One-way or Two-way Factor Model for Matrix Sequences?

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Abstract: This paper investigates the issue of determining the dimensions of row and column factor spaces in matrix-valued data. Exploiting the eigen-gap in the spectrum of sample second moment matrices of the data, we propose a family of randomised tests to check whether a one-way or two-way factor structure exists or not. Our tests do not require any arbitrary thresholding on the eigenvalues, and can be applied with no restrictions on the relative rate of divergence of the cross-sections to the sample sizes as they pass to infinity. Although tests are based on a randomization which does not vanish asymptotically, we propose a de-randomized, “strong” (based on the Law of the Iterated Logarithm) decision rule to choose in favor or against the presence of common factors. We use the proposed tests and decision rule in two ways. We further cast our individual tests in a sequential procedure whose output is an estimate of the number of common factors. Our tests are built on two variants of the sample second moment matrix of the data: one based on a row (or column) “flattened” version of the matrix-valued sequence, and one based on a projection-based method. Our simulations show that both procedures work well in large samples and, in small samples, the one based on the projection method delivers a superior performance compared to existing methods in virtually all cases considered.

Key words and phrases: Matrix sequence; Matrix factor model; Principal component analysis; Projection Estimation; Randomised tests.

JEL classification: C23; C33; C38; C55.

1 Introduction

Matrix time series can be defined as a sequence of $p_1 \times p_2$ random matrices $\{X_t, 1 \leq t \leq T\}$, with each random matrix used to model observations that are well structured to be an

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array. An example is a time list of tables recording several macroeconomic variables across a number of countries; other possible examples include applications to marketing (where $X_t$ could be a series of customers’ ratings on a large number of items in an online platform), finance (where one may want to model data recorded for groups of financial factors for several groups of companies), health sciences (examples include electronic health records and ICU data), and 2-D image data processing - see also Chen and Fan (2021) and Gao et al. (2021) for further discussion and examples.

When dealing with such complex datasets, exploring the possibility of dimensionality reduction is of pivotal importance. In this context, several questions naturally arise: *is there a common, latent factor structure in the rows and/or columns of $X_t$? How many row and/or column factors are there, if they do exist?* Considering the macroeconomic example above, this entails checking the existence of country and/or index factors, and determining their numbers. The present paper is aimed at the solution to these two problems.

In recent years, the literature has paid increasing attention to factor analysis applied to matrix time series. The first contribution to propose and systematically study a matrix factor model is the one by Wang et al. (2019), who consider

$$
(X_t)_{p_1 \times p_2} = (R)_{p_1 \times k_1} (F_t)_{k_1 \times k_2} (C')_{k_2 \times p_2} + (E_t)_{p_1 \times p_2}, \ 1 \leq t \leq T, \ k_1, \ k_2 > 0, \quad (1.1)
$$

where $R$ is the $p_1 \times k_1$ row factor loading matrix exploiting the variations of $X_t$ across the rows, $C$ is the $p_2 \times k_2$ column factor loading matrix reflecting the differences across the columns of $X_t$, $F_t$ is the common factor matrix for all cells in $X_t$, and $E_t$ is the idiosyncratic component. In (1.1), we restrict that both $k_1$ and $k_2$ are positive, demonstrating a collaborative dependence between the cross-row section and the cross-column section, and we name it as a two-way factor structure. Since we interpret $k_1$ and $k_2$ as the numbers of row and column factors, we let $k_1 = 0$ and $k_2 = 0$ correspond to the scenarios without row factors and without column factors, respectively. For $k_2 = 0$ but $k_1 > 0$, it means
the presence of one-way factor structure along the row dimension, and all columns of the whole matrix sequence could be modeled by a $p_1$ dimensional vector factor model with effective sample size $T p_2$. Similar interpretation applies for the scenario of $k_1 = 0$ but $k_2 > 0$. For $k_1 = k_2 = 0$, the matrix-valued data is simply a noise matrix. Henceforth, we make the following convention

\[
(X_t)_{p_1 \times p_2} = \begin{cases} 
(R)_{p_1 \times k_1}(F_t)_{k_1 \times p_2} + (E_t)_{p_1 \times p_2}, & k_1 > 0, k_2 = 0, \\
(F_t)_{p_1 \times k_2}(C')_{k_2 \times p_2} + (E_t)_{p_1 \times p_2}, & k_2 > 0, k_1 = 0, \\
(E_t)_{p_1 \times p_2}, & k_1 = k_2 = 0,
\end{cases} \tag{1.2}
\]

where the first case refers to a one-way factor model along the row dimension (all columns form a vector factor model), the second case is a one-way factor model along the column dimension (all rows form a vector factor model), and the third case means absence of any factor structure. Notice that any factor structure makes sense only if $k_1 \ll p_1$ and $k_2 \ll p_2$, we abandoned the setting where $k_2 = p_2$ & $C = I_{p_2}$ (resp. $k_1 = p_1$ & $R = I_{p_1}$) though it is mathematically equivalent to the first (resp. the second) case in (1.2). Our set-up notation in (1.1)-(1.2) defines four mutually exclusive hypotheses.

As far as inference on matrix factor model is concerned, Wang et al. (2019) propose estimators of the factor loading matrices (and of numbers of the row and column factors) based on an eigen-analysis of the auto-cross-covariance matrix. From a distinct perspective and assuming cross-sectional pervasiveness along the row and column dimensions, Chen and Fan (2021) propose an estimation technique based on an eigen-analysis of a weighted average of the mean and the column (row) covariance matrix of the data; Yu et al. (2021) improve the estimation efficiency of the factor loading matrices with iterative projection algorithms. Extensions and applications of the basic set-up in (1.1) include the constrained version by Chen et al. (2020), the semiparametric estimators by Chen et al. (2020), and the estimators developed in Chen et al. (2021); see also Han et al. (2020). Chen and Chen (2020) apply (1.1) to the dynamic transport network in the con-
text of international trade flows, and Chen et al. (2020) consider applications to financial datasets. However, no works were done so far to seriously test the existence of the two-way factor structure in (1.1). Being able to discern whether a genuine matrix factor structure exists or not is a crucial point in the analysis of matrix-valued data. As Chen and Fan (2021) put it, “... analyzing large scale matrix-variate data is still in its infancy, and as a result, scientists frequently analyze matrix-variate observations by separately modeling each dimension or ‘flattening’ them into vectors. This destroys the intrinsic multi-dimensional structure and misses important patterns in such large scale data with complex structures, and thus leads to sub-optimal results”. Having a test to verify whether a (one-way or two-way) matrix factor structure exists or not serves as a good model checking tool to draw practical implications, e.g. on the estimation technique to be employed. To the best of our knowledge, this is the first work with a hypothesis testing procedure to discern between a genuine two-way matrix factor model (i.e. (1.1)), a one-way matrix factor structure (i.e. first two cases of (1.2)), or no factors at all (i.e. last case of (1.2)).

The aforementioned problem can be answered by testing the following general hypotheses:

\[ H_{i0} : k_i \geq k_{i0}, \quad \text{v. s.} \quad H_{i1} : k_i < k_{i0}, \quad i = 1, 2, \]  

(1.3)

where \( k_{i0}^1 \) and \( k_{i0}^2 \) are the hypothesised numbers of row and column factors, respectively. Our tests exploit the eigen-gap property of the second moment matrix of the matrix series: if there are \( k_i^0 \) common row (or column) factors, then the largest \( k_i^0 \) eigenvalues diverge ALMOST SURELY, as the matrix dimensions increase, at a faster rate than the remaining ones. To the best of our knowledge, for the first time in the literature of matrix factor analysis, this paper obtains an almost-sure (not just in probability) diverging lower bound of the largest \( k_i^0 \) eigenvalues of the column (or row) covariance matrix with and without projection, and an almost-sure upper bound of the remaining eigenvalues. With the exploited almost-sure eigen gap, we construct a randomised test in a similar manner.
to Trapani (2018). In order to avoid the non-reproducibility issue of randomised tests, we propose a “strong” rule to decide between $H_{i0}$ and $H_{i1}$, inspired by the Law of the Iterated Logarithm. Our approach has several desirable features. First, it is based on testing, and therefore it does not suffer from the arbitrariness in thresholding the eigenvalues, which is typical of information criteria. Second, it is also valid for tests for $H_{i0}: k_i \geq 1$ versus $H_{i1}: k_i = 0$, thus avoiding the arbitrariness of having to create an “artificial” eigenvalue, which is typically used to initialise procedures based on eigenvalue ratios. Third, our tests - and therefore our decision rules - do not require any restrictions on the relative divergence rates of $p_1, p_2$ & $T$ as they pass to infinity, nor do they require the white noise assumption on the idiosyncratic error matrix as in Wang et al. (2019). As far as the last point is concerned, we would like to mention that the set-up by Wang et al. (2019) (see also Lam and Yao, 2012) assumes that $E_t$ is white noise, although, as a trade-off, less restrictive assumptions are needed on the cross-sectional correlation among the components of $E_t$. In the context of such a set-up, the factor model can be validated by using existing high-dimensional white noise tests. Conversely, in the context of an approximate factor model like ours, the issue of model validation has not been fully investigated, i.e. no test exists to check that there is indeed a factor structure. Our paper fills the gap in literature, and, in general, is applicable to a wide variety of datasets.

In addition to diagnosing matrix structures, tests for (1.3) can be cast in a sequential procedure, as in Onatski (2009) and Trapani (2018), thereby obtaining an estimator for the number of common row (and/or column) factors. Specifically designed for large matrix sequence, to the best of our knowledge, this is the first non-eigen-thresholding estimator of the numbers of row and/or column factors. After determining the common factor dimensions, it is possible to apply the inferential theory developed e.g. in Yu et al. (2021) or Chen and Fan (2021).

We propose two methodologies to test for (1.3), based on the eigenvalues of two different sample second moment matrices. Our first procedure is based on evaluating the
\(k_i^0\)-th largest eigenvalues of the row (when \(i = 1\)) and column (when \(i = 2\)) “flattened” sample covariance matrices, defined as

\[
M_c := \frac{1}{Tp_2} \sum_{t=1}^{T} X_t X'_t = \frac{1}{Tp_2} \sum_{t=1}^{T} \sum_{i=1}^{p_2} X_{i,t} X'_{i,t},
\]

and

\[
M_r := \frac{1}{Tp_1} \sum_{t=1}^{T} X'_t X_t = \frac{1}{Tp_1} \sum_{t=1}^{T} \sum_{j=1}^{p_1} X_{j,t} X'_{j,t},
\]

respectively, where \(X_{i,t}\) denotes the \(i\)-th column of \(X_t\), and \(X_{j,t}\) its \(j\)-th row. This testing procedure is computationally straightforward, and it requires only one step. On the other hand, using \(M_c\) and \(M_r\) ignores the two-way factor structure in model (1.1). Hence, we also propose a second methodology which, at the price of a slightly more convoluted, two-step, procedure, makes full use of the low-rank structure of the common component matrix in (1.1). In particular, we test for (1.3) based on the column covariance matrix of a projected matrix time series. Our numerical results demonstrate that making use of the two-way factor structure in (1.1) substantially enhances the performance of our procedures.

The rest of the paper is organized as follows. Section 2 presents the main assumptions and results on the spectra of \(M_c\) and \(M_r\), as well as the projection-based second moment matrices. Section 3 gives two hypotheses testing procedures for (1.3), and the sequential testing methodology to determine \(k_i\) for \(i = 1\) and 2; in particular, our “strong” rule to decide between \(H_{i0}\) and \(H_{i1}\) is given in Section 3.2. We evaluate our theory through an extensive simulation exercise in Section 4, and further illustrate our findings through two empirical applications in Section 5. Section 6 concludes the paper and discusses some future research problems.

To end this section, we introduce some notations. Positive finite constants are denoted as \(c_0, c_1, \ldots\), and their values may change from line to line. Throughout the paper, we use the short-hand notation “a.s.” for “almost sure(ly)”. Given two sequences
a_{p_1,p_2,T} and b_{p_1,p_2,T}, we say that \(a_{p_1,p_2,T} = o_{a.s.}(b_{p_1,p_2,T})\) if, as \(\min\{p_1, p_2, T\} \to \infty\), it holds that \(a_{p_1,p_2,T} b_{p_1,p_2,T}^{-1} \to 0\) a.s.; we say that \(a_{p_1,p_2,T} = O_{a.s.}(b_{p_1,p_2,T})\) to denote that as \(\min\{p_1, p_2, T\} \to \infty\), it holds that \(a_{p_1,p_2,T} b_{p_1,p_2,T}^{-1} \to c_0 < \infty\) a.s.; and we use the notation \(a_{p_1,p_2,T} = \Omega_{a.s.}(b_{p_1,p_2,T})\) to indicate that as \(\min\{p_1, p_2, T\} \to \infty\), it holds that \(a_{p_1,p_2,T} b_{p_1,p_2,T}^{-1} \to c_0 > 0\) a.s.

Given an \(m \times n\) matrix \(A\), we denote its transpose as \(A'\) and its element in position \((i, j)\) as \(A_{ij}\) or \(a_{ij}\), i.e. using either upper or lower case letters. Further, we denote the spectral norm as \(\|A\|\); we use \(\|A\|_{\max}\) to denote the maximum of the absolute values of \(A\)'s elements; \(A_{\cdot j}\) and \(A_{i \cdot}\) denote the \(j\)-th column and \(i\)-th row of \(A\), respectively; finally, we let \(\lambda_i(A)\) be the \(i\)-th largest eigenvalue of \(A\).

## 2 Spectra

We study the eigenvalues of the column covariance matrix \(M_c\) and \(M_r\), and of the projected versions (denoted as \(\tilde{M}_1\) and \(\tilde{M}_2\)). In both cases, we find that the matrices have an eigen-gap between the first \(k_1\) (resp. \(k_2\)) eigenvalues and the remaining ones. As the cross-sectional sample size \(p_1\) (resp. \(p_2\)), increases, the first \(k_1\) (resp. \(k_2\)) eigenvalues diverge at a faster rate than the remaining ones.

### 2.1 Assumptions

The following assumptions are borrowed from the paper by Yu et al. (2021), to which we refer for detailed explanations and discussions.

**Assumption B1.** (i) (a) \(E(F_t) = 0\), and (b) \(E\|F_t\|^{4+\epsilon} \leq c_0\), for some \(\epsilon > 0\); (ii) \(\frac{1}{T} \sum_{t=1}^{T} F_t F_t' \overset{a.s.}{\to} \Sigma_1\) and \(\frac{1}{T} \sum_{t=1}^{T} F_t' F_t \overset{a.s.}{\to} \Sigma_2\), \(2.4\)

where \(\Sigma_i\) is a \(k_i \times k_i\) positive definite matrix with distinct eigenvalues and spectral decomposition \(\Sigma_i = \Gamma_i \Lambda_i \Gamma_i'\), \(i = 1, 2\). The factor numbers \(k_1\) and \(k_2\) are fixed as \(\min\{T, p_1, p_2\} \to \infty\).
\( \infty; (iii) \) it holds that, for all \( 1 \leq h_1, l_1 \leq k_1 \) and \( 1 \leq h_2, l_2 \leq k_2 \)
\[
E \max_{1 \leq t \leq T} \left( \sum_{t=1}^{T} (F_{h_1 h_2, t} F_{l_1 l_2, t} - E (F_{h_1 h_2, t} F_{l_1 l_2, t})) \right)^2 \leq c_0 T.
\]

**Assumption B2.** (i) \( \|R\|_{\text{max}} \leq c_0 \), and \( \|C\|_{\text{max}} \leq c_1 \); (ii) as \( \min\{p_1, p_2\} \to \infty \), \( \|p_1^{-1} R' R - I_{k_1}\| \to 0 \) and \( \|p_2^{-1} C' C - I_{k_2}\| \to 0 \).

Assumptions B1 and B2 are standard in large factor models, and we refer, for example, to Chen and Fan (2021). In Assumption B1(i)/b), note the (mild) strengthening of the customarily assumed fourth moment existence condition on \( F_t \) - this is required in order to prove our results, which rely on almost sure rates. Similarly, the maximal inequality in part \( (iii) \) of the assumption is usually not considered in the literature, and it can be derived from more primitive dependence assumptions: for example, it can be shown to hold under various mixing conditions (see e.g. Rio, 1995; and Shao, 1995); in Appendix A, we show its validity for the very general class of decomposable Bernoulli shifts (see e.g. Wu, 2005). Finally, we point out that, according to Assumption B2, the common factors are pervasive. Extensions to the case of “weak” factors go beyond the scope of this paper, but are in principle possible.

**Assumption B3.** (i) \( a) \) \( E(e_{ij,t}) = 0 \), and \( b) \) \( E(e_{ij,t}^8) \leq c_0 \); (ii) for all \( 1 \leq t \leq T \), \( 1 \leq i \leq p_1 \) and \( 1 \leq j \leq p_2 \),
\[
\begin{align*}
(a) &. \sum_{s=1}^{T} \sum_{l=1}^{p_1} \sum_{h=1}^{p_2} |E(e_{ij,t} e_{lh,s})| \leq c_0, \\
(b) &. \sum_{l=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} |E(e_{ij,t} e_{ih,t})| \leq c_0;
\end{align*}
\]

\( (iii) \) for all \( 1 \leq t \leq T \), \( 1 \leq i \leq p_1 \) and \( 1 \leq j \leq p_2 \),
\[
\begin{align*}
(a) &. \sum_{s=1}^{T} \sum_{l=1}^{p_1} \sum_{h=1}^{p_2} |Cov(e_{ij,t}, e_{lh,s}, e_{l_1 l_2, t})| \leq c_0, \\
&. \sum_{s=1}^{T} \sum_{l=1}^{p_1} \sum_{h=1}^{p_2} |Cov(e_{ij,t}, e_{lh,s}, e_{ij,s})| \leq c_0, \\
&. \sum_{s=1}^{T} \sum_{l=1}^{p_1} \sum_{h=1}^{p_2} |Cov(e_{ij,t}, e_{ij,s})| \leq c_0, \\
(b) &. \sum_{s=1}^{T} \sum_{l=1}^{p_1} \sum_{h=1}^{p_2} |Cov(e_{ij,t}, e_{l_1 h_1, t}, e_{l_1 h_2, t}) + Cov(e_{l_1 h_1, t}, e_{l_1 h_2, t})| \leq c_0;
\end{align*}
\]
(iv) it holds that $\lambda_{\text{min}} \left[ E \left( \frac{1}{p_2 T} \sum_{t=1}^{T} E_t E_t' \right) \right] > 0$.

Assumption B3 ensures the (cross-sectional and time series) summability of the idiosyncratic terms $E_t$. The assumption allows for (weak) dependence in both the space and time domains, and - as also mentioned in the introduction - it can be read in conjunction with the paper by Wang et al. (2019), where $E_t$ is assumed to be white noise, but no structure is assumed on its covariance matrix. In Appendix A, we show that the time-series properties of $E_t$ (in particular parts (ii) and (iii), which are high-level assumptions) is satisfied, similarly to Assumption B1, by the wide class of decomposable Bernoulli shifts.

**Assumption B4.**

(i) For any deterministic vectors $v$ and $w$ satisfying $\|v\| = 1$ and $\|w\| = 1$,

$$E \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (F_t v' E_t w) \right\|^2 \leq c_0;$$

(ii) for all $1 \leq i, l_1 \leq p_1$ and $1 \leq j, h_1 \leq p_2$,

\[
\begin{align*}
(a). & \left\| \sum_{h=1}^{p_2} E(\tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{ih}) \right\|_{\max} \leq c_0, & \left\| \sum_{l=1}^{p_1} E(\tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{lj}) \right\|_{\max} \leq c_0, \\
(b). & \left\| \sum_{l_1=1}^{p_1} \sum_{h_2=1}^{p_2} \text{Cov}(\tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{il_1, h_1}, \tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{il_2, h_2}) \right\|_{\max} \leq c_0, & \left\| \sum_{l_2=1}^{p_1} \sum_{h=1}^{p_2} \text{Cov}(\tilde{\zeta}_{ij} \otimes \tilde{\zeta}_{il_1, h}, \tilde{\zeta}_{ih} \otimes \tilde{\zeta}_{il_2, h}) \right\|_{\max} \leq c_0,
\end{align*}
\]

where $\tilde{\zeta}_{ij} = \text{Vec}(\sum_{t=1}^{T} F_t \epsilon_{ij,t}/\sqrt{T})$.

