PRICING VULNERABLE OPTIONS UNDER A JUMP-DIFFUSION MODEL WITH FAST MEAN-REVERTING STOCHASTIC VOLATILITY

WAN-HUA HE
Advanced Modeling and Applied Computing Laboratory, Department of Mathematics
The University of Hong Kong, Pokfulam Road, Hong Kong, China

CHUFANG WU*
Department of Mathematics, The University of Hong Kong
Pokfulam Road, Hong Kong, China
Department of Mathematics, Southern University of Science and Technology
Shenzhen, China

JIA-WEN GU
Department of Mathematics, Southern University of Science and Technology
Shenzhen, China

WAI-KI CHING AND CHI-WING WONG
Department of Mathematics, The University of Hong Kong
Pokfulam Road, Hong Kong, China

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Abstract. In this paper, we propose a model to price vulnerable European options where the dynamics of the underlying asset value and the counter-party’s asset value follow two jump-diffusion processes with fast mean-reverting stochastic volatility. First, we derive an equivalent risk-neutral measure and transfer the pricing problem into solving a partial differential equation (PDE) by the Feynman-Kac formula. We then approximate the solution of the PDE by pricing formulas with constant volatility via multi-scale asymptotic method. The pricing formula for vulnerable European options is obtained by applying a two-dimensional Laplace transform when the dynamics of the underlying asset value and the counter-party’s asset value follow two correlated jump-diffusion processes with constant volatilities. Thus, an analytic approximation formula for the vulnerable European options is derived in our setting. Numerical experiments are given to demonstrate our method by using Laplace inversion.

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* Corresponding author: Chufang Wu.
1. **Introduction.** A vulnerable option is an option on defaultable instruments and it is subject to the credit default risk, which is traded in over-the-counter (OTC) markets. In OTC markets, option holders are exposed to credit default risk because necessary collateral is not required. Once the option writer (i.e., the counter-party) defaults, the option holder can only receive a proportion of its non-default asset value. There are many examples of vulnerable options, especially those privately written and not guaranteed by a third party, e.g., the currency options, options on precious metals and real estate options, etc. Furthermore, some contracts can also be viewed as vulnerable options if there are concerns about the financial health of their contract writing companies, e.g., insurance contracts and forward contracts between manufacturers and suppliers.

Vulnerable options pricing model was first proposed by Johnson and Stulz [9] in 1987, incorporating credit default risk into options pricing under the assumption that the option itself is the only liability of the counter-party. They defined that an event of default occurs when the counter-party’s asset value at maturity (i.e. terminal asset value) falls below some prescribed boundaries. When a default occurs, the payoff is the counter-party’s asset. However, this assumption is only appropriate when the option’s payoff is larger than the counter-party’s asset, which is not true in most business settings. Another definition of a default event is that the counter-party’s asset value falls below a specific boundary any time before the maturity and the payoff ratio in default is assumed to be independent of the counter-party’s asset value. Jarrow and Turnbull [8], Hull and White [7] extended the model and obtained the pricing formula for vulnerable options by adopting the second approach. But it is inappropriate to assume independence between payoff in default and the counter-party’s asset value because counter-party may recover and pay the claims in full on the exercising date. In 1996, Klein [10] extended Johnson and Stulz’s model by allowing the counter-party to have other liabilities, and the underlying asset and the counter-party’s asset to have a correlation. More precisely, the payoff in default depends on the counter-party’s terminal asset value and the payoff ratio in default is endogenous. Following the framework in [10], Klein and Inglis [11] derived an analytical pricing formula for vulnerable European options subject to interest rate risk. They developed a new approach [12] where the default boundary depends on both potential liability of the option and other liability of the counter-party. Hung and Liu [6] priced vulnerable options in an incomplete market under the assumption that the option also faces the risk of illiquidity. Liu and Liu [15] used binomial tree algorithm to price vulnerable options. Klein and Yang [13] studied the vulnerable American options, where the default barrier depends on the payoff of the option before maturity.

Recently, many studies on options pricing incorporate stochastic volatility to explain the phenomena in real market, such as volatility smile and the mean-reverting features of implied volatility. The Heston model [5] and the Fouque-Papanicolaou-Sircar (FPS) model ([4] and [2]) are intensively used in these literature. Fouque et al. [3] found that volatility reverts slowly to its mean in comparison to the tick-by-tick timescale, but it is fast mean-reverting over the lifetime of a derivative contract (a few months). The distinction between these time scales motivates an asymptotic analysis of the PDE satisfied by derivative prices. Following Klein’s framework [10], Yang et al. [28] derived an analytical approximation formula for the price of vulnerable options under the FPS model via asymptotic analysis; Wang
et al. [26] investigated vulnerable options pricing with stochastic volatility by decomposing stochastic volatility into long-term constant volatility and short-term mean-reverting stochastic volatility, and derived the pricing formula in a special case. Following the existing literature, we assume that the volatility of the underlying asset and counter-party’s asset follow a fast mean-reverting Ornstein-Uhlenbeck (OU) process (see [4] and [28]) and allow correlations among the volatility, underlying asset and counter-party’s asset in our model.

This paper aims to extend Klein’s model to a jump-diffusion model with arbitrary jump size distributions. Jumps can be used to model sudden changes in stock price and the jump-diffusion process can explain why the empirical asset return distributions have heavier tail than the normal distributions [19]. Similar conclusion can be found in Eraker’s work [1], where they examined the empirical performance of jump-diffusion models of stock prices from joint options and stock market data and found that these models can fit the options and stocks data simultaneously. There are two popular jump-diffusion models, Merton’s model [18] with normally distributed jump size and double-exponential jump-diffusion model proposed by Kou [14]. Most of the existing literature on vulnerable options subject to jump-diffusion processes assume that the volatility is constant and both the underlying asset and the counter-party’s asset follow jump-diffusion processes with specific jump size distributions, see, for instance Xu et al. [27], Tian et al. [25], Niu et al. [20, 21]. This paper investigates the vulnerable options pricing problem under a jump-diffusion model with fast mean-reverting stochastic volatility. In order to study the approximation formula for the price of vulnerable options, we also investigate vulnerable options pricing with constant volatility and assume that both the dynamics of the underlying asset and the counter-party’s asset follow jump-diffusion processes. Consequently, we obtain the analytical pricing formula for the vulnerable European options via two-dimensional Laplace transforms, which can be numerically implemented through Euler inversion method [22].

Ma, Yue and Ren [16] also studied valuations of vulnerable European options where the dynamics of underlying asset value and counter-party’s asset value follow two correlated exponential Lévy processes. In their work, the diffusion process of the underlying asset and the counter-party’s asset are stochastic and have the same formula of stochastic volatility process. We extend their model to the general case, allowing the diffusion processes to be any bounded function of stochastic volatility. Then we derive the analytical approximation formula for the price of vulnerable options via asymptotic analysis, while they do it by using the Fourier inversion formula for distribution functions.

