Frequency–energy plot and targeted energy transfer analysis of coupled bistable nonlinear energy sink with linear oscillator

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Abstract The nonlinear energy sink (NES), which is proven to perform rapid and passive targeted energy transfer (TET), has been employed for vibration mitigation in many primary small- and large-scale structures. Recently, the feature of bistability, in which two nontrivial stable equilibria and one trivial unstable equilibrium exist, is utilized for passive TET in what is known as bistable NES (BNES). The BNES generates a nonlinear force that incorporates negative linear and multiple positive or negative nonlinear stiffness components. In this paper, the BNES is coupled to a linear oscillator (LO) where the dynamic behavior of the resulting LO-BNES system is studied through frequency–energy plots (FEPs), which are generated by analytical approximation using the complexification-averaging method and by numerical continuation techniques. The effect of the length and stiffness of the transverse coupling springs is found to affect the stability and topology of the branches and indicates the importance of the exact physical realization of the system. The rich nonlinear dynamical behavior of the LO-BNES system is also highlighted through the appearance of multiple symmetrical and unsymmetrical in- and out-of-phase backbone branches, especially at low energy levels. The superimposed wavelet frequency spectrums of the LO-BNES response on the FEP have verified the robustness of the TET mechanism where the role of the unsymmetrical NNM backbones in TET is clearly observed.

Keywords Nonlinear energy sink · Nonlinear normal modes · Frequency–energy plot · Bistable nonlinear energy sink

1 Introduction

The NES is a light-weight nonlinear dynamical oscillator that employs an essentially nonlinear coupling element to engage in optimum passive TET from an associated primary structure to be localized or dissipated by the NES. Due to single or cascade of resonance captures, the NES can perform rapid, passive, and nearly irreversible TET in a broadband frequency–energy fashion. The essentially nonlinearizable coupling element significantly alters the global dynamics of the integrated system resulting in nonlinear interactions between structural modes. This causes passive energy exchange, a phenomenon that is not possible in linear dynamical absorbers except in the case of carefully tuned frequencies. Consequently, the NES has been intensively studied in the literature for shock and seismic mitigation purposes to rapidly
dissipate a significant portion of the energy induced into a structure. Generally, the NESs can be categorized into translational and rotational depending on the nature of their motion. The first NES considered in the literature in [1–9], usually called Type I NES, only incorporates purely nonnegative cubic restoring stiffness in its coupling force. A variety of subtypes branch out from this nonlinear coupling method based on the nature of the coupling damping element, the characteristics of the nonlinear force or the number of nonlinearly attached masses [1–9]. Other types of translational NESs include the magnetic NES which is realized by a nonlinear magnetic force [10–12] and the vibro-impact NES which is realized by non-smooth nonlinearities imposed via impact surfaces to an otherwise linear tuned mass damper [1, 13–21]. The other category, the rotational NESs, is realized by inertial coupling between the NES mass and the primary structure by means of a rigid rotating arm to generate a strongly nonlinear coupling force [22–27]. Additional research works have investigated using an oscillating arm or incorporating non-smooth nonlinearities to the rotational NESs attempting to enhance the irreversible nonlinear energy transfers [28, 29].

Numerous research works show that both categories and many subtypes of the NESs yield enhanced “utilization” of the inherent structural dynamics toward more effective response mitigation, without the necessity of adding more damping.

Recently, the BNES has been proposed and studied for vibration suppression in [30–32]. Unlike the previously mentioned Type I NES, this double-well potential BNES incorporates negative linear and negative high order nonlinear stiffness terms besides to nonnegative high order nonlinear stiffness terms. Furthermore, the BNES possesses two nontrivial stable equilibria and one trivial unstable equilibrium. A detailed numerical and analytical study of the BNES having combined negative linear stiffness and positive cubic nonlinear stiffness terms shows that this type of NESs can achieve efficient TET for impulsively excited structures [30, 31]. Following the analytical study in [30], it is found that the mechanisms leading to the energy exchanges between a low-energy impulsively excited linear oscillator and the BNES are a periodic alternating in-well and cross-well oscillations of the BNES and secondary nonlinear beats occurring when the dynamics evolves solely into well [31]. In addition, the physical realization of the BNES is addressed in [32, 33] where the nonlinearity is obtained by transverse linear springs that are compressed in their vertical position. Using Taylor-expansions, the generated nonlinear force incorporates negative linear and multiple positive or negative nonlinear terms. This resulted in a unique representation of the nonlinear force as it encloses up to seventh-order nonlinear stiffness terms as opposed to cubic nonlinear terms only in most other works in the literature. The resulting system is highly efficient compared to other existing NESs in the literature for a wide range of initial impulsive energies [32]. The efficiency of the BNES is further analyzed using slow invariant manifold, asymptotic analysis and Melnikov analysis to predict the response regimes and their thresholds levels [34]. To optimize the performance of the BNES, a design criterion with a corresponding parameter is proposed in [34] and a novel tuning method using Lyapunov exponents is proposed in [35] which allows the BNES to avoid chaotic behavior and enhances TET. The BNES has also been employed to coupled linear symmetrical oscillators with ungrounded configuration in [36] and with bistable grounding in [37]. In both cases, the capability of the BNES to achieve highly efficient TET is demonstrated. Given this high performance, the BNES was applied to cantilever beams [38], rotor systems [39, 40] and pipes conveying fluids [41] for vibration suppression and incorporated with magnetic effects for structural seismic control [42].

