We globally embed (3+1)-dimensional Schwarzschild and Schwarzschild-AdS black holes in massive gravity into (5+2)-dimensional flat spacetimes. Making use of embedding coordinates, we directly obtain the generalized Hawking, Unruh and freely falling temperatures in a Schwarzschild and Schwarzschild-AdS black hole due to massive graviton effects.

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I. INTRODUCTION

It has long been known that any $d$-dimensional Riemannian manifold can be locally isometrically embedded in an $N$-dimensional flat one with $N > d$ \[1, 3\]. In this respect, it has been known that the Hawking effect \[4\] may be related to the Unruh effect \[5\], i.e., the Hawking effect for a fiducial observer in a curved spacetime can be considered as the Unruh effect for a uniformly accelerated observer in a higher-dimensional global embedding Minkowski spacetime (GEMS). Starting from the works of Deser and Levin \[6–8\], the GEMS approach has been used to provide a unified derivation of Unruh equivalents for Hawking thermal properties \[9–23\]. Moreover, a local temperature measured by a freely falling observer has been introduced by Brynjolfsson and Thoralcus \[24\] using the traditional GEMS method. Here, a freely falling local temperature is defined at special turning points of radial geodesics where a freely falling observer is momentarily at rest with respect to a black hole. After the work, we have extended the results to other interesting curved spacetimes \[25, 26\] to investigate local temperatures of corresponding spacetimes.

On the other hand, Einstein’s general relativity (GR) is a relativistic theory of gravity where the graviton is massless. However, for many decades, attempts to generalize the Fierz-Pauli theory \[29\] to a massive gravity theory, which is reduced to the GR in the massless limit, have been suffered from difficulty in the presence of the Boulware-Deser ghost \[30\]. Recently, de Rham, Gabadadze, and Trolley (dRGT) \[31, 32\] have obtained a ghost free massive gravity having a particular type of nonlinear interaction involving the metric coupled with a symmetric background tensor, called the reference metric, to create mass terms. In the dRGT massive gravity, the nondynamical reference metric is set to be the Minkowskian one, thus breaking the diffeomorphism invariance which is preserved by GR. It has also been shown that ghost free massive theories can be obtained using a general background \[33, 34\]. For more details, the readers can refer to the reviews \[35, 36\]. Later, a nonlinear massive gravity with a special singular reference metric \[37\] has been studied extensively in the gauge/gravity duality since the theory breaks the spatial translational symmetry at the boundary, which is dual to the gravity theory with broken diffeomorphism invariance in the bulk \[38–40\]. Massive gravity theories with a singular reference metric have also been exploited to investigate many interesting black hole models \[41–50\].
Very recently, in order to see how massive gravitons affect the GEMS embeddings and free fall temperatures, we have studied a charged Bañados-Teitelboim-Zanelli (BTZ) black hole in the (2+1)-dimensional massive gravity \[51\]. As a result, we have explicitly shown that GEMS embedding dimensions are differently given by a mass parameter \[52\]. We have also obtained the generalized Hawking, Unruh, and freely falling temperatures of the charged BTZ black hole in massive gravity theory with massive graviton effects.

In this paper, we will study GEMS embedding of the Schwarzschild-anti de Sitter (AdS) black hole in a (3+1)-dimensional massive gravity, and generalize the Hawking, Unruh, and freely falling temperatures of the Schwarzschild-AdS black hole in massless gravity to those in massive gravity with ansätze in the GEMS approach. In Sec. II, we will study GEMS embeddings and freely falling temperatures of the Schwarzschild black hole in massive gravity by comparing it with the Schwarzschild black hole in massless gravity. In Sec. III, we present GEMS embeddings of a (3+1)-dimensional Schwarzschild-AdS black hole in massive gravity into a (5+2)-dimensional flat spacetime and then newly obtain desired temperatures \(T_U\) and \(T_{FF\,AR}\) of the black holes, which are measured by uniformly accelerated observers and freely falling ones, respectively. Finally, conclusions are drawn in Sec. IV.

### II. GEMS OF SCHWARZSCHILD BLACK HOLE IN MASSLESS/MASSIVE GRAVITY

#### A. GEMS of Schwarzschild black hole in massless gravity

In this section, we briefly recapitulate the GEMS embedding \[1-3\] and freely falling temperature \[24\] of the Schwarzschild black hole in massless gravity\(^1\), which is described by

\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\]

(2.1)

with the metric function for the massless graviton case

\[
f(r) = 1 - \frac{2m}{r}.
\]

(2.2)

From the metric, one can find the surface gravity \[53\] as

\[
k_H = \left. \sqrt{-\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)} \right|_{r=\tau_H} = \frac{1}{2\tau_H},
\]

(2.3)

where \(\xi^\mu\) is a Killing vector. Then, the Hawking temperature \(T_H\), which is the temperature of the radiation as measured by an asymptotic observer, is given by

\[
T_H = \frac{k_H}{2\pi} = \frac{1}{4\pi \tau_H}.
\]

(2.4)

On the other hand, a local fiducial temperature, measured by an observer who rests at a distance from a black hole, is given by

\[
T_{\text{FID}}(r) = \frac{T_H}{\sqrt{f(r)}} = \frac{r^{1/2}}{4\pi \tau_H (r - \tau_H)^{1/2}}.
\]

(2.5)

Note that the fiducial temperature \(T_{\text{FID}}\) diverges at an event horizon of a black hole, while it becomes the Hawking temperature asymptotically far away from a black hole.

