EINSTEIN HYPERSURFACES OF $S^n \times \mathbb{R}$ AND $H^n \times \mathbb{R}$

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Abstract. In this paper, we classify the Einstein hypersurfaces of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$. We use the characterization of the hypersurfaces of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ whose tangent component of the unit vector field spanning the factor $\mathbb{R}$ is a principal direction and the theory of isoparametric hypersurfaces of space forms to show that Einstein hypersurfaces of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ must have constant sectional curvature.

1. Introduction

A Riemannian manifold $(M^n, g)$ is said to be Einstein if its Ricci tensor is proportional to the metric, i.e., if $\text{Ric}_M = \rho g$, for some constant $\rho \in \mathbb{R}$. Equivalently, $(M^n, g)$ is an Einstein manifold if it has constant Ricci curvature and, according to Besse [3], constant Ricci curvature could be considered as a good generalization of the concept of constant sectional curvature. Also, as pointed out in [3], there are several results in the literature justifying that an Einstein metric is a good candidate for a “best” metric on a given manifold. When $n = 2$, the Einstein condition means constant Gaussian curvature whereas a simple calculation shows that, when $n = 3$, a manifold $(M^n, g)$ is Einstein if and only if it has constant sectional curvature.

This paper aims to prove that an isometric immersion of an Einstein manifold $M^n$ as a hypersurface of the Riemannian products $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ only occur when $M^n$ has constant sectional curvature. More precisely, let us denote by $Q^n(\varepsilon)$ the unit sphere $S^n$, if $\varepsilon = 1$, or the hyperbolic space $H^n$, if $\varepsilon = -1$. With this notation, our main theorem is given as the following:

Theorem 1. Let $f : M^n \to Q^n(\varepsilon) \times \mathbb{R}$, $n > 3$, be an isometric immersion of an Einstein manifold. Then $M^n$ is a manifold with constant sectional curvature.

Isometric immersions of Einstein manifolds into space forms were considered initially in codimension 1 by Thomas [19], followed by Fialkow [14] and the full classification in this case was concluded by Ryan [18]. Briefly, an Einstein hypersurface of a space form of curvature $\varepsilon$ must have constant sectional curvature, except for the case $\varepsilon = 1$, where we can find a product of spheres as Einstein hypersurfaces. For arbitrary codimensions, Einstein submanifolds of space forms were considered recently under the hypothesis of having flat normal bundle. Onti [17] classified such submanifolds with parallel mean curvature, whereas Dajczer, Onti and Vlachos [8] proved that Einstein submanifolds of space forms with flat normal bundle are locally holonomic.

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The study of the intrinsic geometry of hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ and $\mathbb{S}^n \times \mathbb{R}$ has been drawn much attention in recent years [1 2 3 4 5 6 15 16 21]. Particularly, hypersurfaces with constant sectional curvature were considered by Aledo, Espinar and Galvez [1 2], for the two-dimensional case and by Manfio and Tojeiro, [15], for higher dimensions. When $n \geq 4$, Manfio and Tojeiro have proved that a hypersurface with constant sectional curvature $c$ only exists when $c \geq \varepsilon$ and it must be an open part of a complete rotation hypersurface. When $n = 3$, $c \in (0, 1)$ if $\varepsilon = 1$ and $c \in (-1, 0)$ if $\varepsilon = -1$. In this case, the hypersurface is constructed explicitly using parallel surfaces in $Q^3(\varepsilon)$. Consequently, the results given by Manfio and Tojeiro in [15] and Theorem 1 completely solve the problem of the classification of Einstein hypersurfaces in $Q^n(\varepsilon) \times \mathbb{R}$.