The underlying nonlinear dynamical behavior has been revealed for some types of NESs based on studying frequency–energy dependences of Hamiltonian versions of linear systems attached to NESs. Therefore, FEPs have been generated via numerical and analytical methods. The FEPs of Hamiltonian versions of linear systems attached to NESs with odd-power stiffness coupling have been generated in several publications using analytical and numerical methods [1–5, 43–48]. Similarly, FEPs of linear systems attached to rotating NES were obtained in [22, 23] and systems attached to vibro-impact and piecewise NESs in [49–53]. In these sets of publications, fundamental backbone curves of 1:1 in-phase and anti-phase resonances were obtained. In addition, several bifurcated subharmonic branches from fundamental backbones were also generated. The damped dynamics of considered systems have been studied by
imposing wavelet-transform frequency spectra on the obtained FEPs. Accordingly, the NES nonlinear action of rapid energy transfer has been found to take place through single and cascade of resonance captures between the NESs and the associated linear structures responses. These studies have sufficiently revealed the underlying Hamiltonian and damped dynamics of aforementioned NESs.

In another set of publications [54, 55], the damped dynamics of the NES itself has been studied. Therefore, formulas of displacement, velocity, frequency and energy decay curves have been obtained for odd-power stiffness NES. Based on these formulas, a new formula for the relationship between displacement and velocity damping contents has been introduced in [54]. In [56], a method for generating plots of frequency versus nonlinear energy content (FNLPs) was introduced and directly applied into equations of motion where numerical simulation is not required a priori. This proposed method was employed later in [57] to reveal the underlying nonlinear dynamical behavior of modal damping content of some linear systems attached to NES.

In this paper, we discuss the BNES which has been realized in [32] by transversely coupled linear springs which are compressed at the trivial equilibrium position and neither compressed nor elongated at the two nontrivial stable equilibrium positions. At the point where the linear springs are aligned, the unstable equilibrium exists at which both springs induce a pre-stored potential energy. We consider this BNES to be attached to LO where the resulting nonlinear coupling force consists of several negative and nonnegative linear and nonlinear stiffness components. We focus in particular on the FEP which is a representation of the nonlinear normal modes (NNMs) depicting the frequency content of the system with the corresponding energies. Accordingly, the dynamic behavior of the proposed BNES is investigated here on a FEP generated analytically using complexification-averaging technique (CX–A) as well as via numerical continuation methods [58] of the undamped LO–BNES system to obtain the fundamental backbone branches of the NNMs. Consequently, the damped dynamics of the considered system is analyzed by imposing the wavelet transform of the obtained response into the obtained FEP.

2 System description and governing equations

In the BNES, negative stiffness components appear when the transverse springs are neither compressed nor elongated at their stable equilibrium positions as shown in Fig. 1a. In this configuration, both springs have a presorted potential energy at zero displacements of the NES and the LO masses which in turn generates negative stiffness components. Accordingly, the BNES imposes two stable equilibrium positions at \( z = \pm z_c \) when attached to LO. At \( z = \pm z_c \), both springs are neither compressed nor elongated at their original physical length \( L \). Furthermore, the BNES is unstable at \( z = 0 \). According to [32], the nonlinear restoring force of the springs can be approximated by the Taylor series (T–S) expansion about \( z = z_c \) for \( L_0 < L \). Therefore, the obtained (T–S) expansion of the exact force up to fifteenth order is obtained as

\[
(F_{nl})_{\text{exact}} \approx F_{nl} = -2k \left( \frac{1}{L_0} \right) z - \frac{kl}{L_0^2} \ddot{z}^3 + \frac{3kl}{4L_0^2} z^4 - \frac{5kl}{8L_0^2} z^7 + \frac{35kl}{64L_0^2} z^9 - \frac{63kl}{128L_0^2} z^{11} + \frac{231kl}{512L_0^2} z^{13} - \frac{429kl}{1024L_0^2} z^{15}
\]

\[
= k_1 z - k_2 z^3 + k_3 z^5 - k_4 z^7 + k_5 z^9 - k_6 z^{11} + k_7 z^{13} - k_8 z^{15}
\]

(1)

The bistable nonlinear attachment in Fig. 1a of mass \( m \) is coupled with LO of mass \( M(m << M) \) as shown in Fig. 2. Therefore, the equations of motions at the relative BNES oscillation \( z = x_1 - x_2 \) with respect to the LO oscillation are written as

\[
mx_1 + \lambda \ddot{x}_1 - k_1 z + k_2 z^3 - k_3 z^5 + k_4 z^7 - k_5 z^9 + k_6 z^{11} - \gamma z^{13} + k_8 z^{15} = 0
\]

\[
M\ddot{x}_2 + \lambda_x \ddot{x}_2 - \ddot{z} + k_1 x_1 - k_2 z^3 + k_3 z^5 - k_4 z^7 + k_5 z^9 - k_6 z^{11} + k_7 z^{13} - k_8 z^{15} = 0
\]

(2)

where \( x_1 \) is the NES mass displacement, \( x_2 \) is LO mass displacement, \( \lambda \) is the damping of the NES attachment, \( \lambda_x \) is the damping of the linear structure and \( k_p \) is the stiffness of the linear structure. For \( z_c = \sqrt{(L^2 - L_0^2)} \) and \( z(0) = x_1(0) - x_2(0) \), the initial energy equation
in the physical coordinates of the LO-BNES system is governed by

\[
E_0 = \frac{1}{2} M \dddot{x}_2(0)^2 + \frac{1}{2} m \dddot{x}_1(0)^2 + \frac{1}{2} k_p x_2(0)^2 - \frac{1}{2} k_1 z(0)^2 \\
+ \frac{1}{4} k_2 z(0)^4 - \frac{1}{6} k_3 z(0)^6 + \frac{1}{8} k_4 z(0)^8 \\
- \frac{1}{10} k_5 z(0)^{10} + \frac{1}{12} k_6 z(0)^{12} - \frac{1}{14} k_7 z(0)^{14} \\
+ \frac{1}{16} k_8 z(0)^{16} + k(L - L_0)^2
\]

It is clear from the above equation that when \( x_1(0) = x_2(0) = 0 \) and \( \dot{x}_1(0) = \dot{x}_2(0) = 0 \), the initial presorted energy is obtained as \( E_0 = k(L - L_0)^2 \).

