The vortex-driven dance of droplets within droplets

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Understanding the fluid-structure interaction is crucial for an optimal design and manufacturing of soft mesoscale materials. Multi-core emulsions are a class of soft fluids assembled from cluster configurations of deformable oil-water double droplets (cores), often employed as building-blocks for the realisation of devices of interest in bio-technology, such as drug-delivery, tissue engineering and regenerative medicine. Here, we study the physics of multi-core emulsions flowing in microfluidic channels and report numerical evidence of a surprisingly rich variety of new driven non-equilibrium states (NES), whose formation is caused by a dipolar fluid vortex triggered by the sheared structure of the flow carrier within the microchannel. The observed dynamic regimes range from long-lived NES at low core-area fraction, characterised by a planetary-like motion of the internal drops, to short-lived ones at high core-area fraction, in which a pre-chaotic motion results from multi-body collisions of inner drops, as combined with self-consistent hydrodynamic interactions. The onset of pre-chaotic behavior is marked by transitions of the cores from one vortex to another, a process that, in formal analogy with level crossing in quantum systems, we interpret as manifestations of such functionalized materials. The rate of release of the drug carried by the cores, for example, is significantly influenced by surfactant concentration and hydrodynamic interactions. The latter ones, even when moderate, can foster drop collisions as well as shape deformations that may ultimately compromise the release and the delivery towards targeted diseased tissues. Controlling mechanical properties as well as long-range hydrodynamic interactions of the liquid film formed among cores is of paramount importance for ensuring the prolonged stability of food-grade multiple emulsions [31]. This is essential in high internal phase emulsions extensively used in tissue engineering, where such forces can considerably alter pore size and rate of polydispersity [28, 32], thus jeopardizing the structural homogeneity of the material.

Building ad hoc mathematical and computational models is therefore fundamental to make progress along this direction. Indeed, in stark contrast with the impressive advances in the experimental realisation of multi-core emulsions [9, 10, 33, 34], it is only recently that efforts have been directed to the theoretical investigation of the rheology resulting from the highly non-linear fluid-structure interactions taking place in such systems [35, 37].

Continuum theories, combined with suitable numerical approaches (such as lattice Boltzmann methods [35, 39] and boundary integral method [37]) have proven capable to capture characteristic features observed in double emulsion experiments, such as their production within microchannels [39], the typical shape deformations of the capsule, elliptical and bullet-like, under moderate shear flows [10, 42], as well as more complex dynamic...
behaviors, such as the breakup of the enveloping shell occurring under intense fluid flows [33]. However, much less is known about the dynamics of more sophisticated systems, such as multiple emulsions with distinct inner cores, whose physics has been theoretically investigated only by a few authors to date [37, 38].

In this paper, we present a numerical study of the pressure-driven flow of multi-core droplets confined in a microchannel, following a design directly inspired to actual microfluidic devices. Our equations of motion, as well as numerical method, are based on established continuum principles and detailed in Method Section. In short, we describe the physics of our multi-core emulsions by using a phase field approach [28, 41-43], in which a set of passive scalar fields \( \phi_i(\mathbf{r}, t) \) (\( i = 1, \ldots, N \)) account for the density of each droplet, while the a vector field \( \mathbf{v}(\mathbf{r}, t) \) represents the global fluid velocity. The dynamics of each field \( \phi_i \) is governed by a set of Cahn-Hilliard equations, while the velocity obeys the Navier-Stokes equations [44, 45].

