MODULI OF AFFINE SCHEMES WITH REDUCTIVE GROUP ACTION

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Abstract. For a connected reductive group $G$ and a finite-dimensional $G$-module $V$, we study the invariant Hilbert scheme that parameterizes closed $G$-stable subschemes of $V$ affording a fixed, multiplicity-finite representation of $G$ in their coordinate ring. We construct an action on this invariant Hilbert scheme of a maximal torus $T$ of $G$, together with an open $T$-stable subscheme admitting a good quotient. The fibers of the quotient map classify affine $G$-schemes having a prescribed categorical quotient by a maximal unipotent subgroup of $G$. We show that $V$ contains only finitely many multiplicity-free $G$-subvarieties, up to the action of the centralizer of $G$ in $GL(V)$. As a consequence, there are only finitely many isomorphism classes of affine $G$-varieties affording a prescribed multiplicity-free representation in their coordinate ring.

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0. Introduction

A fundamental result in the classification theory of projective algebraic varieties is the existence of the Hilbert scheme, a projective scheme parameterizing closed subschemes of a fixed projective space, having a fixed Hilbert polynomial. In this paper, we study two versions of the Hilbert scheme,

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relevant to the classification of affine algebraic varieties $X$ equipped with an action of a reductive group $G$.

In this setting, an analog of the Hilbert polynomial is the representation of $G$ in the coordinate ring of $X$. This rational, possibly infinite-dimensional $G$-module is the direct sum of simple modules, with multiplicities that we assume to be all finite. Now the closed $G$-stable subschemes of a fixed finite-dimensional $G$-module $V$, having fixed multiplicities in their coordinate ring, are parameterized by a quasi-projective scheme: the invariant Hilbert scheme $\text{Hilb}_h^G(V)$ (where $h$ encodes the multiplicities).

For subschemes of finite length, the invariant Hilbert scheme may be realized in a punctual Hilbert scheme $\text{Hilb}_h(V)$, as a union of connected components of the $G$-invariant subscheme. This construction is well known in the case of finite groups, see e.g. [Nak01]. But its generalization to arbitrary subschemes is problematic, since there is no reasonable moduli space for closed subschemes of infinite length in affine space. In that setting, the existence of the invariant Hilbert scheme was proved by Haiman and Sturmfels [HS02] for diagonalizable groups, and (building on their work) in [AB02] for connected reductive groups. The connectedness assumption for $G$ is harmless and will be made throughout this paper.

We first present an alternative proof for the existence of $\text{Hilb}_h^G(V)$, by realizing it as a closed subscheme of $\text{Hilb}_h^T(V//U)$; here $T \subseteq G$ is a maximal torus, $U \subset G$ is a maximal unipotent subgroup normalized by $T$, and $V//U$ is the categorical quotient. (Since $V//U$ is an affine $T$-variety, the existence of $\text{Hilb}_h^T(V//U)$ follows from [HS02].)

This enables us to construct another version of the Hilbert scheme: it classifies affine $G$-schemes $X$ equipped with a $T$-equivariant isomorphism $X//U \rightarrow Y$, where $Y = \text{Spec}(A)$ is a fixed affine $T$-scheme (we then say that $X$ has type $Y$). This moduli scheme $M_Y$ may be described more concretely, in terms of the rational $G$-module $R$ such that the $U$-fixed subspace in $R$ (the span of the highest weight vectors) equals $A$: in fact, $M_Y$ parameterizes those algebra multiplication laws on $R$ that are $G$-equivariant and extend the multiplication of $A$.

We show that $M_Y$ is an affine scheme of finite type, and we equip it with an action of the adjoint torus $T_{ad}$ (the quotient of $T$ by the center of $G$) such that all orbit closures have a common point $X_0$. The latter corresponds to the “most degenerate multiplication law” on $R$, where the product of any two simple submodules is generated by the product of their highest weight vectors. Equivalently, $X_0$ is horospherical in the sense of [Kno90].

These properties of $M_Y$ follow easily from its description in terms of multiplication laws, except for the crucial fact that $M_Y$ is of finite type. So we use an indirect approach which relates $M_Y$ to an invariant Hilbert scheme: every algebra $R$ as above is generated by any $G$-submodule containing a set of generators for $A$, so that $X = \text{Spec}(R)$ is equipped with a closed immersion into a fixed $G$-module $V$. And, of course, the multiplicities of the $G$-module $R$ are fixed, since its highest weight vectors are.
Specifically, we define an open subscheme $\text{Hilb}_G^h(V)_0$ of the invariant Hilbert scheme, equipped with a morphism $f : \text{Hilb}_G^h(V)_0 \to \text{Hilb}_T^h(V_U)$ where $V_U$ is the space of co-invariants of $U$ in $V$. We show that $M_Y$ identifies with the fiber of $f$ at $Y$, for any closed $T$-stable subscheme $Y$ of $V_U$. Then we define an action of $T_{\text{ad}}$ on $\text{Hilb}_G^h(V)$ that stabilizes $\text{Hilb}_G^h(V)_0$; it turns out that $f$ is a good quotient in the sense of geometric invariant theory. In addition, the “most degenerate multiplication law” yields a $T_{\text{ad}}$-invariant section $s$ of $f$.

The construction of $M_Y$ gives new insight into several methods and results concerning reductive group actions. For example, the degeneration of an affine $G$-variety $X$ to a horospherical one $X_0$, which plays an important role in [Pop86], [Vin86], [Kno00], [Gro97], is obtained by taking the $T_{\text{ad}}$-orbit closure of $X$ in $M_Y$ where $Y = X//U$. Consider the normalization of this orbit closure, an affine toric variety; then a deep result of Knop [Kno96] implies that the corresponding monoid (of characters) is generated by all simple roots of a certain root system $\Phi_X$. As a consequence, the normalization of any $T_{\text{ad}}$-orbit in $M_Y$ is isomorphic to a $T_{\text{ad}}$-module.

Of special interest are the multiplicity-free $G$-varieties, i.e., those affine $G$-varieties such that all multiplicities of their coordinate ring are 0 or 1. If $G$ is a torus, then these varieties are the (possibly non-normal) toric varieties, and the corresponding invariant Hilbert scheme is the toric Hilbert scheme of Peeva and Stillman [PS02], see also [HS02]. For arbitrary $G$, we show that every finite-dimensional $G$-module $V$ contains only finitely many multiplicity-free subvarieties, up to the action of the centralizer of $G$ in $\text{GL}(V)$. As a consequence, $M_Y$ contains only finitely may $T_{\text{ad}}$-orbits, for any multiplicity-free $Y$.

In fact, those moduli schemes arising from reductive varieties (a nice class of multiplicity-free varieties, studied in [AB02]) turn out to be isomorphic to affine spaces, via the Vinberg family of [AB02, 7.5]. This will be developed elsewhere.

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1. Existence of moduli spaces

1.1. Notation and preliminaries. We will consider algebraic groups and schemes over a fixed algebraically closed field $k$, of characteristic zero. In addition, all schemes will be assumed to be Noetherian.

Let $G$ be a connected reductive group, $B \subseteq G$ a Borel subgroup with unipotent radical $U$, and $T \subseteq B$ a maximal torus; then $B = TU$. Let $B^- = TU^-$ be the opposite Borel subgroup, that is, $B^- \cap B = T$.

The character group $X(T)$ is the weight lattice of $G$, denoted $\Lambda$ and identified with the character group of $B$. The center $Z(G)$ is contained
in $T$; the quotient $T/Z(G)$ is denoted $T_{\text{ad}}$ and called the adjoint torus, a maximal torus of the adjoint group $G_{\text{ad}} = G/Z(G)$. The character group $X(T_{\text{ad}}) \subseteq \Lambda$ is generated by the set $\Phi = \Phi(G, T)$ of roots, that is, of non-zero weights of $T$ in the Lie algebra $\mathfrak{g}$ of $G$. The choice of $B$ defines the subset $\Phi^+ = \Phi(B, T)$ of positive roots; the corresponding subset $\Pi$ of simple roots is a basis of $X(T_{\text{ad}})$. We write $X(T_{\text{ad}}) = \mathbb{Z}\Pi$, and denote $\mathbb{N}\Pi$ the submonoid generated by $\Pi$. This defines a partial ordering $\leq$ on $\Lambda$, where $\mu \leq \lambda$ if and only if $\lambda - \mu \in \mathbb{N}\Pi$.

A rational $G$-module is a $k$-vector space $V$ equipped with a group homomorphism $G \to \text{GL}(V)$ such that any $x \in V$ is contained in some finite-dimensional $G$-stable subspace $V_x$, where the restriction $G \to \text{GL}(V_x)$ is a morphism of algebraic groups. Then $V$ is a direct sum of simple modules, and these are finite-dimensional.

Recall that any simple $G$-module $V$ contains a unique line of eigenvectors of $B$. Moreover, $V$ is uniquely determined by the corresponding weight $\lambda \in \Lambda$; we write $V = V(\lambda)$. Then $\lambda$ is the (unique) highest weight of $V$ (for the partial ordering $\leq$), and occurs with multiplicity 1. The assignment $V(\lambda) \mapsto \lambda$ is a bijection from the set of isomorphism classes of simple $G$-modules, to the subset $\Lambda^+ \subseteq \Lambda$ of dominant weights; the trivial one-dimensional $G$-module is $V(0)$. The dual module $V(\lambda)^*$ is simple with highest weight $-w_\circ \lambda$, where $w_\circ$ is the longest element of the Weyl group of $(G,T)$.

We have an isomorphism

$$V(\lambda)^* \simeq \{ f \in k[G] \mid f(gb) = \lambda(b)f(g) \ \forall g \in G, \ b \in B\},$$

where $k[G]$ denotes the algebra of regular functions on $G$. The evaluation at the identity element $e \in G$ yields a linear form on $V(\lambda)^*$, eigenvector of $B$ of weight $\lambda$, that is, a highest weight vector $v_\lambda \in V(\lambda)$. This choice of highest weight vectors in simple $G$-modules is compatible with tensor products, in the following sense: the multiplication in $k[G]$ restricts to a surjective map of $G$-modules $V(\lambda)^* \otimes_k V(\mu)^* \to V(\lambda + \mu)^*$, compatible with evaluations at $e$. This yields an injective map

$$V(\lambda + \mu) \hookrightarrow V(\lambda) \otimes_k V(\mu), \ v_{\lambda+\mu} \mapsto v_\lambda \otimes v_\mu.$$

For any rational $G$-module $M$, we have an equivariant isomorphism

$$\bigoplus_{\lambda \in \Lambda^+} \text{Hom}^G(V(\lambda), M) \otimes_k V(\lambda) \to M, \quad \sum u_\lambda \otimes x_\lambda \mapsto \sum u_\lambda(x_\lambda).$$

So the dimension of the $k$-vector space $\text{Hom}^G(V(\lambda), M)$ is the multiplicity of $V(\lambda)$ in $M$; the image in $M$ of $\text{Hom}^G(V(\lambda), M) \otimes_k V(\lambda)$ is the isotypical component of $M$ of type $V(\lambda)$, denoted $M(\lambda)$. The set of all $\lambda \in \Lambda^+$ such that $M(\lambda) \neq 0$ is the weight set of $M$, denoted $\Lambda^+_M$.

The multiplicity of $V(\lambda)$ in $M$ can also be read off the subspace $M^U$ of $U$-fixed points in $M$. For $M^U$ is a rational $T$-module, and $V(\lambda)^U = kv_\lambda$; thus, $\text{Hom}^G(V(\lambda), M)$ is isomorphic to $M^U_\lambda$, the $\lambda$-weight space of $M^U$, via
evaluation at the highest weight vector $v_\lambda$. Note that
\[ M_0^U = M^B = M^G = M_{(0)}. \]
We say that $M$ is multiplicity-finite (resp. multiplicity-free) if each $k$-vector space $M_0^U$ is finite-dimensional (resp. of dimension at most 1).

An affine $G$-scheme is an affine scheme $X = \text{Spec}(R)$, of finite type, equipped with an action of $G$; then $R = k[X]$ is called a $G$-algebra, i.e., $R$ is a finitely generated $k$-algebra equipped with an action of $G$ by rational automorphisms. We put $\Lambda^+_X = \Lambda^+_R$, the weight set of $X$. For example, any finite-dimensional $G$-module $V$ defines an affine $G$-scheme, with corresponding algebra $\text{Sym}_k(V^*)$. By abuse of notation, we still denote that subscheme by $V$.

There are obvious notions of multiplicity-finite (resp. multiplicity-free) affine $G$-schemes; for simplicity, we will often drop the adjective “affine”.

As an example, since $k(G)^U = V(\lambda)^*$ for all $\lambda$, the group $G$ is multiplicity-finite (for its action on itself by right multiplication), with weight monoid $\Lambda^+$. It follows that any affine $G$-variety containing a dense $G$-orbit is multiplicity-finite. On the other hand, the multiplicity-free $G$-varieties are those containing a dense $B$-orbit or, equivalently, a dense spherical $G$-orbit.

The above definitions extend to families, in the following sense.

**Definition 1.1.** Given a scheme $S$, a family of affine $G$-schemes over $S$ is a scheme $X$ equipped with an action of $G$ and with a morphism $\pi : X \to S$, such that $\pi$ is affine, of finite type, and $G$-invariant.

(In other words, $X$ is an affine $S$-scheme of finite type, equipped with an action $a : G_S \times_S X \to X$ of the constant group scheme $G_S = G \times_{\text{Spec}(k)} S$.) Then the sheaf of $O_S$-algebras $R = \pi_* O_X$ is equipped with a compatible $G$-action; we say that $R$ is a sheaf of $O_S$-$G$-algebras. We may now formulate a basic (and well-known) finiteness result.

**Lemma 1.2.** With the preceding notation, we have an isomorphism of $O_S$-$G$-modules:
\[ R \simeq \bigoplus_{\lambda \in \Lambda^+} R_{(\lambda)}, \]
where $R_{(\lambda)} \simeq R^U_{(\lambda)} \otimes_k V(\lambda)$ as $O_S$-$G$-modules. Moreover, both $R_{(0)} = R^G$ and $R^U$ are sheaves of finitely generated $O_S$-$G$-algebras, and every $R^U_{(\lambda)}$ is a coherent sheaf of $R^G$-modules.

**Proof.** We may assume that $S$, and hence $X$, are affine. Then $R$ is a rational $G$-module by [Mum94, I.1], and the first assertion follows from the structure of these modules.

