SOLVING THE YAMABE PROBLEM ON CLOSED MANIFOLDS BY ITERATION SCHEMES AND PERTURBATION METHODS

JIE XU

Abstract. We introduce a double iterative scheme and a perturbation method to solve the Yamabe equation \( \Box_g u := -\frac{4(n-1)}{n-2} \Delta_g u + S_g u = \lambda u^{\frac{4n}{n-2}} \) on closed manifolds \((M, g)\) with \(\dim M \geq 3\). Thus \(g\) admits a conformal change to a constant scalar curvature metric. In contrast to the traditional method, the Yamabe problem is completely solved in five cases classified by the sign of the scalar curvature \(S_g\) and the sign of the first eigenvalue of conformal Laplacian. In contrast to the traditional proof, we focus on local analysis; both Weyl tensor and positive mass theorem are not required, and the proof is essentially dimension independent. To treat the most difficult case \(\lambda > 0\), we construct the lower and upper solutions to solve a perturbed Yamabe equation \(\Box_g u_\beta + \beta u_\beta = (\lambda_\beta - \kappa) u_\beta^{\frac{4n}{n-2}}\) with \(\kappa > 0\), \(\beta < 0\) and some specific choices of \(\lambda_\beta\)'s, then take a limit by letting \(\beta \to 0\).

1. Introduction

In this paper, we will solve the Yamabe problem by a double iterative method inspired by [23], in which the Yamabe equation can be solved by an iterative method in a small enough bounded, open subset of \(\mathbb{R}^n\) equipped with some metric \(g\) and with Dirichlet boundary conditions. This iterative method is developed for hyperbolic operators [11, 12] and elliptic operators [23, 27] with a long history in PDE theory dating back to [21, 22]. Instead of a global analysis in terms of calculus of variation, this article applies the local analysis developed especially in [23] and [26] to the Yamabe equation. The local analysis allows us to prove our results in all dimensions by one method, in contrast to previous proofs that separately treated low dimensions and locally conformally flat manifolds. When \(\lambda < 0\), a pair of global sub- and super-solutions are constructed; when \(\lambda > 0\), a perturbation method is applied and a local test function is good enough for the existence of the solution of a perturbed Yamabe equation \(-\frac{4(n-1)}{n-2} \Delta_g u + (S_g + \beta) u = \lambda u^{\frac{4n}{n-2}}\) on any closed manifolds \(M\) with \(\dim M \geq 3\) and \(\beta < 0\), we then show the existence of Yamabe equation by letting \(\beta \to 0\). In particular, the local analysis gives us the flexibility to choose \(\lambda\) in perturbed Yamabe equation such that \(\lambda < \lambda(S^n)\). Here \(\lambda(S^n)\) is the Yamabe invariant with respect to the standard sphere.

In 1960, Yamabe proposed the following generalization of the classical uniformization theorem for surfaces:

**The Yamabe Conjecture.** Given a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\), there exists a metric conformal to \(g\) with constant scalar curvature.

Let \(S_g\) be the scalar curvature of \(g\), and let \(\tilde{S}_g\) be the scalar curvature of the conformal metric \(\tilde{g} = e^{2f} g\). Set \(e^{2f} = u^{p-2}\), where \(p = \frac{2n}{n-2}\) and \(u > 0\). Then

\[
S_{\tilde{g}} = u^{1-p} \left( -4 \cdot \frac{n-1}{n-2} \Delta_g u + S_g u \right). \tag{1}
\]

Setting \(a = 4 \cdot \frac{n-1}{n-2} > 0\), we have that \(\tilde{g} = u^{p-2} g\) has constant scalar curvature \(\lambda\) if and only if \(u > 0\) satisfies the Yamabe equation

\[-a \Delta_g u + S_g u = \lambda u^{p-1}, \tag{2}\]

\(1\)
where $\Delta_g = -d^*d$ is negative definite. was solved in several steps by important work of Yamabe, Trudinger, Aubin and Schoen, with the final cases solved in 1984. A thorough presentation of the historical development of the solution is in [18]. There are also results for manifolds with boundary [16, 6, 7, 20] and open manifolds [3, 10, 14] with certain restrictions.

For a more precise statement of the cases, let $\eta_1$ be the first eigenvalue of the conformal Laplacian $\Box_g := -a\Delta_g + S_g$ on $(M, g)$. The main theorem is the following:

**Theorem.** Let $(M, g)$ be a closed Riemannian manifold with $\dim M \geq 3$. Then

(i) If $\eta_1 = 0$, (2) has a real, positive solution $u \in C^\infty(M)$ with $\lambda = 0$;
(ii) If $\eta_1 < 0$ and $S_g \leq 0$ everywhere or $S_g > 0$ somewhere, (2) has a real, positive solution $u \in C^\infty(M)$ with some $\lambda < 0$;
(iii) If $\eta_1 > 0$ and $S_g \leq 0$ somewhere or $S_g > 0$ everywhere, (2) has a real, positive solution $u \in C^\infty(M)$ with some $\lambda > 0$.

These cover all cases of the Yamabe problem. Case (i) is a trivial eigenvalue problem; Case (ii) is solved in Theorem 4.1 and 4.2 below; Case (iii) is solved in Theorem 4.4 and 4.6. Theorem 4.3, in which a perturbed Yamabe equation is solved, plays a central role in Case (iii).

Historically, the PDE (2) is analyzed as the Euler-Lagrange equation of the functional

$$
\lambda(M) = \inf_{u \neq 0} \frac{\int_M a|\nabla_g u|^2 + S_g u^2 dVol_g}{(\int_M u^p dVol_g)^{\frac{2}{p}}} = \inf_{u \neq 0} Q(u)
$$

(3)

and the Yamabe Conjecture reduced to proving a minimizer exists. After results showing the existence the solution provided $\lambda(M) < \lambda(S^n)$, the problem is equivalent to the nontrivial task of finding a test function $\phi$ such $Q(\phi) < \lambda(S^n)$ in [3].

There are some critical technical differences between our approach and the classical one. In a small enough domain $\Omega$ with radius $r$, a local test function $\phi(\epsilon, x)$ is chosen to show that the Yamabe quotient $Q(\phi(\epsilon, x))$ satisfies

$$
Q(\phi(\epsilon, x)) \leq (1 + O(r^2))(aT + O(\epsilon)),
$$

with best Sobolev constant $T$ for Euclidean Sobolev embedding $L^p \hookrightarrow H^1$. Since we want $\lambda(M) < aT$, the key is to control the smallness of $O(r^2)$. In the classical approach, a local test function $\phi \in C^\infty_c(\Omega)$ and a particular Riemannian metric within the conformal class were carefully chosen to show that $Q(\phi) < \lambda(S^n) = aT$ when $n \geq 6$ and the manifold is not locally conformally flat; Schoen successfully applied the positive mass theorem to the remaining cases. In this article, especially for the cases when $\lambda > 0$, we apply a direct analysis, which can be found in Appendix A to show that a perturbed functional

$$
Q^\beta(u) = \frac{\int_M a|\nabla_g u|^2 + (S_g + \beta) u^2 dVol_g}{(\int_M u^p dVol_g)^{\frac{2}{p}}}, \beta < 0
$$

satisfies $Q^\beta(\phi) < aT$ locally in $\Omega$ for all metric $g$ with $n \geq 3$, provided that $\Omega$ is small enough. This results in the existence of a local solution [36] of the form $Lu = b(x)u^{p-1} + f(x, u)$ on $\Omega$ with trivial Dirichlet boundary condition. We then apply monotone iteration scheme, e.g. [3, 16], to show the existence of a positive solution of the perturbed Yamabe equation

$$
-a\Delta_g u + (S_g + \beta) u = (\lambda - \kappa) u^{p-1} \text{ on } M.
$$

(4)

Here $\lambda_\beta = \inf_{u \neq 0} Q^\beta u$ and $\kappa > 0$ is some fixed constant. The local analysis and monotone iteration scheme give us the flexibility to choose the coefficient, especially the introduction of $\kappa$, ahead of the nonlinear term; the introduction of $\kappa$ cannot be achieved by classical method. A limiting argument shows $u = \lim_{\beta \to 0} u_\beta$ exists for a subsequence of $\{u_\beta\}$ and solves the Yamabe equation (2). The inequality $Q^\beta(\phi) < aT$ holds for all $n \geq 3$, and is independent of the Weyl tensor precisely
because of the introduction of $\beta$. This method bypasses the subcritical exponent analysis of uniform boundedness of the sequence $\{\varphi_s\}$ when $s \to p$, where $\varphi_s$ solves $\Box_g \varphi_s = \lambda_s \varphi_s^{-\frac{4}{n-2}}$ for $s \in (2, p)$.

In particular, we constructed sub-solutions $u_-$ and super-solutions $u_+$ of the Yamabe equation (2) or the perturbed Yamabe equation (72), then applied monotone iterative method—see e.g. 3, 16—to get a solution, this is one layer of iteration. The essential difficulty is to show that $u_+ \neq 0$. Historically the cases when $\eta_1 < 0$ is easy to handle. We applied local analysis in [23] to give a different way to construct the sub-solutions/super-solutions, this is the second layer of iteration. For the cases when $\eta_1 > 0$, barely no one constructed the sub-solution/super-solution before, based on the best understanding of us. The full power of this iteration scheme can be seen in solving the boundary Yamabe problem on compact manifolds with boundary. The size of the local domain we choose is also important: in particular, the scalar curvature at each point tells us the deviation of the volume of small enough geodesic ball centered at the same point from the volume of Euclidean ball with same radius. Weyl tensor cannot provide this type of information.

The paper is organized as follows. In §2, the preliminaries and required results for double iteration scheme are listed. In §3, we constructed local solutions on a small enough single chart of $(M, g)$ with Dirichlet boundary conditions for different cases classified by $\mathrm{sgn}(S_g)$ and $\mathrm{sgn}(\eta_1)$: when $\eta_1 < 0$, the local analysis in [23] is applied, see Proposition 3.1 and 3.2; when $\eta_1 > 0$, a local calculus of variation is applied, see Proposition 3.3. When $\eta_1 > 0$, the region in which $S_g < 0$ plays a central role in the local analysis.

In §4, sub-solutions $u_-$ and super-solutions $u_+$ are constructed in various cases: (i) Theorem 4.1 and 4.2 are related to the case $\eta_1 < 0$. As discussed before, they are historically easy case, here a new construction of sub- and super- solutions is applied. (ii) Theorem 4.3 and 4.4 are related to the case $\eta_1 > 0$. For the case $\eta_1 > 0$, Theorem 4.3 plays a central role for the existence of solution of a perturbed Yamabe equation. The constructions of sub- and super- solutions rely on the local analysis given in Proposition 3.3. To handle the cases when $S_g \leq 0$ everywhere and $S_g \geq 0$ everywhere, we shows in Theorem 4.5 and Corollary 4.1 that they are equivalent to the cases $S_g > 0$ somewhere, $\eta_1 < 0$ and $S_y < 0$ somewhere, $\eta_1 > 0$, respectively. Note that the last two results mentioned above relates to the question: whether a function $f$ can be the prescribed scalar curvature of some metric under conformal change?

2. The Preliminaries

In this section, we list definitions and results required for our analysis.

Let $\Omega$ be a connected, bounded, open subset of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ equipped with some Riemannian metric $g$ that can be extended smoothly to $\bar{\Omega}$. We call $(\Omega, g)$ a Riemannian domain. Furthermore, let $(\Omega, g)$ be a compact manifold with smooth boundary extended from $(\Omega, g)$. Let $(M, g)$ be a closed manifold with $\dim M \geq 3$. Let $(M', g)$ be a general compact manifold with interior $M'$ and smooth boundary $\partial M'$. Throughout this article, we denote the space of smooth functions with compact support by $C_c^\infty$, smooth functions by $C^\infty$, and continuous functions by $C_0$.

We define two equivalent versions of the $L^p$ norms and two equivalent versions of the Sobolev norms on $(\Omega, g)$. We also define global $L^p$ norms and Sobolev norms on $(M, g)$.

**Definition 2.1.** Let $(\Omega, g)$ be a Riemannian domain. Let $(M, g)$ be a closed Riemannian $n$-manifold with volume density $d\omega$ with local expression $d\operatorname{Vol}_g$. Let $u$ be a real valued function. Let $\langle v, w \rangle_g$ and $|v|_g = \langle v, v \rangle_g^{1/2}$ denote the inner product and norm with respect to $g$. 
(i) For $1 \leq p < \infty$,
\[ \mathcal{L}^p(\Omega) \] is the completion of \( \left\{ u \in C^\infty_0(\Omega) : \|u\|_p^p := \int_\Omega |u|^p \ dx < \infty \right\} \),
\[ \mathcal{L}^p(\Omega, g) \] is the completion of \( \left\{ u \in C^\infty_0(\Omega) : \|u\|_{p,g}^p := \int_\Omega |u|^p \ dVol_g < \infty \right\} \),
\[ \mathcal{L}^p(M, g) \] is the completion of \( \left\{ u \in C^\infty(M) : \|u\|_{p,g}^p := \int_M |u|^p \ dVol_g < \infty \right\} \).

(ii) For \( \nabla u \) the Levi-Civita connection of \( g \), and for \( u \in C^\infty(\Omega) \) or \( u \in C^\infty(M) \),
\[ |\nabla^k u|_g^2 := (\nabla^{\alpha_1} \ldots \nabla^{\alpha_k} u)(\nabla^{\alpha_1} \ldots \nabla^{\alpha_k} u). \] (5)
In particular, \( |\nabla^0 u|_g^2 = |u|^2 \) and \( |\nabla^1 u|_g^2 = |\nabla u|_g^2 \).

(iii) For \( s \in \mathbb{N}, 1 \leq p < \infty \),
\[ W^{s,p}(\Omega) = \left\{ u \in \mathcal{L}^p(\Omega) : \|u\|_{W^{s,p}(\Omega)}^p := \sum_{j=0}^s \int_\Omega |D^j u|^p \ dx < \infty \right\} , \] (6)
\[ W^{s,p}(\Omega, g) = \left\{ u \in \mathcal{L}^p(\Omega, g) : \|u\|_{W^{s,p}(\Omega,g)}^p := \sum_{j=0}^s \int_\Omega |\nabla^j u|_g^p dVol_g < \infty \right\} , \]
\[ W^{s,p}(M, g) = \left\{ u \in \mathcal{L}^p(M, g) : \|u\|_{W^{s,p}(M, g)}^p := \sum_{j=0}^s \int_M \nabla^j u|_g^p d\omega < \infty \right\} . \]

Here \( |D^j u|^p := \sum_{|\alpha|=j} |\partial^\alpha u|^p \) in the weak sense. Similarly, \( W^{s,p}_0(\Omega) \) is the completion of \( C^\infty_0(\Omega) \) with respect to the \( W^{s,p} \)-norm. In particular, \( H^{s}(\Omega) := W^{s,2}(\Omega) \) and \( H^s(\Omega, g) := W^{s,2}(\Omega, g) \), \( H^s(M, g) := W^{s,2}(M, g) \) are the usual Sobolev spaces. We similarly define \( H^s_0(\Omega), H^s_0(\Omega, g) \).
(iv) We define the \( W^{s,p} \)-type Sobolev space on \( (M', g) \) the same as in (iii) when \( s \in \mathbb{N}, 1 \leq p < \infty \).

The elliptic regularity for Riemannian domain \((\Omega, g)\) and closed manifold \((M, g)\), and for \( W^{s,p} \)-type and \( H^s \)-type, are listed below, respectively.

**Theorem 2.1.** Let \((\Omega, g)\) be a Riemannian domain. Let \((M, g)\) be a closed Riemannian manifold and \((M', g)\) be a compact Riemannian manifold with smooth boundary.

(i)\,[24] Ch. 5, Thm. 1.3] Let \( L \) be a second order elliptic operator of the form \( Lu = -\Delta_g u + Xu \) where \( X \) is a first order differential operator with smooth coefficients on \( M' \). For \( f \in L^2(M', g) \), a solution \( u \in H^1_0(\Omega, g) \) to \( Lu = f \) in \( M' \) with \( u \equiv 0 \) on \( \partial M' \) belongs to \( H^2(M', g) \), and
\[ \|u\|_{H^2(M', g)} \leq C^* \left( \|f\|_{L^2(M', g)} + \|u\|_{H^1(M', g)} \right) . \] (7)

\( C^* = C^*(L, M', g) \) depends on \( L \) and \( (M', g) \).

(ii)\,[13] Vol. 3, Ch. 17] Let \( -\Delta_g \) be the Laplacian with respect to Riemannian metric \( g \) on \( \Omega \), and let \( u \in H^1_0(\Omega, g) \) be a weak solution of \(-\Delta_g u = f\).
(Interior Regularity) If \( f \in W^{s,p}_0(\Omega, g) \), then \( u \in W^{s+2,p}_0(\Omega, g) \). Also, if \( u \in \mathcal{L}(\Omega, g) \), then
\[ \|u\|_{W^{s+2,p}_0(\Omega, g)} \leq D_1 \left( \|\Delta_g u\|_{W^{s,p}_0(\Omega,g)} + \|u\|_{\mathcal{L}(\Omega,g)} \right), \] (8)
for some \( D_1 > 0 \) where \( D_1 \) only depends on \( -\Delta_g, \Omega \) and \( \partial \Omega \).
(Schauder Estimates) If \( f \in C^{s,\alpha}(\Omega) \), then \( u \in C^{s+2,\alpha}(\Omega) \). Also, if \( u \in C^{0,\alpha}(\Omega) \), then
\[ \|u\|_{C^{s+2,\alpha}(\Omega)} \leq D_2 \left( \|\Delta_g u\|_{C^{s,\alpha}(\Omega)} + \|u\|_{C^{0,\alpha}(\Omega)} \right), \] (9)
for some \( D_2 > 0 \) where \( D_2 \) only depends on \( -\Delta_g, \Omega \) and \( \partial \Omega \).
(iii) [8] Thm. 2.4] Let \(-\Delta_g\) be the Laplace-Beltrami operator, and let \(u \in H^1(M, g)\) be a weak solution of \(-\Delta_g u = f\). If \(f \in W^{s,p}(M, g)\), then \(u \in W^{s+2,p}(M, g)\). Also, if \(u \in L^p(M, g)\), then above estimates in [3] and [9] also hold when replacing \((\Omega, g)\) by \((M, g)\).

Sobolev embeddings, both on \((\Omega, g)\) and \((M, g)\), play an important role in this article.

**Theorem 2.2.** [1] Ch. 4] (Sobolev Embeddings) Let \(\Omega \in \mathbb{R}^n\) be a bounded, open set with smooth boundary \(\partial \Omega\) and equipped with any Riemannian metric \(g\).

(i) For \(s \in \mathbb{N}\) and \(1 \leq p \leq p' < \infty\) such that
\[
\frac{1}{p} - \frac{s}{n} \leq \frac{1}{p'},
\]
\(W^{s,p}(\Omega)\) continuously embeds into \(L^{p'}(\Omega)\) with the following estimates:
\[
\|u\|_{L^{p'}(\Omega, g)} \leq K \|u\|_{W^{s,p}(\Omega, g)}.
\]

(ii) For \(s \in \mathbb{N}\), \(1 \leq p < \infty\) and \(0 < \alpha < 1\) such that
\[
\frac{1}{p} - \frac{s}{n} \leq -\frac{\alpha}{n},
\]
Then \(W^{s,p}(\Omega)\) continuously embeds in the Hölder space \(C^{0,\alpha}(\Omega)\) with the following estimates:
\[
\|u\|_{C^{0,\alpha}(\Omega)} \leq K' \|u\|_{W^{s,p}(\Omega, g)}.
\]

(iii) The above results and estimates also hold when applying to the closed manifold \((M, g)\).

A more precise version of Sobolev embedding is provided by Gagliardo-Nirenberg interpolation inequality.

**Proposition 2.1.** [2] Thm. 3.70] Let \(M\) be either \(\mathbb{R}^n\), or a compact Riemannian manifold with or without boundary. Let \(q, r, l\) be real numbers with \(1 \leq q, r, l \leq \infty\), and let \(j, m\) be integers with \(0 \leq j < m\). Define \(\alpha\) by solving
\[
\frac{1}{l} = \frac{j}{n} + \alpha \left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1 - \alpha}{q},
\]
as long as \(l > 0\). If \(\alpha \in \left[\frac{j}{m}, 1\right]\), then there exists a constant \(C_0\), depending only on \(n, m, j, q, r, \alpha\) and on the manifold, such that for all \(u \in C^\infty_c(M)\) (with \(\int_M u = 0\) in the compact case without boundary),
\[
\|\nabla^j u\|_{L^r(\Omega, g)} \leq C_0 \|\nabla^m u\|_{L^r(\Omega, g)}^{\alpha} \|u\|_{L^q(\Omega, g)}^{1 - \alpha}.
\]
(If \(r = \frac{n}{m-j} \neq 1\), then (14) is not valid for \(\alpha = 1\).)

To see how different choices of \(\lambda\) affect the behaviors of solutions of Yamabe equation \(-a\Delta_g u + S_g u = \lambda u^{p-1}\) in \((\Omega, g)\) and \(u \equiv c > 0\) on \(\partial \Omega\), we need both strong and weak maximum principles.

**Theorem 2.3.** (i) [8] Cor. 3.2] (Weak Maximum Principle) Let \(L\) be a second order elliptic operator of the form
\[
Lu = -\sum_{|\alpha|=2} a_\alpha(x) \partial^\alpha u + \sum_{|\beta|=1} b_\beta(x) \partial^\beta u + c(x) u
\]
where \(a_\alpha, b_\beta, c \in C^\infty(\Omega)\) are smooth and bounded real-valued functions on the bounded domain \(\Omega \subset \mathbb{R}^n\). Let \(u \in C^2(\Omega)\). Denote \(u^- := \min(u, 0)\), and \(u^+ := \max(u, 0)\). Then we have
\[
Lu \geq 0, c(x) \geq 0 \Rightarrow \inf_{\Omega} u = \inf_{\partial \Omega} u^-;
\]
\[
Lu \leq 0, c(x) \geq 0 \Rightarrow \sup_{\Omega} u = \sup_{\partial \Omega} u^+.
\]
(ii) [8] Thm. 3.1] Let \( u, L \) and its coefficients be the same as in (i) above. Then we have
\[
Lu \geq 0, c(x) = 0 \Rightarrow \inf_{\Omega} u = \inf_{\partial \Omega} u;
\]
\[
Lu \leq 0, c(x) = 0 \Rightarrow \sup_{\Omega} u = \sup_{\partial \Omega} u.
\]

(17)

(iii) [8] Thm. 3.5] (Strong Maximum Principle) Let \( u \in C^{2}(\Omega) \). Assume that \( \partial \Omega \in C^{\infty} \). Let \( L \) be a second order uniformly elliptic operator as above. If \( Lu \geq 0, c(x) \geq 0 \), and if \( u \) attains a nonpositive minimum over \( \Omega \) in an interior point, then \( u \) is constant within \( \Omega \); If \( Lu \leq 0, c(x) \geq 0 \), and if \( u \) attains a nonnegative maximum over \( \Omega \) in an interior point, then \( u \) is constant within \( \Omega \).

(iv) [8] Ch. 8] All weak and strong maximum principles above hold when \( u \in H^{1}(\Omega, g) \) or \( u \in H^{1}(M, g) \), respectively, provided that \( L \) is uniformly elliptic with some other restrictions.

