Invariant theory of symplectic and orthogonal groups

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In this paper we study the structure of \((\wedge(M^+_{2n})^* \otimes M_n)^G\), the space of the \(G\)-equivariant matrix valued alternating multilinear maps on the space of (skew-symmetric or symmetric with respect to a specific involution) matrices, where \(G\) is the symplectic group or the odd orthogonal group.

We prove that these algebras are free modules on a subalgebra of the exterior algebra of invariant multilinear alternating functions generated by traces. Further we prove new polynomial trace identities.

The even orthogonal case can also be studied but it presents additional complications and will be addressed in a separate note.

1 Symplectic case

We denote, for all \(k\), by \(1_k\) the identity matrix of order \(k\) and by \(A^t\) the usual transposition of a matrix \(A\).

We consider the skew-symmetric matrix \(J := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}\).

On the space \(V := \mathbb{C}^{2n}\) we have the skew-symmetric form \((u, v) := u^t J v\), where \(u\) and \(v\) are column vectors in \(\mathbb{C}^{2n}\).

The symplectic transposition, \(A \mapsto A^s\), is defined by

\[
(Au, v) = (u, A^s v) \iff (Au)^t J v = u^t JA^s v \iff A^t J = JA^s \iff A^s := -JA^t J.
\]

Explicitly we have that, if \(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) with \(A, B, C, D \in \mathbb{C}^{n \times n}\), its symplectic transposed is

\[
M^s = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}.
\]

Let \(M^+_{2n} = \{ A \mid A^* = A \}\) be the space of symmetric matrices with respect to the symplectic transposition. Notice that the map \(A \mapsto AJ\) gives a linear isomorphism of the space \(M^+_{2n}\) onto the space \(\wedge^2 V\) of skew-symmetric matrices (with respect to the usual transposition). The group \(GL(2n, \mathbb{C}) = GL(V)\) acts on \(\wedge^2 V\) by \(X \circ A := XAX^t\), for \(X \in GL(V)\) and \(A \in \wedge^2 V\).

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Let $G \subset GL(V)$ be the symplectic group. By definition $G$ is the group of transformation preserving the symplectic form, that is

$$G := \{ X \in GL(V) \mid (u, v) = (Xu, Xv), \forall u, v \iff u^t J v = u^t X^t J X v \iff J = X^t J X \}$$

so $X^{-1} = X^*$. 

The Lie algebra $M_{2n}^-$ of $G$ is the space of the skew-symmetric matrices, with respect to the symplectic form, $M_{2n}^- := \{ A \mid A^* = A \}$. The map $A \mapsto AJ$ gives a linear isomorphism of $M_{2n}^-$ onto the space $S^2(V)$ of symmetric matrices (with respect to the usual transposition). Also in this case $GL(V)$ acts on $S^2(V)$ by $X \circ A := XAX^t$, for $X \in GL(V)$ and $A \in S^2(V)$.

The conjugation action of $G$ commutes with the map $A \mapsto AJ$. In fact if $X \in G$ we have $XJX^{-1} = J$, so we can state that

**Proposition 1.1.** The action of the symplectic group $G$ on $M_{2n}^+$ (resp. on $M_{2n}^-$) can be identified with the restriction to $G$ of the usual action of the linear group $GL(V)$ on $\wedge^2 V$ (resp. on $S^2(V)$).

Remark that with respect to the action of $G$, while the Lie algebra $L := M_{2n}^- \simeq S^2(V)$ is irreducible, $M_{2n}^+$ is not irreducible. Indeed it decomposes as the direct sum of the one dimensional space of scalar matrices and of the space $P_0$ of traceless matrices. Under the isomorphism with $\wedge^2 V$ the space of scalar matrices maps to the space spanned by $J$.

## 2 Invariants of the representation $\wedge(M_{2n}^+) \otimes M_{2n}$

### 2.1 Dimension

Here and below we index the irreducible representations of $GL(V)$ by Young diagrams with at most $n$ columns (the row of length $k$ corresponds to $\wedge^k V$). The irreducible corresponding to the diagram $\lambda$ will be denoted by $S_\lambda(V)$.

By the *Plethysm formulas* (see § pag. 161) we know how to decompose $\wedge[\wedge^2 V]$ with respect to the action of the linear group $GL(V)$.

We have that $\wedge[\wedge^2 V]$ is the direct sum of the irreducible representations

$$H_{a_1, a_2, \ldots, a_k} (V) = H_{\underline{\omega}_\lambda} (V) := S_{\lambda(\underline{\omega})}(V),$$

with $2n > a_1 > a_2 \ldots > a_k > 0$. Notice that $H_{\underline{\omega}}$ is the trivial one dimensional representation.