Extensive lattice Boltzmann simulations provide evidence of a rich variety of new driven nonequilibrium states (NES), from long-lived ones at low droplet area fraction, characterised by a highly correlated, planetary-like motion of the cores, to short-lived ones at high droplet area fraction, in which multiple collisions and intense fluid flows trigger a chaotic-like dynamics. Central to each of these non-equilibrium states is the onset of a vorticity dipole within the capsule, which arises as an inevitable consequence of the shear structure of the velocity field within the microchannel. Such dipole structure naturally invites a classification of these states in close analogy with the statistical mechanics of occupation numbers. Even though our system is completely classical, such representation discloses a transparent and insightful interpretation of the transition from periodic to quasi-chaotic steady-states, in terms of level crossings between the occupation numbers in the two vortex structures. Such level crossings are interpreted as manifestations of the system to maximise its entropy by filling voids which arise dynamically within the multi-body structure resulting from the self-consistent motion of the cores within the capsule. This is consistent with the notion of entropy as propensity to motion rather than microscopic disorder [46].

RESULTS

Fluid-structure interaction in a free-core emulsion

We start by describing droplet shape and pattern of the fluid velocity at the steady state in a free-core emulsion under a Poiseuille flow. Once this is imposed, the droplet, initially at equilibrium (Fig.1a), acquires motion, driven by a constant pressure gradient \( \Delta p \) applied across the longitudinal direction of the microfluidic channel. The resulting fluid flow as well as the shape of the emulsion are controlled by the capillary and the Reynolds numbers, defined as \( Ca = v_{max} \eta / \sigma \) and \( Re = \rho v_{max} D_O / \eta \). Here \( v_{max} \) is the maximum value of the droplet speed, \( \eta \) is the shear viscosity of the fluid, \( \sigma \) is the surface tension, \( \rho \) is the fluid density and \( D_O \) is the diameter of the shell (taken as characteristic length of the multi-core emulsion). In our simulations \( Ca \) roughly varies between 0.02 and 1 and \( Re \) may range from 0.02 and 5, hence inertial effects are negligible and the laminar regime is preserved.

In Fig.1b-c we show an example of shape of a free-core emulsion at the steady state (see movie M1 for the full dynamics). In agreement with previous studies [50-54], the droplet attains a bullet-like form, more stretched along the flow direction for higher values of \( Re \) and \( Ca \) (i.e., larger pressure gradients). Such shape results from the parabolic structure of the flow profile (see the Supplemental Material in [55] for further details about the steady state velocity profile), moving faster in the center of the channel and progressively slower towards the wall (Fig.1b). If computed with respect the droplet frame, the flow field exhibits two symmetric counterrotating eddies, resulting from the confining interface of the capsule and whose direction of rotation is consistent with a droplet moving forward (rightwards here, see Fig.1f). These structures are due to the gradient of \( v_y \) along the \( z \) direction, a quantity positive within the lower half of the emulsion and negative in the upper. As long as the droplet remains core-free, such fluid recirculations (also observed, for instance, in micro-emulsions propelled either through Marangoni effect [50, 55] or by means of an active material, such as actomyosin proteins [59, 61], dispersed within) are stable, and their steady-state angular velocity can be approximately estimated as \( \omega \sim v_{max} / L_z \) (Fig.1).

Does this picture still hold when smaller cores are encapsulated? Or, in other words, how does the coupling between the fluid velocity \( \mathbf{v} \) and a number of passive scalar fields \( \phi_i \) affect the dynamics of the multi-core emulsion? To what extent the dipolar fluid flow structure is stable when the effective area fraction of the internal droplets \( A_e = \frac{N \pi R_i^2}{\pi R_O^2} \) (where \( R_i \) and \( R_O \) are the radii of the cores and of the shell and \( N \) the number of cores) increases?

In the next sections we show that the two-eddy fluid structure is essentially preserved, although modifications to this pattern occur when \( A_e \) is larger than \( \sim 0.35 \). More specifically, while in a double emulsion (\( N = 1 \)) the stream in the middle of the droplet drives the core at the front-end of the shell where the core gets stuck, when \( N > 1 \) the vortices trigger and sustain a persistent periodic motion of the cores within each half of the emulsion. As long as \( A_e < 0.35 \) (achieved with \( N = 3 \)), cores
remain confined within either the upper or the lower part of the emulsion giving rise to long-lived nonequilibrium steady states. When $A_e > 0.35$ ($N \geq 4$) droplet crossings between the two regions may occur, and short-lived aggregates of three or more cores only temporarily survive. Hence, the whole emulsion can be effectively visualized as a two-state system in which the two eddies foster the motion of the cores and crucially affect the duration of the states.