Let’s denote the conformal Laplacian by
\[
\Box_{g} u := -a \Delta_{g} u + S_{g} u, u \in C_{c}^{\infty}(\Omega) \text{ or } u \in C^{\infty}(M).
\]

The next two results from Kazdan and Warner [16] for closed manifold, and from Escobar [7] for compact manifold with boundary below, respectively, show that the sign of \( \lambda \) is determined by the sign of \( \eta_{1} \).

**Theorem 2.4.**

(i) [16] Lemma 2.5] Let \((M, g)\) be a closed manifold. Let \( \eta_{1} \) be the first nonzero eigenvalue of the conformal Laplacian \( \Box_{g} \). Then the sign of \( \lambda \), the constant scalar curvature with respect to \( \bar{g} = u^{p-2}g \), must be the same as the sign of \( \eta_{1} \).

(ii) [7] Lemma 1.1] Let \((M', g)\) be a compact manifold with boundary. Then the same conclusion above holds on \((M', g)\) with the boundary condition \( \frac{\partial u}{\partial \nu} + \frac{n+2}{2} h_{g} u = 0 \), where \( h_{g} \) is the mean curvature with respect to \( \partial M' \).

**Remark 2.1.** Note that \( \eta_{1} \) satisfies the PDE
\[
-a \Delta_{g} \varphi + S_{g} \varphi = \eta_{1} \varphi.
\]

It is well-known that the first eigenfunction \( \varphi > 0 \) on closed manifolds \( M \), or on \( M' \) on compact manifold with smooth boundary and the same boundary condition in Theorem 2.4(ii).

The key step that converts our local solutions on Riemannian domain to manifolds is the application of monotone iteration method, in terms of obtaining solutions of Yamabe equation/perturbed Yamabe equation provided the existences of subsolution and supersolution. Here we cite a result due to Kazdan and Warner, see [15] and [16].

**Theorem 2.5.** [16] Lemma 2.6] Let \((M, g)\) be a closed manifold with \( \dim M \geq 3 \). Let \( h, H \in C^{\infty}(M) \) for some \( p > n = \dim M \). Let \( m > 1 \) be a constant. If there exists functions \( u_{-}, u_{+} \in C_{0}(M) \cap H^{1}(M, g) \) such that
\[
-a \Delta_{g} u_{-} + h u_{-} \leq H u_{-}^{m} \text{ in } (M, g);
\]
\[
-a \Delta_{g} u_{+} + h u_{+} \geq H u_{+}^{m} \text{ in } (M, g),
\]
hold weakly, with \( 0 \leq u_{-} \leq u_{+} \) and \( u_{-} \neq 0 \), then there is a \( u \in W^{2,p}(M, g) \) satisfying
\[
-a \Delta_{g} u + h u = H u^{m} \text{ in } (M, g).
\]

In particular, \( u \in C^{\infty}(M) \).

**Remark 2.2.** (i) In the proof of Kazdan and Warner [15], [16], they constructed the monotone iteration steps as
\[
-a \Delta_{g} u_{j+1} + k u_{j+1} = -f(x, u_{j}) + k u_{j}, j \in \mathbb{Z}_{\geq 0}.
\]
where
\[
f(x, u) := h u - H u^{m}, k = \sup_{x \in M, u_{-}(x) \leq u(x) \leq u_{+}(x)} \frac{\partial f(x, u)}{\partial u}, k > 0, u_{0} := u_{+}.
\]
The requirement of \( u_-, u_+ \in W^{2,p}(M, g) \) for \( p \) large enough implies that \( u_-, u_+ \in C_0(M) \), due to Sobolev embedding in Theorem 2.2(iii). Starting with \( u_0 = u_+ \), the function \( f(x, u_0) \in L^p(M, g) \), hence \( u_1 \in W^{2,p}(M, g) \) by elliptic regularity. Inductively with \( f(x, u_j) \in L^p(M, g) \), elliptic regularity in Theorem 2.1(iii) shows that \( u_{j+1} \in W^{2,p}(M, g) \), hence \( u_{j+1} \in C_0(M) \) and thus \( f(x, u_{j+1}) \in L^p(M, g) \) by the same Sobolev embedding argument. It follows that Theorem 2.3 still holds if we start with \( u_-, u_+ \in C_0(M) \). In addition, \( u_-, u_+ \in H^1(M, g) \) are required, since then the sub-solution and super-solution at least hold in the weak sense and a maximum principle in weak sense applies. We see this by claiming that \((-a\Delta_g + k)(u - v) \geq 0\) with \( u, v \in H^1(M) \cap C_0(M) \) implies \( u \geq v \). We choose \( w = \max\{0, v - u\} \), then \( w \in H^1(M) \) and \( w \geq 0 \). Observe also that \((-a\Delta_g + k)w \leq 0\). Thus

\[
0 \geq \int_M w(-a\Delta_g w + kw) d\omega = \int_M (a|\nabla_g w|^2 + kw^2) d\omega \geq 0 \Rightarrow w = 0.
\]

Thus we must have \( u \geq v \).

(ii) The relation \( u_- \leq \ldots \leq u_j \leq \ldots \leq u_+ \) is proved by taking subtraction of two adjacent iterations above, applying mean value theorem and maximum principle in \( H^1\)-spaces. It then follows from 16 that \( u \in W^{2,p}(M, g) \) solves (20). Therefore \( u \in C^{1,\alpha}(M) \) by Sobolev embedding. When \( h, H \in C_0(M) \), the requirement of \( u_- > 0 \) can be replaced by \( u_- \geq 0 \) and \( u_- \not\equiv 0 \).

If \( u_- \) satisfies the restrictions above, it follows that \( u \) has the same property. Choosing \( M = \max_{x \in M}\{h - Hu^{m-1}, 0\} \), (20) implies that

\[-a\Delta_g u + Mu \geq -a\Delta_g u + (h - Hu^{m-1}) u = 0.\]

By strong maximum principle in Theorem 2.3 we conclude that if \( u(x_0) = 0 \) for some \( x_0 \in M \) then \( u \equiv 0 \), contradicting to \( u \geq u_- \) and \( u_- \not\equiv 0 \). Hence \( u > 0 \) on \( M \).

(iii) In later sections, Theorem 2.5 are used by taking \( h = S_g \) or \( h = S_g + \beta \) with some \( \beta < 0 \), \( H = \lambda \) or \( H = \lambda - \kappa \) with some \( \kappa > 0 \) and \( m = p - 1 = \frac{n+2}{n-2} \) with \( u_-, u_+ \in C_0(M) \cap H^1(M, g) \).

**Theorem 2.6.** [19] Thm. 7 Let \((M, g)\) be a compact manifold with smooth boundary, let \( r_{\text{inj}} \) be the injectivity radius of \( M \), and let \( h_g \) be the minimum of the mean curvature of \( \partial M \). Choose \( K \geq 0 \) such that \( \text{Ric}_g \geq -(n-1)K \). If \( \lambda_1 \) is the first nonzero eigenvalue

\[-\Delta_g u = \lambda_1 u \text{ in } M', u \equiv 0 \text{ on } \partial M'.\]

Then

\[
\lambda_1 \geq \frac{1}{\gamma} \left( \frac{1}{4(n-1)r_{\text{inj}}^2} (\log \gamma)^2 - (n-1)K \right),
\]

where

\[
\gamma = \max \left\{ \exp\left(1 + (1 - 4(n-1)^2r_{\text{inj}}^2K)^{\frac{1}{2}}\right), \exp\left[-2(n-1)h_g r_{\text{inj}}\right] \right\}.
\]

**Remark 2.3.** We apply Theorem 2.6 above for Riemannian domain \((\Omega, g)\), for which \((\Omega, g)\) is considered to be a compact manifold with boundary. In Riemannian normal coordinates centered at \( P \in \Omega, g \) agrees with the Euclidean metric up to terms of order \( O(r^2) \). Thus if \( \Omega \) is a \( g \)-geodesic ball of radius \( r \), the mean curvature of \( \partial \Omega \) is close \((n-1)/r\), the mean curvature of a Euclidean \( r \)-ball in \( \mathbb{R}^n \). In [22], as \( r \to 0 \), \( K \) can be taken to be unchanged (since \( g \) is independent of \( r \)), \( r_{\text{inj}} \to 0 \), and \( h_{r_{\text{inj}}} \to n-1 \). Thus \( \gamma \to e^{\frac{1}{2}} \), the right hand side of (21) goes to infinity as \( r \to 0 \), and \( \lambda_1 \to \infty \). If \( \Omega \) is a general, small enough Riemannian domain with a small enough injectivity radius, then \( \Omega \) sits inside a \( g \)-geodesic ball \( \Omega'' \) of small radius. By the Rayleigh quotient characterization of \( \lambda_1 \), we have \( \lambda_{1,\Omega''} \leq \lambda_{1,\Omega} \). Thus for all Riemannian domains \((\Omega, g), \lambda_1^{-1} \to 0 \) as the radius of \( \Omega \) goes to zero.
3. The Local Analysis of Yamabe Equation

In this section, we consider the solvability of the Yamabe equation
\[ -a\Delta_g u + (S_g + \beta) u = \lambda u^{p-1} \text{ in } (\Omega, g); \]
\[ u \equiv c \text{ on } \partial \Omega \]  \( (23) \)
with some constant \( c > 0 \) and \( \beta = 0 \) in Proposition 3.1 and 3.2, due to an iterative scheme from [23]. We then consider the case \( c = 0 \) and \( \beta < 0 \) in Proposition 3.3, due to a local analysis in calculus of variations. The small enough Riemannian domain \((\Omega, g)\) is chosen to be a subset of the closed manifold \((M, g)\). Note that all results are dimensional specific, i.e. different dimensions of manifolds may lead to different choices of \( \lambda \).

In this section, we introduce local analysis of Yamabe equation/perturbed Yamabe equation for (i) \( S_g < 0 \) everywhere and \( \lambda < 0 \) in Proposition 3.1; (ii) \( S_g > 0 \) somewhere and \( \lambda < 0 \) in Proposition 3.2; (iii) \( S_g < 0 \) somewhere and \( \lambda > 0 \) in Proposition 3.3. In the proofs of Proposition 3.1 and 3.2 we refer to Theorem 2.3, Theorem 2.8 and Theorem 3.1 in [23]. We then consider the case \( c > 0 \) in Proposition 3.3, due to a local analysis in [23]. The first result is for the case \( S_g < 0 \) everywhere and \( \lambda < 0 \). Note that \( S_g < 0 \) implies \( \int_M S_g < 0 \), hence \( \eta_1 \) in (18) is negative. Therefore by Theorem 2.4 we search for the solution of Yamabe equation with \( \lambda < 0 \).

**Proposition 3.1.** [23] Let \((\Omega, g)\) be Riemannian domain in \( \mathbb{R}^n \), \( n \geq 3 \), with \( C^\infty \) boundary, and with \( \text{Vol}_g(\Omega) \) and the Euclidean diameter of \( \Omega \) sufficiently small. Assume \( S_g < 0 \) on \( \Omega \). Then for some \( \lambda < 0 \), [23] has a real, positive, smooth solution \( u \in C^\infty(\Omega) \) such that \( \inf_{x \in \Omega} u = c > 0 \).

**Proof.** As in [23], this result is proved by applying the following iterative scheme
\[ au_0 - a\Delta_g u_0 = f_0 \text{ in } (\Omega, g), u_0 \equiv c \text{ on } \partial \Omega; \]
\[ au_k - a\Delta_g u_k = au_{k-1} - S_g u_{k-1} + \lambda u_{k-1}^{p-1} \text{ in } (\Omega, g), u_k \equiv c \text{ on } \partial \Omega, k \in \mathbb{N}. \]
\( (24) \)
Setting \( \tilde{u}_k = u_k - c, k \in \mathbb{N} \), the above iterations are equivalent to
\[ a\tilde{u}_0 - a\Delta_g \tilde{u}_0 = f_0 - ac \text{ in } (\Omega, g), \tilde{u}_0 \equiv 0 \text{ on } \partial \Omega; \]
\[ a\tilde{u}_k - a\Delta_g \tilde{u}_k = a\tilde{u}_{k-1} - S_g \tilde{u}_{k-1} + \lambda \tilde{u}_{k-1}^{p-1} - ac \text{ in } (\Omega, g), \tilde{u}_k \equiv 0 \text{ on } \partial \Omega, k \in \mathbb{N}. \]
\( (25) \)
The solvability of each linear elliptic PDE in (24) and (25) is achieved by Lax-Milgram ([17, Ch. 6]). Thus \( u_k, \tilde{u}_k \in H^1(\Omega, g) \cap H^2(\Omega, g), \forall k \), due to elliptic regularity in Theorem 2.1(i). With appropriate choice of \( f_0 > 0, f_0 \in C^\infty(M) \), we conclude that \( u_0, \tilde{u}_0 \in C^\infty(\Omega) \). By maximum principle in Theorem 2.3
\[ \|u_0\|_{H^2(\Omega, g)} \leq 1, u_0 > 0 \text{ in } \bar{\Omega}, |u_0| \leq C_{M_n}, \forall x \in \bar{\Omega}. \]
\( (26) \)
The upper bound for \( |u_0| \) is obtained by appropriate choice of \( f_0 \) and repeated use of \( W^{s,p} \)-type elliptic regularity in Theorem 2.1(ii) and Sobolev embedding in Theorem 2.2(i). We refer [23, Thm. 3.1] for details. Choosing \( \lambda \) here such that
\[ |\lambda| C_1^{p-2} \leq \inf(-S_g) \Rightarrow |\lambda| \sup u_0^{p-2} \leq \inf(-S_g) \Rightarrow -S_g u_0 + \lambda u_0^{p-1} \geq 0. \]
\( (27) \)
With [27] and the same argument in Theorem 2.3 and Theorem 3.1 of [23], we conclude that
\[ u_1, \tilde{u}_1 \in C^\infty(\Omega), \|u_1\|_{H^2(\Omega, g)} \leq 1, u_1 > 0 \text{ in } \bar{\Omega}, |u_1| \leq C_{M_n}, \forall x \in \bar{\Omega}. \]
\( (28) \)
Inductively, we assume
\[ u_{k-1}, \tilde{u}_{k-1} \in C^\infty(\Omega), \|u_{k-1}\|_{H^2(\Omega, g)} \leq 1, u_{k-1} > 0 \text{ in } \bar{\Omega}, |u_{k-1}| \leq C_{M_n}, \forall x \in \bar{\Omega}, k \in \mathbb{N}. \]
\( (29) \)
By choosing \( \lambda \) the same as in (27),
\[ |\lambda| C_1^{p-2} \leq \inf(-S_g) \Rightarrow |\lambda| \sup u_{k-1}^{p-2} \leq \inf(-S_g) \Rightarrow -S_g u_{k-1} + \lambda u_{k-1}^{p-1} \geq 0. \]
It follows that
\[ u_k, \tilde{u}_k \in C^\infty(\Omega), \|u_k\|_{H^2(\Omega, g)} \leq 1, u_k > 0 \text{ in } \bar{\Omega}, |u_k| \leq C_{M_k}, \forall x \in \bar{\Omega}. \] (30)

Note that \( C_{M_n} \) is uniform in \( k \). In addition, (24) implies that
\[ au_k - a\Delta_g u_k \geq au_{k-1} \]
based on the choice of \( \lambda \) in (27). If the infimum of \( u_k \) is attained on \( \partial \Omega \), then \( \inf_{x \in \Omega} u_k = c \). If the infimum of \( u_k \) is attained at some interior point \( x_0 \), then we have \(-a\Delta_g u_k(x_0) \leq 0\). It follows that
\[ a \inf_{x \in \Omega} u_k = au_k(x_0) \geq a\Delta_g u_k(x_0) \geq au_{k-1}(x_0) \geq a \inf_{x \in \Omega} u_{k-1}. \]

If \( \inf_{x \in \Omega} u_0 = c \), which can be done by setting \( \inf_{x \in \Omega} f_0 = c \), then \( \inf_{x \in \Omega} u_k = c, \forall k \). Hence
\[ C_{M_n} \geq c. \]

Therefore the sequence \( \{u_k\} \) is uniformly bounded in \( H^2 \)-norm and is convergent to some \( u \in H^2(\Omega, g) \) which solves (29). By Sobolev embedding and elliptic regularity again, it follows that \( u \in C^\infty(\Omega) \) and \( u \geq 0 \). Due to strong maximum principle in Theorem 2.3(iii) we conclude that \( u > 0 \) since \( u \equiv c > 0 \) on \( \partial \Omega \). Observe that
\[ |\lambda|u_k^{p-2} \leq -S_g, \forall x \in \Omega \Rightarrow -S_g u + \lambda u^{p-1} \geq 0 \Rightarrow -a\Delta_g u \geq 0. \]

By Theorem 2.3(ii) we conclude that \( u \) achieves its minimum at \( \partial \Omega \). Note that in this proof, we need to choose \( \lambda \) small enough twice so that (30) holds, uniformly for all \( k \).

Next result corresponds to the case \( S_g > 0 \) somewhere in \( M \) and \( \lambda < 0 \), the proof is essentially the same as in [23]. In the next proposition, we assume, without loss of generality, that \( S_g \leq \frac{a}{2} \). This can be done by scaling the metric \( g \). We start with this metric.

**Proposition 3.2.** [23] Let \( (\Omega, g) \) be Riemannian domain in \( \mathbb{R}^n, n \geq 3 \), with \( C^\infty \) boundary, and with Vol\( g(\Omega) \) and the Euclidean diameter of \( \Omega \) sufficiently small. Assume \( S_g(x) \leq \frac{a}{2}, \forall x \in M \), and \( S_g > 0 \) within the small enough closed domain \( \Omega \). Then for some \( \lambda < 0 \), (23) has a real, positive, smooth solution \( u \in C^\infty(\Omega) \) such that \( \sup_{x \in \Omega} u = c > 0 \).

**Proof.** We apply the same iterative scheme as in (24) and (25) as above. In this case we choose \( f_0 = 0 \) in (24), thus we have
\[ au_0 - a\Delta_g u_0 = 0 \text{ in } (\Omega, g), u \equiv c > 0 \text{ on } \partial \Omega. \]

It follows immediately from the PDE above that \(-\Delta_g u \leq 0\), therefore \( u_0 > 0 \) by weak maximum principle in Theorem 2.3(i). Then by strong maximum principle in Theorem 2.3(ii), we conclude that \( \sup_{x \in \Omega} u_0 = c \). For the first iteration \( k = 1 \) in (24), we choose \( \lambda \) so that
\[ |\lambda|u_1^{p-2} \leq a + \inf_{\Omega}(-S_g) \Rightarrow |\lambda|\sup_{\Omega} u_0^{p-2} \leq a - S_g \Rightarrow au_0 - S_g u_0 + \lambda u_0^{p-1} \geq 0. \] (31)

By the same argument as in Theorem 2.3, Theorem 2.8, Theorem 3.1 of [23] and Proposition 3.1 above, we conclude that (26) holds with \( 0 \leq u_0 \leq c \). Inductively we assume that (28) holds with \( 0 \leq u_{k-1} \leq c \). Observe that since \( S_g > 0, \lambda < 0 \) and \( u_{k-1} \geq 0 \), it follows from the \( k \)th iteration of (24) that
\[ au_k - a\Delta_g u_k \leq au_{k-1} - S_g u_{k-1} + \lambda u_{k-1}^{p-1} \leq au_{k-1}. \]

If the supremum is attained at the boundary, then we have \( \sup_{x \in \Omega} u_k = c \). If the supremum is attained in some interior point \( x_0 \), at this point we have \(-a\Delta_g u_k(x_0) \geq 0\) hence
\[ a \sup_{x \in \Omega} u_k = au_k(x_0) \leq au_k(x_0) - a\Delta_g u_k(x_0) \leq au_{k-1}(x_0) \leq a \sup_{x \in \Omega} u_{k-1} = ac. \]

Choosing the same \( \lambda \) as in (31), it follows that
\[ au_{k-1} - S_g u_{k-1} + \lambda u_{k-1}^{p-1} \geq 0. \]
Therefore (30) holds with $0 \leq u_k \leq c$. Due to the same argument as above, we conclude that the smooth sequence $\{u_k\} \subset C^\infty(\Omega)$ converges to some $u \in H^2(\Omega, g)$. Since $u \equiv c$ on $\partial \Omega$, it is not a trivial solution. By bootstrapping strategy with respect to Sobolev inequality (11), (13) and elliptic regularity (8), we conclude that $u$ is the minimizer of the functional

$$J(u) = \int_\Omega \left( \frac{1}{2} \sum_{i,j} a_{ij}(x) \partial_i u \partial_j u - \frac{b(x)}{p} u_+^p - F(x, u) \right) dx,$$

where $u_+ = \max\{u, 0\}$ and $F(x, u) = \int_0^u f(x, t) dt$. Set

$$A(D) = \operatorname{essinf}_{x \in D} \frac{\det(a_{ij}(x))}{|b(x)|^{n-2}}, \forall D \subset \Omega;$$

$$T = \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |Du|^2 dx}{\left( \int_\Omega |u|^p dx \right)^{\frac{1}{p}}};$$

$$K = \inf_{u \neq 0} \sup_{t > 0} J(t u), K_0 = \frac{1}{n} T^\frac{2}{n} (A(\Omega))^{\frac{1}{2}};$$

$$T_L = \inf_{u \neq 0} \frac{\int_\Omega a_{ij} \partial_i u \partial_j u dx}{\left( \int_\Omega b(x)|u|^{p+1} dx \right)^{\frac{1}{2}}}.$$

**Theorem 3.1.** ([26] Thm. 1.1, Thm. 1.4) Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n, n \geq 3$. Let $L u = - \sum_{i,j} \partial_i (a_{ij}(x) \partial_j u)$ be a second order elliptic operator in divergence form. Let $\operatorname{Vol}_g(\Omega)$ and the diameter of $\Omega$ be sufficiently small. Let $b(x) \neq 0$ be a nonnegative bounded measurable function. Let $f(x, u)$ be measurable in $x$ and continuous in $u$. Assume

1. There exist $c_1, c_2 > 0$ such that $c_1 |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq c_2 |\xi|^2, \forall x \in \Omega, \xi \in \mathbb{R}^n$;
2. $\lim_{u \to +\infty} \frac{f(x, u)}{u} = 0$ uniformly for $x \in \Omega$;
3. $\lim_{u \to 0} \frac{f(x, u)}{u} < \lambda_1$ uniformly for $x \in \Omega$, where $\lambda_1$ is the first eigenvalue of $L$;
(4) There exists \( \theta \in (0, \frac{1}{2}) \), \( M \geq 0, \sigma > 0 \), such that \( F(x, u) = \int_0^u f(x, t)dt \leq \theta u^p_f(x, u) \) for any \( u \geq M, x \in \Omega(\sigma) = \{ x \in \Omega, 0 \leq b(x) \leq \sigma \} \).

Furthermore, we assume that \( f(x, u) \geq 0, f(x, u) = 0 \) for \( u \leq 0 \). We also assume that \( a_{ij}(x) \in C_0(\Omega) \). If

\[
K < K_0 \quad \text{or} \quad K < \frac{1}{n} T_L^n,
\]

then the Dirichlet problem (**) possesses a solution \( u \) which satisfies \( J(u) \leq K \).

