Projective analysis and preliminary group classification of the nonlinear fin equation $u_t = (E(u)u_x)_x + h(x)u$

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Abstract

In this paper we investigate for further symmetry properties of the nonlinear fin equations of the general form $u_t = (E(u)u_x)_x + h(x)u$ rather than recent works on these equations. At first, we study the projective (fiber–preserving) symmetry to show that equations of the above class can not be reduced to linear equations. Then we determine an equivalence classification which admits an extension by one dimension of the principal Lie algebra of the equation. The invariant solutions of equivalence transformations and classification of nonlinear fin equations among with additional operators are also given.

Key words: Nonlinear fin equations, Lie symmetries, Optimal system.

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1 Introduction

Investigations for symmetry properties of mathematical models of heat conductivity and diffusion processes [9] are traditionally formulated in terms of nonlinear differential equations which often envisage us with difficulties in studying. To solve this problem, symmetry methods play a key role for finding their exact solutions, similar solutions [2,3,15,17,18] and invariants.

In this study we generalize the study of a class of nonlinear fin equations which has been recently studied in some references and specially in [17,18]. So, we are dealing with the class of nonlinear fin equations of the general form

$$u_t = (E(u)u_x)_x + h(x)u,$$

(1.1)

in which we assumed that $E_u \neq 0$, $u$ is treated as the dimensionless temperature, $t$ and $x$ the dimensionless time and space variables, $E$ the thermal conductivity, $h = -N^2 f(x)$, $N$ the fin parameter and $f$ the heat transfer coefficient [2].

The Lie point symmetry in linear and nonlinear case, the condition $E_u = 0$ corresponds to the linear case, the class of nonlinear one-dimensional diffusion equations when $h = 0$, the class of diffusion-reaction equations when $h = \text{cons.}$, the case which the thermal conductivity is a power function of the temperature and additional equivalence transformations, conditional equivalence groups and nonclassical symmetries have all investigated and listed in Table 1 of [18]. The point symmetry group of nonlinear fin equations of class (1.1) were considered in a number of papers (e.g. see [2] for the physical meaning and applications of the equation). The Lie algebra of the point symmetry group of Eq. (1.1) is

$$\mathfrak{g}_1 := \left\{ \frac{\partial}{\partial u} \right\},$$

(1.2)

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In the next section, we concern with the problem of finding projective symmetry group of Eq. (1.1) as a special case of the point symmetry group; since it may have important information about the equation.

The equivalence classification of Eq. (1.1) in the special case

$$u_t = (E(u) u_x)_x,$$  \hspace{1cm} (1.3)

has performed by L.V. Ovsiannikov [14]. In [17] authors carried out the another special case of equivalence group

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4, \quad \tilde{u} = \delta_5 u, \quad \tilde{E} = \delta_1^{-1} \delta_2^2 E, \quad \tilde{h} = \delta_1^{-1} h,$$  \hspace{1cm} (1.4)

of Eq. (1.1) when \(\delta_i, \ i = 1, \cdots, 5,\) are arbitrary constants and \(\delta_1 \delta_3 \delta_5 \neq 0.\) The more general class of nonlinear fin equations is the nonlinear heat conductivity equations of the form

$$u_t = F(t, x, u, u_x)u_{xx} + G(t, x, u, u_x),$$  \hspace{1cm} (1.5)

which admits non-trivial symmetry group. The group classification of (1.5) is presented in some references [1,11]. However, since the equivalence group of (1.5) is essentially wider than those for particular cases, the results of [1,11] cannot be directly used for symmetry classification of particular ones. Nevertheless, these results are useful for finding additional equivalence transformations in the class of our problem. Therefore in contrast to the above works, in the last two sections of this paper, we study group classification of Eq. (1.1) under equivalence transformations in the general case. Furthermore, a number of nonlinear invariant models which have nontrivial invariance algebras are obtain.

