Coulomb actuated microbeams: A Chebyshev-Edgeworth approach to highly efficient lumped parameter models

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(Dated: 6 January 2022)
In a previous publication we demonstrated that the stable and unstable equilibrium states of prismatic Coulomb actuated Euler-Bernoulli micro-beams, clamped at both ends, can successfully be simulated combining finite element analysis (FEM) with continuation methods. Simulation results were experimentally scrutinised by combining direct optical observations with a modal analysis regarding Euler-Bernoulli eigenmodes. Experiment and simulation revealed convincing evidence for the possibility of modelling the physics of such a micro-beam by means of lumped parameter models involving only a single degree of freedom, the Euler-Bernoulli zero mode. In this paper we present the corresponding analytical single degree of freedom lumped parameter model (LPM). This comprehensive model demonstrates the impact of the beam bending on the nature of the Coulomb singularity, allows for an easy and accurate computation of the pull-in voltage in the presence of stress stiffening and is apt for efficient frequency response computations. Our method to derive the zero-mode LPM is based on a Chebyshev-Edgeworth type method as is common in analytical probability theory. While used here for a very particular purpose, this novel approach to non-linear dynamic systems has a much broader scope. It is apt to analyse different boundary conditions, electrostatic fringe field corrections and squeeze film damping, to name a few applications.

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I. INTRODUCTION

Coulomb-actuated microbeams play a crucial role in many Micro-Electro-Mechanical Systems (MEMS) applications. They enable actuation using electrostatic forces and capacitive sensing, give rise to pioneering applications in medicine, communications, sensing, and consumer products. To meet the needs of recent developments, such as 5G Internet of Things (5G-IoT), augmented reality, and Green ICT (information and communications technology), a system level consideration of a high number of electro-mechanical components is necessary. This is only possible, if accurate and highly efficient lumped parameter models of the components are available.

In this paper we systematically derive a single degree of freedom lumped-parameter model (LPM), describing the physics of prismatic clamped-clamped Coulomb actuated microbeams with high precision as compared to FEM simulations and in line with experimental findings. In a previous publication, Melnikov et al. demonstrated that the stable and unstable states of prismatic Coulomb actuated Euler-Bernoulli micro-beams, clamped at both ends, can be successfully simulated combining FEM with arc-length solvers. The resulting model predictions were experimentally scrutinised by combining direct optical observations with a modal analysis regarding Euler-Bernoulli eigenmodes. Both approaches revealed convincing evidence for an almost perfect congruence of the respective bending profile and the shape of the lowest Euler-Bernoulli eigenmode (the zero-mode). It was shown that this is true for the entire applicable voltage range within very small error margins. The observation suggests the possibility to model the physics of such a micro-beam by means of a lumped parameter model involving only a single degree of freedom, amenable to direct physical interpretation.

Studies analytically deriving lumped parameter models, e. g. Nayfeh, Younis, and Rahman typically begin with the non-linear Euler-Bernoulli beam equation for the bending profile $w(\xi, \tau)$. In its dimensionless form used by Nayfeh et al. this equation reads

$$\frac{\partial^2 w}{\partial \tau^2} + c \frac{\partial w}{\partial \tau} + \frac{\partial^4 w}{\partial \xi^4} = \gamma[w] + N \frac{\partial^2 w}{\partial \xi^2} + \alpha^2 \frac{v(\tau)^2}{(1 - w)^2}. \quad (1)$$

Here $\xi$ and $\tau$ are the dimensionless beam coordinate and the dimensionless time. The
dynamic damping coefficient is denoted by $c$. The geometry dependent parameters $\alpha_1$ and $\alpha_2$ are given in Eq. (47) below. $N$ is an dimensionless axial stress and $v(\tau)$ is the dimensionless drive voltage. The beam is assumed to be clamped at $\xi = -\frac{1}{2}$ and at $\xi = +\frac{1}{2}$, where the usual clamped-clamped boundary conditions Supplementary Eq. (S7) apply. The non-local functional $\gamma[w]$ models the stress stiffening of the clamped-clamped beam,

$$\gamma[w] = \alpha_1 \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left( \frac{\partial w}{\partial \xi} \right)^2 \, d\xi . \tag{2}$$

Nayfeh et al. expand the bending profile with respect to a complete ortho-normal Hilbert space base $\psi_n(\xi)$,

$$w(\xi, \tau) = \sum_{n=0}^{\infty} \hat{w}_n(\tau) \psi_n(\xi) . \tag{3}$$

Upon insertion into Eq. (1), the partial differential equation Eq. (1) is converted into an infinite set of coupled nonlinear ordinary differential equations of the form:

$$\frac{\partial^2 \hat{w}_n}{\partial \tau^2} + c \frac{\partial \hat{w}_n}{\partial \tau} + \sum_{m=0}^{\infty} k_{n,m}[w] \hat{w}_m = \alpha_2 v^2 F_n[\tau, w] . \tag{4}$$

Unlike Nayfeh et al., we select $\{\lambda_n, \psi_n(\xi)\}_{n \in \mathbb{N}}$ to be the Euler-Bernoulli eigen system and can therefore be a little more specific,

$$k_{n,m}[w] = \lambda_n \delta_{n,m} + (\gamma[w] + N) \chi_{n,m} , \tag{5}$$

$$\chi_{n,m} = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial \psi_n}{\partial \xi} \frac{\partial \psi_m}{\partial \xi} \, d\xi . \tag{6}$$

