CONNECTED SUM OF SPHERICAL CR MANIFOLDS WITH POSITIVE CR YAMABE CONSTANT

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ABSTRACT. Suppose $M_1$ and $M_2$ are two closed (compact with no boundary) spherical CR manifolds with positive CR Yamabe constant. In this note, we show that the connected sum of $M_1$ and $M_2$ also admits a spherical CR structure with positive CR Yamabe constant.

1. Introduction and statement of the results

In Riemannian geometry, we have the following fact about positive scalar curvature. Namely, the connected sum of two closed (compact with no boundary) manifolds of positive scalar curvature has a metric of positive scalar curvature (see Corollary 3 in Schoen-Yau’s paper [8] or Gromov-Lawson’s paper [2]). This fact has been generalized to surgeries in codimension $\geq 3$ and discussed in (spin or not spin) cobordism theory in dimension $\geq 5$ (see [2]). As an interesting result in confo-ormal geometry, the connected sum of two closed conformally flat manifolds with positive Yamabe constant is still a conformally flat manifolds with positive Yamabe constant (see Corollary 5 in [8]). In this note, we are going to prove an analogue in CR geometry.

For basic material in CR and pseudohermitian geometry, we refer the reader to [5] or [10]. Let $(M, J)$ be a closed, strictly pseudoconvex CR manifold of dimension $2n+1$. Take a contact form $\theta$, so we can talk about $L^p$ norm $\| \cdot \|_p$, Levi metric $| \cdot |$, subgradient $\nabla b$ and Tanaka-Webster scalar curvature $R$ or $R_{J, \theta}$ on the pseudohermitian manifold $(M, J, \theta)$. Take the volume form $dV := \theta \wedge (d\theta)^n$. Then we can write the CR Yamabe constant (or invariant) $\lambda(M, J)$ or $\lambda(M)$ (if $J$ is clear in the context) as

$$\lambda(M, J) = \inf_{u > 0} \frac{E_\theta(u)}{\|u\|_{2+2/n}^2}$$

where

$$E_\theta(u) := \int_M [(2 + \frac{2}{n})|\nabla b u|^2 + Ru^2]dV$$

and

$$\|u\|_{2+2/n}^2 := (\int_M |u|^{2+\frac{2}{n}}dV)^\frac{n+1}{2+2/n}$$

(see [3] for more details).

**Theorem A.** Suppose $(M_1, J_1)$ and $(M_2, J_2)$ are two closed, spherical CR manifolds of dimension $2n + 1$ with $\lambda(M_k, J_k) > 0$ for $k = 1, 2$. Then their connected sum $M_1 \# M_2$ admits a spherical CR structure $\tilde{J}$ with $\lambda(M_1 \# M_2, \tilde{J}) > 0$.

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The idea of the proof was motivated by the work of O. Kobayashi [4] (note that we do not follow the approach of either Schoen-Yau or Gromov-Lawson). Kobayashi's short proof is specially suitable for conformally flat manifolds with positive Yamabe constant. Due to different nature of cylinders structure between conformal and CR geometries, we modify the original idea of Kobayashi for the CR case (after we posted our paper on the ArXiv, we learned from Yun Shi that a similar argument in [7] works for the original approach of Kobayashi in the CR case. This fills up the gap of the proof in [9]).

Theorem A is used to construct many examples in the study of positive mass theorem for 5 dimensional closed, strictly pseudoconvex CR manifolds $M$ [1]. In [1], we assume further $M$ is spin, spherical with positive CR Yamabe constant. Then we have positive mass theorem for $M$. According to Theorem A, we have the following examples:

$$m_1(S^5/Z_{p_1}) \# l_1(S^4 \times S^1_{(a_1)}) \# m_2(S^5/Z_{p_2}) \# l_2(S^4 \times S^1_{(a_2)}) \# ...$$

(connected sum of finite number of manifolds such as $S^5/Z_p$ or $S^4 \times S^1_{(a_j)}$, $a_j > 1$) for $p_j$ odd (noting that $S^5/Z_2$ is not spin, but still spherical with positive CR Yamabe constant). See the end of Section 2 for more details.

