Numerical study on subsonic-supersonic Laval nozzle

Xingyue Ji1, a, †, Junshen Zhi2, b, † and Hongyu Pan3, c, †
1School of Mechanical and Power Engineering, Zhengzhou University, Zhengzhou, Henan,045000, China
2Vehicle Engineering, Hunan University, Changsha, Hunan, 410006, China
3Electronic and Electrical Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan Province, 610054, China
Corresponding author's e-mail: jiixingyue@stu.zzu.edu.cn, kamiyamarin@hnu.edu.cn, 2429609P@student.gla.ac.uk
†These authors contributed equally.

Abstract. Because subsonic-supersonic Laval nozzles are widely used in many fields, the flow characteristics of the subsonic-supersonic Laval nozzle are studied by numerical simulation. First, the MacCormack scheme is used to discrete the non-conservation and conservation forms of the governing equations of the flow in the subsonic-supersonic Laval nozzle. The flow states inside the nozzle in both formats are then compared. The results show that the numerical results of the two schemes converge well. The variation of the fluid velocity and outlet pressure is more pronounced in the non-conservative form. For the conservation form, the error of the mass flow rates at different positions is constant and acceptable, which is about 1%. For the non-conservation form, the error at different locations within the nozzle is different, reaching a minimum error at the throat. In addition, studies on grid length have demonstrated that the finer the grid, the more accurate and stable the results. Moreover, for the subsonic, supersonic isentropic flow in the quasi-one-dimensional nozzle, the Mach number and velocity become larger along with the nozzle. In contrast, other variables such as pressure and temperature become smaller. There is a subsonic region in front of the throat and a supersonic region behind the throat. The Mach number of the nozzle exit can be very high. Finally, the study reveals the internal flow state of the subsonic, supersonic nozzle, which can provide theoretical reference for the application of the nozzle.

1. Introduction
Laval nozzle was invented by a Swedish named Laval. The front half of the nozzle shrinks from large to small, and there is a narrow throat in the middle. The narrow throat then expanded from small to large to the bottom of the nozzle. Laval nozzle is widely used, such as in rocket engines and missiles. The gas in the rocket flows into the front part of the nozzle under high pressure and then escapes from the other part after passing through the narrow throat. This structure can make the airflow velocity vary with the change of the jet cross-sectional area so that the airflow can be accelerated from subsonic to sonic then to supersonic.

In the research of Laval nozzle, there are three common methods: experimental research, theoretical research, and numerical simulation. Wang et al. used the dimensionless method to simulate the convergent section subsonic flow and divergent section hypersonic flow in Laval nozzle [1]. More, Dawud H. Tan et al. mainly talked about the influence of Laval nozzles on the airflow field in melt
blowing apparatus. Simulations and experiments were performed to examine: (1) the influence of increasing air pressure inside the melt blowing die (Pinlet) on the air jet in a typical melt blowing process and (2) the influence of a Laval nozzle (a converging-diverging nozzle) on the air jet [2]. Also, Yin et al. put forward a structure design of filling thin-layer explosives in the multi-layer shell. It initiated a thin-layer explosive to control the deformation of the shell and obtained an approximate function with a Laval Nozzle structure which can control the gas product when charging detonation [3]. Wang and Xi’ work are smooth transonic flows of Meyer type in De Laval Nozzles. Through numerical analysis, they concluded that if such a flow exists, its sonic curve must be located at the throat of the nozzle, and the nozzle should be suitably flat at its throat [4]. Furthermore, Li et al. did a simulation of the gas flow field in Laval Nozzle and Straight Nozzle for powder metallurgy and spray forming. The results showed that the flow generated by the Laval nozzle had a higher exit velocity in the nozzle's vicinity, compared with that of the straight nozzle, that is to say, a Laval nozzle was more efficient than a straight one in disintegrating the melt stream and was apt to produce finer powders [5]. Moreover, Dykas et al. present the results of research on the wet steam flow with spontaneous condensation in the de Laval nozzle. They compared the results of numerical modelling performed for two cases of boundary conditions obtained using an in-house CFD code and the Ansys CFX commercial package. The results show the importance of further studies in this field. [6]. Nistor et al. applied gas dynamics algorithms to analyze fluid subsonic and supersonic movement through the Laval Nozzle. In their paper, a Laval nozzle or a propelling nozzle is the component of a jet engine that operates to constrict the flow, form an exhaust jet, and maximize the velocity of propelling gases from the engine. Propelling nozzles can be subsonic, sonic, or supersonic [7]. What's more, Dushkov and Dzerzhinsky introduced combining methods for end-to-end gas flow calculation in a Laval nozzle. The paper solves the problem of selecting optimal methods and combining them when performing a complex calculation of all sections of the Laval nozzle [8]. Jun Li et al. did a numerical analysis in the Euler system. They found that the existence and uniqueness of three-dimensional steady subsonic Euler flow in rectangular nozzles were obtained when prescribing normal momentum components at both the entrance and exit [9]. Garcia et al. used the MacCormack scheme to develop a two-dimensional (in plan) hydraulic simulation model. The method has been found to be computationally efficient and warrants further development [10]. Besides, there are a lot of studies on the flow characteristics of the Laval nozzle [11-13].

