Geometric Discord Of the Jaynes-Cummings Model: Pure Dephasing Regime

Sina Hamedani Raja*, 1 Hamidreza Mohammadi†,1,2 and S. Javad Akhtarshenas‡3

1Department of Physics, University of Isfahan, Isfahan, Iran
2Quantum Optics Group, University of Isfahan, Isfahan, Iran
3Department of Physics, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract

In this paper, we study dynamical behavior of geometric discord of a system including a two-level atom interacting with a quantized radiation field, described by the Jaynes-Cummings Hamiltonian. The evolution of the system has been considered in the pure dephasing regime with the field, initially, in a general pure state and the atom in a mixed state. We compare dynamics of geometric discord, as a measure of non-classical correlation, with dynamics of negativity, as a measure of quantum entanglement. In particular, we exemplify the effects of different system parameters such as detuning and mixedness of the initial atomic state on the dynamics of geometric discord for two important initial state of the field, i.e. coherent and number states. It is shown that for asymptotically large times the steady state geometric discord of the system gains an optimum value at some values of detuning.

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* s.hamedani@sci.ui.ac.ir
† h.mohammadi@sci.ui.ac.ir
‡ akhtarshenas@um.ac.ir
I. INTRODUCTION

Nowadays, quantum discord is proved to be a resource to enhance the quality of quantum information and computation processing [1]. This approach of quantum correlation expresses a different type of correlation [2, 3] in comparison with entanglement in which we concern about separability. Quantum discord sheds a new light on the concept of correlation in composite systems and reveals that there is some separable, i.e. disentangled, multipartite states which possess quantum correlation and hence they can be employed as a resource for improving the quantum information processes, speeding up the quantum computation algorithms and/or performing the quantum communication protocols. Despite of entanglement which looks on correlation form the separability point of view, quantum discord captures quantum correlation from the measurement perspective [4, 5]. Indeed, calculations of quantum discord requires to find the best (optimized) measurement performing on the one part of the system, making therefore, calculation of quantum discord more complicated and prevents one to obtain an analytical closed formula for quantum discord, in general. Quantum discord is analytically calculated only for a few families of two-qubit states [6] and for some reduced two-qubit states of pure three-qubit states and also for a class of rank-2 mixed state of $4 \otimes 2$ systems [7]. Nevertheless, the geometric measure of quantum discord is in general rather easier to calculate non-classical correlations [8].

Geometric discord is defined as the squared Hilbert-Schmidt distance between the state of the quantum system and the closest zero-discord state. For a bipartite state $\rho$ on the Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$, geometric discord $D_G$ is defined as [8]

$$D_G = \min_{\chi \in \chi_0} \|\rho - \chi\|_2^2,$$  \hspace{1cm} (1)

where minimization is taken over the set of all zero discord states $\chi_0$, and $\|\rho - \chi\|_2^2 = \text{Tr}((\rho - \chi)^2)$ is the squared norm in the Hilbert-Schmidt space, so this quantity vanishes on the classical-quantum states which are quantum correlation-free.

An exact expression of $D_G$ for pure $N \otimes N$ and arbitrary $2 \otimes N$ states are obtained [21, 22]. Consider a bipartite density matrix $\rho$ acting on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^N$. One can write $\rho$ in the Bloch representation as

$$\rho = \frac{1}{2N} \left( \mathbb{I}^A \otimes \mathbb{I}^B + \sum_{i=1}^{3} x_i \sigma_i \otimes \mathbb{I}^B + \mathbb{I}^A \otimes \sum_{j=1}^{N^2-1} y_j \lambda_j + \sum_{i=1}^{3} \sum_{j=1}^{N^2-1} t_{ij} \sigma_i \otimes \lambda_j \right),$$  \hspace{1cm} (2)
where $\{\sigma_i\}_{i=1}^3$ are the usual Pauli matrices, and $\{\lambda_j\}_{j=1}^{N^2-1}$ are generators of $SU(N)$, fulfilling the following equations

$$\text{Tr}\lambda_i = 0, \quad \text{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}. \quad (3)$$

Also $\vec{x} = (x_1, x_2, x_3)^t$ with $x_i = \text{Tr}[(\sigma_i \otimes I_B)\rho]$, and $\vec{y} = (y_1, \ldots, y_{N^2-1})^t$ with $y_j = \frac{N}{2} \text{Tr}[(I_A \otimes \lambda_j)\rho]$ denote local coherence vectors of two subsystems, and $T = (t_{ij})$ with $t_{ij} = \frac{N}{2} \text{Tr}[(\sigma_i \otimes \lambda_j)\rho]$ is the correlation matrix of the state. It is shown that $D_G$ of $2 \otimes N$ states can be written as [22]

$$D_G = \frac{1}{2N}\left(\|\vec{x}\|^2 + \frac{2}{N}\|T\|^2 - \xi_{\text{max}}\right) = \frac{1}{2N}(\xi_2 + \xi_3), \quad (4)$$

where $\{\xi_k\}_{k=1}^3$ are eigenvalues of $(\vec{x}\vec{x}^t + \frac{2}{N}TT^t)$, in non-increasing order, and $\xi_{\text{max}} = \max\{\xi_k\} = \xi_1$.

