SUBLATTICES OF FINITE INDEX

CHUNLEI LIU

Abstract. Assuming the Gowers Inverse conjecture and the Möbius conjecture for the finite parameter $s$, Green-Tao verified Dickson’s conjecture for lattices which are ranges of linear maps of complexity at most $s$. In this paper, we reformulate Green-Tao’s theorem on Dickson’s conjecture, and prove that, if $L$ is the range of a linear map of complexity $s$, and $L_1$ is a sublattice of $L$ of finite index, then $L_1$ is the range of a linear map of complexity $s$.

1. Introduction

A lattice in an Euclidean vector space is an additive discrete subgroup other than \{0\}. For example, $q \mathbb{Z}$ ($0 \neq q \in \mathbb{Z}$) is a sublattice of $\mathbb{Z}$, the set $\mathbb{Z}(1, 1) = \{(n, n) \mid n \in \mathbb{Z}\}$ is a sublattice of $\mathbb{Z}^2$, the set $\mathbb{Z}(1, 0, -1) + \mathbb{Z}(0, 1, -1) = \{(n_1, n_2, -n_1 - n_2) \mid n_1, n_2 \in \mathbb{Z}^2\}$ is a sublattice of $\mathbb{Z}^3$, and

$\mathbb{Z}(1, 1, \cdots, 1) + \mathbb{Z}(0, 1, \cdots, t - 1) = \{(n, n + r, \cdots, n + (t - 1)r) \mid n, r \in \mathbb{Z}\}$,

which is the set of vectors $(n_1, \cdots, n_t)$ in $\mathbb{Z}^t$ such that $n_1, \cdots, n_t$ form an arithmetic progression, is a sublattice of $\mathbb{Z}^t$.

An affine sublattice of $\mathbb{Z}^t$ is a translation of a sublattice of $\mathbb{Z}^t$. For example, $q \mathbb{Z} + a$ ($q, a \in \mathbb{Z}, q \neq 0$), which is an arithmetic progression, is an affine sublattice of $\mathbb{Z}$, the set $\mathbb{Z}(1, 1) + (0, 2) = \{(n, n + 2) \mid n \in \mathbb{Z}\}$ is an affine sublattice of $\mathbb{Z}^2$, and the set $\mathbb{Z}(1, 0, -1) + \mathbb{Z}(0, 1, -1) + (0, 0, N) = \{(n_1, n_2, N - n_1 - n_2) \mid n_1, n_2 \in \mathbb{Z}^2\}$ is an affine sublattice of $\mathbb{Z}^3$. Note that a one-point subset of $\mathbb{Z}^t$ is not an affine sublattice of $\mathbb{Z}^t$ since we have excluded \{0\} from being a lattice.

A subset of $\mathbb{Z}^t$ is called (Zariski) closed if it is the set of common zeros in $\mathbb{Z}^t$ of a system of polynomials in $t$-variables. For example, the set of integral points on the $X_1$-axis of the Euclidean plane $\mathbb{R}^2$ is a Zariski closed subset of $\mathbb{Z}^t$ since the $X_1$-axis is defined by the equation $x_2 = 0$. In general, since a hyperplane orthogonal to the $X_t$-axis of $\mathbb{R}^t$ is defined by an equation of the form $x_i = a_i$ for some $a_i \in \mathbb{R}$, the set of integral points in

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such a hyperplane is a Zariski closed subset of \( \mathbb{Z}' \). It is an easy exercise to show that a subset of \( \mathbb{Z} \) is Zariski closed if and only if it is \( \mathbb{Z} \) or finite.

Let \( A \) be a subset of \( \mathbb{Z}' \). A subset \( B \) of \( A \) is said to be (Zariski) dense in \( A \) if every Zariski closed subset of \( \mathbb{Z}' \) which contains \( B \) contains \( A \). For example, a subset of \( \mathbb{Z} \) is Zariski dense in \( \mathbb{Z} \) if and only if it is infinite.

Following Bourgain-Gamburd-Sarnak [BGS], we denote by \( \mathcal{P} = \{ \pm 2, \pm 3, \pm 5, \pm 7, \cdots \} \) the set of integers which are prime numbers up to a sign, and call a point in \( \mathbb{Z}' \cap \mathcal{P} \) a prime point. The Bourgain-Gamburd-Sarnak’s formulation of Dickson’s conjecture [BGS], asks when an affine sublattice of \( \mathbb{Z}' \) has a Zariski dense subset of prime points.

