NON-NOETHERIANITY OF DENJOY-CARLEMAN RINGS OF GERMS

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ABSTRACT. It is shown that Denjoy-Carleman quasi analytic rings of germs of functions in two or more variables either complex or real valued that are stable under derivation and strictly larger than the ring of real-analytic germs are not Noetherian rings. The failure of Weierstrass division on these Denjoy-Carleman classes yields a contradiction to Noetherianity via a stronger version of Artin Approximation due to Popescu as well as results on projective modules.

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1. INTRODUCTION

The Denjoy-Carleman quasi analytic classes are local rings of smooth function germs satisfying bounds on their derivatives that fulfill certain conditions. By the Denjoy-Carleman Theorem, these conditions on the bounds make the Taylor morphism injective on the class, so no flat functions are contained therein. The Denjoy-Carleman quasi analytic classes are standard classes of functions in analysis and
partial differential equations, and they have become more interesting recently in an algebraic geometric context as well as in the theory of o-minimality [20].

Each Denjoy-Carleman quasi analytic class we will consider contains all real-analytic functions but is strictly larger, so it also contains functions with non-convergent Taylor expansions. By Torsten Carleman [8], there exist non-convergent formal power series whose coefficients satisfy the Denjoy-Carleman bounds but which do not belong to the image of the Denjoy-Carleman class under the Taylor morphism.

The property that makes these Denjoy-Carleman quasi analytic classes both peculiar and interesting is that they fail to satisfy the Weierstrass Division Theorem, as Childress proved in 1976 in [12]. This left wide open the question whether these were Noetherian rings or not. On one hand, the injectivity of the Taylor morphism makes the \( m \)-adic topology separable for the ring of Denjoy-Carleman germs, a property characteristic of Noetherian rings that does not hold for the ring of germs of smooth functions. On the other hand, the failure of Weierstrass Division makes it impossible to carry out the same argument for Noetherianity as in the case of holomorphic or real-analytic germs.

Each Denjoy-Carleman class we will consider satisfies the implicit function theorem, hence it is a Henselian ring. With an additional assumption on the sequence of bounds, such a class is closed under derivation, which implies it is also closed under composition, division by a coordinate, and inverse. In [3] Edward Bierstone and Pierre Milman were able to extend the resolution of singularities algorithm to all real-valued Denjoy-Carleman quasi analytic classes satisfying these additional assumptions. Therefore, in every such Denjoy-Carleman quasi analytic class, they proved all three Lojasiewicz inequalities for each function as well as topological Noetherianity. The latter condition is weaker than Noetherianity and means that descending chains of varieties corresponding to ideals in such a Denjoy-Carleman quasi analytic ring must stabilize. These results make Denjoy-Carleman quasi analytic rings very interesting from an algebraic-geometric point of view as in [16].
In this paper we use Dorin Popescu’s generalization of Artin approximation in [18] but in its more detailed presentation given by Richard Swan in [21] to show:

1.1. Theorem. Let \( M = \{M_0, M_1, M_2, \ldots \} \) be an increasing sequence of positive real numbers that is assumed to be logarithmically convex and to satisfy

\[
\sum_{k=0}^{\infty} \frac{M_k}{(k + 1) M_{k+1}} = \infty.
\]

Let \( C_M^n(0) \) be the ring of germs at 0 of Denjoy-Carleman complex-valued functions in \( n \) variables determined by the sequence \( M = \{M_0, M_1, M_2, \ldots \} \). \( C_M^n(0) \) is assumed to be closed under derivation and strictly larger than the ring of real-analytic germs at 0. \( C_M^n(0) \) is not a Noetherian ring for \( n \geq 2 \).

When \( n = 1 \), \( C_M^1(0) \) is a principal ideal domain hence Noetherian. Furthermore, using the fact that the real and imaginary parts of germs in \( C_M^n(0) \) are also in \( C_M^n(0) \), these results immediately translate to real-valued Denjoy-Carleman germs:

1.2. Corollary. Let \( M \) satisfy the same assumption as in the previous theorem, and let \( C_M^{n,\mathbb{R}}(0) \) be the ring of germs at 0 of Denjoy-Carleman real-valued functions in \( n \) variables. If \( C_M^{n,\mathbb{R}}(0) \) is closed under derivation and strictly larger than the ring of real-analytic real-valued germs at 0, then \( C_M^{n,\mathbb{R}}(0) \) is not a Noetherian ring for \( n \geq 2 \).

Popescu’s generalization of Artin approximation [18] states that a system of polynomial equations with coefficients in a Henselian excellent ring that has a formal power series solution must also have an actual solution in the ring that equals the formal power series solution to every \( m \)-adic order. Our Denjoy-Carleman ring of germs is Henselian, and we show via a Jacobian criterion due to Matsumura that it would be excellent, if it were Noetherian. The failure of Weierstrass division with respect to a strictly regular Weierstrass polynomial yields a linear relation involving the quotient and the remainder that are power series of the type exhibited by Carleman, namely whose coefficients satisfy the Denjoy-Carleman bounds but do not
correspond to germs. The existence of a second polynomial equation involving either
the quotient and remainder or just the quotient and with coefficients in the ring of
germs is proven using results from module theory. These two equations together give
a system of polynomial equations with a unique solution lying in the completion but
not in the ring itself, therefore yielding a contradiction to Popescu’s generalization
of Artin approximation and thus to Noetherianity.

