Relativistic Heavy Ion Collisions:
Viscous Hydrodynamic Simulations and Final State Interactions

Matthew W. Luzum

A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

University of Washington

2009

Program Authorized to Offer Degree: Department of Physics
University of Washington
Graduate School

This is to certify that I have examined this copy of a doctoral dissertation by

Matthew W. Luzum

and have found that it is complete and satisfactory in all respects, and that any and all revisions required by the final examining committee have been made.

Chair of the Supervisory Committee:

Gerald A. Miller

Reading Committee:

Gerald A. Miller

Paul Romatschke

Andreas Karch

Date:
In presenting this dissertation in partial fulfillment of the requirements for the doctoral degree at the University of Washington, I agree that the Library shall make its copies freely available for inspection. I further agree that extensive copying of this dissertation is allowable only for scholarly purposes, consistent with “fair use” as prescribed in the U.S. Copyright Law. Requests for copying or reproduction of this dissertation may be referred to Proquest Information and Learning, 300 North Zeeb Road, Ann Arbor, MI 48106-1346, 1-800-521-0600, to whom the author has granted “the right to reproduce and sell (a) copies of the manuscript in microform and/or (b) printed copies of the manuscript made from microform.”

Signature___________________________________________

Date_______________________________________________
In this dissertation I introduce relativistic heavy ion collisions and describe theoretical approaches to understanding them—in particular, viscous hydrodynamic simulations and investigations of final state interactions.

The successful ideal hydrodynamic models of the collisions at the Relativistic Heavy Ion Collider (RHIC) were extended by performing viscous hydrodynamic simulations. This was done by making use of the recently derived full conformally invariant second order relativistic viscous hydrodynamic equations. Results for multiplicity, radial flow and elliptic flow in $\sqrt{s_{NN}} = 200$ GeV Au+Au RHIC collisions are presented and the range of the ratio of shear viscosity over entropy density $\eta/s$ for which our hydrodynamic model is consistent with experimental data is quoted.

In addition, simulations were performed of the planned $\sqrt{s_{NN}} = 5.5$ TeV Pb+Pb and $\sqrt{s} = 14$ TeV p+p collisions at the Large Hadron Collider (LHC). The elliptic flow coefficient $v_2$ is predicted to be 10% larger for the Pb+Pb collisions compared to top energy RHIC collisions, and is predicted to be consistent with zero for proton collisions unless $\eta/s < 0.08$.

Finally, final state interactions were investigated within the distorted wave emission function (DWEF) model. Work is presented on an improved understanding of the DWEF model, and the potential effect of final state interactions in the form of a pion optical potential on the elliptic flow coefficient $v_2$ was calculated to be at the ~20% level.
# TABLE OF CONTENTS

List of Figures ................................................................. iv
List of Tables ................................................................. vi

Chapter 1: Prologue: Introduction, Motivation and Philosophy .............. 1
  1.1 Introduction .................................................................. 1
  1.2 Strong Interactions, QCD Phase Diagram and the Quark-Gluon Plasma ...
    1.2.1 What do we know about the world? ......................... 2
    1.2.2 The strong force and quantum chromodynamics .......... 3
    1.2.3 Confinement, temperature, and phase transitions ....... 4

Chapter 2: Relativistic Heavy Ion Collisions ................................... 6
  2.1 AGS, SPS, RHIC and LHC ............................................. 6
  2.2 Some Relevant Observables .......................................... 7
    2.2.1 Single particle spectra and elliptic flow ................. 8
    2.2.2 Two-particle correlations—Hanbury Brown/Twiss interferometry ... 10

Chapter 3: Hydrodynamics .......................................................... 12
  3.1 Introduction ................................................................ 12
  3.2 Non-Relativistic Fluid Dynamics ................................... 13
  3.3 Relativistic Ideal Hydrodynamics .................................. 13
  3.4 Relativistic Viscous Hydrodynamics ............................... 15
    3.4.1 Relativistic Navier-Stokes equations .................... 16
    3.4.2 Causality restored: second-order relativistic viscous hydrodynamics .... 17
    3.4.3 Müller-Israel-Stewart theory .............................. 19
    3.4.4 Conformal relativistic viscous hydrodynamics .......... 21

Chapter 4: Modeling Heavy Ion Collisions Using Viscous Hydrodynamics .. 23
  4.1 Anatomy of a Heavy Ion Collision .................................. 23
  4.2 Setup .................................................................. 26
**LIST OF FIGURES**

| Figure Number | Page |
|---------------|------|
| 2.1           | 7    |
| 2.2           | 8    |
| 4.1           | 23   |
| 4.2           | 34   |
| 4.3           | 36   |
| 4.4           | 40   |
| 4.5           | 45   |
| 4.6           | 46   |
| 4.7           | 48   |
| 4.8           | 51   |
| 4.9           | 53   |
| 4.10          | 55   |
| 4.11          | 57   |
| 5.1           | 66   |
| 5.2           | 67   |
| 5.3           | 68   |
| 5.4           | 69   |
| 5.5           | 70   |
| 6.1           | 85   |
| 6.2           | 86   |

iv
7.1 DWEF model calculated $v_2$ as a function of momentum
# LIST OF TABLES

| Table Number | Description                                                                 | Page |
|--------------|-----------------------------------------------------------------------------|------|
| 4.1          | Parameters used for the viscous hydrodynamics simulations of RHIC collisions | 52   |
| 5.1          | Parameters used for the viscous hydrodynamics simulations of RHIC heavy ion collisions and LHC heavy ion and proton-proton collisions | 63   |
| 6.1          | Parameter sets for DWEF central collision calculations                       | 84   |
| 7.1          | DWEF model best fit parameter values used in $v_2$ calculation              | 91   |
ACKNOWLEDGMENTS

The author wishes to express sincere appreciation to all those who helped make graduate school bearable:

- to my advisor Jerry Miller for his patient guidance

- to my collaborator and mentor Paul Romatschke for (also patiently) putting up with all my stupid questions and his willingness to share his knowledge

- to my fellow physics graduate students for making grad student life much more interesting
DEDICATION

to my parents
Chapter 1
PROLOGUE: INTRODUCTION, MOTIVATION AND PHILOSOPHY

The structure of this dissertation is as follows. This chapter consists of a nontechnical and general discussion of the motivation behind relativistic heavy ion collisions. Chapter 2 briefly describes the experiments and introduces a few measured quantities that will be important for the theoretical work that is presented in the remaining chapters. The framework of hydrodynamic theory in general is introduced in chapter 3. Following these introductory chapters is the main body, which presents original work (collaboratively) done by the author. In chapter 4, hydrodynamic models of collisions at the Relativistic Heavy Ion Collider are introduced and results of these simulations are presented (corresponding to Refs. [1, 2]). Simulations of collisions at the Large Hadron Collider are given in chapter 5 (corresponding to Ref. [3]). Chapters 6 and 7 describe the DWEF model investigations of final state interactions (roughly corresponding to Refs. [4, 5]). Appendix A offers additional details of the DWEF calculation of $v_2$ while appendix B contains a list of the conventions and notation used throughout this manuscript.

A digital version of this manuscript, including high quality color figures, will be available online at http://arxiv.org/a/luzum_m_1. At the time of this writing, source code and results from the viscous hydrodynamic simulations presented in chapters 4 and 5 can be found on Paul Romatschke’s webpage: http://hep.itp.tuwien.ac.at/~paulrom/

1.1 Introduction

The goal of the work described in this dissertation is to better understand how the world works on the most fundamental levels. By studying very small and/or simple physical systems, we can extract information about the fundamental laws that govern the world we live in—including (presumably) the behavior of the much more complex systems that we
typically encounter in day-to-day life. We do this because of simple human curiosity and a natural desire for knowledge, but also because this knowledge tends to be very useful. When we have a detailed understanding of how the world works, we can often manipulate it for our benefit.

Of course this particular line of scientific inquiry is only one of many that are both useful and necessary. More complex systems must be studied on their own as well. For example, it is neither interesting nor practically possible to calculate the fundamental interactions of every molecule in a bridge when trying to determine whether it will support a load without collapsing (let alone all the atoms and electrons or quarks and gluons). Even in the work described herein, hydrodynamic equations will be used extensively. These equations describe a sort of coarse-grained behavior on a scale that is large compared to the fundamental microscopic dynamics to which hydrodynamic behavior is largely insensitive. Indeed, many would argue that the study of larger scale and perhaps less fundamental behavior—e.g., chemistry, biology, materials science, medicine, etc.—is more important. Nevertheless, I would argue that it is still very important to study these fundamental laws of nature—even in such exotic regimes as extremely high temperature nuclear matter, and even if it doesn’t seem to have any obvious practical applications. A hundred years ago, there was no reason to think that understanding the weird quantum mechanical behavior of tiny particles would be of any practical use. On the contrary, almost none of the current technology that we all take for granted—and that enable much of the progress in other sciences—would be possible without the insights gained from these seemingly esoteric studies.

1.2 Strong Interactions, QCD Phase Diagram and the Quark-Gluon Plasma

1.2.1 What do we know about the world?

The world as we know it is made up of matter and the forces that interact with and hold this matter together. These interactions are usually classified into four known fundamental forces: gravity, electricity and magnetism (electromagnetism), the weak nuclear force, and the strong nuclear force. These fundamental forces are listed here in order of increasing strength, and therefore also generically of increasing importance as one considers behavior
at smaller and smaller length scales.

For example, gravity is important for describing the movement of large collections of matter that have essentially no net electric charge, such as planets moving through the solar system. If we want to study how atoms form into molecules and solids, on the other hand, gravity has essentially no effect because electromagnetic interactions are so much stronger and are much more important to the movement of electrically charged matter. Going further down in scale, the structure of nuclei inside atoms is dominated by the strong and weak nuclear forces.

The goal here will be to study particular aspects of the strong nuclear force, often referred to by physicists simply as strong interactions. Correspondingly, it will be necessary to look at very very small length scales. This will—due to Heisenberg’s uncertainty principle—involve studying the behavior of matter at very large energies that are obtained, perhaps unsurprisingly, by smashing things together in accelerators.

1.2.2 The strong force and quantum chromodynamics

The Standard Model of elementary particle physics is composed of well-tested quantum field theories that describe all of the fundamental forces except gravity. The strong interactions in particular are well described by a theory called Quantum Chromodynamics (QCD). QCD describes the interaction of fundamental fields called quarks and gluons. These quarks and gluons form the protons and neutrons that, along with electrons, form almost all of the matter we see around us.

In any particular interaction between fundamental particles in a quantum field theory such as QCD, there is a characteristic energy scale, and the strength of the interaction (quantified by a “coupling” $g$) depends on this scale. QCD has an unusual property called “asymptotic freedom”, which means that the strength of the interaction decreases as this energy scale increases, and vice versa. An intrinsic energy scale for the strong interactions is $\Lambda_{QCD}$, where the coupling becomes order one. At energies much larger than this, the coupling is small and one can usually use the familiar methods of perturbation theory to calculate various quantities in QCD. Most of the precision tests of QCD have been done in
this regime and it is relatively well understood. When there are energy scales in a particular problem that are near or below $\Lambda_{QCD}$ (even if the energies are very large compared to atomic energy scales), things typically become much more difficult, and this will be a hindrance in the study of heavy ion collisions.

### 1.2.3 Confinement, temperature, and phase transitions

Related to asymptotic freedom (but on the other end of the energy spectrum) is the concept of color confinement, another property of the strong interactions. Particles that participate in strong interactions have what’s called a “color” charge, analogous to electric charge for electromagnetic behavior (and completely unrelated to the color of visible light). In loose terms, confinement means that it is impossible to isolate a color-charged particle such as a quark. They are only found tightly bound together with other colored objects in overall color-neutral states.

As an example, think of a color-neutral pair of a quark and an antiquark. Asymptotic freedom implies that if they are very close together they interact very weakly. If one was to try to pull them apart, however, the attraction would become stronger and stronger such that it would in principle take an infinite amount of energy to completely separate them. In reality there would eventually be so much energy between them that more quark-antiquark pairs would be created out of the vacuum, and you would just be left with multiple color-neutral states instead of the one you started with.

One can imagine, however, collecting together some strongly interacting matter and raising the temperature. Asymptotic freedom implies that there exists some (extremely large) temperature at which the strong interactions would become so weak that individual quarks and gluons would no longer be confined. Thus, there is expected to exist a deconfinement phase transition, where the color-neutral hadrons would “melt” into a deconfined state of matter called the quark-gluon plasma (QGP).

How might one go about studying such an exotic regime of physics? Such huge temperatures are hard to come by in the current universe—the temperature required (a few trillion degrees) is perhaps 100000 times larger than that at the center of the sun. The idea, then,
is to actually create conditions in the laboratory sufficient to create a QGP. By accelerating heavy ions (nuclei of large atoms) to extremely high energies and letting them collide, it was hoped that—if only for a tiny fraction of a second—the “fireball” created would be hot and dense enough to create this deconfined phase. By carefully studying what comes out of such a collision, one could then in principle discern many properties of such a state of matter and of the strong interactions in general.
Chapter 2

RELATIVISTIC HEAVY ION COLLISIONS

2.1 AGS, SPS, RHIC and LHC

Once it was realized that QCD implies a deconfined state of matter, the idea of colliding heavy ions to study it was quickly adopted. There already were relativistic heavy ion collisions being studied at the BEVALAC at Berkeley, but these were not at a sufficiently high energy to potentially create a QGP. The first ultrarelativistic heavy ion collisions that were done to investigate the quark-gluon plasma were performed at existing particle colliders that were modified to accept heavy ion beams, most notably the Alternating Gradient Synchrotron (AGS) at Brookhaven and the Super Proton Synchrotron (SPS) at CERN \[6\]. These were both fixed-target experiments that, among other things, collided Au+Au at up to 11 GeV per nucleon beam energy (AGS) and Pb+Pb at up to 160 Ag GeV (SPS). These experiments revealed tantalizing evidence of a hot and dense state of matter that had not previously been seen \[7\].

The first facility specifically designed for colliding ultrarelativistic heavy ions was the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory. There, collisions were performed by colliding beams with a center of mass energy of up to 200 GeV per nucleon pair, leading to much more energy potentially being deposited in the collision region as well as the possibility of the resulting fireball remaining in the deconfined state for a longer period of time, compared to the earlier lower energy collisions. All the experimental data analyzed in this dissertation are from runs at RHIC, and for simplicity the rest of chapter will focus on what has been done there.

The Large Hadron Collider (LHC) at CERN is the next generation collider. Although most people know of it as a proton-proton collider, it was also designed to run heavy ion collisions part-time and the first runs should begin before long. Ultimately it is planned to collide Pb+Pb beams at up to 5.5 TeV per nucleon pair.
2.2 Some Relevant Observables

All the information that can be obtained from a given collision event comes from studying the thousands of produced particles that emerge from the collision region. Any information about the evolution of the fireball system and its medium properties must be inferred by looking at these final products well after they have stopped interacting with each other, as they stream into one of the detectors surrounding the collision region (see Figure 2.1). Much of what a theorist would ideally like to measure, therefore, may be inaccessible to direct measurement. However, a surprising amount about the collision can still be learned this way (see Refs. [9, 10, 11, 12] for an overview from each of the main detector collaborations at RHIC of the first four years of results). The following describes the particular measured quantities that will be important for the work comprising this dissertation.
2.2.1 Single particle spectra and elliptic flow

First, note some standard definitions: The beam direction defines the $z$-axis, and the $x$ direction is defined such that the $x$-$z$ plane is the collision plane as seen in Figure 2.2. It is often useful to use polar coordinates in the transverse ($x$-$y$) plane, where the azimuthal angle $\phi$ is measured with respect to the $x$-axis. One can also define an impact parameter $b$ connecting the lines-of-flight of the centers of mass of the colliding nuclei.

The detected particles are characterized by their momentum $p$ after they exit the collision region. Instead of the longitudinal component of momentum $p_z$ or the polar angle with respect to the beam $\theta$, what is more commonly reported is the particle rapidity $Y \equiv \text{arctanh}(p_z/E)$ or pseudorapidity $\eta \equiv -\ln(\tan(\theta/2))$. (Note that rapidity and pseudorapidity are equivalent for a relativistically moving particle ($m \to 0$) since $Y = \frac{1}{2} \ln \left[ \frac{E+p_z}{E-p_z} \right]$ and $\eta = \frac{1}{2} \ln \left[ \frac{|p|^2+p_z^2}{|p|^2-p_z^2} \right]$. Thus they are often used interchangeably, although one should always keep in mind the limits to the validity of this equivalence.)
The probability of detecting a particle in a given event with a given rapidity and transverse momentum is given by the distribution
\[ \frac{dN}{dY \, d^2p_T} = E \frac{dN}{dp} \]
such that
\[ N = \int dY \, d^2p_T \frac{dN}{dY \, d^2p_T}, \tag{2.1} \]
where \( N \) is the total number of particles in the event and \( p_T = \sqrt{p_x^2 + p_y^2} \) is the transverse momentum. This distribution can refer to a particular species of identified particle (e.g., protons) or a composite measurement such as all charged hadrons combined. From this, one can calculate the mean transverse momentum
\[ \langle p_T \rangle = \left( \int d^2p_T \, p_T \frac{dN}{dY \, d^2p_T} \right) / \left( \int d^2p_T \, \frac{dN}{dY \, d^2p_T} \right). \tag{2.2} \]

It is useful to break up the distribution into its Fourier components with respect to the azimuthal angle of the outgoing particle’s momentum \( \phi_p \equiv \tan^{-1}(p_y/p_x) \):
\[ \frac{dN}{dY \, d^2p_T} = v_0 \left[ 1 + \sum_{n=1}^{\infty} 2v_n \cos(n \phi_p) \right]. \tag{2.3} \]
The sine terms vanish due to the reflection symmetry about the collision \((x-z)\) plane, and for collisions of identical nuclei the reflection symmetry about the \(y-z\) plane causes the odd cosine moments to vanish. \( v_0(p_T) \) is referred to as radial flow and the next lowest non-vanishing coefficient \( v_2(p_T) \) is called the elliptic flow coefficient. Explicitly we have
\[ v_0 \equiv \int \frac{d\phi_p}{2\pi} \frac{dN}{dY \, d^2p_T}, \tag{2.4} \]
\[ v_2 \equiv \langle \cos(2\phi_p) \rangle = \frac{1}{v_0} \int \frac{d\phi_p}{2\pi} \cos(2\phi_p) \frac{dN}{dY \, d^2p_T}. \tag{2.5} \]

The momentum integrated elliptic flow coefficient is denoted
\[ v_2^{\text{int}} \equiv \int \frac{d^2p_T \, v_2 \, v_0}{dp_T \, v_0}, \tag{2.6} \]
and the minimum bias \( v_2 \) is defined by averaging over all collisions in a given run—i.e., integrating over impact parameter \( b \):
\[ v_2^{\text{mb}} = \frac{\int db \, v_2(b) \, v_0(b)}{\int db \, v_0(b)}. \tag{2.7} \]
One should note that the colliding nuclei consist of a collection of well-localized nucleons and so are not as smoothly distributed as Figure 2.2 might suggest. Therefore there can be some ambiguity in defining, e.g., the collision plane. (Also note that the odd moments of the particle distribution are not exactly zero). The theoretical calculations presented here, however, will model the collisions with smooth initial conditions and these definitions are then completely unambiguous. The difficulty then comes in knowing which experimental results to compare the theoretical results to. (See, e.g., chapter 4 comparing minimum bias \(v_2\) results to different experimental extractions that attempt to remove “non-flow” effects).

### 2.2.2 Two-particle correlations—Hanbury Brown/Twiss interferometry

In the 1950’s, Robert Hanbury Brown and Richard Q. Twiss began using a method of intensity interferometry to measure the sizes of various objects—most notably measuring the angular size of the star Sirius in 1956 [14, 15]. It turns out that by correlating the intensity of light emitted incoherently from a source—even without measuring any information about phase (a mandatory ingredient of typical amplitude interferometry)—one can directly measure information about its size, as well the time dependence of a source that varies in time. This effect can be thought of as being caused by the symmetrization of the wavefunction for identical bosons, such as the photons emitted from a star, which causes enhancement of coincident measurement of pairs of these bosons with small momentum difference. (Similarly, an “anti-bunching” effect is present for identical fermions because of the antisymmetric nature of their wavefunction).

Despite early skepticism, this technique was quickly employed to measure space-time properties of collision systems in the laboratory by analyzing two-particle correlations of identical particles emitted from such collisions [16], and has since been used extensively to study heavy ion collisions [17]. Such analyses of two-particle correlations are often generically referred to as Hanbury Brown/Twiss (HBT) interferometry, or simply femtoscopy.

Explicitly, the quantity constructed to analyze RHIC events is the ratio of the two-particle inclusive and single-particle inclusive spectra:

\[
C(p_1, p_2) \equiv \frac{dN/(d^3p_1d^3p_2)}{(dN/d^3p_1)(dN/d^3p_2)}. \tag{2.8}
\]
The numerator measures the probability that two particles of momentum $p_1$ and $p_2$ are detected in the same event, while the denominator is the product of the familiar spectra from section 2.2.1.

Define the average momentum $K \equiv (p_1 + p_2)/2$ and the momentum difference $q \equiv (p_1 - p_2)$. Then one can define the directions $L$ (longitudinal), $O$ (out), and $S$ (side) as the directions parallel to the beam, parallel to $K_T \equiv \sqrt{K_T^2 + K_T^2}$, and perpendicular to both the beam and $K_T$, respectively.

The correlation function can then be parameterized as a Gaussian with parameters that are fit to data:

$$C(p_1, p_2) = C(K, q) \approx 1 + \lambda \exp(-R_O^2 q_O^2 - R_S^2 q_S^2 - R_L^2 q_L^2), \quad (2.9)$$

or, for small $q$

$$C(K, q) \approx 1 + \lambda (1 - R_O^2 q_O^2 - R_S^2 q_S^2 - R_L^2 q_L^2). \quad (2.10)$$

For a static Gaussian source, these HBT radii $(R_O, R_S, R_L)$ would be independent of $|\vec{K}|$, and would reveal the spatial extent of the source. For a general dynamic source, the radii can be complicated functions of $K$, and their interpretation more complicated.
Chapter 3

HYDRODYNAMICS

3.1 Introduction

Before delving into the specifics of viscous hydrodynamic models of relativistic heavy ion collisions, it is useful to know a bit about the theory of hydrodynamics in general and in particular the development of relativistic viscous hydrodynamics (see Paul Romatschke’s lecture notes [18] for a nice, more detailed, treatment).

