ON THE MAXIMAL FUNCTION FOR THE GENERALIZED ORNSTEIN-UHLENBECK SEMIGROUP.

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Abstract. In this note we consider the maximal function for the generalized Ornstein-Uhlenbeck semigroup in $\mathbb{R}$ associated with the generalized Hermite polynomials $\{H_\mu^n\}$ and prove that it is weak type $(1,1)$ with respect to $d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx$, for $\mu > -1/2$ as well as bounded on $L^p(d\lambda_\mu)$ for $p > 1$.

1. Introduction and Preliminaries

The generalized Hermite polynomials were defined by G. Szegö in [14] (see problem 25, pag 380) as being orthogonal polynomials with respect to the measure $d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx$, with $\mu > -1/2$. In his doctoral thesis T. S. Chihara [2] (see also [3]) studied them in detail. In this paper we consider the definition of the generalized Hermite polynomials given by M. Rosenblum in [10].

Let us denote by $H_\mu^n$ this generalized Hermite polynomial of degree $n$, then for $n$ even

$$H_{2m}^\mu(x) = (-1)^m (2m)! \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + \mu + \frac{1}{2})} L_m^{\mu - \frac{1}{2}}(x^2)$$

and for $n$ odd

$$H_{2m+1}^\mu(x) = (-1)^m (2m + 1)! \frac{\Gamma(m + \frac{3}{2})}{\Gamma(m + \mu + \frac{3}{2})} x L_m^{\mu + \frac{1}{2}}(x^2),$$

$L_\gamma^m$ being the $\gamma$-Laguerre polynomial of degree $m$.

Thus, for every $n \in \mathbb{N}$,

$$\|H_n^\mu\|_{L^2(d\lambda)} = \left( \frac{2^n (n!)^2 \Gamma(m + 1/2)}{\gamma_\mu(n)} \right)^{1/2},$$

where $\gamma_\mu(m)$ is a generalized factorial defined by,

$$\gamma_\mu(2m) = \frac{2^{2m} m! \Gamma(m + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} = (2m)! \frac{\Gamma(m + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{3}{2})} \Gamma(\frac{1}{2}) \Gamma(\mu + \frac{1}{2})$$

$$\gamma_\mu(2m + 1) = \frac{2^{2m+1} m! \Gamma(m + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} = (2m)! \frac{\Gamma(m + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{3}{2})} \Gamma(\frac{1}{2}) \Gamma(\mu + \frac{1}{2}).$$

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The generalized Hermite polynomials \( \{H_n^\mu\} \) have a generating function (2.5.8) of [10] which involves the generalized exponential function \( e_\mu \) defined by

\[
e_\mu(z) = \sum_{m=0}^{\infty} \frac{z^m}{\gamma_\mu(m)}.
\]

On the other hand each generalized Hermite polynomial satisfies the following differential equation, see [3],

\[
(H_n^\mu)''(x) + 2(\mu x - x)(H_n^\mu)'(x) + 2(n - \mu \frac{\theta_n}{x^2})H_n^\mu(x) = 0,
\]

with

\[
\theta_n = \begin{cases} 1 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}
\]

and \( n \geq 0 \).

Therefore, by considering the (differential-difference) operator

\[
L_\mu = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{\mu}{x} - x \right) \frac{d}{dx} - \mu \frac{I - \tilde{J}}{2x^2},
\]

where \( I f(x) = f(x) \) and \( \tilde{J} f(x) = f(-x) \), \( H_n^\mu \) turns out to be an eigenfunction of \( L_\mu \) with eigenvalue \(-n\).

Now we can define a Markov semigroup, see D. Bakry [1], by

\[
Pt(x, dy) = \sum_{n=0}^{\infty} \frac{\gamma_\mu(n)}{2^n n!^2} H_n^\mu(x) H_n^\mu(y) e^{-nt} \lambda(dy).
\]

This semigroup is entirely characterized by the action on positive or bounded measurable functions by

\[
T_t^\mu f(x) = \int_{-\infty}^{\infty} f(y) P_t(x, dy).
\]

Thus the family of operators \( \{T_t^\mu\}_{t \geq 0} \) is then a conservative semigroup of operators with generator \( L_\mu \), that we will call the generalized Ornstein-Uhlenbeck semigroup.