Let $R = \Gamma(X, O_X)$ and $A = \Gamma(S, O_S)$. Then $R$ is a finitely generated $A$-$G$-algebra. Thus, it is generated by a finite-dimensional $G$-submodule $V \subseteq R$. Then the $k$-algebra $\text{Sym}_k(V)^G$ is finitely generated, by the Hilbert-Nagata theorem. Since the $A$-algebra $R^G$ is a quotient of $A \otimes_k \text{Sym}_k(V)^G$, ...
it is finitely generated. Likewise, each $R^U_\lambda$ is a finitely generated $A$-module, and $R^U$ is a finitely generated $A$-algebra, see e.g. [Gro97, Theorems 5.6, 9.1]. \qed

Next put $\mathcal{X}/G = \text{Spec}_S(R^G)$ and let

$$p = p_{\mathcal{X},G} : \mathcal{X} \to \mathcal{X}/G$$

be the morphism corresponding to the inclusion $R^G \subseteq R$. Then $p$ is a good quotient by $G$, that is: $p$ is affine and $G$-invariant, and induces an isomorphism $\mathcal{O}_{\mathcal{X}/G} \to (p_*\mathcal{O}_\mathcal{X})^G$. It follows e.g. that $p$ is surjective, and sends any closed $G$-stable subset to a closed subset; as a consequence, $p$ is universally submersive (for these facts, see [Mum94, I.2]). Moreover, $\pi = (\pi/G) \circ p$, where

$$\pi/G : \mathcal{X}/G \to S$$

is an affine morphism of finite type.

Likewise, we obtain a factorization $\pi = (\pi/U) \circ p_{\mathcal{X},U}$, where

$$p_{\mathcal{X},U} : \mathcal{X} \to \mathcal{X}/U$$

is a good quotient by $U$, and

$$\pi/U : \mathcal{X}/U \to S$$

is a family of affine $T$-schemes. Moreover, $(\mathcal{X}/U)/T = \mathcal{X}/B$ is mapped isomorphically to $\mathcal{X}/G$.

Many properties of an affine $G$-scheme $X$ can be read off $\mathcal{X}/U$, e.g., $X$ is reduced (resp. a variety, normal) if and only if $\mathcal{X}/U$ is, see [Pop86]. Since $\Lambda^+_X = \Lambda^+_{\mathcal{X}/U}$, it follows that $\Lambda^+_X$ is a finitely generated monoid, for any affine $G$-variety $X$.

Given two families of affine $G$-schemes $\pi : \mathcal{X} \to S$ and $\pi' : \mathcal{X}' \to S$, we denote by $\text{Hom}_S^G(\mathcal{X}, \mathcal{X}')$ the set of $G$-equivariant morphisms $\mathcal{X} \to \mathcal{X}'$ over $S$; these will just be called morphisms. Any morphism $\gamma : \mathcal{X} \to \mathcal{X}'$ yields a $T$-equivariant morphism $\gamma/U : \mathcal{X}/U \to \mathcal{X}'/U$ over $S$.

With these notations, we may record another easy finiteness result.

**Lemma 1.3.** (i) The map $\text{Hom}_S^G(\mathcal{X}, \mathcal{X}') \to \text{Hom}_S^T(\mathcal{X}/U, \mathcal{X}'/U)$ is injective.

(ii) For any two multiplicity-finite $G$-schemes $X, X'$, the contravariant functor $(\text{Schemes}) \to (\text{Sets}), S \mapsto \text{Hom}_S^G(X \times S, X' \times S)$ is representable by an affine scheme of finite type denoted $\text{Hom}^G(X, X')$.

Moreover, the group functor $S \mapsto \text{Aut}_S^G(X \times S)$ is representable by a linear algebraic group $\text{Aut}^G(X)$, open in $\text{Hom}^G(X, X)$. In particular, the Lie algebra of $\text{Aut}^G(X)$ is the space $\text{Der}^G(X)$ of $G$-equivariant derivations of $X$.

If, in addition, $X$ is multiplicity-free, then $\text{Aut}^G(X)$ is diagonalizable.

**Proof.** (i) follows at once from the corresponding statement for rational $G$-modules.
(ii) Choose a $G$-equivariant closed immersion $X' \hookrightarrow V$, where $V$ is a finite-dimensional $G$-module. Let $R = \Gamma(X, \mathcal{O}_X)$ and $A = \Gamma(S, \mathcal{O}_S)$. Then

$$\text{Hom}_S^G(X \times S, V \times S) = \text{Hom}_A^G(A \otimes_k \text{Sym}_k(V^*), A \otimes_k R) \simeq A \otimes_k (R \otimes_k V)^G.$$ 

So $S \mapsto \text{Hom}_S^G(X \times S, V \times S)$ is represented by the affine space $(R \otimes_k V)^G$. The latter is finite-dimensional, since $X$ is multiplicity-finite. Moreover, denoting by $I$ the ideal of $X'$ in $\text{Sym}_k(V^*)$, the restriction to $I$ yields a morphism

$$\varphi : (R \otimes_k V)^G = \text{Hom}^G_{\text{alg}}(\text{Sym}_k(V^*), R) \to \text{Hom}^G_{\text{mod}}(I, R),$$

with evident notation. Further, $\text{Hom}^G_{\text{mod}}(I, R)$ is an affine space, possibly infinite-dimensional. Clearly, the scheme-theoretic fiber $\varphi^{-1}(0)$ represents $S \mapsto \text{Hom}_S^G(X \times S, X' \times S)$.

In particular, $\text{Hom}^G(X, X)$ is a closed subscheme of the affine space

$$\text{End}^G(\bigoplus_{\lambda \in F} R_{(\lambda)}) = \text{End}^G(\bigoplus_{\lambda \in F} R_{\lambda}^U \otimes_k V(\lambda)) = \prod_{\lambda \in F} \text{End}(R_{\lambda}^U),$$

where $F$ is any finite subset of $\Lambda^+$ such that the algebra $R$ is generated by its isotypical components $R_{(\lambda)}$, $\lambda \in F$. By Lemma 1.2, $\text{Aut}^G(X)$ is the intersection of $\text{Hom}^G(X, X)$ with the open subset $\prod_{\lambda \in F} \text{GL}(R_{\lambda}^U) \subset \prod_{\lambda \in F} \text{End}(R_{\lambda}^U)$. So $\text{Aut}^G(X)$ is a linear algebraic group; arguing as above, we see that it represents $S \mapsto \text{Aut}_S^G(X \times S)$.

Finally, if $X$ is multiplicity-free, then each non-zero $R_{\lambda}^U$ is a line, so that $\text{Aut}^G(X)$ is diagonalizable. 

1.2. The invariant Hilbert scheme. Consider a family $\pi : \mathcal{X} \to S$ of affine $G$-schemes and put $R = \pi_* \mathcal{O}_X$. By Lemma 1.2, the morphism $\pi$ is flat if and only if $\pi//U$ is flat; then $\pi//G$ is flat as well. If, in addition, $\pi//G$ is finite, then every $R_{\lambda}^U$ is a flat, coherent sheaf of $\mathcal{O}_S$-modules, and hence locally free of finite rank. This motivates the following

**Definition 1.4.** Given a function $h : \Lambda^+ \to \mathbb{N}$, a family of multiplicity-finite $G$-schemes $\pi : \mathcal{X} \to S$ has Hilbert function $h$, if every sheaf of $\mathcal{O}_S$-modules $R_{\lambda}^U$ is locally free of constant rank $h(\lambda)$.

Then $\pi$ is flat, and $\pi//U : \mathcal{X}//U \to S$ is a family of multiplicity-finite $T$-schemes with the same Hilbert function $h$. Note that $h(0) \geq 1$, so that $\pi//G$ is finite and surjective. As a consequence, $\pi = (\pi//G) \circ p$ is surjective and maps closed $G$-stable subsets to closed subsets.

**Definition 1.5.** Given an affine $G$-scheme $X$ and a function $h : \Lambda^+ \to \mathbb{N}$, the Hilbert functor is the contravariant functor $\text{Hilb}^G_h(X) : (\text{Schemes}) \to (\text{Sets})$ assigning to any scheme $S$ the set of closed $G$-stable subschemes $\mathcal{X} \subseteq X \times S$ such that the projection $\pi : \mathcal{X} \to S$ is a family of multiplicity-finite $G$-schemes with Hilbert function $h$. 

In the case where $G = T$ is a torus and $X$ is a finite-dimensional $T$-module, the functor $\text{Hilb}_h^T(X)$ is represented by a quasi-projective scheme $\text{Hilb}_h^T(X)$, as follows from [HS02, Theorem 1.1]. We will need the following straightforward consequence of this result.

**Lemma 1.6.** For any affine $T$-scheme $Y$ and any function $h : \Lambda^+ \to \mathbb{N}$, the functor $\text{Hilb}_h^T(Y)$ is represented by a quasi-projective scheme $\text{Hilb}_h^T(Y)$.

**Proof.** Choose a $T$-equivariant closed immersion $Y \hookrightarrow E$, where $E$ is a finite-dimensional $T$-module; let $I \subset \text{Sym}_k(E^*)$ be the corresponding ideal. Then any family $\pi : X \to S$ in $\text{Hilb}_h^T(Y)(S)$ yields a family in $\text{Hilb}_h^T(E)(S)$, that is, a morphism $S \to \text{Hilb}_h^T(E)$ such that $X$ is the pull-back of the universal family

$$\text{Univ}_h^T(E) \subseteq E \times \text{Hilb}_h^T(E).$$

Let $\pi : \text{Univ}_h^T(E) \to \text{Hilb}_h^T(E)$ be the projection. Then each eigenspace

$$\mathcal{F}_\lambda = (\pi_*\mathcal{O}_{\text{Univ}_h^T(E)})_\lambda$$

is a locally free sheaf of rank $h(\lambda)$ on $\text{Hilb}_h^T(E)$, equipped with a map

$$q_\lambda : \text{Sym}_k(E^*)_\lambda \to \Gamma(\text{Hilb}_h^T(E), \mathcal{F}_\lambda).$$

Moreover, the image of $q_\lambda(I_\lambda)$ under the pull-back

$$\Gamma(\text{Hilb}_h^T(E), \mathcal{F}_\lambda) \to \Gamma(S, \mathcal{R}_\lambda)$$

is zero, since $X \subseteq Y \times S$. The intersection over all $\lambda$ of the zero subschemes of the spaces $q_\lambda(I_\lambda)$ is a closed subscheme of $\text{Hilb}_h^T(E)$, that clearly represents $\text{Hilb}_h^G(Y)$.

We now come to our first main result.

**Theorem 1.7.** For any affine $G$-scheme $X$ and any function $h : \Lambda^+ \to \mathbb{N}$, the functor $\text{Hilb}_h^G(X)$ is represented by a closed subscheme $\text{Hilb}_h^G(X)$ of $\text{Hilb}_h^T(X//U)$.

**Proof.** Let $S$ be a scheme. Then $\text{Hilb}_h^G(X)(S)$ is the set of those $G$-stable ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_S \otimes_k \Gamma(X, \mathcal{O}_X)$ such that each sheaf $(\mathcal{O}_S \otimes_k \Gamma(X, \mathcal{O}_X)/\mathcal{I})^U_\lambda$ is locally free of rank $h(\lambda)$. Such an ideal sheaf $\mathcal{I}$ is uniquely determined by $\mathcal{I}^U$; the latter is a $T$-stable ideal sheaf of $\mathcal{O}_S \otimes_k \Gamma(X//U, \mathcal{O}_{X//U})$, which yields a point of $\text{Hilb}_h^T(X//U)(S) = \text{Hom}(S, \text{Hilb}_h^T(X//U))$.

Conversely, a given $\mathcal{J} \in \text{Hom}(S, \text{Hilb}_h^T(X//U))$ equals $\mathcal{I}^U$ for some $\mathcal{I} \in \text{Hilb}_h^G(X)(S)$, if and only if the span

$$\langle G \cdot \mathcal{J} \rangle \subseteq \mathcal{O}_S \otimes_k \Gamma(X, \mathcal{O}_X)$$

of all $G$-translates of $\mathcal{J}$, is a sheaf of ideals. Since $\langle G \cdot \mathcal{J} \rangle$ is a sheaf of $\mathcal{O}_S$-$G$-modules, this amounts to

$$((\langle G \cdot \mathcal{J} \rangle \cdot \Gamma(X, \mathcal{O}_X))^U) \subseteq \mathcal{J}.$$
To express this condition in more concrete terms, consider three dominant weights $\lambda$, $\mu$, $\nu$, and a $B$-eigenvector $v \in V(\lambda) \otimes_k V(\mu)$, of weight $\nu$. We may write

$$v = \sum_{i \in F} a_i (g_i \cdot v_\lambda) \otimes (h_i \cdot v_\mu),$$

where $F$ is a finite set, $a_i \in k$, and $g_i, h_i \in G$. Then, for any local section $\varphi$ of $J_\lambda$ and for any $\psi \in \Gamma(X, \mathcal{O}_X)^U_\mu$, we obtain a local section $\sum_{i \in F} a_i (g_i \cdot \varphi)(h_i \cdot \psi)$ of $\mathcal{O}_S \otimes_k \Gamma(X, \mathcal{O}_X)^U_\mu$, and hence a morphism

$$J_\lambda \otimes_k \Gamma(X, \mathcal{O}_X)^U_\mu \to \mathcal{O}_S \otimes_k \Gamma(X, \mathcal{O}_X)^U_\mu$$

of sheaves over $\mathcal{O}_S$. Composing with the quotient by $J_\nu$ yields a morphism of $\mathcal{O}_S$-sheaves

$$J_\lambda \otimes_k \Gamma(X//U, \mathcal{O}_X//U)_\mu \to \mathcal{O}_S \otimes_k \Gamma(X//U, \mathcal{O}_X//U)_\mu/J_\nu.$$

Now our condition is the vanishing of all such morphisms. The latter may be regarded as sections of certain locally free sheaves over $\text{Hilb}_h(X//U)$, arising from its universal family. So their vanishing defines a closed subscheme of $\text{Hilb}_h(X//U)$, which represents $\text{Hilb}_h^G(X)$. \hfill $\square$

We say that $\text{Hilb}_h^G(V)$ is the invariant Hilbert scheme. Assigning to any family its quotient by $G$, we obtain a morphism

$$\eta : \text{Hilb}_h^G(V) \to \text{Hilb}_h(0)(V//G)$$

to the punctual Hilbert scheme that parameterizes closed subschemes of length $h(0)$ in $V//G$. One may check that $\eta$ is proper; it generalizes the Nakamura morphism for finite groups, see e.g. [Nak01].

