The Absolute Center of $p$-Groups of Maximal Class

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Abstract. The purpose of this paper is to determine $L(G)$, the absolute center of the group $G$, when $G$ is a $p$-group of maximal class. Particularly we find $L(G)$ for metabelian $p$-groups of maximal class, all $p$-groups of maximal class of order less than $p^6$ and $p$-groups of maximal class for $p = 2, 3$.

1. Introduction

In 1994, Hegarty [6] introduced $L(G)$, the absolute center of a group $G$ as follows: $L(G) = \{g \in G \mid g^a = g \text{ for all } a \in \text{Aut}(G)\}$. As we see there is an analogue between $L(G)$ and $Z(G)$, on the other hand we may define $Z(G) = \{g \in G \mid g^a = g \text{ for all } a \in \text{Inn}(G)\}$. Obviously $L(G) \leq Z(G)$. Hegarty [6] proved an analogue of Schur’s theorem for the absolute center, that is, if $G$ is a group such that $G/L(G)$ is finite, then $\langle g^{-1}g^a \mid g \in G, a \in \text{Aut}(G)\rangle$ is also finite. Moreover Meng and Guo [12] explore the relationship between $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group $G$, they also determine the structure of the absolute center of all finite minimal non-abelian $p$-groups.

In this paper we study $L(G)$ for $p$-groups of maximal class. As the definition of $L(G)$ shows, studying $L(G)$ directly depends on the structure of $\text{Aut}(G)$. Therefore we use a structure of the Sylow $p$-subgroup of $\text{Aut}(G)$ for metabelian $p$-groups of maximal class from our paper [5] and also the structure of $p$-automorphism of $p$-groups of maximal class from [13] to prove our main theorem. Moreover we need the concept of the degree of commutativity of $p$-groups of maximal class. Specially we prove that $|L(G)| = 2$ for all $2$-groups of maximal class, $L(G) = 1$ for all $3$-groups of maximal class and also $L(G) = 1$ for $p$-groups of maximal class of order $p^6$. Moreover we show that there is only one group of maximal class of order $p^5$ with $|L(G)| = p$ and all other groups of maximal class of order $p^5$ have trivial absolute center (See Theorem 2.12). Furthermore we determine the absolute center for all metabelian $p$-groups of maximal class (See corollaries 2.5, 2.6 and Theorem 2.8).

Throughout this paper the following notation is used. The terms of the lower and the upper central series of $G$ are denoted by $\gamma_i(G)$ and $Z_i(G)$, respectively. The centre of $G$ is denoted by $Z = Z(G)$. If $a$ is an automorphism of $G$ and $x$ is an element of $G$, we write $x^a$ for the image of $x$ under $a$. For a normal subgroup $N$ of $G$, we let $\text{Aut}^N(G)$ denote the group of all automorphisms of $G$ centralizing $G/N$. Let $H \leq G$ and $A \leq \text{Aut}(G)$, we note that $C_A(H) = \{a \in A \mid h^a = h, \forall h \in H\}$ and $C_H(A) = \{h \in H \mid h^a = h, \forall a \in A\}$. The Frattini subgroup of $G$ is denoted by $\Phi = \Phi(G)$ and $\text{Aut}_p(G)$ for the Sylow $p$-subgroup of $\text{Aut}(G)$. Also we use the notation $x \equiv y \pmod H$ to indicate that $Hx = Hy$, where $H$ is a subgroup of a group $G$ and $x, y \in G$. Let $(a, p) = 1$, we note that $\text{ord}_p(a)$ is the smallest positive integer $t$ such that $a^t \equiv 1 \pmod p$. All unexplained notation is standard and follows that of [9].
2. Main results

Let $G$ be a $p$-group of maximal class of order $p^n$ ($n \geq 3$), where $p$ is a prime. We note that if $n = 3$, then $L(G) = 1$ for $p > 2$ and $L(G) = Z(G)$ for $p = 2$. Therefore in the rest of the paper we assume that $n \geq 4$. Following [9], we define the 2-step centralizer $K_i$ in $G$ to be the centralizer in $G$ of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)/P_{i-1}(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of $G$ is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j}$ for all $i, j \geq 1$ if $P_1$ is not abelian and $l = n - 2$ if $P_1$ is abelian.

