SKEW-SYMMETRIC OPERATORS AND REFLEXIVITY

CHAFIQ BENHIDA*, KAMILA KLIŚ-GARLICKA**, AND MAREK PTAK**,***

Abstract. In contrast to the subspaces of all C-symmetric operators, we show that the subspaces of all skew-C symmetric operators are reflexive and even hyperreflexive with the constant $\kappa(C^*) \leq 3$.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and let $B(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$.

Recall that the space of trace class operators $\tau_c$ is predual to $B(\mathcal{H})$ with the dual action $\langle T, f \rangle = \text{tr}(Tf)$, for $T \in B(\mathcal{H})$ and $f \in \tau_c$. The trace norm in $\tau_c$ will be denoted by $\| \cdot \|_1$. By $F_k$ we denote the set of all operators of rank at most $k$. Often rank-one operators are written as $x \otimes y$, for $x, y \in \mathcal{H}$, and $(x \otimes y)z = \langle z, y \rangle x$ for $z \in \mathcal{H}$. Moreover, $\text{tr}(T(x \otimes y)) = \langle Tx, y \rangle$. Let $S \subset B(\mathcal{H})$ be a closed subspace. Denote by $S_\perp$ the preanihilator of $S$, i.e., $S_\perp = \{ t \in \tau_c : \text{tr}(St) = 0 \text{ for all } S \in S \}$. A weak* closed subspace $S$ is $k$-reflexive iff rank-k operators are linearly dense in $S_\perp$, i.e., $S_\perp = [S_\perp \cap F_k]$ (see [5]). $k$-hyperreflexivity introduced in [1, 6] is a stronger property than $k$-reflexivity, i.e., each $k$-hyperreflexive subspace is $k$-reflexive. A subspace $S$ is called $k$-hyperreflexive if there is a constant $c > 0$ such that

$$\text{dist}(T, S) \leq c \cdot \sup \{ \| \text{tr}(Tt) \| : t \in F_k \cap S_\perp, \| t \|_1 \leq 1 \},$$

for all $T \in B(\mathcal{H})$. Note that $\text{dist}(T, S)$ is the infinum distance. The supremum on the right hand side of (1) will be denoted by $\alpha_k(T, S)$. The smallest constant for which inequality (1) is satisfied is called the $k$-hyperreflexivity constant and is denoted $\kappa_k(S)$. If $k = 1$, the letter $k$ will be omitted.

Recall that $C$ is a conjugation on $\mathcal{H}$ if $C : \mathcal{H} \rightarrow \mathcal{H}$ is an antilinear, isometric involution, i.e., $(Cx, Cy) = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator $T$ in $B(\mathcal{H})$ is said to be C–symmetric if $CTC = T^*$. C–symmetric operators have been intensively studied by many authors in the last decade (see [2, 3, 4,

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It is a wide class of operators including Jordan blocks, truncated Toeplitz operators and Hankel operators.

Recently, in [5], the authors considered the problem of reflexivity and hyperreflexivity of the subspace $\mathcal{C} = \{ T \in B(\mathcal{H}) : CTC = T^* \}$. They have shown that $\mathcal{C}$ is transitive and 2-hyperreflexive. Recall that $T \in B(\mathcal{H})$ is a skew-$C$ symmetric iff $CTC = -T^*$. In this paper, $\mathcal{C}^s = \{ T \in B(\mathcal{H}) : CTC = -T^* \}$ – the subspace of all skew-$C$ symmetric operators will be investigated from the reflexivity and hyperreflexivity point of view. It follows directly from the definition that $\mathcal{C}$ and $\mathcal{C}^s$ are weak$^*$ closed.

We emphasize that the notion of skew symmetry is linked to many problems in physics and that any operator $T \in B(\mathcal{H})$ can be written as a sum of a $C$–symmetric operator and a skew–$C$ symmetric operator. Indeed, $T = A + B$, where $A = \frac{1}{2}(T + CT^*C)$ and $B = \frac{1}{2}(T - CT^*C)$.

The aim of this paper is to show that $\mathcal{C}^s$ is reflexive and even hyperreflexive.

2. Preanihilator

Easy calculations show the following.

**Lemma 2.1.** Let $C$ be a conjugation in a complex Hilbert space $\mathcal{H}$ and $h, g \in \mathcal{H}$. Then

1. $C(h \otimes g)C = Ch \otimes Cg$,
2. $h \otimes g - Cg \otimes Ch \in \mathcal{C}^s$.

In [2, Lemma 2] it was shown that

$$\mathcal{C} \cap F_1 = \{ \alpha \cdot h \otimes Ch : h \in \mathcal{H}, \alpha \in \mathbb{C} \}.$$  

We will show that it is also a description of the rank-one operators in the pre-anihilator of $\mathcal{C}^s$.

