New Curves from Branes

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Abstract

We consider configurations of Neveu-Schwarz fivebranes, Dirichlet fourbranes and an orientifold sixplane in type IIA string theory. Upon lifting the configuration to M-theory and proposing a description of how to include the effects of the orientifold sixplane we derive the curves describing the Coulomb branch of $\mathcal{N} = 2$ gauge theories with orthogonal and symplectic gauge groups, product gauge groups of the form $\bigotimes_i SU(k_i) \otimes SO(N)$ and $\bigotimes_i SU(k_i) \otimes Sp(N)$. We also propose new curves describing theories with unitary gauge groups and matter in the symmetric or antisymmetric representation.

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1. Introduction

The M-theory/type IIA fivebrane has recently become a very important tool in the study of supersymmetric four dimensional gauge theories. The basic mechanism was first explained in [1]. The fivebrane is wrapped over a Riemann surface such that the theory living on the flat four dimensional part of the fivebrane becomes a supersymmetric gauge theory. Which kind of gauge theory one obtains depends on the particular Riemann surface. For $\mathcal{N} = 2$ theories these Riemann surfaces are precisely the ones found in the exact solutions for the low energy effective action on the Coulomb branch in [2–12]. Witten showed in [13] that these Riemann surfaces can actually be derived with the help of some simple rules. He considered configurations of fourbranes stretched between fivebranes in type IIA string theory and lifted this to M-theory. In M-theory the fourbranes are themselves fivebranes wrapped around the eleventh dimension. What appeared in type IIA theory as a configuration of intersecting flat four- and fivebranes becomes a single fivebrane wrapped around a Riemann surface. Witten considered $SU(N)$ theories and products thereof with matter in the fundamental or bifundamental representations. The role the Riemann surfaces play for the brane configurations was noticed also independently in [16] where an orientifold fourplane parallel to the fourbranes was added and theories with orthogonal and symplectic gauge groups were considered. This was combined with Witten’s approach in [17] where it was shown that one could derive the Riemann surfaces for a wide variety of theories based on orthogonal and symplectic gauge groups. The same brane configurations were investigated also in [18]. The Seiberg-Witten differential and the BPS spectrum in the context of the M-theory fivebrane have been studied in [19–21].

Orientifold fourplanes parallel to the fourbranes are however not the only possibility of obtaining orthogonal or symplectic gauge groups. In fact it has been pointed out in the context of brane realizations of $\mathcal{N} = 1$ theories already in [16] [22] that one can also consider orientifold sixplanes.

In this paper we will study theories that arise by including orientifold sixplanes into brane configurations. The basic configuration will consist of fourbranes stretched between

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1 We follow in our nomenclature [13] and will speak only of four-, five- or sixbranes dropping the specification Neveu-Schwarz or Dirichlet since in type IIA the dimension of the brane determines if it is NS or D.

2 These theories have recently also been derived using geometrical engineering [14] in [15].
fivebranes and an orientifold sixplane in between the fivebranes such that \( \mathcal{N} = 2 \) supersymmetry in four dimensions is preserved. There are two cases to consider. In the first one the orientifold projects onto symplectic gauge groups on the fourbranes. In \cite{37, 38} it has been shown that such an orientifold plane is described in M-theory by the Atiyah-Hitchin manifold \cite{39} times flat seven dimensional Minkowski space. It has further been studied by Sen in \cite{40} and before in a T-dual F-theory setup as seven orientifold in \cite{41}. In the second case the orientifold sixplane projects onto orthogonal gauge groups on the fourbranes. No exact M-theory description has been proposed for this case until now. We will thus start out in section two by reviewing some facts about orientifold projections and in section three we will propose an M-theory description for the orientifold projecting onto orthogonal gauge groups on the fourbranes. We show that the Riemann surfaces for the \( \mathcal{N} = 2 \) theories with SO-gauge groups follow immediately. Taking this as a convincing consistency check for the proposed M-theory description we go on in section four to derive the curves for theories with product gauge groups of the form \( \bigotimes_i SU(k_i) \otimes SO(N) \). In section five we put a fivebrane on top of the orientifold sixplane. Considering configurations with three fivebranes and fourbranes stretched between them, we derive curves which we interpret as describing theories with \( SU(N) \) gauge group and matter transforming in the symmetric representation. We test these curves by performing several physical consistency checks. In section six we collect some observations concerning the case with symplectic gauge group on the fourbranes. We use these observations to propose curves for \( SU(N) \) with matter in the antisymmetric and establish the relation to the Atiyah-Hitchin manifold in section seven. In section eight we consider product gauge groups of the form \( \bigotimes_i SU(k_i) \otimes Sp(N) \).

2. Orientifold Projections

When Dirichlet branes come together the world-volume theory gets promoted from a \( U(1)^n \) gauge theory to \( U(n) \). Orthogonal and symplectic gauge groups can be induced by considering Dirichlet branes in an orientifold background \cite{42}. Type IIA configurations of fourbranes in the presence of an orientifold fourplane parallel to them have been analyzed in detail in \cite{16, 22, 17, 18}. We want to consider the case of fourbranes in the background of an orientifold sixplane.

\footnote{Although we will not consider \( \mathcal{N} = 1 \) in this paper it should be noted that there is a rapidly growing amount of literature and powerful new tools have also been developed for understanding these theories from the M-theory fivebrane \cite{16, 22, 23, 33}.}
The fourbranes will lie along the directions $x_0, x_1, x_2, x_3, x_6$ being points in $x_4, x_5, x_7, x_8, x_9$. The orientifold sixplane will extend along the $x_0, x_1, x_2, x_3, x_7, x_8, x_9$ directions and sit at the point $x_4 = x_5 = x_6 = 0$. It corresponds to mod out by the space-time transformation

$$ h : (x_4, x_5, x_6) \rightarrow (-x_4, -x_5, -x_6), \quad (2.1) $$

together with the world-sheet parity projection $\Omega$ and $(-1)^F_L$, which changes the sign of all Ramond states on the left \[40\]. In order to obtain a single orientifold plane and avoid charge cancellation conditions, we work in non-compact space. Every fourbrane which does not pass through $x_4 = x_5 = 0$ must have a mirror image.

The described configuration breaks $1/4$ of the initial supersymmetry, twice more than if the projection is induced by an orientifold fourplane parallel to the fourbranes. Let us concentrate on the massless modes of open strings with Dirichlet boundary conditions on the fourbranes. The $\Omega h$-eigenvalue of the massless vertex operators $\partial_t X^{1,2,3,4}$ and $\partial_n X^{4,5}$ is $-1$. This induces for the Chan-Paton part of the corresponding string state $\lambda^a_{ij} |\psi, a>$ the projection

$$ \lambda^a = -\gamma_{\Omega h} \lambda^{aT} \gamma_{\Omega h}^{-1}. \quad (2.2) $$

For the vertex operator $\partial_t X^6$ and $\partial_n X^{7,8,9}$ it induces

$$ \lambda^a = \gamma_{\Omega h} \lambda^{aT} \gamma_{\Omega h}^{-1}. \quad (2.3) $$

Depending on the choice of $\gamma_{\Omega h}$ \[2.2\] leaves us with $\lambda$-matrices forming the adjoint of $SO(N)$ or the adjoint of $Sp(N)$ \[42\]. Upon dimensional reduction along the $x_6$-direction we obtain from \[2.2\] the bosonic fields of an $\mathcal{N} = 2$ vector multiplet whereas the states from \[2.3\] give rise to hypermultiplets in representations other than the adjoint.

We can eliminate the matter coming from $x_6, x_7, x_8, x_9$ by making the fourbranes end on fivebranes that extend along $x_0, x_1, x_2, x_3, x_4, x_5$ and are points in $x_6, x_7, x_8, x_9$ \[43\]. The inclusion of such fivebranes does not break any further supersymmetry and effectively reduces the world-volume of the fourbranes to have only four macroscopic dimensions, i.e. $x_0, x_1, x_2, x_3$.

Preserving $1/4$ of the supersymmetry allows also to include sixbranes parallel to the orientifold sixplane. By T-dualizing in $x_3, x_6$, a configuration of sixbranes and fourbranes occupying directions as described is dual to a collection of parallel sixbranes and twobranes. An orientifold plane parallel to the sixbranes induces different projections on six- and twobranes \[44\], therefore the same is true for the T-dual system of six- and fourbranes.
If we obtain an orthogonal group on the fourbranes we will have a symplectic one on the sixbranes and vice-versa. We work with the convention to count the charges of branes and their mirror images separately. The sixbrane charge of the orientifold is then $+4$ if it projects onto symplectic groups on sixbranes (orthogonal groups on fourbranes) and $-4$ if it projects onto orthogonal groups on sixbranes (symplectic groups on fourbranes).

