OPENNESS OF MOMENTUM MAPS AND PERSISTENCE OF EXTREMAL RELATIVE EQUILIBRIA

JAMES MONTALDI AND TADASHI TOKIEDA

Abstract. We prove that for every proper Hamiltonian action of a Lie group $G$ in finite dimensions the momentum map is locally $G$-open relative to its image (i.e. images of $G$-invariant open sets are open). As an application we deduce that in a Hamiltonian system with continuous Hamiltonian symmetries, extremal relative equilibria persist for every perturbation of the value of the momentum map, provided the isotropy subgroup of this value is compact. We also demonstrate how this persistence result applies to an example of ellipsoidal figures of rotating fluid, and provide an example with plane point vortices which shows how the compactness assumption is related to persistence.

Mathematics Subject Classification: 53D20, 37J15

1. Introduction

In a Hamiltonian system, nondegenerate equilibria are isolated; in particular, they do not persist from one energy level to nearby levels. In this paper we prove that in a symmetric Hamiltonian system, every extremal relative equilibrium persists to nearby levels of the momentum map, provided the isotropy subgroup of its momentum value is compact. The crucial ingredient in the proof is a generalisation of Sjamaar’s result on the openness of momentum maps.

Let $M$ be a symplectic manifold with a proper and symplectic action of a connected Lie group $G$ and let $h$ be a $G$-invariant Hamiltonian. We suppose that the action of $G$ is Hamiltonian, in that it is infinitesimally generated by a momentum map $\Phi : M \to \mathfrak{g}^*$. A trajectory of the Hamiltonian vector field $X_h$ of $h$ is a relative equilibrium if its image in the orbit space $M/G$ is a single point; such a relative equilibrium with momentum value $\mu$ is extremal if its image in the reduced space $\Phi^{-1}(\mu)/G_\mu$ is a local extremum for the reduced Hamiltonian. $G_\mu$ denotes the isotropy subgroup of $\mu$ for the possibly modified coadjoint action of $G$ on $\mathfrak{g}^*$ (Section 2).

Let $f : X \to Y$ be a $G$-equivariant map. We say that $f$ is $G$-open if the image of any $G$-invariant open set is open in $Y$. This is equivalent to the orbit map $\bar{f} : X/G \to Y/G$ being open.

Suppose that $\gamma$ is an extremal relative equilibrium with momentum value $\mu$. We wish to prove that under certain hypotheses such a relative equilibrium persists to all nearby values of the momentum map. However, since this is really a local result in the phase space $M$, the question arises as to what is meant by all nearby values. If the momentum map $\Phi$ is proper then a result of Sjamaar [13] (see also [1]) says that the momentum map is $G$-open, so that images of $G$-invariant neighbourhoods of $\gamma$ are open in the image of $\Phi$, and the phrase ‘all nearby values’ means just that: a full $G$-invariant neighbourhood of $\mu$ in $\Phi(M)$. On the other hand, if $\Phi$ is not proper, then it may not be $G$-open (for an example, see [6, Example 3.10]), so the image of a $G$-invariant neighbourhood of $\gamma$ may not be open in $\Phi(M)$. Our first

Date: January, 2002.

The research of JM was partially supported by the European Union through the Research Training Network MASIE.
result shows that there is always a $G$-invariant neighbourhood $U_0$ of $\gamma$ restricted to which the momentum map $U_0 \to \Phi(U_0)$ is $G$-open relative to its image. This neighbourhood $U_0$ is a tubular neighbourhood of the group orbit containing $\gamma$ whose existence follows from the Marle-Guillemin-Sternberg normal form for symplectic group actions (Section 3). Of course, if $\Phi$ is proper one can take $U_0 = M$.

**Theorem 1.** Let $M$ be a symplectic manifold with a proper Hamiltonian action of a connected Lie group $G$ and a momentum map $\Phi : M \to \mathfrak{g}^*$. Suppose that $x \in M$ has momentum value $\mu = \Phi(x)$ whose isotropy subgroup $G_{\mu}$ for the modified coadjoint action is compact. Then there exists a $G$-invariant neighbourhood $U_0$ of $x$ such that the restriction $\Phi|_{U_0} : U_0 \to \Phi(U_0)$ is $G$-open, where $\Phi(U_0)$ is given the subspace topology induced from $\mathfrak{g}^*$.

This result then allows us to state the persistence theorem for extremal relative equilibria.

**Theorem 2.** Let $M, G, \Phi, x, \mu$ and $U_0$ be as in Theorem 1; let $h \in C^\infty(M)$ be a $G$-invariant Hamiltonian, and suppose that $\gamma$ is an extremal relative equilibrium for the given Hamiltonian system, with $x \in \gamma$. Then there exists a $G$-invariant neighbourhood $V$ of $\mu$ in $\Phi(U_0)$ such that for every $\mu' \in V$, there is a relative equilibrium in $\Phi^{-1}(\mu') \cap U_0$.

When the $G$-action is trivial, $\Phi^{-1}(\mu') = M$, so the theorem becomes trivial, too. Theorems in Hamiltonian systems often have natural generalisations to those in symmetric Hamiltonian systems, when a group action is thrown in. Theorem 2 is an instance of a theorem in the latter that has no nontrivial specialisation in the former.

In [12], it was shown that extremal relative equilibria are Lyapunov-stable relative to $G$. Also in [12] appeared a version of persistence, but the proof is incomplete. The present version is stronger, first because it does not require $G$ to be compact, but just $G_{\mu}$, which actually suffices to reduce to the compact case (Section 3), second because it proves persistence to a full neighbourhood of $\mu$. To our knowledge, Theorem 2 is the first application of this topological property of the $G$-openness of momentum maps to problems of Hamiltonian dynamical systems.

