INJECTIVE METRIZABILITY AND THE DUALITY THEORY OF CUBINGS

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ABSTRACT. In his pioneering work on injective metric spaces Isbell attempted a characterization of cellular complexes admitting the structure of an injective metric space, following his discovery that finite metric spaces have injective envelopes naturally admitting a polyhedral structure. Considerable advances in the understanding, classification and applications of injective envelopes have been made by Dress, Huber, Sturmfels and collaborators (producing, among other results, many specific examples of injective polyhedra), and most recently by Lang, yet a combination theory explaining how to glue injective polyhedra together to produce large families of injective spaces is still unavailable.

In this paper we apply the duality theory of cubings – simply connected non-positively curved cubical complexes – to provide a more principled and accessible proof of a result of Mai and Tang on the injective metrizability of collapsible simplicial complexes. Our viewpoint allows an easy extension of their result to:

Main Theorem. Any pointed Gromov-Hausdorff limit of locally-finite piecewise-$\ell^\infty$ cubings is injective.

In addition to providing earlier work on two-dimensional complexes with a proper context, our result expands on the natural link between the methods of geometric group theory and the study of general metric spaces exposed by Lang, shedding some light on the role of non-positive curvature in the combination theory of injective metric spaces.

Keywords: Injective metric space; cubing; poc set; median algebra.

1. INTRODUCTION

1.1. Injective Envelopes. The study of injective metric spaces arose from the study of hyper-convexity in functional analysis (see Theorem 1.1.2). Most notably, the characterization of hyper-convexity by Aronszajn and Panitchpakdi led to Isbell’s study of this class of spaces from a categorical viewpoint.

Definition 1.1.1 (Injective Metric Space [15]). A metric space $X$ is said to be injective, if for any isometric embedding $i: A \to B$ and any non-expansive map $f: A \to X$ there exists a non-expansive $F: B \to X$ satisfying $F \circ i = f$.

Theorem 1.1.2 (Aronszajn-Panitchpakdi, [1]). A metric space $(X, d)$ is injective if and only if it is hyper-convex: every finite collection of closed balls $\{B(p_i, r_i)\}_{i=1}^n$ in $X$ satisfying $r_i + r_j \geq d(p_i, p_j)$ for all $1 \leq i, j \leq n$ has a common point.

Isbell introduces the category of metric spaces with non-expansive maps as morphisms, and considers the injective objects of this category with respect to the class
of isometric embeddings\[1\] Expanding on the results of Aroszajn and Panitchpakdi (who demonstrated, among other things, that an injective space is necessarily complete and geodesic), Isbell attempts two tasks that are natural in the categorical setting: the construction of injective envelopes and the classification of injective objects. For a modern account of Isbell’s categorical construction and its variants, see [17].

For the first task, Isbell proves that every metric space $X$ admits an isometric embedding $e_X$ into an injective metric space $\epsilon X$ with the property that any embedding of $X$ in an injective metric space factors through $e_X$. Isbell’s construction is very explicit—in fact, explicit enough to demonstrate that $\epsilon X$ has a polyhedral structure when $X$ is finite. This leads to a natural question, which is a part of the classification task.

**Question 1.1.3.** Which polyhedra can be endowed with an injective metric?

Isbell shows that in order to support an injective metric, a (simplicial) polyhedron $X$ must satisfy some basic topological requirements. For example: $X$ needs to be collapsible.

Isbell’s construction of envelopes has been independently discovered and expanded upon by Dress [10]. This has since been applied with much success to expanding the theory (and practice) of the very practical problem of distance-based clustering [3, 2, 9]. Other related work deals with calculating features of injective envelopes such as cut-points [11], explicit embeddings in the $\ell^1$ plane [12] and Gromov–Hausdorff distance estimates [18]. A detailed analysis of envelopes for discrete infinite spaces has been carried out by Lang in [17] with applications to group theory in mind, where injective envelopes are proposed as an alternative to standard polyhedral model spaces (e.g. Rips complex) for some classes of finitely generated groups (e.g. Gromov-hyperbolic groups).

Throughout this body of work we find hints to a connection of this theory with the notion of metric non-positive curvature in the sense of Alexandrov [4]:

- Much of the cited work relies to a varying extent on a combination theorem stating that gluing a pair of injective spaces along a point results in an injective space; a more developed combination theory seems to be missing, however.
- Dress offers a characterization of the injective envelope $\epsilon X$ of $X$ as the geodesic extension of $X$ where for every pair $x', y' \in \epsilon X$ there exist points $x, y \in X$ satisfying

$$\text{dist}(x, x') + \text{dist}(x', y') + \text{dist}(y', y) = \text{dist}(x, y)$$

In other words, $\epsilon X$ is a minimal geodesically complete extension of $X$ where $X$ shows up in the role of a ‘boundary.’

- Finally, Isbell’s construction of an injective metric on a collapsible 2-dimensional simplicial complex makes explicit use of combinatorial non-positive curvature conditions. Mai and Tang’s extension of Isbell’s result [19] makes a similar use of non-positive curvature, though only in combinatorial form.

\[1\]Note that the requirement from $i : A \rightarrow B$ to be an isometric embedding rather than just a monic map in the category produces a notion of injectivity that is weaker than monic-injectivity.
and without realizing the potential for constructing a whole deformation space of injective metrics on the given complex.

Let us now study Isbell’s argument in more detail as we analyze the connection with non-positive curvature and introduce our own results.

1.2. Results. In [15], Isbell proves that a collapsible 2-dimensional cellular complex $X$ admits an injective metric by explicitly constructing a hyper-convex metric on $X$ as follows: taking a triangulation of $X$, he subdivides its triangles into squares so as to form what he calls a *collapsible cubical 2-complex*, $\Delta$. He then metrizes $X$ as a geometric realization of $\Delta$, having first realized each 2-cube as a copy of the unit cube in $(\mathbb{R}^2, \|\cdot\|_{\infty})$ and endowing the resulting 2-dimensional piecewise-$\ell^\infty$ polyhedron with the associated quotient metric. Isbell’s verification of the injectivity criterion then proceeds in two steps:

1. Reduction to the case where all the $p_i$ are vertices of the cubical subdivision and all the $r_i$ are integers;
2. Applying the properties of a ‘collapsible cubical 2-complex’ to verify the result.

Among Isbell’s requirements of a collapsible cubical 2-complexes one immediately notices Gromov’s ‘no-triangle’ condition for non-positively curved cubical complexes. Indeed, upon closer inspection it becomes clear that the notion of a collapsible cubical 2-complex is exactly a 2-dimensional cubing in the language of modern geometric group theory.

Definition 1.2.1 (Gromov [13], Sageev [23]). A cubical complex $X$ is non-positively curved (NPC) if the link of each vertex in $X$ is a simplicial flag complex. The complex $X$ said to be a cubing, if it is non-positively curved and simply-connected.

Fairly unrestrictive conditions are known for when the piecewise-Euclidean metric on a cubing is complete and geodesic. For example, this will happen when the cubing is locally finite-dimensional, as follows almost immediately from Bridson’s theorem on shapes ([4], theorem I.7.50). Under these conditions one applies the Cartan–Hadamard theorem to conclude that $X$ is CAT(0) and therefore contractible — a necessary condition for injectivity (see, e.g. [10]).

Bridson’s theorem on shapes applies equally well (though with some care) to the setting when the cubes of a locally finite cubing are metrized as axis-parallel parallelopipeds in $(\mathbb{R}^n, \|\cdot\|_{\infty})$, where $n$ may vary. The edge lengths of the parallelopipeds may be chosen with some degree of freedom, subject to the gluing constraints of the complex. We call the resulting geodesic spaces piecewise-$\ell^\infty$ cubings. Our central result is:

**Theorem 1.2.2.** Every locally-finite piecewise-$\ell^\infty$ cubing is injective.

Thus, not only is it true that any locally finite combinatorial cubing is injectively metrizable (Mai and Tang [19]), but, in fact, it carries a whole deformation space of injective metrics. Moreover, observing that the class of injective metric spaces is closed under pointed Gromov-Hausdorff limits (see lemma below) extends the scope of the above theorem to give the main result stated in the abstract.

The same lemma plays a crucial role in our reduction of the general case to the case of finite unit cubings:
Figure 1. The injective envelope of the 5-point space in Example 1.3.1 (a) may be obtained as a limit of $\ell^\infty$-cubings of the form (b).

Lemma 1.2.3 (“Limit Lemma”). A complete metric space arising as a pointed Gromov–Hausdorff limit of proper injective metric spaces is itself injective.

Recall that a metric space is said to be proper if closed bounded subsets thereof are compact. Applied to our setting, this lemma implies the sufficiency of proving only the following theorem.

Theorem 1.2.4. Every finite piecewise-$\ell^\infty$ cubing $X$ is injective.

We explain how to see any such space $X$ as a pointed Gromov–Hausdorff limit of a sequence of the form $(\frac{1}{n}X^{(n)}, v)_{n \in \mathbb{N}}$ where $X^{(n)}$ is a cubical subdivision of $X$ obtained by cutting the cubes of $X$ in a grid-like fashion (parallel to their faces), with unit weights. Applying the limit lemma once again we see that it now suffices to prove one final lemma.

Proposition 1.2.5. Every finite unit piecewise-$\ell^\infty$ cubing is injective.

This is, essentially, the original result proved by Mai and Tang in [19]. In this paper, we propose an alternative proof using the full power of the structure theory of cubings. Roughly speaking, the idea is to prove that the balls in the zero-skeleton $X^0$ of a unit piecewise-$\ell^\infty$ cubing $X$ are convex subsets of $X^0$ with respect to its unit piecewise-$\ell^1$ metric. Once this is known, the same property is inherited by all piecewise-$\ell^\infty$ cubings via the limit lemma. This finishes the proof of injectivity.

As cubings are known to be geodesic median spaces (see below) with respect to their piecewise-$\ell^1$ metric, they satisfy a 1-dimensional Helly theorem: every finite collection of pairwise-intersecting convex sets has a common point. In particular, any collection of pairwise-intersecting $\ell^\infty$-balls in a cubing must have a common point, and injectivity is proved.

