Spectral Theory of Substitutions in $\mathbb{Z}^d$

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Abstract

In this paper, we generalize and develop results of Queffelec allowing us to characterize the spectrum of an aperiodic substitution in $\mathbb{Z}^d$ by describing the Fourier coefficients of mutually singular measures of pure type giving rise to the maximal spectral type of its Koopman representation. We note that this is done without the assumptions of primitivity or trivial height, and provides a simple algorithm for determining singularity to Lebesgue spectrum for such substitutions. This is used to show that the spectrum of any aperiodic bijective and commutative $\mathbb{Z}^d$ substitution on a finite alphabet is purely singular.

Additionally, we use the algorithm to show singularity of the spectrum for Queffelec’s noncommutative bijective substitution, as well as the Table tiling, answering an open question of Solomyak.

1 Introduction

Substitutions are of interest to us as models of aperiodic phenomena - for example, mathematical quasicrystals, see [5]. Here, we consider higher dimensional analogues of substitutions of constant length, or substitutions on $\mathbb{Z}^d$ which replace letters in a finite alphabet with a rectangular block of letters, and which we call $\mathbf{q}$-substitutions. As usual, we study the translation operator on the hull of a substitution, or the collection of all $\mathbb{Z}^d$-indexed sequences whose local patterns can be produced by the substitution. Our primary interest is the spectrum of the translation action, and our analysis is based on the work of Queffelec, which describes the spectrum of one-sided primitive and aperiodic constant length substitutions of trivial height in one dimension ($\mathbb{N}$-indexed sequences.) Our formulation emphasizes the arithmetic properties of such substitutions and the recognizability properties afforded by aperiodicity. Using only these two assumptions, we are able to describe the spectrum of such a substitution to the degree that we can identify its discrete, singular continuous, and absolutely continuous spectral components as a sum of mutually singular measures of pure type with readily computable Fourier coefficients. This is done by using a representation consistent with the abelianization (which gives rise to the substitution matrix) and separates the analysis into two parts: the correlation measures, which depend on the position of letters in substituted blocks (which we call the configuration), and the spectral hull, a convex set which depends only on the abelianization.

An important notion in the theory of mathematical quasicrystals is the usage of local rules for assembling structures which achieve global (aperiodic) order, related to the local assembly of materials at the atomic level. A result of Mozés [16], extended by Goodman-Strauss [11], asserts that self-similar tilings of $\mathbb{Z}^d$ for $d > 1$ can be obtained by local pattern matching rules, whereas when $d = 1$ this assertion is false. Thus, higher dimensional substitution systems are naturally of interest as models of aperiodic physical phenomena. Moreover, an argument of Dworkin [7] shows that the mathematical diffraction spectra of a dynamical system is included in the dynamical spectrum (that of the Koopman operator, which we study here), and so our extension of Queffelec’s results to $\mathbb{Z}^d$ substitutions is of interest as it allows us to describe diffraction (and dynamical) properties of higher dimensional mathematical quasicrystals.

As Queffelec’s work in [18] is nearing three decades in age, and the study of substitutions an active area of research, it is surprising that the latter half of her work has received so little attention. Although a second edition was printed in 2010, little revision was given to chapters 7 thru 11, on which much of our research is based. Moreover, the text contains several gaps and errors which while none are serious (aside from her analysis of a substitution we discuss in example 5.3 still create a significant roadblock to understanding what we feel are the key ideas. Therefore, one of the goals of this paper is to provide a streamlined and self contained analysis in the hopes that her ideas can be further developed. As the proof of our main theorem [1.8] requires details unrelated to the application of the results we aim to emphasize, we have extended and condensed the relevant parts of [18] into proofs appearing in an appendix, allowing the reader to apply the results contained in this paper more readily to the task of computing the spectrum of aperiodic $\mathbf{q}$-substitutions.

The paper is organized as follows: in the next section, we discuss some preliminaries for the theory of $\mathbb{Z}^d$-indexed substitution dynamical systems, including their invariant measures and spectrum. In [8] we
develop some arithmetic properties of \(q\)-substitutions which, along with an aperiodicity result of Mossé extended by Solomyak, allow us to prove a recursive identity related to invariant measures which arise when studying their spectral theory. In \[\text{[11]}\] we use two key results generalized from Queffelec’s work (proved in \[\text{[6]}\]) to separate the spectral characterization problem of an aperiodic \(q\)-substitution into a study of its correlation measures and its spectral hull; see theorems \[\text{[4.4]}\] and \[\text{[4.8]}\]. There, we use the measure recursion to explicitly compute Fourier coefficients of the correlation measures (theorem \[\text{[4.9]}\]), and describe an algorithm for characterizing the spectral hull (lemma \[\text{[4.7]}\]) of a \(q\)-substitution. These combine (via Queffelec’s theorems) to give explicit formulae for the Fourier coefficients of measures of pure type which determine the spectrum of an aperiodic \(q\)-substitution. Finally, in \[\text{[15]}\] we compute several examples, including the spectrum of the Table substitution tiling \[\text{[21]}\] as well as correct an example of Queffelec, mistakenly identifying Lebesgue component in its spectrum (see example \[\text{[5.3]}\]). Further, we show that aperiodic bijective and commutative \(q\)-substitutions have purely singular (to Lebesgue) spectrum, generalizing a result of Baake and Grimm in \[\text{[1]}\]. In \[\text{[6]}\] we prove a diagonalization result for operator valued measures ergodic for a transformation of a compact abelian group, allowing us to prove theorem \[\text{[4.8]}\]. There, we also briefly discuss matrix Riesz Products, and show a singularity result which is used in the proof of theorem \[\text{[5.2]}\].

Before beginning, we make a remark on a particularly important notational convention. We will be working extensively with the ring \(\mathbb{Z}^d\), and wish to do so by interpreting all operations and relations on \(\mathbb{Z}^d\) coordinatwise. We represent \(\mathbb{Z}^d\) integers in boldface \(i, j, k\), etc., and denote the components of \(k\) with \(k_i\) for \(1 \leq i \leq d\). Note that notation such as \(k_0\) refers to a \(\mathbb{Z}^d\) integer; should we need to refer to its coordinates, we will use \((k_n)_j\) for \(1 \leq j \leq d\). The symbols \(0, 1\) represent the \(\mathbb{Z}^d\) integers all of whose coordinates are \(0, 1\) respectively. For \(1 \leq i \leq d\), let \(1_i\) be the integer \(0\) in all coordinates but the \(i\)-th, where it is \(1\), so that 
\[1 = \sum_{i=1}^{d} 1_i.\] 
For \(a, b \in \mathbb{Z}^d\), the inequalities \(a < b\) or \(a \leq b\) should be interpreted as holding in each coordinate simultaneously, i.e. \(a_i < b_i\), for \(1 \leq i \leq d\), and defines a partial order on \(\mathbb{Z}^d\). Additionally, whenever \(a \leq b\), the interval notation \([a, b]\) or \([a, b)\) should be interpreted componentwise in the usual way, giving rise to (semi-)rectangles in \(\mathbb{Z}^d\). For \(t \in \mathbb{Z}\), we have \(ta = (ta_1, \ldots, ta_d)\), with \(a + b\) and \(ab\) representing the usual sum and componentwise product, and we define \(\frac{a}{b} \in \mathbb{Q}^d\) as the componentwise quotient for \(b \geq 1\). The dot product will be denoted using the notations \((a, b) = a \cdot b = \sum a_i b_i\), as usual. Finally, for \(z \in \mathbb{T}^d\) write \(z^n = (z_{1}^n, \ldots, z_d^n)\), so that \(z^n \in \mathbb{T}^d\). With these notations established, we proceed to the next section where we describe the full \(\mathbb{Z}^d\)-shift on a finite alphabet and substitutions of constant length in \(\mathbb{Z}^d\), which we call \(q\)-substitutions.

## 2 Substitution Dynamical Systems

Fix a dimension \(d \geq 1\). An **alphabet** is a finite set \(\mathcal{A}\) consisting of at least 2 **letters** which we will frequently denote with symbols \(\alpha, \beta, \gamma, \delta\). Often, we will consider alphabets of the form \([0, 1, \ldots, s-1]\), and \(s\) will always refer to the size of the alphabet. Consider the collection of all functions from \(\mathbb{Z}^d \to \mathcal{A}\), which can be identified with the product space \(\mathcal{A}^{\mathbb{Z}^d}\) of all \(\mathbb{Z}^d\)-indexed sequences with values in \(\mathcal{A}\). We refer to elements of \(\mathcal{A}^{\mathbb{Z}^d}\) as sequences, denoting them with letters \(\mathbf{A}, \mathbf{B}, \mathbf{C}\), etc. Endowing \(\mathcal{A}\) with the discrete topology, consider the topology of pointwise convergence on \(\mathcal{A}^{\mathbb{Z}^d}\), or equivalently the product topology when viewed as a sequence space. Addition on \(\mathbb{Z}^d\) gives rise to a \(\mathbb{Z}^d\)-action of commuting automorphisms, which act by translation on \(\mathcal{A}^{\mathbb{Z}^d}\) sending \(k \mapsto T^k\) where \(T^k : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}\) sends \(\mathbf{A}\) to the function \(T^k \mathbf{A}\) defined by \(T^k \mathbf{A}(j) := \mathbf{A}(j + k)\). We call this action the **shift** and denote it by \(T\). The pair \((\mathcal{A}^{\mathbb{Z}^d}, T)\) is an invertible topological dynamical system, the **full shift**, and we let \(\mathcal{B}\) denote its Borel \(\sigma\)-algebra of measurable sets.

By a **block**, we mean a map \(\omega\) from a finite subset of \(\mathbb{Z}^d\) into \(\mathcal{A}\), letting \(\mathcal{A}^+\) denote the collection of all blocks in \(\mathcal{A}^{\mathbb{Z}^d}\). Here, blocks differ from convention in two significant ways: they need not be contiguous, and they need not start at \(0\) index. Loosely, the reader can identify blocks with **compactly supported functions** in \(\mathbb{Z}^d\), a notion that can be made rigorous by adjoining an **empty letter** \(\ast\) to \(\mathcal{A}\), and so we define the **support** of a block \(\omega\) as the domain of \(\omega\) in \(\mathbb{Z}^d\), denoted as \(\text{supp}(\omega)\). We will routinely identify blocks with their **graphs**, or the image of the map \(j \mapsto (j, \omega(j)) \in \mathbb{Z}^d \times \mathcal{A}\), as this is often convenient and coincides with the tiling perspective of substitutions, as treated by Radin \[\text{[13]}\]. For \(\omega \in \mathcal{A}^+\) and \(\mathbf{A} \in \mathcal{A}^+ \cup \mathcal{A}^{\mathbb{Z}^d}\), we say \(\omega\) is **extended by** \(\mathbf{A}\) whenever \(\mathbf{A}\) extends \(\omega\) as a function into \(\mathcal{A}\), and say \(\omega\) is a **subword** of \(\mathbf{A}\) if \(T^k \omega\) is extended by \(\mathbf{A}\), for some \(k \in \mathbb{Z}^d\). Note that the subword relation is translation stable in either of its arguments, where as the
extends relation is not. If \( \omega \in \mathcal{A}^+ \) is a block, the cylinder over \( \omega \) is the collection

\[
[\omega] := \{ C \in \mathcal{A}^{zd} : C \text{ extends } \omega \} = \{ C \in \mathcal{A}^{zd} : C(j) = \omega(j) \text{ for } j \in \text{supp}(\omega) \}
\]

so that cylinders over blocks \( \mathcal{A}^+ \) correspond to the standard basis for the topology of \( \mathcal{A}^{zd} \). By discreteness of \( \mathcal{A} \), every cylinder is both open and closed making \( \mathcal{A}^{zd} \) totally disconnected, and compact by the Tychonoff theorem. As every \( \mathcal{A} \in \mathcal{A}^{zd} \) is in the intersection of the cylinders \( [\mathcal{A}_n] \) where \( \mathcal{A}_n \) is the subblock of \( \mathcal{A} \) supported in \([-n,1] \), this implies \( \mathcal{A}^{zd} \) is a perfect set. Thus, the full shift \( (\mathcal{A}^{zd}, T) \) is an invertible topological dynamical system, consisting of a \( \mathbb{Z}^d \)-action of commuting automorphisms acting on a Cantor set of functions \( \mathbb{Z}^d \to \mathcal{A} \).

A substitution is a map \( \mathcal{S} : \mathcal{A} \to \mathcal{A}^+ \), replacing each letter by a block. In the \( d = 1 \) case, substitutions of many types are considered, but we wish to consider the class of substitutions which generalize substitutions \( \mathcal{S} \) all the blocks \( \mathcal{Z} \) of \( \alpha, \gamma \) square matrix whose entry is the number of times \( \alpha \) appears in the block \( \mathcal{S} \) and subdivides mod 1, allowing \( \mathcal{S} \) to replace the letter at each index with the \([0, q]\)-block associated with it. This inflate and subdivision process extends \( \mathcal{S} \) to sequences over arbitrary subsets of \( \mathbb{Z}^d \), and identifies \( \mathbb{Z}^d \) with \( q\mathbb{Z}^d \times [0, q] \) and gives rise to many arithmetic properties of substitutions, which we discuss in \([3]\). We now show how to associate a subshift to a given substitution, after which we detail some essential results and preliminaries for the spectral theory of substitution subshifts.

The language \( L_S \subset \mathcal{A}^+ \) of a substitution is the collection of all blocks \( \omega \in \mathcal{A}^+ \) which appear as subwords of \( S^n(\gamma) \) for some \( n \in \mathbb{N} \) and \( \gamma \in \mathcal{A} \). Associating blocks with their graphs, this can be thought of as the collection of all possible finite patterns attainable by the substitution; by definition, it is both shift-invariant and closed under the action of \( \mathcal{S} \). The substitution subshift of \( \mathcal{S} \) is the collection \( X_S \) consisting of those sequences in \( \mathcal{A}^{zd} \), all of whose subblocks appear in the language of \( \mathcal{S} \). Note that \( X_S \) is always nonempty, see \([3,3]\). By the reduced language of \( \mathcal{S} \), we mean the collection of all blocks in \( \mathcal{A}^+ \) that appear in some word \( \mathcal{A} \in X_S \), and can be strictly smaller than the language; as this has no effect on the measure theoretic or topological structure of \( X_S \), we will often assume our language is reduced.

One checks that \( X_S \) is a nonempty closed and shift-invariant subset of \( \mathcal{A}^{zd} \), and \((X_S, T)\) forms a topological subshift of the full shift \( \mathcal{A}^{zd} \) called a substitution dynamical system. A useful property of substitution subshifts is their independence of the iterate used: for \( n > 0 \), \( X_{S^n} = X_S \), which is sometimes referred to as telescope invariance, see \([3]\) Lemma 2.9]. Note that \( \mathcal{S} \) restricts from \( \mathcal{A}^{zd} \) to a map on \( X_S \), and the Borel \( \sigma \)-algebra for \( X_S \) is generated by the cylinders over blocks in the reduced language \( L_S \). As our goal is to study the spectral theory of substitution dynamical systems, we adopt the convention that all cylinders are intersected with \( X_S \). Finally, let \( \mathcal{M}(X_S) \) denote the space of \( T \)-invariant (\( \mu = \mu \circ T^k \) for all \( k \in \mathbb{Z}^d \)) Borel probability measures on \( X_S \); note that \( \mathcal{M}(X_S) \) is a compact convex set, the extreme points of which are ergodic and denoted by \( \mathcal{E}_S \).

### 2.1 Invariant Measures

Consider the vector space \( \mathbb{C}^\mathcal{A} \) of formal linear combinations in the letters of \( \mathcal{A} \), with standard basis \( e_\alpha \), for \( \alpha \in \mathcal{A} \). Given a substitution \( \mathcal{S} \) on \( \mathcal{A} \), its substitution matrix \( M_S \in \mathbb{M}_\mathcal{A}(\mathbb{C}) \) is the nonnegative \( \mathcal{A} \)-indexed square matrix whose \( \alpha, \gamma \) entry is the number of times \( \alpha \) appears in the word \( \mathcal{S}(\gamma) \). This representation is often called the abelianization of \( \mathcal{S} \). The expansion of a substitution is the spectral radius of its substitution matrix; in the case of \( q \)-substitutions, the expansion is \( Q = \text{Card}(0, q) \) as the substitution matrix is \( Q \)-column stochastic, see \([3,2]\). A substitution is primitive if for some \( n > 0 \), \( \alpha \) appears in the block \( \mathcal{S}^n(\gamma) \), for every \( \alpha, \gamma \in \mathcal{A} \). Thus, the substitution matrix of a primitive substitution is a primitive matrix: there exists some \( n > 0 \) so that \( M_S^n \) has strictly positive entries. By the Perron-Frobenius theorem, the spectral radius of a primitive matrix is a simple eigenvalue with a strictly positive eigenvector \( u \), normalized to a probability vector and called the Perron vector of \( \mathcal{S} \), see \([10]\) Theorem 8.1.2.2.

The following result of Michel \([14]\) relates the invariant measures of a substitution subshift to the eigenspace of its substitution matrix in the primitive case, see \([14]\) Lemma 1.5] for the extension to \( \mathbb{Z}^d \).

**Theorem 2.1** (Michel). If \( \mathcal{S} \) is a primitive \( q \)-substitution, then \((X_S, T)\) is uniquely ergodic with measure \( \mu \). If \( u = (u_\alpha)_{\alpha \in \mathcal{A}} \) is the Perron vector of \( M_S \), we have \( \mu(\mathcal{S}^n(\alpha)) = Q^{-n}u_\alpha \) for every \( n \geq 0, \alpha \in \mathcal{A} \).
As the collection of cylinders over the language of a substitution generate the Borel $\sigma$-algebra of its subshift, the measure $\mu$ is uniquely determined by the above formula. In the case of non-primitive substitutions, there will be a nonempty proper subset $A_0 \subsetneq A$ such that $S^h$ restricts to a substitution on the subalphabet $A_0$, for some $h > 0$. Using this, we can express any substitution in a primitive reduced form, see [13 §10.1.1].

Proposition 2.2. Let $S$ be a substitution on $A$. There is an $h > 0$ and partition $\{E_1, \ldots, E_K, T\}$ of $A$ with

- $S^h : E_j \rightarrow E_j^+$ is primitive for each $1 \leq j \leq K$,

- $\gamma \in T$ implies $S^h(\gamma) \notin T^+$

Clearly, $K = 1$ and $T = \emptyset$ if and only if $S$ is primitive. We call the partition $\{E_1, \ldots, E_K, T\}$ an ergodic decomposition of $S$, its members $E_j$ the ergodic classes of $S$, and the remaining letters form $T$, its transient part, compare [13 §10.1]. As the transient part $T = A \setminus \bigcup E_j$, it suffices to specify the ergodic classes $E = \{E_1, \ldots, E_K\}$, and we will do this frequently; that is, if the ergodic classes do not exhaust the alphabet, we simply include the remaining letters in the transient part. Restricting $S$ to each ergodic class gives the primitive components of $S$, and we call $h$ its index of imprimitivity.

Note that the primitive reduced form can be derived via the substitution matrix, using Perron-Frobenius theory as these matrices are nonnegative, see [10, §8]. This perspective has a useful consequence in the case of $q$-substitutions: let $M$ be the substitution matrix of a $q$-substitution $S$ and assume its index of primitivity $h = 1$. The above proposition states that by a permutation of the alphabet, we can write $M$ in the following block form

$$M_S = \begin{pmatrix}
M_1 & \cdots & M_1 \\
& \ddots & \\
& & M_K \\
& & & M_T
\end{pmatrix}$$

where all unrepresented blocks are $0$, and the $M_1, \ldots, M_K$ are primitive square matrices. As this is obtained via a permutation of the basis, this matrix remains $Q$-column stochastic and an application of [13 Thm 8.6] tells us that the spectral radius of $M_T$ is strictly less than $Q$. As a result, the only $Q$-eigenvectors of $M_S$ are those corresponding to the Perron vectors of the diagonal blocks $M_1, \ldots, M_K$ which correspond to the primitive components of $S$, a fact which will be useful in the proof of lemma 4.7.

A substitution $S$ is aperiodic if $X_S$ contains no shift-periodic elements, or if $T^kA = A$ implies $k = 0$ for all $A \in X_S$. In [4], Cortez and Solomyak use results of Bezuglyi and others in [4] characterizing the invariant measures on aperiodic and stationary Bratteli diagrams, to extend Michel’s theorem 2.1 to aperiodic $q$-substitutions. The result as stated here is a corollary of [4 Theorem 3.8] and is adjusted for our purposes:

Theorem 2.3 (Cortez and Solomyak). Let $S$ be an aperiodic $q$-substitution on $A$. Then the ergodic measures of $X_S$ are the uniquely ergodic measures of its primitive components.

Proof. As $M_S$ is $Q$-column stochastic, we apply [10 §8 Thm 6] which tells us that the eigenvalues corresponding to the transient diagonal blocks are strictly dominated by the expansion of the substitution. Thus, the distinguished eigenvalues, see [4], are those corresponding to $M_S$ restricted to its ergodic classes, which is the substitution matrix of the primitive components of $S$, and the result follows from [4 Corollary 5.6].

In particular, there are at most $\text{Card}(A)$ ergodic measures for any substitution subshift and the study of invariant measures for $q$-substitutions reduces to the primitive case. Note that one can express an invariant measure as a convex sum of the uniquely ergodic measures for the primitive components and thus extend the identity of theorem 2.1 to nonprimitive subshifts. Moreover, the vector $u = (\mu([a]))_{a \in A}$ will be the same convex combination of mutually orthogonal Perron vectors corresponding to the primitive components of $S$.

2.2 Spectral Theory

We now discuss some basics of spectral theory for substitution subshifts, and show that a similar relationship to primitivity holds in that context as well. Given a measure $\mu \in \mathcal{M}(X_S)$, the Koopman representation of $(X_S, T, \mu)$ is the unitary operator $f \mapsto f \circ T$ on $L^2(\mu)$, the space of complex valued ($\mu$) square integrable functions on $X_S$, the substitution subshift. For each pair $f, g \in L^2(\mu)$, Bochner’s theorem gives us a complex
Borel measure $\sigma_{f,g}$ on the $d$-Torus $\mathbb{T}^d$ (the $d$-fold product of the unit circle) called the spectral measure for $f, g$, with Fourier coefficients satisfying the identity
\[
\hat{\sigma}_{f,g}(k) := \int_{\mathbb{T}^d} z_1^{-k_1} \cdots z_d^{-k_d} d\sigma_{f,g} = \int_X f \circ T^{-k} d\mu
\]
and for which $\sigma_f := \sigma_{f,f}$ is a positive measure for every nonzero $f \in L^2(\mu)$. By the spectral theorem for unitary operators, there is a maximal function $F \in L^2(\mu)$ such that for any $g \in L^2(\mu)$, the spectral measure for $g$ is absolutely continuous with respect to that of $F$, or $\sigma_g \ll \sigma_F$, and every measure $\nu \ll \sigma_F$ is the spectral measure of some $g \in L^2(\mu)$. Although the maximal function is not in general unique, all maximal functions have equivalent spectral measures, and thus have the same type. Here, the type of a measure is precisely the set of (complex valued) Borel measures equivalent to it, with equivalence defined relative to total variations of the respective measures. We denote the type of $\sigma_F$ by $\sigma_{\text{max}}(\mu)$, and it is called the maximal spectral type of the Koopman representation of $(X_S, T, \mu)$. Let $\mathcal{E}_S$ denote the extreme points of $\mathcal{M}(X_S)$. In light of theorem 2.3 we define the spectrum of $S$ as the sum of the maximal spectral types of its primitive components, writing
\[
\sigma_{\text{max}} := \sum_{\mu \in \mathcal{E}_S} \sigma_{\text{max}}(\mu)
\]
noting that this notion is well defined up to measure equivalence. Using identity (2) and the Krein-Milman theorem, one can see that $\sigma_{f,g}(\mu) \ll \sigma_{\text{max}}$ for every $\mu \in \mathcal{M}(X_S)$ and $f, g \in L^2(\mu)$, justifying the definition. By the Radon-Nikodym theorem, we can write $\sigma_{\text{max}} \sim \sigma_d + \sigma_{ac} + \sigma_{sc}$, separating $\sigma_{\text{max}}$ into discrete, singular continuous, and absolutely continuous components with respect to normalized Lebesgue measure on $\mathbb{T}^d$. 

In the case of $q$-substitutions and their substitution subshifts, the discrete component is well understood: it is a multiplicative subgroup of $\mathbb{T}^d$ corresponding to $q$ and the height of a substitution, as described by Dekking in [6] where he also provided a complete classification of the pure discrete case: see section 5.2 for more details on Dekking’s criteria. The study of the continuous spectrum is largely based on the work of Queffelec, in which she relates the maximal spectral type of a substitution to correlation measures: for $\alpha, \beta \in \mathcal{A}$, the correlation measure $\sigma_{\alpha,\beta}$ is the spectral measure for the pair of indicator functions of cylinders $[\alpha]$ and $[\beta]$, relative to the Koopman representation.

