On geometric interpretation of Berry’s phase and the Aharonov–Bohm effect

M. O. Katanaev *

Steklov Mathematical Institute,
ul. Gubkina, 8, Moscow, 119991, Russia

November, 18 2009

Abstract

A geometric interpretation of Berry’s phase, its Wilczek–Zee non-Abelian generalization, and Aharonov–Bohm effect is given in terms of connections on principal fiber bundles. It is shown that a principal fiber bundle in all cases can be trivial while the connection and its holonomy group are nontrivial. Therefore the main role is played not by topological effects but geometrical ones.

1 Introduction

The Aharonov–Bohm effect [1] and Berry’s phase [2] attract much interest of theoreticians and experimentalists for a long time. The interest is due to two circumstances. Firstly, the U(1)-connection arises in a natural way in solving the Schrödinger equation in both cases. Secondly, there is a widespread opinion in gauge field theory that only gauge field strength can lead to observable effects but not gauge potentials themselves which are not gauge invariant. In spite of this judgement, Aharonov–Bohm and Berry have shown that the integral of a gauge field along a closed loop can produce observable effects. These conclusions were soon confirmed experimentally.

The notion of the Berry phase was shortly generalized to the non-Abelian case corresponding to degenerate energy levels of a Hamiltonian by Wilczek and Zee [3]. In this case, non-Abelian gauge fields naturally arise in solving the Schrödinger equation.

In all above mentioned cases, the observable effects are produced by elements of the holonomy groups of corresponding connections which are gauge invariant objects. There is no any disagreement at this point. However, the uniform opinion on the geometric interpretation is absent. In the paper by B. Simon [4] and subsequent publications by other authors, the gauge field was considered as a connection on an associated fiber bundle. Since the typical fiber of an associated fiber bundle in a general case is an infinite dimensional Hilbert space, specific difficulties arose. Important and interesting constructions connected to characteristic classes are related to the existence of global sections of associated fiber bundles but not to the Berry phase itself. There arise definite topological obstructions for the existence of global sections. Hence the judgement that

*E-mail: katanaev@mi.ras.ru
the Berry phase and Aharonov–Bohm effect have its origin in topology is widely spread in literature.

In the present paper, we give another geometric interpretation of the Berry phase and Aharonov–Bohm effect in terms of the connection theory on principal fiber bundles. There are no difficulties related to infinite dimensional manifolds in this approach because typical fibers are \( \mathbb{U}(1) \) or \( \mathbb{SU}(r) \) groups which are finite dimensional Lie groups. It is shown that the principal fiber bundle can be trivial while the connection arising on it has nontrivial holonomy group in a general case and therefore leads to observable effects. As the consequence, the Berry phase and Aharonov–Bohm effect have the origin in geometry rather then in topology. Moreover the existence of global sections on an associated fiber bundles is not a necessary condition. If a global section is absent, then the local connection forms are defined on a coordinate covering of the base of the principal fiber bundle defining the unique connection on the principal fiber bundle up to an isomorphism.

The paper consists of two parts. Firstly, we consider the Berry phase and its generalization to the case of degenerate energy levels of a Hamiltonian. In the second part, we give geometrical interpretation of Aharonov–Bohm effect.

2 Berry’s phase

Nontrivial geometric structures, in particular, we are talking about nontrivial connection on a principal fiber bundle, often arise in solving equations of mathematical physics. In this section, we show how nontrivial connection on a principal fiber bundle with the structure group \( \mathbb{U}(1) \) or \( \mathbb{SU}(r) \) appears in nonrelativistic quantum mechanics in the process of solution of the Schrödinger equation. The surprise is based not on the natural appearance of the principal fiber bundle but on the experimental observation of the predicted effects.

In nonrelativistic quantum mechanics, the state of a system is described by the vector of the Hilbert space (the wave function) \( \psi \in \mathbb{H} \) which depends on time and some set of other variables depending on the considered problem. The evolution of a quantum system in time \( t \) is described by the Schrödinger equation \([5, 6]\\]

\[
\frac{i\hbar}{\partial t} \psi = H \psi,
\]

where \( H \) is a selfadjoint operator acting in the Hilbert space \( \mathbb{H} \) which is called Hamiltonian of a system, and \( \hbar \) is the Planck constant.

For simplicity, we put \( \hbar = 1 \) and denote partial derivative on time by a dot, \( \dot{\psi} = \partial_t \psi \), in what follows.

The scalar product is given in the Hilbert space which we denote by parenthesis

\[
\mathbb{H} \times \mathbb{H} \ni \psi, \phi \mapsto \langle \psi, \phi \rangle \in \mathbb{C}.
\]

By definition, the scalar product is linear in the second argument \( \phi \) and the equality \( \langle \psi, \phi \rangle^\dagger = \langle \phi, \psi \rangle \) holds where the symbol \( ^\dagger \) denote complex conjugation. The square of a vector in the Hilbert space \( \langle \psi, \psi \rangle \) is a real number, and we require the quadratic form \( \langle \psi, \psi \rangle \) to be strictly positive definite, i.e. \( \langle \psi, \psi \rangle \geq 0 \), and \( \langle \psi, \psi \rangle = 0 \) if and only if \( \psi = 0 \). Then the scalar product defines the norm of a vector in the Hilbert space

\[
\| \psi \| = \sqrt{\langle \psi, \psi \rangle}.
\]

The Schrödinger equation is linear in \( \psi \), and state vectors can be normalized. Usually, state vectors are assumed to be normalized on unit,

\[
\| \psi \| = 1.
\]
The norm of a state vector is preserved in time due to the self-adjointness of the Hamiltonian. Normalization of a state vector do not eliminate the arbitrariness in choosing the state vector in the Hilbert space because the arbitrariness in choosing a phase factor still remains.

We use the symbol $\dagger$ to denote Hermitian conjugation of matrices, i.e. transposition of a matrix and taking complex conjugation of all elements. In particular, when a matrix consists of one element the Hermitian conjugation coincides with complex conjugation.

Now we describe the problem which was considered by M. Berry [2] in its simplest form. Assume for simplicity that the Hilbert space is a finite dimensional complex space $\mathbb{H} = \mathbb{C}^n$ of complex dimensionality $\dim \mathbb{H} = n$. Then the Hamiltonian is given by the Hermitian $n \times n$ matrix and a state vector is represented by a column of $n$ components,

$$
\psi = \begin{pmatrix} 
\psi_1 \\
\vdots \\
\psi_n 
\end{pmatrix},
$$

where $\psi_1, \ldots, \psi_n$ are complex valued functions on some set of variables which will be specified later. The scalar product is defined by the equality

$$(\psi, \phi) = \psi^\dagger \phi = \psi_1^\dagger \phi_1 + \ldots + \psi_n^\dagger \phi_n.$$ 

Let the Hamiltonian $H = H(\lambda)$ to depend differentiably on a point of a manifold $\lambda \in \mathbb{M}$ of dimension $\dim \mathbb{M} = n$. If we choose a coordinate neighborhood $\mathbb{U} \subset \mathbb{M}$ on $\mathbb{M}$, then the Hamiltonian depends on $n$ parameters $\lambda^k$, $k = 1, \ldots, n$, (coordinates of a point $\lambda$). Assume that position of a point $\lambda$ on $\mathbb{M}$ depends on time $t$ according to a given prescription, i.e. the Hamiltonian depends on a curve $\lambda(t)$, $t \in [0,1]$. We assume also that the Hamiltonian depends on time only through the point $\lambda(t) \in \mathbb{M}$. We also presuppose that components of a vector in the Hilbert space $\psi = \psi(\lambda)$ depend on a point of a parameter manifold $\lambda \in \mathbb{M}$ as well. Then any solution of the Schrödinger equation (1) with normalization condition (2) is defined up to a constant phase factor $e^{i\Theta_0}$, $\Theta_0 = \text{const}$.