The Jaynes-Cummings model (JCM), which describes the interaction of a single two-level atom with the quantized radiation field [9], has been the subject of numerous studies in the field of quantum optics due to its bright expressing of physical concepts such as collapse and revival. Furthermore, it has attracted much attention in order to study the quantum correlations. A lot of studies have been devoted to create entanglement between the atom-field state and to control and quantify it. For example, the entanglement properties of a system described by the JCM has studied in the situation where the atom is initially in a mixed state and the field is initially in a squeezed and coherent state [10]. The entanglement properties of the JCM has also investigated where atom is initially in a mixed state and field has considered a thermal state [11]. In addition, the entanglement of the JCM is measured by employing the negativity as a degree of entanglement in [12], for which initial state of atom is a mixed state and field is initially in a coherent state and the system of atom-field undergoes a unitary evolution.

The evolution of the JCM under decoherence has also been the subject of some studies. Such evolution is considered in [13] where the authors assumed the atom initially be in a pure state and the field initially be in a number state. Also in [14] Buzek et al. has investigated the dephasing evolution of the JCM where initial state of atom is a mixed state and the field is initially in a vacuum state.

Furthermore, the stationary entanglement of the JCM has studied in dephasing condition in [15] where the authors investigated the negativity of the JCM regarding the situation in which atom is initially in a mixed state and field is initially in a number state and they
have shown that we can find an optimum detuning in which the steady state negativity is maximum.

Also, the atom-field interaction recently has received attention in order to investigate quantum discord. For instance, the quantum discord dynamics of a closed system described by the JCM has studied in [16], where the atom is in a thermal equilibrium mixed state and field is in a number state. However the dynamics of discord of the open systems described by JCM for an initially mixed state of the atom and a coherent initial state of the field has not yet considered.

In this paper, we first study the pure dephasing evolution of the JCM where the atom is assumed to be initially in an arbitrary mixed state and the field is initially in a general pure state. As we mentioned, this evolution has been the subject of some studies such as [14] [15] but in comparison, this paper presents the analytic calculation of time evolution of the JCM in pure dephasing condition with a more general set of initial states. By achieving the state of system at any time $t$, we calculate the geometric discord of the JCM in two important cases number state and coherent state as the initial states of the field. We investigate the behavior of geometric discord of the JCM in these cases and show how geometric discord of system at large enough times, steady state $D_G$, dependeds on detuning and mixedness parameters of the system. It is shown that similar to the negativity, the steady state $D_G$ of the system gains an optimum amount at some values of detuning.

The paper is organized as follows. In section II, we briefly review the Jaynes-Cummings Model. Pure dephasing evolution of the system is studied in Section III. Section IV includes the calculation of geometric discord for two distinct initial states of the field i.e. number and coherent states. In Section V, we present our results and conclude the paper with some discussions.

II. THE JAYNES-CUMMINGS MODEL

The Jaynes-Cummings model (JCM) describes the interaction of a two-level atom with a single-mode quantized radiation field [9]. The Hamiltonian of this model is given by

$$H = H_0 + H_I,$$  \hspace{1cm} (5)
where $H_0$ and $H_I$ are, respectively, the internal and interaction energy of the atom-field system, defined by (for $\hbar = 1$)

$$H_0 = \frac{1}{2} \omega_A \sigma_z + \omega_F a^\dagger a,$$

(6)

$$H_I = g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger).$$

(7)

Here $\omega_A$ and $\omega_F$ are the transition frequency of the two-level atom and the frequency of the radiation field, respectively, and $g$ denotes the atom-field coupling. Also $\sigma_+$, $\sigma_- = \sigma_+^\dagger$ are atomic spin-flip operators and $\sigma_z$ is the atomic inversion operator which act on the atom Hilbert space $\mathcal{H}^A$. Also $a$ and $a^\dagger$ are annihilation and creation operators of the field acting on the field Hilbert space $\mathcal{H}^F$. The two-dimensional Hilbert space of the atom is spanned by two orthonormal states $|g\rangle \rightarrow (0,1)^t$ and $|e\rangle \rightarrow (1,0)^t$ and the field Hilbert space is spanned by photon number states $\{|n\rangle = (a^\dagger)^n |0\rangle \}_{n=0}^{\infty}$, where $|0\rangle$ is the vacuum state of the radiation field.