The challenge with this approach however is that the resulting stiffness matrix $k_{n,m}[w]$ and the force components $F_n[\tau, w]$ are rather intricate, non-linear, singular and time dependent functionals of the entire infinite set of the coefficient functions $\{\hat{w}_n(\tau)\}_{n \in \mathbb{N}}$:

$$k_{n,m}[w] = k_{n,m}[\hat{w}_0(\tau), \ldots, \hat{w}_n(\tau), \ldots] , \tag{7}$$

$$F_n[\tau, w] = F_n[\tau, \hat{w}_0(\tau), \ldots, \hat{w}_n(\tau), \ldots] .$$

This circumstance makes it in general very challenging to obtain any elucidating results from Eq. (4). As can be see from literature, the complexity of the functionals $k_{n,m}[w]$ and $F_n[\tau, w]$ leads to a tedious computational task, even after introducing well considered simplifications, e.g. see Younis et al. The resulting computations seem neither more attractive than direct
numerical methods, nor is the need for the number of degrees of freedom, required to obtain satisfactory accuracy, amenable to direct physical interpretation. In fact the number of modes required in Nayfeh’s _et al._ approach turns out to be an artefact, essentially reflecting their comparatively straight forward attempt to technically cope with the singular nature of the Coulomb force, as we will see.

The picture substantially changes however with the observation of Melnikov _et al._\textsuperscript{16} that the lowest Euler-Bernoulli eigenmode \(\psi_0(\xi)\) is by far dominating the physics of Coulomb actuated prismatic clamped-clamped micro-beams in practical applications. This observation implies that the use of higher modes in a LPM for a prismatic Euler-Bernoulli beam is hardly justified, unless higher kinetic energies are involved. Due to the large spectral distance, typically a multiple of the elastic energy corresponding to the considered deflection of the zero-mode is required for significant effects involving higher modes.

The observation of Melnikov _et al._\textsuperscript{16} essentially allows to reduce the Eq. (3) to the single term

\[
  w(\xi, \tau) \approx z(\tau) \frac{\psi_0(\xi)}{\psi_0(0)}, \quad 0 \leq z(\tau) \leq 1 ,
\]

\[
  \psi_0(\xi) = \frac{\cosh(\beta_0 \xi)}{\cosh(\beta_0 / 2)} - \frac{\cos(\beta_0 \xi)}{\cos(\beta_0 / 2)} .
\]  

Here \(\beta_0\) is the smallest solution to the equation

\[
  0 = \tanh(\beta/2) + \tan(\beta/2) .
\]

In zero-mode approximation Eq. (4) simplifies to the quite handy form

\[
  \frac{\partial^2}{\partial \tau^2} z + c \frac{\partial}{\partial \tau} z + k_0 z + \kappa z^3 = u^2 f_0(z) ,
\]  

(10)

The parameters \(\kappa, k_0\) and \(u\) are defined as

\[
  \kappa = \alpha_1 \left( \frac{\chi_0}{\psi_0(0)} \right)^2 ,
\]

\[
  k_0 = \lambda_0 + N \chi_0 , \quad u = \psi_0(0) \sqrt{\alpha_2} v ,
\]

\[
  \chi_0 = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left( \frac{\partial \psi_0}{\partial \xi} \right)^2 d\xi
\]  

(12)

and the force term is

\[
  f_0(z) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\psi_0(\xi)}{\psi_0(0)} \frac{\psi_0(\xi)}{\psi_0(0)} \left( 1 - z \frac{\psi_0(\xi)}{\psi_0(0)} \right)^2 d\xi .
\]  

(13)
The remaining key challenge, and the prime topic of this paper, is of course evaluating the Coulomb integral \( f_0(z) \). This requires a non-pertubative treatment of the Coulomb singularity. The ad-hoc approach of Younis \textit{et al.}\cite{20} essentially creates an artificial need for higher modes and therefore enforces dealing with a coupled system of non-linear ordinary differential equations (ODE). This is far from satisfactory. It is the purpose of this paper to demonstrate, in contrast, that the physics of a Coulomb actuated prismatic Euler-Bernoulli is contained in the single ODE Eq. \((10)\) to an extend sufficient for most practical purposes in MEMS technology. To this end we devise a non-pertubative strategy of dealing with the Coulomb integral, based on a Chebyshev-Edgeworth type expansion\cite{21,22,23}. As a result we arrive at a highly accurate analytical expression for \( f_0(z) \). Finally, the application of our zero-mode LPM Eq. \((47)\) to the simulation results and experimental findings of Melnikov \textit{et al.}\cite{10}, reveal a very good agreement.