1.3. Moduli $\text{M}_Y$ of affine schemes of type $Y$. To construct this moduli scheme, we need a relative version of the invariant Hilbert scheme. Consider two affine $G$-schemes $X$, $Y$, an equivariant morphism $f : X \to Y$, and a function $h : \Lambda^+ \to \mathbb{N}$. Let $\text{Hilb}_h^G(f)$ be the functor assigning to each scheme $S$ the subset of $\text{Hilb}_h^G(X)(S)$ consisting of those $\mathcal{X} \subseteq X \times S$ such that the morphism

$$f \times \text{id} : \mathcal{X} \to Y \times S$$

is a closed immersion.

**Lemma 1.8.** The functor $\text{Hilb}_h^G(f)$ is represented by an open subscheme $\text{Hilb}_h^G(f)$ of $\text{Hilb}_h^G(X)$, equipped with a morphism $\text{Hilb}_h^G(f) \to \text{Hilb}_h^G(Y)$.

**Proof.** Consider a family $\pi : \mathcal{X} \to S$ in $\text{Hilb}_h^G(X)(S)$; put $\mathcal{R} = \pi_* \mathcal{O}_\mathcal{X}$. By Theorem 1.7, we have a morphism $S \to \text{Hilb}_h^T(X//U)$ such that the family $\pi//U : \mathcal{X}//U \to S$ is the pull-back of the universal family $\text{Univ}_h^T(X//U) \to \text{Hilb}_h^T(X//U)$. As a consequence, there exists a finite subset $F \subset \Lambda^+$ (depending only on $X$ and $h$) such that the sheaf of $\mathcal{O}_S$-algebras $\mathcal{R}^U$ is generated by its weight spaces $\mathcal{R}^U_\lambda$, $\lambda \in F$. 
Now \( \pi \) lies in \( \mathcal{H}ilb_h^G(f)(S) \) if and only if the map
\[
(f \times \text{id})^\#: \mathcal{O}_S \otimes_k \Gamma(Y, \mathcal{O}_Y) \to \mathcal{R}
\]
is surjective; equivalently, its restriction
\[
\mathcal{O}_S \otimes_k \Gamma(Y, \mathcal{O}_Y)|^U \to \mathcal{R}_U^U
\]
is surjective for all \( \lambda \in F \). This yields finitely many open conditions on \( \mathcal{H}ilb_h^G(X) \subseteq \mathcal{H}ilb_h^T(X//U) \), and hence an open subset \( \mathcal{H}ilb_h^G(f) \) that represents \( \mathcal{H}ilb_h^G(f) \). Moreover, assigning to \( X \) its image in \( Y \times S \) yields a natural transformation \( \mathcal{H}ilb_h^G(f) \to \mathcal{H}ilb_h^G(Y) \), and hence the desired morphism. \( \Box \)

Next let \( V \) be a finite-dimensional \( G \)-module and let \( V_U \) be the space of co-invariants of \( U \), i.e., \( V_U \) is the quotient of \( V \) by the span of the elements \( u \cdot v - v \), where \( u \in U \) and \( v \in V \). In other words, the quotient map
\[
q : V \to V_U
\]
is the universal \( U \)-invariant linear map with source \( V \). Since \( T \) normalizes \( U \), it acts on \( V_U \), and \( q \) is equivariant.

For any \( \lambda \in \Lambda^+ \), the \( T \)-module \( V(\lambda)_U \) is one-dimensional with weight \( w_0 \lambda \), the lowest weight of \( V(\lambda) \). Moreover, \( q \) maps isomorphically \( V(\lambda)_U \) to \( V(\lambda)_U \). It follows that the restriction \( V^{U^-} \to V_U \) is an isomorphism. Thus, the category of finite-dimensional \( G \)-modules is equivalent to that of finite-dimensional \( T \)-modules with weights in \( -\Lambda^+ \), via \( V \mapsto V_U \).

The map \( q : V \to V_U \) factors as \( f \circ p \), where \( p = p_{\lambda,U} : V \to V//U \) is the quotient, and
\[
f : V//U \to V_U
\]
is a surjective, \( T \)-equivariant morphism.

Let \( \mathcal{H}ilb_h^G(V)_0 \) be the functor assigning to every scheme \( S \) the subset of \( \mathcal{H}ilb_h^G(V)(S) \) consisting of those subfamilies \( \mathcal{X} \subseteq V \times S \) such that the morphism
\[
f//U \times \text{id} : \mathcal{X}//U \to V_U \times S
\]
is a closed immersion. Then Theorem 1.7 and Lemma 1.8 immediately imply the following result.

**Theorem 1.9.** The functor \( \mathcal{H}ilb_h^G(V)_0 \) is represented by an open subscheme \( \mathcal{H}ilb_h^G(V)_0 \) of \( \mathcal{H}ilb_h^G(V) \), equipped with a morphism
\[
f : \mathcal{H}ilb_h^G(V)_0 \to \mathcal{H}ilb_h^T(V_U).
\]

**Definition 1.10.** Given an affine \( T \)-scheme \( Y \), a family of affine \( G \)-schemes of type \( Y \) over a scheme \( S \) consists of a family of affine \( G \)-schemes \( \pi : \mathcal{X} \to S \), together with an isomorphism \( \varphi : \mathcal{X}//U \to Y \times S \) of families of affine \( T \)-schemes over \( S \).

We say that the families of type \( Y \) \( (\pi : \mathcal{X} \to S, \varphi : \mathcal{X}//U \to Y \times S) \) and \( (\pi' : \mathcal{X}' \to S, \varphi' : \mathcal{X}'//U \to Y \times S) \) are equivalent if there exists a morphism \( \psi : \mathcal{X} \to \mathcal{X}' \) such that \( \varphi' \circ \psi//U = \varphi \).
With the preceding notation, \( \pi \) is flat, since the family \( \pi//U \) is trivial. Moreover, \( \pi \) is a family of multiplicity-finite \( G \)-schemes, if and only if \( Y \) is multiplicity-finite; then \( \pi \) and \( Y \) have the same Hilbert function. On the other hand, \( \psi \) is an isomorphism, since \( \psi//U \) is. Likewise, any self-equivalence is the identity.

**Definition 1.11.** Let \( Y \) be a multiplicity-finite \( T \)-scheme. The *moduli functor of affine \( G \)-schemes of type \( Y \)* is the contravariant functor \( M_Y : (\text{Schemes}) \to (\text{Sets}) \) assigning to any \( S \) the set of equivalence classes of families of affine \( G \)-schemes of type \( Y \) over \( S \).

By Lemma 1.3, the group \( \text{Aut}^T_S(Y \times S) \) equals \( \text{Hom}(S, \text{Aut}^T(S)) \). It acts naturally on \( M_Y(S) \), and the isotropy group of the equivalence class of \( (\pi : \mathcal{X} \to S, \varphi : \mathcal{X}//U \to Y \times S) \) is the image of the map

\[
\text{Aut}^G_S(\mathcal{X}) \to \text{Aut}^T_S(\mathcal{X}//U) \cong \text{Aut}^T_S(Y \times S) = \text{Hom}(S, \text{Aut}^T(S))
\]

(which is injective, by Lemma 1.3). The orbit space \( M_Y(S)/\text{Aut}^T_S(Y \times S) \) is the set of isomorphism classes of families \( \mathcal{X} \) over \( S \), such that \( \mathcal{X}//U \) is isomorphic to the trivial family \( Y \times S \).

We may now describe \( M_Y \) in terms of invariant Hilbert schemes.

**Theorem 1.12.** Let \( Y \) be a multiplicity-finite \( T \)-scheme, with Hilbert function \( h \). Choose a finite-dimensional \( T \)-module \( E \) such that \( Y \) admits a closed \( T \)-equivariant immersion into \( E \). Let \( V \) be the \( G \)-module such that \( V_U = E \), and let \( M_Y \) be the fiber at \( Y \) of the morphism \( f : \text{Hilb}_h^G(V)_0 \to \text{Hilb}_h^T(V_U) \). Then \( M_Y \) represents the functor \( M_Y \).

In particular, \( M_Y \) is independent of \( E \) and admits an action of \( \text{Aut}^T(S) \); the orbits are in bijection with the isomorphism classes of affine \( G \)-schemes \( X \) such that \( X//U \simeq Y \).

**Proof.** Let \( (\pi : \mathcal{X} \to S, \varphi : \mathcal{X}//U \to Y \times S) \) be a family of affine \( G \)-schemes of type \( Y \); let \( \mathcal{R} = \pi_* \mathcal{O}_X \). Then we have a morphism of \( T \)-modules \( E^* \to \Gamma(S, \mathcal{R}^U) \) that extends uniquely to a morphism of \( G \)-modules \( V^* \to \Gamma(S, \mathcal{R}) \). This yields a morphism of sheaves of \( \mathcal{O}_S \)-\( G \)-algebras

\[
\mathcal{O}_S \otimes_k \text{Sym}_k(V^*) \to \mathcal{R}
\]

lifting \( \mathcal{O}_S \otimes_k \text{Sym}_k(E^*) \to \mathcal{R}^U \). Since the latter is surjective, we obtain a closed \( G \)-equivariant immersion \( \mathcal{X} \hookrightarrow V \times S \) over \( S \), such that the composition

\[
\mathcal{X}//U \to V//U \times S \to V_U \times S = E \times S
\]

is nothing but \( \varphi : \mathcal{X}//U \to Y \times S \) followed by the inclusion \( Y \times S \subseteq E \times S \). So, by Theorem 1.9, the image of \( \mathcal{X} \) in \( V \times S \) yields an \( S \)-point of the fiber \( f^{-1}(Y) \). Clearly, this point only depends on the equivalence class of the family \( (\pi, \varphi) \).

For the converse, note that an \( S \)-point of \( \text{Hilb}_h^G(V) \) is nothing but an equivalence class of pairs \( (\pi, \iota) \), where \( \pi : \mathcal{X} \to S \) is a family of multiplicity-finite \( G \)-schemes with Hilbert function \( h \), and \( \iota : \mathcal{X} \to V \times S \) is a closed
etale cover whose objects are families of affine $G$-schemes. For every scheme $S$, let $M'_{Y}(S)$ be the category whose objects are families of affine $G$-schemes $\pi : X \to S$ such that for some etale cover $\{S_i \to S\}$ the schemes $(X//U) \times S_i$ are isomorphic to $Y \times S_i$ as $T$-schemes over $S_i$. Morphisms in $M'_Y(S)$ are just isomorphisms of families of affine $G$-schemes. It is immediate that $M'_Y$ is the quotient stack of $M_Y$ by the linear algebraic group $\text{Aut}^T(Y)$. It is certainly not a Deligne-Mumford stack in general, since the stabilizers may be infinite. However, it is an algebraic Artin stack, and it is smooth if the scheme $M_Y$ is.

1.4. The tangent space of the invariant Hilbert scheme. We begin with a description of the tangent space of $\text{Hilb}^G_h(V)$, completely analogous to that of the classical Hilbert scheme; the tangent space of $M_Y$ will be determined in 2.3 below.

**Proposition 1.13.** Consider a finite-dimensional $G$-module $V$, a function $h : \Lambda^+ \to \mathbb{N}$, and a closed point $X \in \text{Hilb}^G_h(V)$, that is, a $G$-stable subscheme $X \subseteq V$ with Hilbert function $h$. Let $I \subseteq \text{Sym}_k(V^*)$ be the ideal of $X$, and let $R = \text{Sym}_k(V^*)/I$. Then $\text{Hom}_R(I/I^2, R)$ is a multiplicity-finite $G$-module, and the Zariski tangent space $T_X \text{Hilb}^G_h(V)$ is canonically isomorphic to $\text{Hom}_R^G(I/I^2, R)$.

Moreover, the space $T^1(X)$ of infinitesimal deformations of $X$ is also a multiplicity-finite $G$-module, and we have an exact sequence of finite-dimensional $k$-vector spaces

$$0 \to \text{Der}^G(X) \to \text{Hom}^G(X, V) \to T_X \text{Hilb}^G_h(V) \to T^1(X)^G \to 0.$$

**Proof.** Let $D = k[t]/(t^2) = k \oplus k\varepsilon$, where $\varepsilon^2 = 0$. Then $T_X \text{Hilb}^G_h(V)$ is the fiber at $X$ of the map $\text{Hilb}^G_h(V)(\text{Spec} D) \to \text{Hilb}^G_h(V)(\text{Spec} k)$. In other words, $T_X \text{Hilb}^G_h(V)$ consists of those $G$-stable ideals $J \subset D \otimes_k \text{Sym}_k(V^*)$ such that: $D \otimes_k \text{Sym}_k(V^*)/(J, \varepsilon) = R$, and $D \otimes_k \text{Sym}_k(V^*)/J$ is flat over $D$. Equivalently, $J \cap \text{Sym}_k(V^*) = I$ and $J \cap \varepsilon \text{Sym}_k(V^*) = \varepsilon I$. For any $u + \varepsilon v \in J$, it follows that: $u \in I$, and the class $v + I \in R$ is uniquely determined by $u$. This yields a map $\varphi : I \to R$, $u \mapsto v + I$. One easily checks that $\varphi$ lies in $\text{Hom}_R^G(I \oplus \varepsilon \text{Sym}_k(V^*), R) = \text{Hom}_R^G(I/I^2, R)$, and that $J$ is the preimage in $I \oplus \varepsilon \text{Sym}_k(V^*)$ of the graph of $\varphi$ in $I \oplus \varepsilon R$. Moreover, any $\varphi \in \text{Hom}_R^G(I/I^2, R)$ arises from an ideal $J \in T_X \text{Hilb}^G_h(V)$. This proves the first assertion.

Next recall the exact sequence of Kähler differentials (over $k$)

$$I/I^2 \to \Omega^1_{\text{Sym}_k(V^*)} \otimes_{\text{Sym}_k(V^*)} R \to \Omega^1_R \to 0,$$
that is,
\[ I/I^2 \to R \otimes_k V^* \to \Omega^1_R \to 0. \]
Applying \( \text{Hom}_R(-, R) \), we obtain a exact sequence of \( R\)-modules
\[ 0 \to \text{Der}(R) \to R \otimes_k V \to \text{Hom}_R(I/I^2, R) \to T^1(X) \to 0, \]
see e.g. [Har77, Exercise II.9.8]. Since \( R \otimes_k V = \text{Hom}(X, V) \), this yields our exact sequence by taking \( G \)-invariants.