Now, exploiting the GEMS approach given by the coordinate transformations for \(r \geq \tau_H\), one can embed the (3+1)-dimensional Schwarzschild spacetime in the massless gravity \[2.1\] into a (5+1)-dimensional flat spacetime as

\[
ds^2 = \eta_{IJ}dz^I dz^J, \text{ with } \eta_{IJ} = \text{diag}(-1,1,1,1,1,1).
\]

(2.6)

---

\(^1\) In particular, we will call it massless when \(\tilde{m}\) is zero in this work. See the action \[2.15\] and below.
Fronsdal \[1\] concretely obtained embedding functions as follows

\[
\begin{align*}
  z^0 &= k_H^{-1} f^{1/2}(r) \sinh k_H t, \\
  z^1 &= k_H^{-1} f^{1/2}(r) \cosh k_H t, \\
  z^2 &= r \sin \theta \cos \phi, \\
  z^3 &= r \sin \theta \sin \phi, \\
  z^4 &= r \cos \theta, \\
  z^5 &= \int dr \left( \frac{r_H (r^2 + r_H r + r_H^2)}{r^3} \right)^{1/2}
\end{align*}
\]  

(2.7)

In static detector \((r, \theta, \phi = \text{constant})\) described by a fixed point in the \((z^2, z^3, z^4, z^5)\) plane on the GEMS embedded spacetime, an observer, who is uniformly accelerated in the \((5+1)\)-dimensional flat spacetime, follows a hyperbolic trajectory in \((z^0, z^1)\) described by

\[
a_6^{-2} = (z^1)^2 - (z^0)^2 = \frac{f(r)}{k_H^2},
\]

(2.8)

Thus, one can arrive at the Unruh temperature for the uniformly accelerated observer in the \((5+1)\)-dimensional flat spacetime

\[
T_U = \frac{a_6}{2\pi} = \frac{k_H}{2\pi \sqrt{f(r)}}.
\]

(2.9)

This corresponds to the fiducial temperature \[2\] in the original Schwarzschild black hole spacetime for an observer locating at a distance from the black hole. Then, the Hawking temperature \(T_H\) seen by an asymptotic observer can be obtained as

\[
T_H = \sqrt{-g_{00}} T_U = \frac{k_H}{2\pi}.
\]

(2.10)

As a result, one can say that the Hawking effect for a fiducial observer in the black hole spacetime is equal to the Unruh effect for a uniformly accelerated observer in a higher-dimensional flat spacetime.

Now, we assume that an observer at rest is freely falling from the radial position \(r = r_0\) at \(\tau = 0\) \[24–28\]. The equations of motion for the orbit of the observer are given by

\[
\begin{align*}
  \frac{dt}{d\tau} &= \frac{\sqrt{f(r_0)}}{f(r)}, \\
  \frac{dr}{d\tau} &= -[f(r_0) - f(r)]^{1/2}.
\end{align*}
\]

(2.11)

Exploiting the embedding functions in Eq. (2.7), one can easily obtain a freely falling acceleration \(\bar{a}_6\) in the \((5+1)\)-dimensional GEMS embedded spacetime as

\[
\bar{a}_6^2 = \frac{r^3 + r_H r^2 + r_H^2 r + r_H^3}{4r_H^2 r^3},
\]

(2.12)

which gives the freely falling temperature at rest (FFAR) measured by the freely falling observer as

\[
T_{\text{FFAR}} = \frac{\bar{a}_6}{2\pi}.
\]

(2.13)

Then, by introducing a dimensionless parameter \(x = r_H / r\), one can rewrite a squared freely falling temperature \(T_{\text{FFAR}}^2\) as

\[
T_{\text{FFAR}}^2 = \frac{1 + x + x^2 + x^3}{16\pi^2 r_H^3}.
\]

(2.14)

Note that at the event horizon the freely falling temperature \(\text{(2.13)}\) is finite, while the fiducial temperature \(\text{(2.9)}\) diverges \[24\].
B. GEMS of Schwarzschild black hole in massive gravity

In this section, we will newly study the GEMS embedding of the (3+1)-dimensional Schwarzschild black hole in massive gravity, which is described by the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ \mathcal{R} + \tilde{m}^2 \sum_{\alpha=1}^{4} c_{\alpha} U_{\alpha}(g_{\mu\nu}, f_{\mu\nu}) \right],$$

(2.15)

where $\mathcal{R}$ is the scalar curvature, $\tilde{m}$ is the graviton mass, $c_{\alpha}$ are constants, and $U_{\alpha}$ are symmetric polynomials of the eigenvalue of the matrix $\mathcal{K}_{\mu\nu} \equiv \sqrt{g^{\alpha\beta} f_{\alpha\beta}}$ as

$$U_{1} = [\mathcal{K}],$$
$$U_{2} = [\mathcal{K}]^2 - [\mathcal{K}^2],$$
$$U_{3} = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3],$$
$$U_{4} = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}^2] + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4].$$

(2.16)

Here, the square root in $\mathcal{K}$ means $(\sqrt{\mathcal{A}})_{\mu}^{\nu} (\sqrt{\mathcal{A}})^{\nu}_{\alpha} = A_{\mu}^{\nu}$ and $[\mathcal{K}]$ denotes the trace $\mathcal{K}_{\mu\nu}$. Finally, $f_{\mu\nu}$ is a non-dynamical, fixed symmetric tensor, called the reference metric, introduced to construct nontrivial interaction metric terms in massive gravity.

