Mixing Constant Sum and Constant Product Market Makers

Alexander Port Neelesh Tiruviluamala*

April 7, 2022

Abstract

Two popular forms of automated market makers are constant sum and constant product (CSMM and CPMM respectively). Each has its advantages and disadvantages: CSMMs have stable exchange rates but are vulnerable to arbitrage and can sometimes fail to provide liquidity, while a CPMM can have large impermanent loss due to exchange rate changes but are always able to provide liquidity to participants.

A significant amount of work has been done in order to get the best of both constant sum and constant product characteristics. Perhaps most the relevant to this paper is Stableswap, which has an “amplification coefficient” parameter controlling the balance between the two types of behavior [Ego19]. Alternative approaches, such as in [AEC21], involve constructing AMMs using portfolio value functions. However, there is still much work to be done on these fronts. This paper presents multiple novel methods for mixing market makers and demonstrates new tools for designing markets with specific features.

1 Basic Construction

The simplest CSMM and CPMM curves are given by $x + y = k_1$ and $xy = k_2$; here one thinks of $x$ and $y$ as being the amounts of the different currencies in the market. For simplicity, let’s assume that $k_1 = 2$ and $k_2 = 1$ so that both curves contain the point $(x_0, y_0) = (1, 1)$. Note that this means that the CSMM can be rewritten as $\frac{x+y}{2} = 1$; the advantage here is that now both curves are defined with a constant of 1 on the righthand side and taking combinations of the two is very easy.

For reasons that will be apparent soon, denote $A_0(x, y) = \frac{x+y}{2}$ and $A_1(x, y) = xy$. The goal is to find a family of functions $A_t(x, y)$ for $0 \leq t \leq 1$ that smoothly transitions from $A_0(x, y)$ to $A_1(x, y)$. Perhaps the easiest way is to take an arithmetic weighted mean of the two functions:

$$A_t^{arith}(x, y) = A_0(x, y)(1-t) + A_1(x, y)t$$  \hspace{1cm} (1)

Part of the reason this works as well as it does is because $A_0(x, y) = 1$ and $A_1(x, y) = 1$; thus, any weighted average of the two will also be equal to 1.

*Emails: alex@thrackle.io, neel@thrackle.io
A concrete example of this construction is seen in Stableswap ([Ego19]). In an \( n \)-currency Stableswap market the initial amounts \( X^i = (x^i_1, \ldots, x^i_n) \) of each currency are given by \( x^i_j = \frac{D}{n} \) for some value of \( D \) and each value of \( j \). The sum and product of these values are then given by the following:

\[
\sum_{j=1}^{n} x^i_j = D \quad \prod_{j=1}^{n} x^i_j = \left( \frac{D}{n} \right)^n
\]

The Stableswap system has a self-proclaimed “leverage” parameter \( \chi \) where \( \chi = 0 \) corresponds to CPMM and \( \chi = \infty \) to CSMM; if the quantities of each currency are given by \( X = (x_1, \ldots, x_n) \) then the AMM is

\[
\chi D^{n-1} \sum_{j=1}^{n} x_j + \prod_{j=1}^{n} x_j = \chi D^n + \left( \frac{D}{n} \right)^n
\]

The claim here is that these formulas can be represented using the above \( A_{\text{arith}} \) format. To see this, note that the CSMM and CPMM equations can written as the following:

\[
A_0(X) = \frac{1}{D} \sum_{j=1}^{n} x_j 
A_1(X) = \left( \frac{D}{n} \right)^n \prod_{j=1}^{n} x_j
\]

Each of these equations is normalized so that the surfaces are defined by \( A_0(X) = 1 \) and \( A_1(X) = 1 \). This in turn allows the defining mixed equation to be written as

\[
1 = \frac{\chi D^{n-1} \sum_{j=1}^{n} x_j + \prod_{j=1}^{n} x_j}{\chi D^n + \left( \frac{D}{n} \right)^n}
\]

\[
= A_0(X) - \chi D^n + \left( \frac{D}{n} \right)^n + A_1(X) - \chi D^n + \left( \frac{D}{n} \right)^n
\]

\[
= A_0(X) \frac{\chi}{\chi + n^{-n}} + A_1(X) \frac{n^{-n}}{\chi + n^{-n}}
\]

Thus, the Stableswap equations can be reparametrized to the arithmetic mixing format where the relation between \( \chi \) and \( t \) is given by

\[
t = \frac{n^{-n}}{\chi + n^{-n}} \tag{2}
\]

While the arithmetic mean is a simple way of combining the curves, it is far from the only way. Intuitively, the curve \( A_{\text{arith}}(x, y) \) is just an arithmetic weighted mean of the CSMM and CPMM curves. A common alternative to the arithmetic mean is the geometric mean. For example, recall that the arithmetic and geometric means of the numbers 3 and 5 are \( \frac{1}{2}(3 + 5) = 4 \) and \( (3 \cdot 5)^{\frac{1}{2}} = \sqrt{15} \) respectively. One can define a geometric weighted mean of the CSMM and CPMM curves in a similar way:

\[
A_{\text{geo}}(x, y) = A_0(x, y)^{1-t} A_1(x, y)^t \tag{3}
\]

This can appear more complicated than the arithmetic version, but it also offers several computational
advantages. For example, $A_0(x, y) = \frac{x + y}{2}$ and $A_1(x, y) = xy$ are homogeneous functions of degree 1 and 2 respectively, i.e. $A_0(\lambda x, \lambda y) = \lambda A_0(x, y)$ and $A_1(\lambda x, \lambda y) = \lambda^2 A_0(x, y)$. There are known benefits to having a homogenous AMM curve ([TPL22]) and one can show that $A_t^{geo}(x, y)$ is homogenous for all values of $t$ while $A_t^{arith}(x, y)$ is non-homogeneous.

A more general construction is needed to fully describe the pros and cons of different mixing methods. The purpose is to find an optimal balance of price stability, impermanent loss, and ability to provide liquidity.

2 General Construction: Homotopy

A classical mathematical construction of a “homotopy” between two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^n$ is a weighted average $H(s, t) = (1 - t) \cdot f(s) + t \cdot g(s)$ depending on some parameter $t$. Fixing a value of $t$ gives a very precise blending of $f$ and $g$ in that each resulting point $H(s, t)$ is exactly 100$t$ percent of the way along the line segment connecting $f(s)$ and $g(s)$. Despite their equations being visually similar, the mixings above do not satisfy this property for a variety of reasons; values of $t$ in arithmetic and geometric mixings are only geometrically meaningful in this way when $t$ is 0 or 1. The idea of the proceeding “homotopy mixing” is that $t$ will always represent the percent movement from CSMM to CPMM in the above mathematical way (see Figure 1 for a visual representation). Note that the current and initial states will be denoted as $X = (x_1, ..., x_n)$ and $X' = (x'_1, ..., x'_n)$ for shorthand throughout this section.