To achieve the time evolution of the system, it is convenient to work with dressed-state representation of the Hamiltonian $[5]$. Regarding the fact that Hamiltonian (5) conserves the total number of excitation $K = (a^\dagger a + 1/2 \sigma_z)$, one can decompose the atom-field Hilbert space $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^F$ as $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ such that $\mathcal{H}_0 = \text{span}\{|g,0\}\}$ and $\mathcal{H}_{n+1} = \text{span}\{|e,n\},|g,n+1\}\}$ for $n \in \{0,1,2,...\}$, are eigen-subspaces of $K$ with corresponding eigenvalues $-1/2$ and $(n + 1/2)$, respectively. Accordingly, Hamiltonian $[5]$ has the eigenvalues

$$E_0 = -\frac{1}{2} \omega_A, \quad E^{(n)}_{\pm} = \omega_F(n + \frac{1}{2}) \pm \Omega_n,$$

(8)

with the corresponding eigenvectors

$$|\Phi_0\rangle = |g,0\rangle,$$

$$|\Phi^{(n)}_+\rangle = \sin \theta_n |e,n\rangle + \cos \theta_n |g,n+1\rangle,$$

$$|\Phi^{(n)}_-\rangle = \cos \theta_n |e,n\rangle - \sin \theta_n |g,n+1\rangle,$$

(9)

where $\tan \theta_n = \frac{2g\sqrt{n+1}}{\Delta + 2\Omega_n}$, $\Omega_n = \sqrt{(\Delta/2)^2 + g^2(n+1)}$ is the Rabi frequency, and $\Delta = \omega_A - \omega_F$ is the detuning parameter of the system.
III. PURE DEPHASING EVOLUTION OF THE SYSTEM

In the following, we assume that our system undergoes a pure dephasing evolution which obeys the following equation\[17–19\]

$$\frac{d}{dt}\rho(t) = -i[H,\rho(t)] - \frac{\gamma}{2}[H,[H,\rho(t)]],$$  \hspace{1cm} \text{(10)}

where $\gamma$ is the dephasing parameter and $\rho(t)$ is the density matrix of the atom-field in any given time $t \geq 0$. Equation (10) has the following formal solution [13, 20]

$$\rho(t) = \sum_{k=0}^{\infty} \frac{(\gamma t)^k}{k!} M^k(t)\rho(0)M^k(t),$$  \hspace{1cm} \text{(11)}

such that $\rho(0)$ is the initial state of the system and

$$M^k(t) = H^k \exp(-iHt) \exp(-\frac{\gamma t}{2}H^2).$$  \hspace{1cm} \text{(12)}

In order to find the time-evolved state $\rho(t)$, we need to expand the initial state $\rho(0)$ in terms of the dressed-states, i.e. the Hamiltonian eigenbasis [0], as

$$\rho(0) = |\Phi_0\rangle\langle \Phi_0|\rho(0)|\Phi_0\rangle\langle \Phi_0| + \sum_{\alpha=\pm} \sum_{n=0}^{\infty} |\Phi^{(n)}_\alpha\rangle\langle \Phi^{(n)}_\alpha|\rho(0)|\Phi_0\rangle\langle \Phi_0|$$

$$+ \sum_{\beta=\pm} \sum_{m=0}^{\infty} |\Phi^{(m)}_\beta\rangle\langle \Phi^{(m)}_\beta|\rho(0)|\Phi_0\rangle\langle \Phi_0| + \sum_{\alpha,\beta=\pm} \sum_{m,n=0}^{\infty} |\Phi^{(n)}_\alpha\rangle\langle \Phi^{(n)}_\alpha|\rho(0)|\Phi^{(m)}_\beta\rangle\langle \Phi^{(m)}_\beta|.$$  \hspace{1cm} \text{(13)}

