Energy Minimizing Configurations for Highly Deformable Single-Director Elastic Surfaces

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Abstract

We consider a class of single-director Cosserat models for highly deformable elastic surfaces. We account for both curvature and finite mid-plane strains. We assume a polyconvexity condition for the stored-energy function that reduces to a physically correct membrane model in the absence of bending. With appropriate growth conditions, we establish the existence of energy minimizers. The local orientation of a minimizing configuration is maintained via the blowup of the stored energy as a version of the local volume ratio approaches zero. Finally, we specialize our results to three constrained versions of the theory commonly employed in the subject.

Keywords— Nonlinear elasticity, Polyconvexity, Energy minimization, Thin structures

Mathematics Subject Classification (2020) 74B20, 35D99, 49K20

1 Introduction

We consider a class of single-director Cosserat models for highly deformable elastic surfaces in this work. We are motivated in part by the wrinkling of highly stretched, thin elastomer sheets, e.g., [NRCH11], [LH16]. In addition, we note that “direct” plate/shell models incorporating finite mid-plane strains, mostly based on [SFR90], are widely available in commercial finite-element codes such as ABAQUS. For instance, the latter was employed in [NRCH11]. Most rigorous existence results for plate/shell models rely on small-thickness expansions from 3D elasticity, leading to small (or even zero) mid-plane strains, e.g., [FJM06]. Clearly these are inappropriate for the phenomena considered here. In contrast, we provide an existence theorem for a general class of direct models in the spirit of nonlinear elasticity, accounting for both curvature and finite mid-plane strains. In particular, a physically correct mid-plane or membrane energy is incorporated. Local orientation of configurations is maintained via the blowup of the stored-energy function as a version of the local volume ratio approaches zero. As such, we obtain global energy minimizers but are generally unable to ensure that these correspond to weak solutions of the Euler-Lagrange equations.

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A general approach to polyconvexity in the context of higher-order elasticity problems is presented in [BCOS1]. Among other examples, the set-up for single-director Cosserat continua is outlined there. On that basis, the complete details for energy minimization in single-director elastic surfaces are presented in [CGM13]. Local orientation preservation of minimizers is carried out in the same manner indicated above. However, the model in [CGM13] includes a “relaxed” mid-plane or membrane energy: In the absence of bending, the stored-energy function remains polyconvex in the sense of [BCO81]. While a nonlinearly elastic membrane energy should be polyconvex when restricted to planar behavior, this cannot be the case for deformations in \( \mathbb{R}^3 \); In lieu of sustained compression, arbitrarily finer and finer out-of-plane oscillations (wrinkles) drive the energy lower and lower towards its infimum [Pip86]. This is precisely why bending energy must be taken into account in order to rectify wrinkling patterns [HP92]. We elaborate on this in Section 2, Remark 1.

The outline of the work is as follows: We formulate the class of problems considered and state our hypotheses in Section 2. The latter includes growth conditions, a polyconvexity condition distinct from that employed in [CGM13], and the blow-up of the stored-energy function as a measure of the local volume ratio approaches zero. In Section 3, we establish weak lower semicontinuity of the energy functional, and we prove our existence theorem in Section 4. We specialize our results to three constrained versions of single-director theories in Section 5: (1) the so-called special theory characterized by a unit director field (shearable without thickness change); (2) the director field is normal to the surface (unshearable with thickness change); (3) the classical Kirchhoff-Love hypothesis whereby the director field coincides with a unit normal field on the surface (unshearable without thickness change). We make some concluding remarks in Section 6.