This construction is best presented in an abstract setting where the number of currencies is arbitrary, even though the focus of this paper is for 2D AMMs. Suppose the CSMM and CPMM surfaces are given by

$$A_0(X) = \frac{\sum_{j=1}^{n} a_j x_j}{\sum_{j=1}^{n} a_j x'_j} = 1 \quad A_1(X) = \prod_{j=1}^{n} \left( \frac{x_j}{x'_j} \right)^{\alpha_j} = 1$$

Here the $x_j$’s are the current quantities of the currencies in the AMM and the $x'_j$’s are the initial quantities. Both of these surfaces pass through the same initial point $X'$; however, it is also important for the surfaces to share the same exchange rates at that initial state. To find the condition for this, differentiation shows that the gradients of these CSMM and CPMM equations are given by

$$\nabla A_0(X) = \frac{1}{\sum_{j=1}^{n} a_j x'_j} (a_1, ..., a_n) \quad \nabla A_1(X) = \prod_{j=1}^{n} \left( \frac{x_j}{x'_j} \right)^{\alpha_j} \left( \frac{\alpha_1}{x_1}, ..., \frac{\alpha_n}{x_n} \right)$$

One may think of the gradient as giving the prices of the two currencies, at least up to some common multiple. One can get the exchange rate from one currency to another by looking at the quotient of the two corresponding gradient values. For example, the exchange rate of exchanging the $x_1$ currency for the $x_2$ currency is given by $\frac{a_2}{a_1}$ and $\frac{\alpha_{x_2}}{\alpha_{x_1}}$ in $A_0$ and $A_1$ respectively. Intuitively this makes sense, especially for $A_1$; as the amount of the $x_2$ currency grows its value will go down and so it costs fewer of the $x_1$ currency to get a particular amount of $x_2$ currency. If one assumes that the $A_0$ and $A_1$ exchange rates must be equal at
the initial point $X^i$ then

$$(a_1, \ldots, a_n) \text{ must be parallel to } \left(\frac{\alpha_1}{x_1}, \ldots, \frac{\alpha_n}{x_n}\right)$$

Note that a very natural and completely acceptable choice from a mathematical standpoint would be to simply take $\alpha_j = a_j x_j^i$ for each $j$. However, practically speaking it is best to normalize these values to be much smaller in order to avoid blowups; a normalization condition of $\sum_{j=1}^n \alpha_j = 1$ helps fix this issue.

There is a very natural bijective association between the CSMM and CPMM surfaces. More specifically, any point on the CPMM surface has a unique corresponding point on the CSMM surface, and the same correspondence is true in the other direction as long as none of the quantities $x_j$ are 0. To construct this:

- Suppose $v = (v_1, \ldots, v_n)$ is a point in the state space $\mathbb{R}_{>0}^n$, i.e. where $v_j > 0$ for all $j$.
- There exists some scalar function $\lambda_0 = \lambda_0(v)$ such that the point $\lambda_0 v = (\lambda_0 v_1, \ldots, \lambda_0 v_n)$ lies on the CSMM surface. In particular:
  $$A_0(\lambda_0 v) = 1 \implies \lambda_0 = \sum_{j=1}^n \frac{a_j x_j^i}{\sum_{j=1}^n a_j v_j}$$

- Similarly, there exists some scalar function $\lambda_1 = \lambda_1(v)$ such that the point $\lambda_1 v = (\lambda_1 v_1, \ldots, \lambda_1 v_n)$ lies on the CPMM surface. By denoting $\text{deg}(A_1) = \sum_{j=1}^n \alpha_j$:
  $$A_1(\lambda_1 v) = 1 \implies \lambda_1 = \prod_{j=1}^n \left(\frac{x_j^i}{v_j}\right)^{\frac{\alpha_j}{\text{deg}(A_1)}}$$

The correspondence is then given by associating $\lambda_0 v$ with $\lambda_1 v$. This method creates a pairing between the surfaces as a whole because one could always choose $v$ to be an arbitrary point on the CSMM or the CPMM.

This correspondence between CSMM and CPMM state points allows for a natural description of a smooth transition from one to the other. This method is inspired by the notion of homotopy in algebraic topology, and therefore the constructed surface will be denoted $A^\text{hom}_t(x_1, \ldots, x_n) = 1$. This function is constructed as follows. Suppose $x = (x_1, \ldots, x_n)$ is a point on this homotopy surface for the given parameter $t$. The assumption in this construction is that the point $x$ lies $100t$ percent of the way along the line segment connecting $\lambda_0 x$ to $\lambda_1 x$. The equation form of this assumption is given by

$$x = (1-t)\lambda_0 x + t\lambda_1 x = ((1-t)\lambda_0 + t\lambda_1) x$$

Clearly if $t = 0$ then $x = \lambda_0 x$ and so $x$ must lie on the CSMM surface; similarly, if $t = 1$ then $x = \lambda_1 x$ and so $x$ is on the CPMM surface. Note that the 2D version of this argument is presented in Figure [I].
Figure 1: Visualization in 2D of the line segment connecting the CSMM and CPMM curves. The point $(x_0, y_0)$ is at the intersection of blue and red, $(\lambda_0 x, \lambda_0 y)$ at the intersection of blue and black, and $(\lambda_1 x, \lambda_1 y)$ at the intersection of red and black.

More generally, the only way the above equation $x = ((1-t)\lambda_0 + t\lambda_1)x$ holds is if $(1-t)\lambda_0 + t\lambda_1 = 1$ (because all values $x_1, ..., x_n$ are assumed strictly positive). This equation is exactly the needed one as original the goal was to find a function that was equal to 1 that described points on the surface. Upon simplifying, the result becomes

$$A_t^{\text{hom}}(X) = (1-t)\frac{\sum_{j=1}^{n} a_j x_j}{\sum_{j=1}^{n} a_j^2 x_j} + t \prod_{j=1}^{n} \left( \frac{x_j^{\alpha_j}}{x_j^{\alpha_j}} \right)^{\frac{\alpha_j}{\deg(A_1)}}$$

$$= A_0(X)^{-1}(1-t) + A_1(X)^{-\frac{1}{\deg(A_1)}}t$$

The remainder of this section is concerned with this specifics of this construction in 2D. The parameter $t$ along with the addition of another parameter $s$ allows for a complete parametric description of the region between the CSMM and CPMM curves in 2D. As seen in Section 4 such a parametrization is a very powerful tool for designing and building mixing methods that satisfy any range of desired properties.

