POTENTIAL POLYNOMIALS AND MOTZKIN PATHS

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Abstract. A Motzkin path of length $n$ is a lattice path from $(0,0)$ to $(n,0)$ in the plane integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of horizontal-steps $(1,0)$, up-steps $(1,1)$, and down-steps $(1,-1)$, which never passes below the $x$-axis. A u-segment (resp. h-segment) of a Motzkin path is a maximum sequence of consecutive up-steps (resp. horizontal-steps). The present paper studies two kinds of statistics on Motzkin paths: "number of u-segments" and "number of h-segments". The Lagrange inversion formula is utilized to represent the weighted generating function for the number of Motzkin paths according to the statistics as a sum of the partial Bell polynomials or the potential polynomials. As an application, a general framework for studying compositions are also provided.

Keywords: Partial Bell polynomials, potential polynomials, Motzkin paths, compositions

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1. Introduction

A Motzkin path of length $n$ is a lattice path from $(0,0)$ to $(n,0)$ in the plane integer lattice $\mathbb{Z} \times \mathbb{Z}$ consisting of up-steps $u = (1,1)$, horizontal-steps $h = (1,0)$, and down-steps $d = (1,-1)$. Denote by $\mathcal{M}_n$ the set of Motzkin paths of length $n$. Let $\mathcal{M}_n^{m,k}$ denote the set of Motzkin paths of length $n$ (i.e. $n = 2m + k$) with $m$ up steps and $k$ horizontal steps and $\mathcal{D}_m$ denote the set of Dyck paths, namely, Motzkin paths in $\mathcal{M}_n^{0,0}$. Let $P$ be any Motzkin path in $\mathcal{M}_n$, a u-segment (resp. h-segment) of $P$ is a maximum sequence of consecutive up-steps (resp. horizontal-steps) in $P$ and define $u_i(P)$ (resp. $h_i(P)$) to be the number of u-segments (resp. h-segments) of length $i$ in $P$ and call $P$ having the u-segments (h-segments) of type $1^{u_1(P)}2^{u_2(P)}\ldots$ (resp. $1^{h_1(P)}2^{h_2(P)}\ldots$).

In two previous papers [15, 16], we study two kinds of statistics on ($k$-generalized) Dyck paths: "number of u-segments" and "number of internal u-segments". In this paper, we consider the statistics "number of u-segments" and "number of h-segments". In order to do this we present two tools we will use: the Lagrange inversion formula and the potential polynomials.

Lagrange Inversion Formula [19]. If $f(x) = \sum_{n \geq 1} f_n x^n$ with $f_1 \neq 0$, then the coefficients of the composition inverse $g(x)$ of $f(x)$ (namely, $f(g(x)) = g(f(x)) = x$) can be given by

$$[x^n]g(x) = \frac{1}{n} [x^{n-1}]\left(\frac{x}{f(x)}\right)^n. \tag{1.1}$$

More generally, for any formal power series $\Phi(x)$,

$$[x^n]\Phi(g(x)) = \frac{1}{n} [x^{n-1}]\Phi'(x)\left(\frac{x}{f(x)}\right)^n, \tag{1.2}$$

for all $n \geq 1$, where $\Phi'(x)$ is the derivative of $\Phi(x)$ with respect to $x$. 


The Potential Polynomials \cite{S} pp. 141,157. The potential polynomials $P_n^{(\lambda)}$ related to a given sequence $(f_n)_{n \geq 1}$ are defined for each complex number $\lambda$ by

$$1 + \sum_{n \geq 1} P_n^{(\lambda)} \frac{x^n}{n!} = \left\{ 1 + \sum_{n \geq 1} f_n \frac{x^n}{n!} \right\}^\lambda,$$

which can be represented by Bell polynomials

$$P_n^{(\lambda)} = P_n^{(\lambda)}(f_1, f_2, f_3, \ldots) = \sum_{1 \leq k \leq n} \binom{\lambda}{k} k! B_{n,k}(f_1, f_2, f_3, \ldots),$$

or if $\lambda = r$ is a positive integer, then

$$P_n^{(r)} = P_n^{(r)}(f_1, f_2, f_3, \ldots) = \binom{n + r}{r}^{-1} B_{n+r,r}(1, 2f_1, 3f_2, 4f_3, \ldots),$$

where $B_{n,i}(x_1, x_2, \ldots)$ is the partial Bell polynomial on the variables $\{x_j\}_{j \geq 1}$ \cite{S}, which has the explicit formula

$$B_{n,r}(x_1, x_2, \ldots) = \sum_{\sigma_n(r)} \frac{n!}{r_1! r_2! \cdots r_n!} \binom{x_1}{1!}^{r_1} \binom{x_2}{2!}^{r_2} \cdots \binom{x_n}{r_n}^{r_n},$$

where the summation $\sigma_n(r)$ is for all the nonnegative integer solutions of $r_1 + r_2 + \cdots + r_n = r$ and $r_1 + 2r_2 + \cdots + nr_n = n$.

