Joint Sensor Node Selection and State Estimation for Nonlinear Networks and Systems

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Abstract

State estimation and sensor selection problems for nonlinear networks and systems are ubiquitous problems that are important for control, monitoring, analysis, and prediction of a large number of engineered and physical systems. In this manuscript, we propose three algorithms for joint state estimation and sensor selection for nonlinear network dynamics. The motivation for developing these methods stems from the following facts. Sensor selection problems are extensively studied for linear networks. However, less attention has been dedicated to networks with nonlinear dynamics. Furthermore, widely used sensor selection methods relying on structural (graph-based) observability approaches might produce far from optimal results when applied to nonlinear network dynamics. Finally, state estimation and sensor selection problems are often treated separately, and this might decrease the overall estimation performance. Our main idea is to incorporate the sensor selection problem into an initial state estimation problem. The resulting mixed-integer nonlinear optimization problem is approximately solved using three hybrid algorithms. Good numerical performance of our approach is demonstrated by testing the algorithms on prototypical Duffing oscillator and associative memory networks. The developed codes are available online.

Index Terms

state and parameter estimation, observability, sensor selection

I. INTRODUCTION

In a large variety of engineering and scientific fields, we are often faced with the problem of estimating states of networks with nonlinear dynamics. For example, this problem is crucial for identification, estimation, monitoring, and control of power systems, communication, traffic, biochemical, biophysical, combustion reaction, and ecological networks, as well as for other systems with nonlinear dynamics [1]-[10].

Generally speaking, state estimation for nonlinear networks consists of two steps. In the first step, which is often referred to as the sensor node selection step, we are interested in selecting a subset of network nodes whose states or output variables should be observed such that from this limited information we can accurately reconstruct the global

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network state. In the second step, which is often referred to as the observer design step or state reconstruction step, we are interested in designing an algorithm (observer) to reconstruct the global network state using the information collected from the sensor nodes assigned in the first step. The estimated states might be used for system monitoring, prediction, or control. Similarly to the design of state estimators, the design of network controllers consists of control node selection and control action design. These problems are dual to the above described estimation steps, for more details see for example [8], [11]–[14].

There is a large body of literature on sensor/control node selection for network dynamics. A detailed literature survey of all methods and approaches goes well beyond the scope and length limits of this manuscript. Consequently, in the sequel, we mention general lines of research and recent contributions. Widely and arguably most popular approaches for sensor (control) node selection are relying on graph-based methods stemming from structured control theory [15]. These approaches have been used in [2], [3] and in a large number of follow-up contributions, to devise methods for control and sensor node selection for complex networks. For example, the main idea of the approach proposed in [3], is to select sensor nodes by searching for strongly connected components in a graph describing node connections. This is a relatively simple and computationally inexpensive approach that can provide us with an initial guess of the location of sensor nodes. However, as correctly observed and numerically investigated by various authors [11]–[14], such graph-based approaches might produce far from optimal solutions, or sensor nodes selected using these graph-based approaches might lead to numerically ill-conditioned estimation problems [8].

Another line of research is to select sensor (control) nodes for linear networks by optimizing suitable observability (controllability) performance criteria depending on system Gramians, see for example [11], [12] and follow-up approaches. Such methods are mainly designed for linear network dynamics and might not be applicable to the nonlinear case. Empirical Gramians [16] of nonlinear systems have been used in [17], [18] to select actuators (control nodes) and sensor nodes for nonlinear networks and systems. The main limitation of these approaches is that the computation of empirical Gramians is a computationally challenging task even for small size networks. In [8], we have developed a sensor selection approach for nonlinear networks. This approach is applicable to a broad class of nonlinear systems, including chemical reaction networks with stiff dynamics. However, the main prerequisite for applying this method and empirical Gramian approaches is a priori information on the initial state for which observability/controllability metrics are defined. In practice, we can rarely obtain accurate a priori information on the network initial state. An approach for sensor selection of nonlinear networks has been presented in [19]. This method assumes a specific form of nonlinear dynamics and its applicability to a broader class of nonlinear networks requires further theoretical and numerical investigations. Sensor and actuator placement problems for linear systems and systems described by partial differential equations have been considered in [20]–[25]. The applicability of these approaches to nonlinear networks requires further adaptation and investigations.

Finally, to the best of our knowledge, sensor selection, and state estimation problems are usually treated separately. Our numerical experience and results, which are partly reported in [8], show that in some cases, this approach might lead to far from optimal estimation performance.

Motivated by the above-described limitations of the existing approaches, in this manuscript, we propose three algorithms for joint state estimation and sensor selection for nonlinear network dynamics. We develop our approaches
by incorporating the sensor selection problem into an initial state estimation problem. The resulting Mixed-Integer Nonlinear Optimization (MINO) problem is approximately solved using three hybrid algorithms. In the initial phase, all three algorithms solve a relaxed MINO problem obtained by replacing the integer constraints by constraints involving continuous variables. The first algorithm uses the solution of the relaxed problem as an initial guess for the Mesh Adaptive Direct Search (MADS) algorithm [26] for solving the MINO problem. This is a derivative-free optimization method requiring only a procedure (function) to compute the cost function. The second and third algorithms use the solution of the relaxed problem to formulate and solve Mixed-Integer Linear Programming (MILP) problems. These MILP problems are solved by using a hybrid branch and bound method. The second and third algorithms are partly inspired by mixed-integer optimal control solution approaches reported in [27]–[29]. All three approaches produce optimal sensor locations and an estimate of the global network state.

Although in the general case the formulated MINO problem is NP-hard and non-convex, our extensive numerical experiments show that developed approximation algorithms can produce relatively good results. Good numerical performance of our methods is demonstrated by testing them on Duffing oscillator networks that are prototypical models of a number of dynamical systems, as well as on associative memory networks representing simplified memory models. We compare our methods with exhaustive search and random selection of sensor nodes. The developed codes are available online [30]. These codes can be effortlessly adapted to other network models.

Compared to other methods for solving MINO problems [31], our approach is relatively easy to implement, does not rely upon convexification procedures that are usually case dependent, and is applicable to a broad class of nonlinear network dynamics.

This manuscript is organized as follows. In Section II we present the problem formulation. In Section III we present the sensor selection algorithms. In Section IV we derive expressions for the derivatives of cost functions used in this paper and present some generalizations of the developed approaches. Numerical results and conclusions are presented in Section V and Section VI respectively.