1.3. Remaining Questions. The class of piecewise-$\ell^\infty$ cubings does not coincide with the class of injective metric spaces. This follows directly from the limit lemma and the example below (see ex. 1.3.1 and fig. 1). It would be interesting to quantify the discrepancy, perhaps in terms of the classification by Lang [17].
Example 1.3.1 (Injective Envelope of 5 Points). It is shown in [3] that the metric space \( X = \{1, 2, 3, 4, 5\} \) with
\[
\begin{align*}
\text{dist}(1, 2) &= \text{dist}(1, 3) = \text{dist}(1, 4) = 1, \\
\text{dist}(5, 2) &= \text{dist}(5, 3) = \text{dist}(5, 4) = 1, \\
\text{dist}(1, 5) &= \text{dist}(2, 3) = \text{dist}(2, 4) = \text{dist}(3, 4) = 2,
\end{align*}
\]
has the injective envelope depicted in Figure 1(a), where one should think of the triple fin depicted there as the result of gluing three unit squares cut out the \( \ell^1 \) plane and glued together to overlap along the (filled-in) triangles with sides \([1, 5]\), \([1, x]\) and \([x, 5]\) for \( x \in \{2, 3, 4\} \). Each of these triangular fins is, in fact the limit of a sequence of piecewise-\( \ell^\infty \) cubings, resulting in a sequence of approximations for \( \epsilon X \) of the form shown in Figure 1(b).

While it is very possible that the class of limits of piecewise-\( \ell^\infty \) cubings is still too narrow to exhaust all injective metric spaces, our results seem to suggest that the combinatorial structure of a cubing is nothing more than a set of explicit gluing instructions following which one could create a ‘big’ injective space out of small, standardized pieces, namely: \( \ell^\infty \)-cubes. Thus, our results are nothing more than a glimpse of a combination theory for constructing ‘big’ injective spaces out of ‘small’/‘simple’ ones. This motivates the following question.

**Question 1.3.2.** Is there a combination theory for injective metric spaces? If two injective spaces are glued along a convex (injective?) subspace, when is the resulting space injective?

A well-developed combination theory should simplify the proofs of the existing results as well as contribute to the understanding of the problem of characterizing injective spaces in constructive terms.

### 2. Preliminaries

We use this section to recall some of the language required for the rigorous development of the main result. Some of the facts presented in this section seem to be common knowledge, yet new in the sense that they are not easily found in the literature — we thought it better to include them here due to their elementary nature, as well as for the sake of giving a self-contained exposition.

#### 2.1. Piecewise-(your favorite geometry here) Complexes

The purpose of this section is to recall the necessary technical language for dealing with geometric complexes. We do not intend to treat the subject at any level of generality beyond what is required for dealing with cubings—see Chapter I.5 of [4] for a careful and detailed exposition of this topic.

**Definition 2.1.1 (Quotient Metric).** Let \((X, d)\) be a metric space, where \( d \) is allowed infinite values. Suppose \( P \) is a partition of \( X \). Then the quotient (semi-)distance \( \bar{d}(\xi, \eta) \) for \( \xi, \eta \in P \) is defined as the greatest lower bound on expressions of the form
\[
\sum_{i=0}^{n} d(x_i, y_i)
\]
where \( x_0 \in \xi, y_n \in \eta \) and \( y_{i-1} \sim_{\rho} x_i \) for all \( i = 1, \ldots, n \).
In the case of a cubical complex, one considers the complex $X$ as a quotient of a disjoint union $\tilde{X}$ of a collection of cubes—copies of $[0,1]^n \subset \mathbb{R}^n$ for possibly varying values of $n \in \mathbb{N}$—with isometric identifications along some of the faces generating the equivalence relation defining the partition $P$. While the distance in $\tilde{X}$ between points of distinct cubes is set to be infinite, each cube is endowed with the metric it inherits from $\mathbb{R}^n$.

In this paper we consider the three cases where all cubes at the same time inherit their geometry from one of the $\ell_2$, the $\ell_1$ or the $\ell_\infty$ norms. The resulting metric space $(X,d)$ is then referred to as a unit piecewise-$\ell_2$, $\ell_1$ or $\ell_\infty$ cubical complex.

A slight variation on this construction is achieved by putting weights on the co-ordinate axes of the individual cubes. One needs to make sure that the weights match, in the sense that any two cubes sharing a common face in $X$ do have their axes weighted in a way that keeps the identified faces isometric. As we shall see in Section 3.5, Roller’s duality theory between cubings and discrete poc sets is ideally suited for the purpose of maintaining the necessary records.

It is not hard to verify (though it does require some work which, thankfully, had been done in [4]) that the quotient metric on a piecewise-$\ell_p$ ($p = 1, 2, \infty$) cubical complex takes the following form:

**Definition 2.1.2** (broken path, length). A broken path in a cubing $X$ is a sequence of points $p = (x_0, \ldots, x_n)$ where $x_0 = x$, $x_n = y$ and every consecutive pair of points is contained in a face of $X$. The length of $p$ is defined to be:

$$(3) \Lambda(p) := \sum_{i=1}^{n} \text{dist}(x_{i-1}, x_i)$$

where $\text{dist}(x_{i-1}, x_i)$ is measured in a face containing the points $x_{i-1}, x_i$.

With this language, of course, the piecewise-$\ell_p$ distance between two points of a cubical complex becomes the greatest lower bound on the length of a broken path between them. There are some obvious optimizations to this picture.

**Definition 2.1.3.** The broken path $p$ is said to be taut if no consecutive triple of points along $p$ is contained in a cube.

Since the individual cubes in $\tilde{X}$ are geodesic metric spaces, tightening a broken path locally will never increase its length. This leads to the formula:

$$(4) \text{dist}(x, y) = \inf \left\{ \Lambda(p) \mid p \in \mathcal{P}(x, y) \text{ is taut} \right\}$$

We postpone the rest of this discussion till a time when we have a better way of representing piecewise-$\ell_p$ cubings, in section 3.5.

### 2.2. Gromov–Hausdorff limits.**

We refer the reader to Chapter I.5 of [4] for a detailed introduction and discussion of Gromov–Hausdorff convergence of proper metric spaces. Recall that a binary relation $R \subset X \times Y$ has projections

$$(5) \pi_X(R) = \{ x \in X \mid \exists y \in Y \ (x,y) \in R \}$$

and $\pi_Y(R) = \{ y \in Y \mid \exists x \in X \ (x,y) \in R \}$

and that a subset $Z$ of a metric space $X$ is said to be $\epsilon$-dense (in $X$), if the collection of closed balls of radius $\epsilon$ about the points of $Z$ cover $X$. We will use the following convergence criterion for our technical work.
Definition 2.2.1. Let $X, Y$ be metric spaces and let $\epsilon > 0$. A relation $R \subseteq X \times Y$ is said to be an $\epsilon$-approximation between $X$ and $Y$, if:

1. the projections $\pi_X(R)$ and $\pi_Y(R)$ are $\epsilon$-dense;
2. $|\text{dist}(x, x') - \text{dist}(y, y')| < \epsilon$ holds for all $(x, y), (x', y') \in R$.

An $\epsilon$-approximation is surjective, if $\pi_X(R) = X$ and $\pi_Y(R) = Y$.

One of the ways to define the Gromov–Hausdorff distance $\text{GH}(X, Y)$ is as (half, sometimes) the greatest lower bound of the set of $\epsilon > 0$ admitting a surjective $\epsilon$-approximation between $X$ and $Y$. It is worth noting that estimation of Gromov–Hausdorff distances in the context of injective spaces and the use of $\epsilon$-approximations are not new to this field, for example: in [13] Lang, Pavón and Züst estimate the change in Gromov–Hausdorff distance between spaces as one passes from the spaces to their injective envelopes. Thus, we expect that neither our limit lemma nor its proof below come as a surprise to the experts. We include the proof for convenience.

Using $\epsilon$-approximations, Gromov–Hausdorff convergence is characterized as follows.

Lemma 2.2.2. Let $X_n, n \in \mathbb{N}$ and $X$ be metric spaces. The sequence $(X_n)_{n=1}^{\infty}$ converges to $X$ in the Gromov–Hausdorff topology, if for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that $\epsilon$-approximations $R_n(\epsilon) \subset X_n \times X$ exist for all $n \geq N$.

This kind of convergence is most meaningful in the category of compact metric spaces. Pointed Gromov–Hausdorff convergence (of proper metric spaces) extends this idea and is defined as follows:

Definition 2.2.3 ([3], Definition 1.8). A sequence $(X_n, b_n)_{n \in \mathbb{N}}$ of pointed proper metric spaces is said to converge to the pointed space $(X, b)$ if for every $R > 0$ the sequence of closed metric balls $B_{X_n}(b_n, R), n \in \mathbb{N}$, converges to the closed ball $B_X(x, R)$ in the Gromov–Hausdorff topology.

Note that the completion of a pointed Gromov–Hausdorff limit of a sequence of spaces is itself a limit of the same sequence.

We are ready to prove the limit lemma from the introduction.

Lemma 2.2.4. Let $(X_n, b_n)$ be a sequence of pointed proper injective metric spaces converging to the space $(X, b)$ in the sense of pointed Gromov–Hausdorff convergence. Then the metric completion of $X$ is injective.

Proof. We may assume $X$ is complete. We first make an independent observation: let $X, Y$ be metric spaces with $X$ injective and suppose that $R \subset X \times Y$ is an $\epsilon$-approximation. Given a collection $\{y_i\}_{i=1}^{k}$ of points in $Y$ and positive numbers $\{r_i\}_{i=1}^{k}$ satisfying

$$r_i + r_j \geq \text{dist}(y_i, y_j)$$

for all $i, j \in \{1, \ldots, k\}$. We find points $x_i \in X$, $i = 1, \ldots, k$, such that $(x_i, y_i) \in R$ and we set $R_i = r_i + \epsilon$. It then follows that

$$R_i + R_j \geq \text{dist}(x_i, x_j)$$

for all $i, j$. Applying theorem [1.1.2] we find a point $x \in X$ with $\text{dist}(x, x_i) \leq R_i$ for all $i$, and conclude the existence of a point $y \in Y$ which then must satisfy $\text{dist}(y, y_i) \leq r_i + 2\epsilon$ for all $i$. 