Variation of the action with respect to $g^{\mu\nu}$ gives us the equations of motion as

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} + \tilde{m}^2 \chi_{\mu\nu} = 0,$$

(2.17)

where

$$\chi_{\mu\nu} = -\frac{c_1}{2} (U_{1} g_{\mu\nu} - \mathcal{K}_{\mu\nu}) - \frac{c_2}{2} (U_{2} g_{\mu\nu} - 2U_{1} \mathcal{K}_{\mu\nu} + 2\mathcal{K}_{\mu\nu}^2) - \frac{c_3}{2} (U_{3} g_{\mu\nu} - 3 U_{2} \mathcal{K}_{\mu\nu} + 6 U_{1} \mathcal{K}_{\mu\nu}^2 - 6 \mathcal{K}_{\mu\nu}^3) - \frac{c_4}{2} (U_{4} g_{\mu\nu} - 4U_{3} \mathcal{K}_{\mu\nu} + 12 U_{2} \mathcal{K}_{\mu\nu}^2 - 24 U_{1} \mathcal{K}_{\mu\nu}^3 + 24 \mathcal{K}_{\mu\nu}^4).$$

(2.18)

We consider the spherically symmetric black hole solution ansatz of

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(2.19)

and the special form of the reference metric as

$$f_{\mu\nu} = \text{diag}(0,0,c_0^2, c_0^2 \sin^2 \theta),$$

(2.20)

where $c_0$ is a positive constant $[37-42]$.

It seems appropriate to comment on the action having this particular reference metric form. Although such a theory breaks global Lorentz invariance and Copernican principle, this action is still invariant under diffeomorphism in the ($t,r$) plane, but not in the ($\theta, \phi$) plane. This implies at the dual side that the theory has conserved energy but no conserved momentum $[38-42]$.

Now, one can find symmetric potentials as

$$U_{1} = \frac{2c_0}{r}, \quad U_{2} = \frac{2c_0^2}{r^2}, \quad U_{3} = U_{4} = 0.$$  

(2.21)

Then, the $tt(rr)$- and $\theta\theta(\phi\phi)$-components of the equations of motion $(2.17)$ are reduced to

$$f'(r) + \frac{f(r)}{r} - \frac{c_0 c_1 \tilde{m}^2}{r} - \frac{1 + c_0^2 c_2 \tilde{m}^2}{r} = 0,$$

(2.22)

$$f''(r) + \frac{2}{r} f'(r) - \frac{c_0 c_1 \tilde{m}^2}{r} = 0,$$

(2.23)

respectively. Here we note that the $\theta\theta(\phi\phi)$-components can be obtained by differentiating the $tt(rr)$-components with respect to $r$. As a result, we have the solution as follows

$$f(r) = 1 - \frac{2m}{r} + \frac{c_0 c_1 \tilde{m}^2}{2} - \frac{c_0^2 c_2 \tilde{m}^2}{r},$$

(2.24)

where $m$ is the mass of the black hole.
where \( m \) is an integration constant related to the mass of the black hole. Without loss of generality, one can define \( R = c_0c_1\tilde{m}^2/4 \) and \( C = c_0^2c_2\tilde{m}^2 \), and finally obtain the solution of

\[
f(r) = 1 - \frac{2m}{r} + 2Rr + C.
\]

Now, from the solution (2.25), one can find the surface gravity

\[
k_H = \frac{1 + C}{2r_H} + 2R,
\]

the Hawking temperature \( T_H \)

\[
T_H = \frac{1 + C}{4\pi r_H} + \frac{R}{\pi},
\]

and the local fiducial temperature \( T_{FID} \)

\[
T_{FID} = \frac{T_H}{\sqrt{f(r)}} = \frac{r^{1/2}(1 + C + 4Rr_H)}{4\pi r_H(r - r_H)^{1/2}[1 + C + 2R(r + r_H)]^{1/2}}.
\]

According to the values of \( C \) and \( R \), the Hawking temperature (2.27) in the massive gravity differently behaves, which is shown in Fig. 1. Note that first \( R \) gives a constant contribution to the Hawking temperature. The Hawking temperature is mostly proportional to \( 1/r_H \). When \( C > -1 \), it decreases as \( r_H \) increases. When \( C = -1 \), it is just a constant given by \( R \). However, when \( C < -1 \), it is negatively proportional to \( 1/r_H \). Since the Hawking temperature is flipped according to the sign of \( C \), it should be careful to embed this spacetime into a flat one by using the GEMS approach, which will be now discussed in the following.

Now, let us newly exploit the GEMS approach given by the coordinate transformations for \( r \geq r_H \) and \( R \geq 0 \) as in Sec. II.A. As a result, we have found that the (3+1)-dimensional Schwarzschild spacetime in the massive gravity (2.19) can be embedded into a (5+2)-dimensional flat one, depending on the massive parameter \( C \) as

\[
ds^2 = \eta_{ij}dz^idz^j, \quad \text{with} \quad \eta_{ij} = \text{diag}(-1, 1, 1, 1, 1, -1),
\]

where the embedding functions are found to be

\[
\begin{align*}
z^0 &= k_H^{-1}f^{1/2}(r) \sinh k_H t, \\
z^1 &= k_H^{-1}f^{1/2}(r) \cosh k_H t, \\
z^2 &= r \sin \theta \cos \phi, \\
z^3 &= r \sin \theta \sin \phi, \\
z^4 &= r \cos \theta
\end{align*}
\]
with

\[ z^5 = \int dr \left( \frac{r_H (1 + h_1 + Ch_2) h_5}{r^3 h_6 h^2_7} \right)^{1/2}, \]

\[ z^6 = \int dr \left( \frac{2R(r + r_H)(1 + h_1 + Ch_2) + 4r_H^2 (h_3 + Ch_4)}{h_6 h^2_7} \right)^{1/2}, \]  