### 2.1 Abelian case: nondegenerate state

We consider the eigenvalue problem

$$H\phi = E\phi, \quad E = \text{const},$$

where $\phi \in \mathbb{H}$ for all $\lambda \in \mathbb{M}$. Suppose there exists nondegenerate energy eigenvalue $E$ which depends on $\lambda \in \mathbb{M}$ differentiably. The eigenfunction $\phi(\lambda)$ is also assumed to be a differentiable function on $\lambda$. Without loss of generality, we suppose that the eigenfunction $\phi$ is normalized on unit, $(\phi, \phi) = 1$. Then it is unique up to a multiplication on a phase factor which may be $\lambda$ dependent.

Next we solve the Cauchy problem for the Schrödinger equation (1) with the initial condition

$$\psi|_{t=0} = \phi_0,$$

where $\phi_0 = \phi(\lambda(0))$. In the adiabatic approximation, for slowly varying Hamiltonian, the system remains in its instantaneous eigenstate. Therefore we look for the solution in the form

$$\psi = e^{i\Theta}\phi,$$
where $\Theta = \Theta(\lambda)$ is an unknown function on $\lambda \in \mathbb{M}$. Substitution of this expression into the Schrödinger equation yields the equation for the phase factor

$$-\dot{\Theta} \phi + i \phi = E \phi,$$

where we dropped the common phase factor $e^{i\Theta}$ and used the commutativity of matrices $H e^{i\Theta} = e^{i\Theta} H$. Now we take the scalar product of left and right sides of the derived equation with $\phi$. As a result, we obtain the equation for the phase

$$\dot{\Theta} = i(\phi, \dot{\phi}) - E \phi,$$

with the initial condition $\Theta|_{t=0} = 0$. Since $\dot{\phi} = \lambda^k \partial_k \phi$, the solution of the Cauchy problem for equation (1) is

$$\Theta = \int_0^t dt \lambda^k A_k - \int_0^t dt E = \int_{\lambda(0)}^{\lambda(t)} d\lambda^k A_k - \int_0^t ds E(s),$$

where we introduced the notation

$$A_k(\lambda) = i(\phi, \partial_k \phi)$$

and the integral on $\lambda$ is taken along the curve $\lambda(t)$.

Hence the integral (5) yields the solution for the Cauchy problem for the Schrödinger equation (1) with the initial condition (3) in the adiabatic approximation. The first term in Eq.(5) is called the geometric or Berry’s phase and the second term is called the dynamical phase.

Note that components (6) are real because of normalization of the wave function. Indeed, differentiation of the normalization condition $(\phi, \phi) = 1$ yields the equality

$$(\partial_k \phi, \phi) + (\phi, \partial_k \phi) = (\phi, \partial_k \phi)^\dagger + (\phi, \partial_k \phi) = 0.$$ 

It implies the reality of components (6) and subsequently the reality of Berry’s phase.

We consider now a set of closed curves $\lambda \in \Omega(\mathbb{M}, \lambda_0)$ on a parameter manifold $\mathbb{M}$ with the beginning and end at a point $\lambda_0 \in \mathbb{M}$. Then the total change in the phase of the wave function is equal to the integral

$$\Theta = \Theta_B - \int_0^1 dt E,$$

where

$$\Theta_B = \oint_{\lambda} d\lambda^k A_k.$$

In this form, we are able to give the geometrical interpretation of the Berry phase $\Theta_B$ which is given by the first term in the obtained expression. Namely, we have a principal fiber bundle $\mathbb{P}(\mathbb{M}, \pi, \mathbb{U}(1))$ which base is the parameter manifold $\lambda \in \mathbb{M}$ and the structure group is $\mathbb{U}(1)$ (geometrical phase $e^{i\Theta}$). The vector in the Hilbert space $\phi \in \mathbb{H}$ represents a local cross section of the associated bundle $\mathbb{E}(\mathbb{M}, \pi_E, \mathbb{H}, \mathbb{U}(1), \mathbb{P})$ which typical fiber is the Hilbert space $\mathbb{H}$.

Consider a change in the local cross section of the associated bundle which is produced by multiplication of a vector in the Hilbert space on a phase factor (vertical automorphism)

$$\phi' = e^{i\phi},$$

where $\Theta = \Theta(\lambda)$ is an unknown function on $\lambda \in \mathbb{M}$. Substitution of this expression into the Schrödinger equation yields the equation for the phase factor

$$-\dot{\Theta} \phi + i \phi = E \phi,$$

where we dropped the common phase factor $e^{i\Theta}$ and used the commutativity of matrices $H e^{i\Theta} = e^{i\Theta} H$. Now we take the scalar product of left and right sides of the derived equation with $\phi$. As a result, we obtain the equation for the phase

$$\dot{\Theta} = i(\phi, \dot{\phi}) - E \phi,$$

with the initial condition $\Theta|_{t=0} = 0$. Since $\dot{\phi} = \lambda^k \partial_k \phi$, the solution of the Cauchy problem for equation (1) is

$$\Theta = \int_0^t dt \lambda^k A_k - \int_0^t dt E = \int_{\lambda(0)}^{\lambda(t)} d\lambda^k A_k - \int_0^t ds E(s),$$

where we introduced the notation

$$A_k(\lambda) = i(\phi, \partial_k \phi)$$

and the integral on $\lambda$ is taken along the curve $\lambda(t)$.

Hence the integral (5) yields the solution for the Cauchy problem for the Schrödinger equation (1) with the initial condition (3) in the adiabatic approximation. The first term in Eq.(5) is called the geometric or Berry’s phase and the second term is called the dynamical phase.

Note that components (6) are real because of normalization of the wave function. Indeed, differentiation of the normalization condition $(\phi, \phi) = 1$ yields the equality

$$(\partial_k \phi, \phi) + (\phi, \partial_k \phi) = (\phi, \partial_k \phi)^\dagger + (\phi, \partial_k \phi) = 0.$$ 

It implies the reality of components (6) and subsequently the reality of Berry’s phase.