Therefore,

\[
\frac{\partial T_t^\mu f(x)}{\partial t} = L_\mu T_t^\mu f(x).
\]

For \( \mu = 0 \), \( \{T_t^0\} \) reduces to the Ornstein-Uhlenbeck semigroup whose behavior on \( L^p \) was studied by B. Muckenhoupt in [7] for the one-dimensional case. By using the generalized Mehler’s formula (2.6.8) of [10]: for \( x, y \in \mathbb{R} \) and \( |z| < 1 \),

\[
\sum_{n=0}^{\infty} \frac{\gamma_\mu(n)}{2^n n!^2} H_n^\mu(x) H_n^\mu(y) z^n = \frac{1}{(1 - z^2)^{\mu+1/2}} e^{\frac{2(x^2 + y^2)}{1 - z^2}} e_\mu \left( \frac{2xyz}{1 - z^2} \right).
\]

we can obtain the following integral expression of this generalized Ornstein-Uhlenbeck semigroup \( \{T_t^\mu\} \),

\[
T_t^\mu f(x) = \frac{1}{(1 - e^{-2t})^{\mu+1/2}} \int_{-\infty}^{\infty} e^{-\frac{2(x^2 + y^2)}{1 - e^{-2t}}} e_\mu \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right) f(y)|y|^{2\mu} e^{-|y|^2} dy.
\]

In the following section we will consider the maximal operator associated with \( \{T_t^\mu\}_{t > 0} \), and prove it is weak type \((1, 1)\) with respect to the measure \( \lambda \), bounded in \( \ell^\infty \) and therefore \( L^p \) bounded for \( 1 < p < \infty \) with respect to \( \lambda \). It is important to
observe that since \( \{T^\mu_t\}_{t>0} \) is not a convolution semigroup, its associated maximal operator is not bounded by the Hardy-Littlewood maximal operator and therefore in order to prove the weak \( (1,1) \) inequality with respect to \( \lambda \) it is needed to develop new techniques. The case \( \mu = 0 \), that as we already said corresponds to the maximal operator of the Ornstein-Uhlenbeck semigroup, was proved by Sjögren in [11] in any dimension.

We will use repeatedly that
\[
|x|^k e^{-x^2} \leq Ce^{-x^2/2} \leq C, \quad \forall \ x \in \mathbb{R}.
\]

The constant \( C \) which will appear throughout this paper may be different on each occurrence.

2. THE MAXIMAL FUNCTION OF THE GENERALIZED ORNSTEIN-UHLENBECK SEMIGROUP.

Let us define the generalized Ornstein-Uhlenbeck maximal function as
\[
T^\mu_\ast f(x) = \sup_{t>0} |T^\mu_t f(x)|, \tag{2.1}
\]
for each \( x \in \mathbb{R} \). Taking \( r = e^{-t} \), we can write
\[
T^\mu_\ast f(x) = \sup_{0 < r < 1} \left| \int_{-\infty}^\infty K_r(x,y) f(y) d\lambda(y) \right|,
\]
with
\[
K_r(x,y) = \frac{1}{\Gamma\left(\mu + \frac{1}{2}\right)} e^{-(x^2+y^2)\frac{r^2}{1-r^2}} e^\mu \frac{2xyr}{1-r^2}.
\]

The main result of this paper is summarized in

**Theorem 2.1.** For \( \mu > -1/2 \),

i) \( T^\mu_\ast \) is weak type \((1,1)\) with respect to \( \lambda \), i.e. there exists a real constant \( C > 0 \) such that for every \( \eta > 0 \)
\[
\lambda \{ x \in \mathbb{R} : T^\mu_\ast f(x) > \eta \} \leq \frac{C}{\eta} \| f \|_{L^1,\lambda}, \tag{2.2}
\]
where \( \| f \|_{L^1,\lambda} = \int_{\mathbb{R}} |f(y)| d\lambda(y) \).

ii) \( T^\mu_\ast \) is bounded in \( L^\infty \), i.e. there exists a real constant \( C > 0 \) such that
\[
\| T^\mu_\ast f \|_{L^\infty} \leq C \| f \|_{L^\infty}, \tag{2.3}
\]
where \( \| f \|_{L^\infty} \) represents the \( L^\infty \) norm.

