A Dissipation Theory for Potentials-Based FDTD for Lossless Inhomogeneous Media

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Abstract—A dissipation theory is proposed for the potentials-based FDTD algorithm for the case of inhomogeneous lossless media. We show that under the Courant-Friedrichs-Lewy (CFL) limit, the equations describing the time evolution of scalar and vector potentials can be seen as a lossless system. The developed theory provides insights into how electromagnetic energy and power flow are approximated in FDTD schemes. It can also be used to create new algorithms with guaranteed stability.

Index Terms—Energy conservation, finite-difference time-domain method, scalar potential, vector potential, stability.

I. INTRODUCTION

T

HE finite-difference time-domain (FDTD) algorithm [1], [2] is widely used for solving Maxwell’s equations. Recently, there has been an increased interest in integrating FDTD with quantum models [7]–[8]. Interaction between electromagnetic signals and quantum states plays an essential role in many architectures being considered for quantum computing [7], calling for algorithms that can jointly simulate quantum and electromagnetic phenomena. Quantum models of field-particle interaction typically involve potentials, as opposed to the fields computed in the traditional FDTD. The requirement of the knowledge of potentials in quantum modeling makes potentials a natural choice of unknowns for FDTD in quantum applications [7], [8]. Earlier, potentials-based FDTD (P-FDTD) formulations have been investigated, for example, as means of reducing computational requirements [10], [11].

P-FDTD methods still lack many of the advancements that have been proposed for the fields-based FDTD, including subgridding [12], model order reduction [13], and many others. One of the difficulties in creating such new schemes is the complexity of ensuring stability. In the case of the traditional fields-based FDTD, one needs to select the time step below the Courant-Friedrichs-Lewy (CFL) limit [2], [14].

\[ \Delta t < \frac{\sqrt{\mu \varepsilon}}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}. \]  

The same stability requirement has been used for P-FDTD algorithms in [7], [11], [15]. Various techniques have been proposed for ensuring stability of new FDTD schemes, such as von Neumann analysis [14], the iteration matrix method [16], as well as methods based on analyzing a linear operator involved in the second order formulation of FDTD equations [13], [17].

In this work, we propose a dissipation-based framework for ensuring stability of P-FDTD schemes, extending previous works on traditional FDTD [18]. The dissipation framework takes root in the theory of dissipative systems [19] and is a generalization of the energy method [20]. The present work provides proper expressions to compute the energy stored in a region and power absorbed through its boundary in P-FDTD. It also gives insights into how electromagnetic energy is approximated in P-FDTD. The proposed approach can facilitate the stability analysis in complex setups involving many parts, such as grids of different resolution in subgridding [12], different types of equations in hybrid methods [21] or in multi-physics simulations [7], [22].

II. P-FDTD EQUATIONS IN STATE SPACE FORM

We consider a rectangular region filled with an inhomogeneous isotropic dielectric with permittivity \( \varepsilon \) and permeability \( \mu \). In this section, we discretize the wave equations governing the scalar potential \( \phi \) and vector potential \( \mathbf{A} \) and cast the discretized equations in the form of a state space dynamical system suitable for analyzing energy dissipation.

A. The Continuous State Equations for \( \phi \) and \( \mathbf{A} \)

Under the generalized Lorenz gauge [11], [23], [24], the wave equation for scalar potential is

\[ \chi \frac{\partial^2 \phi}{\partial t^2} = \nabla \cdot \varepsilon \nabla \phi \]  

where \( \chi = \mu \varepsilon \). Equation (2) can be written in a form involving only first order time derivatives by introducing the following variables: \( [\nabla \phi] \) for the gradient of scalar potential and \( [\partial_t \phi] \) for the temporal derivative of scalar potential. The square brackets are used to indicate that we take the values of \( \nabla \phi \) and \( \partial_t \phi \) as unknowns. Hence, (2) can be rewritten as

\[ \chi \frac{\partial}{\partial t} [\partial_t \phi] = \nabla \cdot \varepsilon [\nabla \phi] \]  

\[ \varepsilon \frac{\partial}{\partial t} [\nabla \phi] = \varepsilon \nabla [\partial_t \phi]. \]  

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samples outside the region, which allows precisely quantifying the amount of electromagnetic energy stored in the region.