In this paper, using the Lagrange inversion formula, we can represent the generating functions for the number of Motzkin paths according to our statistics (see Sections 2) as a sum of partial Bell polynomials or the potential polynomials, for example

$$\sum_{P \in \mathcal{M}_{n,k}} \prod_{i \geq 1} t_i^{u_i(P)} \prod_{i \geq 1} s_i^{h_i(P)} = \sum_{j=0}^{k} \sum_{\ell=j}^{k} (-1)^{\ell-j} \binom{\ell - 1}{j} \binom{m + j}{\ell - j} P_m^{(m+j+1)}(1! t_1, 2! t_2, \ldots) \ell! B_{k,\ell}(1! s_1, 2! s_2, \ldots).$$

Many important special cases are considered which generate several surprising results. As an application (see Section 3), compositions can be regarded as a kind of special Motzkin paths, which leads to a general framework to studying compositions by specializing the parameters. Moreover, in the last section we generalize compositions to matrix compositions.

2. "$u$-segments" and "$h$-segments" statistics

We start this section by studying the ordinary generating function for the number of Motzkin paths of length $n$ according to the statistics $u_1, u_2, \ldots$ and $h_1, h_2, \ldots$, that is,

$$M(x, y; t; s) = M(x, y; t_1, t_2, \ldots; s_1, s_2, \ldots) = \sum_{m,k \geq 0} x^m y^k \sum_{P \in \mathcal{M}_{m,k}} \prod_{i \geq 1} t_i^{u_i(P)} \prod_{i \geq 1} s_i^{h_i(P)}.$$

**Proposition 2.1.** The generating function $M(x, y; t; s)$ satisfies the functional recurrence relation

$$M(x, y; t; s) = \frac{T(z)}{1 - S(y) - 1 \cdot T(z)},$$

where $T(x) = 1 + \sum_{i \geq 1} t_i x^i$, $S(y) = 1 + \sum_{i \geq 1} s_i y^i$ and $z = xM(x, y; t; s)$. 

Proof. Let \( P := P(x, y; t; s) \) and \( Q := Q(x, y; t; s) \) be the ordinary generating functions for the set of Motzkin paths beginning with up-steps and with horizontal steps respectively, according to the statistics \( u_1, u_2, \ldots \) and \( h_1, h_2, \ldots \). Then \( M(x, y; t; s) \) satisfies
\[
(2.2) \quad M(x, y; t; s) = 1 + P + Q.
\]
Note that \( P(x, y; t; s) \) can be written as
\[
P(x, y; t; s) = \sum_{j \geq 1} P_j(x, y; t; s),
\]
where \( P_j(x, y; t; s) \) is the ordinary generating function for the number of Motzkin paths starting with \( j \) up-steps according to the statistics \( u_1, u_2, \ldots \) and \( h_1, h_2, \ldots \). An equation for the generating function \( P_j(x, z; t; s) \) is obtained from the first return decomposition of a Motzkin path \( M \) starting with a \( u \)-segment of length \( j \): either
\[
M = u^i dP^{(j)} d \ldots dP^{(2)} dP^{(1)} \quad \text{or} \quad M = u^i Q^{(j+1)} dP^{(j)} d \ldots dP^2 dP^1,
\]
where \( P^{(1)}, \ldots, P^{(j)} \) are Motzkin paths and \( Q^{(j+1)} \) is a Motzkin path beginning with horizontal steps, see Figure 1.

Thus \( P_j(x, y; t) = (1 + Q)x^j t_j M^j(x, y; t; s) \). Hence, \( P(x, y; t; s) \) satisfies
\[
(2.3) \quad P(x, y; t; s) = (1 + Q) \sum_{j \geq 1} t_j x^j M^j(x, y; t; s).
\]
Similarly, one can derive that \( Q(x, y; t; s) \) satisfies
\[
(2.4) \quad Q(x, y; t; s) = (1 + P) \sum_{j \geq 1} s_j y^j.
\]
Define \( T(x) = 1 + \sum_{i \geq 1} t_i x^i \), \( S(y) = 1 + \sum_{i \geq 1} s_i y^i \) and \( z = xM(x, y; t; s) \). By \( (2.2), (2.4) \), one can deduce \( (2.1) \), as required.

