Algebras Describing Pseudocomplemented, Relatively Pseudocomplemented and Sectionally Pseudocomplemented Posets

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Abstract: In order to be able to use methods of universal algebra for investigating posets, we assigned to every pseudocomplemented poset, to every relatively pseudocomplemented poset and to every sectionally pseudocomplemented poset, a certain algebra (based on a commutative directoid or on a λ-lattice) which satisfies certain identities and implications. We show that the assigned algebras fully characterize the given corresponding posets. A certain kind of symmetry can be seen in the relationship between the classes of mentioned posets and the classes of directoids and λ-lattices representing these relational structures. As we show in the paper, this relationship is fully symmetric. Our results show that the assigned algebras satisfy strong congruence properties which can be transferred back to the posets. We also mention applications of such posets in certain non-classical logics.

Keywords: pseudocomplemented poset; relatively pseudocomplemented poset; sectionally pseudocomplemented poset; Stone poset; commutative directoid; λ-lattice; congruence permutability; congruence distributivity; weak regularity

1. Introduction

When investigating algebras, researchers usually apply well-known algebraic methods and results. Unfortunately, this does not work in the case of partially ordered sets (posets). The reason for this is that posets need not have operations and hence basic theorems of general algebra cannot be applied. In 1990, J. Ježek and R. Quackenbush [1] showed that if a poset $P$ is up-directed or down-directed, then a certain algebra with one binary operation, a so-called directoid, can be assigned to $P$. This assignment is in general not unique, but, conversely, from every such assigned directoid, $P$ can be reconstructed in a unique way. This fact allows to convert directed posets into algebras which bear all the information on the given poset.

In a similar way to a given poset that is both up- and down-directed, one can assign an algebra with two binary operations $\sqcup$ and $\sqcap$ as shown by V. Snášel [2]. Such an algebra is called a λ-lattice. An overview concerning the results on directoids and λ-lattices can be found in our monograph [3].

The aforementioned approach was also used for bounded complemented posets by the authors and M. Kolařík in [4]. In fact, the class of directoids assigned to such posets forms a variety of algebras. This is of great advantage since there exist many methods and results for studying varieties in general algebra. The same method was used in [5], where so-called orthoposets and orthomodular posets (used in the formalization of the logic of quantum mechanics) were converted into algebras forming a variety.

The natural question arises whether such a method can also be applied to pseudocomplemented, relatively pseudocomplemented and sectionally pseudocomplemented...
posets (the last were recently introduced by the authors and J. Paseka in [6]). We solve this question by using commutative directoids and \( \lambda \)-lattices. We characterize the assigned algebras by means of relatively simple conditions. Unfortunately, not all of these conditions can be expressed in the form of identities or quasi-identities. Thus, the corresponding classes of algebras do not form varieties or quasivarieties. On the other hand, these algebras still share nice congruence properties as we will show.

We believe that our approach can bring new insight into the study of pseudocomplemented, relatively pseudocomplemented and sectionally pseudocomplemented posets, since we provide a purely algebraic description of them, thus enabling the application of algebraic tools for their investigation. A certain kind of symmetry can be seen in the relationship between the classes of mentioned posets and the classes of directoids and \( \lambda \)-lattices representing these relational structures. As we show in the paper, this relationship is fully symmetric.

2. Preliminaries

Let \( P := (P, \leq) \) be a poset, \( A, B \subseteq P \) and \( a, b \in P \). Then, \( A \leq B \) means that \( x \leq y \) for all \( x \in A \) and \( y \in B \). Instead of \( A \leq \{b\} \) and \( \{a\} \leq B \), we simply write \( A \leq b \) and \( a \leq B \). The sets:

\[
L(A) := \{ x \in P \mid x \leq A \},
\]

\[
U(A) := \{ x \in P \mid A \leq x \}
\]

are called the lower and upper cone of \( A \), respectively. Instead of \( L(\{a\}) \) and \( L(\{a, b\}) \), we simply write \( L(a) \) and \( L(a, b) \), respectively. In a similar way, we proceed for \( U \). It is easy to see that \( A \leq L(b) \) if and only if \( A \leq b \). If the infimum \( \inf(a, b) \) of \( a \) and \( b \) exists in \( P \), then we will denote it by \( a \land b \). The poset \( P \) is called:

- down-directed if \( L(x, y) \neq \emptyset \) for all \( x, y \in P \);
- up-directed if \( U(x, y) \neq \emptyset \) for all \( x, y \in P \);
- directed if it is both down- and up-directed.

Of course, if \( P \) has a top element 1 then it is up-directed, and if it has a bottom element 0, then it is down-directed.

The concept of a commutative directoid was introduced by J. Ježek and R. Quackenbush [1], see also [3] for details and elementary theory. A commutative meet-directoid is a groupoid \((D, \sqcap)\) satisfying the following identities:

- \( x \sqcap x = x \),
- \( x \sqcap (x \sqcap y \sqcap z) \approx (x \sqcap y) \sqcap z \),
- \( x \sqcap y \approx y \sqcap x \).

If \( D = (D, \sqcap) \) is a commutative meet-directoid and one defines a binary relation \( \leq \) on \( D \) by \( x \leq y \) if \( x \sqcap y = x \) then \( P(D) := (D, \leq) \) is a down-directed poset, called the poset induced by \( D \).

Dually, one can define a commutative join-directoid \((D, \sqcup)\). If \( D = (D, \sqcup) \) is a commutative join-directoid and one defines a binary relation \( \leq \) on \( D \) by \( x \leq y \) if \( x \sqcup y = y \) then \( P(D) := (D, \leq) \) is an up-directed poset, called the poset induced by \( D \).

The concept of a commutative directoid is very useful because it enables to translate ideas and problems in directed posets into groupoids where one can use standard algebraic tools for solutions, see e.g., [5]. The important concept of a commutative directoid is potentially applicable in the study of hierarchical lattices (see [7]).