II. PROBLEM FORMULATION

In this section, we incorporate the sensor selection problem into an open-loop initial state estimation problem. This enables us to formulate the sensor selection and initial state estimation problems as a single MINO problem. Three algorithms for approximating the solution of this problem are stated in Section III. First, we explain the used notation.

A. Notation

Let $w_k \in \mathbb{R}^n$ denote an arbitrary $n$-dimensional discrete-time vector at discrete-time instant $k \in \mathbb{Z}_0^+$. The notation $w_{0:S}$ denotes a lifted column vector $w_{0:S} := \text{col}(w_0, w_1, \ldots, w_S)$, $w_{0:S} \in \mathbb{R}^{(S+1)n}$, where $S \in \mathbb{Z}_0^+$. The notation $\text{col}(w_0, w_1, \ldots, w_S)$ denotes an operator defined by $\text{col}(\cdot) = [w_0^T, w_1^T, \ldots, w_S^T]^T$. The notation $g[w]$ denotes a vector function $g$ of a vector argument $w$. The notation $\text{diag}(w) \in \mathbb{R}^{n \times n}$ denotes a diagonal matrix with the entries of the vector $w$ on its main diagonal. The notation $I_S$ denotes an $S \times S$ identity matrix.
B. Problem Statement

We consider a nonlinear state-space model

\[ \dot{x} = f(x), \]
\[ y = Cx, \]

where \( x \in \mathbb{R}^{Nn} \) is a global network state, \( f(\cdot) : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn} \) is a nonlinear vector function describing the system dynamics, \( y \in \mathbb{R}^{M} \) is an observed global output vector, and \( C \in \mathbb{R}^{M \times Nn} \) is an output matrix. We assume that the network consists of \( N \) nodes, where a local state of each node is \( n \)-dimensional. Consequently, we have

\[ x = \text{col}(x^{(1)}, x^{(2)}, \ldots, x^{(N)}), \]

where \( x^{(i)} \in \mathbb{R}^{n}, i = 1, \ldots, N \), is the local state of the \( i \)-th node. We assume that output vector \( y \) consists of \( M \) entries \( y_j \in \mathbb{R} \). A node is a sensor node if its output is being observed. The entry \( y^{(j)} \) is a local observation at a prescribed sensor node. Without loss of generality, we assume that only a scalar variable is observed at every node. The integer \( M \) denotes the number of sensor nodes. Next, let

\[ z = \text{col}(y^{(1)}, y^{(2)}, \ldots, y^{(N)}), \]

\[ z \in \mathbb{R}^{N}, \]

denote an output vector obtained assuming that all network nodes are sensor nodes.

Since in practice time series of sensor node outputs are collected at discrete-time instants, we assume that the sensor node outputs are available at discrete time samples \( t = kh \), where \( k = 0, 1, 2, \ldots \), is a discrete time instant, and \( h \) is a discretization constant. Furthermore, let \( y_k := y[kh], z_k := z[kh], \) and \( x_k := x[kh] \).

The locations of the sensor nodes are encoded in the structure of the matrix \( C \). In addition to the properties of the function \( f(\cdot) \), the structure of the matrix \( C \) determines the observability properties of the system.

Knowing the model of the system dynamics, given by the equation (1), our goal is to determine an optimal (sparsity) structure of the matrix \( C \), under the condition that sensors can be placed on a fixed number of nodes that is usually smaller than \( N \). At the same time, we want to ensure that the selected matrix \( C \) will produce accurate results when used to design an estimator or an observer.

More precisely, we consider the sensor selection and initial state estimation problems under the following scenarios. In the first scenario, in the sensor selection step, having the model of the network dynamics (1), we can simulate this model for a user-defined initial condition. In this way, we can obtain the state trajectory \( \{x_k \mid k = 0, 1, 2, \ldots\} \). Then, using this state trajectory, and taking into account physical constraints on how local information is collected by a sensor at a local node, we can construct the sequence \( \{z_k \mid k = 0, 1, 2, \ldots\} \). This procedure can be repeated many times, for different user-defined initial conditions, to obtain a number of sequences of \( z_k \). Then, using the sequence \( \{z_k\} \), and the model (1), the goal is to select the optimal structure of the matrix \( C \) such that the initial state of the real system can be accurately reconstructed. After we have determined the sensor locations, we can place sensors on predetermined locations, and these sensors can be used to collect the output sequence \( \{y_k \mid k = 0, 1, 2, \ldots\} \) from the real system. Then, in our final step, the goal is to estimate the unknown initial state of the system.

In the second scenario, instead of simulating the system response (for a selected initial condition), we can place sensors on all nodes of the real system to obtain the sequence \( \{z_k\} \). Then, using this sequence collected from the real system, we can find the optimal sensor locations under the constraint that we can only use \( M \) sensors. After completing the sensor selection procedure, all the sensors except \( M \) selected sensors should be removed from the
network, and the initial state should be estimated. This scenario is only feasible for small or medium-sized networks due to practical and economical constraints.

It should be emphasized that in the first scenario, the user-selected initial condition necessary to obtain the data sequence \( \{z_k\} \) is usually different from unknown initial conditions of the real physical system. There are two methods to minimize this uncertainty. The first method is to choose the user-selected initial conditions to be in the expected range of the initial conditions of the system. The second approach is discussed in Section V-B.

We can formally define the joint problem of selecting the sensor nodes and estimating the initial states as follows.

**Problem 1 (Joint Sensor Selection and Initial State Estimation):** Given the maximal number of sensor nodes, denoted by \( M_{\text{max}} \), \( M_{\text{max}} \leq N \), and given the length of the observation horizon, denoted by \( L \), determine:

1) Locations of sensor nodes using the sequence of data samples \( \{z_0, z_1, \ldots, z_L\} \). These data samples can be obtained by using the two approaches. In the first approach, the data samples are obtained by simulating the system (1) for the user-selected initial condition. In the second approach, the samples are collected by placing sensors on all nodes of the real system. The second approach is practically feasible for small or possibly medium-sized networks.

2) An estimate of the real system initial state \( x_0 \) determined from the sequence of output data samples \( \{y_0, y_1, \ldots, y_L\} \) collected from the real system by the assigned sensor nodes.