By using the tools of statistical mechanics, we propose a classification of such states in terms of the occupation number formalism, where $\langle \alpha_1, \ldots, \alpha_j|\alpha_{j+1}, \ldots, \alpha_N \rangle$ represents a state in which $j$ and $N-j$ distinct cores occupy, respectively, the upper and the lower half of the emulsion, with $j = 1,\ldots,N$. This is analogous to determine the number of ways $N$ distinguishable particles can be placed in two boxes.

Classification of states

In Fig.2(a)-(f), we show the equilibrium configurations of a single core ((a), $N = 1$), a two-core ((b), $N = 2$), a three-core ((c), $N = 3$), a four-core ((d), $N = 4$), a five-core ((e), $N = 5$) and a six-core ((f), $N = 6$) emulsion. Once a Poiseuille flow is applied, the emulsions attain a steady state in which, unlike the core-free droplet, the dynamics of the cores is crucially influenced by the effective area fraction $A_e$, besides the two-eddy flow structure. In Fig.3 we show a selection of the nonequilibrium states observed.

Long-lived nonequilibrium states. The simplest configuration is the one in which $N = 1$. In this case the core and the shell are initially advected forward (rightwards in the figure) and, at the steady state, the former gets stuck at the front-end of the latter (see, for example, movie M2). In Fig.3 two examples for slightly different values of $Re$ and $Ca$ are shown. Note incidentally that, in agreement with previous studies [38–40, 42, 50, 51, 62], as long as $Re \approx 1$, the internal core, unlike the interface of the external shell, is only mildly affected by the fluid flow. This is due to its higher surface tension induced by the smaller curvature radius, which prevents relevant shape deformations.

This picture is dramatically altered when the number of cores increases. If $N = 2$, for example, two nonequilibrium long-lived states emerge. In the first one (Fig.3 and movie M3), the two cores remain locked in the upper (or lower) part of the emulsion, where the fluid eddy fosters a periodic motion in which each core chases the other one in a coupled-dance fashion (see the next section for a detailed description of this dynamics). In the second one (Fig.3 and movie M4), the two cores remain separately confined within the top and the bottom of the emulsion, and move, weakly, along circular trajectories. By using the statistics of occupation number, we indicate with $\langle 1,2|0 \rangle$ the state in which cores 1 and 2 are in the upper region while the lower one is empty, and with $\langle 2|1 \rangle$ the state where cores 2 and 1 occupy, separately, each half.

More complex effects emerge when $A_e$ is further increased. If $N = 3$, once again we find two different long-lived nonequilibrium states, namely $\langle 3|1,2 \rangle$ and $\langle 1,2,3|0 \rangle$. In the first one (Fig.3 and movie M5), the core 3 is confined at the top of the emulsion and moves following a circular path, while cores 1 and 2 remain at the bottom of the emulsion and reproduce the coupled-dance dynamics observed for $N = 2$. In the second one (Fig.3 and movie M6), the three cores, sequestered in the upper region, exhibit a more complex three-body periodic motion, whose dynamics is discussed later. However, although long-lasting, such state may turn unstable due to either hydrodynamic interactions or to direct collisions with other cores. This is precisely what happens when $N$ is further augmented.