The rest of the paper is organized as follows. In Section 2, we present a jump-diffusion model with fast mean-reverting stochastic volatility and obtain the equivalent martingale measure for vulnerable option pricing. In Section 3, we derive the PDE for the option price and employ multi-scale asymptotic analysis to approximate the option price by the pricing formula under jump-diffusion model with constant volatilities. Section 4 presents the analytical vulnerable options pricing formula under jump-diffusion model with constant volatilities via a two-dimensional Laplace transform. In Section 5, numerical experiments are given by Euler inversion algorithm. Finally, Section 6 concludes the paper.

2. The basic model. Following Klein’s framework [10], the price of vulnerable option is determined by both the underlying asset value and the counter-party’s
asset value. In reality, bad news may trigger sudden jumps on the asset value, such as stock price. Therefore, a jump-diffusion process with arbitrary jump size and stochastic volatility is used to model the underlying asset value and the counter-party’s asset value. Besides, our model allows correlations among volatility, the underlying asset value and the counter-party’s asset value.

2.1. An asset model. Given a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\mathbb{P}\) is a real-world probability measure, the dynamics of the underlying asset \(S_t\) and the counter-party’s asset \(V_t\) are given by:

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu_s S_t dt + g(Y_t) S_t dB^S_t + S_t d\left( \sum_{i=1}^{N^S_t} (e^{\xi_i} - 1) - \lambda^S \beta^S_t \right), \\
\frac{dV_t}{V_t} &= \mu_v V_t dt + f(Y_t) V_t dB^V_t + V_t d\left( \sum_{i=1}^{N^V_t} (e^{\nu_i} - 1) - \lambda^V \beta^V_t \right),
\end{align*}
\]

where \(\mu_s, \mu_v\) are constant return rates for the underlying asset and the counter-party’s asset, respectively. Here \(B^S_t, B^V_t\) are standard Brownian motions with correlation \(\rho_{sv} dB^S_t dB^V_t = \rho_{sv} dt, dB^S_t dB^V_t = \rho_{sv} dt\) and \(dB^S_t dB^V_t = \rho_{vy} dt\), where \(\rho_{sv}, \rho_{sy}\) and \(\rho_{vy}\) are constants\(^1\). Moreover, \(g(\cdot)\) and \(f(\cdot)\) are nonnegative bounded functions. \(N^S_t\) and \(N^V_t\) are Poisson processes with density \(\lambda^S\) and \(\lambda^V\), respectively. \(\Xi(\xi)\) and \(\Upsilon(\nu)\) are the density function of \(e^{\xi} - 1\) and \(e^{\nu} - 1\), with expectation of \(\beta^S = \mathbb{E}[e^{\xi} - 1]\) and \(\beta^V = \mathbb{E}[e^{\nu} - 1]\).

We apply the analytical approximation formula under the FPS model derived by Yang et al.\(^2\) to describe \(Y_t := \log \sigma_t\),

\[
dY_t = \frac{1}{\epsilon} (m - Y_t) dt + \frac{u \sqrt{2}}{\sqrt{\epsilon}} dB^V_t,
\]

where \(m\) is long-term mean, \(\frac{1}{\epsilon}\) is mean-reverting speed parameter, and \(u\) is standard deviation. Then the volatility is the exponential of a mean-reverting OU process.

2.2. A vulnerable European option model. We adopt Klein’s model\(^1\) and allow the counter-party to have other liabilities, then the option holder only receives a proportion of the remaining asset once the default occurs. The payoff of a vulnerable call option is given by

\[
c(S_T, V_T) = (S_T - K)^+ \left(1_{\{V_T \geq \hat{D}\}} + 1_{\{V_T < \hat{D}\}} \frac{(1 - \gamma)V_T}{D}\right),
\]

where \(K\) is the strike price and \(\gamma\) is the deadweight costs associated with bankruptcy expressed as a percentage of the counter-party’s asset value. \(\hat{D}\) is a fixed default boundary such that a credit loss occurs if the value of the option writer’s asset \(V_T\) falls below \(\hat{D}\). The default boundary \(\hat{D}\) is related to the value of total liabilities \(D\) and maybe less than \(D\) because the counter-party may keep operating even while \(V_T < D\).

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\(^1\)\(\rho_{sv}\) should be negative because when volatility goes up, the stock price tends to go down and vice-versa.

\(^2\)Mean-reverting is one of the observed volatility features, which is commonly used in literature, see \([4], [2]\) and \([17]\).
2.3. An equivalent martingale measure. In this section, we present an equivalent martingale measure under a jump-diffusion model with stochastic volatility. We assume that the jump parts are independent of the diffusion parts. Following [4], we first orthogonalize \( \{ B_i^S, B_i^Y, B_i^V \} \) into three independent standard Brownian motions and each of them is independent of \( \sum_{i=1}^{N^S} \xi_i, \sum_{i=1}^{N^Y} \eta_i, \sum_{i=1}^{N^V} \nu_i \), which are compound Poisson processes with intensity \( \tilde{\lambda}^S = \lambda^S, \tilde{\lambda}^Y = \lambda^Y \) and the mean jump size \( \tilde{\beta}^S = \beta_S, \tilde{\beta}^V = \beta_V \), respectively. The dynamics of \( S_t, Y_t \) and \( V_t \) are given as follows:

\[
\begin{align*}
\frac{dS_t}{S_t} &= rS_t\,dt + g(Y_t)S_t\,dB_t^S + S_t\,\left( \sum_{i=1}^{N^S} (e^{\xi_i} - 1) - \tilde{\lambda}^S \tilde{\beta}^S t \right), \\
\frac{dY_t}{Y_t} &= \left( \frac{1}{\epsilon} (m - Y_t) - \frac{u\sqrt{2}}{\sqrt{\epsilon} \Lambda(Y_t)} \right) dt + \frac{u\sqrt{2}}{\sqrt{\epsilon}} dB_t^Y, \\
\frac{dV_t}{V_t} &= rV_t\,dt + f(Y_t)V_t\,dB_t^V + V_t\,\left( \sum_{i=1}^{N^V} (e^{\nu_i} - 1) - \tilde{\lambda}^V \tilde{\beta}^V t \right),
\end{align*}
\]

where

\[
\begin{align*}
B_t^S &= B_t^S, \\
B_t^Y &= \rho_{xy} B_t^S + \sqrt{1 - \rho_{xy}^2} B_t^Y, \\
B_t^V &= \rho_{xy} B_t^S + \rho_{yx} \rho_{xy} B_t^Y + \frac{(1 - \rho_{xy}^2)(1 - \rho_{yx}^2) - (\rho_{xy} \rho_{yx} a)^2}{\sqrt{1 - \rho_{xy}^2}} B_t^V.
\end{align*}
\]

We then define

\[
\begin{align*}
\theta_t^S &= \frac{\mu_s - r}{g(Y_t)}, \\
\theta_t^Y &= \frac{\sqrt{1 - \rho_{xy}^2}}{\sqrt{(1 - \rho_{xy}^2)(1 - \rho_{yx}^2) - (\rho_{xy} \rho_{yx} a)^2}} \left( \frac{\mu_v - r}{f(Y_t)} - \rho_{xy} \frac{\mu_s - r}{g(Y_t)} - \rho_{yx} \rho_{xy} a \phi_t \right), \\
Z_t &= \exp \left\{ -\frac{1}{2} \int_0^t \sum_{i=S,Y,V} (\theta_s^i)^2 ds - \sum_{i=S,Y,V} \int_0^t \theta_s^i dB_s^i \right\},
\end{align*}
\]

where \( r \) is the risk-free rate and \( \phi_t \) is the volatility risk premium. For the conciseness of the paper, we adopt \( \phi_t = \phi \) (a constant). Fix a positive \( T \) and define

\[
\mathbb{P}^*(A) = \int_A Z_T \, d\mathbb{P}.
\]

We have the following proposition.

