Collective modes of trapped spinor Bose–Einstein condensates

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Abstract

We study the structure of quasi-one-dimensional Bogoliubov–de Genes collective excitations of inhomogeneous \( F = 1 \) spinor Bose–Einstein condensates confined by a harmonic trap potential and loaded into an optical lattice. Employing a perturbative method, we report general analytical expressions for the polar and ferromagnetic confined collective oscillations. In both cases, the excited eigenfrequencies are given as functions of the effective 1D coupling constants, the trap frequency and the optical lattice parameters. While the presence of the trap yields quantized longitudinal collective modes, whose spectra depends on the collision scattering lengths, the optical lattice shifts their frequencies and, more importantly, yields tunable oscillations of the spin populations, for the even modes, as experimentally observed.

Keywords: Bose–Einstein condensates, collective excitations, optical lattices

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the pioneering works of Ho [1] and Ohmi and Machida, [2] significant effort has been devoted to study \( F = 1 \) spinor Bose–Einstein condensates (BECs). Due to the internal degrees of freedom, the condensate presents a vectorial character and the wavefunction is described by three components in the hyperfine state with magnetic quantum number \( m = -1, 0, 1 \). Accordingly, and depending on spin-exchange interaction values, [3, 4] two phases are predicted: ferromagnetic \((^{87}\text{Rb}) [5–7] \) and polar or antiferromagnetic \((^{23}\text{Na}) [8, 9] \). The experimental realization of spinor BECs, typically produced in optically trapped dilute gases, allows the study of several interesting problems, such as the observation of metastable states, [9] the formation of spin domains in an external magnetic field, [10] spin dynamics, [11] and the miscibility of the spinor components, [10] as well as the nature of the ground state spinor condensates (see the recent overviews in [12] and [13] and references therein). We mention the following relevant experimental results which are of interest to the present study. In [14] coherent spin dynamics of \(^{87}\text{Rb} \) atoms in optical lattices have been reported. It is found that for high lattice depths, atom pairs within the same lattice site show the coupling of two-particle Zeeman states induced by spin-changing interactions, and measurement of the corresponding Rabi frequency allows for the determination of the spin-changing coupling strengths and the interatomic collisions scattering lengths. Zhao et al [15] have demonstrated that the phase diagram of spinor \( F = 1 \) BEC can be tuned by the presence of a two-dimensional optical lattice. Further, in [16], evidence of first-order superfluid-to-Mott-insulator quantum phase transitions in \(^{23}\text{Na} \) antiferromagnetic spinor condensates, confined by a 3D optical lattice, have been reported. Also, the presence of hysteresis effects and change in spin populations, due the presence of spin singlets in the Mott-insulator phase, have been addressed.
Recently, in [17] coherent spin-population oscillations of antiferromagnetic spin-1 of $^{23}\text{Na}$ atoms have been measured and assuming a single-mode antiferromagnetic spinor BEC approach, the dynamics of the measured spin population evolution has been tested theoretically. And, in [18], a suggestion of nonlinear three-mode interferometers based on the unstable spinor BECs is analysed, where it is claimed that low noise sensitivity can be reached with respect to the total number of particles.

An analysis of the collective modes is an important step for a comprehensive study of the dynamics for both polar and ferromagnetic phases. Phenomena, such as quantized vortices, [5] superfluidity, spin-domains [10] or damping processes, require an exhaustive knowledge of the dynamic processes involved, see [19–23]. The behaviour of collective excitations, as a function of the atom–atom self-interaction parameters and of the confined external potential, is a fundamental brick to build the dynamics of the phenomena mentioned above. The nature and evaluation of the collective oscillations, so far, has been tackled assuming a spatial homogeneous BEC [1, 2, 12, 13, 24]. Thus, due to the concomitant translational symmetry, the wavevector $\mathbf{k}$ is a good quantum number for Bogoliubov typical excitations $\omega = \omega(\mathbf{k})$. However, this approach is not longer valid when the condensate is loaded into a confined spatially inhomogeneous trap potential.

In particular, the collective excitations must show a discrete set of modes or confined states.

Starting now from the 1D spinor Gross–Pitaevskii equation (GPE), as described below in section 2, the main contribution of the present work is the description of the ensuing collective longitudinal modes, providing explicit expressions for the corresponding excited wavefunctions, $\psi_m^{(k)}(x; t)$, and their eigenfrequencies, $\omega_m^{(k)}$, where $x$ is the longitudinal coordinate and $k = 1, 2, 3, \ldots$ label the excitations. In section 3 we consider the quasi-1D generalized Bogoliubov–de Gennes equations (B-dGEEs), which allow for the analysis of the polar and ferromagnetic phases. In particular, we study the interplay between the non-linear atomic interactions terms, on the one hand, and the harmonic trapping together with the optical lattice external potentials, on the other. As we stated above, the system is considered as spatially inhomogeneous and, as a consequence, faces the presence of confined collective oscillations with a confined phonon-like spectrum. Section 4 is devoted to analyzing the excitation amplitudes for both the phases considered, and our conclusions are listed in section 5. Finally, some constant values, involved for the numerical evaluation of the phonon eigenfrequencies $\omega_m^{(k)}$ are reported in the appendix.