The Young diagram $\lambda(\underline{\omega})$ is built by nesting the *hook* diagrams $h_{a_i}$, whose column is of length $a_i$ and whose row is of the length $a_i + 1$. As an example, the diagram $\lambda(4, 3, 1)$ is

```
+---+---+---
|   |   |   |
|   |   |   |
|   |   |   |
+---+---+---
```

Using this, we can compute the dimension of the invariants for the symplectic group $G$ in $\wedge M_{2n}^+$. We know that, for each diagram $\lambda$, $\dim(S_\lambda(V))^G \leq 1$ and $\dim(S_\lambda(V)^G) = 1$ if and only if every row of $\lambda$ is even. (see §).
The rows of a representation $H^-(V)$ are even if and only if $a = b_1, b_1 - 1, b_2, b_2 - 1, \ldots, b_s, b_s - 1, \ldots$, or $a = b_1, b_1 - 1, b_2, b_2 - 1, \ldots, b_s, 1$ with the $b_j$’s odd and $b_s > 1$. So we have to compute the number of decreasing sequences of odd numbers smaller than $2n$. This is the same to compute the number of decreasing sequences of odd numbers taken from $1, \ldots, n$, that is the number of subsets of $\{1, \ldots, n\}$. Thus

**Proposition 2.1.** The dimension of the space of invariants $(\bigwedge M_{2n}^+)^G$ is $2^n$.

We now pass to determine the dimension of the space $(\bigwedge M_{2n}^+ \otimes M_{2n})^G$. Before we proceed, let us make a few remarks. We know that to study an isotypic component relative to an irreducible representation $N$ in a representation $M$ of a reductive group $G$, we need to study the space $\text{hom}_G(N,M)$ that we can identify to $(N^* \otimes M)^G$. The dimension of this space will be the number of copies of $N$ in $M$.

We decompose the space of matrices $M_{2n} = M_{2n}^+ \oplus M_{2n}^- = C \oplus P_0 \oplus M_{2n}^-$. Thus we have

$$
\left(\bigwedge M_{2n}^+ \otimes M_{2n}\right)^G = \left(\bigwedge M_{2n}^+ \otimes C\right)^G \oplus \left(\bigwedge M_{2n}^+ \otimes P_0\right)^G \oplus \left(\bigwedge M_{2n}^+ \otimes M_{2n}^-\right)^G
$$

The space $(\bigwedge M_{2n}^+ \otimes C)^G$ is the space of invariants whose dimension we have already computed. Since $P_0$ and $M_{2n}^-$ are both irreducible and self dual, our computation will give us information on the isotypic components relative to these two representations.

Now identify, as a representation of $G$, $M_{2n} \simeq V \otimes V$. We thus have to study the $G$-invariants in

$$
\bigwedge (M_{2n}^+ \otimes V) \otimes V = \oplus_\lambda H^- (V) \otimes V \otimes V.
$$

By Pieri’s formulas we know how to decompose $S_\lambda(V) \otimes V \otimes V$. First we decompose $S_\lambda(V) \otimes V$. We have that $S_\lambda(V) \otimes V = \oplus S_\lambda(V)$ where $\lambda_i$ runs along the diagrams whose first row is of length at most $2n$ and which are obtained from $\lambda$ by adding one box. So, iterating the same process, to compute the dimension of the invariants we have to study when, adding two squares, starting from one of the diagrams $H^-$ we obtain a diagram with even rows.

**Proposition 2.2.** The dimension of $(\bigwedge (M_{2n}^+)^* \otimes M_{2n})^G$ is $(2n - 1)2^n$.

**Proof.** We proceed by induction on $n$. The case $n = 1$ is trivial so we assume $n > 1$. We have to compute the sequences of type:

$$
2n > a_1 > a_2 > \ldots > a_k > 0,
$$

such that when we add two squares to the corresponding diagram we obtain a diagram with even rows.

We can have four different situations.

Let us start by considering the case $a_1 < 2(n - 1) = 2n - 2$. By induction the number of diagrams with even rows whose first row has at most $2(n - 1)$ boxes is $(2(n - 1) - 1)2^{n-1}$. However if $a_1 = 2n - 3$ we can add two squares to the first row so that its length is $(a_1 + 3 = 2n)$. Thus we have a contribution from each sequence $a = 2n - 3 > 2n - 4 > b_2 > b_2 - 1 > \ldots > b_s - 1$ or $a = 2n - 3 > 2n - 4 > b_2 > b_2 - 1 > \ldots > b_s - 1, 1$ with the $b_j$’s odd and $b_s > 1$. Reasoning exactly as in the proof of Proposition 2.1 we see that there are exactly $2^{n-2}$ such diagrams.