Hydrodynamics—also known as fluid dynamics—is the theory governing the motion of fluids. As the name implies, it was initially developed to describe the dynamics of water, but can be applied to fluid behavior of a wide range of materials. It can be thought of as an effective theory describing the long wavelength behavior of a system that has sufficient separation of scales such that this macroscopic motion is so slowly varying in space and time so as to be insensitive to the microscopic dynamics. In the case of water, for example, if the macroscopically-averaged quantities such as pressure and temperature change very slowly in space compared to the average distance between molecules and very slowly in time compared to the scattering rate of the individual molecules, it will behave according to the equations of hydrodynamics.

Likewise, there should be hydrodynamic regimes for systems consisting of a collection of hadrons or even a quark gluon plasma. One can imagine a hypothetical large system that has had enough time to everywhere come very close to thermal equilibrium, yet still has a temperature gradient that slowly varies across the system. This system would behave hydrodynamically, even if it consists of a very hot and dense collection of strongly interacting matter.

It is a more difficult question, however, whether the medium created in a relativistic heavy ion collision interacts strongly enough to behave like a fluid for the short period of time before it flies apart. Although it now appears from the success of hydrodynamical
models that this is likely the case, it was not at all obvious that the hydrodynamic description should be correct and it was originally believed by many to be unlikely.

3.2 Non-Relativistic Fluid Dynamics

The conventional (non-relativistic) formulation of the hydrodynamic equations describes the evolution of the fluid velocity $\vec{v}(t, \vec{x})$, the pressure $p(t, \vec{x})$, and the mass density $\rho(t, \vec{x})$ of a fluid at each point in space and time via the equations

$$\partial_t \rho + \rho \vec{v} \cdot \vec{\nabla} \vec{v} + \vec{v} \cdot \vec{\nabla} \rho = 0,$$

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p,$$

These equations are referred to as the continuity equation (3.1) and Euler equations (3.2), respectively, and are simply the statements of conservation of mass and momentum for a continuous fluid without dissipation (an “ideal” fluid). To close the set of equations another relation is needed—usually given as an equation of state of the material $p = p(\rho)$. The Euler equations can be generalized to treat dissipative effects

$$\partial_t v^i + v^k \partial_i v^k = -\frac{1}{\rho} \partial_i p + \frac{1}{\rho} \partial_k \Pi_{ki},$$

$$\Pi_{ki} = -\eta \left( \frac{\partial v^i}{\partial x^k} + \frac{\partial v^k}{\partial x^i} - \frac{2}{3} \delta_{ki} \frac{\partial v^l}{\partial x^l} \right) - \zeta \delta_{ik} \frac{\partial v^l}{\partial x^l},$$

These are called the Navier-Stokes equations (3.3). The coefficients in the viscous stress tensor $\Pi_{ki}$ are termed the shear viscosity ($\eta$) and bulk viscosity ($\zeta$). Their values, like the form of the equation of state, depend on the specific fluid in question and encode information about the microscopic dynamics of that material.

3.3 Relativistic Ideal Hydrodynamics

Let us generalize this so that it can be applied to a relativistic system. Any system can be characterized by its energy-momentum tensor $T^{\mu\nu}(x)$, which is a symmetric tensor that describes the distribution of energy and momentum in the system. In a given reference frame the time-time component $T^{00}$ is the energy density, the time-space component $T^{0i} = T^{i0}$ is the $i$'th component of the momentum density, and the space-space component $T^{ik}$ is the flux of $i$'th momentum across the $x^k$ surface.
In relativistic notation, the statement of conservation of energy and momentum is simply
\[ \partial_\mu T^{\mu \nu} = 0. \quad (3.5) \]
Any other conserved quantity (e.g., electric charge, baryon number, etc.) is characterized by a conserved current \( j^\mu(x) \) that describes its charge density and current. The conservation equations then include an additional equation,
\[ \partial_\mu j^\mu_n = 0, \quad (3.6) \]
for each conserved quantity (labeled by \( n \)). For simplicity it will be assumed that these additional conservation equations are unimportant to the motion of the fluid and so will be neglected in the following.

Define the local fluid rest frame at each point in space-time as the zero-momentum frame, \( T^{0i}(x) = 0 \). The velocity of this local rest frame with respect to a fixed lab frame defines a fluid 4-velocity \( u^\mu(x) \) such that in the rest frame \( u^\mu_{\text{rest}} = (1, 0, 0, 0) \) (recall that for a 4-velocity, \( u^2 \equiv u^\mu u_\mu = 1 \). So then \( u_\mu T^{\mu \nu} = \epsilon u^\nu \), where \( \epsilon(x) \) is defined as the energy density in this fluid rest frame.

The equations of ideal (relativistic) hydrodynamics then emerge from the conservation equations \( 3.5 \) when the energy-momentum tensor has the property that it is isotropic (i.e., rotationally invariant) in the local rest frame:
\[
T^{\mu \nu}_{\text{ideal,rest}} = \begin{pmatrix}
\epsilon & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p \\
\end{pmatrix}.
\]
In an arbitrary fixed reference frame, in covariant notation, this is
\[
T^{\mu \nu}_{\text{ideal}} = (\epsilon + p) \ u^\mu u^\nu - p \ g^{\mu \nu} = \epsilon \ u^\mu u^\nu - p \ \Delta^{\mu \nu}, \quad (3.7)
\]
where \( p(x) \) is then the isotropic pressure in the rest frame and \( g^{\mu \nu} = \text{diag}(1, -1, -1, -1) \) is the metric tensor. \( \Delta^{\mu \nu} \equiv (g^{\mu \nu} - u^\mu u^\nu) \) is a projection operator on the space orthogonal to the fluid velocity. It has the properties \( \Delta^{\mu \nu} u_\mu = \Delta^{\mu \nu} u_\nu = 0 \) and \( \Delta^{\mu \nu} \Delta^{\rho \sigma} = \Delta^{\mu \rho} \). It is often
useful to use this projector to express the hydrodynamic equations projected into directions parallel ($u_\nu \partial_\mu T^{\mu\nu}$) and perpendicular ($\Delta^\alpha_\nu \partial_\mu T^{\mu\nu}$) to the fluid velocity. Explicitly they are:

\[
\begin{align*}
    u_\nu \partial_\mu T^{\mu\nu}_{\text{ideal}} &= (\epsilon + p) \partial_\mu u^\mu + u^\mu \partial_\mu \epsilon = (\epsilon + p) \partial_\mu u^\mu + D \epsilon = 0, \\
    \Delta^\alpha_\nu \partial_\mu T^{\mu\nu}_{\text{ideal}} &= (\epsilon + p) u^\mu \partial_\mu u^\alpha - \Delta^{\alpha\mu} \partial_\mu p = (\epsilon + p) D u^\alpha - \nabla^\alpha p = 0,
\end{align*}
\]

(3.8) \hspace{1cm} (3.9)

where we have also introduced shorthand notation for the projection of derivatives parallel ($D \equiv u^\mu \partial_\mu$) and perpendicular ($\nabla^\alpha = \Delta^{\alpha\mu} \partial_\mu$) to the fluid velocity.

This system of equations is closed by specifying the equation of state $p = p(\epsilon)$. If this is the equilibrium equation of state for the system in question, these equations describe a system that is everywhere in local thermal equilibrium. Thus, the language used when talking about hydrodynamics is often that of thermal equilibrium, but note that all that is required for Equations 3.8 and 3.9 to be valid is isotropy in the fluid rest frame.

For fluid velocities much less than the speed of light [$u^\mu \simeq (1, \vec{v})$], and when the energy density is dominated by the mass density ($\epsilon \simeq \rho$), these relativistic ideal hydrodynamic equations reduce to the non-relativistic Euler and continuity equations from section 3.2 [18].

It should be noted that these (non-linear) ideal hydrodynamic equations contain instabilities that make them impossible to solve numerically—at least using the most naïve of algorithms. To avoid these problems, numerical algorithms are used that contain what amounts to “numerical viscosity” that dampens the instabilities [18]. Adding viscous terms to the equations also fixes these instabilities, and so solving the viscous hydrodynamic equations will not require these algorithms.

### 3.4 Relativistic Viscous Hydrodynamics

These equations can be generalized to include dissipative, or viscous, effects. This is done by allowing for a more general energy-momentum tensor:

\[
T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu}.
\]

(3.10)
\( \Pi^{\mu \nu} \) is the viscous stress tensor that includes the contributions to \( T^{\mu \nu} \) from dissipation. The hydrodynamic equations then become

\[
D \epsilon + (\epsilon + p) \partial_{\mu} u^{\mu} - \Pi^{\mu \nu} \nabla_{(\mu} u_{\nu)} = 0 ,
\]
\[
(\epsilon + p) D u^{\alpha} - \nabla^{\alpha} p + \Delta^{\alpha}_{\mu} \partial_{\mu} \Pi^{\mu \nu} = 0 ,
\]

where the (\ldots) denote symmetrization, e.g.,

\[
\nabla_{(\mu} u_{\nu)} = \frac{1}{2} (\nabla_{\mu} u_{\nu} + \nabla_{\nu} u_{\mu}) .
\]

Of course, one must still specify the form of \( \Pi^{\mu \nu} \). Determining the correct form to use turns out to be less straightforward than in the non-relativistic case, and until recently there has been a number of versions in use.

### 3.4.1 Relativistic Navier-Stokes equations

If the macroscopically-averaged quantities from ideal hydrodynamics (\( \epsilon, p, u^{\mu} \)) vary slowly in space and time, it can be useful to build a controlled gradient expansion of \( T^{\mu \nu} \); i.e., an expansion in powers of derivatives of these quantities. Using this perspective, \( T^{\mu \nu}_{\text{ideal}} = T^{\mu \nu}_{0} \) is just the zeroth order term in such an expansion, and \( \Pi^{\mu \nu} \) contains first and higher order derivative terms.

To first order in gradients, there are only two independent terms that are consistent with the symmetries of \( \Pi^{\mu \nu} \) (it must be symmetric, \( \Pi^{\mu \nu} = \Pi^{\nu \mu} \), so that \( T^{\mu \nu} \) also remains symmetric, and it must be transverse to the fluid velocity, \( u_{\mu} \Pi^{\mu \nu} = 0 \), to retain the definition of the zero-momentum rest frame). They are usually separated into a traceless term \( \pi^{\mu \nu} \) and the remainder \( \Pi \)

\[
\Pi^{\mu \nu} = T^{\mu \nu}_{1} = \pi^{\mu \nu} + \Delta^{\mu \nu} \Pi = \eta \nabla^{(\mu} u^{\nu)} + \zeta \Delta^{\mu \nu} \nabla_{\alpha} u^{\alpha}.
\]

The angle brackets define a quantity that is traceless, symmetric, and transverse

\[
\nabla_{(\mu} u_{\nu)} = 2 \nabla_{(\mu} u_{\nu)} - \frac{2}{3} \Delta_{\mu \nu} \nabla_{\alpha} u^{\alpha} ,
\]

or, in general

\[
A^{(\alpha} B^{\beta)} = P^{\alpha \beta}_{\mu \nu} A^{\mu} B^{\nu} ,
\]

(3.13)
with \( P_{\alpha\beta}^{\mu\nu} = \Delta_{\alpha}^{\mu}\Delta_{\beta}^{\nu} + \Delta_{\beta}^{\mu}\Delta_{\alpha}^{\nu} - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta} \). Upon plugging into the conservation equations, this results in what are termed the relativistic Navier-Stokes equations, since they reduce to Navier-Stokes in the non-relativistic limit. \( \eta \) is then identified as the shear viscosity and \( \zeta \) the bulk viscosity. Ideal hydrodynamics is recovered when these transport coefficients are set to zero (appropriate when gradients are so small compared to the coefficients that the terms can be neglected). These terms are often derived in the literature by demanding that the second law of thermodynamics always be obeyed, \( \partial_{\mu}s^{\mu} \geq 0 \), where \( s^{\mu} = s u^{\mu} \) is the entropy density current \([18]\), rather than using the perspective of a gradient expansion to first order.

### 3.4.2 Causality restored: second-order relativistic viscous hydrodynamics

There are problems with the relativistic Navier-Stokes equations, however, which in general make them difficult to solve numerically. This is caused by the presence of acausal signal propagation, and associated instabilities.

The problem can be illustrated by considering small perturbations of the energy density and fluid velocity, and tracking how these disturbances travel through the medium. If one decomposes the perturbations into Fourier modes, one finds that the diffusion speed of a particular mode increases linearly with wavenumber. A mode with arbitrarily large wavenumber will have an arbitrarily large speed (larger than the speed of light), and causality is violated \([18]\).

On the surface this shouldn’t necessarily be worrisome. Large wavenumber (or short wavelength) modes are outside the realm of applicability of hydrodynamics—if there is significant short distance behavior, the gradient expansion will not converge and hydrodynamics is not an appropriate description of the system anyway. In practice, however, this acausal behavior causes instabilities that make constructing a numerical solution with arbitrary initial conditions impossible \([22]\).

It turns out that, as with the (unrelated) instabilities of ideal hydrodynamics, these instabilities can be removed by adding higher order gradient terms to the equations.

At second order in gradients, one can construct 15 independent terms (in addition to
the zeroth and first order terms already mentioned) [23]. In the shear (traceless) sector we have

\[ \pi^{\mu\nu} = \eta \sigma^{\mu\nu} - \eta \tau_\pi \left[ (D\sigma^{\mu\nu}) + \frac{4}{3}(\nabla \cdot u)\sigma^{\mu\nu} \right] + \frac{\kappa}{2} \left[ R(\mu\nu) + 2 u_\alpha u_\beta R^{\alpha(\mu\nu)\beta} \right] \]

\[- \frac{\lambda_1}{2} \sigma^{(\mu}_{\lambda} \sigma^{\nu)\lambda} + \frac{\lambda_2}{2} \sigma^{(\mu}_{\lambda} \omega^{\nu)\lambda} - \frac{\lambda_3}{2} \omega^{(\mu}_{\lambda} \omega^{\nu)\lambda} \]

\[- \kappa^* u_\alpha u_\beta R^{(\alpha\mu\nu)\beta} - \eta \tau_\pi^* \frac{4}{3}(\nabla \cdot u)\sigma^{\mu\nu} + \frac{\lambda_4}{2} \nabla(\mu \ln s \nabla^\nu) \ln s, \]

(3.14)

while in the bulk sector the most general form is

\[ \Pi = \zeta (\nabla \cdot u) - \zeta \tau_\Pi D (\nabla \cdot u) - \xi_1 \sigma^{\mu\nu} \sigma_{\mu\nu} - \xi_2 (\nabla \cdot u)^2 \]

\[- \xi_3 \omega^{\mu\nu} \omega_{\mu\nu} + \xi_4 \nabla_\mu \ln s \nabla^\mu \ln s + \xi_5 R - \xi_6 u^\alpha u^\beta R_{\alpha\beta}, \]

(3.15)

where \( s \) is the entropy density. Since we are assuming vanishing conserved charge densities (i.e., zero chemical potential), this is given by \( s = \frac{\epsilon + p}{T} \). Here we have also introduced the notation \( \sigma^{\mu\nu} \equiv \nabla^{(\mu} u^{\nu)} \), and the fluid vorticity is defined as \( \omega^{\mu\nu} \equiv -\nabla^{\mu} u^{\nu} \). Square brackets indicate an antisymmetrized quantity

\[ A^{[\mu B^\nu]} \equiv \frac{1}{2} (A^\mu B^\nu - A^\nu B^\mu). \]

In a general curved space-time the Riemann tensor is non-zero

\[ R^\lambda_{\mu\sigma\nu} \equiv \partial_\sigma \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\kappa_{\mu\sigma} \Gamma^\lambda_{\kappa\nu} - \Gamma^\kappa_{\mu\nu} \Gamma^\lambda_{\kappa\sigma}, \]

with \( \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda \rho} \left( \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \), as well as the Ricci tensor \( R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \) and Ricci scalar \( R = R^\mu_{\mu} \).

Therefore, without any additional information, there are in principle 15 possible second-order transport coefficients multiplying these terms: \( \tau_\pi, \tau_\pi^*, \kappa, \kappa^*, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \tau_{11}, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \) and \( \xi_6 \), in addition to the first order transport coefficients \( \eta \) and \( \zeta \). If one is lucky, these transport coefficients can be calculated from the underlying microscopic theory (e.g., QCD in the case of relativistic heavy ion collisions). Often, however, this is not feasible (as is the case for QCD at the moment, although there is ongoing work). If they cannot be computed from first principles, the transport coefficients can be treated as free parameters that are then constrained by experimental data. In this case, it becomes important to
understand which terms are necessary (or appropriate) to keep in a hydrodynamic simulation of a given problem.

One can use general arguments to reduce the number of free parameters: requiring the positivity of the divergence of the entropy current provides 2 extra constraints, leaving only 13 completely independent transport coefficients \[23\]. Also, when space-time can be assumed flat, the terms involving the Riemann tensor and the Ricci tensor and scalar drop out, reducing the number of independent second order transport coefficients to 11, in addition to the 2 first order coefficients.

For most problems, however, this situation is still not satisfactory. Having so many free parameters (as well as the fact that their temperature dependence is \textit{a priori} unknown) would significantly reduce the predictive power of any simulation. On the other hand, while it is true that one can eliminate the problems of the relativistic Navier-Stokes equations by adding only a single second order term, the precise form of the term used can affect how sensitive the results are to the value of the corresponding transport coefficient (as well as affecting the interpretation of the resulting value in the context of the underlying theory), and so one must be convinced that keeping only that part of the second order expansion is justified.

### 3.4.3 Müller-Israel-Stewart theory

This issue of a causal set of relativistic viscous hydrodynamic equations was originally studied in depth by Müller \[24, 25\], and separately by Israel and Stewart \[26, 27, 28\]. Thus, many of the second order extensions to viscous hydrodynamics are commonly referred to as Müller-Israel-Stewart theory, although the exact form of the equations that have been used varies. An extensive history of the various forms used and derivations thereof is beyond the aim of this summary, although some generic comments can be made.

In all cases one has a term whose coefficient acts as a relaxation time which, when it is larger than a certain value, eliminates any possible acausality. In the shear sector this relaxation time is usually called $\tau_\pi$, and it typically multiplies some combination that includes the term $\langle D\sigma^{\mu\nu} \rangle$ (e.g. in Equation 3.14).
Besides causality, another important consideration is the second law of thermodynamics,

$$\partial_\mu s^\mu \geq 0.$$ \hfill (3.16)

which should be obeyed in any physical system. In the absence of dissipation, the entropy current is given by $s^\mu = s u^\mu$. In ideal hydrodynamics, the inequality is saturated, and the entropy does not increase with time. Assuming this form for the entropy current, one can show that the relativistic Navier-Stokes equations always obey Equation 3.16 for any (non-negative) value of the transport coefficients [18].

When there is dissipation, however, the entropy current can also have gradient terms. If one assumes that the entropy current has to be algebraic in the hydrodynamic degrees of freedom, and that the deviations from equilibrium are small enough that higher order corrections can be neglected, it can be shown that the entropy current has to be of the form

$$s^\mu = su^\mu - \frac{\beta_0}{2T} u^\mu \Pi^2 - \frac{\beta_2}{2T} u^\mu \pi_{\alpha\beta} \pi^{\alpha\beta} + O(\Pi^3).$$ \hfill (3.17)

One can then plug this into Equation 3.16 and derive a form for the second order hydrodynamic terms that is then guaranteed to always obey the second law of thermodynamics [18]. There is doubt, however, that the assumption that the entropy current has to be algebraic in the hydrodynamic degrees of freedom is necessarily true and therefore also that Equation 3.17 is the really the most general form [23, 30]. Also, it may be an unnecessarily strong requirement to demand that the structure of the hydrodynamic equations must be such that Equation 3.16 is satisfied in any regime and for any combination of values of the transport coefficients—it is only necessary that the second law of thermodynamics be obeyed in actually physically realized (or at least realizable) systems.

Another tack that is often taken is to look to the kinetic theory of gases for guidance. Kinetic theory is a description of a system in terms of collisions of dilute particles (or quasiparticle states that are long lived compared to the scattering rate). By considering small departures from equilibrium, one can derive another version of Müller-Israel-Stewart theory [18]. Unfortunately, although kinetic theory has a large range of applicability and can thus often provide much insight into various physical behavior, such a particle-based description is not valid for, e.g., the strongly coupled non-abelian plasma which may exist for part of
the evolution of a relativistic heavy ion collision system. Therefore, it is uncertain that the resulting second-order viscous hydrodynamic equations are general enough to describe the evolution of such a system.

3.4.4 Conformal relativistic viscous hydrodynamics

If there is reason to believe that a system is approximately scale-invariant, the form of the hydrodynamic equations are greatly restricted. To be precise, if one assumes a conformal symmetry in the underlying physics, the total number of possible independent first and second order transport coefficients is reduced to 6—and only 4 in the case of flat space.

The conformal group is the set of symmetry transformations that consists of scale transformations and special conformal transformations. A theory that is conformally symmetric (a.k.a. “conformal”) has the property that its action is invariant under a Weyl transformations of the metric,

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{-2w(x)} g_{\mu\nu}, \quad (3.18)$$

where $w(x)$ is an arbitrary function of $x$.

