Hamiltonian structure of classical $N$-body systems of finite-size particles subject to EM interactions

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I. INTRODUCTION

In classical physics the formulation of the Hamiltonian mechanics of $N$-body systems composed of interacting particles is still incomplete. This includes, in particular, the case of charged particles subject to EM interactions, including both binary and self interactions (*EM-interacting $N$-body systems*). The correct solution to the question represents an overriding prerequisite for the consistency between classical and quantum mechanics. In this paper it is shown that such a description can be consistently obtained in the context of classical electrodynamics, for the case of a $N$-body system of classical finite-size charged particles. A variational formulation of the problem is presented, based on the $N$-body hybrid synchronous Hamilton variational principle. Covariant Lagrangian and Hamiltonian equations of motion for the dynamics of the interacting $N$-body system are derived, which are proved to be delay-type ODEs. Then, a representation in both standard Lagrangian and Hamiltonian forms is proved to hold, the latter expressed by means of classical Poisson Brackets. The theory developed retains both the covariance with respect to the Lorentz group and the exact Hamiltonian structure of the problem, which is shown to be intrinsically non-local. Different applications of the theory are investigated. The first one concerns the development of a suitable Hamiltonian approximation of the exact equations that retains finite delay-time effects characteristic of the binary and self EM interactions. Second, basic consequences concerning the validity of Dirac generator formalism are pointed out, with particular reference to the instant-form representation of Poincarè generators. Finally, a discussion is presented both on the validity and possible extension of the Dirac generator formalism as well as the failure of the so-called Currie “no-interaction” theorem for the non-local Hamiltonian system considered here.
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this assumed to hold, with \( t \) and \( t' \) denoting respectively the “present” and “retarded” coordinate times, both defined with respect to a suitable Laboratory frame. This motivates the introduction of asymptotic approximations, both for the EM self 4-potential and the corresponding self-force, which are based on power-series expansions in terms of the dimensionless parameter \( \epsilon \) and are performed in a neighborhood of the present coordinate time \( t \) (see related discussion in Paper II). Nevertheless, previous approaches of this type have lead in the past to intrinsically non-variational and therefore non-Hamiltonian equations of motion [2-9]. These are exemplified by the well-known LAD and LL RR equations, due respectively to Lorentz, Abraham and Dirac (Lorentz, 1985 [10]; Abraham, 1905 [11] and Dirac [4]) and Landau and Lifschitz [12]. Both features make these treatments incompatible with the physical prerequisites indicated above.

The second motivation arises in reference to the so-called “no-interaction” theorem proposed by Currie [13, 14]. According to its claim, isolated N-body systems, formed by at least two point particles, which are subject to mutual EM binary interactions and in which the canonical coordinates are identified with the space parts of the particle position 4-vectors, cannot define a Hamiltonian system with manifestly covariant canonical equations of motion. The correctness of such a statement has been long questioned (see for example Fronsdal, 1971 [13] and Komar, 1978-1979 [10, 12]). In particular, the interesting question has been posed whether the “no-interaction” theorem can actually be eluded in physically realizable classical systems. Interestingly, Currie approach is based on the well-known generator formalism formulated originally by Dirac (DGF; Dirac, 1949 [3]). Therefore, related preliminary questions concern

\[
\frac{dx^{(i)}}{dt^{(i)}} = [x^{(i)}, H_N].
\] (1)

Here the notation is standard. Thus, \((s_{(1)}, \ldots, s_{(N)})\) and \([\eta, \xi] \equiv [\eta, \xi]_{[x]}\) are respectively the particles proper times and the local Poisson brackets (PBs). The latter are defined in terms of the super-abundant canonical state \( x \) as

\[
[\eta, \xi] = (\frac{\partial \eta}{\partial x})^T \, J \cdot (\frac{\partial \xi}{\partial x}),
\] (2)

with all components of \( x \) to be considered independent (i.e., \( x \) as unconstrained). Furthermore, \( J \) is the canonical Poisson matrix [2], while \( \eta(x) \) and \( \xi(x) \) denote two arbitrary smooth phase-functions.

Evidently, the above prerequisites should be regarded as overriding conditions for the transition from classical to quantum theory of the N-body dynamics to be possible. However, despite notable efforts (see for example Dirac, 1949 [3]) the solution to the problem of fitting them together has remained still incomplete to date, at least in the case of systems of charged particles subject to EM interactions.

From the point of view of classical physics the reason is related to the nature of EM interactions occurring in N-body systems. These can be carried respectively both by external sources (umary interactions, due to prescribed external EM fields) and by the particles themselves of the system (internal interactions). The latter include both binary EM interactions acting between any two arbitrary charged particles and the self EM interaction, usually known as the EM radiation-reaction (RR; Dirac [4], Pauli [5], Feynman [6]). As earlier pointed out (see Refs.[7] and [8], hereon referred to as Paper I and Paper II respectively) a rigorous treatment of the EM self-interaction consistent with prerequisites #1-#5 can only be achieved for extended classical particles, i.e., particles characterized by mass and charge distributions with finite support. In particular, a convenient mathematical model is obtained by assuming that these classical particles are non-rotating and their mass and charge distributions are quasi-rigid in their rest frames (see Paper I for a detailed discussion on this point). In Papers I and II the dynamics of single extended particles (1-body problem) in the presence of their EM self-fields was systematically investigated in the context of classical electrodynamics, by means of a variational approach based on a Hamilton variational principle. As a result, the Hamiltonian description for isolated particles subject to the combined action of the external and the self EM interaction has been established.

However, fundamental issues still remain unanswered regarding the analogous formulation of a consistent dynamical theory holding for classical N-body systems of finite-size charges subject to only EM interactions (EM-interacting N-body systems). In fact, it is well known that traditional formulations of the relativistic dynamics of classical charged particles are unsatisfactory, at least because of the following main reasons.

The first one is related to the approximations usually adopted in classical electrodynamics for the treatments of RR phenomena. In most of previous literature charged particles are regarded as point-like and the so-called short delay-time ordering

\[
\epsilon \equiv \frac{t - t'}{t} \ll 1
\] (3)

is assumed to hold, with \( t \) and \( t' \) denoting respectively the “present” and “retarded” coordinate times, both defined with respect to a suitable Laboratory frame. This motivates the introduction of asymptotic approximations, both for the EM self 4-potential and the corresponding self-force, which are based on power-series expansions in terms of the dimensionless parameter \( \epsilon \) and are performed in a neighborhood of the present coordinate time \( t \) (see related discussion in Paper II). Nevertheless, previous approaches of this type have lead in the past to intrinsically non-variational and therefore non-Hamiltonian equations of motion [2-9]. These are exemplified by the well-known LAD and LL RR equations, due respectively to Lorentz, Abraham and Dirac (Lorentz, 1985 [10]; Abraham, 1905 [11] and Dirac [4]) and Landau and Lifschitz [12]. Both features make these treatments incompatible with the physical prerequisites indicated above.

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the conditions of validity of DGF itself and, in particular, whether such an approach actually applies at all to EM-interacting $N$-body systems.

Goals of the paper

Put all the previous motivations in perspective, in this paper a systematic solution to these issues is presented, for the case of EM-interacting $N$-body systems of classical finite-size charged particles. The theory is developed in the framework of classical electrodynamics and special relativity (i.e., assuming a flat Minkowski space-time) and is shown to satisfy all the prerequisites indicated above (#1-#5).

Starting point is the determination of both Lagrangian and Hamiltonian equations of motion for classical charged particles belonging to an EM-interacting $N$-body system. By construction (see Section 2) the latter can be considered as a system of smooth hard sphere, namely in which hard collisions occurring between the particles, when their boundaries $\partial \Omega_i$ (see below) come into contact, conserve each particle angular momentum. In particular, for simplicity, in the following all extended particles will be considered as acted upon only by EM interactions, thus ignoring the effect of hard collisions on the $N$-body dynamics. As in previous Papers I and II, where the 1-body problem was investigated, the derivation is based on the variational formulation of the problem. This requires, in particular, the determination of the appropriate variational functionals required for the description of both binary and self EM interactions. The resulting action functional is found to be expressed as a line integral in terms of a suitable non-local variational Lagrangian, the non-locality being associated both to the finite-extension of the charge distributions and to delay-time effects arising in binary EM interactions. Based on the Hamilton variational principle expressed in superabundant variables, the resulting variational equations of motion (for the $N$-body system) are then proved to be necessarily delay-type ODEs. As a consequence, based on the definition of suitable effective Lagrangian and Hamiltonian functions, a manifestly-covariant representation of these equations in both standard Lagrangian and Hamiltonian forms is reached. The main goal of the paper is then to show that these features allow a Hamiltonian structure $\{x,H_N\}$ to be properly defined in terms of a suitable superabundant canonical state $x$ and a non-local system Hamiltonian function $H_N$. The result follows by noting that the Hamiltonian equations in standard form admit also a representation in terms of local PBs, defined with respect to the super-abundant canonical state $x$.

A remarkable development concerns the construction of an approximation of the Hamilton equations in standard form. This holds in validity of both the short delay-time and large-distance orderings, namely under the same asymptotic conditions usually invoked in the literature for the asymptotic treatment of the RR problem. Based on the analogous approach developed in Paper II, it is shown that a suitable $N$-body Hamiltonian approximation [of the exact problem] can actually be reached, which preserves its Hamiltonian structure. In particular it is proved that, unlike in the LAD and LL equations, the asymptotic approximation obtained in this way keeps the variational character of the exact theory, retaining the standard Lagrangian and Hamiltonian forms of the $N$-body dynamical equations as well as the delay-time contributions arising from the various EM interactions.

Further interesting conclusions are drawn concerning the validity of DGF in the present context. This refers, in particular, to the so-called instant-form representation of Poincaré generators for infinitesimal transformations of the inhomogeneous Lorentz group. It is pointed out that DGF in its original formulation only applies to local Hamiltonian systems and therefore is inapplicable to (the treatment of) the EM-interacting $N$-body systems considered here. For definiteness, the correct set of Poincaré generators, corresponding to the exact non-local Hamiltonian structure $\{x,H_N\}$ determined here, together with their instant-form representation, are also provided. This permits to develop a modified formulation of DGF, denoted as non-local generator formalism, which overcomes the previous limitations and is applicable also to the treatment of non-local Hamiltonians in terms of essential (i.e., constrained) canonical variables.

Finally, on the same ground, the Currie “no-interaction” theorem is proved to be violated in any case by the Hamiltonian structure, i.e., both by its exact realization $\{x,H_N\}$ and its asymptotic approximation. Counter-examples which overcome the limitations stated by the “no-interaction” theorem are explicitly provided. In particular, the purpose of this discussion is to prove that indeed a standard Hamiltonian formulation for the $N$-body system of EM-mutually-interacting charged particles can be consistently obtained. The main cause of the failure of the Currie theorem is identified in the conditions of validity of DGF on which the proof of the theorem itself is based.

Scheme of the presentation

The paper is organized as follows. In Section 2 the expressions for the EM current density and self 4-potential for extended charged particles are recalled. In Section 3 the $N$-body action integral is constructed, based on the results of Papers I and II and focusing in particular on the derivation of the action integral for the EM binary interactions.
Section 4 deals with the dynamical equations of motion in Lagrangian form for the interacting particles, derived from the $N$-body hybrid synchronous Hamilton variational principle (THM.1). A standard Lagrangian form of the same equations is obtained, by introducing an effective non-local Lagrangian function (Corollary to THM.1). In Section 5 the corresponding non-local Hamiltonian theory is developed in terms of a variational Hamiltonian function (THM.2). A standard Hamiltonian form of the $N$-body dynamical equations is constructed and the associated non-local Hamiltonian structure is proved to exist (Corollary to THM.2). General implications of the non-local $N$-body theory are presented in Section 6, while in Section 7 the $N$-body Hamiltonian asymptotic approximation of the exact solution is developed (THM.3). In Section 8 the validity of DGF is investigated, in connection with the non-local Hamiltonian structure determined here and the corresponding instant-form representation of Poincaré generators implied by it. The extension of DGF to non-local system is presented in Section 9, by means of the introduction of the notion of non-local Poisson brackets. Then, an overview of the basic features of the “no-interaction” theorem is given in Section 10. Explicit counter-examples to the theorem are determined, which are provided by the existence of standard Hamiltonian forms for locally-isolated 1-body systems and globally-isolated $N$-body systems (THM.4). The failure of the “no-interaction” theorem and the non-applicability of its statements to the non-local Hamiltonian $N$-body system considered here are discussed in Section 11. Finally, Section 12 provides a closing summary of the main results, while the mathematical details concerning the derivation of the $N$-body action integral for the binary interaction are reported in the Appendix.

II. $N$-BODY EM CURRENT DENSITY AND SELF 4-POTENTIAL

For the sake of clarity, let us first briefly recall the key points of the formulation presented in Papers I and II dealing with the definitions of extended charged particle and of the corresponding 4-current and self 4-potential. We shall assume that each finite-size particle is characterized by a positive constant rest mass $m_0^{(i)}$ and a non-vanishing constant charge $q^{(i)}$, for $i = 1, N$, both distributed on the same support $\partial\Omega^{(i)}$ (particle boundary). More precisely, for the $i$-th particle, the mass and charge distributions can be defined as follows. Assuming that initially in a time interval $[-\infty, t_o]$ the $i$-th particle is at rest with respect to an inertial frame (particle rest-frame $R_o$ where the external forces acting on the particle vanish identically), we shall assume that:

1) In this frame there exists a point, hereafter referred to as center of symmetry (COS), whose position 4-vector $r^{(u)}_{\text{COS}} \equiv (ct, r_o)$ spans the Minkowski space-time $M^4 \subseteq \mathbb{R}^4$ with metric tensor $g_{\mu\nu} = \text{diag}(+1, -1, -1)$. With respect to the COS the support $\partial\Omega^{(i)}$ is a stationary spherical surface of radius $\sigma^{(i)} > 0$ of equation $(r - r_o)^2 = \sigma^{(i)}$. The $i$-th particle is quasi-rigid, i.e., its mass and charge distributions are stationary and spherically-symmetric on $\partial\Omega^{(i)}$ (concerning the physical requirements assuring the condition of rigidity we refer to the related discussion in Paper I).

2) Mass and charge densities do not possess pure spatial rotations. Therefore, introducing for each particle the Euler angles $\alpha(s^{(i)}) \equiv \{\varphi, \theta, \psi\}(s^{(i)})$ which define its spatial orientation (see definitions in Ref.[20]), the condition of vanishing spatial rotation is obtained imposing that $\alpha(s^{(i)}) = \text{const}.$ holds identically. As a consequence, only the translational motion of charged particles need to be taken into account.

In addition, as stated before, hard collisions occurring between the particles are considered ignorable. As a consequence, for all particles the equations of motion (1) are assumed to hold identically, namely for all $s^{(i)} \in I \subseteq \mathbb{R}$.

For each extended particle the covariant expressions for the corresponding charge and mass current densities readily follow (see Paper I). In particular, these can be expressed in integral form respectively as:

$$j^{(i)\mu}(r) = \frac{q^{(i)} c}{4\pi \sigma^{(i)}} \int_{-\infty}^{+\infty} ds u^{(i)\mu}(s) \delta\left(x^{(i)} - \sigma^{(i)}\right) \delta(s - s^{(i)}),$$

$$j'^{(i)\mu}(r) = \frac{m^{(i)} c}{4\pi \sigma^{(i)}} \int_{-\infty}^{+\infty} ds u'^{(i)\mu}(s) \delta\left(x^{(i)} - \sigma^{(i)}\right) \delta(s - s^{(i)}),$$

where by definition $s^{(i)}$ is the root of the algebraic equation

$$u^{(i)\mu}(s^{(i)}) \left[r^{\mu} - v^{(i)\mu}(s^{(i)})\right] = 0.$$  

Here the notations are analogous to those given in Paper I, so that in particular $u^{(i)\mu}(s^{(i)}) \equiv \frac{dr^{(i)\mu}(s^{(i)})}{ds^{(i)}}$ is the 4-velocity of the COS for the $i$-th particle, while

$$x^{(i)} = r^{\mu} - v^{(i)\mu}(s^{(i)}).$$
Finally, following the equivalent derivations given in Papers I and II, the non-divergent EM self 4-potential $A_{\mu}^{\text{self}(i)}(r)$ for the single (namely, $i$-th) extended particle can be readily obtained as well. It is sufficient to report here the solution for $A_{\mu}^{\text{self}(i)}(r)$ which is valid in the external domain with respect to the spherical shell of the same particle. For a generic displacement 4-vector $X^{(i)\mu} \in M^4$ of the form
\[ X^{(i)\mu} = r^{\mu} - r^{(i)\mu}(s_{(i)}) , \]
which is subject to the constraint
\[ X^{(i)\mu}u_{\mu}(s_{(i)}) = 0 , \]
this sub-domain is defined by the inequality
\[ X^{(i)\mu}X^{(i)\mu} \leq -\sigma^2_{(i)} . \]
In such a set, $A_{\mu}^{\text{self}(i)}(r)$ is expressed in integral form by the equation
\[ A_{\mu}^{\text{self}(i)}(r) = 2q^{(i)} \int_1^2 dr'_\mu \delta(\hat{R}^{(i)\alpha}_\alpha \hat{R}^{(i)\alpha}_\alpha) , \]
where $\hat{R}^{(i)\alpha}$ is the bi-vector
\[ \hat{R}^{(i)\alpha} = r^{\alpha} - r^{(i)\alpha}, \]
with $r^{(i)\alpha} = r^{(i)\alpha}(s'_{(i)})$ being the $i$-th particle COS 4-vector evaluated at the retarded proper time $s'_{(i)}$, obtained as the causal root of the equation $\hat{R}^{(i)\alpha}_\alpha \hat{R}^{(i)\alpha}_\alpha = 0$. As remarked in Paper II, Eq. (11) formally coincides with the analogous solution holding for a point particle. However, in difference with the point-particle case, $A_{\mu}^{\text{self}(i)}(r)$ can be defined everywhere in $M^4$ in such a way to be non-divergent (see Paper I).