By the prime number theorem, \( \mathcal{P} \) is Zariski dense in \( \mathbb{Z} \). Let \( q \neq 0, b \) be integers. By the prime number theorem in arithmetic progressions, \( (q\mathbb{Z} + b) \cap \mathcal{P} \) is Zariski dense in \( q\mathbb{Z} + b \) if and only if \( \gcd(q, b) = 1 \).

Let \( L \) be an affine sublattice of \( \mathbb{Z}' \). If \( t > 1 \) and the projection of \( L \) to some axis of \( \mathbb{Z}' \) is a constant prime point of \( \mathbb{Z} \), then we call \( L \) degenerate. For example, the affine sublattice \( \{(-2, 3, 2n + 1) \mid n \in \mathbb{Z}\} \) is a degenerate lattice of \( \mathbb{Z}' \) since its projection to the first axis is the constant prime point \(-2\) of \( \mathbb{Z} \). Let \( A \) be the set of numbers \( i \) from \( 1, 2, \cdots, t \) such that the projection of \( L \) to the \( X_t \)-axis of \( \mathbb{Z}' \) is a constant prime point of \( \mathbb{Z} \). It is a proper subset of \( \{1, 2, \cdots, t\} \) since \( L \) contains more than one point. It is easy to see that the projection of \( L \) to \( \prod_{1 \leq j \leq t, j \notin A} \mathbb{Z} \) is a non-degenerate affine sublattice of \( \prod_{1 \leq j \leq t, j \notin A} \mathbb{Z} \), we call it the non-degenerate sublattice associated to \( L \). For example, the affine sublattice \( 2\mathbb{Z} + 1 \) is a non-degenerate affine sublattice of \( \mathbb{Z} \) associated to the degenerate affine sublattice \( \{(-2, 3, 2n + 1) \mid n \in \mathbb{Z}\} \) of \( \mathbb{Z}' \).

Let \( p \) be a prime number, and \( \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z} \) be the field of \( p \) elements. If \( L \) is a non-degenerate affine sublattice of \( \mathbb{Z}' \), then we denote by \( L_p \) the image of \( L \) under the natural map \( \mathbb{Z}' \to \mathbb{Z}'_p \), and call the quantity

\[
\alpha_p(L) := \left(\frac{p}{p - 1}\right)^t |L_p \cap (\mathbb{Z}'_p)^t| / |L_p|
\]

the local factor of \( L \) at \( p \). If \( L \) is a degenerate affine sublattice of \( \mathbb{Z}' \), and \( L^* \) the associated non-degenerate sublattice, then we call the quantity \( \alpha_p(L) := \alpha_p(L^*) \) the local factor of \( L \) at \( p \).

**Lemma 1.1.** Let \( L \) be an affine sublattice of \( \mathbb{Z}' \). If \( L \cap \mathcal{P} \) is Zariski dense in \( L \), then all local factors of \( L \) are nonzero.

**Proof.** We may assume that \( L \) is a non-degenerate. Suppose that \( L \cap \mathcal{P} \) is Zariski dense in \( L \), but there is a prime number \( p \) such that \( \alpha_p(L) = 0 \). Then \( L_p \cap (\mathbb{Z}'_p)^t = \emptyset \). So \( p |(n_1 \cdots n_t) \) whenever \( (n_1, \cdots, n_t) \in L \). It follows that if \( (n_1, \cdots, n_t) \in L \cap \mathcal{P} \), then \( n_i = \pm p \) for some \( i \). That is any \( \vec{n} \in L \cap \mathcal{P} \) must lie on one of the hyperplanes \( x_i = \pm p, i = 1, \cdots, t \). So \( L \cap \mathcal{P} \) is contained in the union of these hyperplanes, which is Zariski closed. As \( L \cap \mathcal{P} \) is Zariski dense in \( L \), \( L \) is also contained in that union. It follows that \( L \) must be contained in one of the hyperplanes, contradicting the non-degenerateness of \( L \). The lemma is proved.

\[\square\]
Justified by that lemma, we call the vanishing of $\alpha_p(L)$ a local obstruction at $p$ to the Zariski density of $L \cap \mathcal{P}^t$ in $L$.