From [19] we know that if the partial differential equation possesses non-trivial symmetry, then it is invariant under some finite-dimensional Lie algebra of differential operators which is completely determined by its structural constants. In the event that the maximal algebra of invariance is infinite–dimensional, then it contains, as a rule, some finite-dimensional Lie algebra. Also, if there are local non-singular changes of variables which transform a given differential equation into another, then the finite-dimensional Lie algebra of invariance of these equations are isomorphic, and in the group-theoretic analysis of differential equations such equations are considered to be equivalent. To realize the group classification, we use of the proposed approach consists in the implementation of an algorithm explained and performed in references [1,10,14,16]. For this goal, our method is completely similar to the way of [6] for the nonlinear wave equation \(u_{tt} = f(x, u)u_{xx} + g(x, u).\)

2 Projective symmetries of nonlinear fin equations

In this section, we are concerning with group classification of nonlinear fin equations by projective transformations group as a special case of the point symmetry group. Our study is based on the method of [13] for Lie infinitesimal method.

The equation is a relation among with the variables of 2–jet space \(J^2(\mathbb{R}^2, \mathbb{R})\) with (local) coordinate

\[
(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}),
\]

(2.6)

where this coordinate involving independent variables \(t, x\) and dependent variable \(u\) and derivatives of \(u\) in respect to \(t\) and \(x\) up to order 2 (each index will indicate the derivation with respect to it, unless we specially state otherwise). Let \(\mathcal{M}\) be the total space of independent and dependent variables resp. \(t, x\) and \(u\). The solution space of Eq. (1.1), (if it exists) is a subvariety \(\mathcal{S}_\Delta \subset J^2(\mathbb{R}^2, \mathbb{R})\) of the second order jet bundle of 2-dimensional sub-manifolds of \(\mathcal{M}\). If we wish to preserve the bundle structure of the space \(\mathcal{M}\), we must restrict to the class of fiber–preserving transformations in which the changes in the independent variable are unaffected by the dependent variable. Projective or fiber–preserving symmetry group on \(\mathcal{M}\) is introduced by transformations in the form of

\[
\tilde{x} = \phi(t, x), \quad \tilde{t} = \chi(t, x), \quad \tilde{u} = \psi(t, x, u),
\]

(2.7)

for arbitrary smooth functions \(\phi, \chi, \psi\). Also assume that

\[
v := \xi^1(t, x) \frac{\partial}{\partial t} + \xi^2(t, x) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u},
\]

(2.8)
with coefficients as arbitrary smooth functions be the general form of infinitesimal generators which signify the Lie algebra $g$ of the projective symmetry group $G$ of Eq. (1.1). The second order prolongation of $v$ \cite{13,14} as a vector field on $J^2(\mathbb{R}^2,\mathbb{R})$ is as follows

$$v^{(2)} := v + \eta^t \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^{tt} \frac{\partial}{\partial u_{tt}} + \eta^{tx} \frac{\partial}{\partial u_{tx}} + \eta^{xx} \frac{\partial}{\partial u_{xx}},$$  \hspace{1cm} (2.9)$$

where $\eta^t, \eta^x$ and $\eta^{tt}, \eta^{tx}, \eta^{xx}$ are arbitrary smooth functions depend to variables $t, x, u, p, q$ and (2.6) resp. These coefficients are computed as following

$$\eta^J = D_J(Q) + \xi^1 u_{f,t} + \xi^2 u_{f,x}$$  \hspace{1cm} (2.10)$$

where $D$ is total derivative, $J$ is a multi-index with length $1 \leq |J| \leq 2$ of variables $t, x$ and $Q = u - \xi^1 u_t - \xi^2 u_x$ is characteristic of $v$ \cite{13}. According to \cite{13}, $v$ is a projective infinitesimal generator of Eq. (1.1) if and only if $v^{(2)}[\text{Eq. (1.1)}] = 0$. By applying $v^{(2)}$ on the equation we have the following equation

$$\xi^2 h_x u + \eta(D_{uu} q^2 + D_u q_x) - \eta^2 + 2 \eta^x D_u q + \eta^{xx} D = 0,$$  \hspace{1cm} (2.11)$$

