Lacunary eta-quotients modulo powers of primes

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Abstract
An integral power series is called lacunary modulo $M$ if almost all of its coefficients are divisible by $M$. Motivated by the parity problem for the partition function, Gordon and Ono studied the generating functions for $t$-regular partitions, and determined conditions for when these functions are lacunary modulo powers of primes. We generalize their results in a number of ways by studying infinite products called Dedekind eta-quotients and generalized Dedekind eta-quotients. We then apply our results to the generating functions for the partition functions considered by Nekrasov, Okounkov, and Han.

Keywords Partitions · Eta-quotients · Nekrasov Okounkov formula · Lacunary · Eta-function · Modular forms · Generalized eta-quotients · Powers of primes

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1 Introduction

A partition of a positive integer \(n\) is a nonincreasing sequence of positive integers whose sum is \(n\). For example, the set of partitions of 4 is

\[
\{4, \ 3 + 1, \ 2 + 2, \ 2 + 1 + 1, \ 1 + 1 + 1 + 1\}.
\]

The partition function \(p(n)\) counts the number of partitions of \(n\). From the above example we see that \(p(4) = 5\). The generating function for \(p(n)\) satisfies the identity

\[
P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}.
\]

The partition function has many congruence properties modulo primes and powers of primes, the most famous of which are the Ramanujan congruences [11]:

\[
p(5n + 4) \equiv 0 \pmod{5},
p(7n + 5) \equiv 0 \pmod{7},
p(11n + 6) \equiv 0 \pmod{11}
\]

for all \(n \geq 0\). Few results of this form were known until Ahlgren and Ono [1,8] showed that there are infinitely many congruences of the form

\[
p(An + B) \equiv 0 \pmod{m}
\]

for any integer \(m\) relatively prime to 6.

Although no analogous theorem exists for the partition function modulo the primes 2 and 3, Parkin and Shanks conjectured that half of the values for \(p(n)\) are even and half are odd [10]. To be precise, given an integral power series \(F(q) := \sum_{n \gg -\infty} a(n)q^n\), we define

\[
\delta(F, M; X) := \frac{\# \{n \leq X : a(n) \equiv 0 \pmod{M} \}}{X}.
\]

It is conjectured that \(\delta(P, 2; X)\) and \(\delta(P, 3; X)\) tend to \(\frac{1}{2}\) and \(\frac{1}{3}\), respectively, as \(X\) approaches infinity. Table 1 contains values of \(\delta(P, 2; X)\) and \(\delta(P, 3; X)\) for \(X\) up to 500,000.

Although calculations strongly support this conjecture, it remains unproven. Moreover, it remains unknown whether a positive proportion of the values of \(p(n)\) are even (resp. odd), or whether an infinite number of the values of \(p(n)\) lie in any fixed residue class modulo 3.

In contrast to equal distribution, \(F(q)\) is called lacunary modulo \(M\) if

\[
\lim_{X \to \infty} \delta(F, M; X) = 1,
\]
Table 1 Data for \( p(n) \)

| \( X \) | \( \delta(P, 2; X) \) | \( \delta(P, 3; X) \) |
|-------|----------------|----------------|
| 100,000 | 0.4980... | 0.3334... |
| 200,000 | 0.5012... | 0.3332... |
| 300,000 | 0.5008... | 0.3335... |
| 400,000 | 0.5000... | 0.3339... |
| 500,000 | 0.5000... | 0.3343... |

i.e. if almost all of the coefficients of \( F \) are divisible by \( M \). In [4], Gordon and Ono studied the lacunarity of the generating function \( G_t(\tau) \) of the \( t \)-regular partition function \( b_t(n) \) defined by

\[
G_t(\tau) := \sum_{n=0}^{\infty} b_t(n)q^n = \prod_{n=1}^{\infty} \left( \frac{1-q^n}{1-q^n} \right) = q \frac{1-\eta(t\tau)}{\eta(\tau)},
\]

where \( q := e^{2\pi i \tau} \) and the Dedekind eta-function \( \eta(\tau) \) is the weight \( \frac{1}{2} \) modular form

\[
\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n)
\]

(combinatorially, the function \( b_t(n) \) can be interpreted as counting the number of partitions of \( n \) with no summands divisible by \( t \)). The main result of [4] states that if \( p \) is a prime divisor of \( t \) such that \( p^a \mid t \) and \( p^a \geq \sqrt{t} \), then \( G_t(\tau) \) is lacunary modulo \( p^j \) for any positive integer \( j \).

