How far will you walk to find your shortcut: Space Efficient Synopsis Construction Algorithms

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Abstract

In this paper we consider the wavelet synopsis construction problem without the restriction that we only choose a subset of coefficients of the original data. We provide the first near optimal algorithm. We arrive at the above algorithm by considering space efficient algorithms for the restricted version of the problem. In this context we improve previous algorithms by almost a linear factor and reduce the required space to almost linear. Our techniques also extend to histogram construction, and improve the space-running time tradeoffs for V-Opt and range query histograms. We believe the idea applies to a broad range of dynamic programs and demonstrate it by showing improvements in a knapsack-like setting seen in construction of Extended Wavelets.

1 Introduction

Wavelet synopsis techniques have become extremely popular in query optimization, approximate query answering and a large number decision support systems. Wavelets, specially Haar wavelets, are one-one mappings and admit a natural multi-resolution interpretation, as well as fast algorithms for the forward and inverse transforms.

Given a set of $n$ numbers $X = x_1, \ldots, x_n$ the wavelet synopsis construction problem seeks to choose a synopsis vector $Z$ with at most $B$ non-zero entries, such that the inverse wavelet transform of $Z$ (denoted by $W^{-1}(Z)$) gives a good estimate of the data. The typical objective measures are (suitably weighted*) $\ell_k$ norm of $X - W^{-1}(Z)$. In an early paper [15], demonstrated a number of different applications for wavelet synopsis and proposed greedy algorithms. However for objective measures other than the $\ell_2$ measure, the greedy algorithm does not necessarily provide the optimum solution. The problem is quite non-trivial, primarily due the fact that the Wavelet basis vectors overlap and cancellations (subtractions) occur. This means that we can have two coefficients that cancel out each other leaving a significantly (exponentially) smaller contribution, which needs to be accounted for. The precision of the coefficients in the optimum solution can be much larger than the precision of the data. In fact there are no known bounds or promising techniques for quantifying the precision - this is the biggest stumbling block in the synopsis construction.

Most of the literature focuses on the Restricted case where the non-zero entries of $Z$ are equal to the corresponding entries in the transform of the original data, $W(X)$. A natural question remains: why should we be optimizing under the restriction of retaining the coefficients of the data – with no guarantees that such a restriction does not compromise the quality of the final synopsis? This is clearly suboptimal – a comparable example would be to optimize the synopsis for point queries, and use it for range queries.

A simple example renders the discussion concrete; $X = \{1, 2, 3, 7\}$ and $B = 1$ illustrates that choosing any single coefficient of $W(X) = \{3.25, -1.75, -0.5, -2\}$ (non-normalized) does not give the optimum

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*The weighted $\ell_k$ with weights $\pi_i > 0$ and wlog $\sum_i \pi_i = n$ minimizes $(\sum_i (\pi_i(x_i - W^{-1}(Z)_i))^k)^{\frac{1}{k}}$. 

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answer for $\ell_1$ or $\ell_\infty$ norm. Normalization does not help. The normalized transform is $\{4.55, -2.45, -0.5, -2\}$ – but choosing the first coefficient as 4.55 in the normalized setting implies assigning $4.55/\sqrt{2} = 3.25$ everywhere. Thus dynamic program approaches that seek to see the effect of the coefficient on the data come to the same conclusion in both settings. The optimum choices of $Z$ are $\{z, 0, 0, 0\}$ for any $2 \leq z \leq 3$ and $\{4, 0, 0, 0\}$ for $\ell_1$ and $\ell_\infty$ respectively. The same example applies to weighted $\ell_2$, e.g., if $\pi = \{2, 1, 3, 3\}$ then the best error achieved by retaining any single entry of $W(X)$ is 5.78 whereas $Z = \{4.65, 0, 0, 0\}$ gives an error of 4.87. The example can be extended to any $B$ (by repetition and scaling). The restriction of only retaining the coefficients of the data is significantly self defeating.

However the restriction does ease the search for a solution, and as this paper shows, is an important stepping stone towards the final result. For the restricted case, [5] gave a probabilistic scheme (the space constraint is preserved in expectation only, along with the error) and very recently [4] gave an optimal solution. This has been extended and improved in [17]. However, the solution to the unrestricted case has remained elusive and we provide the first near optimal solutions. In the process, we also improve upon previous algorithms for the restricted case as well. However our algorithm is best explained by taking a different path, which brings us to the major theme of the paper.

Synopsis construction is perhaps most relevant in context of massive data sets. In some scenarios we can justify that the synopsis is created using a “scratch” space larger than the synopsis and stored. However a quadratic or extremely superlinear space complexity is near infeasible for large $n$. The dependence on synopsis size $B$ is also important in this context – the smaller the dependence is, the larger is the synopsis that can be computed in the environment of a particular system. Further, space is typically a more inflexible resource, and not just a matter of wait. However a natural conceptual question arises: We are only given $n$ numbers, – do we really need to save so much information to compute the optimum answer?