The significance of this result is twofold: Firstly, it shows Denjoy-Carleman rings
constitute examples of non-excellent rings that admit resolution of singularities by
the Bierstone-Milman result [3]. Secondly, it yields that Artin Approximation cannot
be expected to hold on non-Noetherian rings, even if they have rather nice properties.

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2. Denjoy-Carleman Quasianalytic Classes

We will follow here the set-up common to the Bierstone-Milman paper [3], the
Childress paper [12], and the Chaumat-Chollet paper [11]. We direct the reader to
Vincent Thilliez’s set of notes [22] for a clear exposition of properties of more general
quasi analytic local rings.

Setting:

Let $M = \{M_0, M_1, M_2, \ldots \}$ be an increasing sequence of positive real numbers,
where $M_0 = 1$.

2.1. Definition. Let $U$ be a connected open set in $\mathbb{R}^n$. Denote by $C_M(U) = C^\infty_M(U)$
the set of $\mathbb{C}$-valued infinitely differentiable $f \in C^\infty(U)$ satisfying that for every
compact set $K \subset U$, there exist constants $A, B > 0$ such that

\[
\left| \frac{1}{\alpha!} D^\alpha f(x) \right| \leq A B^{\|\alpha\|} M_{\|\alpha\|}
\]
for any \( x \in K \), where \( \alpha \) is a multi-index in \( \mathbb{N}^n \). The set \( C_M(U) \) is called \textit{quasi analytic} if the Taylor morphism assigning to each \( f \in C_M(U) \) its Taylor expansion at \( a \in U \) is injective for all \( a \in U \).

\textbf{Notation:} For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) we denote

\[
|\alpha| := \alpha_1 + \cdots + \alpha_n,
\]
\[
\alpha! := \alpha_1! \cdots \alpha_n!,
\]
\[
D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},
\]
\[
x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.
\]

\textit{Remark.} Note that for an open set \( \tilde{U} \subset U \), if \( f \in C_M(U) \), then the restriction \( f \big|_{\tilde{U}} \in C_M(\tilde{U}) \).

2.2. \textbf{Definition.} The set of germs \( C^n_M(0) \) of Denjoy-Carleman functions at 0 given by the sequence \( M \) consists of all germs of \( \mathbb{C} \)-valued smooth functions at the origin \( f \in C^\infty(0) \) such that there exist an open set \( U \ni 0 \) and a representative \( \tilde{f} \) in the equivalence class of \( f \) satisfying that \( \tilde{f} \in C_M(U) \). We say that \( C^n_M(0) \) is \textit{quasi analytic} if the Taylor morphism \( f \mapsto \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha f(0)x^\alpha}{\alpha!} \) assigning to each \( f \in C_M(0) \) its Taylor expansion at 0 is injective.

\textit{Remark.} Note that the real and imaginary parts of any germ in \( C^n_M(0) \) must themselves be elements of \( C^n_M(0) \) since by the linearity of differentiation, they have to satisfy the Denjoy-Carleman derivative bounds.

2.3. \textbf{Definition.} Let \( C^n_{M,\mathbb{R}}(0) \) be the ring of germs at 0 of Denjoy-Carleman real-valued functions in \( n \) variables. By the previous remark, \( C^n_M(0) \simeq C^n_{M,\mathbb{R}}(0) \oplus i C^n_{M,\mathbb{R}}(0) \).

2.4. \textbf{Definition.} Let \( \mathcal{F} = \mathcal{F}^n \) denote the set of formal power series in \( n \) variables with complex coefficients, \( F = \sum_{\alpha \in \mathbb{N}^n} F_\alpha x^\alpha \), where \( F_\alpha \in \mathbb{C} \) for all \( \alpha \in \mathbb{N}^n \). Let \( \mathcal{F}_\mathbb{R} = \mathcal{F}^n_\mathbb{R} \) denote the set of formal power series in \( n \) variables with real coefficients, i.e. \( F_\alpha \in \mathbb{R} \) for all \( \alpha \in \mathbb{N}^n \).
2.5. **Definition.** Let $F_M = F^n_M$ denote the set of formal power series in $n$ variables with complex coefficients, $F = \sum_{\alpha \in \mathbb{N}^n} F_\alpha x^\alpha$, for which there exist positive constants $A$ and $B$ such that

$$|F_\alpha| \leq AB^{||\alpha||}M_{||\alpha||},$$

for all $\alpha \in \mathbb{N}^n$.

We denote by $F_{M,\mathbb{R}} = F^n_{M,\mathbb{R}}$ the corresponding set of formal power series with real coefficients subject to the same bounds.