**Theorem 2.2.** For any integers \( n, m, k \geq 0 \), there holds
\[
\sum_{P \in A_{n,k}} \prod_{i \geq 1} t_{u_i(P)} \prod_{i \geq 1} s_{h_i(P)} = \sum_{j=0}^{k} \sum_{\ell=j}^{k} (-1)^{\ell-j} \binom{\ell-1}{\ell-j} \binom{m+j}{j} P_m^{(m+j+1)} \frac{(1! t_1, 2! t_2, \ldots)}{(m+1)!} \frac{\ell! B_{k,\ell}(1! s_1, 2! s_2, \ldots)}{k!}.
\]

Proof. Applying the Lagrange inversion formula \( (1.2) \) to \( (2.1) \), by the identity
\[
(2.5) \quad \sum_{i=0}^{j} (-1)^i \binom{j}{i} \binom{-i}{\ell} = (-1)^{\ell-j} \binom{\ell-1}{\ell-j},
\]
we obtain
\[
\sum_{P \in \mathcal{M}^{m,k}} \prod_{i \geq 1} t_{i}^{u_{i}(P)} \prod_{i \geq 1} s_{i}^{h_{i}(P)} = [x^{m+1}y^{k}]xM(x, y; \mathbf{t}; \mathbf{s}) = [x^{m+1}y^{k}]z
\]
\[
= \frac{1}{m+1} [x^{m}y^{k}] \left( \frac{T(x)}{1 - S(y)^{-1}T(x)} \right)^{m+1}
\]
\[
= \frac{1}{m+1} \sum_{j=0}^{k} \binom{m+j}{j} [x^{m}y^{k}]T(x)^{m+j+1} \left( \frac{S(y) - 1}{S(y)} \right)^{j}
\]
\[
= \frac{1}{m+1} \sum_{j=0}^{k} \sum_{i=0}^{j} (-1)^{i} \binom{m+j}{j} \binom{j}{i} [x^{m}]T(x)^{m+j+1} [y^{i}]S(y)^{-i}
\]
\[
= \sum_{j=0}^{k} \binom{m+j}{j} \frac{P_{m}^{m+j+1}(1!t_{1}, 2!t_{2}, \ldots)}{(m+1)!} \sum_{i=0}^{j} (-1)^{i} \binom{j}{i} \frac{P_{k}^{i}(1!s_{1}, 2!s_{2}, \ldots)}{k!}
\]
\[
= \sum_{j=0}^{k} \binom{m+j}{j} \frac{P_{m}^{m+j+1}(1!t_{1}, 2!t_{2}, \ldots)}{(m+1)!} \sum_{i=0}^{j} (-1)^{i} \binom{j}{i} \sum_{\ell=0}^{k} (-\ell) \frac{\ell!B_{k,\ell}(1!s_{1}, 2!s_{2}, \cdots)}{k!}
\]
\[
= \sum_{j=0}^{k} \binom{m+j}{j} \frac{P_{m}^{m+j+1}(1!t_{1}, 2!t_{2}, \ldots)}{(m+1)!} \sum_{\ell=0}^{k} (-\ell)^{\ell-j} \frac{\ell!B_{k,\ell}(1!s_{1}, 2!s_{2}, \cdots)}{k!}
\]
\[
= \sum_{j=0}^{k} \binom{m+j}{j} \frac{P_{m}^{m+j+1}(1!t_{1}, 2!t_{2}, \ldots)}{(m+1)!} \prod_{\ell=0}^{k} (-\ell)^{\ell-j} \frac{\ell!B_{k,\ell}(1!s_{1}, 2!s_{2}, \cdots)}{k!},
\]
as claimed. \hfill \Box

Let \(\mathcal{M}^{m,k}_{n,r,\ell}\) be the subset of \(\mathcal{M}^{m,k}_{n}\) with \(r\) number of \(u\)-segments and \(\ell\) number of \(h\)-segments. Note that \(B_{m,r}(1!t_{1}, 2!t_{2}, \cdots) = q^{r}B_{m,r}(1!t_{1}, 2!t_{2}, \cdots)\) by (1.5), combining (1.3) and (2.5). Then Theorem 2.2 produces

**Corollary 2.3.** For any integers \(n, m, r, k, \ell \geq 0\), there holds
\[
\sum_{P \in \mathcal{M}^{m,k}_{n,r,\ell}} \prod_{i \geq 1} t_{i}^{u_{i}(P)} \prod_{i \geq 1} s_{i}^{h_{i}(P)} = \frac{r!q^{r}\ell!V_{m,k}^{r,\ell}}{k!(m+1)!} B_{m,r}(1!t_{1}, 2!t_{2}, \cdots) B_{k,\ell}(1!s_{1}, 2!s_{2}, \cdots),
\]
where
\[
V_{m,k}^{r,\ell} = \sum_{j=0}^{k} (-1)^{\ell-j} \binom{\ell-j}{\ell-j} \binom{m+j}{m} \binom{m+j+1}{r}.
\]

Recall that
\[
B_{m,r}(x_{1}, x_{2}, \cdots) = \sum_{\sigma_{m}(r)} \frac{m!}{r_{1}!r_{2}! \cdots r_{m}!} \left( \frac{x_{1}}{1!} \right)^{r_{1}} \left( \frac{x_{2}}{2!} \right)^{r_{2}} \cdots \left( \frac{x_{m}}{m!} \right)^{r_{m}},
\]
where the summation \(\sigma_{m}(r)\) is for all the nonnegative integer solutions of \(r_{1} + r_{2} + \cdots + r_{m} = r\) and \(r_{1} + 2r_{2} + \cdots + mr_{m} = m\).

