Nonlocal brackets and integrable string models

V.D. Gershun

Institute of Theoretical Physics, NSC Kharkov Institute of Physics and Technology, Kharkov, UA

Abstract

The closed string model in the background gravity field is considered as a bi-
Hamiltonian system in assumption that string model is the integrable model for partic-
ular kind of the background fields. The dual nonlocal Poisson brackets (PB), depending
of the background fields and of their derivatives, are obtained. The integrability condi-
tion is formulated as the compatibility of the bi-Hamiltonity condition and the Jacobi
identity of the dual Poisson bracket. It is shown that the dual brackets and dual Hamil-
tonians can be obtained from the canonical PB and from the initial Hamiltonian by
imposing the second kind constraints on the initial dynamical system, on the closed
string model in the constant background fields, as example. Two types of the nonlocal
brackets are introduced. Constant curvature and time-dependent metrics are consid-
ered, as examples. It is shown, that the Jacobi identities for the nonlocal brackets
have particular solution for the space-time coordinates, as matrix representation of the
simple Lie group.

1. Introduction.
The bi-Hamiltonian approach [1, 2, 3] to the integrable systems was initiated by Magri [4]
for the investigation of the integrability of the KdV equation.

Definition 1. A finite dimensional dynamical system with $2N$ degrees of freedom $x^a, a = 1, \ldots, 2N$ is integrable, if it is described by the set of the $n$ integrals of motion $F_1, \ldots, F_n$ in involution under some Poisson bracket (PB)

$$\{F_i, F_k\}_PB = 0.$$ 

The dynamical system is completely solvable, if $n = N$. Any of the integral of motion (or any linear combination of them) can be considered as the Hamiltonian $H_k = F_k$.

Definition 2. The bi-Hamiltonity condition has following form:

$$\dot{x}^a = \frac{dx^a}{dt} = \{x^a, H_1\}_1 = \cdots = \{x^a, H_N\}_N.$$  \hspace{1cm} (1)

The hierarchy of new PB is arose in this connection:

$$\{,\}_1, \{,\}_2, \cdots, \{,\}_N.$$ 

The hierarchy of new dynamical systems arises under the new time coordinates $t_k$.

$$\frac{dx^a}{dt_{n+k}} = \{x^a, H_n\}_{k+1} = \{x^a, H_k\}_{n+1}.$$  \hspace{1cm} (2)

The new equations of motion describe the new dynamical systems, which are dual to the original system, with the dual set of the integrals of motion. The dual set of the integrals of motion can be obtained from original one by the mirror transformations and by contraction of the integrals of motion algebra. The contraction of the integral of motion algebra means

\hspace{1cm} 1 e-mail:gershun@kipt.kharkov.ua
The second kind constraints take part in deformation of the coordinate functions. Hamiltonian hydrodynamical models. The PB of the hydrodynamical type for the phase Novikov have considered the local dual PB of the similar type in application to the closed string model. They led to some restrictions on the background fields. Dubrovin and Jacobi identities for the dual brackets were considered as the integrability condition for a background gravity field and antisymmetric B-field. The bi-Hamiltonicity condition and the approach to the investigation of the integrability of the closed string model in the arbitrary imposition of the second kind constraints. We have applied bi-Hamiltonian condition leads to the dual PB that are nonlinear and nonlocal brackets as a rule. As rule, such deformation leads to nonlinear and nonlocal brackets. The bi-Hamiltonity algebra. The second kind constraints is defined by the condition

\[ \{ f_i, f_k \} = C_{ik} f_l \approx 0. \]

The structure functions \( C_{ik} \) may be functions of the phase space coordinates in general case. The second kind constraints \( f_k(x^a) \approx 0 \) are the representations of the first kind constraints algebra. The second kind constraints is defined by the condition

\[ \{ f_i, f_k \} = C_{ik} \neq 0. \]

