Approximation by $(p, q)$-Lupaş–Schurer–Kantorovich operators

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1 Introduction

In 1962, Bernstein–Schurer operators were identified in the paper of Schurer [25]. In 1987, Lupaş [16] initiated the $q$-generalization of Bernstein operators in rational form. Some other $q$-Bernstein polynomial was defined by Phillips [22] in 1997. The development $q$-calculus applications established a precedent in the field of approximation theory. We may refer to some of them as Durmeyer variant of $q$-Bernstein–Schurer operators [2], $q$-Bernstein–Schurer–Kantorovich type operators [3], $q$-Durrmeyer operators [8], $q$-Bernstein–Schurer–Durrmeyer type operators [12], $q$-Bernstein–Schurer operators [19], King’s type modified $q$-Bernstein–Kantorovich operators [20], $q$-Bernstein–Schurer–Kantorovich operators [23]. Lately, Mursaleen et al. [17] pioneered the research of $(p, q)$-analogue of Bernstein operators which is a generalization of $q$-Bernstein operators (Philips). The application of $(p, q)$-calculus has led to the discovery of various modifications of Bernstein polynomials involving $(p, q)$-integers. For instance, Mursaleen et al. [18] constructed $(p, q)$-analogue of Bernstein–Kantorovich operators in 2016, and Khalid et al. [15] generalised $q$-Bernstein–Lupaş operators. In the $(p, q)$-calculus, parameter $p$ provides suppleness to the approximation. Some recent articles are [1, 4–6, 9, 10, 13], and [21]. Motivated by the work of Khalid et al. [15], now we define a Kantorovich type Lupaş–Schurer operators based on the $(p, q)$-calculus.

First of all, we introduce some important notations and definitions for the $(p, q)$-calculus, which is a generalization of $q$-oscillator algebras. For $0 < q < p \leq 1$ and $m \geq 0$, the $(p, q)$-
number of \( m \) is denoted by \([m]_{p,q}\) and is defined by

\[
[m]_{p,q} := p^{m-1} + p^{m-2} q + \ldots + pq^{m-2} + q^{m-1} = \begin{cases} 
\frac{p^m - q^m}{p^q} & \text{if } p \neq q \neq 1, \\
\frac{1-q^m}{1-q} & \text{if } p = 1, \\
m & \text{if } p = q = 1.
\end{cases}
\]

The formula for the \((p,q)\)-binomial expansion is defined by

\[
(cx + dy)^{m}_{p,q} := \sum_{l=0}^{m} \left[ \begin{array}{c} m \\ l \end{array} \right]_{p,q} p^{m-l} d^{l} x^{m-l} y^{l},
\]  

(1)

where

\[
\left[ \begin{array}{c} m \\ l \end{array} \right]_{p,q} = \frac{[m]_{p,q}!}{[l]_{p,q}! [m-l]_{p,q}!}
\]

are the \((p,q)\)-binomial coefficients. From Eq. (1) we get

\[
(x + y)^{m}_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{m-1}x + q^{m-1}y)
\]

and

\[
(1 - x)^{m}_{p,q} = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{m-1} - q^{m-1}x).
\]

The \((p,q)\)-Jackson integrals are defined by

\[
\int_{0}^{a} f(x) \, d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f \left( \frac{p^k}{q^{k+1}} a \right), \quad \left| \frac{p}{q} \right| < 1
\]

and

\[
\int_{0}^{a} f(x) \, d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} F \left( \frac{q^k}{p^{k+1}} a \right), \quad \left| \frac{q}{p} \right| < 1.
\]

For detailed information about the theory of \((p,q)\)-integers, we refer to [11] and [24].

2 Construction of the operator

**Definition 1** For any \( 0 < q < p \leq 1 \), we construct a \((p,q)\)-analogue of Kantorovich type Lupǎș–Schurer operator by

\[
K_{m,s}^{(p,q)}(f; x) = [m]_{p,q} \sum_{l=0}^{m+s} B_{m,s}^{p,q}(x) \int_{0}^{\frac{l}{m+s}[p,q]} f(t) \, d_{p,q}t, \quad x \in [0,1],
\]  

(2)

where \( m \in \mathbb{N}, f \in C[0, s+1], s > 0 \) is a fixed natural number and

\[
B_{m,s}^{p,q}(x) = \left[ \begin{array}{c} m+s \\ l \end{array} \right]_{p,q} \frac{p^{l+m+s-l-1}}{q^{l+m+s-l}} \frac{q^{l+m+s-l}}{p^{l+m+s-l}} \prod_{j=1}^{l+m+s-l} \left[ p^{j-1} (1-x) + q^{j-1} x \right].
\]  

(3)
After some calculations we obtain

\[ K^{(p,q)}_{n,m}(f;x) = \sum_{l=0}^{m+s} B^{p,q}_{m,l,s}(x) \int_0^1 f \left( \frac{p[l]_p q^l t}{p[l-m[m]_p q} \right) d_p q t. \] (4)

In the following lemma, we present some equalities for the \((p,q)\)-analogue of Lupaş–Schurer–Kantorovich operators.

