Abstract

The standard model is reconstructed by new method to incorporate strong interaction into our previous scheme based on the non-commutative geometry. The generation mixing is also taken into account. Our characteristic point is to take the fermion field so as to contain quarks and leptons all together which is almost equal to that of SO(10) grand unified theory (GUT). The space-time $M_4 \times Z_2$; Minkowski space multiplied by two point discrete space is prepared to express the left-handed and right-handed fermion fields. The generalized gauge field $A(x, y)$ written in one-differential form extended on $M_4 \times Z_2$ is well built to give the correct Dirac Lagrangian for fermion sector. The fermion field is a vector in 24-dimensional space and gauge and Higgs fields are written in $24 \times 24$ matrices. At the energy of the equal coupling constants for both sheets $y = \pm$ expected to be amount to the energy of GUT scale, we can get $\sin^2 \theta_W = 3/8$ and $m_H = \sqrt{2} m_W$. In general, the equation $m_H = (4/\sqrt{3}) m_W \sin \theta_W$ is followed. Then, it should be noticed that the same result as that of the grand unified theory such as SU(5) or SO(10) GUT is obtained without GUT but with the approach based on the non-commutative geometry and in addition the Higgs mass is related to other physical quantities as stated above.
1 Introduction

Higgs mechanism is crucially important for all spontaneously broken gauge theory such as the standard model, SU(5) or SO(10) grand unified theory. Much efforts have been devoted to understand the essence of this mechanism. However, not so much progress had been achieved so far before the approach in non-commutative geometry (NCG) on the discrete space $M_4 \times \mathbb{Z}_2$; Minkowski space multiplied by two points space was proposed by Connes[1]. In this approach, Higgs field due to this mechanism is regarded as a kind of the gauge field on the discrete space, so that the origin of Higgs particle becomes evident, and so any other extra physical modes are never needed.

Standard model which has cleared all experimental tests so far has been reconstructed many times based on NCG from respective standpoints [1]～[3],[5]. We also reconstructed the standard model based on our formalism which is the generalization of usual differential geometry on the ordinary manifold. The extra differential one-form $\chi$ is introduced in addition to the usual one-form $dx^\mu$ and so our formalism is very familiar with the ordinary differential geometry, though original Connes’one is very difficult to understand. Our method is so flexible that it is easily extended to reconstruct the gauge theory with more complex structure such as the left-right symmetric gauge theory, SU(5) or SO(10) GUT [7],[11]. We have made attempts several times [6],[8],[9],[12] to reconstruct the standard model in which how to incorporate the strong interaction was different with each other.

The direct product method of gauge fields was adopted in Ref.[12] to take account of the strong interaction. In addition, after the specifications of the fermion field on the two point discrete space are determined the generalized gauge field is defined to yield the correct Dirac Lagrangian for fermion sector. This approach was suggested by Sogami [13] who started from the Dirac Lagrangian in the standard model and defined the generalized covariant derivative with gauge and Higgs fields operating on quark and lepton fields to reproduce the correct Dirac Lagrangian. The field strengths for both the gauge and Higgs fields are defined by the commutators of the covariant derivative by which he could obtain the Yang-Mills Higgs Lagrangian in the standard model with the extra restriction on the coupling constant of the Higgs potential. Recently, his method is developed to SU(5) GUT[14].

In this article, we also adopt this approach to reconstruct the standard model and improve our previous article [12] by taking the fermion field $\psi(x,y)$ with the variable $y = \pm$ of two point discrete space to be the fermion field of SO(10) GUT with anti-fermion part discarded. $\psi(x,y)$ is a vector in 24 dimensional space since the three fermion families are also taken into account. Then, we will decide the formations of gauge and
Higgs fields which are all expressed in 24 × 24 matrices. It becomes possible to take the left-handed gauge field, U(1) gauge field and color gauge field so as to commute with each other. According to our algebraic rules in NCG on the discrete space $M_4 \times Z_2$, we can get Yang-Mills-Higgs Lagrangian with the extra restriction on the coupling of Higgs self interaction term within the assumption that the Sitarz term is neglected. In the limit of the equal coupling constants for both sheets $y = \pm$ which is amount to the energy of GUT scale, we can get $\sin^2 \theta_W = 3/8$ and $m_H = \sqrt{2m_W}$. In general, the equation $m_H = (4/\sqrt{3})m_W \sin \theta_W$ is hold. Then, it should be noticed that the same result as that of the grand unified theory such as SU(5) or SO(10) GUT is obtained without GUT but with the approach based on the non-commutative geometry and in addition, the Higgs mass is related to other physical quantities as stated above.

This article is divided into four sections. The next section presents the modifications of our previous formalism based on the generalized differential calculus on $M_4 \times Z_2$ so as to incorporate the generation mixing mechanism and color symmetry. In this section a geometrical picture for the unification of the gauge and Higgs fields is realized, which is the ultimate understanding in this field. The third section is the application to the standard model which leads to the quite different predictions for particle masses from the Sogami’one[13]. This is because the complex Yukawa coupling constants written in matrix form in the generation space is not contained in the generalized gauge field in our formalism. The last section is devoted to concluding remarks. Appendix includes some remarks of our algebraic rules in NCG.

2 Basic Settings

In this section we slightly change our previous formulation [8] to construct the gauge theory in NCG on the discrete space in order to correspond with the incorporation of the strong interaction in the standard model. This theme has been already treated in Ref.[12] with the formalism in which the generalized gauge field has the direct product form of the color and flavour gauge fields, and the fermion field is taken for each sheet of the discrete space to contain leptons and quarks all together in one form. The idea to take the fermion field in such a form is inspired by Sogami [13] which defines the generalized covariant derivative with gauge and Higgs fields operating on quark and lepton fields. In this article, we change the formation of leptons and quarks in a fermion field to match with the fermion field of SO(10) GUT though the anti-fermions are discarded. As a result, we have to devise to construct the generalized gauge field in order to give the correct Dirac
Lagrangian for fermion sector, which is the purpose of this section.

