Nonperturbative Ground State of the Stochastic Stabilization of 2D Quantum Gravity

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Abstract

I construct the ground state, up to first nonperturbative order, of the stochastic stabilization of the zero dimensional matrix model which defines 2D Quantum Gravity. It is given by the linear combination of a perturbative wave function and a nonperturbative one. The nonperturbative behaviour which arise from the stabilized model and from the string equation are similar. I show the modification of the loop equation by nonperturbative contribution.

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1. Introduction

The zero-dimensional one matrix model leads to a nonperturbative definition of simplicial 2D quantum gravity coupled to conformal matter with $c < 1$ [1, 2, 3]. In the continuum limit, the perturbative topological expansion in the genus can be evaluated by solving some differential equation: the string equation [3]. The Schwinger-Dyson equations of the matrix model defines the loop equation of the 2D quantum gravity [2]. The matrix model defines also the nonperturbative behaviour up to the ambiguity of the boundary condition of the string equation. Unfortunately, in the case of pure gravity, the nonperturbative real solutions of the string equation for every boundary condition, can not be solutions of the loop equation [4].

The stochastic stabilization of the zero-dimensional matrix model is defined by a one-dimensional matrix model with the same perturbative expansion in $1/N$, and perhaps it provides a consistent definition of nonperturbative effects [5, 6]. In reference [8] the scaling part of the stabilized hamiltoninan is solved numerically. In this paper I propose an analytic approach in order to show the relationship with the nonperturbative behaviour of the original matrix model.

In section two, I review some results about the nonperturbative effects of the matrix model in order to compare it with the stabilized model.

In section three, the stochastic stabilization of the matrix model is introduced, the relationship between the eigenvalue density of the matrix model and the fermionic density of the stabilized model, and between the Schwinger-Dyson equation of the matrix model and the Ward identities of the stabilized model are showed.
In section four, I perform the calculation of the nonperturbative contribution in the case of pure gravity. This calculation is performed using the old WKB method. Finally, some conclusions are presented.

2. Nonperturbative 2D quantum gravity

In this section I review some results of references[9, 10].

The nonperturbative effects of the zero-dimensional matrix model are given by the tunnelling of one eigenvalue outside the main well of the potential. The potential of pure gravity is unbounded from below and the tunnelling takes place between the well and the unbounded region.

In the large $N$ limit, the free energy is

$$\ln Z = N^2 \left( \int d\lambda d\mu \rho(\lambda) \rho(\mu) \ln |\lambda - \mu| - \int d\lambda \rho(\lambda) W(\lambda) \right)$$

(1)

where $\rho(\lambda)$ is the eigenvalue density and $W(\lambda)$ is the potential. The variation of the free energy when one eigenvalue is moved outside the support of $\rho(\lambda)$ is

$$\Gamma(\lambda_f) = \delta \ln Z = N^2 \int d\lambda \delta \rho(\lambda) \left( 2 \int d\mu \rho(\mu) \ln |\lambda - \mu| - W(\lambda) \right)$$

(2)

where $\delta \rho(\lambda) = N^{-1} \{ \delta(\lambda - \lambda_f) - \delta(\lambda - \lambda_i) \}$. One can interpret $\Gamma(\lambda_f)$ as the effective potential for one eigenvalue in the background created by the $N - 1$ remaining eigenvalues[10]. The formula (2) can be written[9, 11]

$$\Gamma(\lambda_f) = N \int_{\lambda_i}^{\lambda_f} d\lambda \left( 2 \int d\mu \frac{\rho(\mu)}{\lambda - \mu} - W'(\lambda) \right)$$

$$= -2N \int_{\lambda_i}^{\lambda_f} d\lambda U(\lambda) \sqrt{\lambda^2 - a^2}$$

(3)

where $a$ is the cut of the support of $\rho(\lambda)$ and $U(\lambda)$ is a polynomial given by the saddle point approximation. The maxima of the effective potential $\Gamma(\lambda_f)$
are given by the zeros of $U(\lambda)$. One may consider the instanton configuration where one eigenvalue sits at the top of $\Gamma(\lambda_f)$. Taken into account that the eigenvalue density is

$$\rho(\lambda) = U(\lambda) \sqrt{a^2 - \lambda^2}, \quad \lambda \in (-a, a)$$

one can write the instantonic contribution as an integral of the imaginary part of the analytic continuation of the eigenvalue density

$$\Gamma_{\text{matrix}} = -2N \int_a^b d\lambda \Im(\rho(\lambda))$$

where $a$ is the cut of the support of $\rho(\lambda)$ and $b$ is the top of the effective potential $\Gamma(\lambda_f)$. In the double scaling limit this contribution is finite and is the source of all the troubles of pure quantum gravity. This nonperturbative contribution arise also from the string equation[9].

