Optimal control of a quantum sensor: From an analytic solution to a fast algorithm

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Quantum sensors can show unprecedented sensitivities, provided they are controlled in a very specific, optimal way. Here, we consider a spin sensor of time-varying fields in the presence of dephasing noise, and we show that the problem of finding the optimal pulsed control field can be mapped to the determination of the ground state of a spin chain. We find an approximate but analytic solution of this problem, which provides a lower bound for the sensor sensitivity, and a pulsed control very close to optimal, which we further use as initial guess for realizing a fast simulated annealing algorithm. We experimentally demonstrate the sensitivity improvement for a spin-qubit magnetometer based on a nitrogen-vacancy center in diamond.

Introduction. — Quantum systems are notoriously sensitive to external influences. This sensitivity is the core element in the development of quantum technologies, as it is the case for quantum sensing, i.e. the engineering of a sensor containing (possibly many) microscopic objects whose dynamics is dominated by quantum effects. Although a good sensitivity is in general a positive feature, it causes the sensor to couple with detrimental noise sources that induce decoherence on the quantum sensor, therefore limiting the available time for the sensor to interact with the target signal [1].

Here we introduce a method to find optimal control protocols for ac quantum sensing in the presence of dephasing noise. Such optimization problem is in general a complex classical problem. Our method, that draws from an analogy between pulsed dynamical decoupling protocols and spin glass systems, maximizes the phase acquired by the quantum sensor due to the target ac field while minimizing the noise detrimental effect. The optimal control pulsed fields yield an improved sensitivity, as we experimentally demonstrate using a spin-qubit magnetometer based on a Nitrogen-Vacancy (NV) center in diamond.

In particular, we find that the problem of optimizing the control sequence for our quantum sensor is homologous to that of finding the ground state of the classical Hamiltonian of a fictitious Ising spin glass system. The connection between optimization problems and spin glasses is a large field of research in classical disordered systems, with far-reaching connections within the physics of spin glasses [2, 3] and other frustrated models [4–6]. A general remark on the fact that optimal quantum control problems could give rise to optimization problems that fall into this category has been presented recently [7]. Indeed, optimization problems in quantum control show some degree of frustration, with terms that compete in a similar way in which ferromagnetic and anti-ferromagnetic plaquettes compete in spin glasses. We find, however, that in the specific case of finding the optimal control sequence of a qubit sensor, by trading the Ising $\mathbb{Z}_2$ spins for the continuous spins of a spherical model one gets rid of frustration altogether, and the model shows little signs of competing equilibria at low temperature, typical of replica-symmetry broken phases [8]. Since the ground state of the spherical model can be found analytically if the spectra of the signal and of the noise are known, we obtain from this both a lower bound for the sensitivity, and a quasi-optimal $\pi$-pulse sequence. The latter can be fed to a simulated annealing algorithm [9–11], in order to find the optimal sequence. Such optimal sequence shows, in agreement with the experiments, a very good sensitivity (only about 20% higher than the bound). To show the unparalleled performance of the algorithm, we run it on a Raspberry Pi microcomputer, where it takes milliseconds to find an optimal dynamical decoupling (DD) sequence which improves by a factor of 2, at least, on a standard Carr-Purcell (CP) sequence.

Sensing ac signals in noisy environments. — We consider a single spin-qubit sensor of time-varying magnetic fields, in the presence of dephasing noise. This quantum sensing task can be described as a compromise between spin phase accumulation due to the external field to be measured (target field, $b(t) \equiv bf(t)$, being $f(t)$ a normalized function), and refocusing of the non-markovian noise, obtained via dynamical decoupling (DD) protocols [12]. As in the Hahn’s echo, a DD sequence is implemented by applying sets of $\pi$ pulses that act as time reversal for the phase acquired by the qubit during the free evolution between the pulses, and can be described by a modulation function $g(t) : [0,T] \ni t \mapsto \{-1,1\}$.
characterized by the number $n$ and time distribution of \( \pi \) pulses (see Fig. 1b). The DD sequence is embedded within a Ramsey interferometer, hence its coherence is mapped onto the probability of the qubit to populate the excited state [1],

\[
P(T, b) = \text{Tr}[\rho |1⟩⟨1|] = \frac{1}{2} \left( 1 + \cos \varphi(T, b) e^{-\chi(T)} \right). \tag{1}
\]