In Ref. [12], the matrix $K$ in Ref. [2] was introduced to explain the generation mixing between leptons and quarks. However, $K$ has no necessity to be introduced if we regard $c_Y$ in the covariant spinor one-form is related to the Yukawa coupling constant expressed in matrix form of generation space, and moreover $K$ spoils the beautiful relation between the Higgs and charged gauge boson masses in the limit of equal coupling constants for both sheets. Thus, we exile in this article the generation mixing matrix $K$ which is indispensable in Ref. [2] to obtain the meaningful Higgs potential.

Let us start with the equation of the generalized gauge field $A(x, y)$ written in one-form on the discrete space $M_4 \times Z_2$. We modify it in the original form [6] to take account of the strong interaction in such a way that

$$A(x, y) = \sum_i a_i^\dagger(x, y) d a_i(x, y) + \sum_j b_j^\dagger(x, y) d b_j(x, y),$$

(2.1)

where $x$ and $y (= \pm)$ are the variables in the Minkowski space $M_4$ and in the discrete space $Z_2$, respectively. $a_i(x, y)$ and $b_j(x, y)$ are the square-matrix-valued functions and commute with each other. $i$ and $j$ are variables of the extra internal space which we can not now identify what it is. As stated later, $\sum_i a_i^\dagger(x, y) d a_i(x, y)$ and $\sum_j b_j^\dagger(x, y) d b_j(x, y)$ correspond with the flavor and color gauge fields, respectively. These equations of gauge fields are very similar to the effective gauge field in Berry phase [10], which might lead to the identification of this internal space. Now, we simply regard $a_i(x, y)$ and $b_j(x, y)$ as the more fundamental fields to construct gauge and Higgs fields though they have only mathematical meaning. $a_i(x, y)$ and $b_j(x, y)$ never appear in final stage. $d$ in Equation (2.1) is the generalized exterior derivative defined as follows.

$$d a_i(x, y) = (d + d_\chi) a(x, y) = (d + d_\chi) a(x, y),$$

$$d a_i(x, y) = \partial_\mu a_i(x, y) d x^\mu,$$

$$d_\chi a_i(x, y) = [-a_i(x, y) M(y) + M(y) a_i(x, -y)] \chi,$$

$$d b_j(x, y) = \partial_\mu b_j(x, y) d x^\mu$$

(2.2)

$$d b_j(x, y) = \partial_\mu b_j(x, y) d x^\mu$$

(2.3)

$$d_\chi b_j(x, y) = 0.$$  

(2.4)

Here $d x^\mu$ is ordinary one-form basis, taken to be dimensionless, in Minkowski space $M_4$, and $\chi$ is the one-form basis, assumed to be also dimensionless, in the discrete space $Z_2$. We have introduced $x$-independent matrix $M(y)$ whose hermitian conjugation is given by $M(y)^\dagger = M(-y)$. The matrix $M(y)$ turns out to determine the scale and pattern of the spontaneous breakdown of the gauge symmetry. Thus, Equation (2.3) means that the color symmetry of the strong interaction does not break spontaneously.
In order to find the explicit forms of gauge and Higgs fields according to Equations (2.1) and (2.3), we need the following important algebraic rule of non-commutative geometry.

\[ f(x,y)\chi = \chi f(x,-y), \]  

(2.6)

where \( f(x,y) \) can be the quantity defined on the discrete space such as \( a_i(x,y) \), gauge field, Higgs field or fermion fields. It should be noticed that Equation (2.6) never expresses the relation between the matrix elements of \( f(x,+) \) and \( f(x,-) \) but insures the product between the fields expressed in differential form on the discrete space, which can be easily seen in the calculation of the wedge product \( A(x,y) \wedge A(x,y) \). Eq.(2.6) realizes the non-commutativity of our algebra in the geometry on the discrete space \( M_4 \times Z_2 \). In Appendix we will explain the meaning of Eq.(2.6) and some other rules in our NCG scheme in more detail. Using Eq.(2.6) and some other algebraic rules, \( A(x,y) \) is rewritten as

\[ A(x,y) = A_\mu(x,y)dx^\mu + \Phi(x,y)\chi + G_\mu(x)dx^\mu, \]  

(2.7)

where

\[
A_\mu(x,y) = \sum_i a_i^\dagger(x,y)\partial_\mu a_i(x,y), \\
\Phi(x,y) = \sum_i a_i^\dagger(x,y)\left(-a_i(x,y)M(y) + M(y)a_i(x,-y)\right), \\
G_\mu(x) = \sum_j b_j^\dagger(x)\partial_\mu b_j(x).
\]  

(2.8)

\( A_\mu(x,y), \Phi(x,y) \) and \( G_\mu(x) \) are identified the gauge field in the flavor symmetry, Higgs fields, and the color gauge field in the strong interaction, respectively. As known in Section 3, it should be noted that \( G_\mu(x) \) is built to commute with \( A_\mu(x,y) \) and \( \Phi(x,y) \) because \( b_j(x) \) commutes with both \( a_i(x,y) \) and \( M(y) \).

In order to identify \( A_\mu(x,y) \) and \( G_\mu(x) \) as true gauge fields, the following conditions have to be imposed.

\[
\sum_i a_i^\dagger(x,y)a_i(x,y) = 1, \\
\sum_j b_j^\dagger(x)b_j(x) = \frac{1}{g_s}.
\]  

(2.9)

where \( g_s \) is a constant related to the corresponding coupling constant as shown later. In general, we can put the right hand side of the first equation in Eq.(2.9) to be \( 1/g_y \). However, we put it as it is to avoid the complexity.
Before constructing the gauge covariant field strength, we address the gauge transformation of \(a_i(x, y)\) and \(b_j(x)\) which is defined as
\[
a_i^g(x, y) = a_i(x, y)g_f(x, y),
\]
\[
b_j^g(x) = b_j(x)g_c(x),
\]  
where \(g_f(x, y)\) and \(g_c(x)\) are the gauge functions with respect to the corresponding flavor unitary group and the color SU(3)_c group, respectively. It should be noticed that \(g_c(x)\) can be taken to commute with \(a_i(x, y)\) and \(M(y)\) and at the same time \(g_f(x, y)\) is taken to commute with \(b_j(x)\). \(g_f(x, y)\) and \(g_c(x)\) commute with each other. Then, we can get the gauge transformation of \(A(x, y)\) to be
\[
A^g(x, y) = g_f^{-1}(x, y)g_c^{-1}(x)A(x, y)g_f(x, y)g_c(x)
\]
\[+ g_f^{-1}(x, y)dg_f(x, y) + \frac{1}{g_s} g_c^{-1}(x)dg_c(x),
\]
where use has been made of Eq.(2.1) and Equation (2.10), and as in Eq.(2.3),
\[
dg_f(x, y) = \partial_\mu g_f(x, y)dx^\mu + [-g_f(x, y)M(y) + M(y)g_c(x, -y)]\chi = \partial_\mu g_f(x, y)dx^\mu + \partial_y g_f(x, y)\chi.
\]

