A note on bases of admissible rules of proper axiomatic extensions of Łukasiewicz logic

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Abstract

In this note we prove that single-conclusion admissible rules of any proper axiomatic extension of the infinite valued Łukasiewicz logic are finitely based. The proof strongly relies on the characterization of least \( V \)-quasivarieties given in [10].

Introduction.

Admissible rules of a logic are those rules under which the set of theorems are closed. If \( L \) is a logic, an \( L \)-unifier of a formula \( \varphi \) is a substitution \( \sigma \) such that \( \vdash_L \sigma \varphi \). A single-conclusion rule is an expression of the form \( \Gamma / \varphi \) where \( \varphi \) is a formula and \( \Gamma \) is a finite set of formulas. As usual \( \Gamma / \varphi \) is derivable in \( L \) iff \( \Gamma \vdash_L \varphi \). The rule \( \Gamma / \varphi \) is admissible in \( L \) iff every common \( L \)-unifier of \( \Gamma \) is also an \( L \)-unifier of \( \varphi \). \( \Gamma / \varphi \) is passive \( L \)-admissible iff \( \Gamma \) has no common \( L \)-unifier. We say that a logic is structurally complete iff every admissible rule is a derivable rule. Roughly speaking, a logic is structurally complete iff every proper finitary extension must contain new theorems, as opposed to nothing but new rules of inference (see for instance [1, 17, 22]). Every logic \( L \) has a unique structurally complete extension \( L' \) with same theorems of \( L \) [1]. In particular, structural completeness can be seen as a kind of maximality condition on a logic. We say that a logic is almost structurally complete iff every admissible rule is either derivable rule or passive. It follows from a result of Dzik [7] that every finite valued Łukasiewicz logic is almost structurally complete. Jeřábek uses this result [14, Corollary 3.7] to obtain for every \( n > 2 \), one rule that axiomatizes admissible rules of the \( n \)-valued Łukasiewicz logic. In [15] the same author proves that admissible rules of the infinite valued Łukasiewicz calculus \( L_\infty \) are not finitely based, moreover
he explicitly constructs an infinite base of single-conclusion admissible rules for \( L_\infty \). The purpose of this work is to obtain a basis of single-conclusion rules of every proper axiomatic extension of \( L_\infty \). Admissibility theory normally uses proof-theoretic techniques, however in this case we will take an algebraic approach taking advantage of the algebraization of \( L_\infty \). In fact for algebraizable logics there is a analogous algebraic notion of admissible quasiequations and structurally complete and almost structurally complete quasivarieties (see for instance [1, 22]).

It is well known that \( L_\infty \) is algebraizable and the class of MV-algebras \( \text{MV} \) is its equivalent algebraic quasivariety semantics [21]. It follows from the algebraization, that quasivarieties of \( \text{MV} \) are in 1-1 correspondence with finitary extensions of \( L_\infty \). Actually, there is a dual isomorphism from the lattice of all quasivarieties of \( \text{MV} \) and the lattice of all finitary extensions of \( L_\infty \). Moreover if we restrict this correspondence to varieties of \( \text{MV} \) we get the dual isomorphism from the lattice of all varieties of \( \text{MV} \) and the lattice of all axiomatic extensions of \( L_\infty \). Given an axiomatic extension \( L \) of \( L_\infty \) whose equivalent variety semantics is \( \mathcal{V}_L \), single-conclusion admissible rules of \( L \) can be seen as valid quasiequations in the \( \mathcal{V}_L \)-free algebra under a countable set of generators \( F_{\mathcal{V}_L}(\omega) \) in the following sense: \( \gamma_1,\ldots,\gamma_n/\varphi \) is \( L \)-admissible if and only if \( \gamma_1 \approx 1 \& \cdots \& \gamma_n \approx 1 \Rightarrow \varphi \approx 1 \) is valid in \( F_{\mathcal{V}_L}(\omega) \) [22].

Hence the algebraic study of admissible rules of \( L \) is the study of \( \mathcal{Q}(F_{\mathcal{V}_L}(\omega)) \) the quasivariety generated by \( F_{\mathcal{V}_L}(\omega) \). The quasivariety \( \mathcal{Q}(F_{\mathcal{V}_L}(\omega)) \) is the least quasivariety that generates \( \mathcal{V}_L \) as a variety. These quasivarieties were studied in [10], where a Komori’s type characterization is accomplished. In this paper we will prove that \( \mathcal{Q}(F_{\mathcal{V}_L}(\omega)) \) is the class of bipartite algebras of the least quasivariety generated by MV-chains that generates \( \mathcal{V}_L \) as a variety. We use this result to prove that admissible rules for any proper axiomatic extension of \( L_\infty \) are finitely based and finally we give an effective axiomatization for each one.