We consider now a set of closed curves $\lambda \in \Omega(\mathbb{M}, \lambda_0)$ on a parameter manifold $\mathbb{M}$ with the beginning and end at a point $\lambda_0 \in \mathbb{M}$. Then the total change in the phase of the wave function is equal to the integral

$$\Theta = \Theta_B - \int_0^1 dt E,$$

where

$$\Theta_B = \oint_{\lambda} d\lambda^k A_k.$$

In this form, we are able to give the geometrical interpretation of the Berry phase $\Theta_B$ which is given by the first term in the obtained expression. Namely, we have a principal fiber bundle $\mathbb{P}(\mathbb{M}, \pi, \mathbb{U}(1))$ which base is the parameter manifold $\lambda \in \mathbb{M}$ and the structure group is $\mathbb{U}(1)$ (geometrical phase $e^{i\Theta}$). The vector in the Hilbert space $\phi \in \mathbb{H}$ represents a local cross section of the associated bundle $\mathbb{E}(\mathbb{M}, \pi_E, \mathbb{H}, \mathbb{U}(1), \mathbb{P})$ which typical fiber is the Hilbert space $\mathbb{H}$.

Consider a change in the local cross section of the associated bundle which is produced by multiplication of a vector in the Hilbert space on a phase factor (vertical automorphism)

$$\phi' = e^{i\phi},$$
where \( a = a(\lambda) \in C^1(M) \) is an arbitrary differentiable function. Then components (6) transform according to the rule

\[ A'_k = A_k - \partial_k a. \]

Comparing this rule with the transformation of components of a local connection form [7], we see that the fields \( A_k(\lambda) \) can be interpreted as components of a local connection form for \( \mathbb{U}(1) \) group. In other words, \( A_k(\lambda) \) is the gauge field for one dimensional unitary group \( \mathbb{U}(1) \). If the base of the associated fiber bundle \( E(M, \pi_E, H, \mathbb{U}(1), P) \) is covered by some set of coordinate charts, \( M = \bigcup_j U_j \), then a set of sections given on each coordinate chart \( U_j \) defines a family of local connection forms on the principal fiber bundle \( P(M, \pi, \mathbb{U}(1)) \).

A family of local connection forms \( d\lambda^k A_k \) defines the unique connection on \( P \) up to an isomorphism [7].

Let us recall the expression for an element of the holonomy group in terms of the T-exponent [8]. In the present case, the group \( \mathbb{U}(1) \) is Abelian, and the T-exponent coincides with the usual exponent. Therefore the Berry phase (7) defines the element \( e^{i\Theta_b} \) of the holonomy group \( \Phi(\lambda_0, e) \subset \mathbb{U}(1) \) of the principal fiber bundle at the point \((\lambda_0, e) \in P\) corresponding to zero cross section \( M \ni \lambda \mapsto (\lambda, e) \in P \) where \( \lambda_0 = \lambda(0) \) and \( e \) is the unit of the structure group \( \mathbb{U}(1) \). The cross section is the zero one because at the initial moment of time the Berry phase vanish, \( \Theta_b|_{t=0} = 0 \). The connection local form \( d\lambda^k A_k \) corresponds also to zero cross section.

The expression for the Berry phase (7) can be rewritten as a surface integral of the components of the local curvature form. Using Stoke’s formulae, we obtain the following expression

\[ \Theta_b = \frac{1}{2} \int \int_S d\lambda^k \wedge d\lambda^l F_{kl}, \quad (8) \]

where \( S \) is a surface in \( M \) with the boundary \( \lambda \in \Omega(M, \lambda_0) \) and \( F_{kl} = \partial_k A_l - \partial_l A_k \) are components of the local curvature form (gauge field strength).

### 2.2 1/2-spin particle in the magnetic field

As an example, we calculate the Berry phase for a spin 1/2 particle in an external homogeneous magnetic field. In nonrelativistic quantum mechanics, a spin 1/2 particle is described by a two component wave function

\[ \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \]

We assume that it is situated in the space \( \mathbb{R}^3 \) with a given homogeneous magnetic field. Let the strength of the magnetic field \( H^k(t), k = 1, 2, 3 \), do not depend on space point but change in time \( t \) according to some prescribed way. For simplicity, we disregard also the kinetic energy of a particle and assume that other fields are absent. In this case, the Hilbert space \( \mathbb{H} \) is two dimensional, and the Hamiltonian of a particle consists of one term which is equal to the interaction term the magnetic momentum of a particle with external magnetic field (see., i.e. [9, 10]),

\[ H = -\mu H^k \sigma_k, \]

where \( \sigma_k \) are Pauli matrices and \( \mu \) is the magneton (dimensionfull constant which is equal to the ratio of the magnetic momentum of a particle to its spin). To write the Hamiltonian in the form considered earlier, we introduce new variables \( \lambda^k = -\mu H^k \). Then the Hamiltonian is

\[ H = \lambda^k \sigma_k = \begin{pmatrix} \lambda^3 & \lambda^- \\ \lambda^+ & -\lambda^3 \end{pmatrix}, \quad (9) \]
where $\lambda^\pm = \lambda^1 \pm i\lambda^2$.

Eigenvalues of Hamiltonian (9) are found from the equation
\[ \det(H - E\mathbb{I}) = 0, \]
which has two real roots
\[ E_{\pm} = \pm |\lambda|, \quad (10) \]
where
\[ |\lambda| = \sqrt{(\lambda^1)^2 + (\lambda^2)^2 + (\lambda^3)^2} \]
is the length of the vector $\lambda = \{\lambda^k\} \in \mathbb{R}^3$. It can be easily shown, that the equation for eigenfunction
\[ H\phi_{\pm} = E_{\pm}\phi_{\pm}, \]
has two solutions
\[ \phi_{\pm} = \frac{1}{\sqrt{2|\lambda|}} \left( \pm \frac{\lambda^\pm}{\sqrt{|\lambda| + \lambda^3}} \right), \quad (11) \]
The factor in this expression is chosen in such a way that the eigenfunctions are normalized on unit,
\[ (\phi_{\pm}, \phi_{\pm}) = 1. \]
Hence the Hamiltonian (9) for a spin 1/2 particle in the external homogeneous magnetic field has two nondegenerate eigenstates (11) corresponding to energy levels (10).

