GENERAL SCALAR EXCHANGE IN AdS_{d+1}*

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ABSTRACT

The scalar field exchange diagram for the correlation function of four scalar operators is evaluated in anti-de Sitter space, AdS_{d+1}. The conformal dimensions Δ_i, i = 1, . . . , 4 of the scalar operators and the dimension Δ of the exchanged field are arbitrary, constrained only to obey the unitarity bound. Techniques similar to those developed earlier for gauge boson exchange are used, but results are generally more complicated. However, for integer Δ_i, Δ, the amplitude can be presented as a multiple derivative of a simple universal function. Results simplify if further conditions hold, such as the inequalities, Δ < Δ₁ + Δ₃ or Δ < Δ₂ + Δ₄. These conditions are satisfied, with < replaced by ≤, in Type IIB supergravity on AdS₅ × S₅ because of selection rules from SO(6) symmetry. A new form of interaction is suggested for the marginal case of the inequalities. The short distance asymptotics of the amplitudes are studied. In the direct channel the leading singular term agrees with the double operator product expansion. Logarithmic singularities occur at sub-leading order in the direct channel but at leading order in the crossed channel. When the inequalities above are violated, there are also (log)^2 singularities in the direct channel.

* Research supported in part by the National Science Foundation under grants PHY-95-31023 and PHY-97-22072.
I. INTRODUCTION

Many correlation functions have been evaluated in the study of the AdS/CFT correspondence [1,2,3]. The 2- and 3-point functions appear to have non-renormalization properties [4,5] which, although remarkable, suggest that there is little dynamical content at this level. On the other hand, 4-point functions are intrinsically more complex in a conformal field theory and are thus expected to contain more information about the dynamics of the AdS/CFT correspondence. Studies of their structure have begun both from the AdS side [6,7,8,9,10,11,12,13,14] and from the viewpoint of the $N = 4$ super-Yang-Mills boundary theory [14,15,16].

A first question is whether the CFT has a simple $t$-channel OPE, so that the 4-point correlation function admits a convergent power expansion of the form [10]

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_p \frac{\gamma_{13p}}{(x_1 - x_3)^{\Delta_1 + \Delta_3 - \Delta_p}} \frac{\gamma_{24p}}{(x_2 - x_4)^{\Delta_2 + \Delta_4 - \Delta_p}}$$

containing the contribution of a finite number of primary operators (and descendents). Explicit calculations of parts of these correlation functions involving contact interactions [9] or gauge boson exchange [11], show that logarithmic singularities occur as well as the expected leading powers. Thus far, no full correlator of chiral primary operators has been computed, so it is not yet known whether logarithms will cancel or survive when all contributions are assembled. Logarithms need not be inconsistent with (1.1) and it as recently conjectured [17] that they are the effects of anomalous dimensions of double trace operators in the boundary theory.

A second question is whether the 4-point function, presumably upon including full towers of string excitation states, exhibits crossing symmetry, i.e. duality. Crossing symmetry holds universally for the 4-point functions of CFT in 2 space-time dimensions, and plays a crucial role there. To investigate this issue, one needs to be able to study the OPE for exchange states of arbitrary masses and spins which are not necessarily chiral primary or descendents thereof, but may be non-chiral primary.

In the present paper, we consider the correlation function of 4 scalar operators $O_{\Delta_i}$ of arbitrary scaling dimension $\Delta_i$, $i = 1, \cdots, 4$, and study the corresponding Witten diagram in which a scalar field of arbitrary dimension $\Delta$ is exchanged. (See Fig. 1.) We assume throughout that all scaling dimensions obey the unitarity bound $\Delta_i \geq d/2$ and $\Delta \geq d/2$. 
We generalize the expansion and resummation techniques developed in [11] to evaluate the integrals over the interaction points $z$ and $w$ in Fig. 1.

For general integer values of $d$, $\Delta_1$, and $\Delta$, the amplitude is more complicated than in [11], and is expressed as a double infinite series. If the combination

$$\delta = \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)$$

is an integer, then one can introduce a simple universal function $I(\tau; s, t)$ of a parameter $\tau$ and two conformal invariant combinations $s$ and $t$ of the four points $x_i$, but independent of $\Delta$, $\Delta_i$ and $d$. The correlator can then be expressed as a multiple derivative with respect to $x_i$ of the integral in $\tau$ of a product of $I(\tau; s, t)$ and a “spectral weight function” $\rho(\tau)$, which in turn is defined by the double infinite series above.

If one further condition is satisfied, namely that $\delta - \Delta_3 - d/2$ is a non-negative integer, then one of the infinite series above truncates to a finite sum. If the second condition $\Delta < \Delta_1 + \Delta_3$ holds, then the second infinite series also truncates to a finite sum. These are very significant simplifications, and it is quite remarkable to observe that all three conditions above hold in the application of the AdS/CFT correspondence of greatest current interest, namely Type IIB superstring theory on $AdS_5 \times S_5$, as a consequence of the $SO(6) \sim SU(4)/Z_2$ symmetry of the sphere $S_5$.

These truncations take place because the Kaluza-Klein scalar fields on $AdS_5 \times S_5$ involve [18,4] $S_5$ scalar spherical harmonics $Y^k$ (and generalizations) where $k$ denotes a $k$-fold product of the vector 6-dimensional representation of $SO(6)$. One may define the parity of $Y^k$ to be $(-)^k$, a quantity related to the $SU(4)$ quadrality of the representation. Invariant interactions necessarily involve the contraction of an even number of vector indices, so the net parity of each vertex in the diagram of Fig. 1 must be even. It is also the case [18] that scale dimensions are linearly related to $k$ such that $(-)^{\Delta} = (-)^k$ for every scalar field. The parity selection rule then guarantees that $\Delta + \Delta_1 + \Delta_3$ and $\Delta + \Delta_2 + \Delta_4$ are both even, so the first condition of the previous paragraph automatically holds (since $d = 4$).

The basic reason why an inequality of the form $\Delta < \Delta_1 + \Delta_3$ might hold in the $AdS_5 \times S_5$ theory is that the tensor product $Y^{k_1}Y^{k_3}$ of two harmonics contains only representations which satisfy $|k_1 - k_3| \leq k \leq k_1 + k_3$. To use this information in a more precise way, we note that each infinite Kaluza-Klein tower of scalar fields in [18] is characterized by a non-negative integer $s$, such that $\Delta = 2s + k$ for all fields in the tower. The cases $s = 0, 1, 2, 3$ occur, and $s = 0$ holds for the Kaluza-Klein tower corresponding
to the chiral primary operators $\text{tr}X^k$ of the boundary super Yang-Mills theory. For this family of scalars, the Clebsch-Gordon limit above immediately gives $\Delta \leq \Delta_1 + \Delta_3$. (The marginal case requires special discussion; see Appendix 2.) For cubic vertices involving other Kaluza-Klein families a precise statement requires further study.

In Sec. 2 below, we discuss the propagators and vertices which are the ingredients of the diagram of Fig. 1. In Sec. 3, we define the amplitudes precisely and carry out the $z$- and $w$-integrals of Fig. 1, and derive a two-dimensional parametric integral representation of the amplitude for general $\Delta_i$, $\Delta$ and $d$. In Sec. 4, we discuss the simplifications that take place when special conditions on $\Delta_i$, $\Delta$ and $d$ are satisfied and we show how to present the amplitude in terms of a single parametric integral with a universal function. In Sec. 5, we first apply a simple method to obtain the leading singularity in the direct channel, as $x_2 - x_4 \to 0$ and then use general methods to derive the full asymptotic behavior of the amplitude as $x_2 - x_4 \to 0$ and $x_2 - x_3 \to 0$. In Sec. 6, we present brief conclusions. In the Appendix we propose a new cubic interaction for fields satisfying $\Delta = \Delta_1 + \Delta_3$.

![Figure 1](image)

2. THE SCALAR PROPAGATOR

We work on the Euclidean continuation of AdS$_{d+1}$ viewed as the upper half space in $z_\mu \in \mathbb{R}^{d+1}$, with $z_0 > 0$, and metric $g_{\mu\nu}$ of constant negative curvature $R = -d(d + 1)$, given by

$$ds^2 = \sum_{\mu, \nu=0}^{d} g_{\mu\nu}dz_\mu dz_\nu = \frac{1}{z_0^2}(dz_0^2 + \sum_{i=1}^{d} dz_i^2). \quad (2.1)$$

It is well-known that AdS-invariant functions, such as scalar propagators, are simply expressed [19] as functions of (the chordal distance) $u$, defined by

$$u = \frac{(z - w)^2}{2z_0 w_0}, \quad (z - w)^2 = \delta_{\mu\nu}(z - w)_\mu(z - w)_\nu \quad (2.2)$$
We shall consider a general field theory of real scalars $\Phi_\Delta$ with masses

$$m^2 = \Delta(\Delta - d), \quad (2.3)$$

which are sources for the boundary operators $\mathcal{O}_\Delta(x)$ of dimensions $\Delta$ obeying the unitarity bound $\Delta \geq d/2$. For the sake of simplicity, we shall assume that the couplings of the scalar fields are non-derivative and trilinear. Amplitudes with derivative couplings may be reduced to amplitudes with non-derivative couplings only, as shown in [8,9]. Besides trilinear couplings, only quadrilinear couplings might be required in evaluating the four point function, and contact contributions with these interactions were evaluated in [9].

The action then takes the form

$$S = \sum_\Delta \int d^{d+1}z \sqrt{g} \left[ \Phi_\Delta(-\Box_g + m^2)\Phi_\Delta \right] + \sum_{\Delta, \Delta', \Delta''} \gamma_{\Delta\Delta'\Delta''} \Phi_\Delta\Phi_{\Delta'}\Phi_{\Delta''}, \quad (2.4)$$

where $\Box_g = D^\mu D_\mu$ is the scalar Laplacian with metric $g$ on AdS, and $\gamma_{\Delta\Delta'\Delta''}$ are trilinear couplings, which we set to unity.

The scalar bulk to boundary propagator for dimension $\Delta$ is given by [3]

$$K_\Delta(z_0, \vec{z}, \vec{x}) = C_\Delta \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \quad (2.5)$$

with the following normalizations for the constant prefactors [20],

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - d/2)} \quad \text{for} \quad \Delta > d/2, \quad C_{d/2} = \frac{\Gamma(d/2)}{2\pi^{d/2}}. \quad (2.6)$$

The scalar bulk-to-bulk propagator for dimension $\Delta$ was obtained in [19],

$$G_\Delta(u) = \tilde{C}_\Delta(2u^{-1})^\Delta \frac{\Gamma(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}; 2\Delta - d + 1; -2u^{-1})}{\Gamma(\Delta) \Gamma(\Delta - d/2 - \frac{1}{2})} \quad (2.7)$$

where $\tilde{C}_\Delta = \frac{\Gamma(\Delta) \Gamma(\Delta - d/2 - \frac{1}{2})}{(4\pi)^{(d+1)/2} \Gamma(2\Delta - d + 1)}$.