We now apply Theorem 3.1 to show the following result. Unfortunately we cannot show

\[-a\Delta_g u + S_g u = \lambda u^{p-1} \quad \text{in} \quad \Omega, u = 0 \quad \text{on} \quad \partial \Omega \]

directly. Instead we perturb \( S_g \) by \( S_g + \beta \) for some \( \beta < 0 \), and show the existence of the positive solution of

\[-a\Delta_g u + (S_g + \beta) u = \lambda u^{p-1} \quad \text{in} \quad \Omega, u = 0 \quad \text{on} \quad \partial \Omega. \]

**Proposition 3.3.** Let \( (\Omega, g) \) be Riemannian domain in \( \mathbb{R}^n \), \( n \geq 3 \), with \( C^\infty \) boundary, and with \( \text{Vol}_g(\Omega) \) and the Euclidean diameter of \( \Omega \) sufficiently small. Let \( \beta < 0 \) be any constant. Assume \( S_g < 0 \) within the small enough closed domain \( \bar{\Omega} \). Then for any \( \lambda > 0 \), the following Dirichlet problem

\[-a\Delta_g u + (S_g + \beta) u = \lambda u^{p-1} \quad \text{in} \quad \Omega, u = 0 \quad \text{on} \quad \partial \Omega. \]

has a real, positive, smooth solution \( u \in C^\infty(\Omega) \cap H^1_0(\Omega, g) \) in a very small domain \( \Omega \) that vanishes at \( \partial \Omega \).

**Proof.** We show this by showing that (**7**) can be converted to a special case in Theorem 3.1 and thus a positive solution is admitted. Without loss of generality, we may assume \( 0 \in \Omega \). Under \( W^{s, p}(\Omega, g) \)-norms, (**7**) is the Euler-Lagrange equation of the functional

\[
J^*(u) = \int_{\Omega} \left( \frac{1}{2} a \sqrt{\det(g)} g^{ij} \partial_i u \partial_j u - \frac{\sqrt{\det(g)}}{p} \lambda u^{p} - \int_0^u \sqrt{\det(g)(x)} (-S_g(x) - \beta)tdt \right) dx. \]

It is straightforward to check that

\[
\frac{d}{dt} \bigg|_{t=0} J^*(u + tv) = \int_{\Omega} \left( \left( -a\partial_i(\sqrt{\det(g)} g^{ij} \partial_j u) - \sqrt{\det(g)} \lambda u^{p} - \int_0^u \sqrt{\det(g)(x)} (-S_g(x) - \beta)tdt \right) dx \right)
\]

\[
\quad = \int_{\Omega} \left( -a \frac{1}{\sqrt{\det(g)}} \left( \partial_i(\sqrt{\det(g)} g^{ij} \partial_j u) - \lambda u^{p} + (S_g + \beta) u \right) v \sqrt{\det(g)} \right) dx
\]

\[
\quad = (-a\Delta_g u - \lambda u^{p} + (S_g + \beta) u, v)_g.
\]

Hence \( u > 0 \) in \( \Omega \) minimizes \( J^*(u) \) if and only if (**7**) admits a positive solution. Note that \( J^*u \) is just a special expression of (**34**) with

\[
a_{ij}(x) = a \sqrt{\det(g)} g^{ij}(x), b(x) = \lambda \sqrt{\det(g)(x)}, f(x, u) = \sqrt{\det(g)(x)} (-S_g(x) - \beta) u.
\]

We check that hypotheses in Theorem 3.1 are satisfied.

First of all, \( b(x) \geq 0 \) on \( \Omega \) clearly. As mentioned in Remark 2.3, we can choose \( \Omega \) small enough so that it is a geodesic ball centered at some point \( p \in M \). At the point \( p \), \( g^{ij} = \delta^{ij} \) and it follows that \( g^{ij} \) is a small perturbation of \( \delta^{ij} \) within small enough \( \Omega \). Thus the first hypothesis in Theorem 3.1 holds. With the expressions if \( b \) and \( f \) in (**39**), we have

\[
\lim_{u \to +\infty} \frac{f(x, u)}{u^{p-1}} = \lim_{u \to +\infty} \frac{\sqrt{\det(g)(x)} (-S_g(x) + \beta)u^{2-p}}{u^{p-1}} = 0.
\]

Hence the second hypothesis is satisfied.
Hypothesis (3) is for the positivity of $J(u)$. In the general case, we choose $\varphi > 0$ to be the first eigenfunction of the Dirichlet problem $L\varphi = \lambda_1 \varphi$ in $\Omega$, $\varphi = 0$ on $\partial \Omega$, we observe that if $u > 0$ solves

$$\lambda_1 \int_{\Omega} u \varphi dx = \int_{\Omega} u \cdot L\varphi dx = \int_{\Omega} Lu \cdot \varphi dx = \int_{\Omega} (b(x)u^{p-1} + f(x, u)) \varphi d\text{Vol}_g$$

$$= \int_{\Omega} \left( b(x)u^{p-1} + \frac{f(x, u)}{u} \cdot u \right) \varphi dx \geq \int_{\Omega} \frac{f(x, u)}{u} \cdot u \varphi dx.$$ 

It follows that $\lim_{u \to 0} \frac{f(x, u)}{u} > \lambda_1$ then it is impossible to get a positive solution $u$. In our case for $J^*$, we choose $\varphi > 0$ and $\lambda_1 > 0$ be the first eigenfunction and first nonzero eigenvalue of $-\Delta_g$, respectively. It follows from (37) that

$$a \lambda_1 \int_{\Omega} \varphi d\text{Vol}_g = \int_{\Omega} u \cdot (-a\Delta_g \varphi) d\text{Vol}_g = \int_{\Omega} -a\Delta_g u \cdot \varphi dx = \int_{\Omega} (\lambda u^{p-1} + (S_g - \beta)u) \varphi d\text{Vol}_g$$

$$\geq \int_{\Omega} (S_g - \beta) \cdot u \varphi d\text{Vol}_g.$$ 

It follows that Hypothesis (3) holds if $\sup_{x \in \Omega} (-S_g) - \beta \leq a \lambda_1$. For later use, we further require

$$\frac{a}{n} - \left( \frac{n-2}{2n} + \frac{1}{2} \right) \left( \sup_{x \in \Omega} (-S_g) - \beta \right) \cdot \lambda_1^{-1} > 0. \quad (40)$$

by making $\lambda_1$ large enough. Due to the argument of first eigenvalue in Remark 2.3, this can be achieved when $\Omega$ is small enough. Fix this $\Omega$ from now on. It follows that for any $\beta' < 0$ with $\beta < \beta'$, the same $\Omega$ applies.

For the last hypothesis, Wang [26] mentioned that if $b(x) > 0$ a.e. in $\Omega$ then (4) in Theorem 3.1 is not needed. Here our $b(x) = \sqrt{\det(g)} \lambda > 0$ as always since in a small geodesic ball, $\sqrt{\det(g)} \approx 1 + O(\text{diam}(\Omega)^2)$ and $\lambda$ is chosen to be positive.

Lastly, we check (36) by checking $K < K_0$. This can be done by showing that there exists some $u > 0$ in $\Omega$, $u \equiv 0$ on $\partial \Omega$ such that $\sup_{t > 0} J^*(tu) < K_0$. Assume such an $u$ does exist, then we have

$$J^*(tu) = \int_{\Omega} \left( t^2 \frac{1}{2} a \sqrt{\det(g)} g^{ij} \partial_i u \partial_j u - t^p \frac{\sqrt{\det(g)} \lambda}{p} u^p - t^2 \frac{1}{2} \sqrt{\det(g)} (S_g(x) - \beta) u^2 \right) dx.$$ 

It follows that $J^*(tu) < 0$ when $t$ large enough. Hence $\sup_{t > 0} J^*(tu) \geq 0$ is achieved for some finite $t_0 \geq 0$. If $t_0 = 0$, then $\sup_{t > 0} J^*(tu) = 0$ and thus $K \leq 0$, again $K < K_0$ holds trivially. Assume now $t_0 > 0$. In this case, we have

$$\frac{d}{dt} \bigg|_{t=t_0} J^*(tu) = 0$$

$$\Rightarrow t_0 \int_{\Omega} a \sqrt{\det(g)} g^{ij} \partial_i u \partial_j u dx - t_0^{p-1} \int_{\Omega} \sqrt{\det(g)} \lambda u^p dx - t_0 \int_{\Omega} \sqrt{\det(g)} (S_g(x) - \beta) u^2 dx = 0. \quad (41)$$

Denote

$$V_1 = \int_{\Omega} a \sqrt{\det(g)} g^{ij} \partial_i u \partial_j u dx, V_2 = \int_{\Omega} \sqrt{\det(g)} (S_g(x) - \beta) u^2 dx, W = \left( \int_{\Omega} \sqrt{\det(g)} \lambda u^p dx \right)^{\frac{1}{p}}.$$ 

Since $t_0 > 0$, (41) implies

$$t_0^{p-2} = \frac{V_1 - V_2}{W^p} \Rightarrow t_0 = \frac{(V_1 - V_2)^{\frac{1}{p-2}}}{W^{\frac{p}{p-2}}} \quad (42).$$
By $(42)$, $J^*(t_0u)$ is of the form

$$J^*(t_0u) = \frac{(V_1 - V_2)^{\frac{p^2}{p^2-2}}}{W^{\frac{p^2}{p^2-2}}}V_1 - \frac{(V_1 - V_2)^{\frac{p^2}{p^2-2}}}{W^{\frac{p^2}{p^2-2}}} \cdot \frac{1}{W^p} \cdot \frac{(V_1 - V_2)^{\frac{p^2}{p^2-2}}}{W^{\frac{p^2}{p^2-2}}}V_2.$$  

Note that $p = \frac{2n}{n-2}$ hence we have

$$\frac{2}{p-2} = \frac{n-2}{2}, \quad \frac{p}{p-2} = \frac{n}{2}, \quad p - \frac{p^2}{p-2} = p \left(1 - \frac{p}{p-2}\right) = \frac{2p}{p-2} - \frac{n}{2} = \frac{p^2 - p}{p-2}.$$  

In terms of $n$, $J^*(t_0u)$ becomes

$$J^*(t_0u) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{(V_1 - V_2)^{\frac{n}{2}}}{W^n} = \frac{1}{n} \frac{(V_1 - V_2)^{\frac{n}{2}}}{W^n}$$

$$= \frac{1}{n} \left(\int_\Omega \alpha \sqrt{\det(g)} \partial_i u \partial_j u dx - \int_\Omega \sqrt{\det(g)} (-S_g - \beta) u^2 dx\right)^{\frac{1}{2}}.$$  

The inequality $K < K_0$ reduces to find some $u \in H^1(\Omega)$, $u > 0$ such that (43) is less than $K_0$. We check the expression of $K_0$ first. By $a_{ij}, b$ in (39), we have

$$A(\Omega) = \text{essinf}_\Omega \frac{\det(a_{ij}(x))}{|b(x)|^{n-2}} = \text{essinf}_\Omega \frac{\det(\alpha \sqrt{\det(g)} g^{ij})}{\sqrt{\det(g)}} a^n \frac{\det(g^{ij})}{\sqrt{\det(g)}}$$

$$= \text{essinf}_\Omega \alpha^n \lambda^2 a^n \sqrt{\det(g)}^2 \det(g^{ij}) = \lambda^{2-n} a^n.$$  

It follows that

$$K_0 = \frac{1}{n} T^\frac{n}{2} A(\Omega)^\frac{1}{2} = \frac{1}{n} \lambda^\frac{2-n}{2} a^\frac{n}{2} T^\frac{n}{2}.$$  

Compare expressions in (43) and (44), showing $K < K_0$ is equivalent to find some $u > 0$ in $\Omega$ such that

$$\int_\Omega \alpha \sqrt{\det(g)} \partial_i u \partial_j u dx - \int_\Omega \sqrt{\det(g)} (-S_g - \beta) u^2 dx < \lambda^\frac{2-n}{2} a T$$

$$\left(\int_\Omega \sqrt{\det(g)} \lambda u^p dx\right)^\frac{1}{2}$$

$\Leftrightarrow \int_\Omega \sqrt{\det(g)} g^{ij} \partial_i u \partial_j u dx - \frac{1}{n} \int_\Omega \sqrt{\det(g)} (-S_g - \beta) u^2 dx < T$.  

$\Leftrightarrow \left(\int_\Omega \sqrt{\det(g)} u^p dx\right)^\frac{1}{2}$

Claim: When both $\Omega$ and $\epsilon$ are small enough, the function $u_{\epsilon, \Omega}(x) = \frac{\varphi_{\Omega}(x)}{\epsilon + |x|^2}^{\frac{2}{2-n}}$ satisfies

$$\frac{\|\nabla u_{\epsilon, \Omega}\|^2_{L^2(\Omega, g)}}{\|u_{\epsilon, \Omega}\|^2_{L^p(\Omega, g)}} + \frac{1}{\alpha} \int_\Omega (S_g + \beta) u_{\epsilon, \Omega}^2 \sqrt{\det(g)} dx < T.$$  

This claim is proved in Appendix A.

Finally we conclude that

$$K < K_0.$$  

Note that this result is peculiar for our special PDE in (38) with small enough $\Omega$. The regularity of $u$ follows exactly the same as in [Sec, 1][5]. We would like to denote here that when $u \in \mathcal{L}^\infty(\Omega)$ then $u^{p-1} \in \mathcal{L}^p(\Omega, g)$ for all $p$, and by bootstrapping method, it follows that $u \in C^\infty(\Omega)$, provided that $\partial \Omega$ is smooth. In addition, $u \in C^{1,\alpha}(\Omega)$, the constant $\alpha$ is determined by the nonlinear term $u^{p-1}.$

$\square$
Remark 3.2. The proof of Theorem A.1 in Appendix A plays a central role for the validity of Proposition 3.3. This proof is essentially due to Brezis and Nirenberg [5], but they are not the same. The proof in Theorem A.1 is a sharp estimate in the sense that, for example, if any negative term with order $\varepsilon^{\frac{4}{n}} - \frac{2}{n}$ on numerator and any positive term of order $\varepsilon^{\frac{4}{n}} - \frac{2}{n}$ on denominator is dropped when $n \geq 5$, then the estimate fails. A detailed analysis with respect to Beta and Gamma functions are required to connect the best Sobolev constant, the area of unit sphere, and $K_2$ defined in Appendix A.

Remark 3.3. The quantity $Q_{\epsilon, \Omega}$ we estimate in Appendix A is a perturbation of Yamabe quotient, and hence is not conformally invariant. We show that $Q_{\epsilon, \Omega} < T$ for all $n \geq 3$, no matter the ambient manifold is locally conformally flat or not.

4. Yamabe Equation on Closed Manifolds

In this section, we apply the sub-solution and super-solution technique in Theorem 2.5 and Remark 2.2 to solve Yamabe equation for five cases:

(A). $S_g \leq 0$ everywhere;
(B). $S_g > 0$ somewhere and the conformal Laplacian $\Box_g$ admits a negative first eigenvalue $\eta_1$;
(C). $S_g < 0$ somewhere and the conformal Laplacian $\Box_g$ admits a positive first eigenvalue $\eta_1$;
(D). $S_g \geq 0$ everywhere;
(E). $\eta_1 = 0$.

When $\eta_1 = 0$ in case (E), we can solve Yamabe problem trivially with $\lambda = 0$, this is just an eigenvalue problem. Note that generically zero is not an eigenvalue of conformal Laplacian $\Box_g$, see [9].

For the rest four cases above, the sub-solutions and super-solutions will be constructed due to Propositions 3.1 through 3.3 respectively. By Theorem 2.4 (i), we know that (A) and (B) are equivalent to find a solution of (2) with $\lambda < 0$; (C) and (D) are equivalent to find a solution of (2) with $\lambda > 0$. In contrast to the classical calculus of variation approach–in which they find the minimizer of (3) first, and conclude that the desired $\lambda$ is automatically determined by some functional with the minimizer of (3)–this article determines an appropriate choice of $\lambda$ first, and then conclude that the Yamabe equation admits a real, smooth, positive solution associated with the choice of $\lambda$.

Recall the eigenvalue problem of conformal Laplacian on $(M, g)$.

$$- a \Delta_g \varphi + S_g \varphi = \eta_1 \varphi \text{ on }(M, g). \quad (47)$$

The first nonzero eigenvalue is characterized by

$$\eta_1 = \inf_{\varphi \neq 0} \frac{\int_M (a|\nabla_g \varphi|^2 + S_g \varphi^2) d\omega}{\int_M \varphi^2 d\omega} := \inf_{\varphi \neq 0} L(\varphi). \quad (48)$$

Observe that $L(\varphi) = L(|\varphi|)$, we may characterize $\eta_1$ by nonnegative functions, for details, see e.g. [16]. By bootstrapping and maximum principle, we conclude that the first eigenfunction $\varphi > 0$ is smooth.

The first result concerns the case globally when $S_g < 0$ everywhere. Note that taking $u = 1$ in either (3) or (15), we have $\lambda(M) < 0$, or $\eta_1 < 0$, respectively.

Theorem 4.1. Let $(M, g)$ be a closed manifold with $n = \dim M \geq 3$. Assume the scalar curvature $S_g$ to be negative everywhere on $M$. Then there exists some $\lambda < 0$ such that the Yamabe equation (2) has a real, positive, smooth solution.
Proof. Due to Theorem 2.5 and Remark 2.2, we construct a subsolution $u_- \in C_0(M) \cap H^1(M,g)$ and a super-solution $u_+ \in C_0(M) \cap H^1(M,g)$.

First of all, we choose a finite cover $\{U_\alpha, \phi_\alpha\}$ of $(M,g)$. Pick up a small enough Riemannian domain $(\Omega, g) \subset (\phi_\alpha(U_\alpha), g)$. We can definitely take $\Omega \subset \phi_\alpha(U_\alpha)$ to be a small enough closed ball. Take any constant $c > 0$.

We construct the sub-solution first. By Proposition 3.1, there exists some $\lambda \in (\eta_1,0)$ such that $u_1 \in C^\infty(\Omega)$ solves the Yamabe equation (23) locally with $\beta = 0$ and $u_1 \equiv c > 0$ on $\partial \Omega$. Fix the $\lambda$. Extend $u_1$ to $u_-$ by setting

$$u_-(x) = \begin{cases} u_1(x), & x \in \Omega; \\ c, & x \in M \setminus \Omega, \end{cases}$$

Clearly $u_- \in C_0$. Observe that $u_- - c$ is the extension of $u_1 - c$ by zero outside $\Omega$. Due to extension theorem, it follows that $u_1 - c \in H_1(\phi_\alpha(U_\alpha), g)$ since $u_1 - c \in H^1_0(\Omega, g)$. To see this, we can approximate by $u_1 - c$ by a sequence of smooth functions $\{v_k\}$ with compact support in $\Omega$, then extend $v_k$ by zero to the $\phi_\alpha(U_\alpha)$, then take the limit of $\{v_k\}$ with $H^1(\phi_\alpha(U_\alpha), g)$-norm. It follows that the limit $u_1 - c \in H^1(\phi_\alpha(U_\alpha), g)$. Since $M$ is compact, we can extend $u_1 - c$ by zero to the whole manifold. Thus we conclude that

$$u_- \in C_0(M) \cap H^1(M,g).$$

In addition, $u_- \geq 0$ and $u_- \neq 0$. We check that $u_-$ is a sub-solution of Yamabe equation. Within $\Omega$, $u_-$ gives an equality of (2), outside $\Omega$, we observe that

$$-a \Delta_g u_- + S_g u_- - \lambda u_-^{p-1} = -a \Delta_g c + S_g c - \lambda c^{p-1} = c(S_g - \lambda c^{p-2}).$$

As we mentioned in Proposition 3.1, $C_{M_n} \geq c$ and thus

$$|\lambda|^c^{p-2} \leq |\lambda|C_{M_n}^{p-2} \leq \inf(-S_g).$$

It follows that $S_g - \lambda c^{p-2} \leq 0, \forall x \in M$, hence the left side of (25) is nonpositive for $x \in M \setminus \Omega$. In conclusion, $u_-$ is a sub-solution of Yamabe equation in the weak sense.

The super-solution is constructed as follows. For the fixed $\lambda$, pick a constant $C$ such that

$$\inf_{x \in M} C \geq \max\left\{ \left( \frac{\inf_{x \in M} S_g}{\lambda} \right)^{\frac{1}{p-2}}, \sup_{x \in M} u_- \right\}.$$

Denote

$$u_+ := C.$$

Immediately, $u_+ \in C^\infty(M)$. Since $\lambda < 0$, we have

$$C \geq \left( \frac{\inf_{x \in M} S_g}{\lambda} \right)^{\frac{1}{p-2}} \Rightarrow S_g \geq \lambda C^{p-2} \Rightarrow S_g C \geq \lambda C^{p-1}$$

$$\Rightarrow -a \Delta_g C + S_g C \geq \lambda C^{p-1} \Rightarrow -a \Delta_g u_+ + S_g u_+ \geq \lambda u_+^{p-1}.$$

Thus $u_+$ is a super-solution of the Yamabe equation. Due to the choice of $C$, we conclude that $0 \leq u_- \leq u_+$ on $M$. It then follows from Theorem 2.5 that there exists some real, positive function $u \in C^\infty(M)$ that solves (2).

Remark 4.1. Note that $u_1$ above, which is constructed in Proposition 3.1, achieves its minimum $c$ on $\partial \Omega$, and hence $u_- = \max\{u_1, c\}$. It is easy to see that the maximum of two sub-solutions is again a sub-solution in the weak sense.

Next result deals with the case $S_g > 0$ somewhere and conformal Laplacian $\Box_g$ has a negative first eigenvalue. In the next theorem, we require $S_g(x) \in [-\frac{a}{1-t}, \frac{a}{1-t}], \forall x \in M$. This can be done by scaling the original metric $g$. 


**Theorem 4.2.** Let \((M, g)\) be a closed manifold with \(n = \dim M \geq 3\). Assume the scalar curvature
\[ S_g(x) \in [-\frac{4}{n-2}, \frac{4}{n-2}], x \in M, \] which is positive somewhere on \(M\). Let the first eigenvalue of conformal Laplacian \(\Delta_g\) satisfying \(\eta_1 < 0\). Then there exists some \(\lambda < 0\) such that the Yamabe equation \((2)\) has a real, positive, smooth solution.

**Proof.** Since \(\eta_1 < 0\), we solve the Yamabe problem for some \(\lambda < 0\). We construct both the sub-solution and super-solution again here.

We construct the super-solution first. Pick up some \(c > 0\) which will be determined later. Since \(S_g > 0\) somewhere, we take a small enough Riemannian domain \((\Omega, g)\) in which \(S_g > 0\) and \(\max_{x \in M} S_g\) is achieved in the interior of \(\Omega\). As above, we can take \(\bar{\Omega} \subset \phi_\alpha(U_\alpha)\) to be a small enough closed ball for some cover \(\{U_\alpha, \phi_\alpha\}\) of \((M, g)\). By Proposition \(3.2\) there exists some \(\lambda < 0\) when \(|\lambda|\) is small enough such that \(u_2 \in C^\infty(\Omega)\) solves \((24)\) locally in \((\Omega, g)\) with \(u \equiv c > 0\) on \(\partial\Omega\). Since \(S_g \geq -\frac{4}{n}\), we further choose \(\lambda\) such that
\[ \frac{a}{4} \leq |\lambda|c^{p-2} \leq \frac{a}{2} \implies -\frac{a}{4} \geq \lambda c^{p-2} \implies S_g \geq \lambda c^{p-2} \implies S_g c \geq \lambda c^{p-1}, \forall x \in M. \tag{50} \]

Note that the choice of \(\lambda\) in \((50)\) is compliant with the choice of \(\lambda\) in \((31)\) in Proposition \(3.2\) due to Remark \(3.3\). Note also that we need \(|\lambda|\) to be small enough so that the requirement in Theorem 2.3 of \(23\) is also satisfied. In addition, let’s require \(|\lambda| \leq |\eta_1|\), where \(\eta_1\) is the first eigenvalue of \(\Delta_g\) on \(M\) in this situation. This can be done by choose an appropriate \(c\). Setting
\[ u_+(x) = \begin{cases} u_2(x), & x \in \Omega; \\ c, & x \in M \setminus \Omega, \end{cases} \]

Clearly \(u_+ > 0\) on \(M\). By the same argument in Theorem 4.1, we conclude that \(u_+ \in C_0(M) \cap H^1(M, g)\) is a super-solution of Yamabe equation in the weak sense. To see this, we check that within \(\Omega\), \(u_+ = u_2\) and the equality of Yamabe equation holds. When outside \(\Omega\), we have
\[ -a \Delta_g u_+ + S_g u_+ - \lambda u_+^{p-1} = S_g c - \lambda c^{p-1} \geq 0, \]
due to \((50)\).