Assuming \( P = (P, \leq) \) is to be down-directed and define a binary operation \( \sqcap \) on \( P \) by the following prescription: \( x \sqcap y := \min(x, y) \) if \( x \) and \( y \) are comparable, and \( x \sqcap y = y \sqcap x \) should be an arbitrary element of \( L(x, y) \) otherwise. Thus, \( x \sqcap y \in L(x, y) \) in any case. Then, \( D := (P, \sqcap) \) is a commutative meet-directoid, called a commutative meet-directoid assigned to \( P \), and \( P(D) = P \). Hence, the down-directed poset \( P \) is uniquely determined by an assigned commutative meet-directoid \( D \) in contrast to the fact that \( D \) is in general not uniquely determined by \( P \).
Dually, we define a commutative join-directoid \( D = (P, \sqcup) \) assigned to an up-directed poset \( P = (P, \leq) \). We then have \( Q(D) = P \).

Using an assigned commutative meet- and join-directoid, respectively, we can describe lower and upper cones of a given poset as follows.

**Lemma 1.** Let \( P = (P, \leq) \) be a poset and \( a, b, c \in P \). If \( P \) is down-directed and \( (P, \sqcap) \) an assigned commutative meet-directoid then:

(a) \( L(a, b) = \{ (a \sqcap x) \sqcap (b \sqcap x) \mid x \in P \} \),

(b) \( c \in L(a, b) \) if and only if \( (a \sqcap c) \sqcap (b \sqcap c) = c \).

If \( P \) is up-directed and \( (P, \sqcup) \) is an assigned commutative join-directoid then:

(c) \( U(a, b) = \{ (a \sqcup x) \sqcup (b \sqcup x) \mid x \in P \} \),

(d) \( c \in U(a, b) \) if and only if \( (a \sqcup c) \sqcup (b \sqcup c) = c \).

**Proof.** First, we assume \( P \) to be down-directed and \( (P, \sqcap) \) to be an assigned commutative meet-directoid. Now, let us show (a) and (b). If \( c \in L(a, b) \) then \( c = c \sqcap c = (a \sqcap c) \sqcap (b \sqcap c) \).

Conversely, \( (a \sqcap c) \sqcap (b \sqcap c) \leq a \sqcap c \leq a \), and analogously, \( (a \sqcap c) \sqcap (b \sqcap c) \leq b \), i.e., \( (a \sqcap c) \sqcap (b \sqcap c) \in L(a, b) \). The rest of the lemma follows by duality which means that we replace \( L(a, b) \) by \( U(a, b) \) and \( \sqcap \) by \( \sqcup \).

**3. Pseudocomplemented Posets**

Pseudocomplemented posets were introduced and studied by O. Frink [8], see also [9,10] for further development. Let us recall the definition.

**Definition 1.** A pseudocomplemented poset is an ordered quadruple \( (P, \leq, *, 0) \), such that \( (P, \leq, 0) \) is a poset with bottom element 0 and * is a unary operation on \( P \) such that for all \( x \in P \), \( x^* \) is the greatest element \( y \in P \) satisfying \( L(x, y) = \{ 0 \} \). The element \( x^* \) is called the pseudocomplement of \( x \). A Stone poset is a pseudocomplemented poset \( (P, \leq, *, 0) \) satisfying \( U(x^*, x^{**}) = \{ 0^* \} \) for all \( x \in P \).

Pseudocomplemented posets are important in particular in certain non-classical logics which are not based on a lattice structure but only on the structure of a poset. In such structures, one can study the properties of pseudocomplementation as a non-classical negation without the influence of lattice operations (usually used for conjunction and disjunction).

It is worth noticing that \( L(x, y) = \{ 0 \} \) means the same as \( x \land y = 0 \). Hence, for every \( x \in P \), we have \( x \land x^* = 0 \). Moreover, every pseudocomplemented poset \( (P, \leq, *, 0) \) has a top element, namely \( 0^* = 1 \).

**Example 1.** If \( (P, \leq) \) denotes the poset visualized in Figure 1:
and the unary operation \(^*\) on \(P\) is defined by

\[
\begin{array}{c|cccccc}
  x & 0 & a & b & c & d & 1 \\
  \hline
  x^* & 1 & b & a & 0 & 0 & 0 \\
  x^{**} & 0 & a & b & 1 & 1 & 1 \\
\end{array}
\]

then \((P, \leq, ^*, 0)\) is a pseudocomplemented poset which is neither a lattice nor a Stone poset since:

\[
\begin{array}{c|cccccc}
  x & 0 & a & b & c & d & 1 \\
  \hline
  x^* & 1 & b & a & 0 & 0 & 0 \\
  x^{**} & 0 & a & b & 1 & 1 & 1 \\
\end{array}
\]

and \(U(a^*, a^{**}) = \{c, d, 1\} \neq \{1\}\).

**Example 2.** If \((P, \leq)\) denotes the poset visualized in Figure 2:

\[
\begin{array}{c|cccccc}
  x & 0 & a & b & c & d & e & f & 1 \\
  \hline
  x^* & 1 & f & c & f & 0 & 0 & c & 0 \\
\end{array}
\]
then \((P, \leq, , 0)\) is not a lattice, but a Stone poset since:

\[
\begin{array}{c|cccccc}
 x & 0 & a & b & c & d & e \\
\hline
 x^\ast & f & c & f & 0 & 0 & c \\
 x^{**} & c & f & c & 1 & 1 & f \\
\end{array}
\]

and \(U(x^\ast, x^{**}) = \{1\}\) for all \(x \in P\).

Now, we show how a pseudocomplemented poset can be characterized by an assigned commutative meet-directoid equipped with a unary operation \(\ast\) and a nullary operation \(0\).

**Theorem 1.** Let \((P, \leq)\) be a down-directed poset, \(\ast\) a unary operation on \(P\), \(0 \in P\) and \((P, \sqcap)\) a commutative meet-directoid assigned to \((P, \leq)\). Then, \(P = (P, \leq, \ast, 0)\) is a pseudocomplemented poset if and only if \(A = (P, \sqcap, \ast, 0)\) satisfies the following conditions:

(i) \(0 \sqcap x \approx 0\);
(ii) \((x \sqcap y) \sqcap (x^\ast \sqcap y) \approx 0\);
(iii) \((x \sqcap z) \sqcap (y \sqcap z) = 0 \forall z \in P \Rightarrow y \sqcap x^\ast = y\).