Since \( I/I^2 \) is a finitely generated \( R\)-module, it is generated as an \( R\)-module by a finite-dimensional \( G\)-submodule \( E \). Then \( \text{Hom}_R(I/I^2, R) \) identifies to a \( R\)-submodule of \( R \otimes_k E^* \), and the latter is multiplicity-finite as a \( G\)-module, since \( R \) is. So \( \text{Hom}_R(I/I^2, R) \), and hence \( T^1(X) \), are multiplicity-finite \( G\)-modules; in particular, \( \text{Hom}_R^G(I/I^2, R) \) and \( T^1(X)^G \) are finite-dimensional. The same holds for \( \text{Hom}^G(X, V) = (R \otimes_k V)^G \). Thus, \( \text{Der}^G(X) \) is finite-dimensional as well (this also follows from Lemma 1.3). \( \Box \)

Denote by \( \text{Aut}^G(V) \) the automorphism group of the \( G\)-variety \( V \). This group acts on \( \text{Hilb}^G_h(V) \), and we may think of \( T^1(X)^G \) as the normal space at \( X \) to the orbit \( \text{Aut}^G(V) \cdot X \). Indeed, we may regard \( \text{Der}^G(V) \) as the Lie algebra of the (possibly infinite-dimensional) algebraic group \( \text{Aut}^G(V) \). Further, \( \text{Der}^G(V) = (\text{Sym}_k(V^*) \otimes_k V)^G \) maps surjectively to \( (R \otimes_k V)^G = \text{Hom}^G(X, V) \), so that the image of \( \text{Hom}^G(X, V) \) in \( T_X \text{Hilb}^G_h(V) \) is \( \text{Der}^G(V) \cdot X \), the tangent space to the orbit. Under additional assumptions, we will see that \( \text{Aut}^G(V) \) may be replaced with the centralizer \( \text{GL}(V)^G \). The latter is isomorphic to the product of the \( \text{GL}(V^U) \)'s, and hence is a connected reductive algebraic group.

Let \( X \subseteq V \) be a multiplicity-free subvariety, that is, the closure of a spherical orbit. Let \( S = \Lambda^+_X \) be the corresponding weight monoid, then \( h(\lambda) = 1 \) if \( \lambda \in S \), and \( h(\lambda) = 0 \) otherwise. Thus, the data of \( S \) and of \( h \) are equivalent; we will write \( \text{Hilb}^G_S(V) \) for the invariant Hilbert scheme \( \text{Hilb}^G_h(V) \). The group \( \text{GL}(V)^G \) acts on that scheme; thus, we can consider the orbit \( \text{GL}(V)^G \cdot X \), where \( X \) is regarded as a closed point of \( \text{Hilb}^G_S(V) \).

**Definition 1.14.** Given a finite-dimensional \( G\)-module \( V \) and a closed \( G\)-stable subvariety \( X \), we say that \( X \) is non-degenerate, if its projections to the isotypical components of \( V \) are all non-zero.

(In particular, the non-degenerate subvarieties of a multiplicity-free \( G\)-module \( V \) are those that span \( V \).) Now Proposition 1.13 may be refined as follows.

**Proposition 1.15.** Let \( X \) be the closure of a spherical orbit \( G \cdot x \in X \) in a finite-dimensional \( G\)-module \( V \).
(i) If \( X \) is non-degenerate, then the normal space at \( X \) to \( \text{GL}(V)^G \cdot X \) is isomorphic to \( T^1(X)^G \).
(ii) If, in addition, the isotropy group \( G_x \) is reductive, then \( T^1(X)^G = 0 \). In other words, \( \text{GL}(V)^G \cdot X \) is open in \( \text{Hilb}^G_S(V) \).
(iii) On the other hand, if $X$ is normal and if the boundary $X - G \cdot x$ has codimension at least 2 in $X$, then the exact sequence of Proposition 1.13 identifies with

$$0 \to (\mathfrak{g}/\mathfrak{g}_x)^G \to V^G \to (V/\mathfrak{g} \cdot x)^G \to T^1(X)^G \to 0,$$

where $\mathfrak{g}_x$ denotes the isotropy Lie algebra of $x$. As a consequence, we have an exact sequence

$$0 \to T^1(X)^G \to H^1(G_x, \mathfrak{g}/\mathfrak{g}_x) \to H^1(G_x, V).$$

Proof. (i) The Lie algebra of $GL(V)^G$ is $\text{End}^G(V)$, so that the tangent space to $GL(V)^G \cdot X$ at $X$ equals $\text{End}^G(V) \cdot X$. With the notation of Proposition 1.13, this subspace of $\text{Hom}^G_R(I/I^2, R)$ is the image of the composition

$$\text{End}^G(V) \xrightarrow{r_X} \text{Hom}^G(X, V) \xrightarrow{\cdot} \text{Hom}^G_R(I/I^2, R),$$

where $r_X$ denotes the restriction map. By the exact sequence of that proposition, it suffices to show that $r_X$ is surjective. And since the restriction $\text{Hom}^G(X, V) \to \text{Hom}^G(G \cdot x, V) = V^G$ is injective, it suffices in turn to check the surjectivity of the composition

$$\text{End}^G(V) \to V^G, \ f \mapsto f(x).$$

Write

$$V = \bigoplus_{\lambda \in F} M_\lambda \otimes_k V(\lambda),$$

where $F$ is a finite subset of $\Lambda^+$, and $M_\lambda = \text{Hom}^G(V(\lambda), V)$. Then

$$\text{End}^G(V) \simeq \prod_{\lambda \in F} \text{End}(M_\lambda).$$

On the other hand, we have $\dim V(\lambda)^G \leq 1$ for any $\lambda \in \Lambda^+$, since $G_x$ is a spherical subgroup of $G$. And since $X$ has a nonzero projection on each isotypical component $V(\lambda) = M_\lambda \otimes V(\lambda)$, we have

$$x = \sum_{\lambda \in F} m_\lambda \otimes x_\lambda,$$

where each $m_\lambda$ is nonzero, and each $x_\lambda$ spans $V(\lambda)^G$. It follows that

$$\text{End}^G(V)(x) = \sum_{\lambda \in F} M_\lambda \otimes V(\lambda)^G = V^G,$$

as desired.

(ii) Since $G \cdot x$ is dense in $X$, the restriction map $\text{Hom}_R(I/I^2, R) \to H^0(G \cdot x, N_{G,x})$ is injective, where $N_{G,x}$ denotes the normal sheaf. The latter is the $G$-linearized sheaf on $G/G_x$ associated with the $G_x$-module $V/\mathfrak{g} \cdot x$. This yields an injection

$$T_X \text{Hilb}^G_S(V) = \text{Hom}^G_R(I/I^2, R) \to H^0(G \cdot x, N_{G,x})^G = (V/\mathfrak{g} \cdot x)^G.$$
But since $G_x$ is reductive, the natural map $V^{G_x} \to (V/\mathfrak{g} \cdot x)^{G_x}$ is surjective. Together with the proof of (i), it follows that $\text{End}^G(V)X$ equals $T_X \text{Hilb}^G_S(V)$.

(iii) Since $X = \text{Spec}(R)$ is normal, the $R$-modules

$$\text{Der}(X) = \text{Hom}(\Omega^1_X, R), \quad \text{Hom}(X, V) = R \otimes_k V, \quad \text{Hom}_R(I/I^2, R)$$

are all reflexive. And since $\text{codim}_X (X - G \cdot x) \geq 2$, the exact sequence

$$0 \to \text{Der}(X) \to \text{Hom}(X, V) \to \text{Hom}_R(I/I^2, R)$$

identifies with

$$0 \to H^0(G \cdot x, \mathcal{T}_{G,x}) \to H^0(G \cdot x, \mathcal{O}_{G,x} \otimes_k V) \to H^0(G \cdot x, \mathcal{N}_{G,x}),$$

where $\mathcal{T}_{G,x}$ denotes the tangent sheaf. Further, $\mathcal{T}_{G,x}$ (resp. $\mathcal{O}_{G,x} \otimes_k V$) is the $G$-linearized sheaf on $G/G_x$ associated with the $G_x$-module $\mathfrak{g}/\mathfrak{g}_x$ (resp. $V$). Now taking $G$-invariants in the former exact sequence yields our assertion.

In the case where $G = T$ is a torus, the multiplicity-free varieties are those containing a dense orbit; these are the (possibly non-normal) affine toric varieties. They are classified by finitely generated submonoids of $\Lambda^+ = \Lambda$, via $X \mapsto \Lambda_X$ and $\mathcal{S} \mapsto \text{Spec } k[\mathcal{S}]$. Note that the map $T \mapsto \text{Aut}^T(X)$ is surjective with kernel the intersection of the $\ker(\lambda), \lambda \in \Lambda_X$.

By Proposition 1.15 (ii), $T^1(X)^T = 0$ for any multiplicity-free $T$-variety $X$ (in fact, it is easy to show that every family of such varieties is locally trivial, see e.g. [AB02, Lemma 7.4]; we refer to [Alt94] and [Alt97] for a study of arbitrary deformations of affine toric varieties). So we obtain the following result, which generalizes [PS02, Theorem 1.2].

**Corollary 1.16.** Let $X$ be an orbit closure in a finite-dimensional $T$-module $V$. If $X$ is non-degenerate, then the orbit $\text{GL}(V)^T \cdot X$ is open in $\text{Hilb}^G_S(V)$, where $\mathcal{S}$ denotes the submonoid of $\Lambda$ generated by the opposites of the weights of $V$.

If, in addition, $X$ spans $V$, then all weight subspaces have dimension 1, so that $\text{GL}(V)^T$ is a maximal torus of $\text{GL}(V)$. In that case, $\text{Hilb}^G_S(V)$ is the toric Hilbert scheme of [PS02], and the closure of $\text{GL}(V)^T \cdot X$ is its main component.

We now extend this construction to multiplicity-free $G$-varieties. Consider a finite-dimensional $G$-module $V$, and denote by $F$ the set of weights of $(V^*)^U$. Let $\mathcal{S}$ the submonoid of $\Lambda^+$ generated by $F$, and let $M_{\mathcal{S}} = M_Y$ where $Y = \text{Spec } k[\mathcal{S}]$; then $Y$ is a non-degenerate, multiplicity-free subvariety of the $T$-module $V_U$.

**Corollary 1.17.** With this notation, the non-degenerate multiplicity-free subvarieties $X \subseteq V$, having $\mathcal{S}$ as their weight monoid, are parameterized by an open subscheme of $\text{Hilb}^G_S(V)$, stable under $\text{GL}(V)^G$. Moreover, this open subscheme $\text{Hilb}^G_S$ is isomorphic to $(\text{GL}(V)^G \times M_{\mathcal{S}})/T$, where $T$ acts on $\text{GL}(V)^G$ via $t \mapsto (\lambda(t))_{\lambda \in F}$, and on $M_{\mathcal{S}}$ via its action on $k[\mathcal{S}]$. 
is spanned by the image of $x$ denotes the group of $n$-tuples. We determine $\text{Hilb}^G\mathcal{Y}$, and let $x = \sum_{\lambda \in F} x_\lambda$. Then $X_0 = G \cdot x$ is a non-degenerate $G$-subvariety of $V$; one checks that it is also multiplicity-free with weight monoid $\mathcal{S}$.

Clearly, the $GL(V)^G$-orbit of $X_0$ in $\text{Hilb}^G\mathcal{Y}$ is obtained by varying the choices of the $x_\lambda$, and the corresponding $T$-orbit in $M_\mathcal{S}$ is just a fixed point that we still denote by $X_0$. We will see in Theorem 2.7 that the scheme $M_\mathcal{S}$ is affine, with $X_0$ as its unique closed orbit of $T$; as a consequence, $\text{Hilb}^G\mathcal{Y}$ is affine as well, and $GL(V)^G \cdot X_0$ is its unique closed $GL(V)^G$-orbit.

In the case where $G$ is a torus, $\text{Hilb}^G\mathcal{Y}$ is a unique orbit of $GL(V)^G$. But this may fail in the general case, as shown by the following examples.

**Example 1.** Let $G = SL(2)$. The simple $G$-modules are the symmetric powers of the defining module $k^2$; they are indexed by the non-negative integers. We determine $\text{Hilb}^G\mathcal{Y}$, where $F = \{n\}$; then $\text{Hilb}^G\mathcal{Y} \simeq M_{k^n}$ by Corollary 1.17.

The space $V = V(n)$ has basis the monomials $x^n, x^{n-1}y, \ldots, y^n$, where $x^n$ is the highest weight vector. So $X_0 = G \cdot x^n$ is a normal surface, the cone over the rational normal curve in $\mathbb{P}^n$, and $X_0 - G \cdot x^n$ is just the origin. Thus, we may apply Proposition 1.15. Note that $G_{x^n} = U \cdot \mu_n$, where $\mu_n$ denotes the group of $n$-th roots of unity in $T \simeq \mathbb{G}_m$; it acts on $V(n)$ via $t \cdot (x, y) = (tx, t^{-1}y)$. Hence $g \cdot x^n$ is spanned by $x^n, x^{n-1}y$, and $(V(n)/g \cdot x^n)^U$ is spanned by the image of $x^{n-2}y^2$ if $n \geq 2$, whereas $(V(1)/g \cdot x^n)^U$ is zero. But $x^{n-2}y^2$ is fixed by $\mu_n$ if and only if $n$ divides 4. So $T_{X_0} \text{Hilb}^G\mathcal{Y}$ is a line if $n = 2$ or $n = 4$, and vanishes otherwise.

It follows that $\text{Hilb}^G\mathcal{Y}$ is the affine line if $n = 2$ or $n = 4$, and is a point otherwise. The corresponding families are the orbit closures of $x^2 + ty^2$ in $V(2)$, and of $x^4 + tx^2y^2$ in $V(4)$, where $t \in k$; for $t \neq 0$, they are isomorphic to $G/T$, resp.$G/N_G(T)$, where $N_G(T)$ denotes the normalizer of $T$.

These results also follow from work of Pinkham [Pin74, Chapter 8] describing all (possibly non-invariant) deformations of $X_0$.