In the extended form of the latter equation, functions $\xi^1, \xi^2$ and $\eta$ only depend to $t, x, u$ rather than other variables, i.e. $u_t, u_x, u_{tt}, u_{tx}, u_{xx}$, hence the equation will be satisfied if and only if the individual coefficients of the powers of $u_t, u_x$ and their multiplications vanish. This tends to the following over-determined system of determining equations

$$E_u \xi^1_1 = 0, \quad E_x \xi^1_2 = 0, \quad \eta_u - \xi^1_1 + E \xi^1_2 = 0,$$
$$E_u \eta + 2E \xi^2_2 = 0, \quad \xi^2 h_x u - \eta_t + E \eta_{xx} + h \eta = 0,$$
$$E_{uu} \eta + 2E_u (\eta_{u} - \xi^2_x) + E \eta_{uu} = 0, \quad \xi^2 \xi^2 + 2E_u \eta_x + E(2 \eta_{xx} - \xi^2_{xx}) = 0.$$  \hspace{1cm} (2.12)$$

By solving Eq. (2.12), the general solution to these differential equations for $\xi^1, \xi^2$ and $\eta$ will be found:

$$\xi^1(t, x) = c, \quad \xi^2(t, x) = 0, \quad \eta(t, x, u) = 0,$$  \hspace{1cm} (2.13)$$

with arbitrary constant $c$. Therefore the Lie algebra $g$ spanned by projective infinitesimal generators of Eq. (1.1) is $g = \langle \frac{\partial}{\partial t} \rangle$ and the projective symmetry group is nothing but the time translation group.

**Theorem 1** A complete set of all infinitesimal generators of nonlinear fin equation up to projective transformations admits the structure of one-dimensional Lie algebra and so is isomorphic to $\mathbb{R}$. The point symmetry group and projective symmetry of nonlinear fin equation are equal.

It is well–known that the existence of a non–fiber–preserving symmetry usually indicates that one can significantly simplify the equation by some kind of hodograph–like transformation interchanging the independent and dependent variables. But in the case of our problem, we can not use this advantage for simplifying nonlinear fin equations.

According to \cite{13}, any system of partial differential equations which has only a finite–dimensional symmetry group is certainly not linearizable, that is, for every change of variables, it can not be mapped to an inhomogeneous form of the linear system $D[u] = f$, where $D$ is a second order linear differential operator, $u$ indicates dependent variables and $f$ denotes smooth functions of independent variables.

**Remark 2** A system of nonlinear fin equations in the form of Eq. (1.1) can not be reduced into an inhomogeneous form of a linear system.

### 3 Equivalence transformations

At the present section we follow the method of L.V. Ovsiannikov \cite{14} for partial differential equations. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study. It is possible to modify Lie’s algorithm in order to make it applicable for the computation of this group \cite{1,10,14,16}. Next we construct the optimal system of subgroups of the equivalence group.
where invariance conditions of the system (3.15) are infinitesimal method.

We investigate for an operator of the group \( G_C \) in the general form

\[
Y := \xi^t(t,x) \frac{\partial}{\partial t} + \xi^x(t,x) \frac{\partial}{\partial x} + \eta(t,x,u) \frac{\partial}{\partial u} + \varphi(t,x,u,E,h) \frac{\partial}{\partial E} + \chi(t,x,u,E,h) \frac{\partial}{\partial h}.
\]  

(3.14)

from the invariance conditions of Eq. (1.1) written as the system

\[
u_t = (E(u)u_x)_x + h(x)u, \quad E_t = E_x = 0, \quad h_t = h_u = 0,
\]  

(3.15)

where we assumed that \( u, E, h \) are differential variables: \( u \) on the base space \( (t,x) \) and \( E, h \) on the total space \( (t,x,u) \). Also in Eq. (3.14) the coefficients depend to variables \( t, x, u \) and the two last ones, in addition, depend to \( E, h \). The invariance conditions of the system (3.15) are

\[
\tilde{Y} [u_t - (E(u)u_x)_x - h(x)u] = 0, \quad \tilde{Y} [E_t] = \tilde{Y} [E_x] = 0, \quad \tilde{Y} [h_t] = \tilde{Y} [h_u] = 0,
\]  