Now we apply the preceding observation to a fixed sequence of pointed injective spaces \((X_n,b_n)\) converging to a space \((X,b)\). Given finite collections \(\{x_i\}_{i=1}^k\) of points in \(X\) and \(\{r_i\}_{i=1}^k\) of positive reals satisfying \(\sum_{i=1}^k r_i > 0\), find \(R > 0\) large enough so that \(B_X(x_i,r_i + 2)\) contains the union of all the balls \(B_X(x_i,r_i + 2)\), \(i = 1,\ldots,k\). By the preceding argument, for every \(\epsilon \in (0,1)\), the closed ball \(B_X(b,R)\) contains a point \(x_\epsilon\) satisfying \(d(x_\epsilon,x_i) \leq r_i + 2\epsilon\) for \(i = 1,\ldots,k\). The existence of a point \(x \in \bigcap B_X(x_i,r_i)\) now follows from the compactness of \(B_X(b,R)\). \(\square\)

Remark 2.2.5. Observe that a different choice of the \(R_i\)—any choice, in fact, of \(r_i + \epsilon \leq R_i \leq r_i + K\epsilon\) for a value of \(K\) that is fixed in advance—would do. This enables a weakening of the assumptions (on the \(X_n\)) under which the preceding argument remains intact. Instead of assuming injectivity of each \(X_n\) one may assume that the Aronszajn-Panitchpakdi condition holds for:

- Finite subsets of a fixed \(\epsilon\)-dense subset \(A_n\) of \(X_n\),
- Radii restricted to, say, values in the set \(\frac{1}{n}\mathbb{N}\).

The injectivity of \(X\) would follow from these assumptions by the same argument as before.

2.3. Median Spaces. The hyper-convexity criterion of Aronszajn and Panitchpakdi cannot help but remind one of a similar phenomenon in the realm of median spaces. In this short review we follow the exposition of [6]. Let \((X,d)\) be a semi-metric space.

Definition 2.3.1 (Intervals and Medians). For \(x,y \in X\), the interval \(I(x,y)\) is defined to be

\[
I(x,y) = \left\{ z \in X \left| d(x,z) + d(z,y) = d(x,y) \right. \right\}.
\]

For \(x,y,z \in X\), we say that a point \(m \in X\) is a median of the triple \((x,y,z)\) if

\[
m \in M(x,y,z) := I(x,y) \cap I(x,z) \cap I(y,z).
\]

By definition, \(M(x,y,z)\) is independent of the ordering of \(x,y,z\). We also recall the following definitions.

Definition 2.3.2 (Convexity and Half-spaces). A subset \(K\) in a semi-metric space \((X,d)\) is convex, if \(I(x,y) \subseteq K\) holds for every \(x,y \in K\). \(K\) is said to be a half-space of \(X\) if both \(K\) and \(X \setminus K\) are convex subspaces of \(X\).

The notion of median points is significant for a large class of spaces.

Definition 2.3.3 (Median Space). A semi-metric space is a median space, if \(M(x,y,z) \neq \emptyset\) and \(\text{diam}(M(x,y,z)) = 0\) for all \(x,y,z \in X\). A map \(f : X \to Y\) between median spaces is said to be a median morphism if \(f(M(x,y,z)) \subseteq M(f(x),f(y),f(z))\) for all \(x,y,z \in X\).

Recall that any semi-metric can be Hausdorffified: a semi-metric space \(X\) gives rise to a metric space \(X_{\text{hua}}\) by forming the quotient of \(X\) by the equivalence relation \(x \sim y \iff d(x,y) = 0\). It is clear then that the median structure on \(X\) descends to the quotient.

Lemma 2.3.4. Suppose \((X,d)\) is a median semi-metric space. Then \(X_{\text{hua}}\) is a median metric space where \(M(x,y,z)\) is a singleton for all \(x,y,z \in X\). When this happens, we write

\[
M(x,y,z) = \{\text{med}_X(x,y,z)\}
\]

and say that \(\text{med}_X(x,y,z)\) is the median point of the triple \((x,y,z)\).
We provide a few simple examples of median spaces to illustrate this structure.

Example 2.3.5 (The real line is median). It is easy to verify that \( \mathbb{R} \) is a complete geodesic median space, with

\[
\mathrm{med}_\mathbb{R}(x, y, z) = y \quad \text{whenever} \quad x \leq y \leq z
\]

Example 2.3.6 (\( l_1 \)-type normed spaces are geodesic median spaces). Suppose \((T, \mu)\) is a measure space. Then \( X = L^1(T, \mu) \) is a median space and

\[
\mathrm{med}(x, y, z) = (\mathrm{med}_\mathbb{R}(x(t), y(t), z(t)))_{t \in T}
\]

In the case when \( T \) is finite, we see that \( X \) is simply isometric to \( \ell_1(T) := (\mathbb{R}^T, \| \cdot \|_1) \), with medians computed coordinatewise.

Two things make median spaces highly relevant to this paper: the first is the fact that every cubing can be metrized to become a median space (and in more than one way as we shall see in section 3.5); the second is the following theorem we have already mentioned in the introduction.

Theorem 2.3.7 (Helly Theorem, [22] Theorem 2.2). Suppose \( \mathcal{C} = \{ C_i \}_{i=1}^n \) is a family of convex subsets of a median space \( X \). If any two elements of \( \mathcal{C} \) have a point in common, then all elements of \( \mathcal{C} \) have a point in common.

Half-spaces play a crucial role in the theory of median spaces. With some modifications to known proofs for the more general setting of median algebras (see section 4 of [6]), one can prove the following.

Proposition 2.3.8. Let \( X \) be a complete median space. Then every convex subset of \( X \) is an intersection of open half-spaces. Any closed convex subset of \( X \) is an intersection of closed half-spaces.

As explained in the introduction, our strategy for verifying the injectivity of a piecewise-\( \ell_\infty \) cubing \( X \) is to show that closed balls in such a cubing are convex with respect to the piecewise-\( \ell_1 \) metric on the same cubing. By Helly’s theorem it will suffice to demonstrate that any closed \( \ell_\infty \)-ball is the intersection of closed half-spaces of \( X \) when \( X \) is viewed as a median space (using its piecewise-\( \ell_1 \) metric). The convexity of \( \ell_\infty \)-balls will result from us presenting any such ball as the intersection of a suitably chosen family of (closed) \( \ell_1 \)-halfspaces.

The halfspace structure of a median space is fundamental to our technique. In fact, we will make good use of an isometric median embedding \( \rho_X \) of a median space \( X \) into an \( L^1 \) space, constructed as follows (Theorem 5.1 and corollary 5.3 in [6]):

- **Construction of a ‘Transverse Measure’, \( \mu \):** one starts by constructing the set \( \mathcal{W}(X) \) of all pairs of the form \( \{ h, X \setminus h \} \) where \( h \) ranges over the halfspaces of \( X \); it then turns out that a natural \( \sigma \)-algebra \( \mathcal{B}(X) \) exists on \( \mathcal{W}(X) \), together with a measure \( \mu \), such that \( \mu(\mathcal{W}(x|y)) = \mathrm{dist}(x, y) \) for all \( x, y \in X \), where \( \mathcal{W}(x|y) \) denotes the subset of all pairs \( \{ h, X \setminus h \} \in \mathcal{W}(X) \) satisfying \( x \in h \) and \( y \in X \setminus h \).

- **Embedding in \( L^1(\mu) \):** fixing a base-point \( b \in X \), map a point \( x \in X \) to the function \( x_{\mathcal{W}(x|b)} \). This map is a median-preserving isometric embedding.

The case of locally-finite (or, equivalently, proper) cubings is well known ([23, 22, 7, 21]) to fall under the purview of this construction while lending itself to analysis by
simple combinatorial (rather than measure-theoretic) tools. We seek to capitalize on this fact in Section 3, where we use this embedding to study the geometry of cubings endowed with a piecewise-$\ell^\infty$ geometry.

Finally, some quick observations regarding the ambiguous relationship between median spaces and injective spaces are in order. Analyzing Isbell’s construction of injective envelopes it is easy to verify that, if $X$ is an injective space and $Y$ is a subspace of $X$ then the inclusion of $Y$ into $X$ extends to an embedding of the envelope $\epsilon Y$ in $X$. The same construction enables one to prove that the injective envelope of three points is either an interval (degenerate or non-degenerate) or a tripod—a space isometric to a metric tree with four vertices and three leaves standing in bijective correspondence with the initial triple of points. Thus, for any triple of points $x,y,z$ in an injective space $X$, the median set $M(x,y,z)$ is non-empty!

While medians do exist in injective spaces, there is no guarantee of uniqueness of the median, as one can easily observe in the normed space $\ell^\infty(S)$ for any set $S$ of cardinality greater than 2. In fact, even cases so simple as the case of four points demonstrate the complicated relationship between the two classes—consider the following example.

**Example 2.3.9 (Non-Injective Median Spaces).** Let $t \in [0,1]$ be a real parameter, and let $Q_t$ denote the three-dimensional cube $[0,t]^3 \subset \mathbb{R}^3$, endowed with the $\ell^1$ metric. Denote:

$$a = (0,0,0), \ b = (t,t,0), \ c = (t,0,t), \ d = (0,t,t)$$

and let $J_\sigma, x \in \{a,b,c,d\}$ denote pairwise disjoint copies of the the interval $[0,1-t] \subset \mathbb{R}$ endowed with the standard metric. We form a space $X_t$ as the quotient of $Q_t \cup \{J_a, J_b, J_c, J_d\}$ by identifying the point $0 \in J_\sigma$ with the point $\sigma \in Q_t$ for every $\sigma \in \{a,b,c,d\}$, and endow $X_t$ with the quotient metric. Finally, we denote the point in $X_t$ corresponding to $1-t \in J_\sigma$ with the capital letter variant of $\sigma \in \{a,b,c,d\}$; see Figure 2.