Here, $F$ is the standard hypergeometric function $\text{}_2F_1$. It will be very convenient for later use to make a quadratic transformation of the hypergeometric function [21], and to recast it in terms of the variable

$$\xi = \frac{1}{1 + u} = \frac{2z_0w_0}{(z_0^2 + w_0^2 + (\vec{z} - \vec{w})^2)}. \quad (2.8)$$
We obtain
\[
G_\Delta(u) = 2^\Delta \tilde{C}_\Delta \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \xi^2 \right) .
\]
(2.9)

Since from (2.8) it is clear that \(|\xi| \leq 1\), it follows immediately that the form (2.9) of the scalar propagator will have a uniformly convergent expansion in powers of \(\xi\). Thus, the summation of the series expansion of \(F\) in powers of \(\xi\) and any convergent integration of \(G(u)\) may be freely interchanged. This property will be very useful later on.

3. SCALAR EXCHANGE INTEGRALS

We factor out the normalization constants and define the amplitude to be studied by
\[
\langle O_{\Delta_1}(x_1)O_{\Delta_2}(x_2)O_{\Delta_3}(x_3)O_{\Delta_4}(x_4) \rangle = 2^\Delta \tilde{C}_\Delta C_{\Delta_1}C_{\Delta_2}C_{\Delta_3}C_{\Delta_4}A(x_1, x_2, x_3, x_4) \quad (3.1a)
\]
with
\[
A(x_i) = \int \frac{d^{d+1}z}{z_0^{d+1}} \int \frac{d^{d+1}w}{w_0^{d+1}} \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \xi^2 \right) \left(\frac{z_0}{(z-x_1)^2}\right)^{\Delta_1} \left(\frac{w_0}{(w-x_2)^2}\right)^{\Delta_2} \left(\frac{w_0}{(w-x_3)^2}\right)^{\Delta_3} \left(\frac{w_0}{(w-x_4)^2}\right)^{\Delta_4} \quad (3.1b)
\]

The convergence conditions of this integral, for \(x_i\) fixed and separated from one another, are easily obtained by inspection. First, assuming the unitarity bound on all dimensions \(\Delta, \Delta_i \geq \frac{d}{2}\), it is clear that the only divergences arise from when one or two interaction points approach one of the boundary points \(x_i\). Convergence when one of the interaction points approaches any one of the boundary points is guaranteed by the conditions
\[
|\Delta_1 - \Delta_3| < \Delta \quad |\Delta_2 - \Delta_4| < \Delta , \quad (3.2a)
\]
while convergence when both interactions points approach any one of the boundary points requires
\[
2\Delta_i < \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 , \quad i = 1, 2, 3, 4 . \quad (3.2b)
\]
Conditions (3.2a) are familiar from the case of the three point functions [20]. Throughout, we shall assume that the above convergence conditions hold.

The first step [9,11,20] is to simplify the integral by setting \(x_1 = 0\), and by changing integration variables using the inversion isometry of AdS, namely \(z_\mu = z_\mu'/(z')^2\) and \(w_\mu = w_\mu'/(w')^2\), with boundary points \(x_2, x_3, x_4\) referred to their inverses by \(x_i = x_i'/(x_i')^2\). Once this inversion is carried out and \(x_1' = \infty\), the reduced function is invariant under
simultaneous translations of \( x'_2, x'_3 \) and \( x'_4 \), and thus only depends upon \( x \equiv x'_4 - x'_3 \) and \( y \equiv x'_2 - x'_3 \). The net result of the inversion and change of variables is

\[
A(x_1, x_2, x_3, x_4) = |x'_2|^{2\Delta_2} |x'_3|^{2\Delta_3} |x'_4|^{2\Delta_4} B(x, y). \tag{3.3}
\]

The function \( B(x, y) \) is then easily found from (3.1), and is given by

\[
B(x, y) = \int d^{d+1}w \frac{w_0^{\Delta_2+\Delta_4-d-1}(w-x)^{2\Delta_1}(w-y)^{2\Delta_2}}{(w-x)^{2\Delta_1}(w-y)^{2\Delta_2}} R(w) \tag{3.4a}
\]

\[
R(w) = \int d^{d+1}z \frac{z_0^{\Delta_1+\Delta_3-d-1} \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2}; \Delta - \frac{d}{2} + 1; \xi^2\right)}{(z_0^2 + w_0^2 + (z - w)^2)^{\Delta + 2k}} \tag{3.4b}
\]

where we have dropped the primes on the integration variable \( s \) defined by (2.8), as a function of \( z \) and \( w \), and \( \xi \) continues to be defined by (2.8), as a function of \( z \) and \( w \).

**a. Integrals over the interaction point \( z \)**

We now evaluate the \( z \)-integrals, which define \( R(w) \), by using the uniformly convergent expansion of the hypergeometric function in (3.4b) in powers of \( \xi \).

\[
R(w) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{\Delta}{2}) \Gamma(k + \frac{\Delta}{2} + \frac{1}{2}) \Gamma(\Delta - \frac{d}{2} + 1)}{\Gamma\left(\frac{\Delta}{2} + \frac{1}{2}\right) \Gamma(k + \Delta - \frac{d}{2} + 1)k!} (2w_0)^{\Delta+2k} R_k(w) \tag{3.5a}
\]

\[
R_k(w) = \int_0^\infty dz_0 \int d^{d}z \frac{z_0^{\Delta_1+\Delta_3-d-1+2k}}{[z_0^2 + w_0^2 + (z - w)^2]^{\Delta+2k}} \frac{1}{(z_0^2 + w_0^2)^{\Delta}} \tag{3.5b}
\]

The spatial integral in (3.5b) is convergent for \( \Delta_1, \Delta_3 \geq \frac{d}{2} \) and may be carried out by introducing a Feynman parameter \( \alpha \), yielding

\[
\frac{\pi^{d/2} \Gamma(\Delta + \Delta_3 + 2k - \frac{d}{2})}{\Gamma(\Delta + 2k) \Gamma(\Delta_3)} \int_0^1 d\alpha \frac{\alpha^{\Delta+2k-1}(1-\alpha)^{\Delta_3-1}}{[z_0^2 + \alpha(1-\alpha)w^2 + \alpha w_0^2]^{\Delta+\Delta_3+2k-\frac{d}{2}}}.
\]

The \( z_0 \) integral is then straightforward, and we find

\[
R_k(w) = \frac{\pi^{d/2} \Gamma(k + a) \Gamma(k + b)}{2\Gamma(\Delta + 2k) \Gamma(\Delta_3)} \int_0^1 d\alpha \frac{\alpha^{k+\Delta-b-1}(1-\alpha)^{\Delta_3-1}}{[(1-\alpha)w^2 + w_0^2]^{k+b}}. \tag{3.6}
\]

Here, we define the following abbreviations (\( c \) will enter shortly)

\[
a \equiv \frac{1}{2}(\Delta + \Delta_1 + \Delta_3) - \frac{d}{2}
\]

\[
b \equiv \frac{1}{2}(\Delta - \Delta_1 + \Delta_3)
\]

\[
c \equiv \Delta - \frac{d}{2} + 1.
\]
Actually, this integration will be convergent for all $k \geq 0$ only for $b > 0$, but this is guaranteed by (3.2a).

Putting together this result and the summation over $k$ at first sight yields individual terms that have four $k$-dependent $\Gamma$ factors in the numerator, and three others in the denominator. Happily however, the following special ratio always occurs and may be simplified with the help of the doubling formula of the Euler $\Gamma$-function.

$$\frac{\Gamma(k + \frac{\Delta}{2})\Gamma(k + \frac{\Delta}{2} + \frac{1}{2})\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta}{2} + \frac{1}{2})\Gamma(2k + \Delta)} = \frac{1}{2^{2k}}.$$  \hspace{1cm} (3.8)

Using this result, the summation is now over terms with two $k$-dependent $\Gamma$ functions in the numerator, one in the denominator, as well as a factor of $k!$ in the denominator, times a $k$-power of a composite variable

$$\gamma \equiv \frac{\alpha w_0^2}{w_0^2 + (1 - \alpha)w^2}.$$  \hspace{1cm} (3.9)

This expression thus resums to a standard hypergeometric function, and we may recast $R(w)$ in the form

$$R(w) = 2^{\Delta - \frac{3}{2}} \pi^\frac{\Delta}{2} w_0^{\Delta_1 - \Delta_3} \frac{\Gamma(a)\Gamma(b)}{\Gamma(\Delta_3)\Gamma(\Delta)} \int_0^1 d\alpha \frac{\alpha^{\Delta_1 - \Delta_3 - 1}(1 - \alpha)^{\Delta_3 - 1}\gamma^b F(a, b; c; \gamma)}{w_0^2 + (1 - \alpha)w^2}.$$  \hspace{1cm} (3.10)

where the combinations $a$, $b$, $c$ were defined in (3.7), and $\gamma$ is given by (3.9) in terms of $\alpha$. Notice that as $\alpha$ runs from 0 to 1, so does $\gamma$, and the integral may be rewritten in terms of an integration over $\gamma$.

$$R(w) = 2^{\Delta - \frac{3}{2}} \pi^\frac{\Delta}{2} \frac{w_0^{\Delta_1 + \Delta_3}}{w^{2\Delta_3 - 2\Delta_1}} \frac{\Gamma(a)\Gamma(b)}{\Gamma(\Delta_3)\Gamma(\Delta)} \int_0^1 d\gamma \frac{\gamma^{\Delta - b - 1}(1 - \gamma)^{\Delta_3 - 1} F(a, b; c; \gamma)}{[(1 - \gamma)w_0^2 + \gamma w^2]^{\Delta_1}}.$$  \hspace{1cm} (3.11)

In principle, the $\gamma$-integration may be carried out in terms of a generalized hypergeometric function $_4F_3$, but expressions of this type tend to be less useful than explicit series expansions, which we shall use instead.

b. Integrals over the interaction point $w$

Following (3.4a), the reduced amplitude $B(x, y)$, which gives the full amplitude via (3.3), is obtained from an integration over $w$. Interchanging the convergent $\gamma$ and $w$-integrations, we have

$$B(x, y) = 2^{\Delta - \frac{3}{2}} \pi^\frac{\Delta}{2} \frac{\Gamma(a)\Gamma(b)}{\Gamma(\Delta_3)\Gamma(\Delta)} \int_0^1 d\gamma \gamma^{\Delta - b - 1}(1 - \gamma)^{\Delta_3 - 1} F(a, b; c; \gamma) W(x, y; \gamma).$$  \hspace{1cm} (3.12)
where the function $W(x, y; \gamma)$ is defined by

$$W(x, y; \gamma) = \int d^{d+1}w \frac{w_0^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - d - 1}}{(w - x)^{2\Delta_4} (w - y)^{2\Delta_2}} \frac{w^{2\Delta_1 - 2\Delta_3}}{[(1 - \gamma)w_0^2 + \gamma w^2]^\Delta_1}. \quad (3.13)$$

The $w$-integral is convergent for all $x$ and $y$, and $0 \leq \gamma \leq 1$. To evaluate it, it is most convenient to proceed by Taylor series expansion in powers of $w_0^2/w^2$ under the integration of the denominator $(1 - \gamma)w_0^2 + \gamma w^2$. However, such an expansion is convergent only when $\frac{1}{2} < \gamma \leq 1$. We shall evaluate the function $W$ in this range of $\gamma$ (actually, $\frac{1}{2} < \text{Re}(\gamma) \leq 1$) and then analytically continue to the full range of $\gamma$. We shall be able to see explicitly that the analytic continuation encounters no singularities, and is in fact automatic.