3. The Geometrical Set-up

We would like to determine the non-perturbative corrections to various brane configurations and from that derive new Seiberg-Witten curves for $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. In [13] it was shown that a very efficient way of achieving this is to lift type IIA brane configurations to M-theory. Both four- and fivebranes of type IIA derive from a single object in M-theory, the M-fivebrane. Their non-perturbative description will correspond to wrap the M-fivebrane around a certain Riemann surface. The analysis of the low energy degrees of freedom shows that this Riemann surface is the Seiberg-Witten curve for the effective gauge theory living on the world-volume of the wrapped M-fivebrane.

Type IIA sixbranes correspond to Kaluza-Klein monopoles in M-theory, or Taub-NUT spaces [45]. A collection of parallel sixbranes will be given by the product of a multi-Taub-NUT space with $\mathbb{R}^7$. The multi-Taub-NUT metric [46] takes the form

$$ds^2 = \frac{V}{4}d\vec{r}^2 + \frac{V^{-1}}{4}(d\tau + \vec{\omega} \cdot \vec{r})^2 ,$$

(3.1)

where

$$V = 1 + \sum_{a=1}^{n} \frac{q_a}{|\vec{r} - \vec{x}_a|} ,$$

(3.2)

$$\nabla \times \vec{\omega} = \nabla V .$$

The $\vec{x}_a$ are the positions of the sixbranes and $q_a = 1$ their charges.

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4 Open strings in the 4-6 sector provide matter in the fundamental representation for the gauge theory on the fourbranes. The sixbrane gauge group acts then as flavor group. Let us dimensionally reduce one direction of the fourbranes. Since the flavor group for a four dimensional $\mathcal{N} = 2$ gauge theory based on an orthogonal group is symplectic and vice-versa [12], this gives another argument that the orientifold must acts with different projections on four- and sixbranes.
Much information can be obtained however by concentrating on the structure of the multi-Taub-NUT spaces as complex manifolds. In one of its complex structures (3.1) can be described \[13\] by the equation
\[
xy = \prod_{i=1}^{n}(v - m_i),
\]
(3.3)
in \(\mathbb{C}^3\), where \(n\) is the number of sixbranes present. (3.3) suppresses however the positions of the sixbranes in the \(x_6\) direction. Later on, when including an orientifold and using (3.3) we will always tacitly assume that the sixbranes are mirror symmetric also in \(x_6\).

Recalling the brane configurations introduced in the past section \(x_4, x_5, x_6, x_{10}\) are the directions transversal to the sixbranes, with \(x_{10}\) referring to the eleventh dimension of M-theory. These are the directions described by the multi-Taub-NUT space or alternatively (3.3). Its convenient to identify \(v = x_4 + ix_5\), while \(x, y\) are single-valued complex variables associated with \(x_6\) and \(x_{10}\). Let us denote by \(R\) the radius of the 11-th dimension. For large \(y\) with \(x\) fixed \(y\) tends to \(t = exp(-(x_6 + ix_{10})/R)\), while for large \(x\) with \(y\) fixed we have \(x \sim t^{-1}\).

For brane configurations breaking \(1/4\) of the initial supersymmetry (3.3) is all the necessary information from the sixbrane sector \([13]\). When \(m_i = 0\) (3.3) reduces to the equation for an \(A_{n-1}\) singularity, representing in M-theory the \(U(n)\) enhancement of gauge symmetry when the \(n\) sixbranes are coincident.

In addition to sixbranes we want to consider the inclusion of an orientifold sixplane. The M-theory metric representing an orientifold sixplane of charge \(-4\) was shown in \([38]\) to be the Atiyah-Hitchin space \([39]\). The inclusion of sixbranes was discussed recently in \([40]\). Far from the orientifold the proposed metrics reduce to a multi-Taub-NUT space containing a charge \(q = -4\) at the origin and charges \(q = 1\) at the positions of the sixbranes. When the sixbranes get close to the orientifold the space develops a \(D_n\) singularity \([37] [38]\).

We will be interested in configurations including an orientifold sixplane of charge \(+4\), which induces symplectic gauge groups on the world-volume of sixbranes parallel to it. In analogy with the previous cases we would like to argue that non-perturbatively this configuration corresponds to M-theory compactified on a 4-dimensional space developing at some point of its moduli space a \(C_n\) type singularity. There does not exists a direct formulation of such an space. We will assume that the charge \(+4\) orientifold affects the geometry in the same way as the presence of 4 coincident sixbranes. Symplectic groups correspond to boundary singularities of \(C_n\)-type \([47]\). We associate the boundary (or fixed
point) of the $C_n$ singularity with the point where the orientifold sits. Thus in one of its complex structures this space should be described by

$$xy = v^4,$$

for the case without sixbranes. The orientifold projection allows only configurations of objects invariant under the $\mathbb{Z}_2$ transformation (2.1), or its lifting to M-theory

$$(x_4, x_5, x_6, x_{10}) \rightarrow (-x_4, -x_5, -x_6, -x_{10}).$$

In terms of the complex variables $(x, y, v)$ this transformation reads

$$v \rightarrow -v,
\quad x \leftrightarrow y. \quad (3.5)$$

![Diagram](image.png)

**Fig. 1:** A configuration two fivebranes connected by parallel fourbranes and an orientifold sixplane.

We consider now configurations of four- and fivebranes invariant under (3.5) in the geometry defined by (3.4). A test of the validity of (3.4) to describe an orientifold sixplane of charge +4 will be to recover the Seiberg-Witten curves for orthogonal groups from a configuration of fourbranes suspended between two fivebranes as in fig. 1.

Following [13], the exact description of this configuration will be given in terms of a Riemann surface $F(y, v) = 0$ immersed in (3.4)

$$F(y, v) = A(v)y^2 + B(v)y + C(v). \quad (3.6)$$

We do not include semi-infinite fourbranes to the left of the first fivebrane or to the right of the second. This implies that $y$ or $x$ can not go to infinity for finite values of $v$. Therefore we can set $A(v) = 1$. We rewrite now (3.6) as $F'(x, v) = 0$

$$C(v)x^2 + B(v)P(v)x + P(v)^2 = 0, \quad (3.7)$$
with $P = v^4$. The absence of semi-infinite fourbranes implies that both $BP$ and $P^2$ must be divisible by $C$. At the same time we have to impose invariance under (3.3), which translates into $F'(v) = F(-v)$. These conditions have the two possible solutions

$$
I: \quad C(v) = v^4, \quad B(v) = B(-v) \\
II: \quad C(v) = -v^4, \quad B(v) = -B(-v).
$$

Substituting option $I$ into (3.6) we obtain

$$y^2 + B(v^2)y + v^4 = 0,$$

(3.9)

with $B$ a generic polynomial in $v^2$ of degree $k$. Equation (3.9) coincides with the Seiberg-Witten curves for $N = 2$ pure Yang-Mills with even orthogonal gauge groups $SO(2k)$.

The option $II$ implies $B(v) = v\tilde{B}(v^2)$ for $\tilde{B}$ a polynomial of degree $k$ in $v^2$. We then get

$$y^2 + v\tilde{B}(v^2)y - v^4 = 0.$$  

(3.10)

This curve is representing a collection of $2k+1$ fourbranes. One of the fourbranes is stuck at $v = 0$, intersecting the orientifold, since it does not have a mirror image. The curve (3.10) is always singular as a result of this. By redefining $y$ (and $x$) this singularity can be absorbed, obtaining the Seiberg-Witten curves for $SO(2k+1)$ gauge theories without matter.

It is worth noting that both curves, for even and odd orthogonal gauge groups, have singularities where branch points meet at $v = 0$. They appear when one sets the constant term in $B$ or $\tilde{B}$ to zero. These singularities correspond to lines in the moduli space that are present in the semi-classical regime as well as in the quantum case. The situation was analyzed in detail in [9]. For even orthogonal gauge groups the corresponding singularity turned out to be an unphysical one with trivial monodromy. For odd orthogonal gauge groups it was shown that, although the singularity at $v = 0$ exists already in the semi-classical limit, its interpretation changes in the quantum case, such that it can be attributed to a massless monopole. A priori we could expect (3.4) to describe the orientifold only in the semi-classical regime. It is however enough to recover the correct curves for $SO(N)$ gauge groups. Furthermore these curves indicate the presence of the orientifold through the singularities at $v = 0$, both in the semiclassical and fully quantum corrected regimes. This shows that (3.4) is valid even in the strong coupling regime.