*In this case, we call \(A\) an algebra assigned to \(P\).*

**Proof.** \(P\) is a pseudocomplemented poset if and only if the following hold:

(i') \(0 \leq x\),
(ii') \(L(x, x^\ast) = \{0\}\),
(iii') \(L(x, y) = \{0\} \Rightarrow y \leq x^\ast\).

Now, \(P\) satisfies (i') if and only if \(A\) satisfies (i). Because of Lemma 1 (a), \(P\) satisfies (ii') if and only if \(A\) satisfies (ii), and \(P\) satisfies (iii') if and only if \(A\) satisfies (iii). \(\Box\)

The concept of a \(\lambda\)-lattice was introduced by V. Snášel [2], see also [3]. A \(\lambda\)-lattice is an algebra \((L, \sqcup, \sqcap)\) of type \((2, 2)\) satisfying the following identities:

- \(x \sqcup y \approx y \sqcup x, x \sqcap y \approx y \sqcap x\);
- \(x \sqcup ((x \sqcup y) \sqcap z) \approx (x \sqcup y) \sqcup z, x \sqcap ((x \sqcap y) \sqcap z) \approx (x \sqcap y) \sqcap z\);
- \((x \sqcup y) \sqcap x \approx x, (x \sqcap y) \sqcup x \approx x\).

The concept of a \(\lambda\)-lattice was introduced in order to translate the language of directed posets into the language of algebras. One can easily see that \(\lambda\)-lattices differ from lattices only in one property, namely the associativity of lattice operations, which is replaced by the so-called weak associativity. However, the induced relation is again an order relation. This concept was already also applied in computer science (see [2]), but this is not the subject of our paper.

Let \(L = (L, \sqcup, \sqcap)\) be a \(\lambda\)-lattice. Then, \((L, \sqcup)\) and \((L, \sqcap)\) are commutative join- and meet-directoids, respectively, and \(x \sqcap y = x\) if and only if \(x \sqcup y = y\). If one defines a binary relation, \(\leq\) on \(L\) by \(x \leq y\) if \(x \sqcap y = x\) (or, equivalently, \(x \sqcup y = y\)) then \(\mathbb{R}(L) := (L, \leq)\) is a directed poset, called the *poset induced* by \(L\).

Let \(P = (P, \leq)\) be a directed poset and define binary operations \(\sqcup\) and \(\sqcap\) on \(P\) by the following prescription: \(x \sqcup y := \max(x, y)\) and \(x \sqcap y := \min(x, y)\) if \(x\) and \(y\) are comparable, and \(x \sqcup y = y \sqcup x\) and \(x \sqcap y = y \sqcap x\) should be arbitrary elements of \(U(x, y)\) and \(L(x, y)\), respectively, otherwise. Then, \(L := (P, \sqcup, \sqcap)\) is a \(\lambda\)-lattice, called a *\(\lambda\)-lattice assigned to \(P)*, and \(\mathbb{R}(L) = P\). Hence, a given directed poset \(P\) is uniquely determined by an assigned \(\lambda\)-lattice \(L\), in contrast to the fact that \(L\) may be assigned to \(P\) in a non-unique way. It is easy to see that \((P, \sqcup)\) and \((P, \sqcap)\) are commutative join- and meet-directoids assigned to \(P\), respectively.

As remarked above, every Stone poset \(P\) has the top element \(0^\ast\), thus it is also upward-directed and hence directed. Due to this, we can assign to \(P\) a \(\lambda\)-lattice.
Theorem 2. Let \((P, \leq)\) be a directed poset, \(^*\) a unary operation on \(P\), \(0 \in P\) and \((P, \sqcup, \sqcap)\) a \(\lambda\)-lattice assigned to \((P, \leq)\). Then, \(P = (P, \leq, ^*, 0)\) is a Stone poset if and only if \(A = (P, \sqcup, \sqcap, ^*, 0)\) satisfies (i)-(iii) of Theorem 1 as well as

(iv) \((x^* \sqcup y) \sqcup (x^{**} \sqcup y) \approx 0^*\).

In this case, we call \(A\) an algebra assigned to \(P\).

Proof. \(P\) is a Stone poset if and only if it is a pseudocomplemented poset satisfying:

(iv') \(U(x^*, x^{**}) = \{0^*\}\).

Because of Lemma 1 (c), \(P\) satisfies (iv') if and only if \(A\) satisfies (iv). The rest follows from Theorem 1.

Let us recall that a poset \((P, \leq)\) is called distributive if it satisfies the equality:

\[
U(L(x, y), z) = UL(U(x, z), U(y, z))
\]

or, equivalently:

\[
L(U(x, y), z) = LU(L(x, z), L(y, z))
\]

for all \(x, y, z \in P\). We are going to show that distributive pseudocomplemented posets satisfying \(U(x, x^*) = \{0^*\}\) can be characterized by means of equalities only.

Theorem 3. Let \((P, \leq, ^*, 0, 1)\) be a bounded distributive poset with a unary operation \(^*\) satisfying \(U(x, x^*) = \{1\}\) for all \(x \in P\). Then, \(P = (P, \leq, ^*, 0)\) is pseudocomplemented if and only if it satisfies the following equalities for all \(x, y \in P\):

(i) \(L(x, x^*) = \{0\}\),
(ii) \(U(x^*, L(x, y)) = U(x^*, y)\).

Proof. Let \(a, b \in P\). If \(P\) is pseudocomplemented, then (i) follows from Definition 1 and (ii) follows from:

\[
U(a^*, L(a, b)) = UL(U(a^*, a), U(a^*, b)) = UL(1, U(a^*, b)) = UUL(a^*, b) = U(a^*, b)
\]

by using the distributivity of \((P, \leq)\) and \(U(a, a^*) = \{1\}\). If, conversely, \(P\) satisfies (i) and (ii), then \(L(a, a^*) = \{0\}\) by (i) and if \(L(a, b) = \{0\}\), then:

\[
a^* \in U(a^*) = U(a^*, 0) = U(a^*, L(a, b)) = U(a^*, b) \subseteq U(b)
\]

by (ii) whence \(b \leq a^*\). □

An example of a poset satisfying the assumptions of Theorem 3 is the following.

