Testing Dark Energy and Cardassian Expansion for Causality

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Abstract

Causality principle is a powerful criterion that allows us to discriminate between what is possible or not. In this paper we study the transition from decelerated to accelerated expansion in the context of Cardassian and dark energy models. We distinguish two important events during the transition. The first one is the end of the matter-dominated phase, which occurs at some time $t_{eq}$. The second one is the actual crossover from deceleration to acceleration, which occurs at some $t_T$. Causality requires $t_T \geq t_{eq}$. We demonstrate that dark energy models, with constant $w$, and Cardassian expansion, are compatible with causality only if $(\Omega_M - \bar{q}) \leq 1/2$. However, observational data indicate that the most probable option is $(\Omega_M - \bar{q}) > 1/2$. Consequently, the transition from deceleration to acceleration in dark energy and Cardassian models occurs before the matter-dominated epoch comes to an end, i.e., $t_{eq} > t_T$. Which contradicts causality principle.

1 Introduction

Currently, there is a general agreement among cosmologists that: (i) the universe is speeding up, instead of slowing down; (ii) the accelerated expansion is a recent phenomenon; (iii) the universe is spatially flat; (iv) ordinary matter in the universe, including dark matter, can only account for 30% of the critical density. Evidence in favor of these results is provided by observations of high-redshift supernovae Ia [1]-[6], as well as other observations, including the cosmic microwave background and galaxy power spectra [7]-[13].

Since the gravity of both matter and radiation is attractive, the accelerated expansion requires either modified Einstein equations or, in the context of general relativity, the presence of a mysterious form of matter, which accounts for 70% of the total content of the universe and remains unclustered on all scales where gravitational clustering of ordinary matter is seen. Well-known theories that illustrate these alternatives are Cardassian expansion and dark energy models, respectively.

The universe was decelerating and dominated by matter and radiation in the past. The energy densities driving the current accelerated expansion became dominant only recently. In this paper, we distinguish to important events in the process of transition from deceleration to acceleration. The first event is the end of the matter-dominated phase, which occurs at some time $t_{eq}$. The second event is the actual crossover from deceleration to acceleration, which occurs at some $t_T$. Since the transition to an accelerated epoch is a consequence of the domination of the repulsive component, it follows that there is a causal connection between these two times; namely, $t_T \geq t_{eq}$. In terms of the redshift, this is equivalent to $z_T \leq z_{eq}$ because $z$ decreases monotonically with time.

The purpose of this paper is to test dark energy and Cardassian models for causality. We show that, in these models, the transition to an accelerated phase is not causal. Namely, we calculate $z_{eq}$ and $z_T$ (or equivalently $t_{eq}$ and $t_T$) and find that $z_{eq} < z_T$ ($t_T < t_{eq}$), which means the transition to an accelerated phase occurs during the matter-dominated phase, long before the source of acceleration starts to dominate over ordinary matter.

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2 End of the matter-dominated phase

In this section we write the equations of Cardassian and dark energy models in terms of the parameter $z_{eq}$, which marks the end of the matter-dominated phase. We calculate the acceleration of the universe at the end of this phase in terms density and deceleration parameters.

**Cardassian expansion:** In Cardassian models the FRW equation is modified from its usual form, $H^2 = (8\pi G/3)\rho$, to

$$H^2 = A\rho + B\rho^n,$$  \hspace{1cm} (1)

where $A$, $B$ and $n$ are constants. The second term causes accelerated cosmic expansion at late times if $n < 2/3$. \hspace{1cm} (2)

The auxiliary parameter $z_{eq}$ is the redshift at which the two terms in the r.h.s. of (1) become equal to each other, viz., $A\rho(z_{eq}) = B\rho^n(z_{eq})$. Thus,

$$B = A[z(z_{eq})]^{(1-n)}.$$  \hspace{1cm} (3)

Since $(n-1) < 0$, it follows that $A\rho > B\rho^n$ for $z > z_{eq}$ and $A\rho < B\rho^n$ for $z < z_{eq}$. We impose no restrictions on the parameter $z_{eq}$; we obtain its value from our study here.