**Corollary 2.2.** For \( \mu > -1/2 \) and \( p > 1 \),
\[
\| T^\mu_\ast f \|_{L^p,\lambda} \leq C \| f \|_{L^p,\lambda}, \tag{2.4}
\]
where \( \| f \|_{L^p,\lambda} = \int_{\mathbb{R}} |f(y)|^p d\lambda(y) \).

This corollary follows from Marcinkiewicz interpolation theorem between the weak type \((1,1)\) and the boundedness in \( L^\infty \) which will be proved in Theorem 2.1. In order to prove Theorem 2.1 we will introduce well known bounds for the functions \( e_\mu \) and prove two propositions. The first one due to I. P. Natanson and B. Muckenhoupt (8 and 7) is a sort of a generalized Young’s inequality for Borel measures, that we will write it only for the particular case of the measure \( \lambda \) and the other
one has to do with the biggest function whose density distribution as a function of 
\( \eta \) with respect to \( \lambda \) is bounded by \( C/\eta \).

**Properties of \( e_\mu \)**

It can be proved, see (2.2.3) of [10], that the generalized exponential function \( e_\mu \) can be written as,

\[
e_\mu(x) = \Gamma(\mu + 1/2)(2/x)^{\mu-1/2}(I_{\mu-1/2}(x) + I_{\mu+1/2}(x)),
\]

where \( I_\nu \) denotes the modified Bessel function. Then, according to [16] (2), p. 77, and (2), p. 203], we have the following estimates that will be useful in the sequel

\[
|e_\mu(x)| \leq C(1 + |x|)^{-\mu |x|}, \quad x \in \mathbb{R}.
\]

Also, \( e_\mu \) admits the following integral representations depending on the values of \( \mu \) [10],

1. if \( \mu > 0 \) then

\[
e_\mu(x) = \frac{1}{B(\frac{1}{2}, \mu)} \int_{-1}^{1} e^{xt}(1-t)^{\mu-1}(1+t)^{\mu} dt,
\]

2. if \( \mu = 0 \) then

\[
e_0(x) = e^x,
\]

3. if \( -\frac{1}{2} < \mu < 0 \) then

\[
e_\mu(x) = e^x + \frac{\mu}{\mu + 1/2 B(1/2, \mu + 1)} \int_{-1}^{1} (e^{xt} - e^x)(1-t)^{\mu-1}(1+t)^{\mu} dt
\]

According to [24] it is clear that \( e_\mu(x) \geq 0 \), for \( \mu \geq 0, x \in \mathbb{R} \). However, this one is not the case when \( -1/2 < \mu < 0 \). Indeed, assume that \( -1/2 < \mu < 0 \). Since \( e^u - 1 \geq u, u > 0 \), we can write

\[
e^{-x}e_\mu(x) = 1 + \frac{\mu}{\mu + 1/2 B(1/2, \mu + 1)} \int_{-1}^{1} (e^{x(t-1)} - 1)(1-t)^{\mu-1}(1+t)^{\mu} dt
\]

Hence, there exists \( x_0 > 0 \) such that \( e_\mu(x) < 0 \) for every \( x < -x_0 \).

From the above we infer that the generalized Ornstein-Uhlenbeck semigroup \( \{T^t_\mu\}_{t>0} \) is a positive one when \( \mu \geq 0 \) but it is not when \( -1/2 < \mu < 0 \).