Deferring the proof that the $X_t$ are median spaces until Section 3.5, Example 3.5.5, let us focus on explaining why $X_t$ is not injective for any $t > 0$. Observe that the subspace $Y = \{A,B,C,D\} \subset X_t$ satisfies $\text{dist}(x,y) = 2$ for all $x,y \in Y$. It follows that $X_0$ is the injective envelope of $Y$ with this metric. If $X_t$ were injective for some $t > 0$, the inclusion map $\text{inc}: Y \to X_t$ would have extended to an isometric embedding $j : X_0 \to X_t$. Since both $X_0$ and $X_t$ are median spaces, $j$ is then a
median-preserving map. Denoting the median map of $X_t$ by $\text{med}_t$, we observe in $X_t$ that

$$j(\text{med}_0(A, B, C)) = \text{med}_t(A, B, C) = c,$$

$$j(\text{med}_0(A, B, D)) = \text{med}_t(A, B, D) = d,$$

while at the same time in $X_0$ one has

$$\text{med}_0(A, B, C) = c = d = \text{med}_0(A, B, D) \text{ in } X_0.$$ 

Since $c \neq d$ in $X_t$ for $t > 0$, we have arrived at a contradiction and we conclude $X_t$ cannot be injective for $t > 0$.

The last example hints that failure of injectivity is fairly common among median spaces. We find it intriguing, then, that a mere change of the geometry of the vector space on which the individual cubes are modeled should result in a change so radical as turning all the spaces in question into injective spaces. This kind of behaviour seems to hint at the existence of a general principle loosely formulated as a non-positively curved combination of injective spaces is injective. The extent to which such a statement is true remains to be verified.

3. Piecewise-$\ell^1$ Cubings and their Geometry

For a very serious and inspiring recent account of the theory of non-positively curved cubical complexes please see [24]. Piecewise-$\ell^2$ cubings and their applications to group theory are discussed in great detail in [4].

Beyond these references, we will not delve into any detail regarding cubical complexes from the topological viewpoint. Instead we will focus on a very direct construction of cubings, due initially to Sageev [23], and developed by Roller in [22] based on Isbell’s duality [16] between poc sets and median algebras. Let us just clarify what we mean by a cube, to avoid any confusion.

**Definition 3.0.10.** Let $S$ be a set. The *standard $S$-cube* is the set $\square^S$ of all functions from $S$ into $[0, 1]$. When $S = \emptyset$, the 0-dimensional cube $\square^\emptyset$ is defined to equal the one-point set $\{\ast\}$. More generally, we say that $\square^S$ is a $|S|$-dimensional cube.

Much of the material present in this section has been known to geometric group theorists for quite some time, either formally or as folklore. Unfortunately, the literature on the technical tools we are using is in a state of slight disarray preventing immediate application to the specific problem at hand. We therefore decided to gather the necessary material here, filling in some of the gaps and formalizing the folklore.

3.1. Poc Sets. We recall some definitions and examples.

**Definition 3.1.1 (Poc Set, [22]).** A *poc set* is a poset $(P, \leq)$ with a minimum element denoted $0 \in P$ and endowed with an order-reversing involution $a \mapsto a^*$ satisfying the additional requirement

$$(\dagger) \quad a \leq a^* \Rightarrow a = 0.$$ 

We say that $P$ is discrete, if the poset $(P, \leq)$ is discrete, that is: order intervals

$$[a, b] = \left\{x \in \big| a \leq x \leq b\right\}$$

are discrete.
are finite for all $a, b \in P$.

Working with poc sets requires some additional jargon.

**Definition 3.1.2.** Let $P$ be a poc set.
- Elements of $P$ are often referred to as halfspaces;
- The elements $0, 0^* \in P$ are said to be trivial;
- The non-trivial halfspaces are said to be proper;
- A complementary pair $\{a, a^*\}$ of proper halfspaces is called a wall of $P$;
- A pair $\{a, b\}$ with $a \notin \{b, b^*\}$ will be called a proper pair.

The meaning of (†) is summarized in the observation that any proper pair $a, b \in P$ satisfies at most one of the following relations:

\[ a \leq b, \quad a^* \leq b, \quad a \leq b^*, \quad a^* \leq b^* . \]

The above relations (if they hold) are called nesting relations.

**Definition 3.1.3.** Let $P$ be a poc set. A pair of elements $a, b \in P$ satisfying one of the relations (11) is said to be nested. More generally, a set $A \subset P$ is said to be nested pairwise. The set $A$ is said to be transverse if no two of its elements are nested. A transverse pair $a, b \in P$ is often denoted with $a \lessdot b$.

**Example 3.1.4 (Power Set).** The power set $2^S$ of a non-empty set $S$ is a poc set with respect to inclusion and complementation. In particular, the power set of one point, denoted $2$, is the trivial poc set.

**Example 3.1.5 (Linear Poc Set).** Let $(P, \leq)$ be a totally ordered set. Then the set $Q = P \sqcup P^* \sqcup \{0, 0^*\}$ -- where $P^*$ is the set of symbols of the form $p^*$ for $p \in P$ -- and subject to the relations $0 \leq x$ and $x \leq 0^*$ for all $x \in Q$, as well as the relations $p^* \leq q^*$ iff $q < p$ in $P$, is a poc set.

**Example 3.1.6 (Poc Set of a Tree).** Let $T$ be a tree with edge set $E$ and vertex set $V$. Let $P$ be the set of vertex-sets of connected components of all possible $T - e$, where $e$ ranges over $E$. Then $P$ is a discrete nested poc set with respect to inclusion and complementation.

Of course, one should also mention the morphisms in this category:

**Definition 3.1.7.** A function $f : P \to Q$ between poc sets is a poc morphism, if it is order-preserving and $*$-equivariant, that is:

\[ a \leq b \Rightarrow f(a) \leq f(b), \quad f(a^*) = f(a)^* \]

for all $a, b \in P$.

### 3.2. The Dual Cubing of a Poc Set.

The following construction is due to Sageev in the specific context of relative ends of groups, and to Isbell and Roller in the current level of generality.

**Definition 3.2.1 (Dual of a Poc Set).** Let $P$ be a finite poc set. An ultra-filter $U$ on $P$ is a subset of $P$ satisfying:

1. for all $a \in P$ one has $a \in U$ or $a^* \in U$, but not both;
2. for all $a, b \in U$ one never has $a \leq b^*$.
The set of all ultra-filters on $P$ is denoted by $P^\omega$ and may be naturally identified with $\text{Hom}(P, 2)$. A subset $U \subset P$ satisfying $\square$ is called a complete $*$-selection on $P$. A subset $U \subset P$ satisfying $\square$ is said to be coherent. We will denote the set of all $*$-selections by $S(P)^0$.

Picking a basepoint $B \in S(P)^0$ makes it easy to identify $S(P)^0$ with the vertex set of the $B$-cube $\square^B$: simply observe that the map
\[ \rho : U \mapsto 1 - \chi_{B \setminus U} \]
viewed as a map of $S(P)^0$ into $\square^B$ is injective, with image precisely the set of $\{0,1\}$-valued elements of the (possibly infinite-dimensional) standard unit cube:
\[ \square^B := \{ x : B \to [0,1] \}. \]
Note that $B$ is mapped to the origin and that the $\ell^1$ metric on $\square^B$ induces the following combinatorial metric on $S(P)^0$.

**Definition 3.2.2** (Metric on $(\ast)$-selections).
\[ \Delta^1(U, V) = |U \setminus V| = |V \setminus U| = \frac{1}{2} |U \triangle V|. \]
Due to symmetry, this metric is clearly independent of the choice of basepoint $B$. Thus, $\square^B$ effectively endows $S(P)^0$ with the structure of a graph $S(P)^1$ if we set two vertices $U, V \in S(P)^0$ to be joined by an edge iff $\Delta^1(U, V) = 1$. With a little bit of additional work one observes that a sub-graph $L$ of $S(P)^1$ is isomorphic to the 1-skeleton of a $d$-cube if and only if $L$ is mapped by $\rho$ onto the vertex set of some appropriate $d$-face of $\square^B$. Thus, $S(P)^1$ may be thought of as the 1-skeleton of a cubical complex $S(P)$ which is isomorphic to $\square^B$ under the affine extension $\rho$. It is easy to verify that $\square^B$ is homogeneous, making the choice of basepoint $B$ immaterial (for now).

**Definition 3.2.3.** Let $P$ be a discrete poc set. Then $\text{Cube}(P)$ is defined to be the sub-complex of $S(P)$ of all faces not incident to an incoherent vertex.

A few simple examples are as follows.

**Example 3.2.4** (Dual of a Transverse Poc Set). When the poc set $P$ has no non-trivial nesting relations it is clear that $\text{Cube}(P)$ coincides with the cube $S(P)$.

**Example 3.2.5** (Dual of a Linear Poc Set). It is not hard to see that $\text{Cube}(P)$ for a linear poc set $P$ (see Example 3.1.5) is the extension of $P$ by Dedekind cuts.

**Example 3.2.6** (Dual of a nested Poc Set). It is a more involved computation to verify that the dual of a finite nested Poc Set $P$ is a finite tree. Moreover, $P$ is naturally isomorphic to the poc set constructed from this tree as described in example 3.1.6. This is a particular case of [8], II.1.8-10.

Finally, we give an example we will use later on.

**Example 3.2.7** (Cartesian Products). Suppose $P$ and $Q$ are discrete poc sets, and let $P \cup Q$ denote the poc set obtained from $P \cup Q$ by identifying $0_P$ with $0_Q$ (and hence also $\bar{0}_P$ with $\bar{0}_Q$). Then any proper element of $P$ is transverse to any proper element of $Q$ and it is easily shown that $\text{Cube}(P \cup Q)$ is naturally isomorphic to $\text{Cube}(P) \times \text{Cube}(Q)$.

Summarizing all the above is the surprising result anticipated by Sageev and proved by Roller and Chepoi.
Theorem 3.2.8 (Sageev–Roller, Chepoi). Let \( P \) be a discrete poc set. Then every connected component of \( \text{Cube}(P) \) is a cubing and \( \Delta^1 \) coincides with the combinatorial path metric on its 1-dimensional skeleton. Conversely, every cubing \( X \) is a connected component of \( \text{Cube}(P) \) for an appropriately chosen discrete poc set \( P \).

When \( P \) is infinite, \( \text{Cube}(P) \) inevitably forms a disconnected space. The connected components of \( \text{Cube}(P) \) are spanned by the almost-equality classes of its vertices: recall that two subsets \( U, W \) of a set \( S \) are said to be almost equal if the symmetric difference \( U \triangle W \) is finite; by the definition of the metric \( \Delta^1 \) on \( \text{Cube}(P) \), a pair of vertices in \( \text{Cube}(P) \) is joined by an edge-path if and only if they are almost-equal to each other as subsets of \( P \). For example, if \( P \) is nested, \( \text{Cube}(P) \) will consist of a tree whose poc set of halfspaces is naturally isomorphic to \( P \), together with its space of ends, each end being the only point of its component in \( \text{Cube}(P) \) ([8]).