The paper is organized as follows. First, in Section 1 we introduce the necessary definitions, notation and preliminary results on Universal Algebra and MV-algebra Theory that we use throughout the paper. Section 2 is first devoted to survey already existing results on varieties and quasivarieties of MV-algebras and later to obtain principal algebraic results: Theorems 2.4 and 2.6. Finally, Section 3 contains the main theorem, Theorem 3.4, where we built a finite basis for each axiomatic extension of \( L_\infty \).
1 Definitions and first properties.

We assume the reader is familiar with Universal Algebra [2, 13]. To fix notation we denote by $I$, $H$, $S$, $P$, $P_R$ and $P_U$ the operators isomorphic image, homomorphic image, substructure, direct product, reduced product and ultraproduct respectively. We recall that a class $K$ of algebras is a variety if and only if it is closed under $H$, $S$ and $P$. A class $K$ of algebras is a quasivariety if and only if it is closed under $I$, $S$ and $P_R$, or equivalently, under $I$, $S$, $P$ and $P_U$. A class $K$ of algebras is a universal class if and only if it is closed under $I$, $S$ and $P_U$. Given a class $K$ of algebras, the variety generated by $K$, denoted by $V(K)$, is the least variety containing $K$. Similarly, the quasivariety generated by a class $K$, which we denote by $Q(K)$, is the least quasivariety containing $K$. $U(K)$ denotes the universal class generated by $K$, that is the least universal class containing $K$. We also recall that a class $K$ of algebras is a variety if and only if it is an equational class; $K$ is a quasivariety if and only if it is a quasiequational class; $K$ is a universal class if and only if $K$ is definable by first order universal sentences.

An MV-algebra is an algebra $\langle A, \oplus, \neg, 0 \rangle$ satisfying the following equations:

- **MV1** $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$
- **MV2** $x \oplus y \approx y \oplus x$
- **MV3** $x \oplus 0 \approx x$
- **MV4** $\neg(\neg x) \approx x$
- **MV5** $x \oplus \neg 0 \approx \neg 0$
- **MV6** $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

We write 1 instead of $\neg 0$, $x \odot y$ instead of $\neg(\neg x \oplus \neg y)$ and $a \rightarrow b$ instead of $\neg a \oplus b$. Further, for all $n \in \omega$, where $\omega$ is the set of all natural numbers, and $x \in A$, the MV-operations $nx$ and $x^n$ are inductively defined by

$$0x = 0, \quad (n + 1)x = x \oplus (nx)$$

and

$$x^0 = 1, \quad x^{n+1} = x \odot (x^n).$$

Following tradition we assume that the operation $x^n$ takes precedence over any other operation; also $\neg$ takes precedence over $\odot$, $\odot$ takes precedence over $\oplus$, and $\oplus$ takes precedence over $\rightarrow$. 

3
As shown by Chang [3], for any MV-algebra $A$, the stipulation $a \leq b$ iff $a \rightarrow b = 1$ endows $A$ with a bounded distributive lattice-order $\langle A, \lor, \land, 0, 1 \rangle$, called the natural order of $A$.

\[ x \lor y \overset{\text{def}}{=} \neg(\neg x \oplus y) \oplus y. \]

\[ x \land y \overset{\text{def}}{=} \neg(\neg x \lor \neg y). \]

An MV-algebra whose natural order is total is said to be an MV-chain.