Short-lived nonequilibrium states. Indeed, when $N = 4$, we find a state in which three cores move in one region and a single core within the other one. This is analogous to determine as $\langle 1,2,3|4 \rangle$ (Fig.3). However, this state lives for a short period of time since droplet 3 crosses from the top towards the bottom of the emulsion and produces a novel long-lived nonequilibrium state $\langle 1,2,3,4 \rangle$, in which couples of cores ceaselessly dance within two separate regions (Fig.3 and movie M7). Such transition, indicated as $\langle 1,2,3|4 \rangle \rightarrow \langle 1,2,3,4 \rangle$, occurs due to the high values of $A_e$, larger than 0.35. This means that, within half of the emulsion, the effective area fraction is even higher (more than 0.5), and the three-core state would be unstable to changes of the flow direction and to unavoidable collisions with the other cores. This is the reason why, for example, the state $\langle 1,2,3,4|0 \rangle$, although realizable in principle, has not been observed.

For higher values of $N$, $A_e$ further increases and multiple crossings occur. If $N = 5$ for instance, short-lived dynamical states appear, such as $\langle 3,5|1,2,4 \rangle$ or $\langle 2|1,3,4,5 \rangle$, in which four cores are temporarily packed within half of the emulsion (Fig.3 and movie M8). However, these states are highly unstable since cores cross the two regions of the emulsion multiple times. Finally, more complex short-lived states are observed when $N = 6$, such as $\langle 3,5,6|1,2,4 \rangle$ and $\langle 5,6|1,2,3,4 \rangle$ (Fig.3 and movie M9).

In the next section we discuss more specifically the dynamics of these states, focussing, in particular, on how the fluid-structure interaction affects the droplets motion.

Droplet dynamics in a multi-core emulsion

Two-core emulsion. We begin from the two-core emulsion, in which two droplets are initially encapsu-
fluid vortices sustain a persistent and periodic circular interface of the cores. Emulsion, an effect resulting from its interaction with the local variation of the velocity field in the middle of thewards (a-c) or downwards (b-d), and is fostered by the droplet (i.e., droplet 1). Such motion occurs either up-circulating patterns, which trigger the motion of the rear the latter, once again, exhibits two counter-rotating re-

\[ \text{Ca} \approx 0.52, \quad \text{Re} \approx 3, \quad \text{Ca} \approx 0.8. \]

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Figure 3. Nonequilibrium state of multi-core emulsions under Poiseuille flow. Blue arrow indicates its direction (which applies to all cases) while $N$ represents the number of cores encapsulated. (a) is obtained for $Re \simeq 3$ and $Ca \simeq 0.85$, while (b) for $Re \simeq 1.2$ and $Ca \simeq 0.35$. All the other cases correspond to $Re \simeq 3$ and $Ca \simeq 0.85$. As long as $A_c < 0.35$, the nonequilibrium states are long-lived (l.l.), while if $A_c > 0.35$ they turn into short-lived (s.l.). Here, crossings of cores from one region (say top) towards the other one start to occur. Magenta arrows indicate the direction of a crossing. Multiple crossings are observed as $A_c$ increases. In each snapshot, states are indicated by means of the occupation number classification.

Figure 4. Onset of periodic motion in a two-core emulsion. (a)-(c) $Re \simeq 3$ and $Ca \simeq 0.52$. Red arrows indicate the direction of the fluid flow in the lab frame and (c) in the external droplet frame. (b)-(d) $Re \simeq 1.2$ and $Ca \simeq 0.2$. Green arrows in (a) and (b) denote the direction, perpendicular to the flow, along which the droplet 1 starts its motion, while white arrows in (c) and (d) bespeak the direction of the fluid recirculations.

If cores 1 and 2 are initially aligned vertically, rather than horizontally, the state $\langle 2|1 \rangle$ is obtained. Once the flow is applied, they acquire a circular motion triggered by the eddies but, since they are placed within two separate sectors (top and bottom of the emulsion) from the beginning, interact only occasionally along the middle region of the emulsion.

Finally note that, unlike the free-core emulsion, here at the onset of the instability fluid flows, mainly occurring longitudinally, break the symmetric structure of the two eddies. In spite of this deviation (due to the coupling with the interfaces), an approximate evaluation of the angular velocity can be computed as $\omega \sim \frac{v_{max}}{L_z} \simeq 1.5 \times$...
Three-core emulsion. We first discuss the dynamics of the state (3|1, 2). In Fig. 7 we show the time evolution of the displacement of three cores, initially placed as in Fig. 2. While droplets 1 and 2, initially carried rightwards by the fluid, are then driven towards the upper part of the emulsion where an intense flow current pushes it forward, but is rapidly reshaped when the core, once again, migrates back upwards.