**Lemma 1** Let \(K^{(p,q)}_{n,m}(\cdot;\cdot)\) be given by Eq. (4). Then we have

\[
K^{(p,q)}_{n,m}(1;x) = 1, \quad (5)
\]

\[
K^{(p,q)}_{n,m}(t;x) = \left( \frac{m[s]_p q}{[m]_p q} \right) p^{-1} \left( \frac{m[s]_p q}{[m]_p q} + \frac{q^{m+s}[p(1-x) + qx]}{[2]_p q[m]_p q} \right) x + \frac{p^m}{[2]_p q[m]_p q}, \quad (6)
\]

\[
K^{(p,q)}_{n,m}(t^2;x) = \left( \frac{m[s]_p q}{[m]_p q} \right) p^{-1} \left( \frac{m[s]_p q}{[m]_p q} + \frac{q^{m+s}[p(1-x) + qx]}{[2]_p q[m]_p q} \right) x^2 + \frac{2[m+s]_p q[p^{m+s}(1-x) + q^{m+s}x]}{[2]_p q[m]_p q} x + \frac{p^m}{[2]_p q[m]_p q}, \quad (7)
\]

\[
K^{(p,q)}_{n,m}(t-x;x) = \left( \frac{m[s]_p q}{[m]_p q} \right) p^{-1} \left( \frac{m[s]_p q}{[m]_p q} + \frac{q^{m+s}[p(1-x) + qx]}{[2]_p q[m]_p q} - 1 \right) x + \frac{p^m}{[2]_p q[m]_p q}, \quad (8)
\]

\[
K^{(p,q)}_{n,m}(t-x^2;x) = \left( \frac{m[s]_p q}{[m]_p q} \right) p^{-1} \left( \frac{m[s]_p q}{[m]_p q} + \frac{2q^{m+s}p^m - 2q^{m+s}p^s}{[2]_p q[m]_p q} + 1 \right) x^2 + \left( \frac{m[s]_p q[p^{m+s}(1-x) + q^{m+s}x]}{[2]_p q[m]_p q} \right) x + \left( \frac{2p^m}{[2]_p q[m]_p q} \right) x + \frac{p^{-2}(p^{m+s}(1-x) + q^{m+s}x)(p^{m+s}(1-x) + q^{m+s}x)}{[3]_p q[m]_p q[p(1-x) + qx]} \right). \quad (9)
\]

**Proof** (i) From the definition of the operators in (4), we can easily prove the first claim as follows:

\[
K^{(p,q)}_{n,m}(1;x) = \sum_{l=0}^{m+s} B^{p,q}_{m,l,s}(x) \int_0^1 d_p q t
\]

\[
= \sum_{l=0}^{m+s} \left[ \frac{m+s}{l} \right]_p q^{m+s-l} \frac{(l-1)!}{\prod_{j=1}^{m+s} (p^{-1}(1-x) + q^{-1}x)}
\]

\[
= 1. \quad (10)
\]
Thus, (6) is obtained.

(iii) For the third identity involving \(K_{m,s}^{(p,q)}(t^2; x)\), we write

\[
K_{m,s}^{(p,q)}(t^2; x) = \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \alpha_{l}^{2} \int_{0}^{1} p(l)_{p,q} q^{l} t \, d_{p,q}t + 2 \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \beta_{l}^{2} \int_{0}^{1} p(l)_{p,q} q^{l} t \, d_{p,q}t
\]

\[
= \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \alpha_{l}^{2} \int_{0}^{1} p(l)_{p,q} q^{l} t \, d_{p,q}t + \sum_{l=0}^{m+s} B_{m,l,s}^{p,q}(x) \beta_{l}^{2} \int_{0}^{1} p(l)_{p,q} q^{l} t \, d_{p,q}t.
\]
Firstly, we calculate $B_1$ as

$$B_1 = \sum_{l=0}^{m+s} p_{m,s}^{p,q}(x) \frac{p^2[l]^2_{p,q}}{p^{2l-2m}[m]^2_{p,q}} \left[ \prod_{j=1}^{m+s-1} \left[ p^{l-1}(1-x) + q^{l-1}x \right] \right].$$

Now by using the equality

$$[l + 1]_{p,q} = p^l + q[l]_{p,q},$$

we acquire

$$B_1 = \frac{[m + s]_{p,q}}{[m]^2_{p,q}} \sum_{l=0}^{m+s-1} p^{2m-l} \left[ \prod_{j=1}^{m+s-1} \left[ p^{l-1}(1-x) + q^{l-1}x \right] \right].$$

$$= \sum_{l=0}^{m+s-1} p^{2m-2l}[l+1]_{p,q}[m+s]_{p,q} \frac{[m+s-1]_{p,q} \left[ \prod_{j=1}^{m+s-1} \left[ p^{l-1}(1-x) + q^{l-1}x \right] \right]}{[m]^2_{p,q}}.$$