**Proposition 1.** Under the probability measure \( \mathbb{P}^* \), the processes

\[
\tilde{B}_t^i := B_t^i + \int_0^t \theta_s^i ds, \quad \text{for } i = S, Y, V,
\]

are mutually independent Brownian motions and each of them is independent of \( \sum_{i=1}^{N^S} (e^{\xi_i} - 1) \) and \( \sum_{i=1}^{N^Y} (e^{\eta_i} - 1) \), which are compound Poisson processes with intensity \( \tilde{\lambda}^S = \lambda^S, \tilde{\lambda}^Y = \lambda^Y \) and the mean jump size \( \tilde{\beta}^S = \beta_S, \tilde{\beta}^V = \beta_V \), respectively.
and $\Lambda$ is given by

$$\Lambda(y) = \left(\mu_s - r\right)\rho_{sy} + \phi \sqrt{1 - \rho_{sy}^2}.$$  

**Proof.** See Appendix 7.1. \hfill \Box

3. **An asymptotic analysis.** Under the risk neutral measure $\mathbb{P}^*$, the price of a vulnerable call option is given by

$$P(t, s, v, y) = \mathbb{E}^{\mathbb{P}^*}\left[e^{-r(T-t)}c(S_T, V_T)|S_t = s, V_t = v, Y_t = y\right]. \quad (3)$$

Based on the Feynman-Kac formula, $P(t, s, v, y)$ is the solution of the following partial differential equation (PDE):

$$-rP + \frac{\partial P}{\partial t} + (r - \tilde{\lambda}^S \tilde{\beta}^S) s \frac{\partial P}{\partial s} + \frac{s^2 g(y)^2}{2} \frac{\partial^2 P}{\partial s^2} + (r - \tilde{\lambda}^V \tilde{\beta}^V) v \frac{\partial P}{\partial v} + \frac{v^2 f(y)^2}{2} \frac{\partial^2 P}{\partial v^2}$$

$$+ \rho_s v f(y) \frac{\partial^2 P}{\partial s \partial v} + \tilde{\lambda}^S \int_{\mathbb{R}} P(t, (s + 1), v, y) - P(t, s, v, y)) \Xi(\xi) d\xi$$

$$+ \tilde{\lambda}^V \int_{\mathbb{R}} P(t, s, (v + 1), y) \Upsilon(k) dk = 0,$$

with final condition

$$P(T, s, v, y) = (s - K)^+ \left(1_{\{v \geq D\}} + 1_{\{v < D\}} \frac{1 - \gamma}{D} \right).$$

To solve Eq. (4), we adopt the multi-scale asymptotic analysis for continuous case from [4] and [28]. Define the operator $I$ by

$$(IP)(t, s, v, y) = -(r + \tilde{\lambda}^S + \tilde{\lambda}^V) P(t, s, v, y) + \tilde{\lambda}^S \int_{\mathbb{R}} P(t, (s + 1), v, y) \Xi(\xi) d\xi$$

$$+ \tilde{\lambda}^V \int_{\mathbb{R}} P(t, s, (v + 1), y) \Upsilon(k) dk,$$

and assume that the solution of the PDE is in the form

$$P^* = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 P_3 + \ldots,$$

with $0 < \epsilon \ll 1$. Then we rearrange the PDE as follows

$$\left(\frac{1}{\epsilon} L_0 + \frac{1}{\sqrt{\epsilon}} L_1 + L_2\right) P^*(t, s, v, y) = 0,$$

where

$$L_0 = u^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y},$$

$$L_1 = \sqrt{2} w \rho_{sy} g(y) \frac{\partial^2}{\partial s \partial y} + \sqrt{2} w v \rho_{sv} f(y) \frac{\partial^2}{\partial y \partial v} - \sqrt{2} u \Lambda(y) \frac{\partial}{\partial y},$$

$$L_2 = \frac{\partial}{\partial t} + \frac{s^2 g(y)^2}{2} \frac{\partial^2}{\partial s^2} + \frac{v^2 f(y)^2}{2} \frac{\partial^2}{\partial v^2} + \rho_{sv} v f(y) g(y) \frac{\partial^2}{\partial s \partial v} + \frac{(r - \tilde{\lambda}^S \tilde{\beta}^S)s}{\partial s} + \frac{(r - \tilde{\lambda}^V \tilde{\beta}^V)v}{\partial v} + I.$$

By substitution, one can obtain

$$\frac{1}{\epsilon} L_0 P_0 + \frac{1}{\sqrt{\epsilon}} (L_0 P_1 + L_1 P_0) + (L_0 P_2 + L_1 P_1 + L_2 P_0) + \sqrt{\epsilon} (L_0 P_3 + L_1 P_2 + L_2 P_1) + \ldots = 0.$$

In order to get the leading order term $P_0$ and the correction term $P_1$, we first present the following lemma about the solvability (centering) condition for Poisson
equations. Denote $< q >$ the expectation of function $q$ with respect to normal distribution $\mathcal{N}(m, u)$:

$$< q > := \frac{1}{\sqrt{2\pi u^2}} \int_\mathbb{R} q(y)e^{-\frac{(y-m)^2}{2u^2}}dy.$$  

Then the following lemma is obtained immediately by applying Fredholm alternative theorem ([23]) to the Poisson equation.

**Lemma 3.1.** If $q \in C^2(\mathbb{R})$ and solution $X$ of the Poisson equation $\mathcal{L}_0X + q = 0$ exists, then the condition $< q > = 0$ holds.

With lemma 3.1, the characteristics of the leading order term $P_0$ and the correction term $P_1$ are presented in Theorem 3.2 and 3.3.

