MODULI SPACE OF A PLANAR POLYGONAL LINKAGE: A COMBINATORIAL DESCRIPTION

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Abstract. We explicitly describe a structure of a regular cell complex $CWM(L)$ on the moduli space $M(L)$ of a planar polygonal linkage $L$. The combinatorics is very much related (but not equal) to the combinatorics of the permutahedron. In particular, the cells of maximal dimension are labeled by elements of the symmetric group. For example, if the moduli space $M(L)$ is a sphere, the complex $CWM(L)$ is dual to the boundary complex of the permutahedron.

The dual complex $CWM^*$ is patched of Cartesian products of permutahedra and carries a natural PL-structure. It can be explicitly realized as a polyhedron in the Euclidean space via a surgery on the permutahedron.

1. Preliminaries and notation

A polygonal $n$-linkage is a sequence of positive numbers $L = (l_1, \ldots, l_n)$. It should be interpreted as a collection of rigid bars of lengths $l_i$ joined consecutively in a chain by revolving joints.

We assume that $L$ satisfies the triangle inequality.

Definition 1.1. A configuration of $L$ in the Euclidean plane $\mathbb{R}^2$ is a sequence of points $R = (p_1, \ldots, p_n)$, $p_i \in \mathbb{R}^2$ with $l_i = |p_i, p_{i+1}|$, and $l_n = |p_n, p_1|$. We also call $P$ a polygon.

As follows from the definition, a configuration can be self-intersecting.

The set $M(L)$ of all configurations modulo orientation preserving isometries of $\mathbb{R}^2$ is the moduli space, or the configuration space of the linkage $L$.

Throughout the paper (except for the concluding remarks) we assume that no configuration of $L$ fits a straight line. This assumption implies that the moduli space $M(L)$ is a closed $(n - 3)$-dimensional manifold (see [2]).

The manifold $M(L)$ is already well studied, see [2, 3, 7], and many other papers. Explicit descriptions of $M(L)$ existed for $n = 4, 5$, and 6 (see [2, 7, 11]).

There also exist various results for polygonal linkages in 3D (see [8] for example).

The paper presents an explicit combinatorial description of $M(L)$ as a regular cell complex $CWM(L)$. We study $CWM(L)$ and the dual complex $CWM^*(L)$; the latter is geometrically realized as a polyhedron in the Euclidean space. The combinatorics of the complexes is very much related (but not equal) to the combinatorics of the permutahedron.

Key words and phrases. Polygonal linkage, cell complex, CW-complex, configuration space, moduli space, permutahedron, cyclic polytope.
In a sense, our approach is a very elementary application of Gelfand-McPherson ideas from \[5\].

The complex \(CWM(L)\) already appeared in \[7\] in a slight disguise, where it was mentioned as a ”tiling of \(M(L)\)”. Moreover, basing on Deligne-Mostow map, Kapovich and Millson deduced that \(CWM(L)\) can be realized as a piecewise linear manifold in the hyperbolic space.

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We start with necessary preliminaries:

**Convex configurations.**

**Definition 1.2.** A configuration \(P\) is convex if it is a convex (piecewise linear) curve, and all the angles at the vertices \(p_i\) are strictly less than \(\pi\).

The set \(M_{\text{conv}}(L)\) is the set of all convex configurations such that the orientation induced by the numbering goes counterclockwise.

The set \(\overline{M}_{\text{conv}}(L)\) is the closure of \(M_{\text{conv}}(L)\) in \(M(L)\).