Using Equations (2.8) and (2.10), we can find the gauge transformations of gauge and Higgs fields as
\[
A_\mu^g(x, y) = g_f^{-1}(x, y)A_\mu(x, y)g_f(x, y) + g_f^{-1}(x, y)\partial_\mu g_f(x, y),
\]
\[
\Phi^g(x, y) = g_f^{-1}(x, y)\Phi(x, y)g_f(x, y) + g_f^{-1}(x, y)\partial_y g_f(x, y),
\]
\[
G_\mu^g(x) = g_c^{-1}(x)G_\mu(x)g_c(x) + \frac{1}{g_s} g_c^{-1}(x)\partial_\mu g_c(x).
\]

Equation (2.14) is very similar to other two equations and so it strongly indicates that the Higgs field is a kind of gauge field on the discrete space \(M_4 \times Z_2\). From Eqs.(2.12) and (2.14) it is rewritten as
\[
\Phi^g(x, y) + M(y) = g_f^{-1}(x, y)(\Phi(x, y) + M(y))g_f(x, -y),
\]
which makes obvious that
\[
H(x, y) = \Phi(x, y) + M(y)
\]
is un-shifted Higgs field whereas \(\Phi(x, y)\) denotes shifted one with the vanishing vacuum expectation value.
In addition to the algebraic rules in Equation (2.3) we add one more important rule that
\[ d_{\chi} M(y) = M(y) M(-y) \chi \]  
which yields together with Eq. (2.3) the nilpotency\(^{2}\) of the generalized exterior derivative \( d \):
\[ d^2 f(x, y) = (d^2 + d_{\chi}^2) f(x, y) = 0. \]  
With these considerations we can construct the gauge covariant field strength.
\[ \mathcal{F}(x, y) = F(x, y) + \mathcal{G}(x), \]  
where \( F(x, y) \) and \( \mathcal{G}(x) \) are the field strengths of flavor and color gauge fields, respectively and given as
\[ F(x, y) = dA(x, y) + A(x, y) \wedge A(x, y), \]
\[ \mathcal{G}(x) = dG(x) + g_s G(x) \wedge G(x), \]  
where it should be noted that \( dA(x, y) = \sum_i d a_i^\dagger(x, y) \wedge d a_i(x, y) \) and \( dG(x) = \sum_j d b_j^\dagger(x) \wedge d b_j(x) \) are followed because of the nilpotency of \( d \) and \( d_{\chi} \). We can easily find the gauge transformation of \( \mathcal{F}(x, y) \) as
\[ \mathcal{F}^g(x, y) = g^{-1}(x, y) \mathcal{F}(x, y) g(x, y), \]  
where \( g(x, y) = g_f(x, y) g_c(x) \). The algebraic rules defined in Equations (2.3), (2.6) and (2.9) yield
\[ F(x, y) = \frac{1}{2} F_{\mu\nu}(x, y) dx^\mu \wedge dx^\nu + D_{\mu} \Phi(x, y) dx^\mu \wedge \chi + V(x, y) \chi \wedge \chi, \]  
where
\[ F_{\mu\nu}(x, y) = \partial_{\mu} A_{\nu}(x, y) - \partial_{\nu} A_{\mu}(x, y) + [A_{\mu}(x, y), A_{\nu}(x, y)], \]
\[ D_{\mu} \Phi(x, y) = \partial_{\mu} \Phi(x, y) + A_{\mu}(x, y)(M(y) + \Phi(x, y)) - (\Phi(x, y) + M(y)) A_{\mu}(x, -y), \]
\[ V(x, y) = (\Phi(x, y) + M(y)) (\Phi(x, -y) + M(-y)) - Y(x, y). \]  
\(^{2}\) In calculating the nilpotency, the following rule is taken into account that whenever the \( d_{\chi} \) operation jumps over \( M(y) \), minus sign is attached, for example
\[ d_{\chi} \{ a(x, y) M(y) b(x, -y) \} = (d_{\chi} a(x, y)) M(y) b(x, -y) + a(x, y) (d_{\chi} M(y)) b(x, -y) - a(x, y) M(y) (d_{\chi} b(x, -y)) \]
$Y(x,y)$ in Eq.(2.20) is auxiliary field and expressed as

$$Y(x,y) = \sum_i a_i^\dagger(x,y)M(y)M(-y)a_i(x,y), \tag{2.25}$$

which may be independent or dependent of $\Phi(x,y)$ and/or may be a constant field. In contrast to $F(x,y)$, $G(x)$ is simply denoted as

$$G(x) = \frac{1}{2} G_{\mu\nu}(x)dx^\mu \wedge dx^\nu = \frac{1}{2} \{\partial_\mu G_\nu(x) - \partial_\nu G_\mu(x) + g_s[G_\mu(x), G_\mu(x)]\} dx^\mu \wedge dx^\nu. \tag{2.26}$$