(2.31)

when \( C > 0 \), and

\[ z^5 = \int dr \left( \frac{r_H (1 + h_1) h_5}{r^3 h_6 h^2_7} - \frac{2CR(r + r_H) h_2 + 4Cr^2_H h_4}{h_6 h^2_7} \right)^{1/2}, \]

\[ z^6 = \int dr \left( \frac{2(R + r_H)(1 + h_1) + 4r^2_H h_3 - Cr^2_h h_5}{h_6 h^2_7} \right)^{1/2}, \]  

(2.32)

when \( C < 0 \). Here, we have defined some functions associated with \( z^5 \) and \( z^6 \) as follows

\[ h_1 = 8Rr_H (1 + 2Rr_H) + C^2, \]
\[ h_2 = 2(1 + 4Rr_H), \]
\[ h_3 = \frac{R^2 (r^2 + 3r^2_H)(r + r_H)}{r^3}, \]
\[ h_4 = \frac{Rr_H (r + r_H)}{r^3} + \frac{(1 + 4Rr_H)^2}{4r^2_H} + \frac{C^2}{4r^2_H}, \]
\[ h_5 = r^2 + r_H r + r^2_H, \]
\[ h_6 = 1 + C + 2R(r + r_H), \]
\[ h_7 = 1 + C + 4Rr_H. \]  

(2.33)

Moreover, we have imposed the restriction that the function \( h_6 \) is positive definite. Note that in the limit of both \( C \to 0 \) and \( R \to 0 \), the timelike embedding coordinates \( z^6 \) in Eqs. (2.31) and (2.32) become zero, regardless of the sign of \( C \). As a result, the (5+2)-dimensional flat spacetimes in the Schwarzschild black hole in the massive gravity are reduced to the (5+1)-dimensional flat ones in the massless Schwarzschild black hole exactly [8, 11, 24, 25].

Now, in static detectors \((r, \theta, \phi = \text{const})\) described by a fixed point in the \((z^2, z^3, z^4, z^5, z^6)\) plane, a uniformly accelerated observer in the (5+2)-dimensional flat spacetime, follows a hyperbolic trajectory in \((z^0, z^1)\) described by a proper acceleration \( \bar{a}_7 \) as follows

\[ a^{-2}_7 = (z^1)^2 - (z^0)^2 = \frac{4r^2_H (r - r_H)[1 + C + 2R(r + r_H)]}{r (1 + C + 4Rr_H)^2}. \]

(2.34)

Thus, we arrive at the Urruh temperature for the uniformly accelerated observer in the (5+2)-dimensional flat spacetime

\[ T_U = \frac{a_7}{2\pi} = \left( \frac{1}{4\pi r_H (r - r_H)^{1/2}} \right)^{1/2}. \]

(2.35)

This is exactly the same with the local fiducial temperature \( [24, 28] \) measured by a local observer rest at a distance from the black hole.

Now, we assume that an observer at rest is freely falling from the radial position \( r = r_0 \) at \( \tau = 0 \) [24, 28]. The equations of motion for the orbit of the observer are given by Eq. (2.11). Exploiting Eqs. (2.30), (2.31) for \( C > 0 \) (or, \( 2.32 \) for \( C < 0 \)) with Eq. (2.11), we obtain a freely falling acceleration \( \bar{a}_7 \) in the (5+2)-dimensional GEMS embedded spacetime for the Schwarzschild black hole in the massive gravity as

\[ \bar{a}^2_7 = \frac{(r + r_H)(1 + C + 2Rr_H)(r^2 + r^2_H)(1 + C + 2Rr_H) + 4r^2_H r^2}{4r^2_H r^2 [1 + C + 2R(r + r_H)]}, \]

(2.36)

which gives us a freely falling local temperature measured by the freely falling observer as

\[ T_{FFAR} = \frac{\bar{a}_7}{2\pi}. \]

(2.37)
FIG. 2: Squared freely falling temperatures for Schwarzschild black hole in massive gravity for \((C, d) = (0.1, 0.1), (0.5, 0.1), (0.5, 1.0)\). The dashed line is for Schwarzschild black hole in massless gravity.

By using the dimensionless parameters \(x = r_H/r\) and \(d = Rr_H\), the squared freely falling temperature \(T_{\text{FFAR}}^2\) becomes

\[
T_{\text{FFAR}}^2 = \frac{(1 + C + 2d)(1 + x + \sqrt{x^2 + x^3} + 4d(1 + x)x)}{16\pi^2 r_H^2 [(1 + C + 2d)x + 2d]},
\]

which remains finite at the event horizon. In Fig. 2, we have depicted the ratio of the squared freely falling temperature \(T_{\text{FFAR}}^2/T_H^2\) for the Schwarzschild black hole in the massless/massive gravity in the range of \(0 < x \leq 1\). Note that at the event horizon of \(x = 1\) \((r = r_H)\) the freely falling temperatures are all finite.

