Critical behaviors of high-degree adaptive and collective-influence percolation

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How the giant cluster of a network disappears under removing nodes or links addresses key aspect of network robustness, which can be framed into percolation problems. Various strategies to select removing node have been studied in the literature; for instance, a simple random failure or high-degree adaptive (HDA) percolation. Recently a new attack strategy based on a quantity called collective-influence (CI) has been proposed from the perspective of optimal percolation. By successively eliminating the node having the largest CI value, it was shown to be able to dismantle a network more quickly and abruptly than many of existing methods. In this paper, we focus on the critical behaviors of the percolation process following degree-based attack and CI-based attack on random networks. Through extensive Monte Carlo simulations assisted by numerical solutions, we estimate various critical exponents of the HDA percolation and those of the CI percolation with $\ell = 1, 2$. Results show that these attack-type percolation processes, despite displaying apparently more abrupt collapse, nevertheless exhibit standard mean-field critical behaviors at the percolation transition point. We further discover that extensive degeneracy in top-degree and top-CI nodes may provide an underlying reason of the observed results.

Network robustness is a longstanding problem in complex systems. Among various heuristics of optimal percolation proposed to effectively disintegrate a network, high-degree adaptive (HDA) attack and collective-influence (CI) attack are simple heuristics that uses local information but can nevertheless destroy a network much more effectively than random percolation. It is, however, largely unknown as yet how the critical behavior of percolation transition (i.e., the manner in which the giant component disappears) is affected by various attack strategies. In this work, it is revealed that despite apparently more abrupt disintegration the critical behavior of HDA and CI percolation transitions remains of standard mean-field type, supported by the massive degeneracy of maximum degree and CI value near the critical point in these processes.

I. INTRODUCTION

Statistical physics is the study of many-body problems. Complex networks, in which many nodes interact via links, have been the subject of research by many statistical physicists [1]. Understanding the robustness of complex networks is a longstanding problem [2]. Observing how complex networks respond to intentional attacks is a common way to approach this problem. The giant cluster of the network under intentional attack gets smaller and finally disappears at critical point. This can map to an optimal percolation problem that destroys the giant cluster of network by removing a minimal set of nodes. The minimal set of removed nodes is considered to be important nodes for maintaining the function of the network. These important nodes are also called influencers.

Unfortunately, finding the minimal set of nodes that can destroy the giant cluster in network, is an NP-hard problem. In the early research stages, nodes with high-degree or high-betweenness centrality were considered influencers [3,4]. Since then, a number of strategies have been proposed to find near-optimal influencers more effectively [5–14]. These strategies are summarized in Ref. [15]. In general, each strategy assigns a centrality value to nodes in its own way, and the node with highest centrality value is deleted one by one. Centrality value of nodes can either be kept fixed as the initial value during the removal process or be updated in every step “adaptively” as the nodes are removed. However, there is no integrated perspective of various optimal percolation strategies yet. An important step towards integrated understanding from the perspective of statistical physics is to characterize the percolation phase transition properties of each strategy. Recently, critical behaviors of phase transition in high-degree and high-betweenness centrality percolation have been reported [16].

The percolation process we chose to study critical behaviors of phase transition in this work is the collective-influence (CI) percolation. CI percolation includes also the high-degree adaptive (HDA) percolation as a special case. Principal reason why we chose CI percolation is that it is known to be able to attain near-optimal percolation results by straightforward and fast algorithm of time complexity $O(N \log N)$ [5,17]. For this reason, it has been actively studied [18,19] and refined [17,20]. Detailed model description will be given in Sec. II.

As shown in the Fig. 1, in the case of CI percolation, the order parameter seems to disappear rapidly at the critical point. This raises the question that it might possess order-parameter critical exponent $\beta$ different from that of random percolation. To answer this question,
we performed finite-size scaling analysis using extensive Monte Carlo simulations. Despite the apparently abrupt change at critical point, we obtained standard mean-field critical exponents in both the HDA and CI percolations. To understand this result, we analyzed the degeneracy of centrality. We shall argue that the mean-field critical behaviors appear due to massive degeneracy in highest-centrality nodes in both HDA and CI percolations.

II. MODELS

As the percolation process proceeds, a node’s degree undergoes change. In the following, we denote node \( i \)'s degree, \( k_i \), as the number of node \( i \)’s neighbors in the remaining network. In HDA percolation, one removes the node with highest degree and updates its neighbors' degree in every step. If there are more than one node with the same highest degree, one of them is randomly removed.

For CI percolation, we follow the definition of CI value in Ref. [7], given as follows:

\[
CI_l(i) = (k_i - 1) \sum_{j \in \partial B(i, l)} (k_j - 1).
\]

Here \( CI_l(i) \) is the CI value of node \( i \) with distance parameter \( l \), and \( \partial B(i, l) \) is set of nodes which are in distance \( l \) (disallowing backtracking) from the node \( i \). CI value of node \( i \) incorporates not only its own degree but also the degrees of nodes which are in distance \( l \) away from it. Therefore, CI percolation does not simply remove high-degree nodes themselves, but tends rather to remove “bridges” between high-degree nodes [7]. In CI percolation, one removes the node with highest CI value and update the CI values within the distance \( (l + 1) \) from the removed node each step. Similarly to HDA percolation, if there are multiple nodes with same CI value, then a randomly-chosen node among them is removed. Note that by considering the distance-0 neighbor to be the node itself, we have \( CI_0(i) = (k_i - 1)^2 \). Thus HDA percolation can be treated as the special case of CI percolation with \( l = 0 \).