2.6. **Definition.** The sequence $M = \{M_0, M_1, M_2, \ldots\}$ is called logarithmically convex if

$$\frac{M_{j+1}}{M_j} \leq \frac{M_{j+2}}{M_{j+1}} \text{ for all } j \geq 0,$$

i.e. the sequence of subsequent quotients is increasing.

Assumption 2 implies that $C^n_M(0)$ is a ring. Furthermore, it is a local ring with maximal ideal

$$m = \{f \in C^n_M(0) : f(0) = 0\}.$$

The same is true of $C^{n,\mathbb{R}}_M(0)$, $F_M$, and $F_{M,\mathbb{R}}$. For more information on assumption 2 and its implications, see [22]. Another consequence of assumption 2 is the Implicit Function Theorem in $C^n_M(0)$ and $C^{n,\mathbb{R}}_M(0)$ (and in both $C^n_M(U)$ and $C^{n,\mathbb{R}}_M(U)$) first proven in [13] in a form closest to the Inverse Function Theorem. The version stated here is from [3]:

2.7. **Theorem.** (Implicit Function Theorem [13], [3]) Let $M$ be a logarithmically convex increasing sequence of positive real numbers. Suppose that $U \subset \mathbb{R}^n \times \mathbb{R}^k$ with product coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_k)$. Suppose that $f_1, \ldots, f_k \in C^{n,\mathbb{R}}_M(U)$ or $C_M(U)$ such that for $f = (f_1, \ldots, f_k)$, $f(a, b) = 0$, and $\frac{\partial f}{\partial y}(a, b)$ is invertible.

Then there is a product neighbourhood $V \times W$ of $(a, b)$ in $U$, and a mapping $g : V \to W$, $g = (g_1, \ldots, g_n)$; $g_i \in C^{n,\mathbb{R}}_M(V)$ for $i = 1 \ldots n$, such that $g(a) = b$ and $f(x, g(x)) = 0 \ \forall x \in V$. 
2.8. Corollary. Let the sequence $M = \{M_0, M_1, M_2, \ldots\}$ be logarithmically convex. Then the rings $C_M^n(0)$ and $C_M^{n,R}(U)$ are Henselian.

This corollary follows from Michel Raynaud’s arguments on p.78-79 of [19]. Furthermore, Jacques Chaumat and Anne-Marie Chollet proved a version of Hensel’s lemma directly, lemma 7.1 of [11], which verifies one of the equivalent statements of Henselianity, part (2) of Proposition 3 on p.76 of [19].

One of the consequences of (2) along with the assumption that $M_0 = 1$ is that $\{ (M_j)^{\frac{1}{j}} \}_{j \geq 1}$ is an increasing sequence of positive real numbers greater than or equal to 1. Since real-analytic functions satisfy estimate (1) with $M_{|\alpha|} = 1$ for all $\alpha$, it follows that $C_M(U)$ contains all $\mathbb{C}$-valued real-analytic functions on $U$ and $C_M^n(0)$ contains all germs of $\mathbb{C}$-valued real-analytic functions. In fact, a corollary of classical results of Cartan and Mandelbrojt [9] gives a condition on $M$ equivalent to having that $C_M(U)$ is exactly the ring of $\mathbb{C}$-valued real-analytic functions on $U$, $C^\omega(U)$ and $C_M^n(0)$ is exactly the ring of $\mathbb{C}$-valued real-analytic germs $C^\omega(0)$, see [22, Cor. 1 of Thm. 5]:

2.9. Proposition. $C_M(U) = C^\omega(U)$ and $C_M(0) = C^\omega(0)$ iff $\sup_{j \geq 1} (M_j)^{\frac{1}{j}} < \infty$.

Given that the sequence $\{ (M_j)^{\frac{1}{j}} \}_{j \geq 1}$ is increasing, this proposition implies that we must assume

\[(3) \quad \lim_{j \to \infty} (M_j)^{\frac{1}{j}} = \infty\]

in order to guarantee the strict inclusions $C_M(U) \subsetneq C^\omega(U)$ and $C_M(0) \subsetneq C^\omega(0)$ and the strict inclusion of real-valued real-analytic functions in $C_M^{n,R}(U)$ and of real-valued real-analytic germs in $C_M^{n,R}(0)$.

Another corollary of the results of Cartan and Mandelbrojt is the following equivalence; see Corollary 2 to Theorem 5 of [22]:

2.10. **Proposition.** $C_M(U)$ and $C^n_M(0)$ are closed under derivation iff
\[
\sup_{j \geq 1} \left( \frac{M_{j+1}}{M_j} \right)^{\frac{1}{j}} < \infty.
\]

Émile Borel was the first to give examples of quasi analytic functions that were not also real-analytic in two papers [4] and [5]. The reader should consult either these two papers of Borel or [22] for such examples. This work by Borel led Jacques Hadamard to pose the question in 1912 as to whether there exists a condition on the sequence $M$ that ensures the quasi analyticity of the corresponding class. The answer to Hadamard’s question is the Denjoy-Carleman Theorem. This result appears as Theorem 2 of [22], where Vincent Thilliez cites other historically relevant sources.