The reversible matrix \( C_{ik} \) is not constraint and also it is a function of phase space coordinates. The second kind constraints take part in deformation of the \( \{ , \}_PB \) to the Dirac bracket \( \{ , \}_D \). As rule, such deformation leads to nonlinear and to nonlocal brackets. The bi-Hamiltonity condition leads to the dual PB that are nonlinear and nonlocal brackets as a rule. We suppose, that the dual brackets can be obtained from the initial canonical bracket under the imposition of the second kind constraints. We have applied bi-Hamiltonian approach to the investigation of the integrability of the closed string model in the arbitrary background gravity field and antisymmetric B-field. The bi-Hamiltonity condition and the Jacobi identities for the dual brackets were considered as the integrability condition for a closed string model. They led to some restrictions on the background fields. Dubrovin and Novikov have considered the local dual PB of the similar type in the application to the Hamiltonian hydrodynamical models. The PB of the hydrodynamical type for the phase coordinate functions \( u^i(x, t) \) is defined by the formula

\[ \{ u^i(x), u^k(y) \} = g^{ik}(x) \frac{\partial}{\partial x} \delta(x - y) - g^{ij} \Gamma^k_{jl}(u(x)) u^l_x \delta(x - y). \]

There \( g_{ik}(u), \Gamma^k_{jl}(u) \) are the arbitrary functions of the phase space coordinates and \( u_x = \partial_x u \). The Jacobi identity is satisfied if \( g_{ik} \) is the Riemann metric without torsion, the curvature tensor is equal to zero. The metric tensor is constant, locally. Mokhov and Ferapontov introduced the nonlocal PB. The Ferapontov nonlocal PB is:

\[ \{ u^i(x), u^k(y) \} = g^{ik}(u) \frac{\partial}{\partial x} \delta(x - y) - g^{ij} \Gamma^k_{jl} u^l_x \delta(x - y) + \omega^{(s)l}_{k} (u(x)) u^l_x \nu(x - y) \omega^{(s)k}_{l} (u(y)) u^d_y, \]

where \( \nu(x - y) = sgn(x - y) = \left( \frac{d}{dx} \right)^{-1} \delta(x - y) \). This PB was used for description of the Hamiltonian system of the hydrodynamical type. There are systems with functionals of the hydrodynamical type. The density of this functionals does not depend on the derivatives \( u^k_x, u^k_{xx}, ... \) and Hamiltonian is also functional of the hydrodynamical type. On the contrary to these models, the functionals of the closed string model is depended of the derivatives of the string coordinates. The plan of the paper is the following. In the second section we considered closed string model in the arbitrary background gravity field. We suppose that this model is an integrable model for some configurations of the background fields. The bi-Hamiltonity condition and the Jacobi identities for the dual PB must result in to the integrability condition, which restrict the possible configurations of the background fields.
As examples we considered constant curvature space and time-dependent metric space. In the third section we considered closed string model in the constant background gravity field.

2. Closed string in the background fields.

The string model in the background gravity field is described by the system of the equations:

\[ \ddot{x}^a - \xi^a + \Gamma^a_{bc}(x)(\dot{x}^b \dot{x}^c - \dot{x}^b x^c) = 0, \quad g_{ab}(x)(\dot{x}^a \dot{x}^b + \dot{x}^a x^b) = 0, \quad g_{ab}(x)\dot{x}^a \dot{x}^b = 0, \]

where \( \dot{x}^a = \frac{dx^a}{d\sigma} \), \( x^a = \frac{dx^a}{d\sigma} \). We will consider the Hamiltonian formalism. The closed string in the background gravity field is described by first kind constraints in the Hamiltonian formalism:

\[ h_1 = \frac{1}{2} g^{ab}(x)p_a p_b + \frac{1}{2} g_{ab}(x)x^a x^b \approx 0, \quad h_2 = p_a x^a \approx 0, \] (3)

where \( a, b = 0, 1, ... D - 1 \), \( x^a(\tau, \sigma), p_a(\tau, \sigma) \) are the periodical functions on \( \sigma \) with the period on \( \pi \). The original PB are the symplectic PB

\[ \{x^a(\sigma), p_b(\sigma')\}_1 = \delta^a_b \delta(\sigma - \sigma'), \{x^a(\sigma), x^b(\sigma')\}_1 = \{p_a(\sigma), p_b(\sigma')\}_1 = 0. \]

