Abstract. In a recent work the present authors have shown that the eigenvalue probability density function for Dyson Brownian motion from the identity on $U(N)$ is an example of a newly identified class of random unitary matrices called cyclic Pólya ensembles. In general the latter exhibit a structured form of the correlation kernel. Specialising to the case of Dyson Brownian motion from the identity on $U(N)$ allows the moments of the spectral density, and the spectral form factor $S_N(k; t)$, to be evaluated explicitly in terms of a certain hypergeometric polynomial. Upon transformation, this can be identified in terms of a Jacobi polynomial with parameters $(N(\mu-1), 1)$, where $\mu = k/N$ and $k$ is the integer labelling the Fourier coefficients. From existing results in the literature for the asymptotics of the latter, the asymptotic forms of the moments of the spectral density can be specified, as can $\lim_{N\to\infty} \frac{1}{N} S_N(k; t)|_{\mu=k/N}$. These in turn allow us to give a quantitative description of the large $N$ behaviour of the average $\langle |P_N l=1 e^{ikx} l|^2 \rangle$. The latter exhibits a dip-ramp-plateau effect, which is attracting recent interest from the viewpoints of many body quantum chaos, and the scrambling of information in black holes.

1. Introduction

1.1. Focus of the paper. Dyson Brownian motion on the group $U(N)$ of $N \times N$ complex unitary matrices is of long standing interest in random matrix theory. This process is contained in, and gets its name from, Dyson’s 1962 paper [35]. Actually Dyson’s paper is better known for the construction of a Brownian motion model for the eigenvalues of particular spaces of Hermitian Gaussian random matrices; see the monographs [38, 67] for developments in relation to this topic. With $U(N)$ being a Lie group, the notion of a corresponding Brownian motion and its theoretical development can be found in earlier works of Itô [65], Yoshida [104] and Hunt [64]; we owe our knowledge of these references to the Introduction given in [4]. While Dyson’s objective was a Brownian motion theory of the eigenvalues, the objective of the earlier works related to the group elements $U_t$ say; recent works which relate to this latter aspect include [10, 61, 69, 88]. The matrix evolution can be approximated numerically by the specification

$$U_{t+\delta t}(N) = U_t e^{i\sqrt{\delta t} M}, \quad 0 < \delta t \ll 1,$$

where $M$ is an element of the Gaussian unitary ensemble of $N \times N$ complex Hermitian matrices; see e.g. [87] or [42, Eq. (11.26)]. Wolfram Mathematica, with the support of random matrix theory added in version 11 or later, can easily implement (1.1) in a few lines of code [102]. This gives a matrix Brownian motion path on $U(N)$ sampled at discrete time intervals $\delta t$. This path is simple to display in terms of the corresponding eigenvalues; see Figure 1 for an example.
In this paper our aim is to begin with Dyson’s specification of the Brownian motion theory of the eigenvalues on $U(N)$, to specialise to the case that the initial condition is the identity, and to quantify specific statistical properties of the corresponding dynamical state, most notably the dip-ramp-plateau effect; see Section 1.3 in relation to this. Particular attention will be payed to the Fourier components of the eigenvalue density at a specific time, and to the spectral form factor (structure function). This line of study is enabled by the recent identification [70] of a cyclic Pólya ensemble structure in relation to the eigenvalue probability density function (PDF) for this model, supplemented by transformation and asymptotic theory associated with certain hypergeometric polynomials. Insights from other sources have seen progress on many fronts in the study of the eigenvalues for Dyson Brownian motion — interpreted broadly — in recent years; references include [5, 14, 31, 58, 63, 71, 98].

1.2. Definition of key statistical quantities. Unitary matrices have eigenvalues on the unit circle and so can be parametrised as $\{e^{ix_j(t)}\}_{j=1}^N$, with $-\pi < x_j(t) \leq \pi$ angles. Let $\langle \cdot \rangle$ denote an ensemble average with respect to the joint PDF for these angles. The eigenvalue density $\rho^{(1),N}(x; t)$ and the two-point correlation function $\rho^{(2),N}(x, y; t)$ are specified as such ensemble averages of sums of Dirac delta functions

$$\rho^{(1),N}(x; t) = \langle \sum_{j=1}^N \delta(x - x_j(t)) \rangle, \quad \rho^{(2),N}(x, y; t) = \langle \sum_{p \neq q=1}^N \delta(x - x_p(t))\delta(y - x_q(t)) \rangle. \quad (1.2)$$

The Fourier components of the spectral density are, up to proportionality, given by

$$\int_{-\pi}^{\pi} e^{-ikx} \rho^{(1),N}(x; t) \, dx = \sum_{l=1}^N e^{-i\lambda_l(t)} \langle \sum_{l=1}^N \rangle. \quad (1.3)$$
The spectral form factor is specified by
\[
S_N(k; t) := \text{Cov} \left( \sum_{l=1}^{N} e^{ikx_l(t)}, \sum_{l=1}^{N} e^{-ikx_l(t)} \right) := \left\langle \sum_{l,l'=1}^{N} e^{ik(x_l(t)-x_{l'}(t))} \right\rangle - \left\langle \sum_{l=1}^{N} e^{ikx_l(t)} \right\rangle \left\langle \sum_{l=1}^{N} e^{-ikx_l(t)} \right\rangle. \tag{1.4}
\]

Denote the truncated two-point correlation (also referred to as the two-point cluster function [81]) by
\[
\rho_T^{(2)}(x_1, x_2; t) := \rho_T^{(2)}(x_1, x_2; t) - \rho_T^{(1)}(x_1; t)\rho_T^{(1)}(x_2; t). \tag{1.5}
\]
A straightforward calculation (see e.g. [49, proof of Prop. 2.1]) shows that the spectral form factor can be written in terms of the truncated two-point correlation and the density according to
\[
S_N(k; t) = \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy e^{ik(x-y)} \left( \rho_T^{(2)}(x, y; t) + \rho_T^{(1)}(x; t)\delta(x-y) \right)
= N + \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy e^{ik(x-y)} \rho_T^{(2)}(x, y; t), \tag{1.6}
\]
where underlying the second line is the requirement that the density integrated over its support equals $N$.

The Fourier components of the spectral density, and the spectral form factor, show themselves in the consideration of the ensemble average of the quantity
\[
\left| \sum_{l=1}^{N} e^{ikx_l(t)} \right|^2, \quad k \in \mathbb{Z}. \tag{1.7}
\]
Thus from (1.4) one has the decomposition
\[
\left\langle \left| \sum_{l=1}^{N} e^{ikx_l(t)} \right|^2 \right\rangle = S_N(k; t) + \left\langle \left| \sum_{l=1}^{N} e^{ikx_l(t)} \right|^2 \right\rangle. \tag{1.8}
\]
For $\{x_l\}$ viewed as scaled energy levels, both the spectral form factor and the average (1.8) were put forward as probes of quantum chaos in the early literature on the subject [9, 72]. In particular, in relation to the average (1.7), the work [72] drew attention to an effect referred to as a correlation hole in the corresponding graphical shape as a function of $k$, as distinguished from its absence for integrable spectra. The essential point is that for a chaotic system the spectral form factor goes to zero for $k$ small enough and then increases before saturating (measured on some $N$ dependent scale), while the second term on the RHS of (1.8) continues to decrease on this same $N$ dependent scale, with its initial decay as a function of $N$ and $k$ characterised by a distinct $N$ dependent scale. Under the name dip-ramp-plateau, or more accurately slope-dip-ramp-plateau [23, 90, 103], this effect has received renewed attention as a probe of many body quantum chaos [21, 22, 28, 29, 75, 93], and also in relation to the scrambling of information in black holes [18, 27]. This in turn has prompted a revival of interest in the consideration of analytic properties of model systems for which (1.7) exhibits a dip-ramp-plateau shape [15, 23, 24, 44, 45, 84, 85, 96, 97].

In the present work, we identify Dyson Brownian motion on $U(N)$ from the identity as one of the rare known solvable cases of the dip-ramp-plateau effect, joining the Gaussian unitary ensemble (GUE) [15] and Laguerre unitary ensemble (LUE) [48]. Moreover, our exact solution exhibits analytic properties not seen in any of the previous solvable cases. To appreciate this point, general features of this effect, albeit based on non-rigorous analysis and illustrated to the extent possible by GUE and LUE, should first be revised.
The dip-ramp-plateau effect occurs for large $N$ and relates to several regimes as $k$ varies. These are as $k$ varies from order unity to of order proportional to $N$ (this relates to the slope, dip and ramp), and as $k$ varies to be proportional to $N$ (this relates to the ramp and plateau) giving rise to a well defined limiting quantity in $\mu$. Specifically, in relation to the latter we write $\mu = k/N$, and assume $k > 0$. The significance of this scaling is that we have $kx_j(t) = \mu X_j(t)$, where $X_j(t) = N x_j(t)$. Since the spacing between the angles $\{x_j(t)\}$ in the bulk region is $O(1/N)$, the spacing between $\{X_j(t)\}$ in the bulk is $O(1)$. Hence in the variable $\mu$ one would expect a well defined $N \to \infty$ limit of the spectral form factor, corresponding to a deformation of the Fourier transform of the bulk scaled limiting form of the term in the big brackets of the integrand of the first line of (1.6). The deformation is due to the variation of the spacing between $\{X_j(t)\}$ away from their bulk scaled value across all of the spectrum (in particular near the edges). In the ensuing discussion the fact that $k$ is also an integer plays no role, so we can take $\mu$ as continuous.

For large $N$ and an appropriate scaling of time, the eigenvalue density for Dyson Brownian motion on $U(N)$ from the identity has the large $N$ form [10]

$$
\rho_{(1),N}(x; t) \sim \frac{N}{2\pi} \rho_{(1),\infty}(x; t),
$$

(1.9)

where $\rho_{(1),\infty}(x; t)$ is independent of $N$. It follows that the second term in (1.8) can be approximately expressed for large $N$ by

$$
\left( \frac{N}{2\pi} \right)^2 \left| \int_0^\infty \rho_{(1),\infty}(x; t) e^{i\mu N x} \, dx \right|^2.
$$

(1.10)

For $0 < t < 4$ it is known [10] (see also §4.1) that the support of $\rho_{(1),\infty}(x; t)$ is $(-L_0(t), L_0(t))$ for some $0 < L_0(t) < \pi$ and that $\rho_{(1),\infty}(x; t)$ goes to zero as a square root at the end points $\pm L_0(t)$ with some amplitude $A(t)$. According to Fourier transform theory [77], the integral in (1.10) then displays a leading order decay

$$
\sqrt{\pi} A(t) \frac{\cos(\mu N L_0(t) - 3\pi/4)}{(\mu N)^{3/2}}.
$$

(1.11)

Hence (1.10) itself has the decay for $N\mu$ large

$$
\frac{(A(t))^2}{4\pi} \frac{\cos^2(\mu N L_0(t) - 3\pi/4)}{N\mu^3}.
$$

(1.12)

On the other hand, for $t > 4$ the support of $\rho_{(1),\infty}(x; t)$ is all of $(-\pi, \pi]$ and it is a periodic analytic function. As such the integral in (1.10) decays exponentially fast in $N$, and so in this regime (1.12) is replaced by

$$
N^2 e^{-c(t)\mu N}
$$

(1.13)

for some $c(t) > 0$. Since (1.10) takes on the value $N^2$ for $\mu = 0$ and then decreases according to (1.12) or (1.13), its decay as $\mu$ increases is responsible for the initial slope in the terminology slope-dip-ramp-plateau. A significant qualitative feature is that the functional form of the slope changes from being algebraic for $t < 4$ to decaying exponentially for $t > 4$.