**Proposition 2.2.** Let $C$ be a conjugation in a complex Hilbert space $\mathcal{H}$. Then

$$\mathcal{C}_1^s \cap F_1 = \mathcal{C} \cap F_1 = \{ \alpha \cdot h \otimes Ch : h \in \mathcal{H}, \alpha \in \mathbb{C} \}.$$  

**Proof.** To prove ”$\supset$” let us take $T \in \mathcal{C}^s$ and $h \otimes Ch \in \mathcal{C} \cap F_1$. Then

$$\langle T, h \otimes Ch \rangle = \langle Th, Ch \rangle = \langle h, CTh \rangle = \langle h, -T^*Ch \rangle = -\langle Th, Ch \rangle = -\langle T, h \otimes Ch \rangle.$$  

Hence $\langle T, h \otimes Ch \rangle = 0$ and $h \otimes Ch \in \mathcal{C}_1^s \cap F_1$.

For the converse inclusion let us take a rank-one operator $h \otimes Cg \in \mathcal{C}_1^s$. Since $Cg \otimes h - Ch \otimes g \in \mathcal{C}^s$, by Lemma 2.1 we have

$$0 = \langle Cg \otimes h - Ch \otimes g, h \otimes Cg \rangle = \langle (Cg \otimes h)h, Cg \rangle - \langle (Ch \otimes g)h, Cg \rangle = \|h\|^2 \cdot \|Cg\|^2 - \langle h, g \rangle \langle Ch, Cg \rangle = \|h\|^2 \cdot \|g\|^2 - |\langle h, g \rangle|^2.$$
Hence \(|\langle h, g \rangle| = \|h\| \|g\|\), i.e., there is equality in Cauchy-Schwartz inequality. Thus \(h, g\) are linearly dependent and the proof is finished. \(\square\)

**Lemma 2.3.** Let \(C\) be a conjugation in a complex Hilbert space \(\mathcal{H}\). Then 
\[
\mathcal{C}_1^* \cap \mathcal{F}_2 \supset \{h \otimes g + Cg \otimes C^*: h, g \in \mathcal{H}\}.
\]

**Example 2.4.** Note that for different conjugations we obtain different subspaces. Let \(C_1(x_1, x_2, x_3) = (\bar{x}_3, \bar{x}_2, \bar{x}_1)\) be a conjugation on \(\mathbb{C}^3\). Then 
\[
C_1^* = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}
\]
and 
\[
C_1 = \left\{ \begin{pmatrix} a & b * \\ c & * b \\ * c & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.
\]

Rank-one operators in \(C_1\) and in \((C_1^*)_\perp\) are of the form \(\alpha(x_1, x_2, x_3) \otimes (\bar{x}_3, \bar{x}_2, \bar{x}_1)\) for \(\alpha \in \mathbb{C}\).

If we now consider another conjugation \(C_2(x_1, x_2, x_3) = (\bar{x}_2, \bar{x}_1, \bar{x}_3)\) on \(\mathbb{C}^3\), then 
\[
C_2^* = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & -a & c \\ -c & -b & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\},
\]
and 
\[
C_2 = \left\{ \begin{pmatrix} a & * & b \\ * & a & c \\ c & b & * \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.
\]

Similarly, rank-one operators in \(C_2\) and in \((C_2^*)_\perp\) are of the form \(\alpha(x_1, x_2, x_3) \otimes (\bar{x}_2, \bar{x}_1, \bar{x}_3)\).

**Example 2.5.** Let \(C\) be a conjugation in \(\mathcal{H}\). Consider \(\tilde{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}\) the conjugation in \(\mathcal{H} \oplus \mathcal{H}\) (see [7]). An operator \(T \in B(\mathcal{H} \oplus \mathcal{H})\) is skew-\(\tilde{C}\) symmetric, if and only if \(T = \begin{pmatrix} A & B \\ D & -CA^*C \end{pmatrix}\), where \(A, B, D \in B(\mathcal{H})\) and \(B, D\) are skew-\(C\) symmetric. Moreover, rank-one operators in \(\tilde{C}_1^*\) are of the form \(\alpha(f \oplus g) \otimes (Cg \oplus Cf)\) for \(f, g \in \mathcal{H}\) and \(\alpha \in \mathbb{C}\).