Similarly to the approach we pursued in [8], to formulate the sensor selection problem, we represent the continuous-time dynamics (1) in the discrete-time domain. This is a widely used approach for developing estimators and observers, see for example [32]–[34]. The simplest discretization approach is based on the Forward Euler (FE) method [35]. The FE discretized dynamics takes the following form:

\[
x_k = x_{k-1} + h f[x_{k-1}],
\]

where \( x_k := x[kh] \). The FE dynamics (3) approximates relatively well the continuous-time dynamics when \( h \) is small and when the dynamics is not stiff. Stiff network dynamics are characterized by time constants of local nodes that significantly differ in magnitude. A large number of systems such as reaction-diffusion systems, chemical reaction networks, and other systems coupling multi-physics phenomena have stiff dynamics. To allow for larger values of the discretization constant and to be able to accurately represent stiff network dynamics, we employ the Trapezoidal Implicit (TI) discretization method [35]. By applying this method to the dynamics (1), we obtain

\[
x_k = x_{k-1} + 0.5h (f[x_k] + f[x_{k-1}]).
\]

Another option is to use the implicit Runge-Kutta method [8], [35], however, for brevity, we do not use such a method in this manuscript. To solve the sensor selection method we have to simulate the system dynamics many times. The main disadvantage of the TI method over the FE method is that in every simulation step \( k \), we need to solve the system of nonlinear equations (4) for \( x_k \). This generally results in \( O(N^3 n^3) \) computational complexity. In sharp contrast, in every step \( k \), the computational complexity of the FE method is \( O(N n) \). This implies that...
the methods presented in the next section will have a larger computational complexity for the TI dynamics. For notational brevity, both FE and TI discretized dynamics are denoted by the following equation

\[ x_k = g[x_k, x_{k-1}] . \] (5)

Next, we introduce a parametrized output equation relating states and \( z_k \):

\[ z_k = C_\theta \theta \otimes x_k , \] (6)

where \( \theta \in \{0,1\}^N \) is a binary parametrization vector and \( C_\theta \theta \otimes \) is a parametrized output matrix. In this paper, depending on the structure of a system used for performing numerical experiments, we assume two parmetrization forms of \( C_\theta \theta \otimes \). For networks for which \( n = 1 \), we assume \( C_\theta \theta \otimes = \text{diag}(\theta) \). For \( n > 1 \), \( C_\theta \theta \otimes \) is a block diagonal matrix with the \( i \)-th block equal to \( C_i = [\theta_i 0 \ldots 0] \in \mathbb{R}^{1 \times n} \), where \( \theta_i \) is the \( i \)-th entry of \( \theta \).

Our idea for developing the sensor selection method originates from the observability definition for discrete-time systems \[8\], \[36\]. Namely, the observed output sequence \( y_{0:L} \) and the initial state \( x_0 \) are related by a nonlinear function \( w \)

\[ y_{0:L} = w[x_0] , \] (7)

where the notation \( y_{0:L} \) is defined in Section II-A and the nonlinear function \( w[x_0] : \mathbb{R}^{Nn} \rightarrow \mathbb{R}^{L+1} \) is defined by

\[ w[x_0] := (I_{L+1} \otimes C)x_{0:L} = (I_{L+1} \otimes C)x_{0:L}[x_0] , \] (8)

where \( I_{L+1} \) is the notation for an identity matrix defined in Section II-A and \( \otimes \) denotes the Kronecker product. The state sequence \( x_{0:L}[x_0] \) is only a function of the initial state \( x_0 \) since from (5) we obtain telescopic equalities

\[ x_1 = g[x_1, x_0] , \] (9)
\[ x_2 = g[x_2, x_1] , \] (10)
\[ \ldots \]
\[ x_L = g[x_L, x_{L-1}] . \] (11)

Consequently, for known \( x_0 \) we can compute \( x_k, k = 1, 2, \ldots, L \), by solving the nonlinear equations (for the TI dynamics) or by simply propagating the telescopic equations (for the FE dynamics). Let us recall the uniform observability definition. A discrete-time dynamical system is uniformly observable on a set if there exists \( L > 0 \) such that \( w[x_0] \) is an injective function with respect to initial state \( x_0 \) \[36\]. From the practical point of view, this means that the system is observable if we can uniquely solve the system of nonlinear equations (7) for \( x_0 \). This problem can be reformulated as a nonlinear optimization problem

\[ \min_{x_0} \| y_{0:L} - w[x_0] \|^2 , \] (12)

subject to equations (9)-(11). (13)

Our main idea is to incorporate the sensor selection problem into the problem (12) and to jointly solve the resulting problem for the initial state and sensor locations. Following this idea, and by using the parametrization
of the output matrix, we formulate the joint problem of selecting sensor nodes and estimating the initial state as the solution of the following MINO problem:

\[
\begin{align*}
(P1) \quad \min_{\mathbf{x}_0, \theta} & \quad \| \mathbf{z}_{0:L} - (I_{L+1} \otimes C_{\theta}(\theta)) \mathbf{x}_{0:L}[\mathbf{x}_0] \|_2^2, \\
\text{subject to} & \\
\mathbf{x}_i = & \quad g(\mathbf{x}_i, \mathbf{x}_{i-1}), \quad i = 1, 2, \ldots, L, \quad (14b) \\
\sum_{l=1}^{N} \theta_l \leq & \quad M_{\text{max}}, \quad \left( \sum_{l=1}^{N} \theta_l = M_{\text{max}} \right), \quad (14c) \\
\theta_l \in & \quad \{0, 1\}, \quad l = 1, 2, \ldots, N, \quad (14d) \\
\mathbf{x}_0 \leq & \quad \mathbf{x}_0 \leq \mathbf{x}_0. \quad (14e)
\end{align*}
\]

The constraints \((14c)\) limits the number of selected sensor nodes. We have two options for incorporating these constraints into the optimization problem. The first option, represented by less-than-equal relation, ensures that the number of selected nodes is \textit{smaller than or equal} to the desired number of sensor nodes \(M_{\text{max}}\). The second option, represented by the equality relation (in the brackets), is used to ensure that the number of sensor nodes is precisely equal to \(M_{\text{max}}\). The second option is introduced since in our simulations we want to compare the developed algorithms with random sensor selection, and in order to ensure that the comparison is fair, we need to ensure that the selected number of sensor nodes is always constant. Also, we noticed that nonlinear solvers occasionally perform better in the case of the second option. In \((14)\), the notation \(\leq\) in \((14e)\) denotes element-wise less than equal relation. \(\mathbf{x}_0 \in \mathbb{R}^{Nn}\) and \(\mathbf{x}\) are lower and upper bounds on \(\mathbf{x}_0\).

