Numerical Quantum Field Theory on the Continuum and a New Look at Perturbation Theory

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The Source Galerkin method finds approximate solutions to the functional differential equations of field theories in the presence of external sources. While developing this process, it was recognized that approximations of the spectral representations of the Green’s functions by Sinc function expansions are an extremely powerful calculative tool. Specifically, this understanding makes it not only possible to apply the Source Galerkin method to higher dimensional field theories, but also leads to a new approach to perturbation theory calculations in scalar and fermionic field theories. This report summarizes the methodologies for solving quantum field theories with the Source Galerkin method and for performing perturbation theory calculations using Sinc approximations.

1. SOURCE GALERKIN METHOD

To analyze a theory, we start with a Lagrangian in the presence of external sources. The functional differential equations satisfied by $Z$ are generated in the usual way. The Source Galerkin technique is a systematic, iterative algorithm to solve these equations to increasing accuracy. The lowest order solution is given by an ansatz which is an expansion over the external sources of the theory,

$$Z^* = \exp \left( \int j_x G_{xy} j_y + \int j_\omega J_x H_{\omega xy} j_y J_z + \cdots \right)$$

We choose $G_{xy}$ and $H_{\omega xy}$ so that the residual associated with the equation(s) of motion, $\hat{E}Z^* = R(j)$, is as small as possible. We do this systematically by using weighted residuals. We introduce a set of test functions $\{j(x), j(x)j(y), j(x)j(y)j(z), \ldots \}$ and the measure

$$[dj] \exp \left[ \int dx j(x)^2/\epsilon^2 \right].$$

We determine parameters of our expansion by imposing the constraints,

$$\int [dj] e^{[j^2/\epsilon^2]} j(x_1) \cdots j(x_j) \left[ \hat{E}Z^* \right] = 0,$$

where the number of constraints matches the number of free parameters. If $Z^*$ is equal to the exact generating functional, $Z$, then these conditions are always true.

The integrals we encounter are of the form

$$\int [dj] e^{j^2/\epsilon^2} P(j) = \int [dj] P \left( \frac{\partial}{\partial B} \right) e^{-\frac{\epsilon^2}{4} + jB} \bigg|_{B=0}$$

Fermions are dealt with in essentially the same way, using anticommuting sources. This method can be used in two or fewer dimensions, but memory requirements become outrageous in higher dimensions. To surmount this obstacle, the problem needs to be broken into smaller pieces and we must utilize the symmetries of the action. We do this by rewriting the usual two point Green’s function, observing that any exact two point function can be represented as a sum of free two point functions:

$$G(x, y) = \int dk^2 G_{\text{free}}(k^2, x - y) B(k^2).$$
Introducing $\Lambda$ as a cutoff, these are represented as

$$G_{\text{free}}(x) = \frac{1}{(2\pi)^d} \int dp \frac{e^{-p^2/\Lambda^2 + ipx}}{p^2 + m^*}.$$ 

Using Sinc functions $[8,9]$, we can write this in a more numerically tractable form,

$$G_{\text{free}}(x) \approx G_0 \sum_{k=-N}^N c(k) \exp \left[ -\frac{x^2}{4(ckh + 1/\Lambda^2)} \right],$$

$$c(k) = \frac{1}{e^{kh}} \left[ e^{kh} + 1/\Lambda^2 \right]^{d/2} \exp \left[ -e^{kh}m^* \right].$$

Reducing the free propagator to sums over Gaussians suggests a parameterization of $Z^*$. We write $\ln Z^*$ as a Feynman sum of graphs with external legs replaced by sources but, unlike a perturbation theory expansion, the free propagators have arbitrary mass and normalization. The method has been applied to various field theories on the lattice and on the continuum. The ansatz is Poincaré invariant and will obey other symmetries of the theory by construction, giving ample freedom to span the solution space. The procedure can be summarized as follows:

1. Start with an ansatz solution $Z^*$ and derive the Source Galerkin equations by inserting the ansatz into the Schwinger–Dyson equations, and set the parameters. This involves functional and space-time integrals and produces nonlinear equations.

2. To improve the ansatz, expand $Z^*$ with more terms and “iterate”.

A full account of the Source Galerkin approach is currently in preparation. We have made significant progress beyond the initial lattice implementation of Source Galerkin method $[8,9]$. We now can calculate on the continuum. Applications to the $\Phi^4$ in $D = 1$ and 2, and four fermion calculations $[8,9]$ give excellent results, even with simple $Z^*$’s which include only two terms. The two-dimensional nonlinear $\sigma$-model $[9]$ results are compared to $1/N$ calculations and the leading order $\beta$ function calculations are identical. Moreover, treating this model provided a laboratory for investigating renormalization issues in Source Galerkin calculations. A study of the symmetry-breaking sector of Higgs model is planned for the future.

