The oscillations of cylindrical drop under the influence of a nonuniform alternating electric field

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Abstract. The forced oscillations of incompressible fluid drop under the alternating electric field are considered. In equilibrium, the drop has the form of a cylinder bounded axially parallel solid planes and contact angle is right. The drop is surrounded by an incompressible fluid with another density. The external nonuniform electric field acts as an external force that causes motion of the contact line. In order to describe this contact line motion the modified Hocking boundary condition is applied: the velocity of the contact line is proportional to the deviation of the contact angle and the speed of the fast relaxation processes, which frequency is proportional to twice the frequency of the electric field. It is shown that under uniform electric field the oscillation amplitude at the resonance frequency increases with the increase of the Hocking parameter. In the case of nonuniform electric field the oscillation amplitude is always limited.

1. Introduction
The stability and dynamics of fluid drops subjected to external electrical forces continue to attract research efforts [1-4]. One of the most important factor influencing the dynamics and manipulation processes is electrowetting (EW) [4,5]. This branch of science has been actively developed recently when electrowetting-on-dielectric (EWOD) was proposed [6-8]. Now EWOD has been put into applications in various fields, such as digital (droplet) microfluidic devices for bioanalysis (lab-on-a-chip) [9, 10], variable-focus liquid lenses [11,12], electronic display technology [13,14] etc.

The typical electrowetting devices are shown in figure 1. When a direct current (DC) voltage is applied between the drop and the electrode, the varying contact angle $\theta$ of the drop can be expressed by Young–Lippmann equation [6,7]

$$\cos \theta = \cos \theta_0 + \frac{\varepsilon \varepsilon_0 V^2}{2d\sigma},$$  \hspace{1cm} (1)

where $V$ is the value of the applied voltage, $\theta_0$ is the contact angle without applied voltage – equilibrium contact angle, which is defined by well-known Young’s equation, $\sigma$ is the interfacial tension between the droplet and the surrounding fluid, $d$ is the thickness of the dielectric film, $\varepsilon_0$ and $\varepsilon$ are the permittivity of vacuum and the dielectric layer, respectively. The last term in the equation (1) is the modified term of Lippmann equation [16]. Note, that $C = \varepsilon\varepsilon_0 / d$ is capacitance per unit area [5] and $E_w = CV^2 / 2\sigma$ is EW number (it represents the ratio of the electrostatic energy to the liquid – surrounding fluid interfacial energy [8]). For alternating current (AC) instead of $V^2$ in equation (1) the square value of the voltage, or effective voltage, $U^2$ is used [5].

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The Young–Lippmann equation (1) can be derived by several approaches [4,5,8]. However this equation is adequate for the voltage $V$ which is smaller than the critical voltage $V_c$ [4,5,7,15]. When $V \geq V_c$, the electrowetting contact angle does not increase with increasing of applied voltage, this effect is known as contact angle saturation. Also article [17] states that the Young–Lippmann equation surprisingly well agrees with the experimental data for the cases when local contact angle $\theta_0$ (without electric field) is varied between $38^\circ$ and $120^\circ$.

The contact angle $\theta$ depends on the time when AC is used. We propose to change the equation (1), which describes the dynamics of the contact angle, because it is not always correct. The most frequently used Hocking condition (by virtue of its simplicity) is the one employed in a study [18] of the damping of standing waves between two vertical walls. This condition assumes a linear dependence between the velocity of motion of the contact line and the contact angle (in the case of right equilibrium contact angle):

$$\frac{\partial \zeta^*}{\partial t} = \Lambda \vec{k} \cdot \vec{\nabla} \zeta^*,$$

(2)

where $\zeta^*$ is the deviation of the interface from the equilibrium position, $\vec{k}$ is the external normal to the solid surface, $\Lambda$ is a phenomenological constant (the so-called wetting parameter or Hocking parameter), having the dimension of velocity. The special cases of boundary condition (2) are the requirement of the fixed contact line ($\zeta^* = 0$, the so-called pinned-end edge condition) and the constant contact angle ($\vec{k} \cdot \vec{\nabla} \zeta^* = 0$). For example, the equation (2) one can find in the studies of oscillations of a sessile drop and bubble [19-21], cylindrical drop and bubble [22,23], “sandwiched” drop [24] and capillary bridge [25].

In the present paper, we consider the behavior of a droplet sandwiched between two flat plates under AC. In order to describe the motion of the contact line the modified boundary condition (2) is used: the velocity of the contact line is proportional to the deviation of the contact angle and the speed of the fast relaxation processes, whose frequency is proportional to twice of the electric field frequency. In future we plan to compare the results with experimental data [26].