**Theorem 3.2.** Assume that, for each $i = 0, 1, 2, \ldots$, $P_i$ does not grow as much as $\frac{\partial P_i}{\partial y} \sim e^{\frac{y^2}{2}}$ when $y$ goes to infinity, then the leading order term $P_0$ is independent of variable $y$ and satisfies the PDE:

$$\frac{\partial P_0}{\partial t} + \frac{s^2\beta^2}{2} \frac{\partial^2 P_0}{\partial t^2} + \frac{s^2f^2}{2} \frac{\partial^2 P_0}{\partial x^2} + \rho_{su}vus < f g > \frac{\partial^2 P_0}{\partial x \partial v} + (r - \lambda S \beta^2) \frac{\partial P_0}{\partial x} + (r - \lambda S \lambda^2) \frac{\partial P_0}{\partial v} - (r + \lambda S + \lambda^2) P_0 + \lambda S \int_\mathbb{R} P_0(t, (\xi + 1)s, v, y) \Xi(\xi) d\xi + \lambda^2 \int_\mathbb{R} P_0(0, t, (k + 1)v, y) \Upsilon(k) dk = 0$$

with final condition:

$$P_0(T, s, v) = (s - K)^+ \left(1_{\{v \leq D\}} + 1_{\{v < D\}} \frac{(1 - \gamma)D}{D} \right).$$

**Proof.** See Appendix 7.2.

**Theorem 3.3.** Assume that, for each $i = 0, 1, 2, \ldots$, $P_i$ does not grow as much as $\frac{\partial P_i}{\partial y} \sim e^{\frac{y^2}{2}}$ when $y$ goes to infinity, then the correction term $P_1$ is independent of variable $y$ and satisfies the PDE:

$$\frac{\partial P_1}{\partial t} + \frac{s^2\beta^2}{2} \frac{\partial^2 P_1}{\partial t^2} + \frac{s^2f^2}{2} \frac{\partial^2 P_1}{\partial x^2} + \rho_{su}vus < f g > \frac{\partial^2 P_1}{\partial x \partial v} + (r - \lambda S \beta^2) \frac{\partial P_1}{\partial x} + (r - \lambda S \lambda^2) \frac{\partial P_1}{\partial v} - (r + \lambda S + \lambda^2) P_1 + \lambda S \int_\mathbb{R} P_1(t, (\xi + 1)s, v, y) \Xi(\xi) d\xi + \lambda^2 \int_\mathbb{R} P_1(0, t, (k + 1)v, y) \Upsilon(k) dk = G(t, s, v),$$

where

$$G(t, s, v) := \frac{1}{\sqrt{2\pi}} \rho_{su}us^3 < g \psi_1' > \frac{\partial^3 P_0}{\partial t^3} + \frac{\partial^3 P_0}{\partial t^2} \rho_{su}uv^3 < f \psi_2' > \frac{\partial^3 P_0}{\partial t^2} + \frac{\partial^3 P_0}{\partial t} \rho_{su}uv^3 < f \psi_2' > \frac{\partial^3 P_0}{\partial t} - \frac{\partial^3 P_0}{\partial s \partial t} < f \psi_2 > \frac{\partial^3 P_0}{\partial s \partial t}.$$

with the final condition $P_1(T, s, v) = 0$, and $\psi_1(y), \psi_2(y)$ and $\psi_3(y)$ are solutions of

$$\mathcal{L}_0\psi_1(y) = g(y)y^2 - < g^2 >, \quad \mathcal{L}_0\psi_2(y) = f(y)y^2 - < f^2 >, \quad \mathcal{L}_0\psi_3(y) = f(y)y(g(y) - < fg >.)$$

**Proof.** See Appendix 7.3.
Theorem 3.2 indicates that the leading order term \( P_0 \) is the vulnerable option price with constant volatility, and the correction term \( P_1 \) in Theorem 3.3 suggests the impact of stochastic volatility. Therefore, we can obtain values for \( P_0 \) and \( P_1 \) through the analytical pricing formula of vulnerable options under our model setting. In next section, we derive the pricing formula with constant volatility using Laplace transform techniques.

4. Pricing vulnerable options under a jump-diffusion model. In this section, we derive the well-known result for general jump diffusion model with constant volatility, to which we relate the leading order term \( P_0 \) and the correction term \( P_1 \).

Under the risk-neutral measure \( \mathbb{P}^* \), we assume the dynamics of the underlying asset value \( \tilde{S}_t \) and the counterparty’s asset value \( \tilde{V}_t \) are governed respectively by the following jump-diffusion processes:

\[
\begin{align*}
\frac{d\tilde{S}_t}{\tilde{S}_{t-}} &= (r - \lambda^S \beta^S)dt + \sigma_s dB_t^{S*} + d\left( \sum_{i=1}^{N^S_t} \xi_i - 1 \right) \\
\frac{d\tilde{V}_t}{\tilde{V}_{t-}} &= (r - \lambda^V \beta^V)dt + \sigma_v dB_t^{V*} + d\left( \sum_{i=1}^{N^V_t} \nu_i - 1 \right)
\end{align*}
\]

According to Itô formula, the processes \( \ln \frac{\tilde{S}_T}{\tilde{S}_t} \) and \( \ln \frac{\tilde{V}_T}{\tilde{V}_t} \) are given by

\[
\begin{align*}
\ln \frac{\tilde{S}_T}{\tilde{S}_t} &= \mu^S(T-t) + \sigma_s (B_T^{S*} - B_t^{S*}) + Q^S, \\
\ln \frac{\tilde{V}_T}{\tilde{V}_t} &= \mu^V(T-t) + \sigma_v (B_T^{V*} - B_t^{V*}) + Q^V,
\end{align*}
\]

where

\[
\begin{align*}
\mu^S &= r - \lambda^S \beta^S - \frac{1}{2} \sigma_s^2, \\
\mu^V &= r - \lambda^V \beta^V - \frac{1}{2} \sigma_v^2,
\end{align*}
\]

and their jump parts are

\[
Q^S = \sum_{i=N^S_t}^{N^S_T} \xi_i, \quad Q^V = \sum_{i=N^V_t}^{N^V_T} \nu_i.
\]

Thus \( \mu^S(T-t) + \sigma_s (B_T^{S*} - B_t^{S*}) \) and \( \mu^V(T-t) + \sigma_v (B_T^{V*} - B_t^{V*}) \) follow the bivariate normal distribution:

\[
\mathcal{N}_2 \left( \mu^S(T-t), \mu^V(T-t), \sigma_s \sqrt{T-t}, \sigma_v \sqrt{T-t}, \rho_{sv} \right)
\]

and the density function is

\[
\phi(a, b) = \frac{1}{2\pi\sigma_s \sigma_v (T-t) \sqrt{1 - \rho_{sv}^2}} \exp \left\{ - \frac{m(a, b)}{2(1 - \rho_{sv}^2)} \right\},
\]

where

\[
m(a, b) = \frac{(a - \mu^S(T-t))^2}{\sigma_s^2 (T-t)} - \frac{2 \rho(a - \mu^S(T-t))(b - \mu^V(T-t))}{\sigma_s \sigma_v (T-t)} + \frac{(b - \mu^V(T-t))^2}{\sigma_v^2 (T-t)}.
\]
4.1. Solutions by Laplace transform. In this section, we obtain the pricing formula of vulnerable option using the Laplace transform. We represent the price of vulnerable European options at time $t$ as follows:

$$C(t, s, v) = E^P \left[ e^{-r(T-t)} c(S_T, V_T) | S_0 = s, V_0 = v \right].$$

By Feynman-Kac formula, $C(t, s, v)$ satisfies the following PDE:

$$\frac{\partial C}{\partial t} + (r - \lambda^S \beta^S s) \frac{\partial C}{\partial s} + \frac{\sigma^2 s^2 \partial^2 C}{2 \sigma^2 s^2} + \rho_{sv} v \sigma_s \sigma_v \frac{\partial^2 C}{\partial s \partial v} + \left( r - \lambda^V \beta^V v \right) C + \lambda^S \int_R C(t, (\xi + 1)s, v) \Xi(\xi) d\xi$$

$$+ \lambda^V \int_R C(t, s, (k + 1)v) \Psi(k) d\xi = 0,$$

with final condition

$$C(T, s, v) = (s - K)^+ \left( 1_{\{s \geq D\}} + 1_{\{s < D\}} \frac{(1 - \gamma)v^2}{D} \right).$$

