1. Introduction

Given a pair of physical states it is fundamental to determine whether two states can be converted each other or there is some (dis)order in conversion associated with a certain set of physical quantities. Such a question plays an essential role in thermal physics and information sciences. In quantum information science, the convertibility between maximally entangled states by local unitary operations is intensively used as a basic tool to accomplish quantum communication and computation tasks. Moreover, the convertibility under local operation and classical communications (LOCC) characterizes the strength of quantum entanglement.

Nielsen’s majorization theorem states that an entangled pure state can be transformed into another entangled pure state by an LOCC protocol iff the probability distribution of Schmidt basis is majorized by that of the target state. This theorem was extended for pure states (1) and finite dimensional states. In this report we find two examples of majorization relations for a set of TMSNS.

2. Majorization and LOCC conversion between bipartite states

Let us consider the form of d-dimension probability distribution \( p = (p_1, p_2, ..., p_d)^T \) satisfying \( \sum_i p_i = 1 \) and \( p_i \geq 0 \). For such forms, we say \( p \) majorizes \( q \) if it holds

\[
\sum_{n=1}^{m} p_i^\downarrow \geq \sum_{n=1}^{m} q_i^\downarrow \quad \forall m < d,
\]

where the symbol \( \downarrow \) means that the elements are rearranged in the decreasing order \( p_1^\downarrow \geq p_2^\downarrow \geq \cdots \geq p_d^\downarrow \). We denote the majorization condition as

\[
p \succ q \iff \sum_{n=1}^{m} p_i^\downarrow \geq \sum_{n=1}^{m} q_i^\downarrow \quad \forall m < d. \quad (2)
\]

We say that a pure entangled state \( |\phi_p\rangle \) majorizes a pure entangled state \( |\phi_q\rangle \) if there exists local unitary operators \( u, u', v, v' \) and orthonormal bases \( \{|e_i\}, \{|f_i\}\} \) in such a way that they have Schmidt forms

\[
|u_A \otimes v_B|\phi_p\rangle = \sum_i \sqrt{p_i} |e_i\rangle \otimes |f_i\rangle,
\]

\[
|v_A \otimes u_B|\phi_q\rangle = \sum_i \sqrt{q_i} |e_i\rangle \otimes |f_i\rangle,
\]

and the majorization relation Eq. (1) holds for the coefficients \( \{p_i\} \) and \( \{q_i\} \). We may also denote this majorization condition for quantum states as

\[
|\phi_p\rangle \succ |\phi_q\rangle \iff \sum_{m=1}^{m} p_i^\downarrow \geq \sum_{m=1}^{m} q_i^\downarrow \quad \forall m < d. \quad (4)
\]

Nielsen’s theorem proves that an LOCC process can convert \( |\phi_q\rangle \) into \( |\phi_p\rangle \) if and only if \( |\phi_p\rangle \succ |\phi_q\rangle \) holds for finite dimensional states \( d < \infty \). We may write symbolically,

\[
|\phi_q\rangle \xrightarrow{\text{LOCC}} |\phi_p\rangle \iff |\phi_p\rangle \succ |\phi_q\rangle. \quad (5)
\]

3. A class of infinite dimensional pure entangled states

Let us write number states \( |n\rangle \) and assume ordinary commutation relations for annihilation and creation operators of two modes \( [a, a^\dagger] = [b, b^\dagger] = i \) and \( [a, b] = [a, b^\dagger] = 0 \). Let be \( \lambda \in (0, 1) \) and define coupled annihilation operators as

\[
\hat{A}_\lambda := \frac{a - \lambda b^\dagger}{\sqrt{1 - \lambda^2}}, \quad \hat{B}_\lambda := \frac{b - \lambda a^\dagger}{\sqrt{1 - \lambda^2}}. \quad (6)
\]

