A Moser/Bernstein type theorem in a Lie group with a left invariant metric under a gradient decay condition

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Abstract. We say that a PDE on the hyperbolic space $\mathbb{H}^n$ of constant sectional curvature $-1$, $n \geq 2$, is geometric if, whenever $u$ is a solution of the PDE on a domain $\Omega$ of $\mathbb{H}^n$, the composition $u_\phi := u \circ \phi$ is also a solution on $\phi^{-1}(\Omega)$ for any isometry $\phi$ of $\mathbb{H}^n$. We prove that if $u \in C^1(\mathbb{H}^n)$ is a solution of a geometric PDE satisfying the comparison principle and if
\[
\limsup_{r \to \infty} \left( e^{2r} \sup_{S_r} \|\nabla u\| \right) = 0,
\]
where $S_r$ is a geodesic sphere of $\mathbb{H}^n$ centered at a fixed point $o \in \mathbb{H}^n$ with radius $r$, then $u$ is constant. However, given $C > 0$, there exists a bounded non-constant harmonic function $v \in C^\infty(\mathbb{H}^n)$ such that
\[
\lim_{r \to \infty} \left( e^r \sup_{S_r} \|\nabla v\| \right) = C.
\]
We prove (1) by showing a similar result for left invariant PDEs on a Lie group and by endowing $\mathbb{H}^n$ with a Lie group structure.

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1. Introduction. Finding a characterization of entire minimal graphs on $\mathbb{R}^n$ is a long standing problem in differential geometry. The first well known result, due to Bernstein [3], asserts that the only entire solutions of the minimal surface equation (MSE) in $\mathbb{R}^2$ are affine functions. Simons [13] extended Bernstein’s result to $\mathbb{R}^n$ for $2 \leq n \leq 7$ and a celebrated work of Bombieri et al. [1] proved that Bernstein’s theorem is false in $\mathbb{R}^n$ for $n \geq 8$. A full characterization of entire minimal graphs of $\mathbb{R}^n$ was given in terms of the gradient at
infinity of the solutions. Indeed, J. Moser proved that an entire solution for the MSE in $\mathbb{R}^n$, $n \geq 2$, is an affine function if and only if the norm of the gradient of the solution is bounded [8, corollary of Theorem 6]. A thoroughly nice account about entire graphs with prescribed mean curvature on a Riemannian manifold can be found in the work [2].

Moser’s result is not true in the hyperbolic space: there are entire solutions of the MSE in $\mathbb{H}^n$, $n \geq 2$, which assume a prescribed continuous non-constant value at infinity and with bounded gradient (it is a consequence, for instance, of [12, Theorem 3.14]). In this note, we prove that if the norm of the gradient converges exponentially to zero, then the solution must be constant.

Our main result applies to the solutions of a quite broad class of partial differential equations, according to the following definitions.

**Definition 1.** We say that a partial differential equation (PDE) on a complete Riemannian manifold $M$ satisfies the *comparison principle* if, given a bounded domain $\Omega \subset M$, if $u$ and $v$ are solutions of the PDE in $\Omega$ and $u \leq v$ in $\partial \Omega$ that is

$$\limsup_k (u(x_k) - v(x_k)) \leq 0$$

for any sequence $x_k \in \Omega$ that leaves any compact subset of $\Omega$, then $u \leq v$ in $\Omega$.

**Definition 2.** We say that a PDE in a Riemannian manifold $M$ is a *geometric* PDE if, whenever $u$ is a solution of the PDE on a domain $\Omega$ of $M$, the composition $u_\phi := u \circ \phi$ is also a solution on $\phi^{-1}(\Omega)$ for any isometry $\phi$ of $M$.

Any PDE depending only on $u$, $\nabla u$, $\nabla^2 u$, and other geometric operators as $\Delta$ and div are invariant by isometries and hence are geometric PDEs. Those of the form

$$\text{div} \left( \frac{a(\|\nabla u\|)}{\|\nabla u\|} \nabla u \right) + C = 0, C \text{ constant},$$

where $a \in C^1((0, \infty))$, $a' > 0$, $a(0) = 0$ also satisfy the comparison principle [12, Proposition 3.1]. This class includes the minimal surface equation with