The second case is

$$
\begin{align*}
\text{a}_1 &= 2n - 1 \text{ and } \text{a}_2 = 2n - 2,
\end{align*}
$$

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In this case we have two possibilities. We can add two squares at the bottom

and we get a contribution from each sequence \( a = 2n - 3 > 2n - 4 > b_2 > b_2 - 1 > \ldots > b_s - 1 \) or \( a = 2n - 1 > 2n - 2 > b_2 > b_2 - 1 > \ldots > b_s - 1 > 1 \) with the \( b_j \)'s odd and \( b_s > 1 \) and there are \( 2^{n-1} \) such sequences.

Otherwise we add the two squares to the diagram associated to the sequence \( 2(n-1) > a_3 > a_4 > \ldots > a_k > 0 \), so that, by induction on \( n \), we get \( (2(n-1)-1)2^{n-1} \) contributions.

The third case is:

\[ a_1 = 2n - 1 \text{ and } a_2 < 2n - 2, \]

in this case the only way to add two squares is to add one on the second row and one to the second column

so we would get \( 2^{n-2} \) contributions, but we can exchange the order in which we add the squares, so we get \( 2 \cdot 2^{n-2} \) contributions.

In the case we have \( a_1 = 2n - 2 \), so that in order to get a diagram with even rows, we need to add a box to the first row. If \( a_2 < 2n - 4 \), there are at least two rows of length 1 so that we cannot obtain a diagram with even rows by adding a single box. If \( a_2 = 2n - 3 \) we need to add the second square to the second row

and one easily sees that we get \( 2^{n-2} \) contributions.

If \( a_2 = 2n - 4 \) we need to add a box to the second column:

and in this case we get \( 2 \cdot 2^{n-2} \) contributions, since we can exchange the order in which the boxes are inserted.

Finally adding all the contributions we obtain

\[ (2(n-1)-1)2^{n-1} + 2^{n-2} + 2^{n-1} + (2(n-1)-1)2^{n-1} + 2^{n-1} + 2^{n-2} + 2^{n-1} = (2n-1) \cdot 2^n. \]

This is our claim.
2.2 Structure of algebras of invariants

Let $M_{2n}$ be, as in the previous section, the space of complex matrices $2n \times 2n$ and let $M_{2n}^+ := \{x \in M_{2n} | x = x^t\}$ be the symmetric part with respect to the symplectic involution. Since both $\bigwedge^*(M_{2n}^+)$ and $\bigwedge^*(M_{2n}^+ \otimes M_{2n})$ have a structure of graded associative algebras, also the algebras

$$A := \left(\bigwedge^*(M_{2n}^+)^G\right)$$

and

$$B := \left(\bigwedge^*(M_{2n}^+ \otimes M_{2n})^G\right)$$

have a natural structure of graded associative algebras. Furthermore $B$ is clearly an $A$-module. These are the structures we want to investigate.

Identifying $\bigwedge^*(M_{2n}^+)$ with its dual and thus $\bigwedge^*(M_{2n}^+ \otimes M_{2n})$ with $\text{Hom}(\bigwedge^*(M_{2n}^+), M_{2n})$, we can define the elements

$$T_h := \text{Tr}(S_{4h+1}(x_1, \ldots, x_{4h+1})) \in A_{4h+1},$$

where $S_{4h+1}$ is the skew-symmetric standard polynomial, and

$$X \in B_1$$

by $X(x) = x$ for any $x \in M_{2n}^+$ (recall that $M_{2n}^+ \subset M_{2n}$).

We want to give an explicit description of the algebra $B$. This algebra can be thought in a natural way as a module on the algebra $A$. As an algebra $A$ is generated by the element $\text{Tr}(S_{2h+1}(x_1, \ldots, x_{2h+1}))$. Indeed by classical invariant theory, we know that the polynomial invariant functions on the space $M_{2n}$ are generated by traces of the monomials in the variables $x_i, x_i^t$ (FFT for matrices, see [2] pag. 437); furthermore since $M_{2n}^+$ is a $G$-stable subspace of $M_{2n}$, we have that every $G$-invariant polynomial function on $M_{2n}^+$ is the restriction of a $G$-invariant polynomial on $M_{2n}$ ($G$ is a linear reductive algebraic group). So by multilinearizing and alternating we have our claim (recall that $\text{Tr}(S_{2h}(x_1, \ldots, x_{2n})) = 0$).

Since $M_{2n} = M_{2n}^- \oplus M_{2n}^+$ we have that $\bigwedge(M_{2n}^+) \otimes M_{2n} = (\bigwedge(M_{2n}^+) \otimes M_{2n}^-) \oplus (\bigwedge(M_{2n}^+) \otimes M_{2n}^+)$ and, passing to the invariants, $B = B^- \oplus B^+$, with $B^+ = (\bigwedge(M_{2n}^+) \otimes M_{2n}^+)^G$.

The following simple Proposition gives some crucial properties of $A$ and $B M_{2n}^+$.

