On Fractional Quantum Hall Effect (FQHE): A Chern-Simons and nonequilibrium quantum transport Weyl transform approach

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Abstract

We give a simple macroscopic phase-space explanation of fractional quantum Hall effect (FQHE), in a fashion reminiscent of the Landau-Ginsburg macroscopic symmetry breaking analyses. This is in contrast to the more complicated microscopic wavefunction approaches. Here, we employ a nonequilibrium quantum transport in the lattice Weyl transform formalism. This is coupled with the Maxwell Chern-Simons gauge theory for defining fractional filling of Landau levels. Flux attachment concept is inherent in fully occupied and as well as in partially occupied Landau levels.

We derived the $k$-factor scaling hierarchy in Chern-Simons gauge theory, as the scaling hierarchy of the magnetic fields or magnetic flux in FQHE. This is crucial in our simple explanation of FQHE as a topological invariant in phase space. For the fundamental scaling hierarchy, the integer $k$ must be a prime number, and for fractions both the numerator and denominator of $k$ must also be prime numbers. The assumption in the literature that a hierarchy of denominators of $v = \frac{1}{k}$ is given by the expression, $(2n + 1)$, is wrong. Furthermore, even denominators for $v$ cannot belong to fundamental scaling hierarchy and is often absent or less resolved in the experiments.

Keywords: FQHE, Flux attachments, Chern-Simons $k$-factor, lattice Weyl transform, nonequilibrium quantum superfield transport
I. INTRODUCTION

In previous papers [1-4], we make use of the gapped energy-band structure of solids under external electric field to derive the integer quantum Hall effect (IQHE) of Chern insulator. We employ the real-time superfield and lattice Weyl transform nonequilibrium Green’s function (SFIWT-NEGF) [5] quantum transport formalism [6, 7] to the first-order gradient expansion to derive the topological Chern number of the IQHE for two-dimensional systems, as an integral multiple of quantum conductance, also known as the minimal contact conductance in mesoscopic physics [5].

We find that the quantization of Hall effect occurs strictly not to first order in the electric field per se but rather to first-order gradient expansion in the nonequilibrium quantum transport equation. The Berry connection and Berry curvature is the fundamental physics [8] behind the exact quantization of Hall conductance in units of $\frac{e^2}{h}$, which also happens to coincide with the source and drain contact conductance per spin in a closed circuit of mesoscopic quantum transport [5].

In Ref.[1], we have shown that the $(p,q; E,t)$ phase-space is renormalized to that of $(\vec{K}, E)$ phase-space, where in the absence of magnetic fields,

$$\vec{K} = \vec{p} + e\vec{F}t, \quad (1)$$

and

$$E = E_0 + e\vec{F} \cdot \vec{q} \quad (2)$$

Here, the uniform electric field, $\vec{F}$, is in the $x$-direction, and the Hall current is in the $y$-direction.

We have identified the topological invariant in $(\vec{K}, E)$ phase-space nonequilibrium quantum transport equation leading to IQHE. Moreover, the formula is also applicable to gapped Landau-level structure of a free electron gas in intense magnetic field [9] since the variable $\vec{K}$ can incorporates the external vector potential, $\frac{e}{c}A$, and its corresponding parallel transport, if present. Note that the change of variables from $\vec{K}$ to $\vec{p}$ in the integration over the whole Brillouin zone has a Jacobian unity.

It was shown in previous paper [1-4] that the correct expression for the IQHE conductivity given by

$$\sigma_{yx} = \frac{e^2}{h} \sum_{\alpha} \frac{\Delta\phi_{total}}{2\pi} = \sum_{\alpha} \frac{e^2}{h} n_{\alpha}$$

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A. Fractional quantum Hall effect

Here, we give a simple explanation for FQHE effect conductance given by

$$\sigma_{yx} = \frac{e^2}{h} v$$

where $v$ is a fraction with prime number denominator in a hierarchy of scaling factors. We remark that the number 2 is a prime number and should not be considered an even number. Moreover, prime integers exclude many odd integers. All other non-prime integers can be decomposed into products of prime numbers, this we refer to as higher-order scaling, i.e., higher-order power of primes. Thus, if one looks for a fundamental hierarchy of scaling patterns, only fundamental prime number scaling can represent a primary hierarchy.

Although even denominators for $v$ are speculated in the literature, these do not belong to a fundamental hierarchy of scaling in our present analysis for the same reducibility or factorization reason into power of primes, and therefore represent higher-order scaling. Moreover, although factorizable scaling factors can appear in the measurements, these do not belong to a fundamental hierarchy and we believe that these 'higher-order' type of scaling hierarchy should appear very weak or less resolved in the measurements. This is discussed in more detail in Sec. IIIA.

II. EFFECTS OF MAGNETIC FIELDS

For free electrons, the presence of increasing magnetic field has two effects; firstly it increases the energy levels in magnetic fields, $E_n = \hbar \omega_c (n + 1/2)$, $n = 1, 2, 3, ...$, where $\omega_c$ equals the cyclotron frequency, $\frac{eB}{m}$, and their separations and secondly, it increases the degeneracy of Landau levels and corresponding density of states, and hence the flattening of the magnetic sub-band [10, 11] of the lowest Landau level.

In the present paper, we show that the idea of flux attachment is inherent in fully occupied as well as in partially occupied Landau levels, which results in scaling factor closely related to the scaling $k$-factor in Chern-Simons gauge theory. This is shown to be crucial in giving a simple explanation of fractional quantum Hall effect (FQHE), using nonequilibrium quantum transport in the lattice Weyl transform formalism [5, 7] used in previous papers [1].
A. Magnetic sub-bands

1. Low finite fields

The sharp highly degenerate Landau levels for free electrons in a magnetic field is broadened into magnetic energy sub-bands by the presence of periodic atomic lattice sites. At values of the magnetic fields where the effect of the periodic atomic lattice sites dominates, the effect of the magnetic fields can be described simply in terms of the dynamics of the Bloch energy bands \[12, 13\]. For example, for finite magnetic fields, this is manifested in de Haas-van Alphen effects \[14, 15\] due to Landau orbits at the Fermi surface, and magnetic breakdown between orbits in complicated Fermi surfaces in metals \[16, 17\]. For smaller magnetic fields, the Bloch-band dynamics is manifested in such phenomena as paramagnetic and diamagnetic susceptibility. Indeed, interband coupling in narrow-gap alloyed semimetals and semiconductors is responsible for the giant diamagnetism of graphite, bismuth and Bi-Sb alloys \[18, 19\].

2. Large magnetic fields

However, when the effect of large magnetic fields dominates over the effect of periodic atomic lattice sites, then the dynamics is now dictated by the magnetic sub-bands. New length scales and time scales now dominate the dynamics. These are the following,

| Operator                               | Expression                                      |
|----------------------------------------|-------------------------------------------------|
| Cyclotron frequency \(\omega_c\)      | \(\frac{eB}{m^*c}\)                           |
| Magnetic length \(l_B\)                | \(\sqrt{\frac{\hbar c}{eB}}\)                  |
| Quantum flux \(\phi_o\)                | \(\frac{2\pi\hbar c}{e}\)                     |
| Hall conductivity \(\sigma_{yx}\)     | \(\frac{e^2}{2\pi\hbar}\nu\), (\(\nu\) is an integer or a fraction) |

The major quantum mechanical operator in phase space is now played by

\[
\tilde{K} = \tilde{p} + \frac{e}{c}\tilde{A}
\]

which obeys the commutation relation

\[
[K_x, K_y] = \frac{e\hbar}{ic}B_z
\]
This is more transparent if written as

\[ [K_x, K_y] = \frac{e\hbar}{ic} B_z = \frac{e\hbar^2}{ich} = -i \hbar^2 l_B^2 \]

indicating that \( l_B \) characterizes the wavelengths in a magnetic sub-bands. For most magnetic field strengths, \( l_B > a \) where \( a \) is the atomic sites lattice constant. This clearly supports the idea of magnetic sub-bands.