All previous algorithms (for the restricted case) are expensive in space (see table below). This (superlinearity in $n, B$) is also seen in context of histogram construction (we provide a detailed table in Section 4.1). To avoid this expensive space complexity, several researchers have introduced the notion of working space, which is the amount of space required to compute the error – the rest of the space is used to construct the answer (coefficients, representatives, etc.). In case of wavelets the working space used by previous algorithms is $O(nB)$. In case of histograms, known algorithms reconstruct the answer only using the $O(n)$ working space, but with a penalty of an extra factor of $B$ in the running time. In this paper, we reduce the space for wavelets and eliminate the penalty for histograms, in fact our results show that the working space notion is not needed for a wide range of problems. To summarize Our contributions:

- We provide the first near optimum algorithm for the wavelet synopsis construction problem. The algorithm naturally extends to multiple dimensions.
- For the restricted case [5] provided approximation algorithms, however the space constraints were obeyed in expectation. The results for (optimum) algorithms with strict space bounds are $^\dagger$:

| Paper | Error  | Time          | Space                        | Working Space |
|-------|--------|---------------|------------------------------|---------------|
| [4]   | $\ell_\infty$ | $O(n^2B\log B)$ | $O(n^2B)$                    | $O(nB)$       |
|       | $\ell_k$   | $O(n^2B^2)$   | $O(n^2B)$                    | $O(nB)$       |
| [17]  | (weighted) $\ell_k$ | $O(n\log(n/B))$ | $?$                          | $?$           |
| This Paper | (weighted) $\ell_\infty$ | $O(n^2)$     | $O(n + B\log(n/B)) = O(n)$ | $O(n) + B\log(n/B)) = O(n)$ |
|       | (weighted) $\ell_k$ | $O(n^2 \log B)$ | $O(n)$                  | $O(n)$       |

$^\dagger$In [14] the space bounds are not explicitly provided, but the total space appears to be $O(n^2B/\log B)$ as well. The authors of [14] consider the same problem for a non-Haar basis, and is excluded from the discussion here.
We improve several histogram construction algorithms, e.g., V-Opt histograms, range query histograms, by simultaneously achieving the best known running time and space bounds. The results and a table comparing the results are presented in Section 4.1. Due to lack of space omit the improvements for the range query histograms, which are similar.

We believe the space efficient paradigm is applicable to other dynamic programs as well, and we demonstrate the improvements in case of Extended Wavelets in Section 4.2.

2 The Restricted (Haar) Wavelet Synopsis construction Problem

We will work with non-normalized wavelet transforms where the inverse computation is simply adding the coefficients that affect a coordinate\(^2\). The wavelet basis vectors are defined as (assume \(n\) is a power of 2):

\[
V_0(j) = \begin{cases} 
1 & \text{for all } j \\
1 & \text{for } (t-1)\frac{n}{2^r} + 1 \leq j \leq \frac{t}{2^r} - \frac{n}{2^{r+1}} \\
-1 & \text{for } \frac{nt}{2^r} - \frac{n}{2^{r+1}} + 1 \leq j \leq \frac{n}{2^r} (1 \leq t \leq \frac{n}{2^r}, 1 \leq s \leq \log n) 
\end{cases}
\]

The above definitions ensure \(W^{-1}(Z) = \sum_s Z_s V_s\). To compute \(W(X)\), the algorithms computes the average \(\frac{x_{t+1} + x_{t+2}}{2}\) and the difference \(\frac{x_{t+1} - x_{t+2}}{2}\) for each pair of consecutive elements as \(i\) ranges over 0, 2, 4, 6, . . . . The difference coefficients form the last \(n/2\) entries of \(W(X)\). The process is repeated on the \(n/2\) average coefficients - their difference coefficients yield the \(n/4 + 1, \ldots, n/2\)th coefficients of \(W(Z)\).

The process stops when we compute the overall average, which is the first element of \(W(Z)\). The wavelet basis functions naturally form a complete binary tree since their support sets are nested and are of size powers of 2 (with one additional node as a parent of the tree, see Figure 1). The \(x_j\) correspond to the leaves, denoted by boxes, and the coefficients correspond to the non-leaf nodes of the tree. This tree of coefficients is termed as the error tree (following [4]). Likewise assigning a value \(c_i\) to the coefficient corresponds to assigning \(+c_i\) to all leaves \(j\) that are left descendants (descendants of the left child) and \(-c_j\) to all right descendants. The leaves that are descendants of a coefficient are termed as the support of the coefficient. Recall that the Restricted (Haar) Wavelet construction problem is that given a set of \(n\) numbers \(X = x_1, \ldots, x_n\) the problem seeks to choose at most \(B\) terms from the wavelet representation \(W(X)\) of \(X\), say denoted by \(Z_R\), such that a (weighted) \(\ell_k\) norm of \(X - W^{-1}(Z_R)\) is minimized.

