ON THE $K$-THEORETIC HALL ALGEBRA OF A SURFACE

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Abstract. In this paper, we define the $K$-theoretic Hall algebra for 0-dimensional coherent sheaves on a smooth projective surface, prove that the algebra is associative and construct a homomorphism to a redefined shuffle algebra analogous to Negut [9].

1. Introduction

Let $S$ be a smooth surface. We consider the stack $\text{Coh}_n$ of 0-dimensional length $n$ coherent sheaves on $S$, which can be represented by a quotient stack:

$$\text{Coh}_n = [\text{Quot}_n^{\circ}/\text{GL}_n]$$

where $\text{Quot}_n^{\circ}$ is studied in Section 3. Therefore, the Grothendick group of $\text{Coh}_n$ can be represented by

$$K(\text{Coh}_n) = K^{\text{GL}_n}(\text{Quot}_n^{\circ})$$

and we denote $K\text{Coh}$ the abelian group

$$K\text{Coh} = \bigoplus_{n=0}^{\infty} K(\text{Coh}_n).$$

Schiffmann and Vasserot ([11]) expect there is an algebra structure on $K\text{Coh}$, which is called the $K$-theoretic Hall algebra of $S$. The general principle for constructing the $K$-theoretic Hall algebra is to consider the stack $\text{Corr}_{n,m}$ of short exact sequences

$$0 \to E_n \to E_{n+m} \to E_m \to 0$$

where $E_n \in \text{Coh}_n$, $E_m \in \text{Coh}_m$ and $E_{n+m} \in \text{Coh}_{n+m}$, for any two non-negative integers $n, m$. There is a natural diagram:

$$\{0 \to E_n \to E_{n+m} \to E_m \to 0\}$$

which induces morphisms:

$$\text{Corr}_{n,m}$$

where $q$ is a proper map. They expect that there is an appropriate definition of pull back map $p^! : K(\text{Coh}_n \times \text{Coh}_m) \to K(\text{Corr}_{n,m})$ and push forward
$q_*: K(\text{Corr}_{n,m}) \to K(\text{Coh}_{n+m})$ such that $q_* \circ p^!$ induces an associative algebra structure of $K\text{Coh}$.

In the case $S = k^2$, Schiffmann and Vasserot (11) studied an equivariant version of $K$-theoretic Hall algebra. In the case $S = T^* C$ where $C$ is a smooth curve, Minets (7) studied an analogous moduli stack, the moduli stack of Higgs sheaves. Their work rely on a canonical embedding of $\text{Quot}_n^\circ$ into some smooth schemes, which seems difficult to be generalized to all the surfaces.

In this paper, we represent $\text{Corr}_{n,m}$ as a quotient stack $\text{Corr}_{n,m} = [\text{Flag}_{n,m}^\circ/P_{m,n}]$ in Section 3. Instead of embedding $\text{Quot}_n^\circ$ into a smooth scheme, we consider a resolution of universal quotients $0 \to V_m \to W_m \to \mathcal{O}^m \to \mathcal{E}_m \to 0$

over $\text{Quot}_n^\circ \times S$ and define two vector bundles $V_{n,m}$ and $W_{n,m}$ over $\text{Quot}_n^\circ \times \text{Quot}_m^\circ$. We observe there is a Cartesian diagram (3.3):

$$
\begin{array}{ccc}
\text{Flag}_{n,m}^\circ & \to & W_{n,m} \\
\downarrow & & \downarrow \psi_{n,m} \\
\text{Quot}_n^\circ \times \text{Quot}_m^\circ & \to & V_{n,m}
\end{array}
$$

where $\psi_{n,m}$ is a locally complete intersection morphism. We use $\psi_{n,m}^!$ to define $p^!$ and prove that it does not depend on the choice of resolutions. Hence we give the appropriate definition of $K$-theoretic Hall algebra in Definition 4.2. This generalize the work of Schiffmann and Vasserot (11) and Minets (7). Moreover, based on the techniques of refined Gysin maps between two vector bundles which we develop in Section 2.3, we prove that the $K$-theoretic Hall algebra is associative.

**Theorem 1.1 (Theorem 4.4).** The $K$-theoretic Hall algebra $(K\text{Coh}(S), \ast K\text{Coh})$ is associative.

Another question we are considering in the paper is the relation between the $K$-theoretic Hall algebra and the shuffle algebra. The shuffle algebra is considered by Schiffmann and Vasserot (12) for quivers and Negut (5) for surfaces.

The idea is to consider $T_n \subset GL_n$ which is the maximal torus formed by diagonal matrices. The fixed locus $(\text{Quot}_n^\circ)^{T_n} = (\text{Quot}_1^\circ)^n = S^n$ in Lemma 3.14. By Theorem 2.8 and Thomason localization theorem 2.10, we have

$$K^{G_n}(\text{Quot}_n^\circ) = (K^{T_n}(\text{Quot}_n^\circ))^{\sigma_n}$$

and

$$K^{T_n}(\text{Quot}_n^\circ)|_{\text{loc}} = K^{T_n}(S^n)|_{\text{loc}}$$

where $\sigma_n$ is the permutation group of order $n$.

Let

$$K\text{Sh} = \bigoplus_{n=0}^{\infty} K^{T_n}(S^n)^{\sigma_n}_{\text{loc}}.$$

Then it is natural to expect their is a algebra on $K\text{Sh}$ and the localization theorem will induce an algebra homomorphism from $K\text{Coh}$ to $K\text{Sh}$.

For this question, we redefined the shuffle algebra in Definition 5.1 and prove there is a homomorphism $\tau$ in (5.2) from $K\text{Coh}$ to $K\text{Sh}$:
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Theorem 1.2 (Theorem 5.3). \( \tau \) is an algebra homomorphism between \((K\text{Coh}, \star^{K\text{Coh}})\) and \((K\text{Sh}, \star^{K\text{Sh}})\).

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2. Equivariant K-theory

In this section, we recall some basic facts about equivariant K-theory. There are many references for this material, like \[2\], \[3\], \[1\], and \[10\]. We work in the category of separated schemes of finite type over a field \(k\), equipped with an action of a simply connected, reductive group \(G\). All morphisms are equivariant with respect to the action of \(G\).

2.1. Grothendieck groups. The Grothendieck group of equivariant coherent sheaves \(K^G(X)\) is generated by classes \([F]\) for each \(G\)-equivariant coherent sheaf \(F\) on \(X\), subject to the relation \([F] = [G] + [H]\) for any exact sequence of \(G\)-equivariant coherent sheaves

\[0 \to G \to F \to H \to 0\]

Given \(E\) a rank \(n\) locally free sheaf over \(X\), we define \([\wedge^\bullet(E)] = \sum_{i=0}^n (-1)^i [\wedge^i E]\).

2.2. Locally Complete Intersection Morphisms and Refined Gysin Maps.

Definition 2.1. We define \(f : X \to Y\) to be a locally complete intersection morphism (short for l.c.i. morphism) if \(f\) is the composition of a regular embedding and a smooth morphism.

Given a Cartesian diagram

\[
\begin{array}{ccc}
X' & \overset{g'}{\longrightarrow} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \overset{f'}{\longrightarrow} & Y
\end{array}
\]

(2.1)

where \(f\) is a l.c.i. morphism, then \(f\) has finite Tor-dimension, i.e. for any \(F \in \text{Coh}(Y)\), \(\text{Tor}_i^O_Y(O_X, F) = 0\) for \(i \gg 0\) if regarded \(O_Y\) as a \(O_Y\)-module. We define the refined Gysin map \(f^! : K(Y') \to K(X')\) by

\[ [F] \to \sum_{i=0}^\infty (-1)^i [\text{Tor}_i^Y(O_X, F)]. \]

This construction also holds if \(F\) has an equivariant structure.

The refined Gysin map has the following properties:

Lemma 2.2 (Lemma 3.1 of \[1\]). Consider following Cartesian diagrams

\[
\begin{array}{ccc}
X'' & \overset{h'}{\longrightarrow} & X' \\
\downarrow{h} & & \downarrow{g'} \\
Y'' & \overset{h}{\longrightarrow} & Y'
\end{array}
\]

(2.2)
where \( h \) is proper and \( f \) is a l.c.i. morphism. Then \( f^! h_* = h'_* f^! : K(Y'') \to K(X') \).

**Lemma 2.3** (Lemma 3.2 of [1]). Consider following Cartesian diagrams \( 2.3 \):

\[
\begin{array}{ccc}
X'' & \rightarrow & Y'' \rightarrow Z'' \\
\downarrow & & \downarrow h \\
X' & \rightarrow & Y' \rightarrow Z' \\
\downarrow & & \downarrow \\
X & \rightarrow & Y.
\end{array}
\]

(2.3)

such that \( h \) and \( f \) are l.c.i. morphisms. Then \( f^! h_* = h'_* f^! \) as two morphisms from \( K(Y') \) to \( K(X'') \).

**Lemma 2.4.** Consider following Cartesian diagrams:

\[
\begin{array}{ccc}
X'' & \rightarrow & X' \rightarrow X \\
\downarrow & & \downarrow f' \\
Y'' & \rightarrow & Y' \rightarrow Y \\
\downarrow & & \downarrow s \\
\end{array}
\]

where \( f \) and \( f' \) are l.c.i. morphisms. If one of the \( f \) and \( s \) is flat, then \( f^! = f''^! \) as two morphisms from \( K(Y'') \) to \( K(X'') \). Moreover, if \( f \) is flat, then \( f^! = f''^! \).