### 2. 1D spinor Bose–Einstein condensates

We focus our attention on the behavior of the phonon modes for $F = 1$ spinor BECs confined in a cigar-shape geometry. The experimental realizations of two- and one-dimensional condensates in dilute ultracold atoms, employing optical and magnetic traps, are very well established techniques [25–27]. In general, however, the three-dimensional nonlinear GPE cannot be decoupled into transversal and longitudinal motions. Nevertheless, in the presence of highly anisotropic trap potentials, the problem can be handled as being tightly confined in the transversal plane and with an independent 1D motion in the longitudinal direction. [28] Ho, in [1], has shown that the three-dimensional ground state $\psi_m$ of the alkali atoms of the condensate in the hyperfine state $|m\rangle$ ($m = -1, 0, 1$) is ruled by the set of GP equations,

$$
\frac{i\hbar}{\partial t} \psi_m(x, \mathbf{r}, t) = \left[ \frac{p^2}{2M} + U_{3D} + \bar{\epsilon}_0 \phi_m \right] \psi_m(x, \mathbf{r}, t) + \bar{\epsilon}_r \phi_m \cdot \left[ F \psi(x, \mathbf{r}, t) \right]_m, \tag{1}
$$

where is the 2D coordinate, $M$ the mass of the atom, $F$ the total hyperfine spin operator for $F = 1$, $\bar{\epsilon}_0 = (\bar{\epsilon}_x + 2\bar{\epsilon}_y)/3$, $\bar{\epsilon}_r = (\bar{\epsilon}_x - \bar{\epsilon}_y)/3$, $\bar{\epsilon}_r = 4\pi \hbar^2 a_F N/M$ ($F_r = 0, 2$) the atom–atom self-interaction constants related to the s-wave scattering length $a$, $M$ in the total spin $F_r$ channel. $N$ is the number of particles. In equation (1) and below, Einstein summation convention over latin indices is assumed. The spinor state is normalized $\int |\psi(x, \mathbf{r}, t)|^2 \, dx \, dr = 1$. We consider a condensate confined in an anisotropic harmonic trapping potential and loaded into an optical lattice as given by the external potential

$$
U_{3D}(x, \mathbf{r}) = \frac{1}{2} M \left( \omega_0^2 x^2 + \omega_r^2 r^2 \right) - V_L \cos^2 \left( \frac{2\pi}{d} x \right). \tag{2}
$$

where $\omega_0$ and $\omega_r$ are the longitudinal and transverse harmonic oscillator frequencies, respectively; $V_L$ is proportional to the laser intensity, and $d$ the laser wavelength. Assuming now that the longitudinal motion is adiabatic with respect to the transverse one, and considering a highly anisotropic harmonic trapping, with $\omega_0 \ll \omega_r$, the 3D order parameter can be factorized into $\phi_m(x, \mathbf{r}, t) = \phi_m(x; t) \chi(x, \mathbf{r})$ [29]. We note two important consequences of these assumptions: first, that the transverse motion is independent of the hyperfine state $|m\rangle$ and, second, all the time evolution occurs along the 1D longitudinal coordinate $x$. It then follows from equations (1) and (2) that the motion in the plane is given by the equation

$$
\frac{p^2}{2M} + \frac{M}{2} \omega_r^2 r^2 + \left( \frac{1}{\phi_m^2} \frac{\partial^2}{\partial x^2} \phi_m + \frac{1}{\phi_m^2} \frac{\partial^2}{\partial y^2} \phi_m \right) \chi(x, \mathbf{r}) = \mu_r [\phi_m] \chi, \tag{3}
$$

where the transverse chemical potential, $\mu_r [\phi_m]$, is a functional of the 1D order parameter $\phi_m(x; t)$. For the longitudinal dynamical part we have

$$
\frac{i\hbar}{\partial t} \phi_m(x; t) = \left[ \frac{p^2}{2M} + \frac{M}{2} \omega_0^2 x^2 - V_L \cos^2 \left( \frac{2\pi}{d} x \right) + \mu_r [\phi_m] \right] \phi_m(x; t). \tag{4}
$$

In a first approach the function $\chi(x, \mathbf{r})$ can be described by the ground state of a 2D harmonic oscillator with frequency $\omega_r$. Thus, expanding $\mu_r$ as a Taylor series of the wavefunction $\phi_m$
and following the same trend as given in [31] and [32] we have
\[
\mu_I |\phi_m\rangle |\phi_m\rangle \approx \hbar \omega_I |\phi_m\rangle + C_m |\phi_m\rangle + \ldots, \tag{5}
\]
with \( C_m |\phi_0\rangle = c_0 |\phi_0\rangle |\phi_0\rangle + c_2 \phi_0^* F_{ab} \phi_b \cdot (F |\phi_m\rangle \). Here, \( c_0 = M_2 \omega_0 \gamma_0 / (3 \hbar \pi) \) and \( c_2 = M_2 \omega_0 \gamma_0 / (3 \hbar \pi) \) and become the effective self-interaction constants for the 1D cigar-shape spinor BEC. Hence, equation (4) is reduced to
\[
h \frac{\partial \phi_m(x; t)}{\partial t} = \left[ \frac{p_x^2}{2M} + \frac{M}{2} d^2 x^2 \right. - V_0 \cos \left( 2 \mu t \right) \left( \phi^2 \phi_m \right) \phi_m(x; t) + c_2 \phi_0^* F_{ab} \phi_b \cdot (F |\phi_m\rangle \). \tag{6}
\]
This is the 1D system that represents the longitudinal motion of the condensate in the trap. In the following section we will briefly review its ground stationary state first, and then will study its collective excitations in detail.