For the sub-solution. We apply the result of eigenvalue problem, which says that there exists some function \(\varphi > 0\) that solves
\[ -a \Delta_g \varphi + S_g \varphi = \eta_1 \varphi \text{ on } (M, g). \]

Observe that we have chosen \(\lambda\) such that \(|\lambda| \leq |\eta_1| \Leftrightarrow \eta_1 \leq \lambda \leq 0\). Scaling \(\varphi\) by \(\delta \varphi\) with \(\delta < 1\) small enough such that
\[ \sup_{x \in M} \delta \varphi \leq \min \{ \inf_{x \in M} u_+, 1 \}. \]

Denote
\[ u_- := \delta \varphi. \]

Observe that \(0 \leq u_- \leq 1\) implies that \(0 \leq u_+^{p-1} \leq u_-\) since \(p - 1 = \frac{n+2}{n-2} > 1\). Note that \(\lambda < 0\), we then have
\[ -a \Delta_g \varphi + S_g \varphi = \eta_1 \varphi \leq \lambda \varphi \implies -a \Delta_g \delta \varphi + S_g \delta \varphi \leq \lambda \delta \varphi \]
\[ \implies -a \Delta_g u_- + S_g u_- \leq \lambda u_- \leq \lambda u_-^{p-1}. \]

Hence \(u_- \geq 0\) is a sub-solution of the Yamabe equation and \(u_- \neq 0\). Obviously \(u_- \in C^\infty(M)\). In addition, \(0 \leq u_- \leq u_+\) on \(M\). It then follows from Theorem 2.5 that there exists some real, positive function \(u \in C^\infty(M)\) that solves \((2)\). \(\square\)

**Remark 4.2.** Note that \(u_2\) above, which is constructed in Proposition \(3.2\) achieves its maximum \(c\) on \(\partial \Omega\), and hence \(u_+ = \min \{ u_2, c \}\). It is easy to see that the minimum of two super-solutions is again a super-solution in the weak sense.
Remark 4.3. The case when \( S_g \leq 0 \) everywhere—slightly different from the hypothesis in Theorem 4.1—can be converted to the case when \( S_g > 0 \) somewhere and \( \eta_1 < 0 \). This can be done by applying Corollary 4.1 below. Note the sign of first eigenvalue of \( \Box_g \) is a conformal invariant.

Remark 4.4. Historically the cases when \( \eta_1 < 0 \) is easy to handle. On one hand, there are many other ways to construct sub-solutions and super-solutions, for instance, see [16] on closed manifolds and \([3]\) on non-compact manifolds. On the other hand, when \( S_g \leq 0 \) or \( \int_M S_g < 0 \), the functional \([3]\) satisfies \( \lambda(M) < 0 \) and hence \( \lambda(M) < \lambda(S^n) \) holds trivially. Thus by Yomabe, Trudinger and Aubin’s result, the Yomabe equation has a positive solution.

Next we show the case when \( S_g < 0 \) somewhere and the conformal Laplacian \( \Box_g \) has a positive first eigenvalue. This means we are looking for solutions of Yomabe equation with \( \lambda > 0 \). Let \( \beta < 0 \) be any constant. Denote

\[
\lambda_\beta = \inf_{u \neq 0, u \in H^1(M)} \left\{ \frac{\int_M a |\nabla_g u|^2 \, d\omega + \int_M (S_g + \beta) u^2 \, d\omega}{(\int_M u \, d\omega)^{\frac{2}{n}}} \right\}.
\]  

(51)

Recall that

\[
\lambda(M) = \inf_{u \neq 0, u \in H^1(M)} \left\{ \frac{\int_M a |\nabla_g u|^2 \, d\omega + \int_M S_g u^2 \, d\omega}{(\int_M u \, d\omega)^{\frac{2}{n}}} \right\} > 0.
\]  

(52)

It is crucial to assume \( S_g < 0 \) somewhere in order to construct a sub-solution within this region. We cannot work on the Yomabe equation directly. Instead, we solve

\[-a \Delta_g u + (S_g + \beta) u = \lambda u^{p-1}\]

for some specific choice of \( \lambda \) with any \( \beta < 0 \). Before showing this, we need the following lemma, which indicates that a small perturbation of conformal Laplacian will not change the sign of \( \lambda_\beta \) in \([51]\) provided that \( \lambda(M) > 0 \).

**Lemma 4.1.** Let \( \eta_1 > 0 \) be the first eigenvalue of \( \Box_g \) on close manifolds \((M, g)\) with \( \dim M \geq 3 \). Let \( \beta < 0 \) be a constant. Then \( \lambda_\beta \) defined in \([51]\) is also positive provided that \( |\beta| \) is small enough.

**Proof.** It is immediate that if \( \eta_1 > 0 \) then \( \lambda(M) > 0 \). It is due to the characterization of \( \eta_1 \)

\[
\eta_1 = \inf_{\varphi \neq 0} \frac{\int_M (a |\nabla_g \varphi|^2 + S_g \varphi^2) \, d\omega}{\int_M \varphi^2 \, d\omega}
\]

as well as the characterization of \( \lambda(M) \) in \([52]\).

Due to the characterization of \( \lambda_\beta \) in \([51]\), we conclude that for each \( \epsilon > 0 \), there exists a function \( u_0 \) such that

\[
\frac{\int_M (a |\nabla_g u_0|^2 + (S_g + \beta) u_0^2) \, d\omega}{(\int_M u_0^2 \, d\omega)^{\frac{2}{n}}} \leq \lambda_\beta + \epsilon.
\]

It follows that

\[
0 < \lambda(M) \leq \frac{\int_M (a |\nabla_g u_0|^2 + S_g u_0^2) \, d\omega}{(\int_M u_0^2 \, d\omega)^{\frac{2}{n}}} \leq \frac{\int_M (a |\nabla_g u_0|^2 + (S_g + \beta) u_0^2) \, d\omega}{(\int_M u_0^2 \, d\omega)^{\frac{2}{n}}} - \beta \frac{\int_M u_0^2 \, d\omega}{(\int_M u_0^2 \, d\omega)^{\frac{2}{n}}} \cdot \operatorname{Vol}_g(M)^{\frac{n-2}{n}} + \epsilon = \lambda_\beta - \beta \operatorname{Vol}_g(M)^{\frac{n-2}{n}} + \epsilon
\]

Since \( \epsilon \) can be arbitrarily small, it follows that

\[
\lambda_\beta \geq \lambda(M) + \beta \operatorname{Vol}_g(M)^{\frac{n-2}{n}}.
\]

Thus \( \lambda_\beta > 0 \) if \( |\beta| \) is small enough. □
We now show the existence of the solution of the perturbed Yamabe equation with a special choice of constant in front of the nonlinear term $wu^{p-1}$. One of the advantages from the local analysis is the flexibility of the choice of this constant, without knowing $\lambda(M) < \lambda(S^n)$. We would like to mention here that if we already know $\lambda(M) < \lambda(S^n)$, then we can apply the analysis developed by Aubin and Trudinger to show the existence result directly, see [18, §4]. With $\lambda(M) < \lambda(S^n)$, no local analysis developed here is required anymore.

**Theorem 4.3.** Let $(M, g)$ be a closed manifold with $n = \dim M \geq 3$. Let $S_g$ be the scalar curvature with respect to $g$ which is negative somewhere on $M$. Assume the first eigenvalue of conformal Laplacian $\square_g$ satisfying $\eta_1 > 0$. Let $\beta < 0$ be a small negative constant such that $\lambda_\beta > 0$ and $\eta_1 + \beta > 0$, and let $\kappa > 0$ small enough so that $\lambda_\beta - \kappa > 0$. Then the following PDE

$$-a\Delta_g u + (S_g + \lambda) u = (\lambda_\beta - \kappa) u^{p-1} \text{ in } M$$

has a real, positive, smooth solution with $\lambda_s$ given in (57).

**Proof.** We construct both the sub-solution and super-solution again here.

Let $\beta < 0$ be a small enough negative constant such that the first eigenvalue $\eta_1$ of the conformal Laplacian admits a positive solution $\phi \in C^\infty(M)$ satisfying

$$-a\Delta_g \phi + (S_g + \beta) \phi = \eta_1 \phi + \beta \phi \text{ in } M.$$  

Note that any scaling $\delta \phi$ also solves (54). For the given $\lambda_\beta$ and $\kappa$, we want

$$(\eta_1 + \beta) \inf_M (\delta \phi) > 2^{p-2} (\lambda_\beta - \kappa) \sup_M (\delta \phi^{p-1}) \Leftrightarrow \frac{(\eta_1 + \beta)}{2^{p-2} (\lambda_\beta - \kappa)} > \frac{\sup_M \phi^{p-1}}{\inf_M \phi}.$$  

For fixed $\eta_1, \lambda_\beta, \phi, \kappa, \beta$, this can be done by letting $\delta$ small enough. We denote $\phi = \delta \phi$. It follows that

$$-a\Delta_g \phi + (S_g + \beta) \phi = (\eta_1 + \beta) \phi \text{ in } M;$$

$$(\eta_1 + \beta) \inf_M \phi > 2^{p-2} (\lambda_\beta - \kappa) \sup_M \phi^{p-1} \geq 2^{p-2} (\lambda_\beta - \kappa) \phi^{p-1} > (\lambda_\beta - \kappa) \phi^{p-1} \text{ in } M.$$  

Set

$$\beta' = (\eta_1 + \beta) \sup_M \phi - 2^{p-2} (\lambda_\beta - \kappa) \inf_M \phi^{p-1} > (\eta_1 + \beta) \phi - 2^{p-2} (\lambda_\beta - \kappa) \phi^{p-1} \text{ pointwisely.}$$

Thus we have

$$-a\Delta_g \phi + (S_g + \beta) \phi = (\eta_1 + \beta) \phi > 2^{p-2} (\lambda_\beta - \kappa) \phi^{p-1} > (\lambda_\beta - \kappa) \phi^{p-1} \text{ in } M \text{ pointwisely.}$$

For sub-solution, we apply Proposition [33] within the small enough region $(\Omega, g)$ in which $S_g < 0$. Since $S_g$ is smooth on $M$ hence it’s bounded below. Thus based on hypothesis (3) in Theorem 3.1 we choose $\Omega$ small enough in which the first eigenvalue $\lambda_1$ of $-\Delta_g u$ is larger than $a (\sup_{x \in M} (-S_g) + \beta)$.

By the result of (33), we have $u_3 \in C_0(\bar{\Omega}) \cap H^1_0(\Omega, g)$, $u_3 > 0$ in $\Omega$ solves (37) for $\lambda_\beta - \kappa > 0$, i.e.

$$-a\Delta_g u_3 + (S_g + \beta) u_3 = (\lambda_\beta - \kappa) u_3^{p-1} \text{ in } \Omega, u_3 = 0 \text{ on } \partial \Omega;$$

$$\Leftrightarrow -a\Delta_g \left(\frac{u_3}{2}\right) + (S_g + \beta) \frac{u_3}{2} = 2^{p-2} (\lambda_\beta - \kappa) \left(\frac{u_3}{2}\right)^{p-1} \text{ in } \Omega, u_3 = 0 \text{ on } \partial \Omega.$$  

Define

$$u_-(x) = \begin{cases} 
    u_3(x), & x \in \Omega \\
    0, & x \in M \setminus \Omega. 
\end{cases}$$
By the same argument in Theorem 4.1, \( u_- \in C_0(M) \cap H^1(M, g) \). We show that \( u_- \) is a sub-solution of (53). In local coordinates, the outward normal derivative \( \nu = -\frac{\nabla u_3}{|\nabla u_3|} \) along \( \partial \Omega \), thus locally taking some \( \Omega' \supset \Omega \) and any test function \( v \geq 0 \) we have

\[
\int_{\Omega'} \left( a \nabla_g u_- \cdot \nabla_g v + (S_g + \beta) u_- v - \lambda u_-^p v \right) \, dVol_g
\]

\[
= \int_{\Omega} \left( -a \Delta_g u_3 + (S_g + \beta) u_3 - \lambda u_3^p \right) v \, dVol_g + \int_{\partial \Omega} \frac{\partial u_3}{\partial \nu} v \, dS
\]

\[
= \int_{\partial \Omega} (\nabla u_3 \cdot \nu) \, dS = \int_{\partial \Omega} \nabla u_3 \cdot \left( -\frac{\nabla u_3}{|\nabla u_3|} \right) v \, dS = \int_{\partial \Omega} -|\nabla u_3| \, dS \leq 0.
\]

This argument holds on \( M \) also since \( u_- \) is trivial outside \( \Omega \). Thus we conclude

\[-a \Delta_g u_- + (S_g + \beta) u_- - (\lambda_3 - \kappa) u_-^p \leq 0 \text{ weakly in } M.\]

We now construct the super-solution. Pick up \( \gamma \ll 1 \) such that

\[
0 < 20 (\lambda_3 - \kappa) \gamma + 2\gamma \cdot \left( \sup_M |S_g| + |\beta| \right) < \beta' \cdot 31 (\lambda_3 - \kappa) (\phi + \gamma)^{p-2} \gamma < \frac{\beta'}{2},
\]

(59)

This is dimensional specific. Set

\[
V = \{ x \in \Omega : u_3(x) > \phi(x) \}, \quad V' = \{ x \in \Omega : u_3(x) < \phi(x) \}, \quad D_0 = \{ x \in \Omega : u_3(x) = \phi(x) \},
\]

\[
D_0' = \{ x \in \Omega : \text{dist}(x, D_0) < \delta \}, \quad D_0'' = \left\{ x \in \Omega : \text{dist}(x, D_0) > \frac{\delta}{2} \right\},
\]

\[
\partial D_0' = \{ x \in \Omega : \text{dist}(x, D_0) = \delta \}, \quad \partial D_0'' = \left\{ x \in \Omega : \text{dist}(x, D_0) = \frac{\delta}{2} \right\}.
\]

The constant \( \delta > 0 \) is chosen so that

\[
\sup_{x \in D_0'} |u_3(x) - \phi(x)| < \gamma.
\]

If \( \phi \geq u_3 \) pointwisely, then \( \phi \) is a super-solution. If not, a good candidate of super-solution will be \( \max\{u_3, \phi\} \) in \( \Omega \) and \( \phi \) outside \( \Omega \), this is an \( H^1 \cap C_0 \)-function. Let \( \nu \) be the outward normal derivative of \( \partial V \) along \( D_0 \). If \( \frac{\partial u_3}{\partial \nu} = \frac{\partial \phi}{\partial \nu} \) on \( D_0 \) then the super-solution has been constructed. However, this is in general not the case. If not, then \( \frac{\partial u_3 - \partial \phi}{\partial \nu} \neq 0 \), which follows that 0 is a regular point of the function \( u_3 - \phi \) and hence \( D_0 \) is a smooth submanifold of \( \Omega \). Define

\[
\Omega_1 = V \cap D_0'', \quad \Omega_2 = V' \cap D_0'', \quad \Omega_3 = D_0'.
\]

(60)

Clearly \( \Omega_i, i = 1, 2, 3 \) are open sets, \( \bigcup_i \Omega_i = \Omega \) and \( \Omega_1 \cap \Omega_2 = \emptyset \). Note that \( D_0, D_0' \) will never intersect \( \partial \Omega \). Without loss of generality, we may assume that all \( \Omega_i, i = 1, 2, 3 \) are connected, since if not, then we do local analysis in all components where \( u_3 - \phi \) changes sign and combine results together. For any \( 0 < \epsilon < \frac{1}{4} \), set

\[
D_1 = \{ x \in V : \text{dist}(x, \partial D_0') < \epsilon \}, \quad D_1' = \{ x \in V' : \text{dist}(x, \partial D_0') < \epsilon \},
\]

\[
D_2 = \{ x \in V : \text{dist}(x, \partial D_0'') < \epsilon \}, \quad D_2' = \{ x \in V' : \text{dist}(x, \partial D_0'') < \epsilon \},
\]

\[
E_1 = \partial D_1 \cap \Omega_1 \cap \Omega_3, \quad E_2 = \partial D_2 \cap \Omega_1, \quad E_1' = \partial D_1' \cap \Omega_2 \cap \Omega_3, \quad E_2' = \partial D_2' \cap \Omega_3;
\]

\[
F_1 = (\Omega_1 \cap \Omega_3) \setminus (D_1 \cup D_2), \quad F_2 = (\Omega_2 \cap \Omega_3) \setminus (D_1' \cup D_2').
\]

In conclusion, \( D_1, D_2, E_1, E_2, F_1 \) are subsets of \( V \), in which \( u_3 > \phi \). \( D_1', D_2', E_1', E_2', F_2 \) are subsets of \( V' \), in which \( u_3 < \phi \). Note that \( \partial F_1 = E_1 \cup E_2, \partial F_2 = E_1' \cup E_2' \). We now construct appropriate
partition of unity subordinate to \( \Omega_i, i = 1, 2, 3 \). Choose a local coordinate system \( \{x_i\} \) on \( \Omega \), we consider the PDE

\[
- \sum_{i,j} g^{ij}(u_3 - \phi - \gamma) \frac{\partial^2 v_1}{\partial x_i \partial x_j} - 2 \sum_{i,j} g^{ij} \left( \frac{\partial u_3}{\partial x_j} - \frac{\partial \phi}{\partial x_j} \right) \frac{\partial v_1}{\partial x_i} - \sum_{i,j,k} (u_3 - \phi - \gamma) g^{ij} \Gamma_{ij}^{k} \frac{\partial v_1}{\partial x_k} = 0 \quad \text{in} \ F_1;
\]

\( v_1 = 0 \) on \( E_2 ; v_1 = 1 \) on \( E_1 \).

This seemingly complicated PDE (61) is a linear, second order elliptic PDE. For each fixed \( \epsilon > 0 \), this differential operator is uniformly elliptic. Since two parts of boundaries do not intersect, (61) has a unique smooth solution \( v_1 \) due to [13 Vol. 3]. Since the zeroth order term of this differential operator is zero, it follows from maximum principle that both maximum and minimum of \( v_1 \) are achieved at \( \partial F_1 \), it follows that

\[
0 \leq v_1 \leq 1 \quad \text{in} \ F_1.
\]

Let \( \psi \) be the standard mollifier defined on \( B_0(1) \). Note that \( \psi \) can be defined as a radial function in terms of \(|x|\). Scaling this we get

\[
\psi_\epsilon(x) = \frac{1}{\epsilon^n \varphi \left( \frac{x}{\epsilon} \right)}, x \in B_0(\epsilon).
\]

Define

\[
v''_1 := \min\{1, \max\{v_1, 0\}\}, v'_1(x) := v''_1 \ast \psi_\epsilon(x), \forall x \in \overline{\Omega_1 \cap \Omega_3};
\]

\[
\chi_1(x) := \begin{cases} 1, & x \in \Omega_1 \setminus \Omega_3 \\ v'_1(x), & x \in \Omega_1 \cap \Omega_3. \\ 0, & x \in \Omega \setminus \Omega_1 \end{cases}
\]

(62)

Since \( v_1 \) is only nontrivial within \( F_1 \), we conclude that \( \chi_1(x) = 0 \) for all \( x \in \partial D''_0 \cap \Omega_1 \) since \( E_2 \) has an \( \epsilon \)-gap away from \( \partial D'_0 \) due to construction. By the same argument, \( \chi_1(x) = 1 \) for all \( x \in \Omega_1 \setminus \Omega_3 \). In particular, \( \chi_1(x) = 1 \) on \( \partial D'_0 \cap \Omega_1 \). Thus \( \chi_1 \) is continuous. Actually \( \chi_1 \) is smooth. For instance, we pick up a point \( x \in \partial D''_0 \cap \Omega_1 \), then we have

\[
\partial^\alpha v'_1(x) = \int_{B_\epsilon(x)} v''_1(y) \partial^\alpha \psi_\epsilon(x - y) dy \equiv 0
\]

since \( v''_1(y) \equiv 0 \) on \( B_\epsilon(x) \) due to the construction. The smoothness at \( \partial D'_0 \cap \Omega_1 \) is almost the same. Lastly, \( \chi_1 \in [0, 1] \). For example, we pick up a point \( x \in \Omega_1 \cap \Omega_3 \), and have

\[
1 - \chi_1(x) = 1 - v'_1(x) = \int_{B_\epsilon(x)} 1 \cdot \psi_\epsilon(y) dy - \int_{B_\epsilon(x)} v''_1(x - y) \psi_\epsilon(y) dy
\]

\[
= \int_{B_\epsilon(x)} (1 - v''_1(x - y)) \cdot \psi_\epsilon(y) dy \geq 0.
\]

The last inequality is immediate from the construction of \( v''_1 \). By a similar argument, we can see that \( \chi_1(x) \geq 0 \). We then consider the PDE

\[
- \sum_{i,j} g^{ij} \frac{\partial^2 v_2}{\partial x_i \partial x_j} - \sum_{i,j,k} g^{ij} \Gamma_{ij}^{k} \frac{\partial v_2}{\partial x_k} = 0 \quad \text{in} \ F_2;
\]

\( v_2 = 0 \) on \( E'_2 ; v_2 = 1 \) on \( E'_1 \).
By the same argument as above, we conclude that (63) has a unique smooth solution \( v_2 \in [0, 1] \) in \( F_2 \). Define
\[
v''_2 := \min\{1, \max\{v_2, 0\}\}, \quad v'_2(x) := v''_2 * \psi(x), \quad \forall x \in \Omega_2 \cap \Omega_3;
\]
\[
\chi_2(x) := \begin{cases} 1, & x \in \Omega_2 \setminus \Omega_3 \\ v'_2(x), & x \in \Omega_2 \cap \Omega_3 \\ 0, & x \in \Omega \setminus \Omega_2 \end{cases}.
\]
(64)

In the same manner, we conclude that \( \chi_2 \in [0, 1] \) is smooth on \( \Omega \). Lastly we define
\[
\chi_3(x) = 1 - \chi_1(x) - \chi_2(x), \quad \forall x \in \Omega.
\]
(65)

Observe that the smooth function \( \chi_3 \equiv 1 \) in \( \Omega_3 \setminus (\Omega_1 \cup \Omega_2) \), and vanishes outside \( \Omega_3 \). We use \( \chi_1, \chi_2, \chi_3 \) defined in (62), (64) and (65) as a smooth partition of unity subordinate to \( \Omega_i, i = 1, 2, 3 \). Define
\[
\bar{u} = \chi_1 u_3 + \chi_2 \phi + \chi_3 (\phi + \gamma).
\]
(66)

We show that \( \bar{u} \in C^\infty(\Omega) \) is a classical super-solution of Yamabe equation in \( \Omega \). This verification is classified in five different cases. Note that
\[
\Omega = \bigcup_{i=1}^3 \Omega_i = \left( \Omega_1 \setminus \Omega_3 \right) \cup \left( \Omega_2 \setminus \Omega_3 \right) \cup \left( \Omega_3 \setminus (\Omega_1 \cup \Omega_2) \right) \cup (\Omega_1 \cap \Omega_3) \cup (\Omega_2 \cup \Omega_3).
\]
(67)