The circumstance of a square root singularity at the boundary of the eigenvalue support is a feature of both the GUE and LUE. In both cases the second term in (1.8) permits a special function evaluation from which the analogue of (1.11) is readily verified. Considering the GUE for
where \( L_n^{(a)}(x) \) denotes the Laguerre polynomial. To proceed further the analogue of the scaled wavenumber \( \mu \) is required. This is the variable \( \tau_b \) specified by \( k = 2\sqrt{2N}\tau_b \). Analogous to the discussion at the beginning of this subsection, the justification is that then that \( k\lambda_j = \tau_b X_j \), where \( X_j = 2\sqrt{2N}\lambda_j \) has the property that in the bulk the eigenvalue spacings are of order unity; see e.g. [44, §3.3]. Plancherel-Rotach asymptotics of the Laguerre polynomial can be used to exhibit definiteness, one has [94]

\[
\left\langle \sum_{j=1}^{N} e^{ik\lambda_j} \right\rangle_{\text{GUE}} = e^{-k^2/4} L_{N-1}^{(1)}(k^2/2),
\]

(1.14)

where \( L_n^{(a)}(x) \) denotes the Laguerre polynomial. To proceed further the analogue of the scaled wavenumber \( \mu \) is required. This is the variable \( \tau_b \) specified by \( k = 2\sqrt{2N}\tau_b \). Analogous to the discussion at the beginning of this subsection, the justification is that then that \( k\lambda_j = \tau_b X_j \), where \( X_j = 2\sqrt{2N}\lambda_j \) has the property that in the bulk the eigenvalue spacings are of order unity; see e.g. [44, §3.3]. Plancherel-Rotach asymptotics of the Laguerre polynomial can be used to exhibit definiteness, one has [94]

\[
\left\langle \sum_{j=1}^{N} e^{i2\sqrt{2N}\tau_b\lambda_j} \right\rangle_{\text{GUE}} \sim \frac{1}{2\sqrt{2}\pi^{3/2}N\tau_b^{3/2}} \cos(4N\tau_b - 3\pi/4).
\]

(1.15)

A heuristic understanding of the large \( N \) form of spectral form factor for \( \mu \) fixed is based on Dyson Brownian motion on \( U(N) \) being a member of the class of random matrix models which have a unitary symmetry. Thus the matrix distribution at a given time is unchanged by conjugation with a fixed \( U \in U(N) \). A universal feature of such matrix ensembles is the functional form of the bulk truncated two-point correlation function (see e.g. [42, rewrite of (7.2)])

\[
\rho^{T,\text{bulk}}(x, y) = -\frac{(\pi\rho(x-y))}{(\pi(x-y))^2}.
\]

(1.16)

The quantity \( \rho \) is the local eigenvalue density, the precise value of which depends on the choice of units underlying the bulk scaling, which in broad terms requires a centring of the coordinates away from the boundary of support, and a rescaling so that the eigenvalue density is of order unity. According to (1.6) the spectral form factor is determined by \( \rho^{T,\text{bulk}}(2) \). However the average must be taken over the entire spectrum simultaneous to the scaling of \( k \), or equivalently the fixing of \( \rho \) in (1.16) (on this last point recall the discussion of the first paragraph of this subsection). This average was first computed in a form suitable for taking the large \( N \) scaled limit of the corresponding spectral form factor \( S_N^{(G)}(k) \) for the GUE by Brézin and Hikami [15], with the result

\[
\lim_{N \to \infty} \frac{1}{N} S_N^{(G)}(2\sqrt{2N}\tau_b) = \begin{cases} \frac{2}{\pi} (\tau_b \sqrt{(1 - \tau_b^2)} + \text{Arcsin} \, \tau_b), & 0 < \tau_b < 1, \\ 1, & \tau_b > 1. \end{cases}
\]

(1.17)

Moreover it was shown in [15] that a heuristic analysis based on (1.16) could reproduce the exact result (1.17). Later this same argument was shown to also reproduce a newly obtained exact evaluation of the analogous limit of the spectral form factor \( S_N^{(L)}(k) \) for the LUE with Laguerre parameter \( a \) fixed [48, Eq. (1.26)],

\[
\lim_{N \to \infty} \frac{1}{N} S_N^{(L)}(k) = \text{Arctan} k
\]

(1.18)

(here no simultaneous scaling of \( k \) is required since a feature of the LUE is that for large \( N \) the average spacing between eigenvalues in the bulk is of order unity).

The basic idea applied in the present context is to replace the bulk density \( \rho \) in (1.16) by the asymptotic global density \( N\rho_{(1)}(x+y)/2(\pi) \); recall (1.9). Substituting in the second expression of (1.6) and changing variables \( w = N(x-y) \), \( u = (x+y)/2 \) then gives the large \( N \) prediction

\[
\frac{1}{N} S_N(\mu N; t) \sim 1 - \frac{1}{\pi^2} \int_{-\infty}^{\infty} dw \int_{-\pi}^{\pi} du \frac{\sin^2(\rho_{(1)} (u) w/2)}{w^2} e^{i w \mu}.
\]

(1.19)
Changing the order of integration, the integral over \( w \) can be computed explicitly, showing that for \( \mu > 0 \)

\[
\frac{1}{N} S_N(\mu N; t) \sim 1 - \frac{1}{\pi} \int_0^{u^*} (\rho_{(1),\infty}(u; t) - \mu) \, du.
\]

(1.20)

Here use has been made of the fact that \( \rho_{(1),\infty}(u; t) \) is even in \( u \), and \( u^* = u^*(\mu) \) is such that

\[
\rho_{(1),\infty}(u^*; t) = \mu.
\]

(1.21)

If (1.21) has no solution with the LHS always larger than the RHS, (1.19) implies that \( u^* \) in (1.20) is to be replaced by \( \pi \). Then the first integral in (1.20) evaluates to \( \pi \), so implying

\[
\frac{1}{N} S_N(\mu N; t) \sim \mu.
\]

(1.22)

If there is no solution with the LHS always smaller than the RHS then (1.19) implies that the integral in (1.20) is absent, and so

\[
\frac{1}{N} S_N(\mu N; t) \sim 1.
\]

(1.23)

The behaviour (1.22) is referred to as a ramp, and (1.23) as a plateau, in the terminology dip-ramp-plateau.

Equating the RHS of (1.22) multiplied by \( N \) with (1.12) gives \( \mu \asymp N^{-1/2} \) or equivalently \( k \asymp N^{1/2} \). The latter is referred to as the dip-"time". It quantifies the order in \( N \) of the minimum of the graph of (1.7). We put time within quotation marks here as the terminology is confusing in the present setting of Dyson Brownian motion on \( U(N) \), with the term time already being used in the context of the evolution. Instead we will refer to this as the dip-wavenumber. Its calculated value, which is valid for \( 0 < t < 4 \) is the same as for the GUE; see e.g. [23, second last paragraph of §1.1]. This follows by equating (1.15) with (1.17) turned into a large \( N \) statement by replacing the equals by asymptotically equals, and multiplying by \( N \). On the other hand, as is relevant for \( t > 4 \), equating the RHS of (1.22) multiplied by \( N \) with (1.13) gives \( Ne^{-c(t)\mu N} = \mu \). This equation relates to the Lambert \( W \)-function and implies \( k \asymp \log N \) for the dip-wavenumber.

In quantitative terms, it follows from the predictions of the paragraph two above that there will be a deviation from the ramp and plateau functional forms whenever the global density is not a constant. However, if the global density is strictly positive, then the prediction is that the ramp will be unaltered in the range \( 0 < \mu < \mu_r \) for some \( \mu_r \). If the global density is bounded from above, this argument gives instead that the plateau will be unaltered in the range \( \mu_p < \mu < \infty \) for some \( \mu_p \). Aspects of these predictions have previously been confirmed from the exact calculation of the (scaled) limit of \( S_N(k) \) for the GUE as seen by (1.17) and for the LUE as seen by (1.18). For the GUE the global density is given by the Wigner semi-circle law, which vanishes at the endpoints. Hence we can anticipate that the ramp will always be deformed. On the other hand the Wigner semi-circle is bounded, so it is predicted that the plateau is unaltered beyond a critical value \( \mu_p \). Both these quantitative features are indeed seen in the exact functional form (1.17). In the case of the LUE with Laguerre parameter \( a \) fixed, the global density follows a particular Marčenko-Pastur functional form proportional to \( x^{-1/2}(1-x)^{1/2}\chi_{0<x<1} \), which goes to infinity at the origin, before decreasing monotonically to zero at the right hand endpoint of the support. In this setting the heuristic working predicts that the ramp and plateau are both always deformed, which is indeed a feature of the exact solution (1.18).
As commented, Dyson Brownian motion on $U(N)$ from the identity has the feature that for $t > 4$, the global density is nonzero. This is in distinction to the global density for both the GUE and LUE. The significance of this feature in relation to dip-ramp-plateau is the prediction noted above that the ramp will be unaltered in the range $0 < \mu < \mu_r$ for some $\mu_r$. Indeed we will find that this is a feature of our exact solution for the global scaling limit of $S_N(k; t)$ in the range $t > 4$.

**Remark 1.1.**

1. Adding the square of (1.14) with $k = 2\sqrt{2N} \tau_b$ to the asymptotic form of $S_N^{(G)}(2\sqrt{2N} \tau_b)$ as implied by (1.17) gives a graphically accurate approximation to

$$\langle \left| \sum_{l=1}^{N} e^{i2\sqrt{2N} \tau_b x_l} \right|^2 \rangle_{\text{GUE}}$$

as defined by the GUE analogue of (1.8). A numerical plot — see Figure 2 — using a log-log scale of each axis exhibits the dip-ramp-plateau effect.

2. In general Dyson Brownian motion on $U(N)$ is dependent on the initial condition, with the particular choice of the identity matrix and thus all eigenvalues having angle $x_l(0) = 0$ being the subject of the present work. From the above discussion, relevant questions in relation to dip-ramp-plateau of a more general choice (say with an eigenvalue density supported on an interval strictly within $(-\pi, \pi)$) are the functional form of the singularity of the boundary of support for $t > 0$, and the existence of a time such that the eigenvalue density is strictly positive for all angles in $(-\pi, \pi]$.

3. There is some interest in the functional form of the amplitude $A(t)$ in (1.12) and exponent $c(t)$ in (1.13) from the viewpoint of the dip-wavenumber discussed in the paragraph below (1.23). First, for $t < 4$ we see by equating the RHS of (1.22) multiplied by $N$ with (1.12) that a refinement of the asymptotic bound $k \asymp N^{1/2}$ for the dip-wavenumber is to include the dependence on $t$ by way of the amplitude $A(t)$ to obtain $k \asymp (A(t)N)^{1/2}$. The explicit functional form of $A(t)$ given in (4.13) below shows that $A(t) \asymp (1 - t/4)^{-1/4}$ for $t \to 4^-$. This divergence indicates a breakdown in the asymptotic dependence on $N$ for $t \geq 4$. In fact at $t = 4$, the dip-wavenumber relates to $N$ through the asymptotic relation $k \asymp N^{6/11}$ — see Remark 4.8 below. In the case $t > 4$, equating the RHS of (1.22) multiplied by $N$ with (1.13) we see that by including the exponent $c(t)$ in the former the dip-wavenumber dependence on both $t$ and $N$ reads $k \asymp \frac{1}{c(t)} \log N$. The exponent $c(t)$ is given explicitly as $t\gamma(4/t)$ in (4.4) below. We calculate from (4.5) that for $t \to 4^+$, $c(t) \sim \frac{8}{3}(1 - 4/t)^{3/2}$. Hence its reciprocal diverges in this limit, again in keeping with the distinct dip-wavenumber asymptotic relation for $t = 4$. In the limit $t \to \infty$ we read off from (4.5) that $c(t) \sim t$ and thus the dip-wavenumber asymptotic relation $k \asymp \frac{1}{t} \log N$. The factor $1/t$ acts as a damping of the dip-wavenumber for large $t$. This is in keeping with their being no dip effect in the $t \to \infty$ state of Dyson Brownian motion — referred in random matrix theory as the circular unitary ensemble (CUE); see e.g. [42, Ch. 2] — due to the eigenvalue density then being rotationally invariant.

1.4. **Layout of the paper and main results.** Section 2 presents a self contained derivation of the joint eigenvalue PDF for Dyson Brownian motion on $U(N)$ from the identity. As noted in the recent work [70], the functional form can be identified as an example of a cyclic Polyá ensemble. The significance of this is that the correlation kernel determining the general $k$-point correlation then admits a special structured form — the corresponding theory is revised in Section 3.
Section 4 relates to the density $\rho_{(1),N}(x; t)$ and its moments. Since the former is a periodic function of $x$, period $2\pi$, and even in $x$ it admits the Fourier expansion

$$
\frac{2\pi}{N} \rho_{(1),N}(x; t) = \sum_{k=-\infty}^{\infty} m_k^{(N)}(t)e^{ikx} = 1 + 2 \sum_{k=1}^{N} m_k^{(N)}(t) \cos kx,
$$

(1.25)

where the $\{m_k^{(N)}(t)\}$ are real and specified by

$$
m_k^{(N)}(t) := \frac{1}{N} \int_{-\pi}^{\pi} \rho_{(1),N}(x; t) e^{-ikx} dx.
$$

In this context, the Fourier coefficients $\{m_k^{(N)}(t)\}$ are referred to as the moments of the eigenvalue density. Our first new result is to detail the derivation of an exact evaluation of the moments, stated in some early literature, but without derivation.

**Proposition 1.2.** (Onofri [86] and Andrews and Onofri [3], both after correction) We have

$$
m_k^{(N)}(t) = q^{k(N+k+1)} \ {}_{2}F_{1}(1-N,1-k;2;1-q^{-2k}),
$$

(1.26)

where $\ {}_{2}F_{1}(a,b;c;z)$ denotes the Gauss hypergeometric function and $q = e^{-t/2N}$.