$$a(s) = \frac{s}{\sqrt{1 + s^2}}$$

and the $p$-Laplace PDE where

$$a(s) = s^{p-1}, p > 1.$$
Theorem 3. Let $H^n$ be the hyperbolic space of constant sectional curvature $-1$, $n \geq 2$. Let $u \in C^1 (H^n)$ be a solution of a geometric PDE satisfying the comparison principle. If
\[
\limsup_{r \to \infty} \left( e^{2r} \sup_{S_r} \| \nabla u \| \right) = 0,
\]
where $S_r$ is a geodesic sphere of $H^n$ centered at a fixed point $o \in H^n$ with radius $r$, then $u$ is constant. Moreover, given $C > 0$, there is a bounded non-constant harmonic function $v \in C^\infty (H^n)$ such that
\[
\lim_{r \to \infty} \left( e^{r} \sup_{S_r} \| \nabla v \| \right) = C.
\]

Theorem 3 raises the following questions: can the condition on the decay of the gradient be improved? Are there examples of harmonic functions having faster gradient decay which are not constant?

The first part of Theorem 3 is actually a consequence of a more general result that applies to a Lie group with a left invariant metric and to the solutions of a broader class of partial differential equations which we call left invariant PDEs. To state it, we first recall and introduce some notations and definitions.

Let $G$ be a Lie group, $e$ its neutral element, and $\mathfrak{g} = T_e G$ its Lie algebra. Given $g \in G$, denote by $R_g$ and $L_g$ the right and left translations of $G$, $R_g(x) = xg$ and $L_g(x) = gx$, $x \in G$, and by $C_g = L_g \circ R_g^{-1}$ the conjugation. Let $\text{Ad}_g = d(C_g)_e : \mathfrak{g} \to \mathfrak{g}$ be the adjoint map of $G$. Given an inner product $\langle \cdot, \cdot \rangle_e$ in $\mathfrak{g}$, let $\langle \cdot, \cdot \rangle$ be the left invariant metric of $G$ determined by $\langle \cdot, \cdot \rangle_e$, namely
\[
\langle u, v \rangle_g := \langle d(L_g^{-1})_g u, d(L_g^{-1})_g v \rangle_e, g \in G, u, v \in T_g G.
\]

Definition 4. We say that a PDE in $G$ is a left invariant PDE if, whenever $u$ is a solution of the PDE on a domain $\Omega$ of $G$, the composition $u_g := u \circ L_g$ is also solution on $L_g^{-1}(\Omega)$ for any $g \in G$.

Clearly geometric PDEs on $G$ are left invariant. However, the converse is not true. For example, considering $\mathbb{R}^n$ as a commutative Lie group, the PDE $\Delta u = X(u)$, where $X$ is a vector field of $\mathbb{R}^n$ is left invariant but not geometric in general.

Theorem 5. Let $\Omega$ be a bounded $C^1$ domain on a Lie group $G$ with a left invariant metric and let $u \in C^1(\overline{\Omega})$ be a solution of a left invariant PDE satisfying the comparison principle on $\Omega$. Then the following estimates for the gradient hold:
\[
\sup_{\Omega} \| \nabla u \| \leq \sup_{g \in \Omega \cup \Omega^{-1}} \| \text{Ad}_g \|^2 \sup_{\partial \Omega} \| \nabla u \|, \tag{3}
\]
\[
\sup_{\Omega} \| \nabla u \| \leq \sup_{g \in B_{\text{diam}(\Omega)}} \| \text{Ad}_g \| \sup_{\partial \Omega} \| \nabla u \|. \tag{4}
\]
where $B_{\text{diam}(\Omega)}$ is the geodesic ball centered at $e$ with radius
\[
\text{diam}(\Omega) := \sup \{ d(x, y) \mid x, y \in \Omega \}.
\]
Clearly $\|\text{Ad}_g\| = 1$ for any $g \in G$ if the metric of $G$ is bi-invariant. In this case, then, the solution satisfies the maximum principle for the gradient, that is, equality holds in (3).

**Corollary 6.** Let $u \in C^1(G)$ be a solution of a left invariant PDE satisfying the comparison principle on a non-compact Lie group $G$ with a left invariant metric. If one of the equalities below holds

$$\limsup_{r \to \infty} \left( \sup_{g \in B_{2r}} \|\text{Ad}_g\| \sup_{S_r} \|\nabla u\| \right) = 0,$$

$$\limsup_{r \to \infty} \left( \sup_{g \in B_r} \|\text{Ad}_g\|^2 \sup_{S_r} \|\nabla u\| \right) = 0,$$

where $B_r$ is the geodesic ball of $G$ centered at $e$ with radius $r$ and $S_r = \partial B_r$, then $u$ is constant. Moreover, if $G$ has negative sectional curvature, then one can replace $\sup_{g \in B_{2r}} \|\text{Ad}_g\|$ and $\sup_{g \in B_r} \|\text{Ad}_g\|$ by $\sup_{g \in S_{2r}} \|\text{Ad}_g\|$ and $\sup_{g \in S_r} \|\text{Ad}_g\|$ respectively.