In first set of (67), we have
\[
\bar{u} = \chi_1 u_3 = u_3 \in \Omega_1 \setminus \Omega_3 \Rightarrow -a\Delta_g \bar{u} + (S_g + \beta) \bar{u} - (\lambda_\beta - \kappa) (\bar{u})^{p-1} = 0 \text{ in } \Omega_1 \setminus \Omega_3.
\]
due to (63). Similarly, in the second set of (67), we have
\[
\bar{u} = \chi_2 \phi = \phi \in \Omega_2 \setminus \Omega_3 \Rightarrow -a\Delta_g \bar{u} + (S_g + \beta) \bar{u} - (\lambda_\beta - \kappa) (\bar{u})^{p-1} > 0 \text{ in } \Omega_2 \setminus \Omega_3.
\]
due to (63). In the third set of (67), we have
\[
\bar{u} = \chi_3 (\phi + \gamma) = \phi + \gamma \text{ in } \Omega_3 \setminus (\Omega_1 \cap \Omega_2)
\Rightarrow -a\Delta_g \bar{u} + (S_g + \beta) \bar{u} - (\lambda_\beta - \kappa) (\bar{u})^{p-1} = -a\Delta_g \phi + (S_g + \beta) (\phi + \gamma) - (\lambda_\beta - \kappa) (\phi + \gamma)^{p-1}
\geq - (S_g + \beta) \phi + (\eta_1 + \beta) \phi + (S_g + \beta) (\phi + \gamma) - 2^{p-2} (\lambda_\beta - \kappa) \phi^{p-1} - 2^{p-2} (\lambda_\beta - \kappa) \gamma^{p-1}
\geq \beta' + (S_g + \beta) \gamma - 16 (\lambda_\beta - \kappa) \gamma > 0;
\Rightarrow -a\Delta_g \bar{u} + (S_g + \beta) \bar{u} - (\lambda_\beta - \kappa) (\bar{u})^{p-1} > 0 \text{ in } \Omega_3 \setminus (\Omega_1 \cup \Omega_2).
\]
The last line is due to the choices of \( \beta', \gamma \) in (59) and the observation that \( 2^{p-2} = 2^{\frac{4-2}{2}} \leq 16, \forall n \geq 3 \). In the fourth set of (67), we have
\[
\bar{u} = \chi_1 u_3 + \chi_3 (\phi + \gamma) \text{ in } \Omega_1 \cap \Omega_3.
\]
Note that in \( \Omega_1 \cap \Omega_3 \subset \Omega, S_g < 0, \lambda > 0, \) note that \( \phi > u_3 - \gamma \) and \( u_3 > \phi \) in \( \Omega_1 \cap \Omega_3 \), hence we have
\[
\bar{u} = \chi_1 u_3 + \chi_3 (\phi + \gamma) < \chi_1 u_3 + \chi_3 (u_3 + \gamma) = u_3 + \chi_3 \gamma;
\Rightarrow (S_g + \beta) \bar{u} = (S_g + \beta) u_3 + (S_g + \beta) \chi_3 \gamma;
\]
\[
- (\lambda_\beta - \kappa) (\bar{u})^{p-1} \geq - (\lambda_\beta - \kappa) (u_3 + \chi_3 \gamma)^{p-1} \geq - (\lambda_\beta - \kappa) u_3^{p-1} - (2^{p-1} - 1) (\lambda_\beta - \kappa) u_3^{p-2} \chi_3 \gamma
\geq - (\lambda_\beta - \kappa) u_3^{p-1} - (2^{p-1} - 1) (\lambda_\beta - \kappa) (\phi + \gamma)^{p-2} \chi_3 \gamma
\geq - (\lambda_\beta - \kappa) u_3^{p-1} - 31 (\lambda_\beta - \kappa) (\phi + \gamma)^{p-2} \chi_3 \gamma.
\]
Here we use the inequality \( (a + b)^{p-1} \leq a^{p-1} + (2^{p-1} - 1) a^{p-2} b \) when \( 0 \leq b \leq a \). Here \( a = u_3, b = \chi_3 \gamma \) in \( \Omega_1 \cap \Omega_3 \). The constant 31 is the largest possible value of \( (2^{p-1} - 1) \) for all \( n \geq 3 \).
The hard part is $-a\Delta_g u$. Define the commutator between $-a\Delta_g$ and any smooth function $f$ by
\[ [-a\Delta_g, f]u = -a\Delta_g(fu) - f(-a\Delta_g u) = -a\Delta_g(fu) + f(a\Delta_g u). \] (68)
Immediately, $[-a\Delta_g, 1] = 0$. In local coordinates, we see that
\[ -a\Delta_g u = -a\Delta_g (\chi_1 u_3) - a\Delta_g (\chi_3 (\phi + \gamma)) = \chi_1 (-a\Delta_g u_3) + [a\Delta_g, \chi_1] u_3 + \chi_3 (-a\Delta_g (\phi + \gamma)) + [-a\Delta_g, \chi_3] (\phi + \gamma) := V_1 + V_2 + W_1 + W_2. \]
For $V_1 + W_1$, we use $u_3 < \phi + \gamma$ in $\Omega_1 \cap \Omega_3$ and have
\[ V_1 + W_1 = \chi_1 (-a\Delta_g u_3) + \chi_3 (-a\Delta_g \phi) \]
\[ = \chi_1 \left(- (S_g + \beta) u_3 + (\lambda_\beta - \kappa) u_3^{p-1}\right) + \chi_3 \left(- (S_g + \beta) \phi + (\eta_1 + \beta) \phi\right) \]
\[ \geq -\chi_1 (S_g + \beta) u_3 + \chi_1 (\lambda_\beta - \kappa) u_3^{p-1} - \chi_3 (S_g + \beta) u_3 + \chi_3 (\lambda_\beta - \kappa) u_3^{p-1} \]
\[ + \chi_3 \left((S_g + \beta) \gamma + (\eta_1 + \beta) \phi - (\lambda_\beta - \kappa) (\phi + \gamma)^{p-1}\right) \]
\[ \geq - (S_g + \beta) u_3 + (\lambda_\beta - \kappa) u_3^{p-1} \]
\[ + \chi_3 \left((S_g + \beta) \gamma + (\eta_1 + \beta) \phi - 2^{\beta-2} (\lambda_\beta - \kappa) \phi^{p-1} - 2^{\beta-2} (\lambda_\beta - \kappa) \gamma^{p-1}\right) \]
\[ \geq - (S_g + \beta) u_3 + (\lambda_\beta - \kappa) u_3^{p-1} \]
\[ + \chi_3 \left((S_g + \beta) \gamma + (\eta_1 + \beta) \phi - 2^{\beta-2} (\lambda_\beta - \kappa) \phi^{p-1} - 16 (\lambda_\beta - \kappa) \gamma\right). \]
To check $V_2 + W_2$, we recall that $\chi_1 + \chi_3 = 1$ in $\Omega_1 \cap \Omega_3$, hence
\[ 0 = [-a\Delta_g, 1] = [-a\Delta_g, \chi_1] + [-a\Delta_g, \chi_3] \Rightarrow [-a\Delta_g, \chi_1] = [-a\Delta_g, \chi_3]. \]
It follows that
\[ V_2 + W_2 = [-a\Delta_g, \chi_1] u_3 + [-a\Delta_g, \chi_3] (\phi + \gamma) = [-a\Delta_g, \chi_1] (u_3 - \phi - \gamma) \]
\[ = \chi_1 (a\Delta_g) (u_3 - \phi - \gamma) - a\Delta_g (\chi_1 (u_3 - \phi - \gamma)) \]
\[ = - \sum_{i,j} g^{ij} (u_3 - \phi - \gamma) \frac{\partial^2 \chi_1}{\partial x_i \partial x_j} - \sum_{i,j} 2g^{ij} \left( \frac{\partial u_3}{\partial x_j} - \frac{\partial \phi}{\partial x_j} \right) \frac{\partial \chi_1}{\partial x_i} \]
\[ - \sum_{i,j,k} g^{ij} \Gamma^{k}_{ij} (u_3 - \phi - \gamma) \frac{\partial \chi_1}{\partial x_k}. \]
Next we show that
\[ V_2 + W_2 \geq -Le' \] (69)
in $\Omega_1 \cap \Omega_3$ for some constant $L > 0$ which is independent of $e'$. Here $e' > 0$ can be arbitrarily small. We verify (71) in three types of points. Set
\[ G_1 := \{ x \in \Omega : \text{dist}(x, E_1) \leq \epsilon\}, G_2 := \{ x \in \Omega : \text{dist}(x, E_2) \leq \epsilon\}. \]
For points $F_1 \setminus (G_1 \cup G_2)$, i.e. points in $\Omega_1 \cap \Omega_3$ that is at least distance $\epsilon$ from either $E_1$ or $E_2$, we have
\[ \chi_1(x) = \int_{B_0(\epsilon)} v_1(x - y) \psi_\epsilon(y) dy. \]
Since $v_1 \in C^\infty$, it follows that
\[ |\partial^\alpha \chi_1(x) - \partial^\alpha v_1(x)| \leq e', \forall |\alpha| \leq 2. \]
Here $\epsilon' \to 0$. Hence in this case,

$$V_2 + W_2 \geq - \sum_{i,j} g^{ij} (u_3 - \phi - \gamma) \frac{\partial^2 v_1}{\partial x_i \partial x_j} - \sum_{i,j} 2g^{ij} \left( \frac{\partial u_3}{\partial x_j} - \frac{\partial \phi}{\partial x_j} \right) \frac{\partial v_1}{\partial x_i} - \sum_{i,j,k} g^{ij} \Gamma_{ijk}^k (u_3 - \phi - \gamma) \frac{\partial v_1}{\partial x_k} - L\epsilon'$$

$$= -L\epsilon'.$$

Here $L$ depends on $u_3, \phi, g^{ij}, \Gamma_{ij}^k$ and is independent of $\epsilon'$. When $x$ in $\Omega_1 \cap \Omega_3 \cap G_1$ or $\Omega_1 \cap \Omega_3 \cap G_2$, we conclude that

$$\partial^\alpha \chi_1 = O(1), \forall x \in \Omega_1 \cap \Omega_3 \cap (G_1 \cup G_2).$$

This is due to the smoothness of $\chi_1$ and $\partial^\alpha \chi_1 = 0$ on $\partial D_0' \cap \Omega_1$ and $\partial D_0' \cap \Omega_1$. To be precise, when $x \in \Omega_1 \cap \Omega_3 \cap G_1$, $u_3 - \phi - \gamma \to 0$ when $\epsilon \to 0$; furthermore, the first order derivative satisfies

$$\frac{\partial \chi_1}{\partial x_j} = \int_{B_{x}(\epsilon)} v'_1(y) \frac{\partial \psi_0(x-y)}{\partial x_j} dy = \int_{B_{x}(\epsilon)} v'_1(y) (-1)^2 \frac{\partial \psi_0(x-y)}{\partial y_j} dy.$$

The boundary terms on $\partial D_1$ are killed since on $\partial D_1$ we have $v_1 = 1$. The second order derivative is of the form

$$\frac{\partial^2 \chi_1}{\partial x_i \partial x_j} = \int_{B_{x}(\epsilon)} v'_1(y) \frac{\partial^2 \psi_0(x-y)}{\partial x_i \partial x_j} dy = \int_{B_{x}(\epsilon)} v'_1(y) (-1)^2 \frac{\partial^2 \psi_0(x-y)}{\partial y_i \partial y_j} dy$$

$$= \int_{B_{x}(\epsilon) \cap \Omega_1 \cap \Omega_3} \frac{\partial^2 \psi_0(x-y)}{\partial y_i \partial y_j} \psi_0(x-y) dy - \int_{\partial B_{x}(\epsilon) \cap \partial D_1} \frac{\partial v_1}{\partial x_i} \nu_j \psi_0(x-y) dS.$$

The first term above can be controlled by the PDE [61]; for the the second term above, we observe that $u_3 - \phi - \gamma = O(\epsilon)$ and $\frac{\partial v_1}{\partial x_i} = O(\gamma)$ due to the construction of PDE, thus by the fact that $\epsilon \ll \gamma$,

$$-g^{ij} (u_3 - \phi - \gamma) \int_{\partial B_{x}(\epsilon) \cap \partial D_1} \frac{\partial v_1}{\partial x_i} \nu_j \psi_0(x-y) dS \to 0$$

as $\epsilon \to 0$. When $x \in \Omega_1 \cap \Omega_3 \cap G_2$, the first order derivatives are controlled exactly as just discussed. Note that $u_3 - \phi - \gamma < 0$ here, the second order derivative will have

$$\frac{\partial^2 \chi_1}{\partial x_i \partial x_j} = \int_{B_{x}(\epsilon)} v'_1(y) \frac{\partial^2 \psi_0(x-y)}{\partial x_i \partial x_j} dy = \int_{B_{x}(\epsilon)} v'_1(y) (-1)^2 \frac{\partial^2 \psi_0(x-y)}{\partial y_i \partial y_j} dy$$

$$= \int_{B_{x}(\epsilon) \cap \Omega_1 \cap \Omega_3} \frac{\partial^2 \psi_0(x-y)}{\partial y_i \partial y_j} \psi_0(x-y) dy - \int_{\partial B_{x}(\epsilon) \cap \partial D_2} \frac{\partial v_1}{\partial x_i} \nu_j \psi_0(x-y) dS$$

The last inequality is due to the fact that $v_1$ is descending to zero, therefore the derivative is negative when very close to $\partial D_2$. It follows that the boundary part above is positive. Within the $2\epsilon$-strip of $\partial D_0'$ and $\partial D_0''$, we have

$$V_2 + W_2 = - \sum_{i,j} g^{ij} (u_3 - \phi - \gamma) \frac{\partial^2 \chi_1}{\partial x_i \partial x_j} - \sum_{i,j} 2g^{ij} \left( \frac{\partial u_3}{\partial x_j} - \frac{\partial \phi}{\partial x_j} \right) \frac{\partial \chi_1}{\partial x_i}$$

$$- \sum_{i,j,k} g^{ij} \Gamma_{ijk}^k (u_3 - \phi - \gamma) \frac{\partial \chi_1}{\partial x_k} = o(1) \geq -L\epsilon'.$$
Remark 4.5. Recall that the local PDE \((37)\) holds for all \(\lambda > 0\). This is crucial since it gives us the flexibility to choose the constant associate with \(u^{p-1}\) term, hence we can introduce the extra constant \(\kappa > 0\) here.
We now apply Theorem 4.3 to show the existence of the solution of Yamabe equation with \( \eta_1 > 0 \). Note that from perturbed Yamabe equation (53) to the Yamabe equation, we need to pass the limit by letting \( \beta \to 0^- \). Yamabe tried to let \( u^{p-1} \to u^{p-1} \) when \( s \to p \), and the uniform boundedness of the sequence is not that easy. Due to the local construction of our perturbed Yamabe equations in (53), we obtain the uniformly positive upper and lower bounds of \( \|u_{\beta}\|_{L^p(M,g)} \) when \( \beta \in [\beta_0, 0] \), with some fixed \( \beta_0 < 0 \). With the introduction of the extra positive constant \( \kappa > 0 \), a similar argument due to Trudinger and Aubin then applies to show the uniform boundedness of \( \|u_{\beta}\|_{L^r(M,g)} \) for all \( \beta \in [\beta_0, 0] \) for some \( r > p \). In contrast to classical method, we obtain this without using \( \lambda(M) < \lambda(\mathbb{S}^n) \).

**Theorem 4.4.** Let \((M,g)\) be a closed manifold with \( n = \dim M \geq 3 \). Let \( \kappa > 0 \) be some small enough positive constant. Assume the scalar curvature \( S_g < 0 \) somewhere on \( M \) and the first eigenvalue \( \eta_1 > 0 \). Then there exists some \( \lambda > 0 \) such that the Yamabe equation (22) has a real, positive, smooth solution.

**Proof.** By Theorem 4.3 we have a sequence of real, positive, smooth solutions \( \{u_{\beta}\} \) when \( \beta < 0 \) and \(|\beta|\) is small enough, i.e.

\[
-\Delta_g u_{\beta} + (S_g + \beta) u_{\beta} = (\lambda_{\beta} - \kappa) u_{\beta}^{p-1} \text{ in } M. \quad (72)
\]

We show first that \( \{\lambda_{\beta}\} \) is bounded above, and is increasing and continuous when \( \beta \to 0^- \). It then follows that we can choose \( \kappa \) uniformly so that

\[
\lambda_{\beta} - \kappa > 0, \forall \beta \in [\beta_0, 0].
\]

We may assume \( \int_M d\omega = 1 \) for this continuity verification, since otherwise only an extra term with respect to Vol\(_g\) will appear. Recall that

\[
\lambda_{\beta} = \inf_{u \neq 0, u \in H^1(M)} \left\{ \int_M a|\nabla_g u|^2 d\omega + \int_M (S_g + \beta) u^2 d\omega \right\} \left( \int_M u^p d\omega \right)^{\frac{2}{p}}.
\]

It is immediate that if \( \beta_1 < \beta_2 < 0 \) then \( \lambda_{\beta_1} \leq \lambda_{\beta_2} \). For continuity we assume \( 0 < \beta_2 - \beta_1 < \gamma \). For each \( \epsilon > 0 \), there exists a function \( u_0 \) such that

\[
\frac{\int_M a|\nabla_g u_0|^2 d\omega + \int_M (S_g + \beta_1) u_0^2 d\omega}{\left( \int_M u_0^p d\omega \right)^{\frac{p}{2}}} < \lambda_{\beta_1} + \epsilon.
\]

It follows that

\[
\lambda_{\beta_2} \leq \frac{\int_M a|\nabla_g u_0|^2 d\omega + \int_M (S_g + \beta_2) u_0^2 d\omega}{\left( \int_M u_0^p d\omega \right)^{\frac{p}{2}}} \leq \frac{\int_M a|\nabla_g u_0|^2 d\omega + \int_M (S_g + \beta_1) u_0^2 d\omega}{\left( \int_M u_0^p d\omega \right)^{\frac{p}{2}}} + \frac{(\beta_2 - \beta_1) \int_M u_0^2 d\omega}{\left( \int_M u_0^p d\omega \right)^{\frac{p}{2}}}
\]

\[
\leq \lambda_{\beta_1} + \epsilon + \beta_2 - \beta_1 < \lambda_{\beta_1} + \epsilon + \beta_2 - \beta_1.
\]

Since \( \epsilon \) is arbitrarily small, we conclude that

\[ 0 < \beta_2 - \beta_1 < \gamma \Rightarrow |\lambda_{\beta_2} - \lambda_{\beta_1}| \leq 2\gamma. \]

By the argument in Appendix A we conclude that \( \lambda_{\beta} < aT \), where \( T \) is the best Sobolev constant for the embedding \( L^p(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n) \). Fix some \( \beta_0 < 0 \) with \( \eta_{1,\beta_0} > 0 \), we have

\[
\lambda_{\beta_0} \leq \lambda_{\beta} \leq aT, \forall \beta \in [\beta_0, 0], \lim_{\beta \to 0^-} \lambda_{\beta} - \kappa := \lambda. \quad (73)
\]
Next we show \(\|u_\beta\|_{H^1(M)} \leq C\) uniformly when \(\beta \in [\beta_0, 0]\). Pairing the PDE (72) with \(u_\beta\), it follows that

\[
a\|\nabla_g u_\beta\|^2_{L^2(M,g; \mathbb{R})} = (\lambda_\beta - \kappa) \|u_\beta\|^p_{L^p(M,g)} - \int_M (g_s + \beta) u_\beta^2 d\omega
\]

\[
\leq (\lambda_\beta - \kappa) \|u_\beta\|^p_{L^p(M,g)} + \left( \sup_M |S_g| + |\beta| \right) \|u_\beta\|^2_{L^2(M,g)}. \tag{74}
\]

Thus it suffices to show

\[
\|u_\beta\|_{L^p(M,g)} \leq C', \forall \beta \in [\beta_0, 0].
\]

Denote the corresponding local solutions of (37) by \(\tilde{u}_\beta\), i.e.

\[-a \Delta_g \tilde{u}_\beta + (S_g + \beta) \tilde{u}_\beta = (\lambda_\beta - \kappa) \tilde{u}_\beta^p - 1 \text{ in } \Omega, \tilde{u}_\beta = 0 \text{ on } \partial \Omega \]

with fixed domain \(\Omega\). Again we have

\[
a\|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} = (\lambda_\beta - \kappa) \|\tilde{u}_\beta\|^p_{L^p(\Omega,g)} - \int_\Omega (g_s + \beta) \tilde{u}_\beta^2 d\text{Vol}_g
\]

\[
\Rightarrow (\lambda_\beta - \kappa) \|\tilde{u}_\beta\|^p_{L^p(\Omega,g)} \leq a\|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} + \left( \sup_M |S_g| + |\beta| \right) \|u_\beta\|^2_{L^2(\Omega,g)}. \tag{75}
\]

Recall the functional \(J(u)\) in (44) and the quantity \(K_0\) in (45), due to Theorem 1.1 of [26], each solution \(\tilde{u}_\beta\) satisfies

\[
J(\tilde{u}_\beta) \leq K_0 \Rightarrow \frac{a}{2} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} - \frac{(\lambda_\beta - \kappa)}{p} \|\tilde{u}_\beta\|^p_{L^p(\Omega,g)} - \frac{1}{2} \int_\Omega (S_g + \beta) \tilde{u}_\beta^2 d\text{Vol}_g \leq K_0. \tag{76}
\]

Note that since \(\lambda_\beta\) is bounded above and below when \(\beta \in [\beta_0, 0]\), the quantity \(K_0\) is uniformly bounded above, and we denote it by \(K_0\) again. Apply the estimate (75) into (76), we have

\[
\frac{a}{2} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} \leq K_0 + \frac{1}{p} \left( a\|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} + \left( \sup_M |S_g| + |\beta| \right) \|u_\beta\|^2_{L^2(\Omega,g)} \right)
\]

\[
+ \left( \sup_M |S_g| + |\beta| \right) \|u_\beta\|^2_{L^2(\Omega,g)}
\]

\[
\leq K_0 + \frac{a(n - 2)}{2n} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)}
\]

\[
+ \left( \frac{n - 2}{2n} + 1 \right) \left( \sup_M |S_g| + |\beta| \right) \lambda_1^{-1} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)}
\]

\[
\Rightarrow \frac{a}{n} - \left( \frac{n - 2}{2n} + 1 \right) \left( \sup_M |S_g| + |\beta| \right) \lambda_1^{-1} \|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} \leq K_0.
\]

We have already chosen \(\Omega\) small enough so that (40) holds for \(\beta_0\), thus for all \(\beta \in [\beta_0, 0]\), (40) holds. It follows from above that there exists a constant \(C'_0\) such that

\[
\|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} \leq C'_0, \forall \beta \in [\beta_0, 0].
\]

Apply (75) with the other way around, we conclude that

\[
(\lambda_\beta - \kappa) \|\tilde{u}_\beta\|^p_{L^p(\Omega,g)} \leq a\|\nabla_g \tilde{u}_\beta\|^2_{L^2(\Omega,g)} + \left( \sup_M |S_g| + |\beta| \right) \|\tilde{u}_\beta\|^2_{L^2(\Omega,g)}
\]

\[
\leq \left( a + \left( \sup_M |S_g| + |\beta| \right) \lambda_1^{-1} \right) \|\nabla_g u_\beta\|^2_{L^2(\Omega,g)}.
\]