3. \( \text{von Neumann lattice in phase space} \)

Equation (3) suggest the following relation

\[ \Delta K_x \Delta K_y \geq \frac{1}{2} \hbar^2 l_B^2 = \frac{1}{2} \frac{e\hbar}{c} B_z \]

(4)
as a consequence of Heisenberg’s uncertainty principle. Thus for the minimal value of \( \Delta K_x \Delta K_y = \frac{1}{2} \hbar^2 l_B^2 \) a von Neumann lattice in 2-D phase space of \((K_x, K_y)\) has a lattice constant of the order \( l_B \). For our purpose in what follows, it is important to note that \( \Delta K_x \Delta K_y \sim \delta K_x \delta K_y \) is proportional to the perpendicular magnetic field, \( B_z \). Therefore, we may also write a Poisson differential operator in a form,

\[ \frac{\partial^{(a)}}{\partial K_x} \frac{\partial^{(b)}}{\partial K_y} \sim \frac{1}{B_z}, \]

(5)
as inversely proportional to the magnetic field. This equation means that if we increase the magnetic field by a factor \( k \), then from Eq. (5), we can scale the Poisson partial derivatives by \( \frac{1}{k} \), resulting in \( \frac{1}{k} \left( \frac{\partial^{(a)}}{\partial K_x} \frac{\partial^{(b)}}{\partial K_y} \right) = \left( \frac{\partial^{(a)}}{\partial K'_x} \frac{\partial^{(b)}}{\partial K'_y} \right) \) where the primes embodied the updated magnetic field. This observation is crucial in our macroscopic analysis of FQHE.

III. \( \text{LANDAU LEVEL DEGENERACY} \)

Classically, the Landau-level (LL) degeneracy may be approximated by

\[ \frac{A}{\pi r^2} = \frac{R^2}{\pi^2} \]

where \( r \) is the classical radius of tiny Landau orbits in a uniform magnetic fields and the system area is given by \( A = \pi R^2 \). Equation (5) has the dimensional units of total flux divided by the dimensional units of ‘quantum’ flux, \( \frac{(\pi R^2)B}{\pi(\frac{\Phi}{\phi_0})} \Rightarrow \frac{\Phi}{\phi_0} \), where \( E \) stands for
units of energy. In this classical analysis it is thus implied that the magnetic flux is equally divided or attached with each tiny cyclotron orbits of the same energy for a fully occupied occupied Landau level.

Quantum mechanically, the more accurate expression for the Landau-level degeneracy, $N_{LL}$ is $N_{LL} = \frac{\Phi}{(2\pi\hbar e)} = \frac{\Phi}{\phi_o}$. This can be inferred simply by a Bohr-Sommerfeld quantization condition \[8\] which amounts to counting of Planck states (‘pixels’ of action) in phase space using Berry’s curvature and connection, i.e., magnetic field and vector potential, respectively. In Gaussian units, we have,

$$L-L\text{ Degeneracy} = \frac{1}{2\pi\hbar} \int \int \vec{\nabla} \times \vec{K} \cdot d\vec{a}$$

$$= \frac{1}{2\pi\hbar} \int \int \vec{\nabla} \times \vec{e} \frac{c}{c} \vec{A} \cdot d\vec{a}$$

$$= \frac{1}{2\pi\hbar} \frac{|e|}{c} \int \int \vec{B} \cdot d\vec{a}$$

$$= \frac{\Phi}{\phi_o} \epsilon \mathbb{Z}, \text{ (the same degeneracy for all Landau levels)}$$

$$= \frac{1}{2\pi\hbar} \int \int \vec{e} \frac{c}{c} \vec{A} \cdot d\vec{q} = N_{LL} \epsilon \mathbb{Z},$$

(6)

where $\vec{K} = \vec{p} + \frac{e}{c} \left( \vec{A} + \vec{F}ct \right)$ \[4\], where $\vec{F}$ is the uniform electric field. Here, $\Phi$ is the total magnetic flux, and $\phi_o = \frac{hc}{|e|}$ is the quantum flux which one may considered as quantum mechanically attached to each quasiparticle (note that instead of classical tiny cyclotron orbits, we now refer these as quasiparticles) in a fully occupied lowest Landau level (LLL). This is actually a precursor of the concept of flux attachment afforded by the Chern-Simons $U(1)$ gauge theory in 2-dimensions. Moreover, this is also realization of the B-S quantization condition given in Eq. (6) \[8\]. The resulting quantization of orbital motion leads to edge states and integer quantum Hall effect under uniform magnetic fields. The general analysis of edge states marks the works of Laughlin \[20, 21\], and Halperin \[22\].

1. Flux attachments

From Eq. (6), we have for a fully occupied lowest Landau level at $v = 1$ of the experiments,

$$\frac{1}{2\pi\hbar} \int \int \vec{\nabla} \times \vec{K} \cdot d\vec{a} = \frac{\Phi}{\phi_o} = N_o = N_{LLL}$$

(7)

where as before, $N_o$ is the total number of available electrons, e.g., for a given gate voltage bias of a Si MOSFET or GaAs MESFET heterostructures, and $N_{LLL}$ is the degeneracy of
the LLL.

For fractionally occupied, say $\frac{1}{3}$ of LL at higher magnetic fields or higher degeneracies, $\frac{N_{LLL}}{N_{LL}} = \frac{N_0}{N_{LL}} = \frac{1}{3}$, so we can simply re-express Eq (7), on the spirit of the flux-attachment concept as

$$\frac{k}{2\pi\hbar} \int \nabla \times \vec{K} \cdot \vec{dA} = \Phi / \phi_o = k N_o = N_{LL}$$

(8)

where we have indicated the scaling factor by $k$, e.g., $k = 3$.

2. Either flux attachments or fractional charge

Equation (8) means that naively, either $3\phi_o = \frac{3hc}{|e|}$ is attach to each quasiparticle or that $|e|$ in $\phi_o = \frac{hc}{|e|}$ is $\frac{|e|}{3}$, a fractional charge. These two alternatives carry different physical meanings, flux attachment means the number of electrons is not enough to fully occupy the new density of states or LL degeneracy, whereas fractional charge means the new quasiparticles fully occupy the new Landau level degeneracy, $N_{LL}$, but with fractional charge.

A. Hierarchy of primes for the scaling factor $k$

Since the magnetic flux of the LLL is our reference point for scaling by a factor $k$, in order to form a hierarchy of scaling, this cannot be expressed as product of prime numbers, otherwise we will eventually be scaling a different magnetic flux after the first factor of the product. Thus, factorizable numbers and fractions cannot represent as members of a fundamental hierarchy of scaling factor $k$. Thus, for integer $k$ this must be a prime number, whereas for fractions the numerator and denominator of $k$ must also be a prime numbers. This scaling should constitute the principal scaling hierarchy, i.e., first-order in primes.