Secondly, we work out $B_2$ as follows:

$$B_2 = \frac{2}{[2]_{p,q}} \sum_{l=0}^{m+s} p_{m,s}^{p,q}(x) \frac{p[l]^2_{p,q}}{p^{2l-2m}[m]^2_{p,q}} \left[ \prod_{j=1}^{m+s-1} \left[ p^{l-1}(1-x) + q^{l-1}x \right] \right].$$

$$= \frac{2[m + s]_{p,q}}{[2]_{p,q}[m]^2_{p,q}} \sum_{l=0}^{m+s} q^{l-1} \left[ \prod_{j=1}^{m+s-1} \left[ p^{l-1}(1-x) + q^{l-1}x \right] \right].$$

$$= \sum_{l=0}^{m+s-1} q^{l-1} \left[ \prod_{j=1}^{m+s-1} \left[ p^{l-1}(1-x) + q^{l-1}x \right] \right].$$
Thirdly, we deal with B3 as

\[
B3 = \frac{1}{[3]_{p,q}} \sum_{l=0}^{m+s} \binom{m+s}{l} p_{m,s}^{2l} q_{m,s}^{2l-2m} \frac{q^{2l}}{p^{2l-2m}[m]_{p,q}^2}
\]

\[
= \frac{p^{2m}}{[3]_{p,q} [m]_{p,q}^2} \sum_{l=0}^{m+s} \binom{m+s}{l} q_{m,s}^{2l} \frac{q^{2l}}{p^{2l-2m}} \frac{q^{2l}}{p^{2l-2m}} \frac{(1-x)^{m+s}}{p(1-x) + qx}
\]

\[
= \frac{p^{-2l}}{[3]_{p,q} [m]_{p,q}^2} \frac{(p^{m+s}(1-x) + q^{m+s})x}{p(1-x) + qx} (p^{m+s+1}(1-x) + q^{m+s+1}x).
\]  

(15)

As a consequence, \(K_{m,s}^{(p,q)}(t^2; x)\) is found as

\[
K_{m,s}^{(p,q)}(t^2; x) = \frac{[m+s]_{p,q} p^{m+s+1}}{[m]_{p,q}^2} x + \frac{[m+s]_{p,q} [m+s-1]_{p,q} p^{2-2s} q^2}{[m]_{p,q}^2 (p(1-x) + qx)} x^2
\]

\[
+ \frac{2[m+s]_{p,q} p^{4m+2s-3}}{[2]_{p,q} [m]_{p,q}^2} \frac{(p^{m+s}(1-x) + q^{m+s})x}{p(1-x) + qx} \frac{(p^{m+s+1}(1-x) + q^{m+s+1}x)}{p(1-x) + qx}
\]

If we reorganize, we obtain

\[
K_{m,s}^{(p,q)}(t^2; x) = \frac{[m+s]_{p,q} [m+s-1]_{p,q} q^{2s-2s}}{[m]_{p,q}^2} x^2 + \frac{[m+s]_{p,q} p^{m+s+1}}{[m]_{p,q}^2} x
\]

\[
+ \frac{2[m+s]_{p,q} p^{4m+2s-3}}{[2]_{p,q} [m]_{p,q}^2} \frac{(p^{m+s}(1-x) + q^{m+s})x}{p(1-x) + qx} \frac{(p^{m+s+1}(1-x) + q^{m+s+1}x)}{p(1-x) + qx}
\]

(16)

as desired.

(iv) By using the linearity of the operators \(K_{m,s}^{(p,q)}\), we acquire the first central moment \(K_{m,s}^{(p,q)}(t - x; x)\) as

\[
K_{m,s}^{(p,q)}(t - x; x) = K_{m,s}^{(p,q)}(t; x) - x K_{m,s}^{(p,q)}(1; x)
\]

\[
= \left( \frac{[m+s]_{p,q} - p^m}{[m]_{p,q} p^{m-1}} - \frac{p^m}{[2]_{p,q} [m]_{p,q} p^2} - \frac{q^{m+s}}{[2]_{p,q} [m]_{p,q} p^2} - 1 \right) x
\]

\[
+ \frac{p^m}{[2]_{p,q} [m]_{p,q}}.
\]  

(17)
Similarly, we write the second central moment $K_{m,s}^{(p,q)}((t-x)^2;x)$ as

$$K_{m,s}^{(p,q)}((t-x)^2;x) = K_{m,s}^{(p,q)}((t^2;x)-2xK_{m,s}^{(p,q)}(t;x) + x^2K_{m,s}^{(p,q)}(1;x).$$ \hspace{1cm} (18)