We conclude that

\[
\|u_\beta\|^p_{L^p(\Omega,g)} \leq C'_1, \forall \beta \in [\beta_0, 0]. \tag{77}
\]
Recall the construction of super-solution of each $u_\beta$ in Theorem 1.3, we have

$$0 < u_\beta \leq u_{+\beta} = \begin{cases} \bar{u}_\beta, & \text{in } \Omega \\ \phi, & \text{in } M \setminus \Omega. \end{cases}$$

where $\bar{u}_\beta$ is of the form

$$\bar{u}_\beta = \chi_1 \bar{u}_\beta + \chi_2 \phi + \chi_3 (\phi + \gamma).$$

It follows from the uniform upper bound of (77) that

$$\|u_\beta\|_{L^p(M,g)}^p \leq \|u_{+\beta}\|_{L^p(M,g)}^p \leq E_0 \left( \|\bar{u}_\beta\|_{L^p(\Omega,g)}^p + \|\phi\|_{L^p(M,g)}^p \right).$$

Recall in determining $\phi$ we require

$$\frac{(\eta_1 + \beta)}{2p - 2} (\lambda_\beta - \kappa) > \delta^p \frac{\sup_M \varphi^{p-1}}{\inf_M \varphi}.$$ 

Since when $\beta \in [\beta_0,0]$, we have $\lambda_\beta \in [\lambda_{0aT},aT]$, we can choose a fixed scaling $\delta$ for all $\beta$, thus $\|\phi\|_{L^p(M,g)}^p$ is uniformly bounded. It follows that there exists a constant $C'$ such that

$$\|u_\beta\|_{L^p(M,g)}^p \leq C', \forall \beta \in [\beta_0,0].$$

Due to H"older’s inequality, we conclude that

$$\|u_\beta\|_{L^2(M,g)} \leq \|u_\beta\|_{L^p(M,g)}^{2p} \cdot \operatorname{Vol}_g(M)^{\frac{2p - 2}{p}}.$$

Thus by (74) we conclude that

$$\|u_\beta\|_{H^1(M,g)} \leq C, \forall \beta \in [\beta_0,0]. \quad (78)$$

According to the definition of $\lambda_\beta$, we apply the solution of (72) with the form (74), it follows that for all $\beta \in [\beta_0,\beta]$,

$$\lambda_\beta \leq \frac{\int_M g|\nabla u_\beta|^2 d\omega + \int_M (S_g + \beta) u_\beta^2 d\omega}{\left( \int_M u_\beta^p d\omega \right)^{2/p}} = (\lambda_\beta - \kappa) \cdot \frac{\|u_\beta\|_{L^p(M,g)}^p}{\left( \int_M u_\beta^p d\omega \right)^{2/p}} \Rightarrow \frac{\lambda_\beta}{\lambda_\beta - \kappa} \leq \|u_\beta\|_{L^p(M,g)}^{-\frac{2}{p}} \Rightarrow \|u_\beta\|_{L^p(M,g)} \geq 1.$$

The last cancellation is due to the fact that $\lambda_\beta \geq \lambda_{0aT} - \kappa \geq \lambda_{0aT} - \kappa > 0, \forall \beta \in [\beta_0,0]$. It follows that

$$\|u_\beta\|_{H^1(M,g)} \leq C, 1 \leq \|u_\beta\|_{L^p(M,g)} \leq C', \forall \beta \in [\beta_0,0]. \quad (79)$$

Due to uniform boundedness of $L^p$-norms of $u_\beta$, we can scale the metric $g \mapsto \gamma g$, $\gamma < 1$ uniformly for all $\beta \in [\beta_0,0]$ such that

$$0 < C'' \leq \|u_\beta\|_{L^p(M,g)} \leq 1, \forall \beta \in [\beta_0,0]. \quad (80)$$

We still denote the new metric by $g$. This scaling does not affect the local estimates in Appendix A since $\beta < 0$, and hence does not affect the local solvability of the PDE in Proposition 3.3. This scaling does not affect the fact that $\lambda_{0aT} \leq \lambda_\beta \leq aT$ since again $\beta < 0$ and hence does not affect the choice of $\kappa$. We now apply the analysis, essentially due to Trudinger and Aubin, to show that there exists some $r > p$, such that

$$\|u_\beta\|_{L^r(M,g)} \leq K, \forall \beta \in [\beta_0,0]. \quad (81)$$
Let $\delta > 0$ be some constant. Pairing (72) with $u_\beta^{1+2\delta}$ and denote $w_\beta = u_\beta^{1+\delta}$, it follows that

$$\int_M a \nabla_g u_\beta \cdot \nabla_g \left( u_\beta^{1+2\delta} \right) d\omega + \int_M (S_g + \beta) u_\beta^{2+2\delta} d\omega = (\lambda_\beta - \kappa) \int_M u_\beta^{p+2\delta} d\omega;$$

$$\Rightarrow \frac{1 + 2\delta}{1 + \delta^2} \int_M a |\nabla_g w_\beta|^2 d\omega = (\lambda_\beta - \kappa) \int_M w_\beta^2 u_\beta^{-2} d\omega - \int_M (S_g + \beta) w_\beta^2 d\omega.$$ 

According to the sharp Sobolev inequality on closed manifolds [18 Thm. 2.3, Thm. 3.3], it follows that for any $\epsilon > 0$,

$$\|w_\beta\|^2_{L^p(M,g)} \leq (1 + \epsilon) \frac{1}{aT} \left( 1 + \frac{\delta^2}{1 + \delta^2} \int_M a |\nabla_g w_\beta|^2 d\omega \right) + C_\epsilon \|w_\beta\|^2_{L^2(M,g)}$$

$$= (1 + \epsilon) \frac{\lambda_\beta - \kappa}{aT} \cdot \frac{1 + \delta^2}{1 + 2\delta} \int_M w_\beta^2 u_\beta^{-2} d\omega + C_\epsilon \|w_\beta\|^2_{L^2(M,g)}$$

$$\leq (1 + \epsilon) \frac{\lambda_\beta - \kappa}{aT} \cdot \frac{1 + \delta^2}{1 + 2\delta} \|w_\beta\|^2_{L^p(M,g)} \|u_\beta\|^p_{L^p(M,g)} + C_\epsilon \|w_\beta\|^2_{L^2(M,g)}$$

$$\leq (1 + \epsilon) \frac{\lambda_\beta - \kappa}{aT} \cdot \frac{1 + \delta^2}{1 + 2\delta} \|w_\beta\|^2_{L^p(M,g)} + C_\epsilon \|w_\beta\|^2_{L^2(M,g)}.$$ 

Note that $C_\epsilon$ is uniform over all $\beta$. The last two lines above are due to Hölder’s inequality and [80]. Since $\lambda_\beta \leq aT$, $\lambda_\beta - \kappa < aT$; thus we can choose $\epsilon, \delta$ small enough such that

$$(1 + \epsilon) \frac{\lambda_\beta - \kappa}{aT} \cdot \frac{1 + \delta^2}{1 + 2\delta} < 1.$$ 

It follows that

$$\|w_\beta\|^2_{L^p(M,g)} \leq \mathcal{K}_1 \|w_\beta\|^2_{L^2(M,g)} \leq \mathcal{K}_2 \|u_\beta\|^1_{L^p(M,g)} \leq \mathcal{K}_3, \forall \beta \in [\beta_0, 0].$$ 

Note that $\|w_\beta\|_{L^p(M,g)} = \|u_\beta\|^{1+\delta}_{L^{p(1+\delta)}(M,g)}$, it follows that (81) holds. Since $r > p$, by bootstrapping method we conclude that

$$\|u_\beta\|_{C^{2,\alpha}(M)} \leq \mathcal{K}', \forall \beta \in [\beta_0, 0].$$ 

Due to Arzela-Ascoli, we pass the limit by letting $\beta \to 0^-$ on both sides of (73), it follows that

$$-a\Delta_g u + S_g u = \lambda u^{p-1}$$

on $M$.

Another bootstrapping method shows that $u \in C^\infty(M)$ and $u \geq 0$ since $u$ is the limit of a positive sequence. Lastly we show that $u > 0$. Due to uniform boundedness in (82) and lower bound of $L^p$-norms in [80], we conclude that

$$\|u\|_{L^p(M,g)} \geq C'' > 0.$$ 

It follows from maximum principle that $u > 0$ on $M$. \hfill $\square$

**Remark 4.6.** With the introduction of $\kappa > 0$ in (72), we can apply this method for all cases, even when the manifold is conformal to the standard sphere $\mathbb{S}^n$, since we still have $\lambda_\beta - \kappa < \lambda(\mathbb{S}^n) = aT$.

Lastly we show the case when $S_g \geq 0$ everywhere. Due to [16], this amounts to say $\eta > 0$. Therefore we consider the solution of Yamabe equation when $\lambda > 0$. We show this by showing that there exists some $\tilde{g} = e^{2\eta g} = u^{p-2}g$ for some positive $u \in C^\infty(M)$ such that $S_{\tilde{g}} < 0$ somewhere. Obviously, $S_{\tilde{g}}$ cannot be too negative too often, actually due to Theorem 2.3 (i) and taking $\varphi = 1$ in (48), a necessary condition is $\int_M S_g > 0$. Consequently we can apply Theorem 4.3 to obtain some metric with positive constant scalar curvature.
Theorem 4.5. Let \((M, g)\) be a closed manifold with \(n = \dim M \geq 3\). Assume the scalar curvature \(S_g \geq 0\) everywhere on \(M\). Then there exists some smooth function \(H \in C^\infty(M)\) which is negative somewhere such that \(H^2\) is the scalar curvature with respect to some conformal change of \(g\).

Proof. We show this by constructing the smooth function \(H\) with desired property for which the following PDE

\[-a\Delta_g u + S_g u = H u^{p-1}\text{ in } (M, g)\] (83)

admits a smooth, real, positive solution \(u\). Without loss of generality, we can assume \(\sup_x (S_g) \leq 1\) for some \(g\) after scaling.

To construct \(H\), we first construct a smooth function \(F \in C^\infty(M)\) such that: (i) \(F\) is very negative at some point in \(M\); (ii) \(\int_M F d\omega = 0\); and (iii) \(\|F\|_{H^{s-2}(M, g)}\) is very small simultaneously. Here \(s = n/2 + 1\) if \(n\) is even and \(s = n+1\) if \(n\) is odd. For simplicity, we discuss the even dimensional case, the odd dimensional case is exactly the same except the order \(s\).

Pick up a normal ball \(B_r\) in \(M\) with radius \(r < 1\) centered at some point \(q \in M\). Choose a function

\[f(q) = -C < -1, f \leq 0, |f| \leq C, f \in C^\infty(M), f \equiv 0 \text{ on } M \setminus B_r.\] (84)

Such an \(f\) can be chosen as any smooth bump function or cut-off function in partition of unity multiplying by \(-1\). For \(f\), we have

\[|\partial^\alpha f| \leq C' r^n \text{, } \forall |\alpha| \leq \frac{n}{2} - 1 = s - 2.\] (85)

By (85), we have

\[\left| \int_M f d\omega \right| = \left| \int_{B_r} f d\Vol_g \right| \leq C \Vol_g(B_r) \leq C' \Gamma r^n.\] (86)

Here \(C\) is the upper bound of \(|f|\). The hypothesis of the statement says that \(S_g \geq 0\) everywhere, hence \(\Gamma\) depends only on the dimension of \(M\) for every geodesic ball centered at any point of \(M\), due to Bishop-Gromov comparison inequality. Specifically, the volume of a geodesic ball of radius \(r \ll 1\) centered at \(p\) is given by \(\Vol_g(B_r) = \left(1 - \frac{S_g(0)}{n(n-2)} r^2 + O(r^4)\right) \Vol_g(B_r)\). Thus \(\Gamma\) is fixed when \((M, g)\) is fixed. Now we define

\[F := f + \epsilon\] (87)

where \(0 < \epsilon < C\) is determined later. First we show that for any \(C\), we can choose \(\epsilon\) arbitrarily small so that \(\int_M F d\omega = 0\). Set

\[\epsilon := \frac{1}{\Vol_g(M)} \int_{B_r} - f d\Vol_g \Rightarrow \int_M F d\omega = \int_M (f + \epsilon) d\omega = \int_{B_r} f d\Vol_g + \epsilon \int_M d\omega = 0.\]

As we have shown in (86),

\[\epsilon \leq \frac{1}{\Vol_g(M)} \left| \int_{B_r} f d\Vol_g \right| \leq \frac{1}{\Vol_g(M)} C \Gamma r^n = O(r^n).\]

Thus if we choose \(r\) small enough then \(\epsilon\) is small enough.

Secondly we show that for any \(C, C'\), we can make \(\|F\|_{H^{s-2}\text{(M, g)}}\) arbitrarily small. By Bishop-Gromov inequality and (85), we observe that

\[\|F\|_{H^{s-2}(M, g)}^2 = \int_M |f + \epsilon|^2 d\omega + \sum_{1 \leq |\alpha| \leq s-2} \int_{B_r} |\partial^\alpha f|^2 d\Vol_g \leq C_0 r^n + \sum_{1 \leq |\alpha| \leq s-2} \frac{(C')^2}{r^{n-2}} \Gamma r^n \leq (C')^2 C_0' r^{n-2}\]
for some $C'_0$ depends only on the order $s - 2 = \frac{n}{2} - 1$, the lower bound of Ricci curvature of $M$, and the diameter of $(M,g)$. Therefore, for any $C,C'$, we can choose $r$ small enough so that $\|F\|_{H^{s-2}(M,g)}^2$ is arbitrarily small.

It follows that this smooth function $F$ we constructed satisfies all requirements mentioned above. Now we construct $u \in C^\infty(M)$. Let $u'$ be the solution of

$$-a\Delta_g u' = F \text{ in } (M,g).$$

(88)

Due to Lax-Milgram [17, Ch. 6] and hypothesis (i) of $F$, there exists such an $u'$ that solves (88). Since $F \in C^\infty(M)$, standard elliptic regularity implies that $u' \in C^\infty(M)$. Recall the Sobolev embedding in Theorem 2.2(iii) and $H^s$-type elliptic regularity in Theorem 2.1(i), we have fixed constants $K'$ and $C^\ast$ such that

$$|u'| \leq \|u'\|_{C^0,M} \leq K'|u'|_{H^s(M,g)}, \frac{1}{2} - \frac{s}{n} = \frac{1}{2} - \frac{1}{2} - \frac{1}{n} < 0;$$

$$\|u'\|_{H^s(M,g)} \leq C^\ast \left( \|F\|_{H^{s-2}(M,g)} + \|u'\|_{C^2(M,g)} \right) \leq C^\ast \|F\|_{H^{s-2}(M,g)}.$$ 

Note that the last inequality comes from the PDE (88) by pairing $u'$ on both sides, and apply Poincaré inequality. However, we should have $\int_M u'd\omega = 0$ to apply Poincaré inequality, this can be done by taking $u' \mapsto u' + \epsilon$ if necessary. Thus $K'$ depends only on $n, \alpha, s, M, g$ and $C^\ast$ depends only on $s, -a\Delta_g, \lambda_1, M, g$, all of which are fixed once $M, g, s$ are fixed.

Note that $f(q) = -C$ for some $C > 1$ at the point $q \in M$. First choose $r$ small enough so that $\epsilon < \frac{1}{2}$, it follows that

$$F(q) = f(q) + \epsilon \leq -\frac{C}{2}.$$ 

(89)

Now we further shrink $r$ such that

$$C^\ast K' \|F\|_{H^{s-2}(M,g)} \leq C^\ast K' \cdot C' \left( C'_0 \right)^{1/2} r \leq \frac{C}{8}.$$ 

Thus by Sobolev embedding and elliptic regularity, it follows that

$$|u'| \leq C^\ast K' \|F\|_{H^{s-2}(M,g)} \leq \frac{C}{8} \Rightarrow u' \in \left[ -\frac{C}{8}, \frac{C}{8} \right], \forall x \in M.$$ 

(90)

We now define $u$ to be

$$u := u' + \frac{C}{4}.$$ 

(91)

By (91), it is immediate to see that $u > 0$ on $M$ and $u \in C^\infty(M)$ due to definition and smoothness of $u'$. In particular $u \in \left[ \frac{C}{8}, \frac{3C}{8} \right]$. Furthermore, $u$ solves (88). Finally, we define our $H$ to be

$$H := u^{1-p} (-a\Delta_g u + S_g u), \forall x \in M.$$ 

(92)

Clearly $H \in C^\infty(M)$. By $-a\Delta_g u = F$, (89), (90) and (91), we observe that

$$H(q) = u(q)^{1-p} (-a\Delta_g u(q) + S_g (u(q))) = u(q)^{1-p} (F(q) + S_g (u(q))) \leq u(q)^{1-p} \left( -\frac{C}{2} + \frac{3C}{8} \right) < 0.$$ 

Based on our constructions of $u$ and $H$, (92) is equivalent to say that the following PDE

$$-a\Delta_g u + S_g u = Hu^{p-1} \text{ in } (M,g)$$

has a real, positive, smooth solution. It follows that $H$ is the scalar curvature of the metric $u^{p-2}g$ on $M$. $\square$

**Remark 4.7.** We see from above that

$$H(q) \leq u(q)^{1-p} \left( -\frac{C}{2} + \frac{3C}{8} \right) < 0.$$
Furthermore, we observe from above that

\[ |u'| \leq C^{**}K\|F\|_{H^{s,2}(M,g)} \leq C^{**}K \cdot C'u'(C'_0)^{\frac{1}{2}} r. \]

It follows that we can make \( |u'| < \epsilon' \) for any \( \epsilon' \) small enough. Thus taking

\[ u = u' + 2\epsilon' \Rightarrow |u| \leq 3\epsilon'. \]

With fixed \( C \), we can make \( H(q) \) arbitrarily large since \( u^{1-p} = u^{-\frac{n+2}{n-2}} \) would be arbitrarily large with arbitrarily small \( \epsilon' \).

An immediate analogy of Theorem 4.5 is that when \( S_g \leq 0 \) everywhere and hence \( \eta_1 < 0 \), there exists some \( H' \in C^\infty(M) \), positive somewhere, that is the prescribed scalar curvature of some \( \tilde{g} \) under conformal change.

**Corollary 4.1.** Let \((M, g)\) be a closed manifold with \( n = \text{dim } M \geq 3 \). Let \( S_g \leq 0 \) everywhere on \( M \) be the scalar curvature with respect to \( g \). Then there exists some smooth function \( H' \in C^\infty(M) \) which is positive somewhere such that \( H' \) is the scalar curvature with respect to some conformal change of \( g \).

**Proof.** Taking \( f \geq 0 \) with properties as in Theorem 4.5 and do the same argument. \( \square \)

With the help of Theorem 4.5, we can solve the Yamabe equation for the case \( S_g \geq 0 \) everywhere. Note that this amounts to say \( \eta_1 > 0 \) due to \( (48) \).

**Theorem 4.6.** Let \((M, g)\) be a closed manifold with \( n = \text{dim } M \geq 3 \). Assume the scalar curvature \( S_g \geq 0 \) everywhere on \( M \). Then there exists some \( \lambda > 0 \) such that the Yamabe equation \( (2) \) has a real, positive, smooth solution.

**Proof.** After scaling we can, without loss of generality, assume \( \sup M S_g \leq 1 \). We start with this metric. By Theorem 4.4, we conclude that there exists some smooth function \( H < 0 \) somewhere such that it is the scalar curvature with respect to the metric \( \tilde{g}_1 = u^{p-2}g \) for some smooth \( u > 0 \) on \( M \). By Theorem 2.3(i) the first eigenvalue for the conformal Laplacian \(-a\Delta_{\tilde{g}_1} + H\) is positive.

Consequently by Theorem 4.3 there exists some \( \lambda > 0 \) such that \( (2) \) has some real, positive solution \( v \in C^\infty(M) \), i.e. \( \lambda \) is the scalar curvature of the metric \( \tilde{g} = v^{p-2}\tilde{g}_1 \). Combining both conformal changes together, we see that

\[ \tilde{g} = v^{p-2}\tilde{g}_1 = v^{p-2}u^{p-2}g = (uv)^{p-2}g. \]

Clearly \( uv > 0 \) and smooth on \( M \). We now conclude that \( \tilde{g} \) is associated with the positive constant scalar curvature \( \lambda \) under the conformal change \( \tilde{g} = (uv)^{p-2}g \). \( \square \)

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**Appendix A. Proof of The Claim**

In this section, we show that the claim \((46)\) in Proposition 3.3 holds. The key is a very careful analysis in terms of local expression of \( \sqrt{\det(g)} \) and \( g^{ij} \). This estimate is sharp in the sense that if we drop any negative term in numerator or any positive term in denominator with some specific order then we cannot detect the desired upper bound, see Remark 3.2. Note that the quantity \( Q_\epsilon, \Omega \) is not a Yamabe quotient because of the perturbed term \( \beta \). Without loss of generality, we choose \( \Omega = B_0(r) \) be a ball of radius \( r \) and centered at 0. We fix the radius \( r \) due to the discussion in Proposition 3.3. We interchangeably use \( \Omega \) and \( B_0(r) \) when there will be no confusion.
When \( n = 3 \), we set \( \Omega = B_0(1) \) and thus we choose the cut-off function as 
\[
\varphi_\Omega(x) = \cos \left( \frac{\pi |x|}{2} \right).
\]

When \( n \geq 4 \), we choose \( \varphi_\Omega(x) = \varphi_\Omega(|x|) \) to be any radial bump function supported in \( \Omega \) such that \( \varphi_\Omega(x) \equiv 1 \) on \( B_0 \left( \frac{r}{2} \right) \). Therefore, the test function we choose in \( \Omega \) is 
\[
u_{\varepsilon,\Omega}(x) = \frac{\varphi_\Omega(x)}{(\varepsilon + |x|^2)^{n-2}}, n \geq 3.
\]

We will simply denote \( u_{\varepsilon,\Omega} = u \) when there would be no confusion.

When \( n \geq 4 \), denote 
\[
K_1 = (n - 2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} dy;
\]
\[
K_2 = \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^n} dy \right)^{\frac{2}{p}};
\]
\[
K_3 = \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} dy.
\]

We will keep using \( K_1, K_2 \) when \( n = 3 \), but \( K_3 \) is not integrable when \( n = 3 \). The best Sobolev constant \( T \) satisfies 
\[
T = \frac{K_1}{K_2}, n \geq 3.
\]

We use the notation \( T \) for all cases \( n \geq 3 \).

**Theorem A.1.** Let \( \Omega = B_0(r) \) for small enough \( r \) with \( \dim \Omega \geq 3 \). Let \( \beta < 0 \) be any negative constant and \( T \) be the best Sobolev constant in (94). The quantity \( Q_{\varepsilon,\Omega} \) satisfies 
\[
Q_{\varepsilon,\Omega} := \frac{\|\nabla g u_{\varepsilon,\Omega}\|_{L^2(\Omega,g)}}{\|u_{\varepsilon,\Omega}\|_{L^p(\Omega,g)}} + \frac{1}{a} \int_\Omega (S_g + \beta) u_{\varepsilon,\Omega}^2 \sqrt{\det(g)} dx < T.
\]

with the test functions given in (93).

The theorem will be proved in three cases: (i) \( n \geq 5 \); (ii) \( n = 4 \); (iii) \( n = 3 \).