The generating functions of many other partition functions can be expressed in terms of \( \eta(\tau) \). Motivated by this, we study here the lacunarity of eta-quotients of the form

\[
G(\tau) := \frac{\eta(\delta_1 \tau)^{r_1} \eta(\delta_2 \tau)^{r_2} \cdots \eta(\delta_u \tau)^{r_u}}{\eta(\gamma_1 \tau)^{s_1} \eta(\gamma_2 \tau)^{s_2} \cdots \eta(\gamma_t \tau)^{s_t}} = q^{\frac{E_G}{24}} \sum_{n=0}^{\infty} b(n)q^n,
\]

where \( r_i, s_i, \delta_i, \) and \( \gamma_i \) are positive integers with \( \delta_1, \ldots, \delta_u, \gamma_1, \ldots, \gamma_t \) distinct, \( u, t \geq 0 \), and

\[
E_G := \sum_{i=1}^{u} \delta_i r_i - \sum_{i=1}^{t} \gamma_i s_i
\]

(note: we will say that \( G(\tau) \) is lacunary modulo \( M \) when the series \( \sum_{n=0}^{\infty} b(n)q^n \) has that property, and similarly for \( H(\tau) \) defined below). Since \( \eta(\tau) \) has weight \( \frac{1}{2} \), the weight of \( G(\tau) \) is...
\[
\frac{1}{2} \left( \sum_{i=1}^{u} r_i - \sum_{i=1}^{t} s_i \right).
\]

Define \( \mathcal{D}_G := \gcd(\delta_1, \ldots, \delta_u) \). Generalizing the result from [4] stated above, we prove the following theorem.

**Theorem 1.1** Suppose \( G(\tau) \) is an eta-quotient of the form (1.1) with integer weight. If \( p \) is a prime such that \( p^a \) divides \( \mathcal{D}_G \) and

\[
p^a \geq \sqrt{\sum_{i=1}^{u} \frac{r_i}{\delta_i} - \sum_{i=1}^{t} \frac{\gamma_i s_i}{\delta_i}},
\]

then \( G(\tau) \) is lacunary modulo \( p^j \) for any positive integer \( j \). Moreover, there exists a positive constant \( \alpha \), depending on \( p \) and \( j \), such that the number of integers \( n \leq X \) with \( p^j \) not dividing \( b(n) \) is \( O \left( \frac{X \log^\alpha X}{p^{j+1}} \right) \).

**Remark 1.1** For primes \( p \) that do not satisfy the conditions of Theorem 1.1, numerical data (e.g. see Sect. 4) suggest that \( 1/p \) of the coefficients of \( G(\tau) \) are divisible by \( p \).

**Remark 1.2** If we apply Theorem 1.1 to the generating function for the \( t \)-regular partition function \( G_t(\tau) \), we recover the result from [4].

**Remark 1.3** A deep theorem of Serre (see Theorem 2.2) asserts that holomorphic integer weight modular forms with integer coefficients are lacunary modulo any positive integer. Since \( \eta(\tau) \) is holomorphic and nonvanishing on the complex upper half plane \( \mathbb{H} \), the same is true of functions of the form (1.1); however, these eta-quotients typically have poles at cusps. Therefore, Theorem 1.1 applies to many functions excluded by Serre’s Theorem.

Here we give an application of Theorem 1.1. We may represent a partition \( \lambda_1 + \cdots + \lambda_k \) with summands \( \lambda_i \) written in nonincreasing order using a Ferrers diagram, which consists of \( k \) rows of boxes with the \( i \)-th row containing \( \lambda_i \) boxes. The *hook length* of each box counts the number of boxes to the right of the given box and the number of boxes below the given box, plus one for the box itself. Below we show the Ferrers diagram of \( 4 + 2 + 1 \), with each box labeled with its hook length.

\[
\begin{array}{cccc}
6 & 4 & 2 & 1 \\
3 & 1 \\
1
\end{array}
\]