These researches on nozzles further illustrate that the study of the flow state inside the nozzle is of great significance for applying the nozzle. Therefore, the convergent and divergent nozzle numerical study is also carried out in the present work. MacCormack scheme is used to solve the non-conservation governing equations and conservation governing equations in the present work. Then, the flow state and characteristics are analyzed and compared in detail to understand the flow characteristics of the nozzle. The rest of the paper is organized as follows: The second section introduces the method for numerical simulation. The third section presents the results and discussion. And in the last section, the conclusions are drawn.

2. Method

2.1. Governing Equations
Consider a steady isentropic flow through a convergent-divergent nozzle, as shown in Fig. 1. The nozzle length is L, and the velocity in the nozzle is defined as V. The cross-sectional area of the nozzle is \( A(x) = 1 + 2.1(x - 1.5)^2 \), and the sonic throat area is defined as \( A_0 \).

The flow at the inlet to the nozzle comes from a reservoir where the pressure and the temperature are denoted by \( \rho_0 \) and \( T_0 \). The cross-sectional area of the reservoir is large, and hence the velocity is very small. \( \rho_0 \) and \( T_0 \) are the stagnation values, or total pressure and total temperature, respectively. The flow expands isentropically to supersonic speed at the nozzle exit, where the exit pressure, temperature, velocity, and Mach number are denoted by \( p_{\text{out}} \), \( T_{\text{out}} \), \( V_{\text{out}} \) and \( M_{\text{out}} \).

It is assumed that the flow characteristics of a given cross section are uniform. Therefore, although the nozzle area varies with the distance \( x \) along the nozzle, it is a function of the two-dimensional flow field. If we assume that the flow characteristics only vary with \( x \), it is equivalent to assuming that the flow characteristics on any given cross section are uniform. This kind of flow can be considered as quasi one-dimensional flow.

The use of dimensionless variables can avoid large differences in calculation and increase errors. The definition of nondimensional variables is shown below.

\[
T' = \frac{T}{T_0}, \quad \rho' = \frac{\rho}{\rho_0}, \quad x' = \frac{x}{L}, \quad V' = \frac{V}{A_0}, \quad A' = \frac{A}{A_0}, \quad t' = \frac{t}{L/a_0}
\]  

Where \( a_0 \) is the speed of sound in the reservoir defined as \( a_0 = \sqrt{\gamma R T_0} \).

For the convenience of writing, the upper right corner of the dimensionless variable is cancelled. In the following, if not emphasized, the variables used are dimensionless.

From the point of view of numerical simulation, the governing equations of quasi one-dimensional steady isentropic flow have conservation form and non-conservation form.

