On the stability of fuzzy set-valued functional equations

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Abstract: We introduce some fuzzy set-valued functional equations, i.e. the generalized Cauchy type (in \( n \) variables), the Quadratic type, the Quadratic-Jensen type, the Cubic type and the Cubic-Jensen type fuzzy set-valued functional equations and discuss the Hyers-Ulam-Rassias stability of the above said functional equations. These results can be regarded as an important extension of stability results corresponding to single-valued and set-valued functional equations, respectively.

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh (1965). After the pioneering work of Zadeh, there has been a great effort to obtain analogues of classical theories. It is a very powerful handset for modeling uncertainty and vagueness in various problems arising in the field of science and
engineering. It has also very useful applications in various fields, e.g., population dynamics, chaos control, computer programming, non-linear dynamical systems and control theory etc.

Set-valued functional equations in Banach spaces have received a lot of attention in the literature (see Arrow & Debreu, 1954; Aumann, 1965; Debreu, 1966). The pioneering papers by Aumann (1965) and Debreu (1966) were inspired by problems arising in Control Theory and Mathematical Economics.

The concept of stability for functional equations arises when one replaces a functional equation by an inequality which acts a perturbation in the equation. The study of stability problems for functional equations is related to a question posed by Ulam (1960) in a conference at Wisconsin University, Madison in 1940: “Let $G_1$ be a group and $G_2$ a metric group with the metric $\rho(\cdot, \cdot)$. Given a number $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $\rho(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G_1$, then there exist a homomorphism $h : G_1 \to G_2$ with $\rho(f(x), h(x)) \leq \delta$ for all $x \in G_1$?” If the answer is affirmative the equation $f(xy) = f(x)f(y)$ of the homomorphism is called stable (see Brzdek & Jung, 2011; Hyers, Isac, & Rassias, 1998).

In other words, the equation of homomorphism is stable if every “approximat” solution can be approximated by a solution of this equation.

The first answer to Ulam’s problem was given by Hyers (1941) for the Cauchy functional equation in Banach spaces. In fact, he proved: “Let $X$, $Y$ be Banach spaces, $\varepsilon$ a non-negative number, $f : X \to Y$ a function satisfying $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$, then there exists a unique additive mapping $h : X \to Y$ with the property $\|f(x) - h(x)\| \leq \varepsilon$ for all $x \in X$.” Due to the question of Ulam and the result of Hyers this type of stability is called today Hyers-Ulam stability of functional equations. So the Cauchy functional equation $f(x + y) = f(x) + f(y)$ is Hyers-Ulam stable. After Hyers result a large number of literature was devoted to study the Hyers-Ulam stability for various functional equations. A new type of stability for functional equations was introduced by Aoki (1950) (for some historical comments regarding the work of Aoki, see Moslehian and Rassias (2007)) for additive mappings and by Rassias (1978) for linear mappings in which the Cauchy difference is allowed to be unbounded by replacing $\varepsilon$ with a function depending on $x$ and $y$ in the Hyers theorem. However, the paper of Rassias (1978) has provided a lot of influence in the development of generalizations of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Until now, the stability problems for different types of functional equations in various spaces have been extensively studied. Gajda (1988) determined the stability of Cauchy equation on semigroups. The stability of approximately additive mappings have been studied by Gavruta (1994). For more details, the reader can refer to Rassias (1989), Rassias (1999), Xu, Rassias, and Xu (2012), Jung (2011) and Jung (1998). Among the studies of these problems, it is worth mentioning that Radu (2003) proposed a novel method to establish the stability of Cauchy functional equation via fixed point approach in lieu of the direct method which was frequently in used (see also Gordji, Park, & Savadkouhi, 2010). Recently, Cieplinski (2012) summarized some applications of several types of fixed point theorems to the Hyers-Ulam stability of functional equations. As of now, this method has been successfully used in the study of stability problems of many types of functional equations in abstract spaces.

In Mirmostafaeef and Moslehian (2008) initiated the study of stability problems of functional equations in fuzzy setting. Specifically, they considered the stability of the cauchy functional equation in fuzzy normed spaces. Since then, the fuzzy stability problems of various types of functional equations have been extensively investigated by different authors Jang, Lee, Park, and Shin (2009) and Lee, Jang, Park, and Shin (2010). At the same time, the fixed point method has been widely used to prove the fuzzy stability of several types of functional equations (Mohiuddine & Alotaibi, 2012; Park, 2009).

For other results on the Hyers-Ulam stability of functional equations in intuitionistic fuzzy/random normed spaces, one can refer to Al-Fhaid and Mohiuddine (2013), Alotaibi and Mohiuddine (2012), Mohiuddine (2009), Mohiuddine, Alotaibi and Obaid (2012), Mohiuddine, Cancan and Şevli (2011),
Mohiuddine and Şevli (2011), Mursaleen and Ansari (2013), Mursaleen and Mohiuddine (2009) and Mohiuddine and Alghamdi (2012).

In summary, one can see that the (fuzzy) stability for a single-valued functional equation is whether, for a given mapping satisfying almost a functional equation (which means that the mapping is close to a solution of the functional equation), there exists an exact solution of the functional equation which can be used to approximate the given mapping. Typically, a metric associated with the corresponding space is chosen to characterize the functional inequality. In Nikodem and Popa (2009) considered the general solution of set valued maps satisfying linear inclusion relation, which can be regarded as a generalization of the additive single-valued functional equation. By means of the inclusion relation, Lu and Park (2011) and Park, Regan, and Saadati (2011) investigated the stability of several types of set-valued functional equations. However, it should be pointed out that, in their studies, the inclusion relation is applied to characterize the set-valued functional inequality rather than an appropriate metric. Recently, similar to the method is used to deal with the single-valued functional equations, Kenary, Rezaei, Gheisari, and Park (2012) proved the stability of several types of set-valued functional equations via the fixed point approach, in which the Hausdorff metric is adopted to characterize the set-valued functional inequality.

Recently, Shen, Lan, and Chen (2014) extended the set-valued functional equations and establish some stability results for these fuzzy set-valued functional equations. Notice that the supremum metric, as a generalization of the Hausdorff metric is applied to characterize the fuzzy set-valued functional inequality. They discussed the Hyers-Ulam-Rassias stability of those fuzzy set-valued functional equations which are of additive type.

The aim of this paper is to introduce some fuzzy set-valued functional equations, i.e. the generalized Cauchy type (in \( n \) variables), the Quadratic type, the Quadratic-Jensen type, the Cubic type, the Cubic-Jensen type fuzzy set-valued functional equations and then discuss the Hyers-Ulam-Rassias stability of the above said functional equations. Interestingly, the corresponding single-valued and set-valued functional equations acted as special cases will be included in our results.

2. Preliminaries
In what follows, we begin with some related concepts and fundamental results, which are mainly derived from Castaing and Valadier (1977), Diamond and Kloeden (1994) and Inoue (1991). Let \( \mathbb{R} \), \( \mathbb{R}^+ \), and \( \mathbb{R}^n \) denote the set of all real numbers, the set of all non-negative real numbers and the \( n \)-dimensional Euclidean space, respectively.

Let \( Y \) be a separable Banach space with the norm \( \| \cdot \|_Y \). We denote the set of all nonempty compact subsets and the set of all nonempty compact convex subsets of \( Y \) by \( \mathcal{K}_Y \) and \( \mathcal{K}_C(Y) \), respectively.