Denote by \( \mathcal{P} \) the set of all partitions, and by \( \mathcal{H}(\lambda) \) the multi-set of hook lengths for the partition \( \lambda \). Let

\[
\mathcal{H}_t(\lambda) = \{ h \in \mathcal{H}(\lambda) : h \equiv 0 \pmod{t} \}.
\]
Nekrasov and Okounkov [7] established the identity
\[
\sum_{\lambda \in \mathcal{P}} q^{\lambda_{i}} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^{2}}\right) = \prod_{n \geq 1} \left(1 - q^{n}\right)^{z^{-1}},
\]
for \( z \in \mathbb{C} \), where \( |\lambda| = \sum_{i} \lambda_{i} \). This formula, which has significance in mathematical physics, combinatorics, and number theory, relates partition hook lengths to powers of \( \eta(\tau) \). Han [5] provided the \((t, y)\)-extension
\[
G_{y, t, z}(\tau) := \sum_{\lambda \in \mathcal{P}} q^{\lambda} \prod_{h \in \mathcal{H}_{t}(\lambda)} \left(y - \frac{tyz}{h^{2}}\right) = \prod_{n \geq 1} \frac{(1 - q^{tn})^{t}}{(1 - (yz^{t}q^{n})^{t-z}(1 - q^{n})^{z})},
\]
where \( y \in \mathbb{C} \) and \( t \) is a positive integer. One can verify that
\[
G_{1, t, z}(\tau) = q^{\frac{1-tz}{24}} \frac{\eta^{2t-z}(\tau)\eta^{t-z}(4t \tau)}{\eta^{24}(\tau)} \quad \text{and} \quad G_{-1, t, z}(\tau) = q^{\frac{1-tz}{24}} \frac{\eta^{2t-z}(\tau)\eta^{t-z}(4t \tau)}{\eta^{24}(\tau)\eta^{3(t-z)}(2t \tau)}
\]
(note that \( G_{1, t, 1}(\tau) \) is the generating function for \( b_{t}(n) \)). When \( z \) is an odd integer greater than one with \( z \geq t \), both \( G_{1, t, z}(\tau) \) and \( G_{-1, t, z}(\tau) \) are holomorphic positive integral weight modular forms, and hence Serre’s theorem gives that these two functions are lacunary modulo any positive integer \( M \). The following result addresses the case \( z < t \).

**Corollary 1.2** Suppose \( z \) is an odd positive integer with \( z < t \), and \( p \) is a prime divisor of \( t \).

1. If \( p^{a} \mid t \) and
   \[
p^{a} \geq \sqrt{\frac{t}{z}},
   \]
   then \( G_{1, t, z}(\tau) \) is lacunary modulo \( p^{j} \) for any positive integer \( j \).

2. If \( p^{a} \mid t \) and
   \[
p^{a} \geq 2 \sqrt{\frac{t + 6t^{3} - 6t^{2}z}{9t - 5z}},
   \]
   then \( G_{-1, t, z}(\tau) \) is lacunary modulo \( p^{j} \) for any positive integer \( j \).

Theorem 1.1 is a special case of a more general theorem involving the generalized eta-function, \( \eta_{\delta, g}(\tau) \) (see [12]) defined by
\[
\eta_{\delta, g}(\tau) := e^{\pi i P_{2}(\frac{\tau}{\delta})} \prod_{n \equiv g \pmod{\delta}} (1 - q^{n}) \prod_{n \equiv -g \pmod{\delta}} (1 - q^{n}),
\]
where $\delta, g \in \mathbb{Z}$, $\delta > 0$, and $P_2(x) := (x - \lfloor x \rfloor)^2 - (x - \lfloor x \rfloor) + \frac{1}{6}$ is the second Bernoulli polynomial. Note that

$$
\eta_{\delta, 0}(\tau) = \eta(\delta \tau)^2 \quad \text{and} \quad \eta_{\delta, \frac{1}{2}}(\tau) = \frac{\eta(\frac{\delta}{2} \tau)^2}{\eta(\delta \tau)^2}.
$$

Every other generalized eta-function is a meromorphic modular form of weight zero (i.e. a modular function) on a congruence subgroup which is holomorphic on $\mathbb{H}$. Note also that

$$
\eta_{N\delta, N\delta, g}(\tau) = \eta_{\delta, g}(N \tau)
$$

for any positive integer $N$.