2. Preliminary notions and results

In this section we will present some preliminary notions and results that will be used in the proof of Theorem 1. Let us first establish some notation. As said before, we will denote by $Q^n(\varepsilon)$ the unit sphere $\mathbb{S}^n$, if $\varepsilon = 1$, or the hyperbolic space $\mathbb{H}^n$ if $\varepsilon = -1$. The Riemannian manifold $Q^n(\varepsilon) \times \mathbb{R}$ will be given in the following models:

$$
\begin{align*}
\mathbb{S}^n \times \mathbb{R} &= \left\{ (x_1, \ldots, x_{n+2}) \in \mathbb{E}^{n+2} \mid x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 \right\}, \\
\mathbb{H}^n \times \mathbb{R} &= \left\{ (x_1, \ldots, x_{n+2}) \in \mathbb{L}^{n+2} \mid -x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = -1, x_1 > 0 \right\},
\end{align*}
$$

with the metric induced by the ambient space. Here $\mathbb{E}^{n+2}$ is the $(n+2)$-dimensional Euclidean space and $\mathbb{L}^{n+2}$ is the $(n+2)$-dimensional Lorentzian space with the canonical metric $ds^2 = -dx_1^2 + dx_2^2 + \ldots + dx_{n+2}^2$.

Let $f : M^n \to Q^n(\varepsilon) \times \mathbb{R}$ be a hypersurface. Denote by $N$ its unit normal and let $\partial_{x_{n+2}}$ be the coordinate vector field of the factor $\mathbb{R}$. Also, let us denote by $T$ the orthogonal projection of $\partial_{x_{n+2}}$ onto the tangent space of $M^n$. With this notation, we have the following decomposition

$$
\partial_{x_{n+2}} = T + \nu N,
$$

where $\nu$ is a smooth function defined in $M^n$, called angle function. Let $\nabla$ and $R$ be the Riemannian connection and the curvature tensor of a hypersurface $f : M^n \to Q^n(\varepsilon) \times \mathbb{R}$, respectively. It will be considered the following sign convention: $R(\nabla_X Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$. If we denote by $S$ its shape operator, the Gauss equation is given by

$$
\begin{align*}
\langle R(X, Y)Z, W \rangle &= \varepsilon (\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle) \\
&+ \langle X, T \rangle \langle Z, T \rangle \langle Y, W \rangle + \langle Y, T \rangle \langle W, T \rangle \langle X, Z \rangle \\
&- \langle Y, T \rangle \langle Z, T \rangle \langle X, W \rangle - \langle X, T \rangle \langle W, T \rangle \langle Y, Z \rangle \\
&+ \langle SX, W \rangle \langle SY, Z \rangle - \langle SX, Z \rangle \langle SY, W \rangle.
\end{align*}
$$

Moreover, since the vector field $\partial_{x_{n+2}}$ is parallel in $Q^n(\varepsilon) \times \mathbb{R}$, we have

$$
\begin{align*}
\nabla_X T &= \nu SX, \\
X[\nu] &= -\langle X, ST \rangle.
\end{align*}
$$

At this point, we present a fundamental result that will be used in the proof of Theorem 1. In [20], Tojeiro presented a characterization of the hypersurfaces for which $T$ is principal direction. Such characterization is given as follows.
Let \( g : M^{n-1} \to Q^n(\varepsilon) \) be a hypersurface and let \( g_s : M^{n-1} \to Q^n(\varepsilon), s \in I \subset \mathbb{R} \), be its family of parallel hypersurfaces, given by
\[
g_s(x) = C_\varepsilon(s)g(x) + S_\varepsilon(s)N(x),
\]
where \( x \in M^{n-1} \), \( N \) is a unit normal vector field to \( g \) and the functions \( C_\varepsilon \) and \( S_\varepsilon \) are given by
\[
C_\varepsilon(s) = \begin{cases} 
\cos(s), & \text{if } \varepsilon = 1, \\
\cosh(s), & \text{if } \varepsilon = -1
\end{cases}
\]
and
\[
S_\varepsilon(s) = \begin{cases} 
\sin(s), & \text{if } \varepsilon = 1, \\
\sinh(s), & \text{if } \varepsilon = -1.
\end{cases}
\]