**Proposition 2.3.** We have:

1. The element $\text{Tr}(S_{4k+3}(x_1, \ldots, x_{4k+3}))$ is zero for each $k \geq 0$.
2. $X^k \in B^+$ if and only if $k \equiv 0, 1$ modulo 4.
3. $X^k \in B^-$ if and only if $k \equiv 2, 3$ modulo 4.

**Proof.** We prove the first fact. The proof of the rest being identical is left to the reader.

By the explicit form of the sympletic transposition (1) we have $\text{Tr}(x) = \text{Tr}(x^*)$, so

$$\text{Tr}(S_{2h+1}(x_1, \ldots, x_{2h+1})) = \text{Tr}(S_{2h+1}(x_1, \ldots, x_{2h+1})^*),$$

but

$$S_{2h+1}(x_1, \ldots, x_{2h+1})^* = \sum_{\sigma \in S_{2h+1}} \epsilon_{\sigma}(x_{\sigma(1)} \cdots x_{\sigma(2h+1)})^*$$

$$= \sum_{\sigma \in S_{2h+1}} \epsilon_{\sigma}x_{\sigma(2h+1)} \cdots x_{\sigma(1)};$$
and if we consider the permutation $\eta : (1, \ldots, 2h+1) \to (2h+1, \ldots, 1)$ we see that it has the same sign of the parity of $h$. So everything follows.

We deduce the following well know result (see for example [2]):

**Theorem 2.4.** The algebra $A$ is the exterior algebra, of dimension $2^n$, in the elements $T_0, T_1, \ldots, T_{n-1}$, where $T_h := \text{Tr}(S_{4h+1}(x_1, \ldots, x_{4h+1})) \in A_{4h+1}$.

*Proof.* By the previous Proposition, $A$ is generated by the elements $\text{Tr}(S_{4h+1}(x_1, \ldots, x_{4h+1}))$. By the Amitzur-Levitzki theorem, $\text{St}_r(x_1, \ldots, x_r) = 0$ for $r \geq 4n$. It follows that $A$ is generated by $T_0, T_1, \ldots, T_{n-1}$, so it is the quotient of an exterior algebra on $n$ generators. Since $\dim A = 2^n$, our claim follows. □

Notice that we can define the trace function $\text{Tr} : \bigwedge M_{2n} \otimes M_{2n} \to \bigwedge M_{2n}$ by extending the invariant function on matrices. By equivariance, on $B$ the trace function takes values in $A$. In particular notice that $\text{Tr}(X^h)(x_1, \ldots, x_k) = \text{Tr}(\text{St}_h(x_1, \ldots, x_k))$.

**Theorem 2.5.** We have:

1. As a algebra $B$ is generated by $A$ and the element $X$.

2. $B$ is a free module on the the exterior algebra $A_{n-1} \subset A$ generated by $T_0, \ldots, T_{n-2}$, with basis $1, X, \ldots, X^{4n-3}$.

*Proof.*

1) follows from classical invariant theory (see [10],[9],[3]).

2) By Proposition 2.2, it suffices to see that the elements $1, X, \ldots, X^{4n-3}$ are linearly independent over $A_{n-1}$. So, let

$$\sum_{h=0}^{4n-3} P_h X^j = 0,$$

$P_h \in A_{n-1}$ for each $h = 0, \ldots, 4n-3$. Assume by contradiction that not all $P_h$’s are 0. Let $j$ be the minimum such that $P_j \neq 0$. Multiply by $X^{4n-3-j}$ and take traces. We get $P_j T_{n-1} = 0$, since for $h \geq 4n$, $X^h = 0$, $\text{Tr}(X^{4n-2}) = \text{Tr}(X^{4n-1}) = 0$ by Proposition 2.3 and the fact that $4n-2$ is even. $P_j \in A_{n-1}$, so $P_j T_{n-1} = 0$ if and only if $P_j = 0$, a contradiction. □

Consider now the graded super algebra $A[t]$, with deg $t = 1$. Notice that for each $i$, $T_i t = -t T_i$. Define the graded algebra homomorphism $\pi : A[t] \to B$ by

$$\sum_j a_j t^j \mapsto \sum_j a_j X^j,$$

$a_j \in A$. By the previous theorem we see that $\pi$ is surjective. Let us describe its kernel.

