HYPERCYCLICITY OF SHIFTS ON WEIGHTED \( L^p \) SPACES OF DIRECTED TREES

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Abstract. In this paper, we study the hypercyclicity of forward and backward shifts on weighted \( L^p \) spaces of a directed tree. In the forward case, only the trivial trees may support hypercyclic shifts, in which case the classical results of Salas \cite{Salas} apply. For the backward case, nontrivial trees may support hypercyclic shifts. We obtain necessary conditions and sufficient conditions for hypercyclicity of the backward shift and, in the case of a rooted tree on an unweighted space, we show that these conditions coincide.

In memory of Jaime Cruz Sampedro, mathematician, teacher, colleague, and friend.

1. Introduction

A bounded operator on a Banach space is called hypercyclic if there exists a vector such that its orbit under the operator is dense in the space. The study of hypercyclicity (in other types of spaces) can be traced back to the first half of the 20th century, to the papers of Birkhoff \cite{Birkhoff} and MacLane \cite{MacLane}. The first example of a hypercyclic operator on a Banach space was given by Rolewicz \cite{Rolewicz} in 1969, but it was not until the last two decades of the 20th century that the study of hypercyclicity really took off. Instead of giving here the detailed history of the advances in hypercyclicity in the past 35 years, we refer the reader to the excellent books by Grosse-Erdmann and Peris \cite{Bayart-Matheron} and Bayart and Matheron \cite{Bayart-Matheron2}, where the reader can find more information about this concept and its importance.

One large source of examples and counterexamples in the study of bounded operators is the class of weighted shifts. The study of weighted shifts was initiated in the now classical paper of Shields \cite{Shields} and continued by many authors. The characterization of the hypercyclicity of weighted shifts is due to Salas \cite{Salas} (see the books \cite{Bayart-Matheron} and \cite{Bayart-Matheron2} for an alternative statement of this characterization). Many other classes of operators on Banach spaces have been shown to be hypercyclic, under certain conditions. Two other famous families of operators on Hilbert spaces that

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contain hypercyclic operators, are adjoints of multiplication operators (see [13]) and composition operators on spaces of holomorphic functions (see, e.g., [22]).

The interest of the study of operators on infinite trees is motivated mainly by the research in harmonic analysis dealing with the Laplace operator on discrete structures, perhaps initiated in the papers [7, 8]. In particular, infinite trees can be seen as the natural discretizations of the hyperbolic disk. Much more information about these topics can be found in the papers [1, 2, 3, 4, 9, 10, 11, 19]. We should mention that the paper [19] studies hypercyclicity for composition operators defined on the boundary of nondirected trees.

In [16], Jabłoński, Jung and Stochel initiated the study of weighted shifts on Hilbert spaces of functions defined on infinite directed trees. In their paper, they study many operator-theoretic properties of these operators, such as boundedness, hyponormality, subnormality and spectral properties.

Motivated by the work in [16], in this paper we study the hypercyclicity of shifts on directed trees on the weighted $L^p$ space of a directed tree. In concrete, we show in Section 3 that “forward” shifts are never hypercyclic unless they reduce to the classical cases, in which the characterization by Salas mentioned above can be applied. In Section 4 we find a concrete form for the adjoint of the shift and define what we mean by a “backward shift”. More interestingly, we show in Sections 5 and 6 that this backward shift on weighted directed trees, may be hypercyclic if some conditions are satisfied. In concrete, the main results of this paper provide necessary conditions and sufficient conditions for hypercyclicity of the backward shift in the case where the tree has a root. These two conditions coincide when the space is unweighted, in which case the hypercyclicity of the operator depends on a simple property of the tree, that of having no “free ends”. In the case of the unrooted tree, we only give necessary conditions and show an example when these conditions are satisfied. When applied to the classical backward shifts, all of these conditions reduce to the ones obtained by Salas.

Before we begin, we should mention that in [16], the authors study the weighted shift on an unweighted $L^2$ space of the tree. In this paper we prefer to concentrate on the unweighted shift on the weighted spaces $L^p$ of the tree. The reason for this is that the results are cleaner in the case of the weighted space, as is also the case of the classical shifts. Results for the hypercyclicity of the weighted shift can be obtained by similarity with the shift of the weighted space, as it is done in, for example, [6, 15]. We leave these results as an exercise for the interested reader.

2. Definitions and Notation

In this section, we set the basic definitions and notation needed for the rest of the paper. We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ the sets of natural numbers, the nonnegative integers, the integers, the real numbers, the positive real numbers, and the complex numbers, respectively.

**Hypercyclicity.** We first state the main definition in this paper and a few comments about it. After that, we present the main tool used to prove that operators are hypercyclic.

**Definition 2.1.** Let $\mathcal{B}$ be a Banach space and $S : \mathcal{B} \rightarrow \mathcal{B}$ a bounded operator. We say that $S$ is hypercyclic if there exists a vector $x \in \mathcal{B}$ such that

$$\{S^n x : n \in \mathbb{N}_0\}$$
is dense in $\mathcal{B}$. The (necessarily) nonzero vector $x$ is called a hypercyclic vector.

Observe that if the operator $S : \mathcal{B} \to \mathcal{B}$ is hypercyclic, the Banach space $\mathcal{B}$ is necessarily separable. Observe also that, if $x$ is a hypercyclic vector for $S$, then, for each $n \in \mathbb{N}$, the vector $S^n x$ is also a hypercyclic vector and hence the hypercyclic vectors form a dense set (in fact, as is well-known, they form a dense $G_δ$-subset of $\mathcal{B}$).

One does not need to explicitly find a vector $x$ that satisfies the definition above to show that an operator $S : \mathcal{B} \to \mathcal{B}$ is hypercyclic. The following theorem gives an extremely useful sufficient condition for hypercyclicity.

**Theorem 2.2 (Hypercyclicity Criterion).** Let $\mathcal{B}$ be a separable Banach space and $S : \mathcal{B} \to \mathcal{B}$ a bounded operator. Assume there exists a dense subset $X \subseteq \mathcal{B}$, an increasing sequence of natural numbers $\{n_k\}$, and a sequence of functions $T_{n_k} : X \to \mathcal{B}$ such that

1. $S^{n_k} x \to 0$ for each $x \in X$,
2. $T_{n_k} x \to 0$ for each $x \in X$, and
3. $S^{n_k} T_{n_k} x \to x$ for each $x \in X$.

Then $S$ is a hypercyclic operator.

There are several versions of this criterion, which is also referred to as the Kitai–Gethner–Shapiro Criterion (for the history of this criterion and its importance in the development of the field, the reader is referred to the Sources and Comments section of Chapter 3 in [15]). A proof of the version presented here can be found in [5, Theorem 1.6] and [15, Theorem 3.12].

**Trees.** We now state the relevant definitions and notations for directed trees and directed graphs introduced in [10], which we will use in this paper (we assume the reader is familiar with the basic definition of a graph). A directed graph $G = (V, E)$ is a pair consisting of a countably-infinite set $V$ (called the set of vertices of $G$) and a subset $E$ of $V \times V$ (called the set of directed edges). The indegree of a vertex $v$ is the cardinality of the set $\{u \in V : (u, v) \in E\}$ and the outdegree of $v$ is the cardinality of the set $\{u \in V : (v, u) \in E\}$. We only consider graphs $G = (V, E)$ which are locally finite; i.e., both the indegree and the outdegree of each vertex are finite.