The curves for orthogonal groups with fundamental matter can be derived by including semi-infinite fourbranes or by adding sixbranes parallel to the orientifold sixplane. We will treat the inclusion of sixbranes in the next section, when considering more general brane configurations.
Fig. 2: A configuration of fivebranes connected by parallel fourbranes, sixbranes and an orientifold sixplane.

4. Product Groups: $\prod_i SU(k_i) \times SO(k)$

Let us analyze a configuration of $2n$ fivebranes with $k_i$ fourbranes suspended between the $i$-th and $i+1$-th fivebranes in the presence of the orientifold sixplane. We also allow for $d_i$ sixbranes placed between the $i$-th and $i+1$-th fivebranes. This collection of branes must be arranged in a symmetric way under the space-time projection induced by the orientifold (see fig. 2), therefore

$$k_i = k_{2n-i} , \quad d_i = d_{2n-i}.$$  \hspace{1cm} (4.1)

The fluctuations of the $k_i$ fourbranes for $i = 1, \ldots n - 1$ are not constrained by the orientifold but merely related to the mirror images. Therefore on the world-volume of the fourbranes this configuration will induce the product gauge group $\prod_{i=1}^{n-1} SU(k_i) \times SO(k_n)$\footnote{We get special unitary groups because the overall $U(1)$ factor is frozen out by the finite energy condition on the fivebranes as explained in \cite{13}.}. The matter content consists of $n - 2$ hypermultiplets transforming in the bifundamental representation $(k_i, \bar{k}_{i+1})$ of the $SU$ factors, for $i = 1, \ldots, n - 2$, and one hypermultiplet in the $(k_{n-1}, k_n)$, with $k_n$ denoting now the vector representation of $SO(k_n)$. In addition there are $d_i$ fundamental hypermultiplets for each factor $SU(k_i)$, and $d_n/2$ fundamentals for $SO(k_n)$. 

\footnote{We get special unitary groups because the overall $U(1)$ factor is frozen out by the finite energy condition on the fivebranes as explained in \cite{13}.}
We want to derive the Riemann surface encoding the exact solution for the described $\mathcal{N}=2$ gauge theory. In order to take into account the effect of the sixbranes we generalize (3.4) to
\[ xy = P(v) = (-1)^{2d} v^4 \prod_{i=1}^{2d} (v - m_i), \quad (4.2) \]
with $2d = \sum d_i$ and for each mass parameter $m_i \neq 0$ there exits $m_j = -m_i$. We have introduced the factor $(-1)^d$ for convenience. We now define
\[ J_i = A_i \prod_{j=l_i-1+1}^{l_i} (v - m_j) \quad (4.3) \]
with $l_i = \sum_{j=1}^{i} d_j$ and
\[
\begin{align*}
A_i &= 1 & i &= 1, \ldots, n - 1 \\
A_n &= (-1)^{d_n/2} v^4 \\
A_i &= (-1)^{d_i} & i &= n + 1, \ldots, 2n - 1.
\end{align*}
\quad (4.4)
\]
The polynomials $J_i$ represent the contribution to the space (4.2) from the sixbrane charge sources, i.e. sixbranes and orientifold sixplane, between the $i$-th and $i+1$-th fivebranes. They satisfy $P = \prod_{i=1}^{2n-1} J_i$. The $\mathbb{Z}_2$ invariance with respect to the orientifold implies $m_j^{(2n-i)} = -m_j^{(i)}$. Using this, the polynomials $J_i$ verify
\[ J_i(v) = J_{2n-i}(-v). \quad (4.5) \]

In [13] the Riemann surface representing a collection of fivebranes with fourbranes suspended between them and immersed in an space of the form (4.2) was determined. When no semi-infinite fourbranes are present we have
\[ y^{2n} + g_1(v)y^{2n-1} + \ldots + g_i(v) \prod_{s=1}^{i-1} J_s^{i-s} y^{2n-i} + \ldots + f \prod_{s=1}^{2n-1} J_s^{2n-s} = 0, \quad (4.6) \]
with $f$ a constant and $g_i$ polynomials in $v$ of degree $k_i$ given by the number of fourbranes suspended between the $i$-th and $i+1$-th fivebranes
\[
g_i(v) = c_{0,i}v^{k_i} + c_{1,i}v^{k_i-1} + c_{2,i}v^{k_i-2} + \ldots + c_{k_i,i}. \quad (4.7) \]
In our case we have in addition to impose symmetry under \((v, y, x) \rightarrow (-v, x, y)\). This, together with the property (\ref{symmetry_property}) of the polynomials \(J_i\), allows for the following two sets of solutions

\[
I: \quad f = 1, \quad g_i(v) = g_{2n-i}(-v) \\
II: \quad f = -1, \quad g_i(v) = -g_{2n-i}(-v).
\] (4.8)

These conditions leave the polynomials \(g_i(v)\) for \(i = 1, \ldots, n-1\) unconstrained. This is necessary in order they represent \(SU\) group factors. The solution \(I\) implies \(g_n(v) = g_n(-v)\) corresponding to product gauge groups of unitary groups with an even orthogonal group, \(\prod_{i=1}^{n-1} SU(k_i) \times SO(2k)\). The solution \(II\) implies \(g_n(v) = -g_n(-v)\) and is associated with chains including an odd orthogonal group, \(\prod_{i=1}^{n-1} SU(k_i) \times SO(2k+1)\).

The coefficients \(c_{\alpha,i}, \alpha \geq 2\), in (4.7) describe the Coulomb branch of the obtained gauge theory. Equations (4.8) reduce the number of independent \(c_{1,i}\) to \(n - 1\), which represent bare masses for the \(n - 1\) hypermultiplets in the bifundamental representations. The coefficients \(c_{0,i}\) are associated with ratios of gauge couplings. Rescaling \(v\) and \(y\) we obtain \(n - 1\) independent \(c_{0,i}\) as corresponds to a product gauge group with \(n\) factors.

\(\mathcal{N} = 2\) gauge theories based on orthogonal groups with matter content in the vector representation have symplectic flavor symmetry groups \([12]\). Let us consider our brane configuration in the absence of sixbranes. The matter content for the orthogonal factor group \(SO(k_n)\) consists of \(k_{n-1}\) \(\mathcal{N} = 2\) hypermultiplets in the vector representation. The associated flavor group is \(Sp(2k_{n-1})\). In the brane configuration only a \(SU(k_{n-1})\) subgroup is manifest, which is given by the next factor group in the chain.

5. \(SU(N)\) with matter in the symmetric

Until now we always placed the orientifold in between two fivebranes. This is however not the only possibility. In this section we will identify and study the theory that arises when we place a fivebrane on top of the orientifold sixplane. Such a configuration has been studied from the conformal field theory point of view recently in \([48]\). More precisely we place one fivebrane on top of the orientifold sixplane and another fivebrane to the right of it. Further there be \(N\) fourbranes stretching in between the fivebranes. To the left of the orientifold sixplane we have of course the mirror image of this brane configuration. We do

\[\text{We can not restrict to } n - 2 \text{ independent } c_{0,i} \text{ by rescaling separately } v \text{ and } y \text{ since this will introduce an additional parameter in } [12].\]
not include sixbranes in this configuration. Again we assume that the orientifold can be described by (3.4). The polynomial describing the wrapped M-theory fivebrane will now be of third order in $y$. For brane configurations with an odd number of fivebranes the options $I$ and $II$ in (4.8) are equivalent, but just related by $y \rightarrow -y$. Thus we obtain

$$y^3 + y^2 \prod_{i=1}^{N} (v - a_i) + (-)^N y v^2 \prod_{i=1}^{N} (v + a_i) + v^6 = 0.$$  \hspace{1cm} (5.1)

One can readily check that this curve is symmetric under the simultaneous replacements $v \rightarrow -v$ and $y \rightarrow v^4/x$.