This state may however turn unstable and decay into a long-lived state when $A_c$ augments. This is what happens if four cores are included.

Four cores: Droplet crossings and short-lived clusters. In Fig. 8 we show the time evolution of the displacement $\Delta z$ of four cores originally encapsulated as in Fig. 2. While droplets 1 and 2, initially carried rightwards by the fluid, are then driven towards the upper part of the emulsion where an intense flow current pushes it forward, but is rapidly reshaped when the core, once again, migrates back upwards.

At the steady-state, the four cores exhibit a long-lived coupled motion in pairs of two, occurring without further crossings and within two separate regions of the emulsion. These results suggest that, although for a short period of time, the periodicity of motion can be temporarily lost when $A_c$ is sufficiently high. In the next section we show that when $A_c$ attains a value equal to (or higher than) 0.4, the dynamics of the cores lacks of any periodic regularity and their motion turns to chaotic.

Five and six cores: Onset of non-periodic dynamics. In Fig. 10 we show the time evolution of displacement $\Delta z$ in a five (top) and six-core (bottom) emulsion, where $A_c \simeq 0.46$ and $A_c \simeq 0.55$ respectively (see also movies M8 and M9).

In such systems, only short-lived states are observed (such as those shown in Fig. 9), since multi-
ple crossings occur between the top and the bottom of the emulsion. An enduring dance-like dynamics, for instance, is observed only for pair of cores and lasts for relatively short times, interrupted by crossings taking place within the emulsion. When this event occurs, the incoming droplet temporarily binds with the others, yielding to a short-lived nonequilibrium steady state such as those shown in Fig.3 in turn destroyed as soon as a further core approaches. Importantly, such complex dynamics almost completely removes any periodicity of the motion of the internal droplets. This results from the non-trivial coupling between fluid velocity and internal cores (Fig.11); continuous changes of droplet positions lead to significant variations of the local velocity field which, in turn, further modifies the motion of the cores in a typical self-consistent fluid-structure interaction loop.

Conclusions and Outlook

Summarising, we have investigated the physics of a multi-core emulsion within a pressure-driven flow for values of $Re$ and $Ca$ typical of microfluidic experiments. We have shown that, as long as the area fraction occupied by the cores is sufficiently low, the latter exhibit a periodic steady-state dynamics confined within a sector of the emulsion, reminiscent of a dancing couple, in which each dancer chases the partner. Our results strongly suggest that this peculiar behaviour is triggered and sus-
Figure 9. Top: Time evolution of the center of mass component of four cores. Insets show the position of the internal droplets during the crossing of droplet 3 from the top towards the bottom of the emulsion. They are taken at $t = 2 \times 10^5$ (a) and $t = 4 \times 10^5$ (b). White arrow indicates the direction of motion of droplet 3. Bottom: Velocity field at $t = 2 \times 10^5$. An intense flow current drives core 3 downwards. The two eddies undergo significant modifications.

Figure 10. Top: Time evolution of the displacement $\Delta z$ of the cores in a five-core emulsion. Regular periodic motion survives for short periods of time and is temporarily broken by repeated crossings of the cores towards either the top or the bottom of the emulsion. Bottom: Time evolution of $\Delta z$ in a six-core emulsion. Although some cores travel along approximately similar trajectories (such as 1, red, and 6, black), there is no evidence of a persistent coupled periodic motion involving two (or more) cores.

tained by the internal vorticity which forms within the external droplet, whose interface acts as an effective bag confining the cores. The internal vorticity, in turn, is sustained by the heterogeneity of the micro-confined carrier flow, under the effect of the pressure drive.