The instantonic action (2) can be interpreted as an effective potential as follows: a source coupled to an eigenvalue is added to the original potential

$$\sum_{i=1}^N W(\lambda_i) \rightarrow \sum_{i=1}^N W(\lambda_i) - J\lambda_N$$

At leading order the source term only affect the eigenvalue $\lambda_N$, so that the eigenvalue density is now

$$\tilde{\rho}(\lambda) = \rho(\lambda) + \frac{1}{N} \{\delta(\lambda - \tilde{\lambda}_N) - \delta(\lambda - \bar{\lambda}_N)\}$$

where $\tilde{\lambda}_N$ is the $N$ eigenvalue of the saddle point configuration in the presence of the source and $\bar{\lambda}_N$ is the eigenvalue without the source. In the large $N$ limit the classical field $\langle \lambda_N \rangle$ is given by the eigenvalue $\lambda_N$. Hence, the effective potential

$$\Gamma(\langle \lambda_N \rangle) = \ln Z(J) - \langle J\lambda_N \rangle$$
becomes the instantonic action (2) when the classical field and \( \lambda_f \) are identified.

It is well know that \( SU(N) \) symmetry is broken if there exist non trivial solution of

\[
\frac{\delta \Gamma}{\delta \lambda_N} = 0.
\]

Henceforth, the instantonic configurations with an eigenvalue at the top of the effective potential break the \( SU(N) \) symmetry.

Therefore, \( SU(N) \) symmetry is restored like follow: there are \( N \) vacua \(|i\rangle\) with an eigenvalue \( \lambda_i \) at the top of the effective potential. \( SU(N) \) vacua are given by linear combination of \(|i\rangle\) which must be invariant under permutation of the \( N \) eigenvalues. There are only one linear combination and does not exist arbitrary parameter like the \( \theta \) parameter of gauge theories.

3. Stochastic stabilization of the zero-dimensional matrix model

Stochastic stabilization\[5, 6\] provides a nonperturbative definition of 2D quantum gravity, while reproducing the perturbative expansion in powers of \( 1/N \) of the original matrix model and, therefore, the topological expansion of the discretized quantum gravity.

The stochastic stabilization introduces a positive definite hamiltonian

\[
H = \frac{1}{2} Tr \left\{ -\frac{1}{N^2} \frac{\partial^2}{\partial \Phi^2} + \frac{1}{4} \left( \frac{\partial W}{\partial \Phi} \right)^2 - \frac{1}{2N} \frac{\partial^2 W}{\partial \Phi^2} \right\}
\]

this hamiltonian is well defined for all values of \( N \) and coupling constants. The zero mode of the stabilized theory is

\[
\Psi(\Phi) \sim \exp \left\{ -N \frac{W(\Phi)}{2} \right\}
\]

its norm being the partition function of the original matrix model. Hence, the matrix model is well defined if and only if this zero energy state is a
normalizable state. When this is the case, corresponding observables in both theories coincide

\[
\langle Q \rangle_{\text{stab}} = \int Q \mid \Psi \mid^2 \, d\Phi = \frac{1}{Z} \int Q \exp\{-NW(\Phi)\} \, d\Phi = \langle Q \rangle_{\text{matrix}}
\]  

(12)

In the \(\lambda_n\) variables, where \(\{\lambda_n\}\) are the eigenvalues of the matrix \(\Phi\), the zero energy state is the Slater determinant

\[
\Psi(\{\lambda_n\}) = \prod_{i<k}^N (\lambda_i - \lambda_j) \exp\left\{-\frac{N}{2} \sum_{n=1}^N W(\lambda_n)\right\}
\]  

(13)

and, hence, the stabilized model is a Fermi gas.

In the planar limit, the condition (12) becomes

\[
\int_a^b d\lambda \lambda^n \rho_{\text{matrix}} = \int_a^b d\lambda \lambda^n \rho_{\text{stabilized}}
\]  

(14)

where \(\rho_{\text{matrix}}\) and \(\rho_{\text{stabilized}}\) are the density of eigenvalues of the zero dimensional matrix model and the fermionic density of the stabilized model. And taking into account that both densities are the square root of a polynomial which is zero at the points \(a\) and \(b\) it is not difficult to prove that both densities must be equal\(^7\). 