3. Excited states

Before analyzing the excited states of equation (6), we must first establish the structure of the ground state. For this, we seek a solution of the type \( \phi_m(x; t) = \phi_m(x) \exp(-i \mu t / \hbar) \), where \( \mu \) is the chemical potential. Following Ho [1], one can show that the spatial and spinor part decouple as \( \phi_m(x) = \phi(x) \zeta_m \), with \( \zeta_m \) a pure normalized spinor and \( \phi(x) \) a scalar spatial function. There are two type of solutions depending on the sign of \( c_2 \), the coefficient of the spinor coupling part in (6). If \( c_2 > 0 \), the polar (P) phase develops, which corresponds, up to a spin rotation and a gauge transformation, to \( \zeta_0 = 0 \) and \( \zeta_1 = 1 \). For \( c_2 < 0 \) the system falls into the ferromagnetic (Fe) phase, with \( \zeta_1 = 1 \), \( \zeta_0 = 0 \). The corresponding order parameters \( \phi(x) \), labeled accordingly, satisfy the scalar GPE,
\[
\left[ \frac{1}{2M^2} p_x^2 + U(x) + g_2 |\phi^2| \right] |\phi\rangle \mu = \mu |\phi\rangle \mu, \tag{7}
\]
with \( U(x) = M_2 \omega_0 d^2 / 2 - V_0 \cos^2(2 \pi x / d) \), and the coupling constants given by,
\[
g_2 = \begin{cases} c_0 & \text{for } J = P \\ g_2 & \text{for } J = \text{Fe} \end{cases},
\]
where \( g_2 = M_2 \omega_0 \gamma_0 / (3 \hbar \pi) \). We search for a solution of equation (7), with the boundary conditions \( |\phi\rangle \mu \rightarrow 0 \) as \( x \rightarrow \pm \infty \), and normalized to unity,\( |\phi^2| dx = 1 \). As we see below, the collective excitations show strong dependence on its polar or ferromagnetic character.

Using the Bogoliubov approximation at very low temperatures [33], the collective excitation states of the generalized 1D GPE, equation (6), can be represented by the following wavefunction,
\[
|\phi_m^{(k)}(x; t) \rangle = \exp(-i \mu t / \hbar) |\phi_m(x; t) \rangle + |\phi_m^{(k)}(x; t) \rangle \times \exp(-i \omega_{m^{(k)}} t) + |\phi_m^{(k)}(x; t) \rangle \exp(i \omega_{m^{(k)}} t)]. \tag{8}
\]
The functions \( |\phi_m^{(k)} \rangle \) represent small perturbations, or fluctuations, over the stationary solution \( |\phi_m^{(k)}(x) \rangle \exp(-i \mu t / \hbar) \). The notation \( \pm m \) corresponds to the polar (\(- m \)) or ferromagnetic (+m) phases. The corresponding collective excitation frequencies are \( \omega_{m^{(k)}} \) (\( k = 1, 2, \ldots \)).

By substituting equation (8) into equation (6), one finds that, at first order, the wavefunctions of the collective modes for the condensate satisfy the generalized 1D B-dGPEs [1, 33],
\[
h \frac{\partial}{\partial t} \left( \begin{array}{c} \varphi_m^{(+)} \exp(-i \omega_{m^{(k)}} t) \\ \varphi_m^{(-)} \exp(-i \omega_{m^{(k)}} t) \\ \varphi_m^{(+)} \exp(i \omega_{m^{(k)}} t) \\ \varphi_m^{(-)} \exp(i \omega_{m^{(k)}} t) \end{array} \right) = \frac{L_0^{(j)}}{\lambda_m |\phi|^2} \left( \begin{array}{c} \varphi_m^{(+)} \exp(-i \omega_{m^{(k)}} t) \\ \varphi_m^{(-)} \exp(-i \omega_{m^{(k)}} t) \\ \varphi_m^{(+)} \exp(i \omega_{m^{(k)}} t) \\ \varphi_m^{(-)} \exp(i \omega_{m^{(k)}} t) \end{array} \right), \tag{9}
\]
where we have defined the operator \( L_0^{(j)} \) as,
\[
L_0^{(j)} = \frac{1}{2M^2} p_x^2 + U - \mu + \lambda_m |\phi|^2. \tag{10}
\]
In both equations (9) and (10), the coupling parameters are given by,
\[
\lambda_m^{(P)} = \begin{cases} 2c_0 & \text{if } m = 0 \\ g_2 & \text{if } m = \pm 1 \end{cases}, \quad \lambda_m^{(Fe)} = \begin{cases} 0 & \text{if } m = 0 \\ g_2 + 2|c_2| & \text{if } m = -1 \\ 2g_2 & \text{if } m = 1 \end{cases}, \tag{11}
\]
and
\[
\lambda_m^{(P)} = \begin{cases} 0 & \text{if } m = 0 \\ g_2 & \text{if } m = \pm 1 \end{cases}. \tag{12}
\]
Equations (7) and (9) form an independent 3 \times 3 system of equations for \( m = 0 \) and \( \pm 1 \). Assuming the non-linear term \( g_2 |\phi|^2 \) and the periodic potential \( V_0 \cos^2(2 \pi x / d) \), appearing in equation (7), are small with respect to the harmonic trap potential \( M_2 \omega_0 d^2 / 2 \), the chemical potential and the order parameter \( \varphi_m^{(+)} \) can be sought as Taylor polynomials of \( \Lambda_{g} = g/(l_0 h \omega_0) \), with \( l_0 = \sqrt{\hbar / (M_2 \omega_0)} \) the oscillator length and \( V_0 = V_2 / (l_0^2 \omega_0) \). Up to second order terms in \( \varphi_m^{(+)} \) and \( V_0 \), it is possible to show that the chemical potential is given by
\[
\mu = \frac{1}{2} \left( 1 + \frac{\Lambda_{g}}{\sqrt{2 \pi}} \right) - \frac{\exp(-\alpha^2)}{\sqrt{2 \pi}} \times \left[ \frac{\alpha^2}{2} - C - \ln \left( \frac{\alpha^2}{2} \right) \right]
\]
where \( \alpha = 2 \pi l_0 / d \), \( C \) the Euler’s constant, \( Ei(z) \) and \( \text{Chi}(z) \) the exponential and cosine hyperbolic integral, respectively [37, 38]. For the dimensionless order parameter, \( \varphi_m^{(+)} \), considering corrections up to first order in \( \Lambda_{g} \) and \( V_0 \), we have,
\( \phi^{(i)}(x/l_0) = \phi_0(x/l_0) + \Lambda_{g_i} \mathcal{G}(x/l_0; \sqrt{2}) + V_0 \mathcal{F}(x/l_0, \alpha), \quad (15) \)

where \( \phi_0(x) \) is the ground state of the harmonic oscillator wavefunction [39]. The functions \( \mathcal{G}(x/l_0; \sqrt{2}) \) and \( V_0 \mathcal{F}(x/l_0, \alpha) \) are reported in [35, 40].