Due to Dekking’s work, our interest is primarily in the continuous spectrum of $S$, and distinguishing the purely singular case from those with Lebesgue components in their spectrum. Note that this is a nontrivial problem: the Thue-Morse substitution is an example of a substitution of constant length 2 on 2 letters with purely singular spectrum, possessing both discrete and continuous components, whereas the Rudin-Shapiro substitution has a discrete and absolutely continuous component, with no singular continuous spectrum. Moreover, one can form a substitution product of Thue-Morse and Rudin-Shapiro to obtain a substitution with discrete, singular continuous, and absolutely continuous components in the spectrum, which we consider in example 5.10; see also [1] §2. For higher dimensional examples, we have the work of Baake and Grimm which shows a large class of substitutions on 2 symbols to be purely singular to Lebesgue spectrum, see [2], as well as Frank’s paper [4] describing another large class of substitutions with Lebesgue component.

3 Analysis of q-Substitutions

In this section, we discuss some properties of $q$-substitutions which simplify their spectral analysis significantly. We begin by establishing some basic arithmetic tools which provide for an alternative definition for $q$-substitutions based on their arithmetic properties, and in the one-dimensional case is the most significant distinction between constant and nonconstant length substitutions. Then, we discuss some topological and measurable consequences of aperiodicity allowing us to relate the maximal spectral type to the correlation measures $\sigma_{\alpha,\beta}$, which we examine in [4].

3.1 Arithmetic Base $q$ in $\mathbb{Z}^d$

We take a moment to establish some basic arithmetic notions, most of which are consequences of the classical division algorithm on $\mathbb{Z}$. It may be helpful to think of the case $q = 10$, where much of these concepts coincide with notions in decimal arithmetic. Recall that, for $a \geq 0$,
\[
[0,a) = \{ j = (j_i)_{1 \leq i \leq d} \in \mathbb{Z}^d : 0 \leq j_i < a_i \text{ for } 1 \leq i \leq d \}.
\]
Fix $q > 1$, then for every $n \in \mathbb{N}$ and each $k \in \mathbb{Z}^d$, the division algorithm applied componentwise to $\mathbb{Z}^d$ provides a unique pair $[k]_n \in [0, q^n)$ and $[k]_n \in \mathbb{Z}^d$ satisfying
\[ k = [k]_n + [k]_n q^n \]

The map $[\cdot]_n : \mathbb{Z}^d \rightarrow [0, q^n)$ denotes the remainder, and $[\cdot]_n : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ the quotient, under division modulo $q^n$. For $k \in \mathbb{Z}^d$ and $n \geq 0$, letting
\[ k_n := [[k]_n]_1 = [[k]_{n+1}]_n \]
gives a unique digit sequence $(k_j)_{j \in \mathbb{N}} \in [0, q)^\mathbb{N}$ such that for $n \geq 1$,
\[ k = k_0 + k_1 q + \cdots + k_{n-1} q^{n-1} + [k]_n q^n \]
referred to as the $n$-th $q$-adic expansion of $k$, and we call $k_n$ the $n$-th digit of $k$. Thus, for $k \in \mathbb{Z}^d$, $[k]_n$ can be represented by the digits of $k$ below the $n$-th place, and $[k]_n$ by the digits at $n$-th place and above. The power of $k$, denoted $p := p(k)$, is the minimal $p \geq 0$ with $k \in (\mathbf{Z} \setminus q^p \mathbf{Z})$ and is such that $k_n = k_p$ for $n \geq p$. One checks that $k_p$ is 0 in any coordinate where $k$ is positive, and $q_i - 1$ where $k$ is negative, so that for $k \in \mathbb{N}^d$ we have $k_p = 0$, and this terminal digit indicates the quadrant in which $k$ sits in $\mathbb{Z}^d$, or its sign.

Consider the action of translation by $\mathbf{Z}^d$ on itself and the action this induces on digit sequences. An important aspect of this action are carry operations, i.e. when addition at the $p$-th digit gives rise to a number larger than $q_i$, in some coordinate. The $p$-carry function $c_p : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow [0, 1]$ is defined by
\[ c_p(j, k) := [j]_p + [k]_p - q^p c_p(j, k) = [j_0 + j_1 + \cdots + j_{p-1} q^{p-1} + k_0 + \cdots + k_{p-1} q^{p-1}]_p \]
and one can check that
\[ [j + k]_p = [j]_p + [k]_p - q^p c_p(j, k) \quad \text{and} \quad [j + k]_p = c_p(j, k) + [j]_p + [k]_p \quad (4) \]
which follows by uniqueness of the division algorithm. Computing the digits of $j + k$ using the identities (3) is precisely the process of $q$-adic addition, so that $c_p$ is the amount carried at the $p$-th place in each coordinate when adding $j$ and $k$. For $p \geq 0$, and $k \in \mathbb{Z}^d$, let the $p$-carry set for $k$ be the collection
\[ \Delta_p(k) := \{ j \in [0, q^n] : j + k \not\in [0, q^n] \} = \{ j \in [0, q^n] : c_p(j, k) \neq 0 \} \]
Thus, the collection $\Delta_p(k) + q^p \mathbb{Z}^d$ consists of all $j$ for which $q$-adic addition with $k$ will require a carry operation at the $p$-th place. The following statistical property of the carry sets is used often:

**Lemma 3.1.** For $q > 1$, the frequency of carries at the $n$-th place of $q$-adic addition with $k$ goes to 0 as $n \to \infty$, or
\[ \lim_{n \to \infty} \frac{1}{Q^n} \text{Card} \Delta_n(k) = 0 \]
for every $k \in \mathbb{Z}^d$, and this convergence is exponentially fast.

**Proof.** As it is clear that the cardinality of $\Delta_n(k)$ does not depend on the signs of the components of $k$, we can assume without loss of generality that $k \in \mathbb{N}^d$. Then for $n \geq p := p(k)$, we have
\[ \Delta_n(k) \subset \Delta_n(q^p) = [0, q^n] \setminus [0, q^n - q^p) \]
so that taking cardinalities, as $Q = q_1 \cdots q_d$ we obtain
\[ \text{Card} \Delta_n(k) \leq Q^n \left( 1 - \prod_{i=1}^d \left( 1 - \frac{q^p}{q_i} \right) \right) = Q^n \sum_{j=1}^d q_j^{p-n} \prod_{i \neq j} (1 - q_i^{p-n}) \leq Q^n \sum_{j=1}^d q_j^{p-n} \]
Dividing by $Q^n$ and letting $n \to \infty$ gives the desired result. \(\square\)

These estimates track the frequency of a boundary term that arises in multiple contexts, and the carry sets themselves play an important role in the spectral theory of $q$-substitutions. In the next section, we give a description of $q$-substitutions emphasizing their arithmetic properties which complements the abelianization.
3.2 Instructions and Configurations

Let $\mathcal{S}$ be a $\mathbf{q}$-substitution on $\mathcal{A}$. For each $j \in \{0, \mathbf{q}\}$, the map sending $\gamma$ to $\mathcal{S}\gamma(j)$, the $j$-th letter of the word $\mathcal{S}(\gamma)$, is a map $\mathcal{A} \rightarrow \mathcal{A}$ called the $j$-th instruction of $\mathcal{S}$, and is denoted by $R_j$ for $j \in \{0, \mathbf{q}\}$. The following proposition permits an alternate characterization of $\mathbf{q}$-substitutions and exposes their arithmetic properties, see also [13 §5.1].

**Proposition 3.2** (Adic Description of $\mathbf{q}$-Substitution).

Let $\mathcal{S}$ be a $\mathbf{q}$-substitution on $\mathcal{A}$. For each $A \in \mathcal{A}^{\mathbb{Z}^d}$, $n > 0$ and $j = j_0 + j_1 \mathbf{q} + \ldots + j_{n-1} \mathbf{q}^{n-1} + j_n \mathbf{q}^n \in \mathbb{Z}^d$

$$(S^n A)(j) = R_{j_0} R_{j_1} \cdots R_{j_{n-1}} (A([j]_n)),$$

where $R_j$ are the instructions of $\mathcal{S}$, and $j_i \in \{0, \mathbf{q}\}$ the digits of $j$, as is our convention.

**Proof.** The proof follows by a simple inductive argument on the $n = 1$ case. Fix a $\mathbf{q}$-substitution $\mathcal{S}$ and $A \in \mathcal{A}^{\mathbb{Z}^d}$. The sequence $\mathcal{S}A$ is obtained by concatenating the blocks $\mathcal{S}(A(a))$ at the coordinates $a \mathbf{q}$ for $a \in \mathbb{Z}^d$. As $R_{\mathcal{S}(\alpha)}(\alpha) = S(\alpha)_b$ for $b \in \{0, \mathbf{q}\}$, it follows that the letter in the $\mathbf{q}$-th position of $\mathcal{S}A$ comes from the $a$-th letter of $\mathcal{S}(A(a))$, so that $\mathcal{S}A(a + a \mathbf{q}) = R_b(A(a))$. This proves the $n = 1$ case, as $b = |b + a \mathbf{q}|_1$ and $a = |b + a \mathbf{q}|_1$. Writing $S^n A = S(S^{n-1} A)$ gives the inductive step necessary to prove the result.

Denote the instructions of $S^n$ by $R^{(n)}$, which we call the generalized instructions of $\mathcal{S}$. Then the above proposition allows us to write for $j \in \mathbb{Z}^d$

$$R^{(n)}_j = R_{j_0} \cdots R_{j_{n-1}}, \text{ equivalently } T^j S^n = T^{j_0} S \cdots T^{j_{n-1}} S \circ T^{[j]_n} = T^{[j]_n} S^n T^{[j]_n} \quad (5)$$

where we have extended the definition of $R^{(n)}_j$ to all $j \in \mathbb{Z}^d$ by reducing $j$ modulo $\mathbf{q}^n$. This has no effect on the instructions or substitution, however, as the instructions for $S^n$ depend only on the first $n$ digits of $j$. As we are only concerned with $\mathbf{q}$-substitutions, we will use proposition 3.2 over the definition given in 2.

More generally, let $\mathcal{I}$ be a collection of $Q$ instructions, or maps $\mathcal{A} \rightarrow \mathcal{A}$, where we allow for repetition. Now, if $\mathbf{q} \in \mathbb{Z}^d$ satisfies $Q = q_1 q_2 \cdots q_d$, we can associate to each $j \in \{0, \mathbf{q}\}$ a map $R_j \in \mathcal{I}$ in a bijective way. We call such bijection $\mathcal{R} : [0, \mathbf{q}] \rightarrow \mathcal{I}$ a configuration of the instructions $\mathcal{I}$, and note that every configuration determines a $\mathbf{q}$-substitution (and conversely) via the above proposition. We refer to the induced substitution as the $\mathbf{q}$-substitution with configuration $\mathcal{R}$, and $[0, \mathbf{q}]$ as the location set. We will always represent the instructions of a substitution with the symbol of its configuration: $j \mapsto R_j$, the $j$-th instruction. We say two substitutions are configuration equivalent provided they are configurations of the same instructions, or if they can be obtained by rearranging the indices of the instructions defining the substitution. Thus, configuration equivalent substitutions can be of different dimensions, or different shapes within the same dimension.

**Example 3.3.** The Thue-Morse substitution is a 2-substitution on the alphabet $\{0, 1\}$ and sends $0 \mapsto 01$ and $1 \mapsto 10$. The instructions are the maps

$$\mathcal{R}_0 = \begin{cases} 0 &\mapsto 0 \\ 1 &\mapsto 1 \end{cases} \quad \text{and} \quad \mathcal{R}_1 = \begin{cases} 0 &\mapsto 1 \\ 1 &\mapsto 0 \end{cases}$$

and is discussed further in example 5.1. The 2-substitution $\tilde{\mathcal{S}}$ on $\mathcal{A}$ sending $0 \mapsto 10$ and $1 \mapsto 01$ has instructions $\tilde{\mathcal{R}}_0 = \mathcal{R}_1$ and $\tilde{\mathcal{R}}_1 = \mathcal{R}_0$. As they have the same instructions, they are configuration equivalent.

If the instructions are all bijections on $\mathcal{A}$, we say $\mathcal{S}$ is a bijective substitution; if they all commute with each other, we say $\mathcal{S}$ is a commutative substitution. Formally, we can view an instruction as a 1-substitution on $\mathcal{A}$, and so we compute its substitution matrix $R \in M_\mathcal{A}(\mathbb{R})$ which we refer to as an instruction matrix, and we will always conflate the notation of an instruction and its instruction matrix, extending this to the generalized instructions as well. Note that one can compute the instruction matrix from the instruction via the relation (noting that instruction matrices have coefficients 0 or 1)

$$R_{\alpha \gamma} = 1 \iff \alpha = R(\gamma)$$

As they represent functions, the instruction matrices are naturally column-stochastic: their column sums are all 1. Note that the substitution matrix is the sum of its instruction matrices: $M_S = \sum_{j \in \{0, \mathbf{q}\}} R_j$ so that the substitutions matrices of $\mathbf{q}$-substitutions are $\mathbf{Q}$-column stochastic.
Example 3.4. For the Thue-Morse substitution, the instruction matrices are

\[ R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{so that} \quad M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \]

We now describe a \emph{product} for two \( q \)-substitutions, based on a construction of Queffelec. Given two alphabets \( A \) and \( \hat{A} \), let \( A \otimes \hat{A} \) denote their \emph{product alphabet}, or letter pairs \( \alpha \hat{\gamma} \) with \( \alpha \in A, \hat{\gamma} \in \hat{A} \).

**Definition 3.5.** Let \( S, \hat{S} \) be \( q \)-substitutions on alphabets \( A, \hat{A} \) respective. Their \emph{substitution product} \( S \otimes \hat{S} \) is the \( q \)-substitution on \( A \otimes \hat{A} \) with configuration \( R \otimes \hat{R} \) whose \( j \)-th instruction is the map

\[ (R \otimes \hat{R})_j : A \otimes \hat{A} \rightarrow A \otimes \hat{A} \quad \text{with} \quad (R \otimes \hat{R})_j : \alpha \hat{\gamma} \mapsto \hat{R}(\alpha)R_j(\hat{\gamma}), \]

for \( j \in [0, q] \) and where \( R, \hat{R} \) are the configurations of \( S, \hat{S} \).

In the event that \( \hat{A} = A \), we denote the product alphabet by \( A^2 \), and if \( \hat{S} = S \), the substitution product \( S \otimes S \) is called the \emph{bisubstitution} of \( S \), after Queffelec, see [18 §10]. The following should offer justification for the notation used: for two matrices \( A, B \in M_A(\mathbb{C}) \), their \emph{Kronecker product} is the matrix \( A \otimes B \in M_{A^2}(\mathbb{C}) \) whose \( (\alpha \beta, \gamma \delta) \in \mathbb{A}^2 \times \mathbb{A}^2 \) entry is \( A_{\alpha \gamma}B_{\beta \delta} \). Additionally, for \( \alpha, \beta \in A \), note that \( e_{\alpha \beta} = e_{\alpha} \otimes e_{\beta} \) is the standard unit vector in \( \mathbb{C}A^2 \) corresponding to the word \( \alpha \beta \). With this notation, observe that

\[ (A \otimes B)_{\alpha \beta, \gamma \delta} = e^t_{\alpha \beta}A \otimes B e_{\gamma \delta} = (e^t_{\alpha}Ae_{\gamma})(e^t_{\beta}Be_{\delta}) = A_{\alpha \gamma}B_{\beta \delta} \]

where \( ^t \) signifies the transpose operation on matrices, exchanging row and column vectors.

Using this, one checks that the instruction matrices of a substitution product are the Kronecker products of the instruction matrices of the factors, explaining the notation. Note that \( \alpha \hat{\alpha} = R \otimes \hat{R}(\alpha \hat{\gamma}) \) if and only if \( \alpha = R(\gamma) \) and \( \hat{\alpha} = \hat{R}(\hat{\gamma}) \), so that for us the Kronecker product is just formalizing the conjunction and within a linear algebraic context. For more discussion of the Kronecker product, see [12 §4]. The bisubstitution will be revisited in (2) where we describe Queffelec’s application to the spectrum of \( S \).

The substitution product is always aperiodic stable as periodic sequences in the hull of a substitution product would necessarily be periodic in both factors. On the other hand, it is not always primitive stable: the bisubstitution of a primitive substitution is in general itself not primitive, although once telescoped properly it will be primitive on its ergodic classes as guaranteed by proposition 2.2. In (5) we discuss an example of a substitution product of Thue-Morse and Rudin-Shapiro which is primitive, raising an interesting question on the relationship of the spectrum of a substitution product to its factors.

### 3.3 Aperiodicity and the Subshift

Recall that a \( q \)-substitution is aperiodic if \( X_S \) contains no periodic points. By proposition 2.2, it suffices to check this condition on the primitive components, and so for the time being we restrict our attention to primitive \( q \)-substitutions. Let \( S \) be a primitive \( q \)-substitution on \( A \) with configuration \( R \). As the alphabet is finite, we can always find some \( h > 0 \) such that for each \( c \in [-1, 0] \) there is a letter \( \alpha_c \in A \) with \( R^{(h)}_c(\alpha_c) = \alpha_c \), recalling that the generalized configuration \( R^{(h)} \) is defined modulo \( q^h \). Let \( \eta \in A^+ \) be the block \( \eta : [-1, 0] \rightarrow A \) given by \( \eta(c) = \alpha_c \). Then \( \eta \) acts as a \emph{seed} for the subshift in the following sense: by construction, the sequence of blocks \( S^{nh} \eta \) defined on \( [-q^{nh}, q^{nh}] \) are nested, as \( S^{nh} \eta \) extends \( S^{nh} \eta \) whenever \( n \geq m \). As \( A^{Z^2} \) is endowed with the topology of point-wise convergence, the sequence \( S^{nh} \eta \) converges to a unique limit point \( D_\eta \in A^{Z^2} \), which will also be in \( X_S \) by construction. By primitivity, the collection of subblocks of \( D_\eta \), as a language, will generate the same subshift as \( L_S \), except that all subblocks of \( D_\eta \) are trivially subblocks of some sequence in \( X_S \), whereas this need not be \emph{a priori} true for \( L_S \). As we can always replace \( L_S \) with the language of \( D_\eta \) on primitive components of a \( q \)-substitution without affecting the subshift \( X_S \), we will often assume this is the case, referring to the language of \( D_\eta \) as the \emph{reduced language} of \( S \). Note that the reduced language is closed under the action of \( S \), not just its \( h \)-th iterates: as \( S(D_\eta) \) is expressible in \( L_S \), it is in the subshift, so that \( S(D_\eta) \) is expressible in the reduced language as well.

We now describe a criterion for the \( d = 1 \) case which is useful for checking aperiodicity in primitive \( q \)-substitutions, based on a result of Pansiot [17 Lemma 1]. The specific advantage in the \( Z \) setting is provided by the following equivalence: a primitive substitution \( (d = 1) \) is periodic if and only if its subshift
is finite, and hence has a finite word which generates the language. Here, word means a block $\mathbb{Z} \to A$, and its length is the cardinality of its domain. By a $(S,\gamma)$-neighborhood of $\alpha$, we mean a word $\gamma \alpha \delta$ which appears in the reduced language, or equivalently as a subword of $D_n$ as constructed above.

**Lemma 3.6** (Pansiot’s Lemma). A primitive $q$-substitution (the $\mathbb{Z}$ case) which is one-to-one on $A$ is aperiodic if and only if $S$ has a letter with at least two distinct neighborhoods.

**Proof.** First, suppose every $\alpha$ has a unique neighbor on both sides. This clearly reduces $X_S$ to a finite set, as each sequence would then be determined by its value at the origin, and so $S$ is periodic. Thus aperiodicity implies the neighborhood condition.

Inversely, suppose that $\alpha$ has two different neighborhoods, which means that there are distinct $\gamma, \delta \in A$ appearing on the same side of $\alpha$; assume they are neighbors to the right, the left side is a symmetric argument. Now, as $S : A \to A^+$ is one-to-one, the words $S(\gamma)$ and $S(\delta)$ are distinct - let $\omega_1$ denote their maximal common initial word, with $0 \leq |\omega_1| < q$. Then we can write

$$S(\alpha \gamma) = S(\alpha) \omega_1 \gamma_1 u \quad \text{and} \quad S(\alpha \delta) = S(\alpha) \omega_1 \delta_1 v$$

with distinct $\gamma_1, \delta_1 \in A$ and $u, v \in A^+$. Thus $S(\alpha) \omega_1$ can be extended within the language by adjoining either $\gamma_1$, or $\delta_1$, on the right. Applying $S$ again, this gives the sequences

$$S^2(\alpha \gamma) = S^2(\alpha) S(\omega_1) S(\gamma_1) S(u) \quad \text{and} \quad S^2(\alpha \delta) = S^2(\alpha) S(\omega_1) S(\delta_1) S(v)$$

Again, we let $\omega_2$ be the maximal common initial word of $S(\gamma_1)$ and $S(\delta_1)$, with $0 \leq |\omega_2| < q$ as $S$ is one-to-one on $A$, and obtain distinct $\gamma_2, \delta_2 \in A$ such that the word $S^2(\alpha) S(\omega_1) \omega_2$ can be extended within $L_S$ by adjoining either $\gamma_2$, or $\delta_2$, on the right. Continuing this way, we obtain a sequence of words $\omega_n$ with lengths less than $q$ such that the words

$$\Omega_n := S^n(\alpha) S^{n-1}(\omega_1) \cdots S(\omega_{n-1}) \omega_n$$

for each $n > 0$ can be extended within $L_S$ by adjoining either of the distinct letters $\gamma_n$, or $\delta_n$, on the right.

Let $L_S(n)$ be the words in the $L_S$ of length $n$, and let $c(n) = \text{Card} L_S(n)$. Naturally, $c(n)$ is nondecreasing, as every word of length $n + 1$ can be restricted to a word of length $n$. As $\Omega_n$ has two unique extensions to words in $L_S(|\Omega_n| + 1)$ this shows that $c(|\Omega_n| + 1) \geq c(|\Omega_n|) + 1$. As the length of $\Omega_n$ is at least $q^n$, the sequence $c(n)$ is unbounded as $n \to \infty$. This, however, implies $S$ is aperiodic, as a primitive substitution is periodic if and only if $c(n)$ remains bounded as $n \to \infty$.

As $S : A \to A^+$ is a map on a finite set, injectivity is trivial to verify, and primitivity can easily be checked by taking powers of the substitution matrix, or by iterating the substitution a few times. Then one examines the iterates $S^{n\gamma}(\gamma)$ for neighbor pairs; as soon as a letter with multiple same sided neighbors appears, aperiodicity is verified. At the moment, we have no clear criteria for extending Pansiot’s result to the case $d > 1$, as the above proof relies on the fact that, for $d = 1$, periodicity is equivalent to finiteness of the hull, and in higher dimensions this is not the case.

The most significant benefit of aperiodicity comes from the following theorem and is known as recognizability. It was originally proven by Mossé [13] in the one-dimensional case, and extended by Solomyak [23] to higher dimensions; it is stated here in the context of $q$-substitutions. Note that primitivity is not an issue here, as aperiodicity can always be determined for general substitutions by looking at primitive components.

**Theorem 3.7** (Mossé (96), Solomyak (98)). A $q$-substitution $S$ is aperiodic if and only if for every $A \in X_S$ there exists a unique $k \in [0, q]$ and $B \in X_S$ with $T^k S(B) = A$.

Note that for every $n \geq 0$, this theorem gives unique $k \in [0, q^n]$ and $B \in X_S$ with $A = T^k S^n(B)$. For each $\alpha \in A$ and $n \geq 0$, define the collection of pairs $S^{-n}(\alpha) \subset A \times [0, q^n]$, and cylinders $P_n$ given by

$$S^{-n}(\alpha) := \{ (\gamma, j) \in A \times [0, q^n] : S^n(\gamma)_j = R_j^{(n)}(\gamma) = \alpha \} \quad \text{and} \quad P_n := \{ T^j S^n[\gamma] : (\gamma, j) \in A \times [0, q^n] \}.$$
Corollary 3.8. Let $S$ be an aperiodic $q$-substitution on $A$, and let $B_S$ be the Borel subsets of $X_S$. Then for every $\alpha \in A$ and $n \geq 0$,

$$[\alpha] = \bigcup_{\gamma \in S^{-n}(\alpha)} T^j S^n[\gamma]$$

where the above union is disjoint. Moreover, the subshift is given by

$$X_S = \bigcup_{j \in [0, q^n]} T^j S(X_S) = \bigcap_{n \in \mathbb{N}} \bigcup_{\gamma \in A} \bigcup_{j \in [0, q^n]} T^j S^n[\gamma]$$

In this last equality, the cylinders $[\gamma]$ may be taken as subsets of $A^{Z^d}$.