Example 3. If \((P, \leq)\) denotes the poset visualized in Figure 3:
Figure 3. A distributive Stone poset satisfying \( U(x, x^*) = \{1\} \).

and the unary operation \( ^* \) on \( P \) is defined by

\[
\begin{array}{c|cccccccccc}
  x & 0 & a & b & c & d & e & f & g & h & i & j & 1 \\
  \hline
  x^* & 1 & j & i & h & g & f & e & d & c & b & a & 0 \\
  x^{**} & 0 & a & b & c & d & e & f & g & h & i & j & 1 \\
\end{array}
\]

then \((P, \leq, ^*, 0)\) is a distributive Stone poset (which is not a lattice) satisfying \( U(x, x^*) = \{1\} \) for all \( x \in P \) since:

\[
\begin{array}{c|cccccccccc}
  x & 0 & a & b & c & d & e & f & g & h & i & j & 1 \\
  \hline
  x^* & 1 & j & i & h & g & f & e & d & c & b & a & 0 \\
  x^{**} & 0 & a & b & c & d & e & f & g & h & i & j & 1 \\
\end{array}
\]

4. Relatively Pseudocomplemented Posets

Relatively pseudocomplemented posets were studied by numerous authors with different points of view, e.g., as a base of some non-classical logics where the relative pseudocomplementation is considered as the connective implication (see e.g., [11]) or as an approach to Hilbert algebras (see e.g., [12,13]), or for purely algebraic reasons, see e.g., [14]. We recall the following definition.

**Definition 2.** A relatively pseudocomplemented poset is an ordered quadruple \((P, \leq, ^*, 1)\) such that \((P, \leq)\) is a poset and \( ^* \) is a binary operation on \( P \) such that for all \( x, y \in P \), \( x \ast y \) is the greatest element \( z \) of \( P \) satisfying \( L(x, z) \subseteq L(y) \) and 1 denotes the top element of \((P, \leq)\). (The existence of such an element follows from the fact that \( x \ast x \) is the greatest element \( y \) of \((P, \leq)\) satisfying \( L(x, y) \subseteq L(x) \).) The element \( x \ast y \) is called the relative pseudocomplement of \( x \) with respect to \( y \).

It is clear that a relatively pseudocomplemented poset having a bottom element 0 is pseudocomplemented since \( x^* = x \ast 0 \). As it was shown by the authors in [14], every relatively pseudocomplemented poset is distributive.

It is well known that relatively pseudocomplemented semilattices or lattices play important roles in the axiomatization of intuitionistic logics. They are known under the names *Brouwerian (semi-)*-lattices or *Heyting algebras*, see e.g., [11]. However, also relatively
pseudocomplemented posets can be recognized as a formalization of certain logics of this sort, where conjunction is unsharp in the following sense. Directly by Definition 2 we have:

\[ L(a, c) \leq b \text{ if and only if } c \leq a * b. \]

The binary operation * can be considered as the logical connective of implication. From the above we have:

\[ L(a, a * b) \leq b. \]

This can be transferred to the language of a propositional calculus as follows: If we know truth values of \( a \) and \( a * b \), then the truth value of \( b \) cannot be less than that of \( a \) and \( a * b \). This expresses an unsharp version of the derivation rule Modus Ponens.

A typical example of a relatively pseudocomplemented poset is the following.

**Example 4.** If \((P, \leq)\) denotes the poset from Example 1 and the binary operation \(*\) on \( P \) is defined by

\[
\begin{array}{cccccc}
* & 0 & a & b & c & d \\
0 & 1 & 1 & 1 & 1 & 1 \\
a & 1 & b & 1 & 1 & 1 \\
b & a & a & 1 & 1 & 1 \\
c & 0 & a & b & 1 & d \\
d & 0 & a & b & c & 1 \\
1 & 0 & a & b & c & d \\
\end{array}
\]

then \((P, \leq, *, 1)\) is a relatively pseudocomplemented poset which is not a lattice.

Down-directed relatively pseudocomplemented posets can also be characterized by means of assigned directoids equipped with a binary operation.

**Theorem 4.** Let \((P, \leq)\) be a down-directed poset, \(*\) a binary operation on \( P, 1 \in P \) and \((P, \sqcap)\) a commutative meet-directoid corresponding to \((P, \leq)\). Then, \( P = (P, \leq, *, 1) \) is a relatively pseudocomplemented poset if and only if \( A = (P, \sqcap, *, 1) \) satisfies the following conditions:

(i) \( x \sqcap 1 \approx x \);

(ii) \( (x \sqcap t) \sqcap ((x * y) \sqcap z) \sqcap y = (x \sqcap z) \sqcap ((x * y) \sqcap z) \);

(iii) \( (x \sqcap t) \sqcap (z \sqcap t) \sqcap y = (x \sqcap t) \sqcap (z \sqcap t) \forall t \in P \Rightarrow z \sqcap (x * y) = z. \)

In this case, we call \( A \) an algebra assigned to \( P \).

**Proof.** \( P \) is a relatively pseudocomplemented poset if and only if the following hold:

(i') \( x \leq 1; \)

(ii') \( L(x, x * y) \leq y; \)

(iii') \( L(x, z) \leq y \Rightarrow z \leq x * y. \)

Obviously, \( P \) satisfies (i') if and only if \( A \) satisfies (i). Because of Lemma 1 (a); \( P \) satisfies (ii') if and only if \( A \) satisfies (ii); and \( P \) satisfies (iii') if and only if \( A \) satisfies (iii).

As shown above, a relatively pseudocomplemented poset \( P \) is determined by its assigned algebra \( A \) which is a commutative meet-directoid with constant 1 and equipped with a binary operation \(*\). Hence, this algebra shares all the properties of \( P \), but expressed in the language of \( A \). In the following, we show how some properties of \( A \) can be derived directly from conditions (i)-(iii) of Theorem 4.