Using this in Eq. (11) we get

$$\rho(t) = |\Phi_0\rangle\langle \Phi_0|\rho(0)|\Phi_0\rangle\langle \Phi_0|$$

$$+ \sum_{\alpha=\pm} \sum_{n=0}^{\infty} |\Phi^{(n)}_\alpha\rangle\langle \Phi^{(n)}_\alpha| \exp\left(-i(\omega^{(n)}_\alpha)t - \frac{\gamma t}{2}(\omega^{nm}_{\alpha\beta})^2\right) \langle \Phi^{(n)}_\alpha|\rho(0)|\Phi_0\rangle\langle \Phi_0|$$

$$+ \sum_{\beta=\pm} \sum_{m=0}^{\infty} |\Phi^{(m)}_\beta\rangle\langle \Phi^{(m)}_\beta| \exp\left(i(\omega^{(m)}_\beta)t - \frac{\gamma t}{2}(\omega^{nm}_{\beta\alpha})^2\right) \langle \Phi^{(m)}_\beta|\rho(0)|\Phi_0\rangle\langle \Phi_0|$$

$$+ \sum_{\alpha,\beta=\pm} \sum_{m,n=0}^{\infty} |\Phi^{(n)}_\alpha\rangle\langle \Phi^{(n)}_\alpha| \exp\left(-i(\omega^{nm}_{\alpha\beta})t - \frac{\gamma t}{2}(\omega^{nm}_{\alpha\beta})^2\right) \langle \Phi^{(n)}_\alpha|\rho(0)|\Phi^{(m)}_\beta\rangle\langle \Phi^{(m)}_\beta|,$$

where we have defined

$$\omega^{nm}_{\alpha\beta} = E^{(n)}_\alpha - E^{(m)}_\beta, \hspace{1cm} \omega^{(n)}_\alpha = E^{(n)}_\alpha - E_0.$$  \hspace{1cm} \text{(15)}

Now let us suppose that, initially at $t = 0$, the system is found in the product state

$$\rho(0) = \rho^A(0) \otimes \rho^F(0),$$  \hspace{1cm} \text{(16)}
such that $\rho^A(0)$ is the initial state of the atom and is considered to be a mixed state
\[
\rho^A(0) = p|e\rangle\langle e| + (1 - p)|g\rangle\langle g|, \quad 0 \leq p \leq 1,
\] (17)
and $\rho^F(0)$, the initial state of the field, is assumed to be a pure state
\[
\rho^F(0) = |\eta\rangle\langle \eta|,
\] (18)
with $|\eta\rangle = \sum_{n=0}^{\infty} b_n |n\rangle$, where the complex coefficients $b_n$s satisfy the normalizing condition $\sum_{n=0}^{\infty} |b_n|^2 = 1$. In section [IV], we will fix the coefficients $b_n$ for two special cases, namely, number states and coherent states. Accordingly, inserting Eqs. (16)-(18) into Eq. (14) and using orthonormal basis $\{|e_1\rangle \equiv |e\rangle, |e_2\rangle \equiv |g\rangle\}$ for the atomic Hilbert space, one can find after tedious but straightforward calculations the following representation for the density matrix $\rho(t)$
\[
\rho(t) = \begin{pmatrix}
\hat{A}(t) & \hat{C}(t) \\
\hat{C}^\dagger(t) & \hat{B}(t)
\end{pmatrix},
\] (19)
where $\hat{A}(t)$, $\hat{B}(t)$, and $\hat{C}(t)$, operators acting on the field Hilbert space, are defined by
\[
\hat{A}(t) = p\hat{A}^{(e)}(t) + (1 - p)\hat{A}^{(g)}(t),
\] (20)
\[
\hat{B}(t) = p\hat{B}^{(e)}(t) + (1 - p)\hat{B}^{(g)}(t),
\] (21)
\[
\hat{C}(t) = p\hat{C}^{(e)}(t) + (1 - p)\hat{C}^{(g)}(t),
\] (22)
where matrix elements of operators $\hat{A}^{(e,g)}(t)$, $\hat{B}^{(e,g)}(t)$, and $\hat{C}^{(e,g)}(t)$ in the Fock basis $\{|n\rangle\}_{n=0}^{\infty}$ are defined as
\[
A_{nm}^{(e)}(t) = b_n b_m^* \left[ \sin^2(\theta_n) \left[ \sin^2(\theta_m) \exp \left( - i\omega_{++}^n t - \frac{\gamma t}{2} (\omega_{++}^n)^2 \right) 
\right.ight.
\sin^2(\theta_m) \exp \left( - i\omega_{++}^m t - \frac{\gamma t}{2} (\omega_{++}^m)^2 \right)
\left. \right] + \right.
\cos^2(\theta_m) \exp \left( - i\omega_{+-}^m t - \frac{\gamma t}{2} (\omega_{+-}^m)^2 \right) + \cos^2(\theta_m) \exp \left( - i\omega_{-+}^m t - \frac{\gamma t}{2} (\omega_{-+}^m)^2 \right) + \right.
\cos^2(\theta_m) \exp \left( - i\omega_{-+}^m t - \frac{\gamma t}{2} (\omega_{-+}^m)^2 \right)
\right)
\right),
\] (23)
\[
A_{nm}^{(g)}(t) = \frac{1}{4} b_{n+1} b_{m+1}^* \sin (2\theta_n) \sin (2\theta_m) \left[ \exp \left( - i\omega_{++}^n t - \frac{\gamma t}{2} (\omega_{++}^n)^2 \right) - \exp \left( - i\omega_{++}^m t - \frac{\gamma t}{2} (\omega_{++}^m)^2 \right)
\right.
\left. - \exp \left( - i\omega_{--}^n t - \frac{\gamma t}{2} (\omega_{--}^n)^2 \right) + \exp \left( - i\omega_{--}^m t - \frac{\gamma t}{2} (\omega_{--}^m)^2 \right) \right],
\] (24)
\[ B^{(e)}_{nm}(t) = \frac{1}{4} b_{n-1} b^*_m \sin (2\theta_{n-1}) \sin (2\theta_m) \left[ \exp \left(- i \omega_{++}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{++}^{n-1,m-1})^2 \right) 
- \exp \left(- i \omega_{+-}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{+-}^{n-1,m-1})^2 \right) 
- \exp \left(- i \omega_{-+}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{-+}^{n-1,m-1})^2 \right) 
+ \exp \left(- i \omega_{--}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{--}^{n-1,m-1})^2 \right) \right], \tag{25} \]

