Granular metamaterials for seismic protection. Hyperelastic and hypoelastic models

Sergey Kuznetsov$^{1,2,3,*}$ and Hubert Maigre$^4$

$^1$Moscow State University of Civil Engineering, Moscow, Russia
$^2$Bauman Moscow State Technical University, Moscow, Russia
$^3$Institute for Problems in Mechanics, Moscow, Russia
$^4$INSA de Lyon, Villeurbanne, France

$kuzn-sergey@yandex.ru$

Abstract. The hyperelastic potentials along with hypoelastic models applicable to describe dynamic behaviour of the granular metamaterials at wave propagation in their structure under seismic activity are studied. The specially constructed potentials suitable for bimodular materials are analysed. Effects of possible mechanical energy transformation due to formation and propagation of the shock wave fronts are discussed.

1. Introduction
Granular metamaterials used for fillings of seismic barriers and seismic pads are of extensive research at different laboratories, institutes, and companies across the globe. For describing dynamic properties of these materials several approaches are developed, including elastic, elastic-plastic, viscoelastic-plastic, hydrodynamic equations of state, etc. Herein, various nonlinear elastic models are considered.

For elastic models that do not exhibit loss of mechanical energy at the reversal loadings, the following implications hold

\[ \text{hyperelastic} \subset \text{elastic} \subset \text{hypoelastic} \subset \text{elasto-plastic}. \] (1)

All the elastic models considered below are confined to the infinitesimal relations, however, with minor modifications the presented analysis can be extrapolated to the case of large deformations.

2. Hyperelastic potentials
2.1. General equations
The strain-stress relation for these models takes the form

\[ \sigma = \lambda (I_\varepsilon, II_\varepsilon, III_\varepsilon) I_\varepsilon + 2\mu (I_\varepsilon, II_\varepsilon, III_\varepsilon) \cdot \varepsilon, \] (2)

where Lame’s \( \lambda \) and \( \mu \) are functions of strain invariants

\[ I_\varepsilon \equiv \text{tr}(\varepsilon), \quad II_\varepsilon \equiv \frac{1}{2} (I_\varepsilon^2 - \varepsilon : \varepsilon), \quad III_\varepsilon \equiv \text{det}(\varepsilon). \] (3)

In the equation of state (2) \( 1 \) is the unit diagonal matrix. It is usually assumed that the strain energy related to the constitutive relation (2), is positive definite. That imposes the following restriction on Lame’s constants

\[ \mu > 0, \quad 3\lambda + 2\mu > 0 \] (4)
2.2. Equations of motion
Substituting equation of state (2) into (infinitesimal) equation of motion, yields
\[(\lambda + 2\mu) \nabla \chi \text{div} \mathbf{u} - \mu \text{rot} \nabla \chi \text{rot} \mathbf{u} + (\nabla \chi \lambda) \text{div} \mathbf{u} + \nabla \chi \mu \cdot (\nabla \chi \mathbf{u} + \nabla \chi \mathbf{u}^T) = \rho \ddot{u}, \tag{5}\]
where in view of (2)
\[
\nabla \chi \lambda = \left( \frac{\partial \lambda}{\partial I} \nabla I + \frac{\partial \lambda}{\partial II} \nabla II + \frac{\partial \lambda}{\partial III} \nabla III \right). \tag{6}\]
The gradient \(\nabla \chi \mu\) is defined similarly.

In addition to Eq. (2) for a hyperelastic material it is assumed the potential \(\Psi(I_\epsilon, II_\epsilon, III_\epsilon)\) exists, such that (Truesdell and Toupin, 1960)
\[
\sigma = \nabla \epsilon \Psi(I_\epsilon, II_\epsilon, III_\epsilon). \tag{7}\]

Accounting relations (3), the condition (7) can be rewritten as (Ericksen, 1960)
\[
\sigma = \frac{\partial \Psi}{\partial I} I + \frac{\partial \Psi}{\partial II} (II - I) + \frac{\partial \Psi}{\partial III} (\epsilon - I_\epsilon + II_\epsilon I). \tag{8}\]

Comparing Eqs. (2) and (8) yields the following representation of Lame’s constants in terms of potential
\[
\lambda(I_\epsilon, II_\epsilon, III_\epsilon) = \frac{\partial \Psi}{\partial I} I^{-1} + \frac{\partial \Psi}{\partial II} II^{-1} - \frac{\partial \Psi}{\partial III} III^{-1} \tag{9}\]
\[2\mu(I_\epsilon, II_\epsilon, III_\epsilon) = -\frac{\partial \Psi}{\partial II} + \frac{\partial \Psi}{\partial III} (\epsilon^{-1} - I_\epsilon) .\]