Proof. Observe that by proposition 3.2 and the identities in (5) we have

$$P_{r,j} \text{ and } P_{r,j} \text{, so that the first identity follows by eliminating those pairs } j, \gamma \text{ for which } P_{j}^{(n)}(\gamma) \neq \alpha. \text{ As } P_n \text{ is a partition for all } n > 0, \text{ the first equality in the second identity follows immediately. Suppose } A \times X_S, \text{ then for each } n \geq 0, \text{ theorem 3.7 gives us } k \in [0, q^n] \text{ and } A' \times X_S \text{ with } A = T_k S^n A' \text{ and so } A \in T^k S^n[A'(0)], \text{ exhibiting } X_S \text{ as a subset of the right hand side. Conversely, let } B \subset Z^d \text{ finite, then choose } m \geq 0 \text{ large enough so that } B \subset [0, q^n] - j, \text{ for some } j \in [0, q^n). \text{ Now, if } A \in A^{Z^d} \text{ is in the right hand side, then there exists a } \alpha \in A \text{ and } k \in [0, q^n] \text{ with } A \in T_k S^n[\alpha], \text{ so that } A | B = T_k S^n[\alpha] | B, \text{ and equality follows.}$$

Thus, for $j \in Z^d$ and $\gamma, \delta \in A$, we have for all $n > 0$

$$T^j S^n[\gamma] \cap S^n[\delta] = T^{[j_1]} S^n T^{[j_2]}[\gamma] \cap S^n[\delta] = \begin{cases} S^n(T^{[j]}[\gamma] \cap [\delta]) & \text{if } |j|_n = 0 \\ \emptyset & \text{if } |j|_n \neq 0 \end{cases}$$

(7)

by (3). One checks that the maps $T^k S$ for $k \in [0, q^n]$ are (nonstrict) contractions on $A^{Z^d}$, a fact which follows from the expansiveness of the times $q$ map on $Z^d$. Thus, $X_S$ is like an attractor for an iterated function system on $A^Z$ determined by the collection of substitutions $\{T^k S : k \in [0, q^n]\}$. Moreover, using the above and theorem 2.1, we can obtain a recursive identity for the measure of cylinders specified by two letters (or any finite block, really) for any aperiodic $q$-substitution on $A$.

Corollary 3.9. Let $S$ be an aperiodic $q$-substitution on $A$ and $\mu \in \mathcal{M}(X_S)$. For $\alpha, \beta \in A$, $k \in Z^d$, and $n > 0$

$$\mu(T^k[\alpha] \cap [\beta]) = \sum_{\gamma, j, \delta \in D_n} \frac{1}{Q_n} \mu(T^{[j+k]}[\gamma] \cap [\delta]),$$

where $D_n = D_n(\alpha, \beta) := \{ (\gamma, \delta, j) \in A^2 \times [0, q^n] : R_{j}^{(n)}(\gamma) = \alpha \text{ and } R_{j+k}^{(n)}(\delta) = \beta \}.$

Proof. For each $\alpha, \beta \in A$ and $n > 0$, write

$$D = \{ (\gamma, \delta, j) \in A^2 \times [0, q^n] \times [0, q^n] : R_j^{(n)}(\gamma) = \alpha \text{ and } R_{j+k}^{(n)}(\delta) = \beta \}.$$

so that for $(\gamma, \delta, j) \in D$, we have $S^n(\gamma)_j = \alpha$ and $S^n(\delta)_j = \beta$, so that $D_n^k$ corresponds to those $(\gamma, \delta, j) \in D$ for which $i = [j + k]_n$. Using corollary 3.8 to decompose the cylinders $[\alpha]$ and $[\beta]$, we can write for $n \geq 0$

$$\mu(T^k[\alpha] \cap [\beta]) = \sum_{\gamma, j, \delta \in D} \mu(T^{[j+k]} S^n[\gamma] \cap T^j S^n[\delta]) = \sum_{\gamma, j, \delta \in D} \mu(T^{[j+k]} S^n(T^{[j]}[\gamma]) \cap T^j S^n[\delta])$$

where in the second identity we use identity (3). Writing $T^{[j+k]}[\gamma] = \bigcup_{\kappa \in A} [\kappa] \cap T^{[j+k]}[\gamma]$, we obtain

$$\mu(T^k[\alpha] \cap [\beta]) = \sum_{\kappa \in A} \left( \sum_{\gamma, j, \delta \in D} \mu(T^{[j+k]} S^n([\kappa] \cap T^{[j+k]}[\gamma]) \cap T^j S^n[\delta]) \right),$$
as the above decomposition of \( T^{[j+k]} \) is disjoint. Now, using that \( P_n \) is a partition, we have

\[
T^{[j+k]} \cap T^3S^n = \begin{cases} \emptyset & \text{if } \kappa \neq \delta \text{ or } i \neq [j+k]_n \\ T^1S^n & \text{if } \kappa = \delta \text{ and } i = [j+k]_n \end{cases},
\]

so that as \( \mu \) is \( T \)-invariant, the above reduces to

\[
\mu(T^k \cap [\beta]) = \sum_{\gamma, \delta \in P_n^k} \mu(S^n([\delta] \cap T^{[j+k]} \cap [\gamma]) \cap S^n([\delta])) = \sum_{\gamma, \delta \in P_n^k} \frac{1}{q^n} \mu(T^{[j+k]} \cap [\delta]),
\]

as desired.

Note that for \( n \geq p(k) \), the term \([j+k]_n\) above is in \([-1,1]\), with the sign corresponding to \( k \). In the next section, we use the above lemma to compute the Fourier coefficients of some spectral measures for the Koopman representation of \((X_S,T,\mu)\), for \( \mu \in M(X_S) \).

## 4 Spectral Theory

The goal of this section is to state the central result of this paper, theorem \[4.8\] which allows us to identify the maximal spectral type of \( S \) with a collection of measures parametrized by a correlation vector \( \Sigma \) consisting of complex Borel measures on the circle, and its spectral hull \( K \) which is a convex cone in \( \mathbb{C}^{A^2} \) associated to the abelianization of \( S \).

The \( q \)-shift is the topological dynamical system on \( T^d \) given by the map \( S_q: (z_1, \ldots, z_d) \mapsto (z_1^q, \ldots, z_d^q) \) which is topologically conjugate to the times \( q \) map \( x \mapsto qx \) (mod 1) on \( \mathbb{R}^d/\mathbb{Z}^d \). For a measure \( \mu \in M(T^d) \) (see \[6.1\]) its Fourier coefficients are

\[
\hat{\mu}(k) = \int_{T^d} z^{-k} d\mu \quad \text{where} \quad z^{-k} := z_1^{-k_1}z_2^{-k_2} \cdots z_d^{-k_d}
\]

as in equation \[2\]. Normalized Lebesgue measure (denoted \( m \)) on \( T^d \) is strongly mixing for the \( q \)-shift, which can be checked via the following proposition - the proof is standard, see \[18\]§3.1.1 for the \( d = 1 \) case.

**Proposition 4.1.** A measure \( \nu \in M(T^d) \) is invariant for the \( q \)-shift provided

\[
\hat{\nu}(aq) = \hat{\nu}(a),
\]

for every \( a \in \mathbb{Z}^d \). and is strong-mixing for the \( q \)-shift provided it is invariant and

\[
\lim_{p \to \infty} \hat{\nu}(b + aq^p) = \hat{\nu}(b)\hat{\nu}(a),
\]

for every \( a, b \in \mathbb{Z}^d \).

By the \( h \)-th roots of unity \( U_h \subset T^d \), we mean the collection of \( z \in T^d \) satisfying \( z^h_i = 1 \) for \( 1 \leq i \leq d \). It forms a subgroup of the torus and can be identified with the group of integers \( \mathbb{Z}^d \) modulo \( h \), which we represent as \([0,h)\) with addition modulo \( h \). Let \( \nu_h \) denote the Haar measure for all of these groups, so that \( \nu_h \) assigns equal mass to each \( h \)-th root of unity: \( \frac{1}{\pi} \) for \( H := h_1 \cdots h_d \). The following will be useful in the next section: for \( k \in \mathbb{Z}^d \),

\[
\hat{\nu}_h(k) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{h} \\ 0 & \text{if } k \not\equiv 0 \pmod{h} \end{cases}
\]

by usual properties of roots of unity. For \( q \geq 1 \), define the discrete measure

\[
\omega_q := \sum_{n \geq 1} 2^{-n} \nu_{q^n},
\]

so that \( \omega_q \) is probability measure supported by the \( q \)-adic roots of unity, corresponding to the \( q \)-adic rationals \( \mathbb{R}^d/\mathbb{Z}^d \): those rationals which can be expressed with denominator \( q^n \) for some \( n \).
4.1 The Correlation Vector - Σ

For $E \subset X_S$, let $1_E \in L^2(X_S, \mu)$ denote the indicator function of $E$. Recall that for $\alpha, \beta \in A$, the \textit{correlation measure} $\sigma_{\alpha, \beta} \in M(\mathbb{T}^d)$ is the spectral measure for the pair $1_{[\alpha]}$ and $1_{[\beta]}$.

\textbf{Definition 4.2.} The correlation vector $\Sigma$ is the $\mathbb{C}^{d^2}$-vector valued measure whose $\alpha \beta$-entry is $\sigma_{\alpha, \beta}$.

Queffelec defines $\Sigma$ as an $A \times A$ matrix valued measure see [13] §7.1.3, although it is much more convenient for us as a (column) vector. Using identity (2), we compute the Fourier coefficients of the correlation measure $\sigma_{\alpha, \beta}$, and obtain for $k \in \mathbb{Z}^d$

$$\tilde{\sigma}_{\alpha, \beta}(k) = \int_{X_S} 1_{[\alpha]} \circ T^{-k} \cdot 1_{[\beta]} d\mu = \mu(T^k[\alpha] \cap [\beta])$$  \hspace{1cm} (9)

The following result allows us to compute the Fourier coefficients of the correlation measures explicitly.

\textbf{Theorem 4.3.} Let $S$ be an aperiodic $q$-substitution on $A$. Then for $p > 0$

$$\hat{\Sigma}(k) = \frac{1}{Q^p} \sum_{j \in [0, q^n]} R_j^{(p)} \otimes R_{j+k}^{(p)} \hat{\Sigma}([j + k]_p) = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{j \in [0, q^n]} R_j^{(n)} \otimes R_{j+k}^{(n)} \hat{\Sigma}(0),$$

where $[j + k]_p$ is the quotient of $j + k$ under division modulo $q^p$, see [6.7] and $R^{(n)}$ are as in [6].

\textit{Proof.} First, note that $\hat{\Sigma}(0) = \sum_{\gamma \in A} \mu([\gamma]) e_{\gamma \gamma}$, by [2] applied to $\sigma_{\alpha, \beta}(0)$. Using identity (3), one checks that

$$e_{\alpha, \beta}^t R_j^{(n)} \otimes R_{j+k}^{(n)} e_{\gamma \delta} = \begin{cases} 1 & \text{if } \alpha = R_j^{(n)}(\gamma) \text{ and } \beta = R_{j+k}^{(n)}(\delta) \\ 0 & \text{otherwise} \end{cases}$$

so that we can index over $\gamma \delta, j \in A^2 \times [0, q^n)$, and using the above identity for $\tilde{\sigma}_{\alpha, \beta}(k) = e_{\alpha, \beta}^t \hat{\Sigma}(k)$ we obtain

$$\tilde{\sigma}_{\alpha, \beta}(k) = \sum_{j \in [0, q^n]} \sum_{\gamma \delta \in A^2} \frac{1}{Q^n} e_{\alpha, \beta}^t R_j^{(n)} \otimes R_{j+k}^{(n)} \beta_\gamma(\beta_\delta)[j + k]_n e_{\gamma \delta} = e_{\alpha}^t \sum_{j \in [0, q^n]} \frac{1}{Q^n} R_j^{(n)} \otimes R_{j+k}^{(n)} \hat{\Sigma}([j + k]_n),$$

as a consequence of corollary 3.9 and establishing the first identity. For the second equality use the first and write (for $p \geq p(k)$)

$$\hat{\Sigma}(k) = \frac{1}{Q^p} \sum_{[j+k]_p = 0} R_j^{(p)} \otimes R_{j+k}^{(p)} \hat{\Sigma}(0) + \frac{1}{Q^p} \sum_{[j+k]_p \neq 0} R_j^{(p)} \otimes R_{j+k}^{(p)} \hat{\Sigma}([j + k]_p),$$

$$= \frac{1}{Q^p} \sum_{j \in [0, q^n]} R_j^{(p)} \otimes R_{j+k}^{(p)} \hat{\Sigma}(0) + \frac{1}{Q^p} \sum_{j \in \Delta_p(k)} \frac{1}{Q^p} R_j^{(p)} \otimes R_{j+k}^{(p)} \left( \hat{\Sigma}([j + k]_p) - \hat{\Sigma}(0) \right),$$

so that, letting $p \to \infty$ and using [3.1] gives the desired result, as $|\hat{\Sigma}(a)| \leq \text{Card}(A)$ which follows from the fact that the $R_j^{(n)} \otimes R_{j+k}^{(n)}$ are column stochastic, and by [4].

Note that $\hat{\Sigma}(0)$ can be computed from the substitution matrix $M_S$ using theorems 2.1 and 2.3. By the remark following corollary 3.9 the first equality in the above theorem for $p = 1$ allows one to solve for $\hat{\Sigma}(c)$ algebraically for $c \in [-1, 1]$, which we now describe. For $1 \leq i \leq d$, note that $[j + c]_p = 0$ or $1_j$, where $1_j$ is the $j$-th coordinate vector in $\mathbb{Z}^d$. Thus, $\hat{\Sigma}(1_j) = \left( Q1 - \sum_{j \in \Delta_1(1_j)} R_j \otimes R_{j+1}^{(1)} \right)^{-1} \sum_{j \in \Delta_1(1_j)} R_j \otimes R_{j+1}^{(1)} \hat{\Sigma}(0)$

Using this, we can recursively compute $\hat{\Sigma}(c)$ for other $c \in [0, 1]$ of increasing norms, as $0 \leq |j + c|_1 \leq c$ for all $j \in [0, q]$. For example in the $d = 2$ case, we can compute $\hat{\Sigma}(0, 0)$ directly, then use the recursion to compute $\hat{\Sigma}(1, 0)$ (which depends only on $\hat{\Sigma}(0, 0)$ and $\hat{\Sigma}(1, 0)$) and $\hat{\Sigma}(0, 1)$ separately, and use these together to compute $\hat{\Sigma}(1, 1)$. The Fourier coefficients for $k \in \mathbb{Z}^d$ can then be computed explicitly using the above recursion by taking $p \geq p(k)$. The following generalization of Queffelec’s result follows from corollary 3.8 and describes the most important property of the correlation measures; see [13] §7.1.2
Theorem 4.4. If $\mathcal{S}$ is an aperiodic $\mathbf{q}$-substitution, the maximal spectral type of $(X_{\mathcal{S}}, T, \mu)$ is

$$\sigma_{\text{max}} \sim \sum_{\alpha \in \mathcal{A}} \sum_{n \geq 0} 2^{-n} \sigma_{\mathcal{S}^n[\alpha]} \sim \omega_\mathbf{q} * \sum_{\alpha \in \mathcal{A}} \sigma_{\alpha \alpha}$$

where $\sim$ denotes equivalence of measures.

Proof. For $n \geq 1$, let $H_n$ be the cyclic subspace

$$H_n := \text{Span}\{1_{\mathcal{S}^n[\gamma]} \circ T^k : \gamma \in \mathcal{A}, \ k \in \mathbb{Z}^d\}$$

so that $H_n \subset H_{n+1}$ by corollary 4.3, and moreover $L^2(\mu) = \bigcup H_n$ as the partitions $\mathcal{P}_n$ generate the Borel $\sigma$-algebra $\mathcal{B}_\mathcal{S}$ of the substitution subshift. Let $F$ be a maximal function for the Koopman representation of the shift operator, and let $F_n \in H_n$ be a sequence of finite linear combinations of indicator functions for the $T^j \mathcal{S}^n[\delta]$ for which $F_n$ converges to $F$ in $L^2(\mu)$, As the spectral map $\sigma : L^2(\mu) \to \mathcal{M}(\mathbb{T}^d)$ taking $f \mapsto \sigma_f$ is continuous, it follows that $\sigma_{F_n} \to \sigma_F$ in norm. Using bilinearity of $\sigma$

$$F_n = \sum_{\mathcal{A} \times \mathbb{Z}^d} c_{\gamma, k} 1_{\mathcal{S}^n[\gamma]} \circ T^k$$

implies $\sigma_{F_n, F_n} = \sum_{\gamma, \mathbf{k}, \mathbf{j}} c_{\gamma, \mathbf{k}} c_{\delta, \mathbf{j}} 1_{\mathcal{S}^n[\gamma]} \circ T^k, 1_{\mathcal{S}^n[\delta]} \circ T^j$ and, as $\mu$ is $T$-invariant, $\sigma_{f \circ T} = \sigma_f$ and we have

$$\sigma_{1_{\mathcal{S}^n[\gamma]} \circ T^j, 1_{\mathcal{S}^n[\delta]} \circ T^j} \ll \sigma_{1_{\mathcal{S}^n[\gamma]}}$$

and the first equivalence follows from positivity of the measures $\sigma_f$ for $f \in L^2(\mu)$.

We conclude by showing that for $n \geq 1$ and $\alpha \in \mathcal{A}$, we have $\sigma_{1_{\mathcal{S}^n[\alpha]}} = \nu_\mathbf{q}_n * \sigma_{1[\alpha]}$, which is accomplished by computing Fourier coefficients and using corollary 4.3 as well as an invariance property for the correlation measures. Using identities (3) and (4) we have for all $\mathbf{k} \in \mathbb{Z}^d$

$$\sigma_{1_{\mathcal{S}^n[\alpha]}(\mathbf{k})} = \mu(T^{\mathbf{k}} \mathcal{S}^n[a] \cap \mathcal{S}^n[a]) = \begin{cases} \frac{1}{Q^n} \mu(T^{\mathbf{k}} \alpha_n \cap \alpha) & \text{if } |\mathbf{k}|_n = 0 \\ 0 & \text{if } |\mathbf{k}|_n \neq 0 \end{cases}$$

using the scaling property of $\mu$ of theorem 4.1. Finally, one checks using theorem 4.3 that $\sigma_{\alpha \alpha}(\mathbf{a}) = \sigma_{\alpha \alpha}(\mathbf{q}^n)$, so that as $|\mathbf{k}|_n = 0$ implies $\mathbf{k} = |\mathbf{k}|_n \mathbf{q}^n$, the above allows us to write

$$\sigma_{1_{\mathcal{S}^n[\alpha]}(\mathbf{k})} = \begin{cases} 1/Q^n \sigma_{\alpha \alpha}(\mathbf{q}^n) & \text{if } \mathbf{k} \equiv 0(\text{mod } \mathbf{q}^n) \\ 0 & \text{otherwise} \end{cases} = 1/Q^n \sigma_{\alpha \alpha} * \nu_{\mathbf{q}^n}(\mathbf{k})$$

using identity (5) which gives the final equivalence as we are only identifying these measures up to type.

\[\square\]

Combined with theorem 4.3 this allows us to compute Fourier coefficients for $\sigma_{\text{max}}$. Although Fourier coefficients are not a type invariant, one has several tools available for detecting the various pure types (discrete, continuous, singular, or absolutely continuous to Lebesgue) via their Fourier coefficients (Wiener’s criterion, Riemann-Lebesgue lemma) and all of these are $a$ priori type invariants. Unfortunately, many substitutions exhibit mixed spectra and the measures $\sigma_{\alpha \alpha}$ themselves are not in general pure types. An interesting problem, therefore, is to find other measures in the span of the $\sigma_{\alpha \beta}$ giving rise to the maximal spectral type (via convolution with $\omega_\mathbf{q}$) and it is not unrealistic to expect some of them to share properties with $\sum \sigma_{\alpha \alpha}$. The most interesting property for us, known to Queffelec in the $\mathbb{Z}$ case, is a particular invariance which we now discuss.

4.2 The Spectral Hull - $\mathcal{K}$

For an aperiodic $\mathbf{q}$-substitution $\mathcal{S}$ on $\mathcal{A}$, recall that its coincidence matrix $C_\mathcal{S}$ is the substitution matrix of the bisubstitution $\mathcal{S} \otimes \mathcal{S}$. By proposition 2.2 there is an $h > 0$ such that $(\mathcal{S} \otimes \mathcal{S})^h = \mathcal{S}^h \otimes \mathcal{S}^h$ is primitive on its ergodic classes; using telescope invariance of $X_\mathcal{S}$, we can assume that $h = 1$, and will do so in what follows. Queffelec makes the following observation of the correlation measures

$$\hat{\Sigma}(\mathbf{q} \mathbf{a}) = \frac{1}{Q} \sum_{\mathbf{j} \in [0, \mathbf{q}]} \mathcal{R}_j^{(1)} \otimes \mathcal{R}_j^{(1)} \hat{\Sigma}(|j + \mathbf{a}|_1) = \frac{1}{Q} C_\mathcal{S} \hat{\Sigma}^{(1)}(\mathbf{a})$$

for $\mathbf{a} \in \mathbb{Z}^d$ (11)
which follows from theorem 1.3 as $|j + aq|_1 = a$ and $|j + aq|_1 = j$ for $j \in [0, q)$, and as $C_S = \sum \mathcal{R}_j \otimes \mathcal{R}_j$ by definition of the bisubstitution; here we use that the generalized instructions are defined modulo $q^\mathbb{Z}$. Writing

$$\lambda_v = v^t \Sigma := \sum_{\alpha, \beta \in A^2} v_{\alpha, \beta} \sigma_{\alpha, \beta}$$

(12)
defines a linear map associating to every vector in $\mathbb{C}^A$ a complex Borel measure on $\mathbb{T}^d$. Observe that, if $v$ is a left $Q$-eigenvector of $C_S$, then $v^t C_S = Q v^t$, so that

$$\lambda_v(aq) = v^t \Sigma (aq) = \lambda_v(a),$$

and so $\lambda_v$ is invariant for the $q$-shift by proposition 4.6, although it is not in general a positive probability measure. Now, for $v = (v_{\alpha, \beta})_{\alpha, \beta \in A^2} \in \mathbb{C}^A$, let $\Sigma = (v_{\alpha, \beta})_{\alpha, \beta \in A}$ be the matrix of $v$, and write $v \geq 0$ whenever $v$ is Hermitian positive definite; Queffélec calls this condition strong positivity, see [Queffelec, Prop 10.3].

**Lemma 4.5.** If $v, w$ are strongly positive, then $v^t w \geq 0$

**Proof.** By the Schur product theorem, the Hadamard product of two positive definite matrices is positive definite, see [12, §3]. Letting $\circ$ denoting the Hadamard product of two matrices, we have

$$v^t w = \sum v_i w_i = 1^t (\Sigma \circ \Sigma) 1 \geq 0 \quad \square$$

Using bilinearity of the spectral map $f, g \mapsto \sigma_{f, g}$ and the spectral theorem, one checks that $\Sigma \geq 0$, so that $\lambda_v$ is a positive measure whenever $v \geq 0$ by the above lemma.

For $\mathcal{S}$ an aperiodic $q$-substitution on $A$, we define its spectral hull

$$\mathcal{K}(\mathcal{S}) := \{ v \in \mathbb{C}^A : C_S^2 v = Q v, \ w \geq 0 \}$$

Note that $\mathcal{K}(\mathcal{S})$ is a nonempty cone in $\mathbb{C}^A$ so that it is closed under strictly positive linear combinations. The following is a consequence of the above discussion.

**Proposition 4.6.** If $v \in \mathcal{K}(\mathcal{S})$ then $\lambda_v$ is a positive measure, invariant for the $q$-shift.