2.11. **Theorem.** (Denjoy-Carleman Theorem) Let the sequence $M = \{M_0, M_1, M_2, \ldots \}$ be logarithmically convex, then $C_M(U)$ and $C^n_M(0)$ are quasi analytic iff
\[
\sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} = \infty.
\]

The following result of Carleman announced in [7] and carefully proven in [8] is important for the proof of our main theorem. See also Theorem 3 in [22].

2.12. **Theorem.** Assume that $C^n_M(0)$ is quasi analytic, and that $C^\omega(0) \subsetneq C^n_M(0)$. Then the Taylor morphism $C^n_M(0) \rightarrow \mathcal{F}_M$ is not surjective.

2.1. **Weierstrass Division Property.** We say that a polynomial $\varphi \in C^n_M(0)$ is a distinguished Weierstrass polynomial if $\varphi \in C^{n-1}_M(0)[x_n]$ is of the form:
\[
\varphi(x) = x_n^d + a_1(x')x_n^{d-1} + \ldots + a_d(x'),
\]
with $a_j \in C^{n-1}_M(0)$ and $a_j(0) = 0$, for all $1 \leq j \leq d$, where $x = (x', x_n)$.

Such a polynomial is called hyperbolic if there exists a neighbourhood $U$ of 0 in $\mathbb{R}^{n-1}$ such that $\forall x' \in U$, all the roots of $\varphi(x', \cdot)$ are real.
2.13. **Example.** The coordinate projections \(x_1, \ldots, x_n \in \mathcal{C}^n_M(0)\) are hyperbolic.

2.14. **Example.** The polynomial \(\varphi(x, y) = x^2 + y^2 \in \mathcal{C}^2_M(0)\) is not hyperbolic.

   A germ \(f(x', x_n) \in \mathcal{C}^n_M(0)\) is **regular in** \(x_n\) **of order** \(d\) if there exists a unit \(u(x', x_n)\) in the ring \(\mathcal{C}^n_M(0)\) such that \(f(0, x_n) = u(0, x_n) x_n^d\). Note this is equivalent to saying that \(f(0, 0) = \frac{\partial}{\partial x_n} f(0, 0) = \cdots = \frac{\partial}{\partial x_n}^d f(0, 0) = 0\), while \(\frac{\partial}{\partial x_n}^d f(0, 0) \neq 0\).

   A related notion is that of strict regularity. A germ \(f(x', x_n) \in \mathcal{C}^n_M(0)\) is **strictly regular in** \(x_n\) **of order** \(d\) if \(\frac{\partial}{\partial x_n}^d f(0, 0) \neq 0\), but for any multi-index \(\alpha \in \mathbb{N}^n\) such that \(|\alpha| \leq d - 1\), \(D\alpha f(0, 0) = 0\).

2.15. **Example.** The polynomial \(\varphi(x, y) = x + x^2 \in \mathcal{C}^2_M(0)\) is regular of order 2 in \(x\) but not strictly regular, whereas it is strictly regular of order 1 in \(x\).

   **Remark.** Note that any germ and any formal power series can be made strictly regular with respect to any chosen variable via a linear change of variables.

2.16. **Definition.** We say that the **Weierstrass Division Property** holds in \(\mathcal{C}^n_M(0)\) for a function \(f\) that is regular in \(x_n\) of order \(d\) if for every \(g \in \mathcal{C}^n_M(0)\), there exist \(q \in \mathcal{C}^{n-1}_M(0)\) and \(h_0, \ldots, h_{d-1} \in \mathcal{C}^{n-1}_M\) such that

   \[
   g = fq + h,
   \]

   where

   \[
   h(x', x_n) = \sum_{j=0}^{d-1} h_j(x') x_n^j.
   \]

2.17. **Theorem.** (Childress [12]) Assume that \(\mathcal{C}^n_M(0)\) is quasi analytic and stable under derivation, \(M\) is logarithmically convex, and \(\mathcal{C}^\omega(0) \subsetneq \mathcal{C}^n_M(0)\). If the Weierstrass division property holds for a function \(f\) that is regular in \(x_n\) of order \(d\), then there exist a unit \(u\) in the ring \(\mathcal{C}^n_M(0)\) and a distinguished Weierstrass polynomial \(\varphi\) of degree \(d\) such that \(f = u \varphi\) and \(\varphi\) is hyperbolic.
Remark. For $\varphi$ a distinguished Weierstrass polynomial, Childress’ theorem says that if $\varphi$ has the Weierstrass division property, then $\varphi$ must be hyperbolic.

2.18. **Theorem.** (Chaumat-Chollet [11]) Assume that $C^n_M(0)$ is quasi analytic and stable under derivation, $M$ is logarithmically convex, and $C^\omega(0) \subsetneq C^n_M(0)$. If $\varphi$ is a hyperbolic distinguished Weierstrass polynomial, then the Weierstrass division property holds for $\varphi$ in $C^n_M(0)$.