We now plug into equation (18) expressions (5), (6) and (7). Then we get

$$\begin{align*}
K_{m,s}^{(p,q)}((t-x)^2;x) &= \left( \frac{[m+s]_{p,q}[m+s-1]_{p,q}q^2p^{2-2s}}{[m]_{p,q}^2(p(1-x)+qx)} \right. \\
& \quad + \left. \frac{-2[2]_{p,q}[m+s]_{p,q}p^{1-s}+2p^m-2qm^{s}p^{-s}}{[2]_{p,q}[m]_{p,q}} + 1 \right) x^2 \\
& \quad + \frac{[m+s]_{p,q}p^{m+s+1}}{[2]_{p,q}[m]_{p,q}} + \frac{2[m+s]_{p,q}q^2p^{m+2s-3}(p^{m+s}(1-x)+q^{m+s})x}{[2]_{p,q}[m]_{p,q}^2(p(1-x)+qx)} \\
& \quad - \frac{2p^m}{[2]_{p,q}[m]_{p,q}} x \\
& \quad + \frac{p^{-2}(p^{m+s}(1-x)+q^{m+s})(p^{m+s+1}(1-x)+q^{m+s+1})}{[3]_{p,q}q(p(1-x)+qx)[m]_{p,q}^2}. \\
\end{align*}$$ \hspace{1cm} (19)

We can easily see that $K_{m,s}^{(p,q)}(f;x)$ are linear positive operators.

**Remark 1** [15] Let $p,q$ satisfy $0 < q < p \leq 1$ and $\lim_{m \to \infty}[m]_{p,q} = \frac{1}{p-q}$. To obtain the convergence results for operators $K_{m,s}^{(p,q)}(f;x)$, we take sequences $q_m \in (0,1)$, $p_m \in (q_m, 1]$ such that $\lim_{m \to \infty} p_m = 1$, $\lim_{m \to \infty} q_m = 1$, $\lim_{m \to \infty} p_m^m = 1$ and $\lim_{m \to \infty} q_m^m = 1$. Such sequences can be constructed by taking $p_m = 1 - 1/m^2$ and $q_m = 1 - 1/2m^2$.

Now we will present the next theorem, which ensures the approximation process according to Korovkin’s approximation theorem.

**Theorem 1** Let $K_{m,s}^{(p,q)}(f;x)$ satisfy the conditions $p_m \to 1$, $q_m \to 1$, $p_m^m \to 1$ and $q_m^m \to 1$ as $m \to \infty$ for $q_m \in (0,1)$, $p_m \in (q_m, 1]$. Then for every monotone increasing function $f \in C[0,s+1]$, operators $K_{m,s}^{(p,q)}(f;x)$ converge uniformly to $f$.

**Proof** By the Korovkin theorem, it is sufficient to prove that

$$\lim_{m \to \infty} \| K_{m,s}^{(p,q)} e_k - e_k \| = 0, \quad k = 0, 1, 2,$$

where $e_k(x) = x^k$, $k = 0, 1, 2$.

(i) By using Eq. (5), it can be clearly seen that

$$\lim_{m \to \infty} \| K_{m,s}^{(p,q)} e_0 - e_0 \| = \lim_{m \to \infty} \sup_{x \in [0,1]} | K_{m,s}^{(p,q)}(1;x) - 1 | = 0.$$

(ii) By Eq. (6), we write

$$\lim_{m \to \infty} \| K_{m,s}^{(p,q)} e_1 - e_1 \| = \lim_{m \to \infty} \sup_{x \in [0,1]} | K_{m,s}^{(p,q)}(t;x) - x |$$
Let $f$ be a continuous function on $C[0,s+1]$. The modulus of continuity of $f$ is denoted by $w(f,\sigma)$ and given as

$$w(f,\sigma) = \sup_{y \neq x \leq x+\sigma \leq y} |f(y) - f(x)|.$$  

(20)
Then we know from the properties of modulus of continuity that for each $\sigma > 0$, we have
\[
|f(y) - f(x)| \leq w(f, \sigma)\left(\frac{|y - x|}{\sigma} + 1\right), \quad x, y \in [0, 1]. \tag{21}
\]

And also, for $f \in C[0, s + 1]$ we have $\lim_{\sigma \to 0^+} w(f, \sigma) = 0$. First of all, we begin by giving the rate of convergence of the operators $K_{m,s}^{(p,q)}(f;x)$ by using the modulus of continuity.

**Theorem 2** Let the sequences $p := (p_m)$ and $q := (q_m)$, $0 < q_m < p_m \leq 1$, satisfy the conditions $p_m \to 1$, $q_m \to 1$, $p_m^m \to 1$ and $q_m^m \to 1$ as $m \to \infty$. Then for each $f \in C[0,s + 1]$,\[
\|K_{m,s}^{(p,q)}f - f\|_{C[0,s+1]} \leq 2\omega(f; \sigma_m(x)),
\]
where
\[
\sigma_m(x) = \sqrt{K_{m,s}^{(p,q)}((t-x)^2;x)} \tag{22}
\]
and $K_{m,s}^{(p,q)}((t-x)^2;x)$ is as given by (19).