The Hamiltonian equations of motion of the closed string, in the arbitrary background gravity field under the Hamiltonian \( H_1 = \int_0^\pi h_1 d\sigma \) and PB \( \{,\}_1 \), are

\[ \dot{x}^a = g^{ab}p_b, \quad \dot{p}_a = -\frac{1}{2} \frac{\partial g^{bc}}{\partial x^a} p_b p_c - \frac{1}{2} \frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b}. \]

The dual PB are obtained from the bi-Hamiltonity condition

\[ \dot{x}^a = \left\{ x^a, \int_0^\pi h_1(\sigma') d\sigma' \right\}_1 = \left\{ x^a, \int_0^\pi h_2(\sigma') d\sigma' \right\}_2, \]

\[ \dot{p}_a = \{ p_a, \int_0^\pi h_1(\sigma') d\sigma' \}_1 = \{ p_a, \int_0^\pi h_2(\sigma') d\sigma' \}_2. \] (4)

They have the following form:

**Proposition 1.**

\[ \{A(\sigma), B(\sigma')\}_2 = \frac{\partial A}{\partial x^a} \frac{\partial B}{\partial x^b} [\omega^{ab}(\sigma) + \omega^{ab}(\sigma')][\nu(\sigma') - \sigma] + [\Phi^{ab}(\sigma) + \Phi^{ab}(\sigma')] \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma) \]

\[ + [\Omega^{ab}(\sigma) + \Omega^{ab}(\sigma')] \delta(\sigma - \sigma') + \frac{\partial A}{\partial p_a} \frac{\partial B}{\partial p_b} [\omega_{ab}(\sigma) + \omega_{ab}(\sigma')] \nu(\sigma' - \sigma) + \]

\[ + [\Phi_{ab}(\sigma) + \Phi_{ab}(\sigma')] \frac{\partial}{\partial \sigma'} \delta(\sigma - \sigma) + [\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')] \delta(\sigma' - \sigma) + \]

\[ + [\Omega_{ab}(\sigma) + \Phi_{ab}(\sigma')] \frac{\partial}{\partial \sigma'} \delta(\sigma - \sigma) + [\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')] \delta(\sigma' - \sigma) + \]

\[ + [\Omega_{ab}(\sigma) + \Phi_{ab}(\sigma')] \frac{\partial}{\partial \sigma'} \delta(\sigma - \sigma) + [\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')] \delta(\sigma' - \sigma) \]

The arbitrary functions \( A, B, \omega, \Phi, \Omega \) are the functions of the \( x^a(\sigma), p_a(\sigma) \). The functions \( \omega^{ab}, \omega_{ab}, \Phi^{ab}, \Phi_{ab} \) are the symmetric functions on \( a, b \) and \( \Omega^{ab}, \Omega_{ab} \) are the antisymmetric functions to satisfy the condition \( \{A, B\}_2 = -\{B, A\}_2 \). The equations of motion under the Hamiltonian \( H_2 = \int_0^\pi h_2(\sigma') d\sigma' \) and PB \( \{,\}_2 \) are

\[ \dot{x}^a = -\omega^a_{\ b} x^b + 2\omega^{ab} p_b + 2\Phi^{ab} p_b - 2\Phi^a_{\ b} x^b + 2\Omega^a_{\ b} x^b - 2\Omega^{ab} p_b + \]

\[ + [\Omega^{ab}(\sigma) + \Phi_{ab}(\sigma')] \frac{\partial}{\partial \sigma'} \delta(\sigma - \sigma) + [\Omega_{ab}(\sigma) + \Omega_{ab}(\sigma')] \delta(\sigma' - \sigma) \]
The bi-Hamiltonity condition is led to the two constraints

\[ -\omega^a_b x^b + 2\omega^{ab} p_b - \Phi^{a} b x^b - 2\Omega^{a}_b x^b - 2\Omega^a_b p_b + \]
\[ + \int_0^\pi d\sigma'[\omega^a_b x^b + d\frac{\omega^{ab}}{d\sigma} p_b] \nu(\sigma' - \sigma) - \frac{d\Phi^a_b}{d\sigma} p_b + \frac{d\Phi^a_b}{d\sigma} p_b. \]