The nondimensional governing equations of non-conservation form are:

\[
\frac{\partial (\rho A)}{\partial t} + \rho A \frac{\partial V}{\partial x} + \rho V \frac{\partial A}{\partial x} + VA A \rho = 0
\]  

\[
\rho \frac{\partial V}{\partial t} + \rho V \frac{\partial V}{\partial x} = -R \left( \rho \frac{\partial T}{\partial x} + \frac{\partial \rho}{\partial x} \right)
\]  

\[
\rho c_v \frac{\partial T}{\partial t} + \rho V c_v \frac{\partial T}{\partial x} = -\rho R T \left( \frac{\partial V}{\partial x} + V \frac{\partial (\ln A)}{\partial x} \right)
\]

Then the nondimensional governing equations of conservation form are:
\[ \frac{\partial (\rho A)}{\partial t} + \frac{\partial (\rho AV)}{\partial x} = 0 \]  
\[ \frac{\partial (\rho AV)}{\partial t} + \frac{\partial (\rho AV^2 + pA)}{\partial x} = \frac{\partial A}{\partial x} \]  
\[ \frac{\partial}{\partial t} \left[ \rho \left( e + \frac{V^2}{2} \right) A \right] + \frac{\partial}{\partial x} \left[ \rho \left( e + \frac{V^2}{2} \right) AV \right] = - \frac{\partial (pAV)}{\partial x} \]  

Where \( c_v \) is the thermal capacity, \( \gamma \) the specific heat ratio, \( \rho \) the density and \( p \) the pressure and \( e \) the internal energy.

For the equation of conservation form, it can be written as:
\[ \frac{\partial U}{\partial t} = - \frac{\partial F}{\partial x} + J \]  

Where \( U = \begin{pmatrix} \rho A \\ \rho AV \\ \rho \left( \frac{e}{\gamma - 1} + \frac{V^2}{2} \right) A \end{pmatrix}, F = \begin{pmatrix} \rho AV \\ \rho AV^2 + \frac{1}{\gamma} pA \\ \rho \left( \frac{e}{\gamma - 1} + \frac{V^2}{2} \right) VA + pAV \end{pmatrix}, J = \begin{pmatrix} 0 \\ \frac{1}{\gamma} p \frac{\partial A}{\partial x} \end{pmatrix} \)

2.2. MacCormack Scheme and Discrete governing equations.

To realize the finite difference solution, the nozzle is divided into several discrete grid points along the x-axis. MacCormack scheme is a two-step numerical method that can discretize nonlinear equations and hyperbolic partial differential equations.

2.2.1. Non-conservation form of the governing equations.

Consider the predictor step and apply forward difference method for non-conservation form.
\[ \frac{\partial (\rho)}{\partial t} |_{i}^{t+\Delta t} = - \rho_i^{t+\Delta t} v_i^{t+\Delta t} \Delta x - \rho_i^{t} v_i^{t} \frac{ln A_{i+1}^{t+\Delta t} - ln A_i^t}{\Delta x} - \rho_i^{t} \frac{\rho_i^{t+\Delta t} - \rho_i^{t}}{\Delta x} \]  
\[ \frac{\partial v}{\partial t} |_{i}^{t+\Delta t} = - v_i^{t+\Delta t} \frac{ln A_{i+1}^{t+\Delta t} - ln A_i^t}{\Delta x} - \rho_i^{t} \frac{\rho_i^{t+\Delta t} - \rho_i^{t}}{\Delta x} \]  
\[ \frac{\partial T}{\partial t} |_{i}^{t+\Delta t} = - v_i^{t+\Delta t} \frac{ln A_{i+1}^{t+\Delta t} - ln A_i^t}{\Delta x} - \rho_i^{t} \frac{\rho_i^{t+\Delta t} - \rho_i^{t}}{\Delta x} \]  

Then the predicted values of \( \rho, V \) and \( T \) are obtained by
\[ \bar{\rho}_i^{t+\Delta t} = \rho_i^{t} + \left( \frac{\partial \rho}{\partial t} |_{i}^{t} \right) \Delta t \]  
\[ \bar{v}_i^{t+\Delta t} = v_i^{t} + \left( \frac{\partial v}{\partial t} |_{i}^{t} \right) \Delta t \]  
\[ \bar{T}_i^{t+\Delta t} = T_i^{t} + \left( \frac{\partial T}{\partial t} |_{i}^{t} \right) \Delta t \]  

