Recent progress in isoparametric functions and isoparametric hypersurfaces

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Abstract This paper gives a survey of recent progress in isoparametric functions and isoparametric hypersurfaces, mainly in two directions.

(1) Isoparametric functions on Riemannian manifolds, including exotic spheres. The existences and non-existences will be considered.

(2) The Yau conjecture on the first eigenvalues of the embedded minimal hypersurfaces in the unit spheres. The history and progress of the Yau conjecture on minimal isoparametric hypersurfaces will be stated.

1 Introduction

E. Cartan was the pioneer who made a comprehensive study of isoparametric functions (hypersurfaces) on the unit spheres. In the past decades, the study of isoparametric functions (hypersurfaces) has become a highly influential field in differential geometry. For a systematic and complete survey of isoparametric functions (hypersurfaces) and their generalizations, we recommend [43] and [7]. Very recently, Cecil-Chi-Jensen, Immervoll and Chi obtained classification results for isoparametric hypersurfaces with four distinct principal curvatures in the unit spheres, except for one case (c.f. [8], [22] and [10]). As for that with six distinct principal curvatures, Miyaoka showed the homogeneity and hence the classification (c.f. [28]).

The note is organized as follows. In Section 2, we first recall some basic notations and fundamental theory of isoparametric functions on Riemannian manifolds. Next
we introduce exotic spheres and investigate the existences and non-existences of isoparametric functions on exotic spheres. Section 3 will be concerned with the progress of the well known Yau conjecture on the first eigenvalues of embedded minimal hypersurfaces in the unit spheres, especially on the minimal isoparametric case (being isoparametric implies embedding). Moreover, the first eigenvalues of the focal submanifolds are also taken into account. In the end, related topics and applications are described in Section 4.

2 Exotic spheres and isoparametric functions

We start with definitions. Let \( N \) be a connected complete Riemannian manifold. A non-constant smooth function \( f : N \to \mathbb{R} \) is called transnormal if there is a smooth function \( b : \mathbb{R} \to \mathbb{R} \) such that

\[
|\nabla f|^2 = b(f),
\]

where \( \nabla f \) is the gradient of \( f \). If moreover there is a continuous function \( a : \mathbb{R} \to \mathbb{R} \) such that

\[
\Delta f = a(f),
\]

where \( \Delta f \) is the Laplacian of \( f \), then \( f \) is called isoparametric (cf. [44]). Each regular level hypersurface is called an isoparametric hypersurface. The two equations of the function \( f \) mean that regular level hypersurfaces are parallel and have constant mean curvatures. According to Wang [44], a transnormal function \( f \) on a complete Riemannian manifold has no critical value in the interior of \( \text{Im} f \). The preimage of the maximum (resp. minimum), if it exists, of an isoparametric (or transnormal) function \( f \) is called the focal set of \( f \), denoted by \( M_+ \) (resp. \( M_- \)).

Since the work of Cartan ([5], [6]) and Münzner ([32]), the subject of isoparametric hypersurfaces in the unit spheres is rather fascinating to geometers. We refer to [9] for the development of this subject. Up to now, the classification has almost been completed as mentioned in Section 1.

In general Riemannian manifolds, the classification problem is far from being touched. Wang [44] firstly took up a systematic study of isoparametric functions on general Riemannian manifolds, and similar to the case in a unit sphere, proved or claimed a series of beautiful results. The structural result for transnormal functions is stated as follows.

**Theorem 1.** ([44]) Let \( N \) be a connected complete Riemannian manifold and \( f \) a transnormal function on \( N \). Then

- The focal sets of \( f \) are smooth submanifolds (may be disconnected) of \( N \);
- Each regular level set of \( f \) is a tube over either of the focal sets (the dimensions of the fibers may differ on different connected components).