1) Scalar Potential: The variables \( \nabla \phi \) and \( [\partial_t \phi] \) are sampled at the edges and nodes of the primary grid, respectively, as shown in Fig. 1 in blue. By approximating the gradient operator in \([15]\) via centered finite differences, we obtain

\[
\Delta x \Delta y \Delta z \frac{\varepsilon}{\Delta t} \left( [\partial_t \phi]_{i+\frac{1}{2},j,k} - [\partial_t \phi]_{i-\frac{1}{2},j,k} \right) = \\
\Delta x \Delta y \Delta z \left( [\partial_t \phi]_{i+\frac{1}{2},j+\frac{1}{2},k} - [\partial_t \phi]_{i-\frac{1}{2},j-\frac{1}{2},k} \right) 
\]

for the \( x \)-directed edges strictly inside the region. Equation (5) can be seen as a discrete integration of (3b) over the volume associated with the primary edge where the \([\partial_t \phi]\) sample is located. In particular, \( \Delta x \) is the length associated with that primary edge and \( \Delta y \Delta z \) is the area of the secondary face pierced by that edge. For edges on the bottom face of the boundary \((k = 1)\), we take the same equation as (5), but with \( \Delta z \) replaced by \( \Delta z/2 \), since the area of the corresponding secondary face is \( \Delta y \Delta z/2 \). Analogous modifications are made for the edges shared between two faces of the boundary.

With reference to Fig. 2, (3a) is discretized as

\[
\Delta x \Delta y \Delta z \frac{\chi}{\Delta t} \left( [\partial_t \phi]_{i+\frac{1}{2},j,k} - [\partial_t \phi]_{i-\frac{1}{2},j,k} \right) = \\
\Delta y \Delta z \left( \varepsilon [\partial_t \phi]_{i,\frac{1}{2},j+k} - \varepsilon [\partial_t \phi]_{i,\frac{1}{2},j-k} \right) \\
+ \Delta x \Delta z \left( \varepsilon [\partial_t \phi]_{\frac{1}{2},j+\frac{1}{2},k} - \varepsilon [\partial_t \phi]_{\frac{1}{2},j-\frac{1}{2},k} \right) \\
+ \Delta x \Delta y \left( \varepsilon [\partial_t \phi]_{\frac{1}{2},j+\frac{1}{2},\frac{1}{2},k} - \varepsilon [\partial_t \phi]_{\frac{1}{2},j-\frac{1}{2},\frac{1}{2},k} \right) 
\]

for the nodes strictly inside the region. On the right hand side of (6) is the discrete flux of \( \varepsilon \nabla \phi \) through the faces of the secondary cell that has the node \((i, j, k)\) at its center. For nodes on the \( k = 1 \) boundary, (6) would involve samples outside the boundary, which the model aims to avoid. Instead, we introduce a hanging variable \([\partial_t \phi]|_{i,j,1}\) on the boundary of the region \([18], [25]\), as shown in Fig. 2 in red. The equation is written over the \( \Delta x \times \Delta y \times \Delta z/2 \) secondary cell adjacent to the \( k = 1 \) boundary

\[
\Delta x \Delta y \Delta z \frac{\chi}{2 \Delta t} \left( [\partial_t \phi]_{i+\frac{1}{2},j,\frac{1}{2},k} - [\partial_t \phi]_{i-\frac{1}{2},j,\frac{1}{2},k} \right) = \\
\Delta y \Delta z \left( \varepsilon [\partial_t \phi]_{i,\frac{1}{2},j+\frac{1}{2},k} - \varepsilon [\partial_t \phi]_{i,\frac{1}{2},j-\frac{1}{2},k} \right) \\
+ \Delta x \Delta z \left( \varepsilon [\partial_t \phi]_{\frac{1}{2},j+\frac{1}{2},\frac{1}{2}} - \varepsilon [\partial_t \phi]_{\frac{1}{2},j-\frac{1}{2},\frac{1}{2}} \right) \\
+ \Delta x \Delta y \left( \varepsilon [\partial_t \phi]_{\frac{1}{2},j+\frac{1}{2},\frac{1}{2},\frac{1}{2}} - \varepsilon [\partial_t \phi]_{\frac{1}{2},j-\frac{1}{2},\frac{1}{2},\frac{1}{2}} \right) 
\]

Equations for the nodes shared between two or three sides of the boundary involve, respectively, two and three hanging variables. The hanging variables will be crucial to quantifying the power absorbed by the region through the boundary \([18]\).