Then the predicted values are corrected in the corrected step.
\[ \frac{\partial (\rho)^{t+\Delta t}}{\partial t} |_{i} = - \rho_i^{t+\Delta t} v_i^{t+\Delta t} - \rho_i^{t+\Delta t} v_i^{t+\Delta t} \frac{ln A_{i+1}^{t+\Delta t} - ln A_i^t}{\Delta x} - \rho_i^{t+\Delta t} \frac{\rho_i^{t+\Delta t} - \rho_i^{t}}{\Delta x} \]  
\[ \frac{\partial (v)^{t+\Delta t}}{\partial t} |_{i} = - v_i^{t+\Delta t} \frac{ln A_{i+1}^{t+\Delta t} - ln A_i^t}{\Delta x} - \rho_i^{t+\Delta t} \frac{\rho_i^{t+\Delta t} - \rho_i^{t}}{\Delta x} \]  
\[ \frac{\partial (T)^{t+\Delta t}}{\partial t} |_{i} = - v_i^{t+\Delta t} \frac{ln A_{i+1}^{t+\Delta t} - ln A_i^t}{\Delta x} - \rho_i^{t+\Delta t} \frac{\rho_i^{t+\Delta t} - \rho_i^{t}}{\Delta x} \]  

Next, calculate the average time derivative
\[ \frac{\partial (\rho)}{\partial t} |_{av} = 0.5 \left[ \left( \frac{\partial (\rho)}{\partial t} |_{i}^{t} \right) + \left( \frac{\partial (\rho)}{\partial t} |_{i}^{t+\Delta t} \right) \right] \]  
\[ \frac{\partial v}{\partial t} |_{av} = 0.5 \left[ \left( \frac{\partial v}{\partial t} |_{i}^{t} \right) + \left( \frac{\partial v}{\partial t} |_{i}^{t+\Delta t} \right) \right] \]
Finally, the value of each variable in the next time step are obtained.

\[
\rho_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_a V \Delta t
\]

(21)

\[
v_i^{t+\Delta t} = v_i^t + \left(\frac{\partial v}{\partial t}\right)_a V \Delta t
\]

(22)

\[
T_i^{t+\Delta t} = T_i^t + \left(\frac{\partial T}{\partial t}\right)_a V \Delta t
\]

(23)

### 2.2.2. Conservation form of the governing equations.

In the predicted step, \( U_{ij}^t \) represents the value of \( U \) at grid \( i \) at time \( t \) on the grid as shown in Fig.1.

\[
U_{i+\Delta x, j}^{t+\Delta t} = U_i^t - \Delta t \left( F_i^{t+\Delta t} - F_i^{t-\Delta t} \right)
\]

(24)

Where \( U_{i+\Delta x, j}^{t+\Delta t} \) is the predicted value of \( U \) at grid \( i \) at the time \( t + \Delta t \). Then \( F_i^{t+\Delta t}, J_i^{t+\Delta t} \) can be obtained from \( U_{i+\Delta x, j}^{t+\Delta t} \).

In the corrected step, the predicted value \( U_{i+\Delta x, j}^{t+\Delta t} \) is corrected according to the following equation

\[
U_{i+\Delta x, j}^{t+\Delta t} = U_i^{t+\Delta t} = \frac{1}{2} \left( F_i^{t+\Delta t} + F_i^{t-\Delta t} \right) + \frac{\Delta t}{2} \left( F_i^{t+\Delta t} - F_i^{t-\Delta t} \right)
\]

(25)

Note that the corrector step uses backward finite difference scheme for spatial derivative. Also, the time-step used in the corrector step is \( \Delta t \) in contrast \( \Delta t \) in predictor step.

Replacing the \( U_i^{t+\Delta t} \) by the temporal average

\[
U_i^{t+\Delta t} = \frac{U_i^{t+\Delta t} + U_i^t}{2}
\]

(26)

Finally, the corrector can be obtained

\[
U_i^{t+\Delta t} = \frac{U_i^{t+\Delta t} + U_i^t}{2} - \Delta t \left( \frac{F_i^{t+\Delta t} - F_i^{t-\Delta t}}{\Delta x} \right) + J_i^{t+\Delta t}
\]

(27)

### 2.3. Boundary Conditions

For subsonic inflow boundary, the velocity at the boundary is allowed to change, and the boundary conditions as follows,

\[
V_i = 2V_2 - V_3
\]

