D branes in 2d string theory and Classical Limits

Sumit R. Das

Department of Physics and Astronomy,
University of Kentucky, Lexington, KY 40506 USA

das@pa.uky.edu

Abstract

In the matrix model formulation of two dimensional noncritical string theory, a D0 brane is identified with a single eigenvalue excitation. In terms of open string quantities (i.e fermionic eigenvalues) the classical limit of a macroscopically large number of D0 branes has a smooth classical limit: they are described by a filled region of phase space whose size is $O(1)$ and disconnected from the Fermi sea. We show that while this has a proper description in terms of a single bosonic field at the quantum level, the classical limit is rather nontrivial. The quantum dispersions of bosonic quantities survive in the classical limit and appear as additional fields in a semiclassical description. This reinforces the fact that while the open string field theory description of these D-branes (i.e. in terms of fermions) has a smooth classical limit, a closed string field theory description (in terms of a single boson) does not.


d\textsuperscript{1}Based on talks at “Third International Symposium on Quantum Theory and Symmetries”, Cincinnati, September 2003 and “Workshop of Branes and Generalized Dynamics”, Argonne, October 2003.
1 Overview

D-branes of string theory are usually described in terms of excitations of open strings. In recent years it has become clear that they also appear as classical solutions of open string field theory. Their description in a closed string theory is, however, rather obscure. Such a closed string description is desirable from various points of view. In particular, we would like to understand better how D-branes appear as states in a quantum theory of gravity. We know that collections of large number of D-branes appear as classical solutions in supergravity, most often with sources. In a complete theory there cannot be any external source: in supergravity these sources of course represent the effects of stringy degrees of freedom and we should be able to do without them if we have a satisfactory off shell description of closed strings.

In almost all cases of interest, however, we do not have a tractable off shell description of closed strings. The most interesting exception is noncritical string theory in two space-time dimensions [1]. Here, there is a complete nonperturbative description in terms of the double scaling limit of matrix quantum mechanics - a subject which has been vigorously pursued in the 1990’s. It has been long suspected - ever since the original work of ‘t Hooft - that theories of $N \times N$ matrices (e.g. gauge theories) become, at large N, string theories. The new element which was learnt in the 1990’s is that these noncritical string theories typically live in one more dimension [2, 3]. For matrix quantum mechanics the space of eigenvalues of the matrix in fact provides the space dimension in which the string moves [4]. This allows a non-gravitational theory in $0 + 1$ dimensions to be dual to a theory which contains gravity in $1 + 1$ dimensions, thus providing the earliest example of holographic correspondence. In this case, the string theory is rather trivial: for the bosonic string the only dynamical degree of freedom is a single massless scalar, which is related to the collective field [5] - the density of eigenvalues. Recall that the eigenvalues of this theory behave as fermions and the collective field may be regarded as a bosonization of these fermions [6], albeit the bosonization is rather complicated and subtle. Thus we have a well defined closed string field theory as well as a well defined holographic dual.

There were several disappointing features of the matrix model description of two dimensional string theory. Even though gravity is not dynamical in two dimensions, there can be nontrivial gravitational backgrounds - in fact there is a black hole solution [7]. Despite considerable effort, no convincing description of black holes emerged in the matrix model. Secondly, as is well known, the double scaling limit becomes a theory of fermions moving in an inverted harmonic oscillator potential. It was clear that the ground state obtained by filling all the levels below the Fermi surface on one side of the potential corresponds to the ground state of the bosonic string, while small excitations represented a single massless scalar. However this is a nonperturbatively unstable situation. The stable situation is one obtained by filling both sides of the potential: however the corresponding string theory for this was never clear [8]. Thirdly after the discovery of D-branes in usual critical string theory, it became clear that these are generic states in all string theories. Indeed worldsheet descriptions of D0 and D1 branes in the model were discovered [9]. However the matrix model description of these were not clear.

Recent progress has improved the situation considerably. In a very interesting development, matrix quantum mechanics was reinterpreted as a theory of $N$ D0 branes [10]. The worldline gauge field forces all physical states to be singlets. A single unstable D0 brane is interpreted as a single eigenvalue excitation over the filled Fermi sea, whose worldsheet description is a product of the D0 boundary state with Sen’s rolling tachyon [11, 12]. The model with only one
side of the potential filled is the usual bosonic string theory, while the model with both sides filled is the 0B theory [13]. In the latter case, the small excitations of the Fermi level on both sides are linear combinations of the two massless scalars in the perturbative spectrum. There is a related matrix model for the Type 0A theory as well [14]. Black holes, however, continue to be a mystery. Single eigenvalue excitations have played an important role in the subject, e.g. in the discovery that nonperturbative effects in string theory behave as $e^{-1/g_s}$ rather than $e^{-1/g_s^2}$ [15]. Single particle solitonic excitations also appear as states in exact collective field theory as a separate branch [16]. Explicit single particle states, particularly corresponding to tunnelling, have been studied some time ago in [17] and [18].