Before deriving any equations using the parameter $s$, denote the following for simplicity moving forward:

$$A_0(x, y) = \frac{ax + by}{ax_0 + by_0}, \quad A_1(x, y) = \frac{x^\alpha y^\beta}{x_0^\alpha y_0^\beta}$$

The above is a $t$-parametric description of the line segment connecting the pair of points on the CSMM and CPMM surfaces; one can describe the original CSMM curve given by $A_0(x, y) = 1$ in a similar parametric
way. The $y$-intercept and $x$-intercept of this line are given by $(0, \frac{a}{b}x_0 + y_0)$ and $(x_0 + \frac{b}{a}y_0, 0)$ respectively. As above, one can connect these two points using a line segment parametrized by the variable $s$. The CSMM position corresponding to the value of $s$ is the point

$$(1 - s) \left(0, \frac{a}{b}x_0 + y_0\right) + s \left(x_0 + \frac{b}{a}y_0, 0\right) \text{ or equivalently } \left(a x_0 + b y_0\right) \left(s, \frac{1 - s}{b}\right)$$

One may think of $s$ as determining the slope of the dotted line in Figure 1 or equivalently the angle that the dotted line makes with the x-axis. Next, suppose $(x, y)$ is some point on this curve, i.e. $A^\text{hom}(x, y) = 1$. Given a fixed point $(x, y)$ on the curve $A^\text{hom}_i$, one can solve for the $s$ corresponding to $(x, y)$ using that fact that $(x, y)$ and $(\frac{s}{a}, \frac{1-s}{b})$ lie on the same line passing through the origin. In particular, the ratios of the two coordinates in each point must be the same:

$$\frac{y}{x} = \frac{1-s}{s}$$

$$\implies s = \frac{ax}{ax + by} \quad (5)$$

### 3 Comparison of Arithmetic, Geometric, and Homotopy Mixings

Section 2 demonstrates a rigorous construction of a method for mixing CSMM and CPMM curves that is more grounded in measuring the literal distance travelled from one curve to the another than previous methods have been. This concreteness comes with the trade-off of having a potentially more complicated defining equation. As such, it is important to directly compare the three above mixing methods to see and analyze all the pros and cons. The most relevant properties are as follows: existence of a convenient parametrization, ability to provide liquidity, whether or not the curve is exchange rate level independent, and the amount of stability in exchange rates given changes in currency quantity.

#### 3.1 Parametrization

The above parametric homotopy construction allows for the easy plotting of the homotopy mixing curves $A^\text{hom}_i(x, y) = 1$. The point $(\frac{s}{a}, \frac{1-s}{b})$ can be thought of as the base parametrization of a line segment. As seen in Section 2 multiplying this by $ax_0 + by_0$ moves that point to the CSMM curve. Similarly, multiplying by $(\frac{ax_0}{s})^\alpha (\frac{by_0}{1-s})^{\beta} \frac{1}{\alpha + \beta}$ places the point on the CPMM curve. Thus, one may parametrize the scalar function needed to move this point to $A^\text{hom}_t$ in the following way:

$$\lambda^\text{hom}(s, t) = (ax_0 + by_0)(1-t) + \left(\frac{ax_0}{s}\right)^\alpha \left(\frac{by_0}{1-s}\right)^\beta \frac{1}{\alpha + \beta} t$$

This equation provides a convenient closed form equation for moving points into the homotopy mixing curve.
| Mixing Type | Equation in 2D |
|-------------|----------------|
| Arithmetic  | $A^\text{arith}_t(x, y) = \left( ax + by \atop ax_0 + by_0 \right) (1 - t) + \left( x^\alpha y^\beta \atop x_0^\alpha y_0^\beta \right) t$ |
| Geometric   | $A^\text{geo}_t(x, y) = \left( ax + by \atop ax + by \right)^{1-t} \left( x^\alpha y^\beta \atop x_0^\alpha y_0^\beta \right)^t$ |
| Homotopy    | $A^\text{hom}_t(x, y) = \left( ax_0 + by_0 \atop ax_0 + by_0 \right) (1 - t) + \left( x^\alpha y^\beta \atop x_0^\alpha y_0^\beta \right)^\frac{1}{1-t} t$ |

| Mixing Type | Equation in General |
|-------------|---------------------|
| Arithmetic  | $A^\text{arith}_t(x_1, ..., x_n) = A_0(x_1, ..., x_n)(1 - t) + A_1(x_1, ..., x_n)t$ |
| Geometric   | $A^\text{geo}_t(x_1, ..., x_n) = A_0(x_1, ..., x_n)^{1-t}A_1(x_1, ..., x_n)^t$ |
| Homotopy    | $A^\text{hom}_t(x_1, ..., x_n) = A_0(x_1, ..., x_n)^{-1}(1 - t) + A_1(x_1, ..., x_n)^{-1}\frac{1}{(1-t)^{\deg A_1}} t$ |

Table 1: The three main types of CSMM and CPMM mixings considered in this paper. The AMM is always defined by $A_t(x, y) = 1$. The parameter $t$ controls the transition from CSMM at $t = 0$ to CPMM at $t = 1$. The top table gives the equations in this specific 2D context; the bottom table provides a more abstract presentation distanced from the specifics of the CSMM and CPMM surfaces in higher dimensions.

One can show that a similar equation can be used to move points onto the geometric mixing curve $A^\text{geo}_t$:

$$A^\text{geo}_t(x, y) = \left( ax + by \atop ax_0 + by_0 \right)^{1-t} \left( x^\alpha y^\beta \atop x_0^\alpha y_0^\beta \right)^t = 1 \quad (x, y) = \lambda^\text{geo}(s, t) \left( \frac{s}{a}, \frac{1-s}{b} \right)$$

$$\Rightarrow \lambda^\text{geo}(s, t) = (ax_0 + by_0)^{1-t} \left( \frac{ax_0}{s} \right)^\alpha \left( \frac{by_0}{1-s} \right)^\beta \left( \frac{1-t}{1-t^{\alpha+\beta}} \right)^{(1-t)^{\alpha+\beta}} \tag{7}$$

In contrast, it is generally not possible to find such a function with the arithmetic curve. The problem, as seen below, is that it is not possible to directly solve for $\lambda^\text{arith}(s, t)$ because doing so amounts to solving a polynomial of arbitrary and possibly non-integer degree. While this can certainly be approximated, doing so would be slower and less accurate than having an explicit closed-form solution. The polynomial in question is seen below:

$$A^\text{arith}_t(x, y) = \left( ax + by \atop ax_0 + by_0 \right) (1 - t) + \left( x^\alpha y^\beta \atop x_0^\alpha y_0^\beta \right) t = 1 \quad (x, y) = \lambda^\text{arith}(s, t) \left( \frac{s}{a}, \frac{1-s}{b} \right)$$

$$\Rightarrow \lambda^\text{arith}(s, t) \frac{1-t}{ax_0 + by_0} + \lambda^\text{arith}(s, t)^{\alpha+\beta} \left( \frac{s}{ax_0} \right)^\alpha \left( \frac{1-s}{by_0} \right)^\beta t = 1$$

7
Figure 2: Comparison of arithmetic, geometric, and homotopy curves where each diagram has the same value of $t$ but different curve types. Important observations include: (1) arithmetic supports a finite range of exchange rates that increases in size with $t$, (2) geometric and homotopy support all exchange rates, and (3) the exchange rate of homotopy the most stable of the three types for each fixed value of $t$.