As the area fraction (i.e., number of cores) is increased, a more complex multi-body dynamics emerges. Due to unavoidable collisions between droplets, as combined with self-consistent hydrodynamic interactions, cores may leave the confining vortex and switch to the other one. Whenever this occurs, they either restore the planetary-like dynamics or temporarily aggregate with other cores to form unstable multi-droplets chains, which are repeatedly destroyed and re-established, due to the non-trivial coupling between the flow field and local variations of the phase field.

The jumps from one vortex to another are interpreted as entropic events, expressing the tendency of the system to maximise its entropy (propensity to motion) by filling voids, as they dynamically arise in this complex multi-body fluid-structure interaction.

Drawing inspiration from the occupation number formalism in statistical mechanics, we propose a classification of the non-equilibrium states that provides a transparent interpretation of their intricate dynamic behaviour. By denoting each dynamical state as $\langle \alpha_1, ..., \alpha_j | \alpha_{j+1}, ..., \alpha_N \rangle$, in which $j$ and $N - j$ distinct
Figure 11. Instantaneous velocity field in the state \( \langle 3, 5 | 1, 2, 4 \rangle \) (top) and \( \langle 3, 5, 6 | 1, 2, 4 \rangle \) (bottom). In both cases a chaotic-like fluid velocity structure dominates the dynamics of the internal cores.

cores occupy the two sectors of the emulsion, respectively, this occupation-number formalism provides an elegant and transparent tool to i) classify the various non-equilibrium states of the system and ii) to describe the dynamic transitions among them.

From the theoretical standpoint, it would be of great interest to explore whether this formal analogy can be taken one step further, and exploring whether multi-core emulsions may provide further instances of hydrodynamic quantum analogues, along the lines pioneered by previous authors for the case of bouncing droplets on vibrating baths. For instance, may the core transitions from one vortex to another correspond to quantum transitions between excited states of composite molecules? If so, one may even hope that multi-core emulsions may find use not only to manufacture new soft mesoscale materials but also to provide hydrodynamic analogues of quantum materials [63 [64].

On a more experimental side, it is hoped that the insights on the aforementioned non-equilibrium states delivered by the present study may prove useful to gain a deeper understanding of the basic mechanisms which control the behaviour of confined droplets (mimicking, for instance, biological bodies like cells) as they move within capsules flowing in long capillaries. Of particular interest is the case of Janus particles, which, owing to the takeover phenomena discussed in the text, would be periodically exposed to both front1-rear2 and front2-rear1 contacts.

**METHODS**

Following the approach of Ref. \([38 [44]\), we describe the physics of a multi-core emulsion in terms of (i) a set of scalar phase field variables \( \phi_i(\mathbf{r}, t) \), \( i = 1, \ldots, N_d \) (where \( N_d \) is the total number of droplets) accounting for the density of each droplet, and (ii) the global fluid velocity \( \mathbf{v}(\mathbf{r}, t) \). The equilibrium properties are captured by a coarse-grained free-energy density

\[
f = \frac{a}{4} \sum_i N_i (\phi_i - \phi_0)^2 + \frac{k}{2} \sum_i (\nabla \phi_i)^2 + \epsilon \sum_{i,j,i<j} \phi_i \phi_j,
\]

in which the first term guarantees the existence of two coexisting minima, \( \phi_i = \phi_0 \) inside the \( i \)-th droplet and \( \phi_i = 0 \) outside, while the second one determines the interfacial tension. The two positive constants \( a \) and \( k \) set the surface tension \( \sigma = \sqrt{8a k / 9} \) and the interface thickness \( \xi = 5 \sqrt{k / 2a} \) of the droplets. Finally, the last contribution is a soft-core repulsive term whose magnitude is controlled by the (positive) constant \( \epsilon \).