2) Vector Potential: As shown in Fig. 1 \( \vec{B} \) is sampled on the secondary edges. The variables \([\partial_t \vec{A}]\) and \( \kappa \) are sampled on the primary edges and primary nodes, respectively. Equations (4a)–(4c) are discretized analogously to (3a)–(3b). For edges and nodes strictly inside the region, the discrete equations are equivalent to those in (7). In order to discretize equation (4c) on boundary nodes, we introduce a hanging
variable for the normal component of the vector potential. For example, the equation on the \( k = 1 \) boundary is written in an analogous way to (7), involving a hanging variable \( [\partial n A_j]_{1, i, j, 1} \). Likewise, the discretized equation (4b) for \( [\partial n A_j] \) on primary edges tangential to the boundary requires a hanging variable for the tangential magnetic flux density. For instance, on the \( k = 1 \) boundary, the resulting equation reads

\[
\Delta y \Delta z_2 \frac{\varepsilon}{\Delta t} \left( [\partial A_x]_{i+1/2, j, z, 1}^{n+1} - [\partial A_x]_{i+1/2, j, z, 1}^{n} \right) =
- \Delta x \left( \frac{\Delta z}{2} \mu_{\varepsilon}^{-1} B_z^{n+1/2} - \frac{\Delta z}{2} \mu_{\varepsilon}^{-1} B_z^{n+1/2} \right)
- \Delta y \mu_{\varepsilon}^{-1} B_y^{n+1/2} + \Delta y \mu_{\varepsilon}^{-1} B_y^{n+1/2}
\]

\[
- \Delta y \frac{\Delta z}{2} \varepsilon \left( [\kappa_{i+1, j, 1}^{n+1/2} - [\kappa_{i+1, j, 1}^{n+1/2}] \right) \tag{8}
\]

The expression in the first bracket on the right hand side approximates the circulation of the magnetic field around the path in Fig. [2]. The hanging variable for tangential boundary magnetic flux density \( B_y^{i+1/2, 0, 0, 1} \) is introduced to avoid using examples outside of the region and later devise an expression for the energy entering the region through the boundary.

**C. State Equations in Matrix Form**

State equations for scalar potential, such as (5)–(7), written out for all internal and boundary nodes and edges can be collected in convenient matrix form

\[
\Lambda_n^{\varepsilon} A_n^{\varepsilon} \frac{\Delta z}{\Delta t} \left( [\nabla \phi]^n + 1 - [\nabla \phi]^n \right) = -\Lambda_n^{\varepsilon} A_n^{D} D^T [\partial \phi]^{n+1/2} \tag{9a}
\]

\[
\Lambda_n^{\varepsilon} \frac{\Delta x}{\Delta t} \left( [\partial \phi]^{n+1/2} - [\partial \phi]^{n-1} \right) = -D A_n^{\varepsilon} \Lambda_n^{\varepsilon} [\nabla \phi]^{n} + L_{\perp} A_{(n)}^{\varepsilon} A_{(n)}^{S} A_{(n)}^{\varepsilon} [\nabla \phi]^{n} \tag{9b}
\]

where \([\partial \phi]\) is a column vector collecting all \([\partial \phi]_n\) samples shown in blue in Fig. [1] and Fig. [2]. Vector \([\nabla \phi]\) collects the conventional \([\nabla \phi]\) samples, such as \([\partial x \phi]\), also shown in blue in the figures. Vector \([\nabla \phi]_n\) collects the corresponding hanging variables, shown in red in the figures. Symbol \(A\) denotes diagonal matrices. In particular, \(A_n^{\varepsilon}\) and \(A_n^{\varepsilon}\) contain secondary face areas and primary edge lengths. Similarly, \(A_n^{\varepsilon}\) contains volumes of secondary cells associated with the samples in \([\partial \phi]\). Matrices \(A_n^{\varepsilon}\) and \(A_n^{\varepsilon}\) contain, respectively, the values of \(\varepsilon\) and \(\chi\) on the primary edges and nodes. Matrix \(D\) contains \(1\)’s, \(-1\)’s, and \(0\)’s needed for the computation of discrete divergence or flux. Elements of matrix \(L_{\perp}\) are \(0\)’s and \(1\)’s, with 1 if the corresponding hanging variable in \([\nabla \phi]_{\perp}\) is located at the node corresponding to the row of \(L_{\perp}\). Matrix \(A_{(n)}^{\varepsilon}\) contains \(1\) or \(-1\) on the diagonal, depending on the sign of the dot product between the outward normal vector \(n\) to the boundary and the Cartesian unit vector associated with the additional edge where the hanging variable is located. Matrices \(A_n^{\varepsilon}\) and \(A_n^{\varepsilon}\) contain boundary face areas and permittivities on the edges associated with the hanging variables in \([\nabla \phi]_n\). In an analogous way, one can write the discretized equations for vector potential in matrix form.