(3.16)

where

\[
\tilde{Y} := Y + \eta^t \frac{\partial}{\partial u_x} + \eta^x \frac{\partial}{\partial u_t} + \eta^u \frac{\partial}{\partial u_{xt}} + \eta^{xx} \frac{\partial}{\partial u_{tx}} + \varphi^t \frac{\partial}{\partial E_t} + \varphi^x \frac{\partial}{\partial E_x} + \chi^t \frac{\partial}{\partial h_t} + \chi^u \frac{\partial}{\partial h_u}.
\]  

(3.17)

is the prolongation of the operator (3.14). Coefficients \( \eta^J \) for multi–index \( J \) (with length \( 1 \leq |J| \leq 2 \)) have given in section 2 and by applying the prolongation procedure to differential variables \( E, h \) with independent variables \( (t, x, u) \) we have

\[
\begin{align*}
\varphi^t &= \bar{D}_t(\varphi) - E_t \bar{D}_t(\xi^t) - E_x \bar{D}_t(\xi^x) - E_u \bar{D}_t(\eta) - \bar{D}_t(\varphi) - E_u \bar{D}_t(\eta), \\
\varphi^x &= \bar{D}_x(\varphi) - E_x \bar{D}_x(\xi^t) - E_x \bar{D}_x(\xi^x) - E_u \bar{D}_x(\eta) - \bar{D}_x(\varphi) - E_u \bar{D}_x(\eta), \\
\chi^t &= \bar{D}_t(\chi) - h_t \bar{D}_t(\xi^t) - h_x \bar{D}_t(\xi^x) - h_u \bar{D}_t(\eta) - \bar{D}_t(\chi) - h_x \bar{D}_t(\eta), \\
\chi^x &= \bar{D}_u(\chi) - h_t \bar{D}_u(\xi^t) - h_x \bar{D}_u(\xi^x) - h_u \bar{D}_u(\eta) - \bar{D}_u(\chi) - h_x \bar{D}_u(\eta),
\end{align*}
\]  

(3.18)

where in view of Eq. (3.15) we have

\[
\begin{align*}
\bar{D}_t := \frac{\partial}{\partial t} + E_t \frac{\partial}{\partial E} + h_t \frac{\partial}{\partial h} = \frac{\partial}{\partial t}, \\
\bar{D}_x := \frac{\partial}{\partial x} + E_x \frac{\partial}{\partial E} + h_x \frac{\partial}{\partial h} = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h}, \\
\bar{D}_u := \frac{\partial}{\partial u} + E_u \frac{\partial}{\partial E} + h_u \frac{\partial}{\partial h} = \frac{\partial}{\partial u} + E_u \frac{\partial}{\partial E}.
\end{align*}
\]  

(3.19)

Substituting (3.17) in (3.16) we tend to the following system

\[
\begin{align*}
\varphi u_{xx} + h \eta + \chi u - \eta^t + 2E_u u_x \eta^x + \varphi^u u_x^2 + E \eta^{xx} &= 0, \\
\varphi^t &= 0, \quad \varphi^x = 0, \\
\chi^t &= 0, \quad \chi^u &= 0.
\end{align*}
\]  

(3.20)

(3.21)

(3.22)

Replacing relations \( \eta^t \) (for multi–index \( J \) with length \( 1 \leq |J| \leq 2 \)) and \( \chi^t, \chi^x \) in Eqs. (3.20)–(3.22) and then introducing the relation \( u_t = (E(u)u_x)_x + hu \) to eliminate \( u_t \), we have five relations which are called determining equations. The four last ones are determining equations associated with Eqs. (3.21), (3.22), i.e.,

\[
\begin{align*}
\varphi_t - E_u \eta_t &= 0, \\
\varphi_x + h_x \varphi_h - E_u \eta_x &= 0, \\
\chi_t - h_x \xi^t &= 0, \\
\chi_u + E_u \chi_E - h_x \xi^2_u &= 0.
\end{align*}
\]  