Thus, for $\frac{1}{2} < \gamma \leq 1$, we evaluate $W$ by the following Taylor series expansion

$$W(x, y; \gamma) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \Delta_1)}{\Gamma(\Delta_1) k!} (\gamma - 1)^k \gamma^{-k - \Delta_1} W_k(x, y) \quad (3.14a)$$

$$W_k(x, y) = \int d^{d+1}w \frac{w_0^{2k + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - d - 1}}{(w - x)^{2\Delta_4} (w - y)^{2\Delta_2}} \frac{w^{2\Delta_1 - 2\Delta_3}}{w_0^{2\Delta_3 + 2k}}. \quad (3.14b)$$

The integrals $W_k(x, y)$ can now be done by the following standard steps: (a) combine $w^2$ and $(w - y)^2$ denominators with Feynman parameter $\alpha$, (b) combine the composite denominator from the previous step with the $(w - x)^2$ denominator using Feynman parameter $\beta$, (c) carry out the $d^d w$ integral, (d) do the $dw_0$ integral. We suppress details and directly give the result in the form of the following combination of dimensions that will enter throughout

$$\delta \equiv \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4). \quad (3.15)$$

After several simplifications, we get

$$W_k(x, y) = \frac{\pi^{d/2} \Gamma(k + \delta - \frac{d}{2}) \Gamma(\delta - \Delta_1)}{2 \Gamma(\Delta_2) \Gamma(\Delta_3) \Gamma(k + \Delta_3)} \cdot \int_0^1 d\alpha \int_0^1 d\beta \frac{\alpha^{\Delta_2 - 1}(1 - \alpha)^{k + \Delta_3 - 1} \beta^{\Delta_4 - 1}(1 - \beta)^{k + \delta - \Delta_4 - 1}}{[\beta(x - \alpha y)^2 + \alpha(1 - \alpha)y^2]^\delta - \Delta_1}. \quad (3.16)$$

Notice that (3.14) was convergent only when $\delta - \Delta_1 > 0$, but this is precisely the convergence condition (3.2b).

Assembling this result for $W_k(x, y)$ in the sum (3.14a), we notice that the summation over $k$ becomes proportional to a hypergeometric function, so that the function $W(x, y; \gamma)$ takes on the form

$$W(x, y; \gamma) = \frac{\pi^{d/2} \Gamma(\delta - \frac{d}{2}) \Gamma(\delta - \Delta_1)}{2 \Gamma(\Delta_2) \Gamma(\Delta_3) \Gamma(\Delta_4)} \gamma^{-\Delta_1} \int_0^1 d\alpha \int_0^1 d\beta \frac{\alpha^{\Delta_2 - 1}(1 - \alpha)^{\Delta_3 - 1} \beta^{\Delta_4 - 1}(1 - \beta)^{\delta - \Delta_4 - 1}}{[\beta(x - \alpha y)^2 + \alpha(1 - \alpha)y^2]^\delta - \Delta_1} \cdot \frac{F(\Delta_1, \delta - \frac{d}{2}; \Delta_3; -\sigma)}{\sigma^{\Delta_1}}. \quad (3.17)$$
where $\sigma$ is defined by
\begin{equation}
\sigma = (1 - \alpha)(1 - \beta)(1 - \gamma)/\gamma \tag{3.18}
\end{equation}

Now, originally, this expression was valid only for $\frac{1}{2} < \gamma \leq 1$, and the function $W(x, y; \gamma)$ for the full range of integration of $\gamma$ was to be defined by analytic continuation in $\gamma$. The dependence of $\sigma$ on $\gamma$ makes it clear that for any value of $\gamma$ in the integration range, we have $\sigma \geq 0$, so that the hypergeometric function is well-defined for all $0 \leq \gamma \leq 1$, and the only singularity in (3.17) is the power prefactor $\gamma^{-\Delta_1}$. Thus, the analytic continuation in $\gamma$ is straightforward, and (3.17) is the correct expression for $W(x, y; \gamma)$ for all values of $0 < \gamma \leq 1$.

c. Combining all integrals

The expression in (3.17) may be recast in a more useful way by performing an analytic continuation [22] on the hypergeometric function, in such a way that its new argument is brought into the standard range $[0, 1]$. The transformation takes the form
\begin{equation}
F(\Delta_1, \delta - \frac{d}{2}; \Delta_3; -\sigma) = (1 - \eta)^{\Delta_1} F(\Delta_1, \Delta_3 - \delta + \frac{d}{2}; \Delta_3; \eta) \tag{3.19}
\end{equation}

where $\sigma$ was defined in (3.18) and $\eta = \sigma/(1 + \sigma)$, with $0 \leq \eta \leq 1$.

The best formulas for deriving asymptotics are those where the integration range is scale invariant, so we change integration variables for the Feynman parameters $\alpha$ and $\beta$ to variables $u$ and $v$,* and express also $\eta$ in terms of $u$ and $v$
\begin{equation}
\alpha = \frac{1}{1 + u}, \quad \beta = \frac{u}{uv + u + v}, \quad \eta = \frac{(1 - \gamma)uv}{uv + \gamma(u + v)} \tag{3.20}
\end{equation}

After a number of simplifications, the final expression for $W$ is given by
\begin{equation}
W(x, y; \gamma) = \frac{\pi^{d/2} \Gamma(\delta - \frac{d}{2}) \Gamma(\delta - \Delta_1)}{2 \Gamma(\Delta_2) \Gamma(\Delta_3) \Gamma(\Delta_4)} \int_0^\infty du \int_0^\infty dv \ F(\Delta_1, \Delta_3 - \delta + \frac{d}{2}; \Delta_3; \eta) \cdot \\
\frac{1}{[uv + \gamma(u + v)]^{\Delta_1}} \cdot \frac{u^{\delta - \Delta_2 - 1} v^{\delta - \Delta_4 - 1}}{[ux^2 + vy^2 + (x - y)^2]^{\delta - \Delta_1}}. \tag{3.21}
\end{equation}

Finally, the hypergeometric function $F(a, b; c; \gamma)$ in (3.12) may be transformed into a more useful form by the following formula
\begin{equation}
F(a, b; c; \gamma) = (1 - \gamma)^{c-a-b} F(c - a, c - b; c; \gamma) \tag{3.22}
\end{equation}

* This change of variables was carried out in 3 stages in [11] to obtain (3.29) of [11].
The advantage of the form on the right hand side is two-fold. First, when \( \Delta < \Delta_1 + \Delta_3 \), this form will truncate to a finite polynomial, in analogy with the case of the photon exchange [11]. Second, for the parameter values of \( a, b, c \) of (3.7), the series expansion of \( F(a, b; c; \gamma) \) is not uniformly convergent, while that of \( F(c - a, c - b; c; \gamma) \) is uniformly convergent for \( \gamma \in [0, 1] \). Uniform convergence will allow us to systematically expand \( F \) in a power series and treat the series term by term.

We are now ready to combine all contributions and present a completely general formula for the scalar exchange four point function in arbitrary dimension. The reduced amplitude \( B(x, y) \) of (3.3) is given by

\[
B(x, y) = \frac{\pi^d \Gamma(\delta - \frac{d}{2}) \Gamma(\delta - \Delta_1) \Gamma(a) \Gamma(b)}{2^{2-\Delta} \Gamma(\Delta) \Gamma(\Delta_2) \Gamma(\Delta_3)^2 \Gamma(\Delta_4)} B_R(x, y)
\]

\[
B_R(x, y) = \int_0^\infty du \int_0^\infty dv \frac{u^{\delta - \Delta_1 - \Delta_2 - 1} v^{\delta - \Delta_1 - \Delta_4 - 1}}{(ux^2 + vy^2 + (x - y)^2)^{\delta - \Delta_1}} \rho \left( \frac{uv}{u + v + uv} \right)
\]

with the “spectral density” function \( \rho \) given by

\[
\rho(\tau) = \tau^{\Delta_1} \int_0^1 d\gamma \gamma^{\Delta - b - 1} F(c - a, c - b; c; \gamma) \frac{F(\Delta_1, \Delta_3 - \delta + \frac{d}{2}; \Delta_3; \eta)}{(\gamma + \tau - \gamma \tau)^{\Delta_1}}.
\]

The function \( \rho \) depends only on the combination \( uv/(u + v + uv) \) and on the dimensions. Here, \( \eta \) of (3.20) is expressed in terms of \( \tau \) and \( \gamma \) by the relation

\[
\eta = \frac{(1 - \gamma)\tau}{\gamma + \tau - \gamma \tau}.
\]

Notice that the denominator of \( \eta \) is the same as the one occurring in (3.24b), and that the numerator factorizes; this will be very useful in the next section.