III. THE NON-LOCAL N-BODY ACTION INTEGRAL

In this section we formulate the $N$-body Hamilton action functional suitable for the variational treatment of a system of $N$ finite-size charged particles subject to external, binary and self EM interactions. In such a case, in analogy to Paper II, the action integral can be conveniently expressed in hybrid super-abundant variables as follows:
\[ S_N(r, u, [r]) = \sum_{i=1,N} \left[ S_M^{(i)}(r, u) + S_C^{(\text{ext})(i)}(r) + S_C^{(\text{self})(i)}(r,[r]) + S_C^{(\text{bin})(i)}(r,[r]) \right] \]
(non-local action integral). As in Papers I and II, $r$ and $u$ represent local dependences with respect to the 4-vector position and the velocity, while $[r]$ stands for non-local dependences with respect to the 4-vector position. In particular, the latter are included only via the functionals produced by the EM-coupling with the self and binary EM fields for the $i$-th particle, namely $S_C^{(\text{self})(i)}$ and $S_C^{(\text{bin})(i)}$. Instead, $S_M^{(i)}$ and $S_C^{(\text{ext})(i)}$ identify for each particle the functionals produced by the inertial mass and by the EM-coupling with the external EM field. We stress that the functionals $S_M^{(i)}(r, u)$, $S_C^{(\text{ext})(i)}(r)$ and $S_C^{(\text{self})(i)}(r,[r])$ are formally analogous to the case of a 1-body problem treated in Papers I and II and can be represented as line-integrals (see below). We now proceed evaluating explicitly the new contribution $S_C^{(\text{bin})(i)}(r,[r])$.

A. $S_C^{(\text{bin})(i)}(r,[r])$: EM coupling with the binary-interaction field

The action integral $S_C^{(\text{bin})(i)}(r,[r])$ containing the coupling between the EM field generated by particle $j$, for $j = 1, N$, and the electric 4-current of particle $i$ is of critical importance. Its evaluation is similar to that of the action integral of the self-interaction outlined in Paper I. For the sake of clarity, in this subsection we present the relevant results,
where the details of the mathematical derivation are reported in the Appendix. According to the standard approach [12], \( S_C^{(bin)(i)}(r, [r]) \) is defined as the 4-scalar

\[
S_C^{(bin)(i)}(r, [r]) = \sum_{j=1,N \atop i \neq j} S_C^{(bin)(ij)}(r, [r]),
\]

(14)

where \( S_C^{(bin)(ij)}(r, [r]) \) is defined as

\[
S_C^{(bin)(ij)}(r, [r]) = \int_1^2 d\Omega \frac{1}{c^4} A^{(self)(i)\mu}(r) j_{\mu}^{(j)}(r),
\]

(15)

with \( A^{(self)(i)\mu}(r) \) being the EM 4-potential generated by particle \( i \) at 4-position \( r \), whose expression is given by Eq. (11). In addition, \( j_{\mu}^{(j)}(r) \) is the 4-current carried by particle \( j \) evaluated at the same 4-position and given by Eq. (9), while \( d\Omega \) is the invariant 4-volume element. In particular, in an inertial frame \( S_f \) with Minkowski metric tensor \( \eta_{\mu\nu} \), this can be represented as \( d\Omega = cdt dx dy dz \), where \((x, y, z)\) are orthogonal Cartesian coordinates. As shown in the Appendix, an explicit evaluation of the action integral [13] yields the following representation:

\[
S_C^{(bin)(ij)}(r, [r]) = \frac{2q_i q_j}{c} \int_1^2 dr_{\mu}(s(i)) \int_1^2 dr_{\mu}(s(j)) \delta(\tilde{R}^{(ij)\alpha}_{\alpha} - \sigma^2_{(ij)}),
\]

(16)

where \( s(i) \) and \( s(j) \) are respectively the proper times of particles \( i \) and \( j \), while \( \tilde{R}^{(ij)\alpha} \) denotes

\[
\tilde{R}^{(ij)\alpha} = r^{(j)\alpha}(s(j)) - r^{(i)\alpha}(s(i)).
\]

(17)

It is worth pointing out the following basic properties of the functional \( S_C^{(bin)(i)}(r, [r]) \). First, it is a non-local functional in the sense that it contains a coupling between the “past” and the “future” of the particles of the N-body system. In fact it can be equivalently represented as

\[
S_C^{(bin)(ij)}(r, [r]) = \frac{2q_i q_j}{c} \int_{-\infty}^{+\infty} ds_i \frac{dr_{\mu}(s(i))}{ds_i} \int_{-\infty}^{+\infty} ds_j \frac{dr_{\mu}(s(j))}{ds_j} \delta(\tilde{R}^{(ij)\alpha}_{\alpha} - \sigma^2_{(ij)}).
\]

(18)

Furthermore, the N-body system functional [14] is symmetric, namely it fulfills the property

\[
\sum_{i,j=1,N} S_C^{(bin)(ij)}(r_A, [r_B]) = \sum_{i,j=1,N} S_C^{(bin)(ji)}(r_B, [r_A]),
\]

(19)

where \( r_A \) and \( r_B \) are two arbitrary curves of the N-body system.

B. The non-local N-body variational Lagrangian

In this section we provide a line-integral representation of the Hamilton functional \( S_N \) in the form

\[
S_N = \sum_{i=1,N} \int_{-\infty}^{+\infty} ds_i Y(i)_1 \equiv \sum_{i=1,N} \int_{-\infty}^{+\infty} Y(i)(r, u, [r]),
\]

(20)

where \( Y(i)(r, u, [r]) \) and \( L_1^{(i)}(r, u, [r]) \) are respectively the \( i \)-th particle non-local contributions to the fundamental Lagrangian differential form and to the corresponding non-local variational Lagrangian. Invoking Eq. (18) and recalling also the results of Paper II, \( L_1^{(i)}(r, [r], u) \) can be written as

\[
L_1^{(i)}(r, u, [r]) = L_M^{(i)}(r, u) + L_C^{(ext)(i)}(r) + L_C^{(self)(i)}(r, [r]) + L_C^{(bin)(i)}(r, [r]),
\]

(21)

where \( L_M^{(i)}(r, u) \), \( L_C^{(ext)(i)}(r) \) and \( L_C^{(self)(i)}(r, [r]) \), \( L_C^{(bin)(i)}(r, [r]) \) denote respectively the local and non-local terms. In particular, the first one is the contribution carried by the inertial term, while \( L_C^{(ext)(i)} \), \( L_C^{(self)(i)} \) and \( L_C^{(bin)(i)} \) identify
respectively the external, self and binary EM-field-coupling Lagrangians. These are defined as follows:

\[ L_M^{(i)}(r, u) = m_c^{(i)} c s^{(i)} \left[ \frac{dr^{(i)\mu}}{ds^{(i)}} \frac{1}{2} \tilde{u}^{(i)\mu} \right], \]  

\[ L_C^{(ext)(i)}(r) = \frac{q^{(i)}}{c} \frac{dr^{(i)\mu}}{ds^{(i)}} A^{(ext)(i)}_\mu (r^{(i)}(s^{(i)}), \sigma^{(i)}), \]  

\[ L_C^{(self)(i)}(r, [r]) = \frac{q^{(i)}}{c} \frac{dr^{(i)\mu}}{ds^{(i)}} A^{(self)(i)}_\mu, \]  

\[ L_C^{(bin)(i)}(r, [r]) = \sum_{j=1, N} L_C^{(bin)(j)}(r, [r]) = \frac{q^{(i)}}{c} \frac{dr^{(i)\mu}}{ds^{(i)}} \sum_{j=1, N} A^{(bin)(i)}_\mu (\sigma^{(j)}). \]  

Here, \( \overline{A}_\mu^{(ext)(i)} \), \( \overline{A}_\mu^{(self)(i)} \) and \( \overline{A}_\mu^{(bin)(i)} \) denote the surface-averages performed on the \( i \)-th particle boundary \( \partial \Omega^{(i)} \) (see Paper I) respectively of the external, self and binary EM 4-potentials. In particular, \( \overline{A}_\mu^{(self)(i)} \) and \( \overline{A}_\mu^{(bin)(i)} \) are defined as

\[ \overline{A}_\mu^{(self)(i)} = 2q^{(i)} \int_1^2 dr^{(i)\mu} \delta (\tilde{R}^{(i)\mu} - \sigma^{(i)}), \]  

\[ \overline{A}_\mu^{(bin)(i)} (\sigma^{(j)}) = 2q^{(j)} \int_{-\infty}^{+\infty} ds^{(j)} \frac{dr^{(i)\mu}}{ds^{(j)}} \delta (\tilde{R}^{(i)\alpha} - \sigma^{(j)}). \]  

In addition, \( \tilde{R}^{(i)\mu} \) is the bi-vector

\[ \tilde{R}^{(i)\mu} \equiv \tilde{r}^{(i)\mu} (s^{(i)}) - \tilde{r}^{(i)\mu} (s'_{(i)}), \]  

with \( s^{(i)} \) and \( s'_{(i)} \) denoting respectively “present” and “retarded” proper times of the \( i \)-th particle.

**IV. NON-LOCAL N-BODY VARIATIONAL PRINCIPLE AND STANDARD LAGRANGIAN FORM**

Let us now proceed constructing the explicit form of the \( N \)-body relativistic equations of motion for each extended charged particle in the presence of EM interactions (i.e., including external, binary and self EM interactions). This is achieved by adopting for the \( N \)-body problem a synchronous variational principle \[ 21, 22 \] which, in analogy with the approach developed in Papers I and II, can be expressed in terms of the super-abundant hybrid (i.e., generally non-Lagrangian) variables

\[ f^{(i)}(s^{(i)}) \equiv \left[ r^{(i)\mu}(s^{(i)}), u^{(i)\mu}(s^{(i)}) \right], \]  

and for a suitable functional class of variations \( \{ f \} \). The latter is identified with the set of real functions of class \( C^k(\mathbb{R}) \), with \( k \geq 2 \), and fixed endpoints which are prescribed for each particle \( i = 1, N \) at suitable proper times \( s^{(i)1} \) and \( s^{(i)2} \), with \( s^{(i)1} < s^{(i)2} \), i.e.,

\[ \{ f \} = \left\{ f^{(i)}(s^{(i)}) : f^{(i)}(s^{(i)}) \in C^k(\mathbb{R}); \ f^{(i)}(s^{(i)}) = f^{(i)}(s^{(i)}) ; \ i = 1, N ; \ j = 1, 2 \ and \ k \geq 2 \right\}. \]  

It follows that by construction the variational derivatives of the Hamilton functional \( S_N \) (see Eq. \[ 20 \]) are performed in terms of synchronous variations, i.e., by keeping constant the \( i \)-th particle proper time \( s^{(i)} \). The result is expressed by the following theorem.

**THM.1 - \( N \)-body hybrid synchronous Hamilton variational principle**

Given validity of the prerequisites \#1-\#5 for the \( N \)-body system, let us assume that:

1. The Hamilton action \( S_N(r, u, [r]) \) is defined by Eq. \[ 20 \].
2. The real functions $f^{(i)}(s^{(i)})$ in the functional class \{f\} [see Eq. (30)] are identified with the super-abundant variables \[ \mathcal{A} \] which are subject to synchronous variations \[ \delta f^{(i)}(s^{(i)}) \equiv f^{(i)}(s^{(i)}) - f_{1}^{(i)}(s^{(i)}) \]. The latter belong to the functional class of synchronous variations \{\delta f^{(i)}\}, with

\[
\delta f_{k}^{(i)}(s^{(i)}) = f_{k}^{(i)}(s^{(i)}) - f_{1}^{(i)}(s^{(i))}, \tag{31}
\]

for \( k = 1, 2, \forall f^{(i)}(s^{(i)}), f_{1}^{(i)}(s^{(i)}) \in \{f\} \).

3. The extremal curves \( f^{(i)}(s^{(i)}) \in \{f\} \) for \( S_N \), which are solutions of the E-L equations

\[
\frac{\delta S_{N}(r, u, [r])}{\delta f^{(i)}(s^{(i)})} = 0, \tag{32}
\]

exist for arbitrary variations \( \delta f^{(i)}(s^{(i)}) \) (hybrid synchronous Hamilton variational principle).

4. If the curves \( r^{(i)\mu}(s^{(i)}) \), for \( i = 1, N \) are all extremal, each line element \( ds^{(i)} \) satisfies the constraint

\[
ds_{(i)}^{2} = \eta_{\mu\nu}dr^{(i)\mu}(s^{(i)})dr^{(i)\nu}(s^{(i)}). \tag{33}\]

5. The 4-vector field \( A^{(ext)}_{\mu}(r) \) is suitably smooth in the whole Minkowski space-time \( M^{4} \).

6. The E-L equations for the extremal curves \( r^{(i)\mu}(s^{(i)}) \) are determined consistently with the Einstein causality principle.

7. All the synchronous variations \( \delta f_{k}^{(i)}(s^{(i)}) \) (\( k=1,2 \) and \( i = 1, N \)) are considered as being independent.

It then follows that the E-L equations for \( u^{(i)\mu} \) and \( r^{(i)\mu} \) following from the synchronous hybrid Hamilton variational principle \[ \mathcal{A} \] give respectively

\[
\frac{\delta S_{N}}{\delta u^{(i)\mu}} = m_{o}^{(i)}cdr^{(i)\mu} - m_{o}^{(i)}cu^{(i)\mu}ds^{(i)} = 0, \tag{34}\]

\[
\frac{\delta S_{N}}{\delta r^{(i)\mu}(s^{(i)})} = -m_{o}^{(i)}cd_{\mu}(s^{(i)}) + \frac{q^{(i)}_{\mu}}{c} F^{(tot)\mu\nu}(s^{(i)}) = 0, \tag{35}\]

where \( F^{(tot)\mu\nu} \) is the total Faraday tensor acting on particle \( i \) and given by

\[
F_{\mu\nu}^{(tot)(i)} = F_{\mu\nu}^{(ext)(i)} + F_{\mu\nu}^{(self)(i)} + F_{\mu\nu}^{(bin)(i)}, \tag{36}\]

where all quantities are intended as surface-averages on the \( i \)-th particle shell-surface \( \partial\Omega_{(i)} \). Eqs. (34) and (35) are hereon referred to as \( N \)-body equations of motion. In particular:

1) \( F_{\mu\nu}^{(ext)(i)} \) \( \left( r^{(i)}, [r^{(i)}] \right) = \partial_{\mu}A_{\nu}^{(ext)}(s^{(i)}) \) is the antisymmetric Faraday tensor of the external EM field evaluated on the extremal curve \( r^{(i)\mu} = r^{(i)\mu}(s^{(i)}) \).

2) \( F_{\mu\nu}^{(self)(i)} \) \( \left( r^{(i)}, [r^{(i)}] \right) \equiv F_{\mu\nu}^{(self)(i)} \left( r^{(i)}(s^{(i)}), r^{(i)}(s^{(i)}) \right) \) is the non-local antisymmetric Faraday tensor produced by the EM self-field of the \( i \)-th particle and acting on the same particle. This is given by

\[
F_{\mu\nu}^{(self)(i)} = 2 \left[ \partial_{\mu}A_{\nu}^{(self)} - \partial_{\nu}A_{\mu}^{(self)} \right], \tag{37}\]

namely

\[
F_{\mu\nu}^{(self)(i)} = \left[ \frac{2(q^{(i)} \left| R^{(i)\alpha}\hat{u}_{\alpha}^{(i)}(s^{(i)}) \right| ds^{(i)}}{R^{(i)\alpha}(s^{(i)})} \right] \left[ \frac{u_{\mu}^{(i)}(s^{(i)}) - u^{(i)\mu}(s^{(i)})}{R^{(i)\alpha}(s^{(i)})} \right] \left[ \frac{u^{(i)\mu}(s^{(i)})}{R^{(i)\alpha}(s^{(i)})} \right], \tag{38}\]

where the delay-time \( s^{(i)\text{ret}} \) is the positive (causal) root of the 1-particle delay-time equation

\[
R^{(i)\alpha}R_{\alpha}^{(i)} - \sigma^{2} = 0. \tag{39}\]
3) \( \mathbf{F}^{(\text{bin})}_{\mu\nu}(i) \) is the non-local antisymmetric Faraday tensor produced on particle \( i \) by the action of all the remaining particles, i.e. \( \mathbf{F}^{(\text{bin})}_{\mu\nu}(i) \equiv \sum_{j=1,N, j \neq i} \mathbf{F}^{(\text{bin})}_{\mu\nu}(j) \left( r^{(i)}, [r^{(j)}], \sigma^{(i)}, \sigma^{(j)} \right) \), where

\[
\mathbf{F}^{(\text{bin})}_{\mu\nu}(j) \left( r^{(i)}, [r^{(j)}], \sigma^{(i)}, \sigma^{(j)} \right) = \left[ H^{(i)}_{\mu\nu} \left( s^{(i)}, s^{(j)} \right) \right]_{s^{(j)} = s^{(i)}(\sigma^{(i)})} + \left[ H^{(j)}_{\mu\nu} \left( s^{(i)}, s^{(j)} \right) \right]_{s^{(i)} = s^{(i)}(\sigma^{(j)})}. \tag{40}
\]

Here the notation is as follows. \( H^{(i)}_{\mu\nu} \) is defined as

\[
H^{(i)}_{\mu\nu} \left( s^{(i)}, s^{(j)} \right) = -\frac{q^{(j)}}{R^{(ij)}_{ij} u^{(ij)}_{\alpha}(s^{(j)})} \frac{d}{ds^{(j)}} \left\{ \frac{u^{(j)}_{\mu}(s^{(j)}) R^{(ij)}_{ij} - u^{(j)}_{\nu}(s^{(j)}) R^{(ij)}_{ij}}{R^{(ij)}_{ij} u^{(ij)}_{\alpha}(s^{(j)})} \right\}, \tag{41}
\]

while the delay-time \( s^{(j)} = s^{(A)}(\sigma^{(j)}) \) and \( s^{(j)} = s^{(B)}(\sigma^{(j)}) \) are respectively the positive (causal) roots of the 2-particle delay-time equations

\[
\begin{align*}
\tilde{R}^{(ij)}_{ij} - \sigma^{2} & = 0, \tag{42} \\
\tilde{R}^{(ij)}_{ij} - \sigma^{2} & = 0. \tag{43}
\end{align*}
\]

Therefore, \( s^{(A)} \) and \( s^{(B)} \) depend respectively on \( \sigma^{(i)} \) and \( \sigma^{(j)} \).