Let \( A \) and \( B \) be two nonempty subsets of \( Y \) and let \( \lambda \in \mathbb{R} \). The (Minkowski) addition and scalar multiplication can be defined by

\[
A + B = \{ a + b | a \in A, b \in B \},
\]

\[
\lambda A = \{ \lambda a | a \in A \}. \tag{2.1}
\]

Notice that the sets \( \mathcal{K}_Y, \mathcal{K}_C(Y) \) are closed under the operations of addition and scalar multiplication. In fact, these two operations induce a linear structure on \( \mathcal{K}_Y \) and \( \mathcal{K}_C(Y) \) with zero element \( \{ 0 \} \), respectively. It should be noted that this linear structure is just a cone rather than a vector space because, in general, \( A + (-1)A \neq \{ 0 \} \). Moreover, for all \( \lambda, \mu \in \mathbb{R} \), it follows that

\[
\lambda(A + B) = \lambda A + \lambda B, \quad (\lambda + \mu)A \subseteq \lambda A + \mu A. \tag{2.2}
\]

In particular, if \( A \) is convex and \( \lambda \mu \geq 0 \), then \( (\lambda + \mu)A = \lambda A + \mu A \).
Furthermore, we can define the Hausdorff separation of $B$ from $A$ by

$$d_H^*(B, A) = \inf\{\varepsilon > 0 : B \subseteq A + \varepsilon S_1\},$$

(2.3)

where $S_1$ denotes the closed unit ball in $Y$; i.e. $S_1 = \{y \in Y | \|y\|_Y \leq 1\}$. Meantime, the Hausdorff separation of $A$ from $B$ can also be defined in a similar way.

Based on these two types of separations the Hausdorff distance between two nonempty subsets $A$ and $B$ is defined by

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\}.$$  

(2.4)

In general, if $A, B \in \mathcal{K}(Y)$ or $\mathcal{K}_c(Y)$, then $d_H(\lambda A, \lambda B) = |\lambda|d_H(A, B)$ for all $\lambda \in \mathbb{R}$. In addition, according to some of the properties of Hausdorff distance, if we restrict our attention to the nonempty closed subsets $C(Y)$ of $Y$, then it can be verified that $(C(Y), d_H)$ is a metric space. In fact, it follows from Cădariu and Radu (2003) that $(C(Y), d_H)$ is a complete metric space. Clearly, $\mathcal{K}(Y)$ and $\mathcal{K}_c(Y)$ are closed subsets of $C(Y)$. Hence, $(\mathcal{K}(Y), d_H)$ and $(\mathcal{K}_c(Y), d_H)$ are also complete metric spaces.

In Inoue (1991) introduced the concept of Banach space valued fuzzy sets in order to extend the usual fuzzy sets defined on $\mathbb{R}$ or $\mathbb{R}^n$. In other words, the base space of a fuzzy set is replaced by a more general Banach space.

For a given real separable Banach space $Y$, a fuzzy set defined on $Y$ is a mapping $u:Y \to [0, 1]$. Denote by $\mathcal{F}(Y)$ the set of all fuzzy sets defined on $Y$. Let $\mathcal{F}_X(Y)$ denote the class of fuzzy sets $u:Y \to [0, 1]$ with the following properties:

(i) $u$ is normal, i.e. $[u]^+ = \{y \in Y | u(y) \geq 1\}$ is nonempty;

(ii) $u$ is upper semi-continuous;

(iii) $[u]^+ = \{y \in Y | u(y) \geq \alpha\}$ is compact for each $\alpha \in (0, 1]$;

(iv) $[u]^0 = \bigcup_{\alpha \in (0, 1]} [u]^\alpha$.

Notice that the conditions (ii) and (iv) imply that $[u]^0$ is also compact. Moreover, we use the notation $\mathcal{F}_K(Y)$ to denote the subspace of $\mathcal{F}(Y)$ whose members also satisfy

(v) $u$ is fuzzy convex; i.e. $[u]^\alpha$ is convex for each $\alpha \in (0, 1]$.

A linear structure can be defined in $\mathcal{F}(Y)$ in a similar way to fuzzy sets in $\mathbb{R}$ or $\mathbb{R}^n$ by

$$(u \oplus v)(z) = \sup_{x+y=z} \min\{u(x), v(y)\},$$

(2.5)

$$(u \odot v)(z) = \bigvee_{x+y=z} (u(x) \wedge v(y)) \ (\forall \ x, y, z \in \mathbb{R}^+)$$

(2.6)

$$(y \cdot u)(z) = \begin{cases} u\left(\frac{z}{y}\right) & \text{if } y \neq 0, \\ I_0(z) & \text{if } y = 0, \end{cases}$$

(2.7)

for $u, v \in \mathcal{F}(Y)$ and $y \in \mathbb{R}$, where $I_0(z) = 0$ if $z \neq 0$ and $I_0(0) = 1$. Then $\mathcal{F}(Y)$ is closed under these operations and level set-wise

$$[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha, \ [u \odot v]^\alpha = [u]^\alpha \cdot [v]^\alpha, \ [\lambda u]^\alpha = \lambda [u]^\alpha$$

(2.8)

for each $\alpha \in (0, 1]$ and $\lambda \in \mathbb{R}$. Similar to the closeness of $\mathcal{K}_c(Y)$, it is easy to know that $\mathcal{F}_K(Y)$ is closed under these operations. Based on the statement mentioned above, we can easily obtain the following lemma.
**Lemma 2.1** For any \( u, \nu \in T_{nc}(Y) \) and \( \lambda, \mu \in \mathbb{R} \), the following equalities hold:

(i) \( \lambda(u \oplus \nu) = \lambda u \oplus \lambda \nu \);
(ii) \( \lambda(\mu u) = (\lambda \mu) u \);
(iii) \( (\lambda + \mu) u = \lambda u \oplus \mu u \) for any \( \lambda, \mu \geq 0 \).

Integer powers of a fuzzy number can be defined as follows (Kaufmann & Gupta, 1984):

Let \( u \) be a fuzzy number in \( \mathbb{R} \):

\[
 u^n = u \odot u \odot \cdots \odot u
\]  

(2.9)

(1) First case: \( u \in \mathbb{R}^+ \); here \( \forall \alpha \in [0,1] \) therefore from \( [u]^\alpha = [u^\alpha, u^\alpha] \) we have \( ([u]^\alpha)^n = ([u^\alpha]^n, [u^\alpha]^n) \).

(2) Second and general case: \( u \in \mathbb{R} \). This is a much more complicated case and requires the examination of the parity of \( n \) (the evenness or oddness). For \( n \in \mathbb{N}^* \)

(i) \( u_+ \leq u_- \leq 0 \):

\[
 [u_-, u_+]^n = [u_-^n, u_+]^n, \quad \text{if } n \text{ is odd},
\]

\[
 = [u_-^n, u_+]^n, \quad \text{if } n \text{ is even}.
\]

(ii) \( 0 \leq u_- \leq u_+, \ |u_-| \leq |u_+| \):

\[
 [u_-, u_+]^n = [u_- u_-^{-1} u_+, u_+]^n.
\]

(iii) \( 0 \leq u_- \leq u_+, \ |u_-| \geq |u_+| \):

\[
 [u_-, u_+]^n = [u_- u_-^{-1} u_+, u_+]^n, \quad \text{if } n \text{ is odd},
\]

\[
 = [u_-^{-1} u_-, u_+]^n, \quad \text{if } n \text{ is even}.
\]

(iv) \( 0 \leq u_- \leq u_+ \):

\[
 [u_-, u_+]^n = [u_-^n, u_+]^n.
\]

**Remark 2.2** The Lemma 2.1 shows that \( T_{nc}(Y) \) is just a cone defined on \( Y \) rather than a vector space.

As a generalization of the Hausdorff metric \( d_h \) in \( \mathcal{K}(Y) \), we will define the supremum metric \( d_{\infty} \) in \( T_{C}(Y) \). For \( u, \nu \in T_{C}(Y) \), the supremum metric is defined by

\[
 d_{\infty}(u, \nu) = \sup_{\alpha \in (0,1)} d_h([u]^\alpha, [\nu]^\alpha)
\]  

(2.10)

**Remark 2.3** Every ordinary crisp subset \( A \) of \( Y \) can be identified with with the fuzzy set on \( Y \) by its characteristic function \( \chi_A : Y \to \{0,1\} \), i.e. with \( \chi_A(y) = 1 \) if \( y \in A \) and \( \chi_A(y) = 0 \) if \( y \notin A \). Therefore, if \( A \in \mathcal{K}(Y) \) (or \( A \in \mathcal{K}_{c}(Y) \)), then \( \chi_A \in T_{C}(Y) \) (or \( T_{K_{c}}(Y) \)), and vice versa.