The fermionic density in the large \(N\) limit for the stabilized hamiltonian is

\[
\rho(\lambda) = \frac{1}{\pi} \sqrt{2(E_F - V)}
\]  

(15)

where \(V\) is the stabilized potential\(^1\) and \(E_F\) is the Fermi level. From the equality between (13) and (4), \(E_F - V\) must has a simple zero for \(\lambda = a\) and double zeros corresponding to zeros of \(U(\lambda)\), in particular, a double zero for \(\lambda = b\). Therefore, in the simplest case of a cuartic potential\(^2\), the stabilized

\(^1\)in the large \(N\) limit, \(V\) is a one particle potential given by the Hartree aproximation
model is given by a Fermi gas placed at the main well of the stabilized potential, where the classical turning point of the Fermi level is given by the cut of the eigenvalue density of the matrix model: a. The stabilized potential has also a local minimum b, which is reached by the Fermi level, and it is given by the top of the effective potential of the matrix model. Hence, the main well of the stabilized potential defines the perturbative expansion of the matrix model and I will show how the local minimum defines the nonperturbative effects.

The stabilized hamiltonian \([I0]\) is the supersymmetric hamiltonian of reference\([3]\) restricted to the bosonic sector. The Schwinger-Dyson equation of the zero-dimensional matrix model becomes the Ward identities of the supersymmetric matrix model. The Ward identities of the stabilized hamiltonian are given by the condition\([13]\)

\[
\frac{\partial E}{\partial g_n} = 0
\]  

(16)

where \(g_n\) are the coupling constants of the zero-dimensional matrix model and \(E\) is the ground energy of the stabilized hamiltonian. In appendix 2, I extract from this condition the Virasoro constraints.

4. Nonperturbative stochastic pure gravity

Let us consider the matrix model

\[
Z = \int d\Phi \exp\left\{-N\text{Tr}W(\Phi)\right\}
\]  

(17)

where

\[
W(\Phi) = \text{Tr}\Phi^2 - \frac{g}{2}\text{Tr}\Phi^4,
\]  

(18)

and \(\Phi\) is an \(N\) dimensional hermitian matrix. This model corresponds to a discretized formulation of pure quantum gravity\([3]\).
The Fokker-Planck hamiltonian of the stabilized model is the sum of $N$ one particle hamiltonians\[12\]

$$h_n = -\frac{1}{2} \frac{1}{N^2} \frac{\partial^2}{\partial \lambda_n^2} + \frac{1}{2} \left\{ g^2 \lambda_n^6 - 2g \lambda_n^4 + (1 + 2g) \lambda_n^2 - 1 \right\}$$  \hspace{1cm} (19)

and an interacting term

$$\frac{1}{2} \left\{ \frac{g}{N} \sum_{n,m} \lambda_n \lambda_m \right\}$$  \hspace{1cm} (20)

where $\{\lambda_n\}$ are the eigenvalues of the matrix $\Phi$. The interacting term is subleading in the $1/N$ expansion and I discard them. The one particle stabilized potential (19) has a main well and a local minimum below some critical coupling constant and only one well above it.

The $1/N$ expansion is equivalent to the WKB aproximation, and the quantization condition is\[6\]

$$\frac{N}{\pi} \int d\lambda \sqrt{2(E_n - V)} + \cdots = n + \frac{1}{2}$$  \hspace{1cm} (21)

The Fermi level $E_{N-1}$ in the large $N$ limit is the value of the one particle potential (19) in its local minimum\[12\]. The first correction to the Fermi level is negative (appendix 3). Henceforth, in the perturbative expansion $1/N$, the quantization condition (21) restricted to the main well gives the correct perturbative quantization of energy levels.

However, the perturbative WKB wave function has a singularity in the local minimum. It is well know that the WKB aproximation break down near the turning points where the variation of the local wavelength is not small

$$\left| \frac{1}{2} \frac{d}{dx} \left( \frac{\hbar}{\sqrt{2(E - V)}} \right) \right| \sim 1$$  \hspace{1cm} (22)

At those points one needs the solution of the Schrödinger equation in order to give the connection formulas and the quantization condition.
The local wavelength near the local minimum of the stabilized potential \( \Lambda(x) \) is
\[
\Lambda(x) \sim \frac{\hbar}{\sqrt{2(e_1\hbar + x^2)}}
\] (23)

where \( e_1\hbar \) is the perturbative shift of the Fermi level and \( e_1 > 0 \) because the Fermi level is placed below the local minimum. The origin of coordinates is placed at the local minimum and \( x = \lambda - b \).