In short, the homotopy and geometric mixings have very convenient parametric descriptions while the arithmetic mixing does not. There are several benefits to having such a function $\lambda(s,t)$. Perhaps the most important reason is that it allows for easy computation of quantity changes. Keep in mind that the primary purpose of this work is to construct an automated market maker where users can exchange currencies. A fundamental part of such a task is providing the user with an estimate of the total cost of exchanging for a given amount of the desired. This is non-trivial because the AMM is defined implicitly, so efficient computation is key here.

3.2 Ability to Provide Liquidity

The CSMM and CPMM curves have several important differences, but perhaps the most significant is the range of exchange rates that each is able to support. One of the main defining characteristics of a CPMM is its ability to provide liquidity at all possible exchange rates. In such a market, as the quantity of an item goes down to zero it becomes asymptotically more valuable; this item will never be fully extracted because its exchange rate gets arbitrarily large. In contrast, a CSMM only supports a single exchange rate; one can
Important observations include: (1) price stability decreases and impermanent loss gets worse as $t$ increases for all three curve types, and (2) curves are more evenly distributed between CSMM and CPMM in homotopy than they are for geometric.

Another benefit of this $\lambda(s, t)$ function is that one can examine the asymptotic behavior near the endpoints of mixed curves; such asymptotes are related to the derivatives of the curve and therefore the range of exchange rates supported by the AMM. For example, when $s = 0$ or $1$ and $t = 0$ then the corresponding points are the endpoints of the CSMM line segment; these points are finitely far away from the origin therefore $\lim_{s \to 0^+} \lambda(s, 0)$ and $\lim_{s \to 1^-} \lambda(s, 0)$ must be finite for all three types of $\lambda$ functions. On the other hand, these same limits for $\lambda(s, 1)$ must approach infinity because the CPMM “endpoints” are infinitely far away due to being the asymptotes of a hyperbola.

It is important to examine the behavior of these limits when $0 < t < 1$. Thankfully, the asymptotes in Figures 2 and 3 make this more clear and show that the following limits hold:

$$
\begin{align*}
\lim_{s \to 0^+} \lambda_{\text{arith}}(s, t) &< \infty & \lim_{s \to 0^+} \lambda_{\text{geo}}(s, t) & = \infty & \lim_{s \to 0^+} \lambda_{\text{hom}}(s, t) & = \infty \\
\lim_{s \to 1^-} \lambda_{\text{arith}}(s, t) &< \infty & \lim_{s \to 1^-} \lambda_{\text{geo}}(s, t) & = \infty & \lim_{s \to 1^-} \lambda_{\text{hom}}(s, t) & = \infty
\end{align*}
$$
One might expect that the limits in the homotopy case would be infinite because $\lambda^{\text{Hom}}$ will describe a weighted average between a finite value and infinity; the other cases are less intuitive without the figures.

In short, the above limits show that arithmetic mixings share the CSMM property of not always being able to provide liquidity, and they also show that geometric and homotopy mixings more closely resemble a CPMM in this respect. Thus, these latter two mixings are better able to match a wide range of exchange rates and this affects how arbitrageurs interact with these markets.

### 3.3 Exchange Rate Level Independence for Impermanent Loss

The idea of impermanent loss is that liquidity providers for AMMs can be disincentivized from investing due to changes in exchange rates. Suppose the initial quantities of assets provided to the AMM by the investor are $X^i = (x_1^i, \ldots, x_n^i)$ and their initial prices are $P^i = (p_1^i, \ldots, p_n^i)$. After a period of time, the investor may want to withdraw their percent ownership of the AMM and reclaim their assets. Due to changes in the market state, they will be able to obtain quantities $X^f = (x_1^f, \ldots, x_n^f)$ of each currency and their respective prices would be $P^f = (p_1^f, \ldots, p_n^f)$. The initial value of their assets is given by $P^i \cdot X^i$ and the final value of their assets is $P^f \cdot X^f$. However, one can also consider the value they would have had if they’d simply held onto their assets instead of investing; in that case, their held asset value would be $P^f \cdot X^i$. The reader can refer to [TPL22] for more details, but the equation for impermanent loss is as follows:

$$IL = \frac{P^f \cdot X^f}{P^i \cdot X^i} - 1$$

Roughly speaking, impermanent loss compares the relative values of investing and holding assets. One can show that the impermanent loss is always less than or equal to 0 and becomes more negative as prices drift further from initial values. In other words, the investor has no reason to provide liquidity for the AMM unless they benefit from collection of transaction fees that cancel out the difference in these two values.

Figures 2 and 3 demonstrate the impermanent loss curves for the mixed AMMs. There are several aspects of these figures that are worth noting. First, all curves demonstrate the property that impermanent loss is worse the more the exchange rates change from the initial state at the time of investment. Second, the curves closer to CPMM with larger values of $t$ have less severe impermanent loss than those close to CSMM. Third, it is common practice to describe impermanent loss in terms of exchange rates even though the above formula for $IL$ is in terms of prices and quantities.

An AMM is “exchange rate level independent” (i.e. ERLI) if its equation for impermanent loss can be written purely in terms of the ratios of final to initial exchange rates of the currencies in the market. The paper [TPL22] gives a variety of lemmas and theorems related to when an AMM is ERLI; note that many of these statements relate to homogeneity. However, the application of these claims to the equations of Table 1 show that none of the three mixings satisfy this condition when $0 < t < 1$ and therefore none of these mixings are ERLI.
### 3.4 Slippage and Exchange Rate Stability

Perhaps the central purpose of a platform like Stableswap is to ensure that AMM exchange rates closely match external exchange rates when there are small changes in AMM currency quantities. A CSMM is the best in this respect; the exchange rate is fixed regardless of the quantities of currencies. On the other hand, even relatively small transactions can have sizable affects on exchange rates in a CPMM. This price stability is relevant for users of the AMM because it helps to minimize price slippage (i.e. the difference between the price users expect and the actual price given by a variety of factors).