In order to identify the gauge theory whose Coulomb branch is described by (5.1), we note that the gauge theory will be $SU(N)$. This is because the orientifold does not constrain the fluctuations of the fourbranes but relates them to their mirror images on the other side of the middle fivebrane. The endpoints of fourbranes to the right of the orientifold sixplane will give rise to hypermultiplets. If we pretend for a moment that the two fourbrane groups were independent of each other, we would obtain a product gauge group with matter transforming in the fundamental with respect to each factor. Since in our case both gauge groups factors have to be identified, the bifundamental turns into a two index representation of $SU(N)$. It can not be the adjoint since not all degrees of freedom of the adjoint are really available. A string stretching from one fourbrane on the right to a fourbrane on the left comes inevitably with its mirror image. Thus we conclude that the gauge theory on the M-theory fivebrane will contain a hypermultiplet transforming in either the symmetric or antisymmetric representation of $SU(N)$. This is possible because hypermultiplets contain simultaneously fields transforming under the fundamental and anti-fundamental of $SU(N)$.

We can use the symmetry properties of (5.1) in order to decide which case is realized. It is well known that the curves describing the Coulomb branch of $\mathcal{N} = 2$ gauge theories have to reflect the breaking of the $U(1)_R$ symmetry to $\mathbb{Z}_{b_0}$, with $b_0$ being the one-loop beta function coefficient. This $U(1)_R$ symmetry appears as rotations in the $v$-plane. It is spontaneously broken if the fourbranes are not all placed at the origin. Let us assign now $R$-charge 1 to $v$ and $a_i$ and $R$-charge $N$ to $y$. If we perform now such a transformation with $x \rightarrow \exp(i\alpha)$ and $y \rightarrow \exp(iN\alpha)$ we find that the curve (5.1) stays invariant if we chose $\alpha = 2\pi \frac{m}{N-2}$, showing a $\mathbb{Z}_{N-2}$ symmetry. The one loop beta function coefficient of the gauge theory is thus $b_0 = N - 2$. This has to equal $b_0 = 2N - 2I_m$ where $I_m$ is the
index of the representation of the hypermultiplet. We find thus \( I_m = (N + 2)/2 \) which is the index of the symmetric representation of \( SU(N) \).

In the remainder of the section we will concentrate in performing several consistency checks of the proposed curves. In order to do so it is convenient to introduce the scale \( \Lambda \) of the gauge theory. The space (3.4) reads then \( xy = \Lambda^{2N-4}v^4 \). We further observe that (5.1) is singular at \( v = y = 0 \) since the middle fivebrane passes through the orientifold. We can remove this singularity by redefining \( y \rightarrow v^2y \). The precise statement is now as follows. The subspace of the Jacobian selected by the cycles invariant under \( (v,y) \rightarrow (-v, \Lambda^{2N-4}/y) \) of the curve

\[
v^2y^3 + y^2 \prod_{i=1}^{N} (v - a_i) + (-)^N \Lambda^{N-2}y \prod_{i=1}^{N} (v + a_i) + \Lambda^{3N-6}v^2 = 0.
\] (5.2)

parameterizes the Coulomb branch of the moduli space of \( \mathcal{N} = 2 \) gauge theories with a hypermultiplet in the symmetric representation. That the invariant subspace of the Jacobian is the relevant one can be seen by noting that the Seiberg-Witten differential \( \lambda_{SW} = v \frac{dy}{y} \) stays invariant. The \( N \) parameters \( a_i \) represent the positions of the fourbranes in the classical brane configuration. The distance between the average position of the fourbranes on the left and the average position of the fourbranes on the right is the mass of the hypermultiplet in the symmetric

\[
m = \frac{2}{N} \sum_{i=1}^{N} a_i.
\] (5.3)

Thus we see that the curve has just the correct number of free parameters.

A rather nontrivial consistency check for the curves (5.2) is as follows. On the classical Coulomb moduli space there are submanifolds where the \( SU(N) \) group is only partially broken and factorizes as \( SU(N) \rightarrow \bigotimes_{i=1}^{l} SU(n_i) \otimes U(1)^{l-1} \). The hypermultiplet in the symmetric will split up in various matter fields transforming under \( SU(n_i) \) factors according to the decomposition of the symmetric representation. More precisely, if we parameterize the Coulomb branch by \( \Phi \) being a matrix in the fundamental of \( SU(N) \) then the above breaking pattern corresponds to choosing \( \Phi = \text{diag}(e_1, \cdots, e_1; e_2, \cdots, e_2; \cdots; e_l, \cdots, e_l) \). Here each \( e_i \) appears \( n_i \) times, \( e_i \neq e_j \), and \( \sum n_ie_i = 0 \). The weights of the symmetric representation can be obtained as the symmetric product of the weights of the fundamental of \( SU(N) \). We denote the latter by \( \lambda_I, I = 1, \cdots, N \). Under the symmetry breaking
the fundamental of $SU(N)$ decomposes into the fundamentals of the $SU(n_i)$ factors whose weights we denote by $\lambda_i$. The decomposition of the symmetric is thus given by

$$\lambda_I \bigotimes_{\text{sym}} \lambda_J = \bigoplus_{i \leq j} \lambda_i \bigotimes_{\text{sym}} \lambda_j . \quad (5.4)$$

The masses of the various matter fields are given by $m_i = 2e_i$ for a hypermultiplet in the symmetric of $SU(n_i)$ and $m_{ij} = e_i + e_j$ for a hypermultiplet in the bifundamental of $SU(n_i) \otimes SU(n_j)$. We can make these matter fields massless by choosing $m$ to cancel either $m_i$ or $m_{ij}$. The light degrees of freedom are then the vector multiplets of the gauge group factors and a hypermultiplet in the symmetric of one $SU(n_i)$ or in the bifundamental of $SU(n_i) \otimes SU(n_j)$. This symmetry breaking pattern has to be respected by the curve (5.2).

If we take the symmetry breaking scale to be large, then the curve should reduce to the curves describing the gauge groups factors with the corresponding matter content after performing an appropriate scaling limit.

Let us investigate these scaling limits in detail now. We start with scaling to a smaller gauge group $SU(n_i)$ with matter in the symmetric. This can be achieved by choosing $e_i = E, E$ large, and $m = -2E$. Taking into account (5.3) this can be translated into the positions of the fourbranes $a_i = e_i + \frac{m_i}{2}$. We also allow for small fluctuations $\delta a_k$ in the positions of the fourbranes corresponding to the $SU(n_i)$ factor but suppress these fluctuations for the rest of the fourbranes. The curve takes the form

$$v^2 y^3 + y^2 \prod_{j \neq i} (v + E - e_j)^{n_j} \prod_{k=1}^{n_i} (v - \delta a_k) +$$

$$(-)^N \Lambda^{-2} y \prod_{j \neq i} (v - E + e_j)^{n_j} \prod_{k=1}^{n_i} (v + \delta a_k) + \Lambda^{3N-6} v^2 = 0 . \quad (5.5)$$

After rescaling $y \to E^{N-n_i} y$ this becomes

$$v^2 y^3 + y^2 \prod_{j \neq i} \left( \frac{v}{E} + 1 - \frac{e_j}{E} \right)^{n_j} \prod_{k=1}^{n_i} (v - \delta a_k) +$$

$$(-)^N E^{-N+n_i} \Lambda^{-2} y \prod_{j \neq i} \left( \frac{v}{E} - 1 + \frac{e_j}{E} \right)^{n_j} \prod_{k=1}^{n_i} (v + \delta a_k) + E^{-3N+3n_i} \Lambda^{3N-6} v^2 = 0 . \quad (5.6)$$

Now we can perform a renormalization group matching

$$\left( \frac{\Lambda}{E} \right)^{N-2} = \left( \frac{\tilde{\Lambda}}{E} \right)^{n_i-2} . \quad (5.7)$$
Taking $E$ to infinity but keeping $\tilde{\Lambda}$ fixed \((5.6)\) reproduces \((5.2)\) with $N = n_i$.