The last assertion of Corollary 6 is a consequence of the following general fact, namely, if $G$ has negative sectional curvature, then the norm of the adjoint map satisfies the strong maximum principle (Proposition 11). A well known paper of Milnor [7] studies Lie groups with a left invariant metric. More recently, 3-dimensional Lie groups with a left invariant metric have been classified and an explicit description is given in [9]. Finally we mention that it follows by the Iwasawa decomposition that any symmetric space of non-compact type is isometric to a Lie group with a left invariant metric [5].

2. Preliminary facts, proofs of Theorem 5 and Corollary 6. We introduce some notation and prove some preliminary results to be used in the proofs of Theorem 5 and Corollary 6. Given $h$ and $g$ in $G$, we denote

$$R(g)_h = \|(R_g)_{*h}\|$$

where $(R_g)_{*h}$ is the derivative of $R_g$ at the point $h$.

**Lemma 7.** The quantity $R(g)_h$ does not depend on $h$.

So, from now on we denote

$$R(g) = \|(R_g)_{*e}\|$$

and introduce, for a subset $S$ of $G$,

$$R_S = \max_{g \in S} R(g).$$

**Proof.** We first notice that the left and right translations commute, i.e., if $g_1, g_2, h \in G$,

$$L_{g_1} \circ R_{g_2}(h) = R_{g_2} \circ L_{g_1}(h) = g_1hg_2.$$
Let $h, g \in G$. Then

$$R(g)_h^2 = \sup_{X \in T_h G, \|X\|=1} \langle (R_g)_*X, (R_g)_*X \rangle_{h g}$$

$$= \sup_{X \in T_h G, \|X\|=1} \langle (L_{(h g)^{-1}})_*(R_g)_*X, (L_{(h g)^{-1}})_*(R_g)_*X \rangle_e$$

$$= \sup_{X \in T_h G, \|X\|=1} \langle (L_1)_*(L_{h^{-1}})_*(R_g)_*X, (L_1)_*(L_{h^{-1}})_*(R_g)_*X \rangle_e$$

$$= \sup_{X \in T_h G, \|X\|=1} \langle (L_{h^{-1}})_*(R_g)_*(L_{h^{-1}})_*X, (L_{h^{-1}})_*(R_g)_*(L_{h^{-1}})_*X \rangle_e \tag{5}$$

For $X \in T_h G$, $Z = (L_{h^{-1}})_*X$ belongs to $T_e G$; moreover

$$\|X\|_h = \|(L_{h^{-1}})_*X\|_e$$

and we can rewrite (5) as

$$\tag{5} = \sup_{Z \in T_e G, \|Z\|=1} \langle (L_{h^{-1}})_*(R_g)_*Z, (L_{h^{-1}})_*(R_g)_*Z \rangle_e$$

$$= \sup_{Z \in T_e G, \|Z\|=1} \langle (R_g)_*Z, (R_g)_*Z \rangle_g = \|(R_g)_*e\|^2.$$ 

\[\square\]

**Lemma 8.** If $g \in G$,

$$R(g) = \| \text{Ad}_g \|.$$

**Proof.** If $X \in T_e G$, by definition of the metric,

$$\langle (R_g)_*X, (R_g)_*X \rangle_g = \langle (L_{h^{-1}})_*(R_g)_*X, (L_{h^{-1}})_*(R_g)_*X \rangle_e$$

$$= \langle \text{Ad}_g(X), \text{Ad}_g(X) \rangle_e.$$

\[\square\]

Denote by $d$ the Riemannian distance in $G$. The next lemma is elementary and its proof is therefore omitted:

**Lemma 9.** Let $\Omega$ be a $C^1$ open subset of $G$. Let $u \in C^1(\overline{\Omega})$ and $x_0 \in \overline{\Omega}$ be given and assume that $\|\nabla u\|(x_0) \neq 0$. Let $\gamma : [0, \varepsilon) \rightarrow \Omega$ be an arc length geodesic such that $\gamma(0) = x_0$ and

$$\gamma'(0) = \frac{\nabla u(x_0)}{\|\nabla u(x_0)\|} \text{ or } \gamma'(0) = -\frac{\nabla u(x_0)}{\|\nabla u(x_0)\|}.$$