Proof - The proof is analogous to the corresponding 1-body problem detailed in THM.1 of Paper I. Indeed, since the Dirac-deltas \( \delta \tilde{R}^{(ij)}_{ij} \) and \( \delta (\tilde{R}^{(ij)}_{ij} - \sigma^{2}) \) are independent of particle 4-velocities, the variations with respect to \( u^{(i)}_{\mu} \) deliver necessarily the E-L equation (3). To prove also Eq. (3), we notice that the synchronous variations of the functionals \( S_{M}^{(i)}(r, u) \), \( S_{C}^{(ext)}(r) \) and \( S_{C}^{(ext) f}(r, [r]) \) necessarily coincide with those of the 1-body problem (see Papers I and II). Therefore, it is sufficient to inspect the variational derivative of the non-local binary-interaction functional \( S_{C}^{(bin)}(r, [r]) \). Its variation with respect to \( \delta s^{(i)}(s^{(i)}) \) takes the form

\[
\delta S_{C}^{(bin)}(i) = \sum_{j=1, N, j \neq i} \left\{ [\delta A + \delta B]_{ij} + [\delta A + \delta B]_{ji} \right\}, \tag{44}
\]

where

\[
\begin{align*}
\delta A_{ij} & \equiv -\frac{2}{c} \int_{1}^{2} \delta (r^{(j)} - r^{(i)}) d\tilde{R}^{(ij)}_{ij} \delta \tilde{R}^{(ij)}_{ij} \left[ \int_{1}^{2} dr^{(j)} \delta (\tilde{R}^{(ij)}_{ij} - \sigma^{2}) \right] \\
\delta B_{ij} & \equiv \frac{2}{c} \int_{1}^{2} \delta (r^{(j)} - r^{(i)}) d\tilde{R}^{(ij)}_{ij} \left[ \int_{1}^{2} dr^{(j)} \delta \tilde{R}^{(ij)}_{ij} \right] \\
& \quad \times \frac{1}{\gamma^{(i)}} \delta \gamma^{(j)} \left[ \int_{1}^{2} dr^{(j)} \delta \tilde{R}^{(ij)}_{ij} \right] \frac{1}{\gamma^{(j)}} \delta \gamma^{(i)} \left[ \int_{1}^{2} dr^{(j)} \delta \tilde{R}^{(ij)}_{ij} \right]
\end{align*} \tag{45}
\]

and the second term \( [\delta A + \delta B]_{ji} \) follows by exchanging the particle indices. Then, using the chain rule and integrating by parts, after elementary algebra Eqs. (40) and (41) follow (for details see Appendix A in Paper I). In agreement with the Einstein causality principle the positive roots of the delay-time equations (12) and (13) must be selected.

Q.E.D.

A few comments are here in order regarding the implications of THM.1.

1. Coordinate-time parametrization of the N-body equations of motion

It is important to stress that for each \( i \)-th particle, its equations of motion (in particular the E-L Eqs. (3) and (4)) can be parametrized in terms of the single coordinate time \( t \) rather than the corresponding particle proper time \( s^{(i)} \). This is obtained introducing the representations in terms of the single coordinate (i.e., Laboratory) time \( t \in I \in \mathbb{R} \), namely letting for all \( i = 1, N \)

\[
\begin{align*}
\left( r^{(i)} \right)_{\mu}(t) & \equiv (ct, r^{(i)}), \\
\left( s^{(i)} \right) & \equiv \frac{ct}{\gamma^{(i)}}
\end{align*} \tag{46}
\]

with \( \gamma^{(i)} \) and \( \beta^{(i)} \) denoting the usual relativistic factors

\[
\begin{align*}
\gamma^{(i)} & = \left( 1 - \beta^{2(i)} \right)^{-1/2}, \\
\beta^{(i)} & = \frac{v^{(i)}}{c}.
\end{align*} \tag{47} \tag{48}
\]
This implies also that the 4-velocity can be represented as $u^{(i)\mu} = \frac{1}{c} \gamma^{(i)} v^{(i)\mu}$ with $v^{(i)\mu} = (c, v^{(i)})$. Hence, equations (54) and (55) become respectively

$$m_o^{(i)} cd^{(i)\mu} - m_o^{(i)} c v^{(i)\mu} dt = 0, \quad (49)$$

$$-m_o^{(i)} cd \left[ \frac{\gamma^{(i)}}{c} v^{(i)}(s^{(i)}) \right] + \frac{q^{(i)}}{c} F_{\mu\nu}^{(tot)(i)} d\tau^{(i)\nu}(s_i) = 0. \quad (50)$$

2. Delay-time effects

Delay-time effects which appear both in the EM RR and binary interactions are due to the extended size of the charged particles. In particular, the delay-time characterizing the self-interaction acting on particle $i$ depends only on the radius of the charge distribution of the same particle. Instead, the delay-time appearing in the binary interaction experienced by particle $i$ depends either on the radius $\sigma^{(i)}$ of particle $i$ or on the radii $\sigma^{(j)}$ of all the remaining particles. In the case of $N$-body system of like particles, such that $\sigma^{(i)} = \sigma^{(j)} = \sigma$, the two terms on the r.h.s. of Eq. (44) coincide yielding a single delay-time contribution in Eq. (40). Explicit evaluation of delay times involves the construction of the positive (causal) roots of the equations (42) and (43). Based on the coordinate-time parametrization (40), these can be solved explicitly for the coordinate delay-time $t^{(i)\text{ret}} = t^{(i)} - t$. The causal roots are in the two cases respectively

$$t^{(i)\text{ret}}(t) = \frac{1}{c} \sqrt{\left[ r^{(i)}(t) - r^{(i)}(t - t^{(i)\text{ret}}(t)) \right]^2 + \sigma^2_{(i)}}, \quad (51)$$

$$t^{(i)\text{ret}}(t) = \frac{1}{c} \sqrt{\left[ r^{(i)}(t) - r^{(j)}(t - t^{(i)\text{ret}}(t)) \right]^2 + \sigma^2_{(j)}}. \quad (52)$$

Notice that the same roots can also be equivalently represented in terms of the corresponding particle proper times ($s^{(i)}$ for $i = 1, N$). For this purpose it is sufficient to introduce for the $i$-th particle proper time the parametrization $s^{(i)} = s^{(i)}(t)$ which is determined in terms of the coordinate time $t$ by means of the equations (43). It follows, in particular, that the proper delay-times corresponding to (51) and (52) become respectively $s^{(i)\text{ret}}(s^{(i)}) = s(t^{(i)\text{ret}}(t))$ and $s^{(ij)\text{ret}}(s^{(j)}) = s(t^{(ij)\text{ret}}(t))$.

Along the lines of the approach given in Paper II, it is immediate to show that the hybrid-variable variational principle given in THM.1 can be given an equivalent Lagrangian formulation. An elementary consequence is provided by the following proposition.

**Corollary to THM.1 - Standard Lagrangian form of the $N$-body equations of motion**

*Given validity of THM.1, let us introduce the non-local real function*

$$L_{\text{eff},N} = \sum_{i=1,N} L_{\text{eff}}^{(i)}(r, u, [r]), \quad (53)$$

*where $L_{\text{eff},N}$ is denoted as $N$-body effective Lagrangian, while $L_{\text{eff}}^{(i)}(r, u, [r])$ is defined as*

$$L_{\text{eff}}^{(i)}(r, u, [r]) = L_M^{(i)}(r, u) + L_C^{(\text{ext})(i)}(r) + 2L_C^{(\text{self})(i)}(r, [r]) + L_{\text{eff}}^{(\text{bin})(i)}(r, [r]). \quad (54)$$

*Here $L_M^{(i)}$, $L_C^{(\text{ext})(i)}$ and $L_C^{(\text{self})(i)}$ coincide with the variational Lagrangians defined above (see Eqs. 22-24), while $L_{\text{eff}}^{(\text{bin})(i)}$ is given by*

$$L_{\text{eff}}^{(\text{bin})(i)}(r, [r]) = \sum_{j=1,N \atop j \neq i} 2q^{(i)} q^{(j)} \frac{d\rho^{(i)}}{ds^{(i)}} \int_{-\infty}^{+\infty} ds^{(j)} \frac{d\rho^{(j)}}{ds^{(j)}} K^{(ij)}, \quad (55)$$

*with $K^{(ij)}$ being the sum of Dirac-deltas*

$$K^{(ij)} = \delta(\tilde{R}^{(ij)\alpha} - \sigma^2_{(j)}) + \delta(\tilde{R}^{(ij)\alpha} - \sigma^2_{(i)}). \quad (56)$$
Then, the E-L equations (44) and (55) coincide with the E-L equations in standard form (see Papers I and II) determined in terms of the N-body effective Lagrangian \( L_{\text{eff}, N} \). In particular, the E-L equations in standard form for the i-th particle become

\[
\frac{\partial L_{\text{eff}, N}}{\partial u^i_\mu(s_{(i)})} = 0, \\
F^i_\mu(r)L_{\text{eff}, N} = 0,
\]

(57), (58)

where

\[
F^i_\mu(r) \equiv \frac{d}{ds_{(i)}} \left( \frac{\partial}{\partial r^i_\mu(s_{(i)})} - \frac{\partial}{\partial r^{(i)\mu}(s_{(i)})} \right)
\]

(59)
denotes the E-L differential operator.

Proof - The proof is based on THM.2 of Paper I and by noting that the E-L differential operator acts only on local quantities. Hence, the equivalence of Eqs. (57)–(58) with the corresponding E-L equations (44) and (55) follows by elementary algebra and in view of the identity \( F^i_\mu(r)L_{\text{eff}, N} = F^i_\mu(r)L^i_{\text{eff}} \).

Q.E.D.

To conclude this Section a final remark is necessary regarding the functional setting of the N-body equations of motion given above.

We first notice that the E-L equations (51) and (55), and the equivalent Lagrange equations in standard form (58), imply for all \( i = 1, N \) the second-order delay-type ODEs

\[
m^i_0 c \frac{d^2 r^i_\mu(s_{(i)})}{ds_{(i)}^2} = \frac{d}{ds_{(i)}} F^i_\mu(r) dr^{(i)\mu}(s_{(i)}).
\]

(60)

Let us introduce the Lagrangian state \( \mathbf{w} \equiv \left( (r^i_\mu, u^i_\mu), i = 1, N \right) \), with \( u^i_\mu \equiv \frac{dr^{(i)\mu}}{ds_{(i)}} \), spanning the N-body phase-space \( \Gamma_N = \prod_{i=1}^{N} \Gamma_{1(i)} \), where \( \Gamma_{1(i)} = M^{(4)}_{1(i)} \times U^{(4)}_{1(i)} \) and \( U^{(4)}_{1(i)} \equiv \mathbb{R}^4 \) indicate respectively the corresponding 1-body phase and velocity spaces, the latter endowed with a metric tensor \( \eta_{\mu\nu} \). To define the initial conditions, let us make use of the coordinate-time parametrization (10), denoting \( \mathbf{w}(t) \equiv \left( (\tilde{r}^i(t), \tilde{u}^i(t)), i = 1, N \right) \) and \( \tilde{r}^i(t) \equiv r^i(s_{(i)}(t)) \), \( \tilde{u}^i(t) \equiv u^i(s_{(i)}(t)) \). Then, a well-posed problem for Eqs. (51) is obtained prescribing the initial history set \( \{\mathbf{w}\}_{t_0} \subset \Gamma_N \). For an arbitrary coordinate initial time \( t_0 \in I = \mathbb{R} \), this is defined as the ensemble of initial states

\[
\{\mathbf{w}\}_{t_0} = \left\{ \left( (\tilde{r}^i(t), \tilde{u}^i(t)) \in C^{(k-1)}(I), i = 1, N \right), \forall t \in [t_0 - t^\text{max}_\text{ret}(t_0), t_0], k \geq 2 \right\}.
\]

(61)

Here, for a given initial (coordinate) time \( t_0 \in I \), \( t^\text{max}_\text{ret}(t_0) \) denotes the maximum (for all particles) of the delay-times \( t^{(i)\text{ret}}(t_0) \) and \( t^{(ij)\text{ret}}(t_0) \), namely

\[
t^\text{max}_\text{ret}(t_0) = \max \left\{ t^{(i)\text{ret}}(t_0), t^{(ij)\text{ret}}(t_0), \forall i, j = 1, N \right\}.
\]

(62)

Solutions of Eqs. (60) fulfilling the initial conditions defined by the history set \( \{\mathbf{w}\}_{t_0} \) are sought in the functional class of smooth 4-vector solutions of the form \( r^{(i)\mu} \equiv r^{(i)\mu}(s_{(i)}) \), with \( s_{(i)} = s_{(i)}(t) \) and \( t \in I_0 \equiv (t_0, \infty) \), which belong to the functional class

\[
\{r(s)\} \equiv \left\{ (r^{(i)\mu}, i = 1, N) \mid r^{(i)\mu}(s_{(i)}) \in C^{(k)}(I), \forall s_{(i)} = s_{(i)}(t) \in C^{(k)}(I), k \geq 2, \forall t \in I_0 \right\}.
\]

(63)

In the following we shall assume that in the setting defined by Eq. (63) with the history set (61), the ODEs (60) admit a unique global solution of class \( C^{(k-1)}(I_0) \), with \( k \geq 2 \).

V. N-BODY NON-LOCAL HAMILTONIAN THEORY

Based on THM.1 and its Corollary, an equivalent non-local Hamiltonian formulation can be given for the hybrid and Lagrangian-variable approaches stated in THM.1 and Corollary. The strategy is similar to that developed in Paper II.
This permits us to represent the Hamilton action functional in terms of the hybrid state \( y \equiv (r, p) \equiv (r^{(i)} \mu, p^{(i)} \mu, i = 1, N) \) and the related non-local variational Hamiltonian \( H_1^{(i)} = H_1^{(i)}(r, p, [r]) \), identified, as usual, with the Legendre transformation of the corresponding non-local variational Lagrangian \( L_1^{(i)} \). Hence, for all \( i = 1, N \):

\[
H_1^{(i)} = p^{(i)}_\mu \frac{d r^{(i)} \mu}{ds^{(i)}} - L_1^{(i)},
\]

(64)

while \( p^{(i)}_\mu \) is the \( i \)-th particle conjugate momentum defined in terms of \( L_1^{(i)} \) as

\[
p^{(i)}_\mu = \frac{\partial L_1^{(i)}}{\partial \dot{r}^{(i)} \mu}. \tag{65}
\]

From THM.1 it follows that

\[
p^{(i)}_\mu = m_o^{(i)} c u^{(i)}_\mu + \frac{q^{(i)}}{c} A^{(tot)(i)}_\mu,
\]

(66)

where \( A^{(tot)(i)}_\mu \) is given by

\[
A^{(tot)(i)}_\mu (r; [r]) = A^{(ext)(i)}_\mu + A^{(self)(i)}_\mu + \sum_{j=1, N \neq j} A^{(bin)(i)}_{ij},
\]

(67)

according to the definitions given in Eqs. (26)-(27). As a consequence, \( H_1^{(i)} \) becomes simply

\[
H_1^{(i)}(r, p, [r]) = \frac{1}{2 m_o^{(i)} c} \left[ p^{(i)}_\mu - \frac{q^{(i)}}{c} A^{(tot)(i)}_\mu \right] \left[ p^{(i)}_\mu - \frac{q^{(i)}}{c} A^{(tot)(i) \mu} \right].
\]

(68)

This permits us to represent the Hamilton action functional in terms of the hybrid state \( y \), yielding

\[
S_{H_N}(r, p, [r]) = \sum_{i=1, N} S_{H_1^{(i)}},
\]

(69)

where

\[
S_{H_1^{(i)}} = \int_{s^{(i)}(1)}^{s^{(i)}(2)} ds^{(i)} \left[ p^{(i)}_\mu \frac{d r^{(i)} \mu}{ds^{(i)}} - H_1^{(i)} \right]
\]

(70)

represents the \( i \)-th particle contribution. Of course, as an alternative, analogous dynamical variables can be defined also in terms of the effective Lagrangian \( L_{e f f}^{(i)} \) [see Eq. (63)]. This yields the notion of effective Hamiltonian \( H_{e f f}^{(i)} \) and of the corresponding state \( x \equiv (r, P) \equiv (r^{(i)} \mu, P^{(i)} \mu, i = 1, N) \), which will be shown below to identify a (super-abundant) canonical state (see Corollary to THM.2). Thus, \( H_{e f f}^{(i)} \) - to be considered a non-local function of the form \( H_{e f f}^{(i)} = H_{e f f}^{(i)}(r, P, [r]) \) - is prescribed in terms of the Legendre transformation with respect to \( L_{e f f}^{(i)} \), namely letting:

\[
H_{e f f}^{(i)} \equiv P^{(i)}_\mu \frac{d r^{(i)} \mu}{ds^{(i)}} - L_{e f f}^{(i)},
\]

(71)

while \( P^{(i)}_\mu \) denotes the effective canonical momentum

\[
P^{(i)}_\mu = \frac{\partial H_{e f f}^{(i)}}{\partial \dot{r}^{(i)} \mu}. \tag{72}
\]

From the Corollary to THM.1 it follows immediately that

\[
P^{(i)}_\mu = m_o^{(i)} c u^{(i)}_\mu + \frac{q^{(i)}}{c} A^{(tot)(i)}_{(e f f) \mu},
\]

(73)
where \( A^{(tot)(i)}_{(eff)\mu} \) is given by

\[
A^{(tot)(i)}_{(eff)\mu} = \overline{A}^{(ext)(i)}_{\mu} + 2\overline{A}^{(eff)}_{\mu} + \sum_{j=1,N \atop i \neq j} \overline{A}^{(bin)(ij)}_{(eff)\mu},
\]  
(74)

and

\[
\overline{A}^{(bin)(ij)}_{(eff)\mu} = 2\eta^{(j)} \int_{-\infty}^{+\infty} ds_{(j)} \frac{dv_{(j)}}{ds_{(j)}} K^{(ij)},
\]  
(75)

with \( K^{(ij)} \) being given by Eq.(66). Finally, \( H^{(i)}_{eff} \) becomes

\[
H^{(i)}_{eff} = \frac{1}{2m^{(i)}_{\mu}} \left[ p^{(i)}_{\mu} - \frac{q^{(i)}}{c} A^{(tot)(i)}_{(eff)\mu} \right] - \frac{q^{(i)}}{c} A^{(tot)(i)\mu}_{(eff)}.
\]  
(76)

Therefore, by direct comparison with Eq.(68) it follows identically that

\[
H^{(i)}_{eff} \equiv H^{(i)}_{1}.
\]  
(77)

Then the following theorem, casting the Hamilton variational principle of THM.1 in terms of the state \( \mathbf{x} \), holds.