In fact two different derivations of (1.26) are given. The first involves Schur function averages, and is the one stated in [3, 86] as giving rise to (1.26) but with the details omitted. A novel aspect of our presentation is the use of the cyclic Pólya ensemble structure to compute the Schur function average. The second is to make use of the cyclic Pólya ensemble structure for the density itself. This is both quicker and more straightforward than using Schur function averages. The exact formula (1.26) allows for the large $N$ asymptotic form of the $m_k^{(N)}(t)$ to be determined in the regime that $k/N = \mu > 0$ is fixed.

**Corollary 1.3.** Fix $k/N = \mu > 0$. Define

$$
t^* = t^*(\mu) := \frac{2}{\mu} \log \left| \frac{1 + \mu}{1 - \mu} \right|,
$$

(1.27)
For $0 < t < t^*$ we have

$$m_k^{(N)}(t) = \frac{(-1)^N}{N^{3/2}} \sqrt{\frac{2}{\pi}} e^{-\mu t/2} \left(\frac{(1 - e^{-\mu t})\mu}{((1 - e^{-\mu})(\mu + 1)^2 e^{-\mu t} - (\mu - 1)^2))^{1/4}} \times \cos(N\hat{h}(t, \mu) + \pi/4) + O\left(\frac{1}{N^{5/2}}\right)\right), \quad (1.28)$$

where $\hat{h}(t, \mu) = h(\lambda, \mu)|_{\lambda=e^{-\mu t/2}}$ with $h(\lambda, \mu)$ given by (4.30) below.

On the other hand, for $t > t^*$ we have that $m_k^{(N)}(t)$ decays exponentially fast in $N$.

The proof is given in Section 4.4. The relevance of Corollary 1.3 to the slope regime comes about by its validity in an extended regime $k, N \to \infty$ with $k \ll N$, which is also proved in Section 4.4.

**Corollary 1.4.** Suppose $t < 4$, and let $A(t)$ be the amplitude of the square root singularity at the endpoints of the support of $\rho_{(1),N}(x;t)$; recall the text below (1.10). For $k, N \to \infty$ and $k \ll N$ we have

$$m_k^{(N)}(t) \sim \frac{\sqrt{\pi}}{(N\mu)^{3/2}} A(t) \cos(kL_0(t) - 3\pi/4 + O(k^2/N)), \quad (1.29)$$

as is in agreement with (1.11).

The topic of Section 5 is the calculation of the spectral form factor $S_N(k,t)$, and its $N \to \infty$ asymptotic limits, first for $k$ fixed, and then for $k$ proportional to $N$. In Proposition 5.3, Eq. (5.12), $S_N(k,t)$ is expressed in terms of the an integral over a variable $s$, where the key factor in the integrand can be identified with $(m_k^{(N)}(s + t))^2$, known explicitly according to (1.26). From this, the $N \to \infty$ limit is almost immediate. Further simplification then leads to our first limit formula in relation to the structure function.

**Theorem 1.5.** We have

$$\lim_{N \to \infty} S_N(k,t) = k - e^{-kt} \sum_{s=0}^{k-1} (k-s)\left(L_s^{(-1)}(kt)\right)^2, \quad (1.30)$$

where $L_s^{(a)}(z)$ denotes the Laguerre polynomial.

Next considered is the regime that $k/N =: \mu$ is fixed for $N \to \infty$, as is relevant for dip-ramp-plateau. The same strategy as used to deduce Corollary 1.3, which proceeds by rewriting the $2F_1$ function in (1.26) in terms of a Jacobi polynomial with parameters $(N(\mu-1),1)$, then makes use of known asymptotics of the latter reported in the literature [92], suffices to deduce the corresponding limit theorem.

**Theorem 1.6.** Define $t^* = t^*(\mu)$ by (1.27) and require that $\mu > 0$. For $0 < t < t^*$ we have

$$\tilde{S}_\infty(\mu; t) := \lim_{N \to \infty} \frac{1}{N} S_N(k,t)\bigg|_{\mu=k/N} = \min(\mu, 1) - \frac{\mu^3}{\pi(\mu + 1)} e^{-\mu t} \int_0^{(t^*-t)_+} \frac{se^{-\mu s}}{(1 - e^{-\mu(s+t)})^{3/2} \sqrt{e^{-\mu(s+t)} - e^{-\mu t}}} ds, \quad (1.31)$$

where $(t^*-t)_+ = t^* - t$ for $t^* - t > 0$, and $(t^*-t)_+ = 0$ otherwise.
Figure 3. [Colour online] Log-log plot of (1.8), converted to a continuous function of $\mu = k/N$ for large $N$ according to the procedure of the text, with $N = 20$ and various values of $t$. In order of the maximum height of the oscillations, or equivalently sharpest dip, these are $t = 6$ (green), $t = 4$ (yellow) and $t = 2$ (blue).

The (continuous) dip-ramp-plateau effect is evident; cf. Figure 2.

As must be, $\tilde{S}_\infty(\mu; t)$ is well defined for continuous $\mu > 0$. Less immediate, but similarly true, is that this is a feature of the asymptotic formula (1.28) for $m_k^{(N)}(t)$, even though for finite $N$ the Fourier coefficient label $k$ must be discrete. Consequently a continuous version of dip-ramp-plateau can be defined, which is the (log-log) plotting of (1.8) as a function of $\mu$, with the first term on the RHS replaced by $N$ times the limit formula of Theorem 1.6, and the second term replaced by $N^2$ times the large $N$ asymptotics of $(m_k^{(N)}(t))^2$. From a practical viewpoint of performing the plot, since the latter has not been made explicit for $t \geq 4$, we replace this term by the finite $N$ Jacobi polynomial form (4.26) below with $k = \mu N$, as this prescription leads to the same asymptotic formulas; see Figure 3 for some examples.

2. The joint eigenvalue PDF

Dyson showed that Brownian motion associated with the group $U(N)$ gives that the eigenvalue PDF, $p_\tau = p_\tau(e^{ix_1}, \ldots, e^{ix_N})$ — this quantity is a function too of the initial conditions — evolves according to the Fokker-Planck equation (referred to by Dyson as the Smoluchowski equation)

$$
\gamma \frac{\partial p_\tau}{\partial \tau} = \mathcal{L} p_\tau, \quad \mathcal{L} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( \frac{\partial W}{\partial x_j} + \frac{1}{\beta} \frac{\partial}{\partial x_j} \right),
$$

(2.1)

where $\gamma$ is a scale for the time like parameter $\tau$, $\beta = 2$, and

$$
W = - \sum_{1 \leq j < k \leq N} \log |e^{ix_k} - e^{ix_j}|.
$$

(2.2)

As pointed out in [35], this equation permits the interpretation of an $N$ particle system on the unit circle, with particles interacting pairwise via the potential $-\log |e^{ix} - e^{iy}|$, and executing
overdamped Brownian motion in a fictitious viscous fluid with friction coefficient $\gamma$ at inverse temperature $\beta$. The statistical properties of the eigenvalues at a particular time like parameter $\tau$ are completely determined by $p_\tau$, which in turn requires solving (2.1) subject to a prescribed initial condition. This is an essential point of Dyson’s original work [32]. With the Fokker-Planck equation written in its equivalent form as coupled stochastic differential equations, this relation was further studied in [19]; see too [13, §2.2].

A fundamental result of Sutherland [91] identifies a similarity transformation that maps the Fokker-Planck operator $\mathcal{L}$ to a Schrödinger operator $H$. Thus we have

$$-e^{\beta W/2} \mathcal{L} e^{-\beta W/2} = \frac{2}{\beta} (H - E_0)$$

with

$$E_0 = \left( \frac{\beta}{2} \right)^2 \frac{N(N^2 - 1)}{24}$$

and

$$H = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\beta}{2}(\beta/2 - 1) \sum_{1 \leq j < k \leq N} \frac{1}{(2 \sin(x_k - x_j)/2)^2}.$$  

Notice that for $\beta = 2$, as required for Dyson Brownian motion on $U(N)$, the interaction term in (2.5) vanishes.

In light of (2.3) and (2.5), knowledge of the free fermion Green function solution of the imaginary time Schrödinger equation on a circle allows for the computation of $p_\tau$ in the case of the initial condition

$$p_\tau(x) \big|_{\tau=0} = \prod_{i=1}^{N} \delta(x_i - x_i^{(0)}), \quad (-\pi < x_1^{(0)} < \cdots < x_N^{(0)} \leq \pi).$$

Let us write $p_\tau(x; x^{(0)})$ to indicate this initial condition. We will compute the functional form of $p_\tau(x; x^{(0)})$ by first calculating $p_\tau(x; x^{(0)})$ in terms of a determinant and then taking the limit $x^{(0)} \to 0$. In preparation, introduce the Jacobi theta functions

$$\theta_2(z; q) := \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} e^{2i\pi(n-1/2)}, \quad \theta_3(z; q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{2i\pi n}. \quad (2.7)$$

Scale $\tau$ by setting $\tau/\gamma = t/N$, and relate $q$ in (2.7) to $t/N$ by setting

$$q = e^{-t/2N}. \quad (2.8)$$

**Proposition 2.1.** (Liechty and Wang [76], Kieburg et al. [70]) With $\kappa = 2$ for $N$ odd and $\kappa = 3$ for $N$ even and $q$ as in (2.8) we have

$$p_\tau(x; 0) = \frac{q^{N(N^2-1)/12}}{(2\pi)^N N!} \prod_{1 \leq j < k \leq N} \sin(x_k - x_j)/2 \det \left[ \left( -2 \frac{\partial}{\partial x_j} \right)^{k-1} \theta_{k}(x_j/2; q) \right]_{j,k=1,\ldots,N}. \quad (2.9)$$

**Proof.** The solution of the single particle imaginary time Schrödinger equation

$$\frac{\gamma}{\partial \tau} g_\tau = -\frac{1}{2 \partial^2 x^2} g_\tau, \quad -\pi < x < \pi$$

with initial condition $g_\tau|_{x=x^{(0)}} = \delta(x - x^{(0)})$ as $\tau \to 0^+$ is

$$g_\tau(x; x^{(0)}) = \begin{cases} \frac{1}{2\pi} \theta_3((x - x^{(0)})/2; q), & \text{periodic b.c.} \\ \frac{1}{2\pi} \theta_2((x - x^{(0)})/2; q), & \text{anti-periodic b.c.} \end{cases}.$$


where $q = e^{-\tau/2\gamma}$. This can be verified directly, with the initial condition following from the functional form of the theta functions written in conjugate modulus form; see [99].

Forming a Slater determinant gives that the Green function free fermion solution of the $N$-particle imaginary time free Schrödinger equation

$$
\gamma \frac{\partial}{\partial \tau} g_\tau = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} g_\tau, \quad -\pi < x_j < \pi
$$

is

$$
g_\tau(x; x^{(0)}) = \begin{cases}
\det \left[ \frac{1}{2\pi} \theta_3((x_j - x_k^{(0)})/2; q) \right]_{j<k=1}^{N}, & \text{periodic b.c.} \\
\det \left[ \frac{1}{2\pi} \theta_2((x_j - x_k^{(0)})/2; q) \right]_{j<k=1}^{N}, & \text{anti-periodic b.c.}
\end{cases} \quad (2.10)
$$

The significance of knowledge of $g_\tau(x; x^{(0)})$ is that it follows from (2.3) with $\beta = 2$ that

$$
p_\tau(x; x^{(0)}) = e^{E_0|\beta=2\tau/\gamma|} \left( \prod_{1 \leq j < k \leq N} \sin((x_k - x_j)/2) \right) g_\tau(x; x^{(0)}). \quad (2.11)
$$

Consequently, requiring that $p_\tau(x; x^{(0)})$ exhibits periodic boundary conditions with respect to the translation $x_j \mapsto x_j + 2\pi$ and setting $\tau/\gamma = t/N$ we have [40, 41, 87]

$$
p_t(x; x^{(0)}) = q^{N(N^2-1)/12} \left( \prod_{1 \leq j < k \leq N} \sin((x_k - x_j)/2) \right) \det \left[ \frac{1}{2\pi} \theta_3((x_j - x_k^{(0)}/2); q) \right]_{j<k=1,\ldots,N}, \quad (2.11)
$$

where $\kappa = 2$ for $N$ even and $\kappa = 3$ for $N$ odd and now $q$ is given by (2.8). Here the different functional forms depending on the parity of $N$ can be traced back to the product over pairs in the numerator of the RHS of (2.11) being multiplied by the sign $(-1)^{N-1}$ upon the mapping $x_j \mapsto x_j + 2\pi$. With $\theta_2$ for $N$ even, and $\theta_3$ for $N$ odd, the determinant factor on the RHS of (2.11) has the same property as noted in (2.10), implying that $p_\tau$ itself is always periodic. Also, in keeping with the ordering in (2.6) and thus the implied normalisation, the normalisation associated with (2.11) is so that $\int_R p_\tau(x) \, dx = 1$, where the region $R$ is specified by $-\pi < x_1 < \cdots < x_N < \pi$.

