Codes with structured Hamming distance in graph families

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Abstract

We investigate the maximum size of graph families on a common vertex set of cardinality $n$ such that the symmetric difference of the edge sets of any two members of the family satisfies some prescribed condition. We solve the problem completely for infinitely many values of $n$ when the prescribed condition is connectivity or 2-connectivity, Hamiltonicity or the containment of a spanning star. We give lower and upper bounds when it is the containment of some fixed finite graph concentrating mostly on the case when this graph is the 3-cycle or just any odd cycle. The paper ends with a collection of open problems followed by an important update in this new version of the manuscript.

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1 Introduction

Celebrated problems of extremal combinatorics may get an exciting new flavour when the presence of some special structure is imposed in the condition. A prominent example is the famous Simonovits-Sós conjecture [27] proven by Ellis, Filmus and Friedgut [8] in which the maximum number of subgraphs of a complete graph was asked for provided that the intersection of any two contains a triangle. (The result of [8] shows, along with several far reaching generalizations, that the best is to take all graphs containing a given triangle, just as it was conjectured in [27]. This is clearly reminiscent of the Erdős-Ko-Rado theorem [9,].) As another example we can also mention the Ramsey type problem investigated in [18] that was also initiated by a question of Vera Sós and can be considered as a graph version of the first unsolved case of the so-called perfect hashing problem. (For details we refer to [18]).

In this paper we study several problems we arrive to if the basic code distance problem (how many binary sequences of a given length can be given at most if any two differ in at least a given number of coordinates) is modified so that we not only prescribe the minimum distance of any two codewords but also require that they differ in some specific structure. In particular, just as in the Simonovits-Sós problem we seek the largest family of (not necessarily induced) subgraphs of a complete graph such that the symmetric difference of the edge sets of any two graphs in the family has some required property. We will consider properties like connectedness, Hamiltonicity, containment of a triangle and some more. Formally all these can be described by saying that the graph defined by the symmetric difference of the edge sets of any two of our graphs belongs to a prescribed family of graphs (namely those that are connected, contain a Hamiltonian cycle, or contain a triangle, etc.)

Let $\mathcal{F}$ be a fixed class of graphs. A graph family $\mathcal{G}$ on $n$ labeled vertices is called $\mathcal{F}$-good if for any pair $G, G' \in \mathcal{G}$ the graph $G \oplus G'$ defined by

$$V(G \oplus G') = V(G) = V(G') = [n],$$

where $[n] = \{1, \ldots, n\}$ and

$$E(G \oplus G') = \{ e : e \in (E(G) \setminus E(G')) \cup (E(G') \setminus E(G)) \}$$

belongs to $\mathcal{F}$.

Let $M_\mathcal{F}(n)$ denote the maximum possible size of an $\mathcal{F}$-good family on $n$ vertices. We are interested in the value of $M_\mathcal{F}(n)$ for various classes $\mathcal{F}$. We will give exact answers or both lower and upper bounds in several cases.

We mention that codes where the codewords are described by graphs already appear in the literature. In [28], for example, Tonchev looked at the usual code distance problem restricted to codes whose codewords are characteristic vectors of edge sets of graphs. Gray codes on graphs are also considered, see [23], where the graphs representing the codewords should have some similarity properties if they are consecutive in a certain listing. Problems analogous to the present ones though restricted to special graph classes
were also considered in [19] and [5]. A very interesting result along these lines is the one in [21].

The paper is organised as follows. In Section 2 we give a general upper bound that will turn out to be sharp in several of the cases we consider. In Section 3 we consider classes \( F \) defined by some global criterion as connectivity or 2-connectivity, Hamiltonicity or containing a full star, that is, a vertex of degree \( n-1 \). We determine \( M_F(n) \) for infinitely many values of \( n \) and for all \( n \) in the first and the last case. In Section 4 we consider classes \( F \) defined by containing a certain fixed finite subgraph. (All graphs in this paper are considered simple.) We consider the problem also in general but most of our efforts are concentrated on the special case when this subgraph is a triangle or just any odd cycle. In these cases we were not able to determine \( M_F(n) \) exactly but give some nontrivial bounds. The final section contains a collection of open problems followed by an important update in this new version of the manuscript.

2 A general upper bound

To bound \( M_F(n) \) for various graph classes \( F \) it will often be useful to also consider the related problem of constructing large graph families in which no pair satisfies the condition prescribed by \( F \).

**Definition 1.** For a class of graphs \( F \) let \( D_F(n) \) denote the maximum possible size of a graph family on \( n \) labeled vertices (that is, each member of the family has \([n] = \{1, \ldots, n\}\) as vertex set), the symmetric difference of no two members of which belongs to \( F \). Determining \( D_F(n) \) will be referred to as the dual problem of determining \( M_F(n) \).

**Lemma 1.** For any graph class \( F \) we have

\[
M_F(n) \cdot D_F(n) \leq 2\binom{n}{2}.
\]

**Proof.** Let us define a graph \( H_F \) whose vertices are all the possible (simple) graphs on the vertex set \([n]\). Connect two such vertices if and only if the corresponding pair of graphs have their symmetric difference belonging to \( F \). Then by definition we have

\[
M_F(n) = \omega(H_F) \quad \text{and} \quad D_F(n) = \alpha(H_F),
\]

where \( \omega(H) \) and \( \alpha(H) \) denote the clique number and the independence number of graph \( H \), respectively. Observe that \( H_F \) is vertex-transitive. Indeed, if \( G_1 \) and \( G_2 \) are two graphs forming vertices of \( H_F \) then taking the symmetric difference of all \( n \)-vertex graphs forming vertices of \( H_F \) with the graph \( G_1 \oplus G_2 \) is an automorphism of \( H_F \) that maps \( G_1 \) to \( G_2 \). Since a vertex-transitive graph \( H \) always satisfies \( \alpha(H) \omega(H) \leq |V(H)| \) (this can be seen by using that the fractional chromatic number \( \chi_f(H) \) always satisfies \( \omega(H) \leq \chi_f(H) \)), while if \( H \) is a vertex-transitive graph we also have \( \chi_f(H) = \frac{|V(H)|}{\alpha(H)} \), cf. [26], the statement follows. \( \square \)
The above lemma makes it possible to bound $M_F(n)$ from above by bounding $D_{\bar{F}}(n)$ from below. In particular, whenever we construct two families of graphs $A$ and $B$ on $[n]$ such that $A, A' \in A$ implies $A \oplus A' \in \mathcal{F}$ and $B, B' \in B$ implies $B \oplus B' \notin \mathcal{F}$, while $|A||B| = 2^{(n/2)}$, then we know that $|A|$ and $|B|$ realize the optimal values $M_F(n)$ and $D_{\bar{F}}(n)$ for such families. Below we will see several cases when this simple technique can indeed be used to obtain these optimal values.

Remark 1. It is worth noting that Lemma 1 can be proven in a different way, with no reference to the fractional chromatic number. Indeed, if $G_1, \ldots, G_k$ is an $\mathcal{F}$-good family, while $T_1, \ldots, T_m$ is a family satisfying the conditions of the dual problem, then all the symmetric differences of the form $G_i \oplus T_j$ are different, implying $km \leq 2^{(n/2)}$. This is true because if $G_i \oplus T_j$ and $G_r \oplus T_s$ would be the same for some $\{i, j\} \neq \{r, s\}$, then $(G_i \oplus T_j) \oplus (G_r \oplus T_s)$ would be the empty graph that could also be written (by commutativity and associativity of the symmetric difference) as $(G_i \oplus G_r) \oplus (T_j \oplus T_s)$. This would mean that $G_i \oplus G_r$ and $T_j \oplus T_s$ are two identical graphs. But if one of them is the empty graph, then the other cannot be empty and if both are nonempty, then one of them belongs to $\mathcal{F}$ while the other one does not, so this is impossible. ♦

3. Global conditions

3.1 Connectivity

When we speak about the class of connected graphs in the following theorem, we mean that these graphs span all their vertex set, that is, no isolated vertices are allowed either.