In the modern interpretation, the matrix degrees of freedom, or equivalently fermions, are the open string degrees of freedom and the collective field is the closed string degree of freedom. Open-closed duality is then bosonization [20]. The open string description is simple since the fermions are non-interacting. Near the hump of the potential, the string coupling is large and the bosonic description is complicated. However in the asymptotic region the bosons are free massless particles. The late time decay product of a D0 brane can be seen to agree with a coherent state of these massless particles, as expected from the worldsheet picture [11, 21].

This note deals with the nature of classical limit of D-brane states from the point of view of closed string theory, largely based on work done in collaboration with S.D. Mathur and P. Mukhopadhyay.

In the limit $g_s \to 0$ each of the fermions may be thought of as particles moving classically in the inverted harmonic oscillator potential, subject to the Pauli principle [22]. In the phase space of fermions, the ground state is the filled Fermi sea. In the bosonized theory, this limit corresponds to the classical limit of the field theory. Thus, small deformations of the Fermi sea correspond to classical waves of the bosonic theory with small amplitudes. The classical dynamics of such small ripples is given by classical solutions of collective field theory, which may be obtained directly from the dynamics of the phase space density of fermions. The time evolution of a single unstable D-brane is then represented by the trajectory of a single fermion which is disconnected from the Fermi sea. A state with many D0 branes is represented as a disconnected blob of fermi fluid. As we will soon see, and as has been noted earlier [18], the resulting density of fermions do not obey the classical equations of collective field theory. In fact, the density of fermions $u(x, p, t)$ in phase space is equivalent to an infinite number of fields in configuration space, e.g. the various moments $\int dpp^nu(x, p, t)$ and for a generic configuration these fields are independent. For quadratic profiles of the Fermi surface, i.e. for profiles in which a constant $x$ line intersects the Fermi surface precisely twice, these various moments are not independent of each other and the configuration can be determined by the collective field and its conjugate momentum [23]. This would seem to indicate that the “closed string” description of D0 branes necessarily involve an infinite number of bosonic fields at the classical level.

We will argue, however, that at the full quantum level these states are still described by a single scalar field, using a key result of [24] about the physics of formation of folds on the fermi surface. If we start out with some ripple on the Fermi surface - which is represented by a classical configuration of the collective field - time evolution generically results in folds which make the Fermi profile non-quadratic. In [24] it was argued that this is still described as a state in a quantum theory involving a single scalar field. What happens is that an initial coherent state representing a classical wave evolves into a state in which quantum dispersions become
rather than $O(h)$ after a certain time. This is precisely the time at which a fold forms on the Fermi surface.

For D0 branes, the fermi surface has disconnected pieces to begin with and the same phenomenon occurs. For a single D0 brane we find that quantum fluctuations of closed string quantities diverge in the classical limit. This is entirely expected. Just as in critical string theory, we do not expect that a state of a small number of D-branes can be described as a classical configuration of closed string fields. On the other hand, one would expect that the state of a large number of D0 branes (i.e. $O(1/g_s)$ in number) should be a classical state of the closed string theory, i.e. the bosonic field should have vanishing quantum dispersion in the $g_s \to 0$ limit. We will show, however, that this is not true. Quantum fluctuations of quantities in the bosonic field theory do not vanish in the $g_s \to 0$ limit and are in fact of the same order as the expectation values of the fields themselves. Consequently, these fluctuations enter the effective classical equations as independent fields. It would be interesting to find the relationship between the description of D0 brane states in terms of bosons as discussed above and the solitonic states found in an exact treatment of collective field theory in [16].

The upshot of this is that while the open string field theory description of D0 branes in this model has a smooth classical limit in which Ehrenfest’s theorem holds (as argued in [20]), a description in terms of closed strings necessarily involves states which have large quantum fluctuations.

## 2 Preliminaries

Matrix quantum mechanics is described by a Hamiltonian

$$H = -\frac{1}{2\beta N} \sum_{ij} \frac{\partial^2}{\partial M_{ij}^2} + \beta N \text{tr} V(M)$$

(1)

where $V(M)$ is some potential for a $N \times N$ hermitian matrix $M$ which has a maximum at some value of $M$. We will work in the singlet sector of the theory. In this sector one has the dynamics of $N$ eigenvalues which behave as fermions - so that one has a set of $N$ fermions in an external potential. The double scaling limit is obtained by tuning the coupling $\beta$ to some critical value such that the fermi level $\epsilon_F$ reaches the maximum of the potential, $\epsilon_c$ and taking $N \to \infty$ keeping

$$\mu = \beta N(\epsilon_c - \epsilon_F)$$

(2)

fixed. As is well known, in this limit only the quadratic part of the potential survives. After a suitable rescaling of the space coordinate $x$, the theory may be written in terms of a second quantized fermionic field $\psi(x,t)$ with a Hamiltonian

$$H = \int dx \left[ -\frac{1}{2\mu} \partial_x \psi^\dagger \partial_x \psi - \frac{1}{2} \psi^\dagger \psi \right]$$