After its proof in Section 3, Theorem 2 is applied in Section 4 to the problem of ellipsoidal figures of rotating fluid (affine rigid bodies). In Section 5 we check that the compactness hypothesis on the isotropy subgroup $G_{\mu}$ is essential in Theorem 2 by analysing point vortices on the plane. Finally, in Section 6 we explain how reduction by stages yields a partial persistence result even in the case of noncompact momentum isotropy. For complementary results on persistence of relative equilibria for noncompact group actions, see Wulf [14].

2. **Modification of coadjoint action**

Theorem 1 does not assume the equivariance of the momentum map $\Phi$ with respect to the standard coadjoint action. However, Souriau [17] showed that the momentum map can always be made equivariant by modifying the coadjoint action, as follows.

Let a Lie group $G$ act in a Hamiltonian manner on a connected symplectic manifold $M$ with a momentum map $\Phi : M \to \mathfrak{g}^*$. Define the cocycle $\theta : G \to \mathfrak{g}^*$ by

$$
\theta(g) = \Phi(g \cdot x) - \text{Coad}_g(\Phi(x))
$$

(which is independent of the choice of $x \in M$); $\text{Coad}_g = \text{Ad}_{g^{-1}}^*$ is our notation for the standard coadjoint action of $g \in G$ on $\mathfrak{g}^*$. The **modified coadjoint action** is

$$
\text{Coad}^\theta_g(\mu) = \text{Coad}_g(\mu) + \theta(g), \quad \mu \in \mathfrak{g}^*.
$$

With respect to this shifted affine action $\Phi$ becomes equivariant. All the usual properties of standard coadjoint action continue to hold for the modified actions [17]: for example
the momentum map is Poisson for a suitably modified Poisson structure on $g^*$, and the symplectic leaves of the modified Poisson structure are the modified coadjoint orbits.

Throughout this paper the reduced space at $\mu \in g^*$ is understood to be $\Phi^{-1}(\mu)/G_\mu$, where $G_\mu$ is the isotropy subgroup of $\mu$ for the modified coadjoint action.

3. Reduction to Actions of Compact Groups

Theorem 3 does not assume the compactness of the symmetry group $G$, but only the compactness of the isotropy subgroup $G_\mu$. The reduction to compact group actions is based on the Marle-Guillemin-Sternberg normal form for symplectic actions and momentum maps, which we now recall [3, 4]. Let again a connected Lie group $G$ act in a Hamiltonian manner on a symplectic manifold $M$ with a momentum map $\Phi : M \to g^*$ and the corresponding cocycle $\theta : G \to g^*$ (Section 2). At $x \in M$, consider the four spaces

$$
\begin{align*}
T_0 &= T_x(G \cdot x) \cap \ker d\Phi(x) = T_x(G_\mu \cdot x), \\
T_1 &= T_x(G \cdot x)/T_0, \\
N_1 &= \ker d\Phi(x)/T_0, \\
N_0 &= T_xM/(T_x(G \cdot x) + \ker d\Phi(x)).
\end{align*}
$$

Since $\ker d\Phi_x$ is the symplectic complement to $T_x(G \cdot x)$, these spaces depend only on the $G$-action and not on the choice of $\Phi$. Using the compactness of $G/C(\mu \mid x)$, we can realise the quotients $T_1$, $N_1$, $N_0$ as $G$-invariant subspaces of $T_xM$ satisfying

$$
T_0 \oplus T_1 = T_x(G \cdot x), \quad T_0 \oplus N_1 = \ker d\Phi(x), \quad T_0 \oplus T_1 \oplus N_1 \oplus N_0 = T_xM
$$

(the so-called Witt or Moncrief decomposition). $N_1$ is the symplectic slice to the action at $x$. With respect to such a decomposition, the symplectic form $\omega$ has the matrix

$$
[\omega]_x = \begin{bmatrix} 0 & 0 & 0 & \mu \\
0 & \omega_{T_1} & 0 & * \\
0 & 0 & \omega_{N_1} & * \\
-A^t & * & * & * \end{bmatrix},
$$

where $\omega_{T_1}$ and $\omega_{N_1}$ are the restrictions of $\omega$ to $T_1$ and $N_1$, $A$ is nondegenerate, and the *'s are of no interest. The Marle-Guillemin-Sternberg normal form theorem states that in a $G$-invariant neighbourhood $U$ of $x$, the symplectic $G$-action is isomorphic to that on $G \times C(\mu \mid x)$, where $m^* = g^*_x \cap g^*_\mu$, so that $g_\mu \simeq m \oplus g_x$ and $g^*_\mu \simeq m^* \oplus g^*_x$. ($g^*_x \cap g^*_\mu$ is the annihilator of $g_x$ in $g^*_\mu$.) The momentum map has the explicit form

$$
\Phi : G \times C(\mu \mid x) \to g^* : [g, \nu, v] \mapsto \text{Coad}_g(\mu + (\nu \oplus \Phi_G(v))).
$$

Now we reduce the problem to the case where the whole group $G$ is compact. (This part of the argument is similar to the beginning of Section 2 in [6].) Since the isotropy subgroup $G_\mu$ of $\mu$ is compact, we can choose a momentum map so that the restriction of $\theta$ to $G_\mu$ vanishes (essentially by averaging [2]), so that for $g \in G_\mu$ we have $\Phi(g \cdot x) = \text{Coad}_g(\Phi(x))$. There is therefore an inner product on $g^*$, invariant under $\text{Coad}(g_\mu)$, inducing a $G_\mu$-equivariant splitting $g = g_\mu \oplus h$. Then a small enough $G_\mu$-invariant neighbourhood $B$ of $\mu$ in the affine plane $\mu + h^\circ$ is transverse to the momentum map. ($h^\circ$ is the annihilator of $h$ in $g^*$). Hence $R := \Phi^{-1}(B)$ is a $G_\mu$-invariant submanifold of $M$ containing the given relative equilibrium $\gamma$.