Here, $\varphi$ is the phase acquired by the qubit during the sensing time $T$

\[
\varphi(T, b) = b \gamma \int_0^T dt f(t) y(t), \tag{2}
\]

with $\gamma$ the coupling to the field (e.g., the electronic gyromagnetic ratio for the NV spin.) The noise-induced decoherence function

\[
\chi(T) \equiv \int d\omega \frac{S(\omega)}{\pi \omega^2} |Y(T, \omega)|^2 \tag{3}
\]

is the convolution between the noise spectral density (NSD) $S(\omega)$ and the filter function $Y(T, \omega) = i \omega \int_0^T dt e^{-i \omega t} y(t)$. Note that we neglect the effect of the target field on the noise source [13].

**Mapping to a spin glass problem.** — The DD protocol is a very versatile control technique, especially if we consider that the space of degrees of freedom spanned by all the possible distributions of $\pi$ pulses is virtually infinite, even for a finite sensing time $T$. One of the most common DD sequences is the Carr-Purcell (CP) sequence [14], formed by a set of equidistant pulses that act as a quasi monochromatic filter. Other non equidistant sequences have been proposed and experimentally tested [15–17]. Each of these mono- and poly-chromatic filter sequences have internal degrees of freedom, that can be tuned to increase the sensing capabilities for specific target fields. However, as the complexity of the target field increases, also increases the difficulty to find a pulse sequence that successfully filters out the noise components, while still maintaining the sensitivity to the target field.

A possible approach is to use an optimization algorithm to find a sequence that optimizes a desired figure of merit, for example the sensitivity, i.e. the smallest detectable signal. This concept was proposed and demonstrated experimentally [18] for an NV center used as a quantum magnetometer. In this case the sensitivity is expressed as

\[
\eta = \frac{e^{\chi(T)}}{|\varphi(T)/b| \sqrt{T}}, \tag{4}
\]

and it encompasses both the effect of the detrimental noise, and the effect of the target ac field. Despite the achieved improvements [18], the computational complexity of the optimization problem limited its applicability.

Here, we present a new approach in which the cost function $\eta$ is redefined to mimic the Hamiltonian of a spin glass system. In this way, the continuous optimization problem that minimizes the sensitivity of the NV center as a magnetometer, is re-interpreted as a spin glass energy minimization problem. Specifically, we propose as new cost function the adimensional quantity

\[
\epsilon = \log \left( \eta \gamma \sqrt{T} \right) = \chi(T) - \log \frac{\varphi(T)}{T \gamma b}, \tag{5}
\]

which is the quantity to be minimized, and, as we will show below, it can be rewritten as a spin glass Hamiltonian with long range interaction and a peculiar logarithmic field-spin coupling.

Let us discretize the sensing time $T$ into small time intervals $\Delta t$, to obtain a sequence of times $t_i$ with $i \in 1, ..., N$ where $N = T/\Delta t$. The discretization in intervals $\Delta t$ is the smallest time separation we allow the $\pi$-pulses to be separated. Apart from the physical limit given by the experimental apparatus which gives a possible $\Delta t$, one does not expect to need in the optimal solution $\pi$-pulses separated by less than the minimum period of the waves present in the signal $b(t)$.

As introduced in the previous section, the modulation function at each of these times is $y(t_i) = \pm 1$, which dictates the sign of the phase acquired by the spin qubit during the time interval $[t_i - \Delta t, t_i]$. We can therefore write the modulation function as

\[
y(t) = \sum_{i=1}^{N=T/\Delta t} s_i H_{(i-1)\Delta t,i\Delta t}(t), \tag{6}
\]

where $s_i = \pm 1$, and $H_{[a,b]}(t)$ is the characteristic function of an interval $[a, b]$. Writing the modulation function in this way, allows us to recast Eqs. (2) and (3) respectively as $b \varphi(T) = T \gamma b \sum_{i=1}^{N} h_i s_i$, and $\chi(T) = \frac{1}{T} \sum_{i=1}^{N} J_{ij} s_i s_j$, where $h_i = \frac{1}{T} \int_{(i-1)\Delta t}^{i\Delta t} f(t) dt$ represents the interaction with a normalized target ac field, and $J_{ij} \equiv \frac{1}{2} \int d\omega \frac{1-\cos(\omega \Delta t)}{\omega^2} \cos(\omega(j-i)\Delta t) S(\omega)$ represents
the interaction with the detrimental noise. We can now express the new cost function as

$$\epsilon = \frac{1}{2} \sum_{i,j=1}^{N} J_{ij} s_i s_j - \log \sum_{i=1}^{N} h_i s_i, \quad (7)$$

that closely resembles the Hamiltonian of the Ising spin glass problem for a set of $N$ spins $s_i$. The ground state for this Hamiltonian can be used to obtain a modulation function, therefore a DD sequence, that minimizes the sensitivity $\eta$.