2. Problem formulation

We consider the dynamic behavior of an incompressible liquid drop of density $\rho_i^*$ surrounded by other liquid of density $\rho_e^*$ (here and in the following, quantities with subscript $i$ refer to the drop, and those with subscript $e$ to the surrounding liquid). The system is bounded by two parallel solid surfaces (see figure 2) separated by a distance $h^*$. The equilibrium contact angle $\theta_0$ between the side surface of the drop and the solid surface is equal to $\pi/2$. The external nonuniform alternating electric field acts as an external force that causes the contact line motion.

According to the problem symmetry it is convenient to introduce the cylindrical coordinates $r^*$, $\alpha$, $z^*$. The azimuthal angle $\alpha$ is reckoned from the x-axis. Let the surface of the droplet be described by the equation $r^* = R_0^* + \zeta^*(\alpha, z^*, t^*)$. Assuming potential liquid motion, we introduce the velocity potential $\vec{v}^* = \vec{V} \varphi^*$. By choosing the length $R_0^*$, height $h^*$, the density $\rho_i^* + \rho_e^*$, the time $\sigma^{-1/2} \sqrt{(\rho_i^* + \rho_e^*) R_0^*}$, the velocity potential $A^* \sqrt{\sigma \left( (\rho_i^* + \rho_e^*) R_0^* \right)^{3/2}}$, the pressure $A^* \sigma (R_0^*)^2$, and the...
deviation of the surface $A$ as characteristic quantities, we pass to dimensionless variables and obtain the following linear problem

$$
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \phi_j + \frac{1}{r^2} \frac{\partial^2}{\partial \alpha^2} \phi_j + b^2 \frac{\partial^2}{\partial z^2} \phi_j = 0 , \quad j = i, e ,
$$

(3)

where $p$ is the fluid pressure, $f(\alpha)$ is the function of nonuniform electric field, the square brackets denote the jump in the quantity at the interface between the external liquid and the drop. The second term in boundary condition (6) is external action and written similarly to the Young–Lippmann equation (1): $\cos \theta = E_u = \tilde{E}_u V^2$ as $\cos \theta = 0$ in our problem, therefore $\zeta_z \sim \text{ctg} \theta = E_u / \sqrt{1 - E_u^2} = E_u V^2 / \sqrt{1 - E_u^2 V^2} \approx E_u V^2 + O(V^4) \approx E_u V^2 - V^2$. The boundary-value problem (3)–(6) involves six parameters: the aspect ratio, the dimensionless density, the wetting parameter, the AC frequency and amplitude

$$
b = R_b h^{-1}, \quad \rho_i = \rho_i^* \left( \rho_i^* + \rho_e^* \right)^{-1}, \quad \rho_e = \rho_e^* \left( \rho_i^* + \rho_e^* \right)^{-1}, \quad \lambda = \Lambda \sigma^{-1/2} \sqrt{\left( \rho_i^* + \rho_e^* \right) R_b} , \quad \\
\omega = \omega \sigma^{-1/2} \sqrt{\left( \rho_i^* + \rho_e^* \right) R_b^3} \quad a = A \sigma^{-3/2} \sqrt{\left( \rho_i^* + \rho_e^* \right) R_b^3} / 2 .
$$

In view of axial symmetry, the solution of the Laplace equation (6) is written as

$$
\phi_i (r, z, t) = \text{Re} \left( \sum_{k=0}^{\infty} a_k I_0 \left[ (2k+1) \pi b r \right] \sin \left[ (2k+1) \pi z \right] e^{i \omega t} \right) , \quad (7)
$$

$$
\phi_e (r, z, t) = \text{Re} \left( \sum_{k=0}^{\infty} b_k K_0 \left[ (2k+1) \pi b r \right] \sin \left[ (2k+1) \pi z \right] e^{i \omega t} \right) , \quad (8)
$$

where $I_0$ and $K_0$ are modified Bessel functions. The kinematic condition on the free surface (the second condition in (4)) gives the expression for the surface deviation

$$
\zeta (z, t) = \text{Re} \left( \sum_{k=0}^{\infty} c_k \cos \left( 2 \pi k z \right) + d \cos \left( \frac{\pi z}{b} \right) e^{i \omega t} \right) , \quad (9)
$$

Figure 2. Problem geometry (1 – electrode, 2 – dielectric layer).

3. Uniform electric field

In order to investigate the problem it is convenient to begin with a consideration of the uniform electric field, i.e. $f(\alpha) = 1$. Note that the natural oscillations of a cylindrical drop and those eigen frequencies were investigated in [22]. Our interest deals with the axisymmetric oscillation modes, therefore system of the equations and boundary conditions (3)–(6) are axially symmetric at $f(\alpha) = 1$.

In view of axial symmetry, the solution of the Laplace equation (6) is written as
Substituting solutions (7)–(9) into (3) –(6) we obtain expressions for the unknown amplitudes $a_i$, $b_i$, $c_i$ and $d$. These expressions are not given due to their cumbersome forms.

