Desperately Seeking Chiral Fermions

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Chiral fermions can (presumably) be constructed by introducing two regulators, one for the gauge fields (e.g., a lattice), and another for the fermion functional integrals in a fixed (regulated) gauge field. This talk discusses cutoff effects arising from the regulator of the fermions.

1. INTRODUCTION

It has proven difficult to couple chiral fermions to lattice gauge fields [1]. One approach introduces an interpolation of the lattice gauge field to a continuum gauge field. Then the fermion functional integrals are regulated, for each individual gauge field, by another regulator, which will be denoted here by $M$. Examples include Pauli-Villars or a finer lattice of spacing $1/M$. The idea is not that new [2–4]—indeed, for some time Jan Smit has described the approach as desperate [5]. But, spurred by ref. [6], the approach has recently attracted more attention [7–10].

With two cutoffs the first question is whether they are taken away together,

$$M \to \infty, \quad a \to 0, \quad Ma \text{ fixed;}$$

(1)

or separately,

$$M \to \infty \text{ with } a \text{ fixed, } \quad a \to 0 \text{ later.}$$

(2)

Limiting procedure (1) is close to the strategy with fermions on the same lattice as the gauge field: even if there are two technical regulators, there is essentially only one cutoff scale. This is the goal that seems so difficult to reach. On the other hand, limiting procedure (2) is obviously extremely costly from a practical (i.e. numerical) point of view: the limit $M \to \infty$ is taken for each gauge field separately.

The “well-known” properties of chiral fermions are usually established in perturbation theory by a procedure (2). One introduces a cutoff $M$ for the fermions, presuming gauge fields to be smooth and weak on the scale $M$. Only after taking $M \to \infty$ does one contemplate a regulator for the gauge field. Thus, one might expect a two-cutoff scheme starting with interpolated lattice gauge fields to work, even if it’s expensive, provided the details are worked out [3].

Be that as it may, with two cutoffs coupling-constant renormalization is not carried out in a satisfactory way. In a nonperturbative renormalization one would take $M \to \infty$, holding physical quantities (or a fiducial physical quantity) constant. But physics comes after integrating over all gauge fields, which contradicts the notion of taking the limit gauge field by gauge field. The alternative is a perturbative renormalization, introducing, say, a partially renormalized coupling

$$g_{0}^{-2}(a) = g_{00}^{-2}(a; M) + \beta_{f} \log Ma$$

(3)

where the coefficient $\beta_{f}$ comes from one fermion loop, and holding $g_{0}^{-2}(a)$ constant as $M \to \infty$. Such a set-up is not nonperturbative in the same sense as vector-like lattice gauge theories.

The aim of this talk is to characterize terms in the fermion effective action that are suppressed by $1/M^{2n}$. In a vector-like theory these terms are not very interesting, because their effect on physical observables is (after renormalization) nil. In a chiral theory, however, they can break gauge symmetry—because the regulator always breaks chiral gauge symmetry.

My original goal was to control the cutoff artifacts to the extent that it would be permissible (for some specific regulator(s)) to implement limiting procedure (1). This has not (yet) proven
possible. The analysis does, however, suggest a kind of Symanzik improvement to limiting procedure \([\hat{\Gamma}]\). This might prove valuable to a desperate numerical simulation. Furthermore, many properties of the cutoff artifacts are common to a wide class of regulators, and it might be fruitful to check how the gauge-breaking terms can affect physical quantities, for example in perturbation theory.

2. INTERPOLATION

There are several properties that one would like from an interpolation. First, one would like the only singularities in the field strength \(F_{\mu\nu}(x)\) to be instanton-like directional singularities. Second, the interpolation scheme should also generate an interpolation \(g(x)\) of lattice gauge transformations, so that the gauge potential transforms as \(gA_{\mu}(x) = g(x)(\partial_{\mu} + A_{\mu}(x))g^{-1}(x)\). Third, a composition of lattice gauge transformations should be interpolated as

\[
g(s) = g_2(s)g_1(s) \Rightarrow g(x; U) = g_2(x; s_2 U)g_1(x; U),
\]

where the dependence of the interpolation on the underlying lattice gauge field is emphasized. Finally, the interpolation ought to provide an explicit form for \(A_{\mu}(x)\). It is needed for computing of eigenvalues (in a Pauli-Villars scheme) or for constructing a finer-grained lattice gauge field.

The interpolation given in ref. \([1]\) satisfies these requirements.