Let \( f : M^n := M^{n-1} \times I \to Q^n(\varepsilon) \times \mathbb{R} \) be a hypersurface defined by
\[
f(x, s) = g_s(x) + a(s)\partial_{n+2},
\]
for a smooth function \( a : I \to \mathbb{R} \) with positive derivative. In this context, the following theorem provides the mentioned characterization:

**Theorem 2** (20). Let \( f \) be the map given in (2.6), where \( g_s \) is defined by (2.4). Then the map \( f \) defines, at regular points, a hypersurface that has \( T \) as a principal direction. Conversely, any hypersurface \( f : M^n \to Q^n(\varepsilon) \times \mathbb{R}, n \geq 2, \) with nowhere vanishing angle function that has \( T \) as a principal direction is locally given in this way.

**Remark 3.** The hypersurfaces with the property of the vector field \( T \) being a principal direction constitute an important class of hypersurfaces of \( Q^n(\varepsilon) \times \mathbb{R} \). This class of hypersurfaces includes the rotation hypersurfaces \([9]\), the hypersurfaces with constant sectional curvature \([15]\) and the hypersurfaces whose normal direction makes a constant angle with the vector field \( \partial_{x_{n+2}} \) \([12, 13, 15, 20]\). Besides that, it was proved in [20] that such a property is equivalent to \( M^n \) has flat normal bundle as a submanifold into \( E^{n+2} \), resp. \( L^{n+2} \). This fact was also obtained for the two-dimensional case in \([10, 11]\), where surfaces of \( Q^2(\varepsilon) \times \mathbb{R} \) having \( T \) as a principal direction were considered.

For a hypersurface given locally by (2.6), one has:

\[
|T| = \frac{a'(s)}{\sqrt{1 + a'(s)^2}},
\]
\[
\nu = \frac{1}{\sqrt{1 + (a'(s))^2}}.
\]

Also, the principal curvatures are given by

\[
\lambda_i = -\frac{a'(s)}{\sqrt{1 + a'(s)^2}}\lambda_i^s, \quad 1 \leq i \leq n - 1,
\]
\[
\lambda_n = \frac{\nu}{(\sqrt{1 + a'(s)^2})^3},
\]
where \( \lambda_n \) is the principal curvature associated to \( T \) and \( \lambda_i^s, 1 \leq i \leq n - 1, \) are the principal curvatures of \( g_s, \) i.e.,

\[
\lambda_i^s = \frac{\varepsilon S_\varepsilon(s) + \lambda_i^gC_\varepsilon(s)}{C_\varepsilon(s) - \lambda_i^gS_\varepsilon(s)}.
\]
where $\lambda_i^g$, $1 \leq i \leq n - 1$, are the principal curvatures of $g$. Finally, let us observe that, by equations (2.7) and (2.9) we have

$$\lambda_n = \frac{d|T|}{ds}.$$  

We also present in this section two results regarding isoparametric hypersurfaces in space forms. We may suggest to the reader as references the survey [6] or Section 3.1 in [7]. Let us recall that $g : \overline{M}^{n-1} \rightarrow Q^n(\epsilon)$ is said to be an isoparametric hypersurface if it has constant principal curvatures. In [4], Cartan proved that a hypersurface $g : \overline{M}^{n-1} \rightarrow Q^n(\epsilon)$ is isoparametric if and only if each parallel hypersurface $g_s$ as given in (2.4) has constant mean curvature, i.e., the mean curvature of $g_s$ depends only on $s$ (see Theorem 3.6 in [7]). In the same paper, Cartan established an important relation between the principal curvatures of isoparametric hypersurfaces. This relation is known as Cartan’s identity (or Cartan’s formula, following [2], page 91) and it is given as follows: let $g : \overline{M}^{n-1} \rightarrow Q^n(\epsilon)$ be an isoparametric hypersurface with $d$ distinct principal curvatures and respective multiplicities $m_1, \ldots, m_d$. If $d > 1$, for each $i$, $1 \leq i \leq d$ one has

$$\sum_{j \neq i} m_j \frac{\epsilon + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0.$$  

In order to prove Theorem 1, we will need the following lemmas. The first will establish the Ricci tensor on a hypersurface $f : M^n \rightarrow Q^n(\epsilon) \times \mathbb{R}$ while the second will show that, on an Einstein hypersurface, the vector field $T$ is an eigenvector of the shape operator at $p \in M$, as long as $T \neq 0$ at $p$.