II. RESULTS

A. Chebyshev’s argument

Our evaluation the of integral \( f_0[z] \) begins with the series representation

\[
f_0(z) = \sum_{n=1}^{\infty} n I_n z^{n-1},
\]

where the integrals \( I_n \) are defined as

\[
I_n = \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \left( \frac{\psi_0(\xi)}{\psi_0(0)} \right)^n d\xi.
\]

Note that because \(|I_n| < 1\) we can infer by means of the Cauchy-Hadamard theorem that the series Eq. \((14)\) is absolutely convergent in the open disc \(|z| < 1\), as required for our purposes. The integrals \( I_n \) can be cast into the form,

\[
I_n = \frac{\sqrt{2\pi}}{\sigma \sqrt{n}} \int_{-\frac{1}{\sigma \sqrt{n}}}^{\frac{1}{\sigma \sqrt{n}}} \Phi_n(\xi) d\xi,
\]

where \( \Phi_n(\xi) \) is defined as

\[
\Phi_n(\xi) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\psi_0(0)} \psi_0 \left( \frac{\xi}{\sigma \sqrt{n}} \right) \right)^n,
\]

\[
\sigma^2 = -\frac{\psi_0^{(2)}(0)}{\psi_0(0)}.
\]
Our strategy now is to evaluate the limiting function $\Phi_\infty(\xi)$ of the sequence $\{\Phi_n(\xi)\}_{n\in\mathbb{N}}$ and subsequently to expand $\Phi_n(\xi)$ around $n = \infty$ with respect to $n^{-1}$. This allows us to explicitly perform the integration Eq. (16). As a result we can perform the summation Eq. (14). This way we arrive at the targeted formula for $f_0(z)$.

The crucial observation regarding the limiting function $\Phi_\infty(\xi)$ is that the sequence $\{\Phi_n(\xi)\}_{n\in\mathbb{N}}$ uniformly converges to the shape of the Gauss bell curve,

$$\lim_{n\to\infty} \Phi_n(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right).$$

This important fact is illustrated in Fig. 1. To motivate how this comes about, we remind the reader of Euler’s elementary definition of the exponential function, presented here in a form suitable for our purposes,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \left(1 - \frac{\xi^2}{2n}\right)^n = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right).$$

The quite perplexing idea that Eq. (19) holds for a much broader class of functions, inserted into its left hand side, dates back to the groundbreaking contributions of P.L. Chebyshev to the field of analytical probability theory. In fact Eq. (18) and Eq. (19) essentially are special case of the celebrated central limit theorem (CLT). The reader acquainted with the CLT is reminded, that the operations of multiplication and convolution in function space interchange their roles when subjected to a Fourier transformation. There is however no need to discuss the details of the proof of the CLT here: Luckily, our mechanical setting allows for a pedestrian’s approach to verify Eq. (18).

The proof starts with the observation that the normalized bending profile of a fully concentrated load (only the right hand side of the symmetric profile is given),

$$g(\xi) = (1 - 2\xi)^2(1 + 4\xi), \quad 0 \leq \xi \leq \frac{1}{2}$$

and the normalized bending profile of the fully distributed, i.e. constant load,

$$h(\xi) = (1 - 4\xi^2)^2$$

provide an upper and a lower bound for $\phi_n(\xi)$,

$$G_n(\xi) \geq \phi_n(\xi) \geq H_n(\xi).$$
Here $G_n(\xi)$ and $H_n(\xi)$ are defined analogously to Eq. (17), i.e. by replacing $g(\xi)$ and $h(\xi)$ respectively for $\psi_0(\xi)$ in that equation (also the respective $\sigma$ needs to be calculated),

$$
G_n(\xi) = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\xi}{\sqrt{6n}} \right)^{2n} \left( 1 + \frac{2\xi}{\sqrt{6n}} \right)^n,
$$

$$
H_n(\xi) = \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\xi^2}{4n} \right)^{2n}.
$$

The relation Eq. (22) is easily verified by establishing the assertion for $n = 1$ first, and then using the positivity of the functions involved when raising to the $n$-th power. Note that the relation Eq. (22) also is invariant under the scaling of the $\xi$-axis, required when progressing from $n$ to $n + 1$. Computing the limiting function of the sequence $\{H_n(\xi)\}_{n \in \mathbb{N}}$ is a simple application of Eq. (19),

$$
\lim_{n \to \infty} H_n(\xi) = \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\xi^2}{4n} \right)^{2n} = \lim_{m \to \infty} \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\xi^2}{2m} \right)^{m} = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\xi^2}{2} \right).
$$

Computing the limiting function of the sequence $\{G_n(\xi)\}_{n \in \mathbb{N}}$ is little more challenging,

$$
\lim_{n \to \infty} G_n(\xi) = \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{\xi^2}{2n} + \frac{\xi^3}{3\sqrt{6n^3}} \right)^n = \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \left( 1 + \frac{\xi^2}{2n} \right)^n \left( 1 + O \left( \frac{1}{n} \right)^{\frac{1}{2}} \right) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\xi^2}{2} \right).
$$

Dini’s theorem\textsuperscript{[20]} asserts the uniformity of the convergence in both cases. Now since both, the upper and the lower bound of $\Phi_n(\xi)$ uniformly converge to the Gaussian, the same holds true for the sequence $\{\Phi_n(\xi)\}_{n \in \mathbb{N}}$ itself, establishing Eq. (18).