Theorem 2. Let $\mathcal{F}_c$ denote the class of connected graphs. Then

$$M_{\mathcal{F}_c}(n) = 2^{n-1}.$$ 

Proof. First we give a very simple dual family $\mathcal{B}_c$. Let it consist of all graphs on $[n]$ in which the vertex labeled $n$ is isolated. Clearly $|\mathcal{B}_c| = 2^{(n-1)}$ (that is, the number of all graphs on $[n-1]$) and $n$ is also isolated in the symmetric difference of any two of them, so no such symmetric difference is connected, This gives $D_{\bar{\mathcal{F}_c}}(n) \geq 2^{(n-1)/2}$ and thus by Lemma 1 we have

$$M_{\mathcal{F}_c}(n) \leq 2^{(n-1)/2} - 2^{(n-1)/2} = 2^{n-1}.$$ 

Now we show that this upper bound can be attained. Let the family $\mathcal{A}_c$ consist of all those graphs on $[n]$ that are the vertex-disjoint union of two complete graphs (with no isolated vertices remaining) including the case when one of the two is on the empty set. Clearly, the number of these graphs is just half the number of subsets of $[n]$, that is exactly $2^{n-1}$. All we have to show is that the symmetric difference of any two of these graphs is connected. Choose two arbitrary graphs $G$ and $G'$ from our family. Let $G$ be
the union of complete graphs on the complementary vertex sets $K$ and $L$, while $G'$ be the same on $K'$ and $L'$. Let $A = K \cap L'$, $B = L' \cap L$, $C = L \cap K'$ and $D = K' \cap K$. It is possible that one, but only one of $A, B, C, D$ is empty. The edges of $G \oplus G'$ are all the edges of the complete bipartite graph with partite classes $A \cup C$ and $B \cup D$, so it must be connected. □

With just a little more consideration one can also treat the case of 2-connectedness at least for even $n$.

**Theorem 3.** Let $\mathcal{F}_{2c}$ denote the class of 2-connected graphs. Then if $n$ is even, we have

$$M_{\mathcal{F}_{2c}}(n) = 2^{n-2}.$$

**Proof.** The proof is a modification of the previous one, therefore we use the notation introduced there. The construction given there may result in symmetric differences that are not 2-connected only if $A \cup C$ or $B \cup D$ contains only one element. For even $n$ this can be avoided if we consider only such graphs in our construction where the bipartition of $[n]$ defining the individual graphs has an even number of elements in both partite classes $K$ and $L$. This proves the lower bound.

For the upper bound we consider all graphs in which the vertex $n$ is either isolated or it has one fixed neighbor, say $n - 1$. The symmetric difference of any two such graphs is not 2-connected, since $n$ has at most one neighbor in it. The number of such graphs is just twice the number of graphs in which $n$ is an isolated point, that is, $2^{\binom{n}{2} + 1}$ proving the matching upper bound by Lemma 1. □

**Remark 2.** The upper bound proven in Theorem 3 clearly holds also for odd $n$ but we have not found a matching construction in general. For $n = 3$ a triangle and the empty graph would do, still achieving the upper bound. But for larger odd $n$ the best we could do is to take only those graphs from our construction for which in the corresponding bipartition the smaller partition class has an odd number of elements if $n \equiv 1 \pmod{4}$ and it has an even number of elements if $n \equiv 3 \pmod{4}$. The number of graphs obtained this way is $2^{n-2} - \binom{n-2}{(n-3)/2}$. ♦

**Remark 3.** Changing the graphs to their complements in the proofs of Theorems 2 and 3 makes these graph families vector spaces over the 2-element field, while they still satisfy the conditions as the symmetric differences do not change by complementation (or by taking the symmetric difference of all elements with any fixed graph which is the complete graph in case of complementation). ♦

It does not sound surprising that if we step further on to $k$-connectedness for $k > 2$ then the problem becomes rather more complicated. Nevertheless, if we insist on linear codes, that is graph families closed under the symmetric difference operation then for $k = 3$ we can still determine the largest possible cardinality for infinitely many values on $n$ using Hamming codes.
Theorem 4. Let \( \mathcal{F}_{3c} \) be the class of 3-connected graphs and let \( M_{\mathcal{F}_{3c}}^{(t)}(n) \) denote the size of a largest possible linear graph family on vertex set \([n]\) any two members of which has a 3-connected symmetric difference. If \( n = 2^k - 1 \) for some integer \( k \geq 2 \), then

\[
M_{\mathcal{F}_{3c}}^{(t)}(n) = 2^{n-1}.
\]

Proof. First we prove that \( D_{\mathcal{F}_{3c}}(n) \geq n2^{(n-2)/2} \) holds in general. Consider the family of all graphs on vertex set \([n]\) in which the degree of vertex \( n \) is at most 1. There are exactly \( n2^{(n-2)/2} \) such graphs. The symmetric difference of any two of these graphs is at most 2-connected, since the vertex \( n \) has degree at most 2 in all these symmetric differences. This proves the claimed inequality and by Lemma 1 this implies \( M_{\mathcal{F}_{3c}}(n) \leq 2^{n-1}/n \).

It is well-known that if a family of subsets of a finite set contains the empty set and is closed under the symmetric difference operation then the cardinality of this set must be a power of 2. This follows immediately from linear algebra and the fact that such a family forms a vector space over \( GF(2) \), cf. e.g. Lemma 3.1 in Kozlov’s book [22] where a simple combinatorial proof of this fact is also presented. Since a linear graph family code on \([n]\) can be viewed as a collection of subsets of \( E(K_n) \), this implies that \( M_{\mathcal{F}_{3c}}^{(t)}(n) \) is a power of 2. Since we obviously have \( M_{\mathcal{F}_{3c}}^{(t)}(n) \leq M_{\mathcal{F}_{3c}}(n) \), the upper bound proved above implies \( M_{\mathcal{F}_{3c}}^{(t)}(n) \leq 2^d \) with \( d = \lfloor \log_2 2^{n-1} \rfloor \) giving

\[
M_{\mathcal{F}_{3c}}^{(t)}(n) \leq 2^{n-1}
\]

for \( n = 2^k - 1, k \geq 2 \), which proves the required upper bound.

For the lower bound consider the Hamming code \( C_H(n) \) with length \( n = 2^k - 1 \) that exists for every \( k \geq 2 \). (For a nice quick account on Hamming codes see e.g. [3].) It is a linear code with minimum distance 3 that consists of \( 2^{n-k} \) binary codewords having the property that if \( c = (c_1, \ldots, c_n) \) belongs to the code then so does also \( \bar{c} = (\bar{c}_1, \ldots, \bar{c}_n) \) where \( \bar{c}_i = 1 - c_i \). For each codeword \( c \in C_H(n) \) consider the bipartition of \([n]\) into the subsets \( K_c, L_c \), where \( K_c = \{i : c_i = 0\}, L_c = \{i : c_i = 1\} \) and the complete bipartite graph \( G_{K_c,L_c} \) with partite classes \( K_c, L_c \). Note that by the above mentioned property of Hamming codes we have \( c \in C_H(n) \) if and only if \( \bar{c} \in C_H(n) \) and thus since \( G_{K_c,L_c} = G_{K_c',L_c'} \), we get \( \lfloor 1/2 \rfloor C_H(n) = 2^{n-k-1} \) different complete bipartite graphs this way. All we have to prove is that the symmetric difference of any two of our graphs is 3-connected. This is equivalent to show that if \( c \neq \bar{c} \), then the cardinality of both partite classes of \( G_{K_c,L_c} \), that is of \( (K_c \cap K_c') \cup (L_c \cap L_c') \) and \( (K_c \cap L_c') \cup (K_c' \cap L_c) \) is at least 3. However, this immediately follows from the fact that the codeword \( c' \) must differ from both \( c \) and \( \bar{c} \) in at least 3 coordinates. This completes the proof. \( \square \)

3.2 Hamiltonicity

A graph is connected if and only if it contains a spanning tree. Next we consider what happens if we require the containment of specific spanning trees: a path in this subsection and a star in the next one.
Theorem 5. Let $\mathcal{F}_{Hp}$ denote the class of graphs containing a Hamiltonian path. Then for infinitely many values of $n$ we have

$$M_{\mathcal{F}_{Hp}}(n) = 2^{n-1}.$$ 

In particular, this holds whenever $n = p$ or $n = 2p - 1$ for some odd prime $p$.