In the absence of an optical lattice, namely, for \( V_0 = 0 \), in [34–36] it was shown that the range of validity of the GPE perturbation approach, equation \((7)\), is limited by a universal criterion given in terms of the dimensionless self-interaction parameter \( |\Lambda_{g_i}| < 3 \). In the presence of the optical lattice, \( V_0 \neq 0 \), the range of validity of equation \((14)\) depends additionally on two parameters, \( V_0 \) and \( d/l_0 \), yielding a two-dimensional parameter map. In a typical experimental setup, the parameter \( \alpha = 2 \pi l_0/d \) runs between 10 and 50 [41, 42]. Hence, the terms linear in \( V_0 \) and in \( V_0 \Lambda_{g_i} \) do not play a significant role in determining the chemical potential. In consequence, the main contribution of the lattice intensity in equation \((14)\) corresponds to the quadratic function in \( V_0 \). By comparison with the numerical solution of equation \((7)\) it can be shown that the perturbative method is good enough for \( V_0 < 200 \), in the range \(-2 < \Lambda_{g_i} < 2 \) and \( \alpha = 16 \) (see figure 3 in [35]). Higher \( \alpha \) values increase the validity range for the reduced laser intensity \( V_0 > 200 \) and the applicability of equations \((14)\) and \((15)\).

Under the above conditions, we are able to report analytical solutions for the collective modes of trapped spinor BECs, as given by equation \((9)\), for polar and ferromagnetic phases. By following an analogous perturbative scheme as in [23, 43], one can now solve the eigenfrequency problem posed by the generalized 1D B-dGEs, equation \((9)\), and obtain the involved frequencies \( \omega_m^{(k)} \) for both phases, \( P \) and \( Fe \), as a perturbation series up to second order in \( \Lambda_c \) and \( V_0 \), where \( \Lambda_c = C/(l_0 \omega_0) \) with \( C = c_0, c_2, g_2 \).

3.1. Polar phase

3.1.1. Homogeneous system. For comparison purposes, we consider first the case in the absence of trap potential and optical lattice, \( U(x) = 0 \), yielding a homogeneous gas with translational invariance. Under these conditions, the linear momentum \( p_k = \hbar k = \hbar k \) is a good quantum number. Further, for a repulsive self-interaction parameter, \( c_0 > 0 \), and assuming the density \( n_x \) to be constant everywhere, the solution of equation \((7)\) for the order parameter is given by a traveling wavefunction \( \phi^{(P)} = \exp(i p_k x) \) with \( \mu = p_k^2/2M + \bar{n}_0 c_0 \). \( ^6 \) As a consequence, solving the system of equations \((9)\), we find the expected gapless phonon-like Bogoliubov excitation spectrum in the low-momentum regime \( \omega_0 = \sqrt{\varepsilon_p(c_p + 2 \bar{n}_0 n_0)} \) and \( \omega_{\pm 1} = \sqrt{\varepsilon_p(c_p + 2 \bar{n}_0 n_0)} \) with \( c_{p} = p_k^2/2M \) [33, 44, 45].

3.1.2. Confined phonons. By confining the gas now with an external 1D trap potential \( M \omega_0^2 x^2/2 = 0 \), the spatial symmetry is broken and one finds a discrete set of confined phonon-like modes with frequencies \( \omega_m^{(k)} \) (\( k = 1, 2, \ldots \)).

Accordingly to the values of the interaction constant \( \lambda_{m}^{(P)} \), the inherent symmetry of the system of equations \((9)\) shows that the states with \( m = \pm 1 \) are degenerate. Using the definition \((18)\), the polar phonon modes frequencies with \( m = 0 \) are given by

\[
\omega_{0}^{(k)} = \omega^{(k)}(\Lambda_{c_0}, 2 \Lambda_{c_1}, \Lambda_{c_2}), \quad (16)
\]

while for \( m = \pm 1 \) we have,

\[
\omega_{\pm 1}^{(k)} = \omega^{(k)}(\Lambda_{c_0}, \Lambda_{g_2}, \Lambda_{c_3}). \quad (17)
\]

The function \( \omega^{(k)}(z_1, z_2, z_3) \) is given by

\[
\frac{\omega^{(k)}(z_1, z_2, z_3)}{\omega_0} = k - \frac{z_1}{\sqrt{2} \pi} + \frac{\Gamma(k + \frac{1}{2})}{(2\pi)^{\frac{1}{2}}k!} - \frac{V_0}{2} \exp(-\alpha^2 L_k(2 \alpha^2) - 1) - \frac{z_1 V_0}{\sqrt{2} \pi} \exp(-\alpha^2) \left[ Ei(\frac{\alpha^2}{2}) - \ln(\alpha^2) \right] - \frac{z_2 V_0}{\sqrt{2} \pi} \exp(-\alpha^2) \delta(\alpha) + \frac{V_0^2}{4} \exp(-2 \alpha^2) \times \left[ \chi(2 \alpha^2) - \ln(2 \alpha^2) - C + \rho(\alpha) \right] + 0.033106 z_1^2 + \left( \gamma_1^{(k)} + \frac{\gamma_2^{(k)}}{4} z_2^2 \right) \frac{1}{2 \pi^2}, \quad (18)
\]

with \( \Gamma(z) \) the Gamma function, and \( L_0(z) \) the Laguerre polynomials. The functions \( \delta(\alpha) \) and \( \rho(\alpha) \) are reported in [23] and the values of \( \gamma_1^{(k)} \) and \( \gamma_2^{(k)} \) for \( k = 1, 2, \ldots, 6 \) are listed in appendix.