From Remark 2.3, for any \( A, B \in \mathcal{K}(Y) \) (or \( T_{K_{c}}(Y) \)), it follows that

\[
 d_{\infty}(\chi_A, \chi_B) = \sup_{\alpha \in (0,1)} d_h([\chi_A]^\alpha, [\chi_B]^\alpha) = d_h(A, B).
\]  

(2.11)

In particular, if \( A \) and \( B \) degenerate into two singleton sets \( \{a\} \) and \( \{b\} \), then we can infer from equality (2.11) that \( d_{\infty}(\chi_a, \chi_b) = d(a, b) \), where \( d \) denotes the usual metric between \( a \) and \( b \).
In view of the property of the Hausdorff metric, it is easy to see that \( d_{\infty}(\lambda u, \lambda v) = \lambda d_{\infty}(u, v) \) for any \( \lambda \geq 0 \). Restricting attention to the set \( T_{KC}(Y) \), we can prove that \((T_{KC}(Y), d_{\infty})\) is a complete metric space by the method analogous to that used in Diamond and Kloeden (1994) (see Proposition 7.2.3). 

Finally, we quote a fundamental result in fixed point theory.

**Theorem 2.4** (Diaz & Margolis, 1968) Let \((X, d)\) be a complete generalized metric space, i.e. one for which \( d \) may assume infinite values. Suppose that \( JX \to X \) is a strictly contractive mapping with Lipschitz constant \( L < 1 \). Then, for each given element \( x \in X \), either

\[
d(J^n x, J^{n+1} x) = \infty
\]

for all \( n \geq 0 \) or there exists an \( n_0 \in \mathbb{N} \) such that

(i) \( d(J^n x, J^{n+1} x) < \infty \) for all \( n \geq n_0 \)

(ii) the sequence \( \{J^n x\} \) converges to a fixed point \( y^* \) of \( J \);

(iii) \( y^* \) is the unique fixed point of \( J \) in the set \( \text{ran} J = \{ y \in X | d(J^n x, y) < \infty \} \);

(iv) \( d(y, y^*) \leq (1/(1-L))d(y, Jy) \) for all \( y \in Y \).

### 3. Stability of the generalized Cauchy type additive fuzzy set-valued functional equation in \( n \) variables

In this section, we will establish the Hyers-Ulam-Rassias stability of the generalized Cauchy type additive fuzzy set-valued functional equation by using fixed point alternative approach.

**Definition 3.1** Let \( X \) be a cone with the vertex \( 0 \) and let \( f : X \to T_{KC}(Y) \) be a fuzzy set-valued mapping. The generalized Cauchy type additive fuzzy set-valued functional equation in \( n \) variables is defined by

\[
f \left( \sum_{i=1}^{k} a_i x_i \right) = \bigoplus_{i=1}^{k} a_i f(x_i)
\]  

(3.1)

for all \( x, y \in X \) and for some \( a_i > 0 \) with \( \sum_{i=1}^{k} a_i \neq 1 \).

Especially, if \( a_i = 1 \), for each \( i \in \{1, 2, ..., k\} \), then (3.1) is called the standard Cauchy type additive fuzzy set-valued functional equation in \( n \) variables. Every solution of (3.1) is called a generalized Cauchy type additive fuzzy set-valued mapping.

**Example 3.2** Let \( X = \mathbb{R}^+ \) and \( Y = \mathbb{R} \). Suppose that \( f : \mathbb{R}^+ \to T_{KC}(\mathbb{R}) \) is a triangular fuzzy set-valued mapping, i.e. for every \( t \in \mathbb{R}^+ \), \( f(t) \) is a triangular fuzzy number in \( \mathbb{R} \), which is defined by

\[
f(t) = (t - \alpha t, t + \beta t), \quad t \in \mathbb{R}^+,
\]

(3.2)

where \( \alpha \) and \( \beta \) are two non-negative real numbers. By the definition of \( a \)-level set, we can obtain that

\[
[f(t)]^a = [t - \alpha t(1-a), t + \beta t(1-a)]
\]

(3.3)

for every \( t \in \mathbb{R}^+ \). Then, for every \( a \in [0,1] \), it is easy to verify that

\[
\left[ f \left( \sum_{i=1}^{k} a_i x_i \right) \right]^a = \sum_{i=1}^{k} a_i[f(x_i)]^a
\]

(3.4)

for all \( x_1, x_2, ..., x_k \in \mathbb{R}^+ \) and \( a_i > 0 \) with \( \sum_{i=1}^{k} a_i \neq 1 \). That is, \( f \) is a solution of (3.1) in \( \mathbb{R}^+ \).

**Remark 3.3** A triangular fuzzy number \( u \in T_{KC}(\mathbb{R}) \) is characterized by an ordered triple \((x_l, x_c, x_r) \in \mathbb{R}^3 \) with \( x_l \leq x_c \leq x_r \) such that the support set \([u]^0 = \{x_l, x_r\}\) and 1-level set \([u]^1 = \{x_c\}\).
Remark 3.4 More generally, if $X = \mathbb{R}^+$, by Lemma 2.1, it is easy to see that $f(t) = tu_0$ is a solution of (3.1) for any $t \in \mathbb{R}^+$ and any fixed $u_0 \in F_{KC}(Y)$.

**Theorem 3.5** Let $\varphi: X \rightarrow [0, +\infty)$ be a function such that there exists a positive constant $L < 1$ satisfying

$$\varphi(x_1, x_2, \ldots, x_k) \leq \frac{L}{\left(\sum_{i=1}^{k} a_i\right)} \varphi\left(\left(\sum_{i=1}^{k} a_i\right)x_1, \left(\sum_{i=1}^{k} a_i\right)x_2, \ldots, \left(\sum_{i=1}^{k} a_i\right)x_k\right)$$

(3.5)

for all $x_i \in X$ and for some $a_i > 0$ with $\sum_{i=1}^{k} a_i \neq 1$, $1 \leq i \leq k$. Suppose that $f: X \rightarrow F_{KC}(Y)$ is a mapping satisfying

$$d_w\left(f\left(\sum_{i=1}^{k} a_i x_i\right), \bigoplus_{i=1}^{k} f(x_i)\right) \leq \varphi(x_1, x_2, \ldots, x_k)$$

(3.6)

for all $x_i \in X$, $1 \leq i \leq k$. Then

$$A(x) = \lim_{n \to \infty} \left(\sum_{i=1}^{k} a_i\right)^n f\left(\frac{x}{\left(\sum_{i=1}^{k} a_i\right)}\right)$$

(3.7)

exists for each $x \in X$ and defines a unique generalized Cauchy type additive fuzzy set-valued mapping $A: X \rightarrow F_{KC}(Y)$ such that

$$d_w\left(f(x), A(x)\right) \leq \frac{L}{\left(\sum_{i=1}^{k} a_i\right)(1 - L)} \varphi(x_1, x_2, \ldots, x_k)$$

(3.8)

for all $x$.