A directed graph has a directed circuit if there exist $n \in \mathbb{N}$, $n \geq 2$, and a set of distinct vertices

$$\{u_1, u_2, \ldots, u_n\},$$

such that $(u_j, u_{j+1}) \in E$ for each $j = 1, 2, \ldots, n-1$, and $(u_n, u_1) \in E$. Furthermore, we also say that the directed graph has a directed circuit if there exists $v \in V$ with $(v, v) \in E$.

The underlying graph of a directed graph $G = (V, E)$ is the graph $\tilde{G} = (V, \tilde{E})$ with vertex set $V$ and (undirected) edge set $\tilde{E} := \{(u, v) : (u, v) \in E\}$. Recall that, given distinct vertices $v$ and $w$ in a graph $\tilde{G}$, a path is a finite set of distinct vertices

$$\{v = u_1, u_2, u_3, \ldots, u_n = w\}$$

such that $(u_j, u_{j+1}) \in E$ for each $j = 1, 2, \ldots, n-1$; we say such a path has length $n$. A graph $\tilde{G}$ is connected if for each pair of vertices $v$ and $w$ there exists a path between $v$ and $w$. We denote by $\text{dist}(v, w)$ the length of the shortest path in the
undirected graph joining \( v \) and \( w \) and by \( \text{dist}(v, H) \) the infimum of the distances of \( v \) to the nonempty set of vertices \( H \).

We can now say what we mean by a directed tree.

**Definition 2.3.** A directed graph \( T = (V, E) \) is a directed tree if

- it has no directed circuits,
- the underlying graph of \( T \) is connected, and
- the indegree of every vertex is either zero or one.

If \( T = (V, E) \) is a directed tree, we say that \( v \in V \) is a root if its indegree is zero. A vertex \( v \in V \) is called a leaf if its outdegree is zero.

It can be easily seen \([16]\) that if a directed tree has a root, then the root is unique. If there is a root, we call the tree rooted and we denote the root by the symbol \( \text{root} \). If there is no root, we say the tree is unrooted.

We also need to set some notation. The third condition in the definition above is necessary for the following definition to make sense.

**Definition 2.4.** Let \( T = (V, E) \) be a directed tree. Given a vertex \( v \in V \), \( v \neq \text{root} \), we define its parent as the unique vertex \( u \) such that \((u, v) \in E\) and we denote it by \( u := \text{par}(v) \). For an integer \( n \geq 2 \), we inductively define the operator \( \text{par}^n \) as \( \text{par}^n(v) := \text{par}(\text{par}^{n-1}(v)) \), whenever \( \text{par}^{n-1}(v) \neq \text{root} \). In such a case, we say that \( v \) has an \( n \)-ancestor and we denote the set of all vertices that have \( n \)-ancestors as \( V^n \). Also, if \( u = \text{par}(v) \) we say that \( v \) is a child of \( u \) and we denote the set of children of \( u \) by \( \text{Chi}(u) \). For \( n \in \mathbb{N} \) and \( u \in V \), we also define the set \( \text{Chi}^n(u) := \{ v \in V : v \text{ has an } n \text{-ancestor and } \text{par}^n(v) = u \} \).

If \( v \in \text{Chi}^n(u) \) for some \( n \in \mathbb{N} \), we say that \( v \) is a descendant of \( u \).

We will use frequently, and without mentioning it, the equivalence between \( u = \text{par}^n(v) \) and \( v \in \text{Chi}^n(u) \). Also, observe that if the tree \( T \) is unrooted, then \( V^n = V \) for all \( n \in \mathbb{N} \).

**L^p space of a tree.** Lastly, we define the Banach spaces we will be dealing with. First, we mention that given a countable set \( V \), we will refer (abusing the notation) to the set \( \lambda = \{ \lambda_v \in \mathbb{R}_+ : v \in V \} \), indexed by \( V \) as a sequence, and we will denote it by \( \lambda = \{ \lambda_v \}_{v \in V} \). The Banach spaces we use throughout this paper are always vector spaces over the complex numbers.

**Definition 2.5.** Let \( 1 \leq p < \infty \) and let \( T = (V, E) \) be a directed tree. Let \( \lambda = \{ \lambda_v \}_{v \in V} \) be a sequence of positive numbers. We denote by \( L^p(T, \lambda) \) the space of complex-valued functions \( f : V \to \mathbb{C} \) such that

\[
\sum_{v \in V} |f(v)|^p \lambda_v < \infty.
\]

This is a Banach space if we endow it with the norm

\[
\|f\|_p = \left( \sum_{v \in V} |f(v)|^p \lambda_v \right)^{1/p}.
\]

If \( p = 2 \), this is also a Hilbert space (with the obvious inner product). We do not consider in this paper the space \( L^\infty(T, \lambda) \), since it is not separable, regardless of the choice of positive weights \( \lambda \) (e.g., \([18]\ p. 115\)). Observe that the only case of
interest for us is when the sequence $\lambda$ is composed of strictly positive numbers: if $\lambda_v = 0$ for some $v$, it is easily seen that the space can then be written as a direct sum of (perhaps infinitely many) spaces on smaller directed trees.

The graph structure, obviously, has nothing to do with Banach space structure of $L^p(T, \lambda)$ itself. What is interesting in this setting, are the operators that we can build here, which we define in the next section.

3. The Shift Operator and its Hypercyclicity

We can now introduce one of the main objects of study of this paper.

**Definition 3.1.** Let $T = (V, E)$ be a directed tree, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence and let $1 \leq p < \infty$. The shift $S : L^p(T, \lambda) \to L^p(T, \lambda)$ is the operator defined as

$$(Sf)(v) = \begin{cases} f(\text{par}(v)), & \text{if } v \neq \text{root}, \\ 0, & \text{if } v = \text{root}. \end{cases}$$

The following proposition was established for weighted shifts (instead of shifts on weighted spaces) in the case $p = 2$ in [16]. The proof is the same for our case, but we include it here for the sake of completeness.

**Proposition 3.2.** Let $T = (V, E)$ be a directed tree, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence and let $1 \leq p < \infty$. The operator $S : L^p(T, \lambda) \to L^p(T, \lambda)$ is bounded if and only if

$$\sup_{u \in V} \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u} < \infty.$$ 

In either case,

$$\|S\| := \left( \sup_{u \in V} \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u} \right)^{1/p}.$$ 

**Proof.** Assume $M := \sup_{u \in V} \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u} < \infty$. Let $f \in L^p(T, \lambda)$. Then

$$\|Sf\|_p = \sum_{v \in V} |(Sf)(v)|^p \lambda_v$$

$$= \sum_{v \in V; v \neq \text{root}} |f(\text{par}(v))|^p \lambda_v$$

$$= \sum_{u \in V} |f(u)|^p \lambda_u \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u}$$

$$\leq M \sum_{u \in V} |f(u)|^p \lambda_u$$

$$= M \|f\|_p^p.$$ 

Hence, $\|Sf\|_p \leq M^{1/p} \|f\|_p$. Thus $S$ is bounded and $\|S\| \leq M^{1/p}$.

On the other hand, assume $S$ is bounded. Let $u \in V$, and denote by $\chi_u$ the characteristic function of the vertex $u$. Define $f_u := \frac{1}{\lambda_u^{1/p}} \chi_u$. Clearly, $\|f_u\|_p = 1$. 