**THM.2 - N-body non-local Hamiltonian variational principle**

Given validity of THM.1 with Corollary and the definitions \([63]-[70]\) as well as \([71]-[75]\), let us assume that the curves \( f^{(i)}(s_{(i)}) \equiv y^{(i)} = (r^{(i)\mu} , p^{(i)}_{\mu})(s_{(i)}) \) belong to the functional class \( \{ f \} \) of \( C^2 \)-functions subject to the boundary conditions

\[
y^{(i)}(s_{(i)k}) = y^{(i)}_{k},
\]  
(78)

for \( k = 1,2, s_{(i)1}, s_{(i)2} \in I \subseteq \mathbb{R} \) and with \( s_{(i)1} < s_{(i)2} \). Then the following proposition holds:

The modified Hamilton variational principle

\[
\delta S_{H_{N}} = 0
\]  
(79)

subject to independent synchronous variations \( \delta f^{(i)}(s_{(i)}) \equiv \left( \delta r^{(i)\mu}(s_{(i)}), \delta p^{(i)}_{\mu}(s_{(i)}) \right) \) performed in the functional class indicated above, yields, for all \( i = 1,N \), the E-L equations

\[
\frac{\delta S_{H^{(i)}_{1}}}{\delta p^{(i)}_{\mu}} = 0,
\]  
(80)

\[
\frac{\delta S_{H^{(i)}_{1}}}{\delta r^{(i)\mu}} = 0.
\]  
(81)

These equations coincide identically with the N-body variational equations of motion \([74]\) and \([76]\). Hence, the set \( \{ y, H_{N} \} \equiv \{ y^{(i)}, H^{(i)}_{1}, i = 1,N \} \) defines a non-local Hamiltonian system.

**Proof** - The proof is analogous to that given in THM.2 of Paper II. In particular, it can be reached, after elementary algebra, by invoking the symmetry properties of the variational functional \( S_{H_{N}} \), namely

\[
S_{H_{N}}(r_{A} , p_{A} , [r_{B}]) = S_{H_{N}}(r_{B} , p_{B} , [r_{A}]),
\]  
(82)

where again \( r_{A} \) and \( r_{B} \) are two N-body arbitrary curves of the functional class \( \{ y \} \). It follows that the variational derivative in the E-L equation Eq.(81) becomes

\[
\frac{\delta S_{H^{(i)}_{1}}}{\delta r^{(i)\mu}} = \frac{\delta S_{H^{(i)}_{1}}}{\delta r^{(i)\mu}} \bigg|_{[r]} + \sum_{j=1,N} \frac{\delta S_{H^{(j)}_{1}}}{\delta [r^{(j)\mu}]} \bigg|_{r} = 0,
\]  
(83)
where the summation is performed only on the non-local contributions. As a consequence, the E-L equations (80) and (81) yield

\[
\frac{\delta S_{H_{i}^{(i)}}}{\delta p_{\mu}^{(i)}} = m_{y}^{(i)} \frac{d r_{\mu}^{(i)\mu}}{d s_{(i)}} - \left[ p_{\mu}^{(i)} - q_{\mu}^{(i)} A_{\mu}^{(\text{tot}(i))} \right] = 0, \quad (84)
\]

\[
\frac{\delta S_{H_{i}^{(i)}}}{\delta r_{\mu}^{(i)\mu}} = -\frac{d p_{\mu}^{(i)}}{d s_{(i)}} + \frac{q_{\mu}^{(i)}}{c} \frac{d r_{\mu}^{(i)\nu}(s_{i})}{d s_{(i)}} \left[ \frac{\partial A_{\mu}^{(\text{tot}(i))}}{\partial r_{\nu}^{(i)}} + F_{\mu}^{(\text{tot}(i))} \right] = 0. \quad (85)
\]

Taking into account the definitions given by Eq.(66) the equivalence with Eqs.(34) and (35) is immediate. Q.E.D.

Let us now pose the problem of the construction of the corresponding N-body Hamiltonian equations in standard form, as suggested by the results of Paper II. The non-local Hamiltonian system \{y, H_{N}\} is said to admit a standard Hamiltonian form \(\{x, H_{\text{eff}}^{(1)}, \ldots, H_{\text{eff}}^{(N)}\}\) if the N-body equations of motion can be cast, for all \(i = 1, N\), in the form

\[
\frac{d r_{\mu}^{(i)\mu}}{d s_{(i)}} = \frac{\partial H_{\text{eff}}^{(i)}}{\partial P_{\mu}^{(i)}}, \quad (86)
\]

\[
\frac{d P_{\mu}^{(i)}}{d s_{(i)}} = -\frac{\partial H_{\text{eff}}^{(i)}}{\partial r_{\mu}^{(i)\mu}}. \quad (87)
\]

in terms of a suitably-defined effective particle Hamiltonian \(H_{\text{eff}}^{(i)}\), to be identified with Eq.(76). In particular, a N-body system with state \(x \equiv \{x^{(i)}, i = 1, N\}\) is said to be endowed with a Hamiltonian structure \(\{x, H_{N,\text{eff}}\}\) if, for all particles belonging to the N-body system, the equations of motion for the \(i\)-th canonical particle state \(x^{(i)}\) can be represented in the PBs notation (2) in terms of a single Hamiltonian function \(H_{N,\text{eff}}\), i.e., for all \(i = 1, N\)

\[
\frac{dx^{(i)}}{d s_{(i)}} = \left[ x^{(i)}, H_{N,\text{eff}} \right], \quad (88)
\]

with \(H_{N,\text{eff}}\) denoting a still to be determined, appropriate system effective Hamiltonian. Extending the treatment holding for the 1-body problem (see Paper II), here we intend to prove that the Hamiltonian structure \(\{x, H_{N,\text{eff}}\}\) holds also in the case of EM-interacting N-body systems. The following proposition holds.

**Corollary to THM.2 - Standard Hamiltonian form and Hamiltonian structure of the N-body equations of motion**

*Given validity of THM.2 and the definitions given by Eqs.(71)-(76), it follows that:

TC1) The non-local Hamiltonian system \(\{x, H_{N}\}\) admits a standard Hamiltonian form defined in terms of the set \(\{x, H_{\text{eff}}^{(1)}, \ldots, H_{\text{eff}}^{(N)}\}\), with \(H_{\text{eff}}^{(i)}\) the \(i\)-th particle effective Hamiltonian [given by Eq.(76)]; furthermore \(x \equiv \{x^{(i)}, i = 1, N\}\) is the super-abundant canonical state, while

\[
x^{(i)} \equiv \left( r_{\mu}^{(i)\mu}, P_{\mu}^{(i)} \right), \quad (89)
\]

\[
r_{\mu}^{(i)\mu} = \left( r_{\mu}^{(i)0}, r^{(i)} \right), \quad (90)
\]

\[
P_{\mu}^{(i)} \equiv \left( p^{(i)0}, p^{(i)} \right), \quad (91)
\]

are respectively the \(i\)-th particle canonical state, 4-position and effective canonical momentum [defined by Eq.(73)]. As a consequence, Eqs.(80) and (81) can be cast in the standard Hamiltonian form (86) and (87).

TC2) The equations (86) and (87) admit also the equivalent representation (88) and hence the set \(\{x, H_{N,\text{eff}}\}\) defines a Hamiltonian structure, with \(H_{N,\text{eff}}\) being the effective N-body Hamiltonian function

\[
H_{N,\text{eff}} \equiv \sum_{i=1,N} H_{\text{eff}}^{(i)}. \quad (92)
\]
Introducing the system Hamiltonian
\[ H_N = \sum_{i=1}^{N} H^{(i)} \]  \hspace{1cm} (93)
defined in terms of the variational i-th particle variational Hamiltonian \( H^{(i)} \) [see Eq.(63)], it follows identically that
\[ H_N = H_{N,\text{eff}}. \]  \hspace{1cm} (94)

**Proof - TC2** The proof follows from straightforward algebra. The first equation manifestly reproduces Eq.(50), because of the definition of \( H_{\text{eff}}^{(i)} \) given above. Similarly, in the second equation the partial derivative of \( H_{\text{eff}}^{(i)} \) recovers the correct form of the total EM force expressed in terms of \( A_{(\text{eff})\mu}^{(\text{tot})(i)} \) TC2 To prove the existence of the Hamiltonian structure, it is sufficient to notice that \( \frac{\partial H_{\text{eff}}^{(i)}}{\partial P_{\mu}^{(i)}} = [x^{(i)\mu}, H_{\text{N,eff}}] \) and \( \frac{\partial H_{\text{eff}}^{(i)}}{\partial r^{(i)\mu}} = - \left[ P_{\mu}^{(i)}, H_{\text{N,eff}} \right] \) TC2 By construction [see Eqs.(68) and (76)] for all \( i = 1, N \) the effective and variational Hamiltonians coincide [see Eq.(77)]. This implies the validity of Eq.(94) too, namely \( H_N \) identifies also the system Hamiltonian. It follows that for all particles \( i = 1, N \) the canonical equations of motion recover the standard Hamiltonian form expressed in terms of the PBs with respect to the the system Hamiltonian, i.e., Eqs.(1), so that \( \{x, H_N\} \) identifies the Hamiltonian structure of the EM-interacting N-body system.

**Q.E.D.**

VI. GENERAL IMPLICATIONS OF THE NON-LOCAL N-BODY THEORY

Let us now comment on the general implications of the previous theorems.

**Remark #1 - Difference form of the Hamilton equations of motion.** The canonical equations (1) imply the following difference equations, i.e., the infinitesimal canonical transformation generated by \( H_N \):
\[ dx^{(i)} = ds^{(i)} \left[ x^{(i)}, H_N \right]. \]  \hspace{1cm} (95)

**Remark #2 - Coordinate-time representation of the Hamilton equations of motion.** Introducing the coordinate-time parametrization (60), Eqs.(65) become
\[ dx^{(i)} = \frac{cdt}{\gamma^{(i)}} \left[ x^{(i)}, H_N \right], \]  \hspace{1cm} (96)
with \( \gamma^{(i)} \) denoting again the relativistic factor (17), while \([\cdot, \cdot]\) are the local PBs evaluated with respect to the superabundant canonical state \( x \). These yield explicitly
\[ dr^{(i)\mu} = \frac{dt}{\gamma^{(i)}} \frac{1}{m_0^{(i)}} \left( P^{(i)\mu} - \frac{q^{(i)}}{c} A_{(\text{eff})\mu}^{(\text{tot})(i)} \right) = \frac{dt}{\gamma^{(i)}} m_0^{(i)} \equiv d\tau^{(i)\mu}. \]  \hspace{1cm} (97)
\[ dP^{(i)\mu} = \frac{dt}{\gamma^{(i)}} \frac{q^{(i)}}{m_0^{(i)}} \frac{\partial A_{(\text{eff})\mu}^{(\text{tot})(i)}}{\partial r^{(i)\mu}} \left( P^{(i)\nu} - \frac{q^{(i)}}{c} A_{(\text{eff})\nu}^{(\text{tot})(i)} \right) = \frac{cdt}{\gamma^{(i)}} \frac{q^{(i)}}{c} \frac{\partial A_{(\text{eff})\mu}^{(\text{tot})(i)}}{\partial r^{(i)\mu}} \equiv d\lambda^{(i)\nu}. \]  \hspace{1cm} (98)

**Remark #3 - Well-posedness of the N-body equations of motion.** All the equations of motion indicated above [see THMs.1 and 2 and their Corollaries] are equivalent to each other and are manifestly Lorentz-covariant (see also related discussion in Paper II). Then, a well-posed problem for the Hamiltonian equations in standard form (88) can be obtained in analogy to the problem defined by Eqs.(60),(61). This is achieved, first, by prescribing the appropriate initial history set \( \{\mathbf{x}\} \in \Gamma_N \). For an arbitrary coordinate initial time \( t_0 \in I \equiv \mathbb{R} \), this is defined as the ensemble of initial states
\[ \{\mathbf{x}\} \equiv \left\{ x(t) \equiv \left( \hat{r}^{(i)}(t), \hat{p}^{(i)}(t) \right) \in C^{(k-1)}(I), i = 1, N \right\}, \forall t \geq t_0 \]  \hspace{1cm} (99)
standard Hamiltonian form (1) is that the fundamental local PBs for the state \( x \equiv \text{canonical flow} \) are identically satisfied for all \( i,j \) (denoting \( \hat{C} \) functional class (30) by identifying (88) admit a unique global solution of class \( C^{k-1}(I_0) \) with \( k \geq 2 \).

Remark #4 - Extremant and extremal curves. In all cases indicated above the solutions of N-body E-L equations of motion (extremal curves) and of the Lagrangian and Hamiltonian equations in standard form given by the Corollaries to THMs.1 and 2 (extremant curves), satisfy identically, for all \( i = 1, N \), the kinematic constraints

\[
    u^{(i)}_\mu u^{(i)\mu} = 1 \tag{101}
\]

(velocity constraints) and

\[
    ds^2_{(i)} = \eta_{\mu\nu} dr^{(i)\mu} dr^{(i)\nu} \tag{102}
\]

(line-element constraints). The first constraint implies that the time-components of the 4-velocity depend on the corresponding space components, while the second constraint requires that the particles' proper times are uniquely related to the corresponding coordinate times. In particular, we shall denote as extremant canonical curves

\[
    x(s_{(1)}, \ldots s_{(N)}) = \{ x^{(1)}(s_{(1)}), \ldots x^{(N)}(s_{(N)}) \}, \tag{103}
\]

with \( x^{(i)} = x^{(i)}(s_{(i)}) \) for all \( i = 1, N \), arbitrary particular solutions of the canonical equations (123).

Remark #5 - Unconstrained varied functions. By assumption, both the varied functions \( f^{(i)} = \left[ r^{(i)\mu}, u^{(i)}_\mu \right]_{(s_{(i)})} \), \( y^{(i)} = (r^{(i)\mu}, p^{(i)}_\mu)_{(s_{(i)})} \) and \( x^{(i)} = (r^{(i)\mu}, p^{(i)}_\mu)_{(s_{(i)})} \) entering respectively THMs.1 and 2 as well as the Corollary of THM.2 are unconstrained, namely they are solely subject to the requirement that end points and boundary values are kept fixed (and therefore do not fulfill the previous kinematic constraints). This implies, in particular, that all of the (8) components of \( f^{(i)} \), \( y^{(i)} \) and \( x^{(i)} \) must be considered independent. On the other hand, both the extremal and extremant curves satisfy all of the required kinematic constraints, so that only (6) of them are actually independent for each particle (see also discussion in Paper I).

Remark #6 - Non-local Hamiltonian structure and unconstrained canonical state. Thanks to proposition TC2_3 of the Corollary to THM.2 the Hamiltonian structure \( \{ x, H_{N,eff} \} \) coincides with \( \{ x, H_N \} \), \( H_N \) denoting the non-local system Hamiltonian defined by Eq.(32). We remark, however, that in the PBs given by Eq.(1) the partial derivatives must be evaluated with respect to the unconstrained states \( x^{(i)} \) and not \( y^{(i)} \) indicated above. This means that \( H_N \) must be considered a function of \( x \). It is immediate to prove that the same Hamiltonian structure \( \{ x, H_N \} \) holds provided the super-abundant canonical state \( x \equiv (x^{(i)}, i = 1, N) \) is considered unconstrained. In fact, as shown by the Corollary to THM.2, in such a case the canonical equations in standard form (36) and (37) admit a PB-representation of the form (38). For this purpose let us make use of the coordinate-time parametrization \( x \equiv \tilde{x}(t) \), denoting \( \tilde{x}(t) = x_o + dx \), with \( x_o \equiv \tilde{x}(t_o) \) and \( dx \equiv (dx^{(1)}, \ldots dx^{(N)}) \). Furthermore let us require that the initial history set \( \{ \tilde{x} \}_{t_0} \) is prescribed. Then, a necessary and sufficient condition for the equations of motion to admit the standard Hamiltonian form (11) is that the fundamental local PBs for the state \( x \equiv \tilde{x}(t) \), defined with respect to the same state \( x_o \equiv \tilde{x}(t_o) \), namely

\[
    \left[ r^{(i)\mu}, r^{(j)\nu} \right]_{(x_o)} = 0, \\
    \left[ p^{(i)}_\mu, p^{(j)}_\nu \right]_{(x_o)} = 0, \\
    \left[ r^{(i)\mu}, p^{(j)}_\nu \right]_{(x_o)} = \delta^{ij} \delta^{\mu}_\nu, \tag{104}
\]

are identically satisfied for all \( i, j = 1, N \) and \( \mu, \nu = 0, 3 \). This is realized only when the super-abundant variables \( x \equiv (x^{(i)}, i = 1, N) \) are considered independent.