(3)

This form of the hamiltonian clearly shows that the coupling constant

$$g_s = \frac{1}{\mu}$$

(4)
acts like a $\hbar$ so that the classical limit is $g_s \to 0$. The singlet sector states are best described in terms of the density of fermions or the collective field $\rho(x, t)$. The effective theory for $\rho$ can be obtained directly from the matrix Hamiltonian (1) by a change of variables to

$$\rho(x, t) = \text{Tr} \delta(x \cdot I - M(t))$$

(5)

with the result

$$H = \frac{1}{2} \int dx \left[ \frac{1}{3} (P^3_+ - P^3_-) - (x^2 - 2\mu)(P_+ - P_-) \right]$$

(6)

where $P_\pm$ are

$$P_\pm = \partial_x \Pi_\rho \pm \pi \rho$$

(7)

Here $\Pi_\rho$ is the momentum conjugate to $\rho$. The classical solution is given by

$$\Pi = 0 \quad \rho = \frac{1}{\pi} P_0 = \frac{1}{\pi} \sqrt{x^2 - 2\mu}$$

(8)

Expanding around the classical solution as

$$P_\pm = \pm P_0 + \frac{1}{\sqrt{2} P_0} [\Pi_\phi \pm \partial_q \phi]$$

(9)

where we have defined a new space variable $q$ by

$$\partial_q = P_0 \partial_x$$

(10)

one can then easily see that the quadratic part of the Hamiltonian is

$$H_2 = \frac{1}{2} \int dq [\Pi^2_\phi + (\partial_q \phi)^2]$$

(11)

which identifies $\phi$ as a massless scalar field and $\Pi_\phi$ its canonically conjugate momentum.

On the left part of the potential one has

$$x = -\sqrt{2\mu} \cosh q$$

(12)

However this choice of a space coordinate is valid only in the classically allowed region. For the region inside the potential hump one has to use a different parametrization. A description of nonperturbative physics is of course best done in terms of the fermionic theory formulated in the full $x$ space.

3 Classical bosonization

At the classical level, the collective Hamiltonian can be alternatively obtained by considering the phase space dynamics of the fermions. Let $u(x, p, t)$ denote the density of fermions in phase space. In the classical limit $g_s \to 0$, Pauli principle dictates that $u = \frac{1}{2\pi g_s}$ in regions where the fermions are present and $u = 0$ elsewhere. The ground state is thus given by

$$u(\rho, x, t) = \frac{1}{2\pi g_s} \theta(x^2 - p^2 - 2\mu)$$

(13)
which corresponds to the Fermi sea filled on both sides of the potential. Consider now a small fluctuation which may be thought of a small ripple on the Fermi sea. We will first restrict our attention to perturbations which produce a quadratic profile. This means that a $x=\text{constant}$ line intersects the Fermi surface $x^2 - p^2 = 2\mu$ precisely twice, at $P_{\pm}(x, t)$. In this special case, the density of fermions in space at some given time, $\rho(x, t)$, may be obtained as [22]

$$\rho(x, t) = \int dp \ u(x, p, t) = \frac{1}{2\pi g_s} [P_+(x, t) - P_-(x, t)]$$

(14)

while the momentum density $\nu(x, t)$ is given by

$$\nu(x, t) = \int dp \ p \ u(x, p, t) = \frac{1}{2\pi g_s} \frac{1}{2} [P_+^2 - P_-^2]$$

(15)

Identifying $\nu = \rho \partial_x \Pi$ one gets the standard relationship between collective field (and its conjugate momentum) with $P_{\pm}$. The hamiltonian (6) then follows by noting that

$$H = \int dx dp \frac{1}{2} [p^2 - x^2 + 2\mu] u(x, p, t)$$

(16)

As is well known, however, the quadratic profile approximation made above is valid only for very small and shallow ripples on the fermi sea. A generic initial quadratic deformation would inevitably form “folds” rendering the profile non-quadratic.

Of particular interest are initial states which correspond to non-quadratic profiles. Single eigenvalue excitations or excitations of a bunch of eigenvalues are examples of these since the filled fermi sea now has disconnected pieces. The classical dynamics is still described by the equation of motion for the phase space density $u(x, p, t)$

$$(\partial_t + x \partial_p + p \partial_x) u(x, p, t) = 0$$

(17)
and a constraint which implements Pauli principle. However this equation cannot be any longer reduced to classical collective field theory equations [18]. A complete solution of $u(x, p, t)$ corresponding to the rolling tachyon has been obtained recently in [19].