Then, if \( v \in V, v \neq \text{root} \), we have \( Sf_u(v) = f_u(\text{par}(v)) \) and this is zero unless \( u = \text{par}(v) \). Therefore,

\[
\|Sf_u\|_p^p = \sum_{v \in V} |(Sf_u)(v)|^p\lambda_v = \sum_{v \in V, v \neq \text{root}} |f_u(\text{par}(v))|^p\lambda_v = \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u}.
\]

Hence

\[
\sup_{u \in V} \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u} \leq \|S\|_p^p,
\]

which implies that \( \sup_{u \in V} \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u} < \infty \) and

\[
\|S\| = \left( \sup_{u \in V} \sum_{v \in \text{Chi}(u)} \frac{\lambda_v}{\lambda_u} \right)^{1/p},
\]

concluding the proof. \( \square \)

Our first observation is that the shift \( S \) cannot be hypercyclic if the tree \( T \) has a root (for example, if \( S \) is the unilateral forward shift on the tree \( \mathbb{N}_0 \)). Note that

\[
(S^nf)(v) = \begin{cases} f(\text{par}^n(v)), & \text{if } v \in V^n, \text{ and} \\ 0, & \text{if } v \notin V^n, \end{cases}
\]

for any \( f \in \mathbf{L}^p(T, \lambda) \) and any \( n \in \mathbb{N} \).

**Proposition 3.3.** Let \( T = (V, E) \) be a directed tree, let \( \lambda = \{\lambda_v\}_{v \in V} \) be a positive sequence and let \( 1 \leq p < \infty \). If \( T \) has a root, then \( S : \mathbf{L}^p(T, \lambda) \to \mathbf{L}^p(T, \lambda) \) is not hypercyclic.

**Proof.** Since \( (Sf)(\text{root}) = 0 \) for every \( f \in \mathbf{L}^p(T, \lambda) \), it follows that \( (S^nf)(\text{root}) = 0 \) for all \( n \in \mathbb{N} \). Let \( \chi_{\text{root}} \) denote the characteristic function of \( \text{root} \). If \( f \) were a hypercyclic vector, it would follow that there exists an increasing sequence \( \{n_k\} \) of positive integers such that

\[
\|S^{n_k}f - \chi_{\text{root}}\|_p \to 0,
\]

as \( k \to \infty \). But since

\[
(\lambda_{\text{root}})^{1/p} = |0 - 1| (\lambda_{\text{root}})^{1/p} = |(S^{n_k}f)(\text{root}) - \chi_{\text{root}}(\text{root})| (\lambda_{\text{root}})^{1/p} \leq \|S^{n_k}f - \chi_{\text{root}}\|_p,
\]

this is a contradiction. Hence \( f \) cannot be a hypercyclic vector and \( S \) cannot be a hypercyclic operator. \( \square \)

The next observation is that the shift \( S \) cannot be hypercyclic if the tree \( T \) has a vertex with outdegree larger than 1.

**Proposition 3.4.** Let \( T = (V, E) \) be a directed tree, let \( \lambda = \{\lambda_v\}_{v \in V} \) be a positive sequence and let \( 1 \leq p < \infty \). If \( T \) has at least one vertex of outdegree at least 2, then \( S : \mathbf{L}^p(T, \lambda) \to \mathbf{L}^p(T, \lambda) \) is not hypercyclic.
Proof. Let \( w \) be the vertex with outdegree \( n \), with \( n \geq 2 \) and let \( v_1 \) and \( v_2 \) be two different elements in \( \text{Chi}(w) \). Observe that \( \text{par}(v_1) = \text{par}(v_2) \) and hence

\[
(S^k f)(v_1) = f(\text{par}^k(v_1)) = f(\text{par}^k(v_2)) = (S^k f)(v_2)
\]

for all \( k \in \mathbb{N} \) such that \( v_1 \) and \( v_2 \) have \( k \)-ancestors. If \( v_1 \) and \( v_2 \) do not have \( k \)-ancestors, then \( (S^k f)(v_1) = 0 = (S^k f)(v_2) \). Thus

\[
(S^k f)(v_1) = (S^k f)(v_2)
\]

for every \( k \in \mathbb{N} \).

Let \( \epsilon > 0 \) such that \( \epsilon < \frac{\lambda_{v_1}^{-1/p} + (\lambda_{v_2})^{-1/p}}{p} - 1 \). If \( f \) were a hypercyclic vector for \( S \), there would exist \( N \in \mathbb{N} \) such that

\[
\|S^N f - \chi_{v_1}\|_p < \epsilon.
\]

We have then that

\[
|(S^N f)(v_1) - \chi_{v_1}(v_1)|((\lambda_{v_1})^{1/p} \leq \|S^N f - \chi_{v_1}\|_p < \epsilon,
\]

and

\[
|(S^N f)(v_2) - \chi_{v_1}(v_2)|((\lambda_{v_2})^{1/p} \leq \|S^N f - \chi_{v_1}\|_p < \epsilon.
\]

And hence, we have

\[
|(S^N f)(v_1) - 1|(\lambda_{v_1})^{1/p} < \epsilon \quad \text{and} \quad |(S^N f)(v_2)|((\lambda_{v_2})^{1/p} < \epsilon.
\]

Define \( z := (S^N f)(v_1) = (S^N f)(v_2) \). We then have

\[
|z - 1| < \epsilon(\lambda_{v_1})^{-1/p} \quad \text{and} \quad |z| < \epsilon(\lambda_{v_2})^{-1/p}.
\]

But the first inequality above implies that

\[
1 - |z| < \epsilon(\lambda_{v_1})^{-1/p}
\]

and hence that

\[
1 - \epsilon(\lambda_{v_1})^{-1/p} < |z| < \epsilon(\lambda_{v_2})^{-1/p}
\]

which in turn implies that

\[
1 - \epsilon(\lambda_{v_1})^{-1/p} < \epsilon(\lambda_{v_2})^{-1/p},
\]

which contradicts the choice of \( \epsilon \). Hence \( f \) cannot be a hypercyclic vector and \( S \) cannot be a hypercyclic operator.

The previous two propositions imply that in order for \( S \) to be hypercyclic the tree \( T \) cannot have a root and cannot have vertices of outdegree larger than 1. It is easy to see, then, that the tree must be either isomorphic to the directed graph \((\mathbb{Z}, \{(n, n + 1) : n \in \mathbb{Z}\})\) if \( T \) has no leaf (see the picture below),

\[
\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots
\]

or isomorphic to the directed graph \((\mathbb{N}_0, \{(n + 1, n) : n \in \mathbb{N}_0\})\) if \( T \) has a leaf (see the picture below).

\[
0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots
\]
In the former case, $S$ is just a bilateral shift on a weighted $\ell^p(\mathbb{Z})$ space and in the latter case $S$ is just a unilateral backward shift on a weighted $\ell^p(\mathbb{N}_0)$ space. The hypercyclicity in these two cases has been characterized by Salas [21] (see [5, Theorems 1.38 and 1.40] and [15, Theorems 4.3 and 4.12] for alternative statements of Salas’ result).