Alternatively we could have chosen $m = 0$. The light degrees of freedom consist then only of the vector multiplets of the gauge group factors. We concentrate now on the $i$-th factor. If we introduce $\xi = v - e_i$, rescale $y \rightarrow E^{N - n_i - 2}y$ and suppress fourbrane fluctuations outside of this factor. We further assume that $e_j \neq -e_i$ since then we have additional massless matter in the bifundamental of $SU(k_i) \otimes SU(k_j)$. The curve can be written as

\[
\left(\frac{\xi}{E} + 1\right)^2 y^3 + y^2 \prod_{k=1}^{n_i} (\xi - \delta a_k) \prod_{j \neq i} \left(\frac{\xi}{E} + 1 - \frac{e_j}{E}\right)^{n_j} +
\]

\[
(-)^N \Lambda^{N - 2} y E^{-N + 2} \prod_{k=1}^{n_i} \left(\frac{\xi}{E} + 2 + \frac{\delta a_k}{E}\right)^{n_j} \prod_{j \neq i} \left(\frac{\xi}{E} + 1 + \frac{e_j}{E}\right)^{n_j} + \Lambda^{3N - 6} E^{-3N + 3n_i + 6} = 0.
\]

After a RG-matching

\[
\left(\frac{\Lambda}{E}\right)^{N - 2} = \left(\frac{\tilde{\Lambda}}{E}\right)^{2n_i},
\]

and taking $E$ to infinity this takes the form of the curve for pure $SU(n_i)$ gauge theory.

Next we chose $m = -e_i - e_j$. We expect to find the curve for $SU(n_i) \otimes SU(n_j)$. We set now $\frac{e_j - e_i}{2} = E$. The curve takes the form

\[
v^2 y^3 + y^2 \prod_{k=1}^{n_i} (v + E - \delta a_k) \prod_{l=1}^{n_j} (v - E - \delta b_l) \prod_{p \neq i, j} (v - a_p)^{n_p} +
\]

\[
(-)^N \Lambda^{N - 2} y \prod_{k=1}^{n_i} (v + E - \delta a_k) \prod_{l=1}^{n_j} (v + E + \delta b_l) \prod_{p \neq i, j} (v + a_p)^{n_p} + \Lambda^{3N - 6} v^2 = 0.
\]

Since below the scale $E$ we have two gauge groups factors the RG-matching has to fulfill now

\[
\left(\frac{\Lambda}{E}\right)^{N - 2} = \left(\frac{\tilde{\Lambda}_i}{E}\right)^{2n_i - n_j} = \left(\frac{\tilde{\Lambda}_j}{E}\right)^{2n_j - n_i}.
\]

From this we see that as long as $n_i \neq n_j$, in the limit $E \rightarrow \infty$ one of the two gauge group factors inevitably ends up at infinite or vanishing coupling. In order to avoid this we set $n_i = n_j = n$. Upon introducing $\xi = v + E$, rescaling $y \rightarrow E^{N - n - 2}y$ and absorbing some numerical factors into $\tilde{\Lambda}$ \((5.10)\) becomes

\[
y^3 + y^2 \prod_{k=1}^{n} (\xi - \delta a_k) + \tilde{\Lambda}^n y \prod_{l=1}^{n} (\xi - \delta b_l) + \tilde{\Lambda}^{3n} = 0.
\]
This is the correct form of the curve for $SU(n) \otimes SU(n)$ with matter transforming in the bifundamental $[13] [15]$.

As a final example for scaling limits of (5.2) let us take the mass $m$ to infinity. It can be seen easily that in this limit, and after introducing $\xi = v - \frac{m}{2}$, $y \to m^{-2}y$ and the RG-matching $\Lambda^{N-2}m^{N+2} = \tilde{\Lambda}^{2N}$, the curve

$$v^2 y^3 + y^2 \prod_{i=1}^{N} (v - \frac{m}{2} - e_i) + (-)^N \Lambda^{N-2} y \prod_{i=1}^{N} (v + \frac{m}{2} + e_i) + \Lambda^{3N-6} v^2 = 0,$$

reproduces the one for the $SU(N)$ theory without matter.

The case of $N = 2$ is special. First we can not eliminate all possible numerical factors by scalings of $y$ or $v$. Moreover the symmetric of $SU(2)$ is actually the adjoint. Therefore the curve

$$v^2 y^3 + y^2 (v^2 - mv - u) + ey(v^2 + mv - u) + e^3 v^2 = 0,$$

should describe the softly broken $\mathcal{N} = 4$ $SU(2)$ theory. That we can not eliminate the dimensionless complex number $e$ by rescalings is then the expected behavior. It determines the gauge coupling of the softly broken $\mathcal{N} = 4$ theory. The discriminant of (5.14) is given by

$$\Delta = e(1 + e)mu \left( m^4 + 8m^2u + 40em^2u + 16u^2 - 96eu^2 + 144e^2u^2 \right).$$

Since the curve (5.14) is at most quadratic in $v$, it is hyperelliptic. Its genus is two. If we write it as $v^2 A + vB + C = 0$ and introduce the new coordinate $\tilde{v} = \frac{v - B}{2A}$, the curve takes the form

$$\tilde{v}^2 = y^2 m^2 (y - e)^2 + 4(y + e)^2 (y^2 + y(1 - e) + e^2) uy.$$

At $m = 0$ after absorbing the overall $y + e$ factor into $\tilde{v}$ we are left with a torus. The complex structure of this torus depends only on $e$ and its degenerations are located at $\Delta_e = e(1 + e)(1 - 3e) = 0$. Notice that there is a new factor $(1 - 3e)$ that did not appear in (5.13). We can make its appearance manifest by redefining $m = (1 - 3e)\tilde{m}$. Remember that in the case of theories with vanishing beta function one can reparameterize the moduli space with the help of arbitrary functions of the dimensionless coupling constant with the only condition that the new moduli coincide with the classical ones in the weak coupling limit [7] [12]. In our case this sets the weak coupling limit at $e = 0$. That this is so can be deduced from the double scaling limit to obtain pure $SU(2)$, which works with $e \to 0$ and $m \to \infty$ as in (5.13). The discriminant (5.13) is given now by $\Delta = \tilde{m} . \Delta_e . \Delta_u$ where we set

$$\Delta_u = u \left( \tilde{m}^4 (1 - 3e)^2 + 8\tilde{m}^2u + 40e\tilde{m}^2u + 16u^2 \right).$$
The singularities in the $u$-plane are located at $u = 0$ and $u = u_\pm$ with

$$u_\pm = \left[-(1 + 5e)/4 \pm \sqrt{e(1 + e)}\right]\tilde{m}^2. \quad (5.18)$$

We want to compare this now with the singularities of the curve for the softly broken $\mathcal{N} = 4$ theory [3] [19]

$$y^2 = \prod_{i=1}^{3}(x - e_iU - \frac{1}{4}e_i^2M^2). \quad (5.19)$$

For $M = 0$ the $e_i(\tau)$ are the roots of a cubic describing a torus of modular parameter $\tau$. Their relation to Jacobi theta functions is given by $e_1 - e_2 = \theta_3(\tau)^4$, $e_3 - e_2 = \theta_1(\tau)^4$ and $e_1 - e_3 = \theta_2(\tau)^4$. Its singularities are located at $\Delta_U, \Delta_x = 0$ with

$$\Delta_x = \prod_{i \leq j}(e_i - e_j), \quad (5.20)$$

and

$$\Delta_U = \prod_{i=1}^{3}(U - e_i(M/2)^2). \quad (5.21)$$

If we shift $U \to U + e_1(M/2)^2$ and identify $U$ with $u$ we obtain the equations

$$e_2 - e_1 = \left[-(1 + 5e) + 4\sqrt{e(1 + e)}\right](\tilde{m}/M)^2$$
$$e_3 - e_1 = \left[-(1 + 5e) - 4\sqrt{e(1 + e)}\right](\tilde{m}/M)^2. \quad (5.22)$$

After a little algebra we extract from that

$$e = \frac{-(5h^2 - 8) \pm 8\sqrt{1 - h^2}}{25h^2 - 16}, \quad (5.23)$$

where

$$h = \frac{e_2 - e_3}{3e_1}. \quad (5.24)$$

The zeroes of $\Delta_x$ are at $\tau = (i\infty, 0, 1)$. These values correspond to $e_2 = e_3$, $e_1 = e_3$ and $e_1 = e_2$. Thus at weak coupling $\tau = i\infty$ we have $h = 0$ and $e = 0$ or $e = -1$. The values $\tau = 0$ and $\tau = 1$ are mapped onto $h = 1$ and correspond to $e = 1/3$. These values are the zeroes of $\Delta_e$. It is a nontrivial check of consistency that after matching of $\Delta_U$ to $\Delta_u$ we obtain automatically the matching between $\Delta_x$ and $\Delta_e$.