We recall that a lattice-ordered abelian group (for short, ℓ-group) is an algebra $\langle G, \land, \lor, +, -, 0 \rangle$ such that $\langle G, \land, \lor \rangle$ is a lattice, $\langle G, +, -, 0 \rangle$ is an abelian group and satisfies the following equation:

\[ (x \lor y) + z \approx (x + z) \lor (y + z) \]

For any ℓ-group $G$ and element $0 < u \in G$, let $\Gamma(G, u) = \langle [0, u], \oplus, \neg, 0 \rangle$ be defined by

\[ [0, u] = \{ a \in G \mid 0 \leq a \leq u \}, \quad a \oplus b = u \land (a + b), \quad \neg a = u - a. \]

Then, $\langle [0, u], \oplus, \neg, 0 \rangle$ is an MV-algebra. Further, for any ℓ-groups $G$ and $H$ with elements $0 < u \in G$ and $0 < v \in H$, and any ℓ-group homomorphism $f : G \to H$ such that $f(u) = v$, let $\Gamma(f)$ be the restriction of $f$ to $[0, u]$. An element $0 < u \in G$ is called a strong unit iff for each $x \in G$ there is an integer $n \geq 1$ such that $x \leq nu$. Then, as proved in [19], (see also [3]) $\Gamma$ is a categorical equivalence from the category of ℓ-groups with strong unit, with ℓ-homomorphisms that preserve strong units, onto the category of MV-algebras with MV-homomorphisms. Moreover the functor $\Gamma$ preserves embeddings and epimorphisms.

The following MV-algebras play an important role in the paper.

- $[0, 1] = \Gamma(\mathbb{R}, 1)$, where $\mathbb{R}$ is the totally ordered group of the reals.
- $[0, 1] \cap \mathbb{Q} = \Gamma(\mathbb{Q}, 1) = \langle \{ \frac{k}{m} : k \leq m < \omega \}, \oplus, \neg, 0 \rangle$, where $\mathbb{Q}$ is the totally ordered abelian group of the rationals.
- $L_n = \Gamma(\mathbb{Z}, n) = \langle \{ 0, 1, \ldots, n \}, \oplus, \neg, 0 \rangle$, where $\mathbb{Z}$ is the totally ordered group of all integers. Notice that $L_n$ is isomorphic to the subalgebra of $[0, 1]$ given by $\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1 \}$. 

4
\[ L_\omega^n = \Gamma(\mathbb{Z} \times_{\text{lex}} \mathbb{Z}, (n,0)) = \langle \{(k,i) : (0,0) \leq (k,i) \leq (n,0)\}, \oplus, \neg, 0 \rangle, \]
where \( \times_{\text{lex}} \) denotes the lexicographic product.

\[ L_s^n = \Gamma(\mathbb{Z} \times_{\text{lex}} \mathbb{Z}, (n,s)) = \langle \{(k,i) : (0,0) \leq (k,i) \leq (n,s)\}, \oplus, \neg, 0 \rangle, \]
where \( s \in \mathbb{Z} \) such that \( 0 \leq s < n \). Notice that \( L_\omega^n = L_0^n \).

As usual, by an ideal of an MV-algebra \( A \) we mean the kernel \( I \) of a homomorphism \( h \) of \( A \) into some MV-algebra \( B \). In other words, \( 0 \in I \), \( I \) is closed under the \( \oplus \) operation, and \( x \leq y \in I \) implies \( x \in I \). We denote by \( I(A) \) the set of all ideals of \( A \).

An ideal is prime iff it is the kernel of a homomorphism of \( A \) into an MV-chain. We denote by \( \text{Spec}(A) = \{ I \in I(A) : I \text{ is prime} \} \). An ideal is maximal iff it is the kernel of a homomorphism of \( A \) into \([0,1]\). We denote by \( \mathcal{M}(A) = \{ I \in I(A) : I \text{ is maximal} \} \). The radical of \( A \) denoted by \( \text{Rad}(A) \) is the intersection of all maximal ideals of \( A \). Notice that when \( A \) is an MV-chain \( \mathcal{M}(A) = \{ \text{Rad}(A) \} \) and \( \text{Rad}(A) = \{ a \in A : a^k \neq 0 \text{ for all } k > 0 \} \).

An MV-algebra is said to be bipartite iff there is \( I \in I(A) \) such that \( A/I \cong L_1 \).