**Proof.** Let \( \Omega \) be a small enough geodesic ball centered at 0 with radius \( r \), as discussed above. It is well known that [13 §5], [7 §3] on normal coordinates \( \{x_i\} \) within a small geodesic ball of radius \( r \), we have:
\[
g^{ij} = \delta^{ij} - \frac{1}{3} \sum_{r,s} R_{ijrs} x_r x_s + O(|x|^3);
\]
\[
\sqrt{\det(g)} = 1 - \frac{1}{6} \sum_{i,j} R_{ij} x_i x_j + O(|x|^3).
\]

Here \( R_{ij} \) are the coefficients of Ricci curvature tensors. In local expression of \( \sqrt{\det(g)} \), the term \( \sum_{i,j} R_{ij} x_i x_j = \text{Ric}(x,x) \). Since the Ricci curvature tensor is a symmetric bilinear form, there exists an orthonormal principal basis \( \{v_1, \ldots, v_n\} \) on tangent space of each point in the geodesic ball such that its corresponding eigenvalues \( \{\lambda_i\} \) satisfies 
\[
\sum_i \lambda_i = S_g(0).
\]
Let \( \{y_1, \ldots, y_n\} \) be the corresponding normal coordinates with respect to the principal basis, the Ricci curvature tensor is diagonalized at the center 0, thus in a very small ball we have

\[
R_{ij}(y) = \delta_{ij} R_{ij}(0) + o(1) = \delta_{ij} R_{ij}(0) + \sum_{|\alpha|=1} \partial^{\alpha} R_{ij}(0) y^{\alpha} + O(|y|^2)
\]

\[
\Rightarrow S_g(y) = g^{ij} = S_g(0) + \sum_{i,|\alpha|=1} \partial^{\alpha} R_{ii}(0) y^{\alpha} + O(|y|^2);
\]

\[
R_{ij}(y) = \delta_{ij} R_{ij}(0) + \sum_{|\alpha|=1} \partial^{\alpha} R_{ij}(0) y^{\alpha} + O(|y|^2)
\]

\[
\Rightarrow \sqrt{\det(g)} = 1 - \frac{1}{6} \sum_i \left( R_{ii}(0) + \sum_{|\alpha|=1} \partial^{\alpha} R_{ii}(0) y^{\alpha} \right) y_i^2 - \frac{1}{6} \sum_{i \neq j,|\alpha|=1} \partial^{\alpha} R_{ij}(0) y^{\alpha} y_i y_j + o(|y|^3).
\]

The smallness \( o(1) \) above only depends on the smallness of \( r \) and is no larger than \( O(r) \) due to the Taylor expansion of the smooth functions \( R_{ij}(y) \) near \( y = 0 \):

\[
R_{ij}(y) = R_{ij}(0) + \sum_{|\alpha|=1} \partial^{\alpha} R_{ij}(0) y^{\alpha} + O(|y|^2).
\]

Since \( \varphi_\Omega(y) = \varphi_\Omega(|y|) \) is radial; note also that in normal coordinates \( \{y_i\} \), we also have

\[
s^2 = |y|^2 \Rightarrow g^{ss} = 1 \Rightarrow |\nabla_g u_c, \Omega|^2 = |\partial_s u|^2.
\]

We apply (97) and have

\[
\|\nabla_g u\|^2_{L^2(\Omega, g)} = \int_\Omega |\nabla_g u|^2 \sqrt{\det(g)} dy
\]

\[
\leq \int_\Omega \left( 1 - \frac{1}{6} \sum_i \left( R_{ii}(0) + \sum_{|\alpha|=1} \partial^{\alpha} R_{ii}(0) y^{\alpha} \right) y_i^2 \right) |\partial_s u|^2 dy
\]

\[
- \int_\Omega \left( \frac{1}{6} \sum_{i \neq j,|\alpha|=1} \partial^{\alpha} R_{ij}(0) y^{\alpha} y_i y_j + o(|y|^3) \right) |\partial_s u|^2 dy
\]

\[
\leq \int_\Omega |\partial_s u|^2 dy - \frac{1}{6n} \int_\Omega S_g(0) |y|^2 |\partial_s u|^2 dy + \tilde{C}_1 \int_\Omega |y|^3 |\partial_s u|^2 dy
\]

\[
:= \|\partial_s u\|^2_{L^2(\Omega)} + A_1 + A_2;
\]

\[
\frac{1}{a} \int_\Omega (S_g + \beta) u^2 \sqrt{\det(g)} dx \leq \frac{1}{a} \int_\Omega (S_g(0) + \beta) |u|^2 dx + \frac{1}{a} \int_\Omega \sum_{i,|\alpha|=1} \partial^{\alpha} R_{ii}(0) y^{\alpha} |u|^2 dx
\]

\[
+ \tilde{C}_2 \int_\Omega |y|^2 |u|^2 dx + \tilde{C}_3 \int_\Omega |y|^3 |u|^2 dx
\]

\[
:= \frac{1}{a} \int_\Omega (S_g(0) + \beta) |u|^2 dx + B_1 + B_2 + B_3;
\]

\[
\|u\|_{L^p(\Omega, g)}^p = \int_\Omega |u|^p \sqrt{\det(g)} dx = \int_\Omega |u|^p dx - \frac{1}{6n} \int_\Omega S_g(0) |y|^2 |u|^p dy
\]

\[
+ \tilde{C}_4 \int_\Omega |y|^3 |u|^p dy := \|u\|_{L^p(\Omega)}^p + C_1 + C_2.
\]
Thus the terms \( \| \partial_s u \|_{L^2(\Omega)}^2 \), \( A_1 \), \( A_2 \) and \( A_3 \) can be estimated as

\[
\| \partial_s u \|_{L^2(\Omega)}^2 = \int_{\Omega} \frac{|\partial_s \varphi_\Omega|^2}{(\epsilon + |y|^2)^{n-2}} dy - 2(n - 2) \int_{\Omega} \frac{\varphi_\Omega (\partial_s \varphi_\Omega \cdot y)}{(\epsilon + |y|^2)^{n-1}} dy + (n - 2)^2 \int_{\Omega} \frac{\varphi_{\Omega}^2 |y|^2}{(\epsilon + |y|^2)^n} dy
\]

\[
= (n - 2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{\epsilon + |y|^2} dy + \int_{\Omega} \frac{|\partial_s \varphi_\Omega|^2}{(\epsilon + |y|^2)^n-2} dy - 2(n - 2) \int_{\Omega} \frac{\varphi_\Omega (\partial_s \varphi_\Omega \cdot y)}{(\epsilon + |y|^2)^{n-1}} dy
\]

\[
+ (n - 2)^2 \int_{\Omega} \frac{(\varphi_{\Omega}^2 - 1) |y|^2}{(\epsilon + |y|^2)^n} dy - (n - 2)^2 \int_{\mathbb{R}^n \setminus \Omega} \frac{|y|^2}{(\epsilon + |y|^2)^n} dy
\]

\[
:= (n - 2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(\epsilon + |y|^2)^n} dy + A_{0.1} + A_{0.2} + A_{0.3} + A_{0.4};
\]

\[
A_1 = -\frac{S_g(0)}{6n} \int_{\mathbb{R}^n} \frac{|y|^2 |\partial_s \varphi_\Omega|^2}{(\epsilon + |y|^2)^{n-2}} dy + \frac{S_g(0)(n - 2)}{3n} \int_{\Omega} \frac{\varphi_\Omega |y|^2 (\partial_s \varphi_\Omega \cdot y)}{(\epsilon + |y|^2)^{n-1}} dy
\]

\[
- \frac{S_g(0)(n - 2)^2}{6n} \int_{\mathbb{R}^n} \frac{\varphi_{\Omega}^2 |y|^4}{(\epsilon + |y|^2)^n} dy
\]

\[
:= A_{1.1} + A_{1.2} + A_{1.3};
\]

\[
A_2 = \tilde{C}_1 \int_{\Omega} |y|^3 |\partial_s u_{\epsilon, \Omega}|^2 dy = \tilde{C}_1 \int_{\Omega} \frac{|y|^3 |\partial_s \varphi_\Omega|^2}{(\epsilon + |y|^2)^{n-2}} dy - 2 \tilde{C}_1 (n - 2) \int_{\Omega} \frac{\varphi_\Omega |y|^3 (\partial_s \varphi_\Omega \cdot y)}{(\epsilon + |y|^2)^{n-1}} dy
\]

\[
+ \tilde{C}_1 (n - 2)^2 \int_{\Omega} \frac{\varphi_{\Omega}^2 |y|^5}{(\epsilon + |y|^2)^n} dy
\]

\[
:= A_{2.1} + A_{2.2} + A_{2.3}.
\]
The other two terms above can be estimated as
\[ \frac{1}{a} \int_{\Omega} (S_g(0) + \beta) u_{x,\Omega}^2 \, dx = \frac{1}{a} \int_{\Omega} (S_g(0) + \beta) \frac{\varphi^2}{(\epsilon + |y|^2)^{n-2}} \, dy = \frac{1}{a} \int_{\mathbb{R}^n} (S_g(0) + \beta) \frac{1}{(\epsilon + |y|^2)^{n-2}} \, dy \]
\[ - \frac{1}{a} \int_{\mathbb{R}^n \setminus \Omega} (S_g(0) + \beta) \frac{\varphi^2}{(\epsilon + |y|^2)^{n-2}} \, dy \]
\[ + \frac{1}{a} (S_g(0) + \beta) \int_{\Omega} \frac{\varphi^2}{(\epsilon + |y|^2)^{n-2}} \, dy \]
\[ =: \frac{1}{a} \int_{\mathbb{R}^n} (S_g(0) + \beta) \frac{1}{(\epsilon + |y|^2)^{n-2}} \, dy + B_{0,1} + B_{0,2}; \]

\[ ||u_{\epsilon,\Omega}||_{L^p(\Omega)}^p = \int_{\Omega} \frac{\varphi^p}{(\epsilon + |y|^2)^n} \, dy = \int_{\mathbb{R}^n} \frac{1}{(\epsilon + |y|^2)^n} \, dy - \int_{\mathbb{R}^n \setminus \Omega} \frac{1}{(\epsilon + |y|^2)^n} \, dy \]
\[ + \int_{\Omega} \frac{\varphi^p}{(\epsilon + |y|^2)^n} - \frac{1}{\epsilon} \, dy \]
\[ := \int_{\mathbb{R}^n} \frac{1}{(\epsilon + |y|^2)^n} \, dy + C_{0,1} + C_{0,2}. \]

**Case I:** \( n \geq 5 \). Fix \( r \) to be small enough, independent of the choice of \( \epsilon \). Claim that

\[ ||\partial_x u_{\epsilon,\Omega}||_{L^2(\Omega)}^2 + \frac{1}{a} \int_{\Omega} S_g(0) u_{x,\Omega}^2 \, dx + A_1 + A_2 + B_1 + B_2 + B_3 \]
\[ \leq (n - 2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(\epsilon + |y|^2)^n} \, dy + \frac{(n - 2)\beta}{4(n - 1)} \epsilon^{-\frac{2n}{n - 2}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy + O \left( \epsilon^{\frac{2n}{n - 2}} \right) \]
\[ - \epsilon^{\frac{2n}{n - 2}} \frac{S_g(0)(n - 2) n - 2}{6} \int_{B_{\epsilon^{\frac{1}{2}}}} \frac{|y|^2}{(1 + |y|^2)^n} \, dy \]
\[ = (n - 2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(\epsilon + |y|^2)^n} \, dy + \frac{(n - 2)\beta}{4(n - 1)} \epsilon^{-\frac{2n}{n - 2}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy + O \left( \epsilon^{\frac{2n}{n - 2}} \right) + \Sigma_2; \]

\[ ||u_{\epsilon,\Omega}||_{L^p(\Omega)}^p + C_1 + C_2 = \int_{\mathbb{R}^n} \frac{1}{(\epsilon + |y|^2)^n} \, dy + O \left( \epsilon^{\frac{2n}{n - 2}} \right) + \Sigma_1. \]

Recall that \( \beta < 0 \) is some fixed constant, independent of \( \epsilon \).
Let’s estimate the rest $B_i, C_i, A_{i,j}, B_{i,j}, C_{i,j}$ except the crucial term $A_{1,3}$. Let $\Omega' = B_0 \left( r e^{-\frac{t}{2}} \right)$.

Since $\varphi_\Omega \equiv 1$ on $B_0 \left( \frac{r}{2} \right)$, $|\partial_\nu \varphi_\Omega| \leq \frac{L}{r}$ for some fixed constant $L$, we observe that

$A_{0,1} = O(1), A_{0,2} = O(1), A_{0,3} \leq 0, A_{0,4} \leq 0$;

$A_{1,1} = O(1), |A_{1,2}| = O(1), A_{2,1} = O(1), |A_{2,2}| = O(1)$;

$A_{2,3} = \tilde{C}_1(n-2)^2 \int_{\Omega'} \frac{\varphi_\Omega^2 |y|^5}{(\epsilon + |y|^2)^n} dy = \epsilon^\frac{4-n}{2} \tilde{C}_1(n-2)^2 \int_{\Omega'} \frac{\varphi_\Omega^2 |y|^5}{(1 + |y|^2)^n} dy \leq L_1 \epsilon^\frac{4-n}{2}$

$\Rightarrow A_{2,3} = O \left( \epsilon^\frac{4-n}{2} \right)$;

$|B_1| \leq \frac{1}{a} \int_{\Omega} \sum_{i, |\alpha| = 1} \partial_\alpha R_{ii}(0) |y|^n |u|^2 dy \leq L_2 \int_{\Omega} \frac{|y|}{(\epsilon + |y|^2)^{n-2}} dy$

$\leq L_2 \epsilon^\frac{4-n}{2} \int_{\Omega'} \frac{|y|}{(1 + |y|^2)^{n-2}} dy \Rightarrow B_1 = O \left( \epsilon^\frac{4-n}{2} \right)$;

$B_2 = \tilde{C}_2 \int_{\Omega} \frac{\varphi_\Omega^2 |y|^2}{(\epsilon + |y|^2)^n} dy = \epsilon^\frac{4-n}{2} \tilde{C}_2 \int_{\Omega'} \frac{\varphi_\Omega^2 |y|^2}{(1 + |y|^2)^n} dy \leq L_3 \epsilon^\frac{4-n}{2}$

$\Rightarrow B_2 = O \left( \epsilon^\frac{4-n}{2} \right); B_3 \leq O \left( \epsilon^\frac{7-n}{2} \right)$;

$B_{0,1} = O(1), B_{0,2} = O(1); C_{0,1} = O(1), C_{0,2} = O(1)$;

$C_1 + C_2 = -\frac{1}{6n} \int_{\Omega} S_g(0) |y|^2 |u|^p dy + \tilde{C}_4 \int_{\Omega} |y|^3 |u|^p dy$

$= -\frac{S_g(0)}{6n} \int_{\Omega} \frac{|y|^2 \varphi_\Omega}{(\epsilon + |y|^2)^n} dy + O \left( \epsilon^\frac{3-n}{2} \right)$

$= -\epsilon^\frac{2-n}{2} \frac{S_g(0)}{6n} \int_{B_0 \left( \frac{r}{2} e^{-\frac{t}{2}} \right)} \frac{|y|^2}{(1 + |y|^2)^n} dy + O(1) + O \left( \epsilon^\frac{3-n}{2} \right)$

$:= \Sigma_1 + O \left( \epsilon^\frac{3-n}{2} \right)$.

Here $L_i, i = 1, \ldots, 3$ are constants independent of $r, \epsilon$.

For the crucial terms $A_{1,3}$, we show that

$A_{1,3} + \frac{1}{a} \int_{\mathbb{R}^n} (S_g(0) + \beta) \frac{1}{(\epsilon + |y|^2)^{n-2}} dy$

$\leq \beta \epsilon^\frac{4-n}{2} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} dy - \epsilon^\frac{4-n}{2} \frac{n}{6} S_g(0)(n-2)^2 \int_{B_0 \left( \frac{r}{2} e^{-\frac{t}{2}} \right)} \frac{|y|^2}{(1 + |y|^2)^n} dy$

$+ O(1)$

$:= \frac{(n-2)\beta}{4(n-1)} \epsilon^\frac{4-n}{2} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} dy + \Sigma_2 + O(1)$. (100)
The term $A_{1,3}$ can be bounded as

$$A_{1,3} = -\frac{S_g(0)(n-2)^2}{6n} \int_{\Omega} \frac{\varphi_\Omega^2 |y|^4}{(e + |y|^2)^n} dy = -\varepsilon \frac{S_g(0)(n-2)^2}{6n} \int_{\Omega} \frac{\varphi_\Omega^2 |y|^4}{(1 + |y|^2)^n} dy$$

$$= -\varepsilon \frac{S_g(0)(n-2)^2}{6n} \int_{\Omega} \frac{\varphi_\Omega^2 |y|^2 \left(|y|^2 - \frac{n}{n-2}\right)}{(1 + |y|^2)^n} dy - \varepsilon \frac{S_g(0)(n-2)^2}{6n} \int_{\Omega} \frac{\varphi_\Omega^2 |y|^2 \cdot \frac{n}{n-2}}{(1 + |y|^2)^n} dy$$

$$\leq -\varepsilon \frac{S_g(0)(n-2)^2}{12n} \int_{\mathbb{R}^n} |y|^2 \cdot \frac{2(n-2)^2 \left(|y|^2 - \frac{n}{n-2}\right)}{(1 + |y|^2)^n} dy$$

$$- \varepsilon \frac{S_g(0)(n-2)^2}{6n} \int_{\Omega} \varphi_\Omega^2 |y|^2 \cdot \frac{n}{n-2} dy := \Gamma_1 + \Gamma_2.$$

The above inequality holds since the radius of $\Omega'$ is much larger than 1, which follows that $|y|^2 - \frac{n}{n-2} > 0$ for $y \in \Omega'$, $\Omega' \in (1 + |y|^2)^{-1}$. It is direct to check that for Euclidean Laplacian $-\Delta$, the function $\frac{1}{(1 + |y|^2)^{n-2}}$ satisfies

$$\Delta \left( \frac{1}{(1 + |y|^2)^{n-2}} \right) = \frac{2(n-2)^2 \left(|y|^2 - \frac{n}{n-2}\right)}{(1 + |y|^2)^n}. $$

Using this, we estimate $\Gamma_1$ as

$$\Gamma_1 = -\varepsilon \frac{S_g(0)}{12n} \int_{\mathbb{R}^n} |y|^2 \cdot \frac{2(n-2)^2 \left(|y|^2 - \frac{n}{n-2}\right)}{(1 + |y|^2)^n} dy$$

$$= -\varepsilon \frac{S_g(0)}{12n} \int_{\mathbb{R}^n} |y|^2 \cdot \Delta \left( \frac{1}{(1 + |y|^2)^{n-2}} \right) dy = -\varepsilon \frac{S_g(0)}{12n} \int_{\mathbb{R}^n} \left( \Delta |y|^2 \right) \cdot \frac{1}{(1 + |y|^2)^{n-2}} dy$$

$$= -\varepsilon \frac{S_g(0)}{6} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} dy.$$

For $\Gamma_2$, we see that

$$\Gamma_2 = -\varepsilon \frac{S_g(0)(n-2)^2}{6n} \int_{\Omega'} \varphi_\Omega^2 |y|^2 \cdot \frac{n}{n-2} dy$$

$$= -\varepsilon \frac{S_g(0)(n-2)^2}{6} \int_{B_0 \left( \frac{r_0}{2} \right)} \frac{|y|^2}{(1 + |y|^2)^n} dy - \varepsilon \frac{S_g(0)(n-2)^2}{6} \int_{\Omega' \setminus B_0 \left( \frac{r_0}{2} \right)} \frac{\varphi_\Omega^2 |y|^2}{(1 + |y|^2)^n} dy$$

$$= -\varepsilon \frac{S_g(0)(n-2)^2}{6} \cdot \frac{n-2}{n} \int_{B_0 \left( \frac{r_0}{2} \right)} \frac{|y|^2}{(1 + |y|^2)^n} dy$$

$$- \varepsilon \frac{S_g(0)(n-2)^2}{6} \cdot \frac{2}{n} \int_{B_0 \left( \frac{r_0}{2} \right)} \frac{|y|^2}{(1 + |y|^2)^n} dy + O(1)$$

$$< -\varepsilon \frac{S_g(0)(n-2)^2}{6} \cdot \frac{n-2}{n} \int_{B_0 \left( \frac{r_0}{2} \right)} \frac{|y|^2}{(1 + |y|^2)^n} dy$$

$$- \varepsilon \frac{S_g(0)(n-2)^2}{6} \cdot \frac{2(n-2)}{n} \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} dy + O(1)$$

$$:= \Sigma_2 + \Gamma_{2,1} + O(1).$$
Consider the integration \( \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} \, dy \) in \( \Gamma_{2,1} \). Recall that \( \omega_n \) is the area of the unit \( n-1 \)-sphere. We check with change of variables \( s = \tan(\theta) \) in some middle step that

\[
\int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} \, dy = \omega_n \int_0^\infty \frac{s^{n+1}}{(1 + s^2)^n} ds = \omega_n \int_0^{\pi/2} \frac{\tan^{n+1}(\theta) \sec^2(\theta)}{(1 + \tan^2(\theta))^n} \, d\theta
\]

\[
= \omega_n \int_0^{\pi/2} \sin^2(\frac{n+1}{2}) \cos^{n-1}(\frac{n-1}{2}) \, d\theta = \omega_n B\left(\frac{n}{2} + 1, \frac{n}{2} - 1\right)
\]

Here \( B(a,b) \) is the Beta function with parameters \( a, b \) and \( \Gamma(a) \) is the Gamma function with parameter \( a \). The term below satisfies

\[
\frac{1}{a} \int_{\mathbb{R}^n} S_g(0) \frac{1}{(\epsilon + |y|^2)^{n-2}} \, dy = \epsilon^{4-n} \frac{1}{a} \cdot S_g(0) \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy.
\]

Check the integration \( \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} \, dy \) above, we have

\[
\int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} \, dy = \omega_n B\left(\frac{n}{2}, \frac{n}{2} - 2\right) = \omega_n \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n}{2} - 2\right)}{\Gamma(n - 2)}.
\]

By the standard relation \( \Gamma(z + 1) = z\Gamma(z) \) we conclude that

\[
\int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y|^2)^n} \, dy = \frac{n(n-4)}{4(n-1)(n-2)} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy.
\]  

Note that \( \frac{1}{a} = \frac{n-2}{4(n-1)} \). Applying (101) as well as the estimates of \( \Gamma_1 \) and \( \Gamma_2 \) above, we conclude that

\[
A_{1,3} + \frac{1}{a} (S_g(0) + \beta) \int_{\mathbb{R}^n} \frac{1}{(\epsilon + |y|^2)^{n-2}} \, dy \leq \Gamma_1 + \Gamma_2 + \epsilon^{4-n} \frac{1}{a} \cdot (S_g(0) + \beta) \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy
\]

\[
= -\frac{1}{6} \epsilon^{4-n} S_g(0) \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy + \Sigma_2
\]

\[
+ \frac{n(n-4)}{4(n-1)(n-2)} \left( -\frac{4-n}{6} \cdot S_g(0) + \frac{2(n-2)}{n} \right) \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy
\]

\[
+ \epsilon^{4-n} \frac{n-2}{4(n-1)} \cdot (S_g(0) + \beta) \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy + O(1)
\]

\[
= \frac{(n-2)\beta}{4(n-1)} \epsilon^{4-n} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{n-2}} \, dy + \Sigma_2 + O(1).
\]
It follows that (100) holds. Therefore we conclude that the estimates in (99) hold by combining all estimates above together. Recall $K_1, K_2$ defined above, we apply estimates in (99), we get

\[
Q_{\epsilon, \Omega} \leq \frac{(n-2)^2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy + \frac{(n-2)\beta}{4(n-1)} \epsilon^{\frac{4-n}{2}} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{n-2}} dy + O\left(\epsilon^{\frac{5-n}{2}}\right) + \Sigma_2}{(n-2)^2 \epsilon^{\frac{2-n}{2}} \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^n} dy + \frac{(n-2)\beta}{4(n-1)} \epsilon^{\frac{4-n}{2}} \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{n-2}} dy + O\left(\epsilon^{\frac{5-n}{2}}\right) + \Sigma_2} = \frac{K_1 + \frac{(n-2)\beta}{4(n-1)} K_3 + O\left(\epsilon^{\frac{3}{2}}\right) + \epsilon^{\frac{n-2}{2}} \Sigma_2}{K_2 + O\left(\epsilon^{\frac{3}{2}}\right) + \frac{n-2}{n} \epsilon^{\frac{n-2}{2}} \Sigma_1 K_2^{-\frac{n-2}{n}}}.
\]

