank}(\hat{A}_i)\).

4. Compute \(M = \begin{pmatrix} I_{k_1 + \ldots + k_{i-1}} & 0 \\ 0 & \tilde{P} \end{pmatrix}\) and set \(P^T := M^T P^T\). (If \(i = 1\), take \(M = P\))

5. Let \(A_i\) be any matrix in \(P^TLP\) such that \(\bar{\pi}_{k_1 + \ldots + k_{i-1}}(A_i) = \tilde{P}^T \tilde{A}_i \tilde{P}\). For each \(1 \leq j < i\) exchange \(A_j\) for \(M^T A_j M\).

6. Set \(\tilde{L} := \bar{\pi}_{k_1}(P^T \tilde{L} \tilde{P}), \tilde{c} := \bar{\pi}_{k_1}(P^T \tilde{c} \tilde{P}), K := \bar{\pi}_{k_1}(K_n), i := i + 1\) and return to Step 2. (This step is just to pick the lower-right block after the congruence transformation.)

Proposition 6. Suppose that \((K_n, L, c)\) is such that there is a nonzero element in \(K_n \cap L\). Applying FP to \((K_n, L, c)\) we have that:
1. \((K_n, L, c)\) is strongly feasible if and only if \(\pi_{k_1+\ldots+k_m}(K_n, P^T L P, P^T cP)\) is.

2. \((K_n, L, c)\) is strongly infeasible if and only if \(\pi_{k_1+\ldots+k_m}(K_n, P^T L P, P^T cP)\) is.

3. \((K_n, L, c)\) is in weak status if and only if \(\pi_{k_1+\ldots+k_m}(K_n, P^T L P, P^T cP)\) is weakly feasible.

**Proof.** If \(m = 0\), then the proposition follows because \(\pi_0\) is equal to the identity map. In the case \(m = 1\), the result follows from Theorem 5.

Note that at the \(i\)-th iteration, if a direction \(A_i\) is found then, after applying the congruence transformation \(\hat{P}, \pi_k(K, P^T L P, \hat{P}^T cP)\) preserves feasibility properties in the sense of Theorem 1. Note that it is a SDFP over \(\mathbb{S}_{n-k_1-\ldots-k_i}\). Also, due to the way \(M\) is selected, we have that equation \(\pi_k(K, \hat{P}^T L P, \hat{P}^T cP) = \pi_{k_1+\ldots+k_i}(K_n, P^T L P, P^T cP)\) holds after Line 4 and before \(L\) and \(K\) are updated. This justifies items 1. and 2.

Consider the case where \((K_n, L, c)\) is in weak status. When \((K, \hat{L}, \hat{c})\) is weakly infeasible we can always find a new direction \(A_i\) and the size of problem decreases by a positive amount, so that \((K, \hat{L}, \hat{c})\) cannot be weakly infeasible for all iterations. The only other possibility is weak feasibility, which justifies item 3. \(\square\)

The matrices \(A_1, \ldots, A_m\) obtained through FP have the shape

\[
\begin{pmatrix}
\hat{A}_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
* & * & * & * \\
* & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
* & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
* & * & * & * \\
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & * & 0 & 0
\end{pmatrix},
\ldots
\]

where \(\hat{A}_1, \hat{A}_2, \hat{A}_3, \ldots\) are positive definite. The matrix \(A_i\) are referred to as sub-hyper feasible directions, since the \(\hat{A}_i\) are hyper feasible directions. The problem \(\pi_{k_1+\ldots+k_m}(K_n, P^T L P, P^T cP)\) will be referred to as the last subproblem of \((K_n, L, c)\).

**Example 7.** Let

\[
L + c = \begin{pmatrix}
t & u & v & 1 \\
v & z & z + 2 & v + 1 \\
u & z + 1 & u - 1 & s \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad t, u, v, s, z \in \mathbb{R}
\]

and let us apply FP to \((K_4, L, c)\). The first direction can be, for instance, \(A_1 = \begin{pmatrix}1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1\end{pmatrix}\). Then \(k_1 = 1\) and \(\hat{P}\) is the identity, at this step. At next iteration, we have \(K = K_3\) and \(\hat{L} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}\).