3. GENERAL FEATURES

The fermion Boltzmann factor is given (formally) by

\[
e^{-\Gamma(A)} = \int \mathcal{D}\psi_+ \mathcal{D}\psi_+^\dagger e^{-S(A, \psi_+, \psi_+^\dagger)}\]

where the action \(S = \int d^4x \psi_+^\dagger \mathcal{D}^{\dagger} \psi_+\). Since \(\psi_+\) is an element of a positive-chirality vector space, \(\mathcal{D}^{\dagger}\) is not a determinant. Nevertheless, from eq. \((8)\) one can abstract the feature of fermion-number nonconservation \([1]\): if the vector-like Dirac operator \(\mathcal{D}\) has positive (negative) chirality zero modes, the Boltzmann factor vanishes; inserting, however, enough factors of \(\psi_+ (\psi_+^\dagger)\) yields a nonvanishing integral. One would like to preserve this property in the regulated theory. One would also like to find

1. \(\text{Re } \Gamma(gA) = \text{Re } \Gamma(A)\),
2. \(\text{Im } \Gamma(gA) = \text{Im } \Gamma(A)\) in an anomaly-free representation,
3. \(\text{Im } \Gamma(A) \neq 0\) in a complex representation.

4. A DEFINITION OF \(\Gamma\)

One way to define a regulated fermion Boltzmann factor starts by considering the operator

\[
\hat{D} = \gamma + \frac{1}{2}(1 - \gamma_5)A.
\]

The action is now \(S = \int d^4x \bar{\psi} \hat{D}\psi\), and \(\hat{D}\) has different left and right eigenvectors. If \(\lambda_n \neq 0\)

\[
i\hat{D}\eta_n = \lambda_n \eta_n, \quad (i\hat{D})^\dagger \chi_n = \lambda_n^* \chi_n.
\]

In addition, \(\hat{D} (\hat{D}^\dagger)\) annihilates the positive-(negative-)chirality zero modes of \(\mathcal{D}\), denoted \(\varphi_\pm\). The functional integral over \(\psi\) or \(\bar{\psi}\) is defined through the eigenmodes \(\eta_n, \chi_m\), and \(\varphi_\pm\). After integrating over the nonzero modes one obtains the Boltzmann factor

\[
e^{-\hat{\Gamma}} = \prod_{n: \lambda_n \neq 0} (-i\lambda_n) = \prod_{n: \text{Re } \lambda_n > 0} \lambda_n^2
\]

Integration over zero modes (when needed) yields zero, unless enough powers of \(\psi\) or \(\bar{\psi}\) are inserted. But even then \(\hat{\Gamma}\) must be computed.

From eq. \((8)\), one posits a class of regulated effective actions is given by

\[
\hat{\Gamma}_L = -\frac{1}{2} \sum_n \infty L(\lambda_n^2 / M^2)
\]

where the cutoff function \(L\) satisfies

\[
\lim_{x \to 0} \frac{L(x)}{\log x} = 1,
\]

\[
L(\infty) = L'(\infty) = L''(\infty) = \cdots = 0.
\]

Examples include Fujikawa’s regulator \([12]\) and’t Hooft’s Pauli-Villars regulator \([3]\).
As with all regulators of chiral gauge theories, the ones given in eq. (3) break gauge symmetry. After a gauge transformation $g = e^{\varphi}$ the eigenvalues of $\hat{D}$ change. As a consequence, neither $\text{Re} \hat{\Gamma}$ nor $\text{Im} \hat{\Gamma}$ is gauge invariant.

If the regulator function $L(x)$ is smooth and analytic everywhere, the manipulations of the appendices of ref. [8] show that the gauge variation can be organized by powers of $M^2$. One finds

$$\delta_\omega \hat{\Gamma}_L = \sum_{n=-1}^{\infty} M^{-2n} f_L^{(n)}(0) \int d^4 x \omega^a \alpha_{4+2n}^a, \quad (12)$$

where $f_L(x) = xL'(x)$ and the superscript on $f_L$ denotes differentiation with respect to $x$. (And $f^{(-1)}(x) = - \int_x^\infty dy f_L(y)$ is the first antiderivative.) The gauge field comes in through the $\alpha_{4}^a$, described in a bit more detail below.

The utility of eq. (12) is that it disentangles the $L$-dependent coefficients $f_L^{(n)}(0)$ from the operators $\alpha_{4+2n}^a$, which are the same for all cutoffs in the class defined by eq. (4). Thus, for all these cutoffs the dimension-four ($n = 0$) operator is multiplied by the universal $f_L(0) = 1$, but the $M$-dependent terms are multiplied by nonuniversal derivatives. Moreover, an operator that causes difficulty in one cutoff will cause the same difficulty for others, unless by design or good fortune the appropriate derivative $f_L^{(n)}(0)$ vanishes.