From an analytic solution to an efficient algorithm. — Minimizing $\epsilon$ in Eq. (7) with $\{s_i\}$ on the hypercube $\{-1, 1\}^N$ can be in general a difficult problem, since the couplings $J_{ij}$ can be of both signs. Prima facie, this is a typical spin-glass problem, and therefore one is tempted to use a simulated annealing (SA) minimization algorithm [9, 10] to find the minimum of the energy $\epsilon$. However, the performance of SA is strongly affected (if not in the final value, certainly in the time to reach it) by the starting point of the dynamics (for more details see Supplemental Material at [19]), and with this in mind it is instructive to solve the corresponding spherical model analytically.

Let us trade Ising spins $s_i = \pm 1$ for continuous spins $s_i \in \mathbb{R}$, subjected to the spherical constraint

$$\sum_i s_i^2 = N. \quad (8)$$

The cost function is still given by the same Eq. (7). Spherical models are often good mean field models of spin glasses and of their dynamics [20, 21], and this case will prove to be of similar nature despite the unusual logarithmic field coupling term. The ground state of the spherical model will then provide us with a putative minimum by the mapping $s_i \rightarrow \pi_j (s_i) \in \{-1, 1\}$. It should be clear that there is no guarantee that this is actually a minimum (not even a local minimum) of the original function on the hypercube. The minimization of Eq. (7) with the constraint in Eq. (8) gives:

$$s_j = \left( \sum_{k'} |\hat{h}_{k'}|^2 J_{k'j} + \lambda \right)^{-1/2} \sum_{k=1}^{N} e^{i \frac{2\pi k}{N} j} \frac{\hat{h}_k}{\sqrt{N}} \frac{\hat{h}_k}{J_{k'j} + \lambda} \quad (9)$$

See Supplemental Material at [19] for all the details of the spherical model. Above, we introduced the Fourier transform of the signal and noise terms: respectively, they are $\hat{h}_k = \frac{1}{\sqrt{N}} \sum_j e^{-i \frac{2\pi k}{N} j} h_j$ and $\hat{J}_{k} = \frac{1}{\sqrt{N}} \sum_j e^{-i \frac{2\pi k}{N} j} J_{j-i-j}$. The value of $\lambda$ is chosen to enforce the spherical constraint Eq. (8). Notice that the form of the solution Eq. (9) is quite intuitive: in Fourier space, the component of the spin $s_j$ is proportional to the field $\hat{h}_k$, and the coefficient of proportionality decreases precisely where the noise has more spectrum, viz. where $|\hat{J}_{k}|$ is larger. In this sense, the optimal solution is aligned with the field and orthogonal to the noise. An example solution is shown in Fig. 2. The values of $s_i$ do not form a sequence of $\pm 1$, but $\{s_i\}$ is reasonably close to the minimum of the original functional Eq. (7) over the hypercube $\{-1, 1\}^N$.

We can then use the solution in Eq. (9) as a starting point to find the optimal sequence $s_i \in \{-1, 1\}$. To do so, we first map $s_i \rightarrow \pi_j (s_i) \in \{-1, 1\}$ and then run few steps of simulated annealing algorithm moving only the domain walls, i.e. flipping only spins which are on a sign change: $s_i = -s_{i+1}$. The $\pi$-pulse sequence is, as before, the sequence of times where the spins change sign (the position of the domain walls in the spin chain). This very quickly gives very good results, as shown in Fig. 4.