2.1 Reviewing Previous Algorithm(s)

It is immediate that the value of \(W^{-1}(Z_R)\) is fixed by the choices of all coefficients \(i\) such that \(j\) belongs to the support of \(i\). Suppose \(S\) is a subset of the ancestors of a coefficient \(i\). Thus a natural dynamic program emerges where we define \(E[i, b, S]\) to be the minimum contribution to the error from all \(j\) in the support of \(i\), such that exactly \(b\) coefficients that are descendants of \(i\) are chosen along with the coefficients of \(S\). The algorithm is given in Figure 1(b). Clearly the number of entries in the array \(E[]\) is \(Bn\) times \(2^r\) where \(r\) is the maximum number of ancestors of any node. It is easy to see that \(r = \log n + 1\) and thus the number of entries is \(n^2B\). For \(\ell_1\) measure we need to spend \(O(B)\) time in the minimization giving a running time of \(O(n^2B^2)\). For \(\ell_\infty\), we may perform binary search and only need \(\log B\) time (see [4]).

2.2 A Simple Improvement

Observation 1 A node \(i\) at level \(t_i\) can have at most \(2^{t_i} - 1\) descendants. Thus \(E[i, b, S]\) is meaningful only for \(2^i\) values of \(b\) (including \(b = 0\)). Further, the number of nodes at level \(t_i\) is \(\lceil \frac{n}{2^{t_i}} \rceil\) and the number of possible subsets of ancestors of a node is \(2^{\log n + 1 - t_i}\).

\(^2\)For normalized wavelets the normalization constant appears both in forward and inverse transform, all the results in the paper will carry over in that setting as well, with the introduction of the normalization constants at several places.
At each internal node $i$ to compute $E[i, b, S]$:

- We determine if we are choosing the coefficient $i$.
- Assuming we are, we decide how the remaining $b - 1$ coefficients are to be allocated between the two subtrees. If the children are $i_L$ and $i_R$, we are interested in
  \[
  \min_b E[i_L, b, S \cup \{i\}] + E[i_R, b - 1 - b', S \cup \{i\}]
  \]
- Assuming that we do not choose $i$ we are interested in a similar expression giving the overall minimization to be
  \[
  \min_{b'} \left\{ E[i_L, b', S \cup \{i\}] + E[i_R, b - 1 - b', S \cup \{i\}], E[i_L, b', S] + E[i_R, b - b', S] \right\}
  \]

Thus the number of $E[]$ entries to fill corresponding to $i$ is $2^{\log n + 1 - t_i} \min\{B, 2^{t_i}\}$. The time takes $2^{\log n + 1 - t_i} \min\{B^2, 2^{2t_i}\}$. Thus one way of computing the total time taken is

\[
\sum_{t_i=1}^{\log n} \frac{n}{2^{t_i}} 2^{\log n + 1 - t_i} \min\{B^2, 2^{2t_i}\} + B^2 = \sum_{t_i=1}^{\log n} \frac{n}{2^{t_i}} 2^{\log n + 1 - t_i} 2^{2t_i} + \sum_{t_i=\log B+1}^{\log n} \frac{n}{2^{t_i}} 2^{\log n + 1 - t_i} B^2 + B^2
\]

\[
= \sum_{t_i=1}^{\log B} 2n^2 + \sum_{u=1}^{\log n - \log B} \frac{n}{2^{u+\log B}} 2^{\log n + 1 - u - \log B} B^2 + B^2 = 2n^2 \log B + 2n^2 \sum_{u=1}^{\log n - \log B} \frac{1}{4^u} + B^2
\]

which is $O(n^2 \log B)$. In case of $\ell_\infty$, the expression $\sum_{t_i=1}^{\log n} \frac{n}{2^{t_i}} 2^{\log n + 1 - t_i} \min\{B \log B, t_i 2^{t_i}\} + B^2$ can be shown to be $O(n^2)$ using the same scheme and change of variables as above.

2.3 The Intuition and the new algorithm

The properties that stand out from the above dynamic program are

- There is no connection between $E[i, b, S]$ and $E[i', b', S']$ as long as $S \neq S'$.
- We do not need $E[i, b, S]$ while computing $E[i', b', S']$ unless $i$ is a child of $i'$ and either $S = S'$ or $S = S' \cup \{i'\}$.
- And finally, there is no need to allocate space for $E[i, b, S]$ while computing $E[i', b', S']$ if $i$ is an ancestor (not a descendant) of $i'$.