**Proof** Replacing $x_1, x_2, \ldots, x_k$ by $x$ in (3.6), by Lemma 2.1, we get

$$d_w\left(f\left(\sum_{i=1}^{k} a_i x\right), \bigoplus_{i=1}^{k} f(x)\right) \leq \varphi(x, x, \ldots, x)$$

(3.9)

for all $x \in X$. Thus we can obtain

$$d_w\left(f(x), \sum_{i=1}^{k} a_i f\left(\frac{x}{\left(\sum_{i=1}^{k} a_i\right)}\right)\right) \leq \varphi\left(\frac{x}{\left(\sum_{i=1}^{k} a_i\right)}, \frac{y}{\left(\sum_{i=1}^{k} a_i\right)}\right) \leq \frac{L}{\left(\sum_{i=1}^{k} a_i\right)} \varphi(x, x, \ldots, x)$$

(3.10)

for all $x \in X$.

Consider the set $E = \{g: X \rightarrow F_{KC}(Y), g(0) = I_0\}$ and introduce the generalized metric $D$ on $E$, which is defined by
\[ D(g, h) = \inf\{ \mu \in (0, \infty) \mid d_{\infty}(g(x), h(x)) \leq \mu \varphi(x, x, \ldots, x), \forall x \in X \} \]  \hspace{1cm} (3.11)

where, as usual, \( \inf \emptyset = \infty \). It can be easily verified that \((E, D)\) is a complete generalized metric space (see Radu, 2003).

Now, we consider the linear mapping \( J : E \to E \) such that

\[ Jg(x) = \sum_{i=1}^{k} a_i g \left( \frac{x}{\sum_{i=1}^{k} a_i} \right) \]  \hspace{1cm} (3.12)

for all \( x \in X \).

Let \( g, h \) be given such that \( D(g, h) \leq \varepsilon \). Then

\[ d_{\infty}(g(x), h(x)) \leq \varepsilon \varphi(x, x, \ldots, x) \]  \hspace{1cm} (3.13)

for all \( x \in X \). Hence, we have

\[
d_{\infty}(Jg(x), Jh(x)) = d_{\infty} \left( \sum_{i=1}^{k} a_i g \left( \frac{x}{\sum_{i=1}^{k} a_i} \right), \sum_{i=1}^{k} a_i h \left( \frac{x}{\sum_{i=1}^{k} a_i} \right) \right) \\
= \sum_{i=1}^{k} a_i d_{\infty} \left( g \left( \frac{x}{\sum_{i=1}^{k} a_i} \right), h \left( \frac{x}{\sum_{i=1}^{k} a_i} \right) \right) \\
\leq \varepsilon \sum_{i=1}^{k} a_i \varphi \left( \frac{x}{\sum_{i=1}^{k} a_i}, \frac{x}{\sum_{i=1}^{k} a_i} \right) \\
\leq \frac{L}{\sum_{i=1}^{k} a_i} \varepsilon \left( \sum_{i=1}^{k} a_i \right) \\
= L \varepsilon \varphi(x, x, \ldots, x)
\]

for all \( x \in X \). So \( D(g, h) \leq \varepsilon \) implies that \( D(Jg, Jh) \leq L \varepsilon \). This means that

\[ D(Jg, Jh) \leq LD(g, h) \]  \hspace{1cm} (3.15)

for all \( g, h \in E \). Evidently, \( J \) is a strictly contractive self-mapping on \( E \) with the Lipschitz constant \( L < 1 \).

Moreover, it follows from (3.10) that \( D(f, Jf) \leq L / \left( \sum_{i=1}^{k} a_i \right) \). According to Theorem 2.4, there exists a mapping \( A : X \to P_{\mathcal{C}}(Y) \) satisfying the following.

(i) \( A \) is a fixed point of \( J \); i.e.

\[ \frac{1}{\sum_{i=1}^{k} a_i} A(x) = A \left( \frac{x}{\sum_{i=1}^{k} a_i} \right) \]  \hspace{1cm} (3.16)
for all \( x \in X \). The mapping \( A \) is the unique fixed point of \( J \) in the set

\[
M = \{ g \in E \mid D(f, g) < \infty \},
\]

which implies that \( A \) is the unique mapping satisfying (3.16) such that there exists an \( r \in (0, 1) \)
satisfying

\[
d_\infty (f(x), A(x)) \leq r \varphi(x, x, \ldots, x)
\]

for all \( x \in X \).

(ii) \( D(J^nf, A) \rightarrow 0 \) as \( n \rightarrow \infty \). This implies the equality

\[
\lim_{n \rightarrow \infty} \left( \sum_{i=1}^{k} a_i \right)^n f \left( \left( \sum_{i=1}^{k} \frac{x}{a_i} \right)^n \right) = A(x)
\]

for all \( x \in X \).

(iii) \( D(f, A) \leq \frac{1}{1 - L} D(f, Jf) \), which implies the inequality

\[
D(f, A) \leq \frac{1}{1 - L} \cdot \frac{1}{\left( \sum_{i=1}^{k} a_i \right)^n \left( \sum_{i=1}^{k} a_i \right)^{(1 - L)}}
\]

By (3.5), we can obtain that

\[
d_\infty \left( \sum_{i=1}^{k} a_i \right)^n f \left( \left( \sum_{i=1}^{k} \frac{x_i}{a_i} \right)^n \right) \cdot \left( \sum_{i=1}^{k} a_i \right)^n f \left( \left( \sum_{i=1}^{k} \frac{x_i}{a_i} \right)^n \right) \leq \left( \sum_{i=1}^{k} a_i \right)^n \varphi(x_1, x_2, \ldots, x_k)
\]

\[
\leq \left( \sum_{i=1}^{k} a_i \right)^n \cdot \frac{L^n}{\left( \sum_{i=1}^{k} a_i \right)^n} \varphi(x_1, x_2, \ldots, x_k)
\]

\[
= L^n \varphi(x_1, x_2, \ldots, x_k)
\]

which tends to zero as \( n \rightarrow \infty \) for all \( x_1, x_2, \ldots, x_k \). Thus,

\[
A \left( \sum_{i=1}^{k} a_ix_i \right) = \bigoplus_{i=1}^{k} a_iA(x_i)
\]

for all \( x_1, x_2, \ldots, x_k \in X \) and therefore the mapping \( A : X \rightarrow T^\infty(Y) \) is a generalized Cauchy type additive fuzzy set-valued mapping as desired.
Corollary 3.6  Let $X$ be a cone with the vertex 0 contained in a real normed space and let $p, \theta$ be positive real numbers with $p > 1$ (resp. $0 < p < 1$). Suppose that $f: X \to \mathcal{F}_{KC}(Y)$ is a mapping satisfying

$$d_w \left( f \left( \sum_{i=1}^{k} a_i x_i \right), \bigoplus_{i=1}^{k} a_i f(x_i) \right) \leq \theta \left( \sum_{i=1}^{k} \|x_i\|^p \right)$$

(3.23)

for all $x_1, x_2, \ldots, x_k \in X$ and for some $a_i > 0$ with $\sum_{i=1}^{k} a_i > 1$ (resp. $< 1$). Then

$$A(x) = \lim_{n \to \infty} \left( \sum_{i=1}^{k} a_i \right)^n f \left( \sum_{i=1}^{k} \frac{x_i}{a_i} \right)$$

(3.24)

exists for each $x \in X$ and defines a unique generalized Cauchy type additive fuzzy set-valued mapping $A: X \to \mathcal{F}_{KC}(Y)$ such that

$$d_w \left( f(x), A(x) \right) \leq \frac{\theta k \|x\|^p}{\left( \sum_{i=1}^{k} a_i \right)^p - \left( \sum_{i=1}^{k} a_i \right)^p}$$

(3.25)

for all $x \in X$.