By relaxing the binary constraints and by eliminating the constraints on the maximal number of sensor nodes, from the MINO problem \((14)\) we obtain a relaxed problem:

\[
\begin{align*}
(P2) \quad \min_{\mathbf{x}_0, \theta} & \quad \| \mathbf{z}_{0:L} - (I_{L+1} \otimes C_{\theta}(\theta)) \mathbf{x}_{0:L}[\mathbf{x}_0] \|_2^2, \\
\text{subject to} & \\
\mathbf{x}_i = & \quad g(\mathbf{x}_i, \mathbf{x}_{i-1}), \quad i = 1, 2, \ldots, L, \quad (15b) \\
0 \leq & \quad \theta \leq 1, \quad (15c) \\
\mathbf{x}_0 \leq & \quad \mathbf{x}_0 \leq \mathbf{x}_0. \quad (15d)
\end{align*}
\]

In \((15)\), \(0 \in \mathbb{R}^{N}\) and \(1 \in \mathbb{R}^{N}\) are the vectors of zeros and ones, respectively.

In the next section, we present three algorithms for approximating the solution of the MINO problem \((14)\). These algorithms are partly based on solving the relaxed problem \((15)\).
III. SENSOR SELECTION ALGORITHMS

In this section, we introduce three algorithms for approximating the solution of the MINO problem \( P_1 \) in (14). The first algorithm is based on solving the problem using the MADS algorithm that is also known as NOMAD [37]. The NOMAD algorithm is initialized with a solution guess generated by solving the relaxed problem \( P_2 \) defined in (15). The second and third algorithms are based on computing the initial solution guess by solving the relaxed problem and then using this initial solution to formulate and solve MILPs that are defined in the sequel. The developed codes are available online [30].

In Section [IV] we derive expressions for derivatives of the cost functions. In Section [IV-B] we present some generalizations of the presented algorithms. Also in Section [IV-B], we explain how to increase the robustness and to decrease the sensitivity of the sensor selection procedure with respect to uncertainties and lack of knowledge of the operating range of initial states.

A. Algorithm 1 - Mesh Adaptive Direct Search Approach

This method is summarized in Algorithm 1. The MADS algorithm (also known as NOMAD) is a derivative-free optimization method. We use a version of the MADS algorithm implemented in the OPTI MATLAB toolbox [37]. The advantage of this approach is that it can easily be integrated with all MATLAB nonlinear solvers that are necessary to solve the system of equations (14b). Furthermore, to implement this method we only need a numerical procedure to evaluate the cost function.

In step 1 of Algorithm 1, we solve the relaxed problem \( P_2 \) defined in (15). This problem is solved using the interior point method implemented in the MATLAB function fmincon(·). We use a recursive approach for solving the relaxed problem. The recursive approach is originally used for solving model predictive control problems, see Chapter 10 in [38]. However, with minor modifications, it is also applicable to our problem. The recursive approach does not consider intermediate states \( x_1, x_2, \ldots, x_L \) in (15b) as explicit optimization variables. Instead, in every iteration, the state sequences are computed by either forward propagation of the discrete-time dynamics (3) (in the case of the FE method) or by solving the nonlinear system of equations (9)-(11) (in the case of the TI method). The nonlinear system of equations is solved using the trust-region dogleg method implemented in the MATLAB function fsolve(·).

The solution of the relaxed problem is used in step 2 of Algorithm 1 as an initial guess for the MADS algorithm. By generating the initial guess in this way, we significantly decrease the number of iterations of the NOMAD algorithm. To implement the NOMAD algorithm, we use the previously explained recursive approach. Once the state sequence is computed, we can evaluate the cost function (14a), and this value is given to the NOMAD solver. Finally, in step 3, using the sensor nodes computed by the NOMAD algorithm, we form the matrix \( \hat{C} \) by eliminating the zero rows (corresponding to sensor nodes that are not selected) of the matrix \( C_0[\hat{\theta}] \). For this \( \hat{C} \), we form and solve the optimization problem (12) to compute the initial state estimate \( \hat{x}_0 \). This problem is solved using the quasi-Newton method implemented in the MATLAB function fminunc(·).
Algorithm 1 Sensor Selection Using NOMAD Method

inputs: In the first phase (steps 1-2), the inputs are the output sequence \( \{z_0, z_1, \ldots, z_L\} \) and the maximal number of sensor nodes \( M_{\text{max}} \). In the second phase (step 3), the input is the output sequence \( \{y_0, y_1, \ldots, y_L\} \) collected by the assigned sensor nodes.

outputs: The optimal sensor selection vector \( \hat{\theta} \) and the initial state estimate \( \hat{x}_0 \).

1. initial solution: Solve the relaxed problem \( P_2 \) in (15). Let the solution of this problem be denoted by \( (\tilde{\theta}, \tilde{x}_0) \).

2. solve: Using \( (\tilde{\theta}, \tilde{x}_0) \) as an initial guess, solve the MINO problem \( P_1 \) in (14) by using the NOMAD solver. Let the solution of this problem be denoted by \( (\hat{\theta}, \hat{x}_0^{(1)}) \).

3. solve: Implement the sensor locations on the physical system. Using the installed sensors, collect the output sequence \( \{y_0, y_1, \ldots, y_L\} \). Form the matrix \( \hat{C} \) by eliminating the zero rows of the matrix \( C_{\theta}(\hat{\theta}) \). Setting \( C := \hat{C} \), solve (12) to compute \( \hat{x}_0 \).

B. Algorithm 2 - Solving the Relaxed and MILP Problems

Here we present an algorithm for approximating the solution of the MINO problem \( P_1 \) in (14) that is based on solving the relaxed problem \( P_2 \) in (15) and an MILP that is defined in the sequel.

Let the solution of the relaxed problem be denoted by \( x_0 \). Using this solution, we define the following cost function

\[
J[\theta] = \|G[\theta]\|_1, \quad (16)
\]

\[
G[\theta] = z_{0:L} - (I_{L+1} \otimes C_{\theta}[\theta])x_{0:L}[\tilde{x}_0]. \quad (17)
\]

It should be noted that this cost function is similar to the cost functions in (14a) and (15a), except for the \( \ell_1 \) norm and substituted initial state. We determine optimal sensor locations as the solution of the following optimization problem:

\[
\min_{\theta} J[\theta], \quad (18)
\]

subject to

\[
\sum_{l=1}^{N} \theta_l \leq M_{\text{max}}, \quad \left( \sum_{l=1}^{N} \theta_l = M_{\text{max}} \right), \quad (19)
\]

\[
\theta_l \in \{0, 1\}, \quad l = 1, 2, \ldots, N. \quad (20)
\]

To solve (18)-(20), we first transform this problem into an MILP problem. We present the procedure for the less than equal case in (19), and it can be easily generalized for the equality case. By using the parametrization of the matrix \( C_{\theta}[\theta] \) introduced in Section II-B, we can transform the cost function (18) into the following form

\[
J[\theta] = \sum_{j=0}^{L} \sum_{i=1}^{N} |z_j^{(i)} - \hat{\theta}_j x_j^{(i)}|, \quad (21)
\]
where $z_j^{(i)}$ is the $i$-th entry of $z_j$, and $\tilde{x}_j^{(i)}$ is the first entry of $\tilde{x}_j^{(i)}$, and this vector is the $i$-th entry of $\tilde{x}_j$. The state sequence \{$\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_L$\} is obtained by propagating the equations (9)-(11) from $\tilde{x}_0$ in the case FE dynamics or by solving these equations starting from $\tilde{x}_0$ in the case of the TI dynamics.