**Proof.** Recall the Tor spectral sequence:

\[
\text{Tor}^Y_{p+q}(\mathcal{F}, \mathcal{O}_X) \Rightarrow \text{Tor}^Y_{p}(\mathcal{F}, \text{Tor}^Y_{q}(\mathcal{O}_{Y'}, \mathcal{O}_X)).
\]

(2.4)

Since \( f \) or \( s \) is flat, we have \( \text{Tor}^Y_q(\mathcal{O}_{Y'}, \mathcal{O}_X) = 0 \) for \( q > 0 \) and \( \text{Tor}^Y_0(\mathcal{O}_{Y'}, \mathcal{O}_X) = \mathcal{O}_{Y'} \). Hence

\[
\sum_{i=0}^\infty (-1)^i \text{Tor}^Y_i(\mathcal{F}, \mathcal{O}_X) = \sum_{p=0}^\infty \sum_{q=0}^\infty (-1)^{p+q} \text{Tor}^Y_p(\mathcal{F}, \text{Tor}^Y_q(\mathcal{O}_{Y'}, \mathcal{O}_X)) = \sum_{i=0}^\infty (-1)^i \text{Tor}^Y_i(\mathcal{F}, \mathcal{O}_{Y'}),
\]

for any \( \mathcal{F} \in \text{Coh}(Y'') \). \( \square \)

**Lemma 2.5.** Consider the following Cartesian diagram:

\[
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \rightarrow & Y \\
\downarrow & & \downarrow g \\
Z' & \rightarrow & Z
\end{array}
\]

where \( f, g \) and \( h = g \circ f \) are l.c.i morphisms. Then \( h^! = f^! \circ g' \) as two morphisms from \( K(Z') \) to \( K(X') \).

**Proof.** It is a corollary of Tor spectral sequence \( 2.4 \) \( \square \)
2.3. **Morphisms Between Vector Bundles.** Now we assume $V$ and $W$ are two locally free sheaves over $X$. Their total spaces are to two vector bundles and we still denote them $V$ and $W$.

**Remark 2.6.** In this paper, we will not distinguish locally free sheaves and vector bundles which are their total spaces.

Let $\phi : V \to W$ be a linear morphism, and let $Y = \phi^{-1}(0)$ be the pre-image of zero section of $W$. Then we have the following cartesian diagram:

$$
\begin{array}{ccc}
Y & \longrightarrow & V \\
\downarrow & & \downarrow \rho \\
V & \xrightarrow{pr_V} & V \times_X W \\
\downarrow pr_V & & \downarrow pr_W \\
X & \longrightarrow & W \\
\end{array}
$$

where $pr_V$ and $pr_W$ are projection maps and $i_W$ is the zero section. $\rho = (id, \phi)$. Then $\phi = pr_W \circ \rho$ is a l.c.i. map, and $\phi^! : K(X) \to K(Y)$ is well defined.

**Lemma 2.7.** Let

$$
\begin{array}{ccc}
0 & \longrightarrow & W_1 \xrightarrow{g'} W_2 \xrightarrow{f'} W_3 \longrightarrow 0 \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
0 & \longrightarrow & V_1 \xrightarrow{g} V_2 \xrightarrow{f} V_3 \longrightarrow 0 \\
\end{array}
$$

be a commutative diagram of vector bundles where all the rows are exact sequences. Let $X_3 = \phi_3^{-1}(0)$ and $X_2 = \phi_2^{-1}(0)$. Let $Y = f'^{-1}(X_3)$, then $Y$ is an affine bundle over $X_2$. Moreover we have a cartesian diagram:

$$
\begin{array}{ccc}
X_2 & \longrightarrow & Y \\
\downarrow & & \downarrow \psi \\
X_3 & \xrightarrow{i_{V_1} \times id} & V_1 \times_X X_3 \\
\end{array}
$$

where $\psi$ is a l.c.i. morphism, and $\psi \circ \phi_3^! = \phi_2^!$ as morphisms from $K(X)$ to $K(X_2)$.

**Proof.** We have the following two Cartesian diagrams:

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
W_2 & \xrightarrow{f'} & W_3 \xrightarrow{\phi_3} V_3 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X \\
\downarrow & & \downarrow \phi_3^! \\
Y & \xrightarrow{pr_{V_1}} & V_1 \longrightarrow X \\
\downarrow & & \downarrow \phi_2 \\
W_2 & \xrightarrow{\phi_2} & V_2 \xrightarrow{f} V_3 \\
\end{array}
$$
Then we have
\[ \phi_2^i = \phi_2^i \circ i_{V_1}^i \circ pr_{V_1}^i \]
\[ = i_{V_1}^i \circ \phi_2^i \circ f^* \quad \text{by Lemma 2.3} \]
\[ = i_{V_1}^i \circ f_\gamma^i \circ \phi_3^i \quad \text{by Lemma 2.5} \]
as morphisms \( K(X) \to K(X_2) \). And we also have the following Cartesian diagram:

\[
\begin{array}{ccc}
X_2 & \longrightarrow & X_3 \longrightarrow & X \\
\downarrow & & \downarrow \quad i_{V_1} & \\
Y & \xrightarrow{\psi} & V_1 \times_X X_3 \longrightarrow & V_1 \\
\downarrow & & \downarrow \quad pr_{V_1} & \\
X_3 & \longrightarrow & \phantom{V_1} & X \\
\end{array}
\]

Thus
\[ \psi^i = \psi^i \circ i_{V_1}^i \circ pr_{V_3}^i \]
\[ = i_{V_1}^i \circ \psi^i \circ pr_{V_3}^i \quad \text{by Lemma 2.5} \]
\[ = i_{V_1}^i \circ ((pr_{V_1} \times_X id) \circ \psi)^! \quad \text{by Lemma 2.5} \]
\[ = i_{V_1}^i \circ f_\gamma^i \]
as morphisms from \( K(X_3) \) to \( K(X_2) \). Hence we have \( \psi^i \circ \phi_3^i = \phi_2^i \). \( \square \)

2.4. Localization Theorem. Let \( T = (m)^n \) be the maximal torus of \( G \). Then following theorems compare \( K^G(X) \) and \( K^T(X^T) \), where \( X^T \) is the fixed locus of \( X \) with respect to the \( T \) action. We denote \( i_X \) the closed embedding from \( X^T \) to \( X \). We also denote the representation ring of \( T \) by \( \text{Rep}(T) \).

**Theorem 2.8** (Theorem 6.1.22 of [2]). If \( G \) is simply connected, then the natural restriction map \( K^G(X) \to K^T(X^T) \) gives rise to an isomorphism \( K^G(X) = K^T(X)^W \).

**Definition 2.9.** Given a \( \text{Rep}(T) \) module \( M \), we define \( M_{\text{loc}} = M \otimes_{\text{Rep}(T)} \text{Frac}(\text{Rep}(T)) \), where \( \text{Frac}(\text{Rep}(T)) \) is the fraction field of \( \text{Rep}(T) \).

**Theorem 2.10** (Thomason localization theorem, Theorem 2.2 of [13]). The map:

\[ K^T(X^T)_{\text{loc}} \xrightarrow{i_X} K^T(X)_{\text{loc}} \]

is an isomorphism. Moreover, if \( i_X \) is a regular embedding, then

\[ K^T(X)_{\text{loc}} \xrightarrow{i_X^*} K^T(X^T)_{\text{loc}} \]

is also an isomorphism.

One corollary of the localization theorem is that

**Lemma 2.11** (Lemma 5.1.1 of [3]). Let \( X \) be a quasi-projective scheme and \( T \) acts on \( X \) trivially. Then for every \( T \)-equivariant vector bundle \( E \), identifying it with its total space, satisfying \( E^T = X \), the element \([\wedge^\bullet(E)]\) is invertible in the localization \( K^T(X)_{\text{loc}} \).
Lemma 2.12 (Proposition 5.4.10 of [2]). Let $i : N \hookrightarrow M$ be a $G$-equivariant closed embedding of a smooth $G$-variety as a submanifold of a smooth $G$-variety $M$. Then the composite map $K^G(N) \xrightarrow{i^*} K^G(M) \xrightarrow{i^*} K^G(N)$ is given by the formula $i^*i^*F = [\wedge^\bullet T^*_NM] \otimes F$, for any $F \in K^G(N)$.

Remark 2.13. This lemma still holds even if $N$ or $M$ are not smooth, but $i$ is a regular embedding.

One application of Lemma 2.12 is that

Lemma 2.14. Let $X$ be a quasi-projective scheme with trivial $T$ action. Let $W$ and $V$ be two $T$ equivariant vector bundles satisfying $W^T = X$ and $V^T = X$, and $f : W \to V$ a $T$-equivariant linear morphism. Let $Y = f^{-1}(0)$, i.e. we have the following fiber square:

$$
\begin{array}{ccc}
Y & \xrightarrow{i'} & W \\
\downarrow f' & & \downarrow f \\
X & \xrightarrow{iv} & V
\end{array}
$$

where $iv$ is the zero section in $V$. Then $Y^T = X$, and if we denote $X \xrightarrow{j} Y$ the embedding of fixed points, we have

$$f^*F = j_*([\wedge^\bullet V][\wedge^\bullet W]F)$$

for all $F \in K^T(Y)_\text{loc}$.