(3.23)
But these relations must hold for every \( E \) and \( h \) and this fact results in the following conditions
\[
\xi_t^2 = \xi_u^2 = 0, \quad \eta_t = \eta_u = 0, \quad \varphi_t = \varphi_x = \varphi_h = 0, \quad \chi_t = \chi_u = \chi_E = 0,
\]
so, we find that
\[
\xi^2 = \xi^2(x), \quad \eta = \eta(u), \quad \varphi = \varphi(u, E), \quad \chi = \chi(x, h).
\] (3.25)

Adding these conditions to the remained determining equation, since \( u_t, u_x, u_{tt}, u_{tx}, u_{xx} \) are considered to be independent variables, we lead to the following system of equations
\[
\begin{align*}
    h \eta + \chi u + h u (\xi_1^2 - \eta_u) &= 0, \quad E_u (\eta_u + \xi_1^2 - 2 \xi_2^2) + E \eta_{uu} + E_{uu} \eta = 0, \\
    \varphi + \varphi_u + E (\xi_1^2 - 2 \xi_2^2) &= 0, \quad E \xi_1^2 = 0, \quad E \xi_1^2 = 0, \quad E \xi_2^2 = 0.
\end{align*}
\] (3.26)

This system follows
\[
\xi_1 = 2 c_1 t + c_2, \quad \xi_2 = c_1 x + c_3, \quad \eta = 0, \quad \varphi = e^{-u} F(E), \quad \chi = -2 c_1 h,
\] (3.27)

with arbitrary function \( F = F(E) \) and constants \( c_1, c_2, c_3 \). Therefore the class of Eqs. (1.1) has an infinite continuous group of equivalence transformations generated by infinitesimal operators
\[
Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2 t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2 h \frac{\partial}{\partial h}, \quad Y_F = e^{-u} F(E) \frac{\partial}{\partial E}.
\] (3.28)

Moreover, in the group of equivalence transformations are included also discrete transformations, i.e., reflections
\[
t \mapsto -t, \quad x \mapsto -x, \quad u \mapsto -u, \quad E \mapsto -E, \quad h \mapsto -h.
\] (3.29)

The communication relations between these vector fields is given in Table 1. The Lie algebra \( g := \langle Y_F, Y_i : i = 1, \cdots, 3 \rangle \) is solvable since the descending sequence of derived subalgebras of \( g: g \supset g^{(1)} = \langle 2 Y_1, Y_2 \rangle \supset g^{(2)} = \{0\} \), terminates with a null ideal. But its Killing form: \( K(v, w) = \text{tr}(\text{ad}(v) \circ \text{ad}(w)) = 5 a_3 b_3 \) for each \( v = \sum_i v_i Y_i + v_F Y_F \) and \( w = \sum_j w_j Y_j + w_F Y_F \) in \( g \) is degenerate and hence \( g \) is neither semisimple nor simple.

**Theorem 3** Let \( G_i \) be the one–parameter groups generated by the \( Y_i \), then we have
\[
G_1 : (t, x, u, E, h) \mapsto (t + s, x, u, E, h), \quad G_2 : (t, x, u, E, h) \mapsto (t, x + s, u, E, h), \quad G_3 : (t, x, u, E, h) \mapsto (t e^{2 s}, x e^s, u, E, h e^{2 s}), \quad G_4 : (t, x, u, E, h) \mapsto (t, x, u, \tilde{E}, h),
\] (3.30)

when \( \tilde{E} \) is the solution of equation \( \int_{\tilde{E}}^E d\alpha / F(\alpha) = e^{-u} s \) for a constant \( c \). Furthermore, if \( u = f(t, x) \) for functions \( E \) and \( h \) be a solution of nonlinear fin equation, so are the following functions
\[
u_1 = f(t + s, x), \quad \nu_2 = f(t, x + s),
\] (3.31)

for the same functions \( E \) and \( h \), \( \nu_3 = f(t e^{2 s}, x e^s, u) \) for the same functions \( E \) and \( h = h e^{2 s} \) and also \( u_4 = f(t, x, u) \) for \( \tilde{E} \) and the same \( h \).