To prove the above theorem we will refer to the following old conjecture that is known to be true in several special cases. To state it we need the notion of perfect 1-factorization. It means the partition of the edge set of a graph into perfect matchings such that the union of any two of them is a Hamiltonian cycle.

Perfect 1-factorization conjecture (P1FC) (Kotzig [20]). The complete graph $K_n$ has a perfect 1-factorization for all even $n > 2$.

This conjecture is still open in general, however it is known to hold in several special cases, for example, whenever $n = p + 1$ (Kotzig [20]) or $n = 2p$ for some odd prime $p$ (Anderson [2] and Nakamura [24], cf. also Kobayashi [17]). For a recent survey, see Rosa [25], according to which the smallest open case of the conjecture is $n = 64$.

Proof of Theorem 5. Since Hamiltonian paths are connected, it follows from the proof of Theorem 2 that $2^{n-1}$ is again an upper bound. Now we show that it is also a lower bound whenever the Perfect 1-factorization conjecture holds for $n + 1$. (Note that if the conjecture is true, then this means all odd numbers at least 3, while for 1 our statement is void.)

Let $n$ be an odd number for which $K_{n+1}$ has a perfect 1-factorization $\mathcal{M}$ and $v$ a fixed vertex of $K_{n+1}$. Note that deleting the edge incident to $v$ from all matchings belonging to $\mathcal{M}$ we obtain $n$ matchings of $K_n$ such that the union of any two of them is a Hamiltonian path in $K_n := K_{n+1} \setminus \{v\}$. Now consider all those subgraphs of $K_n$ that can be obtained as the union of an even number of these $n$ matchings. Clearly, the symmetric difference of any two of them is also the union of at least two of these matchings and thus contains a Hamiltonian path. The number of graphs obtained this way is $2^{n-1}$, matching the upper bound. \(\square\)

The case of Hamiltonian cycles can be treated essentially the same way.

Theorem 6. Let $\mathcal{F}_{Hc}$ denote the class of graphs containing a Hamiltonian cycle. For all even values of $n$ for which the P1FC holds, we have

$$M_{\mathcal{F}_{Hc}}(n) = 2^{n-2}.$$ 

In particular, this is the case if $n = p + 1$ or $n = 2p$ for some odd prime $p$.

Proof. Since Hamiltonian cycles are 2-connected, it follows from the proof of Theorem 3 that $2^{n-2}$ is again an upper bound.
Let \( n \) be an even number for which the P1FC holds and let \( \mathcal{M} \) be a perfect 1-factorization of \( K_n \). Note that \( \mathcal{M} \) contains \( n - 1 \) matchings (as it is well-known that the edge-chromatic number of \( K_n \) for even \( n \) is \( n - 1 \)). Now consider the \( 2^{n-2} \) graphs we can obtain as the union of an even number of matchings from \( \mathcal{M} \). Clearly, the symmetric difference of any two of them contains a Hamiltonian cycle. \( \square \)

**Remark 4.** Since Hamiltonian cycles are 2-connected graphs the proof of Theorem 6 obviously gives an alternative proof of Theorem 3 for those values of \( n \) for which the Perfect 1-factorization conjecture is known to hold. (The situation is similar for Theorems 5 versus 2.) On the other hand, the construction in the proof of Theorem 3 utterly fails to give a good lower bound for the value of \( M_{F_{hc}}(n) \) investigated in Theorem 6. Indeed, the symmetric difference of two graphs in the construction given in the proof of Theorem 3 contains a Hamiltonian cycle if and only if the sets denoted by \( A \cup C \) and \( B \cup D \) in that proof both have cardinality \( \frac{n}{2} \) and this happens exactly when the partition classes of the partitions \((K, L)\) and \((K', L')\) are orthogonal in the sense that \( |K \cap K'| = |K \cap L'| = \frac{n}{4} \) (that clearly implies the similar property if \( K \) is exchanged with \( L \)). This means that representing these bipartitions by characteristic vectors consisting of +1 and −1 coordinates in the obvious way, we get a collection of vectors that are pairwise orthogonal, so their number cannot be more than just \( n \) and we can give exactly \( n \) such vectors if and only if an \( n \times n \) Hadamard matrix exists.

It might be interesting to note also that if we consider such a restricted version of the dual problem, that is, the question of how many (spanning) complete bipartite graphs can be given on \( n \) vertices so that the symmetric difference of no two of them contains a Hamiltonian cycle then we arrive to a problem that came up in Kahn and Kalai’s celebrated paper [15] disproving Borsuk’s conjecture. Indeed, their construction presented there is based on a collection of complete bipartite graphs with this property and on a famous result by Frankl and Wilson [12] who showed that the number of graphs in such a family is exponentially smaller than even the number of balanced complete bipartite graphs on \( n \) vertices whenever \( n = 4k \) for some prime power \( k \). \( \diamond \)

### 3.3 Containing a spanning star

We have seen in the previous subsection that if we want every symmetric difference to contain a spanning tree which is a path, then for infinitely many values of \( n \) our family can be just as large as if we did not want more than just the connectedness of these symmetric differences. In this subsection we show that if the required spanning tree is a star, then the largest possible family is drastically smaller.

**Theorem 7.** Let \( \mathcal{F}_S \) denote the class of graphs containing a spanning star, that is a vertex connected to all other vertices in the graph. Then we have

\[
M_{\mathcal{F}_S}(n) = \begin{cases} 
  n + 1 & \text{if } n \text{ is odd} \\
  n & \text{if } n \text{ is even.}
\end{cases}
\]
Proof. First we prove the upper bound. Let $G_1, \ldots, G_m$ be an $\mathcal{F}_S$-good family on the vertex set $[n]$. Consider the complete graph $K_m$ whose vertices are labeled with the graphs $G_1, \ldots, G_m$. For each edge $\{G_i, G_j\}$ of this graph assign an element of $h \in [n]$ for which $h$ is adjacent to all other elements of $[n]$ in the graph $G_i \oplus G_j$. By the definition of $\mathcal{F}_S$-goodness such a $h$ exists for every pair of our graphs. Now observe that if an element $a \in [n]$ is assigned to two distinct edges $e$ and $f$ of our graph $K_m$, then $e$ and $f$ must be independent edges. Indeed, if that was not the case then we would have $e = \{G_i, G_j\}, f = \{G_k, G_l\}$ for some $i, j, k \in [n]$ and $a$ would be a full-degree vertex (that is one, connected to all other vertices) in both of the graphs $G_i \oplus G_j$ and $G_k \oplus G_l$. But since $G_j \oplus G_k = (G_i \oplus G_j) \oplus (G_i \oplus G_k)$, that would mean that $a$ is an isolated vertex in $G_j \oplus G_k$, so no vertex of this latter graph can have full degree contradicting the $\mathcal{F}_S$-goodness of our family. Thus our assignment of vertices from $[n]$ to the edges of our $K_m$ partitions the edge set of $K_m$ into sets of independent edges (every partition class consisting of the edges with the same assigned label), in other words, it defines a proper edge-coloring of $K_m$. This means that the number of possible labels, which is $n$, should be at least as large as the edge-chromatic number $\chi_e(K_m)$ of $K_m$. Since the latter is $m - 1$ for even $m$ and $m$ for odd $m$, turning it around we obtain that for odd $n$ we must have $m \leq n + 1$ and for even $n$ we must have $m \leq n$.