Note added: The referee has kindly pointed out to us that in [14], Grosse-Erdmann has considered what he calls “weighted pseudo-shifts” on sequence spaces, and characterized their hypercyclicity. It is not hard to see that the shifts considered in Section 3 are weighted pseudo-shifts and, therefore, Grosse-Erdmann’s characterization applies. Nevertheless, we should point out that, for the case where the structure of directed trees is available, our results give an easier-to-check formulation of the characterization, since we are able to tell that the shift may be hypercyclic only if the tree reduces to one of the cases already considered by Salas in [21]. We thank the referee for the observation.

4. The Adjoint of the Shift Operator and the Backward Shift

Our goal in this section is to identify the adjoint operator of the shift operator on a directed tree. The case for the Hilbert space adjoint ($p = 2$) was done in [16]. After we identify the adjoint, we will define the backward shift.

The following result is standard in the theory of $L^p$ spaces.

**Proposition 4.1.** Let $T = (V, E)$ be a directed tree, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence, let $1 < p < \infty$ and let $q = \frac{p}{p-1}$. For $g \in L^q(T, \lambda)$ define $\Phi_g : L^p(T, \lambda) \to \mathbb{C}$ as

$$\Phi_g(f) = \sum_{v \in V} f(v)g(v)\lambda_v.$$ 

Then $\Phi_g$ is a bounded linear functional on $L^p(T, \lambda)$. Conversely, if $\Phi$ is a bounded linear functional on $L^p(T, \lambda)$, there exists $g \in L^q(T, \lambda)$ such that $\Phi = \Phi_g$. Moreover, $\|\Phi_g\| = \|g\|_q$.

Henceforth, we identify the dual space of $L^p(T, \lambda)$ with $L^q(T, \lambda)$ and we will use the identification of the vector $g \in L^q(T, \lambda)$ with the functional $\Phi_g$ on $L^p(T, \lambda)$. We can now compute the adjoint of a shift.

**Proposition 4.2.** Let $T = (V, E)$ be a directed tree, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence, let $1 < p < \infty$, and let $q := \frac{p}{p-1}$. Assume the operator $S : L^p(T, \lambda) \to L^p(T, \lambda)$ is bounded. Then $S^* : L^q(T, \lambda) \to L^q(T, \lambda)$ is given by

$$(S^* g)(u) = \sum_{v \in \text{Chi}(u)} g(v)\frac{\lambda_v}{\lambda_u},$$

for each $g \in L^q(T, \lambda)$ and $u \in V$.

**Proof.** Let $g \in L^q$ and define $h : T \to \mathbb{C}$ as

$$h(u) := \sum_{v \in \text{Chi}(u)} g(v)\frac{\lambda_v}{\lambda_u},$$

where, as usual, we define a sum over an empty set to be zero. We first show that $h \in L^q(T, \lambda)$. 

Let us define
\[ M := \sup_{u \in V} \sum_{v \in \text{Chi}(u)} \lambda_v \lambda_u^{-1}. \]

Observe that, since \( S \) is bounded, \( M < \infty \). Let \( u \in V \). Clearly
\[ |h(u)| \leq \sum_{v \in \text{Chi}(u)} |g(v)| \frac{\lambda_v}{\lambda_u} \]
(note that if \( u \) is a leaf then \( h(u) = 0 \)). By the classical inequality of Jensen we have
\[ \left( \sum_{v \in \text{Chi}(u)} |g(v)| \lambda_v \lambda_u^{-1} \right)^q \leq \frac{\sum_{v \in \text{Chi}(u)} |g(v)|^q \lambda_v \lambda_u^{-1}}{\sum_{v \in \text{Chi}(u)} \lambda_v \lambda_u^{-1}}, \]
which simplifies to
\[ \left( \sum_{v \in \text{Chi}(u)} |g(v)| \lambda_v \lambda_u^{-1} \right)^q \leq \left( \sum_{v \in \text{Chi}(u)} \lambda_v \lambda_u^{-1} \right)^{q-1} \left( \sum_{v \in \text{Chi}(u)} |g(v)|^q \lambda_v \lambda_u^{-1} \right), \]
and hence we have
\[ |h(u)|^q \leq M^{q-1} \left( \sum_{v \in \text{Chi}(u)} |g(v)|^q \lambda_v \lambda_u^{-1} \right). \]
Therefore, multiplying by \( \lambda_u \) and summing over all vertices \( u \in V \), we get
\[ \sum_{u \in V} |h(u)|^q \lambda_u \leq M^{q-1} \sum_{u \in V} \left( \sum_{v \in \text{Chi}(u)} |g(v)|^q \lambda_v \right). \]
The right-hand side of the previous expression is no larger than
\[ M^{q-1} \sum_{v \in V} |g(v)|^q \lambda_v, \]
and hence
\[ \sum_{u \in V} |h(u)|^q \lambda_u \leq M^{q-1} \sum_{v \in V} |g(v)|^q \lambda_v \]
which shows that \( h \in L^q(T, \lambda) \) and, in fact, \( \|h\|_q \leq M^{\frac{q-1}{q}} \|g\|_q \).
Now, let $f \in L^p(T, \lambda)$. Then
\[
(S^* \Phi_g)(f) = \Phi_g(Sf) = \sum_{v \in V} (Sf)(v)g(v)\lambda_v = \sum_{v \in V, v \neq \text{root}} f(\text{par}(v))g(v)\lambda_v = \sum_{u \in V} f(u) \left( \sum_{v \in \text{Chi}(u)} g(v)\lambda_v \right)\lambda_u = \sum_{u \in V} f(u)h(u)\lambda_u = \Phi_h(f),
\]
thus $S^* \Phi_g = \Phi_h$. If we identify, as usual, $\Phi_g$ with $g$ and $\Phi_h$ with $h$, we have
\[
S^*g = h,
\]
which is what we wanted to prove. □

We now study the hypercyclicity of $S^*$. The first part of the following proposition, for the case $p = 2$, is proven implicitly in [16, Proposition 3.1.7].

**Proposition 4.3.** Let $T = (V, E)$ be a directed tree, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence, let $1 < p < \infty$, and let $q := \frac{p}{p-1}$. Assume the operator $S : L^p(T, \lambda) \to L^p(T, \lambda)$ is bounded. If the directed tree $T$ has a leaf, then the operator $S^* : L^q(T, \lambda) \to L^q(T, \lambda)$ does not have dense range. Hence, $S^*$ is not hypercyclic.

**Proof.** Let $w$ be a leaf and let $f := \chi_w$. Since $f \neq 0$, we can choose a functional $\Phi$ on $L^p(T, \lambda)$ such that $\Phi(f) = 1$. By Proposition 4.1 there exists $h \in L^q(T, \lambda)$ such that $\Phi = \Phi_h$. If $S^*$ had dense range, we could choose functions $g_n \in L^q(T, \lambda)$ such that $S^*g_n \to h$, and, hence, such that $\Phi(S^*g_n) \to \Phi_h$. But
\[
\Phi_{S^*g_n}(f) = \sum_{u \in V} f(u)(S^*g_n)(u)\lambda_u = (S^*g_n)(w)\lambda_w = \sum_{v \in \text{Chi}(w)} g_n(v)\lambda_v = 0,
\]
since $w$ is a leaf, and hence it has no children. But this implies that
\[
1 = |0 - 1| = |\Phi_{S^*g_n}(f) - \Phi(f)| \leq ||\Phi_{S^*g_n} - \Phi|| \|f\|_p \to 0,
\]
which is a contradiction. Hence $S^*$ does not have dense range. Since every hypercyclic operator must have dense range, the second part of the proposition follows. □

Observe that if we were to define an operator $S^* : L^1(T, \lambda) \to L^1(T, \lambda)$ by the expression
\[
(S^*f)(u) = \sum_{v \in \text{Chi}(u)} f(v)\frac{\lambda_v}{\lambda_u},
\]
for each $u \in V$, then this operator would be bounded. Indeed, the proof of Proposition 4.2 shows that if we set

$$h(u) := \sum_{v \in \text{Chi}(u)} g(v) \frac{\lambda_v}{\lambda_u},$$

for each $u \in V$, then $\|h\|_1 \leq \|g\|_1$. Hence $\|S^*g\|_1 \leq \|g\|_1$ and thus $S^*$ is a contraction. Therefore $S^*$ is not hypercyclic. Nevertheless, we will denote this operator on $L^1(T, \lambda)$ by $S^*$, when the occasion arises.