Application of L'Hôpital's rule in (2.11) to take $x_j^{(0)} \to 0$ for $j = 1, 2, \ldots, N$ in succession gives (2.9). A further factor of $1/N!$ relative to (2.11) has been included to allow the normalisation condition to become $\int_{[-\pi, \pi]^N} p_t(x; 0) \, dx = 1$.

3. Cyclic Pólya ensemble structure

3.1. Definition of a cyclic Pólya ensemble. The eigenvalue PDF (2.9) has the structural property of being proportional to

$$
\left( \prod_{1 \leq j < k \leq N} \sin(x_k - x_j)/2 \right) \det \left[ \frac{\partial^{k-1}}{\partial x_j^{k-1}} \hat{w}(x_j) \right]_{j,k=1,\ldots,N}, \quad (3.1)
$$

for a particular $\hat{w}(x)$. Such a form has been isolated in the recent work [70]. This was in the context of a study of the implications of the theory of matrix spherical transforms on $U(N)$, as applied to multiplicative convolutions conserving a determinant structure, first presented in [105].

By setting $z_j = e^{ix_j}$ it is easy to see that (3.1) has the equivalent complex form

$$
\frac{1}{Z_N} \Delta_N(z) \det \left[ \left( -z_j \frac{\partial}{\partial z_j} \right)^{k-1} w(z_j) \right]_{j,k=1,\ldots,N}, \quad (3.2)
$$
Here \( w(z) = z^{-(N-1)/2} \hat{w}(z) \), \( Z_N \) is the normalisation constant which can be determined for general \( N \) (see (3.9) below), and \( \Delta_N(z) \), which is referred to as the Vandermonde product, is specified by

\[
\Delta_N(z) := \prod_{1 \leq j < k \leq N} (z_k - z_j). \tag{3.3}
\]

The range of \( z \) is

\[
z \in \mathbb{S}_1^N := \{ z \in \mathbb{C}^N : \forall j = 1, \ldots, N, |z_j| = 1 \}, \tag{3.4}
\]

where the permutation invariance relaxes the ordering of eigenvalues. By requiring \( w(z) \) to be a \((2M + \chi - 1)\)-differentiable cyclic Pólya frequency function of order \( N \) (odd with \( \chi = 1 \) or even with \( \chi = 0 \), multiplied by \( z^{-M-\chi+1} \) [105, Sec. 2.5], one can ensure the positivity of (3.2), and hence specify an ensemble with its eigenvalue PDF of this general form; (3.2) is called a cyclic Pólya ensemble. For technical purposes, we further restrict the weight function \( w \) to be in the set

\[
\tilde{L}^1(\mathbb{S}_1) := \{ w \in L^1(\mathbb{S}_1) : w(z)^* = z^{-N} \hat{w}(z) \text{ and } \partial_z^2 w(z) \in L^1(\mathbb{S}_1) \text{ for } j = 0, \ldots, N-1 \}; \tag{3.5}
\]

see [70] for further details. This allows us to construct bi-orthogonal systems of functions corresponding to each cyclic Pólya ensemble. However, there are further aspects of the theory that need to be revised before doing this.

3.2. Spherical transform and closure under multiplicative convolution. One distinctive feature of cyclic Pólya ensembles is that they are closed under multiplicative convolution. Thus a product of two random matrices drawn from this class of ensembles is still a random matrix drawn from this class of ensembles. To revisit the underlying theory [70], we first introduce the spherical transform of a \( U(N) \) random matrix with eigenvalue PDF \( f \). This is defined as

\[
f_S(s) := \int_{\mathbb{S}_1^N} \left( \prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j} \right) f(z) \Phi(z, s), \quad s_j \neq s_k \text{ for any } j \neq k, s_j \in \mathbb{Z}, \tag{3.6}
\]

where \( \frac{dz_j}{2\pi i z_j} \) is the uniform measure on the \( j \)-th unit circle \( \mathbb{S}_1 \), and \( \Phi \) is the normalised character of irreducible representations of \( U(N) \). The latter has the explicit form

\[
\Phi(z, s) := \left( \prod_{j=0}^{N-1} j! \right) \frac{\det[z_j^{s_k}]_{j,k=1}^N}{\Delta_N(z) \Delta_N(s)}. \tag{3.7}
\]

There are two important properties of the spherical transform. First, it possesses an inversion formula [70, (2.2.12)], and therefore every \( f_S \) is uniquely associated with the eigenvalue PDF \( f \). Second, it also admits a convolution formula. Thus, for independent random matrices \( U_1, U_2 \) with eigenvalue PDFs \( f_{U_1} \) and \( f_{U_2} \), the spherical transform of the eigenvalue PDF \( f_{U_1 U_2} \) of the product \( U_1 U_2 \) has the factorisation

\[
f_{U_1 U_2}(s) = f_{U_1}(s) f_{U_2}(s). \tag{3.8}
\]

These two properties allow us to obtain the eigenvalue PDF of the product \( U_1 U_2 \) by inverting the right side of (3.8).

We are now in a position to address the question of closure with respect to multiplicative convolution. First of all, integrating (3.2) shows that the normalisation constant is given by

\[
Z_N = N! \prod_{j=0}^{N-1} j! \hat{w}(j). \tag{3.9}
\]
Here $S_w$ is the one-dimensional spherical transform of the weight function $w$, which correspond to the coefficients of the Fourier series expansion of $w$. To show closure under multiplicative convolution, an explicit calculation tells us that the spherical transform of a cyclic Pólya ensemble of the form (3.2) is given by
\[
Sf(s) = \prod_{j=1}^{N} \frac{Sw(s_j)}{Sw(j-1)},
\] (3.10)
(see [70, Prop. 4]). From the convolution formula, it is now not difficult to see that
\[
S_{U_1U_2}(s) = \prod_{j=1}^{N} \frac{Sw_{U_1}(s_j)Sw_{U_2}(s_j)}{Sw_{U_1}(j-1)Sw_{U_2}(j-1)} = \prod_{j=1}^{N} \frac{Sw_{U_1} * Sw_{U_2}}{Sw_{U_1} * Sw_{U_2}}(j-1),
\] (3.11)
where $w_{U_1} * w_{U_2}$ denotes the univariate multiplicative convolution of $w_{U_1}$ and $w_{U_2}$. Explicitly,
\[
 w_{U_1} * w_{U_2}(z) := \int_{S_1} w_{U_1}(zy^{-1})w_{U_2}(y) \frac{dy}{2\pi i}.
\] (3.12)
As (3.11) has the same structure as (3.10), one concludes that the product $U_1 U_2$ is also a cyclic Pólya ensemble with a new weight function $w_{U_1} * w_{U_2}$.

3.3. Biorthogonal system. Another important feature of the cyclic Pólya ensemble is that it corresponds to a determinantal point process, which in turn can be characterised by a bi-orthogonal system. With the formula for the ratio of gamma functions
\[
\frac{\Gamma(j-l)}{\Gamma(-l)} = (-1)^{j-1} \frac{\Gamma(l+1)}{\Gamma(l-j+1)}, \quad l \in \mathbb{Z},
\] (3.13)
we introduce a set of pairs of functions $\{(P_j, Q_j)\}_{j=0,\ldots,N-1}$, where
\[
P_j(z_1) := \sum_{k=0}^{j} \frac{1}{(j-k)!} \frac{(-z_1)^k}{Sw(k)},
\]
\[
Q_j(z_2) := z_2^j \partial_{z_2}^j z_2^{-1} w(z_2) = \lim_{\epsilon \to 0^+} \sum_{l \in \mathbb{Z} \setminus \{0,\ldots,j-1\}} \frac{\Gamma(j-l)}{\Gamma(-l)} Sw(l) z_2^{-l} e^{-\epsilon(l+1-N)l}
\] (3.14)
for $j = 1,\ldots,N-1$. In the formula for $Q_j$ the factor $e^{-\epsilon(l+1-N)l}$ is a regularisation. This is required to ensure convergence of the infinite sum in the case of general weights $w(z)$. It is shown in [70] that both $\{P_j(z_1)\}_{j=0,\ldots,N-1}$ and $\{z_1^{-l}\}_{j=1,\ldots,N}$ span the same vector space, and so do both $\{Q_j(z_2)\}_{j=0,\ldots,N-1}$ and $\{(z_2 \partial_{z_2})^{j-1} w(z_2)\}_{j=1,\ldots,N}$. Moreover $\{P_j\}$ and $\{Q_j\}$ have been constructed to form a bi-orthogonal set with respect to the Haar measure on $S_1$. Thus
\[
\int_{S_1} \frac{dz}{2\pi i z} P_a(z) Q_b(z) = \delta_{a,b},
\] (3.15)
where $\delta_{a,b}$ denotes the Kronecker delta function. For this reason, we say that $\{(P_j, Q_j)\}_{j=0,\ldots,N-1}$ is the corresponding bi-orthogonal system for the cyclic Pólya ensemble.

From the linearity of determinants in (3.2), the span property of the polynomials $\{P_j(z_1)\}_{j=0,\ldots,N-1}$ and the functions $\{Q_j(z_2)\}_{j=0,\ldots,N-1}$ enables us to rewrite the PDF as
\[
\frac{1}{N!} \operatorname{det} [P_{j-1}(z_k)]_{j,k=1}^N \operatorname{det} [Q_{j-1}(z_k)]_{j,k=1}^N.
\] (3.16)
We can check that (3.16) is correctly normalised. For this we recall Andréief’s identity (see e.g. [43]), which gives
\[
\frac{1}{N!} \int_{\mathbb{S}^1} \left( \prod_{j=1}^N \frac{dz_j}{2\pi i z_j} \right) \det[P_{j-1}(z_k)]_{j,k=1}^N \det[Q_{j-1}(z_k)]_{j,k=1}^N = \det \left[ \int_{\mathbb{S}^1} \frac{dz_j}{2\pi i z_j} P_{j-1}(z) Q_{k-1}(z) \right]_{j,k=1,\ldots,N}.
\]
Application of (3.15) shows that the matrix on the RHS is in fact the identity, and thus the determinant is equal to unity, as required.

3.4. **Correlation kernel.** Associated with any bi-orthogonal system for a determinantal point process is its correlation kernel \(K_N(z, z').\) By definition, the corresponding \(m\)-point correlation function can be expressed as an \(m\)-dimensional determinant of this kernel,
\[
\rho_{(m),N}(z_1, \ldots, z_m) = \det[K_N(z_j, z_k)]_{j,k=1}^m. \tag{3.17}
\]
Starting from the expression (3.16), it is a standard fact that \(K_N(z, z') = \sum_{j=1}^N P_{j-1}(z) Q_{j-1}(z').\) \(\tag{3.18}\)
It is shown in [70] that with the substitution (3.14), (3.18) permits the rewrite
\[
K_N(z_1, z_2) = \sum_{k=0}^{N-1} (z_1 z_2^{-1})^k + \lim_{\epsilon \to 0^+} \sum_{k=0}^{N-1} \sum_{l \in \mathbb{Z} \setminus \{0, \ldots, N-1\}} \frac{\Gamma(N-l)}{\Gamma(-l) \Gamma(N-k) \Gamma(k+1)} \frac{S_{w(l)} (-z_1)^k z_2^{-l}}{S_{w(k)} k-l} e^{-\epsilon(l+1-N)l}, \tag{3.19}
\]
where as in the second expression of (3.14) the factor involving \(\epsilon\) is included to ensure convergence of the infinite sum for general weights \(w(z).\)

**Remark 3.1.** Using the identity
\[
\prod_{1 \leq j < k \leq N} \sin(x_k - x_j)/2 \propto \prod_{i=1}^N z_i^{-(N-1)/2} \Delta_N(z),
\]
as implicit in the passage from (3.1) to (3.2), shows that for general initial conditions \(z^{(0)}\) the PDF (2.11) has the form (3.16). From the general theory of biorthogonal ensembles in random matrix theory [12] the correlation functions are again of the form (3.17), albeit with \(K_N\) now dependent on \(z^{(0)};\) see [95] for results in this direction.

3.5. **Application to the PDF** (2.11). We conclude this section by presenting the cyclic Pólya ensemble structure and the correlation kernel for (2.11), which is a direct corollary of the theory presented in [70], as revised above.