Substituting (3) into (1) and evaluating today, we get

$$A = \frac{\dot{H}^2}{\bar{\rho} \left[1 + \rho(1-n)(z_{eq})\bar{\rho}^{(n-1)}\right]},$$  \hspace{1cm} (4)

where $\dot{H}$ and $\bar{\rho}$ represent the current values of the Hubble constant and matter density, respectively. Now, using (3) and (4) in (1), we find

$$H^2 = \frac{\dot{H}^2}{F} \left[1 + (F - 1) \left(\frac{\rho}{\bar{\rho}}\right)^{(n-1)}\right],$$  \hspace{1cm} (5)

where

$$F = 1 + \rho^{(1-n)}(z_{eq})\bar{\rho}^{(n-1)}.$$  \hspace{1cm} (6)

In Cardassian models, by assumption, there is no vacuum contribution: only radiation and ordinary matter\(^1\) contribute to the expansion of the universe. Thus, in units of the critical density, we have\(^2\) $\Omega_{Total} = \Omega_M + \Omega_R$, with $\Omega_M = \Omega_B + \Omega_{WIMP}$. Current observations suggest $\Omega_{WIMP} \approx 0.35$, $\Omega_B \approx 0.02h^{-2}$ and $\Omega_R \approx 2.5 \times 10^{-5}h^{-2}$, where $h$ is the Hubble constant today in units of 100 km/s/Mpc. Thus, $\Omega_B \approx 800\Omega_R$. Neglecting radiation we have $\rho = \rho_M = \bar{\rho}(1+z)^3$. Therefore, in Cardassian models the Friedmann equation for the expansion rate becomes

$$H^2 = \frac{\dot{H}^2}{F} \left[(1+z)^3 + (F - 1)(1+z)^{3n}\right],$$  \hspace{1cm} (7)

where $F$ now is

$$F(z_{eq}, n) = \frac{[1 + (1 + z_{eq})3(n-1)]}{(1 + z_{eq})3(n-1)}.$$  \hspace{1cm} (8)

So far there is no unique explanation for the origin of the term $B\rho^n$ in (1). It may arise either as a consequence of embedding our universe as a brane in higher dimensions, or from some (yet unknown) modified Einstein’s equation.

\(^1\)Ordinary matter includes baryonic and non-baryonic dark matter.

\(^2\)Here $\Omega_M$, $\Omega_R$, $\Omega_B$ and $\Omega_{WIMP}$ denote the density parameters of ordinary matter, radiation, baryonic matter and non-baryonic dark matter, respectively.
**Dark energy:** Within the context of four-dimensional general relativity, the source of cosmic acceleration is usually called dark energy. The simplest candidate for dark energy is the cosmological constant \( \Lambda \) \cite{14-15}. In this approach, \( \Lambda \) is introduced by “hand” as a parameter in Einstein’s theory of gravity. However, if \( \Lambda \) remains constant one faces the problem of fine tuning or “cosmic coincidence problem” \cite{16}, which refers to the coincidence that \( \rho_\Lambda \) and \( \rho_M \) are of the same order of magnitude today.

A phenomenological solution to this problem is to consider a time dependent cosmological term, or an evolving scalar field known as quintessence \cite{16}-\cite{19}. In these models, the dark energy component can be considered to be a smooth fluid characterized by the equation of state \( p_D = w \rho_D \), where \( D \) stands for dark energy and \( w \) may vary with time. If the dark energy is the vacuum energy, i.e. \( \Lambda \), then \( w = -1 \).

Neglecting radiation, the Friedmann equation in dark energy models with constant \( w \) is given by

\[
H^2 = \frac{\Omega_M (1+z)^3 + \Omega_D (1+z)^{3(1+w)}}{1 + (1+z)^{3w}},
\]

where

\[
-1 \leq w < -1/3.
\]

In this range, dark energy violates the strong energy condition, but satisfies the dominant energy condition\(^3\).