By introducing the slack variable vector $b \in \mathbb{R}^{N(L+1)}$, and by taking into account (21), we can transform the optimization problem (18)-(20) as follows

\[
\text{(MILP1) } \min_{b, \theta} \sum_{j=0}^{L} \sum_{i=1}^{N} b_j^{(i)}, \quad (22a)
\]

subject to

\[
\begin{align*}
z_j^{(i)} - \theta_i \tilde{x}_j^{(i)} &\leq b_j^{(i)}, \quad \theta_i \tilde{x}_j^{(i)} - z_j^{(i)} &\leq b_j^{(i)}, \quad b_j^{(i)} \geq 0, \\
\sum_{i=1}^{N} \theta_i &\leq M_{\max}, \quad \sum_{i=1}^{N} \theta_i = M_{\max}, \quad \theta_i \in \{0, 1\}, (22b) \\
i = 1, \ldots, N, \quad j = 0, 1, \ldots, L, (22c)
\end{align*}
\]

where $b_j^{(i)}$ is the $i$-th entry of $b_j$, and $b_j$ is the $j$-th entry of $b = \text{col}(b_0, b_1, \ldots, b_L)$.

The problem (22) is an MILP problem that we solve using the MATLAB function intlinprog(·). The MATLAB function intlinprog(·) implements a hybrid method partly based on a branch and bound method. Algorithm 2 summarizes this approach for approximating the solution of the MINLP problem.

**Algorithm 2 Sensor Selection by Solving the Relaxed Problem P2 and MILP1**

**inputs:** In the first phase (steps 1-3), the inputs are the output sequence \{$z_0, z_1, \ldots, z_L$\} and the maximal number of sensor nodes $M_{\max}$. In the second phase (step 4), the input is the output sequence \{$y_0, y_1, \ldots, y_L$\} collected by the assigned sensor nodes.

**outputs:** The optimal sensor selection vector $\hat{\theta}$ and the initial state estimate $\hat{x}_0$.

1. **initial solution:** Solve the relaxed problem P2 in (15). Let the solution of this problem be denoted by $(\hat{\theta}, \hat{x}_0)$.
2. **compute state sequence:** Using $\hat{x}_0$ compute the state sequence \{$\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_L$\} by solving the telescopic equations (9)-(11).
3. **solve:** Using $\hat{\theta}$ and \{$\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_L$\}, form and solve the MILP1 in (22). Let the solution of this problem be denoted by $\hat{\theta}$.
4. **solve:** Implement the sensor locations on the physical system. Using the installed sensors, collect the output sequence \{$y_0, y_1, \ldots, y_L$\}. Form the matrix $\hat{C}$ by eliminating the zero rows of the matrix $C_{\theta}(\hat{\theta})$. Setting $C := \hat{C}$, solve (12) to compute $\hat{x}_0$. 


C. Algorithm 3 - Solving Relaxed and Binary MILP Problem

Algorithm 2 is based on computing the state sequence once the relaxed problem is solved. Due to the specific structure of the cost function (16), we can create another cost function in which the initial state is eliminated. This cost function penalizes the difference between the solution \( \tilde{\theta} \) of the relaxed problem \( P_2 \) in (15) and the optimization variable \( \theta \). The third algorithm computes optimal sensor selection by minimizing this cost function. This approach is inspired by an approach for solving mixed-integer optimal control problems in [27]–[29]. However, our problem formulation and introduced cost function differ from the ones considered in [27]–[29].

Generally speaking, the entries of the solution \( \tilde{\theta} \) are in the interval \([0, 1]\). Using the rounding strategy [27]–[29] we can set the entries of this vector to either 0 or 1. However, this approach does not produce the solution that satisfies the constraints on the maximal number of sensor nodes (14c). Consequently, we need to find a way to incorporate this constraint into an optimization function.

Consider expression \( G \) in (17). The difference between the value of \( G \) for the relaxed solution \( \tilde{\theta} \) and for the binary solution \( \theta \) that we want to determine is given by

\[
G[\tilde{\theta}] - G[\theta] = \left( I_{L+1} \otimes (C_0[\theta] - C_0[\tilde{\theta}]) \right)x_{0:L}[\tilde{x}_0].
\]  

(23)

Our goal is to find \( \theta \) such that an upper bound on the \( \ell_1 \) norm of this difference is minimized while ensuring that the constraints on the total number of sensor nodes are satisfied. From (23) we have

\[
\|G[\tilde{\theta}] - G[\theta]\|_1 \leq \left\| I_{L+1} \otimes (C_0[\theta] - C_0[\tilde{\theta}]) \right\|_1 \|x_{0:L}[\tilde{x}_0]\|_1.
\]  

(24)

This upper bound can be minimized by minimizing the first term on the right-hand-side of (24). This term is minimized if \( \|C_0[\theta] - C_0[\tilde{\theta}]\|_1 \) is minimized. Recalling the definition of the \( \ell_1 \) norm, we conclude that in order to minimize \( \|C_0[\theta] - C_0[\tilde{\theta}]\|_1 \), we need to minimize the maximum of all the column sums of matrix difference \( C_0[\theta] - C_0[\tilde{\theta}] \). Since the matrix \( C_0[\theta] \) has a (block) diagonal structure and due to the parametrization introduced in Section II the problem boils down to the problem of minimizing the maximal difference between the entries of \( \theta \) and \( \tilde{\theta} \). Consequently, to compute optimal sensor locations, we formulate the following optimization problem:

\[
\min_{\theta} \max_{i} |\theta_i - \tilde{\theta}_i|, 
\]  

(25)

subject to

\[
\sum_{l=1}^{N} \theta_l \leq M_{\text{max}}, \quad \left( \sum_{l=1}^{N} \theta_l = M_{\text{max}} \right),
\]  