III. MONTE CARLO SIMULATION METHODS AND NUMERICAL SOLUTIONS

We applied HDA percolation and CI percolation with \( l = 1, 2 \) on Erdős-Rényi (ER) networks. The control parameter is the fraction of removed nodes \( q \) (\( 0 \leq q \leq 1 \)). Primary quantity of interest is the the probability that a randomly chosen node belongs to giant cluster, \( P_\infty \), which serves as the order parameter in the percolation process. Numerically we calculate it by the largest connected component fraction averaged over Monte Carlo configurations. We also examine the average cluster size \( \chi \) that a randomly-chosen node belongs to, which plays the role of susceptibility, defined as follows:

\[
\chi(q) = \frac{\sum_{s, \ finite} s^2 n(s, q)}{\sum_{s, \ finite} sn(s, q)},
\]

where \( s \) is the cluster size, \( n(s, q) \) is the number of clusters with size \( s \) at removal fraction \( q \), and the summation runs over only finite sizes. Numerically we calculate Eq. (2) using the cluster configurations obtained by Monte Carlo simulations by omitting the largest component. We have performed Monte Carlo simulations on ER networks with different mean degrees to find consistent results and in the following will present the results specifically for the mean degree \( z = 7/2 \).

We also formulate the numerical solution for the HDA percolation. For a random network, various quantities can be calculated by generating function method with degree distribution \( p_k \) \([3, 21]\]. Fortunately, we can calculate degree distribution \( p_k \) as a function of removal fraction \( q \) for HDA percolation using iteration method. To apply iteration method, we need initial condition and recurrence relation. Degree distribution of ER network of mean degree \( z \) is well known to be the Poisson distribution \( p_k = z^k e^{-z}/k! \) \([22]\).

To obtain the recurrence relations for the degree distribution as the nodes are removed, let the degree distribution \( p_k(q) \) be the probability that the randomly chosen node has degree \( k \) at removal fraction \( q \) on HDA percolation. The degree distribution after removal additional \( dq \) fraction of nodes, \( p_k(q + dq) \), can be obtained as follows. First, as \( q \) increases by \( dq \), then \( p_0 \) increases by \( dq \) and \( p_k \) decreases by \( dq \) where \( K = k_{max}(q) \), the maximum degree in the network at the removal fraction \( q \). At the same time, when the node with degree \( K \) is deleted, degree of nodes linked to that deleted node decreases by 1. The probability that a node with degree \( k \) is connected to deleted node is \( kp_k / \sum_{k'=0}^\infty k'p_{k'} \). Combining the two
effects, we obtain the complete set of recurrence relations as follows.

\[
\begin{align*}
p_0(q + dq) &= p_0(q) + K \frac{p_1}{\sum_{k' = 1}^{K} k'p_{k'}} dq + dq , \\
p_1(q + dq) &= p_1(q) + K \frac{2p_2 - p_1}{\sum_{k' = 1}^{K} k'p_{k'}} dq , \\
&\vdots \\
p_k(q + dq) &= p_k(q) + K \frac{(k + 1)p_{k+1} - kp_k}{\sum_{k' = 1}^{K} k'p_{k'}} dq , \\
&\vdots \\
p_K(q + dq) &= p_K(q) + K \frac{-Kp_K}{\sum_{k' = 1}^{K} k'p_{k'}} dq - dq .
\end{align*}
\]

The right hand side of each line contains a gain term due to the nodes with degree \((k+1)\) linked to the deleted node and the loss term due to nodes with degree \(k\) linked to the deleted node, except for the \(k = 0\) (\(k = K\)) equation that contains two gain (loss) terms. Theoretically, the maximum degree \(K\) is not limited; however, when numerically solving Eq. (3) we can practically introduce a finite maximum degree \(K\) without sacrificing precision. In our numerical calculation we set the cutoff probability density to be \(10^{-10}\), meaning that the degree \(k\) with \(p_k < 10^{-10}\) is considered nonexistent in the network. In this way, the initial value of \(K\) was obtained to be \(K = 21\) for the ER network with mean-degree \(z = 7/2\). As Eq. (3) is iterated, the maximum degree decreases when \(p_K\) drops below the cutoff.

Once the \(p_k(q)\) is obtained by solving the recurrence relations Eq. (3), one can apply the standard generating function technique [4, 21] to calculate \(P_N(q)\) and \(\chi(q)\) as a function of the removal fraction \(q\). The underlying assumption behind this procedure is that after the HDA percolation, the remaining network can be characterized solely by \(p_k(q)\) like random networks. The agreement of
the numerical solution with the Monte Carlo simulations, as shown in Fig. 1 validates this assumption.