4 Preliminary group classification

In many applications of group analysis, most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra \( g_E \). These extensions are called \( E \)-extensions of the principal Lie algebra. The classification of all nonequivalent equations (with respect to a given equivalence group \( G_E \)) admitting \( E \)-extensions of the principal Lie algebra is called a **preliminary group classification** [10]. We can take
any finite-dimensional subalgebra (desirable as large as possible) of an infinite-dimensional algebra with basis (3.28) and use it for a preliminary group classification. We select the subalgebra \( g_4 \) spanned on the following operators:

\[
\begin{align*}
Y_1 &= \frac{\partial}{\partial t}, \\
Y_2 &= \frac{\partial}{\partial x}, \\
Y_3 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2h \frac{\partial}{\partial h}, \\
Y_4 &= e^{-u} E \frac{\partial}{\partial E}.
\end{align*}
\]  

(4.32)

It is well-known that the problem of classifying invariant solutions is equivalent to the problem of classifying subgroups of the full symmetry group under conjugation in which itself is equivalent to determining all conjugate subalgebras \([13,14]\). The latter problem, tends to determine a list (that is called an optimal system) of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation i.e. \( \text{Ad}(g) \) for some \( g \) of a considered Lie group. Thus we will deal with the construction of the optimal system of subalgebras of \( g_4 \).

The adjoint action is given by the Lie series

\[
\text{Ad}(\exp(sY_i))Y_j = Y_j - s [Y_i, Y_j] + \frac{s^2}{2} [Y_i, [Y_i, Y_j]] - \cdots,
\]

(4.33)

where \( s \) is a parameter and \( i, j = 1, \cdots, 4 \). The adjoint representations of \( g_4 \) is listed in Tables 2; it consists the separate adjoint actions of each element of \( g_4 \) on all other elements.

**Theorem 4** An optimal system of one-dimensional Lie subalgebras of nonlinear fin equation (1.1) is provided by those generated by

1) \( A^1 = Y_1 = \partial_t \),
2) \( A^2 = Y_2 = \partial_x \),
3) \( A^3 = Y_3 = 2t \partial_t + x \partial_x - 2h \partial_h \),
4) \( A^4 = Y_4 = e^{-u} E \partial_E \),
5) \( A^5 = \alpha Y_1 + Y_2 = \alpha \partial_t + \partial_x \),
6) \( A^6 = \beta Y_3 + Y_4 = 2\beta t \partial_t + \beta x \partial_x - 2\beta h \partial_h + e^{-u} E \partial_E \),

(4.34)

for nonzero constants \( \alpha, \beta \).

**Proof.** Let \( g_4 \) is the symmetry algebra of Eq. (1.1) with adjoint representation determined in Table 2 and

\[
Y = a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4
\]

(4.35)
is a nonzero vector field of $\mathfrak{g}$. We will simplify as much of the coefficients $a_i$ as possible through proper adjoint applications on $Y$. We follow our aim in the below easy cases.

**Case 1** At first, assume that $a_4 \neq 0$. Scaling $Y$ if necessary, we can consider $a_4$ to be 1 and so follow the problem with

$$Y = a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + Y_4. \quad (4.36)$$

According to Table 2, if we act on $Y$ by $\text{Ad}(\exp(\frac{1}{a_1} Y_1))$, the coefficient of $Y_1$ can be vanished:

$$Y' = a_2 Y_2 + a_3 Y_3 + Y_4. \quad (4.37)$$

Then we apply $\text{Ad}(\exp(a_2 Y_2))$ on $Y'$ to cancel the coefficient of $Y_2$:

$$Y'' = a_3 Y_3 + Y_4. \quad (4.38)$$

**Case 1a** If $a_3 \neq 0$ then we can not simplify the coefficient of $Y_3$ to be either $+1$ or $-1$. Thus any one–dimensional subalgebra generated by $Y$ with $a_3, a_4 \neq 0$ is equivalent to one generated by $\beta Y_3 + Y_4$ which introduce parts 6) of the theorem for constant $\beta \neq 0$.