In what follows, the Ricci tensor is given by

$$\text{Ric}(Y, Z) = \text{trace} \{ X \mapsto R(X, Y)Z \}.$$  

**Lemma 4.** Let $M^n$ be a hypersurface in $Q^n(\epsilon) \times \mathbb{R}$, then the Ricci tensor of $M^n$ is given by

$$\text{Ric}(Y, Z) = \epsilon(n - 1 - |T|^2)\langle Y, Z \rangle + \epsilon(2 - n)\langle Y, T \rangle\langle Z, T \rangle + nH\langle SY, Z \rangle - \langle SY, SZ \rangle,$$

where $Y, Z$ are arbitrary vector fields on $M^n$ and $H$ is the mean curvature.

**Proof.** Let $\{e_i\}_{i=1}^n$ an orthonormal basis of principal directions, with $Se_i = \lambda_ie_i$.

If we write $Y = \sum_{k=1}^n y_k e_k$, $Z = \sum_{k=1}^n z_k e_k$ and $T = \sum_{k=1}^n t_k e_k$, it follows by Gauss Equation (2.2) that

$$\langle R(e_k, Y)Z, e_k \rangle = \epsilon \left[ \langle Y, Z \rangle - y_k z_k + t_k y_k \langle Z, T \rangle + t_k z_k \langle Y, T \rangle - \langle Y, T \rangle \langle Z, T \rangle - t_k^2 \langle Y, Z \rangle \right] + \lambda_k \langle SY, Z \rangle - \langle e_k, SZ \rangle \langle SY, e_k \rangle.$$  

Consequently, by (2.10) and (2.11), the Ricci tensor is given by (2.12). \hfill \Box

The next lemma give a characterization of Einstein hypersurfaces with $T \neq 0$.

**Lemma 5.** Let $M^n$, $n > 3$, be an Einstein hypersurface in $Q^n(\epsilon) \times \mathbb{R}$. If $T \neq 0$ at $p \in M^n$, then $T$ is an eigenvector for the shape operator at $p$. 

Proof. Let \( \{e_i\}_{i=1}^n \) an orthonormal basis of principal directions, with \( S e_i = \lambda_i e_i \).

Let us write \( T = \sum_{k=1}^n t_k e_k \). If \( T \neq 0 \) at \( p \in M^n \), there is at least one coefficient \( t_k \neq 0 \). Since \( M^n \) is an Einstein manifold, its Ricci tensor satisfy

\[
\text{Ric}(e_i, e_j) = \rho \delta_{ij},
\]

for some constant \( \rho \). When we consider the Ricci tensor applied on the orthonormal basis \( \{e_i\}_{i=1}^n \), we have

\[
\text{Ric}(e_i, e_j) = [\varepsilon(n - 1 - |T|^2) + nH\lambda_i - \lambda_i\lambda_j] \delta_{ij} + \varepsilon(2 - n)t_it_j.
\]

By Equation (2.16) we must have

\[
(2.17) \quad [\varepsilon(n - 1 - |T|^2) + nH\lambda_i - \lambda_i\lambda_j - \rho] \delta_{ij} + \varepsilon(2 - n)t_it_j = 0
\]

and we conclude that \( t_it_j = 0 \), for all \( i, j \), with \( i \neq j \). Consequently, there is only one coefficient \( t_k \neq 0 \) and therefore \( T = t_k e_k \) at \( p \).

\[ \square \]