The above theorem shows that the existence of a transnormal function on a Riemannian manifold \( N \) restricts strongly its topology.
In the first part of \[16\], Ge and Tang improved the fundamental theory of isoparametric functions on Riemannian manifolds. Given a transnormal function \( f : N \rightarrow \mathbb{R} \), we denote by \( C_1(f) \) the set where \( f \) attains its global maximum value or global minimum value, by \( C_2(f) \) the union of singular level sets of \( f \), i.e., \( C_2(f) = \{ p \in N | \nabla f(p) = 0 \} \), and for any regular value \( t \) of \( f \), by \( C_3^t(f) \) the focal set of the level hypersurface \( M_t := f^{-1}(t) \), i.e., the set of singular values of the normal exponential map. From \[44\], it follows that \( C_1(f) = C_2(f) = M_{-} \cup M_{+} \), and for any two regular level hypersurfaces which will be thus denoted simply by \( C_3(f) \). Moreover, one can see that \( C_3(f) \subseteq C_1(f) = C_2(f) \). Then Ge and Tang proved

**Theorem 2. (16)** Each component of \( M_\pm \) has codimension not less than 2 if and only if \( C_3(f) = C_1(f) = C_2(f) \). Moreover in this case, each level set \( M_t \) is connected. If in addition \( N \) is closed and \( f \) is isoparametric, then at least one isoparametric hypersurface is minimal in \( N \).

Indeed, there exists example of an isoparametric function \( f \) satisfying \( C_3(f) \supsetneq C_1(f) = C_2(f) \) (c.f. \[16\]). For this case, the focal sets of the isoparametric function are not really focal sets of the level hypersurface. Hence, in \[16\], a transnormal (isoparametric) function \( f \) is called *proper* if the focal sets have codimension not less than 2. It seems that a properly transnormal (isoparametric) function is exactly what we should concern in geometry. Furthermore, in \[15\], they observed three elegant ways to construct examples of isoparametric functions, i.e.,

- For a Riemannian manifold \((N, ds^2)\) with an isoparametric function \( f \), take a special conformal deformation \( \tilde{ds}^2 = e^{2u(f)} ds^2 \). Then \( f \) is also isoparametric on \((N, \tilde{ds}^2)\);
- For a cohomogeneity one manifold \((N, G)\) with a \( G \)-invariant metric, one can get isoparametric functions on \( N \);
- For a Riemannian submersion \( \pi : E \rightarrow B \) with minimal fibers, if \( f \) is an isoparametric function on \( B \), then so is \( F := f \circ \pi \) on \( E \).

Applying these methods, interesting results and abundant examples are acquired, especially, isoparametric functions on Brieskorn varieties and on isoparametric hypersurfaces of spheres are obtained.

As a continuation of \[16\], they made new contributions in \[17\]. First, for a properly isoparametric function, they proved that at least one isoparametric hypersurface is minimal if the ambient space \( N \) is closed in Theorem. By using the Riccati equation, they can further show that such a minimal isoparametric hypersurface is also unique if \( N \) has positive Ricci curvature. Next, by expressing the shape operator \( S(t) \) of \( M_t \) as a power series, they gave a complete proof to Theorem D of \[44\] (no proof there; compare with \[33\] and \[27\]).

**Theorem 3. (17)** The focal sets \( M_\pm \) of an isoparametric function \( f \) on a complete Riemannian manifold \( N \) are minimal submanifolds.

Meanwhile, Ge and Tang also established the following theorem, which is a generalization of the spherical case to general Riemannian manifolds.
Theorem 4. (17) Suppose that each isoparametric hypersurface $M_t$ has constant principal curvatures with respect to the unit normal vector field in the direction of $\nabla f$. Then each of the focal sets $M_{\pm}$ has common constant principal curvatures in all normal directions, i.e., the eigenvalues of the shape operator are constant and independent of the choices of the point and unit normal vector of $M_{\pm}$.

Owning to the rich and beautiful topological and geometric properties of isoparametric functions on Riemannian manifolds, Ge and Tang initiated the study of isoparametric functions on exotic spheres in [16].