Equations (9) impose some restrictions on behavior of the potential \(\Psi\). In particular, since Lame’s constants assumed to be continuous with respect to strain invariants, should be bounded at \(I_\epsilon \to 0, \epsilon \to 0\), Eqs. (9) yield
\[
\frac{\partial \Psi}{\partial I} = O(I_\epsilon), \quad I_\epsilon \to 0; \quad \frac{\partial \Psi}{\partial II} = O(I_\epsilon), \quad I_\epsilon \to 0, \quad \frac{\partial \Psi}{\partial III} = O(III_\epsilon), \quad III_\epsilon \to 0. \tag{10}\]

At modeling both statics and dynamics of granular materials the hyperelastic constitutive equations are applied quite often (Wang, Truesdell, 1973; Jackson, 1983; Nesterenko, 2001; 2008; Herbold et al., 2008; Sen, 2008; Molinari and Daraio, 2009; Sun and Sundaresan, 2013). It should be noted that in most of these works a concept of multimodulus, actually, bi-modulus material, was applied (Lomakin and Rabotnov, 1978) with elastic potential
\[
\Psi(I_\epsilon, II_\epsilon) \equiv \alpha I_\epsilon^2 + \beta II_\epsilon + \gamma I_\epsilon \sqrt{II_\epsilon}, \tag{11}\]
where \(\alpha, \beta, \gamma\) are elastic material constants, independent of invariants \(I_\epsilon, II_\epsilon\)
\[
II_\epsilon^- = -II_\epsilon + I_\epsilon^2. \tag{12}\]
Introducing parameter \(\gamma\) allows one to account dependence of material properties on sign of the first invariant.

It should also be noted that with introduction (Volokh, 2005) of the potential
\[
\Psi(I_\epsilon, II_\epsilon^-) = \Psi_1(I_\epsilon, II_\epsilon^-)(1 - \exp(-\chi(II_\epsilon^-))) \tag{13}\]
\[
\chi(II_\epsilon^-) \to 0 \quad @ \quad II_\epsilon^- \to 0 \quad \& \quad \chi(II_\epsilon^-) \to \infty \quad @ \quad II_\epsilon^- \to \infty \]
where \(\Psi_1(I_\epsilon, II_\epsilon^-)\) is an arbitrary potential, media with the dropdown (softening) diagrams can be modeled.

3. Elastic models
3.1. General equations
Elastic models are described by the following equation of state
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\[ \mathbf{\sigma} = \lambda (I_\sigma, II_\sigma, III_\sigma) I_{\mathbf{\varepsilon}} + 2\mu (I_\sigma, II_\sigma, III_\sigma) \cdot \mathbf{\varepsilon}. \]  

14. Equations of motion

By analogy with Eq. (5), the linearized equation of motion can be represented in a form

\[ \frac{\lambda + 2\mu}{\rho} \nabla_x \text{div}_x \mathbf{u} - \frac{\mu}{\rho} \text{rot}_x \text{rot}_x \mathbf{u} + \frac{1}{\rho} \left( \nabla_x \lambda \cdot \text{div}_x \mathbf{u} + \nabla_x \mu \cdot (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T) \right) + \mathbf{b} = \mathbf{\dot{u}}. \]

Despite the apparent more generality, the elastic models are rarely used for modeling granular materials. Norris and Johnson (1997) and Coste and Gilles (1999) considered determination of velocities of acoustic waves in a granular media modeled by a system of elastic balls, interacting by the Hertz theory.

4. Hypoelastic models

4.1. General equations

According to Trusiedell (1955, 1963) the speed of the stress tensor \( \mathbf{\sigma} \) for a hypo-elastic medium is determined by speed of the strain tensor \( \mathbf{\varepsilon} \). Assuming infinitesimal strains, the constitutive relation for an isotropic hypo-elastic material can be written in a form

\[ \mathbf{\sigma} = \lambda (I_\sigma, II_\sigma, III_\sigma) I_{\mathbf{\varepsilon}} + 2\mu (I_\sigma, II_\sigma, III_\sigma) \cdot \mathbf{\varepsilon} \]  

where \( \lambda = \frac{\partial\sigma}{\partial\varepsilon} \), \( \lambda \) and \( \mu \) are functions of the corresponding invariants. Comparing the stress-strain relations for hypo-elastic (16) and elastic media (14) reveals, the only difference is in the incremental form of the constitutive relation for the hypo-elastic medium.