Before ending this section, we would like to highlight that Chebyshev’s general argument works in the domain of elasto-mechanics far beyond the simple case presented here and does not require any kind of symmetry. That is because Chebyshev essentially exploits the fact that Hermite polynomials form a complete base of the Hilbert space of functions over the reals, that are square integrable with respect to the measure defined by the Gauss bell curve.
B. The Edgeworth expansion

For the evaluation of the Coulomb integral \( f_0(z) \) we need to know how exactly \( \Phi_n(\xi) \) approaches Gauss’ bell curve as \( n \) grows larger. The answer is provided by the famous Edgeworth expansion: Following the ideas of F.Y. Edgeworth, Eq. (18) warrants the existence of an asymptotic expansion of the form\(^{20,27}\)

\[
\Phi_n(\xi) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\xi^2}{2} \right) \times \left( 1 - \frac{c_1(\xi)}{n} + \frac{c_2(\xi)}{n^2} + O \left( \frac{1}{n} \right)^3 \right). \tag{26}
\]

The explicit version of this asymptotic expansion, is obtained by expanding \( \Phi_n(\xi) \) in a Taylor series at \( n = \infty \) in powers of \( n^{-1} \). The computation of the respective Taylor coefficients is enabled by the use of Eq. (18) and of

\[
\sqrt{2\pi} \Phi_1(\xi) = 1 - \frac{\xi^2}{2} + \frac{\mu_4}{24} \xi^4 - \frac{\mu_6}{720} \xi^6 + O(\xi)^8. \tag{27}
\]

Here we have introduced the following abbreviations related to the derivatives of order \( 2k \),

\[
\mu_{2k} = \frac{(-1)^k}{\sigma^{2k}} \frac{\psi_0^{(2k)}(0)}{\psi_0(0)}, \tag{28}
\]

\[
\mu_{4k} = \mu_{4k+2} = \left( \cosh(\beta_0/2) - \cos(\beta_0/2) \right)^{2k} \tag{29}
\]

The sequence of integers appearing in Eq. (27) is the sequence of the non prime factorials; this jointly with Eq. (29) implies that the expansion Eq. (27) is absolutely convergent within an infinite radius of convergence. Merten’s theorem regarding Cauchy products\(^{28}\) therefore ensures that any integer power of Eq. (27), required for the evaluation of Eq. (17) exists, also possessing an infinite radius of convergence.

The first two coefficients of the Edgeworth expansion obtained following the route outlined here are,

\[
c_1(\xi) = \frac{-3 + \mu_4}{24} \xi^4, \tag{30}
\]

\[
c_2(\xi) = \frac{(-3 + \mu_4)^2}{1152} \xi^8 + \frac{-30 + 15\mu_4 - \mu_6}{720} \xi^6.
\]

Finally we would like to add that in the setting of analytical probability theory, the \( \Phi_1(\xi) \) plays the role of the characteristic function of a probability density and the \( \{\mu_n\}_{n\in\mathbb{N}} \) are its respective moments. In our case, the Fourier transform of \( \Phi_1(\xi) \), which should be a probability density, can adopt negative values. It only is asymptotically a non negative
function. So the notions of analytical probability theory, strictly speaking, do not apply. However the line of arguments of Chebyshev and Edgeworth still hold under our somewhat weaker conditions, as we have explicitly shown above.

C. Evaluating the Coulomb integral

In this section we evaluate Eq. (16) and perform the summation Eq. (14). The last subtlety to cope with, are the finite boundaries of the integral Eq. (16). While it is perfectly possible to analytically perform the integration within these finite boundaries and expand the results in terms of \( n^{-1} \), little is gained by this tedious exercise. Truncating the integrand at order \( O(n)^{-3} \) according to Eq. (26) and extending the integration boundaries of Eq. (16) to infinity generates an overall error, which is negligible for all practical purposes, as we will show in Eq. (33) below. Therefore we evaluate Eq. (16) in the form,

\[
I_n = \frac{\sqrt{2\pi}}{\sigma \sqrt{n}} \int_{-\infty}^{\infty} \Phi_n(\xi) \, d\xi + \Delta_n. \tag{31}
\]

Inserting Eq. (26) and Eq. (30) into Eq. (31) yields

\[
I_n = \frac{\sqrt{2\pi}}{\sigma} \times \left( \frac{1}{n^{\frac{3}{2}}} + \frac{\mu_4 - 3}{8 n^{\frac{3}{2}}} + \frac{75 - 90 \mu_4 + 35 \mu_4^2 - 8 \mu_6}{384 n^{\frac{5}{2}}} \right) + \Delta_n, \tag{32}
\]

where the remainder \( \Delta_n \) in Eq. (32) accounts for both simplifications mentioned above. An upper bound for \( \Delta_n \) can easily be found upon noticing that the maximum remainder occurs for \( n = 2 \). Taylor’s remainder theorem then asserts that according to Eq. (26) and Eq. (31) the remainder decays at least with the power \( n^{-\frac{5}{2}} \),

\[
0 < \Delta_n \leq \Delta_2 \left( \frac{2}{n} \right)^{\frac{5}{2}} < \frac{1}{1956 n^{\frac{5}{2}}}. \tag{33}
\]

The excellent accuracy of the expansion Eq. (32) for \( I_n \) is apparent from Fig. 2a.