**Example 2.6.** Let us consider the classical Hardy space \(H^2\) and let \(\alpha\) be a nonconstant inner function. Define \(K_{\alpha}^2 = H^2 \oplus \alpha H^2\) and \(C_\alpha h(z) = \alpha \overline{h(z)}\). Then \(C_{\alpha}\) is a conjugation on \(K_{\alpha}^2\). By \(S_\alpha\) and \(S_{\alpha}^*\) denote the compressions of the unilateral shift \(S\) and the backward shift \(S^*\) to \(K_{\alpha}^2\), respectively. Recall after [9] that the kernel functions in \(K_{\alpha}^2\) for \(\lambda \in \mathbb{C}\) are projections of appropriate kernel
functions $k_\lambda$ onto $K_2^\alpha$, namely $k_\lambda^\alpha = k_\lambda - \alpha(\lambda)\alpha k_\lambda$. Denote by $k_\lambda^{\alpha} = C_\alpha k_\lambda^\alpha$. Since $S_\alpha$ and $S_\alpha^*$ are $C_\alpha$–symmetric (see [2]), for a skew–$C_\alpha$ symmetric operator $A \in B(K_2^\alpha)$ we have

$$\langle AS_\alpha^n k_\lambda^\alpha, (S_\alpha^n)^m k_\lambda^{\alpha} \rangle = \langle C_\alpha (S_\alpha^n)^m k_\lambda^\alpha, C_\alpha AS_\alpha^m k_\lambda^\alpha \rangle = -\langle S_\alpha^m C_\alpha k_\lambda^{\alpha}, A^* C_\alpha S_\alpha^n k_\lambda^\alpha \rangle = -(AS_\alpha^n k_\lambda^\alpha, (S_\alpha^n)^m k_\lambda^{\alpha}),$$

(2)

for all $n, m \in \mathbb{N}$. Note that if $n = m$, then

$$\langle AS_\alpha^n k_\lambda^\alpha, (S_\alpha^n)^n k_\lambda^{\alpha} \rangle = 0.$$  

(3)

In particular, we may consider the special case $\alpha = z^k, k > 1$. Then the equality (3) implies that a skew–$C_\alpha$ symmetric operator $A \in B(K_2^\alpha)$ has the matrix representation in the canonical basis with 0 on the diagonal orthogonal to the main diagonal. Indeed, let $A \in B(K_2^\alpha)$ have the matrix $(a_{ij})_{i,j=0,...,k-1}$ with respect to the canonical basis. Note that $C_z f = z^{k-1} f$, $k_0 = 1$, $k^{k-1} = z^{k-1}$. Hence for $0 \leq n \leq k-1$ we have

$$0 = \langle AS_\alpha^1, (S_\alpha^n)^z^{k-1} \rangle = \langle A z^n, z^{k-n-1} \rangle = a_{n,k-n-1}.$$  

Moreover, from the equality (2) we can obtain that

$$\langle A z^n, z^{k-m-1} \rangle = -\langle A z^m, z^{k-n-1} \rangle,$$

which implies that $a_{n,k-m-1} = -a_{m,k-n-1}$ for $0 \leq m, n \leq k-1$.

3. Reflexivity

The following theorem can be obtained as a corollary of Theorem [1]. However, we think that the proof presented here is also interesting.

**Theorem 3.1.** Let $C$ be a conjugation in a complex Hilbert space $\mathcal{H}$. The subspace $C^\alpha$ of all skew–$C$ symmetric operators on $\mathcal{H}$ is reflexive.

**Proof.** By Proposition [2] it is necessary to show that if $(T, h \otimes Ch) = \langle Th, Ch \rangle = 0$ for any $h \in \mathcal{H}$, then $CTC = -T^\ast$.

Recall after [2] Lemma 1 that $\mathcal{H}$ can be decomposed into its real and imaginary parts $\mathcal{H} = H_R + i H_I$. Recall also that we can write $h = h_R + i h_I \in \mathcal{H}$ with $h_R = \frac{1}{2}(I + C)h \in H_R$ and $h_I = \frac{1}{2i}(I - C)h \in H_I$. Then $Ch_R = h_R, Ch_I = h_I$ and $Ch = C(h_R + i h_I) = h_R - i h_I$.

Let $T \in B(\mathcal{H})$. The operator $T$ can be represented as

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix},$$

where $W: H_R \rightarrow H_R, Z: H_I \rightarrow H_I, X: H_I \rightarrow H_R, Y: H_R \rightarrow H_I$ and $W, X, Y, Z$ are real linear. The condition $CTC = -T^\ast$ is equivalent to the following: $W = -W^\ast, Z = -Z^\ast, Y = X^\ast$. 

On the other hand, the condition $\langle Th, Ch \rangle = 0$ for any $h = h_R + ih_I$ is equivalent to

$$\langle Wh_R, h_R \rangle + \langle Xh_I, h_R \rangle - \langle Yh_R, h_I \rangle - \langle Zh_I, h_I \rangle = 0$$

(4)

for any $h_R \in H_R, h_I \in H_I$. In particular, $\langle Wh_R, h_R \rangle = 0$ for any $h_R \in H_R$.