With the same metric structure on the discrete space $M_4 \times Z_2$ as in Ref.[6] that

$$<dx^\mu, dx^\nu> = g^{\mu\nu}, \quad g^{\mu\nu} = \text{diag}(1, -1, -1, -1),$$

$$<\chi, dx^\mu> = 0,$$

$$<\chi, \chi> = -\alpha^2 \tag{2.27}$$

we can obtain the gauge invariant Yang-Mills-Higgs Lagrangian (YMH)

$$\mathcal{L}_{YMH}(x) = -\text{Tr} \sum_{y=\pm} \frac{1}{g_y^2} <\mathcal{F}(x,y), \mathcal{F}(x,y)>$$

$$= -\text{Tr} \sum_{y=\pm} \frac{1}{2g_y^2} F_{\mu\nu}^\dagger(x,y)F^{\mu\nu}(x,y)$$

$$+ \text{Tr} \sum_{y=\pm} \frac{\alpha^2}{g_y^2} (D_\mu \Phi(x,y))^\dagger D^\mu \Phi(x,y)$$

$$- \text{Tr} \sum_{y=\pm} \frac{\alpha^4}{g_y^2} V^\dagger(x,y)V(x,y)$$

$$- \text{Tr} \sum_{y=\pm} \frac{1}{2g_y^2} G_{\mu\nu}^\dagger(x,y)G^{\mu\nu}(x), \tag{2.28}$$

where $g_y$ is a constant relating to the coupling constant of the flavor gauge field and $\text{Tr}$ denotes the trace over internal symmetry matrices including the color, flavor symmetries and generation space. The third term in the right hand side is the potential term of Higgs particle.

Here, the remark invoked by Sitarz [4] should be ordered about the term to be able to join Equation(2.28). He defined the new metric $g_{\alpha\beta}$ with $\alpha$ and $\beta$ running over 0, 1, 2, 3, 4 by $g^{\alpha\beta} = \text{diag}(+, -1, -1, -1, -1)$. The fifth index represents the discrete space $Z_2$. Then, $dx^\alpha = (dx^0, dx^1, dx^2, dx^3, \chi)$ is followed. The generalized field strength $F(x,y)$ in Equation(2.23) is written by $F(x,y) = F_{\alpha\beta}(x,y)dx^\alpha \wedge dx^\beta$ where $F_{\alpha\beta}(x,y)$ is denoted in Equation(2.24). Then, it is easily derived that $\text{Tr}\{g^{\alpha\beta}F_{\alpha\beta}(x,y)\}$ is gauge invariant. Thus, the term

$$|\text{Tr}\{g^{\alpha\beta}F_{\alpha\beta}(x,y)\}|^2 = \{\text{Tr}V(x,y)\}^\dagger\{\text{Tr}V(x,y)\} \tag{2.29}$$
can be added to Eq.(2.28). Some comments will be ordered in Section 3.

Let us turn to the fermion sector to construct the Dirac Lagrangian. We start to define the covariant derivative acting on the spinor field $\psi(x,y)$ which is the representation of the semi simple group of the corresponding flavor gauge group and $\text{SU}(3)_c$.

$$D\psi(x,y) = (d + A^f(x,y))\psi(x,y), \tag{2.30}$$

which we call the covariant spinor one-form and $A^f(x,y)$ is chosen to make $D\psi(x,y)$ gauge covariant. As known in Section 3, we specify $A^f_\mu(x,y)$ to be the differential representation for $\psi(x,y)$. Since the role of $d_\chi$ makes the shift $\Phi(x,y) \rightarrow \Phi(x,y) + M(y)$ as shown previously, we define also for fermion field

$$d_\chi\psi(x,y) = M(y)\chi\psi(x,y) = M(y)\psi(x,-y)\chi \tag{2.31}$$

which leads Equation(2.30) to

$$D\psi(x,y) = (\partial_\mu + A_\mu(x,y) + g_s G_\mu(x))\psi(x,y) dx^\mu + H(x,y)\psi(x,-y)\chi. \tag{2.32}$$

In deriving Equation(2.32), Equation(2.6) is used and $g_s$ is necessary for $D\psi(x,y)$ to be gauge covariant. In this context, $A^f_\mu(x,y) = A_\mu(x,y) + g_s G_\mu(x)$ is Lee algebra for the fermion representation of gauge group under consideration. As $\psi(x,y)$ is subjected to the gauge transformation

$$\psi^g(x,y) = g^{-1}(x,y)\psi(x,y), \tag{2.33}$$

$D\psi(x,y)$ becomes gauge covariant thanks to Equations(2.13), (2.14), (2.15) and (2.32).

$$D\psi^g(x,y) = g^{-1}(x,y)D\psi(x,y). \tag{2.34}$$

In addition, $d + A^f(x,y)$ is Lorentz invariant, and so $D\psi(x,y)$ is transformed as a spinor just like $\psi(x,y)$ against Lorentz transformation.

In order to obtain the Dirac Lagrangian for fermion sector, the associated spinor one-form is introduced as the counter-part of Equation(2.30) by

$$\hat{D}\psi(x,y) = \gamma_\mu \psi(x,y) dx^\mu + i O^\dagger(y) c^\dagger_Y(y) \psi(x,y)\chi, \tag{2.35}$$

where $c_Y(y)$ is a matrix assumed to be dimensionless constant related to the Yukawa coupling constant between Higgs field and fermions and $c_Y(y)^\dagger = c_Y(-y)$ is satisfied. The operator $O(y)A$ represents the right operation by $A$ for $y = +$ and the left operation by $A$ for $y = -$, and satisfies $(O^\dagger(y)A^\dagger)^\dagger = O(y)A$, so that

$$\{O(+A)B = BA, \quad \{O(-A)B = AB, \tag{2.36}$$
where $A$ and $B$ are appropriate square matrices. The introduction of the operator $O(y)$ is needed to insure the Hermiticity of the Yukawa coupling between Higgs and fermion fields. With the same inner products for spinor one-forms as in Ref. [6] that

$$
<A(x, y)dx^\mu, B(x, y)dx^{\nu}> = \bar{A}(x, y)B(x, y)g^{\mu\nu}, \\
<A(x, y)^c, B(x, y)^\chi> = -\bar{A}(x, y)B(x, y)\alpha^2,
$$

(2.37)

with vanishing other inner products we can get the Dirac Lagrangian.

$$
\mathcal{L}_D(x, y) = i \text{Tr} \left< \hat{D}\psi(x, y), D\psi(x, y) \right>
= i \text{Tr} \left[ \bar{\psi}(x, y)\gamma^\mu (\partial_\mu + A_\mu(x, y) + g_s G_\mu(x))\psi(x, y) \\
- \alpha^2 \bar{\psi}(x, y)\{O(y)c_{\gamma}(y)\}H(x, y)\psi(x, -y) \right],
$$

(2.38)

where Tr is also the trace over internal symmetry matrices including the color, flavor symmetries and generation space. The total Dirac Lagrangian is the sum over $y$:

$$
\mathcal{L}_D(x) = \sum_{y=\pm} \mathcal{L}_D(x, y),
$$

(2.39)

which is apparently invariant for the Lorentz and gauge transformations. Equations (2.28) and (2.38) along with Equation (2.39) are crucially important to reconstruct the spontaneously broken gauge theory.