It is important to keep in mind that there is a distinction between internal and external changes in exchange rate. Stability in internal exchange rates must be balanced with the ability to provide liquidity at a variety of exchange rates. For example, the CSMM is completely stable with respect to internal changes but can only support a single external exchange rate; if the internal and exchange exchange rates don’t match then arbitrageurs will completely drain the currency that the AMM undervalues and thereby shut down the usefulness of the AMM. As discussed above, a CPMM is different and can support the providing of liquidity at any exchange rate.

With that in mind, the diagrams in Figures 2 and 3 show the relative stabilities of the three mix types. For small changes in quantity at a fixed shared value of $t$, the homotopy curve $A^\text{hom}_t(x, y)$ is the most stable and the geometric curve $A^\text{geo}_t(x, y)$ is the least stable. On the other hand, the arithmetic curve $A^\text{arith}_t(x, y)$ is the most stable for large changes and the homotopy curve $A^\text{hom}_t(x, y)$ is the least. In short, each mixing type appears to have its own unique levels of stability in different regions of the market state. These ideas are also demonstrated in a more direct way in Figure 4.

### 4 Advanced Homotopy Construction

There are two important methods for creating more advanced and dynamic behavior in these homotopy AMMs. As an example, it is possible to think of the parameter $t$ as a function of $x$ and $y$ in order to create a non-uniform homotopy between $A_0(x, y)$ and $A_1(x, y)$. A practical example of this appears for Stableswap
Figure 4: Comparison of exchange rate stability in the three mixed curves. (Top) These images all have the same value of $t$. Homotopy is the most stable for small changes but the least stable for larger ones. Geometric is the least stable for small changes. Arithmetic is the most stable for large changes. (Bottom) These images all have the same derivative for exchange rate at the initial point. For both small and large changes, arithmetic is the most stable while homotopy is the least stable. The arithmetic and geometric curves will always have the same derivative here, but homotopy will be different. Note that the value of 0.1817 for $t$ was determined empirically for this figure.

in [Ego19]. Recall that the defining equation there is

$$\chi D^{n-1} \sum_{j=1}^{n} x_j + \prod_{j=1}^{n} x_j = \chi D^n + \left(\frac{D}{n}\right)^n$$

where initial values $X^i = (x^i_1, ..., x^i_n)$ are chosen such that $x^i_j = \frac{D}{n}$ for each $j$. Initially there is the equality $\prod_{j=1}^{n} x^j_i = \left(\frac{D}{n}\right)^n$, but as the market state falls out of this initial balance the equality will be less and less exact. The proposed solution to counter this lack of balance is to dynamically adjust the leverage parameter $\chi$. For some fixed choice of $A$, the value of of $\chi$ is given by

$$\chi = A \frac{\prod_{j=1}^{n} x_j}{\left(\frac{D}{n}\right)^n}$$
Figure 5: Comparison of the Stableswap non-uniform homotopy curve with a uniform homotopy curve. The left image is the majority of the curve; the right image is a zoomed in section that demonstrates how no single value of $t$ captures the behavior of the Stableswap curve.

When this is substituted into the above equation, the curve is now defined by the following:

$$An^\sum_{j=1}^n x_j + D = ADn^n + \frac{D^{n+1}}{n^n \prod_{j=1}^n x_j}$$

To get more specific with this construction, let’s take $n = 2$ and turn to the above equations as well as those for the homotopy mixing curves:

$$\chi = \frac{4Axy}{D^2}$$

$$t = \frac{D^2}{16Axy + D^2}$$

$$A_{t}^{\text{hom}}(x, y) = \frac{D(1 - t)}{x + y} + \frac{Dt}{2\sqrt{xy}}$$

The above gives a fixed rule for dynamically updating the value of $t$ as the state $(x, y)$ of the AMM changes. Substituting this function for $t$ into $A_{t}^{\text{hom}}(x, y) = 1$ yields the following curve:

$$\frac{16ADxy}{x + y} + \frac{D^3}{2\sqrt{xy}} = 16Axy + D^2$$

As Figure 6 demonstrates, this new curve represents a homotopy between the CSMM and CPMM where the rate of transition from one to the other is non-uniform over the curve. Note that it is possible to solve for $t$ as a function of only $s$, but the result is a rather complicated formula and is not worth presenting here.

4.1 Constructing Specific Non-Uniform Homotopy Behavior

The original definition of the homotopy mixing curve $A_{t}^{\text{hom}}(x, y)$ assumed that $t$ was a uniform constant. However, the above Stableswap construction demonstrates that interesting behavior can appear when $t$ is
non-uniform and is taken to be a function of other variables ([Ego19]). The purpose of this section is (1) to argue that taking \( t \) to be a function of \( s \) is a naturally powerful way for designing non-uniform homotopy curves, (2) to provide required conditions on this function \( t(s) \) to ensure the resulting curve satisfies the axioms of an AMM, and (3) to demonstrate examples of interesting curves using this construction.

In Section 2, a method for parametrizing the family of homotopy AMM curves is presented. In particular, a choice of values of \( s \) and \( t \) returns a point \((x,y)\) according to the following equation:

\[
(x,y) = \left( \frac{ax_0 + by_0}{ax + by} \right) (1 - t) + \left( \frac{x_0}{x} \right)^\alpha t \left( \frac{y_0}{y} \right)^\beta \left( \frac{s}{a}, \frac{1 - s}{b} \right)
\]

\[
= \left( \frac{x_0}{x} \right)^\frac{\alpha}{\alpha + \beta} \left( \frac{y_0}{y} \right)^{\frac{\beta}{\alpha + \beta}} - \left( \frac{ax_0 + by_0}{ax + by} \right) t + \left( \frac{ax_0 + by_0}{ax + by} \right) \left( \frac{s}{a}, \frac{1 - s}{b} \right)
\]

\[
\Rightarrow \lambda(x,y) = \left( \frac{x_0}{x} \right)^\frac{\alpha}{\alpha + \beta} \left( \frac{y_0}{y} \right)^{\frac{\beta}{\alpha + \beta}} - \left( \frac{ax_0 + by_0}{ax + by} \right) t + \left( \frac{ax_0 + by_0}{ax + by} \right)
\]

This idea is further developed using the fact that \( s = \frac{ax}{ax + by} \), meaning that an entire curve is simply given by a single value of \( t \). The above introduction discusses how one can think of the Stableswap curve in this context by taking \( t \) to be a function of \( x \) and \( y \). However, this section reduces the system to a single parameter by taking \( t \) to be a function of \( s \) directly. Of course, because \( s \) can be determined by \( x \) and \( y \), the result is practically no different. The advantage of this new perspective is that it helps in designing more predictable behavior.