2 Varieties and quasivarieties of MV-algebras

Since the class of all MV-algebras is definable by a set of equations, it is a variety that we denote by \( \text{MV} \). By Chang’s Completeness Theorem [4] (see also [5]), \( \text{MV} \) is the variety generated by the MV-algebra \([0,1]\) (or \([0,1]\cap\mathbb{Q})

in symbols,

\[ \text{MV} = \mathcal{V}([0,1]) = \mathcal{V}([0,1]\cap\mathbb{Q}). \]

Proper subvarieties of \( \text{MV} \) are well known. Komori proves the following characterization

**Theorem 2.1** [16] Theorem 4.11] \( V \) is a proper subvariety of \( \text{MV} \) if and only if there exist two disjoint finite subsets \( I,J \) of positive integers, not both empty such that

\[ V = \mathcal{V}(\{ L_i : i \in I \} \cup \{ L_j^\omega : j \in J \}). \]

A pair \((I,J)\) of finite subsets of positive integers, not both empty is said to be reduced iff for every \( n \in I \), there is no \( k \in (I \setminus \{n\}) \cup J \) such that \( n \mid k \) and for every \( m \in J \), there is no \( k \in J \setminus \{m\} \) such that \( m \mid k \). In [20] Panti shows that there is a 1-1 correspondence between proper subvarieties of \( \text{MV} \) and
reduced pairs of finite subsets of positive integers not both empty. Given a reduced pair \((I, J)\), we denote by \(\mathcal{V}_{I,J}\) its associated subvariety. Moreover for every reduced pair \((I, J)\) there is a single equation in just one variable of the form \(\alpha_{I,J}(x)\approx 1\) axiomatizing \(\mathcal{V}_{I,J}\).

Quasivarieties of MV-algebras have been studied by this author in [11][12] [9][10]. Particularly in [9], the author finds a characterization, classification an axiomatization of every quasivariety generated by MV-chains. Moreover he obtains necessary condition for finitely axiomatization that yields to the following result:

**Theorem 2.2** Every quasivariety generated by MV-chains contained in a proper subvariety of MV is finitely axiomatizable

Let \(\mathbf{V}\) a variety of any type of algebras, a quasivariety \(\mathbf{K}\) of same type is a \(\mathbf{V}\)-quasivariety provided that \(\mathcal{V}(\mathbf{K}) = \mathbf{V}\). It follows from the work in [9] that \(Q^1_{I,J} := \mathcal{Q}(\{L_m \mid m \in I\} \cup \{L_n^1 \mid n \in J\})\) is the least \(\mathcal{V}_{I,J}\)-quasivariety generated by chains. However \(Q^1_{I,J}\) is not the least \(\mathcal{V}_{I,J}\)-quasivariety. In fact, for any variety \(\mathbf{V}\), not necessarily of MV-algebras, the least \(\mathbf{V}\)-quasivariety is \(\mathcal{Q}(F_\mathbf{V}(\omega))\). In [10] we study least \(\mathbf{V}\)-quasivarieties and we obtain the following characterization.

**Theorem 2.3** [10] Theorem 4.8 Let \((I, J)\) be a reduced pair. Then \(Q_{I,J} := \mathcal{Q}(\{L_1 \times L_m \mid m \in I\} \cup \{L_1 \times L_n^1 \mid n \in J\})\) is the least \(\mathcal{V}_{I,J}\)-quasivariety.

Next result establishes the relation between least \(\mathbf{V}\)-quasivarieties of MV-algebras and least \(\mathbf{V}\)-quasivarieties generated by chains.

**Theorem 2.4** \(Q_{I,J}\) is the class of all bipartite algebras in \(Q^1_{I,J}\)

**Proof:** Let \(BPQ^1_{I,J}\) be the class of all all bipartite algebras in \(Q^1_{I,J} =\mathcal{Q}(\{L_m \mid m \in I\} \cup \{L_n^1 \mid n \in J\})\). Since being bipartite is preserved under \(\mathcal{I}, \mathcal{S}, \mathcal{P}\) and \(\mathcal{P}_U\) and \(\{L_1 \times L_m \mid m \in I\} \cup \{L_1 \times L_n^1 \mid n \in J\} \subseteq BPQ^1_{I,J}\), then \(Q_{I,J} = \mathcal{ISP}_U(\{L_1 \times L_m \mid m \in I\} \cup \{L_1 \times L_n^1 \mid n \in J\}) \subseteq BPQ^1_{I,J}\).