To compute the Coulomb integral \( f_0(z) \), we need, last not least, to perform the summation according to Eq. (14). The result is given in Eq. (34), which we will call the Chebyshev-Edgeworth projection of the Coulomb force,

\[
f_0(z) = \frac{\sqrt{2\pi}}{\sigma} \frac{1}{z} \left( \text{Li}_{\frac{3}{2}}(z) + \frac{\mu_4 - 3}{8} \text{Li}_{\frac{1}{2}}(z) + \frac{75 - 90 \mu_4 + 35 \mu_4^2 - 8 \mu_6}{384} \text{Li}_{\frac{3}{2}}(z) \right) + \Delta f_0(z). \tag{34}
\]
In Eq. (34) the function \( \text{Li}_s(z) \) denotes Jonquière’s poly-logarithm\(^{20} \), defined for all \( |z| < 1 \) and for all \( s \in \mathbb{C} \) as
\[
\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.
\] (35)

Note the relation with Riemann’s zeta function\(^{20} \) \( \zeta(z) \) relevant to us,
\[
\text{Li}_s(1) = \zeta(s).
\] (36)

Based on Eq. (33) and on Eq. (35) we can evaluate an upper bound for the remainder \( \Delta f_0(z) \)
\[
0 < \Delta f_0(z) < \frac{1}{1956} z \text{Li}_{\frac{3}{2}}(z)
\]
\[
\frac{1}{1956} z \text{Li}_{\frac{3}{2}}(z) \leq \frac{1}{1956} \text{Li}_{\frac{3}{2}}(1) < \frac{1}{1236}.
\] (37)

Fig. 2b shows that the Chebyshev-Edgeworth projection produces excellent results for the Coulomb integral Eq. (13). The underlying reason is that the Chebyshev-Edgeworth expansion maintains the exact singularity structure of the Coulomb integral. This does not hold true for the approach of Younis et al.\(^{20} \).

D. The singular structure of the Coulomb Integral

The Chebyshev-Edgeworth projection formula Eq. (34) may actually appear a bit awkward, from a practitioners point of view. In the sequel we will seek to improve this. The key information contained in Eq. (34) is how exactly to deal with the Coulomb force: The zero-mode approximation is a projection of the beam equation Eq. (1) from the infinite-dimensional Hilbert space of all Euler-Bernoulli eigenmodes onto the one-dimensional subspace, spanned by the zero-mode \( \psi_0(z) \) only. A priori, it is far from obvious what this means for the Coulomb force. Note that this kind of question arises in any type of Galerkin procedure applied to Eq. (1). Eq. (34) allows us to give the answer in case of the zero-mode approximation, leading to a more practical version of our projection formula.

The contact singularity of the Coulomb force obviously causes a singularity of the Coulomb integral \( f_0(z) \) at \( z = 1 \). The information about this singularity is entirely contained
in the poly-logarithms $\text{Li}_{-\frac{1}{2}}(z)$ and $\text{Li}_{\frac{1}{2}}(z)$; all poly-logarithms of larger index are regular,

$$
\text{Li}_{-\frac{1}{2}}(z) = \text{Li}_{-\frac{1}{2}}(1) + \frac{\sqrt{\pi}}{2(1-z)^{\frac{1}{2}}} - \frac{\sqrt{3\pi}}{8(1-z)^{\frac{1}{2}}} + O(1-z)^{\frac{1}{2}},
$$

$$
\text{Li}_{\frac{1}{2}}(z) = \text{Li}_{\frac{1}{2}}(1) + \frac{\sqrt{\pi}}{(1-z)^{\frac{1}{2}}} + O(1-z)^{\frac{1}{2}},
$$

$$
\text{Li}_{n+\frac{1}{2}}(z) = \text{Li}_{n+\frac{1}{2}}(1) + O(1-z)^{\frac{1}{2}}, \quad 0 < n \in \mathbb{N}.
$$

This local analysis reveals that $f_0(z)$ as defined in Eq. (13) has the singular structure,

$$
f_0(z) = \hat{f}_0(z) + O(1-z)^{\frac{1}{2}}, \\
\hat{f}_0(z) = a + b(1-z)^{\frac{1}{2}} + c(1-z)^{\frac{3}{2}}.
$$

We now wish to find an algebraic approximation of the form $\hat{f}_0(z)$ to Eq. (34) that is as accurate as possible over the entire range $0 \leq z \leq 1$. This means we give up a little bit of the achieved accuracy at the singularity, in exchange for a global approximation, that pointwise has a relative error small enough for all practical purposes. To this end we demand that $a$, $b$ and $c$ minimize the distance between $f_0(z)$ and its algebraic approximation $\hat{f}_0(z)$ with respect to a suitable norm in function space. The challenge here is the isolated singularity at $z = 1$. The associated lack of integrability can however be mended by introducing an apt non negative weight function $r(z)$. The weight function should be selected such that it has a zero of sufficiently high degree compensating the singularity. Having said this, we choose the coefficients $a$, $b$ and $c$ to minimize the functional

$$
\mathcal{S}_r(a, b, c) = \int_0^1 r(z) \left( f_0(z) - \hat{f}_0(z) \right)^2 dz.
$$

A suitable $r(z)$ ensures the existence of this functional and of its Hessian as a positive definite matrix. Conceptually, an optimal weight function simultaneously minimizes the relative error. In practice, our simplistic choice, justified in arrears by Eq. (46), is

$$
r(z) = (1-z)^3.
$$

With this weight function the Hessian is

$$
\text{Hess}(\mathcal{S}_r) = \begin{pmatrix} 1/4 & 2/7 & 2/5 \\ 2/7 & 1/3 & 1/2 \\ 2/5 & 1/2 & 1 \end{pmatrix}.
$$
Accordingly, there is a uniquely defined minimum which is found solving for
\[
\frac{\partial}{\partial a} S_r(a, b, c) = 0 \iff H_a = \frac{a}{4} + \frac{2b}{7} + \frac{2c}{5}
\]
\[
\frac{\partial}{\partial b} S_r(a, b, c) = 0 \iff H_b = \frac{2a}{7} + \frac{b}{3} + \frac{c}{2}
\]
\[
\frac{\partial}{\partial c} S_r(a, b, c) = 0 \iff H_c = \frac{2a}{5} + \frac{b}{2} + c
\] (43)