If comparing the coefficient of \(t_{1}^{r_{1}}t_{2}^{r_{2}} \cdots t_{m}^{r_{m}}s_{1}^{s_{1}}s_{2}^{s_{2}} \cdots s_{k}^{s_{k}}\) in Corollary 2.3, one can obtain that
Corollary 2.4. The number of Motzkin paths in $\mathcal{A}_{n,r,\ell}^{m,k}$ with a number $r$ of $u$-segments of type $1^r2^r\cdots m^r m$ and a number $\ell$ of $h$-segments of type $1^\ell2^\ell\cdots k^\ell k$ is

$$\frac{1}{m+1} \binom{r}{r_1,r_2,\ldots,r_m} \binom{\ell}{\ell_1,\ell_2,\ldots,\ell_k} V_{m,k}^{r,\ell}.$$  

Next, specialization for $T(x)$ and $S(y)$ are considered, which generate several interesting results, as described in Examples 2.5-2.6.

Example 2.5. Let $T(x) = e^x$, $S(y) = e^y$, that is, $t_i = s_i = 1/i!$ for all $i \geq 1$. And Stirling numbers $S(k,i)$ of the second kind satisfy $(e^x - 1)^i / i! = \sum_{k \geq i} S(k,i)x^k / k!$. Then Theorem 2.2 gives

$$\sum_{P \in \mathcal{A}_{n}^{m,k}} \prod_{i \geq 1} \left\{ \frac{1}{i!} \right\} u_i(P) + h_i(P) = \sum_{j=0}^{k} (-1)^{k-j} \binom{m+j}{j} j! \frac{(m+j)!}{k!(m+1)!} S(k,j).$$  

Note that $B_{k,i}(1,1,1,\cdots) = S(k,i)$ [8 pp.135], by Corollary 2.3 we have

$$\sum_{P \in \mathcal{A}_{n}^{m,k}} \prod_{i \geq 1} \left\{ \frac{1}{i!} \right\} u_i(P) + h_i(P) = \frac{r! \ell! V_{m,k}^{r,\ell}}{k!(m+1)!} S(m,r) S(k,\ell).$$

Example 2.6. Let $T(x) = f(x), S(y) = g(y)$, where $f(x), g(y)$ are the generating function for the complete b-ary and d-ary plane trees (see, for instance, [11] and [12] pp. 112-113), which satisfies the relations $f(x) = 1 + xf'(x)$ and $g(y) = 1 + yg'(y)$ respectively. By the Lagrange inversion formula (1.2), one can deduce $t_i = \frac{1}{(b+1)i} \binom{bi+1}{i}$ and $s_i = \frac{1}{(di+1)(di+1)}$. Then Theorem 2.2 leads to

$$\sum_{P \in \mathcal{A}_{n}^{m,k}} \prod_{i \geq 1} \left\{ \frac{1}{bi+1} \right\} u_i(P) \prod_{i \geq 1} \left\{ \frac{1}{di+1} \right\} h_i(P) \bigg|_{b=1, d=1} = \frac{1}{m+1} \sum_{j=0}^{k} \binom{m+j}{j} \frac{m+j+1}{(b+1)m+j+1} \frac{dj-j}{dk-j} \binom{dk-j}{k-j},$$

which, in the case $d = 1$, generates

$$\sum_{P \in \mathcal{A}_{n}^{m,k}} \prod_{i \geq 1} \left\{ \frac{1}{bi+1} \right\} u_i(P) = \frac{1}{(b+1)m+k+1} \binom{m+k+1}{m} \binom{(b+1)m+k+1}{k}.$$  

Recently, Abbas and Bouroubi [1] derived two new identities for Bell polynomials, that is,

Lemma 2.7. Let $f(x) = 1 + \sum_{i \geq 1} f_i x^i$ be any analytic function about the origin and define $f_m(i) = D^m[f(x)]^{(i)}|_{x=0}$, where $D$ is the differential operator $d/dx$. Then for any integers $m \geq r \geq 1$, there holds

$$B_{m,r}(1, f_1(2), f_2(3), \cdots) = \binom{m-1}{r-1} f_{m-r}(m).$$

Lemma 2.8. Let $\{\phi_n(x)\}_{n \geq 0}$ be a binomial sequence. Then for any integers $m \geq r \geq 1$, there holds

$$B_{m,r}(1, 2\phi_1(1), 3\phi_2(1), \cdots) = \binom{m}{r} \phi_{m-r}(r).$$
Recall that a sequence of polynomials \( \{ \phi_n(x) \}_{n \geq 0} \) with \( \phi_0(x) = 1 \) is called binomial if
\[
\phi_n(x + y) = \sum_{i=0}^{n} \binom{n}{i} \phi_i(x) \phi_{n-i}(y),
\]
or equivalently, there exists a power series \( \lambda(u) = \sum_{i \geq 1} \lambda_i u^i \) with \( \lambda_1 \neq 0 \) such that
\[
\sum_{n \geq 0} \phi_n(x) \frac{u^n}{n!} = \exp(x \lambda(u)).
\]
For examples, the following binomial sequences are well known [13],
- Power polynomials \( \phi_n(x) = x^n; \)
- Factorial polynomials \( \phi_n(x) = x(x + 1) \cdots (x + n - 1); \)
- Abel polynomials \( \phi_n(x) = x(x - qn)^{n-1} \) for fixed \( q; \)
- Exponential polynomials \( \phi_n(x) = \sum_{i=0}^{n} S(n, i)x^i. \)