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2. Proof of Theorem A

We first discuss the structure of the Heisenberg cylinder. Let $H_n$ denote the Heisenberg group. On $H_n \setminus \{0\}$, the dilations $\tau_a(z,t) = (az,a^2t)$, $z = (z^1, ..., z^n) \in C^n$, $t \in R$, for $a > 0$ and the CR inversion map $(z,t) \rightarrow (z^*, t^*)$ defined by

$$z^* = \frac{z}{w}, \quad t^* = -\frac{t}{|w|^2},$$

where $w := t + |z|^2$, are all CR transformations. The standard contact form $\Theta$ on $H_n$ reads

$$\Theta := dt + i \sum_{a=1}^n (z^a dz^{\bar{a}} - z^{\bar{a}} dz^a).$$

Instead of $\Theta$, we consider the contact form $\Theta \text{ Re} \rho$ where $\rho := |w|^{1/2} = (|z|^4 + t^2)^{1/4}$. Then all these maps also preserve the new contact form $\Theta \text{ Re} \rho$. The space $H_n \setminus \{0\}$ together with the new contact form $\Theta \text{ Re} \rho$ is called the Heisenberg cylinder. Topologically, $H_n \setminus \{0\} = (0, \infty) \times S^{2n}(1)$ where $S^{2n}(1) := \{ \rho = 1 \} \subset H_n$. For fixed $a > 1$, each slice $[a^{m-1}, a^m] \times S^{2n}(1)$ is isomorphic to one another as pseudohermitian manifolds. Consider the quotient space

$$(2.1) \quad S^{2n} \times S^1 \text{ or } S^{2n} \times S^1_{(a)} \text{ (to indicate the dependence on } a)$$

$$= H_n \setminus \{0\} / \{ \cdots, \tau_{a-1}, 1, \tau_a, \tau_{a+1}, \cdots \}.$$
We want to apply the following results to $S^{2n} \times S^1_{(a)}$. The proof is similar as for the analogous statements in the Riemannian case (see, e.g., [6]). For completeness, we give a proof here.

**Lemma 1.** Let $(M, J, \theta)$ be a closed pseudohermitian manifold of dimension $2n+1$.

1. Suppose the Tanaka-Webster scalar curvature $R_{J,\theta} \geq 0 > 0$ somewhere. Then $\lambda(M, J) > 0$.
2. Suppose $\lambda(M, J) > 0$. Then there exists a contact form $\tilde{\theta}$ such that $R_{J,\tilde{\theta}} > 0$.

**Proof.** For (1), suppose $\lambda(M, J) \leq 0$. We can solve the CR Yamabe equation by a theorem in [3] to find $u > 0$ such that

$$-(2+\frac{2}{n})\Delta_b u + R_{J,\theta} u = \lambda(M, J) u^{\frac{n+2}{n}}.$$  

Multiplying (2.2) by $u$ and integrating give

$$0 \leq \int_M [(2+\frac{2}{n})|\nabla_b u|^2 + R_{J,\theta} u^2]dV = \lambda(M, J) \int_M u^{2+\frac{2}{n}}dV \leq 0$$

since $R_{J,\theta} \geq 0$ and $\lambda(M, J) \leq 0$. It follows that $R_{J,\theta} \equiv 0$. Contradicts to $R_{J,\theta} > 0$ somewhere. So we conclude $\lambda(M, J) > 0$.

For (2), let $2 \leq s < p := 2+\frac{2}{n}$, critical exponent. Set $\lambda_s := \inf \{E_{\theta}(\varphi)/||\varphi||^{2s} : \varphi \in C^\infty(M)\}$. There exists a smooth, positive solution $\varphi_s$ to the subcritical equation

$$-p\Delta_b \varphi_s + R_{J,\theta} \varphi_s = \lambda_s \varphi_s^{p-1}$$

(Folland-Stein space $S^2 \subset L^s$ is compact). On the other hand, let $\tilde{\theta} := \varphi_s^{2/n}\theta$. Then by the transformation law, we have

$$-p\Delta_b \tilde{\theta} + R_{J,\tilde{\theta}} \tilde{\theta} = R_{J,\tilde{\theta}} \varphi_s^{p-1}$$

From (2.3) and (2.4), we have

$$R_{J,\tilde{\theta}} = \lambda_s \varphi_s^{s-p}.$$ 

Observe that $\lambda_s$ is continuous in $s$ from the left (cf. Lemma 4.3 in [4]). So $\lambda_s > 0$ for $s$ close to $p$ where $\lambda_p = \lambda(M, J) > 0$ by assumption. Thus $R_{J,\tilde{\theta}} > 0$ in view of (2.5) for $s$ close to $p$.