Detailed expressions for the operators $\alpha_{4}^a$ are too long to present here. They are traces of the $d$-dimensional combination of $\hat{D}$ and $D_\mu$ yielding functionals of $A$, rather than a differential operators. After the (gauge-group and Dirac-matrix) trace one can write $\alpha_{4}^a = \alpha_{dR}^a + i \alpha_{dI}^a$. In this notation the first several imaginary parts can be identified in a less obtuse way: the two-dimensional term $\alpha_{2}^a = 0$, and the four-dimensional term $\alpha_{4}^a$ is the consistent anomaly $[13, 14]$. Before the trace the expression that generates $\alpha_{4}^a$ would generate the consistent anomaly of a six-dimensional theory; since Dirac-matrix traces vary somewhat with dimension, after the trace the expression in four dimensions differs in detail.

The real parts $\alpha_{dR}^a$ are well-known. In general they can be related to local counter-terms $[13]$

$$\int d^3 x \omega^a \alpha_{dR}^a = \delta_{s d},$$

where the $s_d$ are local (gauge noninvariant) interactions. Since the first two do not vanish they must be added explicitly to obtain an effective action

$$\Gamma = \hat{\Gamma} - f^{(-1)}(0) M^2 s_2 - s_4 \quad (13)$$

that is gauge invariant after the taking $M \to \infty$.

5. ANOTHER DEFINITION

Another approach deals with eigenvalues of vector-like Dirac operators only. It would take

$$e^{-\Gamma} = \sqrt{\text{det}_L D} \exp(i\eta_L), \quad (14)$$

where $\text{det}_L D$ is the regulated Boltzmann factor of a vector-like theory, and $\eta_L$ is a certain (UV-regulated) property of a five-dimensional Dirac operator $[15]$. After Pauli-Villars regulators (for example) are removed $\eta_L$ provides the anomalous Wess-Zumino action $[14]$ and gauge-invariant (but parity-violating) terms.

The work of Göckeler and Schierholz $[3]$, as well as some more recent papers $[13, 14]$, introduces a Boltzmann factor with the same properties as eq. (14) under gauge transformations. These $\text{Re} \Gamma$ are all gauge invariant, so their cutoff effects are neither harmful nor interesting. On the other hand, at finite $M$ the imaginary part $\delta_\omega \text{Im} \Gamma \neq 0$, even when anomalies cancel. But, as long as the regulator used for $\eta_L$ reproduces the consistent anomaly, it seems likely that the higher-dimension analogues are the same as $\alpha_{dI}$, etc, described in sect. 4.

6. REDUCTION OF CUTOFF EFFECTS

If there are higher-dimension gauge-breaking terms in $\text{Re} \Gamma$ it is essential to remove them. Since they are equivalent to local counter-terms, they will mix with lower-dimensional terms, leaving the interacting theory with gauge-breaking of $O(1/(aM)^2)$. The first several such terms can be eliminated by choosing the cutoff function so that successive derivatives $f^{(n)}(0) = 0$. It does not seem possible to eliminate all cutoff effects by setting all $f^{(m)}(0) = 0$. (A function like that has an essential singularity at $x = 0$. The manipulations in the appendices of ref. [8] require Taylor expansions of $f(x)$ too close the origin to remain valid in this case.) Barring a more robust
proof, the most optimistic prospect is that an improved cutoff would accelerate the convergence to the limit \( M \to \infty \).

If the gauge breaking comes only from \( \text{Im} \Gamma \) the conclusions are less obvious, because one cannot express these terms as local functionals of \( A_\mu \) and its derivatives. For example, the anomalous part is given by the Wess-Zumino action, an integral over a five-dimensional manifold. If the conclusion of the mixing argument applies in this setting, then the higher-dimensional terms would filter back down to the anomaly, the imaginary part of lowest dimension. It would then seem that an anomaly-free theory contains no residue of the cutoff’s gauge breaking. That seems too good to be true, but it could be examined by inserting \( \alpha_{6I} \) into Feynman diagrams.

7. CONCLUSIONS

This talk has exposed some features of the cutoff effects in chiral gauge theories with two regulators, one for the gauge fields and another for the fermions. In particular, there is a connection between the gauge-breaking cutoff artifacts of the imaginary part and expressions that lead to anomalies in higher-dimensional theories. Inasmuch as the same terms arise in a variety of different regulators, this may be a first step towards deriving some general properties.

For example, there is some disagreement whether the overlap formalism yields chiral fermions. In this formalism the imaginary part of the effective action breaks gauge invariance at finite lattice spacing. One can imagine setting up the overlap in a two-cutoff scheme, and then examining the large \( Ma \) limit, as above. Since the dimension-four gauge breaking is the consistent anomaly, it again seems likely that at dimension six \( \alpha_{6I} \) appears, the same expression as above. Depending on how it (or other gauge breaking) trickles down to physical quantities, one may learn whether the overlap can implement limiting procedure, or whether it too must live with the desperate limiting procedure.

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