**Case 1b** For $a_3 = 0$ we can see that each one–dimensional subalgebra generated by $Y$ is equivalent to $\langle Y_4 \rangle$.

**Case 2** The remaining one–dimensional subalgebras are spanned by vector fields of the form $Y$ with $a_4 = 0$.

**Case 2a** If $a_3 \neq 0$ then by scaling $Y$, we can assume that $a_3 = 1$. Now by the action of $\text{Ad}(\exp(\frac{1}{a_1} Y_1))$ on $Y$, we can cancel the coefficient of $Y_1$:

$$\tilde{Y} = a_2 Y_2 + Y_3. \quad (4.39)$$

Then by applying $\text{Ad}(\exp(a_2 Y_2))$ on $\tilde{Y}$ the coefficient of $Y_2$ can be vanished and we tend to part 3) of the theorem.

**Case 2b** Let $a_3 = 0$ then $Y$ is in the form

$$\tilde{Y} = a_1 Y_1 + a_2 Y_2. \quad (4.40)$$

Suppose that $a_2 \neq 0$ then if necessary we can let it equal to 1. the simplest possible form of $Y$ is equal to $\tilde{Y} = a_1 Y_1 + Y_2$ after taking $a_2 = 1$.

**Case 2b-1** Let $a_1$ be nonzero. In this case we can not make the coefficient of $Y_1$ in $\tilde{Y}$ more simplier and find 5) section of the theorem.

**Case 2b-2** If $a_1$ is zero then by scaling we can make the coefficient of $\tilde{Y}$ equal to 1. Hence this case suggests part 2).

**Case 2c** Finally if in the latter case $a_2$ be zero, then no further simplification is possible and then $Y$ is reduced to 1).

There is not any more possible case for studying and the proof is complete. \[\square\]

The coefficients $E, h$ of Eq. (1.1) resp. depend on the variables $u, x$. Therefore, we take their optimal system’s projections on the space $(x, u, E, h)$. The nonzero in $x$–axis or $u$–axis projections of (4.34) are

$$1) \quad Z^1 = A^2 = A^5 = \partial_x, \quad 3) \quad Z^3 = A^4 = e^{-u} E \partial_E,$$

$$2) \quad Z^2 = A^3 = x \partial_x - 2h \partial_h, \quad 4) \quad Z^4 = A^6 = \beta x \partial_x - 2\beta h \partial_h + e^{-u} E \partial_E. \quad (4.41)$$

From paper 7 of [10] we conclude that

**Proposition 5** Let $\mathfrak{g}_m := \langle Y_i : i = 1, \cdots, m \rangle$ be an $m$–dimensional algebra. Denote by $A^i(i = 1, \cdots, s, 0 < s \leq m, s \in \mathbb{N})$ an optimal system of one–dimensional subalgebras of $\mathfrak{g}_m$ and by $Z^i(i = 1, \cdots, t, 0 < t \leq s, t \in \mathbb{N})$ the projections of $A^i$, i.e., $Z^i = \text{pr}(A^i)$. If equations

$$F = F(u), \quad g = g(x), \quad (4.42)$$


are invariant with respect to the optimal system $Z^i$ then the equation

$$ u_t = (F(u)u_x)_x + g(x)u, \quad (4.43) $$

admits the operators $X^i = \text{projection of } A^i \text{ on } (t,x,u)$.