A second scaling hierarchy would constitute prime numbers for integers and prime-number numerators only for $k$. The third type of scaling hierarchy will constitute prime number denominators only for $k$. Since all these scaling $k$ are factorizable, we consider these as 'higher-order' scaling hierarchy and to be weakly observed. The fourth scaling hierarchy is for both numerators and denominators to be factorizable. This we expect to be rarely seen in the experiments.
1. Higher probability for prime-number numerators for $k$

Therefore, from the fundamental and second scaling hierarchies, we expect the dominance of prime-number numerators for $k$ in the experiments. We summarize these statements by the following table which give some examples of the values of $k$ and its inverse,

| $k \geq 1$ | $k < 1$ |
|------------|----------|
| $k \nu = \frac{1}{k}$ | $k \nu = \frac{1}{k}$ |
| $\nu = 1$ | $\nu = \frac{1}{2}$ |
| $\nu = 1\frac{1}{2}$ | $\nu = \frac{1}{3}$ |
| $\nu = 1\frac{1}{3}$ | $\nu = \frac{1}{4}$ |
| $\nu = 1\frac{1}{5}$ | $\nu = \frac{1}{6}$ |
| $\nu = 1\frac{1}{7}$ | $\nu = \frac{1}{2}$ |
| $\nu = 2\frac{1}{3}$ | $\nu = \frac{2}{3}$ |
| $\nu = 2\frac{1}{5}$ | $\nu = \frac{2}{4}$ |
| $\nu = 2\frac{1}{7}$ | $\nu = \frac{2}{6}$ |
| $\nu = 3\frac{1}{3}$ | $\nu = \frac{3}{5}$ |
| $\nu = 3\frac{1}{5}$ | $\nu = \frac{3}{4}$ |
| $\nu = 3\frac{1}{7}$ | $\nu = \frac{3}{6}$ |
| $\nu = 4\frac{1}{3}$ | $\nu = \frac{4}{5}$ |
| $\nu = 4\frac{1}{5}$ | $\nu = \frac{4}{4}$ |
| $\nu = 4\frac{1}{7}$ | $\nu = \frac{4}{6}$ |
| $\nu = 5\frac{1}{3}$ | $\nu = \frac{5}{5}$ |
| $\nu = 5\frac{1}{5}$ | $\nu = \frac{5}{4}$ |
| $\nu = 5\frac{1}{7}$ | $\nu = \frac{5}{6}$ |
| $\nu = 6\frac{1}{3}$ | $\nu = \frac{6}{5}$ |
| $\nu = 6\frac{1}{5}$ | $\nu = \frac{6}{4}$ |
| $\nu = 6\frac{1}{7}$ | $\nu = \frac{6}{6}$ |
| $\nu = 7\frac{1}{3}$ | $\nu = \frac{7}{5}$ |
| $\nu = 7\frac{1}{5}$ | $\nu = \frac{7}{4}$ |
| $\nu = 7\frac{1}{7}$ | $\nu = \frac{7}{6}$ |
| $\nu = 8\frac{1}{3}$ | $\nu = \frac{8}{5}$ |
| $\nu = 8\frac{1}{5}$ | $\nu = \frac{8}{4}$ |
| $\nu = 8\frac{1}{7}$ | $\nu = \frac{8}{6}$ |
| $\nu = 9\frac{1}{3}$ | $\nu = \frac{9}{5}$ |
| $\nu = 9\frac{1}{5}$ | $\nu = \frac{9}{4}$ |
| $\nu = 9\frac{1}{7}$ | $\nu = \frac{9}{6}$ |

We remark that the number 2 in $\nu = \frac{1}{2}$ is a prime number and should not be considered an even number. The whole number in $\nu = 2$ in Eq. (9) simply means two LL are filled by the reference population of the LLL, $N_0 = N_{LLL}$, and corresponds to the re-emergence of IQHE. Moreover, prime integers exclude many odd integers. Thus, it appears that the assumption in the literature that the denominator of $v$ is given by the expression, $(2n + 1)$, in a hierarchy is wrong, this claim is simply borne out of the dominance of prime number numerators for $k$ (2 is a prime number). Furthermore, although even denominators for $v$ are speculated in the literature, these cannot be members of a fundamental scaling hierarchy for the same reason that these can be decomposed into products of prime numbers. The entries of Eq. (9) actually exist as experimental values [23–25], see Figs. 2 and 3.
IV. RELATION TO CHERN-SIMONS GAUGE THEORIES

We write the Chern-Simons Lagrangian density for $U(1)$ gauge theory for 2-dimensional system of manifold, $M$, as

$$ \mathcal{L}_{CS} = \gamma \varepsilon^{\mu \lambda \nu} A_{\mu} \partial_{\lambda} A_{\nu} - A_{\mu} J^{\mu} $$

(10)

where $J^{\mu} = (\rho, \vec{J})$, $\rho$ is the charge density and $\vec{J}$ is the current density. Later, we will associate the parameter $\gamma$ with our scaling parameter. Equation (10) is often referred to as the Maxwell Chern-Simons theory. The equation of motion is obtained by variation with respect to $A_{\mu}$

$$ \frac{\delta \mathcal{L}_{CS}}{\delta A_{\mu}} = \gamma \varepsilon^{\lambda \nu} \partial_{\lambda} A_{\nu} - J^{\mu} = 0 $$

This gives

$$ \gamma \int_{M} \nabla \times \vec{A} = \int_{M} J^{0} = \int_{M} \rho $$

(11)

For a fully occupied LLL, we may equate the following

$$ \gamma = \frac{1}{2\pi \hbar c} $$

$$ \nabla \times \vec{A} \Rightarrow \nabla \times \frac{e}{c} \vec{A} = \nabla \times \vec{K}. $$

$$ \int_{M} \rho = N_{LL} $$

where $\vec{K} = P + \frac{e}{c} \vec{A} + \vec{E} c t$, [4] where $\vec{E}$ is the uniform electric field. Thus, Eq. (11) reduces to that of Eq. (7). The relation of Eq. (8) with Chern-Simons $U(1)$ gauge theory thus becomes clear. We re-write Eq. (8) with $k = 3$ as

$$ \frac{k}{2\pi \hbar} \int \int \nabla \times \frac{e}{c} \vec{A} \cdot d\vec{a} = \frac{\Phi}{\phi_{o}} = N_{LL} $$

(12)

where $\vec{A}$ is the vector potential. Equation (12) clearly shows that the magnetic field is scaled by an integer factor $k$ to yield a larger number, $N_{LL}$, of degenerate states in a Landau level compared to our reference fully occupied LLL, resulting in fractional filling. Equation (10), with $k\gamma = k \left( \frac{e}{2\pi \hbar c} \right)$, is basically in the Chern-Simons form of $U(1)$ gauge theory.

A. Experiments in IQHE and FQHE

In 1980, Klaus von Klitzing, working with Si MOSFET samples developed by Michael Pepper and Gerhard Dorda, made the unexpected discovery, see Fig. [1] that the Hall
resistance was exactly quantized [9]. Other experiments are done in GaAs MESFETs.

In 1982, a group of physicists lead by Tsui and Stormer, working on GaAs MESFET heterostructures, discovered $\nu = \frac{1}{3}$ state [23]. This means they found a state in which the FQHE conductivity was given by

$$\sigma_{yx} = \frac{e^2}{2\pi\hbar} v$$

as shown in Fig. 2. In general, it was discovered over time that there were several states with fractional values that had plateaus corresponding to values of quantum Hall resistivity that correspond to different $\nu$'s. This was confusing how these states could occur since $\nu = 1$ should be the lowest possible state given the scheme found by the Landau levels above. At that time, it was considered strange since some fractions would occur and others would not. For example there was $\nu = \frac{1}{3}$ but not $\nu = \frac{1}{2}$. There’s $\nu = \frac{3}{5}$, but there is no $\nu = \frac{3}{4}$. In general, it was found at first that no state had an even denominator.

However, more later experiments have presented evidence for even denominator, see Fig. 3 in some expecial cases [24]. More detailed experiments were done [25, 26], see Figs. 4 and 5.