Let \( t_i = \frac{f_{i+1}(i)}{i+1}! \) and \( s_i = \frac{g_{i+1}(i)}{i+1} \), where \( g_m(i) = D^m[g(x)]|_{x=0} \) and \( g(x) = 1 + \sum_{i \geq 1} g_i x^i \), using (1.4) and (2.5), by Theorem 2.2 and Lemma 2.8, one can deduce that

**Corollary 2.9.** For any integers \( n, m, k \geq 0 \), there holds
\[
\sum_{P \in \mathcal{M}_{n,k}} \prod_{i \geq 1} \left( \frac{f_{i+1}(i+1)}{(i+1)!} \right) u_i^{(P)} \prod_{i \geq 1} \left( \frac{g_{i+1}(i)}{i!} \right) h_i^{(P)}
\]

\[
= \sum_{\ell=0}^{k} \sum_{j=0}^{k} (-1)^{\ell-j} \binom{\ell}{j} \binom{k}{\ell} k^{m+j+1} \frac{\sum_{i=0}^{(m+j+1)} f_m(2m+j+1)}{(m+1)!},
\]

which, in the case \( g_m(i) = \binom{i}{m} m! \) for all \( i \geq 1 \), by the identity
\[
\sum_{\ell=0}^{k} (-1)^{\ell-j} \binom{\ell}{j} \binom{k}{\ell} = \delta_{k,j},
\]
leads to

**Example 2.10.** Let \( f_r(x) = \sum_{i \geq 0} (ri+1)^{-1}x^i i! \), which is the exponential generating function for rooted complete \( r \)-ary labeled trees for \( r \geq 0 \) and satisfies the relation \( f_r(x) = e^{x f_r(x)^r} \). By the Lagrange inversion formula, one can deduce \( f_r,i = \binom{i}{m} m! \) Then (2.4) produces
\[
\sum_{P \in \mathcal{M}_{n,k}} \prod_{i \geq 1} \left( \frac{(r+1)i+1)^{-1}}{i!} \right) u_i^{(P)} = \binom{m+k+1}{k} \binom{(r+2)m+k+1}{m+1}.
\]

Let \( t_i = \phi_{i+1}(1) \) and \( s_i = \psi_{i+1}(1) \), where \( \{ \phi_n(x) \}_{n \geq 0} \) and \( \{ \psi_n(x) \}_{n \geq 0} \) are binomial sequences, using (1.4) and (2.5), by Theorem 2.2 and Lemma 2.8, one can deduce that

**Corollary 2.11.** For any integers \( n, m, k \geq 0 \), there holds
\[
\sum_{P \in \mathcal{M}_{n,k}} \left( \frac{\phi_{i}(1)}{i!} \right) u_i^{(P)} \prod_{i \geq 1} \left( \frac{\psi_{i+1}(1)}{(i-1)!} \right) h_i^{(P)}
\]

\[
= \sum_{j=0}^{k} \sum_{\ell=j}^{k} (-1)^{\ell-j} \binom{\ell-j}{\ell-j} \binom{k-1}{k-j} \binom{m+j}{j} \frac{\phi_m(m+j+1)}{(m+1)!}.
\]
Example 2.12. Corollary [2.11] in the case \( \psi_n(x) = x(x+1) \cdots (x+n-1) \), by (2.8), produces

\[
\sum_{P \in \mathcal{C}_{m,k}} \prod_{i \geq 1} \left( \frac{\phi_i(1)}{i!} \right)^{u_i(P)} = \frac{(m+k)(m+k+1)}{(m+1)!}.
\]

In addition, let \( \phi_n(x) = x(x-qn)^{n-1} \) for fixed \( q \). Then (2.10) generates

\[
\sum_{P \in \mathcal{C}_{m,k}} \prod_{i \geq 1} \left( \frac{1-q^n}{i!} \right)^{u_i(P)} = \frac{(m+k+1)}{m!} \left( \frac{(1-q)m+k+1}{m!} \right)^{m-1},
\]

which, in the case \( q = -(r+1) \), coincides with Example [2.10].

Let \( \phi_n(x) = \sum_{i=0}^{n} S(n,i)x^i \), which implies \( \phi_n(1) \) is the \( n \)-th Bell number \( B_n \). Then (2.10) produces

\[
\sum_{P \in \mathcal{C}_{m,k}} \prod_{i \geq 1} \left( B_i \right)^{u_i(P)} = \frac{(m+k)}{m!} \sum_{i=0}^{m} S(m,i)(m+k+1)^i
\]

\( \frac{1}{(m+1)!} \).