The dependence of the amplitude of the surface oscillations and the contact angle at the upper plate on the frequency of the driving force is given in figure 3 for different values of the Hocking parameter and of the aspect ratio. As it was mentioned in [20,22-24], at certain values of eigen frequencies the drop motion is independent of the wetting parameter: at any $\lambda$ the contact line remains motionless. At such points the amplitude of the contact line goes to zero. This is so-called “antiresonance” frequencies. Values of these frequencies do not depend on $\lambda$ and depend on the parameters of the drops (aspect ratio $b$, density $\rho_i$ etc.).

Note one feature of the behavior of the drop side surface. The study of the contact line dynamics applying Hocking boundary condition (2) have shown that this condition leads to the energy dissipation. Non-dissipative cases are only $\zeta = 0$ and $\zeta_z = 0$. For example, the study of forced oscillations under vibrations [18–25] has shown that the oscillations amplitude of the surface tends to infinity in the limiting cases, but bounded at finite values. It can be seen from figure 3 that the amplitude of contact line oscillations increases with the increase of $\lambda$. However the amplitude decreases with increasing aspect ratio $b$ and vibration frequency $\omega$. The cause of the first phenomenon is obvious: the larger drop, the harder it moves. Apparently the second reason is related to the fact that dissipation for high-frequency oscillations is greater than for low-frequency oscillations [19,20,22].

4. Nonuniform electric field

The function $f(\alpha)$ is expanded into the Fourier series in eigen functions of the Laplace operator. Let us considered the particular case of nonuniform electric field: $f(\alpha) = \sin(\kappa x)\sin(\kappa \cos(\alpha))$, where $\kappa$ is wave number. The solutions of the velocity potential and the surface deviation are written as

$$\varphi_i(r,z,t) = \Re\left[\sum_{m=0}^{\infty} a_{i,m} R_m(r) \sin\left((2k+1)\pi z\right) \cos\left((2m+1)\right)^2 + \alpha \ e^{i\alpha z}\right],$$

$$\varphi_z(r,z,t) = \Re\left[\sum_{m=0}^{\infty} b_{i,m} \sin\left((2k+1)\pi z\right) \cos\left((2m+1)\right)^2 + \alpha \ e^{i\alpha z}\right],$$

$$\zeta(z,t) = \Re\left( \sum_{m=0}^{\infty} c_{i,m} \sin\left((2k+1)\pi z\right) \cos\left((2m+1)\alpha\right) + d_{i}z \cos\left(\right) + \sum_{m=1}^{\infty} d_{m} \sin\left(\frac{(2m+1)^2 - 1}{b}\right) \cos\left((2m+1)\alpha\right) e^{i2\alpha z}\right).$$

Figure 3. The maximum deviation of the lateral surface $\zeta_s$ (a,c) and the contact angle $\theta$ (b,d) for three different values of the Hocking parameter $\lambda$ (a,b) and of the aspect ratio $b$ (c,d).

(a,b) $b = 1$, $a = 2.5$, $\rho_i = 0.7$, $\lambda = 0.5$ (solid line), $\lambda = 3.5$ (dashed line), $\lambda = 10$ (dotted line).

(c,d) $\lambda = 10$, $a = 2.5$, $\rho_i = 0.7$, $b = 2$ (solid line), $b = 1.5$ (dashed line), $b = 1$ (dotted line).
where $I_m$ and $K_m$ are modified Bessel functions of m order. Substituting solutions (10)–(12) into (3)–(6), we obtain expressions for the unknown amplitudes $a_{mk}$, $b_{mk}$, $c_{mk}$, $d_0$, and $d_m$.

![Figure 4](image1.png)

**Figure 4.** The maximum deviation of the lateral surface $\zeta_s$ (a) and the contact angle $\theta$ (b) for three different values of Hocking’s constant $\lambda$ (a,b).

$b = 1$, $a = 2.5$, $\rho_i = 0.7$, $\kappa = 1$, $\lambda = 0.1$ (solid line), $\lambda = 1$ (dashed line), $\lambda = 10$ (dotted line).

![Figure 5](image2.png)

**Figure 5.** The maximum deviation of the lateral surface $\zeta_s$, and the contact angle $\theta$ (a), form of later surface (b) and form of contact line (c) ($b = 1$, $a = 2.5$, $\rho_i = 0.7$, $\kappa = 1$, $\lambda = 1$).

(b,c) $\omega = 2$, $T = 2\pi\omega^{-1}$, $dT = 0.125T$, $t = 0$ (solid line), $t = dT$ (dashed line), $t = 2dT$ (dotted line), $t = 3dT$ (dash-dotted line).