It implies the commutation relations \( [\hat{A}, \hat{A}^\dagger] = [\hat{B}, \hat{B}^\dagger] = i \) and \( [\hat{A}, \hat{B}] = [\hat{A}, \hat{B}^\dagger] = 0 \). Let us define the vacuum
\( |\psi_{0,0}\rangle \) associated with \( \hat{A}_\lambda \) and \( \hat{B}_\lambda \) as the state which satisfies
\[
\hat{A}_\lambda |\psi_{0,0}(\lambda)\rangle = 0, \quad \hat{B}_\lambda |\psi_{0,0}(\lambda)\rangle = 0. \tag{7}
\]
This relation gives the familiar form of the two mode squeezed vacuum
\[
|\psi_{0,0}(\lambda)\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n,n\rangle. \tag{8}
\]
TMSNSs can be defined as the simultaneous eigenstates associated with the coupled number operators \( \hat{N}_A = \hat{A}_\lambda^\dagger \hat{A}_\lambda \) and \( \hat{N}_B = \hat{B}_\lambda^\dagger \hat{B}_\lambda \):
\[
|\psi_{N_A,N_B}(\lambda)\rangle = \frac{(\hat{A}_\lambda^\dagger)^{N_A} (\hat{B}_\lambda^\dagger)^{N_B}}{\sqrt{N_A! \sqrt{N_A!}}} |\psi_{0,0}(\lambda)\rangle. \tag{9}
\]
We can find the Schmidt decomposed form of the TMSNSs \( [7] \)
\[
|\psi_{N_A,N_B}(\lambda)\rangle = \sum_{m=0}^{\infty} C_m(N_A, N_B, \lambda) |N_A - N_B + m\rangle_A |m\rangle_B, \tag{10}
\]
where we assume \( N_A \geq N_B \) and the Schmidt coefficients \( \{C_m\}_{m=0,1,2,...} \) are given by
\[
C_m(N_A, N_B, \lambda) = (1 - \lambda^2)^{N_A-N_B} \min\{m,N_B\}
\sum_{k=0}^{\min\{m,N_B\}} \frac{(1 - \lambda^2)^{k} (-\lambda)^{N_B-k}}{\sqrt{N_A! \sqrt{N_A!}}}
\times \lambda^{m-k} \frac{\sqrt{N_A!N_B!(N_A - N_B + m)!m!}}{k!(m-k)!(N_A - N_B + k)!(N_B - k)!}. \tag{11}
\]
For the class of TMSNSs involving single-mode excitations, e.g., \( N_B = 0 \), it was shown \( [8] \) that
\[
|\psi_{n,0}(\lambda)\rangle \succ |\psi_{n+m,0}(\lambda)\rangle, \tag{12}
\]
holds for \( n, m = 0, 1, 2, 3, ... \). Hence, a complete set of ordered convertible property under LOCC was established in this class of TMSNSs
\[
|\psi_{n+m,0}(\lambda)\rangle \overset{\text{LOCC}}{\longrightarrow} |\psi_{n,0}(\lambda)\rangle. \tag{13}
\]
To reach the majorization relation Eq. (12) the following Lemma is essential:

**Lemma. (Lemma 3.1 of [10])** — An existence of a column-stochastic matrix \( D \) that satisfies
\[
q = D p, \tag{14}
\]
is sufficient to the majorization condition
\[
p \succ q. \tag{15}
\]
for positive infinite sequences \( p, q \). Here, a positive real matrix \( D (D_{i,j} \geq 0 \text{ for all } i, j) \) is called column-stochastic if its column sum is one (\( \sum D_{i,j} = 1 \text{ for all } j \)) and row-sum is less than one (\( \sum D_{i,j} \leq 1 \text{ for all } i \)). These relations suggest that \( q \) has a higher entropy than \( p \), and the two probability distributions, \( p \) and \( q \), are actually connected with a stochastic transformation \( D \).