We claim that in some neighborhood of $\gamma$, $R$ is a symplectic submanifold of $M$. Should the momentum map be equivariant already with respect to the standard coadjoint action, this is a consequence of the symplectic cross-section theorem of Guillemin and Sternberg (cf. [6], Corollary 2.3.6). In general, we resort to the Witt-Moncrief decomposition described
above. As $R$ is complementary to $T_1$ by construction, the restriction of $\omega$ to $R$ is obtained by eliminating the second row and the second column of $[\omega]_x$ in $\{3\}$. The resulting matrix is nondegenerate, hence $R$ is symplectic in a neighbourhood of $\gamma$, as claimed.

The action of $G_\mu$ on $R$ is Hamiltonian, and its momentum map is the restriction of $\Phi$ to $R$ followed by the natural projection $\mathfrak{g}^* \to \mathfrak{g}_\mu^*$. Since $\mathfrak{g}^* \to \mathfrak{g}_\mu^*$ restricted to $\mu + h^\circ$ is an isomorphism, the restriction $\Phi|_R$ is a momentum map for the action of $G_\mu$ up to this isomorphism. It follows that

$$\ker d\Phi(y) = \ker d(\Phi|_R)(y) \quad \forall y \in R.$$ 

Moreover, because $h$ is $G$-invariant, the flow of $X_h$ preserves the fibres of the momentum map, and so the flow preserves $R$. It follows that

$$(X_h)|_R = X(h|_R).$$

Another question that requires attending to is whether $\Phi$ inherits openness from $\Phi|_R$. The answer is affirmative in view of

**Lemma 3.** Let $K$ be a closed subgroup of a Lie group $G$ and $H$ be a closed subgroup of $K$. Let $A$ be an $H$-space and $B$ a $K$-space, and let $f : A \to B$ be an $H$-equivariant map. Then the map

$$F : G \times_H A \longrightarrow G \times_K B$$

$$([g, a]|_H) \mapsto [g, f(a)]_K$$

is well-defined and $G$-equivariant. Furthermore, if $f$ is $H$-open, then $F$ is $G$-open.

**Proof.** The only nontrivial conclusion is the $G$-openness of $F$. The diagram below commutes:

$$\begin{array}{ccc}
G \times H A & \longrightarrow & G \times K B \\
\downarrow & & \downarrow \\
G \times H A & \xrightarrow{F} & G \times K B.
\end{array}$$

Here $\pi_1$ and $\pi_2$ are the orbit (quotient) maps, open by definition of the topology on orbit spaces. Let $U \subset G \times_H A$ be $G$-invariant and open. Then $\pi_1^{-1}(U) = G \times U'$, with $U'$ open and $H$-invariant in $A$. Then $F(U)$ is open, since $F(U) = \pi_2 \circ (\text{id} \times f)(\pi_1^{-1}(U)) = \pi_2(G \times f(U'))$. $$\square$$

Thus, we have found a Hamiltonian sub-system $(R, G_\mu, \Phi|_{\mu}, h|_R)$ for which the symmetry group $G_\mu$ is compact. Passing to this sub-system, we may and shall assume without loss of generality that $G$ is compact and $G = G_\mu$.

### 4. Openness of momentum maps

In this Section we establish Theorem 1. By the result of Section 3, we may focus our attention on $\Phi : M \to \mathfrak{g}^*$, a momentum map for a Hamiltonian action of a compact Lie group. Sjamaar [3] proved that if $\Phi$ is proper then it is $G$-open relative to its image: that is, if $U \subset M$ is a $G$-invariant open subset, then $\Phi(U)$ is open in $\Phi(M)$, where $\Phi(M)$ is given the subspace topology induced from $\mathfrak{g}^*$. In this section we deduce from Sjamaar’s theorem that $\Phi$ is locally $G$-open even when it is not proper.

Besides Lemma 3 of Section 3, we need two more lemmas.

**Lemma 4.** Let $\Delta_1$ be a compact convex polytope in $\mathbb{R}^N$, and for $r > 0$ let

$$\Delta_{<r} = \{a\delta \mid 0 \leq a < r, \delta \in \Delta_1\}.$$ 

Then $\Delta_{<r}$ is an open subset of $\Delta = \Delta_{<\infty}$, where the latter has the topology induced from $\mathbb{R}^N$. 

Proof. There are two cases to examine depending on whether or not 0 ∈ Δ1. Only one (0 ∈ Δ1) is needed for the proof of Theorem 1, but we include the other for completeness.

Case 0 ∈ Δ1. Let Δ1 be the open faces of the polytope Δ1, and let
d = \min_j \{\text{dist}(0, Δ^1_j) \mid 0 \notin Δ^1_j\}.

Note that d > 0. Then S_d ∩ Δ1 = S_d ∩ Δ, where S_d is the sphere in \( \mathbb{R}^N \) of radius d.

Let \{x_n\} be any sequence in Δ converging to 0, and suppose that all x_n ≠ 0. Write
\( y_n = d x_n/\|x_n\| \). Then \( y_n \in S_d \cap Δ \) and so \( y_n \in Δ1 \). Therefore \( x_n = (\|x_n\|/d) y_n \in Δ_{<r} \)
provided n is sufficiently large for \( \|x_n\| < rd \) to hold.