3 FEP using complexification-averaging technique

To understand the underlying nonlinear dynamics of the LO-BNES system, we firstly study the FEP of the system using complexification-averaging technique (CX–A) proposed in [58] through slow-fast partition of the dynamics to generate the backbone branches.
S11 ± which correspond to periodic motion where the LO and the BNES possess identical dominant frequency components. Accordingly, Eq. (2) is firstly normalized with respect to the mass \( M \) of the LO which yields the following equations

\[
\begin{align*}
\varepsilon\ddot{x}_1 + \dddot{x}_1 - \dddot{x}_2 - \dddot{k}_1(x_1 - x_2) + \dddot{k}_2(x_1 - x_2)^3 \\
- \dddot{k}_3(x_1 - x_2)^5 + \dddot{k}_4(x_1 - x_2)^7 - \dddot{k}_5(x_1 - x_2)^9 \\
+ \dddot{k}_6(x_1 - x_2)^{11} - \dddot{k}_7(x_1 - x_2)^{13} + \dddot{k}_8(x_1 - x_2)^{15} = 0
\end{align*}
\]

\[
\begin{align*}
\dddot{x}_2 + \dddot{x}_p\dddot{x}_2 - \dddot{l}(\dddot{x}_1 - \dddot{x}_2) + \omega_0^2\dddot{x}_2 + \dddot{k}_1(x_1 - x_2) \\
- \dddot{k}_2(x_1 - x_2)^3 + \dddot{k}_3(x_1 - x_2)^5 - \dddot{k}_4(x_1 - x_2)^7 \\
+ \dddot{k}_5(x_1 - x_2)^9 - \dddot{k}_6(x_1 - x_2)^{11} + \dddot{k}_7(x_1 - x_2)^{13} \\
- \dddot{k}_8(x_1 - x_2)^{15} = 0
\end{align*}
\]

(4)

where

\[
\dddot{x}_p = \frac{\dddot{\lambda}_p}{M}, \quad \dddot{\lambda} = \frac{\dddot{\lambda}}{M}, \quad \varepsilon = \frac{\varepsilon}{M}, \quad \omega_0^2 = \frac{k_p}{M}, \quad \dddot{k}_i = \frac{k_i}{M}
\]

Considering the Hamiltonian version of the system in Eq. (4) in which \( \dddot{\lambda}_p = \dddot{\lambda} = 0 \) and \( \omega_0^2 = 1 \), the complexification of the dynamics is firstly performed by introducing new complex variables as:

\[
\begin{align*}
\psi_1 &= \dot{x}_1 + j\omega x_1 \\
\psi_2 &= \dot{x}_2 + j\omega x_2
\end{align*}
\]

(5)

where \( \omega \) is the dominant fast frequency of oscillation. Given that we are interested in the periodic solutions at the fast frequency \( \omega \) where the LO and the BNES oscillate with the same fast frequency, we can express the complex variables defined in (5) in terms of fast oscillations of frequency \( \omega \), \( e^{j\omega t} \), modulated by slowly varying complex amplitudes \( \phi_i(t) \), \( i = 1, 2 \):

\[
\begin{align*}
\psi_1(t) &= \phi_1(t)e^{j\omega t} \\
\psi_2(t) &= \phi_2(t)e^{j\omega t}
\end{align*}
\]

(6)

By substituting Eqs. (6) and (5) into Eq. (4), the following set of equations are obtained

\[
\varepsilon\left(\phi_1e^{j\omega t} + j\omega\phi_1e^{j\omega t} - \frac{j\omega}{2}(\phi_1e^{j\omega t} + \phi_1^*e^{-j\omega t})\right)
\]

\[
+j\dddot{k}_1\left(\phi_1e^{j\omega t} + \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)
\]

\[
-j\dddot{k}_2\left(\phi_1e^{j\omega t} - \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)^3
\]

\[
+j\dddot{k}_3\left(\phi_1e^{j\omega t} - \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)^5
\]

\[
-j\dddot{k}_4\left(\phi_1e^{j\omega t} - \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)^7
\]

\[
+j\dddot{k}_5\left(\phi_1e^{j\omega t} - \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)^9
\]

\[
-j\dddot{k}_6\left(\phi_1e^{j\omega t} - \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)^{11}
\]

\[
+j\dddot{k}_7\left(\phi_1e^{j\omega t} - \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)^{13}
\]

\[
-j\dddot{k}_8\left(\phi_1e^{j\omega t} - \phi_1^*e^{-j\omega t} - \phi_2e^{j\omega t} + \phi_2^*e^{-j\omega t}\right)^{15} = 0
\]

(7)

The second step of the CX–A technique is the averaging of Eq. (7) with respect to the fast frequency \( \omega \). Therefore, only the terms containing the fast frequency are considered which results in a set of complex modulation equations constituting the approximate slow flow reduction in the dynamics. Following that, a set of polar representations are introduced as

\[
\begin{align*}
\phi_1 &= A e^{j\alpha} \\
\phi_2 &= B e^{j\beta}
\end{align*}
\]