Proof. The fact that $Y^T = X$ is trivial.

Let $iv$ be the zero section of $W$, then $i'_W$ and $i^*_W$ are isomorphisms, by Lemma 2.12 and Theorem 2.10. Thus in order to prove equation 2.5 holds, we only need to prove

$$i^*_W i^* f^* F = i^*_W i'_W j_* ([\wedge^\bullet V][\wedge^\bullet W]F),$$

i.e.

$$i^*_W \left( \sum_{i=0}^{\infty} (-1)^i \text{Tor}^W_i(F, \mathcal{O}_W) \right) = i^*_W i'_W j_* ([\wedge^\bullet V][\wedge^\bullet W]F).$$

Right hand side of equation 2.6 is $[\wedge^\bullet V]F$ by lemma 2.12. The left hand side is

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (-1)^{i+j} \text{Tor}^W_j(F, \mathcal{O}_W, \mathcal{O}_X) = \sum_{i=0}^{\infty} (-1)^i \text{Tor}^V_i(F, \mathcal{O}_X)$$

by the Tor spectral sequence, and it is also $[\wedge^\bullet V]F$ by lemma 2.12.

2.5. Induction. Let $P \subset G$ be a closed algebraic subgroup which has an action on $X$. Then we define the induced space, $G \times_P X$, to be the space of orbits of $P$ acting freely on $G \times X$ by $h : (g, x) \to (gh^{-1}, hx)$. Then by (5.2.17) of [2], we have inverse ring homomorphisms:

$$K^P(X) \xrightarrow{\text{res}} K^G(X \times_G P).$$

Now we assume $T \subset P$ and $P$ is a parabolic subgroup of $G$, and we denote $H$ the Levi subgroup of $P$. We denote $W$ the Weyl group of $G$, and $W_H$ the Weyl
group of $H$. Let $g = \text{Lie}(G)$ and $p = \text{Lie}(P)$, then $g \in \text{Rep}(T)$ and $p \in \text{Rep}(T)$ by adjoint representations.

**Lemma 2.15.** Let $Y = X \times_P G$, then $Y^T = X^T \times_{W_H} W$. Let $s$ to be the canonical projection from $Y^T$ to $X^T$.

Then if $X^T$ is connected, we consider following morphisms:

$$
\begin{array}{ccc}
K^T(Y^T)_{loc} & \xrightarrow{i_Y^*} & K^T(Y)_{loc} \\
\downarrow{s^*} & & \downarrow{j^*} \\
K^T(X^T)_{loc} & \xrightarrow{i_X^*} & K^T(X)_{loc}
\end{array}
$$

(2.7)

where $j$ is the inclusion map from $X$ to $Y$ by map $x$ to $(x,e)$ and $e$ is the identity element of $G$. We will have

$$
i_X^*(\mathcal{F}) = \frac{j^* \circ i_Y^* \circ s^*(\mathcal{F})}{[\wedge^*(g/p)^*]},
$$

and the commutative diagram:

$$
\begin{array}{ccc}
K^G(Y) & \longrightarrow & K^T(Y) \\
\downarrow{\text{res}_G^*} & & \downarrow{j^*} \\
K^P(Y) & \longrightarrow & K^T(X)
\end{array}
$$

(2.9)

**Proof.** Diagram (2.9) is obvious from definition. Let $j$ be the canonical closed embedding of $X$ in $Y$, and $j^T$ the embedding of $X^T$ in $Y^T$. Moreover, we have

$$
Y^T = \bigcup_{i=1}^k Y_i,
$$

where all $Y_i$ are different connected components of $Y^T$ and $Y_1 = X^T$, the only components which has non-empty intersection with $X$. Thus $s^*(\mathcal{F}) = j_{T*}\mathcal{F} + \mathcal{F}'$, where $\mathcal{F}'$ is supported in $\bigcup_{i=2}^k Y_i$, and hence $j^* \circ i_Y^*(\mathcal{F}') = 0$. Now we have

$$
j^* \circ i_Y^* \circ s^*(\mathcal{F}) = j^* \circ i_Y^* \circ j_{T*}(\mathcal{F}).
$$

And we also have the fiber square:

$$
\begin{array}{ccc}
X^T & \xrightarrow{i_X} & X \\
\downarrow{j_P} & & \downarrow{j} \\
Y^T & \xrightarrow{i_Y} & Y
\end{array}
$$

Then $i_Y \circ j_{T*} = j^* \circ i_X$, and

$$
j^* \circ i_Y^* \circ s^*(\mathcal{F}) = j^* \circ i_X^*(\mathcal{F}) = [\wedge^*T_XY^*]i_X^*(\mathcal{F})
$$

by Lemma 2.12.

Now we only need to prove $T_XY = g/p$. Notice that $Y$ is a fiber bundle over $G/P$ with fiber $X$:

$$
\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
\downarrow{\pi} & & \downarrow{} \\
\text{Spec}(k) & \xrightarrow{c} & G/P,
\end{array}
$$
where the bottom line corresponds to the identity element. Hence $T_X Y = \pi^* (g/p) = g/p$.  

2.6. K-theory of Artin stacks. Now we assume $Y$ is an Artin stack, then in [14], Toen defined the K theory of $Y$. A special case of Artin stacks is group quotient, i.e. $Y = [X/H]$, where $X$ is a scheme and $H$ is an algebraic group with group action on $X$, then we have:

**Lemma 2.16** (Lemma 2.11 of [14]).

$$K(X/H) = K^H(X).$$

3. Refined Gysin map of Quot Schemes and Flag Schemes

We work over an algebraically closed field $k$ of characteristic 0. Fix a smooth projective surface $S$ with an ample divisor $O(1)$.

3.1. Geometry of Quot Schemes. Given $d$ a non-negative integer, we let $Quot_d$ be Grothendieck’s Quot scheme which represents the moduli space of quotients of coherent sheaves $\{\phi : k^d \otimes O_S \rightarrow E_d\}$, where $E_d$ has constant Hilbert polynomial $d$.

Moreover, we denote the open subscheme $Quot^d_d \subset Quot_d$ which consists of quotients such that $H^0(\phi) : k^d \rightarrow H^0(E_d)$ is an isomorphism.

**Definition 3.1.** We define $\mathcal{I}_d$ to be the kernel sheaf of universal quotients of coherent sheaves $k^d \otimes O \rightarrow E_d$.

**Lemma 3.2.** $Quot_d$ and $Quot^d_d$ are irreducible.

**Proof.** See [4]. □

**Remark 3.3.** $Quot^d_1 = S$. Let $\Delta : S \rightarrow S \times S$ be the diagonal map, then $E_1 \in Coh(Quot^1_1 \times S)$ is just $O_{\Delta}$.

**Remark 3.4.** For any scheme $X$ and $f : X \rightarrow Quot^d_n$, we will abuse the notation, using $E_d$ to denote $f^* E_d \in Coh(X \times S)$, $\mathcal{I}_d$ to denote $f^* \mathcal{I}_d \in Coh(X \times S)$ and $\pi = \pi_X$ to denote the projection from $X \times S$ to $X$.

3.2. Resolution of The Kernel Sheaf. Now we consider the resolution of kernel sheaf $\mathcal{I}_d$ by locally free sheaves.

**Lemma 3.5.** There exists a short exact sequence

$$0 \rightarrow \mathcal{V}_d \xrightarrow{\nu_d} \mathcal{W}_d \xrightarrow{\mu_d} \mathcal{I}_d \rightarrow 0,$$

over $Quot^d_d \times S$, where $\mathcal{V}_d$ and $\mathcal{W}_d$ are locally free sheaves.

**Proof.** For a sufficient large integer $r$, let $\mathcal{W}_d = \pi^* \pi_* \mathcal{I}_d (r) \otimes O(-r)$, where $\pi$ is the projection map from $Quot^d_d \times S$ to $Quot^d_d$. There is a surjective map $\mathcal{W}_d \rightarrow \mathcal{I}_d$. Let $\mathcal{V}_d$ be its kernel and $\mathcal{V}_d$ is also locally free by Lemma 2.1.7 of [6]. □

Now we assume $n, m$ two non-negative integers and we use $Quot^d_n \times Quot^d_m$ to denote $Quot^d_n \times Quot^d_m$. Given a standard resolution of locally free sheaves of $\mathcal{I}_m$ over $Quot^d_n \times S$

$$0 \rightarrow \mathcal{V}_m \rightarrow \mathcal{W}_m \rightarrow \mathcal{I}_m \rightarrow 0,$$

then we can consider $\text{Hom}(\mathcal{W}_m, \mathcal{E}_n), \text{Hom}(\mathcal{V}_m, \mathcal{E}_n) \in Coh(Quot^d_n \times S)$. We define $W_{n,m} = \pi_* \text{Hom}(\mathcal{W}_m, \mathcal{E}_n) = \pi_* (\mathcal{W}_m \mathcal{E}_n)$.
and
\[ V_{n,m} = \pi_*\text{Hom}(V_m, E_n) = \pi_*(V_m \otimes E_n), \]
where \( \pi \) is the projection map from \( \text{Quot}_{n,m}^o \times S \) to \( \text{Quot}_{n,m}^o \).

