The Gevrey smoothing effect for the spatially inhomogeneous Boltzmann equations without cut-off

Hua Chen\textsuperscript{1,2,*}, Xin Hu\textsuperscript{1}, Wei-Xi Li\textsuperscript{1,2} & Jinpeng Zhan\textsuperscript{3}

\textsuperscript{1}School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China; \\
\textsuperscript{2}Computational Science Hubei Key Laboratory, Wuhan University, Wuhan 430072, China; \\
\textsuperscript{3}Department of Mathematics, School of Science, Wuhan University of Technology, Wuhan 430070, China \\
Email: chenhua@whu.edu.cn, hux@whu.edu.cn, wei-xi.li@whu.edu.cn, jinpeng@whu.edu.cn

Received May 19, 2021; accepted June 25, 2021; published online November 26, 2021

Abstract In this article we study the Gevrey regularization effect for the spatially inhomogeneous Boltzmann equation without angular cut-off. This equation is partially elliptic in the velocity direction and degenerates in the spatial variable. We consider the nonlinear Cauchy problem for the fluctuation around the Maxwellian distribution and prove that any solution with mild regularity will become smooth in the Gevrey class at positive time with the Gevrey index depending on the angular singularity. Our proof relies on the symbolic calculus for the collision operator and the global subelliptic estimate for the Cauchy problem of the linearized Boltzmann operator.

Keywords Boltzmann equation, Gevrey regularity, subelliptic estimate, non cut-off, symbolic calculus

MSC(2020) 35B65, 35H10, 35Q20

Citation: Chen H, Hu X, Li W-X, et al. The Gevrey smoothing effect for the spatially inhomogeneous Boltzmann equations without cut-off. Sci China Math, 2022, 65: 443–470, https://doi.org/10.1007/s11425-021-1886-9

1 Introduction and the main result

The Cauchy problem for the spatially inhomogeneous Boltzmann equation is

\[ \partial_t F + v \cdot \partial_x F = Q(F, F), \quad F\big|_{t=0} = F_0, \]

where \( F(t, x, v) \) is a probability density function with a given datum \( F_0 \) at \( t = 0 \), and \( x \) and \( v \) stand for the spatial and velocity variables, respectively. We consider here the important physical dimension \( n = 3 \) and suppose both \( x \) and \( v \) vary in the whole space \( \mathbb{R}^3 \). When the density function \( F \) does not depend on the spatial variable \( x \), we get the spatially homogeneous Boltzmann equation

\[ \partial_t F = Q(F, F), \quad F\big|_{t=0} = F_0. \]

The bilinear operator \( Q \) on the right-hand side of (1.1) stands for the collision part acting only on the velocity, so the spatially inhomogeneous Boltzmann equation degenerates in \( x \), which is one of the main

*Corresponding author
Furthermore, we are concerned with singular cross-sections, also called non cut-off sections, i.e., the cross-sections between two interacting molecules, then the cross-section behaves like

\[ B'(v, v') = B(v - v'_*, \sigma) \]  

where \( |v| = |v'| \) and \( v, v' \) stand for the velocities of particles before and after collisions, respectively, with the following momentum and energy conservation rules fulfilled:

\[ v' + v'_* = v + v_* \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \]

From the above relations, we have the so-called \( \sigma \)-representation with \( \sigma \in S^2 \):

\[
\begin{align*}
\sigma' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \\
\sigma'_* &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.
\end{align*}
\]

The cross-section \( B(v - v'_*, \sigma) \) in (1.2) depends on the relative velocity \( |v - v_*| \) and the deviation angle \( \theta \) with

\[ \cos \theta = \langle (v - v'_*)/|v - v_*|, \sigma \rangle. \]

Here, we denote by \( \langle \cdot, \cdot \rangle \) the scalar product in \( \mathbb{R}^3 \). Without loss of generality, we may assume that \( B(v - v'_*, \sigma) \) is supported on the set \( \theta \in [0, \pi/2] \) where \( \langle v - v'_*, \sigma \rangle \geq 0 \), since as usual \( B \) can be replaced by its symmetrized version, and furthermore we treat in this paper the classical case when

\[ B(v - v'_*, \sigma) = |v - v'_*|^7 b(\cos \theta), \]  

(1.3)

where \( \gamma \in [0, 1] \). Recall that \( \gamma = 0 \) is the case of Maxwellian molecules and meanwhile the cases of \( -3 < \gamma < 0 \) and \( 0 < \gamma \leq 1 \) are called, respectively, soft potential and hard potential. In this paper, we will restrict our attention to the cases of Maxwellian molecules and hard potential, i.e., \( 0 \leq \gamma \leq 1 \). Furthermore, we are concerned with singular cross-sections, also called non cut-off sections, i.e., the angular part \( b(\cos \theta) \) has singularity near \( 0 \) so that

\[ \int_0^{\pi/2} \sin \theta b(\cos \theta) d\theta = +\infty. \]

Precisely, we suppose that \( b \) has the following expression near \( \theta = 0 \):

\[ 0 \leq \sin \theta b(\cos \theta) \approx \theta^{-1-2s}; \]  

(1.4)

here and throughout the paper \( p \approx q \) means \( C^{-1} q \leq p \leq C q \) for some constant \( C \geq 1 \). In particular, if the interaction potential obeys the inverse power law \( r^{-\gamma(p-1)} \) for \( 2 < p < +\infty \) with \( r \) the distance between two interacting molecules, then the cross-section behaves like

\[ |v - v'_*|^\gamma \theta^{-2-2s}, \quad -3 < \gamma = \frac{p-5}{p-1} < 1, \quad 0 < s = \frac{1}{p-1} < 1. \]

Note that the cross-sections of the type (1.3) include the potential of the inverse power law as a typical physical model.

It is well understood nowadays that whether or not the angular singularity occurs is closely linked with the regularization effect and the propagation property in time. If the angular collision kernel is integrable (also called Grad’s cut-off assumption), then similar to hyperbolic equations, the singularity or regularity of solutions to the Boltzmann equation usually propagates in time, i.e., the solutions should precisely have the same singularity or regularity as initial data. The understanding of this propagation property has made very substantial developments, and we just mention the results of propagation in the Gevrey
class setting by Desvillettes et al. [25] and Ukai [63], the arguments therein working well for both cut-off and non cut-off cases. The subject of the cut-off Boltzmann equation has a long history and there is vast literature on it, investigating the well-posedness, the propagation property, moments and positivity and so on. For the mathematical treatment of the cut-off Boltzmann equation, we refer to the books of Cercignani [17] and Cercignani et al. [18], for example; more classical references, concerned with the cut-off and non cut-off cases, can be found in the surveys of Alexandre [1] and Villani [65].

When the singularity is involved, the properties herein are quite different from the ones observed in the cut-off case. In fact, the regularization effect occurs for the Cauchy problem of the non cut-off Boltzmann equation, due to diffusion properties caused by the angular singularity. Then the solution should become smooth at positive time like the solutions to the heat equation. The mathematical treatment of the regularization properties goes back to Desvillettes [23,24] for a one-dimensional model of the Boltzmann equation. Later on, Alexandre et al. [2] established the optimal regularity estimate in $v$ for the collision operator after the earlier work of Lions [52], and since then substantial developments have been achieved (see, for example, [3,5,7,8,34,35,59,60] and the references therein). These works show that the Boltzmann operator behaves locally as a fractional Laplacian:

$(-\Delta_v)^s + \text{lower-order terms}$,

and more precisely, from a global perspective the linearized Boltzmann operator around the Maxwellian distribution behaves essentially as

$\langle v \rangle^\gamma(-\Delta_v - (v \wedge \partial_v)^2 + |v|^2)^s + \text{lower-order terms}$,

where $v \wedge \partial_v$ is the cross product of vectors $v$ and $\partial_v$. This diffusion property indicates that the spatially homogeneous equation should behave as the fractional heat equation, and we may expect that solutions to the Cauchy problem will enjoy better regularity at positive time than that of initial data. Strongly related to this regularization effect is another well-known Landau equation, taking into account all the grazing collisions. So far there have been extensive works on the regularity, in a wide variety of different settings, of solutions to the homogeneous Boltzmann equation without angular cut-off (see, for example, [2,10,11,19,20,23,24,26], [27,30,41,50,53,54,56,58,64] and the references therein). We refer to the very recent work of Barbaroux et al. [15], where they proved any weak solution of the fully non-linear homogeneous Boltzmann equation for Maxwellian molecules belongs to the Gevrey class at positive time, and the Gevrey index therein is optimal.

Compared with the homogeneous case, the situation becomes more intricate for the spatially inhomogeneous Boltzmann equation, and much less is known for the regularization properties. The main difficulty lies in the degeneracy in the spatial variable since diffusion only occurs in the velocity, and this is quite different from the spatially homogeneous case where we have elliptic properties for solutions to the Boltzmann equation. Nevertheless we may expect some hypoelliptic effects due to the non trivial interaction between the transport operator and the collision operator. To see this let us first mention the velocity-averaging lemma, which is an important tool for transport equations and is also applied extensively to the study of the Boltzmann equation. The velocity-averaging lemma shows the velocity-averages of solutions to transport equations are smoother in the spatial variable than the distribution function itself (see, for example, the works of Golse et al. [32] and Golse et al. [33]). Other tools from microlocal analysis are also developed for the hypoelliptic properties of the Boltzmann equation in the setting of $L^2$ norm, and we refer the interested readers to the works of Bouchut [16] for the use of Hörmander’s techniques and Alexandre et al. [4] for the application of the uncertainty principle to kinetic equations, as well as the work [3] involving the multiplier method. To understand the intrinsic hypoelliptic structure, Morimoto and Xu [55] first began to study the following simplified Boltzmann model:

$\partial_t + v \partial_x + \sigma_0 (-\Delta_v)^s$,

where $\sigma_0 > 0$ is a constant, and using the analysis of the commutator between the transport part and the diffusion part they obtained the subelliptic estimate in time and space variables (see also [21,48] for the
further improvement on the exponent of the subelliptic estimate). Note that the above operator is just a local model of the Boltzmann equation, inspired by the diffusion property in velocity \( v \) obtained in [2]. Furthermore in the joint work [3] of the third author with Alexandre and Hérau, the global sharp estimate is obtained for the linearized Boltzmann operator rather than for the model operator, by additionally using symbolic calculus for the collisional cross-section (see also [37, 38, 51] for the earlier works on the hypoelliptic properties of other related models). Let us mention that the aforementioned works about hypoellipticity do not involve the initial data, and in fact the time variable \( t \) therein is supposed to vary in the whole space so that Fourier analysis can be applied when the subelliptic estimate is derived in the time variable. In this work we are concerned with the hypoelliptic structure for the Cauchy problem of the Boltzmann equation and thus the initial data will be involved in the analysis.

The well-posedness for general initial data is still a mathematically challenging problem, and so far there are very few results. In 1989, DiPerna and Lions [28] established global renormalized weak solutions in the cut-off case for general initial data without a size restriction, and Alexandre and Villani [12] in 2002 proved the existence of DiPerna-Lions’ renormalized weak solutions in the non cut-off case.

Now we mention the regularity results for the spatially inhomogeneous Boltzmann equation without cut-off. Under a mild regularity assumption on the initial data, the local-in-time existence and uniqueness are obtained by Alexandre et al. [5]. We also mention the earlier work of Ukai [63], where the well-posedness in the anisotropic Gevrey space is established by virtue of the Cauchy-Kovalevskaya theorem. When considering the perturbation around the Maxwellian distribution, the well-posedness in the weighted Sobolev space is obtained independently by Gressman and Strain [34] and Alexandre et al. [6, 8], where the novelty is the introduction of a non isotropic triple norm which enables us to capture the sharp estimate to close the energy. We will explain later the non isotropic triple norm in detail. The DiPerna-Lions’ renormalized solutions are quite weak so the uniqueness is unknown. It is natural to expect higher-order regularity of weak solutions and this still remains a challenging problem up to now. We refer to the very recent works of Golse et al. [31], Imbert and Mouhot [42, 43] and Imbert and Silvestre [45] for the progress on this regularity issue, where Hölder continuity of \( L^\infty \) weak solutions is obtained by using the Harnack inequality and the De Giorgi-Nash-Moser theorem. For the spatially inhomogeneous Landau equation the \( C^\infty \) smoothing of bounded weak solutions is obtained by Henderson and Snelson [36] and Snelson [62], where the pointwise Gaussian upper bound plays a crucial role (see also the recent work of Imbert et al. [44] for an attempt to establish upper bounds for the Boltzmann equation). On the other hand, a long lasting conjecture on the smoothing effect expects better regularity of solutions at positive time than that of initial data and asks furthermore how much better. Under a mild regularity assumption on the initial data, the \( C^\infty \) smoothing effect is obtained by Chen et al. [22] for the inhomogeneous Landau equation and by Alexandre et al. [4, 5] for the Boltzmann equation. In this paper we are concerned with higher-order regularity of mild solutions at positive time, inspired by the Gevrey regularization effect for the fractional heat equation, and our main tool here will be the symbolic calculus developed in [3]. Let us mention the Gelfand-Shilov and Gevrey smoothing effect has been obtained by Lerner et al. [49] for the non cut-off Kac equation, a one-dimensional Boltzmann model. Here, we will further investigate the most physical three-dimensional Boltzmann equation. We hope that the present work may give better insights into the regularity issue of the inhomogeneous Boltzmann equation.

We will restrict our attention to the fluctuation around the Maxwellian distribution. Let $\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$ be the normalized Maxwellian distribution. Write the solution $F$ of (1.1) as $F = \mu + \sqrt{\mu}f$ and accordingly $F_0 = \mu + \sqrt{\mu}f_0$ for the initial datum. Then the fluctuation $f$ satisfies the Cauchy problem

$$\begin{cases}
\partial_t f + v \cdot \partial_v f - \mu^{-\frac{1}{2}}Q(\mu, \sqrt{\mu}f) - \mu^{-\frac{1}{2}}Q(\sqrt{\mu}f, \mu) = \mu^{-\frac{1}{2}}Q(\sqrt{\mu}f, \sqrt{\mu}f), \\
f|_{t=0} = f_0.
\end{cases}$$

(1.5)

We will use throughout the paper the notations as follows. Define by $\mathcal{L}$ the linearized collision operator, i.e.,

$$\mathcal{L} f = \mu^{-1/2}Q(\mu, \sqrt{\mu}f) + \mu^{-1/2}Q(\sqrt{\mu}f, \mu),$$

(1.6)
and define
\[ \Gamma(g, h) = \mu^{-1/2}Q(\sqrt{\mu}g, \sqrt{\mu}h). \]
Furthermore, denote by \( P \) the linearized Boltzmann operator
\[ P = \partial_t + v \cdot \partial_x - \mathcal{L}. \]  
(1.7)
So the Cauchy problem (1.5) for the perturbation \( f \) can be rewritten as
\[ Pf = \Gamma(f, f), \quad f|_{t=0} = f_0. \]  
(1.8)
Note that the global existence in the Sobolev space for the above Cauchy problem is obtained by Alexandre et al. [6,8], taking advantage of a triple norm defined by
\[ \|f\|^2 \overset{\text{def}}{=} \int B(v - v_\star, \sigma)\mu_\star(f - f_\star)^2d\sigma dv_\star + \int B(v - v_\star, \sigma)f_\star^2(\sqrt{\mu} - \sqrt{\mu})^2d\sigma dv_\star, \]  
(1.9)
where the integration is over \( \mathbb{R}_v^3 \times \mathbb{R}_v^3 \times \mathbb{S}^2_\sigma \). We refer to the work of Gressman and Strain [34] for the global existence in the Sobolev space when \( x \) varies in a torus. Denote by \( H^k(\mathbb{R}^6) \) the classical Sobolev space. For any \( \ell \in \mathbb{R} \), define
\[ H^k_\ell(\mathbb{R}^6) = \{ u \in H^k(\mathbb{R}^6); \langle \cdot \rangle^\ell u \in H^k(\mathbb{R}^6) \}; \]
here and throughout the paper we use the notation
\[ \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}. \]

**Theorem 1.1 (The global existence in [6]).** Assume that the cross-section satisfies (1.3) and (1.4) with \( 0 < s < 1 \) and \( \gamma + 2s > 0 \). Suppose the initial datum \( f_0 \in H^k_\ell(\mathbb{R}^6) \) with \( k \geq 6 \) and \( \ell > 3/2 + 2s + \gamma \). Then the Cauchy problem (1.8) admits a global mild solution \( f \in L^\infty([0, +\infty[: H^k_\ell(\mathbb{R}^6)) \), provided \( \|f_0\|_{H^k_\ell(\mathbb{R}^6)} \) is small enough.