Recall that an $n$-dimensional smooth manifold $\Sigma^n$ is called an exotic $n$-sphere if it is homeomorphic but not diffeomorphic to $S^n$. It is J. Milnor [26] who firstly discovered an exotic 7-sphere which is an $S^3$-bundle over $S^4$. Later, Kervaire and Milnor [24] computed the group of homotopy spheres in each dimension greater than four which implies that there exist exotic spheres in infinitely many dimensions and in each dimension there are at most finitely many exotic spheres. In particular, ignoring orientation there exist 14 exotic 7-spheres, 10 of which can be exhibited as $S^3$-bundles over $S^4$, the so-called Milnor spheres. However, in dimension four, the question of whether an exotic 4-sphere exists remains open, which is the so called smooth Poincaré conjecture (c.f. [23]).

Since the discovery of exotic spheres by Milnor, a very intriguing problem is to interpret the geometry of them (c.f. [19], [20], [3], and [23]). In [16], using isoparametric (even transnormal) functions to attack the smooth Poincaré conjecture in dimension four, Ge and Tang showed the following theorem.

Theorem 5. (16) Suppose $\Sigma^4$ is a homotopy 4-sphere and it admits a transnormal function under some metric. Then $\Sigma^4$ is diffeomorphic to $S^4$.

Note that a homotopy $n$-sphere is a smooth manifold with the same homotopy type as $S^n$. Freedman [14] showed that any homotopy 4-sphere is homeomorphic to $S^4$. As a result of this, the above Theorem 5 says equivalently that there exists no transnormal function on any exotic 4-sphere if it exists. In contrast to the non-existence result in dimension four, Ge and Tang also constructed many examples of isoparametric functions on the Milnor spheres. Furthermore, by projecting an $S^3$-invariant isoparametric function on the symplectic group $Sp(2)$ with a certain left invariant metric, they constructed explicitly a properly transnormal but not an isoparametric function on the Gromoll-Meyer sphere with two points as the focal sets. Inspired by this example, they posed a question that whether there is an isoparametric function on the Gromoll-Meyer sphere or on any exotic $n$-sphere ($n > 4$) with two points as the focal sets. More generally, they posed the following:

**Problem 1.** (16) Does there always exist a properly isoparametric function on an exotic sphere $\Sigma^n$ ($n > 4$) with the focal sets being those occurring on $S^n$?

To answer the Problem 1, Qian and Tang developed a general way to construct metrics and isoparametric functions on a given manifold in [35], which is based on a simple and useful observation that a transnormal function on a complete Riemannian manifold is necessarily a Morse-Bott function (c.f. [44]). As is well known, a Morse-Bott function is a generalization of a Morse function, and it admits critical
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submanifolds satisfying a certain non-degenerate condition on normal bundles. In [35], the following fundamental construction is given, whose proof depends heavily on Moser’s volume element theorem.

**Theorem 6.** ([35]) Let \( N \) be a closed connected smooth manifold and \( f \) a Morse-Bott function on \( N \) with the critical set \( C(f) = M_+ \sqcup M_- \), where \( M_+ \) and \( M_- \) are both closed connected submanifolds of codimensions more than 1. Then there exists a metric on \( N \) so that \( f \) is an isoparametric function.

It follows from a theorem of S. Smale that

**Corollary 1.** ([35]) Every homotopy n-sphere with \( n > 4 \) admits a metric and an isoparametric function with 2 points as the focal sets.

**Remark 1.** Corollary 1 answers partially the above Problem 1.

Moreover, metrics and isoparametric functions on homotopy spheres and on the Eells-Kuiper projective planes can also be constructed so that at least one component of the critical set is not a single point.

In addition to the above existence theorem on homotopy spheres, on the other side, the following non-existence results was also proved.

**Theorem 7.** ([35]) Every odd dimensional exotic sphere admits no totally isoparametric functions with 2 points as the focal set.