Let $q \neq 0, b$ be integers. Then the local obstructions to the Zariski density of prime points in $q\mathbb{Z} + b$ occur precisely at prime factors of $\gcd(q, b)$. It follows that all the local obstructions to the Zariski density of prime points in $q\mathbb{Z} + b$ are passed if and only if $\gcd(q, b) = 1$. So, by the prime number theorem in arithmetic progressions, the set of prime points in $q\mathbb{Z} + b$ is Zariski dense in $q\mathbb{Z} + b$ if and only if all local obstructions for $q\mathbb{Z} + b$ to have a Zariski dense subset of prime points are passed.

**Conjecture 1.2** (Dickson’s conjecture, Bourgain-Gamburd-Sarnak’s formulation [BGS]).

Let $L$ be an affine sublattice of $\mathbb{Z}^t$. If all local obstructions to the Zariski density of prime points in $L$ are passed, then the set of prime points in $L$ is Zariski dense in $L$.

Let $L$ the sublattice of $\mathbb{Z}^t$ consisting of vectors $(n_1, \ldots, n_t)$ such that $n_1, \ldots, n_t$ form an arithmetic progression. Then for every prime number $p$, $(1, 1+p, 1+2p, \ldots, 1+(t-1)p)$ gives rise to a point in $L_p \cap (\mathbb{Z}_p^\times)^t$. It follows that all local obstructions to the Zariski density of prime points in $L$ are passed. According to Dickson’s conjecture, the prime points in $L$ should be Zariski dense in $L$. This is now a marvelous theorem of Green-Tao [GT4].

**Theorem 1.3** (Green-Tao [GT4]). Let $L$ the sublattice of $\mathbb{Z}^t$ consisting of vectors $(n_1, \ldots, n_t)$ such that $n_1, \ldots, n_t$ form an arithmetic progression. Then the set of prime points in $L$ is Zariski dense in $L$.

Following Bourgain-Gamburd-Sarnak [BGS], we define the von Mangoldt function $\Lambda$ on $\mathbb{Z}$ by setting $\Lambda(n) = \log p$ when $\pm n$ is a power of a prime $p$, and $\Lambda(n) = 0$ otherwise. We define the $t$-dimensional von Mangoldt function on $\mathbb{Z}^t$ by setting

$$\Lambda^{\otimes t}(n_1, \ldots, n_t) = \prod_{i=1}^t \Lambda(n_i).$$

Let $L$ be a sublattice of $\mathbb{Z}^t$. The classical Dickson’s conjecture for $L$ predicts that, if $N$ is large, $K \subseteq [-N, N]^t$ is convex and of volume $\gg N^t$, and $a \in \mathbb{Z}^t$ is of size $O(N)$ such that all local obstructions to the Zariski density of prime points in $a + L$ are passed, then the quantity

$$\frac{1}{|L \cap K|} \sum_{\vec{n} \in L \cap K} \Lambda^{\otimes t}(a + \vec{n}),$$

which is the expectation of the $t$-dimensional von Mangoldt function on $a + L \cap K$, is asymptotically $\prod_p \alpha_p(a + L)$, which, by Lemma 1.3 of [GT3], is always convergent.

**Conjecture 1.4** (Dickson’s conjecture). Let $N$ be a variable. If $N$ is large, $K \subseteq [-N, N]^t$ is convex and of volume $\gg N^t$, and $a \in \mathbb{Z}^t$ is of size $O(N)$ such that all local obstructions to the Zariski density of prime points in $a + L$ are passed. Then

$$\frac{1}{|L \cap K|} \sum_{\vec{n} \in L \cap K} \Lambda^{\otimes t}(a + \vec{n}) = \prod_p \alpha_p(a + L) + o(1).$$
Green-Tao [GT3] also made great advances in the study of Dickson’s conjecture for a general sublattice \( L \). We proceed to introduce their main result in this direction.

A map \( \Psi \) from \( \mathbb{Z}^d \) to \( \mathbb{Z}' \) is called linear if it is a group homomorphism, is called affine-linear if \( \Psi := \Psi - \Psi(0) \) is linear. A linear map from \( \mathbb{Z}^d \) to \( \mathbb{Z} \) is called a linear form on \( \mathbb{Z}^d \). And an affine-linear map from \( \mathbb{Z}^d \) to \( \mathbb{Z} \) is called an affine-linear form on \( \mathbb{Z}^d \).