This implies that (to second order in derivatives) the trace of the energy-momentum tensor vanishes, and that $T^{\mu\nu}$ transforms under a Weyl rescaling as

$$T^{\mu\nu} \rightarrow \tilde{T}^{\mu\nu} = e^{6w(x)} T^{\mu\nu}. \quad (3.19)$$

Imposing these conditions gives as the most general form

$$\Pi^{\mu\nu} = \pi^{\mu\nu} = \eta \sigma^{\mu\nu} - \eta \tau_{\pi} \left[ \langle D\sigma^{\mu\nu} \rangle + \frac{4}{3} (\nabla \cdot u) \sigma^{\mu\nu} \right] + \frac{\kappa}{2} \left[ R^{(\mu\nu)} + 2u_\alpha u_\beta R^\alpha(\mu\nu)\beta \right] - \frac{\lambda_1}{2} \sigma^{(\mu} \sigma^{\nu)} = \frac{\lambda_2}{2} \sigma^{(\mu} \omega^{\nu)\lambda} - \frac{\lambda_3}{2} \omega^{(\mu} \omega^{\nu)\lambda}. \quad (3.20)$$

Note in particular that there is no bulk viscosity. To this order in derivatives, it is valid to replace $\eta \sigma^{\mu\nu} \rightarrow \pi^{\mu\nu}$, and so the form that is more convenient to actually solve numerically
is

\[ \Pi^{\mu\nu} = \eta \nabla^{(\mu} u^{\nu)} - \tau_{\pi} \left[ \Delta_\alpha^\mu \Delta_\beta^\nu D \Pi^{\alpha\beta} + \frac{4}{3} \Pi^{\mu\nu} \nabla_\alpha u^\alpha \right] \\
+ \frac{\kappa}{2} \left[ R^{(\mu\nu)} + 2 u_\alpha R^{\alpha(\mu\nu)} u_\beta \right] \\
- \frac{\lambda_1}{2\eta^2} \Pi^{(\mu,\lambda} \Pi^{\nu)\lambda} + \frac{\lambda_2}{2\eta} \Pi^{(\mu,\lambda} \omega^{\nu)\lambda} - \frac{\lambda_3}{2} \omega^{(\mu,\lambda} \omega^{\nu)\lambda}. \]  \tag{3.21}

In the expectation that the medium created in a relativistic heavy ion collision is approximately conformal, this is the form that will be used in the viscous hydrodynamic simulations of the following chapters.
Chapter 4
MODELING HEAVY ION COLLISIONS USING VISCOUS HYDRODYNAMICS

4.1 Anatomy of a Heavy Ion Collision

![Space-time cartoon diagram of a heavy ion collision](image)

Figure 4.1: Space-time cartoon diagram of a heavy ion collision, indicating the pre-equilibrium stage, hydrodynamic stage, and the system after it has frozen out. Here it is assumed that there is hydrodynamic evolution on both sides of the phase transition as well as approximately boost invariant evolution.

Even if a heavy ion collision system does behave hydrodynamically for a significant period of its evolution, that is not the whole story, of course. Once the proper hydrodynamic equations are set, one must still specify the boundary conditions. There is a finite period of time at the beginning of a collision before which the system can equilibrate (or at least
isotropize—recall section 3.3) and begin behaving hydrodynamically. Likewise, as the system expands and cools, there will be some point at which the system no longer interacts strongly enough for hydrodynamics to be a valid description. Eventually, the particles get so far apart that they completely cease interacting and these free particles are what ultimately get detected. Therefore one must define the process by which the system “freezes out”.

Once the initial conditions and freeze out algorithm have been specified, the simulations can be run numerically and experimental observables calculated. Details of these choices will be given later in this chapter when describing the viscous hydrodynamic simulations done by the author in collaboration with Paul Romatschke.

Ideal hydrodynamic simulations were previously done, however, offering a remarkably good description of the experimental data for bulk properties (multiplicity, radial and elliptic flow) of low \( p_T \) particles for heavy-ion collisions at RHIC [13, 32, 33, 34, 35].

Upon closer inspection, however, not all of this success can be attributed to modeling the system as an ideal fluid. For instance, the energy density distribution which is used as an initial condition for the hydrodynamic equations is customarily chosen such that the output from the hydrodynamic model matches the experimental data for the multiplicity. Furthermore, the time where the hydrodynamic model is initialized as well as the temperature (or energy density) at which the hydrodynamic evolution is stopped are typically chosen such that the model output matches the experimental data for the radial flow. After these parameters have been fixed, only the good description of experimental data for the elliptic flow coefficient can be considered a success for ideal hydrodynamics (in the sense that it is parameter-free).

In order to make progress and learn more about the properties of matter created at RHIC, the task is now to both test and improve this ideal hydrodynamic model. The obvious framework for this task is dissipative hydrodynamics, since it contains ideal hydrodynamics as the special case when all dissipative transport coefficients (such as shear and bulk viscosity and heat conductivity) are sent to zero. If the value of the transport coefficients were known (e.g. by some first principle calculation), then one could use dissipative hydrodynamics to constrain e.g. the initial energy density distribution, which is chosen conveniently in the ideal hydrodynamic models. Or otherwise, choosing again physically acceptable initial
conditions, one is able to constrain the allowed ranges of the transport coefficients. Despite recent progress in first principle calculations \cite{36,37,38,39,40,41,42,43,44,45}, the values of the hydrodynamic transport coefficients for QCD in the relevant energy range are poorly constrained to date, so the second option is currently the only viable possibility.

For RHIC, the first step in this direction was carried out by Teaney \cite{46}, who provided estimates for the sign and size of corrections due to shear viscosity. This famous calculation, however, did not provide a description of experimental data for non-zero viscosity, because it was not dynamic and the initial conditions could not be altered. Only very recently, the first hydrodynamic calculations with shear viscosity describing particle spectra for central and non-central collisions at RHIC have became available \cite{47,48,49}.

Several other groups have produced numerical codes capable of performing similar matching to data \cite{50,51,52,53,54,55,56,57}.

However, the precise formulation of the viscous hydrodynamic equations themselves has long been debated (recall \textsection 3.4). For the case of non-vanishing shear viscosity only, it was shown recently \cite{31} that the most general form implies five independent terms of second order in gradients. This form is general enough to describe the hydrodynamic properties of (conformal) plasmas both for weakly coupled systems describable by the Boltzmann equation as well as infinitely strongly coupled plasmas, which are accessible via Maldacena’s conjecture \cite{58}.

The aim of this chapter is to now apply this new set of equations for relativistic shear viscous hydrodynamics to the problem of heavy-ion collisions at RHIC. In \textsection 4.2, we review the setup of conformal relativistic viscous hydrodynamics and our numerics for the simulation of heavy-ion collisions. In \textsection 4.3, details about the two main models of initial conditions for hydrodynamics are given. Section 4.4 contains our results for the multiplicity, radial flow and elliptic flow in Au+Au collisions at top RHIC energies, as well as a note on the notion of “early thermalization”. We conclude in \textsection 4.5.
4.2 Setup

We use the most general form of the second order viscous hydrodynamic equations, which as a reminder are given by (see section 3.4)

\[
\Pi^{\mu\nu} = \eta \nabla^{\langle \mu u^\nu \rangle} - \tau_\pi \left[ \Delta_\alpha^\mu \Delta_\beta^\nu D\Pi^{\alpha\beta} + \frac{4}{3} \Pi^{\mu\nu} (\nabla_\alpha u^\alpha) \right] + \frac{\kappa}{2} \left[ R^{(\mu\nu)} + 2 u_\alpha R^{\alpha\langle\mu\nu\rangle\beta} u_\beta \right] - \frac{\lambda_1}{2\eta^2} \Pi^{\langle \mu \lambda \Pi^{\nu \rangle \lambda} + \frac{\lambda_2}{2\eta} \Pi^{\langle \mu \lambda \omega \nu \rangle \lambda} - \frac{\lambda_3}{2} \omega^{\langle \mu \lambda \omega \nu \rangle \lambda}. \tag{4.1}
\]

The coefficients \(\tau_\pi, \kappa, \lambda_1, \lambda_2, \lambda_3\) are the five new coefficients controlling the size of the allowed terms of second order in gradients. Having an application to the problem of heavy-ion collisions in mind, the above set of equations can be simplified: for all practical purposes spacetime can be considered flat, such that both the Riemann and Ricci tensors vanish identically. Thus, only the four coefficients \(\tau_\pi, \lambda_1, \lambda_2, \lambda_3\) enter the problem.

4.2.1 A note on bulk viscosity and conformality

Besides shear viscosity, QCD also has non-vanishing bulk viscosity \(\zeta\) which can be related to the QCD trace anomaly \[59\] \[
\zeta \sim T_\mu^\mu = \epsilon - 3p. \tag{4.2}
\]

QCD lattice simulations seem to indicate that the ratio bulk viscosity over entropy density \(s, \zeta/s\), is small compared to \(\eta/s\) except for a small region around the QCD deconfinement transition temperature, where it is sharply peaked \[60, 61, 62\]. If we are interested in describing effects from shear viscosity only, we are led to consider \(\zeta = 0\), or conformal fluids. This has been the main guiding principle in Ref. \[31\] and as a consequence Equation 4.1 obeys conformal invariance, unlike most other second-order theories.\[1\]

4.2.2 First steps: 0+1 dimensions

In order to get a crude estimate of the effect of viscous corrections, let us consider the arguably simplest model of a heavy-ion collision: a system expanding in a boost-invariant

\[1\text{Note that Muronga derived a version of Equation 4.1 in Ref. 29 that turns out to obey conformal symmetry.}\]
fashion along the longitudinal direction and having uniform energy density in the transverse plane. Introducing the Milne variables proper time $\tau = \sqrt{t^2 - z^2}$ and space-time rapidity $\xi = \arctanh(z/t)$, boost invariance simply translates to requiring all hydrodynamic variables $(\epsilon, u^\mu, \Pi^\mu_\xi)$ to be independent of rapidity, and tensor components $u^\xi, \Pi^\mu_\xi$ to vanish. Assuming uniformity in the transverse plane furthermore requires independence from the transverse coordinates $\mathbf{x}_T = (x, y)$. Even though this means that all the velocity components except $u^\tau$ are zero, the system is nevertheless non-trivial in the sense that the sum over velocity gradients does not vanish, $\nabla_\mu u^\mu = \frac{1}{\tau}$, sometimes referred to as “Bjorken flow”.

In a way one has modeled an expanding system in static space-time by a system at rest in an expanding space-time. This has been achieved by transforming to the Milne coordinates $\tau, \xi$, where the metric is $g_{\mu\nu} = \text{diag}(g_{\tau\tau}, g_{xx}, g_{yy}, g_{\xi\xi}) = (1, -1, -1, -\tau^2)$. Note that even though the spacetime in these coordinates is expanding, it is nevertheless flat (e.g. has vanishing Riemann tensor).

In this 0+1 dimensional toy model, the viscous hydrodynamic equations become exceptionally simple \[31\],

$$
\begin{align*}
\partial_\tau \epsilon &= -\frac{\epsilon + p}{\tau} + \frac{\Pi^{\xi}_\xi}{\tau} \\
\partial_\tau \Pi^{\xi}_\xi &= -\frac{\Pi^{\xi}_\xi}{\tau_\pi} + \frac{4\eta}{3\tau_\pi} \frac{\Pi^{\xi}_\xi}{3\tau} - \frac{\lambda_1}{2\tau_\pi \eta^2} \left(\Pi^{\xi}_\xi\right)^2.
\end{align*}
$$

(4.3)

The Navier-Stokes equations are recovered formally in the limit where all second-order coefficients vanish (e.g. $\tau_\pi, \lambda_1 \rightarrow 0$); then, one simply has

$$
\Pi^{\xi}_\xi = \frac{4\eta}{3\tau}.
$$

(4.4)

The equations (4.3) can be solved numerically along the lines of \[63, 64\]. At very early times, where $\Pi^{\xi}_\xi > (\epsilon + p)$, the Navier-Stokes equations indicate an increase in energy density and a negative effective longitudinal pressure $p - \Pi^{\xi}_\xi$. Since gradients $\nabla_\mu u^\mu = 1/\tau$ are strongest at early times, this suggests that one is applying the Navier-Stokes equations outside their regime of validity. Theories including second order gradients may be better behaved at early times, but eventually also have to break down when gradients become too strong. Here we want to study the effects of the second order coefficients on the value of the shear tensor at late times, where a hydrodynamic approach should be valid.
To this end, let us study the deviation of the shear tensor from its first order value, $\delta \Pi = \Pi^\xi_\xi - \frac{4\eta}{3\tau}$. At late times, Equation 4.3 implies $\epsilon \sim \tau^{-4/3}$, so $\eta \sim \tau^{-1}$. Thus, if $\delta \Pi$ is small compared to the first order value, from Equation 4.3 we find

$$
\delta \Pi = \frac{4\eta}{3\tau} \left( \frac{2\tau_\pi}{3\tau} - \frac{2\lambda_1}{3\tau\eta} \right).
$$

For a strongly coupled $\mathcal{N} = 4$ plasma [31, 36, 65, 66], one has

$$
\frac{\eta}{s} = \frac{1}{4\pi}, \quad \tau_\pi = \frac{2 - \ln 2}{2\pi T}, \quad \lambda_1 = \frac{\eta}{2\pi T},
$$

and thus $\Pi^\xi_\xi$ is larger than its first order value by a factor of $1 + \frac{1-\ln 2}{3\pi T \tau}$. For RHIC, $T \tau \gtrsim 1$ is a reasonable estimate, so one finds that the second order corrections to $\Pi^\xi_\xi$ increase its value by a few percent over the first order result.

As an example on the importance of obeying conformal invariance, imagine dropping the term involving $\nabla_\alpha u^\alpha$ in the first line of Equation 4.1. Redoing the above calculation one finds

$$
\delta \Pi_{NC} = \frac{4\eta}{3\tau} \left( \frac{2\tau_\pi}{\tau} - \frac{2\lambda_1}{3\tau\eta} \right),
$$

which indicates a nearly ten-fold increase of the size of $\delta \Pi$ for the non-conformal theory.

For a weakly coupled plasma well described by the Boltzmann equation [31], where one has $\tau_\pi = \frac{6\eta}{sT}$, ($\lambda_1$ is unknown but generally set to zero in Müller-Israel-Stewart theory), the effect may be less pronounced, but still one qualitatively expects second-order effects to be anomalously large if conformal invariance is broken in an “ad-hoc” manner.

Clearly, the above estimates are not meant to be quantitative. Indeed, even the sign of the correction may change when allowing more complicated (e.g., three-dimensional) dynamics. However, the lesson to be learned from this exercise is that second-order gradients can and indeed do modify the shear tensor from its first order (Navier-Stokes) value. This is physically acceptable, as long as the second-order corrections are small compared to the first order ones (otherwise the system is probably too far from equilibrium for even a hydrodynamic description correct to second order in gradients to be valid). A practical means for testing this is calculating physical observables for different values of the second-
order coefficients and making sure that the results do not strongly depend on the choice for these specific values.

4.2.3 Including radial flow: lessons from 1+1 dimensions

Some more insight on the effect of viscous corrections may be gained by improving the model of the previous subsection to allow for radially symmetric dynamics in the transverse plane (but still assuming boost invariance). This is most easily implemented by changing to polar coordinates \((x, y) \rightarrow (r, \phi)\) with \(r = \sqrt{x^2 + y^2}\) and \(\phi = \arctan(y/x)\). In this case, the only non-vanishing velocity components are \(u^r\) and \(u^\phi\), and hence the vorticity \(\omega^{\mu\nu}\) vanishes identically. Although non-trivial, the radially symmetric flow case is still a major simplification over the general form Equation 4.1 since again the terms involving \(\kappa, \lambda_2, \lambda_3\) drop out.

Such a formulation allows both for important code tests [67] as well as realistic simulations of central heavy-ion collisions [47] (note that truncated versions of Equation 4.1 were used in these works). The advantage of this formulation is that since the equations are comparatively simple, it is rather straightforward to implement them numerically and they are not very time consuming to solve since only one dimensional (radial) dynamics is involved. The shortcoming of simulations with radially symmetric flow profiles ("radial flow") is that by construction they cannot be matched to experimental data on the impact-parameter dependence of multiplicity. Thus, the considerable freedom in the initial/final conditions inherent to all hydrodynamic approaches cannot be eliminated in this case.

For this reason, we will choose not to discuss the case of radial flow here in more detail, but rather will comment on it later as a special case of the more general situation.

4.2.4 Elliptic flow: 2+1 dimensional dynamics

Retaining the assumption of boost invariance, but allowing for general dynamics in the transverse plane, it is useful to keep Cartesian coordinates in the transverse plane, and thus \(u^r, u^x, u^y\) are the non-vanishing fluid velocities. The main reason is that e.g. in polar coordinates the equations for the three independent components of \(\Pi^{\mu\nu}\) would involve some
extra non-vanishing Christoffel symbols (other than $\Gamma^\tau_{\xi\xi} = \tau, \Gamma^\xi_{\tau\xi} = 1/\tau$).

Fortunately, the case of two dimensions is special insofar as the only nontrivial component of the vorticity tensor, namely $\omega^{xy}$, fulfills the equation

$$D\omega^{xy} + \omega^{xy} \left[ \nabla_{\mu} u^\mu + \frac{Dp}{\epsilon + p} - \frac{Du^\tau}{u^\tau} \right] = O(\Pi^3), \quad (4.8)$$

which can be derived by forming the combination $\nabla^x D u^y - \nabla^y D u^x$. The expression $O(\Pi^3)$ denotes that the r.h.s. of Equation 4.8 is of third order in gradients, and thus should be suppressed in the domain of applicability of hydrodynamics. For heavy-ion collisions, typically $\nabla_{\mu} u^\mu \geq \frac{1}{\tau}$, so that for an equation of state with a speed of sound squared $c^2_s \equiv \frac{dp(c)}{d\epsilon} \sim \frac{1}{3}$, Equation 4.8 translates to $\frac{D\omega^{xy}}{\omega^{xy}} < 0$ unless $D \ln u^\tau \geq (1 - c^2_s) \nabla_{\mu} u^\mu$. In particular, this implies that in general $\omega^{xy} = 0$ is a stable fix-point of the above equation and hence we expect $\omega^{xy}$ to remain small throughout the entire viscous hydrodynamic evolution if it is small initially.

Generically, one uses $u^{x,y} = 0$ as an initial condition for hydrodynamics [68], which implies $\omega^{xy} = 0$ initially. Therefore, to very good approximation we can neglect the terms involving vorticity in Equation 4.1, such that again only the second-order coefficients $\tau_\pi, \lambda_1$ have to be specified.

The equations to be solved for 2+1 dimensional relativistic viscous hydrodynamics are
then (in components)

\[(\epsilon + p) D u^i = c_s^2 \left( g^{ij} \partial_j \epsilon - u^i u^\alpha \partial_\alpha \epsilon \right) - \Delta^i_\alpha D_\beta \Pi^{\alpha \beta} \]

\[D \epsilon = - (\epsilon + p) \nabla_\mu u^\mu + \frac{1}{2} \Pi^{\mu \nu} \nabla_{(\mu} u_{\nu)} \]

\[D_\beta \Pi^{\alpha \beta} = \Pi^\alpha \partial_\tau \frac{u^i}{u^\tau} + \frac{u^i}{u^\tau} \partial_i \Pi^\alpha + \partial_i \Pi^\alpha + \Gamma^\alpha_{\beta \delta} \Pi^{\beta \delta} + \Gamma^\beta_{\beta \delta} \Pi^{\alpha \delta} \]

\[\partial_\tau \Pi^{\alpha \beta} = - \frac{4}{3 u^\tau} \Pi^{\alpha \beta} \nabla_\beta u^\alpha + \frac{\eta}{\tau u^\tau} \nabla_{(i} u_{\alpha)} - \frac{1}{\tau u^\tau} \Pi^{\alpha \beta} \]

\[\nabla_\mu u^\mu = \partial_\tau u^\tau + \partial_i u^i + \frac{u^\tau}{\tau} \]

\[\nabla_{(x} u_{x)} = 2 \Delta^{x \tau} \partial_\tau u^x + 2 \Delta^{x \tau} \partial_i u^x - \frac{2}{3} \Delta^{x x} \nabla_\mu u^\mu \]

\[\nabla_{(y} u_{y)} = 2 \Delta^{y \tau} \partial_\tau u^y + 2 \Delta^{y \tau} \partial_i u^y + 2 \Delta^{y y} \partial_i u^x + \frac{2}{3} \Delta^{y y} \nabla_\mu u^\mu \]

\[\nabla_{(\xi} u_{\xi)} = 2 \Delta^{\xi \xi} \partial_\tau u^\tau - \frac{2}{3} \Delta^{\xi \xi} \nabla_\mu u^\mu. \quad (4.9) \]

Here and in the following Latin indices collectively denote the transverse coordinates $x, y$ and the relation $u_\mu \Pi^{\mu \nu} = 0$ has been used to derive the above equations (similarly, $u^\mu \nabla_{(\mu} u_{\nu)} = 0$ can be used to obtain the other non-trivial components needed). Note that this particular form of Equation 4.9 has not been simplified further since it roughly corresponds to the equations implemented for the numerics of [48], and is meant to facilitate understanding of the code [69]. A simple algorithm to solve Equation 4.9 has been outlined in [67] and will be reviewed in the next subsection for completeness.

4.2.5 A numerical algorithm to solve relativistic viscous hydrodynamics

The first step of the algorithm consists of choosing the independent degrees of freedom. For boost-invariant 2+1 dimensional dynamics, a sensible choice for this set is e.g. $\epsilon, u^x, u^y, \Pi^{xx}, \Pi^{xy}, \Pi^{yy}$. The pressure is then obtained via the equation of state $p(\epsilon)$, and the only other non-vanishing velocity as $u^\tau = \sqrt{1 + u_x^2 + u_y^2}$. Similarly, the other nonzero components of $\Pi^{\mu \nu}$ are calculated using the equations $\Pi^\mu_\mu = 0, u_\mu \Pi^{\mu \nu} = 0$.