## 2 Problem Formulation

We let \( \mathbb{R}^n \) denote both Euclidean point space and its translate or tangent space, and we henceforth make the identification \( \mathbb{R}^2 \cong \text{span}\{e_1, e_2\} \), where \( \{e_1, e_2, e_3\} \) denotes the standard orthonormal basis for \( \mathbb{R}^3 \). Let \( \Omega \subset \mathbb{R}^2 \) be an open bounded domain with a strongly locally Lipschitz boundary \( \partial \Omega \). We associate \( \overline{\Omega} \) with a reference configuration for a material surface in a “flat” state as follows: A configuration is specified by two fields on \( \overline{\Omega} \): a deformation \( f : \overline{\Omega} \to \mathbb{R}^3 \) and a director field \( d : \overline{\Omega} \to \mathbb{R}^3 \); the reference configuration corresponds to \( f(x) = x \) and \( d(x) = e_3 \). The latter need not be stress free. The gradients or total derivatives of \( f, d \) at \( x \in \Omega \) are denoted \( F(x) := \nabla f(x), G(x) := \nabla d(x) \in L(\mathbb{R}^2, \mathbb{R}^3) \), respectively. We further require that a smooth configuration satisfy the local orientation condition

\[
J(f, d) := d \cdot (f_{,1} \times f_{,2}) > 0 \quad \text{in} \quad \overline{\Omega},
\]

where \( f_{,\alpha}, \alpha = 1, 2, \) denote partial derivatives and \( \alpha \times b \) is the usual right-handed cross product in \( \mathbb{R}^3 \). We list the set of 15 independent \( 2 \times 2 \) sub-determinants, \( \{m_l(F, G)\}_{l=1}^{15} \), of the \( 2 \times 6 \) gradient matrix \( \begin{bmatrix} F^T & G^T \end{bmatrix} \), which play an important role in what follows:

- \( m_1 = F_{31}F_{32} - F_{32}F_{31} \)
- \( m_2 = F_{31}F_{12} - F_{32}F_{11} \)
- \( m_3 = F_{11}F_{22} - F_{12}F_{21} \)
- \( m_4 = F_{11}G_{12} - F_{12}G_{11} \)
- \( m_5 = F_{11}G_{22} - F_{12}G_{21} \)
- \( m_6 = F_{11}G_{32} - F_{12}G_{31} \)
- \( m_7 = F_{31}G_{12} - F_{32}G_{11} \)
- \( m_8 = F_{31}G_{22} - F_{32}G_{21} \)
- \( m_9 = F_{31}G_{32} - F_{32}G_{31} \)
- \( m_{10} = G_{31}G_{12} - G_{32}G_{11} \)
- \( m_{11} = G_{31}G_{22} - G_{32}G_{21} \)
- \( m_{12} = G_{31}G_{32} - G_{32}G_{31} \)
- \( m_{13} = G_{21}G_{32} - G_{22}G_{31} \)
- \( m_{14} = G_{12}G_{31} - G_{11}G_{32} \)
- \( m_{15} = G_{11}G_{22} - G_{12}G_{21} \).

We assume that the surface is equipped with a stored-energy function, \( W(x, d, F, G) \), \( W : \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \).
\[ W(x, Qd, QF, QG) \equiv W(x, d, F, G) \text{ for all } Q \in SO(3). \]

We further assume:

(H1) For \( p, r, q \geq 2 \), there exist constants \( C_1 > 0 \) and \( C_2 \) such that

\[ W(x, d, F, G) \geq C_1 \left( |F|^p + |G|^r + \sum_{l=1}^{15} |m_l|^q \right) + C_2. \]

(H2) There is a \( C^1 \) function \( \Phi : \Omega \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 2} \times \mathbb{R}^{3 \times 2} \times (0, \infty) \times \mathbb{R}^{12} \to (0, \infty) \), such that

\[ (F, G, J, m_4, ..., m_{15}) \mapsto \Phi(x, d, F, G, J, m_4, ..., m_{15}) \text{ is convex} \]

and \( W(x, d, F, G) \equiv \Phi(x, d, F, G, J, m_4, ..., m_{15}) \).

(H3) \( \Phi \to +\infty \) as \( J \to 0^+ \).