In terms of \( z_{eq} \), the density parameters \( \Omega_M \) and \( \Omega_D \), can be written as

\[
\Omega_M = \frac{(1 + z_{eq})^{3w}}{1 + (1 + z_{eq})^{3w}}, \quad \Omega_D = \frac{1}{1 + (1 + z_{eq})^{3w}}.
\]

Therefore, the evolution equation (9) becomes

\[
H^2 = \frac{\Omega_M (1+z)^3 + \Omega_D (1+z)^{3(1+w)}}{1 + (1+z)^{3w}},
\]

If we compare (9) with (11) we see that Cardassian and quintessence models are mathematically equivalent to each other. The following identification connects the two models

\[
n = 1 + w, \quad \Omega_M = \frac{1}{F}, \quad \Omega_D = \frac{F - 1}{F}, \quad F(z_{eq}, w) = \frac{[1 + (1 + z_{eq})^{3w}]}{(1 + z_{eq})^{3w}}.
\]

**Deceleration parameter at the end of matter-dominated phase:** The deceleration parameter, \( q = \ddot{a}/a^2 \), in both Cardassian and quintessence models is given by

\[
q(z, z_{eq}, w) = \frac{[1 + (F - 1)(3w + 1)(1 + z)^{3w}]}{2[1 + (F - 1)(1 + z)^{3w}]}.
\]

Evaluating this expression at \( z_{eq} \) we obtain \( q_{z_{eq}} \), the deceleration at the end of the matter-dominated phase. Using (13) and (14), we find

\[
q_{z_{eq}}(w) = \frac{2 + 3w}{4}.
\]

Now, the current cosmic acceleration \( \ddot{q} \) is obtained from (14) evaluated at \( z = 0 \). Since \( F = 1/\Omega_M \), we get\(^4\)

\[
\ddot{q}(\Omega_M, w) = \frac{3w(1 - \Omega_M) + 1}{2}.
\]

Consequently, \( q_{z_{eq}} \) can be written as

\[
q_{z_{eq}}(\Omega_M, \ddot{q}) = \frac{1 - 2(\Omega_M - \ddot{q})}{4(1 - \Omega_M)}.
\]

Thus, knowing the values of \( \Omega_M \) and \( q \) today we can predict the cosmic acceleration at the end of the matter-dominated phase.

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\(^3\)The case \( w < -1 \), which corresponds to what is called Phantom fields, violate both energy conditions.

\(^4\)We note that accelerated expansion \( \ddot{q} < 0 \) requires \( w < (-0.370, -0.417, -0.476) \), for \( \Omega_M = (0.1, 0.2, 0.3) \), respectively.
3 Causality

Let us now study causality in these models. First, the redshift of transition from deceleration to acceleration, which we denote as $z_T$, is the solution of $q(z_T) = 0$. From (14) we get

$$
\frac{1 + z_T}{1 + z_{eq}} = f(w),
$$

(18)

with

$$
f(w) = \left(\frac{-1}{1 + 3w}\right)^{1/3}.
$$

(19)

Certainly, $f(w)$ must be positive for all values of $z$ and $z_{eq}$. This requires $w < -1/3$, which is compatible with the condition $n < 2/3$ for acceleration in Cardassian models (2). On the other hand, for causality reasons, it is clear that the timing of $z_{eq}$ must be earlier (or not later than) the one for $z_T$. In other words, causality requires $z_{eq} \geq z_T$, i.e., $f(w) \leq 1$, which in turn demands $w \geq -2/3$. This implies that the universe was decelerating at the end of the matter-dominated era, i.e., $q_{z_{eq}} \geq 0$.

Thus, collecting results we have

$$
z_T = f(w)z_{eq} + [f(w) - 1],
$$

(20)

where $z_{eq}$, from (13), is

$$
z_{eq} = \left(\frac{\Omega_M}{1 - \Omega_M}\right)^{1/3} - 1,
$$

(21)

and the values of $w$ compatible with causality are

$$
-2/3 < w < -1/3,
$$

(22)

or equivalently $1/3 < n < 2/3$. For other values the transition is not causal.

