Dissipative dynamics of atom–field entanglement in the ultrastrong-coupling regime

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Abstract
The dynamics of atom–field entanglement for a system composed of two atoms resonantly coupled to a single mode leaky cavity field has been investigated beyond rotating wave approximation (RWA). By using a monogamic relation for the entanglement of formation (EOF) as well as the lower bound of the EOF for bipartite mixed states in higher dimensions, contrary to the RWA case, the atom–field system in the steady states is found to be entangled in the strong-coupling regime and the entanglement can grow as a function of atom–field coupling strength.

Keywords: entanglement, Rabi model, dissipation, rotating wave approximation

(Some figures may appear in color only in the online journal)

1. Introduction

The light–matter interaction is one of the fundamental problems investigated in many aspects of modern physics ranging from quantum optics to quantum information processing and condensed matter physics. The simple description of the quantum light–matter interaction, namely the interaction of a qubit with a harmonic oscillator, is given by the Rabi [1] model. Indeed, this simple model has been used recently to describe many physical phenomena, such as an electron coupled to a phonon mode [2], an atom interacting with an electromagnetic field in a cavity [3], a superconducting qubit interacting with a nanomechanical resonator [4], a transmission line resonator [5] and an LC circuit [6], etc [7]. Due to the excitation non-conserving nature of the Rabi model, it is hard to obtain an elegant formula for its eigen-spectrum; only recently was Braak [8] given the spectrum in terms of a solution of a transcendental equation with a single variable. For the weak atom–field coupling regime and the nearly resonant condition, it is legitimate to drop the excitation non-conserving terms, the so-called counter-rotating terms (CRTs), in the atom–field interaction. This approximation is known as the rotating wave approximation (RWA) and has been well applied recently in most optical settings both experimentally and theoretically. Under the RWA, the Rabi model reduces to the Jaynes–Cummings model [9] which is exactly solvable. On the other hand, with the recent technological advances in the circuit and cavity quantum electrodynamic systems [4–7], it is now possible to engineer the strong atom–field couplings which makes it crucial to consider the effects of CRTs on the dynamics of atom–field systems.

In the present study, we consider a model of two non-interacting qubits strongly coupled to a harmonic oscillator in a dissipative cavity. Our main focus is to investigate the effects of CRTs on the dissipative dynamics of entanglement between one of the atoms and the cavity mode by using the recently derived Lindblad-type quantum optical master equation [10] which is adapted to the strong-coupling regime under Born and Markov approximations. The entanglement between atoms in the same setting has recently been investigated by many groups in the strong-coupling regime by using different master equation approaches [11–17]. It has been shown that the entanglement between the atoms displays richer dynamical features in comparison to the RWA one. In particular, CRTs are found to lead to steady state entanglement between the atoms even for the initial states (including the vacuum state) that have no overlap with the decoherence free subspace (DFS) of the system [11]. In contrast to the previous studies [11–17], we focus on the entanglement for the atom–field system in the strong-coupling regime. Under RWA
conditions, even the entanglement between the atoms can reach a high non-zero value in the long-time limit for certain initial states, and the cavity mode always becomes separable from the atoms in the steady states. By using the monogamic relation for the entanglement of formation (EOF) [18, 19] as well as the lower bound of the EOF for bipartite mixed states in higher dimensions [20], contrary to the RWA case, we show that the atom–field system in the steady states can become entangled in the strong-coupling regime and the entanglement can grow as a function of the atom–field coupling constant.

2. The model

The Rabi [1] Hamiltonian that describes the strong interaction of two identical qubits (A and B) with a quantized cavity field $F$ can be written as ($\hbar = 1$)

$$H_R = \frac{1}{2} \omega_a \sum_{j=A,B} \sigma_j^+ a + \omega a^\dagger a + g \sum_{j=A,B} \sigma_j^z (a + a^\dagger),$$  
(1)