(26)

\[
\theta_l \in \{0, 1\}, \quad l = 1, 2, \ldots, N.
\]  

(27)

By introducing the slack variable \( q \in \mathbb{R}_{0+}^{N} \), the last optimization problem can be reformulated as follows
\[(\text{MILP2}) \quad \min_{q, \theta} q, \quad (28a)\]

subject to

\[
\theta_i - \tilde{\theta}_i \leq q, \quad \tilde{\theta}_i - \theta_i \leq q, \quad i = 1, 2, \ldots, N, \quad (28b)
\]

\[
\sum_{l=1}^{N} \theta_l \leq M_{\text{max}}, \quad \left(\sum_{l=1}^{N} \theta_l = M_{\text{max}}\right), \quad (28c)
\]

\[
\theta_l \in \{0, 1\}, \quad l = 1, 2, \ldots, N. \quad (28d)
\]

Similarly to the MILP1 problem in (22), the MILP2 problem in (28) is solved by using the hybrid method implemented in the MATLAB function intlinprog(). Algorithm 3 summarizes the third approach for selecting the sensor nodes.

**Algorithm 3 Sensor Selection by Solving Relaxed Problem P2 and MILP2**

**inputs:** In the first phase (steps 1-2), the inputs are the output sequence \{\(z_0, z_1, \ldots, z_L\}\) and the maximal number of sensor nodes \(M_{\text{max}}\). In the second phase (step 3), the inputs is the output sequence \{\(y_0, y_1, \ldots, y_L\)\} collected by the assigned sensor nodes.

**outputs:** The optimal sensor selection vector \(\hat{\theta}\) and the initial state estimate \(\hat{x}_0\).

1. **initial solution:** Solve the relaxed problem P2 in (15). Let the solution of this problem be denoted by \((\tilde{\theta}, \tilde{x}_0)\).
2. **solve:** Using \((\tilde{\theta}, \tilde{x}_0)\), solve the LIP (28). Let the solution of this problem be denoted by \(\hat{\theta}\)
3. **solve:** Implement the sensor locations on the physical system. Using the installed sensors, collect the output sequence \{\(y_0, y_1, \ldots, y_L\)\}. Form the matrix \(\hat{C}\) by eliminating the zero rows of the matrix \(C_{\theta}(\hat{\theta})\). Setting \(C := \hat{C}\), solve (12) to compute \(\hat{x}_0\).

In the general case, the formulated MINO problem P1 in (14) is NP-hard and non-convex. On the other hand, the introduced methods are approximating the solution. Theoretical investigation of the degree of the suboptimality of the developed methods is hard. Instead, in Section V we investigate the optimality by performing extensive numerical experiments. The good numerical performance of our methods is demonstrated by testing the methods on Duffing oscillator networks that are prototypical models of a number of dynamical systems, as well as on associative memory networks representing simplified memory models. We compare our methods with exhaustive search and random selection of sensor nodes.

In the next section, we derive expressions for derivatives of the cost functions introduced in this section, and we present some generalizations of the developed approaches.

**IV. DERIVATIVES AND GENERALIZATIONS**

To significantly speed up the computations of the solutions of the optimization problems and the nonlinear equations necessary to approximate the solution of the MINO problem P1 in (14), in this section we derive
expressions for gradients and derivatives of the cost function used in the problems (14) and (15). These gradients and derivatives are used by nonlinear solvers.

In addition, in Section IV-B we briefly present generalizations of the optimization problems (14) and (15) that incorporate a number of output and state sequences. In this way, we can potentially increase the robustness and decrease the sensitivity of the sensor selection procedure with respect to uncertainties and lack of knowledge of the operating range of initial states.

A. Derivatives

First, we start with the derivatives of the FE and TI dynamics in the equations (3) and (4). As it will become clear, similarly to the recursive computation (simulation) of state sequences, derivatives are also computed in a recursive manner. From (3) it follows that for the FE dynamics, we have

$$\frac{\partial x_k}{\partial x_0} = \frac{\partial x_{k-1}}{\partial x_0} + h \frac{\partial x_{k-1}}{\partial x_0} \frac{\partial f[x_{k-1}]}{\partial x_k}. \tag{29}$$

On the other hand, from (4) it follows that for the TI dynamics, we have

$$\frac{\partial x_k}{\partial x_0} = \frac{\partial x_{k-1}}{\partial x_0} + h \left( \frac{\partial x_{k-1}}{\partial x_0} \frac{\partial f[x_{k-1}]}{\partial x_k} + \frac{\partial x_{k-1}}{\partial x_0} \frac{\partial f[x_{k-1}]}{\partial x_{k-1}} \right). \tag{30}$$

Under a mild assumption of invertibility of \((I_{Nn} - (h/2) \partial f[x_k]/\partial x_k)\), from the last expression, we obtain

$$\frac{\partial x_k}{\partial x_0} = \left( I_{Nn} - \frac{h}{2} \frac{\partial f[x_k]}{\partial x_k} \right)^{-1} \frac{\partial x_{k-1}}{\partial x_0} \left( I_{Nn} + \frac{h}{2} \frac{\partial f[x_{k-1}]}{\partial x_{k-1}} \right). \tag{31}$$

By initializing the recursive equations (29) and (31) with \(\partial x_0/\partial x_0 = I_{Nn}\), we can recursively compute the derivatives of state sequences. These derivatives will be used to compute the gradients of the cost function

$$L[\theta, x_0] = \|z_{0:L} - K[\theta]x_{0:L}[x_0]\|^2_2, \quad K[\theta] = (I_{L+1} \otimes C_\theta[\theta]), \tag{32}$$

that appear in the MINO problem P1 defined in (14) and in the relaxed problem P2 defined in (15). The gradient of the cost function is defined by

$$\nabla L = \begin{bmatrix} \nabla x_0 L \\ \nabla \theta L \end{bmatrix}. \tag{33}$$

For brevity we only give final expressions of the gradients. We obtained

$$\nabla x_0 L = -2 \frac{\partial x_{0:L}[x_0]}{\partial x_0} K^T[\theta] \left( z_{0:L} - K[\theta]x_{0:L}[x_0] \right), \tag{34}$$

$$\frac{\partial x_{0:L}[x_0]}{\partial x_0} = \begin{bmatrix} I_{Nn} & \frac{\partial x_1}{\partial x_0} & \cdots & \frac{\partial x_L}{\partial x_0} \end{bmatrix}, \tag{35}$$

and

$$\nabla \theta L = -2 \begin{bmatrix} C_1 x_0 & C_1 x_1 & \cdots & C_1 x_L \end{bmatrix} \left( z_{0:L} - K[\theta]x_{0:L}[x_0] \right), \tag{36}$$

where the matrix \(C_1\) depends on the parametrization of \(C_\theta[\theta]\). In the case when \(C_\theta[\theta] = \text{diag}(\theta)\), we have \(C_1 = \text{diag}(1)\), where \(C_1 \in \mathbb{R}^{N \times N}\). In the case when \(C_\theta[\theta]\) is a block diagonal matrix with the \(i\)-th block \([0 \ldots 0] \in \mathbb{R}^{1 \times n}\), the matrix \(C_1 \in \mathbb{R}^{N \times N n}\) is a block diagonal matrix with the \(i\)-th block \([1 0 \ldots 0] \in \mathbb{R}^{1 \times n}\).