\[ B^{(g)}_{nm}(t) = b_n b^*_m \left[ \cos^2(\theta_{n-1}) \left[ \cos^2(\theta_m) \exp \left(- i \omega_{++}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{++}^{n-1,m-1})^2 \right) 
+ \sin^2(\theta_{m-1}) \exp \left(- i \omega_{+-}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{+-}^{n-1,m-1})^2 \right) \right] 
+ \sin^2(\theta_{n-1}) \left[ \cos^2(\theta_m) \exp \left(- i \omega_{-+}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{-+}^{n-1,m-1})^2 \right) 
+ \sin^2(\theta_{m-1}) \exp \left(- i \omega_{--}^{n-1,m-1} t - \frac{\gamma t}{2} (\omega_{--}^{n-1,m-1})^2 \right) \right] \right], \tag{26} \]

\[ C^{(e)}_{nm}(t) = b_n b^*_m \sin (2\theta_{n-1}) \left[ \frac{1}{2} \sin^2(\theta_n) \left[ \exp \left(- i \omega_{++}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{++}^{n,m-1})^2 \right) 
- \exp \left(- i \omega_{+-}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{+-}^{n,m-1})^2 \right) \right] 
+ \frac{1}{2} \cos^2(\theta_n) \left[ \exp \left(- i \omega_{--}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{--}^{n,m-1})^2 \right) 
- \exp \left(- i \omega_{-+}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{-+}^{n,m-1})^2 \right) \right] \right], \tag{27} \]

\[ C^{(g)}_{nm}(t) = \frac{1}{2} b_n b^*_m \sin (2\theta_n) \left[ \cos^2(\theta_{m-1}) \left[ \exp \left(- i \omega_{++}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{++}^{n,m-1})^2 \right) 
- \exp \left(- i \omega_{+-}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{+-}^{n,m-1})^2 \right) \right] 
+ \sin^2(\theta_{m-1}) \left[ \exp \left(- i \omega_{-+}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{-+}^{n,m-1})^2 \right) 
+ \exp \left(- i \omega_{--}^{n,m-1} t - \frac{\gamma t}{2} (\omega_{--}^{n,m-1})^2 \right) \right] \right]. \tag{28} \]

We will employ this representation to calculate the geometric discord.

IV. GEOMETRIC DISCORD OF THE SYSTEM

In this section, we turn our attention on the geometric quantum discord of the state given by Eq. [19]. To do so, we should first note that the state [19] is supported, actually, on a
2 \otimes \infty \text{ Hilbert space and investigating the correlation properties of this system looks like an impossible task. However, as we will show in the following, by considering some appropriate initial states for the field, our state at a given time } t > 0 \text{ can be represented by a } 2N \times 2N \text{ matrix for a finite } N. \text{ In continue, we express two examples of these preparations.}

### A. Number state as initial state of the field

We first consider the case that the field is prepared, initially, in a given number state \( |k\rangle \), i.e. the coefficients \( b_n \) are set to be \( b_n = \delta_{nk} \). In this particular case the atom-field state \( \rho(t) \) of Eq. \([19]\) is supported on a \( 2 \otimes 3 \) Hilbert space. By setting \( \{|k-1\rangle, |k\rangle, |k+1\rangle\} \) as orthonormal basis of the field, one can represent the corresponding density matrix as

\[
\rho_k(t) = \begin{pmatrix}
(1 - p)B_{k-1}(t) & 0 & 0 & -(1 - p)C_{k-1}(t) & 0 \\
0 & pA_k(t) & 0 & 0 & pC_k(t) \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-(1 - p)C^*_{k-1} & 0 & 0 & (1 - p)(A_{k-1}(t) + \delta_{k,0}) & 0 \\
0 & pC^*_k(t) & 0 & 0 & pB_k(t)
\end{pmatrix},
\]