We make the following observation, which we will use in the proof of the main theorem.

2.19. **Corollary.** Assume that $C^n_M(0)$ is quasi analytic and stable under derivation, $M$ is logarithmically convex, and $C^\omega(0) \subsetneq C^n_M(0)$. Then the maximal ideal $\mathfrak{m} = \{f \in C^n_M(0) : f(0) = 0\}$ is generated by the coordinate projections $\{x_1, \ldots, x_n\}$.

**Proof.** Each of the polynomials $x_i$ is a hyperbolic distinguished Weierstrass polynomial, thus by Theorem 2.18 the Weierstrass division property holds for it. In particular, for $f \in \mathfrak{m}$,

$$f = x_1 g_1 + h(x_2, \ldots, x_n), \text{ with } g_1 \in C^n_M(0) \text{ and } h \in C^{n-1}_M(0).$$

Since $f(0) = 0 = 0 \cdot g_1(0) + h(0)$, we get $h(0) = 0$. Repeating this argument, we get $f = \sum_{i=1}^n x_i g_i$, with $g_i \in C^n_M(0)$. □

We close this section with the result of Jacques Chaumat and Anne-Marie Chollet in [10] that spells out what happens with Weierstrass division in $\mathcal{F}_M$. See also Theorem 6 in [22].

2.20. **Theorem.** (Chaumat-Chollet [10]) The following three properties are equivalent:

(i) The ring $\mathcal{F}_M$ is stable under derivation.

(ii) Weierstrass division holds in $\mathcal{F}_M$ for strictly regular divisors.

(iii) The ring $\mathcal{F}_M$ is Noetherian.
Given our assumptions on $M$, it follows that $\mathcal{F}_M$ is a Noetherian ring and Weierstrass division holds in it for strictly regular divisors. We will use these facts in the proof of the main theorem.

3. Algebraic considerations

Let $M = \{M_0, M_1, M_2, \ldots \}$ be an increasing sequence of positive real numbers with $M_0 = 1$, such that all the above assumptions (2), (3), (5), and (4) are satisfied.

Assuming that $C^n_M(0)$ is Noetherian, we will get a contradiction, applying Popescu’s generalization of Artin Approximation [1], which appears as Theorem 2.4 of Swan’s explanatory paper [21]:

3.1. Theorem. Suppose that $S$ is an excellent Henselian ring. Let $f(\alpha_1, \ldots, \alpha_r) = 0$ be a finite system of polynomial equations over $S$. If these equations have a solution in the completion $\hat{S}$, then they have a solution in $S$.

In our case, the completion $\hat{C^n_M(0)}$ of $C^n_M(0)$ with respect to the $m$-adic topology equals the ring $\mathcal{F}$ of all formal power series. $\mathcal{F}$ is a regular local ring of dimension $n$. The interested reader may consult p. 52 of AC VIII in [6]. Furthermore, the corollary to Proposition 1 on pp. 52-53 of AC VIII in [6] is also relevant to our argument:

3.2. Corollary. Let $S$ be a Noetherian local ring. $S$ is regular iff its completion $\hat{S}$ is regular.

It follows that if we assume $C^n_M(0)$ is Noetherian, then $C^n_M(0)$ is also a regular ring. A slight modification of the proof of Corollary 2.19 shows that $n = \dim_{C^n_M(0)/m}(m/m^2)$, so $\{x_1, \ldots, x_n\}$ forms a regular system of parameters for $C^n_M(0)$ and $\dim C^n_M(0) = n = \dim m$. The dimension here is Krull dimension, i.e., the supremum of the number of strict inclusions in a chain of prime ideals. It should be noted here that the completion $\hat{\mathcal{F}_M}$ of $\mathcal{F}_M$ with respect to the $m$-adic topology also equals $\mathcal{F}$, and $\mathcal{F}_M$ is likewise a regular local ring with $\{x_1, \ldots, x_n\}$ as its regular system of parameters.
By Corollary 2.8, $C^n_M(0)$ is Henselian. We would like to show that if $C^n_M(0)$ is Noetherian, then it is an excellent ring. Too much information is missing to check the definition of excellence directly. Since $C^n_M(0)$ is stable under derivation, we apply instead Matsumura’s Jacobian criterion, Theorem 102 on p.291 of [15].

3.3. Theorem. (Matsumura’s Jacobian criterion) Let $\mathbb{K}$ be a field of characteristic 0, and let $S$ be a regular ring containing $\mathbb{K}$. Suppose that

(1) for any maximal ideal $\mathfrak{m}$ of $S$, the residue field is algebraic over $\mathbb{K}$ and the Krull dimension $\dim \mathfrak{m}$ equals $n$.

(2) there exist $D_1, \ldots, D_n \in \text{Der}_{\mathbb{K}}(S)$ and $x_1, \ldots, x_n \in S$ such that $D_i x_j = \delta_{ij}$.