The variation of the local wavelength \( \Lambda(x) \) is order one when \( x \) is order \( \sqrt{\hbar} \). Henceforth, I perform the change of variables
\[
y = \frac{x}{\sqrt{\hbar}},
\] (24)

and the Schrödinger equation at first order in \( \hbar \) becomes
\[
\frac{d^2}{dy^2} \Psi - (e_1 + C^2 y^2) \Psi = 0
\] (25)

In appendix 1, I show the solutions of this equation and the connection formulas in the local minimum which arise from them.

The more general wave function in the main well of the stabilized potential is the oscillatory solution
\[
\Psi(\lambda) = A \left( \frac{2}{2(E-V)} \right)^{\frac{1}{2}} \cos \left( N \int_0^\lambda dx \sqrt{2(E-V) + \delta} \right).
\] (26)

Because the potential is even, one can construct a wave function with definite parity. For the even wave function \( \delta = 0 \) and \( \delta = \frac{\pi}{2} \) for the odd wave function.

The energy level may be written as the sum of a perturbative and a nonperturbative term: \( E = e_n + \tilde{e}_n \). Hence, the wave function becomes
\[
\Psi(\lambda) = \Psi^p(\lambda) + \Psi^{np}(\lambda),
\] where
\[
\Psi_{np}(\lambda) = \frac{-AN\tilde{e}_n}{2\omega_n(2(E-V))^{\frac{1}{4}}} \sin \left\{ N \int_\lambda^a dx \sqrt{2(E-V)} - \frac{\pi}{4} + \eta \right\}
\]
\[
\Psi^p(\lambda) = \frac{A}{(2(E-V))^{\frac{1}{4}}} \cos \left\{ N \int_\lambda^a dx \sqrt{2(E-V)} - \frac{\pi}{4} + \eta \right\}.
\] (27)

where
\[
\eta = \frac{\pi}{4} - N \int_0^a dx \sqrt{2(e_n - V)} - \delta,
\] (28)
a is the classical turning point and \(\omega_n\) is the classical frequency. The wave amplitude of \(\Psi_{np}(\lambda)\) is suppressed by the nonperturbative term \(\tilde{e}_n\), hence \(\Psi^p(\lambda)\) is the perturbative part of the wave function.

Beyond the local minimum the wave function must be the fall off exponential in the infinity
\[
\Psi(\lambda) = \frac{B}{(2(V-E))^{\frac{1}{4}}} \exp \left\{ -N \int_\lambda^b dx \sqrt{2(V-E)} \right\}
\] (29)
where \(b\) is the local minimum.

The connection formula at the local minimum gives the wave function between the local minimum and the main well (appendix 1)
\[
\Psi(\lambda) \to \Psi(\lambda) = \frac{Bf_1(e_1, x_0)}{(2(E-V))^{\frac{1}{4}}} \exp \left\{ -N \int_\lambda^b dx \sqrt{2(V-E)} \right\} + \frac{Bf_2(e_1, x_0)}{(2(E-V))^{\frac{1}{4}}} \exp \left\{ +N \int_\lambda^b dx \sqrt{2(V-E)} \right\}
\] (30)
where \(x_0\) is an arbitrary constant. This wave function may be written as \(\Psi(\lambda) = \Psi_+(\lambda) + \Psi_-(\lambda)\) where
\[
\frac{\Psi_+(\lambda)}{Bf_1(e_1, x_0)} = \frac{\tilde{A}}{(2(V-E))^{\frac{1}{4}}} \exp \left\{ +N \int_a^\lambda dx \sqrt{2(V-E)} \right\}
\]
\[
\frac{\Psi_-(\lambda)}{Bf_2(e_1, x_0)} = \frac{\tilde{A}^{-1}}{(2(V - E))^\frac{1}{4}} \exp \left\{ -N \int_a^\lambda dx \sqrt{2(V - E)} \right\}
\]

\[
\tilde{A} = \exp \left\{ -N \int_a^b dx \sqrt{2(V - E)} \right\}
\]

(31)

where \(\Psi_+(\lambda)\) is suppressed by a nonperturbative term, hence the perturbative part of the wave function is \(\Psi_-(\lambda)\).