**Proposition 3.6.** \( W_{n,m} \) and \( V_{n,m} \) are locally free sheaves and
\[ R^i\pi_*(\text{Hom}(W_m, E_n)) = 0, \quad R^i\pi_*(\text{Hom}(V_m, E_n)) = 0 \]
for \( i > 0 \).

To prove this theorem, we recall a theorem about cohomology along fibers:

**Theorem 3.7.** (Chapter III, Theorem 12.11 of \([5]\)). Let \( f: X \rightarrow Y \) be a projective morphism of noetherian schemes, and let \( F \) be a coherent sheaf on \( X \), flat over \( Y \). Let \( y \) be a closed point of \( Y \). Then for any non-negative integer, there is a natural map
\[ \phi^i(y): R^i f^*(F) \otimes k(y) \rightarrow H^i(X_y, F_y). \]
If \( \phi^i \) is surjective, then it is an isomorphism. Moreover, if \( R^i f_*(F) \) is locally free in a neighborhood of \( y \), then \( \phi^{i-1}(y) \) is also surjective.

Now we prove the Proposition 3.6.

**Proof.** We assume \( W_m \) has rank \( r \). Let \( X = \text{Quot}_{n,m}^o \times S \), \( Y = \text{Quot}_{n,m}^o \) and \( f = \pi \). Then for any closed point \( b \in Y \), \( X_b = b \times S \), and \( W_m^r \otimes E_n|_{b \times S} \) has dimension 0 and length \( nr \). Thus
\[ h^i(b \times S, W_m^r \otimes E_n|_{b \times S}) = 0 \]
when \( i > 0 \) and
\[ h^0(b \times S, W_m^r \otimes E_n|_{b \times S}) = nr \]
for any point \( b \in B \).

By Theorem 3.7, we have
\[ R^i\pi_*(\text{Hom}(W, E_n)) = 0 \]
for \( i > 0 \). Let us take \( i = 1 \) in the last sentence of Theorem 3.7. \( \phi^0(b) \) is also an isomorphism. Hence \( W_{n,m} \) is locally free.

The analogous proof also holds for \( V \). \qed

The exact sequence (3.1) induces a left exact sequence over \( \text{Quot}_{n,m}^o \times S \)
\[ 0 \rightarrow \text{Hom}(I_m, E_n) \rightarrow \text{Hom}(W_m, E_n) \xrightarrow{\psi_{n,m}} \text{Hom}(V_m, E_n), \]
and thus induces a linear morphism
\[ W_{n,m} = \pi_*\text{Hom}(W_m, E_n) \xrightarrow{\pi_*\psi_{n,m}} V_{n,m} = \pi_*\text{Hom}(V_m, E_n) \]
We abuse the notation \( W_{n,m} \) and \( V_{n,m} \) to denote the total space of \( W_{n,m} \) and \( V_{n,m} \), and \( \psi_{n,m} \) to denote \( \pi_*\psi_{n,m} \). Then
\[ \psi_{n,m}: W_{n,m} \rightarrow V_{n,m} \]
is a linear morphism between two vector bundles.
3.3. Flag Schemes. Let \( d_* = (d_0, d_1, \ldots, d_l) \) be a sequence of non-decreasing integers, such that \( d_0 = 0 \) and \( d_l = d \). Fix a flag \( F = \{ k^{d_0} \subseteq \ldots \subseteq k^{d_l} \} \). We denote

\[
\text{Quot}_{d_*} = \prod_{i=1}^{k} \text{Quot}_{d_i - d_{i-1}}, \quad \text{Quot}_{d_*} = \prod_{i=1}^{k} \text{Quot}_{d_i - d_{i-1}}.
\]

For any closed point \( \{ \phi : k^d \otimes O_S \to E \} \) of \( \text{Quot}_{d_i} \), we denote \( E_i \) the image of \( \phi_i = \phi|_{k^{d_i} \otimes O} \).

**Definition 3.8.** We define the subset \( \text{Flag}_{d_*} \) of \( \text{Quot}_{d_*} \) which consists of quotients such that the map \( H^0(\phi_i) : k^{d_i} \to H^0(E_i) \) is an isomorphism for any \( i \).

**Proposition 3.9.** \( \text{Flag}_{d_*} \) is a closed subscheme of \( \text{Quot}_{d_*} \).

**Proof.** See [7]. \( \square \)

Over \( \text{Flag}_{d_*} \), we also have universal quotients \( \mathcal{O}_{d_i} \to \mathcal{E}_d \). Moreover, fixing an isomorphism \( k^{d_i} - d_{i-1} = k^{d_i} / k^{d_{i-1}} \), then \( \mathcal{O}_{d_i - d_{i-1}} \to \mathcal{E}_d / \mathcal{E}_{d_j} \) is also surjective for any \( i > j \).

Let \( \mathcal{E}_i = \mathcal{E}_i / \mathcal{E}_{i-1} \), then there is a surjective map \( \mathcal{O}_{d_i - d_{i-1}} \to \mathcal{E}_i \), which induces a morphism \( p_{d_*} : \text{Flag}_{d_*} \to \text{Quot}_{d_*} \).

3.4. A Cartesian Diagram Between Flag Schemes and Quot Schemes. Now let \( d_* = (0, n, n + m) \), where \( n \) and \( m \) are two non-negative integers. Then \( \text{Quot}_{d_*} = \text{Quot}_{n,m} \). And we use \( \text{Flag}_{n,m} \) to denote \( \text{Flag}_{d_*} \).

We have a commutative diagram of coherent sheaves over \( \text{Flag}_{n,m} \times S \), which are flat over \( \text{Flag}_{n,m} \)

\[
\begin{array}{ccc}
0 & \to & k^n \otimes \mathcal{O} \\
\downarrow & & \downarrow
\end{array}
\]

\[
\begin{array}{ccc}
k^{n+m} \otimes \mathcal{O} & \to & k^m \otimes \mathcal{O} & \to & 0 \\
\phi_n & & \phi_m
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & \mathcal{E}_n & \to & \mathcal{E}_{n+m} & \to & \mathcal{E}_m & \to & 0 \\
0 & \to & \mathcal{W}_m & \to & \mathcal{E}_n & \to & \mathcal{E}_{n+m} & \to & \mathcal{E}_m & \to & 0
\end{array}
\]

where all the columns and rows are exact, and the first row split by \( \ell(v) = (0, v) \). Then

\[
\phi_{n+m} \circ \ell \circ w_m : \mathcal{W}_m \to \mathcal{E}_{n+m}
\]

has image 0 in \( \mathcal{E}_m \). It induces a morphism from \( \mathcal{W}_m \) to \( \mathcal{E}_n \), i.e. a global section of \( \text{Hom}(\mathcal{W}_m, \mathcal{E}_n) \), which corresponds to a morphism from \( \text{Flag}^0_{n,m} \) to \( W_{n,m} \). Moreover,
the diagram

\[
\begin{array}{ccc}
\text{Flag}_{n,m} & \xrightarrow{t_{n,m}} & W_{n,m} \\
\downarrow & & \downarrow \\
\text{Quot}_{n,m} & \xrightarrow{i_V} & V_{n,m}
\end{array}
\] (3.3)

is commutative, where \(i_V\) is the zero section. Let \(\text{Flag} = W_{n,m} \times_{\text{Quot}_{n,m}} V_{n,m}\), and the above diagram 3.3 induces a morphism

\[
\tau : \text{Flag}_{n,m} \rightarrow \text{Flag}
\]

**Proposition 3.10.** \(\tau\) is an isomorphism.

**Proof.** We construct \(\tau'\) which is an inverse map of \(\tau\), i.e. a natural transformation from the functor \(\text{Hom}(-, \text{Flag})\) to \(\text{Hom}(-, \text{Flag}_{n,m})\).

Given a test scheme \(X\) and a morphism \(f : X \rightarrow \text{Flag}\), we have a long exact sequence

\[
0 \rightarrow V_{m} \rightarrow W_{m} \rightarrow k_{m} \otimes \mathcal{O} \rightarrow E_{m} \rightarrow 0
\]

and the universal coherent sheaf \(E_n\) over \(X \times S\), which are flat over \(X\). Moreover, it induces a morphism \(t : W_m \rightarrow E_n\) such that \(tv_m = 0\). We consider the following morphism:

\[
t : W_m \rightarrow E_n \oplus (k^m \otimes \mathcal{O})
\]

\[a \mapsto (t(a), w_m(a))\]

and denote \(E_{n+m}\) the cokernel of \(t\). Then we have the exact sequence:

\[
0 \rightarrow V_{m} \rightarrow W_{m} \rightarrow E_n \oplus (k^m \otimes \mathcal{O}) \rightarrow E_{n+m} \rightarrow 0
\]

and moreover a exact sequence:

\[
0 \rightarrow I_{m} \rightarrow E_n \oplus (k^m \otimes \mathcal{O}) \rightarrow E_{n+m} \rightarrow 0.
\]

We decompose \(p = \varepsilon_n \oplus p_2\) for \(\varepsilon_n : E_n \rightarrow E_{n+m}\) and \(p_2 : k^m \otimes \mathcal{O} \rightarrow E_{n+m}\). Then we have the following diagram:

\[
\begin{array}{cccccccc}
\varepsilon_n \\
\downarrow & & & & & & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & E_n & \rightarrow & E_n & \rightarrow & 0 \\
\downarrow & & & & & & & \downarrow \\
0 & \rightarrow & I_{m} & \rightarrow & E_n \oplus (k^m \otimes \mathcal{O}) & \rightarrow & E_{n+m} & \rightarrow & 0 \\
\downarrow & & & & & & & \downarrow \\
0 & \rightarrow & I_{m} & \rightarrow & k^m \otimes \mathcal{O} & \rightarrow & E_{m} & \rightarrow & 0 \\
\downarrow & & & & & & & \downarrow \\
0 & & & & & & & 0
\end{array}
\]

where all the rows are exact sequences and all the columns except the right one are also exact. Then
(1) It is easy to construct the dashed morphism $\varepsilon_m$ and prove that the right column is also exact by diagram chasing.