Then $S$ is excellent.

In this case, the field $\mathbb{C}$ of characteristic 0 lies inside $C^n_M(0)$. The residue field $C^n_M(0)/\mathfrak{m}$ equals $\mathbb{C}$, since for all $g \in C^n_M(0)$, $g(0) \in \mathbb{C}$ and $g(x) - g(0) \in \mathfrak{m}$. $\mathbb{C}$ is clearly an algebraic extension of itself, so part (1) holds. By Proposition 2.10, part (2) also holds for $D_i = \frac{\partial}{\partial x_i}$. The same argument with the field $\mathbb{R}$ replacing $\mathbb{C}$ applies to $C^n_{M,\mathbb{R}}(0)$.

Thus, if we assume $C^n_M(0)$ (respectively $C^n_{M,\mathbb{R}}(0)$) is Noetherian, Popescu’s Theorem holds for $C^n_M(0)$ (respectively $C^n_{M,\mathbb{R}}(0)$). In the next section we construct a system of two algebraic equations over $C^n_M(0)$ with two unknowns that has a solution in $\hat{F}_M \subset F = C^n_M(0)$ but not in $C^n_M(0)$. The first equation comes from the failure of Weierstrass division, but the second equation involving one of the unknowns or both cannot be explicitly constructed using any machinery involving Weierstrass division. To prove its existence, we rely on results about projective and free modules in a commutative local ring.

While there is a unique way to define a free module, namely as a module with a basis, a projective module can be defined in several equivalent ways. The interested reader should consult p.137 of [14]. We shall now state the definition that will be employed in our proof:
3.4. Definition. Let $A$ be a ring. An $A$-module $P$ is called projective if there exists an $A$-module $N$ such that $P \oplus N$ is a free $A$-module.

Remark. As on p.20 of [2], we shall view the $A$-module $P \oplus N$ as the set of pairs $(u, v)$ with $u \in P$ and $v \in N$ whereby module operations are defined by

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$$

and

$$a(u, v) = (au, av)$$

for $u_1, u_2 \in P$; $v_1, v_2 \in N$; and $a \in A$.

Finally, the following result is Theorem 4.4. on p.425 of [14]:

3.5. Theorem. Let $A$ be a commutative local ring with maximal ideal $m$. Let $P$ be a finite projective $A$-module. Then $P$ is free. Furthermore, if $x_1, \ldots, x_p$ are elements of $P$ whose residue classes $\bar{x}_1, \ldots, \bar{x}_p$ are a basis of $P/mP$ over $A/m$, then $x_1, \ldots, x_p$ are a basis of $P$ over $A$.

4. Proof of the main theorem

As mentioned in the introduction, the ring $C^1_M(0)$ has a very simple structure. Indeed, any germ in $C^1_M(0)$ is of the form $u(x)x^k$, where $u(x)$ is a unit and $k \geq 0$, simply because $C^1_M(0)$ is a DVR. It follows immediately that $C^1_M(0)$ is a principal ideal domain, so it must be Noetherian.

Next, to prove the main theorem, Theorem 1.1, we need to exploit the failure of Weierstrass division.

Preliminary Step: Assuming $C^2_M(0)$ is Noetherian, we construct an equation $f(x, y) = (y - ix)q(x, y) + yr(x)$ such that $q(x, y)$ and $r(x)$ are non-germs in $F_M$. Consider $\varphi(x, y) = y - ix$. Clearly, $\varphi(x, y)$ is a distinguished Weierstrass polynomial that is non-hyperbolic and strictly regular of order 1. By Theorem 2.17, there exists a germ $\hat{f}(x, y) \in C^2_M(0)$ such that in the Weierstrass division $\hat{f}(x, y) = \ldots$
\((y - ix)\hat{q}(x, y) + \hat{r}(x)\) carried out in the formal power series ring \(\mathcal{F}\), the quotient \(\hat{q}(x, y)\) and the remainder \(\hat{r}(x)\) do not both correspond to germs in \(C^2_M(0)\). By Theorem 2.20, \(\hat{q}(x, y), \hat{r}(x) \in F_M\). We will modify \(\hat{f}(x, y)\) to get \(f(x, y)\) in \(C^2_M(0)\) that equals the sum of non-germs \(q(x, y)\) and \(r(x)\) each multiplied by an element of \(m\), the maximal ideal of \(C^2_M(0)\). This property will become crucial when we apply Theorem 3.5. The construction is based on the following two observations:

- If \(a(x, y)\) is a germ in \(C^2_M(0)\) and \(b(x, y) \in F_M\) does not come from a germ, then their sum \(a(x, y) + b(x, y)\) cannot come from a germ. This is evident from the fact that \(C^2_M(0)\) is a ring.