In analogy with eta-quotients, we define the generalized eta-quotient by

$$
H(\tau) := \frac{\eta_{r_1, s_1}(\tau) \cdots \eta_{r_u, s_u}(\tau)}{\eta_{\gamma_1, h_1}(\tau) \cdots \eta_{\gamma_t, h_t}(\tau)} \cdot \frac{\eta_{r'_1, s'_1}(\tau) \cdots \eta_{r'_v, s'_v}(\tau)}{\eta_{\gamma'_1, h'_1}(\tau) \cdots \eta_{\gamma'_u, h'_u}(\tau)} \cdot \frac{\eta_{r''_1, 0}(\tau) \cdots \eta_{r''_w, 0}(\tau)}{\eta_{\gamma''_1, 0}(\tau) \cdots \eta_{\gamma''_x, 0}(\tau)},
$$

where the $r_i, r'_i, r''_i, s_i, s'_i, s''_i$ are positive, $r_i, s_i \in \mathbb{Z}$, $r'_i, r''_i, s'_i, s''_i \in \frac{1}{2} \mathbb{Z}$, and $u, v, w, t, x, y \geq 0$. If we further assume that $h_i, g_i, \gamma'_i, \delta_i \neq 0, g_i \neq \delta_i/2$, and $h_i \neq \gamma_i/2$, this expression is unique. We may write $H(\tau) = q^{E_H} \sum_{n=0}^{\infty} c(n)q^n$ where

$$
E_H := \frac{1}{2} \left( \sum_{i=1}^{u} \delta_i P_2 \left( \frac{s_i}{\delta_i} \right) r_i - \sum_{i=1}^{v} \gamma_i P_2 \left( \frac{s'_i}{\gamma_i} \right) s_i - \sum_{i=1}^{x} \gamma'_i P_2 \left( \frac{s''_i}{\gamma'_i} \right) \gamma'_i - \sum_{i=1}^{w} \gamma''_i P_2 \left( \frac{s''_i}{\gamma''_i} \right) \gamma''_i \right).
$$

For example, the generalized eta-quotient

$$
\frac{\eta_{5, 2}(\tau)}{\eta_{5, 1}(\tau)} = q^{\frac{1}{2}} \prod_{n=0}^{\infty} \frac{(1 - q^{5n+1})(1 - q^{5n+4})}{(1 - q^{5n+2})(1 - q^{5n+3})}
$$

is related to the famous Rogers–Ramanujan identities [2, p. 158]

$$
\sum_{i=1}^{\infty} \frac{q^{i^2}}{(1 - q) \cdots (1 - q^i)} = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{5n+1})(1 - q^{5n+4})}
$$
and
\[
\sum_{i=1}^{\infty} \frac{q^{i^2+i}}{(1-q)\cdots(1-q^i)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.
\]

**Remark 1.4** Using the first identity in \((1.2)\), one can express any function of the form \((1.1)\) in the form \((1.4)\) with \(u = v = t = x = 0\).

Define
\[
D_H := \gcd(\delta_1, \ldots, \delta_u, \delta'_1, \ldots, \delta'_v, \delta''_1, \ldots, \delta''_w, \gamma'_1, \ldots, \gamma'_x).
\]

**Theorem 1.3** Suppose \(H(\tau)\) is a generalized eta-quotient of the form \((1.4)\), with
\[
-\frac{1}{2} \sum_{i=1}^{u} \delta_i r_i + \frac{1}{2} \sum_{i=1}^{w} \frac{r''_i}{\delta'_i} + \frac{1}{2} \sum_{i=1}^{v} \frac{r'_i}{\delta'_i} - \frac{1}{2} \sum_{i=1}^{x} \gamma'_i s'_i > 0.
\]

If \(p\) is a prime divisor of \(D_H\) such that \(p^a \mid D_H\) and
\[
p^a \geq \sqrt{-\frac{1}{2} \sum_{i=1}^{u} \delta_i r_i + \frac{1}{2} \sum_{i=1}^{w} \frac{r''_i}{\delta'_i} + \frac{1}{2} \sum_{i=1}^{v} \frac{r'_i}{\delta'_i} - \frac{1}{2} \sum_{i=1}^{x} \gamma'_i s'_i},
\]
then \(H(\tau)\) is lacunary modulo \(p^j\) for any positive integer \(j\). Moreover, there exists a positive constant \(\alpha\) depending on \(p\) and \(j\) such that the number of integers \(n \leq X\) with \(p^j\) not dividing \(c(n)\) is \(O\left(\frac{X}{\log^\alpha X}\right)\).

**Remark 1.5** By Remark 1.4, an eta-quotient can be expressed using the generalized eta-quotient form in \((1.4)\). If \(t = u = v = x = 0\) in this expression, applying Theorem 1.3 recovers the results of Theorem 1.1.