Proof  In Theorem 3.5, let $\varphi(x_1, x_2, \ldots, x_k) = \theta \left( \sum_{i=1}^{k} \|x_i\|^p \right)$. Then we can choose $L = \left( \sum_{i=1}^{k} a_i \right)^{1-p}$ and we get the desired result.

Corollary 3.7  Let $X$ be a cone with the vertex 0 contained in a real normed space and let $p, \theta$ be positive real numbers with $p > 1/2$ (resp. $p < 1/2$). Suppose that $f: X \to \mathcal{F}_{KC}(Y)$ is a mapping satisfying

$$d_w \left( f \left( \sum_{i=1}^{k} a_i x_i \right), \bigoplus_{i=1}^{k} a_i f(x_i) \right) \leq \theta \prod_{i=1}^{k} \|x_i\|^p$$

(3.26)

for all $x_1, x_2, \ldots, x_k \in X$ and for some $a_i > 0$ with $\sum_{i=1}^{k} a_i > 1$ (resp. $< 1$). Then

$$A(x) = \lim_{n \to \infty} \left( \sum_{i=1}^{k} a_i \right)^n f \left( \sum_{i=1}^{k} \frac{x_i}{a_i} \right)$$

(3.27)

exists for each $x \in X$ and defines a unique generalized Cauchy type additive fuzzy set-valued mapping $A: X \to \mathcal{F}_{KC}(Y)$ such that

$$d_w \left( f(x), A(x) \right) \leq \frac{\theta \|x\|^{kp}}{\left( \sum_{i=1}^{k} a_i \right)^2p - \left( \sum_{i=1}^{k} a_i \right)^2p}$$

(3.28)

for all $x \in X$.

Proof  In Theorem 3.5, let $\varphi(x_1, x_2, \ldots, x_k) = \theta \prod_{i=1}^{k} \|x_i\|^p$. Then we can choose $L = \left( \sum_{i=1}^{k} a_i \right)^{1-2p}$ and we get the desired result.
\[ d_\alpha \left( f \left( \sum_{i=1}^{k} a_i x_i \right), \bigoplus_{i=1}^{k} a_i f(x_i) \right) \leq \theta \left( \prod_{i=1}^{k} \|x_i\|^p + \sum_{i=1}^{k} \|x_i\|^p \right) \]  

(3.29)

for all \( x_1, x_2, \ldots, x_k \in X \) and for some \( \alpha > 0 \) with \( \sum_{i=1}^{k} \alpha_i > 1 \) (resp. < 1). Then

\[ A(x) = \lim_{n \to \infty} \left( \sum_{i=1}^{k} a_i \right)^n \left( \frac{x}{\sum_{i=1}^{k} a_i} \right)^n \]  

(3.30)

exists for each \( x \in X \) and defines a unique generalized Cauchy type additive fuzzy set-valued mapping \( A: X \to F_{KC}(Y) \) such that

\[ d_\alpha(f(x), A(x)) \leq \frac{\theta(k+1)\|x\|^p}{\left( \sum_{i=1}^{k} a_i \right)^p - \left( \sum_{i=1}^{k} a_i \right)^p} \]  

(3.31)

for all \( x \in X \).

**Proof** In Theorem 3.5, let \( \varphi(x_1, x_2, \ldots, x_k) = \theta \left( \prod_{i=1}^{k} \|x_i\|^p + \sum_{i=1}^{k} \|x_i\|^p \right) \). Then we can choose \( L = \left( \sum_{i=1}^{k} a_i \right)^{1-p} \) and we get the desired result.

4. **Stability of the quadratic type fuzzy set-valued functional equation**

In this section, we will establish the Hyers-Ulam-Rassias stability of the quadratic type fuzzy set-valued functional equation by applying the same method as used in the previous Section 3.

**Definition 4.1** Let \( X \) be a cone with the vertex 0 and let \( f: X \to F_{KC}(Y) \) be a fuzzy set valued mapping. The quadratic type fuzzy set-valued functional equation is defined by

\[ f(x + y) \oplus f(x - y) = 2f(x) \oplus 2f(y) \]  

(4.1)

for all \( x, y \in X \). Every solution of (4.1) is called a quadratic type fuzzy set-valued mapping.

**Remark 4.2** Obviously, it can be checked that for the triangular fuzzy set-valued mapping \( f \) as defined in Example 3.2, \( f(tu) = tu \) is a solution of (4.1) for any \( t \in \mathbb{R}^+ \) and any fixed \( u \in F_{KC}(Y) \).

**Theorem 4.3** Let \( j \in (-1, 1) \) be fixed and let \( \varphi: X \times X \to [0, +\infty) \) be a function such that there exists a positive constant \( L < 1 \) satisfying

\[ \varphi(x, x) \leq 4L\varphi(2^{-j}x, 2^{-j}x) \]  

(4.2)

for all \( x \in X \). Moreover, assume that \( \varphi \) satisfies

\[ \lim_{n \to \infty} 4^{-jn} \varphi(2^n x, 2^n y) = 0 \]  

(4.3)

for all \( x, y \in X \). If a mapping \( f: X \to F_{KC}(Y) \) satisfies \( f(0) = I_0 \) and the inequality

\[ d_\alpha(f(x + y) \oplus f(x - y), 2f(x) \oplus 2f(y)) \leq \varphi(x, y) \]  

(4.4)

for all \( x, y \in X \), then

\[ A(x) = \lim_{n \to \infty} 4^{-jn} f(2^n x) \]  

(4.5)

exists for each \( x \in X \) and defines a unique quadratic type fuzzy set-valued mapping \( A: X \to F_{KC}(Y) \) such that
for all $x \in X$.

**Proof** Letting $y = x$ in (4.4). Since $u \oplus I_0 = u$ for any $u \in R_{KC}(Y)$, we get

$$d_{\infty}(f(x), \frac{1}{4}f(2x)) \leq \frac{1}{4} \phi(x, x)$$

for all $x \in X$. Furthermore, it follows from (4.2) that

$$d_{\infty}(f(x), 4f(\frac{x}{2})) \leq \phi(x/2, x/2)$$

for all $x \in X$.

Consider the set $E = \{ g | g: X \to R_{KC}(Y), g(0) = I_0 \}$ and introduce the generalized metric $D$ on $E$, which is defined by

$$D(g, h) = \inf \{ \mu \in (0, \infty) | d_{\infty}(g(x), h(x)) \leq \mu \phi(x, x), \forall x \in X \}$$

where, as usual, $\inf \emptyset = \infty$. It can be easily verified that $(E, D)$ is a complete generalized metric space (see Radu, 2003).

Now, we consider the linear mapping $J: E \to E$ such that

$$Jg(x) = 4^{-j}g(2^jx)$$

for all $x \in X$.

Moreover, we can infer from (4.7) and (4.8) that

$$D(f, Jf) \leq \left\{ \begin{array}{ll}
\frac{1}{4} & \text{if } j = -1 \\
\frac{1}{4} & \text{if } j = 1.
\end{array} \right.$$  

The rest of the proof is similar to the proof of Theorem 3.5.