**Proof** By the positivity and linearity of the operators $K_{m,s}^{(p,q)}(f;x)$, we get
\[
|K_{m,s}^{(p,q)}f(x) - f(x)| = |K_{m,s}^{(p,q)}(f(t) - f(x);x)| \\
\leq K_{m,s}^{(p,q)}(|f(t) - f(x)|;q;x).
\]

After that we apply (21) and obtain
\[
|K_{m,s}^{(p,q)}f(x) - f(x)| \leq K_{m,s}^{(p,q)}\left(w(f, \sigma_m)\left(\frac{|t-x|}{\sigma_m} + 1\right);x\right) \\
= \frac{w(f, \sigma_m)}{\sigma_m} \sqrt{K_{m,s}^{(p,q)}((t-x)^2;x) + w(f, \sigma_m)} \\
= w(f, \sigma_m)\left(1 + \frac{1}{\sigma_m} \sqrt{K_{m,s}^{(p,q)}((t-x)^2;x)}\right). \tag{23}
\]

Then, taking supremum of the last equation, we have
\[
\|K_{m,s}^{(p,q)}f - f\| = \sup_{x \in [0,1]} |K_{m,s}^{(p,q)}f(x) - f(x)| \\
\leq w(f, \sigma_m)\left(1 + \frac{1}{\sigma_m} \sqrt{K_{m,s}^{(p,q)}((t-x)^2;x)}\right).
\]

Choose
\[
\sigma_m(x) = \left\{ \left(\frac{\alpha^2[m + l]_{p,q}[m + l - 1]_{p,q}}{(m)_{p,q} + \beta}^2(1 - x) + qx \right) - \frac{2[m + l]_{p,q} + 1}{\sigma_m}\right) x^2 \\
+ \left(\frac{2\alpha}{\sigma_m} + \frac{[m + l]_{p,q}}{\sigma_m} \cdot \left(\frac{1}{2} \left(\frac{\alpha^2[m + l]_{p,q}}{(m)_{p,q} + \beta}^2 + qx\right) + \frac{1}{\sigma_m}\right)\right)x + \left(\frac{\alpha}{\sigma_m} \right)^2 \right\}^{1/2}.
\]
Thus, we achieve

$$\|K_{m,s}^{(p,q)} f - f\|_{C([0,s+1])} \leq 2\omega(f;\sigma_m(x)).$$

This result completes the proof of the theorem. \(\Box\)

In what follows, by using Lipschitz functions, we will give the rate of convergence of the operators \(K_{m,s}^{(p,q)}(f;x)\). We remember that if the inequality

$$|f(y) - f(x)| \leq M|y - x|^\alpha; \quad \forall x, y \in [0,1]$$

(24)

is satisfied, then \(f\) belongs to the class \(\text{Lip}_M(\alpha)\).

**Theorem 3** Denote \(p := (p_m)\) and \(q := (q_m)\) satisfying \(0 < q_m < p_m \leq 1\). Then, for every \(f \in \text{Lip}_M(\alpha)\), we have

$$\|K_{m,s}^{(p,q)} f - f\| \leq M\sigma_m^\alpha(x),$$

where \(\sigma_m(x)\) is the same as in (22).

**Proof** Let \(f\) belong to the class \(\text{Lip}_M(\alpha)\) for some \(0 < \alpha \leq 1\). Using the monotonicity of the operators \(K_{m,s}^{(p,q)}(f;x)\) and (24), we obtain

$$|K_{m,s}^{(p,q)}(f;x) - f(x)| \leq K_{m,s}^{(p,q)}(|f(t) - f(x)|;x) \leq MK_{m,s}^{(p,q)}((t-x)^\alpha;x).$$

Taking \(p = \frac{2}{\alpha}\), \(q = \frac{2}{2-\alpha}\) and applying Hölder inequality yields

$$|K_{m,s}^{(p,q)}(f;x) - f(x)| \leq M\left[K_{m,s}^{(p,q)}((t-x)^\alpha; x)\right]^\frac{2}{\alpha} \leq M\sigma_m^\alpha(x).$$

By choosing \(\sigma_m(x)\) as in Theorem 2, we complete the proof as desired. \(\Box\)

Finally, in the light of Peetre-K functionals, we obtain the rate of convergence of the constructed operators \(K_{m,s}^{(p,q)}(f;x)\). We recall the properties of Peetre-K functionals, which are defined as

$$K(f,\delta) := \inf_{g \in C^2([0,s+1])} \{\|f - g\|_{C([0,s+1])} + \delta \|g\|_{C^2([0,s+1])}\}.$$ 