Experimental methods. — While our method is general and applicable to any spin-qubit sensor, we exemplify it through experiments with a single NV center in bulk diamond with naturally abundant $^{13}$C nuclear spins, at room temperature. The ground state electron spin of the NV center can be initialized and measured by exploiting spin-dependent fluorescence, and can be coherently manipulated by microwaves [22]. We consider the two ground-state spin levels, $m_S = 0$ and $m_S = 1$, to form the computational basis of the qubit sensor $\{|0\rangle, |1\rangle\}$ (more details in the Supplemental Material at [19]). The main source of noise for the NV spin qubit derives from the collective effect of $^{13}$C impurities randomly oriented in the diamond lattice. In the presence of a relatively high bias field ($\gtrsim 150$ G), the collective effect of the nuclear spin bath on the NV spin is effectively described as a classical stochastic field, with noise spectral density (NSD) following a gaussian distribution centered at the $^{13}$C Larmor frequency $\nu_L$ [23, 24]. We previously characterized the NSD for our spin sensor using the method described in Ref. [24]. Notice that the direct coupling between the target field and the nuclear spins is negligible due to the small nuclear magnetic moment [13], and the
indirect coupling via the NV electronic spin is also negligible due to the presence of the strong bias field [24]. Therefore, the NV spin dynamics is well described by Eq. (1).

As a test case for our optimal control method versus standard control, we consider a three-chromatic target signal, with \( f(t) = \sum_{i=-1}^{+1} A_i \cos(2\pi \nu_i t) \), where \( A_i \) are the amplitudes of the Fourier components, \( \nu_i \) their frequencies (see Fig. 3b, gray curve).

In Fig. 3 we show an example of \( P(T, b) \) for a reference ac field \( b = (3.647 \pm 0.089) \mu T \), under a Carr-Purcell (CP)-type DD control, formed by \( n \) pulses with uniform interpulse spacing \( \tau = T/n \). The CP pulse sequence acts as a quasi-monochromatic filter centered at \( 1/\tau \), so that a single component of \( b(t) \) can be sensed in each experiment. \( P(n,\tau, b) \) shows collapses occurring at \( \tau \sim 1/2\nu_i \). Notice that the collapse corresponding to the frequency component \( \nu_{+1} \) (\( \tau \sim 3.448 \mu s \)) cannot be resolved from noise since the first harmonic of the filter function roughly coincides with the NSD peak (\( \nu_{+1} \approx \nu_i/3 \) [Fig. 3(b)]. In order to be able to detect the three components of such a target signal, and to filter out the NSD, we need an optimized sequence [Fig. 3(b)]. We can apply the optimization algorithm detailed before to solve this experimental sensing problem.

Results. — In order to determine the capability of the protocol to find an optimal DD sequence, we compare the sensitivity achieved by the optimized sequence with that of a CP sequence. We use three different CP sequences, each with time between pulses \( \tau = \frac{1}{2\nu_i} \), for \( i = -1, 0, +1 \). Having a previous knowledge of the NSD allows us to predict the sensitivity of the spin sensor using equations (3), (2), and (4), for any given DD sequence, and for any target AC signal \( b(t) \). In Fig. 4(a) we show the estimated values for the inverse of the sensitivity as a function of the sensing time \( T = n\tau \). Since \( \tau = \frac{1}{2\nu_i} \) is fixed for each of the CP sequences, the variation of \( T \) corresponds to a variation of the number of pulses \( n \). Notice how for \( \tau = \frac{1}{2\nu_i} \), the inverse of the sensitivity rapidly goes to zero. The estimated inverse sensitivity for the optimized sequences is also shown in Fig. 4(a). The inverse sensitivity increases as a function of \( T \), although we expect it to decrease at longer times due to decoherence. Notice that already at \( T \sim 100 \mu s \) the inverse sensitivity for the optimized sequence is higher than the best possible case achieved by the CP sequences.