In this work we will improve the Sobolev regularity at positive time in the framework of the Gevrey class.

**Definition 1.2.** Let \( \mu \geq 1 \) and we denote by \( G^\mu \) the space of all the \( C^\infty \) functions \( u(x,v) \) satisfying that a constant \( C \) exists such that
\[ \forall \alpha, \beta \in \mathbb{Z}^3_+, \quad \|\partial^\alpha_x \partial^\beta_v u\|_{L^2(\mathbb{R}^6_+,\mu)} \leq C^{(|\alpha| + |\beta| + 1)(|\alpha| + |\beta|)!}\mu. \]
Here, \( \mu \) is called the Gevrey index. We can also define an anisotropic Gevrey space \( G^{\mu_1,\mu_2}; \mu_j \geq 1 \), which consists of all the \( C^\infty \) functions \( u(x,v) \) satisfying that a constant \( C \) exists such that
\[ \forall \alpha, \beta \in \mathbb{Z}^3_+, \quad \|\partial^\alpha_x \partial^\beta_v u\|_{L^2(\mathbb{R}^6_+,\mu)} \leq C^{(|\alpha| + |\beta| + 1)(|\alpha|!)^{\mu_1}(|\beta|!)^{\mu_2}}. \]
Before stating our main result we provide a representation of the triple norm \( \| \cdot \| \) defined by (1.9) in terms of a pseudo-differential operator. Precisely, we have (see Lemma 2.3 in the next section)
\[ \|u\|^2 \approx \|(a^{1/2})^w u\|^2_{L^2(\mathbb{R}^6)}, \]
where \( (a^{1/2})^w \) stands for the Weyl quantization with the symbol \( a^{1/2} \). The definition of \( a^{1/2} \) as well as some basic facts on the symbolic calculus will be given in Subsection 2.1 below and Appendix A.

**Theorem 1.3.** Assume that the cross-section satisfies (1.3) and (1.4) with \( 0 < s < 1 \) and \( \gamma \geq 0 \). Let \( f \in L^\infty([0, +\infty[: H^2(\mathbb{R}^6)) \) be any mild solution to (1.8) such that for any \( 0 < T < +\infty \),
\[ \sup_{t \geq 0} \|f(t)\|_{H^2(\mathbb{R}^6)} + \sum_{|\alpha| + |\beta| \leq 2} \left( \int_0^T \|(a^{1/2})^w \partial^\alpha_x \partial^\beta_v f(t)\|^2_{L^2(\mathbb{R}^6)} dt \right)^{1/2} \leq \epsilon_0 \]  
(1.10)
for some constant $\epsilon_0 > 0$. Suppose that $\epsilon_0$ is small enough. Then $f(t) \in C^{1+2s}_{\text{loc}}$ for all $t > 0$. Moreover, there exists a constant $C \geq 1$ depending only on $s, \gamma$ and $\epsilon_0$ above such that for any multi-indices $\alpha$ and $\beta$ with $|\alpha| + |\beta| \geq 0$,

$$
\sup_{t > 0} \phi(t)^{1+2s(|\alpha|+|\beta|)} \| \partial_x^\alpha \partial_v^\beta f(t) \|_{L^2(\mathbb{R}^6)} \leq C(|\alpha|+|\beta|+1)((|\alpha|+|\beta|)t)^{1+2s}
$$

with $\phi(t) \overset{\text{def}}{=} \min\{t, 1\}$, or equivalently,

$$
\sup_{t > 0} \| e^{\epsilon_0 \phi(t)^{1+2s}(-\Delta_{x,v})^{1+2s/2}} f(t) \|_{L^2(\mathbb{R}^6)} < +\infty
$$

for some constant $\epsilon_0 > 0$.

**Remark 1.4.** Theorem 1.1 guarantees the existence of solutions satisfying the assumption listed in Theorem 1.3, provided the initial datum $f_0 \in H^k_v(\mathbb{R}^6)$ with $k \geq 6$ and $\ell > 3/2 + 2s + \gamma$ and furthermore $\|f_0\|_{H^k_v(\mathbb{R}^6)}$ is small. We believe that it still holds true even if we replace the Sobolev space $H^k_v(\mathbb{R}^6)$ by $H^\ell_v(\mathbb{R}^6)$.

**Remark 1.5.** Theorem 1.3 provides explicit dependence of the Gevrey semi-norms on the short time $0 < t \leq 1$. If the solutions admit additionally some kind of decay for long time, then the dependence on the large time is also variable. Precisely, replacing the estimate (1.10) by the following:

$$
\sup_{t \geq 0} \sum_{|\alpha| + |\beta| \leq 2} \langle t \rangle^{1+2s(|\alpha|+|\beta|)} \| \partial_x^\alpha \partial_v^\beta f(t) \|_{L^2(\mathbb{R}^6)}
$$

$$
+ \sum_{|\alpha| + |\beta| \leq 2} \left( \int_0^{+\infty} \langle t \rangle^{1+2s(|\alpha|+|\beta|)} \| (a^{1/2})^\gamma \partial_x^\alpha \partial_v^\beta f(t) \|_{L^2(\mathbb{R}^6)} dt \right)^{1/2} \leq \epsilon_0,
$$

we can obtain that

$$
\sup_{t > 0} \| e^{ct^{1+2s}(-\Delta_{x,v})^{1+2s/2}} f(t) \|_{L^2(\mathbb{R}^6)} < +\infty
$$

for some constant $\epsilon_0 > 0$, provided the $\epsilon_0$ above is small enough. The argument is quite similar to that for proving Theorem 1.3 with slight modifications, so we leave it to the interested readers. Note that the above estimate means that we have polynomial decay to the equilibrium. The existence of solutions with strong exponential decay is obtained by Gressman and Strain [34].

**Remark 1.6.** The Gevrey index $(1 + 2s)/2s$ is just the same as that obtained by [49] for the Kac equation, the one-dimensional model of the Boltzmann equation. This index is deduced from the sharp subelliptic estimate in the spatial variable. But we do not know whether or not the Gevrey index is optimal. In fact consider the following generalized Kolmogorov equation, which can be seen as a simplified model of (1.8) if ignoring the nonlinear term on the right-hand side:

$$
\begin{aligned}
\partial_t f + v \cdot \nabla_x f + (-\Delta_v)^s f &= 0, \\
|f|_{t=0} &= f_0.
\end{aligned}
$$

A simple application of Fourier analysis shows the solution $f$ to the generalized Kolmogorov equation has an explicit representation and satisfies

$$
e^{c(t(-\Delta_v)^s + \ell^{s+1}(-\Delta_v)^s)} f \in L^2(\mathbb{R}^6)
$$

for some constant $c > 0$. This yields $f \in C^{1/2s}$; we refer to [56] for more detailed analysis on the model equation. So it remains interesting to verify whether or not we can achieve the Gevrey index $1/2s$, which seems to be optimal, in the space-velocity or only velocity variable. We also mention the recent work of Morimoto and Xu [57] where they proved the Landau equation indeed behaves as the heat equation, enjoying the analytic smoothing effect.
Remark 1.7. Here, we consider the Gevrey class regularization of solutions with mild regularity. It is natural to ask what is the minimal regularity required to boot the regularization procedure. It is more interesting to ask the regularization effect of weak solutions satisfying only some kind of physical conditions such as finite mass, energy and entropy. We refer to the recent related work of Duan et al. [29] on the existence of solutions with low regularity.

Remark 1.8. Considered here is a setting of exponential perturbation. The existence theory in a more physical framework of polynomial perturbation was achieved recently by Alonso et al. [13, 14] and Hérau et al. [39]. The hypoelliptic techniques presented here may give insights on the higher-order regularity of these solutions; in fact the regularization for the linear problem has been investigated by Hérau et al. [39].

Notations. If no confusion occurs, we will use \( L^2 \) to stand for the function space \( L^2(\mathbb{R}^6_{v,\eta}) \), and use \( \| \cdot \|_{L^2} \) and \( (\cdot, \cdot)_{L^2} \) to define the norm and the inner product of \( L^2 = L^2(\mathbb{R}^6_{x,v}) \), respectively. We will also use the notations \( \| \cdot \|_{L^2(\mathbb{R}^3^3)} \) and \( (\cdot, \cdot)_{L^2(\mathbb{R}^3^3)} \) when the variables are specified. Similarly we can define \( H^k \) and \( H^k_b \).

Let \( \xi \) and \( \eta \) be the dual variables of \( x \) and \( v \), respectively. We denote by \( \hat{u}(\xi) \) the (partial) Fourier transform in variable \( x \) and denote by \( q(D_v) \) a Fourier multiplier in variable \( x \) with the symbol \( q(\xi) \), i.e.,

\[
q(D_v)u(\xi) = q(\xi)\hat{u}(\xi).
\]

Similarly we can define \( q(D_v) \), a Fourier multiplier in variable \( v \) with the symbol \( q(\eta) \). In particular, let \( (D_v)^\tau \) be the Fourier multiplier with the symbol \( \langle \xi \rangle^\tau \), recalling \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \). Similarly, we can define \( (D_v)^\tau \). Given a function \( p(x, v; \xi, \eta) \), we denote by \( p^w \) and \( p^{\text{Wick}} \) the Weyl and Wick quantizations, respectively, with the symbol \( (v, \eta) \rightarrow p(x, v; \xi, \eta) \), considering \( x \) and \( \xi \) as parameters. The precise definitions of Weyl and Wick quantizations are given in Appendix A.

Given two operators \( T_1 \) and \( T_2 \), we denote by \([T_1, T_2]\) the commutator between \( T_1 \) and \( T_2 \), i.e.,

\[
[T_1, T_2] = T_1T_2 - T_2T_1.
\]

We say that \( T_1 \) commutes with \( T_2 \) if \([T_1, T_2] = 0\).

2 The subelliptic estimate for the Cauchy problem

Let \( P \) be the linearized Boltzmann operator given in (1.7). In this part, we will derive a subelliptic estimate for the linear Cauchy problem of Boltzmann operator \( P \). To do so we need the symbolic calculus developed in [3].

2.1 Some facts on the symbolic calculus

Here, we recall without proof some facts on the symbolic calculus obtained in [3] for the collision operator \( \mathcal{L} \) defined in (1.6). To do so we introduce some notations for the phase space analysis and list in Appendix A the basic properties of the quantization of symbols, and we refer to [47] for the comprehensive discussion.

Let \( \eta \in \mathbb{R}^3 \) be the dual variable of velocity \( v \), and throughout the paper we let \( \tilde{a} \) be defined by

\[
\tilde{a}(v, \eta) = (v)\gamma(1 + |\eta|^2 + |v \wedge \eta|^2 + |v|^2)^s, \quad (v, \eta) \in \mathbb{R}^6,
\]

(2.1)

where \( \gamma \) and \( s \) are the numbers given in (1.3) and (1.4), respectively, and \( v \wedge \eta \) stands for the cross product of two vectors \( v \) and \( \eta \). Direct computation shows that \( \tilde{a} \) is an admissible weight for the flat metric \( |dv|^2 + |d\eta|^2 \) (see [3] for the verification in detail), and we can consider the Weyl quantization \( p^w \) and the Wick quantization \( p^{\text{Wick}} \) of a symbol \( p \) lying in the class \( S(\tilde{a}, |dv|^2 + |d\eta|^2) \) (see Appendix A for the definitions of admissible weights, the symbol class and its quantization).
Proposition 2.1 (See [3, Proposition 1.4 and Lemma 4.3]). Assume that the cross-section satisfies (1.3) and (1.4) with $0 < s < 1$ and $\gamma > -3$. Let $L$ be the linearized collision operator defined by (1.6). Then we can write

$$L = -a^w - R,$$

where $a^w$ stands for the Weyl quantization of the symbol $a$ with the properties listed below fulfilled.

(i) We have $a, a \in S(\mathbb{R}^3, |dv|^2 + |d\eta|^2)$, and moreover there exists a positive constant $C \geq 1$ such that $C^{-1}a(v, \eta) \leq a(v, \eta) \leq Ca(v, \eta)$ for all $(v, \eta) \in \mathbb{R}^6$.

(ii) For any $\epsilon > 0$, there exists a constant $C_\epsilon$ such that

$$\forall f \in S(\mathbb{R}^3), \quad \|Rf\| \leq \epsilon \|a^w f + C_\epsilon \|v\|^\gamma \|s\|^2 f;$$

here and below $S(\mathbb{R}^3)$ stands for the Schwartz space in $\mathbb{R}^3$.

(iii) The operators $a^w$ and $(a^{1/2})^w$ are invertible in $L^2$ and their inverses can be written as

$$(a^w)^{-1} = H_1(a^{-1})^w = (a^{-1})^w H_2$$

and

$$[(a^{1/2})^w]^{-1} = G_1(a^{-1/2})^w = (a^{-1/2})^w G_2$$

with $H_1$ and $G_1$ the bounded operators in $L^2$.

**Remark 2.2.** To simplify the notation, we write $a$ here instead of $a_K$ defined in [3, Proposition 1.4] with $K$ a large positive number. Accordingly $R = K - K(v)^{2s+\gamma}$ with $K$ given in [3, Proposition 1.4] or defined precisely in [3, Equation (53)].

The symbolic calculus above enables us to get the exact diffusion property of the collision operator and provides the following representation of the triple norm defined by (1.9) in terms of $(a^{1/2})^w$.