**Example 2.** Let $G = SL(4)$ and $F = \{\omega_1, \omega_2, \omega_3\}$, where each $\omega_i$ is the highest weight of the simple $G$-module $\wedge^i k^4$. Then $\mathcal{S} = \Lambda^+$, and $\text{Hilb}^G\mathcal{Y} = M_{\Lambda^+}$. We have

$$V = k^4 \times \wedge^2 k^4 \times \wedge^3 k^4.$$
One checks that the multiplicity-free subvarieties of $V$ that span it and have $\Lambda^+$ as their weight monoid are exactly the orbit closures $G \cdot x$, where $x$ is one of the following points:

$$x_0 = (e_1, e_1 \wedge e_2, e_1 \wedge e_2 \wedge e_3), \quad x_1(t) = (e_1, e_2 \wedge e_3, t e_1 \wedge e_2 \wedge e_3),$$

$$x_2(t) = (e_1, e_1 \wedge e_4, t e_1 \wedge e_2 \wedge e_3), \quad x_{12}(t, u) = (e_1, t(e_1 \wedge e_4 + e_2 \wedge e_3), u e_1 \wedge e_2 \wedge e_3).$$

Here $(e_1, e_2, e_3, e_4)$ is an arbitrary basis of $k^4$, and $t, u$ are non-zero scalars.

Fixing this basis, we obtain representatives of the $T$-orbits in $M_{\Lambda^+}$; the dimensions of these orbits are $0, 1, 1, 2$. In particular, $X_0 = G \cdot x_0$ is the $T$-fixed point. The tangent space $T_{X_0} M_{\Lambda^+} = T_{X_0} \text{Hilb}_G^G(V)$ may be determined by using Proposition 1.15: indeed, one may check that $X_0$ is a normal variety, with boundary $X_0 - G \cdot x_0$ of codimension 2. Moreover, $G_{x_0}$ is a maximal unipotent subgroup of $G$, and the images in $V/\mathfrak{g} \cdot x_0$ of

$$(0, e_1 \wedge e_4, 0), \quad (0, e_2 \wedge e_3, 0)$$

are a basis of $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Thus, $T_{X_0} M_{\Lambda^+}$ has dimension 2.

It follows that $M_{\Lambda^+}$ is isomorphic to an affine plane where $T$ acts linearly with weights $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3$ where $\alpha_1, \alpha_2, \alpha_3$ are the simple roots. The $T$-orbit closures of $X_1(1), X_2(1)$ are coordinate lines, and the orbit closure of $X_{12}(1,1)$ is the complement of their union.

2. The action of the adjoint torus

2.1. Definition of the action. Given $\lambda \in \Lambda$, we denote $e^\lambda \in k[T]$ the corresponding regular function on $T$. Then

$$k[T] = \bigoplus_{\lambda \in \Lambda} k e^\lambda \supseteq \bigoplus_{\lambda \in \Pi} k e^\lambda = k[T_\text{ad}] \supseteq \bigoplus_{\lambda \in \Pi} k e^\lambda = k[\mathbb{A}^\Pi],$$

where $\mathbb{A}^\Pi$ is the affine space where $T$ acts linearly with weights being the simple roots. The latter inclusion yields an open immersion

$$T_\text{ad} \hookrightarrow \mathbb{A}^\Pi, \quad t \mapsto (\alpha(t))_{\alpha \in \Pi}.$$

We may identify the center $Z(G)$ with a central subgroup of $G \times T$, via $z \mapsto (z, z^{-1})$. The quotient $\widetilde{G} = (G \times T)/Z(G)$ is a connected reductive group, with maximal torus $\widetilde{T} = (T \times T)/Z(G)$. In fact, $\widetilde{T}$ is isomorphic to $T \times T_\text{ad}$ via $(t_1, t_2)Z(G) \mapsto (t_1 t_2, t_2 Z(G))$; this defines an injective homomorphism

$$T_\text{ad} \hookrightarrow \widetilde{T}, \quad tZ(G) \mapsto (t^{-1}, t)Z(G).$$

On the other hand, we have an injective homomorphism

$$G \hookrightarrow \widetilde{G}, \quad g \mapsto (g, 1)Z(G).$$

Together, these yield an isomorphism of $\widetilde{G}$ with the semi-direct product of $G$ with $T_\text{ad}$, acting on $G$ by conjugation.
We now extend this construction to families of affine \( G \)-schemes. Let \( \pi : \mathcal{X} \to S \) be such a family, then \( Z(G) \) acts on \( \mathcal{X} \times T \) by \( z \cdot (x, t) = (z \cdot x, z^{-1} t) \). This action is free, and \( \pi \) is affine and \( Z(G) \)-invariant; thus, the quotient

\[
\widetilde{\mathcal{X}} = (\mathcal{X} \times T)/Z(G)
\]

is a scheme. The map \( \mathcal{X} \times G \times T \to \mathcal{X} \times T \), \((x, g, t) \mapsto (g \cdot x, t)\) descends to a morphism \( \mathcal{X} \times G \to \widetilde{\mathcal{X}} \) which is \( G \)-invariant, and the induced morphism \( (\mathcal{X} \times G)/G \to \widetilde{\mathcal{X}} \) is an isomorphism. Thus, \( \widetilde{\mathcal{X}} \) is equipped with an action of \( \widetilde{G} \), and hence of \( G \). The morphism \( \pi \times \text{id} : \mathcal{X} \times T \to S \times T \) is affine, of finite type, and \( Z(G) \)-invariant. Hence it descends to an affine morphism of finite type

\[
\tilde{\pi} : \tilde{\mathcal{X}} \to (S \times T)/Z(G) = S \times T_{\text{ad}}
\]

which is \( G \)-invariant and \( T_{\text{ad}} \)-equivariant. In other words, \( \tilde{\pi} \) is a family of affine \( G \)-schemes, with a compatible action of \( T_{\text{ad}} \). Putting \( \tilde{\mathcal{R}} = \tilde{\pi}_* \mathcal{O}_{\tilde{\mathcal{X}}} \), with isotypical components the \( R_{(\lambda)} \), we obtain

\[
\tilde{\mathcal{R}} = (\mathcal{R} \otimes_k k[T])^{Z(G)} = \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda} (R_{(\lambda)} e^\mu)^{Z(G)} = \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda, \lambda - \mu \in \mathbb{Z}^I} R_{(\lambda)} e^\mu.
\]

This yields an isomorphism of \( \mathcal{O}_S[T_{\text{ad}}]-G \)-modules

\[
\tilde{\mathcal{R}} = \bigoplus_{\lambda \in \Lambda^+} R_{(\lambda)} e^\lambda \otimes_k k[T_{\text{ad}}].
\]

Moreover, \( \tilde{\mathcal{X}}//U = (\mathcal{X}//U \times T)/Z(G) \) is mapped isomorphically to \( \mathcal{X}//U \times T_{\text{ad}} \), via \((x, t) \mapsto (t \cdot x, tZ(G))\). In other words, the family \( \tilde{\pi}//U : \tilde{\mathcal{X}}//U \to S \times T_{\text{ad}} \) is equipped with an isomorphism to the pull-back of \( \pi//U : \mathcal{X}//U \to S \) via the projection \( S \times T_{\text{ad}} \to S \).

If the family \( \pi \) is flat, then so is \( \tilde{\pi} \), and its fiber at any \((s, t) \in S \times T_{\text{ad}} \) is \( G \)-equivariantly isomorphic to the fiber of \( \pi \) at \( s \). Moreover, if \( \pi \) is multiplicity-finite, then so is \( \tilde{\pi} \), with the same Hilbert function \( h \).

In the case where \( \mathcal{X} = V(\lambda) \) where \( G \) acts linearly (and \( \pi \) is constant), then \( Z(G) \) acts by the restriction of the character \( \lambda \). Then, for any \( \mu \in \Lambda \) such that \( \mu|_{Z(G)} = \lambda|_{Z(G)} \) (that is, \( \mu - \lambda \in \mathbb{Z}^I \)), we obtain an isomorphism

\[
\overline{V}(\lambda) = (V(\lambda) \times T)/Z(G) \to V(\lambda) \times T_{\text{ad}},
\]

\( (v, t)Z(G) \mapsto (\mu(t)v, tZ(G)) \),

which is \( \tilde{G} \)-equivariant for the action of \( G \) on \( V(\lambda) \times T_{\text{ad}} \) via \( g \cdot (v, s) = (g \cdot v, s) \), and the action of \( T_{\text{ad}} \) via \( t \cdot (v, s) = (\mu(t)t^{-1} v, ts) \). We choose \( \mu = w_0 \lambda \), so that \( T_{\text{ad}} \) acts on \( V(\lambda) \) by

\[
t \cdot v = (w_0 \lambda)(t)t^{-1} v.
\]

Then \( V(\lambda)^{U^-} \) is fixed pointwise, and \( q : V(\lambda) \to V(\lambda)_{U} \) is invariant.

More generally, any finite-dimensional \( G \)-module \( V \) becomes a \( \widetilde{G} \)-module by letting \( T_{\text{ad}} \) act on every submodule \( V(\lambda) \) as above, so that \( V^{U^-} \) and \( q : V \to V_{U} \) are \( T_{\text{ad}} \)-invariant. This yields an \( \widetilde{G} \)-equivariant isomorphism
Thus, if $\mathcal{X}$ is a closed $G$-stable subscheme of $V \times S$, then $\tilde{\mathcal{X}}$ is a closed subscheme of

$$(V \times S \times T)/\Gamma(G) \simeq V \times T_{\text{ad}} \times S,$$

stable under $\tilde{G}$, and hence under $G$. So, given $(\pi : \mathcal{X} \to S)$ in $\text{Hilb}_h^G(V)(S)$, we obtain $(\tilde{\pi} : \tilde{\mathcal{X}} \to T_{\text{ad}} \times S)$ in $\text{Hilb}_h^G(V)(T_{\text{ad}} \times S)$. Applying this to the universal family $\pi : \text{Univ}_h^G(V) \to \text{Hilb}_h^G(V)$, we obtain an element of $\text{Hilb}_h^G(V)(T_{\text{ad}} \times \text{Hilb}_h^G(V))$, that is, a morphism of schemes

$$a : T_{\text{ad}} \times \text{Hilb}_h^G(V) \to \text{Hilb}_h^G(V).$$

**Proposition 2.1.** (i) With the preceding notation, $a$ is an action of $T_{\text{ad}}$ on $\text{Hilb}_h^G(V)$. The open subscheme $\text{Hilb}_h^G(V)_0$ is stable under $T_{\text{ad}}$, and the morphism $f : \text{Hilb}_h^G(V)_0 \to \text{Hilb}_h^G(V)$ is invariant.

(ii) The restriction $T_{\text{ad}} \times \text{Hilb}_h^G(V)_0 \to \text{Hilb}_h^G(V)_0$ extends uniquely to a morphism

$$\mathbb{A}^\pi \times \text{Hilb}_h^G(V)_0 \to \text{Hilb}_h^G(V)_0.$$

Hence this morphism is an action of the multiplicative monoid $\mathbb{A}^\pi$, leaving $f$ invariant.

**Proof.** (i) follows from the preceding discussion; in fact, $a$ comes from the action of $T_{\text{ad}}$ on $V$, normalizing the actions of $G$, $U$, and fixing $q$.

(ii) Let $(\pi : \mathcal{X} \to S)$ in $\text{Hilb}_h^G(V)_0(S)$. For $\mathcal{R}$ and $\tilde{\mathcal{R}}$ as above, consider the sheaf

$$\tilde{\mathcal{R}} = \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda, \lambda - \mu \in \mathbb{N}\Pi} \mathcal{R}(\lambda)e^\mu \subseteq \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda, \lambda - \mu \in \mathbb{Z}\Pi} \mathcal{R}(\lambda)e^\mu = \mathcal{R}.$$ 

Since any simple $G$-submodule $V(\nu)$ of a tensor product $V(\lambda) \otimes_k V(\mu)$ satisfies $\lambda + \mu - \nu \in \mathbb{N}\Pi$, it follows that $\tilde{\mathcal{R}}$ is a sheaf of $\mathcal{O}_S[\mathbb{A}^\pi]$-$G$-subalgebras of $\mathcal{R}$. Moreover,

$$\tilde{\mathcal{R}} = \bigoplus_{\lambda \in \Lambda^+} \mathcal{R}(\lambda)e^\lambda \otimes_k k[\mathbb{A}^\pi].$$

Thus, $\tilde{\mathcal{R}}$ is locally free over $\mathcal{O}_S[\mathbb{A}^\pi]$, and $\tilde{\mathcal{R}}^U$ is isomorphic to $\mathcal{R}^U[\mathbb{A}^\pi]$. This defines a flat family of affine $G$-schemes

$$\tilde{\pi} : \tilde{\mathcal{R}} \to S \times \mathbb{A}^\pi$$

extending $\tilde{\pi} : \tilde{\mathcal{X}} \to S \times T_{\text{ad}}$, such that $\tilde{\pi}/U$ is equipped with an isomorphism to the pull-back of $\pi//U$. By assumption, $\mathcal{X} \subseteq V \times S$, and the composition $\mathcal{X}//U \to V//U \times S = V_U \times S$ is a closed $T$-equivariant immersion. This yields a closed $T$-equivariant immersion $\tilde{\mathcal{X}}//U \simeq \mathcal{X}//U \times \mathbb{A}^\pi \to V_U \times S \times \mathbb{A}^\pi$, and hence a closed $G$-equivariant immersion $\tilde{\mathcal{X}} \hookrightarrow V \times S \times \mathbb{A}^\pi$ extending $\mathcal{X} \hookrightarrow V \times S \times T_{\text{ad}}$. □

The above Proposition defines an action of $T_{\text{ad}}$ on $M_Y$ (where $Y$ is a multiplicity-finite $T$-scheme), that extends to $\mathbb{A}^\pi$. Here is a more direct construction of this action.
Lemma 2.2. Consider the action of $T$ on $M_Y$ via the homomorphism $T \to \text{Aut}^T(Y)$. Then the induced action of $Z(G)$ is trivial, and the corresponding action of $T/Z(G)$ is the same as the preceding $T_{ad}$-action.

Proof. Let $(\pi : X \to S, \varphi : X'/U \to Y \times S)$ be a family of type $Y$ and let $\gamma \in \text{Hom}(S, Z(G))$ act on $Y \times S$. Then $\gamma$ defines $\psi \in \text{Aut}_{G}(X)$, such that $\gamma \circ \varphi = \varphi \circ (\psi//U)$. Thus, the family $(\pi, \gamma \circ \varphi)$ is equivalent to $(\pi, \varphi)$, so that $Z(G)$ acts trivially on $M_Y$.

Choose a closed $T$-equivariant immersion $Y \hookrightarrow V_U$, where $V$ is a finite-dimensional $G$-module. By construction, the action of $T_{ad}$ on $\text{Hilb}_B^G(V)$ arises from the $T$-action on $V^*$, where each isotypical component $V^*_\lambda$ is the eigenspace of weight $\lambda$. This action stabilizes $(V^*)_U = (V_U)^*$, and it restricts to the natural $T$-action on that space, hence on $V_U$ and on $Y$. This completes the proof. \hfill \Box

2.2. Horospherical schemes as fixed points. We will construct a $T_{ad}$-invariant section of the morphism $f : \text{Hilb}_B^G(V)_0 \to \text{Hilb}_B^T(V_U)$. For this, we need some preliminaries on representation theory.