If \( x_n \to x ≠ 0 \), we can write \( x_n = a_n δ_n \) and \( x = a δ \), with \( δ, δ_n \in S_d \cap Δ1 \). Then \( a_n \to a \)
and \( δ_n \to δ \). Rescaling \( δ \) and \( δ_n \) if necessary the result ensues.

Case 0 ∉ Δ1. Let \( x \in Δ_{<r} \), so that \( x = a δ \) with \( 0 ≤ a < r \) and \( δ \in Δ1 \). Let \{x_n\} be
a sequence in Δ converging to x. Write \( x_n = a_n δ_n \), with \( δ_n \in Δ1 \). If \( x = 0 \), then \( a_n \to 0 \)
as \( δ_n \) is bounded away from 0, so that \( x_n \in Δ_{<r} \) for n sufficiently large. If on the other hand \( x ≠ 0 \),
then both sequences \{a_n\} and \{δ_n\} are bounded and bounded away from 0 for sufficiently large n. By rescaling \( δ_n \) if necessary we can arrange for \( a_n \) to converge to \( a \), and so \( x_n \in Δ_{<r} \) for n sufficiently large.

Lemma 5. Let V be a symplectic representation of a compact Lie group G, and let \( \Phi : V → \mathfrak{g}^* \) be the homogeneous quadratic momentum map. Then \( \Phi \) is G-open relative to its image.

Proof. Take a G-invariant Hermitian metric whose imaginary part is the given symplectic structure on V. G acts as unitary transformations on V seen as a complex vector space.

Consider the unit sphere S in V with respect to the real part of the Hermitian metric, and the symplectic action of the circle group U(1). As U(1) is the centre of the unitary group, the actions of G and U(1) commute. The G-action descends to the symplectic manifold \( \mathbb{P}V := S/U(1) \) (diffeomorphic to \( \mathbb{C}^{\dim V}/2-1 \)). Denote its momentum map by \( \Phi_1 : \mathbb{P}V → \mathfrak{g}^* \).

Since the actions of G and U(1) commute, \( \Phi_1 \) can be chosen so that \( \Phi_1(U(1) \cdot x) = \Phi(x) \).

Now let U be a G-invariant open subset of V. We want to show that \( \Phi(U) \) is open in \( \Phi(V) \), and to do so we examine two cases separately: (i) \( 0 ∉ U \) and (ii) \( U = B(0, ε) \); indeed a general open set containing the origin is the union of sets of these types.

Case (i): We exploit a basis for the topology of \( V\setminus\{0\} \simeq S × \mathbb{R}^+ \) (diffeomorphic) consisting of ‘product sets’. Thus let \( U = U_1 × (a, b) ⊂ S × \mathbb{R}^+ \), where \( U_1 \) is a G-invariant open subset of S. Then by the homogeneity of \( \Phi \),
\[
\Phi(U) = \{r^2ν \mid r ∈ (a, b), ν ∈ \Phi(U_1)\}.
\]

By Sjamaar’s theorem, \( \Phi(U_1) \) is open in \( \Phi_1(\mathbb{P}V) = \Phi(S) \), and it follows that \( \Phi(U) \) is open in \( \Phi(V) \). To see this, let \( μ ∈ \Phi(U) \), and let \{μ_n\} be a sequence in \( \Phi(V \setminus \{0\}) \) converging to \( μ \); we may suppose \( μ_n ≠ 0 \) (otherwise it is trivial). Then
\( μ = \Phi(v, r) = r^2Φ_1(v) \) for some \( (v, r) ∈ S × (a, b) \). Since \( μ/r^2 ∈ \Phi_1(\mathbb{P}V) \), there is a sequence \( r_n → r \) satisfying
\( μ_n/r_n^2 ∈ \Phi_1(\mathbb{P}V) \) for all n. \( \Phi_1 \) being open by Sjamaar’s theorem, there is a sequence \( (v_n) \) in S converging to v such that \( μ_n/r_n^2 = Φ_1(v_n) \); in other words \( μ_n = Φ(v_n, r_n) \), and \( (v_n, r_n) → (v, r) \). Consequently \( \Phi(U) \) is open, as required.

Case (ii): Let \( U = B(0, ε) \), the open ball in V with centre 0 and radius ε. Because G is compact, we can and do identify \( \mathfrak{g} \) with \( \mathfrak{g}^* \), and the adjoint action with the coadjoint action. Let \( t^+ \) be a positive Weyl chamber in \( \mathfrak{g} = \mathfrak{g}^* \), and let
\[
Δ_1 = Φ_1(\mathbb{P}V) ∩ t^+,
Δ = Φ(V) ∩ t^+.
\]
By the convexity theorem of Atiyah-Guillemin-Sternberg-Kirwan, $\Delta_1$ is a convex polytope, and by the homogeneity of $\Psi$, $\Delta = \mathbb{R}^+ \Delta_1$. $U$ is $G$-invariant, and

$$\Phi(U) \cap t^+ = \Delta_{< \varepsilon^2} = \bigcup_{0 \leq r < \varepsilon^2} r\Delta_1.$$ 

By Lemma 4, $\Delta_{< \varepsilon^2}$ is open in $\Delta$.

To finish the proof that $\Phi(U)$ is open in $\Phi(V)$, note that $U$ is $G$-invariant, so that both $\Phi(U)$ and $\Phi(V)$ are $G$-invariant subsets of $g^*$. Since $t^+$ and $g^*/G$ are homeomorphic, we have that $\Phi(U)/G$ is open in $\Phi(V)/G$, and the result ensues.