In order to prove the other implication, let \(A \in BPQ^1_{I,J}\). Let \(\Delta = \{I \in Spec(A) : A/I \in \mathcal{Q}(\{L_m \mid m \in I\} \cup \{L_n^1 \mid n \in J\})\}.\) Since \(\mathcal{Q}(\{L_m \mid m \in I\} \cup \{L_n^1 \mid n \in J\})\) is relative congruence distributive quasivariety, \(\Delta\) induces a natural subdirect representation of \(A\): \(A \hookrightarrow_{SD} \prod_{I \in \Delta} A/I : a \mapsto (a/I)_{I \in \Delta}\) where each \(A/I\) is an MV-chain of \(\mathcal{Q}(\{L_m \mid m \in I\} \cup \{L_n^1 \mid n \in J\}).\)

Since \(A\) is bipartite there exists \(I \in \Delta\) such that \(A/I \cong L_1\). Thus \(A \in \mathcal{ISP}(\{L_1 \times B \mid B \in K\})\) where \(K = \{A/I \mid I \in \Delta\}\) and \(A/I \not\cong L_1\). Since
B is an MV-chain of $Q_{I,J}^1$, then $B \in ISP_U(\{L_m : m \in I\} \cup \{L_n^1 : n \in J\}) = \bigcup_{m \in I} IS(L_m) \cup \bigcup_{n \in J} ISP_U(L_n^1)$. Thus for every $B \in K$, $L_1 \times B \in \bigcup_{m \in I} IS(L_1 \times L_m) \cup \bigcup_{n \in J} ISP_U(L_1 \times L_n^1) \subseteq Q_{I,J}$. □

In [18], G. Metcalfe and C. Röthlisberger give the following characterization of almost structurally complete quasivarieties

**Theorem 2.5** [18, Theorem 4.10] Let $K$ be a quasivariety. The following are equivalent for any $B \in S(F_K(\omega))$

1. $K$ is almost structurally complete.
2. $Q(\{A \times B : A \in K\}) = Q(F_K(\omega))$.
3. $\{A \times B : A \in K\} \subseteq Q(F_K(\omega))$.

**Theorem 2.6** Let $(I, J)$ be a reduced pair. Then $Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})$ is almost structurally complete.

**Proof:** Since $F_{V_{I,J}}(\omega) = F_{Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})}(\omega)$ and $L_1$ is a subalgebra of $F_{V_{I,J}}(\omega)$, by previous theorem it is enough to prove that $A \times L_1 \in Q(F_{V_{I,J}}(\omega)) = Q_{I,J}$ for every $A \in Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})$. Trivially $A \times L_1$ is a bipartite member of $Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\}) = Q_{I,J}^1$. Then, by Theorem 2.4, $A \times L_1 \in Q_{I,J}$ concluding the proof. □

### 3 Bases of admissible rules

From the algebraization of $L_\infty$ we obtain a 1 to 1 correspondence between quasivarieties of MV-algebras and finitary extensions of $L_\infty$. In fact given a finitary extension $L$, $L$ is also algebraizable and its equivalent quasivariety semantics is its associated quasivariety. Viceversa if $K$ is a quasivariety of MV-algebras the logic $|=K$ is its associated finitary extension, where $|=K$ is defined as follows $\Gamma |=K \varphi$ iff for every $A \in K$ and every evaluation $e : Prop(\Gamma) \rightarrow A$ if $\{e(\varphi) = 1\}$ then $e(\varphi) = 1$. Moreover there is a translation from formulas to equations and a translation from equations to formulas that allows to obtain an quasiequational axiomatization of a quasivariety $K$ from the axiomatization of its associated finitary extension, and viceversa to get an axiomatization of a finitary extension $L$ from the quasiequational axiomatization of its equivalent quasivariety semantics.

Jerabek in [15] gives an infinite axiomatization for all $L_\infty$-admissible rules ad moreover he proves that they are not finitely based. Our purpose is
to obtain a base of all admissible rules for every proper axiomatic extension of $L_\infty$. By Komori’s classification of axiomatic extensions of $L_\infty$ and Panti’s correspondence [16, 20], every axiomatic extension is given by a reduced pair $(I,J)$. Given a reduced pair $(I,J)$ we denote by $L_{(I,J)}$ its associated axiomatic extension. Notice that $\mathcal{V}_{I,J}$ is the equivalent quasivariety semantics of $L_{I,J}$. Moreover since $\mathcal{V}_{I,J} = \mathcal{Q}(\{L_i : i \in I\} \cup \{L_j^\omega : j \in J\})$ (see [8]), we get the following finite strong completeness theorem:

$\varphi_1, \ldots, \varphi_n \vdash_{L_{I,J}} \varphi$ if and only if $\varphi_1, \ldots, \varphi_n \models_{\{L_i : i \in I\} \cup \{L_j^\omega : j \in J\}} \varphi$.