To illustrate these findings, in figure 1(a) we show \( \omega_m^{(k)} \) in units of \( \omega_0 \) for the first four modes, for \( m = 0 \) and \( \pm 1 \), as functions of the dimensionless interaction parameter, for the repulsive case, \( \Lambda_{c_0} = c_0/(l_0 \omega_0) > 0 \). For simplicity, we fix the intensity of the optical lattice as \( V_0 = 0 \). We observe that \( \omega_{0}^{(k)} = \omega_0 \) is constant, independent of the self-interaction parameters [33], while the other modes, \( \omega_{\pm 1}^{(k)} \), decrease as \( \Lambda_{c_0} \) increases. It is interesting to note that for \( \Lambda_{c_0} > \Lambda_{c_2} = c_2/(l_0 \omega_0) \) the first excited state corresponds to \( \omega_{\pm 1}^{(k)} \) with \( \omega_{\pm 1}^{(k)} < \omega_0^{(k)}, \quad \forall k \) in general (see equations \((16)\) and \((17)\)). Figure 1(b) is devoted to the contour plot for the dimensionless frequency \( \omega_{\pm 1}^{(k)}/\omega_0 \) in terms of the interactions \( \Lambda_{c_0} \) and \( \Lambda_{c_2} \). The isosvalues of the function \( \omega_{\pm 1}^{(k)}(\Lambda_{c_0}, \Lambda_{c_2}/\omega_0) \) are represented by dotted lines, i.e., the set of self-interaction terms \( \Lambda_{c_0} \) and \( \Lambda_{c_2} \) for which \( \omega_{\pm 1}^{(k)}/\omega_0 \) has the same value. For given values of the parameter \( \Lambda_{c_0} \) we observe that the frequency of the collective excitations increases monotonically as \( \Lambda_{c_2} \) increases.

3.2. Ferromagnetic phase

We recall that this phase emerges when \( g_2 < 0 \).

3.2.1. Homogeneous system. As in the polar case for \( g_2 > 0 \), the order parameter is \( \phi^{(o)} = \exp(i p_k x) \) with \( \mu = p_k^2/2M + \bar{n}_0 g_2 \). Here, we have three sets of non-degenerate states \( \varphi_{Fe,m}(x) \) for \( m = -1, 0, 1 \). The energies

\( ^6 \) If \( g_2 < 0 \) the non-linear GPE \((7)\) with \( U(x) = 0 \) admits the bright soliton solution \( \phi_0 = \sqrt{|M|/4 \hbar^2} \text{sech}(s \sqrt{|M|}/4 \hbar^2) \) with \( \mu = -M \phi_0^2/8 \hbar^2 \) [34].
of the excited states are obtained directly from equations (7) and (9) and can be cast as \(\omega_{m}^{\pm} = \pm \sqrt{\varepsilon_{p} + 2c_{n}g_{0}}\), \(\omega_{0} = \varepsilon_{p}\), and \(\omega_{1} = \sqrt{\varepsilon_{p} \varepsilon_{p} + 2g_{2}g_{0}}\). The latter case is the only one with a phonon-like Bogoliubov spectrum.

3.2.2. Confined phonons. In this case, the system of equations (9) is decoupled into two independent equations, one for \(\varphi_0^{(k)}\) and one for \(\varphi_1^{(k)}\), and a \(3 \times 3\) system of equations for the state with \(m = 1\). Following the same procedure mentioned above for the polar phase, we report the analytical solutions for the three independent excited frequencies \(\omega_{m}^{(k)}\) for \(m = 0\) ferromagnetic states are of the form,

\[
\omega_{0}^{(k)} = \omega^{(k)}(\Lambda_{g2}, \Lambda_{c0}, 0),
\]

for \(m = 1\),

\[
\omega_{1}^{(k)} = \omega^{(k)}(\Lambda_{g2}, 2\Lambda_{c0}, \Lambda_{c2}),
\]

and for \(m = -1\),

\[
\omega_{-1}^{(k)} = \omega^{(k)}(\Lambda_{g2}, \Lambda_{c2} + 2|\Lambda_{c2}|, 0),
\]

with \(\omega^{(k)}(z_1, z_2, z_3)\) given by (18).

Figures 1(c) and (d) are devoted to the collective excitations for the ferromagnetic phase. In the upper panel we observe the three independent set \(m = -1, 0, 1\) of confined frequencies for \(k = 1, 2, 3, 4\), as functions of \(\Lambda_{g2}\). All frequencies decrease as \(\Lambda_{g2}\) increases, while the state \((m = 1, k = 1)\) is independent of the interaction parameters. Notice that the states with \(m = -1, 0\) do not fulfill typical properties of B-dGE solutions, for instance, their first excited state is independent of the interaction. This appears to be in agreement with the fact that, in the homogeneous case, their dispersion relations are not linear in the low-momentum regime. In figure 1(d), the characteristic contour map for the reduced confined phonon frequency \(\omega_{-1}^{(k)}/\omega_{0}\) is represented as a function of \(\Lambda_{c0}\) and \(\Lambda_{c2}\). For a given value of \(\Lambda_{c0}\) the
frequency $\omega_{m}^{(k)}$ decreases as $\Lambda_{c_2}\to 0$ in correspondence with the result shown in figure 2, as discussed further below.

