Quantum Prisoner’s Dilemma game on hypergraph networks

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We study the beneficialness of quantum strategies in multi-player evolutionary games. We base our study on the three-player Prisoner’s Dilemma (PD) game. In order to model the simultaneous interaction between three agents we use hypergraphs and hypergraph networks. In particular, we study two types of networks: a random network and a SF-like network. The obtained results show that in the case of a three player game on a hypergraph network, quantum strategies not necessarily are Evolutionary Stable Strategies. In some cases, the defection strategy can be as good as a quantum one.

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I. INTRODUCTION

Game theory is a branch of mathematics broadly applied in a great number of fields, from biology to social sciences and economics. A great deal of effort has gone into the study of evolutionary games on graphs, which was initiated by the work of Nowak and May [1]. Since their work was published, a lot of effort was put into studying the problem [2].

Quantum game theory [3] allows the agents to use quantum strategies. The set of quantum strategies is much larger than a classical one; hence it offers possibility for much more diverse behavior of agents in the network. It has been shown that if only one player is aware of the quantum nature of the system, he/she will never lose in some types of games [4]. Recently, it has been demonstrated that a player can cheat by appending additional qubits to the quantum system [5].

Combining evolutionary games and quantum game theory, has resulted in absorbing results [6]. In some cases the quantum strategies can dominate the entire network, infecting it effectively. In our work we like to focus on introducing additional strategies which the agents can use, since in the multi-player case there exists a Pareto Optimal Nash Equilibrium for the Prisoner’s Dilemma game [7]. Moreover, the PD game is interesting to study, because it was realized experimentally [8].

This paper is organized as follows: Section II describes the types of 3-hypergraph networks used in simulations. Section III introduces the three-player Prisoner’s Dilemma game. In Section IV the simulation setup is described. Section V contains results obtained from computer simulations and their discussion. Finally, in Section VI the final conclusions are drawn.

II. HYPERGRAPHS AND HYPERGRAPH NETWORKS

We assume a hypergraph [9] network $H(X,E)$ where $X$ is a set of nodes and $E$ is a set of non-empty subsets of $X$, $E \subseteq 2^X$. Elements of $E$ are the hyperedges of $H$. We keep within the boundaries of the case when every subset of $X$, $A \in E$ satisfies $|A| = 3$, i.e. every edge of the hypergraph connects three nodes exactly. Hereafter we will refer to this structure as a 3-hypergraph. We set $N = |X|$ – the total number of agents.

We construct two types of networks: a random network, in which all hyperedges connect random nodes and a SF-like [10] network. We set the number of hyperedges in the random case to $|E| = 10000$. The SF-like network is constructed in the following way: First, a network of $m_0 \ll N$ all connected nodes is created. Then a new node with $m < m_0$ links is added to the network. For each of the $m$ links, a pair of unique nodes is chosen from the existing network and a new hyperedge is added. The probability of a node $i$ being chosen is given by:

$$p_{sf}(i) = \frac{k_i}{\sum_{j \in X} k_j},$$

(1)

where $k$ is the degree of a node. This procedure is repeated until the number of nodes of the network reaches $N$.

III. THREE-PLAYER PD GAME

The classical Prisoner’s Dilemma game is as follows: two players can either cooperate (C) or defect (D). When they both cooperate, each receives a payoff of 3. On the other hand, when they both defect, each receives a payoff of 1. When one defects, he/she receives a payoff of 5, while the other gets 0.

This approach can be extended to a greater number of players. In the three-player case, the payoff matrix is
shown in Table I. We can see that every player is better off defecting than cooperating no matter what the other players do. In terms of game theory, \((D, D, D)\) is the unique Nash equilibrium of the game. If any one player deviates from this strategy, he will receive a lower payoff. On the other, we can see that the strategy profile \((C, C, C)\) can yield a higher payoff than \((D, D, D)\). In terms of game theory this profile is Pareto Optimal. In our case the players are rational and the game will end in \((D, D, D)\), not \((C, C, C)\); hence the dilemma.