In Thomas (1955), Green (1956), and Gurtin (1983) it was demonstrated that the special triggering mechanism can be incorporated into equation of state (16) allowing to account different states for active and unloading cases; thus, the general elastic-plastic behavior can be modeled within the hypo-elastic models.

4.2. Equations of motion

For a hypo-elastic medium the equation of motion can be written in the form

\[ \text{div} \mathbf{\sigma} + \rho \mathbf{b} = \rho \mathbf{\ddot{v}} \]  

where \( \rho \) is the material density; it is assumed that \( \rho = 0 \); \( \mathbf{b} \) is the field of body forces. Substituting the equation of state (16) into equation of motion (17) with account of the linearized Cauchy relations

\[ \mathbf{\varepsilon} = \frac{1}{2} \left( \nabla_x \mathbf{v} + \nabla_x \mathbf{v}^T \right) \]  

yields

\[ \frac{\lambda + 2\mu}{\rho} \nabla_x \text{div}_x \mathbf{v} - \frac{\mu}{\rho} \text{rot}_x \text{rot}_x \mathbf{v} + \frac{\lambda}{\rho} \nabla_x \mathbf{v} + \nabla_x \frac{\mu}{\rho} \cdot (\nabla_x \mathbf{v} + \nabla_x \mathbf{v}^T) + \mathbf{b} = \mathbf{\ddot{v}} \]  

where

\[ \nabla_x \frac{\lambda}{\rho} = \frac{1}{\rho} \left( \frac{\partial\lambda}{\partial I_\sigma} \nabla_x I_\sigma + \frac{\partial\lambda}{\partial II_\sigma} \nabla_x II_\sigma + \frac{\partial\lambda}{\partial III_\sigma} \nabla_x III_\sigma \right) \]  

The gradient \( \nabla_x \mu \) is defined analogously.

Despite the obvious generality, the hypo-elastic media are rarely used for modeling granular materials; in this regard it should be mentioned that the hypo-elastic models were used for analyzing propagation of the impact bulk wave fronts propagating in granular materials (Varley, 1965; Nariboli, 1971), and the horizontally polarized surface acoustic waves (Chandrasekharaiah, 1977).
5. Some inelastic models

5.1. General considerations
Along with various elastic models, there is a large number of works accounting inelastic behavior of granular metamaterials. Apparently, one of the simplest inelastic models applicable for static and quasi-static modeling of granular metamaterials are based on various variants of the Mohr – Coulomb and Drucker – Prager theories.

Within the Mohr-Coulomb theory, there are several approaches applied for modeling granular materials; see (Goodman and Cowin, 1977; Massoudi and Mehrabadi, 2001; Nedderman, 2005). The Mohr-Coulomb theory was also applied to modeling effects associated with the early stage of beginning and developing inelastic deformations prior to flow of avalanches (Rajchenbach, 1990).

5.2. Specific inelastic models for dynamics of granular metamaterials
For the considered inelastic models used for dynamics of granular metamaterials apparently, the most widespread is the Cam-Clay (CC), the Modified Cam-Clay (MCC) and the related models; see (Borja et al., 1990, 1998, 2003; Roscoe et al., 1958, 1963), along with some more recent works (Goldstein et al., 2016; Ilyashenko et al., 2017).

6. Concluding remarks
As the literature review shows, currently, the hyperelastic models are the most widely used for characterizing dynamic problems of wave propagation inside granular metamaterials; see Eq. (11).

However, Eq. (11) is not the only equation allowing description of materials exhibiting different behavior in tension and compression. For one dimensional motions various potentials used in atomic and molecular dynamics can also be used, e.g. Lennard-Jones and Morse potentials; see (Zhen and Davies, 1985).

Yet another problem relating to dynamic behavior of wave propagation in granular media associates with possible formation of shock wave fronts at interaction of smooth waves travelling with different velocities when compression wave moves faster than tension wave; such a situation is very common in granular media.

Another problem relating to wave dynamics of granular media relies on the possible inhomogeneity of the physical properties. Some interesting phenomena relating to wave propagation in the functionally graded and stratified media are studied recently; e.g. works by (Ilyashenko et al., 2017, 2018a, b; Kuznetsov et al., 2018, 2019). It can be anticipated that combination of inhomogeneity with physical nonlinearity may result in new peculiarities in respect of wave propagation, shock wave front formation, and loss of the mechanical energy due to the thermodynamic conversion at the shock wave front.

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