3.3. Transition from ferromagnetic to polar phases

Since the sign of the collision parameter $c_2$ is responsible for the polar or ferromagnetic character of the condensate, one can study the transition between these two phases. This is represented in figure 2 for the modes with frequencies $\omega_{m}^{(k)}$ ($k = 1, 2, 3, 4$) as functions of $\Lambda_{c_2}$. In panel (a), for $\Lambda_{c_2} < 0$, the three set of independent modes with $m = -1, 0, -1$ are very well resolved. However, they show different behaviour as $\Lambda_{c_2}$ decreases, with the steepest slope for the phonon modes $\omega_{m}^{(k)}$. For $\Lambda_{c_2} = 0$, the states $m = -1, 0$ become degenerate, while for $\Lambda_{c_2}$ positive, the values of $\omega_{m}^{(k)}$ are closer to $k\omega_0$, (see panel (a)), that is, the influence of $\Lambda_{c_2}$ is negligible and we have that these three states are in quasi-degeneracy.

3.4. Influence of the optical lattice

We now turn our attention to the influence of the optical lattice on the collective excitations. Figure 3 shows a contour plot of the frequencies $\omega_{m}^{(k)}$ for the first two states ($k = 1, 2$) as functions of the dimensionless laser intensity $V_0$ and the parameter $\Lambda_{c_2}$, for the polar state $m = \pm 1$, and for the ferromagnetic one $m = 1$. The main contribution of $V_0$ is to shift the confined phonon frequencies. For larger values of $V_0$, the frequency is almost independent of $\Lambda_{c_2}$, while the mayor modification of $\omega_{m}^{(k)}$ occurs for lower values of the laser intensity, $V_0 \sim 40$. These facts can be explained by observing in equations (17) and (20) the interplay between the self-interaction constant $\Lambda_a$ and the laser intensity $V_0$. For a better visualization of the present behavior, we consider the polar mode frequency $\omega_{\pm 1}^{(k)}$. As we stated above, the parameter $\alpha = 2\pi l_0/d \gg 1$ and the contribution of the linear term in $V_0$ and the cross terms $z_1 V_0, z_2 V_0$ to the excited frequency (18) are negligible. On the other hand the function $\exp(-2\alpha^2)|\text{Chi}(2\alpha^2) - \ln(2\alpha^2) - C + \rho_2(\alpha)|$ approaches $1/(2\alpha^2)$. In consequence, for $\Lambda_{c_2} = 1$ the phonon frequency $\omega_{\pm 1}^{(k)}$ is reduced to

$$\frac{\omega_{\pm 1}^{(k)}}{\omega_0} = 0.63416 - \frac{\Lambda_{c_2}}{2\sqrt{2\pi}} - \frac{1}{2\pi^2}(0.284\Lambda_{c_2}^2 + 0.1215\Lambda_{c_2}^2) + \frac{V_0^2}{8\alpha^2}. \quad (22)$$

From equation (22) it follows that for large values of $V_0$, the dependence of $\omega_{\pm 1}^{(k)}$ is almost independent of the non-linear interaction parameter $\Lambda_{c_2}$; whereas for lower values of $V_0$, the frequency $\omega_{\pm 1}^{(k)}$ is reduced as $\Lambda_{c_2}$ increases. In fact, as the laser wavelength becomes much smaller than the oscillator wavelength, the rapid variation of the function $\cos^2(2\pi x/d)$ for $0 < x < l_0$ effectively provides a constant potential. The same can be argued for the other frequencies drawn in figure 3.

As discussed above, the validity of the perturbative approach for equation (7) is restricted to $|\Lambda_{c_2}| < 3$. This imposes conditions to the solutions of the B-dGEs. Since equation (9) contains spin-mixing terms, the validity of equations (16)–(21) depends on the phase and on the angular quantum number $m$. Employing (9) and considering the excited wavefunctions $\varphi_m^{(k)}$, with an effective Hamiltonian $H_{\text{eff}} = \hbar^2/2M + U + \lambda_{\text{eff}}(m)|\psi^{(1)}|^2$ and $\lambda_{\text{eff}}(m) = \lambda^{(f)} + \overline{\lambda^{(f)}}$, (or equivalently to $\Lambda_{c_2}^{(f)} = \Lambda_{c_2}^{(f)} + \overline{\Lambda_{c_2}^{(f)}}$). The operator $H_{\text{eff}}$ is isomorphic to the GPE Hamiltonian given by equation (7) and, in consequence, we are able to make an estimate of the

![Figure 2](image-url)

**Figure 2.** Evolution of the frequencies modes ($k = 1, 2, 3, 4$) from the ferromagnetic phase to the polar phase, as a function $\Lambda_{c_2}$. Left (right) panel $\Lambda_{c_2} = 1 (\Lambda_{c_2} = 1)$.
Figure 3. Contour plots of $\omega_m^{(k)}/\omega_0$ as functions of the dimensionless parameters $\Lambda_{g_2}$ and $V_0$ for the first two modes $k = 1$ (upper panels) and $k = 2$ (lower panels). Left panel: polar state for $\omega_m^{(k)}/\omega_0$. Right panel: ferromagnetic state for $\omega_m^{(k)}/\omega_0$. Dashes lines represent isovalues of the reduced frequencies $\omega_m^{(k)}/\omega_0$.

interval of validity of equation (16)–(21) in terms of $\Lambda_{c_0}$, $\Lambda_{g_2}$, $\Lambda_{c_0}$, as given by

for polar states

$$\begin{cases} |\Lambda_{c_0}| < 1; & m = 0 \\ |\Lambda_{g_2} + \Lambda_{c_0}| < 3; & m = \pm 1 \end{cases}$$

(23)

for ferromagnetic states

$$\begin{cases} |\Lambda_{g_2}| < 3; & m = 0 \\ |\Lambda_{g_2} + 2|\Lambda_{c_0}|| < 3; & m = -1 \\ |\Lambda_{g_2}| < 1; & m = 1 \end{cases}$$

(24)

For the case $V_0 = 0$, we can extrapolate the conditions discussed for the GPE of equation (7), but taking into account the effective interaction $\Lambda_{g_2}^{[m]}$ for each phase and angular momentum.