**Proposition 2.3.** (Natanson) Let \( f \) and \( g \) be two \( L^1(d\lambda) \) functions. Let us assume that \( g(y) \) is nonnegative and there is an \( x \in \mathbb{R} \) such that \( g(y) \) is monotonically increasing for \( y \leq x \) and monotonically decreasing for \( x \leq y \), then

\[
\left| \int g(y)f(y) \, d\lambda(y) \right| \leq \|g\|_{1, \lambda} \mathcal{M}_\lambda f(x)
\]

where

\[
\mathcal{M}_\lambda f(x) = \sup_{x \in I} \frac{1}{\lambda(I)} \int_I |f(y)| \, d\lambda(y)
\]

is the Hardy-Littlewood maximal function of \( f \) with respect to \( \lambda \). Moreover the Hardy-Littlewood maximal function \( \mathcal{M}_\lambda f \) is weak type \((1,1)\) and strong type \((p,p)\) for \( p > 1 \) with respect to the measure \( \lambda \).
A proof of this proposition can be found in [7].

**Proposition 2.4.** For \( \mu > -1/2 \), there is a real constant \( C > 0 \) such that the distribution function with respect to \( \lambda \) of the function

\[
h(x) = \max \left( \frac{1}{|x|}, |x| \right) \frac{e^{x^2}}{|x|^{2\mu}}
\]

satisfies the inequality

\[
\lambda \{ x \in \mathbb{R} : h(x) > \eta \} \leq \frac{C}{\eta},
\]

for any \( \eta > 0 \).

**Proof.** Since \( \lambda \) is a finite measure, it is enough to prove this result for \( \eta \geq e \).

Besides, due to the fact that \( h \) is even and \( \lambda \) is symmetric, then

\[
\lambda \{ x \in \mathbb{R} : h(x) > \eta \} = 2 \lambda \{ x > 0 : h(x) > \eta \}.
\]

Now

\[
\lambda \{ x > 0 : h(x) > \eta \} \leq \lambda \left\{ 0 < x < 1 : \frac{1}{x^{2\mu+1}} > \eta/e \right\} + \lambda \left\{ x > 1 : \frac{e^{x^2}}{x^{2\mu-1}} > \eta \right\}
\]

\[
= \int_0^{(e/\eta)^{1/(2\mu+1)}} x^{2\mu} e^{-x^2} \, dx + \int_{x_0}^{\infty} x^{2\mu} e^{-x^2} \, dx
\]

\[
= I + II
\]

with \( x_0 > 1 \) and \( \frac{e^{x_0^2}}{x_0^{2\mu-1}} = \eta \). Let us observe that

\[
I \leq \int_0^{(e/\eta)^{1/(2\mu+1)}} x^{2\mu} \, dx = \frac{e}{(1 + 2\mu)\eta},
\]

and

\[
II \leq C x_0^{2\mu-1} e^{-x_0^2} = \frac{C}{\eta}
\]

For last inequality see [1]. From these two bounds the conclusion of this proposition follows. \( \square \)

**Proof. of Theorem 2.1.**

In order to prove this theorem it suffices to show that there exists \( C > 0 \) such that

\[
\lambda \{ x \in (0, \infty) : T_{\mu,+}^* f(x) > \eta \} \leq \frac{C}{\eta} \| f \|_{1,\lambda}, \quad \eta > 0,
\]

and

\[
\| T_{\mu,+}^* f \|_{\infty} \leq C \| f \|_{\infty}
\]

for every \( f \geq 0 \), where

\[
T_{\mu,+}^* f(x) = \sup_{t > 0} |T_{\mu,+}^t f(x)|,
\]
and
\[ T^t_{\mu,+} f(x) = \frac{1}{(1 - e^{-2t})^{\mu+1/2}} \int_0^\infty e^{-e^{-2t} + \frac{x^2 y^2 r^2}{1-r^2}} e^\mu \left( \frac{2xye^{-t}}{1-e^{-2t}} \right) f(y) |y|^{2\mu} e^{-|y|^2} dy. \]

Indeed, let us write \( r = e^{-t} \), with \( t > 0 \). By (2.5), we have that
\[ K_r(x,y) \leq K_r(|x|, |y|), \quad x, y \in \mathbb{R}. \]