Then, \(\hat{A}_2\) can be taken as \(\begin{pmatrix}0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0\end{pmatrix}\) and \(k_2 = 1\). A possible choice of \(\hat{P}\) is \(\begin{pmatrix}1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1\end{pmatrix}\) and we can take \(A_2 = \begin{pmatrix}0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0\end{pmatrix}\). \(\hat{L}\) is then updated and it becomes \(\{(\frac{0}{z} \ 0) \ | \ z \in \mathbb{R}\}\). The procedure stops here, because 0 is the only positive semidefinite matrix in \(\hat{L}\).

Now, \(\pi_2(P^T (L+c) P)\) is \(\{(\frac{z+2}{z+1} \ 0) \ | \ z \in \mathbb{R}\}\), so \(\pi_2(K_4, P^T L P, P^T cP)\) is a weakly feasible system. Therefore, by Proposition 6, \((K_4, L, c)\) has weak status and is either weakly infeasible or weakly feasible. The 0 in the lower right corner of (2) forces \(u = 0\), \(z = -1\) and \(s = 0\), but this assignment produces a negative element in the diagonal. This tells us that \((K_4, L, c)\) is infeasible so it must be weakly infeasible.

**Corollary 8.** The matrices \(A_1, \ldots, A_m, P, \hat{B}\) as described in FP together with a finite weak feasibility certificate for \(\pi_{k_1+\ldots+k_m}(K_n, P^T L P, P^T cP)\) form a finite certificate that \((K_n, L, c)\) is in weak status. If no such a certificate exists, then either item 1 or item 3 of Proposition 1 holds. This shows that deciding whether a SDFP is in weak status is in \(\text{NP} \cap \text{coNP}\).

**Proof.** Follows directly from Proposition 6. \(\square\)

9
3.4 Maximum number of directions required to approach the positive semidefinite cone

According to Proposition 4, there is always a nonzero element in \( K_n \cap L \) when \((K_n, L, c)\) is weakly infeasible. Therefore, a natural question is, given a weakly infeasible \((K_n, L, c)\), whether it is always possible to select a point in \( x \in L + c \) and then a nonzero direction \( d \in K_n \cap L \) such that \( \lim_{t \to +\infty} \text{dist}(x + td, K_n) = 0 \) or not. We call weakly infeasible problems having this property directionally weakly infeasible (DWI). The simplest instance of DWI problem is

\[
\max \ 0 \ \text{s.t.} \ \left( \begin{array}{cc} t & 1 \\ 1 & 0 \end{array} \right) \in K_2, \ t \in \mathbb{R}.
\]

Unfortunately, not all weakly infeasible problems are DWI, as shown in the following instance.

**Example 9** (A weakly infeasible problem that is not directionally weakly infeasible). Let \((K_3, L, c)\) be such that \( L + c = \left\{ \left( \begin{array}{cc} 1 & s \\ s & 1 \end{array} \right) \ | \ t, s \in \mathbb{R} \right\} \) and let \( A_1 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \).

Applying Theorem 1 twice, we see that the problem is in weak status. Looking at its \( 2 \times 2 \) lower right block, we see this problem is infeasible and hence is weakly infeasible. But this problem is not DWI. If \((K_3, L, c)\) were DWI, we would have \( \lim_{t \to +\infty} \text{dist}(tA_1 + c', K_3) = 0 \), for some \( c' \in L + c \). To show this does not hold, we fix \( s \). Regardless of the value of \( t \geq 0 \), the minimum eigenvalue of the matrix is uniformly negative, since its \( 2 \times 2 \) lower right block is strongly infeasible.

Thus, a weakly infeasible problem is not DWI in general. If we let \( s \) sufficiently large in the example, then the minimum eigenvalue of the lower \( 2 \times 2 \) matrix gets very close to zero. This will make \((1,3)\) and \((3,1)\) elements large. But we can let \( t \) much larger than \( s \). Then, the minimum eigenvalue of the submatrix \((t \ s \ s \ t)\) is close to zero. Intuitively, this neutralize the effect of big off-diagonal elements, and we obtain points in \( L + c \) arbitrarily close to \( K_3 \), by taking \( s \) to be large and \( t \) to be much larger than \( s \).

Generalizing this intuition, in the following, we show that \( n - 1 \) directions are enough to approach the positive semidefinite cone. First we discuss how the hyper feasible partition \( \{A_1, \ldots, A_m\}\) of FP fits in the concept of tangent cone. We recall that for \( x \in K_n \) the cone of feasible directions is the set \( \text{dir}(x, K_n) = \{d \in \mathbb{R}_n \ | \ \exists \theta > 0 \ \text{s.t.} \ x + td \in K_n \} \). Then the tangent cone at \( x \) is the closure of \( \text{dir}(x, K_n) \) and is denoted by \( \text{tanCone}(x, K_n) \). It can be shown that if \( d \in \text{tanCone}(x, K_n) \) then \( \lim_{t \to +\infty} \text{dist}(tx + d, K_n) = 0 \).