4. Excitation amplitudes

The dynamics of the condensate particle density for the elementary excitation $\omega_m^{(k)}$ is ruled by the particle excitation density $n_m^{(k)}(x, t) = |\phi_m^{(k)}(x; t)|^2$. As can be seen from equation (8), the main contribution of these excitations can be obtained by analyzing their fluctuations $\delta n_m^{(k)}(x, t)$ over the stationary particle density, $n_f(x) = |\phi_f(x)|^2$, which can be cast as,

$$\delta n_m^{(k)}(x, t) = n_m^{(k)}(x, t) - n_f(x).$$

(25)

As discussed further below, these spin population fluctuations are amenable to be measured [15, 17].

The calculation of the collective modes, for the polar and ferromagnetic phases, $\omega_m^{(k)}(x; t) = \phi_m^{(k)} \exp (-i\omega_m^{(k)}t)$, see equations (8) and (9), are obtained in first order of perturbation for the self-interaction constants $\Lambda_{c_0}$, $\Lambda_{g_2}$, $\Lambda_{c_0}$, and the dimensionless laser intensity $V_0$. For the density profile of the order parameter $\phi_f(x)$ we employ the results of equation (15). Hence, for the polar state the excited wavefunction with $m = 0$ is

$$\phi_m^{(k)} = 0 = \exp(-i\mu t/\hbar)[\phi_f(y) + f_0^{(k)}(y; t; \Lambda_{c_0}, \Lambda_{c_0})].$$

(26)
while for the case of $m = \pm 1$ we obtain
\[
\phi_{m=\pm 1}^{(k)} = \exp(-i\mu t/\hbar) [\phi^{(P)}(y) + \phi^{(k)}_{m}(y, t; \Lambda_{e}, \Lambda_{c})],
\]
where the function $f_{m}^{(k)}(y, t; z, z_{2})$ is given by
\[
f_{m}^{(k)}(y, t; z, z_{2}) = \phi_{m}(y) \exp(-i\omega_{m}^{(k)}t)
+ \sum_{p \neq k} \left\{ \frac{2z_{1}z_{2} f_{k-p}^{(m)}(y) - V_{0} f_{k-p}^{(m)}(y)}{2(k-p)} \phi_{p}(z) \exp(-i\omega_{m}^{(k)}t) \right\}
- \sum_{p=0}^{\infty} \frac{z_{2} f_{k-p}^{(m)}(y)}{k + p} \phi_{p}(y) \exp(i\omega_{m}^{(k)}t).
\]

In these expressions, $y = x/l_{0}$, $\phi_{p}(y)$ are the 1D harmonic oscillator wavefunctions [39] and the functions $f_{k-p}^{(m)}(y)$, $g_{k-p}^{(m)}(y)$ are reported in [43].

Similarly, the excited states for the ferromagnetic phase are described by
\[
\phi_{m=0}^{(k)} = \exp(-i\mu t/\hbar) [\phi^{(F)}(y) + f_{1}^{(k)}(y, t; \Lambda_{e}/2, 0)],
\]
\[
\phi_{m=1}^{(k)} = \exp(-i\mu t/\hbar) [\phi^{(F)}(y) + f_{1}^{(k)}(y, t; \Lambda_{e}, \Lambda_{c})].
\]

It is important to summarize the symmetry properties of the density perturbation $\delta n_{m}^{(k)}$. In equation (28), for a given state $\vert m \rangle$, the matrix elements $f_{k-p}^{(m)}(y)$ and $g_{k-p}^{(m)}(y)$ (see [43]) must fulfill the parity condition $k + p$ = even number. Thus, $f_{k}^{(m)}$ is antisymmetric (symmetric). Hence, the space of solutions $\phi_{m}^{(k)}(x)$ is composed of two independent Hilbert subspaces $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ for odd ($k = 1, 3, \ldots$) and even ($k = 2, 4, \ldots$) wavefunctions with respect to the inversion symmetry $x \rightarrow -x$. In consequence, the density perturbation $\delta n_{m}^{(k)}$ is restricted by the symmetry property of the $f_{k}^{(m)}$ function.

Thus, employing the above results for the wavefunction of the excited states of the polar state with $m = 0$, we obtain an analytical representation for the function $\delta n_{0}^{(k)}$,
\[
\delta n_{0}^{(k)} = 2 \cos(\omega_{0}^{(k)}t) \left[ \phi^{(P)}(x) \phi_{y}(x) - \frac{\exp(-x^{2})}{\sqrt{\pi}} \right]
\times \left\{ \frac{\Lambda_{e}}{k \sqrt{\pi}} \frac{f_{0}^{(k,0)}}{x} + \left[ \sum_{m=1}^{\infty} \frac{\Lambda_{e}}{k - m - \frac{1}{k + m}} \phi_{m}^{(k)}(x) \right] - \frac{V_{0}^{(k,0)}}{2(k - m)} \right\}.
\]