Then

$$\|\nabla u\|(x_0) = \lim_{t \to 0} \frac{|u(\gamma(t)) - u(x_0)|}{d(\gamma(t), x_0)}.$$

**Lemma 10.** Let $a, b, g$ be elements in $G$. Denoting by $d$ the distance on $G$, we have

$$d(ag, bg) \leq R(g)d(a, b),$$

$$d(a, b) \leq R(g^{-1})d(ag, bg).$$
Proof. If $\gamma$ is a minimizing geodesic between $a$ and $b$, then $d(a, b) = \text{length}(\gamma)$. Also, since $R_g \circ \gamma$ is a path between $ag$ and $bg$,

$$d(ag, bg) \leq \text{length}(R_g \circ \gamma) \leq R(g) \text{length}(\gamma) = R(g)d(a, b).$$

We have thus proved the first inequality of the lemma. We derive

$$d(a, b) = d((ag)g^{-1}, (bg)g^{-1}) \leq R(g^{-1})d(ag, bg)$$

which proves the second inequality. \hfill $\square$

Proof of Theorem 5. The proof is an extension of [6, Lemma 12.7] (see also [14]). Let $\Omega$ be a bounded $C^1$ domain on $G$ and let $u \in C^1(\overline{\Omega})$ be a solution of a left invariant PDE satisfying the comparison principle. Take

$$k = \sup \left\{ \frac{|u(x) - u(z)|}{d(x, z)} \middle| x \in \Omega, z \in \partial \Omega \right\}.$$

From Lemma 9, for the estimate (3), it is enough to prove that

$$|u(x) - u(y)| \leq k \sup_{z \in \Omega \cup \Omega^{-1}} \|\text{Ad}_z\|^2 d(x, y)$$

for all $x, y \in \Omega$. Given $x_1, x_2 \in \Omega$, set $z = x_1 x_2^{-1}$,

$$\Omega_z = \{ z^{-1} x \in G \mid x \in \Omega \} = \{ x \in G \mid zx \in \Omega \},$$

and define $u_z \in C^1(\overline{\Omega}_z)$ by $u_z(x) = u(zx)$. We have $\Omega \cap \Omega_z \neq \emptyset$ since $x_2$ belongs to both $\Omega$ and $\Omega_z$. By the comparison principle,

$$\sup \{ u(x) - u_z(x) \mid x \in \Omega \cap \Omega_z \} = \sup \{ u(x) - u_z(x) \mid x \in \partial (\Omega \cap \Omega_z) \}.$$

Then, in particular,

$$u(x_2) - u(x_1) = u(x_2) - u_z(x_2) \leq \sup \{ u(x) - u_z(x) \mid x \in \partial (\Omega \cap \Omega_z) \}.$$

Let $x_0 \in \partial (\Omega \cap \Omega_z)$ be such that

$$u(x_0) - u(zx_0) = \sup \{ u(x) - u(zx) \mid x \in \partial (\Omega \cap \Omega_z) \}.$$

If $x_0 \in \partial (\Omega \cap \Omega_z)$, then either $x_0 \in \partial \Omega$ or $zx_0 \in \partial \Omega$. It then follows from the hypothesis that

$$u(x_2) - u(x_1) \leq kd(x_0, zx_0).$$

(7)

Now, using Lemma 10 twice, we have

$$d(x_0, zx_0) = d(x_0, (x_1 x_2^{-1}) x_0) \leq R(x_0)d(e, x_1 x_2^{-1}) \leq R(x_0)R(x_2^{-1}) d(x_2, (x_1 x_2^{-1}) x_0)$$

$$= R(x_0)R(x_2^{-1}) d(x_2, x_1) \leq \sup_{z \in \Omega \cup \Omega^{-1}} \|\text{Ad}_z\|^2 d(x_2, x_1).$$

With (7) we obtain (6), proving (3).