**Corollary 3.2 ([70, special case of Prop. 19]).** With the change of variable \(z_j = e^{ix_j},\) the PDF (2.9) becomes a cyclic Pólya ensemble with weight function
\[
w(z) = \sum_{s=-\infty}^{\infty} q^{s-(N-1)/2} z^{-s}, \quad q = e^{-t/2N}. \tag{3.20}
\]
The correlation kernel corresponding to (2.9) is

\[
K_N(x, y) = \frac{1}{2\pi} \left( \frac{e^{iN(x-y)}}{e^{i(x-y)}} - 1 \right) + \sum_{k=0}^{N-1} \sum_{l \in \mathbb{Z} \setminus \{0, \ldots, N-1\}} \frac{\Gamma(N-l)q^{(l-(N-1)/2)^2-(k-(N-1)/2)^2} (-1)^ke^{i(2k+1)y}}{\Gamma(l)\Gamma(-k)\Gamma(k+l)}.
\]

(3.21)

Compared to (3.19), here the regularisation is removed.

**Proof.** The spherical transform of \( w \) is read off from the coefficients of its Fourier series form (3.20). Thus

\[
\mathcal{S}w(s) = q^{(s-(N-1)/2)^2}.
\]

(3.22)

With \( z_j = e^{ix_j} \), upon recalling (3.9), we see that (3.2) can be rewritten as

\[
\frac{1}{q^{N(N^2-1)/12}} \prod_{j=1}^{N} \left( e^{ix_j} - e^{ix_j} \right) \det \left[ \left( i \frac{\partial}{\partial x_j} \right)^{k-1} w(e^{ix_j}) \right]_{j,k=1,\ldots,N}.
\]

(3.23)

The Haar measure \( \frac{dx_j}{2\pi} \) is replaced accordingly by \( \frac{dx_j}{2\pi} \), which is responsible for the factor of \( \frac{1}{2\pi} \) in (3.21) since there we take \( dx_j \) as the reference measure. Now one notices that the Vandermonde product can be replaced by

\[
\prod_{j<k} \left( e^{ix_k} - e^{ix_j} \right) = e^{(N-1)i\sum_{j=1}^{N} x_j/2} \prod_{j<k} 2i \sin \left( \frac{x_k - x_j}{2} \right).
\]

(3.24)

Also, it can be checked that the Jacobi theta functions (2.7) have the unified form

\[
\theta_{\kappa} \left( \frac{y}{\kappa} ; q \right) = \exp \left( i y \frac{N-1}{2} \right) w(e^{iy}),
\]

(3.25)

for \( N \) being even with \( \kappa = 2 \) and \( N \) being odd with \( \kappa = 3 \). Therefore taking the derivatives of the theta functions and applying a row reduction to reduce every entry to be the highest derivative of \( w \), one has

\[
\det \left[ \left( i \frac{\partial}{\partial x_j} \right)^{k-1} \theta_{\kappa} \left( \frac{x_j}{2} ; q \right) \right]_{j,k=1,\ldots,N} = e^{(N-1)i\sum_{j=1}^{N} x_j/2} \det \left[ \left( i \frac{\partial}{\partial x_j} \right)^{k-1} w(e^{ix_j}) \right]_{j,k=1,\ldots,N}.
\]

(3.26)

This is (3.2). To obtain the correlation kernel, one substitutes \( z_j = e^{ix_j} \) into (3.19).

It remains to show that the \( l \)-sum is absolutely convergent such that we can drop the limit \( \epsilon \to 0^+ \) in (3.21). After decomposing the \( l \)-sum into two sums, where one consists of positive \( l \) and the other one is for negative \( l \), we need to verify that both series with all positive terms

\[
S_1 := \sum_{l=0}^{N} \frac{\Gamma(l+1)}{\Gamma(l-N+1)} q^{(l-(N-1)/2)^2 - (l-k)^2} \quad S_2 := \sum_{l=1}^{N} \frac{\Gamma(N+l)}{\Gamma(l)} q^{(l+N-1)/2} \frac{(l+N-1)^2}{k+l}
\]

(3.27)

are convergent for \( k \in \{0, \ldots, N-1\} \) and \( q < 1 \). A ratio test suffices to give this claim. \( \square \)

**Remark 3.3.**

1. In relation to multiplicative convolution of two matrices \( U_1, U_2 \) drawn from the PDF (2.9), one with \( q_1 = e^{-t_1/2N} \), and the other with \( q_2 = e^{-t_2/2N} \), the theory in the final paragraph of Section 3.2 is applicable. Thus the PDF corresponding to \( U_1U_2 \) is again a cyclic Pólya ensemble with weight
function computed from (3.12), in keeping with the laws of unitary Brownian motion forming a semigroup under the multiplicative convolution
\[ f^{(U_1)}_N * f^{(U_1)}_N = \int_{U(N)} f^{(U_1)}_N(U) f^{(U_1)}_N(UU') \, d\mu(U'), \]
where \( f^{(U)}_N \) denotes the PDF. This gives back the same weight (3.20), but with parameter \( q = q_1q_2 \), or equivalently with time parameter \( t = t_1 + t_2 \). We remark that this in turn coincides with the fact that in the numerical construction (1.1) the Brownian motions have Gaussian increments with variance depending only on the length of the time interval.

2. Substituting (3.22) in the first equation of (3.14) shows
\[ P_j(z) = q^{-(N-1)^2/4} \sum_{k=0}^j \frac{1}{(j-k)!k!} q^{k(N-1-k)} (-z)^k. \]
One sees in particular that
\[ z^{-n/2} P_n(z) \bigg|_{N=n+1} \propto \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} q^{k(n-k)} z^{(k-n)/2} =: \tilde{H}_n(z; q). \]

The RHS defines the so-called unitary Hermite polynomials (terminology from [66]). They first appeared, in a random matrix context at least, in the thesis of Mirabelli [82] on finite free probability [79]. They are extensively studied in the recent work of Kabluchko [66] from various viewpoints; see Remark 4.1 below for more on this. We can use (3.28) to deduce a formula for \( \tilde{H}_n(z; q) \) expressed as an average with respect to the PDF (2.9). First, in relation to the polynomials \( \{P_j(z)\} \) generally, we observe the representation as an average
\[ P_l(z) = \left( \prod_{k=1}^n (z - z_k) \right). \]
Here \( \langle \cdot \rangle \) is with respect to the PDF (3.2), but where the size of the Vandermonde product and the determinant are reduced from \( N \) to \( l \), although with \( w(z) \) unchanged and thus still dependent on \( N \). Replace this latter \( N \) by \( n+1 \), then set \( l = n \). As in verifying the equality between (3.1) and (3.2), one can then check that the LHS of (3.28), and thus up to proportionality \( \tilde{H}_n(e^{ix}; q) \), admits the form
\[ \langle \prod_{k=1}^n \sin((x - x_k)/2) \rangle, \]
where the average is with respect to the PDF (2.9) with \( N = n \).

4. THE DENSITY AND ITS MOMENTS

4.1. Known results. The eigenvalue density, \( \rho_{(1),N}(x; t) \) say, corresponding to (2.9) is specified in terms of \( p_t(x; 0) \) by
\[ \rho_{(1),N}(x; t) = N \int_{[-\pi,\pi]^{N-1}} p_t(x; 0) \, dx_2 \cdots dx_N. \]
Within random matrix theory the corresponding moments (1.25) were first considered by Biane [10] using methods of free probability theory, and independently by Rains [88]. Specifically the exact evaluation for \( N \to \infty \)
\[ m_k^{(\infty)}(t) := \lim_{N \to \infty} m_k^{(N)}(t) = \frac{e^{-kt/2}}{k} L_k^{(-1)}(kt) = e^{-kt/2} {}_1F_1(1-k;2;kt), \quad k \geq 1, \]
was obtained. Here \( L_n^{(\mu)}(x) \) denotes the Laguerre polynomial of degree \( n \) with parameter \( \mu \), and \( {}_1F_1(a; b; x) \) denotes the confluent hypergeometric function with parameters \( a, b \) and argument \( x \), which like the Laguerre polynomial can be defined by its power series expansion.

It turns out that the same result (4.2) can already be found in the literature on exactly solvable low-dimensional field theories from the early 1980’s [89] (see also [68] for the cases \( n = 2, 3 \). The
point here is that these field theories can be reduced to Dyson Brownian motion on $U(N)$ (also referred to as the heat kernel measure on $U(N)$), with $\langle \text{Tr} U^k \rangle$ an observable relating to Wilson loops. Moreover, soon after the exact formula (1.26) for the finite $N$ case was found [3, 86], although the result was stated without derivation and contains misprints; we reference [57, Eq. (3.14) after correction] for a more recent work relating to this result, also in a field theory context. From the series form of the Gauss hypergeometric function $\, _2F_1(a, b; c; z)$ in (1.26), which with $b = 1 - k$ terminates to give a polynomial in $z$ of degree $k - 1$, the second expression in (4.2) is reclaimed in the limit $N \to \infty$ upon recalling from (2.8) that $q = e^{-t/2N}$.

Defining

$$\rho_{(1),\infty}(x; t) := \lim_{N \to \infty} \frac{2\pi}{N} \rho_{(1),N}(x; t) = 1 + 2 \sum_{k=1}^{\infty} m_k(\infty)(t) \cos kx,$$

(4.3)

substituting (4.2) does not lead to a closed expression by way of evaluation of the summation over $k$. Instead, one source of analytic information comes from consideration of the large $k$ form of $m_k(\infty)$. Thus it follows from (4.2) that for $t < 4$ the rate of decay is $O(k^{-3/2})$ and is consistent with (1.10) [59]. This is no longer true for $t \geq 4$. Explicitly, for $t > 4$ the asymptotic analysis of the Laguerre polynomial given in [59] implies

$$m_k(\infty)(t) \sim \frac{t}{\sqrt{2\pi k}} \left(1 - \frac{4}{t}\right)^{-1/4} e^{-kt/2},$$

(4.4)

where

$$\gamma(x) := \sqrt{1 - x} - \frac{x}{2} \log \frac{1 + \sqrt{1 - x}}{1 - \sqrt{1 - x}},$$

(4.5)

giving that the rate of decay is exponentially fast [59]. At $t = 4$ the decay is $O(k^{-4/3})$ [73]. Moreover, the support of the density for $t < 4$ is $[-L_0(t), L_0(t)]$ where [10, 34]

$$L_0(t) = \frac{1}{2} \sqrt{t(4 - t)} + \arccos (1 - t/2),$$

(4.6)

while for $t > 4$ the support is the full interval $[-\pi, \pi]$ with $\rho_{(1),\infty}(x; t) \to 1$ as $t \to \infty$.

Next we summarise results from [10], following the clear presentation in [62]. Set

$$z = e^{ix}, \quad \rho_{(1),\infty}(x; t) \mapsto \rho_{(1),\infty}(z; t),$$

and introduce the Herglotz transform

$$H_t(w) = \int_{\mathcal{C}} \frac{z + w}{z - w} \rho_{(1),\infty}(z; t) \frac{dz}{2\pi iz},$$

(4.7)

where the contour $\mathcal{C}$ is the unit circle in the complex $z$-plane, traversed anti-clockwise. With

$$\psi_t(w) = \int_{\mathcal{C}} \frac{w}{z - w} \rho_{(1),\infty}(z; t) \frac{dz}{2\pi iz} = \sum_{p=1}^{\infty} w^p m_p(\infty)(t)$$

(4.8)

denoting the moment generating function one sees

$$H_t(w) = 1 + 2\psi_t(w)$$

(4.9)

and thus

$$\rho_{(1),\infty}(w; t) = \text{Re} H_t(w).$$

(4.10)

From knowledge of $\{m_p(\infty)(t)\}$ from (4.2) substituted in (4.8), with the result then substituted in (4.9), we can check that $H_t$ satisfies the partial differential equation

$$\left(\frac{\partial}{\partial t} + w H_{2t}(w) \frac{\partial}{\partial w}\right) H_{2t}(w) = 0,$$

(4.11)
subject to the initial condition $H_0(w) = (1 + w)/(1 - w)$. The equation (4.11) can be identified with the complex Burgers equation in the inviscid limit, and can also be derived by other considerations [50, 87]. It can be checked from (4.11) that $H_t$ satisfies the functional equation

$$\frac{H_t(w) - 1}{H_t(w) + 1} e^{(t/2)H_t(w)} = w. \tag{4.12}$$

Using (4.11), it can be established [11], [1, Appendix A] that for $t < 4$,

$$\rho((1),\infty)(L_0(t) - x; t) \sim A(t) \sqrt{x}, \quad A(t) = \frac{1}{\pi} \sqrt{\frac{2}{\epsilon^{3/2} (4 - \epsilon)^{1/2}}, \tag{4.13}$$

and that for $t = 4$, $\rho((1),\infty)(\pi - x; t) \asymp |x|^{1/3}$ as $x \to 0$. The former is well known in random matrix theory as a characteristic of a soft edge known from the boundaries of the Wigner semi-circle law — see e.g. [42, §1.4] — while the cusp $|x|^{1/3}$ characterises the Pearcey singularity [16, 37, 60]. It is also known that for small $t$ the functional form of the density is well approximated by a Wigner semi-circle functional form, which becomes exact in a scaling limit [66].