Thus, the spectral hull consists of vectors in the span of the correlation vectors which, via $\lambda$, give rise to $q$-shift invariant positive measures. By theorem 2.1 $u = (\mu([\alpha]))_{\alpha \in A}$ and observe that

$$\lambda_v(\mathbb{T}^d) = \lambda_v(0) = v^t \Sigma(0) = \sum_{\alpha \in A} v_{\alpha, \alpha} \mu([\alpha])$$

so that a vector in $\mathcal{K}$ gives rise to a probability measure if and only if $\sum v_{\alpha, \alpha} u_{\alpha} = 1$. In the primitive case, this is equivalent to $v_{\alpha, \alpha} = 1$ for $\alpha \in A$, so that the probability measures arising from $\mathcal{K}$ via $\lambda$ is a bounded convex set with finitely many extreme points. Let $\mathcal{K}^* = \text{Ext} \{ v \in \mathcal{K}(\mathcal{S}) : \sum_{\alpha \in A} v_{\alpha, \alpha} \mu([\alpha]) = 1 \}$, be the extreme points of this collection, then it follows by limiting arguments that the $\lambda_v$ for $v \in \mathcal{K}^*$ are also $q$-shift invariant probability measures which form a convex basis for the $q$-shift invariant probability measures in the span of the correlation measures. We note here that the set $\mathcal{K}$ is an open cone and $\mathcal{K}^*$ is the extreme points of the intersection of $\mathcal{K}$ and the hyperplane $\sum z_i = 1$.

For $E \subset A^2$, let $E := \sum_{\gamma \delta \in E} e_{\gamma \delta} \in \mathbb{C}^A$ correspond to $E$ via the same representation used for the abelianization of $\mathcal{S}$. Let $F$ represent the ergodic classes for the bisubstitution $\mathcal{S} \otimes \mathcal{S}$, and let $P_j$ denote the standard projection onto the span of the $e_{\gamma \delta}$ with $\gamma \delta \in F_j$ and let $P_T$ be projection onto the transient pairs $T$ of $A^2$. For $w_1, \ldots, w_J \in \mathbb{C}$, let

$$\mathcal{V}(w_1, \ldots, w_J) := \sum_j w_j \bar{f}_j \in \mathbb{C}^A$$

The following should be compared to [15, Proposition 10.2] - it allows us to parametrize $\mathcal{K}(\mathcal{S})$ over a convex region of $\mathbb{C}^K$, where $K$ is the number of ergodic classes for the bisubstitution of $\mathcal{S}$. 

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Lemma 4.7. A vector \( \mathbf{v} \in \mathbb{C}^{A^2} \) satisfies \( \mathbf{v} \in K(S) \) if and only if
\[
\mathbf{v} = \mathbf{v} + \mathcal{P}_T \left( QI - \mathcal{P}_T C_S^T \right)^{-1} \mathcal{P}_T C_S^T \mathbf{v}
\]
and \( \mathbf{v} \gg 0 \)
where \( \mathcal{V} = \mathcal{V}(w_1, \ldots, w_J) \) for some \( w_1, \ldots, w_J \in \mathbb{C} \). If the transient class \( \mathcal{T} = \emptyset \), then \( \mathcal{P}_T = 0 \) and \( \mathbf{v} = \mathcal{V} \).

Proof. Order the standard basis for \( \mathbb{C}^{A^2} \) relative to the partition \( \mathcal{F}_1, \ldots, \mathcal{F}_K, \mathcal{T} \) for the primitive reduced form of \( S \otimes S \) given by proposition 22; this is just a permutation of the basis. This allows us to express the matrix \( C \) in block matrix form so that the \( i, j \)-th block \( C_{i,j} \) corresponds to the operator \( \mathcal{P}_j C \mathcal{P}_i \) for \( i, j = 1, \ldots, K \) and, with empty blocks assumed 0,
\[
C = \begin{bmatrix}
C_{1,1} & C_{1,T} \\
\vdots & \vdots \\
C_{J,1} & C_{J,T}
\end{bmatrix}
\]
and \( \mathcal{P}_j C \mathcal{P}_i = \begin{cases} 
\mathcal{P}_j C \mathcal{P}_i = \mathcal{P}_j C \text{ if } i = j \\
\mathcal{P}_j C \mathcal{P}_i = \mathcal{P}_j C \text{ if } i = T \\
0 & \text{otherwise.}
\end{cases} \)

and where the matrices \( C_{i,j} \) associated to the operator \( \mathcal{P}_j C \mathcal{P}_i \) are primitive for \( i \neq T \). As \( S \otimes S \) is a \( q \)-substitution, \( C \) is \( Q \) column-stochastic so that by primitivity, the \( Q \)-eigenspace of \( C \mathcal{P}_i \mathcal{P}_j \) is spanned by \( F_j \), for \( 1 \leq j \leq K \) as these vectors are constant on the ergodic classes of \( S \otimes S \). Using this, we apply a result of Gantmacher [13, Thm 8.6] to show that the spectral radius of \( \mathcal{P}_T C \mathcal{P}_T \) is strictly dominated by \( Q \); this fact will be useful momentarily.

Writing the identity on \( \mathbb{C}^{A^2} \) as \( I = \sum_{j=1}^K \mathcal{P}_j + \mathcal{P}_T \) and as the \( \mathcal{P}_j \) are orthogonal projections, we have
\[
C^T = \left( \sum_{j=1}^K \mathcal{P}_j + \mathcal{P}_T \right) C_S^T \left( \sum_{j=1}^K \mathcal{P}_j + \mathcal{P}_T \right) = \sum_{j=1}^K \mathcal{P}_j C_S^T \mathcal{P}_j + \mathcal{P}_T \left( C_S^T \mathcal{P}_T + \sum_{j=1}^K C_S^T \mathcal{P}_j \right)
\]
from which one can see that \( C^T \mathbf{v} = Q \mathbf{v} \) if and only if
\[
\mathcal{P}_j C_S^T \mathcal{P}_j \mathbf{v} = Q \mathbf{v} \quad \text{for } 1 \leq j \leq K \quad \text{and} \quad \mathcal{P}_T C_S^T \mathcal{P}_T \mathbf{v} + \sum_{j=1}^K \mathcal{P}_T C_S^T \mathcal{P}_j \mathbf{v} = Q \mathcal{P}_T \mathbf{v}
\]
The first condition shows \( \mathcal{P}_j \mathbf{v} = v_j \mathcal{F}_j \) for some \( v_j \in \mathbb{C} \) and the second gives
\[
\mathcal{P}_T (QI - C^T \mathcal{P}_T) \mathcal{P}_T \mathbf{v} = \mathcal{P}_T C_S^T \mathcal{V}(w_1, \ldots, w_K) \quad \implies \quad \mathcal{P}_T \mathbf{v} = \mathcal{P}_T (QI - C^T \mathcal{P}_T)^{-1} \mathcal{P}_T C_S^T \mathcal{V}
\]
as the spectral radius of \( C^T \mathcal{P}_T \) is less than \( Q \), completing the proof. \( \square \)

Note that when the transient class is empty, for example when \( S \) is bijective, the projection \( \mathcal{P}_T = 0 \), and \( \mathbf{v} \in K \) if and only if \( \mathbf{v} = \mathcal{V}(w_1, \ldots, w_J) \).

4.3 Queffelec’s Theorem

Letting \( \mathbf{v} = \sum_{\alpha \in A} e_{\alpha \alpha} \in K(S) \), we could restate theorem 4.4 as \( \sigma_{\max} \sim \omega_q * \lambda_v \) as defined in [12]. The following result shows that this holds for any \( \mathbf{v} \in K(S) \), and those measures arising from the extremal rays enjoy ergodic properties, see [13, Cor 10.2, Thm 10.1].

Theorem 4.8. If \( S \) is an aperiodic \( q \)-substitution on \( A \), then for every \( \mathbf{v} \in K(S) \) we have
\[
\sigma_{\max} \sim \omega_q * \lambda_v
\]
Moreover, the measures \( \lambda_w \) for \( w \in K^* \) are strong-mixing for the \( q \)-shift.

Proof. See Appendix. \( \square \)

An important corollary of the above is that the extremal rays of \( K(S) \) parametrize mutually singular measures on \( T^d \). Being ergodic, one checks using continuity of the \( q \)-shift that these \( \lambda_w \) are of pure type by writing out their discrete, singular continuous and absolutely continuous components with respect to Lebesgue measure \( m \), all of which will be separately invariant. Note that \( m \) is translation invariant (implying \( \omega_q * m = m \)) and ergodic for every \( q \)-shift. As convolution with \( \omega_q \) does not change the purity of a measure, \( \lambda \) is pure discrete, purely singular continuous, or Lebesgue measure \( m \), respectively, if and only if \( \lambda * \omega_q \) is as well. Thus, we obtain the following as a corollary of the above theorem:
Corollary 4.9. The measures \( \lambda_w \) for \( w \in K^* \) are either purely discrete, purely singular continuous, or Lebesgue measure in \( \mathbb{T}^2 \), and separately describe pure components of the spectrum of \( S \), as in theorem 4.8.

Given an aperiodic substitution \( S \) of constant length, one can compute the extremal rays of \( K(S) \) using lemma 4.7 as well as \( \hat{\Sigma}(c) \) for \( c \in [-1, 1] \) using the initial weights \( \mu([\alpha]) \) with \( \alpha \in A \) and formula (10); in the primitive case, the initial weights are unique and can be computed via the Perron vector \( u \) of \( M_S \). Using theorem 4.3 one can then compute \( \hat{\Sigma}(k) \) for any \( k \in \mathbb{Z}^d \) and then we have:

\[
\hat{\lambda}_w(k) = v^T \hat{\Sigma}(k) := \sum_{\alpha \beta \in A^2} v_{\alpha \beta} \hat{e}_{\alpha \beta}(k).
\]

By the above corollary, the spectrum of any aperiodic \( q \)-substitution is singular to Lebesgue unless there exists a \( w \in K^* \) for which the above sequence vanishes identically away from \( k = 0 \). As this requires us to compute \( \hat{\Sigma}(k) \) to verify, one might hope for an upper bound on the number of coefficients that need to be checked in order to guarantee Lebesgue measure. We know of no \( q \)-substitution (the \( \mathbb{Z} \) case) singular to Lebesgue component for which all of the first \( q + 1 \) positive Fourier coefficients vanish, suggesting that it may suffice to compute \( \hat{\lambda}(k) \) for \( k \in [0, q + 1] \) in order to verify the presence of Lebesgue component, but we have not examined this question in detail.

Recall that two \( q \)-substitutions are configuration equivalent if they have the same collection of instructions, counted with multiplicity. A property of a substitution is a configuration invariant if all configuration equivalent substitutions share that property, or if it does not depend on the particular arrangement of the instructions comprising the substitution. The following proposition is immediate, as it follows entirely from properties of the abelianization of \( S \) which is a priori configuration independent:

Proposition 4.10. For a \( q \)-substitution \( S \) on \( A \), the matrices \( M_S \) and \( C_S \), the ergodic decompositions of \( S \) and \( S \otimes S \), as well as \( \Sigma(0) \) and \( K(S) \), are configuration invariants.

By Queffelec’s theorem 4.8 the spectrum of \( S \) is equivalent to the measures \( \lambda_v = v^T \Sigma \), for \( v \in K(S) \), and so the spectrum of \( S \) can be separated into the study of extremal properties of the spectral hull and the correlation vector. As \( K \) is a configuration invariant, however, this shows us that any property of the spectrum which depends on the configuration of \( S \) is determined by the correlation vector \( \Sigma \). It is immediate that the spectrum of a substitution is not invariant with respect to configuration equivalence: not only can the spectrum exist on different dimensional Tori, but the height (see 5.2) of a substitution can vary with configuration. One can, however, use theorem 4.3 to study the effect changes in configuration have on a given substitution, and it is evident from identities such as (10) that the structure of the configuration relative to the carry sets \( \Delta_p(k) \) accounts for much of this difference. In the next section, we summarize an algorithm for the computation of the spectrum of a substitution using these results, compute several examples, and prove that all aperiodic bijective and commutative substitutions have purely singular spectrum.

5 Algorithm and Examples

Let \( S \) be an aperiodic \( q \)-substitution with configuration \( R \). Using proposition 2.2 we can write \( S \) in its primitive reduced form, and using theorem 4.3 with definition 4, we may restrict to the primitive components of \( S \) for the purposes of classifying the spectrum; thus, it suffices to assume \( S \) is primitive. Then:

1. Compute \( M_S = \sum R_j \) and \( C_S = \sum R_j \otimes R_j \). Let \( u \) be the Perron vector of \( M_S \).
2. Then \( \hat{\Sigma}(0) = \sum u_{\alpha} e_{\alpha} \), and \( \hat{\Sigma}(k) \) can be computed with (10) and theorem 4.3 for \( k \in \mathbb{Z}^d \).
3. Compute the ergodic decomposition of \( S \otimes S \) by looking at orbits of the \( R_j \otimes R_j \).
4. Use lemma 4.7 to compute the extreme rays of \( K_S \). One can enforce strong positivity via:
   - Forcing positivity of the principal minors.
   - Write \( V = \sum w_j M_j \), with \( M_j \) constant matrices. Often, these can be simultaneously diagonalized.
   - Using software (sympy for Python) to find eigenvalues of matrices with variable coefficients.
5. The maximal spectral type of \( S \) is the sum of the \( \lambda_v = v^* \Sigma \) as \( v \) ranges over the extremal rays of \( K \).

Our first example is classical, and comes from the Thue-Morse sequence. As its spectrum is well established, it serves as a test case for the algorithm.

**Example 5.1** (Thue-Morse). Let \( \tau \) denote the 2-substitution on \( \mathcal{A} = \{0, 1\} \) given by

\[
\tau : \begin{cases} 
0 \mapsto 01 \\
1 \mapsto 10 
\end{cases}
\]

with instructions \( \mathcal{R}_0, \mathcal{R}_1 \) appearing consecutively in the above sum. Clearly, \( \tau \) is primitive, and as \( \tau^2(0) = 0110 \) with \( \mathcal{R}_0 \) the identity, the symbol 1 can be proceeded by both symbols 0,1 so that it is aperiodic by Pansiot’s lemma. The Perron vector of \( \tau \) is \((\frac{1}{2}, \frac{1}{2})\) so that \( \hat{\Sigma}(0) = \frac{1}{2} \sum_{\gamma \in \mathcal{A}} e_\gamma \). As \( q = 2 \), we have \( \Delta_1(1) = \{1\} \) and so equation (10) gives

\[
\hat{\Sigma}(1) = \left(2I - \mathcal{R}_1 \otimes \mathcal{R}_0\right)^{-1} \mathcal{R}_0 \otimes \mathcal{R}_1 \hat{\Sigma}(0) = \frac{1}{6}(1, 2, 2, 1)^t
\]

where the basis is given the lexicographic order 00, 01, 10, 11.

Now we compute the ergodic classes of the bisubstitution: as \( \tau \) is primitive, \( \mathcal{F}_1 = \{01, 11\} \), and one checks that \( \mathcal{F}_2 = \{01, 10\} \) is also a minimal orbit of the instructions \( \mathcal{R}_j \otimes \mathcal{R}_j \). As these partition, \( \mathcal{F}_1, \mathcal{F}_2 \) form the ergodic classes of \( \tau \otimes \tau \) with empty transient part. By lemma 4.7, \( v \in \mathcal{K} \) is given by \( v = (w_1, w_2, w_1)^t \) and as the matrices \( \mathcal{F}_1, \mathcal{F}_2 \) commute, they can be simultaneously diagonalized, and we obtain

\[
\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} v \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} w_1 - w_2 \\ w_1 + w_2 \end{pmatrix}
\]

so that \( v \gg 0 \) implies \( -w_1 < w_2 < w_1 \). Letting \( w_1 = 1 \), gives us \( -1 < w_2 < 1 \) and \( \mathcal{K} \) has two extreme rays

\[
v_1 = \mathcal{F}_1 + \mathcal{F}_2 = (1, 1, 1, 1)^t \quad \text{and} \quad v_2 = \mathcal{F}_1 - \mathcal{F}_2 = (1, -1, -1, 1)^t
\]

Thus, \( \lambda_{v_1} = \sum_{\alpha, \beta \in \mathcal{A}}^{}\sigma_{\alpha, \beta} = \delta \in \mathcal{M}(\mathcal{F}) \), the Dirac-Delta mass at 1 (all the Fourier coefficients are 1), and

\[
\hat{\lambda}_{v_2}(1) = v_2^* \hat{\Sigma}(1) = -\frac{1}{3} \neq 0
\]

so that \( \lambda_{v_2} \) is not Lebesgue measure, and Thue-Morse has purely singular spectrum. Note that \( \lambda_{v_1} = \delta \) gives rise to the discrete component and \( \lambda_{v_2} \) the singular continuous component and

\[
\sigma_{\max} \sim \omega_2 + \omega_2 * \lambda_{v_2}
\]

Thus the Thue-Morse substitution is a bijective substitution on 2 letters. One can generalize the Thue-Morse sequence by considering all bijective \( q \)-substitutions on 2 letter alphabets. In [2] theorem 3 it is shown that the spectrum of any bijective \( q \)-substitution on 2 letters is singular to Lebesgue spectrum. Note that bijective instructions on 2 letter alphabets always commute, so that such substitutions are also commutative substitutions. In the one-dimensional case, Queffelec showed that all commutative bijective substitutions are singular to Lebesgue spectrum (see [18] Prop 3.19 and Thm 8.2), and this generalizes to \( \mathbb{Z}^d \) substitutions as well, which we briefly discuss.

### 5.1 Aperiodic Bijective and Commutative \( q \)-Substitutions

A useful identity relating Kronecker products to matrix products is

\[
(A \otimes B)v = AvB^t
\]

where \( v = (v_{\alpha, \beta})_{\alpha, \beta \in \mathcal{A}} \in \mathbb{C}^{\mathcal{A}^2} \) and \( A, B \), and \( v = (v_{\alpha, \beta})_{\alpha, \beta \in \mathcal{A}} \) are in \( \mathbb{M}_A(\mathbb{C}) \). If we let \( S = \hat{\Sigma} \) be the matrix of \( \Sigma \), then theorem 4.3 can be expressed in the form

\[
\hat{S}(k) = \lim_{n \to \infty} \frac{1}{q^n} \sum_{\beta \in [0,q^n]} R_j^{(n)} \otimes R_j^{(n)} \hat{S}(0) \quad \text{implies} \quad \hat{S}(k) = \lim_{n \to \infty} \frac{1}{q^n} \sum_{\beta \in \mathcal{Q}^n} R_j^{(n)} \hat{S}(0)(R_j^{(n)})^*
\]
From this, one checks that \( S \) is a matrix Riesz product (using lemma \([7, 1]\) as in the proof of theorem \([13]\))

\[
S = \lim_{n \to \infty} \frac{1}{Q^n} \Pi_n \Pi^* \quad \text{with} \quad \Pi_n(x) = R(q^{-n}x) \cdots R(qx)R(x) \quad \text{and} \quad R(x) = \sum_{j \leq q} R_j e^{2\pi j x}
\]

As the instructions are bijective, the matrices \( R_j \) are permutation matrices and therefore unitary. If they also commute, the instruction matrices are simultaneously diagonalizable, and there exists a unitary matrix \( P \) so that \( PR(x)P^* \) is a diagonal matrix polynomial. Therefore, \( S \) is diagonalizable over \( \mathbb{C} \) and

\[
PSP^* = \lim_{n \to \infty} \frac{1}{Q^n} P \Pi_n P^* P \Pi_n P^* = \lim_{n \to \infty} \frac{1}{Q^n} \Lambda(x) x \cdots \Lambda(q^{-n} x) \Lambda(q^{-n+1} x) \cdots \Lambda(x)
\]

where \( \Lambda(x) \) is the diagonal matrix polynomial \( PR(x)P^* \). Thus, the diagonal entries of \( PSP^* \) are of the form

\[
\lim_{n \to \infty} \frac{1}{Q^n} \prod_{j=0}^{n-1} |\pi(q^j x)|^2
\]

where \( \pi(w) \) is an eigenpolynomial of \( R(x) \), so that the measures on the diagonal are generalized Riesz products (see \([18, \text{§}1.3]\)). These measures, however, are singular to Lebesgue measure which follows from lemma \([6, 13]\) in the appendix. By theorem \([4, 3]\) the maximal spectral type is determined by the trace of \( S \), and so is singular to Lebesgue measure. This gives the following:

**Theorem 5.2.** The spectrum of an aperiodic bijective and commutative \( q \)-substitution is purely singular.

An interesting question is the importance of commutativity. Our next example, due to Queffelec, is bijective but not commutative. In \([18, \text{Examples 9.3, 10.2.2.3, and 11.1.2.3}]\) it was shown to have Lebesgue spectrum, however there were errors in the analysis. As we now show, it is in fact purely singular.

**Example 5.3** (Queffelec’s \( \zeta \)). Let \( \zeta \) be the 3-substitution on \( A = \{0, 1, 2\} \) given by

\[
\zeta : \begin{cases} 
0 \rightarrow 001 \\
1 \rightarrow 122 \\
2 \rightarrow 210
\end{cases}
\]

with the instruction matrices \( R_0, R_1, R_2 \) appearing sequentially above. As \( M_\zeta^2 \) is positive, \( \zeta \) is primitive. As \( R_0 \) is the identity and as \( 1 \) can be followed by \( 0, 1 \) and \( 2 \) in \( \zeta^2(0) \), Pansiot’s lemma applies and shows that \( \zeta \) is aperiodic. One checks that the Perron vector of \( M_\zeta \) is \( \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \), giving \( \Sigma(0) = \frac{1}{3} \sum_{\gamma \in \mathcal{A}} e_{\gamma} \). Using \([10]\),

\[
\hat{\Sigma}(1) = \left( 3I - \sum_{j \in \Delta_1(1)} R_j^{(1)} \otimes R_{j+1}^{(1)} \right)^{-1} \sum_{j \notin \Delta_1(1)} R_j^{(1)} \otimes R_{j+1}^{(1)} \hat{\Sigma}(0)
\]

\[
= (3I - R_2 \otimes R_0)^{-1} (R_0 \otimes R_1 + R_1 \otimes R_2) \hat{\Sigma}(0)
\]

\[
= \frac{1}{39} (5, 6, 2, 6, 2, 5, 2, 5, 6)^f
\]

as \( \Delta_1(1) = \{2\} \) for \( q = 3 \). Here, the basis \( \mathcal{A}^2 \) is given the lexicographic order

\[
(00, 01, 02, 10, 11, 12, 20, 21, 22)
\]

and will be standard for the bialphabet. Computing \( \hat{\Sigma}(2) \) using the recursion of theorem \([4, 3]\) \( (p = 1) \) gives

\[
\hat{\Sigma}(2) = \frac{1}{3} \sum_{j=0}^{2} R_j \otimes R_{j+2} \hat{\Sigma}([j + 2]_1)
\]

\[
= \frac{1}{3} R_0 \otimes R_2 \hat{\Sigma}(0) + \frac{1}{3} (R_1 \otimes R_0 + R_2 \otimes R_1) \hat{\Sigma}(1)
\]

\[
= \frac{1}{117} (7, 7, 25, 25, 7, 7, 7, 25, 7)^f
\]
Now, we compute the ergodic classes of the bisubstitution \( \zeta \otimes \zeta \), and obtain the orbits \( F_1 = \{00, 11, 22\} \) and \( F_2 = \{01, 10, 12, 21, 02, 20\} \) with empty transient class (as \( \zeta \) is bijective), which form the ergodic classes of \( \zeta \otimes \zeta \). Thus, by lemma 4.7

\[ \mathbf{v} = w_1 F_1 + w_2 F_2 = (w_1, w_2, w_2, w_1, w_2, w_2, w_1)^t \]

Performing simultaneous row and column operations, we arrive at

\[ \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2^2 - w_1^2 & 0 \\ 0 & w_2^2 - w_1^2 & 0 \end{pmatrix} \]

and \( \mathbf{v} \gg 0 \) implies

\[ \begin{cases} w_1 > 0, \\ (w_1 + w_2)(w_1 - w_2) > 0, \\ (w_1 - w_2)(w_1 + 2w_2) > 0 \end{cases} \]

so that \( w_1 > 0 \) and \(-1/2 w_1 < w_2 < w_1\), giving \( K \) two extremal rays determined by the vectors

\[ \mathbf{v}_1 = \mathcal{F}_1 + \mathcal{F}_2 = 1, \quad \text{and} \quad \mathbf{v}_2 = \mathcal{F}_1 - \frac{1}{2} \mathcal{F}_2 = (1, -\frac{1}{2}, -\frac{1}{2}, 1, -\frac{1}{2}, -\frac{1}{2}, 1)^t. \]

One checks that \( \lambda_{v_1} = \delta \), and that \( \hat{\lambda}_{v_1}(1) = 0 \), and thus \( \hat{\lambda}_{v_2}(3a) = 0 \) for all \( a \) by \( q \)-shift invariance. Using the value of \( \hat{\Sigma}(2) \) above, however, we obtain

\[ \hat{\lambda}_{v_2}(2) = \frac{1}{117} \left( 1(7 \cdot 3) - \frac{1}{2}(7 \cdot 3 + 25 \cdot 3) \right) \neq 0, \]

so that \( \lambda_{v_2} \) is not Lebesgue measure. As \( \sigma_{\max}(\zeta) = \omega_3 * (\delta + \lambda_{v_1}) \), it follows that the spectrum of \( \zeta \) is purely singular to Lebesgue spectrum on the circle \( \mathbb{T} \), as both \( \omega_3 \) and \( \omega_3 * \lambda_{v_1} \) are singular to Lebesgue measure.

