Free-field realization of the exceptional current superalgebra $D(2, 1; \alpha)_k$

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Abstract
The free-field representations of the $D(2, 1; \alpha)$ current superalgebra and the corresponding energy–momentum tensor are constructed. The related screening currents of the first kind are also presented.

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1. Introduction

The interest in two-dimensional nonlinear sigma models with supermanifold as target space has drastically increased over the last ten years due to their applications in superstring theory [1–5], logarithmic conformal field theory (CFT) and condensed matter physics [6–13]. Among them, the sigma models associated with superalgebras $psl(n|n)$ and $osp(2n + 2|2n)$ stand out as an important class. These supergroups have a vanishing superdimension or vanishing dual Coxeter number [14]. As a result, sigma models based on the supergroups have a vanishing one-loop beta function and are thus expected to be conformal invariant even without adding the Wess–Zumino terms [3]. The sigma models related to the first series superalgebras $psl(n|n)$ have been extensively studied over the last few years [2, 4, 5, 8–11, 13]. It is well known that there exists a parameter deformation of the simplest superalgebra $osp(4|2)$ in the second series, the exceptional Lie superalgebra $D(2, 1; \alpha)$, which shares the very property of having a vanishing dual Coxeter number with $osp(4|2)$. Recent studies show that the superalgebra

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\( D(2, 1; \alpha) \) has played an important role in describing the origin of the Yangian symmetry of AdS/CFT [15] and the symmetry of string theory on \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) [16] (where the parameter \( \alpha \) is related to the relative size of the radii of the geometry [17]). Therefore, similar to the fact that the sigma model associated with \( psl(4|4) \) (or \( su(2, 2|4) \)) is related to the string theory on \( \text{AdS}_3 \times S^3 \) background [3], it might be expected that the understanding of the \( D(2, 1; \alpha) \) model would shed important light on the quantization of the string theory on \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) background. Here we shall study the Wess–Zumino–Novikov–Witten (WZNW) model associated with the exceptional superalgebra \( D(2, 1; \alpha) \).

Free-field realization [18–28] has been proven to be a powerful method in analyzing CFTs such as WZNW models. In this paper, motivated by its potential applications, we obtain the free-field realization of the current superalgebra underlying the \( D(2, 1; \alpha) \) WZNW model. To our knowledge, it might be the first result on the current superalgebra associated with exceptional superalgebra.

2. \( D(2, 1; \alpha) \) current algebra

Let us start with some basic notation of the \( D(2, 1; \alpha) \) current algebra, i.e. the \( D(2, 1; \alpha) \) affine superalgebra [14]. The exceptional Lie superalgebra \( D(2, 1; \alpha) \) with \( \alpha \neq 0, -1, \infty \) forms a one-parameter (i.e. \( \alpha \)) family of superalgebras of rank 3 and dimension 17. The bosonic (or even) part is a direct sum of three \( su(2) \) and the fermionic (or odd) part is a spinorial representation \((2, 2, 2)\) of the even part. Let \( \Pi := \{\alpha_1, \alpha_2, \alpha_3\} \) be the simple roots with \( \alpha_1 \) being fermionic and \( \alpha_2, \alpha_3 \) being bosonic. The Cartan matrix \( a_{ij} \) is given by

\[
a = \begin{pmatrix}
0 & 1 & \alpha \\
-1 & 2 & 0 \\
-1 & 0 & 2
\end{pmatrix}.
\]

The positive even roots \( \Delta^+_0 \) and the positive odd roots \( \Delta^+_1 \) are then given, respectively, by

\[
\Delta^+_0 = \{\alpha_2, \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3\}, \quad \Delta^+_1 = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\},
\]

the set of all positive roots \( \Delta^+ = \Delta^+_0 \cup \Delta^+_1 \). Associated with each positive root \( \delta \), there is a raising operator \( E_\delta \), a lowering operator \( F_\delta \equiv E_{-\delta} \) and a Cartan generator \( H_\delta \). These operators have definite \( \mathbb{Z}_2 \)-gradings:

\[
[H_\delta] = 0, \quad [E_\delta] = [F_\delta] = \begin{cases}
0, & \delta \in \Delta^+_0, \\
1, & \delta \in \Delta^+_1.
\end{cases}
\]