The dynamics of the order parameters \( \phi_i \) is controlled by a set of Cahn-Hilliard equations

\[
\frac{\partial \phi_i}{\partial t} + \nabla \cdot (\phi_i \mathbf{v}) = M \nabla^2 \mu_i,
\]

where \( M \) is the mobility and \( \mu_i = \partial f / \partial \phi_i - \partial_s f / \partial (\partial_s \phi_i) \) is the chemical potential (Greek letters denote Cartesian components).

The fluid velocity \( \mathbf{v} \) is governed by the Navier-Stokes equation which, in the incompressible limit, reads

\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} - \sum_i \phi_i \nabla \mu_i.
\]
In Eq. (3), $\rho$ is the density of the fluid, $p$ is the isotropic pressure and $\eta$ is the dynamic viscosity. Eqs. (2)–(3) are numerically solved by using a hybrid lattice Boltzmann (LB) approach \cite{65, 66}, in which a finite difference scheme, adopted to integrate Eq. (2), is coupled to a standard LB method employed for Eq. (1). Further details about numerical implementation and thermodynamic parameters can be found in the Supplemental Material \cite{55}.

We finally provide an approximate mapping between our simulation parameters and real physical values, such as the ones for a microfluidic channel of length $\sim 1\text{mm}$, in which a droplet, with diameter ranging between 30$\mu\text{m}$ and 100$\mu\text{m}$ and surface tension $\sigma \sim 1\text{mN/m}$, is set in a fluid of viscosity $\sim 10^{-1}\text{ Pa.s}$. By fixing the length scale, the time scale, and the force scale as $L = 1\text{mm}$, $T = 10\mu\text{s}$, and $F = 10\mu\text{N}$, a velocity of $10^{-2}$ in simulation units corresponds approximately to a droplet speed of 1 mm/s. The Reynolds number (defined as $Re = \rho D_O v_{\text{max}}/\eta$, where $D_O$ is the diameter of the shell) ranges approximately between 1 ($\Delta p = 4 \times 10^{-4}$ and $v_{\text{max}} \approx 0.01$) and 5 ($\Delta p = 10^{-3}$ and $v_{\text{max}} \approx 0.025$), while the Capillary number (denied as $Ca = v_{\text{max}}\eta/\sigma$) ranges between 0.1 and 1. These values ensure that inertial effects are negligible and are in good agreement with those reported in previous experiments \cite{1} and simulations \cite{50}.

ACKNOWLEDGMENTS

A. T., A. M., M. L., F. B. and S. S. acknowledge funding from the European Research Council under the European Union’s Horizon 2020 Framework Programme (No. FP/2014-2020) ERC Grant Agreement No.739964 (COMP-MAT).

SUPPLEMENTAL MATERIAL

1. NUMERICAL DETAILS

Here we provide further details about the numerical method and the simulation parameters.

The equations for the order parameter $\phi_i$ (Eq. (2) of the main text) and the Navier-Stokes equation (Eq. (3) of the main text) are solved by using a hybrid lattice Boltzmann (LB) method \cite{70}, in which Eq. (2) is integrated via a finite-difference predictor-corrector algorithm and Eq. (3) via a standard LB approach.

As reported in the main text, simulations are performed on a rectangular lattice with size ratio $\Gamma = L_z/L_y$ ranging from 0.16 to 0.22. More specifically, $\Gamma = 0.167$ ($L_y = 600$, $L_z = 100$) for the free-core droplet, $\Gamma = 0.2$ ($L_y = 600$, $L_z = 120$) for the single-core emulsion and $\Gamma = 0.21$ ($L_y = 800$, $L_z = 170$) for the two-core and higher complex emulsions. Periodic boundary conditions are set along the y-axis and two flat walls along the z-axis, placed at $z = 0$ and $z = L_z$. Here no-slip conditions hold for the velocity field (i.e. $u_i(z = 0, z = L_z) = 0$) and neutral wetting for the order parameters $\phi_i$. The latter ones are achieved by setting

$$\frac{\partial \mu_i}{\partial z}igg|_{z=0,z=L_z} = 0 \quad (4)$$

$$\frac{\partial \nabla^2 \phi_i}{\partial z}igg|_{z=0,z=L_z} = 0. \quad (5)$$

The first one guarantees density conservation (no mass flux through the walls) while the second one imposes the wetting to be neutral.