According to Assumption B4, the common factors $F_t$ and the errors $E_t$ can be weakly correlated. The assumption holds under the more restrictive case that $\{F_t\}$ and $\{E_t\}$ are two mutually independent groups.

### 2.2 The spectra of $M_c$ and $M_r$

To avoid repetitions, we only present results for $M_c$; the spectrum of $M_r$ can be studied exactly in the same way.
We use the short-hand notation $\lambda_j$ to indicate the $j$-th largest eigenvalue of the expectation of $M_c$, and use $\hat{\lambda}_j$ denote the $j$-th largest eigenvalue of $M_c$. Our first theorem provides an a.s. eigen-gap for $M_c$.

**Theorem 1.** Suppose that Assumptions B1-B4 are satisfied. Then it holds that

$$\hat{\lambda}_j = \Omega_{a.s.}(p_1),$$

(2.5)

for all $j \leq k_1$. Also, there exist a constant $c_0 < \infty$ such that

$$\hat{\lambda}_j = c_0 + o_{a.s.} \left( \frac{p_1}{\sqrt{T_p}} (\ln^2 p_1 \ln p_2 \ln T)^{1/2+\epsilon} \right),$$

(2.6)

for all $j > k_1$, and all $\epsilon > 0$.

The eigen-gap in the spectrum of $M_c$ is the building block to construct a procedure to decide between $H_{i0}$ and $H_{i1}$ in (1.3). We point out that, although the result is similar, in spirit, to the one derived by Trapani (2018), we follow a quite different method of proof - using the approach in Trapani (2018), we would be able to show only the rate $o_{a.s.} \left( T^{-1/2} p_1 (\ln^2 p_1 \ln p_2 \ln T)^{1/2+\epsilon} \right)$ in (2.6), thus having a (much) worse rate.

**2.3 The spectra of projected covariance matrices**

The matrices $M_c$ and $M_r$ are straightforward to compute and use, but they are based on the implicit assumption of a “one-way” factor structure, present only in the columns (or rows) of the observations. In order to fully make use of the two-way interactive factor structure in (1.1), we propose studying the spectrum of a projected column (row) covariance matrix, as suggested by Yu et al. (2021). Heuristically, if $C$ is known and satisfies the orthogonality condition $C'C/p_2 = I_{k_2}$, the data matrix can be projected into
a lower dimensional space by setting \( Y_t = X_t \hat{C}/p_2 \). In view of this, we define

\[
\tilde{M}_1 = \frac{1}{T} \sum_{t=1}^{T} \tilde{Y}_t \tilde{Y}_t',
\]

where \( \tilde{Y}_t = X_t \hat{C}/p_2 \) and \( \hat{C} \) is an initial estimator of \( C \) (\( \tilde{M}_2 \) can be defined similarly). As suggested by Yu et al. (2021), the initial estimator can be set as \( \hat{C} = \sqrt{p_2}Q \), where the columns of \( Q \) are the leading \( k_2 \) eigenvectors of \( M_r \).

Let \( \tilde{\lambda}_j \) denote the \( j \)-th largest eigenvalue of \( \tilde{M}_1 \). The following result measures the eigen-gap of \( \tilde{M}_1 \), which strengthens Theorem 1.

**Theorem 2.** We assume that Assumptions B1-B4 are satisfied. Then it holds that

\[
\tilde{\lambda}_j = \Omega_{a.s.}(p_1),
\]

for all \( j \leq k_1 \). Also, there exists a finite constant \( c_1 \) such that

\[
\tilde{\lambda}_j = o_{a.s.}\left(\left(\frac{1}{p_2} + \frac{1}{T} + \frac{p_1}{\sqrt{Tp_2}}\right)\left(\ln^2 p_1 \ln p_2 \ln T\right)^{1+\epsilon}\right),
\]

for all \( j > k_1 \) and all \( \epsilon > 0 \).

Comparing (2.8) with (2.6) in Theorem 1, the eigen-gap of \( \tilde{M}_1 \) is wider than that of \( M_c \). Thus, using \( \tilde{M}_1 \) should yield a higher testing power and a better estimate of \( k_1 \) (and/or \( k_2 \)) if the two-way interactive factor structure is really true in practice. As mentioned after Theorem 1, the rate in (2.8) is sharper than one would find following method of proof in Trapani (2018); even in this case, we would only obtain the rate \( o_{a.s.}\left(T^{-1/2}p_1 \left(\ln^2 p_1 \ln p_2 \ln T\right)^{1+\epsilon}\right)\), which again would be sub-optimal.
3 Inference on the number of factors

In this section, we investigate two related problems, based on determining the dimension of the (row or column) factor structures. For brevity, we only report results concerning $k_1$, but all our procedures can be readily extended to analyse $k_2$.

We begin by presenting the tests for the null that $H_0 : k_1 \geq k_1^0$ for a given $k_1^0$ (we omit the subscript $i$ in $H_{i0}$ for simplicity). We then apply these to determining whether there is a factor structure and, if so, we develop a sequential procedure to determine the dimension of each factor space. Both procedures are based on constructing, as a first step, a test based on the rates of divergence of the eigenvalues of either $M_c$ or $\tilde{M}_1$ (see Section 3.1); and, as a second step, a decision rule to choose between $H_0$ and $H_1$ which is not affected by the randomness added by the researcher (Section 3.2).

3.1 Hypothesis testing and the randomised tests

We consider tests for

$$H_0 : k_1 \geq k_1^0 \text{ v.s. } H_1 : k_1 < k_1^0 \text{ for some } k_1^0 \in \{1, \ldots, k_{\text{max}}\},$$

(3.9)

where $k_{\text{max}}$ is a pre-specified upper bound. The hypothesis in (3.9) is equivalent to the following hypothesis on the eigenvalue $\lambda_{k_1^0}$ (the same holds when using $\tilde{\lambda}_{k_1^0}$)

$$H_0 : \lambda_{k_1^0} \geq c_0 p_1 \text{ v.s. } H_1 : \lambda_{k_1^0} \leq c_0.$$

(3.10)

We propose two types of test statistics for the hypothesis testing problem in (3.10). Let $\beta = \ln p_1 / \ln (p_2 T)$, and let $\delta = \delta (\beta) \in [0, 1)$, such that

$$\delta = \varepsilon \quad \text{if } \beta \leq 1/2$$

$$\delta = 1 - 1/(2\beta) + \varepsilon \quad \text{if } \beta > 1/2$$

(3.11)
where $\varepsilon > 0$ is an arbitrarily small, user-defined number. Given $\delta$, define

$$
\hat{\phi}_k^0 = \exp \left\{ \frac{p_1^{-\delta} \hat{\lambda}_k^0}{p_1 \sum_{j=1}^{p_1} \lambda_j} \right\} - 1 \quad \text{and} \quad \tilde{\phi}_k^0 = \exp \left\{ \frac{p_1^{-\delta} \tilde{\lambda}_k^0}{p_1 \sum_{j=1}^{p_1} \lambda_j} \right\} - 1.
$$

(3.12)

The (individual) randomised tests are similar to the ones proposed in Trapani (2018). Prior to studying the test statistics, we make some comments on (3.12). The sequences $\hat{\phi}_k^0$ and $\tilde{\phi}_k^0$ are transformations of $\hat{\lambda}_k^0$ and $\tilde{\lambda}_k^0$, rescaled by the trace of $M_c$ and $\tilde{M}_1$ respectively. This serves the purpose of making the eigenvalues $\hat{\lambda}_k^0$ and $\tilde{\lambda}_k^0$ scale-invariant. The choice of $\delta$ defined in (3.11) is another important specification. Its purpose, as explained in Trapani (2018), is to make $p_1^{-\delta} \hat{\lambda}_k^0$ (and $p_1^{-\delta} \tilde{\lambda}_k^0$) drift to zero when $\lambda_k^0 = 0$, essentially getting rid of the estimation error, while still allowing $p_1^{-\delta} \hat{\lambda}_k^0$ (and $p_1^{-\delta} \tilde{\lambda}_k^0$) to pass to infinity if $\lambda_k^0$ does diverge.

We do not know the limiting distributions of $\hat{\phi}_k^0$ or $\tilde{\phi}_k^0$. However, on account of the results in Section 2, we can derive their rates under both the cases that $\lambda_k^0 \leq c_0$ and $\lambda_k^0 \geq c_0 p_1$. Exploiting such rates, we propose a randomized version of $\hat{\phi}_k^0$ or $\tilde{\phi}_k^0$ as follows:

**Step 1** Generate i.i.d. samples $\{\eta^{(m)}\}_{m=1}^M$ with common distribution $\mathcal{N}(0, 1)$.

**Step 2** Given $\{\eta^{(m)}\}_{m=1}^M$, construct sample sets $\{\psi_k^0(u)\}_{m=1}^M$ and $\{\tilde{\psi}_k^0(u)\}_{m=1}^M$ as

$$
\tilde{\psi}_k^0(u) = I \left[ \sqrt{\hat{\phi}_k^0} \times \eta^{(m)} \leq u \right], \quad \psi_k^0(u) = I \left[ \sqrt{\tilde{\phi}_k^0} \times \eta^{(m)} \leq u \right].
$$

**Step 3** Define

$$
\hat{\nu}_k(u) = \frac{2}{\sqrt{M}} \sum_{m=1}^M \left[ \psi_k^0(u) - \frac{1}{2} \right], \quad \tilde{\nu}_k(u) = \frac{2}{\sqrt{M}} \sum_{m=1}^M \left[ \tilde{\psi}_k^0(u) - \frac{1}{2} \right].
$$

(3.13)
Step 4. The test statistics are finally defined as

\[
\hat{\Psi}_{k_0} = \int_U \left[ \hat{\nu}_{k_0}(u) \right]^2 dF(u), \quad \tilde{\Psi}_{k_0} = \int_U \left[ \tilde{\nu}_{k_0}(u) \right]^2 dF(u),
\]

where \( F(u) \) is a weight function.

The test described above is similar to the one proposed in Trapani (2018); however, in the construction of \( \hat{\Psi}_{k_0} \) and \( \tilde{\Psi}_{k_0} \), we propose a weighted average across different values of \( u \) through the weight function \( F(u) \). As a consequence, it can be expected that the test will not be affected by an individual value of \( u \), a form of scale invariance which is not considered in Trapani (2018).

Let \( P^* \) denote the probability law of \( \{ \hat{\psi}^{(m)}(u) \}_{m=1}^M \) and \( \{ \tilde{\psi}^{(m)}(u) \}_{m=1}^M \) conditional on the sample \( \{ X_t, 1 \leq t \leq T \} \), and \( \overset{D*}{\rightarrow} \) and \( \overset{P*}{\rightarrow} \) as convergence in distribution and in probability, respectively, according to \( P^* \). We need the following assumption.

Assumption C1. (i) \( \int_U dF(u) = 1 \); (ii) \( \int_U u^2 dF(u) < \infty \).

Assumption C1 is satisfied by several functions, the most “natural” candidates being distribution functions.

Proposition 1. We assume that Assumptions B1-B4 and C1 are satisfied. Then, under \( H_0 : k_1 \geq k_1^0 \), as \( \min\{p_1,p_2,T,M\} \to \infty \) with

\[
M \exp \left\{ -\epsilon p_1^{1-\delta} \right\} \to 0,
\]

for some \( 0 < \epsilon < c_0/\bar{\lambda} \) and \( \bar{\lambda} = 1/p_1 \sum_{j=1}^{p_1} \lambda_j \), it holds that

\[
\hat{\Psi}_{k_1^0} \overset{D^*}{\rightarrow} \chi_1^2,
\]

for almost all realisations of \( \{ X_t, 1 \leq t \leq T \} \). Under the same assumptions, it also holds that \( \tilde{\Psi}_{k_1^0} \overset{D^*}{\rightarrow} \chi_1^2 \) for almost all realisations of \( \{ X_t, 1 \leq t \leq T \} \).
Under $H_1 : \lambda_{k_1} = c_0 < \infty$, as $\min\{p_1, p_2, T, M\} \to \infty$ it holds that

$$M^{-1}\tilde{\Psi}_{k_1}^0 \xrightarrow{P^*} c_1,$$  \hfill (3.16)

for some $0 < c_1 < \infty$ and almost all realisations of $\{X_t, 1 \leq t \leq T\}$. Under the same assumptions, it also holds that $M^{-1}\hat{\Psi}_{k_1}^0 \xrightarrow{P^*} c_1$, for almost all realisations of $\{X_t, 1 \leq t \leq T\}$.

Equation (3.15) states that, under the null, both test statistics $\hat{\Psi}_{k_1}^0$ and $\tilde{\Psi}_{k_1}^0$ converge in distribution to a chi-squared distribution with degree 1. This can be understood heuristically by noting that, under the null, both $\hat{\phi}_{k_1}^0$ and $\tilde{\phi}_{k_1}^0$ go to infinity, thus the variances of $\sqrt{\hat{\phi}_{k_1}^0} \times \eta^{(m)}$ and $\sqrt{\tilde{\phi}_{k_1}^0} \times \eta^{(m)}$ also pass to positive infinity. Thus, heuristically, $\{\hat{\psi}_{k_1}^{(m)}(u)\}_{m=1}^M$ and $\{\tilde{\psi}_{k_1}^{(m)}(u)\}_{m=1}^M$ follow a Bernoulli distribution with success probability 0.5. By the Central Limit Theorem, in (3.13) as $M$ goes to infinity, both $\hat{\nu}_{k_1}^0(u)$ and $\tilde{\nu}_{k_1}^0(u)$ follow the standard normal distribution $N(0, 1)$ (conditional on the sample) asymptotically. The results hold for all samples, save for a zero measure set, and no restrictions are exerted on the relative rate of divergence of $p_1, p_2$ and $T$ as they pass to infinity. By Proposition 1, it follows immediately that, for almost all realisations of $\{X_t, 1 \leq t \leq T\}$

$$\lim_{\min\{p_1, p_2, T, M\} \to \infty} P^* \left( \hat{\Psi}_{k_1}^0 > c_\alpha | H_0 \right) = \alpha,$$  \hfill (3.17)

where $c_\alpha$ is such that $P(\chi_1^2 > c_\alpha) = \alpha$. Similarly

$$\lim_{\min\{p_1, p_2, T, M\} \to \infty} P^* \left( \tilde{\Psi}_{k_1}^0 > c_\alpha | H_1 \right) = 1.$$  \hfill (3.18)

### 3.2 A “strong” rule to decide between $H_0$ and $H_1$

The tests are constructed by using added randomness, $\{\eta^{(m)}\}_{m=1}^M$, whose effect does not vanish asymptotically as would be the case e.g. when using the bootstrap. In turn, this entails that the properties of tests based on $\hat{\Psi}_{k_1}^0$ (and $\tilde{\Psi}_{k_1}^0$) are different to the properties
of “standard” tests. Indeed, equation (3.18) has the classical interpretation: whenever a researcher uses \( \hat{\Psi}_{k_1} \) (and \( \tilde{\Psi}_{k_1} \)), (s)he will reject the null, when false, with probability one. Conversely, the implications of (3.17) are subtler. Due to the artificial randomness \( \{ \eta^{(m)} \}_{m=1}^M \), different researchers using the same data will obtain different values of \( \hat{\Psi}_{k_0} \) and \( \tilde{\Psi}_{k_0} \), and, consequently, different p-values; indeed, if an infinite number of researchers were to carry out the test, the p-values would follow a uniform distribution on \([0, 1]\).

Corradi and Swanson (2006) provide an alternative explanation, writing that “[…] as the sample size gets larger, all researchers always reject the null when false, while \( \alpha \% \) of the researchers always reject the null when it is true”.

In order to address this problem, we propose a further step which, in essence, “de-randomizes” \( \hat{\Psi}_{k_1} \) and \( \tilde{\Psi}_{k_1} \). Each researcher, instead of computing \( \hat{\Psi}_{k_1} \) or \( \tilde{\Psi}_{k_1} \) just once, will compute the test statistic \( S \) times, at each iteration \( s \) generating a statistic \( \hat{\Psi}_{k_0, s} \) (or \( \tilde{\Psi}_{k_0, s} \)) using a random sequence \( \{ \eta^{(m)}_s, 1 \leq m \leq M \} \), independent across \( 1 \leq s \leq S \), and thence defining, for some \( \alpha \in (0, 1) \)

\[
\hat{Q}_{k_1} (\alpha) = S^{-1} \sum_{s=1}^S I \left[ \hat{\Psi}_{k_0, s} \leq c_\alpha \right], \tag{3.19}
\]

and the same when using \( \tilde{\Psi}_{k_0, s} \) (in this case obtaining \( \tilde{Q}_{k_1} (\alpha) \)). A consequence of Proposition 1 is

\[
\lim_{\min(p_1, p_2, T, M, S) \to \infty} P^* \{ \hat{Q}_{k_1} (\alpha) = 1 - \alpha \} = 1 \quad \text{for } H_0 : k_1 \geq k_0^0, \\
\lim_{\min(p_1, p_2, T, M, S) \to \infty} P^* \{ \hat{Q}_{k_1} (\alpha) = 0 \} = 1 \quad \text{for } H_1 : k_1 < k_0^0. \tag{3.20}
\]

Equation (3.20) stipulates that, as \( S \to \infty \), averaging across \( s \) in (3.19) washes out the added randomness in \( \hat{Q}_{k_1} (\alpha) \): all researchers using this procedure will obtain the same value of \( \hat{Q}_{k_1} (\alpha) \), thereby ensuring reproducibility. The function \( \hat{Q}_{k_1} (\alpha) \) corresponds to (the complement to one of) the “fuzzy decision”, or “abstract randomised decision rule” reported in equation (1.1a) in Geyer and Meeden (2005a). Geyer and Meeden (2005a)
(see also Geyer and Meeden, 2005b) provide a helpful discussion of the meaning of $\hat{Q}_{k_1^0}(\alpha)$: the problem of deciding in favour or against $H_0$ may be modelled through a random variable, say $D$, which can take two values, namely “do not reject $H_0$” and “reject $H_0$”. Such a random variable has probability $\hat{Q}_{k_1^0}(\alpha)$ to take the value “do not reject $H_0$”, and probability $1 - \hat{Q}_{k_1^0}(\alpha)$ to take the value “reject $H_0$”. In this context, (3.20) states that (asymptotically), the probability of the event $\{\omega : D = \text{“reject } H_0\text{”}\}$ is $\alpha$ when $H_0$ is satisfied, for all researchers - corresponding to the notion of size of a test; see also the quote from Corradi and Swanson (2006) above. Conversely, under $H_1$, the probability of the event $\{\omega : D = \text{“reject } H_0\text{”}\}$ is 1 (asymptotically), corresponding to the notion of power.

Reporting the value of $\hat{Q}_{k_1^0}(\alpha)$ could be sufficient in some applications. In our case, the individual tests for $H_0 : k_1 \geq k_1^0$ will form the basis of a sequential procedure to provide an estimate of $k_1$, and therefore we also need a decision rule to choose, based on $\hat{Q}_{k_1^0}(\alpha)$, between $H_0$ and $H_1$.

We base such a decision rule on a Law of the Iterated Logarithm for $\hat{Q}_{k_1^0}(\alpha)$.

**Theorem 3.** We assume that Assumptions B1-B4 and C1 are satisfied, and that $M = c_0 T$ and $S = c_1 T$. Then it holds that

$$
\frac{\hat{Q}_{k_1^0}(\alpha) - (1 - \alpha)}{\sqrt{\alpha(1 - \alpha)}} = \Omega_{\text{a.s.}}\left(\sqrt{\frac{2\ln\ln S}{S}}\right),
$$

under $H_0 : k_1 \geq k_1^0$, for almost all realisations of $\{X_t, 1 \leq t \leq T\}$. Also, it holds that

$$
\hat{Q}_{k_1^0}(\alpha) \leq \epsilon,
$$

for all $\epsilon > 0$ under $H_1 : k_1 < k_1^0$, for almost all realisations of $\{X_t, 1 \leq t \leq T\}$.