**Theorem 2.6.** The Kernel $I$ of the homomorphism $\pi$ is the principal ideal generated by the element

$$nt^{4n-3} - \sum_{i=0}^{n-1} \frac{1}{2} T_{n-1-i} t^{4i}.$$
Proof. Let us first start by showing that the element $nt^{4n-3} - \sum_{i=0}^{n-1} \frac{1}{2} T_{n-i-1} t^4 i$ lies in $I$. By the second part of Theorem 2.4, we necessarily have a homogeneous relation

$$T_{n-1} = \sum_{h=0}^{4n-3} P_h X^h$$

with $P_h \in A_{n-1}$ of degree $4n - 3 - h$. Let us compute $P_{4n-3}$. Notice that $Tr(\sum_{h=0}^{4n-4} P_h X^h) \in A_{n-1}$, while $Tr(T_{n-1}) = 2nT_{n-1}$ It follows that $(2n - P_{4n-3})T_{n-1} \in A_{n-1}$ namely $P_{4n-3} = 2n$. Assume now $0 \leq j \leq 4n - 4$ and multiply by $X^{4n-3-j}$. Taking trace and reasoning as in the proof of Theorem 2.5, we get that

$$T_{n-1} (Tr(X^{4n-3-j}) + (-1)^j P_j) \in A_{n-1}$$

If $j = 0$ we get $P_0 = 0$. Assume $j > 0$. This implies $Tr(X^{4n-3-j}) + (-1)^j P_j = 0$ that is

$$P_j = \begin{cases} 0 & \text{if } j = 2h + 1, 4h + 2 \\ T_h & \text{if } i = 4h + 1 \end{cases}$$

If $i = 4n - 3$ we get $P_{4n-3} = 2n$. If we divide by 2 we obtain

$$\frac{1}{2} T_{n-1} = nX^{4n-3} - \sum_{j=1}^{n-1} \frac{1}{2} T_{n-j-1} X^{4j}$$

which is our relation. Denote by $J$ the ideal generated by

$$nt^{4n-3} - \sum_{i=0}^{n-1} \frac{1}{2} T_{n-i-1} t^4 i.$$)

We have seen that $J \subset I$.

Now let us also see that $t^{2n-2} \in J$. To see this let us write the relation (4) as

$$nt^{2n-3} - \sum_{h=0}^{n-1} T_{n-h-1} t^{4h}.$$)

Let us remark that multiplying on the right by $t$ we get

$$nt^{2n-2} - \sum_{h=0}^{n-1} T_{n-h-1} t^{4h+1} \in J.$$)

If we multiply by $t$ on the left we get

$$nt^{2n-2} - \sum_{h=0}^{n-1} tT_{n-h-1} t^{4h} = nt^{2n-2} + \sum_{h=0}^{n-1} T_{n-h-1} t^{4h+1}.$$)

Adding the two relation we thus obtain

$$2nX^{2m-2} \in J.$$ (5)
Denote by $J$ the ideal generated by

$$nt^{4n-3} - \sum_{i=0}^{n-1} \frac{1}{2} T_{n-i-1} t^{4i}.$$  

We have seen that $J \subset I$. Now consider $A/J$. Using (5), it is clear that the image of $1, t, \ldots, t^{4n-3}$ span $A/J$ as an $A$ module. However by (4) we deduce that

$$T_{n-1} = 2nt^{4n-3} - \sum_{i=1}^{n-1} T_{n-i-1} t^{4i},$$

so that, substituting, the same elements span $A/J$ as an $A$ module. It follows that $\dim A/J \leq (4n - 2) 2^{n-1} = (2n - 1) 2^n$. We deduce that $I = J$. 

Remark 2.7. The fact that $X^{2n-2} = 0$ means that the standard polynomial $S_{4n-2}(x_1, \ldots, x_{4n-2})$ is identically 0 for $x_1, \ldots, x_{4n-2} \in M_{2n}^+$. This is a result of Rowen [11].

Finally using Proposition 2.3 we deduce,

Corollary 2.8. 1) The elements $1, X, X^4, X^5, \ldots, X^{4n-3}$ are a basis of $(B^+)^G$ as a free module over the exterior algebra $A_{n-1}$.

2) The elements $X^2, X^3, X^6, X^7, \ldots, X^{4n-5}$ are a basis of $(B^-)^G$ as a free module over the exterior algebra $A_{n-1}$.

3 Invariants of the representation $\wedge (M_{2n}^-)^* \otimes M_{2n}$

We set $L = M_{2n}^-$. We want to study the algebra

$$[\wedge^* L^* \otimes M_{2n}]^G.$$  

Remark 3.1. Let us remark that $L$ is the Lie algebra of type $C_n$ and in this part of the work we will recover (among the others) general results for covariants in the exterior algebra of a Lie algebra $g$ as in [7].

3.1 Dimension

By the Plethysm formulas (see [6]) we know how to decompose $\wedge [S^2V]$ with respect to the action of the linear group $GL(V)$.

We have that $\wedge [S^2V]$ is the direct sum of the irreducible representations

$$H_{\lambda_1, \ldots, \lambda_k}^+(V) = H_{\lambda_1, \ldots, \lambda_k}^-(V) = S_{\lambda(a)}(V),$$

with $2n > a_1 > a_2 \ldots > a_k \geq 0$.