The usual matching condition in the classical turning point \(a\) between the perturbative wave functions \(\Psi^p(\lambda)\) and \(\Psi_-(\lambda)\) gives the perturbative quantization condition (21) and the following relation between the arbitrary constants \(A\) and \(B\)

\[
A = 2Bf_2(e_1, x_0) \exp \left\{ N \int_a^b dx \sqrt{2(V - E)} \right\}
\]

(32)

And the matching condition between \(\Psi^{np}(\lambda)\) and \(\Psi_+(\lambda)\) gives the following value for the nonperturbative part of the energy level

\[
\tilde{\epsilon}_n = \pm \frac{1}{N} f(e_1, x_0) \int_a^b dx \sqrt{2(V - e_n)} \exp \left\{ -2N \int_a^b dx \sqrt{2(V - e_n)} \right\}
\]

(33)

where, \(f(e_1, x_0) \sim x_0\) for \(x_0\) small (appendix 1). This nonperturbative corrections arise only for levels near the local minimum.

Therefore, the total energy of the ground state is

\[
E = \sum_{n=0}^{N-1} \tilde{\epsilon}_n
\]

(34)

where \(\tilde{\epsilon}_n\) is nonzero only for levels near the Fermi energy where

\[
e_n = E_F + O(1/N)
\]

(35)

hence, in the large \(N\) limit, (34) is approached by

\[
E = \frac{x_0}{N} \exp \left( -2N \int_a^b d\lambda \sqrt{2(V - E_F)} \right) \sum_{n=0}^{N-1} \frac{1}{\int_a^b d\lambda \sqrt{2(V - e_n)}}
\]

(36)
The nonperturbative contribution to the observables can be written as

$$K(g) \exp(\Gamma_{\text{stabilized}}),$$

where the instantonic action is given by

$$\Gamma_{\text{stabilized}} = -2N \int_{a}^{b} d\lambda \Im(\rho(\lambda))$$

where \(\rho(\lambda)\) is the analytic continuation of the fermionic density of the stabilized model which is equal to the eigenvalue density of the zero-dimensional matrix model. Hence, the instantonic action of the matrix model (3) and of the stabilized model (38) are equal. (38) also agree with the instantonic action of reference[7], which is given by a succession of instanton-antiinstanton starting and ending at the main well.

The constant \(x_0\) is analogous to the arbitrary constant of the nonperturbative contribution which arise from the string equation [3]. However, in the stabilized model there is not a relationship between \(x_0\) and boundary conditions of some string equation, in fact the ambiguity arise form the connection formulas (appendix 1). In reference[8] the stabilized model is solved numerically and the solution is unique, so the ambiguity which I have found must be a defect of the approach, but the arbitrary constant \(x_0\) must be different from zero because otherwise the WKB wave function is not a solution of the Schrödinger equation near the local minimum.

The important point is that the ground state energy is greater than zero by a nonperturbative correction. From reference[13] and (34) the first loop equation becomes in the double scaling limit

$$\dot{V} \left( \frac{\partial}{\partial L} \right) \langle W(L) \rangle - \int_{0}^{L} dJ \langle W(J)W(L-J) \rangle \sim f(z) \exp \left\{ -\frac{2}{5} \sqrt{6} z^2 \right\} (1 + \cdots)$$

(39)
hence, the nonperturbative behaviour of this new loop equation is compatible with the string equation of the zero dimensional matrix model. But, in the stabilized model the string equation and the KdV flow are lost because the Virasoro constraints also changes by nonperturbative corrections. Is an open problem what replaces the old string equation and the KdV flow.

5. Conclusions

I have found the first nonperturbative correction to the ground state of the stochastic stabilization of zero dimensional matrix model. The nonperturbative behaviour of the original matrix model and its stabilization are similar.

The loop equation of the stabilized model and the zero-dimensional matrix model differ by a nonperturbative term (39), which is similar to the nonperturbative behaviour of the string equation. Therefore, I expect that the stabilized model gives a consistent definition of the simplicial 2D quantum gravity.

From the point of view of the original matrix model, the nonperturbative effects arise from the tunnelling between the well of the potential to the unbounded region. Hence, the perturbative vacuum state is a metastable state and it decay to an ill defined vacuum. But, from the point of view of the stabilized model there is only one perturbative vacuum state and there is not an ill defined vacuum[13, 8]. In ordinary quantum mechanics the nonperturbative effects arise from tunnelling between two or more perturbative vacua, the true nonperturbative vacuum is given by some linear combination of perturbative vacua, and the physical interpretation is that one perturbative vacuum decay by quantum tunnelling into the others. However, the
ground state of the stablized model is given by the sum of a perturbative part, which is the perturbative wave function around the main well, and a nonperturbative term

$$\Psi(\lambda) = \Psi^p(\lambda) + \Psi^{np}(\lambda).$$

(40)