Here \(C^2[0,s + 1] \) defines the space of the functions \(f\) such that \(f, f', f'' \in C[0,s + 1]\). The norm in this space is given by

$$\|f\|_{C^2[0,s+1]} = \|f''\|_{C[0,s+1]} + \|f'\|_{C[0,s+1]} + \|f\|_{C[0,s+1]}.$$

Also we consider the second modulus of smoothness of \(f \in C[0,s + 1]\), namely

$$\omega_2(f,\delta) := \sup_{0 < h \leq \delta} \sup_{x \in [0,s+1]} |f(x + 2h) - 2f(x + h) + f(x)|, \quad \delta > 0.$$
We know from [7] that for $M > 0$

$$K(f; \delta) \leq M \omega_2(f, \sqrt{\delta}).$$

Before giving the main theorem, we present an auxiliary lemma, which will be used in the proof of the theorem.

**Lemma 4** For any $f \in C[0, s + 1]$, we have

$$|K_m(f; x)| \leq \|f\|. \quad (25)$$

**Proof**

$$|K_m(f; x)| = \left| \sum_{l=0}^{m+1} B_{m,l}(x) \int_0^1 f \left( \frac{p[l][q] + q^l t}{p-l-m[m][p,q]} \right) dt \right|$$

$$\leq \sum_{l=0}^{m+1} B_{m,l}(x) \int_0^1 \left| f \left( \frac{p[l][q] + q^l t}{p-l-m[m][p,q]} \right) \right| dt$$

$$\leq \|f\| \|K_m(1; x)\|$$

$$= \|f\|. \quad \square$$

**Theorem 4** Let $0 < q_m < p_m \leq 1$, $m \in \mathbb{N}$ and $f \in C[0, s + 1]$. There exists a constant $M > 0$ such that

$$|K_m(f; x) - f(x)| \leq M \omega_2(f, \alpha_m(x)) + \omega(f, \beta_m(x)),$$

where

$$\alpha_m(x) = \sqrt{K_m(1; x)} \left( (t - x)^2 \right) + \frac{1}{2} \left( \frac{[2]_{p,q} [m+s]_{p,q} p^{1+s} - p^m + p^{-s} q^{m+s} x + p^m}{[2]_{p,q} [m]_{p,q}} - x \right)^2 \quad (26)$$

and

$$\beta_m(x) = \frac{[2]_{p,q} [m+s]_{p,q} p^{1+s} - p^m + p^{-s} q^{m+s} x + p^m}{[2]_{p,q} [m]_{p,q}} - x. \quad (27)$$

**Proof** Define an auxiliary operator $K_{m,s}$ as follows:

$$K_{m,s}(f; x) = K_m(f; x) - f \left( \frac{[2]_{p,q} [m+s]_{p,q} p^{1+s} - p^m + p^{-s} q^{m+s} x + p^m}{[2]_{p,q} [m]_{p,q}} \right) + f(x). \quad (28)$$

From Lemma 1, we have

$$K_{m,s}(1; x) = 1,$$
Using (29) and (28), we obtain

\[ K_{m,s}^*(t-x;x) = K_{m,s}^{(p,q)}((t-x);x) = \left( \frac{([2]_{p,q} [m+s]_{p,q} p^{1-s} - p^m + p^{-s} q^{m+s}) x + p^m}{[2]_{p,q} [m]_{p,q}} \right) \]

\[ \int_x^t \frac{([2]_{p,q} [m+s]_{p,q} p^{1-s} - p^m + p^{-s} q^{m+s}) x + p^m}{[2]_{p,q} [m]_{p,q}} dt \]

\[ = \left( \frac{m+s}{p^m} - \frac{p^m}{[2]_{p,q} [m]_{p,q}} + \frac{q^{m+s}}{[2]_{p,q} [m]_{p,q}} - 1 \right) x + x \]

\[ + \frac{p^m}{[2]_{p,q} [m]_{p,q}} \int_x^t \left( \frac{([2]_{p,q} [m+s]_{p,q} p^{1-s} - p^m + p^{-s} q^{m+s}) x + p^m}{[2]_{p,q} [m]_{p,q}} \right) dt = 0. \quad (29) \]

Taylor’s expansion for a function \( g \in C^2[0, s + 1] \) can be written as follows:

\[ g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u) du, \quad t \in [0, 1]. \quad (30) \]

Then applying operator \( K_{m,s}^* \) to both sides of (30), we get

\[ K_{m,s}^*(g;x) = K_{m,s}^*(g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u) du) \]

\[ = g(x) + K_{m,s}^*(x)g'(x) + K_{m,s}^*(\int_x^t (t-u)g''(u) du). \]

So,

\[ K_{m,s}^*(g;x) - g(x) = g'(x)K_{m,s}^*(x) + K_{m,s}^*(\int_x^t (t-u)g''(u) du). \]