With these preparations, we can apply the formalism proposed in this section to the standard model and compare it with the Sogami’s presentation [13].

### 3 Model Construction

We first specify the fermion field $\psi(x, y)$ in Eq. (2.30) with the existing leptons and quarks and then decide the generalized gauge field $A(x, y)$ in order to give the correct Dirac Lagrangian for the fermion sector in the standard model. Hereafter, the argument $x$ is often abbreviated if there is no confusion.

$$
\psi(x, +) = \begin{pmatrix}
  u_r^L \\
  u_d^L \\
  u_b^L \\
  \nu_e^L \\
  d_e^L \\
  d^b_L \\
  e_e^L
\end{pmatrix}, \quad \psi(x, -) = \begin{pmatrix}
  u_r^R \\
  u_d^R \\
  u_b^R \\
  d_e^R \\
  d^b_R \\
  e_e^R
\end{pmatrix},
$$

(3.1)

where subscripts $L$ and $R$ denote the left-handed and right-handed fermions, respectively and superscripts $r$, $g$ and $b$ represent the color indices. It should be noticed that $\psi(x, y)$
has the index for the three generation and so do the explicit expressions for fermions in the right hand sides of Equation(3.1). In the strict expressions, $u$, $d$, $\nu$ and $e$ in Eq.(3.1) should be replaced by

\[
\begin{align*}
    u & \rightarrow \begin{pmatrix} u \\ c \\ t \end{pmatrix}, &
    d & \rightarrow \begin{pmatrix} d \\ s \\ b \end{pmatrix}, &
    e & \rightarrow \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, &
    \nu & \rightarrow \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix},
\end{align*}
\tag{3.2}
\]

respectively. Thus, $\psi(x, \pm)$ is a vector in the 24-dimensional space.

In order to obtain the Dirac Lagrangian for fermion fields in Eq.(2.38) we specify $A_\mu(x,y)$, $\Phi(x,y)$ and $G_\mu(x)$ in Eq.(2.7) in the following.

\[
A_\mu(x,+) = -\frac{i}{2} \left\{ \sum_{k=1}^{3} \tau^k \otimes 1^4 A^k_{L\mu} + \tau^0 \otimes a B_\mu \right\} \otimes 1^3,
\tag{3.3}
\]

where $A^k_{L\mu}$ and $B_\mu$ are SU(2)$_L$ and U(1) gauge fields, respectively and so $\tau^k$ is the Pauli matrices and $\tau^0$ is $2 \times 2$ unit matrix. $1^3$ represents the unit matrix in the generation space and $a$ is the U(1) hypercharge matrix corresponding to $\psi(x, +)$ in Equation(3.1) and expressed as

\[
a = \begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\tag{3.4}
\]

\[
A_\mu(x,-) = -\frac{i}{2} b B_\mu \otimes 1^3,
\tag{3.5}
\]

where $b$ is the U(1) hypercharge matrix corresponding to $\psi(x, -)$ in Eq.(3.1) and so it is $8 \times 8$ diagonal matrix expressed in

\[
b = \text{diag} \left( \frac{\lambda^1}{3}, \frac{\lambda^2}{3}, \frac{\lambda^3}{3}, 0, -\frac{\lambda^4}{3}, -\frac{\lambda^4}{3}, -\frac{\lambda^5}{3}, -2 \right).
\tag{3.6}
\]

$G_\mu(x)$ is denoted by

\[
G_\mu(x) = -\frac{i}{2} \sum_{a=1}^{8} \tau^0 \otimes \lambda^a G^a_\mu \otimes 1^3,
\tag{3.7}
\]

where $\lambda^a$ is $4 \times 4$ matrix made of the Gell-Mann matrix $\lambda^a$ by adding 0 components to fourth line and column.

\[
\lambda^a = \begin{pmatrix}
\lambda^a & 0 \\
0 & \lambda^a \\
0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\tag{3.8}
\]
Higgs field $\Phi(x, y)$ is represented in $24 \times 24$ matrix by
\[
\Phi(+) = \begin{pmatrix} \phi_0^* & \phi^+ \\ -\phi^- & \phi_0 \end{pmatrix} \otimes 1^4 \otimes 1^3, \quad \Phi(-) = \begin{pmatrix} \phi_0 & -\phi^+ \\ \phi^- & \phi_0^* \end{pmatrix} \otimes 1^4 \otimes 1^3. \tag{3.9}
\]
Corresponding to Equation (3.9), symmetry breaking function $M(y)$ is given by
\[
M(+) = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \otimes 1^4 \otimes 1^3, \quad M(-) = M(+)\dagger. \tag{3.10}
\]
With these specifications, all quantities needed to give the explicit expression to $F(x, y)$ in Equation (2.18) can be explicitly written down as follows.
\[
F_{\mu \nu}(x, +) = -\frac{i}{2} \left\{ \sum_{k=1}^3 \tau^k \otimes 1^4 F^k_{\mu \nu} + \tau^0 \otimes aB_{\mu \nu} \right\} \otimes 1^3, \tag{3.11}
\]
\[
F_{\mu \nu}(x, -) = -\frac{i}{2} bB_{\mu \nu} \otimes 1^3, \tag{3.12}
\]
\[
G_{\mu \nu}(x) = -\frac{i}{2} \sum_a \tau \otimes \lambda^a G^a_{\mu \nu} \otimes 1^3, \tag{3.13}
\]
where
\[
F^k_{\mu \nu} = \partial_\mu A^k_{L \nu} - \partial_\nu A^k_{L \mu} + \epsilon^{klm} A^l_{L \mu} A^m_{L \nu}, \tag{3.14}
\]
\[
B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{3.15}
\]
\[
G^a_{\mu \nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_sf^{abc} G^b_\mu G^c_\nu. \tag{3.16}
\]
$D_\mu \Phi(x, y)$ in Equation (2.24) is represented in
\[
D_\mu \Phi(+) = (D_\mu \Phi(-))\dagger = \left\{ \partial_\mu h' - \frac{i}{2} \left( \sum_{k=1}^3 \tau^k A^k_{L \mu} h' + h'cB_\mu \right) \right\} \otimes 1^4 \otimes 1^3, \tag{3.17}
\]
where $h'$ and $c$ are given as
\[
h' = \begin{pmatrix} \phi_0^* + \mu & \phi^+ \\ -\phi^- & \phi_0 + \mu \end{pmatrix}, \quad c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.18}
\]
The matrix $c$ stems from
\[
\tau^0 \otimes a - b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes 1^4 = c \otimes 1^4 \tag{3.19}
\]
to insure that Higgs doublet $h = (\phi^+, \phi_0 + \mu)^t$ has plus one hypercharge and $\tilde{h} = i\tau^2 h^*$ minus one. $V(x, y)$ in Equation (2.21) is expressed in
\[
V(x, +) = (h'h'^\dagger - \mu^2) \otimes 1^4 \otimes 1^3, \tag{3.20}
\]
\[
V(x, -) = (h'^\dagger h' - \mu^2) \otimes 1^4 \otimes 1^3. \tag{3.20}
\]
It should be noticed that $Y(x, y)$ in Eq.(2.24) can be estimated by use of Eq.(3.10) to be