**Remark 4.1.** One of the results obtained in [66] in relation to the unitary Hermite polynomials (3.28) concerns their zeros. In particular, it is shown that all the zeros of (3.28) are on the unit circle in the complex $z$-plane, and with $w = e^{ix}$, $q = e^{-t/2n}$, their density for $n \to \infty$ is given by $\rho((1),\infty)(x; t)$. Our expression for the $\tilde{H}_n(z; q)$ as an average with respect to the PDF (2.9) obtained in Remark 3.3.2 gives a different viewpoint on the result for the density. Thus for the average one has the limit formula (see e.g. [55, Lemma 3.3] for justification)

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \prod_{k=1}^{n} \sin((x - x_k)/2) \right)_{q = e^{-t/2n}} = \frac{1}{2\pi} \int_I \log[\sin((y - x)/2)] \rho((1),\infty)(y; t) \, dy, \quad x \notin I, \tag{4.14}$$

where $I \subset [-\pi, \pi]$ is the interval of support of $\rho((1),\infty)$, which interpreted in terms of $\tilde{H}_n(z; q)$ implies the limiting density of zeros result from [66].

**4.2. Schur function average.** Let $z_j = e^{ix_j}$, and let $\langle \cdot \rangle$ denote an average with respect to the PDF (2.9), which is dependent of $q = e^{-t/2n}$. The moments of the spectral density then correspond to computing

$$m_k^{(N)}(t) = \frac{1}{N} \langle \sum_{j=1}^{N} z_j^k \rangle. \tag{4.15}$$

In the theory of symmetric polynomials, $\sum_{j=1}^{N} z_j^k$ is referred to as the power sum, which for $N$ large enough can be used to form a basis [78]. Another prominent choice of basis for symmetric polynomials is the Schur polynomials $\{S_\kappa\}$, labelled by a partition $\kappa = (\kappa_1, \ldots, \kappa_N)$, where $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N \geq 0$. They can be defined in terms of a determinant according to

$$S_\kappa(z_1, \ldots, z_N) = \frac{\det[z_k^{N-j+\kappa_j}]_{j,k=1}^{N}}{\Delta_N(-z)}, \tag{4.16}$$

with $\Delta_N(z)$ specified according to (3.3). In fact one has the identity (see e.g. [78])

$$\sum_{j=1}^{N} z_j^k = \sum_{r=0}^{\min(k-1, N-1)} (-1)^r S_{(k-r,1^r)}(z_1, \ldots, z_N), \tag{4.17}$$
where \((k - r, 1^r)\) denotes the partition with largest part \(\kappa_1 = k - r\), \(r\) parts \((r \leq N - 1)\) equal to 1 and the remaining parts equal to 0. Thus
\[
\left\langle \sum_{j=1}^{N} z_j^k \right\rangle = \min \{ (k-1,N-1) \} \sum_{r=0}^{\min\{k,N-1\}} (-1)^r \langle S_{(k-r,1^r)}(z_1, \ldots, z_N) \rangle. \tag{4.18}
\]
Evaluating the RHS of (4.18) is the method sketched in [3, 86] to obtain (1.26) — details of the calculation were not given. Here we provide the details, showing too that the crucial step of evaluating the Schur polynomial average can be carried out within the general formalism based on cyclic Pólya ensembles.

On this, define the complex form of \(p_t(x;0)\) as implied by (3.2), and denote this by \(f_U\). Let \(s_k = N - k + \kappa_k\) \((k = 1, \ldots, N)\). Recalling the definition of the spherical transform (3.6), we see that
\[
\langle S^\kappa(z_1, \ldots, z_N) \rangle = \frac{\prod_{j=0}^{N-1} j!}{\Delta_N(-s)|_{s_k = N-k+\kappa_k}} \langle S^\kappa_{\kappa}(z_1, \ldots, z_N) \rangle. \tag{4.19}
\]
Hence knowledge of \(\langle S^\kappa_{\kappa}(z_1, \ldots, z_N) \rangle\), which is known for general cyclic Pólya ensembles from (3.10), suffices for the calculation of the Schur polynomial average.

**Proposition 4.2.** In relation to the complex form of \(p_t(x;0)\) as implied by (3.2), we have
\[
\langle S^\kappa(z_1, \ldots, z_N) \rangle = \prod_{1 \leq j < k \leq N} \frac{k - j + \kappa_j - \kappa_k}{k-j} \prod_{j=1}^{N} q^{\kappa_j^2 + (N-2j+1)\kappa_j}, \tag{4.20}
\]
In particular
\[
\langle S_{(k-r,1^r)}(z_1, \ldots, z_N) \rangle = q^{k^2+k(N-1)-2kr} \frac{(N-1+k)!}{(N-1)!!} \frac{(-1)^r (-k+1)_r (-N+1)_r}{r! (-k+1+N)_r}, \tag{4.21}
\]
where \((u)_r := u(u+1) \cdots (u+r-1)\) denotes the increasing Pochhammer symbol.

**Proof.** The result (4.20) follows immediately from (4.19), upon noting
\[
\Delta_N(-s)|_{s_k = N-k+\kappa_k} = \prod_{1 \leq j < k \leq N} (k - j + \kappa_j - \kappa_k)
\]
and noting from (3.10) and (3.22) that
\[
\prod_{j=1}^{N} S(N-j + \kappa_j) S(N-j) = \prod_{j=1}^{N} q^{\kappa_j^2 + (N-2j+1)\kappa_j}.
\]
With \(\kappa = (k-r,1^r)\), a direct calculation shows
\[
\prod_{j=1}^{N} q^{\kappa_j^2 + (N-2j+1)\kappa_j} = q^{k^2+k(N-1)-2kr},
\]
and furthermore
\[
\prod_{1 \leq j < k \leq N} \frac{k - j + \kappa_j - \kappa_k}{k-j} = A_1 A_2 A_3,
\]
As remarked in Section 1.3, the distribution of the ensemble of matrices obtained from Dyson ensembles have a property not shared by (4.22), whereby the sequence of moments satisfy a three expression in terms of hypergeometric polynomials \[6, 30, 53, 100\]. However, these classical ensembles all share the property seen in (4.22) that their moments can be expressed in terms of \(P_{N-1}^{(a,b)}(z)\) and \(\text{Cauchy unitary ensembles all share the property seen in (4.22) that their moments can be expressed in terms of hypergeometric polynomials \[6, 30, 53, 100\]. However, these classical ensembles have a property not shared by (4.22), whereby the sequence of moments satisfy a three

\[
A_1 := \prod_{q=2}^{r+1} \frac{q-2+k-r}{q-1} = \frac{(-1)^r}{r!} (-k+1)_r, \\
A_2 := \prod_{p=2}^{r+1} \prod_{q=r+2}^{N} \frac{q-p+1}{q-p} = \frac{(-1)^r}{r!} (-N+1)_r, \\
A_3 := \prod_{q=r+2}^{N} \frac{q-1+k-r}{q-1} = (-1)^r \frac{r!}{(N-1)! k!} \frac{(N-1+k)!}{(k+1-N)_r}.
\]

The result (4.21) now follows.

\[\square\]

**Corollary 4.3.** With \(q = e^{-t/2N}\) we have

\[
m_k^{(N)}(t) = q^{k^2+k(N-1)} \frac{(N-1+k)!}{k!N!} 2F_1(1-N,1-k;-(k-1+N);q^{-2k}) \\
= q^{k^2+k(N-1)} 2F_1(1-N,1-k;2;1-q^{-2k}). \tag{4.22}
\]

**Proof.** The series expansion of the Gauss hypergeometric function is

\[
2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n. \tag{4.23}
\]

With this noted, the first equality follows by substituting (4.21) in (4.18) then substituting the result in (4.15). To obtain the second equality, we make use of the polynomial identity

\[
2F_1(-a,-b;c;z) = \frac{(c+a+b-1) \cdots (c+b)}{(a+c-1) \cdots c} 2F_1(-a,-b;c-a-b+1+c;1-z), \tag{4.24}
\]

valid for \(a \in \mathbb{Z}_{\geq 0}\). This can be checked by verifying that both sides satisfy the Gauss hypergeometric function differential equation, and agree at \(z = 0\). In fact (4.24) is a special case of the connection formula expressing a solution analytic about \(z = 1\) as a linear combination of the two linearly independent solutions about \(z = 0\); see e.g. [2, Th. 2.3.2]. \(\square\)

**Remark 4.4.**

1. Applying one of the standard Pfaff transformations to the second of the hypergeometric formulas in (4.22) gives a third form

\[
m_k^{(N)}(t) = q^{k^2-k(N-1)} 2F_1(1-N,1+k;2;1-q^{2k}). \tag{4.25}
\]

All of the three forms can be equivalently expressed in terms of Jacobi polynomials \(P_{N-1}^{(a,b)}(z)\) for certain parameters \((a,b)\) and argument \(z\). For future reference, we make explicit note of the Jacobi polynomial form equivalent to (4.25),

\[
m_k^{(N)}(t) = \frac{1}{N} q^{k^2-k(N-1)} P_{N-1}^{(1-k-N)}(2q^{-2k}-1) = \frac{(-1)^{N-1}}{N} q^{k^2-k(N-1)} P_{N-1}^{(k-N,1)}(1-2q^{2k}). \tag{4.26}
\]

2. As remarked in Section 1.3, the distribution of the ensemble of matrices obtained from Dyson Brownian motion on \(U(N)\) at a given time is unchanged by conjugation with a fixed \(U \in U(N)\), telling us that this ensemble exhibits unitary symmetry. The classical Gaussian, Laguerre, Jacobi and Cauchy unitary ensembles all share the property seen in (4.22) that their moments can be expressed in terms of hypergeometric polynomials \[6, 30, 53, 100\]. However, these classical ensembles have a property not shared by (4.22), whereby the sequence of moments satisfy a three
term recurrence. Hypergeometric polynomials and their basic $q$ extensions have also been shown to provide the closed form expression for the moments of certain discrete and $q$ generalisations of ensembles of random matrices with unitary symmetry [24, 25, 46, 52].

4.3. Moment evaluation from the cyclic Pólya density formula. According to (3.17) in the case $m = 1$ and (3.21), the density $\rho_{(1),N}$ for the PDF (2.9) has the explicit functional form

\[
\rho_{(1),N}(x; t) = \frac{1}{2\pi} \left( N + \sum_{k=0}^{N-1} \sum_{l \in \mathbb{Z}\setminus\{0, \ldots, N-1\}} \frac{\Gamma(N-l)q^{l-(N-1)/2^2} - (k-(N-1)/2^2)}{\Gamma(-l)\Gamma(N-k)\Gamma(k+1)} \frac{(-1)^ke^{ix(k-l)}}{k-l} \right).
\]

This provides an alternative starting point to deduce the first equality in (4.22) for $m_k^{(N)}(t)$.

**Proposition 4.5.** For $k \geq 1$ we have

\[
m_k^{(N)}(t) = \frac{1}{kN} q^{k^2+(N-1)k} \sum_{p=0}^{N-1} (-1)^p \frac{\Gamma(N+k-p)}{\Gamma(k-p)\Gamma(N-p)\Gamma(p+1)} q^{-2kp}.
\]

This sum can be identified with the first hypergeometric form in (4.22).

**Proof.** We replace the summation label $k$ in (4.27) by the symbol $p$. Then we write $l = p-k$, where $k$ is an independent positive integer. Since $m_k^{(N)}(t)$ is the coefficient of $e^{ik}$ in $\rho_{(1),N}(x; t)/N$, the formula (4.28) results. Using the formula

\[
\frac{\Gamma(u)}{\Gamma(u-p)} = (-1)^p(1-u)_p
\]
to rewrite the gamma functions in the summand gives the series form of the hypergeometric polynomial in the first line of (4.22).

\[\square\]

**Remark 4.6.** The expression on the RHS of (4.28) was first obtained in a field theory context in [8, Eq. (17)]. It was related to the result (1.26) of [3, 86] in [57].