Similar results can be obtained for the listed Bogoliubov-type excitation amplitudes (27)–(31). In figure 4 a contour plot of the condensate density perturbation $\delta n_{0}^{(k)}(x; t)$ is shown, for both polar and ferromagnetic phases. Here, we consider the first two excited states, the first one with $k = 1$ belongs to the Hilbert subspaces $\mathcal{H}_{1}$, while for $k = 2$ belongs to $\mathcal{H}_{2}$. The antisymmetric and symmetric character of $\delta n_{0}^{(k)}(x; t)$, for both phases, are clearly seen in the figure. In general, the evolution from one phase to the other, as a function of the parameter $\Lambda_{e}$, does not change the parity of the density perturbation $\delta n_{m}^{(k)}(x; t)$. Also, from density perturbation $\delta n_{0}^{(k)}(x; t)$, it follows that, for a certain instant of time $t/\tau_{0}$, there are pronounced oscillations of the density perturbation $\delta n_{m}^{(k)}(x; t)$, $k = 1, 2$; furthermore, along the coordinate $x$ the density is quenched according to the exponential behaviour $\exp(-x^{2})$, see equation (32). From the symmetry consideration of the order parameter $\phi^{(P)}(x)$ and the density perturbation $\delta n_{m}^{(k)}(x; t)$, immediately, follows that, for the antisymmetric excited states (k odd), the spinor dynamics of the number of particle $N(t)$ is constant, while for the even modes $N(t)$ presents spin waves or oscillatory population with frequency $\omega_{m}^{(k)}$. These facts are clearly reflected in figure 4 for the antisymmetric and symmetric excited collective modes $k = 1$ and $k = 2$, respectively. Hence, by choosing the optical lattice intensity, $V_{0}$, the frequency $\omega_{m}^{(k)}$ value is changed (see figures 1–4), and so, we are able to tune the spin population and the fraction of population for each quantum number $m$. In [15], employing a two-dimensional optical lattice, similar effects have been clearly demonstrated in antiferromagnetic spinor $^{23}$Na BEC, where the spin dynamics can be tuned as a function of the lattice depth. Also, similar experimental observations of coherent spin-population oscillations are reported in [17].

5. Discussion of the results and conclusions

To summarize, we have found the multicomponent order parameter of the coupled B-DEGs, equation (9), for the one dimensional cigar-shaped BECs, with $F = 1$ spin degrees of freedom. We have provided a relatively simple mathematical tool for the description of the collective confined modes for the ferromagnetic and polar phases. An examination of the collective excitations shows that the phonon energies are proportional to the longitudinal harmonic trap frequency. We conclude that the phonon modes are weakly dependent on the interaction constants for the antiferromagnetic states, while a more pronounced structure is reached in the case of a BEC in the ferromagnetic phase, see figures 1 and 2. Moreover, we have found the existence of a set of values of the self-interaction parameters for which the lower frequency lies below the harmonic oscillator frequency $\omega_{0}$. The polar $m = 0$ and ferromagnetic $m = -1$ states coincide with the oscillation of the center of mass and are independent of the atom-atom interaction [33]. It is interesting to compare the main qualitative differences between our perturbative results with the opposite limit, that is that of the Thomas–Fermi approximation where
the self-interaction constant is so strong, \( \Lambda c_1 \) or \( \Lambda c_2 \), that we can omit the kinetic energy in equation (7). In such a case, the excitation spectrum and the density of excited polar modes are interaction independent (see [1] and [46]). This is in definite contrast to the present case of weak interactions studied here, where the condensate densities, \( d_n(k,t) \), show a structure which clearly depends on the \( g_0 \) and \( g_2 \) atom–atom self-interaction terms.

The present analytical results indicate that one can quantitatively study, by manipulating in a controllable way, several quantum phenomena, such as spin dynamics and superfluid properties of the 1D cigar-shape spinor BECs. For instance, with a similar experimental set up to that used in [14], and by measuring the relative population or spin oscillations, one can obtain information on the effective self-interaction constants \( \Lambda c_1 \) and \( \Lambda c_2 \), for the 1D cigar-shape spinor BEC.

It is worth noting that an estimation of spin-orbit interactions for \(^{23}\text{Na}\) antiferromagnetic condensates, and employing the inequalities given in (23), bring up the interval of the number of particles to \( 250 \leq N \leq 1.5 \times 10^4 \) for \( m = 0 \), under which the reliability of our perturbative approach can be trusted. For the case of \( m = \pm 1 \), we obtain the range \( 700 \leq N \leq 4.0 \times 10^4 \). If we choose \(^{27}\text{Rb}\) atoms for the ferromagnetic phase, the inequalities (24) provide that the reliability of equations (19)–(21) and (29)–(31) are valid in the range \( 220 \leq N \leq 2.2 \times 10^3 \) for \( m = -1, 0 \), and \( 75 \leq N \leq 7.5 \times 10^3 \) for \( m = 1 \).

A straightforward application of the present formalism is the theoretical estimation of the damping process of the collective excitations [47]. The damping rate, \( \gamma \), is a crucial magnitude in the dynamic of BECs. Typically, \( \gamma \) is assumed as a phenomenological parameter without any structure, for example see [14]. Nevertheless, an intrinsic issue is knowing

![Figure 4](image_url)  
**Figure 4.** Contour map of the 1D condensate density perturbation, \( \delta n^m_k(x,t) \), for the first two excited states \( k = 1 \) (upper panels) and \( k = 2 \) (lower panels). Left panel: polar modes for \( \Lambda c_1 = 0.5 \) and \( \Lambda c_2 = 2 \). Right panel: ferromagnetic modes for \( \Lambda c_1 = 1 \). In the calculation \( \tau_0 = 2\pi/\omega_0 \) and \( V_L = 0 \).
the dependence of the damping rate on the temperature and on the condensate parameters. Following the procedure of [23, 48], we can obtain $\gamma$ for the Landau or Beliaev processes of the spinor BECs.

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**Appendix. Excited frequencies**

Table A1 shows the values of the constants $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$, satisfying $\gamma_k = \gamma_k^{(1)} + \gamma_k^{(2)}$ with $\gamma_k$ given in [43].

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**Table A1. Values of constants $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$.**

| $k$ | 1   | 2   | 3   | 4   | 5   | 6   |
|-----|-----|-----|-----|-----|-----|-----|
| $\gamma_k^{(1)}$ | −0.284 | −0.620 | 0.142 | 0.015 | 0.093 | 0.050 |
| $\gamma_k^{(2)}$ | −0.486 | −0.165 | −0.162 | −0.095 | −0.079 | −0.058 |

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