**Theorem 5.** Let \( A = (P, \sqcap, *, 1) \) be an algebra assigned to a relatively pseudocomplemented poset \((P, \leq, *, 1)\). Then, it satisfies the following identities:

(a) \( x * x \approx 1; \)

(b) \( 1 * x \approx x; \)
(c) \( x \sqcap ((x \ast y) \ast y) \approx x \).

**Proof.**
(a) We have:
\[
(x \sqcap t) \sqcap (1 \sqcap t) \leq x \sqcap t \leq x \quad \forall t \in P
\]
and hence:
\[
((x \sqcap t) \sqcap (1 \sqcap t)) \sqcap x = (x \sqcap t) \sqcap (1 \sqcap t) \quad \forall t \in P
\]
which implies:
\[
x \ast x = 1 \sqcap (x \ast x) = 1
\]
by (i) and by (iii) of Theorem 4 when replacing \( y \) and \( z \) by \( x \) and 1, respectively.
(b) We have:
\[
1 \ast x = 1 \sqcap (1 \ast x) = (1 \sqcap 1) \sqcap ((1 \ast x) \sqcap 1) = \left( (1 \sqcap 1) \sqcap ((1 \ast x) \sqcap 1) \right) \sqcap x \leq x
\]
by (i) and by (ii) of Theorem 4 when replacing \( x \), \( y \) and \( z \) by 1, \( x \) and 1, respectively. Conversely, we have:
\[
((1 \sqcap t) \sqcap (x \sqcap t)) \sqcap x = (t \sqcap (x \sqcap t)) \sqcap (x \sqcap t) \sqcap x = x \sqcap t = t \sqcap (x \sqcap t) = (1 \sqcap t) \sqcap (x \sqcap t) \quad \forall t \in P
\]
and hence:
\[
x = x \sqcap (1 \ast x) \leq 1 \ast x
\]
by (iii) of Theorem 4 when replacing \( x \), \( y \) and \( z \) by 1, \( x \) and \( x \), respectively. Altogether, we obtain \( 1 \ast x \approx x \).
(c) We have:
\[
\left( (x \sqcap t) \sqcap ((x \ast y) \sqcap t) \right) \sqcap y = (x \sqcap t) \sqcap ((x \ast y) \sqcap t) \quad \forall t \in P
\]
by (ii) of Theorem 4 when replacing \( z \) by \( t \) and hence because of:
\[
(x \sqcap t) \sqcap ((x \ast y) \sqcap t) = ((x \ast y) \sqcap t) \sqcap (x \sqcap t),
\]
\[
\left( ((x \ast y) \sqcap t) \sqcap (x \sqcap t) \right) \sqcap y = ((x \ast y) \sqcap t) \sqcap (x \sqcap t) \quad \forall t \in P
\]
which implies:
\[
x \sqcap ((x \ast y) \ast y) = x
\]
by (iii) of Theorem 4 when replacing \( x \) and \( z \) by \( x \ast y \) and \( x \), respectively.
\[\square\]

**5. Sectionally Pseudocomplemented Posets**

Sectionally pseudocomplemented posets were recently introduced by the authors and J. Paseka in [6].

**Definition 3.** A sectionally pseudocomplemented poset is an ordered triple \((P, \leq, \circ)\) such that \((P, \leq)\) is a poset and \(\circ\) is a binary operation on \(P\) such that for all \(x, y \in P\), \(x \circ y\) is the greatest element \(z\) of \(P\) satisfying \(L(U(x, y), z) = L(y)\). The element \(x \circ y\) is called the sectional pseudocomplement of \(x\) with respect to \(y\). A sectionally pseudocomplemented poset with 1 is an ordered quadruple \((P, \leq, \circ, 1)\) such that \((P, \leq, \circ)\) is a sectionally pseudocomplemented poset and 1 is the top element of \((P, \leq)\). A strongly sectionally pseudocomplemented poset (see [6]) is a sectionally pseudocomplemented poset \((P, \leq, \circ, 1)\) with 1 satisfying \(x \leq (x \circ y) \circ y\) for all \(x, y \in P\).
If a sectionally pseudocomplemented poset \((P, \leq, \circ)\) has a top element 1, then \(1 = x \circ x\) for all \(x \in P\). The following example shows a sectionally pseudocomplemented poset which is neither a lattice nor strongly sectionally pseudocomplemented.

**Example 5** (cf. [6]). If \(P = (P, \leq)\) denotes the poset visualized in Figure 4:

![Figure 4. A sectionally pseudocomplemented poset.](image)

and the binary operation \(\circ\) on \(P\) is defined by

| \(\circ\) | \(a\) | \(b\) | \(c\) | \(d\) | \(e\) | \(f\) | \(g\) | \(1\) |
|---|---|---|---|---|---|---|---|---|
| \(a\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(b\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(c\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(d\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(e\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(f\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(g\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

then \((P, \leq, \circ)\) is a sectionally pseudocomplemented poset which is neither a lattice nor strongly sectionally pseudocomplemented since \(c \not\leq a = f \circ a = (c \circ a) \circ a\).

**Example 6** (cf. [6]). If \(P = (P, \leq)\) denotes the poset visualized in Figure 5:

![Figure 5. A strongly sectionally pseudocomplemented poset.](image)
and the binary operation $\circ$ on $P$ is defined by

| $\circ$ | 0 | a | b | c | d | e | 1 |
|---------|---|---|---|---|---|---|---|
| 0       | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| a       | c | 1 | 1 | c | 1 | 1 | 1 |
| b       | c | a | 1 | c | 1 | 1 | 1 |
| c       | b | a | b | 1 | 1 | 1 | 1 |
| d       | 0 | a | b | c | 1 | e | 1 |
| e       | 0 | a | b | c | d | 1 | 1 |
| 1       | 0 | a | b | c | d | e | 1 |

then $(P, \leq, \circ, 1)$ is a strongly sectionally pseudocomplemented poset which is neither a lattice nor relatively pseudocomplemented since the relative pseudocomplement of $b$ with respect to $a$ does not exist. Since:

$$L(U(a,c),b) = LU(b) = L(b) \neq L(a) = LUL(a) = LU(L(a,b),L(c,b)),$$

$P$ is not distributive.