We remark that if \( x = (D \ 0) \), where \( D \) is positive definite \( k \times k \) matrix, then \( \text{tanCone}(x, K_n) \) consists of all symmetric matrices \((\begin{array}{cc} A & E \\ E & \end{array})\), where \( A \) denotes arbitrary entries and \( E \) is a positive semidefinite \((n-k) \times (n-k)\) matrix. See [8] for more details.

The output \( \{A_1, \ldots, A_m\}\) of FP is such that \( A_2 \in \text{tanCone}(A_1, K_n) \). This is clear from the shape of \( A_1 \) and \( A_2 \), and from a simple argument using the Schur Complement. Now, \( A_3 \) is such that \( \pi_{k_1+k_2}(A_3) \) is positive semidefinite. We have \( A_2 = \begin{pmatrix} * & * & * \\ * & A_2 & 0 \\ * & 0 & 0 \end{pmatrix} \end{pmatrix} A_3 = \begin{pmatrix} * & * & * \\ * & A_3 & 0 \\ * & 0 & 0 \end{pmatrix} \). Then \( \begin{pmatrix} * & * & * \\ * & A_3 & 0 \\ * & 0 & 0 \end{pmatrix} \) \( \in \text{tanCone}(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_n-k_1 \end{pmatrix}, K_n-k_1) \), i.e., \( \pi_{k_1}(A_3) \in \text{tanCone}(\pi_{k_1}(A_2), K_n-k_1) \). Denote \( k_1 + \ldots + k_i \) by \( N_i \) and set \( N_0 = 0 \). Then, for \( i > 2 \), we have:

\[
\pi_{N_{i-2}}(A_i) \in \text{tanCone}(\pi_{N_{i-2}}(A_{i-1}), K_n-N_{i-2}).
\]

Moreover, if the last subproblem \( \pi_{N_m}(K_n, L, c) \) has a feasible solution, we can pick some \( c' \) such that \( \pi_{N_m}(c') \) is positive semidefinite. Then \( \pi_{N_{m-1}}(c') \in \text{tanCone}(\pi_{N_{m-1}}(A_m), K_n-N_{m-1}) \). Given \( \epsilon > 0 \), by picking \( \alpha_m > 0 \) sufficiently large we have \( \text{dist}(\pi_{N_{m-1}}(c' + \alpha_m A_m), K_n-N_{m-1}) < \epsilon \). Now, \( \pi_{N_{m-2}}(x + \alpha_m A_m) \) does not necessarily lie on the tangent cone of \( \pi_{N_{m-2}}(A_{m-1}) \) at \( K_n-N_{m-2} \), but still it is possible to pick \( \alpha_{m-1} > 0 \) such that

\[
\text{dist}(\pi_{N_{m-2}}(c' + \alpha_m A_m, K_n-N_{m-2}) < 2\epsilon.
\]

In order to show this, let \( h \in K_n-N_{m-1} \) be such that

\[
\|\pi_{N_{m-1}}(c' + \alpha_m A_m) - h\| = \text{dist}(\pi_{N_{m-1}}(c' + \alpha_m A_m), K_n-N_{m-1}).
\]
Now, define $\tilde{h}$ to be the matrix $\pi_{N_{m-2}}(c' + \alpha_m A_m)$, except that the lower right $(n-k_m) \times (n-k_m)$ block is replaced by $h$. It follows readily that $\tilde{h}$ lies on the tangent cone of $\pi_{N_{m-2}}(A_{m-1})$. Then, we may pick $\alpha_{m-1} > 0$ sufficiently large such that $\text{dist}(\pi_{N_{m-2}}(\alpha_m A_m) + h, K_{n-N_{m-2}}) < \epsilon$. Let $y_1 = \pi_{N_{m-2}}(c' + \alpha_m A_m)$, $y_2 = \pi_{N_{m-2}}(\alpha_{m-1} A_{m-1})$. We then have the following implications:

$$\text{dist}(y_1 + y_2, K_{n-N_{m-2}}) \leq \text{dist}(y_1 - \tilde{h}, K_{n-N_{m-2}}) + \text{dist}(y_2 + \tilde{h}, K_{n-N_{m-2}}) \leq \|\pi_{N_{m-1}}(c' + \alpha_m A_m) - h\| + \epsilon \leq 2\epsilon.$$ 

If we continue in this way, it becomes clear that $\alpha_1, \ldots, \alpha_m$ can be selected such that $\text{dist}(c' + \alpha_m A_m + \alpha_{m-1} A_{m-1} + \ldots + \alpha_1 A_1, K_n) < m\epsilon$. This shows how the directions $\{A_1, \ldots, A_m\}$ can be used to construct points that are arbitrarily close to $K_n$, when the last subproblem is feasible. This leads to the next theorem.