Equations (9a)–(9b) can be rewritten in the generalized state space form

\[
(R + F)x^{n+1/2} = (R - F)x^{n-1/2} + Bu^n \tag{10a}
\]

\[
y^{n-1/2} = LTx^{n-1/2} \tag{10b}
\]

where the state, input, and output vectors are given by

\[
x^{n-1/2} = \begin{bmatrix} [\nabla \phi]_n \\ [\partial \phi]_{\perp}^{n-1/2} \end{bmatrix}, y^n = [\nabla \phi]_n, y^{n-1/2} = (\partial \phi)_{\perp}^{n-1/2}
\]

and \([\partial \phi]_n\) is a vector collecting the \([\partial \phi]_n\) samples located on the boundary. Matrices \(R, F, B,\) and \(L\) are given by

\[
R = \begin{bmatrix} R_{11} & R_{21} \\ R_{21} & R_{22} \end{bmatrix}, F = \begin{bmatrix} 0 & R_{21} \\ -R_{21} & 0 \end{bmatrix}, B = LS, L = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{12}
\]

where \(R_{11} = (\Delta t)^{-1} A_n^{\varepsilon} A_n^{\varepsilon} A_n^{\varepsilon}, R_{21} = \frac{1}{2}DA_n^{\varepsilon} A_n^{\varepsilon}, R_{22} = (\Delta t)^{-1} A_n^{\varepsilon} A_n^{\varepsilon}, S = A_{(n)}^{S} A_{(n)}^{S} A_{(n)}^{\varepsilon}. One can also formulate a dynamical system of the form (10) for the vector potential. Writing the FDTD equations in this form enables the application of many theorems from systems theory, including those that investigate whether the system is lossless or not.

**III. DISSIPATIVITY ANALYSIS**

**A. Scalar Potential**

A dynamical system in the form (10) is said to be lossless with supply rate \(s\) if there is a storage function \(E\), satisfying

\[
E(x^{n+1/2}) \geq 0, E(0) = 0 \quad \forall n \tag{13a}
\]

\[
E(x^{n+1/2}) - E(x^{n-1/2}) = s(u^n, y^{n+1/2}, y^{n+1/2}) \quad \forall n \tag{13b}
\]

for all \(u^n\). As storage function and supply rate, we propose to use the following expressions, based on [18], [20]

\[
E(x^{n+1/2}) = \frac{\Delta t}{2} \left( x^{n+1/2} \right)^T R x^{n+1/2} \tag{14}
\]

\[
s(u^n, y^{n+1/2}, y^{n+1/2}) = \Delta t \left( y^{n+1/2} + \frac{y^{n+1/2}}{2} \right)^T S u^n. \tag{15}
\]

When (14) is expanded, it is revealed to be a discrete counterpart of the continuous expression for the stored energy found in literature [27], [28].

\[
E_{cont} = \int \int_V \left[ \frac{\varepsilon}{2} \nabla \phi \cdot \nabla \phi + \frac{\chi}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 \right] dV. \tag{16}
\]

Expression (15) is a discrete approximation of the continuous energy entering the region during one time step [28]

\[
s_{cont} = \int \int_S \int \int_S s dS dV \tag{17}
\]

where \(-\frac{\partial \phi}{\partial t}\) is the power flux density similar to the Poynting vector.

Using the structure of the matrices in (12), we were able to prove that the system (10) satisfies (13) if \(R\) is positive definite. For homogeneous media, it can be shown that ensuring the CFL limit (1) guarantees the positive definiteness of \(R\) and thus ensures that (10) conserves energy.