Equations (3.21) and (3.22) complement (3.20), and quantify the gap in the asymptotic behaviour of $\hat{Q}_{k_1^0}(\alpha)$ according as the null $H_0$, or the alternative $H_1$, is satisfied. This
gap can be exploited to construct a decision rule based on \( \hat{Q}_{k_1^0} (\alpha) \), not rejecting the null when \( \hat{Q}_{k_1^0} (\alpha) \) exceeds a threshold, and rejecting otherwise. In theory, one could use the threshold defined in (3.21), but this, albeit valid asymptotically, is likely to be overly conservative in finite samples. An alternative, less conservative, a decision rule in favour of the null could be

\[
\hat{Q}_{k_1^0} (\alpha) \geq (1 - \alpha) - f (S),
\]

with \( f (S) \) a user-specified, non-increasing function of \( S \) such that

\[
\lim_{S \to \infty} f (S) = 0 \quad \text{and} \quad \limsup_{S \to \infty} \left( f (S) \right)^{-1} \sqrt{\frac{2 \ln \ln S}{S}} = 0.
\]

We call such a family of rules “strong rules”, since they originate from a “strong” result (the Law of the Iterated Logarithm). In Sections 4 and 5 we discuss possible choices of \( f (S) \), offering guidelines based on synthetic and real data, respectively.

### 3.2.1 Determining the number of common factors

The output of the decision rules proposed in (3.23) can be used for two purposes. Firstly, it is possible to check whether \( k_1 = 0 \) or \( k_2 = 0 \): this entails that there exists no factor structure in the rows or columns. In turn, this entails that there is no Kronecker structure \( \text{vec}(X_t) = (C \otimes R) \times \text{vec}(F_t) \) in \( \{X_t, 1 \leq t \leq T\} \), and that \( \text{vec}(X_t) \) can be represented as a standard one-way factor model. Similarly, finding \( k_1 = k_2 = 0 \) implies that there is no factor structure in the dataset. As a second application of (3.23), upon finding that \( k_1 > 0 \) (or \( k_2 > 0 \)), the individual decision rules proposed above can be cast in a sequential procedure to determine the number of common row and column factors, based on \( M_c \) (and \( M_r \)) and \( \tilde{M}_1 \) (and \( \tilde{M}_2 \)) respectively. As above, in the interest of brevity, we only report results for the estimation of \( k_1 \).

The estimator of \( k_1 \) (denoted as \( \tilde{k}_1 \) when using \( \hat{\Psi}_1 \) and \( \hat{Q}_{k_1^0} (\alpha) \), and \( \tilde{k}_1 \) when using \( \hat{\Psi}_1 \) and \( \tilde{Q}_{k_1^0} (\alpha) \)) is the output of the following algorithm:
**Step 1** Run the test for $H_0 : k_1 \geq 1$ based on either $\hat{Q}_1 (\alpha)$ or $\tilde{Q}_1 (\alpha)$. If the null is rejected with (3.23), set $\hat{k}_1 = 0$ (resp. $\tilde{k}_1 = 0$) and stop, otherwise go to the next step.

**Step 2** For $j = 2$, run the test for $H_0 : k_1 \geq j$ based on either $\hat{Q}_j (\alpha)$ or $\tilde{Q}_j (\alpha)$, constructed using an artificial sample $\{\eta_{j,s}^{(m)} , 1 \leq m \leq M\}$ generated independently across $1 \leq s \leq S$, and independently of $\{\eta_{1,s}^{(m)} , 1 \leq m \leq M\}$, $\ldots$, $\{\eta_{j-1,s}^{(m)} , 1 \leq m \leq M\}$.

If the null is rejected with (3.23), set $\hat{k}_1 = j-1$ (resp. $\tilde{k}_1 = j-1$) and stop; otherwise repeat step 2 until the null is rejected, or until a pre-specified value $k_{\text{max}}$ is reached.

The consistency of $\hat{k}_1$ and $\tilde{k}_1$ is stated in the next theorem.

**Theorem 4.** We assume that the assumptions of Theorem 3 are satisfied, and that $k_1 \leq k_{\text{max}}$. Then, as $\min \{p_1, p_2, T, M, S\} \to \infty$, it holds that $P^* (\hat{k}_1 = k_1 ) = 1$ and $P^* (\tilde{k}_1 = k_1 ) = 1$ for almost all realisations of $\{X_t, 1 \leq t \leq T\}$.

### 4 Simulation studies

In this section, we evaluate the finite sample performances of our strong rule to determine whether there is a factor structure, and of the sequential procedure to estimate the number of common factors. As far as the latter is concerned, we compare our Sequential Testing Procedure (henceforth denoted as STP) with the $\alpha$-PCA Eigenvalue-Ratio method (denote as $\alpha$-PCA) by Chen and Fan (2021) and Iterative Eigenvalue-Ratio (denoted as IterER) by Yu et al. (2021).

As far as implementation is concerned, recall that for the proposed STP, two different kinds of test statistics can be adopted, according to whether the projection technique is applied or not. We denote the method in which the test statistics are constructed using the eigenvalues of $M_c$ (or $M_r$) as STP$_1$, while STP$_2$ refers to the method in which the test statistics are constructed using the the eigenvalues of $\tilde{M}_1$ (or $\tilde{M}_2$). In order to obtain the eigenvalues of $\tilde{M}_1$ (or $\tilde{M}_2$), an initial estimator for $C$ (or $R$) is required. This entails
having to determine $k_2$ ($k_1$) in advance. In our simulations, we simply use $k_1$ and $k_2$ set equal to $k_{\text{max}}$, when computing the initial estimate for $R$ or $C$. When computing integrals such as $\int_{-\infty}^{\infty} \left[ \hat{\nu}_{k_1}(u) \right]^2 dF(u)$, we use the distribution of the standard normal as weight function $F(u)$, using a Gauss-Hermite quadrature with

$$\hat{\Psi}_{k_1} = \sum_{s=1}^{n_S} w_s \hat{\nu}_{k_1}(\sqrt{2}z_s),$$

where the $z_s$, $1 \leq s \leq n_S$, are the zeros of the Hermite polynomial $H_{n_S}(z)$ and the weights $w_s$ are defined as

$$w_s = \frac{2^{n_S-1}(n_S-1)!}{n_S[H_{n_S-1}(z_s)]^2}.$$

Thus, when computing $\hat{\nu}_{k_1}(u)$ in Step 2 of the algorithm, we construct $n_S$ of these statistics, each using $u = \pm \sqrt{2}z_s$. The values of the roots $z_s$, and of the corresponding weights $w_s$, are tabulated e.g. in Salzer et al. (1952). In our case, we have used $n_S = 4$, which corresponds to $w_1 = w_4 = 0.05$ and $w_2 = w_3 = 0.45$, and $u_1 = -u_4 = 2.4$ and $u_2 = -u_3 = 0.7$. Finally, we note that in the simulations in Yu et al. (2021), the $\alpha$-PCA method with $\alpha = \pm 1$ is found to have comparable finite sample performances with the case $\alpha = 0$; hence, we only report the results for $\alpha$-PCA with $\alpha = 0$.

Data generation

We generate the observed data matrices according to model (1.1), following the strategy in Yu et al. (2021). Specifically, when $k_1 \neq 0$ or $k_2 \neq 0$, i.e. when there is a factor structure, we generate the entries of $R$ and $C$ independently from the uniform distribution $U(-1, 1)$, and we let

$$\text{Vec}(F_t) = \phi \times \text{Vec}(F_{t-1}) + \sqrt{1 - \phi^2} \times \epsilon_t, \quad \epsilon_t \overset{i.i.d.}{\sim} \mathcal{N}(0, I_{k_1 \times k_2}),$$

$$\text{Vec}(E_t) = \psi \times \text{Vec}(E_{t-1}) + \sqrt{1 - \psi^2} \times \text{Vec}(U_t), \quad U_t \overset{i.i.d.}{\sim} \mathcal{MN}(0, U_E, V_E),$$

where $U_t$ is from a matrix-normal distribution, i.e., $\text{Vec}(U_t) \overset{i.i.d.}{\sim} \mathcal{N}(0, V_E \otimes U_E)$, and $U_E$
and $V_E$ are matrices with ones on the diagonal, and the off-diagonal entries are $a/p_1$ and $a/p_2$, respectively. The parameter $a$ controls cross-sectional dependence, with larger $a$ leading to stronger cross-dependence. In the case of no factor structure exists, we simply let $X_t = E_t$, with $E_t$ generated in the same way as in (4.24). The parameters $\phi$ and $\psi$ control both the temporal and cross-sectional correlations of $X_t$; with nonzero $\phi$ and $\psi$, the generated factors are temporally correlated while the idiosyncratic noises are both temporally and cross-sectionally correlated. Finally, in all the simulation settings, the reported results are based on 500 replications.

*Determining whether there is a factor structure*

We investigate the finite sample performance of our “strong” rule to determine whether a factor structure exists in the matrix time-series data. This offers a solution to the question in the discussion section of Yu et al. (2021): *is the matrix factor structure true for the time series?*

We provide results for $p_1 = T = \{100, 150\}$ and $p_2 = \{15, 20, 30\}$. In all our simulations, we use $k_{\text{max}} = 8$, although we tried different values for $k_{\text{max}}$ and the results show that the proposed methods are not sensitive to the choice of it. For the parameters $\phi$ and $\psi$ in (4.24), we set $\phi = \psi = 0.1$. As far as our decision rule is concerned, we have used $\alpha \in \{0.01, 0.05, 0.10\}$, $M = 300$ and $S = 300$; in (3.23), we have used $f(S) = S^{-1/4}$. Results using different combinations of $(\alpha, M, S)$ and different choice of $f(S)$ are in the supplement.

Firstly, we consider the case of no matrix factor structure by setting $(k_1, k_2) = \{(1, 1), (1, 3)\}$ in (4.24). We report the proportions of correctly claiming that there exists factor structure by the “strong” rule in Table 1, in which $\tilde{\Psi}_1^S$ and $\hat{\Psi}_1^S$ denote our procedure with and without projection technique, respectively. The results indicate that, when using $\tilde{\Psi}_1^S$, a factor structure is found more than 95% of the times, with the sole exception of the (small sample) case $(p_1, p_2, T) = (100, 15, 100)$. Results are always worse when $\hat{\Psi}_1^S$.

\footnote{We have also tried $\phi = \psi = 0.3$, and results are essentially the same.}
Table 1: Proportions of correctly determining whether there exists factor structure using (3.23) \( \hat{\Psi}_1 \) and \( \tilde{\Psi}_1 \) over 500 replications with \( M = S = 300, \phi = \psi = 0.1, f(S) = S^{-1/4} \).

| \( \alpha \) | \( (k_1, k_2) \) | Method | \( (p_1, T) = (100, 100) \) | \( (p_1, T) = (150, 150) \) |
|-----------|-------------|--------|-----------------|-----------------|
| \( (0,0) \) | \( \hat{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 |
| \( \tilde{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( (1,1) \) | \( \hat{\Psi}_1 \) | 0.61 | 0.826 | 0.974 | 0.828 | 0.944 | 1 |
| \( \tilde{\Psi}_1 \) | 0.97 | 1.000 | 1.000 | 0.994 | 1.000 | 1 | 1 |
| \( (1,3) \) | \( \hat{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \tilde{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( (0,0) \) | \( \hat{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \tilde{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( (1,1) \) | \( \hat{\Psi}_1 \) | 0.340 | 0.59 | 0.908 | 0.618 | 0.826 | 0.99 |
| \( \tilde{\Psi}_1 \) | 0.888 | 1.00 | 1.000 | 0.986 | 1.000 | 1.00 |
| \( (1,3) \) | \( \hat{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \tilde{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( (0,0) \) | \( \hat{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \tilde{\Psi}_1 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( (1,1) \) | \( \hat{\Psi}_1 \) | 0.182 | 0.418 | 0.834 | 0.452 | 0.714 | 0.962 |
| \( \tilde{\Psi}_1 \) | 0.816 | 0.988 | 1.000 | 0.956 | 1.000 | 1.00 |
| \( (1,3) \) | \( \hat{\Psi}_1 \) | 0.998 | 1 | 1 | 1 | 1 | 1 |
| \( \tilde{\Psi}_1 \) | 1.000 | 1 | 1 | 1 | 1 | 1 | 1 |

is used, although improvements are seen for larger sample sizes: this reinforces the case in favour of the projection method developed by Yu et al. (2021), especially when \( p_2 \) is small.

Secondly, we investigate the performance of our proposed methodology when there is no factor structure in the matrix time series \( X_t = E_t \), i.e., \( (k_1, k_2) = (0, 0) \), with \( E_t \) generated in the same way as in (4.24). In Table 1, we report the proportions of correctly claiming that there exists no factor structure, from which we can conclude that our proposed methodology is extremely powerful in identifying no factor structure, even in the small sample case \( (p_1, p_2, T) = (100, 15, 100) \), and with or without projection.

In Table C.3 in the supplement, we report results based on different combinations of
(\(\alpha, M, S\)). Results are essentially the same when using \(\hat{\Psi}_1^S\), whereas \(\hat{\Psi}_1^S\) is more sensitive (at least in small samples) to the choice of \(M\) and - albeit to a lesser extent - \(S\). In particular, as far as the former is concerned, smaller values of it seem to yield better results in small samples. In Table C.4, we assess the sensitivity to \(f(S)\); as can be expected, results are affected by the choice of the threshold, but this is only marginal when using \(\hat{\Psi}_1^S\) and, again, more pronounced when using \(\hat{\Psi}_1^S\).

Determining the number of common factors

We investigate the finite sample performances of the sequential testing procedure introduced in Section 3.2.1. We use essentially the same design as in the previous set of experiments, using \(p_2 = \{15, 20, 30\}\) and \(p_1 = T = \{100, 150\}\) and setting \(k_{\text{max}} = 8\) - as in the previous section, we tried different values for \(k_{\text{max}}\) and the results show that the STP methods are not sensitive to the choice of it. In order to evaluate the sequential procedure, we use it to estimate the number of row factors \(k_1\) considering the cases \((k_1, k_2) = \{(1, 1), (1, 3), (3, 1), (3, 3)\}\). As in the previous section, we use \(M = 300\) and \(S = 300\), and \(f(S) = S^{-1/4}\), and we only report results for the case \(\alpha = 0.01\).

Results are in Table 2, from which we can draw the following three conclusions. First, especially for the case \(k_1 = k_2 = 1\), and especially when \(p_2\) is small, the STP_2 procedure clearly dominates existing procedures, which have a pronounced tendency to understate the number of common factors - thus, in this case, mistakenly finding no evidence of a row factor structure and, consequently, mistakenly indicating a vector, as opposed to a matrix, factor model. In general, the STP_2 procedure dominates over all other procedures in almost all cases considered. We also note that the STP_1 method performs comparably with the \(\alpha\)-PCA method, but it is inferior to the IterER method, albeit with some exceptions - e.g. when \(k_1 = k_2 = 1\), and \(p_2 = 15\). Second, all procedures seem to improve as \(k_2\) increases; in such cases, the STP_2 procedure still retains its advantage, but less evidently than in the previous cases. Third, confirming what also found in the previous section, the STP_2 methodology always outperforms the STP_1 one. Indeed, the
Table 2: Simulation results for estimating $k_1$ in the form $x(y|z)$, $x$ is the sample mean of the estimated factor numbers based on 500 replications $\alpha = 0.01$, $M = S = 300$, $\phi = \psi = 0.1$, $f(S) = S^{-1/4}$, $y$ and $z$ are the numbers of underestimation and overestimation, respectively.

| $(k_1, k_2)$ | Method | $(p_1, T) = (100, 100)$ | $(p_1, T) = (150, 150)$ |
|-------------|--------|------------------|------------------|
|             |        | $p_2 = 15$     | $p_2 = 20$   | $p_2 = 30$ | $p_2 = 15$ | $p_2 = 20$ | $p_2 = 30$ |
| (1,1)       | STP$_1$ | 0.59(205|0) | 0.824(88|0) | 0.978(11|0) | 0.892(54|0) | 0.956(22|0) | 1(0|0) |
|             | STP$_2$ | 0.954(23|0) | 1(0|0) | 1(0|0) | 0.994(3|0) | 1(0|0) | 1(0|0) |
|             | IterER | 0.426(287|0) | 0.774(113|0) | 0.984(8|0) | 0.56(220|0) | 0.832(84|0) | 0.998(1|0) |
|             | $\alpha$-PCA | 0.732(134|0) | 826(87|0) | 878(61|0) | 964(18|0) | 962(19|0) | 996(2|0) |
| (1,3)       | STP$_1$ | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) |
|             | STP$_2$ | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) |
|             | IterER | 0.978(11|0) | 0.988(6|0) | 1(0|0) | 0.998(1|0) | 0.998(1|0) | 1(0|0) |
|             | $\alpha$-PCA | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) | 1(0|0) |
| (3,1)       | STP$_1$ | 2.172(171|0) | 2.736(54|0) | 2.984(4|0) | 2.674(59|0) | 2.938(11|0) | 3(0|0) |
|             | STP$_2$ | 2.926(13|0) | 3(0|0) | 3(0|0) | 2.988(2|0) | 3(0|0) | 3(0|0) |
|             | IterER | 2.958(7|0) | 3(0|0) | 3(0|0) | 2.982(3|0) | 3(0|0) | 3(0|0) |
|             | $\alpha$-PCA | 2.45(92|1) | 2.528(79|1) | 2.758(42|5) | 2.79(35|0) | 2.91(15|0) | 2.988(2|0) |
| (3,3)       | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|             | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|             | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|             | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |

STP$_1$ procedure works well for large sample sizes, but it is dominated even by the IterER methodology in small samples (with few exceptions), and, to a lesser extent, by the $\alpha$-PCA one. This suggests that the gains observed with STP$_2$ arise from two equally important sources: the use of the projection method proposed by Yu et al. (2021), and the use of our randomised tests cum the decision rule advocated in (3.23). In the supplement, we report further simulations to assess the impact of $(\alpha, M, S)$ and of $f(S)$ for the case $(k_1, k_2) = (3,3)$. The broad conclusion, even in this case, is that our procedure is not affected by these specifications, reinforcing the message that although some specifications need to be chosen by the researcher, the impact thereof is negligible.
5 Empirical studies

We illustrate our procedure through two applications: we firstly present an application to a set of macroeconomic indices (Section 5.1), and then consider a 3D image recognition dataset (Section 5.2).

5.1 Multinational macroeconomic indices

Inspired by Chen and Fan (2021), we investigate the presence (and dimension) of a matrix factor structure in a time series of macroeconomic indices. In our application, we use the dataset employed by Yu et al. (2021), containing records of $p_2 = 10$ macroeconomic indices across $p_1 = 8$ OECD countries over $T = 130$ quarters, ranging from 1988Q1 to 2020Q2. Whilst we refer to Yu et al. (2021) for details, the countries are the United States, the United Kingdom, Canada, France, Germany, Norway, Australia and New Zealand, which can be naturally divided into three groups as North American, European and Oceania based on their geographical locations. The indices are from 4 major groups, namely consumer price, interest rate, production, and international trade.\(^2\) As in Yu et al. (2021), we use the log-differences of each index, and each series is standardised.

We begin with testing whether there exists a matrix factor structure in the data. Results are in Tables 3 and 4; as also demonstrated by Chen and Fan (2021), there is overwhelming evidence in favour of a matrix structure in the data, for all test specifications considered, which corresponds to not rejecting the null hypotheses that $p_1, p_2 \geq 1$.