\[
A^{hom}(x,y) = \left( \frac{ax_0 + by_0}{ax + by} \right) \left( 1 - t \left( \frac{ax}{ax + by} \right) \right) + \left( \frac{x_0}{x} \right)^\alpha t \left( \frac{y_0}{y} \right)^\beta \left( \frac{ax}{ax + by} \right)
\]

\[
= \left( \frac{x_0}{x} \right)^\frac{\alpha}{\alpha + \beta} \left( \frac{y_0}{y} \right)^{\frac{\beta}{\alpha + \beta}} - \frac{ax_0 + by_0}{ax + by} \left( \frac{ax}{ax + by} \right) t + \frac{ax_0 + by_0}{ax + by}
\]

In the development of this paper, it became apparent that even simple equations for \( t(s) \) can sometimes fail to provide proper AMMs; the repeated issue is that the curve can fail to be convex but it was unclear why this was happening. The goal here is to provide a simple condition on \( t(s) \) to ensure that the resulting AMM is convex. The second derivative of this curve must be computed to find the necessary condition because a twice differentiable function is convex if and only if its second derivative is non-negative.

To find this second derivative, note that the above equation for \( A^{hom}(x,y) \) gives an implicit definition of the curve. In general it is not possible to solve for \( y \) as a function of \( x \), but it is still possible to implicitly compute the first and second derivatives of this homotopy curve. For these computations think of \( x \) and \( y \) as functions of \( s \) and use the following shorthand for simplicity:

\[
C = ax_0 + by_0 \quad P(s) = \left( \frac{ax_0}{s} \right)^\frac{\alpha}{\alpha + \beta} \left( \frac{by_0}{1 - s} \right)^{\frac{\beta}{\alpha + \beta}}
\]
Next, denoting $\lambda(s) = \lambda^{\text{hom}}(s)$ and differentiating yields the following:

\[
\lambda(s) = (P(s) - C)t(s) + C
\]

\[
\lambda'(s) = (P(s) - C)t'(s) + P'(s)t(s)
\]
\[
= (P(s) - C)t'(s) - \frac{\alpha(1 - s) - \beta s}{s(1 - s)(\alpha + \beta)} P(s)t(s)
\]

\[
\lambda''(s) = (P(s) - C)t''(s) + 2P'(s)t'(s) + P''(s)t(s)
\]
\[
= (P(s) - C)t''(s) - 2 \frac{\alpha(1 - s) - \beta s}{s(1 - s)(\alpha + \beta)} P(s)t'(s)
\]
\[
+ \frac{2\alpha^2(1 - s)^2 + \alpha\beta(1 - 2s) + 2\beta^2 s^2}{s^2(1 - s)^2(\alpha + \beta)^2} P(s)t(s)
\]

The functions $x = x(s)$ and $y = y(s)$ and their derivatives can now be written in terms of $\lambda$ and its derivatives:

\[
x(s) = \frac{s}{a} \lambda(s) \quad y(s) = \frac{1 - s}{b} \lambda(s)
\]
\[
x'(s) = \frac{1}{a} \lambda(s) + \frac{s}{a} \lambda'(s) \quad y'(s) = \frac{1}{b} \lambda(s) + \frac{1 - s}{b} \lambda'(s)
\]
\[
x''(s) = \frac{2}{a} \lambda'(s) + \frac{s}{a} \lambda''(s) \quad y''(s) = -\frac{2}{b} \lambda'(s) + \frac{1 - s}{b} \lambda''(s)
\]

One can show that the following equations hold as definitions for implicit derivatives in this case:

\[
\frac{dy}{dx}(s) = \frac{y'(s)}{x'(s)}
\]
\[
\frac{d^2y}{dx^2}(s) = \frac{x'(s)y''(s) - x''(s)y'(s)}{x'(s)^3}
\]

The equations for $x$ and $y$ in terms of $\lambda$ can be substituted in here and result can be simplified:

\[
\frac{dy}{dx}(s) = \left( \frac{\lambda'(s)}{\lambda(s) + s\lambda'(s)} - 1 \right) \cdot \frac{a}{b}
\]
\[
\frac{d^2y}{dx^2}(s) = \frac{\lambda(s)\lambda''(s) - 2\lambda'(s)^2}{(\lambda(s) + s\lambda'(s))^3} \cdot \frac{a^2}{b}
\]

Note that $b$ and $x'(s)$ should always be positive by the construction of the homotopy curve as an AMM. Thus, under these assumptions the above can be distilled into a simple convexity condition in terms of $\lambda$:

\[
A^\text{hom}(x, y) = 1 \text{ is convex if and only if } \lambda(s)\lambda''(s) \geq 2\lambda'(s)^2 \text{ for all } s \in (0, 1) \tag{9}
\]

The following is a summary of some basic examples of non-uniform homotopy curves. These examples present a novel way to create specifically designed behavior in the stability and overall shape of the curve. For these curves, take $s_0 = \frac{x_0}{ax_0 + by_0}$ to be the value of $s$ corresponding to the initial point $(x_0, y_0)$. 


The parameter $K$ is thought of as a stability parameter where a larger value of $K$ forces the curve to more closely resemble a CSMM. When comparing to the homotopy curves of Figure 3, it is clear that the advantage of this non-uniformity is a much more stable exchange rate given changes in the quantities of currency.

1. Denote $M = \max(s_0, 1 - s_0)$. Take $K$ to be a “stability” parameter, where $K \geq 0$ and larger values of $K$ give curves with higher price stability. Then the following family of functions are always valid in that the resulting $A^{\text{hom}}$ curves are convex:

$$t(s) = \left|\frac{s - s_0}{M}\right|^K$$

2. Let $B$ be a “bias” parameter where $t(0) = B$ and $t(1) = 1 - B$. Let $T$ be a parameter controlling $t$ at $s = s_0$, i.e. $t(s_0) = T$. A parabolic extension of this yields the following:

$$t(s) = \frac{(1 - 2B)s_0 + (B - T)}{s_0(1 - s_0)}s^2 + \frac{(1 - 2B)s_0^3 + (B - T)}{s_0(1 - s_0)}s + B$$

The values of $B$ and $T$ must be chosen so that $0 \leq t(s) \leq 1$ and so that the test $\lambda(s)\lambda''(s) \geq 2\lambda'(s)^2$ will hold for all $s \in (0, 1)$; the equations for this are complicated to write down, but these values can be easily verified with simple code.

These curves are visualized in Figures 6 and 7 respectively.