In general, however, it may not be possible to naturally select a (distinguished) component \( K \) of \( \text{Cube}(P) \) whose poc set of halfspaces is isomorphic to \( P \). Work done in [22] and in [14] explains how to achieve this under the condition that \( P \) contains no infinite transverse set.

Seeking to avoid such technicalities we will use the trick of restricting attention to a component containing a particular vertex of interest.

Definition 3.2.9. Let \( P \) be a discrete poc set and \( B \in P^o \). The connected component of the complex \( \text{Cube}(P) \) containing the vertex \( B \) will be denoted \( \text{Cube}(P)_B \).

Recall the (piecewise affinely extended) representation map \( \rho: \text{Cube}(P) \to \mathbb{R}^B \) defined in Equation 13, and note that \( \text{Cube}(P)_B \) is precisely the intersection of the image of \( \rho \) with the vector subspace \( \ell^1(B) \) of \( \mathbb{R}^B \) consisting of all vectors of finite 1-norm. This is due to \( P \) being discrete.

3.3. Local Properties of Duals. We will need some technical information about the local structure of \( \text{Cube}(P) \). The following results are well-known and appear in [23, 22].

Lemma 3.3.1 (vertex links in \( \text{Cube}(P) \)). Let \( P \) be a discrete poc set and let \( V, V' \in P^o \) be vertices of \( \text{Cube}(P) \). Then \( V' \) is adjacent to \( V \) in \( \text{Cube}(P) \) if and only if \( V' \) has the form

\[
V' = [V]_a := (V - \{a\}) \cup \{a^*\}
\]

for \( a \in \min(V) \), where \( \min(V) \) denotes the set of minimal elements of \( V \) with respect to the ordering induced on it from \( P \).

The operation of replacing \( V \) by \( [V]_a \) will be called a flip. Clearly, any vertex of \( \text{Cube}(P) \) is connected to any other by a finite sequence of flips. The fact that \( \Delta^1(U, V) \) simply measures the minimal number of flips required to turn \( U \) into \( V \) is central to the theory of discrete median algebras.

A special situation occurs when several flips may be applied to a vertex in different orders of application without affecting the outcome. It is easy to verify that the hyperplanes being flipped form a transverse set in such a case, and, more generally we have the following lemma.
Lemma 3.3.2 (cubes in Cube\((P)\)). Let \(P\) be a discrete poc set and let \(V \in P^0\) be a vertex in Cube\((P)\). Then the set of cubes of Cube\((P)\) incident to \(V\) is in one-to-one correspondence with the transverse subsets of \(\min(V)\). Moreover, for each such subset \(A\) the vertices of the corresponding cube adjacent to \(V\) are all of the form
\[
[V]_{a_1 \cdots a_m} := [\cdots [V]_{a_1} a_2 \cdots a_m]
\]
where \((a_1, \ldots, a_m)\) is a sequence of distinct elements of \(A\) and we agree to denote \([V]_{a_1} \cdots a_m\) is independent of the ordering of the flips.

This lemma is very well illustrated by Example 3.2.7, demonstrating how the dimensions of cubes in the complex grow with the sizes of transverse subsets.

3.4. Cubings as Median Spaces. The following facts were shown in [22, 21].

Proposition 3.4.1 (Poc set duals are median). Let \(P\) be a discrete poc set and let \(B \in P^0\) be a basepoint in Cube\((P)\). Then each component of \((P^0, \Delta^1)\) is a median metric space with the median map given by
\[
\text{med}(U, V, W) = (U \cap V) \cup (V \cap W) \cup (W \cap U).
\]
Moreover, the map \(\rho : \text{Cube}(P)_B \to \ell^\infty(B)\) is a median-preserving isometric embedding.

The explicit form of the median map allows for a complete characterization of halfspaces and convex subsets of \((P^0, \Delta^1)\).

Proposition 3.4.2 (convex subsets of \(P^0\)). Let \(P\) be a discrete poc set. The halfspaces of \((P^0, \Delta^1)\) are precisely the sets of the form
\[
V(a) := \left\{ U \in P^0 \mid a \in U \right\}.
\]
In particular, the convex subsets of \(P^0\) are precisely
\[
V(A) := \bigcap_{a \in A} V(a) = \left\{ U \in P^0 \mid A \subseteq U \right\}
\]
for \(A\) ranging over all coherent subsets of \(P\). Moreover,
\[
a < b \text{ in } P \iff V(a) \subset V(b)
\]
holds for all \(a, b \in P\). Thus \(P\) is naturally isomorphic to the poc set of halfspaces of \(P^0\).

Our current goal is to extend this combinatorial structure theory of abstract cubings to meet the metric theory of median spaces at a point where we can plainly view finite cubings as ‘finitely presented median spaces’, and proper median spaces as pointed Gromov-Hausdorff limits of finite cubings. For example, one would like to be able to reason in a way hinted at by Figure 3.

3.5. Weighted Realizations of Cube\((P)\). Let \(P\) be a discrete poc set, and let \(B\) be a base point in \(P^0\), fixed once and for all. Note that \(0^* \in U\) for all \(U \in P^0\), which makes \(0^*\) an uninformative coordinate of \(\mathbb{R}^B\) when it comes to representing vertices of Cube\((P)\).

A more varied geometric realization of Cube\((P)\) is needed.
Definition 3.5.1 (weight on a poc set). A weight on $P$ is a function $w : P \rightarrow \mathbb{R}_{\geq 0}$ satisfying $w(0) = 0$ and $w(a^*) = w(a)$ for all $a \in P$. We say that a weight is non-degenerate if $w(a) > 0$ for all proper $a \in P$. A weight $w$ is used for defining a map $\operatorname{diag}_w : \mathbb{R}^B \rightarrow \mathbb{R}^B$ via the pointwise product $\operatorname{diag}_w(x) = wx$.

Given a weight $w$ on $P$ we revisit the map $\rho : P^o \rightarrow \mathbb{R}^B$, only that now we view it as a piecewise affine map of $\rho : S(P) \rightarrow \mathbb{R}^B$. Extending the construction from [21] slightly, we define a new embedded complex in $\mathbb{R}^B$.

Definition 3.5.2 ($\ell^1$ realization of $\operatorname{Cube}(P)$). Let $P$ be a discrete poc set with weight $w$ and basepoint $B \in P^o$. The weighted dual $\operatorname{Cube}_w(P)_B$ of $P$ (with weight $w$ and basepoint $B$) is defined to be the image of $\operatorname{Cube}(P)_B$ under the map $\rho_w : \operatorname{Cube}(P) \rightarrow \ell^1(B)$ obtained as the affine extension $\operatorname{diag}_w \circ \rho : P^o \rightarrow \mathbb{R}^B$, endowed with the metric $\Delta^1_w$ induced on it from $\ell^1(B)$.

Some important observations:

Proposition 3.5.3 (Properties of $\operatorname{Cube}_w(P)_B$). Let $P$ be a discrete poc set with non-degenerate weight $w$ and basepoint $B \in P^o$. Then:

1. $\rho_w$ is a homeomorphism of $\operatorname{Cube}(P)_B$ onto $\operatorname{Cube}_w(P)_B$. In particular, $\operatorname{Cube}_w(P)_B$ is a cubing.
2. $\Delta^1_w$ is a piecewise-$\ell^1$ metric: $\Delta^1_w$ coincides with the quotient metric on $\operatorname{Cube}_w(P)_B$ obtained by endowing each cube with the metric induced on it from $\ell^1(B)$ (and then carrying out the appropriate identifications).
3. $\rho_w$ restricts to a median-preserving map of $P^o$ into $\ell^1(B)$.
4. $\operatorname{Cube}_w(P)$ is a median metric space – in fact, the smallest geodesic median subspace of $\ell^1(B)$ containing $\rho_w(P^o)$.

Proof. Property (1) holds by construction, as $\operatorname{Cube}(P)$ and $\operatorname{Cube}_w(P)$ differ by the bijective (modulo collapsing the uninformative $0^*$ coordinate) stretching map $\operatorname{diag}_w$.

To see (2), observe that a cube of $\operatorname{Cube}(P)$ is mapped to a rectangular parallelipiped with edges parallel to the coordinate axes (RAP). Since paths that are piecewise parallel to the coordinate axes are geodesics in $\ell^1(B)$ provided they cross each hyperplane at most once, the quotient metric induced from endowing each RAP with the trace metric from $\ell^1(B)$ coincides with the distance measured along geodesics of $\ell^1(B)$ which do not exit $\operatorname{Cube}_w(P)_B$. The connectedness of $\operatorname{Cube}_w(P)_B = \rho_w(\operatorname{Cube}(P)_B)$ finishes the proof.

Property (3) follows from [6], Lemma 3.12 and Theorem 5.1.
Property (4) is the tricky one. Our argument for (2) explains the fact of $\text{Cube}_w(P)_B$ being a geodesic subspace of $\ell^1(B)$. One observes:

- Since $P$ is discrete we must have $x(a) = y(a) = z(a)$ for all but finitely many $a \in P$, reducing the problem to the case when $P$ is finite;
- For any pair of vertices $U, W \in P^o$ sharing a cube in $\text{Cube}(P)_B$, if $Q$ is the minimal cube containing both $U$ and $W$ then $\rho_w Q = I(\rho_w U, \rho_w W)$, the interval calculated in $\ell^1(B)$, is the unique RAP with antipodal vertices $\rho_w U$ and $\rho_w W$. This explains the minimality property claimed in (4).