Take $s \in G - \bigcup_{i=2}^{n-1} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n - 1$. It is easily seen that $\{s, s_1\}$ is a generating set for $G$ and $P_1(G) = \langle s_1, \ldots, s_{n-1} \rangle$ for $1 \leq i \leq n - 1$ and so $Z(G) = P_{n-1}(G) = \langle s_{n-1} \rangle$. For the rest of the paper we fix the above notation.

By [9, Corollary 3.2.7] and [2, Corollary p.59] we have the following result.

**Lemma 2.1.** Let $G$ be a $p$-group of maximal class of order $p^n$.

(i) The degree of commutativity of $G$ is positive if and only if the 2-step centralizers of $G$ are all equal.

(ii) If $G$ is metabelian then $G$ has positive degree of commutativity.

**Lemma 2.2.** [7, Hilfssatz III. 14.13] If $G$ is a $p$-group of maximal class of order $p^n$ and $s \notin K_i$ for $2 \leq i \leq n-2$, then $C_G(s) = \langle s \rangle P_{n-1}(G)$ and $s' \in P_{n-1}(G)$.

**Theorem 2.3.** [3, Theorem 3.2] Let $G = \langle a, b \rangle$ be a two-generated metabelian group. Then the following are equivalent:

(i) For all $u, v \in G'$, there is an automorphism of $G$ that maps $a$ to $au$ and $b$ to $bv$;

(ii) $G$ is nilpotent.

By the above theorem we see that if $G$ is a metabelian $p$-group of maximal class of order $p^n$, then for any elements $x, y \in G'$ there is an automorphism that maps $s$ to $sx$ and $s_i$ to $s_i y$ hence $| Aut(G)| = p^{2n-4}$. Moreover $\frac{Aut(G)}{Aut^p(G)} \hookrightarrow Aut(\frac{G}{Z(G)})$ and so $| AUT_p(G) : Aut^p(G)|$ divides $p$, since $\frac{G}{Z(G)} \cong Z_p \times Z_p$.

**Lemma 2.4.** If $G$ is a $p$-group of maximal class of order $p^n$, then $AUT_p(G)$ fix $Z(G)$ elementwise.

**Proof.** Consider the action of $AUT_p(G)$ on $Z(G)$. It is obvious that $C_{Z(G)}(AUT_p(G)) \neq 1$ since $AUT_p(G)$ and $Z(G)$ are $p$-groups. As $|Z(G)| = p$, we have $C_{Z(G)}(AUT_p(G)) = Z(G)$, which completes the proof.

**Corollary 2.5.** If $G$ is a $p$-group of maximal class of order $p^n$ and $AUT(G)$ is also a $p$-group, then $L(G) = Z(G)$.

**Proof.** This is obvious by the fact that $L(G) \leq Z(G) \cong Z_p$ and Lemma 2.4.

**Corollary 2.6.** Let $G$ be a 2-group of maximal class of order $2^n$, then $L(G) = Z(G)$.

**Proof.** By [5, Theorem 5.9], we see that $AUT(G)$ is also a 2-group which completes the proof by using Corollary 2.5.

**Lemma 2.7.** Let $G$ be a $p$-group of maximal class of order $p^n$. If $\delta \in AUT(G)$ with $s^\delta = s^\delta x$ and $s_i^\delta = s_i^\delta y$, where $x, y \in \Phi(G)$ and $0 < a, c < p$. Then $s_i^{\delta_{n-1}} = s_i^{\delta_{n-1}} c$.