$$Y(x, \pm) = \sum_i a_i^\dagger(x, \pm) M(\pm) M(\mp) a_i(x, \pm) = \mu^2 \sum_i a_i^\dagger(x, \pm) a_i(x, \pm) = \mu^2 I^{24}, \quad (3.21)$$

where use has been made of Equation(2.9).

The quartic term of Higgs field remarked by Sitarz [4] is expressed in Equation(2.29). In general, this term can be added to Eq.(2.28). However, the minimal Lagrangian in non-commutative geometry seems to be Eq.(2.28), since Sitarz term is introduced by a different way from that in Eq.(2.28). Taking the inner product of field strength is the standard way to get the Lagrangian for gauge field. For the time being, we proceed without the Sitarz term. Putting above equations into Eq.(2.28) and rescaling gauge and Higgs fields we can obtain Yang-Mills-Higgs Lagrangian for the standard model as follows:

$$\mathcal{L}_{YMH} = -\frac{1}{4} \sum_{k=1}^{3} (F_{\mu\nu}^k)^2 - \frac{1}{4} B_{\mu\nu}^2$$

$$+ |D_\mu h|^2 - \lambda (h^\dagger h - \mu^2)^2$$

$$- \frac{1}{4} \sum_{a=1}^{8} G_{\mu\nu}^a G^{a\mu\nu}, \quad (3.22)$$

where

$$F_{\mu\nu}^k = \partial_\mu A^k_\nu - \partial_\nu A^k_\mu + g_\epsilon^{k\ell m} A^\ell_\mu A^m_\nu,$$  \quad (3.23)

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$  \quad (3.24)

$$D^\mu h = [ \partial_\mu - i/2 (\sum_k \tau^k g A^k_{\ell\mu} + \tau^0 g' B_\mu) ] h, \quad h = \left( \begin{array}{c} \phi^+ \\ \phi_0 + \mu \end{array} \right) \quad (3.25)$$

$$G_{\mu\nu}^a = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_c \tau^{abc} G^b_\mu G^c_\nu,$$  \quad (3.26)

with

$$g^2 = \frac{g_s^2}{12}, \quad g'^2 = \frac{2g_s^2 g_\epsilon^2}{3g_s^2 Tr a^2 + 3g_s^2 Tr b^2} = \frac{g_+^2 g_-^2}{16g_+^2 + 4g_-^2},$$  \quad (3.27)

$$\lambda = \frac{g^2 g'^2}{24(g_+^2 + g_-^2)},$$  \quad (3.28)

$$g_\epsilon^2 = \frac{g_+^2 g_-^2}{6(g_+^2 + g_-^2)},$$  \quad (3.29)

Eq.(3.27) yields the Weinberg angle with the parameter $\delta = g_+/g_-$ to be

$$\sin^2 \theta_w = \frac{3}{4(\delta^2 + 1)}.$$  \quad (3.30)

The gauge transformation affords the Higgs doublet $h$ to take the form that

$$h = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ \eta + v \end{array} \right) \quad (3.31)$$
which makes possible along with Equations (3.22)∼(3.28) to expect the gauge boson and Higgs particle masses.

\[ m_w^2 = \frac{1}{48} g^2 v^2, \]  
\[ m_Z^2 = \frac{1 + \delta^2}{12(4\delta^2 + 1)} g^2 v^2, \]  
\[ m_H^2 = \frac{1}{12(\delta^2 + 1)} g^2 v^2. \]  
(3.32)  
(3.33)  
(3.34)

These estimations are only valid in the classical level. However, we are tempted to compare them with the experimental values. At first, in the limit of equal coupling constant \( g_+ = g_- \), \( \sin^2 \theta_w = 3/8 \) is followed, which is the same value as in \( SU(5) \) or \( SO(10) \) GUT. This fact makes possible to expect that the limit of \( g_+ = g_- \) or \( \delta = 1 \) yields the relations that hold at the grand unification scale. At the same limit, Equations (3.32)∼(3.34) become

\[ m_w = \frac{1}{\sqrt{48}} g_+ v, \quad m_Z = \frac{1}{\sqrt{30}} g_+ v, \quad m_H = \frac{1}{\sqrt{24}} g_+ v. \]  
(3.35)

From Equation (3.35) we can get the mass relation \( m_H = \sqrt{2} m_w \) which is expected to hold at the energy of GUT scale. In general, by eliminating \( \delta \) and \( g_+v \) from Equations (3.30), (3.32) and (3.34) the following interesting relation is extracted.

\[ m_H = \frac{4}{\sqrt{3}} m_w \sin \theta_w. \]  
(3.36)

Inserting the experimental values \( m_w = 79.9 \text{GeV} \) and \( \sin^2 \theta_w = 0.233 \) we can obtain the tentative value of the Higgs mass \( m_H = 89 \text{GeV} \) which is sufficiently low that it is within the energy range of the accelerator in very near future.