□

We compute the Tanaka-Webster scalar curvature $R = R_{\tilde{\theta}}$ on $S^{2n} \times S^1_{(a)}$ as follows:

$$R_{\tilde{\theta}} = \frac{n(n+1)|z|^2}{2\rho^2}.$$ 

Observe that $R_{\tilde{\theta}} \geq 0$ and $> 0$ if $z \neq 0$. From Lemma [1] (1) and (2), we can find a contact form $\tilde{\theta}$ such that the Webster curvature $R_{\tilde{\theta}}$, with respect to this contact form $\tilde{\theta}$, is positive on $S^{2n} \times S^1$. Therefore if we consider the lifting $\tilde{\Theta}$ of $\tilde{\theta}$ by the covering map $H_n \setminus \{0\} \to (S^{2n} \times S^1, \tilde{\theta})$ then the Webster curvature $R_{\tilde{\Theta}} > c$ for some positive constant $c$. In addition, $\tau_a$ also defines a symmetry on $(H_n \setminus \{0\}, \tilde{\Theta})$, which
implies that each slice $[a^{n-1}, a^n] \times S^{2n}(1)$ with respect to this new contact form $\Theta$, instead of $\frac{\Theta}{e^\rho}$, is also isomorphic to one another as pseudohermitian manifolds.

**Lemma 2.** We have

\[(2.6) \quad \lambda((M_1, J_1) \amalg (M_2, J_2)) = \min\{\lambda(M_1, J_1), \lambda(M_2, J_2)\},\]

provided that both $\lambda(M_1, J_1)$ and $\lambda(M_2, J_2)$ are positive.

**Proof.** Let $M = (M_1, J_1) \amalg (M_2, J_2)$ and $f = f_1 \amalg f_2$ be a $C^\infty$ function on $M$, where $f_1$ and $f_2$ are $C^\infty$ functions on $M_1$ and $M_2$, resp. If we take $f_2$ to be zero, it is easy to see that $\lambda(M) \leq \lambda(M_1, J_1)$. Similarly, taking $f_1$ to be zero, we have $\lambda(M) \leq \lambda(M_2, J_2)$.

On the other hand, suppose we choose $f = f_1 \amalg f_2$ such that

\[(2.7) \quad \int_M |f|^{2+\frac{2}{n}} dV = 1.\]

We compute

\[ \begin{align*}
(2 + \frac{2}{n}) \int_M |\nabla_b f|^2 dV + \int_M R f^2 dV \\
= 2 \sum_{j=1}^2 (2 + \frac{2}{n}) \int_{M_j} |\nabla_b f_j|^2 dV_j + \int_{M_j} R_j f_j^2 dV_j \\
\geq 2 \sum_{j=1}^2 \lambda(M_j, J_j) \left( \int_{M_j} |f_j|^{2+\frac{2}{n}} dV_j \right) \\
\geq 2 \sum_{j=1}^2 \lambda(M_j, J_j) \left( \int_{M_j} |f_j|^{2+\frac{2}{n}} dV_j \right) = 2 \sum_{j=1}^2 \lambda(M_j, J_j) \alpha_j,
\end{align*}\]

where $\alpha_j = \int_{M_j} |f_j|^{2+\frac{2}{n}} dV_j > 0$, and by \[(2.7), \quad \alpha_1 + \alpha_2 = 1. \] This shows that $\lambda(M) \geq \min\{\lambda(M_1, J_1), \lambda(M_2, J_2)\}$.

\[\square\]

**Proof. (of Theorem A)** Let $M_j, j = 1, 2$, be two differentiable manifolds and let $M = M_1 \amalg M_2$ be the disjoint union of $M_1$ and $M_2$. Fix $p_j \in M_j, j = 1, 2$. We take off two small balls around $p_1$ and $p_2$, and then attach a cylinder, which is topologically the product of a line segment and $S^{2n}$. The new manifold obtained in this way is called the connected sum of $M_1$ and $M_2$ denoted by $M_1 \# M_2$. If, in addition, assume that $M_j, j = 1, 2$, are two spherical CR manifolds with pseudohermitian structures $(J_j, \theta_j)$, then we will use (part of) the Heisenberg cylinder to glue them together in order that the result manifold is also spherical.