Recall that a **totally isoparametric** function is an isoparametric function so that each regular level hypersurface has constant principal curvatures, as defined in [18]. As it is well known, an isoparametric function on a unit sphere must be totally isoparametric.

**Remark 2.** According to [21] and [37], there exists at least one exotic Kervaire sphere \( \Sigma^{4m+1} \) which has a cohomogeneity one action. Consequently, \( \Sigma^{4m+1} \) admits a totally isoparametric function \( f \) under an invariant metric (c.f. [16]). However, each component of the focal set of \( f \) is not just a point, but a smooth submanifold. Hence, the assumption on the focal set in Theorem 7 is essential.

In light of the above Theorem 7, it is reasonable to ask

**Problem 2.** Does there exist an even dimensional exotic sphere \( \Sigma^{2n} (n > 2) \) admitting a metric and a totally isoparametric function with 2 points as the focal set?

In the last section of [35], both existence and non-existence results of isoparametric functions on some homotopy spheres which also have SC\(^p\)-property were investigated. A Riemannian manifold has SC\(^p\)-property if every geodesic issuing from the point \( p \) is closed and has the same length ([21]). For some even dimensional homotopy spheres, the existence theorem in [35] improves a beautiful result of Bérard-Bergery [1].

Recently, Tang and Zhang [42] solved a problem of Bérard-Bergery and Besse. That is, they showed that every Eells-Kuiper quaternionic projective plane carries a Riemannian metric with SC\(^p\)-property for a certain point \( p \). Thus, it is interesting to know whether there is a metric on every Eells-Kuiper quaternionic projective plane which not only has the SC\(^p\)-property, but also admits a certain isoparametric function (c.f. [35]).
3 Yau conjecture on the first eigenvalue and isoparametric foliations

The Laplace-Beltrami operator is one of the most important operators acting on \( C^\infty \) functions on a Riemannian manifold. Over several decades, research on the spectrum of the Laplace-Beltrami operator has always been a core issue in the study of geometry. For instance, the geometry of closed minimal submanifolds in the unit sphere is closely related to the eigenvalue problem.

Let \((M^n, g)\) be an \( n \)-dimensional compact connected Riemannian manifold without boundary and \( \Delta \) be the Laplace-Beltrami operator acting on a \( C^\infty \) function \( f \) on \( M \) by \( \Delta f = -\text{div}(\nabla f) \), the negative of divergence of the gradient \( \nabla f \). It is well known that \( \Delta \) is an elliptic operator and has a discrete spectrum \( \{ 0 = \lambda_0(M) < \lambda_1(M) \leq \cdots \leq \lambda_k(M), \cdots \uparrow \infty \} \) with each eigenvalue repeated a number of times equal to its multiplicity. As usual, we call \( \lambda_1(M) \) the first eigenvalue of \( M \). When \( M^n \) is a minimal hypersurface in the unit sphere \( S^{n+1}(1) \), it follows from Takahashi Theorem that \( \lambda_1(M) \) is not greater than \( n \). Consequently, S.T. Yau posed in 1982 the following conjecture:

Yau conjecture (45): The first eigenvalue of every closed embedded minimal hypersurface \( M^n \) in the unit sphere \( S^{n+1}(1) \) is just \( n \).

In 1983, Choi and Wang made the most significant breakthrough to this conjecture (11). To be precise, they showed that the first eigenvalue of every (embedded) closed minimal hypersurface in \( S^{n+1}(1) \) is not smaller than \( \frac{n}{2} \). Usually, the calculation of the spectrum of the Laplace-Beltrami operator, even of the first eigenvalue, is rather complicated and difficult. Up to now, the Yau conjecture is far from being solved even in dimension two.

It was proved in (29) that if the Yau conjecture is true for the torus of dimension two, then the Lawson conjecture holds, that is to say, the only minimally embedded torus in \( S^3(1) \) is the Clifford torus. In fact, the Lawson conjecture has been a challenging problem for more than 40 years, and recently it was solved by S. Brendle (c.f. 4).