In Sect. 2, we give preliminaries on modular forms, Serre’s theorem on the lacunarity of holomorphic positive integer weight modular forms with integer coefficients, as well as Dedekind eta-quotients and generalized eta-quotients. We utilize this in Sect. 3, where we provide the proofs of Theorem 1.1, Corollary 1.2, and Theorem 1.3. Lastly in Sect. 4, we conclude with specific examples of functions that empirically demonstrate our theorems as well as the potentially slow convergence of \(\delta(G, p; X)\) when \(G(\tau)\) is lacunary modulo \(p\).
2 Preliminaries

2.1 Modular Forms

For $N$ a positive integer, define the level $N$ congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ of $\text{SL}_2(\mathbb{Z})$ [9, p. 1] by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N}, \ a \equiv d \equiv 1 \pmod{N} \right\}.$$

Let $f(\tau)$ be a meromorphic function on $\mathbb{H}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $k \in \mathbb{Z}$ define the weight-$k$ “slash” operator $|_k$ by

$$(f|_k \gamma)(\tau) := (\text{det} \gamma)^{\frac{k}{2}} (c \tau + d)^{-k} f\left(\frac{a \tau + b}{c \tau + d}\right).$$

Then $f$ is a meromorphic modular form of weight $k$ on a congruence subgroup $\Gamma$, i.e. is modular on $\Gamma$, if (1) for all $\gamma \in \Gamma$ we have $(f|_k \gamma)(\tau) = f(\tau)$, and (2) for any $\gamma \in \text{SL}_2(\mathbb{Z})$, $(f|_k \gamma)(\tau)$ has a Fourier expansion of the form

$$(f|_k \gamma)(\tau) = \sum_{n \geq n_\gamma} a_\gamma(n) q^N_n,$$

with $q_N := e^{2\pi i \tau/N}$ and $a_\gamma(n_\gamma) \neq 0$ [9, p. 3].

Define a cusp of $\Gamma$ to be an equivalence class of $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ under the action of $\Gamma$. We say that $f(\tau)$ is a holomorphic modular form on $\Gamma$ if it is holomorphic on $\mathbb{H}$ and at all the cusps of $\Gamma$ (the latter occurring if $n_\gamma \geq 0$ for every $\gamma$). The quantity $n_\gamma$, called the order of vanishing at $\frac{-d}{c}$, depends only on the equivalence class of the cusp $-\frac{d}{c}$. The following proposition gives complete sets of representatives for the cusps of $\Gamma_0(N)$ and $\Gamma_1(N)$ (see [3, p. 99] and [12]).

Proposition 2.1 Let

$$C_0(N) := \left\{ \frac{c}{d} : d \mid N, \ (c, N) = 1 \right\},$$

where $c$ runs through a complete residue system modulo $d$, $\frac{N}{d}$, and

$$C_1(N) := \left\{ \frac{\lambda}{\mu} : \epsilon \mid N, \ 1 \leq \lambda, \mu \leq N, \ (\mu, \lambda) = (\lambda, N) = (\mu, N) = 1 \right\}.$$

Then $C_0(N)$ (resp. $C_1(N)$) is a complete set of representatives of the cusps on $\Gamma_0(N)$ (resp. $\Gamma_1(N)$). Moreover, $C_0(N)$ is minimal.
2.2 Serre’s Theorem

The following deep theorem of Serre [6,13], which is proved using the theory of Galois representations, is essential for the proofs of Theorems 1.1 and 1.3.

Theorem 2.2 (Serre) Let \( f(\tau) \) be a holomorphic modular form of positive integer weight with Fourier expansion

\[
f(\tau) = \sum_{n=0}^{\infty} c(n)q^n,
\]

where \( c(n) \) is an integer for all \( n \). If \( M \) is a positive integer, then there exists a positive constant \( \alpha \) such that the number of integers \( n \leq X \) with \( M \) not dividing \( c(n) \) is \( O\left(\frac{X}{\log^2 X}\right) \).

Remark 2.1 Serre proves a more general result addressing the case where the \( c(n) \) lie in the ring of integers of an algebraic number field.

2.3 Dedekind Eta-Quotients

The following theorems (see [9, p. 18]) give conditions for an eta-quotient to be a modular form on \( \Gamma_0(N) \).