We can find contact forms $\tilde{\theta}_j$ on $M_j \setminus \{p_j\}, j = 1, 2$, such that it is part of the Heisenberg cylinder on a punched neighborhood of $p_j$. Precisely, we can choose $\tilde{\theta}_j$ such that

\[M_j \setminus \{p_j\} = \tilde{M}_j \cup [1, \infty) \times S^{2n}(1)\]

or equivalently

\[M_j \setminus \{p_j\} = \tilde{M}_j \cup (0, 1] \times S^{2n}(1)\]
For convenience, we write

\[ (M \setminus \{p_1, p_2\}, \tilde{J}, \tilde{\theta}) = \left( [1, \infty) \times S^{2n}(1) \cup \tilde{M}_1 \right) \cup \left( \tilde{M}_2 \cup (0, 1] \times S^{2n}(1) \right), \]

where

\[ \tilde{J}|_{M \setminus \{p_1, p_2\}} = J, \quad \tilde{\theta}|_{M \setminus \{p_1, p_2\}} = \tilde{\theta}. \]

Fix \( a > 1 \) and \( l \in \mathbb{N} \), a positive integer. Let \( \tilde{M} = M_1 \# M_2 \) be the connected sum of \( M_1 \) and \( M_2 \) obtained by cutting ends \( (a', \infty) \times S^{2n}(1) \) and \( (0, a^{-1}) \times S^{2n}(1) \) out of \( M \setminus \{p_1, p_2\} \) and gluing the left two parts by means of the dilation \( \tau_{a^{-1}} : [1, a'] \times S^{2n}(1) \to [a^{-l}, 1] \times S^{2n}(1) \). We hence get a new spherical CR manifold \((\tilde{M}, J_l, \theta_l)\) such that

\begin{equation}
(2.12)
\end{equation}

Moreover, instead of \( \frac{\rho}{n} \), we can choose \( \theta_1 \) such that \( \theta_1|[1, a'] \times S^{2n}(1) = \Theta \) so that the Webster curvature is strictly positive on each part \( [a^{m-1}, a^m] \times S^{2n}(1), \) for \( m = 1 \cdots l \).

Recall that

\[ \lambda(\tilde{M}, J_l) = \inf_{f > 0} \left( \frac{2 + \frac{2}{n}}{f^2} \int_{\tilde{M}} |\nabla_b f|^2 dV_i + \int_{\tilde{M}} R_i f^2 dV_i \right)^{-\frac{n}{n+2}}, \]

where \( dV_i = \theta_1 \wedge (d\theta_1)^n \). So, take a positive function \( f_1 \in C^\infty(\tilde{M}) \) such that

\begin{equation}
(2.9)
\end{equation}

\[ (2 + \frac{2}{n}) \int_{\tilde{M}} |\nabla_b f_1|^2 dV_i + \int_{\tilde{M}} R_i f_1^2 dV_i \leq \lambda(\tilde{M}, J_l) + \frac{1}{l} \]

and

\begin{equation}
(2.10)
\end{equation}

\[ \int_{\tilde{M}} |f_1|^{2+ \frac{2}{n}} dV_i = 1. \]

Lemma 3. There is an integer \( m \in \{1, \cdots, l\} \) such that

\begin{equation}
(2.11)
\end{equation}

\[ \int_{[a^{m-1}, a^m] \times S^{2n}(1)} (|\nabla_b f_1|^2 + f_1^2) dV_i \leq \frac{A}{l}, \]

where \( A \) is a constant independent of \( l \in \mathbb{N} \).