(8)

where \( A \) and \( B \) are real amplitudes, and \( x \) and \( \beta \) are real phases. After rearranging and using Euler’s formula \( e^{j\theta} = \cos(\theta) + jsin(\theta) \), the real and imaginary parts of the resulting equations can be separately set to zero which represent the slow evolution of the real amplitudes and phases of modulation. To impose stationary conditions on the modulation equations, the derivatives with respect to time are set to zero. Hence, the periodic solutions on the backbone branches can be computed. Considering the trivial solution by assuming identity of phases (\( \alpha = \beta \)), the following equations are obtained
\[
\frac{\varepsilon \omega B}{2} + \frac{\kappa_1 (A - B)}{2 \omega} + \frac{3\kappa_2 (A - B)^3}{(2 \omega)^3} + \frac{10\kappa_3 (A - B)^5}{(2 \omega)^5} \\
+ \frac{35\kappa_4 (A - B)^7}{(2 \omega)^7} + \frac{126\kappa_5 (A - B)^9}{(2 \omega)^9} \\
+ \frac{462\kappa_6 (A - B)^{11}}{(2 \omega)^{11}} + \frac{1716\kappa_7 (A - B)^{13}}{(2 \omega)^{13}} \\
+ \frac{6435\kappa_8 (A - B)^{15}}{(2 \omega)^{15}} = 0
\]

\[
\frac{\omega A}{2} - \frac{\kappa_1 (A - B)}{2 \omega} - \frac{3\kappa_2 (A - B)^3}{(2 \omega)^3} \\
- \frac{10\kappa_3 (A - B)^5}{(2 \omega)^5} - \frac{35\kappa_4 (A - B)^7}{(2 \omega)^7} \\
- \frac{126\kappa_5 (A - B)^9}{(2 \omega)^9} - \frac{462\kappa_6 (A - B)^{11}}{(2 \omega)^{11}} \\
- \frac{1716\kappa_7 (A - B)^{13}}{(2 \omega)^{13}} - \frac{6435\kappa_8 (A - B)^{15}}{(2 \omega)^{15}} = 0
\]

(9)

The resulting equations in (9) are solved for A and B for each value of \( \omega \). Accordingly, the conserved energy of the system can be expressed as

\[
E = \frac{X_2^4}{2} - \frac{\kappa_1 (X_1 - X_2)^2}{2} + \frac{\kappa_2 (X_1 - X_2)^4}{4} \\
- \frac{\kappa_3 (X_1 - X_2)^6}{6} + \frac{\kappa_4 (X_1 - X_2)^8}{8} \\
- \frac{\kappa_5 (X_1 - X_2)^{10}}{10} + \frac{\kappa_6 (X_1 - X_2)^{12}}{12} \\
- \frac{\kappa_7 (X_1 - X_2)^{14}}{14} + \frac{\kappa_8 (X_1 - X_2)^{16}}{16} + k(L - L_0)^2
\]

where \( X_2 = \frac{B}{A} \) and \( X_1 = \frac{A}{B} \).

The periodic motions on the backbones and their associated low and high frequency subharmonic branches are represented by nonlinear normal modes (NNMs) in the configuration space. The frequency content in backbones and the subharmonic branches is characterized here by the frequency ratio between the LO and BNES periodic oscillations. The frequency indices \( S_{nm} \) and \( U_{nm} \) indicate symmetrical and unsymmetrical periodic motions, respectively, on the backbones and subharmonic branches. In \( S_{nm} \) and \( U_{nm} \), \( nm \) indicates the resonance frequency ratio \( (n : m) \) in the periodic motion between the LO and BNES masses where the plus sign (+) indicates in phase and the minus sign (−) indicates anti-phase motions. For example, \( S_{13} \) indicates that the BNES frequency of oscillation is three times the LO oscillation frequency where the associated NNM passes through the origin with a negative slope.

Therefore, the FEPs are generated as shown in Fig. 3 showing the backbone branches \( S_{11} \pm \) for different orders of the truncated nonlinear stiffness force by the T-S expansion at \( M = 1 \) kg and \( k_p = 1 \) N/m of the LO and the optimized BNES parameters \( k = 0.14 \) N/m, \( L = 1 \) m and \( L_0 = 0.882L \) obtained in [32] at BNES mass of \( m = 0.05 \) M (i.e., \( \varepsilon = 0.05 \)). Since the rich underlying dynamics of interest for the BNES attachment is expected to be below or close to the linear oscillator frequency, the results in Fig. 3 provide sufficient convergence at the seventh-order nonlinear stiffness. In addition, avoiding higher order stiffness terms reduces the unnecessary complications in application of CX–A and the numerical continuation methods for generating the FEPs.

The backbone branches obtained by CX–A are plotted in Figs. 4a and 5a for varying BNES length ratio \( L_0/L \) and stiffness \( k \), respectively. First, it is important to note that changing the stiffness and/or the

Fig. 3 Comparison of the backbone branch \( S_{11} \pm \) in the approximate FEP generated by CX–A for a LO–BNES system with \( \varepsilon = 0.05 \), \( k = 0.14 \) N/m and \( L_0 = 0.882L \) when considering up fifteenth-order stiffness terms in (4)
length ratio of the transversely coupled linear springs, which produce the bistability conditions and nonlinearity, has a significant effect on the global dynamics of the LO–BNES system. It is also noted that these changes have an effect on the topology of the unstable branches. This is demonstrated in Figs. 4b, 5b and can be concluded from the width of the energy interval corresponding to the unstable solutions which is realized by the derivative of the total energy with respect to the frequency index (i.e., slope of Figs. 4a, 5a). As shown in [35], this unstable part of the branch has a significant effect on the TET in the system.