Theorem 2.3 If \( f(\tau) = \prod_{\delta|N} \eta(\delta \tau)^{r_\delta} \) is an eta-quotient with \( k = \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z} \) such that

\[
\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}
\]

and

\[
\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},
\]

then

\[
f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau)
\]

for all \( \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_0(N) \), where \( \chi(d) := \left(\frac{-1}{d}\right)^{s_\delta} \right) with \( s := \prod_{\delta|N} \delta^{r_\delta}. \)

Theorem 2.4 Let \( c, d, \) and \( N \) be positive integers with \( d \mid N \) and \( (c, d) = 1 \). If \( f(\tau) \) is an eta-quotient satisfying the conditions of Theorem 2.3, then the order of vanishing of \( f(\tau) \) at the cusp \( \frac{c}{d} \) is given by
\[
\frac{N}{24d \left( \frac{N}{d} \right)} \sum_{\delta \mid N} (d, \delta)^2 r_{\delta}.
\]

### 2.4 Generalized Eta-Quotients

We now recall results on generalized eta-quotients analogous to Theorems 2.3 and 2.4.

**Theorem 2.5** ([12, Theorem 3]) If \( f(\tau) = \prod_{\delta \mid N} \eta_{\delta, g}(\tau) \) is such that

\[
\sum_{\delta \mid N} \delta P_2 \left( \frac{g}{\delta} \right) r_{\delta, g} \equiv 0 \pmod{2}
\]

and

\[
\sum_{\delta \mid N} \frac{N}{6\delta} r_{\delta, g} \equiv 0 \pmod{2},
\]

then \( f(\tau) \) is modular on \( \Gamma_1(N) \).

**Theorem 2.6** ([12, Theorem 4]) If \( f(\tau) \) satisfies the hypotheses of Theorem 2.5, then the order of vanishing at the cusp \( \frac{\lambda}{\mu} \in C_1(N) \) in the uniformizing variable \( q \frac{e}{N} \) is

\[
\frac{N}{2} \sum_{\delta \mid N} \frac{(\delta, \epsilon)^2}{\delta \epsilon} P_2 \left( \frac{\lambda g}{(\delta, \epsilon)} \right) r_{\delta, g}.
\]

### 3 Proof of Main Results

#### 3.1 Proof of Theorem 1.1

We begin by constructing an integer weight holomorphic modular form that is the product of \( G(24\tau) \) and an eta-quotient that is congruent to 1 modulo \( p^j \). For \( a \in \mathbb{Z}^+ \) define

\[
f_{G, p^a}(\tau) := \prod_{i=1}^t \left( \frac{\eta^{p^a}(24\gamma_1 \tau)}{\eta(24p^a\gamma_1 \tau)} \right)^{s_i} = \prod_{i=1}^t \prod_{n=1}^\infty \frac{(1 - q^{-24\gamma_1 n})^{p^a s_i}}{(1 - q^{-24p^a\gamma_1 n})^{s_i}}.
\]
By the binomial theorem we have \( f_{G,p^a}(\tau) \equiv 1 \mod p \), and a straightforward induction argument yields that \( f_{G,p^a}^{p^j} \equiv 1 \mod p^{j+1} \) for all \( j \geq 0 \). Now define

\[
F_{G,p^a,j}(\tau) := G(24\tau) f_{G,p^a}^{p^j}(\tau) = \frac{\prod_{i=1}^{u} \eta^{r_i}(24\delta_i \tau)}{\prod_{i=1}^{t} \eta^{s_i}(24\gamma_i \tau)} \left( \frac{(\eta p^a(24\gamma_i \tau))}{(\eta(24p^a\gamma_i \tau))} \right)^{s_i p^j}.
\]

Since \( f_{G,p^a}^{p^j} \equiv 1 \mod p^{j+1} \), we have that

\[
F_{G,p^a,j}(\tau) \equiv G(24\tau) = q^{E_G} \sum_{n=0}^{\infty} b(n)q^{24n} \mod p^{j+1}.
\] (3.1)

Let

\[
L_G = \text{lcm}(\delta_1, \ldots, \delta_u, \gamma_1, \ldots, \gamma_t).
\]