This time, differently from the symmetric case, the Young diagram $\lambda(a)$ is built by nesting the hook diagrams $h_{a_i}$ whose column is of length $a_i + 2$ and whose row is of the length $a_i + 1$. Now (similarly to the previous case) we want to compute the dimension of the invariants for the
symplectic group.

We have that the rows of a representation $H^+_a(V)$ are even if and only if the sequence is of the type $a, a-1, b, b-1, c, c-1, \ldots$, with $a, b, c, \ldots$ odd. So as before we can state:

**Lemma 3.2.** The dimension of the space of invariants $(\wedge M_{2n})^G$ is $2^n$.

Further we can state an analogous result to the symmetric case for the dimension of covariants:

**Proposition 3.3.** The dimension of $(\wedge (M_{2n})^* \otimes M_{2n})^G$ is $(2n)^2$.

**Proof.** By induction the proof is very similar to the Proposition 2.2.

### 3.2 Structure

By the classical invariant theory we have that the polynomial functions $G$-equivariant from $m$-copies of $L$ to $M_{2n}$, are an associative algebra generated by the coordinates $Y_i$ and by traces of monomials in $Y_i$. So we have that $[\wedge L^* \otimes M_{2n}]^G$ is generated by the element $Y, Y^s = -Y$ and $Tr (Y^i)$. Further we can decompose in a natural way

$$B := [\wedge L^* \otimes M_{2n}]^G = [\wedge L^* \otimes L]^G \oplus [\wedge L^* \otimes M_{2n}^+]^G.$$  

We have an analogous of the Proposition 2.3:

**Proposition 3.4.**

1. The element $Tr(S_{4k+1}(y_1, ..., y_{4k+1}))$ is zero for each $k \geq 0$.

2. $Y^k \in B^+$ if and only if $k \equiv 0, 3 \pmod{4}$

3. $Y^k \in B^-$ if and only if $k \equiv 1, 2 \pmod{4}$.

**Proof.** As in the Proposition 2.3 we prove the first fact. The proof of the rest being identical is left to the reader.

We recall that

$$Tr(S_{2h+1}(y_1, ..., y_{2h+1})) = Tr(S_{2h+1}(y_1, ..., y_{2h+1})^s),$$

but

$$S_{2h+1}(y_1, ..., y_{2h+1})^s = \sum_{\sigma \in S_{2h+1}} \epsilon_{\sigma}(y_{\sigma(1)} \cdots y_{\sigma(2h+1)})^s$$

$$= \sum_{\sigma \in S_{2h+1}} (-1)^{2h+1} \epsilon_{\sigma} y_{\sigma(2h+1)} \cdots y_{\sigma(1)},$$

and if we consider the permutation $\eta : (1, ..., 2h+1) \rightarrow (2h+1, ..., 1)$ we see that it has the same sign of the parity of $h$. So everything follows. \hfill $\Box$

We set

$$T_h := Tr (S_{4h+3}(Y_1, ..., Y_{4h+3})) \quad (7)$$

we have, by the computation of the dimension and by the Amitsur-Levitzki’s Theorem, that
Lemma 3.5. The algebra $A := (\wedge M_{2n}^-)^G$ is the exterior algebra in the elements $T_0, ..., T_{n-1}$.

So we can describe the structure of covariants:

Theorem 3.6. 1. The algebra $B$ is a free module on the subalgebra $A_{n-1} \subset A$ generated by the elements $T_i, i = 0, ..., n-2$, with basis $1, ..., Y^{4n-1}$.

2. With analogous notations to the previous section we have that the kernel of the canonical homomorphism $\pi : A[t] \to B$ is the principal ideal generated by

$$\sum_{i=0}^{n-1} t^{4i} \wedge T_{n-i-1} - 2nt^{4n-1}.$$ 

Proof. By the calculus of dimensions we only need a formula to compute the multiplication by the element $T_{n-1} = Tr(Y^{4(n-1)+3}) = Tr(Y^{4n-1})$, but in this case we can use the general formulas for the matrices (see [2]).

We start from the universal formula for $M_{2n}$:

$$1 \wedge T(4n-1) = T(4n-1) = -\sum_{i=1}^{2n-1} Z^{2i} \wedge T(2(2n-i)-1) + 2nZ^{4n-1},$$

where $Z : M_{2n} \to M_{2n}$ is the identity map and $T(h) := Tr(St_h(x_1, ..., x_h))$.