Remark #7 - The canonical flow is not a dynamical system. A final issue concerns the properties of the flow generated by the canonical problem (38) and (39) - (100) (canonical flow). In the N-body phase-space \( \Gamma_N \) this is an ensemble \( C^{(k-1)} \text{-homeomorphism} \) (with \( k \geq 2 \)) of the type

\[
    \{ \tilde{x} \}_{t_0} \leftrightarrow \tilde{x}(t), \tag{105}
\]
which maps an arbitrary history set \( \{ \tilde{x} \} \subset \Gamma_N \) onto a state \( \tilde{x}(t) \) crossed at a later coordinate time \( t \) (i.e., at \( t > t_0 \)). This map does not generally define a dynamical system. In fact, unless there is a subset on non-vanishing measure of \( \Gamma_N \) in which \( \{ \tilde{x} \}_t \) reduces to the initial instant set

\[
\{ \tilde{x} \}_t \equiv \{ \tilde{x}(t_0) = x_0 \},
\]

the flow \( \tilde{x}^t \) is not a bijection in \( \Gamma_N \). To prove the statement it is sufficient to notice that - in the case of a non-local Hamiltonian structure \( \{ x, H_{N,cff} \} \) - if the history set is left unspecified and only the initial state \( \tilde{x}(t_0) \) is prescribed, the image of the initial state is obviously generally non-unique [and hence it may not coincide with \( \tilde{x}(t) \)]. In fact, while the same initial state \( \tilde{x}(t_0) \) may be produced by different history sets, for example \( \{ \tilde{x} \}_t \) and \( \{ \tilde{x}' \}_t \), the same history sets will generally give rise to different images \( \tilde{x}(t) \) and \( \tilde{x}'(t) \).

We emphasize that for \( N \)-body systems subject to EM interactions the instant set \( \{ x \} \) can be realized only for special initial conditions, i.e., for example, if for all \( t \leq t_0 \) all the particles of the system are in inertial motion with respect to an inertial Lorentz frame. Since, unlike the external EM field, binary and self EM interactions cannot be “turned off”, it follows that the set of initial conditions \( \{ x \} \) has necessarily null measure in \( \Gamma_N \).

\[ \text{VII. THE} \ N \text{-BODY HAMILTONIAN ASYMPTOTIC APPROXIMATION} \]

In this section we want to develop asymptotic approximations for the equations of motion of EM-interacting \( N \)-body systems. This involves different asymptotic conditions to be imposed on both the self and binary interactions. In particular:

1) For the RR self-interaction of each particle \( i \) with itself this is provided by the short delay-time ordering, namely the requirement that the dimensionless parameters \( \epsilon_{(i)} \equiv \frac{\sigma_{(i)} - \sigma'_{(i)}}{\sigma_{(i)}} \), for \( i = 1, N \) are all infinitesimal of the same order \( \epsilon \), i.e., \( \epsilon \sim \epsilon_{(i)} \ll 1 \), \( s_{(i)} - s'_{(i)} \) denoting the \( i \)-th proper-time difference between observation \( (s) \) and emission \( (s') \) of self-radiation.

2) For the binary EM interactions the Minkowski distance \( \left| R^{(ij)} \right| \) between two arbitrary particles of the system is much larger than their radii, in the sense that for all \( i, j = 1, N \), with \( i \neq j \), the large-distance ordering \( 0 < \frac{\sigma_{(i)}}{\left| R^{(ij)} \right|} \sim \frac{\sigma_{(i)}}{R^{(ij)}} \ll \epsilon \) holds.

The fundamental issue arises whether an approximation can be found for the \( N \)-body problem which:

1) is consistent with the orderings 1) and 2);
2) recovers the variational, Lagrangian and Hamiltonian character of the exact theory (see THMs.1 and 2);
3) preserves both the standard Lagrangian and Hamiltonian forms of the equations of motion;
4) retains finite delay-time effects characteristics of both the RR and binary EM interactions, consistent with the prerequisites #1-#5 indicated above.

In this regard, a fundamental result is the discovery pointed out in Paper II of an asymptotic Hamiltonian approximation of this type for single extended particles subject to the EM self-interaction. This refers to the retarded-time Taylor expansion of the Faraday tensor contribution carried by the RR self-force. More precisely, in the case of a single particle, this is obtained by Taylor-expanding the RR self-force

\[
G^{(i)}_{\mu} \equiv \frac{q^{(i)}}{c} T^{(self)(i)}_{\mu \nu} \left( \gamma^{(i)} \left( s_{(i)} \right), r^{(i)} \left( s'_{(i)} \right) \right) \frac{d r^{(i)}_{(k)} \left( s_{(i)} \right)}{d s_{(i)}}
\]

for \( i = 1 \) (see Eq.38) in the neighborhood of the retarded proper-time \( s'_{(i)} \). Here we claim that an analogous conclusion can be drawn also for the corresponding \( N \)-body problem, by introducing the same expansion to all charged particles and invoking the large-distance ordering for the binary interaction. For this purpose, we shall assume that the external force acting on each charged particle is slowly varying in the sense that, denoting \( r' \equiv r^{(i) \mu} \left( s'_{(i)} \right) \) and \( r \equiv r^{(i) \mu} \left( s_{(i)} \right) \),

\[
T^{(ext)}_{\mu \nu} \left( r' \right) - T^{(ext)}_{\mu \nu} \left( r \right) \sim O (\epsilon),
\]

\[
\left( T^{(ext)}_{\mu \nu} \left( r' \right) - T^{(ext)}_{\mu \nu} \left( r \right) \right)_h \sim O (\epsilon),
\]

\[
\left( T^{(ext)}_{\mu \nu} \left( r' \right) - T^{(ext)}_{\mu \nu} \left( r \right) \right)_{,hk} \sim O (\epsilon).
\]
Then, the following proposition holds.

**THM.3 - N-body asymptotic Hamiltonian approximation.**

Given validity of THM.2 and the short delay-time and large-distance asymptotic orderings as well as the smoothness assumptions (108)-(110) for the external EM field, neglecting corrections of order $e^n$, with $n \geq 1$ (first-order approximation), the following results hold:

1. **T3.1** The vector fields (107) describing the RR self-force are approximated in a neighborhood of $s_i'$ as
   \[
   g^{(i)}_\mu \left( r^{(i)} \left( s_i' \right) \right) = \left\{ -m^{(i)}_\text{ext,EM} \frac{d}{ds^{(i)}} g^{(i)}_\mu \left( s_i' \right) + g^{(i)}_{\mu'} \left( r^{(i)} \left( s_i' \right) \right) \right\},
   \]
   to be referred to as retarded-time Hamiltonian approximation for the self-force, in which the first term on the r.h.s. identifies a retarded mass-correction term, $m^{(i)}_{\text{ext,EM}} \equiv \frac{q^{(i)}_0}{c \sigma^{(i)}}$ denoting the leading-order EM mass. Finally, $g^{(i)}_{\mu'}$ are the 4-vectors
   \[
   g^{(i)}_{\mu'} \left( r^{(i)} \left( s_i' \right) \right) = -\frac{1}{3} \frac{q^{(i)}_0}{c} \left[ \frac{d^2}{ds'^{(i)}_\mu} u^{(i)} \left( s_i' \right) - u^{(i)} \left( s_i' \right) u^{(i)k} u^{(i)}_k \frac{d^2}{ds'^{(i)}_\mu} u^{(i)}_k \left( s_i' \right) \right].
   \]

2. **T3.2** The tensor fields $F^{\text{bin}(i,j)}_{\mu\nu}$ for all $i, j = 1, N$, with $i \neq j$, appearing in the binary EM interaction (see Eq.(40)) are approximated by the leading-order (point-particle) terms:
   \[
   F^{\text{bin}(i,j)}_{\mu\nu} \approx F^{\text{bin}(i,j)}_{\mu\nu} \left( r^{(i)}, \left[ \frac{\partial}{\partial s^{(j)}_\nu} \right], \sigma^{(i)} = 0, \sigma^{(j)} = 0 \right).
   \]

3. **T3.3** The corresponding asymptotic N-body equations of motion obtained replacing $G^{(i)}_\mu$ and $F^{\text{bin}(i,j)}_{\mu\nu}$ with the asymptotic approximations (111) and (113) are variational, Lagrangian and admit a standard Lagrangian form. Denoting with $r_0^{(i)'} \equiv r_0 \left( s_i' \right)$ the extremal i-th particle world-line at the retarded proper time $s_i'$, the i-th particle asymptotic variational Lagrangian functions become:
   \[
   L_{1,\text{asym}}^{(i)}(r, u, [r]) = L_M^{(i)}(r, u) + L_{C,\text{asym}}^{(\text{ext})}(r) + L_{C,\text{asym}}^{(\text{self})}(r, r_0^{(i)'}) + L_{C,\text{asym}}^{(\text{bin})}(r, [r]).
   \]
   Here $L_M^{(i)}$ and $L_{C,\text{asym}}^{(\text{ext})}$ remain unchanged (see Eqs.(22) and (23)), while the non-local terms $L_{C,\text{asym}}^{(\text{self})}(r, [r])$ and $L_{C,\text{asym}}^{(\text{bin})}(r, [r])$ are respectively
   \[
   L_{C,\text{asym}}^{(\text{self})}(r, r_0^{(i)'}) = g^{(i)}_\mu \left( r_0^{(i)'} \right) r^{(i)\mu},
   \]
   \[
   L_{C,\text{asym}}^{(\text{bin})}(r, [r]) = \frac{q^{(i)}}{c} \int_{-\infty}^{+\infty} ds^{(j)} \sum_{\substack{1, N \ j \neq j}} \mathcal{A}^{\text{bin}(i,j)}_\mu \left( \sigma^{(j)} = 0 \right),
   \]
   where, from Eq.(37) one obtains
   \[
   \mathcal{A}^{\text{bin}(i,j)}_\mu \left( \sigma^{(j)} = 0 \right) \equiv 2q^{(i)} \int_{-\infty}^{+\infty} ds^{(j)} \frac{dr^{(i)\mu}}{ds^{(j)}} s^{(j)} \tilde{R}^{(i)\alpha} \tilde{R}^{\alpha(j)}. \]

Similarly, the effective particle Lagrangians are, for $i = 1, N$:
   \[
   L_{e,\text{asym}}^{(i)}(r, u, [r]) = L_{M,\text{asym}}^{(i)}(r, u) + L_{C,\text{asym}}^{(\text{ext})}(r) + L_{C,\text{asym}}^{(\text{self})}(r, r_0^{(i)'}) + 2L_{C,\text{asym}}^{(\text{bin})}(r, [r]).
   \]

4. **T3.4** The N-body equations obtained imposing the asymptotic approximations given by Eqs.(111) and (113) are also Hamiltonian. The asymptotic variational and effective Hamiltonian functions are given respectively by
   \[
   H_{1,\text{asym}}^{(i)} = p^{(i)\mu} \frac{dr^{(i)\mu}}{ds^{(i)}} - L_{1,\text{asym}}^{(i)},
   \]
   \[
   H_{e,\text{asym}}^{(i)} = p^{(i)\mu} \frac{dr^{(i)\mu}}{ds^{(i)}} - L_{e,\text{asym}}^{(i)}.
   \]
with \( L_{1,\text{asyg}}^{(i)} \) and \( L_{\text{eff,asyg}}^{(i)} \), defined by Eqs. (117) and (118), while now

\[
\begin{align*}
\frac{p_{\mu}^{(i)}}{p_{\mu}^{(i)}} &= \frac{\partial L_{1,\text{asyg}}^{(i)}}{\partial \dot{r}_{\mu}^{(i)}}, \\
\frac{p_{\mu}^{(i)}}{p_{\mu}^{(i)}} &= \frac{\partial L_{\text{eff,asyg}}^{(i)}}{\partial \dot{r}_{\mu}^{(i)}}.
\end{align*}
\]

(121) (122)

**Proof - T3.1** The proof is analogous to that given in THM.5 of Paper II. T3.2) To prove the validity of Eq. (113), let us recall the definition of \( \mathcal{F}_{\mu}^{(\text{bin})(ij)} \) given by Eq. (30). Then, for each particle, imposing the large-distance ordering and neglecting corrections of order \( e^n \), with \( n \geq 1 \), the leading-order contribution is given by Eq. (113), which depends on a single delay-time determined by the positive root of the equation \( R^{(i)a}(\tilde{R}^{(i)}_{1\text{a}}) = 0 \). T3.3) The proof follows by first noting that the function \( L_{(\text{self})(i)}^{(i)} \) contributes to the \( i \)-th particle E-L equations only in terms of the local dependence in terms of \( r^{(i)} \). Second, in the large-distance ordering, the asymptotic approximation for the \( N \)-body Lagrangian carrying the binary interactions yields a symmetric functional, as the exact one. Therefore, the \( N \)-body asymptotic equations are necessarily variational and Lagrangian. Straightforward algebra shows that the E-L equations determined with respect to the asymptotic variational Lagrangian (114) coincide with the asymptotic approximations proved by propositions T3.1) and T3.2). In a similar way it is immediate to prove the validity of Eq. (113), which shows that the same asymptotic equations admit a standard Lagrangian form. T3.4) Finally, the equivalent \( N \)-body variational and standard Hamiltonian formulations follow by performing Legendre transformations on the corresponding asymptotic variational and effective Lagrangian functions. It follows that the asymptotic \( N \)-body equations of motion can also be represented in the standard Hamiltonian form in terms of \( H_{\text{eff,asyg}}^{(i)} \).

Q.E.D.

It is worth pointing out the unique features of THM.3. These are related, in particular, to the asymptotic expansion performed on the RR self-force alone. In most of the previous literature, the short delay-time expansion is performed with respect to the particle present proper-time. This leads unavoidably to local asymptotic equations (analogous to the LAD and LL equations) which are intrinsically non-variational and therefore non-Lagrangian and non-Hamiltonian. In contrast, the short delay-time expansion adopted here (as in Paper II) approximates the non-local RR vector field in a manner that meets the goals indicated at the beginning of the section. The remarkable consequence is that the asymptotic \( N \)-body equations of motion retain the representation in standard Hamiltonian form characteristic of the corresponding exact equations. Finally, we stress that in all cases both for the exact and asymptotic formulations, the variational and effective Lagrangian and Hamiltonian functions are always non-local functions of the particle states. The non-locality is intrinsic and arises even in the 1-body systems, being due to the functional form of the EM 4-potential generated by each extended particle.

**VIII. ON THE VALIDITY OF DIRAC GENERATOR FORMALISM**

A seminal approach in relativistic dynamics is the Dirac generator formalism developed originally by Dirac (Dirac, 1949 [3]) to describe the dynamics of interacting \( N \)-body systems in the Minkowski space-time. Dirac’s primary goal is actually to determine the underlying dynamical system, exclusively based on DGF. In his words “In setting up such a new dynamical system one is faced at the outset by the two requirements of special relativity and of Hamiltonian equations of motion”. It thus "...becomes a matter of great importance to set up (in this way) new dynamical systems and see if they will better describe the atomic world" (quoted from Ref. [3]).

DGF is couched on the Lie algebra of Poincaré generators for classical \( N \)-body systems. The basic hypothesis behind Dirac approach is that these systems must have a Hamiltonian structure \( \{ \mathbf{z}, K_N \} \) of some sort, with \( \mathbf{z} = \{ \mathbf{z}^{(1)}, ..., \mathbf{z}^{(N)} \} \) and \( K_N \) being a suitable canonical state and a Hamiltonian function of the system. In particular, the canonical states \( \mathbf{z}^{(i)} \) of all particles \( i = 1, N \) must satisfy, by assumption, covariant Hamilton equations of motion of the form

\[
\frac{ds^{(i)}}{ds^{(i)}} = \left[ \mathbf{z}^{(i)}, K_N \right],
\]

(123)

with \( s^{(i)} \) denoting the \( i \)-th particle proper time. However, it must be stressed that certain aspects of Dirac theory remain “a priori” undetermined. This concerns the functional settings both of the canonical state \( \mathbf{z} \) and of the
Hamiltonian function $K_N$. In particular, the system state $z = \{z^{(1)}, \ldots, z^{(N)}\}$ remains in principle unspecified, so that it might be identified either with a set of super-abundant or essential canonical variables. Thus, for example, in the two cases the $i$-the particle state $z^{(i)}$ might be prescribed respectively either as: a) the ensemble of two 4-vectors $z^{(i)} = (r^{(i)\mu}, \pi^{(i)\mu})$, with $r^{(i)\mu} = (r^{(i)0}, r^{(i)}\nu)$ and $\pi^{(i)\mu} = (\pi^{(i)0}, \pi^{(i)}\nu)$ being respectively the particle 4-position and its conjugate 4-momentum; b) the ensemble of the corresponding two 3-vectors obtained taking only the space parts of the same 4-vectors $r^{(i)\nu}$ and $\pi^{(i)\mu}$, namely in terms of $z^{(i)} = (r^{(i)}, \pi^{(i)})$. Thus, depending on the possible prescription, the very definitions of the PBs entering Eqs. (123) and DGF change.