The bi-Hamiltonity condition is led to the two constraints

\[ -\omega^a_b x^b + 2\omega^{ab} p_b - \Phi^{a} b x^b - 2\Omega^{a}_b x^b - 2\Omega^a_b p_b + \]
\[ + \int_0^\pi d\sigma'[\omega^a_b x^b + d\frac{\omega^{ab}}{d\sigma} p_b] \nu(\sigma' - \sigma) - \frac{d\Phi^a_b}{d\sigma} p_b + \frac{d\Phi^a_b}{d\sigma} p_b = g^{ab} p_b, \]

In really, there is the list of the constraints depending on the possible choice of the unknown functions \( \omega, \Omega, \Phi \). In the general case, there are both the first kind constraints and the second kind constraints. Also it is possible to solve the constraints equations as the equations for the definition of the functions \( \omega, \Phi, \Omega \). We considered the latter possibility and we obtained the following consistent solution of the bi-Hamiltonity condition:

\[ \Phi^{ab} = 0, \ \Omega^{ab} = 0, \ \Phi^{a}_b = 0, \ \Omega^{a}_b = 0, \ \frac{\partial \omega^{ab}}{\partial x^c} x^c + 2\omega^{ab} = g^{ab}, \]
\[ \omega^a_b = \frac{1}{2} \frac{\partial^2 \omega}{\partial x^a \partial x^b} p_c p_d, \ \omega^a_b = -\frac{\partial \omega}{\partial x^b} p_c, \]
\[ \Phi_{ab} = \frac{1}{2} g_{ab}, \ \Omega_{ab} = \frac{1}{2} \left( \frac{\partial \Phi}{\partial x^a} - \frac{\partial \Phi}{\partial x^b} \right) x^c, \ \frac{\partial \omega}{\partial x^b} = 0. \]

**Remark 1.** In distinct from the PB of the hydrodynamical type, we need to introduce the separate PB for the coordinates of the Minkowski space and for the momenta because, the gravity field is not depend of the momenta. Although, this difference is vanished under the such constraint as \( f(x^a, p_a) \approx 0 \).

Consequently, the dual PB for the phase space coordinates are

\[ \{x^a(\sigma), x^b(\sigma')\}_2 = [\omega^a_b(\sigma) + \omega^a_b(\sigma')] \nu(\sigma' - \sigma), \]
\[ \{p_a(\sigma), p_b(\sigma')\}_2 = \left[ \frac{\partial^2 \omega}{\partial x^a \partial x^b} p_c p_d + \frac{\partial^2 \omega}{\partial x^a \partial x^b} p_c p_d \right] \nu(\sigma' - \sigma) - \frac{1}{2} [g_{ab}(\sigma) + g_{ab}(\sigma')] \frac{\partial}{\partial \sigma} \delta(\sigma' - \sigma) + \left[ \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^a} \right] x^c(\sigma) \delta(\sigma' - \sigma) \]
\[ \{x^a(\sigma), p_b(\sigma')\}_2 = -\frac{\partial \omega}{\partial x^b} p_c + \frac{\partial \omega}{\partial x^b} p_c \nu(\sigma' - \sigma), \]
\{p_a(\sigma), x^b(\sigma')\}_2 = -\left[\frac{\partial \omega^{bc}(\sigma)}{\partial x^a} p_c + \frac{\partial \omega^{bc}(\sigma')}{\partial x^a} p_c\right] \nu(\sigma' - \sigma). \tag{5}

The function \(\omega^{ab}(x)\) is satisfied on the equation:

\[\frac{\partial \omega^{ab}}{\partial x^c} x^c + 2\omega^{ab} = g^{ab}. \tag{6}\]

The Jacobi identities for the PB \{,\}_2 are led to the nonlocal consistence conditions on the unknown function \(\omega^{ab}(\sigma)\). We can calculate unknown metric tensor \(g^{ab}(\sigma)\) by substitution of the solution of the consistence condition for \(\omega^{ab}\) to the equation (6). The Jacobi identity

\[\{x^a(\sigma), x^b(\sigma')\} \equiv \{x^a(\sigma), x^b(\sigma')\}x^c(\sigma'') + \{x^c(\sigma''), x^a(\sigma)\}x^b(\sigma') + \{x^b(\sigma'), x^c(\sigma'')\}x^a(\sigma) = 0 \tag{7}\]