In the experiment, we measure the quantity \( E \equiv \max (\partial_b P(T, b))/\sqrt{T} \), which is proportional to \( 1/\eta \) [18]. \( P(T, b) \) is an oscillating function of the field amplitude \( b \) at fixed sensing time (see Eq. 1). An example of measurements of \( P(T, b) \) as a function of \( b \) is shown in Fig. 4(b). From such measurements we extract \( E \). Note that the scaling factor \( C = \eta E \), including readout efficiency and calibration factors, is independent of the DD sequence. We estimate its value from the comparison between experimental data \( E \) and predicted values of \( \eta \) for the CP sequences, and use this value to obtain \( \eta \) for any optimized sequences. The sensitivity measured experimentally shows an excellent agreement with the expected simulated values (see Fig 4a). This demonstrates that the optimization protocol is able to find a sequence with a sensitivity that is much improved with respect to any CP sequence. See the Supplemental Material at [19] for an additional test case.
FIG. 4. (a) Inverse sensitivity in the case of a three-chromatic target signal. Blue data corresponds to the optimized sequences obtained with simulated annealing (SA). Orange, red and purple data correspond to the CP sequences each with $\tau$ as indicated in the label. The predicted values (dotted lines) for the inverse sensitivity $1/\eta$ as a function of the sensing time $T$ are shown together with the experimental values (bullets with errorbars) of $E/C$. The scaling factor $C$ was obtained from the comparison between $E$ and $1/\eta$ for the CP sequences. The good agreement between experiment and theory validates the fact that the optimized sequences improve the sensing capabilities of our spin sensor. The black dashed line corresponds to the theoretical upper bound of $1/\eta$, obtained from the solution of the spherical model in the continuum limit, i.e. $N \to \infty$ (see Supplemental Material at [19] for details on the continuum limit). (b) Probability $P(T, b)$ [Eq. (1)] as a function of the amplitude of the external magnetic field $b$, for fixed sensing times $T \simeq 152 \mu s$. Same color code as in (a). A cosine function is fitted (solid lines) to the experimental data (bullets with errorbars) in order to obtain $E = C/\eta$ (see text).

Conclusions. — We have shown that the problem of finding an optimal solution to quantum-control a single spin system for quantum sensing can be done by first finding the ground state of a solvable spherical model of classical spins and then using this as a starting point for a simulated annealing algorithm. In this way the problem of finding the optimal DD sequence of a given signal $b(t)$ can be solved in a few milliseconds on a Raspberry Pi. This opens the door to miniaturization of the control electronics, using for example a low-power processor with limited performance. Fast optimization would also enable the implementation of adaptive protocols for sensing and spectroscopy.

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I. SPHERICAL MODEL

In this section we present the solution for the ground state of the spherical model, which gives a theoretical lower bound for the sensitivity and can also be used as a *bona-fide* sequence for the dynamic decoupling, or as a starting point for the simulated annealing algorithm to run.

Let us define the Fourier transform with the symmetric convention

\[ \tilde{s}_k := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-i \frac{2\pi k j}{N}} s_j, \quad s_j = \frac{1}{\sqrt{N/2}} \sum_{k=-N/2}^{N/2} e^{i \frac{2\pi k j}{N}} \tilde{s}_k. \]  

(10)

By the reality of \( s_i \) we have \( \tilde{s}_k = \tilde{s}_{-k}^* \), so the number of independent degrees of freedom is halved (but they are now complex rather than real).

We can rewrite the cost function in terms of Fourier-transformed variables as

\[ \epsilon = \frac{1}{2} \sum_k \tilde{J}_k |\tilde{s}_k|^2 - \log \left| \sum_k \tilde{h}_k \tilde{s}_k \right|. \]  

(11)

where in particular

\[ \tilde{J}_k = \frac{1}{\sqrt{N}} \sum_n J_{i,n} e^{-i \frac{2\pi k n}{N}}. \]  

(12)

Notice that the above definition is independent of \( i \) since \( J_{ij} \equiv f(i - j) \).

We add a Lagrange multiplier term to enforce the spherical constraint and define the *free energy* to be minimized:

\[ F = \epsilon + \frac{\lambda}{2} \left( \sum_k |\tilde{s}_k|^2 - N \right). \]  

(13)

The stationarity conditions for \( F \), considering that \( \tilde{J}_k = \tilde{J}_{-k}^* \), are

\[ 0 = (\tilde{J}_k + \lambda) \tilde{s}_k - \frac{\tilde{h}_k}{\sum_{k'} \tilde{h}_{k'} \tilde{s}_{k'}}, \]  

(14)

\[ 0 = \sum_k |\tilde{s}_k|^2 - N. \]  

(15)

Defining

\[ D \equiv \sum_k \tilde{h}_k \tilde{s}_k, \]  

(16)

we solve Eq. (14) for \( \tilde{s}_k \) obtaining

\[ \tilde{s}_k = \frac{\tilde{h}_k}{D(\tilde{J}_k + \lambda)}. \]  

(17)

Using this result, we get a self-consistent equation for \( D \):

\[ D = \sum_k \frac{|\tilde{h}_k|^2}{D(\tilde{J}_k + \lambda)} \quad \Rightarrow \quad D = \sqrt{\sum_k \frac{|\tilde{h}_k|^2}{\tilde{J}_k + \lambda}}. \]  

(18)
and as a consequence

\[ \tilde{s}_k = \left( \sum_{k'} \frac{|	ilde{h}_{k'}|^2}{J_{k'} + \lambda} \right)^{-1/2} \frac{\tilde{h}_k}{J_k + \lambda}. \]  \tag{19} \]

Finally, by the inverse Fourier transform we get a set of \( s_t \in \mathbb{R} \) (Eq. (9) of the main text).