For proving (4), we observe that

$$d(x_0, zx_0) = d(x_0, (x_1 x_2^{-1}) x_0)$$

$$= d(x_2 x_2^{-1} x_0, (x_1 x_2^{-1}) x_0) \leq R(x_2^{-1} x_0) d(x_2, x_1).$$
Since
\[ d(x_2^{-1}x_0, e) = d(x_0, x_2) \leq \text{diam}(\Omega) \]
since
\[ x_0, x_2 \in \Omega, \]
it follows that \( x_2^{-1}x_0 \in B_{\text{diam}(\Omega)} \) and
\[ R(x_2^{-1}x_0) \leq \sup_{z \in B_{\text{diam}(\Omega)}} \| \text{Ad}_z \|. \]
From (7), we then obtain
\[ u(x_2) - u(x_1) \leq k \sup_{z \in B_{\text{diam}(\Omega)}} \| \text{Ad}_z \| d(x_1, x_2) \]
which, by Lemma 9, proves (4). \( \Box \)

**Proof of Corollary 6.** Given \( r > 0 \), if \( g \in S_r \), then
\[ r = d(e, g) = d(g^{-1}, e) = d(e, g^{-1}) \]
and hence \( S_r^{-1} = S_r \). It follows that \( B_r^{-1} = B_r \), where \( B_r \) is the geodesic ball centered at \( e \) with radius \( r \). The estimates of Corollary 6 then follow from Theorem 5. \( \Box \)

We close this section by proving a strong maximum principle for the adjoint map.

**Proposition 11.** Let \( G \) be a Lie group with negative sectional curvature and let \( \Lambda \) be any open subset of \( G \). Then
\[ \| \text{Ad}_h \| < \sup_{g \in \partial \Lambda} \| \text{Ad}_g \| \]
for all \( h \in \Lambda \). In particular,
\[ \sup_{g \in \Lambda} \| \text{Ad}_g \| = \sup_{g \in \partial \Lambda} \| \text{Ad}_g \|. \]

**Proof.** By contradiction, assume that
\[ \| \text{Ad}_h \| = \sup_{g \in \Lambda} \| \text{Ad}_g \| \]
for some \( h \in \Lambda \). There exists \( x \in T_e G, \| x \| = 1 \), such that \( \| \text{Ad}_h \| = \| \text{Ad}_h(x) \| \). Let \( X \) be the right invariant vector field of \( G \) such that \( X(e) = x \). Choose \( u \in T_h G, \| u \| = 1 \), such that \( u \neq X(h) \) and let \( \gamma(t), t \geq 0 \), be the geodesic parametrized by arc length in \( G \) such that \( \gamma(0) = h \) and \( \gamma'(0) = u \). Then
\[ \frac{d}{dt} \| \text{Ad}_{\gamma(t)}(x) \|^2 \bigg|_{t=0} = 0, \quad (8) \]
\[ \frac{d^2}{dt^2} \| \text{Ad}_{\gamma(t)}(x) \|^2 \bigg|_{t=0} \leq 0. \quad (9) \]
We have
\[ \| \text{Ad}_{\gamma(t)}(x) \|^2 = \langle \text{Ad}_{\gamma(t)}(x), \text{Ad}_{\gamma(t)}(x) \rangle \]
\[ = \left\langle d \left( L_{\gamma(t)}^{-1} \right) (x), d \left( L_{\gamma(t)}^{-1} \right) (x) \right\rangle \]
\[ = \langle d \left( R_{\gamma(t)}(x) \right), d \left( R_{\gamma(t)}(x) \right) \rangle, \quad t \geq 0. \]

Then
\[ \| \text{Ad}_{\gamma(t)}(x) \|^2 = \langle X(\gamma(t)), X(\gamma(t)) \rangle = \| X(\gamma(t)) \|^2, \quad t \geq 0. \]

Since right invariant vector fields are Killing fields and Killing fields restricted to geodesics are Jacobi vector fields,
\[ J(t) := X(\gamma(t)) \]
is a Jacobi field along \( \gamma \), \( t \geq 0 \). Thus, \( \| \text{Ad}_{\gamma(t)}(x) \|^2 \) is equal to the square of the norm of the Jacobi field \( J(t) \) along \( \gamma \) satisfying the initial conditions
\[ J(0) = X(h), \]
\[ J'(0) = \nabla_{\gamma'(0)} X = \nabla_u X. \]

We have, from (8),
\[ \frac{d}{dt} \| J(t) \|^2 \bigg|_{t=0} = 2 \langle J'(0), J(0) \rangle = 0. \]
And, using the Jacobi equation,
\[ \frac{d^2}{dt^2} \| J(t) \|^2 = 2 \langle J''(t), J(t) \rangle + 2 \langle J'(t), J'(t) \rangle \]
\[ = -2 \langle R(\gamma', J) \gamma', J(t) \rangle + 2 \| J'(t) \|^2. \]