Let $E$ be a rational $B$-module. Put

$$\text{Ind}_B^G(E) = \text{Hom}^B(G, E) = (k[G] \otimes_k E)^B,$$

where $B$ acts by right multiplication on $k[G]$, and simultaneously on $E$. Then the action of $G$ on $k[G]$ by left multiplication equips $\text{Ind}_B^G(E)$ with the structure of a rational $G$-module, the induced module of $E$. The evaluation at the identity $e \in G$ yields a $B$-equivariant morphism

$$\varepsilon : \text{Ind}_B^G(E) \to E,$$

which is universal for $B$-equivariant morphisms from rational $G$-modules to $E$. If, in addition, $E$ is a $B$-algebra, then so is $\text{Ind}_B^G(E)$, and $\varepsilon$ is an algebra homomorphism.

In the case where $E = k_\lambda$, the one-dimensional $B$-module with weight $\lambda \in \Lambda$, then $\text{Ind}_B^G(E) = V(-\lambda)^*$ if $\lambda \in -\Lambda^+$; otherwise, $\text{Ind}_B^G(E) = 0$. As a consequence, if $E$ is finite-dimensional, then so is $\text{Ind}_B^G(E)$.

Recall that any rational $G$-module $V$ defines the space $V_U$ of its co-invariants, a rational $B$-module with trivial action of $U$ and weights in $-\Lambda^+$. The $B$-equivariant map $q : V \to V_U$ factors uniquely through a $G$-equivariant map

$$V \to \text{Ind}_B^G(V_U), \ v \mapsto (g \mapsto q(g \cdot v))$$

which is in fact an isomorphism. The assignments $E \mapsto \text{Ind}_B^G(E)$ and $V \mapsto V_U$ yield an equivalence of the category of rational $G$-modules, with the category of rational $B$-modules with trivial $U$-action and weights in $-\Lambda^+$.

We define the (restricted) dual $V^*$ of a rational module $V$ as the largest rational submodule of the space of linear forms on $V$. Then we define the co-induced module of a rational $B$-module $E$:

$$\text{Coind}_B^G(E) = (\text{Ind}_B^G(E^*))^*.$$
This is a rational $G$-module, equipped with a $B$-equivariant map

$$\iota : E \to \text{Coind}_B^G(E)$$

which is universal for maps from $E$ to rational $G$-modules. Note that $\text{Coind}_B^G(\mathbb{k}\lambda)$ equals $V(\lambda)$ if $\lambda \in \Lambda^+$, and vanishes otherwise. Moreover, $\text{Coind}_B^G(E)$ is finite-dimensional if $E$ is. Note also that

$$(V^U)^* = (V^*)_U.$$ 

So the assignments $V \to V^U$ and $E \to \text{Coind}_B^G(E)$ yield an equivalence of the category of rational $G$-modules, with the category of rational $T$-modules with weights in $\Lambda^+$. 

Another relation between induced and co-induced modules arises from the isomorphism $V^U - \to V^U$, where $V$ is any rational $G$-module. This implies an isomorphism

$$\text{Coind}_B^G(E) \simeq \text{Ind}_B^G(E),$$

where $E$ is any rational $T$-module (regarded as a trivial $U^-$-module on the left-hand side, and as a trivial $U$-module on the right-hand side). If $E$ is a $T$-algebra, this yields a $G$-algebra structure on $\text{Coind}_B^G(E)$.

Next we extend these constructions to families. Let $\pi : Y \to S$ be a family of affine $T$-schemes. Put

$$A = \pi_* \mathcal{O}_Y$$

and

$$R = \text{Coind}_B^G(A),$$

where $B$ acts on $A$ via its quotient $T$; then the canonical map $\iota : A \to R$ is $U$-invariant. Note that $R$ only depends on $\bigoplus_{\lambda \in \Lambda^+} A_\lambda$, a sheaf of finitely generated $\mathcal{O}_S$-$T$-subalgebras of $A$. Thus, we may and will assume that all weights of $A$ lie in $\Lambda^+$. We have

$$R \simeq \text{Ind}_B^G(A) = (k[G]^{U^-} \otimes_k A)^T,$$

where $T$ acts diagonally on $k[G]^{U^-} \otimes_k A$. It follows that $R$ is a sheaf of finitely generated $\mathcal{O}_S$-$G$-algebras; $\iota : A \to R$ is an injective homomorphism of $\mathcal{O}_S$-$T$-algebras identifying $A$ with $R^U$; and $\varepsilon : R \to A$ is a surjective homomorphism of $\mathcal{O}_S$-$B^-$-algebras identifying $A$ with $R^-_U$.

**Definition 2.3.** A family of affine $G$-schemes $\pi : \mathcal{X} \to S$ is **horospherical** if $\pi_* \mathcal{O}_\mathcal{X} = \text{Coind}_B^G(A)$ for some sheaf $A$ of finitely generated $\mathcal{O}_S$-$T$-algebras with weights in $\Lambda^+$. Denoting $\mathcal{Y} = \text{Spec}_S(A)$, a family of affine $T$-schemes, we say that $\mathcal{X}$ is **induced from** $\mathcal{Y}$ and write $\mathcal{X} = \text{Ind}_B^G(\mathcal{Y})$.

Then $\mathcal{X}$ is equipped with an isomorphism $\mathcal{X}/U \to \mathcal{Y}$. Further, the map $\text{Aut}_S^G(\mathcal{X}) \to \text{Aut}_S^T(\mathcal{X}/U) = \text{Aut}_S^T(\mathcal{Y})$ is an isomorphism. In particular, for any multiplicity-finite $T$-scheme $Y$, we may regard $\text{Ind}_B^G(Y)$ as a closed point of $\mathcal{M}_Y$, fixed by $\text{Aut}_S^T(Y)$.

In [Kno90], a $G$-variety $X$ is said to be horospherical if $X = G \cdot X^U$. We now show that both definitions agree.
Lemma 2.4. The following conditions are equivalent, for a family of affine $G$-schemes $\pi : X \to S$ with $U^-$-fixed point subscheme $X^{U^-}$:

(i) $X$ is horospherical.

(ii) $G \cdot X^{U^-} = X$ (as schemes).

(iii) The product in $R = \pi_*\mathcal{O}_X$ of any two isotypical components $R_\chi, R_\mu$ is contained in $R_{\chi + \mu}$.

Then the map $X^{U^-} \to X//U$ is an isomorphism.

Proof. We may assume that $X$ is affine and we put $R = \Gamma(X, \mathcal{O}_X)$. Note that $G \cdot X^{U^-} = G \cdot X^U$ (since $U^-$ and $U$ are conjugate in $G$). Further, $X^U$ is a closed $B$-stable subscheme of $X$, and hence $G \cdot X^U$ is also closed in $X$ (since $G/B$ is complete).

(i)$\Rightarrow$(ii) We have $R \cong \text{Ind}_{B^-}^G(R_{U^-})$, so that the quotient map $q^- : R \to R_{U^-}$ is an algebra homomorphism: its kernel $\ker(q^-)$ is an ideal of $R$. But the ideal of the fixed point subscheme $X^{U^-}$ is generated by the $g \cdot f - f$ for $g \in U^-$ and $f \in R$; hence this ideal equals $\ker(q^-)$. Now $\ker(q^-)$ contains no simple $G$-submodule of $R$, so that $\bigcap_{g \in G} g \cdot \ker(q^-) = \{0\}$. It follows that $G \cdot X^{U^-} = X$.

(ii)$\Rightarrow$(iii) The group $G \times B^-$ acts on $G \times X^{U^-}$ via $(g_1, \gamma) \cdot (g_2, y) = (g_1g_2\gamma^{-1}, \gamma \cdot y)$, and the surjective morphism $G \times X^{U^-} \to X$ is $G$-equivariant and $B^-$-invariant. Thus, it yields an injective homomorphism of $G$-algebras

$$R \hookrightarrow (k[G]^{U^-}_1 \otimes_k A)^T,$$

where $A = \Gamma(X^{U^-}, \mathcal{O}_{X^{U^-}})$. Since each weight space $k[G]^{U^-}_\chi$ is a simple $G$-module (with highest weight $-\chi$), it follows that any simple $G$-submodule of $R$ can be written as $k[G]^{U^-}_{-\lambda} \otimes f$, for some $\lambda \in \Lambda^+$ and some $f \in A_\lambda$. Now (iii) follows from the fact that

$$k[G]^{U^-}_{-\lambda} k[G]^{U^-}_{-\mu} = k[G]^{U^-}_{-\lambda-\mu}$$

for all $\lambda, \mu \in \Lambda^+$.

(iii)$\Rightarrow$(i) It suffices to show that the kernel $\ker(q^-)$ of the quotient map $q^- : R \to R_{U^-}$ is an ideal of $R$, i.e., is stable under multiplication by any simple $G$-submodule $V(\mu) \subseteq R$. For this, note that the intersection of $\ker(q^-)$ with any simple $G$-submodule $V(\lambda) \subseteq R$ is the kernel of $V(\lambda) \hookrightarrow V(\lambda)_{U^-}$, that is, the sum of all weight subspaces $V(\lambda)_\chi$ where $\chi \neq \lambda$; then $\chi < \lambda$. Thus, $\chi + \eta < \lambda + \mu$, for any weight $\eta$ of $V(\mu)$. Since the product $V(\lambda)V(\mu)$ is either 0 or $V(\lambda + \mu)$, this implies our assertion. $\square$

Next we relate the induction $Y \mapsto \text{Ind}_B^G(Y)$ to invariant Hilbert schemes.

Proposition 2.5. (i) The induction of families of affine $T$-schemes yields a closed immersion

$$s : \text{Hilb}_h^T(V_U) \to \text{Hilb}_h^G(V)_0$$

which is a section of $f : \text{Hilb}_h^G(V)_0 \to \text{Hilb}_h^T(V_U)$. 
(ii) The image of $s$ is contained in the fixed point subscheme of $T_{ad}$ in $\text{Hilb}^G_h(V)_0$. Thus, $s$ is invariant under $\mathbb{A}^\Pi$.

(iii) For any closed point $Y$ of $\text{Hilb}^T_h(V_U)$, the origin $o \in \mathbb{A}^\Pi$ acts on $M_Y = f^{-1}(Y)$ as the constant map to the closed point $\text{Ind}^G_B(Y)$.

Proof. (i) Let $\pi : \mathcal{X} \to S$ be induced from a family $\rho : \mathcal{Y} \to S$ of affine $T$-schemes, such that all weights of $\rho_*\mathcal{O}_Y$ are in $\Lambda^+$. Then $\pi$ is flat if and only if $\rho = \pi//U$ is. If, in addition, $\rho$ is multiplicity-finite with Hilbert function $h$, then $\pi$ is multiplicity-finite with Hilbert function $h|_{\Lambda^+}$. Finally, if $\mathcal{Y}$ is a closed $T$-stable subscheme of $V_U \times S$, then the $B$-equivariant map $(V_U)^* \to A$ yields a $G$-equivariant map

$$V^* = \text{Coind}^G_B((V_U)^*) \to \mathcal{R} = \pi_*\mathcal{O}_\mathcal{X},$$

and hence a homomorphism $\mathcal{O}_S \otimes_k \text{Sym}_k(V^*) \to \mathcal{R}$. The latter is surjective, since the corresponding homomorphism

$$\mathcal{O}_S \otimes_k \text{Sym}_k(V^*)_U = \mathcal{O}_S \otimes_k \text{Sym}_k(V_U)^* \to \mathcal{R}^U = \mathcal{A}$$

is. So we obtain a closed $G$-equivariant immersion $\mathcal{X} \hookrightarrow V \times S$ lifting the immersion $\mathcal{Y} \hookrightarrow V_U \times S$, and hence an element of $\text{Hilb}^G_h(V)_0(S)$ mapped to $\mathcal{Y}$ under $f : \text{Hilb}^G_h(V)_0(S) \to \text{Hilb}^G_h(V_U)(S)$. This constructs the section $s$, which is automatically a closed immersion.

(ii) With the preceding notation, one has by Lemma 2.4:

$$\mathcal{X} = G \cdot \mathcal{X}^U \subseteq G \cdot V^{U^-*} \times S,$$

and $T_{ad}$ acts on $V$ by normalizing the $G$-action and fixing $V^{U^-*}$ pointwise. Thus, $\mathcal{X}$ is invariant under $T_{ad}$.

(iii) Let $\pi : \mathcal{X} \to S$ be an $S$-point of $M_Y$. By definition of the action, $o$ maps this point to the pullback to $S \times \{o\}$ of the family $\hat{\pi} : \mathcal{X} \to S \times \mathbb{A}^\Pi$ constructed in the proof of Proposition 2.1. Let $o \cdot \mathcal{X}$ be the corresponding scheme, and $o \cdot \mathcal{R}$ the corresponding sheaf of $\mathcal{O}_S$-$G$-algebras. Since

$$\hat{\pi}_*\mathcal{O}_{\hat{\mathcal{X}}} = \bigoplus_{\lambda \in \Lambda^+} \mathcal{R}_{(\lambda)} e^\lambda \otimes_k k[\mathbb{A}^\Pi]$$

as sheaves of $\mathcal{O}_S[\mathbb{A}^\Pi]$-$G$-modules, and

$$\mathcal{R}_{(\lambda)} e^\lambda : \mathcal{R}_{(\mu)} e^\mu \subseteq \mathcal{R}_{(\lambda+\mu)} e^{\lambda+\mu} + \sum_{\gamma \in \text{NH}, \gamma \neq 0} \mathcal{R}_{(\lambda+\mu-\gamma)} e^{\lambda+\mu-\gamma} e^\gamma,$$

we see that $o \cdot \mathcal{R}$ satisfies condition (iii) of Lemma 2.4. By that Lemma, $o \cdot \mathcal{X}$ is horospherical. Now the isomorphism $o \cdot \mathcal{X}//U \to Y \times S$ implies that $o \cdot \mathcal{X} = \text{Ind}^G_B(Y) \times S$. $\square$

2.3. Structure and tangent space of $M_Y$. We begin with a general result on torus actions which is probably known, but for which we could not find a reference.
Lemma 2.6. Let $G^n_m \times X \to X, (t, x) \mapsto t \cdot x$ be an action of the $n$-dimensional torus $G^n_m$ on a scheme $X$, which extends to a morphism $\mathbb{A}^n \times X \to X$ under the natural inclusion $G^n_m \subset \mathbb{A}^n$; let $o$ be the origin of $\mathbb{A}^n$. Then the image of the morphism $o : X \to X, x \mapsto o \cdot x$ is the fixed point subscheme $X^{G^n_m}$, and the corresponding morphism $f : X \to X^{G^n_m}$ is a good quotient by $G^n_m$.