Then
\[ |T^t_{\mu} f(x)| \leq T^t_{\mu,+} f(|x|) + T^t_{\mu,+} |\tilde{f}|(|x|), \quad x \in \mathbb{R}, \]
being \( \tilde{f} = f(-x) \), \( x \in \mathbb{R} \). Hence,
\[ T^t_{\mu} f(x) \leq T^t_{\mu,+} f(|x|) + T^t_{\mu,+} |\tilde{f}|(|x|), \quad x \in \mathbb{R}, \]
and we can write, for every \( \eta > 0 \),
\[ \lambda \{ x \in \mathbb{R} : T^t_{\mu} f(x) > \eta \} \leq \lambda \{ x \in \mathbb{R} : T^t_{\mu,+} f(|x|) > \eta/2 \} + \lambda \{ x \in \mathbb{R} : T^t_{\mu,+} |\tilde{f}|(|x|) > \eta/2 \} \leq 2(\lambda \{ x \in (0, \infty) : T^t_{\mu,+} f(x) > \eta/2 \} + \lambda \{ x \in (0, \infty) : T^t_{\mu,+} |\tilde{f}|(x) > \eta/2 \}). \]

Thus (2.12) follows from (2.10), (2.11) and the fact that \( \|f\|_{1,\lambda} = \|\tilde{f}\|_{1,\lambda} \) and \( \|f\|_{\infty} = \|\tilde{f}\|_{\infty} \).

From now on let us assume \( f \geq 0 \) and \( x > 0 \). First let us prove the weak type (1, 1) inequality.

(1) Case \( \mu = 0 \). This case corresponds to the Ornstein-Uhlenbeck maximal operator which proved to be weak type (1, 1) by B. Muckenhoupt in [7].

(2) Case \( \mu > -1/2 \). By using (2.5) we can write
\[ T^t_{\mu,+} f(x) \leq \frac{C}{(1 - r^2)^{\mu+1/2}} \int_0^\infty e^{-\frac{(x^2 + y^2 r^2)}{1-r^2}} \left( 1 + \frac{2xyr}{1-r^2} \right)^{-\mu} f(y) d\lambda(y) \]
\[ = \frac{Ce^{x^2}}{(1 - r^2)^{\mu+1/2}} \int_0^\infty e^{-\frac{|x-y|r^2}{1-r^2}} \left( 1 + \frac{2xyr}{1-r^2} \right)^{-\mu} f(y) d\lambda(y) \]
\[ = \frac{Ce^{x^2}}{(1 - r^2)^{\mu+1/2}} \left( \int_0^{x/2r} + \int_{x/2r}^{4x/r} + \int_{4x/r}^{\infty} \right) e^{-\frac{|x-y|r^2}{1-r^2}} \left( 1 + \frac{2xyr}{1-r^2} \right)^{-\mu} f(y) d\lambda(y) \]
\[ = C(K_{1,r} f(x) + K_{2,r} f(x) + K_{3,r} f(x)). \]

Let us observe that if \( 0 < y < x/2r \), then \( x - ry > x/2 \) and
\[ \frac{1}{(1 - r^2)^{\mu+1/2}} \left( 1 + \frac{2xyr}{1-r^2} \right)^{-\mu} \leq \frac{1}{(1 - r^2)^{\mu+1/2}} + \frac{x^{-2\mu}}{(1 - r^2)^{1/2}}, \]
thus
\[ K_{1,r} f(x) \leq Ce^{x^2} \left( \frac{1}{(1 - r^2)^{\mu+1/2}} + \frac{x^{-2\mu}}{(1 - r^2)^{1/2}} \right) e^{-\frac{4x^2}{1-r^2}} \|f\|_{1,\lambda} \leq C e^{x^2} \|f\|_{1,\lambda} \]
where last inequality is obtained as an application of (1.10).
On the other hand, if \( y > \frac{4x}{r} \), then \( ry - x > x \), and again by applying (1.9) repeatedly in the sequel below