3. Special Motzkin paths: Compositions

A composition of nonnegative integer \( \lambda \) into \( j \) parts is an ordered sequence \( \lambda_1, \lambda_2, \ldots, \lambda_j \) of length \( j \) such that \( \lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_j \) with each \( \lambda_i \geq 0 \). Each \( \lambda_i \) is called the \( i \)-th summand of the composition. Compositions are well known combinatorial objects [2, 5, 8] and several of their properties have been discussed in some recent papers, as in [4, 9, 10, 13, 14, 17].

A composition can be regarded as a special Motzkin path if each summand \( \lambda_i \) is replaced by \( u^{\lambda_i}d^{\lambda_i} \) when \( \lambda_i \geq 1 \) and by a \( h \) when \( \lambda_i = 0 \). A \( u \)-segment or \( h \)-segment of a composition is defined to be that of its corresponding Motzkin path.

Let \( \mathcal{C}_{m,k,j} \) denote the set of compositions of \( m \) with \( j \) parts and \( k \) zero summands, so any \( C \in \mathcal{C}_{m,k,j} \) has \( j-k \) \( u \)-segments. Define the ordinary generating functions for weighted compositions according to the statistics \( u_1, u_2, \ldots \) and \( h_1, h_2, \ldots \) as follows

\[
C_j(x, y; t; s) = \sum_{m,k \geq 0} x^m y^k \sum_{C \in \mathcal{C}_{m,k,j}} \prod_{i \geq 1} t_i^{u_i(C)} \prod_{i \geq 1} s_i^{h_i(C)}
\]

\[
C(x, y; t; s; q) = \sum_{j \geq 0} q^j C_j(x, y; t; s).
\]

Proposition 3.1. The explicit formula for \( C(x, y; t; s; q) \) is

\[
C(x, y; t; s; q) = \frac{S(qy)}{1 + qS(qy) - qS(qy)T(x)},
\]

where \( T(x) = 1 + \sum_{i \geq 1} t_i x^i \), \( S(y) = 1 + \sum_{i \geq 1} s_i y^i \).

Proof. A recurrence relation for \( C_j(x, y; t; s) \) can be derived as follows

\[
C_j(x, y; t; s) = s_j y^j + \sum_{i=1}^{j} s_{j-i} y^{j-i} C_{i-1}(x, y; t; s)(T(x) - 1),
\]
for \( j \geq 1 \) and \( C_0(x, y; t; s) = 1 \) if one notices that a composition begins with a \( h \)-segment of length \( i \) for \( 0 \leq i \leq j \) or a \( u \)-segment of length \( r \) for \( r \geq 1 \). Then

\[
C(x, y; t; s; q) = \sum_{j \geq 0} q^j C_j(x, y; t; s)
\]

\[
= 1 + \sum_{j \geq 1} q^j \left\{ s_j y^j + \sum_{i=1}^{j} s_{j-i} y^{j-i} C_{i-1}(x, y; t; s)(T(x) - 1) \right\}
\]

\[
= S(qy)(1 + q(T(x) - 1)C(x, y; t; s; q)),
\]

which leads to (3.1).

**Theorem 3.2.** For any integers \( m, k, j \geq 0 \), there holds

\[
\sum_{C \in \mathcal{C}_{m,k,j}} \prod_{i \geq 1} \binom{\mu_i(C)}{t_i} \prod_{i \geq 1} s_i^{h_i(C)} = \frac{P_{k}^{(j-k+1)}(1! s_1, 2! s_2, \ldots)(j-k)! B_{m,j-k}(1! t_1, 2! t_2, \ldots)}{k! m!}.
\]

**Proof.** By the definition of potential polynomials and (3.1), using the identity

\[
\sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} \binom{j-k-i}{r} = \delta_{r,j-k},
\]

where \( \delta_{r,j-k} \) is the Kronecker symbol, we have

\[
\sum_{C \in \mathcal{C}_{m,k,j}} \prod_{i \geq 1} \binom{\mu_i(C)}{t_i} \prod_{i \geq 1} s_i^{h_i(C)} = [x^m y^k q^j] C(x, y; t; s; q)
\]

\[
= [x^m y^k q^j] \frac{S(qy)}{1 + qS(qy) - qS(qy)T(x)} = [x^m y^k q^j] \sum_{i \geq 0} q^i S(qy)^{i+1}(T(x) - 1)^i
\]

\[
= [x^m (qy)^k] S(qy)^{j-k+1}(T(x) - 1)^{j-k} = [(qy)^k] S(qy)^{j-k+1} [x^m](T(x) - 1)^{j-k}
\]

\[
= [(qy)^k] S(qy)^{j-k+1} \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} [x^m] T(x)^{j-k-i}
\]

\[
= \frac{P_{k}^{(j-k+1)}(1! s_1, 2! s_2, \ldots)}{k!} \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} \frac{P_{m}^{(j-k-i)}(1! t_1, 2! t_2, \ldots)}{m!}
\]