The value of \( \lambda \) has to be chosen to enforce the spherical constraint, Eq. (8). Substituting \( \tilde{s}_k \) from Eq. (17) and \( D \) from Eq. (18) in Eq. (15):

\[ \left( \sum_{k'} \frac{|	ilde{h}_{k'}|^2}{J_{k'} + \lambda} \right)^{-1} \sum_k \left| \frac{\tilde{h}_k}{J_k + \lambda} \right|^2 = N. \]  \tag{20} 

This equation has a unique solution for \( \lambda \).

The minimum value for the cost function is obtained by substituting the solution, Eq. (19), into Eq. (11), to obtain:

\[ \epsilon_{\text{min}} = \frac{1}{2}(4 - \lambda N) - \frac{1}{2} \log \left( \sum_k \frac{|	ilde{h}_k|^2}{J_k + \lambda} \right). \]  \tag{21} 

Note also that, if we take at this point the stationary point with respect to \( \lambda \), we find again the equation for the Lagrange multiplier:

\[ 0 = \frac{\partial \epsilon_{\text{min}}}{\partial \lambda} = \frac{N}{2} + \frac{1}{2} \left( \sum_{k'} \frac{|	ilde{h}_{k'}|^2}{J_{k'} + \lambda} \right)^{-1} \sum_k \left| \frac{\tilde{h}_k}{J_k + \lambda} \right|^2. \]  \tag{22} 

It is instructive to notice that, if the signal \( b(t) \) possesses only one frequency:

\[ \tilde{h}_k = \tilde{h}_{k_0} \delta_{kk_0}, \]  \tag{23} 

then the solution of the spherical model is

\[ \tilde{s}_k = \sqrt{N} \delta_{kk_0}. \]  \tag{24} 

Therefore, the DD sequence is equally spaced with times \( t_i \) such that \( t_{i+1} - t_i = \tau \) where \( 2\tau \) is the period of \( b(t) \).

**II. CONTINUUM LIMIT AND THEORETICAL BOUND**

In the limit of \( \Delta t \to 0 \) with \( T \) finite (hence \( N \to \infty \)) we can write a functional for the function \( s : [0, T] \to \{-1, 1\} \).

In this case,

\[ J_{ij} = \frac{4}{\pi} \int d\omega \frac{1 - \cos(\omega \Delta t)}{\omega^2} \cos(\omega(i\Delta t - j\Delta t))S(\omega) \to 2 \int \frac{d\omega}{\pi} \cos(\omega(t' - t))S(\omega) =: dt'J(t, t'), \]  \tag{25} 

where \( t = i\Delta t, t' = j\Delta t \), and the energy to minimize becomes

\[ \epsilon[s] = \frac{1}{2} \int_0^T dt \int_0^T dt' s(t)J(t, t')s(t') - \log \left| \int_0^T dt h(t)s(t) \right|. \]  \tag{26} 

The corresponding spherical model is defined on the functions \( s : [0, T] \to \mathbb{R} \). To take into account the spherical constraint, the energy to be minimized becomes a free energy:

\[ F[s] = \frac{1}{2} \int_0^T dt \int_0^T dt' s(t)J(t, t')s(t') - \log \left| \int_0^T dt h(t)s(t) \right| + \frac{\lambda}{2} \left( \int_0^T dt s^2(t) - T \right). \]  \tag{27} 

The (unique) minimum is found by

\[ 0 = \frac{\delta F}{\delta s(t)} = \int_0^T dt' J(t, t')s(t') - \frac{1}{\int_0^T dt h(t')s(t')} h(t) + \lambda s(t), \]  \tag{28}
which is solved as

\[ s(t) = \frac{1}{D^{1/2}} \int_0^T dt' (J + \lambda)^{-1}(t, t') h(t'). \]  

(29)