Remark 1. In our notation, the stored-energy function employed in [CGM13] takes the form \( W(x, d, F, G) = \Psi(x, d, F, G, m_1, ..., m_{15}) \), where \( (F, G, m_1, ..., m_{15}) \mapsto \Psi \) is convex. In the absence of bending energy, the stored-energy depends only on the deformation gradient, i.e., \( \tilde{W}(x, F) = \tilde{\Psi}(x, F, m_1, m_2, m_3) \) with \( (F, m_1, m_2, m_3) \mapsto \tilde{\Psi} \) convex. That is, \( \tilde{W} \) is polyconvex as in [BCOS1]. As a consequence, energy minimization for this model is well posed for a very large class of membrane problems formulated in \( \mathbb{R}^3 \). In contrast, the membrane version of (2) reduces to \( \tilde{W}(x, F) = \tilde{\Phi}(x, F, J) \), with \( (F, J) \mapsto \tilde{\Phi} \) convex. Here, \( J \) denotes the local area ratio of the surface, i.e.,

\[ J := n \cdot (f_1 \times f_2) = |f_1 \times f_2| = |\det(F^T F)|^{1/2}, \]

where \( n \) denotes the unit normal field parallel to \( f_1 \times f_2 \). Observe that \( \tilde{W} \) is polyconvex for planar deformations only, viz., \( F \in \mathbb{R}^{2 \times 2} \). Indeed, it can be shown that \( \tilde{W} \) is not even rank-one convex for \( F \in \mathbb{R}^{3 \times 2} \).

Let \( L^p(\Omega, \mathbb{R}^3) \) denote the space of \( L^p \)-integrable 3-vector valued functions on \( \Omega \) and let \( W^{1,p}(\Omega, \mathbb{R}^3) \subset L^p(\Omega, \mathbb{R}^3) \) denote the Sobolev space of vector fields whose weak partial derivatives are also \( L^p \)-integrable. When the context is clear, we will avoid writing the domain and co-domain in our notation and simply refer to these spaces as \( L^p \) and \( W^{1,p} \). The norms on these spaces are defined by

\[ \|f\|_{L^p(\Omega, \mathbb{R}^3)}^p = \int_{\Omega} |f|^p \, dx, \]
\[ \|f\|_{W^{1,p}(\Omega, \mathbb{R}^3)}^p = \|f\|_{L^p(\Omega, \mathbb{R}^3)}^p + \int_{\Omega} |\nabla f|^p \, dx. \]

Consider a subset \( \Gamma \subset \partial \Omega \) with positive length, i.e., \( |\Gamma|_{\partial \Omega} > 0 \), and define

\[ W^{1,p}_\Gamma(\Omega, \mathbb{R}^3) = \{ u \in W^{1,p}(\Omega, \mathbb{R}^3) : u = 0 \text{ a.e. on } \Gamma \}, \]
where $\mathbf{u}$ on the boundary is understood in the sense of trace. We define the *admissible set*

$$
\mathcal{A} := \{ (f, d) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3) : m_l \in L^q, l = 1, \ldots, 15; J \in L^1; J > 0 \text{ a.e. in } \Omega; f - f_o \in W^{1,p} \Gamma, \mathbb{R}^3); d - d_o \in W^{1,r} \Gamma, \mathbb{R}^3) \},
$$

where $(f_o, d_o) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3)$ are prescribed and satisfy $d_o \cdot (f_o,1 \times f_o,2) > 0$ a.e.

**Remark 2.** We note that when $p, r > 2$, a weakened version of (H1) viz.,

$$
W(x, d, F, G) \geq C_1 \{ |F|^p + |G|^r \} + C_2,
$$

is sufficient to establish the results that follow. In this case, the requirement $m_l \in L^q$ can be dropped from the definition of $\mathcal{A}$ as well.

The total potential energy of the surface is given by

$$
E[f, d] = \int_{\Omega} W(x, d(x), \nabla f(x), \nabla d(x)) \, dx - L(f, d),
$$

where $L$ is a bounded linear functional on $W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3)$ representing “dead” loading. For example,

$$
L(f, d) = \int_{\Omega} (b \cdot f + g \cdot d) \, dx + \int_{\Gamma^c} [\tau \cdot f + \mu \cdot d] \, ds,
$$

where $b, g \in L^\infty(\Omega, \mathbb{R}^3), \tau, \mu \in L^\infty(\Gamma^c, \mathbb{R}^3)$ are prescribed loadings and $\Gamma^c := \partial \Omega \setminus \Gamma$.