Since \( u \neq X(h) \), the vector fields \( J(t) = X(\gamma(t)) \) and \( \gamma'(t) \) are linearly independent along \( \gamma \) and we then have
\[ \frac{d^2}{dt^2} \| J(t) \|^2 = -2K(\gamma', J) \| \gamma' \wedge J \|^2 + 2 \| J' \|^2. \]

Since \( K < 0 \), it follows that
\[ \frac{d^2}{dt^2} \| J(t) \|^2 \bigg|_{t=0} > 0 \]
contradicting (9). This proves the proposition. \( \square \)

3. The hyperbolic space. Proof of Theorem 3. We begin by calculating the norm of the adjoint map in the hyperbolic space to apply Corollary 6. We present an explicit construction of the Lie group structure of the hyperbolic space, that comes from the Iwasawa decomposition [5].

In the half-space model
\[ \mathbb{H}^n = \{(x_1, ..., x_n) \mid x_n > 0 \}, \quad ds^2 = \delta_{ij} \frac{dx_i^2}{x_n^2}, \]
of the hyperbolic \( n \)-dimensional space, \( n \geq 2 \), given \( s > 0 \) and \( t := (t_1, ..., t_{n-1}) \in \mathbb{R}^{n-1} \), define

\[
a_s, n_t : \mathbb{H}^n \to \mathbb{H}^n,
\]

\[
a_s (x_1, ..., x_n) = s (x_1, ..., x_n),
\]

\[
n_t (x_1, ..., x_n) = (x_1 + t_1, ..., x_{n-1} + t_{n-1}, x_n).
\]

Set

\[
G := AN = \{ a_s \circ n_t \mid s > 0, t \in \mathbb{R}^{n-1} \}
\]

where

\[
A = \{ a_s \mid s > 0 \},
\]

\[
N = \{ n_t \mid t \in \mathbb{R}^{n-1} \}.
\]

Given \( p \in \mathbb{H}^n \), there is one and only one \( g_p \in G \subset \text{Iso} (\mathbb{H}^n) \) such that \( p = g_p ((0, ..., 0, 1)) \). Indeed: if \( p = (x_1, ..., x_n) \in \mathbb{H}^n \), define \( n := n(x_1, ..., x_{n-1}, a), a := ax_n \in \text{Iso} (\mathbb{H}^n) \), that is,

\[
n (z_1, ..., z_n) = (z_1 + x_1, ..., z_{n-1} + x_{n-1}, z_n)
\]

\[
a = ax_n (z_1, ..., z_n), (z_1, ..., z_n) \in \mathbb{H}^n.
\]

Then, taking

\[
g_p = n \circ a,
\]

we have

\[
g_p (0, ..., 0, 1) = n (a (0, ..., 0, 1)) = n (0, ..., 0, x_n) = (x_1, ..., x_n) = p.
\]

Note that \( n \) and \( a \) are not uniquely determined by \( p \), but \( g_p \) is.

One may see that with the operation

\[
p \cdot q := (g_p \circ g_q) ((0, ..., 0, 1))
\]

\( \mathbb{H}^n \) is a Lie group (a solvable Lie group indeed). This is a general fact that holds for symmetric spaces of non-compact type \([5, \text{Chapter VI}]\)). Moreover, given \( p \in \mathbb{H}^n \), we have, for any \( q \in \mathbb{H}^n \),

\[
L_p (q) = p \cdot q = (g_p \circ g_q) ((0, ..., 0, 1)) = g_p (g_q ((0, ..., 0, 1))) = g_p (q),
\]

that is, \( L_p = g_p \). Since \( g_q \in \text{Iso} (\mathbb{H}^n) \), it follows that the left translation \( L_p \) is an isometry of \( \mathbb{H}^n \) with respect to the hyperbolic metric, that is, the hyperbolic metric is left invariant with respect to the Lie group structure of \( \mathbb{H}^n \).