**Remark** This lemma illustrates why it is important to consider $G$-openness rather than openness, for momentum maps are not in general open. For example the momentum map for the $SO(3)$ action on $T^*\mathbb{R}^3$, namely $\Phi(q,p) = q \times p$, is not open. For example, the images of sufficiently small neighbourhoods of $(q,p) = (e_1, e_1)$ are not neighbourhoods of $0$: in particular they do not contain nonzero points of the $e_1$-axis.

Equipped with these lemmas, we are in a position to prove Theorem 1.

**Proof.** Let $x \in M$ and $\mu = \Phi(x)$, and let $U_1$ be the $G$-invariant neighbourhood of $x$ whose existence is guaranteed by the Marle-Guillemin-Sternberg normal form (Section 3). Let $V_1$ be a $G$-invariant tubular neighbourhood of $\mu$, so that

$$V_1 \simeq G \times_{G_\mu} \mathcal{O},$$

where $\mathcal{O}$ is a neighbourhood of $0$ in the slice $g^*_0$ (the fixed-point set of the action of the centre of $G_\mu$ on $g^*$, which is isomorphic to $g^*_0$). Finally, let $U = U_1 \cap \Phi^{-1}(V_1)$. Thus, as symplectic $G$-spaces, $U \simeq G \times_{G_x} (m^* \times Y)$, where $Y$ is a $G_x$-invariant neighbourhood of $0$ in $N_1$, and it therefore suffices to show that the momentum map (3.2) is open. Since $G$ is compact, we can take the cocycle $\theta$ in (3.2) to vanish.

By Lemma 3, the quadratic momentum map $\Phi_{G_x} : N_1 \to g^*_0$ is open relative to its image. It follows that the restriction of $\Phi$ to the slice $m^* \times Y \to g^*_0$ is open relative to its image. Lemma 3 now applies, with $f$ replacing the restriction of $\Phi$ to the slice and $F$ replacing $\Phi$.

**5. Persistence of extremal relative equilibria**

In this section we establish Theorem 2, using Theorem 1 which was proved in Section 4. By the results of Section 3 we may assume $G$ to be compact.

We treat the case when $\gamma$ is a minimal relative equilibrium: the maximal case is similar. Let $U_0$ be the $G$-invariant neighbourhood of $\gamma$ guaranteed by Theorem 1. The minimality of $\gamma$ means that there is a precompact $G$-invariant neighbourhood $U \subset U_0$ of $\gamma$ such that

$$\left|\Phi^{-1}(\mu)\cap U\right| \geq h(\gamma) \quad \text{with equality only on} \quad G \cdot \gamma \cap U.$$

Suppose Theorem 2 is false. Let $\{\mu_n\}$ be a sequence of points in $\Phi(U)$ (which is open in $\Phi(U_0)$) converging to $\mu$, such that the restriction of $h$ to $\Phi^{-1}(\mu_n)\cap U$ has no minimum. However, by compactness, the restriction of $h$ to $\Phi^{-1}(\mu_n)\cap U$ has a minimum, say at $y_n \in U \setminus U_0$. Also by compactness, $y_n \to y$, with $y \in \overline{U} \setminus U$ (possibly after passing to a subsequence).

We claim that there is a sequence $\{x_n\}$ converging to some $x \in G \cdot \gamma$, with $\Phi(x_n) = \mu_n$. Granted that claim, we have $h(x) = h(G \cdot \gamma) < h(y)$ by construction of $y$. On the other hand, for each $n$, $h(y_n) < h(x_n)$. In the limit we get $h(y) \leq h(x)$, which is a contradiction.

The existence of the sequence $\{x_n\}$ is a consequence of the openness property. Indeed, we can choose a nested sequence $\{U_n\}$ of $G$-invariant neighbourhoods of $G \cdot \gamma$ whose intersection
is \( G \cdot \gamma \), such that \( \mu_n \in \Phi(U_n) \). Choosing \( x_n \in \Phi^{-1}(\mu_n) \cap U_n \) gives (after passing to a subsequence if necessary) a sequence converging to a point \( x \in G \cdot \gamma \), as claimed.

6. An Example: The Affine Rigid Body

The problem of affine rigid bodies (alias Riemann ellipsoids) has a long and important history, dating back perhaps to when Newton correctly suggested that the Earth was an oblate spheroid. Since then, it has been studied by such illustrious figures as Maclaurin, Jacobi, Dirichlet, Riemann, and Poincaré. A classical discussion can be found in the book of Chandrasekhar [2]; for a recent account from the symmetry perspective, we refer to Roberts and Sousa Dias [14].

An affine rigid body models a mass of ideal fluid evolving in time in such a manner that it always remains an ellipsoid. This is a Hamiltonian system whose configuration space is either \( Q = \text{SL}(3, \mathbb{R}) \) or \( \text{GL}(3, \mathbb{R}) \) depending on whether one is modelling incompressible or compressible fluids. The matrix \( Q \in Q \) represents the configuration that is the image of a sphere under \( Q \), an ellipsoid whose semi-axes are given by the singular values of \( Q \). It is supposed that the potential energy depends only on the shape of the ellipsoid, and so is invariant under the symmetry group \( G = \text{SO}(3) \times \text{SO}(3) \), the first copy of \( \text{SO}(3) \) acting by multiplication on the left, and the second by multiplication on the right.