We now turn to determining the dimensions of the row and column factor spaces $k_1$ and $k_2$. The empirical exercise in Chen and Fan (2021) demonstrates that, possibly owing to the small cross-sectional sample sizes, the estimated number of common factors differs considerably depending on the estimation method employed. Table 5 reports the

\(^2\)In particular, we have considered the following indices, grouped by family: productivity (comprising: Total Index excluding Construction, Total Manufacturing, and GDP), CPI (comprising Food, Energy, and CPI Total), interest rates (long-term government bond yields, and 3-month Interbank rates and yields), and international trade (comprising total exports and total imports, both measured by value).
Table 3: Testing the null hypothesis $H_{01}: k_1 \geq 1$. Tests are based on $\alpha = 0.01$, $M = 100$, using the following thresholds: $f_1(S) = 1 - \alpha - \sqrt{2 \ln S/S}$, $f_2(S) = 1 - \alpha - S^{-1/3}$, $f_3(S) = 1 - \alpha - S^{-1/4}$, $f_4(S) = 1 - \alpha - S^{-1/5}$, $f_5(S) = (1 - \alpha)/2$.

| $S$ | $f_1(S)$ | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ | $f_1(S)$ | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 200 | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   |
| 300 | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   |
| 400 | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   |

Table 4: Testing the null hypothesis $H_{01}: k_2 \geq 1$. Tests are based on $\alpha = 0.01$, $M = 100$, using the following thresholds: $f_1(S) = 1 - \alpha - \sqrt{2 \ln S/S}$, $f_2(S) = 1 - \alpha - S^{-1/3}$, $f_3(S) = 1 - \alpha - S^{-1/4}$, $f_4(S) = 1 - \alpha - S^{-1/5}$, $f_5(S) = (1 - \alpha)/2$.

| $S$ | $f_1(S)$ | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ | $f_1(S)$ | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ |
|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 200 | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   |
| 300 | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   |
| 400 | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   | Accept   |

estimated numbers of common factors in the form of $(a, b)$ where $a$ denotes the number of common row factors, and $b$ the number of column factors. Using the results in Section 4 as guidelines, we have used $M = 100$, and $\alpha = 0.01$; by way of robustness check, we have also considered different values of $\alpha$, noting that, as $\alpha$ increases, our proposed procedure is more and more in favor of rejecting the existence of factors.\(^3\) We try different combinations of $S$ and $f(S)$ to shed further light on the impact of these specifications; in particular, we use: the thresholds employed also in Section 4 - i.e. $f(S) = S^{-a}$ with $a = \frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$; a conservative threshold, $f(S) = \sqrt{2 \ln S/S}$; and a very “liberal” one, with $f(S) = (1 - \alpha)/2$.

\(^3\)Unreported results show that, for $\alpha \geq 0.05$, one would find $k_1 = 1$ and $k_2 = 0$, thus rejecting a matrix factor structure altogether. This reinforces the findings in the previous section, where it was noted that our procedure, in small samples, requires a smaller $\alpha$ in order to estimate the factor dimensions correctly.
Table 5: Estimated numbers of row and column factors for macroeconomic indices data set. Five ways to select the threshold: $f_1(S) = 1 - \alpha - \sqrt{2 \ln S/S}$, $f_2(S) = 1 - \alpha - S^{-1/3}$, $f_3(S) = 1 - \alpha - S^{-1/4}$, $f_4(S) = 1 - \alpha - S^{-1/5}$, $f_5(S) = (1 - \alpha)/2$.

|       | No-projection |       |       |       |       |
|-------|---------------|-------|-------|-------|-------|
| $S$   | $f_1(S)$      | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ |
| 200   | (1,4)         | (1,4)  | (1,4)  | (1,4)  | (2,4)  |
| 300   | (1,3)         | (1,0)  | (1,4)  | (1,4)  | (2,4)  |
| 400   | (1,3)         | (1,3)  | (1,4)  | (1,4)  | (2,4)  |

|       | Projection    |       |       |       |       |
|-------|---------------|-------|-------|-------|-------|
| $S$   | $f_1(S)$      | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ |
| 200   | (1,4)         | (1,3)  | (1,4)  | (2,4)  | (2,4)  |
| 300   | (1,3)         | (1,3)  | (1,3)  | (1,4)  | (2,4)  |
| 400   | (1,3)         | (1,3)  | (1,3)  | (1,4)  | (2,4)  |

Results are only partly affected by the choice of $S$ and $f(S)$, which play a very minor role (a desirable form of robustness). As pointed out in Section 4, the projection technique should work better in finite samples, but in our application results are actually comparable between the two techniques. According to Table 5, the number of row factors is at most $k_1 = 2$: whilst there is strong evidence in favour of at least one common factor (thus confirming that there is a matrix factor structure, as also found by Chen and Fan (2021) using the eigenvalue ratio approach), the second factor seems to be weaker, and deciding whether $\hat{k}_1 = 1$ or $2$ can be done on account of the researcher’s preference for (possible) underestimation versus overestimation. Reading these results in conjunction with Table 9 in Yu et al. (2021) would suggest choosing $\hat{k}_1 = 2$: factors broadly represent the different geographical locations, but European countries (particularly the largest economy, Germany) seem to also share a common factor structure with North America, speaking to the integration between the two economic areas. As far as $\hat{k}_2$ is concerned,
using the majority vote when applying the projection technique suggests \( \hat{k}_2 = 4 \); even in this case, there seems to be some evidence in favour of \( \hat{k}_2 = 3 \) also, again suggesting that, possibly, the fourth common factor is weaker than the others. Interestingly, the results in Chen and Fan (2021) using two different techniques (respectively, a scree-plot and an eigenvalue ratio approach) indicate that \( k_2 \) may range between 2 and 6, so our proposed approach offers a considerable narrowing down; Yu et al. (2021) also find \( \hat{k}_2 = 4 \) or 5, but their results with \( \hat{k}_2 = 4 \) show that this estimate explains the data very well, and it matches the four groups in which the indices belong.

### 5.2 MNIST: handwritten digit numbers

In our second example, we apply matrix time series to an image recognition dataset, namely the Modified National Institute of Standards and Technology (MNIST) dataset, which has been analysed in numerous applications of classification algorithms and machine learning, and which consists of images of handwritten digit numbers from 0 to 9. As is typical in these applications, each single (gray-scale) image represents the matrix \( X_t \), whose elements are the pixels of the image. We only use the training set, which contains \( T = 10,000 \) images; in our dataset, the digits have been size-normalized and centered in a fixed-size image with \( 28 \times 28 \) pixels, thus having \( p_1 = p_2 = 28 \). We standardize the pixels at each location.

The estimated numbers of row and column factors are reported in Table 6 with multiple combinations of \( \alpha, S \) and \( f(S) \), as in the previous section. Since \( p_1 \) and \( p_2 \) are larger than those in our previous example, we use \( M = 200 \) in the testing (we tried \( M = 100 \) and 300 and results are essentially the same).

Results and conclusions are similar to those in the previous section. In particular, a bigger difference emerges in the performance of projection versus non-projection based estimation, with the former offering a performance which is more robust across the different specifications. In light also of the results in Section 4, the findings in this section
Table 6: Estimated numbers of row and column factors for handwritten digit number data set. Five ways to select the threshold. Q1: $1 - \alpha - \sqrt{2\ln \ln S/S}$; Q2: $1 - \alpha - S^{-1/3}$; Q3: $1 - \alpha - S^{-1/4}$; Q4: $1 - \alpha - S^{-1/5}$; Q5: $(1 - \alpha)/2$.

| $S$ | $\alpha$ | $f_1(S)$ | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ | $f_1(S)$ | $f_2(S)$ | $f_3(S)$ | $f_4(S)$ | $f_5(S)$ |
|-----|---------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 200 | 0.01    | (0,3)     | (0,3)     | (0,3)     | (0,4)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     |
| 300 | 0.01    | (0,3)     | (0,3)     | (0,3)     | (0,4)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     |
| 400 | 0.01    | (0,3)     | (0,3)     | (0,3)     | (0,3)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     | (4,5)     |
| 200 | 0.05    | (0,0)     | (0,0)     | (0,0)     | (0,1)     | (0,1)     | (4,3)     | (4,3)     | (4,3)     | (4,3)     | (4,3)     |
| 300 | 0.05    | (0,0)     | (0,0)     | (0,0)     | (0,1)     | (0,1)     | (4,3)     | (4,3)     | (4,3)     | (4,3)     | (4,3)     |
| 400 | 0.05    | (0,0)     | (0,1)     | (0,1)     | (0,1)     | (4,3)     | (4,3)     | (4,3)     | (4,3)     | (4,3)     | (4,3)     |
| 200 | 0.1     | (0,0)     | (0,0)     | (0,0)     | (0,0)     | (0,0)     | (4,3)     | (0,3)     | (4,3)     | (4,3)     | (4,3)     |
| 300 | 0.1     | (0,0)     | (0,0)     | (0,0)     | (0,0)     | (0,0)     | (0,3)     | (0,3)     | (4,3)     | (4,3)     | (4,3)     |
| 400 | 0.1     | (0,0)     | (0,0)     | (0,0)     | (0,0)     | (0,0)     | (0,3)     | (0,3)     | (0,3)     | (4,3)     | (4,3)     |

strengthen the case in favour of the projection-based estimator. We note that, when using this technique, the number of row factors is almost always (save for some exceptions, based on a large $\alpha$ and a high threshold $f(S)$) estimated as $\hat{k}_1 = 4$. As far as $k_2$ is concerned, all results indicate that this is not smaller than 3, and the most conservative approach (based on using $\alpha = 0.01$) indicates the possibility of having $k_2 = 5$. This may suggest that two common factors are less pervasive than the others. In order to avoid underestimation, we recommend taking $\hat{k}_1 = 4$ and $\hat{k}_2 = 5$ in this example. Indeed, in any real applications, we suggest the readers to try different combinations of $\alpha$ and threshold, and select the numbers of factors based on the real tolerance of underestimation and overestimation errors. Smaller $\alpha$ and threshold are in favor of $H_0$, but in higher risk of overestimation. Larger $\alpha$ and threshold will lead to opposite results.

For this example, we further compare the results for different digit numbers in Table 7 using only a small part of the images associated with a specific number. In this table,
we report results corresponding to $S = 400$, $\alpha = 0.01$ and $f_5(S)$; we point out however that using different specifications leaves the results virtually unchanged. Results are remarkably stable across the different digits.

**Table 7:** Estimated numbers of row and column factors for different digit numbers.

| Projection | “0” | “1” | “2” | “3” | “4” | “5” | “6” | “7” | “8” | “9” |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| No         | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,4) |
| Yes        | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) | (4,5) |

6 Conclusions

In this contribution, we studied the important issue of determining the presence and dimension of the row and column factor structures in a series of matrix-valued data. Our methodology allows to check whether there is a factor structure in either dimension (row and column), thus helping the researcher decide whether data should be studied using the techniques developed by the literature for a standard vector factor model, or whether different techniques should be employed that are specific to tensor-valued data. In addition to finding evidence of a factor structure, we also proposed a methodology to estimate the number of common row and column factors.

Technically, our methodology is based on exploiting the eigen-gap which is found, in the presence of common factors, in the sample second moment matrix of the series. For each eigenvalue, we propose a test for the null that it diverges (as opposed to being bounded). Our tests are similar to the randomised tests (designed for vector factor models) proposed in Trapani (2018), but - crucially - we propose a “strong”, Law-of-the-Iterated-Logarithm-inspired, decision rule which does away with the randomness, thus ensuring that all researchers using the same datasets will obtain the same results. In our paper, we proposed two procedures, based on two different ways of computing the sample second
moment matrix: specifically, we use a “flattened” version of the matrix-valued series, and a projected version thereof, as proposed in Yu et al. (2021). We found that both techniques work very well in large samples, but, in small samples, the projection-based method is superior in all scenarios considered, also outperforming other existing methods.

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A Discussion: Assumptions B1-B4

As mentioned in the main text, Assumptions B1-B4 hold for a wide variety of serial dependence assumptions. In this section, we go one step further and provide some examples of data generating processes that satisfy our assumptions. Our examples pertain to the time series dimension, where there is a natural ordering for the observations. In particular, we assume that the common factors $\text{vec}(F_t)$ and the individual errors $e_{ij,t}$ can be represented as causal processes, viz.

$$F_t = g^F(\eta_t, \ldots, \eta_{-\infty}),$$  \hfill (A.1)

$$e_{ij,t} = g^e_{ij}(\eta_{ij,t}, \ldots, \eta_{ij,-\infty}),$$  \hfill (A.2)

where $g^F : \mathbb{R}^{(-\infty, \infty) \times \cdots \times (-\infty, \infty)} \rightarrow \mathbb{R}^{k_1 \times k_2}$ and $g^e_{ij} : \mathbb{R}^{(-\infty, \infty)} \rightarrow \mathbb{R}$ are measurable functions such that the random variables $\text{vec}(F_t)$ and $e_{ij,t}$ are well-defined, and $\{\eta_t, -\infty < t < \infty\}$ and $\{\eta_{ij,t}, -\infty < t < \infty\}$ are i.i.d., zero mean sequences. According to (A.1) and (A.2), $\text{vec}(F_t)$ and $e_{ij,t}$ are stationary and ergodic processes; the causal process/"physical system" representation in (A.1) and (A.2) is now a standard way of modelling dependence, after the seminal contribution by Wu (2005) (see also Aue et al., 2009 and Berkes et al., 2011).

We define the quantities

$$\delta_{t,p}^F = \left| g^F(\eta_t, \ldots, \eta_0, \ldots, \eta_{-\infty}) - g^F(\eta_t, \ldots, \eta_0', \ldots, \eta_{-\infty}) \right|_p,$$

$$\delta_{t,p}^{e,ij} = \left| g^e_{ij}(\eta_{ij,t}, \ldots, \eta_{ij,0}, \ldots, \eta_{ij,-\infty}) - g^e_{ij}(\eta_{ij,t}, \ldots, \eta'_{ij,0}, \ldots, \eta_{ij,-\infty}) \right|_p,$$

where $\eta_0'$ and $\eta'_{ij,0}$ are independent copies of $\eta_0$ and $\eta_{ij,0}$ respectively, such that $\eta_0' \overset{D}{=} \eta_0$ and $\eta'_{ij,0} \overset{D}{=} \eta_{ij,0}$ and $\eta_0'$ and $\eta'_{ij,0}$ are independent of $\{\eta_t, -\infty < t < \infty\}$ and $\{\eta_{ij,t}, -\infty < t < \infty\}$ respectively.

**Lemma A.1.** We assume that (A.1) holds, with $|F_{ij,i}|_p < \infty$ for all $1 \leq i \leq k_1$ and
$1 \leq j \leq k_2$, and $p = 4 + \epsilon$ for some $\epsilon > 0$ and $\sum_{t=0}^{\infty} \delta_{i,p}^F < \infty$. Then Assumption B1(iii) is satisfied.

**Proof.** For simplicity, we let $E (F_{h_1 h_2,t} F_{l_1 l_2,t}) = 0$, and we define the $(i,j)$-th component of $g^F$ as $g^F_{i,j}$. We have

$$F_{h_1 h_2,t} F_{l_1 l_2,t} = g^F_{h_1 h_2} (\eta_t, \ldots, \eta_0, \ldots, \eta_{-\infty}) g^F_{l_1 l_2} (\eta_t, \ldots, \eta_0, \ldots, \eta_{-\infty})$$

$$= h (\eta_t, \ldots, \eta_0, \ldots, \eta_{-\infty}).$$

Define now

$$F'_{h_1 h_2,t} = g^F_{h_1 h_2} (\eta_t, \ldots, \eta_0', \ldots, \eta_{-\infty})$$

$$F'_{l_1 l_2,t} = g^F_{l_1 l_2} (\eta_t, \ldots, \eta_0', \ldots, \eta_{-\infty})$$

and note that

$$F'_{h_1 h_2,t} F'_{l_1 l_2,t} = h (\eta_t, \ldots, \eta_0', \ldots, \eta_{-\infty}).$$

It is easy to see that

$$|F_{h_1 h_2,t} F_{l_1 l_2,t} - F'_{h_1 h_2,t} F'_{l_1 l_2,t}|_{p/2}$$

$$\leq |F'_{h_1 h_2,t} (F_{l_1 l_2,t} - F'_{l_1 l_2,t})|_{p/2} + |F'_{l_1 l_2,t} (F_{h_1 h_2,t} - F'_{h_1 h_2,t})|_{p/2}$$

$$+ |(F_{h_1 h_2,t} - F'_{h_1 h_2,t}) (F_{l_1 l_2,t} - F'_{l_1 l_2,t})|_{p/2}$$

$$\leq |F'_{h_1 h_2,t}|_p |F_{l_1 l_2,t} - F'_{l_1 l_2,t}|_p + |F'_{l_1 l_2,t}|_p |F_{h_1 h_2,t} - F'_{h_1 h_2,t}|_p$$

$$+ |F_{h_1 h_2,t} - F'_{h_1 h_2,t}|_p |F_{l_1 l_2,t} - F'_{l_1 l_2,t}|_p$$

$$\leq c_0 |F_{l_1 l_2,t} - F'_{l_1 l_2,t}|_p + c_1 |F_{h_1 h_2,t} - F'_{h_1 h_2,t}|_p$$

$$+ |F_{h_1 h_2,t} - F'_{h_1 h_2,t}|_p |F_{l_1 l_2,t} - F'_{l_1 l_2,t}|_p$$

$$\leq c_0 \delta^F_{i,p}.$$

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Lemma A.2 in Liu and Lin (2009) entails that

\[
E \left[ \sum_{t=1}^{T} (F_{h_1 h_2, t} F_{l_1 l_2, t} - E(F_{h_1 h_2, t} F_{l_1 l_2, t})) \right]^{p/2} \leq c_0 T^{p/4}.
\]

Noting that \( p > 4 \), Theorem 1 in Móricz (1976) immediately yields the desired result (see also Corollary 1 in Berkes et al., 2011).

\[\square\]

**Lemma A.2.** We assume that (A.2) holds, with \( |e_{ij,t}|_4 < \infty \) for all \( 1 \leq i \leq p_1 \) and \( 1 \leq j \leq p_2 \), and \( \sum_{l=1}^{T} \sum_{m=1}^{\infty} \delta_{m,q}^{e_{ij,t}} < \infty \) for all \( q \leq 4 \). Then it holds that

\[
\sum_{s=1}^{T} |E(e_{ij,t} e_{lh,s})| \leq c_0, \tag{A.3}
\]

for all \( 1 \leq i, l \leq p_1 \) and \( 1 \leq j, h \leq p_2 \), and

\[
\sum_{s=1}^{T} |\text{Cov}(e_{ij,t} e_{i_1 j_1, t}, e_{lh,s} e_{l_1 h_1, s})| \leq c_0, \tag{A.4}
\]

for all \( 1 \leq i, l, i_1, l_1 \leq p_1 \) and \( 1 \leq j, h, j_1, h_1 \leq p_2 \).