**Lemma 1.3.** \[6\]

1. The set \(M_{\text{conv}}(L)\) is an open subset of \(M(L)\) homeomorphic to the open \((n - 3)\)-dimensional ball.

2. The closure \(\overline{M}_{\text{conv}}(L)\) is homeomorphic to the closed \((n - 3)\)-dimensional ball.

Proof. In view of the paper \[7\], the space \(M(L)\) is the space of stable configurations of \(n\) marked points \((x_1, \ldots, x_n)\) in \(\mathbb{RP}^1\) modulo the diagonal action of \(PSL(2, \mathbb{R})\). A configuration of points yields a convex polygon whenever the numbering \((1, \ldots, n)\) goes counterclockwise. Therefore \(M_{\text{conv}}(L)\) is identified with the set of \(n\)-tuples of cyclically ordered distinct points \(x_i\) in \(S^1 = \mathbb{RP}^1\) modulo \(PSL(2, \mathbb{R})\). We can omit the action of the group by assuming that the first three points are 0, 1, and \(\infty\). The rest of the points are then given by linear inequalities

\[1 < x_4 < x_5 < \ldots < x_n < \infty,\]

which implies the statement (1). The statement (2) is now straightforward. \(\Box\)
Polytopes. We shall use the combinatorial structure of the following polytopes:

The permutohedron $\Pi_n$ (see [10]) is defined as the convex hull of all points in $\mathbb{R}^n$ that are obtained by permuting the coordinates of the point $(1, 2, \ldots, n)$. It has the following properties:

1. $\Pi_n$ is an $(n-1)$-dimensional polytope.
2. The $k$-faces of $\Pi_n$ are labeled by ordered partitions of the set $\{1, 2, \ldots, n\}$ into $(n-k)$ non-empty parts.
   
   In particular, the vertices are labeled by the elements of the symmetry group $S_n$. For each vertex, the label is the inverse permutation of the coordinates of the vertex.
3. A face $F$ of $\Pi_n$ is contained in a face $F'$ iff the label of $F$ is finer than the label of $F'$.
4. A face of $\Pi_n$ is a Cartesian product of permutohedra of smaller dimensions.
5. The permutohedron is a zonotope, that is, a Minkowski sum of line segments.
6. The permutohedra $\Pi_1$, $\Pi_2$, and $\Pi_3$ are a one-point polytope, a segment, and a regular hexagon respectively.

The permutohedron $\Pi_4$ (with its vertices labeled) is depicted in Fig. 1.
The cyclic polytope $C(n, d)$ (see [10]) is the convex hull of $n$ distinct points $x_1, ..., x_n$ on the moment curve in $\mathbb{R}^d$. Its combinatorics is completely defined by the following property (Gale evenness condition): a $d$-subset $F \subset \{x_1, ..., x_n\}$ forms a facet of $C(n, d)$ iff any two elements of $\{x_1, ..., x_n\} \setminus F$ are separated by an even number of elements from $F$ in the sequence $x_1, ..., x_n$.

2. The complex $CW(M(L))$

Definition 2.1. A partition of the set $\{l_1, \ldots, l_n\}$ is called admissible if the total length of any part does not exceed the total length of the rest.

In the terminology of paper [3], all parts of an admissible partition are short sequences.

Instead of partitions of $\{l_1, \ldots, l_n\}$ we shall speak of partitions of the set $\{1, 2, \ldots, n\}$, keeping in mind the lengths $l_i$.

Definition 2.2. Two edges $p_ip_{i+1}$ and $p_jp_{j+1}$ of a configuration $P$ are called parallel if the vectors $\overrightarrow{p_ip_{i+1}}$ and $\overrightarrow{p_jp_{j+1}}$ are parallel and codirected.

For instance, in Fig. 3 the blue and the red edges are parallel.

Given a configuration $P$ of $L$ without parallel edges, there exists a unique convex polygon $\overline{P}$ such that

1. The edges of $P$ are in one-to-one correspondence with the edges of $\overline{P}$. The bijection preserves the directions of the vectors.
2. The induced orientations of the edges of $\overline{P}$ give the counterclockwise orientation of $\overline{P}$.

In other words, the edges of $\overline{P}$ are the edges of $P$ ordered by the slope (see Fig. 2). Obviously, $\overline{P} \in M_{conv}(\lambda L)$ for some permutation $\lambda \in S_n$. The permutation is defined up to some power of the cyclic permutation $(2, 3, 4, \ldots, n, 1)$. We consider $\lambda$ as a cyclic ordering on the set $\{1, 2, \ldots, n\}$. Alternatively, $\lambda$ can be identified with an element of $S_{n-1}$.