- Assuming that \(C^2_M(0)\) is Noetherian, if \(a(x, y)\) is a germ in \(C^2_M(0)\) and \(b(x, y) \in F_M\) does not come from a germ, then their product \(a(x, y)b(x, y)\) cannot come from a germ. Assume it did. Then there would exist some germ \(c(x, y) \in C^2_M(0)\) such that \(a(x, y)b(x, y) = c(x, y)\). This is an equation involving one unknown \(a(x, y)X = c(x, y)\) with coefficients in \(C^2_M(0)\) and solution in \(\mathcal{F}\) but not in \(C^2_M(0)\) in violation of Popescu’s version of Artin Approximation, Theorem 3.1.

As a result, if \(C^2_M(0)\) is Noetherian, and one of the quotient and the remainder does not correspond to a germ then both do not correspond to germs. Also, without loss of generality, we can assume there exists some \(u(x, y)\) a unit in \(C^2_M(0)\) that fails to satisfy Weierstrass division by \(y - ix\). This holds because given any function \(h(x, y)\) not a unit that fails to satisfy Weierstrass division by \(y - ix\), we can set \(u(x, y) = h(x, y) + 1\); by the first observation, the remainder of the division of \(h(x, y)\) by \(y - ix\) to which we add the constant \(1\) is still not a germ.

Next, consider \(f(x, y) = u(x, y) y\). Let \(u(x, y) = (y - ix)\hat{q}(x, y) + \hat{r}(x)\) be the Weierstrass division of \(u(x, y)\) by \(y - ix\) in \(\mathcal{F}_M\). It follows

\[f(x, y) = (y - ix)q(x, y) + yr(x),\]
where \( r(x) = \tilde{r}(x) \) and \( q(x, y) = y \tilde{q}(x, y) \). By the second observation, the last equality, and the assumption on \( \tilde{q}(x, y) \), \( q(x, y) \) cannot come from a germ. Therefore, both \( q(x, y) \) and \( r(x) \) are formal power series in \( \mathcal{F}_M \) that do not come from germs.

**Proof of Theorem 1.1:**

\( C^2_M(0) \) is not Noetherian.

Assume that \( C^2_M(0) \) is Noetherian. The \( q(x, y) \) and \( r(x) \) in the equation we have constructed, \( f(x, y) = (y - ix)q(x, y) + y r(x) \) such that \( q(x, y) \) and \( r(x) \) are not germs can be viewed as solutions of a polynomial equation in two unknowns \( f(x, y) = (y - ix)X + y Y \) with coefficients in \( C^2_M(0) \). Now, the proof splits into two cases:

**Case 1:** There exists a second equation \( a(x, y)X + b(x, y)Y = c(x, y) \) such that \( a(x, y), b(x, y), \) and \( c(x, y) \) are all germs in \( C^2_M(0) \), this second equation is not obtained from the first by multiplication by an element in \( C^2_M(0) \) (so it is algebraically independent), and the formal power series \( q(x, y) \) and \( r(x) \) that are not germs satisfy this equation when we let \( q(x, y) = X \) and \( r(x) = Y \). In this case, the system of two polynomial equations given by \( f(x, y) = (y - ix)X + y Y \) and \( a(x, y)X + b(x, y)Y = c(x, y) \) has a uniquely determined solution, \( q(x, y) = X \) and \( r(x) = Y \), consisting of formal power series but not germs in contradiction to Popescu’s version of Artin Approximation, Theorem 3.1. Therefore, \( C^2_M(0) \) is not a Noetherian ring.

**Case 2:** No such second algebraically independent equation as in Case 1 exists.

In other words, the module of relations of the \( C^2_M(0) \)-module \((q(x, y), r(x))_{C^2_M(0)} \) is generated by \( f(x, y) = (y - ix)q(x, y) + y r(x) \). Let us mod out by \( C^2_M(0) \) and consider the \( C^2_M(0) \)-module \( \mathcal{P} = (q(x, y), r(x))_{C^2_M(0)} / C^2_M(0) \cap (q(x, y), r(x))_{C^2_M(0)} \). \( \mathcal{P} \) is finite since it is generated by two elements \( q(x, y) \) and \( r(x) \) modulo \( C^2_M(0) \). This case splits into two further sub-cases depending on whether \( \mathcal{P} \) is a projective module or not. To test projectivity, let

\[
N = (q(x, y), (q(x, y))^2, (q(x, y))^3, \ldots)_{C^2_M(0)} / C^2_M(0) \cap (q(x, y), (q(x, y))^2, (q(x, y))^3, \ldots)_{C^2_M(0)}
\]
consisting of all positive powers of $q(x, y)$ considered as an $C^2_M(0)$-module and modulo $C^2_M(0)$ itself.