Hence, in addition to the well-accepted connection to stability [2], [7], we interpret the CFL limit as a condition under
We consider a particular solution of the wave equation in the region given by
\[ \phi = C_\phi \sin(k_x x) \sin(k_y y) \sin(k_z z) \cos(\omega t + \frac{\pi}{3}) \]  
\[ \vec{A} = C_A \cos(k_x x) \sin(k_y y) \sin(k_z z) \sin(\omega t + \frac{\pi}{3}) \cdot \hat{x} \]  
\[ k_x = \frac{3\pi}{a}, \quad k_y = k_z = \frac{\pi}{a}, \quad \omega = \sqrt{\frac{\epsilon_0}{\chi_0} (k_x^2 + k_y^2 + k_z^2)} \]  
where \( C_A = 10^{-9} \text{ Tm} \), \( C_\phi = -C_A \varepsilon_0 k_x / (\chi_0 \omega) \), \( \varepsilon_0 \) and \( \chi_0 \) are values of free-space \( \varepsilon \) and \( \chi \). From (16), (18), (20), and (21), one can calculate the energy stored in the region for the scalar and vector potential
\[ E_{\phi,\text{cont}} = \frac{1}{16} \chi_0 C_\phi^2 \omega^2 a^3, \quad E_{A,\text{cont}} = \frac{1}{16} \varepsilon_0 C_A^2 \omega^2 a^3. \]  
The region was divided into \( n_x = 90 \) and \( n_y = n_z = 30 \) cells. The potentials were updated based on the equations in Sec. III. The initial conditions on the variables were set in accordance with (20)–(21). The hanging variables on the boundary ([\( \phi \mid_x \), \( \vec{B}_\perp \), and \( \vec{A}_\perp \)] were set by solving for their values from (20)–(21). For instance, the \( [\vec{B}_\perp] \) on the \( i = 1 \) boundary was found by differentiating (20) with respect to \( x \) and evaluating the result at \( x = 0 \).

The top row of Fig. 3 shows the evolution of the potentials over time for time steps 0.1% below and 0.1% above the CFL limit [1]. The simulation run below the CFL limit shows no sign of instability. The simulation performed above the CFL limit (1). The simulation performed above the CFL limit is unstable, as a result of the algorithm violating the principle of energy conservation. The two plots in the middle row of Fig. 3 confirm that the storage functions for scalar and vector potentials are approximately equal to their continuous counterparts ([23]).

From (13b), the storage function should be equal to the energy initially present in the region at the beginning of the simulation plus the total energy absorbed through the boundaries. This is consistent with observation for the test performed under the CFL limit in the second row of Fig. 3 where the two curves match with a maximum error comparable to machine precision: \( 1.8 \times 10^{-15} \) pJ and \( 2.1 \times 10^{-15} \) pJ for the scalar and vector potentials, respectively. Interestingly, when the CFL limit is violated (the last row of Fig. 3), the two curves deviate substantially, showing that the numerical algorithm violates energy conservation.

V. Conclusion

We showed that the update equations of the P-FDTD method ([7]) can be cast in the form of a discrete-time dynamical system. We proposed suitable discrete expressions to compute the energy stored in the scalar and vector potential present in a region, as well as the power absorbed by the region through its boundary. Through these equations, we identified the precise condition under which the update equations of this method conserve energy in discrete time domain, and discussed its relation to the CFL stability limit.

The proposed developments have multiple uses. The proposed expressions for energy and absorbed power can be used to reliably estimate these quantities, which are useful in characterizing resonators, antennas, and superconductive devices.
This study also provided insight into how electromagnetic energy is approximated and behaves in FDTD-like methods based on potentials. Finally, conditions for energy conservation can be used to devise new FDTD schemes based on potentials with guaranteed stability.

REFERENCES

[1] K. S. Yee, “Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media,” IEEE Trans. Antennas Propag., vol. 14, no. 3, pp. 302–307, 1966.

[2] S. D. Gedney, Introduction to the Finite-Difference Time-Domain (FDTD) Method for Electromagnetics. Morgan & Claypool Publishers, 2011.