**Vacuum energy:** As an example, let us consider the models where the dark energy is the cosmological constant (i.e., $w = -1$). From (20) and (21) we find

$$
z_T - z_{eq} = \left(\frac{1 - \Omega_M}{\Omega_M}\right)^{1/3} (2^{1/3} - 1) > 0.
$$

(23)

If we take $\Omega_M = 0.3$, then

$$
q_{z_{eq}} = -1/2, \quad z_{eq} \approx 0.326, \quad t_{eq} \approx 0.702/\bar{H}, \quad z_T \approx 0.671, \quad t_T \approx 0.534/\bar{H}.
$$

(24)

According to this, in models with vacuum energy (or Cardassian expansion with $n = 0$), the universe starts accelerating long before ($t_T \approx 0.534/\bar{H}$) the matter-dominated era comes to an end ($teq \approx 0.702/\bar{H}$).

3.1 Cosmological parameters and causality

Let us find the condition for causality in terms of $\Omega_M$ and $\bar{q}$. From (10) we obtain

$$
w = \frac{(2\bar{q} - 1)}{3(1 - \Omega_M)}.
$$

(25)

Substituting this expression into (19), (20) and (21), we find

$$
z_{eq} = \left(\frac{1 - \Omega_M}{\Omega_M}\right)^{(1-\Omega_M)/(1-2\bar{q})} - 1,
$$

$$
z_T = \left(\frac{\Omega_M - 2\bar{q}}{\Omega_M}\right)^{(1-\Omega_M)/(1-2\bar{q})} - 1.
$$

(26)

\footnote{Since $dz/dt = -(1 + z)H$, and $H > 0$, it follows that $z$ decreases \textit{monotonically} with time. In other words, if $t_2 > t_1$, then $z_2 < z_1$.}
Now, causality condition $z_{eq} > z_T$, requires

$$\left(\Omega_M - \bar{q}\right) < \frac{1}{2}.$$  \hspace{1cm} (27)

Which, by virtue of (17) implies that the universe is decelerating at the end of the matter dominated era.

### 3.2 Fitting observational data for $q$

For $\Omega = 0.3$ causality requires $\bar{q} \geq -0.2$. However, all estimates of the deceleration parameter today; direct, indirect and theoretical, predict $\bar{q} < -0.2$.

Direct determination of $q$, from high red-shift supernovae, has recently been provided by John [20] and Daly and Djorgovski [21]. John obtains $\bar{q} \approx -0.77$ in a model-independent cosmographic evaluation of SNe data, without reference to the specific cosmic inventory of energy densities. Daly and Djorgovski obtain $\bar{q} = -0.35 \pm 0.15$ directly from combinations of the first and second derivatives of the coordinate distance and observational data, without invoking any theory of gravity.

Indirect evidence about $q$ can be obtained from studies of the equation of state for dark energy. Current observations indicate [1]-[6], [22]-[26]

$$-1.48 < w < -0.72,$$

or equivalently $-1 < \bar{q} < -0.26$, which is outside of the range permitted by causality.

Theoretical models of quintessence, in the framework of flat FRW cosmologies, like tracker fields [27], indicate $w \approx -0.7$. Similarly, in braneworld theories, where our universe is embedded in a bulk with more than four dimensions, the predicted values for $\bar{q}$ are in the interval $(-0.50, -0.32)$ [28]-[30].

### 4 Conclusion

From the above discussion it follows that the most probable option is $(\Omega_M - \bar{q}) > 1/2$. Consequently, in these models the transition from deceleration to acceleration occurs before the matter-dominated epoch comes to an end, i.e., $t_{eq} > t_T$. Which contradicts the most basic of all the principles of physics, namely, the causality principle. A clear example of this is provided by the cosmological constant [24]. We advise in-depth theoretical work and a reexamination of observational basics.

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### References

[1] A.G. Riess *et al.*, Supernova Search Team Collaboration, *Astron. J.*, 116, 1009 (1998); astro-ph/9805201

[2] S. Perlmutter *et al.*, Supernova Cosmology Project Collaboration, *Astrophys. J.*, 517, 565 (1999); astro-ph/9812133

[3] Andrew R Liddle, *New Astron. Rev.*, 45, 235(2001), astro-ph/0009491

[4] N. Seto, S. Kawamura and T. Nakamura, *Phys.Rev.Lett.* 87, 221103(2001), astro-ph/0108011