Note that \( \zeta \) is an example of a bijective substitution, as all of its instructions are bijections of \( A \). Correcting this mistake of Queffelec’s is significant, as it represented the only known example of a bijective substitution with Lebesgue component (for more examples with Lebesgue component, see 24). Our first computed example with \( d > 1 \) comes from a substitution tiling system in the plane known as the Table. In 21, Robinson showed that it could be viewed as a \( \mathbb{Z}^2 \) substitution on 4 symbols.

**Example 5.4 (The Table).** Let \( T \) be the (2,2)-substitution on \{0,1,2,3\} given by

\[
\begin{align*}
0 &\rightarrow \begin{pmatrix} 3 & 0 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} & 1 &\rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 2 \end{pmatrix} \\
2 &\rightarrow \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} & 3 &\rightarrow \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}
\end{align*}
\]

with the instruction matrices \( R_{(0,0)}, R_{(1,0)}, R_{(0,1)}, \) and \( R_{(1,1)} \) appearing consecutively above. It is clearly primitive, and aperiodicity follows from recognizability or theorem 3.7, see 21. Note that \( R_{(0,0)} \) and \( R_{(0,1)} \) do not commute, so that \( T \) is not a commutative substitution, and theorem 5.2 does not apply. One checks that the Perron vector of \( M_T \) is \( \mathbf{u} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^t \) and so \( \hat{\Sigma}(0,0) = \frac{1}{4} \sum_{\gamma \in A} e_{\gamma\gamma} \). Using (10), we obtain

\[ \hat{\Sigma}(1,0) = \frac{1}{20} (0, 2, 1, 2, 0, 2, 2, 1, 5, 0, 0, 0, 1, 2, 2)^t, \]

where \( A^2 \) is given the lexicographic order on pairs, as in example 5.3. We now compute its spectral hull.

One checks (by looking at letter orbits) that the ergodic classes of the bisubstitution are: \( F_1 = \{00, 11, 22, 33\} \) and \( F_2 = \{01, 02, 03, 10, 12, 13, 20, 21, 23, 30, 31, 32\} \), which partition \( A^2 \) completely, leaving no transient part as is always the case for bijective substitutions. Using lemma 4.7, we have \( \mathbf{v} \in K \) if and only if

\[ \mathbf{v} = \begin{pmatrix} w_1 & w_2 & w_2 & w_2 \\ w_2 & w_1 & w_2 & w_2 \\ w_2 & w_2 & w_1 & w_1 \end{pmatrix} \gg 0 \implies \begin{cases} w_1 > 0, \\ (w_1 + w_2)(w_1 - w_2) > 0, \\ w_1(w_1 + w_2)(w_1 - w_2) > 0, \\ (w_1 - w_2)^3(w_1 + 3w_2) > 0 \end{cases} \]
as \( \psi \gg 0 \) if and only if its principal minors are positive definite. These inequalities show us that \( \mathcal{K} \) has two extremal rays determined by

\[
\nu_1 = \mathcal{F}_1 + \mathcal{F}_2 \quad \text{and} \quad \nu_2 = \mathcal{F}_1 - \frac{1}{3} \mathcal{F}_2
\]

and so, along with the above, we have

\[
\lambda_{\nu_1}(1, 0) = 1 \quad \text{and} \quad \lambda_{\nu_2}(1, 0) = \nu_2(\mathcal{E}(1, 0)) = -\frac{1}{15},
\]

so that none of the measures coming from the spectral hull are Lebesgue, and thus the spectrum of the Table is singular to Lebesgue measure on \( \mathbb{T}^2 \), the two Torus.

We now take a brief detour to discuss a particular aspect of substitutions which affects their discrete component, and its impact on the spectrum of a given substitution.

### 5.2 Height and Dekking’s Criterion

Most of our understanding of the discrete spectrum of constant length substitutions is due to the work of Dekking and the notion of height of a substitution, see [8]. The following description of height in the \( \mathbb{Z}^d \) case is based on [8] §3.1 and [13] §6.1.1. Given a primitive and aperiodic \( \mathbf{q} \)-substitution \( \mathcal{S} \) on \( \mathcal{A} \), consider the sequence \( \mathbf{D}_n \in \mathcal{A}^{\mathbb{Z}^d} \) constructed in [8] and which was used to construct the reduced language of \( \mathcal{S} \). Let \( \mathcal{L} \) be a sublattice of \( \mathbb{Z}^d \), and let \( \mathcal{L}_0 \) be a set of class representatives for this lattice, so that \( \mathbb{Z}^d \) is the disjoint union of \( \mathbf{j} + \mathcal{L} \) for \( \mathbf{j} \in \mathcal{L}_0 \); let \( \mathbf{D}_n(\mathbf{j} + \mathcal{L}) \) denote the letters appearing in positions \( \mathbf{j} + \mathbf{l} \) for \( \mathbf{l} \in \mathcal{L} \). By primitivity we know \( \mathcal{A} = \bigcup_{\mathbf{j} \in \mathcal{L}_0} \mathbf{D}_n(\mathbf{j} + \mathcal{L}) \), though in some cases the \( \mathbf{D}_n(\mathbf{j} + \mathcal{L}) \) may form a partition of \( \mathcal{A} \). When that happens, one can identify all the letters in each \( \mathbf{D}_n(\mathbf{j} + \mathcal{L}) \) with a single representative, and this gives a map \( \mathcal{A} \rightarrow \mathcal{A} \). When extended pointwise to \( \mathcal{A}^{\mathbb{Z}^d} \), this map takes \( \mathbf{D}_n \) to a periodic sequence. By primitivity, this identification takes \( \mathcal{X}_\mathcal{S} \) onto a finite set. If we let \( \mathcal{L} = \mathbb{Z}^d \), this reduces \( \mathcal{X}_\mathcal{S} \) to a singleton. The height lattice of \( \mathcal{S} \) is the smallest lattice \( \mathcal{L} \) (in the subset order) for which the above partition property holds. In this case, the factor map which sends each \( \mathbf{D}_n(\mathbf{j} + \mathcal{L}) \) to a representative gives rise to the maximal equicontinuous factor of \( (\mathcal{X}_\mathcal{S}, T, \mu) \), see [8].

The following theorem of Dekking completely characterizes the discrete spectrum for us in the case of aperiodic substitutions of constant length, see [8] Thm 6.1, 6.2 for the \( d = 1 \) case, and [8] §3.1 for the general case. For a sublattice \( \mathcal{L} \subset \mathbb{Z}^d \), let \( \nu_\mathcal{L} \) be the uniform probability measure supporting the quotient \( \mathbb{Z}^d/\mathcal{L} \subset \mathbb{T}^d \). In the case \( \mathcal{L} = h\mathbb{Z}^d \) for \( h \gg 1 \), the measure \( \nu_\mathcal{L} = \nu_h \).

**Theorem 5.5** (Dekking). If \( \mathcal{S} \) is a primitive and aperiodic \( \mathbf{q} \)-substitution with height lattice \( \mathcal{L} \), then the discrete component of \( \sigma_{\text{max}} \) is equivalent to \( \omega_{\mathbf{q}} \ast \nu_\mathcal{L} \).

If \( \mathcal{L} = \mathbb{Z}^d \), then the spectrum is pure discrete if and only if some generalized instruction of \( \mathcal{S} \) is constant.

The statement for the pure discrete case is referred to as trivial height (\( \mathcal{L} = \mathbb{Z}^d \)) and the coincidence condition (a constant generalized instruction). In the primitive case the coincidence condition is equivalent to the bisubstitution possessing only one ergodic class, as was observed by Queffelec. More generally, the discrete spectrum of an aperiodic \( \mathbf{q} \)-substitution is given by the sum of \( \omega_{\mathbf{q}} \ast \nu_\mathcal{L} \) as \( \mathcal{L} \) ranges over the height lattices of its primitive components.

We now describe a class of commutative and bijective substitutions which attain any height lattice of the form \( h\mathbb{Z}^d \). Fix \( h \gg 1 \) in \( \mathbb{Z}^d \) and let \( \mathcal{A} = \mathcal{A}_h := \mathbb{Z}^d/2h\mathbb{Z}^d \) be the quotient ring of \( \mathbb{Z}^d \) integers modulo \( 2h \), using the residue class \([0, 2h] \) to represent the letters. For each \( \mathbf{k} \in \mathbb{Z}^d \) let \( \pi_\mathbf{k} : \mathcal{A} \rightarrow \mathcal{A} \) be the map

\[
\pi_\mathbf{k} : \alpha \mapsto \alpha + \mathbf{k} \pmod{2h}.
\]

Note that each \( \pi_\mathbf{k} \) is bijective as a map on \( \mathbb{Z}^d/2h\mathbb{Z}^d \), and they form the commutative group generated by \( \pi_i := \pi_1 \) for \( 1 \leq i \leq d \), which is a subgroup of permutations of \( \mathcal{A} \), with \( \pi_0 \) giving the identity map on \( \mathcal{A} \).

Let \( \mathbf{q} := h + 1 \), so that \( [0, \mathbf{q}] = [0, h] \). Consider the \( \mathbf{q} \)-substitution \( \mathcal{H} := \mathcal{H}_h \) on \( \mathcal{A}_h \) determined by the instructions \( \mathcal{R}_\mathbf{k} = \pi_\mathbf{k} \) for \( \mathbf{k} \in [0, h] \). As the instructions of \( \mathcal{H} \) are a subset of a commutative group of permutations, \( \mathcal{H} \) is a commutative and bijective \( \mathbf{q} \)-substitution. As a map \( \mathcal{H} : \mathcal{A} \rightarrow \mathcal{A}^+ \), the \( \mathbf{q} \)-substitution \( \mathcal{H} \) takes the letter \( \alpha \) to the block on \([0, h]\) whose value at \( \mathbf{j} \) is \( \pi_\mathbf{j}(\alpha) = \alpha + \mathbf{j} \pmod{2h} \). Therefore, as
\{\alpha + j + k : j, k \in [0, h]\} covers an entire equivalence class modulo 2h, it follows that \(H^2(\alpha)\) covers \(A\) for every \(\alpha\), and \(H\) is primitive.

Using proposition [3.2] and the above, we have for \(n > 0\) and \(j = j_0 + j_1q + \ldots + j_{n-1}q^{n-1} \in \mathbb{Z}_d\)

\[
R_j^{(n)} = R_{j_0} \cdots R_{j_{n-1}} \implies R_j^{(n)}(\alpha) \equiv \alpha + j_0 + \ldots + j_{n-1} \pmod{2h}
\]

so that

\[
H^\alpha(\alpha)^j = \beta \iff j_0 + \ldots + j_{n-1} \equiv \beta - \alpha \pmod{2h}
\]

Using this, one checks that both the return times and the correlation vector are \emph{independently invariant} under permutations of coordinates in \(\mathbb{Z}_d^n\) and order of q-adic digits.

We now compute the height \(h\) of \(H_h\); let \(D_\eta \in \mathcal{X}_H\) be the sequence generating rise to the reduced language, generated by telescoping \(H\) and iterating on a seed patch about the origin: in other words, \(D_\eta\) is a fixed point of some iterate of \(H\). Writing for \(a > 1\)

\[
C^\alpha_\eta := D_\eta (j + a\mathbb{Z}_d^d) \quad \text{and} \quad C^\alpha := \{C^\alpha_\eta : j \in [0, a]\}
\]

so that the height \(h\) is maximal amongst \(a\) with \(a\mathbb{Z}_d^d + q\mathbb{Z}_d^d = \mathbb{Z}_d^d\) for which \(C^\alpha\) partitions \(A\). Using (13), one can show that \(H^\alpha\) partitions \(A\) as well, and thus \(h \geq h\). Alternatively, if \(2 \leq k < n\) and \(i \in [0, h]\) we have

\[
H^(\alpha)(0) = H^n(\alpha)(h+i(q)+iq) = H^n(\alpha)(h+i(q)+iq) = \alpha
\]

as the q-adic digits of the indices add up to \(0\) modulo \(2h\). As the difference between the second pair of indices is \(ih\), it follows that \(h\) divides \(h\) as \(C^\alpha\) partitions \(A\) by definition, and \(h = h\), as desired.

We now briefly examine the bisubstitution \(H \otimes H\). As \(H\) is bijective, it has no transient part, and the subalphabets

\[
A^2_j := \{\alpha \beta \in A^2 : \alpha - \beta \equiv j \pmod{2h}\}
\]

for \(j \in [0, 2h]\) form a partition of the bisubstitution \(A^2\) into \(2h\) sets of size \(\text{Card}(0, 2h)\), and with respect to which \(H \otimes H\) is primitive. It follows that the \(A_j^2\) partition \(A^2\) and thus form the ergodic classes of the bisubstitution. Thus, if \(v\) is a left \(q\)-eigenvector of \(C_S\), we can write \(v\) in matrix form

\[
v \approx \sum_{j \in [0, 2h]} w_j R_j,
\]

as \(R_j\) corresponds to the ergodic class \(A_j^2\); note also that \(R_0 = I\).

Note that in the one-dimensional setting, the \(h = 1\) case gives the Thue-Morse substitution. As all of these are examples of bijective and commutative substitutions, theorem [5.2] shows they are purely singular. What is not clear, however, is the role played by height. In the next example, we use the algorithm to compute the spectrum for the case \(h = 3\), and show how the height comes out in this computation.

**Example 5.6.** For \(h = 3\), this gives us the substitution \(H\), whose bisubstitution has the ergodic classes (represented in matrix form)

\[
H : \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{bmatrix}
\]

and \(v = \sum_{j=0}^{5} w_j R_j\)

the eigenvalues of which are (using Python’s \texttt{sympy} toolbox to compute eigenvalues of symbolic matrices)

\[
\begin{bmatrix}
w_0 + w_1 + w_2 + w_3 + w_4 + w_5 \\
w_0 + w_1 + w_2 - w_3 + w_4 - w_5 \\
w_0 + \frac{1}{2}w_1 - \frac{1}{2}w_2 - w_3 - \frac{1}{2}w_4 + \frac{1}{2}w_5 - \frac{2}{\sqrt{2}}(w_1 + w_2 - w_4 - w_5) \\
w_0 + \frac{1}{2}w_1 - \frac{1}{2}w_2 - w_3 - \frac{1}{2}w_4 + \frac{1}{2}w_5 + \frac{2}{\sqrt{2}}(w_1 + w_2 - w_4 - w_5) \\
w_0 + \frac{1}{2}w_1 - \frac{1}{2}w_2 + w_3 - \frac{1}{2}w_4 - \frac{1}{2}w_5 - \frac{2}{\sqrt{2}}(w_1 - w_2 + w_4 - w_5) \\
w_0 - \frac{1}{2}w_1 - \frac{1}{2}w_2 + w_3 - \frac{1}{2}w_4 + \frac{1}{2}w_5 + \frac{2}{\sqrt{2}}(w_1 - w_2 + w_4 - w_5)
\end{bmatrix}
\]
so that enforcing positive definiteness forces \( w_4 = \overline{w_2} \) and \( w_5 = \overline{w_1} \) as it must be Hermitian, and we write \( w_1 = \alpha + \beta i \) and \( w_2 = a + bi \). Positivity of eigenvalues gives the equations:

\[
\begin{align*}
& w_0 + w_3 + 2\alpha + 2a > 0 \\
& w_0 - w_3 - 2\alpha + 2a > 0 \\
& w_0 - w_3 + \alpha - a - \sqrt{3}(\beta + b) > 0 \\
& w_0 - w_3 + \alpha - a + \sqrt{3}(\beta + b) > 0 \\
& w_0 + w_3 - \alpha - a - \sqrt{3}(\beta - b) > 0 \\
& w_0 + w_3 - \alpha - a + \sqrt{3}(\beta - b) > 0
\end{align*}
\]

with extremal rays

\[
\begin{align*}
(1,1,1,0,0) & \\
(1,-1,-1,0,0) & \\
(1,-1,1,-\frac{1}{\sqrt{3}},-\frac{\sqrt{3}}{3}) & \\
(1,-1,\frac{1}{\sqrt{3}},\frac{\sqrt{3}}{3}) & \\
(1,1,-\frac{1}{\sqrt{3}},\frac{\sqrt{3}}{3}) & \\
(1,1,\frac{1}{\sqrt{3}},-\frac{\sqrt{3}}{3})
\end{align*}
\]

where the vectors are \((w_0, w_3, \alpha, a, \beta, b)\), found by intersecting any 5 of them and letting \( w_0 = 1 \). Thus, by lemma 4.7 the extreme rays of \( K \) are obtained from the above via the identification \( w_1 = \alpha + \beta i \) and \( w_2 = a + bi \) and we obtain \( \mathbf{v} = \sum_{j=0}^{5} w_j \mathbf{R}_j \) for \((w_0, w_1, w_2, w_3, w_4, w_5)\):

\[
\begin{align*}
& \mathbf{v}_1 \approx (1,1,1,1,1,1) \\
& \mathbf{v}_2 \approx (1,-\frac{1}{2} - \frac{\sqrt{3}}{2} i, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, 1, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, -\frac{1}{2} + \frac{\sqrt{3}}{2} i) \\
& \mathbf{v}_3 \approx (1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i) \\
& \mathbf{v}_4 \approx (1,-1,-1,1,1,1) \\
& \mathbf{v}_5 \approx (1,\frac{1}{2} - \frac{\sqrt{3}}{2} i, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, -1, -\frac{1}{2} + \frac{\sqrt{3}}{2} i, 1 + \frac{\sqrt{3}}{2} i) \\
& \mathbf{v}_6 \approx (1,\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, 1 + \frac{\sqrt{3}}{2} i)
\end{align*}
\]

Now, as \( \mathcal{H} \) is bijective, \( \frac{1}{2} M \mathcal{H} \) is both row and column stochastic, so its Perron vector is \( \frac{1}{3} \mathbf{1} \), and \( \hat{\Sigma}(0) = \sum_{\gamma \in \Delta} \frac{1}{3} \mathcal{E}_{\gamma} \). Using equation (10) and theorem 4.3 we obtain for the Fourier coefficients of \( \Sigma \) (in matrix form)

\[
\begin{align*}
\hat{\Sigma}(0) &= \frac{1}{6} \mathbf{R}_0, \\
\hat{\Sigma}(1) &= \frac{1}{30} \mathbf{R}_2 + \frac{4}{30} \mathbf{R}_5, \\
\hat{\Sigma}(2) &= \frac{2}{30} \mathbf{R}_1 + \frac{3}{30} \mathbf{R}_4, \\
\hat{\Sigma}(3) &= \frac{3}{30} \mathbf{R}_0 + \frac{2}{30} \mathbf{R}_3
\end{align*}
\]

Letting \( \lambda_j := \lambda_{\nu_j} \), this gives \( \lambda_1 = \delta \) as usual, and

\[
\begin{align*}
\hat{\lambda}_1(1) &= 1, & \hat{\lambda}_1(2) &= 1, & \hat{\lambda}_1(3) &= 1 \\
\hat{\lambda}_2(1) &= -\frac{1}{2} - \frac{\sqrt{3}}{2} i, & \hat{\lambda}_2(2) &= -\frac{1}{2} - \frac{\sqrt{3}}{2} i, & \hat{\lambda}_2(3) &= 1 \\
\hat{\lambda}_3(1) &= -\frac{1}{2} + \frac{\sqrt{3}}{2} i, & \hat{\lambda}_3(2) &= -\frac{1}{2} + \frac{\sqrt{3}}{2} i, & \hat{\lambda}_3(3) &= 1
\end{align*}
\]

and one checks from Fourier unicity and equation (5) that \( \lambda_1 + 2\lambda_2 + 2\lambda_3 = 5\nu_3 \), which by Dekking’s theorem is the discrete spectrum of \( \mathcal{H} \). Additionally, we have

\[
\begin{align*}
\hat{\lambda}_4(1) &= -\frac{3}{5}, & \hat{\lambda}_4(2) &= \frac{1}{5}, & \hat{\lambda}_4(3) &= \frac{1}{5} \\
\hat{\lambda}_5(1) &= \frac{3}{10} + \frac{3\sqrt{3}}{10} i, & \hat{\lambda}_5(2) &= -\frac{1}{10} + \frac{\sqrt{3}}{10} i, & \hat{\lambda}_5(3) &= \frac{1}{5} \\
\hat{\lambda}_6(1) &= \frac{3}{10} - \frac{3\sqrt{3}}{10} i, & \hat{\lambda}_6(2) &= -\frac{1}{10} - \frac{\sqrt{3}}{10} i, & \hat{\lambda}_6(3) &= \frac{1}{5}
\end{align*}
\]

from which one can see that \( \lambda_4 + \lambda_5 + \lambda_6 \) is equal to \( \nu_3 \ast \lambda \) for some (singular continuous) measure \( \lambda \) on \( \mathbb{T} \), as the Fourier coefficients which are not multiples of 3 all vanish and \( \lambda \) is invariant for the times 3 map on \( \mathbb{T} \). We know that it is continuous as it is singular to \( \nu_3 \) by Queffelec’s theorem, which is the entire discrete spectrum of \( \mathcal{H} \) by Dekking’s theorem, and it is singular as it has nonvanishing Fourier coefficients. Thus, \( \sigma_{\max} \sim \omega_2 \ast \nu_3 + \omega_2 \ast \nu_3 \ast \lambda \), for \( \lambda \) singular continuous on \( \mathbb{T} \).
By theorem 5.5 a primitive $q$-substitution of trivial height has pure discrete spectrum if and only if some
generalized instruction $R(n)$ is constant (Dekking’s coincidence condition). It follows that primitive and
aperiodic bijective $q$-substitutions cannot be pure discrete in the case of trivial height, and moreover, this
implies they have at least two ergodic classes, as noted before. Thus, their spectral hulls will always give rise
to continuous measures. None of the bijective examples we consider here possess Lebesgue component, and
using software to automate the above algorithm, we have excluded Lebesgue component from the spectrum
of all bijective substitutions of constant length (the $Z$ case) 2, 3, 4, and 5 on alphabets of 2, 3, 4, and 5 letters.

Our correction of Queffelec’s example, together with the extension of Baake and Grimm’s result in theorem
5.2 suggests that all bijective $q$-substitution may have spectrum singular to Lebesgue measure.

We now look at an example which is known to have Lebesgue component in its spectrum, the Rudin-
Shapiro substitution. Although this fact is well-established, it is interesting to see how the details work out
allowing for all the terms $v^*\Sigma(k)$ for $k \neq 0$ to vanish. After this example, we discuss a class of generalizations
of Rudin-Shapiro, due to Frank, which share its absolutely continuous spectral components.

**Example 5.7 (Rudin-Shapiro).** The Rudin-Shapiro substitution $\rho$ is the 2-substitution on $\{0, 1, 2, 3\}$

$$\rho : \begin{cases}
0 \mapsto 02 \\
1 \mapsto 32 \\
2 \mapsto 01 \\
3 \mapsto 31
\end{cases}$$

with $M_\rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\end{pmatrix}$

As $M_\rho^2 > 0$, the Rudin-Shapiro substitution $\rho$ is primitive, and as $\rho^2(0) = 0201$ with $\rho(0)_0 = 0$ the symbol 0
can be proceeded by both the symbols 1 and 2, it follows from Pansiot’s lemma that $\rho$ is aperiodic. As $\frac{1}{2}M_\rho$
is both row and column stochastic, its Perron vector is $\frac{1}{2}(1, 1, 1, 1)'$ and so $\hat{\Sigma}(0) = \sum_{\gamma \in A} \frac{1}{2}e_{\gamma\gamma}$. As $q = 2$, we
have $\Delta_1(1) = \{1\}$ and so equation (10) gives us

$$\hat{\Sigma}(1) = (2I - R_1 \otimes R_0)^{-1}R_0 \otimes R_1 \hat{\Sigma}(0) = \frac{1}{8}(0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

where the basis for $\mathbb{C}^4$ is given the lexicographic order: 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33.