For any two homogeneous elements (i.e. elements with definite \( \mathbb{Z}_2 \)-gradings) \( a, b \in D(2, 1; \alpha) \), the Lie bracket is defined by

\[
[a, b] = ab - (-1)^{[a][b]}ba,
\]

this (anti)commutator extends to inhomogeneous elements through linearity. Then the commutation relations of \( D(2, 1; \alpha) \) are [14]

\[
[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij}H_{\alpha_i}, \quad [H_{\alpha_i}, H_{\alpha_j}] = 0, \quad i, j = 1, 2, 3,
\]

\[
[H_{\alpha_i}, E_{\alpha_j}] = \alpha_{ij}E_{\alpha_i}, \quad [H_{\alpha_i}, F_{\alpha_j}] = -\alpha_{ij}F_{\alpha_i}, \quad i, j = 1, 2, 3,
\]

\[
[E_{\alpha_i}, E_{\alpha_j}] = -E_{\alpha_{i+j}}, \quad [E_{\alpha_i}, F_{\alpha_j}] = -E_{\alpha_{i+j}}, \quad [F_{\alpha_i}, F_{\alpha_j}] = -E_{\alpha_{i+j}},
\]

\[
E_{\alpha_{12}}, F_{\alpha_{12}} = 0,
\]

\[
[H_{\alpha_{12}}] = 0, \quad [E_{\alpha_{12}}, F_{\alpha_{12}}] = 0.
\]

\[
\text{Here we only give the relevant relations for our purpose; the others can be found in [14].}
\]
One may introduce a nondegenerate and invariant supersymmetric metric or bilinear form, denoted by \((X, Y)\) for any \(X, Y \in D(2, 1; \alpha)\), as follows:

\[
(E_{a_1}, F_{a_1}) = -(F_{a_1}, E_{a_1}) = -1, \quad (E_{a_1+a_2}, F_{a_1+a_2}) = -(F_{a_1+a_2}, E_{a_1+a_2}) = 1, \quad (E_{a_1+a_2+a_3}, F_{a_1+a_2+a_3}) = -(F_{a_1+a_2+a_3}, E_{a_1+a_2+a_3}) = -1,
\]

\[
(H_{a_1}, F_{a_1}) = (F_{a_1}, E_{a_1}) = \frac{1}{\alpha}, \quad (H_{a_2}, F_{a_2}) = (F_{a_2}, E_{a_2}) = 1, \quad (H_{a_3}, F_{a_3}) = (F_{a_3}, E_{a_3}) = 0, \quad (H_{a_2}, H_{a_2}) = 2, \quad (H_{a_3}, H_{a_3}) = 0, \quad (H_{a_3}, H_{a_3}) = \frac{2}{\alpha},
\]

All the other inner products are zero. With the help of the above inner product, we can construct the corresponding second-order Casimir element

\[
C_2 = (E_{a_1}, F_{a_1} - F_{a_1}, E_{a_1}) - (E_{a_1+a_2}, F_{a_1+a_2} - F_{a_1+a_2}E_{a_1+a_2}) = -(E_{a_1+a_3}, F_{a_1+a_3} - F_{a_1+a_3}E_{a_1+a_3}) + (E_{a_1+a_2+a_3}, F_{a_1+a_2+a_3} - F_{a_1+a_2+a_3}E_{a_1+a_2+a_3}) + (E_{a_1+a_2+a_3}, F_{a_1+a_2+a_3} + E_{a_1+a_2+a_3})
\]

\[
+ (E_{a_1+a_2+a_3}, F_{a_1+a_2+a_3} + E_{a_1+a_2+a_3}) + (E_{a_1+a_2+a_3}, F_{a_1+a_2+a_3} + E_{a_1+a_2+a_3}) + (E_{a_1+a_2+a_3}, F_{a_1+a_2+a_3} + E_{a_1+a_2+a_3}) + (E_{a_1+a_2+a_3}, F_{a_1+a_2+a_3} + E_{a_1+a_2+a_3})
\]

\[
+ \frac{1}{1 + \alpha} \left[ -2H_{a_1}^2 + 2H_{a_1}H_{a_2} + 2\alpha H_{a_1}H_{a_1} + \frac{\alpha}{2} H_{a_1}^2 - \alpha H_{a_1}H_{a_1} + \frac{\alpha}{2} H_{a_1}^2 \right].
\]