Comparing Eq. (5) and Eq. (6) to Eq. (7) and setting $\sigma_s = \sqrt{<g^2}_{s^2}$, $\sigma_v = \sqrt{<g^2}_{v^2}$, we have

$$P_0(t, s, v) = C(t, s, v)$$

$$= E^P \left[ e^{-r(T-t)} (S_T - e^{-k}) + I_{\{V_T = e^{-d}\}} \right]$$

$$+ e^{-r(T-t)} \left[ 1_{\{V_T = e^{-d}\}} \right]$$

$$= P_1^1(k, d) + P_0^2(k, -d),$$

where $k = -\ln K$ and $d = -\ln \tilde{D}$, and

$$P_1^1(k, d) = E^P \left[ \int_t^T e^{r(T-t)} \left\{ \frac{uS^2}{\sqrt{2}} (\Lambda \psi') - \rho_{sv} < g \psi' > \right\} \frac{\partial^2 P_1^1}{\partial s^2} \right]$$

$$- uV^2 S_T \left[ \frac{\partial^3 P_1^1}{\partial s^2 \partial v^2} \right]$$

$$- uV^2 S_T \left[ \frac{\partial^3 P_1^1}{\partial s^2 \partial v^2} \right]$$

$$- \sqrt{2} uV^2 S_T \left[ \frac{\partial^3 P_1^1}{\partial s^2 \partial v^2} \right]$$

Denote the Laplace transforms for $P_0^1$ and $P_0^2$ by $\tilde{P}_0^1$ and $\tilde{P}_0^2$, respectively. We have the following theorem.
Theorem 4.1. The Laplace transform of \( P^1_0(k,d) \) with respect to \( k \) and \( d \) is given by

\[
\hat{P}_1^1(\alpha_1, \alpha_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\alpha_1 k - \alpha_2 d} P^1_0(k,d) \, dk \, dd
\]

\[
= e^{-r(T-t)} \sum_{\alpha_1 \alpha_2} \left\{ \left( T-t \right) \left[ \lambda^S(\varphi_\xi(\alpha_1 + 1) - 1) + \lambda^V(\varphi_\nu(\alpha_2) - 1) \right] + (\alpha_1 + 1) \mu^S + (1 - \alpha_2) \mu^V + V(\varphi_\nu(\alpha_2) - 1) \right\}.
\]

where \( \varphi_\xi \) and \( \varphi_\nu \) are the moment generating function of \( \xi \) and \( \nu \), respectively.

Proof. See Appendix 7.4.

\[\square\]

Corollary 1. The Laplace transform with respect to \( k \) and \( d \) of \( P^1_0(k,d) \) is given by:

\[
\hat{P}_2^1(\alpha_1, \alpha_2)
\]

\[
= \sum_{\alpha_1 \alpha_2} \int_{-\infty}^{+\infty} \left[ -\frac{\alpha_2}{\sqrt{T}} \left( -\Lambda \psi_1' > -\rho_{sy} < g \psi_1' > \right) \right] \frac{1}{\alpha_2}
\]

\[
+ \left[ \frac{\alpha_2}{\sqrt{T}} \left( -\Lambda \psi_2' > -\rho_{vy} < g \psi_2' > \right) \right] \frac{1}{\alpha_2}
\]

\[
+ \left( \sqrt{2\rho_{sy} \rho_{sv}} < g \psi_1' > + \frac{\rho_{sy}}{\sqrt{2}} < f \psi_1' > \right) \frac{\alpha_2}{\alpha_1 + 1} + \frac{\rho_{sv}}{\sqrt{2}} < f \psi_2' > \left( \frac{\alpha_2(\alpha_2 - 1)}{\alpha_1(\alpha_1 + 1)} \right)
\]

\[
+ \left( \sqrt{2\rho_{sy} \rho_{sv}} < g \psi_2' > + \frac{\rho_{sy}}{\sqrt{2}} < f \psi_2' > - < \Lambda \psi_2' > \right) \frac{1}{\alpha_2}
\]

\[
\exp \left\{ (T-t) \left[ \lambda^S(\varphi_\xi(\alpha_1 + 1) - 1) + \lambda^V(\varphi_\nu(\alpha_2) - 1) + (\alpha_1 + 1) \mu^S + (1 - \alpha_2) \mu^V + \frac{1}{2}(\alpha_1 + 1)^2 \sigma_\nu^2 + (\alpha_1 + 1)(1 - \alpha_2) \sigma_\sigma \rho_{sv} + \frac{1}{2}(1 - \alpha_2)^2 \sigma_v^2 \right] \right\}.
\]

The Laplace transform with respect to \( k \) and \( d \) of \( P^2_1(k,d) \) can be obtained as:

\[
\hat{P}_2^1(\alpha_1, \alpha_2)
\]

\[
= \sum_{\alpha_1 \alpha_2} \int_{-\infty}^{+\infty} \left[ \frac{\alpha_2}{\sqrt{T}} \left( -\Lambda \psi_1' > -\rho_{sy} < g \psi_1' > \right) \right] \frac{1}{\alpha_2}
\]

\[
+ \left( \sqrt{2\rho_{sy} \rho_{sv}} < g \psi_1' > + \frac{\rho_{sy}}{\sqrt{2}} < f \psi_1' > \right) \frac{\alpha_2}{\alpha_1 + 1} + \frac{\rho_{sv}}{\sqrt{2}} < f \psi_2' > \left( \frac{\alpha_2(\alpha_2 - 1)}{\alpha_1(\alpha_1 + 1)} \right)
\]

\[
+ \left( \sqrt{2\rho_{sy} \rho_{sv}} < g \psi_2' > + \frac{\rho_{sy}}{\sqrt{2}} < f \psi_2' > - < \Lambda \psi_2' > \right) \frac{1}{\alpha_2}
\]

\[
\exp \left\{ (T-t) \left[ \lambda^S(\varphi_\xi(\alpha_1 + 1) - 1) + \lambda^V(\varphi_\nu(\alpha_2) - 1) + (\alpha_1 + 1) \mu^S + (1 - \alpha_2) \mu^V + \frac{1}{2}(\alpha_1 + 1)^2 \sigma_\nu^2 + (\alpha_1 + 1)(1 - \alpha_2) \sigma_\sigma \rho_{sv} + \frac{1}{2}(1 - \alpha_2)^2 \sigma_v^2 \right] \right\}.
\]

Proof. See Appendix 7.5.

\[\square\]
5. Numerical experiments. In this section, we demonstrate our model with some numerical experiments of vulnerable European option pricing. Table 1 presents a set of parameters adopted from literature based on real market data. The default time to maturity $T$ is set to be one year. We choose a non-negative function $g$ in Eq. (1) such that $<\psi_1'1>_1=0$, $<\psi_2'2>_2=0$, $\sqrt{<g^2>}=.125$ and $<g>=.2$ are satisfied. For more details, one can refer to [4] and [28]. We assume that $\xi_i$ are independent and identically distributed double-exponential jumps having density

$$h_\xi(x) = p e^{-\eta_1 x} 1_{x \geq 0} + (1-p) e^{\eta_2 x} 1_{x < 0},$$

with $0 \leq p \leq 1$, $\eta_1 > 1$, $\eta_2 > 0$. We let $p = 0.5$ and $\eta_1 = \eta_2 = 20$, adopted from [22].