Conversely, each convex polygon from $M_{conv}(\lambda L)$ is the image of some element of $M(L)$ under the above rearranging map. Thus we get a continuous bijection between $M_{conv}(\lambda L)$ and its preimage in $M(L)$.

Our construction assigns to $P \in M(L)$ the label $\lambda$, considered as a cyclically ordered partition of the set $\{1, \ldots, n\}$ into $n$ non-empty parts.

If $P$ has parallel edges, a permutation which makes $P$ convex is not unique. Indeed, one can choose any ordering on the set of parallel edges.

The label assigned to $P$ is a cyclically ordered partition of the set $\{1, 2, \ldots, n\}$. Fig. 3 gives an example with one two-element set and three one-element sets. It is convenient to write such a partition as a (linearly ordered) string of sets where the set containing "n" stands on the last position.

We stress once again that there is no ordering inside a set, that is, we identify two labels whenever they differ on permutations of the elements inside the
Figure 2. Labeling of a polygon with no parallel edges

Figure 3. Labeling of a polygon with parallel edges

parts. For instance,

\[(\{1\}\{3\}\{4256\}) \neq (\{3\}\{1\}\{4256\}) = (\{3\}\{1\}\{2456\}).\]

By the triangle inequality, all labels are admissible partitions.
Definition 2.3. Two points from \( M(L) \) (that is, two configurations) are said to be equivalent if they have one and the same label. Equivalence classes of \( M(L) \) we call the open cells. The closure of an open cell in \( M(L) \) is called a closed cell. By Lemma 1.3 all cells are ball homeomorphic.

For a cell \( C \), either closed or open, its label \( \lambda(C) \) is defined as the label of its interior point.

Before we formulate the main theorem, remind that a CW-complex can be constructed inductively by defining its skeleta. Once the \((k - 1)\)-skeleton is constructed, we attach a collection of closed \( k \)-balls \( C_i \) by some continuous mappings \( \varphi_i \) from their boundaries \( \partial C_i \) to the \((k - 1)\)-skeleton. For a regular complex, each of the mappings \( \varphi_i \) is injective, and \( \varphi_i \) maps \( \partial C_i \) to a subcomplex of the \((k - 1)\)-skeleton (see [4]).

Regularity guarantees the existence of well-defined barycentric subdivision and (for manifolds) the well-defined dual complex.

Theorem 2.4. The above described collection of open cells yields a structure of a regular CW-complex \( CWM(L) \) on the moduli space \( M(L) \). Its complete combinatorial description reads as follows:

1. \( k \)-cells of the complex \( CWM(L) \) are labeled by cyclically ordered admissible partitions of the set \( \{1, 2, ..., n\} \) into \( k + 3 \) non-empty parts.
   In particular, the facets of the complex (that is, the cells of maximal dimension \( n - 3 \)) are labeled by cyclic orderings of the set \( \{1, 2, ..., n\} \).
2. A closed cell \( C \) belongs to the boundary of some other closed cell \( C' \) iff the partition \( \lambda(C') \) is finer than \( \lambda(C) \).

For the complex \( CWM(L) \), we have:

1. The intersection of two closed cells \( C \) and \( C' \) is a cell labeled by the finest cyclically ordered admissible partition \( \nu \) such that both partition \( \lambda(C) \) and \( \lambda(C') \) refine \( \nu \). If such a partition \( \nu \) does not exist, the intersection is empty.
2. The vertex figure of any vertex \( v \) of the complex \( CWM(L) \) is combinatorially dual to the Cartesian product of three permutohedra.
   More precisely, if the three parts of \( \lambda(v) \) have \( k \), \( l \), and \( m \) elements, then the vertex figure of \( v \) is combinatorially dual to \( \Pi_k \times \Pi_l \times \Pi_m \).
3. The face figure of any \( k \)-dimensional face is combinatorially dual to the Cartesian product of \( k + 3 \) permutohedra. (Some of these permutohedra can be \( \Pi_1 \), and thus degenerate to a point.)