Let \(X(z)\) be the current associated with the generator \(X \in D(2, 1; \alpha)\). For an example, denote the currents corresponding to \(E_{i\delta}, H_{i\delta}, F_{i\delta} (\forall \delta \in \Delta^\perp)\) by \(E_{i\delta}(z), H_{i\delta}(z), F_{i\delta}(z)\), respectively. The \(D(2, 1; \alpha)\) current algebra is generated by a set of currents \(\{X(z)|X \in D(2, 1; \alpha)\}\), which obey the following OPEs \([18]\):

\[
X(z)Y(w) = k \frac{(X, Y)}{(z - w)^2} + \frac{[X, Y](w)}{(z - w)},
\]

where \(k\) is the level (or central charge) of the current algebra.

### 3. Free-field realization

To obtain a free-field realization of the \(D(2, 1; \alpha)\) currents, one needs \([19–21, 23–28]\) firstly to construct the differential operator representation of the corresponding finite-dimensional superalgebra \(D(2, 1; \alpha)\). Let \(\langle \Lambda \rangle\) be the lowest weight vector in a representation of \(D(2, 1; \alpha)\), satisfying the following conditions:

\[
\langle \Lambda | F_{a_1} = 0,
\]

\[
\langle \Lambda | H_{a_1} = \lambda_i \langle \Lambda \rangle.
\]
An arbitrary vector in this representation space can be parametrized by bosonic variables $x_\alpha$ and fermionic variables $\theta_\alpha$.

$$\langle \Lambda, x, \theta \rangle = \langle \Lambda | G_+ (x, \theta) \rangle.$$ \hspace{1cm} (3.3)

However, it might be very involved, if not impossible, to apply the general procedure proposed in \cite{19,20,23} to derive explicit expressions of the differential operator representation for higher rank algebras such as $D(2,1;\alpha)$. It was shown \cite{28} that a particular ordering of all positive roots of a higher rank (super) algebra could drastically simplify the computation of its explicit differential operator expression. Here we find that for the $D(2,1;\alpha)$ case the very ordering of the positive roots is given by

$$\alpha_3, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1.$$ \hspace{1cm} (3.4)

Such an ordering allows us to construct the explicit expression of the differential operator representation of $D(2,1;\alpha)$ by the method developed in \cite{28}. Based on the above ordering of the positive roots, we can construct the corresponding $G_+ (x, \theta)$ as follows:

$$G_+(x, \theta) = G_{2\alpha_1} G_{\alpha_2} G_{\alpha_1 + \alpha_2 + \alpha_3} G_{\alpha_1 + \alpha_3} G_{2\alpha_1 + \alpha_2 + \alpha_3} G_{\alpha_1 + \alpha_2} G_{\alpha_1},$$ \hspace{1cm} (3.5)

the associated $G_i$ are given by

$$G_i = \begin{cases} e^{\theta_\alpha E_\alpha}, & \text{if } [E_\alpha] = 0, \\ e^{\theta_\alpha E_\alpha}, & \text{if } [E_\alpha] = 1. \end{cases}$$ \hspace{1cm} (3.6)

Here $x_\alpha$ is the bosonic variable, while $\theta_\alpha$ is the fermionic one. Define a differential operator realization $\rho^{(d)}$ of the generators of $D(2,1;\alpha)$ by the following relation:

$$\rho^{(d)}(g)\langle \Lambda, x, \theta \rangle \equiv \langle \Lambda, x, \theta \rangle | g = \epsilon g \in D(2,1;\alpha).$$ \hspace{1cm} (3.7)