For a notation convention, let us write the probability distribution associated with the Schmidt basis of \( |\psi_{N_A,N_B}(\lambda)\rangle \) as
\[
(p_{N_A,N_B})_m := |(N_A + N_B - m, m \psi_{N_A,N_B}(\lambda))|^2
= |C_m(N_A, N_B, \lambda)|^2 \tag{16}
\]
where the Schmidt coefficient \( C_m \) is given in Eq. (11). It was shown in \( [8] \)
\[
p_{n+m,0} = D^n p_{n,0} \tag{17}
\]
is fulfilled with the following column-stochastic matrix
\[
D = (1 - \lambda^2) \begin{pmatrix}
1 & 0 & 0 \\
\lambda^2 & 1 & 0 \\
\lambda^4 & \lambda^2 & 1 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots
\end{pmatrix}
= \begin{pmatrix}
a_0 & 0 & 0 & \cdots \\
a_1 & a_0 & 0 & \cdots \\
a_2 & a_1 & a_0 & \cdots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}, \tag{18}
\]
for any \( n, m = 0, 1, 2, ... \) and \( \lambda \in [0, 1) \). The final expression is a form of the Toeplitz matrix. The relation Eq. (17) implies the majorization relation Eq. (12) due to the above Lemma. Then, an extension of Nielsen’s theorem for infinite dimension implies the capability of the LOCC transformation Eq. (13).

4. Main results

In what follows, we find the majorization relations between the three TMSNSs \( \psi_{0,0}, \psi_{1,0}, \) and \( \psi_{1,1} \). Our method has basically two steps: (i) We empirically find a Toeplitz matrix in the form of Eq. (13) that connects the probability distributions of Schmidt bases. (ii) Then we determine a parametric regime of \( \lambda \) where the matrix becomes column-stochastic. Thereby, the majorization condition is established in the specified regime.

The squared Schmidt coefficients of the three states \( \psi_{1,1}, \psi_{1,0}, \) and \( \psi_{0,0} \) are respectively given by
\[ \mathbf{p}_{11} = (1 - \lambda^2) \begin{pmatrix} \lambda^2 \\ (1 - 2\lambda^2)^2 \\ \lambda^2 (2 - 3\lambda^2)^2 \\ \lambda^4 (3 - 4\lambda^2)^2 \\ \vdots \\ \lambda^{2(n-1)} (n - (n + 1)\lambda^2)^2 \end{pmatrix}, \quad \mathbf{p}_{10} = (1 - \lambda)^2 \begin{pmatrix} 1 \\ 2\lambda^2 \\ 3\lambda^4 \\ 4\lambda^6 \\ \vdots \\ (n + 1)\lambda^{2n} \end{pmatrix}, \quad \mathbf{p}_{00} = (1 - \lambda)^2 \begin{pmatrix} 1 \\ \lambda^4 \\ \lambda^6 \\ \vdots \\ \lambda^{2n} \end{pmatrix} \] (19)

Let us consider \( \psi_{1,1} \) and \( \psi_{1,0} \). One may empirically find that their probability distributions fulfill

\[ \mathbf{p}_{11} = A\mathbf{p}_{10} \] (20)

where \( A \) is a Toeplitz matrix defined as follows:

\[
A = (1 - \lambda^2)^{-1} \begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 \\ 2\lambda^4 - 4\lambda^2 + 1 & \lambda^2 & 0 & 0 & 0 \\ 2\lambda^6 - 4\lambda^4 + 2\lambda^2 & 2\lambda^4 - 4\lambda^2 + 1 & \lambda^2 & 0 & 0 \\ 2\lambda^8 - 4\lambda^6 + 2\lambda^4 & 2\lambda^6 - 4\lambda^4 + 2\lambda^2 & 2\lambda^4 - 4\lambda^2 + 1 & \lambda^2 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \end{pmatrix} = (1 - \lambda^2)^{-1} \begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 \\ 2(1 - \lambda^2)^2 - 1 & \lambda^2 & 0 & 0 & 0 \\ 2\lambda^4(1 - \lambda^2)^2 & 2(1 - \lambda^2)^2 - 1 & \lambda^2 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \] (21)