The simplest view of the new algorithm that computes the same table (but it is not stored in entirety at any time) is a parallel algorithm, where there is a processor at each node of the error tree. The algorithm at a node $i$ with children $i_L, i_R$ can be described as follows:

1. The node $i$ receives $S$ from its parent and seeks to return an array of size $B$ (or less) corresponding to $E[i, b, S]$ for $0 \leq b \leq B$. It actually receives

\[
v(i, S) = \sum_{i' \in S : \text{left descendant of } i'} c_{i'} - \sum_{i' \in S : \text{right descendant of } i'} c_{i'}
\]

2. To evaluate $\min_{b'} E[i_L, b', S \cup \{i\}] + E[i_R, b - 1 - b', S \cup \{i\}]$ the node $i$ passes $S \cup \{i\}$ to both of its children, i.e., $v(i_L, S \cup \{i\})$ and $v(i_R, S \cup \{i\})$. The children return the two arrays of size $B$ (or less), and the $\min_{b'}$ is performed for each $b$. Note that the right child can reuse the same space needed by the left child.

Figure 1: The Error Tree and the previous algorithm.
3. Now \( i \) passes \( S \) to the children and asks for \( E[i_L, b, S] \) for all \( b \) and likewise for \( i_R \).

4. The node \( i \) can now compute all \( E[i, b, S] \). The entire time spent at this node is \( \min\{2^{2t_i}, B^2\} \).

5. If \( i \) is the overall root, then \( i \) also performs a minimization over all \( b \) to find the solution with at most \( B \) coefficients.

**Lemma 1.1** No node receives the value \( v(i, S) \) twice for the same set \( S \).

The above shows that the algorithm is correct and runs in time \( O(n^2 \log n) \) (and \( O(n^2) \) for \( \ell_\infty \)). The next lemma is also immediate from the description of the algorithm:

**Lemma 1.2** The space required at node \( i \) is \( \min\{B, 2^{t_i}\} \), since this space is used for all \( S \).

Thus the total space required is \( O(B \log(n/B)) \) (the last \( \log B \) levels use geometrically decreasing space which sums to \( O(B) \) and \( \log n - \log B = \log(n/B) \)). Therefore if we consider the algorithm that simulates the parallel algorithm, we can conclude with

**Theorem 2** We can compute the error of optimum \( B \) term wavelet synopsis in time \( O(n^2 \log B) \) (and \( O(n^2) \) for \( \ell_\infty \)) using overall space \( O(n + B \log(n/B)) = O(n) \).

Observe that we can only compute the error, and we do not know which coefficients are in the synopsis.

### 2.4 How do we find the coefficients?

We now show how to retrieve the coefficients after finding the total error. When we find the optimum error, we also resolve (i) if the topmost coefficient is present or not and (ii) what is the allocation of the coefficients to the left and right children. Armed with these two pieces of information, we simply recurse/recompute, i.e., we pass the appropriate set (or \( v(i, S) \) values) to the two children and their respective allocations. Each child now finds the total error restricted to its subtree and each decides on the two pieces of information to set up the recursive game.

**Analysis:** Let the running time of the recompute strategy be \( f(n) \). To find the optimum error, we spend \( cn^2 \log B \) time and therefore we have the recursion:

\[
f(n) = cn^2 \log B + 2f(n/2)
\]

If we unroll the recursion one step, we see that \( f(n) = cn^2 \log B + 2c(n/2)^2 \log B + 4f(n/4) \). We can immediately observe that we are setting up a geometric sum and we can bound \( f(n) \) by \( 2cn^2 \log B \).

Therefore we conclude:

**Theorem 3** We can compute the complete solution, i.e., total error and the stored coefficients of the optimum \( B \) term wavelet synopsis in \( O(n^2 \log B) \) time \( (O(n^2) \text{ for } \ell_\infty) \) using overall space \( O(n + B \log(n/B)) \).

**Caveat:** We have to be careful and ensure that when we output the coefficients recursively, we output all the coefficients of the first half before outputting all the coefficients of the next half. In the process, we need to remember the partition of the buckets, the parameter \( b' \), for \( \log n \) levels. But since we have to remember only 1 number, the total space is \( O(n + B \log(n/B) + \log n) = O(n + B \log(n/B)) \).
3 Unrestricted Wavelet Synopsis construction Algorithms

We now show how to obtain an approximation algorithm for the general/unrestricted wavelet synopsis construction problem. We focus our attention on $\ell_k$ error, we indicate the changes necessary for the weighted case appropriately. Recall that the Wavelet synopsis problem is: Given a set of $n$ numbers $X = x_1, \ldots, x_n$, find a $Z \in \mathcal{R}^n$ with at most $B$ non-zero entries such that $\|X - W^{-1}(Z)\|_k$ is minimized.