Here, \((J + \lambda)^{-1}\) is the inverse of the real and symmetric operator \(J(t, t') + \lambda \delta(t - t')\), and

\[ D = \int_0^T dt \int_0^T dt' h(t)(J + \lambda)^{-1}(t, t') h(t'). \]  

(30)

The Lagrange multiplier \(\lambda\) is obtained by solving the equation:

\[ \frac{1}{D} \int dt \int dt' h(t) (J + \lambda)^{-2}(t, t') h(t') = T. \]  

(31)

The minimum can also be written, using the solution, as

\[ \epsilon_{\text{min}} = \frac{1}{2} (1 - \lambda T) - \frac{1}{2} \log(D). \]  

(32)

Again, taking the stationary point with respect to \(\lambda\) we obtain the defining equation for the Lagrange multiplier:

\[ 0 = \frac{\partial \epsilon_{\text{min}}}{\partial \lambda} = -\frac{T}{2} + \frac{1}{2D} \int dt \int dt' h(t) (J + \lambda)^{-2}(t, t') h(t') \]  

(33)

which is equivalent to Eq. (31).

The spherical model in the continuum limit provides a theoretical lower bound for the sensitivity, because it enables the minimization of the original sensitivity Eq. (4), viz. with a control continuous in time. In principle the bound is not sharp, however, since the target space of the minimization is bigger: \(s(t)\) assumes values in \(\mathbb{R}\), and not only in \([-1, 1]\).

III. PERFORMANCE OF SIMULATED ANNEALING

Here we study the performance of Simulated Annealing (SA) algorithms for the minimization on the hypercube \(s_i \in \{-1, 1\}\) of the cost function (Eq. (7) of the main text)

\[ \epsilon = \frac{1}{2} \sum_{i,j=1}^{N} J_{ij} s_i s_j - \log \sum_{i=1}^{N} h_i s_i. \]  

(34)

The strategies that one may follow are virtually infinite; we have implemented two.

- **Algorithm 1: Vanilla Simulated Annealing.** Start at infinite temperature (i.e. from a uniformly random state) and gradually lower the temperature to zero. The temperature ramp is adjustable (we have considered several decreasing functions). At each Montecarlo step, the proposed move is a single spin flip, of a spin chosen at random uniformly.

- **Algorithm 2: Simulated Annealing guided from the solution of the Spherical Model.** Solve numerically Eq. (20) to get the ground state of the spherical model; Map this solution to a \(\{-1, +1\}\) sequence using the sign function; Start from this the annealing process with the temperature already low. At each Montecarlo step, the proposed move is a domain wall translation, i.e. the flip of a spin \(s_i\) having \(s_i - 1 = -s_i\) or \(s_i + 1 = -s_i\).

For algorithm 1, we noticed that a significant speed-up and improvement of the minimum found come from the modification of the cost function to

\[ \epsilon = \frac{1}{2} \sum_{i,j=1}^{N} J_{ij} s_i s_j - \log \sum_{i=1}^{N} h_i s_i - K \sum_{i=1}^{N-1} s_i s_{i+1} \]  

(35)

with \(K > 0\), viz. introducing a nearest-neighbour ferromagnetic term that favors the formation of aligned spin bubbles. This is reasonable, since it helps to filter out high frequencies of little physical meaning. Besides, it also reduces the
number of π pulses needed to control the quantum device, thus providing a DD sequence easier to be implemented in experiments. Notice that the ferromagnetic term is irrelevant in algorithm 2, since the number of domain walls is conserved.

Algorithm 2 is clearly preferable in terms of results and performance and we present the results for algorithm 1 (which is anyway orders of magnitude better than those previously used) to show an alternative, less guided, approach. We show the details in Fig. 5. For what concerns the final sensitivity found by the algorithm (Fig. 5a), algorithm 2 performs better: indeed, it starts already close to a minimum on the hypercube (see Fig. 2 in the main text). Besides, algorithm 2 suffers of much less fluctuations, thus it needs fewer repetitions (if not only one) to get a very good DD sequence, and does not need the determination of the optimal value of $K$. For what concerns instead the execution time in Fig. 5b, again algorithm 2 is better, since in general much less Montecarlo steps are needed, and the numerical solution of Eq. (20) can be obtained quickly thanks to the monotonicity of the functions involved. To show the superior performance of our technique we have written a C++ code that performs all these tasks on a Raspberry Pi microcomputer in less than a hundredth of a second.