**Proof.** We only show (A.3), as (A.4) follows from essentially the same arguments. By stationarity

\[
\sum_{s=1}^{T} |E(e_{ij,t} e_{lh,s})| = |E(e_{ij,0} e_{lh,0})| + \sum_{m=1}^{T} |E(e_{ij,0} e_{lh,m})|,
\]

and our assumptions immediately yield \( |E(e_{ij,0} e_{lh,0})| \leq |e_{ij,0}|_2 |e_{lh,0}|_2 < \infty \). Define now the sigma-field \( \mathcal{F}_{ijlh, -k}^{0} = \mathcal{F} \{ \eta_{i_j,0}, \ldots, \eta_{i_{j-1}, -k}; \eta_{lh,0}, \ldots, \eta_{lh, -k} \} \), and consider

\[
\sum_{m=1}^{T} |E(e_{ij,0} e_{lh,m})| \leq \sum_{m=1}^{T} |E(e_{lh,m} (e_{ij,0} - E(e_{ij,0} | \mathcal{F}_{ijlh, -[am]}^{0})))| + \sum_{m=1}^{T} |E(e_{lh,m} E(e_{ij,0} | \mathcal{F}_{ijlh, -[am]}^{0})))| = I + II,
\]

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for some $0 < a < 1$. We have

$$\sum_{m=1}^{T} |E (e_{lh,m} (e_{ij,0} - E (e_{ij,0} | F_{ijlh,-[am]} )))| \leq \sum_{m=1}^{T} \| e_{lh,m} \|_2 \| e_{ij,0} - E (e_{ij,0} | F_{ijlh,-[am]} ) \|_2 \leq c_0 \sum_{m=1}^{T} \| e_{ij,0} - E (e_{ij,0} | F_{ijlh,-[am]} ) \|_2^2.$$

Consider now the projections $Q_j X_0 = E (X_0 | F_{X,-j})$, for a generic process $X_t = g (\eta_t, ..., \eta_{-\infty})$, where $F_{X,-k} = \{ \eta_0, ..., \eta_k \}$, and define $\tilde{Q}_j X_0 = Q_j X_0 - Q_{j-1} X_0$. Clearly, $X_0 = \lim_{j \to \infty} Q_j X_0$. We have

$$e_{ij,0} - E (e_{ij,0} | F_{ijlh,-[am]} ) = \sum_{k=[am]}^{\infty} \tilde{Q}_k e_{ij,0},$$

so that

$$\sum_{m=1}^{T} \| e_{ij,0} - E (e_{ij,0} | F_{ijlh,-[am]} ) \|_2 \leq \sum_{m=1}^{T} \sum_{k=[am]}^{\infty} \| \tilde{Q}_k e_{ij,0} \|_2 \leq \sum_{m=1}^{T} \sum_{k=[am]}^{\infty} \| \tilde{Q}_k e_{ij,0} \|_2^2.$$

Following the proof of Theorem 1(iii) in Wu (2005), it is immediate to verify that $\| \tilde{Q}_k e_{ij,0} \|_2 \leq \delta_{k,2} e_{ij}$, and therefore term $I$ in (A.5) is bounded. Also

$$\left| E (e_{lh,m} E (e_{ij,0} | F_{ijlh,-[am]} )) \right| = \left| E (E (e_{lh,m} | F_{ijlh,-[am]} ) E (e_{ij,0} | F_{ijlh,-[am]} )) \right| \leq \| e_{ij,0} \|_2 \| E (e_{lh,m} | F_{ijlh,-[am]} ) \|_2 \left| E (e_{lh,m} | F_{ijlh,-[am]} ) \right| \leq c_0 \| E (e_{lh,m} | F_{ijlh,-[am]} ) \|_2.$$
Consider now the projections $P_0 X_k = E \left( X_k | F_{X_k,-\infty} \right) - E \left( X_k | F_{X_k,-\infty}^{-1} \right)$, and note

$$E \left( e_{lh,m} | F_{ijlh,-[am]} \right) = \sum_{r=m-[am]}^{\infty} P_{m-r} e_{lh,m} = \sum_{r=m-[am]}^{\infty} P_0 e_{lh,r},$$

by stationarity. Again, using Theorem 1(i) in Wu (2005), $|P_0 e_{lh,r}|_2 \leq \delta_{e,2}^{lh}$, so that

$$\sum_{m=1}^{T} \left| E \left( e_{lh,m} | F_{ijlh,-[am]} \right) \right|_2 \leq \sum_{m=1}^{T} \sum_{r=m-[am]}^{\infty} \left| P_0 e_{lh,r} \right|_2 < \infty,$$

which gives the desired result. □

Representation (A.1) and (A.2) - together with the assumption that, essentially, the dependence measures $\delta_{F,t,p}$ and $\delta_{e,ij,t}$ are summable - allows for a wide variety of data generating processes. Indeed, as shown e.g. in Barigozzi and Trapani (2021), many commonly employed models result in having $\delta_{F,t,p}$ and $\delta_{e,ij,t}$ exponentially decay in $t$. Hereafter, we mention the most important examples of DGPs that satisfy our assumptions.

Liu and Lin (2009) show that (A.1) and (A.2), together with the summability conditions, hold for (multivariate) linear processes: hence, all our theory can be used if either $F_t$ or $e_{ij,t}$ (or both) follow a stationary (V)ARMA model. In addition, Liu and Lin (2009) (see their Corollaries 3.1-3.7) prove that (A.1) and (A.2), and the summability conditions, hold for several nonlinear models, e.g. for stationary Random Coefficient AutoRegressions, nonlinear transformations of VARMA models, threshold autoregressive models, and other non-linear mappings. Aue et al. (2009) study several multivariate GARCH processes, concluding that (A.1) and (A.2), together with the summability conditions, hold for e.g. the CCC-GARCH of Bollerslev (1990) (and for the CCC-GARCH studied by Jeantheau, 1998), and for the multivariate exponential GARCH (Kawakatsu, 2006) - see also Barigozzi and Trapani (2021). In all the cases mentioned above, the summability conditions hold with $\delta_{F,t,p}$ and $\delta_{e,ij,t}$ decaying exponentially in $t$.

We also point out that, as far as univariate GARCH models are concerned, Barigozzi and Trapani (2021) (see their Corollary 9) have shown the validity of our assumptions for the broad
class of augmented GARCH models (including standard GARCH, the Threshold GARCH model, and the Exponential GARCH model).
B Technical lemmas

Given an $m \times n$ matrix $A$, recall that $\|A\|$ denotes its spectral norm, $a_{ij}$ its element in position $(i, j)$, and we use $\|A\|_{\text{max}}$ to denote the largest $|a_{ij}|$. Further, we denote the Frobenious norm as $\|A\|_F = \left( \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right)^{1/2}$ and the $L_1$ and $L_\infty$ induced norms as $\|A\|_1 = \max_{1 \leq i \leq n} \sum_{i=1}^m |a_{ij}|$ and $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{ij}|$ respectively.

Lemma B.1. Consider a multi-index partial sum process $U_{S_1, \ldots, S_h} = \sum_{i_1=1}^{S_1} \cdots \sum_{i_h=1}^{S_h} \xi_{i_1, \ldots, i_h}$, and assume that, for some $p \geq 1$

$$E|U_{S_1, \ldots, S_h}|^p \leq c_0 \prod_{j=1}^h S_j^{d_j},$$

where $d_j \geq 1$ for all $1 \leq j \leq h$. Then it holds that

$$\limsup_{\min(S_1, \ldots, S_h) \to \infty} \frac{|U_{S_1, \ldots, S_h}|}{\prod_{j=1}^h S_j^{d_j/p} \left( \ln S_j \right)^{1+\frac{1}{p}+\epsilon}} = 0 \text{ a.s.},$$

for all $\epsilon > 0$.

Proof. We begin by showing that the function $f(x_1, \ldots, x_h) = \prod_{j=1}^h x_j^{d_j}$ is superadditive. Consider the vector $(y_1, \ldots, y_h)$, such that $y_j \geq x_j$ for all $1 \leq j \leq h$, and, for any two non-zero numbers $s$ and $t$ such that

$$x_i + s \leq y_i + t.$$

We show that

$$\frac{1}{s} \left[ f(x_1, \ldots, x_i + s, \ldots, x_h) - f(x_1, \ldots, x_h) \right] \leq \frac{1}{t} \left[ f(y_1, \ldots, y_i + t, \ldots, y_h) - f(y_1, \ldots, y_h) \right].$$

(B.1)
Indeed,

\[ f(x_1, \ldots x_i + s, \ldots, x_h) - f(x_1, \ldots, x_h) = \left( \prod_{j \neq i} x_j^{d_j} \right) \left( (x_i + s)^{d_i} - x_i^{d_i} \right), \]

and, by construction, \( \prod_{j \neq i} x_j^{d_j} \leq \prod_{j \neq i} y_j^{d_j} \). Also, note that the function \( g(x_i) = x_i^{d_i} \) is convex, and therefore it holds that (see Potra, 1985) there exists a mapping \( \delta(x_i + s, x_i) \) such that \( \delta(u, v) \leq \delta(p, q) \) whenever \( u \leq p \) and \( v \leq q \), such that

\[ \frac{1}{s} \left( (x_i + s)^{d_i} - x_i^{d_i} \right) = \delta(x_i + s, x_i). \]

Thus it follows immediately that (B.1) holds, for every \( 1 \leq i \leq h \). This entails that \( f(x_1, \ldots, x_h) \) is an \( S \)-convex function (see Definition 2.1 and Proposition 2.3 in Potra, 1985), with \( f(0, \ldots, 0) = 0 \). Hence, by Proposition 2.9 in Potra (1985), \( f(x_1, \ldots, x_h) \) is superadditive. We can now apply the maximal inequality for rectangular sums in Corollary 4 of Moricz (1983), which stipulates that

\[ E \max_{1 \leq i_1 \leq S_1, \ldots, 1 \leq i_h \leq S_h} |U_{i_1, \ldots, i_h}|^p \leq c_0 \prod_{j=1}^h S_j^{d_j} (\ln S_j)^p, \]

so that

\[
\sum_{S_1=1}^{\infty} \cdots \sum_{S_h=1}^{\infty} \frac{1}{\prod_{j=1}^h S_j} P \left( \max_{1 \leq i_1 \leq S_1, \ldots, 1 \leq i_h \leq S_h} |U_{i_1, \ldots, i_h}| \geq \varepsilon \prod_{j=1}^h S_j^{d_j/p} (\ln S_j)^{1+\frac{1}{p}+\varepsilon} \right) \\
\leq \sum_{S_1=1}^{\infty} \cdots \sum_{S_h=1}^{\infty} \frac{1}{\prod_{j=1}^h S_j} \left( E \max_{1 \leq i_1 \leq S_1, \ldots, 1 \leq i_h \leq S_h} |U_{i_1, \ldots, i_h}|^p \right) \varepsilon^{-p} \left( \prod_{j=1}^h S_j^{d_j/p} (\ln S_j)^{1+\frac{1}{p}+\varepsilon} \right)^{-p} \leq c_0.
\]

From here onwards, the desired result follows from Lemma A.1 in Barigozzi and Trapani (2021). \( \square \)
Lemma B.2. We assume that Assumptions B1-B4 are satisfied. It holds that

$$\|\hat{C} - CH_1\|_F = o_{a.s.}\left(\frac{1}{p_2} \left(\frac{1}{p_2} + \frac{1}{\sqrt{T} p_1}\right) l'_{p_1,p_2,T}\right),$$

where

$$H_1 = \left(\frac{1}{T} \sum_{t=1}^T F_t F_t'\right) \frac{C'\hat{C} \hat{\Lambda}_{c}^{-1}}{p_2},$$

$$l'_{p_1,p_2,T} = \left(\ln T \ln^2 p_1 \ln^2 p_2\right)^{1/2+\epsilon}, \quad (B.2)$$

for all $\epsilon > 0$.

Proof. The proof is related to the proof of Theorem 3.3 in Yu et al. (2021), and some passages are omitted when possible; also, to avoid overburdening the notation, we let $k_1 = k_2 = 1$. Note that, by construction

$$\hat{C} = \hat{M}_c \hat{\Lambda}_{c}^{-1},$$

where $\hat{\Lambda}_{c}$ is the diagonal matrix containing the leading $k_2$ eigenvalues of $\hat{M}_c = (Tp_1p_2)^{-1} \sum_{t=1}^T X_t' X_t$.

Therefore

$$\hat{C} = C \left(\frac{1}{T} \sum_{t=1}^T F_t F_t'\right) \frac{C'\hat{C} \hat{\Lambda}_{c}^{-1}}{p_2} + \frac{1}{Tp_1p_2} C \sum_{t=1}^T F_t R_t E_t' \hat{C} \hat{\Lambda}_{c}^{-1}$$

$$+ \frac{1}{Tp_1p_2} \sum_{t=1}^T E_t' R_t C' \hat{C} \hat{\Lambda}_{c}^{-1} + \frac{1}{Tp_1p_2} \sum_{t=1}^T E_t' E_t C' \hat{C} \hat{\Lambda}_{c}^{-1}$$

$$= CH_1 + (II + III + IV) \hat{C} \hat{\Lambda}_{c}^{-1},$$

having defined

$$H_1 = \left(\frac{1}{T} \sum_{t=1}^T F_t F_t'\right) \frac{C'\hat{C} \hat{\Lambda}_{c}^{-1}}{p_2}.$$

We begin by using exactly the same logic as in the proof of Theorem 1, it can be shown
that there exist a positive constant $c_0$ such that

$$\lambda_j \left( \frac{1}{Tp_1p_2} \sum_{t=1}^{T} X_t'X_t \right) = \Omega_{a.s.} \left( c_0 \right),$$

for all $j \leq k_2$; in turn, this entails that $\hat{\Lambda}_c^{-1}$ exists. Also, recall that $\|\hat{C}\|_F = p_2^{1/2}$. Consider now

$$\frac{1}{p_2} \left\| \frac{1}{Tp_1p_2} C \sum_{t=1}^{T} F_t R_t^* E_t \hat{C} \right\|_2^2 \leq \frac{1}{T^2 p_1^2 p_2^2} \|C\|_2^2 \left\| \sum_{t=1}^{T} F_t R_t^* E_t \right\|_2^2,$$

where $E_{j,t}$ denotes the $j$-th column of $E_t$. Hence

$$\frac{1}{p_2} \left\| \frac{1}{Tp_1p_2} C \sum_{t=1}^{T} F_t R_t^* E_t \hat{C} \right\|_2^2 \leq c_0 \frac{1}{T^2 p_1^2 p_2^2} \sum_{i=1}^{p_2} \left( \sum_{j=1}^{p_1} R_{j_i} e_{j_i, t} F_t \right)^2.$$

Given that

$$E \left( \frac{1}{T^2 p_1^2 p_2^2} \sum_{i=1}^{p_2} \left( \sum_{j=1}^{p_1} R_{j_i} e_{j_i, t} F_t \right)^2 \right)$$

$$= E \left( \frac{1}{T^2 p_1^2 p_2^2} \sum_{j=1}^{p_1} \sum_{h=1}^{p_1} R_j R_h \left( \sum_{t=1}^{T} e_{j_i, t} F_t \right) \left( \sum_{t=1}^{T} e_{h_i, t} F_t \right) \right)$$

$$\leq c_0 \frac{1}{T^2 p_1^2 p_2^2} \sum_{i=1}^{p_2} \sum_{j=1}^{p_1} \sum_{h=1}^{p_1} R_j E \left( \sum_{t=1}^{T} e_{j_i, t} F_t \right) \left( \sum_{t=1}^{T} e_{h_i, t} F_t \right)$$

$$\leq c_0 \frac{1}{T p_1},$$

by Assumption B4(i), Lemma B.1 entails that

$$\frac{1}{p_2} \left\| (II) \hat{\Lambda}_c^{-1} \right\|_2^2 = o_{a.s.} \left( \frac{1}{T p_1} \left( \ln T \ln p_1 \ln p_2 \right)^{1+\epsilon} \right),$$

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for all $\epsilon > 0$. Similarly

$$
\frac{1}{p_2} \left\| \frac{1}{Tp_1p_2} \sum_{t=1}^{T} E_t' R F_t C' \right\|_2^2 \leq \frac{1}{T^2p_1^2p_2^2} \left\| \sum_{t=1}^{T} E_t' R F_t \right\|_2^2 \left\| C \right\|_2^2 \left\| \hat{C} \right\|_2^2
$$

and therefore by the same passages as above, it follows that

$$
\frac{1}{p_2} \left\| (III) \hat{C} \right\|_2^2 = o_{a.s.} \left( \frac{1}{T^2p_1^2} \ln T \ln p_1 \ln p_2 \right)^{1+\epsilon}.
$$

We note that the rates for $II$ and $III$ are not necessarily sharp (see the proof of Theorem 3.3 in Yu et al., 2021), but they suffice for our purposes. Finally, we have

$$
\frac{1}{p_2} \left\| \frac{1}{Tp_1p_2} \sum_{t=1}^{T} E_t' E_t \hat{C} \right\|_2^2 \leq \frac{1}{T^2p_1^2p_2^2} \left\| \sum_{t=1}^{T} (E_t' E_t) \right\|_2^2 \left\| C \right\|_2^2 \left\| \hat{C} \right\|_2^2 + \frac{1}{T^2p_1^2p_2^3} \left\| \sum_{t=1}^{T} (E_t' - E_t'E_t) (C H_1) \right\|_2^2 \left\| \hat{C} \right\|_2^2
$$

We have

$$
\frac{1}{T^2p_1^2p_2^2} \left\| \sum_{t=1}^{T} (E_t' E_t) \right\|_2^2 \leq \frac{1}{T^2p_1^2p_2^2} \left\| \sum_{t=1}^{T} (E_t' E_t) \right\|_2^2 \left\| \hat{C} \right\|_2^2 \leq c_0 \frac{1}{T^2p_1^2p_2^2} \left\| \sum_{t=1}^{T} (E_t' E_t) \right\|_2^2;
$$

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\[
\| (E (E'_t E_t)) \|_2^2 \leq \| (E (E'_t E_t)) \|_1^2, \quad \text{and}
\]
\[
\| (E (E'_t E_t)) \|_1^2 \leq \max_{1 \leq h \leq p_2} \sum_{k=1}^{p_1} \sum_{i=1}^{p_1} \left( E(e_{ih,t} e_{ik,t}) \right) \leq c_0 p_1^2,
\]
by Assumption B3(ii)(a), so that
\[
\frac{1}{T^2 p_1^2 p_2^2} \left\| \sum_{t=1}^{T} (E (E'_t E_t)) \hat{C} \right\|_2^2 \leq c_0 \frac{1}{p_2^2},
\]
Also
\[
\frac{1}{T^2 p_1^2 p_2^2} \left\| \sum_{t=1}^{T} (E'_t E_t - E (E'_t E_t)) \hat{C} \right\|_2^2 \leq c_0 \frac{1}{T^2 p_1^2 p_2^2} \left\| \sum_{t=1}^{T} (E'_t E_t - E (E'_t E_t)) \right\|_2^2
\]
\[
= c_0 \frac{1}{T^2 p_1^2 p_2^2} \sum_{i,j=1}^{p_2} \left( \sum_{h=1}^{p_1} \sum_{t=1}^{T} \left( e_{hi,t} e_{hj,t} - E(e_{hi,t} e_{hj,t}) \right) \right)^2,
\]
and
\[
\frac{1}{T^2 p_1^2 p_2^2} E \left( \sum_{i,j=1}^{p_2} \sum_{h=1}^{p_1} \sum_{t=1}^{T} \left( e_{hi,t} e_{hj,t} - E(e_{hi,t} e_{hj,t}) \right) \right)^2
\]
\[
\leq \frac{1}{T^2 p_1^2 p_2^2} \sum_{i,j=1}^{p_2} \sum_{h=1}^{p_1} \sum_{t=1}^{T} \left| Cov\left(e_{hi,t} e_{hj,t}, e_{ki,s} e_{kj,s}\right) \right|
\]
\[
\leq c_0 \frac{1}{T^2 p_1^2 p_2^2} (E_\epsilon T p_1),
\]
by Assumption B3(ii). Thus, by Lemma B.1, this term is bounded by
\[
\frac{1}{T^2 p_1^2 p_2^2} \left\| \sum_{t=1}^{T} (E'_t E_t - E (E'_t E_t)) \hat{C} \right\|_2^2 = o_{a.s.} \left( \frac{1}{T p_1} \left( \ln T \ln p_1 \ln^2 p_2 \right)^{1+\epsilon} \right),
\]
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for all $\epsilon > 0$; again this bound is not necessarily sharp, but it suffices for our results. Putting all together, the desired rate now follows.

\[ \bar{\lambda} = p_1^{-1} \sum_{j=1}^{p_1} \hat{\lambda}_j. \]

**Lemma B.3.** We assume that Assumptions B1-B4 are satisfied, and let $\bar{\lambda} = p_1^{-1} \sum_{j=1}^{p_1} \hat{\lambda}_j$. Then it holds that there exist two constants $0 < c_0 \leq c_1 < \infty$ such that $\bar{\lambda} = \Omega_{a.s.} (c_0)$ and $\bar{\lambda} = O_{a.s.} (c_1)$. The same result holds for $\bar{\tilde{\lambda}} = p_1^{-1} \sum_{j=1}^{p_1} \tilde{\lambda}_j$.