4 FEP via numerical continuation method

The FEP backbones and their associated subharmonic resonance branches are generated here based on the numerical continuation method described in [47, 48]. All backbones and subharmonic branches are obtained at nonzero initial displacements and zero initial velocities of the system in Eq. (2). Therefore, we firstly compare in Fig. 6 the obtained backbone branches $S_{11}^+$ and $S_{13}^-$ of the LO-BNES system generated by the CX-A technique to the exact FEPs obtained by the numerical continuation method at the same physical parameters of the LO-BNES system used in Sect. 3 (The derivation of $S_{13}^-$ branch using CX-A is given in the appendix). The log-scale in energy axis is avoided in this figure to provide feasible comparison. The figure indicates an acceptable agreement between CX-A and the numerical continuation backbones.

The fundamental FEP backbone of 1:1 in-phase resonance $S_{11}^+$ and its associated tongues of subharmonic resonances are shown in Figs. 7 and 8. In
addition, several unsymmetrical (A–F) and symmetrical (G–I) periodic oscillation backbones are also shown in Fig. 7. The stability of periodic NNMs solution is analyzed by applying the Floquet’s theory where the numerical simulation periodic response is considered to be stable when two Floquet’s multipliers coincide at either +1 or −1 [1]. Therefore, the unstable portions of NNMs backbones and the subharmonic branches are obtained by the numerical continuation method and imposed into the FEP plot as shown in Fig. 7.

The symmetrical in-phase and anti-phase subharmonic resonance branches on the fundamental backbone shown in Fig. 8 appear near to the LO natural frequency ($f_{LO} = 0.1592$ Hz) and to $1/3$ of this frequency. These branches are associated with NNMs of $S13\pm, S15+$ and $S17\mp$. In addition, $S31\pm$ and $S51+$ are other bifurcated branches from the fundamental $S11+$ backbone as shown in the FEP. Samples of NNMs are shown in Fig. 8 where the LO oscillation is plotted in $x$-axis and the BNES oscillation in $y$-axis in NNMs configuration space. These NNMs are associated with full BNES oscillation through the stable and unstable equilibrium positions. Examples of time histories of the LO-BNES periodic response and the corresponding NNMs at $S51+$ and $S31−$ branches are plotted in Fig. 9. It is observed that the LO oscillates five times faster than the BNES as shown in Fig. 9a and three times faster as shown in Fig. 9c.

Another $S11+$ backbone curve of similar behavior of the fundamental $S11+$ backbone has been obtained and plotted in Fig. 10. Symmetrical in-phase and anti-phase subharmonic resonance branches are also obtained on this backbone as shown in the figure near to $1/3$ of the LO natural frequency where these branches are represented by $S13\pm$ and $S15+$. These
Fig. 6 Comparison of S11+ and S13− backbones in the FEP generated by the CX−A and numerical continuation methods of the LO–BNES at $\epsilon = 0.05, k = 0.14\text{N/m and } L_0 = 0.882L$ for $L = 1\text{m}$.

Fig. 7 The Floquet's stability analysis of the frequency–energy plot of the symmetrical and unsymmetrical NNM backbones of the LO–BNES system.
backbones have not been observed with Type I NES FEPs. Examples of periodic oscillation on these subharmonic branches and their associated NNMs are shown in Fig. 11 where full BNES oscillation through the equilibrium positions takes place.

Unlike the NES with purely nonnegative cubic coupling stiffness, the BNES is found to be associated with several unsymmetrical backbones ($U_{11}^+$ and $U_{11}^-/C_0$) of 1:1 resonance which appear to the left zone of $S_{11}^+$ and $S_{11}^-/C_0$ backbones at low energy as shown in the FEP in Fig. 12. It is also observed that at these unsymmetrical backbones, the periodic oscillation of the BNES mass at 1:1 resonance occurs only about one of its stable equilibrium positions ($z = \pm z_c$) which is called as in-well oscillation [30]. This means that, passing through the unstable equilibrium position at ($z = 0$) during this in-well periodic oscillation does not exist at these backbones. Furthermore, these unsymmetrical NNM backbones converge to specific frequency values near to their left ends where the oscillation becomes closely linear. Therefore, the linearization of the equations of motion in Eq. (4) about the stable equilibrium positions $z = \pm z_c$ is used to approximate the frequencies at the left side of these backbones. Accordingly, the following stiffness matrix of the linearized LO-BNES system is obtained

$$K = \begin{bmatrix}
    -\bar{K}_1 + 3\bar{K}_2 z_c^2 - 5\bar{K}_3 z_c^4 + 7\bar{K}_4 z_c^6 & -(-\bar{K}_1 + 3\bar{K}_2 z_c^2 - 5\bar{K}_3 z_c^4 + 7\bar{K}_4 z_c^6) \\
    -(-\bar{K}_1 + 3\bar{K}_2 z_c^2 - 5\bar{K}_3 z_c^4 + 7\bar{K}_4 z_c^6) & \nu_0^2 - \bar{K}_1 + 3\bar{K}_2 z_c^2 - 5\bar{K}_3 z_c^4 + 7\bar{K}_4 z_c^6
\end{bmatrix}$$

(11)

The natural frequencies of the linearized system are obtained from calculating the eigenvalues of $M^{-1/2}K M^{-1/2}$ where $M$ is the mass matrix of the LO-BNES system. The obtained natural frequencies are $f_A = 0.196$ Hz and $f_B = 0.1486$ Hz. The frequency $f_A$ represents the LO-BNES oscillation frequency near to the left end of $U_{11}^-$ backbone A where the corresponding exact one obtained from the FEP in Fig. 12 is $\bar{f}_A = 0.1945$ Hz. This frequency is nearly equivalent to the LO frequency. In addition, the frequency $f_B$ represents the LO-BNES oscillation frequency near to the left end of $U_{11}^+$ backbone B where the exact one obtained from the FEP is $\bar{f}_B = 0.1483$ Hz. Accordingly, the frequency relations between the $U_{11}^-$ backbones A and C, and $U_{11}^+$ backbones B, D, E and F are provided in Table 1.