Proof. In the case where $X = \text{Spec}(R)$ is affine, the assumption that the action extends to $\mathbb{A}^n$ means that all weights of $G^n_m$ in $R$ lie in $\mathbb{N}^n$. Then the projection $R \to R^{G^n_m} = R_0$ is an algebra homomorphism; regarded as a self-map of $R$, it is the comorphism of $o$. This implies easily our statements.

In the general case, note that the image of $o$ is contained in $X^{G^n_m}$. Moreover, the resulting morphism $f : X \to X^{G^n_m}$ restricts to the identity on $X^{G^n_m}$; in particular, $f$ is surjective. By the first step of the proof, it suffices to show that $f$ is affine. For this, we may assume that $X$ is a variety.

Let $\nu : \bar{X} \to X$ be the normalization map. Then the map

$$\mathbb{A}^n \times \bar{X} \to X, (z, \xi) \mapsto z \cdot \nu(\xi)$$

is dominant, and hence factors through a morphism $\mathbb{A}^n \times \bar{X} \to \tilde{X}$. The latter is an action of the multiplicative monoid $\mathbb{A}^n$, since $\nu$ is birational; this yields a morphism $\tilde{f} : \bar{X} \to \tilde{X}^{G^n_m}$ lifting $f$. And since $\nu$ is finite surjective, its fiber at any fixed point is stable under this action, and hence under $G^n_m$. Thus, $\nu$ restricts to a finite surjective morphism $\eta : \tilde{X}^{G^n_m} \to X^{G^n_m}$.

By a theorem of Chevalley [Har77, Exercise III.4.2], an open subset of $X$ is affine if and only if its preimage under $\nu$ is. Thus, it suffices to show that the composition $f \circ \nu$ is affine. Since $f \circ \nu = \eta \circ \tilde{f}$, we may further assume that $X$ is normal.

By [Sum74], $X$ is covered by open affine $G^n_m$-stable subsets $X_i$. Then the $X_i^{G^n_m}$ form an open affine covering of $X^{G^n_m}$. Thus, to show that $o$ is affine, it suffices to check that the preimage of each $X_i^{G^n_m}$ is affine. Note that this preimage is contained in $X_i$ (namely, $o \cdot x \in X_i$ implies that the closure of $G^n_m \cdot x$ meets $X_i$, whence $x \in X_i$ since $X_i$ is open and $G^n_m$-stable). Thus, replacing $X$ with $o^{-1}(X_i^{G^n_m})$, we may assume that: $X$ is contained in an affine $G^n_m$-variety $V$ as a locally closed $G^n_m$-stable subvariety, and $X^{G^n_m}$ is closed in $V$. We may further assume that $V$ is a $G^n_m$-module.

Let $V_0 = V^{G^n_m}$ and $V_+ = \{v \in V \mid o \cdot v = 0\}$ (the sum of all “positive” $G^n_m$-eigenspaces). By our assumptions, $X^{G^n_m} = X \cap V_0$ is closed in $V_0$, and $X$ is contained in $(X \cap V_0) \times V_+$ as a locally closed $G^n_m$-stable subvariety. So the boundary $\bar{X} - X$ is a closed $G^n_m$-stable subvariety of $(X \cap V_0) \times V_+$, disjoint from $X \cap V_0$: this boundary must be empty. Hence $X$ is closed in $V$ and, in particular, affine.

Combining Propositions 2.1 and 2.5 with Lemma 2.6, we obtain
\textbf{Theorem 2.7.} The morphism \( f : \text{Hilb}_h^G(V)_0 \to \text{Hilb}_h^T(V_U) \) is a good quotient by the action of \( T_{\text{ad}} \), and the image of the section \( s \) is the fixed point subscheme.

As a consequence, for any multiplicity-finite \( T \)-scheme \( Y \), the scheme \( M_Y \) is affine and connected; its fixed point \( \text{Ind}_G^T(Y) \) is the unique closed orbit of \( T_{\text{ad}} \).

Next we describe the tangent space to \( M_Y \) at any closed point \( X \). Since \( X \) is equipped with an isomorphism \( X//U \to Y \), we have restriction maps

\[ r^0 : \text{Der}^G(X) \to \text{Der}^T(Y) \quad \text{and} \quad r^1 : T^1(X)^G \to T^1(Y)^T. \]

We may regard \( r^0 \) as the differential of the inclusion \( \text{Aut}^G(X) \hookrightarrow \text{Aut}^T(Y) \) (Lemma 1.3).

\textbf{Proposition 2.8.} With the preceding notation, the restriction maps fit into an exact sequence

\[ 0 \to \text{Der}^G(X) \to \text{Der}^T(Y) \to T_X M_Y \to T^1(X)^G \to T^1(Y)^T \to 0. \]

Thus, \( \ker(r^1) \) identifies with the normal space at \( X \) to the orbit \( \text{Aut}^T(Y) \cdot X \) in \( M_Y \); in particular, if \( X = \text{Ind}_G^T(Y) \) then \( \ker(r^1) = T_X M_Y \).

\textbf{Proof.} With the notation of Theorem 1.12, consider the differential

\[ df_X : T_X \text{Hilb}_h^G(V) \to T_Y \text{Hilb}_h^T(V_U). \]

It is surjective (since \( f \) admits a section), with kernel \( T_X f^{-1}(Y) = T_X M_Y \). Moreover, Proposition 1.13 yields a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \text{Der}^G(X) & \to & \text{Hom}^G(X,V) & \to & T_X \text{Hilb}_h^G(V)_0 & \to & T^1(X)^G & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \text{Der}^T(Y) & \to & \text{Hom}^T(Y,V_U) & \to & T_X \text{Hilb}_h^T(V_U) & \to & T^1(Y)^T & \to 0
\end{array}
\]

Here the map \( \text{Hom}^G(X,V) \to \text{Hom}^T(Y,V_U) \) is the composition

\[ \text{Hom}^G(X,V) = \text{Hom}^G(V^*,R) \to \text{Hom}^T((V^*)^U,R^U) = \text{Hom}^T(Y,V_U) \]

(where \( R = \Gamma(X,\mathcal{O}_X) \), so that \( R^U = \Gamma(Y,\mathcal{O}_Y) \)). Hence \( \text{Hom}^G(X,V) \) is mapped isomorphically to \( \text{Hom}^T(Y,V_U) \); this implies our exact sequence. \( \square \)

If, in addition, \( X \) is nonsingular, then \( T^1(X) = 0 \), see e.g. [Har77, Example III.9.13.2]. This yields

\textbf{Corollary 2.9.} Let \( X \) be a nonsingular affine \( G \)-variety, and put \( Y = X//U \). Then the orbit \( \text{Aut}^T(Y) \cdot X \) is open in \( M_Y \).

As a consequence, there are only finitely many isomorphism classes of nonsingular affine \( G \)-varieties of fixed type.

In particular, there are only finitely many isomorphism classes of nonsingular multiplicity-free \( G \)-varieties with a prescribed weight monoid. (This will be generalized to possibly singular varieties in Corollary 3.4 below.) In fact, a conjecture of Knop asserts that all nonsingular multiplicity-free \( G \)-varieties are classified by their weight monoid. This conjecture has been
established by Camus [Cam01] for $G$ of type $A$, building on Luna’s classification of spherical varieties for such $G$ [Lun01].

2.4. Orbit closures. Let $A$ be a multiplicity-finite $T$-algebra with weights in $A^+$, and let $Y = \text{Spec}(A)$. We will study the $T_{\text{ad}}$-orbits in the moduli space $M_Y$; we begin with a more concrete description of that space.

**Proposition 2.10.** The functor $\mathcal{M}_Y$ is isomorphic to the contravariant functor $(\text{Schemes}) \to (\text{Sets})$ assigning to $S$ the set of those $\mathcal{O}_S$-$G$-algebra morphisms $S$, which extend uniquely to an isomorphism

$$\varphi^\# : \mathcal{O}_S \otimes_k A \to \mathcal{R}^U$$

of $\mathcal{O}_S$-$T$-algebras (where $\mathcal{R} = \pi_+\mathcal{O}_X$), which extends uniquely to an isomorphism

$$\text{Coind}_B^G(\varphi^\#) : \mathcal{O}_S \otimes_k \text{Coind}_B^G(A) \to \mathcal{R} = \text{Coind}_B^G(\mathcal{R}^U)$$

of $\mathcal{O}_S$-$G$-modules. Hence we obtain a structure of $\mathcal{O}_S$-$G$-algebra on $\mathcal{O}_S \otimes_k \text{Coind}_B^G(A)$, extending the $T$-algebra structure of $A$.

Clearly, any such structure arises from some family. Moreover, two families $(\pi, \varphi)$ and $(\pi', \varphi')$ give rise to the same structure if and only if there exists a $G$-equivariant isomorphism $\psi : X \to X'$ over $S$ such that $\text{Coind}_B^G(\varphi'^\#) \circ \psi^\# = \text{Coind}_B^G(\varphi^\#)$, that is, $\varphi = \varphi' \circ (\psi//U)$. In other words, these families are equivalent in the sense of Definition 1.10.

Consider a $\mathcal{O}_S$-$G$-algebra multiplication law

$$m : \mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{R} \to \mathcal{R}.$$ 

Since $\mathcal{R} = \bigoplus_{\lambda \in A^+} \mathcal{R}(\lambda)$, it follows that

$$m = \sum_{\lambda, \mu, \nu \in A^+} m_{\lambda, \mu, \nu}^\nu,$$

where every $m_{\lambda, \mu}^\nu$ lies in $\text{Hom}_G^\nu(\mathcal{R}(\lambda) \otimes_{\mathcal{O}_S} \mathcal{R}(\mu), \mathcal{R}(\nu))$. And since $\mathcal{R}(\lambda) \simeq \mathcal{O}_S \otimes_k A_\lambda \otimes_k V(\lambda)$ as $\mathcal{O}_S$-$G$-modules, we obtain

$$m_{\lambda, \mu}^\nu \in \Gamma(S, \mathcal{O}_S) \otimes_k \text{Hom}(A_\lambda \otimes_k A_\mu, A_\nu) \otimes_k \text{Hom}^G(V(\lambda) \otimes_k V(\mu), V(\nu))$$

for all triples $(\lambda, \mu, \nu)$.

As a consequence, $m_{\lambda, \mu}^\nu = 0$ unless $\nu \leq \lambda + \mu$, and $m_{0, \mu}^\nu = 0$ unless $\nu = \mu$. Moreover, the condition that $m$ extends the multiplication of $\mathcal{O}_S \otimes_k A$ is equivalent to

$$m_{\lambda, \mu}^{\lambda + \mu} = 1 \otimes \text{mult}_{\lambda, \mu} \otimes p_{\lambda, \mu},$$

where $\text{mult}_{\lambda, \mu}$ denotes the multiplication $A_\lambda \otimes_k A_\mu \to A_{\lambda + \mu}$, and $p_{\lambda, \mu} : V(\lambda) \otimes_k V(\mu) \to V(\lambda + \mu)$ is the unique morphism of $G$-modules sending $v_\lambda \otimes v_\mu$ to $v_{\lambda + \mu}$. Under this condition, the identity element $1 \in A$ is also the identity for $m$. On the other hand, the commutativity (resp. associativity)
of \( m \) translates into a family of linear (resp. quadratic) relations between the \( m_{\lambda,\mu}^{\nu} \)'s.

Taking \( S = M_Y \), we see that any isotypical component \( m_{\lambda,\mu}^{\nu} \) yields a morphism from \( M_Y \) to some affine space. For simplicity, we still denote \( m_{\lambda,\mu}^{\nu} \) this morphism, and \( m \) the product of all the \( m_{\lambda,\mu}^{\nu} \). Clearly, \( m \) is universally injective.

**Proposition 2.11.** Every isotypical component \( m_{\lambda,\mu}^{\nu} \) is an eigenvector of \( T_{ad} \) of weight \( \lambda + \mu - \nu \), and these generate the algebra \( \Gamma(M_Y, \mathcal{O}_{M_Y}) \).

**Proof.** By Lemma 2.2, the \( T_{ad} \)-action on \( M_Y \) comes from the \( T \)-action on \( A = \bigoplus_{\lambda \in \Lambda^+} A_{\lambda} \). This implies the first assertion.

By that assertion or Lemma 2.4, \( m_{\lambda,\mu}^{\nu} \) vanishes at the \( T_{ad} \)-fixed point \( X_0 = \text{Ind}^G_B(Y) \) whenever \( \lambda + \mu - \nu \neq 0 \). Moreover, the tangent space \( T_{X} M_Y \) is spanned by the images of all such components, since \( m \) is universally injective. But the algebra \( \Gamma(M_Y, \mathcal{O}_{M_Y}) \) is \( \mathbb{N} \Pi \)-graded, with homogeneous maximal ideal corresponding to \( X_0 \). So the \( m_{\lambda,\mu}^{\nu} \) generate this algebra, by the graded Nakayama lemma. \( \square \)

The above Proposition, together with Proposition 2.10, yields another, more direct proof of the second part of Theorem 2.7. It also motivates the following

**Definition 2.12.** Let \( X = \text{Spec}(R) \) be an affine \( G \)-scheme, and denote by \( m = \sum m_{\lambda,\mu}^{\nu} : R \otimes_k R \rightarrow R \) the multiplication. The root monoid of \( X \) is the submonoid \( \Sigma_X \subseteq \Lambda \) generated by the \( \lambda + \mu - \nu \), where \( \lambda, \mu, \nu \in \Lambda^+ \) and \( m_{\lambda,\mu}^{\nu} \) is nonzero.

Note that \( \Sigma_X \) is contained in \( \mathbb{N} \Pi \). Now Proposition 2.11 easily implies

**Proposition 2.13.** For any affine \( G \)-scheme \( X \), the root monoid \( \Sigma_X \) is the weight monoid of the \( T_{ad} \)-orbit closure of \( X \) (regarded as a closed point of \( M_Y \)). As a consequence, the monoid \( \Sigma_X \) is finitely generated.