**Lemma 2.3.** Assume that the cross-section satisfies (1.3) and (1.4) with $0 < s < 1$ and $\gamma \geq 0$. Then for all $l \in \mathbb{R}$ with $l \leq \gamma/2 + s$ and for any $u \in S(\mathbb{R}^3)$, we have

$$\|u\|^2 \approx -(Lu, u)_{L^2(\mathbb{R}^3)} + \|v\|^2 u_{L^2(\mathbb{R}^3)} \approx \|(a^{1/2})^w u\|^2_{L^2(\mathbb{R}^3)},$$

where in the first equivalence the constant depends only on $l$.

**Proof.** The first equivalence is obtained by [8, Proposition 2.1] and the second one follows from [3, Lemmas 4.7 and 4.9].

As a result of Lemma 2.3, the following upper bound for the trilinear term (see [9, Theorem 1.2])

$$\forall g, h, \psi \in S(\mathbb{R}^3), \quad |(\Gamma(g, h), \psi)_{L^2(\mathbb{R}^3)}| \leq C \|g\|_{L^2(\mathbb{R}^3)} \|h\|_{L^2(\mathbb{R}^3)} \|\psi\|_{L^2(\mathbb{R}^3)}$$

can be re-written as

$$\forall g, h, \psi \in S(\mathbb{R}^3), \quad |(\Gamma(g, h), \psi)_{L^2(\mathbb{R}^3)}| \leq C \|g\|_{L^2(\mathbb{R}^3)} \|(a^{1/2})^w h\|_{L^2(\mathbb{R}^3)} \|(a^{1/2})^w \psi\|_{L^2(\mathbb{R}^3)}.$$

Note that it follows from the above estimate as well as the density argument that for any $g, h, \psi \in L^2_v$ with $(a^{1/2})^w h, (a^{1/2})^w \psi \in L^2_v$, we have

$$\left| \int_{\mathbb{R}^3} \Gamma(g, h, \psi) dv \right| \leq |(a^{1/2})^w| \Gamma(g, h, (a^{1/2})^w \psi)_{L^2(\mathbb{R}^3)}| \leq C \|g\|_{L^2(\mathbb{R}^3)} \|(a^{1/2})^w h\|_{L^2(\mathbb{R}^3)} \|(a^{1/2})^w \psi\|_{L^2(\mathbb{R}^3)}.$$

This gives that for any $g, h, \psi \in H^2_v L^2_v$ with $(a^{1/2})^w h, (a^{1/2})^w \psi \in H^2_v L^2_v$,

$$|(\Gamma(g, h), \psi)_{L^2}| \leq C \sum_{|\beta| \leq 2} \|\partial^\beta_x g\|_{L^2} \|(a^{1/2})^w h\|_{L^2} \|(a^{1/2})^w \psi\|_{L^2},$$

$$|(\Gamma(g, h), \psi)_{L^2}| \leq C \|g\|_{L^2} \left( \sum_{|\beta| \leq 2} \|(a^{1/2})^w \partial^\beta_x h\|_{L^2} \right) \|(a^{1/2})^w \psi\|_{L^2}.$$
and
\[
|\langle \Gamma(g, h), \psi \rangle|_{L^2} \leq C \|g\|_{L^2}(a^{1/2})^w h\|_{L^2}(\sum_{|\beta| \leq 2} \|\partial_\beta^2 (a^{1/2})^w \psi\|_{L^2})
\]  
(2.5)
by recalling \( L^2 = L^2(\mathbb{R}^6) \). Moreover, we have the following inequality: for any \( u, w \in H^1(\mathbb{R}^3) \),
\[
\|uw\|_{L^2(\mathbb{R}^3)} \leq C\left( \sum_{|\beta| \leq 1} \|\partial_\beta^2 u\|_{L^2(\mathbb{R}^3)} \right) \sum_{|\beta| \leq 1} \|\partial_\beta^2 w\|_{L^2(\mathbb{R}^3)}.
\]  
(2.6)
To see this, we observe
\[
\|uw\|_{L^2(\mathbb{R}^3)} \leq \|u\|_{L^4(\mathbb{R}^3)}\|w\|_{L^4(\mathbb{R}^3)}
\]
and the Gagliardo-Nirenberg-Sobolev inequality gives
\[
\|u\|_{L^4(\mathbb{R}^3)} \leq \|u\|_{L^2(\mathbb{R}^3)}^{1/4}\|u\|_{L^6(\mathbb{R}^3)}^{3/4} \leq C\|u\|_{L^2(\mathbb{R}^3)}^{1/4}\|\partial_x u\|_{L^2(\mathbb{R}^3)}^{3/4} \leq C \sum_{|\beta| \leq 1} \|\partial_\beta^2 u\|_{L^2(\mathbb{R}^3)}.
\]
It is similar to the treatment of \( \|w\|_{L^4(\mathbb{R}^3)} \). Then (2.6) follows. Combining (2.6) and (2.2) implies
\[
|\langle \Gamma(g, h), \psi \rangle|_{L^2} \leq C\left( \sum_{|\beta| \leq 1} \|\partial_\beta^2 g\|_{L^2} \right) \|(a^{1/2})^w h\|_{L^2}\left( \sum_{|\beta| \leq 1} \|\partial_\beta^2 (a^{1/2})^w \psi\|_{L^2} \right).
\]  
(2.7)

2.2 The subelliptic estimate for the Cauchy problem of the linear Boltzmann equation

The non isotropic triple norm given in the previous subsection is not enough for the Gevrey regularity, since we cannot get any regularity in the spatial variable \( x \). When the time varies in the whole space, the sharp regularity in all the variables is obtained by using the Fourier transform in the time and space variables [3]. Here, we will derive a subelliptic estimate involving the initial data, following the multiplier method in [3].

**Proposition 2.4** (The elliptic estimate in velocity). Assume that the cross-section satisfies (1.3) and (1.4) with \( 0 < s < 1 \) and \( \gamma \geq 0 \). Then there exists a constant \( C \geq 1 \) such that for any given \( r \geq 1 \) and any function \( u \in L^2 \) satisfying that \( Pu \in L^2([0, 1] \times \mathbb{R}^6) \) and that
\[
t^r Pu(t) \in L^\infty([0, 1]; L^2) \quad \text{and} \quad t^{r(1/2)} u(t) \in L^2([0, 1] \times \mathbb{R}^6),
\]
we have that for any \( 0 < t \leq 1 \),
\[
t^{2r}\|u(t)\|_{L^2}^2 + \int_0^1 t^{2r}\|(a^{1/2})^w u(t)\|_{L^2}^2 dt \leq C \int_0^1 t^{2r}\|(Pu, u)_{L^2}\| dt + rC \int_0^1 t^{2r-1}\|u\|_{L^2}^2 dt.
\]

**Proof.** By density we may assume that \( u \) is rapidly decreasing on \( \mathbb{R}^6 \). Using the second equivalence in Lemma 2.3 as well as the fact that
\[
-(Lu, u)_{L^2} = Re(Pu, u)_{L^2} - \frac{1}{2} \frac{d}{dt}\|u\|_{L^2}^2,
\]
we conclude a small constant \( 0 < c_1 < 1 \) exists such that
\[
\frac{1}{2} \frac{d}{dt}\|u\|_{L^2}^2 + c_1\|(a^{1/2})^w u\|_{L^2}^2 \leq |(Pu, u)_{L^2}| + \|u\|_{L^2}^2.
\]  
(2.8)
Thus for any \( 0 < t \leq 1 \),
\[
\frac{1}{2} \frac{d}{dt}(t^{2r}\|u\|_{L^2}^2) + c_1 t^{2r}\|(a^{1/2})^w u\|_{L^2}^2 \leq t^{2r}|(Pu, u)_{L^2}| + t^{2r}\|u\|_{L^2}^2 + rt^{2r-1}\|u\|_{L^2}^2.
\]
Integrating both sides over the interval \([0, t]\) with any \( 0 < t \leq 1 \) and observing \( t^{r-1/2} u \in L^\infty([0, 1]; L^2) \) which implies
\[
\lim_{t \to 0} t^{2r}\|u\|_{L^2}^2 = 0,
\]
we obtain the estimate as desired for variable \( v \). The proof is completed. \( \square \)
Proposition 2.5 (The subelliptic estimate in space). Assume that the cross-section satisfies (1.3) and (1.4) with $0 < s < 1$ and $\gamma \geq 0$. Then we can find a constant $C \geq 1$ and a bounded operator $A$ in $L^2$ with the following properties:

$$\forall u \in L^2 \text{ with } (a^{1/2})^w u \in L^2, \quad \begin{cases} \|Au\|_{L^2} \leq C_{s, \gamma} \|u\|_{L^2}, \\ |[A, q(D_x)]| = 0, \\ \|[A, (a^{1/2})^w u]\|_{L^2} \leq C_{s, \gamma} \|(a^{1/2})^w u\|_{L^2}, \end{cases}$$

(2.9)

fulfilled for some constant $C_{s, \gamma}$ depending only on $s$ and $\gamma$ and for any Fourier multiplier $q(D_x)$ in only variable $x$ such that for any given $r \geq 1$, the following two estimates hold:

(i) For any function $u \in L^2$ satisfying that $Pu \in L^2([0, 1] \times \mathbb{R}^6)$ and that

$$t^{-\frac{1}{2}} u \in L^\infty([0, 1]; L^2) \quad \text{and} \quad t^r \langle D_x \rangle^{-r} u, t^r (a^{1/2})^w u \in L^2([0, 1] \times \mathbb{R}^6),$$

we have that for any $0 < t \leq 1$,

$$t^{2r} \|u(t)\|_{L^2}^2 + \int_0^1 t^{2r} \|t^r \langle D_x \rangle^{-r} u(t)\|_{L^2}^2 dt + \int_0^1 t^{2r} \|(a^{1/2})^w u(t)\|_{L^2}^2 dt \leq C \int_0^1 t^{2r} \|(Pu, u)_{L^2}\| dt + C \int_0^1 t^{2r} \|(Pu, Au)_{L^2}\| dt + r C \int_0^1 t^{2r-1} \|u\|_{L^2}^2 dt. \quad (2.10)$$

(ii) For any function $u$ satisfying that $Pu \in L^2([1, +\infty] \times \mathbb{R}^6)$ and that

$$u \in L^\infty([1, +\infty]; L^2) \quad \text{and} \quad \langle D_x \rangle^{-r} u, (a^{1/2})^w u \in L^2([1, +\infty] \times \mathbb{R}^6),$$

we have that for any $t \geq 1$,

$$\|u(t)\|_{L^2}^2 + \int_1^\infty \|\langle D_x \rangle^{-r} u(t)\|_{L^2}^2 dt + \int_1^\infty \|(a^{1/2})^w u(t)\|_{L^2}^2 dt \leq \|u(1)\|_{L^2}^2 + C \int_1^\infty \|(Pu, u)_{L^2}\| dt + C \int_1^\infty \|(Pu, Au)_{L^2}\| dt + C \int_1^\infty \|u\|_{L^2}^2 dt. \quad (2.11)$$

Note that the constant $C$ in (2.10) is independent of $r$.

Proof. We adopt the idea used for proving [3, Lemma 4.12]. Let $u$ be an arbitrarily given function satisfying the assumption above. By density we may assume that $u$ is rapidly decreasing on $\mathbb{R}^6$. Recall that $\xi$ is the dual variable of $x$ and $\tilde{u}(\xi, v)$ is the partial Fourier transform of $u(x, v)$ with respect to $x$. Then we have

$$\tilde{F}_u(t, \xi, v) = (\partial_t + iv \cdot \xi - \mathcal{L})\tilde{u}(t, \xi, v).$$

Let $\lambda^{\text{Wick}}$ be the Wick quantization (see Appendix A for the definition of the Wick quantization) of the symbol $\lambda$ with

$$\lambda(v, \eta) = \lambda_\xi(v, \eta) = \frac{d(v, \eta)}{\tilde{a}(v, \xi)} \chi \left( \frac{\tilde{a}(v, \eta)}{\tilde{a}(v, \xi)} \right),$$

where $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ on $[-1, 1]$ and supp$\chi \subset [-2, 2]$ and

$$d(v, \eta) = \langle v \rangle^2 (1 + |v|^2 + |\xi|^2 + |v \wedge \xi|^2)^{s-1} (\xi \cdot \eta + (v \wedge \xi) \cdot (v \wedge \eta)).$$

Recall that $\tilde{a}$ is defined in (2.1). Direct computation shows

$$\lambda \in S(1, |dv|^2 + |d\eta|^2)$$

uniformly with respect to $\xi$. As a result, $\lambda^{\text{Wick}}$ is a bounded operator in $L^2$:

$$\forall u \in L^2, \quad \|\lambda^{\text{Wick}} u\|_{L^2} \leq C_{\gamma, s} \|u\|_{L^2} \quad (2.12)$$
for some constant $C_{s,\gamma}$ depending only on $s$ and $\gamma$. The advantage of $\lambda^{\text{Wick}}$ lies in the fact that the interaction between $\lambda^{\text{Wick}}$ and the transport part will yield the regularity in $x$. Precisely, observing $v \cdot \xi = (v \cdot \xi)^{\text{Wick}}$ with $\xi$ a parameter, we use the relationship (A.2) in Appendix A to get

$$\text{Re}(i(v \cdot \xi)\hat{u}, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} = (\{\lambda, v \cdot \xi\}^{\text{Wick}}\hat{u}, \hat{u})_{L^2(\mathbb{R}_x^3)}$$

with $\{\cdot, \cdot\}$ the Poisson bracket defined by (A.3). Moreover, using the positivity property of the Wick quantization (see Appendix A.2) yields

$$\{\lambda, v \cdot \xi\}^{\text{Wick}}\hat{u}, \hat{u})_{L^2(\mathbb{R}_x^3)} \geq C_2 \|\xi\|_{L^2(\mathbb{R}_x^3)}^2$$

here and throughout the proof $0 < c_2 < 1$ and we use $C_j$ $(j \geq 1)$ to define different constants depending on $s$ and $\gamma$ (see [3, Lemma 4.13] for proving the above inequalities in detail). Combining these estimates and using the fact that $\gamma \geq 0$, we conclude

$$c_2 \|\xi\|_{L^2(\mathbb{R}_x^3)}^2 \leq \text{Re}(i(v \cdot \xi)\hat{u}, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} + C_2 \|(a^{1/2})^w\hat{u}\|_{L^2(\mathbb{R}_x^3)}.$$  \hspace{1cm} (2.13)

As for the first term on the right-hand side, we have

$$\text{Re}(i(v \cdot \xi)\hat{u}, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} = \text{Re}(\hat{P}_u, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} - \text{Re}(\hat{\partial}_u\hat{u}, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} + \text{Re}(\hat{L}\hat{u}, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)}$$

the last line holding because $\lambda^{\text{Wick}}$ is self-adjoint in $L^2(\mathbb{R}_x^3)$. Moreover, observing $\hat{L}\hat{u} = \Gamma(\mu^{1/2}, \hat{u}) + \Gamma(\mu^{1/2})$ and then using (2.2) as well as the assertions (i) and (iii) in Proposition 2.1, we obtain

$$\text{Re}(\hat{L}\hat{u}, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} = |\text{Re}([\lambda^{1/2}]\hat{u}, a^{1/2}\lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)}| \leq C_3 \|(a^{1/2})^w\hat{u}\|_{L^2(\mathbb{R}_x^3)}.$$