(2) $E_{n+m}$ is also flat over $X$, by the fact $E_m$ and $E_n$ are flat over $X$.

(3) $\phi_n \oplus id : O^n \oplus O^m \to E^n \oplus O_m$ induced a map $\phi_{n+m} = p(\phi_n \oplus id) : k^{n+m} \otimes \mathcal{O} \to E_{n+m}$, and we have the following diagram:

\[
\begin{array}{ccccccccc}
0 & \to & k^n \otimes \mathcal{O} & \to & k^{n+m} \otimes \mathcal{O} & \to & k^m \otimes \mathcal{O} & \to & 0 \\
& & \downarrow{\phi_n} & & \downarrow{\phi_{n+m}} & & \downarrow{\phi_m} & & \downarrow & \\
0 & \to & E_n & \to & E_{n+m} & \to & E_m & \to & 0 \\
& & 0 & & 0 & & 0 & & 0, \\
\end{array}
\]

where all the columns and rows are exact, and thus it induces a morphism $f' : X \to Flag^o_{n,m}$.

Hence we construct $\tau'$ and it is easy to check that $\tau'$ is the inverse of $\tau$.

\[\square\]

3.5. Refined Gysin Map. By the Cartesian diagram \ref{3.3} we have the refined Gysin map:

\[\psi^!_{n,m} : K(Quot^o_{n,m}) \to K(Flag^o_{n,m}).\]

Moreover, for any Cartesian diagram

\[
\begin{array}{ccc}
X & \to & Flag^o_{n,m} \\
\downarrow & & \downarrow \\
Y & \to & Quot^o_{n,m}
\end{array}
\]

we also have the refined Gysin map:

\[\psi^!_{n,m} : K(Y) \to K(X).\]

**Proposition 3.11.** $\psi^!_{n,m}$ does not depend on the choice of resolutions of the kernel sheaf $\mathcal{I}_m$.

**Proof.** Let

\[0 \to V_m \to W_m \xrightarrow{w} \mathcal{I}_m \to 0\]

and

\[0 \to V'_m \to W'_m \xrightarrow{w'} \mathcal{I}_m \to 0\]

be two resolutions of $\mathcal{I}_m$. Then $W_m \oplus W'_m \xrightarrow{w \oplus w'} \mathcal{I}_m$ is also surjective, which also forms a resolution of $\mathcal{I}_m$. So we can assume $W_m$ is a direct summand of $W'_m$. Then $V_m$ is also a subsheaf of $V'_m$.

Then we have the following exact sequence:

\[0 \to V'_m \to W'_m \oplus V_m \to W_m \to 0\]

and thus

\[0 \to W_{n,m} \to W'_{n,m} \oplus V_{n,m} \to V'_{m,n} \to 0.\]
So we have the Cartesian diagram:

\[
\begin{array}{ccc}
\text{Flag}^\circ_{n,m} & \longrightarrow & W_{n,m} \\
& \downarrow \psi_{n,m} & \downarrow \psi'_{n,m} \\
\text{Quot}^\circ_{n,m} & \longrightarrow & V_{n,m} \\
\end{array}
\]

Moreover, \(f\) is a smooth morphism. Then by Lemma 2.4 we have that \(\psi^!_{n,m} = \psi^!_{n,m} \).

Now we assume \(d\) has length 3, i.e. \(d = (0, n, n + m, n + m + l)\) for three different non-negative integers \(n, m, l\). We use \(\text{Flag}^\circ_{n,m,l}\) and \(\text{Quot}^\circ_{n,m,l}\) to denote \(\text{Flag}^\circ_{d}\) and \(\text{Quot}^\circ_{d}\). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Flag}^\circ_{n,m,l} & \longrightarrow & \text{Flag}^\circ_{n,m} \times \text{Quot}^\circ_l \\
& \downarrow & \downarrow \\
\text{Quot}^\circ_{n,m} \times \text{Flag}^\circ_{m,l} & \longrightarrow & \text{Quot}^\circ_{n,m,l} \\
\end{array}
\]

Moreover we have the Cartesian diagram:

\[
\begin{array}{ccc}
\text{Flag}^\circ_{n,m,l} & \longrightarrow & \text{Flag}^\circ_{n,m,l+1} \\
& \downarrow & \downarrow \\
\text{Quot}^\circ_{n,m} \times \text{Flag}^\circ_{m,l} & \longrightarrow & \text{Quot}^\circ_{n,m,l+1} \\
\end{array}
\]

Hence we have

\[
\psi^!_{n,m+l} : K(\text{Quot}^\circ_{n,m+l} \times \text{Flag}^\circ_{m,l}) \rightarrow K(\text{Flag}^\circ_{n,m,l})
\]

and

\[
\psi^!_{n+m+l} : K(\text{Flag}^\circ_{n,m+l} \times \text{Quot}^\circ_l) \rightarrow K(\text{Flag}^\circ_{n,m,l}).
\]

**Proposition 3.12.** \(\psi^!_{n,m+l}\psi^!_{m,l} = \psi^!_{n,m+l}\psi^!_{n,m}\) as two morphisms from \(K(\text{Quot}^\circ_{n,m+l})\) to \(K(\text{Flag}^\circ_{n,m,l})\).

**Proof.** Given a sufficient large \(r\), similar to Lemma 3.5, we define \(W_l = \pi^* \pi_*(I_l(r)) \otimes \mathcal{O}(-r)\), \(W_m = \pi^* \pi_*(I_m(r)) \otimes \mathcal{O}(-r)\) and \(W_n = \pi^* \pi_*(I_n(r)) \otimes \mathcal{O}(-r)\) over \(\text{Quot}^\circ_{n,m+l}\times S\). We also define \(W_{m+n} = \pi^* \pi_*(I_{m+n}(r)) \otimes \mathcal{O}(-r)\) over \(\text{Flag}^\circ_{n,m+l} \times \text{Quot}^\circ_l\), and \(W_{m+l} = \pi^* \pi_*(I_{m+l}(r)) \otimes \mathcal{O}(-r)\). Then have surjective maps to \(I_l, I_m, I_n, I_{m+n}, I_{n+l}\) respectively, and we define \(V_l, V_m, V_n, V_{m+n}, V_{n+l}\) to be their kernels respectively.

Then we have the exact sequence:

\[
\begin{array}{cccccc}
0 & \longrightarrow & V_n & \longrightarrow & V_{m+n} & \longrightarrow & V_m & \longrightarrow & 0 \\
& \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \\
0 & \longrightarrow & W_n & \longrightarrow & W_{m+n} & \longrightarrow & W_m & \longrightarrow & 0
\end{array}
\]
over $\text{Flag}_{n,m} \times \text{Quot}^o_l \times S$ and exact sequences.

$$
\begin{array}{ccccccccc}
0 & \rightarrow & V_m & \rightarrow & V_{m+l} & \rightarrow & V_l & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & W_m & \rightarrow & W_{m+l} & \rightarrow & W_l & \rightarrow & 0
\end{array}
$$

over $\text{Quot}^o_n \times \text{Flag}^o_{m,l} \times S$. Similar to Section 3.2, there are locally free sheaves $V_{n,l}, V_{m,l}, V_{n,m}, W_{n,l}, W_{m,l}, W_{n,m}$ over $\text{Flag}^o_{n,m,l}, V_{n+m+l}, W_{n+m,l}$ over $\text{Flag}^o_{n,m,l}$.

We define the scheme $\text{Corr}^o_{n,m,l}$ by the following Cartesian diagram:

$$
\begin{array}{cccccc}
\text{Corr}^o_{n,m,l} & \rightarrow & \text{Flag}^o_{n,m} \times \text{Quot}^o_l \\
\downarrow & & \downarrow \\
\text{Quot}^o_n \times \text{Flag}^o_{m,l} & \rightarrow & \text{Quot}^o_{n,m,l}
\end{array}
$$

By Lemma 2.3, we have

$$
\psi^o_{n,l} \psi^o_{m,l} = \psi^o_{m,l} \psi^o_{n,l} : K(\text{Quot}^o_{n,m,l}) \rightarrow K(\text{Corr}^o_{n,m,l}).
$$

We have the following exact sequences of locally free sheaves over $\text{Quot}^o_n \times \text{Flag}^o_{m,l}$:

$$
\begin{array}{cccccc}
0 & \rightarrow & W_{n,l} & \rightarrow & W_{n,l+m} & \rightarrow & W_{n,m} & \rightarrow & 0 \\
\downarrow & & \downarrow \psi_{n,l+m} & & \downarrow \psi_{n,m} & & \\
0 & \rightarrow & V_{n,l} & \rightarrow & V_{n,l+m} & \rightarrow & V_{n,m} & \rightarrow & 0
\end{array}
$$

where $\text{Corr}^o_{n,m,l} = \psi^{-1}_{n,m}(0)$ and $\text{Flag}^o_{n,m,l} = \psi^{-1}_{n,l+m}(0)$. Let $Y_1 = f^{-1}(\text{Corr}^o_{n,m,l})$.