4.4. Asymptotics. According to the form (4.26) we see that knowledge of the large $k, N$ asymptotic form of $m_k^{(N)}(t)$ is reliant on knowledge of the large $k, N$ asymptotic form of the Jacobi polynomial $P_{N-1}^{(k-1)}(1 - 2\lambda^2)$, with $0 < \lambda < 1$. This has been investigated in the literature some time ago [20], but the formulas obtained therein contain some inaccuracies [39]. Fortunately an accurate and user friendly statement is available in the recent work [92]. The statement of the result requires some notation. Introduce the complex number $z_+ = z_+(\mu, \lambda)$ by

\[
z_+ = u(\mu, \lambda) + i\sqrt{1 - (u(\mu, \lambda))^2}, \quad u(\mu, \lambda) = \frac{(\mu - 1)(1 + \lambda^2) + 2\lambda^2}{2\lambda\mu},
\]

where it is assumed the parameters $\mu, \lambda$ are such that $|u(\mu, \lambda)| \leq 1$ so that $z_+$ has unit modulus and is in the upper half plane. Now define $\phi_+ = \phi_+(\mu, \lambda) \in [0, \pi]$ as the argument of $z_+$ so that $e^{i\phi_+} = z_+$. In terms of $\phi_+$ define

\[
h(\mu, \lambda) = \arg\left(\frac{z_+^{\mu}(1-\lambda z)}{\lambda - z}\right)\bigg|_{z=e^{i\phi_+}} = \mu\phi_+ + \arg\left(\frac{1 - \lambda e^{i\phi_+}}{e^{i\phi_+} - \lambda}\right),
\]

where we are free to take the argument function as multivalued.

**Proposition 4.7.** ([92, Th. 1]) Set $\mu = k/N$, and require that for $k, N \to \infty$ we have $0 < \mu < 1$.

For

\[
1 > \lambda > \frac{1 - \mu}{1 + \mu},
\]

\[\square\]
the large $N$ expansion

$$\lambda^{k-N} P^{(k-N, 1)}_{N-1} (1 - 2\lambda^2) = \sqrt{\frac{2}{N\pi}} \frac{(1 - \lambda^2)\mu^{-1/2}}{((1 - \lambda^2)((\mu + 1)^2 \lambda^2 - (\mu - 1)^2))^{1/4}} \times \cos(Nh(\mu, \lambda) + \pi/4) + O(\frac{1}{N^{3/2}}) \quad (4.32)$$

holds true, and moreover holds uniformly in $\lambda$ for $\lambda$ a compact interval in $((1 - \mu)/(1 + \mu), 1)$. (On this latter point see [26, §4.1].)

In contrast, for $0 \leq \lambda < (1 - \mu)/(1 + \mu)$, the LHS of (4.32) decays exponentially fast in $N$.

Suppose instead that $\mu > 1$. For $0 < \lambda < \frac{\mu - 1}{\mu + 1}$ (4.33)

the asymptotic formula (4.32) again remains valid, while for

$$\frac{\mu - 1}{\mu + 1} < \lambda < 1 \quad (4.34)$$

the LHS of (4.32) decays exponentially fast in $N$.

We are now in a position to establish the sought asymptotic forms of $m^{(N)}_k(t)$.

**Proof of Corollary 1.3**. With $\lambda = e^{-\mu t/2}$, the inequalities (4.31) and (4.33) can be recast in terms of $t$ and $\mu$ to become equivalent to the requirement that $0 < t < t^*$ where $t^*$ is given by (1.27). In these cases we substitute (4.32) for the Jacobi polynomial in the second expression of (4.26), to obtain (1.28). Outside of this interval, Proposition 4.7 tells us that the Jacobi polynomial, and thus the moments, decay exponentially fast. □

**Proof of Corollary 1.4**. Inspection of the working of [92] for the derivation of (4.32) shows that the leading term, appropriately expanded, remains valid in an extended regime with $k, N \to \infty$ and $k \ll N$, and is thus uniform for $\mu \to 0^+$ provided $N\mu \to \infty$. The keys are the validity of the inequality (4.31), and that $u(\mu, \lambda)$ in (4.29) remains well defined. On the former point, substitute $\lambda = e^{-\mu t/2}$ and expand for small $\mu$. We see that (4.31) is valid provided $t < 4$. On the latter point, doing the same in (4.29) shows

$$u(\mu, \lambda) \to 1 - t/2, \quad (4.35)$$

and thus $z_+ = (1 - t/2) + i\sqrt{t(1 - t/4)}$. The significance of $z_+$ in the working of [92] is that it is the saddle point in the method of stationary phase.

Again setting $e^{i\phi_+} = z_+$, we thus have $\cos \phi_+ = 1 - t/2$, $\sin \phi_+ = \sqrt{t(1 - t/4)}$, which in turn allows us to compute from (4.30) the small $\mu$ expansion

$$h(\mu, \lambda)|_{\lambda = e^{-\mu t/2}} \sim \pi + \mu L_0(t) + O(\mu^2), \quad (4.36)$$

where $L_0(t)$ is specified by (4.6). Also, for small $\mu$,

$$\left. \frac{(1 - \lambda^2)\mu^{-1/2}}{((1 - \lambda^2)((\mu + 1)^2 \lambda^2 - (\mu - 1)^2))^{1/4}} \right|_{\lambda = e^{-\mu t/2}} \sim \frac{1}{\mu^{3/2} \sqrt{t^{3/2}(4 - t)^{1/2}}}, \quad (4.37)$$

Recalling (4.26) allows us to conclude from (4.32) with $\lambda = e^{-\mu t/2}$, and expanded for small $\mu$, the leading $k, N \to \infty$ and $k \ll N$ expansion of $m^{(N)}_k(t)$ as specified by (1.29). □
Remark 4.8. The asymptotics of the Jacobi polynomial in Proposition 4.7 is also known in the cases \(\lambda = [(\mu - 1)/(\mu + 1)]\), with the leading order \(N\) dependence then proportional to \(N^{-1/3}\) [92]. Hence for \(t = t'\) the moments \(m_k^{(N)}(t)\) decay as \(N^{-4/3}\). Note that the exponent \(-4/3\) is the same as that for the asymptotic decay of \(m_k^{(\infty)}(t)\) at \(t = 4\), as noted above (4.6) from [73], as we would expect. Furthermore, repeating the reasoning of the first sentence in the paragraph below (1.23) gives \(k \asymp N^{6/11}\).

5. The spectral form factor

5.1. Evaluation of \(S_N(k; t)\). According to (1.6) and (3.18) we have

\[
S_N(k; t) = N - \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} dy \, e^{i(k(x-y))} K_N(x, y) K_N(y, x) \tag{5.1}
\]

The exact expression (3.21) allows this double integral to be evaluated in terms of a sum.

Proposition 5.1. For \(j, l\) integers, define

\[
a_j = \frac{(-1)^j q^{-(j-(N-1)/2)^2}}{\Gamma(N-j)\Gamma(j + 1)}, \quad b_l = \frac{\Gamma(N-l)q^{-(N-1)/2}}{\Gamma(-l)} \tag{5.2}
\]

For these quantities to be nonzero we require \(0 \leq j \leq N - 1\) and \(l \notin \{0, \ldots, N - 1\}\) respectively. For \(k\) a non-negative integer, we have

\[
S_N(k; t) = \min(k, N) + \sum_{j=0}^{\min(k-1, N)} \sum_{l=\max(k, N)}^{N+1} \frac{a_j b_{j-k} a_{l-k} b_l}{(j - l)^2} \tag{5.3}
\]

Proof. In terms of the notation (5.2), the correlation kernel (3.21) can be written as

\[
2\pi K_N(x, y) = \sum_{l=0}^{N-1} e^{il(x-y)} + \sum_{j=0}^{N-1} \sum_{l \in \mathbb{Z} \setminus \{0, \ldots, N-1\}} \frac{a_j b_l}{j-l} e^{(x-y)l} \tag{5.4}
\]

For \(s\) a non-negative integer and \(f(x, y)\) a \(2\pi\)-periodic function in both \(x\) and \(y\), introduce the notation \([e^{is(x-y)}] f(x, y)\) to denote the coefficient of \(e^{is(x-y)}\) in the corresponding double Fourier series. It follows from (5.4) that

\[
(2\pi)^2 [e^{is(x-y)}] K_N(x, y) K_N(y, x) = \max(N - s, 0) - \sum_{j=0}^{\min(s-1, N)} \sum_{l=\max(s, N)}^{N+s-1} \frac{a_j b_{j-s} a_{l-s} b_l}{(j - l)^2} \tag{5.5}
\]

Using this result in (5.1), and noting too from this that \(S_N(k; t) = S_N(-k; t)\), (5.3) results. \(\square\)

For \(k \leq N\) in (5.3) we have the rewrite

\[
S_N(k; t) = k + \sum_{j=0}^{k-1} \sum_{l'=0}^{k-1} \frac{a_j b_{j-k} a_{l'-N-k-1-l'} b_{N+k-1-l'}}{(j + l' - N - k + 1)^2} \tag{5.6}
\]

Substituting according to (5.2) and simplifying then gives

\[
S_N(k; t) = k - q^{2k^2 + 2k(N-1)} \sum_{j=0}^{k-1} \sum_{l'=0}^{k-1} q^{-2k(j+l')(-1)^{j+l'}} \times \frac{\Gamma(N+k-j)\Gamma(N+k-l')}{\Gamma(N-j)\Gamma(j+1)\Gamma(l'+1)\Gamma(N-l')\Gamma(k-j)\Gamma(k-l')\Gamma(k-l')} \frac{1}{(j + l' - N - k + 1)^2} \tag{5.7}
\]
In the case \( k > N \), this expression again holds with the modifications that the first \( k \) on the RHS is to be replaced by \( N \), as is each \( k \) in the upper terminals of the sums. The fact that at \( t = 0 \) the initial matrix is the identity, and so all eigenvalues are unity, implies from the definition (1.4) that

\[
S_N(k; t)\bigg|_{t=0} = 0; \tag{5.8}
\]

since \( q = 1 \) for \( t = 0 \), note that according to (5.7) this implies an identity for the double sum.

For \( k = 1 \) the double sum in (5.7) consists of a single term, which after simplification and substituting for \( q \) according to (2.8) reads

\[
S_N(k; t)\bigg|_{k=1} = 1 - e^{-t}. \tag{5.9}
\]

The next simplest case is \( k = 2 \), when (5.7) consists of four terms. However the summand is symmetric in \( j, l' \) so this can immediately be reduced to three terms,

\[
S_N(k; t)\bigg|_{k=2} = 2 - e^{-2t}\left( N^2 e^{-2t/N} - 2(N^2 - 1) + N^2 e^{2t/N} \right). \tag{5.10}
\]

Expanding for large \( N \) gives

\[
S_N(k; t)\bigg|_{k=2} = 2 - 2e^{-2t}\left( 1 + 2t^2 \right) + O\left( \frac{1}{N^2} \right). \tag{5.11}
\]

**Remark 5.2.** For Dyson Brownian motion from the identity in the case of \( SU(N) \) as distinct from \( U(N) \) (in \( SU(N) \) the determinant is constrained to be unity) a formula similar in appearance to (5.7), derived using a Schur polynomial/heat kernel expansion of the corresponding eigenvalue PDF, can be found in [74, Th.10.2].

### 5.2. Large \( N \) limit of \( S_N(k; t) \) for fixed \( k \)

For general \( k \) and fixed \( q \), the summand in (5.7) appears to be a polynomial in \( N \) of degree \( 2(k-1) \). Yet we expect the large \( N \) limit of \( S_N(k; t) \) to be well defined. A similar feature is already present in the first form of \( m_k^{(N)}(t) \) from (4.22), or alternatively in (4.28), in which the summand is a polynomial in \( N \) of degree \( k \). It is only after application of the identity (4.24) that the second form in (4.22) is obtained, which allows for the computation of the large \( N \) limit (4.2). In fact (5.7) can be written in terms of an integral of two \( _2F_1 \) polynomials of the same type as appearing in the first line of (4.22). After applying the transformation (4.24) an explicit expression for the \( N \rightarrow \infty \) limit can be obtained.