The phase space is then the cotangent bundle \( T^*Q \), and the group \( G \) acts by cotangent lift on \( T^*Q \). Accordingly, the momentum map has two ‘components’

\[ \Phi_L, \Phi_R : T^*Q \to \mathfrak{so}(3)^* \]
given by

\[ \Phi_L(Q, P) = \frac{1}{2} (PQ^T - QP^T), \quad \Phi_R(Q, P) = \frac{1}{2} (P^TQ - Q^TP). \]

The particular example of the potential energy function used by Dirichlet, Riemann and others is the self-gravitating potential. Other potentials arise in linear elasticity theory.

Corollary 6. Suppose the potential energy has a minimum at the point \((I, 0) \in T^*Q\). Then there exist \( \varepsilon_L, \varepsilon_R > 0 \) and a \( G \)-invariant neighbourhood \( U \) of \((I, 0) \) in \( T^*Q \) such that for all \((\mu_L, \mu_R) \in \mathfrak{so}(3)^* \times \mathfrak{so}(3)^* \) with \( \|\mu_L\| < \varepsilon_L, \|\mu_R\| < \varepsilon_R \) there is an extremal relative equilibrium of the affine rigid body in \( U \) with momentum \((\Phi_L, \Phi_R) = (\mu_L, \mu_R)\).

Recall [12] that extremal relative equilibria are Lyapunov-stable relative to \( G \), and by [6] they are then Lyapunov-stable relative to \( G_\mu \) as well.

7. A Counter-example: Plane Point Vortices

In this section we give an example to the effect that the hypothesis on the compactness of the momentum isotropy subgroup \( G_\mu \) is necessary in Theorem 2. Consider the symplectic manifold

\[ M = \mathbb{C}^N \setminus \bigcup_{k \neq l} \{z_k = z_l\}, \quad \omega = \frac{i}{2} \sum_{k=1}^N \Gamma_k dz_k \wedge d\overline{z}_k, \quad \Gamma_1, \ldots, \Gamma_N \in \mathbb{R} \]
on which the Euclidean group \( G = \text{SE}(2) = \mathbb{R}^2 \rtimes \text{SO}(2) \) acts diagonally. This action is free, proper, and Hamiltonian, and has a momentum map \( \Phi : M \to g^* \). We use the identification
\[ g^* \simeq \mathbb{C} \times \mathbb{R} \] denoting the components of \( \Phi \) by
\[
(\Phi_C, \Phi_R) : (z_1, \ldots, z_N) \mapsto \left( \sum_{k=1}^{N} \Gamma_k z_k, \sum_{k=1}^{N} \Gamma_k \frac{|z_k|^2}{2} \right).
\]

It can be shown that \( \Phi \) is equivariant with respect to the standard (unmodified) coadjoint action if and only if \( \sum_{k=1}^{N} \Gamma_k = 0 \). As the \( G \)-invariant Hamiltonian we take
\[
h(z_1, \ldots, z_N) = -\frac{1}{2\pi} \sum_{k<l} \Gamma_k \Gamma_l \log |z_k - z_l|.
\]

Hamilton’s equation reads
\[
\frac{dz_k}{dt} = \frac{2}{i} \frac{\partial h}{\partial (\Gamma_k z_k)} = -\frac{1}{2\pi i} \sum_{l \neq k} \frac{\Gamma_l}{z_k - z_l} \quad (k = 1, \ldots, N).
\]

This system describes the motion of \( N \) interacting plane point vortices with vorticities \( \Gamma_1, \ldots, \Gamma_N \). See for example [\ref{1}].

We study the case of 3 vortices with vorticities 1, 1, −2. Let us call the axis the subset \( 0 \times \mathbb{R}^+ \subset \mathbb{C} \times \mathbb{R} \simeq g^* \). A theorem of Synge, [\ref{18}, Theorem 6], tells us that the relative equilibria \( \gamma = (z_1, z_2, z_3) \) for \( h \) are of two types:

1. \( \Phi(\gamma) \) is on the axis, i.e. \( \Phi_C(\gamma) = 0 \), in which case \( z_1, z_2, z_3 \) are collinear, with \( z_3 \) midway between \( z_1 \) and \( z_2 \);
2. \( \Phi(\gamma) \) is off the axis, i.e. \( \Phi_C(\gamma) \neq 0 \), in which case \( z_1, z_2, z_3 \) form an equilateral triangle.

In an equilateral relative equilibrium (type 2), \( z_2 - z_3 = e^{\pm i\pi/3}(z_1 - z_3) \), from which we calculate easily that
\[
e^{2\pi h(\gamma)} = \frac{1}{3 \sqrt{3}} |\Phi_C(\gamma)|^3.
\]

**Proposition 7.** In a system of 3 vortices with vorticities 1, 1, −2, let \( \gamma \) be a collinear relative equilibrium for \( h \) (type 1), \( U \) the \( G \)-invariant neighbourhood of \( \gamma \) in \( M \) defined by \( e^{2\pi h(U)} > e^{2\pi h(\gamma)}/3 \), and \( V = (D \times \mathbb{R} \setminus \text{axis}) \) a punctured neighbourhood of \( \mu = \Phi(\gamma) \) in \( g^* \) where \( D \subset \mathbb{C} \) is the disc of radius \( \sqrt{3} e^{2\pi h(\gamma)}/3 \) centred at 0. Then \( \gamma \) is extremal, but \( \Phi^{-1}(V) \cap U \) contains no relative equilibrium for \( h \).