If \( \psi_1, \ldots, \psi_t \) is a system of affine-linear forms on \( \mathbb{Z}^d \), then
\[
(\psi_1, \ldots, \psi_t)(\vec{n}) := (\psi_1(\vec{n}), \ldots, \psi_t(\vec{n}))
\]
is an affine-linear map from \( \mathbb{Z}^d \) to \( \mathbb{Z}' \). Conversely, if \( \Psi : \mathbb{Z}^d \to \mathbb{Z}' \) is an affine-linear map, then there is a unique system \( \psi_1, \ldots, \psi_t \) of affine-linear forms on \( \mathbb{Z}^d \) such that \( \Psi = (\psi_1, \ldots, \psi_t) \). Moreover, \( \Psi \) is linear if and only if all \( \psi_i \) (\( 1 \leq i \leq t \)) are linear.

If \( \Psi : \mathbb{Z}^d \to \mathbb{Z}' \) is a linear map, then the range \( \Psi(\mathbb{Z}^d) \) is a sublattice of \( \mathbb{Z}' \). Conversely, if \( L \) is a sublattice of \( \mathbb{Z}' \), then there is a linear map \( \Psi : \mathbb{Z}^d \to \mathbb{Z}' \) such that \( L = \Psi(\mathbb{Z}^d) \). Similarly, if \( \Psi : \mathbb{Z}^d \to \mathbb{Z}' \) is an affine-linear map, then the range \( \Psi(\mathbb{Z}^d) \) is an affine sublattice of \( \mathbb{Z}' \). And, if \( L \) is an affine sublattice of \( \mathbb{Z}' \), then there is an affine-linear map \( \Psi : \mathbb{Z}^d \to \mathbb{Z}' \) such that \( L = \Psi(\mathbb{Z}^d) \).

**Definition 1.5** (Green-Tao partition). Let \( \psi_1, \ldots, \psi_t \) be a system of linear forms on \( \mathbb{Z}^d \), and \( \Psi = (\psi_1, \ldots, \psi_t) \). A Green-Tao partition of size \( s \geq 0 \) of \( \Psi \) at \( i \) (\( 1 \leq i \leq t \)) is a partition of the \( t - 1 \) forms \( \{\psi_j : j \neq i\} \) into \( s + 1 \) classes such that \( \psi_i \) does not lie in the \( \mathbb{Q} \)-linear span of any class.

**Lemma 1.6** (Green-Tao, Lemma 1.6 in [GT3]). Let \( \psi_1, \ldots, \psi_t \) be a system of linear forms on \( \mathbb{Z}^d \), and \( \Psi = (\psi_1, \ldots, \psi_t) \). Let \( 1 \leq i \leq t \). A Green-Tao partition of \( \Psi \) at \( i \) exists if and only if \( \psi_i \) is not a rational multiple of any other form.

**Proof.** First suppose that \( \psi_i \) is not a rational multiple of any other form. Then the singleton partition \( \{\psi_j : j \neq i\} = \cup_{j \neq i} \{\psi_j\} \) is a Green-Tao partition of \( \Psi \) at \( i \). Secondly suppose that \( \psi_i \) is a rational multiple of some other form, say \( \psi_{i_0} \), and \( \{\psi_j : j \neq i\} = \cup_{j \neq i} A_k \) is any partition. Then \( \psi_i \) must lie in the \( \mathbb{Q} \)-linear span of \( A_k \) which contains \( \psi_{i_0} \). Therefore that partition is not a Green-Tao partition of \( \Psi \) at \( i \).

**Definition 1.7** (Green-Tao complexity). Let \( \psi_1, \ldots, \psi_t \) be a system of linear forms on \( \mathbb{Z}^d \), and \( \Psi = (\psi_1, \ldots, \psi_t) \). Let \( 1 \leq i \leq t \). If \( \psi_i \) is not a rational multiple of any other form, the \( i \)-complexity of \( \Psi \) is defined to be the least of the sizes of all Green-Tao partitions of \( \Psi \) at \( i \). If \( \psi_i \) is a rational multiple of some other form, the \( i \)-complexity of \( \Psi \) is defined to be \( \infty \). The Green-Tao complexity of \( \Psi \) is defined to be the largest of all \( i \)-complexities of \( \Psi \) with \( 1 \leq i \leq t \).