Furthermore, it remains “a priori” unspecified whether $K_N$ is actually intended as a local or a non-local function of the canonical state $z$. In Ref. [3], however, certain restrictions on the nature of the set $\{z, K_N\}$ are actually implied. These will be discussed below, leaving aside for the moment further discussions on this important issue.

Provided the Hamiltonian structure $\{z, K_N\}$ exists, any set of smooth dynamical variables $\eta, \xi$ and $\zeta$ depending locally on the canonical state $z$ necessarily fulfills the following laws

\[
\begin{align*}
\{\eta, \zeta\} & = -\{\xi, \eta\}, \\
\{\eta, \xi + \zeta\} & = \{\eta, \xi\} + \{\eta, \zeta\}, \\
\{\xi, \eta\zeta\} & = \{\xi, \eta\} \zeta + \eta \{\xi, \zeta\}, \\
\{\xi, \eta\}, \zeta & + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0.
\end{align*}
\]  

(124)

DGF relies on the Lie transformation formalism and is based on the representation of the Lorentz transformation group in terms of the corresponding generator algebra. This is defined as the set of phase-functions (Poincaré algebra generators) $\{F\}$ given by

\[
F = -\hat{p}^\mu a_\mu + \frac{1}{2} \hat{M}^{\mu\nu} b_{\mu\nu},
\]  

(125)

with $a_\mu$, $b_{\mu\nu}$ being suitable real constant infinitesimals and $\hat{p}^\mu$, $\hat{M}^{\mu\nu} = -\hat{M}^{\nu\mu}$ appropriate local phase-functions obeying the PBs (Lorentz conditions)

\[
\begin{align*}
[\hat{p}_\mu, \hat{p}_\nu] & = 0, \\
[\hat{M}_{\mu\nu}, \hat{p}_\alpha] & = -\eta_{\alpha\omega} \hat{p}_\omega + \eta_{\nu\omega} \hat{p}_\mu, \\
[\hat{M}_{\mu\nu}, \hat{M}_{\alpha\beta}] & = -\eta_{\mu\alpha} \hat{M}_{\nu\beta} + \eta_{\nu\alpha} \hat{M}_{\mu\beta} - \eta_{\mu\beta} \hat{M}_{\nu\alpha} + \eta_{\nu\beta} \hat{M}_{\mu\alpha}.
\end{align*}
\]  

(126)

Hence, $F$ as given by Eq. (125) generate respectively infinitesimal 4-translations (for $b_{\mu\nu} \equiv 0$ and $a_\mu \neq 0$) and 4-rotations (for $b_{\mu\nu} \neq 0$ and $a_\mu \equiv 0$, corresponding either to Lorentz-boosts or spatial rotations) via infinitesimal canonical transformations of the type

\[
z \rightarrow z' + \delta_o z,
\]  

(127)

with $\delta_o z \sim O(\delta)$, $\delta > 0$ denoting a suitable infinitesimal. $\delta_o z$ is determined identifying it with

\[
\delta_o z \equiv [z, F],
\]  

(128)

to be referred to as the local variation of $z$. Hence, according to DGF an arbitrary dynamical variable $\xi$ depending locally and smoothly on the canonical state $z$ transforms in terms of the law

\[
\begin{align*}
\xi & \rightarrow \xi' = \xi + \delta_o \xi, \\
\delta_o \xi(z) & \equiv [\xi, F],
\end{align*}
\]  

(129)

(130)

where $\delta_o \xi(z) = [[\xi(z'), \xi(z)] \{1 + O(\delta)\}$. It is immediate to determine an admissible representation for the generators $\{F\} \equiv \{\hat{p}^\mu, \hat{M}^{\mu\nu}\}$. Let us first consider a relativistic 1-body system represented by a single particle in the absence of external forces. For definiteness, let us assume that $s_{(1)}(t) \in I \equiv \mathbb{R}$ the particle 4-velocity is constant. Then $\hat{p}^\mu$ and $\hat{M}^{\mu\nu}$ are manifestly given by:

\[
\begin{align*}
\hat{p}_\mu & = \pi^{(1)}_\mu, \\
\hat{M}^{\mu\nu} & = q^{(1)}_\mu \pi^{(1)}_\nu - q^{(1)}_\nu \pi^{(1)}_\mu,
\end{align*}
\]  

(131)

where respectively the 4-vectors $q^{(1)}_\mu$ and $\pi^{(1)}_\mu$ are to be identified with the 1-body coordinates and momenta. In the case of the corresponding $N$-body problem for relativistic interacting particles, in Dirac paper three different
realizations of \( \{ \hat{p}^\mu, \hat{M}^{\mu\nu} \} \) were originally proposed, which are referred to as the instant, point and front forms. All of them follow by imposing the velocity kinematic constraints \([101]\). In particular, the instant form is realized by prescribing the reference frame in such a way to set \( r_0^{(i)} = 0 \), for all \( i = 1, N \), namely describing each particle position only in terms of the space components \( r^{(i)} \equiv r_l^{(i)} \) of its position 4-vector, for \( l = 1, 3 \). In detail, recalling Eq.\([100]\) and introducing the notation

\[
\pi^{(i)} = \left( \pi_0^{(i)}, \pi^{(i)} \right),
\]

according to Dirac the instant form (for \( N \)-body systems of interacting particles) is obtained by imposing the velocity kinematic constraints \([101]\) on the free-particle canonical momenta \( \pi^{(i)}_{\text{free, } \mu} \equiv m_o \pi^{(i)}_\mu = \left( \pi^{(i)}_{\text{free},0}, \pi^{(i)}_{\text{free}} \right) \) [i.e., in the absence of an external EM field] such that

\[
\pi^{(i)}_{\text{free, } 0} = \sqrt{m_o^2 c^2 + \pi^{(i)}_{\text{free}}^2},
\]

and then introducing a suitable interaction 4-potential \( V_\mu \equiv (V_0, V) \) taking into account all the particle interactions. Letting \( l, m = 1, 3 \), this yields the \( N \)-body Dirac constrained instant-form generators \( \left( \hat{p}_0, \hat{p}_l, \hat{M}_{lm}, \hat{N}_{0l} \right) \) \([3]\), represented in terms of the constrained states \( \mathbf{z}^{(i)} = (\mathbf{r}^{(i)}, \pi^{(i)}) \) (for \( i = 1, N \)):

\[
\hat{p}_0 = \sum_{i=1,N} p_0^{(i)} = \sum_{i=1,N} \sqrt{m_o^2 c^2 + \pi^{(i)}_{\text{free}}^2} + V_0,
\]

\[
\hat{p}_l = \sum_{i=1,N} \pi^{(i)}_l,
\]

\[
\hat{M}_{lm} = \sum_{i=1,N} \left[ r_l^{(i)} \pi^{(i)}_m - r_m^{(i)} \pi^{(i)}_l \right],
\]

\[
\hat{N}_{0l} = \sum_{i=1,N} r_l^{(i)} \sqrt{m_o^2 c^2 + \pi^{(i)}_{\text{free}}^2} + V_l,
\]

with \( V_l \equiv V_0 \sum_{i=1,N} r_l^{(i)} \) and \( V_0 \) denoting the time-component of a suitable interaction potential 4-vector \( V_\mu \equiv (V_0, V) \). Here both \( \hat{p}_0 \) and \( \hat{N}_{0l} \) are still expressed in terms of the free-particle canonical momentum \( \pi^{(i)}_{\text{free}} \), while \( \hat{M}_{lm} \) differs from \( \hat{M}_{0l} \) because of the imposed kinematic constraint. Therefore, if interactions occur, their contribution show up only in \( \hat{p}_0 \) and \( \hat{N}_{0l} \). In Ref.\([3]\) the interaction-dependent Poincaré generators were called “Hamiltonians”.

Nevertheless, for the validity of the transformation laws \([129]\) as well as of the Lorentz conditions \([120]\), Eqs.\([134]-[137]\) are actually to be cast in terms of the 4-momenta of the interacting system \( \pi^{(i)} \) (rather than the free particle momenta \( \pi^{(i)}_{\text{free}} \)). This means that in general \( \pi^{(i)} \) should be considered as suitably-prescribed functions of \( \pi^{(i)}_{\text{free}} \) and of the interaction 4-potential \( V_\mu \). For definiteness, let us consider the case of an isolated \( N \)-body system subject only to binary interactions occurring between point particles of the same system. In such a case the interaction potential 4-vector \( V_\mu \) is necessarily separable \([20]\), i.e., such that

\[
V_\mu \equiv (V_0, V) = \sum_{i=1,N} V^{(i)}_\mu,
\]

with \( V^{(i)}_\mu \) denoting the \( i \)-th particle interaction potential 4-vector. Then, assuming that \( V^{(i)}_\mu \) are only position-dependent, in view of Eqs.\([134]\) and \([137]\), for each particle the canonical 4-momentum of interacting particles \( \pi^{(i)}_{\mu} \) must depend linearly on \( V^{(i)}_\mu \) and \( \pi^{(i)}_{\text{free, } \mu} \), namely it takes the form

\[
\pi^{(i)}_{\mu} = \pi^{(i)}_{\text{free, } \mu} - V^{(i)}_\mu,
\]

which implies in turn that necessarily \( \pi^{(i)}_0 = \sqrt{m_o^2 c^2 + \left( \pi^{(i)} - V^{(i)} \right)^2} + V_0^{(i)} \). As a consequence, Eqs.\([134]-[137]\) are
actually replaced with
\[
\begin{align*}
\widehat{p}_0 &= \sum_{i=1,N} p_0^{(i)} = \sum_{i=1,N} \sqrt{m_0^{(i)} c^2 + (\pi^{(i)} - \mathbf{V}(i))^2} + V_0, \\
\widehat{\pi}_l &= \sum_{i=1,N} \pi_l^{(i)}, \\
\widehat{M}_{lm} &= \sum_{i=1,N} \left[ r_l^{(i)} \pi_m^{(i)} - r_m^{(i)} \pi_l^{(i)} \right], \\
\widehat{N}_{l0} &= \sum_{i=1,N} r_l^{(i)} \sqrt{m_0^{(i)} c^2 + (\pi^{(i)} - \mathbf{V}(i))^2} + V_{0l},
\end{align*}
\]

where (141) and (142) retain their free-particle form. In order that the Poincarè generators \( \widehat{p}_0 \) and \( \widehat{\pi}_l \) commute [in accordance with Eqs. (120), with PBs now defined in terms of the constrained state \( \mathbf{z}' \)], then it follows necessarily that the 4-vectors \( V^{(i)}_\mu \), for all \( i = 1, N \), must be local functions of the 4-positions of the particles of the \( N \)-body system, namely
\[
V^{(i)}_\mu = V^{(i)}_\mu (r_1^{(i)}, ..., r_N^{(i)}).
\]

The explicit proof of this statement is given below [see the subsequent subsection #3 and in particular the inequality (164)]. Hence, by construction, in the Dirac approach the “Hamiltonians” \( \widehat{p}_0 \) and \( \widehat{N}_{l0} \) are necessarily local functions too. It follows that DGF applies only to local Hamiltonian systems.

It is worth noting that the same conclusion follows directly also from Dirac’s claim (see the quote from his paper on the \( \text{non-local Hamiltonian structure} \)). According to the discussion reported above (see Remark #7 in Section 7), it implies again that \( K_N \) must only be a local function of the system canonical state \( \mathbf{z} \).

Let us now analyze, for comparison, the implications of the theory developed in the present work for EM-interacting \( N \)-body systems.

1. **Non-local Hamiltonian structure**

The first issue is related to the Hamiltonian structure \( \{ \mathbf{z}, K_N \} \) which characterizes these systems. According to the Corollary of THM.2 this should be identified with \( \{ \mathbf{x}, H_N \} \). Thus, the super-abundant canonical state \( \mathbf{z} \) should coincide with \( \mathbf{x} \equiv \{ x^{(i)}, i = 1, N \} \) spanning the 8\( N \)-dimensional phase-space \( \Gamma_N \equiv \Pi_{i=1,N} \Gamma_{1}^{(i)} \) (with \( \Gamma_{1}^{(i)} \subset \mathbb{R}^8 \)), while \( K_N \) is identified with the non-local Hamiltonian \( H_N \equiv H_N(r, P, [r]) \). In particular, \( \mathbf{x}^{(i)} = \left( r^{(i)}_\mu, P^{(i)}_\mu \right) \) is the corresponding \( i \)-th particle canonical state, with \( r^{(i)}_\mu \) and \( P^{(i)}_\mu \) denoting respectively its position and canonical momentum 4-vectors. In the present case and in contrast to DGF, it follows that:

1. **Property #1**: the system Hamiltonian \( K_N \equiv H_N \) must be necessarily non-local.

2. **Property #2**: the super-abundant state \( \mathbf{x} \equiv \{ x^{(i)}, i = 1, N \} \) is canonical, namely it satisfies the canonical equations (1) in terms of the local PBs defined with respect to the same state. This occurs if \( \mathbf{x} \) is considered unconstrained, i.e., when the \( i \)-th particle state \( x^{(i)} \) is identified with \( \mathbf{x}^{(i)} = \left( r^{(i)}_\mu, P^{(i)}_\mu \right) \). Hence, the Hamiltonian structure \( \{ \mathbf{x}, H_N \} \) holds in the unconstrained 8\( N \)-dimensional phase-space \( \Gamma_N \equiv \Pi_{i=1,N} \Gamma_{1}^{(i)} \), with \( \Gamma_{1}^{(i)} \subset \mathbb{R}^8 \). As a consequence, also the PBs (including the fundamental PBs (103) and the Lorentz conditions (120)) are defined with respect to the unconstrained state \( \mathbf{x} \).

3. **Property #3**: only the extremal or extremant canonical curves \( \mathbf{x}(s_{(1)}, ..s_{(N)}) \) [see Eq. (103)] and not the varied functions satisfy identically the kinematic constraints (104) and (102).
4. Property #4: the 4-potential \( V_\mu \) must be necessarily a non-local function. In particular, for binary EM interactions it must be a separable function, i.e., of the form \([135]\):

\[
V_\mu(r, [r]) = \sum_{i=1, N} V_\mu^{(i)}(r, [r]),
\]

\[
V_\mu^{(i)}(r, [r]) \equiv \frac{q^{(i)}}{c} A_{(\text{eff})\mu}^{(tot)(i)},
\]

with \( A_{(\text{eff})\mu}^{(tot)(i)} \) being defined by Eq.(147).

2. Conditions of validity of Dirac instant-form generators

A further issue is related to the representation of the Poincaré generators and, in particular, to the instant form representation given by Dirac and usually adopted in the literature. The latter is based on Eqs.(134)-(137), rather than on Eqs.(140)-(143), in which \( \pi^{(i)2} \) replaces \( \pi^{(i)2, \text{res}} \), under the square root on the r.h.s. of Eqs.(134) and (137). On the other hand, based on the non-local Hamiltonian structure \( \{x, H_N\} \) expressed in terms of the unconstrained superabundant canonical state \( x \) [see Eqs.(89), (90) and (91)], an admissible realization for \( \{\hat{p}, \hat{M}^{\mu\nu}\} \) can be determined which holds for an arbitrary \( N \geq 1 \). In fact, it is immediate to verify that the phase-functions

\[
\hat{p}_\mu = \sum_{i=1, N} P_\mu^{(i)},
\]

\[
\hat{M}_{\mu\nu} = \sum_{i=1, N} \left[ P_\mu^{(i)} P_\nu^{(i)} - P_\nu^{(i)} P_\mu^{(i)} \right],
\]

satisfy identically the PBs \([126]\) expressed in terms of the same state \( x \). In view of Property #2 this requires that the canonical generators \( \{\hat{p}, \hat{M}^{\mu\nu}\} \) defined by Eq.(148) must be considered independent. Hence, no constraints (on them) can possibly arise by imposing the validity of the PBs \([126]\).

Another possibility, however, lies in the adoption of a constrained formulation. This is obtained imposing the kinematic constraints \([101]\) and identifying the canonical state with the constrained vector \( x' \equiv (x^{(i)}, i = 1, N) \) with \( x^{(i)} = (r^{(i)}, p^{(i)}) \). Recalling again Eqs.(90) and (91), here \( r^{(i)} \) and \( p^{(i)} \) denote respectively the space parts of the corresponding \( i \)-th particle 4-vectors.

To carry out a detailed comparison with Dirac, let us consider in particular the instant-form representation of \( \{\hat{p}, \hat{M}^{\mu\nu}\} \) as given by Eq.(148). In such a case the generators are represented by the set, defined for \( l, m = 1, 3 \):

\[
\hat{p}_0 = \sum_{i=1, N} P_0^{(i)},
\]

\[
\hat{p}_l = \sum_{i=1, N} P_l^{(i)},
\]

\[
\hat{M}_{lm} = \sum_{i=1, N} \left[ P_l^{(i)} P_m^{(i)} - P_m^{(i)} P_l^{(i)} \right],
\]

\[
\hat{N}_{l0} = \sum_{i=1, N} \left[ P_l^{(i)} P_0^{(i)} \right],
\]

where \( \hat{p}_0, \hat{p}_l, \hat{M}_{lm} \) and \( \hat{N}_{l0} \) must all be considered as independent. The corresponding constrained (representation of the) instant-form generators, with \( \hat{p}_0 \) and \( \hat{N}_{l0} \) expressed in terms of the constrained state \( x' \), become therefore

\[
\left. \hat{p}_0 \right|_{x'} = \sum_{i=1, N} \left[ \sqrt{m_0^{(i)} c^2} + \left( P^{(i)} - \frac{q^{(i)}}{c} A_{(\text{eff})\mu}^{(tot)(i)} \right)^2 + \frac{q^{(i)}}{c} A_{(\text{eff})\mu}^{(tot)(i)} \right],
\]

\[
\left. \hat{N}_{l0} \right|_{x'} = \sum_{i=1, N} \left[ r^{(i)} \sqrt{m_0^{(i)} c^2} + \left( P^{(i)} - \frac{q^{(i)}}{c} A_{(\text{eff})\mu}^{(tot)(i)} \right)^2 + \frac{q^{(i)}}{c} r^{(i)} A_{(\text{eff})\mu}^{(tot)(i)} \right].
\]
where we have represented \( A_{(eff)\mu}^{(tot)(i)} \equiv A_{(eff)\mu}^{(tot)(i)} \cdot A_{(eff)\mu}^{(tot)(i)} \), with \( A_{(eff)\mu}^{(tot)(i)} \) being defined by Eq. (174) setting \( A_{\mu}^{(ext)(i)} = 0 \).