Our analysis in this paper is not based on variational wavefunction microscopic approaches, for example, using Laughlin wavefunctions [21], More-Read wavefunctions [27], or Jain wavefunctions [28-30], but is based on macroscopic phase-space analysis reminiscent of the macroscopic Landau-Ginsburg analysis of phase transitions in matter via symmetry breaking, employing minimal description using order parameters. Here, our 'order' parameter is the magnetic flux, $\frac{\Phi}{\phi_0}$, or more precisely the prime number scaling factor of the magnetic flux of a fully occupied lowest Landau level (LLL) for a given geometrical 2-D
FIG. 2. The dashed diagonal line represents the classical Hall resistance and the full drawn diagonal stepped curve the experimental results. The magnetic fields causing the steps are marked with arrows. The step first discovered by Störmer and Tsui [23] at the highest value of the magnetic field and the steps earlier discovered by von Klitzing (integers) with a weaker magnetic field. Here we consider the $v = 1$ as our reference in calculating the $v = \frac{1}{3}$ by employing our novel approach.

FIG. 3. Graph showing the diagonal resistivity, $\rho_{xx}$, at $T = 80$ mK showing strong transport anomalies at $\nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$ [Reproduced from Ref. [24]].

feature size, i.e., channel area, of either Si MOSFET or GaAs MESFET heterostructures used in the experiments [23–25].

Further experiments in GaAs/AlGaAs MESFET revealed strong dominance of prime number denominators for the filling factor $\nu$, see Figs. 4 and 5.

Another interesting approach to FQHE, which makes use of Bohr-Sommerfeld quantization, was given by Jacak [31, 32]. This approach may be related to our use here of Maxwell Chern-Simons $U(1)$ gauge theory, which may also related to the Bohr-Sommerfeld quanti-
FIG. 4. In the present analysis, \( v = \frac{10}{21} \), \( \frac{3}{8} \), and \( \frac{3}{10} \), which are weakly resolved in the figure, do not belong to a hierarchy of scaling factors, since the denominators are factorizable and not prime numbers. Figure reproduced from Ref. [25].

V. TOPOLOGICAL INVARIANT IN PHASE SPACE AND FQHE

The following transport analysis give a simple account of the experimental results [23, 25] with the fully occupied lowest Landau level as a reference point, i.e., the point \( v = 1 \) moving towards successive \( v \)'s < 1 with increasing magnetic fields, or moving backward of successive \( v \)'s > 1 with decreasing magnetic fields. In particular, we will focus on \( v = \frac{1}{3} \) or \( k = 3 \) as a particular case since this looks like a very defined state in the experiments, although the following analysis holds for any values of \( k \).

The topologically invariant result for \( \sigma_{yx} \) will thus come out to be,

\[
\frac{1}{(2\pi\hbar)^2} \int \int \int d\mathbf{k}_x d\mathbf{k}_y dt \left( \frac{1}{3} \right) \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathbf{K}_x \partial \mathbf{K}_y} - \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathbf{K}_y \partial \mathbf{K}_x} \right] \times H^{(a)} \left( \tilde{\mathbf{K}}', \mathcal{E} \right) \left( -iG^{(b)} \left( \tilde{\mathbf{K}}', \mathcal{E} \right) \right)
\]

where \( \tilde{\mathbf{K}}' \) comes from the actual lattice Weyl transformation with updated magnetic field. The factor \( \left( \frac{1}{3} \right) \) comes from the relation of Eq. [5], with \( k = 3 \). Note that the integrals have to
FIG. 5. A quantum Hall data showing plateaus in the second Landau level. Note that many of the plateaus are labeled RIQHE (reentrant integer quantum Hall effect). Note that only prime number denominators are resolved. Figure reproduced From Ref. [26]

do with the counting of the number of occupied states, moderated by the energy-dependent Wigner distribution, $-iG^{<b}(\vec{K}', \mathcal{E})$, of particle population in phase space.

Thus, in place of the previous expression for fully occupied LLL for $k = 1$, we now have, for any value of $k$,

$$
\sigma_{yx} = \left( \frac{1}{k} \right) \frac{e^2}{(2\pi \hbar)^2} \int \int \int dp_x dp_y dt \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \vec{K}_x \partial \vec{K}_y} - \frac{\partial^{(a)} \partial^{(b)}}{\partial \vec{K}_y \partial \vec{K}_x} \right] \times H^{(a)} \left( \vec{K}', \mathcal{E} \right) \left( -iG^{<b} \left( \vec{K}', \mathcal{E} \right) \right)
$$

(14)

1. Magnetic sub-band representation of magnetic quantum states

Now the common representation of $\vec{K}$ and $\vec{K}'$ of the differential operator in Eq. [13] is the $\vec{p}$ of their magnetic sub-bands. Therefore this will yield the same result if we write as

$$
\left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial p_x \partial p_y} - \frac{\partial^{(a)} \partial^{(b)}}{\partial p_y \partial p_x} \right] H^{(a)} \left( \vec{K}', \mathcal{E} \right) \left( -iG^{<b} \left( \vec{K}', \mathcal{E} \right) \right),
$$
since
\[
\frac{\partial^{(a)}}{\partial \vec{K}_x} \equiv \frac{\partial \vec{K}_x}{\partial p_x} \frac{\partial^{(a)}}{\partial \vec{K}_x} = \frac{\partial^{(a)}}{\partial p_x}
\]
and so on.

Therefore, by going over the analogous procedure used before \[\text{[1]}\], following Eq. (13) we finally obtain,
\[
\sigma_{yx} = \left(\frac{e^2}{h}\right) v \quad \left( v = \frac{1}{3} \right)
\]
(15)

Generalizing, we have
\[
\sigma_{yx} = \left(\frac{e^2}{h}\right) v \quad \left( v = \frac{1}{k} \right)
\]
(16)

For convenience, the derivation is given in some details in the Appendix A. The readers is also referred to Ref. \[\text{[1]}\] for more details. In Eq. (15), there is no summation over occupied levels since we are here mainly concerned with scaling hierarchy of fractionally occupied lowest Landau level with increasing magnetic field beyond the field at the fully occupied LLL.

VI. CONCLUDING REMARKS

Our analysis of FQHE parallels that of our previous papers \[\text{[1-4]}\] on IQHE. The only new theoretical ingredient that we need is Eq. (6) for the magnetic field, which is equivalent to the Chern-Simons $U(1)$ gauge theory equation of motion, Eq. (12). The $k$-parameter essentially serves as our order parameter analogous to Landau-Ginsburg order parameter. This naturally brings in the flux attachment concept that is concomitant to new statistics of quasiparticles, the so-called anyons. Physically, this means that instead of talking about filled gapped energy levels in IQHE, we are treating fractionally filled gapped energy levels at different values of the magnetic field, specifically using a fully filled LLL as the point of reference in FQHE analysis of this paper. This will naturally produced hierarchies of scaling factors, $k$. We observe that experiments are done on heterostructured semiconductors, e.g., Si MOSFET or GaAs MESFET, where electrons can be considered 2-D Bloch electrons in magnetic sub-bands.

We have shown that the scaling $k$-factor in Chern-Simons gauge theory corresponds to the scaling of the magnetic fields in FQHE. This leads us to account for the flattening or deformation of the magnetic sub-band of the LLL upon increasing the magnetic field.
Following similar procedure using the nonequilibrium quantum transport in a lattice Weyl transform formalism, we determined the exact expression for the Hall conductivity in FQHE found in experiments.