Combining the above estimate with (2.13), we obtain

$$c_2 \|\xi\|_{L^2(\mathbb{R}_x^3)}^2 \leq \text{Re}(\hat{P}_u, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} - \frac{d}{dt} \|\hat{u}, \lambda^{\text{Wick}}\hat{u})_{L^2(\mathbb{R}_x^3)} + C_4 \|(a^{1/2})^w\hat{u}\|_{L^2(\mathbb{R}_x^3)}^2. $$

Define the operator $\mathcal{A}$ by

$$\mathcal{A}\xi(v) = \lambda^{\text{Wick}}\hat{u}(\xi, v).$$

Then it follows from the above inequality and the Plancherel formula that

$$c_2 \|\langle D_x \rangle^{\frac{1}{2}} u\|_{L^2}^2 \leq \|(P_u, A_u)\|_L^2 - \frac{d}{dt} \|u, A_u\|_{L^2} + C_4 \|(a^{1/2})^w u\|_{L^2}^2$$  \hspace{1cm} (2.14)

and

$$\|Au\|_{L^2} = \|\lambda^{\text{Wick}}\hat{u}\|_{L^2(\mathbb{R}_x^3)} \leq C_{s,\gamma} \|\hat{u}\|_{L^2(\mathbb{R}_x^3)} \leq C_{s,\gamma} \|u\|_{L^2}. $$  \hspace{1cm} (2.15)

Now we choose an $N$ such that

$$N = \max \left\{2C_4/c_1, 2C_{s,\gamma}\right\} + 4$$  \hspace{1cm} (2.16)

with $C_4$ given in (2.14), $c_1$ the number in (2.8) and $C_{s,\gamma}$ the constant in (2.12). Then we multiply both sides of (2.8) by $N$ and then add the result to (2.14); this gives

$$\frac{N d}{dt} \|u\|_{L^2}^2 + c_2 \|\langle D_x \rangle^{\frac{1}{2}} u\|_{L^2}^2 + \frac{Nc_1}{2} \|(a^{1/2})^w u\|_{L^2}^2 \leq N\|(P_u, u)\|_{L^2} + \|(P_u, A_u)\|_{L^2} - \frac{d}{dt} \|u, A_u\|_{L^2} + N\|u\|_{L^2}^2,$$  \hspace{1cm} (2.17)
and thus
\[
\frac{N}{2} \frac{d}{dt} \left( t^{2r} \| u \|_{L^2}^2 \right) + c_2 t^{2r} \| (D_x + \lambda) \frac{t^{2r}}{2} u \|_{L^2}^2 + \frac{N c_1}{2} t^{2r} (a^{1/2})^w u \|_{L^2}^2
\]
\[
\leq N t^{2r} \| (Pu, u)_{L^2} \| + t^{2r} \| (Pu, Au)_{L^2} \| - \frac{1}{2} \frac{d}{dt} t^{2r} (u, Au)_{L^2} + N t^{2r} \| u \|_{L^2}^2
\]
\[
+ N t^{2r-1} \| u \|_{L^2}^2 + rt^{2r-1} (u, Au)_{L^2}.
\]
Integrating both sides over the interval \([0, t]\) for any \(0 < t \leq 1\) and observing \(t^{r-1/2} u \in L^\infty([0, 1]; L^2)\) which along with (2.15) implies
\[
0 \leq \lim_{t \to 0} t^{2r} \| (u, Au)_{L^2} \| \leq C_{\gamma, s} \lim_{t \to 0} t^{2r} \| u \|_{L^2}^2 = 0,
\]
we obtain, using (2.15) again,
\[
\frac{N}{4} \int_0^t t^{2r} \| u(t) \|_{L^2}^2 \, dt + c_2 \int_0^t t^{2r} \| (D_x + \lambda) \frac{t^{2r}}{2} u(t) \|_{L^2}^2 \, dt + \frac{N c_1}{2} \int_0^t t^{2r} (a^{1/2})^w u(t) \|_{L^2}^2 \, dt
\]
\[
\leq N \int_0^t t^{2r} \| (Pu, u)_{L^2} \| \, dt + \int_0^t t^{2r} \| (Pu, Au)_{L^2} \| \, dt + \frac{C_{\gamma, s}}{2} t^{2r} \| u(t) \|_{L^2}^2 \, dt
\]
\[
+ N \int_0^t t^{2r} \| u \|_{L^2}^2 \, dt + (N + C_{\gamma, s}) r \int_0^t t^{2r-1} \| u \|_{L^2}^2 \, dt
\]
for any \(0 < t \leq 1\). Thus, by observing \(r \geq 1\) and \(C_{\gamma, s} \leq N/2\) due to (2.16),
\[
\frac{N}{4} \int_0^1 t^{2r} \| u \|_{L^2}^2 \, dt + c_2 \int_0^1 t^{2r} \| (D_x + \lambda) \frac{t^{2r}}{2} u(t) \|_{L^2}^2 \, dt + \frac{N c_1}{2} \int_0^1 t^{2r} (a^{1/2})^w u(t) \|_{L^2}^2 \, dt
\]
\[
\leq N \int_0^1 t^{2r} \| (Pu, u)_{L^2} \| \, dt + \int_0^1 t^{2r} \| (Pu, Au)_{L^2} \| \, dt + 3 N r \int_0^1 t^{2r-1} \| u \|_{L^2}^2 \, dt.
\]
The above inequality holds for all \(0 < t \leq 1\). Thus the desired (2.10) follows if we choose \(C = 24 \max\{\frac{1}{c_1}, \frac{N}{2}\}\). Similarly integrating (2.17) over \([1, t]\) for any \(t > 1\), we obtain the estimate (2.11).

It remains to prove the assertions in (2.9), and the first one follows from (2.15). The second assertion in (2.9) is obvious since the spatial variable \(x\) is not involved in the symbol \(\lambda\). To prove the last assertion, we only need work with the \(L^2(\mathbb{R}^6_{\xi, v})\)-norm by the Plancherel formula. The symbol of the commutator
\[
[\lambda \text{Wick}, (a^{1/2})^w]
\]
belongs to \(S(\tilde{a}^{1/2}, |dv|^2 + |d\eta|^2)\) since \(\lambda \in S(1, |dv|^2 + |d\eta|^2)\) uniformly for \(\xi\) and \(a^{1/2} \in S(\tilde{a}^{1/2}, |dv|^2 + |d\eta|^2)\).

As a result, we can write
\[
[\lambda \text{Wick}, (a^{1/2})^w] = \left[\lambda \text{Wick}, (a^{1/2})^w]\right]\left((a^{1/2})^w\right)^{-1} (a^{1/2})^w
\]
da bounded operator
due to the conclusions (i) and (iii) in Proposition 2.1, and thus the third assertion in (2.9) follows. The proof is then completed.

\[\square\]

### 3 Gevrey regularity in the spatial variable

This part is devoted to proving the Gevrey smoothing effect in the spatial variable \(x\), i.e., we have the following theorem.

**Theorem 3.1.** Under the same assumption as in Theorem 1.3, we can find a positive constant \(C_0\) depending only on \(s, \gamma\) and the constant \(c_0\) in (1.10) such that
\[
\forall \alpha \geq 0, \quad \sup_{t > 0} \phi(t) \frac{d^{2r} \| u \|_{L^2}^2}{dt} \leq C_0^{\alpha+1} (|\alpha|!)^{\frac{2r}{2s}}.
\]
Recall \(\phi(t) = \min \{t, 1\} \).
We will use induction to prove the above theorem, and the following proposition is crucial.

**Proposition 3.2.** Define $\kappa = (1 + 2s)/2s$. Let $f \in L^\infty([0, +\infty[; H^2)$ be any solution to the Cauchy problem (1.8) satisfying the condition (1.10). In addition, suppose that there exists a positive constant $C_s \geq 1$ depending only on $s$, and the constant $\epsilon_0$ in (1.10) such that for any multi-index $\beta$ with $2 \leq |\beta| \leq 3$, we have

$$
\begin{align*}
\sup_{t > 0} \phi(\kappa(|\beta|-2)\partial_x^2 f(t))_L^2 + \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\beta|-2)} \| (D_x) \frac{\partial^\beta}{\partial t^\beta} f(t) \|_{L^2}^2 dt \right)^{1/2} \\
+ \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\beta|-2)} \| (a^{1/2})^w \partial_x^3 f(t) \|_{L^2}^2 dt \right)^{1/2} \leq C_s.
\end{align*}
$$

(3.1)

Let $m \geq 5$ be an arbitrarily given integer. Then we can find a positive constant $C_0 \geq C_s$, depending only on $s$, and the constant $\epsilon_0$ in (1.10) but independent of $m$, such that if the following estimate

$$
\begin{align*}
\sup_{t > 0} \phi(t)^{\kappa(|\beta|-2)} \| \partial_x^m f(t) \|_{L^2} + \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\beta|-2)} \| (D_x) \frac{\partial^\beta}{\partial t^\beta} f(t) \|_{L^2}^2 dt \right)^{1/2} \\
+ \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\beta|-2)} \| (a^{1/2})^w \partial_x^3 f(t) \|_{L^2}^2 dt \right)^{1/2} \leq C_0^{[\kappa(|\beta|-4)|]^{1/2}}
\end{align*}
$$

(3.2)

holds for any $\beta$ with $4 \leq |\beta| \leq m - 1$, then for any multi-index $\alpha$ with $|\alpha| = m$, we have $\phi(t)^{\kappa(m-2)} \partial_x^\alpha f \in L^\infty([0, +\infty[; L^2)$ and

$$
\phi(t)^{\kappa(m-2)} (D_x \frac{\partial^\alpha}{\partial x^\alpha} f(t)) \in L^2([0, +\infty[ \times \mathbb{R}^d_{x,v}),
$$

and moreover,

$$
\begin{align*}
\sup_{t > 0} \phi(t)^{\kappa(m-2)} \| \partial_x^m f(t) \|_{L^2} + \left( \int_0^{+\infty} \phi(t)^{2\kappa(m-2)} \| (D_x) \frac{\partial^\beta}{\partial t^\beta} f(t) \|_{L^2}^2 dt \right)^{1/2} \\
+ \left( \int_0^{+\infty} \phi(t)^{2\kappa(m-2)} \| (a^{1/2})^w \partial_x^3 f(t) \|_{L^2}^2 dt \right)^{1/2} \leq C_0^{m-3([\kappa(|\beta|-4)|]^{1/2}}
\end{align*}
$$

(3.3)

Before proving the above proposition, we first state the interpolation inequality in the Sobolev space which is to be used frequently. Given three numbers $r_j$ with $r_1 < r_2 < r_3$, we have

$$
\forall \varepsilon > 0, \quad \forall u \in H^{r_3}, \quad \| (D_x)^{r_2} u \|_{L^2} \leq \varepsilon \| (D_x)^{r_3} u \|_{L^2} + \varepsilon^{-\frac{r_3 - r_2}{r_3}} \| (D_x)^{r_1} u \|_{L^2}.
$$

(3.4)

**Proof of Proposition 3.2 (The case of $0 < t \leq 1$).** We first consider the case where $t \in [0, 1]$, and in this part we will prove the fact that $t^{\kappa(m-2)} \partial_x^m f \in L^\infty([0, 1]; L^2)$ and

$$
t^{\kappa(m-2)} (D_x \frac{\partial^\alpha}{\partial x^\alpha} f(t)) \in L^2([0, 1] \times \mathbb{R}^d_{x,v}),
$$

for any $\alpha$ with $|\alpha| = m$, and moreover the norms of these quantities are controlled by the right-hand side of (3.3).

To do so we define the regularization $f_\delta$ of $f$ with $0 < \delta \leq 1$ by setting

$$
f_\delta = \Lambda_\delta^{-2} f, \quad \Lambda_\delta = (1 + \delta |D_x|)^{1/2}.
$$

(3.5)

Note that $\Lambda_\delta$ is just the Fourier multiplier with the symbol $(1 + \delta |\xi|^2)^{1/2}$. We have $[T, \Lambda_\delta^{-2}] = 0$ for any operator $T$ acting only on $v$, and $\Lambda_\delta^{-2}$ is uniformly bounded in $L^2$ for $\delta$:

$$
\forall u \in L^2, \quad \| \Lambda_\delta^{-2} u \|_{L^2} \leq \| u \|_{L^2}.
$$

Observe that the Fourier multiplier $\Lambda_\delta^{-2}$ is a bounded operator from $L^2(\mathbb{R}^d_x) \to H^2(\mathbb{R}^d_x)$ with the norm depending on $\delta$. As a result, it follows from the assumption (3.2) that

$$
\begin{align*}
\left\{ \begin{array}{l}
t^{\kappa(m-2)-1/2} \frac{\partial^m}{\partial t^m} f_\delta \in L^\infty([0, 1]; L^2), \\
t^{\kappa(m-2)} (D_x \frac{\partial^\alpha}{\partial x^\alpha} f_\delta) \in L^2([0, 1] \times \mathbb{R}^d).
\end{array} \right.
\end{align*}
$$

(3.6)
To simplify the notation we will use $C$ in the following discussion to denote different suitable constants which depend only on $s, \gamma$ and the constant $\epsilon_0$ in (1.10), but are independent of $m$ and the number $\delta$ in the Fourier multiplier $\Lambda_\delta^{-2}$, and moreover denote by $C_s$ different constants depending on $\epsilon$ additionally.