Here $\rho^{(d)}(g)$ is a differential operator of the bosonic and fermionic coordinates $[x, \theta]$ associated with the generator $g$, which can be obtained from (3.7). The defining relation also ensures that the differential operator realization is actually a representation of $D(2,1;\alpha)$. By using (2.6) and (3.7) and the Baker–Campbell–Hausdorff (BCH) formula, after some manipulations, we have obtained \cite{32} the following differential operator representations of generators associated with the simple roots of $D(2,1;\alpha)$:

$$\rho^{(d)}(E_{\alpha_1}) = \delta_{\theta_\lambda z},$$ \hspace{1cm} (3.8)

$$\rho^{(d)}(E_{\alpha_2}) = -\theta_{\alpha_2} \delta_{\theta_{\alpha_1}} + \theta_{\alpha_1 + \alpha_2} \delta_{\theta_{\alpha_1 + \alpha_2 + \alpha_3}} + \delta_{\theta_{\alpha_3}},$$ \hspace{1cm} (3.9)

$$\rho^{(d)}(E_{\alpha_3}) = -\theta_{\alpha_3} \delta_{\theta_{\alpha_1}} + \theta_{\alpha_1 + \alpha_2} \delta_{\theta_{\alpha_1 + \alpha_2 + \alpha_3}} - \theta_{\alpha_1 + \alpha_3} \delta_{\theta_{\alpha_1 + \alpha_2 + \alpha_3}} + \delta_{\theta_{\alpha_2}},$$ \hspace{1cm} (3.10)

$$\rho^{(d)}(H_{\alpha_1}) = -\theta_{\alpha_1 + \alpha_2} \delta_{\theta_{\alpha_1}} + (1 + \alpha) \theta_{\alpha_1 + \alpha_2} \delta_{\theta_{\alpha_1 + \alpha_2 + \alpha_3}} - \theta_{\alpha_1 + \alpha_3} \delta_{\theta_{\alpha_1 + \alpha_2 + \alpha_3}} + \delta_{\theta_{\alpha_2}},$$ \hspace{1cm} (3.11)

$$\rho^{(d)}(H_{\alpha_2}) = \theta_{\alpha_2} \delta_{\theta_{\alpha_1}} - \theta_{\alpha_1 + \alpha_2} \delta_{\theta_{\alpha_1}} + \theta_{\alpha_1 + \alpha_3} \delta_{\theta_{\alpha_1 + \alpha_3}} - \theta_{\alpha_1 + \alpha_2 + \alpha_3} \delta_{\theta_{\alpha_1 + \alpha_2 + \alpha_3}} - 2\lambda_{\alpha_2} \delta_{x_{\alpha_2}},$$ \hspace{1cm} (3.12)

$$\rho^{(d)}(H_{\alpha_3}) = \theta_{\alpha_3} \delta_{\theta_{\alpha_1}} - \theta_{\alpha_1 + \alpha_3} \delta_{\theta_{\alpha_1}} + \theta_{\alpha_1 + \alpha_2 + \alpha_3} \delta_{\theta_{\alpha_1 + \alpha_2 + \alpha_3}} - 2\lambda_{\alpha_3} \delta_{x_{\alpha_3}},$$ \hspace{1cm} (3.13)
\[
\rho^{(d)}(F_{\alpha}) = -\partial_{\alpha+i} \partial_{\alpha+i} \partial_{\alpha+i+2+\alpha} - \partial_{\alpha+i} \partial_{\alpha+i} + x_{2\alpha+i+2+\alpha} \partial_{\alpha+i+2+\alpha} \\
- \alpha \partial_{\alpha+i} \partial_{\alpha+i} - \partial_{\alpha+i} \partial_{\alpha+i} - (1 + \alpha) \partial_{\alpha+i} x_{\alpha+i+2+\alpha} \partial_{\alpha+i+2+\alpha} \\
- \partial_{\alpha+i} \partial_{\alpha+i} \partial_{\alpha+i+2+\alpha} - (1 + \alpha) \partial_{\alpha+i} \partial_{\alpha+i+2+\alpha} \partial_{\alpha+i+2+\alpha} \\
- \partial_{\alpha+i} \partial_{\alpha+i} \partial_{\alpha+i+2+\alpha} - \partial_{\alpha+i} \lambda_{\alpha+i}.
\]
(3.14)