**d. Infinite series expansions**

It will be advantageous to express the second hypergeometric function in (3.23b) and (3.24b) in terms of an infinite sum. In particular, since \( \eta \) is a composite variable that depends on all \( u, v \) and \( \gamma \), this expansion will better allow us to carry out the integrations. Also, as we shall demonstrate in the subsequent section, under certain mild conditions (which are always satisfied in the case of AdS\(_5 \times S_5\)) on the dimensions \( d \) and \( \Delta_1 \), this infinite series truncates to a sum over a finite number of terms. We have

\[
\rho(\tau) = \sum_{\ell=0}^\infty \frac{\Gamma(\ell + \Delta_1) \Gamma(\ell + \Delta_3 - \delta + \frac{d}{2}) \Gamma(\Delta_3)}{\Gamma(\Delta_1) \Gamma(\Delta_3 - \delta + \frac{d}{2}) \Gamma(\ell)} \rho_\ell(\tau),
\]
with the functions $\rho_\ell$ given by
\[
\rho_\ell(\tau) = \tau^{\ell+\Delta_1} \int_0^1 d\gamma \, \gamma^{\Delta-b-1}(1-\gamma)^\ell \frac{F(c-a,c-b;c;\gamma)}{(\gamma + \tau - \gamma \tau)^{\Delta_1 + \ell}}. \tag{3.27}
\]
A simple relation gives $\rho_\ell$ in terms of $\rho_0$,
\[
\rho_\ell(\tau) = (-)^\ell \tau^{\ell+\Delta_1} \frac{\Gamma(\Delta_1)}{\Gamma(\Delta_1 + \ell)} \frac{\partial^\ell}{\partial \tau^\ell} \{\tau^{-\Delta_1} \rho_0(\tau)\}. \tag{3.28}
\]
It remains to study $\rho_0$.

Using the property of uniform convergence of the series expansion of the hypergeometric function in powers of $\gamma$ for $1 + a + b - c = \Delta_3 \geq 2$ throughout $\gamma \in [0,1]$, we obtain
\[
\rho_\ell(\tau) = \sum_{k=0}^\infty \frac{\Gamma(c-a+k)\Gamma(c-b+k)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)\Gamma(c+k)\Gamma(k+1)} \rho_\ell^{(k)}(\tau) \tag{3.29}
\]
The function $\rho_\ell^{(k)}(\tau)$ is clearly proportional to a hypergeometric function, given by
\[
\rho_\ell^{(k)}(\tau) = \tau^{\Delta_1 + \ell} \frac{\Gamma(\ell+1)\Gamma(\Delta - b + k)}{\Gamma(\Delta - b + k + \ell + 1)} F(\ell + 1, \ell + \Delta_1; \Delta - b + k + \ell + 1; 1 - \tau) \tag{3.30}
\]
From the form (3.30) it is clear that $\rho_\ell^{(k)}(\tau)$ is a (complicated) elementary function, since the hypergeometric function for the arguments of (3.30) is degenerate and reduces to a combination of rational and logarithmic contributions [21].

For $|\tau| < 1$, the hypergeometric function of (3.30) also admits an absolutely convergent series expansion in $1 - \tau$, which may be combined with (3.29) into a double series. Exchanging the orders of summation, and generalizing to the case of arbitrary value of $\ell$, we get
\[
\rho_\ell(\tau) = \tau^{\Delta_1 + \ell} \sum_{p=0}^\infty Q_{p+\ell}(1-\tau)^p \frac{\Gamma(\Delta_1)\Gamma(p+1)}{\Gamma(\Delta_1 + \ell) \, p!}. \tag{3.31}
\]
The coefficients $Q_p$ depend only upon the dimensions and are given by
\[
Q_p \equiv \sum_{k=0}^\infty \frac{\Gamma(c-a+k)\Gamma(c-b+k)\Gamma(c)\Gamma(\Delta-b+k)\Gamma(p+\Delta_1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(c+k)\Gamma(k+1)\Gamma(\Delta-b+k+p+1)\Gamma(\Delta_1)}, \tag{3.32a}
\]
\[
= \frac{\Gamma(\Delta-b)\Gamma(p+\Delta_1)}{\Gamma(\Delta-b+p+1)\Gamma(\Delta_1)} \, _3F_2(c-a,c-b,\Delta-b;c,\Delta-b+p+1;1) \tag{3.32b}
\]
which is convergent for all $p \geq 0$, and for large $p$ behaves like $Q_p \sim p^{\min(0,a-c)}$. The function $_3F_2$ is the standard generalized hypergeometric function.


4. SPECIAL CONDITIONS ON $\Delta_i, \Delta$

Henceforth, we shall restrict attention to the case of four point functions in which the dimensions $\Delta$ and $\Delta_i$, $i = 1, \ldots, 4$ are integers and larger than or equal to the unitarity bound $\frac{d}{2}$. Under these restrictions, $a$, $b$, $c$ and $\delta$ are half integers. We shall now show that if certain combinations of these numbers are actually integers (possibly of definite sign), considerable simplifications occur in (3.23), (3.26) and (3.29), and we shall work out the simplified expressions explicitly in those cases.

The following cases below are mutually independent from one another.

a. **Integer valued $\delta - \frac{d}{2} - \Delta_3 \geq 0$ : Truncating the $\ell$-series.**

When $\delta - \frac{d}{2} - \Delta_3$ is integer and $\geq 0$, it is straightforward to see that the infinite series over $\ell$ in (3.26) truncates to a finite sum, with $\delta - \frac{d}{2} - \Delta_3 + 1$ terms. For the case of most physical urgency AdS$_5 \times S_5$ with $d = 4$, the remaining condition that $\delta$ is integer and that $\delta > \Delta_3 + 1$ is automatically satisfied thanks to the $R$-parity considerations given in the introduction, and the convergence conditions of (3.2b).

b. **The case $\Delta < \Delta_1 + \Delta_3$ : Truncating the $k$-series**

When $c - a = 1 + (\Delta - \Delta_1 - \Delta_3)/2 \leq 0$, the infinite series expansion of the spectral density $\rho_0$ and thus of $\rho_\ell$ truncates to a finite sum, with $(\Delta_1 + \Delta_3 - \Delta + 1)/2$ terms. In other words, this is the case when a triangle inequality holds

$$\Delta < \Delta_1 + \Delta_3$$

which is analogous to (3.2a). More generally, if either $\Delta_1 + \Delta_3 > \Delta$ or $\Delta_2 + \Delta_4 > \Delta$, then by interchanging the role of the pairs (13) and (24), one may always assume that $\Delta_1 + \Delta_3 > \Delta$. Thus, only the case where both $\Delta_1 + \Delta_3 \leq \Delta$ and $\Delta_2 + \Delta_4 \leq \Delta$ will not lead to a truncated $k$-series.

The structure of quantum field theory on AdS space by itself does not guarantee a condition like (4.1). However, quantum field theory on AdS $\times S$ will be restricted by the tensor product rules of the finite-dimensional representations of the sphere isometry group, and this produces inequalities such as (4.1). Notice that the case of gauge boson exchange falls into this category [11].

c. **Integer $\delta$ : Amplitudes from a Universal Function**

The amplitude $B_R(x, y)$ of (3.24) may be expressed in terms of $\rho(\tau)$ and a universal function that will not depend upon any of the dimensions of the problem. To see this, we introduce the notation

$$X = x^2, \quad Y = y^2, \quad Z = \frac{1}{2}(x - y)^2$$

(4.2)
and temporarily declare the variables $X$, $Y$ and $Z$ independent, setting them to the values of (4.2) only later. Defining also the combinations

$$n_x = \delta - \Delta_1 - \Delta_2 = \frac{1}{2}(-\Delta_1 - \Delta_2 + \Delta_3 + \Delta_4)$$

$$n_y = \delta - \Delta_1 - \Delta_4 = \frac{1}{2}(-\Delta_1 + \Delta_2 + \Delta_3 - \Delta_4)$$

$$n_z = \delta - \Delta_3 - 1 = \delta - \Delta_1 - n_x - n_y - 1,$$

we have

$$B_R(x, y) = \frac{(-)^{\delta+\Delta_1+1}}{2^{n_x} \Gamma(\delta - \Delta_1)} \frac{\partial^{n_x+n_y+n_z} \hat{B}_R(X, Y, Z)}{\partial X^{n_x} \partial Y^{n_y} \partial Z^{n_z}} \bigg|_{X=x^2, \ Y=y^2, \ Z=\frac{1}{2}(x-y)^2}$$

(4.4)

where the function $\hat{B}_R(X, Y, Z)$ is defined by

$$\hat{B}_R(X, Y, Z) = \int_0^\infty du \int_0^\infty dv \ \rho\left(\frac{u}{u+v+uv}\right)$$

$$\rho\left(\frac{u}{u+v+uv}\right)$$

(4.5)

For expression (4.4) to make sense, the number of derivatives with respect to each variable $X$, $Y$ and $Z$ must be positive or zero, so that we must have $n_x \geq 0$, $n_y \geq 0$ and $n_z \geq 0$. From the convergence condition (3.2b), it is clear that $n_z \geq 0$. The conditions $n_x \geq 0$ and $n_y \geq 0$ require that $\Delta_3 - \Delta_1 \geq |\Delta_4 - \Delta_2|$, which may always be achieved simply by interchanging the roles of the external legs of the amplitude.

There is a special case (which does not occur in the $AdS_5 \times S^5$ theory) where the assumptions of § b above may conflict with the assumptions of the preceding paragraph. This is when $\Delta_1 + \Delta_3 > \Delta \geq \Delta_2 + \Delta_4$. The roles of the pairs (13) and (24) may have to be interchanged in order to obtain the inequalities in this ordering, and it may not be possible then to choose $\Delta_3 - \Delta_1 \geq |\Delta_4 - \Delta_2|$ as well. If needed, this case is best handled separately with methods parallel to the ones to be used below.

To calculate $B_R$, it suffices to obtain $\hat{B}_R$, a task on which we now concentrate. Performing a change of variables $\tau = uv/(u+v+uv)$ and $\lambda = (v-u)/(v+u)$, one finds that $\hat{B}_R$ may be expressed in terms of a single integral over $\tau$ of $\rho(\tau)$ and a universal function $I(\tau; s, t)$, as follows

$$\hat{B}_R(X, Y, Z) = \frac{1}{X+Y} \int_0^1 \frac{d\tau}{\tau} \rho(\tau) \ I(\tau; \frac{Z}{X+Y}, \frac{X-Y}{X+Y}).$$

(4.6)

The universal function is elementary and given by

$$I(\tau; s, t) = \int_{-1}^{+1} d\lambda \ \frac{1}{\tau(1-\lambda t) + s(1-\lambda^2)(1-\tau)}$$

$$= \frac{1}{\sqrt{\omega^2 - (1-t^2)\tau^2}} \left\{- \ln(1-t^2)\tau^2 + 2 \ln(\omega + \sqrt{\omega^2 - (1-t^2)\tau^2})\right\},$$

(4.7a)

(4.7b)
where we have set \( \omega = 2s(1 - \tau) + \tau \) for short. This form of the amplitude will be very useful when deriving convergent series expansions and asymptotics.