Unlike the $S_{11}^+$ and $S_{11}^-$ backbones, the frequency on $U_{11}^+$ and $U_{11}^-$ backbones decreases with
increasing the energy. These interesting findings regarding to the unsymmetrical backbones show the rich nonlinear dynamical behavior of the BNES compared with the purely cubic stiffness Type I NES.

Examples of the periodic motion of the LO-BNES system at the points \( e, f, \) and \( g \) which are shown at \( U_{11}^+ \) and \( U_{11}^- \) backbones A, B and C in Fig. 12, respectively, are shown in Fig. 13. Both masses depict an anti-phase periodic oscillation at the \( U_{11}^- \) backbone A as shown from the response and the NNM in Fig. 13a, b, respectively. Similar observations are depicted from Fig. 13c, d of point \( f \) at \( U_{11}^+ \) backbone B and from Fig. 13e, f of point \( g \) at \( U_{11}^- \) backbone C. In these figures, the BNES mass exhibits an in-well oscillation about its left side stable equilibrium position where the passage through the unstable equilibrium position does not occur.

Fig. 9  Periodic motions in plots a and c and their corresponding NNMs in plots b and d of points a and b in Fig. 8, respectively, at the given initial conditions in the appendix (Table 2)
Finally, the damped dynamics of the LO-BNES system is investigated by superimposing the wavelet frequency spectrum (WFS) content of the NES relative displacement with respect to the LO on the full FEP. The WFS content is superimposed into the FEP for different initial conditions as shown in Fig. 14 by assuming that the BNES is initially setting at one of its stable equilibrium positions (i.e., $x_1(0) = z_c = 0.4712$). The optimized BNES parameters in [32] which are used in generating the FEPs in Sects. 3 and 4 have been obtained by optimizing the energy dissipation by the BNES at LO initial velocity of 0.1 m/s and zero BNES initial velocity. This optimized BNES was found to be providing higher performance in energy transfer and dissipation than the corresponding optimized Type I NES for a broadband energy range. The optimized BNES damping coefficient is $\lambda = 0.01$ N s/m where the optimized Type I NES damping and stiffness coefficients are $\lambda = 0.01$ N s/m and $k_{nes} = 1.6$ N/m$^3$, respectively, at LO damping coefficient $\lambda_p = 0.001$ N s/m. Accordingly, the TET of the optimized BNES and Type I NES is investigated and compared by superimposing the WFS content of the NES relative displacement on the numerical continuation FEPs. Therefore, at low, intermediate and relatively high input energies, the results are shown in Figs. 14 and 15 for the BNES and Type I NES, respectively. In these figures, the time response of the LO for NES locked and unlocked cases, the relative displacement response of the NES and the corresponding superimposed WFS on the FEP are shown at each considered initial velocity input to the LO. In Fig. 14a, c, at initial velocity $\dot{x}_2(0) = 0.1$ m/s of the LO mass, the maximum energy transfer and dissipation was achieved to be 99% of the input energy into the LO-BNES system [32]. This amount of energy is rapidly transferred and dissipated by the BNES nonlinear action. It is interesting to find that this TET takes place at multiple response captures with the left unsymmetrical NNM backbones, especially the stable portions of $U_{11}^{1}$ backbone $A$ and $U_{11}^{1} +$ backbone $B$. For the corresponding results of Type I NES in Fig. 15a, c, the TET is only achieved by single strong resonance capture with the $S_{11}^{*}$ + fundamental backbone. According to [1], the optimization energy of the NES parameters can be considered as a threshold energy for efficient TET. Therefore, above this threshold energy, significant portion of the induced energy into the LO is assumed to be rapidly absorbed by the nonlinear TET action and passively...
dissipated by the NES damping. Unlike Type I NES, at or below this threshold energy, the BNES maintains a nonlinear action by scattering energy through multiple resonance captures at the unsymmetrical NNM branches in the FEP. This highlights the contribution of these unsymmetrical NNM backbones in achieving higher TET by the BNES than Type I NES according to [32] for energies below or above the threshold energy. It is important to indicate that these unsymmetrical NNM backbones of the BNES have not been observed to exist with Type I NES. For an intermediate energy input induced into the LO at $\dot{x}_{2}(0) = 0.25 \text{ m/s}$, the time histories and the superimposed WFS of the BNES relative displacement on the FEP are shown in Fig. 14d–f for the BNES and in Fig. 15d–f for the corresponding Type I NES. It is also observed that the TET is still achieved at single resonance capture with the fundamental

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**Fig. 11** Periodic motions in plots a and c and their corresponding NNMs in plots b and d of points c and d in Fig. 10, respectively, of the second $S11+$ backbone at the given initial conditions in the appendix (Table A1)
S11 + backbone by Type I NES where the TET is achieved at multiple resonance captures by the BNES. The strongest resonance capture by the BNES is observed to take place on the unstable S13− subharmonic branch that bifurcates from the fundamental S11+ backbone. After significant energy decay, this resonance capture continues at the stable portion of the unsymmetrical U11+ backbone E. Another resonance captures are observed on the unstable portion of U11− backbone B. These resonance captures with unstable NNM backbone branches explain the observed small reduction in TET performance at initial conditions near to 0.25 m/s of the LO in [32]. Furthermore, at relatively high initial energy input induced into the LO at \(\dot{x}_2(0) = 0.4\) m/s, the high TET performance of the BNES is maintained due to the cascade of resonance captures as shown in Fig. 14g–i where it is deteriorated for Type I NES as shown in Fig. 15g–i at single resonance capture with the S11 + backbone. The TET at \(\dot{x}_2(0) = 0.4\) m/s is observed to take place through a strong resonance capture with the stable portion of the fundamental S11+ backbone. In addition, less intensity resonance