**Proof.** By induction on $m$ we have $\langle s_i^m, s \rangle \equiv s_i^{m+1} \pmod{\gamma_{i+2}(G)}$ and so $\langle s_i^m, s \rangle \equiv s_i^{m+1} \pmod{\gamma_{i+2}(G)}$ for $\ell, i \geq 1$. Therefore by induction on $i$ we see that $s_i^\delta \equiv s_i^{\delta_i} \pmod{\gamma_{i+1}(G)}$, as required.
Now for the rest of the paper by using corollaries 2.5 and 2.6 we may assume that $G$ is a metabelian $p$-group of maximal class of order $p^n(p > 2)$ and $\text{Aut}(G)$ is not $p$-group. It is straightforward to see that when $p$ is odd, $\text{Aut}(G)$ is supersolvable and is a split extension of $\text{Aut}_p(G)$ by a subgroup of the direct product of two cyclic groups of order $p - 1$. On the other hand, if $H$ be a $p'$-subgroup of $\text{Aut}(G)$, then we have $\text{Aut}(G) = \text{Aut}_p(G) \rtimes H$ and $H$ is embedded in $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$ (see [1] Section 1). Since $P_1(G)$ and $\Phi(G)$ are characteristic subgroups of $G, G/\Phi(G)$ and $P_1/\Phi(G)$ are invariant under $H$. So by Maschke’s Theorem there exists $s \in G - P_1$ such that $G/\Phi(G) = P_1(\Phi(G)) \times (\Phi(G), s)/\Phi(G)$ and $(\Phi(G), s)/\Phi(G)$ is invariant under $H$. In the rest of the paper $s$ will be as above. Therefore if $\delta \in H$ then $s^\delta = s^\alpha x$ and $s^\delta = s^\beta y$, where $x, y \in \Phi(G)$ and $0 < a, c < p$. We recall that if $G$ is metabelian $p$-group of maximal class, then $G$ has positive degree of commutativity and $|s|$ divides $p^2$ by Lemma 2.2. In the next theorem we find the absolute center for finite metabelian $p$-group of maximal class when $H \neq 1$.

**Theorem 2.8.** Let $G$ be a metabelian $p$-group of maximal class of order $p^n(p > 2)$ and $H \neq 1$. If $H$ is not cyclic, then $L(G) = 1$. Let $H$ be cyclic such that $H = \langle \delta \rangle$ with $s^\delta = s^\alpha x$, $s^\delta = s^\beta y$, where $1 \leq a, c < p$ and $x, y \in \Phi(G)$.

(i) If $|s| = p^2$, then $L(G) = 1$.

(ii) If all elements out of $P_1$ have order $p$, then

(a) if $\text{ord}_p(c) \nmid \text{ord}_p(a)$, then $L(G) = 1$.

(b) if $\text{ord}_p(c) | \text{ord}_p(a)$, then there exists $0 \leq r < \text{ord}_p(a)$ such that $c \equiv a^r \pmod{p}$. On setting $\text{ord}_p(a) = t$ we have $L(G) = \mathbb{Z}(G)$ when $t | n - 2 + r$ and $L(G) = 1$ when $t \nmid n - 2 + r$.

**Proof.** By [13, Theorem A], we have $C_H(Z(G))$ is cyclic. Hence there exists $\alpha \in H$ such that $C_Z(G)(\alpha) \neq Z(G)$. As $|Z(G)| = p$ we deduce that $C_Z(G)(\alpha) = 1$, which completes the proof, since $L(G) \leq C_Z(G)(\alpha)$.

(i) Since $\delta \notin \text{Aut}^G(G)$, we have $(a, c) \neq (1, 1)$. By Lemma 2.7, if $a = 1$ then $s^\delta_{n-1} = s^\beta_{n-1} = s_{n-1} - 1$, as desired.

If $a > 1$, then by Theorem 2.3, the map $\beta$ defined by $s^\delta = s^{\alpha} x^{-1}$ and $s^\beta = s_1 w^{-1}$, where $u^\delta = x$ and $w^\delta = y$, is an automorphism of $G$ lying in $\text{Aut}^G(G)$. On setting $\alpha = \beta \delta$, we see that $s^\alpha = s^\delta x$ and $s^\alpha_1 = s^\beta_1$ and so $(s^\beta)^t = s^{\beta t} \neq s^\delta$. Moreover by Lemma 2.2, $Z(G) = (s^\delta)$, which completes the proof.

(ii) The map $\beta$ defined by $s^\delta = s^{\alpha} x^{-1}$ and $s^\beta = s_1 w^{-1}$, where $u^\delta = x$ and $w^\delta = y$, is an automorphism of $G$ lying in $\text{Aut}^G(G)$. On setting $\alpha = \beta \delta$ and $\text{ord}_p(a) = t$, we have $s^{\alpha t} = s$ and $s^{\alpha t}_1 = s^\beta_1$ and so by Lemma 2.7, $s^\alpha_{n-t} = s_{n-t} - 1$ since $\text{ord}_p(c) \nmid t$.