Given the value of the set of independent components at some time $\tau = \tau_0$, the aim is then to construct an algorithm from Equation 4.9 such that the new values of the set
can be calculated as time progresses. Note that in Equation 4.9, time derivatives of the independent component set enter only linearly. Therefore, Equation 4.9 may be written as a matrix equation for the derivatives of the independent component set,

\[
\begin{pmatrix}
    a_{00} & a_{01} & \cdots & a_{05} \\
    a_{10} & a_{11} & \cdots & a_{15} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{50} & a_{51} & \cdots & a_{55}
\end{pmatrix}
\begin{pmatrix}
    \partial_\tau \varepsilon \\
    \partial_\tau u^x \\
    \vdots \\
    \partial_\tau \Pi^{yy}
\end{pmatrix}
= \begin{pmatrix}
    b_0 \\
    b_1 \\
    \vdots \\
    b_6
\end{pmatrix}.
\]  

(4.10)

Denoting the above matrix and vector as \( A \) and \( b \), respectively, a straightforward way to obtain the time derivatives is via numerical matrix inversion,

\[
\begin{pmatrix}
    \partial_\tau \varepsilon \\
    \partial_\tau u^x \\
    \vdots \\
    \partial_\tau \Pi^{yy}
\end{pmatrix}
= A^{-1} \cdot b.
\]  

(4.11)

Choosing a naive discretization of derivatives

\[
\partial_\tau f(\tau) = \frac{f(\tau + \delta \tau) - f(\tau)}{\delta \tau}, \quad \partial_x f(x) = \frac{f(x + a) - f(x - a)}{2a},
\]  

(4.12)

which is first order accurate in the temporal grid spacing \( \delta \tau \) and second order accurate in the spatial grid spacing \( a \), one can then directly calculate the new values of the independent component set from Equation 4.11.

Note that for ideal hydrodynamics, the algorithm Equation 4.11 would fail for this naive discretization [70]. The reason is that ideal hydrodynamics is inherently unstable to high wavenumber fluctuations (which can be thought of as the basis for turbulence). For ideal hydrodynamics, one thus has to use a discretization which amounts to the introduction of numerical viscosity to dampen these fluctuations. Luckily, viscous hydrodynamics does not suffer from this problem because it has real, physical viscosity inbuilt. It is because of this reason that the naive discretization can be used in the algorithm Equation 4.11 without encountering the same problems as in ideal hydrodynamics, as long as a finite value for the viscosity \( \eta \) is used\(^3\). While applicable to sufficiently smooth initial conditions, the above

---

\(^3\) In practice we have used \( \frac{\eta}{s} > 10^{-4} \). Typically, between \( \frac{\eta}{s} = 10^{-2} \) and \( \frac{\eta}{s} = 10^{-4} \) there are no significant changes in the hydrodynamic results and we refer to \( \frac{\eta}{s} = 10^{-4} \) as “ideal hydrodynamics.”
algorithm is too simple to treat strong gradients such as the propagation of shocks, and should be replaced by a more involved scheme in such cases.

Since matrix inversions are computationally intensive, one can speed up the numerics by expressing $\partial_\tau \Pi^{\mu\nu}$ in terms of $\partial_\tau u^i, \partial_\tau \epsilon$. Inserting these in the equations for $Du^\mu$ and $D\epsilon$, one only needs to invert a $3 \times 3$ matrix to obtain the new values of the energy density and fluid velocities. This approach has been used in [47, 48, 67].

4.2.6 Initial conditions and equation of state

As outlined at the beginning of the chapter, any hydrodynamic description of RHIC physics relies on given initial energy density distributions. Two main classes of models for boost-invariant setups exist: the Glauber models and the Color-Glass-Condensate (CGC) models.

As will be shown in the following, both model classes can give a reasonable description of the experimentally found multiplicity distribution, but they differ by their initial spatial eccentricity. A detailed discussion of the initial conditions will be given in subsequent sections.

Besides an initial condition for the energy density, one also needs to specify an initial condition for the independent components of the fluid velocities and the shear tensor. For the fluid velocities we will follow the standard assumption that these vanish initially [68]. Finally, when using the set of equations (4.9), one also has to provide initial values for the independent components of $\Pi^{\mu\nu}$. Extreme choices are $\Pi^{\mu\nu} = 0$ and a shear tensor so large that a diagonal component of the energy-momentum tensor vanishes in the local rest frame (e.g. $\Pi^{\xi\xi} = p$, or zero longitudinal effective pressure), with the physical result expected somewhere in between (see e.g. the discussion in [71]).

Once the initial conditions for the independent hydrodynamic variables have been specified, one needs the equation of state to solve the hydrodynamic equations (4.9). Aiming for a description of deconfined nuclear matter at zero chemical potential, a semi-realistic equation of state has to incorporate evidence from lattice QCD calculations [72] that the transition from hadronic to deconfined quark matter is probably an analytic crossover, not a first or second order phase transition as often used in ideal hydrodynamic simulations. On the
other hand, continuum extrapolations for the value of the energy density and pressure for physical quark masses are still not accessible with high precision using current lattice methods. For this reason, we will employ the equation of state by Laine and Schröder \[73\], which is derived from a hadron resonance gas at low temperatures, a high-order weak-coupling perturbative QCD calculation at high temperatures, and an analytic crossover regime interpolating between the high and low temperature regime, respectively. For hydrodynamics, an important quantity is the speed of sound squared extracted from the equation of state, \( c_s^2 = \frac{dp}{d\epsilon} \). For completeness, we reproduce a plot of this quantity in Figure 4.2.

### 4.2.7 Freeze-out

At some stage in the evolution of the matter produced in a heavy-ion collision, the system will become too dilute for a hydrodynamic description to be applicable. This “freeze-out” process is most probably happening gradually, but difficult to model realistically. A widely-used approximation is therefore to assume instantaneous freeze-out whenever a certain fluid cell cools below a certain predefined temperature or energy density (see \[56, 74\] for different approaches). The standard prescription for this freeze-out process is the Cooper-Frye formula \[75\], which allows conversion of the hydrodynamic variables (energy density, fluid velocity,...) into particle distributions.
Specifically, in the case of isothermal freeze-out at a temperature $T_f$, the conversion from hydrodynamic to particle degrees of freedom will have to take place on a threedimensional freeze-out hypersurface $\Sigma$, which can be characterized by its normal four-vector, and parametrized by three space-time variables $[76, 77]$. The spectrum for a single particle on mass shell with four momentum $p^\mu = (E, \mathbf{p})$ and degeneracy $d$ is then given by

$$E \frac{d^3N}{d^3p} = \frac{d}{(2\pi)^3} \int p_\mu d\Sigma^\mu f(x^\mu, p^\mu),$$

where $d\Sigma^\mu$ is the normal vector on the hypersurface $\Sigma$ and $f$ is the off-equilibrium distribution function.

Originally, the Cooper-Frye prescription was derived for systems in thermal equilibrium, where $f$ is built out of a Bose or Fermi distribution function $f_0(x) = (\exp[(x) \pm 1]^{-1})$, depending on the statistics of the particle under consideration. In order to generalize it to systems out of equilibrium, one customarily relies on the ansatz used in the derivation of viscous hydrodynamics from kinetic theory $[78]$,

$$f(x^\mu, p^\mu) = f_0 \left( \frac{p_\mu u^\mu}{T} \right) + f_0 \left( \frac{p_\mu u^\mu}{T} \right) \left[ 1 \mp f_0 \left( \frac{p_\mu u^\mu}{T} \right) \right] \frac{p_\mu p_\nu \Pi^{\mu\nu}}{2T^2(\epsilon + p)}.$$  

For simplicity, in the following we approximate $f_0(x) \sim \exp(-x)$, so similarly

$$f(x^\mu, p^\mu) = \exp(-p_\mu u^\mu/T) \left[ 1 + \frac{p_\mu p_\nu \Pi^{\mu\nu}}{2T^2(\epsilon + p)} \right].$$

The effect of this approximation will be commented on in the following sections.

In practice, for boost-invariant 2+1 dimensional hydrodynamics, the freeze-out hypersurface $\Sigma^\mu = (\Sigma^t, \Sigma^x, \Sigma^y, \Sigma^z) = (t, x, y, z)$ can be parametrized either by $\tau, \xi$ and the polar angle $\phi$, or by $x, y, \xi:

$$
\begin{align*}
&t = \tau \cosh \xi & t = \tau(x, y) \cosh \xi \\
&x = x(\tau, \phi) & x = x \\
&y = y(\tau, \phi) & y = y \\
&z = \tau \sinh \xi & z = \tau(x, y) \sinh \xi
\end{align*}
$$

The normal vector on $\Sigma^\mu$ is calculated by

$$d\Sigma_\mu(\tau, \phi, \xi) = \varepsilon_{\mu_1\mu_2\mu_3} \frac{\partial \Sigma^{\alpha}}{\partial \tau} \frac{\partial \Sigma^{\beta}}{\partial \phi} \frac{\partial \Sigma^{\gamma}}{\partial \xi} d\tau d\phi d\xi$$

$$d\Sigma^\mu(\tau, \phi, \xi) = -\tau \left( \cosh \xi \left( \frac{\partial x}{\partial \tau} \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \tau} \frac{\partial x}{\partial \phi} \right), \frac{\partial y}{\partial \phi}, \frac{\partial x}{\partial \phi}, \sinh \xi \left( \frac{\partial x}{\partial \tau} \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \tau} \frac{\partial x}{\partial \phi} \right) \right) d\tau d\phi d\xi$$

For simplicity, in the following we approximate $f_0(x) \sim \exp(-x)$, so similarly

$$f(x^\mu, p^\mu) = \exp(-p_\mu u^\mu/T) \left[ 1 + \frac{p_\mu p_\nu \Pi^{\mu\nu}}{2T^2(\epsilon + p)} \right].$$

The effect of this approximation will be commented on in the following sections.
Figure 4.3: Space-time cut through the three-dimensional hypersurface for a central collision within the Glauber model. Simulation parameters used were $a = 1 \text{ GeV}^{-1}$, $\tau_0 = 1 \text{ fm/c}$, $T_i = 0.36 \text{ GeV}$, $T_f = 0.15 \text{ GeV}$, $\tau_\pi = 6 \frac{\eta}{s}$ and $\lambda_1 = 0$ (see next sections for definitions). As can be seen from the figure, inclusion of viscosity only slightly changes the form of the surface.

and similarly for the other parametrization [79].

For a realistic equation of state, at early times the freeze-out hypersurface will contain the same transverse coordinate values $(x, y)$ for different times $\tau$ (see Figure 4.3). Therefore, the parametrization in terms of $(x, y, \xi)$ cannot be used for early times. On the other hand, the parametrization in terms of $(\tau, \phi, \xi)$ contains derivatives of $(x, y)$ with respect to $\tau$, which become very large at late times (see Figure 4.3). Numerically, it is therefore not advisable to use this parametrization at late times. As a consequence, we use the one parametrization at early times but switch to the other parametrization at late times, such that the integral in Equation 4.13 is always defined and numerically well-behaved.

In order to evaluate the integral (4.13), it is useful to express $p^\mu$ also in Milne coordinates,

$$
p^\mu = (p^\tau, p^x, p^y, p^\xi) = (m_T \cosh(Y - \xi), p^x, p^y, \frac{m_T}{\tau} \sinh(Y - \xi)),
$$

(4.17)

It may be possible that other parametrizations may turn out to be more convenient. For instance, it is conceivable that performing a triangulation of the three-dimensional hypersurface and replacing the integral in Equation 4.13 by a sum over triangles could turn out to be numerically superior to our method.
where \( m_T = \sqrt{m^2 + p_x^2 + p_y^2} = \sqrt{E^2 - p_z^2} \). Here and in the following \( Y = \text{arctanh}(p^z/E) \) is the rapidity, and \( m \) is the rest mass of the particle under consideration. Then the \( \xi \) integration can be carried out analytically using

\[
\frac{1}{2} \int_{-\infty}^{\infty} d\xi \cosh^n(Y - \xi) \exp[-z \cosh(Y - \xi)] = (-1)^n \frac{\partial^n}{\partial^2 z} K_0(z) \equiv K(n, z),
\]

(4.18)

where \( K_0(z) \) is a modified Bessel function. One finds

\[
\frac{E d^3N}{d^3p} = \frac{2d}{(2\pi)^3} \int d\tau d\phi \exp \left[ (p^x u^x + p^y u^y)/T \right] \times
\]

\[
\left[ m_T \left( \frac{\partial x}{\partial \tau} \frac{\partial y}{\partial \phi} - \frac{\partial y}{\partial \tau} \frac{\partial x}{\partial \phi} \right) (T_1 K(1, m_T u^\tau/T) + T_2 K(2, m_T u^\tau/T) + T_3 K(3, m_T u^\tau/T))
\right.
\]

\[
\left. - \left( p^x \frac{\partial y}{\partial \phi} - p^y \frac{\partial x}{\partial \phi} \right) (T_1 K(0, m_T u^\tau/T) + T_2 K(1, m_T u^\tau/T) + T_3 K(2, m_T u^\tau/T)) \right]
\]

T_1 = 1 + \frac{m_T^2 \Pi^\xi + p_x^2 \Pi^{xx} + p_y^2 \Pi^{yy} + 2p_x p_y \Pi^{xy}}{2T^2(\epsilon + p)},

T_2 = -\frac{2m_T p^x \Pi^{xt} + p^y \Pi^{yt}}{2T^2(\epsilon + p)},

T_3 = \frac{m_T^2 \Pi^{xt} - \Pi^\xi}{2T^2(\epsilon + p)},

(4.19)

for the \((\tau, \phi, \xi)\) parametrization, and a similar result for the other parametrization of the hypersurface. The remaining integrals for the particle spectrum have to be carried out numerically unless one is considering the case of a central collision \[47, 67\] where the integral has an additional symmetry in \( \phi \).

For the simulation of a heavy-ion collision, one then also needs to take into account the feed-down process of particle resonances that decay into lighter, stable particles \[80, 81\]. Therefore, we calculate the spectra for particle resonances with masses up to \( \sim 2 \) GeV and then use available routines from the AZHYDRO package \[82\] to determine the spectra of stable particles including these feed-down contributions. Ultimately, one would be interested in describing the last stage of the evolution by coupling the hydrodynamics to a hadronic cascade code \[83, 84, 85, 86\]. We leave this for future work.

The particle spectra \( E \frac{d^3N}{d^3p} \) including feed-down contributions can then be used to calculate experimental observables at central rapidity \( Y = 0 \), such as radial and elliptic flow coefficients, \( v_0, v_2 \), respectively, and the mean transverse momentum \( \langle p_T \rangle \), as defined in section 2.2.1.
4.2.8 Code tests

It is imperative to subject the numerical implementation of the relativistic viscous hydrodynamic model to several tests. The minimal requirement is that the code is stable for a range of simulated volumes and grid spacings $a$, such that an extrapolation to the continuum may be attempted (keeping the simulated volume fixed but sending $a \to 0$). Our code fulfills this property.

Furthermore, one has to test whether this continuum extrapolation corresponds to the correct physical result in simple test cases. One such test case is provided by the 0+1 dimensional model discussed in section 4.2.2. Using initial conditions of uniform energy density in the 2+1 dimensional numerical code, the temperature evolution should match that of Equation 4.3 for which it is straightforward to write an independent numerical solver. Our 2+1 dimensional code passes this test, for small and large $\eta/s$ and different values for $\tau_\pi, \lambda_1$.

The above test is non-trivial in the sense that it allows to check the implementation of nonlinearities in the hydrodynamic model. However, it does not probe the dynamics of the model, since, e.g., all velocities are vanishing. Therefore, another test that one can (and should!) conduct is to study the dynamics of the model against that of linearized hydrodynamics (this test was first outlined in Ref. [67]; see [87] for similar considerations). More specifically, let us consider a viscous background “solution” with $u^i = 0$ but non-vanishing $\epsilon(\tau), \Pi^\xi_\xi(\tau)$ obeying Equation 4.3. To first order in small fluctuations $\delta \epsilon, \delta u^\mu, \delta \Pi^{\mu\nu}$
around this background the set of equations (4.9) become

\[
\begin{align*}
&\left[ c_s^2 \partial_r \epsilon + \frac{1}{2} \partial_r \Pi_\xi^\xi + \frac{3}{2} \frac{\Pi_\xi^\xi}{\tau} + (\epsilon + p + \frac{1}{2} \Pi_\xi^\xi) \partial_r \right] \delta u^x + c_s^2 \partial_x \delta \epsilon + \partial_t \delta \Pi^{xi} = 0 \\
&\left[ c_s^2 \partial_r \epsilon + \frac{1}{2} \partial_r \Pi_\xi^\xi + \frac{3}{2} \frac{\Pi_\xi^\xi}{\tau} + (\epsilon + p + \frac{1}{2} \Pi_\xi^\xi) \partial_r \right] \delta u^y + c_s^2 \partial_y \delta \epsilon + \partial_t \delta \Pi^{yi} = 0 \\
&\left[ \partial_r + \frac{1}{\tau} + \frac{c_s^2}{\tau} \right] \delta \epsilon + \left[ (\epsilon + p) + \frac{1}{2} \Pi_\xi^\xi \right] \partial_t \delta u^i - \frac{1}{\tau} \delta \Pi_\xi^\xi = 0 \\
&\left[ \partial_r + \frac{1}{\tau} + \frac{c_s^2}{\tau} \right] \delta \Pi^{xx} - \left[ \frac{2\eta}{3\tau} + \frac{1}{4\tau} \Pi^{xx} \right] \frac{\delta \epsilon}{\epsilon} + \frac{2\eta}{\tau} \partial_x \delta \epsilon + \left[ -\frac{2\eta}{3\tau} + \frac{4}{3} \Pi^{xx} \right] \partial_t \delta u^i = 0 \\
&\left[ \partial_r + \frac{1}{\tau} + \frac{c_s^2}{\tau} \right] \delta \Pi^{yy} + \frac{\eta}{\tau} \left( \partial_x \delta u^y + \partial_y \delta u^x \right) = 0,
\end{align*}
\]

where we have put \( \lambda_1 = 0 \) and assumed a constant \( c_s^2 \) for simplicity. Noting that \( \delta \Pi^{yy} = \delta \Pi_\xi^\xi - \delta \Pi^{xx} \) from \( \delta \Pi_\mu^\mu = 0 \), Equation 4.20 are a closed set of linear, but coupled differential equations for the fluctuations \( \delta \epsilon, \delta u^x, \delta u^y, \delta \Pi_\xi^\xi, \delta \Pi^{xx}, \delta \Pi^{xy} \). Doing a Fourier transform,

\[
\delta \epsilon(\tau, x, y) = \int \frac{d^2 k}{(2\pi)^2} e^{ikx+i ky} \delta \epsilon(\tau, k^x, k^y)
\]

(and likewise for the other fluctuations), Equation 4.20 constitute coupled ordinary differential equations for each mode doublet \( k = (k^x, k^y) \), which again are straightforward to solve with standard numerical methods [69] (and analytically for ideal hydrodynamics).

A useful test observable is the correlation function

\[
f(\tau, x_1, x_2) = \frac{\langle \delta \epsilon(\tau, x_1) \delta \epsilon(\tau, x_2) \rangle}{\epsilon(\tau)^2},
\]

where \( \langle \rangle \) denotes an ensemble average over initial conditions \( \delta \epsilon |_{\tau=\tau_0} \). In particular, let us study initial conditions where \( \delta \epsilon \) is given by Gaussian random noise with standard deviation \( \Delta \),

\[
f(\tau_0, x_1, x_2) = \Delta^2 \delta^2(x_1 - x_2)
\]

and all other fluctuations vanish initially. These initial conditions are readily implemented both for the full 2+1 dimensional hydrodynamic code as well as for the linearized system Equation 4.20 As the system evolves to finite time \( \tau \), both approaches have to give the
Figure 4.4: The correlation function \( f(\tau, k) \) as a function of momentum \( k = |k| \) for a lattice with \( a = 1 \text{ GeV}^{-1} \), 64\(^2\) sites and averaged over 30 initial configurations (symbols), compared to the result from the linearized hydrodynamic equations (lines).

The same correlation function \( f \) as long as the linearized treatment is applicable, and hence Equation 4.20 can be used to test the dynamics of the full numerical code.

In practice, note that for the above construction \( f \) can only depend on the difference of coordinates,

\[
\left\langle \delta \epsilon(\tau, x_1) \delta \epsilon(\tau, x_2) \right\rangle \epsilon(\tau)^2 = f(\tau, x_1 - x_2) = \int \frac{d^2k}{(2\pi)^2} e^{ik \cdot (x_1 - x_2)} f(\tau, k) \quad (4.24)
\]

and therefore in Fourier-space

\[
f(\tau, k) \delta^2(k') = \frac{\left\langle \delta \epsilon(\tau, k) \delta \epsilon(\tau, k' - k) \right\rangle}{(2\pi)^2 \epsilon(\tau)^2}. \quad (4.25)
\]

In the full 2+1 dimensional numerical code which is discretized on a space-time lattice, \( \delta^2(k') \) is regular for any finite \( a \), and one can maximize the signal for \( f(\tau, k) \) by calculating the r.h.s. of Equation 4.25 for \( k' = 0 \). Similarly, one solution \( \delta \epsilon(\tau, k) \) per \( k \) mode is sufficient calculate \( f(\tau, k) \) for the linearized system Equation 4.20.

The above initial conditions imply \( f(\tau = \tau_0, k) = \Delta^2 \), but for finite times characteristic peaks develop as a function of \( |k| \), whose position, height and width are sensitive to the values of \( c_s^2, \tau_\pi, \eta/s \) and of course the correct implementation of the hydrodynamic equations.
The comparison between full numerics and linearized treatment shown in Figure 4.4 suggests that our code also passes this test.

Finally, for the case of ideal hydrodynamics, analytic solutions to the hydrodynamic equations are known \[88, 89, 90\]. Specifically, the code for central collisions \[67\] has been found to agree with the results from Ref. \[88\] for ideal hydrodynamics. Since our code agrees with Ref. \[67\] in the case of central collisions and when dropping the appropriate terms in the equations \[4.1\], this provides yet another test on our numerics.

To summarize, after conducting the above tests we are reasonably confident that our numerical 2+1 dimensional code solves the relativistic viscous hydrodynamic equations \[4.9\] correctly. This completes the setup of a viscous hydrodynamic description of relativistic heavy-ion collisions. In the following sections, we will review comparisons of viscous hydrodynamic simulations to experimental data, for both Glauber and CGC initial conditions.