**Lemma 3.1** Let $(I,J)$ be a reduced pair and $n = \max\{\max I, \max J + 1\}$. Then $\neg p^n \vdash_{L_{I,J}} \neg p^n$ for every $m > 0$.

**Proof:** By completeness $\neg p^n \vdash_{L_{I,J}} \neg p^n$ is equivalent to the following statement: For every $A \in \{L_i : i \in I\} \cup \{L_j^\omega : j \in J\}$ and any $a \in A$, $a^n = 0$ implies $a^m = 0$ which is valid because $\text{Rad}(A) = \{a \in A : a^n \neq 0\}$ for every $A \in \{L_i : i \in I\} \cup \{L_j^\omega : j \in J\}$. $\square$

In [14] the author gives a basis of single conclusion passive rules for every extension of BL.

**Theorem 3.2** [14, Theorem 3.6]

If $L$ is an extension of BL then $CC^1 = \{\neg (p \lor \neg p)^n / \bot : n > 1\}$ is a basis of single-conclusion passive $L$-admissible rules.

**Theorem 3.3** Admissible rules for proper axiomatic extensions of $L_\infty$ are finitely based.

**Proof:** By Theorem 2.6, every $L_{I,J}$-admissible rule is either derivable in $\models_{Q^1_{I,J}}$ or it is a passive $L_{I,J}$-admissible rule. By Theorem 2.7, derivable rules in $\models_{Q^1_{I,J}}$ are finitely axiomatizable. Moreover by Theorem 3.2 and Lemma 3.1 passive $L_{I,J}$-admissible rules are axiomatizable by $\neg (p \lor \neg p)^n / \bot$ where $n = \max\{\max I, \max J + 1\}$. $\square$

**Theorem 3.4** Let $(I,J)$ be a reduced pair, then a base of admissible rules for $\vdash_{I,J}$ is given by

- $L_1$, $L_2$, $L_3$, $LA + M.P.$
- $\alpha_{I,J}(\gamma)$.
- $\Delta(Q_p) := [(\neg \varphi)^{p-1} \leftrightarrow \varphi] \lor [\psi \leftrightarrow \chi] / \psi \leftrightarrow \chi$ for every prime number $p \in \text{Div}(J) \setminus \text{Div}(I)$
\[
\Delta(U_q) := \left( (\neg \varphi)^{q-1} \leftrightarrow \varphi \right) \lor \left[ \psi \leftrightarrow \chi \right] / \alpha_{I_q,\emptyset}(\gamma) \lor (\psi \leftrightarrow \chi) \\
\text{for every prime number } q \in \text{Div}(I), \text{ where } I_q = \{ n \in I : q|n \}\]

\[
CC^1_n := (\varphi \lor \neg \varphi)^n / \bot \\
\text{where } n = \max\{ \max I, \max J + 1 \}\]

Proof: Following the proof of the previous theorem it is enough to put together the axiomatization of \(\models_{Q^I_{I,J}}\) plus \(CC^1_n\) where \(n = \max\{ \max I, \max J + 1 \}\). It follows from [9, Theorem 4.5] that \(\models_{Q^I_{I,J}}\) is axiomatized by

- L1, L2, L3, L4 + M.P.
- \(\alpha_{I,J}(\gamma)\).
- \(\Delta(Q_n)\) for every \(n \in \text{Div}(J) \setminus \text{Div}(I)\)
- \(\Delta(U_m)\) for every \(m \in \text{Div}(I)\)
- \(Q_n := (\neg \varphi)^{p-1} \leftrightarrow \varphi / \psi\) \\
  \text{for every } n \in \text{Div}(J) \setminus \text{Div}(I)\)
- \(U_m := (\neg \varphi)^{q-1} \leftrightarrow \varphi / \alpha_{I_m,\emptyset}(\gamma)\) for every \(m \in \text{Div}(I)\)

We can avoid \(Q_n\) and \(U_m\), since they are passive \(L_{I,J}\)-admissible rules, therefore derivable from \(CC^1_n\). It enough to take \(\Delta(Q_p)\) for every prime number \(p \in \text{Div}(J) \setminus \text{Div}(I)\), and \(\Delta(U_q)\) for every prime number \(q \in \text{Div}(I)\) because \(\Delta(Q_n)\) is derivable from \(\Delta(Q_p)\) if \(p|n\) and \(\Delta(U_m)\) is derivable from \(\Delta(U_q)\) if \(q|m\).

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