(ii) The first we see that $1, a, \ldots, a^{-1}$ are all distinct roots of the equation $x^t \equiv 1 \pmod{p}$. Therefore there exists $0 \leq r < t$ such that $c \equiv a^r \pmod{p}$. Now by Lemma 2.7, $s^\alpha_{n-t} = s^\alpha_{n-t-r}$, which completes the proof.

In what follows first we find the absolute center for all finite 3-groups of maximal class and finally we obtain the absolute center for all $p$-groups of maximal class of order $p^n$, where $4 \leq n \leq 5$.

**Lemma 2.9.** Let $G$ be a $p$-group of maximal class of order $p^n(p > 2)$ and $H \neq 1$. If $P_1$ is abelian, then $L(G) = 1$.

**Proof.** First we may assume that $|s| = p$ by Theorem 2.8. Now we see that any element of $G$ is uniquely determined by $s^t u$, where $0 \leq t < p$ and $u \in P_1$. Assume that $1 < b < p$, we define $\beta : G \to G$ by $(s^t u)^\delta = s^t u^\delta$, and we show that $\beta$ is an automorphism. Let $g_1 = s^t u$ and $g_2 = s^{t'} u'$, where $0 \leq t, t' < p$ and $u, u' \in P_1$. We may write $g_1 g_2 = s^{t+t'}[s^t, u^t] u^{t'}$. If $t + t' \equiv r \pmod{p}$, then $s^{t+t'} = s^r$ since $|s| = p$ and so $(g_1 g_2)^\delta = s^{t+t'}[s^t, u^t] u^{t'}$. Moreover $g_1 g_2 = s^{t+t'}[s^t, u^t] u^{t'}$. We have $[s^t, u^t] = [s^t, u^t]$ since $P_1$ is abelian and also $\beta$ is a homomorphism. Also $\beta$ is onto since $G = \langle s, s^t \rangle$. Thus $\beta$ is an automorphism. Furthermore $s^\delta_{n-1} = s^\beta_{n-1} - 1$, which completes the proof since $L(G) \leq Z(G) = \langle s_{n-1} \rangle$.

**Lemma 2.10.** Let $G$ be a 3-group of maximal class of order $3^n (n \geq 4)$, then $L(G) = 1$.
Proof. First we see that for \( n = 4 \), \( G \) is metabelian; and for \( n \geq 5 \), \( G \) has degree of commutativity \( n - 4 \) by [2, Theorem 3.13] and so is metabelian. Moreover by [5, Theorem 5.8], \( L(G) = 1 \). Furthermore if \( P_1 \) is not abelian, then by observing the proof of [5, Theorem 5.6 (i)], we have \( H = \langle \beta_3 \rangle \) when \( n \) is odd and \( H = \langle \beta_2 \rangle \) when \( n \) is even, where \( \phi = s^{-1}, s_1^4 = s_1 \) and \( \phi^5 = s^{-1} \). Note that \( s^{-1} = s^2s^{-3} \) and \( s^{-3} \in \Phi(G) \). Therefore Lemma 2.7 completes the proof. \( \square \)

Lemma 2.11. Let \( G \) be a p-group of maximal class of order \( p^4 \) (\( p > 2 \)). Then \( L(G) = 1 \).

Proof. First we see that \( H \neq 1 \) by [11, Lemma 9]. Since \( P_1 = C_G(\gamma_2(G)) \), we have \( \gamma_2(G) \leq Z(P_1) \leq P_1 \) which implies that \( P_1 / Z(P_1) \) is cyclic and so \( P_1 \) is abelian, as desired. \( \square \)

Now for \( p > 3 \), Curran [4, Corollary 5] shown that there is only one group of order \( p^5 \) whose automorphism group is also a p-group in which \((p - 1, 3) = 1 \). The presentation of this group is as follows:

\[
G_0 = \langle a_1, a | a^2 = [a_1, a]^p = [a_1, a, a]^p = [a_1, a, a, a]^p = [a_1, a, a, a] = 1 \rangle
\]

\[
a_1^p = [a_1, a, a, a] = [a_1, a_1, a_1]^{-1}
\]

We note that \( G_0 \) is of maximal class. By this observation we state the following theorem.

Theorem 2.12. Let \( G \) be a p-group of maximal class of order \( p^5 \) with \( p > 3 \). If \( G = G_0 \) then \( L(G) = Z(G) \), for otherwise \( L(G) = 1 \).