The following will be an important observation leading towards a suitable algorithm: If we observe the previous algorithm based on assigning a processor to each coefficient in the error tree, we immediately observe that if for different subsets of ancestors, we receive the same value, i.e., $v(i, S) = v(i, S')$ for $S' \neq S$, we need not redo the computation. Note: that the savings cannot be guaranteed and in order to achieve the savings we have to increase the space bound.

Overview: The above will form a kernel of our algorithm for the (unrestricted) wavelet synopsis construction problem. We would actually perform the computation for all possible, anticipated values of $v(i, S)$. However, non-zero elements of $Z$ can have any real value and it is not clear how to restrict the set of values.

In what follows, we first describe the algorithm assuming that the wavelet coefficients belong to a set of zero entries such that $\|X - W^{-1}(Z)\|_k$ is minimized.

3.1 The Algorithm

**Definition 3.1** Let $E[i, v, b]$ be the minimum possible contribution to the overall error from all descendants of $i$ using exactly $b$ coefficients, under the assumption that the combined value of all ancestors chosen is $v$.

The overall answer is clearly $\min_b E[root, 0, b]$. A natural dynamic program is immediate, to compute $E[i, v, b]$ if we decide the best choice is to allocate $b'$ coefficients to the left and let the $i^{th}$ coefficient be $r$, then we need to add $E[i_L, v + r, b']$ and $E[i_R, b - b' - 1, v - r]$. The overall algorithm is:

1. The number of $b$ that are relevant to $i$ is $\min\{B, 2^{r_i}\}$. The node receives the $E[i_L, v', b'], E[i_R, v'', b'']$ from its children.
2. A non-root node computes $E[i, v, b]$ as follows:

   $$E[i, v, b] = \min \left\{ \begin{array}{ll} \min_{r, b'} E[i_L, v + r, b'] + E[i_R, v - r, b - b' - 1] & \text{ith coefficient is } r \\ \min_{b'} E[i_L, v, b'] + E[i_R, v, b - b'] & \text{ith coefficient not chosen} \end{array} \right. $$

3. If $i$ is the root, then $i$ computes

   $$\min_b \left\{ \begin{array}{ll} \min_{r, b'} E[i_L, 0, b'] + E[i_R, r, b - b' - 1] & \text{root coefficient is } r \\ \min_{b'} E[i_L, 0, b'] + E[i_R, 0, b - b'] & \text{root coefficient not chosen} \end{array} \right. $$

Note that the root can figure out (i) the optimum error (ii) if any coefficient corresponding to it is chosen and (iii) the value $r$ of the coefficient. After the final solution is computed, we apply the recompute strategy, and each node in the tree finds out if it has a coefficient in the answer and its value. The running time is

$$\sum_i |R| \min\{2^{r_i}, B\} \cdot |R| \min\{2^{r_i}, B\} = \sum_i |R|^2 \frac{n}{2^t} \min\{2^{2t}, B^2\} = |R|^2 n B$$

For $\ell_\infty$, the bound is $\sum_i |R|^2 \frac{n}{2^t} \min\{t 2^t, B \log B\} = O(n |R|^2 \log^2 B)$. The required space can be shown to be $O(RB \log(n/B))$ ensuring that the computation resembles a post-order traversal of the tree and we do not the tables of the children nodes once we are done. Thus for each level we may need at most 2 tables of size $R \min\{B, 2^t\}$, which sums to the above.
3.2 Computing $R$

Lemma 3.1 If the $\max_i |x_i|$ is $M$ then $\max_i |W(X)_i| \leq M$.

Proof: The $1^{st}$ coefficient is the average of all values and therefore cannot exceed $M$. Every other coefficient is half the average value of left half (of the support) minus half the average value of right half. Each cannot be more than $M$ in absolute value.

Lemma 3.2 If the optimum solution is $Z^*$ then $\max_i |Z^*_i| \leq 2n^{\frac{1}{2}} M$.

Proof: If $\max_i |W^{-1}(Z^*)_i| \geq 2n^{\frac{1}{2}} M$ then $\|X - W^{-1}(Z^*)\|_k \geq \|W^{-1}(Z^*)\|_k - \|X\|_k$ and

$$\|W^{-1}(Z^*)\|_k - \|X\|_k \geq \|W^{-1}(Z^*)\|_k - M n^{\frac{1}{2}} \geq \max_i |W^{-1}(Z^*)_i| - M n^{\frac{1}{2}} \geq M n^{\frac{1}{2}} \geq \|X\|_k$$

The all zero solution is a better solution, which is a contradiction. Now we apply Lemma 3.1 and get $\max_i |W(W^{-1}(Z^*)_i)| = \max_i |Z^*_i| \leq 2n^{\frac{1}{2}} M$, which proves the lemma.