This seems to indicate that at the classical level there are an infinite number of bosonic fields since we can define an independent bosonic field by taking some moment of $u$. A useful definition of the fields is provided by the following [25]

$$
\int dp \ u(x, p, t) = \frac{1}{2\pi g_s}[\beta_+(x, t) - \beta_-(x, t)]
$$

$$
\int dp \ p \ u(x, p, t) = \frac{1}{2\pi g_s}\left[\frac{1}{2}(\beta_+^2(x, t) - \beta_-^2(x, t)) + (w_{1+}(x, t) - w_{1-}(x, t))\right]
$$

$$
\int dp \ p^2 \ u(x, p, t) = \frac{1}{2\pi g_s}\left[\left(\frac{1}{3}\beta_+^3(x, t) + \beta_+ w_{1+}(x, t) + w_{2+}(x, t)\right)
- \left(\frac{1}{3}\beta_-^3(x, t) + \beta_- w_{1-}(x, t) + w_{2-}(x, t)\right)\right]
$$

The $\beta_{\pm}, w_{n\pm}$ satisfy $w_{\infty}$ algebras under usual Poisson brackets.

In this parametrization, the presence of nonzero $w_n$ signifies that the profile is not quadratic. For example if the fermi sea consists of two disconnected pieces (Fig.2) and a given $x = \text{constant}$ line intesects the fermi surface at four points $P_{\pm \pm}$ and $P_{2\pm}$ one has

$$
\begin{align*}
\beta_+ & = P_{1+} - P_{1-} + P_{2+} \\
\beta_- & = P_{2-} \\
w_{1+} & = (P_{1+} - P_{1-})(P_{1-} - P_{2+}) \\
w_{1-} & = 0 \cdots
\end{align*}
$$

The equations of motion for these various fields $\beta_{\pm}(x, t), w_{\pm i}(x, t)$ (as usual half of them should be regarded as conjugate momenta) may be obtained from (17). These equations couple the $w_{\pm i}$ with the $\beta_{\pm i}$. As a result, even if we start at $t = 0$ with all the $w_{\pm i} = 0$ time evolution may result in a nonzero $w_{\pm i}$ at a later time. This is the physics of fold formation from unfolded configurations.

We will be interested in rolling tachyon states representing decaying D0 branes. The semi-classical description of a single D0 brane at some instant of time is a filled region in phase space of size $O(g_s)$ which is disconnected from the fermi sea, just like in Fig 2. Equation (19) then implies that $w_{1+}$ is of $O(g_s)$ and hence subleading in the classical limit. Note that each of $P_{1\pm}$ are of $O(1)$ so that this state has an energy of order $O(1/g_s)$. As expected a single D0 brane is a highly quantum state.

However, when we have a macroscopically large number of D0 branes we expect a classical state in terms of bosons. Indeed, in this case the disconnected region has a size of $O(1)$, the various $w_i$ are also of $O(1)$ and contribute to the classical limit.

In terms of an open string (fermionic) description this collection of D0 branes has a smooth classical limit since the phase space density $u(x, p, t)$ has small quantum fluctuation, implying that in phase space the boundaries of the filled region are sharp. In terms of a closed string (bosonic) description we also have a classical limit. However it appears that such a state necessarily involves an infinite number of fields.
4 Quantum bosonization

The conclusion of the preceding section is puzzling. The discussion of classical bosonization using phase space can be repeated for relativistic fermions as well. Consider for example the right moving part of a massless relativistic fermion. Now the fermi sea has only one edge - all states with negative momentum are filled. This in fact simplifies the discussion of classical bosonization. Instead of having two copies of each field labelled by ± we have a single copy. For a state with some disconnected filled region, we have Figure (3) instead of Figure (2).

The formulae are the same as in (18) except that we have to set $\beta_-=w_{i-}=0$ and rename $\beta_+ \rightarrow \beta$ and $w_{i+} \rightarrow w_i$.

$$\int dp \ u(x,p,t) = \frac{1}{2\pi g} \beta(x,t)$$
$$\int dp \ p \ u(x,p,t) = \frac{1}{2\pi g} \left[ \frac{1}{2} \beta^2 + w_1(x,t) \right]$$

and so on. The dynamics is of course different, governed by a hamiltonian

$$H_{rel} = \int dpdx \ p \ u(x,p,t)$$

This in fact ensures that the equations for $\beta$ and $w$ do not mix with each other, so that if we have a state with $w_i = 0$ at some time, $w_i$'s remain zero at all future times. Thus folds cannot form out of unfolded configuration - a simple consequence of the fact that the velocity of all fermions is the speed of light and independent of the energy.

However, we can certainly have a state which corresponds to Figure (3) to begin with, and we would conclude - as in the previous section - that we need an infinite number of bosonic fields to describe this classically. On the other hand we certainly have an exact bosonization at
the full quantum level in terms of single (chiral) boson. Consequently all states of the theory can be written down exactly in terms of this single boson. The puzzle is: how is it that one needs an infinite number of fields when one passes to the classical limit?