Since the magnetic flux of the LLL is our reference point for scaling by a factor $k$, this cannot be expressed as product of numbers within the fundamental hierarchy, otherwise we will eventually be scaling a different magnetic flux after the first factor of the product. Thus, the integer $k$ this must be a prime number, whereas for fractions the numerator and denominator of $k$ must also be prime numbers. Indeed, the experiments clearly show that the value of the magnetic flux scaling number $k$ has a prime number numerator, or FQHE scaling fraction $\nu$ with prime number denominator. These are what is expected to have higher probability of occurring and is mostly seen in experiments. Prime integers exclude many odd integers. Thus, it appears that the assumption in the literature that the denominator of $\nu$ is given by the expression, $(2n + 1)$, in a fundamental scaling hierarchy is wrong. Furthermore, although even denominators for $\nu$ are speculated in the literature, these cannot be members of a fundamental hierarchy for reason of factorizability. Therefore, these prime numbers in our theory must be the controlling parameters of stable FQHE conductance well-resolved in the experiments [26].

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Appendix A: REVERTING TO MATRIX ELEMENTS

1. Lattice Weyl transform

We refer the readers to [1–4] for more details of this section. Denoting the operation of taking the lattice Weyl transform by the symbol $\mathcal{W}$ then the lattice Weyl transform of total partial derivatives is given by [1],

$$
\mathcal{W} \left( \frac{\partial}{\partial q^r} + \frac{\partial}{\partial q^s} \right) \langle q', \lambda' | \hat{\mathbf{A}} | q'', \lambda'' \rangle = \frac{\partial}{\partial q} \sum_v e^{\left(\frac{\pi i}{\hbar}\right) p^v} \langle q - v, \lambda' | \hat{\mathbf{A}} | q + v, \lambda'' \rangle ,
$$

$$
= \frac{\partial}{\partial q} A_{\lambda', \lambda''}(p, q) .
$$

(A1)
Similarly
\[
W \left( \frac{\partial}{\partial p'} + \frac{\partial}{\partial p''} \right) \langle p', \lambda' \mid \hat{A} \mid p'', \lambda'' \rangle = \frac{\partial}{\partial p} \sum_u e^{\left( \frac{2}{\hbar} \right) q u} \langle p + u, \lambda \mid \hat{A} \mid p - u, \lambda' \rangle,
\]
\[
= \frac{\partial}{\partial p} A_{\lambda' \lambda''} (p, q).
\] (A2)

In applying to problems in uniform electromagnetic fields, the form
\[
\langle \vec{q}_1, t_1 \mid \hat{H}^{(1)} \mid \vec{q}_2, t_2 \rangle \rightarrow e^{-i \pi \left( \vec{A}(q) + \vec{F} \cdot ct \right) - \vec{q}_1 - \vec{q}_2} e^{-i \vec{F} \cdot \vec{q} (t_1 - t_2)} \hat{H}^{(1)} (\vec{q}_1 - \vec{q}_2, t_1 - t_2),
\] (A3)
where
\[
\vec{q} = \frac{1}{2} (\vec{q}_1 + \vec{q}_2),
\]
\[
t = \frac{1}{2} (t_1 + t_2).
\]
of matrix elements in Eq. (A3), we have
\[
\left\langle \vec{q} - \vec{v}; t - \frac{\tau}{2}, \lambda \mid \hat{A} \mid \vec{q} + \vec{v}; t + \frac{\tau}{2}, \lambda' \right\rangle = e^{-i \vec{F} \cdot (\vec{q} - \vec{q}_v) - \vec{q} (t_1 - t_2)} A (\vec{q}_1 - \vec{q}_2, t_1 - t_2)
\]
\[
= e^{i \vec{F} \cdot (2 \vec{v})} A_{\lambda' \lambda} (\vec{q}_1 - \vec{q}_2, t_1 - t_2).
\] (A4)

Thus
\[
A_{\lambda' \lambda} (\vec{p}, \vec{q}; E, t) = \sum_{\vec{v}, \tau} e^{\left( \frac{2}{\hbar} \right) (\vec{p} + \vec{e} F t \cdot \vec{v}) - \vec{e} (\vec{F}) \tau} e^{i \vec{F} \cdot (2 \vec{v})} e^{i \vec{F} \cdot \vec{q} \tau} A_{\lambda' \lambda} (\vec{q}_1 - \vec{q}_2, t_1 - t_2),
\]
\[
= \sum_{\vec{v}, \tau} e^{\left( \frac{2}{\hbar} \right) (\vec{p} + \vec{e} F t \cdot \vec{v}) - \vec{e} (\vec{F}) \tau} e^{i \vec{F} \cdot (2 \vec{v})} A_{\lambda' \lambda} (\vec{q}_1 - \vec{q}_2, t_1 - t_2),
\]
\[
= A_{\lambda' \lambda} \left( (\vec{p} + \frac{e}{c} \left( \vec{A} + \vec{F} \cdot ct \right)) ; \left( E + e \vec{F} \cdot q \right) \right),
\]
\[
= A_{\lambda' \lambda} \left( \vec{K} ; E \right).
\] (A5)

Hence the relevant dynamical variables in the phase space including the time variable occurs in particular combinations of \( \vec{K} \) and \( E \). Therefore, besides the crystal momentum varying in time as
\[
\vec{K} = \vec{p} + \frac{e}{c} \left( \vec{A} + \vec{F} \cdot ct \right),
\] (A6)
the energy variable vary with \( \vec{q} \) due to the electric field as
\[
E = E_o + e \vec{F} \cdot \vec{q}.
\] (A7)

Note that the derivatives on the LHS of Eqs. (A1) and (A2) obviously operate only on the wavefunctions or state vectors not on the operator. In Eqs. (A2) and (A2), we use the four dimensional notation, e.g, \( q = (\vec{q}, t) \), and \( p = (\vec{p}, E) \).
a. Derivation of FQHE

The Hall current in the $y$-direction maybe written as \[1–4\], after dividing by $\frac{1}{k}$ the Poisson operator as,

\[
\frac{a^2}{(2\pi\hbar)^2} \int \int dp_x dp_y \left( \frac{e}{a^2} \frac{\partial \mathcal{E}}{\partial \mathcal{K}_y} \right) \left(-iG^{<} \left( \mathcal{K}', \mathcal{E} \right) \right)
\]

\[
= e^2 \left| \vec{F} \right|^2 \frac{1}{(2\pi\hbar)^2} \int \int \int dp_x dp_y dt \times \left( \frac{1}{k} \right) \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathcal{K}_x \partial \mathcal{K}_y} \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathcal{K}_y \partial \mathcal{K}_x} H^{(a)} \left( \mathcal{K}', \mathcal{E} \right) \left(-iG^{<(b)} \left( \mathcal{K}', \mathcal{E} \right) \right) . \tag{A8}
\]

Now to show that the Eq. \([14]\) gives $\sigma_{yx} = \frac{\epsilon^2}{k} \nu$, where $\nu = \frac{1}{k}$, we need to transform the integral of Eq. \([14]\) to the integral of the curvature of Berry connection in a closed loop, which is quantized by the winding number. This necessitates a 'pull back' (i.e., undoing) the lattice transformation of Eq. \([14]\), i.e., we revert to corresponding matrix-element expressions.