Then by Lemma 2.5, there is a Cartesian diagram:

$$
\begin{array}{cccccc}
\text{Flag}^o_{n,m,l} & \rightarrow & Y_1 \\
\downarrow & & \downarrow \psi_1 \\
\text{Corr}^o_{n,m,l} & \rightarrow & V_{n,l} \times \text{Quot}^o_{n,m,l} \text{ Corr}^o_{n,m,l}.
\end{array}
$$

Similarly, we have the following exact sequences of locally free sheaves over $\text{Flag}^o_{n,m} \times \text{Quot}^o_l$

$$
\begin{array}{cccccc}
0 & \rightarrow & W_{n,l} & \rightarrow & W_{n+l,m} & \rightarrow & W_{m,l} & \rightarrow & 0 \\
\downarrow & & \downarrow \psi_{n+m,l} & & \downarrow \psi_{m,l} & & \\
0 & \rightarrow & V_{n,l} & \rightarrow & V_{n+l,m} & \rightarrow & V_{m,l} & \rightarrow & 0
\end{array}
$$

It is obvious $\text{Corr}^o_{n,m,l} = \psi^{-1}_{n,m}(0)$ and $\text{Flag}^o_{n,m,l} = \psi^{-1}_{n+m,l}(0)$. Let $Y_2 = f'^{-1}(\text{Corr}^o_{n,m,l})$.

And we also have the Cartesian diagram

$$
\begin{array}{cccccc}
\text{Flag}^o_{n,m,l} & \rightarrow & Y_2 \\
\downarrow & & \downarrow \psi_2 \\
\text{Corr}^o_{n,m,l} & \rightarrow & V_{n,l} \times \text{Quot}^o_{n,m,l} \text{ Corr}^o_{n,m,l}.
\end{array}
$$
Now we prove that $Y_1$ and $Y_2$ are isomorphic, and $\psi_1 = \psi_2$. Recall that over $\text{Corr}_{n,m,l}^\circ \times S$, we have following diagrams:

![Diagram]

and

![Diagram]

By Section 3.4 we have a global section $t_{n,m} : \text{Corr}_{n,m,l}^\circ \rightarrow W_{n,m}$ and $t_{m,l} : \text{Corr}_{n,m,l}^\circ \rightarrow W_{m,l}$. And $Y_1 = f^{-1}(t_{n,m}(\text{Corr}_{n,m,l}^\circ)), Y_2 = f'^{-1}(t_{m,l}(\text{Corr}_{n,m,l}^\circ))$. Then for every closed point $x \in Y_1$, which corresponds to a morphism from $W_l$ to $\mathcal{E}_{n+m}$, where $h_{n+m} \circ x = t_{m,l}$. Then $h_{n+m} \circ x \circ g_{m+l} = t_{m,l} \circ g_{m+l} = 0$, thus there exists a unique morphism $x'$ from $W_{m+l}$ to $\mathcal{E}_n$ such that $x \circ g_{m+l} = h_{m} \circ x'$. By diagram chasing, we can prove that $x' \circ g_{m} = t_{n,m}$. Hence we construct an isomorphism between $Y_1$ and $Y_2$, and $\psi_1^l = \psi_2^l$.

Thus by Lemma 2.7 and Equation 3.3 we have $\psi_{n,m,l}^l \circ \psi_{n,m}^l = \psi_1^l \circ \psi_{m,l}^l \circ \psi_{n,m}^l = \psi_2^l \circ \psi_{n,m}^l \circ \psi_{m,l}^l \circ \psi_{n,m,l}^l = \psi_{n,m,l}^l$.

3.6. Equivariant Refined Gysin Map. Let $G_d$ be short for $GL_d(k)$. $G_d$ has a natural action on $\text{Quot}^\circ_d$. Moreover, let $P_{d_n}$ the parabolic group of $G_d$ which preserves the flag $F$ and $G_{d_n} = \prod_{i=1}^k G_{d_i-d_{i-1}}$ the Levi subgroup of $P_{d_n}$. Then $P_{d_n}$ has a natural action on $\text{Flag}^\circ_d$ and $G_{d_n}$ has a natural action on $\text{Quot}^\circ_d$. 

\[\phi\] 

\[\phi\]
In Section 3.3 we defined a morphism \( p_d : \text{Flag}^\circ_{d} \to \text{Quot}^\circ_{d} \), which is in fact \( P_d \) equivariant. The inclusion map from \( \text{Flag}^\circ_{d} \) to \( \text{Quot}^\circ_{d} \) is also \( P_d \) equivariant, which induces a \( G_d \) equivariant map \( q_d \) from \( \text{Flag}^\circ_{d} \times p_d \) to \( \text{Quot}^\circ_{d} \).

We will also use notations \( P_{m,n}, P_{m,n} \) and \( q_{m,n} \) to denote \( P_d \) and \( q_d \) where \( d = (0, n, n + m) \). And \( P_{m,n}, q_{m,n} \) are short for \( P_d \) and \( q_d \) where \( d = (0, n, n + m, n + m + l) \).

**Remark 3.13.** All the refined Gysin map on the above sections can be defined equivariantly by same constructions, and Proposition 3.11 and Proposition 3.12 also have equivariant version.

Let \( T_d \subset G_d \) be the maximal torus formed by diagonal matrices. Now we study\( T_d \) action on \( \text{Quot}^\circ_{d} \).\( \Box \)

**Lemma 3.14.** \((\text{Quot}^\circ_{d})^{T_d} = (\text{Quot}^\circ_{d})^d = S_d \).

Let \( pr_{ij} : S^{d+1} \to S \times S \) be the projection to \( i \)-th and \( j \)-th factors. Then the universal sheaf \( \mathcal{E}_d \) over \( (\text{Quot}^\circ_{d})^d \times S = S^{d+1} \) has the following formula:

\[
\mathcal{E}_d = \bigoplus_{i=1}^{d} pr_{i,d+1}^* (\mathcal{O}_\Delta),
\]

and also

\[
\mathcal{I}_d = \bigoplus_{i=1}^{d} pr_{i,d+1}^* (\mathcal{I}_\Delta),
\]

where \( \Delta \) is the diagonal map \( S \to S \times S \), \( \mathcal{I}_\Delta \) is the ideal sheaf of \( \Delta \) and \( \mathcal{O}_\Delta \) is the structure sheaf of diagonal.

**Proof.** See Lemma 3.1 of [7] \( \Box \)

Next we study some properties of \( T_{m+n} \)-fixed locus of \( \text{Quot}^\circ_{n,m} \). First let \( m = 1 \) and \( n = 1 \), we have \( \text{Quot}^\circ_{1} \times \text{Quot}^\circ_{1} \times S = S^3 \) and still let \( pr_{ij} : S^3 \to S \times S \) be the projection to the \( i \)-th and \( j \)-th factor. Then we have the following projection lemma:

**Lemma 3.15 (Projection Lemma).**

\[
pr_{12} \bullet \text{Hom}(pr_{13}^* \mathcal{W}_1, pr_{23}^* \mathcal{O}_\Delta) = \mathcal{W}_1
\]

and

\[
pr_{12} \bullet \text{Hom}(pr_{13}^* \mathcal{V}_1, pr_{23}^* \mathcal{O}_\Delta) = \mathcal{V}_1
\]

**Proof.** Let \( \Delta_{23} = id \times \Delta : S \times S \to S \times S \times S \). Then \( pr_{23}^* \mathcal{O}_\Delta = \mathcal{O}_{\Delta_{23}} \), and \( pr_{13} \Delta_{23} = pr_{23} \Delta_{23} = id \). Then

\[
\text{Hom}(pr_{13}^* \mathcal{W}_1, pr_{23}^* \mathcal{O}_\Delta) = pr_{13}^* \mathcal{W}_1 \otimes pr_{23}^* \mathcal{O}_\Delta
\]

\[
= pr_{13}^* \mathcal{W}_1 \otimes \Delta_{23}
\]

\[
= \Delta_{23} (\Delta_{23} pr_{13}^* \mathcal{W}_1)
\]

\[
= \Delta_{23} \mathcal{W}_1.
\]

Hence

\[
pr_{12} \bullet \text{Hom}(pr_{13}^* \mathcal{W}_1, pr_{23}^* \mathcal{O}_\Delta) = pr_{12} \bullet \Delta_{23} \mathcal{W}_1 = \mathcal{W}_1
\]

And the similar results also hold for \( \mathcal{V} \). \( \Box \)
which induces the following equation