Proof. From \( (2.12) \), we have

\begin{equation}
(2.12)
\end{equation}

\[ (2 + \frac{2}{n}) \int_{[1, a'] \times S^{2n}(1)} |\nabla_b f_1|^2 dV_i + \int_{[1, a'] \times S^{2n}(1)} R_i f_1^2 dV_i \leq \lambda(\tilde{M}, J_l) + \frac{1}{l} + \int_{\tilde{M}} (-R_i f_1^2) dV_i. \]

On the other hand, using \( (2.10) \) and Hölder inequality, we have

\begin{equation}
(2.13)
\end{equation}

\[ \int_{\tilde{M}} -R_i f_1^2 dV_i \leq \left( \int_{\tilde{M}} |R_i|^{n+1} dV_i \right)^{\frac{1}{n+1}} \left( \int_{\tilde{M}} |f_1|^{2+ \frac{2}{n}+1} dV_i \right)^{\frac{n}{n+2}} \leq \left( \max_{\tilde{M}} \left| R_i \right| \right) \left( \text{vol}(\tilde{M}) \right)^{\frac{n}{n+2}}. \]
where $\tilde{R}$ is the Webster curvature with respect to $(\tilde{J}, \tilde{\theta})$. Substituting (2.13) into (2.12) and noticing that $R_t$ has an uniform lower bound $c > 0$ on the Heisenberg cylinder, we have
\[
(2 + \frac{2}{n}) \int_{[1,a']^2 \times S^{2n}(1)} |\nabla_b f_t|^2 dV + c \int_{[1,a']^2 \times S^{2n}(1)} f_t^2 dV \\
\leq \lambda(S^{2n+1}) + \frac{1}{l} + A_1,
\]
where $\lambda(S^{2n+1})$ is the Yamabe constant of the standard sphere and
\[
A_1 = \left( \max_{\tilde{M}} |\tilde{R}| \right) \left( \text{vol}(\tilde{M}) \right)^{\frac{1}{n+1}},
\]
which is independent of $l$. Let $C = \min\{2 + \frac{2}{n}, c\}$ and $A = \frac{\lambda(S^{2n+1}) + 1 + A_1}{C}$, and let $m \in \{1, \cdots, l\}$ be chosen so that the energy $\int_{[a^{m-1}, a^m] \times S^{2n}(1)} (|\nabla_b f_t|^2 + f_t^2) dV_t$ on the interval $[a^{m-1}, a^m]$ is the smallest among those on all the $l$ intervals. Then this energy satisfies the assertion of this lemma. We have completed the proof.

Now we cut off $\tilde{M}$ on the section $\{a^{2m-1} \} \times S^{2n}(1)$, where $a^{2m-1} = \sqrt{a^{m-1}a^m}$, and attach respectively the two cylinders $[a^{2m-1}, \infty) \times S^{2n}(1)$ and $(0, a^{2m-1-2l}] \times S^{2n}(1)$ to it, precisely, to $[1, a^{2m-1}] \times S^{2n}(1)$ and $[a^{2m-1-2l}, 1] \times S^{2n}(1)$. Then we obtain again the manifold
\[
(M \setminus \{p_1, p_2\}, \tilde{J}, \tilde{\theta}) = (M \setminus \{(a^{2m-1} \} \times S^{2n}(1)) \cup [a^{2m-1}, \infty) \times S^{2n}(1) \cup (0, a^{2m-1-2l}] \times S^{2n}(1).
\]
We think of the function $f_t$ as defined on $M \setminus \{(a^{2m-1} \} \times S^{2n}(1))$, and extend it to the whole space $(M \setminus \{p_1, p_2\}, \tilde{J}, \tilde{\theta})$ as follows: Let $F_t$ be the $C^\infty$ function on $M \setminus \{p_1, p_2\}$ such that
\[
F_t = \begin{cases} 
  f_t & \text{on } M \setminus \{(a^{2m-1} \} \times S^{2n}(1) \\
  \chi_1 f_t & \text{on } [a^{2m-1}, \infty) \times S^{2n}(1) \\
  \chi_2 f_t & \text{on } (0, a^{2m-1-2l}] \times S^{2n}(1),
\end{cases}
\]
where both $\chi_1$ and $\chi_2$ are cut off functions on the cylinder defined by
\[
\chi_1(t, x) = \begin{cases} 
  1, & t \leq a^{2m-1}; \\
  0, & t \geq a^m,
\end{cases}
\]
and
\[
\chi_2(t, x) = \begin{cases} 
  1, & t \geq a^{2m-1-2l}; \\
  0, & t \leq a^{m-1-2l}.
\end{cases}
\]
Notice that each part $[a^{m-1}, a^m] \times S^{2n}(1)$ is isomorphic to $S^{2n} \times S^1$, so that $\chi_1$ and $\chi_2$ can be chosen so that $|\nabla_b \chi_1|$ and $|\nabla_b \chi_2|$ have upper bound $c_1$ and $c_2$ independent of $m$. Now it is easy to see from (2.9) and (2.11)
\[
(2 + \frac{2}{n}) \int_{M \setminus \{p_1, p_2\}} |\nabla_b F_t|^2 d\tilde{V} + \int_{M \setminus \{p_1, p_2\}} \tilde{R} F_t^2 d\tilde{V} \leq \lambda(M, J_t) + \frac{B}{l},
\]
where \(d\tilde{V} = \tilde{\theta} \wedge (d\tilde{\theta})^n\) and \(\tilde{R}\) is the Webster curvature with respect to \((\tilde{J}, \tilde{\theta})\), and \(B\) is a constant independent of \(l\). Obviously from (2.10)