The form of the operator $S^*$ on $L^q(T, \lambda)$ suggests that a natural candidate for study is the operator defined below. It will turn out that $S^*$ will be unitarily equivalent to the following operator, with appropriate weights. We will show this in the last section of this paper.

**Definition 4.4.** Let $T = (V, E)$ be a directed tree, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence and let $1 \leq q < \infty$. The **backward shift** is defined as the operator $B : L^q(T, \lambda) \to L^q(T, \lambda)$ given by the expression

$$(Bf)(u) = \sum_{v \in \text{Chi}(u)} f(v),$$

for each $u \in V$. In the expression above, as it is usual, the sum over an empty set is defined to be zero.

From now on, we will deal with the operator $B$, since the hypercyclicity results we obtain are cleaner for $B$ than they are for $S^*$. Let us show that, under certain conditions, $B$ is a bounded operator. We denote by $\gamma(u)$ the cardinality of $\text{Chi}(u)$.

**Proposition 4.5.** Let $T = (V, E)$ be a directed tree, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence and let $1 \leq q < \infty$. If

$$\sup_{w \in V \setminus \text{root}} \gamma(\text{par}(w))^{q-1} \frac{\lambda_{\text{par}(w)}}{\lambda_w} < \infty,$$

then the backward shift operator $B : L^q(T, \lambda) \to L^q(T, \lambda)$ is bounded.

**Proof.** Let

$$M := \sup_{w \in V \setminus \text{root}} \gamma(\text{par}(w))^{q-1} \frac{\lambda_{\text{par}(w)}}{\lambda_w},$$

and let $f \in L^q(T, \lambda)$. Proceeding as in the proof of Proposition 4.2 by Jensen’s inequality, for every $u \in V$ we have

$$\left( \sum_{v \in \text{Chi}(u)} |f(v)| \right)^q \leq (\gamma(u))^{q-1} \sum_{v \in \text{Chi}(u)} |f(v)|^q.$$
It then follows that

\[ \|Bf\|_q^q = \sum_{u \in V} |Bf(u)|^q \lambda_u \]

\[ \leq \sum_{u \in V} \left( \sum_{v \in \text{Chi}(u)} |f(v)| \right)^q \lambda_u \]

\[ \leq \sum_{u \in V} \left( \gamma(u) \right)^{q-1} \lambda_u \sum_{v \in \text{Chi}(u)} |f(v)|^q \]

\[ = \sum_{w \in V, w \neq \text{root}} |f(w)|^q \left( \gamma(\text{par}(w)) \right)^{q-1} \lambda_{\text{par}(w)} \]

\[ \leq M \sum_{w \in V, w \neq \text{root}} |f(w)|^q \lambda_w \]

\[ \leq M \|f\|_q^q, \]

and therefore \( B \) is bounded.

The special case of the unweighted space is simpler.

**Corollary 4.6.** Let \( T = (V, E) \) be a directed tree, and let \( 1 \leq q < \infty \). Let \( \lambda \) be the constant sequence defined by \( \lambda_v = 1 \) for each \( v \in V \). If \( q = 1 \), then the backward shift operator \( B \) is bounded on \( L^1(T, \lambda) \). If \( 1 < q < \infty \), then \( B \) is bounded on \( L^q(T, \lambda) \) if the set \( \{\gamma(u) : u \in V\} \) is bounded; i.e., if the outdegrees of the tree are bounded.

Let \( \lambda \) be the constant sequence defined by \( \lambda_v = 1 \) for each \( v \in V \). It turns out that the if \( q = 1 \), the operator \( B \) has norm equal to one. If \( 1 < q < \infty \), the condition in Corollary 4.6 is not only sufficient, but also necessary. We investigate these matters in a different paper [12].

As it was the case in Proposition 4.3, if the tree has leaves, the backward shift operator is never hypercyclic.

**Proposition 4.7.** Let \( T = (V, E) \) be a directed tree, let \( \lambda = \{\lambda_v\}_{v \in V} \) be a positive sequence, let \( 1 \leq q < \infty \). Assume the backward shift \( B \) is bounded. If the directed tree \( T \) has a leaf, then the operator \( B : L^q(T, \lambda) \to L^q(T, \lambda) \) is not hypercyclic.

**Proof.** Let \( w \in V \) be a leaf. Since the sum over an empty set is zero, for every \( g \in L^q(T, \lambda) \) we have

\[ (Bg)(w) = \sum_{v \in \text{Chi}(w)} g(v) = 0. \]

Hence \( (B^n g)(w) = 0 \) for every \( n \in \mathbb{N} \). If \( g \) were a hypercyclic vector for \( B \), there would exist an increasing sequence of natural numbers \( \{n_k\} \) such that \( \|B^{n_k} g - \chi_w\|_q \to 0 \). But then

\[ \lambda^{1/q}_w = |0 - 1|^{1/q}_w = \|(B^{n_k} g)(w) - \chi_w(w)\|^{1/q}_w \leq \|B^{n_k} g - \chi_w\|_q \to 0 \]

which is impossible. \( \square \)
5. Hypercyclicity of the Backward Shift: Rooted Directed Trees

We need to distinguish two cases to study the hypercyclicity of $B$. In this section we deal with the case where the tree $T$ has a root. Observe that in this case, for every vertex $u \in V$, there exists a unique path (in the underlying graph of the tree $T$) starting from root and ending at $u$. We denote by $|u|$ the length of such a path.

For every $u \in V$, and every $n \in \mathbb{N}$, we denote by $\gamma(u, n)$ the cardinality of the set $\text{Chi}^n(u)$. Observe that $\gamma(u, n) > 0$ for every $u \in V$ and every $n \in \mathbb{N}$ if $T$ is leafless.

For every $u \in V$, for $1 \leq q < \infty$, and every $n \in \mathbb{N}$ we denote by $\Omega(u, n)$ the number

$$\Omega(u, n) := \frac{1}{(\gamma(u, n))^q} \sum_{v \in \text{Chi}^n(u)} \lambda_v.$$  

The next theorem gives a sufficient condition for hypercyclicity of $B$, in terms of the numbers defined above.

**Theorem 5.1.** Let $T = (V, E)$ be a leafless directed tree with a root, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence, and let $1 \leq q < \infty$. Assume the backward shift operator $B : L^q(T, \lambda) \to L^q(T, \lambda)$ is bounded. If there exists an increasing sequence of natural numbers $\{n_k\}$ such that, for all $u \in V$ we have

$$\Omega(u, n_k) \to 0$$

as $k \to \infty$, then $B$ is hypercyclic.