2. Perturbation Theory

The Sinc function representation used above also leads to a new method for numerical evaluation of Feynman diagrams in conventional perturbation theory. In turn, the perturbative analysis is a powerful guide to “exact” Source Galerkin calculations. This approach uses generalized Sinc functions to approximate the propagators as infinite sums whose spatial or momentum dependence appears only in terms like $\exp(-x^2)$. Consequently, all integrations over internal momenta or vertex locations are reduced to Gaussian integrals, which can be performed analytically. To compute the diagram one has to numerically evaluate a rapidly converging sum with dimensionality equal to the number of propagators in the graph. The Sinc function representation has been applied to scalar and fermionic field theories in $[8,9]$. Here we review the application of this approach to QED calculations.

As shown in $[11]$, the fermion propagator is approximated as

$$S_{\Phi h}(p, \Lambda) = \frac{h}{M^2} \sum_{k_1 = -\infty}^{+\infty} \exp \left( k_1 h - e^{k_1 h} m^2 M^2 \right) \times \exp \left( -\frac{p^2}{M^2} \alpha(k_1) \right) \times \exp \left( -\frac{p^2}{M^2} \alpha(k_1) \right).$$

while the Feynman gauge photon propagator is

$$D_{\mu \nu}^{\mu \nu}(p, \Lambda) = \delta^{\mu \nu} \frac{h}{M^2} \sum_{k_1 = -\infty}^{+\infty} \exp \left( k_1 h - e^{k_1 h} m^2 M^2 \right).$$

The accuracy and speed of any computation performed using these expressions are governed by three factors. First, some error is introduced through the Sinc function method itself. This error decreases exponentially with $h$ and for $h \to 0$ the approximation becomes exact. Additional error comes from the truncation of the infinite sums in the approximate versions of the propagators. The general term of the sum, and thus the
Table 1
Correction to the magnetic moment calculated with different values of $h$.

| $h$  | Correction $\times \alpha/\pi$ | Error $\times \alpha/\pi$ |
|------|---------------------------------|---------------------------|
| 0.2  | 0.4999999999949221              | $5.1 \times 10^{-12}$    |
| 0.4  | 0.500000000008583              | $-8.6 \times 10^{-13}$   |
| 0.6  | 0.500000008001                 | $-8.0 \times 10^{-9}$    |
| 0.8  | 0.499999900285                 | $9.97 \times 10^{-8}$    |
| 1.0  | 0.499998                      | $2.3 \times 10^{-6}$     |
| 1.2  | 0.49977                       | 0.00023                   |
| 1.4  | 0.4987                        | 0.0012                    |

truncation error, decreases exponentially with the summation parameter $k$. Finally, finite numerical precision limits the maximum accuracy achievable in any computation. Setting a value of $h$ fixes the expected precision of the result. Since both errors due to truncation of sums and the Sinc expansion decrease exponentially with corresponding parameters, the accuracy of the whole computation improves exponentially as $h$ decreases while amount of work necessary to perform the summations with required precision increases linearly.

In order to test this method we have evaluated several second and fourth order diagrams. The results of computation of second order contribution to the anomalous magnetic moment of an electron are shown in Table 1. This illustrates that accuracy as high as one part in $10^{13}$ can be achieved using the method in question on 64-bit hardware, and this limit dominates for $h \leq 0.3$. Performing the same calculation with 32-bit accuracy yields a maximum accuracy of one part in $10^6$, for $h = 0.6$.

Our initial tests show that the Sinc representation method is capable of producing results whose accuracy is limited only by numerical precision of the hardware in a fraction of the time required to obtain matching results by more conventional methods.

3. SUMMARY

The Source Galerkin method promises to tackle a range of problems which are difficult or impossible to formulate using conventional Monte Carlo approaches. Our methods work on the continuum and avoid most of the usual fermionic problems. They are well suited for calculation in any phase of a theory and are not susceptible to the “sign problem”.

Sinc function representations can be used as a universal tool for performing conventional perturbation theory calculations. Accuracy as high as one part in $10^{13}$ was achieved during testing. The method is extremely fast and can be used to quickly obtain estimates of the final result. High precision computations done using the Sinc representation take significantly less time than comparable calculations performed by with Monte Carlo methods. We are working on the tasks of automating this method and applying it to higher order diagrams, as well as higher iterations of Source Galerkin.

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