is led to the following nonlocal analogy of the WDVV \[\text{[3]} \text{[4]}\] consistence condition:

\[\frac{\partial \omega^{ab}(\sigma)}{\partial x^d} [\omega^{dc}(\sigma) + \omega^{dc}(\sigma'') - \omega^{db}(\sigma) + \omega^{db}(\sigma')] \nu(\sigma' - \sigma) - \nu(\sigma' - \sigma') + \frac{\partial \omega^{cb}(\sigma')}{\partial x^d} [\omega^{da}(\sigma) + \omega^{da}(\sigma'') - \omega^{db}(\sigma) + \omega^{db}(\sigma')] \nu(\sigma' - \sigma) - \nu(\sigma' - \sigma') + \frac{\partial \omega^{ac}(\sigma'')}{\partial x^d} [\omega^{db}(\sigma) + \omega^{db}(\sigma'') - \omega^{da}(\sigma) + \omega^{da}(\sigma')] \nu(\sigma' - \sigma) - \nu(\sigma' - \sigma') = 0. \tag{8}\]

This equation has the particular solution of the following form:

\[\frac{\partial \omega^{ab}(\sigma)}{\partial x^d} [\omega^{dc}(\sigma) + \omega^{dc}(\sigma'') - \omega^{db}(\sigma) + \omega^{db}(\sigma')] = [T^b, T^c]T^a f(\sigma, \sigma', \sigma'') \nu(\sigma'' - \sigma) - \nu(\sigma' - \sigma), \]

where \(T^a, a = 0, 1, \ldots, D - 1\) is the matrix representation of the simple Lie algebra and \(f(\sigma, \sigma', \sigma'')\) is arbitrary function. The Jacobi identity is satisfied on the Jacobi identity of the simple Lie algebra in this case:

\[(\{T^a, T^b\}T^c) + [T^c, T^a]T^b + [T^b, T^c]T^a) f(\sigma, \sigma', \sigma'') = 0 \]

and we used the relation \(\nu^2(\sigma' - \sigma) = 1\). The local solution of the Jacobi identities leads to the constant metric tensor. The rest Jacobi identities are cumbersome and we do not reduce this expressions here. The symmetric factor of \(\sigma, \sigma'\) of the antisymmetric functions \(\nu(\sigma' - \sigma), \frac{\partial}{\partial x^a} \delta(\sigma - \sigma')\) in the right side of the PB can be both sum of the functions of \(\sigma, \sigma'\), and production of them. Last possibility can be used in the vielbein formalism.

**Proposition 2.** The bi-Hamiltonity condition can be solved in the terms PB \{,\}_2, which have the following form:

\[\{x^a(\sigma), x^b(\sigma')\}_2 = e^a_\mu(\sigma) e^b_\mu(\sigma') \nu(\sigma' - \sigma), \]

\[\{x^a(\sigma), p_b(\sigma')\}_2 = -e^a_\mu(\sigma) \frac{\partial e^c_\mu(\sigma')}{\partial x^b} p_c(\sigma') \nu(\sigma' - \sigma), \]

\[\{p_a(\sigma), p_b(\sigma')\}_2 = \frac{\partial e^c_\mu(\sigma)}{\partial x^a} p_c(\sigma) - \frac{\partial e^d_\mu(\sigma')}{\partial x^b} p_d(\sigma') \nu(\sigma' - \sigma) - e^a_\mu(\sigma) e^b_\mu(\sigma') \frac{\partial \nu(\sigma' - \sigma)}{\partial x^a} + \frac{\partial e^a_\mu}{\partial x^c} e^b_\mu - \frac{\partial e^b_\mu}{\partial x^c} e^a_\mu - \frac{\partial e^c_\mu}{\partial x^a} e^b_\mu + \frac{\partial e^c_\mu}{\partial x^b} e^a_\mu] x^c(\sigma') \delta(\sigma' - \sigma), \tag{9}\]
where vielbein $e^a_\mu$ is satisfied on the additional conditions:

$$g^{ab} = \eta^{\mu\nu} e^a_\mu e^b_\nu, \quad g_{ab} = \eta_{\mu\nu} e^a_\mu e^b_\nu$$

and $\eta^{\mu\nu}$ is the metric tensor of the flat space.