It remains to verify that the $\ell^1(B)$-median $\text{med}(x, y, z)$ of a triple of points $x, y, z \in \text{Cube}_w(P)_B$ is also contained in $\text{Cube}_w(P)_B$. This verification is purely technical, and we omit the details in the interest of saving space, leaving only an outline of the argument: knowing that $\rho_w : P^o \to \ell^1(B)$ is median-preserving, one considers the set $A$ of all $a \in B$ for which at least one of $x(a), y(a), z(a)$ does not belong to $\{0, w(a)\}$. If $A$ is empty, then $x, y, z$ are vertices of $\text{Cube}_w(P) -$ images of points in $P^o$ under $\rho_w$, that is $-$ and there is nothing to prove. If $A$ is non-empty, observe that $A$ is a transverse set. Considering $m = \text{med}(x, y, z)$ as a real-valued function of $B$ and recalling medians in $\ell^1(B)$ are computed coordinate-by-coordinate, we conclude that the value of $m$ on $B \setminus A$ is determined by majority (as in the $A = \emptyset$ case), thus forcing $m$ into the unique cube whose vertices coincide with $m$ on $B \setminus A$ and have arbitrary values $m(a) \in \{0, w(a)\}$ for all $a \in A$.

**Definition 3.5.4.** By a piecewise-$\ell^1$ cubing we mean a space of the form $\text{Cube}_w(P)_B$ endowed with the metric $\Delta^1_w$. This metric space will be denoted $\text{Cube}^1_w(P)$. In the special case when $w(a) = 1$ for all proper $a \in P$, we will identify $\text{Cube}_w^1(P) = \text{Cube}^1(P)$, and refer to $\Delta^1_w$ simply as $\Delta^1$.

**Example 3.5.5.** Revisiting Example 2.3.9 we represent the space $X_t$ (figure 2) from that example as a piecewise-$\ell^1$ cubing. First, the poc-set structure may be chosen to have the form $P = S \cup S^*$ where

$$S = \{\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{A, B\}, \{A, C\}, \{A, D\}\}$$
with $P$ considered a sub poc set of $2^{\{A, B, C, D\}}$. Next we set the weights to equal

$$w(\text{singleton}) = 1 - t, \quad w(\text{pair}) = t.$$ 

Note how the weights of walls separating any given pair of points in $\{A, B, C, D\}$ add up to 2—see figure 3.

### 3.6. Halfspaces and Walls of the Weighted Realization.

Understanding the open (closed) half-spaces of $\text{Cube}_w(P)$ for non-degenerate $w$ is easy: they are the intersections of $\text{Cube}_w(P)$ with the open (closed) halfspaces of $\ell^1(B)$. The latter all have the form

$$h_b(t) := \{ x \in \text{Cube}_w(P) \ | \ x(b) < t \}$$

or

$$h_b^*(t) := \{ x \in \text{Cube}_w(P) \ | \ x(b) > t \}.$$

Note how, for any choice of $0 < t < w(b)$ and $b \in B$, one has

$$\rho_w(P^o) \cap h_b(t) = V(b)$$

so that the ‘abstract’ halfspaces—the elements of $P$—are reconstructed from the ‘visual’ half-spaces of the realization. It is common in the field to refer to the sets of the boundaries of half-spaces

$$w_b(t) = w_b^*(t) = \{ x \in \text{Cube}_w(P) \ | \ x(b) = t \}, \quad 0 < t < w(b)$$

as the walls, or hyperplanes, of the cubical complex $\text{Cube}_w(P)$, noting that every wall separates $V(b)$ from $V(b^*)$ in $\text{Cube}_w(P)$, while having a ‘thickness’ of $w(b)$.

The distances between vertices are influenced accordingly: the following expression for the distance between vertices $U, W \in P^o$ is derived directly from property (2) stated in Proposition 3.5.3:

$$\Delta^1_w(\rho_w U, \rho_w W) = \sum_{a \in W \setminus U} w(a) = \sum_{b \in U \setminus W} w(b),$$

as $U \setminus W$ indexes the set of walls separating $U$ from $W$.

For a general pair of points $x, y \in \text{Cube}_w(P)_B$ a similar formula may be written down.

**Definition 3.6.1** (separators). Let $x, y \in \text{Cube}_w(P)_B$. The separator of $x$ and $y$ is the set—denoted $x \setminus y$ by abuse of notation—of all $a \in P$ such that either

- $a \in B$ and $x(a) < y(a)$;
- $a^* \in B$ and $x(a) > y(a)$.

**Remark 3.6.2.** Note how, if $w$ is non-degenerate, one always has $\rho_w U \setminus \rho_w W = U \setminus W$ for $U, W \in P^o$. Thus, the separator of a pair of vertices is the same as their combinatorial separator in $P$.

The distance formula then trivially becomes:

$$\Delta^1_w(x, y) = \sum_{a \in x \setminus y} |y(a) - x(a)|.$$

There are two cases to consider for the summands:
- $x(a) = 0$ and $y(a) = w(a)$ produces a summand of $w(a)$. In this case we see that every wall of the form $\Delta^w(t)$ separates $X$ from $Y$ — hence the contribution of $w(a)$ to the distance between the points.
- $0 < x(a) < y(a) < w(a)$. In this case, the two points are contained in a thickened hyperplane of $\text{Cube}_w(P)_B$, which is merely a direct product of an interval with the dual cubing of the sub poc set $P_{b,a}$ of all $b \in P$ satisfying $b \cap a$.

3.7. Degenerate Weights. Finally, the degenerate case is essentially identical to the non-degenerate one, as one may think of a null weight assigned to a wall as the limiting result of shrinking the thickness of that wall to zero. More formally, if the given weight $w$ is degenerate, the metric $\Delta^w_ω$ becomes a semi-metric. To obtain the reduction to the non-degenerate case, set:

\[
Z = \left\{ a \in P \mid a \neq 0, 0^* \text{ and } w(a) = 0 \right\}, \quad \bar{P} := P \setminus Z, \quad \bar{w} = w|_{\bar{P}}.
\]

It is straightforward to prove that the (unique) piecewise affine extension of the co-restriction map $P^o \to \bar{P}^o$ defined by $U \mapsto U \setminus Z$ induces an isometry of $\text{Cube}_w(P)_{b,a}$ onto $\text{Cube}_w(P)$.

4. Piecewise-$\ell^\infty$ Cubings

We have finally arrived at the point where a formal and workable construction of piecewise-$\ell^\infty$ cubings is given.

4.1. Generalities. In view of Sageev–Roller duality (Theorem 3.2.8) the definition of a piecewise-$\ell^\infty$ cubing from Section 2.1 may be rewritten as follows.

Definition 4.1.1. By a piecewise-$\ell^\infty$ cubing we mean a metric space of the form $\text{Cube}_w(P)_B$, where $P$ is a discrete poc-set, $w$ is a non-degenerate weight and $B \in P^o$ is a basepoint. This time we endow $\text{Cube}_w(P)_B$ with the quotient metric $\Delta^w_ω$ obtained by endowing each of the cubes of $\text{Cube}_w(P)$ with the metric induced on it from $(R^B, \| \cdot \|_\infty)$. The resulting quotient space will be denoted $\text{Cube}^\infty_ω(P)_B$. By a unit piecewise-$\ell^\infty$ cubing we mean a space of the form $\text{Cube}^\infty_ω(P)_B$ with $w(a) = 1$ for every proper $a \in P$. We denote this space with $\text{Cube}^\infty(P)_B$, and its metric with $\Delta^\infty$.

A little bit more can be said about piecewise-$\ell^\infty$ cubings by applying results on quotient metrics from [4], Chapter I.5.

Lemma 4.1.2. Let $P$ be a finite poc set with weight $w$ and basepoint $B \in P^o$. Then the metric $\Delta^\infty_ω$ on $\text{Cube}_w(P)_B$ is the length metric induced on $\text{Cube}_w(P)_B$ from $(R^B, \| \cdot \|_\infty)$. Furthermore, $\text{Cube}^\infty_ω(P)_B$ is a complete geodesic metric space.□

In fact, a little more can be said by applying pointed Gromov–Hausdorff convergence to adequately selected finite exhaustions of a locally finite cubing.

Corollary 4.1.3. Let $P$ be a discrete poc set with weight $w$ and basepoint $B \in P^o$. If the dual $\text{Cube}(P)_B$ is locally finite, then $\Delta^\infty_ω$ is the length metric on $\text{Cube}_w(P)_B$ induced on it from $(R^B, \| \cdot \|_\infty)$, and $\text{Cube}^\infty_ω(P)_B$ is a complete geodesic metric space. □
4.2. Lower Bound on $\Delta_w^\infty$. The following technical lemma places a lower bound on distances in $\text{Cube}_w(P)$.

**Lemma 4.2.1.** Let $U, W \in P^\circ$ be vertices of $\text{Cube}_w(P)$ and let $N$ be a finite nested subset of $U \setminus W$. Then

$$\Delta_w^\infty(\rho_w U, \rho_w W) \geq \sum_{a \in N} w(a) := w(N).$$

**Proof.** For any $a \in P$, the sub-complex $X(a)$ of $X = \text{Cube}_w(P)$ induced by the vertex set $V(a) \subset P^\circ$ is itself a cubing. We start by verifying that $\Delta_w^\infty(x,y) \geq w(a)$ for any $x \in X(a)$ and $y \in X(a^*)$: since the walls $w_a(t), t \in (0,w(a))$ separate $X(a)$ from $X(a^*)$ in $X$, every broken path $q = (x_0, x_1, \ldots, x_n)$ from $x$ to $y$ in $X$ must satisfy

$$\Lambda(q) \geq \sum_{i=1}^n |x_i(a) - x_{i-1}(a)| \geq |x_0(a) - x_n(a)| \geq w(a),$$

by the triangle inequality.

Now, if $N \subset U \setminus W$ is nested, then for any $a, b \in N$ we cannot have $a \leq b^*$, since $a, b \in U$ and $U$ is coherent. Neither can we have $a^* \leq b$, for $a^*, b^* \in W$ and $W$ is coherent. We are left with $a \leq b$ or $b \leq a$ for all $a, b \in N$. Thus, when $N$ is finite we may write $N = \{a_1, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$, which gives:

$$\begin{align*}
\rho_w U &\subset X(a_1) \subset X(a_2) \subset \cdots \subset X(a_n) \\
X(a_1^*) \supset X(a_2^*) \supset \cdots \supset X(a_n^*) \supset \rho_w W.
\end{align*}$$

The fact that each $X(a_i)$ and $X(a_i^*)$ is an $\ell^\infty$-cubing in its own right allows us to repeatedly apply the preceding argument to conclude

$$\Delta_w^\infty(\rho_w U, \rho_w W) \geq \sum_{i=1}^n w(a_i)$$

as desired. \hfill $\square$

An immediate corollary is the following.