The discriminant (5.15) contains however also a singularity at $\tilde{m} = 0$. We can study it by looking at the holomorphic differentials on the curve (5.10). Since it has genus...
two $\omega_1 = \frac{dv}{v}$ and $\omega_2 = \frac{w dw}{v}$ form a basis of holomorphic differentials. The orientifold $\mathbb{Z}_2$ involution acts as $(\tilde{v}, y) \to (-e^3/y^3 \tilde{v}, e^2/y)$ in the new coordinates. It also acts as $(A, C) \to e^3/y^3 (A, C)$ and $B \to -e^3/y^3 B$, where $A, B$ and $C$ as defined above. On the holomorphic differentials it acts as $e \omega_1 \leftrightarrow \omega_2$. $\Omega_\pm = e \omega_1 \pm \omega_2$ form a basis of even and odd holomorphic differentials. As needed we find $\partial u \lambda = \Omega_+$. When $\tilde{m}$ goes to zero the genus of the curve drops. Then we can absorb the overall factor of $(y + e)$ into the definition of $\tilde{v}$. Doing this we notice that $\Omega_+$ descends to the unique holomorphic differential on the remaining torus! Since the physically relevant periods stem only from $\Omega_+$ none of them vanishes at this particular degeneration. Thus we do not expect any new massless state at $\tilde{m} = 0$.

We think that the above arguments are rather strong evidence in favor of the conjecture that (5.14) describes the softly broken $\mathcal{N} = 4$ theory correctly.

### 6. $SU(N)$ with matter in the antisymmetric

In the previous sections we have analyzed brane configurations in the background defined by an orientifold sixplane of charge +4. Now we want to consider situations including an orientifold sixplane of the opposite charge, i.e. −4. Some configurations in the presence of a negative charge orientifold sixplane, or its $T$-dual orientifold sevenplane, have been studied in [41] [37] [38].

In the absence of branes that probe the geometry induced by the orientifold, the sign of the orientifold charge is a matter of convention. In [38] it was shown that the geometry seen by a twobrane probe parallel to an orientifold sixplane which projects onto symplectic groups on the twobrane, is given by the Atiyah-Hitchin space [39]. Far from the orientifold this space reduces to a multi-Taub-NUT space containing a charge $q = -4$ at the origin [50] [38] [40]. An approximate description of this space as a complex manifold, valid far from the orientifold, will be given by

$$xy = \Lambda^{2N+4} v^{-4}. \tag{6.1}$$

We consider now a configuration of two fivebranes and fourbranes in the space defined by (6.1). As before, the presence of the orientifold forces any collection of objects to be arranged in a symmetric way under the $\mathbb{Z}_2$ transformation (3.5) (see fig. 1). Imposing
invariance under \((v, y, x) \rightarrow (-v, x, y)\) and following the same steps as in section 3, we obtain the following curve associated to our configuration

\[
y^2 + y \left( b(v^2) + A v^{-2} \right) + \Lambda^{2N+4} v^{-4} = 0.
\] (6.2)

We have reduced ourselves to the option \(I\) in (1.8). This constrains the number of four-branes \(N\) to be even. We notice two different contributions to (6.2), \(b(v^2)\) and \(A v^{-2}\). \(b\) is a polynomial in \(v^2\) of degree \(k = N/2\) whose coefficients, according to [13], we associate with Casimirs of a gauge theory. \(A\) is an undetermined constant. The additional term \(A v^{-2}\) is allowed due to the \(v^{-4}\) contribution of the orientifold. For large values of \(v\) this term is irrelevant, however it dominates for small \(v\).

We interpret the extra term \(A v^{-2}\) as a manifestation that close to \(v = 0\) the space (6.1) does not provide a good description of the orientifold background. It is missing strong coupling effects that would modify the geometry into that of the Atiyah-Hitchin space. However when formulating curves in (6.1) the possibility to take into account strong coupling effects reappears. It is encoded in the presence of additional monomials in negative powers of \(v\), as it is the case of \(A v^{-2}\) in (6.2). Indeed, by setting \(A = 2\Lambda^{N+2}\) and rescaling \(y \rightarrow v^{-2} y\) we recover the Seiberg-Witten curves for \(N = 2\) pure Yang-Mills theory with gauge group \(Sp(2k)\) [11] [12] [51]

\[
y^2 + y \left( v^2 b(v^2) + 2\Lambda^{N+2} \right) + \Lambda^{2N+4} = 0.
\] (6.3)

We observe the direct dependence of the constant \(A\) of strong coupling effects whose characteristic scale is set by \(\Lambda\).

A similar situation was encountered [17] when analyzing brane configurations in the presence of an orientifold fourplane. For configurations providing symplectic gauge groups, strong coupling effects associated with the orientifold led to the additional factor encountered in (6.2). In that case the analysis of the theory on the world-volume of the fivebranes allowed to fix the constant \(A\) and recover the symplectic curves. In the present case we do not know how to carry out such an analysis. In order to fix this ambiguity we use as guideline the matching with expected results from field theories.

Let us analyze next a configuration containing three fivebranes. Invariance under (3.3) leads to the curve

\[
y^2 + y^2 \left( p(v) + B v^{-1} + A v^{-2} \right) + \Lambda^{N+2} v^{-2} y \left( q(v) - B v^{-1} + A v^{-2} \right) + \Lambda^{3N+6} v^{-6} = 0,
\] (6.4)
where \( p(v) = q(-v) = \prod_{i=1}^{N} (v - a_i) \) and \( N \) can be even or odd now. In the present case two additional factors in negative powers of \( v \) are allowed, \( Av^{-2} \) and \( Bv^{-1} \) with \( A, B \) constants. Using the same reasoning as in the beginning of section 5, we conclude that this curve should describe an \( SU(N) \) theory with matter in a tensor representation. The natural candidate for the matter representation in the present case is the antisymmetric.

In order to check this we analyze the breaking of the \( U(1)_R \) symmetry to a residual \( \mathbb{Z}_b \). We assign \( R \)-charge 1 to \( v \) and \( a_i \), and \( N \) to \( y \). Under a \( U(1)_R \) transformation the curve (6.4) remains invariant only if the angle verifies \( \alpha = 2\pi \frac{n}{N+2} \), showing a \( \mathbb{Z}_{N+2} \) symmetry. Since for \( N = 2 \) theories this residual symmetry is directly related to the one-loop beta function coefficient we have

\[
b_0 = 2N - 2I_m = N + 2, \tag{6.5}
\]

where \( I_m \) is the index of the matter representation. Therefore we obtain \( I_m = (N - 2)/2 \) which is the index of the antisymmetric representation of \( SU(N) \).

In the previous argument we have ignored the terms associated to the constants \( A \) and \( B \). Equation (6.4) can be made \( U(1)_R \) invariant if we assign convenient \( R \)-charges to \( \Lambda, A \) and \( B \). The parameter \( \Lambda \) must be assigned charge 1. The \( R \)-charge of \( A \) and \( B \) has to be \( N + 2 \) and \( N + 1 \) respectively. We have argued that \( A \) and \( B \) are associated with strong coupling effects. Based on this we set \( A \sim \Lambda \) and \( B \sim \Lambda/m \) with \( m \) the mass of the antisymmetric hypermultiplet. Such a \( B \) coefficient will introduce large effects at \( m \to 0 \) even for small \( \Lambda \). Since no such effects are expected in gauge theories we set \( B = 0 \).