One can verify that for all \( j \geq 1 \), \( F_{G,p^a,j} \) satisfies the hypotheses of Theorem 2.3 with \( k = \frac{1}{2} \left( \sum_{i=1}^{u} r_i - t \sum_{i=1}^{t} s_i + \sum_{i=1}^{t} s_i p^j (p^a - 1) \right) \) and \( N = 576L_G^2 \). To conclude that \( F_{G,p^a,j} \) is a holomorphic modular form we need only check its behavior at the cusps, and by Proposition 2.1 we may restrict our attention to the cusps of the form \( \frac{c}{d} \) with \( d \mid 576L_G^2 \). By Theorem 2.4, we have that \( F_{G,p^a,j} \) is holomorphic at a cusp \( \frac{c}{d} \) if and only if

\[
p^a \frac{\sum_{i=1}^{u} r_i (d, 24\delta_i)^2}{\sum_{i=1}^{t} \frac{s_i}{\gamma_i} (d, 24p^a\gamma_i)^2} + p^a \frac{\sum_{i=1}^{t} \frac{s_i}{\gamma_i} (d, 24\gamma_i)^2 (p^{a+j} - 1)}{\sum_{i=1}^{t} \frac{s_i}{\gamma_i} (d, 24p^{a}\gamma_i)^2} \geq p^{j}.
\]

Since \( (d, 24p^a\gamma_i)^2 \leq p^{2a} (d, 24\gamma_i)^2 \) for each \( 1 \leq i \leq t \), it is sufficient to show

\[
p^a \frac{\sum_{i=1}^{u} r_i (d, 24\delta_i)^2}{\sum_{i=1}^{t} \frac{s_i}{\gamma_i} (d, 24p^{a}\gamma_i)^2} + p^j - \frac{1}{p^a} \geq p^{j}.
\]
By assumption \( p^a \) divides every \( \delta_i \) so \((d, 24p^a) \leq (d, 24\delta_i) \). Since we have \((d, \gamma_i) \leq \gamma_i \), it follows that \( F_{G, p^a, j}(\tau) \) is holomorphic at \( \frac{c}{d} \) if

\[
p^a \sum_{i=1}^{u} \frac{r_i}{\delta_i} - \frac{1}{p^a} \geq 0,
\]

that is, if

\[
p^a \geq \sqrt{\frac{\sum_{i=1}^{t} s_i \gamma_i}{\sum_{i=1}^{u} r_i / \delta_i}}.
\]

By Theorem 2.2, it follows that for \( F_{G, p^a, j}(\tau) = \sum b'(n)q^n \) there is a positive constant \( \alpha \) such that there are at most \( O\left(\frac{X}{\log x} X\right) \) integers \( n \leq X \) for which \( b'(n) \) is not divisible by \( p^{j+1} \). By (3.1), the same holds for \( b(n) \).

**Proof of Corollary 1.2** Recall that

\[
G_{1,t,z}(\tau) = q^{\frac{1-tz}{24}} \frac{\eta^t(t\tau)}{\eta(t)} \quad \text{and} \quad G_{-1,t,z}(\tau) = q^{\frac{1-tz}{24}} \frac{\eta^{2t-z}(t\tau)\eta^{t+z}(4t\tau)}{\eta(t)\eta^{3(t-z)}(2t\tau)}.
\]

Thus \( \mathcal{D}_{G_{1,t,z}} = \mathcal{D}_{G_{-1,t,z}} = t \). Applying Theorem 1.1, we have that if \( p \) is a prime divisor of \( t \) such that \( p^a \mid t \) and

\[
p^a \geq \sqrt[4]{\frac{t}{z}} = \sqrt[4]{\frac{t}{z}}.
\]

then \( G_{1,t,z}(\tau) \) is lacunary modulo \( p^{j} \) for any \( j \geq 1 \) which proves (1). If \( p \) is a prime divisor of \( t \) such that \( p^a \mid t \) and

\[
p^a \geq \sqrt[4]{\frac{1 + 2t(3(t-z))}{\frac{2t-z}{t} + \frac{t-z}{4t}}} = 2 \sqrt{\frac{t + 6t^3 - 6t^2z}{9t - 5z}},
\]

then \( G_{-1,t,z}(\tau) \) is lacunary modulo \( p^{j} \) for any \( j \geq 1 \), completing the proof of (2).
3.2 Proof of Theorem 1.3