### 4.3 Initial Conditions: Glauber Model vs. CGC

#### 4.3.1 The Glauber model

In the Glauber model \[13\], the starting point is the Woods-Saxon density distribution for nuclei,

\[
\rho_A(x) = \frac{\rho_0}{1 + \exp\left(\frac{|x| - R_0}{\chi}\right)},
\]

where for a gold nucleus with weight \(A = 197\) we use \(R_0 = 6.4\) fm and \(\chi = 0.54\) fm. The parameter \(\rho_0\) is chosen such that \(\int d^3x \rho_A(x) = A\). One can then define the nuclear thickness function

\[
T_A(x^i) = \int_{-\infty}^{\infty} dz \rho_A(x)
\]

and subsequently the number density of nucleons participating in the collision \(n_{\text{Part}}\) and the number density of binary collisions \(n_{\text{Coll}}\). For a collision of two nuclei with weight \(A\)

---

\(^5\)Note that a small numerical error occurred in the linearized hydrodynamic solver and the corresponding figure in Ref. \[48\]. This error has been corrected in Figure 4.4.
at an impact parameter $b$, one has

$$n_{\text{Part}}(x, y, b) = TA\left(x + \frac{b}{2}, y\right) \left[1 - \left(1 - \frac{\sigma TA(x - \frac{b}{2}, y)}{A}\right)^A\right] + TA\left(x - \frac{b}{2}, y\right) \left[1 - \left(1 - \frac{\sigma TA(x + \frac{b}{2}, y)}{A}\right)^A\right],$$

$$n_{\text{Coll}}(x, y, b) = \sigma TA\left(x + \frac{b}{2}, y\right) TA\left(x - \frac{b}{2}, y\right),$$

where $\sigma$ is the nucleon-nucleon cross section. We assume $\sigma \simeq 40$ mb for Au+Au collisions at $\sqrt{s} = 200$ GeV per nucleon pair.

While the total number of participating nucleons $N_{\text{Part}}(b) = \int dxdy n_{\text{Part}}(x, y, b)$ will be used to characterize the centrality class of the collision, as an initial condition for the energy density we will only use the parametrization

$$\epsilon(\tau = \tau_0, x, y, b) = \text{const} \times n_{\text{Coll}}(x, y, b),$$

since it gives a sensible description of the multiplicity distribution of experimental data, as will be discussed later on. In the following, “Glauber-model initial condition” is used synonymous to Equation 4.29.

The constant in Equation 4.29 is chosen such that the central energy density for zero impact parameter, $\epsilon(\tau = \tau_0, 0, 0, 0)$ corresponds to a predefined temperature $T_i$ via the equation of state. This temperature will be treated as a free parameter and is eventually fixed by matching to experimental data on the multiplicity.

### 4.3.2 The CGC model

The other model commonly used to obtain initial conditions for hydrodynamics is the so-called Color-Glass-Condensate approach, based on ideas of gluon saturation at high energies. In particular, we use a modified version of the KLN (Kharzeev-Levin-Nardi) $k_T$-factorization approach [91], due to Drescher et al. [92]. We follow exactly the procedure described in [71], and in fact we use the same numerical code, provided to us by the authors and only slightly modified to output initial conditions suitable for input into our viscous hydrodynamics program.
In this model, the number density of gluons produced in a collision of two nuclei with atomic weight $A$ is given by

$$\frac{dN_g}{d^2x_T dY} = \mathcal{N} \int \frac{d^2p_T}{p_T^2} \int_{p_T^2}^{p_T^2} d^2k_T \alpha_s(k_T) \phi_A(x_1, (p_T + k_T)^2/4; x_T) \phi_A(x_2, (p_T - k_T)^2/4; x_T)$$

(4.30)

where $p_T$ and $Y$ are the transverse momentum and rapidity of the produced gluons, respectively. $x_{1,2} = p_T \times \exp(\pm Y)/\sqrt{s}$ is the momentum fraction of the colliding gluon ladders with $\sqrt{s}$ the center of mass collision energy and $\alpha_s(k_T)$ is the strong coupling constant at momentum scale $k_T \equiv |k_T|$.

The value of the normalization constant $\mathcal{N}$ is unimportant here, since as for Glauber initial conditions, we treat the overall normalization of the initial energy density distribution as a free parameter. The unintegrated gluon distribution functions are taken as

$$\phi(x, k_T^2; x_T) = \frac{1}{\alpha_s(Q_s^2)} \frac{Q_s^2}{\max(Q_s^2, k_T^2)} P(x_T)(1 - x)^4,$$

(4.31)

$P(x_T)$ is the probability of finding at least one nucleon at transverse position $x_T$, taken from the definition for $n_{\text{Part}}$

$$P(x_T) = 1 - \left(1 - \frac{\sigma T_A}{A}\right)^A,$$

(4.32)

where $T_A$ and $\sigma$ are as defined in the previous section.

The saturation scale at a given momentum fraction $x$ and transverse coordinate $x_T$ is given by

$$Q_s^2(x, x_T) = 2 \text{GeV}^2 \left(\frac{T_A(x_T)/P(x_T)}{1.53/\text{fm}^2}\right) \left(\frac{0.01}{x}\right)^\lambda.$$

(4.33)

The growth speed is taken to be $\lambda = 0.288$.

The initial conditions for hydrodynamic evolution require that we specify the energy density in the transverse plane at some initial proper time $\tau_0$ at which the medium has thermalized. Equation 4.30, on the other hand, is in principle valid at a time $\tau_s = 1/Q_s$ at which the medium is likely not yet in thermal equilibrium. To obtain the desired initial conditions, we again follow [71] and assume that the number of gluons is effectively conserved during the evolution from $\tau_s$ to $\tau_0$ and so the number density profile is the same at both times, scaled by the one-dimensional Bjorken expansion $n(\tau_0) = \frac{\tau_0}{\tau_s} n(\tau_s)$. The energy
density can then be obtained from the number density through thermodynamic relations—it is proportional to the number density to the 4/3 power. Again, we take the overall normalization as a free parameter, so the initial energy density is finally given as

\[
\epsilon(\tau = \tau_0, x_T, b) = \text{const} \times \left[ \frac{dN_g}{d^2x_TdY(x_T, b)} \right]^{4/3} \tag{4.34}
\]

where the number density is given by Equation 4.30 evaluated at central rapidity \( Y = 0 \).

As a final comment, it should be pointed out that the original version of the CGC, the McLerran-Venugopalan model \[93, 94\], differs from the KLN ansatz we used here, as will be discussed in the next section.

### 4.3.3 Spatial and momentum anisotropy

One of the key parameters discussed in the following is the eccentricity (or spatial anisotropy) of the collision geometry. Following \[13\], we define it as

\[
e_x \equiv \frac{\langle y^2 - x^2 \rangle_\epsilon}{\langle y^2 + x^2 \rangle_\epsilon}, \tag{4.35}
\]

where \( \langle \rangle_\epsilon \) denotes an averaging procedure over space with the energy density \( \epsilon \) as a weighting factor. Shown in Figure 4.5a, a plot of \( e_x \) for different centralities highlights the quantitative difference between the initial energy density from the Glauber and CGC model, Equation 4.29 and Equation 4.34, respectively. As can be seen from this figure, the CGC model generally gives a higher spatial anisotropy than the Glauber model. Note that the results for the CGC model shown here are extreme in the sense that the McLerran-Venugopalan model gives spatial eccentricities which essentially match the ones from the Glauber model \[95\]. This allows us to use the difference between the CGC and Glauber models as an indication of the systematic theoretical error stemming from the ignorance of the correct physical initial condition.

Hydrodynamics converts pressure gradients into fluid velocities, and hence one expects the spatial anisotropy to decrease at the expense of a momentum anisotropy (which is related to the magnitude of the elliptic flow). We follow \[96\] in defining a momentum anisotropy according to

\[
e_p \equiv \frac{\langle T^{xx} - T^{yy} \rangle}{\langle T^{xx} + T^{yy} \rangle}, \tag{4.36}
\]
Figure 4.5: Left: The initial spatial anisotropy for the Glauber and CGC model. Right: The time evolution of the spatial and momentum anisotropy for a collision with $b = 7$ fm in ideal hydrodynamics.

where we stress that here $\langle \rangle$ denotes spatial averaging with weight factor unity. Figure 4.5b shows the time evolution in ideal hydrodynamics ($\eta/s \ll 1$) of both the spatial and momentum anisotropies for a heavy-ion collision at $b = 7$ fm modeled through Glauber and CGC initial conditions. As one can see, for the same impact parameter, the higher initial spatial anisotropy for the CGC model eventually leads to a higher momentum anisotropy than the Glauber model. Using a quasiparticle interpretation where the energy momentum tensor is given by

$$T^{\mu\nu} \propto \int \frac{d^3 P}{(2\pi)^3} \frac{p^\mu p^\nu}{E} f (x^\mu, p^\nu),$$

(4.37)

the momentum anisotropy $e_p$ can be approximately related to the integrated elliptic flow $v_2^{\text{int}}(b)$, with a proportionality factor of $\sim 2$ [96, 97]. We find this proportionality to be maintained even for non-vanishing shear viscosity, as can be seen in Figure 4.9.

4.4 Results

4.4.1 Which parameters matter?

In the following, we will attempt to obtain limits on the mean value (throughout the hydrodynamic evolution) of the ratio $\eta/s$ from experimental data. While e.g. temperature
Figure 4.6: Spatial and momentum anisotropy for the Glauber model at $b = 7$ fm with $T_i = 0.353$ GeV, $\tau_0 = 1$ fm/c and various values for the viscosity (grid spacing $a = 2$ GeV$^{-1}$).

(a): The dependence on the initialization value of the shear tensor: shown are results for vanishing initial value ($\Pi^\mu_\nu_{\text{init}} = 0$) and Navier-Stokes initial value ($\Pi^\mu_\nu_{\text{init}} \neq 0$), given in Equation 4.4.

(b): The dependence on the choice of value for $\tau_\pi, \lambda_1$: shown are results for $\tau_\pi = \frac{6 \eta}{T_s}, \lambda_1 = 0$ (labelled “IS”) and $\tau_\pi = \frac{2(2-\ln 2) \eta}{T}, \lambda_1 = \frac{\eta}{2\pi T}$ (labelled “AdS”). For $\tau_\pi = \frac{2(2-\ln 2) \eta}{T}$, the results for $\lambda_1 = 0$ (not shown) would be indistinguishable by bare eye from those for $\lambda_1 = \frac{\eta}{2\pi T}$.
variations of $\eta/s$ are to be expected in the real physical systems, probing for such variations would invariably force us to introduce more unknown parameters. We prefer to leave this program for future studies once robust results for the mean value of $\eta/s$ exist. Having fixed the equation of state and the freeze-out procedure as explained in the previous sections, the remaining choices that have to be made in the hydrodynamic model are the

- Initial energy density profile: Glauber or CGC
- Initial value of shear tensor: vanishing or Navier-Stokes value
- Hydrodynamic starting time $\tau_0$
- Second-order coefficients: relaxation time $\tau_{\pi}$ and $\lambda_1$
- Ansatz for non-equilibrium particle distribution

where it is to be understood that we fix the initial energy density normalization ($T_i$) and the freeze-out temperature $T_f$ such that the model provides a reasonable description of the experimental data on multiplicity and $\langle p_T \rangle$. Historically, a strong emphasis has been put on requiring a small value of $\tau_0$ for ideal hydrodynamics. For this reason, we will discuss the dependence on $\tau_0$ separately in section 4.4.4. A good indicator for which parameters matter is the momentum anisotropy since it is very sensitive to the value of $\eta/s$. From Figure 4.5 one therefore immediately concludes that the choice of Glauber or CGC initial conditions is important since it has a large effect on $e_p$. Fortunately, most of the other choices turn out not to have a strong influence on the resulting $v_2$ coefficient and hence the extracted $\eta/s$. In the following we test for this sensitivity by studying $e_p$ for a “generic” heavy-ion collision of two gold nuclei, modeled by Glauber initial conditions at an initial starting temperature of $T_i = 0.353$, an impact parameter of $b = 7$ fm, and various choices of the above parameters.

Figure 4.6 shows the time evolution of $e_x, e_p$ for various values of $\eta/s$. From these plots, it can be seen that $e_p$ (and hence $v_2$) clearly is sensitive to the value of $\eta/s$, suggesting that it can be a useful observable to determine the viscosity of the fluid from experiment.
However, in order to be a useful probe of the fluid viscosity, the dependence of the final value of $e_p$ on other parameters should be much weaker than the dependence on $\eta/s$. In Figure 4.6a we show $e_p$, calculated for $\Pi^\mu\nu(\tau_0) = 0$ and $\Pi^\mu\nu(\tau_0)$ equal to the Navier-Stokes value, Equation 4.4. As can be seen from this figure, the resulting anisotropies are essentially independent of this choice, corroborating the finding in Ref. [55, 57]. Similarly, in Figure 4.6b we show $e_p$ calculated in simulations where the values of the second-order transport coefficients were either those of a weakly-coupled Müller-Israel-Stewart theory ($\tau_\pi = 6\frac{n}{sT}$, $\lambda_1 = 0$) or those inspired by a strongly coupled $\mathcal{N} = 4$ SYM plasma ($\tau_\pi = 2(2 - \ln 2)\frac{n}{sT}$, $\lambda_1 = \frac{n}{2\pi T}$). Again, the dependence of $e_p$ on the choice of the values of $\tau_\pi, \lambda_1$ can be seen to be very weak for the values of $\eta/s$ shown here. This result is in stark contrast to the findings of Ref. [55], where a large sensitivity on the value of $\tau_\pi$ was found. However, recall that Ref. [55] used evolution equations that differ from Equation 4.1 and in particular do not respect conformal invariance. As argued in section 4.2.2, it is therefore expected to encounter anomalously large sensitivity to the value of the second order transport coefficients.

To study the dependence of results on the ansatz of the non-equilibrium particle distribution function (4.14), one would want to quantify the effect of neglecting terms of higher order
in momenta in Equation 4.14. To estimate this, let us rewrite $E d^3 N/d^3 p = E d^3 N^{(0)}/d^3 p + E d^3 N^{(1)}/d^3 p$, where $N^{(0)}$ contains only the equilibrium part where $f(x^\mu, p^\mu) = f_0 \left( \frac{p_\mu u_\mu}{T} \right)$, and perform a Padé-type resummation,

$$E \frac{d^3 N_{\text{Pade}}}{d^3 p} \equiv E \frac{d^3 N^{(0)}}{d^3 p} \frac{1}{1 - \frac{d^3 N^{(1)}}{d^3 p} \frac{d^3 p}{d^3 N^{(0)}}}. \quad (4.38)$$

Since Equation 4.38 contains powers of momenta to all orders when re-expanded, the difference between the ansatz (4.14) and the Padé resummed particle spectra can give a handle on the systematic error of the truncation used in Equation 4.14. Shown in Figure 4.7 this difference suggests that this systematic error is small for momenta $p_T \lesssim 2.5$ GeV. Therefore, we do not expect our results to have a large systematic uncertainty coming from the particular ansatz (4.14) for these momenta.

To summarize, for values of $\eta/s \lesssim 0.2$, the results for the momentum anisotropy are essentially insensitive to the choices for the second-order transport coefficients $\tau_\pi, \lambda_1$ and the initialization of the shear tensor $\Pi^{\mu \nu}(\tau = \tau_0)$. Conversely, $\epsilon_p$ is sensitive to the value of viscosity and the choice of initial energy density profile (initial eccentricity). Since the physical initial condition is currently unknown, this dependence will turn out to be the dominant systematic uncertainty in determining $\eta/s$ from experimental data.

### 4.4.2 Multiplicity and radial flow

As outlined in the introduction, we want to match the hydrodynamic model to experimental data for the multiplicity, thereby fixing the constant in Equations 4.29 and 4.34. This translates to fixing an initial central temperature $T_i$ for $b = 0$, which we will quote in the following.

For a constant speed of sound, the evolution for ideal hydrodynamics is isentropic, while for viscous hydrodynamics additional entropy is produced. Since the multiplicity is a measure of the entropy of the system, one expects an increase of multiplicity for viscous compared to ideal hydrodynamic evolution. This increase in final multiplicity has been measured as a function of $\eta/s$ for the semi-realistic speed of sound Figure 4.2 in central heavy-ion collisions in Ref. [47], and found to be approximately $\frac{1}{6}$ for $\eta/s \lesssim 0.75$. (See

---

6 The quoted fraction is for a hydrodynamic starting time of $\tau_0 = 1$ fm/c. Reducing $\tau_0$ leads to consid-
Ref. [71, 100] for related calculations in simplified models.) Reducing $T_i$ accordingly therefore ensures that for viscous hydrodynamics, the multiplicity in central collisions will stay close to that of ideal hydrodynamics.

Hydrodynamics gradually converts pressure gradients into flow velocities, which in turn relate to the mean particle momenta. Starting at a predefined time $\tau_0$ and requiring the hydrodynamic model spectra to match the experimental data on particle $\langle p_T \rangle$ then fixes the freeze-out temperature $T_f$.

For both Glauber-type and CGC-type model initial conditions, the experimental impact parameter dependence of the multiplicity and $\langle p_T \rangle$ is reasonably well parametrized for both ideal hydrodynamics as well as viscous hydrodynamics provided $T_i$ is adjusted accordingly (see Figure 4.8). The values for $T_i$ used in the simulations are compiled in Table 4.1. We recall that no chemical potential is included in our equation of state, prohibiting a distinction between particles and anti-particles, and chemical and kinetic freeze-out of particles occurs at the same temperature. Furthermore, approximating the equilibrium particle-distributions for bosons by a Boltzmann distribution (4.14) leads to small, but consistent underestimation of the multiplicity of light particles, such as pions. For these reasons, it does not make sense to attempt a precision fit to experimental data, especially for pions and protons. Rather, we have aimed for a sensible description of the overall centrality dependence of multiplicity and $\langle p_T \rangle$ of kaons.

Note that in particular for the CGC model one could achieve a better fit to the data on mean $\langle p_T \rangle$ by increasing the freeze-out temperature by $\sim 10$ MeV. This would also lead to a decrease in elliptic flow for this model. However, to facilitate comparison between the CGC and Glauber initial conditions, we have kept $T_f$ the same for both models.

4.4.3 Elliptic flow

Having fixed the parameters $\tau_0, T_i, T_f$ for a given $\eta/s$ to provide a reasonable description of the experimental data, a sensible comparison between the model and experimental results for the elliptic flow coefficient can be attempted. For charged hadrons, the integrated and

erably larger entropy production.
Figure 4.8: Centrality dependence of total multiplicity \( dN/dY \) and \( \langle p_T \rangle \) for \( \pi^+, \pi^-, K^+, K^-, p \) and \( \bar{p} \) from PHENIX \[101\] for Au+Au collisions at \( \sqrt{s} = 200 \) GeV, compared to the viscous hydrodynamic model and various \( \eta/s \), for Glauber initial conditions and CGC initial conditions. The model parameters used here are \( \tau_0 = 1 \) fm/c, \( \tau_\pi = 6\eta/s \), \( \lambda_1 = 0 \), \( T_f = 140 \) MeV and adjusted \( T_i \) (see Table 4.1).
Table 4.1: Summary of parameters used for the viscous hydrodynamics simulations

| Initial condition | $\eta/s$ | $T_i$ [GeV] | $T_f$ [GeV] | $\tau_0$ [fm/c] | $a$ [GeV$^{-1}$] |
|-------------------|---------|------------|------------|-----------------|----------------|
| Glauber           | $10^{-4}$ | 0.340      | 0.14       | 1               | 2              |
| Glauber           | 0.08    | 0.333      | 0.14       | 1               | 2              |
| Glauber           | 0.16    | 0.327      | 0.14       | 1               | 2              |
| CGC               | $10^{-4}$ | 0.310      | 0.14       | 1               | 2              |
| CGC               | 0.08    | 0.304      | 0.14       | 1               | 2              |
| CGC               | 0.16    | 0.299      | 0.14       | 1               | 2              |
| CGC               | 0.24    | 0.293      | 0.14       | 1               | 2              |

minimum-bias $v_2$ coefficients are shown in Figure 4.9 for Glauber and CGC initial conditions. As noted in section 4.3.3, charged hadron $v^\text{int}_2$ turns out to be very well reproduced by the momentum eccentricity $\frac{1}{2} e_p$, evaluated when the last fluid cell has cooled below $T_f$. This agreement is independent from impact parameter or viscosity and hence may serve as a more direct method on obtaining an estimate for $v^\text{int}_2$ if one cannot (or does not want to) make use of the Cooper-Frye freeze-out procedure described in section 4.2.7.

The comparison of the hydrodynamic model to experimental data with 90% confidence level systematic error bars from PHOBOS [102] for the integrated elliptic flow in Figure 4.9 suggests a maximum value of $\eta/s \sim 0.16$ for Glauber-type and $\eta/s \sim 0.24$ for CGC-type initial conditions. Whereas for Glauber initial conditions, ideal hydrodynamics ($\eta/s \sim 0$) gives results consistent with PHOBOS data, for CGC initial conditions zero viscosity does not give a good fit to the data, which is consistent with previous findings [85].

For minimum-bias $v_2$, to date only experimental data using the event-plane method are available, where the statistical, but not the systematic error of that measurement is directly accessible. The dominant source of systematic error is associated with the presence of so-called non-flow effects [105]. Recent results from STAR suggest that removal of these non-flow effects imply a reduction of the event-plane minimum bias $v_2$ by 20 percent [103, 104]. For charged hadrons, a comparison of both the event-plane and the estimated non-
Figure 4.9: Comparison of hydrodynamic models to experimental data on charged hadron integrated (left) and minimum bias (right) elliptic flow by PHOBOS [102] and STAR [103], respectively. STAR event plane data has been reduced by 20 percent to estimate the removal of non-flow contributions [103, 104]. The line thickness for the hydrodynamic model curves is an estimate of the accumulated numerical error (due to, e.g., finite grid spacing). The integrated $v_2$ coefficient from the hydrodynamic models (full lines) is well reproduced by $\frac{1}{2} c_p$ (dots); indeed, the difference between the full lines and dots gives an estimate of the systematic uncertainty of the freeze-out prescription.
flow corrected experimental data from STAR with the hydrodynamic model is shown in Figure 4.9.