**Proof.** We will verify each of the conditions of the Hypercyclicity Criterion (Theorem 2.2). Define $X$ as the set $X := \{g \in L^q(T, \lambda) : g$ is finitely supported $\}$. Clearly $X$ is dense in $L^q(T, \lambda)$.

First, it can easily be seen that, for every $f \in L^q(T, \lambda)$ and every $u \in V$, we have

$$(B^n f)(u) = \sum_{v \in \text{Chi}^n(u)} f(v).$$

(1) If $g$ is finitely supported, there exists $N \in \mathbb{N}$ such that $g(v) = 0$ for all $v$ with $|v| \geq N$. For all $u \in V$, we have that if $v \in \text{Chi}(u)$ then $|v| = |u| + 1$ and hence that if $v \in \text{Chi}^n(u)$, then $|v| = |u| + n$. Therefore, for any $u \in V$, if $n \geq N$ and $v \in \text{Chi}^n(u)$ then $g(v) = 0$. It follows that $(B^n g)(u) = 0$ for all $u \in V$ as soon as $n \geq N$. Thus the function $B^n g$ is identically zero if $n \geq N$. Therefore $B^n g \to 0$ for all $g \in X$, as $k \to \infty$.

(2) Given $g \in X$ and $n \in \mathbb{N}$, we define the complex-valued function $T_n g$ as

$$(T_n g)(v) := \begin{cases} \frac{1}{\gamma(v, n)} g(\text{par}^n(v)), & \text{if } v \in V_n, \text{ and } \\ 0, & \text{if } v \notin V_n, \end{cases}$$
where, as before, \( V^n \) denotes the set of vertices that have \( n \)-ancestors. It follows that
\[
\|T_n g\|_q = \sum_{v \in V} |(T_n g)(v)|^q \lambda_v
\]
\[
= \sum_{v \in V^n} \frac{1}{(\gamma(\text{par}^n(v), n))^q} g(\text{par}^n(v))^q \lambda_v
\]
\[
= \sum_{u \in V} |g(u)|^q \sum_{v \in \text{Chi}^n(u)} \lambda_v
\]
\[
= \sum_{u \in V} |g(u)|^q \Omega(u, n).
\]

Evaluating at the sequence \( \{n_k\} \), remembering that \( g \) is finitely-supported (and hence the last expression has only finitely-many summands) and recalling that for every \( u \in V \) we have \( \Omega(u, n_k) \to 0 \) as \( j \to \infty \), it follows that
\[
T_{n_k} g \to 0,
\]
as \( k \to \infty \).

(3) Lastly, we show that \( B^n(T_n g) = g \) for all \( g \in X \) and \( n \in \mathbb{N} \). Indeed, if \( g \in X \), and \( u \in V \), then
\[
(B^n(T_n g))(u) = \sum_{v \in \text{Chi}^n(u)} (T_n g)(v)
\]
\[
= \sum_{v \in \text{Chi}^n(u)} \frac{1}{\gamma(\text{par}^n(v), n)} g(\text{par}^n(v))
\]
\[
= \sum_{v \in \text{Chi}^n(u)} \frac{1}{\gamma(u, n)} g(u)
\]
\[
= g(u),
\]
as desired. Hence \( B^{n_k} T_{n_k} g \to g \) as \( k \to \infty \), for each \( g \in X \).

Therefore, by the Hypercyclicity Criterion, \( B \) is hypercyclic.

We will need the following definition to state some of the coming results.

**Definition 5.2.** Let \( T \) be a leafless directed tree. We say that \( T \) has a free end if there exists a vertex such that all of its descendants have degree one.

The fact that the following corollary does not apply for \( q = 1 \) is not surprising, given the comment after Corollary 4.6.

**Corollary 5.3.** Let \( T = (V, E) \) be a leafless directed tree with a root, let \( 1 < q < \infty \), let \( \lambda \) be the constant sequence defined by \( \lambda_v = 1 \) for each \( v \in V \), and assume that the backward shift \( B \) is bounded on \( L^q(T, \lambda) \). If the tree \( T \) has no free end, then \( B \) is hypercyclic.

**Proof.** Let \( u \in V \) be fixed. Clearly, the sequence \( \{\gamma(u, n)\}_{n \in \mathbb{N}} \) is a nondecreasing sequence of natural numbers. We claim that \( \gamma(u, n) \to \infty \) as \( n \to \infty \). Indeed,
the only possible way the sequence would not go to infinity is if it became eventually constant, say after $N$ steps. But this would mean that each of the vertices in $\text{Chi}^N(u)$ has the property that all of its descendants have outdegree one, contradicting the fact that $T$ has no free ends.

It then follows that
$$\Omega(u, n) := \frac{1}{(\gamma(u, n))q} \sum_{v \in \text{Chi}^n(u)} \lambda_v = \frac{1}{(\gamma(u, n))q-1} \rightarrow 0$$
as $n \rightarrow \infty$. Applying the previous theorem to the full sequence of natural numbers, we obtain that $B$ is hypercyclic. \hfill $\Box$

We now study a necessary condition for hypercyclicity of $B$.

**Theorem 5.4.** Let $T = (V, E)$ be a leafless directed tree with a root, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence, let $1 \leq q < \infty$ and assume the backward shift $B$ is bounded on $L^q(T, \lambda)$. If $B$ is hypercyclic, then for each $u \in V$ there exists an increasing sequence of nonnegative integers $\{n_k\}$ such that
$$\sum_{v \in \text{Chi}^{n_k}(u)} \lambda_v^{-1/q} \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

**Proof.** Let $u \in V$ be fixed. We proceed inductively. Let $k \in \mathbb{N}$ and assume that for each $j < k$ we have chosen $n_1 < n_2 < \cdots < n_j$ such that
$$j < \sum_{v \in \text{Chi}^{n_j}(u)} \lambda_v^{-1/q}$$
(no assumption is needed if $k = 1$).

Define $\delta := (k + \lambda_u^{-1/q} - 1)$. Since $B$ is hypercyclic, we can choose $f$ a hypercyclic vector such that
\begin{equation}
\|f\| < \delta.
\end{equation}

We now choose an integer $n_k > n_{k-1}$ (or $n_1 \in \mathbb{N}$, if $k = 1$) such that
\begin{equation}
\|B^{n_k} f - \chi_u\| < \delta,
\end{equation}
where $\chi_u$ is the characteristic function of $u$.

From inequality (2), we get
$$\left| \left( B^{n_k} f - \chi_u \right)(u) \right|^q \lambda_u < \delta^q,$$
which is
$$\left| \sum_{v \in \text{Chi}^{n_k}(u)} f(v) - 1 \right|^q \lambda_u < \delta^q,$$
and hence
$$1 - \left| \sum_{v \in \text{Chi}^{n_k}(u)} f(v) \right| < \frac{\delta}{\lambda_u^{1/q}}.$$

Therefore
$$1 - \frac{\delta}{\lambda_u^{1/q}} \leq \sum_{v \in \text{Chi}^{n_k}(u)} |f(v)|.$$

By inequality (1) we have that for each $v \in V$
$$|f(v)|^q \lambda_v < \delta^q,$$
and hence combining the last two inequalities we obtain
\[
1 - \frac{\delta}{\lambda_u^{1/q}} < \sum_{v \in \text{Chi}^{n_k}(u)} |f(u)| < \sum_{v \in \text{Chi}^{n_k}(u)} \frac{\delta}{\lambda_v^{1/q}},
\]
which simplifies to
\[
k = \frac{1}{\delta} - \frac{1}{\lambda_u^{1/q}} < \sum_{v \in \text{Chi}^{n_k}(u)} \frac{1}{\lambda_v^{1/q}},
\]
which is what we wanted. Therefore, we have chosen an increasing sequence of natural numbers \(\{n_k\}\) such that
\[
k < \sum_{v \in \text{Chi}^{n_k}(u)} \lambda_v^{-1/q},
\]
which proves the theorem. \(\square\)

In the following corollary, we exclude the case \(q = 1\), given the comment after Corollary 4.6.