Using theorem 4.3 for $k = 2$ and $p = 1$ we obtain

$$\hat{\Sigma}(2) = \frac{1}{2}(R_0 \otimes R_0 + R_1 \otimes R_1)\hat{\Sigma}(0) = \frac{1}{8}(1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1)$$

and one can check that $\hat{\Sigma}(2n) = \hat{\Sigma}(2)$ for $n \neq 0$ and $\hat{\Sigma}(2n + 1) = \hat{\Sigma}(1)$ for $n \in \mathbb{Z}$. Note that $\hat{\Sigma}(1) \perp \hat{\Sigma}(2)$

We now compute the ergodic decomposition of $\rho \otimes \rho$, the bisubstitution. As usual for primitive substitutions, $F_1 = \{00, 11, 22, 33\}$, and in this case the only other ergodic class is $F_2 = \{03, 12, 21, 30\}$, so that

$T = \{01, 02, 10, 13, 20, 23, 31, 32\}$ is the transient part. Using lemma 4.7 we have $v \in \mathcal{K}$ if and only if

$$\hat{v} = \begin{pmatrix} w_1 & 0 & 0 & w_2 \\
0 & w_1 & w_2 & 0 \\
0 & w_2 & w_1 & 0 \\
w_2 & 0 & 0 & w_1 \\
\end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & w_1 + w_2 & w_1 + w_2 & 0 \\
0 & w_1 + w_2 & 0 & w_1 + w_2 \\
0 & w_1 + w_2 & 0 & w_1 + w_2 \\
w_1 + w_2 & 0 & w_1 + w_2 & 0 \\
\end{pmatrix} > 0$$

As the matrices

$$\begin{pmatrix} 1 & 1 & 1 \\
1 & 1 \\
1 \\
\end{pmatrix}, \begin{pmatrix} 1 & 1 \\
1 \\
1 \\
\end{pmatrix}, \begin{pmatrix} 1 & 1 \\
1 \\
1 \\
\end{pmatrix}$$

commute and are diagonalizable, one checks that for

$$S = \begin{pmatrix} 1 & 0 & -1 & 1 \\
-1 & -1 & 0 & 1 \\
-1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
\end{pmatrix}$$

we have $S^{-1}\hat{v}S = \begin{pmatrix} 0 & w_1 - w_2 & w_1 - w_2 \\
w_1 - w_2 & 2w_1 + 2w_2 \\
\end{pmatrix}$
so that $\mathbf{v}$ is strongly positive if and only if $-w_1 < w_2 < w_1$, the extreme points of which are given by the vectors $(w_1, w_2) = (1, 1)$ or $(1, -1)$. Thus, the extreme rays of $K$ are given by

$$\mathbf{v}_1 = 1 \quad \text{and} \quad \mathbf{v}_2 = (1, 0, 0, -1, 0, 1, -1, 0, 0, -1, 0, 0, 1)^T$$

As usual, $\lambda_{\mathbf{v}_1} = \delta$, and using the computed values of $\hat{\Sigma}(k)$, one checks that $\hat{\lambda}_{\mathbf{v}_2}(k) = 0$ for $k \neq 0$, and so $\lambda_{\mathbf{v}_2}$ is Lebesgue measure. As $m$ is $q$-invariant for all $q$, we have $\sigma_{\max} \sim \omega_2 + m$.

We now briefly discuss Frank’s generalizations of the Rudin-Shapiro sequence, see [9]. Consider the alphabet on $2Q$ letters, represented by $A = \{1, \ldots, Q,-1, \ldots,-Q\}$ and consider the collection of instructions

$$\mathcal{F}_Q := \{\mathcal{R} : A \to A \mid \exists \, \gamma \text{ with } \mathcal{R} : A \to \{\gamma, -\gamma\} \text{ and } \mathcal{R}(-\alpha) = -\mathcal{R}(\alpha) \, \forall \, \alpha \in A\}$$

so that $\mathcal{F}_Q$ consists of those instructions which are morphisms for negation and take on exactly one value in $1, \ldots, Q$. If $\mathcal{R} \in \mathcal{F}_Q$ takes on the values $\gamma$ and $-\gamma$, its sign vector is a $\pm 1$ vector in $\mathbb{C}^Q$ whose $\alpha$ component is 1 if $\mathcal{R}(\alpha) = \gamma$ and -1 if $\mathcal{R}(\alpha) = -\gamma$. As instructions in $\mathcal{F}_Q$ preserve negation, every $\mathcal{R} \in \mathcal{F}_Q$ is determined by a letter in $1, \ldots, Q$ and a sign vector. For example, if $Q = 4$, the letter 3 and sign vector $(+1, -1, -1, +1)$ determine the instruction

$$\mathcal{R} : \begin{cases} 1 \mapsto 3, & 1 \mapsto -3 \\ 2 \mapsto -3, & 2 \mapsto 3 \\ 3 \mapsto -3, & 3 \mapsto 3 \\ 4 \mapsto 3, & 4 \mapsto -3 \end{cases}$$

A Hadamard matrix is a square $\pm 1$-matrix whose rows (and columns) are orthogonal, and are necessarily even dimensional. Every $Q \times Q$ Hadamard matrix determines $Q$ instructions in $\mathcal{F}_Q$ in the following way: as the columns of a Hadamard matrix are sign vectors, the $i$-th column paired with the letter $i$ in $\{1, \ldots, Q\}$ gives an instruction via the above association. For example $Q = 2$ corresponds to the Rudin-Shapiro substitution via the Hadamard matrix $H = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Here, the instructions are represented by

$$\mathcal{R}_0 : \begin{cases} 1 \mapsto 1, & 1 \mapsto -1 \\ 2 \mapsto -1, & 2 \mapsto 1 \end{cases} \quad \text{and} \quad \mathcal{R}_1 : \begin{cases} 1 \mapsto -2, & 1 \mapsto 2 \\ 2 \mapsto 2, & 2 \mapsto -2 \end{cases}$$

For a Hadamard matrix $H$, let $\mathcal{I}(H)$ be the corresponding instructions induced by $H$. If $q \geq 1$ in $\mathbb{Z}^d$ has expansion $Q$, then any configuration $\mathcal{R} : \{0, q\} \to \mathcal{I}(H)$ gives rise to a $q$-substitution.

**Theorem 5.8** (Frank). *Let $H$ be a $Q \times Q$ Hadamard matrix and $q > 1$ in $\mathbb{Z}^d$ such that $Q = q_1 \cdots q_d$. Let $\mathcal{R} : \{0, q\} \to \mathcal{I}(H)$ be any configuration on the instructions induced by $H$. The $q$-substitution determined by the configuration $\mathcal{R}$ has Lebesgue spectral components, with multiplicity $Q$.*

As suggested in [9] [5.1], any configuration of the instructions $\mathcal{I}(H)$ give rise to substitutions in $\mathbb{Z}^d$ with Lebesgue spectrum. As convolution with $\omega_q$ has no effect on absolutely continuous spectrum, theorem 5.8 tells us that presence of Lebesgue component is determined by the spectral hull and the correlation vector, the first of which is already configuration invariant. Thus, for all substitutions of the above type, any difference in the correlation vector arising from a change in configuration is orthogonal to the vectors in the spectral hull giving rise to Lebesgue measure. This raises an interesting question: is presence of absolutely continuous component a configuration invariant in general? If the configuration $\mathcal{R}$ gives a $q$-substitution with Lebesgue component in its spectrum, then there is a $\mathbf{v}$ in the spectral hull for which $\hat{\Sigma}(k)$ is orthogonal to $\mathbf{v}$ for all $k \neq 0$. As a change in configuration is a relabeling of the indices of the instructions, one can use theorem 5.8 to compare correlation vectors for substitutions with the same set of instructions, but with different configurations.

### 5.3 The Substitution Product

Given the role the Kronecker product plays in our analysis, we take a moment to consider some of its properties. For example, it is an associative product, and while not commutative we still have $A \otimes B \approx B \otimes A$. 
where ≈ means permutation equivalent: there exists a 0, 1 unitary matrix $P$ with $P(A \otimes B)P^* = (B \otimes A)$, and this permutation depends only on the dimensions of $A$ and $B$. Moreover, the spectrum of Kronecker products are closely related to their factors: if $\lambda_i$ and $\mu_j$ are the eigenvalues of $A$ and $B$, respective, with corresponding eigenvectors $x_i$ and $y_j$, then the eigenvalues of $A \otimes B$ are the pairs $\lambda_i\mu_j$ corresponding to the eigenvector $x_i \otimes y_j$. This follows from the Kronecker products mixed product property:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

As the instructions for a substitution product are the Kronecker products of the instructions of its factors, and ergodic classes depend only on the orbits of the generalized instructions, it follows from the mixed product property that the the ergodic classes for the bisubstitution of a substitution product are given by the pairwise ergodic classes of the bisubstitutions of its factors. This gives the following:

**Proposition 5.9.** Let $S$ and $\tilde{S}$ be $q$-substitutions on the alphabets $A$ and $\tilde{A}$. Let $E$ and $\tilde{E}$ denote the ergodic classes of their respective bisubstitutions, with transient classes $T$ and $\tilde{T}$. Then the ergodic classes of the bisubstitution of $S \otimes \tilde{S}$ are given by the collections $E_i \otimes \tilde{E}_j$, as $i, j$ range over the indices for the respective bisubstitutions; here the transient part is given by $A \otimes \tilde{T} \cup T \otimes \tilde{A}$.

Our last example is a substitution of constant length 2 on 8 symbols possessing all three pure types in its spectrum and allows us to illustrate an interesting property of the substitution product, see also [1, \( \bullet \) 2].

**Example 5.10.** Consider the Thue-Morse and Rudin-Shapiro substitutions of constant length 2 represented on the alphabets $A_\tau = \{\circ, \_\}$ and $A_\rho = \{a, b, c, d\}$, respectively, by

$$\tau : \begin{cases} \circ \mapsto \circ \\ \_ \mapsto \_ \end{cases} \quad \text{and} \quad \rho : \begin{cases} a \mapsto ac, & b \mapsto dc \\ c \mapsto ab, & d \mapsto db \end{cases}$$

and consider the substitution $S$ of constant length 2 on $A = \{a, b, c, d, a, b, c, d\}$, given by

$$S : \begin{cases} a \mapsto ac & a \mapsto ac \\ b \mapsto dc & b \mapsto dc \\ c \mapsto ab & c \mapsto ab \\ d \mapsto db & d \mapsto db \end{cases}$$

which is equivalent to the substitution product $\tau \otimes \rho$ via the obvious map $A_\tau \otimes A_\rho \to A$. Note that it is aperiodic being the substitution product of aperiodic substitutions, and primitivity follows as $M_S^2$ is positive. The Perron vector of $M_S$ is the vector all of whose entries are 1/8, so that $\Sigma(0) = \frac{1}{8}I$, in matrix form.

The ergodic classes for the bisubstitutions of $\tau$ and $\rho$ are given by

$$\tau : \begin{cases} E_1^\tau = \{\circ, \_\} \\ E_2^\tau = \{\_\, \circ\} \end{cases} \quad \text{and} \quad \rho : \begin{cases} E_1^\rho = \{aa, bb, cc, dd\} \\ E_2^\rho = \{ad, bc, cb, da\} \end{cases} \quad \text{with} \quad T^\rho = \{ab, ac, ba, bd, ca, cd, db, dc\}$$

so that one can see the relationship between the ergodic decompositions of the bisubstitutions of $\tau, \rho$ and $\tau \otimes \rho$ indicated by the above proposition:

$$\begin{cases} E_1 = E_1^\tau E_1^\rho \\ E_2 = E_1^\tau E_2^\rho \end{cases} \quad \text{and} \quad T = E_1^\tau T^\rho \cup E_2^\tau T^\rho.$$ 

Using lemma 5.7 we find that $v \in K$ if $\hat{v} \gg 0$ where

$$\hat{v} = \begin{pmatrix} w_1 & 1/2 w_1 + 1/2 w_2 & 1/2 w_1 + 1/2 w_2 & w_2 & w_3 & w_4 & 1/2 w_3 + 1/2 w_4 & 1/2 w_3 + 1/2 w_4 & w_4 \\ 1/2 w_1 + 1/2 w_2 & w_1 & w_2 & 1/2 w_1 + 1/2 w_2 & w_3 & w_4 & w_3 & w_4 & 1/2 w_3 + 1/2 w_4 \\ 1/2 w_1 + 1/2 w_2 & w_2 & w_2 & 1/2 w_1 + 1/2 w_2 & 1/2 w_3 + 1/2 w_4 & w_4 & w_3 & w_4 & 1/2 w_3 + 1/2 w_4 \\ w_2 & w_3 & 1/2 w_3 + 1/2 w_4 & 1/2 w_3 + 1/2 w_4 & w_4 & w_3 & w_4 & w_3 & 1/2 w_3 + 1/2 w_4 \\ w_3 & 1/2 w_3 + 1/2 w_4 & w_4 & 1/2 w_3 + 1/2 w_4 & 1/2 w_1 + 1/2 w_2 & w_1 & w_1 & w_2 & 1/2 w_1 + 1/2 w_2 \\ 1/2 w_3 + 1/2 w_4 & w_4 & w_3 & 1/2 w_3 + 1/2 w_4 & 1/2 w_1 + 1/2 w_2 & w_2 & w_2 & w_1 & 1/2 w_1 + 1/2 w_2 \\ 1/2 w_3 + 1/2 w_4 & w_4 & w_3 & 1/2 w_3 + 1/2 w_4 & 1/2 w_1 + 1/2 w_2 & w_2 & w_2 & w_1 & 1/2 w_1 + 1/2 w_2 \\ w_4 & 1/2 w_3 + 1/2 w_4 & 1/2 w_3 + 1/2 w_4 & w_3 & w_2 & w_2 & 1/2 w_1 + 1/2 w_2 & 1/2 w_1 + 1/2 w_2 & w_1 \end{pmatrix}$$

25
and with basis ordered: \((a, b, c, d, a, b, c, d)\). Note that \(\tilde{\nu}\) can be expressed as the Kronecker product of the \(\nu\) for Thue-Morse (example 5.1) and Rudin-Shapiro (example 5.7):

\[
\tilde{\nu} = \begin{pmatrix} \nu_1^\tau & \nu_2^\tau \\ \nu_2^\tau & \nu_1^\tau \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2}(\nu_1^p + \nu_2^p) & \frac{1}{2}(\nu_1^p + \nu_2^p) & \nu_2^p \\ \frac{1}{2}(\nu_1^p + \nu_2^p) & \frac{1}{2}(\nu_1^p + \nu_2^p) & \nu_2^p \\ \frac{1}{2}(\nu_1^p + \nu_2^p) & \frac{1}{2}(\nu_1^p + \nu_2^p) & \nu_2^p \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} w_1 = w_1^2 w_1^p \\ w_2 = w_1^2 w_1^p \\ w_3 = w_2^2 w_1^p \\ w_4 = w_2^2 w_1^p \end{pmatrix}
\]

As the eigenvalues of a Kronecker product are the product of the eigenvalues of its factors, it follows that the spectrum of \(\tilde{\nu}\) is positive if and only if

\[
\begin{align*}
(w_1^2 - w_2^2)(w_1^p - w_2^p) > 0 & \quad \Rightarrow \quad |w_1 - w_2| > |w_3 - w_4| > 0 \\
(w_1^2 - w_2^2)(w_1^p + w_2^p) > 0 & \quad \Rightarrow \quad w_1 + w_2 > w_3 - w_4 > 0 \\
(w_1^2 + w_2^2)(w_1^p - w_2^p) > 0 & \quad \Rightarrow \quad w_1 - w_2 > w_3 - w_4 > 0 \\
(w_1^2 + w_2^2)(w_1^p + w_2^p) > 0 & \quad \Rightarrow \quad w_1 + w_2 > w_3 + w_4 > 0
\end{align*}
\]

are the extreme rays of this cone, which are the Kronecker products of the extreme rays of this cone for Thue-Morse and Rudin-Shapiro, see the relevant examples. It follows that the extreme rays of \(K\) are the Kronecker products of the extreme rays of the cones \(K_\tau\) and \(K_\rho\). Let \(\nu_1, \nu_2, \nu_3, \nu_4\) be the respective extreme rays of \(K\) corresponding to the four vectors \((w_1, w_2, w_3, w_4)\) indicated above.

Let \(\delta\) be the usual Dirac mass at 1, \(m\) Lebesgue measure on the circle, and \(\lambda_\tau\) the singular continuous measure in the spectrum of \(\tau\) identified in example 4.1. Using software with [14] and theorem 4.9 we can compute \(\tilde{\Sigma}(k)\) and using theorem 4.8 we can compute the Fourier coefficients of the measures \(\lambda_j = \nu_j^* \Sigma\); they are vectors in \(C^{64}\) so we do not include the computations here. As usual, \(\lambda_1 = \delta\), and comparing the first 100 Fourier coefficients suggests that \(\lambda_2 = \lambda_4 = m\) and \(\lambda_3 = \lambda_\tau\). In [14], Baake, Gähler, and Grimm show that \(S\) has pure discrete, singular continuous, and absolutely continuous spectrum, strengthening this hypothesis. This suggests that \(\sigma_{\tau \gamma_{\rho}} \sim \sigma_\tau + \sigma_\rho\), and raises the following interesting question: in general, how is the spectrum of a substitution product related to the spectra of its factors?

6 Appendix - Proofs

The appendix is in three sections, the purpose of the first two is to prove theorem 5.8 extended from Quefflec’s results for substitutions of constant length on \(A^\mathbb{N}\). First, however, we need to extend [13] Corollary 7.1 of Quefflec which allows for the diagonalization of a matrix of \(K_\tau\) over a compact abelian group, which we treat in some detail as Quefflec’s treatment is scattered throughout her text and omits enough details that an a priori extension to our setting is not possible. Moreover, we attempt to simplify her argument by removing the need for any character theory and instead rely on the Hahn-Banach theorem. This result is used in the following section to diagonalize a bicorrelation matrix - a matrix of measures related to the bisubstitution - which gives rise to the spectrum of the underlying substitution and allows us to prove the main result. We note here that the exposition contained here is more direct and linear than that contained in [13]. Finally, we briefly discuss some aspects of Riesz products used in the proof of theorem 5.8.

6.1 Generalized Functionals on \(\mathcal{M}(X)\)

Let \(X\) be a metrizable compact space. By a measure \((X)\) we mean a complex Borel measure of finite total variation, and denote by \(\mathcal{M} := \mathcal{M}(X)\) the Banach space of measures on \(X\) under the total variation norm. Here, the norm \(\|\mu\|\) is given by

\[
\|\mu\| := \sup \left\{ \sum_{i=1}^N |\mu(A_i)| : A_1, \ldots, A_N \text{ partitions } X \right\}
\]

and the total variation of \(\mu\) is the positive measure \(|\mu|(E) := \|1_E \mu\|\), where \(1_E\) is the indicator function of the set \(E\). Let \(\mathcal{M}^e\) denote the Banach space dual of \(\mathcal{M}\) consisting of continuous linear functionals \(\mathcal{M} \to \mathbb{C}\).
Proposition 6.2, which is the content of the following proposition of Sreider.

We say that a map 

\[ \psi \]

is the content of the following proposition of Sreider. The action of 

\[ \mu \]

maps 

\[ B \]

onto 

\[ \nu \]

and those about variations. The measures 

\[ L \]

are continuous with respect to 

\[ M \]

the disjoint union of the spaces 

\[ L^p(\mu) \]

for 

\[ \mu \in M \]

referring to 

\[ L^1 \]

as the integrable functions and 

\[ L^\infty \]

as the essentially bounded functions on 

\[ X \]

We now describe a localization of 

\[ M \]

in integrable functions, and of 

\[ M^* \]

in essentially bounded functions on 

\[ X \]

For 

\[ \mu \in M \]

let 

\[ L(\mu) := \{ \nu \in M : \nu \ll \mu \} \]

be the \( \mathcal{L} \)-space for 

\[ \mu \]

consisting of all measures absolutely continuous with respect to 

\[ \mu \]

The Lebesgue decomposition theorem implies that 

\[ L(\mu) \]

is a closed subspace, and that 

\[ M = L(\mu) \oplus L(\mu)^\perp \]

where 

\[ L(\mu)^\perp \]

is the set of all measures mutually singular to 

\[ \mu \]

Let 

\[ D_\mu \]

denote the projection (provided by the decomposition) of 

\[ M \]

onto 

\[ L(\mu) \]

The following is based on the classical Radon-Nikodym theorem, see [Benedetto and Czaja, Chapter 5] for details extending it to complex measures.

**Theorem 6.1.** For each 

\[ \mu \in M \]

there exist isometric isomorphisms

\[ \tilde{\mu} : L(\mu) \rightarrow L^1(\mu) \quad \text{with} \quad d\nu = \tilde{\mu}_\nu d\mu \quad \text{and} \quad \tilde{\mu}_\nu | L(\nu) = \tilde{\mu}_\nu \nu \cdot \tilde{\nu} \]

and

\[ \tilde{\mu}^* : L(\mu)^* \rightarrow L^\infty(\mu) \quad \text{with} \quad F(\nu) = \int_X \tilde{\mu}^*_\nu F d\nu \quad \text{and} \quad \tilde{\mu}^*_\nu | L(\nu)^* = \tilde{\mu}^* \]

The statements about 

\[ \tilde{\mu} \]

follow immediately from the Radon-Nikodym theorem for complex measures and those about 

\[ \tilde{\mu}^* \]

follow as

\[ L(\mu)^* \]

is isometrically isomorphic to 

\[ L^\infty(\mu) \]

when 

\[ \mu \]

has finite total variation.

Here, the identity 

\[ d\nu = \tilde{\mu}_\nu d\mu \]

holds in the sense of the Riesz Representation theorem - as continuous linear functionals on 

\[ C(X) \]

Using density arguments, this extends to integration of 

\[ L^1(\nu) \]

functions, so that 

\[ \nu(A) = \int_A \tilde{\mu}_\nu d\mu \]

for 

\[ A \]

Borel, and they agree pointwise as measures. In this sense, we can think of 

\[ L(\mu) \]

as 

\[ L^1(\mu) d\mu \]

and one can use this to show that multiplication by 

\[ \tilde{\mu}_\nu \]

is a map from 

\[ L(\nu) \rightarrow L(\mu) \]

We now use the maps 

\[ \tilde{\mu}^* \]

to describe a similar localization of 

\[ M^* \]

due to Sreider [23], giving rise to an action on 

\[ M \]

used by Queffelec.

A generalized functional 

\[ \varphi \]

on 

\[ M \]

is an association 

\[ \mu \rightarrow \varphi_\mu \in L^\infty(\mu) \]

from 

\[ M \]

to essentially bounded functions on 

\[ X \]

such that for all 

\[ \mu, \nu \in M \]

\[ \nu \ll \mu \implies \varphi_\mu = \varphi_\nu \quad \text{\( \nu \)-ae} \quad \text{and} \quad \| \varphi \| := \sup_{\| \mu \| = 1} \| \varphi_\mu \|_{L^\infty(\mu)} < \infty \]

By the above theorem, it is clear that 

\[ \mu \rightarrow \tilde{\mu}^*_\nu F \]

takes every 

\[ F \in M^* \]

to a generalized functional. Moreover, each generalized functional 

\[ \varphi \]

determines a functional on 

\[ M \]

given by 

\[ F(\nu) := \int_X \varphi_\nu d\nu \]

with 

\[ \| \varphi \| = \| F \| \]

which is the content of the following proposition of Sreider.

**Proposition 6.2 (Sreider).** The Banach dual 

\[ M^* \]

coincides with the generalized functionals on 

\[ X \]

As all our measures have finite total variation, 

\[ L^\infty(\mu) \subset L^1(\mu) \]

for all 

\[ \mu \in M \]

so that we can compose the maps 

\[ \tilde{\mu}^{-1} \circ \tilde{\mu}^* : L(\mu)^* \rightarrow L(\mu) \]

sends 

\[ F \rightarrow \tilde{\mu}^*_\mu F d\mu \]

This gives rise to an action of 

\[ M^* \]

on 

\[ M \]

\[ M^* \times M \rightarrow M \]

sending 

\[ F, \mu \rightarrow F \cdot \mu \]

with 

\[ d(F \cdot \mu) = \tilde{\mu}^*_\mu F d\mu \]

We say that a map 

\[ \psi : M \rightarrow M \]

is absolutely continuous if 

\[ \psi : L(\mu) \rightarrow L(\mu) \]

for all 

\[ \mu \in M \]

**Proposition 6.3.** The action of 

\[ M^* \]

on 

\[ M \]

is by absolutely continuous commuting operators.

**Proof.** That the action is absolutely continuous follows as 

\[ \tilde{\mu}^{-1} \circ \tilde{\mu}^* \]

is a map into 

\[ L(\mu) \]

and so 

\[ \nu \ll \mu \]

imply 

\[ F \cdot \nu \ll \nu \ll \mu \]

so that the map 

\[ \mu \rightarrow F \cdot \mu \]

is absolutely continuous. Moreover, for 

\[ F, G \in M^* \]

theorem 6.1 tells us that 

\[ \tilde{\mu}^*_G F = \tilde{\mu}^*_F G \in L^1(\mu) \]

and so

\[ d(F \cdot G \cdot \mu) = \tilde{\mu}^*_G F d(G \cdot \mu) = \tilde{\mu}^*_F d(G \cdot \mu) = \tilde{\mu}^*_F \tilde{\mu}^*_G F d\mu = \tilde{\mu}^*_G \tilde{\mu}^*_F G d\mu = d(G \cdot F \cdot \mu) \]

so that the action is commutative, completing the proof.

\[ \square \]
We say $F \in \mathcal{M}^*$ acts invariantly on $\mu \in \mathcal{M}$ if there exists $F_\mu \in \mathcal{C}$ so that $F_\mu \mu = F_\mu \mu$, a constant multiple of $\mu$. We refer to $F_\mu$ as the eigenvalue for $F$ on $\mu$. If $\nu \in \mathcal{L}(\mu)$ then $F_\mu = F_\nu \nu$-ae by theorem 6.3 so that if $F$ acts invariantly on $\mu$, it also acts invariantly on $\nu$ with the same eigenvalue.