We reformulate Green-Tao’s theorem on Dickson’s conjecture in [GT3] as follows.

**Theorem 1.8** (Green-Tao, [GT3]). Let \( \Psi : \mathbb{Z}^d \to \mathbb{Z}' \) be a linear map of complexity at most \( s < \infty \). Assume that the Gowers Inverse conjecture GI(s) and the M"obius and nilsequences conjecture MN(s) are true for the parameter \( s \). Then Dickson’s conjecture is true for \( \Psi(\mathbb{Z}^d) \).
The conjectures GI(s) and MN(s) were formulated by Green-Tao [GT3]. The conjecture GI(1) is easy. Green-Tao [GT2] reduced the conjecture MN(1) to a classical result of Davenport [Da]. But, according to Green-Tao [GT3], MN(1) was essentially already present in the work of Hardy-Littlewood [HL] and Vinogradov [Vi]. The conjectures GI(2) and MN(2) were also proved by Green-Tao [GT1, GT2].

The above reformulation of Green-Tao’s theorem on Dickson’s conjecture will be proved in the next section. It needs some knowledge of the local factors of affine sublattices.

To state the main result of this paper, we introduce the following definition.

**Definition 1.9.** Let $T : \mathbb{Z}^d \to \mathbb{Z}^d$ be a linear map. We call $T$ complexity preserving if for every linear map $\Psi : \mathbb{Z}^d \to \mathbb{Z}^t$, the Green-Tao complexities of $\Psi$ and $\Psi \circ T$ are equal.

The following theorem is the main result of this paper.

**Theorem 1.10.** Let $\Psi : \mathbb{Z}^d \to \mathbb{Z}^t$ be a linear map, and $L$ be a sublattice of $\Psi(\mathbb{Z}^d)$ of finite index. Then there is a complexity-preserving linear map $T : \mathbb{Z}^d \to \mathbb{Z}^d$ such that $T(\mathbb{Z}^d) = \Psi^{-1}(L)$.

One can infer the following corollary from Theorems 1.10 and 1.8.

**Corollary 1.11.** Let $\Psi : \mathbb{Z}^d \to \mathbb{Z}^t$ be a linear map of complexity at most $s < \infty$. Let $L$ be a sublattice of $\Psi(\mathbb{Z}^d)$ of finite index. Suppose that the Gowers Inverse conjecture GI(s) and the M"obius and nilsequences conjecture MN(s) are true for the parameter s. Then Dickson’s conjecture is true for $L$.

**Proof.** By Theorem 1.10, there is a complexity-preserving linear map $T : \mathbb{Z}^d \to \mathbb{Z}^d$ such that $T(\mathbb{Z}^d) = \Psi^{-1}(L)$. So $\Psi \circ T$ is of complexity at most s. By Theorem 1.8 Dickson’s conjecture is true for $\Psi \circ T(\mathbb{Z}^d) = L$. The corollary is proved. 

A great deal of work were done for special sublattices of the range of the linear map

$$\Psi : \mathbb{Z}^2 \to \mathbb{Z}^3, \ (n_1, n_2) \mapsto (n_1, n_2, -n_1 - n_2).$$

The pioneer is Rademacher [Ra]. The followers are Ayoub [AV], Liu-Tsang [LT], Liu-Wang [LW], Liu-Zhan [LZ], Bauer [Ba], and etc..

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2. **Green-Tao’s theorem**

Let $p$ be a prime number. Following Green-Tao [GT3], we denote the function $\frac{p}{p-1} \mathbf{1}(\mathbb{Z}_p)^s$ on $\mathbb{Z}_p$ as $\Lambda_p$. So $\Lambda_p(b) = 0$ when $b = p\mathbb{Z}$ and $\Lambda_p(b) = \frac{p}{p-1}$ otherwise. We define the
(t-dimensional) local von Mangoldt function at \( p \) on \((\mathbb{Z}_p)^t\) by setting
\[
\Lambda^\otimes_t(n_1, \ldots, n_t) = \prod_{i=1}^t \Lambda_p(n_i).
\]

In this section we deduce Theorem 1.8 from the following theorem, which is more close to Green-Tao’s original formulation.