In the quantum case the setup is as follows. Each player is sent a qubit and can locally operate on it, using any unitary operator \(U \in SU(2)\). The initial state of the system is entangled:

\[
|\psi\rangle = J|000\rangle,
\]

where \(J\) is the entangling operator [12]:

\[
J = \frac{1}{\sqrt{2}} \left( I^\otimes N + i \sigma^N_x \right).
\]

The quantum circuit for the game is shown in Figure 1. After the players have applied their respective strategies, the untangling gate, \(J^\dagger\), is applied to the system, hence the final state of the game is

\[
|\psi_f\rangle = J^\dagger (U_A \otimes U_B \otimes U_C) J|000\rangle,
\]

where \(U_A, U_B, U_C\) are the players strategies. The payoff of the first player (Alice) amounts to:

\[
S_A = \sum_{i,j,k \in \{0,1\}} p_{ijk} \langle \psi_f | ijk \rangle,
\]

where \(p_{ijk}\) are numbers corresponding to the possible classical payoffs of Alice, defined in Table I.

| Charlie C | Bob  |
|-----------|------|
| C         | D    |
| (6, 6, 6) | (3, 9, 3) |
| D         | C    |
| (9, 3, 3) | (5, 0, 5) |

| Charlie D | Bob  |
|-----------|------|
| C         | D    |
| (3, 3, 9) | (0, 5, 5) |
| D         | C    |
| (5, 0, 5) | (1, 1, 1) |

TABLE I: The payoff matrix of the three-player PD game (after [11]). The first entry is the payoff of Alice, the second denotes the payoff of Bob and the third represents the payoff of Charlie.

IV. SIMULATIONS

We assume an initial population of 2500 agents, located at the nodes of the hypergraph. The SF-like network is constructed with initial size \(m_0 = 3\), and the number of links of each new node is \(m = 2\). The set of allowed strategies is as follows [6]:

\[
S = \{C, D, H, Q\},
\]

where the unitary operators corresponding to each of the strategies take the form of:

\[
C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

In the two-player case the strategy profile \((Q, Q)\) is a Nash Equilibrium of the system in. These strategies are randomly assigned to agents in the network in such a way that the initial fractions of strategies \(C, D, H, Q\) are 49\%, 49\%, 1\%, 1\% respectively.

Next, we introduce an additional strategy \(\Sigma\), defined as:

\[
\Sigma = i \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The strategy profile \((\Sigma, \Sigma, \Sigma)\) is Pareto Optimal and a Nash Equilibrium [7]. We assign the strategies \(C, D, H, Q, \Sigma\) with frequencies 48\%, 48\%, 2\%, 1\%, 1\%.

Finally, we do not assign strategies randomly, but choose to allocate the strategy \(Q\) in the first case and \(\Sigma\) in the second one to nodes with the highest degree.

The PD game is played by all agents on both networks. We study the impact of the value of the parameter \(T\) (moral hazard) on the final state of the population. This parameter is defined as the first players payoff when other players use the \(C\) strategy. Its interpretation is as follows. Suppose the prisoners had a chance to discuss a strategy. It is evident that they should decide for a Pareto Optimal profile \((C, C, C)\). However, if Alice decides to defect, she receives a higher payoff. Thus this parameter measures, how much Alice is tempted to betray the other prisoners.

The game is played for 10000 generations and the last 1000 results are stored. Average frequencies of strategies are used as the final results. If a population does not change for 500 generations, the state is considered to be an equilibrium state of the system.

V. RESULTS AND DISCUSSION

In the case with four possible strategies, the results of computer simulations are depicted in Figure 2a. Figure 2a shows the results for a random network, whereas the results for the SF-like network are shown in Figure 2b.

In the case of a random network, we see that strategy \(C\)
is the dominant one, until \( T = 5.64 \), when the network starts shifting between strategies \( C \) and \( D \). It settles down at \( T = 6 \), where about half the agents use strategy \( C \). As \( T \) increases, strategies \( C \) and \( D \) slowly lose their significance in favour of strategy \( Q \). For \( T > 8 \) the system reaches another equilibrium state, where strategies \( D \) and \( Q \) have the same frequency.