**Theorem 10.** If $(K_n, L, c)$ is weakly infeasible then there exists an affine space of dimension at most $n - 1$ such that $L' + c' \subseteq L + c$ and $(K_n, L', c')$ is weakly infeasible.

**Proof.** The construction above shows that if $L'$ is the space spanned by $\{A_1, \ldots, A_m\}$ and $c'$ is taken as above, then $(K_n, L', c')$ is weakly infeasible. As $(K_n, L, c)$ is weakly infeasible, we have $m > 0$. We also have $k_1 + \ldots + k_m \leq n$, which implies $m \leq n$. Notice that $\pi_m(K_n, P^TLP, P^TCP)$ is strongly feasible, because it is equal to the system $\{(0), \{0\}, 0\}$. Therefore $k_1 + \ldots + k_m < n$, which forces $m < n$. 

4 Backward Procedure

In this section, we discuss a “backward procedure” for distinguishing the 4 different feasibility statuses. The main difficulty is when the problem is in weak status. In that case, due to Proposition 6, the last subproblem is weakly feasible. This offers the opportunity to shrink both the last subproblem and the whole problem, as discussed in our next theorem.

**Theorem 11.** Let $(K_n, L, c)$ be a given SDFP, satisfying the following assumptions:

1. for some $k > 0$, $\pi_k(K_n, L, c)$ is weakly feasible.
2. for some $l$ such that $0 \leq l < n - k$, the face $F = \{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} | A \in K_l\}$ contains the feasible region of $\pi_k(K_n, L, c)$.

$$\begin{pmatrix} * & * & \pi_k(L + c) \\ * & * \end{pmatrix} = \begin{pmatrix} * & * & A \\ * & 0 & 0 \end{pmatrix},$$

Furthermore, let $E$ be the set of $(k + l) \times (k + l)$ upper left principal submatrices in $S_n$, i.e., $E = \{(B \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) | B \in S_{k+l}\}$, and define $\bar{L}$ and $\bar{c}$ as the vector subspace and a vector such that $E \cap (L + c)$ is equal to $\bar{L} + \bar{c}$, i.e.,

$$E \cap (L + c) = \bar{L} + \bar{c} = \begin{pmatrix} \pi_{k+l}(\bar{L} + \bar{c}) & 0 \\ 0 & 0 \end{pmatrix}.$$ \hspace{1cm} (3)

($\bar{L} + \bar{c}$ is an affine subspace of $L + c$ where all but the upper left block is set to zero.)

Then, the following holds: If (3) is empty, $(K_n, L, c)$ is infeasible. Otherwise, $(K_n, L, c)$ is feasible if and only if the SDFP subproblem $\pi_{k+l}(K_n, \bar{L}, \bar{c})$ is feasible.
Proof. Due to assumption 1, \( \pi_k(K_n, L, c) \) is weakly feasible, so if \( x \in L + c \) and \( \pi_k(x) \) is positive semidefinite then \( x \) has the format \((\ast \ 0)\), where \( C \) is a \((n - k) \times (n - k)\) positive semidefinite definite and \( \ast \) denotes arbitrary entries. Now, due to Assumption 2, \( C \) itself has the format \((\ast \ 0 \ 0 \ 0)\), where \( C \) is a \( l \times l \) positive semidefinite matrix. So, actually, \( x \) has the format \((\ast \ 0 \ 0 \ 0)\).

In order for \( x \) itself to be positive semidefinite, the elements in the upper right and lower left must be 0. In other words, \( x \) must have the format \((\ast \ 0 \ 0 \ 0)\), where \((\ast \ 0)\) is a \((k + l) \times (k + l)\) positive semidefinite matrix. Therefore, if \( E \cap (L + c) = \emptyset \) there is no way \((K_n, L, c)\) could be feasible.