### 3 Weak Lower Semicontinuity

We show that (H2) implies weak lower semicontinuity of $E[\cdot]$ in the following sense:

**Proposition 1.** The energy functional $E[\cdot]$ is weakly lower semicontinuous, i.e.,

$$
\liminf_{k \to \infty} E[f^k, d^k] \geq E[f, d],
$$

whenever $f^k \rightharpoonup f$ weakly in $W^{1,p}, d^k \rightharpoonup d$ weakly in $W^{1,r}$. $J^k := J(f^k, d^k) \rightarrow J := J(f, d)$ and $m_l^k := m_l(\nabla f^k, \nabla d^k) \rightarrow m_l := m_l(\nabla f, \nabla d)$, weakly in $L^q, l = 1, \ldots, 15$, for $p, r, q \geq 1$.

**Proof.** Since $L$ is weakly continuous, we focus on the internal energy

$$
I[f, d] := \int_{\Omega} W(x, d, \nabla f, \nabla d) \, dx.
$$

Assume (by passing a subsequence, if necessary) that

$$
\lim_{k \to \infty} I[f^k, d^k] = \liminf_{k \to \infty} I[f^k, d^k].
$$

From compact embedding (Bre10), $d^k \rightharpoonup d$ in $W^{1,r} \implies d^k \rightharpoonup d$ strongly in $L^r$. Consequently, for some subsequence (without relabelling) $d^k \rightarrow d$ pointwise a.e. By Egorov’s theorem, for $\epsilon > 0$, there is a set $\mathcal{O}_\epsilon$ with $|\Omega \setminus \mathcal{O}_\epsilon| \leq \epsilon$ such that $d^k \rightarrow d$ uniformly on $\mathcal{O}_\epsilon$. We also define $\mathcal{Y}_\epsilon := \{ x \in$
\( \Omega : |d| + |\nabla f| + |\nabla d| \leq 1/\epsilon, J \geq \epsilon \) and \( \Omega_\epsilon := \mathcal{O}_\epsilon \cap \Upsilon_\epsilon \). Hence, \( |\Omega \setminus \Omega_\epsilon| \to 0 \) as \( \epsilon \to 0 \). By virtue of [H2], we then find

\[
\int_{\Omega} W(x, d^k, \nabla f^k, \nabla d^k) \, dx \geq \int_{\Omega} W(x, d^k, \nabla f^k, \nabla d^k, m_4^k, \ldots, m_{15}^k) \, dx
\]

\[
= \int_{\Omega} \Phi(x, d^k, \nabla f^k, \nabla d^k, J^k, m_4^k, \ldots, m_{15}^k) \, dx
\]

\[
\geq \int_{\Omega} \Phi(x, d^k, \nabla f, \nabla d, J, m_4^k, \ldots, m_{15}^k) \, dx
\]

\[
+ \int_{\Omega} D_F \Phi(x, d^k, \nabla f, \nabla d, J, m_4^k, \ldots, m_{15}^k) \cdot (\nabla f^k - \nabla f) \, dx
\]

\[
+ \int_{\Omega} D_G \Phi(x, d^k, \nabla f, \nabla d, J, m_4^k, \ldots, m_{15}^k) \cdot (\nabla d^k - \nabla d) \, dx
\]

\[
+ \int_{\Omega} D_J \Phi(x, d^k, \nabla f, \nabla d, J, m_4^k, \ldots, m_{15}^k)(J^k - J) \, dx
\]

\[
+ \int_{\Omega} \sum_{l=4}^{15} D_{m_l} \Phi(x, d^k, \nabla f, \nabla d, J, m_4^k, \ldots, m_{15}^k)(m_l^k - m_l) \, dx,
\]

where, \( D_{\rho}, \rho = F, G, J, m_4, \ldots, m_{15} \) denotes the partial derivatives.