2. 'Pull back' of the lattice Weyl transformation

The pull-back process means we have to undo the lattice transformation of SFLWT-NEGF transport equation, to return to its equivalent matrix element expressions. Consider the integrand in Eq. \([14]\) given by the partial derivatives of lattice Weyl transformed quantities.

\[
\left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathcal{K}_x \partial \mathcal{K}_y} \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathcal{K}_y \partial \mathcal{K}_x} \right] H^{(a)} \left( \mathcal{K}', \mathcal{E} \right) \left(-iG^{<(b)} \left( \mathcal{K}', \mathcal{E} \right) \right)
\]

\[
= \frac{\partial H^{(a)} \left( \mathcal{K}', \mathcal{E} \right)}{\partial \mathcal{K}_x} \frac{\partial G^{<(b)} \left( \mathcal{K}', \mathcal{E} \right)}{\partial \mathcal{K}_y} - \frac{\partial H^{(a)} \left( \mathcal{K}', \mathcal{E} \right)}{\partial \mathcal{K}_y} \frac{\partial G^{<(b)} \left( \mathcal{K}', \mathcal{E} \right)}{\partial \mathcal{K}_x} \right] . \tag{A9}
\]

Take first the term of Eq. \([A9]\), where,

\[
\frac{\partial H^{(a)} \left( \mathcal{K}', \mathcal{E} \right)}{\partial k_x} = \hbar \frac{\partial H^{(a)} \left( \mathcal{K}', \mathcal{E} \right)}{\partial \mathcal{K}_x} . \tag{A10}
\]
From Eq. (A2) this can be written as a lattice Weyl transform $W$ in the form,

$$\frac{\partial H^{(a)}(\vec{K}', \mathcal{E})}{\partial \vec{K}_x} = W \left\{ \left( \frac{\partial}{\partial \vec{K}_x^\alpha} + \frac{\partial}{\partial \vec{K}_x^\beta} \right) \langle \alpha, \vec{K}', \mathcal{E} \left| \hat{H} \right| \beta, \vec{K}', \mathcal{E} \rangle \right\},$$

$$= W \left\{ \langle \alpha, \frac{\partial}{\partial \vec{K}_x} \vec{K}', \mathcal{E} \left| \hat{H} \right| \beta, \vec{K}', \mathcal{E} \rangle + \langle \alpha, \vec{K}', \mathcal{E} \left| \hat{H} \right| \beta, \frac{\partial}{\partial \vec{K}_x} \vec{K}', \mathcal{E} \rangle \right\},$$

$$= W \left\{ E_\beta (\vec{K}', \mathcal{E}) \langle \alpha, \frac{\partial}{\partial \vec{K}_x} \vec{K}', \mathcal{E} \left| \beta, \vec{K}', \mathcal{E} \rangle + E_\alpha (\vec{K}', \mathcal{E}) \langle \alpha, \vec{K}', \mathcal{E} \left| \beta, \frac{\partial}{\partial \vec{K}_x} \vec{K}', \mathcal{E} \rangle \right\},$$

$$= W \left\{ \left( E_\beta (\vec{K}', \mathcal{E}) - E_\alpha (\vec{K}', \mathcal{E}) \right) \langle \alpha, \frac{\partial}{\partial \vec{K}_x} \vec{K}', \mathcal{E} \left| \beta, \vec{K}', \mathcal{E} \rangle, \right\} \right\} = (A11),$$

where we defined

$$\langle \alpha, \frac{\partial}{\partial \vec{K}_x} \vec{K}', \mathcal{E} \rangle \equiv \frac{\partial}{\partial \vec{K}_x} \langle \alpha, \vec{K}', \mathcal{E} \rangle.$$

We also have

$$\frac{\partial G^{< (b)}(\vec{K}', \mathcal{E})}{\partial \vec{K}_y} = W \left\{ \left( \frac{\partial}{\partial \vec{K}_y^\beta} + \frac{\partial}{\partial \vec{K}_y^\alpha} \right) \langle \beta, \vec{K}', \mathcal{E} \left| (i\hat{\rho}) \right| \alpha, \vec{K}', \mathcal{E} \rangle \right\},$$

where $\hat{\rho}$ is the density matrix operator. The density operator in the Heisenberg representation is given by,

$$\hat{\rho} (t) = e^{-\frac{i}{\hbar} \hat{H} t} \hat{\rho} (0) e^{\frac{i}{\hbar} \hat{H} t},$$

$$= \hat{U} (t) \hat{\rho} (0) \hat{U}^\dagger (t),$$

which provides the major time dependence in our transport equation that follows. From Eq. (A12), we take the time dependence of

$$\langle \beta, \vec{K}', \mathcal{E} \left| (i\hat{\rho}) \right| \alpha, \vec{K}', \mathcal{E} \rangle$$

to be given by

$$i \langle \beta, \vec{K}', \mathcal{E} \left| \hat{\rho} (0) \right| \alpha, \vec{K}', \mathcal{E} \rangle e^{i\omega_{\alpha \beta} t}.$$

We have

$$\frac{\partial G^{< (b)}(\vec{K}, \mathcal{E})}{\partial \vec{K}_y} = W \left\{ \langle \beta, \frac{\partial}{\partial \vec{K}_y} \vec{K}', \mathcal{E} \left| (i\hat{\rho} \hat{\rho}_0) \right| \alpha, \vec{K}', \mathcal{E} \rangle \right\} e^{i\omega_{\alpha \beta} t}.$$
The density matrix operator $\hat{\rho}_0$ is of the form,

$$\hat{\rho}_0 = \sum_m \rho \left| m \right\rangle \left\langle m \right|$$

$$\hat{\rho}_0 \left| m \right\rangle = \rho \left| m \right\rangle = f \left( E_m \right) \left| m \right\rangle$$

$$\left\langle m \right| \hat{\rho}_0 \left| n \right\rangle = \rho_{mm} = f \left( E_n \right) \delta_{mn} \text{ or } f \left( E_m \right) \delta_{mn}$$

where the weight function is the Fermi-Dirac function,

$$\rho \left| m \right\rangle_0 = f \left( E_m \right)$$

Hence

$$i \hat{\rho}_0 \left| \alpha, \vec{K}', E \right\rangle = i \sum_\gamma \left| \gamma, \vec{K}', E \right\rangle \hat{\rho}_0 \left\langle \gamma, \vec{K}', E \right| \left| \alpha, \vec{K}', E \right\rangle$$

$$= i \left| \alpha, \vec{K}', E \right\rangle f \left( E_{\alpha} \right).$$

Similarly,

$$i \left\langle \beta, \vec{K}', E \right| \left( \hat{\rho}_0 \right) = i \left\langle \beta, \vec{K}', E \right| \sum_\gamma \left| \gamma, \vec{K}', E \right\rangle \rho \left| \gamma, \vec{K}', E \right\rangle$$

$$= if \left( E_{\beta} \right) \left| \beta, \vec{K}', E \right\rangle.$$

Hence

$$\frac{\partial G^{< (b)} \left( \vec{K}', E \right)}{\partial \vec{K}_y} = \mathcal{W} \left\{ i \left[ \beta, \frac{\partial}{\partial \vec{K}_y} \vec{K}', E \left| \alpha, \vec{K}', E \right] \rho_0^\alpha \right\} e^{i\omega_{\alpha \beta} t}. \right.$$  

Shifting the first derivative to the right, we have

$$\frac{\partial G^{< (b)} \left( \vec{K}', E \right)}{\partial \vec{K}_y} = \mathcal{W} \left\{ -i \left[ \beta, \vec{K}', E \left| \alpha, \frac{\partial}{\partial \vec{K}_y} \vec{K}', E \right] f \left( E_{\alpha} \right) \right\} e^{i\omega_{\alpha \beta} t}$$

$$= \mathcal{W} \left\{ i \left[ f \left( E_{\beta} \right) - f \left( E_{\alpha} \right) \right] \left| \beta, \vec{K}, E \left| \alpha, \frac{\partial}{\partial \vec{K}_y} \vec{K}', E \right] \right\} e^{i\omega_{\alpha \beta} t}. \right.$$  