**Case 2.1:** $P \oplus N$ is not free. Note that $P \oplus N$ is generated by

$$\{(q(x, y), q(x, y)^j), (r(x), q(x, y)^k)\}_{j, k \geq 1}$$

over $C^2_M(0)$ modulo $C^2_M(0)$. If $P \oplus N$ is not free, then there exists a relation among these generators

$$\sum_{j=1}^J a_j (q(x, y), q(x, y)^{p_j}) + \sum_{k=1}^K b_k (r(x), q(x, y)^{l_k})$$

$$= \left(\sum_{j=1}^J a_j q(x, y) + \sum_{k=1}^K b_k r(x), \sum_{j=1}^J a_j q(x, y)^{p_j} + \sum_{k=1}^K b_k q(x, y)^{l_k}\right)$$

$$= (0, 0) \text{ mod } C^2_M(0)$$

where $J, K \geq 1$ and $a_j, b_k \in C^2_M(0)$ and $\neq 0$ for all $j, k$. The entry in $P$,

$$\sum_{j=1}^J a_j q(x, y) + \sum_{k=1}^K b_k r(x) = 0 \text{ mod } C^2_M(0)$$

can simply amount to the relation $f(x, y) = (y - ix)q(x, y) + yr(x)$, but it is clear that the entry in $N$,

$$\sum_{j=1}^J a_j q(x, y)^{p_j} + \sum_{k=1}^K b_k q(x, y)^{l_k} = 0 \text{ mod } C^2_M(0)$$

gives a polynomial equation in $q(x, y)$ with coefficients in $C^2_M(0)$ that is algebraically independent from $f(x, y) = (y - ix)q(x, y) + yr(x)$. We thus have a system of two polynomial equations in two unknowns with coefficients in $C^2_M(0)$ whose unique solution given by $q(x, y), r(x)$ non-germs yields a contradiction to Popescu’s version of Artin Approximation, Theorem 3.1. Therefore, $C^2_M(0)$ is not a Noetherian ring.

**Case 2.2:** $P \oplus N$ is free, so $P$ is projective. $P$ is finite as observed above and $C^2_M$ is a commutative local ring. Theorem 3.5 yields that $P$ is a free module. Because
$f(x, y) = (y - ix)q(x, y) + yr(x)$ is the only relation between $q(x, y)$ and $r(x)$ with coefficients in $C^2_M(0)$ and $y - ix$ and $y$ are elements of the maximal ideal of $C^2_M(0)$ thus non-units, $q(x, y)$ and $r(x)$ are a basis of $P/\mathfrak{m}P$ over $C^2_M(0)/\mathfrak{m}$. The last part of Theorem 3.5 implies $q(x, y)$ and $r(x)$ must be a basis of $P$ over $C^2_M(0)$. This is impossible because $(y - ix)q(x, y) + yr(x) \equiv f(x) \equiv 0 \mod C^2_M(0)$, so $q(x, y)$ and $r(x)$ are linearly dependent over $C^2_M(0)$. We have obtained a contradiction, which means Case 2.2 cannot take place. There is always a second polynomial equation involving both $q(x, y)$ and $r(x)$ as in Case 1 or just $q(x, y)$ alone as in Case 2.1.

Remark. The reader should note that it is essential for this case that there be one and only one relation between $q(x, y)$ and $r(x)$ with coefficients in $C^2_M(0)$. Furthermore, the coefficients of $q(x, y)$ and $r(x)$ should be in the maximal ideal of $C^2_M(0)$; otherwise, the conclusion that $q(x, y)$ and $r(x)$ are a basis of $P/\mathfrak{m}P$ over $C^2_M(0)/\mathfrak{m}$ might not hold. This justifies the initial division of the argument into Case 1 and Case 2. Additionally, it has to be mentioned that the argument in Case 2.2 uses the ideas applied by Serge Lang on p.614 of [14] to show the maximal ideal of the formal power series ring in two or more variables cannot be flat as a module over the ring itself.

Note that both cases lead to the conclusion that $C^2_M(0)$ is not Noetherian.

$C^n_M(0)$ is not Noetherian for $n \geq 3$. We use an elementary result from [14], namely Proposition 1.5 on p.415 that says that if $A$ is a Noetherian ring and $\varphi : A \to B$ is a surjective ring homomorphism, then $B$ is Noetherian. We know $C^3_M(0)$ maps surjectively to $C^2_M(0)$, so if it were Noetherian, then $C^2_M(0)$ would be Noetherian, which is a contradiction. Inductively, $C^n_M(0)$ is Noetherian for every $n \geq 2$. \hfill \Box

Proof of Theorem 1.2:

Since we can write $C^n_M(0) \simeq C^{n,\mathbb{R}}_M(0) \oplus i C^{n,\mathbb{R}}_M(0)$ (see the remark following Definition 2.2), and it is an elementary fact that direct sums of Noetherian rings are Noetherian,
if $C_M^{m,R}(0)$ were Noetherian, then $C_M^n(0)$ would be Noetherian, which is a contradiction.

\[\square\]

Remark. We conclude by noting that Krzysztof Nowak recently announced the following result in [17]: If a first degree equation with coefficients in $C_M^{m,R}$ has a solution over $\mathcal{F}$, then it has a solution in $C_M^{m,R}$. His argument proceeds by sharpening some of the conclusions that Edward Bierstone and Pierre Milman obtained when they extended resolution of singularities to the Denjoy-Carleman real-valued quasi analytic functions.

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