In the limit \( k \to \infty \), weak convergence implies that the last three integrals in the final inequality of (5) all vanish, while the first converges to \( \int_{\Omega} \chi_{\Omega} W(x, d, \nabla f, \nabla d) \, dx \), where \( \chi_{\cdot} \) denotes the characteristic function. Taking \( \epsilon \to 0 \), the desired result then follows from the monotone convergence theorem, viz., \( \lim_{k \to \infty} I[f_k, d_k] \geq \int_{\Omega} W(x, d, \nabla f, \nabla d) \, dx = I[f, d] \). \( \square \)

### 4 Energy Minimizers

Before proving our main result, we need the following:

**Proposition 2.** Suppose \( f^k \rightharpoonup f^* \) weakly in \( W^{1,p} \) and \( d^k \rightharpoonup d^* \) weakly in \( W^{1,r} \), where \( p, r \geq 2 \). Denote \( m_4^k := m_l(\nabla f^k, \nabla d^k) \) and \( m_4^* := m_l(\nabla f^*, \nabla d^*) \). Then, \( m_4^k \) converges distributionally to \( m_4^* \in L^1 \), for \( l = 1, \ldots, 15 \), i.e.,

\[
\int \Omega \phi \, dx \to \int \Omega m_4^* \phi \, dx \quad \forall \phi \in C_c^\infty(\Omega).
\]

**Proof.** The well-known argument is the same for each \( l = 1, \ldots, 15 \), and we provide it here for completeness for \( l = 4 \) only. Recall that,

\[
m_4^k = f_{1,1}^k d_{1,2}^k - f_{1,2}^k d_{1,1}^k = \det \left[ \nabla (f_{1,1}^k, d_{1,1}^k) \right].
\]

By density [Cha88], we have the identity,

\[
\int_{\Omega} \det \nabla w \phi \, dx = -\frac{1}{2} \int_{\Omega} \langle \text{Cof} \nabla w | \nabla \phi \rangle \cdot w \, dx,
\]

for any \( w \in W^{1,p}(\Omega, \mathbb{R}^2) \) \( p \geq 2 \) and for all \( \phi \in C_c^\infty(\Omega) \). Setting \( w = [f_{1,1}^k, d_{1,1}^k]^T \), we then have,

\[
\int_{\Omega} \det \left[ \begin{array}{cc}
f_{1,1}^k & f_{1,2}^k \\
d_{1,1}^k & d_{1,2}^k
\end{array} \right] \phi \, dx = -\int_{\Omega} \left[ \begin{array}{cc}
f_{1,1}^k & -d_{1,1}^k \\
-f_{1,2}^k & d_{1,2}^k
\end{array} \right] \left[ \begin{array}{c}
\phi, 1 \\
\phi, 2
\end{array} \right] \left[ \begin{array}{c}
f_{1,1}^k \\
d_{1,1}^k
\end{array} \right] \, dx
\]
Each of these terms can be shown to converge. For instance, consider the first term, \( \int_{\Omega} f_{1}^{k} \varphi_{1} f_{1}^{k} \). By compact embedding, \( f_{1}^{k} \rightarrow f_{1}^{*} \) in \( W^{1,p} \) \( \implies f_{1}^{k} \rightarrow f_{1}^{*} \) in \( L^{2} \). Thus,

\[
\int_{\Omega} f_{1,1}^{k} \varphi_{1} f_{1}^{k} \rightarrow \int_{\Omega} f_{1,1}^{*} \varphi_{1} f_{1}^{*} \text{ for all } \varphi \in C_{c}^{\infty}(\Omega).
\]

\[\square\]

**Theorem 3.** Suppose that \( \mathcal{A} \) is non-empty with \( \inf_{\mathcal{A}} E[f,d] < \infty \). Then there exists \( (f^{*},d^{*}) \in \mathcal{A} \) such that \( E[f^{*},d^{*}] = \inf_{\mathcal{A}} E[f,d] \).