From the asymptotic analysis in Section 3, the option price $P^\epsilon$ can be approximated by

$$P^\epsilon \approx P_0 + \sqrt{\epsilon} P_1.$$

We adopt the two-dimensional Euler inversion algorithm in [22] based on Theorem 4.1 and Corollary 1. As illustrated in [22], this algorithm is faster and more stable than the original Euler inversion algorithm.

To begin with, we compute the correction term $P_1$ numerically under different conditions to demonstrate its convergence. Figure 1 illustrates the relationship between the correction term $P_1$ and the initial value of the underlying asset $S_0$ for different values of the strike price $K$. We find out that for every given $K$, the correction term $P_1$ is always positive and bounded, suggesting the convergence of $P_1$ and the overprice effect of stochastic volatility. And the overprice effect is highest near the strike price as shown in the Figure. However, the correction term $P_1$ is positive correlated with the counter-party’s initial asset value $V_0$ under different total liability $D$, which can be seen from Figure 2. One can observe that the overprice effect is higher with smaller $D$ or greater $V_0$. It implies that the impact of stochastic volatility is greater when the counter-party is less likely to default. Figure 3 shows how $P_1$ behaves under different correlation parameters $\rho_{sy}, \rho_{sv}, \rho_{vy}$. With these graphical evidence, we can draw the same conclusion as in [4] and [28]: the correction term $P_1$ can be well controlled. Furthermore, one can observe from Figure 1, 2 and 3 that $P_1$ is positive always, which means the price of vulnerable option with stochastic volatility and jumps is higher than the price of the vulnerable option with constant volatility and jumps.

The numerical approximation of vulnerable European option price $P^\epsilon$ is given in Table 2 and Table 3. Table 2 shows that vulnerable options have time value: the

| Parameter                  | Value | Parameter                  | Value |
|----------------------------|-------|----------------------------|-------|
| Initial stock price $S_0$  | 40    | Correlation($S$ & $Y$)     | $\rho_{sy} = -0.1$ |
| Strike price $K$           | 40    | Correlation($S$ & $V$)     | $\rho_{sv} = 0.2$ |
| Initial asset price $V_0$  | 100   | Correlation($Y$ & $V$)     | $\rho_{vy} = 0.1$ |
| Total liability $D$        | 100   | Intensity of Poisson process $N^S_t\lambda^S = 1$ |
| Default boundary $\tilde{D}$| 100   | Intensity of Poisson process $N^V_t\lambda^V = 0$ |
| Deadweight of bankruptcy $\gamma$ | 0.6 | Inverse mean-reverting speed $\epsilon = 0.001$ |
| Asset volatility $\sigma$  | 0.2   | Total risk premium $\Lambda = 2$ |
| Risk-free rate $r$         | 0.05  | Standard deviation of $Y$  | $u = \frac{1}{\sqrt{2}}$ |
| Time to maturity $T$       | 1     |                            |
option price increases as the term to maturity goes up. Moreover, we find that for a given initial value of the underlying asset, the option price decreases as the strike price $K$ increases. It means that vulnerable options behaves more like a call option.
Table 2. Numerical approximation for the option price $P^*$. 

| Strike price | $T=0.5$ | $T=1$ | $T=1.5$ | $T=2$ |
|--------------|---------|-------|---------|-------|
| 30           | 7.9925  | 8.6168| 9.2444  | 9.8764|
| 35           | 4.7952  | 5.4263| 6.0566  | 6.6868|
| 40           | 2.2740  | 2.8658| 3.4625  | 4.0619|
| 45           | 0.8294  | 1.2551| 1.7232  | 2.2222|
| 50           | 0.2386  | 0.4569| 0.7373  | 1.0737|

Table 3. Numerical approximation for the option price $P^*$. 

| Strike price | $\lambda^S = 0.5$ | $\lambda^S = 1$ | $\lambda^S = 1.5$ | $\lambda^S = 2$ | $\lambda^S = 2.5$ |
|--------------|-------------------|-----------------|-------------------|-----------------|-----------------|
| 30           | 8.6090            | 8.6168          | 8.6257            | 8.6355          | 8.6462          |
| 35           | 5.3874            | 5.4263          | 5.4653            | 5.5045          | 5.5438          |
| 40           | 2.7907            | 2.8658          | 2.9387            | 3.0095          | 3.0785          |
| 45           | 1.1744            | 1.2551          | 1.3336            | 1.4098          | 1.4839          |
| 50           | 0.3964            | 0.4569          | 0.5167            | 0.5756          | 0.6338          |

though it consists of a call and a put option. Data in Table 3 tells that stronger jump intensities increases the probability of executing an option and thus pushes the option price up. All these results are consistent with the results in the existing literature with constant volatility and jump diffusion, or stochastic volatility and no jumps.

6. Conclusions. In this paper, we propose a model to price vulnerable European options. We incorporate stochastic volatility, which has correlations with underlying asset and the counter-party’s asset, to explain real market phenomena like volatility smile and the mean-reverting features of implied volatility. Besides, we allow sudden changes in stock price, reflected in the model that the underlying asset price and the counter-party’s asset price have jumps with arbitrary jump size. Based on this model, we provide an equivalent martingale measure and approximate the option price by the multi-scale asymptotic analysis. The analytical pricing formula is obtained by the two dimensional Laplace transform. We also derive the analytical pricing formula for vulnerable European options with constant volatility and with general jumps on both underlying asset and the counter-party’s asset. Finally, we find the option price numerically by implementing Euler inversion. The numerical results are consistent with the results in the existing literature.