### III. GEMS OF SCHWARZSCHILD-ADS BLACK HOLE IN MASSLESS/MASSIVE GRAVITY

#### A. GEMS of Schwarzschild-AdS black hole in massless gravity

In this section, we also briefly summarize the GEMS embeddings \([1, 8]\) and freely falling temperature \([24]\) of the Schwarzschild-AdS black hole in massless gravity, which is described by

\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

with the metric function for the massless graviton case

\[
f(r) = 1 - \frac{2m}{r} + \frac{r^2}{l^2}.
\]

This Schwarzschild-AdS solution has the surface gravity

\[
k_H = \frac{1}{2r_H} + \frac{3r_H}{2l^2},
\]

the Hawking temperature \(T_H\)

\[
T_H = \frac{1}{4\pi r_H} + \frac{3r_H}{4\pi l^2},
\]

and the local fiducial temperature \(T_{\text{FID}}\) as

\[
T_{\text{FID}} = \frac{r^{1/2}(3r_H^2 + l^2)}{4\pi r_H[r - r_H]^{1/2}(r^2 + r r_H + r_H^2 + l^2)^{1/2}}.
\]

Exploiting the GEMS approach given by the coordinate transformations for \(r \geq r_H\), we can embed the \((3+1)\)-dimensional Schwarzschild-AdS black hole in the massless gravity \([5, 1]\) into a \((5+2)\)-dimensional flat spacetime as

\[
ds^2 = \eta_{IJ}dz^I dz^J, \text{ with } \eta_{IJ} = \text{diag}(-1, 1, 1, 1, 1, -1),
\]
where the embedding functions are given by

\[
\begin{align*}
    z^0 &= k_H^{-1} f^{1/2}(r) \sinh k_H \tau, \\
    z^1 &= k_H^{-1} f^{1/2}(r) \cosh k_H \tau, \\
    z^2 &= r \sin \theta \cos \phi, \\
    z^3 &= r \sin \theta \sin \phi, \\
    z^4 &= r \cos \theta, \\
    z^5 &= \int dr \, \frac{1}{3 r_H^2 + l^2} \left( \frac{r_H h_8}{r^3 \left(h_5 + l^2\right)} \right)^{1/2}, \\
    z^6 &= \int dr \, \frac{1}{3 r_H^2 + l^2} \left( \frac{h_5 h_8}{r^3 \left(h_5 + l^2\right)} \right)^{1/2},
\end{align*}
\]

(3.7)

where \( h_5 \) is given by Eq. (2.33) and \( h_8 \) is defined as

\[
h_8 = 9 r_H^4 + 10 r_H^2 l^2 + l^4.
\]

(3.8)

In static detector (\( r, \theta, \phi = \text{constant} \)) described by a fixed point in the \((z^2, z^3, z^4, z^5, z^6)\) plane, a uniformly accelerated observer in the \((5+2)\)-dimensional flat spacetime, follows a hyperbolic trajectory in \((z^0, z^1)\) described by

\[
\alpha_r^2 = \left( z^1 \right)^2 - \left( z^0 \right)^2 = \frac{4 r_H^2 l^2 (r - r_H) \left[r^2 + r_H r + (r_H^2 + l^2)\right]}{r^3 \left[3 r_H^2 + l^2\right]^2}.
\]

(3.9)

Thus, we arrive at the Unruh temperature for a uniformly accelerated observer in the \((5+2)\)-dimensional flat spacetime

\[
T_U = \frac{\alpha_r}{2 \pi} = \frac{r/r_H}{4 \pi r_H l (r - r_H) \left[3 r_H^2 + l^2\right]^{1/2} \left[r^2 + r_H r + (r_H^2 + l^2)\right]^{1/2}}.
\]

(3.10)

This corresponds to the local fiducial temperature \((\text{FFAR})\) in the original Schwarzschild-AdS black hole spacetime for an observer locating at a distance from the black hole, and one can find the Hawking temperature \(T_H\) seen by an asymptotic observer as

\[
T_H = \sqrt{-g_{00} T_U} = \frac{k_H}{2 \pi}.
\]

(3.11)

As before, one can see that the Hawking effect for a fiducial observer in the black hole spacetime is equal to the Unruh effect for a uniformly accelerated observer in a higher-dimensional flat spacetime.

Now, we assume that an observer at rest is freely falling from the radial position \( r = r_0 \) at \( \tau = 0 \) \([24, 28]\). The equations of motion for the orbit of the observer are given by Eq. (2.11), and by exploiting Eq. (3.7), we obtain a freely falling acceleration \( \dot{a}_r \) in the \((5+2)\)-dimensional GEMS embedded spacetime as

\[
\dot{a}_r^2 = \frac{[(r + r_H)(r_H^2 + l^2) - 2 r_H r^2] \left[(r^2 + r_H^2)(r_H^2 + l^2) + 2 r_H r^2 (r + r_H)\right]}{4 r_H^2 r^3 l^2 (r^2 + r_H r + r_H^2 + l^2)^2}.
\]

(3.12)

This gives us a freely falling temperature measured by the freely falling observer as

\[
T_{\text{FFAR}} = \frac{\dot{a}_r}{2 \pi}.
\]

(3.13)

By using the dimensionless parameter \( x = r_H/r, \ c = l/r_H \), the squared freely falling temperature \( T_{\text{FFAR}}^2 \) can be written as

\[
T_{\text{FFAR}}^2 = -\frac{4(1 + x) + (c^2 + 1)(c^2 + 5) x^2 + (c^2 + 1)^2 (1 + x + x^2) x^3}{16 \pi^2 l^2 [1 + x + (c^2 + 1)x^2]}.
\]

(3.14)

In Fig. 3, we have depicted squared freely falling temperatures for the Schwarzschild-AdS black hole in the massless gravity. Note that in the limit of \( x \to 1 \) (or, \( r \to r_H \)), the freely falling temperatures \( T_{\text{FFAR}}^2 \) are finite, while the Unruh temperature \((\text{FFAR})\) diverges. Moreover, at the spatial infinity limit of \( x \to 0 \), one can obtain imaginary temperature as

\[
T_{\text{FFAR}}^2 = \frac{-1}{4 \pi^2 l^2} < 0,
\]

(3.15)

which is seemingly unphysical. However, it is allowed for a geodesic observer who follows a spacelike motion \([0, 24, 25]\).
FIG. 3: Squared freely falling temperatures for the Schwarzschild-AdS black hole in massless gravity with $c = 2, 5, 10, 100$ from bottom to top. Here, the dashed line is for the Schwarzschild black hole in massless gravity.