**Proposition 6.4.** If $F \in \mathcal{M}^*$ acts invariantly on $\mu, \nu \in \mathcal{M}^*$ and $F_\mu \neq F_\nu$, then $\mu \perp \nu$.

**Proof.** If $\rho \in \mathcal{L}(\mu) \cap \mathcal{L}(\nu)$ is nonzero, then $F$ acts invariantly on $\rho$ and $F_\mu = F_\rho = F_\nu$, a contradiction. \(\square\)

We now discuss some preliminaries allowing us to extend the action of $\mathcal{M}^*$ to matrices with entries in $\mathcal{M}$. For $n \geq 1$, let $\mathcal{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices and $\mathcal{M}_n := \mathcal{M}_n(X)$ the collection of $\mathcal{M}_n(\mathbb{C})$-valued Borel measures of finite total variation. For $\mathcal{W} = (\omega_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(X)$ we write $|\mathcal{W}| := \sum_{i,j} |\omega_{ij}|$ for the total variation of $\mathcal{W}$, so that $A$ is $|\mathcal{W}|$-null if and only if $\mathcal{W}(B) = 0$ for every $B \subset A$. The following shows that the total variation’s type is a similarity invariant (over $\mathbb{C}$.)

**Proposition 6.5.** If $S \in \mathcal{M}_n(\mathbb{C})$ is invertible and $\mathcal{W} \in \mathcal{M}_n(X)$, then $|\mathcal{W}| \sim |SWS^{-1}|$.

**Proof.** If $A$ is $|\mathcal{W}|$-null then $A$ is $\omega_{ij}$-null for $1 \leq i,j \leq n$, so that $\mathcal{W}(B) = 0$ for every $B \subset A$ and $SW(B)S^{-1} = 0$ for all $B \subset A$. Thus, $|SWS^{-1}| \ll |\mathcal{W}|$, and the result follows as $S^{-1}SWS^{-1}S = \mathcal{W}$. \(\square\)

Integration with respect to $\mathcal{W}$ is done as follows: for $f \in L^1(\omega)$ the integral $\int_X f d\mathcal{W} \in \mathcal{M}_n(\mathbb{C})$ is the matrix with the integrals $\int_X f d\omega_{ij}$ as its components. All of the above notions can be extended to rectangular matrices as well as vectors, although the square case is of primary interest to us and we will not need the others beyond formalities. Finally, we abuse notation by using the symbol $0$ to denote both the zero matrix and zero matrix of measures, for any dimension of matrices.

Extend the action of $\mathcal{M}^*$ to $\mathcal{M}_n$ componentwise (as $\mathcal{M}^*\mathcal{M}_n$ is a scalar valued function for $F \in \mathcal{M}^*$)

$$\mathcal{M}^* \times \mathcal{M}_n \longrightarrow \mathcal{M}_n \quad \text{sending} \quad F, \mathcal{W} \longrightarrow F \cdot \mathcal{W} \quad \text{with} \quad d(F \cdot \mathcal{W}) = \mathcal{M}^*_n F d\mathcal{W}$$

We say that $F$ acts invariantly on $\mathcal{W} \in \mathcal{M}_n$ if there exists a $F_\mathcal{W} \in \mathcal{M}_n(\mathbb{C})$ such that $F \cdot \mathcal{W} = \mathcal{W} F_\mathcal{W}$ (or $= F_\mathcal{W} \mathcal{W}$) and refer to $F_\mathcal{W}$ as a right (or left) eigenmatrix for $F$ on $\mathcal{W}$. As $(F \cdot \mathcal{W})^* = \mathcal{W}^* F_\mathcal{W}^*$ we have

$$F \cdot \mathcal{W} = F_\mathcal{W} \mathcal{W} \quad \iff \quad \mathcal{W}^* F_\mathcal{W} = \mathcal{W}^* F_\mathcal{W}^*$$

so that we may restrict to pairs $F, \mathcal{W}$ for which the eigenmatrices act from the right. Although in general eigenmatrices for a given functional $F$ need not be unique, they must be whenever $\mathcal{W}(A)$ is invertible for some $A \subset X$, as

$$F \cdot \mathcal{W} = \mathcal{W} F_\mathcal{W} \quad \implies \quad F_\mathcal{W} = \mathcal{W}(A)^{-1} \int_A \mathcal{M}^*_n F d\mathcal{W}$$

Let $\mathcal{M}^*_n$ denote the collection of such matrix measures, or

$$\mathcal{M}^*_n := \{ \mathcal{W} \in \mathcal{M}_n : \text{there exists} A \subset X \text{ Borel with } \mathcal{W}(A) \text{ invertible} \}$$

Thus when $F$ acts invariantly on $\mathcal{W} \in \mathcal{M}^*_n$ it makes sense to speak of the eigenmatrix $F_\mathcal{W}$ for the action of $F$ on $\mathcal{W}$. For the action of $\mathcal{M}^*$ on a matrix measure $\mathcal{W}$, we have the collection of (right) eigenmatrices of $\mathcal{W}$, or

$$\text{Eig}_R(\mathcal{W}) := \{ B \in \mathcal{M}_n(\mathbb{C}) : F \cdot \mathcal{W} = \mathcal{W} B \}$$

**Proposition 6.6.** For $\mathcal{V} \in \mathcal{M}^*_n$ the matrices in $\text{Eig}_R(\mathcal{V})$ commute and are simultaneously diagonalizable.

**Proof.** If $F,G \in \mathcal{M}^*$ act invariantly on $\mathcal{V} \in \mathcal{M}^*_n$, then as the action of $\mathcal{M}^*$ on $\mathcal{M}_n$ is componentwise, we have $F \cdot G \cdot \mathcal{V} = G \cdot F \cdot \mathcal{V}$ by proposition 6.3 and so

$$\mathcal{V} F_\mathcal{V} G_\mathcal{V} = G_\mathcal{V} F_\mathcal{V} = F_\mathcal{V} G_\mathcal{V} = \mathcal{V} G_\mathcal{V} F_\mathcal{V}$$

so that, as $\mathcal{V}(A)$ is invertible for some $A$ Borel, we can cancel and the eigenmatrices commute. Now, suppose $F \in \mathcal{M}^*$ acts invariantly on $\mathcal{V} \in \mathcal{M}^*_n$. We show that $F_\mathcal{V}$ is diagonalizable. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $F_\mathcal{V}$ and $\nu \in \mathbb{C}^n$ a generalized eigenvector of degree $k > 0$, so that

$$(F_\mathcal{V} - \lambda)^k \nu = 0$$

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The claim will follow by showing that \( k = 1 \), as this implies the eigenvalues of \( F_V \) are simple, and thus diagonalized by putting into Jordan form. As \( F \cdot \mathcal{V} = \mathcal{V} F_V \), we have

\[
\mathcal{V}(F_V - \lambda I)^k \nu = (F - \lambda)^k \cdot \mathcal{V} \nu = 0
\]

where \((F - \lambda)^k\) means the action of the functional \( \mu \mapsto F(\mu) - \lambda \mu(X) \) on \( \mathcal{V} \) done \( k \) consecutive times, and

\[
(F - \lambda)^k \cdot \mathcal{V} \nu = 0 \implies \hat{\mathcal{V}}_\nu^k ((F - \lambda)^k) = (\hat{\mathcal{V}}_\nu F - \lambda)^k = 0 \quad \mathcal{V}\text{-ae} \implies \hat{\mathcal{V}}_\nu^k F = \lambda \quad \mathcal{V}\text{-ae}
\]

In particular, this shows \( \mathcal{V} F_V \nu = F \cdot \mathcal{V} \nu = \lambda \mathcal{V} \nu \) and we have \( F_V \nu = \lambda \nu \) as \( \nu \in \mathcal{M}_n^\circ \). Thus, \( \nu \) is an eigenvector for \( F_V \) and the generalized eigenspaces of \( F_V \) are all one-dimensional, so that \( F_V \) is diagonalizable.

Thus \( \text{Eig}_\nu(V) \) is a commuting family of diagonalizable matrices, and therefore can be simultaneously diagonalized, as commuting matrices share invariant subspaces and diagonalizable matrices have one-dimensional invariant subspaces spanning their domain.

### 6.1.1 Invariant Measures

For a continuous function \( S : X \to X \), the Koopman operator is the map \( U_S : f \mapsto f \circ S \) acting on measurable functions. We say a function \( f \) is \( S \)-invariant if \( f \circ S = f \), or if it is a fixed point of the Koopman operator. A measure is \( S \)-invariant if \( \mu \circ S^{-1} = \mu \), and we let \( \mathcal{M}(X, S) \) denote the collection of \( S \)-invariant measures. As simple functions are dense in integrable functions, one checks that for \( \nu \in \mathcal{M}(X, S) \),

\[
\int_X U_S(f) d\mu = \int_X f d\mu \circ S^{-1}
\]

so that invariant measures are those measures in \( \mathcal{M} \) for which integration is invariant under the action of the Koopman operator. Restricting to \( S \)-invariant probability measures gives a compact convex set, the extreme points of which are ergodic and mutually singular. By ergodic decomposition, \( \mathcal{M}(X, S) \) is the \( \mathbb{C} \)-span of \( \mathcal{E} = \mathcal{E}(X, S) \), the ergodic \( S \)-invariant probability measures on \( X \), see [20].

For each \( \mu \in \mathcal{M}(X, S) \) we have the \( \sigma \)-algebra \( B_\mu = \{ A \subset X \text{ Borel and } A \Delta S^{-1} A = \mu \text{-null} \} \) consisting of the \( \mu \)-ae \( S \)-invariant Borel subsets of \( X \). The simple functions over this \( \sigma \)-algebra generate a closed subspace of \( L^2(\mu) \) which we denote \( L^2(\mu, S) \) and the orthogonal projection \( \mathbb{E}_\mu : L^2(\mu) \to L^2(\mu, S) \) sends \( f \) to \( \mathbb{E}_\mu(f) = \mathbb{E}(f | B_\mu) \) is the conditional expectation of \( f \) given the \( \sigma \)-algebra of \( \mu \)-ae \( S \)-invariant Borel sets. By the ergodic averages of \( f \), we mean the averages

\[
A_n(f) := \frac{1}{n} \sum_{k=1}^n U_S^k f
\]

for \( n \) positive. The following is a corollary of the mean ergodic theorem of von Neumann.

**Theorem 6.7** (Mean Ergodic Theorem).

For every \( \mu \in \mathcal{M}(X, S) \) and \( f \in L^2(\mu) \), the ergodic averages \( A_n(f) \) converge to \( \mathbb{E}_\mu(f) \) in \( L^2(\mu) \).

Using this, we construct a collection of functionals on \( \mathcal{M} \) determined by a continuous function on \( X \). For \( \gamma \in C(X) \), the ergodic averages \( A_n(\gamma) \) are all bounded by the supremum norm of \( \gamma \), so that \( \mathbb{E}_\mu(\gamma) \in L^\infty(\mu) \) and \( \| \mathbb{E}_\mu(\gamma) \|_X \leq \| \mu \|_X \). As all our measures have finite total variation, \( L^2 \) convergence implies convergence in \( L^1 \), and so we have subsequences of ergodic averages converging pointwise almost everywhere with respect to an invariant measure. Thus, if \( \nu \ll \mu \) are both invariant, we can pass to the almost everywhere pointwise subsequential limits of \( A_n(\gamma) \) and show that \( \mathbb{E}_\mu(\gamma) = \mathbb{E}_\nu(\gamma) \) \( \nu \)-ae. Thus, for each \( \gamma \in C(X) \), the map \( \mu \mapsto \mathbb{E}_\mu(\gamma) \) gives rise to a generalized functional on \( \mathcal{M}(X, S) \). By proposition 6.2, this gives rise to a functional on \( \mathcal{M}(X, S) \), a closed subspace of \( \mathcal{M} \), and so this functional can be extended to \( \mathcal{M} \) via the Hahn-Banach theorem; let \( [\gamma] \) denote the collection of all such extensions. Thus, we obtain:

**Proposition 6.8.** For each \( \gamma \in C(X) \), there exists a nonempty collection of functionals \( [\gamma] \subset \mathcal{M}^* \) such that

\[
\Gamma \in [\gamma] \text{ and } \mu \in \mathcal{M}(X, S) \implies \hat{\mathcal{V}}_\nu^k \Gamma = \mathbb{E}_\mu(\gamma) \quad \mu \text{-ae}
\]

Note that for every \( \Gamma \in [\gamma] \), we have for every \( \mu \in \mathcal{M}(X, S) \)

\[
\Gamma(\mu) = \int_X \mathbb{E}_\mu(\gamma) d\mu = \int_X \gamma d\mu
\]
as $X$ is always an invariant set, and $\mathbb{E}_\mu(\gamma)$ is conditional expectation. We now apply the above to the case when $X$ is a compact abelian group $G$, and prove a diagonalization result for ergodic matrices of measures relative to $S : G \to G$ continuous. Let $M_n(X, S)$ denote those measures in $M_n$ all of whose components are in $M(X, S)$. A matrix of measures $W \in M_n(X, S)$ with total variation $\omega$ is

- ergodic if $\frac{1}{n} \sum_{k<n} \int_X U_S^k(f) g dW \to \int_X f dW \int_X g dW$
- strong mixing if $\int_X U_S^k(f) g dW \to \int_X f dW \int_X g dW$

as $n \to \infty$, for all $f, g \in L^2(\omega)$. Just as in the $n = 1$ case, mixing implies ergodic by looking at component measures. Note that there are corresponding statements for when the above limits are equal to $\int_X g dW \int_X f dW$ as in general these integrals need not commute.

### 6.1.2 Diagonalization of Ergodic Measures

Let $X = G$ be a metrizable compact abelian group written additively and $\hat{G}$ the Pontryagin dual of $G$, consisting of the continuous group homomorphisms $G \to \mathbb{T}$, the unit circle group in $\mathbb{C}$. For $W \in M_n(G)$, the Fourier-Stieltjes coefficients of $W$ are the integrals

$$\hat{W}(\gamma) = \int_G \gamma dW$$

and Fourier unicity allows us to characterize measures via their Fourier coefficients, as $W \equiv 0$ if and only if $\hat{W}(\gamma) = 0$ for all $\gamma \in \hat{G}$, see [20] for details. For each $\gamma \in \hat{G}$, the identification

$$\gamma^* : M \to \mathcal{C} \quad \text{where} \quad \gamma^* : \mu \mapsto \hat{\mu}(\gamma) = \int_G \gamma d\mu$$

determines an element in $M^*$ and this is an embedding of $\hat{G} \hookrightarrow M^*$.

Note that the characterizations of ergodicity and mixing given in the previous section can be reduced to $f, g \in \hat{G}$ by Fourier unicity, and moreover, the action of $M^*$ on $M_n$ can be characterized by their Fourier-Stieltjes coefficients, in that the integrals $F \hat{W}(\varphi) = \int_G \hat{\varphi}(\gamma) F^\gamma dW$ determine the matrix of measures $F W$. We say a matrix of measures $W$ is diagonalizable if there exists a $Q \in M_n(\mathbb{C})$ invertible with $Q W Q^{-1}$ a diagonal matrix of measures; the measures appearing on the diagonal are called eigenmeasures of $W$.

**Theorem 6.9.**

*If $W \in M_n(G, S) \cap M_\infty^n$ is ergodic, then $W$ is diagonalizable with eigenmeasures ergodic in $M(G, S)$.***

**Proof.** The result is almost an immediate consequence of propositions [6.6 and 6.8]. If $W$ is ergodic, then for every $\gamma, \varphi \in \hat{G}$ and $\Gamma \in \{\gamma\}$, we have

$$\Gamma \hat{W}(\varphi) = \int_G \mathbb{E}_\omega(\gamma) \varphi dW = \lim_{\Gamma_n \to \infty} \int_G A_n(\gamma) \varphi dW = \hat{W}(\varphi) \hat{W}(\gamma) \quad \text{or} \quad \hat{W}(\gamma) \hat{W}(\varphi)$$

by the Lebesgue dominated convergence theorem and the mean ergodic theorem. Thus, we have $\hat{W}(\gamma) \in \text{Eig}_\Gamma(W)$ for all $\gamma \in \hat{G}$, and the Fourier-Stieltjes coefficients of $W$ are simultaneously diagonalizable over $\mathbb{C}$ by proposition [6.6]. Thus, there exists a matrix $Q \in M_n(\mathbb{C})$ such that $Q \hat{W}(\varphi) Q^{-1}$ is diagonal for all $\varphi \in \hat{G}$, and $Q W Q^{-1}$ is a diagonal matrix of measures by Fourier unicity. The claim of ergodicity and invariance follows by observing that

$$Q \left( \int f dW \right) Q^{-1} = \int f d(Q W Q^{-1})$$

and using the Fourier descriptions of invariance and ergodicity given earlier. 

### 6.2 Proof of Theorem 4.8

The results and proofs in this section are largely generalized from Queffelec’s work in [18]. For a $q$-substitution $S$ on $A$, we can telescope and so assume that its index of imprimitivity is 1, see proposition [2.2]. With this assumption, note that both $\lim_{n \to \infty} \frac{1}{Q^n} M^n_S$ and

$$P := \lim_{n \to \infty} \frac{1}{Q^n} C^n_S = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{j < q^n} R_j(n) \otimes R_j(n)$$

(14)
exist. This follows from the primitive reduced form of $M_S$ provided by proposition 22 and results in [10 Chap III §7]. Note that, as defined, $P$ is a (nonorthogonal) projection onto the $Q$-eigenspace of $C_S$.

For $B, C \in A^+$, let

$$L_B(C) := \text{Card} \{ j \in \mathbb{Z}^d : T^j B \text{ is extended by } C \} \quad \text{and} \quad h_n = \frac{1}{Q^n} \left( L_B(S^n(\gamma)) \right)_{\gamma \in A}$$

which give the occurrence number of $B$ inside $C$, and the frequency of $B$ in blocks $S^n(\gamma)$ for each $\gamma \in A, n > 0$.

**Lemma 6.10.** For $S$ as above, $\lim_{n \to \infty} h_n$ exists for every $B \in A^+$

**Proof.** See also [18, Proof of Prop 10.4]. If $B$ is a subblock of $S^{n+p}(\gamma)$ for $n, p > 0$, then either:

1. $B$ appears as a subblock of $S^n(\alpha)$ for some $\alpha$ appearing in $S^p(\gamma)$, or
2. $B$ appears as a subblock of a union of (two or more) such blocks.

The number of ways $\alpha$ can appear in $S^p(\gamma)$ is given by $e^* M_S^p e_\gamma$, and the number of ways $B$ can appear in the second category above is bounded by

$$Q^p \text{Card}(\Delta_p(-k) \cup \Delta_p(k)) = 2Q^p \text{Card}(\Delta_n(k))$$

where $k$ is such that the support of the block $B$ is contained in $[j, j + k)$ for some $j \in \mathbb{Z}^d$. From this, we have

$$\sum_{\alpha \in A} L_B(S^n(\alpha)) e^* M_S^p e_\gamma \leq L_B(S^{n+p}(\gamma)) \leq 2Q^p \text{Card}(\Delta_n(k)) + \sum_{\alpha \in A} L_B(S^n(\alpha)) e^* M_S^p e_\gamma$$

Dividing by $Q^{n+p}$ gives the (componentwise) inequality

$$\frac{1}{Q^p} h_n M_S^p \leq h_{n+p} \leq 2 \frac{\text{Card}(\Delta_n(k))}{Q^n} + 1 + \frac{1}{Q^p} h_n M_S^p$$

(15)

where $1$ is again the vector of 1’s. Letting $p \to \infty$ we have

$$h_n \mathcal{P} \leq \lim \inf_{p \to \infty} h_{n+p} \leq \lim \sup_{p \to \infty} h_{n+p} \leq 2 \frac{\text{Card}(\Delta_n(k))}{Q^n} + h_n \mathcal{P}$$

Moreover, the first inequality in (15) together with the identity $M_S \mathcal{P} = Q \mathcal{P}$ and $\mathcal{P} \geq 0$ gives $h_n \mathcal{P} \leq h_{n+p} \mathcal{P}$, so that as $0 \leq h_n \leq 1$,

$$h^* := \lim_{n \to \infty} h_n \mathcal{P}$$

exists. Letting $n \to \infty$ in the above inequalities and using lemma 3.1 we obtain

$$h^* \leq \lim \inf_{p \to \infty} h_p \leq \lim \sup_{p \to \infty} h_p \leq h^*$$

so that the limit exists, completing the proof. \( \square \)

**Corollary 6.11.** Let $\mathcal{R}$ be a configuration for a $q$-substitution on $A$. For each $k \in \mathbb{Z}^d$, the limit

$$\lim_{n \to \infty} \frac{1}{q^n} \sum_{j < q^n} \mathcal{R}^{(n)}_j \otimes \mathcal{R}^{(n)}_{j+k}$$

exists, and the $\mathbb{Z}^d$ sequence in $k$ forms the Fourier coefficients for a matrix of complex measures on $\mathbb{T}^d$, which we denote by $Z$.

**Proof.** See also [18, Prop 10.4]. First, as $e^* \mathcal{R}^{(n)}_j \otimes \mathcal{R}^{(n)}_{j+k} e_\gamma \delta = (e^* \mathcal{R}^{(n)}_j e_\gamma)(e^* \mathcal{R}^{(n)}_{j+k} e_\delta)$ we have

$$\sum_{j < q^n} e^* \mathcal{R}^{(n)}_j \otimes \mathcal{R}^{(n)}_{j+k} e_\gamma \delta = \text{Card} \{ j \in [0, q^n) : \mathcal{R}^{(n)}_j(\gamma) = \alpha \text{ and } \mathcal{R}^{(n)}_{j+k}(\delta) = \beta \}$$
so that
\[
\left| \text{Card}\{ j < q^n : j + k < q^n, S^n(\gamma)_j = \alpha \text{ and } S^n(\delta)_{j+k} = \beta \} - e_{\alpha\beta}^* \left( \sum_{j<q^n} R_{j}^{(n)} \otimes R_{j+k}^{(n)} \right) e_{\gamma\delta} \right| \leq \text{Card}(\Delta_n(k))
\]

Writing \( \eta = \mathcal{S} \otimes \mathcal{S} \),
\[
\{ j < q^n : j + k < q^n, S^n(\gamma)_j = \alpha, S^n(\delta)_{j+k} = \beta \} = \bigsqcup_{\alpha', \beta'} \{ j < q^n : j + k < q^n, \eta^n(\gamma\delta)_j = \alpha\alpha', \eta^n(\gamma\delta)_{j+k} = \beta'\beta \}
\]
so that, if \( B(\alpha', \beta') \) is the block sending \( 0 \mapsto \alpha\alpha' \) and \( k \mapsto \beta'\beta \), this gives us
\[
\text{Card}\{ j < q^n : j + k < q^n, S^n(\gamma)_j = \alpha, S^n(\delta)_{j+k} = \beta \} = \sum_{\alpha', \beta'} L_B(\alpha', \beta')(\eta^n(\gamma\delta))
\]

Dividing by \( q^n \) and using lemmas 3.3 and 3.10 combined with the inequality at the beginning show that the required limits exist. Now, for \( w \in \mathbb{T}^d \) and \( k \in \mathbb{Z}^d \), write
\[
Z_n(w) = \frac{1}{q^n} \sum_{j < q^n} R_j^{(n)} \otimes R_j^{(n)} w^{i\cdot j} \quad \text{where} \quad w^k = (w_{k_1}^1, \ldots, w_{k_d}^d)
\]

Viewing \( Z_n = Z_n(w)dw \) as a measure on \( \mathbb{T}^d \), the total variation of \( Z_n \) is
\[
|Z_n| = \sum_{\alpha\beta\gamma\delta} |e_{\alpha\beta}^* Z_n e_{\gamma\delta}| = \sum_{\gamma} \int_{\mathbb{T}^d} |e_{\alpha\beta}^* Z_n(w)e_{\gamma\delta}|dw = \frac{1}{q^n} \sum_{\gamma} \int_{\mathbb{T}^d} \left| \sum_{j < q^n} (e_{\alpha\beta}^* R_j^{(n)} e_{\gamma\delta}) w^{-j} \right| dw
\]
\[
\leq \frac{1}{q^n} \sum_{\gamma} \left( \int_{\mathbb{T}^d} \left| \sum_{j < q^n} e_{\alpha\beta}^* R_j^{(n)} e_{\gamma\delta} \right|^2 dw \right)^{1/2} \left( \int_{\mathbb{T}^d} \left| \sum_{j < q^n} e_{\beta\gamma}^* R_j^{(n)} e_{\delta\alpha} \right|^2 dw \right)^{1/2}
\]
by Cauchy-Schwartz
\[
= \frac{1}{q^n} \sum_{\alpha\beta\gamma\delta} \left( \sum_{j < q^n} (e_{\alpha\beta}^* R_j^{(n)} e_{\gamma\delta}) \right)^{1/2} \left( \sum_{j < q^n} (e_{\beta\gamma}^* R_j^{(n)} e_{\delta\alpha}) \right)^{1/2} \leq \frac{\text{Card}(A)^4}{q^n} (Q^n/2)(Q^n/2) = \text{Card}(A)^4
\]
as \( e_{\alpha\beta}^* R_j^{(n)} e_{\gamma\delta} = 0 \) or 1. It follows that the measures \( Z_n \) are uniformly bounded, and so a subsequence converges in the weak-star topology to some measure \( Z \). However, as
\[
\mathcal{Z}_n(k) = \frac{1}{q^n} \sum_{j < q^n} R_j^{(n)} \otimes R_j^{(n)} k
\]
the Fourier coefficients all converge to the same limit by the first half of the proof, so \( Z \) is the unique limit point of this sequence, and satisfies
\[
\mathcal{Z}(k) = \lim_{n \to \infty} \mathcal{Z}_n(k) = \lim_{n \to \infty} \frac{1}{q^n} \sum_{j < q^n} R_j^{(n)} \otimes R_j^{(n)} k
\]
completing the proof.