**Proof.** We study the case $k_1 = k_2 = 1$ for simplicity and with no loss of generality. It holds that

\[
\bar{\lambda} = \frac{1}{p_1} tr \left( \frac{1}{p_2 T} \sum_{i=1}^{p_2} \sum_{t=1}^{T} X_{i,t}X'_{i,t} \right)
= \frac{1}{p_1 p_2 T} \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} X_{ji,t}^2 = \frac{1}{p_1 p_2 T} \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} (R_j F_i C_i + e_{ji,t})^2
= \frac{1}{p_1 p_2 T} \sum_{j=1}^{p_1} R_j^2 \sum_{i=1}^{p_2} C_i^2 \sum_{t=1}^{T} F_i^2 + \frac{1}{p_1 p_2 T} \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} e_{ji,t}^2
+ \frac{2}{p_1 p_2 T} \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} R_j F_i C_i e_{ji,t}
= I + II + III.
\]

Assumptions B1(i) and B2(ii) entail that, whenever $k_1 > 0$ and $k_2 > 0$, $I = c_0 + o_{a.s.} (1)$, with $c_0 > 0$. Consider now $II$

\[
II = \left( \frac{1}{p_1 p_2 T} \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} E(e_{ji,t}^2) \right) + \left( \frac{1}{p_1 p_2 T} \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} (e_{ji,t}^2 - E(e_{ji,t}^2)) \right)
= II_a + II_b.
\]

Note that

\[
E \left( \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} (e_{ji,t}^2 - E(e_{ji,t}^2)) \right)^2 = \sum_{i,j=1}^{p_1} \sum_{h,k=1}^{p_2} \sum_{l,s=1}^{T} Cov \left( e_{ih,t}^2, e_{jk,s}^2 \right) \leq c_0 p_1 p_2 T
\]
having used Assumption B3(iii)(a). Hence, by Lemma B.1

\[ II_b = o_a.s. \left( \frac{1}{\sqrt{p_1 p_2 T}} \left( \ln p_1 \ln p_2 \ln T \right)^{1/2 + \epsilon} \right), \]

for all \( \epsilon > 0 \). Also, using Assumption B3(iv)

\[
\frac{1}{p_1 p_2 T} \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} E \left( \epsilon_{ji,t}^2 \right) \\
= \frac{1}{p_1} tr \left( E \left( \frac{1}{p_2 T} \sum_{t=1}^{T} E_t E_t' \right) \right) \\
\geq \lambda_{\min} \left( E \left( \frac{1}{p_2 T} \sum_{t=1}^{T} E_t E_t' \right) \right) > 0,
\]

which entails that \( II > 0 \) a.s. Finally, recall that, by Assumption B4(i)

\[
E \left( \sum_{j=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} F_t \frac{R}{p_1^{1/2}} E_t C_t' \frac{1}{p_2^{1/2}} \right)^2 \leq c_0,
\]

which entails that

\[ III = o_a.s. \left( \frac{1}{\sqrt{p_1 p_2 T}} \left( \ln p_1 \ln p_2 \ln T \right)^{1/2 + \epsilon} \right), \]

for all \( \epsilon > 0 \). Putting all together, the desired result now follows. The result for \( \bar{\lambda} \) follows from the same passages, and from the same arguments as in the proof of Theorem 2.
C Proofs

Henceforth, we let $E^*$ and $V^*$ denote the expectation and variance with respect to $P^*$ respectively.

Proof of Theorem 1. Some arguments in the proof of this theorem are very similar, and in fact easier, than the ones in the next proof, to which we refer for further details. Let $C_i$ denote the $i$-th column of $C'$, and note

$$\frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} X_{i,t} X'_{i,t}$$

$$= \frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} (RF_tC_i + E_{i,t}) (RF_tC_i + E_{i,t})'$$

$$= \frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} RF_tC_i C_i' F_t R' + \frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} E_{i,t} E_{i,t}'$$

$$+ \frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} RF_tC_i E_{i,t} + \frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} E_{i,t} C_i' F_t R'$$

$$= I + II + III + IV,$$

so that, by Weyl’s inequality

$$\lambda_j (I) + \lambda_{\text{min}} (II + III + IV) \leq \lambda_j \left( \frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} X_{i,t} X'_{i,t} \right) \leq \lambda_j (I) + \lambda_{\text{max}} (II) + \lambda_{\text{max}} (III + IV).$$

Note that

$$I = \frac{1}{p_2 T} \sum_{t=1}^T \sum_{i=1}^{p_2} RF_tC_i C_i' F_t R'$$

$$= R \left( \frac{1}{T} \sum_{t=1}^T F_t F_t' \right) R',$$

by Assumption B2(ii). It is easy to see that, for all $j > k_1$, $\lambda_j \left( R \left( \frac{1}{T} \sum_{t=1}^T F_t F_t' \right) R' \right) = 0$; when $j \leq k_1$, the multiplicative Weyl’s inequality (see Theorem 7 in Merikoski and Kumar,
2004) yields
\[
\lambda_j \left( R \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t \right) R' \right) = \lambda_j \left( \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t \right) R' R \right) \geq \lambda_j \left( R' R \right) \lambda_{\min} \left( \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t \right) \right);
\]
by Assumption B1 (i), \( \lambda_{\min} \left( \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t \right) \right) \geq c_0 + o_{a.s.} \) (1) with \( c_0 > 0 \); using Assumption B2 (ii), it follows that
\[
\lambda_j \left( R \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t \right) R' \right) = \Omega_{a.s.} \left( p_1 \right).
\]
Consider now II
\[
\lambda_{\max} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} E_{i,t} E'_{i,t} \right) = \lambda_{\max} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} E \left( E_{i,t} E'_{i,t} \right) \right) + \lambda_{\max} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} \left( E_{i,t} E'_{i,t} - E \left( E_{i,t} E'_{i,t} \right) \right) \right).
\]
Using Weyl’s inequality and Assumption B3 (ii)(b)
\[
\lambda_{\max} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} E \left( E_{i,t} E'_{i,t} \right) \right) \\
\leq \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} \lambda_{\max} \left( E \left( E_{i,t} E'_{i,t} \right) \right) \\
\leq \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} \max_{1 \leq h \leq p_1} \sum_{k=1}^{p_1} \left| E \left( e_{hi,t} e_{ki,t} \right) \right| \leq c_0.
\]
Also
\[
\lambda_{\max} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} \left( E_{i,t} E'_{i,t} - E \left( E_{i,t} E'_{i,t} \right) \right) \right) \\
\leq \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} \left( e_{hi,t} e_{ki,t} - E \left( e_{hi,t} e_{ki,t} \right) \right) \right)^{2 \frac{1}{2}}.
\]
Consider now

\[ E \sum_{h,k=1}^{p_1} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} (e_{hi,t}e_{ki,t} - E(e_{hi,t}e_{ki,t})) \right)^2 = \frac{1}{p_2 T^2} \sum_{h,k=1}^{p_1} \sum_{i,j=1}^{p_2} \sum_{t,s=1}^{T} \text{Cov}(e_{hi,t}e_{ki,t}, e_{hj,s}e_{kj,s}) \leq c_0 \frac{p_1^2}{p_2 T}, \]

having used Assumption B3(iii) in the last passage. Thus, by Lemma B.1

\[ \lambda_{\text{max}} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} \sum_{i=1}^{p_2} (E_i e_{i,t} - E(E_i e_{i,t})) \right) = o_{a.s.} \left( \frac{p_1}{\sqrt{p_2 T}} \left( \ln^2 p_1 \ln p_2 \ln T \right)^{1/2 + \epsilon} \right), \]

for all \( \epsilon > 0 \). Thus, there exist a positive, finite constants \( c_0 \) such that

\[ \lambda_{\text{max}} (II) = c_0 + \Omega_{a.s.} \left( \frac{p_1}{\sqrt{p_2 T}} \left( \ln^2 p_1 \ln p_2 \ln T \right)^{1/2 + \epsilon} \right). \]

Finally, note that

\[ \lambda_{\text{max}} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} (RF_t C'E_t' + E_t'CF_t'R_t') \right) = 2\lambda_{\text{max}} \left( \frac{1}{p_2 T} \sum_{t=1}^{T} RF_t C'E_t' \right) \leq 2 \left\| \frac{1}{p_2 T} \sum_{t=1}^{T} F_t C'E_t'R_t' \right\|_F, \]

and

\[ E \left\| \frac{1}{p_2 T} \sum_{t=1}^{T} F_t C'E_t'R_t' \right\|_F^2 = \frac{1}{p_2 T^2} E \sum_{h,k=1}^{p_1} \left( \sum_{i=1}^{p_2} \sum_{t=1}^{T} R_h F_t C_i e_{ki,t} \right)^2 = \frac{1}{p_2 T^2} \sum_{h,k=1}^{p_1} \sum_{i,j=1}^{p_2} \sum_{t,s=1}^{T} R_h C_i C_j E_{hi} e_{ki,t} e_{kj,s} \leq \frac{p_1}{\sqrt{p_2 T}} \]

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having used Assumptions B4(i)-(ii), so that, using Lemma B.1, it follows that

\[ \lambda_{\text{max}} (III + IV) \leq o_{\text{a.s.}} \left( \frac{p_1}{\sqrt{p_2 T}} \left( \ln^2 p_1 \ln p_2 \ln T \right)^{1/2+\epsilon} \right), \]

for all \( \epsilon > 0 \). The desired result follows from putting all together.

Proof of Theorem 2. In the proof of the theorem, we assume, for the sake of notational simplicity and no loss of generality, that \( k_1 = k_2 = 1 \); and we omit the rotation matrix \( H_1 \) defined in Lemma B.2. Further, some passages have already been shown by Yu et al. (2021), and when possible we omit them to save space. Recall that

\[ \tilde{Y}_t = \frac{1}{p_2} X_t \hat{C}, \]
\[ \tilde{M}_1 = \frac{1}{p_1 T} \sum_{t=1}^{T} \tilde{Y}_t \tilde{Y}_t'. \]

Then we can write

\[
\tilde{M}_1 = \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} \left( R F_t C' \hat{C} + E_t \hat{C} \right) \left( R F_t C' \hat{C} + E_t \hat{C} \right)' \tag{B.1}
\]
\[
= \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_t C' \hat{C} \hat{C}' C F_t' R' + \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_t C' \hat{C} \hat{C}' E_t'
+ \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E_t \hat{C} \hat{C}' C F_t' R' + \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E_t \hat{C} \hat{C}' E_t'
= \tilde{M}_{1,1} + \tilde{M}_{1,2} + \tilde{M}_{1,3} + \tilde{M}_{1,4}.
\]

We begin by noting that, by Weyl’s inequality

\[
\lambda_j \left( \tilde{M}_{1,1} \right) + \lambda_{\text{min}} \left( \sum_{k=2}^{4} \tilde{M}_{1,k} \right) \leq \lambda_j \left( \tilde{M}_{1,1} \right) \leq \lambda_j \left( \tilde{M}_{1,1} \right) + \lambda_{\text{max}} \left( \sum_{k=2}^{4} \tilde{M}_{1,k} \right). \tag{B.2}
\]
By construction, it holds that, for all $j > k_1$

$$
\lambda_j \left( \hat{M}_{1,1} \right) = 0.
$$

We now study the case $j \leq k_1$; we will use the decomposition

$$
\hat{M}_{1,1} = \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_i C' C' \hat{C} C F'_t R' + \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_i C' \left( \hat{C} - C \right) \hat{C} C F'_t R' \\
+ \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_i C' \hat{C} \left( \hat{C} - C \right) C F'_t R' + \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_i C' \left( \hat{C} - C \right) \left( \hat{C} - C \right) C F'_t R' \\
= \hat{M}_{1,1,1} + \hat{M}_{1,1,2} + \hat{M}_{1,1,3} + \hat{M}_{1,1,4}.
$$

We have

$$
\lambda_j \left( \hat{M}_{1,1,1} \right) = \lambda_j \left( \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_i C' C' \hat{C} C F'_t R' \right) \\
= \lambda_j \left( p_1^{-1} R \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t \right) R' \right) \\
\leq \lambda_j \left( p_1^{-1} R \left( \frac{1}{T} \sum_{t=1}^{T} E \left( F_t F'_t \right) \right) R' \right) + \lambda_{\max} \left( p_1^{-1} R \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t - E \left( F_t F'_t \right) \right) R' \right).
$$

By the multiplicative Weyl inequality (Theorem 7 in Merikoski and Kumar, 2004), we have

$$
\lambda_j \left( p_1^{-1} R \left( \frac{1}{T} \sum_{t=1}^{T} E \left( F_t F'_t \right) \right) R' \right) \geq \lambda_j \left( p_1^{-1} R' R \right) \lambda_{\min} \left( \left( \frac{1}{T} \sum_{t=1}^{T} E \left( F_t F'_t \right) \right) \right) \geq c_0,
$$

having used Assumptions B1 and B2. Also

$$
\lambda_{\max} \left( p_1^{-1} R \left( \frac{1}{T} \sum_{t=1}^{T} F_t F'_t - E \left( F_t F'_t \right) \right) R' \right) \\
\leq \lambda_{\max} \left( p_1^{-1} R' R \right) \lambda_{\max} \left( \frac{1}{T} \sum_{t=1}^{T} \left( F_t F'_t - E \left( F_t F'_t \right) \right) \right) \\
\leq c_0 \lambda_{\max} \left( \frac{1}{T} \sum_{t=1}^{T} \left( F_t F'_t - E \left( F_t F'_t \right) \right) \right).
$$
We now have
\[ \lambda_{\max} \left( \frac{1}{T} \sum_{t=1}^{T} (F_t F_t' - E(F_t F_t')) \right) \leq \left\| \frac{1}{T} \sum_{t=1}^{T} (F_t F_t' - E(F_t F_t')) \right\|_F, \]
and
\[ \left\| \frac{1}{T} \sum_{t=1}^{T} (F_t F_t' - E(F_t F_t')) \right\|_F = \left( \sum_{h,k=1}^{k_1} \left( \frac{1}{T} \sum_{t=1}^{T} (F_{h,t} F_{k,t} - E(F_{h,t} F_{k,t})) \right)^2 \right)^{1/2}. \]
Assumption B1 (iii) yields
\[ \sum_{h,k=1}^{k_1} E \max_{1 \leq t \leq T} \left| \sum_{t=1}^{T} (F_{h,t} F_{k,t} - E(F_{h,t} F_{k,t})) \right|^2 \leq c_0 T, \]
which in turn, by Lemma B.1, entails
\[ \sum_{h,k=1}^{p_1} \left( \frac{1}{T} \sum_{t=1}^{T} (F_{h,t} F_{k,t} - E(F_{h,t} F_{k,t})) \right)^2 = o_{a.s.} \left( T \left( \ln T \right)^{1+\epsilon} \right), \]
for all \( \epsilon > 0 \), whence
\[ \lambda_{\max} \left( \frac{1}{T} \sum_{t=1}^{T} (F_t F_t' - E(F_t F_t')) \right) = o_{a.s.} \left( T^{1/2} \left( \ln T \right)^{1/2+\epsilon} \right). \]
Thus, it follows that there exists a positive constant \( c_0 \) such that \( \lambda_j (\tilde{M}_{1,1,1}) = \Omega_{a.s.} (c_0) \).

Also
\[ \| \tilde{M}_{1,1,2} \|_F = \left\| \frac{1}{p_1 p_2 T} \sum_{t=1}^{T} R F_t C' (\hat{C} - C) C' C F_t' R' \right\|_F \]
\[ \leq \frac{1}{p_1 p_2} \left( \frac{1}{T} \sum_{t=1}^{T} \| F_t \|_F^2 \right) \| R \|_F^2 \| \hat{C} - C \|_F \| C \|_F. \]

By Assumption B2, \( \| R \|_F^2 \leq c_0 p_1 \) and \( \| C \|_F \leq c_0 p_2^{1/2} \); further, by similar arguments as above, it is easy to see that \( T^{-1} \sum_{t=1}^{T} \| F_t \|_F^2 = O_{a.s.} (1) \). Finally, applying Lemma B.2, it
follows that
\[ \| \tilde{M}_{1,1,2} \|_F = o_{a.s.} \left( \left( \frac{1}{p_2} + \frac{1}{\sqrt{T \pi}} \right) l'_{p_1 , p_2 , T} \right), \] (B.4)
where \( l'_{p_1 , p_2 , T} \) is defined in (B.2). The same result holds for \( \| \tilde{M}_{1,1,3} \|_F \), by exactly the same passages. We now turn to
\[ \| \tilde{M}_{1,1,4} \|_F \leq \frac{1}{p_1 p_2^2} \left( \frac{1}{T} \sum_{t=1}^{T} \| F_t \|_F^2 \right) \| R \|_F^2 \| C - \hat{C} \|_F^2 \| C \|_F^2 ; \]
noting that \( \| R \|_F^2 \leq c_0 p_1 \) and \( \| C \|_F^2 \leq c_0 p_2 \), and using again Lemma B.2, it immediately follows that
\[ \| \tilde{M}_{1,1,4} \|_F = o_{a.s.} \left( \left( \frac{1}{p_2} + \frac{1}{\sqrt{T \pi}} \right)^2 \left( l'_{p_1 , p_2 , T} \right)^2 \right) . \]
These results entail that there exists a positive constant \( c_0 \) such that \( \lambda_j (\tilde{M}_{1,1}) = \Omega_{a.s.} (c_0) \), whenever \( j \leq k_1 \).

We now bound the other terms in (B.1), using the bound
\[ \lambda_{\max} \left( \sum_{k=2}^{4} \tilde{M}_{1,k} \right) \leq \left\| \sum_{k=2}^{4} \tilde{M}_{1,k} \right\|_2 \leq \sum_{k=2}^{4} \| \tilde{M}_{1,k} \|_2 . \]
Consider \( \tilde{M}_{1,2} \) first, and note that
\[ \| \tilde{M}_{1,2} \|_F \leq \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_t C' C' E_t' \right\|_F + \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_t C' (\hat{C} - C) C' E_t' \right\|_F \]
\[ + \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_t C' (\hat{C} - C)' E_t' \right\|_F + \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} R F_t C' (\hat{C} - C)' E_t' \right\|_F \]
\[ = \| \tilde{M}_{1,2,1} \|_F + \| \tilde{M}_{1,2,2} \|_F + \| \tilde{M}_{1,2,3} \|_F + \| \tilde{M}_{1,2,4} \|_F . \]

It holds that
\[ \| \tilde{M}_{1,2,1} \|_F = \left( \sum_{i,j=1}^{p_1} \left( \sum_{l=1}^{p_2} C_{i} R_{i} F_{i} e_{j,l} \right)^2 \right)^{1/2} , \]
and, expanding the square and taking the expectation, we now study

\[
\sum_{i,j=1}^{p_1} \sum_{t,s=1}^{T} \sum_{h_1,h_2=1}^{p_2} C_{h_1} C_{h_2} R_i^2 E \left( F_t F_s e_{j h_1, t} e_{j h_2, s} \right)
\]

\[
\leq c_0 \sum_{i,j=1}^{p_1} \sum_{t,s=1}^{T} \sum_{h_1,h_2=1}^{p_2} |E \left( F_t F_s e_{j h_1, t} e_{j h_2, s} \right)|.
\]

Using Assumptions B4(i) and B4(ii)(a), it now follows that

\[
E \left( \sum_{i,j=1}^{p_1} \sum_{t,s=1}^{T} \sum_{h_1,h_2=1}^{p_2} p_1^2 p_2 T \right)
\]

so that, by Lemma B.1, it follows that

\[
\| \tilde{M}_{1,2} \|_F = o_{a.s.} \left( \frac{1}{p_2} \left( \ln T \ln^2 p_1 \ln p_2 \right)^{1+\epsilon} \right),
\]

for all \( \epsilon > 0 \). Consider now

\[
\| \tilde{M}_{1,2} \|_2 \leq \frac{1}{p_1 p_2 T} \left\| \sum_{t=1}^{T} RF_t C' \left( \tilde{C} - C \right) C' E_t \right\|_F
\]

\[
\leq \frac{1}{p_1 p_2 T} \left| C' \left( \tilde{C} - C \right) \right| \left( \sum_{i,k=1}^{p_1} \left( \sum_{t=1}^{T} \sum_{h=1}^{p_2} R_i C_h F_{t e_{kh,t}} \right) \right)^{1/2}.
\]

Using the Cauchy-Schwartz inequality and Lemma B.2, it follows that

\[
\left| C' \left( \tilde{C} - C \right) \right| = o_{a.s.} \left( p_2 \left( \frac{1}{p_2} + \frac{1}{\sqrt{T p_1}} \right) t'_{p_1, p_2, T} \right).
\]

Also note that

\[
\sum_{i,k=1}^{p_1} E \left( \sum_{t=1}^{T} \sum_{h=1}^{p_2} R_i C_h F_{t e_{kh,t}} \right)^2
\]

\[
\leq \sum_{i,k=1}^{p_1} \sum_{t,s=1}^{T} \sum_{h_1,h_2=1}^{p_2} E \left( R_i^2 C_{h_1} C_{h_2} F_t F_s e_{kh_1,t} e_{kh_2,s} \right)
\]

\[
\leq c_0 p_1^2 p_2 T,
\]