### B. GEMS of Schwarzschild-AdS black hole in massive gravity

In this section, we will finally study the following (3+1)-dimensional massive gravity with a negative cosmological constant as \[37, 41, 43, 51\]

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \Lambda + \tilde{m}^2 \sum_{i=1}^{4} c_i U_i(g_{\mu\nu}, f_{\mu\nu}) \right],
\]  
(3.16)

where $\Lambda = -6/l^2$ and others are the same as before. As in Sec.II.B, from the same ansatz of the reference metric $f_{\mu\nu} = \text{diag}(0, 0, c_0^2, c_0^2 \sin^2 \theta)$ and the spherically symmetric metric ansatz of

\[
d s^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]  
(3.17)

one has the $tt$- and $\theta\theta$-components of the equations of motion as follows

\[
f'(r) + \frac{f(r)}{r} - \frac{3r}{l^2} - c_0 c_1 \tilde{m}^2 - \frac{1 + c_0^2 c_2 \tilde{m}^2}{r} = 0,
\]  
(3.18)

\[
f''(r) + \frac{2}{r} f'(r) - \frac{c_0 c_1 \tilde{m}^2}{r} - \frac{6}{l^2} = 0,
\]  
(3.19)

respectively. These equations include the cosmological constant contribution. Note also that (3.19) can be obtained by differentiating (3.18) with respect to $r$. As a result, the solution can be found as

\[
f(r) = 1 - \frac{2m}{r} + \frac{r^2}{l^2} + \frac{c_0 c_1 \tilde{m}^2}{2} \frac{r}{2} + c_0^2 c_2 \tilde{m}^2. 
\]  
(3.20)

As before, one can define $R = c_0 c_1 \tilde{m}^2/4$ and $C = c_0^2 c_2 \tilde{m}^2$, and finally obtain the solution as follows

\[
f(r) = 1 - \frac{2m}{r} + \frac{r^2}{l^2} + 2Rr + C.
\]  
(3.21)

This spacetime is asymptotically described by the AdS. However, the vacuum solution with $m = 0$ is not an AdS unless $\tilde{m} = 0$ (i.e. $R = C = 0$) in Eq. (3.16). Since the event horizon is determined by $f(r)|_{r=r_H} = 0$, the mass $m$ can be written in terms of the event horizon $r_H$ as

\[
m(r_H) = \frac{(1 + C)r_H}{2} + Rr_H^2 + \frac{r_H^3}{2l^2}.
\]  
(3.22)

In Fig. 4, we have drawn the metric $f(r)$ and mass functions $m(r_H)$ to compare the features of the Schwarzschild-AdS black hole in the massive gravity with those in the massless gravity.
FIG. 4: For the Schwarzschild-AdS black hole in massive gravity, (a) metric function with $m = 100$, $l = 1$, $R = 5$, and $C = 1$, and (b) mass function $m(r_H)$ with $l = 1$, $R = 5$, and $C = 1$. For comparison, metric and mass functions for the Schwarzschild-AdS black hole in massless gravity are drawn by dashed lines with $l = 1$, $R = 0$, and $C = 0$.

FIG. 5: Hawking temperatures for the Schwarzschild-AdS black hole in massive gravity with varying $C$ and $l = 1$, $R = 5$, comparing with the one in massless gravity.

From the solution, one can find the surface gravity as

$$k_H = \frac{1 + C}{2r_H} + 2R + \frac{3r_H}{2l^2},$$

the Hawking temperature $T_H$

$$T_H = \frac{1 + C}{4\pi r_H} + \frac{R}{\pi} + \frac{3r_H}{4\pi l^2},$$

and a local fiducial temperature $T_{\text{FID}}$ as

$$T_{\text{FID}} = \frac{r^{1/2}[3r_H^2 + (1 + C + 4Rr_H)^2]}{4\pi r_H l[(r - r_H)^{1/2} + r_H r + r_H^2 + (1 + C)^2 + 2R(r + r_H)^2]^{1/2}}.$$  

In Fig. 5, we have drawn the Hawking temperatures for the Schwarzschild-AdS black hole in the massive gravity, compared with the case of the massless gravity. For the case of $C > -1$, the Hawking temperature in the massive gravity has a minimum at $r_H = r_m = l\sqrt{1 + C}/3$. When $C = -1$, the Hawking temperature is a straight line having a minimum value $R/\pi$ at $r_H = 0$. When $C < -1$, since the first term in Eq. (2.4) has a minus sign, the Hawking temperature monotonically decreases as $r_H$ decreases, and finally becomes zero at $r_H = r_0 = \frac{2}{3}R l^2 \left(\sqrt{1 - \frac{3(1+C)}{4R^2l^2}} - 1\right)$.

In short, the Hawking temperature in the massive gravity can be classified by the values of $C$ and $R$: when $C > -1$, the Hawking temperature behaves as the Schwarzschild-AdS black hole in the massless gravity. When $C = -1$, 

the Hawking temperature is a straight line, and when $C < -1$, the Hawking temperature becomes a monotonically decreasing function as $r_H$ decreases. Additionally, $R$ just shifts the Hawking temperatures vertically up and down.