**Proposition 12.** Consider \( \mathbb{H}^n \) as a Lie group and let \( e \) be its neutral element. If \( B_r \) is the closed geodesic ball of \( \mathbb{H}^n \) centered at \( e \) with radius \( r \), then

\[
\max_{g \in B_r} \| \text{Ad}_g \| = \cosh r + \sinh r.
\]

**Proof.** We claim that it is enough to consider the 2-dimensional case. Indeed: let \( g \in B_r \) and \( x \in T_e \mathbb{H}^n \) be such that

\[
\| \text{Ad}_g (x) \| = \max_{h \in B_r} \| \text{Ad}_h \|.
\]
Let $y \in T_e \mathbb{H}^n$ be a nonzero vector tangent to the geodesic from $e$ to $g$. There exists a totally geodesic hyperbolic plane $\mathbb{H}^2$ of $\mathbb{H}^n$ such that $e \in \mathbb{H}^2$ and $x, y \in T_e \mathbb{H}^2$. Since $\mathbb{H}^2$ is a Lie subgroup of $\mathbb{H}^n$, we have

$$\|\text{Ad}_y (x)\| = \max_{h \in D_e} \|\text{Ad}_h\|$$

where $D_e = B_e \cap \mathbb{H}^2$ is the closed geodesic disk centered at $e$ with radius $r$. It is clear that the maximum of the norm of the adjoint map does not depend on the hyperbolic plane containing $e$. This proves our claim.

Considering the half-plane model for $\mathbb{H}^2$ with $e = (0, 1)$, given $p \in \mathbb{H}^2$, the conjugation $C_p : \mathbb{H}^2 \to \mathbb{H}^2$ is given by

$$C_p (q) = (g_p \circ g_q \circ g_p^{-1}) (0, 1) = g_p \left( g_q \left( g_p^{-1} (0, 1) \right) \right). \quad (12)$$

Expanding (12), we arrive at

$$C_{(x,y)} (z,w) = (-xw + yz + x,w)$$

from which we obtain, at a given $X := (a,b) \in T_{(0,1)} \mathbb{H}^2$,

$$\text{Ad}_{(x,y)} (a,b) = d(C_{(x,y)})_e (a,b) = (-xb + ya, b).$$

Since at $(0, 1)$ the hyperbolic metric coincides with the Euclidean metric and since $\text{Ad}_{(x,y)} (X) \in T_{(0,1)} \mathbb{H}^2$, we obtain

$$\|\text{Ad}_{(x,y)} (X)\| = \sqrt{(-xb + ya)^2 + b^2}. \quad (13)$$

Since the identity of $\mathbb{R}^2_+$ is a conformal map between the Euclidean and hyperbolic geometries, the hyperbolic geodesic circle in the half plane model of $\mathbb{H}^2$ centered at $(0, 1)$ with hyperbolic radius $r$ is also an Euclidean circle. Moreover, because an Euclidean symmetry with respect to a vertical line is also a hyperbolic isometry, the Euclidean center of the circle is at some point $(0, y_0)$ of the vertical line $x = 0$. The Euclidean circle is parametrized by

$$(0, y_0) + r_E (\cos \theta, \sin \theta), \theta \in [0, 2\pi), \quad (14)$$

where $r_E$ is the Euclidean radius. Since $\gamma (t) = (0, e^t)$ is an arc length hyperbolic geodesic such that $\gamma (0) = (0, 1)$, we have $t = d_{\mathbb{H}^2} ((0, 1), (0, e^{\pm t}))$. In particular,

$$r = d_{\mathbb{H}^2} ((0, 1), (0, e^r)) = d_{\mathbb{H}^2} ((0, 1), (0, e^{-r})).$$

It follows that the points $(0, e^r)$ and $(0, e^{-r})$ are both in the hyperbolic circle centered at $(0, 1)$ and with hyperbolic radius $r$. Then the Euclidean circle must contain both points $(0, e^r)$ and $(0, e^{-r})$. Since these points are in the same vertical straight line, the Euclidean center of the Euclidean circle is

$$\frac{(0, e^r) + (0, e^{-r})}{2} = (0, \cosh r).$$

And the Euclidean radius $r_E$ of this hyperbolic circle is just half of the Euclidean distance between the points $(0, e^r)$ and $(0, e^{-r})$, that is,

$$r_E = \frac{1}{2} d_{\mathbb{H}^2} ((0, e^r), (0, e^{-r})) = \frac{e^r - e^{-r}}{2} = \sinh r.$$
It follows from \((14)\) that the geodesic disk \(D_r\) of \(\mathbb{H}^2\) centered at \((0,1)\) with radius \(r\) is given by
\[
D_r = \{ (\sinh r_E \cos \theta, \cosh r_E + \sinh r_E \sin \theta) \mid \theta \in \mathbb{R}, 0 \leq r_E \leq R \}.
\]
From \((13)\),
\[
\| \text{Ad}_{(x,y)}(X) \|^2 = (\cosh r_E + \sinh r_E \sin \theta)^2 \left( \frac{-b \sinh r_E \cos \theta}{\cosh r_E + \sinh r_E \sin \theta} + a \right)^2 + b^2
\]
and we may see that the biggest value of the right hand side of the last inequality occurs at \(\theta = \pi/2\) and \(a = 1, b = 0\). From Proposition \(11\) (which, in the case of the hyperbolic space, can also be directly confirmed from the expression above), we obtain \((11)\). \(\square\)