**Corollary 5.5.** Let \(T = (V, E)\) be a leafless directed tree with a root, let \(1 < q < \infty\), let \(\lambda\) be the constant sequence defined by \(\lambda_v = 1\) for each \(v \in V\), and assume that the backward shift \(B\) is bounded on \(L^q(T, \lambda)\). If the tree \(T\) has a free end then \(B\) is not hypercyclic.

**Proof.** If \(T\) has a free end, there exists a vertex such that all its descendants have degree one. Let \(w^*\) be one of these descendants. It is then clear that \(\gamma(w^*, n) = 1\) for all \(n\). But then, for any sequence \(\{n_k\}\) we have
\[
\sum_{v \in \text{Chi}^{n_k}(w^*)} \lambda_v^{-1/q} = \gamma(w^*, n_k) = 1.
\]
The previous theorem then assures that \(B\) is not hypercyclic. \(\square\)

Putting together Corollaries 5.3 and 5.5, we obtain the following characterization of hypercyclicity of the backward shift for the unweighted case.

**Corollary 5.6.** Let \(T = (V, E)\) be a leafless directed tree with a root, let \(1 < q < \infty\), let \(\lambda\) be the constant sequence defined by \(\lambda_v = 1\) for each \(v \in V\), and assume that the backward shift \(B\) is bounded on \(L^q(T, \lambda)\). The operator \(B\) is hypercyclic if and only if the tree \(T\) has no free ends.

We have not been able to obtain a condition that is both necessary and sufficient for hypercyclicity of \(B\) on \(L^q(T, \lambda)\) for the case when the sequence \(\lambda\) is not constant. We leave the question open for future research.

### 6. Hypercyclicity of the Backward Shift: Unrooted Directed Trees

We deal now with the case where the tree \(T\) does not have a root. For each \(u \in V\) and \(n \in \mathbb{N}\), recall that \(\gamma(u, n)\) denotes the cardinality of the set \(\text{Chi}^n(u)\) and that \(\Omega(u, n)\) denotes the number
\[
\Omega(u, n) := \frac{1}{(\gamma(u, n))^q} \sum_{v \in \text{Chi}^n(u)} \lambda_v.
\]
We also define for each \(u \in V\) and each \(n \in \mathbb{N}\) the number
\[
\Theta(u, n) := (\gamma(\text{par}^n(u), n))^{q-1} \lambda_{\text{par}^n(u)}.
\]
We first give a necessary condition for hypercyclicity of the backward shift.

**Theorem 6.1.** Let $T = (V, E)$ be a leafless directed tree with no root, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence, and let $1 \leq q < \infty$. Assume the backward shift operator $B : L^q(T, \lambda) \to L^q(T, \lambda)$ is bounded. If there exists an increasing sequence of natural numbers $\{n_k\}$ such that, for all $u \in V$ we have

$$\Theta(u, n_k) \to 0 \quad \text{and} \quad \Omega(u, n_k) \to 0$$

as $k \to \infty$, then the operator $B : L^q(T, \lambda) \to L^q(T, \lambda)$ is hypercyclic.

**Proof.** As done in the proof of Theorem 5.1, we apply the Hypercyclicity Criterion (Theorem 2.2) to $B$. Again, $X$ denotes the set $X := \{g \in L^q(T, \lambda) : g \text{ is finitely supported}\}$, which is dense in $L^q(T, \lambda)$. Also, recall that for every $g \in L^q(T, \lambda)$ and every $u \in V$ we have

$$B^n g)(u) = \sum_{v \in \text{Chi}^n(u)} g(v).$$

(1) Let $g \in X$. For every $u \in V$ we have

$$|(B^n g)(u)|^q \leq \left( \sum_{v \in \text{Chi}^n(u)} |g(v)| \right)^q.$$

As before, by Jensen’s inequality, we have

$$\left( \sum_{v \in \text{Chi}^n(u)} |g(v)| \right)^q \leq \left( \gamma(u, n) \right)^{q-1} \left( \sum_{v \in \text{Chi}^n(u)} |g(v)|^q \right),$$

and hence

$$|(B^n g)(u)|^q \leq \sum_{v \in \text{Chi}^n(u)} (\gamma(u, n))^{q-1} |g(v)|^q.$$

Multiplying by $\lambda_u$ and summing over all $u \in V$, we obtain

$$\|B^n g\|_q^q = \sum_{u \in V} |(B^n g)(u)|^q \lambda_u \leq \sum_{u \in V} \left( \sum_{v \in \text{Chi}^n(u)} (\gamma(u, n))^{q-1} \lambda_u |g(v)|^q \right) = \sum_{u \in V} (\gamma(\text{par}^n(w), n))^{q-1} \lambda_{\text{par}^n(u)} |g(w)|^q = \sum_{u \in V} \Theta(w, n) |g(w)|^q.$$

Since $g \in X$, this sum is finite. Evaluating at the sequence $n_k$ and taking the limit as $k \to \infty$, we obtain

$$B^{n_k} g \to 0.$$

This shows that the first condition of the Hypercyclicity Criterion (Theorem 2.2) is satisfied.
(2) Now, given \( g \in X \) and \( n \in \mathbb{N} \), we define the complex-valued function \( T_n g \) as

\[
(T_n g)(v) := \frac{1}{\gamma(\text{par}^n(v), n)} g(\text{par}^n(v)),
\]

for \( v \in V \). The rest of the proof of this part is the same as that of Part (2) of Theorem 5.1.

(3) This is the same as Part (3) of Theorem 5.1.

Since all the conditions of the Hypercyclicity Criterion are satisfied, the operator \( B \) is hypercyclic. \( \square \)

We show an example of an unrooted tree where hypercyclicity occurs.

**Example 6.2.** Let \( 1 \leq q < \infty \), let \( r \in \mathbb{N} \) and let \( s \in \mathbb{R} \) with \( s \geq r^q - 1 \). Let \( T \) be the infinite unrooted leafless directed tree such that each vertex has outdegree \( r \).

Select an arbitrary fixed vertex and call it \( w^* \). Consider the set \( H \) defined as the set of all vertices that share a common \( n \)-ancestor with \( w^* \), that is

\[
H := \{ w \in V : w \in \text{Chi}^n(\text{par}^n(w^*)) \ \text{for some} \ n \in \mathbb{N} \}.
\]

For each \( u \in V \), set \( \lambda_u = \frac{1}{s^d} \) for some \( d \in \mathbb{N}_0 \). Also, let \( n > d \).