5. SHORT DISTANCE EXPANSIONS

The main purpose of this section is to study the short distance limits of scalar exchange correlation in the direct channel \( x_2 - x_4 \to 0 \), and the crossed channel \( x_2 - x_3 \to 0 \). We are interested in the sense in which the amplitude admits double OPE expansions of the type (1.1) in these limits. This question will be analyzed using the series expansion of \( \rho(\tau) \) of (3.24) and then estimating the contribution of each term to the amplitude \( B_R(x_i, y_i) \) via (3.23). However, we will first discuss the leading term in the direct channel using the simple method discussed in [7] which works from the original expression (3.1b) for the amplitude, before any processing. This discussion will bring in some physical aspects of the AdS/CFT correspondence, and it will provide checks on the intricate manipulations used to derive (3.23) and (3.24) and then short distance expansions.

a. Leading term in direct channel \( x_{24} \to 0 \).

The intuitive idea behind the simple method of [7] is that the region of \( \text{AdS}_{d+1} \) which gives the dominant contribution in the direct channel limit is the region where \( \vec{z} \) is near \( \vec{x}_1 \) and \( \vec{w} \) near \( \vec{x}_2 \). Thus, we consider an expansion of the bulk-to-bulk scalar propagator in (3.1b) about these points. The leading term for \( |x_{12}| \gg |x_{13}|, |x_{24}| \) is obtained from the term in the series expansion of the hypergeometric function with slowest fall-off as \( u \to \infty \), so that our approximation is

\[
\xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta + 1}{2}; \Delta - \frac{d}{2} + 1; \xi^2\right) \sim \xi^\Delta \sim \frac{(2z_0w_0)^\Delta}{|x_{12}|^{2\Delta}}.
\] (5.1)

We insert this in (3.1b) and obtain the factorized form

\[
A(x_i) \sim \frac{2^\Delta}{|x_{i2}|^{2\Delta}} V(\Delta_1, \Delta_3, \Delta, |x_{13}|) V(\Delta_2, \Delta_4, \Delta, |x_{24}|), \tag{5.2a}
\]

where

\[
V(\Delta_i, \Delta_j, \Delta, |x_{ij}|) = \int \frac{d^{d+1}z}{z_0^d+1} \frac{z_0^{\Delta_i+\Delta_j+\Delta}}{(z - x_i)^{2\Delta_i}(z - x_j)^{2\Delta_j}}
\] (5.2b)

with

\[
T = \Gamma\left(\frac{\Delta}{2}(\Delta_i + \Delta_j - \Delta)\right) \Gamma\left(\frac{\Delta}{2}(\Delta + \Delta_i - \Delta_j)\right) \Gamma\left(\frac{\Delta}{2}(\Delta + \Delta_j - \Delta_i)\right).
\] (5.2c)
It is important that \( V(\Delta_i, \Delta_j, \Delta, |x_{ij}|) \) differs from the 3-point correlator of the CFT operators \( O_{\Delta_i}, O_{\Delta_j} \) and \( O_{\Delta} \) only by normalization factors, including those factored out in (3.1a), and that the result above for the integral in (5.2) was obtained directly from (22-23) of [20], where the 3-point correlators were systematically studied. In fact, when normalization factors in (3.1a) are included, one can show that the leading term in the 4-point correlator is exactly the result expected for the contribution of the primary operator \( O_{\Delta} \) in the double OPE of (1.1). This confirms that the AdS/CFT correspondence satisfies expected non-linear properties in the boundary theory.

We observe that poles in \( \Gamma \)-functions in (5.2) correspond to divergences in the AdS integral for the correlator \( \langle O_{\Delta}, O_{\Delta}, O_{\Delta} \rangle \), and in particular this correlator diverges unless \( |\Delta_i - \Delta_j| < \Delta < \Delta_i + \Delta_j \). We have already observed (see (3.2a)) that the 4-point amplitude also diverges when the lower limit is not obeyed, and we now see that the upper limit is the triangle inequality (4.1), which is the condition for truncation of the infinite series in the spectral density. The 4-point correlator converges if this inequality is violated, but we cannot expect that the direct channel limit conforms to a double OPE when the 3-point function diverges. Thus, we should expect a qualitatively different asymptotic behavior, and this will be confirmed by our systematic analysis below.

b. Expansion in the Direct Channel for \( \Delta < \Delta_1 + \Delta_3 \).

Here, we shall concentrate on the direct channel OPE, in which \( x - y \to 0 \), while keeping \( x \) and \( y \) finite. In this regime, \( X - Y \) and \( Z \) both tend to 0, so that \( s \to 0 \) and \( t \to 0 \) in (4.6b). We begin by carrying out a convergent expansion in powers of \( (X - Y) \), i.e. in terms of \( t \), which is valid for both \( \Delta < \Delta_1 + \Delta_3 \) (this subsection) and \( \Delta \geq \Delta_1 + \Delta_3 \) (subsection c.).

Expansion in powers of \( t \)

The universal function \( I(\tau; s, t) \) of (4.6b) admits a simple convergent series expansion in powers of \( t \), given as follows

\[
I(\tau; s, t) = \sum_{m=0}^{\infty} t^{2m} I_{2m}(\tau; s) \quad (5.3a)
\]

\[
I_{2m}(\tau; s) = 2 \int_{0}^{1} d\lambda \frac{\lambda^{2m}}{[\tau + s(1 - \tau)(1 - \lambda^2)]^{2m+1}}. \quad (5.3b)
\]

Since \( I_{2m}(\tau; s) \leq I_0(\tau; s) \) for all \( \tau \in [0, 1] \) and \( s \geq 0 \), the radius of convergence in \( t \) of (5.3a) is larger or equal to 1.
As a result, the function \( \hat{B}_R(X, Y, Z) \) (and thus \( B_R(x, y) \)) admits an expansion in powers of \( t \) as well, and we have

\[
\hat{B}_R(X, Y, Z) = \frac{1}{X+Y} \sum_{m=0}^{\infty} t^{2m} B^{(2m)}(\frac{Z}{X+Y})
\]

\( B^{(2m)}(s) = \int_0^1 d\tau \rho(\tau) I_{2m}(\tau; s). \) (5.4b)

It remains to compute the partial amplitudes \( B^{(2m)}(s) \).

**The Exact form of \( \rho(\tau) \).**

In order to analyze the \( s \)-dependence of the partial amplitudes \( B_{2m}(s) \), we need good control over the \( \tau \)-dependence of \( \rho(\tau) \). We consider first the case where the \( k \)-sum of \( \rho_0(\tau) \) truncates to a finite sum, which occurs when \( c-a \leq 0 \), or equivalently \( \Delta < \Delta_1 + \Delta_3 \). In this case, the infinite series of (3.29) collapses to a finite sum. Also, precisely for these values does the hypergeometric function in (3.30) become a rational function of \( \tau \), and \( \rho^{(k)}(\tau) \) a polynomial in \( \tau \) of degree \( \Delta_1 - 1 \). Both are given by

\[
\rho_\ell(\tau) = \sum_{k=0}^{a-c} \frac{\Gamma(1+a-c)\Gamma(c-b+k)\Gamma(c)}{\Gamma(1+a-c-k)\Gamma(c-b)\Gamma(c+k)\Gamma(k+1)} (-)^k \rho^{(k)}(\tau)
\]

\[
\rho^{(k)}(\tau) = \sum_{n=0}^{a-c-k} \frac{\Gamma(a-c+1-k)\Gamma(\Delta-b+k+n)\Gamma(1+a-c+\ell-n-k)}{\Gamma(\Delta_1+\ell)\Gamma(1+a-c-n-k)\Gamma(n+1)} \tau^{\Delta_1-b+k+n}
\]

where the latter may be established with the help of [23]. Finally, putting together (3.26) with (5.5), we obtain a complete expression for \( \rho(\tau) \), as follows,

\[
\rho(\tau) = \sum_{k=0}^{a-c} \sum_{n=0}^{a-c-k} \frac{(-)^k E_{k+n}\Gamma(c-b+k)\Gamma(\Delta-b+k+n)}{\Gamma(c+k)\Gamma(c-b)\Gamma(k+1)\Gamma(n+1)} \tau^{\Delta_1-b+k+n}
\]

\[
E_k = \frac{\Gamma(\delta - \frac{d}{2} - 1 - a + c + k)\Gamma(\Delta_3)\Gamma(a-c+1)\Gamma(c)}{\Gamma(\Delta_1)\Gamma(\delta - \frac{d}{2})\Gamma(b+k)}
\]

Polynomial behavior of \( \rho(\tau) \) is ideal for the evaluation of the asymptotics of the amplitudes, as we shall see next.

**The Calculation of \( B^{(2m)}(s) \).**

For \( \Delta < \Delta_1 + \Delta_3 \), we have \( c-a \leq 0 \), and thus \( \rho_\ell(\tau) \) and \( \rho(\tau) \) are polynomial in \( \tau \), given by (5.5-6). Notice also from these expressions that \( \rho^{(k)}(\tau) \) and thus \( \rho(\tau) \) vanishes
at $\tau = 0$, so that the integral in (5.4b) is always convergent. It remains to evaluate, for $p \geq 1$, the integrals
\[
B_p^{(2m)}(s) \equiv \int_0^1 d\tau \ \tau^{p-1}I_{2m}(\tau; s).
\] (5.7)

Using (5.3b) and interchanging the orders of integration of $\tau$ and $\lambda$, we find
\[
B_p^{(2m)}(s) = 2\int_0^1 d\lambda \ \lambda^{2m} \int_0^1 d\tau \ \tau^{2m+p-1}[\tau + s(1-\tau)(1-\lambda^2)]^{-1-2m}.
\] (5.8)

The $\tau$-integral is readily recognized as a hypergeometric function,
\[
B_p^{(2m)}(s) = \frac{2}{2m + p} \int_0^1 d\lambda \ \lambda^{2m} F\left(1, 2m + 1; 2m + p + 1; 1 - s(1 - \lambda^2)\right).
\] (5.9)