If, in addition, \( Y \) is a variety, then so is \( X \). Let \( \Sigma_X \) be the saturation of \( \Sigma_X \), i.e., the intersection of the cone and the lattice generated by that monoid (the corresponding multiplicity-free \( T_{ad} \)-variety is the normalization of that associated with \( \Sigma_X \)). By [Kno96, Theorem 1.3], the monoid \( \Sigma_X \) is freely generated by a basis of a certain root system \( \Phi_X \). Together with Propositions 2.11 and 2.13, this implies

**Corollary 2.14.** The normalization of every \( T_{ad} \)-orbit in \( M_Y \) is isomorphic to a \( T_{ad} \)-module, for any multiplicity-finite \( T \)-variety \( Y \).

### 3. Finiteness results for multiplicity-free varieties

It is easy to see that only finitely many subgroups of a torus \( T \) occur as isotropy groups of points of a given finite-dimensional \( T \)-module \( V \); it follows that \( V \) contains only finitely many multiplicity-free subvarieties, up to
the action of $GL(V)^T$. In contrast, there are examples of finite-dimensional $G$-modules where infinitely many pairwise non-isomorphic isotropy groups occur, see [Ric68] 1.3 and [Ric72] 12.4.2. But finiteness still holds for spherical isotropy groups:

**Theorem 3.1.** For any $G$-scheme $X$ of finite type, only finitely many conjugacy classes of spherical subgroups of $G$ occur as isotropy groups of points of $X$.

**Proof.** It proceeds through several reductions; we divide it into five steps.

**Step 1.** We reduce to the case of a nonsingular $G$-variety $X$ equipped with a smooth $G$-invariant morphism

$$\pi : X \to S$$

such that the fiber $X_s$ at any closed point $s \in S$ is a unique $G$-orbit.

Indeed, it suffices to prove the theorem for a $G$-variety $X$ (of finite type). By a classical result of Rosenlicht [Ros63], there exists a dense open subset $X_0 \subseteq X$ admitting a geometric quotient $\pi : X_0 \to S$, that is, $\pi$ is surjective, its fibers are the orbits, and the map $O_S \to (\pi_*O_{X_0})^G$ is an isomorphism. Shrinking $S$, we may further assume that it is nonsingular, and that $\pi$ is smooth. Thus $X_0$ is smooth as well. Now, using induction over the dimension, we may replace $X$ with $X_0$.

**Step 2.** After shrinking $S$ again, we may assume that $X$ admits a smooth compactification $\overline{X}$, that is, a nonsingular $G$-variety containing $X$ as a dense open $G$-stable subset, such that $\pi$ extends to a smooth projective morphism

$$\overline{\pi} : \overline{X} \to S$$

(then $\overline{\pi}$ is $G$-invariant).

Indeed, by [Sum75], there exists a $G$-variety $\overline{X}$ over $S$, containing $X$ as a dense open $G$-stable subset. Using equivariant desingularization [EV00], we may assume that $\overline{X}$ is nonsingular. Then, by generic smoothness, we may also assume that $\overline{\pi}$ is smooth.

**Step 3.** After shrinking $S$ again, we may further assume that the compactification is “regular”, that is, the boundary

$$\partial \overline{X} = \overline{X} - X$$

is a union of irreducible nonsingular divisors $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ with normal crossings, and each restriction

$$\overline{\pi}_i : \mathcal{Y}_i \to S$$

is smooth.

Indeed, we have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X \otimes_k \mathfrak{g} \to T_{X/S} \to 0,$$
where $\mathcal{T}_{X/S}$ is the relative tangent sheaf and $\mathcal{E}$ is a locally free sheaf on $X$ such that every fiber $\mathcal{E}_x$ identifies with the isotropy Lie algebra $g_x$. This yields a morphism

$$\varphi : X \to \text{Grass}(g), \ x \mapsto g_x$$

where Grass$(g)$ denotes the Grassmannian of subspaces of $g$. Replacing $X$ with the closure in $X \times \text{Grass}(g)$ of the graph of $\varphi$, and resolving singularities, we may assume that $\varphi$ extends to $\varphi : X \to \text{Grass}(g)$.

So the relative tangent bundle $T_{X/S}$ extends to a vector bundle $\tilde{T}_{X/S}$ over $X$, the pull-back of the tautological bundle of Grass$(g)$. By [BB96], it follows that every fiber $\tilde{X} = X_s$ is a regular compactification of its open orbit $X = X_s$. This implies easily our reduction.

Step 4. By [BB96, Proposition 2.5], the pull-back of $\tilde{T}_{X/S}$ to any fiber $\tilde{X}$ is the logarithmic tangent bundle $T_{\tilde{X}}(-\log \partial \tilde{X})$, associated with the sheaf $\mathcal{T}_{\tilde{X}}(-\log \partial \tilde{X})$ of derivations of $\tilde{X}$ preserving the ideal sheaf of $\partial \tilde{X}$. The space of infinitesimal deformations of the pair $(\tilde{X}, \partial \tilde{X})$ is the first cohomology group $H^1(\tilde{X}, \mathcal{T}_{\tilde{X}}(-\log \partial \tilde{X}))$, see e.g. [Kaw85, Proposition 3.1]. And this group vanishes, as a special case of [Kno94, Theorem 4.1]. Now deformation theory should tell us that the equivariant isomorphism class of $(\tilde{X}, \partial \tilde{X})$ is independent of the fiber, which should complete the proof. But for lack of an appropriate reference, we will provide an alternative argument based on nested Hilbert schemes; for the latter, see [Che98].

Since the morphism $\pi$ is projective, we may regard $\tilde{X}$ as a $G$-stable subvariety of $\mathbb{P}(V) \times S$, where $V$ is a finite-dimensional $G$-module. Replacing $V$ with a symmetric power, we may assume additionally that the restriction map

$$V^* = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is surjective for all fibers $X$.

Let Hilb be the nested Hilbert scheme that parameterizes those families $(Z, Z_1, \ldots, Z_n)$ of closed subschemes of $\mathbb{P}(V)$ such that $Z_1 \cup \cdots \cup Z_n \subseteq Z$, and the Hilbert polynomial of $Z$ (resp. $Z_1, \ldots, Z_n$) equals that of $\tilde{X}$ (resp. $Y_1, \ldots, Y_n$). Then we have a morphism $\psi : S \to \text{Hilb}$ such that $\tilde{X}$ is the pull-back under $\psi$ of the universal family. The group GL$(V)$, and hence $G$, acts on Hilb, and $\psi$ is $G$-invariant.

The differential of the GL$(V)$-action yields a linear map

$$f : \text{End}(V) \to T_{(\tilde{X}, Y_1, \ldots, Y_n)} \text{Hilb}$$

which is $G$-equivariant. We will check in Step 5 below that $f$ is surjective.

As a consequence, the restriction

$$f^G : \text{End}^G(V) \to (T_{(\tilde{X}, Y_1, \ldots, Y_n)} \text{Hilb})^G$$
is surjective as well. Since the subscheme of $G$-invariants Hilb$^G$ is stable under the group GL$(V)^G$ with Lie algebra End$^G(V)$, and
\[ T(\pi_{Y_1,\ldots,Y_n}) \subseteq (T(\pi_{X,Y_1,\ldots,Y_n}) \operatorname{Hilb})^G, \]
the orbit GL$(V)^G \cdot (\pi_{X,Y_1,\ldots,Y_n})$ is open in Hilb$^G$. It follows that all fibers of $\pi$ are isomorphic in a neighborhood of $s$, as desired.

**Step 5.** To complete the proof, we deduce the surjectivity of $f$ from Knop’s vanishing theorem stated in Step 4. For this, we recall the description of the tangent space to the nested Hilbert scheme [Che98].

Let $N_{\overline{X}}$ (resp. $N_{Y_1}$) be the normal bundle to $\overline{X}$ (resp. $Y_i$) in $\mathbb{P}(V)$; then we have restriction maps $p_i : N_{Y_i} \to N_{\overline{X}}|_{Y_i}$. Now $T(\pi_{X,Y_1,\ldots,Y_n}) \operatorname{Hilb}$ equals
\[
\{(s,t_1,\ldots,t_n) \in H^0(N_{\overline{X}}) \oplus \bigoplus_{i=1}^n H^0(N_{Y_i}) \mid s|_{Y_i} = p_i(t_i) \ (i = 1,\ldots,n)\}.
\]

We now construct a sheafified version of $T(\pi_{X,Y_1,\ldots,Y_n}) \operatorname{Hilb}$, as follows. Denote by $N_{\overline{X}}$ (resp. $N_{Y_1}$) the normal sheaf to $\overline{X}$ (resp. $Y_i$) in $\mathbb{P}(V)$. We have exact sequences of sheaves on $Y_i$:
\[
0 \to N_{Y_i}/X \to N_{Y_i} \to N_{\overline{X}}|_{Y_i} \to 0,
\]
where $N_{Y_i}/X$ denotes the normal sheaf to $Y_i$ in $X$. Denote by $j_i : Y_i \to X$ the inclusions and let
\[
N_{\overline{X},Y_1,\ldots,Y_n} \subseteq N_{\overline{X}} \oplus \bigoplus_{i=1}^n j_i^*N_{Y_i}
\]
be the subsheaf consisting of those tuples of local sections $(s,t_1,\ldots,t_n)$ such that $s|_{Y_i} = p_i(t_i)$ for all $i$. Then $N_{\overline{X},Y_1,\ldots,Y_n}$ is a sheaf on $\overline{X}$, endowed with a morphism
\[
\eta : T_{\mathbb{P}(V)}|_{\overline{X}} \to N_{\overline{X},Y_1,\ldots,Y_n}
\]
that sends any tangent vector to the collection of the corresponding normal vectors to $\overline{X}, Y_1,\ldots,Y_n$. The first projection $N_{\overline{X},Y_1,\ldots,Y_n} \to N_{\overline{X}}$ is surjective, with kernel the direct sum of the $N_{Y_i}/\overline{X}$. In addition, $\eta$ fits into a commutative diagram
\[
\begin{array}{cccccc}
0 & \to & T_{\overline{X}} & \to & T_{\mathbb{P}(V)}|_{\overline{X}} & \to & N_{\overline{X}} & \to & 0 \\
& & \downarrow \eta & & \downarrow \operatorname{id} & & \downarrow \operatorname{id} & & \\
0 & \to & \bigoplus_{i=1}^n N_{Y_i}/\overline{X} & \to & N_{\overline{X},Y_1,\ldots,Y_n} & \to & N_{\overline{X}} & \to & 0.
\end{array}
\]

Together with the exact sequence
\[
0 \to T_{\overline{X}}(-\log \partial \overline{X}) \to T_{\overline{X}} \to \bigoplus_{i=1}^n N_{Y_i}/\overline{X} \to 0,
\]
this yields an exact sequence
\[
0 \to T_{\overline{X}}(-\log \partial \overline{X}) \to T_{\mathbb{P}(V)}|_{\overline{X}} \xrightarrow{\eta} N_{\overline{X},Y_1,\ldots,Y_n} \to 0.
\]
Since $H^1(X, T_X(-\log \partial X)) = 0$, we obtain a surjection
$$H^0(\eta) : H^0(X, T_P(V)) \to H^0(X, N_{X/Y_1,\ldots,Y_n}) = T_{(X,Y_1,\ldots,Y_n)}\text{Hilb}.$$Note that $f$ factors as the restriction $\rho : \text{End}(V) \to H^0(X, T_P(V))$, followed by $H^0(\eta)$. So it remains to check surjectivity of $\rho$.

For this, we use the exact sequence
$$0 \to \mathcal{O}_X \to V \otimes_k \mathcal{O}_X(1) \to T_P(V)|_{X} \to 0,$$and the vanishing of $H^1(X, \mathcal{O}_X)$ (since $X$ is nonsingular, projective and rational). This yields a surjection
$$V \otimes\! H^0(X, \mathcal{O}_X(1)) \to H^0(X, T_P(V)).$$Since the restriction $V^* \to H^0(X, \mathcal{O}_X(1))$ is surjective as well, the composition $\text{End}(V) = V \otimes V^* \to H^0(X, T_P(V)|_{X})$ is surjective as desired. □

Consider the action of $G$ on the Grassmannian $\text{Grass}(g)$ of subspaces of its Lie algebra. If $H$ is a spherical subgroup of $G$ with Lie algebra $\mathfrak{h}$, then the isotropy group of $\mathfrak{h} \in \text{Grass}(g)$ is the normalizer $N_G(\mathfrak{h})$. The latter equals $N_G(H)$, by [BP87, §5]. So Theorem 3.1 implies

**Corollary 3.2.** There exist only finitely many conjugacy classes of spherical subgroups $H \subseteq G$ having finite index in their normalizer.

(Note that there exist infinitely many conjugacy classes of spherical subgroups of $G$, for example, those containing $U$ as a subgroup of finite index.) For $G$ of type $A$, Corollary 3.2 is a consequence of Luna’s classification of spherical varieties by combinatorial invariants [Lun01]. Extending this classification to arbitrary $G$ would give another proof of that corollary.

Another direct consequence of Theorem 3.1 is

**Corollary 3.3.** Any finite-dimensional $G$-module $V$ contains only finitely many multiplicity-free subvarieties, up to the action of $\text{GL}(V)^G$.

**Proof.** Let $X$ be a multiplicity-free subvariety of $V$, with open orbit $G \cdot x$. By Theorem 3.1, we may fix the isotropy group $G_x = H$. Write
$$V = \bigoplus_{\lambda \in F} M_\lambda \otimes_k V(\lambda)$$where $F$ is a finite subset of $\Lambda^+$, and $M_\lambda = \text{Hom}^G(V(\lambda), V)$. Then
$$\text{GL}(V)^G \simeq \prod_{\lambda \in F} \text{GL}(M_\lambda),$$and $x \in V^H = \bigoplus_{\lambda \in F} M_\lambda \otimes V(\lambda)^H$ where every $V(\lambda)^H$ is at most one-dimensional. Thus,
$$x = \sum_{\lambda \in F} m_\lambda \otimes x_\lambda.$$
where every nonzero $x_\lambda$ is a generator of $V(\lambda)^H$. It follows that there are only finitely many $x$ up to the action of $GL(V)^G$. So the same holds for $X = G \cdot x$.

Together with Theorem 1.12, this implies

**Corollary 3.4.** Given a submonoid $S \subseteq \Lambda^+$, there exist only finitely many isomorphism classes of multiplicity-free $G$-varieties with weight monoid $S$. Equivalently, the moduli scheme $M_S$ contains only finitely many $T_{ad}$-orbits.

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