The particular solution of the Jacobi identity is

$$\left\{ \frac{\partial e^a_\mu}{\partial x^d} e^b_\nu(\sigma') e^c_\nu(\sigma'') - \frac{\partial e^a_\mu}{\partial x^d} e^c_\nu(\sigma'') e^b_\nu(\sigma'') \right\} = [T^b, T^c] T^a \cdot f(\sigma, \sigma', \sigma'') \nu(\sigma'' - \sigma) \nu(\sigma' - \sigma).$$

As example let me consider the the constant curvature space.

**Example 1.** The constant curvature space is described by the metric tensor $g_{ab}(x(\sigma))$ and by it inverse tensor $g^{ab}$:

$$g_{ab} = \eta_{ab} + \frac{kx_a x_b}{1 - kx^2}, \quad g^{ab} \equiv g^{-1}_{ab} = \eta_{ab} - kx_a x_b.$$ 

**Proposition 3.** Dual (PB){,} are:

$$\left\{ x_a(\sigma), x_b(\sigma') \right\} = [\eta_{ab} - kx_a(\sigma)x_b(\sigma')] \nu(\sigma' - \sigma),$$

$$\left\{ x_a(\sigma), p_b(\sigma') \right\} = kx_a(\sigma)p_b(\sigma') \nu(\sigma' - \sigma),$$

$$\left\{ p_a(\sigma), p_b(\sigma') \right\} = -k p_a(\sigma)p_b(\sigma') \nu(\sigma' - \sigma)$$

$$-\frac{1}{2} [2\eta_{ab} + \frac{kx_a x_b}{1 - kx^2}(\sigma) + \frac{kx_a x_b}{1 - kx^2}(\sigma')] \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma) + \frac{x_a x_b - x_b x_a}{2(1 - kx^2)} \delta(\sigma' - \sigma).$$ (10)

The Jacobi identity (11) is led to the equation

$$[\eta_{ab} x_c(\sigma'') - \eta_{ac} x_b(\sigma')] \nu(\sigma' - \sigma) \nu(\sigma - \sigma'' + [\eta_{bc} x_a(\sigma) - \eta_{ba} x_c(\sigma'')] \nu(\sigma' - \sigma') \nu(\sigma' - \sigma'') +$$

$$[\eta_{ca} x_b(\sigma') - \eta_{cb} x_a(\sigma)] \nu(\sigma' - \sigma'') \nu(\sigma'' - \sigma) = 0.$$ (11)

The particular solution of this equation is:

$$\eta_{ab} x_c(\sigma'') - \eta_{ac} x_b(\sigma') = [T_b, T_c] T_a \cdot f(\sigma, \sigma', \sigma'') \nu(\sigma'' - \sigma) \nu(\sigma' - \sigma).$$ (11)

Consequently, the space-time coordinate $x_a(\sigma)$ is the matrix representation of the simple Lie algebra. The Jacobi identity $\{x_a(\sigma), x_b(\sigma')\} T_c(\sigma'')$ is led to the equation

$$k \eta_{ab} p_c(\sigma'') \nu(\sigma' - \sigma) [\nu(\sigma'' - \sigma) + \nu(\sigma'' - \sigma')] = 0.$$ (12)

These results can be obtained from the vielbein formalism under the following ansatz for the vielbein of the constant curvature space:

$$e^{(s)}_\mu = n_\mu (m_1^{(s)} n^a + \sqrt{km_2^{(s)} x_a}), \quad e^{(s)}_a = n^\mu g_{ab} (m_1^{(s)} n^b + \sqrt{km_2^{(s)} x^b}),$$

where $n_\mu^2 = 1$, $m_1^{(s)} m_1^{(s)} = 1$, $m_2^{(s)} m_2^{(s)} = 1$, $m_1^{(s)} m_2^{(s)} = 0$, $n_a n_b = \delta^{ab}$ and $(s)$ is number of the solution of the equations

$$e^{(s)}_\mu e^{(s)}_\mu = g^{ab}, \quad e^{(s)}_a e^{(s)}_b = g_{ab}, \quad e^{(s)}_\mu e^{(s)}_\mu = \delta^{ab}. $$


The following example is time-dependent metric space.