\[
\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}} \left( 1 + \frac{2rxy}{1-r^2} \right)^{-\mu} = \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}} \left( 1 + \frac{2r(ry-x) + 2x^2}{1-r^2} \right)^{-\mu} \\
\leq C \frac{e^{-\frac{x^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}} \left( 1 + \frac{1}{1-r^2} \mu + \frac{x^{-\mu}}{(1-r^2)^{1/2}} + \frac{x^{-2\mu}}{(1-r^2)^{1/2}} \right) \\
\leq C \frac{e^{-x^2}}{x^{2\mu+1}},
\]

we get

\[
K_{3,r} f(x) \leq C \frac{e^{x^2}}{x^{2\mu+1}} \| f \|_{1,\lambda}.
\]

Finally for \( \frac{x}{2r} \leq y \leq \frac{4x}{r} \) we have the following estimate

\[
(2.12) \quad \frac{1}{(1-r^2)^{\mu+1/2}} \left( 1 + \frac{2rxy}{1-r^2} \right)^{-\mu} \leq \frac{1}{x^{2\mu+1}} + \frac{x^{-2\mu}}{(1-r^2)^{1/2}},
\]

which is immediate for \( \mu \geq 0 \) and for \( \mu < 0 \) one has to argue between \( \frac{2rxy}{1-r^2} \leq 1 \) and its complement. Now by taking into account inequality (2.12) we are ready to estimate \( K_{2,r} f(x) \) and for that we consider two cases. If \( 0 < r \leq 1/2 \) we have

\[
K_{2,r} f(x) \leq C \left( \frac{1}{x} + 1 \right) \frac{e^{x^2}}{x^{2\mu}} \| f \|_{1,\lambda},
\]

and, if \( 1/2 < r < 1 \) then

\[
K_{2,r} f(x) \leq C \left( \frac{e^{x^2}}{x^{2\mu+1}} \| f \|_{1,\lambda} + \frac{e^{x^2}}{(1-r^2)^{1/2} x^{2\mu}} \int_0^\infty N(r, x, y) f(y) d\lambda(x) \right),
\]

with

\[
(2.13) \quad N(r, x, y) = \begin{cases} 
1 & \text{if } y \in [x, \frac{x}{r}] \\
e^{-\frac{|x-ry|^2}{1-r^2}} & \text{if } y \in \left[ \frac{x}{2r}, \frac{4x}{r} \right) \setminus \left[ x, \frac{x}{r} \right] \\
0 & \text{otherwise}.
\end{cases}
\]

Since \( N(r, x, \cdot) \) is a Natanson kernel (see (2.9)), we get

\[
K_{2,r} f(x) \leq C \left( \frac{e^{x^2}}{x^{2\mu+1}} \| f \|_{1,\lambda} + \frac{e^{x^2}}{x^{2\mu}(1-r^2)^{1/2}} \| N(r, x, \cdot) \|_{1,\lambda} \mathcal{M}_\lambda f(x) \right),
\]

Let us prove that

\[
(2.14) \quad \| N(r, x, \cdot) \|_{1,\lambda} \leq C x^{2\mu}(1-r^2)^{1/2} e^{-x^2}.
\]
Indeed,
\[
\int_{\mathbb{R}} N(r, x, y) \, d\lambda(y) = \int_x^{x/r} e^{-y^2} y^{2\mu} \, dy + \int_{x/2r}^x e^{-\frac{|r-y|^2}{1-r^2}} e^{-y^2} y^{2\mu} \, dy \\
+ \int_{x/r}^{4x/r} e^{-\frac{|r-y|^2}{1-r^2}} e^{-y^2} y^{2\mu} \, dy \\
\sim x^{2\mu} \left( \int_x^{x/r} e^{-y^2} \, dy + e^{-x^2} \int_{x/2r}^x e^{-\frac{|r-y|^2}{1-r^2}} \, dy \right) \\
+ e^{-x^2} \int_{x/r}^{4x/r} e^{-\frac{|r-y|^2}{1-r^2}} \, dy \\
\leq C x^{2\mu} e^{-x^2} \left( \min \left( \frac{1}{x}, (1-r)x \right) \\
+ \int_{\mathbb{R}} e^{-\frac{|r-y|^2}{1-r^2}} \, dy \right) \\
\leq C x^{2\mu} (1-r^2)^{1/2} e^{-x^2}.
\]
Now gathering together all the bounds obtained above, we get
\[
T_{\mu, +}^t f(x) \leq C(h(x)) \|f\|_{1, \lambda} + M_\lambda f(x),
\]
for all \( t > 0 \), where \( h \) is the function defined in Proposition 2.4. Thus the weak type \((1, 1)\) of \( T_{\mu, +}^* \) follows from propositions 2.3 and 2.4.