IV. CRITICAL BEHAVIORS

As one removes nodes (increasing $q$), the giant component size decreases and becomes zero at the critical point $q_c$. Typical of continuous phase transitions, the average cluster size $\chi$ and the correlation length $\xi$ diverge at the critical point in the thermodynamic limit. In the vicinity of critical point, the order parameter, the average cluster size and the correlation length are known to exhibit power laws with critical exponents $\beta$, $\gamma$ and $\nu$, respectively:

\begin{align}
    P_\infty(q) &\propto (q_c - q)^\beta \quad \text{with } q \rightarrow q_c^- , \\
    \chi(q) &\propto |q_c - q|^{-\gamma} \quad \text{with } q \rightarrow q_c , \\
    \xi(q) &\propto |q_c - q|^{-\nu} \quad \text{with } q \rightarrow q_c .
\end{align}

(4) (5) (6)

We use finite-size scaling ansatz and Monte Carlo simulations to obtain these critical exponents. According to finite-size scaling theory, percolation quantities scale at the critical point with the size of the system $N$ as

\begin{align}
    P_\infty(N, q_c) &\propto N^{-\beta/\tilde{\nu}} , \\
    \chi(N, q_c) &\propto N^{\gamma/\tilde{\nu}} , \\
    q_c - q_c(N) &\propto N^{-1/\tilde{\nu}} ,
\end{align}

(7) (8) (9)

where $\tilde{\nu} = d\nu$ with $d$ being effective dimension [22], which is $d = 6$ for ER network with random percolations. In Eq. (9), $q_c(N)$ is the value of $q$ at which $\chi(q)$ displays maximum value in the networks of finite size $N$.

To apply the finite-size scaling theory, we first need to know the critical point $q_c$. Using numerical solution, the critical point on HDA percolation on ER network with mean degree $z = 7/2$ is obtained to be $q_c = 0.235550$. For CI percolation, we estimated $q_c$ that best fits Eqs. (7) using Monte Carlo simulation results. For the ER networks with the same mean degree $z = 7/2$, we obtained the critical point of CI percolation to be $q_c = 0.211\ 61(1)$, and for CI$_2$ percolation $q_c = 0.206\ 01(1)$.

Table I. Critical points and critical exponents of various percolation processes on Erdős-Rényi networks with mean degree $z = 7/2$.

| Model          | $q_c$  | $\beta$ | $\gamma$ | $\tilde{\nu}$ |
|----------------|--------|---------|----------|----------------|
| HDA percolation| 0.235 550 | 0.99(4) | 1.00(4) | 2.97(13)       |
| CI$_1$ percolation | 0.211\ 61(1) | 1.02(5) | 1.03(5) | 3.05(16)       |
| CI$_2$ percolation | 0.206\ 01(1) | 1.03(6) | 1.02(6) | 3.04(16)       |
| Random percolation | 0.714\ 285 | 1      | 1        | 3              |

TABLE I. Critical points and critical exponents of various percolation processes on Erdős-Rényi networks with mean degree $z = 7/2$.

VI. DISCUSSION

In this paper, we have studied critical behaviors of HDA and CI percolation transitions on random networks using Monte Carlo simulations and numerical solutions. We found that the critical behavior of the two attack-based processes is in the same universality class as the random percolation, despite their more abrupt, near-optimal disruption of network connectivity than random percolation. We uncovered the massive degeneracy of maximum degree and CI values near the critical point, which might contribute to render the transition of mean-field type. Recently, a study reported [16] non-standard mean-field critical exponent for HDA percolation, specifically $\tilde{\nu} = 2.53(1)$, which contradicts with our mean-field-consistent result $\tilde{\nu} = 2.97(13)$ in this work. Our analysis uses much larger network size (up to $O(10^8)$) compared to $1.6 \times 10^4$ in Ref. [16] and is supported also by the numerical solutions.

In a broader perspective, our work initiates studies of attack-type and optimal percolation processes for deeper understanding from the viewpoint of critical behaviors at the percolation transition point. Understanding the reasons why HDA and CI percolations would belong to the same universality as random percolation.
nature of critical behaviors will help for example devise detection and protection strategy upon the attacks on the network, as different criticality entails different ‘early warnings’ [24]. Another interesting standing question is the condition for non-mean-field criticality in the optimal percolation processes. The HDA and CI percolation processes implement local heuristics for selecting nodes to be removed. There exist approaches based on global optimization methods [9–14] and improved local heuristics [20] that reportedly produce discontinuous disappearance of the giant component in the attacked network. Identifying minimal condition for the non-standard mean-field criticality and discontinuity in the network attack process remains a theoretically intriguing problem, to be explored in future works.

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AUTHOR CONTRIBUTIONS

J-HK, S-JK and K-IG conceived the project; J-HK, S-JK performed the simulations; all authors interpreted the results and contributed to the writing of the manuscript.

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