**Corollary 4.4** Let $j \in \{-1, 1\}$ be fixed and let $p, \theta$ be positive real numbers with $p \neq 2$, and let $X$ be a cone with the vertex 0 contained in a real normed space. Suppose that $f: X \to R_{KC}(Y)$ is a mapping satisfying

$$d_{\infty}(f(x + y) \oplus f(x - y), 2f(x) \oplus 2f(y)) \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then

$$A(x) = \lim_{n \to \infty} 4^{-jn}f(2^nx)$$

exists for each $x \in X$ and defines a unique quadratic type fuzzy set-valued mapping $Q: X \to R_{KC}(Y)$ such that

$$d_{\infty}(f(x), Q(x)) \leq \left\{ \begin{array}{ll}
\frac{\theta \|x\|^p}{2^{\frac{p}{p-2}}}, & \text{if } j = -1, \; p > 2 \\
\frac{\theta \|x\|^p}{2^{\frac{p}{p-2}}}, & \text{if } j = 1, \; 0 < p < 2
\end{array} \right.$$
Proof In Theorem 4.3, let \( \varphi(x, y) = \theta(\|x\|^p + \|y\|^q) \). Then we can choose \( L = 2^{(p-2)} \) and we get the desired result.

Corollary 4.5 Let \( j \in \{-1, 1\} \) be fixed and let \( p, \theta \) be positive real numbers with \( p \neq 1 \), and let \( X \) be a cone with the vertex 0 contained in a real normed space. Suppose that \( f:X \to \mathcal{F}_{KC}(Y) \) is a mapping satisfying
\[
d_m(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \leq \theta \|x\|^p \|y\|^q
\]
for all \( x, y \in X \). Then
\[
A(x) = \lim_{n \to \infty} 4^{-p}f(2^nx)
\]
exists for each \( x \in X \) and defines a unique quadratic type fuzzy set-valued mapping \( Q:X \to \mathcal{F}_{KC}(Y) \) such that
\[
d_m(f(x), Q(x)) \leq \begin{cases} 4^{1-\frac{1}{p}} \|\varphi(x)\|, & \text{if } j = -1, p > 1 \\ 4^{1-\frac{1}{q}} \|\varphi(x)\|, & \text{if } j = 1, 0 < p < 1 \end{cases}
\]
for all \( x \in X \).

Proof In Theorem 4.3, let \( \varphi(x, y) = \theta(\|x\|^p + \|y\|^q) \). Then we can choose \( L = 2^{(p-1)} \) and we get the desired result.

Corollary 4.6 Let \( j \in \{-1, 1\} \) be fixed and let \( p, \theta \) be positive real numbers with \( p + q \neq 2 \), and let \( X \) be a cone with the vertex 0 contained in a real normed space. Suppose that \( f:X \to \mathcal{F}_{KC}(Y) \) is a mapping satisfying
\[
d_m(f(x+y) \oplus f(x-y), 2f(x) \oplus 2f(y)) \leq \theta \left( \|x\|^p \|y\|^q + \|x\|^{p+q} + \|y\|^{p+q} \right)
\]
for all \( x, y \in X \). Then
\[
A(x) = \lim_{n \to \infty} 4^{-p}f(2^nx)
\]
exists for each \( x \in X \) and defines a unique quadratic type fuzzy set-valued mapping
\[
Q:X \to \mathcal{F}_{KC}(Y)
\]
such that
\[
d_m(f(x), Q(x)) \leq \begin{cases} 4^{1-\frac{1}{p+q}} \|\varphi(x)\|, & \text{if } j = -1, p + q > 2 \\ 4^{1-\frac{1}{p+q}} \|\varphi(x)\|, & \text{if } j = 1, 0 < p + q < 2 \end{cases}
\]
for all \( x \in X \).

Proof In Theorem 4.3, let \( \varphi(x, y) = \theta(\|x\|^p \|y\|^q + \|x\|^{p+q} + \|y\|^{p+q}) \). Then we can choose \( L = 2^{(p+q-2)} \) and we get the desired result.

5. Stability of the quadratic-Jensen type fuzzy set-valued functional equation
In this section, we will prove the Hyers-Ulam-Rassias stability of the quadratic-Jensen type fuzzy set-valued functional equation by using the same method as applied in Section 4.
Definition 5.1 Let $X$ be a cone with the vertex 0 and let $f:X \to T_{KC}(Y)$ be a fuzzy set valued mapping. The quadratic-Jensen type fuzzy set-valued functional equation is defined by

$$2f\left(\frac{x+y}{2}\right) \oplus 2f\left(\frac{x-y}{2}\right) = f(x) \oplus f(y)$$

(5.1)

for all $x, y \in X$. Every solution of (5.1) is called a Jensen type quadratic fuzzy set-valued mapping.

Remark 5.2 Obviously, it can be checked that for the triangular fuzzy set-valued mapping $f$ as defined in Example 3.2, $f(t) = tu_0$ is a solution of (5.1) for any $t \in \mathbb{R}^+$ and any fixed $u_0 \in T_{KC}(Y)$.

Theorem 5.3 Let $j \in \{-1, 1\}$ be fixed and let $\varphi:X \times X \to [0, +\infty)$ be a function such that there exists a positive constant $L < 1$ satisfying

$$\varphi(x, 0) \leq 4^{-1}L\varphi(2^j x, 0)$$

for all $x \in X$. Moreover, assume that $\varphi$ satisfies

$$\lim_{n \to \infty} \varphi(2^{j-n} x, 2^{-j} y) = 0$$

(5.3)

for all $x, y \in X$. If a mapping $f:X \to T_{KC}Y$ satisfies $f(0) = I_0$ and the inequality

$$d_\infty(2f\left(\frac{x+y}{2}\right) \oplus 2f\left(\frac{x-y}{2}\right), f(x) \oplus f(y)) \leq \varphi(x, y)$$

(5.4)

for all $x, y \in X$, then

$$A(x) = \lim_{n \to \infty} 4^{-n}f(2^n x)$$

(5.5)

exists for each $x \in X$ and defines a unique quadratic-Jensen type fuzzy set-valued mapping $A:X \to T_{KC}(Y)$ such that

$$d_\infty(f(x), A(x)) \leq \begin{cases} 
\frac{1}{2^{-j}} \varphi(x, 0) & \text{if } j = 1 \\
\frac{1}{2^{-j}} \varphi(x, 0) & \text{if } j = -1 
\end{cases}$$

(5.6)

for all $x \in X$.

Proof Letting $y = 0$ in (5.4). Since $u \oplus I_0 = u$ for any $u \in T_{KC}(Y)$, we get

$$d_\infty(f(x), 4f\left(\frac{x}{2}\right)) \leq \varphi(x, 0)$$

(5.7)

for all $x \in X$. Furthermore, it follows from (5.7) that

$$d_\infty(f(x), \frac{1}{2} f(2x)) \leq L\varphi(x, 0)$$

(5.8)

for all $x \in X$.

Consider the set $E = \{g | g:X \to T_{KC}(Y), g(0) = I_0\}$ and introduce the generalized metric $D$ on $E$, which is defined by

$$D(g, h) = \inf\{\mu \in (0, \infty) | d_\infty(g(x), h(x)) \leq \mu \varphi(x, 0), \forall x \in X\}$$

(5.9)

where, as usual, $\inf\emptyset = \infty$. It can be easily verified that $(E, D)$ is a complete generalized metric space (see Radu, 2003).
Now, we consider the linear mapping $J: E \rightarrow E$ such that
\[ Jg(x) = 4^{-j}g(2^j x) \]  
for all $x \in X$.

Moreover, we can infer from (5.7) and (5.8) that
\[ D(f, Jf) \leq \left\{ \begin{array}{ll}
L & \text{if } j = 1 \\
1 & \text{if } j = -1.
\end{array} \right. \]  
(5.11)

The rest of the proof is similar to the proof of Theorem 3.5.