\[
= \frac{P_{k}^{(j-k+1)}(1! s_1, 2! s_2, \ldots)}{k!} \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} \sum_{r=0}^{m} \frac{r!}{m!} \binom{j-k-i}{r} B_{m,r}(1! t_1, 2! t_2, \ldots)
\]

\[
= \frac{P_{k}^{(j-k+1)}(1! s_1, 2! s_2, \ldots)}{k!} \sum_{r=0}^{m} \frac{r!}{m!} B_{m,r}(1! t_1, 2! t_2, \ldots) \sum_{i=0}^{j-k} (-1)^i \binom{j-k}{i} \binom{j-k-i}{r}
\]

\[
= \frac{P_{k}^{(j-k+1)}(1! s_1, 2! s_2, \ldots)}{k!} (j-k)! B_{m,j-k}(1! t_1, 2! t_2, \ldots) \sum_{j=1}^{m} \frac{r!}{m!} B_{m,r}(1! t_1, 2! t_2, \ldots),
\]

as claimed.

**Remark:** Theorem 3.2 provides a unified method to investigate compositions. This very general framework can be applied to many special cases by specializing the parameters. For example, let \( T(x) = \frac{1}{1-x} - x^r \), i.e., \( t_r = 1 \) except for \( t_r = 0 \) for \( r \geq 1 \), then Theorem 3.2 in the case \( k = 0 \) leads to compositions without occurrences of \( r \); More generally, let \( T(x) = 1 + \sum_{i \in A} x^i \), where \( A \) is a given set of positive integers, then Theorem 3.2 in the case \( k = 0 \) leads to compositions with summands in a given set \( A \).
Recall that any $C \in \mathcal{C}_{m,k,j}$ has $j - k$ u-segments. Let $\mathcal{C}^\ell_{m,k,j}$ be the subset of $\mathcal{C}_{m,k,j}$ with $\ell$ number of h-segments. Note that $\mathbf{B}_{m,r}(1!q_1t_1, 2!q_2t_2, \cdots) = q^r \mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots)$ by (1.5), combining (1.3) with Theorem 3.2, we have

**Proposition 4.1.** The explicit formula for

$$C \in \mathcal{C}^\ell_{m,k,j}$$

for some $0 \leq \ell \leq m$, $k = 0, \ldots, m$, is given by

$$\sum_{C \in \mathcal{C}^\ell_{m,k,j}} \prod_{i \geq 1} t_i^{u_i(C)} \prod_{i \geq 1} s_i^{h_i(C)} = \binom{j - k + 1}{\ell} \binom{j - k}{r_1, r_2, \ldots, r_m} \binom{\ell}{\ell_1, \ell_2, \ldots, \ell_k} \cdot \mathbf{B}_{m,j-k}(1!t_1, 2!t_2, \cdots) \mathbf{B}_{k,\ell}(1!s_1, 2!s_2, \cdots).$$

Using (1.5) and then comparing the coefficient of $t_1^{r_1}t_2^{r_2} \cdots t_m^{r_m} s_1^{\ell_1} s_2^{\ell_2} \cdots s_k^{\ell_k}$ in Corollary 3.3 one can obtain that

**Corollary 3.4.** The number of compositions in $\mathcal{C}^\ell_{m,k,j}$ with u-segments of type $1^{r_1} 2^{r_2} \cdots m^{r_m}$ and h-segments of type $1^{\ell_1} 2^{\ell_2} \cdots k^{\ell_k}$ is

$$C_{j}(x, y; t; s) = \sum_{i=0}^{j} \frac{y^i x^{m-i} t^{j+i} (1!s_1, 2!s_2, \cdots)}{(j - i)!} (T(x) - 1)^i.$$ 

However, it seems that the coefficients of $[x^m y^k]$ in $C_j^p(x, y; t; s)$ have no simple explicit formulas. For the sake of this, we can consider a kind of special matrix compositions, called *bipartite matrix compositions*, namely, each row has the type $(a_1, a_2, \ldots, a_i, 0, \ldots, 0)$ for some $0 \leq i \leq j$, where $a_1, \ldots, a_i \geq 1$. If $a_1 = \cdots = a_i = 1$, then we call it a *bipartite* $(0, 1)$-matrix.

Let $\mathcal{B}_{m,p,j}$ denote the set of $p \times j$ bipartite matrix compositions of $m$ and let $B_{p,j}(x; t)$ denote the ordinary generating functions for weighted $p \times j$ bipartite matrix compositions according to the statistics $u_1, u_2, \ldots$ and $h_1, h_2, \ldots$ is just $C_j^p(x, y; t; s)$. From Proposition 3.1 one can deduce easily that

$$B_{p,j}(x; t) = \sum_{m \geq 0} x^m \sum_{B \in \mathcal{B}_{m,p,j}} \prod_{i \geq 1} t_i^{u_i(B)}.$$ 

**Proposition 4.1.** The explicit formula for $B_{p,j}(x; t)$ is

$$B_{p,j}(x; t) = \left\{ \frac{1 - \{(T(x) - 1)^{j+1}\}}{1 - \{(T(x) - 1)^j\}} \right\}^p,$$

where $T(x) = 1 + \sum_{i \geq 1} t_i x^i$.