In this note, we pay attention to a little more restricted problem of the Yau conjecture for closed minimal isoparametric hypersurfaces \( M^n \) in \( S^{n+1}(1) \).

Recall that an isoparametric hypersurface \( M^n \) in the unit sphere \( S^{n+1}(1) \) must have constant principal curvatures (c.f. 5, 6, 9). Let \( \xi \) be a unit normal vector field along \( M^n \) in \( S^{n+1}(1) \), \( g \) the number of distinct principal curvatures of \( M \), \( \cot \theta_\alpha (\alpha = 1, \ldots, g; 0 < \theta_1 < \cdots < \theta_g < \pi) \) the principal curvatures with respect to \( \xi \) and \( m_\alpha \) the multiplicity of \( \cot \theta_\alpha \). Using a brilliant topological method, Münzner (c.f. 32) proved the remarkable result that the number \( g \) must be 1, 2, 3, 4 or 6; \( m_\alpha = m_{\alpha+2} \) (indices mod \( g \)); \( \theta_\alpha = \theta_1 + \frac{\alpha - 1}{g} \pi \) (\( \alpha = 1, \ldots, g \)) and when \( g \) is odd, \( m_1 = m_2 \).

In order to attack the Yau conjecture, Muto-Ohnita-Urakawa (31) and Kotani (25) made a breakthrough for some minimal homogeneous (automatically isopara-
metric) hypersurfaces. More precisely, they verified the Yau conjecture for all minimal homogeneous hypersurfaces with $g = 1, 2, 3, 6$. However, when it came to the case $g = 4$, they were only able to deal with the cases $(m_1, m_2) = (2, 2)$ and $(1, k)$.

Furthermore, Muto ([30]) proved that the Yau conjecture is also true for some families of minimal inhomogeneous isoparametric hypersurfaces with $g = 4$. This remarkable result contains many inhomogeneous isoparametric hypersurfaces. However, there is no result in [30] for isoparametric hypersurfaces with $\min(m_1, m_2) > 10$.

Based on all results mentioned above and the classification of isoparametric hypersurfaces in $S^{g+1}(1)$ (c.f. [8], [22], [10], [12] and [28]), Tang and Yan [39] completely solved the Yau conjecture on the minimal isoparametric case by establishing the following

**Theorem 8.** ([39]) Let $M^n$ be a closed minimal isoparametric hypersurface in the unit sphere $S^{g+1}(1)$ with $g = 4$ and $m_1, m_2 \geq 2$. Then

$$\lambda_1(M^n) = n.$$ 

**Remark 3.** Theorem 8 depends only on the values of $(m_1, m_2)$. In particular, it covers the unclassified case $g = 4$, $(m_1, m_2) = (7, 8)$.

**Remark 4.** A purported conjecture of Chern states that a closed, minimally immersed hypersurface in $S^{g+1}(1)$, whose second fundamental form has constant length, is isoparametric (c.f. [15]). If this conjecture is proven, Theorem 8 would have settled the Yau conjecture for the minimal hypersurface whose second fundamental form has constant length, which gives more confidence in the Yau conjecture.

Indeed, the more fascinating part of [39] was to determine the first eigenvalues of the focal submanifolds in $S^{g+1}(1)$. To state their result clearly, let us recall some notations. Given an isoparametric hypersurface $M^n$ in $S^{g+1}(1)$ and a smooth field $\xi$ of unit normals to $M$, for each $x \in M$ and $\theta \in \mathbb{R}$, we can define $\phi_\theta : M^n \rightarrow S^{g+1}(1)$ by