[3] W. Sui, J. Yang, X. Yun, and C. Wang, “Including quantum effects in electromagnetic systems—An FDTD solution to Maxwell–Schrödinger equations,” in IEEE MTT-S Int. Microw. Symp., 2007, pp. 1979–1982.

[4] I. Ahmed, E. H. Khoo, E. Li, and R. Mittra, “A hybrid approach for solving coupled Maxwell and Schrödinger equations arising in the simulation of nano-devices,” IEEE Antennas Wireless Propag. Lett., vol. 9, pp. 914–917, 2010.

[5] S. Ohnuki, T. Takeuchi, T. Sako, Y. Ashizawa, K. Nakagawa, and M. Tanaka, “Coupled analysis of Maxwell–Schrödinger equations by using the length gauge: Harmonic model of a nanoplate subjected to a 2D electromagnetic field,” Int. J. Numer. Model. Electron. Dev. Devices, vol. 26, no. 6, pp. 533–544, 2013.

[6] T. Takeuchi, S. Ohnuki, and T. Sako, “Maxwell–Schrödinger hybrid simulation for optically controlling quantum states,” IEEE Antennas Propag., vol. 11, no. 9, pp. 40–47, 2016.

[7] C. I. Byrnes, F. Kong, K. Li, and M. Liu, “A quantum state controller based on the electromagnetic potentials,” IEEE Access, vol. 7, pp. 42006–42013, 2019.

[8] F. Jazaeri, A. Beckers, A. Tajaddini, and J.-M. Sallese, “A review on quantum computing: From qubits to front-end electronics and cryogenic MOSFET physics,” in Proc. 26th Int. Conf. Mixed Des. Integr. Circuits Syst. (MIXDES), Jun. 2019, pp. 15–25.

[9] K. S. Kunz and R. J. Luebbers, The Finite-Difference Time-Domain Method for Electromagnetics. Boca Raton, Florida: CRC Press, 1993.

[10] F. De Flaviis, M. G. Gorò, R. E. Diaz, G. Franceschetti, and N. G. Alexopoulos, “A time-domain vector potential formulation for the solution of electromagnetic problems,” IEEE Microw. Guided Wave Lett., vol. 8, no. 9, pp. 310–312, 1998.

[11] M. Okoniewski, E. Okoniewska, and M. A. Stuchly, “Three-dimensional subgridding algorithm for FDTD,” IEEE Trans. Antennas Propag., vol. 45, no. 3, pp. 422–429, 1997.

[12] L. Kulas and M. Mrozowski, “Stability of the FDTD scheme containing macromodels,” IEEE Microw. Wireless Compon. Lett., vol. 14, no. 10, pp. 484–486, 2004.

[13] I. S. Kim and W. J. R. Hoefer, “The numerical energy conservation of the TD-FD method,” J. Comput. Phys., vol. 27, no. 5, pp. 4056–4060, 1991.

[14] G. Xie, Z. Huang, J. W. You, Z. Lan, N. C. Panoiu, and W. E. I. Sha, “Universal vector–scalar potential formulation for inhomogeneous electromagnetic system and its application in semiclassical quantum electromagnetic,” IEEE Plasma Sci., vol. 49, no. 11, pp. 3459–3471, 2021.

[15] R. F. Remis, “On the stability of the finite-difference time-domain method,” J. Comput. Phys., vol. 163, no. 1, pp. 249–261, 2000.

[16] M. Mrozowski, “Stability condition for the explicit algorithms of the time domain analysis of Maxwell’s equations,” IEEE Microw. Guided Wave Lett., vol. 4, no. 8, pp. 279–281, 1994.

[17] F. Bekmambetova, X. Zhang, and P. Triverio, “A dissipation theory for three-dimensional FDTD with application to stability analysis and subgridding,” IEEE Trans. Antennas Propag., vol. 66, no. 12, pp. 7156–7170, 2018.

[18] J. C. Willems, “Dissipative dynamical systems part I: General theory,” Arch. Ration. Mech. Anal., vol. 45, no. 5, pp. 312–351, 1972.

[19] F. De Flaviis, R. Schuhmann, and T. Weiland, “A general stability analysis of FIT/FDTD applied to lossy dielectrics and lumped elements,” Int. J. Numer. Model., Electron. Dev. Devices Fields, vol. 17, no. 4, pp. 407–419, 2004.