7. Appendix.

7.1. Proof of Proposition 1.

Proof. By the multiple dimensional Girsanov theorem [24], one can verify that $\mathbb{P}^*$ and $\mathbb{P}$ are equivalent, and $\{B_{i,\ast}^\ast\}_{i=S,Y,V}$ are Brownian motions under $\mathbb{P}^*$. Since the processes $\{\theta_i^\ast\}_{i=S,Y,V}$ are independent of jump parts $\sum_{i=1}^{N^S_i} (e^{\xi_i} - 1)$ and $\sum_{i=1}^{N^V_i} (e^{\nu_i} - 1)$,

$$
\mathbb{E}^{\mathbb{P}^*} \left[ e^{\sum_{i=S,Y,V} a_i B_{i,\ast}^\ast + b_i \sum_{i=1}^{N^S_i} (e^{\xi_i} - 1) + c_i \sum_{i=1}^{N^V_i} (e^{\nu_i} - 1)} \right]
= \mathbb{E}^{\mathbb{P}} \left[ e^{\sum_{i=S,Y,V} a_i B_{i,\ast} + b_i \sum_{i=1}^{N^S_i} (e^{\xi_i} - 1) + c_i \sum_{i=1}^{N^V_i} (e^{\nu_i} - 1)} Z(t) \right]
$$
Poisson processes with intensity $\lambda$, and respectively, the moment generating functions for 

$$\mathcal{PDE}$$

where $\mathcal{L} P = 0$ for some $y$-independent functions $k_1$ and $k_2$. From the growth rate condition, $k_1 = 0$ so that $P_0$ is independent of the $y$ variable. This is expressed as $P_0 = P_0(t, s, v)$.

Similarly, eliminating the terms of order $1$ for $\mathcal{L} P_0 = 0$, we have $\mathcal{L} P_0 = 0$, solving the equation, one can obtain

$$P_0(t, s, v) = k_1(t, s, v) \int_0^v e^{-2\frac{(m-s)}{\eta_2}}dz + k_2(t, s, v),$$

for some $y$-independent functions $k_1$ and $k_2$. From the growth rate condition, $k_1 = 0$ so that $P_0$ is independent of the $y$ variable. This is expressed as $P_0 = P_0(t, s, v)$.

Similarly, eliminating the terms of order $1$ for $\mathcal{L} P_0 = 0$, we have $\mathcal{L} P_0 = 0$ and therefore $P_1 = P_1(t, s, v)$.

The $O(1)$ term gives $\mathcal{L} P_0 + \mathcal{L} P_1 + \mathcal{L} P_0 = 0$. Since $\mathcal{L} P_0 = 0$, it becomes $\mathcal{L} P_0 + \mathcal{L} P_0 = 0$. By Lemma 3.1, the leading order term $P_0(t, s, v)$ satisfies the PDE

$$\frac{\partial P_0}{\partial t} + \frac{x^2+hx^2}{2} \frac{\partial^2 P_0}{\partial x^2} + \frac{v^2+fv^2}{2} \frac{\partial^2 P_0}{\partial v^2} + \rho_s v s <f g > \frac{\partial^2 P_0}{\partial s \partial v} + \frac{r - \lambda^S \beta^S}{\partial P_0} + \frac{(r - \lambda^V \beta^V)}{\partial P_0} - (r + \lambda^S + \lambda^V) P_0 + \lambda^S \int_0^s P_0(t, \xi + 1, s, v) \Xi(\xi) d\xi + \lambda^V \int_0^v P_0(t, s, k + 1, v) \Upsilon(k) dk = 0$$

with final condition

$$P_0(T, s, v) = (s - K)^+ \left(1_{(v \geq \delta)} + 1_{(v < \delta)} \frac{1 - \gamma(v)}{\delta^2} \right).$$

7.3. Proof of Theorem 3.3.

Proof. By Appendix 7.2, we have

$$\mathcal{L}_2 P_0 + \mathcal{L}_2 P_0 = 0.$$

By Lemma 3.1, $\mathcal{L}_2 P_0 > 0$ holds so that

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - <\mathcal{L}_2 P_0> = \frac{1}{2} s^2 (g(y)^2 - <g^2>) \frac{\partial^2 P_0}{\partial s^2} + \frac{1}{2} v^2 (f(y)^2 - <f^2>) \frac{\partial^2 P_0}{\partial v^2} + \rho_s v s (f(y)g(y) - <fg>) \frac{\partial^2 P_0}{\partial s \partial v}.$$ 

Then the solution of the Poisson equation $\mathcal{L}_2 P_0 + \mathcal{L}_2 P_0 = 0$ for $P_2$ is given by

$$P_2 = - \mathcal{L}_0^{-1} (\mathcal{L}_2 P_0) = - \frac{s^2}{2} \mathcal{L}_0^{-1} (g(y)^2 - <g^2>) \frac{\partial^2 P_0}{\partial s^2} - \frac{v^2}{2} \mathcal{L}_0^{-1} (f(y)^2 - <f^2>) \frac{\partial^2 P_0}{\partial v^2}.$$
Thus, we obtain

$$- \rho_{sv} u s L_0^{-1} (f(y) g(y) - < f g >) \frac{\partial^2 P_0}{\partial s \partial v}$$

$$= - \frac{s^2}{2} (\psi_1(y) + c_1(t, s, v)) \frac{\partial^2 P_0}{\partial s^2} - \frac{v^2}{2} (\psi_2(y) + c_2(t, s, v)) \frac{\partial^2 P_0}{\partial v^2}$$

$$- \rho_{sv} u s \psi_3(y) + c_3(t, s, v)) \frac{\partial^2 P_0}{\partial s \partial v}$$

for arbitrary function $c_1(t, s, v), c_2(t, s, v)$ and $c_3(t, s, v)$ independent of $y$. From the order $\sqrt{v}$ terms, we have

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$$

which is a Poisson equation for $P_3$ with respect to $\mathcal{L}_0$, so $< \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 > = 0$, then

$$< \mathcal{L}_2 P_1 >$$

$$= - < \mathcal{L}_1 P_2 >$$

$$= \frac{1}{\sqrt{2}} \rho_{sv} u s^3 < g \psi' > \frac{\partial^3 P_0}{\partial s^3} + \frac{1}{\sqrt{2}} \rho_{sv} u v^3 < f \psi' > \frac{\partial^3 P_0}{\partial v^3}$$

$$+ u v s^2 \left( \sqrt{2} \rho_{sv} \rho_{sv} < g \psi'_3 > + \frac{1}{\sqrt{2}} \rho_{sv} < f \psi'_1 > \right) \frac{\partial^3 P_0}{\partial s^2 \partial v}$$

$$+ u v s^2 \left( \sqrt{2} \rho_{sv} \rho_{sv} < f \psi'_3 > + \frac{1}{\sqrt{2}} \rho_{sv} < g \psi'_1 > \right) \frac{\partial^3 P_0}{\partial s \partial v^2}$$

$$+ \sqrt{2} u v s^2 \rho_{sv} (\rho_{sv} < g \psi'_3 > + \rho_{sv} < f \psi'_1 >) \frac{\partial^2 P_0}{\partial s \partial v}$$

$$- \frac{u v s^2}{\sqrt{2}} < \Lambda \psi'_1 > - \rho_{sv} < g \psi'_1 > \frac{\partial^2 P_0}{\partial s^2} - \frac{u v s^2}{\sqrt{2}} < \Lambda \psi'_2 > - \rho_{sv} < f \psi'_2 > \frac{\partial^2 P_0}{\partial v^2}.$$}

Thus, we obtain $< \mathcal{L}_2 > P_1 = < \mathcal{L}_2 P_1 > = - < \mathcal{L}_1 P_2 >$.  