(29)

where

\[
A_k(t) = \frac{1}{4}\left(2 + \frac{\Delta^2}{2\Omega_k^2} + \left(2 - \frac{\Delta^2}{2\Omega_k^2}\right) \cos (2\Omega_k t) \exp (-2\gamma t \Omega_k^2)\right),
\]

(30)

\[
C_k(t) = \frac{g\sqrt{k+1}}{4\Omega_k} \left(\frac{\Delta}{\Omega_k} (1 - \cos (2\Omega_k t) \exp (-2\gamma t \Omega_k^2)) + 2i \sin (2\Omega_k t) \exp (-2\gamma t \Omega_k^2)\right),
\]

(31)

\[
B_k(t) = \frac{g^2(k+1)}{2\Omega_k^2} \left(1 - \cos (2\Omega_k t) \exp (-2\gamma t \Omega_k^2)\right).
\]

(32)

This density matrix can be used to obtain coherence vector \( \vec{x} \) and correlation matrix \( T \) and hence the geometric discord. In comparison with the results of \([15]\) where the authors considered the atom initially in the exited state, we can set \( p = 1 \) in Eq. \((29)\) obtaining the dephasning evolution of the JCM when the atom is initially in the exited state.

### B. Coherent state as initial state of the field

As a second example, we consider the case that the field is initially in a coherent state \( |\alpha\rangle \) i.e. \( b_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \). Although this initial condition implies a \( 2 \otimes \infty \) support for \( \rho(t) \),
the Poissonian distribution of photon numbers of coherent states allows us to truncate the dimension of the field Hilbert space to a finite one. Numerically, it is seen that if we set \( |\alpha| = \sqrt{5} \) the coefficients \( b_n \)s will be negligible for \( n > 30 \); Indeed, the ratio \( \frac{|b_{n+1}|^2}{|b_n|^2} \) becomes less than \( 10^{-10} \) for large enough values of \( n \), so it is reasonable to restrict the dimension of the field Hilbert space to an appropriate \( N \). To fulfill the adequate accuracy in the case of \( |\alpha| = \sqrt{5} \), numerical calculations have shown that \( N = 30 \) is sufficient enough such that by limiting the summation on \( n \) to \( N = 30 \) the trace of \( \rho(t) \) will be 1 up to \( 10^{-15} \) amount of error, i.e. \( \text{Tr}[\rho(t)] \cong 1 \), so we truncate our field Hilbert space to a 30-dimensional one. This approach has also employed in [12] to calculate negativity of the JCM where the system undergoes a unitary evolution and the consideration of the initial states of atom and field are the same as the present paper. We use the same approach to evaluate the dephasing evolution of the JCM and it can be seen that just by setting the dephasing parameter \( \gamma \) equal to zero, i.e. the unitary evolution, the results of [12] for the state of the system can be achieved. Nevertheless, in the case of dephasing evolution the 25-dimensional field Hilbert space which is used in [12] were no longer accurate in numerical calculations, so we have expanded this dimension to thirty.

Now we focus on calculating the \( D_G \) of the atom-field state while the field is initially in the mentioned coherent state. In this case, it is by far more convenient to use the atomic representation of \( \rho(t) \) as it is argued in [19], hence, each block of the density matrix [19] will be a \( 30 \times 30 \) matrix acting on the field. If we let \( \{\lambda_j\}_{j=1}^{30^2-1} \) to be the generators of \( SU(30) \) and use the Pauli matrices as the generators of the \( SU(2) \) and utilize the representation [19] we have

\[
\begin{align*}
  x_1(t) &= \text{Tr}_F \left[ \hat{C}^\dagger(t) + \hat{C}(t) \right], \\
  x_2(t) &= \text{Tr}_F \left[ -i\hat{C}^\dagger(t) + i\hat{C}(t) \right], \\
  x_3(t) &= \text{Tr}_F \left[ \hat{A}(t) - \hat{B}(t) \right],
\end{align*}
\]