It is useful to introduce spherical coordinates $|\lambda|, \theta, \varphi$:
\[ \lambda^1 = |\lambda| \sin \theta \cos \varphi, \]
\[ \lambda^2 = |\lambda| \sin \theta \cos \varphi, \]
\[ \lambda^3 = |\lambda| \cos \theta. \]
in the parameter space $\lambda \in \mathbb{R}^3$ for the calculations. Then the eigenfunctions take the form
\[ \phi_{\pm} = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\varphi} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad \phi_{-} = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\varphi} \\ \cos \frac{\theta}{2} \end{pmatrix}. \]

We assume that the experimentalist observing the particle varies homogeneous magnetic field differentiably in time. That is the parameters $\lambda^k(t)$ in the Hamiltonian depend on time differentiably. Assume also that the particle was in the state $\phi_{\pm}$ at the initial moment of time $t = 0$. The corresponding solution of the Schrödinger equation (1) is
\[ \psi = e^{i\Theta}\phi_{\pm}, \]
where the phase $\Theta$ satisfies Eq.(4). Components of the local connection form $A_k = i(\phi_{\pm}, \partial_k \phi_{\pm})$ for the eigenstate $\phi_{\pm}$ are easily calculated
\[ A_{|\lambda|} = 0, \quad A_{\theta} = 0, \quad A_{\varphi} = \cos^2 \frac{\theta}{2}. \]
The respective local form of the curvature has only two nonzero components:
\[ F_{\theta\varphi} = -F_{\varphi\theta} = -\frac{1}{2} \sin \theta. \]
Now we calculate the Berry phase for a closed curve in the parameter space \( \lambda = \lambda(t) \in \mathcal{M} \),

\[
\Theta_u = \oint_{\lambda} d\lambda^k A_k = \int \int_{\mathcal{S}} d\lambda^k \wedge d\lambda^l F_{kl} = \\
= \int \int_{\mathcal{S}} d\theta \wedge d\varphi F_{\theta\varphi} = -\frac{1}{2} \int \int_{\mathcal{S}} d\theta \wedge d\varphi \sin \theta = -\frac{1}{2} \Omega(\lambda),
\]

(12)

where \( S \) is a surface in \( \mathbb{R}^3 \) with the boundary \( \lambda(t) \) and \( \Omega(\lambda) \) is the solid angle of a surface \( S \) as it looks from the origin of the coordinate system.

If the particle is in the state \( \phi_- \) at the initial moment of time, then calculations are similar. In this case

\[
A_{\lambda} = 0, \quad A_\theta = 0, \quad A_\varphi = \sin^2 \frac{\theta}{2},
\]

and components of the local form of the curvature differ in sign

\[
F_{\theta\varphi} = -F_{\varphi\theta} = \frac{1}{2} \sin \theta.
\]

Therefore the Berry phase differs only by the sign.

So, if a particle was in one of the states \( \phi_\pm \) in the initial moment of time then after variation of the homogeneous magnetic field along a closed curve \( \lambda(t) \) its wave function acquires the phase factor, the geometrical part of which is

\[
\Theta_u^\pm = \mp \frac{1}{2} \Omega(\lambda),
\]

(13)

where \( \Omega(\lambda) \) is the solid angle at which the closed contour \( \lambda(t) \) is seen from the origin. This result does not depend on how parameters \( \lambda(t) \) depend on time.

Expression for the Berry’s phase (13) was confirmed experimentally [11] for the scattering of polarized neutrons in spiral magnetic field.

The homogeneous magnetic field in the considered example may have an arbitrary direction and magnitude. Therefore the base \( \mathcal{M} \) of the principal fiber bundle \( P(\mathcal{M}, \pi, \mathbb{U}(1)) \) coincides with the Euclidean space \( \mathcal{M} = \mathbb{R}^3 \). Hence the principal fiber bundle \( P \) is trivial, \( P \approx \mathbb{R}^3 \times \mathbb{U}(1) \). As the consequence, the Berry phase is not a topological notion but a geometrical one because the topology of the principal fiber bundle is trivial. It arises due to the nontrivial connection defined by cross sections of the associated fiber bundle.

### 2.3 Non-Abelian case: degenerate state

The notion of the Berry phase was generalized for the case when energy levels of a Hamiltonian are degenerate [13]. In this case, the principle fiber bundle \( P(\mathcal{M}, \pi, \mathbb{SU}(r)) \) with the structure group \( \mathbb{SU}(r) \), where \( r \) is the number of independent eigenfunctions corresponding to the degenerate energy level \( E \), appears in the process of solution of the Schrödinger equation. Here we describe this construction in detail.

We suppose that the Hamiltonian of a quantum system depends on a point of some manifold \( \lambda(t) \in \mathcal{M} \) as was done earlier. Let \( E \) be a degenerate eigenvalue of the Hamiltonian \( H \) with \( r \) independent eigenfunctions \( \phi^a, a = 1, \ldots, r \),

\[
H \phi^a = E \phi^a.
\]

Eigenfunctions can be chosen orthonormalized,

\[
(\phi_a, \phi^b) = \delta^b_a,
\]
where \( \delta^b_a \) is the Kronecker symbol. We look for solution \( \psi^a \) of the Cauchy problem for the Schrödinger equation (1) with the initial condition

\[
\psi^a|_{t=0} = \phi^a(\lambda(0)).
\]

That is the system is in one of the eigenstates \( \phi^a \) at the initial moment of time. In the adiabatic approximation, we look for the solution in the form

\[
\psi^a = \phi^b U^{-1a}_b,
\]

where \( U^b_a(\lambda) \in SU(r) \) is some unitary matrix which depends differentiably on a point \( \lambda \in \mathbb{M} \). The unitarity is needed for the orthonormalization of the solutions in the case under consideration, \( (\psi_a, \psi^b) = \delta^b_a \) because orthonormalization is preserved in time. Then the Schrödinger equation is reduced to the equation

\[
i\phi^b \dot{U}^{-1a}_b + i\phi^b U^{-1a}_b = H \phi^b U^{-1a}_b.
\]

Take the scalar product of left and right hand sides with \( \phi_b \). As a result, we obtain equation for the unitary matrix

\[
\dot{U}^{-1b}_a = \dot{\lambda}^k A_{ka} c U^{-1a}_c - i E U^{-1b}_a,
\]

where we introduced notation

\[
A_{ka}^b = - (\phi_a, \partial_k \phi^b).
\]

The orthonormalization of the eigenfunctions \( \phi^a \) implies antiunitarity of the components \( A_{ka}^b \) for all \( k = 1, \ldots, n \) when indices \( a, b \) are considered as matrix ones. Indeed, differentiation of the orthonormalization condition \( (\phi_a, \phi^b) = \delta^b_a \) yields the equality

\[
(\partial_k \phi_a, \phi^b) + (\phi_a, \partial_k \phi^b) = (\phi_b, \partial_k \phi^a)^\dagger + (\phi_a, \partial_k \phi^b) = 0.
\]

That is matrices \( A_k \) are antiunitary and therefore lie in the Lie algebra \( su(r) \). Hence matrices \( A_k \) define 1-forms on \( \mathbb{M} \) with values in the Lie algebra as components of a local connection form.

The initial condition for the unitary matrix is

\[
U^{-1b}_a|_{t=0} = \delta^b_a.
\]

The solution of the Cauchy problem for Eq.(15) can be written as the T-product

\[
U^{-1}(t) = T \exp \left( \int_0^t ds \dot{\lambda}^k(s) A_k(s) - i \int_0^t ds E(s) \right),
\]

\[
= T \exp \left( \int_{\lambda(0)}^{\lambda(t)} d\lambda^k A_k \right) \times \exp \left( -i \int_0^t ds E(s) \right),
\]

where we omitted matrix indices for simplicity.

The first factor is the generalization of the Berry phase to the case of degenerate states and the second one is the dynamical phase. The dynamical phase has the same form as for the nondegenerate state.