In our case $Z \sim Y$ and we have (by 3.3) that $T(2(2n-i)-1) = 0$ unless that $-2i-1 = 3 \mod(4)$. So $i$ is even. We deduce:

$$T_{n-1} = -\sum_{i=1}^{n-1} Y^{4i} \wedge T_{n-i-1} + 2nY^{4n-1},$$

so

$$Y^j \wedge T_{n-1} = -\sum_{i=1}^{n-1} Y^{4i+j} \wedge T_{n-i-1}.$$ 

So by this last result and from the Proposition 3.4 we have (consistently to the general theory for Lie algebras developed in [7]):

Corollary 3.7. 1. The $2n$ elements $Y, Y^2, Y^4, Y^5, Y^6, ..., Y^{4n-3}, Y^{4n-2}$ are a basis of $[\wedge L^* \otimes L]^G$ on the exterior algebra generated by the $n-1$ elements $T_i, i = 0, ..., n-2$.

2. The $2n$ elements $1, Y^3, Y^4, Y^7, ..., Y^{4n-4}, Y^{4n-1}$ are a basis of $[\wedge L^* \otimes M_{2n}^+]^G$ on the exterior algebra generated by the $n-1$ elements $T_i, i = 0, ..., n-2$. 

10
4 The skewsymmetric orthogonal case

In the last part of the work we want to generalize our considerations to the case in which the group $G$ considered is not $Sp(2n)$, but $O(2n + 1)$. Results are very similar and, almost always, the proof is the same of the symplectic case, so we will leave details to the reader.

In this case we start to investigate the space $(\bigwedge (M_{2n+1}^-) \otimes M_{2n+1})^G$, where $M_{2n+1}$ indicates the space of the skew-symmetric matrices with respect to the usual transposition, similarly we will also study the symmetric case.

4.1 Dimension

We start analyzing the skew-symmetric case. We recall the isomorphism $M_{2n+1}^- \simeq \bigwedge^2 V$, where $V = \mathbb{C}^{2n+1}$. We have the usual decomposition:

$$\bigwedge \left( \bigwedge^2 V \right) = \bigoplus H_{a_1,a_2,...,a_k}(V),$$

with $2n + 1 > a_1 > \cdots > a_k > 0$. The only difference with the symplectic case is into the invariants. This time, there is the invariant only if the diagram has even columns (see [9]).

We have the analogous result:

**Proposition 4.1.** The dimension of the space of invariants $\bigwedge (M_{2n+1}^-)^G$ is $2^n$.

**Proof.** We want that the diagram has even columns. So the sequence $a_1 > a_2 > \cdots > a_k$ must be of type $2n + 1 > b_1 > b_1 - 1 > b_2 > b_2 - 1 > \cdots$, with $b_i$ even (not zero). So the number of these sequences is the number of different sequences composed by even numbers minor than $2n + 1$, therefore likewise the symplectic case we have our claim. \qed

Furthermore we can compute the dimension of covariants.

**Proposition 4.2.** The dimension of the space $(\bigwedge (M_{2n+1}^-)^G \otimes M_{2n+1})^G$ is $n2^{n+1}$.

**Proof.** The proof will follow directly from the the Proposition [3.3] with same devices.

We can reinterpret the orthogonal case from the point of view of the symplectic case. By transposing diagrams we are in the situation of the skewsymmetric symplectic case for $2n$ unless to:

1. subtract the contribution given by the case in which the first two columns have maximal length and we add two boxes to these.

2. add the contribution given by the case in which the first row has maximal length and we add two boxes to this.

So we have that the dimension is $n2^{n+1} - 2^{n-1} + 2^{n-1} = n2^{n+1}$.

\qed
4.2 Structure

We proceed to the explicit description of the space \( B := \left( \bigwedge (M_{2n+1}^-)^* \otimes M_{2n+1}^- \right)^G \). We can see such algebra as a left module on the algebra \( A := \left( \bigwedge (M_{2n+1}^-)^* \right)^G \) which is, as an algebra, generated by the elements \( Tr(S_{2k+1}) \) and similarly to the symplectic case can be described as the exterior algebra.

**Proposition 4.3.** The algebra \( A = \left( \bigwedge (M_{2n+1}^-)^* \right)^G \) is the exterior algebra in the elements \( T_h = Tr(S_{4h+3}) \), with \( h = 0, 1, \ldots, n-1 \).

**Proof.** By the proof of Proposition 3.4 and the Amitsur-Levitzki’s Theorem we have that the elements \( T_h \), \( h = 0, \ldots, n-1 \) generate the invariants. Since the dimension is exactly \( 2^n \), we have our claim.

To study covariants let us recall the by classical invariant theory we have that the space \( B = \left( \bigwedge (M_{2n+1}^-)^* \otimes M_{2n+1}^- \right)^G \) is generated as a module on \( A = \left( \bigwedge (M_{2n+1}^-)^* \right)^G \) by the elements \( Y^i \). So reasoning as in 2.5 and 2.6 we can state the following.