If \( E \cap (L + c) \) is not empty, since \( \pi_{k+1}(x) = (\ast \ 0) \), it is clear that the feasibility of \((K_n, L, c)\) is equivalent to the feasibility of \( \pi_{k+1}(K_n, L, c) \).

We remark that whenever \((K_n, L, c)\) satisfies Assumption 1, it is possible to apply a congruence transformation to \((K_n, L, c)\) in order to meet Assumption 2. Using Theorem 11.3 of [18], the weak feasibility of \( \pi_k(K_n, L, c) \) implies the existence of \( w \neq 0 \) such that \( \langle w, \pi_k(l + c) \rangle \leq \langle w, x \rangle \), for every \( l \in L \) and \( x \in K_n - k \). The only way this inequality can hold is if \( w \in K_n - k \cap \pi_k(L)^{\perp} \) and \( \langle w, \pi_k(c) \rangle \leq 0 \). As \( \pi_k(K_n, L, c) \) is not strongly infeasible, we have \( \langle w, \pi_k(c) \rangle = 0 \). Changing \( L + c \) and \( w \) by a congruence transformation if necessary, we may assume that \( w = (0 \ 0 \ w) \), where \( \bar{w} \) is a \((n - k - l) \times (n - k - l)\) positive definite matrix and \( l < n - k \). Then, it is clear that Assumption 2 holds for the transformed problem.

The search for a \( w \) as above is essentially one step of a facial reduction algorithm [3, 10, 21]. Each iteration of a facial reduction algorithm aims to find a proper face of \( K_n \) that still contains the feasible region. Usually, however, the search is done on the whole problem. Let \( w' = (0 \ 0 \ w) \), where \( w' \in K_n \), then \( w' \in K_n \cap L^\perp \cap \{c\}^\perp \). Which means that \( w' \) can be used to perform a step of facial reduction on the whole problem. In particular, if \( x \) is a feasible point, since \( \langle x, w \rangle = 0 \), it must be true that \( x = (0 \ 0 \ 0) \), where \( D \) is a positive semidefinite \((k + l) \times (k + l)\) matrix. The idea is that the knowledge of a weakly feasible subproblem makes it possible to confine the search to a smaller subproblem and still find a smaller face of \( K_n \) that contains the feasible region of the original problem.

If we apply Theorems 5 and 11 repeatedly, we obtain a facial reduction-like procedure which is able to determine the feasibility status of a given \((K_n, L, c)\) as shown below.

[Procedure BP]

Step 1. Apply FP to \((K_n, L, c)\). If the last subproblem \( \pi_{k_1 + \ldots + k_m}(K_n, P^T L P, P^T c P) \) is strongly infeasible, then \((K_n, L, c)\) is also strongly infeasible. If \( \pi_{k_1 + \ldots + k_m}(K_n, P^T L P, P^T c P) \) is strongly feasible, then \((K_n, L, c)\) is also strongly feasible. In both cases we stop the procedure. Otherwise set \( i = 0, F_0 = K_n, L_0 = L, c_0 = c = c \).

Step 2. If we reach this step, \( \pi_{k_1 + \ldots + k_m}(F_i, P^T L P, P^T c P) \) is weakly feasible, i.e., \((F_i, L_i, c_i)\) is in weak status. Applying a congruence transformation to \((F_i, P^T L P, P^T c P)\), if necessary, both assumptions of Theorem 11 can be met. Let \( K_{k+i}, L, \bar{c} \) and \( E \) be as in Theorem 11. If \( E \cap P^T(L_i + c_i)P \) is empty, we stop and declare \((K_n, L, c)\) to be weakly infeasible. Otherwise, we obtain \( L + \bar{c} \) such that \( E \cap P^T(L_i + c_i)P = L + \bar{c} \) and a projection \( \pi_{k+i} \).

Step 3. Apply FP to \( \pi_{k+i}(F_i, L, \bar{c}) \) and obtain a new projection \( \pi_{k_1 + \ldots + k_m} \). If \( \pi_{k_1 + \ldots + k_m}(K_{k+i}, \pi_{k+i}(L), \pi_{k+i}(c)) \) is strongly infeasible, then \((K_n, L, c)\) is weakly feasible. If \( \pi_{k_1 + \ldots + k_m}(K_{k+i}, \pi_{k+i}(L), \pi_{k+i}(c)) \) is strongly infeasible, then \((K_n, L, c)\) is weakly infeasible. In both cases, we end the procedure. Otherwise, set \( F_{i+1} := K_{k+i}, L_{i+1} := \pi_{k+i}(L), c_{i+1} := \pi_{k+i}(c), i := i + 1 \) and return to Step 2.