\[
\rho^{(d)}(F_{\alpha}) = -\theta_{\alpha+i} - \theta_{\alpha+i+2+\alpha} \partial_{\alpha+i+2+\alpha} - \partial_{\alpha+i} \lambda_{\alpha+i}.
\]
(3.15)

\[
\rho^{(d)}(F_{\alpha}) = -\theta_{\alpha+i} + (1 + \alpha) \partial_{\alpha+i+2+\alpha} \partial_{\alpha+i+2+\alpha} \\
- \theta_{\alpha+i} \partial_{\alpha+i} + \partial_{\alpha+i+2+\alpha} \partial_{\alpha+i+2+\alpha}.
\]
(3.16)

With the help of the differential representation (3.8)–(3.16) we can obtain the free-field realization of the \(D(2, 1; \alpha)\) current algebra in terms of three bosonic \(\beta-\gamma\) pairs \(((\beta_{\alpha+i}(z), \gamma_{\alpha+i}(z)), (\beta_{\alpha+i}(z), \gamma_{\alpha+i}(z)))\) and \((\beta_{2\alpha+i+2+\alpha}(z), \gamma_{2\alpha+i+2+\alpha}(z))\), four fermionic \(b-c\) pairs \(((\Psi_{\alpha+i}(z), \Psi_{\alpha+i}(z)), (\Psi_{\alpha+i}(z), \Psi_{\alpha+i}(z)), (\Psi_{\alpha+i}(z), \Psi_{\alpha+i}(z))\) and \((\Psi_{\alpha+i+2+\alpha}(z), \Psi_{\alpha+i+2+\alpha}(z))\) and three free scalar fields \((\phi_i(z), \text{for } i = 1, 2, 3)\). These free fields obey the following OPEs:

\[
\beta_j(z)\gamma_j(w) = -\gamma_j(z)\beta_j(w) = \frac{\delta_{ij}}{z - w}, \quad i, j \in \Delta^+_0,
\]
(3.17)

\[
\Psi_i(z)\Psi_j^j(w) = \Psi_j(z)\Psi_i(w) = \frac{\delta_{ij}}{z - w}, \quad i, j \in \Delta^+_1,
\]
(3.18)

\[
\phi_1(z)\phi_i(w) = -\ln(z - w),
\]
(3.19)

\[
\phi_i(z)\phi_j(w) = \delta_{ij}\ln(z - w), \quad i, j = 2, 3,
\]
(3.20)

and the other OPEs are trivial. Here we present the results for the currents associated with the simple roots,

\[
E_{\alpha+i}(z) = \Psi_{\alpha+i}(z),
\]
(3.21)

\[
E_{\alpha+i}(z) = -\Psi_{\alpha+i}^+(z)\Psi_{\alpha+i}(z) - \Psi_{\alpha+i}^+(z)\Psi_{\alpha+i}(z) + \beta_{\alpha+i}(z),
\]
(3.22)

\[
H_{\alpha+i}(z) = -\Psi_{\alpha+i}^+(z)\Psi_{\alpha+i}(z) - (1 + \alpha)\gamma_{\alpha+i}(z)\beta_{2\alpha+i+2+\alpha}(z) \\
- \Psi_{\alpha+i}^+(z)\Psi_{\alpha+i}(z) + \beta_{\alpha+i}(z),
\]
(3.23)