The perturbative wave function is given by an oscillatory solution at the main well and a fall off exponential at the classical forbidden region

$$\Psi^p(\lambda) = \begin{cases} 
A \left( 2(E - V) \right)^{1/4} \cos \left\{ N \int_\lambda^a dx \sqrt{2(E - V)} - \frac{\pi}{4} + \eta \right\} & \lambda < a \\
A \left( 2(V - E) \right)^{1/4} \exp \left\{ -N \int_\lambda^a dx \sqrt{2(V - E)} \right\} & \lambda > a 
\end{cases}$$

(41)

Therefore, the particle density at the main well is given by

$$\rho(\lambda, E) = \frac{1}{\sqrt{2(E - V(\lambda))}} \cos^2 \left( N \int_{0}^{\lambda} dy \sqrt{2(E - V(y))} \right).$$

(42)

In the large $N$ limit the $\cos^2$ must be replace by $1/2$ and the particle density is given by the WKB formula

$$\rho(\lambda, E) = \frac{1}{\sqrt{2(E - V(\lambda))}}$$

(43)

But, in the double scaling limit (42) becomes

$$\rho(x, e) = \frac{1}{\sqrt{2(e - v(x))}} \cos^2 \left( \frac{1}{\hbar} \int_{-\infty}^{x} dy \sqrt{2(e - v(y))} \right)$$

(44)

where $e$ and $v$ are the scaling part of the energy and the potential, and $\hbar$ is the scaling coupling constant

$$\hbar^2 = \frac{4g^2}{N^2(g - \bar{g})^{3/2}}$$

(45)
where $g_c$ is the critical coupling constant. The argument of the $\cos^2$ is a function of the scaling constant $\hbar$, and now the particle density becomes an oscillatory function if $\hbar$ is finite. The amplitude is an increasing function of the position and go to infinity when the position approach the classical turning point because the WKB approach break down at the turning points. In fact the exact particle density is computed numerically in $[8]$ and it is given by an oscillatory solution with an increasing amplitude when the position approach the turning point, but, of course, is finite at the turning point. Therefore, the oscillations of the particle density, which are nonperturbative in nature, appears at the first WKB approximation of the wave function.

The nonperturbative part is given by the same oscillatory solution at the main well with a phase sifted by $\pi/2$, and an increasing exponential between the cut $a$ and the local minimum $b$, and beyond the local minimum there is only the perturbative wave function.

$$
\Psi_{np}(\lambda) = \frac{-\tilde{A}N}{2\omega_n(2(E - V))^{\frac{3}{4}}} \sin \left\{ N \int_{\lambda}^{a} dx \sqrt{2(E - V)} - \frac{\pi}{4} + \eta \right\} \lambda < a
$$

$$
\Psi_{np}(\lambda) = \frac{\tilde{A}f(e_1, x_0)}{(2(V - E))^{\frac{3}{4}}} \exp \left\{ +N \int_{a}^{x} dx \sqrt{2(V - E)} \right\} a < \lambda < b \tag{46}
$$

The amplitude of the nonperturbative wave function is suppressed by a nonperturbative term

$$
\tilde{A} \sim \exp \left\{ -2N \int_{a}^{b} dx \sqrt{2(V - E)} \right\} A \tag{47}
$$

In reference $[8]$ the particle density beyond the local minimum $b$ decrease very fast. In the double scaling limit the perturbative wave function $[41]$ decrease as

$$
\Psi \sim \exp \left\{ -\frac{1}{\hbar} \int_{a}^{x} \sqrt{2(v - e)} \right\} \sim \exp \left\{ -\frac{1}{\hbar} x^{\frac{3}{2}} \right\} \tag{48}
$$
Between the cut $a$ and the local minimum $b$ the increasing exponential of the nonperturbative wave function gives some contribution to the particle density. Hence, the particle density decrease more slowly between $a$ and $b$. However, the nonperturbative wave function is suppressed by a nonperturbative term, and the particle density also decrease exponentially between $a$ and $b$. This is in agreement with\cite{.external_ref} where the particle density decrease more slowly between the cut $a$ and the local minimum $b$ than beyond the local minimum.

The above discussion suggest that there are only one perturbative vacuum, given by $\Psi^p$, and almost one nonperturbative vacuum, given by $\Psi^{np}$. So the perturbative vacuum decay into a nonperturbative vacuum. The nonperturbative vacuum break down the symmetries of the perturbative one, so the Ward identities of the model (loop equation and Virasoro constraints) must change. Is an open question what kind of symmetries replaces the old one.