Now we show that the upper bound we proved is sharp. First assume that $n$ is odd and consider a complete graph $K_{n+1}$ on the vertices $v_1, \ldots, v_{n+1}$ along with an optimal edge-coloring $c : E(K_{n+1}) \to [n]$ of this graph. This edge-coloring partitions $E(K_{n+1})$ into $n$ disjoint matchings $M_1, \ldots, M_n$, where $M_j$ consists of the edges colored $j$ for every $j \in [n]$. Now we construct the graphs $G_1, \ldots, G_{n+1}$ by telling for each potential edge $ij$ of the complete graph on $[n]$ which $G_k$’s will contain it and which ones will not. Consider the edge $ij$ and the union of the matchings $M_i$ and $M_j$ (note that these matchings are in the ”other” complete graph on $n+1$ vertices). This union is a bipartite graph on the vertex set $\{v_1, \ldots, v_{n+1}\}$ with two equal size partite classes $A$ and $B$. Let $ij$ be an edge of the graph $G_k$ if and only if $v_k \in A$. (So $ij$ will be an edge of exactly half of our graphs $G_1, \ldots, G_{n+1}$.) Do this similarly for all edges of $K_n$, the complete graph on vertex set $[n]$. This way we defined our $n+1$ graphs. We have to show that they form an $\mathcal{F}_S$-good family.

To this end consider two of our graphs, say $G_h$ and $G_k$. The edge $\{v_h, v_k\}$ has got some color in our coloring $c$, call this color $j$. This means that $\{v_h, v_k\}$ belongs to the matching $M_j$. We claim this means that $j \in [n]$ is a full-degree vertex of $G_h \oplus G_k$. The latter is equivalent to the statement that every edge $ji$ incident to the point $j$ appears in exactly one of the graphs $G_h$ and $G_k$. But this follows from the way we constructed our graphs: when we decided about the edge $ji$ we considered the matchings $M_i$ and $M_j$ and the bipartite graph their union defines. Since $\{v_h, v_k\} \in M_j$, the points $v_h$ and $v_k$ are always in different partite classes of this bipartite graph, so whichever was called $A$, exactly one of $v_h$ and $v_k$ belonged to it. Thus the edge $ij$ was declared to be an edge of exactly one of $G_h$ and $G_k$. Since this is so for every $i \neq j$, $j$ is indeed a full-degree vertex.
in $G_h \oplus G_k$.

Assume now that $n$ is even. Then $n-1$ is odd and we can construct graphs $G_1, \ldots, G_n$ on vertex set $[n-1] = \{1, \ldots, n-1\}$ as given in the previous paragraph. These are not yet good, however, since we have an $n$th vertex that does not appear yet in any of the graphs. Note that we have $n-1$ matchings $M_1, \ldots, M_{n-1}$ involved in the construction so far whose indices are just the first $n-1$ vertices of our graphs. Think about the additional vertex $n$ as the index of an additional “matching” $M_n$ that has no edges at all. We decide about the involvement of the edges $ni$ ($i < n$) in our graphs analogously as we did for the earlier edges: Consider the bipartite graph $M_i \cup M_n$, that consists of just the edges of $M_i$, so it is a perfect matching on the vertex set $\{v_1, \ldots, v_n\}$. Let the two partite classes defined by this perfect matching be $A$ and $B$ and add the edge $ni$ to the graph $G_h$ if and only if $v_h$ belongs to $A$. Now we can prove analogously to the odd case that the symmetric difference of any two of our graphs contains a vertex of degree $n-1$. Consider $G_h$ and $G_k$. The edge between $v_h$ and $v_k$ in the auxiliary complete graph belongs to exactly one of the matchings $M_j$ and every edge $ij$ is in exactly one of the graphs $G_h$ and $G_k$ if $i \in \{1, \ldots, j-1, j+1, \ldots, n\}$. This completes the proof. □

4 Local conditions

In the previous section we investigated $M_{\mathcal{F}}(n)$ in cases when the required symmetric differences contain specific spanning subgraphs, therefore to check whether these conditions are satisfied we have to consider our graphs on the whole vertex set. Now we turn to families $\mathcal{F}$ whose member graphs contain fixed small finite graphs, so the nature of these conditions will be local.

4.1 Containing a prescribed subgraph

Let $L$ be a fixed finite simple graph. We will now focus on the case $\mathcal{F} = \{L\}$ and to make the notation more transparent we will denote $M_{\mathcal{F}}(n)$ as $M_L(n)$ in these cases.

In the next theorem we give straightforward lower and upper bounds on the value of $M_L(n)$. The upper bound is in terms of $ex(n, L)$ that, as usually in extremal graph theory, denotes the maximum number of edges a graph on $n$ vertices can have without containing $L$ as a subgraph. The lower bound will use the following celebrated theorem due to Wilson [31].

**Wilson’s theorem.** ([31]) For every finite simple graph $L$ there exists a threshold $n_0(L)$ such that if $n > n_0(L)$ and the following two conditions hold then the edge set of the complete graph $K_n$ can be partitioned into subgraphs each of which is isomorphic to $L$. The two conditions are:

1. $\binom{n}{2}$ is divisible by $|E(L)|$;
2. $n-1$ is divisible by the greatest common divisor of the degrees of vertices in $L$. 

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Note that the two conditions in the above theorem are obviously necessary. The decomposition of $K_n$ in the conclusion of the theorem is called an $L$-design when it exists, cf. [1].

**Theorem 8.** For any fixed finite simple graph $L$

$$M_L(n) \leq 2^{\binom{n}{2} - \text{ex}(n, L)}.$$

If $n$ is sufficiently large and satisfies the conditions given in Wilson’s theorem, then we also have

$$M_L(n) \geq 2^{\binom{n}{2}/|E(L)|}.$$

**Proof.** For the upper bound consider an $n$-vertex graph $H$ satisfying $|E(H)| = \text{ex}(n, L)$ and containing no subgraph isomorphic to $L$. The family of all subgraphs of $H$ clearly satisfies the dual problem of $M_L(n)$ (no two graphs in that family can have a symmetric difference containing $L$) and the family has size $2^{\text{ex}(n, L)}$. Thus the claimed upper bound follows from Lemma 1.

For the lower bound let $n$ be larger than the value of $n_0(L)$ appearing in Wilson’s theorem and further let $n$ satisfy the two conditions therein. Then by the theorem $E(K_n)$ can be partitioned into subgraphs isomorphic to $L$. Fix such an edge decomposition of $K_n$ and let the subgraphs in it be denoted by $T_1, \ldots, T_m$, where we obviously have $m = \binom{n}{2}/|E(L)|$.

For each $I \subseteq [m]$ consider the graph $G_I$ on the vertex set $[n]$ of our $K_n$ with $E(G_I) := \bigcup_{i \in I} E(T_i)$. The symmetric difference of any two of these graphs contains at least one of the $T_i$’s, which is a graph isomorphic to $L$. So this family satisfies the requirements and its size is $2^{\binom{n}{2}/|E(L)|}$ proving the lower bound. □

Theorem 8 suggests to introduce the rate of our graph family codes in the current setting.

**Definition 2.** The rate $R_L(n)$ of an optimal graph family code for graph $L$ on $n$ vertices is defined as

$$R_L(n) := \frac{2}{n(n-1)} \log_2 M_L(n).$$

It follows immediately from the above that the value $\limsup_{n \to \infty} R_L(n)$ is strictly positive for any $L$. Wilson’s theorem also implies, however, that the limit exists. This is what we prove next.

**Theorem 9.** Let $L$ be an arbitrary fixed finite simple graph. Then the value $\lim_{n \to \infty} R_L(n)$ always exists and is bounded from below by $R_L(n)$ for every $n$.

**Proof.** Let $n$ be an arbitrary natural number and $G = \{G_1, \ldots, G_m\}$ be an optimal graph family code for $L$ with $V(G_i) = [n], i \in \{1, \ldots, m\}$, that is one with $m = M_L(n)$. By Wilson’s theorem a $K_n$-design exists for $K_N$, whenever $N$ is large enough and both $n - 1$ divides $N - 1$ and $\binom{n}{2}$ divides $\binom{N}{2}$. Take such an $N$ and consider the $K_n$-design on $K_N$
consisting of the subgraphs \(K^{(1)}, \ldots, K^{(r)}\), where \(r = \frac{N(N-1)}{n(n-1)}\) and each \(K^{(i)}\) is isomorphic to \(K_n\). Now let \(G_j := \{G^{(j)}_1, \ldots, G^{(j)}_m\}\) be an optimal graph family code for \(L\) on \(V(K^{(j)})\) for every \(j \in \{1, \ldots, r\}\). (Obviously, we can choose each \(G_j\) to be isomorphic to \(G\).)