In case of weighted $\ell_k$ the above is modified to $\max_i |Z^*_i| \leq 2n^{\frac{1}{2}} M \frac{1}{\min_i x_i}$. The next lemma follows from triangle inequality.

Lemma 3.3 If we round each non-zero value of the optimum $Z^*$ to the nearest multiple of $\delta$ thereby obtaining $\hat{Z}$, then $\|X - W^{-1}(\hat{Z})\|_k \leq \|X - W^{-1}(Z^*)\|_k + \delta n^{\frac{1}{2}}$ and $|R| \leq \frac{2n^{\frac{1}{2}} M}{\delta}$.

Therefore if we set $\delta = \epsilon M / n^{\frac{1}{2}}$ we can say that we have an additive approximation of $\epsilon M$ as well as $|R| = O(\epsilon n^{\frac{1}{2}})$. Therefore we conclude the following:

Theorem 4 We can solve the Wavelet Synopsis Construction problem with $\ell_k$ error with an additive approximation of $\epsilon M$ where $M = \max_i |x_i|$ in time $O(n^{1 + \frac{1}{2}} B e^{-2})$ and space $O(n + n^{\frac{1}{2}} \epsilon^{-1} B \log(n/B))$. For $\ell_\infty$ the running time is $O(n \epsilon^{-2} \log^2 B)$.

4 The theme of space efficiency and applications

A natural paradigm emerges from inspecting the above: If we can compute the total error and the best way to partition the problem into two halves of $\frac{n}{2}$ elements, we do not need to store the entire dynamic programming table – and thereby save space. If we can compute the overall error in time $f(n) = An^\alpha$ where $A$ is independent of $n$, then the time taken by the Recompute strategy is $g(n) = f(n) + 2g(n/2)$. The solution to the recurrence is $O(A n^\alpha)$ if $\alpha > 1$ and $O(A n \log n)$ if $\alpha = 1$.

We demonstrate the above idea in two examples. First, we show its impact in space efficient V-Opt histogram construction. Second, we show the applicability in a new synopsis technique, Extended Wavelets.

The idea also improves several results on range query histograms – however those algorithms are quite similar in spirit to the V-Opt histogram construction and we relegate the discussion to a fuller version of the paper. However the idea does help in reducing the space bound across the board – in fact for a large variety of problems it is immediate that the notion of working space, the space necessary to compute the value of the final answer, is not required any more. We can compute the entire answer, in the aforementioned working space.
4.1 V-Opt Histograms

The V-Opt histogram is a classic problem in synopsis construction. Given a set of \( n \) numbers \( X = x_1, \ldots, x_n \) the problem seeks to construct a \( B \) piecewise constant representation \( H \) such that \( \| X - H \|_2 \) (or its square) is minimized. Since their introduction in query optimization in \([11]\), and subsequently in approximate query answering ( \([11]\) among others), histograms have accumulated a rich history \([12]\). Several different optimization criteria have been proposed for histogram construction, e.g., \( \epsilon_1 \), relative error, \( \epsilon_{\infty} \), to name a few. However most of them are based on a dynamic program similar to the V-Opt case. Thus the V-Opt histograms provide an excellent foil to discuss all of the measures at the same time. As mentioned in the introduction, \([13]\) gave a \( O(n^2B) \) time algorithm to find the optimum histogram using \( O(nB) \) space. They observed that the space could be reduced to \( O(n) \) at the expense of increasing the running time to \( O(n^2B^2) \). The data stream algorithms\(^5\) of \([9]\) (extended in \([8]\)) represent sparse dynamic tables – but the space is still \( \tilde{O}(B^2) \), a quadratic in \( B \). In those algorithms the \( \tilde{O}(B^2) \) space performs a double role of storing the coefficients as well as maintaining a frontier.

This is somewhat remedied in \([7, 18]\), where a robust wavelet representation of \( \tilde{O}(B) \) coefficients is constructed and then a dynamic program in the fashion of \([13]\) or \([9]\) restricted to the endpoints of the support regions is used. The dynamic program of \([13]\) can be used to compute the answer in \( O(B) \) space, but with an extra factor of \( \tilde{B} \) in running time. Therefore, irrespective of offline or streaming computation there was a tradeoff between large space and an increased running time – this is the penalty referred to in the introduction. This is the first paper which removes that penalty and gives an algorithm that simultaneously achieves the best known space and time bounds.