(28)

For supersonic outflow boundary, all factors have to be allowed to change. Use extrapolation to obtain their values likewise,

\[
V_N = 2V_{N-1} - V_{N-2}
\]

(29)

\[
\rho_N = 2\rho_{N-1} - \rho_{N-2}
\]

(30)

\[
p_N = 2p_{N-1} - p_{N-2}
\]

(31)

The subscript N represents the node number at the nozzle outlet. For the initial conditions of the variables in the nozzle, the settings are as follows

\[
\rho = 1 - 0.3x
\]

(32)

\[
T = 1 - 0.25x
\]

(33)

\[
V = 1 + 0.8x
\]

(34)

### 3. Results and Discussion

The use of sections to divide the text of the paper is optional and left as a decision for the author. Where the author wishes to divide the paper into sections the formatting shown in table 2 should be used.
3.1. Convergence of numerical simulations

Fig. 2 shows the convergence of the flow field with time in both conservation and non-conservation forms. A 1200-time step numerical simulation is carried out to observe the history curves of Ma and P at the nozzle throat with a simulation time-step. It is easy to find that under the conservation form, the speed of the fluid changes with the nozzle, while under the non-conservation form, the speed of the fluid changes more obvious. Both of them converge at the end of the nozzle. Obviously, the flow field in the early stage of simulation is not convergent, but it will eventually converge as time goes on. In comparison, the numerical simulation with a conservation scheme is easier to converge. Although the error of the non-conservation form is larger in the early stage of simulation, the final results of the conservation form and non-conservation form are almost identical.

3.2. Comparison of analytic and numerical solutions

Figure 3. Comparison of analytic and numerical solutions.
Fig. 3 compares the mass flow rates at different positions of the nozzle in the conservation form, the non-conservative form, and the analytic solutions. From this figure, both the conservation and non-conservation forms are a little larger than the analytic solutions. For the conservation form, the error is constant, which is about 0.005(1%). Then for the non-conservation form, the error changes with the nozzle. Obviously, under the non-conservation form, the error becomes smaller at the center of the nozzle, which is about 0.004(0.7%). But at the beginning or end, the error is much bigger, for about 0.008 at the beginning and 0.006 at the end.

Overall, the errors between the analytic solutions and the numerical results of the two forms are very small, which shows that the numerical results in the present work are accurate.

3.3. Mesh generation study

(a) Mach number
(b) Pressure
(c) Density
(d) Temperature
Mesh generation study is conducted for different values of the grid length Δx in the conservation governing situation. Three different values of Δx (0.05, 0.1, and 0.3) are generated to evaluate the precision. Comparing the difference of each variable that using diverse Δx for simulation. The result of M-aM,a,P,ρ,T is similar in these three situations, shown in Fig. 4. However, when simulating Mass flow, there is an obvious difference among those situations, as shown in Fig. 4(f). There is a large fluctuation as Δx=0.3, and a smaller one as Δx=0.1. However, only as Δx=0.05, the simulation is able to be found a stable result. Therefore, the grid length of 0.005 is chosen for numerical simulation.

3.4. Study on the flow characteristics

The results of P,ρ,V,T are the same whenever simulating in non-conservation governing or conservation governing form, as shown in Fig. 5.

The initial value of ρ, and P were set to 1.0, as shown in the Figs. 5 (a), (b), and (d), the value remains steadily at the initial stage from x=0.0 to about 1.0, then there is a significant decrease of ρ, and P until around x=2.0, which cause the value to fell to 0.25 and 0.15 respectively. However, the downtrend slows down after the above-mentioned stage. The variable T also shows a similar trend.