The last term above is due to the Taylor expansion of the function $f(x) = x^\frac{2}{n}$. To control the last terms in numerator and denominator above, respectively, we first recall that the ratio $\frac{K_1}{K_2}$ is the square of the reciprocal of the best Sobolev constant of

\[
\|u\|_{L_\rho(\mathbb{R}^n)} \leq E\|Du\|_{L^2(\mathbb{R}^n)} \Rightarrow \inf_E E^{-2} = T = \frac{K_1}{K_2}.
\]

Aubin and Talenti \cite{2} showed that

\[
T^{-\frac{3}{2}} = \pi^{-\frac{1}{2}} n^{-\frac{1}{2}} \left(\frac{1}{n-2}\right)^{\frac{3}{2}} \left(\Gamma(n)\right)^{\frac{1}{2}} \left(\Gamma\left(\frac{n}{2}\right)\right)^{-\frac{1}{2}}.
\]

Recall that $\beta < 0$, independent of $\epsilon$, thus $Q_{\epsilon, \Omega}$ can be estimated as

\[
Q_{\epsilon, \Omega} \leq \frac{K_1 + \frac{(n-2)\beta}{4(n-1)} K_3 + O\left(\epsilon^{\frac{3}{2}}\right) + \epsilon^{\frac{n-2}{2}} \Sigma_2}{K_2 + O\left(\epsilon^{\frac{3}{2}}\right) + \frac{n-2}{n} \epsilon^{\frac{n-2}{2}} \Sigma_1 K_2^{-\frac{n-2}{n}}} \leq \frac{K_1 + \epsilon^{\frac{n-2}{2}} \Sigma_2}{K_2 + \frac{n-2}{n} \epsilon^{\frac{n-2}{2}} \Sigma_1 K_2^{-\frac{n-2}{n}}} + O\left(\epsilon^{\frac{3}{2}}\right) - \epsilon \frac{(n-2)\beta}{4(n-1)} K_3 K_2^{-\frac{n-2}{n}}.
\]

The last inequality is due to the fact that both $\epsilon^{\frac{n-2}{2}} \Sigma_1 K_2^{-\frac{n-2}{n}}$ and $\epsilon^{\frac{n-2}{2}} \Sigma_2$ are bounded above and below by constant multiples of $\epsilon$. Lastly we must show that

\[
\frac{K_1 + \epsilon^{\frac{n-2}{2}} \Sigma_2}{K_2 + \frac{n-2}{n} \epsilon^{\frac{n-2}{2}} \Sigma_1 K_2^{-\frac{n-2}{n}}} \leq \frac{K_1}{K_2}.
\]

(102)

By the best Sobolev constant, we observe that

\[
\frac{K_1}{K_2} = T \Rightarrow K_1 = K_2 T.
\]
Thus (102) is equivalent to

\[
\frac{K_1 + \epsilon^{\frac{n-2}{2}} \Sigma_2}{K_2 + \frac{n-2}{n} \epsilon^2 \Sigma_1 K_2^{-\frac{n-2}{2}}} \leq \frac{K_1 + \epsilon^{\frac{n-2}{2}} \Sigma_2}{K_2 + \frac{n-2}{n} \epsilon^2 \Sigma_1 K_2^{-\frac{n-2}{2}}} - K_1 \leq 0
\]

\[
\frac{K_2^{-\frac{n-2}{2}} + \frac{n-2}{n} \epsilon^2 \Sigma_1 K_2^{-\frac{n-2}{2}}}{K_2} \leq \frac{K_2^{-\frac{n-2}{2}} + \frac{n-2}{n} \epsilon^2 \Sigma_1 K_2^{-\frac{n-2}{2}}}{K_2} - K_1 \leq 0
\]

\[
\frac{K_2^{-\frac{n-2}{2}} + \frac{n-2}{n} \epsilon^2 \Sigma_1 K_2^{-\frac{n-2}{2}}}{K_2} \leq 0
\]

\[
\frac{K_2^{-\frac{n-2}{2}} + \frac{n-2}{n} \epsilon^2 \Sigma_1 K_2^{-\frac{n-2}{2}}}{K_2} \leq 0
\]

\[
\frac{(n-2) - \frac{T}{n}}{K_2^{-\frac{n-2}{2}}} \leq 0.
\]

Recall that the best Sobolev constant \(T\) is of the form

\[
T = \pi n (n-2) \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma(n)} \right)^{\frac{2}{n}}
\]

For the term \(K_2^{-\frac{n-2}{2}}\), we have

\[
K_2^{-\frac{n-2}{2}} = \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^\frac{n}{2}} dy \right) = \left( \omega_n \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^\frac{n}{2}} dr \right)^{-\frac{2}{n}} = \omega_n^{-\frac{2}{n}} = \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( 2n/2 \right)} \right)^{\frac{2}{n}} = \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( n/2 \right)} \right)^{\frac{2}{n}} 2^{-\frac{n}{2}} \pi^{-1}.
\]

For \(D_1\), we use change of variables \(r = \tan \theta, \theta \in [0, \frac{\pi}{2}]\), we have

\[
D_1 = \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^\frac{n}{2}} dr = \int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}(\theta)}{(1 + \tan^2(\theta))^\frac{n}{2}} \sec^2(\theta) d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^{n-1}(\theta) \sec^{n+1}(\theta)}{\sec^{2n}(\theta)} d\theta = \int_0^{\frac{\pi}{2}} \sin^{n-1}(\theta) \cos^{n-1}(\theta) d\theta = 2^{1-n} \int_0^{\frac{\pi}{2}} \sin^{n-1}(2\theta) d\theta = 2^{-n} \int_0^{\pi} \sin^{n-1}(x) dx.
\]

We use \(2\theta = x\) in the last equality. It is well-known that the last integration above is symmetric with respect to \(x = \frac{\pi}{2}\) and is connected to the Gamma function. To be precise,

\[
D_1 = 2^{-n} \int_0^{\pi} \sin^{n-1}(x) dx = 2^{1-n} \int_0^{\frac{\pi}{2}} \sin^{n-1}(x) dx = 2^{1-n} \sqrt{n \pi} \left( \frac{n-1}{2} + 1 \right) \Gamma \left( \frac{n-1}{2} + 1 \right) \]

\[
\Rightarrow D_1^{-\frac{2}{n}} = 2^n \pi^{\frac{n-1}{2}} \Gamma \left( \frac{n}{2} + 1 \right)^{\frac{2}{n}} = 2^n \pi^{n-1} \Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n}{2} \right)^{-\frac{2}{n}}.
\]
Packing terms together, we have

\[ \frac{TK_2^{n/2}}{n} = n^{-1} \pi n(n - 2) \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma(n)} \right)^{\frac{2}{n}} \cdot \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \right)^{-\frac{2}{n}} 2^{-\frac{2}{n}} \pi^{-1} \]

\[ \cdot 2^{\frac{2}{n}} \pi^{-\frac{1}{n}} \left( \Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n}{2} \right) \right)^{\frac{2}{n}} \Gamma(n)^{-\frac{2}{n}} \]

\[ = (n - 2)2^{\frac{2}{n}} \pi^{-\frac{1}{n}} \left( \Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n}{2} \right) \right)^{\frac{2}{n}} \Gamma(n)^{-\frac{2}{n}} \]

\[ = (n - 2). \]

Hence we conclude that

\[ (n - 2) - \frac{TK_2^{n/2}}{n} = (n - 2) - (n - 2) = 0 \Rightarrow \frac{K_1 + \epsilon^{\frac{n+3}{2}} \Sigma_2}{K_2 + \frac{n-2}{n} \epsilon \Sigma_1 K_2^{-\frac{2}{n}}} \leq \frac{K_1}{K_2}. \]

Therefore (102) holds. It follows that when \( \epsilon \) is small enough,

\[ Q_{\epsilon, \Omega} \leq \frac{K_1}{K_2} \left( \epsilon^{\frac{3}{2}} \right) - \epsilon \cdot \frac{(n - 2)\beta}{4(n - 1)} \frac{K_3}{K_2} < \frac{K_1}{K_2} = T. \]

Thus the inequality in (35) holds when \( n \geq 5 \).

**Case II:** \( n = 4 \). In this case, we keep notations to be the same as \( n \geq 5 \) cases. Again we fix \( r \).

Most estimates of \( A_{i,j}, B_i, B_{i,i}, C_i, C_{i,j} \) with \( n = 4 \) are the same as above:

- \( A_{0,1} = O(1), A_{0,2} = O(1), A_{0,3} \leq 0, A_{0,4} \leq 0; \)
- \( A_{1,1} = O(1), |A_{1,2}| = O(1), A_{2,1} = O(1), |A_{2,2}| = O(1); A_{2,3} = O \left( \epsilon^\frac{1}{2} \right); \)
- \( |B_1| = O \left( \epsilon^{\frac{1}{2}} \right); B_2 = O(\epsilon); B_3 = O \left( \epsilon^{\frac{3}{2}} \right); \)
- \( C_{0,1} = O(1), C_{0,2} = O(1); C_1 \geq 0, C_2 = O \left( \epsilon^{-\frac{1}{2}} \right). \)

Here \( C_1 > 0 \) due to the fact that \( S_g(0) < 0 \). We do not need \( B_{0,1} \) and \( B_{0,2} \) since the key term

\[ \frac{1}{a} \cdot (S_g(0) + \beta) \int_\Omega \frac{\varphi_\Omega^2}{(\epsilon + |y|)^2} dy \]

is not integrable. Note that the term \( A_{1,3} \) with \( n = 4 \) can be estimated as

\[ A_{1,3} = -\frac{S_g(0)(n - 2)^2}{6n} \int_\Omega \frac{\varphi_\Omega^2 |y|^4}{(\epsilon + |y|)^2} dy = -\frac{S_g(0)}{6} \int_\Omega \frac{\varphi_\Omega^2 |y|^4}{(\epsilon + |y|^2)^2} dy \]

\[ \leq -\frac{S_g(0)}{6} \int_\Omega \frac{\varphi_\Omega^2}{(\epsilon + |y|^2)^2} dy \]
since the ratio \( \frac{|y|^2}{\epsilon + |y|^2} < 1 \) as always. Thus with \( a = \frac{n-2}{4(n-1)} = \frac{1}{6} \) when \( n = 4 \), we can see
\[
A_{1,3} + \frac{1}{a} \cdot (S_g(0) + \beta) \int_\Omega \frac{\varphi_\Omega^2}{(\varphi_\Omega^2)^2} dy \leq \frac{\beta}{6} \int_\Omega \frac{\varphi_\Omega^2}{(\varphi_\Omega^2)^2} dy.
\]
Applying exactly the same calculation as in [5, Lemma 1.1], we conclude that
\[
\int_\Omega \frac{\varphi_\Omega^2}{(\varphi_\Omega^2)^2} dy \geq L_5 |\log \epsilon|
\]
with some positive constant \( L_5 \) independent of \( \epsilon \). It follows that
\[
A_{1,3} + \frac{1}{a} \cdot (S_g(0) + \beta) \int_\Omega \frac{\varphi_\Omega^2}{(\varphi_\Omega^2)^2} dy \leq -L_5 |\beta||\log \epsilon|.
\] (103)
Applying (103) and all other estimates above, we have
\[
Q_{\epsilon, \Omega} \leq \frac{4 \int_{\mathbb{R}^n} \frac{|y|^2}{(\varphi_\Omega^2)^2} dy - L_5 |\beta||\log \epsilon| + O(1) + O \left( \epsilon^{\frac{1}{2}} \right)}{\left( \int_{\mathbb{R}^n} \frac{1}{(\varphi_\Omega^2)^2} dy + O(1) + O \left( \epsilon^{-\frac{1}{2}} \right) \right)^{\frac{1}{2}}}
\]
\[
\leq \frac{4 \epsilon^{-1} \int_{\mathbb{R}^n} \frac{|y|^2}{(\varphi_\Omega^2)^2} dy - L_5 |\beta||\log \epsilon| + O(1)}{\left( \epsilon^{-2} \int_{\mathbb{R}^n} \frac{1}{(\varphi_\Omega^2)^2} dy + O \left( \epsilon^{-1} \right) \right)^{\frac{1}{2}}}
\]
\[
\leq \frac{K_1 - L_5 |\beta||\log \epsilon| + O \left( \epsilon^{\frac{3}{2}} \right)}{K_2 + O \left( \epsilon^{\frac{3}{2}} \right)}
\]
\[
= \frac{K_1}{K_2} + O \left( \epsilon \right) - \frac{L_5 |\beta|}{K_2} |\log \epsilon|.
\]
It follows that when \( \epsilon \) small enough, we have
\[
Q_{\epsilon, \Omega} < T.
\]
Thus (95) holds for \( n = 4 \).

**Case III :** \( n = 3 \). This case is inspired by the argument in Brezis and Nirenberg [5, Lemma 1.3], but not the same. We have to do some conformal change so that
\[
\frac{S_g(0)}{8} + \frac{\pi^2}{4} = 0 \text{ on } B_0(1).
\] (104)
This can be done due to Theorem 4.5 and Remark 4.7. Thus
\[
\frac{S_g(0)}{8} + \frac{\pi^2}{4} < 0 \text{ on } B_0(1)
\]
since \( \beta < 0 \). Note that the first eigenvalue \( \lambda_1 \) of \(-\Delta_g\) is almost the same as the first eigenvalue of Euclidean Laplacian when the geodesic ball is small enough. Since \( S_g \lambda_1^{-1} \) is scaling invariant, we can scale the metric, i.e. scale the radius of the ball without changing the relation in (104) as well as the relation in (10) with a small enough \( \beta \). Thus it suffices to set \( \Omega = B_0(1) \). We would like to point out that the choice of \( S_g(0) \) in (104) is compatible with the choice of \( S_g(0) \) in (10). We estimate \( A_1 \) first. Recall that
\[
A_1 = -\frac{S_g(0)}{18} \int_\Omega \frac{|y|^2}{\epsilon + |y|^2} \varphi_\Omega dy + \frac{S_g(0)}{9} \int_\Omega \frac{\varphi_\Omega |y|^2 (\partial_\varphi \Omega \cdot y)}{(\varphi_\Omega^2)^2} dy - \frac{S_g(0)}{18} \int_\Omega \frac{\varphi_\Omega^2 |y|^4}{(\varphi_\Omega^2)^2} dy
\]
\[
:= A_{1,1} + A_{1,2} + A_{1,3}.
\]
For $A_{1,1}$, we have

$$A_{1,1} = -\frac{S_g(0)}{18} \int_{\Omega} \frac{|y|^2 |\partial_s \varphi_1|^2}{\epsilon + |y|^2} dy = \frac{\pi^2}{9} \omega_3 \cdot \frac{\pi^2}{4} \int_0^1 \sin^2 \left(\frac{\pi s}{2}\right) s^4 ds \leq \frac{\pi^2}{9} \omega_3 \cdot \frac{\pi^2}{4} \int_0^1 \sin^2 \left(\frac{\pi s}{2}\right) s^2 ds$$

$$= \omega_3 \cdot \frac{\pi^4}{36} \left(\frac{1}{6} + \frac{1}{\pi^2}\right).$$

For $A_{1,2}$, we have

$$A_{1,2} = \frac{S_g(0)}{9} \int_{\Omega} \frac{\varphi_1|y|^2 (\partial_s \varphi_1 \cdot y)}{(\epsilon + |y|^2)^2} dy = 2 \frac{\pi^2}{9} \omega_3 \int_0^1 \frac{(\sin(\pi s))^3}{(\epsilon + s^2)^2} ds \leq \frac{2 \pi^2}{9} \omega_3 \int_0^1 \sin(\pi s) ds$$

$$= \omega_3 \frac{\pi^3}{18} \cdot \frac{1}{\pi} = \omega_3 \frac{\pi^2}{18}.$$

For $A_{1,3}$, we have

$$A_{1,3} = -\frac{S_g(0)}{18} \int_{\Omega} \frac{\varphi_1 |y|}{(\epsilon + |y|^2)^3} dy = \frac{\pi^2}{9} \omega_3 \int_0^1 \frac{\cos^2 \left(\frac{\pi s}{2}\right)}{(\epsilon + s^2)^3} ds \leq \frac{\pi^2}{9} \omega_3 \int_0^1 \cos^2 \left(\frac{\pi s}{2}\right) ds$$

$$= \omega_3 \frac{\pi^2}{18}.$$

Here $L_6 > 0$ is some constant independent of $\epsilon, r$. For $A_2, C_1, C_2, C_{0,1}, C_{0,2}$, a simple estimate shows that

$$A_2 = O \left(\frac{\epsilon^3}{2}\right), C_1 > 0, C_2 = O(1), C_{0,1} = O(1), C_{0,2} = O(1).$$

Based on Brezis and Nirenberg [5], the term $\|\partial_s u\|_{L^2(\Omega)}^2$ can be written as

$$\|\partial_s u\|_{L^2(\Omega)}^2 = \omega_3 \frac{\pi^2}{4} \int_0^1 \frac{\sin^2 \left(\frac{\pi s}{2}\right)}{\epsilon + s^2} ds + 3 \omega_3 \epsilon \int_0^1 \frac{\cos^2 \left(\frac{\pi s}{2}\right)}{(\epsilon + s^2)^3} ds := E_1 + E_2.$$

Due to Brezis and Nirenberg again, we have

$$E_1 = \omega_3 \frac{\pi^2}{4} \int_0^1 \sin^2 \left(\frac{\pi s}{2}\right) ds + O(\epsilon).$$

For the term $E_2$, we estimate as

$$E_2 = 3 \omega_3 \epsilon \int_0^1 \frac{\cos^2 \left(\frac{\pi s}{2}\right)}{(\epsilon + s^2)^3} s^2 ds$$

$$= 3 \omega_3 \epsilon \int_0^1 \frac{s^2}{(\epsilon + s^2)^3} ds - 3 \omega_3 \epsilon \int_1^\infty \frac{s^2}{(\epsilon + s^2)^3} ds - 3 \omega_3 \epsilon \int_0^1 \frac{\sin^2 \left(\frac{\pi s}{2}\right)}{(\epsilon + s^2)^3} ds$$

$$:= 3 \omega_3 \epsilon \int_0^\infty \frac{s^2}{(\epsilon + s^2)^3} ds + E_{2,1} + E_{2,2}.$$

Using polar coordinates $s = \frac{1}{r^2} \tan(\theta)$, the term $E_{2,1}$ can be computed as

$$E_{2,1} = -3 \omega_3 \epsilon \int_1^\infty \frac{s^2}{\epsilon + s^2} ds = -3 \omega_3 \epsilon^\frac{1}{2} \int_\arctan(\epsilon^{-\frac{1}{2}})^{\frac{\pi}{4}} \frac{\tan^2(\theta) \sec^2(\theta)}{\sec^6(\theta)} d\theta$$

$$=-3 \omega_3 \epsilon^\frac{1}{2} \int_\arctan(\epsilon^{-\frac{1}{2}})^{\frac{\pi}{4}} \frac{1}{4} d\theta + o(1)$$

$$= -\frac{3}{4} \omega_3 \epsilon^{-\frac{1}{2}} \left(\frac{\pi}{2} - \arctan(\epsilon^{-\frac{1}{2}})\right) + o(1) = -\frac{3}{4} \omega_3 + o(1).$$
The last inequality is due to the Taylor expansion of \( \arctan(x) \) at \( x = +\infty \), which is
\[
\arctan(x) \bigg|_{x=+\infty} = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right).
\]
Again by Brezis and Nirenberg, we know that
\[
\frac{3\pi}{16} = \int_0^\infty \frac{s^4}{(1 + s^2)^3} ds = 3 \int_0^\infty \frac{s^2}{(1 + s^2)^3} ds.
\] (105)
Thus with (105), the Taylor expansion of \( \sin(x) \) at origin, we have
\[
E_{2,2} = -3\omega_3 \epsilon \int_0^1 \sin^2 \left(\frac{\pi s}{\epsilon + s^2}\right) s^2 ds = -3\omega_3 \epsilon \frac{\pi^2}{4} \int_0^1 \frac{s^4}{(\epsilon + s^2)^3} ds + O(\epsilon)
\]
\[
\geq -\frac{9\pi^3}{64} \omega_3 + O(\epsilon).
\]
By a direct calculation, it follows that
\[
A_1 + E_{2,1} + E_{2,2} \leq O(\epsilon).
\]
By (105), we can see that
\[
3\omega_3 \epsilon \int_0^\infty \frac{s^2}{(\epsilon + s^2)^3} ds = 3\omega_3 \epsilon \frac{1}{2} \int_0^\infty \frac{s^2}{(1 + s^2)^3} ds = \omega_3 \epsilon \frac{1}{2} \int_0^\infty \frac{s^4}{(1 + s^2)^3} ds = \epsilon \frac{1}{2} K_1.
\]
Exactly following Brezis and Nirenberg \[5, \text{Lemma 1.3}\], we have
\[
\|u\|_{C^0(\Omega)}^2 = \epsilon \frac{1}{2} K_2 + O \left(\epsilon^\frac{1}{2}\right);
\]
\[
\frac{S_g(0) + \beta}{8} \int_\Omega \frac{\varphi_\Omega^2(y)}{\epsilon + |y|^2} dy = \frac{S_g(0) + \beta}{8} - \omega_3 \epsilon \int_0^1 \cos^2 \left(\frac{\pi s}{2}\right) ds + O \left(\epsilon^\frac{1}{2}\right).
\]
Lastly, note that
\[
\int_0^1 \cos^2 \left(\frac{\pi s}{2}\right) ds = \int_0^1 \sin^2 \left(\frac{\pi s}{2}\right) ds.
\]
Thus the quantity \( Q_{\epsilon, \Omega} \) can be estimated as
\[
Q_{\epsilon, \Omega} \leq \frac{\epsilon \frac{1}{2} K_1 + \omega_3 \int_0^1 \cos^2 \left(\frac{\pi s}{2}\right) ds \left(\frac{\pi^2}{4} + \frac{S_g(0) + \beta}{8}\right) + O \left(\epsilon^\frac{1}{2}\right)}{\epsilon \frac{1}{2} K_2 + O \left(\epsilon^\frac{1}{2}\right)}
\]
\[
\leq \frac{K_1}{K_2} + \frac{\epsilon \frac{1}{2} \omega_3 \int_0^1 \cos^2 \left(\frac{\pi s}{2}\right) ds \left(\frac{\pi^2}{4} + \frac{S_g(0) + \beta}{8}\right) + O(\epsilon)}{K_2}.
\]
By (104), we know that \( \frac{\pi^2}{4} + \frac{S_g(0) + \beta}{8} < 0 \) hence
\[
Q_{\epsilon, \Omega} < T
\]
When \( \epsilon \) is small enough. All cases are now proven.
\[\square\]

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