3. PROOF OF THE MAIN RESULT

Proof of Theorem 7. If \( T = 0 \), then \( M^n \) is an open part of a slice \( Q^n(\varepsilon) \times \{t_0\} \), where \( t_0 \in \mathbb{R} \). Since the slices are isometric to \( Q^n(\varepsilon) \), \( M^n \) is a manifold with constant sectional curvature \( \varepsilon \). Otherwise, let \( \Omega \) be the open, non-empty subset where \( |T| > 0 \). By Lemma 8 \( T \) is a principal direction in \( \Omega \). Without loss of generality we can write \( T = t_ne_n \) and \( ST = \lambda_n T \). Since \( M^n \) is Einstein, we have from Equation (2.17) that

\[
(3.1) \quad \varepsilon(n - 1 - |T|^2) + nH\lambda_i - \lambda_i^2 - \rho = 0, \quad \text{for} \quad 1 \leq i \leq n - 1
\]

\[
(3.2) \quad \varepsilon(n - 1)(1 - |T|^2) + nH\lambda_n - \lambda_n^2 - \rho = 0.
\]

Equation (3.1) implies that we have at most two distinct principal curvatures among the \((n - 1)\) first principal curvatures. In fact, from (3.1) we have

\[
(3.3) \quad (\lambda_i - \lambda_j)(nH - \lambda_i - \lambda_j) = 0, \quad \text{for} \quad 1 \leq i, j \leq n - 1.
\]

Let us suppose by contradiction that there are three distinct principal curvatures \( \lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3} \). It follows by equation of (3.3) that

\[
\lambda_{i_1} + \lambda_{i_2} = nH,
\]

\[
\lambda_{i_2} + \lambda_{i_3} = nH,
\]

\[
\lambda_{i_3} + \lambda_{i_1} = nH.
\]

The equations above implies that \( \lambda_{i_1} = \lambda_{i_2} = \lambda_{i_3} \), which is a contradiction.

If \( \lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = \mu \), the sectional curvature is constant. In fact, by equation (2.14) we have

\[
(3.4) \quad \langle R(e_i, e_j)e_i, e_j \rangle = \varepsilon + \mu^2, \quad 1 \leq i, j \leq n - 1
\]

\[
(3.5) \quad \langle R(e_i, e_n)e_n, e_i \rangle = \varepsilon(1 - |T|^2) + \mu\lambda_n.
\]

It follows from (3.2) that

\[
(3.6) \quad \varepsilon(1 - |T|^2) + \mu\lambda_n = \frac{\rho}{n - 1},
\]

therefore equation (3.5) implies that \( \langle R(e_i, e_n)e_n, e_i \rangle \) is constant. By equation (3.1) we have

\[
(3.7) \quad \varepsilon(1 - |T|^2) + \mu\lambda_n = \rho - (n - 2)(\mu^2 + \varepsilon).
\]
When we combine equations (3.6) and (3.7), we have from (3.4) that
\[ \langle R(e_i, e_j)e_j, e_i \rangle = \rho \frac{n}{n - 1} \]
and, consequently, the sectional curvature is equal to \( \rho \frac{n}{n - 1} \) in \( \Omega \).

Next we will show that the possibility of two distinct principal curvatures does not occur. In this case, we can consider as the two distinct principal curvatures \( \lambda_1 \) and \( \lambda_2 \) and therefore there are \( p \) principal curvatures equal to \( \lambda_1 \) and \( q \) principal curvatures equal to \( \lambda_2 \), with \( \lambda_1 \neq \lambda_2 \) and \( p + q = n - 1 \). By equation (3.8) we have
\[ \lambda_1 + \lambda_2 = nH, \]
consequently,
\[ \lambda_1 \lambda_2 = \rho - \varepsilon(n - 1) - |T|^2. \]
where (3.9) is obtained when we substitute \( \lambda_1 + \lambda_2 = nH \) into (3.1).

We will show that \( \lambda_n \equiv 0 \) in \( \Omega \) and this fact will lead us to a contradiction. If \( \nu \equiv 0 \), it follows by (2.11) that \( \lambda_n \equiv 0 \) in \( \Omega \). Otherwise, there is a point \( p_0 \) where \( \nu(p_0) \neq 0 \) and an open neighborhood \( \Omega_0 \subset \Omega \) of \( p_0 \) such that \( \nu \neq 0 \). Therefore, by Lemma 5 we can apply Theorem 2 to conclude that \( \Omega_0 \) is given locally by (2.6).