Now define a graph family code on \(K_N\) for \(L\) as the collection of graphs that can be written in the form of \(G_a := \cup_{j=1}^r G^{(j)}_a\) where \(a = (a_1, \ldots, a_r)\) runs through all possible sequences satisfying \(a_i \in \{1, \ldots, m\}\) for every \(i\). Since there are \(m^r\) such \(a\), this way we have \(m^r\) different graphs in our family. They form indeed a graph family code for \(L\) since for any two of them, \(G_a\) and \(G_b\) there is some \(j\) for which \(a_j \neq b_j\) and thus \(G_a \oplus G_b \supseteq G_{a_j} \oplus G_{b_j} \supseteq L\). This implies \(M_L(N) \geq m^r\) and thus

\[
R_L(N) \geq \frac{2}{N(N-1)} \log_2 m^r = \frac{2}{n(n-1)} \log_2 M_L(n) = R_L(n).
\]

The requirements for \(N\) are satisfied if \(N = kn(n-1) + 1\) and \(k\) is large enough. (Also for \(N = kn(n-1) + n\) and large enough \(k\) but considering the former is enough for our argument.) Since \(M_L(n)\) is clearly monotone nondecreasing in \(n\) (as we can always ignore some vertices and consider a graph family code only on the rest), we can write that for any \(kn(n-1) + 1 \leq i \leq (k+1)n(n-1)\) we have \(M_L(i) \geq m^r\) for \(r = \binom{kn(n-1)+1}{k}/2\). Introducing the sequence \(b_i := m^r\) for \(r = \binom{kn(n-1)+1}{k}/2\) whenever \(kn(n-1) + 1 \leq i \leq (k+1)n(n-1)\) we can write

\[
\liminf_{i \to \infty} \frac{2}{i(i-1)} \log_2 M_L(i) \geq \liminf_{i \to \infty} \frac{2}{i(i-1)} \log_2 b_i \geq
\]

\[
\liminf_{k \to \infty} \frac{1}{(k+1)n(n-1)/2} \log_2 m \binom{kn(n-1)+1}{k}/2 = \liminf_{k \to \infty} \frac{2}{(k+1)n(n-1)/2} \log_2 m = R_L(n).
\]

This proves that \(\lim_{n \to \infty} R_L(n)\) exists and is equal to \(\sup_n R_L(n)\).

**Remark 5.** The above proof is similar to proving that the limit defining the Shannon capacity of graphs exists which is usually done using Fekete’s Lemma. Here, however, there are some technical subtleties (because of the divisibility requirements for \(N\)) that made it simpler to present a full proof than to refer simply to Fekete’s Lemma.

In view of Theorem 9 the following definition is meaningful.

**Definition 3.** The **distance capacity** (or **distancity for short**) of a graph \(L\) is defined as

\[
DC(L) := \lim_{n \to \infty} R_L(n).
\]
Remark 6. Note that distancity is not necessarily a new graph parameter as we cannot exclude the possibility that it might always be equal to its upper bound implied by Theorem 8. If this is indeed the case, then it should be equal to a simple function of the chromatic number as we will see in Corollary 11 below. This is so for all bipartite graphs following from classical results as we will show next. ♦

Turán’s famous theorem [29] (cf. also e.g. in [7]) gives the exact values \( ex(n, K_r) \) along with the unique \( K_r \)-free graphs of that many edges. Based on a celebrated theorem of Erdős and Stone [11], Erdős and Simonovits [10] determined the order of magnitude of \( ex(n, L) \) for any \( L \) as follows:

\[
\lim_{n \to \infty} \frac{ex(n, L)}{\binom{n}{2}} = 1 - \frac{1}{\chi(L) - 1},
\]

where \( \chi(F) \) stands for the chromatic number of graph \( F \). In particular, this formula implies \( ex(n, L) = o\left(\binom{n}{2}\right) \) whenever \( L \) is bipartite.

The well-known Gilbert-Varshamov bound [13, 30] in coding theory (stated here only in the binary case) tells us that the number of possible codewords in a largest binary code of length \( t \) and minimum distance \( d \) is at least \( \sum_{j=0}^{d-1} \binom{t}{j} \). It is also well-known (cf. e.g. Lemma 2.3 in [6]) that

\[
\left( \frac{t}{\alpha t} \right) \leq 2^{h(\alpha)},
\]

where \( h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x) \) is the binary entropy function and \( 0 \leq \alpha \leq 1 \) is meant to be such that \( \alpha t \) is an integer. Applying this for \( t = \binom{n}{2} \) we obtain that for any \( 0 \leq \alpha < \frac{1}{2} \) we can give at least \( 2^{\binom{n}{2}(1-h(\alpha))}/\alpha \binom{n}{2} \) binary sequences of length \( \binom{n}{2} \) such that the minimum Hamming distance between any two of them is at least \( \alpha \binom{n}{2} \). Considering these binary sequences as characteristic vectors of subgraphs of the \( n \)-vertex complete graph in the natural way, we get that we can give at least \( 2^{\binom{n}{2}(1-h(\alpha))}/\alpha \binom{n}{2} \) graphs any two of which has a symmetric difference containing at least \( \alpha \binom{n}{2} \) edges. Using the consequence of the Erdős-Simonovits formula mentioned above that for bipartite \( L \) we have \( ex(n, L) = o\left(\binom{n}{2}\right) \) the foregoing implies that for large enough \( n \) all these symmetric differences with at least \( \alpha \binom{n}{2} \) edges must contain a subgraph isomorphic to \( L \). Since \( \alpha > 0 \) can be chosen to be arbitrarily small and \( \lim_{x \to 0} h(x) = 0 \) there follows the next result we formulate.

**Corollary 10.** If \( L \) is any fixed bipartite graph, then

\[
DC(L) = 1.
\]

\[ \square \]
The Erdős-Simonovits formula also tells us that the upper bound Theorem 8 implies on $DC(L)$ depends only on the chromatic number of $L$. In particular, we obtain the following.

**Corollary 11.**

$$DC(L) \leq \frac{1}{\chi(L) - 1}.$$  

**Proof.** The claimed inequality follows by writing

$$DC(L) = \lim_{n \to \infty} \frac{1}{\binom{n}{2}} \log_2 M_L(n) \leq \lim_{n \to \infty} \frac{1}{\binom{n}{2}} \log_2 2^{\binom{n}{2} - \text{ex}(n, L)} = \frac{1}{\chi(L) - 1},$$

where the last equality follows from the Erdős-Simonovits formula.  

As we have seen above this inequality stands with equality in case $L$ has chromatic number $2$. In the next subsection we focus on the next case when $\chi(L) = 3$.

### 4.2 Containing a triangle or an odd cycle

In this subsection we are investigating $M_L(n)$ and $DC(L)$ for the simplest 3-chromatic graph, which is the triangle $K_3$. We will also look at the analogous notions when $K_3$, the cycle of length 3 is replaced by the family of all odd cycles.

For $L = K_3$ the bounds of Theorem 8 give us $2^{n(n-1)/6} \leq M_{K_3}(n) \leq 2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$ that implies $\frac{1}{3} \leq DC(K_3) \leq \frac{1}{2}$. Note that when Wilson’s theorem is applied for $L = K_3$ then the conclusion is that Steiner triple systems exist for all large $n$ satisfying that $n \equiv 1, 3 \pmod{6}$. In fact, it is known that Steiner triple systems exist for all such $n$ since the work of Kirkman [16] from 1847, cf. [1].

We will improve the above lower bound on $DC(K_3)$. This will be done by showing that for some small $n$ the upper bound

$$M_{K_3}(n) \leq 2^{\binom{n}{2} - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$$

is sharp. In view of Theorem 9 for $n \geq 5$ this will imply some improvement right away. The next Proposition is very simple and we present it only for the sake of completeness.