Using (29) and (28), we obtain

\[ K_{m,s}^*(g;x) - g(x) \]

\[ = K_{m,s}^*(\int_x^t (t-u)g''(u) du) \]

\[ = K_{m,s}^{(p,q)}(\int_x^t (t-u)g''(u) du) \]

\[ - \int_x^t \frac{([2]_{p,q} [m+s]_{p,q} p^{1-s} - p^m + p^{-s} q^{m+s}) x + p^m}{[2]_{p,q} [m]_{p,q}} \]

\[ \cdot \frac{([2]_{p,q} [m+s]_{p,q} p^{1-s} - p^m + p^{-s} q^{m+s}) x + p^m}{[2]_{p,q} [m]_{p,q}} du \]

\[ - u \right) g''(u) du \]

\[ + \int_x^t \left( \frac{([2]_{p,q} [m+s]_{p,q} p^{1-s} - p^m + p^{-s} q^{m+s}) x + p^m}{[2]_{p,q} [m]_{p,q}} \right) - u \right) g''(u) du. \quad (31) \]

Moreover,

\[ \left| \int_x^t (t-u)g''(u) du \right| \leq \int_x^t |t-u||g''(u)| du \leq \|g''\| \int_x^t |t-u| du \leq (t-x)^2 \|g''\| \quad (32) \]
and

\[
\left| \int_{x}^{(2)_{p,q}(m+s)p^{1-s}m+q^{m+q}x} \left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right) - u \right) g''(u) \, du \right| \\
\leq \| g'' \| \int_{x}^{(2)_{p,q}(m+s)p^{1-s}m+q^{m+q}x} \left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right) - u \right) du \\
= \frac{\| g'' \|}{2} \left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right)^{2}. \tag{33}
\]

Let us employ (32) and (33) when taking the absolute value of (31). We obtain

\[
|K^*_{m,s}(g; x) - g(x)| \leq \| g'' \| K^{(p,q)}_{m,s}((t - x)^{2}; x) \\
+ \frac{\| g'' \|}{2} \left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right)^{2}.
\]

where

\[
\alpha_{m}(x) \]

\[
= \sqrt{K^{(p,q)}_{m,s}((t - x)^{2}; x) + \frac{1}{2} \left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right)^{2}}. \tag{34}
\]

We now give an upper bound for the auxiliary operator $K^*_{m,s}(f; x)$. From Lemma 4 we get

\[
|K^*_{m,s}(f; x)| = |K^{(p,q)}_{m,s}(f; x) - f\left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right) + f(x)| \\
\leq |K^{(p,q)}_{m,s}(f; x)| + \left| f\left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right) \right| + |f(x)| \\
\leq 3\| f \|. 
\]

Accordingly,

\[
\left| K^{(p,q)}_{m,s}(f; x) - f(x) \right| \\
= \left| K^*_{m,s}(f; x) - f(x) + f\left( \frac{([2]_{p,q}[m+s]_{p,q}p^{1-s} - p^{m} + p^{-s}q^{m+q})x + p^{m}}{[2]_{p,q}[m]_{p,q}} \right) - f(x) \right| \\
\pm g(x) \mp K^*_{m,s}(g; x), 
\]
\[ |K_{m,s}^{(p,q)}(f;x) - f(x)| \leq |K_{m,s}^{*}(f-g;x) - (f-g)(x)| + |K_{m,s}^{*}(g;x) - g(x)| \]
\[ + \left| f\left(\frac{([2]_{p,q} m + s) p^{1-s} - p^{m} + p^{-s} q^{m+s} x + p^{m}}{[2]_{p,q} m_{p,q}} - x\right)\right| + \left| f\left(\frac{([2]_{p,q} m + s) p^{1-s} - p^{m} + p^{-s} q^{m+s} x + p^{m}}{[2]_{p,q} m_{p,q}} - x\right)\right| \]
\[ \leq 4\|f - g\| + \|g''\| \alpha_{m}^{2}(x) + \alpha(f, \beta_{m}(x)) \left(\frac{([2]_{p,q} m + s) p^{1-s} - p^{m} + p^{-s} q^{m+s} x + p^{m}}{[2]_{p,q} m_{p,q}} - x + 1\right) \]
\[ + 2\alpha\left(f, \left(\frac{([2]_{p,q} m + s) p^{1-s} - p^{m} + p^{-s} q^{m+s} x + p^{m}}{[2]_{p,q} m_{p,q}} - x\right)\right), \]

where
\[ \beta_{m}(x) = \frac{([2]_{p,q} m + s) p^{1-s} - p^{m} + p^{-s} q^{m+s} x + p^{m}}{[2]_{p,q} m_{p,q}} - x. \]