It is well known that every relatively pseudocomplemented lattice is distributive. Sectional pseudocomplementation was introduced by the first author [15] in order to extend the concept of relative pseudocomplementation to non-distributive lattices. It was shown that sectionally pseudocomplemented lattices form a variety and considered as posets, they are strongly sectionally pseudocomplemented. It is a natural question if these posets can also be considered as a formalization of certain unsharp propositional logic. We can consider the binary operation $\circ$ of sectional pseudocomplementation as the logical connective of implication. Having two propositions with truth values $a$ and $b$ such that $b \leq a$, the formula:

$$L(U(a,b),c) = L(b)$$

can be rewritten as

$$a \land c = b.$$

Hence, by Definition 3, for $b \leq a$:

$$a \land c = b \text{ if and only if } c \leq a \circ b.$$

Therefore, for $b \leq a$ we have a form of adjointness of operations $\land$ and $\circ$. Moreover, we have:

$$a \land (a \circ b) = b.$$

Thus, the truth value of $b$ is the same as the truth value of conjunction $a \land (a \circ b)$ provided $b \leq a$. This is again a version of Modus Ponens restricted to elements $a, b$ with $b \leq a$.

Directed sectionally pseudocomplemented posets can be also characterized by assigned $\lambda$-lattices with a binary operation as follows.

**Theorem 6.** Let $(P, \leq)$ be a directed poset, $\circ$ a binary operation on $P$ and $(P, \sqcup, \sqcap)$ a $\lambda$-lattice corresponding to $(P, \leq)$. Then, the following hold:

(a) $P = (P, \leq, \circ)$ is a sectionally pseudocomplemented poset if and only if $A = (P, \sqcup, \sqcap, \circ)$ satisfies the following conditions:

(i) $y \sqcap (x \circ y) \approx y$,

(ii) $(z \sqcap ((x \sqcup t) \cup (y \sqcup t))) \sqcap ((x \circ y) \sqcap z) = z \forall t \in P \Rightarrow z \sqcap y = z,$
As we remarked above, in a $\lambda$-lattice

**Theorem 7.** Let $A = (P, \cup, \cap, \wedge, 1)$ be an algebra assigned to a sectionally pseudocomplemented poset $(P, \leq, \circ, 1)$ with 1. Then, it satisfies the following identities:

(a) $x \circ x = 1$,
(b) $1 \circ x = x$. 

In both cases, we call $A$ an algebra assigned to $P$.

**Proof.**

(a) $P$ is a sectionally pseudocomplemented poset if and only if the following hold:

(i) $L(U(x, y), x \circ y) = L(y)$;

(ii) $L(U(x, y), z) = L(y) \Rightarrow z \leq x \circ y$.

Now, $P$ satisfies (a) and (iii) if and only if it satisfies (i') and (ii') where (i') and (ii') denote the following conditions:

(i') $y \leq x \circ y$;

(ii') $L(U(x, y), x \circ y) \leq y$.

This can be seen as follows: If $P$ satisfies (a) and (iii') then it satisfies (ii') and because of $L(U(x, y), y) = L(y)$ and (iii'), it satisfies (i'). If, conversely, $P$ satisfies (i') and (ii'), then because of:

$L(y) = L(U(x, y), y) \subseteq L(U(x, y), x \circ y) \subseteq L(y)$

it satisfies (a). This shows that $P$ is a sectionally pseudocomplemented poset if and only it satisfies (i') and (ii'). Now, $P$ satisfies (i') if and only if $A$ satisfies (i). According to Lemma 1 (c):

$U(x, y) = \{(x \cup t) \cup (y \cup t) \mid t \in P\}$

and hence, by Lemma 1 (b):

$z \in L(U(x, y), x \circ y) \Leftrightarrow \big((x \cup t) \cup (y \cup t) \big) \cap \big((x \circ y) \cap z\big) = z \forall t \in P,$

$s \in L(U(x, y), z) \Leftrightarrow \big((x \cup t) \cup (y \cup t) \big) \cap \big(s \cap z\big) = s \forall t \in P.$

This shows that $P$ satisfies (i') if and only if $A$ satisfies (ii), and that $P$ satisfies (iii') if and only if $A$ satisfies (iii).

(b) $P$ is a sectionally pseudocomplemented poset with 1 if and only if $(P, \leq, \circ)$ is a sectionally pseudocomplemented poset satisfying:

(i') $x \leq 1$.

Now, $P$ satisfies (i') if and only if $A$ satisfies (iv). The rest follows from (a).

In the next theorem, we show how some properties of algebras assigned to sectionally pseudocomplemented posets with 1 can be derived from conditions (i)--(iv) of Theorem 6. As we remarked above, in a $\lambda$-lattice $(A, \cup, \cap)$ the identities $x \cap 1 \equiv x$ and $x \cup 1 \equiv 1$ are equivalent.

**Theorem 7.** Let $A = (P, \cup, \cap, \wedge, 1)$ be an algebra assigned to a sectionally pseudocomplemented poset $(P, \leq, \circ, 1)$ with 1. Then, it satisfies the following identities:

(a) $x \circ x = 1$,
(b) $1 \circ x = x$. 

In both cases, we call $A$ an algebra assigned to $P$. 

Proof. 

(a) $P$ is a sectionally pseudocomplemented poset if and only if the following hold:

(i) $L(U(x, y), x \circ y) = L(y)$;

(ii) $L(U(x, y), z) = L(y) \Rightarrow z \leq x \circ y$.

Now, $P$ satisfies (a) and (iii) if and only if it satisfies (i') and (ii') where (i') and (ii') denote the following conditions:

(i') $y \leq x \circ y$;

(ii') $L(U(x, y), x \circ y) \leq y$.