**Theorem 2.1 (Green-Tao, [GT3]).** Let \( \Psi : \mathbb{Z}^d \to \mathbb{Z}^t \) be a linear map of complexity at most \( s < \infty \). Assume that the Gowers Inverse conjecture \( \text{GI}(s) \) and the M"obius and nilsequences conjecture \( \text{MN}(s) \) are true for the parameter \( s \). If \( N \) is large, \( K \subseteq [-N,N]^d \) is convex and of volume \( \gg N^d \), and \( a \in \mathbb{Z}^t \) is of size \( O(N) \) such that all local obstructions to the Zariski density of prime points in \( a + \Psi(\mathbb{Z}^d) \) are passed. Then
\[
\frac{1}{|K \cap \mathbb{Z}^d|} \sum_{\vec{n} \in K \cap \mathbb{Z}^d} \Lambda^\otimes_t(a + \Psi(\vec{n})) = \prod_p \gamma_p(a + \Psi) + o(1),
\]
where
\[
\gamma_p(a + \Psi) := \frac{1}{p^d} \sum_{\vec{n} \in \mathbb{Z}^d_p} \Lambda^\otimes_t(a + \Psi(\vec{n})).
\]

First we reformulate that theorem as follows.

**Theorem 2.2 (Green-Tao, [GT3]).** Let \( \Psi : \mathbb{Z}^d \to \mathbb{Z}^t \) be a linear map, and \( L \) be a sublattice of \( \Psi(\mathbb{Z}^d) \) of finite index. Let \( a \in \mathbb{Z}^t \) be such that \( a + L \) is non-degenerate. Then
\[
\frac{1}{|K \cap \mathbb{Z}^d|} \sum_{\vec{n} \in K \cap \mathbb{Z}^d} \Lambda^\otimes_t(a + \Psi(\vec{n})) = \prod_p \alpha_p(a + L) + o(1),
\]
where \( L = \Psi(\mathbb{Z}^d) \).

**Proof.** Since \( s \) is finite, \( a + L \) is non-degenerate. The theorem then follows from the following one.

**Theorem 2.3.** Let \( \Psi : \mathbb{Z}^d \to \mathbb{Z}^t \) be a linear map, and \( L \) be a sublattice of \( \Psi(\mathbb{Z}^d) \) of finite index. Let \( a \in \mathbb{Z}^t \) be such that \( a + L \) is non-degenerate. Then
\[
\alpha_p(a + L) = \frac{1}{p^d \left| \Psi(\mathbb{Z}^d_p) \right|} \sum_{\vec{n} \in \mathbb{Z}^d_p, \Psi(\vec{n}) \in L_p} \Lambda^\otimes_t(a + \Psi(\vec{n})).
\]

**Proof.** Since the sequence
\[
0 \to \{ \vec{n} \in \mathbb{Z}^d_p \mid \Psi(\vec{n}) = 0 \} \to \mathbb{Z}^d_p \xrightarrow{\Psi^t} \mathbb{Z}^t_p \to \mathbb{Z}^t_p / \Psi(\mathbb{Z}^d_p) \to 0
\]
is exact, we have
\[
|\{ \vec{n} \in \mathbb{Z}^d_p \mid \Psi(\vec{n}) = 0 \}| = \frac{p^d}{\left| \Psi(\mathbb{Z}^d_p) \right|}.
\]
Therefore we have

\[
\sum_{\vec{n} \in \mathbb{Z}^d, \Psi(\vec{n}) \in L_p} \Lambda_p^{\otimes t}(a + \Psi(\vec{n})) = \sum_{\vec{m} \in L_p} \Lambda_p^{\otimes t}(\vec{m}) \sum_{\vec{n} \in \mathbb{Z}^d, \Psi(\vec{n}) = \vec{m}} 1
\]

\[
= \sum_{\vec{m} \in L_p} \Lambda_p^{\otimes t}(\vec{m}) \sum_{\vec{n} \in \mathbb{Z}^d, \Psi(\vec{n}) = 0} 1 = \frac{N^d}{|\Psi(\mathbb{Z}_p^d)|} \sum_{\vec{n} \in \mathbb{L}_p} \Lambda_p^{\otimes t}(\vec{n}).
\]

The lemma now follows from the definition of local factors.

We now prove Theorem 1.8.