**Proposition 6** Let Eq. (4.43) and the equation

$$ u_t = (\bar{F}(u)u_x)_x + \bar{g}(x)u, \quad (4.44) $$

be constructed according to Proposition 5 via optimal systems $Z^i$ and $\bar{Z}^i$ resp. If the subalgebras spanned on the optimal systems $Z^i$ and $\bar{Z}^i$ resp. are similar in $g_m$, then Eqs. (4.43) and (4.44) are equivalent with respect to the equivalence group $G_m$ generated by $g_m$.

Now by applying Propositions 5 and 6 for the optimal system (4.41), we want to find all nonequivalent equations in the form of Eq. (1.1) admitting $E$-extensions of the principal Lie algebra $g_E$, by one dimension, i.e, equations of the form (1.1) such that they admit, together with the one basic operators (1.2) of $g_1$, also a second operator $X^{(2)}$. In each case which this extension occurs, we indicate the corresponding coefficients $E, h$ and the additional operator $X^{(2)}$.

We perform the algorithm passing from operators $Z^i (i = 1, \cdots, 4)$ to $E, h$ and $X^{(2)}$ via the following example.

Let consider the vector field

$$ Z^4 = -x \frac{\partial}{\partial x} + 2h \frac{\partial}{\partial h} + e^{-u}E \frac{\partial}{\partial E}, \quad (4.45) $$

then the characteristic equation corresponding to $Z^4$ is

$$ \frac{dx}{\beta x} = \frac{dh}{2\beta h} = \frac{dE}{e^{-u}E}, \quad (4.46) $$

which determines invariants. Invariants can be taken in the following form

$$ I_1 = u, \quad I_2 = h^{1/2}x, \quad I_3 = x^{-e^{-u}E}. \quad (4.47) $$

In this case there are no invariant equations because the necessary condition for existence of invariant solutions (see [14], Section 19.3) is not satisfied, i.e., invariants (4.47) cannot be solved with respect to $E$ and $h$ since $I_3$ is not an invariant function of $I_1$ and $I_2$ to derive a function in term of $u$ for $E$.

Considering $Z^2$ we have the below characteristic equation

$$ \frac{dx}{-x} = \frac{dh}{2h}, \quad (4.48) $$

This equation suggest the following invariants

$$ I_1 = u, \quad I_2 = h^{1/2}x, \quad I_3 = E. \quad (4.49) $$

From the invariance equations we can write

$$ I_2 = \phi(I_1), \quad I_3 = \psi(I_1), \quad (4.50) $$

provided that $\phi(I_1)$ is a function in term of variable $u$ and $\psi(I_1)$ an arbitrary function. The first condition occurs when $\phi(I_1) = c$ for constant $c$. It results in the forms

$$ E = \phi(u), \quad h = \left(\frac{c}{x}\right)^2. \quad (4.51) $$
Table 3
The result of the classification

| N  | Z   | Invariant | Equation                      | Additional operator $X^{(2)}$ |
|----|-----|----------|-------------------------------|-------------------------------|
| 1  | $Z^1$ | $u$      | $u_t = (\phi(u) u_x)_x + c u$ | $\frac{\partial^2}{\partial t^2} \alpha \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$ |
| 2  | $Z^2$ | $u$      | $u_t = (\phi(u) u_x)_x + (\xi)^2 u$ | $2 t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}$ |

From Proposition 5 applied to the operator $Z^4$ we obtain the additional operator

$$X^{(2)} = 2 \beta t \frac{\partial}{\partial t} + \beta x \frac{\partial}{\partial x}. \quad (4.52)$$

One can perform the algorithm for other $Z^i$ s of (4.41) similarly. The preliminary group classification of nonlinear fin equation (1.1) admitting an extension $g_2$ of the principal Lie algebra $g_1$ is listed in Table 3.

5 Conclusion

Projective analysis as a new symmetry property of equations $u_t = (E(u) u_x)_x + h(x) u$ rather than previous results on this equation [17,18], is carried out exhaustively. Also, equivalence classification is given of the equation admitting an extension by one of the principal Lie algebra of the equation. The paper is one of few applications of a new algebraic approach to the problem of group classification: the method of preliminary group classification. Derived results are summarized in Table 3.

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