For energy scale it is convenient to chose $f \left( E_{\alpha} \right)$ in the above equation, with the viewpoint that $\alpha$-state, i.e., LLL sub-band is far remove from the $\beta$-state in gapped states, so that we can set $f \left( E_{\beta} \right) \simeq 0$. In the case that several sub-bands are occupied in FQHE, we can assume that the last occupied sub-band is fractionally occupied. Thus unoccupied sub-bands will
Since it appears as a product of two Weyl transforms, it must be a trace formula in the untransformed or pulled back version, i.e., for the remaining indices \( \alpha \) and \( \beta \) we must be a summation,

\[
\frac{\partial H}{\partial \tilde{K}_x} \frac{\partial G^<}{\partial \tilde{K}_y} \frac{\partial H}{\partial \tilde{K}_y} \frac{\partial G^<}{\partial \tilde{K}_x} = \mathcal{W} \left[ \sum_{\alpha,\beta} \left\{ \left\langle E_\beta (\tilde{K}', \mathcal{E}) - E_\alpha (\tilde{K}', \mathcal{E}) \right| \left| \alpha, \frac{\partial}{\partial \tilde{K}_y}\tilde{K}', \mathcal{E} \right\rangle \right\} \left\{ \left\langle \beta, \tilde{K}', \mathcal{E} \right| \left| \alpha, \frac{\partial}{\partial x}\tilde{K}', \mathcal{E} \right\rangle \right\} e^{i\omega_{\alpha\beta}t} \right] .
\]

Therefore we obtain,

\[
\left[ \frac{\partial H}{\partial \tilde{K}_x} \frac{\partial G^<}{\partial \tilde{K}_y} - \frac{\partial H}{\partial \tilde{K}_y} \frac{\partial G^<}{\partial \tilde{K}_x} \right] = \mathcal{W} \left[ \sum_{\alpha,\beta} \left\{ \left\langle E_\beta (\tilde{K}', \mathcal{E}) - E_\alpha (\tilde{K}', \mathcal{E}) \right| \left| \alpha, \frac{\partial}{\partial x}\tilde{K}', \mathcal{E} \right\rangle \right\} \left\{ \left\langle \beta, \tilde{K}', \mathcal{E} \right| \left| \alpha, \frac{\partial}{\partial y}\tilde{K}', \mathcal{E} \right\rangle \right\} e^{i\omega_{\alpha\beta}t} \right] .
\]

Now the LHS of Eq. (A13), namely

\[
\left( \frac{a}{2\pi \hbar} \right)^2 \int dp_x dp_y \frac{e}{a^2} \frac{\partial \mathcal{E}}{\partial \tilde{K}_y} G^< (\tilde{K}', \mathcal{E})
\]

\[
= \left( \frac{a}{2\pi \hbar} \right)^2 \int dp_x dp_y \frac{e}{a^2} \frac{\partial H}{\partial \tilde{K}_y} G^< (\tilde{K}', \mathcal{E}).
\]
Using the result of Eq. (A11), we have

\[
\frac{\partial H (\vec{K}', \mathcal{E})}{\partial \vec{K}'_y} = W \left\{ \left[ \left( E_\alpha (\vec{K}', \mathcal{E}) - E_\beta (\vec{K}', \mathcal{E}) \right) \right] \langle \alpha, \vec{K}', \mathcal{E} | \beta, \frac{\partial}{\partial \vec{K}'_y} \vec{K}', \mathcal{E} \rangle \right\},
\]

\[
= W \left\{ \left[ \left( E_\beta (\vec{K}', \mathcal{E}) - E_\alpha (\vec{K}', \mathcal{E}) \right) \right] \langle \alpha, \vec{K}', \mathcal{E} | \partial_{\vec{K}'_y} \vec{K}', \mathcal{E} \rangle \right\},
\]

\[
= W \left\{ \omega_\beta \langle \alpha, \frac{\partial}{\partial k_y} \vec{K}', \mathcal{E} | \beta, \vec{K}', \mathcal{E} \rangle \right\} = \langle \alpha, \vec{K}', \mathcal{E} | v_{y,y} | \beta, \vec{K}', \mathcal{E} \rangle,
\]

where we use the identity

\[
\omega_\beta \langle \alpha, \nabla_{\vec{p}} \vec{p} | \beta, \vec{p} \rangle = \langle \alpha, \vec{p} | \vec{v}_{g,y} | \beta, \vec{p} \rangle.
\]

Likewise

\[
G^< (\vec{K}, \mathcal{E}) = i W \left( \langle \beta, \vec{K}', \mathcal{E} | \vec{p} \rangle \langle \alpha, \vec{K}', \mathcal{E} \rangle \right)
\]

Again, since Eq. (A13) is a product of lattice Weyl transform, it must be a trace in the untransformed version, i.e.,

\[
\left( \frac{a}{(2\pi \hbar)} \right)^2 \int dp_x dp_y e \frac{\partial H}{\partial K'_y} G^< (\vec{K}, \mathcal{E})
\]

\[
= W \left\{ i \int \left( \frac{a}{(2\pi \hbar)} \right)^2 dp_x dp_y \sum_{\alpha, \beta} \langle \alpha, \vec{K}', \mathcal{E} | e_a v_y | \beta, \vec{K}', \mathcal{E} \rangle \langle \beta, \vec{K}', \mathcal{E} | \hat{\rho} | \alpha, \vec{K}', \mathcal{E} \rangle \right\},
\]

\[
= W \left\{ iTr \left( \frac{e}{a^2} \hat{v}_{y,y} \right) \hat{\rho} \right\} = i W \{ Tr (j_y \rho) \} = i W \{ Tr (j_y \rho) \},
\]

For calculating the conductivity we are interested in the term multiplying the first-order in electric field. We can now convert the quantum transport equation in the transformed space, Eq. (A8),

\[
\left( \frac{a}{(2\pi \hbar)} \right)^2 \int dp_x dp_y e \frac{\partial E}{\partial K'_y} \left[ -i G^< (\vec{K}', \mathcal{E}) \right]
\]

\[
= \left( \frac{1}{k} \right) e^2 |\vec{F}| \int \int dp_x dp_y dt \left[ \frac{\partial^{(a)}}{\partial \vec{K}'_x} \frac{\partial^{(b)}}{\partial \vec{K}'_y} \right] \left[ \frac{\partial^{(a)}}{\partial \vec{K}'_x} \frac{\partial^{(b)}}{\partial \vec{K}'_y} \right] \times H^{(a)} (\vec{K}', \mathcal{E}) \left( -i G^< (\vec{K}', \mathcal{E}) \right),
\]
to the equivalent matrix element expressions by undoing the lattice Weyl transformation \( \mathcal{W} \), which amounts to canceling \( \mathcal{W} \) in both side of the equation given by,

\[
\mathcal{W} \{ \langle \hat{\mathcal{J}}_y (\vec{t}) \rangle \} = \mathcal{W} \left[ \sum_{\alpha, \beta} \left( \frac{1}{k} \right) \epsilon^2 \left| \vec{F} \right| \frac{1}{(2\pi) \hbar} \int \int dp_x dp_y dt \right]
\]

\[
\left[ \langle \alpha, \frac{\partial}{\partial k_x} \vec{K}', \vec{E} \rangle \langle \beta, \vec{K}', \vec{E} \rangle - \langle \alpha, \frac{\partial}{\partial k_y} \vec{K}', \vec{E} \rangle \langle \beta, \vec{K}', \vec{E} \rangle \right] \left( E_{\beta} (\vec{K}', \vec{E}) - E_{\alpha} (\vec{K}', \vec{E}) \right)
\]

\[
\times e^{i\omega_{\alpha\beta} t} f (E_{\alpha}) \right] .
\]

(A14)