4.2 Dynamic Adjustments to Match External Prices

One key point of this construction is that the exchange rate is designed to be stabilized about the exchange rate of the initial point. There are other constructions that do this as well, particularly in the area of AMMs used for stablecoins. For example, the Curve v1 system from the Stableswap white paper [Ego19] was designed to always be stabilized to any equal amount of each currency; this is seen by the fact that
Figure 7: Examples of asymmetric non-uniform homotopy curves and their corresponding exchange rate curves. (Left) The AMM curves demonstrate that a high value of the bias parameter $B$ creates a curve where the behavior more closely resembles a CSMM when the quantity of Currency 1 is higher than the initial quantity. In addition, the parameter $T$ controls the overall transition from CSMM to CPMM as the value of $t$ does in the uniform homotopy case. (Right) The exchange rate curves demonstrate that the bias parameter controls the exchange rate near the initial point. In particular, the exchange rates to both the left and right of the initial point are lower for the $B = 0.2$ curve than they are for the $B = 0.8$ curve.

Each value is initialized to $x_i^j = \frac{D}{n}$. The follow up paper [Ego21] for Curve v2 allows for stabilizing around arbitrary states.

One potential function to add into this homotopy model is the ability to dynamically adjust the stabilized exchange rate of the model. Recall that the initial state of the model is given by $(x_0, y_0)$ and that the initial internal exchange rate of the price of Currency 1 in terms of Currency 2 is given by $\frac{a}{b}$. If the current state of the AMM given by $A^{hom}(x, y) = 1$ is $(x_1, y_1)$ then the current internal exchange rate of the system is given by the following:

$$\text{current internal exchange rate} = \frac{\partial A^{hom}_{x}(x_1, y_1)}{\partial A^{hom}_{y}(x_1, y_1)}$$

In principal, the internal exchange rate of the system is driven by the external exchange rate; arbitrageurs
will tend to make exchanges biased toward shifting the two exchange rates together. If the current internal exchange rate gets too far away from the initial exchange rate then the system could become unstable. One way to try to relieve this pressure is to update the curve itself; one could (1) update the initial point to be \((x_1, y_1)\), (2) choose new \(a\) and \(b\) so that \(\frac{y_2 - y_1}{x_2 - x_1}\) is equal to the external exchange rate, and (3) update the values of \(\alpha\) and \(\beta\) accordingly.

As a note before proceeding, the notion of slippage is used precisely here and should be defined. Let \(p_1\) denote the current internal exchange rate or “estimated spot price”. Suppose a transaction in the AMM moves the state from \((x_1, y_1)\) to \((x_2, y_2)\). The “actual spot price” would then be given by \(p_2 = -\frac{y_2 - y_1}{x_2 - x_1}\). Slippage here is defined to be

\[
\text{slippage} = \left| \frac{\text{estimated spot price} - \text{actual spot price}}{\text{estimated spot price}} \right| = \left| \frac{p_1 - p_2}{p_1} \right|
\]  

(10)

In other words, slippage is taken to be the magnitude of the percent difference between the estimated and actual spot prices of the transaction. This definition is used because the notion of stability for a homotopy curve is the ability of the curve to maintain the initial exchange rate given small changes in currency quantity in the AMM; a very stable curve should have very small differences in spot prices when in the stable region of the curve.

Simulations show that a curve with high price stability will reduce slippage when near the stable point but also make it more difficult for the AMM to match the external exchange rate. The goal of these simulations was to have a very rough approximation of arbitrageurs attempting to match external exchange rates in a non-uniform homotopy AMM. Note that the exact parameters for this simulation are fairly simplistic and arbitrary; however, changing these parameters did not yield significantly different takeaways. These simulations were run with the following conditions:

- each simulation ran for 500 time steps
- the initial state was always set to be \((3000, 1000)\)
- a randomly generated external exchange rate curve is reused every simulation where
  - the initial exchange rate was set to 0.5
  - the exchange rate could increase or decrease by up to 20% every 80 time steps
- \(t(s) = \left| \frac{s - s_0}{M} \right|^K\) was used where \(K\) was equal to 8 times a stability parameter between 0 and 1
- users could extract up to 2% of the total amount of that currency in the AMM with each transaction
- transactions favored moving the internal exchange rate toward the external exchange rate with a likelihood of 90%
- the underlying curve parameters where never updated over the course of the simulation
Figure 8: The stabilities used were all multiples of 0.05 between 0.05 and 0.95 inclusive. Each stability was run 100 times and the results were averaged to create this figure. (Left) Mean-squared difference between the internal and external rates as a function of the stability. This demonstrates that the arbitrageurs have a harder time matching external exchange rates when the stability of the AMM is high. (Center) The quantities and exchange rate remain close to the initial values in the first 100 time steps. Higher price stability in this region of the curve means that price slippage is lower. (Right) The homotopy curve model can support any exchange rate, but stability is lost as the exchange rate diverges from the initial one. By time step 500 the external exchange rate has changed too much and the stability is lost, and this loss is more prevalent the higher this original stability was; this is an argument for why dynamically adjusting the curve is important.

A future goal for this work is to create more sophisticated simulations with a more in-depth analysis of non-uniform homotopy curve behavior.

5 Portfolio Value Functions and Stability

As introduced in [AEC21] and discussed in [TPL22], there is an elegant mathematical relationship between AMMs and their corresponding portfolio value functions. For completeness, an adapted version of that work is presented here. Suppose $P = (p_1, ..., p_n)$ is a price vector and the AMM is defined by $A(x_1, ..., x_n) = k$ using the function $A : \mathbb{R}^n_+ \rightarrow \mathbb{R}$. The value of the portfolio at the price vector $P$ is defined to be

$$V(P) = \inf\{P \cdot X : A(X) = k\} \quad (11)$$

The idea of this definition is that arbitrageurs will extract value from the market by making transactions that adjust the currency balances; this process continues until the value remaining in the market has been minimized and the arbitrageurs can’t extract any more.