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Appendix 1

In a neighbourhood of size $\sqrt{\hbar}$ around the local minimum the Schrödinger equation is given by

$$\frac{d^2\Psi}{dx^2} - (e_1 + C^2x^2)\Psi = 0 \quad (49)$$

where $e_1$ is the first perturbative correction to the Fermi level and is positive because the perturbative Fermi level is placed below the local minimum. I have placed the origin of coordinates at the local minimum of the potential and $x = \lambda - b$.

Solutions of (49) are given by linear combinations of the even and odd functions

$$\Psi_{even}(x) = e^{-\frac{1}{4}Cx^2}F(\frac{1}{4} + \frac{e_1}{4C} | \frac{1}{2} | \sqrt{C}x^2)$$

$$\Psi_{odd}(x) = \sqrt{C}xe^{-\frac{1}{4}Cx^2}F(\frac{3}{4} + \frac{e_1}{4C} | \frac{3}{2} | \sqrt{C}x^2) \quad (50)$$

where $F(a \mid c \mid z)$ is the Hypergeometric degenerate functions and is the solution of

$$zF'' + (c - z)F' - aF = 0 \quad (51)$$

The asymptotic behaviour of (50) can not match the WKB wave function. However for large values of $x$, $e_1$ is small in comparison with the potential, and one can approach (50) by Bessel functions of order $\frac{1}{4}$ at zero order in $e_1$. The asymptotic behaviour of Bessel functions match the WKB wave functions.

Hence, I construct a wave function, with the correct asymptotic behaviour and given by (50) around the local minimum, as follows.

The real axis is divided into three regions: the neighbourhood of the local minimum, where the wave function is given by Hypergeometric degenerate...
functions

\[ \Psi_1 = \alpha(\Psi_{\text{even}} + \Psi_{\text{odd}}) + \beta(\Psi_{\text{even}} - \Psi_{\text{odd}}). \]  \hspace{1cm} (52)

An intermediate region, where the wave function is given by Bessel functions

\[ \frac{\Psi_2(-x)}{\sqrt{-x}} = A_2 \left[ I_{\frac{1}{4}}\left(\frac{Cx^2}{2}\right) + I_{-\frac{1}{4}}\left(\frac{Cx^2}{2}\right) \right] + B_2 \left[ I_{\frac{1}{4}}\left(\frac{Cx^2}{2}\right) - I_{-\frac{1}{4}}\left(\frac{Cx^2}{2}\right) \right] \]

\[ \frac{\Psi_2^+(x)}{\sqrt{x}} = A_1 \left[ I_{\frac{1}{4}}\left(\frac{Cx^2}{2}\right) + I_{-\frac{1}{4}}\left(\frac{Cx^2}{2}\right) \right] + B_1 \left[ I_{\frac{1}{4}}\left(\frac{Cx^2}{2}\right) - I_{-\frac{1}{4}}\left(\frac{Cx^2}{2}\right) \right] \]  \hspace{1cm} (53)

The continuity of the wave function and its derivative is required in the boundary \(x_0\) of the above regions

\[ \Psi_1(x_0) = \Psi_2^+(x_0) \]

\[ \Psi_1(-x_0) = \Psi_2^-(x_0) \]

\[ \left( \frac{d\Psi_1}{dx} \right)_{x=x_0} = \left( \frac{d\Psi_2^+}{dx} \right)_{x=x_0} \]

\[ \left( \frac{d\Psi_1}{dx} \right)_{x=-x_0} = \left( \frac{d\Psi_2^-}{dx} \right)_{x=-x_0} \]  \hspace{1cm} (54)

And the asymptotic region, where the Bessel functions becomes WKB wave functions. The asymptotic behaviour of Bessel functions are

\[ I\frac{1}{4}(z) + I\frac{1}{4}(-z) \rightarrow \frac{2}{\sqrt{2\pi z}} e^z \]

\[ I\frac{1}{4}(z) - I\frac{1}{4}(-z) \rightarrow \frac{-2}{\sqrt{2\pi z}} \sin\left(\frac{\pi}{4}\right) e^{-z} \]  \hspace{1cm} (55)

The set of equations (54) gives the connections formulas between the asymptotic WKB wave functions.
The wave function must be the fall off exponential for $x \gg 0$. Hence, $A_1 = 0$, and $B_1 = B$ is an arbitrary constant, which can be fixed by normalization. For $x \ll 0$ the wave function is

$$\Psi(\lambda) = \frac{Bf_1(e_1, x_0)}{(2(V - E))^{\frac{1}{4}}} \exp \left\{ -N \int_0^x dx \sqrt{2(V - E)} \right\}$$

$$+ \frac{Bf_2(e_1, x_0)}{(2(V - E))^{\frac{1}{4}}} \exp \left\{ +N \int_0^x dx \sqrt{2(V - E)} \right\}$$

(56)

where $f_1$ and $f_2$ are given by the condition (54).