Using [24], we see that this (somewhat complicated) elementary function $F$ has a simple convergent expansion for small $s$. We recall from [24] that
\[
F\left(1, 2m + 1; 2m + p + 1; 1 - z\right) = \frac{2m + p}{\Gamma(p)\Gamma(2m + 1)} \left\{ \sum_{n=0}^{p-2} (-)^n \Gamma(2m + n + 1)\Gamma(p - n - 1)z^n + (-)^{p-1} \sum_{n=0}^{\infty} \frac{\Gamma(2m + p + n)}{\Gamma(n + 1)} \left(\psi(n + 1) - \psi(2m + p + n) - \ln z\right)z^{n+p-1} \right\}.
\] (5.10)

To compute $B_p^{(2m)}(s)$, we set $z = s(1 - \lambda^2)$ and perform the $\lambda$-integrals explicitly, with the help of
\[
\int_0^1 d\lambda \lambda^{2m}(1 - \lambda^2)^n = \frac{\Gamma\left(m + \frac{1}{2}\right)\Gamma(n + 1)}{2\Gamma\left(m + n + \frac{3}{2}\right)}
\] and
\[
\int_0^1 d\lambda \lambda^{2m}(1 - \lambda^2)^n \ln(1 - \lambda^2) = \frac{\Gamma\left(m + \frac{1}{2}\right)\Gamma(n + 1)}{2\Gamma\left(m + n + \frac{3}{2}\right)} \left(\psi(n + 1) - \psi(m + n + \frac{3}{2})\right)
\] (5.11)

Putting all together, we have
\[
B_p^{(2m)} = \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(p)\Gamma(2m + 1)} \left\{ \sum_{n=0}^{p-2} (-)^n \frac{\Gamma(2m + n + 1)\Gamma(p - n - 1)\Gamma(n + 1)}{\Gamma(m + n + \frac{3}{2})} s^n + (-)^{p-1} \sum_{n=0}^{\infty} \frac{\Gamma(2m + p + n)\Gamma(n + p)}{\Gamma(m + n + p + \frac{1}{2})\Gamma(n + 1)} \left(h_n - \ln s\right)s^{n+p-1} \right\},
\] (5.12)

where the combinations $h_n$ are defined by
\[
h_n = \psi(n + 1) - \psi(n + p) + \psi(m + n + p + \frac{1}{2}) - \psi(2m + p + n).
\] (5.13)
The coefficient function that multiplies the \( \ln s \) contribution is manifestly a hypergeometric function, given by

\[
B^{(2m)}_p(s) \bigg|_{\ln s} = (-)^p s^{p-1} \ln s \frac{\Gamma(m + \frac{1}{2})\Gamma(2m + p)}{\Gamma(2m + 1)\Gamma(m + p + \frac{1}{2})} F(2m + p, p; m + p + \frac{1}{2}; s) \quad (5.14)
\]

Combining (5.4), (5.6) and (5.12), we finally obtain a complete expression for \( B^{(2m)}(s) \) in terms of \( s \), by

\[
B^{(2m)}(s) = \sum_{k=0}^{a-c} \sum_{n=0}^{a-c-k} (-)^k E_{k+n} \frac{\Gamma(c - b + k)\Gamma(\Delta - b + k + n)}{\Gamma(c + k)\Gamma(c - b)\Gamma(k + 1)\Gamma(n + 1)} B^{(2m)}_{\Delta - b + k + n}(s). \quad (5.15)
\]

The complete logarithmic contribution may easily be read off from (5.14) and (5.15), and we find

\[
B^{(2m)}(s) \bigg|_{\ln s} = \ln s \sum_{k=0}^{a-c} \sum_{n=0}^{a-c-k} \frac{E_{k+n} \Gamma(c - b + k)\Gamma(\Delta - b + k + n)}{\Gamma(c + k)\Gamma(c - b)\Gamma(k + 1)\Gamma(n + 1)}
\]

\[
(-)^\Delta - b + n s^{\Delta - b + k + n - 1} \frac{\Gamma(m + \frac{1}{2})\Gamma(2m + \Delta - b + k + n)}{\Gamma(2m + 1)\Gamma(m + \Delta - b + k + n + \frac{1}{2})} F(2m + \Delta - b + k + n; m + \Delta - b + k + n + \frac{1}{2}; s). \quad (5.16)
\]

The contribution of \( B^{(2m)}(s) \) to the full amplitude is then gotten by combining (5.15) and (5.13) with (5.4a) and (4.4), in particular, all the logarithmic contributions arise from (5.15).

c. Matching of Direct Channel Results with OPE Predictions

It is worthwhile to calculate the leading term of \( A(x_i) \) directly from the formulas above as a check on the long development of the previous sections. The process is tedious so we shall present only an outline of the steps to be followed.

(i) The relation [11] between the variables \( s, t \) used above and conformal invariant functions of the original coordinates \( x_i \) is

\[
s = \frac{1}{2} \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \quad t = \frac{x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \quad (5.17)
\]

In the direct channel limit, \( x_{13} \) and \( x_{24} \) are small, and all other large \( x_{ij}^2 \) are equal to \( x_{12}^2 \) to leading order. The leading behaviors of \( s \) and \( t \) are

\[
s \sim \frac{x_{13}^2 x_{24}^2}{4 x_{12}^4} \quad t \sim \frac{1}{x_{12}^4} \frac{(x_{31})_{\mu} J_{\mu \nu}(x_{12})(x_{42})_{\nu}}{x_{12}^4} \quad (5.18)
\]
where \( J_{\mu\nu}(y) = \delta_{\mu\nu} - 2y_\mu y_\nu / y^2 \) is the well-known inversion Jacobian. We see that \( s \) and \( t \) both vanish in the limit, but \( t \) vanishes more slowly than \( s \). For the leading term, it is thus sufficient to consider only the \( m = 0 \) term in (5.1a) and only \( B^{(0)}(s) \) and \( B^{(0)}_p(s) \) in (5.12) and (5.16).

(ii) To simplify somewhat we consider only the special case where \( \Delta_1 = \Delta_3 \) and \( \Delta_2 = \Delta_4 \), and we further assume that \( \Delta \) is an even integer, as implied by the selection rules of the AdS\(_5\) \( \times \) S\(_5\) theory discussed in Sec 1. It is then the case that the minimum value of the subscript \( p \) in \( B^{(2m)}_p \) in (5.12) and (5.15) is \( p_0 = \Delta/2 \), and the entire singular contribution as \( x_{24} \to 0 \) is given by the derivatives of (4.4) applied to \( B^{(2m)}_p(s) \) \( \big|_{\ln s} \) in (5.16). For the leading singularity we can further restrict to \( B^{(0)}(s) \) \( \big|_{\ln s} \), as shown in (i) above, and we can keep only the \( k = n = 0 \) contribution in (5.16) with the hypergeometric function replaced by \( F \to 1 \). The higher powers of \( s \) we have dropped give only non-leading terms in the direct-channel limit.

(iii) Turning to (4.3-4.4) we see that \( n_x = n_y = 0 \) and \( n_z = \Delta_2 - 1 \) with our assumptions. We must then compute the \( Z \)-derivatives in (4.4), whose leading singularity at small \( s \) is given by

\[
\frac{\partial^{n_z}}{\partial z^{n_z}} \left( \ln s \ s^{p_0-1} \right) = \frac{(-)^{\Delta_2} \Gamma(\frac{1}{2} \Delta) \Gamma(\Delta_2 - \frac{1}{2} \Delta)}{(X + Y)^{\Delta_2 - 1}} \frac{1}{s^{\Delta_2 - \frac{1}{2} \Delta}}
\]  

(5.19)

One can see that this has the same singularity as (5.2) as \( x_{24} \to 0 \). Note that

\[
X + Y = (x'_4 - x'_3)^2 + (x'_2 - x'_3)^2 = \frac{x_{12}^2 x_{34}^2 + x_{23}^2 x_{14}^2}{x_{13}^2 x_{12}^2 x_{14}^2} \]

(5.20)

The coordinate dependence of the leading term is obtained by combining the factors in (5.19) with those from inversion in (3.3). The result is

\[
\frac{1}{|x_{12}|^{2\Delta} |x_{13}|^{2(\Delta_1 - \Delta)} |x_{24}|^{2\Delta_2 - 1}}
\]

(5.21)

in agreement with (5.2). To compute its coefficient one must combine all \( \Gamma \)-functions and powers of 2 and \( (-) \) from (5.16), (5.5b), (5.2a), (4.4), and (3.23a). The result agrees precisely with (5.2) and the double OPE.

To close this subsection we note that not only the leading term, but also the full asymptotic series of the direct channel limit of the correlation function is contained in the formulas (5.2) and (5.16). In particular the most singular logarithmic term is generically

\[
A(x_i) \sim \frac{1}{|x_{13}|^{2(\Delta_1 - \Delta_2)} |x_{12}|^{4\Delta_2}} \ln \frac{x_{13}^2 x_{24}^2}{x_{12}^4} \]

(5.22)
Note that the leading logarithmic singularity always arises multiplied by a regular power term in the expansion.

**d. Expansion in the Direct Channel** $x_{24} \to 0$ for $\Delta \geq \Delta_1 + \Delta_3$.

Recall that the expansions in powers of $t$ in (5.3) and (5.4) continue to hold in this case.

In this case, we have $c - a \geq 1$ and the series in (3.29) for $\rho(\tau)$ truly has an infinite number of terms. The coefficients $Q_p$ of (3.32) in the series expressions for $\rho_\ell$ of (3.31) behave as $Q_p \sim 1/p^{c-a}$ as $p \to \infty$, so that the series of (3.31) will be uniformly convergent throughout $\tau \in [0, 1]$ for $c - a \geq 2$, and uniformly convergent throughout $\tau \in [0, 1 - \epsilon]$, $\epsilon > 0$ for $c - a = 1$. Clearly, the series defines a function that is analytic around $\tau = 1$ by (3.31). At $\tau = 0$, $\tau$-derivatives of $\rho_\ell$ up to order $c - a + \Delta_1 - 1$ will be finite, but an extra derivative leads to a divergence, signaling non-analyticity at $\tau = 0$. Thus, the analytic continuation of the sum (3.31) and its asymptotics around $\tau = 0$ have to be extracted with care.

We use standard methods of analytic continuation and asymptotics, applied to string loop amplitudes in [25], to obtain the desired asymptotics. For simplicity, we restrict attention to $\rho_0(\tau)$, the generalization to other values of $\ell$ being available from (3.31).