There are a couple observations to make regarding the properties of this function. First, the use of Lagrange multipliers shows that if the function $A$ is differentiable then $P$ will be parallel to $\nabla A(X)$; i.e. one must find the point on the AMM where the price vector is normal to the surface at that point. This provides a nice geometric description of the solution to this minimization problem. Second, as pointed out in [TPL22], one may think of the portfolio value function as being related to a Legendre transformation. One definition of this transformation is as follows. Suppose $f : (a, b) \rightarrow \mathbb{R}$ is a differentiable function; the Legendre transformation of $f$ (if it exists) is denoted by $f^*$ and satisfies the property that $(f^*)'$ is the inverse of $f'$. Note that convexity of $f$ is enough to guarantee that such a transformation exists. To see how portfolio
value functions and Legendre transformations are related by an example:

- Suppose \( f : (a, b) \to \mathbb{R} \) satisfies the property \( A(x, f(x)) = k \)

- The value at price vector \((p, q)\) is given by

\[
V(p, q) = \inf \{px + qy : A(x, y) = k\} = \inf \{px + qf(x) : x \in (a, b)\}
\]

- In order to minimize the function \( g(x) = px + qf(x) \):

\[
g'(x) = p + qf'(x) = 0 \implies f'(x) = -\frac{p}{q}
\]

- By assumption \( f' \) is invertible and therefore

\[
x = (f')^{-1}\left(-\frac{p}{q}\right) = (f^*)'\left(-\frac{p}{q}\right)
\]

- Denoting \( L = f^* \) for simplicity, \( V(p, q) \) can be written using the Legendre transformation:

\[
V(p, q) = pL'\left(-\frac{p}{q}\right) + qf\left(L'\left(-\frac{p}{q}\right)\right)
\]

These two notions are closely related in a broader context for more general AMMs but that material will not be presented here.

One relevant simplification is made to the portfolio value function in [AEC21] and [TPL22]. It is clear that \( V(P) \) is 1-homogeneous because if \( \lambda > 0 \) then

\[
V(\lambda P) = \inf \{\lambda P \cdot X : A(X) = k\} = \lambda \inf \{P \cdot X : A(X) = k\} = \lambda V(P)
\]

Thus, it makes sense to rewrite the function in a way that extracts only the relevant information. The reduced portfolio value function \( U : \mathbb{R}^{n-1}_{\geq 0} \to \mathbb{R} \) is defined to satisfy the following property:

\[
V(p_1, ..., p_n) = p_n U \left(\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}\right) \tag{13}
\]

In other words, \( V \) is defined in terms of prices \( p_i \) while \( U \) is defined in terms of exchange rates \( r_i \) (where the \( n^{th} \) currency is thought of as the num\( \circ \)raire) ([AEC21]). It is clear that \( U(r_1, ..., r_{n-1}) = V(r_1, ..., r_{n-1}, 1) \)
Figure 9: (Left) Reduced portfolio value functions of uniform homotopy curves $A^\text{hom}_t(x, y) = 1$ for different stability values. Note that stability here is equal to $1 - t$; thus, stabilities of 0 and 1 correspond to CPMM and CSMM respectively. (Middle and Right) Portfolio value functions of asymmetric non-uniform homotopy curves $A^\text{hom}(x, y) = 1$ where $t(s) = (1 - 2B)s_0 + (B - T)s_0(1 - s_0)s^2 - (1 - 2B)s_0^2 + (B - T)s_0(1 - s_0)s + B$. The middle and right figures have $B = 0.2$ and 0.8 respectively. The value of $T$ is given by $0.9 \ast (1 - \text{stability}) + 0.1 \ast \text{stability}$. These curves demonstrate that (1) an increase in stability causes a decrease in value and (2) an increase in bias causes an increase in value.

and therefore this offers the convenience of being easily graphed in the 2-dimensional case (see Figure 9).

As far as the mixing curves are concerned, the above Figure 9 demonstrates the effects that stability has on the portfolio value functions. The most notable phenomenon is the fact that the portfolio value increases as the stability decreases. To see why this is the case, note that Figure 10 shows the following rather non-intuitive fact: given a fixed price vector $P$, the point $X$ where $A^\text{hom}_t(X) = k$ and $\nabla A^\text{hom}_t(X) = \lambda P$ will be closer to the initial point $X_0$ when stability is lower. In other words, one must be careful when thinking about the meaning of stability. An increase in stability means that a change in quantity vector $X$ results in a smaller change in the corresponding price vector $P$. Conversely, a decrease in stability means that a change in price vector $P$ results in a smaller change in the corresponding quantity vector $X$. Because lower stability means higher slippage, the takeaway here is that (1) an AMM with higher slippage will tend to have higher portfolio value functions and (2) AMMs with greater sensitivity to user behavior are better able to hold value. Further work is needed to show that this notion holds true in a broader context, but this and Figure 10 are helpful in initial understanding.

6 Conclusion

This work presents a novel method for blending the benefits of constant sum and constant product market makers. In particular, the non-uniform homotopy construction of a CSMM-to-CPMM blend opens many doors for designing interesting and intuitive AMM behavior. The homotopy construction gives a very concrete geometric meaning to $t$ as the blending weight. The $\lambda(s, t)$ homotopy parametrization allows for very exact computations and prevents drift due to rounding errors. In addition, thinking of $t$ as a function of $s$ allows for easy customizability of non-uniform curves. More investigation is needed to see the full behavior of these non-uniform curves, but already it is clear that the balance between stability and slippage is significant factor in designing market makers.
Figure 10: A visual demonstration of why lower stability results in an increase in the portfolio value function. $X_1$ and $X_2$ are the vectors pointing to the locations on the AMM curves where the gradient is parallel to the price vector $P$, and therefore these are the locations that minimize the remaining value in the market. This figure shows that a decrease in stability moves the minimizing point closer to the initial point $X_0$. The claim is that the lower stability curve has a higher value, i.e. $P \cdot X_1 > P \cdot X_2$. This is equivalent to the condition that $P \cdot (X_1 - X_2) > 0$, and the pair of arrows in the upper right indicate that this is indeed the case (since the angle between them is less than 90°).

Future directions for this work include a variety of possible options. There are many designs and behavior types to consider in designing new non-uniform homotopy markets, especially in a higher dimensional setting. There is plenty of room for more sophistication in the models weighing the balance of stability and slippage. Determining how to algorithmically and dynamically adjust the parameters of the AMM could be helpful in maintaining stable behavior even under large external price changes. Portfolio value function behavior as it relates to AMM parameters like stability and bias should be investigated in more depth. In short, this work lays a foundation for the initial design of a new class of market makers with a wide variety of possible desired behaviors.
References

[AEC21] Angeris, Guillermo; Evans, Alex; Chitra, Tarun: Replicating Market Makers. March 26, 2021
(accessed November 17, 2021)

[Ego19] Egorov, Michael: StableSwap - efficient mechanism for Stablecoin liquidity. November 10, 2019
(accessed November 5, 2021)

[Ego21] Egorov, Michael: Automatic market-making with dynamic peg. June 9, 2021 (accessed November
10, 2021)

[EH21] Engel, Daniel; Herlihy, Maurice: Composing Networks of Automated Market Makers. August 31,
2021 (accessed November 18, 2021)

[TPL22] Tiruviluamala, Neelesh; Port, Alexander; Lewis, Erik: A General Framework for Impermanent
Loss in Automated Market Makers. March 23, 2022 (arXiv:2203.11352)