**Corollary 5.4** Let $j \in \{-1, 1\}$ be fixed and let $p, \vartheta$ be positive real numbers with $p \neq 2$, and let $X$ be a cone with the vertex 0 contained in a real normed space. Suppose that $f: X \rightarrow F_{sc}(Y)$ is a mapping satisfying
\[ d_\infty(2f(\frac{x+y}{2}) \oplus 2f(\frac{x-y}{2}), f(x) \oplus f(y)) \leq \theta(\|x\|^p + \|y\|^p) \]  
(5.12)

for all $x, y \in X$. Then
\[ A(x) = \lim_{n \rightarrow \infty} 4^{-jn}f(2^j x) \]  
(5.13)

exists for each $x \in X$ and defines a unique quadratic type fuzzy set-valued mapping $Q: X \rightarrow F_{sc}(Y)$ such that
\[ d_\infty(f(x), Q(x)) \leq \left\{ \begin{array}{ll}
\frac{\|x\|^p}{1 - 2^{-p}} & \text{if } j = 1, p > 2 \\
\frac{\|x\|^p}{2^{-p} - 1} & \text{if } j = -1, 0 < p < 2
\end{array} \right. \]  
(5.14)

for all $x \in X$.

**Proof** In Theorem 5.3, let $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$. Then we can choose $L = 2^{2j - 2p}$ and we get the desired result.

**6. Stability of the cubic type fuzzy set-valued functional equation**

In this section, we will prove the Hyers-Ulam-Rassias stability of the cubic type fuzzy set-valued functional equation by using the same method as applied in Section 6.

**Definition 6.1** Let $X$ be a cone with the vertex 0 and let $f: X \rightarrow F_{sc}(Y)$ be a fuzzy set-valued mapping. The cubic type fuzzy set-valued functional equation is defined by
\[ f(2x + y) \oplus f(2x - y) = 2f(x + y) \oplus 2f(x - y) \oplus 12f(x) \]  
(6.1)

for all $x, y \in X$. Every solution of (6.1) is called a cubic type fuzzy set-valued mapping.

**Remark 6.2** Obviously, it can be checked that for the triangular fuzzy set-valued mapping $f$ as defined in Example 3.2, $f(t) = tu_0^+$ is a solution of (6.1) for any $t \in \mathbb{R}^+$ and any fixed $u_0 \in F_{sc}(Y)$. 
Theorem 6.3  Let \( j \in \{-1, 1\} \) be fixed and let \( \varphi: X \times X \to [0, +\infty) \) be a function such that there exists a positive constant \( L < 1 \) satisfying

\[
\varphi(x, 0) \leq 8L \varphi(2^jx, 0)
\]

for all \( x \in X \). Moreover, assume that \( \varphi \) satisfies

\[
\lim_{n \to \infty} 8^{-jn} \varphi(2^n x, 2^n y) = 0
\]

for all \( x, y \in X \). If a mapping \( f: X \to \mathcal{T}_\infty(Y) \) satisfies \( f(0) = I_0 \) and the inequality

\[
d_\infty(f(2x + y) \oplus f(2x - y), 2f(x + y) \oplus 2f(x - y) \oplus 2f(x)) \leq \varphi(x, y)
\]

for all \( x, y \in X \), then

\[
A(x) = \lim_{n \to \infty} 8^{-jn} f(2^n x)
\]

exists for each \( x \in X \) and defines a unique cubic type fuzzy set-valued mapping \( C: X \to \mathcal{T}_\infty(Y) \) such that

\[
d_\infty(f(x), A(x)) \leq \begin{cases} 
\frac{1}{8^{1-j}} \varphi(x, 0) & \text{if } j = 1 \\
\frac{1}{8^{1-j}} \varphi(x, 0) & \text{if } j = -1 
\end{cases}
\]

for all \( x \in X \).

Proof  Letting \( y = 0 \) in (6.4). Since \( u \oplus I_0 = u \) for any \( u \in \mathcal{T}_\infty(Y) \), we get

\[
d_\infty(f(x), \frac{1}{8} f(2x)) \leq \frac{1}{8} \varphi(x, 0)
\]

for all \( x \in X \). Furthermore, it follows from (6.2) that

\[
d_\infty(f(x), 8f(\frac{x}{2})) \leq \frac{1}{8} \varphi(x, 0)
\]

for all \( x \in X \).

Consider the set \( E = \{ g: g: X \to \mathcal{T}_\infty(Y), g(0) = I_0 \} \) and introduce the generalized metric \( D \) on \( E \), which is defined by

\[
D(g, h) = \inf\{ \mu \in (0, \infty) \mid d_\infty(g(x), h(x)) \leq \mu \varphi(x, 0), \forall x \in X \}
\]

where, as usual, \( \inf \emptyset = \infty \). It can be easily verified that \( (E, D) \) is a complete generalized metric space (see Radu, 2003).

Now, we consider the linear mapping \( J: E \to E \) such that

\[
Jg(x) = 8^{-1} g(2^jx)
\]

for all \( x \in X \).

Moreover, we can infer from (6.7) and (6.8) that

\[
D(f, Jf) \leq \begin{cases} 
\frac{1}{8} & \text{if } j = 1 \\
\frac{1}{8} & \text{if } j = -1
\end{cases}
\]

The rest of the proof is similar to the proof of Theorem 3.5.
Corollary 6.4 Let \( j \in \{-1, 1\} \) be fixed and let \( p, \theta \) be positive real numbers with \( p \neq 3 \), and let \( X \) be a cone with the vertex 0 contained in a real normed space. Suppose that \( f : X \to F_{KC}(Y) \) is a mapping satisfying
\[
d_{\infty}(f(2x + y) \oplus f(2x - y), 2f(x + y) \oplus 2f(x - y) \oplus 12f(x)) \leq \theta(\|x\|^p + \|y\|^p)
\]
(6.12)
for all \( x, y \in X \). Then
\[
A(x) = \lim_{n \to \infty} 8^{-jn} f(2^n x)
\]
(6.13)
exists for each \( x \in X \) and defines a unique cubic type fuzzy set-valued mapping \( C : X \to F_{KC}(Y) \) such that
\[
d_{\infty}(f(x), C(x)) \leq \begin{cases} 
\frac{\theta \|x\|^p}{8^{-jn}}, & \text{ if } j = 1, 0 < p < 3 \\
\frac{\theta \|x\|^p}{2^{-jn}}, & \text{ if } j = -1, p > 3
\end{cases}
\]
(6.14)
for all \( x \in X \).

Proof. In Theorem 6.3, let \( \varphi(x, y) = \theta(\|x\|^p + \|y\|^p) \). Then we can choose \( L = 2^{(p-3)} \) and we get the desired result.

7. Stability of the Jensen type cubic fuzzy set-valued functional equation

In this section, we will prove the Hyers-Ulam-Rassias stability of the Jensen type cubic fuzzy set-valued functional equation by using the same method as applied in Section 6.

Definition 7.1 Let \( X \) be a cone with the vertex 0 and let \( f : X \to F_{KC}(Y) \) be a fuzzy set valued mapping. The Jensen type cubic fuzzy set-valued functional equation is defined by
\[
f\left(\frac{3x + y}{2}\right) \oplus f\left(\frac{x + 3y}{2}\right) = 12f\left(\frac{x + y}{2}\right) \oplus 2f(x) \oplus 2f(y)
\]
(7.1)
for all \( x, y \in X \). Every solution of (7.1) is called a Jensen type cubic fuzzy set-valued mapping.

Remark 7.2 Obviously, it can be checked that for the triangular fuzzy set-valued mapping \( f \) as defined in Example 3.2, \( f(t) = t u_0^j \) is a solution of (6.1) for any \( t \in \mathbb{R}^+ \) and any fixed \( u_0 \in F_{KC}(Y) \).