Non-analyticity at $\tau = 0$ arises from the slow rate of convergence of the $p$-series. From (3.32a), it is manifest that the large $p$ behavior of the contribution of a single term of order $k$ in the series (3.32a) is given by $1/p^{c-a+k}$. Thus, the worst non-analyticities in $\rho_0(\tau)$ arise from the terms in the series (3.32a) with the lowest values of $k$. Assuming that we are interested in the asymptotics of $\rho(\tau)$ up to a given order, say $K$, we may isolate the first $K$ terms in the series (3.32a), treating these exactly, and then bound the remaining infinite series, which now has improved analytic behavior at $\tau = 0$. Thus, we define

$$Q_p = Q^K_p + \sum_{k=0}^{K-1} \frac{\Gamma(c - a + k)\Gamma(c - b + k)\Gamma(c)\Gamma(\Delta - b + k)\Gamma(p + \Delta_1)}{\Gamma(c - a)\Gamma(c - b)\Gamma(c + k)\Gamma(k + 1)\Gamma(\Delta - b + k + p + 1)\Gamma(\Delta_1)}$$ (5.23)

where the large $p$ behavior of $Q^K_p$ is now given by $Q^K_p \sim 1/p^{c-a+K}$. As a result, the contribution to $\rho_0(\tau)$ from $Q^K_p$ will have improved analytic behavior at $\tau = 0$ and will be differentiable $c - a + \Delta_1 + K - 1$ times.

The leading asymptotics of $\rho_0(\tau)$ may now be worked out by obtaining that of the leading $K$ terms of (5.23). (Equivalently, one can proceed to obtain the asymptotics from (3.30), and making use of (5.10), but we find it instructive to follow the route below.)
each $k$ with $0 \leq k \leq K - 1$, we perform the $p$-sum exactly. Thus, we need to resum the series

$$
\sum_{p=0}^{\infty} \frac{\Gamma(p + \Delta_1)}{\Gamma(\Delta - b + k + p + 1)} (1 - \tau)^p, \quad (5.24)
$$

which is most easily carried out by decomposing the ratio of $\Gamma$-functions in terms of its simple poles, as follows

$$
\frac{\Gamma(p + \Delta_1)}{\Gamma(\Delta - b + k + p + 1)} = \sum_{q=0}^{\Delta - b - \Delta_1 + k} \frac{(-)^q}{\Gamma(\Delta - b - \Delta_1 + k - q + 1) q!} \frac{1}{\Delta_1 + q + p}. \quad (5.25)
$$

Making use of the following $p$-sums

$$
\sum_{p=0}^{\infty} \frac{(1 - \tau)^p}{\Delta_1 + q + p} = (1 - \tau)^{-\Delta_1 - q} \left( -\ln \tau - \sum_{p=1}^{\Delta_1 + q - 1} \frac{1}{p} (1 - \tau)^p \right), \quad (5.26)
$$

we may readily extract the logarithmic behavior of (5.24) by resumming in (5.25) only the logarithmic contributions of (5.26). Thus, putting all together, we have

$$
\sum_{p=0}^{\infty} \frac{\Gamma(p + \Delta_1)}{\Gamma(\Delta - b + k + p + 1)} (1 - \tau)^p = \frac{(-\tau)^{\Delta - b - \Delta_1 + k} (1 - \tau)^{-\Delta + b - k}}{\Gamma(\Delta - b - \Delta_1 + k + 1)} \ln \tau + \text{analytic}
$$

$$
\sim \frac{(-\tau)^{\Delta - b - \Delta_1 + k}}{\Gamma(\Delta - b - \Delta_1 + k + 1)} \ln \tau \quad (5.27)
$$

Despite its appearance, expression (5.26) is perfectly regular and analytic in $\tau$ around $\tau = 1$, since the original sums were. Thus, the novel feature of $\rho(\tau)$ when $\Delta \geq \Delta_1 + \Delta_3$ is the appearance of logarithmic singularities around $\tau = 0$. Also, from the arguments above, we see that no other non-analyticities take place for $\tau \in [0, 1]$.

Calculation of $B^{(2m)}(s)$

The novel feature in dealing with the case $\Delta \geq \Delta_1 + \Delta_3$ is the appearance of the integrals (for $p \geq 1$) of the type

$$
\tilde{B}^{(2m)}_p(s) \equiv \int_0^1 d\tau \tau^{p-1} \ln \tau I_{2m}(\tau; s), \quad (5.28)
$$

where $p$ takes on the values of $\Delta - b - \Delta_1 + k$ of (5.27). They may be evaluated by starting from the function $B^{(2m)}_p(s)$ for arbitrary real values of $p$ and then taking the derivative with respect to $p$:

$$
\tilde{B}^{(2m)}_p(s) = \frac{\partial}{\partial p} B^{(2m)}_p(s). \quad (5.29)
$$
Eqs. (5.8) and (5.9) still hold when $p$ is real instead of integer, but an analytic continuation that is more general than (5.10) is now needed. This is given by [26], and may be re-expressed by expanding the hypergeometric functions in a power series, as follows

$$
\frac{1}{2m+p} F(1, 2m + 1; 2m + p + 1; 1-z) = \frac{\Gamma(1-p)}{\Gamma(2m+1)} \left\{ - \sum_{n=0}^{\infty} \frac{\Gamma(2m + n + 1)}{\Gamma(2 - p + n)} \frac{\Gamma(2m + p + n)}{\Gamma(n + 1)} z^n + \sum_{n=0}^{\infty} \frac{\Gamma(2m + p + n)}{\Gamma(n + 1)} z^{n+p-1} \right\}.
$$

To compute $B_p^{(2m)}(s)$ for arbitrary real $p$, we set $z = s(1-\lambda^2)$ and perform the $\lambda$-integrals explicitly, using (5.11),

$$
B_p^{(2m)} = \frac{\Gamma(m + \frac{1}{2}) \Gamma(1-p)}{\Gamma(2m+1)} \left\{ - \sum_{n=0}^{\infty} \frac{\Gamma(2m + n + 1) \Gamma(n + 1)}{\Gamma(2 - p + n) \Gamma(m + n + \frac{3}{2})} s^n + \sum_{n=0}^{\infty} \frac{\Gamma(2m + p + n) \Gamma(n + p)}{\Gamma(m + n + p + \frac{1}{2}) \Gamma(n + 1)} s^{n+p-1} \right\}.
$$

The prefactor $\Gamma(1-p)$ will produce a pole at positive integer values of $p$, but the combined expression is regular in view of the fact that the two series inside of the brace of (5.31) cancel one another when $p$ is a positive integer. It is straightforward to obtain the derivative with respect to $p$ of this expression, and then to set $p$ to a positive integer. The result is as follows

$$
\tilde{B}_p^{(2m)} = \frac{\Gamma(m + \frac{1}{2})}{\Gamma(p) \Gamma(2m+1)} \left\{ \sum_{n=0}^{p-2} (-s)^n \frac{\Gamma(2m + n + 1) \Gamma(p - n - 1) \Gamma(n + 1)}{\Gamma(m + n + \frac{3}{2})} (\psi(p - n - 1) - \psi(p)) \right. + \left. (-)^p \sum_{n=0}^{\infty} \frac{\Gamma(2m + p + n) \Gamma(n + p)}{\Gamma(m + n + p + \frac{1}{2}) \Gamma(n + 1)} H_n(s) s^{n+p-1} \right\}.
$$

where the functions $H_n(s)$ are defined by

$$
H_n(s) = \frac{1}{2} (\ln s)^2 + [h_n + \psi(n+1) - \psi(p)] \ln s + \frac{1}{2} h'_n + \frac{1}{2} h_n [h_n + 2\psi(n+1) - 2\psi(p)]
$$

where $h_n$ was defined in (5.13) and $h'_n$ is given by

$$
h'_n = \psi'(n+1) - \psi'(n+p) + \psi'(m + n + p + \frac{1}{2}) - \psi'(2m + p + n).
$$

In particular the logarithm square term may be isolated and yields

$$
\tilde{B}_p^{(2m)}(s) \bigg|_{(\ln s)^2} = (-)^p s^{p-1} (\ln s)^2 \frac{\Gamma(m + \frac{1}{2}) \Gamma(2m + p)}{\Gamma(2m+1) \Gamma(m + p + \frac{1}{2})} F(2m + p; m + p + 1; s).
$$

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We see that the presence of logarithmic singularities in $\rho(\tau)$ generates logarithm squared terms in the amplitude when $\Delta_1 + \Delta_3 \geq \Delta$.

**e. Expansions in the Crossed Channel $x_3 - x_2 \to 0$**

In the crossed channel, the expansion is given in terms of $y \to 0$, while $x$ is kept fixed, corresponding to $s \to \frac{1}{2}$ and $t \to +1$. We shall show here that the amplitude behaves in a universal way in this limit, as long as the dimensions $\Delta_i$ and $\Delta$, as well as $\delta$ are integers. In particular, there is no need here to make assumptions about the relative magnitudes of the dimensions, as long as they obey the convergence requirements of (3.2).

Our starting point is the universal formula (4.7). Around $s = \frac{1}{2}$ and $t = 1$, the composite variable $\omega$ behaves as $\omega \sim 1$. As a result, the universal function $I(\tau; s, t)$ may be split into a logarithmically singular and a regular analytic part, $I = I_{\text{sing}} + I_{\text{reg}}$ as follows,

$$I_{\text{sing}}(\tau; s, t) = -\ln(1 - t^2) \sqrt{\omega^2 - (1 - t^2)\tau^2}$$

$$I_{\text{reg}}(\tau; s, t) = \frac{-\ln \tau^2 + 2 \ln \{\omega + \sqrt{\omega^2 - (1 - t^2)\tau^2}\}}{\sqrt{\omega^2 - (1 - t^2)\tau^2}}$$

where $\omega = 1 + (2s - 1)(1 - \tau)$ for short, as in (4.7). The square root in $I_{\text{sing}}$ as well as the full function $I_{\text{reg}}$ admits convergent series expansions in powers of $(1 - t^2)$ and $(2s - 1)$, with coefficients which are regular functions of $\tau$, leading to convergent $\tau$-integrations versus the spectral function $\rho(\tau)$. As a result, $I_{\text{reg}}$ contributes to the amplitude $B_R(x, y)$ through a convergent power series in $(1 - t^2)$ and $(2s - 1)$. $I_{\text{sing}}$ on the other hand, has an overall factor of $\ln(1 - t^2)$, and multiplies a function that also admits a convergent power series in $(1 - t^2)$ and $(2s - 1)$. The $\ln(1 - t^2)$ factor simply gives the factor $\ln Y/X$ in $\hat{B}_R(X, Y, Z)$ and higher powers of $(1 - t^2)$ can be dropped if we are interested only in the leading term as $Y \to 0$. The power series in $(2s - 1)$ is then a geometric series, and we obtain

$$\hat{B}_R(X, Y, Z) \approx -\frac{1}{X} \ln \frac{Y}{X} \sum_{n=0}^{\infty} (1 - 2s)^n \int_0^1 \frac{d\tau}{\tau} \rho(\tau)(1 - \tau)^n + \frac{1}{X} O(Y/X).$$