![Fig. 12 The frequency–energy plot of U11+, U11− and S11− backbones](image)

![Fig. 13 Periodic motions and their corresponding NNMs.](image)

### Table 1 The frequency relations between U11± FEP backbones

| Backbone Name                  | FEP frequency (Hz) | Linearized EOM Frequency |
|--------------------------------|--------------------|--------------------------|
| (U11−) backbone A             | \(f_A = 0.1945\)   | \(f_A = 0.1960\)          |
| (U11+) backbone B             | \(f_B = 0.1483\)   | \(f_B = 0.1486\)          |
| (U11−) backbone C             | \(f_C \approx \frac{1}{2}f_A = 0.0975\) | \(f_C = \frac{1}{2}f_A = 0.0973\) |
| (U11+) backbone D             | \(f_D \approx \frac{1}{2}f_B = 0.0742\) | \(f_D = \frac{1}{2}f_B = 0.0743\) |
| (U11+) backbone E             | \(f_E \approx \frac{1}{2}f_B = 0.0494\) | \(f_E = \frac{1}{2}f_B = 0.0495\) |
| (U11+) backbone F             | \(f_F \approx \frac{1}{4}f_B = 0.0370\) | \(f_F = \frac{1}{4}f_B = 0.0372\) |
Frequency–energy plot and targeted energy transfer

(a) LO mass displacement
- NES mass displacement
- Stable equilibrium positions

(b) NNM on U11- backbone A
- Stable equilibrium positions

(c) LO mass displacement
- NES mass displacement
- Stable equilibrium positions

(d) NNM on U11+ backbone B
- Stable equilibrium positions

(e) LO mass displacement
- NES mass displacement
- Stable equilibrium positions

(f) NNM on U11- backbone C
- Stable equilibrium positions
Table 2  The initial conditions of the NNMs

| Number of the Figure | $x_1(0)$  | $x_2(0)$  |
|----------------------|-----------|-----------|
| Figure 9 a, b        | 0.1671    | 0.3234    |
| Figure 9 c, d        | –0.1493   | –0.2165   |
| Figure 11 a, b       | –2.0401   | –0.6707   |
| Figure 11 c, d       | –2.3898   | –0.7957   |
| Figure 13 a, b       | –0.5780   | 0.0540    |
| Figure 13 c, d       | –0.0555   | 0.0218    |
| Figure 13 e, f       | –0.5500   | 0.0165    |

capture is observed to take place on the unstable portion of the secondary $S11+$ backbone. These resonance captures also expand to the left energy zone of the stable branches of the unsymmetrical NNM backbones. The results shown in Fig. 14 explain the BNES TET mechanism on the FEP where the role of unsymmetrical NNM backbones is clearly observed. In addition, the influence of resonance capture on unstable branches of the FEP backbones on slowing the TET is also observed. Moreover, the comparison between the BNES and Type I NES shows that the TET is more efficient at multiple resonance captures than a single resonance capture. At multiple resonance captures, the induced energy into the LO is immediately scattered by the BNES nonlinear action to different nonlinear frequency modes rather than a single mode resonance capture as in Type I NES case.

6 FEP based on the exact form of the BNES force

The analytical CX-A analysis and the numerical continuation method which have been employed for generating the FEPs in Sects. 3 and 4 are based on the truncated T-S expansion of the exact BNES force. This has been a common practice in the literature [1, 8] to approximate the exact force of the NES transverse coupling springs by a polynomial form using the T-S expansion. However, the exact force of the BNES in Fig. 2 is given as [32]

$$ (F_{ni})_{\text{exact}} = -2k_{z}\left(1 - L(z^2 + L_z^2)^{-1/2}\right) $$

(12)

Therefore, the FEP is also generated here based on the exact BNES force in Eq. (12) where the obtained FEP backbones are compared with those of the T-S expanded force in Eq. (1) as shown in Fig. 16. This comparison shows that both exact and T-S expanded forces reveal similar backbone curves of nearly similar nonlinear dynamical behavior. At nonlinear frequencies below the natural frequency of the LO, slight difference between the exact and approximate backbones is observed. In addition, the range of input energies induced into the LO-BNES system that previously applied in Fig. 14a, d, g can be considered according to the literature in [1–9] as of low, intermediate and high energy inputs, respectively. Consequently, the accuracy of the backbones of the T–S expanded BNES force can be acceptable within this range of input energies. Moreover, this noticeable difference is also affected by the log scale of the energy axis and the dependence of the frequency on the relative BNES amplitude. Even though employing CX-A with the exact force of the BNES could induce much more difficulties to analytically obtain the FEPs, the comparison in Fig. 16 suggests that using the exact force in the numerical continuation method should be preferred for high-frequency energy levels to guarantee sufficient accuracy and results validity. The same can be considered with Type I NES at high-frequency energy levels.