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having used again Assumptions B4(i) and B4(ii)(a). Thus, using Lemma B.1 and putting all together, it holds that

$$
\|\bar{M}_{1,2,2}\|_F = o_{a.s.} \left( \frac{1}{p_2 T^{1/2}} \left( \frac{1}{p_2} + \frac{1}{\sqrt{T p_1}} \right) T'_{p_1,p_2,T} \left( \ln T \ln^2 p_1 \ln p_2 \right)^{1/2+\epsilon} \right).
$$

Also

$$
\|\bar{M}_{1,2,3}\|_F = \frac{1}{p_1 p_2 T} \left\| \sum_{t=1}^T R F_t C' C \left( \hat{C} - C \right)' E_t \right\|_F
= \frac{1}{p_1 p_2 T} \left\| \sum_{t=1}^T R F_t \left( \hat{C} - C \right)' E_t \right\|_F
\leq \frac{1}{p_1} \|R\|_F \frac{1}{p_2 T} \sum_{t=1}^T F_t \left( \hat{C} - C \right)' E_t' \right\|_F.
$$

Following the proof of Theorem 3.3 in Yu et al. (2021), it can be show that

$$
E \left\| \frac{1}{p_2 T} \sum_{t=1}^T F_t \left( \hat{C} - C \right)' E_t \right\|_F^2 \leq c_0 p_1 \left( \frac{1}{T p_2^2} + \frac{1}{(T p_1)^2} \right),
$$

and therefore, by the same arguments as above, it follows that

$$
\left\| \frac{1}{p_2 T} \sum_{t=1}^T F_t \left( \hat{C} - C \right)' E_t' \right\|_F = o_{a.s.} \left( p_1^{1/2} \left( \frac{1}{T p_2^2} + \frac{1}{(T p_1)^2} \right)^{1/2} \left( \ln p_1 \ln p_2 \ln T \right)^{1/2+\epsilon} \right),
$$

for all $\epsilon > 0$, so that ultimately

$$
\|\bar{M}_{1,2,3}\|_F = o_{a.s.} \left( \left( \frac{1}{T^{1/2} p_2} + \frac{1}{T p_1} \right) \left( \ln p_1 \ln p_2 \ln T \right)^{1/2+\epsilon} \right).
$$
Finally, by the same logic we have that

\[
\|\tilde{M}_{1,2}\|_F = \frac{1}{p_1 p_2^2 T} \left\| \sum_{t=1}^T R_F C' (\hat{C} - C)' E_t' \right\|_F \\
\leq \frac{1}{p_1 p_2} \|R\|_F \|C\|_F \left\| (\hat{C} - C)' E_t' \right\|_F \\
= \Omega_{a.s.} \left[ p_1^{-1/2} p_2^{-1/2} \left( p_2^2 \left( \frac{1}{p_2} + \frac{1}{\sqrt{T p_1}} \right) T'_{p_1,p_2,T} \right) \times \left( p_1^{1/2} \left( \frac{1}{T p_2^2} + \frac{1}{(T p_1)^2} \right)^{1/2} \left( \ln p_1 \ln p_2 \ln T \right)^{1/2+\varepsilon} \right) \right] \\
= \Omega_{a.s.} \left[ \left( \frac{1}{p_2} + \frac{1}{\sqrt{T p_1}} \right) \left( \frac{1}{T^{1/2} p_2^2} + \frac{1}{T p_1} \right) T'_{p_1,p_2,T} \left( \ln p_1 \ln p_2 \ln T \right)^{1/2+\varepsilon} \right].
\]

Putting all together, it follows that

\[
\|\tilde{M}_{1,2}\|_F = o_{a.s.} \left( \left( \frac{1}{\sqrt{T p_2}} + \frac{1}{T p_1} \right) \left( \ln^2 p_1 \ln p_2 \ln T \right)^{1+\varepsilon} \right);
\]
the same holds for \( \|\tilde{M}_{1,3}\|_F \). Finally, consider

\[
\tilde{M}_{1,4} = \frac{1}{p_1 p_2^2 T} \sum_{t=1}^T E_t C C' E_t' + \frac{1}{p_1 p_2 T} \sum_{t=1}^T E_t C (\hat{C} - C)' E_t' \\
+ \frac{1}{p_1 p_2 T} \sum_{t=1}^T E_t (\hat{C} - C)' C E_t' + \frac{1}{p_1 p_2 T} \sum_{t=1}^T E_t (\hat{C} - C)' (\hat{C} - C)' E_t' \\
= \tilde{M}_{1,4,1} + \tilde{M}_{1,4,2} + \tilde{M}_{1,4,3} + \tilde{M}_{1,4,4}.
\]

We have

\[
\tilde{M}_{1,4,1} = \frac{1}{p_1 p_2^2 T} \sum_{t=1}^T E (E_t C C' E_t') + \frac{1}{p_1 p_2 T} \sum_{t=1}^T (E_t C C' E_t' - E (E_t C C' E_t')) \\
= \tilde{M}_{1,4,1,1} + \tilde{M}_{1,4,1,2}.
\]
Then
\[
\lambda_{\text{max}} \left( \begin{bmatrix} \tilde{M}_{1,4,1,1} \end{bmatrix} \right) \leq \frac{1}{p_1 p_2} \max_{1 \leq j \leq p_1} \sum_{i=1}^{p_1} \left| E \left( \frac{1}{T} \sum_{t=1}^{T} \sum_{h_1, h_2=1}^{p_2} C_{h_1} C_{h_2} e_{ih_1, t} e_{j h_2, t} \right) \right|
\]
\[
\leq \frac{1}{Tp_1 p_2} \sum_{t=1}^{T} \sum_{h_1, h_2=1}^{p_2} \sum_{i=1}^{p_1} \left| E \left( e_{ih_1, t} e_{j h_2, t} \right) \right|
\]
\[
\leq c_0 p_2 \frac{1}{p_1 p_2^2} = c_0 \frac{1}{p_1 p_2},
\]
having used Assumption B3(ii). Also, letting \( \eta_{i,j,h_1,h_2,t} = e_{ih_1, t} e_{j h_2, t} - E \left( e_{ih_1, t} e_{j h_2, t} \right) \)
\[
\left\| \begin{bmatrix} \tilde{M}_{1,4,1,2} \end{bmatrix} \right\|_F = \frac{1}{p_1 p_2^2 T} \left( \sum_{i,j=1}^{p_1} \left( \sum_{h_1, h_2=1}^{p_2} \sum_{t=1}^{T} C_{h_1} C_{h_2} \eta_{i,j,h_1,h_2} \right)^2 \right)^{1/2},
\]
and
\[
E \sum_{i,j=1}^{p_1} \left( \sum_{t=1}^{T} \sum_{h_1, h_2=1}^{p_2} C_{h_1} C_{h_2} \eta_{i,j,h_1,h_2} \right)^2
\]
\[
\leq c_0 \sum_{i,j=1}^{p_1} \sum_{t,s=1}^{T} \sum_{h_1, h_2, h_3=1}^{p_2} \left| \text{Cov} \left( e_{ih_1, t} e_{j h_2, t}, e_{ih_3, s} e_{j h_4, s} \right) \right|
\]
\[
\leq c_0 p_1 T p_3^2,
\]
in light of Assumption B3(ii). Using Lemma B.1, we therefore have
\[
\sum_{i,j=1}^{p_1} \left( \sum_{t=1}^{T} \sum_{h_1, h_2=1}^{p_2} C_{h_1} C_{h_2} \eta_{i,j,h_1,h_2} \right)^2 = o_{a.s.} \left( p_1 T p_3^2 \left( \ln T \ln^2 p_1 \ln^2 p_2 \right)^{1+\epsilon} \right),
\]
so that
\[
\left\| \begin{bmatrix} \tilde{M}_{1,4,1,2} \end{bmatrix} \right\|_F = o_{a.s.} \left( \frac{1}{\sqrt{p_1 p_2 T}} \left( \ln T \ln^2 p_1 \ln^2 p_2 \right)^{1/2+\epsilon} \right),
\]
whence
\[
\left\| \begin{bmatrix} \tilde{M}_{1,4,1} \end{bmatrix} \right\|_F = O \left( \frac{1}{p_1 p_2} \right) + o_{a.s.} \left( \frac{1}{\sqrt{p_1 p_2 T}} \left( \ln T \ln^2 p_1 \ln^2 p_2 \right)^{1/2+\epsilon} \right).
\]
Let now \( E_{i,j,t} \) represent the \( j \)-th column of \( E_t \), and \( (\hat{C} - C)_j \) be the \( j \)-th element of
(\hat{C} - C); we have

$$\|\tilde{M}_{1.4.3}\|_2 = \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E_t (\hat{C} - C) C' E'_t \right\|_2$$

$$\leq \left\| \frac{1}{p_1 p_2^2 T} \sum_{j=1}^{p_2} (\hat{C} - C)_j \sum_{t=1}^{T} E_{j,t} C' E'_t \right\|_2$$

$$\leq \sum_{j=1}^{p_2} \left\| \frac{1}{p_1 p_2^2 T} (\hat{C} - C)_j \sum_{t=1}^{T} E_{j,t} C' E'_t \right\|_2$$

$$\leq \sum_{j=1}^{p_2} (\hat{C} - C)_j \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E_{j,t} C' E'_t \right\|_2 .$$

Thus, by the Cauchy-Schwartz inequality

$$\|\tilde{M}_{1.4.3}\|_2 \leq \left( \sum_{j=1}^{p_2} \left( (\hat{C} - C)_j \right)^2 \right)^{1/2} \left( \sum_{j=1}^{p_2} \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E_{j,t} C' E'_t \right\|_2^2 \right)^{1/2} .$$

We know from Lemma B.1 that

$$\left( \sum_{j=1}^{p_2} \left( (\hat{C} - C)_j \right)^2 \right)^{1/2} = \|\hat{C} - C\|_F = o_{a.s.} \left( p_2^{1/2} \left( \frac{1}{p_2} + \frac{1}{\sqrt{T p_1}} \right) t'_{p_1 p_2 T} \right) .$$

Further

$$\sum_{j=1}^{p_2} E \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E_{j,t} C' E'_t \right\|_2^2$$

$$= \sum_{j=1}^{p_2} \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} (E_{j,t} C' E'_t) \right\|_2^2 + \sum_{j=1}^{p_2} E \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} (E_{j,t} C' E'_t - E(E_{j,t} C' E'_t)) \right\|_2^2 ;$$

we have

$$\left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E (E_{j,t} C' E'_t) \right\|_2^2 \leq \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E (E_{j,t} C' E'_t) \right\|_1 \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E (E_{j,t} C' E'_t) \right\|_\infty ,$$

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and
\[
\left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E \left( E_{.,j,t} C' E'_t \right) \right\|_1 \leq \frac{1}{p_1 p_2^2 T} \max_{1 \leq t \leq p_2} |C_t| \max_{1 \leq k \leq p_2} \sum_{h=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t=1}^{T} |E(e_{h,j,t} e_{k,i,t})| \leq c_0 \frac{1}{p_1 p_2^2},
\]
by Assumption B3(ii), so that
\[
\left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E \left( E_{.,j,t} C' E'_t \right) \right\|_1 \leq c_0 \frac{1}{p_1 p_2^2};
\]
using the same logic, it follows that
\[
\left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E \left( E_{.,j,t} C' E'_t \right) \right\|_\infty \leq c_0 \frac{1}{p_1 p_2^2}.
\]
Hence
\[
\sum_{j=1}^{p_2} \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} E \left( E_{.,j,t} C' E'_t \right) \right\|_2^2 \leq c_0 p_2 \left( \frac{1}{p_1 p_2^2} \right)^2.
\]
Also
\[
E \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} (E_{.,j,t} C' E'_t - E \left( E_{.,j,t} C' E'_t \right)) \right\|_2^2 \leq \left( \frac{1}{p_1 p_2^2 T} \right)^2 E \sum_{h,k=1}^{p_1} \sum_{t=1}^{T} \sum_{i=1}^{p_2} \left( e_{h,j,t} C_i e_{k,i,t} - E \left( e_{h,j,t} C_i e_{k,i,t} \right) \right)^2 \leq \left( \frac{1}{p_1 p_2^2 T} \right)^2 T p_2 \sum_{h,k=1}^{p_1} \sum_{i=1}^{p_2} \sum_{t,s=1}^{T} \left| E(\eta_{h,j,k,i,t} \eta_{h,j,k,i,s}) \right|,
\]
with \( \eta_{h,j,k,i,t} = (E_{h,j,t} C_i E_{k,i,t} - E(E_{h,j,t} C_i E_{k,i,t})) \). Assumption B3(iii) entails that
\[
E \left\| \frac{1}{p_1 p_2^2 T} \sum_{t=1}^{T} (E_{.,j,t} C' E'_t - E \left( E_{.,j,t} C' E'_t \right)) \right\|_2^2 \leq c_0 \frac{1}{p_2^3 T}.
\]
Putting all together, it follows that
\[
E \left\| \tilde{M}_{1.4.3} \right\|_2 \leq c_0 p_2^{1/2} \left( \frac{1}{p_2} + \frac{1}{\sqrt{T p_1}} \right) \left( \frac{1}{T p_2} + \frac{1}{p_1^2 p_2^2} \right)^{1/2},
\]

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so that Lemma B.1 yields

\[
\|\tilde{M}_{1,4,3}\|_2 = o_a.s. \left( p_2^{1/2} \left( \frac{1}{p_2} + \frac{1}{\sqrt{T_p1}} \right) \right)^{t'_{p_1,p_2,T}} \times \left( \frac{1}{p_1 p_2^{3/2}} + p_2^{1/2} \left( \frac{1}{p_2} + \frac{1}{T_p1} \right) \left( \frac{1}{T p_2^2} + \frac{1}{p_1^2 p_2^3} \right)^{1/2} \left( \ln T \ln^2 p_1 \ln^2 p_2 \right)^{1/2+\varepsilon} \right),
\]

for all \( \varepsilon > 0 \). The same rate, by symmetry, is found for \( \|\tilde{M}_{1,4,2}\|_2 \). Finally, note that

\[
\{ \tilde{M}_{1,4,4} \}_{hk} = \frac{1}{p_1 p_2^T} \sum_{t=1}^{T} \left( \sum_{j=1}^{p_2} E_{h,j,t} \left( \hat{C} - C \right)_j \right) \left( \sum_{j=1}^{p_2} \left( \hat{C} - C \right)_j E_{k,j,t} \right) + \frac{1}{p_1^2 p_2^2 T} \sum_{t=1}^{T} \sum_{m,n=1}^{p_1} E \left( e_{hm,t} e_{kn,t} \right) \left( \hat{C} - C \right)_m \left( \hat{C} - C \right)_n.
\]

Note now that, using Assumption B3(ii)

\[
\max_{1 \leq k \leq p_1} \frac{1}{p_1 p_2^T} \sum_{t=1}^{T} \sum_{h=1}^{p_1} \sum_{m,n=1}^{p_2} |E \left( e_{hm,t} e_{kn,t} \right)\left( \hat{C} - C \right)_m \left( \hat{C} - C \right)_n| \leq c_0 \frac{1}{p_1 p_2} \sum_{m,n=1}^{p_1} \left( \hat{C} - C \right)_m \left( \hat{C} - C \right)_n^2 = c_0 \frac{1}{p_1^2 p_2} \left( \sum_{m=1}^{p_2} \left( \hat{C} - C \right)_m \right)^2 \leq c_0 \frac{1}{p_1 p_2} \sum_{m=1}^{p_2} \left( \hat{C} - C \right)_m^2 = c_0 \frac{1}{p_1^2} \| \hat{C} - C \|_F^2,
\]

which, by Lemma B.2, yields that this term is bounded by \( o_a.s. \left( \frac{1}{p_1^2} \left( \frac{1}{p_2} + \frac{1}{T_p1} \right) \right)^{t'_{p_1,p_2,T}} \).
Finally, letting \( \eta_{h,m,k,n,t} = e_{hm,t} e_{kn,t} - E(e_{hm,t} e_{kn,t}) \), we have

\[
\begin{align*}
\frac{1}{p_1 p_2 T} \sum_{m,n=1}^{p_2} (\hat{C} - C)_m (\hat{C} - C)_n \left( \sum_{t=1}^{T} \eta_{h,m,k,n,t} \right) \\
\leq \frac{1}{p_1 p_2 T} \left( \sum_{m,n=1}^{p_2} (\hat{C} - C)_m^2 (\hat{C} - C)_n^2 \right)^{1/2} \left( \sum_{m,n=1}^{p_2} \left( \sum_{t=1}^{T} \eta_{h,m,k,n,t} \right)^2 \right)^{1/2}.
\end{align*}
\]

It is easy to see that \( E\sum_{m,n=1}^{p_2} \left( \sum_{t=1}^{T} \eta_{h,m,k,n,t} \right)^2 \leq c_0 p_2^2 T \), so that, by using Lemma B.1

\[
\left( \sum_{m,n=1}^{p_2} \left( \sum_{t=1}^{T} \eta_{h,m,k,n,t} \right)^2 \right)^{1/2} = o_{a.s.} \left( p_2 T^{1/2} \left( \ln T \ln^2 p_2 \right)^{1/2+\epsilon} \right),
\]

for all \( \epsilon > 0 \). Also

\[
\left( \sum_{m,n=1}^{p_2} (\hat{C} - C)_m^2 (\hat{C} - C)_n^2 \right)^{1/2} = \sum_{m=1}^{p_2} (\hat{C} - C)_m^2 = \|\hat{C} - C\|_F^2.
\]

Using Lemma B.2 and putting all together, it follows that

\[
\|\hat{M}_{1,4,4}\|_2 = o_{a.s.} \left( \frac{1}{p_1 p_2 T^{1/2}} \left( \frac{1}{p_2^2} + \frac{1}{T p_1} \right) \left( \ln T \ln^2 p_2 \right)^{1/2+\epsilon} p_2^2 p_1, p_2, T \right),
\]

for all \( \epsilon > 0 \). The proof of the final result is now complete. \( \square \)

**Proof of Proposition 1.** We report the proof based on \( \hat{\Psi}_{k_1} \) only; the case of \( \hat{\Psi}_{k_0} \) follows exactly from the same arguments. Theorem 1 entails that, under the null and for all \( 0 < \epsilon < 1/\lambda \)

\[
P \left( \omega : \lim_{\min(p_1, p_2, T) \to \infty} \exp \left( -\epsilon p_1^{-\delta} \right) \hat{\phi}_{k_1} = \infty \right) = 1, \quad \text{(B.5)}
\]

and therefore we can assume from now on that

\[
\lim_{\min(p_1, p_2, T) \to \infty} \exp \left( -\epsilon p_1^{-\delta} \right) \hat{\phi}_{k_1} = \infty.
\]
Consider, for simplicity, the case \( u \geq 0 \). It holds that

\[
M^{-1/2} \sum_{m=1}^{M} \left[ \hat{\psi}_{k_1^0} (u) - \frac{1}{2} \right] = M^{-1/2} \sum_{m=1}^{M} \left[ \hat{\psi}_{k_1^0} (0) - \frac{1}{2} \right] + M^{-1/2} \sum_{m=1}^{M} \left[ \left( \hat{\psi}_{k_1^0} (u) - \hat{\psi}_{k_1^0} (0) \right) - \left( G \left( \frac{u}{\phi_{k_1^0}} \right) - \frac{1}{2} \right) \right]
\]

where \( G (\cdot) \) denotes the standard normal distribution. By definition, it holds that

\[
E^* (\hat{\psi}_{k_1^0} (u)) = G \left( \frac{u}{\phi_{k_1^0}} \right),
\]

\[
V^* (\hat{\psi}_{k_1^0} (u)) = G \left( \frac{u}{\phi_{k_1^0}} \right) \left( 1 - G \left( \frac{u}{\phi_{k_1^0}} \right) \right).
\]