\[\square\]

7.4. Proof of Theorem 4.1.

Proof. The Laplace transform of $P_1^0(k, d)$ with respect to $k$ and $d$ is given by

$$\hat{P}_1^0(\alpha_1, \alpha_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\alpha_1 k - \alpha_2 d} P_0^1(k, d) dk \, dd$$

$$= e^{-r(T-t)} \frac{\mu_1 \rho_{sv} \alpha_1^2 \alpha_2 + 1}{\alpha_1 \alpha_2 (\alpha_1 + 1)} \mathbb{E}^{* \rho_{sv}}_{t,s,v} \left[ e^{(\alpha_1 + \gamma) s_T + \alpha_2 d_T} \right],$$

where $s_T := \frac{S_T}{S}$ and $\nu_T := \frac{\nu_T}{\nu}$, while the Laplace transform of $P_0^2(k, -d)$ with respect to $k$ and $-d$ is given by

$$\hat{P}_0^2(\alpha_1, \alpha_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\alpha_1 k + \alpha_2 d} P_0^2(k, -d) dk \, dd(-d)$$

$$= e^{-r(T-t)} (1 - \gamma) \frac{\mu_1 \rho_{sv} \alpha_1^2 \alpha_2 + 1}{\alpha_1 \alpha_2 (\alpha_1 + 1)D} \mathbb{E}^{* \rho_{sv}}_{t,s,v} \left[ e^{(\alpha_1 + \gamma) s_T + (1 - \alpha_2) d_T} \right].$$

We wish to obtain a closed-form for the expectation:

$$\mathbb{E}^{* \rho_{sv}}_{t,s,v} \left[ e^{a s_T + b d_T} \right]$$

$$= \mathbb{E}^{* \rho_{sv}}_{t,s,v} \left[ e^{a (s_T - \sigma_s (B_T^s - B_0^s)) + b (\nu (T-t) + \sigma_s (B_T^s - B_0^s))} \right] \mathbb{E}^{* \rho_{sv}}_{t,s,v} \left[ e^{\sigma Q + b Q^V} \right].$$
Since $\mu^S(T-t) + \sigma_1 B_{S_{T-t}}^S$ and $\mu^V(T-t) + \sigma_2 B_{V_{T-t}}^V$ follow the bivariate normal distribution, we have
\[
\mathbb{E}_{t,s,v}^*\left[e^{-\alpha(T-t)}\mathbb{E}_{t,s,v}^*\left[e^{\sum_{m=0}^{\infty} P^S_{t,s,v} \left[ N_t^S - N_t^S = m, N_t^V - N_t^V = n \right]} \right] \right]
\]
and
\[
\mathbb{E}_{t,s,v}^*\left[e^{-Q^S + k^V}\right]
\]
where $\phi_1$ and $\phi_2$ are the moment generating function of $\xi$ and $v$, respectively. 

7.5. Proof of Corollary 1.

Proof. The Laplace transform of $P_t^1(k,d)$ with respect to $k$ and $d$ is given by
\[
\mathbb{E}_{t,s,v}^*\left[-\int_0^T e^{-\tau(T-t)} \left\{-\frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_1^2} (\tau, S_r, V_r) \right. \right.
\]
\[
-\frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_2^2} (\tau, S_r, V_r) + \frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_1 \partial \alpha_2} (\tau, S_r, V_r) \right)
\]
\[
\mathbb{E}_{t,s,v}^*\left[e^{-Q^S + k^V}\right]
\]
which is denoted by $P_t^1(\alpha_1, \alpha_2)$, where
\[
\alpha_1 \in (0,1) \quad \alpha_2 \in (1,1) + \frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_1 \partial \alpha_2} (\tau, S_r, V_r)
\]
\[
\frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_1 \partial \alpha_2} (\tau, S_r, V_r)
\]
\[
\frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_1^2} (\tau, S_r, V_r)
\]
\[
\frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_2^2} (\tau, S_r, V_r)
\]
\[
\frac{\partial^2 P_t^1(\alpha_1, \alpha_2)}{\partial \alpha_1 \partial \alpha_2} (\tau, S_r, V_r)
\]
\[
\frac{\partial^3 P^1_{\mu,s}(\alpha_1, \alpha_2)}{\partial s^2 \partial \mu} \left( \tau, S^0, V^0 \right) = \frac{\partial^3 P^2_{\mu,s}(\alpha_1, \alpha_2)}{\partial s^2 \partial \mu} \left( \tau, S^0, V^0 \right) \\
= \frac{\alpha_1}{\alpha_2} - e^{-\kappa S^0 \kappa S^0} \exp \{(T - \tau) \lambda S^0 (\varphi_0(1 + 1) - 1) + \lambda V(\varphi_v(1 + 1) - 1) + (\alpha_1 + 1) \mu^2 + \alpha_2 \mu^2 + \frac{1}{2} (\alpha_1 + 1)^2 \sigma_s^2 + (\alpha_1 + 1) \alpha_2 \sigma_s \sigma_v \rho_{sv} + \frac{1}{2} \alpha_2^2 \sigma_v^2) \}.
\]

Thus we have
\[
P^1_{\mu,s}(\alpha_1, \alpha_2) = \mathbb{E}_{t,s,u} \left[ - \int_T^T V^0_\tau S^0_{\tau + \alpha} \exp \{(T - \tau) \lambda S^0 (\varphi_0(1 + 1) - 1) + \lambda V(\varphi_v(1 + 1) - 1) + (\alpha_1 + 1) \mu^2 + \alpha_2 \mu^2 + \frac{1}{2} (\alpha_1 + 1)^2 \sigma_s^2 + (\alpha_1 + 1) \alpha_2 \sigma_s \sigma_v \rho_{sv} + \frac{1}{2} \alpha_2^2 \sigma_v^2) \} \right].
\]

Similarly, the Laplace transform of \( P^2_{\mu,s}(k, -d) \) with respect to \( k \) and \( -d \), denoted as \( P^2_{\mu,s}(\alpha_1, \alpha_2) \), can be obtained as follows:
\[
P^2_{\mu,s}(\alpha_1, \alpha_2) = \mathbb{E}_{t,s,v} \left[ - \int_T^T V^0_\tau S^0_{\tau + \alpha} \exp \{(T - \tau) \lambda S^0 (\varphi_0(1 + 1) - 1) + \lambda V(\varphi_v(1 + 1) - 1) + (\alpha_1 + 1) \mu^2 + \alpha_2 \mu^2 + \frac{1}{2} (\alpha_1 + 1)^2 \sigma_s^2 + (\alpha_1 + 1) \alpha_2 \sigma_s \sigma_v \rho_{sv} + \frac{1}{2} \alpha_2^2 \sigma_v^2) \} \right].
\]

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E-mail address: hevanhua2015@gmail.com
E-mail address: wucf@connect.hku.hk
E-mail address: gujw@sustech.edu.cn
E-mail address: wching@hku.hk
E-mail address: cwwongab@hku.hk