For Glauber-type initial conditions, the data on minimum-bias $v_2$ for charged hadrons is consistent with the hydrodynamic model for viscosities in the range $\eta/s \in [0, 0.1]$, while for the CGC case the respective range is $\eta/s \in [0.08, 0.2]$. It is interesting to note that for Glauber-type initial conditions, experimental data for both the integrated as well as the minimum-bias elliptic flow coefficient (corrected for non-flow effects) seem to be reproduced best by a hydrodynamic model with $\eta/s \approx 0.08 \simeq \frac{1}{14\pi}$. This number has first appeared in the gauge/string duality context \cite{36} and has been conjectured to be the universal lower bound on $\eta/s$ for any quantum field theory at finite temperature and zero chemical potential \cite{106}. For CGC-type initial conditions, the charged hadron $v_2$ data seems to favor a hydrodynamic model with $\eta/s \sim 0.16$, well above this bound.

4.4.4 Early vs. late thermalization

Currently, there seems to be a common misunderstanding in the heavy-ion community that hydrodynamic models can universally only reproduce experimental data if they are initialized at early times $\tau_0 < 1$ fm/c. This notion has been labeled “early thermalization” and continues to create a lot of confusion. In this section, we argue that the matching of hydrodynamics to data itself does not require $\tau_0 < 1$ fm/c. It is the additional assumptions about pre-equilibrium dynamics that lead to this conclusion for the Glauber initial conditions.

Performing hydrodynamic simulations in the way we have described earlier, the energy density distribution is specified by either the Glauber or CGC model at an initial time $\tau_0$. In Figure 4.10 we show the result for the elliptic flow coefficient (or the momentum anisotropy) for three different values of $\tau_0$, namely 0.5, 1 and 2 fm/c, where also $T_i$ and $T_f$ have been changed in order to obtain roughly the same multiplicity and mean $p_T$ for each $\tau_0$. As can be seen from this figure, the resulting final elliptic flow coefficient is essentially independent of the choice of $\tau_0$. In particular, this implies that experimental data for bulk quantities

\footnote{In Ref. \cite{48} a lower value of $\eta/s$ for the Glauber model was reported. The results for viscous hydrodynamics shown in Figure 4.9 are identical to Ref. \cite{48}, but the new STAR data with non-flow corrections became available only after \cite{48} had been published.}
Figure 4.10: Momentum anisotropy (a) and elliptic flow for charged hadrons (b) for $b = 7$ fm, $\eta/s = 0.08$ and different hydrodynamic initialization times $\tau_0$. Horizontal light gray lines in (a) are visual aids to compare the final value of $e_p$. As can be seen from these plots, neither the final $e_p$ nor the charged hadron $v_2$ depend sensitively on the value of $\tau_0$ if the same energy distribution is used as initial condition at the respective initialization times. Simulation parameters were $T_i = 0.29$ GeV, $T_f = 0.14$ GeV for $\tau_0 = 2$ fm/c, $T_i = 0.36$ GeV, $T_f = 0.15$ GeV for $\tau_0 = 1$ fm/c, and $T_i = 0.43$ GeV, $T_f = 0.16$ GeV for $\tau_0 = 0.5$ fm/c.
can be reproduced by hydrodynamic models also for large initialization times, so no early thermalization assumption is needed.

However, it is true that the above procedure assumes that the energy density distribution remains unchanged up to the starting time of hydrodynamics, which arguably becomes increasingly inaccurate for larger $\tau_0$. It has therefore been suggested to mimic the pre-hydro time evolution of the energy density distribution by assuming free-streaming of partons. Assuming free-streaming gives the maximal contrast to assuming hydrodynamic evolution, since the latter corresponds to very strong interactions while the former corresponds to no parton interactions at all. Indeed, one can calculate the effect of the free-streaming evolution on the spatial anisotropy, finding

$$e_x(\tau) = \frac{e_x(0)}{1 + \frac{\tau^2}{3\langle R^2 \rangle}}, \quad \langle R^2 \rangle = \frac{\int d^2x \epsilon(x=0)}{\int d^2x \frac{(x^2+y^2)}{2} \epsilon(x=0)}. \quad (4.39)$$

This implies that the spatial anisotropy decreases with time, whereas one can show that free-streaming does not lead to a build-up of $e_p$. In other words, the eccentricity gets diluted without producing elliptic flow, such that once hydrodynamic evolution starts, it will not lead to as much $v_2$ as it would have without the dilution effect. It is tempting to conclude from this that by comparing to experimental data on elliptic flow one could place an upper bound on the maximally allowed dilution time, and interpret this as the thermalization time of the system. One should be aware, however, that this bound will depend on the assumption made about the pre-hydro evolution. Furthermore, one should take into account the fact that the initial state of the system remains unknown. For instance, the system could start with an energy density distribution similar to the CGC model, which has a fairly large eccentricity. Figure 4.11 shows that when allowing the eccentricity to get diluted according to Equation 4.39, it takes a time of $\tau \sim 1.5$ fm/c until the eccentricity has shrunk to that of the Glauber model. This implies that even when assuming no particle interactions (no elliptic flow build-up) for the first stage of the system evolution, one can

---

8 It seems that if one forces the energy-momentum tensor at the end of free-streaming period to match to that of ideal hydrodynamics (instantaneous thermalization), the resulting fluid velocities are anisotropic, i.e. correspond to a non-vanishing elliptic flow coefficient. It is possible that this effect stems from neglecting velocity gradients (viscous hydrodynamic corrections) in the matching process. We ignore the complications of the detailed matching from free-streaming to hydrodynamics in the following.
Figure 4.11: Spatial eccentricity for the Glauber and CGC model compared to evolving the CGC model according to Equation 4.39 for $\tau = 1.5$ fm/c. This implies that starting with Glauber-type initial conditions at $\tau_0 > 1$ fm/c may not be unreasonable.

get eccentricities which are Glauber-like after waiting for a significant fraction of the system life time. Allowing at least some particle interactions (which is probably more realistic), one expects some build-up of elliptic flow already in the dilution (or pre-equilibrium) phase, and therefore dilution (or “thermalization”) times of $\tau \sim 2$ fm/c seem not to be incompatible with the observed final elliptic flow even for non-vanishing viscosity.

4.5 Summary and Conclusions

We applied conformal relativistic viscous hydrodynamics to simulate Au+Au collisions at RHIC at energies of $\sqrt{s} = 200$ GeV per nucleon pair. Besides one first-order transport coefficient (the shear viscosity) in general there are five second-order transport coefficients in this theory, for which one would have to supply values. We provided arguments that physical observables in the parameter range accessible to hydrodynamics (low momenta, central to semi-central collisions) do not seem to be strongly dependent on specific (reasonable) choices for any of these second-order coefficients. On the other hand, we do find a pronounced dependence of the elliptic flow coefficient on the ratio of shear viscosity over entropy density, which suggests that by combining viscous hydrodynamics and experimen-
tal data a measurement of the quark-gluon plasma viscosity may not be futile. However, we have shown that our ignorance about the precise distribution of energy density at the earliest stages of a heavy-ion collision introduces a large systematic uncertainty in the final elliptic flow of the hydrodynamic model. Adding to this is the considerable experimental uncertainty pertaining to the removing of non-flow contributions to the elliptic flow. For these reasons, we are unable to make precise statements about the value of the shear viscosity of the quark-gluon plasma and in particular cannot place a firm lower bound on $\eta/s$.

Indeed, our hydrodynamic models seem to be able to consistently describe experimental data for multiplicity, radial flow and elliptic flow of bulk charged hadrons for a wide range of viscosity over entropy ratios,

$$\frac{\eta}{s} = 0.1 \pm 0.1\text{(theory)} \pm 0.08\text{(experiment)},$$

where we estimated the systematic uncertainties for both theory and experiment from the results shown in Figure 4.9. We stress that Equation 4.40 does not account for physics not included in our model, such as finite chemical potential, bulk viscosity, heat flow, hadron cascades, three-dimensional fluid dynamic effects and possibly many more. Consistent inclusion of all these may result in changes of the central value and theory uncertainty in Equation 4.40. Nevertheless, none of the mentioned refinements is currently expected to dramatically increase the elliptic flow coefficient (though some increase may be expected when e.g. implementing partial chemical equilibrium [109]). Therefore, we seem to be able to exclude viscosities of $\eta/s \gtrsim 0.5$ with high confidence, which indicates that the quark-gluon plasma displays less friction than any other known laboratory fluid [106, 110]. Other groups have come to similar conclusions [111, 112, 113].

To better quantify the shear viscosity of the quark-gluon plasma at RHIC calls for more work, both in theory and experiment. On the theory side, a promising route seems to be the study of fluctuations and comparing to existing experimental data [102, 111, 114, 115, 116, 117, 118, 119]. For instance, it might be interesting to investigate the critical value of $\eta/s$ for the onset of turbulence in heavy-ion collisions and explore possible consequences of fully developed turbulence [120]. However, maybe most importantly, a more thorough understanding of the earliest stages of a heavy-ion collision, in particular
thermalization, could fix the initial conditions for hydrodynamics and hence dramatically reduce the theoretical uncertainty in final observables.

Leaving these ideas for future work, we stress that with the advent of conformal relativistic viscous hydrodynamics at least the uncertainties of the hydrodynamic evolution itself now seem to be under control. We hope that this serves as another step towards a better understanding of the dynamics of relativistic heavy-ion collisions.
Chapter 5

VISCOS HYDRODYNAMIC PREDICTIONS FOR THE LHC

5.1 Introduction

Using the knowledge gained from viscous hydrodynamic simulations for RHIC, it should be possible to predict experimental results at the Large Hadron Collider (LHC), which will collide lead ions at a maximum center of mass energy of $\sqrt{s} = 5.5$ TeV per nucleon pair compared to $\sqrt{s} = 200$ GeV gold ions at RHIC. If experimental data on, e.g., $v_2$ from LHC is close to the hydrodynamic model prediction, this would confirm that real progress has been made in understanding nuclear matter at extreme energy densities; if far away, it may indicate that the successful hydrodynamic description of experimental data from RHIC was a coincidence.

Regardless of the outcome, the advent of the RHIC experiments clearly has lead to major progress in the theory and application of hydrodynamics to heavy-ion collisions. A few years ago the form of the hydrodynamic equations in the presence of shear viscosity $\eta$ was still unresolved, with different groups keeping some terms while neglecting others [29, 64, 121, 122]. For the case of approximately conformal theories, where the viscosity coefficient for bulk—but not shear—becomes negligible, all possible terms to second order in gradients were derived in Ref. [31] (see chapter 3), and their relative importance investigated in Ref. [1] (chapter 4). Three of the groups performing viscous hydrodynamic simulations now agree on these terms [1, 123, 124], while another group [56] uses a different formalism that gives matching results. While this development still leaves out the consistent treatment of bulk viscosity, the quantitative suppression of elliptic flow by shear viscosity is therefore essentially understood. As shown in chapter 4 from comparison of viscous hydrodynamic simulations to experimental data [102, 103], one can infer an upper limit of the ratio of shear viscosity over entropy density, $\eta/s < 0.5$, for the matter produced in Au+Au collisions at $\sqrt{s} = 200$ GeV [1], which is in agreement to extractions by other methods [111, 112, 113].
A sizeable uncertainty for this limit comes from the fact that the initial conditions for the hydrodynamic evolution are poorly known, with the two main models, the Glauber and Color-Glass-Condensate (CGC) models, giving different results for the elliptic flow coefficient [1]. This difference can be understood to originate from the different initial spatial eccentricity $e_x$ in the Glauber/CGC models, which we recall is defined as

$$e_x \equiv \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle},$$

(5.1)

where the symbols $\langle \rangle$ denote averaging over the initial energy density in the transverse plane, $\epsilon(x, y)$.

Indeed, it had been suggested [125] that the elliptic flow coefficient $v_2$ at the end of the hydrodynamic evolution would be strictly proportional to the initial spatial eccentricity, $v_2/e_x \propto \text{const.}$, if the fluid was evolving without any viscous stresses for an infinitely long time. This is to be contrasted with experimental data indicating a proportionality factor of total multiplicity over overlap area $v_2/e_x \propto dN/dY/S_{\text{overlap}}$ [125]. Total multiplicity $dN/dY$ here refers to the total number of observed particles $N$ per unit rapidity $Y$, while the overlap area is calculated as

$$S_{\text{overlap}} = \pi \sqrt{\langle x^2 \rangle \langle y^2 \rangle}.$$

(5.2)

Ideal fluid dynamics does not adequately describe the latest stage of a heavy-ion collision (the hadron gas), because of the large viscosity coefficient in this stage [126]. Therefore, the hydrodynamic stage lasts only for a finite time (e.g., until all fluid cells have cooled below the decoupling temperature), resulting in a dependence of $v_2/e_x$ on $dN/dY$. Also, viscous effects affect the proportionality between $v_2$ and $e_x$, leading to a behavior that is qualitatively similar to that observed in the data [123].

One of the objectives of this work is to extend the energy range for fluid dynamic results of $v_2/e_x$ from Au+Au collisions at top RHIC to Pb+Pb collisions at top LHC energies, as well as to study the dependence on shear viscosity. If in the future either $e_x$ or the mean $\eta/s$ becomes known, these results can thus be used to constrain the respective other quantity from experimental data. On the other hand, the values of shear viscosity for which the Glauber/CGC models match to experimental data at top RHIC energies have
been extracted in Ref. [1,2] for Au+Au collisions. Since $\eta/s$ averaged over the system evolution is not expected to be dramatically different for Pb+Pb collisions at the LHC, another objective of this work is to obtain a prediction for the elliptic flow coefficient for the LHC based on the best-fit values to RHIC.

Finally, the feasibility of detecting elliptic flow in p+p collisions at $\sqrt{s} = 14$ TeV at the LHC is being discussed [127]. As a reference for other approaches and experiment, it interesting to study the possible size and viscosity dependence of $v_2$ under the hypothetical assumption that the bulk evolution following p+p collisions could be captured by fluid dynamics.

5.2 Setup

To make predictions for nuclear collisions at LHC energies, we use our hydrodynamic model that successfully described experimental data at RHIC [1,2] and make modifications to the input parameters appropriate for the higher collision energies at the LHC.

As a reminder, the hydrodynamic model [1] is based on the conservation of the energy momentum tensor [31]

$$T^{\mu\nu} = \epsilon u^\mu u_\nu - p \Delta^{\mu\nu} + \Pi^{\mu\nu},$$

$$\Pi^{\mu\nu} = \eta \nabla^{(\mu} u^{\nu)} - \tau \left[ \Delta_{\alpha}^{\mu} \Delta_{\beta}^{\nu} D\Pi^{\alpha\beta} + \frac{4}{3} \Pi^{\mu\nu} (\nabla_{\alpha} u^{\alpha}) \right]$$

$$- \frac{\lambda_1}{2\eta^2} \Pi^{(\mu}_{\lambda} \Pi^{\nu)\lambda} + \frac{\lambda_2}{2\eta} \Pi^{(\mu}_{\lambda} \omega_{\nu)\lambda} - \frac{\lambda_3}{2} \omega^{(\mu}_{\lambda} \omega^{\nu)\lambda},$$

where $\epsilon, p$ and $u^\mu$ are the energy density, pressure, and fluid 4-velocity, respectively. $D \equiv u^\mu D_\mu$ and $\nabla_\alpha \equiv \Delta^\alpha_\mu D_\mu$ are time-like and space-like projections of the covariant derivative $D_\mu$, where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and we remind the compact notations

$$A_{(\mu}B_{\nu)} \equiv \left( \Delta^{\alpha}_\mu \Delta^{\beta}_\nu + \Delta^{\alpha}_\nu \Delta^{\beta}_\mu - \frac{2}{3} \Delta^{\alpha\beta} \Delta_{\mu\nu} \right) A_\alpha B_\beta$$

and $\omega_{\mu\nu} \equiv \frac{1}{2} (\nabla_\nu u_\mu - \nabla_\mu u_\nu)$. For relativistic nuclear collisions it is convenient to follow Bjorken [128] and use Milne coordinates proper time $\tau = \sqrt{t^2 - z^2}$ and spacetime rapidity $\xi = \text{atanh} \frac{z}{t}$, in which the metric becomes $g_{\mu\nu} = \text{diag}(1, -1, -1, -\tau^2)$, and assume that close to $\xi = 0$, the hydrodynamic degrees of freedom are approximately boost-invariant ($\xi \simeq Y$).

The hydrodynamic equations $D_\mu T^{\mu\nu} = 0$ then constitute an initial value problem in
Table 5.1: Central collision parameters used for the viscous hydrodynamics simulations ($T_f = 0.14$ GeV for all).

| Beam     | Initial cond. | $\frac{dN_{ch}}{dy}$ | $T_i$ [GeV] | $\sqrt{s}$ [GeV] | $\tau_0$ [fm/c] |
|----------|---------------|-----------------------|-------------|------------------|-----------------|
| Gold     | Glauber       | 800                   | 0.34        | 200              | 1               |
| Gold     | CGC           | 800                   | 0.31        | 200              | 1               |
| Lead     | Glauber       | 1800                  | 0.42        | 5500             | 1               |
| Lead     | CGC           | 1800                  | 0.39        | 5500             | 1               |
| Protons  | Glauber       | 6                     | 0.400       | 14000            | 0.5             |
| Protons  | Glauber       | 6                     | 0.305       | 14000            | 1               |
| Protons  | Glauber       | 6                     | 0.270       | 14000            | 2               |

proper time and transverse space, and are solved numerically (see Ref. [1]). The input parameters for hydrodynamic evolution are the equation of state $p = p(\epsilon)$ and the first (second) order hydrodynamic transport coefficients $\eta_1, \lambda_1, \lambda_2, \lambda_3$. The values for $\lambda_{1,2,3}$ have been found to hardly affect the boost-invariant hydrodynamic evolution for Au+Au collisions at RHIC [1], so here they are generally set to zero.

The mechanisms leading to thermalization (the onset of hydrodynamic behavior) are not well understood in nuclear collisions. Therefore, it is not known how the thermalization time $\tau_0$ at which hydrodynamic evolution is started depends on the collision energy. Barring further insight, we start hydrodynamic evolution for the LHC at the same time as for the RHIC simulations ($\tau_0 = 1$ fm/c). At this time, the initial conditions for the transverse energy density $\epsilon(x,y)$ are given by the Glauber or CGC model, respectively, the fluid velocities are assumed to vanish, and the shear tensor $\Pi^{\mu\nu}$ is set to zero (other values for $\Pi^{\mu\nu}$ do not seem to affect the final results [1, 57]). For brevity, we refer to chapter 4 for the details of the Glauber and CGC models, but note that we use the Woods-Saxon parameters of radius $R_0 = 6.4 (6.6)$ fm and skin depth $\chi = 0.54 (0.55)$ fm for gold (lead), and assume a nucleon-nucleon cross section of $\sigma = 40 (60)$ mb for $\sqrt{s} = 200 (5500)$ GeV collisions.

The overall normalization of the initial energy density (parametrized by the initial tem-
perature at the center $T_i$) was adjusted to match the experimentally observed multiplicity at RHIC; by analogy, for LHC the normalization is adjusted to match the predicted multiplicity [129, 130, 131, 132]. Since we lack detailed knowledge about its temperature dependence, the ratio of shear viscosity to entropy density $\eta/s$ is set to be constant during the hydrodynamic evolution (equal to the average over the spacetime evolution of the system). The relaxation time coefficient $\tau_\pi$ is expected [31, 133] to lie in the range $\tau_\pi (\epsilon + p) \simeq 2.6 - 6$. The equation of state (EoS) can in principle be provided by lattice QCD. While at present there are points of disagreement between lattice groups about, e.g., the precise location of the QCD phase transition, there is consensus that it is an analytic crossover [134, 135]. Therefore, we use a lattice-inspired EoS [73] that is consistent with both the current consensus and perturbative QCD; also, since it resembles [135], we expect that using a different lattice EoS will have a minor effect on our results. In fact, as a preview of work in progress, some of the calculations were also done with an equation of state taken from Ref. [135] (Figure 5.5 at the end of the chapter) and the results compared (see Figure 5.2).

Once a given fluid cell has cooled down to the decoupling temperature $T_f$, its energy and momentum are converted into particle degrees of freedom using the Cooper-Frye freeze-out prescription [75]. A value of $T_f = 0.14$ GeV was determined by matching to RHIC data and will also be used for LHC energies, assuming that it is mostly determined by local conditions, and less so by initial energy density, system size or collision energy. The distribution of the particle degrees of freedom may be further evolved using a hadronic cascade code (as in Ref. [83]), or in a more simple approach the unstable particle resonances are allowed to decay, without further evolving the stable particle distributions. In both cases, the total multiplicity and particle correlations (such as the elliptic flow coefficient) are then calculated from the stable particle distribution (cf. [1]). Surprisingly, it was found in Ref. [1, 96] (see chapter 4) that the momentum integrated elliptic flow coefficient for charged hadrons—to good approximation—is equal to half the momentum anisotropy,

$$v_2 \simeq \frac{1}{2} e_p = \frac{1}{2} \int \frac{dx dy}{x^2 + y^2} \left( T_{xx} - T_{yy} \right).$$

(5.3)

Since the momentum anisotropy is a property of the fluid, it is independent on the details of the freeze-out procedure and only mildly dependent on the choices of $\tau_0, T_f$. Unlike at RHIC
where pairs of $\tau_0$ and $T_f$ could be fine-tuned to fit the particle spectra at central collisions, no such extra information is available for the LHC. Hence Equation 5.3 may provide the most reliable way of determining the elliptic flow of charged hadrons, and will be used in the following. Similarly, one can use the total entropy per unit spacetime rapidity $\frac{dS}{d\xi}$ in the fluid as a proxy for the total (charged hadron) multiplicity per unit rapidity $\frac{dN}{dY} (\frac{dN_{ch}}{dY})$ with a proportionality factor $\frac{dS}{d\xi} \sim \frac{dS}{dY} \simeq 4.87\frac{dN}{dY} \simeq 7.85\frac{dN_{ch}}{dY}$. (5.4)

Note that for a gas of massive hadrons in thermal equilibrium at $T_f = 0.14$ GeV the ratio of entropy to particle density is $\sim 6.41$, but the decay of unstable resonances produces additional entropy, resulting in Equation 5.4. Since results from RHIC suggest there is only approximately 10% viscous entropy production during the hydrodynamic phase $\frac{dS}{d\tau}$, the entropy $\frac{dS}{d\tau}$ at $\tau = \tau_0$ can be used to estimate the final particle multiplicity. In the case of the LHC, the world average for the predicted charged hadron multiplicity for central Pb+Pb collisions at $\sqrt{s} = 5.5$ TeV $\frac{dN_{ch}}{dY} \simeq 1800$, can be used to estimate the total entropy at $\tau = \tau_0$, and hence the overall normalization $T_i$ of the initial energy density (see Table 5.1).