It holds that

\[
E^* \int_{-\infty}^{\infty} \left[ M^{-1/2} \sum_{m=1}^{M} \left( I \left[ 0 \leq \hat{\phi}_{k_1^0}^{1/2} \eta^{(m)} \leq u \right] - \left( G \left( \frac{u}{\phi_{k_1^0}} \right) - \frac{1}{2} \right) \right) \right]^2 dF (u)
\]

\[
= \int_{-\infty}^{\infty} E^* \left[ \left( I \left[ 0 \leq \hat{\phi}_{k_1^0}^{1/2} \eta^{(m)} \leq u \right] - \left( G \left( \frac{u}{\phi_{k_1^0}} \right) - \frac{1}{2} \right) \right) \right]^2 dF (u),
\]

where

\[
E^* \left( I \left[ 0 \leq \hat{\phi}_{k_1^0}^{1/2} \eta^{(m)} \leq u \right] \right) = G \left( \frac{u}{\phi_{k_1^0}} \right) - \frac{1}{2},
\]

\[
V^* \left( I \left[ 0 \leq \hat{\phi}_{k_1^0}^{1/2} \eta^{(m)} \leq u \right] \right) = \left( G \left( \frac{u}{\phi_{k_1^0}} \right) - \frac{1}{2} \right) \left( 3 \frac{1}{2} - G \left( \frac{u}{\phi_{k_1^0}} \right) \right) \leq G \left( \frac{u}{\phi_{k_1^0}} \right) - \frac{1}{2}.
\]
Thus
\[
\int_{-\infty}^{\infty} E^* \left[ \left( I \left[ 0 \leq \hat{\psi}^{1/2}_{k_1^0} \eta^{(m)} \leq u \right] - \left( G \left( \frac{u}{\hat{\phi}_{k_1^0}} \right) - \frac{1}{2} \right) \right) \right]^2 dF(u)
\]
\[
\leq \int_{-\infty}^{\infty} |E^* \left( I \left[ 0 \leq \hat{\psi}^{1/2}_{k_1^0} \eta^{(1)} \leq u \right] \right) | dF(u)
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \hat{\phi}^{-1/2}_{k_1^0} \int_{-\infty}^{\infty} |u| dF(u) = o_p^* (1),
\]

having used (B.5). Also
\[
\int_{-\infty}^{\infty} \left( M^{-1/2} \sum_{m=1}^{M} \left[ G \left( \frac{u}{\hat{\phi}_{k_1^0}} \right) - \frac{1}{2} \right] \right)^2 dF(u)
\]
\[
\leq \frac{M}{2\pi \hat{\phi}_{k_1^0}} \int_{-\infty}^{\infty} u^2 dF(u) \leq c_0 \frac{M}{\hat{\phi}_{k_1^0}} = o_p^* (1),
\]

where the last passage follows from (3.14). Putting all together
\[
\int_{-\infty}^{\infty} \left( \sum_{m=1}^{M} \hat{\psi}_{k_1^0} (u) - \frac{1}{2} \right)^2 dF(u)
\]
\[
= \int_{-\infty}^{\infty} \left( \sum_{m=1}^{M} \hat{\psi}_{k_1^0} (0) - \frac{1}{2} \right)^2 dF(u) + o_p^* (1),
\]

and now equation (3.15) follows from the CLT for Bernoulli random variables. We now turn to (3.16). As in the proof of the previous result, we focus on \( \hat{\psi}_{k_1^0} \) only. By Theorem 1, under the alternative we have
\[
P \left( \omega : \lim_{\min(p_1, p_2, T) \to \infty} \hat{\phi}_{k_1^0} = 0 \right) = 1,
\]

and therefore we can assume from now on that
\[
\lim_{\min(p_1, p_2, T) \to \infty} \hat{\phi}_{k_1^0} = 0.
\]
We begin by noting that
\[ E^* \int_{-\infty}^{\infty} \left[ M^{-1/2} \sum_{m=1}^{M} \left( J \left( \hat{\psi}_{k_1}^{1/2} u^{(1)} \right) - G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) \right) \right]^2 dF(u) \leq c_0. \] (B.8)

Note now that
\[ \int_{-\infty}^{\infty} \left( M^{-1/2} \sum_{m=1}^{M} \left[ \hat{\psi}_{k_1}^0 (u) - G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) \right] \right)^2 dF(u) = \int_{-\infty}^{\infty} \left( M^{-1/2} \sum_{m=1}^{M} \left[ \hat{\psi}_{k_1}^0 (u) - G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) \right] \right)^2 dF(u) + \int_{-\infty}^{\infty} M^{1/2} \left( G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) - \frac{1}{2} \right)^2 dF(u).

Using (B.8) and Markov inequality, it holds that
\[ \int_{-\infty}^{\infty} \left( M^{-1/2} \sum_{m=1}^{M} \left[ \hat{\psi}_{k_1}^0 (u) - G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) \right] \right)^2 dF(u) = O_{P^*} (1); \]
also, using (B.7), it follows immediately that
\[ \lim_{\min(p_1,p_2,T,M) \to \infty} M \int_{-\infty}^{\infty} \left( G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) - \frac{1}{2} \right)^2 dF(u) = \infty. \]

By applying the Cauchy-Schwartz inequality, it also holds that
\[ \int_{-\infty}^{\infty} M^{1/2} \left( G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) - \frac{1}{2} \right) \left( M^{-1/2} \sum_{m=1}^{M} \left[ \hat{\psi}_{k_1}^0 (u) - G \left( \frac{u}{\hat{\phi}_{k_1}^{1/2}} \right) \right] \right) dF(u) = O_{P^*} \left( M^{1/2} \right). \]

The desired result now follows.

\[ \Box \]

Proof of Theorem 3. We begin by noting that \( E^* I \left[ \tilde{\psi}_{k_1}^{s_1} \leq c_\alpha \right] = P^* \left( \tilde{\psi}_{k_1}^{s_1} \leq c_\alpha \right). \) From
the proof of Theorem 1, we know that

\[
\hat{\Psi}_{k_0} = X_{M,s} + Y_{M,s},
\]

where

\[
X_{M,s} = \left( \frac{2}{\sqrt{M}} \sum_{m=1}^{M} \left[ \hat{\psi}^{(m)}_{k_0,s}(0) - \frac{1}{2} \right] \right)^2,
\]

and the remainder \( Y_{M,s} \) is such that \( E^* Y_{M,s}^2 = c_0 \left( \hat{\phi}^{-1/2}_{k_0} + M \hat{\phi}^{-1}_{k_1} \right) \) - henceforth, we omit
the dependence on \( s \) when possible. Let now \( \epsilon = \left( \hat{\phi}^{-1/2}_{k_0} + M \hat{\phi}^{-1}_{k_1} \right)^{1/3} \), and note that, using
elementary arguments

\[
P^* \left( \hat{\Psi}_{k_0} \leq c_0 \right) \leq P^* \left( X_{M,s} \leq c_0 + \epsilon \right) + P^* \left( |Y_{M,s}| \geq \epsilon \right).
\]

Thus

\[
P^* \left( \hat{\Psi}_{k_0} \leq c_0 \right) - P^* \left( X_{M,s} \leq c_0 \right)
\]

\[
\leq P^* \left( X_{M,s} \leq c_0 + \epsilon \right) - P^* \left( X_{M,s} \leq c_0 \right) + P^* \left( |Y_{M,s}| \geq \epsilon \right).
\]

Markov inequality immediately yields \( P^* \left( |Y_{M,s}| \geq \epsilon \right) \leq \epsilon^{-2} E^* Y_{M,s}^2 = c_0 \left( \hat{\phi}^{-1/2}_{k_0} + M \hat{\phi}^{-1}_{k_1} \right)^{1/3} \).

Letting \( Z \) be a \( N(0,1) \) distributed random variable, we can write

\[
P^* \left( X_{M,s} \leq c_0 + \epsilon \right) - P^* \left( X_{M,s} \leq c_0 \right)
\]

\[
= P^* \left( X_{M,s} \leq c_0 + \epsilon \right) \pm P \left( Z^2 \leq c_0 + \epsilon \right) - \left[ P^* \left( X_{M,s} \leq c_0 \right) \leq P \left( Z^2 \leq c_0 \right) \right].
\]

Using the mean value theorem, it is easy to see that

\[
P \left( Z^2 \leq c_0 + \epsilon \right) - P \left( Z^2 \leq c_0 \right) \leq c_0 \epsilon.
\]
Also, by the Berry-Esseen theorem (see e.g. Michel, 1976) it holds that

\[ \left| P^* (X_{M,s} \leq c_\alpha + \epsilon) - P \left( Z^2 \leq c_\alpha + \epsilon \right) \right| \leq c_0 \frac{M^{-1/2}}{1 + |c_\alpha + \epsilon|^{3+\delta}}, \]

\[ \left| P^* (X_{M,s} \leq c_\alpha) - P \left( Z^2 \leq c_\alpha \right) \right| \leq c_0 \frac{M^{-1/2}}{1 + |c_\alpha|^{3+\delta}}, \]

for all \( \delta \geq 0 \). Putting all together, it follows that

\[ \left| P^* \left( \hat{\Psi}_{k_0}^{*} \leq c_\alpha \right) - P \left( Z^2 \leq c_\alpha \right) \right| \leq c_0 \left( M^{-1/2} + \epsilon \right). \]

Hence we have

\[ \sqrt{\frac{S}{2 \ln \ln S}} \frac{\hat{Q}_{k_0}^* (\alpha) - (1 - \alpha)}{\sqrt{\alpha (1 - \alpha)}} = \sqrt{\frac{S}{2 \ln \ln S}} \frac{S^{-1} \sum_{s=1}^{S} Z_s^{(\alpha)} - (1 - \alpha)}{\sqrt{\alpha (1 - \alpha)}} + c_0 \sqrt{\frac{S}{2 \ln \ln S}} \left( M^{-1/2} + \epsilon \right), \]

where \( \{Z_s^{(\alpha)}, 1 \leq s \leq S\} \) is an i.i.d. sequence with common distribution \( Z_s^{(\alpha)} \sim N (1 - \alpha, \alpha (1 - \alpha)) \).

By the fact that \( S = O(M) \), it follows that

\[ \sqrt{\frac{S}{2 \ln \ln S}} \left( M^{-1/2} + \epsilon \right) = o_{a.s.} \left( 1 \right); \]

the desired result now follows from the Law of the Iterated Logarithm. Under \( H_A \), the results in the proof of Proposition 1 show that \( P^* \left( \hat{\Psi}_{k_0}^{*} \leq c_\alpha \right) \leq \epsilon \), for all \( \epsilon > 0 \) and for almost all realisations of \( \{X_t, 1 \leq t \leq T\} \). Hence

\[ \hat{Q}_{k_0}^* (\alpha) = \frac{1}{S} \sum_{s=1}^{S} I \left( \hat{\Psi}_{k_0}^{*} \leq c_\alpha \right) \]

\[ = \frac{1}{S} \sum_{s=1}^{S} E^* I \left( \hat{\Psi}_{k_0}^{*} \leq c_\alpha \right) + \frac{1}{S} \sum_{s=1}^{S} \left( I \left( \hat{\Psi}_{k_0}^{*} \leq c_\alpha \right) - E^* I \left( \hat{\Psi}_{k_0}^{*} \leq c_\alpha \right) \right). \]
Note that

\[
V^\ast \left( \frac{1}{S} \sum_{s=1}^{S} \left( I \left( \hat{\Psi}_{k_1^0,s} \leq c_\alpha \right) - E^\ast \left( \hat{\Psi}_{k_1^0,s} \leq c_\alpha \right) \right) \right) \\
= \frac{1}{S^2} \sum_{s=1}^{S} V^\ast I \left( \hat{\Psi}_{k_1^0,s} \leq c_\alpha \right) \leq \frac{1}{S^2} \sum_{s=1}^{S} E^\ast I \left( \hat{\Psi}_{k_1^0,s} \leq c_\alpha \right) \\
\leq \epsilon S^{-1},
\]

for all \( \epsilon > 0 \) and for almost all realisations of \( \{ X_t, 1 \leq t \leq T \} \). Using Lemma B.1, it follows that

\[
\hat{Q}_{k_1^0}(\alpha) \leq \epsilon_1 + \epsilon_2 \left( \frac{\ln S}{S^{1/2}} \right)^{1+\epsilon},
\]

for all \( \epsilon > 0 \) and for almost all realisations of \( \{ X_t, 1 \leq t \leq T \} \). This shows the desired result.

\[ \square \]

**Proof of Theorem 4.** The proof is very similar to that of Theorem 3.3 in Trapani (2018), and we only sketch its main passages. We begin by noting that, when \( k_1 = 0 \), there is nothing to prove as the desired result is already contained in Theorem 3. When \( k_1 > 0 \), consider the events \( \{ \hat{k}_1 = j \} \), for \( 0 \leq j \leq k_1 - 1 \). By the independence of tests across \( j \) (conditional on the sample), it follows that

\[
P^\ast \left( \hat{k}_1 = j \right) = P^\ast \left( \hat{Q}_j(\alpha) < 1 - \alpha - \sqrt{\alpha (1 - \alpha)} \sqrt{\frac{2 \ln \ln S}{S}} \right) \times \\
\prod_{k=1}^{j-1} P^\ast \left( \hat{Q}_k(\alpha) \geq 1 - \alpha - \sqrt{\alpha (1 - \alpha)} \sqrt{\frac{2 \ln \ln S}{S}} \right).
\]

Note that equation (3.21) immediately implies that \( P^\ast \left( \hat{k}_1 = j \right) = 0 \) as \( \min \{ p_1, p_2, T, M, S \} \rightarrow \)
Similarly, considering the events $\{\hat{k}_1 = k_1 + j\}$, for $1 \leq j \leq k_{\text{max}} - k_1$, it holds that

$$P^* (\hat{k}_1 = k_1 + j) = \left( \prod_{h=1}^{k_1} P^* (\hat{Q}_h (\alpha) \geq 1 - \alpha - \sqrt{\alpha (1 - \alpha)} \sqrt{\frac{2 \ln \ln S}{S}}) \right) \times \left( \prod_{h=k_1+1}^{k_1+j} P^* (\hat{Q}_h (\alpha) \geq 1 - \alpha - \sqrt{\alpha (1 - \alpha)} \sqrt{\frac{2 \ln \ln S}{S}}) \right) \times P^* (\hat{Q}_{k_1+j+1} (\alpha) < 1 - \alpha - \sqrt{\alpha (1 - \alpha)} \sqrt{\frac{2 \ln \ln S}{S}}),$$

for all $j = 1, \ldots, k_{\text{max}} - k_1$; by equation (3.22), it follows that $P^* (\hat{k}_1 = k_1 + j) = 0$ as $\min \{p_1, p_2, T, M, S\} \to \infty$. Putting all together, it is easy to see that $P^* (\hat{k}_1 = k_1).$  \hfill \square


## D Additional Simulation Results

**Table C.1:** Proportions of correctly determining whether there exists factor structure using (3.23) $\hat{\Psi}_1^S$ and $\tilde{\Psi}_1^S$ over 500 replications with $\alpha = 0.01, \phi = \psi = 0.1, f(S) = S^{-1/4}$.

| $(M, S)$ | $(k_1, k_2)$ | Method | $(p_1, T) = (100, 100)$ | $(p_1, T) = (150, 150)$ |
|----------|--------------|--------|-------------------------|-------------------------|
|          |              |        | $p_2 = 15$ | $p_2 = 20$ | $p_2 = 30$ | $p_2 = 15$ | $p_2 = 20$ | $p_2 = 30$ |
| (100,300) | (0,0)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          | (1,1)        | $\hat{\Psi}_1^S$ | 0.956 | 0.99 | 1 | 0.984 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 0.998 | 1.00 | 1 | 1.000 | 1 | 1 |
|          | (1,3)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
| (500,300) | (0,0)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          | (1,1)        | $\hat{\Psi}_1^S$ | 0.42 | 0.666 | 0.922 | 0.69 | 0.882 | 0.984 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 0.92 | 1.000 | 1.000 | 0.98 | 0.998 | 1.000 |
|          | (1,3)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
| (300,100) | (0,0)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          | (1,1)        | $\hat{\Psi}_1^S$ | 0.636 | 0.866 | 0.97 | 0.874 | 0.972 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 0.976 | 1.000 | 1.000 | 0.988 | 1.000 | 1 |
|          | (1,3)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
| (300,500) | (0,0)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          | (1,1)        | $\hat{\Psi}_1^S$ | 0.634 | 0.822 | 0.966 | 0.822 | 0.946 | 0.992 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 0.966 | 1.000 | 1.000 | 0.994 | 1.000 | 1.000 |
|          | (1,3)        | $\hat{\Psi}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
|          |              | $\hat{\tilde{\Psi}}_1^S$ | 1 | 1 | 1 | 1 | 1 | 1 |
Table C.2: Proportions of correctly determining whether there exists factor structure using \((3.23) \hat{\Psi}_1^S\) and \(\tilde{\Psi}_1^S\) over 500 replications with \(\alpha = 0.01, \phi = \psi = 0.1, M = S = 300\).

| \(f(S)\) | \((k_1, k_2)\) | Method | \((p_1, T) = (100, 100)\) | \((p_1, T) = (150, 150)\) |
|---------|----------------|--------|-----------------------------|-----------------------------|
|         |                |        | \(p_2 = 15\) | \(p_2 = 20\) | \(p_2 = 30\) | \(p_2 = 15\) | \(p_2 = 20\) | \(p_2 = 30\) |
|         |                |        | \(p_1 = 15\) | \(p_1 = 20\) | \(p_1 = 30\) | \(p_1 = 15\) | \(p_1 = 20\) | \(p_1 = 30\) |
| \(S^{-1/3}\) | (0,0) | \(\hat{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
|          |        | \(\tilde{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
|          | (1,1)  | \(\hat{\Psi}_1^S\) | 0.548 | 0.778 | 0.97 | 0.804 | 0.936 | 1 |
|          |        | \(\tilde{\Psi}_1^S\) | 0.964 | 1.000 | 1.00 | 0.992 | 1.000 | 1 |
|          | (1,3)  | \(\hat{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
|          |        | \(\tilde{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
| \(S^{-1/5}\) | (0,0) | \(\hat{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
|          |        | \(\tilde{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
|          | (1,1)  | \(\hat{\Psi}_1^S\) | 0.642 | 0.836 | 0.988 | 0.848 | 0.95 | 1 |
|          |        | \(\tilde{\Psi}_1^S\) | 0.976 | 1.000 | 1.000 | 0.996 | 1.000 | 1 |
|          | (1,3)  | \(\hat{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
|          |        | \(\tilde{\Psi}_1^S\) | 1 | 1 | 1 | 1 | 1 | 1 |
Table C.3: Simulation results for estimating $k_1$ in the form $x(y|z)$, $x$ is the sample mean of the estimated factor numbers based on 500 replications with $(k_1, k_2) = (3, 3)$, $\phi = \psi = 0.1$, $f(S) = S^{-1/4}$, $y$ and $z$ are the numbers of underestimation and overestimation, respectively.

| $(\alpha, M, S)$ | Method | $(p_1, T) = (100, 100)$ | $(p_1, T) = (150, 150)$ |
|------------------|--------|-------------------------|-------------------------|
| $\alpha$-PCA     |        | $p_2 = 15$ | $p_2 = 20$ | $p_2 = 30$ | $p_2 = 15$ | $p_2 = 20$ | $p_2 = 30$ |
| $(0.05,300,300)$  | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
| $(0.10,300,300)$  | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
| $(0.01,100,300)$  | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
| $(0.01,500,300)$  | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
| $(0.01,300,100)$  | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
| $(0.01,300,500)$  | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|                  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
Table C.4: Simulation results for estimating $k_1$ in the form $x(y|z)$, $x$ is the sample mean of the estimated factor numbers based on 500 replications with $(k_1, k_2) = (3, 3)$, $\phi = \psi = 0.1, M = S = 300, \alpha = 0.01, y$ and $z$ are the numbers of underestimation and overestimation, respectively.

| $f(S)$ | Method | $(p_1, T) = (100, 100)$ | $(p_1, T) = (150, 150)$ |
|--------|--------|----------------|----------------|
| $S^{-1/3}$ | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
| $S^{-1/5}$ | STP$_1$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|  | STP$_2$ | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|  | IterER | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
|  | $\alpha$-PCA | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) | 3(0|0) |
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