The matrix \( Z = (\sigma_{\gamma\delta})_{\alpha\beta, \gamma\delta \in \mathcal{A}^2} \) is the bicorrelation matrix, and its components bicorrelation measures, after Queffelec. Let \( \mathcal{Z} \) be the matrix of measures \( (\sigma_{\gamma\delta})_{\alpha\gamma, \beta\delta \in \mathcal{A}^2} \), noting the change in the order of indices.

**Lemma 6.12.** The matrix \( \mathcal{Z} \) is a positive definite matrix of measures.

**Proof.** Let \( \{t_{\alpha\gamma}\}_{\alpha\gamma \in \mathcal{A}^2} \subset \mathbb{C} \). Denote by \( \mu \) the measure
\[
\mu := \sum_{\alpha\gamma, \beta\delta} t_{\alpha\gamma} \overline{t_{\beta\delta}} \sigma_{\gamma\delta}^{\alpha\beta}.
\]
We will show that \( \mu \) is a positive measure by showing that its Fourier coefficients form a positive definite sequence, and appealing to Bochner’s theorem. Thus, we fix \( n > 0 \) and let \( \{a_j\}_{j \in [0, q^n]} \) be a sequence of complex numbers: we are interested in the positivity of
\[
\sum_{j, k < q^n} a_j a_k \overline{t_{\alpha\gamma}} \overline{t_{\beta\delta}} \sigma_{\gamma\delta}^{\alpha\beta}(j - k)
\]
For \( j, k \in \mathbb{Z}^d \), note that
\[
\hat{\sigma}_{\alpha\beta}^\gamma(j - k) = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{i < q^n} e_{\alpha\beta} R^{(n)}_{i} \otimes R^{(n)}_{i+j-k} e_{\gamma} = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{i < q^n} (e^*_\alpha e^*_\beta R^{(n)}_{i+k} e_{\gamma})(e^*_\beta R^{(n)}_{i+j} e_{\delta})
\]
as the sum is invariant under translation in its index (\( i \mapsto i + k \)) and the definition of the Kronecker product. Now, for any \( n > 0 \), if \( \{a_k\}_{k < q^n} \subset \mathbb{C} \), then
\[
\sum_{j,k < q^n} a_{j} \overline{a}_{k} \hat{\mu}(j - k) = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{\alpha, \beta, \gamma, j,k < q^n} \left( (a_k t_{\alpha\gamma} e^*_\alpha e^*_\beta (a_k) \overline{a}_{j\beta} e_{\beta} R^{(n)}_{i+j} e_{\delta}) \right)
\]
\[
= \lim_{n \to \infty} \frac{1}{Q^n} \sum_{i,k < q^n} \sum_{\gamma \alpha \in \mathbb{A}^2} a_{t_{\alpha\gamma}} e^*_\alpha e^*_\beta R^{(n)}_{i+k} e_{\gamma} \geq 0,
\]
and so \( \{\hat{\mu}(k)\}_{k \in \mathbb{Z}^d} \) forms a positive definite \( \mathbb{Z}^d \)-sequence. By Bochner’s theorem, \( \mu \) is a positive measure on \( \mathbb{T}^d \), and \( \hat{\Sigma} \) is a positive definite matrix of measures. \( \square \)

The same estimates used in the proof of theorem 4.3 can be used to show that
\[
\hat{\Sigma}(k) = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{j \in (0, q^n)} R^{(n)}_{j} \otimes R^{(n)}_{j+k} = \frac{1}{Q^p} \sum_{j \in (0, q^p)} R^{(p)}_{j} \otimes R^{(p)}_{j+k} \hat{\Sigma}([j + k]_p)
\] (16)
It follows from the first equality for \( \hat{\Sigma}(k) \) above that \( \hat{\Sigma}(0) = \mathcal{P} \). Letting \( p \to \infty \) in the second equality and using lemma 5.1 we obtain \( \mathcal{Z} = \mathcal{ZP} \). As theorem 4.3 gives \( \Sigma = \hat{\Sigma}(0) \), the following corollary is no surprise.

**Corollary 6.13.** The maximal spectral type of \( (X_{\mathcal{Z}}, \mu) \) is equivalent to \( \omega_{\mathcal{Z}} * |\mathcal{Z}| \).

**Proof.** First, we note that for each \( \alpha, \beta, \gamma \in \mathcal{A} \), \( \sigma_{\alpha\beta}^{\gamma} = \sigma_{\alpha\beta} \), which can be seen immediately by comparing the respective Fourier coefficients and using the identities \( \mathcal{Z} = \mathcal{ZP} \) and \( \mathcal{P} e_{\gamma} = \hat{\Sigma}(0) \) for \( \gamma \in \mathcal{A} \). Let \( v = v_{\alpha\beta}^{\gamma} \in \mathbb{C}^{\mathcal{Z}} \) be the vector \( (v_{\alpha'\beta'}^{\gamma}) \) where, for some \( x, y \in \mathcal{C} \) not both 0,
\[
v_{\alpha'\beta'}^{\gamma} = \begin{cases} x & \alpha'\beta' = \gamma\alpha \\ y & \alpha'\beta' = \delta\beta \\ 0 & \text{otherwise}. \end{cases}
\]
Then, as \( \hat{\Sigma} \) is positive definite, we have for all measurable \( A \subset \mathbb{T}^d \),
\[
v^{\ast} \hat{\Sigma}(A)v = |x|^2 \sigma_{\alpha\alpha}^{\gamma}(A) + |y|^2 \sigma_{\beta\beta}^{\delta}(A) + x\overline{y} \sigma_{\beta\alpha}^{\gamma\delta}(A) + y\overline{x} \sigma_{\alpha\beta}^{\delta\gamma}(A)
\]
\[
= |x|^2 \sigma_{\alpha\alpha}(A) + |y|^2 \sigma_{\beta\beta}(A) + x\overline{y} \sigma_{\beta\alpha}(A) + y\overline{x} \sigma_{\alpha\beta}(A) \geq 0
\]
Let \( A \) be such that \( \sigma_{\alpha\alpha}(A) = 0 \), and let \( y = -1 \) in the above, so we obtain:
\[
\sigma_{\beta\beta}(A) - x\sigma_{\beta\alpha}^{\gamma\delta}(A) - y\overline{x} \sigma_{\alpha\beta}^{\delta\gamma}(A) \geq 0
\]
so that, letting \( x \to \infty \) along the real and imaginary axes we obtain \( \sigma_{\beta\alpha}^{\gamma\delta}(A) = \sigma_{\alpha\alpha}^{\gamma\delta}(A) = 0 \), and so \( \sigma_{\alpha\beta}^{\delta\gamma} < \sigma_{\alpha\alpha} \).
Letting \( x = -1 \), and \( y \to \infty \) gives the corresponding result for \( \sigma_{\beta\beta} \). Thus, \( |\mathcal{Z}| = \sum \sigma_{\alpha\alpha}^{\gamma\delta} \sim \sum \sigma_{\alpha\alpha} = \sum \sigma_{\alpha\alpha} \) and the result follows as a corollary of theorem 4.4. \( \square \)

**Corollary 6.14.** The matrix of measures \( \mathcal{Z} \) and the projection matrix \( \mathcal{P} \) preserve strong positivity on \( \mathbb{C}^{\mathcal{A}^2} \).
Proof. As \( P = \hat{\lambda}(0) = Z(T^d) \) and strong positivity of a vector-valued measure is determined pointwise, the statement for \( P \) will follow from the statement for \( Z \). Let \( v \) be strongly positive. We must show that \( Zv = \sum_{\gamma \in \mathcal{E}} v_{\gamma} \delta_{\alpha_\beta} \) gives rise to a positive definite matrix of measures: this means that

\[
z^T(\hat{Z}v)z = \sum_{\alpha, \beta \in \mathcal{E}} \sum_{\gamma \in \mathcal{E}} v_{\gamma} \delta_{\alpha_\beta} \sum_{\delta \in \mathcal{E}} \alpha z_{\beta} \delta_{\gamma} \delta_{\alpha_\beta}
\]

is a positive measure for every \( z \in \mathbb{C}^d \). As \( v \geq 0 \), there exists an orthonormal basis \( \{w_\kappa\}_{\kappa \in \mathcal{E}} \) of eigenvectors with eigenvalues \( \lambda_\kappa \geq 0 \) (for \( \kappa \in \mathcal{E} \)) such that \( v = \sum_{\kappa} \lambda_\kappa w_\kappa \). By the above and positivity of the \( \lambda_\kappa \), it suffices to show for \( w \in \mathbb{C}^d \)

\[
\sum_{\alpha, \beta} z_{\alpha} w_{\beta} \delta_{\alpha_\beta} \delta_{\alpha_\beta}
\]

is a positive measure. This, however, follows immediately from the above lemma by setting \( t_{\alpha \gamma} = z_{\alpha} w_{\gamma} \). \( \square \)

Note that for \( a \in \mathbb{Z}^d \),

\[
\hat{P}Z(\lambda a) = P \left( \frac{1}{Q} C_\delta \hat{Z}(a) \right) = \hat{P}Z(a)
\]

(17)

Now, fix \( a, b \in \mathbb{Z}^d \). Then

\[
\hat{Z}(b + aq^p) = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{i \in [0, q^p]} R_i^{(n+p)} \otimes R_j^{(n+p)} = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{i \in [0, q^p]} R_i^{(p)} \otimes R_j^{(n)}
\]

using the identities in \( 15 \). Using the mixed product property of the Kronecker product:

\[(AB) \otimes (CD) = (A \otimes C)(B \otimes D)\]

we obtain

\[
\hat{Z}(b + aq^p) = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{i, j < q^p} R_i^{(p)} \otimes R_j^{(n)} = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{i, j < q^p} R_i^{(p)} \otimes R_j^{(n)}
\]

Now, as \( Z = ZP \), we take the limit as \( p \to \infty \) and use lemma 15 again, obtaining

\[
\lim_{p \to \infty} \hat{P}Z(b + aq^p) = \hat{P}Z(b)\hat{P}Z(a)
\]

(18)

Corollary 6.15. The matrix of measures \( PZ \) is diagonalizable with strongly-positive eigenvectors and eigenmeasures strong-mixing for the \( q \)-shift.

Proof. As \( PZ = PZP \), the matrix of measures is zero on the kernel of \( P \), and it will diagonalized there with respect to any basis for the kernel of \( P \). Thus, it suffices to restrict to the image of \( P \), where \( P \) is the identity. This, however, implies that \( PZ(T^d) = PZ(0) = P^2 = P \) is invertible, so that we may consider \( PZ \in \mathcal{M}_q \) as a matrix operating on the image of \( P \). The identities \( 17 \) and \( 18 \) above show that \( PZ \) is strong-mixing for the \( q \)-shift, and so theorem 6.13 tells us that \( PZ \) is diagonalizable with respect to the image of \( P \). Thus, \( PZ \) is diagonalized with respect to both the image and kernel of \( P \), which are invariant subspaces for \( P \), and \( PZ \) is diagonalizable. That the eigenmeasures are strong-mixing follows immediately from the strong-mixing property of \( PZ \). All that remains is the choice of eigenvectors.

As \( \hat{P}Z(0) = P \) is the identity on the image of \( P \), it follows that \( \hat{\lambda}(0) = 1 \) for every eigenmeasure \( \lambda \) of \( PZ \). As they are strong-mixing, they are ergodic and thus are constant multiplies of probability measures; it follows that the eigenmeasures are all probability measures, and therefore are either equal or mutually singular by ergodic decomposition. Fix an eigenmeasure \( \lambda \) of \( PZ \), and let \( D_\lambda \) be projection onto \( L(\lambda) \). As \( P \) and \( Z \) preserve strong-positivity, we have \( D_\lambda(PZ) \) preserves strong-positivity as well, which follows as this property is determined pointwise on Borel sets. Thus, the matrix of measures \( D_\lambda(PZ) \) is similar (via the same similarity diagonalizing \( PZ \)) to a diagonal matrix of measures, with \( \lambda \) or \( 0 \) on the diagonal, and so \( D_\lambda(PZ) = \lambda P_\lambda \) for some projection operator \( P_\lambda \in \mathcal{M}_q \). This, however, implies that \( P_\lambda \) preserves strong-positivity, and as strongly-positive vectors span \( \mathbb{C}^d \) (as positive definite matrices span matrices) it follows that the image of \( P_\lambda \) is spanned by strongly-positive vectors. Thus the eigenspace corresponding to each eigenmeasure \( \lambda \) is spanned by strongly-positive vectors, and we can therefore choose a basis of strongly-positive eigenvectors for \( PZ \), as desired. \( \square \)
Proposition 6.16. There is a strongly positive basis with respect to which \( Z \) is similar to

\[
\begin{pmatrix}
\Lambda & 0 \\
W & 0
\end{pmatrix}
\]

where \( \Lambda \) is the diagonal matrix of measures \( \lambda_1, \ldots, \lambda_K \), and \( |W| < \omega_q \cdot |\Lambda| \)

Proof. The specified basis is provided by the above corollary \[6.15\] let \( w_1, \ldots, w_n \) be the strongly positive eigenvectors for nonzero eigenmeasures of \( PZ \), and \( w_{n+1}, \ldots, w_{d^2} \) those corresponding to the zero eigen-measures. Then \( \Lambda \) corresponds to \( Z \) on the span of the \( w_j \) and \( W \) corresponds to \( Z \) on the span of the \( w_j \). That the last block column of \( Z \) is zero follows as \( Z = ZP \). That \( \Lambda \) is diagonal follows as \( P \) is the identity on the span of the \( w_j \), which diagonalize \( PZ \). It remains to show that \( |W| < \omega_q \cdot |\Lambda| \).

Let \( L = L(\omega_q \cdot |\Lambda|) \) be the set of measures absolutely continuous with respect to the measures \( \omega_q \cdot \lambda \) where \( \lambda \) is an eigenmeasure of \( PZ \), and let \( S_q \) be the \( q \)-shift on \( T^d \) sending \( z \mapsto z^q \) or \( t \mapsto qt \mod 1 \) and for measures \( \mu \), write \( S_q \mu := \mu \circ S_q^{-1} \). Note that the support of \( |\Lambda| \cdot \omega_q \) is \( S_q \)-invariant, as the \( q \)-adic rationals and the support of \( \Lambda \) is \( S_q \)-invariant. Similarly, the null sets of \( |\Lambda| \cdot \omega_q \) are also \( S_q \)-invariant, and so it follows that \( L \) and \( L^{\perp} \) are \( S_q \)-invariant \( L \)-spaces of measures; see also \[18, Lemma 10.4\]. Let \( D \) and \( D^\perp \) represent the projections onto the \( L \)-spaces \( L \) and \( L^{\perp} \), respectively.

Let \( w_1, \ldots, w_K \) be a collection of strongly positive eigenvectors corresponding to the mutually singular eigenmeasures \( \lambda_1, \ldots, \lambda_K \) of \( PZ \), and write \( w := \sum w_j \). Then as \( Pw = w \), we have \( C_S w = Qw \) and

\[
Zw = \sum \lambda_i w_i + Ww \implies C_S Zw = Q \sum \lambda_i w_i + C_S Ww \quad \text{and} \quad PZw = 0
\]

Using identity \[10\] and \( \widehat{S_q \mu}(a) = \widehat{\mu}(aq) \), one checks that \( S_q Z = \frac{1}{\lambda} C_S Z \), and so

\[
S_q Zw = \sum \lambda_i w_i + \frac{1}{\lambda} C_S Ww \implies \frac{1}{\lambda} C_S Ww = S_q Ww \implies \frac{1}{\lambda^q} C_S^n D^\perp Ww = S_q^n D^\perp Ww
\]

as \( L \) and \( L^{\perp} \) are \( S_q \)-invariant. As \( PZw = 0 \), this implies that \( S_q^n(D^\perp Ww) \to 0 \) in norm. Thus

\[
S_q^n D^\perp Ww(0) = D^\perp Ww(0) = 0
\]

As strong-positivity is determined pointwise and the \( \lambda_i \in L \), \( D^\perp(Zw) = D^\perp Ww \) is strongly positive, and thus it is identically \( 0 \). It follows that \( Ww = D(Ww) \). As the eigenvectors of \( PZ \) for nonzero eigenmeasures span the image of \( P \), it follows that \( |W| < |\Lambda| \cdot \omega_q \), completing the proof.

The following proposition allows us to compute Fourier coefficients of the eigenmeasures of \( PZ \) explicitly.

Proposition 6.17. The map \( v \mapsto v^* \Sigma \) takes the extreme rays of \( \mathcal{K} \) onto the eigenmeasures of \( PZ \).

Proof. Let \( \lambda \) be an eigenmeasure of \( PZ \). Then \( PZ \) and \( v \) strongly-positive with \( v^* PZ = \lambda v^* \). Then as \( Z = ZP \),

\[
\lambda v^* = v^* PZ = v^* PZ P = \lambda v^* \quad \implies \quad v^* P = v^*
\]

so that \( v \) is a left \( Q \)-eigenvector of \( C_S \) and so \( v_{\alpha \alpha} \) is constant for \( \alpha \in A \) by primitivity and stochasticity of \( C_S \). As \( v \) is strongly positive, this implies that \( v_{\alpha \alpha} \neq 0 \) and moreover, that \( v \in \mathcal{K} \). Now, if \( u \) is the Perron vector of \( M_S \), (or if \( u, = \mu(\gamma) \) for the invariant measure \( \mu \) on \( X_S \) then

\[
\lambda v = v^* \Sigma = v^* P \Sigma = v^* P Z \Sigma(0) = \lambda v^* \hat{\Sigma}(0) = \lambda \sum \gamma \in A v_{\gamma \gamma} u_{\gamma} = \lambda v_{\alpha \alpha} u_{\alpha} v_{\alpha} \lambda
\]

so that \( \lambda = \lambda_w \) for some \( w \in \mathcal{K} \). Now, as \( PZ \) can be diagonalized with strongly positive vectors, we can write \( PZ = \sum \lambda_j P_j \) where \( P_j \) preserves strong positivity as in the proof of corollary \[6.11\]. If \( v \in \mathcal{K} \), we have

\[
v^* \Sigma = v^* P Z \hat{\Sigma}(0) = \sum \lambda_j v^* P_j \hat{\Sigma}(0) = \sum c_j \lambda_j
\]

with \( c_j \geq 0 \) by lemma \[6.15\]. Thus the map \( \lambda \) takes \( \mathcal{K} \) onto the positive span of the eigenmeasures of \( PZ \). As affine maps preserve convexity, this map takes the extreme points of \( \mathcal{K} \) onto the eigenmeasures of \( PZ \).
With the above results established, we can prove Queffelec’s theorem 4.8

**Proof.** By corollary 6.13 combined with propositions 6.5 and 6.16, we have

$$\sigma_{\text{max}} \sim |\omega_q \ast |\mathcal{Z}| \sim |\omega_q \ast |S\mathcal{Z}S^{-1}| = \sum \omega_q \ast \lambda_i + \omega_q \ast |\mathcal{W}| \sim \sum \omega_q \ast \lambda_i$$

Using proposition 6.17 and positivity of the measures in $\lambda(\mathcal{K}^*)$ gives

$$\sum \omega_q \ast \lambda_i \sim \sum_{\omega \in \mathcal{K}^*} \omega_q \ast \lambda_w$$

As every $\nu \in \mathcal{K}$ is a positive linear combination of the extreme rays, this gives $\sigma_{\text{max}} \sim \lambda_v \ast \omega_q$ for $\nu \in \mathcal{K}$. □

### 6.3 Generalized Riesz Products

Fix $q > 1$ in $\mathbb{Z}^d$, and let $\{a_k\}_{-q < k < q} \subset \mathbb{C}$ be a sequence of complex numbers with $a_0 = 1$, and the others not all 0. For $x \in [0, 1]^d$ and $n > 0$, let $P$ and $P_n$ be the trigonometric polynomials

$$P(x) = \sum_{k \in (-q, q)} a_k e^{2\pi i k \cdot x} \quad \text{and} \quad P_n(x) = \prod_{0 \leq j < n} P(q^j x)$$

where

$$k \cdot x = k_1 x_1 + \ldots + k_d x_d \quad \text{and} \quad q^j x = (q^j_1 x_1, \ldots, q^j_d x_d)$$

If $m \, dx$ denotes Lebesgue measure on $[0, 1]^d$, then $P_n \, dx$ determine a (nonzero, complex) measure on $[0, 1]^d$ for every $n > 0$. We would like to consider the weak-star limit points of the sequence $P_n \, dx$, and this is equivalent to convergence of Fourier coefficients, and uniform boundedness of the measures in the total variation norm.

**Lemma 6.18.** If $P_n \, dx$ converges to $\rho$ in the weak-star topology, $\rho$ is singular to Lebesgue measure on $[0, 1]^d$.

**Proof.** Let $x(q) := \left(\frac{x_1}{q}, \ldots, \frac{x_d}{q}\right)$, and $P(x(q))$ denote the push forward of $\rho$ under the map $x \mapsto x(q)$. As $\rho$ is the weak-star limit, it follows that

$$\rho(x(q)) = w^* \lim_{n \to \infty} \prod_{j < n} P(q^j x(q))$$

$$= (1/q) P(x(q)) \left( w^* \lim_{n \to \infty} \prod_{j < n-1} P(q^j x) dx \right)$$

$$= (1/q) P(x(q)) \rho(x)$$

As $P$ vanishes on a set of 0 Lebesgue measure, and the push forward map preserves absolutely continuous components, this equality passes to the absolutely continuous part of $\rho$, so that

$$\rho_{\text{ac}}(x(q)) = (1/q) P(x(q)) \rho_{\text{ac}}(x)$$

(19)

If $S_q : [0, 1]^d \to [0, 1]^d$ is the map $x \mapsto qx(mod\ 1)$, then

$$\rho_{\text{ac}} \circ S_q^{-1} = \sum_{j \in [0, q]} \rho_{\text{ac}} \left( \frac{x+j}{q} \right) = \left( \frac{1}{q} \right) \sum_{j \in [0, q]} P \left( \frac{x+j}{q} \right) \rho_{\text{ac}}(x+j) = \left( \frac{1}{q} \right) \sum_{j \in [0, q]} P \left( \frac{x+j}{q} \right) \rho_{\text{ac}}(x) = \rho_{\text{ac}}(x)$$

as $\rho$ is 1 periodic (all the $P_n(x)$ are, and Lebesgue measure is translation invariant) and as

$$\sum_{j \in [0, q]} e^{2\pi i k \cdot \left( \frac{x}{q} \right)} = 0 \quad \text{for all} \quad k \in \mathbb{Z}^d$$

so that $\rho_{\text{ac}}$ is an $S_q$-invariant measure. As $m$ is ergodic for the $q$-shift, it follows that $\rho_{\text{ac}} = m$ or $\rho_{\text{ac}} = 0$. However, $\rho_{\text{ac}}$ cannot be Lebesgue measure, as $m$ only satisfies (19) for constant $P$. □
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