A characteristic obvious feature of the constrained representations given above is that of the non-local dependences arising both from binary and self EM interactions. Analogous conclusions can be drawn also for the so-called point and front forms of the same generators. This implies that the Lorentz conditions (126), with PBs now defined in terms of the same constrained state \( x' \), are generally violated. Indeed, due to the non-locality of \( A_{(eff)\mu}^{(tot)(i)} \) in this case the PBs-inequalities

\[
[p_{0 x'}, \hat{p}_i]_{(x')} \neq 0 \tag{155}
\]

hold. Hence, if - consistent with DGF - the validity of the Lorentz conditions (126) is imposed, the constrained forms of the Poincaré generators are manifestly not applicable to the treatment of EM-interacting \( N \)-body systems.

Nevertheless, it is immediate to prove that \( \hat{p}_0 |_{x'} \) indeed generates the correct evolution equations for the constrained state \( x' \). In fact, denoting by \([\cdot, \cdot]_{(x')}\) the local PBs evaluated with respect to the constrained state \( x' \), let us determine by means of the PBs

\[
d_\alpha \xi(x) \equiv [\xi, F]_{(x')} \tag{156}
\]

the infinitesimal transformations \( d_\alpha r^{(i)} \) and \( d_\alpha P^{(i)} \) generated by \( F = dt \hat{p}_0 |_{x'} \). It is immediate to prove that these yield respectively

\[
\begin{align*}
d_\alpha r^{(i)\mu} &= dt \left[ r^{(i)\mu}, \hat{p}_0 |_{x'} \right]_{(x')} \equiv dt v^{(i)}, \tag{157} \\
d_\alpha P^{(i)} &= dt \left[ P^{(i)}, \hat{p}_0 |_{x'} \right]_{(x')} \equiv dt \frac{\alpha}{c} \nabla^{(i)} A_{(eff)\mu}^{(tot)(i)} v^{(i)\mu}, \tag{158}
\end{align*}
\]

where the r.h.s. of both equations coincide identically with the spatial parts of the canonical Eqs. (97) and (98). Hence, as expected, the constrained state \( x' \) is indeed canonical. Eqs. (157) and (158) provide the Hamiltonian equations for \( x' \) in terms of the non-local Hamiltonian function \( \hat{p}_0 |_{x'} \). Again, a necessary and sufficient condition for Eqs. (97) and (98) to hold is that the fundamental PBs

\[
\begin{align*}
\left[ r^{(i)}, r^{(j)} \right]_{(x'_0)} &= 0, \\
\left[ P^{(i)}, P^{(j)} \right]_{(x'_0)} &= 0, \\
\left[ u^{(i)}, u^{(j)} \right]_{(x'_0)} &= \delta^{ij} 1,
\end{align*}
\]

are identically satisfied for all \( i, j = 1, N \). Here \( x'_0 \) and \( x' \) are identified respectively with \( x'_0 \equiv \hat{x}'(t_o) \) and \( x' \equiv \hat{x}'(t) = x'_0 + dx', \) with \( dx' = (\delta_o x^{(1)}, ..., \delta_o x^{(N)}) \), while the previous PBs are evaluated with respect to the initial state \( x'_0 \). Furthermore, also in this case the canonical initial history set \( \{ \hat{x}' \}_0 \), to be defined in analogy with Eq. (99), is assumed prescribed.

**IX. NON-LOCAL GENERATOR FORMALISM**

A basic consequence of the previous considerations is that in the case of non-local phase-functions, such as \( H_N \) or \( \hat{p}_0 |_{x'} \), the local transformation law (129) becomes inapplicable.

A suitably-modified formulation of DGF appropriate for the treatment of non-local phase-functions must therefore be developed. This can be immediately obtained. In fact, let us consider an arbitrary non-local function of the form \( \xi = \xi(z, [z]) \), with \( z \) and \( [z] \) denoting respectively local and non-local functional dependences with respect to the canonical state \( z \). Let us consider an arbitrary infinitesimal canonical transformation generated by \( F \) of the form \( z \rightarrow z' = z + [z, F] \), with \( \delta z \) to be considered as infinitesimal (i.e., of \( O(\Delta) \)). Then, requiring that \( \xi \) is suitably smooth both with respect to \( z \) and \([z]\), the corresponding infinitesimal variation of \( \xi \) can be approximated with

\[
\delta \xi(z, [z]) \equiv [\xi(z + \alpha \delta z, [z + \alpha \delta z]) - \xi(z, [z])][1 + O(\Delta)], \tag{160}
\]

\( \delta \xi(z, [z]) \) being the (Frechet) functional derivative of \( \xi(z, [z]) \), namely

\[
\delta \xi(z, [z]) \equiv \lim_{\alpha \to 0} \frac{d}{d\alpha} \xi(z + \alpha \delta z, [z + \alpha \delta z]) = [\xi(z, [z]), F]. \tag{161}
\]
Here \( \{\xi(z, [z]), F\} \) denotes the non-local Poisson brackets (NL-PBs) and generally also \( F \) can be considered a non-local function of the form \( F(z, [z]) \) [i.e., of a type analogous to \( \xi \)]. Such a definition reduces manifestly to (129) in case of local functions.

Let us prove that the transformation law (161) is indeed the correct one. To elucidate this point, let us consider the 4-scalar defined by the Dirac-delta \( \xi(r, [r]) \equiv \delta \left( \bar{R}^{(i)\alpha} \bar{R}_{\alpha}^{(i)} - \alpha^2 \right) \) entering the non-local Lagrangian and Hamiltonian functions in the EM self-interaction, where \( \bar{R}^{(i)\alpha} \) denotes the bi-vector defined by Eq.(28). Let us consider, for example, the action of an arbitrary infinitesimal Lorentz transformation defined by \( \delta_\alpha \eta^{(i)\mu} \). In order that \( \bar{R}^{(i)\alpha} \bar{R}_{\alpha}^{(i)} \) is left invariant by the transformation (Lorentz invariance) it must be

\[
\delta \xi(r, [r]) = \{ \xi(r, [r]), F \} = 0,
\]

with \( F = -\bar{p}^\alpha a_\mu \). This means that the NL-PBs \( \{\xi(r, [r]), F\} \) defined by Eq.(161), rather than the local PBs \( \{\xi(r, [r]), F\} \), must vanish identically. In particular Eq.(161), contrary to the local variation (128), preserves the Lorentz invariance of 4-scalars and hence provides the correct transformations law. Hence, in particular, it follows that for an isolated \( N \)-body system with arbitrary \( N > 1 \):

\[
\delta H_N(r, P, [r]) = 0, \quad \delta \bar{p}_0|_{\mathbf{x}'} = 0.
\]

It is immediate to prove that Eq.(163) holds by construction for all Poincarè generators [see Eqs.(128) above], while - instead - generally

\[
\delta_o H_N(r, P, [r]) = 0, \quad \delta_o \bar{p}_0|_{\mathbf{x}'} = 0.
\]

Hence, consistent with the results indicated above, we conclude that the local transformation laws realized by the Lorentz conditions (126), which are a distinctive feature of DGF, become invalid in the case of non-local Hamiltonians.

The non-local generator formalism is therefore formally achieved by imposing modified Lorentz conditions obtained from Eqs.(126), in which the local PBs are replaced with the non-local PBs defined by Eq.(161).

In particular, the correct transformation laws for the constrained instant-form Poincarè generators [see Eqs.(151), (153) and (153) follow by imposing for the Hamiltonians [see Eqs.(153) and (154)] appropriate non-local transformation laws of the type (162), all defined with respect to the constrained state \( \mathbf{x}' \). Finally, it must be remarked that the non-local generator formalism does not affect the validity of the canonical equations of motion (157) and (158) as well as the fundamental PBs (159) indicated above for the constrained state \( \mathbf{x}' \), which remain unchanged.

**X. COUNTER-EXAMPLES TO THE “NO-INTERACTION” THEOREM**

An open problem in relativistic dynamics is related to the so-called “no-interaction” theorem due to Currie (Currie, 1963 [14]), derived by adopting the DGF, and in particular the instant form representation for the Poincarè generators (see previous Section) given in Ref.3. According to this theorem, an isolated classical \( N \)-body system of mutually interacting particles which admits a Hamiltonian structure in which the coordinate variables of the individual particles coincide with the space parts 3-vectors of the particles 4-positions and the canonical equations of motion are Lorentz covariant, can only be realized by means of a collection of free particles. This requires, in particular that “…it is impossible to set up a canonical theory of two interacting particles in which the individual particle positions are the space parts of 4-vectors”. In other words, according to the theorem, it should be impossible to formulate - in terms of a Hamiltonian system - a covariant canonical theory for an isolated system of \( N > 1 \) classical particles subject to binary interactions (see also Ref.18). The validity of the theorem was confirmed by several other authors (see for example, Beard and Fong, 1969 [23], Kracklauer, 1976 [24], Martin and Sanz, 1978 [23], Mukunda and Sudarshan, 1981 [26], Balachandran et al., 1982 [27]). Its original formulation obtained by Currie for the case of two interacting particles (\( N = 2 \)) was subsequently extended to include the case \( N = 3 \) (Cannon and Jordan, 1964 [23], first-class constraints (see Sudarshan and Mukunda, 1983 [29] and the corresponding Lagrangian proof given by Marmo et al., 1984 [30]) and the treatment of curved space-time (De Bièvre, 1986 [31] and Li, 1989 [32]). Common assumptions to these approaches are that:
1. Hypothesis #1: Both DGF and the Dirac instant form realization of the Poincarè generators apply. In particular, the Poincarè generators in the instant form, corresponding to the constrained Hamiltonian structure \( \{ z, K_N \} \), satisfy identically both to the commutation rules (120) and the kinematical constraints (111).

2. Hypothesis #2: \( K_N \) admits the Poincarè group of symmetry, i.e., it commutes with \( \{ F \} \).

3. Hypothesis #3: All particles, in a suitable proper-time interval, are not subject to the action of an external force (locally/globally isolated \( N \)-body system).

Nevertheless, the theorem has been long questioned (see for example Fronsdal, 1971 [13] and Komar, 1978-1979 [16, 19]). In particular, there remains the dilemma whether the “no-interaction” theorem actually applies at all for \( N \)-body systems subject only to non-local EM interactions. This refers in particular, to extended charged particles in the presence of binary and self EM forces. Another interesting question is whether restrictions placed by the “no-interaction” theorem actually exist for physically realizable classical systems. Several authors have advanced the conjecture that the limitations set by the Currie theorem might be avoided in the framework of constrained dynamics formulated adopting a super-abundant-variable canonical approach (see for example Komar, 1978-1979 [18] and Marmo et al., 1984 [30] and references indicated therein). In particular, to get a better understanding of interacting \( N \)-body systems, Todorov [22] and then Komar [16, 19] developed a manifestly covariant classical relativistic model for two particles, of an action-at-a-distance kind. In the Todorov-Komar model the dynamics is given in terms of two first-class constraints. An equivalent model was discovered by Droz-Vincent [34, 35] based on a two-time formulation of the classical relativistic dynamics. However, the precise identification of the Hamiltonian structure \( \{ z, K_N \} \) pertaining to \( N \)-body systems subject to EM interactions has remained elusive to date.

Here we claim that counter-examples, escaping both the assumptions and the restrictions of the “no-interaction” theorem, can be achieved, based on the classical \( N \)-body system of extended charged particles formulated here. Starting from the Corollary to THM.2, the following theorem applies.

**THM.4 - Standard Hamiltonian form of a locally-isolated 1-body system and a globally-isolated \( N \)-body system.**

In validity of THM.2 and of the definitions given by Eqs. (71)-(76), the following propositions hold:

- **T4.1** The Hamiltonian structure \( \{ x, H_N \} \) of the classical system formed by a single extended charged particle is preserved also in the particular case in which the external EM 4-potential is such that along the particle world-line \( r(1) \):

\[
A^\text{(ext)}_\mu (r(1)) = \begin{cases} 
\neq 0 & \forall s(1) \in (-\infty, s_0] \\
= 0 & \forall s(1) \in [s_0, +\infty) 
\end{cases}
\]

(locally-isolated particle).

- **T4.2** The Hamiltonian structure \( \{ x, H_N \} \) of the classical \( N \)-body system formed by extended charged particles is preserved also in the particular case in which the external EM 4-potential vanishes identically,

\[
A^\text{(ext)}_\mu (r) \equiv 0
\]

(globally-isolated \( N \)-body system).

**Proof - T4.1** The proof is an immediate consequence of the Corollary to THM.2. In fact, in the absence of an external EM field, the effective EM 4-potential \( A^{\text{(tot)}}_{\text{(eff)}\mu} \) (see Eq. (74)) simply reduces to

\[
A^{\text{(tot)}}_{\text{(eff)}\mu} = 2A^\text{(self)}_{\mu},
\]

where, in view of the requirement (167), \( A^\text{(self)}_{\mu} \) is non-vanishing also in the interval \([s_0, +\infty)\) (see related discussion in Paper I). Hence, both the Lagrangian and Hamiltonian equations in standard form [see respectively Eqs. (68) and (80), (81)] are satisfied, with \( H^\text{(i)}_{\text{eff}} \equiv H^\text{N,eff} \equiv H_N \) still defined by Eqs. (76) and (83). T4.2 The proof is similar. In this case, due to assumption (168), \( A^{\text{(tot)}\mu} \) reduces to

\[
A^{\text{(tot)}\mu} = 2A^\text{(self)}_{\mu} + \sum_{j=1,N} A^{\text{(bin)}\mu}_{\text{eff}ij}.
\]

Hence, also in this case both the Lagrangian and Hamiltonian equations in standard form still hold, with \( H^\text{(i)}_{\text{eff}} \) and \( H^\text{N,eff} \) defined by Eqs. (76) and (83).
Q.E.D.

It is clear that both propositions T4₁ and T4₂ indeed escape the “no interaction” theorem (avoiding also the limitations set by its assumptions #1-#4). In fact, concerning the Hamiltonian structure \{x, H_N\} associated to the classical \(N\)-body system of extended charged particles, from THM.4 it follows that:

- The effective Hamiltonian is a non-local function of the canonical state \(x\).
- The canonical particle equations of motion (88) satisfy the correct transformation laws with respect to the Poincaré group, since the non-local system Hamiltonian \(H_N(\mathbf{r}, \mathbf{P}, [\mathbf{r}])\) is by construction a Lorentz 4-scalar.
- The super-abundant canonical state \(x \equiv \{x^{(i)}, i = 1, N\}\) is defined in terms of \(x^{(i)} \equiv \left(r^{(i)\mu}, P^{(i)}_\mu\right)_{(s^{(i)})}\), where \(r^{(i)\mu}\) and \(P^{(i)}_\mu\) are represented by Eqs. (90) and (91).
- The extreant curves \((x^{(1)}(s^{(1)}),...x^{(N)}(s^{(N)})\)) solutions of Eqs. (123) satisfy identically the kinematic constraints (101) and (102). As a consequence, only the space parts of the extreant 4-vectors \(r^{(i)\mu}(s^{(1)})\) and \(P^{(i)}_\mu(s^{(1)})\) are, for all \(i = 1, N\), actually independent.
- In case of T4₁, the single-particle motion is non-inertial for all \(s^{(1)} \in I \equiv \mathbb{R}\). Hence, the instant form of Dirac generators (see Eqs. (134) and (137)) becomes inapplicable even in the case of a 1-body system.

XI. ON THE FAILURE OF THE “NO-INTERACTION” THEOREM

The actual causes of the failure of the “no-interaction” theorem emerge clearly from the analysis of the conditions of validity of DGF and the Dirac instant-form generators (see Section 8). For systems of extended charged particles subject only to EM interactions the previous assumptions #1-#4 (common to all customary approaches [13, 23–31]) which characterize the underlying Hamiltonian structure \(\{z, K_N\}\) make it incompatible with the exact non-local Hamiltonian structure \(\{x, H_N\}\) determined here. In fact, in difference to \(\{z, K_N\}\), the Hamiltonian structure \(\{x, H_N\}\) is characterized by:

- Super-abundant canonical variables \(x \equiv \{x^{(i)}, i = 1, N\}\), with \(x^{(i)} \equiv \left(r^{(i)\mu}, P^{(i)}_\mu\right)_{(s^{(i)})}\) being the \(i\)-th particle canonical state.
- Extreant curves \((x^{(1)}(s^{(1)}),...x^{(N)}(s^{(N)})\)) which satisfy identically the kinematic constraints discussed above. This is a characteristic property of the canonical extreant curves only. In fact, the same constraints are not satisfied by the super-abundant canonical state \(x\).
- Fundamental PBs (104) which are satisfied only by the unconstrained state \(x\). Hence, the non-local Hamiltonian structure \(\{x, H_N\}\) is warranted if all the canonical variables defining the state \(x \equiv \{x^{(i)}, i = 1, N\}\) are considered independent. This means that, in order for the fundamental PBs (104) to be fulfilled, these constraints cannot be imposed “a priori” on the canonical state.
- Poincaré generators (see Eqs. (149)-(152)) which satisfy the commutation rules (126) when they are considered independent, as the super-abundant canonical variables \(x \equiv \{x^{(i)}, i = 1, N\}\), and fulfilling the fundamental PBs. For this reason, the Poincaré generators are necessarily left unconstrained by imposing the validity of the same equations [i.e., Eqs. (126)].
- A non-local Hamiltonian of the form \(H_N(\mathbf{r}, \mathbf{P}, [\mathbf{r}])\). In particular, it follows that \(H_N\) for classical \(N\)-body systems of extended charged particles reduces to a local function only in the case of a single isolated particle which exhibits inertial motion. In view of THM.1 given in Paper I, this requires the external EM 4-potential acting on such a particle to vanish identically along the particle world-line, i.e., \(A^{(\text{ext})}_\mu(r(s_1)) = 0\) for all \(s_1 \in I \equiv \mathbb{R}\).