The first factor in the solution (17) represent the unitary matrix of Wilczek–Zee

\[
U^{-1}_{wz} = T \exp \left( \int_{\lambda(0)}^{\lambda(t)} d\lambda^k A_k \right),
\]

where
which can be given the following geometric interpretation. We have the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \text{SU}(r)) \) with the structure group \( \text{SU}(r) \) (transformation (14)). The set of eigenfunctions \( \phi^a \) is the cross section of the associated fiber bundle \( E(\mathcal{M}, \pi_E, \mathbb{H}^r, \text{SU}(r), \mathbb{P}) \) with the typical fiber being the tensor product of Hilbert spaces
\[
\mathbb{H}^r = \mathbb{H} \otimes \ldots \otimes \mathbb{H}. 
\]

Under the vertical automorphism given by the unitary matrix \( U(\lambda) \in \text{SU}(r) \),
\[
\phi'^a = \phi^b U^{-1a}_b, \quad \phi'_{a} = U_{a}^b \phi_b,
\]
the fields (16) transform according to the rule
\[
A'_k = UA_k U^{-1} + \partial_k UU^{-1}, \quad (19)
\]
where we omitted matrix indices. It implies that the fields \( A_k \) can be interpreted as components of the local connection form or Yang–Mills fields. A set of these components given on a coordinate covering of the base \( \mathcal{M} \) defines uniquely the connection on the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \text{SU}(r)) \).

If the path is closed, \( \lambda \in \Omega(\mathcal{M}, \lambda_0) \), then the unitary Wilczek–Zee matrix (18) represents the element of the holonomy group \( U_{wz}^{-1} \in \Phi(\lambda_0, e) \) at the point \( (\lambda_0, e) \in \mathbb{P} \) corresponding to the zero cross section \( \mathcal{M} \ni \lambda \mapsto (\lambda, e) \in \mathbb{P} \) where \( \lambda_0 = \lambda(0) \) and \( e \) is the unity of the structure group \( \text{SU}(r) \).

So the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \text{SU}(r)) \) arises in the case of the degenerate energy state of the Hamiltonian. In the considered case, the base \( \mathcal{M} \) is the parameter manifold \( \lambda \in \mathcal{M} \), the Hamiltonian being dependent on its point. We suppose that this manifold is finite dimensional. The structure group is the unitary group \( \text{SU}(r) \) which is also finite dimensional. The connection on the principal fiber bundle is defined by cross sections of the associated bundle \( E(\mathcal{M}, \pi_E, \mathbb{H}^r, \text{SU}(r), \mathbb{P}) \). In a general case, a typical fiber of the associated fiber bundle can be infinite dimensional Hilbert space \( \mathbb{H}^r \). In the present paper we do not consider infinite dimensional manifolds to avoid the difficulties which may arise [12]. Nevertheless, in our case, everything that is needed is the transformation formulae for components of the local connection form (19) which can be easily checked in every particular case. If the associated bundle is not diffeomorphic to the direct product \( \mathcal{M} \times \mathbb{H}^r \), then the state of a quantum system is given by a family of local cross sections on a coordinate covering of the base \( \mathcal{M} \). It defines the family of local connection forms (16). In its turn, the family of local connection forms defines connection on the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \text{SU}(r)) \) uniquely up to an isomorphism.

We see once again that principal and associated fiber bundles can be trivial or not depending on the problem under consideration. Connection on the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \text{SU}(r)) \) can be nontrivial and imply nontrivial Wilczek–Zee matrix (18) describing parallel transport of fibers along a path on the base \( \lambda(t) \in \mathcal{M} \) even for trivial bundles. This observation confirms its geometric but not topological origin. For closed paths \( \lambda \in \Omega(\mathcal{M}, \lambda_0) \) with the beginning and end at a point \( \lambda_0 \in \mathcal{M} \), the Wilczek–Zee matrix defines the element of the holonomy group \( U_{wz} \in \Phi(\lambda_0, e) \subset \text{SU}(r) \).

3 Aharonov–Bohm effect

Another example of the appearance of the nontrivial connection on the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \text{U}(1)) \) is given by the Aharonov–Bohm effect [5]. In contrast to the Berry
phase, here the space-time itself in which particle moves appears as the base $\mathcal{M}$ of the principal fiber bundle instead of a parameter space.

Let us consider the Schrödinger equation (1) in which the Hamiltonian describes motion of a free point particle of mass $m$ in three dimensional Euclidean space $\mathbb{R}^3$ with Cartesian coordinates $x^\mu$, $\mu = 1, 2, 3$,

$$
H = -\frac{\eta^{\mu\nu} p_\mu p_\nu}{2m} = -\frac{\hbar^2}{2m} \Delta,
$$

where $p_\mu = i\hbar \partial_\mu$ is the particle momentum operator, $\eta_{\mu\nu} = \text{diag} \, (- - -)$ is the negative definite space metric and $\triangle = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator.

The four dimensional momentum operator has the form $p_\alpha = i\hbar \partial_\alpha$, $\alpha = 0, 1, 2, 3$, the zero component of 4-momentum $p_0 = i\hbar \partial_0 = i\hbar \partial_t$ having the physical meaning of the particle energy operator.

If the particle interacts with the electromagnetic field, then this interaction is described by the minimal substitution for all four components of the momentum

$$
p_\alpha \mapsto i\hbar \partial_\alpha - \frac{e}{c} A_\alpha,
$$

(20)

where $e$ is the particle charge, $c$ is the light velocity, and $A_\alpha$ is the electromagnetic field potential (components of the local form of the $\mathbb{U}(1)$-connection). The zero component divided by the speed of light $A_0/c$ has the physical meaning of the electrical field potential and space components $A_\mu$ are covector components of the magnetic field potential. Thus the point particle moving in the electromagnetic field is described by the Schrödinger equation

$$
i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{\hbar^2}{2m} \eta^{\mu\nu} \left( \partial_\mu + i \frac{e}{\hbar c} A_\mu \right) \left( \partial_\nu + i \frac{e}{\hbar c} A_\nu \right) \right] \psi + \frac{e}{c} A_0 \psi.
$$

(21)

For simplicity, we put $\hbar = 1$ and $c = 1$ in what follows.

From the geometric standpoint, the minimal substitution (20) coincides with the replacement of the partial derivative by the covariant one up to constants

$$
\partial_\alpha \mapsto \partial_\alpha + ieA_\alpha.
$$

We consider two cases.