and

\[
\begin{align*}
  t_{1j}(t) &= 15\text{Tr}_F \left[ \lambda_j \left( \hat{C}^\dagger(t) + \hat{C}(t) \right) \right], \\
  t_{2j}(t) &= 15\text{Tr}_F \left[ \lambda_j \left( -i\hat{C}^\dagger(t) + i\hat{C}(t) \right) \right], \\
  t_{3j}(t) &= 15\text{Tr}_F \left[ \lambda_j \left( \hat{A}(t) - \hat{B}(t) \right) \right],
\end{align*}
\]
where \( j = 1, \ldots, 30^2 - 1 \) and \( \text{Tr}_F \) is trace over the field and readily would be applied on matrix representation of operators \( \hat{A}(t), \hat{B}(t), \hat{C}(t) \) and \( \hat{C}^\dagger(t) \). Knowing the coherent vector \( \vec{x} \) and correlation matrix \( T \) corresponding to the density matrix \( \rho(t) \), enable one to calculate the geometric discord via Eq. (4). Meanwhile, the explicit representation of generators of \( SU(30) \) are required; The method argued in [26] has expressed a set of straightforward equations to achieve both Cartan sub-algebra and non-diagonal generators of \( SU(N) \).

V. RESULTS AND DISCUSSIONS

Based on analytical calculations of pure dephasing evolution of the atom-field system described by the JCM, expressed in previous sections, we have calculated the geometric discord \( D_G \) of the state regarding different amounts of the parameters \( \Delta, p \) and \( \gamma \). In all of calculations the parameters \( g \) and \( \omega_A \) are assumed to be 1. The evolution of the state of the system and hence the dynamics of the quantum correlations strongly depend on the initial state of the system. Therefore, in all of the following calculations we suppose that the atom is initially in a mixture of ground and excited state and the field is in a pure initial state. For the pure state of the field we consider two distinct cases: the field is initially in a number state and the field is initially in a coherent state. Also, we mostly concern our attention on quantum correlations of the JCM in pure dephasing evolution, but it is worthwhile to evaluate our results when the dephasing parameter \( \gamma \) is assumed to be zero, i.e. the unitary evolution, and then compare the result when the dephasing process is present.

A. Initiating the field in a number state

We first consider the case that the field is initially prepared in a number state, so that time behavior of the system can be readily realized by looking to the Eq. (29).

- \textit{Unitary Evolution i.e. } \gamma = 0: \\
  Suppose that the initial state of the field is set to be vacuum state \( |0\rangle \). Figure 1 shows time behavior of \( D_G \) when the atom is initially in the excited state \( |e\rangle \), and for two different values of detuning \( \Delta = 0, 1 \). Since evolution is unitary, the pure initial state of the system will remain pure during the evolution. As it is mentioned in Ref. [27], for a \( m \otimes m \) pure state \( |\Psi\rangle = \sum_{i=1}^{m} \sqrt{s_i} |i\rangle |i\rangle \), geometric discord is related to generalized
concurrency as
\[ D_G(\Psi) = 1 - \sum_{i=1}^{m} s_i^2 = \frac{1}{2} C^2(\Psi), \]
where \( C(\Psi) \) is the generalized concurrence of \(|\Psi\rangle\) \[28\]. The authors of \[15\] have investigated the negativity of the JCM with the same assumptions. Comparing the results, truly expresses this resemblance; where regardless the scale of graphs, the time behavior of \( D_G \) and negativity are the same. We can see that as the collapse and revival occurs, the atom and the field becomes periodically correlated and de-correlated.

- **Dephasing regime i.e. \( \gamma \neq 0 \):**

  When the dephasing parameter \( \gamma \) is not zero, the story is different. Decoherence suppresses the coherence oscillations (collapse and revival) of the evolution and after passing enough time, it leads the system to get a stationary state. In Figs. 2 and 3 we have plotted \( D_G \) versus time for different values of \( p \) and \( \Delta \) when the initial state of the field is \(|1\rangle\). Since elements of the density matrix are proportional to \( e^{-\gamma t} \), they vanish after times \( t \gg \frac{1}{\gamma} \), for value of \( \gamma > 0 \). So the state of the system reach a stationary state, asymptotically. Hence we have the following time-independent elements for the

\[ \text{FIG. 1: (Color online) Time evolution of } D_G \text{ when field is initially in vacuum state } |0\rangle, \text{ and } \gamma = 0, \]

\[ p = 1 \text{ with } \Delta = 0 \text{ [solid line] and } \Delta = 1 \text{ [dashed line].} \]
FIG. 2: (Color online) Time evolution of $D_G$ when field is initially in $|1\rangle$ for $\gamma = 0.1$ and $p = 0.5$ with $\Delta = 0$ [solid line], $\Delta = 5$ [dashed line] and $\Delta = 10$ [dot-dashed line].