4.1 Bosonization at the algebraic level

To see this let us review the bosonization formulae at an algebraic level [26]. Suppose there are an infinite set of (Schrodinger picture) fermionic operators $\psi_n$ and $\psi_n^\dagger$, with $n$ a positive or negative integer or zero, which satisfy the anticommutation relations

$$\{\psi_m^\dagger, \psi_n\} = \delta_{mn} \quad \{\psi_m^\dagger, \psi_n^\dagger\} = 0 \quad \{\psi_m, \psi_n\} = 0$$ (22)

Suppose we also have a vacuum state $|0\rangle$ which obeys

$$\psi_n |0\rangle = 0 \quad \text{for } n > 0 \quad \psi_n^\dagger |0\rangle = 0 \quad \text{for } n \leq 0$$ (23)

This allows the definition of a normal ordering prescription

$$:\psi_m^\dagger \psi_n: = \psi_m^\dagger \psi_n \quad n > 0$$
$$:\psi_m^\dagger \psi_n: = -\psi_n \psi_m^\dagger \quad n \leq 0$$ (24)

Then the following bosonic operators

$$\alpha_m = \sum_n :\psi_{-m+n}^\dagger \psi_n :$$ (25)

satisfy the commutation relations

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}$$ (26)
Note that the above facts do not depend on the dynamics of the theory.

The simplest application of the above algebraic relation is bosonization of a single chiral fermion into a single chiral boson. In this case the integers \( m, n \) may be regarded as labels for momenta in a box. Then the operators \( \psi_n \) may be combined into a single chiral field \( \psi(x) \)

\[
\psi(x) = \frac{1}{\sqrt{L}} \sum \psi_m e^{\frac{2\pi imx}{L}} \quad \psi^\dagger(x) = \frac{1}{\sqrt{L}} \sum \psi_m^\dagger e^{-\frac{2\pi imx}{L}} \tag{27}
\]

while the oscillators \( \alpha_n \) can be combined into a single chiral boson field \( \phi(x) \)

\[
\partial_x \phi(x) = \frac{1}{L} \sum \alpha_m e^{\frac{2\pi imx}{L}} \tag{28}
\]

The bosonization relationship then becomes

\[
\alpha(x) \equiv \partial_x \phi(x) =: \psi^\dagger(x)\psi(x) : \tag{29}
\]

The vacuum \(|0\rangle\) is the Fock vacuum of a free theory. We have chosen the fermion to be right moving so that the energy of a single free fermion is \( E = p \), where \( p \) is the momentum. The vacuum chosen above then corresponds to the standard filled Dirac sea. In the \( L \to \infty \) limit the momentum becomes continuous. The above relations have the standard limits. \( \psi^\dagger(k) \) is a creation operator for \( k > 0 \) and an annihilation operator for \( k \leq 0 \).

The fermion field may be also expressed in terms of the bosonic field by the relationship

\[
\psi(x) =: e^{i\alpha(x)} : \tag{30}
\]

where normal ordering in the bosonic theory is standard, i.e. all the oscillators with negative indices have to be pushed to the right. Using (30) one can express other fermion bilinears in terms of \( \alpha \). One such relation which will be crucial in what follows is

\[
-\frac{i}{2} : [\psi^\dagger \partial_x \psi - (\partial_x \psi^\dagger)\psi] := \pi : \alpha^2(x) : \tag{31}
\]

### 4.2 Relativistic fermions and nonrelativistic fermions in an inverted harmonic oscillator potential

The key ingredient in the above is the presence of an infinite filled Fermi (or Dirac) sea. This is the reason why the above bosonization does not work for free non-relativistic fermions. The energy of a single fermion is \( h = \frac{1}{2m} p^2 \) and now the momentum space Fermi sea has an upper and a lower edge. For example the fermionic modes \( \psi(k) \) are creation operators for \( -k_F < k < k_F \) where \( k_F \) is the Fermi momentum. It is now clear that the momentum modes of the field \( \alpha(x) =: \psi^\dagger \psi : \) do not satisfy the standard bosonic oscillator algebra. It is only for momenta close to the Fermi sea that we can use the standard bosonization formulae. But this is simply because for such momenta nonrelativistic fermions become effective relativistic with an effective velocity of light (\( k_F/m \)).

The situation is, however, quite different for nonrelativistic fermions in an inverted harmonic oscillator potential. In this case the fermion field may be expanded in terms of Parabolic cylinder functions \( \chi_\nu^\pm(x) \) with a continuous index \( \nu \) which runs from \(-\infty\) to \(+\infty\).

\[
\psi(x) = \int d\nu [\psi_\nu^+ \chi_\nu^+(x) + \psi_\nu^- \chi_\nu^-(x)] \tag{32}
\]
where the modes $\psi_{\nu}$ satisfy the standard fermionic anticommutation relations. The functions $\chi_{\nu}^{\pm}(x)$ are eigenstates of the single particle Hamiltonian

$$ h = \frac{1}{2}[p^2 - x^2] $$

(33)

The vacuum is defined by $\psi_{\nu}|0> = 0$ for $\nu > -\mu$ and $\psi_{\nu}^{|0} > 0$ for $-\infty < \nu < -\mu$. Thus the Fermi sea is infinite, pretty much like a relativistic fermion. A bosonization based on the exact eigenstates have been recently used in [27] to investigate interactions involving D0 branes.