The constants \(H_a, H_b\) and \(H_c\) are the integrals
\[
H_a = \int_0^1 (1 - z)^3 f_0(z) dz,
\]
\[
H_b = \int_0^1 (1 - z)^{\frac{3}{2}} f_0(z) dz,
\] (44)
\[
H_c = \int_0^1 (1 - z)^{\frac{3}{2}} f_0(z) dz.
\]

Using suitable integer fractions we find the targeted algebraic expansion for \(\hat{f}_0(z)\) to be,
\[
\hat{f}_0(z) = \frac{1}{77} - \frac{1}{38(1 - z)^{\frac{3}{2}}} + \frac{15}{28(1 - z)^{\frac{3}{2}}}. \] (45)

The upper bound for the maximum relative error regarding this greatly simplified version of the Coulomb integral is easily computed analytically to be
\[
\max_{0 \leq z \leq 1} \left(1 - \frac{\hat{f}_0(z)}{f_0(z)}\right) \leq \lim_{z \to 1} \left(1 - \frac{\hat{f}_0(z)}{f_0(z)}\right),
\]
\[
= 1 - \frac{15\sigma}{14\sqrt{2\pi}} < \frac{1}{494}. \] (46)

This is excellent for all practical purposes. In summary we have shown that the Coulomb singularity of Eq. \([1]\) transforms into the quite different singularity given by the asymptotic expansion of Eq. \([34]\), or for all practical purposes, by the global approximation Eq. \([45]\), when projected onto the one dimensional Hilbert subspace spanned by the Euler-Bernoulli zero-mode. To the best of our knowledge this is a completely new result of substantial practical relevance.

E. Synopsis of the zero-mode LPM

The zero-mode approximation
\[
w(\xi, \tau) \approx z(\tau) \frac{\psi_0(\xi)}{\psi_0(0)},
\]
developed in the previous section, leads upon careful treatment of the Coulomb singularity to the lumped parameter model,

\[
\frac{\partial^2}{\partial \tau^2} z + c \frac{\partial}{\partial \tau} z + k_0 z + \kappa z^3 = u^2 \hat{f}_0(z),
\]

\[
\hat{f}_0(z) = \frac{1}{77} - \frac{1}{38(1-z)^{1/2}} + \frac{15}{28(1-z)^{3/2}},
\]  

\[
\kappa = \alpha_1 \left( \frac{\chi_0}{\psi_0(0)} \right)^2, \quad k_0 = \lambda_0 + N\chi_0,
\]

\[
u = \psi_0(0)\sqrt{\alpha_2} v,
\]

\[
\alpha_1 = 6 \left( \frac{g}{t} \right)^2, \quad \alpha_2 = \frac{6\epsilon l^4}{Et^3 g^3}.
\]

Here \(l\), \(t\), and \(E\) denote length, thickness and Young’s modulus of the beam. \(g\) is the electrode gap. The definitions of \(\psi_0(0)\) and \(\chi_0\) can be found in Eq. (8), Eq. (9) and Eq. (12). For higher precision, Eq. (34) or any refinement thereof, can be used instead of Eq. (45).

The bifurcation diagram, showing the static deflection of the beam center as a function of the drive voltage, is obtained as the set of all points in the \((u, z)\) plane, solving the purely algebraic equation

\[
\kappa z^3 + k_0 z = u^2 \hat{f}_0(z).
\]

Eq. (48) is best used by looking upon the voltage \(u\) as a function of the deflection, i.e. \(u = u(z)\). The static pull-in deflection \(z_{PI}\) is reached at the critical point where

\[
\frac{\partial u}{\partial z}(z_{PI}) = 0.
\]

This condition is conveniently exploited by taking the inverse of the logarithmic derivative of Eq. (48). As shown in Eq. (51) below, within very small error margins, the inverse of the logarithmic derivative of the Coulomb integral is a linear function of the deflection amplitude \(z\),

\[
\left( \frac{\partial}{\partial z} \log(f_0(z)) \right)^{-1} = \frac{64}{97} - \frac{44}{67} z - \Delta_{LOG}(z).
\]

The approximation Eq. (50) is obtained upon inserting Eq. (14) into the left hand side of Eq. (50) and performing a Taylor expansion. The maximum error occurs at \(z = 1\) where the left hand side of Eq. (50) vanishes, due to the nature of its singularity as exhibited in Eq. (39). This puts a tight absolute bound on the remainder \(\Delta_{LOG}(z)\),

\[
0 \leq \Delta_{LOG}(z) < \frac{1}{325}.
\]
It should be emphasised, that the derivation of Eq. (50) does not require using Eq. (45) or any other approximation discussed in this paper. The absolute upper bound of the remainder $\Delta_{LOG}(z)$ is therefore not affected by any choice or error estimate made elsewhere. The highly effective approximation Eq. (50) leads to a simple algebraic equation for the practical evaluation of the pull-in deflection $z_{PI}$,

$$\frac{\kappa z_{PI}^3 + k_0 z_{PI}}{3\kappa z_{PI}^2 + k_0} \approx \frac{64}{97} - \frac{44}{67} z_{PI}.$$  (52)