### 3.1 Electric potential

Suppose that the magnetic potential is zero, $A_\mu = 0$, $\mu = 1, 2, 3$. Let us write the Schrödinger equation in the form

$$
i\dot{\psi} = (H_0 + eA_0)\psi,
$$

(22)

where $H_0$ is the system Hamiltonian in the absence of the electric potential. We introduced new notation for the Hamiltonian because the following construction does not depend on the form $H_0$ of the Hamiltonian. Suppose also that the electric potential depends only on time, $A_0 = A_0(t)$. We look for a solution of the Schrödinger equation (22) in the form

$$
\psi = e^{-i\Theta} \phi
$$

where $\phi$ is a solution of the free Schrödinger equation

$$
i\dot{\phi} = H_0 \phi.$$
and $\Theta = \Theta(t)$ is some phase which does not depend on a space point. Substitution $\psi = e^{-i\Theta} \phi$ in the initial Schrödinger equation (22) leads to the equation for the phase

$$\dot{\Theta} = eA_0,$$

where we dropped a common phase factor $e^{-i\Theta}$ and took the scalar product of both sides of the equation with $\phi$. The solution of this equation has the form

$$\Theta(t) = \Theta_0 + e \int_0^t ds A_0(s),$$

where $\Theta_0$ is the value of the phase of the wave function at the initial moment of time.

Aharonov and Bohm proposed the experiment which scheme is shown in Fig.1. An electron beam is splitted in two beams which pass through two metallic tubes supplied with different potentials. Afterwards the beams are gathered, and the interference picture is observed on the screen. The electric potential transmitted to the tubes depends on time. It is supposed to be zero until both beams are found in their tubes. Then it grows to some values which are different inside the tubes and is lowered to zero before the beams go out from the tubes. Thus the beams interact with the field $A_0$ only inside the tubes. The interference picture depends on the difference between electron phases in the beams which can be written as the integral along a closed contour

$$\Theta_{\text{AB}} = e \int_\gamma dt A_0(t).$$

We assume here that the integration contour is located in the space $\mathbb{R}^3$ and $t$ is some parameter along this contour which is proportional to time and defined by the velocity of electrons.

Figure 1: An electron beam is splitted in two ones which pass through two metallic tubes having different potentials. Afterwards the beams are gathered, and the interference picture which depends on the difference between electron phases in different beams is observed.

Geometric interpretation of this Aharonov–Bohm effect is as follows. We have principal fiber bundle $\mathbb{P}(\mathbb{R} \times S^1, \pi, U(1))$ which base is the direct product of a line (time $t$) and one dimensional circle $S^1$ (contour $\gamma$) and the structure group $U(1)$ (the phase factor $e^{-i\Theta}$). The base coincides with a cylinder because we choose time as a parameter along trajectories of the beams. The local connection form $\omega = dt eA_0$ which is defined by the
external electric field and the electron charge is given on this bundle. The holonomy
group for a given connection is nontrivial, and the Aharonov–Bohm phase \(^{(23)}\) defines
the element of the holonomy group for one rotation around the circle \(S^1\). This is related
to the property that the base is not simply connected because the fundamental group of
a circle is isomorphic to the group of integers, \(\pi(S^1) \cong \mathbb{Z}\).

For this geometric interpretation, the base \(\mathbb{R} \times S^1\) of the principal fiber bundle has
nontrivial topology, and the Aharonov–Bohm effect can be called either topological or
dynamical. It depends on the taste because both topology and geometry are nontrivial.

From our point of view, a different geometric interpretation is more natural. That is,
since the Schrödinger equation \((21)\) is solved in the whole space \(\psi = \psi(t, x)\), it is natural
to choose the four-dimensional space-time \((t, x) \in \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4\) as the base manifold of
fiber bundles. We consider \(\mathbb{R}^4\) simply as the manifold without any metric attached. If
one wish, a metric can be introduced but its existence does not change the structure of
a principal fiber bundle and connection. The metric \(\eta_{\mu\nu}\) is defined only on space sections
\(t = \text{const}\), because it enters the Schrödinger equation. The wave function \(\psi(t, x)\) is the
section of the associated fiber bundle \(E(\mathbb{R}^4, \pi_E, \mathbb{C}, U(1), P')\) with complex plane \(\mathbb{C}\) as the
typical fiber and which is associated with some principal fiber bundle \(P'(\mathbb{R}^4, \pi, U(1))\). This
principal fiber bundle is always trivial \(P' \cong \mathbb{R}^4 \times U(1)\) because the base is the Euclidean
space. We consider not all sections but only a subset consisting of those differentiable
functions \(\psi(t, x)\) which belong to the Hilbert space of square integrable functions \(L^2(\mathbb{R}^3)\) on
space sections \(\mathbb{R}^3\) at each moment of time \(t\). Connection on the principal fiber bundle
\(P'(\mathbb{R}^4, \pi, U(1))\) has only one nonzero component \(A_0\) corresponding to electric potential.

The definition of the integration contour \(\gamma\) in Eq.\((23)\) is more complicated and depends
on the experiment. Suppose that an experimentalist somehow situated two metallic tubes
in space thus defining the contour \(\gamma\) in space \(\mathbb{R}^3\). Mathematically, it can be described by
introduction of some potential in the Hamiltonian \(H_0\) in Eq.\((22)\) which guarantees motion
of electrons along given trajectories with a prescribed accuracy. Assume also for simplicity
that electrons move with constant velocity \(v\). Let the length of upper and lower pathes
of beams be the same and equal \(l\). Then electron needs time \(t_0 = l/v\) to go through its
path. If \(t = 0\) at the beam splitting moment of time, then the phase of the wave function
of the electron in the upper beam when it reaches the screen changes by the value given
by the integral

\[
\Theta_1 = e \int_0^{t_0} dt A_0^{(1)}(t),
\]

where \(A_0^{(1)}(t)\) is the potential of electric field at the time \(t\), i.e. at that point in space
where the electron from the upper beam is found at the moment \(t\). Similarly, the phase
change of the wave function of the electron from the lower beam is

\[
\Theta_2 = e \int_0^{t_0} dt A_0^{(2)}(t),
\]

where \(A_0^{(2)}\) is the potential of the electric field along lower trajectory. Clearly, the difference
in phases of electrons in upper and lower beams \(\Theta_{ab} = \Theta_2 - \Theta_1\) can be written as integral
\((23)\) along closed contour \(\gamma\) when one goes firstly along lower half of the contour and
afterwards the upper half in opposite direction.

Surely, since our starting point is quantum mechanics rather then the classical one,
then the reasoning above is not truly justified because an electron is described by a wave
function \(\psi(t, x)\) defined on the whole space. We assume that each electron is described
by the wave function which is concentrated in sufficiently small region of space such that
we can talk about an electron trajectory. Corresponding solutions of the Schrödinger equation can hardly be found in explicit form because of the complex geometry of the problem. But this is not needed. For our purposes, it is sufficient to suppose that such solution exists.

Let us proceed with the geometric interpretation. The difference in electron phases is given by integral (23) which is uniquely defined by the contour $\gamma$ and potential $A_0$ given on this contour. On the other hand, since the base of the principal fiber bundle is the Euclidean space $\mathbb{R}^4$ in the given interpretation, the connection must be extended on the whole fiber bundle. It can be done in an arbitrary way because integral (23) does not depend on this extension. Assume that on the principal fiber bundle $P(\mathbb{R}^4, \pi, U(1))$ an arbitrary connection with components $(A_0(t, x), 0, 0, 0)$ is given such that its restriction to contour $\gamma$ coincides with potentials $A_0^{(1)}$ and $A_0^{(2)}$ on upper and lower halves of the contour. We assume that the component $A_0(t, x)$ is a differentiable function in the whole $\mathbb{R}^4$. In particular, a metallic tube has a finite thickness because otherwise a singularity appears. The potential $A_0$ on contour $\gamma$ can be continued on the whole space in many ways which depend on the phantasy of experimentalist. Nevertheless the result of the experiment does not depend on this continuation because metallic tubes and their potentials are fixed.