I have constructed a wave function which is given by the usual WKB wave function in the asymptotic region, and in the interval $(-x_0, x_0)$, by the exact solution of the approximate Schrödinger equation (49). However, this wave function is not unique because the point $x_0$ is arbitrary.

In the limit of $x_0$ small, the coefficients of the wave function (56) becomes

$$f_1(x_0) = x_0 \frac{1}{(\Gamma(5/4))^2} \sqrt{C/4} + O(x_0^2)$$

$$f_2(x_0) = 1 + x_0 \frac{\Gamma(3/4)}{\Gamma(5/4)} \sqrt{C/4} + O(x_0^2)$$

(57)

If one set $x_0 = 0$, the wave function for $x \ll 0$ is given only by the increasing exponential. This connection condition also arise from the Schrödinger equation (49) if one set $e_1 = 0$. Hence, the decreasing exponential can be interpreted as the first correction to the approach $e_1 = 0$. 

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Appendix 2

In this appendix I show the proof of the Virasoro constraints in the stochastic stabilization of 2D quantum gravity. The proof of the Loop equation is given in reference [13].

Let be the matrix potential

\[ W = \sum_{n=0}^{\infty} \frac{2g_n}n Tr\Phi^n \]  

(58)

the potential of the stabilized theory becomes now

\[ V = \frac{1}{2} \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_n g_m \text{Tr}\Phi^{n+m-2} - \frac{1}{N} \sum_{n=0}^{\infty} \sum_{p=0}^{n-2} \text{Tr}\Phi^p \text{Tr}\Phi^{k-p-2} \right\} \]  

(59)

and from the Hellmann-Feynman theorem

\[ \frac{\partial E}{\partial g_k} = \frac{1}{2} \left\{ \sum_{n=0}^{\infty} g_n \langle Tr\Phi^{n+k-2} \rangle - \frac{1}{N} \sum_{p=0}^{k-2} \langle Tr\Phi^p \text{Tr}\Phi^{k-p-2} \rangle \right\} \]  

(60)

If the ground energy is zero, (60) becomes the usual discrete Virasoro constraints [16]

\[ \left( \sum_{n=0}^{\infty} g_n (k + n - 2) \frac{\partial}{\partial g_{n+k-2}} + \frac{1}{4N^2} \sum_{p=0}^{k-2} p(k - p - 2) \frac{\partial^2}{\partial g_p \partial g_{k-p-2}} \right) Z = 0 \]  

(61)
Appendix 3

In this appendix I perform the calculation of the first correction to the Fermi level in the stabilized potential of the quartic matrix model and I will show that the Fermi level is placed below the local minimum.

The perturbative quantization condition for the Fermi level is given by

\[
\frac{N}{\pi} \int_{-a}^{a} dx \sqrt{2(E_F - V)} - \frac{g}{2\pi} \int_{-a}^{a} dx \frac{x^2}{\sqrt{2(E_F - V)}} + O(1/N) = N - \frac{1}{2} \tag{62}
\]

where the extra term arise from the Hartree approximation to the interacting term (20). The Fock term is subleading in the quantization condition (62).

The first correction to the Fermi level is given by

\[
E^{(1)}_F = \omega \left\{ -1 + g \frac{1}{\pi} \int_{-a}^{a} d\lambda \frac{\lambda^2}{\sqrt{2(E_0^F - V)}} \right\} \tag{63}
\]

where \(\omega\) is the classical frequency which is positive.

Following reference[22] \(E^{(1)}_F\) becomes

\[
\frac{E^{(1)}_F}{\omega} = -1 + \frac{1}{\pi} \frac{a^2}{2b\sqrt{b^2 - a^2}} \left\{ \pi - 2 \arctan \left( \frac{b}{\sqrt{b^2 - a^2}} \right) \right\} \tag{64}
\]

and is easy to see that for \(b^2 \geq a^2 > 0\), \(E^{(1)}_F\) is negative and the Fermi level is placed below the local minimum.
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