\textbf{Proof of Theorem 3.} The first part of the proof is a direct consequence of Corollary \(6\) and Proposition \(12\). For the second part, consider a polar coordinate system \((r, \theta, \varphi_1, \ldots, \varphi_{n-2})\) of \(\mathbb{H}^n\) centered at a point \(o \in \mathbb{H}^n\), that is, \(\Theta = (\theta, \varphi_1, ..., \varphi_{n-2})\) are spherical coordinates of the unit sphere \(\mathbb{S}^{n-1}\) centered at the origin of \(T_o \mathbb{H}^n\) and \(p \in \mathbb{H}^n\setminus\{o\}\) is parametrized by
\[
p = \exp_o (r \Theta), r > 0, \Theta \in \mathbb{S}^{n-1}.
\]
Let \(v\) be the function that depends only on \(r\) and \(\theta\) given by
\[
v(r, \theta) = \frac{C(n - 1)}{2} \cdot \int_0^r (\sinh s)^{n-1} ds \cos \theta.
\]
Recall that in this coordinate system, the metric has the form
\[
ds^2 = dr^2 + \sinh^2 r d\theta + \sinh^2 r \sin^2 \theta d\varphi_1 + \sinh^2 r \sin^2 \theta \sin^2 \varphi_1 d\varphi_2 + \cdots + \sinh^2 r \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-3} d\varphi_{n-2}
\]
and, since \(v = v(r, \theta)\), the Laplacian of \(v\) can be expressed by
\[
\Delta v = \frac{\partial^2 v}{\partial r^2} + (n - 1) \coth r \frac{\partial v}{\partial r} + (n - 2) \frac{\cot \theta}{\sinh^2 r} \frac{\partial v}{\partial \theta} + \frac{1}{\sinh^2 r} \cdot \frac{\partial^2 v}{\partial \theta^2}.
\]
Hence, by a straightforward calculation, we have that \(\Delta v = 0\). Moreover, since \(v\) depends only on \(r\) and \(\theta\) and the fields \(E_1 := \frac{\partial}{\partial r}\) and \(E_2 := \frac{\partial}{\partial \theta}\) are orthogonal, we have
\[
\| \nabla v(r, \theta) \|^2 = \frac{|E_1(v)|^2}{\|E_1\|^2} + \frac{|E_2(v)|^2}{\|E_2\|^2} = \left( \frac{|\partial_r v|^2}{1} + \frac{|\partial_\theta v|^2}{\sinh^2 r} \right)
\]
\[
= \left( \frac{C(n - 1)}{2} \right)^2 \left( 1 - (n - 1) \cosh r \sinh r \right)^{-2} \left( \int_0^r (\sinh s)^{n-1} ds \right)^2 \cos^2 \theta
\]
\[
+ \left( \frac{C(n - 1)}{2} \right)^2 \left( \sinh r \right)^{-2} \left( \int_0^r (\sinh s)^{n-1} ds \right)^2 \sin^2 \theta.
\]
Then
\[
\| \nabla v(r, \theta) \| = \frac{C}{2(1 + \cosh r)} \text{ for any } r \geq 0 \text{ if } n = 2,
\]
\[
\|\nabla v(r, \theta)\| \approx Ce^{-r} \sin \theta \quad \text{as} \quad r \to +\infty \quad \text{if} \quad n > 2 \quad \text{and} \quad \theta \neq 0.
\]

In both cases
\[
\sup_{S_R} \|\nabla v(R, \theta)\| \approx Ce^{-R} \quad \text{as} \quad R \to +\infty.
\]

Therefore, \(v\) is a bounded non-constant harmonic function such that
\[
\lim_{R \to \infty} \left( e^R \sup_{S_R} \|\nabla v\| \right) = C,
\]
completing the proof. \(\square\)

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