Now, exploiting the GEMS approach given by the coordinate transformations for $r \geq r_H$ and $R \geq 0$, we can embed the (3+1)-dimensional Schwarzschild-AdS in the massive gravity into a (5+2)-dimensional flat spacetime as

$$ds^2 = \eta_{IJ}dz^I dz^J,$$

with $\eta_{IJ} = \text{diag}(-1, 1, 1, 1, 1, 1, -1)$, and newly defined functions as

$$z^0 = k_H^{-1} f^{1/2}(r) \sinh k_H t,$$
$$z^1 = k_H^{-1} f^{1/2}(r) \cosh k_H t,$$
$$z^2 = r \sin \theta \cos \phi,$$
$$z^3 = r \sin \theta \sin \phi,$$
$$z^4 = r \cos \theta$$

with

$$z^5 = l \int dr \left( \frac{(r^2 + l^2)^2 + l^4 h_1 + Ch_2)}{r^3 (h_5 + l^2 h_6) h_9} \right)^{1/2},$$
$$z^6 = l \int dr \left( \frac{(h_7 + l^4 (h_1 + Ch_2)(h_5 + 2Rl^2 (r + r_H))) + 4l^6 H^2 (h_3 + Ch_4)}{(h_5 + l^2 h_6) h_9^2} \right)^{1/2},$$

when $C > 0$, and

$$z^5 = l \int dr \left( \frac{R^2 (r^2 + l^2)^2 + l^4 h_1 + Ch_2)}{r^3 (h_5 + l^2 h_6) h_9} \right)^{1/2},$$
$$z^6 = l \int dr \left( \frac{(h_7 + l^4 (h_1 + Ch_2)(h_5 + 2Rl^2 (r + r_H))) + 4l^6 H^2 (h_3 + Ch_4)}{(h_5 + l^2 h_6) h_9^2} \right)^{1/2},$$

when $C < 0$. Associated with $z^5$ and $z^6$ coordinates in Eqs. (3.28) and (3.29), we have used the functions, $h_5$, $h_6$, and $h_7$ given in Eqs. (2.39) and (3.38), and newly defined functions as

$$h_1 = 8Rr_H \left( 1 + 2Rr_H + \frac{3r_H^2}{l^2} \right) + C^2,$$
$$h_2 = 2 \left( 1 + 4Rr_H + \frac{3r_H^2}{l^2} \right),$$
$$h_3 = \frac{4R^2 r_H (5r_H^2 + l^2)(r + r_H)}{l^2 r^3} + \frac{R^2 r^2 + 3r_H^2 (r + r_H)}{r^3} + \frac{C^2 (3r_H^2 + (1 + 4Rr_H)^2)}{2r_H^2 l^2},$$
$$h_4 = \frac{R r_H^2 + r_H^2 r + r_H}{l^2 r^3} + \frac{r_H (r^2 + r_H r + r_H)}{r^3} + \frac{(3r_H^2 + (1 + 4Rr_H)^2)^2}{4r_H l^4} + \frac{C^2}{4r_H^2},$$
$$h_9 = 3r_H^2 + (1 + C + 4Rr_H)^2.$$

Here, we have imposed the restriction that the function $h_5 + l^2 h_6$ is positive definite.

In static detectors ($\theta$, $\phi$, $r = \text{const}$) described by a fixed point in the ($z^2$, $z^3$, $z^4$, $z^5$, $z^6$) plane, a uniformly accelerated observer in the (5+2)-dimensional flat spacetime follows a hyperbolic trajectory in ($z^0$, $z^1$) described by a proper acceleration $a_7$ as follows

$$a_7^2 = (z^1)^2 - (z^0)^2 = \frac{4r_H^2 l^2 (r - r_H)}{r^3 (3r_H^2 + (1 + C + 4Rr_H)^2)^2}.$$

Thus, we arrive at the Unruh temperature for a uniformly accelerated observer in the (5+2)-dimensional flat spacetime as

$$T_U = \frac{a_7}{2\pi} = \frac{r^{1/2} [3r_H^2 + (1 + C + 4Rr_H)^2]}{4\pi r_H (r - r_H)^{1/2} [r^2 + r_H r + r_H^2 + (1 + C)l^2 + 2R (r + r_H)l^2]^{1/2}}.$$
which gives us the temperature measured by the freely falling observer as

$$T_{\text{FFAR}} = \frac{\tilde{a}_7}{2\pi}.$$  (3.34)

With previously defined massless parameters of $x = r_H/r$, $c = l/r_H$, and $d = Rr_H$, the squared freely falling temperature $T_{\text{FFAR}}^2$ can be rewritten as

$$T_{\text{FFAR}}^2 = \frac{g(x, c, C, d)}{16\pi^2 r_H^2 c^2 (1 + 1 + c^2)x^2 + 2c^2d(1 + x)x + Cc^2x^2},$$  (3.35)

where

$$g(x, c, C, d) = -4(1 + x + 2c^2dx) + (1 + c^2 + 2c^2d + Cc^2)(5 + c^2 + 2c^2d + Cc^2)x^2 + (1 + c^2 + 2c^2d + Cc^2)(1 + x + x^2)x^3 + 4c^2d(1 + c^2 + 2c^2d + Cc^2)(1 + x)x^2.$$  (3.36)

The ratio of the squared freely falling temperature for the Schwarzschild-AdS black hole in the massive gravity to the squared Hawking temperature is plotted in Fig. 6. Note that the freely falling temperatures are finite at the event horizons. In order to see the graviton mass effect clearly, we have redrawn Fig. 6 to Fig. 7 by rescaling ten times $T_{\text{FFAR}}^2/T_H^2$ with $c = 1$. Then, in Fig. 7(a), one can see that the freely falling temperature starts to appear when $C > -1$. One can also see that the graviton mass effect of $(C, d)$ on the freely falling temperatures is increased as the freely falling observer approaches the event horizon. Moreover, the squared freely falling temperatures near the event horizon are more enhanced when $(C, d)$ are relatively small.