| Paper | Stream | Factor | Time | Space | Working space |
|-------|--------|--------|------|-------|--------------|
| \([13]\) | No | Opt | \( O(n^2B) \) | \( O(nB) \) | \( O(n) \) |
| \([9]\) | Yes | \( (1 + \epsilon) \) | \( O(nB^2\epsilon^{-1}\log n) \) | \( O(B^2\epsilon^{-1}\log n) \) | \( O(n) \) |
| \([7]\) | Yes | \( (1 + \epsilon) \) | \( O(n + B^4\epsilon^{-8}\log^4 n) \) | \( O(B^4\epsilon^{-8}\log^2 n) \) | \( O(n) \) |
| \([10]\) | No | \( (1 + \epsilon) \) | \( O(n + B^4\epsilon^{-2}\log n + n) \log n) \) | \( O(n + B^4\epsilon^{-2} \log^2 n) \) | \( O(n + B\epsilon^{-1}) \) |
| \([18]\) | Yes | \( (1 + \epsilon) \) | \( O(n + B^4\epsilon^{-2}\log(1/\epsilon) \log n) \) | \( O(B\epsilon^{-2}(1/\epsilon) \log n + B^2/\epsilon) \) | \( O(n) \) |

| This Paper | No | Opt | \( O(n^2B) \) | \( O(n) \) | \( O(n) \) |
| No | \( (1 + \epsilon) \) | \( O(n + B^3\epsilon^{-2} \log n \log n) \) | \( O(n + B\epsilon^{-1}) \) | \( O(n) \) |
| Yes | \( (1 + \epsilon) \) | \( O(n + B^3\epsilon^{-2}(\log 1/\epsilon) \log n) \) | \( O(B\epsilon^{-2}(1/\epsilon) \log n + B/\epsilon) \) | \( O(n + B\epsilon^{-1}) \) |

Algorithm idea: Due to lack of space, we indicate the modification to the optimum algorithm. The modifications to the approximation and streaming algorithms are similar. The optimal algorithm maintains \( E[i, b] \) which is the minimum error of expressing the interval \([1, i]\) by at most \( b \) buckets (intervals where the representation is constant). A natural dynamic programming arises: \( E[i, b] = \min_{j<i} E[j, b - 1] + e(j + 1, i) \) where \( e(j, i) \) is the minimum error of a single bucket\(^6\). The running time is \( O(n^2B^2) \). If we are interested in computing only the final answer, there is an \( O(n) \) space algorithm which computes \( E[i, 1] \) for all \( i \), and then extends that to \( b = 2, 3 \), etc.

If \( i > \frac{n}{2} \) we maintain \( A[i] \) to be the starting point of the bucket that contains the \( x_i \) for the best representation of \([1, i]\) by \( b \) buckets, and \( B[i] \) to be the ending point of that interval, and \( C[i] \) to be the

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\(^5\)Note that by the streaming model we refer to the “sorted” or “aggregate” model, most useful in time series data, where the input is \( x_i \) in increasing order of \( i \). Only \([8]\) applies to the general “turnstile” or “update” model, but seems to have high polynomial dependence on \( B\epsilon^{-1}\log n \). See \([16, 2]\) for more details on data stream models.

\(^6\)It is straightforward to show that the minimum error is achieved by the mean of \( x_{i+1}, \ldots, x_i \).
number of buckets used before \( A[i] \). This requires \( O(n) \) space, and is updated as shown below. Now, after we compute \( E[n,B] \) we can divide the problem into two parts, representing \([1, A[i]]\) using \( C[i] \) buckets and \([B[i] + 1, n]\) by \( B - C[i] - 1 \) buckets. Note that each subproblem is defined on \( \frac{B}{2} \) or less elements. Therefore the \textit{Recompute strategy} will run in time \( O(n^2 B) \) as well and compute all the coefficients.

1. \( A[i] = 0 \) if \( i \leq \frac{n}{2} \) and 1 otherwise. \( B[i] = 0 \) if \( i \leq \frac{n}{2} \) and \( i \) otherwise. \( c[i] = 0 \) for all \( i \).
2. For \( b = 2 \) to \( B \) do
3. For \( i = 2 \) to \( n/2 \) do
4. \( E[i,b] = \min_{j<i} E[j,b-1] + e(j+1,i) \)
5. For \( i = n/2 \) to \( n \) do
6. \( E[i,b] = \min_{j<i} E[j,b-1] + e(j+1,i) \)
7. If \( j \) (which achieved the minimum) \( \leq 2 \) then \( newA[i] = j+1, newC[i] = b, newB[i] = i \).
8. Else \( newA[i] = A[j], newB[i] = B[j], newC[i] = C[j] \).
9. \( A \leftarrow newA, B \leftarrow newB, C \leftarrow newC \).
10. Recurse using \( A[n], B[n], C[n] \) to compute the coefficients.