A first easy conclusion that can be drawn from Eq. (52) is that the pull-in deflection of a Coulomb actuated clamped-clamped Euler Bernoulli beam varies within the limits

$$0.3982 \leq z_{PI} \leq 0.6664.$$  (53)

The lower bound of Eq. (53) is obtained as the limiting case of Eq. (52), where the stress stiffening (Duffing) coefficient $\kappa$ vanishes. Likewise, the upper bound of Eq. (53) results from Eq. (52) in case of an infinitely large $\kappa$. Within the realm of Euler-Bernoulli theory, these boundaries are independent of the shape of the beam cross section. While this fact certainly is known from numerical studies\textsuperscript{16}, it is derived here based on an analytical model, probably for the first time.

Once we know $z_{PI}$, we can find the respective pull-in voltage $u_{PI}$ using Eq. (48). The simple recipe presented in this section, requires little more than a spreadsheet or a pocket calculator to compute the pull-in data and the entire bifurcation diagram, with the astonishing numerical accuracy exhibited in Fig. 3, Fig. 4 and Fig. 5.

F. LPM analysis of the beam used by Gilbert\textit{ et al.}

As a first application of our single degree of freedom LPM, we use the zero-mode approximation to compute the equilibria of the Coulomb actuated prismatic Euler-Bernoulli beam studied by Gilbert\textit{ et al.}\textsuperscript{31}. For this exercise we apply the formulae compiled in Section II E. Gilbert used the geometrical dimensions: beam length $l = 80 \mu m$, beam width $w = 10 \mu m$, beam thickness $t = 0.5 \mu m$, electrostatic gap $g = 0.7 \mu m$, and stop layer $s = 0.1 \mu m$. For silicon, Gilbert used an isotropic stiffness with a Young’s modulus of $E = 169 \text{ GPa}$ and a Poisson ratio of $\nu = 0.25$.

The zero-mode results are compared to the results of Gilbert\textit{ et al.}\textsuperscript{31} and to the 3D ANSYS simulation by Melnikov\textit{ et al.}\textsuperscript{16} in Fig. 3. Obviously there is a very good agreement.
between our zero-mode approximation based on the Chebyshev-Edgeworth expansion and
the results of Gilbert et al.\textsuperscript{[11]} and Melnikov et al.\textsuperscript{[16]} The deflection profile, the pull-in voltage,
and the pull-out voltage can be reliably determined using our method.

G. Comparison with the numerical and experimental results of Melnikov et al.

Melnikov et al.\textsuperscript{[16]} used a continuation method to extend the reach of FEM simulations
to the entire bifurcation diagram of Coulomb actuated prismatic clamped-clamped Euler-
Bernoulli beams, including all stable and unstable equilibria. They calculated the respective
bifurcation diagrams and pull-in voltages for micro-beams with a length of \( l = 80 \mu m \), a
thickness range between \( t = 0.12 \mu m \) and \( t = 2 \mu m \) and an electrode gap of \( g = 0.7 \mu m \).

Fig. 4 shows the pull-in deflection and the pull-in voltage, respectively. These graphs
demonstrate the excellent match of the zero-mode approximation and the FEM results.
Additionally, Fig. 5 reveals an almost perfect agreement between FEM results and the
zero-mode approximation, regarding the entire deflection profiles, including their unstable
branches. We note that the solution close to the contact singularity at \( z = 1 \) is correctly
reproduced using a single mode.

Melnikov et al.\textsuperscript{[16]} scrutinized their findings by running a MEMS experiment. The basic
experimental set-up is shown in Fig. 5b. A clamped-clamped MEMS micro-beam of length
\( l = 1000 \mu m \), width \( w = 75 \mu m \) and with a measured thickness of \( t = 2.47 \mu m \) was manu-
factured on a Bonded Silicon on Insulator (BSOI) wafer, to perform in-plane movements.
The beam is Coulomb actuated by a planar electrode positioned in front of the beam at
a distance of \( g = 10.15 \mu m \) (fitted electrode gap). The beam movement was enabled by
removing the oxide layer underneath the beam by etching with hydrofloric acid. The details
of the experiment can be found in Melnikov et al.\textsuperscript{[16]} Furthermore, a small compression stress
of 2.6 MPa was used for the zero-mode approximation. The experimental findings are well
reproduced by the simple LPM developed in this paper, as can be seen in Fig. 5c.