**Proposition 12.** We have $M_{K_3}(3) = 2$ and $M_{K_3}(4) = 4$.

**Proof.** For $n = 3$ the statement is trivial: take the empty graph and a triangle on three vertices, this 2-element family already achieves the value of the upper bound which is 2 for $n = 3$.

For $n = 4$ we give the following four graphs on the vertex set $\{1, 2, 3, 4\}$ by their edges sets. Let

$$E(G_0) = \emptyset, E(G_1) = \{12, 23, 13\}, E(G_2) = \{23, 34, 24\}, E(G_3) = \{12, 13, 24, 14\}.$$
It takes an easy checking that the symmetric difference of any two of these graphs contains a triangle. Since the upper bound in Theorem 8 is also 4 in this case, this proves that $M_{K_3}(4) = 4$. □

**Remark 7.** Note that both of the above simple constructions are closed under the symmetric difference operation, that is they form a linear space over $GF(2)$ when the graphs are represented by the characteristic vectors of their edge sets. In fact, the second construction could also be presented as the vector space generated in this sense by any two of the graphs $G_1, G_2, G_3$. ♦

**Lemma 13.**

$$M_{K_3}(5) = 16.$$ 

*Proof.* The value of the upper bound in the inequality

$$M_{K_3}(n) \leq 2^{\binom{n}{2}} - \lceil \frac{n}{4} \rceil \lfloor \frac{n}{2} \rfloor$$

already stated above gives 16 for $n = 5$, so we only have to prove that 16 is also a lower bound. To this end we will give a set of graphs forming a vector space in the sense of Remark 7. We will give this vector space by a set of generators, although in a somewhat redundant way. (Our reason to keep this redundancy is that the construction has more symmetry this way.)

Think about the vertices $\{1, 2, 3, 4, 5\}$ as if they were given on a circle at the vertices of a regular pentagon in their natural order. Consider the graph with edge set

$$E(G_1) := \{12, 23, 35\}.$$

Let $G_2, G_3, G_4, G_5$ be the four graphs we obtain from $G_1$ by rotating it along the circle containing the vertices so that vertex 1 moves to 2, 2 to 3, etc. Thus we have

$$E(G_2) = \{23, 34, 24, 41\}, E(G_3) = \{34, 45, 35, 52\},$$

$$E(G_4) = \{45, 51, 41, 13\}, E(G_5) = \{51, 12, 52, 24\}.$$ 

Now we consider the linear space the characteristic vectors of the edges sets of these five graphs $G_i, i \in \{1, 2, 3, 4, 5\}$ generate. These graphs can be defined as the elements of the family $\mathcal{G} = \{G_I : I \subseteq [5]\}$, where

$$G_I = \oplus_{i \in I} G_i,$$

meaning that $V(G_I) = [5]$ and $E(G_I)$ contains exactly those edges that appear in an odd number of the graphs $G_i$ with $i \in I$.

Note that every edge of the underlying $K_5$ on [5] appears in exactly two of the graphs $G_1, \ldots, G_5$, therefore for $I = [5]$ we have that $G_I$ is the empty graph just as for $G_\emptyset$. 

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This implies that for every \( I \subseteq [5] \) and \( \overline{I} := [5] \setminus I \) we have \( G_I = G_{\overline{I}} \), thus every graph in our graph family has exactly two representations as \( G_I \) for some \( I \subseteq [5] \). (The two representations are given by \( I \) and \( \overline{I} \) as we have seen. It also follows that if \( J \neq I, \overline{I} \) then \( G_J \neq G_I \), otherwise we would have \( G_{J \cup I} \) be the empty graph for \( J \cap I \notin \{\emptyset, [5]\} \) contradicting that every edge appears exactly twice in the sets \( E(G_i), i = 1, \ldots, 5 \).) Thus we have indeed \( \frac{1}{2}2^5 = 16 \) graphs in our family matching our upper bound for \( n = 4 \).

We have to show that the symmetric difference of any two of our graphs contains a triangle. Since our construction is closed for the symmetric difference operation this is equivalent to say that all graphs in our family except the empty graph contains a triangle. Since \( G_I = G_{\overline{I}} \) it is enough to prove that \( G_I \) contains a triangle for all \( 1 \leq |I| \leq 2, I \subseteq [5] \). This is easy to see when \( |I| = 1 \). For subsets with \( |I| = 2 \) it is enough to check this for \( I = \{1, 2\} \) and \( I = \{1, 3\} \) by the rotational symmetry of our construction. But these two cases are easy to check: \( G_{\{1, 2\}} \) contains the triangles on the triples of vertices 1, 2, 4 and 1, 3, 4, while \( G_{\{1, 3\}} \) contains the triangle on vertices 1, 2, 3. \( \square \)

The previous lemma already implies that the natural construction using Steiner triple systems is not asymptotically optimal.

**Theorem 14.**

\[
\frac{5}{2} \leq DC(K_3) \leq \frac{1}{2}.
\]

**Proof.** The upper bound is simply what we get from Corollary [11] Lemma [13] gives us \( R_{K_3}(5) = \frac{1}{10} \log_2 16 = \frac{2}{5} \) and thus the lower bound follows from Theorem [9]. \( \square \)

Note that Lemma [13] also implies that \( R_{K_3}(n) \geq \frac{2}{5} \) whenever \( n \equiv 1, 5 \mod 20 \). Here we do not even have to say “for large enough \( n \)” due to the result of Hanani [14] according to which \( K_5 \)-designs exist for all such \( n \) (cf. also [1]). Although the lower bound stated in Theorem [14] for \( DC(K_3) \) is the best we were able to prove so far, the following lemma can at least be used to show that the numbers for which we can state \( R_{K_3}(n) \geq \frac{2}{5} \) are significantly more frequent. It also makes the possible conjecture that the upper bound in Theorem [8] might perhaps be tight a little (though only a little) less farfetched.

**Lemma 15.**

\( M_{K_3}(6) = 64 \).

**Proof.** The value of the upper bound implied by Theorem [8] is \( 2^6 \) for \( L = K_3 \) and \( n = 6 \), so we need to prove only the lower bound.

To this end we give a construction of 64 graphs forming a graph family code on [6] for \( K_3 \). The construction will have several similarities to that in Lemma [13] though with somewhat less symmetry. But again our graphs will form a vector space in the sense of Remark 7 to be specified through a set of seven generators that altogether cover each one of the edges of the underlying \( K_6 \) exactly twice, so every member of our graph family
will have exactly two representations by the generators just as in the proof of Lemma 13. Here are the details.

Think about the 6 vertices 1, . . . , 6 as being on a circle in the vertices of a regular hexagon in their natural order as we go around the circle. Our first four generator graphs are the following four edge-disjoint triangles (plus three isolated points) given by their edge sets as follows.

\[ E(G_1) = \{12, 23, 13\}, E(G_2) = \{34, 45, 35\}, E(G_3) = \{56, 16, 15\}, E(G_4) = \{24, 46, 26\}. \]

The other three graphs are three \(K_4\)'s (plus two isolated vertices) that are rotations of each other, in particular,

\[ E(G_5) = \{12, 24, 45, 15, 14, 25\}, E(G_6) = \{23, 35, 56, 25, 36\}, \]
\[ E(G_7) = \{34, 46, 16, 13, 36, 14\}. \]

It is easy to check that the above seven graphs cover each edge of the underlying \(K_6\) exactly twice. Just as in the proof of Lemma 13 this implies that the generated family of graphs of the form

\[ G_I = \oplus_{i \in I} G_i \]

where \(I\) runs through all subsets of \([7]\) contains exactly two representations of this form for each of its members, namely

\[ G_I = G_J \] if and only if \(J = [7] \setminus I\).