**Proof of Theorem 1.8.** Let \( r \) be the rank of \( \Psi \). Then there is an automorphism \( T \) of \( \mathbb{Z}^d \) such that

\[
\Psi \circ T(n_1, \ldots, n_t) = \Psi \circ T(n_1, \ldots, n_r, 0, \ldots, 0), \forall (n_1, \ldots, n_t) \in \mathbb{Z}^d.
\]

Let \( K' \subseteq [-N, N]^t \) be convex and of volume \( \gg N^r \), and set \( K = K' \times [-N, N]^{d-r} \). Note that (2.2) is also valid if \( \Psi \) is replaced by \( \Psi \circ T \). It follows that

\[
\frac{1}{|K' \cap \mathbb{Z}^d|} \sum_{\vec{n} \in K' \cap \mathbb{Z}^d} \Lambda^{\otimes t}(a + \Phi(\vec{n})) = \prod_p \alpha_p(a + L) + o(1), \tag{2.2}
\]

where \( L = \Psi(\mathbb{Z}^d) \), and \( \Phi : \mathbb{Z}^d \to \mathbb{Z}^t \) is defined by setting

\[
\Phi(n_1, \ldots, n_t) = \Psi \circ T(n_1, \ldots, n_r, 0, \ldots, 0).
\]

Now we let \( K \subseteq [-N, N]^t \) be convex and of volume \( \gg N^t \), and set \( K' = \Phi^{-1}(K) \). Note that there are positive constants \( c_1, c_2 \) such that

\[
\Phi^{-1}([-N, N]^t) \subseteq [-c_1 N, c_2 N]^r.
\]

It follows that (2.2) is also valid for this new \( K' \). As

\[
|K' \cap \mathbb{Z}^d| = |L \cap K|,
\]

and

\[
\sum_{\vec{n} \in K' \cap \mathbb{Z}^d} \Lambda^{\otimes t}(a + \Phi(\vec{n})) = \sum_{\vec{m} \in L \cap K} \Lambda^{\otimes t}(a + \vec{m}),
\]

(2.2) gives

\[
\frac{1}{|L \cap K|} \sum_{\vec{n} \in L \cap K} \Lambda^{\otimes t}(a + \vec{n}) = \prod_p \alpha_p(L) + o(1).
\]

Theorem 1.8 is proved.

3. Complexity of sublattices of finite index

In this section we prove Theorem 1.10.
Proof of Theorem 1.10. It is easy to see that $\Psi^{-1}(L)$ is a sublattice of $\mathbb{Z}^d$. Since $L$ is of finite rank in $\Psi(\mathbb{Z}^d)$, $\Psi^{-1}(L)$ is also of finite index in $\mathbb{Z}^d$. So $\Psi^{-1}(L)$ can be generated by $d$ vectors in $\mathbb{Z}^d$ which are linearly independent over $\mathbb{Q}$. These $d$ vectors give rise to a linear map $T: \mathbb{Z}^d \to \mathbb{Z}^d$ of rank $d$ such that $T(\mathbb{Z}^d) = \Psi^{-1}(L)$. Theorem 1.10 then follows from the following theorem.

**Theorem 3.1 (Complexity Preserving Theorem).** Let $T: \mathbb{Z}^d \to \mathbb{Z}^d$ be a linear map. Then $T$ is complexity preserving if and only if $T(\mathbb{Z}^d)$ is of rank $d$.

*Proof.* First we show that $T$ is of rank $d$ under the assumption that $T$ is complexity preserving. Let $I: \mathbb{Z}^d \to \mathbb{Z}^d$ be the identity map. Since $T$ is complexity preserving, $T = I \circ T$ is of Green-Tao complexity equal to that of $I$. It is easy to see that $I$ is of Green-Tao complexity 0. So $T$ is of the Green-Tao complexity 0. Let $T_1, \ldots, T_t$ be linear forms on $\mathbb{Z}^t$ such that $T = (T_1, \ldots, T_t)$. Then the system $T_1, \ldots, T_t$ is $\mathbb{Q}$-linearly independent. It follows that $T(\mathbb{Z}^d)$ is of rank $d$.