From the expression of the first line we can readily see that the column sum is one (\( \sum_i A_{i,j} = 1 \)). From the second expression we can see that all elements are non-negative if \( 2(1 - \lambda^2)^2 - 1 \geq 0 \). This condition implies

\[ |\lambda| \leq \lambda_{10-11}^{(0)} := \sqrt{\frac{2 - \sqrt{2}}{2}} = 0.541196... \] (22)

If all elements are non-negative, the row sum \( s_i = \sum_j A_{i,j} \) is an increasing sequence and bounded by one (e.g., \( s_i \leq \sum_j A_{i,1} = 1 \)). Therefore, \( A \) is proven to be column-stochastic if \( \lambda \in [0, \lambda_{10-11}^{(0)}] \). Hence, from the Lemma, we can conclude that the following majorization relation holds

\[ \psi_{10}(\lambda) \succ \psi_{11}(\lambda), \quad \lambda \in [0, \lambda_{10-11}^{(0)}]. \] (23)

Let us consider \( \psi_{1,1} \) and \( \psi_{0,0} \). We can see that the following Toeplitz matrix converts the probability distributions of \( \psi_{1,1} \) and \( \psi_{0,0} \) as

\[ \mathbf{p}_{11} = A\mathbf{p}_{00} \] (24)

where

\[
A = \begin{pmatrix} \lambda^2 & 0 & 0 & \cdots \\ 3\lambda^4 - 4\lambda^2 + 1 & \lambda^2 & 0 & \cdots \\ 5\lambda^6 - 8\lambda^4 + 3\lambda^2 & 3\lambda^4 - 4\lambda^2 + 1 & \lambda^2 & 0 & \cdots \\ 7\lambda^8 - 12\lambda^6 + 5\lambda^4 & 5\lambda^6 - 8\lambda^4 + 3\lambda^2 & 3\lambda^4 - 4\lambda^2 + 1 & \lambda^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \end{pmatrix} = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \] (25)

We can readily confirm that the column-sum condition is satisfied (\( \sum_{i=1}^{\infty} A_{i,j} = 1 \)). The matrix elements can be
written as
\[ A_{i,i} = a_0(\lambda) := \lambda^2, \]
\[ A_{n+i,i} = a_n(\lambda) := \lambda^{2(n-1)}(\lambda^4 + 2n(1 - \lambda^2)^2 - 1) \]
\[ (n = 1, 2, 3, \ldots) \] (26)
and the following recurrence formula holds for \( n \geq 1 \)
\[ a_{n+1}(\lambda) = \lambda^2 a_n(\lambda) + 2\lambda^{2n}(1 - \lambda)^2. \] (27)
This implies \( a_{n+1} \) is positive if \( a_n \) is positive. Hence, we can show all elements are positive if \( a_1 \) is positive. The condition \( a_1(\lambda) \geq 0 \), namely, \( 3\lambda^4 - 4\lambda^2 + 1 \geq 0 \) is fulfilled if
\[ |\lambda| \leq \lambda^{(1)}_{00>11} := 1/\sqrt{3} = 0.57735\ldots \] (28)

Here, the underlines remark the modified elements possibly to be oversight. We can routinely show that \( A' \) is column-stochastic when
\[ |\lambda| \leq \lambda^{(1)}_{00>11} := \sqrt{9 - \sqrt{21}}/10 = 0.6646\ldots \] (31)
This condition comes from \( A'_{5,3} = 5\lambda^4 - 9\lambda^2 + 3 \geq 0 \). Therefore, the majorization relation still holds for this regime:
\[ \psi_{00}(\lambda) \succ \psi_{11}(\lambda), \quad \lambda \in [0, \lambda^{(1)}_{00>11}]. \] (32)

5. Conclusion and Remarks
We have found two examples of majorization relations for the TMSNSs, \( \psi_{11}, \psi_{10}, \) and \( \psi_{00} \) (Eqs. 29 and (29) with the constraints on the squeezing parameter Eq. (29) and (31), respectively). Our approach was mostly heuristic, and seems scarcely be helpful to reach a general theorem. It is desirable to find a systematic method to determine the majorization condition over a general pair of TMSNSs.

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