where $\omega_a$ ($\omega$) is the qubit transition (cavity field) frequency, $a$ ($a^\dagger$) is the annihilation (creation) operator of the cavity field, $\sigma_j^z$ ($\sigma_j^+$) is the Pauli matrix in the $z$ ($x$) direction, and $g$ is the coupling constant between the atom and the cavity field. The terms $\sigma_j^z a$, $\sigma_j^z a^\dagger$ in the interaction part of the Hamiltonian are called CRTs and they do not conserve the excitation number; even the atoms and the field are initially in their ground states, and the time evolution of the mean excitation number is non-zero [12, 21]. On the other hand, in the weak-coupling regime where $g \ll \omega_a$, $\omega$, the effects of CRTs are negligible and these terms can be dropped from the Hamiltonian [12, 22]. In this regime, the mean number of photons and the cavity field is determined by the quantum fields and the atom–field coupling constant. This approximation is known as the RWA which leads to the two-qubit Jaynes–Cummings Hamiltonian [9]

$$H_{JC} = \frac{1}{2} \omega_a \sum_{j=A,B} \sigma_j^+ a + \omega a^\dagger a + g \sum_{j=A,B} \left[ \sigma_j^z a + \sigma_j^z a^\dagger \right].$$  
(2)

The photon loss due to the imperfections of the cavity mirrors should be taken into account. Under Born and Markov approximations\(^1\), the non-unitary dynamics of atom–field system is generally represented by the ‘standard’ Lindblad master equation

$$\dot{\rho} = -i[H_{JC}, \rho] + \kappa D[a] \rho,$$  
(3)

where $D[m] \rho = \frac{1}{2} \left( 2m m^\dagger - m^\dagger m - m m^\dagger \right)$, $\kappa$ is the photon leakage rate. In the weak atom–field coupling regime where the RWA can be applied (equation (2)), the master equation (3) has been well studied and can be used accurately to describe many cavity and circuit QED systems. Contrary to the numerous investigations of the master equation (3) in the strong atom–field coupling regime by using the Rabi Hamiltonian (1) in an ad hoc manner (see [11] and references therein), it is, in fact, at the root of unphysical effects [10]; it creates excess photons in the atom–field system and produces spurious qubit flipping. The main reason

\(^{1}\) The principal system comprised of atoms and the cavity field is assumed to weakly interact with an environment which is the source of non-unitary dynamics. The Born and Markov approximations are considered between the open system and environment that leads to the Lindblad-type master equation.

for the unphysical predictions obtained from the standard Lindblad master equation in the strong-coupling regime is to neglect the qubit–field interaction ($g \to 0$) while deriving the dissipator part of equation (3). Although the error introduced by the approximation ($g \to 0$) is negligible under the RWA, the qubit–field interaction becomes influential in the strong-coupling regime and causes the unphysical effects. Recently, a new Lindblad master equation, which carefully considers the transitions among the eigenstates of the Rabi Hamiltonian (1), has been given [10]. It can be written as [10]

$$\dot{\rho} = -i[H_R, \rho] + \sum_{j,k > f} \Gamma^k_j D[\rho, \omega_j, \omega_f] \rho,$$  
(4)

where $\Gamma^k_j = \kappa \omega_f \omega_j |\langle j | a + a^\dagger | k \rangle|^2$ are the relaxation coefficients [22], $|j \rangle$ and $|k \rangle$ define the spectrum of the Hamiltonian (1), i.e. $H_R |j \rangle = \omega_j |j \rangle$ and the eigenstates should be labeled according to increasing energy (label $j$ such that $\omega_0 > \omega_j$ for $k > j$). Also, note that as $g \to 0$, the standard dissipator (3) can be recovered from equation (4). Here, we call the equation (4) the ‘improved’ Lindblad master equation which, indeed, cures the unphysical effects originated from the use of the ad hoc master equation (3) in the strong atom–field coupling regime [10].

For the numerical solution of the master equation (4) with the Rabi model, we have considered basis vectors of type $|i \rangle_{AB} |n \rangle_F \equiv |i j n \rangle$ where $i, j \in \{ e, g \}$ are the excited and ground states of the qubit and $n = 0, 1, . . . , d - 1$. The field Fock space is truncated for large enough $d$ until the results converge [11]. For the sake of simplicity, we will consider the resonant case, $\omega_f = \omega_a = \omega$, and the field initially prepared in its ground state $|0 \rangle_F$.