$$\phi_\theta(x) = \cos \theta \, x + \sin \theta \, \xi(x).$$

Clearly, $\phi_\theta(x)$ is the point at an oriented distance $\theta$ to $M$ along the normal geodesic through $x$. If $\theta \neq \theta_\alpha$ for any $\alpha = 1, \ldots, g$, $\phi_\theta$ is a parallel hypersurface to $M$ at an oriented distance $\theta$, which we will denote by $M_\theta$ henceforward. If $\theta = \theta_\alpha$ for some $\alpha = 1, \ldots, g$, it is easy to find that for any vector $X$ in the principal distributions $E_\alpha(x) = \{X \in T_xM \mid A_\xi X = \cot \theta_\alpha X\}$, where $A_\xi$ is the shape operator with respect to $\xi$, $(\phi_\theta)_*x = 0$. In other words, in case $\cot \theta = \cot \theta_\alpha$ is a principal curvature of $M$, $\phi_\theta$ is not an immersion, which is actually a focal submanifold of codimension $m_\alpha + 1$ in $S^{g+1}(1)$.

Münzner asserted that there are only two distinct focal submanifolds in a parallel family of isoparametric hypersurfaces, regardless of the number of distinct principal curvatures of $M$; and every isoparametric hypersurface is a tube of constant radius over each focal submanifold. Denote by $M_1$ the focal submanifold in $S^{g+1}(1)$ at an oriented distance $\theta_1$ along $\xi$ from $M$ with codimension $m_1 + 1$, $M_2$ the focal
submanifold in $S^{n+1}(1)$ at an oriented distance $\frac{2}{3} - \theta_1$ along $-\xi$ from $M$ with codimension $m_2 + 1$. In virtue of Cartan’s identity, one sees that the focal submanifolds $M_1$ and $M_2$ are both minimal in $S^{n+1}(1)$ (c.f. [9]).

Another main result of [39] concerning the first eigenvalues of focal submanifolds in the non-stable range, is now stated as follows.

**Theorem 9.** ([39]) Let $M_1$ be the focal submanifold of an isoparametric hypersurface with $g = 4$ in $S^{n+1}(1)$ with codimension $m_1 + 1$. If $\dim M_1 \geq \frac{2}{3}n + 1$, then

$$\lambda_1(M_1) = \dim M_1$$

with multiplicity $n + 2$. A similar conclusion holds for $M_2$ under an analogous condition.

We emphasize that the assumption $\dim M_1 \geq \frac{2}{3}n + 1$ in Theorem 9 is essential. For instance, Solomon [36] constructed an eigenfunction on the specific focal submanifolds $M_2$ of OT-FKM-type (we will explain it immediately), which has $4m$ as an eigenvalue. In some case, $4m$ is less than the dimension of $M_2$.

As an example, Theorem 9 implies that each focal submanifold of isoparametric hypersurfaces with $g = 4$, $(m_1, m_2) = (7, 8)$ has its dimension as the first eigenvalue.

We need to recall the construction of the isoparametric hypersurfaces of OT-FKM-type. For a symmetric Clifford system $\{P_0, \cdots, P_m\}$ on $\mathbb{R}^{2l}$, i.e., $P_i$’s are symmetric matrices satisfying $PP_j + P_jP_i = 2\delta_{ij}I_2$, Ferus, Karcher and Münzner ([13]) constructed a polynomial $F$ on $\mathbb{R}^{2l}$:

$$F : \mathbb{R}^{2l} \to \mathbb{R}$$

$$F(x) = |x|^4 - 2 \sum_{i=0}^{m} (P_i x, x)^2.$$ For $f = F|_{S^{2l-1}}$, define $M_1 = f^{-1}(1)$, $M_2 = f^{-1}(-1)$, which have codimensions $m + 1$ and $l - m$ in $S^{n+1}(1)$, respectively.

For focal submanifold $M_1$ of OT-FKM-type, the only unsettled multiplicities in [39] are $(m_1, m_2) = (1, 1), (4, 3), (5, 2)$. And for the $(4, 3)$ case, there exist only one homogeneous and one inhomogeneous examples.

Finally, Tang and Yan [39] proposed the following problem, which could be regarded as an extension of the Yau conjecture.