![Figure 6](image3.png)

**Figure 6.** The maximum deviation of the lateral surface $\zeta_s$, and the contact angle $\theta$ (a), form of later surface (b) and form of contact line (c) ($b = 1$, $a = 2.5$, $\rho_i = 0.7$, $\kappa = 10$, $\lambda = 1$).

(b,c) $\omega = 2$, $T = 2\pi\omega^{-1}$, $dT = 0.125T$, $t = 0$ (solid line), $t = dT$ (dashed line), $t = 2dT$ (dotted line), $t = 3dT$ (dash-dotted line).

The dependence of the amplitude of the surface oscillations and the contact angle at the upper plate on the frequency of the driving force is given in figure 4 for different values of the Hocking parameter.
and $\kappa = 1$. It is obvious that the translational mode of oscillations is excited. The amplitude of the side surface oscillations has identical local maximum unlike to the uniform electric field. It is obvious that the minimum value of the contact angle achieved when the amplitude deflection is maximum (see figure 4 and 5). In addition, the minimum value of the contact angle is different from the zero and independent on parameter $\lambda$. There is no traveling wave along the lateral surface (see figure 5), the section of the lateral surface is a straight line in the plane $xz$ (fig. 5b).

Azimuthal modes excited by a larger wave number $\kappa = 10$ (see figure 6). The traveling wave propagates along the lateral surface of the drop in this case.

5. Conclusions
The data about deviation of frequency and surface characteristics depending on Hocking parameter, frequency and amplitude of an external nonuniform electric field and the geometric parameters of the system are obtained. It is shown that under uniform electric field the oscillation amplitude at the resonance frequency increases with the increase of the Hocking parameter. It is similar to forced oscillations of the drop under the mechanical axisymmetric vibration [20]. There are also «anti-resonant» frequencies (i.e. the frequencies at which the amplitude of contact line deviation is zero) under the alternating electric field as under the mechanical vibrations. In the case of the nonuniform electric field the oscillations amplitude is always limited similarly to the forced translational vibrations of a drop [19,24]. Note that we consider the weakly nonuniform electric field which leads to translational vibrations, while a strongly nonuniform electric field leads to azimuthal vibrations.

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6. References
[1] Melcher J R and Taylor G I 1969 Annual Review of Fluid Mechanics 1 111
[2] Pelekasis N A, Economou K and Tsamopoulos J A 2001 Phys. of Fluids 13 3564
[3] Shilov A A 2002 Technical Physics 47 1501
[4] Mugele F and Baret J-C 2005 J. Phys.: Condens. Matter. 17 705
[5] Chen L and Bonacurso E 2014 Adv. Colloid Interface Sci. 210 2
[6] Berge B 1993 Comptes Rendus Acad. Sci. II 317 157
[7] Quilliet C and Berge B 2001 Curr. Opin. Colloid Interface Sci. 6 34
[8] Zhao Y-P and Wang Y 2013 Rev. Adhesion Adhesives 114
[9] Hua Z, Rouse J L, Eckhardt A E, Srinivasan V, Pamula V K, Schell W A, Benton J L, Mitchell T G and Pollack M G 2010 Anal Chem 82 2310-16
[10] Li J, Wang Y, Chen H and Wan J 2014 Lab Chip 14 4334-37
[11] Kuiper S and Hendriks B H W 2004 Appl. Phys. Lett. 85, 1128–30
[12] Li C and Jiang H 2014 Micromachines 5 432-41
[13] Hayes R A and Feenstra B J 2003 Nature 425 383–5
[14] Roques-Carmes T, Hayes R A, Feenstra B J and Schlangen L J M 2004 J. Appl. Phys 95 4389–96
[15] Mugele F 2009 Soft Matter 5 3377–84
[16] Lippmann G 1875 Ann. Chim. Phys. 5 494–549
[17] Adamiak K 2006 Microfluid Nanofluid 2 471-80
[18] Hocking L M 1987 J. Fluid Mech. 179 253-66
[19] Lyubimov D V, Lyubimova T P and Shklyaev S V 2004 Fluid Dynamics 39 851–62
[20] Lyubimov D V, Lyubimova T P and Shklyaev S V 2006 Phys. Fluids 18 052102
[21] Shklyaev S and Straube A V 2008 Phys. Fluids 20 052102
[22] Alabuzhev A A and Lyubimov D V 2007 J. Appl. Mech. Tech. Phys. 48 686–93
[23] Alabuzhev A A 2014 Computational Continuum Mechanics 7 151–61 (in Russian)
[24] Alabuzhev A A and Lyubimov D V 2012 J. Appl. Mech. Tech. Phys. 53 9–19
[25] Borkar A and Tsamopoulos J 1991 Phys. Fluids A 3 2866–74
[26] Mampallil D, Eral H B, Staicu A, Mugele F and van den Ende D 2013 Phys. Rev. E 88 053015