**Example 2.** The time-dependent metric in the light-cone variables has form:

\[ ds^2 = g_{ik}(x^+)dx^i dx^k + g_{++}(x^+)dx^+dx^+ + 2g_{+-}dx^+dx^- . \]  

We are used Poisson brackets \((5)\) for the space coordinates \(x^a = \{x^i, x^+, x^-\}, i = 1, 2...D-2\).

We introduced the light-cone gauge as two first kind constraints:

\[ F_1(\sigma) = x'^+ \approx 0, \quad F_2(\sigma) = p'_- \approx 0, \]

and we imposed them on the equations of motion and on the Jacobi identities. The Jacobi identities are reduced to the simple equation

\[ \frac{\partial \omega^{ab}}{\partial x^+} \omega^{+c} - \frac{\partial \omega^{ac}}{\partial x^+} \omega^{+b} = 0. \]

We obtained following result from this equation and additional condition \((6)\): there is constant background gravity field only for the non-degenerate metric.

3. **Constant background fields.**

In this section we are supplemented the bi-Hamiltonity condition \((4)\) by the mirror transformations of the integrals of motion.

\[ \dot{x}^a = \{x^a, \int_0^\pi h_1 d\sigma\} = \{x^a, \int_0^\pi \pm h_2 d\sigma'\} \pm 2. \]

The dual PB are

\[ \{x^a(\sigma), x^b(\sigma')\} \pm 2 = \pm g^{ab} \nu(\sigma' - \sigma), \quad \{x^a(\sigma), p_b(\sigma')\} \pm 2 = 0, \]

\[ \{p_a(\sigma), p_b(\sigma')\} \pm 2 = \mp g_{ab} \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma). \]

The dual dynamical system

\[ \dot{x}^a = \{x^a, \pm H_2\} = \{x^a, H_1\} \pm 2. \]

is the left(right) chiral string

\[ \dot{x}^a = \pm x'^a, \quad \dot{p}_a = \pm p'_a. \]

In the terms of the Virasoro operators

\[ L_k = \frac{1}{4\pi} \int_0^\pi (h_1 + h_2) e^{ik\sigma} d\sigma, \quad \bar{L}_k = \frac{1}{4\pi} \int_0^\pi (h_1 - h_2) e^{ik\sigma} d\sigma, \]

the first kind constraints form the \(Vir \oplus \bar{Vir}\) algebra under the PB \(\{,\}_1\).

\[ \{L_n, L_m\}_1 = -i(n - m)L_{n+m}, \quad \{\bar{L}_n, \bar{L}_m\}_1 = -i(n - m)\bar{L}_{n+m}, \quad \{L_n, \bar{L}_m\}_1 = 0. \]

The dual set of the integrals of motion is obtained from initial it by the mirror transformations

\[ H_1 \to \pm H_2, \quad L_0 \to \pm L_0, \quad \bar{L}_0 \to \mp L_0, \quad \tau \to \sigma. \]

and by the contraction of the first kind constraints algebra \(L_n = 0, \bar{L}_n = 0, n \neq 0\). Another way to obtain the dual brackets is the imposition of the second kind constraints on
the initial dynamical system, by such manner, that $F_i = F_k$ for $i \neq k, i, k = 1, 2, \ldots$ on the constraints surface $f(x^a, p_a) = 0$.

The constraints $f_a^{-}(x, p) = p_a - g_{ab}x^b \approx 0$ or $f_a^{(+)} = p_a + g_{ab}x^b \approx 0$ (do not simultaneously) are the second kind constraints.

$$\{f_a^{(\pm)}(\sigma), f_b^{(\pm)}(\sigma')\}_1 = C_{ab}^{(\pm)}(\sigma - \sigma') = \pm 2g_{ab}\frac{\partial}{\partial \sigma'}\delta(\sigma' - \sigma).$$