To fix the precise value of the constant \( A \) we consider the case in which \( p(v) = q(v) \). This can be achieved only for \( N \) even, by turning off the coefficients in \( p \) that multiply odd powers of \( v \). The curve (6.4) then factorizes in two pieces

\[
(y + \Lambda^{N+2}v^{-2}) \left( y^2 + y \left( p(v^2) + (A - \Lambda^{N+2})v^{-2} \right) + \Lambda^{2N+4}v^{-4} \right) = 0. \tag{6.6}
\]

This corresponds to fourbranes on the left and right of the middle fivebrane recombining into single fourbranes. In this situation the middle fivebrane decouples, can be pulled away, and it is just described by the first factor in (6.6). The second factor will now describe an even number of fourbranes suspended between two fivebranes, and therefore must reproduce the Seiberg-Witten curve for symplectic gauge groups without matter. This condition sets \( A = 3\Lambda^{N+2} \). We will assume that the same value of \( A \) holds for \( N \) odd.
Rescaling \( y \rightarrow y v^{-2} \), we propose finally the following curves for representing \( SU(N) \) with a hypermultiplet transforming in the antisymmetric representation

\[
y^3 + y^2 \left( v^2 \prod_{i=1}^{N} (v - a_i) + 3\Lambda^{N+2} \right) + \Lambda^{N+2} y \left( (-)^N v^2 \prod_{i=1}^{N} (v + a_i) + 3\Lambda^{N+2} \right) + \Lambda^{3N+6} = 0.
\]

(6.7)

These curves have the correct behavior under analogous scaling limits to that studied in section 5. We refer to that section for details. At \( v = 0 \) (6.7) has a triple solution in \( y \). Configurations containing two fivebranes satisfy a similar property, the curve (6.3) has a double solution in \( y \) at \( v = 0 \). As it is the case for the symplectic curves, the point \( v = 0 \) is a singularity of (6.7) for any value of the Casimirs and mass parameter. In the next section we will rewrite (6.7) in a form that eliminates the singularity at \( v = 0 \). We will now analyze in detail the cases \( N = 2, 3 \) as consistency checks of our curves.

For \( N = 2 \) the previous curve reads

\[
y^3 + y^2 \left( v^2 (v^2 + vm + u) + 3\Lambda^4 \right) + \Lambda^4 y \left( v^2(v^2 - vm + u) + 3\Lambda^4 \right) + \Lambda^{12} = 0.
\]

(6.8)

The antisymmetric representation of \( SU(2) \) is just the singlet. Since a singlet will decouple from the theory, the curve (6.8) should reproduce the physics of pure \( SU(2) \) Yang-Mills. Let us set first \( m = 0 \). The condition \( p(v) = q(v) \) is then satisfied and from the way we fixed \( A \) our curve reduces to that of pure \( Sp(2) = SU(2) \) gauge theory. We consider now the limit in which \( m \) is sent to infinite. After shifting \( v \rightarrow v - m/2 \) and rescaling \( y \rightarrow y m^2 \), in this limit our curve becomes

\[
y \left( y^2 + y (v^2 + u) + \Lambda^4 \right) = 0.
\]

(6.9)

The second factor is precisely the Seiberg-Witten curve for pure \( SU(2) \) gauge theory [2] [3] [4] [5].

For \( 0 < m < \infty \) (6.8) does not factorizes. However the fact that we can take \( m \rightarrow \infty \) without having to tune the value of the dynamical scale shows that the field of mass \( m \) does not contribute to the beta-function, and therefore can not be charged under the gauge group. Furthermore we can calculate the discriminant\( \Delta \) of (6.8)

\[
\Delta = m(u - \frac{m^2}{4} + 4\tilde{\Lambda}^2)(u - \frac{m^2}{4} - 4\tilde{\Lambda}^2)(u^3 - 27m^2\tilde{\Lambda}^4).
\]

(6.10)

\footnote{We mean here the loci in moduli space at which (6.8) acquire a non-generic singularity, i.e. excluding the singularity always occurring at \( v = 0 \).}
The first factor is associated with the factorization (6.6). We will assume that the cycle shrinking at this singularity is a non physical one, as was the case for the curve (5.14) of the previous section. If we shift \( u \rightarrow u + \frac{m^2}{4} \) the second and third factors match with the discriminant of \( SU(2) \) without matter. We leave for the end of this section the analysis of the last factor. We will argue that it governs a non-physical singularity and therefore we can discard it.

We study next the case \( N = 3 \). For \( SU(3) \) the antisymmetric representation coincides with the anti-fundamental. \( N = 2 \) hypermultiplets will contain simultaneously fields transforming in the anti-fundamental and fundamental representation. Thus (6.7) for \( N = 3 \) should reproduce \( N = 2 \) \( SU(3) \) gauge theory with one hypermultiplet in the fundamental. The known curve for this case is \( y^2 + y(v^3 + Wv + U) + 4(v + m) = 0 \),

\[
y^2 + y(v^3 + Wv + U) + 4(v + m) = 0, \quad (6.11)
\]

where we have set arbitrarily the dynamical scale \( \Lambda' \) to \( \Lambda'^5 = 4 \). The discriminant of (6.11) is the following rather lengthy expression

\[
\Delta = 216000m^2U + 22680m^2U^2W + 13680W^3mU - 729U^4Wm - 24300U^3m
\]
\[
- 200000 - 10800W^2U^2 + 90000WU - 66000mW^2 - 3552m^2W^4 - 259200m^3W
\]
\[
+ 729U^5 + 216U^3W^3 + 16W^6U - 16mW^7 - 16m^3W^6 - 3456m^4W^3 - 186624m^5
\]
\[
- 64W^5 - 729U^4m^3 - 216W^4U^2m - 216W^3U^2m^3 + 10368m^3W^2U + 23328m^4U^2.
\]

(6.12)

Using Maple we could calculate the discriminant of the proposed curve (6.7) with \( p(v) = v^3 + \frac{3m}{2}v^2 + vw + u \), for the following cases: i) \( (m = 0, u, w) \); ii) \( (m, u = 0, w) \); iii) \( (m, u, w = 0) \). After fixing \( \Lambda = 1 \) and relating \( (U, W) \) and \( (u, v) \) by

\[
U = u - \frac{1}{2}wm + \frac{1}{4}m^3, \\
W = w - \frac{3}{4}m^2.
\]

(6.13)

the obtained discriminant exactly matches (6.12) up to an additional prefactor

\[
u^3 - 27w^2.
\]

(6.14)

Notice that the identifications (6.13) coincide with what is expected from the relation between Higgs vev’s and positions of fourbranes explained in section 5.
For arbitrary $N$ let us denote by $u$ and $w$ the highest and next to highest order Casimirs, $p(v) = v^N + \ldots + wv + u$. The expression (6.14) is part of the discriminant for all the curves (6.7). This can be checked more easily after rewriting the curves in the form (7.3), which is smooth at the origin. In the rest of this section we will be referring to the curves in that formulation. One can then see that when (6.14) equals zero a singularity forms at $v = 0$. In the case of $SU(2)$ the expression we are analyzing corresponds to the last factor in (6.10).

When $N$ is even we can study the singularity in the following way. We consider again the situation $p(v) = q(v)$ in which our cubic curves factorize. One of the pieces reproduces the Seiberg-Witten curves for symplectic groups. The expression (6.14) reduces now to

$$u = 0. \quad (6.15)$$

It was shown in [17] that this describes a non-physical singularity of the symplectic curves. At $u = 0$ a cycle of the opposite behavior under the $\mathbb{Z}_2$ involution that the Seiberg-Witten differential shrinks to zero size. Since at the zeroes of (6.14) only two branch points collide\footnote{The branch points in the $v$-plane are located where (6.7) has double points as a polynomial in $y$.}, the associated vanishing cycle must be non-physical also when we move to a generic situation $p(v) \neq q(v)$. Therefore we can discard this singularity for $N$ even.

Let us redefine $u = 3\rho^2 + \epsilon$ and $w = \rho^3$. Since the singularity governed by (6.14) develops at $v = 0$ for all $N$, in order to analyze it we could approximate $p(v) \sim wv + u$ for sufficiently small $\epsilon$. In an small neighborhood around $v = 0$, for sufficiently small $\epsilon$, all the curves behave then as the case $N = 2$. For $N$ even we could show that (6.14) is associated with a non-physical singularity. Based in the local form of the curve close to the degenerate situation, we can extend this conclusion to all $N$.

7. Relation to the Atiyah-Hitchin Space

We have used the semi-classical description of the orientifold

$$xy = v^{-4} \quad (7.1)$$

as a tool that allowed us to propose new Seiberg-Witten curves for four-dimensional $\mathcal{N} = 2$ gauge theories. Although this description is not exact, the key point is that it allowed for
additional factors in the curves proportional to negative powers of $v$. Using these factors we were able to take into account strong coupling effects that our brane probes are feeling but are not included in (7.1). In this section we will study the relation between the obtained curves and the exact description of the orientifold, the Atiyah-Hitchin space.