Finally, for all \( g \in C^{2}[0,s + 1] \), taking the infimum of (35), we get
\[ |K_{m,s}^{(p,q)}(f;x) - f(x)| \leq 4K\left(f, \alpha_{m}^{2}(x)\right) + \alpha(f, \beta_{m}(x)). \]

Consequently, using the property of Peetre-K functional, we obtain
\[ |K_{m,s}^{(p,q)}(f;x) - f(x)| \leq M\omega_{2}(f, \alpha_{m}(x)) + \alpha(f, \beta_{m}(x)). \]

This completes the proof. \( \square \)

### 4 Graphical illustrations

In this section, we illustrate an approximation of the operators \( K_{m,s}^{(p,q)} \) for a function \( f(x) \) by employing Matlab codes. Let us specially choose
\[ f(x) = \frac{1}{96} \tan\left(\frac{x}{16}\right)\left(\frac{x}{8}\right)^{2} \left(1 - \frac{x}{4}\right)^{3}, \]

and take \( p = 0.8, q = 0.7 \) and \( s = 5 \).

**Algorithm 1**
Initially, we discuss the error estimates of the Kantorovich type Lupaş–Schurer operators based on \((p, q)\)-integers for different values of \(x\) and \(m\) in Table 1 by using Algorithm 1.

And then, we illustrate the convergence of the \((p, q)\)-Lupaş–Schurer–Kantorovich operators \(K^{(p,q)}_{m,s}(f;x)\) for the selected function \(f(x) = \frac{1}{96} \tan \left(\frac{1}{16}x\right) - \frac{1}{8} (1 - \frac{1}{2}x)\) in Fig. 1 for several values of \(m\) by using Algorithm 2. Furthermore, we give the error estimates in Table 2 in order to indicate that the \((p, q)\)-analogue Lupaş–Schurer operators [14] converge and

### Table 1 Error estimates for different values of \(x\) when \(s = 5\), \(p = 0.8\) and \(q = 0.7\)

| \(m\) | \(\text{Error at } x = 0.1\) | \(\text{Error at } x = 0.5\) | \(\text{Error at } x = 0.9\) |
|------|--------------------------|--------------------------|--------------------------|
| 5    | 0.1494 · 10^{-6}         | 0.0583 · 10^{-6}         | 0.0441 · 10^{-6}         |
| 10   | 0.0326 · 10^{-6}         | 0.5298 · 10^{-6}         | 0.1599 · 10^{-6}         |
| 15   | 0.0135 · 10^{-6}         | 0.2398 · 10^{-6}         | 0.0078 · 10^{-6}         |

**Figure 1** Convergence of \((p, q)\)-analogue Lupaş–Schurer–Kantorovich operators \(K^{(p,q)}_{m,s}(f;x)\) for various values of \(p, 1\) and \(m\) with fixed \(s = 5\)
Table 2  Error estimates of \((p,q)\)-Lupaş–Schurer operators for various values of \(x\)

| \(m\) | Error at \(x=0.1\) | Error at \(x=0.5\) | Error at \(x=0.9\) |
|------|-----------------|-----------------|-----------------|
| 5    | \(0.0067 \cdot 10^{-5}\) | \(0.3011 \cdot 10^{-5}\) | \(0.4464 \cdot 10^{-5}\) |
| 10   | \(0.0073 \cdot 10^{-5}\) | \(0.3821 \cdot 10^{-5}\) | \(0.5743 \cdot 10^{-5}\) |
| 15   | \(0.0075 \cdot 10^{-5}\) | \(0.4077 \cdot 10^{-5}\) | \(0.6141 \cdot 10^{-5}\) |

Figure 2  Convergence of the \((p,q)\)-analogue Lupaş–Schurer operators \(L^{(p,q)}_{m,l}(f,x)\) with fixed \(l=5\) for various values of \(p\) and \(q\)

then plot Fig. 2. It can be clearly seen that the \((p,q)\)-Lupaş–Schurer–Kantorovich operators converge faster than the \((p,q)\)-analogue Lupaş–Schurer operators.

5 Conclusion
In this paper, we constructed a new kind of Lupaş operators based on \((p,q)\)-integers to provide a better error estimation. Firstly, we investigated some local approximation results by the help of the well-known Korovkin theorem. Also, we calculated the rate of convergence of the constructed operators employing the modulus of continuity, by using Lipschitz functions and then with the help of Peetre’s K-functional. Additionally, we presented a table of error estimates of the \((p,q)\)-Lupaş–Schurer–Kantorovich operators for a certain function. Finally, we compared the convergence of the new operator to that of the \((p,q)\)-analogue of Lupaş–Schurer operator.

Funding
The authors have not received any research funding for this manuscript.

Competing interests
The authors declare that they have no competing interests regarding the publication of this paper.

Authors’ contributions
The authors declare that they have studied in collaboration and share the same responsibility for this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 10 May 2018  Accepted: 18 September 2018  Published online: 26 September 2018
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