At the classical level this is clear from the fact that there is a canonical transformation in phase space which maps the problem to that of a relativistic fermion. For negative energy orbits on the left side of the potential is this achieved by

$$ x = -\sqrt{2\xi} \cosh \eta \quad p = -\sqrt{2\xi} \sinh \eta $$

(34)

while for positive energy orbits we have a similar parametrization. Here $-\infty < \eta < \infty$ and $0 < \xi < \infty$. In this parametrization the single particle Hamiltonian becomes $h = -\xi$ and the Fermi level is $\xi = \mu$. Here $\eta$ is a coordinate and $\xi$ is a momentum. In the ground state the filled Fermi sea is $\mu < \xi < \infty$.

In any case, this shows that there is an exact bosonization for fermions in an inverted harmonic oscillator potential in terms of a single bosonic field since the filled Fermi sea is infinite. The natural position space is of course not $x$, but the conjugate of the quantum number $\nu$. In the semiclassical level this may be taken to be $\eta$. This is clear from the fact that if we regard $x$ as the coordinate there is an upper as well as a lower edge of the Fermi sea. When $\eta \to \pm \infty$ motion in $\eta$ becomes the same as motion in $x$ so that the bosonization yields a local field in $x$ space.

Such a canonical transformation does not exist for free non-relativistic fermions. or for non-relativistic fermions in a generic potential where there is both a lower and upper edge of the fermi sea.

5 Disconnected fermi fluids and quantum fluctuations

Let us return to the question posed in the beginning of the previous section : how is it that at the classical level one finds that there has to be an infinite number of bosonic fields ? Our discussion of the previous section shows that for the present purpose we can treat free relativistic fermions and non-relativistic fermions in a $-x^2$ potential in a unified way. Let us denote the coordinates in phase space by $p$ and $q$. For relativistic fermions the coordinate $q$ is the standard position $x$, while for the fermions coming from matrix model $q$ stands for the conjugate to the quantum number $\nu$ - which may be considered to the coordinate $\eta$ of (34) at the semiclassical level. With this understanding the bosonization formulae are given by equations (29 - 31) with the variable $x$ replaced by $q$. Our discussion below has nothing to do with the dynamics of the theory.

Consider now a Schrodinger picture state $|\Psi, t >$ of the system which is represented at the semiclassical limit by a single disconnected blob of fermi fluid on top of the infinite filled bulk of the fermi sea, as depicted in Figure 3. Note that the boson $\alpha(q)$ is simply the density of fermions in $q$ space, while the left hand side of (31) is the momentum density of fermions.
Therefore we can combine the operator statements in (29 - 31) with the phase space definitions of semiclassical fields in (20) to obtain at some given time

\[
< \Psi, t | : \psi^\dagger(q) \psi(q) : | \Psi, t > = < \Psi, t | : \alpha(q) : | \Psi, t > = \int dp \ u(q,p,t) = \frac{1}{2\pi g_s} \tilde{\beta}(q,t) \quad (35)
\]

\[-i g_s/2 < \Psi, t | : \psi^\dagger \partial_x \psi - (\partial_x \psi^\dagger) \psi : | \Psi, t > = \pi g_s < \Psi, t | : \alpha^2(q) : | \Psi, t > = \int dp \ p \ u(x,p,t) = \frac{1}{2\pi g_s} (\frac{1}{2} \tilde{\beta}^2 + w_1(q,t)) \quad (36)\]

where \( \tilde{\beta} \) denotes the quantities where the ground state values have been subtracted.

This immediately shows that \( w_1 \) is related to the quantum dispersion of the field \( \alpha(q) \)

\[
< \Psi, t | : \alpha^2(q) : | \Psi, t > = < \Psi, t | : \alpha(q) : | \Psi, t >^2 = \frac{1}{2\pi^2 g_s^2} w_1(q) \quad (37)
\]

A state with \( w_1 \sim O(1) \) (as in Fig. 3) therefore has a rather unconventional property; the quantum dispersion \( \Delta \alpha \) in such a state is of the same order as the value of \( < \alpha > \).

This state represents a large number of fermions which are all excited above the fermi level leaving a gap. From the point of view of fermions, such a state certainly has smooth semiclassical limit since the position and momentum of each fermion is determined upto \( O(g_s) \). This means that the quantum spread of \( u(p,q,t) \) is controlled as usual by \( g_s \) and vanishes in the \( g_s \to 0 \) limit. However quantities which are natural to the bosonized theory, like expectation values of \( \alpha \) have quantum dispersions which do not vanish in the \( g_s \to 0 \) limit. Furthermore these quantum dispersions like \( < \alpha^n > - < \alpha >^n \) are precisely the extra infinite set of fields \( w_n \) which appear in a semiclassical description of the state in terms of bosons.

In contrast, a state with no disconnected pieces or folds - a smooth shallow ripple - is a coherent state of the field \( \alpha \) for which dispersions of operators constructed from \( \alpha \) vanish in the \( g_s \to 0 \) limit. From the point of view of the bosonic theory these are “classical states”. However a state with no folds generically time evolves into a state with folds if the dynamics is appropriate. In [24] it was shown (for free nonrelativistic fermions) that this happens because the dispersions of \( \alpha \), while initially of \( O(g_s) \) become \( O(1) \) after a certain time. This time is precisely the time when a fold forms.