[5] R. A. Knop *et al*, *Astrophys. J.*, 598, 102(2003); astro-ph/0309368

[6] J.L. Tonry et al., *Astrophys. J.*, 594, 1 (2003); astro-ph/0305008

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6The difference between results in these two papers, which are model-independent, is mainly due to technical reasons. Namely, John expands the scale factor about zero redshift into a polynomial of order 5, while Daly and Djorgovski expand about a redshift that systematically increases. However, if we add a fractional error bar to the average value of $\bar{q} = -0.77$ obtained by John, we find $\bar{q} = -0.77 \pm 0.33$, which is easily within one $\sigma$ of the result published by Daly and Djorgovski.
[7] A.T. Lee et al, *Astrophys. J.*, **561**, L1(2001); astro-ph/0104459
[8] R. Stompor et al, *Astrophys. J.*, **561**, L7(2001); astro-ph/0105062
[9] N.W. Halverson et al, *Astrophys. J.*, **568**, 38(2002); astro-ph/0104489
[10] C.B. Netterfield et al, *Astrophys. J.*, **571**, 604(2002); astro-ph/0104460
[11] C. Pryke, *et al.*, *Astrophys. J.*, **568**, 46(2002); astro-ph/0104490
[12] D.N. Spergel *et al.*, *Astrophys. J.Suppl.*, **148**, 175 (2003); astro-ph/0302209
[13] J. L. Sievers, *et al.*, *Astrophys. J.*, **591**, 599(2003); astro-ph/0205387
[14] P. J. E. Peebles and B. Ratra, *Rev.Mod.Phys.*, **75**, 559(2003); astro-ph/0207347
[15] T. Padmanabhan, *Phys.Rept.* **380**, 235(2003), hep-th/0212290
[16] I. Zlatev, L Wang and P. J. Steinhardt, *Phys.Rev.Lett.* **82**, 896(1999), astro-ph/9807002
[17] C. Armendariz, V. Mukhanov, P. J. Steinhardt, *Phys.Rev.Lett.* **85**, 4438(2000), astro-ph/0004134
[18] R.R. Caldwell, R. Dave, P. J. Steinhardt, *Phys.Rev.Lett.* **80**, 1582(1998), astro-ph/9708069
[19] S.E. Deustua, R. Caldwell, P. Garnavich, L. Hui, A. Refregier, “Cosmological Parameters, Dark Energy and Large Scale Structure”, astro-ph/0207293
[20] M. John, “Cosmographic evaluation of deceleration parameter using SNe Ia data”, astro-ph/0406444 To be published in The Astrophysical Journal.
[21] Ruth A. Daly and S. G. Djorgovski, “Direct Determination of the Kinematics of the Universe and Properties of the Dark Energy as Functions of Redshift”, astro-ph/0403664 Will appear in the Astrophysical Journal, v612, September 10, 2004.
[22] U. Alam, V. Sahni, T. Deep Saini and A. A. Starobinsky, *Mon.Not.Roy.Astron.Soc.* **354**, 275(2004); astro-ph/0311364
[23] T. Roy Choudhury and T. Padmanabhan, “Cosmological parameters from supernova observations: A critical comparison of three data sets”, astro-ph/0311622
[24] J. S. Alcaniz and N. Pires, *Phys.Rev.* **D70**, 047303(2004); astro-ph/0404146
[25] Ruth A. Daly and S. G. Djorgovski, “Direct Constraints on the Properties and Evolution of Dark Energy”, astro-ph/0405550
[26] P.S. Corasaniti, M. Kunz, D. Parkinson, E.J. Copeland and B.A. Bassett, *Phys.Rev.* **D70**, 083006(2004); astro-ph/0406608
[27] P.J. Steinhardt, L. Wang and I. Zlatev, *Phys.Rev.* **D59**, 123504(1999), astro-ph/9812313
[28] J. Ponce de Leon, *Class.Quant.Grav.* **20**, 5321(2003); gr-qc/0305041
[29] J. Ponce de Leon, “Accelerated expansion from braneworld models with variable vacuum energy” Will appear in *Gen. Rel. Gravit.* **37**, January 2005; gr-qc/0401026
[30] J. Ponce de Leon, “Transition from decelerated to accelerated cosmic expansion in braneworld universes”, gr-qc/0412005