**Proof.** Since the action of \( G \) is free, \( \Phi \) is a submersion, and \( \text{codim}(\Phi^{-1}(\mu)) = \text{codim}(\mu) = 3 = \dim(G) = \dim(G\mu) \). This means that \( \Phi^{-1}(\mu)/G\mu \) is discrete (by explicit calculation, in fact a single point), hence \( \gamma \) is trivially extremal. Suppose a relative equilibrium \( \gamma' \) exists in \( \Phi^{-1}(V) \cap U \). Since \( \Phi(\gamma') \) is off the axis, \( \gamma' \) is an equilateral triangle (type 2) and
\[
e^{h(\gamma')} = \frac{1}{3 \sqrt{3}} |\Phi_C(\gamma')|^3 < \frac{1}{3 \sqrt{3}} e^{h(\gamma)}. \]

This is incompatible with \( e^{h(\gamma')} > e^{h(\gamma)}/3 \).

**Remark.** In Proposition \[\ref{7} \] \( M, G, \Phi, h, \gamma \) satisfy all the hypotheses in Theorem \[\ref{2} \] except the compactness of the isotropy group of \( \Phi(\gamma) \), which is \( G = \text{SE}(2) \) itself. The failure of \( \gamma \) to persist to nearby levels of the momentum map shows that this compactness hypothesis is essential. As observed in the proof, the momentum map is a submersion and the conclusion of Theorem \[\ref{3} \] is still true.

Note, however, that \( \gamma \) persists when \( \mu \) is perturbed along the axis \( 0 \times \mathbb{R} \), which is the annihilator of the noncompact part \( \mathbb{R}^2 \) of the Lie algebra and so in a natural way the dual of the Lie algebra of the compact subgroup \( \text{SO}(2) \) of \( G \). This partial persistence in ‘compact directions’ is in fact also covered by Theorem \[\ref{2} \] via reduction by stages and Corollary \[\ref{8} \] below.

It is interesting that, in the model of \( N \) plane vortices, if the sum of the vorticities does not vanish, then the momentum map is equivariant with respect to a modified coadjoint
action (with a nontrivial cocycle $\theta$), and for this modified coadjoint action all the isotropy subgroups are isomorphic to $\text{SO}(2)$ and so are compact. Hence Theorem 4 implies that the relative equilibrium persists to nearby values of momentum. This does not contradict the theorem of Synge quoted above, for in the case of nonvanishing total vorticity all nearby values of momentum are realisable by collinear configurations of 3 vortices.

8. Reduction by stages

Let $K$ be a normal subgroup of a Lie group $G$ with quotient $L = G/K$. Roughly speaking, we say that reduction by stages works if reduction by $G$ coincides with reduction first by $K$ and then by $L$. In detail, reduction by stages describes the following general procedure.

At the level of Lie algebras and their duals, we have $\mathfrak{f} \subset \mathfrak{g}$ and $\mathfrak{l}^* \subset \mathfrak{g}^*$. Moreover, the inclusion $\mathfrak{l}^* \hookrightarrow \mathfrak{g}^*$ naturally identifies $\mathfrak{l}^*$ with the annihilator $\mathfrak{f}^\circ$ of $\mathfrak{f}$ in $\mathfrak{g}^\circ$. Let $\pi: \mathfrak{g}^* \to \mathfrak{l}^*$ be the canonical projection.

Suppose $G$ acts in a Hamiltonian manner on a symplectic manifold $M$ with momentum map $\Phi_G: M \to \mathfrak{g}^*$. This action restricts to an action of $K$, and the momentum map is just $\Phi_K = \pi \circ \Phi_G: M \to \mathfrak{l}^*$. $G$ and $K$ act on $\mathfrak{g}^*$ and $\mathfrak{l}^*$ in such a way as to make the momentum maps equivariant. For $\mu \in \mathfrak{g}^*$, we shall use the notation $M/_{\mu}G$ to denote the reduced space $\Phi^{-1}_G(\mu)/G^\circ$.

The modified coadjoint action $\text{Coad}^0$ of $G$ on $\mathfrak{g}^*$ descends to an action on $\mathfrak{l}^*$, and so $L$ acts in a natural way on the set of $K$-orbits in $\mathfrak{l}^*$. Let $L_\nu$ be the subgroup of $L$ that preserves the coadjoint orbit $K \cdot \nu$; one can show that $L_\nu \simeq G_\nu/K_\nu$. It follows that $L_\nu$ acts on $M/_{\nu}K$ in a natural way, preserving the symplectic structure (compare [13]).

The affine subspace $\pi^{-1}(\nu)$ of $\mathfrak{g}^*$ can be identified with $\mathfrak{f}^\circ$, and hence with $\mathfrak{l}^*$, by translation in $\mathfrak{g}^*$. Let $\rho_\nu: \pi^{-1}(\nu) \to \mathfrak{l}^*$ be the composite of such an identification with the natural projection $\mathfrak{l}^* \to \mathfrak{l}^\circ$. Define

$$\Phi_\nu: M/_{\nu}K \to \mathfrak{l}^\circ_\nu$$

by first restricting $\Phi_G$ to $\Phi^{-1}_K(\nu)$ (whose values lie in $\pi^{-1}(\nu)$), passing to the quotient $M/_{\nu}K$ and finally applying the identification $\rho_\nu$. If this map $\Phi_\nu$ is well-defined and is a momentum map for the action of $L_\nu$, then we say reduction by stages works in this context provided in addition that

$$M/_{\mu}G \simeq (M/_{\nu}K)_{\nu}L_\nu$$

at least at the level of connected components, where $\nu = \pi(\mu)$ and $\sigma = \rho_\nu(\mu) = \Phi_\nu(\Phi^{-1}_G(\mu))$. The isomorphism between the two spaces should be as needed in the context; here a homeomorphism is sufficient, though more generally one might require an isomorphism as symplectic stratified spaces [16]. The papers [10, 11] explain the current state of the art on reduction by stages for free actions.