If Sitarz term Eq.(2.29) would be effective, Equations (3.34) and (3.36) would become useless. However, we can consider as follows. We can get the renormalizable Lagrangian for boson sector with no restrictions between coupling constants if the Sitarz term is taken into account. However, as stated above, we can consider Equation (3.35) is still held at the grand unification scale. Then, the renormalization group equation can be used to predict the Higgs mass with the initial condition in Equation (3.35) at the energy of GUT scale. This seems to be interesting theme which will be considered in the different article.

Let us turn to the construction of the Dirac Lagrangian for fermion sector. After the rescaling of the boson fields, corresponding with the specification of Eq.(3.1) we can write the covariant spinor one-form in Eq.(2.32) and the associated spinor one-form in Eq.(2.33) as

\[ D\psi(x, +) = \partial_\mu \otimes 1^8 - \frac{i}{2} (g \sum_{k=1}^3 \tau^k \otimes 1^4 A^i_{\nu, \mu} + \tau^0 \otimes ag'B_\mu). \]
\( \sum_{a=1}^{8} \tau^0 \otimes \lambda^a e G^a_{\mu} \) \( \otimes 1^3 \psi(x, +) dx^\mu + g_\mu h \otimes 1^4 \otimes 1^3 \psi(x, -) \chi \) \hfill (3.37) 

\[
\tilde{D}_L \psi(x, +) = 1^{24} \gamma_\mu \psi(x, +) dx^\mu + i \{ O^\dagger(+) c^\dagger_\nu \} \psi(x, +) \chi,
\]

and

\[
D_\psi(x, -) = \left[ 1^8 \partial_\mu - \frac{i}{2} \{ g \beta B_\mu \psi(x, -) + \sum_{a=1}^{8} \tau^0 \otimes \lambda^a e G^a_{\mu} \} \otimes 1^3 \psi(x, -) dx^\mu \\
+ g_e \sum_{a=1}^{8} \tau^0 \otimes \lambda^a \chi \right].
\]

\[
\tilde{D}_\psi(x, -) = 1^{24} \gamma_\mu \psi(x, -) dx^\mu + i \{ O^\dagger(-) c_\nu \} \psi(x, -) \chi.
\]

According to Equation (2.38) we can find with a special attention on Equation (2.36) the Dirac Lagrangian for the standard model as follows:

\[
L_D = \sum_{y = \pm} i < \tilde{D}_\psi(x, y), D_\psi(x, y) > \\
= i \bar{\psi}(x, +) \gamma^\mu \left[ 1^8 \partial_\mu - \frac{i}{2} \{ g \sum_{k=1}^{3} A_{L \mu}^k \otimes 1^4 + g' \tau^0 \otimes a B_\mu \\
+ g_e \sum_{a=1}^{8} \tau^0 \otimes \lambda^a G^a_{\mu} \} \right] \otimes 1^3 \psi(x, +) \\
+ i \bar{\psi}(x, -) \gamma^\mu \left[ 1^8 \partial_\mu - \frac{i}{2} \{ g' \beta B_\mu + g_e \sum_{a=1}^{8} \tau^0 \otimes \lambda^a G^a_{\mu} \} \right] \otimes 1^3 \psi(x, -) \\
- \bar{\psi}(x, +) h' \otimes 1^4 \otimes 1^3 g_\nu \psi(x, -) - \bar{\psi}(x, -) g_\nu^\dagger h' \otimes 1^4 \otimes 1^3 \psi(x, +),
\]

which is sufficient as the Dirac Lagrangian of the standard model with a Yukawa coupling constants \( g_\nu = \alpha^2 g_\mu c_\nu \) in matrix form given as

\[
g_\nu = \text{diag}(g^u, g^u, g^u, g^d, g^d, g^d, g^e).
\]

\( g^u, g^d, g^e \) in Equation (3.42) are complex Yukawa coupling constant written in \( 3 \times 3 \) matrix in generation space. Equation (3.42) yields the interaction Lagrangian between Higgs and fermion fields as

\[
L_{Yukawa} = - \bar{q}_L h \otimes 1^3 \otimes g^d d_R - \bar{q}_L \tilde{h} \otimes 1^3 \otimes g^u u_R - \bar{l}_L h \otimes g^e e_R - h.c.,
\]

where

\[
q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad l_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}
\]

with color indices abbreviated. In this notation it should be noted again that in the exact notation \( u, d, e \) and \( \nu \) are replaced by

\[
u \rightarrow \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}.
\]

\[
(3.45)
\]
If we diagonalize the complex Yukawa coupling constants $g^u$ and $g^d$, we can obtain the correct Dirac Lagrangian for the standard model which contains the generation mixing through the Kobayashi-Maskawa mixing matrix. $g^\nu$ is taken zero and $g^e$ is a matrix already diagonalized, which yield that there is no generation mixing for lepton family.

### 4 Concluding remarks

We present the new incorporation of the strong interaction in our NCG scheme to reconstruct the standard model. It begins with the formation of the fermion field which is extracted from that of SO(10) GUT by discarding the anti-fermion part. Then, the generalized gauge field including the left-handed, U(1) and color gauge fields is constructed in order to yield the correct Dirac Lagrangian for fermion sector in the standard model. Characteristic points is that all gauge fields and Higgs field are written in $24 \times 24$ matrices according to 24-dimensional vector of the fermion field. The gauge fields are constructed so as to commute with each other.

The generation mixing is taken into account through the associated spinor one-form in Equation(2.35) so that it does not give any effect on the Yang-Mills-Higgs Lagrangian. As a result, we can estimate gauge boson and Higgs masses without any effect of the generation mixing matrices as in Equation(3.35), which is very characteristic point of our approach compared with other approaches [2], [3], [13]. For example, Sogami [13] estimated the Higgs mass around $m_H = \sqrt{2} m_t$ with the top quark mass $m_t$ since his generalized covariant derivative contains the complex Yukawa couplings between fermion and Higgs fields. $m_H = \sqrt{2} m_t$ is a characteristic prediction in such approach, which is contractive to our result $m_H = (4/\sqrt{3}) m_w \sin \theta_w$. Though these predictions are only valid in tree level, it is expected that not so much differences seem to appear even in quantum effect included. Thus, Higgs search experiments in very near future shall judge which is superior to other on the premise that NCG approach is true understanding of the Higgs mechanism.