**Step 1** (The upper bound for the trilinear terms). Recall that $A$ is the bounded operator given in Proposition 2.5 with the properties in (2.9) fulfilled and $\epsilon_0$ is the number in (1.10). Let $f_\delta$ be the regularization of $f$ given by (3.5). In this step, we will show that for any $\epsilon > 0$,

$$
\int_0^1 \int 2^{2(m-2)}|(|P\partial_x^m f_\delta, \partial_x^m f_\delta)|_L^1|dt + \int_0^1 \int 2^{2(m-2)}|(|P\partial_x^m f_\delta, A\partial_x^m f_\delta)|_L^1|dt \\
\leq (\epsilon + \epsilon_0) \int_0^1 \int 2^{2(m-2)}|| \int (a^{1/2})^w \partial_x^m f_\delta||^2_L^2 dt + C_0 2^{(m-4)}[(m-4)]^{1/2}. \quad (3.7)
$$

To confirm this we use the facts that $[P, \partial_x^m \Lambda_\delta^{-2}] = 0$ and that $f$ solves the equation $Pf = \Gamma(f, f)$; by recalling $f_\delta = \Lambda_\delta^{-2}f$, this gives

$$
P\partial_x^m f_\delta = \Lambda_\delta^{-2}\partial_x^m \Gamma(f, f) = \Lambda_\delta^{-2} \sum_{j\leq m} \left(\begin{array}{c} m \\ j \end{array}\right) \Gamma(\partial_x^j f, \partial_x^m f).$$

Thus

$$
\int_0^1 \int 2^{2(m-2)}|(|P\partial_x^m f_\delta, A\partial_x^m f_\delta)|_L^1|dt \leq S_1 + S_2 + S_3 + S_4 \quad (3.8)
$$

with

$$
S_1 = \int_0^1 \int 2^{2(m-2)}|(|\Lambda_\delta^{2} \Gamma(f, \partial_x^m f), A\partial_x^m f_\delta)|_L^1|dt, \\
S_2 = \sum_{1 \leq j < \lfloor m/2 \rfloor} \left(\begin{array}{c} m \\ j \end{array}\right) \int_0^1 \int 2^{2(m-2)}|(|\Lambda_\delta^{2} \Gamma(\partial_x^j f, \partial_x^m f), A\partial_x^m f_\delta)|_L^1|dt, \\
S_3 = \sum_{\lfloor m/2 \rfloor \leq j \leq m-1} \left(\begin{array}{c} m \\ j \end{array}\right) \int_0^1 \int 2^{2(m-2)}|(|\Lambda_\delta^{2} \Gamma(\partial_x^j f, \partial_x^m f), A\partial_x^m f_\delta)|_L^1|dt, \\
S_4 = \int_0^1 \int 2^{2(m-2)}|(|\Lambda_\delta^{2} \Gamma(f, \partial_x^m f, f), A\partial_x^m f_\delta)|_L^1|dt,
$$

where $\lfloor m/2 \rfloor$ stands for the largest integer less than or equal to $m/2$. We first handle $S_1$ and use the fact that $(1 - \delta \Delta_x)\Lambda_\delta^{-2} = 1$ to write

$$
\Gamma(f, \partial_x^m f) = \Gamma(f, \partial_x^m (1 - \delta \Delta_x)f_\delta) \\
= (1 - \delta \Delta_x)\Gamma(f, \partial_x^m f_\delta) + 2 \sum_{k=1}^3 \delta \partial_{x_k} \Gamma(\partial_x f_k, \partial_x^m f_\delta) - \delta \Gamma(\Delta_x f, \partial_x^m f_\delta).
$$

Thus

$$
S_1 \leq \int_0^1 \int 2^{2(m-2)}|(|\Gamma(f, \partial_x^m f_\delta), A\partial_x^m f_\delta)|_L^1|dt \\
+ \sum_{k=1}^3 \int_0^1 \int 2^{2(m-2)}|(|\Gamma(\partial_x f_k, \partial_x^m f_\delta), \delta \partial_{x_k} \Lambda_\delta^{-2} A\partial_x^m f_\delta)|_L^1|dt \\
+ \int_0^1 \int 2^{2(m-2)}|(|\Gamma(\Delta_x f, \partial_x^m f_\delta), \delta \Lambda_\delta^{-2} A\partial_x^m f_\delta)|_L^1|dt \\
= S_{1,1} + S_{1,2} + S_{1,3}.
$$

Using (2.3) and (2.9) gives

$$
S_{1,1} \leq C \int_0^1 \int 2^{2(m-2)} \sum_{|\beta| \leq 2} \|\partial_x^\beta f\|_L^1 \|\partial_x^\beta \partial_x^m f_\delta\|_L^1 \|\partial_x^\beta \Lambda_\delta^{-2} A\partial_x^m f_\delta\|_L^1 dt.
$$
with (1.10) used in the last inequality. As for \( S_{1,2} \), we use (2.7) and the fact that the operators \( \Lambda_{x}^{-2} \) and \( \delta \partial_{x_{s}} \partial_{x_{s}} \Lambda_{s}^{-2} \) are uniformly bounded in \( L^2 \) with respect to \( \delta \) and both commute with \( (a^{1/2})^{w} \) to compute

\[
S_{1,2} \leq C \sup_{0 \leq t \leq 1} \sum_{|\beta| \leq 2} \| \partial_{x}^{\beta} f(t) \|_{L^{2}} \int_{0}^{1} t^{2\kappa(m-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m} f_{s} \right\|_{L^{2}}^{2} dt
\]

\[
\leq C \left( \sum_{|\beta| \leq 2} \| \partial_{x}^{\beta} f(t) \|_{L^{2}} \int_{0}^{1} t^{2\kappa(m-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m} f_{s} \right\|_{L^{2}}^{2} dt \right)^{1/2}
\]

(3.9)

with (1.10) used again in the last inequality. Similarly we use (2.5) to get

\[
S_{1,3} \leq C \sup_{0 \leq t \leq 1} \| \Delta_{x} f(t) \|_{L^{2}} \int_{0}^{1} t^{2\kappa(m-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m} f_{s} \right\|_{L^{2}}^{2} dt
\]

\[
\leq C \left( \sup_{0 \leq t \leq 1} \| \Delta_{x} f(t) \|_{L^{2}} \int_{0}^{1} t^{2\kappa(m-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m} f_{s} \right\|_{L^{2}}^{2} dt \right)^{1/2}
\]

(3.10)

This along with (3.9) and (3.10) gives

\[
S_{1} \leq \epsilon_{0} C \int_{0}^{1} t^{2\kappa(m-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m} f_{s} \right\|_{L^{2}}^{2} dt.
\]

(3.11)

Next, we treat \( S_{2} \) and use (2.3) and (2.9) again to compute

\[
S_{2} \leq C \sum_{1 \leq j < [m/2]} t^{\kappa(j-1)} \left[ \sum_{|\beta| \leq 2} \| \partial_{x}^{\beta} f(t) \|_{L^{2}} \int_{0}^{1} t^{2\kappa(m-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m-j} f_{s} \right\|_{L^{2}} \right] dt
\]

\[
\leq C \sum_{1 \leq j < [m/2]} t^{\kappa(j-1)} \left[ \sum_{|\beta| \leq 2} \| \partial_{x}^{\beta} f(t) \|_{L^{2}} \int_{0}^{1} t^{2\kappa(m-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m-j} f_{s} \right\|_{L^{2}} dt \right]^{1/2}
\]

\[
\times \left( \sum_{|\beta| \leq 2} \| \partial_{x}^{\beta} f(t) \|_{L^{2}} \int_{0}^{1} t^{2\kappa(m-j-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m-j} f_{s} \right\|_{L^{2}} dt \right)^{1/2}
\]

Moreover, using the assumptions (3.1) and (3.2) for \( 1 \leq j < [m/2] \), we compute that for any \( |\beta| \leq 2 \),

\[
\sup_{0 < t < 1} t^{\kappa j} \| \partial_{x}^{\beta} f(t) \|_{L^{2}} \leq \left\{ \begin{array}{ll}
\sup_{0 < t < 1} t^{\kappa(j+|\beta|-2)} \| \partial_{x}^{\beta} f(t) \|_{L^{2}}, & \text{if } j + |\beta| \geq 2,
\sup_{0 < t < 1} \| \partial_{x}^{\beta} f(t) \|_{L^{2}}, & \text{if } j + |\beta| \leq 1
\end{array} \right.
\]

\[
\leq CC_{0}^{-1}[(j-1)\frac{1}{2\kappa}]
\]

and

\[
\int_{0}^{1} t^{2\kappa(m-j-2)} \left\| (a^{1/2})^{w} \partial_{x_{s}}^{m-j} f_{s} \right\|_{L^{2}}^{2} dt \leq C_{0}^{m-j-3}[(m-j-4)\frac{1}{2\kappa}].
\]
As a result, we put these inequalities into the estimate on $S_2$ to obtain
\[
S_2 \leq C \left( \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_x^m f_s \|_{L^2}^2 dt \right)^{1/2} \times \sum_{1 \leq j < [m/2]} \frac{m!}{j!(m-j)!} C_j^{-1} [(j-1)!]^{1+2s} (C_0^{m-j-3} [(m-j-4)!]^{1+2s})^{1/2},
\]
and direct computation shows that
\[
\sum_{1 \leq j < [m/2]} \frac{m!}{j!(m-j)!} C_j^{-1} [(j-1)!]^{1+2s} (C_0^{m-j-3} [(m-j-4)!]^{1+2s})^{1/2} \leq CC_0^{m-4} \sum_{1 \leq j < [m/2]} \frac{1}{jm^{2s}} \leq CC_0^{m-4} [(m-4)!]^{1+2s} / m^{2s},
\]
where the last inequality holds because
\[
\frac{1}{m^{2s}} \sum_{1 \leq j < [m/2]} \frac{1}{j} \leq C_s
\]
with $C_s$ a constant depending only on $s$ but independent of $m$. Thus we combine the above inequalities to obtain that for any $\varepsilon > 0$,
\[
S_2 \leq \varepsilon \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_x^m f_s \|_{L^2}^2 dt + C_\varepsilon C_0^{2(m-4)} [(m-4)!]^{1+2s}. \tag{3.12}
\]
The treatment of $S_3$ is similar to that of $S_2$ by using (2.4) here instead of (2.3). Meanwhile following the argument for handling $S_1$ will yield the upper bound of $S_4$. For brevity, we omit the details and conclude that
\[
S_3 \leq \varepsilon \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_x^m f_s \|_{L^2}^2 dt + C_\varepsilon C_0^{2(m-4)} [(m-4)!]^{1+2s}
\]
and
\[
S_4 \leq \varepsilon C_0 \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_x^m f_s \|_{L^2}^2 dt.
\]
These along with the estimates (3.11)–(3.12) on $S_1$ and $S_2$ as well as (3.8) yield the desired upper bound for the second term on the left-hand side of (3.7). Meanwhile the first term can be handled in the same way with a simpler argument. Then we have proven (3.7).

**Step 2.** In this step we will derive the desired estimate (3.2) for short time $0 < t \leq 1$, i.e., for any multi-index $a$ with $|a| = m$, we have
\[
\sup_{0 < t \leq 1} t^{2\kappa(m-2)} \| \partial_x^a f(t) \|_{L^2}^2 + \left( \int_0^1 t^{2\kappa(m-2)} \| (D_x)^{r/\kappa} \partial_x^m f(t) \|_{L^2}^2 dt \right)^{1/2} + \left( \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_x^m f(t) \|_{L^2}^2 dt \right)^{1/2} \leq \frac{1}{2} C_0^{m-3} [(m-4)!]^{1+2s}. \tag{3.13}
\]
To do so, by (3.6) we can apply the subelliptic estimate (2.10) with $u = \partial_x^m f_s$ and $r = \kappa(m-2)$; this gives that for any $0 < t \leq 1$,
\[
t^{2\kappa(m-2)} \| \partial_x^m f_s \|_{L^2}^2 + \int_0^1 t^{2\kappa(m-2)} \| (D_x)^{r/\kappa} \partial_x^m f_s \|_{L^2}^2 dt + \int_0^1 t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_x^m f_s \|_{L^2}^2 dt \leq \mathcal{M}. \tag{3.14}
\]
where
\[ M = C \int_{0}^{1} t^{2\kappa(m-2)} \| (P \partial_{x_1}^m f, \partial_{x_1}^m f)_{L^2} \| dt \]
\[ + C \int_{0}^{1} t^{2\kappa(m-2)} \| (P \partial_{x_1}^m f, A \partial_{x_1}^m f)_{L^2} \| dt + C m \int_{0}^{1} t^{2\kappa(m-2)-1} \| \partial_{x_1}^m f \|_{L^2}^2 dt. \]
As for the last term above, we use the interpolation inequality (3.4) to get that for any \( \varepsilon > 0 \),
\[ m t^{2\kappa(m-2)-1} \| \partial_{x_1}^m f \|_{L^2}^2 \leq \varepsilon t^{2\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 + C \varepsilon m^{1+2\kappa(m-3)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 \]
\[ \leq \varepsilon t^{2\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 + C \varepsilon m^{1+2\kappa(m-3)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2, \]
recalling \( \kappa = \frac{1+2\alpha}{2s} \). As a result, we use the assumption (3.2) to compute
\[ C m \int_{0}^{1} t^{2\kappa(m-2)-1} \| \partial_{x_1}^m f \|_{L^2}^2 dt \]
\[ \leq \varepsilon \int_{0}^{1} t^{2\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 dt + C \varepsilon m^{1+2\kappa(m-3)} \int_{0}^{1} t^{2\kappa(m-3)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 dt \]
\[ \leq \varepsilon \int_{0}^{1} t^{2\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 dt + C \varepsilon C_0^{2(m-4)} m^{1+2\kappa(m-3)} [(m-4)!]^{1+2\kappa}. \]
Combining the above inequality and (3.7), we get the upper bound of the term \( M \) on the right-hand side of (3.14), i.e.,
\[ M \leq (\varepsilon + \varepsilon_0 C) \int_{0}^{1} t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_{x_1}^m f \|_{L^2}^2 dt \]
\[ + \varepsilon \int_{0}^{1} t^{2\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 dt + C \varepsilon C_0^{2(m-4)} [(m-4)!]^{1+2\kappa}. \]
Suppose that \( \varepsilon_0 \) is small enough and let \( \varepsilon \) be small as well such that the first two terms on the right-hand side of the above inequality can be absorbed by the left ones in (3.14). Thus we conclude that for any \( 0 < t \leq 1 \),
\[ t^{2\kappa(m-2)} \| \partial_{x_1}^m f \|_{L^2}^2 + \int_{0}^{1} t^{2\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 dt \]
\[ + \int_{0}^{1} t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_{x_1}^m f \|_{L^2}^2 dt \leq CC_0^{2(m-4)} [(m-4)!]^{1+2\kappa}. \]
Since the constants \( C \) and \( C_0 \) above are independent of \( \delta \), letting \( \delta \to 0 \) implies that
\[ t^{\kappa(m-2)} \partial_{x_1}^m f \in L^\infty([0, 1]; L^2) \]
and
\[ t^{\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 + t^{\kappa(m-2)} \| (a^{1/2})^w \partial_{x_1}^m f \|_{L^2}^2 \in L^2([0, 1] \times \mathbb{R}^6), \]
and moreover,
\[ \sup_{0 < t \leq 1} t^{\kappa(m-2)} \| \partial_{x_1}^m f \|_{L^2}^2 + \left( \int_{0}^{1} t^{2\kappa(m-2)} \| (D_x) \partial_{x_1}^m f \|_{L^2}^2 \right)^{1/2} \]
\[ + \left( \int_{0}^{1} t^{2\kappa(m-2)} \| (a^{1/2})^w \partial_{x_1}^m f \|_{L^2}^2 \right)^{1/2} \leq CC_0^{m-4} [(m-4)!]^{1+2\kappa}. \]
The above estimate obviously holds with \( \partial_x^m \) replaced by \( \partial_x^m \) (\( j = 2, 3 \)). Then using the fact that \( \| \partial_x^m f \|_{L^2_t}^2 \leq \| \partial_x^m f \|_{L^2_t}^2 + \| \partial_x^m f \|_{L^2_t}^2 + \| \partial_x^m f \|_{L^2_t}^2 \) for any \( |\alpha| = m \), we conclude that for any \( \alpha \) with \( |\alpha| = m \),

\[
\sup_{0 < t < T} \left| T^{r(m-2)} \| \partial_x^m f \|_{L^2_t}^2 + \left( \int_0^1 t^{2a(m-2)|\langle D_x \rangle|^2} \| \partial_x^m f \|_{L^2_t}^2 \right) \right|^{1/2} + \left( \int_0^1 t^{2a(m-2)|\langle (a^{1/2})^x \partial_x^m f \|_{L^2_t}^2} \right)^{1/2} \leq C C_{m-4}^{m-4}[(m-4)!]\frac{1 + 2a}{2}.
\]

Then the desired estimate (3.13) follows if we take \( C_0 \geq 2C \) with \( C \) the constant in the above inequality.