Now let us take care of the boundedness of \( T_{\mu, +}^* \) in \( L^\infty \).
For the case \( \mu \geq 0 \) this boundedness is immediate since its kernel is non-negative and its integral equals 1. Therefore let us study just the case \(-1/2 < \mu < 0\). By using 2.3 and proceeding like in case 2 of the weak type \((1, 1)\) inequality
\[
T_{\mu, +}^* f(x) \leq \frac{C}{(1-r^2)^{\mu+1/2}} \int_0^\infty e^{-\frac{|r-y|^2}{1-r^2}} \left( 1 + \frac{2xyr}{1-r^2} \right)^{-\mu} f(y) \, d\lambda(y) \\
\leq \frac{C}{(1-r^2)^{\mu+1/2}} \int_0^\infty e^{-\frac{|r-y|^2}{1-r^2}} \left( 1 + \frac{2xyr}{1-r^2} \right)^{-\mu} y^{2\mu} \, dy \|f\|_\infty \\
= \frac{C}{(1-r^2)^{\mu+1/2}} \int_0^\infty e^{-\frac{|r-y|^2}{1-r^2}} \left( 1 + \frac{2(rx-y)y}{1-r^2} + \frac{2y^2}{1-r^2} \right)^{-\mu} y^{2\mu} \, dy \|f\|_\infty \\
\leq C \left( \int_0^\infty e^{-\frac{|r-y|^2}{1-r^2}} \left( 1 + \frac{2|rx-y||y|}{1-r^2} \right)^{-\mu} y^{2\mu} \, dy \\
+ \int_0^\infty e^{-\frac{|r-y|^2}{1-r^2}} \, d\lambda(y) \right) \|f\|_\infty \\
\leq C \left( \frac{2|rx-y||y|}{1-r^2} \right)^{-\mu} e^{-\frac{|r-y|^2}{1-r^2}} \leq C \left( \frac{y}{(1-r^2)^{1/2}} \right)^{-\mu} e^{-\frac{|r-y|^2}{2(1-r^2)}},
\]
then we split the integral in two subintervals one from 0 to \( \sqrt{1-r^2} \) and the other from \( \sqrt{1-r^2} \) to \( \infty \) and we call them \( I \) and \( II \). Now we proceed to bound each
part.

\[ I = \int_0^{\sqrt{1-r^2}} e^{-\frac{1}{2} \left(1-r^2\right)} \left(1 + \left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu}\right) y^{2\mu} dy \leq \int_0^{\sqrt{1-r^2}} \frac{y^{2\mu}}{1-r^2} dy + \int_0^{\sqrt{1-r^2}} \frac{y^{\mu}}{(1-r^2)^{\mu/2}} dy \leq C, \]

and

\[ II = \int_0^{\sqrt{1-r^2}} e^{-\frac{1}{2} \left(1-r^2\right)} \left(1 + \left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu}\right) y^{2\mu} dy \leq \int_0^{\sqrt{1-r^2}} \frac{e^{-y^2/2}}{1-r^2} dy + \int_0^{\sqrt{1-r^2}} \frac{e^{-y^2/2}}{1-r^2} y^{\mu} dy \leq 2 \int_0^{\sqrt{1-r^2}} \frac{e^{-y^2/2}}{1-r^2} dy \leq C. \]

This ends the proof of the boundedness of \( T_{\mu,+}^* \) in \( L^\infty \) and at the same time the proof of Theorem 2.1.

\[ \square \]

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