Using Equations 5.3 and 5.4 for the multiplicity and elliptic flow allows to make predictions for the LHC without having to model the hadronic freeze-out, which should make the results more robust. However, as a consequence one does not get information about the momentum dependence of the elliptic flow coefficient, prohibiting detailed comparison with predictions by other groups $\frac{dS}{d\tau}$.

5.3 Results

With the initial energy density distribution fixed at $\tau_0$, the hydrodynamic model then gives predictions for the ratio of $v_2/e_x$ at the LHC. In Figure 5.1 the results are shown for three different values of shear viscosity, for two different initial conditions and two different beams/collision energies (Au+Au at $\sqrt{s} = 200$ GeV, Pb+Pb at $\sqrt{s} = 5.5$ TeV). The resulting values for $v_2/e_x$ seem to be quasi-universal functions of the total multiplicity scaled by the overlap area $S_{\text{overlap}}$, only depending on the value of $\eta/s$ (and, to a lesser
Figure 5.1: Anisotropy (5.3) divided by (5.1), as a function of initial entropy (5.4) divided by (5.2). Shown are results from hydrodynamic simulations for $\sqrt{s} = 200$ GeV Au+Au (RHIC) and $\sqrt{s} = 5.5$ TeV Pb+Pb collisions (LHC). For comparison, experimental data for $v_2$ from RHIC [138], divided by $e_x$ from two models [113], is shown as a function of measured $\frac{dN_{ch}}{dY}$ [101] divided by (5.2). See text for details.

extent, the collision energy). The deviations of the RHIC simulations from the universal curve can be argued to arise from a combination of the finite lifetime of the hydrodynamic phase at $\sqrt{s} = 200$ GeV and the presence of the QCD phase transition, and is strongest for ideal hydrodynamics, in agreement with earlier findings [123].

Also shown in Figure 5.1 is experimental data for the elliptic flow coefficient for Au+Au collisions at RHIC, normalized by $e_x$ from a Monte-Carlo calculation (including fluctuations) in Glauber and CGC models (see Ref. [113] for details). Since $e_x$ is not directly measurable, the differently normalized data gives an estimate of the overall size of $v_2/e_x$ at RHIC. Directly matching experimental data on $v_2$ using a hydrodynamic model with an initial $e_x$ specified by the Glauber or CGC model, a reasonable fit was achieved for a mean value of
Figure 5.2: Anisotropy \( v_2 \) prediction for \( \sqrt{s} = 5.5 \) TeV Pb+Pb collisions (LHC), as a function of centrality. Prediction is based on values of \( \eta/s \) for the Glauber/CGC model that matched \( \sqrt{s} = 200 \) GeV Au+Au collision data from PHOBOS at RHIC [138], shown for comparison.

\[ \eta/s \approx 0.08 \text{ and } \eta/s \approx 0.16, \text{ respectively} \ [1, 2]. \]

Under the assumption that the average \( \eta/s \) is similar for collisions at RHIC and the LHC (along with the assumptions discussed above), one can make a prediction for the integrated elliptic flow coefficient for charged hadrons as a function of impact parameter (or more customarily the number of participants \( N_{\text{part}} \), cf. [1]). The result is shown in Figure 5.2. As can be seen, we expect integrated \( v_2 \) at the LHC to be about ten percent larger than at RHIC, which is less than the increase predicted by ideal hydrodynamics [139], and in agreement with the extrapolations by Drescher et al. [131]. For comparison, Figure 5.3(a) shows these LHC prediction curves along with those with \( \eta/s \) set to 0.0001, corresponding to ideal hydrodynamics and illustrating the larger value of \( v_2 \) predicted by ideal hydrodynamics. Also, as can be seen in Figure 5.3(b), remaining uncertainty in the equation of state seems to have little effect on this prediction.
Finally, using the charge density parametrization of the proton $\rho(b)$ in Ref. [140] as an equivalent of the nuclear thickness function in the Glauber model (cf. [1]) one obtains an estimate for the shape of the transverse energy density following a relativistic p+p collision. Using the predicted multiplicity at mid-rapidity $dN/dY \sim 6$ [129, 132] for $\sqrt{s} = 14$ TeV p+p collisions at the LHC, one can again use Equation 5.4 to infer the overall normalization of the energy density (or $T_i$) at $\tau = \tau_0$ (see Table 5.1). As a “Gedankenexperiment” one can then ask how much elliptic flow would be generated in LHC p+p collisions if the subsequent evolution was well approximated by boost-invariant viscous hydrodynamics. One finds that for ideal hydrodynamics $\frac{dN}{dY} \sim v_2 \sim 0.035$ for integrated $|v_2|$ in minimum bias collisions (cf. (28) in [1]), while for $\eta/s \geq 0.08$, $v_2$ typically changes by almost 100 percent when varying the relaxation time $\tau_0 (\epsilon + p)$ between 2.6 and 6 and varying $\tau_0$ by a factor of two (see Figure 5.4). This indicates that for $\eta/s \geq 0.08$, the hydrodynamic gradient expansion does not converge and as a consequence it is unlikely that elliptic flow develops in p+p collisions at top LHC energies. If experiments find a non-vanishing value for integrated $|v_2| > 0.02$ in minimum bias p+p collisions, this would be an indication for an extremely small viscosity $\eta/s < 0.08$ in deconfined nuclear matter.
Figure 5.4: Bands encompassing the calculated anisotropy curves for $\sqrt{s} = 14$ TeV p+p collisions. The relaxation time $\frac{\tau}{\eta}(\epsilon + p)$ was varied between 2.6 and 6 and $\tau_0$ from 0.5 fm to 2.0 fm for each value of $\eta/s$. (Note that much of the $\eta/s = 0.08$ band is obscured by the $\eta/s = 0.16$ band, as both have significant dependence on the relaxation time and $\tau_0$.)

To conclude, viscous hydrodynamics can be used to make predictions for the ratio of $v_2/e_x$ as a function of multiplicity and $\eta/s$. Assuming a multiplicity of $\frac{dN_{ch}}{dY} \simeq 1800$ for the matter created in Pb+Pb collisions at LHC, as well as $\eta/s$ similar to RHIC, we predict the integrated elliptic flow for charged hadrons to be 10% larger at the LHC than at RHIC. We expect $v_2$ measurements in p+p collisions to be consistent with zero, unless the shear viscosity is extremely small ($\eta/s < 0.08$).
Figure 5.5: Equation of state from Laine and Schröder [73]—used for all the main results of chapters 4 and 5—and from Bazavov et al. ([135], p4+HRG)—used to generate the circles in Figure 5.3(b). Above are the energy density (a) and pressure (b)—scaled by $T^4$—as a function of temperature. Below (c) is the speed of sound squared.
Chapter 6

FINAL STATE INTERACTIONS AND THE DISTORTED WAVE EMISSION FUNCTION

Although much of the work in this chapter was performed before the viscous hydrodynamic simulations described previously, it is most naturally described here, after some understanding of hydrodynamic simulations has been obtained. This is not meant to be a comprehensive discussion of final state interactions. It is only a description of one line of inquiry with which the author of this dissertation participated—namely the distorted wave emission function (DWEF) model.

6.1 The RHIC HBT Puzzle

Despite the success of early ideal hydrodynamical simulations of heavy ion collisions at RHIC, they had much difficulty fitting two-particle correlations while simultaneously matching single-particle data. For example, when the simulation parameters were set such that the experimental multiplicity and mean transverse momentum were at least roughly reproduced, the predictions for Hanbury Brown and Twiss radii (recall section 2.2.2) $R_O$ and $R_L$ are too large, while $R_S$ is too small. In all reasonable cases, it seemed, the emission of particles occurred over a relatively long time period, causing the ratio $R_O/R_S$ to be large. The experimental result, however, showed $R_O/R_S \approx 1$.

This difficulty of describing HBT data with otherwise successful methods was dubbed the “RHIC HBT puzzle” [141]. It should be noted that—although adding viscosity improves the agreement of $R_O/R_S$ in particular—even the most recent viscous hydrodynamic simulations do not completely resolve this puzzle [47]. Shedding light on this was the original motivation behind the development of the distorted wave emission function (DWEF) model, in which final state interactions are introduced in the form of a relativistic optical model formalism [142, 143]. Recent discussions of this puzzle can be found in Refs. [144, 145].
6.2 The DWEF Model

The medium produced in an ultrarelativistic heavy ion collision is very hot and dense. It may be that the particles emitted from this opaque source experience significant refractive effects, which in turn affect measured quantities such as HBT radii in a way that cannot be captured by hydrodynamic models with only a simple Cooper-Frye freeze out mechanism. With this in mind, let us introduce a model for these final state interactions and develop the formalism for calculating various quantities within this model. Then by varying free parameters in the model one can show that it is possible to fit both single- and two-particle data for pions at RHIC, and then analyze the meaning of the values of the parameters necessary to fit the data.

The DWEF formalism (along with many of the results) is laid out in detail in Ref. [142]. Here we first present a general derivation of the formalism, in the manner of Ref. [4], and then we will describe the specific choices for the analytic form of the optical potential and emission function.

6.2.1 Plane wave formalism

For comparison, it is useful to start by deriving the formalism for calculating HBT radii in the absence of final state interactions, given some source function that represents particles that are emitted (incoherently) from the collision medium and which then travel without interaction into the detectors. In principle one could obtain this source function from some Cooper-Frye freeze out surface, but here it will be just a given function (and later an analytic parametrization with tunable parameters). We follow one of the previous derivations [146] and then we can alter it appropriately to add an optical potential that the emitted particles interact with. For simplicity we will specifically consider pions, the dominate hadron produced in a heavy ion collision. The extension to other particles is straightforward.

We want to calculate the correlation function $C(p, q)$

$$C(p, q) \equiv \frac{P(p, q)}{P(p)P(q)},$$  \hspace{1cm} (6.1)

where $P(p_1, \cdots, p_n)$ is the probability of observing pions of momentum $\{p_i\}$ all in the same
event. The identical nature of all pions of the same charge cause \( C(\mathbf{p}, \mathbf{p}) = 2 \).

A state created by a random pion source \(|\eta\rangle\) is described by

\[
|\eta\rangle = \exp \left[ \int d^4 x \eta(x) \hat{\psi}^\dagger(x) \right] |0\rangle = \exp \left[ \int d^3 p d t \eta(\mathbf{p}, t) \gamma(t) c^\dagger(\mathbf{p}) e^{-iE_p t} \right] |0\rangle,
\] (6.2)

where \( \hat{\psi}^\dagger \) is the pion creation operator in the Heisenberg representation, \( \gamma(t) \) is the random phase factor that accounts for the chaotic nature of the source and \( c^\dagger(\mathbf{p}) \) is the creation operator for a pion of momentum \( \mathbf{p} \). In particular, an average over collision events gives

\[
\langle \gamma^*(t) \gamma(t') \rangle = \delta(t - t'),
\] (6.3)

and

\[
\langle \gamma^*(t_1) \gamma^*(t_2) \gamma(t_3) \gamma(t_4) \rangle = \delta(t_1 - t_3) \delta(t_2 - t_4) + \delta(t_1 - t_4) \delta(t_2 - t_3).
\] (6.4)

We note that as written, the state \(|\eta\rangle\) is not normalized to one. However, the normalization constant will divide out of the numerator and denominator of the correlation function. Therefore we do not make the normalization factor explicit here, but note that it enters when calculating the pion spectrum.

For \( \hat{\psi} \) and its time derivative to obey the Heisenberg commutation relation one has

\[
\sqrt{E_p E_{p'}} [c(\mathbf{p}), c^\dagger(\mathbf{p'})] = \delta^{(3)}(\mathbf{p} - \mathbf{p'}). \] (6.5)

Furthermore, we define

\[
\eta(\mathbf{p}, t) \equiv \int d^3 x e^{-i \mathbf{p} \cdot \mathbf{x}} \eta(x).
\] (6.6)

The state \(|\eta\rangle\) is an eigenstate of the destruction operator in the Schroedinger representation, \( c(\mathbf{p}) \):

\[
c(\mathbf{p}) |\eta\rangle = \int dt e^{iE_p t} \frac{\eta(\mathbf{p}, t)}{E_p} \gamma(t) |\eta\rangle.
\] (6.7)

The correlation function is

\[
C(\mathbf{p}, \mathbf{q}) = \frac{\langle \eta | c^\dagger(\mathbf{p}) c^\dagger(\mathbf{q}) c(\mathbf{q}) c(\mathbf{p}) |\eta\rangle}{\langle \eta | c^\dagger(\mathbf{p}) c(\mathbf{p}) |\eta\rangle \langle \eta | c^\dagger(\mathbf{q}) c(\mathbf{q}) |\eta\rangle}.
\] (6.8)

The use of Equation 6.4 and Equation 6.7 in the numerator of Equation 6.8 yields

\[
\langle \eta | c^\dagger(\mathbf{p}) c^\dagger(\mathbf{q}) c(\mathbf{q}) c(\mathbf{p}) |\eta\rangle = \langle \eta | c^\dagger(\mathbf{p}) c(\mathbf{p}) |\eta\rangle \langle \eta | c^\dagger(\mathbf{q}) c(\mathbf{q}) |\eta\rangle + |\langle \eta | c^\dagger(\mathbf{p}) c(\mathbf{q}) |\eta\rangle|^2.
\] (6.9)
Furthermore

\[ \langle \eta | c\dagger(p) c(q) | \eta \rangle = \int dt \exp[-i(E_p - E_q)t] \frac{\eta^\dagger(p, t) \eta(q, t)}{E_p E_q}. \]  

(6.10)

The quantity \( g(x, p) \) is denoted the emission function and is defined as

\[ g(x, p) = \int d^3 x' \eta^\dagger(x + \frac{1}{2} x', t) \eta(x - \frac{1}{2} x', t) e^{ip \cdot x'}. \]  

(6.11)

so that

\[ \int d^3 p \frac{(2 \pi)^3}{2} g(x, p) e^{-ip \cdot z} = \eta^\dagger(x + \frac{1}{2} z, t) \eta(x - \frac{1}{2} z, t) \]  

(6.12)

\[ \int d^3 p \frac{(2 \pi)^3}{2} g((y + y')/2, t, p) e^{-ip \cdot (y - y')} = \eta^\dagger(y, t) \eta(y', t). \]  

(6.13)

The second expression appears in the right-hand-side of Equation 6.10 (if one uses Equation 6.6) so that we may write

\[ \langle \eta | c\dagger(p) c(q) | \eta \rangle = \int d^4 x \exp[-i(p - q) \cdot x] \frac{g(x, (p + q)/2)}{E_p E_q}. \]  

(6.14)

Using Equation 6.14 with \( p = q \) shows that the function \( g(x, p)/E_p^2 \) is the probability of emitting a pion of momentum \( p \) from a space-time point \( x \). Using Equation 6.9 and Equation 6.14 in Equation 6.8 gives the desired expression:

\[ C(p, q) = 1 + \frac{1}{E_p E_q} \int d^4 x \int d^4 x' g(x, \frac{1}{2} K) g(x', \frac{1}{2} K) \exp[-i k \cdot (x - x')] / \int d^4 x \int d^4 x' g(x, p) g(x', q), \]  

(6.15)

where \( K \equiv p + q \) and \( k \equiv (E_p - E_q, p - q) \), and the factors of \( \frac{1}{E_p E_q} \) have canceled out.

From a formal point of view, a key step in the algebra is the relation between the Heisenberg representation pion creation operator \( \hat{\psi}\dagger(x) \) and its momentum-space Schrödinger representation counterpart \( c\dagger(p) \) that appears in Equation 6.2

\[ \hat{\psi}\dagger(x) = \int d^3 p \ c\dagger(p) \frac{e^{-i p \cdot x}}{(2\pi)^3/2} e^{i E_p t} \]  

(6.16)

\[ \hat{\psi}(x) = \int d^3 p \ c(p) \frac{e^{i p \cdot x}}{(2\pi)^3/2} e^{-i E_p t} \]  

(6.17)

The operators \( c\dagger(p) \) (\( c(p) \)) are coefficients of a plane wave expansion for \( \hat{\psi}\dagger(x) \) (\( \hat{\psi}(x) \)), with the plane wave functions \( e^{i p \cdot x}/(2\pi)^3/2 \) being the complete set of basis functions. However, one could re-write \( \hat{\psi}\dagger(x) \) (\( \hat{\psi}(x) \)) as an expansion using any set of complete wave functions. We shall exploit this feature below.
6.2.2 Distorted waves—real potential

We represent the random classical source, emitting pions that interact with a real, time-independent external potential $U$ by the Lagrangian density:

$$- \mathcal{L} = \hat{\psi}^\dagger (-\partial^2 + U + m^2) \hat{\psi} + j(x) \hat{\psi}.$$  \hspace{1cm} (6.18)

The current operator $j(x)$ is closely related to the emission function $g$. In this Lagrangian the terms $U$ and $j(x)$ are independent. Thus the relation between the emission function and $U$ derived in [148] need not be satisfied. Also note that the medium—and therefore also the potential—is in principle a time-dependent quantity. Nevertheless, for simplicity we take $U$ to be time independent and it can be interpreted as a time-averaged quantity.

The field operator $\hat{\psi}^\dagger$ can be expanded in the mode functions $\psi_p^{(-)}$ that satisfy:

$$(-\nabla^2 + U) \psi_p^{(-)}(x) = p^2 \psi_p^{(-)}(x).$$  \hspace{1cm} (6.19)

These wave functions obey the usual completeness and orthogonality relations

$$\int d^3p \psi_p^{(-)*}(x) \psi_p^{(-)}(y) = \delta^{(3)}(x - y)$$  \hspace{1cm} (6.20)

$$\int d^3x \psi_p^{(-)*}(x) \psi_{p'}^{(-)}(x) = \delta^{(3)}(p - p'),$$  \hspace{1cm} (6.21)

so that one may use the field expansion

$$\hat{\psi}(x) = \int d^3p \psi_p^{(-)}(x,t) e^{-iE_p t} d(p),$$  \hspace{1cm} (6.22)

with $d^\dagger(p)$ being the creation operator for pions of momentum $p$ in the basis of Equation 6.19.

These mode functions are the eponymous distorted waves which replace the plane waves of the previous section. The expansion Equation 6.22 assumes that $U$ produces no bound states. If they did exist, the integral term would be augmented by a term involving a sum over discrete states.

The availability of mode expansions when distortion effects are included means that the simplification of the correlation function can proceed as in the previous section. We again
use Equation 6.2 and Equation 6.4. The use of the field expansion Equation 6.22 enables a generalization of the function $\eta(x)$:

$$\eta(x) = \int \, d^3 p \, \psi_p^{(-)}(x, t) \bar{\eta}(p, t), \quad (6.23)$$

with

$$\bar{\eta}(p, t) \equiv \int \, d^3 x \, \psi_p^{(-)}(x, t) \eta(x), \quad (6.24)$$

so that

$$|\eta \rangle = \exp \left[ \int \, d^3 p \, dt \, \bar{\eta}(p, t) \gamma(t) d^d(p) \right] |0 \rangle. \quad (6.25)$$

Note that the ability to obtain a relation between the $\bar{\eta}(p, t)$ and $\eta(x)$ rests on the relations Equation 6.20 and Equation 6.21.

The state $|\eta \rangle$ is an eigenstate of $d(p)$. Thus the result

$$C(p, q) = 1 + \frac{|\langle \eta | d^d(p) d(q) | \eta \rangle|^2}{\langle \eta | d^d(p) d(p) | \eta \rangle \langle \eta | d^d(q) d(q) | \eta \rangle}, \quad (6.26)$$

very similar to Equation 6.8 is obtained. We need the matrix elements appearing in the numerator and find

$$\langle \eta | d^d(p) d(q) | \eta \rangle = \int \, d^3 x \, d^3 x' \, \frac{\exp[-it(E_p - E_q)]}{E_p E_q} \psi_p^{(-)}(x) \psi_q^{(-)*}(x') \eta(x) \eta(x', t). \quad (6.27)$$

and the use of Equation 6.13 allows us to obtain

$$\langle \eta | d^d(p) d(q) | \eta \rangle = \int \, dt \, d^3 x \, d^3 x' \frac{d^d p'}{(2\pi)^d} e^{it(E_q - E_p)} e^{-i p' \cdot x'} \times \psi_p^{(-)}(x + x'/2) \psi_q^{(-)*}(x - x'/2) g(x, p'). \quad (6.28)$$

This result, which can be applied for $p \neq q$ and for $p = q$, specifies the evaluation of the correlation function of Equation 6.26 with the result

$$C(p, q) = 1 + \frac{|S(K, k)|^2}{S(p) S(q)} \quad (6.29)$$

where

$$S(K, k) \equiv \int \, d^4 x \, d^3 x' \frac{d^d p'}{(2\pi)^d} e^{it(E_q - E_p')} e^{-i p' \cdot x'} \psi_p^{(-)}(x + x'/2) \psi_q^{(-)*}(x - x'/2) g(x, p'), \quad (6.30)$$
and
\[
S(p) \equiv \int d^4 x \, d^3 x' \frac{d^3 p'}{(2\pi)^3} e^{-i p' \cdot x'} \psi_p^- (x + x'/2) \psi_p'^{(-)*} (x - x'/2) g(x, p').
\] (6.31)

This is the expression that is used in the DWEF formalism [142, 143]. In principle one could use either Equation 6.15 or Equation 6.29 to analyze data, but the extracted space time properties of the source \(\eta(x)\) would be different.