If the action of $K$ is free, then the partially reduced space $M/_{\nu}K$ is a smooth symplectic manifold, and one can apply Theorem 4 to the resulting $L_\nu$-invariant system, provided $(L_\nu)_\sigma$ is compact. The result is that if the relative equilibrium $\gamma$ in question is extremal in $M/_{\nu}K$ then it persists to nearby values of the $L_\nu$-momentum map $\Phi_\nu$.

It often happens that $L_\nu = L$, in which case $\rho_\nu$ is just a translation of $\mu + \mathfrak{f}^\circ$ to $\mathfrak{f}^\circ$, and one can ask for persistence to all $\mu' \in V \subset (\mu + \mathfrak{f}^\circ)$.

Corollary 8. Let $G, M, \Phi, K, L, \mu, \nu$ and $\sigma$ be as above, with $K$ acting freely on $M$, and such that reduction by stages works. Suppose that $L_\nu = L$, and that $L_\sigma$ is compact. Let $h$ be a $G$-invariant Hamiltonian on $M$ for which $\gamma \subset M$ is an extremal relative equilibrium with $\mu = \Phi(\gamma)$. Let $U_0$ be the $L$-invariant neighbourhood of $\gamma$ in $M/_{\nu}K$ guaranteed by Theorem 4. Then there is an $L$-invariant neighbourhood $V$ of $\mu$ in $\Phi(U_0) \cap \mathfrak{f}^\circ$ such that for every $\sigma' \in V$ there is a relative equilibrium in $\Phi^{-1}(\sigma') \cap U_0$. 
For the vortex system of Section 7 the corollary applies as follows: $G = \text{SE}(2)$, $K = \mathbb{R}^2$, the normal subgroup of $\text{SE}(2)$ consisting of translations, $\mu = (0, r)$, and $\nu = 0$. Then $L_\nu = L = \text{SE}(2)/\mathbb{R}^2 \simeq \text{SO}(2)$, which is compact. The subspace $\mathbb{R}^2$ is a line (the ‘axis’ of Section 7) consisting of the coadjoint orbits that are isolated points.

**Remark.** The freeness hypothesis on the $K$-action could easily be relaxed to local freeness. In the general setting where the action is not locally free, the same argument can be applied to the symplectic stratum in $M/\nu K$ containing the image of $\gamma$.

**Acknowledgement.** We would like to thank Eugene Lerman for many useful discussions, and for pointing out an error in a preliminary draft.

**REFERENCES**

[1] H. Aref, Integrable, chaotic, and turbulent motion in two-dimensional flows. *Ann. Rev. Fluid Mech.* 15 (1983) 345–389.
[2] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*. Dover, 1987.
[3] V. Guillemin, E. Lerman, S. Sternberg, *Symplectic fibrations and multiplicity diagrams*. Cambridge University Press, Cambridge, 1996.
[4] V. Guillemin and S. Sternberg, A normal form for the moment map, in: *Differential Geometric Methods in Mathematical Physics* (ed. S. Sternberg), Reidel, Dordrecht, 1984.
[5] Y. Karshon and E. Lerman, The centralizer of invariant functions and division properties of the moment map. *Illinois J. Math* 41 (1997) 462–487.
[6] E. Lerman, E. Meinrenken, S. Tolman, C. Woodward, Non-abelian convexity by symplectic cuts. *Topology* 37 (1998) 245–259.
[7] E. Lerman and S. F. Singer, Stability and persistence of relative equilibria at singular values of the moment map. *Nonlinearity* 11 (1998), 1637–1649
[8] E. Lerman and T. Tokieda, On relative normal modes. *C. R. Acad. Sci. Paris*, série I 328 (1999) 413–418.
[9] C.-M. Marle, Modèle d’action hamiltonienne d’un groupe de Lie sur une variété symplectique. *Rend. del Seminario Matematico* 43 (1985) 227–251.
[10] J. Marsden, G. Misiołek, M. Perlmutter, T. Ratiu, Symplectic reduction for semidirect products and central extensions. *Diff. Geom. and its Applications* 9 (1998) 173–212.
[11] J. Marsden, G. Misiołek, J-P. Ortega, M. Perlmutter, T. Ratiu, Symplectic reduction by stages. *Preprint* (2001).
[12] J. Montaldi, Persistence and stability of relative equilibria. *Nonlinearity* 10 (1997) 449–446.
[13] J. Montaldi and R. M. Roberts, Note on semisymplectic actions of Lie groups. *C. R. Acad. Sci. Paris*, série I, 330 (2000) 1079–1084.
[14] R. M. Roberts and M. E. R. de Sousa Dias, Symmetries of Riemann ellipsoids. *Resenhas IME-USP* 4 (1999) 183–221.
[15] R. Sjamaar, Convexity of the moment mapping revisited. *Adv. Math.* 138 (1998) 46–91.
[16] R. Sjamaar and E. Lerman, Stratified symplectic spaces and reduction, *Ann. of Math.* 134 (1991) 375–422.
[17] J.-M. Souriau, *Structure des Systèmes Dynamiques*. Dunod, Paris, 1970. [English translation: *Structure of Dynamical Systems*. Birkhauser, Boston, 1997.]
[18] J. L. Synge, On the motion of three vortices. *Canadian J. Math.* 1 (1949) 257–270.
[19] C. Wulff, Persistence of relative equilibria in Hamiltonian systems with noncompact symmetry. *Preprint* (2001).

**Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, UK**

**E-mail address:** j.montaldi@umist.ac.uk

**Département de Mathématiques, Université de Montréal, C.P. 6128, succ. Centre-Ville, Montréal H3C 3J7, Canada**

**E-mail address:** tokieda@dms.umontreal.ca