Remark: The procedure terminates in at most \( n \) iterations, because the size of the problem is reduced at least by one for each iteration.

Example 12. Let \( L \) and \( c \) be as in Example 7 and let us apply BP to \((K_4, L, c)\). At Step 1, we apply to FP and we obtain \( \pi_2(P^T(L + c)P) = \{z \cap z^2 \cap z^3 \} \mid z \in \mathbb{R} \}. And \( \pi_2(K_4, P^T L P, P^T c P) \) is weakly feasible, so we move on to Step 2. The feasible region of \( \pi_2(K_4, P^T L P, P^T c P) \) consists of a single matrix which is \((0 \ 0)\).
We are under the conditions of Theorem 11 and

\[ E \cap P^T(L + c)P = \left\{ \begin{pmatrix} t & 1 & v & 0 \\ 1 & -1 & v + 1 & 0 \\ v & v + 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid t, v \in \mathbb{R} \right\}. \]

Then \( \pi_3(\tilde{L} + \tilde{c}) = \left\{ \begin{pmatrix} \frac{1}{v} & 1 & v + 1 & 0 \\ v - 1 & 0 & 1 & 0 \\ v & v + 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid t, v \in \mathbb{R} \right\}. \) Applying \( \mathbf{FP} \) to \((K_3, \pi_M(\tilde{L}), \pi_M(\tilde{c}))\) we obtain as output \( P = I_3, m = 1, k_1 = 1 \) and \( A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \) Then \( \pi_1(\pi_3(\tilde{L} + \tilde{c})) = \left\{ \begin{pmatrix} -1 & v + 1 \\ v & v + 1 \end{pmatrix} \mid v \in \mathbb{R} \right\}. \) The \(-1\) in the upper left entry ensures that the system \((K_2, \pi_1(\pi_3(\tilde{L}))), \pi_1(\pi_3(\tilde{L})))\) is strongly infeasible, which shows that \((K_4, L, c)\) is weakly infeasible itself.

4.1 Complexity aspects of BP

Let us discuss briefly certificates and complexity issues regarding \( \mathbf{BP} \). Both \( \mathbf{BP} \) and \( \mathbf{FP} \) can be thought as procedures that invoke several oracles. For instance, we can consider that a nonzero hyper feasible direction, as required in \( \mathbf{FP} \), is obtained by querying an oracle. According to the recipe explained in the second paragraph of Section 2.1, we can show that the problem of deciding the feasibility status of \((K_n, L, c)\) has a finite certificate and that \( \mathbf{BP} \) acts as as verifier procedure. All we have to do is argue that all the computations required by \( \mathbf{BP} \) can be checked in polynomial time.

First note that all computations done by \( \mathbf{FP} \) can be checked either by the certificates discussed in Proposition 1 or by Gordan’s Theorem. The same is true for Steps 1 and 3 of \( \mathbf{BP} \). The only part of \( \mathbf{BP} \) that needs further analysis is when Theorem 11 is invoked at Step 2, where we need to check that assumption 2 of Theorem 11 holds. However, we can use as certificate a nonzero element \( w \) satisfying \( w \in F_i \cap (P^T L_i P)^+ \cap \{P^T c_i P\}^+ \), as in the discussion that follows the proof of Theorem 11 and, if necessary, a non-singular matrix which puts the problem in the correct shape.

This provides an alternative proof of the fact that for each different feasibility status, the problem of deciding whether \((K_n, L, c)\) has that status is in \( \mathbf{NP} \cap \mathbf{coNP} \) in the BBS model of real computation.

5 Conclusion

In this article we presented an analysis of weakly infeasible problems via two procedures: \( \mathbf{FP} \) and \( \mathbf{BP} \). The procedure \( \mathbf{FP} \) produces as an output a finite set of directions and for weakly infeasible problem, they can be used to construct \( L + c \) arbitrarily close to \( K_n \). The procedure \( \mathbf{BP} \) uses \( \mathbf{FP} \) and is able to distinguish between the four feasibility statuses. The computations involved in both procedures might be hard, but they are verifiable in polynomial time, in the BSS model. Extension of our analysis to blockwise SDPs and to other classes of conic linear programs is an interesting topic for future research.

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