We now show that $T$ is complexity preserving under the assumption that $T$ is of rank $d$. Let $\Psi: \mathbb{Z}^d \to \mathbb{Z}^t$ be an arbitrary linear map. We must show that the Green-Tao complexity of $\Psi \circ T$ is no less than that of $\Psi$. By the following lemma, it suffices to show that the Green-Tao complexity of $\Psi$ is no less than that of $\Psi \circ T$. As $T(\mathbb{Z}^d)$ is of rank $d$, there is a linear map $Q: \mathbb{Z}^d \to \mathbb{Z}^d$ and a nonzero integer $a$ such that $T \circ Q = aI$. Again, by the following lemma, the Green-Tao complexity of $a\Psi = \Psi \circ T \circ Q$ is no less than that of $\Psi \circ T$. Since scalars are obviously complexity preserving, the Green-Tao complexity of $a\Psi$ is equal to that of $\Psi$. It follows that the Green-Tao complexity of $\Psi$ is no less than that of $\Psi \circ T$. The proposition is proved.

**Lemma 3.2.** Let $T: \mathbb{Z}^d \to \mathbb{Z}^d$ and $\Psi: \mathbb{Z}^d \to \mathbb{Z}^t$ be linear. Then the Green-Tao complexity $\Psi \circ T$ is no less than that of $\Psi$.

*Proof.* Suppose that $\Psi = (\psi_1, \ldots, \psi_t)$, where $\psi_1, \ldots, \psi_t$ is a system of linear forms on $\mathbb{Z}^d$. Let $1 \leq i \leq t$. Let $s_i$ be the $i$-complexity of $\Psi \circ T = (\psi_1 \circ T, \ldots, \psi_t \circ T)$. It suffices to show that the $i$-complexity of $\Psi$ is no greater than $s_i$. We may assume that $s_i < \infty$. Let

$$\{\psi_j \circ T: j \neq i\} = \cup_{k=1}^{s_i} \{\psi_j \circ T: j \in J_k\}$$

be a Green-Tao partition of $\Psi \circ T$ at $i$. Then

$$\{\psi_j: j \neq i\} = \cup_{k=1}^{s_i} \{\psi_j: j \in J_k\}$$

is a Green-Tao partition of $\Psi \circ T$ at $i$. It follows that the $i$-complexity of $\Psi$ is no greater than $s_i$. \qed

**References**

[Ay] R. Ayoub, *On Rademacher’s extension of the Goldbach-Vinogradov theorem*, Trans. AMS 74(1953), 482–491.

[Ba] C. Bauer, *On Goldbach’s conjecture in arithmetic progressions*, Studia Sci. Math. Hungar. 37(2001), no. 1-2, 1–20.
[BGS] J. Bourgain, A. Bourgain and P. Sarnak, *Sieving and expanders*, C. R. Acad. Sci. Paris, Ser. I 343(2006), 155–159.

[Da] H. Davenport, *On some infinite series involving arithmetical functions. II*, Quart. J. Math. Oxf. 8 (1937), 313–320.

[GT1] B. J. Green and T. C. Tao, *An inverse theorem for the Gowers $U^3(G)$ norm*, to appear in Proc. Edin. Math. Soc.

[GT2] B. J. Green and T. C. Tao, *Quadratic uniformity of the Möbius function*, preprint.

[GT3] B. J. Green and T. C. Tao, *Linear equations in primes*, to appear in Annals of Math.

[GT4] B. J. Green and T. C. Tao, *The primes contain arbitrarily long arithmetic progressions*, to appear in Annals of Math.

[HL] G.H. Hardy and J.E. Littlewood *Some problems of “partitio numerorum”; III: On the expression of a number as a sum of primes*, Acta Math. 44 (1923), 1–70.

[LT] M. C. Liu and K. M. Tsang, *Small prime solutions of linear equations*, Théorie des Nombres, 595–624, de Gruyter, Berlin, 1989.

[LW] M. C. Liu and T. Z. Wang, *On the equation $a_1p_1 + a_2p_2 + a_3p_3 = b$ with prime variables in arithmetic progressions*, Number Theory, 243–263, CRM Proc. Lecture Notes, 19, AMS, Providence, RI, 1999.

[LZ] J. Y. Liu and T. Zhan, *Ternary Goldbach problem in arithmetic progressions*, Acta Arith. 82(1997), no. 3, 197–227.

[Ra] H. Rademacher, *Über eine Erweiterung des Goldbachschen Problems*, Math. Z. 25(1926), no. 1, 627–657.

[Vi] I. M. Vinogradov, *Some theorems concerning the primes*, Mat. Sbornik. N.S. 2 (1937), 179–195.

School of Mathematics, Beijing Normal University, Beijing 100875

E-mail address: clliu@bnu.edu.cn