This can be seen as follows: If $P$ satisfies (a) and (iii') then it satisfies (ii') and because of $L(U(x, y), y) = L(y)$ and (iii'), it satisfies (i'). If, conversely, $P$ satisfies (i') and (ii'), then because of:

$L(y) = L(U(x, y), y) \subseteq L(U(x, y), x \circ y) \subseteq L(y)$

it satisfies (a). This shows that $P$ is a sectionally pseudocomplemented poset if and only it satisfies (i') and (ii'). Now, $P$ satisfies (i') if and only if $A$ satisfies (i). According to Lemma 1 (c):

$U(x, y) = \{(x \cup t) \cup (y \cup t) \mid t \in P\}$

and hence, by Lemma 1 (b):

$z \in L(U(x, y), x \circ y) \Leftrightarrow \big((x \cup t) \cup (y \cup t) \big) \cap \big((x \circ y) \cap z\big) = z \forall t \in P,$

$s \in L(U(x, y), z) \Leftrightarrow \big((x \cup t) \cup (y \cup t) \big) \cap \big(s \cap z\big) = s \forall t \in P.$

This shows that $P$ satisfies (i') if and only if $A$ satisfies (ii), and that $P$ satisfies (iii') if and only if $A$ satisfies (iii).

(b) $P$ is a sectionally pseudocomplemented poset with 1 if and only if $(P, \leq, \circ)$ is a sectionally pseudocomplemented poset satisfying:

(i') $x \leq 1$.

Now, $P$ satisfies (i') if and only if $A$ satisfies (iv). The rest follows from (a). 

\[\Box\]
Proof. 

(a) According to (iv) we have:

\[
\left( (x \sqcup t) \sqcup (x \sqcup t) \right) \sqcap (1 \sqcap s) = ((x \sqcup t) \sqcap s) \sqcap s = (x \sqcup t) \sqcap s
\]

and hence the following are equivalent:

\[
\left( ((x \sqcup t) \sqcup (x \sqcup t)) \sqcap s \right) \sqcap (1 \sqcap s) = s \forall t \in P,
\]

\[
(x \sqcup t) \sqcap s = s \forall t \in P,
\]

\[
s \leq x \sqcup t \forall t \in P,
\]

\[
s \leq x,
\]

\[
s \sqcap x = s
\]

which implies:

\[
x \circ x = 1 \sqcap (x \circ x) = 1
\]

by (iv) and by (iii) when replacing \(y\) and \(z\) by \(x\) and 1, respectively.

(b) According to (iv) we have:

\[
\left( ((1 \sqcup t) \sqcup (x \sqcup t)) \sqcap (1 \circ x) \right) \sqcap (1 \circ x) \sqcap (1 \circ x) = (1 \circ x) \sqcap (1 \circ x)
\]

\[
= 1 \circ x \forall t \in P
\]

and hence:

\[
(1 \circ x) \sqcap x = 1 \circ x
\]

(2)

by (ii) when replacing \(x\), \(y\) and \(z\) by 1, \(x\) and \(1 \circ x\), respectively. Moreover, according to (iv) we have:

\[
\left( ((1 \sqcup t) \sqcup (x \sqcup t)) \sqcap s \right) \sqcap (x \sqcap s) = \left( (1 \sqcup (x \sqcup t)) \sqcap s \right) \sqcap (x \sqcap s)
\]

\[
= (1 \sqcap s) \sqcap (x \sqcap s) = s \sqcap (x \sqcap s) = x \sqcap s = s \sqcap x
\]

and hence the following are equivalent:

\[
\left( (x \sqcup t) \sqcup (x \sqcup t) \right) \sqcap x = s \forall t \in P,
\]

\[
s \sqcap x = s.
\]

This implies

\[
x \sqcap (1 \circ x) = x
\]

(3)

by (iii) when replacing \(x\), \(y\) and \(z\) by 1, \(x\) and \(x\), respectively. From (2) and (3) we obtain (b).

\[\square\]

Analogously to the characterizations in Theorem 6, we can also characterize strongly sectionally pseudocomplemented posets.

**Theorem 8.** Let \((P, \leq)\) be a directed poset, \(\circ\) a binary operation on \(P\), \(1 \in P\) and \((P, \sqcup, \sqcap)\) a \(\lambda\)-lattice corresponding to \((P, \leq)\). Then, \(P = (P, \leq, \circ, 1)\) is a strongly sectionally pseudocomplemented poset if and only if \(A = (P, \sqcup, \sqcap, \circ, 1)\) satisfies (i)–(iv) of Theorem 6 as well as

(v) \(x \sqcap ((x \circ y) \circ y) \approx x\).

In this case, we call \(A\) an algebra assigned to \(P\).
Proof. \( P \) is a strongly sectionally pseudocomplemented poset if and only if \((P, \leq, \circ)\) is a sectionally pseudocomplemented poset with 1 satisfying:

\[(v') \quad x \leq (x \circ y) \circ y.\]

Now, \( P \) satisfies \((v')\) if and only if \( A \) satisfies \((v)\). The rest follows from Theorem 6. \( \square \)

6. Congruence Properties

We then consider the congruence properties of algebras. For the convenience of the reader, we recall the corresponding definitions.

An algebra \( A \) is called:

- **congruence permutable** if \( \Theta \circ \Phi = \Phi \circ \Theta \) for all \( \Theta, \Phi \in \text{Con}A \);
- **congruence distributive** if the congruence lattice \( \text{Con}A \) of \( A \) is distributive;
- **arithmetical** if it is both congruence permutable and congruence distributive;
- **weakly regular** (with respect to an equationally definable constant 1) if \( \Theta, \Phi \in \text{Con}A \) and \( [1] \Theta = [1] \Phi \) imply \( \Theta = \Phi \).

The following results are well known (see e.g., [16]):

Let \( A \) be an algebra.