\[
H_{\alpha+i}(z) = \Psi_{\alpha+i}^+(z)\Psi_{\alpha+i}(z) - \Psi_{\alpha+i}^+(z)\Psi_{\alpha+i}(z) + \Psi_{\alpha+i}^+(z)\Psi_{\alpha+i}(z).
\]
It is well known that the energy–momentum tensor associated with a current algebra can be
\[ T_{\alpha}^{\beta}(z) = \Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \Psi_{\alpha+1+2}(z)\beta_{\alpha}(z) + \gamma_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \alpha\Psi_{\alpha+1+2}(z)\beta_{\alpha}(z) - \Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) + \alpha\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \Psi_{\alpha+1+2}(z)(1 + \alpha)\gamma_{\alpha+1+2}(z)\beta_{\alpha+1+2}(z) - \Psi_{\alpha+1+2}(z)(1 + \alpha)\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) + \alpha\gamma_{\alpha+1+2}(z)\beta_{\alpha+1+2}(z) + \sqrt{k}\Psi_{\alpha+1+2}(z) \left\{ \frac{1 + \alpha}{2} \partial\phi_{1}(z) + \sqrt{\frac{\alpha}{2}} \partial\phi_{2}(z) + \sqrt{\frac{\alpha}{2}} \partial\phi_{3}(z) \right\} - k\partial\Psi_{\alpha+1+2}(z), \] (3.24)

\[ T_{\alpha}^{\beta}(z) = -\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \gamma_{\alpha}^{2}(z)\beta_{\alpha}(z) + \sqrt{2k}\gamma_{\alpha+1+2}(z)\partial\phi_{2}(z) + (k - 2)\partial\gamma_{\alpha}(z), \] (3.25)

\[ T_{\alpha}^{\beta}(z) = (1 + \alpha)\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \Psi_{\alpha+1+2}(z)\Psi_{\alpha+1+2}(z) - \gamma_{\alpha}^{2}(z)\beta_{\alpha}(z) + \sqrt{2k}\gamma_{\alpha}(z)\partial\phi_{3}(z) + \left( \frac{k}{\alpha} - 2 \right)\partial\gamma_{\alpha}(z). \] (3.26)

Here normal ordering of the free-field expressions is implied. The free-field realization for currents associated with the non-simple roots can be obtained from the OPEs of the simple ones. It is straightforward to check that (3.21)–(3.26) satisfy the OPEs of the $D(2, 1; \alpha)$ current algebra given in the last section.

4. Energy–momentum tensor

It is well known that the energy–momentum tensor associated with a current algebra can be obtained by means of the Sugawara construction. For the present case, the Sugawara tensor corresponding to the quadratic Casimir $C_{2}$ (2.9) is

\[ T(z) = \frac{1}{2k} \left\{ (E_{\alpha+1+2}(z) - E_{\alpha+1+2}(z)) - (E_{\alpha+1+2}(z) - E_{\alpha+1+2}(z)) - (F_{\alpha+1+2}(z) - F_{\alpha+1+2}(z)) + (F_{\alpha+1+2}(z) - F_{\alpha+1+2}(z)) - (1 + \alpha)(F_{\alpha+1+2}(z) - F_{\alpha+1+2}(z)) + (E_{\alpha+1+2}(z) - F_{\alpha+1+2}(z)) \right\} + \frac{1}{1 + \alpha} \left\{ 2H_{\alpha+1+2}(z)H_{\alpha+1+2}(z) - 2H_{\alpha+1+2}(z)H_{\alpha+1+2}(z) - 2H_{\alpha+1+2}(z)H_{\alpha+1+2}(z) + \frac{\alpha}{2} H_{\alpha+1+2}(z)H_{\alpha+1+2}(z) \right\}. \] (4.1)

Here normal ordering of the free-field expressions is implied. The above expression can be written in terms of the $\beta\gamma$ and $b - c$ pairs, and the scalar fields. After tedious calculation, we obtain

\[ T(z) = -\frac{1}{2} \partial\phi_{1}(z)\partial\phi_{1}(z) - \sqrt{\frac{1 + \alpha}{2k}} \partial^{2}\phi_{1}(z) + \frac{1}{2} \partial\phi_{2}(z)\partial\phi_{2}(z) - \sqrt{\frac{1 + \alpha}{2k}} \partial^{2}\phi_{2}(z) + \frac{1}{2} \partial\phi_{3}(z)\partial\phi_{3}(z) - \sqrt{\frac{1 + \alpha}{2k}} \partial^{2}\phi_{3}(z). \]
where the normal ordering of the free-field expressions is implicit. We find that the Sugawara-type stress tensor \( T(z) \) satisfies the Virasoro algebra

\[
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)},
\]

with the central charge \( c = 1 \). Moreover, it is easy to check that

\[
T(z)X(w) = \frac{X(w)}{z-w} + \frac{\partial X(w)}{z-w} = \partial_w \left\{ \frac{X(w)}{z-w} \right\}, \quad \forall X \in D(2, 1; \alpha).
\]

Namely, all the currents are primary fields of conformal dimension 1.