**Theorem 4.4.** We have:

1. As a algebra \( B \) is generated by \( A \) and the element \( Y \).

2. \( B \) is a free module on the the exterior algebra \( A_{n-1} \subset A \) generated by \( T_0, \ldots, T_{n-2}, \) with basis \( 1, Y, \ldots, Y^{4n-1} \).

With notations of the Section 2 we can consider the canonical surjective homomorphism \( \pi : A[t] \to B \). We have:

3. The Kernel of the homomorphism \( \pi \) is the principal ideal generated by the element

\[
mt^{2m-3} - \sum_{i=0}^{n-1} T_{n-i} \land t^{4i},
\]

where \( m = 2n + 1 \).

**Remark 4.5.** Let us remark that multiplying by \( t \) the generator in (11), we obtain an identity between an element of even degree \( t^{2m-2} \) and elements of odd degree \( t^{4i+1} \). By the relation \( T_{n-i} \land t^{4i+1} = -t^{4i+1} \land T_{n-i} \), we can deduce \( t^{2m-2} = 0 \). We have recovered a well known result of Hutchinson (see [5]).

So from the Proposition 4.4 we have:

**Corollary 4.6.**

1. The algebra \( \left( \bigwedge (M_{2n+1}^-)^* \otimes M_{2n+1}^- \right)^G \) is a free module on the exterior algebra \( A_{n-1}^+ \) with basis \( Y, Y^2, Y^3, \ldots, Y^{4n-3}, Y^{4n-2} \).

2. The algebra \( \left( \bigwedge (M_{2n+1}^-)^* \otimes M_{2n+1}^+ \right)^G \) is a free module on the exterior algebra \( A_{n-1}^- \) with basis \( 1, Y^3, Y^4, Y^7, \ldots, Y^{4n-4}, Y^{4n-1} \).
4.3 Structure of the symmetric case

The last case we study in this work is the symmetric orthogonal case. This is very simple by previous considerations. We have $M_{2n+1}^+ \cong S^2(V)$. So by the decomposition (6), to compute the dimension of invariants we have to consider sequences of type $2n+1 > b_1 > b_1 - 1 \ldots > b_k > b_k - 1 > 0$, with each $b_i$ even or of type $2n+1 > b_1 > b_1 - 1 \ldots b_k - 1 > b_k - 1 - 1 > b_k = 0$. So we have:

**Proposition 4.7.** The dimension of the space of invariants $(\bigwedge M_{2n+1}^+ + 1)^G$ is $2^{n+1}$.

By the same proofs of the previous sections we have:

**Proposition 4.8.** The dimension of the space of invariants $(\bigwedge (M_{2n+1}^+)^* \otimes M_{2n+1}^+)^G$ is $(2n+1)n+1$.

So we can describe easily the structure of invariants. By classical invariant theory we have that this algebra is generated by the elements $Tr(St_{4i+1})$, but by Propositions 2.3, 4.7 and Amitsur-Levitzki we have:

**Lemma 4.9.** The algebra of invariants $A := (\bigwedge (M_{2n+1}^+)^*)^G$ is the exterior algebra in the elements $T_0, \ldots, T_n$.

So we can describe the covariants $B$:

**Theorem 4.10.**

1. As a algebra $B$ is generated by $A$ and the element $X$.

2. $B$ is a free module on the exterior algebra $A_{n-1} \subset A$ generated by $T_0, \ldots, T_{n-1}$, with basis $1, X, \ldots, X^{4n+1}$.

With notations of the Section 2 we can consider the canonical surjective homomorphism $\pi : A[t] \rightarrow B$. We have:

3. The Kernel of the homomorphism $\pi$ is the principal ideal generated by the element

$$ (2n+2)t^{4n+1} - \sum_{i=0}^{n} T_{n-i} \wedge t^{4i+2}, \quad (12) $$

**Proof.** We need to prove only the third part. By the analogous of the general formula we have

$$ 1 \wedge T(4n+1) = T(4n+1) = - \sum_{i=1}^{2n+1} Z^{2i} \wedge T(2(2n+2-i) - 1) + (2n+2)Z^{4n+1}, $$

so we deduce $i$ is odd and we have:

$$ T_n = - \sum_{i=0}^{n} X^{4i+2} \wedge T_{n-i} + (2n+2)X^{4n+1}, $$

which is our claim.

**Corollary 4.11.**

1. The algebra $(\bigwedge (M_{2n+1}^+)^* \otimes M_{2n+1}^+)^G$ is a free left module on the exterior algebra $A_{n-1}$, with basis $X^2, X^3, \ldots, X^{4n-2}, X^{4n-1}$.

2. The algebra $(\bigwedge (M_{2n+1}^+)^* \otimes M_{2n+1}^+)^G$ is a free left module on the exterior algebra $A_{n-1}$, with basis $1, X, \ldots, X^{4n}, X^{4n+1}$.
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