IV. DETAILS ON THE EXPERIMENTAL PLATFORM

The ground state of an NV center is a spin triplet $S = 1$, naturally suited for sensing magnetic fields via Zeeman effect. The NV electronic spin presents extremely long coherence times, of the order of milliseconds at room temperature [25], due to the protective environment provided by the diamond itself. The $S = 1$ electronic spin can be initialized into the $m_S = 0$ state by addressing the NV center with green light (532 nm). This is due to an excitation–decay process involving radiative (637 nm) and non-radiative decay routes, occurring with a probability that depends on the spin projection $m_S$. This same mechanism implies that the red photoluminescence intensity of the $m_S = 0$ state is higher than the one of $m_S = \pm 1$, hence enabling to optically readout the state of the system. In addition, the internal structure of the NV center removes the degeneracy between the $m_S = \pm 1$ states and the $m_S = 0$ state, imposing a zero-field-splitting of $D_y \approx 2.87$ GHz. An external bias field, aligned with the spin quantization axis, removes the degeneracy between the $m_S = \pm 1$ states, allowing to individually address the $m_S = 0 \leftrightarrow m_S = \pm 1$ transition using on-resonance microwave radiation. By using microwave pulses with an appropriate duration, amplitude and phase, it is possible to apply any kind of gate to the single two level system. Therefore, the two level system formed by the
m_S = 0 (|0\rangle) and m_S = +1 (|1\rangle) states fulfills the requirements to be used as a qubit based magnetometer.

V. SECOND TEST CASE: MONOCHROMATIC TARGET SIGNAL

In order to reinforce our results, we repeated the analysis presented in the main text for a different target signal. If we want to detect a monochromatic target signal \( b(t) \), in most cases a Carr-Purcell CP sequence of equidistant pulses is the best way to increase the sensor’s response to that target signal and filter out the noise. This is due to the quasi-monochromatic filter function associated with a CP sequence. Assuming that \( \tau \) is the time between pulses, the filter function shows a peak centered at \( \omega/2\pi = \frac{1}{2\tau} \). However, the filter function is not exactly monochromatic, it shows harmonics at \( \omega/2\pi = \frac{1}{2(2\ell+1)\tau} \), with \( \ell \in \{1, 2, \ldots\} \). Therefore, if the frequency associated with \( b(t) \) is close to \( \nu_L/(2\ell+1) \), then a CP sequence will amplify the effect of both, the target signal and the noise, leading to not-optimal sensitivities.

Here we used the optimization algorithm described in the main text in order to obtain optimal sequences for this problem. In particular, we explored the case of a monochromatic signal with frequency \( \nu_{\text{mono}} = 39.29 \text{ kHz} \), which is close enough to \( \nu_L/11 \) so that the 5-th harmonic of the CP sequence coincides with the noise components. We used the same NSD as in the case of the main text. The experimental observable \( \mathcal{E} \) is obtained from the measurement of \( P(T, b) \) as a function of \( b \). The results for one value of the sensing time \( T \) are shown in Fig. 6(a). The predicted values of the inverse sensitivity, together with their experimental values are shown in Fig. 6(b). Similarly to the case detailed in the main text, the optimal sequences improve the sensitivity of the quantum sensor, resulting in some cases to an inverse sensitivity that is close to a twice the one from the CP sequence. In the monochromatic case explored here, the sensitivity gets worse when increasing the sensing time beyond 100 \( \mu s \). Instead the optimal solutions are able to improve the sensitivity even for times \( T > 300 \mu s \). For \( T \approx 100 \mu s \), and longer sensing times, the optimized sequences achieve higher values of \( 1/\eta \) than the maximum value achieved by a CP sequence.

FIG. 6. Results for the case of a monochromatic target signal. (a) Probability to remain in the state |1\rangle as a function of \( b \), for fixed sensing times \( T \) for an optimal DD sequence (blue), and for a CP sequence (orange). The values of the sensing time and of the number of pulses for both sequences are shown as titles of the plots. A cosine function is fitted (solid lines) to the experimental data (bullets with errorbars) in order to obtain \( 1/\eta \) (see main text). (b) Inverse sensitivity as a function of the sensing time \( T \). Blue data corresponds to the optimized sequences obtained with simulated annealing (SA). Orange data corresponds to the CP sequences with \( \tau = 12.726 \mu s \). We found a good agreement between the predicted values (dotted lines) and the experimental values (bullets with errorbars).