Completeness of the proof of Proposition 3.2 (The case of \( t \geq 1 \)). It remains to prove the validity of (3.3) in Proposition 3.2 when \( t > 1 \). The proof is quite similar to that in the case of \( 0 < t \leq 1 \), and the argument here will be simpler since this part is just the propagation property of Gevrey regularity. Indeed, we apply the estimate (2.11) for \( u = \partial_x^m f \); this gives that for any \( t \geq 1 \),

\[
\| \partial_x^m f(t) \|_{L^2_t}^2 + \int_1^{t+\infty} \| (D_x \partial_x^m f) \|_{L^2_t}^2 dt + \int_1^{t+\infty} \| (a^{1/2} \partial_x^m f) \|_{L^2_t}^2 dt \leq \| \partial_x^m f(1) \|_{L^2_t}^2 + C \int_1^{t+\infty} \| (P \partial_x^m f, \partial_x^m f) \|_{L^2_t}^2 dt + C \int_1^{t+\infty} \| (P \partial_x^m f, \partial_x^m f) \|_{L^2_t}^2 dt.
\]

As for the first term on the right-hand side, we have obtained in (3.16) its upper bound:

\[
\| \partial_x^m f(t) \|_{L^2_t}^2 \leq \| \partial_x^m f(1) \|_{L^2_t}^2 \leq C C_{m-4}^{m-4}[(m-4)!]\frac{1 + 2a}{2}.
\]

Repeating the argument for proving (3.7), we see that the second and third terms on the right-hand side of (3.17) are bounded from above by

\[
(\epsilon + \epsilon_0 C) \int_1^{t+\infty} \| (a^{1/2} \partial_x^m f) \|_{L^2_t}^2 dt + C \int_1^{t+\infty} \| (P \partial_x^m f, \partial_x^m f) \|_{L^2_t}^2 dt.
\]

with \( \epsilon \) arbitrarily small. As for the last term in (3.17), we use the interpolation equality (3.4) and then the assumption (3.2) to obtain

\[
\int_1^{t+\infty} \| \partial_x^m f(t) \|_{L^2_t}^2 dt \leq \epsilon \int_1^{t+\infty} \| (D_x \partial_x^m f) \|_{L^2_t}^2 dt + C \epsilon \int_1^{t+\infty} \| (D_x \partial_x^m f) \|_{L^2_t}^2 dt + C \int_1^{t+\infty} \| (P \partial_x^m f, \partial_x^m f) \|_{L^2_t}^2 dt.
\]

Finally, supposing that \( \epsilon_0 \) is small enough and choosing \( \epsilon \) small as well, we combine the above inequalities to get that for any \( t > 1 \),

\[
\| \partial_x^m f(t) \|_{L^2_t}^2 + \int_1^{t+\infty} \| (D_x \partial_x^m f) \|_{L^2_t}^2 dt + \int_1^{t+\infty} \| (a^{1/2} \partial_x^m f) \|_{L^2_t}^2 dt \leq C C_{m-4}^{m-4}[(m-4)!]\frac{1 + 2a}{2}.
\]

The remaining argument is just the same as that in the previous case of \( 0 < t \leq 1 \), so we omit it here and conclude that for any \( |\alpha| = m \),

\[
\sup_{T \geq 1} \| \partial_x^m f(t) \|_{L^2_t}^2 + \left( \int_1^{t+\infty} \| (D_x \partial_x^m f) \|_{L^2_t}^2 dt \right)^{1/2} + \left( \int_1^{t+\infty} \| (a^{1/2} \partial_x^m f) \|_{L^2_t}^2 dt \right)^{1/2} \leq C_{m-3}^{m-3}[m-4!]\frac{1 + 2a}{2}.
\]

This along with (3.13) yields (3.3) as desired. The proof of Proposition 3.2 is completed.
Proposition 3.3. Define $\kappa = (1+2s)/2s$. Assume that the cross-section satisfies (1.3) and (1.4) with $0 < s < 1$ and $\gamma \geq 0$. Let $f \in L^\infty([0, +\infty); H^2]$ be any solution to (1.8) satisfying (1.10). Then there exists a constant $C_*$ depending only on $s, \gamma$ and the constant $\epsilon_0$ in (1.10) such that for any multi-index $\beta$ with $2 \leq |\beta| \leq 4$, we have

\[
\sup_{t > 0} \phi(t)^{\kappa(|\beta|-2)} \| \partial_{x}^\beta f(t) \|_{L^2} + \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\beta|-2)} \| \langle D_x \rangle \partial_{x}^\beta f(t) \|_{L^2}^2 dt \right)^{1/2} \\
+ \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\beta|-2)} \| (a^{1/2})^w \partial_{x}^\beta f(t) \|_{L^2}^2 dt \right)^{1/2} \leq C_*. \tag{3.18}
\]

Proof. The proof is quite similar to that of Proposition 3.2. In fact, using a similar subelliptic estimate to (2.11) and repeating the procedure for proving Proposition 3.2, we have

\[
\sum_{|\beta|=2} \left( \int_0^{+\infty} \| \langle D_x \rangle \partial_{x}^\beta f(t) \|_{L^2}^2 dt \right)^{1/2} \leq C
\]

for some constant $C$ depending only on $s, \gamma$ and the constant $\epsilon_0$. This along with the assumption (1.10) yields the validity of (3.18) for $|\beta| = 2$. Furthermore, repeating again the argument for proving Proposition 3.2, we can directly verify that (3.18) holds for $|\beta| = 3$ and then for $|\beta| = 4$. Since the argument involved here is direct and simpler than the one in Proposition 3.2, we omit it for brevity.

Now we can prove the main result on the Gevrey regularization in the spatial variable.

Proof of Theorem 3.1. By Proposition 3.3, we see the assumption (3.1) in Proposition 3.2 holds and moreover the induction assumption (3.2) holds for any $\beta$ with $|\beta| = 4$ provided $C_0 \geq C_*$. These along with Proposition 3.2 enable us to use induction to obtain that for any $\alpha \in \mathbb{Z}_+^3$ with $|\alpha| > 4$, we have

\[
\sup_{t > 0} \phi(t)^{\kappa(|\alpha|-2)} \| \partial_{x}^\alpha f(t) \|_{L^2} + \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\alpha|-2)} \| \langle D_x \rangle \partial_{x}^\alpha f(t) \|_{L^2}^2 dt \right)^{1/2} \\
+ \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\alpha|-2)} \| (a^{1/2})^w \partial_{x}^\alpha f(t) \|_{L^2}^2 dt \right)^{1/2} \leq C_0^{(|\alpha|-3)(|\alpha|-4)! \frac{14+2s}{2s}}. \tag{3.19}
\]

As a result, for any $t > 0$ and any $|\alpha| \geq 0$ we have

\[
\phi(t)^{\kappa(|\alpha|)} \| \partial_{x}^\alpha f(t) \|_{L^2} \leq \begin{cases} 
\| \partial_{x}^\alpha f(t) \|_{L^2}, & \text{if } |\alpha| \leq 2 \\
\phi(t)^{\kappa(|\alpha|-2)} \| \partial_{x}^\alpha f \|_{L^2}, & \text{if } |\alpha| \geq 3
\end{cases} \leq C_0^{\kappa(|\alpha|+1)(|\alpha|)! \frac{14+2s}{2s}},
\]

completing the proof of Theorem 3.1.

4 The Gevrey regularization in the velocity variable

As in the previous section, the Gevrey regularization for variable $v$ is just an immediate consequence of the following proposition.

Proposition 4.1. Define $\kappa = (1+2s)/2s$ and let $m \geq 5$ be an arbitrarily given integer. Let $f \in L^\infty([0, +\infty); H^2]$ be any solution to the Cauchy problem (1.8) satisfying the condition (1.10). In addition, suppose that there exists a positive constant $\tilde{C}_* \geq 1$ depending only on $s, \gamma$ and the constant $\epsilon_0$ in (1.10) such that for any multi-index $\beta$ with $2 \leq |\beta| \leq 3$, we have

\[
\sup_{t > 0} \phi(t)^{\kappa(|\beta|-2)} \| \partial_{v}^\beta f(t) \|_{L^2} + \left( \int_0^{+\infty} \phi(t)^{2\kappa(|\beta|-2)} \| (a^{1/2})^w \partial_{v}^\beta f(t) \|_{L^2}^2 dt \right)^{1/2} \leq \tilde{C}_*.
\]

Then we can find a constant $\tilde{C}_0 \geq \max\{\tilde{C}_*, C_0^2\}$ with $C_0$ given in Theorem 3.1, depending only on $s, \gamma$ and the constant $\epsilon_0$ in (1.10) but independent of $m$, such that if for any multi-index $\beta$ with $4 \leq |\beta| \leq m - 1$,
we have
\[
\sup_{t>0} \phi(t)^{\kappa(|\beta|-2)} \|\partial_v^\beta f(t)\|_{L^2} + \left( \int_0^\infty \phi(t)^{2\kappa(|\beta|-2)} \|(a^{1/2})^w \partial_v^\beta f(t)\|^2_{L^2} dt \right)^{\frac{1}{2}} \leq C_0 |\beta| - 3 \begin{pmatrix} |\beta| - 4 \end{pmatrix}^\frac{1+2s}{2},
\]
(4.1)
then the above estimate (4.1) still holds for any \(\beta\) with \(|\beta| = m\).

Proof. Since the proof is quite similar to that of Proposition 3.2, we only give a sketch here and will emphasize the difference. In the following argument, we use the notation
\[
f_{m,\delta} = (1 + \delta |D_v|^2)^{-1} \partial_v^m f.
\]
Apply Proposition 2.4 for \(u = f_{m,\delta}\) and \(r = \kappa(m - 2)\); this gives that for any \(0 < t \leq 1\),
\[
\int_0^1 t^{2\kappa(m-2)} \|f_{m,\delta}(t)\|^2_{L^2} dt + \int_0^1 t^{2\kappa(m-2)} \|(a^{1/2})^w f_{m,\delta}\|^2_{L^2} dt
\]
\[
\leq C \int_0^1 t^{2\kappa(m-2)} \|(Pf_{m,\delta}, f_{m,\delta})_{L^2}\| dt + Cm \int_0^1 t^{2\kappa(m-2)-1} \|f_{m,\delta}\|^2_{L^2} dt.
\]
(4.2)
For the last term on the right-hand side of (4.2), we can repeat the argument for proving (3.15) to obtain that for any \(\varepsilon, \bar{\varepsilon} > 0\),
\[
Cm \int_0^1 t^{2\kappa(m-2)-1} \|f_{m,\delta}\|^2_{L^2} dt
\]
\[
\leq \bar{\varepsilon} \int_0^1 t^{2\kappa(m-2)} \|\langle D_v \rangle \partial_v^m f\|^2_{L^2} dt + C\varepsilon t^{2\kappa(m-3)} \|\langle D_v \rangle \partial_v^{m-1} f\|^2_{L^2} dt
\]
\[
\leq \varepsilon \int_0^1 t^{2\kappa(m-2)} \|(a^{1/2})^w f_{m,\delta}\|^2_{L^2} dt + C\varepsilon m^{\frac{1+2s}{2}} \int_0^1 t^{2\kappa(m-3)} \|(a^{1/2})^w \partial_v^m f\|^2_{L^2} dt
\]
\[
\leq \varepsilon \int_0^1 t^{2\kappa(m-2)} \|(a^{1/2})^w f_{m,\delta}\|^2_{L^2} dt + C\bar{\varepsilon} C_0^{2(m-1)} [(m - 4)!]^{\frac{1+2s}{2}},
\]
(4.3)
where the second inequality holds because we may write
\[
\langle D_v \rangle \partial_v^m = \langle D_v \rangle \partial_v^m [(a^{1/2})^w]^{-1} (a^{1/2})^w
\]
due to the conclusions (i) and (iii) in Proposition 2.1, and the last inequality holds due to the assumption (4.1).

Next, we treat the first term on the right-hand side of (4.2), the main different part from the previous section. Observe
\[
P f_{m,\delta} = (1 + \delta |D_v|^2)^{-1} \partial_v^m P f + (1 + \delta |D_v|^2)^{-1} [P, \partial_v^m] f + |P, (1 + \delta |D_v|^2)^{-1} \partial_v^m f|
\]
and furthermore \(P f = \Gamma(f, f)\). Then
\[
\int_0^1 t^{2\kappa(m-2)} \|(Pf_{m,\delta}, f_{m,\delta})_{L^2}\| dt \leq \sum_{\ell=1}^3 J_\ell
\]
with
\[
J_1 = \int_0^1 t^{2\kappa(m-2)} \|(1 + \delta |D_v|^2)^{-1} \partial_v^m \Gamma(f, f), f_{m,\delta})_{L^2}\| dt,
\]
\[
J_2 = \int_0^1 t^{2\kappa(m-2)} \|(1 + \delta |D_v|^2)^{-1} [P, \partial_v^m] f, f_{m,\delta})_{L^2}\| dt,
\]
\[
J_3 = \int_0^1 t^{2\kappa(m-2)} \|[P, (1 + \delta |D_v|^2)^{-1} \partial_v^m f, f_{m,\delta})_{L^2}\| dt.
\]
The estimate on $\mathcal{J}_1$. The $\mathcal{J}_1$ can be handled in the same way as the terms $S_j$ defined in (3.8). Here, we have to handle the commutator between $(a^{1/2})^w$ and $(1 + \delta |D_v|^2)^{-1}$ and there is no additional difficulty since

\[
[(a^{1/2})^w, (1 + \delta |D_v|^2)^{-1}] = [(a^{1/2})^w, (1 + \delta |D_v|^2)^{-1}] [((a^{1/2})^w)^{-1} (a^{1/2})^w].
\]

Moreover, the Leibniz formula also holds in the form

\[
\partial^m_v \Gamma(f, f) = \sum_{0 \leq j \leq m} \sum_{0 \leq k \leq j} \binom{m}{j} \binom{j}{k} \mathcal{T}(\partial^{m-j}_v f, \partial^{j-k}_v f, \partial^k_v h^{1/2})
\]

with

\[
\mathcal{T}(g, h, \omega) \overset{\text{def}}{=} \int \int B(v - v_*, \sigma) \omega(g' h' - g_\ast h) dv_\ast d\sigma.
\]

Note that $\Gamma(g, h) = \mathcal{T}(g, h, \mu^{1/2})$ and a constant $L > 0$ exists such that

\[
\forall k \geq 0, \quad |\partial^k_v \mu^{1/2}| \leq L^{k+1}! \mu^{1/4}.
\]