FIG. 3: (Color online) Time evolution of $D_G$ when field is initially in $|1\rangle$ for $\gamma = 0.1$ and $p = 1$ with $\Delta = 0$ [solid line], $\Delta = 5$ [dashed line] and $\Delta = 10$ [dot-dashed line].
density matrix of the system at the asymptotically large times:

\[\bar{A}_k(\Delta) = \frac{1}{4}\left(2 + \frac{\Delta^2}{(\Delta^2/2) + 2(k + 1)}\right),\]

\[\bar{C}_k(\Delta) = \frac{\Delta\sqrt{k + 1}}{\Delta^2 + 4(k + 1)},\]

\[\bar{B}_k(\Delta) = \frac{(k + 1)}{(\Delta^2/2) + 2(k + 1)}.\]  

(40)

As it is clear, the value of \(\Delta\) acts an important rule in the value of the asymptotic \(D^\infty_G\).

For instance for the value \(\Delta = 0\), we have \(\bar{A}_k(0) = \frac{1}{2}, \bar{C}_k(0) = 0\) and \(\bar{B}_k(0) = \frac{1}{2}\) i.e the density matrix (29) becomes diagonal and consequently \(D^\infty_G = 0\). So it seems that we need a nonzero \(\Delta\) in order to have nonzero \(D^\infty_G\), but on the other hand, for large values of detuning the atom and the field are far from interaction and therefore they remain uncorrelated. Accordingly, the stationary state value of the geometric discord may have a finite optimum value. Figure 4 expresses the asymptotic geometric discord \(D^\infty_G\) versus \(\Delta\) when the field is initially in \(|1\rangle\) and for different values of \(p\). Figure 5 exhibit a 3D plot of \(D^\infty_G\) versus \(\Delta\) and \(p\). These figures reveal that the value of \(D^\infty_G\) seems to

![](image.png)

**FIG. 4:** (Color online) \(D^\infty_G\) as a function of \(\Delta\) when field is initially in \(|1\rangle\) with \(p = 0\) [solid line], \(p = 0.5\) [dot-dashed line] and \(p = 1\) [dashed line].

be not symmetric with respect to the parameter \(p\). Since the one-dimensional dressed state \(|g, 0\rangle\) does not change through the evolution (because it is a one-dimensional trivial decoherence-free-subspace), it turns out that the rules of \(p\) and \((1 - p)\) differ in
evolution, leads therefore to the above mentioned asymmetry.

FIG. 5: (Color online) $D_G^\infty$ as a function of $\Delta$ and $p$ when field is initially in the number state $|1\rangle$.

B. Initiating the field in a coherent state

We now check our results in the case that the field is prepared initially in a coherent state $|\alpha\rangle$. In what follows we set the field density as $|\alpha| = \sqrt{5}$, so it is enough to use only 30-dimensional truncated subspace of the field Fock space.

- **Unitary evolution i.e $\gamma = 0$:**
  First suppose the evolution is unitary, i.e. $\gamma = 0$. Figure 6 shows $D_G$ of the JCM when the atom is initially in the excited state, i.e. $p = 1$, and in resonance with the field, i.e. $\Delta = 0$. Authors of [12] have calculated the negativity and mutual information of the JCM with the same assumptions. Evidently, without concerning about the scales, Fig. 6 exhibits coincidence of the results: the dynamical behavior of $D_G$ coincides with the dynamical behavior of negativity presented in [12].

- **Dephasing regime i.e $\gamma \neq 0$:**
  The dephasing mechanism tends to demolish the non-diagonal elements of the density
FIG. 6: (Color online) Time evolution of $D_G$ when field is initially in $|\alpha\rangle$ for $\gamma = 0$ and $p = 1$ with $|\alpha| = \sqrt{5}$.

Further calculation reveals that, for the asymptotic case i.e. when $t \rightarrow \infty$, the support of the density operator of the system lies only on a 4-dimensional subspace, irrespective to the initial state. One can manage this support by changing the system parameters e.g. detuning, $\Delta$. On this case the amounts of the asymptotic discord depend on $\Delta$ and $p$. For instance, for the case of $p = 0.5$, the support of the asymptotic states coincides for both
FIG. 7: (Color online) $D_G$ as a function of $\Delta$ when field is initially in the coherent state $|\alpha\rangle$, with $p = 0$ [dashed line], $p = 0.5$ [dot-dashed line] and $p = 1$ [solid line].

coherent and number initial states of the field. This fact is obvious from the Fig. 4 and Fig. 7; these figures are exactly the same. For other values of $p$ the rank of the support remains unchanged but the above mentioned coincidence does not occur.

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