**Corollary 4.2.2.** Let $P$ be a discrete poc set with weight $w$. Then the inclusion map

$$\text{inc}_Q : (Q, \|\cdot\|_\infty) \to (\text{Cube}_w(P), \Delta_w^\infty)$$

is an isometric embedding for every face $Q$ of $\text{Cube}_w(P)$. \hfill $\square$

Recall that a geodesic in a metric space $(X,d)$ from a point $x$ to a point $y$ is an isometric embedding $\gamma : [0,d(x,y)] \to X$ satisfying $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. A discretized version of this notion for our purposes is the following definition.

**Definition 4.2.3.** We say that a broken path $p$ from $x$ to $y$ in $\text{Cube}_w(P)$ is geodesic, if $\Lambda(p) = \Delta_w^\infty(x,y)$.

The proof of the last lemma produces an obvious lower bound on the length of a geodesic between two vertices of $\text{Cube}_w(P)$.

**Corollary 4.2.4.** Let $P$ be a discrete poc set with weight $w$. Then the bound

$$\Delta_w^\infty(\rho_w U, \rho_w W) \geq \max \left\{ w(N) \left| N \text{ is a chain in } U \setminus W \right. \right\}$$

holds for any $U, W \in P^\circ$. \hfill $\square$
4.3. Constructing Geodesics in Unit $\ell^\infty$-cubings. Assume for now that all the weights on $P$ are unity. We now consider a special family of broken paths introduced in [20].

**Definition 4.3.1.** A broken path $(U_0, \ldots, U_m)$ of vertices in $\text{Cube}_w(P)$ is said to be a normal cube path, if
\begin{equation}
U_i \setminus U_{i+1} = \min(U_i \setminus U_m)
\end{equation}
for all $i = 0, \ldots, m - 1$.

It is easy to verify that $\min(U \setminus V)$ is a transverse subset of $P$ for any pair of vertices $U, V \in P^o$. Thus, a normal cube path from $U$ to $V$ has the form:
\begin{equation}
U_0 = U, \ U_1 = [U]_{\min(U \setminus V)}, \ldots, U_{k+1} = [U_k]_{\min(U_k \setminus V)}, \ldots
\end{equation}
and in particular:
- Normal cube paths are taut;
- Each consecutive pair of vertices along a normal cube path are at unit distance from each other (as they span the diagonal of an embedded unit $\ell^\infty$-cube);
- Thus, the length $\Lambda(p)$ of a normal cube path $p$ as in the definition above equals $m$.

These two properties hint at the possibility that a normal cube path is, in fact an $\ell^\infty$-geodesic. In any case, the length of such a path provides one with an upper bound on the distance between its endpoints. We use normal cube paths to prove the following:

**Proposition 4.3.2.** For every $U, W \in P^o$ one has the formula
\begin{equation}
\Delta^\infty(\rho U, \rho W) = \max \{|N| | N \subseteq U \setminus W \text{ is nested} \}.
\end{equation}

**Proof.** Let $p = (U_0, \ldots, U_m)$ be the normal cube path from $U_0 = U$ to $U_m = W$. In $\text{Cube}^\infty(P)$ this implies $\Lambda(p) = m$. Setting $A_{i+1} = U_i \setminus U_{i+1}$ for $i = 0, \ldots, m - 1$ we have:
1. $A_i$ is transverse, for all $i = 1, \ldots, m$;
2. $U \setminus W = \bigcup_{i=1}^m A_i$, and this union is disjoint.

In particular, any nested set $N \subseteq U \setminus W$ intersects every $A_i$ in at most one point, and therefore satisfies $|N| \leq m$.

Now for any $i > 1$ and any $a_i \in A_i$ we observe that $a_i \in U_{i-1} \setminus U_m$. At the same time, by the definition of a normal cube path,
\begin{equation}
A_{i-1} = U_{i-1} \setminus U_i = \min(U_{i-1} - U_m)
\end{equation}
and we conclude that there has to be some $a_{i-1} \in A_{i-1}$ with $a_{i-1} < a_i$. Starting with any $a_m \in A_m$ we thus obtain at least one chain $N = \{a_1, \ldots, a_m\}$ with $a_i \in A_i$ for all $i$. Thus, $U \setminus W$ contains a nested subset of cardinality $m$. We conclude:
\begin{equation}
\Delta^\infty(U, W) \leq \Lambda(p) = \max \{|N| | N \text{ is a nested subset of } U \setminus W \}.
\end{equation}

The reverse inequality was established previously, in Corollary 4.2.4, so we are done. \hfill $\Box$

As a by-product of this proof we also obtain the following corollary.
Corollary 4.3.3. In a unit piecewise-$\ell^\infty$ cubing, normal cube paths are geodesic broken paths.

4.4. Subdivisions of $\ell^\infty$ Cubings. We are looking for an operation on poc sets that would result in subdividing each cube of the dual cubing into a ‘grid’ of smaller cubes.

Definition 4.4.1 (Refinement). We will say that a morphism of poc sets $f : \tilde{P} \rightarrow P$ is a refinement map – and that $\tilde{P}$ is a refinement of $P$—if $f^{-1}(0) = \{0\}$, and for any proper pair $p, q \in P$ one has:

1. $f^{-1}(p)$ is a finite chain in $\tilde{P}$,
2. for any $\tilde{p} \in f^{-1}p$ and $\tilde{q} \in f^{-1}q$ one has $\tilde{p} < \tilde{q} \Leftrightarrow p < q$.

We will now show that this notion of refinement produces the correct notion of subdivision in the dual. The idea is, roughly, that—since each cube was to be replicated into a linear sub poc set (in $\tilde{P}$), then the dual of the union of these linear refinements is the cartesian product of subdivided intervals; i.e., a subdivided cube (see Examples 3.2.7 and 3.2.5 respectively).

Proposition 4.4.2 (subdivision lemma). Let $f : \tilde{P} \rightarrow P$ be a refinement of a discrete poc set $P$ and let $w, \tilde{w}$ be weights on $P$ and on $\tilde{P}$ respectively, satisfying $w(p) = \sum_{a \in \sigma_{-1}(p)} \tilde{w}(a)$ for all $p \in P$. Then for any choice of $B \in P^o$, setting $\tilde{B} = f^{-1}(B) \in \tilde{P}$, there is a bijection $F$ of $\text{Cube}_w(P)_B$ onto $\text{Cube}_{\tilde{w}}(\tilde{P})_\tilde{B}$ with the following properties:

1. $F$ extends the dual map $f^o : P^o \rightarrow \tilde{P}^o$ given by $f^o(U) = f^{-1}(U)$;
2. $F$ is an isometry of $\text{Cube}_w(P)$ onto $\text{Cube}_{\tilde{w}}(\tilde{P})$ (in particular, $F$ is a median isomorphism);
3. For each proper $p \in P$, if one writes $f^{-1}(p) = \{\tilde{p}_1, \ldots, \tilde{p}_n\}$, $n = n(p)$, with $\tilde{p}_1 < \ldots < \tilde{p}_n$ then there are the reals $0 < t_1 < \ldots < t_n < w(p)$ such that $F(\check{\eta}_p(t_k)) = \check{\eta}_{\tilde{p}_k}(\frac{\tilde{w}(\tilde{p}_k)}{2})$ for all $k \in \{1, \ldots, n\}$;
4. $F$ is an isometry of $\text{Cube}_w^\infty(P)$ onto $\text{Cube}_{\tilde{w}}^\infty(\tilde{P})$.

Proof. Let $X = \text{Cube}_w(P)_B$ and $\tilde{X} = \text{Cube}_{\tilde{w}}(\tilde{P})_\tilde{B}$. For each $p \in P$, writing $f^{-1}(p) = \{\tilde{p}_1, \ldots, \tilde{p}_n\}$ as above, set $t_i = \frac{\tilde{w}(\tilde{p}_i)}{2}$ and $t_{k+1} = \sum_{i=1}^k \tilde{w}(\tilde{p}_i) + \frac{\tilde{w}(\tilde{p}_{k+1})}{2}$ for $1 \leq k < n$. Let $\mathcal{H}$ denote the set of half-spaces of $(X, \Delta^1)$ arising as $\check{h}(p,k) := X \cap \check{h}_p(t_k)$ for all $p \in P$ and $\tilde{p}_k \in f^{-1}(p)$, augmented with $\{\emptyset, X\}$ to make it into a sub poc set of $2^X$ with respect to inclusion and the complementation operator $h \mapsto X \setminus h$. Observe that the map sending each $h(p,k)$ to $p$ is a refinement map factoring as the composition of $f : \tilde{P} \rightarrow P$ over the isomorphism sending each $h(p,k) \in \mathcal{H}$ to the appropriate $\tilde{p}_k \in \tilde{P}$. In particular, the poc sets $\tilde{P}$ and $\mathcal{H}$ have the same dual median space. Since $\mathcal{H}$ is a half-space system in the median space $(X, \Delta^1)$, its dual median space is isometric to $(X, \Delta^1)$ (see [6], Theorem 5.12 and Lemma 3.12).

Finally, the piecewise-$\ell^\infty$ distance is preserved because the combinatorial information — the links of the cubical faces — is preserved. \qed
5. Deforming Piecewise-$\ell^\infty$ Cubings

We begin with a rather coarse estimation of the Gromov–Hausdorff distance between $\ell^\infty$-cubings sharing the same combinatorial structure:

**Lemma 5.0.3.** Let $P$ be a finite poc set and $B \in P^0$ be a basepoint. Let $u, w$ be non-degenerate weights on $P$. Then the pair of spaces $X = \text{Cube}_u^\infty(P)_B$ and $Y = \text{Cube}_w^\infty(P)_B$ admits an $\epsilon$-approximation for any $\epsilon > \|u - w\|_1$.