We have the following exact sequence

Thus

By Lemma 3.15, we have \( W_{n,m}, V_{n,m} \in Coh(T^{n+n} \times S^m) \),

and

We have the following exact sequence

which induces the following equation

\[(3.5) \quad [\wedge^* V_{n,m}^\vee] = \prod_{i=1}^{n} \prod_{j=n+1}^{n+m} \frac{\wedge^* [z_j O_{\Delta ij}]}{1 - \frac{z_i}{z_j}} \]
Let $FQ = S^{n+m} \times_{\text{Quot}_{n,m}} \text{Flag}_{n,m}^\circ$, then there is a natural $T^{n+m}$ action on $F$ with fixed locus $S^{n+m}$. And we have the following fiber square:

$$
\begin{array}{ccc}
FQ & \xrightarrow{i_F} & \text{Flag}_{n,m}^\circ \\
\downarrow & & \downarrow \\
S^{n+m} & \xrightarrow{i} & \text{Quot}_{n,m}^\circ
\end{array}
$$

which defines a refined Gysin map $\psi^\circ_{n,m} : K_{T^{n+m}}(S^{n+m}) \to K_{T^{n+m}}(F)$. Moreover, $W_{n,m}^{T_{n+m}} = V_{n,m}^{T_{n+m}} = S^n \times S^m$. Thus by equation (2.4), Lemma 2.2 and Lemma 2.14 we have the following lemma:

**Lemma 3.16.** We have the commutative diagram

$$
\begin{array}{ccc}
K_{T^{n+m}}(S^{n+m}) & \xrightarrow{i_*} & K_{T^{n+m}}(FQ) \\
\downarrow & & \uparrow \psi^\circ_{n,m} \\
K_{T^{n+m}}(S^{n+m}) & \xrightarrow{i_*} & K_{T^{n+m}}(\text{Quot}_{n,m}^\circ)
\end{array}
$$

where

$$
\rho_{n,m}(F) = \prod_{i=1}^n \prod_{j=n+1}^{n+m} \frac{\Lambda^* \Delta_{ij}}{1 - \frac{z_i}{z_j}} F.
$$

Now we consider the scheme $\text{Flag}_{n,m}^\circ = \text{Flag}_{n,m}^\circ \times_{\rho_{n,m}} G_{n,m}$. And let $m$ be the permutation group of order $n$.

**Lemma 3.17.** $(\text{Flag}_{n,m}^\circ)^{T_{n+m}} = S^{n+m} \times_{\sigma_n \times \sigma_n} S^{n+m}$. Moreover, we have the following commutative diagrams

$$
\begin{array}{ccc}
K_{T^{n+m}}(S^{n+m} \times_{\sigma_n \times \sigma_n} S^{n+m}) & \xrightarrow{i_*} & K_{T^{n+m}}(\text{Flag}_{n,m}^\circ) \\
\downarrow & & \downarrow j^\circ_{n,m} \\
K_{T^{n+m}}(S^{n+m}) & \xrightarrow{i_*} & K_{T^{n+m}}(\text{Flag}_{n,m}^\circ)
\end{array}
$$

where $s^\circ(F) = s^\circ(\prod_{i=1}^n \prod_{j=n+1}^{n+m} \frac{F}{1 - \frac{z_i}{z_j}})$ and $j_{n,m}$ is the inclusion map from $\text{Flag}_{n,m}^\circ$ to $\text{Flag}_{n,m}^\circ$.

We also have

$$
\begin{array}{ccc}
K_{T^{n+m}}(S^{n+m} \times_{\sigma_n \times \sigma_n} S^{n+m}) & \xrightarrow{i_*} & K_{T^{n+m}}(\text{Quot}_{n,m}^\circ) \\
\downarrow & & \downarrow q_{n,m} \\
K_{T^{n+m}}(S^{n+m}) & \xrightarrow{i_*} & K_{T^{n+m}}(\text{Quot}_{n,m}^\circ)
\end{array}
$$

**Proof.** Equation (2.8) is induced by Lemma 2.15 and Equation (3.9) is induced by the fact $i_* \circ s_* = (i \circ s)_* = (q_{n+m} \circ i)_* = q_{n+m} \circ i_*$. \(\square\)

**Remark 3.18.** We will use $S^{n+m}$ to denote $S^{n+m} \times_{\sigma_n \times \sigma_n} S^{n+m}$ later.
4. K-Theoretic Hall Algebra of a Surface

4.1. Let $Coh$ be the moduli space of 0-dimensional coherent sheaves over $S$. Then

$$Coh_0 = \cup_{d=0}^{\infty} Coh_d,$$

where $Coh_d$ is the moduli stack of dimension 0, degree $d$ coherent sheaves on $S$. In fact, $[Quot^0_d/Gl_d] = Coh_d$, which follows from the fact that all dimension 0 coherent sheaves are generated by their global sections.

**Definition 4.1.** We define the Abelian group $KCoh(S)$ by

$$KCoh(S) = \bigoplus_{d=0}^{\infty} K(Coh_d(S)) = \bigoplus_{d=0}^{\infty} K^{G_d}(Quot^0_d(S))$$

By Proposition 3.11, we can define the Gysin refined map

$$\psi^j_{n,m} : K^{P_{n,m}}(Quot^0_{n,m}) \to K^{P_{n,m}}(Flag^0_{n,m}).$$

Now we consider the following diagram:

$$(4.1)$$

$$K^{G_n \times G_m}(Quot^0_{n,m}) \xrightarrow{\text{proj}_{n,m}} K^{P_{n,m}}(Quot^0_{n,m}) \xrightarrow{\psi^j_{n,m}} K^{G_{n+m}}(Quot^0_{n+m})$$

where $\text{proj}_{n,m}$ is induced by the natural projection from $P_{n,m}$ to $G_n \times G_m$, and we denote $\tilde{q}_{n,m} = q_{n,m} \circ \text{ind}_{P_{n,m}}^{G_{n+m}}$.

We denote $*_{KCoh}^J$ the compositions of all maps in the above diagram. Then

$$*_{KCoh}^J : K^G(Quot^0_{0,n}) \otimes K^G(Quot^0_{0,m}) \to K^{G_{n+m}}(Quot^0_{0,n+m})$$

induces a morphism $*$:

$$*_{KCoh} : Coh(S) \otimes Coh(S) \to Coh(S).$$

**Definition 4.2.** We define $(KCoh(S), *_{KCoh}^J)$ to be the K-theoretical Hall algebra associated to $S$.

**Remark 4.3.** The analogous definition of cohomological Hall algebra was introduced by Schiffmann and Vasserot’s paper [12]. The case $S = \mathbb{A}^2$ corresponds to the double quivers in their definitions of cohomological Hall algebra.

In this section, we will also prove that $(KCoh(S), *_{KCoh}^J)$ is associative.

**Theorem 4.4.** The K-theoretic Hall algebra $(KCoh(S), *_{KCoh}^J)$ is associative.

**Proof.** Given any three non-negative integers $n, m, l$, then

$$*_{n+m,l} \circ *_{n,m} : K(Coh_{0,n}(S)) \otimes K(Coh_{0,m}(S)) \otimes K(Coh_{0,l}(S)) \to K(Coh_{0, n+m+l}(S))$$
is induced by the diagram:

\[
\begin{array}{ccc}
K^{P_{n,m,l}}(\text{Flag}^\circ_{n,m,l}) & \xrightarrow{\mathbf{q}_{n,m,l}} & K^{P_{n+m,l}}(\text{Flag}^\circ_{n+m,m,l}) \\
\psi^l_{n,m} & & \psi^l_{n+m,m,l} \\
K^{P_{n,m,l}}(\text{Quot}^\circ_{n,m,l}) & \xrightarrow{q_{n,m,l}} & K^{P_{n+m,l}}(\text{Quot}^\circ_{n+m,m,l}) \\
\end{array}
\]

Moreover, we have the following commutative diagram by Lemma 2.2

\[
\begin{array}{ccc}
K^{P_{n,m,l}}(\text{Flag}^\circ_{n,m,l}) & \xrightarrow{\mathbf{q}_{n+m,l}} & K^{P_{n+m,m,l}}(\text{Flag}^\circ_{n+m,m,l}) \\
\psi^l_{n+m,m,l} & & \psi^l_{n+m,m,l} \\
K^{P_{n,m,l}}(\text{Flag}^\circ_{n,m,l} \times \text{Quot}^\circ_{l}) & \xrightarrow{q_{n+m,l}} & K^{P_{n+m,l}}(\text{Quot}^\circ_{n+m,m,l}) \\
\end{array}
\]

So we have \(*_{K^{\text{Coh}}_{n+m,m,l}} \circ *_{K^{\text{Coh}}_{n,m,l}}\) is induced by

\[
\tilde{q}_{n,m,l} \circ \mathbf{q}_{n,m,l} \circ \psi^l_{n+m,m,l} \circ \psi^l_{n,m,l}.
\]

Similarly \(*_{K^{\text{Coh}}_{n+m,m,l}} \circ *_{K^{\text{Coh}}_{n,m,l}}\) is induced by

\[
\tilde{q}_{n,m+l} \circ \mathbf{q}_{n,m+l} \circ \psi^l_{n+m+l,m,l} \circ \psi^l_{n,m+l}.
\]

Notice \(\tilde{q}_{n,m,l} \circ \mathbf{q}_{n,m,l} = \mathbf{q}_{n,m,l} \circ \psi^l_{n+m,m,l} \circ \psi^l_{n,m,l}\), so

\[
\bar{q}_{n,m+l} \circ \mathbf{q}_{n,m+l} = \bar{q}_{n,m+l} \circ \psi^l_{n+m+l,m,l} \circ \psi^l_{n,m+l}.
\]

And \(\psi^l_{n,m+l} \circ \psi^l_{n,m+l} = \psi^l_{n,m+l} \circ \psi^l_{n,m+l}\) by Proposition 3.12. So we have \(*_{K^{\text{Coh}}_{n+m,m,l}} \circ *_{K^{\text{Coh}}_{n,m,l}}\) is associative.

\[\square\]

5. Shuffle Algebra and K-theoretical Hall Algebra

In this section, we study the equivariant K-theory of Quot schemes through the localization theorem 2.10 and construct a homomorphism from the K-theoretical hall algebra to shuffle algebra, which is introduced by [9].