- If there exists a ternary term \( p \) satisfying the identities \( p(x, x, y) \approx p(y, x, x) \approx y \), then \( A \) is congruence permutable.
- If there exists a ternary term \( m \) satisfying the identities \( m(x, x, y) \approx m(y, x, x) \approx m(y, x, x) \approx x \), then \( A \) is congruence distributive.
- If \( A \) has an equationally definable constant 1 and there exists a positive integer \( n \), binary terms \( t_1, \ldots, t_n \) and quaternary terms \( s_1, \ldots, s_n \) satisfying the identities:

\[
\begin{align*}
    t_1(x, x) &\approx \cdots \approx t_n(x, x) \approx 1, \\
    s_1(t_1(x, y), 1, x, y) &\approx x, \\
    s_i(1, t_i(x, y), x, y) &\approx s_{i+1}(t_{i+1}(x, y), 1, x, y) \text{ for } i = 1, \ldots, n-1, \\
    s_n(1, t_n(x, y), x, y) &\approx y
\end{align*}
\]

then \( A \) is weakly regular.

The terms \( p \) and \( m \) are called **Maltsev term** and **majority term**, respectively.

Now, we can prove that:

**Theorem 9.**

(i) Let \((P, \leq, \ast, 0)\) be a Stone poset and \( A = (P, \sqcup, \cap, \ast, 0) \) be an assigned algebra. Then, \( A \) is congruence distributive.

(ii) Let \((P, \leq, \ast, 1)\) be a down-directed relatively pseudocomplemented poset and \( A = (P, \sqcup, \cap, \ast, 1) \) an assigned algebra. Then, \( A \) is congruence permutable and weakly regular.

(iii) Let \((P, \leq, \circ)\) be a directed sectionally pseudocomplemented poset and \( A = (P, \sqcup, \cap, \circ) \) an assigned algebra. Then, \( A \) is congruence distributive.

(iv) Let \((P, \leq, \circ, 1)\) be a down-directed sectionally pseudocomplemented poset with 1 and \( A = (P, \sqcup, \cap, \circ, 1) \) an assigned algebra. Then, \( A \) is congruence distributive and weakly regular.

(v) Let \((P, \leq, \circ, 1)\) be a down-directed strongly sectionally pseudocomplemented poset and \( A = (P, \sqcup, \cap, \circ, 1) \) an assigned algebra. Then, \( A \) is arithmetical and weakly regular.

**Proof.**

(i) If:

\[
m(x, y, z) := ((x \sqcup y) \cap (y \sqcup z)) \sqcup (z \sqcup x)
\]
then:

\[
m(x, y, z) = ((x * y) * z) \sqcap (y * x) \equiv x * (y * x) \equiv x * y \equiv y * x
\]

(iii) This follows like in (i).

(iv) This follows like in (i) and (ii) using Theorem 7 instead of Theorem 5.

(v) This follows like in (i) and (ii) using Theorem 7 and Theorem 8 instead of Theorem 5.

\[\square\]

7. Conclusions and Discussion

The advantage of our approach of introducing algebras \( A \) closely related to given posets \( P \) is that we can introduce congruences in \( P \) by means of \( A \). Namely, we may introduce congruences on \( P \) as congruences on a fixed algebra \( A \) assigned to \( P \). Of course, the assignment of \( A \) to \( P \) is in general not unique as mentioned above and hence for different algebras assigned to \( P \), we could obtain different congruences on \( P \). However, regardless of which algebra \( A \) is assigned to \( P \), the lattice of congruences on \( P \) will be distributive if \( A \) is a \( \lambda \)-lattice and congruences on \( P \) will permute and will be fully determined by their 1-class if \( P \) is down-directed and either relatively pseudocomplemented or strongly sectionally pseudocomplemented. This may shed new light on the concept of congruences on posets.

Another application of our approach can be the following. Given a directed relatively or sectionally pseudocomplemented poset \( P \), we can ask if there are posets \( P_1 \) and \( P_2 \) of the same kind as \( P \) such that \( P_1 \times P_2 \cong P \). For posets, it is not easy to decide whether such a decomposition is possible. However, we can assign to \( P \) an algebra \( A \) as constructed above. Now, we have a simple criterion for deciding whether there exist algebras \( A_1 \) and
$A_2$ with $A_1 \times A_2 \cong A$. Namely, such a decomposition is possible if and only if there exist $\Theta, \Phi \in \text{Con}A$ such that:

\[
\begin{align*}
\Theta \lor \Phi &= \nabla A, \\
\Theta \cap \Phi &= \Delta A, \\
\Theta \circ \Phi &= \Phi \circ \Theta.
\end{align*}
\]

Hence, if such congruences exist, we have $A \cong A_1 \times A_2$ with $A_1 = A/\Theta$ and $A_2 = A/\Phi$. Now, we can conversely assign posets $P_i$ to $A_i$ for $i = 1, 2$ and we can write $P_1 \times P_2 \cong P$.

Moreover, if $P = P_1 \times P_2$, then we can ask if a given congruence $\Theta$ on $P$ is directly decomposable, i.e., if there are $\Theta_i \in \text{Con}P_i$ for $i = 1, 2$ such that $\Theta_1 \times \Theta_2 = \Theta$. If for assigned algebras we have $A = A_1 \times A_2$ then, provided $A$ is congruence distributive, there exist $\Theta_i \in \text{Con}A_i$ for $i = 1, 2$ such that $\Theta_1 \times \Theta_2 = \Theta$ and hence $P$ has directly decomposable congruences.

The structures considered in our paper are often applied in the formalization of some kind of non-classical logics, in particular of some intuitionistic logics. For such a formalization, our description by means of directoids and $\lambda$-lattices brings some advantages because one can work with axioms of these logics by means of algebraic rules.

It is well known that intuitionistic logics are formalized by means of Heyting and/or Brouwer algebras. It is also well known that the logical connectives conjunction, disjunction and implication are independent in these logics and that the connective implication is formalized by the relative pseudocomplementation or by sectional pseudocomplementation. Hence, it is natural to study the formalization of implication in posets where lattice operations have no influence. This motivates the study of relatively and sectionally pseudocomplemented posets. One may discuss the following questions:

- What role is played by the directoid and $\lambda$-lattice operations $\sqcup$ and $\sqcap$ in such a formalization?
- Can these operations be considered logical connectives?
- If these operations would be considered as disjunction and conjunction, respectively, what kind of intuitionistic logic would be obtained?

These questions may be investigated in future research.

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