To compute (34) and (35), we use the recursive expressions (29) and (31).
B. Generalizations

The sensor selection approach presented in Section III is derived on the basis of the output sequence \( \{z_k\} \). In the first sensor selection scenario, explained in Section II-B, this sequence is computed by simulating the dynamics for the user-selected initial condition. This implies that the optimal sensor locations, and consequently, the optimal structure of the matrix \( C_{\theta}(\hat{\theta}) \), depends to some extent on the user-selected initial condition. Furthermore, the sensor selection problem is non-convex and it depends on the initial guess of the state and sensor locations. It might happen that the selected sensor locations do not produce satisfactory results when the initial state of the real system significantly differs from the one used to generate the data sequence \( \{z_k\} \).

If we know an operating range of initial states of the real system, then we can use this knowledge to select the user-defined initial state for generating the sequence \( \{z_k\} \). In this way, we can reduce the sensitivity of the sensor selection procedure with respect to user-selected initial states. However, in some cases, due to the lack of knowledge, this is impossible.

Our approach for dealing with this problem is to generalize the cost functions in (14) and (15), by including many output and state sequences \( \{z_k\} \) generated for different selections of initial conditions. We can replace the cost functions in optimization problems defined in Section III by the following cost function

\[
W[\theta, x_0] = \|Z - K[\theta]X\|_F^2, \tag{37}
\]

\[
Z = \begin{bmatrix} z_{0:L,1} & z_{0:L,2} & \cdots & z_{0:L,P} \end{bmatrix},
\]

\[
X = \begin{bmatrix} x_{0:L,1}[x_{0,1}] & x_{0:L,2}[x_{0,2}] & \cdots & x_{0:L,P}[x_{0,P}] \end{bmatrix}, \tag{38}
\]

where \( z_{0:L,i} \) and \( x_{0:L,i}, i = 1, 2, \ldots, P \), are the output and state sequences, respectively, obtained from the user-selected initial state \( x_{0,i} \), and the matrix \( K[\theta] \) is defined in (32). The number \( P \) is the total number of user-selected initial states and \( \|\cdot\|_F^2 \) is the Frobenius norm. In some sense, this generalization produces an averaged sensor selection \( \hat{\theta} \) that is optimal for a number of initial conditions. The initial conditions \( x_{0,i} \) can be for example selected as corners of a convex hull of the assumed operating region of the initial state of the real system.

The disadvantage of this approach is that the computational complexity of solving sensor selection problems is increased. In our future work, we will present a detailed numerical study of this generalized approach for selecting the sensor nodes.

V. Numerical Results

In this section, we present numerical results of applying the developed algorithms to associative memory and Duffing oscillator networks. All simulations are performed on a computer with 16GB RAM and Intel® Core™ i7-8700 processor. The developed codes are available online [30].

A. Sensor Selection for Associative Memory Networks

Associative memory networks can be seen as simplified memory models [39]–[41]. The idea is to select network parameters such that starting from an initial state, representing a perturbed letter or a binary pattern, the network
state converges to the equilibrium state, representing the memorized letter or the binary pattern. For brevity, we give a final state-space form of the network, for more background information, see [39]–[41] and references therein.

The dynamics of the \( i \)-th node has the following form

\[
\dot{x}_i = \sum_{j=1}^{N} \beta_{ij} \sin(x_j - x_i) + \frac{\gamma}{N} \sum_{j=1}^{N} \sin^2(x_j - x_i)
\]  

(39)

where \( x_i \in \mathbb{R} \), \( \gamma = 0.8 \), \( \beta_{ij} \in \mathbb{R} \) is given by \( \beta_{ij} = (1/N) \sum_{\omega=1}^{p} \zeta_1^\omega \zeta_2^\omega \) (Hebb’s learning rule), where \( \zeta_1^\omega = \pm 1 \), \( \omega = 1, 2, \ldots, p \), and \( p \) denotes the number of binary patterns to be memorized. A binary pattern to be memorized is represented by the vector \( \zeta_1^\omega = \text{col}(\zeta_1^\omega, \zeta_2^\omega, \ldots, \zeta_N^\omega) \). Let \( \zeta_1, \zeta_2, \ldots, \zeta_p \) represent \( p \) desired binary patterns that need to be memorized by the network. Then for such a selection of binary patterns, we can compute \( \beta_{ij} \) using the previously explained formula, and we can construct the network dynamics (39).

We use the following parameters in our simulations \( N = 25 \) and \( h = 10^{-3} \). We choose the FE method to discretize dynamics since as demonstrated in [39] this method produces a small discretization error and the network dynamics is not stiff. The observation horizon is \( L = 21 \). We define binary patterns on a \( 5 \times 5 \) grid, corresponding to the letters “H”, “T”, and “L” \((p = 3)\). The network initial state is the “T” letter that is perturbed by the normal Gaussian noise. Our goal is to estimate this initial state. We test the performance of the method for 40%, 56%, 72%, and 88% of observed nodes (corresponding to \( M_{\text{max}} = 10, 14, 18, 22 \)). For Algorithm 1 we use the less than equal constraints in (14). This is necessary since we have noticed that in the case of equality constraints the NOMAD solver occasionally produces infeasible solutions. On the other hand, in the case of Algorithms 2 and 3 we use the equality constraints.

For the relaxed problem \( P_2 \) in (15), the initial guess of the unknown state is a vector with entries drawn from the Gaussian normal distribution, and the initial guess of the sensor locations is a vector with entries drawn from the uniform distribution on \([0, 1]\). In the relaxed problem, we use the lower and upper bounds equal to \(-5\) and \(5\), respectively, and \(0\) and \(1\) for the relaxed binary constraints.