Thus the terms in the summation of (4.4) enjoy the same upper bounds as in (2.3)–(2.5) and (2.7), so we can follow the argument for handling $\partial^m_v \Gamma(f, f)$ in the previous section to conclude

\[
\mathcal{J}_1 \leq (\varepsilon + \varepsilon_0)\int_0^1 \frac{2s(m-2)}{2} \| (a^{1/2})^w f_{m, \delta} \|_{L^2}^2 dt + C_n \overline{C}_0 \varepsilon^{2(m-4)} [(m - 4)!]^{\frac{1}{2} + \frac{1}{2s}}.
\]

The upper bound of $\mathcal{J}_2$. Note that

\[
[P, \partial^m_v] = -m \partial^m_v \partial_v^{-1} + [a^w + \mathcal{R}, \partial^m_v]
\]

with $a^w$ and $\mathcal{R}$ given in Proposition 2.1. Thus

\[
\mathcal{J}_2 \leq m \int_0^1 \frac{2s(m-2)}{2} \| (1 + \delta |D_v|^2)^{-1} \partial_x \partial_v^{-1} f, f_{m, \delta} \|_{L^2} dt
\]

\[
+ \int_0^1 \frac{2s(m-2)}{2} \| (1 + \delta |D_v|^2)^{-1} [a^w + \mathcal{R}, \partial_v^m f, f_{m, \delta}] \|_{L^2} dt
\]

\[
\overset{\text{def}}{=} \mathcal{J}_{2.1} + \mathcal{J}_{2.2}.
\]

We first estimate $\mathcal{J}_{2.1}$ and use the fact that

\[
\| (1 + \delta |D_v|^2)^{-1} \partial_x \partial_v^{-1} f \|_{L^2} \leq C \| \partial_v^m f \|_{L^2} + C \| (1 + \delta |D_v|^2)^{-1} \partial_v^m f \|_{L^2}
\]

to get

\[
\mathcal{J}_{2.1} \leq C m \int_0^1 \frac{2s(m-2)}{2} \| \partial_v^m f \|_{L^2}^2 dt + C m \int_0^1 \frac{2s(m-2)}{2} \| f_{m, \delta} \|_{L^2}^2 dt
\]

\[
\leq \varepsilon \int_0^1 \frac{2s(m-2)}{2} \| (D_v) \overline{\partial_v^m f} \|_{L^2}^2 \frac{1}{2} dt + \varepsilon \int_0^1 \frac{2s(m-2)}{2} \| (a^{1/2})^w f_{m, \delta} \|_{L^2}^2 dt
\]

\[
+ C_n C_0^{2(m-4)} [(m - 4)!]^{\frac{1}{2s} + 1} + C_n \overline{C}_0 \varepsilon^{2(m-4)} [(m - 4)!]^{\frac{1}{2s}}
\]

\[
\leq \varepsilon C_n^{2(m-3)} [(m - 4)!]^{\frac{1}{2s} + 1} + \varepsilon \int_0^1 \frac{2s(m-2)}{2} \| (a^{1/2})^w f_{m, \delta} \|_{L^2}^2 dt
\]

\[
+ C_n C_0^{2(m-4)} [(m - 4)!]^{\frac{1}{2s} + 1} + C_n \overline{C}_0 \varepsilon^{2(m-4)} [(m - 4)!]^{\frac{1}{2s}}
\]

\[
\leq \varepsilon \int_0^1 \frac{2s(m-2)}{2} \| (a^{1/2})^w f_{m, \delta} \|_{L^2}^2 dt + C_n C_0^{2(m-4)} [(m - 4)!]^{\frac{1}{2s}},
\]
where the second inequality follows from the similar argument for proving (3.15) and (4.3), (3.19) is used in the third inequality and the last inequality holds because of $\tilde{C}_0 > C_0^2$ by assumption. Now we derive the upper bound for $\mathcal{J}_{2.2}$. We observe

$$(a^w + \mathcal{R})f = -\Gamma(\mu^{1/2}, f) - \Gamma(f, \mu^{1/2}) = -\mathcal{T}(\mu^{1/2}, f, \mu^{1/2}) - \mathcal{T}(f, \mu^{1/2}, \mu^{1/2}),$$

recalling the trilinear operator $\mathcal{T}$ is defined by (4.5). Thus using the Leibniz formula again gives

$$[a^w + \mathcal{R}, \partial_{v_1}^m]f = \sum_{1 \leq j \leq m} \sum_{0 \leq k \leq j} \binom{m}{j} \binom{j}{k} \mathcal{T}(\partial_{v_1}^{j-k} \mu^{1/2}, \partial_{v_1}^{m-j} f, \partial_{v_1}^k \mu^{1/2})$$

$$+ \sum_{1 \leq j \leq m} \sum_{0 \leq k \leq j} \binom{m}{j} \binom{j}{k} \mathcal{T}(\partial_{v_1}^{m-j} f, \partial_{v_1}^{j-k} \mu^{1/2}, \partial_{v_1}^k \mu^{1/2}).$$

This along with (4.6) enables us to use the similar upper bounds for the trilinear operator $\mathcal{T}$ as in (2.3) and (2.4) to compute

$$\mathcal{J}_{2.2} \leq C \left( \int_0^1 t^{2\kappa (m-2)} \| (a^{1/2})^w f_{m, \delta} \|^2_{L^2(T)} dt \right)^{1/2}$$

$$\times \sum_{1 \leq j \leq m} \frac{m!}{j!(m-j)!} \tilde{L}^{j+1} \left( \int_0^1 t^{2\kappa (m-j-2)} \| (a^{1/2})^w \partial_{v_1}^{m-j} f \|^2_{L^2(T)} dt \right)^{1/2}$$

with $\tilde{L}$ a constant depending only on the constant $L$ given in (4.6). Moreover as for the last factor in the above inequality, we use the assumption (4.1) to compute

$$\sum_{1 \leq j \leq m} \frac{m!}{j!(m-j)!} \tilde{L}^{j+1} \left( \int_0^1 t^{2\kappa (m-j-2)} \| (a^{1/2})^w \partial_{v_1}^{m-j} f \|^2_{L^2(T)} dt \right)^{1/2}$$

$$\leq C \left\{ \sum_{1 \leq j < |m/2|} + \sum_{|m/2| \leq j \leq m-4} \right\} \frac{m!}{(m-j)!} \tilde{L}^{j+1} \left( \int_0^1 t^{2\kappa (m-j-2)} \| (a^{1/2})^w \partial_{v_1}^{m-j} f \|^2_{L^2(T)} dt \right)^{1/2}$$

$$+ C \sum_{m-4 \leq j \leq m} \frac{m!}{(m-j)!} \tilde{L}^{j+1} \left( \int_0^1 t^{2\kappa (m-j-2)} \| (a^{1/2})^w \partial_{v_1}^{m-j} f \|^2_{L^2(T)} dt \right)^{1/2}$$

$$\leq C \left\{ \sum_{1 \leq j < |m/2|} + \sum_{|m/2| \leq j \leq m-4} \right\} \frac{m!}{(m-j)!} \tilde{L}^{j+1} \tilde{C}_0^{m-j-3} ((m-j-4)!)^{\frac{12+2\kappa}{12}} + C(3\tilde{L})^{m+1}$$

$$\leq C \tilde{C}_0^{m-4} ((m-4)!)^{\frac{12+2\kappa}{12}} \tilde{L}^2 \left( \sum_{1 \leq j < |m/2|} \left( \frac{\tilde{L}}{\tilde{C}_0} \right)^{j-1} + \sum_{|m/2| \leq j \leq m-4} \left( \frac{4\tilde{L}}{\tilde{C}_0} \right)^{j-1} \right) + C(3\tilde{L})^{m+1}$$

$$\leq C \tilde{C}_0^{m-4} ((m-4)!)^{\frac{12+2\kappa}{12}},$$

where the constant $C$ in the last line depends on the constant $L$ given in (4.6) and the last inequality holds since we can choose $\tilde{C}_0 \geq \max \left\{ 8\tilde{L}, (3\tilde{L})^{6} \right\}$. Combining these inequalities, we conclude

$$\mathcal{J}_{2.2} \leq \epsilon \int_0^1 t^{2\kappa (m-2)} \| (a^{1/2})^w f_{m, \delta} \|^2_{L^2(T)} dt + C_2 \tilde{C}_0^{2(m-4)} ((m-4)!)^{\frac{12+2\kappa}{12}}.$$

This along with the upper bound for $\mathcal{J}_{2.1}$ gives that for any $\epsilon > 0,$

$$\mathcal{J} \leq \epsilon \int_0^1 t^{2\kappa (m-2)} \| (a^{1/2})^w f_{m, \delta} \|^2_{L^2(T)} dt + C_2 \tilde{C}_0^{2(m-4)} ((m-4)!)^{\frac{12+2\kappa}{12}}.$$
The estimate on $J_3$. It can be treated in the way similar to $J_2$ but the argument is direct and much simpler. In fact, observe

$$[v \cdot \partial_x, (1 + \delta |D_v|^2)^{-1}D_{v_i}^m = -2i \sum_{1 \leq j \leq 3} (1 + \delta |D_v|^2)^{-1} \delta \partial_x \partial_{v_j} - (1 + \delta |D_v|^2)^{-1} \partial_{v_i} \partial_{v_i}^{-1}.$$ 

This enables us to use the argument for treating $J_{2,1}$ with slight modifications to get

$$\int_0^1 l^{2\kappa(m-2)} \|[v \cdot \partial_x, (1 + \delta |D_v|^2)^{-1}D_{v_i}^m, f, f_m, s]_L^2 dt \leq \varepsilon \int_0^1 l^{2\kappa(m-2)} \|(a^{1/2}w f, f_m, s)_{L^2}^2 dt + C \varepsilon \tilde{C}_0^{2(m-4)} (m - 4)! \frac{1+2s}{m}.$$ 

Moreover, observe

$$[\alpha^w + \mathcal{R}, (1 + \delta |D_v|^2)^{-1}D_{v_i}^m f = -(1 + \delta |D_v|^2)^{-1}[\alpha^w + \mathcal{R}, 1 - \delta \Delta_v](1 + \delta |D_v|^2)^{-1}D_{v_i}^m f = (1 + \delta |D_v|^2)^{-1}[\alpha^w + \mathcal{R}, \delta \Delta_v]f_m, s.]$$

Then following the argument for treating $J_{2,2}$ above and observing $(1 + \delta |D_v|^2)^{-1} \delta \Delta_v$ is uniformly bounded in $L^2$ with respect to $\delta$, we have

$$\int_0^1 l^{2\kappa(m-2)} \|[\alpha^w + \mathcal{R}, (1 + \delta |D_v|^2)^{-1}D_{v_i}^m f, f_m, s]_L^2 dt \leq \varepsilon \int_0^1 l^{2\kappa(m-2)} \|(a^{1/2}w f_m, s)_{L^2}^2 dt + C \varepsilon \tilde{C}_0^{2(m-4)} (m - 4)! \frac{1+2s}{m}.$$ 

As a result, combining the above estimates gives that for any $\varepsilon > 0$,

$$J_3 \leq \varepsilon \int_0^1 l^{2\kappa(m-2)} \|(a^{1/2}w f_m, s)_{L^2}^2 dt + C \varepsilon \tilde{C}_0^{2(m-4)} (m - 4)! \frac{1+2s}{m}.$$ 

It then follows from the upper bounds for $J_1 - J_3$ that

$$\int_0^1 l^{2\kappa(m-2)} \|[P f_m, s]_L^2 dt \leq (\varepsilon + \epsilon_0) \int_0^1 l^{2\kappa(m-2)} \|(a^{1/2}w f_m, s)_{L^2}^2 dt + C \varepsilon \tilde{C}_0^{2(m-4)} (m - 4)! \frac{1+2s}{m}.$$ 

This along with (4.2) and (4.3) yields that for any $0 < t \leq 1$, supposing $\epsilon_0$ is small enough and choosing $\varepsilon$ small as well, we have

$$l^{2\kappa(m-2)} \|f_m, s(t)\|_{L^2}^2 + \int_0^1 l^{2\kappa(m-2)} \|(a^{1/2}w f_m, s)_{L^2}^2 dt \leq C \tilde{C}_0^{2(m-4)} (m - 4)! \frac{1+2s}{m}.$$ 

Thus letting $\delta \to 0$, we obtain

$$\sup_{0 < t \leq 1} l^{\kappa(m-2)} \|\partial_{v_i}^m f(t)\|_{L^2}^2 + \left(\int_0^1 l^{2\kappa(m-2)} \|(a^{1/2}w f_m, s)_{L^2}^2 dt \right)^{1/2} \leq C \tilde{C}_0^{m-4}(m - 4)! \frac{1+2s}{m}.$$ 

The remaining argument is just the same as that in the proof of Proposition 3.2 and we conclude that for any $\alpha$ with $|\alpha| = m$,

$$\sup_{t > 0} \phi(t)^{\kappa(m-2)} \|\partial_{\alpha}^m f(t)\|_{L^2}^2 + \left(\int_0^{+\infty} \phi(t)^{2\kappa(m-2)} \|(a^{1/2}w f_m, s)_{L^2}^2 dt \right)^{1/2} \leq \tilde{C}_0^{m-3}(m - 4)! \frac{1+2s}{m}.$$ 

Thus the proof of Proposition 4.1 is completed. \qed
Proof of Theorem 1.3. By virtue of Proposition 4.1, we can follow the same procedure for proving Theorem 3.1 to obtain
\[
\forall |\alpha| \geq 0, \quad \sup_{t>0} \phi(t)^{1+2s(|\alpha|+|\beta|)} \| \partial_x^\alpha \partial_v^\beta f(t) \|_{L^2} \leq C_0^{\alpha+1} (|\alpha|!)^{1+2s} (4.7)
\]
for some $C_0 > 0$. Let $C_0 > 0$ be the constant given in Theorem 3.1 and choose
\[
C = \max\{2^{1+2s} C_0, 2^{1+2s} \tilde{C}_0\}.
\]
Then for any $\alpha, \beta \in Z_+^1$, we use Theorem 3.1 and (4.7) as well as the fact that $(m+n)! \leq 2^{m+n} m! n!$ for any positive integers $m$ and $n$ to compute
\[
\phi(t)^{1+2s(|\alpha|+|\beta|)} \| \partial_x^\alpha \partial_v^\beta f(t) \|_{L^2} \leq \phi(t)^{1+2s(|\alpha|+|\beta|)} \| \partial_x^{2\alpha} \partial_v^{2\beta} f(t) \|_{L^2}^{1/2} \| \partial_x^{2\alpha} \partial_v^{2\beta} f(t) \|_{L^2}^{1/2} \\
\leq (\phi(t)^{1+2s(|\alpha|+|\beta|)} \| \partial_x^{2\alpha} \partial_v^{2\beta} f(t) \|_{L^2})^{1/2} (\phi(t)^{1+2s(|\alpha|+|\beta|)} \| \partial_x^{2\alpha} \partial_v^{2\beta} f(t) \|_{L^2})^{1/2} \\
\leq (C_0^{\alpha+1} + (2 |\alpha|!)^{1+2s}(C_0^{\beta+1} + (2 |\beta|!)^{1+2s}))^{1/2} \\
\leq C_0^{\alpha+1/2} 2^{1+2s(|\alpha|+|\beta|)1/2} C_0^{\beta+1/2} 2^{1+2s(|\alpha|+|\beta|)1/2} \\
\leq C^{\alpha+|\beta|+1} (|\alpha|+|\beta|)!^{1+2s}.
\]
The proof is thus completed.