Figure 2: The \( O(n) \) space optimum algorithm

Observe that we wave kept the \( E[j,b-1], E[i,b] \) notation, but we can reuse two arrays of size \( n \) for this purpose (and keep switching them as \( newE, E \) etc.) – the overall space required is \( O(n) \). We now know the final solution \( E[n,B] \) and how to partition the problem. For \textit{offline approximation algorithm}, when we recurse, we have to add the approximate error \( E'[B[i] + 1, C[i] + 1] \) to all the elements on the right subproblem (since we build histograms with error increasing by \( 1 + \epsilon \) factor, this ”shift” is needed). Due to lack of space, the details are relegated to the full version.

### 4.2 Extended Wavelets

Extended wavelets were introduced in [8]. The central idea is that in case of multi-dimensional data, there can be significant saving of space if we use a non-standard way of storing the information. There are several standard ways of extending 1-dimensional (Haar) wavelets to multiple dimensions. The wavelet basis corresponds to high-dimensional squares. But irrespective of the number of dimensions, the format of the synopsis is a pair of numbers (\textit{coefficient index, value}). In Extended Wavelets we perform wavelet decomposition independently in each dimension but then we store tuples consisting of the coefficient index, a bitmap indicating the dimensions for which the coefficient in that dimension is chosen, and a list of values. Since the coefficient number and the bitmap is shared across the coefficients, we can store more coefficients than a simple union of unidimensional transforms.

Notice that there is no interaction between the benefits of storing coefficient \( i \) and \( i' \). The problem reduces naturally to a \textit{Knapsack} problem with a twist that each item (coefficient \( i \)) can be present in varying sizes (how many values corresponding to different dimensions are stored). However the variant also has a simplifying feature that the space bound is polynomially bounded, therefore allowing a simple dynamic program. The program estimates \( E[i,b] \) which indicates the minimum error on using \textit{at most} \( b \) space and storing only a subset of the first \( i \) coefficients.

The idea is relatively new, and it remains to be seen if Extended wavelets are applied widely. But it is an intriguing and novel idea in synopsis construction and serve as an example of the broad applicability of the ideas in this paper. This paper is also the first (almost) linear \((O(B), ignoring M)\) space algorithm in the streaming (as well as offline) model. We present the results on the optimum algorithms below.\footnote{The input is \( n \) tuples in \( M \) dimensions and the total synopsis size is \( B \). The papers [3, 8] contain other approximation algorithms that are not relevant to our context. The extended version of [8] reduces Extended Wavelets to a problem similar to \textit{V-Opt} histogram construction and gives a \( O(NM) \) time algorithm using dynamic programming. The ideas of this paper naturally}

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Algorithm Idea: We follow the previous algorithms and introduce a few small changes and a more careful analysis. For each item $i$ we compute the best profit if $i$ is allocated size $j$. This is done in time $O(nM \log M)$ as in [8]. For each $1 \leq j \leq M$ we maintain the top $B/j$ items corresponding to size $j$. For each $j$ we can achieve this in $O(B/j)$ space and $O(n)$ running time (using details from [7]), using overall $O(nM)$ time and $\sum_j (B/j)j = O(BM)$ space. The optimum answer uses items and sizes from this list only. The total number of item-size pairs are $\sum_j (B/j) = O(B \log M)$.

We can sort this list in lexicographic order. Suppose item $i$ has $x_i \geq 1$ occurrences (thus $\sum_i x_i = O(B \log M)$). The dynamic program to extend the answer to $i$ (from the item before $i$) first needs to guess/choose which of the $x_i$ occurrences are used (or none) and compute the best solution for each $B$. The time taken is $c(x_i + 1)B$ at $i$, which totals to at most $2cB^2 \log M$.

We maintain a $O(B)$ array where $P[z]$ corresponds to the best profit for space $z$ up to the current $i$. For space efficiency, for $z \geq B/2$ we keep track of $Q[z]$ which contains the pair $\langle i', r, b' \rangle$ s.t. the optimum solution for space $z$ for current $i$ uses space $b' < B/2$ Upton $i'$ and a size $r$ copy of $i'$ with $b' + r \geq B/2$. In other words, the crossing point where we crossed $B/2$ space for that solution (which remains same even if we extend it later).

We now recurse with $b, b' \leq B/2$ on the two parts. Now each item contributes $c(x_i + 1)B/2$ adding up to less than $cB^2 \log M$. Once again we have a geometric sum which sums up to $O(B^2 \log M)$ for the entire recursion.

Acknowledgments: We would like to thank Hyoongmin Park and Kyuseok Shim for many interesting discussions.

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imply improvements to the space requirement under the assumption that $B \ll NM$. The reduction is somewhat detailed and is omitted in this draft.
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