**Problem 3.** ([39]) Let $M^d$ be a closed embedded minimal submanifold in the unit sphere $S^{n+1}(1)$. If the dimension $d$ of $M^d$ satisfies $d \geq \frac{2}{3}n + 1$, then

$$\lambda_1(M^d) = d.$$ Later, Tang, Xie and Yan [41] took chance to solve the unsolved cases in [39] and considered the case with $g = 6$. First, by applying the similar method as in [39], they got the following theorem for the case with $g = 6$ which contains more information than that in [31] and does not depend on the classification result of Miyaoka ([28]).
Theorem 10. (41) Let $M^{12}$ be a closed minimal isoparametric hypersurface in $S^{13}(1)$ with $g = 6$ and $(m_1, m_2) = (2, 2)$. Then

$$\lambda_1(M^{12}) = 12$$

with multiplicity $14$. Furthermore, the following inequality holds

$$\lambda_k(M^{12}) > \frac{3}{7} \lambda_k(S^{13}(1)), \quad k = 1, 2, \ldots .$$

And for focal submanifolds $M_1$ of OT-FKM-Type, they solved two left cases and proved

Theorem 11. (41) For the focal submanifold $M_1$ of OT-FKM-type in $S^5(1)$ with $(m_1, m_2) = (1, 1)$,

$$\lambda_1(M_1) = \dim M_1 = 3$$

with multiplicity $6$; for the focal submanifold $M_1$ of homogeneous OT-FKM-type in $S^{15}(1)$ with $(m_1, m_2) = (4, 3)$,

$$\lambda_3(M_1) = \dim M_1 = 10$$

with multiplicity $16$.

At last, in the case with $g = 6$, by a deep investigation into the shape operator of the focal submanifolds, they obtained estimates on the first eigenvalue. Particularly, for one of the focal submanifolds with $g = 6$, $m_1 = m_2 = 2$, the first eigenvalue is equal to its dimension. It gives an affirmative answer to Problem 3 in this case.

4 Related topics and applications

The connection between geometry of Riemannian manifolds with positive scalar curvatures and surgery theory is quite a deep subject which has attracted widely attention. The most important aspect of this field is the original discovery of Gromov-Lawson and of Schoen-Yau.

Motivated by the Schoen-Yau-Gromov-Lawson surgery theory on metrics of positive scalar curvature, Tang, Xie and Yan [40] constructed a double manifold associated with a minimal isoparametric hypersurface in the unit sphere. The resulting double manifold carries a metric of positive scalar curvature and an isoparametric foliation as well. To investigate the topology of the double manifolds, they used topological $K$-theory and the representation of the Clifford algebra for the OT-FKM-type, and determined completely the isotropy subgroups of singular orbits for homogeneous case. Here we note that, as it is well known, a homogeneous (isoparametric) hypersurface in the unit sphere can be characterized as a principal orbit of isotropy representation of some symmetric space of rank two.
In the last part of this section, we describe an application of isoparametric foliation to Willmore submanifolds. By definition, a Willmore submanifold (in the unit sphere) is the critical point of the Willmore functional. In particular, every minimal surface in the unit sphere is automatically Willmore; in other words, Willmore surfaces are a generalization of minimal surfaces in the unit sphere. However, examples of Willmore submanifolds in the unit sphere are rare in the literature.

Qian, Tang and Yan ([38], [34]) proved that each focal submanifold of isoparametric hypersurface (not only OT-FKM-type) in the unit sphere with \( g = 4 \) is a Willmore submanifold. For \( g = 1, 2, 3 \), the conclusion above is clearly valid. As for \( g = 6 \), the conclusion should be also true.

Recall that the focal submanifolds are minimal in unit spheres. It is worth noting that an Einstein manifold minimally immersed in the unit sphere is a Willmore submanifold. A natural problem arises: whether the focal submanifolds are Einstein? To this problem with \( g = 4 \), [38] and [34] gave a complete answer, depending on the classification results. In other words, they dealt with this problem in two cases—homogeneous type and OT-FKM-type.

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