Acknowledgements The first author was supported by National Natural Science Foundation of China (Grant No. 11631011). The third author was supported by National Natural Science Foundation of China (Grant Nos. 11961160716, 11871054 and 11771342), the Natural Science Foundation of Hubei Province (Grant No. 2019CFA007) and the Fundamental Research Funds for the Central Universities (Grant No. 2042020kff0210).

References
1 Alexandre R. A review of Boltzmann equation with singular kernels. Kinet Relat Models, 2009, 2: 551–646
2 Alexandre R, Desvillettes L, Villani C, et al. Entropy dissipation and long-range interactions. Arch Ration Mech Anal, 2000, 152: 327–355
3 Alexandre R, Hérau F, Li W-X. Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff. J Math Pures Appl (9), 2019, 126: 1–71
4 Alexandre R, Desvillettes L, Villani C, et al. Entropy dissipation and long-range interactions. Arch Ration Mech Anal, 2000, 152: 327–355
5 Alexandre R, Desvillettes L, Villani C, et al. Entropy dissipation and long-range interactions. Arch Ration Mech Anal, 2010, 198: 39–123
6 Alexandre R, Desvillettes L, Villani C, et al. Entropy dissipation and long-range interactions. Arch Ration Mech Anal, 2011, 202: 599–661
7 Alexandre R, Desvillettes L, Villani C, et al. Entropy dissipation and long-range interactions. Arch Ration Mech Anal, 2010, 198: 39–123
8 Alexandre R, Desvillettes L, Villani C, et al. Entropy dissipation and long-range interactions. Arch Ration Mech Anal, 2011, 202: 599–661
9 Alexandre R, Desvillettes L, Villani C, et al. Entropy dissipation and long-range interactions. Arch Ration Mech Anal, 2010, 198: 39–123
10 Alexandre R, Desvillettes L, Villani C. On the Boltzmann equation for long-range interactions. Comm Pure Appl Math, 2002, 55: 30–70
11 Alexandre R, Desvillettes L, Villani C. On the Boltzmann equation for long-range interactions. Comm Pure Appl Math, 2002, 55: 30–70
12 Alonso R, Morimoto Y, Sun W, et al. De Giorgi argument for weighted $L^2 \cap L^\infty$ solutions to the non-cutoff Boltzmann equation. arXiv:2010.10065, 2020
13 Alonso R, Morimoto Y, Sun W, et al. Non-cutoff Boltzmann equation with polynomial decay perturbations. Rev Mat Iberoam, 2021, 37: 189–292
15 Barbaroux J-M, Hundertmark D, Ried T, et al. Gevrey smoothing for weak solutions of the fully nonlinear homogeneous Boltzmann and Kac equations without cutoff for Maxwellian molecules. Arch Ration Mech Anal, 2017, 225: 601–661
16 Bouchut F. Hypoelliptic regularity in kinetic equations. J Math Pures Appl (9), 2002, 81: 1135–1159
17 Cercignani C. Mathematical Methods in Kinetic Theory, 2nd ed. New York: Plenum Press, 1990
18 Cercignani C, Illner R, Pulvirenti M. The Mathematical Theory of Dilute Gases. Applied Mathematical Sciences, vol. 106. New York: Springer-Verlag, 1994
19 Chen H, Li W-X, Xu C-J. Propagation of Gevrey regularity for solutions of Landau equations. Kinet Relat Models, 2008, 1: 355–368
20 Chen H, Li W-X, Xu C-J. Analytic smoothness effect of solutions for spatially homogeneous Landau equation. J Differential Equations, 2011, 248: 77–94
21 Chen H, Li W-X, Xu C-J. Gevrey hypoellipticity for a class of kinetic equations. Comm Partial Differential Equations, 2011, 36: 693–728
22 Chen Y, Desvillettes L, He L. Smoothing effects for classical solutions of the full Landau equation. Arch Ration Mech Anal, 2009, 193: 21–55
23 Desvillettes L. About the regularizing properties of the non-cut-off Kac equation. Comm Math Phys, 1995, 168: 417–440
24 Desvillettes L. Regularization properties of the 2-dimensional non-radially symmetric non-cut-off spatially homogeneous Boltzmann equation for Maxwellian molecules. Transp Theory Stat Phys, 1997, 26: 341–357
25 Desvillettes L, Furioli G, Terraneo E. Propagation of Gevrey regularity for solutions of the Boltzmann equation for Maxwellian molecules. Trans Amer Math Soc, 2009, 361: 1731–1747
26 Desvillettes L, Villani C. On the spatially homogeneous Landau equation for hard potentials. I: Existence, uniqueness and smoothness. Comm Partial Differential Equations, 2000, 25: 179–259
27 Desvillettes L, Wennberg B. Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff. Comm Partial Differential Equations, 2004, 29: 133–155
28 DiPerna R J, Lions P-L. On the Cauchy problem for Boltzmann equations: Global existence and weak stability. Ann of Math (2), 1989, 130: 321–366
29 Duan R, Liu S, Sakamoto S, et al. Global mild solutions of the Landau and non-cut-off Boltzmann equations. Comm Pure Appl Math, 2021, 74: 932–1020
30 Glangetas L, Li H-G, Xu C-J. Sharp regularity properties for the non-cut-off spatially homogeneous Boltzmann equation. Kinet Relat Models, 2016, 9: 299–371
31 Golse F, Imbert C, Mouhot C, et al. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. Ann Sc Norm Super Pisa Cl Sci (5), 2019, 19: 253–295
32 Golse F, Lions P-L, Perthame B, et al. Regularity of the moments of the solution of a transport equation. J Funct Anal, 1988, 76: 110–125
33 Golse F, Perthame B, Sentis R. Un résultat de compactité pour les équations de transport et application au calcul de la limite de la valeur propre principale d’un opérateur de transport. C R Acad Sci Paris Sér I Math, 1985, 301: 341–344
34 Gressman P T, Strain R M. Global classical solutions of the Boltzmann equation without angular cut-off. J Amer Math Soc, 2011, 24: 771–847
35 Gressman P T, Strain R M. Sharp anisotropic estimates for the Boltzmann collision operator and its entropy production. Adv Math, 2011, 227: 2349–2384
36 Henderson C, Snelson S. $C^\omega$ smoothing for weak solutions of the inhomogeneous Landau equation. Arch Ration Mech Anal, 2020, 236: 113–143
37 Hérau F, Li W-X. Global hypoelliptic estimates for Landau-type operators with external potential. Kyoto J Math, 2013, 53: 533–565
38 Hérau F, Pravda-Starov K. Anisotropic hypoelliptic estimates for Landau-type operators. J Math Pures Appl (9), 2011, 95: 513–552
39 Hérau F, Tonon D, Tristani I. Regularization estimates and Cauchy theory for inhomogeneous Boltzmann equation for hard potentials without cut-off. Comm Math Phys, 2020, 377: 697–771
40 Hörmander L. The Analysis of Linear Partial Differential Operators. III. Pseudo-Differential Operators. Berlin: Springer-Verlag, 1985
41 Huo Z, Morimoto Y, Ukai S, et al. Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff. Kinet Relat Models, 2008, 1: 453–489
42 Imbert C, Mouhot C. Hölder continuity of solutions to hypoelliptic equations with bounded measurable coefficients. arXiv:1505.04608, 2015
43 Imbert C, Mouhot C. The Schauder estimate in kinetic theory with application to a toy nonlinear model. arXiv:1801.07891, 2018
44 Imbert C, Mouhot C, Silvestre L. Decay estimates for large velocities in the Boltzmann equation without cut-off. J Éc Polytech Math, 2020, 7: 143–184
Appendix A  Some facts on symbolic calculus

We recall here some notations and basic facts of symbolic calculus, and refer to [40, Chapter 18] or [47] for detailed discussions on the pseudo-differential calculus.

Appendix A.1  Weyl-Hörmander calculus

Let $M$ be an admissible weight function with respect to the flat metric $|dv|^2 + |d\eta|^2$, i.e., the weight function $M$ satisfies the following conditions:

(a) (The slowly varying condition) There exists a constant $\delta$ such that

$$\forall X, Y \in \mathbb{R}^6_{v,\eta}, \quad |X - Y| \leq \delta \Rightarrow M(X) \approx M(Y).$$

(b) (Temperance) There exist two constants $C$ and $N$ such that

$$\forall X, Y \in \mathbb{R}^6_{v,\eta}, \quad M(X)/M(Y) \leq C(\langle X - Y \rangle^N.$$
Considering the symbol \( q(\xi, v, \eta) \) as a function of \((v, \eta)\) with the parameter \( \xi \), we say that \( q \in S(M, |dv|^2 + |d\eta|^2) \) uniformly with respect to \( \xi \), if

\[
\forall \alpha, \beta \in \mathbb{Z}_+^3, \quad \forall v, \eta \in \mathbb{R}^3, \quad |\partial_\xi^\alpha \partial_v^\beta q(\xi, v, \eta)| \leq C_{\alpha, \beta} M
\]

with \( C_{\alpha, \beta} \) a constant depending only on \( \alpha \) and \( \beta \) but independent of \( \xi \). For simplicity of notations, in the following discussion, we omit the parameter dependence in the symbols, and by \( q \in S(M, |dv|^2 + |d\eta|^2) \) we always mean that \( q \) satisfies the above inequality uniformly with respect to \( \xi \). The space \( S(M, |dv|^2 + |d\eta|^2) \) endowed with the semi-norms

\[
\|q\|_{k; S(M, |dv|^2 + |d\eta|^2)} = \max_{0 \leq |\alpha| + |\beta| \leq k} \sup_{(v, \eta) \in \mathbb{R}^6} |M(v, \eta)^{-1} \partial_\xi^\alpha \partial_v^\beta q(v, \eta)|
\]

becomes a Fréchet space. Let \( q \in S'(\mathbb{R}^3 \times \mathbb{R}^3) \) be a tempered distribution and let \( t \in \mathbb{R} \). The operator \( \text{op}_t q \) is an operator from \( S(\mathbb{R}^3) \) to \( S'(\mathbb{R}^3) \), whose Schwartz kernel \( K_t \) is defined by the oscillatory integral

\[
K_t(z, z') = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i(z \cdot z') - i(t - 1)z + tz' + z \cdot \zeta} d\zeta.
\]

In particular, we define \( q(D_v) = \text{op}_0 q \) and \( q^w = \text{op}_{1/2} q \). Here, \( q^w \) is called the Weyl quantization of the symbol \( q \).

An elementary property to be used frequently is the \( L^2 \) continuity theorem in the class \( S(1, |dv|^2 + |d\eta|^2) \) (see, for example, [47, Theorem 2.5.1]) which says that there exist a constant \( C \) and a positive integer \( N \) depending only on the dimension such that

\[
\forall u \in L^2, \quad \|q^w u\|_{L^2} \leq C\|q\|_{S(1, |dv|^2 + |d\eta|^2)} \|u\|_{L^2}.
\]  

(A.1)

Let us recall the composition formula of the Weyl quantization. Given \( p_i \in S(M_i, |dv|^2 + |d\eta|^2) \), we have

\[
p_1^w p_2^w = (p_1^w p_2)^w
\]

with \( p_1^w p_2 \in S(M_1 M_2, |dv|^2 + |d\eta|^2) \) admitting the expansion

\[
p_1^w p_2 = p_1 p_2 + \int_0^1 \int \frac{2 \sigma(Y - Y_1, Y - Y_Z)}{2i} \frac{1}{\sigma(\partial_{Y_1}, \partial_{Y_2})} p_1(Y_1) p_2(Y_2) dY_1 dY_2 d\theta / (\pi \theta),
\]

where \( \sigma \) is the symplectic form in \( \mathbb{R}^6 \) given by

\[
\sigma((z, \zeta), (\tilde{z}, \tilde{\zeta})) = \zeta \cdot \tilde{z} - \tilde{\zeta} \cdot z.
\]

Finally, we mention that \( q^w \) is self-adjoint in \( L^2 \) if \( q \) is a real-valued symbol.

**Appendix A.2 The Wick quantization**

Finally, let us recall some basic properties of the Wick quantization, which is also called anti-Wick in [61]. The importance in studying the Wick quantization lies in the fact that positive symbols give rise to positive operators. There are several equivalent ways of defining the Wick quantization and one is defined in terms of coherent states. The coherent states method essentially reduces the partial differential operators to ODEs by virtue of the Wick calculus.

Let \( Y = (v, \eta) \) be a point in \( \mathbb{R}^6 \). The Wick quantization of a symbol \( q \) is given by

\[
q^{\text{Wick}} = (2\pi)^{-3} \int_{\mathbb{R}^6} q(Y) \Pi_Y dY,
\]

where \( \Pi_Y \) is the projector associated with the Gaussian \( \varphi_Y \) which is defined by

\[
\varphi_Y(z) = \pi^{-3/4} e^{-\frac{1}{4} |z - v|^2} e^{iz \cdot \eta/2}, \quad z \in \mathbb{R}^3.
\]
The main property of the Wick quantization is its positivity, i.e.,

\[ q(v, \eta) \geq 0 \quad \text{for all} \quad (v, \eta) \in \mathbb{R}^6 \quad \text{implies} \quad q_{\text{Wick}} \geq 0. \]

According to [61, Theorem 24.1], the Wick and Weyl quantizations of a symbol \( q \) are linked by the following identities:

\[ q_{\text{Wick}} = (q \ast \pi^{-3} e^{-|\cdot|^2})_{\text{w}} = q_{\text{w}} + r_{\text{w}} \]

with

\[ r(Y) = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) q''(Y + \theta Z) Z^2 e^{-|Z|^2} dZ d\theta. \]

As a result, \( q_{\text{Wick}} \) is a bounded operator in \( L^2 \) if \( q \in S(1, g) \) due to (A.1) and self-adjoint in \( L^2 \) if \( q \) is a real-valued symbol.

We also recall the following composition formula obtained in the proof of [46, Proposition 3.4]:

\[ q_{1, \text{Wick}} q_{2, \text{Wick}} = \left[ q_1 q_2 - q_1' \cdot q_2' + \frac{1}{i} \{ q_1, q_2 \} \right]_{\text{Wick}} + \mathcal{T} \quad (A.2) \]

with \( \mathcal{T} \) being a bounded operator in \( L^2(\mathbb{R}^{2n}) \), where \( q_1 \in L^\infty(\mathbb{R}^{2n}) \), \( q_2 \) is a smooth symbol whose derivatives of order \( \geq 2 \) are bounded on \( \mathbb{R}^6 \), and the notation \( \{ q_1, q_2 \} \) denotes the Poisson bracket defined by

\[ \{ q_1, q_2 \} = \frac{\partial q_1}{\partial \eta} \cdot \frac{\partial q_2}{\partial v} - \frac{\partial q_1}{\partial v} \cdot \frac{\partial q_2}{\partial \eta}. \quad (A.3) \]