Considering the variable V, because of the decline of ρ,T and P, it remains relatively stable from x=0 to x=0.6, then soar from about 0.3 to around 1.5 as x=2.0. After that, it increases gradually, reaching 1.8 at x=3.3. Then, the physical quantities in the nozzle are shown. In addition, it can be seen from Fig. 4 (a) that there is a subsonic region in front of the throat and a supersonic region behind the throat. The Mach number of nozzle outlets can be as high as 3.5.
4. Conclusion
This study mainly uses the MacCormack scheme to solve the conservation and non-conservation governing equations of the flow in the nozzle. A steady isentropic flow through a convergent-divergent nozzle is considered. Assuming that the flow at the inlet to the nozzle comes from a reservoir where the pressure and temperature are constant. Considering the governing function, it is obvious that the Mach number and the pressure are both converging at the nozzle's throat in the end. Then for both the conservation and non-conservation forms of the mass flow in the nozzle, they are a little larger than the analytic solution, but the error is acceptable. As for the Macmorck scheme, three different values of grid length are considered under the conservation governing situation. It is clear to find that as the grid length becomes smaller, the simulation result becomes more accurate. For the numerical results in the non-conservation form, the results are essentially the same compared to the conservation form, with only small errors. Moreover, for the subsonic, supersonic isentropic flow in a quasi-one-dimensional nozzle, the Mach number and V become larger along with the nozzle. Other variables like P, p, and T become smaller. There is a subsonic region in front of the throat and a supersonic region behind the throat. The Mach number of the nozzle exit can be very high. Finally, the study in this paper fully demonstrates the flow state inside a subsonic, supersonic nozzle and can provide a theoretical reference for nozzle applications.

Overall, this study reflects the importance of applying the nozzle. Considering the necessity of studying laval nozzle, it is significant to do deep research on this basis or in other direction.

References
[1] Wang, Y.D., Jia, H.G. (2013) Numerical simulation of Laval nozzle. Applied Mechanics & Materials, 397-400: 266-269.
[2] Tan, D.H., Herman, P.K., Janakiraman, A., Bates, F.S., Macosko, C.W. (2012) Influence of Laval nozzles on the air flow field in melt blowing apparatus. Chemical Engineering Science, 80: 342–348.
[3] Wang, C., Xin, Z. (2019) Smooth Transonic Flows of Meyer Type in De Laval Nozzles. Archive for Rational Mechanics and Analysis, 232:1597-1647.
[4] Yin, J.T., Li, G. (2015) Numerical simulation on dynamic Laval nozzle under different detonation model. Applied Mechanics & Materials, 799-800: 651-655.
[5] Li, Z., Zhang, G.Q., Zhou, L.I., Zhang, Y., XU W. (2008) Simulation of gas flow field in Laval nozzle and straight nozzle for powder metallurgy and spray forming. Journal of Iron and Steel Research (International). 15(6):44-47.
[6] Dykas, S., Majkut, M., Smolka, K., Strozik, M. (2018) An attempt to make a reliable assessment of the wet steam flow field in the de Laval nozzle. Heat and Mass Transfer. 54:2675-2681.

[7] Nistor, C., Tefan, A.G. (2013) Gas dynamics algorithms applied to the analysis of fluid subsonic and supersonic movement through the Laval nozzles. Annals Computer Science.

[8] Dushkov, R. E., Dzerzhinsky, R.I. (2021) Combining methods for end-to-end calculation of gas flow in a Laval nozzle. IOP Conference Series: Materials Science and Engineering, 1047(1): 012012

[9] Li, J., Xin, Z., Yin, H. (2012) Transonic Shocks for the Full Compressible Euler System in a General Two-Dimensional De Laval Nozzle. Archive for Rational Mechanics and Analysis, 207: 533-581.

[10] Garcia, R., & Kahawita, R. A. (2010). Numerical solution of the st. venant equations with the maccormack finite-difference scheme. International Journal for Numerical Methods in Fluids, 6(5): 259-274.

[11] Liu, Z., Ding, J., Jiang, W., Jian, Z., Feng, Y. (2008). Numerical simulation of highly-swirling supersonic flow inside a laval nozzle. Progress in Computational Fluid Dynamics An International Journal, 8(7-8): 536-540.

[12] Omar, Z., Yusop F., Zain, B.A.M.Z. (2013) Numerical Method for Nozzle Airflow Problem. International Journal of Systems Applications. 5(7):219-226.

[13] Spangenberg, T., Khler, S., Hansmann, B., Wachsmuth, U., Smith, M. A. (2004) Low-temperature reactions of oh radicals with propene and isoprene in pulsed Laval nozzle expansions. Journal of Environmental Engineering, 108(37): 1-7.