After fixing the coefficients $A$ and $B$ in (6.4), we redefined $y \rightarrow yv^{-2}$ in order to eliminate the negative powers of $v$. This led to the curves (6.7), which have a singularity at the origin for any value of the parameters. Let us consider instead the redefinition $y \rightarrow yv^{-1}$, we then have

$$y^3 + y^2(vp(v) + 3v^{-1}) + y(q(v) + 3v^{-2}) + v^{-3} = 0. \quad (7.2)$$

We can eliminate all negative powers in $v$ by shifting $y \rightarrow y - v^{-1}$, obtaining

$$y^3 + vy^2p + y(q - 2p) + v^{-1}(p - q) = 0. \quad (7.3)$$

Notice that $q(v) = p(-v)$ and then $p - q = vf(v^2)$. This form of the curve is regular at $v = 0$.

Substituting the previous redefinition of $y$ and an analogous one for $x$ in (7.1) we get the space $xy - (x + y)v^{-1} = 0$. It is convenient to change once more coordinates to $(\tilde{x}, \tilde{y})$ defined by $y = \tilde{x}$, $x = -\tilde{x} - v\tilde{y}$. We then obtain the space

$$\tilde{x}^2 + v\tilde{x}\tilde{y} - \tilde{y} = 0. \quad (7.4)$$

which contains no negative powers of $v$ and is smooth at $v = 0$. Equation (7.3) can be seen as a curve in this space after just rewriting $\tilde{x}$ instead of $y$. We can use (7.4) to eliminate the highest order in $y$ from these curves. This greatly simplifies (7.3), reducing it to the equation

$$\tilde{x}(p - \tilde{y}) = v^{-1}(p - q). \quad (7.5)$$

Finally let us shift $\tilde{x} = \tilde{x}' - \frac{v\tilde{y}}{2}$ and define $z = v^2$, transforming (7.4) into

$$\tilde{x}'^2 = \frac{z\tilde{y}^2}{4} + \tilde{y}. \quad (7.6)$$

This equation defines the Atiyah-Hitchin space as a complex manifold in one of its complex structures [38] [39]. The coordinates $(z, \tilde{x}', \tilde{y})$ are invariant under the $\mathbb{Z}_2$ transformation.
imposed by the orientifold, that in the initial variables read \((v, y, x) \rightarrow (-v, x, y)\). We substitute now \(\tilde{x}'\) into (7.3) and use (7.6) in order to eliminate this variable. The result is

\[
\tilde{y}(\tilde{y} - p)(\tilde{y} - q) = v^{-2}(p - q)^2.
\]

(7.7)

It is immediate to see that this equation only contains even powers of \(v\) and therefore defines a Riemann surface \(F(z, \tilde{y}) = 0\) in the complex manifold (7.6). What we have achieved by going through all the coordinate changes is the following. We have translated the strong coupling effects encoded in the constants \(A\) and \(B\) of (6.4) into the background geometry describing the orientifold. In this way we have replaced the approximate description of the orientifold (7.1) by the exact one, valid for all \(v\). The key steps were given in (7.2) and (7.3). There it was crucial having \(A\) and \(B\) set to the values deduced in the past section mostly from field theory arguments. This shows once more the remarkable interplay between gauge theory, and strings and M-theory physics.

Before ending this section, let us notice that when the odd powers of \(v\) are switched off in the polynomial \(p\) we have \(p = q\) and equation (7.7) factorizes. The first factor \(\tilde{y} = 0\) maps to the first factor in (6.4). For the configuration of three fivebranes and fourbranes we are describing, this corresponds to the decoupling of the middle fivebrane. The rests reduces to

\[
\tilde{y} - p(z) = 0.
\]

(7.8)

Equation (7.8) was presented in [52], derived from local mirror symmetry [14], for describing the Seiberg-Witten curves of symplectic gauge groups immersed in \(D_n\) spaces. \(D_0\) denotes the complex manifold (7.4). By just substituting (7.8) into it we recover the curves for pure Yang-Mills as formulated in [12]. We could have obtained (7.8) by directly beginning with the curve (6.2) and follow exactly the same steps presented in this section.

Equation (7.1) is associated with an orientifold sixplane of charge \(-4\) in the absence of sixbranes. We can include sixbranes by considering the space \(xy = v^{-4} \prod_{a=1}^{d}(v^2 - m_a^2)\). A direct extension of the considerations in the past section will lead then to the curves for \(SU(N)\) gauge theories with \(d\) hypermultiplets in the fundamental representation and one in the antisymmetric. Following an analogous chain of coordinate changes done in this section, we would have obtained the defining equation of a \(D_n\) space instead of the \(D_0\) space (7.6).

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8. Product Groups: \( SU(N) \times Sp(2k) \)

We will consider briefly a brane configuration containing four fivebranes in the presence of the negative charge orientifold sixplane. We will place \( N \) fourbranes between the first and second fivebrane and \( 2k \) between the second and third. The rest follows from the \( \mathbb{Z}_2 \) invariance imposed by the orientifold (see fig. 2). This configuration induces an \( SU(N) \times Sp(2k) \) gauge theory on the four macroscopic dimensions of the fourbranes. The matter content consists in a \( \mathcal{N} = 2 \) hypermultiplet transforming of the bifundamental \( (N, 2k) \).

Again we will use the space (7.1) as a tool for deriving the curve describing this configuration. Applying (4.6) to our new case, we obtain

\[
\begin{align*}
y_4 + y_3 (p(v) + Dv^{-1} + Cv^{-2}) \\
y_2 \left( b(v^2) + Bv^{-2} + Av^{-4} \right) + v^{-4} y_1 (q(v) - Dv^{-1} + Cv^{-2}) + v^{-8} = 0,
\end{align*}
\]

where \( p \) is a generic polynomial of order \( N \), \( q(v) = p(-v) \) and \( b \) is a polynomial in \( v^2 \) of degree \( k \). The polynomials \( p \) and \( b \) include a total of \( N + k + 2 \) parameters. \( N + k - 1 \) of them will represent expectation values for the \( Sp(2k) \) and \( SU(N) \) Casimirs, and one will correspond to a bare mass term for the \( (N, 2k) \) hypermultiplet. Another one is related to the ratio between the coupling constants of both factor gauge groups and the last one can be eliminated by rescaling \( v \). Thus we have the right number of parameters for describing the Coulomb branch of the product gauge theory.

We could try to fix the constants \( A, B, C, D \) by arguments based on gauge theory. However we will just use the criteria that (8.1) can alternatively be written as a curve in the Atiyah-Hitchin space. This uniquely fixes all the constants. In order to make the structure of the solution more explicit we reintroduce a constant \( \tilde{\Lambda} \) such that \( xy = \tilde{\Lambda}^2 v^{-4} \).

We write the resulting curve after rescaling \( y \rightarrow v^{-2} y \)

\[
\begin{align*}
y_4 + y_3 (e v^2 \prod_{i=1}^{N} (v - a_i) + 4\tilde{\Lambda}) \\
y_2 \left( v^4 \prod_{j=1}^{k} (v - b_i^2) + 2e\tilde{\Lambda}v^2 \prod_{i=1}^{N} (-a_i) + 6\tilde{\Lambda}^2 \right) \\
+ \tilde{\Lambda}^2 y \left( (-)^N e v^2 \prod_{i=1}^{N} (v + a_i) + 4\tilde{\Lambda} \right) + \tilde{\Lambda}^4 = 0.
\end{align*}
\]

As a consistency check of this curve we analyze the limit in which the first and the fourth fivebranes are moved off to infinity. This is achieved by rescaling \( y \rightarrow e^{-1}y \) and sending \( e \rightarrow \infty \) and \( \tilde{\Lambda} \rightarrow 0 \) in such a way that their product remains finite. Notice that the parameter \( e \) represents the ratio between the coupling constants of the \( SU(N) \) and
$Sp(2k)$ groups. The coupling constant of the $SU(N)$ factor is inversely proportional to the distance between the first and second fivebranes. The limit $e \to \infty$ corresponds to send it to zero, decoupling the gauge degrees of freedom of this group factor. In this limit the previous curve reduces to

$$
y^2 \prod_{i=1}^{N} (v - a_i) + y \left( v^2 \prod_{j=1}^{k} (v^2 - b_i^2) + 2e\bar{\Lambda} \prod_{i=1}^{N} (-a_i) \right) + (-)^N (e\bar{\Lambda})^2 \prod_{i=1}^{N} (v + a_i) = 0, \quad (8.3)
$$

This curve is equivalent to the known curve describing an $\mathcal{N} = 2$ gauge theory with group $Sp(2k)$ and $N$ hypermultiplets in the fundamental representation \[12\].

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