The proof follows directly from the above construction. □

Example 2.5. Let \( n = 4 \); \( l_1 = l_2 = l_3 = 1 \), \( l_4 = 1/2 \). The moduli space \( M(L) \) is known to be a disjoint union of two circles (see [2]). The cell complex \( CWM(L) \) is as depicted in Fig. 4.
Example 2.6. Assume that $l_n$ is a "long" edge of an $n$-linkage $L$. This means that

$$\forall i \quad l_n + l_i > \sum_{n \neq j \neq i} l_j.$$ 

It is known (see [2]) that in this case $M(L)$ is an $(n-3)$-sphere.

The complex $CWM(L)$ is dual to the boundary complex of the permutahedron $\Pi_{n-1}$.

Proof. Indeed, each admissible partition is of the type

$$\text{(any ordered partition of } \{1, \ldots, n-1\}, \{n\}).$$

This means that the facets of $CWM(L)$ are in a natural bijection with the vertices of $\Pi_{n-1}$. It remains to observe that the patching rules for $CWM(L)$ are exactly dual to the combinatorics of the permutahedron. \qed

Example 2.7. Let $n = 5$, $L = (1, 1, 1, 1, 1)$. Then $CWM(L)$ is a surface of genus 4 patched of 24 pentagons. Each vertex has 4 incident edges. The
CW-complex is completely transitive. This means that any combinatorial equivalence of any two pentagons extends to an automorphism of the entire CW-complex.

**Example 2.8.** Let \( L = (l_1, l_2, \ldots, l_n) \), \( L' = (l_1, l_2, \ldots, l_n, \varepsilon) \), where \( \varepsilon \) is small. It is known that \( M(L') = M(L) \times S^1 \). On the one hand, this does not extend to CW-complex structure. On the other hand, there is a natural forgetting projection

\[
\pi : CW M(L') \to CW M(L)
\]

which removes the number \( n + 1 \) from the labeling. Fig. 5 depicts the local structure of \( \pi \) for the case \( L = (1, 1, 1, 1/2) \).

![Diagram](https://example.com/diagram.png)

**Figure 5.** We depict the preimage of the cell \((\{1\}\{2\}\{3\}\{4\})\)

**Example 2.9.** Let \( n = 2k + 1, L = (1, 1, \ldots, 1) \). Then \( CW M(L) \) is patched of \((2k)!\) copies of duals to the cyclic polytope \( C(n, n - 3) \).

However, unlike the previous example, the CW-complex is not completely transitive, just facet-transitive: for every two facets there exists an automorphism of the complex mapping one facet to the other.

**Proof.** Fix a facet \( C \) of \( CW M(L) \). Without loss of generality we may assume that its label is \((\{1\}\{2\}\{3\}\{4\}\ldots\{n\})\). Consider the following "starlike" bijection \( \varphi \) which maps the vertices \( x_1, \ldots, x_n \) of the cyclic polytope \( C(n, n - 3) \) to facets of the cell \( C \).

\[
\varphi(x_1) = (\{1\}\{2\}\{3\}\{4\}\ldots\{n\})
\]

\[
\varphi(x_2) = (\{1\}\{2\}\ldots\{k+1,k+2\}\ldots\{n\})
\]
\[ \varphi(x_3) = (\{2\}...\{n-1\}\{1, n\}), \]
\[ \varphi(x_4) = (\{1\}\{2\}...\{k, k+1\}...\{n\}), \]
\[ \varphi(x_5) = (\{1\}\{2\}...\{n-1, n\}). \]

... It is easy to check that \( \varphi \) is a combinatorial duality. \( \square \)

**Proposition 2.10.** \( (1) \) For any \( n \)-linkage \( L = (l_1, \ldots, l_n) \), the boundary complex of each closed cell of \( CWM(L) \) is combinatorially equivalent to a face of the dual to the cyclic polytope \( C(D + 3, D) \) for some even \( D \in \mathbb{N} \). The face is a simple \( k \)-polytope with at most \( k + 3 \) facets.

(2) Conversely, any simple \( k \)-dimensional polytope \( K \) with at most \( k + 3 \) facets arises in this way. That is, there exist a number \( n \), an \( n \)-linkage \( L \), and a cell \( C \) of the complex \( CWM(L) \) such that the boundary complex of \( C \) is combinatorially equivalent to \( K \).