\[
\int_{M\setminus\{p_1, p_2\}} |F|^{2+\frac{2}{n}} d\tilde{V} > 1.
\]

Therefore we have

\[
\inf (2 + \frac{2}{n}) \int_{M\setminus\{p_1, p_2\}} |\nabla F|^2 d\tilde{V} + \int_{M\setminus\{p_1, p_2\}} \tilde{R} F^2 d\tilde{V} \leq \lambda(M, J_1) + \frac{B}{l},
\]

where the infimum is taken over all nonnegative \(C^\infty\) function \(F\) with compact support. It follows from the choice of the contact form \(\tilde{\theta}\) that the left hand side of (2.16) is equal to \(\lambda(M)\), in which \(M = M_1 \amalg M_2\). Therefore, by lemma 2 if \(l\) is large enough, we have \(\lambda(M, J_1) > 0\).

\[
\Box
\]

**Examples.** Let \(p\) be a positive integer. Let \(Z_p\) denote the cyclic group generated by the following \((n + 1) \times (n + 1)\) diagonal matrix:

\[
\begin{pmatrix}
e^{2\pi i/p} & 0 & \cdots & 0 \\
0 & e^{2\pi i/p} & \cdots & : \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & e^{2\pi i/p}
\end{pmatrix}
\]

acting on \(C^{n+1}\) by multiplying by \(e^{2\pi i/p}\). It is clear that \(Z_p\) leaves the unit sphere \(S^{2n+1} \subset C^{n+1}\) invariant and preserves the standard CR structure on \(S^{2n+1}\). Moreover, \(Z_p\) acts on \(S^{2n+1}\) freely. We then have the quotient manifold \(S^{2n+1}/Z_p\) which is closed, spherical and has \(\lambda(S^{2n+1}/Z_p) > 0\) (since \(Z_p\) leaves the standard contact form for \(S^{2n+1}\) invariant, \(S^{2n+1}/Z_p\) has the same Tanaka-Webster scalar curvature as \(S^{2n+1}\), which is a positive constant). On the other hand, we learn from the paragraph before Lemma 2 that \(S^{2n} \times S^1\) (see (2.1)) is also a closed, spherical CR manifold with \(\lambda(S^{2n} \times S^1) > 0\).

Now according to Theorem A, we have the following closed, spherical CR manifolds with positive CR Yamabe constant:

\[
m_1(S^{2n+1}/Z_{p_1}) \# l_1(S^{2n} \times S^1_{(a_1)}) \# m_2(S^{2n+1}/Z_{p_2}) \# l_2(S^{2n} \times S^1_{(a_2)}) \# \ldots
\]

(connected sum of finite number of manifolds such as \(S^{2n+1}/Z_{p_1}\), \(p_j\) being positive integers, or \(S^{2n} \times S^1_{(a_1)}\), \(a_j > 1\) where \(m_1, l_1, m_2, l_2\) are nonnegative integers. In application to construct 5-dimensional CR manifolds for positive mass theorem to hold (see [1]), we need to restrict \(p\) to be odd in order for \(S^5/Z_p\) to be spin.

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