Theorem 7.3 Let \( j \in \{-1, 1\} \) be fixed and let \( \varphi : X \times X \to [0, +\infty) \) be a function such that there exists a positive constant \( L < 1 \) satisfying
\[
\varphi(x, y) \leq 8^{-jn} \varphi(2^j x, 2^j y)
\]
(7.2)
for all \( x \in X \). Moreover, assume that \( \varphi \) satisfies
\[
\lim_{n \to \infty} 8^n \varphi(2^{-jn} x, 2^{-jn} y) = 0
\]
(7.3)
for all \( x, y \in X \). If a mapping \( f : X \to F_{KC}(Y) \) satisfies \( f(0) = I_0 \) and the inequality
\[
d_{\infty}(f\left(\frac{3x + y}{2}\right) \oplus f\left(\frac{x + 3y}{2}\right), 12f\left(\frac{x + y}{2}\right) \oplus 2f(x) \oplus 2f(y)) \leq \varphi(x, y)
\]
(7.4)
for all \( x, y \in X \), then
\[
A(x) = \lim_{n \to \infty} 8^n f\left(\frac{x}{2^n}\right)
\]
(7.5)
exists for each \( x \in X \) and defines a unique cubic type fuzzy set-valued mapping \( C : X \to \mathcal{T}_{KC}(Y) \) such that

\[
d_{\infty}(f(x), A(x)) \leq \begin{cases} 
\frac{1}{8(1-\lambda)} \phi(x, x) & \text{if } j = 1 \\
\frac{1}{8(1-\lambda)} \phi(x, x) & \text{if } j = -1
\end{cases} \quad (7.6)
\]

for all \( x \in X \).

**Proof** Letting \( y = x \) in (7.4). Since \( u \oplus I_0 = u \) for any \( u \in \mathcal{T}_{KC}(Y) \), we get

\[
d_{\infty}(f(x), \frac{1}{8} f(2x)) \leq \frac{1}{8} \phi(x, x) \quad (7.7)
\]

for all \( x \in X \). Furthermore, it follows from (7.2) that

\[
d_{\infty}(f(x), 8f \left( \frac{x}{2} \right)) \leq \frac{L}{8} \phi(x, x) \quad (7.8)
\]

for all \( x \in X \).

Consider the set \( E = \{ g \mid g : X \to \mathcal{T}_{KC}(Y), g(0) = I_0 \} \) and introduce the generalized metric \( D \) on \( E \), which is defined by

\[
D(g, h) = \inf \{ \mu \in (0, \infty) \mid d_{\infty}(g(x), h(x)) \leq \mu \phi(x, x), \forall x \in X \}
\]

(7.9)

where, as usual, \( \inf \emptyset = \infty \). It can be easily verified that \((E, D)\) is a complete generalized metric space (see Radu, 2003).

Now, we consider the linear mapping \( J : E \to E \) such that

\[
Jg(x) = 8g \left( \frac{x}{2} \right)
\]

(7.10)

for all \( x \in X \).

Moreover, we can infer from (7.7) and (7.8) that

\[
D(f, Jf) \leq \begin{cases} 
\frac{L}{8} & \text{if } j = 1 \\
\frac{1}{8} & \text{if } j = -1.
\end{cases}
\]

(7.11)

The rest of the proof is similar to the proof of Theorem 3.5.

**Corollary 7.4** Let \( j \in \{-1, 1\} \) be fixed and let \( p, \theta \) be positive real numbers with \( p \neq 3 \), and let \( X \) be a cone with the vertex 0 contained in a real normed space. Suppose that \( f : X \to \mathcal{T}_{KC}(Y) \) is a mapping satisfying

\[
d_{\infty} \left( f \left( \frac{3x + y}{2} \right) \oplus f \left( \frac{x + 3y}{2} \right), 12f \left( \frac{x + y}{2} \right) \oplus 2f(x) \oplus 2f(y) \right) \leq \psi (\|x\|^p + \|y\|^p)
\]

(7.12)

for all \( x, y \in X \). Then

\[
A(x) = \lim_{n \to \infty} 8^n f \left( 2^{-jn} x \right)
\]

(7.13)

exists for each \( x \in X \) and defines a unique cubic-Jensen type fuzzy set-valued mapping \( C : X \to \mathcal{T}_{KC}(Y) \) such that
\[ d_{m}(f(x), C(x)) \leq \begin{cases} 
\frac{2|x|^p}{2 - p}, & \text{if } j = 1, p > 3 \\
\frac{2|x|^{p^2}}{2 - p^{2}}, & \text{if } j = -1, 0 < p < 3 
\end{cases} \quad (7.14) \]

for all \( x \in X \).

**Proof**  In Theorem 7.3, let \( \varphi(x, y) = \theta(\|x\|^p + \|y\|^q) \). Then we can choose \( L = 2^{(3-p)} \) and we get the desired result.

**Corollary 7.5**  Let \( j \in (-1, 1) \) be fixed and let \( p, \theta \) be positive real numbers with \( p \neq 3/2 \), and let \( X \) be a cone with the vertex 0 contained in a real normed space. Suppose that \( f : X \to F_{KC}(Y) \) is a mapping satisfying

\[ d_{m}(f\left(\frac{3x + y}{2}\right), f\left(\frac{x + 3y}{2}\right), 12f\left(\frac{x + y}{2}\right) \oplus 2f(x) \oplus 2f(y) \) \leq \theta(\|x\|^p \|y\|^q) \quad (7.15) \]

for all \( x, y \in X \). Then

\[ A(x) = \lim_{n \to \infty} 8^n f(2^{-n}x) \quad (7.16) \]

exists for each \( x \in X \) and defines a unique cubic-Jensen type fuzzy set-valued mapping \( C : X \to F_{KC}(Y) \) such that

\[ d_{m}(f(x), C(x)) \leq \begin{cases} 
\frac{2|x|^p}{2 - p}, & \text{if } j = 1, p > 3/2 \\
\frac{2|x|^{p^2}}{2 - p^{2}}, & \text{if } j = -1, 0 < p < 3/2 
\end{cases} \quad (7.17) \]

for all \( x \in X \).

**Proof**  In Theorem 7.3, let \( \varphi(x, y) = \theta(\|x\|^p \|y\|^q) \). Then we can choose \( L = 2^{(3-2p)} \) and we get the desired result.

**Corollary 7.6**  Let \( j \in (-1, 1) \) be fixed and let \( p, \theta \) be positive real numbers with \( p + q \neq 3 \), and let \( X \) be a cone with the vertex 0 contained in a real normed space. Suppose that \( f : X \to F_{KC}(Y) \) is a mapping satisfying

\[ d_{m}(f\left(\frac{3x + y}{2}\right), f\left(\frac{x + 3y}{2}\right), 12f\left(\frac{x + y}{2}\right) \oplus 2f(x) \oplus 2f(y) \) \leq \theta(\|x\|^p \|y\|^q + \|x\|^{p+q} + \|y\|^{p+q}) \quad (7.18) \]

for all \( x, y \in X \). Then

\[ A(x) = \lim_{n \to \infty} 8^n f(2^{-n}x) \quad (7.19) \]

exists for each \( x \in X \) and defines a unique cubic-Jensen type fuzzy set-valued mapping \( C : X \to F_{KC}(Y) \) such that

\[ d_{m}(f(x), C(x)) \leq \begin{cases} 
\frac{2|x|^p}{2 - p}, & \text{if } j = 1, p > 3/2 \\
\frac{2|x|^{p^2}}{2 - p^{2}}, & \text{if } j = -1, 0 < p + q < 3/2 
\end{cases} \quad (7.20) \]

for all \( x \in X \).

**Proof**  In Theorem 7.3, let \( \varphi(x, y) = \theta(\|x\|^p \|y\|^q + \|x\|^{p+q} + \|y\|^{p+q}) \). Then we can choose \( L = 2^{(3-p-3q)} \) and we get the desired result.
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