**Proof.** Integrating the growth condition \([H1]\) yields

\[
\int_{\Omega} W(x,d,\nabla f,\nabla d) \, dx \geq C_{1} \left\{ \|\nabla f\|_{L^{p}}^{p} + \|\nabla d\|_{L^{r}}^{r} + \sum_{l=1}^{15} \|m_{l}\|_{L^{q}} \right\} + C_{2}^{r},
\]

A generalised Poincaré inequality \([MJ09]\) reads

\[
\int_{\Omega} |v|^{p} \, dx \leq C \left\{ \int_{\Omega} |\nabla v|^{p} \, dx + \left| \int_{\Gamma} Tv \, da \right|^{p} \right\}, \quad 1 \leq p < \infty,
\]

where \( T : W^{1,p}(\Omega) \rightarrow L^{p}(\partial\Omega) \) is the trace operator. From this, we find

\[
\int_{\Omega} W(x,d,\nabla f,\nabla d) \, dx \geq C_{1}^{r} \left\{ \|f\|_{W^{1,p}}^{p} + \|d\|_{W^{1,r}}^{r} + \sum_{l=1}^{15} \|m_{l}\|_{L^{q}} \right\} + C_{2}^{r},
\]

with constants \( C_{1}^{r} > 0 \) and \( C_{2}^{r} \). Furthermore, since \( L(f,d) \) is a bounded linear functional on \( W^{1,p} \times W^{1,r} \) we have

\[
|L(f,d)| \leq C_{3} \left( \|f\|_{W^{1,p}} + \|d\|_{W^{1,r}} \right),
\]

Since \( p,r > 1 \), the last two inequalities yield

\[
E[f,d] \geq C \left\{ \|f\|_{W^{1,p}}^{p} + \|d\|_{W^{1,r}}^{r} + \sum_{l=1}^{15} \|m_{l}\|_{L^{q}} \right\} + D,
\]

where \( C > 0 \) and \( D \) are constants.

Let \( \{(f^{n},d^{n})\} \subset \mathcal{A} \) be a minimizing sequence for \( E[\cdot] \), i.e.,

\[
\lim_{k \to \infty} E[f^{n},d^{n}] = \inf_{(f,d) \in \mathcal{A}} E[f,d].
\]

As before, let \( m_{l}^{n} := m_{l}(\nabla f^{n},\nabla d^{n}) \), \( l = 1,...,15 \). By virtue of \([3]\), we see that the sequences \( \{f_{n}\} \), \( \{d_{n}\} \) and \( \{m_{l}^{n}\} \) are bounded in \( W^{1,p}, W^{1,r} \) and \( L^{q} \), respectively, each of which is a reflexive Banach space. Hence, there exist \( f^{*} \in W^{1,p}, d^{*} \in W^{1,r} \) and \( \alpha_{l} \in L^{q} \) and subsequences \( \{f^{k}\}, \{d^{k}\}, \{m_{l}^{k}\} \) (not relabelled) converging weakly in \( W^{1,p}, W^{1,r} \) and \( L^{q} \), respectively, i.e. \( f^{k} \rightharpoonup f^{*}, d^{k} \rightharpoonup d^{*} \) and \( m_{l}^{k} \rightharpoonup \alpha_{l} \) \([Bre10]\). Combining the last of these with Proposition \([2]\) we deduce

\[
\int_{\Omega} (m_{l}^{*} - \alpha_{l}) \varphi \, dx = 0 \quad \text{for all } \varphi \in C_{c}^{\infty}(\Omega).
\]
Since \( m^*_l - \alpha_l \in L^1(\Omega) \), the theorem of DuBois-Reymond [MR09] gives \( m^*_l = \alpha_l \) a.e. Thus, we have established \( m^*_l \to m^*_l \) weakly in \( L^q \).

We now consider the convergence of \( J^k := J(f^k, d^k) \). We first observe that
\[
f^k_1 \times f^k_2 = m^k_1 e_1 + m^k_2 e_2 + m^k_3 e_3.
\]
Thus \( f^k_1 \times f^k_2 \to f^*_1 \times f^*_2 \) weakly in \( L^q \) for \( q \geq 2 \). In addition, \( d^k \to d^* \) weakly in \( W^{1,r} \) \( r \geq 2 \) implies strong convergence in \( L^2 \). Hence, \( \int_\Omega d^k : (f^k_1 \times f^k_2) \varphi \, dx \to \int_\Omega d^* : (f^*_1 \times f^*_2) \varphi \, dx \) for all \( \varphi \in L^\infty \), i.e., \( J^k \to J^* \) weakly in \( L^1 \).