The inverse matrix $(C^{(\pm)})^{-1}$ has following form $C^{(\pm)ab}(\sigma - \sigma') = \pm \frac{1}{2}g^{ab}\nu(\sigma' - \sigma)$. There is only one set of the constraints, because consistency condition

$$\{f^{(\pm)}(\sigma), H_1\}_1 = f^{(\pm)}(\sigma) \approx 0, \ldots, \{f^{(\pm)(n)}(\sigma), H_1\}_1 = f^{(\pm)(n+1)}(\sigma) \approx 0.$$ is not produce the new sets of constraints. By using the standard definition of the Dirac bracket, we are obtained following Dirac brackets for the phase space coordinates.

$$\{x^a(\sigma), x^b(\sigma')\}_D = \pm \frac{1}{2}g^{ab}\nu(\sigma' - \sigma), \{p_a(\sigma), p_b(\sigma')\}_D = \mp \frac{1}{2}g_{ab}\frac{\partial}{\partial \sigma'}\delta(\sigma' - \sigma),$$

$$\{x^a(\sigma), p_b(\sigma')\}_D = \frac{1}{2}\delta_0^a\delta(\sigma' - \sigma).$$

The equations of motion under the Hamiltonians $H_1 = h_1, H_2 = h_2$ and Dirac bracket

$$\dot{x}^a = \{x^a, H_1\}_D = \{x^a, H_2\}_D = g^{ab}p_b = \pm x^a;$$

$$\dot{p}_a = \{p_a, H_1\}_D = \{p_a, H_2\}_D = g_{ab}x^b = \pm p_a'$$ are coincide on the constraints surface. The dual brackets $\{\cdot, \cdot\}_\pm$ are coincide with the Dirac brackets also. The contraction of the algebra of the first kind constraints means that the integrals of motion $H_1 = H_2$ are coincide on the constraints surface too.

**References**

[1] L. D. Fadeev and L. A. Takhtajan, Hamiltonian methods in theory of solitons. Springer, Berlin, 1987.

[2] J.A. Mitropolski, N. N. Bogolubov(Jr), A. K. Prikarpatski, V. G. Samoylenko, Integriruemie dinamicheskije sistemi:spectralno-geometricheskie aspecti. Naukova Dumka, Kiev, 1987. (in Russian).

[3] S. Okubo, A. Das, The integrability condition for dynamical systems. Phys. Lett. B209, 1988, 311–314.

[4] F. A. Magri, A geometrical approach to the nonlinear solvable equations. Lect. Notes Phys., 1980, v. 120, 233–263.

[5] V. D. Gershun, Bihamiltonity as origin T-duality the closed string model. Nucl. Phys. B (Proc. Suppl.), 2001, v. 102&103, 71–76, hep-th/0103097

[6] V. D. Gershun, Bihamiltonian approach to the closed string model in the background fields. hep-th/0112234

[7] V. D. Gershun, T-duality of string models in the background gravity and antisymmetric fields, Gravitation, cosmology and relativistic astrophysics, KNU, Kharkov, 69, 2001.

[8] V. D. Gershun, Bihamiltonian approach to the gauge models: Closed string models, Problems of atomic science and technology, KIPT, Kharkov, 2001, v. 6(1), 65–70.

[9] Bihamiltonian approach to the closed string model and noncommutativity, Proc. Non-Eucl Geometry in Modern Physics (BGL)(Tirgu-Mures, Romania, 2002), Debrecen, Budapest, Hungary, 2003, 51–62.
[10] B. A. Dubrovin, S. P. Novikov, Hydrodynamics of weakly deformed soliton lattices. Russian Math. Surveys, 1989, v.44, No 6, 35–124.

[11] O. I. Mokhov, E. V. Ferapontov, Nonlocal Hamiltonian operators of hydrodynamical type related to metric of constant curvature. Russian Math. Surveys, 1990, v.45, No. 3, 218–219.

[12] E. V. Ferapontov, Differential geometry of nonlocal Hamiltonian of hydrodynamical type. Functional Anal. Appl., 1991, v.25, No. 3, 195–204.

[13] R. Dijkgraaf, E. Witten, Mean field theory, topological field theory, and multi-matrix models. Nucl. Phys. B342, 1990, 486–522.

[14] R. Dijkgraaf, E. Verlinde, H. Verlinde, Topological strings in d=1. Nucl. Phys. B352, 1991, 59–86.