(5.37)

This must be inserted in (4.4) and differentiated, and the result depends on whether $n_y$ is positive or zero. If $n_y = 0$ the leading singularity in $Y$ is logarithmic, and its coefficient comes from the $n = n_x$ term of the series in (5.37). We give the final result for the leading term in the case $\Delta_1 = \Delta_3$ and $\Delta_2 = \Delta_4$, which is

$$A(x_c) \sim \frac{1}{x_{31}^2 \Delta_1} \frac{1}{x_{24}^2 \Delta_2} \ln\left(\frac{x_{23} x_{41}}{x_{21} x_{34}}\right) \int_0^1 \frac{d\tau}{\tau} \rho(\tau)(1 - \tau)^{\Delta_2 - 1}.$$  

(5.38)

When $n_y > 0$, the leading term contains the power singularity $1/(x_{23})^{2n_y}$.
6. CONCLUSIONS AND OUTLOOK

In this paper we have studied the contribution of the exchange of a bulk scalar field in $\text{AdS}_{d+1}$ to the correlation function of four scalar operators of arbitrary scale dimension in the boundary CFT. In the most general case the amplitude is far more complicated than the calculation of gauge boson exchange in [11], but there are simplifications if certain conditions involving the dimensions $\Delta_i$, $\Delta$, and $d$ are satisfied. We note here the role of the triangle inequality $\Delta < \Delta_1 + \Delta_3$. If satisfied, the final amplitude is as simple as for gauge exchange, and the derivation could also be simplified by substituting (3.22) in (3.10) and performing the $\gamma$ integral immediately, roughly as in [11]. It is also striking that the triangle inequality is satisfied, up to marginal cases, for chiral primary operators in the Type IIB $\text{AdS}_5 \times S_5$ theory, and we have proposed a particular interaction which simplifies the marginal cases in the Appendix. When $\Delta > \Delta_1 + \Delta_3$, the amplitude is more complicated, and we have been able to show that its short-distance limit is also different, contains $(\ln s)^2$ terms, and cannot obey the double OPE. We observed that 3-point functions [20] also diverge in this case.

The major question still left open by this work is the question of logarithmic singularities which have now appeared in all recent studies [9,8,11,14] of 4-point correlators. It is still the case that no complete correlator of the $\text{AdS}_5 \times S_5$ theory has been obtained, and we hope that a nearly complete calculation of graviton exchange will achieve that goal [27]. If logarithms survive then the proposed interpretation [17] in terms of anomalous dimensions of double-trace operators can be studied. The results for these dimensions could then be new non-perturbative information from the AdS/CFT correspondence.

A. APPENDIX

In this Appendix we discuss the marginal case $\Delta = \Delta_1 + \Delta_3$ of the triangle inequality $\Delta < \Delta_1 + \Delta_3$. In particular we will discuss the case $\Delta_1 = \Delta_3 = k$ and $\Delta = 2k$ for AdS scalars corresponding to chiral primary operators $\text{tr}X^k$ of the $\text{AdS}_5 \times S_5$ theory. Here, there is the somewhat paradoxical situation in that the super Yang-Mills 3-point function $\langle \text{tr}X^k(x_1)\text{tr}X^{2k}(y)\text{tr}X^k(x_3) \rangle$ has a non-vanishing free-field amplitude shown to be equal to the corresponding supergravity correlator [4] despite the fact that the integrals for 3-point correlators computed in [20] diverge in this case. The finite result in [4] comes from a zero in the cubic Lagrangian coupling which multiplies the infinite value of the integral. Further, the simplifying truncation for 4-point functions in the present paper fails for this case, and the 4-point correlator appears to have $(\ln s)^2$ terms rather than the powers expected in the double OPE.
These paradoxes can be explained by the simple hypothesis that the actual cubic coupling of the bulk fields $\phi_k$ and $\phi_{2k}$ have the particular form

$$\mathcal{L}_k = \frac{1}{2} c [D^m u \phi_{2k} \partial_m u(\phi_k^2) + m_{2k}^2 \phi_{2k} \phi_k^2], \quad (A.1)$$

where $m_{2k}^2 = 2k(2k - d)$ is the mass squared of the heavier field, rather than the non-derivative coupling used in [4]. It is exactly for this interaction that two divergent bulk integrals for the 3-point function formally cancel by partial integration, leaving a boundary term, i.e.

$$V(x_1, x_3, y) \equiv c \int \frac{d^{d+1}z}{z_0}\left\{ z_0^2 \partial_\mu \left( \frac{z_0^2}{(z - x_1)^2(z - x_3)^2} \right)^k \partial_\mu \left( \frac{z_0}{(z - y)^2} \right)^{2k} + \frac{2k(2k - d)z_0^{4k}}{(z - x_1)^2(z - x_3)^2k(z - y)^{4k}} \right\} \quad (A.2)$$

$$= \lim_{z_0 \to 0} c \int d^d z \frac{z_0^{2k-d+1}}{(z - x_1)^{2k}(z - x_3)^{2k}} \partial_0 \left( \frac{z_0}{(z - y)^2} \right)^{2k} \quad (A.3)$$

Proceeding heuristically we note (3.20) that

$$\lim_{z_0 \to 0} \partial_0 \left( \frac{z_0}{(z - y)^2} \right)^{2k} \sim \frac{2k}{C_{2k}} z_0^{d-2k-1} \delta^{(d)}(z - y)$$

where $C_{2k}$ is the normalization constant of (2.6). Thus the heuristic limit of (A.2) is

$$V(x_1, x_3, y) = c \int d^d z \frac{z_0^{2k-d+1}}{(z - x_1)^{2k}(z - x_3)^{2k}} \partial_0 \left( \frac{z_0}{(z - y)^2} \right)^{2k} \quad (A.4)$$

which has the correct coordinate dependence. We cannot be certain that the coefficient is correct because the situation is similar to that of 2-point functions in [20] where it was shown that the heuristic limit gave an incorrect coefficient and that a more careful calculation using Ward identities or momentum space was required. This point needs further study.

Next we discuss a 4-point function in which the interaction $L_k$ of (A.1) appears at the $z$-vertex of Fig. 1 with the field of dimension $2k$ propagating to the $w$-vertex where there is a non-derivative interaction with fields of dimension $\Delta_2$ and $\Delta_4$. We note that the partial integration similar to that in (A.2) can be made with no residual boundary term because $w_0 \neq 0$. Instead we find $(\Box - 2k(2k - d))G_{2k}(u) = \delta^{d+1}(z - w)$ from the bulk-to-bulk propagator. Thus the $z$-integral in the 4-point function can be done immediately leaving the amplitude

$$B(x_1) = \int d^{d+1} w \frac{w_0^{2k+\Delta_2+\Delta_4}}{w_0^{d+1} (w - x_1)^{2k}(w - x_3)^{2k}(w - x_2)^{2\Delta_2}(w - x_4)^{2\Delta_4}} \quad (A.5)$$

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So the 4-point function is described by an effective contact interaction for which a parametric integral representation [6,9] can be obtained. We note that this amplitude has a finite limit as $x_3 \to x_1$ which has the exact form of a 3-point function of operators of dimension $2k, \Delta_2, \Delta_4$, and thus the coordinate dependence

$$
\lim_{x_3 \to x_1} B(x_i) = \frac{c'}{(x_1 - x_2)^{2k+\Delta_2-\Delta_4}(x_1 - x_4)^{2k+\Delta_4-\Delta_2}(x_2 - x_4)\Delta_2+\Delta_4-2k} \quad (A.6)
$$

In the further limit $x_2 \to x_4$, we find exactly the expected contribution to the double OPE of a primary operator of dimension $\Delta = 2k$.

Thus the hypothesis of an interaction of the form (A.1) appears to resolve the paradoxes discussed above. It gives well defined 3- and 4-point correlation functions whose coordinate dependence agrees with expectations. It should be noted that for the case $\Delta > \Delta_1 + \Delta_3$, one can also define a combination of derivative and non-derivative cubic interactions for which the divergent bulk integral for 3-point functions can be formally replaced by a surface term. In this case, however, the surface term is divergent, so it does not seem possible to achieve well-defined 3-point correlators.

There is still another issue to which the interaction (A.1) seems relevant, namely the question of decoupling of fields in the graviton multiplet from higher Kaluza-Klein modes in the $\text{AdS}_5 \times S^5$ theory. Since the correlator $\langle \text{tr}X^2\text{tr}X^4\text{tr}X^2 \rangle$ is non-vanishing, the theory certainly has a cubic coupling $\sim \phi_4 \phi_2 \phi_2$ when expressed in terms of the fields that directly correspond to chiral primary operators. However, one may try to redefine fields to eliminate the coupling. For the Lagrangian

$$
\mathcal{L} = \frac{1}{2} [D^\mu \phi_4 \partial_\mu \phi_4 + D^\mu \phi_2 \partial_\mu u \phi_2 + c'' \phi_4 \phi_2 \phi_2] \quad (A.7)
$$
decoupling can be achieved by redefining $\phi_4 = \Phi_4 + 1/2c'' \Box^{-1} \phi_2^2$ and decoupling by a non-local transformation is generic for a combination of derivative and non-derivative interactions. However, precisely for the interaction $L_k$ of (A.1) with $k = 2$, the decoupling transformation is local, viz. $\phi_4 = \Phi_4 - 1/2c' \phi_2^2$. One should note that other interaction terms such as $\phi_4^4$ or $\phi_4^2 \phi_2^2$ can spoil local decoupling unless they also appear in special combinations. So further study of this and of the interactions of descendent fields is required before the issue of local decoupling can be settled. The decoupling issue is related to recent work [28] exploring classical solutions of the $\text{AdS}_5 \times S^5$ theory which interpolate between critical points. Non-local decoupling is apparently required to justify the restriction to solutions involving only the metric and scalars of the graviton multiplet and no Kaluza-Klein excitations.*

* We thank O. Aharony for this observation, and thank him and also S. Gubser, K.
ACKNOWLEDGMENTS

It is a pleasure to acknowledge helpful conversations with Samir Mathur, Alec Matusis and Leonardo Rastelli. They are our collaborators on related projects and have been generous with their advice for this one as well. E. D. also wishes to thank the members of the Institute for Theoretical Physics, Santa Barbara for their hospitality while part of this work was completed.

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