Thus our family has \(2^6 = 64\) members that matches our upper bound. Now we have to show that the symmetric difference of every pair of our graphs contains a triangle. Since the family is closed under symmetric difference this is equivalent to every \(G_I\) except \(G_\emptyset\) containing a triangle. To show this we consider the representation of each of our graphs as \(G_I\) where \(I\) contains at most one of the three \(K_4\) generators, that is \(|I \cap \{5, 6, 7\}| \leq 1\). When \(I \cap \{5, 6, 7\} = \emptyset\) but \(I\) itself is nonempty then this is trivial as in such a case \(G_I\) is the union of some of the edge-disjoint graphs \(G_1, \ldots, G_4\) each of which is a triangle itself. In case \(|I \cap \{5, 6, 7\}| = 1\), then by symmetry we may assume w.l.o.g. that \(I \cap \{5, 6, 7\} = \{5\}\). Then if we also have \(\{1, 2\} \subseteq I\) then the triangles on vertices 1, 3, 4 and 2, 3, 5 (and two more) will be contained in \(G_I\). So we may assume that at least one of \(G_1\) and \(G_2\) is not part of our representation of \(G_I\) and by symmetry, we may assume \(2 \notin I\). But then to avoid the triangles on vertices 1, 4, 5 and 2, 4, 5 being in \(G_I\) we need both \(3 \in I\) and \(4 \in I\). In this case, however, we will have the triangle on vertices 4, 5, 6 present in \(G_I\). This completes the proof. □

Since \(\frac{1}{15}\log_2 64 = \frac{1}{10}\log_2 16 = \frac{2}{5}\) Lemmas 13 and 15 together imply that \(R_{K_3}(n) \geq \frac{2}{5}\) whenever a \(K_5\)-design or a \(K_6\)-design on \(K_n\) exists. The former is equivalent to \(n \equiv 1, 5 \mod 20\) while for the latter it is sufficient if \(n \equiv 1, 6 \mod 15\) and \(n > 801\), cf. [11]. So for large enough \(n\) we have 13 numbers out of every 60 consecutive integers for which we
can prove $R_{K_3}(n) \geq \frac{5}{2}$. Using, however, the following strengthening of Wilson’s theorem due to Caro and Yuster \cite{CaroYuster}, we can show this for every large enough $n$ for which $\binom{n}{2}$ is divisible by 5.

**Caro-Yuster theorem.** (\cite{CaroYuster}) Let $\{H_1, \ldots, H_r\}$ be a family of graphs having the common gcd property, meaning that the greatest common divisor of the vertex degrees in $H_i$ is the same number $d$ for every $i$. Then there exists an $n_0$ such that if $n > n_0$ and $n - 1$ is divisible by $d$, then for any equality of the form $\alpha_1|E(H_1)| + \cdots + \alpha_r|E(H_r)| = \binom{n}{2}$ with nonnegative integers $\alpha_i, i = 1, \ldots, r$, the edge set of $K_n$ can be partitioned into edge-disjoint copies of the graphs $H_i$ such that for every $i$ exactly $\alpha_i$ copies of $H_i$ appear in the partition.

**Theorem 16.** If $n$ is large enough and satisfies $n \equiv 0$ or 1 (mod 5), then

$$R_{K_3}(n) \geq \frac{5}{2}.$$  

*Proof.* Let $H_1$ be the edge disjoint union of two copies of $K_5$ and one copy of $K_6$, while $H_2$ be the edge-disjoint union of one copy of $K_5$ and two copies of $K_6$. (Whether these copies are also vertex-disjoint or not will not matter, just fix them in some way.) Then clearly the greatest common divisor of the degrees of both $H_1$ and $H_2$ is 1, so the family $\{H_1, H_2\}$ possesses the common gcd property. Since $|E(H_1)| = 35, |E(H_2)| = 40$ and every large enough number (in fact, every number larger than 205) that is divisible by 5 can be written in the form of $35k + 40\ell$ for appropriate nonnegative integers $k$ and $\ell$, the Caro-Yuster theorem implies that if $\binom{n}{2}$ is divisible by 5 for some large enough $n$, then $K_n$ can be decomposed into an appropriate number of copies of $H_1$ and $H_2$. As both $H_1$ and $H_2$ are created as an edge-disjoint union of $K_5$’s and $K_6$’s this gives a decomposition of $K_n$ into copies of $K_5$’s and $K_6$’s. Then using the constructions of Lemma 13 and Lemma 15 on these copies of $K_5$’s and $K_6$’s, respectively, the way we used small constructions in the proof of Theorem 9 the theorem follows. □

Let $C_{\text{odd}}$ be the set of all odd cycles. Since a complete bipartite graph contains no odd cycle, the upper bound $2\binom{n}{2} - \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ proven for $M_{K_3}(n)$ is also valid for $M_{C_{\text{odd}}}(n)$. Since $K_3 \cong C_3$ is an odd cycle, we obviously have $M_{K_3}(n) \leq M_{C_{\text{odd}}}(n)$ and so by Proposition 12 and Lemmas 13 and 15 the previous upper bound is also sharp for $M_{C_{\text{odd}}}(n)$ for $n \in \{3, 4, 5, 6\}$. The next value $n = 7$ is the first one for which a nontrivial Steiner triple system exists. Although we could not prove that $M_{K_3}(7)$ is also equal to its mentioned upper bound in this case, we can show this at least for $M_{C_{\text{odd}}}(n)$. As a by-product we also show that the largest $\{K_3\}$-good family on 7 vertices is strictly larger than $2^7$, so the natural construction based on Steiner triple systems does not only asymptotically fail to be best possible but does so already for the smallest value of $n$ for which a nontrivial Steiner triple system exists.
Lemma 17. 

\[ M_{\text{odd}}(7) = 2^9 \]

and 

\[ M_{K_3}(7) \geq 2^8. \]

Proof. The upper bound \[2^{\binom{n}{2}} - \left\lceil \frac{n}{2} \right\rceil \lfloor \frac{n}{2} \rfloor\] is equal to \(2^9\), so for the first statement it is enough to prove that this is also a lower bound. This we do similarly as in the proofs of Lemmas 13 and 15.

Again, we think about the seven vertices forming the set \([7]\) as the vertices of a regular 7-gon around a cycle in their natural order. We define \(7+3 = 10\) simple graphs \(G_1, \ldots, G_7\) and \(G_8, \ldots, G_{10}\) that will generate our family. Let \(G_1\) be the triangle with edges 12, 24, 14 and \(G_2, \ldots, G_7\) be its six possible rotated versions, that is the triangles with edge sets \(\{23, 35, 25\}, \{34, 46, 36\}, \ldots, \{17, 13, 37\}\), respectively. Note that these seven triangles cover all pairs of vertices exactly once, that is, they form a Steiner triple system. The three other graphs \(G_8, G_9, G_{10}\) are three edge-disjoint seven-cycles, namely those with edge sets

\[\{12, 23, 34, 45, 56, 67, 17\}, \{13, 35, 57, 27, 24, 46, 16\}, \{14, 47, 37, 36, 26, 25, 15\},\]

respectively. Note that these three graphs also cover all pairs of vertices exactly once and that the edge sets of a \(G_i\) for \(i \in [7]\) and \(G_j\) with \(j \in \{8, 9, 10\}\) intersect in exactly one element. Since our ten graphs cover the edges of the underlying \(K_7\) exactly twice, just as in the proofs of Lemmas 13 and 15 the generated family

\[\oplus_{i \in I} G_i\]