**Proof.** \( (1) \) We may assume that all \( l_i \) are integers, and that their sum \( D + 3 = \sum l_i \) is odd. The space \( M(L) \) embeds in a natural way in the moduli space of the equilateral polygon with \( D + 3 \) edges \( M(1, 1, \ldots, 1) \). The embedding maps a polygon with edgelengths \( l_1, \ldots, l_n \) to the equilateral polygon which represents the same curve, that is, with first \( l_1 \) edges parallel, next \( l_2 \) edges parallel, etc. The embedding respects the structure of cell complexes, and therefore, realizes \( CWM(L) \) as a subcomplex of the complex \( CWM(1, 1, \ldots, 1) \), whose facets are combinatorial cyclic polytopes (see Example 2.9).

(2) Assume that \( K \) has \( k + 3 \) facets. Then the dual polytope \( K^* \) has \( k + 3 \) vertices. We shall prove that every simplicial \( k \)-polytope with at most \( k + 3 \) vertices is a face figure of the cyclic polytope \( C(D + 3, D) \) for some even \( D \). The Gale diagram of \( K^* \) (see [10]) is a one-dimensional configuration of distinct black and white points. Remind that the Gale diagram of \( C(D + 3, D) \) is the alternating configuration of distinct black and white points in the straight line. Being translated to the Gale diagram’s language, the statement we need reads as ”any configuration of distinct black and white points in the straight line can be completed to an alternating configuration of distinct black and white points”, which is obvious.

If \( K \) has less than \( k + 3 \) facets, the proof is even simpler. \( \square \)

3. **The dual complex \( CWM^*(L) \). Surgery on the permutohedron**

**Theorem 3.1.** The dual cell complex \( CWM^*(L) \) carries a natural structure of a PL-manifold.

**Proof.** The cells of the dual complex \( CWM^*(L) \) are the duals to the face figures of \( CWM(L) \). By Theorem 2.4 the latter are combinatorially equivalent to Cartesian products of permutohedra. In order to realize \( CWM^*(L) \) as a PL-manifold, for each facet of \( CWM^*(L) \) we take the Cartesian product of
three standard permutohedra. Their faces that are identified via the patching rule are isometric.

Moreover, we describe below a realization of $CWM^*(L)$ as a "polyhedron" in $\mathbb{R}^{n-2}$.

For an $n$-linkage $L$, consider the permutohedron $\Pi_{n-1} \subset \mathbb{R}^{n-1}$, assuming (as usual) that the faces of $\Pi_{n-1}$ are labeled by ordered partitions of the set $(1, ..., n-1)$. In particular, the vertices of $\Pi_{n-1}$ are labeled by permutations of the set $(1, ..., n-1)$. We introduce the following bijection between the vertex sets

$$\psi : Vert(CWM^*(L)) \rightarrow Vert(\Pi_{n-1}).$$

Each vertex of $CWM^*(L)$ (whose label $\lambda$ is a cyclically ordered set $\{1, ..., n\}$) we map by $\psi$ to the vertex of $\Pi_{n-1}$ by cutting $\lambda$ at the position of "$n$" and omitting "$n$" from the label.

Further, take the points

$$Q_{ij} = (0, ..., 0, 1, 0, ..., 0, -1, 0, 0, ..., 0) \in \mathbb{R}^{n-1}, \text{ and}$$

$$R_i = (1, ..., 1, 2-n, 1, ..., 1, 1, 1) \in \mathbb{R}^{n-1}.$$  

The points yield the line segments

$$q_{ij} = [O, Q_{ij}], \quad r_i = [O, R_i], \quad \text{and the vectors} \quad \overrightarrow{q}_{ij} = O\overrightarrow{Q_{ij}}, \quad \overrightarrow{r}_i = O\overrightarrow{R_i}.$$  

Before we formulate the next lemma, remind that the Minkowski sum of the segments $q_{ij}$ is a translate of $\Pi_{n-1}$. Remind also that for convex polytopes we have not only Minkowski addition, but also a well-defined \textit{Minkowski subtractions}, which leads to the group of \textit{virtual polytopes}, see [1, 9]. Below we encounter \textit{virtual zonotopes}, that is, Minkowski sums of line segments with some of the weights negative.