Now, we investigate the dissipative dynamics of entanglement between one of the atoms (say A) and the cavity mode F in the strong-coupling regime. The reduced density matrix of the A–F system can be obtained by taking a partial trace of $\rho$ over the remaining part B: $\rho_{AF} = Tr_B \rho$. In practice, the considered system has a dimension of $(2 \otimes \infty)$. Since we apply a convergence criteria to the field Fock space to solve the master equation (4), the atom–field system becomes a qubit–qudit $(2 \otimes d)$ system.

3. Atom–field entanglement in the ultrastrong-coupling regime

Entangled states are vital for many quantum computation and communication protocols. The EOF is one of the most meaningful and physically motivated measures of entanglement. For two-qubit states, the EOF can be calculated directly through the entanglement monotone, concurrence [23]. For higher dimensional bipartite mixed states, except in some special cases\(^2\), the evaluation of the EOF is a formidable task which can require highly complex optimization process. Therefore, recent efforts have led to several bounds to estimate the amount of entanglement in high dimensional quantum systems [20]. In the present study, we will use the

\(^{2}\) One of the special cases is the analytic calculation of the EOF for a family of $(2 \otimes d)$-dimensional bipartite mixed states reduced from tripartite $(2 \otimes 2 \otimes d)$ pure states [18, 19] which will be discussed later in the paper.
recently proposed lower bound of the EOF [20] to estimate the amount of entanglement for the atom–field states in the strong-coupling regime. Its definition is based on the comparison of two strong separability criteria: the positive partial transpose and the realignment criteria. It was shown that the EOF is tightly bounded for the qubit–qudit system as [20]

\[
E(\rho_{AF}) \geq \begin{cases} 
0 & \text{if } \Lambda = 1, \\
H_2\left[\frac{1}{2} \left(1 + \sqrt{1 - (\Lambda - 1)^2}\right)\right] & \text{if } \Lambda \in [1, 2],
\end{cases}
\]

where \( \Lambda = \max(\|\rho_{AF}^T\|, \|R(\rho_{AF})\|) \), \( H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x) \) and \( \|G\| = \text{Tr}(GG^T) \) is the trace norm. The matrix \( \rho_{AF}^T \) is the partial transpose with respect to the subsystem A. \( \rho_{AF}^T \geq 0 \) is, in fact, sufficient for the separability of bipartite (2 \otimes 2) and (2 \otimes 3) states. \( R(\rho_{AF}) \) is the realigned version of \( \rho_{AF} \) [20] and is another operational criteria for separability which satisfies \( \|R(\rho_j)\| \leq 1 \) for any bipartite separable state \( \rho_j \).

We first investigate the dissipative dynamics of entanglement between the atom A and cavity mode F under RWA conditions. The dynamics of the atom–field system with cavity decay under the RWA given by equation (3) have been thoroughly investigated recently [11, 24]. It was shown that the atom–field steady states have a simple structure determined solely by the overlap of the atomic initial state with the substrad state (see [11, 24] for details). More precisely, the cavity decay tends to drive the atom–field system to the ground state \( |gg0\rangle \) of \( H_{BC} \). On the other hand, the population of the substrad state, \( |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|eg\rangle - |ge\rangle) \), given by \( b = \langle \Phi^- | \rho_{AB} | 0 \rangle |\Phi^-\rangle \) does not decay in the evolution process. Therefore, the atom–field system is driven to a mixed state involving only the two DFSs: the ground state \( |gg0\rangle \), and the dark state \( |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|eg\rangle - |ge\rangle) \). Depending on the atomic initial state\(^3\), the atom–field steady states can be written as [11]

\[
\rho^{SS} = \left[(1 - b) |gg\rangle \langle gg| + b |\Phi^-\rangle \langle \Phi^-| \right] \otimes |0\rangle \langle 0|. 
\]