as \(I\) runs over all subsets of \(\{1, \ldots, 10\}\) will have exactly \(2^9\) distinct members each of which is represented by two subsets of \(\{1, \ldots, 10\}\), some \(I\) and its complement. All we are left to show for proving \(M_{\text{odd}}(7) \geq 2^9\) is that each such \(G_I\) except \(G_\emptyset\) contains an odd cycle. If \(I \subseteq [7]\), this is obvious and so is also if \(I \subseteq \{8, 9, 10\}\). When both \(I \cap [7]\) and \(I \cap \{8, 9, 10\}\) are nonempty, then we consider that representation \(G_I\) which has \(|I \cap [7]| \leq 3\). If we have \(|I \cap \{8, 9, 10\}| = 1\) then whichever 7-cycle we have (that is, whichever of \(G_8, G_9, G_{10}\)) it will have two consecutive edges that do not appear in either of the at most three triangles. If we take the first pair of such edges (as we go along our 7-cycle in an appropriate direction) for which the previous one is an edge of one of our triangles (since we take at least one triangle and each triangle intersects each 7-cycle, such an edge must exist), then the construction ensures that these two consecutive edges close up to a \(K_3\) in our \(G_I\). In case we have two 7-cycles in our \(G_I\) representation, then those create 7 distinct \(K_3\)'s in their union. Each of our triangles intersect exactly three of those seven \(K_3\)'s created, so if we have \(|I \cap [7]| \leq 2\) then at least one of these seven \(K_3\)'s remain untouched. Thus we are left with the case of two 7-cycles and exactly three triangles. For this case let us switch to the complementary representation with four triangles and one 7-cycle. By symmetry, we may assume that our 7-cycle is \(G_8\). If the four triangles
are such that two consecutive edges of \( G_8 \) do not appear in any of them then we can finish the argument as before. If this is not the case, then the four triangles must leave three such edges of \( G_8 \) uncovered which form a matching. Because of symmetry we may assume that these are the edges 12, 34, 56. This also tells us exactly which are the four triangles we have in the representation of \( G_I \), namely those that contain the remaining four edges, that is, \( G_2, G_4, G_6 \) and \( G_7 \). In this case \( G_I \) contains the \( K_3 \), for example, on the vertices 2, 5, 6. Finally, if we have all the three 7-cycles in our representation then the complementary representation has no 7-cycle at all and this case we have already covered. This completes the proof of \( MC_{\text{odd}}(7) = 2^9 \).

To show \( MK_3(7) \geq 2^8 \) first note that the only way we did not find a triangle but only a 7-cycle in one of our \( G_I \) was when \( |I \cap [7]| = 0 \) and \( |I \cap \{8, 9, 10\}| = 1 \). We avoid any symmetric differences resulting in such a graph if we take only those \( G_I \) for which \( |I \cap [7]| \leq 3 \) (this ensures that no complementary sets of our triangles are taken) and \( |I \cap \{8, 9, 10\}| \) is even. This gives us a family of \( 2^8 \) graphs such that the symmetric difference of any two of them contains a \( K_3 \) proving the second statement. □

We may define \( R_{C_{\text{odd}}}(n) = \frac{2}{n(n-1)} \log_2 MC_{\text{odd}}(n) \) and \( DC(C_{\text{odd}}) = \lim_{n \to \infty} R_{C_{\text{odd}}}(n) \) analogously to \( R_L(n) \) and \( DC(L) \). (The existence of the limit defining \( DC(C_{\text{odd}}) \) follows essentially the same way as for \( DC(L) \).) We clearly have \( DC(C_{\text{odd}}) \geq DC(K_3) \) and using that our upper bound is sharp on \( MC_{\text{odd}}(n) \) even in the case of \( n = 7 \), we have a better lower bound for \( DC(C_{\text{odd}}) \) then the one proven in Theorem 14 for \( DC(K_3) \).

**Theorem 18.**

\[
\frac{3}{7} \leq DC(C_{\text{odd}}) \leq \frac{1}{2}
\]

**Proof.** The upper bound is obtained simply by realizing that that the upper bound of Corollary 11 applies as \( K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} \) does not contain any odd cycle. Lemma 17 gives us \( R_{C_{\text{odd}}}(7) = \frac{1}{21} \log_2 2^9 = \frac{3}{7} \) and so the lower bound follows analogously as in the proof of Theorem 9. □

5 Open problems

In this final section we collect some of the many problems in our setting that remained open.

**Problem 1.** Is it true that if \( MF(n) \) is at least exponential in \( n \), then it is always achieved by a linear graph family code, that is one that is closed under the symmetric difference operation?

The construction in the proof of Theorem 2 has the property that for any two of its graphs \( G \) and \( G' \) with an equal number of edges (that is trivially necessary for satisfying the condition in the next problem) their two asymmetric differences

\[
G \setminus G' = ([n], E(G) \setminus E(G')) \quad \text{and} \quad G' \setminus G = ([n], E(G') \setminus E(G))
\]
are isomorphic. This suggests the following question.

**Problem 2.** What is the maximum possible size of a graph family $A$ of graphs on $n$ vertices satisfying that if $A, A' \in A$ then $A \setminus A'$ and $A' \setminus A$ are isomorphic?

Such a family that is larger than the one in the proof of Theorem 2 can be given by taking $\lfloor n/k \rfloor$ vertex-disjoint stars, each on $k$ vertices. This gives a lower bound that is superexponential in $n$ but we do not have any nontrivial upper bound.

The only graph family code proven to be optimal and nonlinear (or not the coset of a linear code) in this paper is that related to the containment of a spanning star in Theorem 7. This was also the only case when the upper bound was proven without the use of Lemma 1. This suggests the question of what could be said about the dual problem in this case. A natural construction for the dual problem considers a subgraph of $K_n$ with the minimum number of edges $\lceil n/2 \rceil$ such that no vertex is isolated and takes all possible subgraphs of the complement. This construction gives $D_{\bar{\mathcal{F}}_S}(n) \geq 2^\left(\binom{n}{2} - \lceil n/2 \rceil \right)$.

**Problem 3.** What is the value of $D_{\bar{\mathcal{F}}_S}(n)$? Is the lower bound $2^\left(\binom{n}{2} - \lceil n/2 \rceil \right)$ sharp?

Theorems 6 and 7 show a huge difference between requiring a spanning path or a spanning star in the symmetric differences. One may wonder what happens “in between”. Note that if we formulate this “in betweenness” so that we want to have a spanning tree with diameter at most $k$, then while with $k = 2$ we are at Theorem 4 and with $k = n - 1$ at Theorem 5, already for $k = 3$ we get the same result as for $k = n - 1$ by the construction in the proof of Theorem 2. (This is simply because complete bipartite graphs contain spanning trees of diameter at most 3.) So it seems plausible to formulate questions in terms of more specific “natural” sequences of spanning trees $T_1, T_2, \ldots$. (In the problem below the notation $M_{T_n}(n)$ is meant to denote the largest possible cardinality of a family of graphs on vertex set $[n]$ such that the symmetric difference of any two of them contains $T_n$ as a subgraph.)

**Problem 4.** For what “natural” sequences $T_1, T_2, \ldots, T_i, \ldots$ of trees (with $T_i$ having exactly $i$ vertices for every $i$) will the value of $M_{\mathcal{F}_T}(n)$ grow only linearly in $n$?

There are several open problems related to the value of $DC(L)$ for various $L$. The following ones are perhaps those for which our lack of knowledge is the most frustrating.

**Problem 5.** Does there exist a graph $L$ for which

$$DC(L) < \frac{1}{\chi(L) - 1}$$
Problem 6. Is
\[ DC(K_3) = \frac{1}{2} ? \]

Even more strongly, does perhaps \( M_{K_3}(n) = 2^{\binom{n}{2} - \lceil \frac{n^2}{2} \rceil \lfloor \frac{n^2}{2} \rfloor} \) always hold?

The paper [8] proving the Simonovits-Sós conjecture mentioned in the Introduction also shows that requiring only an arbitrary odd cycle (instead of insisting on a triangle) in the intersection of any two graphs results in the same extremal family. The result of Kovács and Soltész in [21], answering a question of [19], is that the size of a largest family of Hamiltonian paths of \( K_n \), the union of any two of which contains a triangle is the same what we get if we require only an arbitrary odd cycle in such unions. Our last problem asks whether the analogous statement is true also in our setting at least in the following relaxed form.

Problem 7. Is
\[ DC(K_3) = DC(C_{\text{odd}}) ? \]

UPDATE:

In joint work with Noga Alon and Aleksa Milojevic we have just solved Problems 5, 6 and 7 showing that equality holds in all three cases, and also settled Problem 3 for even values of \( n \). This will appear soon in a manuscript of all five of us.

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