\textbf{Lemma 3.2.} Whenever $\{v_1, ..., v_k\}$ is the vertex set of a $k$-cell $C$ of the complex $CWM^*(L)$,

1. The images $\psi(v_1), ..., \psi(v_k)$ belong to a $k$-plane in $\mathbb{R}^{n-1}$.
2. The images $\psi(v_1), ..., \psi(v_k)$ are the vertices of a parallel translation of the $k$-dimensional virtual zonotope

$$Z(C) = \sum q_{ij} - \sum r_i,$$

(*)&

where

(a) the first (Minkowski) sum extends over all $i < j < n$ such that $i$ and $j$ belong to one and the same part of $\lambda(C)$, and

(b) the second sum extends over all $i < n$ such that $i$ and $n$ belong to one and the same part of $\lambda(C)$.

(c) The sign "$-$" denotes the Minkowski subtraction.

3. If a cell $C'$ is contained in a cell $C$, the virtual polytope $Z(C')$ is a face of $Z(C)$ (up to a parallel translation).
Proof. (1) Take a $k$-face $C$ of $CWM^*(L)$. Denote by $e(C)$ the linear subspace of $\mathbb{R}^{n-1}$ generated by the affine hull of $\psi(\text{Vert}(C))$. It is linearly spanned by the vectors $\overrightarrow{q}_{ij}$ and $\overrightarrow{r}_i$, where the indices range over the same sets as in $(\ast)$. It is easy to observe that taken together, the vectors span a $k$-dimensional linear space. (2) follows from the coordinate analysis, exactly in the same way as it goes for the faces of permutahedron. (An illustration of "what’s going on" is given in Example 3.5.)

The above lemma makes the following surgery algorithm correct:

(1) Start with the complex $CWM^*(L)$ and the boundary complex of the permutahedron $\Pi_{n-1}$.
(2) Realize the vertices of $CWM^*(L)$ as the vertices of $\Pi_{n-1}$ via the above described mapping $\psi$.
(3) For every face $F$ of $\Pi_{n-1}$ do the following. The face is labeled by some ordered partition of $\{1, ..., n-1\}$. Add the one-element set $\{n\}$ to the partition. If the result is non-admissible, remove the face $F$ from the complex.

This step gives a realization of all the cells of $CWM^*(L)$ whose label $\lambda$ contains the one-element set $\{n\}$.
(4) Take all the cells $C$ of $CWM^*(L)$ such that the part of $\lambda(C)$ containing $n$ has more than one element. Patch in a parallel translation of the above introduced zonotope $Z(C)$ to the set of vertices $\psi(\text{Vert}(C))$.

We get a $(n-2)$-dimensional object. Indeed, although we started by $\mathbb{R}^{n-1}$, the entire construction lives in the affine hull of $\Pi_{n-1}$, that is, in a hyperplane.

**Example 3.3.** Let $L$ be as in Example 2.7. The above described surgery leaves the permutahedron as it is. That is, all the faces of $\Pi_{n-1}$ survive on the third step of the surgery algorithm, and nothing is added on the fourth step.

Important is that the "long" edge is the last one. Otherwise we would get another surgery, but, of course, an isomorphic combinatorics.

**Example 3.4.** Let $n = 5$, $L = (1, 2; 1; 1; 0, 8; 2, 2)$. The surgery algorithm starts with the permutahedron $\Pi_4$ (see Fig. 6). The two shadowed faces are labeled by $\{(123)\{4\}\}$ and $\{(4)\{123\}\}$. Since the partitions $\{(123)\{4\}\{5\}\}$ and $\{(4)\{123\}\{5\}\}$ are non-admissible, according to the algorithm, the faces are removed. All other faces of the permutahedron survive the surgery. Step 4 gives six new "diagonal" rectangular faces. They correspond to the cells labeled by $\{(1)\{2\}\{3\}\{45\}\}$, $\{(1)\{3\}\{2\}\{45\}\}$, $\{(2)\{1\}\{3\}\{45\}\}$, $\{(2)\{3\}\{1\}\{45\}\}$, $\{(3)\{1\}\{2\}\{45\}\}$, and $\{(3)\{2\}\{1\}\{45\}\}$.