Proof. First we claim that \( G \) is metabelian. To prove this we have \([\gamma_2(G), Z_2(G)] = 1 \) and so \( \gamma_3(G) = Z_2(G) \leq Z(\gamma_2(G)) \leq \gamma_2(G) \), which implies that \( \gamma_2(G) \) is abelian. If \( G = G_0 \) then Corollary 2.5 completes the proof. Therefore for the rest of the proof we may assume that \( H \neq 1 \). Since \( p > 5 \), by using [9, Proposition 3.3.2] we have \( \exp(G/Z(G)) = \exp(G) = p \) which yields that \( \Omega_1(G) \leq Z(G) \equiv Z_p \). Moreover by [9, Lemma 1.2.11] \( G \) is regular. Now if \( \Omega_1(G) = Z(G) \), then \( |\Omega_1(G)| = p^4 \). Hence \( \Omega_1(G) \) is a maximal subgroup of \( G \) and \( \Omega_1(G) = [x \in G | x^p = 1] \) since \( G \) is regular. On setting \( s \in G - (P_1 \cup \Omega_1(G)) \), we have \( |s| = p^3 \) and so \( L(G) = 1 \) by Theorem 2.8. If \( \Omega_1(G) = 1 \), then \( \exp(G) = p \). Now from James’s list [8], there are only two families \( \Phi_9 \) and \( \Phi_{10} \) of groups of maximal class of order \( p^5 \). By observing the presentation of these groups, we see that only \( \Phi_9(1^5) \) and \( \Phi_{10}(1^5) \) are of exponent \( p \). Now if \( G = \Phi_9(1^5) \) with the following presentation:

\[
\langle s, s_1, \ldots, s_4 \mid [s_i, s] = s_{i+1}, s^p = s_1^p = 1 (1 \leq i \leq 4) \rangle
\]

then obviously \( P_1 \) is abelian and so \( L(G) = 1 \) by Lemma 2.9. Furthermore if \( G = \Phi_{10}(1^5) \) with the presentation

\[
\langle s, s_1, \ldots, s_4 \mid [s_i, s] = s_{i+1}, [s_1, s_2] = s_4, s^p = s_1^p = 1 (1 \leq i \leq 4) \rangle
\]

then the map \( \alpha \) defined by \( s^i = s^{-1}, s_1^4 = s_1 \) is an automorphism of order \( 2 \) and it is easily seen that \( s_4^2 = s_4^{-1} \), completing the proof. \( \square \)

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References

[1] A. H. Baartmans, J. J. Woeppe, The automorphism group of a p-group of maximal class with an abelian maximal subgroup, Fund. Math. 93 (1976) 41–46.

[2] N. Blackburn, On a special class of p-groups, Acta Math. 100 (1958) 45–92.

[3] A. Caranti and C. M. Scoppola, Endomorphisms of two-generated metabelian groups that induce the identity modulo the derived subgroup, Arch. Math. 56 (1991) 218–227.

[4] M. J. Curran, Automorphisms of certain p-groups (p odd), Bull. Aust. Math. Soc. 38 (1988) 299–305.

[5] S. Fouladi and R. Orfi, Automorphisms of metabelian prime power order groups of maximal class, Bull. Aust. Math. Soc. 77 (2008) 261–276.

[6] P. Hegarty, The absolute center of a group, J. Algebra 169 (1994) 929–935.
[7] B. Huppert, Endliche Gruppen , Vol. 1, Springer-Verlag, 1967.
[8] Rodney James, The Groups of Order $p^n$ (p an Odd Prime), Math. Comp. 34 (1980) 613–637.
[9] C. R. Leedham-Green, S. McKay, The structure of groups of prime power order, Oxford University Press, 2002.
[10] H. Liebeck, The automorphism group of finite $p$-groups, J. Algebra 4 (1966) 426–432.
[11] D. MacHale, Some finite groups which are rarely automorphism groups II, Math. Proc. R. Ir. Acad. 83A (1983) 189–196.
[12] H. Meng and X. Guo, The absolute center of finite groups, J. Group Theory 18 (2015) 887–904.
[13] B. Wolf, A note on $p'$-automorphism of $p$-groups $P$ of maximal class centralizing the center of $P$, J. Algebra 190 (1997) 163–171.