**Proposition 5.3.** The double sum formula (5.7) for \( S_N(k; t) \) permits the integral forms

\[
S_N(k; t) = \min(k, N) - q^{2k^2 + 2k(N-1)}\left( \frac{(N-1+k)!}{(k-1)!(N-1)!} \right)^2 \\
\times \int_0^\infty se^{-s(N+k-1)}\left( 2F_1(1 - N, 1 - k; -(k - 1 + N); q^{-2k}e^s) \right)^2 \, ds \\
= \min(k, N) - q^{2k^2 + 2k(N-1)}(kN)^2 \int_0^\infty se^{-s(N+k-1)}\left( 2F_1(1 - N, 1 - k; 2; 1 - q^{-2k}e^s) \right)^2 \, ds. \tag{5.12}
\]

Consequently, for \( k \) fixed with respect to \( N \),

\[
S_\infty(k; t) := \lim_{N \rightarrow \infty} S_N(k; t) = k - e^{-tk}k^2 \int_0^\infty se^{-s}\left( 1F_1(1 - k; 2; kt + s) \right)^2 \, ds \\
= k - e^{-tk} \int_0^\infty se^{-s}\left( L_{k-1}^{(1)}(kt + s) \right)^2 \, ds. \tag{5.13}
\]
Proof. With \( u = N + k - 1 - j - l' \) we rewrite the factor of \( 1/u^2 \) in (5.7) according to the simple integral
\[
\frac{1}{u^2} = \int_0^\infty se^{-us} ds.
\]
Taking the integral outside of the double summation, the latter factorises into the product of two single summations, both of which have an identical structure to that in (4.28), which we know can be identified with the first line of (4.22). This gives the first expression in (5.12). The second follows by applying the transformation (4.24), as is analogous with the second line of (4.22). The limit (5.13) now follows by changing variables \( s \mapsto s/N \) in this latter expression, and proceeding as in deducing (4.2) from (1.26). An additional technical issue is the need to interchange the order of the limit with the integral. Here one first note that the hypergeometric polynomial in the final line of (5.12) is, for fixed \( k \) and \( N \) large enough, of degree \( 2(k - 1) \) in \( (1 - q^{2k}e^s) \). It follows that the task of justifying the interchange of the order can be reduced to justifying this for
\[
\lim_{N \to \infty} \int_0^\infty se^{-s(1+(k-1)/N)}(-N)^p(1 - q^{-2k}e^{s/N})^p ds
\]
with \( p \leq 2(k - 1), p \in \mathbb{Z}^+ \). For \( N \) large enough, \( e^{-s(k-1-p)/N} \leq e^{s/2} \). Also, the elementary inequality \( 1 - e^{-x} \leq x \) shows that \( N^p(1 - e^{-s(k+k)/N})^p \) is bounded by \( (s+k)^p \). Hence for \( N \) large enough the integrand is bounded pointwise by \( se^{-s/2(s+k)^p} \), which is integrable on \( s \in \mathbb{R}^+ \). The justification of the interchange of the order now follows by dominated convergence.

\[\square\]

Remark 5.4. The Laguerre polynomials permit the addition formula
\[
L_p^{(\alpha+\beta+1)}(x+y) = \sum_{s=0}^p L_s^{(\alpha)}(x)L_{p-s}^{(\beta)}(y).
\] (5.14)
Applying this relation to each \( L_{k-1}^{(1)}(kt+s) \) in (5.13) with \( p = k - 1, x = s, y = kt, \alpha = 1, \beta = -1 \), and using the orthogonality
\[
\int_0^\infty se^{-s}L_p^{(1)}(s)L_q^{(1)}(s) ds = (p+1)\delta_{p,q}
\]
shows that the integral in (5.13) permits an evaluation as the finite sum (1.30).

5.3. Large \( N \) limit of \( S_N(k;t) \) for \( k \) proportional to \( N \) — proof of Theorem 1.6. We first consider the case \( k \leq N \). Changing variables \( s \mapsto sk/N \) in (5.12) and using the equality between (4.25) and (4.26) shows
\[
S_N(k;t) = k - \frac{k^4}{N^2} \int_0^\infty se^{-(s+t)(k-N+1)/N} \left( P_{N-1}^{(k-N,1)}(1 - 2e^{-(s+t)/k}/N) \right)^2 ds.
\] (5.15)
We see that the asymptotic formulas for \( P_{N-1}^{(k-N,1)}(1 - 2\lambda^2) \), with \( 0 < \lambda < 1 \), given in Proposition 4.7 are again relevant. It is application of these formulas which allows the sought scaling limit of (5.15), as specified in Theorem 1.6, to be computed.
Suppose first that \( 0 < t < t^* \). In relation to Proposition 4.7, set \( \lambda = e^{-\mu(s+t)/2} \). Taking into consideration the range of \( \lambda \) for which (4.32) holds uniformly, we see that for \( 0 < s < t^* - t - \epsilon \),
$0 < \epsilon \ll 1$, the factor of the integrand

$$\left(\lambda^{k-N} P_{N-1}^{(k-N-1)}(1 - 2\lambda^2)\right)^2$$

(5.16)
can be replaced by the square of the leading term on the RHS of (4.32). Making use too of (1.27) shows

$$\frac{k^4}{N^2} \int_0^{t^* - t - \epsilon} \text{se}^{-(s+t)k(k-N+1)/N} \left(P_{N-1}^{(k-N-1)}(1 - 2\text{e}^{-(s+t)k/N})\right)^2 \text{d}s$$

$$\sim \frac{2}{\pi} \frac{\mu^3}{(1 + \mu)} \int_0^{t^* - t - \epsilon} \text{sf}(\mu; \lambda) \cos^2(N\tilde{h}(t + s, \mu + \pi/4)) \bigg|_{\lambda = \mu^{(s+t)/2}} \text{ds},$$

(5.17)

where

$$f(\mu; \lambda) = \frac{\lambda^2}{(1 - \lambda^2)^{3/2}} \frac{1}{(\lambda^2 - e^{-\mu t^*})^{1/2}}.$$  

Next we make use of the identity $\cos^2(N\tilde{h}(t + s, \mu + \pi/4)) = \frac{1}{2}(1 + \cos 2(N\tilde{h}(t + s, \mu + \pi/4)))$. It shows that to leading order the square of the cosine can be replaced by $\frac{1}{2}$. Thus the term involving $\cos 2(N\tilde{h}(t + s, \mu + \pi/4))$, being multiplied by an absolutely integrable function of $s$ — specifically $sf(\mu; \lambda)$ — must decay in $N$ in accordance with the Riemann-Lebesgue lemma for Fourier integrals and its generalisations [36], and hence vanish in the limit. We therefore conclude that the $N \to \infty$ limit of (5.17) is equal to

$$\frac{1}{\pi} \frac{\mu^3}{(1 + \mu)} \int_0^{t^* - t - \epsilon} \text{sf}(\mu; \lambda) \bigg|_{\lambda = \mu^{(s+t)/2}} \text{ds}.$$  

(5.18)

Taking $\epsilon \to 0$ gives the term involving the integral in (1.31).

The range $s > t^* - t + \epsilon$ does not contribute due to the corresponding exponential decay in $N$ of (5.16), as known from the second statement of Proposition 4.7. Thus

$$\frac{k^4}{N^2} \int_{t^* - t + \epsilon}^{\infty} \text{se}^{-(s+t)k(k-N+1)/N} \left(P_{N-1}^{(k-N-1)}(1 - 2\text{e}^{-(s+t)k/N})\right)^2 \text{ds}$$

$$\leq \mu^4 N^2 \text{e}^{-Nu(\mu, s)} \int_{t^* - t + \epsilon}^{\infty} \text{se}^{-\mu(s+t)} \text{ds}$$

(5.19)

for some $u(\mu, s) > 0$. Since the RHS tends to zero as $N \to \infty$, so must the LHS. The remaining task then is to check that there is no separate contribution from the singularity of (4.32) at $s = t^* - t$.

Setting $t = 0$ to begin, one approach is to verify that the RHS of (1.31) is already identically zero, as required by the sum rule (5.8). This follows from the integration formula

$$\int_0^{T} \frac{se^{-s}}{(1 - e^{-s})^{3/2}} \frac{1}{\sqrt{e^{-s} - e^{-t}}} \text{ds} = \pi \left(1 + \tanh(T/4)\right).$$

(5.20)
as can be validated by computer algebra, used in conjunction with (1.27). As a consequence we that with $t = 0$ in the integral of (5.15), there is no contribution to its $N \to \infty$ form for $s > t^* - \epsilon$ in the limit $\epsilon \to 0^+$, and thus

$$\lim_{\epsilon \to 0^+} \lim_{N \to \infty} \int_{t^* - \epsilon}^{\infty} \text{se}^{-sp(k-N+1)} \left(P_{N-1}^{(k-N-1)}(1 - 2\text{e}^{-sp})\right)^2 \text{ds} = 0.$$  

(5.21)

We claim now that knowledge of (5.21) is sufficient to establish that for general $0 < t < t^*$ fixed in (5.15), there is no contribution to its $N \to \infty$ form for $s > t^* - t - \epsilon$ in the limit $\epsilon \to 0^+$. Note that this is a stronger statement than establishing that there is no separate contribution from the neighbourhood of $s = t^* - t$, and in particular supersedes the need for (5.19). The reason
Figure 4. Plot of $\tilde{S}_\infty(\mu; t)$ with respect to both $\mu$ and $t$, as specified by Theorem 1.6. The ramp and plateau features are evident in the direction of fixed $t$ and varying $\mu$, with the transition being smooth in keeping with Remark 5.5.4.

for its validity stems from the fact that the more general case differs from the case $t = 0$ only by the replacement of the factor of the first factor of $s$ in the integrand by $(s - t)$, as implied by the explicit functional form in (5.15), with the $s$ dependence of all the other terms left unaltered and the terminals of integration $(t^* - \epsilon, \infty)$. Moreover, since we are assuming $t < t^*$ and we have $s \in (t^* - \epsilon, \infty)$, with $\epsilon$ small enough it follows $0 < (s - t) \leq s$. Hence the limit formula (5.21) remains valid with $s$ in the integrand replaced by $(s - t)$, as required for general $0 < t < t^*$, and the claim is established.

The above working involving (5.17) assumed $0 < t < t^*$. For $t > t^*$, the factor of the integrand (5.16) decays exponentially fast in $N$ for all $s > 0$. The inequality (5.19) with the lower terminal of integration replaced by 0 on both sides implies the integral in (5.15) does not contribute to the $N \to \infty$ limit, which implies (1.31) for this range of $t$.

The case $k > N$ remains to be considered. According to (5.12) and (4.26) the formula (5.15) remains valid but with the first $k$ on the RHS replaced by $N$. The quantity $k - N$ in $P_{N-1}^{(k-N,1)}$ is now positive, or equivalently $\mu > 1$, which is also covered by Proposition 4.7. The two cases $0 < t < t^*$ and $t > t^*$ have to again be considered separately. Doing so, and following the strategy of the working for $\mu < 1$ as detailed above, we arrive at (4.29) in both cases. $\Box$

Remark 5.5.
1. As $\mu$ varies from 0 to 1, $t^*$ monotonically increases from the value 4 to $\infty$. In particular, this means that for all $t < 4$ there is always a deformation of the limiting CUE result as given by the first term in (1.31). On the other hand, for $t > 4$ there is always a range of $\mu$ values, $0 \leq \mu \leq \mu_r$, and...
for which the limiting CUE result is valid. Here $\mu_r$ is determined by being the solution of (1.27) with $t^* = t$. We recall from the discussion of §4.1 that $t = 4$ has the significance as the value of $t$ for which the support of the density becomes the whole unit circle.

2. As $\mu$ varies from $1$ to $\infty$, $t^*$ monotonically decreases from the value $\infty$ to $0$. This means that for all $t > 0$ there is always a range of $\mu$ values, $\mu_p < \mu < \infty$, for which the limiting CUE result given by the first term of (1.31) is valid. This is in contrast to the circumstance for $0 < \mu < 1$ as described above. Both features are in agreement with the heuristic predictions of Section 1.3.

3. For a given value of $t$, the critical values of $\mu$ introduced above can be read off from (1.27) to be given by the implicit equations

$$t = \frac{2}{\mu_r} \log \frac{1 + \mu_r}{1 - \mu_r} \quad (t > 4), \quad t = \frac{2}{\mu_p} \log \left| \frac{1 + \mu_p}{1 - \mu_p} \right| \quad (t > 0).$$

(5.22)

In keeping with the heuristic discussion of Section 1.3, we expect that these equations can be rewritten as

$$\mu_r = \rho_{(1),\infty}(x; t) \bigg|_{x=\pi} \quad (t > 4), \quad \mu_p = \rho_{(1),\infty}(x; t) \bigg|_{x=0} \quad (t > 0),$$

(5.23)

with $x = \pi \ (x = 0)$ being the points of least (greatest) value of the density. We see from (4.12) and (4.10) with $w = -1$ (for $x = \pi$) and $w = 1$ (for $x = 0$) that indeed the equations in (5.22) and (5.23) are equivalent.

4. Expanding the integral in (1.31) to leading order in $t^* - t$ shows that it is proportional to $(t^* - t)^{3/2}$. The exponent $3/2$ is the same as that known for the transition from ramp to plateau in the cases of the Gaussian unitary ensemble [15], and the Laguerre unitary ensemble with Laguerre parameter scaled with $N$ [45]. In particular, it implies the function values and first derivatives of $\tilde{S}_\infty(\mu; t)$ agree across the transition.

To use Theorem 1.6 for the computation of $\tilde{S}_\infty(\mu; t)$, with $t$ fixed and as a function of $\mu$, for each $\mu$ we must compute $t^*$ as specified by (1.27). If $0 < t < t^*$ we compute $\tilde{S}_\infty(\mu; t)$ according to the functional form (4.29) which involves a contribution from the integral, whereas for $t \geq t^*$ the term involving the integral is to be set to 0. A plot obtained by the implementation of this procedure, iterated also over $t$, is given in Figure 4.

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