The time integral of the RHS amounts to taking zero-order time dependence [zero electric field] of the rest of the integrand, then we have for the remaining time-dependence, explicitly integrated as,

\[
\int_{-\infty}^{0} dt \exp i\omega_{\alpha\beta} t = \exp \frac{i\omega_{\alpha\beta} \tau}{\omega_{\alpha\beta}} \bigg|_{\tau=-\infty}^{\tau=0} = \frac{1}{i\omega_{\alpha\beta}}
\]

Thus eliminating the time integral we finally obtain.

\[
\langle \hat{\mathcal{J}}_y (t) \rangle = -i \epsilon^2 \left( \frac{1}{k} \right) \left| \vec{F} \right| \frac{1}{(2\pi) \hbar} \int \int dp_x dp_y dt \left[ \sum_{\alpha, \beta} \left( f (E_{\alpha}) \left( \frac{\hbar \omega_{\alpha\beta}}{\omega_{\alpha\beta}} \right) \right) \left\{ \langle \alpha, \frac{\partial}{\partial k} \vec{K}', \vec{E} \rangle \langle \beta, \vec{K}', \vec{E} \rangle \right\} \right]
\]

\[
\times \sum_{\alpha, \beta} \left\{ \langle \alpha, \frac{\partial}{\partial k_x} \vec{K}', \vec{E} \rangle \langle \beta, \vec{K}', \vec{E} \rangle \right\}
\]

which reduces to

\[
\langle \hat{\mathcal{J}}_y (t) \rangle = i \epsilon^2 \left( \frac{1}{k} \right) \left| \vec{F} \right| \frac{1}{(2\pi) \hbar} \int \int dk_x dk_y \left[ \sum_{\alpha, \beta} f (E_{\alpha}) \left\{ \langle \alpha, \frac{\partial}{\partial k_x} \vec{K}', \vec{E} \rangle \langle \beta, \vec{K}', \vec{E} \rangle \right\} \right]
\]

\[
\times \sum_{\alpha, \beta} f (E_{\alpha}) \left\{ \langle \alpha, \frac{\partial}{\partial k_x} \vec{K}', \vec{E} \rangle \langle \beta, \vec{K}', \vec{E} \rangle \right\} \right] .
\]

(A15)
Taking the Fourier transform of both sides, we obtain
\[
\langle \hat{j}_y (\omega) \rangle = \left( \frac{1}{k} \right) i e^2 \frac{\epsilon}{\hbar} |\vec{F}| \frac{\delta (\omega)}{(2\pi)} \int \int dk_x dk_y \\
\times \sum_{\alpha, \beta} \left\{ \langle \alpha, \frac{\partial}{\partial k_x} \vec{K}', \mathcal{E} | \beta, \vec{K}', \mathcal{E} \rangle \langle \beta, \vec{K}', \mathcal{E} | \alpha, \frac{\partial}{\partial k_y} \vec{K}', \mathcal{E} \rangle - \langle \alpha, \frac{\partial}{\partial k_y} \vec{K}', \mathcal{E} | \beta, \vec{K}', \mathcal{E} \rangle \langle \beta, \vec{K}', \mathcal{E} | \alpha, \frac{\partial}{\partial k_x} \vec{K}', \mathcal{E} \rangle \right\} f (E_\alpha). \tag{A16}
\]
Taking the limit \( \omega \rightarrow 0 \) and summing over the states \( \beta \), we readily obtain the conductivity, \( \sigma_{yx} \).
\[
\sigma_{yx} = e^2 \frac{1}{k} \sum_{\alpha} f (E_\alpha) \int \int dk_x dk_y \left[ \langle \alpha, \frac{\partial}{\partial k_x} \alpha, \vec{k} \rangle \frac{\partial}{\partial k_y} | \alpha, \vec{k} \rangle - \langle \alpha, \frac{\partial}{\partial k_y} \alpha, \vec{k} \rangle \frac{\partial}{\partial k_x} | \alpha, \vec{k} \rangle \right]. \tag{A17}
\]
Note the transformation from \( \vec{K} \rightarrow \vec{p} \) in the integration in both Eqs. \( (A16) \) and \( (A17) \) has a Jacobian unity. Equation \( (A17) \) is the same expression that can obtained to derive the integer quantum Hall effect from Kubo formula \[1\].

We now prove that for each statevector, \( | \alpha, \vec{k} \rangle \), the expression,
\[
\left( \frac{1}{k} \right) i \frac{1}{(2\pi)} \int \int d\vec{k} \sum_{\alpha} f (E_\alpha) \left[ \langle \alpha, \frac{\partial}{\partial k_x} \alpha, \vec{k} \rangle \frac{\partial}{\partial k_y} | \alpha, \vec{k} \rangle - \langle \alpha, \frac{\partial}{\partial k_y} \alpha, \vec{k} \rangle \frac{\partial}{\partial k_x} | \alpha, \vec{k} \rangle \right], \tag{A18}
\]
is the winding number around the occupied contour in the Brillouin zone. First we can rewrite the terms within the square bracket as
\[
\left[ \langle \alpha, \frac{\partial}{\partial k_x} \alpha, \vec{k} \rangle \frac{\partial}{\partial k_y} | \alpha, \vec{k} \rangle - \langle \alpha, \frac{\partial}{\partial k_y} \alpha, \vec{k} \rangle \frac{\partial}{\partial k_x} | \alpha, \vec{k} \rangle \right] = \langle \frac{\partial}{\partial k} \alpha, \vec{k} \rangle \times \langle \frac{\partial}{\partial k} \alpha, \vec{k} \rangle = \nabla_{\vec{k}} \times \langle \alpha, \vec{k} \rangle \frac{\partial}{\partial k} | \alpha, \vec{k} \rangle. \tag{A19}
\]
The last term indicates the operation of the curl of the Berry connection which is related to the quantization of Hall conductivity. This quantization is due to the uniqueness of the parallel-transported wavefunction, which may also have bearing on the self-consistent Bohr-Sommerfeld quantization \[8\].

At low temperature, we can just write Eq. \( (A17) \) as,
\[
\sigma_{yx} = \left( \frac{1}{k} \right) e^2 \frac{1}{2\pi \hbar} \int \int_{\text{sub-band}} d\vec{k} \sum_{\alpha} f (E_\alpha) \left[ \nabla_{\vec{k}} \times \langle \alpha, \vec{k} \rangle \frac{\partial}{\partial k} | \alpha, \vec{k} \rangle \right]_{\text{plane}},
\]
\[
= \left( \frac{1}{k} \right) e^2 \frac{1}{2\pi \hbar} \int_{\text{sub-band}} d\vec{k} \left[ \langle LL, \vec{k} \rangle i \frac{\partial}{\partial k_c} | LL, \vec{k} \rangle \right]_{\text{contour}}. \tag{A20}
\]
where the quantity within the parenthesis gives $n_{LL} \in \mathbb{Z}$ which is the winding number or the Chern number. In the experiment, $n_{LL} = 1$ Therefore

$$
\sigma_{yx} = \left( \frac{1}{k} \right) \frac{e^2}{h},
= \frac{e^2}{h} v
$$

(A21)

For lower magnetic fields below the magnetic field strength of the fully occupied LLL, the lowered fully occupied sub-bands simply serve as a background for the Hall conductance, where the last fractionally occupied sub-band dictates the FQHE. Thus the FQHE conductivity is given by $\sigma_{yx} = \frac{e^2}{h} v$ below and above the magnetic fields of the fully occupied LLL. This is what is found in the experiments for magnetic field strength beyond and below the fully occupied LLL. We emphasize that the prime-number remainders in going from $v = 1$ to $v = 2$ in the experiments increases as the magnetic fields decreases.

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