5.1. Localization for K^{\text{Coh}}. Let \(d\) be a non-negative integer, and \(\sigma_d\) the permutation group of order \(d\), which is also the Weyl group of maximal torus \(T_d\) in \(G_d\).

By Theorem 2.3 we have

\[
K^{G_d}(\text{Quot}_{d}) = (K^{T_d}(\text{Quot}_{d}))^{\sigma_d}.
\]

By Lemma 3.14 we have

\[
(\text{Quot}_{d})^{T_d} = S_d,
\]

and \(K^{T_d}(S^d_{\text{loc}}) = K(S^d_{\text{Sym}}).\) Then by localization theorem 2.10 there is an isomorphism

\[
K^{T_d}(S^d_{\text{loc}}) \xrightarrow{id_*} K^{T_d}(\text{Quot}_{d})_{\text{loc}}.
\]

Let \(l_d : K^{T_d}(\text{Quot}_{d})^{\sigma_d} \to (K^{T_d}(\text{Quot}_{d})_{\text{loc}})^{\sigma_d}\) be the natural localization map, and

\[
\tau_d = l_d \circ i^{-1}_{d} : (K^{T_d}(\text{Quot}_{d}))^{\sigma_d} \to K^{\infty}(S^d_{\text{Sym}})(z_1, \ldots, z_d)_{\text{Sym}}
\]

where \(\text{Sym}\) means invariant under the \(\sigma_n\) action. Then \(\tau = \bigoplus_{d=0}^{\infty} \tau_d\) define a linear morphism:

\[
\tau : \bigoplus_{d=0}^{\infty} K^{G_d}(\text{Quot}_{d}) \to \bigoplus_{d=0}^{\infty} K^{\infty}(S^d_{\text{Sym}})(z_1, \ldots, z_d)_{\text{Sym}}
\]
5.2. **Shuffle Algebra Revisited.** Now we recall the definition of shuffle algebra associated to $S$, which is defined in [9].

**Definition 5.1.** Consider the abelian group:

$$KSh = \bigoplus_{d=0}^{\infty} K_{S^d}(z_1, ..., z_d)^{Sym}$$

where the superscript $Sym$ means that we consider rational functions that are symmetric under the simultaneous permutation of the variables $z_i$ and the factors of the $d$-fold product $S^dS$. We endow $\mathcal{V}_{big}$ with the following associative product:

$$R(z_1, ..., z_n) \ast_{KSh} R'(z_1, ..., z_m) =$$

$$= Sym \left( (R \boxtimes 1^m)(z_1, ..., z_n)(1^m \boxtimes R')(z_{n+1}, ..., z_{n+m}) \prod_{i=1}^{n} \prod_{j=n+1}^{n+m} \zeta_{ij}^{S} \left( \frac{z_j}{z_i} \right) \right)$$

where:

$$\zeta_{ij}^{S}(x) = \Lambda^{\ast} \frac{(x \cdot \mathcal{O}_{\Delta_{ij}})}{(1 - x)(1 - \frac{1}{x})} \in K_{S_{n+m}}(x)$$

and $F \boxtimes G = pr_{n}^{\ast}(F) \otimes pr_{m}^{\ast}(G)$ for $F \in K(S^n)$, $G \in K(S^m)$ and $pr_n, pr_m$ the respective projection map from $S^n \times S^m$ to $S^n$ and $S^m$. We call $(KSh, \ast_{KSh})$ the **shuffle algebra** associated to $S$.

**Remark 5.2.** The definition of Shuffle algebra in our paper is slightly different the definition in [9], but they differ by a straightforward automorphism.

**Theorem 5.3.** $\tau$ is an algebra homomorphism between $(Kcoh, \ast_{Kcoh})$ and $(KSh, \ast_{KSh})$.

**Proof.** By Lemmas 3.16 and 3.17 we have the following commutative diagram:

$$
\begin{array}{c}
\begin{array}{c}
K^{T_{n+m}}(S^{n+m})_{loc}^{\sigma_n \times \sigma_m} \leftarrow K^{T_{n+m}}(\text{Quot}_{n,m}^\circ) \leftarrow K^{G_{n+m}}(\text{Quot}_{n,m}^\circ) \\
\downarrow \text{id} \downarrow \quad \downarrow \text{id} \downarrow \\
K^{T_{n+m}}(S^{n+m})_{loc}^{\sigma_n \times \sigma_m} \leftarrow K^{T_{n+m}}(\text{Quot}_{n,m}^\circ) \leftarrow K^{P_{n,m}}(\text{Quot}_{n,m}^\circ) \\
\downarrow \rho_{n,m} \downarrow \quad \downarrow \psi_{n,m} \downarrow \\
K^{T_{n+m}}(S^{n+m})_{loc}^{\sigma_n \times \sigma_m} \leftarrow K^{T_{n+m}}(\text{Flag}_{n,m}^\circ) \leftarrow K^{P_{n,m}}(\text{Flag}_{n,m}^\circ) \\
\downarrow \delta' \downarrow \quad \downarrow j_{n,m}^{\ast} \downarrow \\
K^{T_{n+m}}(S^{n+m})_{loc}^{\sigma_n \times \sigma_m} \leftarrow K^{T_{n+m}}(\text{Flag}_{n,m}^\circ) \leftarrow K^{G_{n+m}}(\text{Flag}_{n,m}^\circ) \\
\downarrow s \downarrow \quad \downarrow g_{n+m}^{\ast} \downarrow \\
K^{T_{n+m}}(S^{n+m})_{loc}^{\sigma_n \times \sigma_m} \leftarrow K^{T_{n+m}}(\text{Quot}_{n,m}^\circ) \leftarrow K^{G_{n+m}}(\text{Quot}_{n,m}^\circ) \\
\end{array}
\end{array}
$$

where $j_{n,m}$ is the embedding from $\text{Flag}_{n,m}^\circ$ to $\text{Flag}_{n,m}$. Notice $\ast_{KSh}^{n,m} = s_{\ast} \circ \delta' \circ \rho_{n,m}$ and $\ast_{Kcoh}^{n,m} = q_{n+m} \circ \text{ind}_{\text{flag}_{n,m}^\circ}^{\text{flag}_{n,m}^\circ} \circ \psi_{n,m}^{\ast}$.
Thus we have the commutative diagram:

\[
\begin{array}{ccc}
K^{T^{n+m}}(S^{n+m})_{\text{loc}} & \leftarrow & K^{G^{n+m}}(\text{Quot}^{n,m}_{n,m}) \\
\downarrow^{K_{\text{Sh}}} & & \downarrow^{K_{\text{Coh}}} \\
K^{T^{n+m}}(S^{n+m})_{\text{loc}} & \leftarrow & K^{G^{n+m}}(\text{Quot}^{n,m}_{n,m})
\end{array}
\]

i.e. \( \tau \) forms a homomorphism between \( *^{K_{\text{Coh}}} \) and \( *^{K_{\text{Sh}}} \).

\[ \square \]

REFERENCES

1. Dave Anderson and Sam Payne, *Operational k-theory*, Doc. Math 20 (2015), no. 357-399, 13.
2. Neil Chriss et al., *Representation theory and complex geometry*, Springer Science & Business Media, 2009.
3. Ionut Ciocan-Fontanine and Mikhail Kapranov, *Virtual fundamental classes via dg-manifolds*, 2007.
4. Geir Ellingsrud and Manfred Lehn, *On the irreducibility of the punctual Quotient Scheme of a Surface*, 1997.
5. Robin Hartshorne, *Algebraic geometry*, vol. 52, Springer Science & Business Media, 2013.
6. Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Cambridge University Press, 2010.
7. Alexandre Minets, *Cohomological Hall algebras for Higgs torsion sheaves, moduli of triples and sheaves on surfaces*, 2018.
8. Andrei Negut, *The Shuffle Algebra Revisited*, 2012.
9. Andrei Negut, *Shuffle algebras associated to surfaces*, arXiv preprint arXiv:1703.02027 (2017).
10. F. Qu, *Virtual pullbacks in K-theory*, 2016.
11. O. Schiffmann and E. Vasserot, *Hall algebras of curves, commuting varieties and Langlands duality*, 2010.
12. Olivier Schiffmann and Eric Vasserot, *On cohomological Hall algebras of quivers : generators*, 2017.
13. Robert W Thomason et al., *Une formule de lefschetz en k-théorie équivariante algébrique*, Duke Mathematical Journal 68 (1992), no. 3, 447–462.
14. Bertrand Toen, *K-theory and cohomology of algebraic stacks: Riemann-Roch theorems, D-modules and GAGA theorems*, 1999.