It is now clear why states with non-quadratic profiles will not evolve according to the classical equations of motion of the bosonic field theory even in the semiclassical limit. The reason is that while there could be an exact quantum bosonization, the states being considered are not those which have a smooth classical limit for natural bosonic variables. In other words Ehrenfest’s theorem does not hold for the equations of motion of the closed string field theory, which is the quantum bosonic theory in this case. Ehrenfest’s theorem of course holds for the equations of motion of the open string field theory, which is the theory of eigenvalues, simply because in the classical limit the fermions have trajectories in phase space with a fuzziness which is at most of order \( g_s \).
6 Explicit calculations

In this section we will perform simple calculations which illustrate the above point. The system
we consider is the infinite set of fermionic oscillators as in equations (22)-(26). As argued
in the previous section, this system can refer to either a single chiral relativistic fermion or
nonrelativistic fermions which arise in the $c = 1$ matrix model or in fact any theory of fermions
where the ground state consists of an infinite Fermi sea. The meaning of the integer index is
of course different in each case. With this understanding we can then use the bosonization
formulae (27 - 33) where $x$, now denoted by $q$, is the position coordinate conjugate to the
quantum number $\nu$ which characterizes the eigenstates in the inverted harmonic oscillator
potential, where we have put this conjugate position in a box of size $L$, thus rendering $\nu$
integral. In the semiclassical limit this is the coordinate $\eta$ defined in (34).

In this section we consider properties of states at some given time $t$. We calculate expectation
values of various bosonic quantities in such states, go over to the classical limit and compare
them with the expressions based on the phase space analysis presented above.

A ripple on the Fermi sea is a coherent state of the bosonic field

$$|C(q)\rangle = \mathcal{N} \prod_{n=1}^{\infty} e^{C_n \alpha_{-n}} |0\rangle$$

where the $C_n$ are the fourier components of the function $C(q)$

$$C(q) = \frac{1}{\sqrt{L}} \sum_n C_n e^{2\pi i n q}$$

and $\mathcal{N}$ is a normalization factor which ensures that $\langle C(q)|C(q)\rangle = 1$. Since $|C(q)\rangle$ is an
eigenstate of $\alpha_n$ for $n > 0$ with eigenvalue $C_n/2\pi g_s$ it is clear that

$$\langle C(q)|\alpha(q)|C(q)\rangle = \frac{1}{2\pi g_s} C(q) \quad \quad \langle C(q)|\alpha^2(q) :C(q)\rangle = \frac{1}{4\pi^2 g_s^2} C^2(q)$$

Comparing with (35)-(36) it is clear that we have $\beta(q) = C(q)$ and $\omega(q) = 0$. This then
 corresponds to a quadratic profile. In fact the state $|C(q)\rangle$ has been chosen so that the
quantum dispersions of all bosonic operators which do not involve $g_s$ explicitly vanish in the
$g_s \to 0$ limit. These “ripple” states thus have smooth classical limits from the point of view of
the bosonic theory.

Consider now a state consisting of $M$ D0 branes, each of which are in a definite energy
eigenstate, and hence delocalized in $q$. The D0 branes occupy successive energy levels between
$n = n_A(>0)$ and $n = n_B = n_A + M - 1$

$$|n_A, M\rangle = \psi^\dagger_{n_A} \psi^\dagger_{n_A+1} \cdots \psi^\dagger_{n_B} |0\rangle$$

Let $\gamma_n$ denote the fourier modes of $T(q) \equiv \alpha^2(q) :$

$$T(q) = \frac{1}{L^2} \sum_n \gamma_n e^{-2\pi i n q}$$

The following relationships then follow from the basic bosonization relation (25)

$$[\alpha_n, \psi^\dagger_m] = \psi^\dagger_{-n+m} \quad \quad [\alpha_n, \psi_m] = -\psi_{n+m}$$
\[
[\gamma_n, \psi_{-m}^\dagger] = (2m - n)\psi^\dagger_{-n+m}, \quad [\gamma_n, \psi_{m}] = -(2m + n)\psi_{n+m}
\]  

(44)

Note that the ground state obeys

\[
\alpha_n|0> = 0 \quad \gamma_n|0> = 0 \quad \text{for } n \geq 0
\]  

(45)

These relations lead to the following results

\[
<n_A, M|\alpha(q)|n_A, M> = \frac{M}{L}
\]

\[
<n_A, M|\alpha^2(q)|n_A, M> = \frac{M}{L^2}(2n_A + M - 1)
\]  

(46)

The result for \(< \alpha >\) simply reflects the fact that we have \(M\) fermions which are completely delocalized in \(q\) space.