One should note that the atom–field steady states (6) have very interesting features: (i) a necessary condition for the long-time asymptotic entanglement between the atoms is the non-zero overlap with the substrad state, i.e. \( b \neq 0 \); (ii) the atom–field steady states only depend on the initial states, so they are independent of the atom–field coupling strength, \( g \); (iii) the field mode always becomes separable from the atoms—even the atoms can reach highly entangled states for \( b \neq 0 \). For instance, for an initial state \( |\Psi(0)\rangle = |ge0\rangle \), we have in the steady states \( E(\rho_{AB}) \approx 0.35 \), while \( E(\rho_{AF}) = E(\rho_{BF}) = 0 \). The breakdown of descriptions (i) and (ii) under non-RWA conditions was analyzed in [11] where the dissipative dynamics of entanglement between the atoms have been investigated in the strong-coupling regime. In the following, we will focus on the breakdown of description (iii) and show that the atom–field system can become entangled in the steady states under non-RWA conditions.

In figure 1, we investigate the dissipative dynamics of atom–field entanglement in the strong-coupling regime by using the improved dissipator (equation (4)). Figure 1(a) shows the dynamics of the lower bound of the EOF (equation (5)) for the initial state \( |\Psi(0)\rangle = |ge0\rangle \) and for several \( g/\omega \) values at \( \kappa = 0.2 \omega \). It can be deduced from the figure that due to the interaction between the atom and field, entanglement can be induced in a short time even from an initial product state. However, the field does not become separable from the atoms in the asymptotic time limit in the strong-coupling regime. The simple example shown in figure 1(a) demonstrates the breakdown of the above description (iii) which states that \( E(\rho_{AF}) = 0 \) in the steady states under the RWA. In fact, it was conjectured [11] for the extended Werner-like atomic initial states that the atom–field steady states under non-RWA dynamics have the same form of equation (6) with just \( |gg0\rangle \) replaced by the ground state \( |\tilde{gg}0\rangle \), of the Rabi Hamiltonian (1)

\[
\rho^{SS} = (1 - b) |\tilde{gg}0\rangle \langle \tilde{gg}0| + b |\Phi^-\rangle \langle \Phi^-|. 
\]

One can immediately see from equation (7) that the atom–field entanglement, more precisely the role of the CRT, in the steady states stems from the ground state of the Rabi

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\(^3\) Here we consider initial states that have no initial coherence between ground and dark states, i.e. \( |gg\rangle / \rho_{AB}(0) |\Phi^-\rangle = 0 \). Such states correspond to a wide range of initial states, including X states [11].
Hamiltonian [25]. In figure 1(b), we analyze the ground state entanglement for the A–F system as a function of coupling strength. One should note that the ground state forms a tripartite pure state of dimensions \((2 \otimes 2 \otimes d)\). For a tripartite pure density matrix, the EOF and quantum discord (or classical correlations [18]) obey an important monogamic relation [19]

\[ E(\rho_{AF}) = D(\rho_{AB}) + S_{AB}, \]

where \(S_{AB} = S(\rho_{AB}) - S(\rho_B)\) (\(S(\rho)\) is the von Neumann entropy) and \(D(\rho_{AB})\) is the quantum discord between the atoms which can be calculated analytically for the ground state [11, 26]. Indeed, equation (8) provides the exact calculation of the EOF for a special class of higher dimensional bipartite mixed states. The line of circles in figure 1(b) displays the ground state EOF as a function of coupling strength \(g/\omega\) obtained from the monogamic relation (equation (8)). The EOF grows from zero (the RWA regime) with the coupling up to \(g/\omega \approx 0.5\) and drops as \(g/\omega\) further increases after it approaches the maximum. In figure 1(b), we also show the lower bound of the EOF (rectangular line) between A and F for the ground state and it is found to be in agreement with the exact EOF [27]. One should note that for \(g/\omega > 0.5\), the number of virtually created photons increases significantly in the ground state as coupling increases. It seems that those virtual photons partially destroy the entanglement between the qubits [11, 26] and between the atom and cavity mode as shown in figure 1(b).

4. Conclusions

The non-RWA dynamics of atom–field entanglement has been investigated for a system composed of two identical qubits resonantly coupled to a single mode leaky cavity field. The strong atom–field interaction is found to induce atom–field entanglement in the steady states contrary to the weak-coupling one where the RWA is valid.

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