5. Screening currents

An important object in applying the free-field realization to the computation of correlation functions of the associated CFT is screening currents. A screening current is a primary field with conformal dimension 1 and has the property that the singular part of the OPE with the affine currents is a total derivative. These properties ensure that integrated screening currents (screening charges) may be inserted into correlators while the conformal or affine Ward identities remain intact. This in turn makes them very useful in the computation of the correlation functions [29, 30]. For the present case, we find the following three screening currents associated with the simple roots:

\[
S_{\alpha_1}(z) = \{ \gamma_{\alpha_1}(z) \gamma_{\alpha_2}(z) \psi_{\alpha_1+\alpha_2}(z) + (1 + \alpha) \gamma_{\alpha_2}(z) \psi_{\alpha_1+\alpha_2}^+(z) \beta_{\alpha_1+\alpha_2} \psi_{\alpha_1+\alpha_2}(z) + \gamma_{\alpha_2}(z) \psi_{\alpha_1+\alpha_2}(z) - \gamma_{\alpha_1}(z) \psi_{\alpha_1+\alpha_2}(z) - \gamma_{\alpha_1}(z) \psi_{\alpha_1+\alpha_2}^+(z) - \gamma_{\alpha_2}(z) \psi_{\alpha_1+\alpha_2}(z) + \psi_{\alpha_1}(z) \} e^{\sqrt{\alpha_1} \phi_1(z) + \sqrt{\alpha_2} \phi_2(z) + \sqrt{\alpha_3} \phi_3(z)},
\]

where the normal ordering of the free-field expressions is implicit in the above equations. The OPEs of the screening currents with the energy–momentum tensor \( T(z) \) and the \( D(2, 1; \alpha) \) currents \((3.21)–(3.26)\) are

\[
T(z)S_{\alpha_j}(w) = \frac{S_{\alpha_j}(w)}{(z-w)^2} + \frac{\partial S_{\alpha_j}(w)}{(z-w)} = \partial_w \left\{ \frac{S_{\alpha_j}(w)}{(z-w)} \right\}, \quad j = 1, 2, 3,
\]

\[
E_{\alpha_i}(z)S_{\alpha_j}(w) = 0, \quad H_{\alpha_i}(z)S_{\alpha_j}(w) = 0, \quad i, j = 1, 2, 3,
\]

\[
F_{\alpha_i}(z)S_{\alpha_j}(w) = \delta_{ij} \partial_w \left\{ \frac{k e^{\sqrt{\alpha_1} \phi_1(w) + \sqrt{\alpha_2} \phi_2(w) + \sqrt{\alpha_3} \phi_3(w)}}{z-w} \right\}, \quad j = 1, 2, 3,
\]
\[ F_{\alpha}(z)S_{\alpha j}(w) = \delta_{2j}\partial_w \left\{ \frac{k e^{-\sqrt{2} \phi_2(w)}}{z - w} \right\}, \quad j = 1, 2, 3, \quad (5.7) \]

\[ F_{\alpha}(z)S_{\alpha j}(w) = \delta_{3j}\partial_w \left\{ \frac{k e^{-\sqrt{3} \phi_3(w)}}{z - w} \right\}, \quad j = 1, 2, 3. \quad (5.8) \]

The screening currents obtained here correspond to the screening currents of the first kind [31]. The screening current \( S_{\alpha j}(z) \) is fermionic and the others are bosonic.

6. Discussions

We have studied the exceptional current algebra \( \hat{D}(2; 1; \alpha) \) at the general level \( k \). The Wakimoto free-field realization of the currents (3.21)–(3.26) and the energy–momentum tensor (4.2) are constructed. We have also found three screening currents, (5.1)–(5.3), of the first kind. With a special choice of \( \alpha = 1 \), the results recover those of \( osp(4|2) \) current algebra [28].

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