In conclusion, contrary to the claim of the “no-interaction” theorem, a Lorentz covariant Hamiltonian formulation for the dynamics of \(N\)-body systems, with \(N \geq 1\), actually exists also for mutually interacting charged particles subject to binary as well as self EM interactions. The result holds even in the presence of an external EM field, for extended classical particles described by the Hamiltonian structure \(\{x, H_N\}\) determined here.
One might conjecture that the validity of the “no-interaction” theorem could be restored by introducing a suitable asymptotic approximation for the $N$-body system dynamics. The latter is related, in particular, to the short delay-time and large-distance approximations (see Section 7), invoked here for the treatment of particle self and binary EM interactions. It is immediate to prove that also this route is necessarily unsuccessful. The reason lays in THM.3 and its consequences. In fact, as shown above, a Hamiltonian structure of the same type of $\{\mathbf{x}, H_N\}$ can be recovered for the asymptotic $N$-body equations of motion determined by the same theorem. This is identified with the set $\{\mathbf{x}, H_{N,\text{eff}}^{\text{asym}}\}$, where $H_{N,\text{eff}}^{\text{asym}} = \sum_{i=1}^N H_{\text{eff,asym}}^{(i)}$ and $H_{\text{eff,asym}}^{(i)}$ is given by Eq. (120). By construction, $\{\mathbf{x}, H_{N,\text{eff}}^{\text{asym}}\}$ inherits the same qualitative properties of the exact Hamiltonian structure $\{\mathbf{x}, H_N\}$. Therefore, in particular, in this approximation $\mathbf{x}$ satisfies the fundamental PBs (104) if it is unconstrained. In addition, since the same definition applies for the Poincaré generators and their representation in the instant form, the same conclusions on the validity of the “no-interaction” theorem follow.

XII. CONCLUSIONS

A formidable open problem in classical mechanics is provided by the missing consistent Hamiltonian formulation for the dynamics of EM-interacting $N$-body systems. This critically affects both classical and quantum mechanics. In this paper a solution to this fundamental issue has been reached exclusively within the framework of classical electrodynamics and special relativity. In particular, the Hamiltonian structure of classical $N$-body systems composed of EM-interacting finite-size charged particles has been explicitly determined and investigated.

Both local and non-local EM interactions have been retained. The former are due to externally-prescribed EM fields, while the latter include both binary and self EM interactions, both characterized by finite delay-time effects. Binary interactions occur between any two charges of the $N$-body systems, while self interactions ascribe to the so-called radiation-reaction phenomena due to action of the EM self-field on a finite-size particle. All of these contributions have been consistently dealt with in the derivation of the $N$-body dynamical equations of motion by means of a variational approach based on the hybrid synchronous Hamilton variational principle.

Both Lagrangian and Hamiltonian covariant differential equations have been obtained, which are intrinsically of delay-type. The same equations have also been proved to admit a representation in both standard Lagrangian and Hamiltonian forms, through the definition of effective non-local Lagrangian and Hamiltonian functions. The property of Hamilton equations of admitting a Poisson bracket representation has lead us to prove the existence of a non-local Hamiltonian structure $\{\mathbf{x}, H_N\}$ for the $N$-body system of EM-interacting particles. This has been shown to be determined by the non-local Hamiltonian function $H_N$ and to hold for the superabundant canonical states $\mathbf{x}$. In particular the correct Hamiltonian equations of motion are obtained considering the same vector $\mathbf{x}$ as unconstrained, the relevant (kinematic) constraints being satisfied identically by the solution of the same equations.

A further interesting development concerns the asymptotic approximation determined for the Hamiltonian structure $\{\mathbf{x}, H_N\}$ of the full $N$-body problem. Here we have shown that consistent with the short delay-time and large-distance asymptotic orderings the latter can be preserved also by a suitable asymptotic Hamiltonian approximation. In particular, the perturbative expansion adopted here permits to retain consistently delay-time contributions, while preserving also the variational character and the standard Lagrangian and Hamiltonian forms of the $N$-body dynamical equations. As a basic consequence the very Hamiltonian structure of the $N$-body problem is warranted. This permits us to overcome the usual difficulties related to the adoption of non-variational and non-Hamiltonian approximations previously developed in the literature.

Two important applications of the theory have been pointed out.

The first one concerns the famous and widely cited (both in the context of classical and quantum mechanics) paper by Dirac (1949) on the generator formalism approach to the forms of the Poincaré generators for the inhomogeneous Lorentz group. Contrary to a widespread belief, we have found out that the Dirac approach is not valid in the case of $N$-body systems subject to retarded, i.e., non-local, interactions. In fact, the Lorentz conditions for the instant-form Poincaré generators are found to be satisfied only in the case of local Hamiltonians. Analogous conclusions can be drawn also for the so-called point and front-forms of the same generators. Due to the non-local character of the Hamiltonian structure $\{\mathbf{x}, H_N\}$ this means that the Dirac generator formalism expressed in terms of the essential (i.e., constrained) canonical state $\mathbf{x}'$ is not “per se” directly applicable to the treatment of EM-interacting $N$-body systems. However, as shown here, in the same variables its extension to non-local Hamiltonians can be readily achieved by suitably modifying the Lorentz conditions so to account for the non-local dependences of the Hamiltonian structure $\{\mathbf{x}, H_N\}$.

Second, the validity of the Currie “no-interaction” theorem, concerning the Hamiltonian description of the relativistic dynamics of isolated interacting particles, has been investigated. It has been proved that the set $\{\mathbf{x}, H_N\}$ violates the statements of the theorem. The cause of the failure of theorem (and its proof) lays precisely in the adoption of the
Dirac generator formalism. Explicit counter-examples which overcome the limitations posed by the “no-interaction” theorem have been issued. Contrary to the claim of the “no-interaction” theorem, it has been demonstrated that a standard Hamiltonian formulation for the N-body system of charged particles subject to EM interactions can be consistently formulated.

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Appendix A: Evaluation of the action integral of the binary interaction

In this Appendix the mathematical details of the calculation of the action integral \( S^{(bin)(ij)}_C(r, [r]) \) are given. The latter contains the information about the binary EM interactions among the charged particles of the N-body system and it has been defined by Eqs. (14) and (15) in Section 4-A. To proceed with the calculation we first notice that, invoking the definition of the current density given by Eq. (4), the functional (15) can be equivalently represented as

\[
S^{(bin)(ij)}_C(r, [r]) = \frac{q^{(ij)}}{4\pi\sigma^2} \int_{1}^{2} d\Omega A^{(self)}(s^{(ij)})(\mu)(r) \int_{-\infty}^{\infty} ds_{2} \delta(s_{2} - s_{1(j)}) \times
\]

\[
\times \int_{-\infty}^{\infty} ds_{(j)} u^{(j)}(\mu)(s_{(j)}) \delta(|x (s_{(j)})| - \sigma_{(j)}) \delta(s_{(j)} - s_{2}),
\]

where \( s_{1(j)} \) is the root of the equation

\[
u^{(j)}_{\mu}(s_{1(j)}) \left[ r^{\mu} - r^{(j)\mu}(s_{1(j)}) \right] = 0.
\]

Furthermore, because of the principle of relativity, the integral (A1) can be evaluated in an arbitrary reference frame. The explicit calculation of the integral (A1) is then achieved, thanks to Lemma 3 given in Paper I, by invoking a Lorentz boost to the reference frame \( S_{N1} \) moving with 4-velocity \( u_{\mu}(s_{2}) \). In this frame, by construction \( d\Omega' = cdt'dx'dy'dz' \equiv d\Omega \). In particular, introducing the spherical spatial coordinates \((ct', \rho', \vartheta', \varphi')\) it follows that the transformed spatial volume element can also be written as \( cdt'dx'dy'dz' \equiv cdt'd\rho'd\vartheta'd\varphi'\rho^{2}\sin\vartheta' \). In such a reference frame the previous scalar equation becomes

\[
u^{(j)'}_{\mu}(s_{1(j)}) \left[ r'^{\mu} - r'^{(j)\mu}(s_{1(j)}) \right] = 0.
\]

On the other hand, performing the integration with respect to \( s_{2} \) in Eq. (A1), it follows that necessarily \( s_{2} = s_{1(j)} \), so that from Eq. (A3) \( s_{1(j)} \) is actually given by

\[s_{1(j)} = ct' = s_{2}.
\]

As a result, the integral \( S^{(bin)(ij)}_C \) reduces to

\[
S^{(bin)(ij)}_C(r', [r']) = \frac{q^{(ij)}}{4\pi\sigma^2} \int_{1}^{2} dx'dy'dz' A^{(self)}(s^{(ij)})(r') \int_{-\infty}^{\infty} ds_{(j)} u^{(j)}(\mu)(s_{(j)}) \delta\left(|x^{(j)}(s_{(j)})| - \sigma_{(j)}\right),
\]

with \( x^{(j)}(s_{(j)}) = r^{\mu} - r^{(j)\mu}(s_{(j)}) \). Moreover

\[
A^{(self)}_{\mu}(s^{(ij)})(r') = 2q^{(ij)} \int_{-\infty}^{\infty} ds_{(j)} u_{\mu}(s_{(j)}) \delta(R^{(i)\alpha}_{\mu} R^{(j)\alpha}_{\mu}),
\]
with \( \hat{R}^{(i)\alpha} = r^{\alpha} - r^{(i)\alpha}(s''_{(i)}) \).

Hence, \( S_{C}^{(bin)(ij)} \) reduces to the functional form:

\[
S_{C}^{(bin)(ij)}(r', [r']) = \frac{2\sigma_{(ij)}^{(i)}}{4\pi \sigma_{(ij)}^{(j)}} \int_{0}^{\pi} d\theta' \sin \theta' \int_{0}^{2\pi} d\varphi' \int_{0}^{+\infty} d\rho' \rho'^{2} \times
\int_{-\infty}^{+\infty} ds''_{(j)} u^{(j)\mu}_{\alpha}(s'') \delta(\hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) \int_{-\infty}^{+\infty} ds_{(j)} u^{(j)\nu}_{\alpha}(s_{(j)}) \delta(x^{(j)\nu}_{\alpha}(s_{(j)}) - \sigma_{(j)}) - \sigma_{(j)}). \tag{A7}
\]

The remaining spatial integration can now be performed letting

\[
\rho' \equiv |x' (s_{(j)})|	ag{A8}
\]

and making use of the spherical symmetry of the charge distribution. The constraints placed by the two Dirac-delta functions \( \delta(\hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}) \) and \( \delta(|x' (s_{(j)})| - \sigma_{(j)}) \) in the previous equation imply that both \( \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} \) and \( |x' (s_{(j)})| \) are 4-scalars. Then, introducing the representation

\[
\hat{R}^{(i)\alpha} \equiv r^{\alpha} - r^{(i)\alpha}(s''_{(i)}) = \hat{R}^{(i)\alpha} + x^{(j)\alpha}(s_{(j)}),
\]

with

\[
\hat{R}^{(i)\alpha} = x^{(j)\alpha}(s_{(j)}) - r^{(i)\alpha}(s''_{(i)}),
\]

\[x^{(j)\alpha}(s_{(j)}) \equiv r^{\alpha} - r^{(j)\alpha}(s_{(j)}),\]

it follows that

\[
\hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} = \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} + x^{(j)\alpha}(s_{(j)}) x^{(j)\nu}_{\alpha}(s_{(j)}) + 2\hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) = \tag{A12}
\]

is necessarily a 4-scalar independent of the integration angles \((\varphi', \theta')\) when evaluated on the hypersurface \( \Sigma : \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} = 0 \). Similarly, the Dirac-delta \( \delta(|x^{(j)\nu}_{\alpha}(s_{(j)})| - \sigma_{(j)}) \) warrants that \( x^{(j)\alpha}(s_{(j)}) x^{(j)\nu}_{\alpha}(s_{(j)}) = -\sigma_{(j)}^{2} \), which is manifestly a 4-scalar too. Let us now prove that necessarily

\[
\hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) \equiv 0. \tag{A13}
\]

In fact, on \( \Sigma \) it must be

\[
\frac{d}{ds_{(j)}} \left[ \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} \right] = \frac{d}{ds_{(j)}} \left[ \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} \right] = 0, \tag{A14}
\]

\[
\frac{d}{ds_{(j)}} \left[ \hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) \right] = u^{(j)\alpha}(s_{(j)}) x^{(j)\nu}_{\alpha}(s_{(j)}) - \hat{R}^{(i)\alpha} u^{(j)\nu}_{\alpha}(s_{(j)}) =
\]

\[
= -\hat{R}^{(i)\alpha} u^{(j)\nu}_{\alpha}(s_{(j)}) = -\frac{1}{2} \frac{d}{ds_{(j)}} \left[ \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} \right], \tag{A15}
\]

\[
\frac{d}{ds''_{(i)}} \left[ \hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) \right] = -u^{(i)\alpha}(s''_{(i)}) x^{(j)\nu}_{\alpha}(s_{(j)}), \tag{A16}
\]

\[
\frac{d}{ds''_{(j)}} \left[ \hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) \right] = -2\hat{R}^{(i)\alpha} u^{(j)\nu}_{\alpha}(s''_{(i)}). \tag{A17}
\]

Therefore,

\[
\frac{d}{ds_{(j)}} \left[ \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} \right] = \frac{d}{ds_{(j)}} \left[ \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} + 2\hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) \right] = 0, \tag{A18}
\]

\[
\frac{d}{ds''_{(i)}} \left[ \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} \right] = \frac{d}{ds''_{(i)}} \left[ \hat{R}^{(i)\alpha} \hat{R}^{(j)\nu}_{\alpha} + 2\hat{R}^{(i)\alpha} x^{(j)\nu}_{\alpha}(s_{(j)}) \right] =
\]

\[
= -2\hat{R}^{(i)\alpha} u^{(j)\nu}_{\alpha}(s''_{(i)}) - 2u^{(i)\alpha}(s''_{(i)}) x^{(j)\nu}_{\alpha}(s_{(j)}) = 0. \tag{A19}
\]
from which it follows that, on $\Sigma$, $\tilde{R}^{(ij)\alpha}$ is a 4-vector, since by definition both $u^{(i)r}_\alpha\left(s''_{(i)}\right)$ and $x^{(j)r}_\alpha\left(s_{(j)}\right)$ are 4-vectors too. Now we notice that

$$\tilde{R}^{(ij)\alpha}\tilde{R}^{(ij)\beta} = f \left(s_{(j)}, s''_{(i)}\right) = f \left(s''_{(i)}, s_{(j)}\right), \quad (A20)$$

with $f$ being a 4-scalar which is symmetric with respect to $s_{(j)}$ and $s''_{(i)}$, while by construction

$$\tilde{R}^{(ij)\alpha}x^{(j)\mu}_\alpha\left(s_{(j)}\right) = g \left(s_{(j)}, s''_{(i)}, \sigma_{(j)}\right) \neq g \left(s''_{(i)}, s_{(j)}, \sigma_{(i)}\right), \quad (A21)$$

where $g$ is a non-symmetric 4-scalar with respect to the same parameters. On the other hand, Eq. $(A14)$ requires that $\tilde{R}^{(ij)\alpha}\tilde{R}^{(ij)\beta}$ must be symmetric in both $s_{(j)}$ and $s''_{(i)}$, so that, thanks to Eqs. $(A20)$ and $(A21)$, we can conclude that $g$ is a constant 4-scalar. To determine the precise value of $\tilde{R}^{(ij)\alpha}x^{(j)\mu}_\alpha\left(s_{(j)}\right)$ we evaluate it in the $j$-th particle COS co-moving reference frame, where by definition $s^{(j)\mu}_{COS}(s_0) = (s_0, 0)$ for all the COS proper times $s_0 \in [-\infty, +\infty]$. In this frame $\tilde{R}^{(ij)\alpha} = \left(s_{(j)} - s''_{(i)}, 0\right)$ has only time component and when $s_0 = s_{(j)}$ we get $g = \tilde{R}^{(ij)\alpha}x^{(j)\mu}_\alpha\left(s_{(j)}\right) = 0$ identically. On the other hand, since $g$ is a 4-scalar, it is independent of both $s_{(j)}$ and $s''_{(i)}$ and it is null when $s_0 = s_{(j)}$, we conclude that it must be null for all $s_0$ and in any reference frame, which proves Eq. $(A13)$.

Hence, as a result of the integration, the action integral $S^{(bin)(ij)}_C$ carrying the interaction of particle $i$ on particle $j$ takes necessarily the expression

$$S^{(bin)(ij)}_C (\rho', [\rho']) = \frac{2q^{(i)}q^{(j)}}{c} \int_1^2 \int_1^2 d\nu^{(i)\nu}_s (s''_{(i)}) \int_1^2 d\nu^{(j)\nu}_s (s'_{(j)}) \delta (\tilde{R}^{(ij)\alpha} - \sigma^\alpha_{(i)})) \quad (A22)$$

Finally, since by construction $S^{(bin)(ij)}_C$ is a 4-scalar, by dropping the primes and replacing $s''_{(i)}$ and $s'_{(j)}$ respectively with $s_{(i)}$ and $s_{(j)}$, the result reported in Eq. $(13)$ is recovered.

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