In this case, (3.2) implies that
\[ \varepsilon(n - 1)(1 - |T|^2) + \lambda_n(n - 1)H_{gs} - \rho = 0, \]
where \( H_{gs} \) is the mean curvature of the parallel \( g_s \).

Let us suppose by contradiction \( \lambda_n \neq 0 \) in \( \Omega_0 \). It follows by (3.10) that \( H_{gs} \) depends only on \( s \), once that equations (2.7) and (2.10) imply that the functions \( |T|^2 \) and \( \lambda_n \) depend only on \( s \). In this case, the mean curvature of the parallel \( g_s \) depends only on \( s \) which implies that \( g \) is an isoparametric hypersurface, with two distinct principal curvatures. By Cartan’s identity (2.12) we must have
\[ \lambda_1 \lambda_2 + \varepsilon = 0. \]
In this case, it follows directly from (2.7), (2.9), (2.10) and (3.11) that
\[ \lambda_1 \lambda_2 = -\varepsilon|T|^2. \]
When we replace (3.12) in (3.9) we have
\[ |T|^2 = \frac{\varepsilon(n - 1) - \rho}{2\varepsilon}, \]
which implies that \( |T|^2 \) is constant. By equation (2.11), it follows that \( \lambda_n = 0 \) in \( \Omega_0 \), which is a contradiction.

Therefore, we have \( \lambda_n \equiv 0 \) in \( \Omega \). It follows by (3.10) that
\[ |T|^2 = \frac{\varepsilon(n - 1) - \rho}{\varepsilon(n - 1)}. \]
In this case, equations (3.8) and (3.9) are rewritten as
\[ (p - 1)\lambda_1 + (q - 1)\lambda_2 = 0, \]
\[ \lambda_1 \lambda_2 = \left(\frac{n - 2}{n - 1}\right)(\rho - \varepsilon(n - 1)). \]

Since \( |T| \neq 0 \), equations (2.7), (2.9), (3.13), (3.14) and (3.15) imply that
\[ (p - 1)\lambda_1^s + (q - 1)\lambda_2^s = 0, \]
\[ \lambda_1^s \lambda_2^s = -\varepsilon(n - 2). \]
We claim that the system above has no solution for \( n > 3 \). In fact, Equations (3.16) and (3.17) imply that \( \lambda_1^2 \) are constants, unless \( p = q = 1 \), which is not the case since \( p + q = n - 1 \). Therefore, evaluating in \( s = 0 \) we conclude that \( q \) is isoparametric. By Cartan’s identity (2.12), \( \lambda_1^2 \lambda_2^2 + \varepsilon = 0 \). This fact with Equation (3.17) in \( s = 0 \) implies \( n = 3 \), which is not the case.

We conclude that, in the open subset \( \Omega \) where \( |T| > 0 \), the sectional curvature is a constant \( K_0 = \frac{\rho}{n-1} \). If \( M^n \setminus \Omega \) has empty interior, we have by continuity that \( M^n \) has constant sectional curvature \( K_0 \). Otherwise, there is an open subset \( \mathcal{O} \subset M^n \setminus \Omega \), where \( T \equiv 0 \). As we saw at the beginning of the proof, \( \mathcal{O} \) is an open part of a slice \( Q^n(\varepsilon) \times \{t_1\} \), for some \( t_1 \in \mathbb{R} \), and the sectional curvature in \( \mathcal{O} \) is constant equal to \( \varepsilon \), which implies \( \rho = (n-1)\varepsilon \). Since \( \rho \) is constant in \( M^n \), we must have \( K_0 = \varepsilon \) and the sectional curvature in \( \Omega \cup \mathcal{O} \) is \( \varepsilon \), for any open subset \( \mathcal{O} \) where \( T \equiv 0 \). Again we use the continuity of the sectional curvature to conclude that \( M^n \) has constant sectional curvature equal to \( \varepsilon \). \( \square \)

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