**Proof.** Denote $X = \text{Cube}_u^\infty(P)$, $Y = \text{Cube}_w^\infty(P)$, $\epsilon = \|u - w\|_1$ and let $\delta > 0$. Without loss of generality we may assume that the diameter of a cube in either space does not exceed $\delta$: otherwise, construct a refinement $f : \tilde{P} \to P$ admitting weights $\tilde{u}$ and $\tilde{w}$ with the property that every cube of $\tilde{X} = \text{Cube}_{\tilde{u}}^\infty(\tilde{P})_B$ and of $\tilde{Y} = \text{Cube}_{\tilde{w}}^\infty(\tilde{P})_B$ has diameter no more than $\delta$; Proposition 4.4.2 tells us we may replace $X$ and $Y$ by the spaces $\tilde{X}$ and $\tilde{Y}$, respectively, while replacing $P$ by $\tilde{P}$ and retaining $\|\tilde{u} - \tilde{w}\|_1 = \epsilon$.

We proceed to define a relation $R \subseteq X \times Y$ by setting $(x, y) \in R$ if and only if the points $\rho_u^{-1}(x)$ and $\rho_w^{-1}(y)$ share a cube in $\text{Cube}(\tilde{P})_B$.

Take points $x, x' \in X$ and $y, y' \in Y$ satisfying $(x, y) \in R$ and $(x', y') \in R$. If $p$ is a taut broken path in $X$ from $x$ to $x'$ of total length $\lambda$, then $p' = \rho_w\rho_u^{-1} p$ is a taut broken path in $y$ of length at most $\lambda + \epsilon$. Replacing the initial and terminal points of $p'$ with $y$ and $y'$ respectively creates a broken path in $Y$ of length at most $\lambda + \epsilon + 2\delta$, providing us with the bound

$$\Delta_w^\infty(y, y') \leq \Delta_u^\infty(x, x') + \epsilon + 2\delta$$

Symmetry takes care of

$$\Delta_w^\infty(x, x') \leq \Delta_u^\infty(y, y') + \epsilon + 2\delta$$

and the result follows. \qed

**Remark 5.0.4.** The same argument works for piecewise-$\ell^1$, as well as piecewise-$\ell^2$ cubings. One only needs to observe a more serious dependence of the $\delta$-dependent overhead in the above estimate on the cardinality of $P$.

Thus, small deformations of the weight on a piecewise-$\ell^\infty$ cubing result in only small changes in the distances between its points. This allows us to apply our slightly deeper understanding of unit cubings to the study of the more general locally-compact case.

**Definition 5.0.5.** Let $P$ be a discrete poc set endowed with a weight $w$. The $n$-th lower rational approximation $[P, w]_n$ of $P$ is obtained as follows: For all $p \in P$ set $u(p) = \lfloor n \cdot w(p) \rfloor$ and let $[P, w]_n$ denote the unique refinement of the weighted poc set $(P, u)$ with unit weights, rescaled by a factor of $\frac{1}{n}$.

An immediate corollary of the preceding lemma is the following approximation lemma.

**Proposition 5.0.6 (Approximation Lemma).** Let $P$ be a discrete poc set with weight $w$ and base point $B \in P^0$, and let $X_n$ denote the dual of its $n$-th lower rational approximation, $n \in \mathbb{N}$, with the appropriate basepoint $B_n$. If $\text{Cube}(P)_B$ is locally finite, then the sequence $(X_n, B_n)$ converges to $(\text{Cube}_w(P)_B, \rho_w B)$ in the
pointed Gromov-Hausdorff topology both as $\ell^\infty$ and as $\ell^1$ cubings. In particular, if $x, y \in \text{Cube}_w(P)_B$ and $x_n, y_n$ are vertices of $X_n$ with $x_n \to x$, $y_n \to y$ and $p_n$ is the normal cube path from $x_n$ to $y_n$, then the sequence $(p_n)_{n=1}^\infty$ converges to a geodesic arc in $X$ joining $x$ with $y$.

Proof. Local finiteness of $\text{Cube}(P)$ implies it is enough to prove both our claims for the case when $P$ is finite. By the preceding lemma, the Gromov–Hausdorff distance between $P$ and $\lfloor P \rfloor_n$ does not exceed $\frac{|P|}{n}$, as required. $\Box$

As a corollary we obtain the promised reduction of our main result to the finite unit case.

**Corollary 5.0.7.** If every finite unit piecewise-$\ell^\infty$ cubing is injective, then every locally finite piecewise-$\ell^\infty$ cubing is injective.

Proof. Simply apply Lemma 2.2.4 to the result of the last proposition. $\Box$

Among the many consequences of Proposition 5.0.6 let us highlight the few most useful to our cause. Having reduced our problem to the case of finite cubings, let us see what we could glean from studying this special case using rational approximations. Clearly, $\lfloor P \rfloor_n$ for a unit-weighted finite poc set $P$ produces a refinement $P_n$ where each $d$-dimensional cube of $\text{Cube}^\infty(P)$ is subdivided into $n^d$ cubes of edge-length $\frac{1}{n}$. As a result, both $\text{Cube}^\infty(P)$ and $\text{Cube}^1(P)$ may be represented as Gromov–Hausdorff limits of the discrete rescaled spaces $K_n := \frac{1}{n}(P_n^\infty, \Delta^\infty)$ and $M_n := \frac{1}{n}(P_n^1, \Delta^1)$, respectively. As a consequence we obtain, for this special setting, the following corollary.

**Corollary 5.0.8.** Every closed ball $B(p,r)$ in $\text{Cube}^\infty(P)$ is the limit of a sequence of balls $B(p_n, r_n) \subset K_n$ with $r_n \geq r$. A closed subset $h \subset \text{Cube}^1(P)$ is a half-space if and only if it is the limit of a sequence of halfspaces $h_n \subset M_n$.

As a result we obtain an even deeper reduction of the main result to a finite, discrete problem.

**Corollary 5.0.9.** If every finite poc set has the property

(†) Any ball of integer radius in $(P^\infty, \Delta^\infty)$ is a convex subset of $(P^\infty, \Delta^1)$

then every locally finite piecewise-$\ell^\infty$ cubing is an injective metric space.

Proof. By the preceding corollary, if every finite poc set has (†), then every ball in any finite piecewise-$\ell^\infty$ cubing $X$ is $\ell^1$-convex. By Helly’s theorem (Theorem 2.3.7), any finite family of pairwise-intersecting balls $X$ has a common point. Since $X$ is a geodesic space, hyper-convexity follows. $\Box$

6. **Proof of the Main Result, Final Remarks**

In view of Corollary 5.0.9 proving our main theorem requires merely the verification of property (†) for finite poc sets $P$.

**Lemma 6.0.10.** Let $P$ be a finite poc set and let $U, W \in P^\infty$ be vertices with $\Delta^\infty(U,W) > n$, $n \in \mathbb{N}$. Then there exists a wall of $P$ separating $W$ from the $\ell^\infty$-ball of radius $n$ about $U$. 


Proof. Let \( B \) denote the \( \ell^\infty \)-ball of radius \( n \) about \( U \). Consider the normal cube path \( p = (U_0, \ldots, U_m) \), \( m = \Delta^\infty(U, W) \) from \( U \) to \( W \). As in the proof of proposition 4.3.2, let us write \( A_i = U_i \setminus U_{i-1} \) while observing that \( U \setminus W = \bigcup_{i=1}^m A_i \). Pick any \( a \in A_m \) and construct a chain \( a_1 < \ldots < a_m = a \) with \( a_i \in A_i \) – again, as in the proof of Proposition 4.3.2.

We contend that the wall \( \{ a, a^* \} \) is a wall of the kind we are looking for, that is:

1. The vertex \( W \) is contained in \( V(a^*) \);
2. The ball \( B \) is contained in \( V(a) \).

Indeed, (1) is satisfied by construction: \( a \notin W \) means \( a^* \in W \), which, in turn, means \( W \in V(a^*) \).

Property (2) holds by Lemma 4.2.1: indeed, any vertex in \( V(a^*) \) is separated from \( U \) by the nested set \( \{ a_1, \ldots, a_n \} \) and is therefore at \( \ell^\infty \)-distance at least \( n \) from \( U \). Since \( V(a) = P^o \setminus V(a^*) \) we conclude \( B \subseteq V(a) \) and we are done.

To close, we would like to draw the reader’s attention to an alternative line of reasoning having the added benefit of clarifying some leftover questions regarding the metric structure of a piecewise-\( \ell^\infty \) cubing. Reading through our own exposition we felt that, in the end, one could not help but wonder at the effort we have put into avoiding a direct computation of \( \Delta^\infty_w \) in the general case. Why only the unit case?

Example 6.0.11. Figure 5 compares a weighted cubing drawn in the \( \ell^\infty \) plane where the normal cube path joining a pair of points fails to be a geodesic broken path to a subdivided version of the same cubing, where the normal cube path between the same two points changes into a geodesic one, due to the weights being more uniformly distributed among the walls of the refined poc set.

This example suggests it should be possible to apply the Approximation Lemma (Proposition 5.0.6) in a proof of an explicit formula for \( \Delta^\infty_w \) based on the formula for unit cubings. One can verify that this is indeed the case:

**Proposition 6.0.12.** Let \( P \) be a discrete poc set with non-degenerate weight \( w \) and basepoint \( B \in P^o \). Then, for any \( x, y \in \text{Cube}_w(P)^o \), one has

\[
\Delta^\infty_w(x, y) = \max \left\{ \sum_{a \in N} |x(a) - y(a)| \mid N \text{ is a nested subset of } x \setminus y \right\}
\]

(see Definition 3.6.1). \( \square \)
Skipping the proof, we would like to observe the fact that the convexity of balls in $\text{Cube}^\infty(P)$ now becomes self-evident: if $y \not\in B(x,r)$ in $\text{Cube}^\infty(P)$, then the above formula provides us with a hyperplane of $\text{Cube}^1(P)$ separating $y$ from $B(x,r)$ using essentially the same procedure as we had used in the proof of Lemma 6.0.10, avoiding the need for a reduction of the statement regarding the injectivity of piecewise-$\ell^\infty$ cubings to the finite, unit, vertex-only case.

This argument seems to apply Gromov–Hausdorff convergence more sparingly, but ultimately it does nothing but shift the weight (of the technical details) around. On an emotional note, we admit our preference of strategy was motivated by a sense of indebtedness to Isbell’s vision: not only did he reveal the way (first followed by Mai and Tang), but he also provided the machinery (the duality theory of median algebras) for the present extension. It would have been ungrateful of us to have picked a different path.

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