The freely falling temperature (3.35) can be further simplified in the two interesting limit: at the spatial infinity $r \to \infty$ ($x \to 0$) and at the event horizon $r = r_H$ ($x = 1$). At the spatial infinity of $x \to 0$, one obtains imaginary temperature as

$$T_{\text{FFAR}}^2 = -\frac{1}{4\pi^2l^2} < 0,$$  (3.37)
FIG. 7: Freely falling temperature for the Schwarzschild-AdS black hole in massive gravity (a) by varying $C$ with $d = 0.1$, and (b) by varying $d$ with $C = 1$. Note that in (a) the dotted line on the $(C, x)$ plane is for $C = -1$. Thick curves represent freely falling temperatures for $(C, d) = (0, 0.1), (1, 0)$.

which is allowed for a geodesic observer who follows a spacelike motion similar to the case of the Schwarzschild-AdS black hole in the massless gravity \cite{8, 24, 25}. On the other hand, at the event horizon $x = 1$, one obtains

$$T_{\text{FFAR}}^2 = \frac{c^2(1 + C + 2d)}{4\pi^2 l^2}. \quad (3.38)$$

It seems appropriate to comment that the freely falling temperatures are finite at the event horizon, but become hotter (colder) in the massive gravity with the condition $2d + C > 0$ ($2d + C < 0$) than in the massless gravity with $d = C = 0$.

On the other hand, by taking the limit of $l^2 \to \infty$ in Eq. (3.35), one can find the freely falling temperature of a Schwarzschild black hole in the massive gravity of Eq. (2.38). In the limit of $C = 0$ while keeping $d \neq 0$, the temperature becomes

$$T_{\text{FFAR}}^2 = \frac{x[(2d(1 + 2d) + (1 + 2d)^2)(1 + x) + (1 + 2d)(x^2 + x^3)]}{16\pi^2 T_H^2 [2d + (1 + 2d)x]}, \quad (3.39)$$

which is drawn in Fig. 6(a) with a thick curve. In the limit of $d = 0$ with keeping $C > -1$ except $C = 0$, we have

$$T_{\text{FFAR}}^2 = \frac{(1 + C)(1 + x + x^2 + x^3)}{16\pi^2 T_H^2} > 0, \quad (3.40)$$

which behaves like the Schwarzschild-AdS black hole in the massless gravity \cite{24, 25}. In Fig. 6(b), we have depicted a thick curve which is for $d = 0$ with $C = 1$. In the limit of both $d = 0$ and $C = -1$, we have

$$T_{\text{FFAR}}^2 = 0. \quad (3.41)$$

Moreover, in the case of $C < -1$ with $d = 0$, we have

$$T_{\text{FFAR}}^2 = \frac{(1 + C)(1 + x + x^2 + x^3)}{16\pi^2 T_H^2} < 0, \quad (3.42)$$

which is also allowed for a geodesic observer who follows a spacelike motion for the case of the Schwarzschild black hole in the massive gravity as expected. On the other hand, in the massless case of both $C = 0$ and $d = 0$, one can easily obtain the squared freely falling temperature (2.14) of the Schwarzschild black hole in the massless gravity \cite{24, 25}.

IV. DISCUSSION

In summary, we have globally embedded the (3+1)-dimensional Schwarzschild(-AdS) black hole in massless/massive gravity into corresponding higher dimensional flat spacetimes. Making use of the embedded coordinates, we have
directly obtained the Hawking, Unruh, and freely falling temperatures in a Schwarzschild(AdS) black hole in massive gravity and have shown that the Hawking effect for a fiducial observer in a curved spacetime is equal to the Unruh effect for a uniformly accelerated observer in a higher-dimensionally embedded flat spacetime.

Moreover, we have shown that the GEMS embeddings of the Schwarzschild-AdS black hole in massive gravity include not only massive graviton effects but also AdS structures. Thus, in the limit of $l^2 \to \infty$, the GEMS embedding of the Schwarzschild-AdS black hole in massive gravity is reduced to the $(5+2)$-dimensional flat spacetime of the Schwarzschild black hole in massive gravity, and furthermore in the massless limit of $C \to 0$ and $R \to 0$, they are reduced to the well-known $(5+1)$-dimensional flat spacetime. We have also obtained their corresponding freely falling temperatures in these limiting cases. Finally, we have found that freely falling temperatures are finite at the event horizon, but become hotter (colder) in massive gravity with the condition $2d + C > 0$ ($2d + C < 0$) than in massless gravity with $d = C = 0$.

Finally, it seems appropriate to comment on thermodynamic stability briefly. The Hawking temperatures in Eqs. (2.27), (3.24) show that they are explicitly modified by massive gravitons. First of all, as for the Schwarzschild black hole in the massive gravity, when $C > -1$, the Hawking temperature (2.27) is exactly the same form with the one in the massless gravity so that it would be unstable. However, when $C < -1$, it appears to have a stable black hole. On the other hand, the Hawking temperature (3.24) in the Schwarzschild-AdS black hole in the massive gravity has a minimum temperature $T_m$ at $r_H = r_m$ as seen in Fig. 4. As a result, when $C > -1$, a large black hole which is defined by the one with $r_H > r_m$ would be stable, while a small black hole with $r_H < r_m$ expects to be unstable. Moreover, when $C \leq -1$, it would be stable since $dT/dr_H > 0$ on the whole range of $r_H$ in Fig. 4. Since the thermodynamic stabilities differ due to the values of the mass parameters, it would be interesting to study this topic more in details for a later work as a further supplement of Ref. [41].

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