To test the performance of the developed algorithms, we compare optimal sensor selections with a sequence of random selections of control nodes. For every random selection, we form the corresponding matrix \( C \), and we estimate the initial state by solving (12). The estimation error is quantified by \( e = \|x_0^{\text{true}} - \hat{x}_0\|_2 / \|x_0^{\text{true}}\|_2 \), where \( x_0^{\text{true}} \) is the “true” value of the initial state. Figure 1 compares the performance of the developed methods with random sensor selections. We have generated 1000 random selections of sensor nodes. Histograms correspond to empirical distributions of the estimation error produced by random sensor selections. The colored vertical lines denote estimation errors produced by optimal sensor selections computed by the developed algorithms. We can notice that the performance of algorithms varies with a fraction of observed nodes. However, for a fixed fraction of sensor nodes, there is at least one algorithm that produces the error that is smaller than most of the errors produced by the random sensor selection. This is a numerical verification of a relatively good performance of the developed approaches.
Fig. 1: The estimation errors for associative memory networks. The vertical colored lines correspond to the final estimation errors produced by the developed algorithms and histograms correspond to empirical error distributions computed by random sensor selection for the fixed fraction of observation nodes. (a) 40\%, (b) 56\%, 72\%, and 88\% of observed nodes.

B. Sensor Selection for Duffing Oscillator Networks

Duffing oscillators are dynamical systems with a nonlinear spring stiffness $F_s = \eta x - \chi x^3$, where $x$ is the spring displacement, $\eta, \chi \in \mathbb{R}$ are the spring constants, and $F_s$ is the spring force (we assume a softening spring). These oscillators are prototypical models of a large number of dynamical systems, such as electrical circuits, nanomechanical resonators, structural beams, cables, etc. An excellent introduction to the theory and applications of Duffing oscillators is given in [42]. We consider a nonlinear network consisting of damped Duffing oscillators with nonlinear connections:

\begin{align}
\dot{x}_{i1} &= x_{i2}, \\
\dot{x}_{i2} &= -\eta_{ii} x_{i1} + \chi_{ii} x_{i1}^3 - \rho_{ii} x_{i2} - \sum_{j \in \mathcal{N}(i)} \eta_{ij} (x_{i1} - x_{j1}) \\
&\quad + \sum_{j \in \mathcal{N}(i)} \chi_{ij} (x_{i1} - x_{j1})^3 - \sum_{j \in \mathcal{N}(i)} \rho_{ij} (x_{i2} - x_{j2}),
\end{align}

where $x_{i1}$ is the displacement and $x_{i2}$ is the velocity of the $i$-th oscillator, $\eta_{ij}$ and $\chi_{ij}$ are previously introduced spring constants, $\rho_{ij}$ is a damping parameter, $\mathcal{N}(i)$ is a set of oscillators that are connected to the $i$-th oscillator. We assume that connections between oscillators are defined by a Geometric Random Graph (GRG). We use the method and codes available in [43] to generate GRGs. GRGs are constructed by randomly placing nodes on a unit square, and connecting nodes according to a user-defined connection radius. In our simulations, we use the radius of $\sqrt{1.44/N}$. The parameters $\eta_{ij}$ are selected from a uniform distribution defined on the interval $[10, 20]$. On the other hand, parameters $\chi_{ij}, \rho_{ij}$ are selected from a uniform distribution defined on $[1, 2]$. The state of the $i$-th node
Our first goal is to construct Duffing oscillator networks with $N = 10$ nodes for which we compare our methods with an exhaustive search for controlled nodes. In this way we can quantify how far are computed solutions from the most optimal ones. In our simulations, we choose $h = 10^{-4}$ and $L = 201$. Due to the fact that the dynamics is stiff, we use the TI method. The entries of a “true” initial state (to be estimated) are generated from the uniform distribution on the interval $[0, 1]$. An initial guess for this solution is generated using the same principle, however, we make sure that the entries of the guess and “true” initial state are different. The initial guess for the sensor nodes for the relaxed problem $P_2$ in (15) is generated as a random vector whose entries are selected from the uniform distribution on $[0, 1]$. We test the performance of the methods for $20\%$, $40\%$, $60\%$ and $80\%$ of observed nodes (corresponding to $M_{\text{max}} = 2, 4, 6, 8$). At the same time, we perform the exhaustive search by generating all the possible combinations for fixed fractions of observation nodes, and by estimating an initial state for every combination by solving (12). The results are shown in Fig. 2.

Finally, we test the computational complexity of the developed methods. The results are shown in Fig. 3. Here we used the TI dynamics and the parameters used to generate the results shown in Fig. 2. That is, to generate the computational complexity results, we were using the approach that requires us to solve the nonlinear system of equations to simulate the dynamics. The computational complexity times are in two orders of magnitude smaller for the FE dynamics. From Fig. 2 we can see that Algorithm 3 produces the lowest computational complexity. This is due to the fact that the MILP2 problem in (28) only contains $N + 1$ optimization variables. On the other hand, as expected, Algorithm 1 produces the highest computational complexity since it is based on the direct search. Also, it should be observed that the slopes of the computational complexity lines start to decrease as the network size
Fig. 3: The computational complexity results of the developed algorithms. The results are obtained for the TI dynamics and the Duffing oscillator networks with $L = 201$ and $h = 10^{-4}$.

decreases for algorithms 2 and 3. These computational complexity results can be improved by employing parallel implementations of the used nonlinear solvers.

VI. CONCLUSIONS AND FUTURE WORK

In this manuscript, we have developed methods for sensor selection for nonlinear networks. The main idea of our approaches is to incorporate the sensor selection problem into the initial state estimation problem. As a result, we obtained a Mixed-Integer Nonlinear Optimization (MINO) problem whose solutions are the optimal location of sensor nodes and the initial state estimate. We developed algorithms that approximate the solution to this problem. In the initial phase, all three algorithms solve a relaxed MINO problem. The first algorithm uses this solution and the mesh adaptive direct search method to approximate the solution. The second and third algorithms use the solution of the relaxed problem and mixed-integer linear optimization techniques to approximate solution. We performed extensive numerical experiments that demonstrate the good performance of the developed methods. We noticed that the performance of the developed approaches largely depends on the parameters such as the length of the observation horizon, discretization constant, and nonlinear solver parameters. In future research, we will systematically investigate the influence of these parameters on the overall performances of the developed methods. Furthermore, due to the nonlinear nature of the problem, optimal sensor selections depend on the output sequence used for optimization. In future research, using the directions explained in Section IV-B, we will generalize the approach by including a number of output and state sequences in the sensor selection problem. In this way, we will further improve the performance of the developed approaches.

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