7 Concluded remarks

The Hamiltonian and damped dynamics of the BNES attachment with a linear oscillator are investigated using FEPs obtained by analytical and numerical methods where the fundamental in-phase and out-of-phase backbones have been obtained. The complexification-averaging (CX–A) technique is applied to the Hamiltonian system, and the characteristics of the transverse springs on the resulting FEPs are found to affect the stability and topology of the branches which indicates the importance of the exact physical realization of the system. Following that, numerical continuation techniques are utilized to generate exact FEPs of the Hamiltonian LO–BNES system which included multiple symmetrical and unsymmetrical in- and out-of-phase backbone branches. It is found that new 1:1 in-phase and out-of-phase periodic oscillations occur on unsymmetrical NNMs backbone branches. On these unsymmetrical NNM backbones, the BNES has been found to periodically oscillate at 1:1 resonance with the linear structure about either its

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left or right stable equilibrium positions. Moreover, passage of the BNES mass through its stable equilibrium position does not occur at these unsymmetrical NNMs backbones. The performed comparison between the BNES and Type I NES by superimposing the wavelet frequency spectrum on FEPs has revealed that the BNES achieves its TET by scattering energy at different frequency levels through multiple of resonance captures where this has not been observed with Type I NES. The benefit of multiple resonance

![Time histories of the LO-BNES system and the corresponding FEPs with the superimposed WFS of the BNES](image)

Fig. 14 Time histories of the LO-BNES system and the corresponding FEPs with the superimposed WFS of the BNES relative displacements in a–c at \( \dot{x}_2(0) = 0.1 \) m/s, in d–f at \( \dot{x}_2(0) = 0.25 \) m/s and in g–i at \( \dot{x}_2(0) = 0.4 \) m/s for \( x_1(0) = z_c \) and zeros of other initial conditions.
Fig. 15 Time histories of the LO with Type I NES and the corresponding FEPs with the superimposed WFS of the NES relative displacements in a–c at $\dot{x}_2(0) = 0.1$ m/s, in d–f at $\dot{x}_2(0) = 0.25$ m/s and in g–i at $\dot{x}_2(0) = 0.4$ m/s for $x_1(0) = z_c$ and zeros of other initial conditions.

$\dot{x}_2(0) = 0.1$ m/s:
- (a) Primary mass displacement
- (b) NES relative displacement
- (c) Frequency vs. Energy

$\dot{x}_2(0) = 0.25$ m/s:
- (d) Primary mass displacement
- (e) NES relative displacement
- (f) Frequency vs. Energy

$\dot{x}_2(0) = 0.4$ m/s:
- (g) Primary mass displacement
- (h) NES relative displacement
- (i) Frequency vs. Energy
captures is the immediate scattering of the input energy through different nonlinear frequency modes which makes the BNES outperforms Type I NES in TET.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Data availability statement No datasets are associated with this manuscript. The datasets used for generating the plots and results during the current study can be directly obtained from the numerical simulation of the related mathematical equations in the manuscript.

Appendix

In this section, we discuss the application of the CX-A method to determine the subharmonic periodic motions where the LO vibrates three times faster than the NES, i.e., branches S13 ±. Therefore, two fast frequencies, \( \omega \) and \( 3\omega \), are necessary for accurately modeling the periodic orbits [1]. Accordingly, considering the Hamiltonian version of the system in Eq. (4) in which \( \overline{T_p} = \overline{L} = 0 \) and \( \omega_0^2 = 1 \), the complexification of the dynamics is firstly performed by introducing four new complex variables

\[
\psi_1 = \dot{x}_{11} + j\omega x_{11} \\
\psi_2 = \dot{x}_{21} + j\omega x_{21} \\
\psi_3 = \dot{x}_{12} + 3j\omega x_{21} \\
\psi_4 = \dot{x}_{22} + 3j\omega x_{22}
\]

where \( \omega \) is the dominant fast frequency of oscillation. Following the same approach to derive the backbone branches S11 ±, we can express the complex variables defined in (5) in terms of fast oscillations of frequencies, \( e^{j\omega t} \) and \( e^{3j\omega t} \), modulated by slowly varying complex amplitudes \( \phi_i(t) \), \( i = 1,2,3,4 \) :

\[
\psi_{1,2}(t) = \phi_{1,2}(t)e^{j\omega t} \\
\psi_{3,4}(t) = \phi_{3,4}(t)e^{3j\omega t}
\]

By substituting Eqs. (6) and (5) into Eq. (4) and averaging of Eq. (7) with respect to the fast frequencies \( \omega \) and \( 3\omega \), a set of polar representations are introduced as

\[ Fig. 16 \] Comparison between the LO-BNES FEPs of the exact and T-S expanded forces of the unsymmetrical backbones \( U11 \pm (A-F) \) and the two fundamental \( S11 \pm \) backbones

Frequency–energy plot and targeted energy transfer
\[ \begin{align*}
\phi_1 &= Ae^{ix} \\
\phi_2 &= Be^{i\beta} \\
\phi_3 &= De^{i\gamma} \\
\phi_4 &= Ge^{i\delta}
\end{align*} \]

where \( A, B, D \) and \( G \) are real amplitudes, and \( x, \beta, \gamma \) and \( \delta \) are real phases. Following the same approach to derive the backbone branches \( S_{11} \pm \), the real and imaginary parts of the resulting equations can be separately set to zero and stationary conditions can be imposed on the modulation equations. Considering the trivial solution by assuming identity of phases \( (x = \beta = \gamma = \delta) \), the resulting equations are solved for \( A, B, D \) and \( G \) for each value of \( \omega \). Hence, given that where \( X_1 = \frac{A}{\omega} + \frac{D}{3\omega} \) and \( X_2 = \frac{B}{\omega} + \frac{G}{3\omega} \), the \( S_{13} \pm \) backbone branches can be plotted (Table 2).

The following table provides the initial conditions that have been used for generating the NNMs in Figs. 7, 9 and 11.

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