Like in previous works \cite{71, 72}, the pressure gradient $\Delta p$ producing the Poiseuille flow is modeled through a body force (force per unit density) added to the collision operator of the LB equation at each lattice node.

Thermodynamic parameters have been chosen as follows: $a = 0.07$, $k = 0.1$, $M = 0.1$ and $\epsilon = 0.05$ (a value larger than 0.005 is enough to prevent droplet merging). These values fix the surface tension and the interface width to $\sigma = \sqrt{8ak/9} \approx 0.08$ and $\xi = 5\sqrt{k/2a} \approx 4$, respectively. Also, the dynamic viscosity $\eta$ of both fluid components is set equal to 5/3. Such approximation, retained for simplicity, may be relaxed by letting $\eta$ depends on $\phi$ \cite{73, 74}. Lattice spacing and integration time-step have been kept fixed to $\Delta x = 1$ and $\Delta t = 1$, while droplet radii are the following: $R = 30$ for the free-core droplet, $R_{in} = 15$ and $R_{out} = 30$ for a single-core emulsion, and $R_{in} = 17$ and $R_{out} = 56$ for emulsions containing more than one core. Here $R_{in}$ is the radius of the cores while $R_{out}$ is the one of the surrounding shell.

2. VELOCITY PROFILE UNDER POISEUILLE FLOW

In Fig.12 we report, for example, the typical steady-state velocity profile observed in a two-core emulsion for different values of the pressure gradient. They are averaged over space and time, i.e. the channel length and approximately $3 \times 10^5$ time steps at the steady state. The curves are compatible with a parabolic profile expected in an isotropic fluid with the same viscosity, and remain essentially unaltered for the other multi-core emulsions considered in this work.

However, substantial modifications occur when instantaneous configurations are considered. In Fig.13 we show, for instance, the instantaneous velocity profile observed in core-free (a), one-core (b), two-core (c) and three-core (d) emulsion calculated along a cross section of the channel where internal cores temporarily accumulate. While in (a) the parabolic profile is only weakly disturbed by the droplet interface, in (b)-(d) it is significantly modified by local bumps and dips caused by internal cores.
Such distortions wash out when these profiles are averaged over space and time.

3. SUPPLEMENTAL MOVIES

Supplementary Movie 1: This movie shows the dynamics of a core-free emulsion under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (see Fig.1 of the main text)

Supplementary Movie 2: This movie shows the dynamics of a one-core droplet under Poiseuille flow at $Re \simeq 1.2$ and $Ca \simeq 0.35$ (Fig.3b of the main text).

Supplementary Movie 3: This movie shows the dynamics of a two-core droplet under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (Fig.3c of the main text).

Supplementary Movie 4: This movie shows the dynamics of a two-core droplet under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (Fig.3d of the main text).

Supplementary Movie 5: This movie shows the dynamics of a three-core droplet under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (Fig.3e of the main text).

Supplementary Movie 6: This movie shows the dynamics of a three-core droplet under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (Fig.3f of the main text).

Supplementary Movie 7: This movie shows the dynamics of a four-core droplet under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (Fig.3g and Fig.3h of the main text).

Supplementary Movie 8: This movie shows the dynamics of a five-core droplet under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (Fig.3i-j of the main text).

Supplementary Movie 9: This movie shows the dynamics of a six-core droplet under Poiseuille flow at $Re \simeq 3$ and $Ca \simeq 0.85$ (Fig.3k-l of the main text).

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Figure 13. Instantaneous velocity profile of a core-free (a), one-core (b), two-core (c) and three-core (d) emulsion computed along the cross section of the channel, indicated by the dotted green line. Insets show the corresponding configuration of the emulsion.