We have that

\[
\lambda_{\text{par}^n(u)} = \frac{1}{s^{n \pm d}},
\]

where the plus sign corresponds to the case where \( u \) has a descendant in \( H \) and the minus sign corresponds to the case where \( u \) has an ancestor in \( H \). Then

\[
\Theta(w, n) = (\gamma(\text{par}^n(w), n))^{q-1} \lambda_{\text{par}^n(w)} = r^{n(q-1)} \frac{1}{s^{n \pm d}} = \left( \frac{r^{q-1}}{s} \right)^n \frac{1}{s^{\pm d}}
\]

which goes to zero as \( n \) goes to infinity.

Also, for each \( v \in \text{Chi}^n(u) \) we have

\[
\lambda_v = \frac{1}{s^{n \pm d}}
\]

where the plus sign corresponds to the case where \( u \) has an ancestor in \( H \) and the minus sign corresponds to the case where \( u \) has a descendant in \( H \). Then

\[
\Omega(u, n) = \frac{1}{(\gamma(u, n))^q} \sum_{v \in \text{Chi}^n(u)} \lambda_v = \frac{1}{r^{nq}} \sum_{v \in \text{Chi}^n(u)} \frac{1}{s^{n \pm d}} = \frac{1}{r^{nq}} \frac{1}{s^{n \pm d}} = \frac{1}{(r^{q-1}s)^n} \frac{1}{s^{\pm d}}
\]

which goes to zero as \( n \) goes to infinity, since \( r^{q-1}s > 1 \). Hence, by the previous theorem applied to the full sequence \( n_k = k \), the operator \( B : L^q(T, \lambda) \rightarrow L^q(T, \lambda) \) is hypercyclic. \( \square \)

It is easy to see that the conditions in the hypothesis of Theorem 6.1 reduce to the conditions given by Salas [21], if \( T \) is the rootless tree \( (\mathbb{Z}, \{(n, n+1) : n \in \mathbb{Z}\}) \). In this case, Salas’ work shows the conditions are necessary and sufficient.

It is not hard to obtain a necessary condition for hypercyclicity in the general case for the case of the unrooted tree, in the style of Theorem 5.4. Nevertheless, we could not find any other family of trees \( T \) or weights \( \lambda \) where such a condition was
also sufficient (except, of course, the case of the trees already described by Salas). Is there a necessary and sufficient condition for hypercyclicity in this case? Or for an “interesting” family? We leave the question open for future research.

7. Relation between $B$ and $S^*$

We now show that the operators $B$ and $S^*$ are indeed unitarily equivalent. We can then translate the sufficient condition for hypercyclicity we found for $B$ to the operator $S^*$. First, we need a lemma.

**Lemma 7.1.** Let $T = (V, E)$ be a directed tree and let $1 \leq q \leq \infty$. Let $\lambda = \{\lambda_u\}_{u \in V}$ and $\mu = \{\mu_u\}_{u \in V}$ be sequences of positive numbers such that $\mu_u \lambda_u^{q-1} = 1$ for every $u \in U$. Define the operator $\Phi : L^q(T, \mu) \to L^q(T, \lambda)$ as

$$\Phi f(u) := \frac{f(u)}{\lambda_u}. $$

Then $\Phi$ is an isometric isomorphism between $L^q(T, \mu)$ and $L^q(T, \lambda)$.

**Proof.** Clearly $\Phi$ is linear. To see that $\Phi$ is isometric, let $f \in L^q(T, \mu)$. Then

$$\|f\|_q^q = \sum_{u \in V} |f(u)|^q \mu_u,$$

and

$$\|\Phi f\|_q^q = \sum_{u \in V} |(\Phi f)(u)|^q \lambda_u = \sum_{u \in V} \frac{|f(u)|^q}{\mu_u} \lambda_u = \sum_{u \in V} \frac{|f(u)|^q}{\lambda_u^{q-1}}.$$

Since $\mu_u = \frac{1}{\lambda_u^{q-1}}$, it follows that $\|\Phi f\|_q = \|f\|_q$. We now show $\Phi$ is surjective. Let $g \in L^q(T, \lambda)$. Define $f(u) := \lambda_u g(u)$. We show that $f \in L^q(T, \mu)$. Indeed, the hypothesis implies that $\lambda_u^q \mu_u = \lambda_u$ and hence we have

$$\|f\|_q^q = \sum_{u \in V} |f(u)|^q \mu_u = \sum_{u \in V} \lambda_u^q |g(u)|^q \mu_u = \sum_{u \in V} |g(u)|^q \lambda_u < \infty,$$

since $g \in L^q(T, \lambda)$. Clearly, $\Phi f = g$, which concludes the proof. 

As we mentioned before, $S^*$ and $B$ turn out to be unitarily equivalent. Recall that in Section 4 we defined $S^*$, even in the case $q = 1$.

**Theorem 7.2.** Let $1 \leq q < \infty$ and let $T = (V, E)$ be a directed tree. Let $\lambda = \{\lambda_u\}_{u \in V}$ and $\mu = \{\mu_u\}_{u \in V}$ be sequences of positive numbers such that $\mu_u \lambda_u^{q-1} = 1$ for every $u \in U$. Then $S^* : L^q(T, \lambda) \to L^q(T, \lambda)$ is unitarily equivalent to $B : L^q(T, \mu) \to L^q(T, \mu)$.

**Proof.** Let $\Phi$ be as in Lemma 7.1. We will show that $S^* \Phi = \Phi B$.

Let $f \in L^q(T, \mu)$. For each $u \in V$ we have

$$(\Phi B f)(u) = \frac{1}{\lambda_u} (B f)(u) = \frac{1}{\lambda_u} \sum_{v \in \Chi(u)} f(v),$$

and, Proposition 4.2 gives

$$(S^* \Phi f)(u) = \sum_{v \in \Chi(u)} (\Phi f)(v) \frac{\lambda_v}{\lambda_u} = \sum_{v \in \Chi(u)} \frac{f(v)}{\lambda_v} \frac{\lambda_u}{\lambda_v} = \sum_{v \in \Chi(u)} f(v) \frac{1}{\lambda_u}. $$

Hence $S^* \Phi = \Phi B$, as desired. Since $\Phi$ is an isometric isomorphism, the result follows. 

□
We can use the previous theorem to give a sufficient condition for the hypercyclicity of $S^*$ in the case of the rooted tree.

**Theorem 7.3.** Let $T = (V,E)$ be a leafless directed tree with a root, let $\lambda = \{\lambda_v\}_{v \in V}$ be a positive sequence, and let $1 < q < \infty$. If there exists an increasing sequence of natural numbers $\{n_k\}$ such that, for all $u \in V$ we have

$$\frac{1}{(\gamma(u,n_k))^q} \sum_{v \in \text{Chi}^k(u)} \lambda_v^{1-q} \to 0$$

as $k \to \infty$, then $S^* : L^q(T,\lambda) \to L^q(T,\lambda)$ is hypercyclic.

**Proof.** For each $u \in V$, define $\mu_u$ as $\mu_u = \lambda_u^{1-q}$. By Theorem 5.1, we have that $B : L^q(T,\mu) \to L^q(T,\mu)$ is hypercyclic. But since $\mu_u\lambda_u^{q-1} = 1$ for each $u \in V$, Theorem 7.2 implies that the operator $S^*$ is unitarily equivalent to $B$. Hence $S^*$ is hypercyclic.

Similar results can be obtained for $S^*$ in the case of the unrooted tree and for necessary conditions for hypercyclicity in the case of the rooted tree. We leave them as an exercise for the interested reader.

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