Let $h'_R, h''_R \in H_R$. Then $\langle Wh'_R, h'_R \rangle = 0$, $\langle Wh''_R, h''_R \rangle = 0$ and

$$0 = \langle W(h'_R + h''_R), h'_R + h''_R \rangle = \langle Wh'_R, h'_R \rangle + \langle Wh''_R, h''_R \rangle.$$

Hence

$$\langle Wh'_R, h''_R \rangle = \langle h'_R, -Wh''_R \rangle$$

and finally $W^* = -W$. Since, by (4), in particular $\langle Zh_I, h_I \rangle = 0$ for any $h_I \in H_I$ we can also get $Z^* = -Z$.

Because $\langle Wh_R, h_R \rangle = 0 = \langle Zh_I, h_I \rangle$ for any $h_R \in H_R, h_I \in H_I$, hence by (4) we get

$$\langle Xh_I, h_R \rangle - \langle Yh_R, h_I \rangle = 0.$$

Thus $Y = X^*$ and the proof is finished. \qed

Recall that a single operator $T \in B(H)$ is called reflexive if the weakly closed algebra generated by $T$ and the identity is reflexive. In [7] authors characterized normal skew symmetric operators and by [10] we know that every normal operator is reflexive. Hence one may wonder, if all skew–C symmetric operators are reflexive. The following simple example shows that it is not true.

**Example 3.2.** Consider the space $\mathbb{C}^2$ and a conjugation $C(x, y) = (\bar{x}, \bar{y})$. Note that operator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is skew-C symmetric. The weakly closed algebra $\mathcal{A}(T)$ generated by $T$ consists of operators of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Hence $\mathcal{A}(T)_\perp = \left\{ \begin{pmatrix} t & s \\ s & \bar{t} \end{pmatrix} : t, s \in \mathbb{C} \right\}$. It is easy to see, that $\mathcal{A}(T)_\perp \cap F_1 = \{0\}$, which implies that $T$ is not reflexive.

### 4. Hyperreflexivity

**Theorem 4.1.** Let $C$ be a conjugation in a complex Hilbert space $\mathcal{H}$. Then the subspace $\mathcal{C}^*$ of all skew–C symmetric operators is hyperreflexive with the constant $\kappa(\mathcal{C}^*) \leq 3$ and 2-hyperreflexive with $\kappa_2(\mathcal{C}^*) = 1$.

**Proof.** Let $A \in B(\mathcal{H})$. Firstly, similarly as in the proof of Theorem 4.2 [5] it can be shown that $A - CA^*C \in \mathcal{C}^*$. It is also shown there that $(CAC)^* = CA^*C$. 

Hence we have
\[
\|A - \frac{1}{2} (A - CA^* C)\| = \frac{1}{2} \|A + CA^* C\| \\
= \frac{1}{2} \|A^* + CAC\| = \frac{1}{2} \sup\{|\langle h, (A^* + CAC)g \rangle| : \|h\|, \|g\| \leq 1\} \\
= \frac{1}{2} \sup\{|\langle A, h \otimes g + Cg \otimes Ch \rangle| : \|h\|, \|g\| \leq 1\} \\
= \frac{1}{2} \sup\{|\langle A, h \otimes Cg + g \otimes Ch \rangle| : \|h\|, \|g\| \leq 1\} = \alpha_2(A, C^*) \\
= \frac{1}{2} \sup\{|\langle A, (h + g) \otimes C(h + g) - h \otimes Ch - g \otimes Cg \rangle| : \|h\|, \|g\| \leq 1\} \\
\leq \frac{1}{2} \sup\{|\langle A, h \otimes Ch \rangle| : \|h\| \leq 1\} + \frac{1}{2} \sup\{|\langle A, g \otimes Cg \rangle| : \|g\| \leq 1\} + \\
+ \frac{1}{2} \sup\{|\langle A, \frac{1}{2}(h + g) \otimes C(\frac{1}{2}(h + g)) \rangle| : \|h\|, \|g\| \leq 1\} \\
\leq 3 \alpha(A, C^*).
\]

We have used the characterization given in Proposition 2.2. 

\[\square\]

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* Universite Lille 1
Laboratoire Paul Painlevé
59655 Villeneuve d’Ascq
France
E-mail address: chafiq.benhida@math.univ-lille1.fr
**Department of Applied Mathematics**  
University of Agriculture  
Balicka 253c  
30-198 Krakow  
Poland  
_E-mail address:_ rmklis@cyf-kr.edu.pl

***Institut of Mathematics***  
Pedagogical University  
ul. Podchorążych 2  
30-084 Kraków  
Poland  
_E-mail address:_ rmptak@cyf-kr.edu.pl