The physical momenta in \(q\) space corresponding to the lowest and the highest filled level above the Fermi sea are given by

\[
p_A = \frac{2\pi g_s n_A}{L}, \quad p_B = \frac{2\pi g_s (n_A + M - 1)}{L}
\]  

(47)

In terms of these quantities

\[
<n_A, M|\alpha(q)|n_A, M> = \frac{1}{2\pi g_s}(p_B - p_A) + \frac{1}{L}
\]

\[
<n_A, M|\alpha^2(q)|n_A, M> = \frac{1}{4\pi^2 g_s^2}(p_B^2 - p_A^2) + \frac{1}{2\pi g_s L}(p_A + p_B)
\]  

(48)

In the limit \(L \to \infty\) the physical momenta become continuous. Furthermore in the classical limit \(g_s \to 0\) the momenta become \(O(1)\) if \(n_A \to \infty\) keeping \(p_A, p_B\) fixed. In this limit we can ignore the \(O(1/L)\) terms and the above results reproduce the expectations from classical Fermi fluid picture. In particular, the quantities defined in (35-36) we have

\[
\tilde{\beta} = (p_B - p_A)
\]

\[
w_1 = (p_B - p_A)p_A
\]  

(49)

as expected.

When the number of D0 branes \(M \sim O(1)\), we have \(w_1 \sim \frac{g_s}{L}\) so that the quantum dispersion of \(\alpha\) is large

\[
\frac{\Delta \alpha}{\alpha} \sim \frac{\sqrt{w_1}}{\tilde{\beta}} \sim \sqrt{\frac{L p_A}{g_s M}}
\]  

(50)

This clearly diverges in the classical limit \(g_s \to \infty\). This is entirely expected. As in critical string theory a state of a finite number of D branes is a highly quantum state from the point of view of closed string theory - we have simply reproduced this result.

In critical string theory, however, closed string theory provides a classical description of a large number of D-branes in the limit \(g_s \to 0\) and \(M \to \infty\) with \(g_s M\) held fixed. This is the limit in which the collection of D branes is described in terms of a classical gravitational background. In the present case this corresponds to the limit in which the quantity \((p_B - p_A) \sim O(1)\). In the \(L \to \infty\) limit this is described by a continuous disconnected region in the Fermi fluid picture.
Since we have considered D0 branes in energy eigenstates, this is a band rather than a blob. Contrary to what one might have expected, this corresponds to a limit in which the quantum dispersions of the bosonic field $\Delta \alpha_o \sim O(1)$ rather than $O(g_s)$! A classical description of such configurations of a large number of D0 branes involves an infinite number of fields, while an exact quantum description involves a single bosonic field. The additional fields are nothing but quantum dispersions which survive the classical limit. In other words, while an open string description of D0 branes has a smooth classical limit in accordance with Ehrenfest’s theorem [20], a closed string description does not.

In order to describe disconnected blobs in phase space corresponding to a large number of D0 branes approximately localized in space at any given time we need to consider wavepackets of individual fermions. For example consider the single fermion states (in the $L \to \infty$ limit)

$$|p_0, q_0 > = \int \frac{dk}{2\pi} \exp \left[ -\frac{1}{2} g_s (k - \frac{p_0}{g_s} )^2 + ikq_0 \right] |k >$$

where

$$|k > = \frac{1}{\sqrt{L}} \psi^\dagger (k) |0 >$$

and the oscillators $\psi(k), \psi^\dagger(k)$ are related to the oscillators with corresponding discrete index $n = \frac{L}{2\pi} k$ by the relation $\psi(k) = \sqrt{L} \psi_n$.

For $p_0 >> 0$ (corresponding to the average energy far above the Fermi sea), and small $g_s$ the above results may be used to see that

$$< \alpha(q) > = \frac{< p_0, q_0 | \alpha(q) | p_0, q_0 >}{< p_0, q_0 | p_0, q_0 >} = \frac{1}{\sqrt{\pi g_s}} e^{-\frac{(q-q_0)^2}{g_s}}$$

while

$$< \alpha^2(q) >= \frac{p_0}{\sqrt{\pi g_s}} e^{-\frac{(q-q_0)^2}{g_s}}$$

in agreement with the results from the phase space picture.

To describe a blob of finite size, corresponding to a large number of D0 branes localized approximately in phase space one can proceed with states which are direct products of states described above. It is clear from the preceding discussion that the conclusion remains unchanged for such configurations, though the calculations are more complicated. These calculations are useful in an understanding of gravitational effects of D-branes [28].

7 Acknowledgements

I would like to thank the organizers of QTS3 and “Workshop on Branes and Generalized Dynamics” for their invitation to present this talk. I also thank Samir D. Mathur and Partha Mukhopadhyay for discussions and collaboration, and A. Jevicki, I. Klebanov and S.R. Wadia for discussions. This work was supported by National Science Foundation grant PHY-0244811 and the Department of Energy grant No. DE-FG01-00ER45832.

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2The integral in (51) is effectively from $k = 0$ to $k = \infty$. However for $p_0$ large and $g_s$ small we may replace this by an integral over the entire range.
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