**Proof.** For any $1 \times j$ bipartite matrix compositions, it has the type $(a_1, a_2, \ldots, a_i, 0, \ldots, 0)$ for some $0 \leq i \leq j$, where $a_1, a_2, \ldots, a_i \geq 1$, then each $a_r$ has the weight $t_{a_r}$ which is a term
of \(T(x) - 1\). Hence we have

\[B_{1,j}(x; t) = \sum_{i=0}^{j} (T(x) - 1)^i = \frac{1 - \{(T(x) - 1)^{j+1}\}}{1 - \{(T(x) - 1)^{j+1}\}}.\]

Then by the relation \(B_{p,j}(x; t) = B_{1,j}(x; t)^p\), we obtain (4.1).

**Theorem 4.2.** For any integers \(m, p, j \geq 0\), there holds

\[
\sum_{B \in \mathbb{B}_{m,p,j}^r} \prod_{i \geq 1} t_i^{u_i(B)} = \sum_{r=0}^{m} \frac{r!}{m!} U_{p,j,r} B_{m,r}(1!t_1, 2!t_2, \cdots),
\]

where

\[U_{p,j,r} = \sum_{i=0}^{r} (-1)^i \binom{p}{i} \frac{(p + r - i(j + 1) - 1)}{p - 1} \cdot \frac{1}{i!} \cdot \frac{1}{(j + 1)!} \cdot \frac{1}{(r - i)!} .\]

**Proof.** Similar to the proof of Theorem 3.2 we have

\[
\sum_{B \in \mathbb{B}_{m,p,j}^r} \prod_{i \geq 1} t_i^{u_i(B)} = [x^{m}] B_{p,j}(x; t) = [x^{m}] \left\{ \frac{1 - \{(T(x) - 1)^{j+1}\}}{1 - \{(T(x) - 1)^{j+1}\}} \right\}^p
\]

\[
= \sum_{r=0}^{m} \frac{r!}{m!} U_{p,j,r} B_{m,r}(1!t_1, 2!t_2, \cdots)
\]

as claimed. \(\square\)

Let \(\mathbb{B}_{m,p,j}^r\) be the subset of \(\mathbb{B}_{m,p,j}\) with \(r\) number of nonzero entries. Note that \(B_{m,r}(1!t_1, 2!t_2, \cdots) = q^r B_{m,r}(1!t_1, 2!t_2, \cdots)\) by (1.5), combining (1.3) with Theorem 4.2 we have

**Corollary 4.3.** For any integers \(m, p, j, r \geq 0\), there holds

\[
\sum_{B \in \mathbb{B}_{m,p,j}^r} \prod_{i \geq 1} t_i^{u_i(B)} = \frac{r!}{m!} B_{m,r}(1!t_1, 2!t_2, \cdots) U_{p,j,r} .
\]

Using (1.5) and comparing the coefficient of \(t_1^{r_1} t_2^{r_2} \cdots t_m^{r_m}\) in Corollary 4.3 one can obtain that

**Corollary 4.4.** The number of \(p \times j\) bipartite matrix compositions of \(m\) in \(\mathbb{B}_{m,p}^{j,r}\) with nonzero entries of type \(1^{r_1} 2^{r_2} \cdots m^{r_m}\) is

\[
\binom{r}{r_1, r_2, \cdots, r_m} \sum_{i=0}^{\lceil \frac{r}{j+1} \rceil} (-1)^i \binom{p}{i} \frac{(p + r - i(j + 1) - 1)}{p - 1} .
\]

**Example 4.5.** Let \(T(x) = 1 + x\), then Theorem 4.2 signifies that the number of \(p \times j\) bipartite matrix compositions of \(m\) with nonzero summands 1 or, in other words, of \(p \times j\) bipartite \((0,1)\)-matrices with \(m\) ones is counted by \(U_{p,j,m}\). Specializing to \(p = m + 1\), we have

\[U_{m+1,j,m} = \sum_{i=0}^{\lceil \frac{r}{j+1} \rceil} (-1)^i \binom{m+1}{i} \binom{2m - i(j + 1)}{m} .
\]

Note that the number \(\frac{1}{m+1} U_{m+1,j,m}\) counts the unlabeled plane trees on \(m+1\) vertices in which every vertex has outdegree not greater than \(j\). Klarner [12] first considered this problem, which
was solved by Chen [6] and later by Mansour and Sun [15]. Then it is clear that \( U_{m+1,j,m} \) counts the unlabeled double rooted plane trees on \( m + 1 \) vertices in which every vertex has outdegree not greater than \( j \). We leave it as an open problem to find the bijection between these two settings.

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