If we include in the Yang-Mills-Higgs Lagrangian the quartic term of Higgs field due to Sitarz [4], Yang-Mills-Higgs Lagrangian is free from any restriction for coupling constants. Then, we can reproduce the renormalizable Lagrangian for boson sector of the standard model on our NCG approach. In the limit of $g_+ = g_-$, we can get the prediction of $\sin^2 \theta_w = 3/8$, which suggests that our Yang-Mills-Higgs Lagrangian in this limit and with the vanishing Sitarz term is the Lagrangian at the energy of GUT scale. The Higgs mass $m_H = \sqrt{2} m_w$ is followed in this limit. Thus, it is expected to calculate the Higgs
mass by use of the renormalization equation in the same way as the Weinberg angle was calculated. This calculation will be performed in very near future.

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A Appendix

In this Appendix, we explain rather unknown property of the extra-differential $\chi$ which is introduced with respect to the discrete space $Z_2$ in more extended version than that necessary in this article. $\chi$ plays a role that connects the fields on $y = +$ and $y = -$ spaces and so exhibits the following equation.

$$f(x, y)\chi = \chi f(x, -y), \quad (A.1)$$

where $f(x, y)$ can be gauge field, Higgs field or fermion field. The generalized gauge fields $A(x, y)$ is expressed as

$$A(x, y) = A_\mu(x, y)dx^\mu + \Phi(x, y)\chi. \quad (A.2)$$

Let $a_i(x, y)$ and $a_i(x, -y)$ in Eq.(2.4) be in general the square matrices with $(p, p)$ and $(q, q)$ types, respectively. Then, $M(y)$ is the $(p, q)$ type matrix to lead to the consistent calculation in Eq.(2.1). To be general, $p \neq q$ is required to reconstruct the realistic gauge theory with the spontaneous symmetry breaking such as the standard model or SU(5) GUT. For example, the symmetry of the standard model is SU(2)$_L \times$U(1)$_Y$, so that $p = 2$ and $q = 1$ should be followed in our previous formulation[1], [2]. Thus, $\Phi(x, y)$ is in general not square matrix. The reader might imagine why $\Phi(x, y)$ is not square matrix in spite that $A_\mu(x, y)$ is square matrix in Eq.(A.2). This is because there is no linear transformation between $dx^\mu$ and $\chi$ and so the components of $A_\mu(x, y)$ and $\Phi(x, y)$ never mix with each others. Our algebra in NCG on the discrete space is consistently constructed in conformity with this fact.

Furthermore, to justify Eq.(A.1), We take the example of $A(x, y) \wedge A(x, y)$. Eq.(A.1) leads this wedge product to

$$A(x, y) \wedge A(x, y) = (A_\mu(x, y)dx^\mu + \Phi(x, y)\chi) \wedge (A_\nu(x, y)dx^\nu + \Phi(x, y)\chi)$$

$$= A_\mu(x, y)A_\nu(x, y)dx^\mu \wedge dx^\nu$$
\[ A\mu(x, y)\Phi(x, y) - \Phi(x, y)A\mu(x, -y))dx^\mu \land \chi + \Phi(x, y)\Phi(x, -y)\chi \land \chi. \]  
\hspace{1cm} (A.3)

We can readily see the consistency of the matrix products in Equation (A.3) for \( y = + \) and \( y = - \) in the case of the standard model stated just above.

Eq. (A.1) was originally introduced by Sitarz [4] without the introduction of the symmetry breaking function \( M(y) \). Ref. [5] reconstructed the Weinberg-Salam theory according to Sitarz’s formalism where all fields such as \( A\mu(x, \pm) \) and \( \Phi(x, \pm) \) are same type square matrices. Recently, Konisi and Saito [14] indicated that the Higgs field \( \Phi(x, y) \) might become the unitary matrix to lead the vanishing Higgs potential. However, their conclusion cannot be applicable to our case because \( A(x, y) \) in our formalism expresses the different kind of gauge field for each sheet \( y = \pm \) and \( \Phi(x, y) \) is not in general the square matrix.

We can formulate our framework in slightly different way though it yields the same results as in this article, which may help one understand the true meaning of Equation (A.1). If we express the generalized gauge field \( A \) without the variable \( y \) in the following matrix form
\[ A = \begin{pmatrix} A\mu(x, +)dx^\mu & \Phi(x, +) \\ \Phi(x, -)\chi & A\mu(x, -)dx^\mu \end{pmatrix}, \]  
\hspace{1cm} (A.4)
we can calculate the wedge product of \( A \land A \) as in
\[ A \land A = \begin{pmatrix} A\mu(x, +)A\nu(x, +)dx^\mu \land dx^\nu + \Phi(x, +)\Phi(x, -)\chi \land \chi \\ -\Phi(x, +)A\mu(x, -)dx^\mu \land \chi \\ (A\mu(x, -)\Phi(x, -) - \Phi(x, -)A\mu(x, +))dx^\mu \land \chi + \Phi(x, -)\Phi(x, +)\chi \land \chi \end{pmatrix} \]  
\hspace{1cm} (A.5)
\[ f(x, y)\chi = \chi f(x, y) \]  
\hspace{1cm} (A.6)
in this case. Since Yang-Mills-Higgs Lagrangian \( L_{YMHH} \) is defined with the field strength \( F \) expressed in
\[ F = dA + A \land A \]  
\hspace{1cm} (A.7)
through the equation
\[ L_{YMHH} = \text{Tr} < F, F >, \]  
\hspace{1cm} (A.8)
we can find the same Lagrangian in Section 3. Thus, we can conclude that Equation (A.1) makes possible to perform the calculations in our formalism without the matrix product
as in Equation (A.3). Then, it becomes evident that Eq. (A.1) is never the relation between the matrix elements of \( f(x, \pm) \) and it realizes the non-commutativity of our algebra in our formalism.

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