![Graph](image1)

**FIG. 2:** Results for PD on hypergraph networks, 4 strategies, strategies assigned at random, according to weights.

In the case of a SF-like network, the agents prefer the \( D \) strategy, which almost never reaches zero frequency. Again, for \( T > 6 \) we have an increase of significance of the quantum strategy \( Q \). Although there are some oscillations of the fraction of strategies as \( T \) increases, again strategies \( D \) and \( Q \) have been adopted by approximately the same fraction of agents. On the basis of the presented figures as well as above discussion it may be inferred that the change of type of the network significantly decreases the importance of strategy \( C \), but does not have a great impact on strategies \( D \) and \( Q \).

![Graph](image2)

**FIG. 3:** Results for PD on hypergraph networks, 5 strategies, strategies assigned at random, according to weights.

Next we move on to the case, where only one agent, with the highest degree was assigned a quantum strategy. For the case of four available strategies, results are shown in Figure 4. Figure 4a shows the results for a random network and Figure 4b shows the results for a SF-like network. In this case the agent with the highest degree was assigned the \( Q \) strategy, all other strategies were distributed to the agents with equal probabilities. We perceive, that for \( T < 6 \) the strategy \( C \) dominates the network. Again, at around \( T = 6 \) there is a shift, but this time the strategy \( H \) increases its significance. The fraction of agents using strategy \( H \) slowly increases with \( T \) increasing. In the case of a SF-like network, we obtain that for \( T < 6.5 \) the strategy \( C \) dominates the network. For greater \( T \) the network starts shifting from strategy \( Q \) to strategy \( D \), and for \( T > 8 \), there is another shift in strategies, and the fraction of strategy \( C \) decreases to zero, and strategies \( D \) and \( Q \) are used by equal fraction of agents. On examining the
C to H, still a small fraction of agents also use the Q strategy. Summing up this case, we can conclude that strategy Q cannot infect any of the networks, but assigning the strategy H to a relatively big fraction of agents allows it to dominate the network for some values of T.

Finally, we show the results for the case with five possible strategies. Now we assign the strategy Σ to the agent with the highest degree. The results obtained in this case are shown in Figure 5a. Figure 5b illustrates the results for a random network and Figure 5c shows the results for a SF-like network. For a random network, as can be seen still that for T < 6 the C strategy is employed by all of the agents. As T increases from 6 to 8, the strategies C, Q and D are used by a significant fraction of agents. At around T = 8 the strategy H dominates the network. Then, just before T reaches 9, there is another sudden shift and strategies D and Q are used by the same fraction of agents, with other strategies being far less significant. In the case of SF-like network, we observe an entirely different behaviour. The strategy Σ always dominates the network, regardless of the value of T. From this discussion it is evident that the Σ strategy can only invade a network of a specific type. A random network is immune to invasion.

VI. CONCLUSIONS

We investigate the evolution of strategies on hypergraph networks when quantum strategies H, Q and Σ are available to the players. Strategies Q and Σ are considered to be invaders in our scenario. Our simulations of the evolution of strategies on a random and SF-like hypergraph network indicate that the structure of the network is a decisive factor. In addition, we discovered that, the strategy Σ, despite being Pareto Optimal and a Nash Equilibrium for the three-player Prisoner’s Dilemma game, does not invade the entire network in all cases. In fact, it can only invade a SF-like network, provided that the agent with the highest degree is assigned this strategy. In other cases, depending on the value of Temptation, the network is dominated by strategy C, what happens for T < 6, or strategies D and Q have equal frequencies what happens for T > 8. The results obtained for the case with four available strategies, are slightly different. In this case the strategy Q is considered to be an invader. The results show that a random network is invaded not by strategy Q, but by strategy H for T > 6. On the other hand the SF-like network constantly shifts between C and H for T > 6.
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