**Example 3.5.** Let $n = 5$, $L = (3, 1, 1, 4, 4)$. Figure 4 presents the permutahedron, the labels of the vertices, and the coordinates of the vertices (in bold). We also depict the hexagonal face labeled by $\{(1)\{4\}\{235\}\}$. It is the Minkowski sum of two "negative" and one "positive" segments.
Figure 6. The complex $CWM^*(L)$ for the 5-linkage $L = (1, 2; 1; 1; 0, 8; 2, 2)$. We remove from the permutahedron the grey facets and patch in the blue cylinder.

Figure 7. A "diagonal" face

It remains an open problem to realize $CWM^*(L)$ as a piecewise linear surface with all faces convex.

4. Concluding remarks

Using the above combinatorial description, it is possible to compute the Euler characteristics of $M(L)$, to observe Morse surgery which occurs at "crossing the walls" in the parameter space (see [2]), to establish a connectivity criterion...
for \( M(L) \), and to prove many other (already known) facts. It seems also possible to visualize the generators of the cohomology groups and intersections in the ring \( H^*(M(L)) \).

A similar cell complex exists also for singular configuration spaces, that is, for the case when \( L \) has aligned configurations.

**Definition 4.1.** For a singular case, a partition of \( L = (l_1, \ldots, l_n) \) is called admissible if one of the two conditions holds:

1. The number of the parts is greater than 2, and the total length of any part is strictly greater than the total length of the rest.
2. The number of parts equals 2, and the lengths of the parts are equal.

The combinatorics of the complex \( CWM(L) \) is literally the same as in Theorem 2.4 except for the following items:

1. Non-singular vertices are labeled by admissible partitions with exactly three parts.
2. Singular vertices are labeled by admissible partitions with exactly two parts.
3. Assume that a singular vertex \( v \) of \( CWM(L) \) corresponds to an ordered partition of \( \{1, 2, \ldots, n\} \) into two non-empty parts, say, with \( k \) and \( l \) elements. Then the vertex figure of \( v \) is combinatorially equivalent to the cone over \((\partial \Pi_k \times \partial \Pi_l)^*\).

**References**

[1] G. Ewald, Combinatorial convexity and algebraic geometry. Springer Verlag, 1996.
[2] M. Farber, Invitation to Topological Robotics. Zuerich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zuerich, 2008.
[3] M. Farber and D. Schütz, Homology of planar polygon spaces. Geom. Dedicata, 125 (2007), 75-92.
[4] A. Hatcher, Algebraic topology. Cambridge University Press, 2002.
[5] I. Gelfand, M. Goresky, R. MacPherson, and V. Serganova, Combinatorial geometries, grassmannians, and the moment map. Advances in Math, 63 (1987), 301-316.
[6] M. Kapovich, personal communications.
[7] M. Kapovich and J. Millson, On the moduli space of polygons in the Euclidean plane. J. Diff. Geom., 42 (1995), 430-464.
[8] A. Klyachko, Spatial polygons and stable configurations of points in the projective line. Tikhomirov, Alexander (ed.) et al., Algebraic geometry and its applications, Proceedings of the 8th algebraic geometry conference, Yaroslavl’, Russia, August 10-14, 1992. Braunschweig: Vieweg. Aspects Math. E 25, 67-84 (1994).
[9] A. Pukhlikov and A. Khovanski, Finitely additive measures of virtual polyhedra. St. Petersburg Math. J., 4, 2 (1993), 337-356.
[10] G. Ziegler, Lectures on polytopes. Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.
[11] D. Zvonkine, Configuration spaces of hinge constructions. Russian J. of Math. Phys., 5, 2(1997), 247-266.

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