Define

\[ L_H := \text{lcm}(\delta_1, \ldots, \delta_u, \gamma_1, \ldots, \gamma_v, \delta_1', \ldots, \delta_v', \gamma_1', \ldots, \delta_1'', \gamma_1'', \ldots, \gamma_v''), \]

and let \( \tilde{N} = 24L_H. \) For any

\[ \delta \in \{ \delta_1, \ldots, \delta_u, \gamma_1, \ldots, \gamma_v, \delta_1', \ldots, \delta_v', \gamma_1', \ldots, \delta_1'', \gamma_1'', \ldots, \gamma_v'' \} \]

and \( g \in \mathbb{Z}, \) we have that \( P_2(\frac{\tilde{N}}{p^i}) \delta \tilde{N} \in 2 \mathbb{Z}. \) For \( a \in \mathbb{Z}^+, \) define

\[
 f_{H, p^a}^j(\tau) := H(\tilde{N}\tau) f_{H, p^a}^{j}(\tau) \]

\[ = \prod_{i=1}^{t} \eta_{\gamma_i, g_i}(\tilde{N}\tau) \cdot \prod_{i=1}^{v} \eta_{\delta_i', \delta_i'}(\tilde{N}\tau) \cdot \prod_{i=1}^{w} \eta_{\gamma_i'', \gamma_i''}(\tilde{N}\tau) \cdot f_{H, p^a}^{j}(\tau). \]

Then

\[ F_{H, p^a, j}(\tau) \equiv H(\tilde{N}\tau) = q^{\tilde{N}E_H} \sum c(n)q^{\tilde{N}n} \pmod{p^{j+1}}, \]

where \( E_H, \) defined after (1.4), is the order of vanishing at infinity of \( H(\tau). \)

Note that by (1.3), \( H(\tilde{N}\tau) \) satisfies the hypothesis of Theorem 2.5 with weight \( k_H = w - y \) and \( N = 576L_H^2. \) Furthermore, one can verify that for any \( j \geq 1, \) \( f_{H, p^a}^{j}(\tau) \) also satisfies the hypotheses of Theorem 2.5 with \( k = \left( \sum_{i=1}^{t} s_i + \sum_{i=1}^{v} r_i' + \sum_{i=1}^{w} s_i'' \right) p^j(p^a - 1) \) and \( N = 576L_H^2. \) Thus, to conclude that \( F_{H, p^a, j} \) is a holomorphic modular form on \( \Gamma(576L_H^2), \) it suffices to check its behavior at the cusps. Theorem 2.6 shows that \( F_{H, p^a, j} \) is holomorphic at a cusp \( \frac{\lambda}{\mu \varepsilon} \in C_1(N) \) if and only if

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Consider the weight one eta-quotient 

\[ J(\tau) := q^{\frac{\epsilon}{2}} G_{1.18.3}(\tau) = \frac{\eta^3(18\tau)}{\eta(\tau)}, \]

which is not holomorphic at the cusp \( \frac{1}{6} \). Since \( 3^2 \) divides \( 18 \) and \( 3^2 \geq \sqrt{6} \), by Corollary 1.2 we have that \( \delta(J, 3; X) \rightarrow 1 \). In contrast, \( 2 < \sqrt{6} \) and \( 5 \mid 18 \), so the corollary does not apply when \( p \in \{2, 5\} \), and we see in Table 2 that \( \delta(J, 2; X) \) and \( \delta(J, 5; X) \)

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tend towards 0.5 and 0.2, respectively. Table 2 shows some values for $\delta(J, p; X)$ for $p \in \{2, 3, 5\}$.

### 4.2 Example of Theorem 1.3

Consider the generalized eta-quotient

$$K(\tau) = \frac{\eta_{9,0}(\tau)}{\eta_{6,1}(\tau)}.$$  

At the cusp $\frac{1}{3}$, $K(\tau)$ is not holomorphic. Since $D_K = 9$, we have that $p^a \mid D_K$ for $p = 3, a = 2$. Furthermore, we have that $K(\tau)$ satisfies the inequalities

$$\frac{r''_1}{\delta''_1} = \frac{1}{9} > 0$$

and

$$p^a = 3^2 = 9 \geq \sqrt{54} = \sqrt{\frac{6 \cdot 1}{\frac{1}{9}}} = \sqrt{\frac{\gamma_1 s_1}{r''_1 \delta''_1}},$$

so by Theorem 1.3, $K(\tau)$ is lacunary modulo $3^j$ for any positive integer $j$. Table 3 illustrates the extremely slow convergence of $\delta(K, 3; X)$ to 1. As before, $5 \nmid D_K$ and $\delta(K, 5; X)$ appears to converge to 0.2 instead of 1.
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