Phase difference (23) defines the holonomy group element $e^{i \Theta(\gamma)} \in \Phi((t_0, x_0), e) \subset U(1)$ at the point $(t_0 = 0, x_0)$ where $x_0$ is the space point where the beam is splitted.

By construction, the base manifold of the principal fiber bundle $P^s$ has trivial topology. Therefore the contour $\gamma$ can be always contracted to a point. During this process, metallic tubes stay on their place, and the contour crosses them continuously. When the contour is contracted, the Aharonov–Bohm phase $\Theta(\gamma)$ goes to zero and the corresponding element of the holonomy group tends to unity $e \in U(1)$. This is well known because the holonomy group is a Lie subgroup in the structure group. Note that contraction of a contour is not at all related to the experiment in which we observe a definite element of the holonomy group corresponding to the initial displacement of the contour.

The last geometric interpretation is simply related to the interpretation which was given earlier. Since the contour $\gamma$ is diffeomorphic to a circle $S^1$, we have the embedding

$$f : \mathbb{R} \times S^1 \supset (t, s) \mapsto (t, x) \in \mathbb{R} \times \mathbb{R}^3,$$

where $s \in S^1$ is a point on a circle. Then the principal fiber bundle $P(\mathbb{R} \times S^1, \pi, U(1))$ coincides with the induced fiber bundle $f^{-1}P(\mathbb{R}^4, \pi, U(1))$ which is obtained by restriction of the base $\mathbb{R} \times \mathbb{R}^3$ to $\mathbb{R} \times S^1$ [7].

The proposed construction is not the only possible. We can build a principal fiber bundle $P(\mathbb{R} \times \mathbb{R} \times S^1, \pi, U(1))$ with an arbitrary base $\mathbb{M}$. To this end, it is sufficient to take the fiber bundle $P(\mathbb{R}^4, \pi, U(1))$ with the given connection and narrow down the base by exception of an arbitrary number of lines in the space $\mathbb{R}^3$ which are perpendicular to the plane of Fig.1 and do not lie on the contour $\gamma$. As the consequence, the base $\mathbb{M}$ is no longer a simply connected space. This procedure appears artificial. Moreover, it makes the problem more complicated because the initial Schrödinger equation (21) has to be solved on a topologically nontrivial manifold in this case.

At the end of this section we note the following. The Schrödinger equation (22) implies that under the vertical automorphism

$$\psi' = e^{ia} \psi,$$

where $a = a(t)$ is a differentiable function of time, components of the local connection form transform according to the rule

$$eA'_0 = eA_0 + \dot{a},$$

13
as components of local $U(1)$-connection behave.

Thus we see that the Aharonov–Bohm effect as well as the Berry phase is based not on a topology but on nontrivial geometry, i.e. on a connection with nontrivial holonomy group. At the same time topology of the space can be arbitrary, in particular, trivial.

### 3.2 Magnetic potential

Now we consider the case when the electric field potential is equal to zero, $A_0 = 0$. Suppose that the magnetic field covector potential depends only on space coordinates $x^\mu$ and does not depend on time $t$ (static field). Then the Schrödinger equation takes the form

$$i\dot{\psi} = \frac{1}{2m}\eta^{\mu\nu}(\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu)\psi. \\
= \frac{1}{2m}\eta^{\mu\nu}(\partial^2_{\mu\nu}\psi + 2ieA_\mu\partial_\nu\psi + ie\partial_\mu A_\nu\psi - e^2A_\mu A_\nu\psi). \\
(24)$$

Let $\phi$ be a solution of the Schrödinger equation in the absence of the magnetic field potential

$$i\dot{\phi} = -\frac{1}{2m}\Delta\phi.$$

Then it is not difficult to verify that the function

$$\psi = e^{-i\Theta}\phi,$$

where the phase $\Theta$ satisfies the equation

$$\partial_\mu\Theta = eA_\mu$$

is the solution of the original Schrödinger equation (24).

Aharonov and Bohm proposed the experiment for determining the phase $\Theta$ which scheme is shown in Fig. 2. In this experiment, the beam of electrons is splitted in two ones bending from different sides a thin solenoid with constant magnetic flux $\Phi$ which is perpendicular to the plane of the figure. Afterwards the beams are gathered together, and the interference picture depending on the phase difference of electrons in different beams is observed on the screen. The difference in phase can be written as the contour integral

$$\Theta_{AB} = e\oint_{\gamma} dx^\mu A_\mu, \\
(25)$$

where the contour $\gamma$ surrounds the solenoid. This integral does not depend on the chosen contour surrounding the solenoid because the magnetic field outside the solenoid vanishes, $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$.

The Aharonov–Bohm phase can be rewritten using the Stokes formulae as the surface integral

$$\Theta_{AB} = \frac{1}{2}e\int_S dx^\mu \wedge dx^\nu F_{\mu\nu} = e\Phi, \\
(26)$$

where $F_{\mu\nu}$ is the magnetic field strength (components of the local curvature 2-form) and $\Phi$ is the total magnetic flux through the solenoid. Note that we have to assume that the magnetic field is defined everywhere in space $\mathbb{R}^3$ including the solenoid axis itself to use of the Stokes formulae.
Figure 2: The electron beam is splitted in two ones which bend the solenoid from different sides. Afterwards the beams are gathered together, and the interference picture depending of the phase difference of electrons in different beams is observed on the screen.

The geometric interpretation of the described Aharonov–Bohm effect is as follows. We have the same principal fiber bundle $P'(\mathbb{R}^4, \pi, U(1))$ as it was for the electric potential, the base being four-dimensional Euclidean space $(t, x) \in \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$ in which electrons move and the structure group is the unitary group $U(1)$ (the phase factor $e^{i\phi}$ of the wave function). However connection is now different: only space components of local connection form $A_\mu$, $\mu = 1, 2, 3$, differ from zero. The Aharonov–Bohm phase difference $\Theta_{ab}$ is uniquely defined by the contour $\gamma$ and values of the connection form $A_\mu$ on it. Writing the contour integral as the surface one $\int$, we must assume that the connection is continued to the whole space $\mathbb{R}^3$ in such a way that components of the local connection form are differentiable functions. It means that we consider solenoid of finite radius to avoid singularities. Of course, the result does not depend on the way of continuation due to the Stokes theorem.

So the Aharonov–Bohm effect must be considered as geometrical rather then topological.

Since the potential of the magnetic field does not depend on time, the base $\mathbb{R}^4$ of the principal fiber bundle $P'$ can be restricted to the space part $\mathbb{R}^3$. The result does not depend on that. We deliberately retained the dependence on time to show that the principal fiber bundle coincides with that in the case of electric potential.