Next, we show that \( (f^*, d^*) \in A \). First, we claim that \( J^* > 0 \) a.e. in \( \Omega \). By virtue of Mazur’s theorem, we can construct a sequence of convex combinations of the sequence \( \{J^k\} \) that converges strongly in \( L^1 \) to \( J^* \). Thus, there is a subsequence converging to \( J^* \) a.e. in \( \Omega \). Since each \( J^k > 0 \) a.e., we deduce that \( J^* \geq 0 \) a.e. Now suppose that that \( J^* = 0 \) a.e. in \( \Omega \subset \Omega \), where \( |\Omega| > 0 \). Employing \( \chi_\Omega \) as a test function, the weak convergence of \( J^k \) implies \( J^k \to 0 \) strongly in \( L^1(\Omega) \).

To complete the proof, we combine the results above with Proposition 1 to conclude
\[
\inf_{k \to \infty} E[f^k, d^k] \leq \inf_{k \to \infty} E[f^k, d^*] \leq \inf_{k \to \infty} E[f^k, d^k] \quad \text{with} \quad (f^*, d^*) \in A, \text{ i.e., } E \text{ attains its infimum on } A.
\]

## 5 Constrained Minimizers

We now explore three different constrained versions of the theory that are common in the study of nonlinearly elastic shells.

### 5.1 Special Theory

Here the director field is constrained to have unit length, i.e.,
\[
|d| = 1 \text{ a.e. in } \Omega.
\]
Again, we assume \([H1],[H3]\) and incorporate \([S]\) into the admissible set:
\[
A := \{(f, d) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3) : m_l \in L^q, l = 1, \ldots, 15;
J \in L^1; J > 0 \text{ a.e. in } \Omega; |d| = 1 \text{ a.e. in } \Omega;
f - f_o \in W^{1,p}_\Gamma(\Omega, \mathbb{R}^3); d - d_o \in W^{1,r}_\Gamma(\Omega, \mathbb{R}^3)\},
\]
where \((f_o, d_o) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3)\) are prescribed and \(|d_o| = 1\) and \(d_o \cdot (f_o,1 \times f_o,2) > 0\) a.e.

The existence of a minimizer follows precisely as before in Theorem 3. We only need to show that \([S]\) is satisfied. This follows from compact embedding: For a minimizing sequence, we have \(d^k \to d^*\) in \(W^{1,r}\) \(\Rightarrow d^k \to d^*\) in \(L^r\). Thus there is a convergent subsequence \(d^{k_n} \to d^*\) a.e. Since \(|d^{k_n}| = 1\) a.e., we have \(|d^*| = 1\) a.e.
5.2 Normal Director Field

We now constrain the director field to be normal to the surface (allowing its length to be variable), viz.,

\[ d \cdot f_{\alpha} = 0 \text{ a.e. in } \Omega, \quad \alpha = 1, 2. \]  

(9)

Again, we assume (H1)-(H3) and incorporate (9) into the admissible set:

\[ A := \{ (f, d) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3) : m_l \in L^q, l = 1, \ldots, 15; \]
\[ J \in L^1; J > 0 \text{ a.e. in } \Omega; d \cdot f_{\alpha} = 0 \text{ a.e. in } \Omega, \alpha = 1, 2; \]
\[ f - f_o \in W^{1,p}_G(\Omega, \mathbb{R}^3); d - d_o \in W^{1,r}(\Omega, \mathbb{R}^3) \}

where \((f_0, d_0) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3)\) are prescribed with the property that \(d_0 \cdot f_{\alpha o} = 0\) for \(\alpha = 1, 2\) and \(d_o \cdot (f_{0,1} \times f_{0,2}) > 0\) a.e.