In summary we find that the zero-mode approximation gives rise to a simple LPM with
a single degree of freedom, well suited to quantitatively describe all stable and unstable
equilibria of clamped-clamped Coulomb actuated prismatic Euler-Bernoulli beams.
III. DISCUSSION

Spitz et al.\textsuperscript{32} observed that the performance of a fairly complex MEMS \textmu Speaker can be successfully modelled by a heuristic single degree of freedom lumped parameter model. Motivated by this research Melnikov et al.\textsuperscript{16} revisited the analysis of the bending profile of a Coulomb-activated prismatic micro beam, clamped at both ends: The study clearly confirms that the bending profile stays almost identical to the shape of the Euler-Bernoulli zero mode, independent of the load. This is true for the entire applicable voltage range within a very small error margin. The observations of Melnikov et al.\textsuperscript{16} allowed us here to develop the single degree of freedom lumped parameter model Eq. (47), capable of accurately describing all stable and unstable equilibria of this highly non-linear electro-mechanical system. To the best of our knowledge, the existence of an accurate single degree of freedom LPM is not reported in the literature. In fact literature claims the need for higher modes.\textsuperscript{19}

The zero-mode approximation requires a method correctly projecting the Coulomb force onto the one dimensional Hilbert subspace, spanned by the Euler-Bernoulli zero-mode Eq. (8). Such projection is a global task in function space, that can not be performed using local techniques, such as a plain Taylor expansion. The ideas of Chebyshev and Edgeworth, underlying the original proof of the celebrated central limit theorem, furnish us here with the required means. As a result we obtain the analytical projection formula Eq. (34) for the Coulomb force. This formula allows us to extract the exact form of the contact singularity of the projected Coulomb force. Based on this knowledge, a global analysis of the Coulomb integral can be performed, leading to the handy algebraic expression Eq. (45). This completes the derivation of our highly accurate and simple to use lumped parameter model. To the best of our knowledge this is the first time, that Chebyshev-Edgeworth methods have been successfully used to solve a non-linear differential equation.

The results presented above now allow to efficiently compute the detailed frequency response and harmonic distortion of electromechanic MEMS transducers, with little computational effort. For practical applications, such dynamic computations are enabled by the large spectral distance of the Euler-Bernoulli zero-mode from higher Euler-Bernoulli modes, see Melnikov et al.\textsuperscript{16}. We note that the approach is applicable not only to clamped-clamped microbeams, but also to other conditions such as pinned-pinned or clamped-free. In such a case, Eq. (8) can stay the same while Eq. (9) changes, resulting in a new beta and new co-
efficient in Eq. (8). Certainly, time dependent FEM simulations will always allow to handle substantially more complex MEMS actuator geometries. However the process of basic actuator design, as well as the circuit simulation of complex systems embracing MEMS actuators, see Monsalve et al.\cite{33}, greatly benefit from the availability of powerful LPM models.

We have presented the use of the Chebyshev-Edgeworth methods in this publications to model a very particular situation. While our focus on a simple case may help to understand the basic principle, it probably is misleading at the same time. Chebyshev-Edgeworth methods apply to far more general situations and allow for a broad range of applications. These include different boundary conditions, non-prismatic beams, the modelling of squeeze film damping, the computation of electric fringe field corrections and of contact forces to name a few. For the sake of clarity, we defer sharing the details of such generalizations to forthcoming publications.

IV. CONCLUSION

All stable and unstable equilibrium states of Coulomb actuated prismatic clamped-clamped Euler-Bernoulli beams can be accurately computed by the simple to use lumped parameter model Eq. (47). This LPM features only one degree of freedom, i.e., the amplitude of the Euler-Bernoulli zero-mode. The contradiction of our results with previous findings of other groups are easily understood in terms of the advanced methods outlined above to adequately treat the Coulomb singularity.

The idea of the Chebyshev-Edgeworth expansion for the solution of nonlinear partial differential equations, which originates from probability theory, is not limited to beam mechanics. We believe that our approach enables new insights into the derivation of highly effective lumped parameter models in a wide range of applications beyond elasticity theory.

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FIG. 1. $\Phi_n(\xi)$ is compared for $n = 1, 2, 4, 8$ (colored lines) to the Gauss bell curve (dashed black line) illustrating the rapid convergence according to Eq. (18).
FIG. 2. (a) Comparison of two methods to compute the integrals $I_n$ according to Eq. (15). Open circles mark the results from direct numerical integration. The red line is the result of the Chebyshev-Edgeworth formula Eq. (32). (b) Comparison of two methods to compute the Coulomb integral $f_0(z)$ as defined by Eq. (13). Open circles mark the results from evaluating the sum Eq. (14) up to a certain maximum number of terms $n_{\text{max}}$ by direct numerical integration. The red line is the Chebyshev-Edgeworth projection Eq. (34) of the Coulomb force, neglecting the remainder $\Delta f_0(z)$.

FIG. 3. Comparison of the deflection curve obtained from the zero-mode approximation Eq. (47) to the 3D simulation results from literature (Gilbert CoSolve-EM$^{31}$ and 3D FEM$^{16}$).
FIG. 4. (a) The pull-in deflection $z$ of a Coulomb actuated beam obtained by ANSYS (blue squares) and by zero-mode approximation Eq. (52) as a function of $\alpha_1$ (solid red line). (b) The pull-in voltage of a Coulomb actuated beam simulated by ANSYS (blue squares) in comparison to simulations based on the zero-mode approximation Eq. (52) and Eq. (48) as a function of $\alpha_1$ (solid red line).
FIG. 5.  (a) The equilibria of a Coulomb actuated beam as obtained by ANSYS (colored solid lines) and by the zero-mode approximation Eq. (47) (dotted black lines) for various thicknesses $t$.  (b) The experimental setup and example frames used for the deflection measurement.  (c) Measured equilibria of a Coulomb actuated beam (filled green circles) compared to the zero-mode approximation (solid red line) according to Eq. (47) as a function of the voltage.