So the principal fiber bundle is trivial and the Aharonov–Bohm phase $\Theta_{ab}$ depending on the connection and the contour uniquely defines the element of the holonomy group. When the contour is contracted to a point the corresponding element of the holonomy group tends to unity $e \in U(1)$ as it must be. Here we assume that the contour goes freely through the solenoid.

The wave function of the electron is the cross section of the associated fiber bundle $E(\mathbb{R}^4, \pi_E, \mathbb{C}, U(1), P')$ with the Euclidean space $\mathbb{R}^4$ as the base, the structure group being the unitary group $U(1)$. Under the vertical automorphism

$$\psi' = e^{ia_\psi},$$

where $a = a(x)$ is a differentiable function of space coordinates $x^\mu$, $\mu = 1, 2, 3$, the magnetic field potential transforms according to the rule

$$eA'_\mu = eA_\mu - \partial_\mu a.$$

It follows from the Schrödinger equation $\psi$. Thus components of the magnetic field potential have indeed the behavior of components of the local connection form.
Many authors consider infinitely thin solenoid and the Euclidean space $\mathbb{R}^3$ with removed axis corresponding to the solenoid as the base $M$. In this case, the base $M$ is not simply connected, and the Aharonov–Bohm effect could be called topological. As was shown earlier, this is not necessary. It is sufficient to assume that the magnetic field differs from zero only in the bounded region on the plane, and the integration contour does not cross this region. Moreover, if we assume that the base $M$ is the Euclidean space $\mathbb{R}^3$ with removed axis then the Stokes formulae is not applicable, and hence the expression for the Aharonov–Bohm phase through the magnetic field flux requires additional assumptions. So the Aharonov–Bohm effect must be considered as geometrical rather than topological.

The Aharonov–Bohm effect with electric as well as magnetic potential attracts much interest of physicists because of the following. According to the contemporary point of view, the gauge invariant quantities are the only observables in gauge models. From this point of view, the electromagnetic field potential $A_{\alpha}$, $\alpha = 0, 1, 2, 3$, itself is not observable because it is not gauge invariant. In the considered cases, the electron beams are not subjected to the electromagnetic field action because the electric and magnetic field strengths are zero in the regions where electrons move. Therefore it seems that the difference in phases of electron beams has to be zero. But it is not so as the consequence of the Schrödinger equation. We have to note that observable is not the electromagnetic field potential itself but its integral along a closed contour defining the element of the holonomy group of the $U(1)$ connection which is the gauge invariant object.

Soon after publication of the paper, the Aharonov–Bohm effect was confirmed experimentally. The effect caused by the magnetic potential was observed in experiments [13, 14, 15].

### 4 Conclusion

In this paper, we give the geometric interpretation of the Berry phase, its non-Abelian generalization by Wilczek and Zee, and the Aharonov–Bohm effect. All these effects are shown to be the consequences of nontrivial connections on the principal fiber bundles which define nontrivial holonomy group. At the same time, the topology of the principal fiber bundle can be trivial. Therefore the considered effects are not topological as they are often called in contemporary physical literature but rather geometrical effects.

The interpretation proposed in the paper is universal because contains nothing except differential geometric notions. In a geometric interpretation of mathematical physics models, one has to take into account that a connection exists on any principal fiber bundle independently on a topology of the base [7]. Moreover, if a family of local connection forms is given on an arbitrary closed submanifold of the base of some principal fiber bundle, then the corresponding connection can be always extended to the whole principal fiber bundle. This can be done in many ways. A connection defines the holonomy group which is nontrivial in a general case. In experiments for testing of the existence of Berry’s phase and the Aharonov–Bohm effect, the observed effects are produced not by the whole holonomy group but a fixed element of the holonomy group which depends on the connection and the closed contour. The topology of the base may be trivial or not, it does not play any role. If the topology is trivial then the contour can be contracted to a point. The effect disappears in this case because the corresponding element of the holonomy group tends to the unity element, and this is quite natural from physical point of view.

A connection on a principal fiber bundle defines connections on all fiber bundles which...
are associated with it. In particular, if the typical fiber is an infinitely dimensional Hilbert space, then the connection is also defined. At present, the interpretation of Berry’s phase and the Aharonov–Bohm effect as a rule reduced to consideration of a connection on a associated fiber bundle, and this forces to consider infinite dimensional manifolds and to take into account the related subtleties. From our point of view, the interpretation of the geometric effects in terms of connections on principal fiber bundles is simpler and more natural.

Is author is grateful to I. V. Volovich for the discussion of the paper. The work is partly supported by the Russian Foundation of Basic Research (Grant No. 08-01-00727), the Program for Supporting Leading Scientific Schools (Grant No. NSh-3224.2008.1), and the program “Contemporary problems of theoretical mathematics” by the Russian academy of science.

References

[1] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. Phys. Rev., 115(3):485–491, 1959.

[2] M. V. Berry. Quantal phase factors accompanying adiabatic changes. Proc. Roy. Soc. London, A392(1802):45–57, 1984.

[3] F. Wilczek and A. Zee. Appearance of gauge structure in simple dynamical systems. Phys. Rev. Lett., 52:2111, 1984.

[4] B. Simon. Holonomy, the quantum adiabatic theorem, and berry’s phase. Phys. Rev. Lett., 51(24):2167–2170, 1983.

[5] E. Schrödinger. Quantizierung als Eigenwertproblem (Erste Mitteilung). Ann. Phys. Leipzig, 79(4):361–376, 1926.

[6] E. Schrödinger. Quantizierung als Eigenwertproblem (Zweite Mitteilung). Ann. Phys. Leipzig, 79(6):489–527, 1926.

[7] S. Kobayashi and K. Nomizu. Foundations of differential geometry, volume 1, 2. Interscience publishers, New York – London, 1963, 1969.

[8] C. J. Isham. Modern Differential Geometry. World Scientific, Singapore, 1999.

[9] V. A. Fock. Fundamentals of Quantume Mechanics. Nauka, Moscow, 2-nd edition, 1976. [In Russian].

[10] A. Messiah. Quantum Mechanics, volume 2. North Holland, Amsterdam, 1962.

[11] T. Bitter and D. Dubbers. Manifestation of berry’s topological phase in neutron spin rotation. Phys. Rev. Lett., 59:251–254, 1987.

[12] V. V. Zharinov. Algebro-geometrical Foundations of Mathematical Physics. Steklov Math. Inst. Publ., Moscow, 2008.

[13] F. G. Werner and D. R. Brill. Significance of electromagnetic potentials on the quantum theory in the interpretation of electron interferometer fringe observations. Phys. Rev. Lett., 4(7):344–347, 1960.
[14] R. G. Chambers. Shift of an electron interference pattern by enclosed magnetic flux. *Phys. Rev. Lett.*, 5(1):3–5, 1960.

[15] H. Boersch, H. Hamisch, D. Wohlleben, and K. Grohmann. Weissche bereiche als bi-prisme für elektroneninterferenzen. *Z. Phys.*, 159:397–404, 1960.