Existence of a minimizer again follows as before. In this case, we only need to show that \((f^*, d^*)\) satisfies (9). We have \(f_{\alpha} \rightarrow f_{\alpha}^*\) weakly in \(L^p\) and by compact embedding \(d^k \rightarrow d^*\) in \(L^2\). By admissibility, \(0 = \int_{\Omega} d^k \cdot f_{\alpha}^* \varphi \, dx = \int_{\Omega} d^* \cdot f_{\alpha}^* \varphi \, dx\) for all \(\varphi \in C_c(\Omega), \alpha = 1, 2\). By the theorem of DuBois-Reymond, we conclude that \((f^*, d^*)\) satisfies (9).

5.3 Kirchoff-Love Theory

In this classical case, the director field is required to coincide with the unit normal field to the deformed surface. This is usually referred to as the Kirchoff-Love theory.

The constraints are now

\[ d \cdot f_{\alpha} = 0 \quad \text{a.e. in } \Omega, \quad \alpha = 1, 2; \]
\[ |d| = 1 \quad \text{a.e. in } \Omega. \]

We modify the admissible set to accommodate these:

\[ A := \{ (f, d) \in W^{1,p}(\Omega, \mathbb{R}^3) \times W^{1,r}(\Omega, \mathbb{R}^3) : m_l \in L^q, l = 1, \ldots, 15; \]
\[ J \in L^1; J > 0 \text{ a.e. in } \Omega; d \cdot f_{\alpha} = 0 \text{ a.e. in } \Omega, \alpha = 1, 2; \]
\[ |d| = 1 \text{ a.e. in } \Omega; f - f_o \in W^{1,p}_G(\Omega, \mathbb{R}^3); \]
\[ d - d_o \in W^{1,r}(\Omega, \mathbb{R}^3) \}

where \((f_0, d_0) \in W^{1,p} \times W^{1,r}\) are prescribed and satisfy \(d_0 \cdot f_{\alpha o} = 0\) for \(\alpha = 1, 2\), \(|d_o| = 1\) and \(d_o \cdot (f_{0,1} \times f_{0,2}) > 0\) a.e.

The existence of a minimizer follows as before, and the arguments used in Section 5.1 and 5.2 imply \(|d^*| = 1\) a.e. and \(d^* \cdot f_{\alpha}^* = 0\) a.e., \(\alpha = 1, 2\), respectively.

6 Concluding Remarks

A rigorous existence theorem for the Kirchoff-Love model, based on energy minimization, is presented in [Ani18]. Local orientation preservation is maintained via a blow-up argument similar to that employed here and in [CGM13], except that the volume ratio employed in [Ani18] takes into account the surface thickness via the Cosserat ansatz, cf. [Ant05]. The definition of polyconvexity used in [Ani18], involving only \(G, F\) and the volume ratio just described, is much more restrictive than ours. However, when reduced to its membrane part (ignoring thickness) it agrees
with ours. Also, in contrast to our approach based on constraints, the unit normal field is
directly parametrized by the surface deformation in Anicic. A distinction between those results and
those of Section 5.3 becomes apparent upon taking a formal first variation (not rigorous): Our
Euler-Lagrange equations at a minimizer would involve Lagrange multiplier fields enforcing the
constraints (representing transverse shears and through-thickness resultants), whereas the latter
would be effectively eliminated from the Euler-Lagrange equations associated with Anicic.

We also mention that Healey is comparable to our results from Section 5.2. In the former,
the normal director field (not necessarily unit) is directly parametrized by the surface deformation.
This entails a full second-gradient surface theory, while local orientation is preserved in a manner
similar to that presented here. For growth conditions on the second gradient with  \( p > 2 \), the first
variation can be taken rigorously at a minimizer, leading to the weak form of the Euler-Lagrange
equations. This follows from the same construction used in HK09. Interestingly, it is not at all
clear how to do so based on the results of Section 5.2.

Acknowledgements

This work was supported in part by the National Science Foundation through grant DMS-2006586,
which is gratefully acknowledged.

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