A comment should be made on the possible momentum and energy dependence of the optical potential. The completeness and orthogonality relations are obtained with any Hermitian \(U\) which can therefore be momentum dependent, but not energy dependent. As explained in section 5 (Equation 43) of Ref. [142], the real part of the potential can and should be thought of as a momentum-dependent, but energy-independent potential. If there were true energy dependence a factor depending on the derivative of the potential with respect to energy [149] would enter into the orthogonality and completeness relations.

### 6.2.3 Coupled channels

If the optical potential \(U\) from the previous section is complex, the derivation above fails. Using the necessary completeness and orthogonality relations to relate \(\eta(x)\) to \(\bar{\eta}(p, t)\) requires the use of a real potential. If we would like to include the effects of an imaginary part of the potential, we should investigate possible corrections to the above formalism.

The optical potential or pion self-energy is an effective interaction between the pion and the medium. The medium is not an eigenstate of the Hamiltonian, but rather of \(H_0\), which is the full Hamiltonian minus the Hermitian operator representing the pionic final state interactions. Eliminating the infinite number of possible states of \(H_0\) and representing these by a single state leads to a self-energy that is necessarily complex. Our procedure here is to specifically consider the infinite number of states of the medium, obtain a Lagrangian density that involves Hermitian interactions, and derive the optical potential formalism and any corrections to it.

Let \(P_n\) denote a projection operator for the medium to be in a given eigenstate of \(H_0\),
These obey

\[ \sum_n P_n = 1, \]
\[ P_n P_m = \delta_{n,m} P_n. \]  

(6.32)

For the case of \( \pi \)-nuclear scattering, \( n \) would represent the nuclear eigenstates. Here \( n \) represents states of the medium in the absence of its interactions with pions. The correlation function is now given by

\[ C(p, q) = \frac{\sum_n P_n(p, q)}{\sum_n P_n(p) \sum_m P_m(q)} \]  

(6.33)

where \( P_n(p) \) is the probability for emission of a pion of momentum \( p \) from the medium in a state \( n \). Similarly \( P_n(p, q) \) is the probability for emission of a pair of pions of momentum \( p, q \) from the medium in a state \( n \). The sums over \( n \) account for the inclusive nature of the process of interest.

It is convenient to define the product of the field operator with the projection operator \( P_n \):

\[ \hat{\psi}_n(x) \equiv \hat{\psi}(x) P_n, \]  

(6.34)

with

\[ \hat{\psi}(x) = \sum_n \hat{\psi}_n(x), \]  

(6.35)

using the complete nature of the set \( n \). The Lagrangian density is given by

\[ -L = \sum_n \partial \hat{\psi}_n^\dagger \cdot \partial \hat{\psi}_n + \sum_{n,m} \hat{\psi}_n^\dagger \left( (m^2_p + M^2_m) \delta_{nm} + U_{nm} \right) \hat{\psi}_m + \sum_n j_n(x) \hat{\psi}_n(x), \]  

(6.36)

where

\[ U_{nm} = U_{mn}^* \equiv (\hat{U})_{nm} \]  

(6.37)

and \( \hat{U} \) is the Hermitian interaction operator and \( M^2_m \), the \( m \) matrix element of the diagonal operator \( M^2 \), represents the effects of the different energies of the states labeled by \( m \). The field operator \( \hat{\psi}_n \) can be expanded in the mode functions \( \psi_{p,m}^{(-)} \):

\[ \sum_{m \neq n} U_{nm}(x) \psi_{p,m}^{(-)}(x, t) = (p^2 + \nabla^2 - M^2_n - U_{nm}(x)) \psi_{p,n}^{(-)}(x, t). \]  

(6.38)
Here the potential $U$ is taken as a local operator in the position space of the outgoing pion.

To see the correspondence between the formulation of Equation 6.36 and Equation 6.38, let $\hat{\psi}_1$ correspond to the field operator (and state) of the previous section and solve formally for $\psi^{(-)}_{p,0}$ in terms of $\psi^{(-)}_{p,1}$. It is convenient to define the operator $\tilde{U}$ with matrix elements given by

$$\tilde{U}_{n,n'} \equiv (1 - \delta_{n,1})(1 - \delta_{n',1})U_{n,n'} \tag{6.39}$$

Then

$$\psi^{(-)}_{p,n \neq 1} = \sum_{m \neq 1} \left( \frac{1}{\nabla^2 + p^2 - M^2 - \tilde{U} - i\epsilon} \right)_{nm} U_{m1}\psi^{(-)}_{p,1}, \tag{6.40}$$

where $$(\nabla^2 + p^2 - M^2)_{nm} \propto \delta_{n,m},$$ and $M^2$ is an operator giving $M_n^2$ when acting on the state $n$. Then rewrite Equation 6.38 in terms of $\psi^{(-)}_{p,1}$ as

$$U_{11}\psi^{(-)}_{p,1} + \sum_{m,n \neq 1} U_{1n} \left( \frac{1}{\nabla^2 + p^2 - M^2 - \tilde{U} - i\epsilon} \right)_{nm} U_{m1}\psi^{(-)}_{p,1} = (p^2 + \nabla^2 - M_n^2)\psi^{(-)}_{p,1} \tag{6.41}$$

The complex object

$$U_{11} + \sum_{m,n \neq 1} U_{1n} \left( \frac{1}{\nabla^2 + p^2 - M^2 - \tilde{U} - i\epsilon} \right)_{m,n} U_{n1},$$

a non-local operator in coordinate space, can be identified with the optical potential, given by the operator $V(Z)$ as a function of a complex variable $Z$:

$$V(Z) = U_{11} + \sum_{m,n \neq 1} U_{1n} \left( \frac{1}{\nabla^2 + Z - M^2 - \tilde{U}} \right)_{m,n} U_{n1}. \tag{6.42}$$

We proceed by employing Equation 6.35 and Equation 6.36 to compute the correlation function. The solutions of Equation 6.38 form a complete orthogonal set:

$$\sum_n \int d^3p \psi^{(-)*}_{p,n}(x) \psi^{(-)}_{p,n}(y) = \delta^{(3)}(x - y) \tag{6.43}$$

$$\sum_n \int d^3x \psi^{(-)*}_{p,n}(x) \psi^{(-)}_{p',n}(x) = \delta^{(3)}(p - p'). \tag{6.44}$$

The field expansion is now

$$\hat{\psi}(x) = \int d^3p \sum_n a(p) P_n \varphi^{(-)}_{p,n}(x)e^{-iE_pt}, \tag{6.45}$$
so that

$$|\eta\rangle = \exp \left[ \sum_n \int d^4 x \eta_n(x) \gamma(t) \int d^3 p \, a^\dagger(p) P_n \psi^{(-)}_{p,n}(x) e^{i E_p t} \right] \sum_m |0, m\rangle, \quad (6.46)$$

where the state $|0, m\rangle$ is the pionic vacuum if the medium is in the state $m$, and $\eta_n(x)$ represents the source for the state $n$. These state vectors obey the relations

$$\langle 0, n | 0, m \rangle = \delta_{n,m} = \langle 0, n | P_n | 0, m \rangle. \quad (6.47)$$

Define

$$\eta_n(p, t) \equiv \int d^3 x \, \eta_n(x, t) \psi^{(-)*}_{p,n}(x), \quad (6.48)$$

so that

$$|\eta\rangle = \exp \left[ \int d^3 p \, dt \, \gamma(t) \sum_n \eta_n(p, t) a^\dagger(p) P_n e^{i E_p t} \right] \sum_m |0, m\rangle, \quad (6.49)$$

$$a(p)|\eta\rangle = \int dt \, \gamma(t) \sum_n \frac{\eta_n(p, t)}{E_p} P_n e^{i E_p t} |\eta\rangle. \quad (6.50)$$

The emission probability is given by

$$E_p E_q \sum_n \langle \eta | a^\dagger(p) P_n a(q) | \eta \rangle = \sum_n \int dt \, d^3 x \, d^3 y \, \eta^*_n(x, t) \eta_n(y, t) \times \psi^{(-)*}_{p,n}(x) \psi^{(-)}_{q,n}(y) e^{i (E_p - E_q) t} \quad (6.51)$$

or using Equation 6.13

$$E_p E_q \langle \eta | a^\dagger(p) a(q) | \eta \rangle = \sum_n \int dt \, d^3 x \, d^3 y \int d^3 p' \, g_n((x + y)/2, t, p') e^{-i p' \cdot (x - y)} \times \psi^{(-)*}_{p,n}(x) \psi^{(-)}_{q,n}(y) e^{i (E_p - E_q) t}, \quad (6.52)$$

where

$$g_n(x, p) = \int d^3 x' \, \eta^*_n(x + \frac{1}{2} x', t) \eta_n(x - \frac{1}{2} x', t) e^{i p \cdot x'}. \quad (6.53)$$

If pionic final state interactions are ignored, the term $\sum_n g_n$ enters and this may be identified with the emission function, $g$ of previous sections.
The expression Equation 6.52 is the same as Equation 6.28 except that now we sum over the channels \( n \). These sums may be expressed in terms of the optical model wave functions of Equation 6.40. The term of Equation 6.52 with \( n = 1 \) corresponds to the DWEF formalism, and the terms with \( n > 1 \) are corrections. As an example of a correction term, suppose part of the imaginary part of the optical potential arises from a pion-nucleon interaction that makes an intermediate \( \Delta \). Then a term corresponding to one of \( n > 1 \) involves the emission of a pion from a nucleon that makes an intermediate \( \Delta \).

It is difficult to assess the importance of the second term in a general way. The only obvious limit is that if states with \( n > 1 \) are not excited then \( Im(V) \) of Equation 6.42 must vanish. Conversely, if \( Im(V) = 0 \), the states \( n > 1 \) must be above the threshold energy and the propagators that appear in the correction terms correspond to virtual propagation over a small distance with limited effect.

Therefore in the following, results will be presented with the imaginary part of of the optical potential set at a vanishing value, and separately with any value allowed. While the former is correct within the model, the latter will have unknown corrections. Nevertheless, it will still be illustrative to look at both in the hope that it will give some idea of the effect of an imaginary part of the optical potential in addition to the more reliable information concerning the real part.

### 6.2.4 Complete DWEF formalism

The key unknown pieces in the expressions above are the emission function \( g \) and the optical potential \( U \). The wavefunctions \( \psi_p^{(-)} \) can be calculated from \( U \), and then integrals can be performed to obtain the quantities of interest. Reiterating the results of section 6.2.2, the correlation function for determining HBT radii is given by (recall Equation 6.29)

\[
C(p, q) = \frac{\left[ \int d^4x d^3x' d^3p' \frac{e^{it(E_q - E_{p'})}}{(2\pi)^3} e^{-ip' \cdot x'} \psi_p^{(-)}(x + x'/2)\psi_q^{(-)*}(x - x'/2)S_0(x, p') \right]^2}{\left( \int d^4x d^3x' d^3p' \frac{e^{-ip' \cdot x'}}{(2\pi)^3} \psi_p^{(-)}(x + x'/2)\psi_q^{(-)*}(x - x'/2)S_0(x, p') \right) (p \to q)},
\]

(6.54)

and single particle observables can be derived from one of the factors in the denominator:

\[
E \frac{dN}{dp} = \int d^4x d^3x' d^3p' \frac{e^{-ip' \cdot x'}}{(2\pi)^3} \psi_p^{(-)}(x + x'/2)\psi_p^{(-)*}(x - x'/2)S_0(x, p').
\]

(6.55)
(In this section, the notation $S_0(x,p) \equiv g(p,x)$ is used to make contact with the notation of Ref. [142]).

We proceed by using an analytic parametrization that is inspired by hydrodynamic freeze out. A more detailed discussion can be found in Ref. [142].

The form used is:

$$g(p,x) \equiv S_0(x,p) = \cosh \eta \frac{e^{-\eta^2}}{(2\pi)^3} \frac{1}{\sqrt{2\pi} \Delta\tau^2} e^{-\frac{(\tau-\eta)^2}{2\Delta\tau^2}} \frac{M_\perp \rho(b)}{e^{(p-\mu)/T} - 1}, \quad (6.56)$$

$$U(b) = -(w_0 + w_2 p^2) \rho(b). \quad (6.57)$$

$p$ is the asymptotic pion momentum and $M_\perp = \sqrt{p_\perp^2 + m_\pi^2}$. Just as in previous chapters, it is natural to use Milne coordinates, although here we will use radial coordinates in the transverse plane:

$$\eta = \text{arctanh}(z/t), \quad \tau = \sqrt{t^2 - z^2}, \quad b = \sqrt{x^2 + y^2},$$

$$\phi = \text{arctan}(y/x), \quad b = (b, \phi). \quad (6.58)$$

Also as above, we will restrict ourselves to mid-rapidity data.

$\rho(b)$ represents the transverse density of the medium and is used for the transverse shape of both the emission function and the optical potential. It is normalized as $\rho(0) = 1$. The original DWEF model was restricted to rotationally symmetric systems (corresponding to central collisions) and used

$$\rho(b) = \rho(b) = \left[ e^{-R_{WS}/a_{WS}} + 1 \right]^2 \left[ e^{(b-R_{WS})/a_{WS}} + 1 \right]^2. \quad (6.59)$$

This distribution has the correct exponential fall-off at large distance, and different choices of the parameters $R_{WS}$ and $a_{WS}$ allow for a variety of shapes. To calculate the elliptic flow coefficient $v_2$, it will be necessary to generalize this form for non-rotationally-symmetric systems (see chapter 7).

The velocity field that describes the dynamics of the expanding source in a central collision event is parametrized by a fluid rapidity $\eta_t(b)$

$$u^\mu(x) = (\cosh \eta_\tau \cosh \eta_t, \ \sinh \eta_\tau \cos \phi, \ \sinh \eta_\tau \sin \phi, \ \sinh \eta_\tau \cosh \eta_t). \quad (6.60)$$
The flow rapidity is taken to have a linear radial profile with strength $\eta_f$

$$\eta_f(b) = \eta_f \frac{b}{R_{WS}}.$$  \hspace{1cm} (6.61)

This also will have to be modified when calculating elliptic flow for a peripheral collision.

The free parameters, then, are $\Delta \eta$, $\Delta \tau$, $\tau_0$, $\mu_x$, $T$, $w_0$, $w_2$, $R_{WS}$, $a_{WS}$, and $\eta_f$. These parameters were varied (with various of them occasionally held fixed) to reproduce the single- and two-particle pion data for $\sqrt{s} = 200$ GeV Au-Au collisions at RHIC [142].

### 6.2.5 Results for Central Collisions

Calculations for central RHIC collisions were originally presented in Refs. [142, 143]. With the above insight, it is instructive to assess the possible importance of the imaginary part of the optical potential.

A variety DWEF fits are performed, see Table 6.1. In Ref. [142] the imaginary part of the optical potential as represented by the term $w_2$ is about one tenth of the real potential. It is therefore possible that, in the limit that $\text{Im}(w_2) = 0$, there would be no significant correction term, so we try to understand if removing the imaginary part of the optical potential can be done without degrading the quality of the fit. The results are shown in Figures 6.1 and 6.2. An example of the previous calculations [142, 143] is shown as the green dotted curve (second line of Table 6.1). The red solid curve (first line of Table 6.1) shows the result of setting the imaginary potential to a vanishingly small value. This results in only a slightly worse description of the data. The changes in the imaginary part of the optical potential $w_2$ are largely compensated by a reduction of the temperature from about 160 MeV to about 120 MeV. We also point out that the length of the flux tube as represented by $\Delta \eta$ is vastly increased, providing greater justification to our previous procedure of taking the length of the flux tube to be infinitely long in the longitudinal direction. However, the emission duration is reduced to 0 fm/c, which is similar to the results of the blast wave model [150]. This means that all of the pionic emission occurs at a single proper time. This value justifies the use of a time-independent optical potential, but does seem to be difficult to understand because some spread of emission times is expected for a long-lived plasma. The results shown by the blue dashed curves (third line of Table 6.1) are obtained
Table 6.1: Four parameter sets obtained with slightly different procedures [4]. The values of $\chi^2$ represents the accuracy of the description of the data.

| $T$ (MeV) | $\eta_f$ | $\Delta \tau$ | $R_{WS}$ (fm) | $a_{WS}$ (fm$^{-2}$) | $w_0$ | $w_2$ | $\tau_0$ | $\Delta \eta$ (fm) | $\mu_\pi$ (MeV) | $\chi^2$ |
|-----------|--------|-------------|---------------|----------------|------|------|--------|----------------|---------------|--------|
| 121       | 1.05   | 0           | 11.7          | 1.11           | 0.495| 0.762| +0.0001i| 9.20           | 70.7          | 139.57 |
| 162       | 1.22   | 1.55        | 11.9          | 1.13           | 0.488| 1.19+0.13i| 9.10           | 1.68          | 139.57 | 117   |
| 121       | 1.04   | 1.5         | 11.7          | 0.905          | 0.564| 0.595| +0.0001i| 8.85           | 70.7          | 139.57 | 451   |
| 144       | 0.990  | 2.07        | 12.57         | 0.876          | 0.0001| 0.0001+0.0001i| 6.85           | $\infty$     | 83.5   | 1068  |

with fixing the emission duration to 1.5 fm/c, which is our previous value [142, 143]. The description of the spectrum is basically unchanged but the radii are less precisely described. The violet long-dashed curves (fourth line of Table 6.1) show the DWEF fit using a vanishing optical potential. This does not give a good description of the momentum dependence of the radii and is associate with the largest deviation between our calculations and the data as represented by the $\chi^2$ values of Table 6.1.

It is clear that the precision of our description of the data is improved by including the imaginary part of the optical potential. However, this is a quantitatively but not a qualitatively important effect. It is also true that including the real part of the optical potential is a qualitatively important effect. These results suggest that the correction terms embodied by the terms with $n \neq 1$ of Equation 6.52 are not very important, but non-negligible. It is also possible that an optical potential with a different geometry than the volume form that we have assumed might be able to account for the the neglected terms. However, an accurate assessment would require the development a theory that involves dealing with explicit models for $g_{n, j_n}$ and $\mathcal{U}$. 
Figure 6.1: Computed pionic spectrum. Red upright triangles: $\pi^-$ spectrum (STAR) Green inverted triangles: $\pi^+$ spectrum points (STAR) [151] Red solid line: DWEF fit with vanishing imaginary part of the optical potential, first line of Table 6.1. Green dotted line: DWEF fit including search on the imaginary part of the optical potential, second line of Table 6.1. Blue dashed line(almost entirely covered by the red solid curve): DWEF fit with vanishing imaginary part of the optical potential, $\Delta \tau = 1.5 \text{ fm/c}$, third line of Table 6.1. Violet long dashed line: DWEF fit including search on $\mu_\pi$, setting the optical potential to essentially 0, fourth line of Table 6.1.
Figure 6.2: HBT radii. Curves are labeled as in Figure 6.1. STAR data [152]
Chapter 7

\( v_2 \) IN THE DWEF MODEL

Once results for central collisions have been calculated, the next interesting quantity is the elliptic flow coefficient \( v_2 \) in non-central collisions. A few pieces need to be generalized for a non-rotationally symmetric system. For the transverse density \( \rho \) we take the modified Woods-Saxon profile from Ref. [150].

\[
\rho(b) = \frac{(\exp[-(1)\frac{R_{WS}}{a_{WS}}] + 1)^2}{(\exp[(b\sqrt{\cos^2 \phi + \sin^2 \phi - 1})\frac{R_{WS}}{a_{WS}}] + 1)^2},
\]

with \( R_{WS} = \sqrt{\frac{1}{2}(R_x^2 + R_y^2)} \). Thus lines of constant density in the transverse plane form ellipses with semimajor to semiminor axis ratio \( \frac{R_x}{R_y} \).

Next we must generalize the fluid velocity \( u \), for which we again defer to Ref. [150].

\[
u_{\mu}(x) = (\cosh \eta \cosh \eta_t, \sinh \eta \cos \phi_b, \sinh \eta \sin \phi_b, \sinh \eta \cosh \eta_t). \]

(7.2)

The transverse direction is taken to be perpendicular to lines of constant density. It can be shown that the angle of such a fluid velocity, \( \phi_b \), obeys

\[
\phi_b(\phi) = \tan^{-1}\left(\frac{R_x^2}{R_y^2} \tan \phi\right).
\]

(7.3)

The transverse fluid rapidity \( \eta_t(b) \) is first taken to have the same elliptic symmetry as the density, increasing linearly with the “radial” coordinate \( \tilde{b} \equiv \sqrt{\frac{(b \cos(\phi))^2}{R_x^2} + \frac{(b \sin(\phi))^2}{R_y^2}} \). Then added to this is a term proportional to \( \cos(2\phi) \) representing the amount of elliptic flow built up before freezeout

\[
\eta_t(b) = \eta_f b \sqrt{\frac{\cos^2 \phi}{R_x^2} + \frac{\sin^2 \phi}{R_y^2}}(1 + a_2 \cos(2\phi)).
\]

(7.4)

The momentum in these coordinates takes the form

\[
p^\mu = (M_{\perp} \cosh Y, p_{\perp} \cos \phi_p, p_{\perp} \sin \phi_p, M_{\perp} \sinh Y). \]

(7.5)
Again we choose to focus on data at midrapidity, \( Y = 0 \), and so

\[
p \cdot u = M_\perp \cosh \eta \cosh \eta_t - p_\perp \sinh \eta_t \cos(\phi_b - \phi_p). \tag{7.6}
\]

Thus there are two extra parameters that characterize the departure from cylindrical symmetry. In all, the parameters involved in this model are: \( \Delta \eta, \Delta \tau, \tau_0, \mu_x, T, w_0, w_2, R_x, R_y, a_{WS}, \eta_f \), and \( a_2 \). Rather than rerunning the fit for peripheral collisions (which would be prohibitively difficult numerically) we will here be interested in the effect of an optical potential like those found to give the best fit to central collision data, and therefore we will only adjust \( \frac{R_y}{R_x} \) and \( a_2 \) to give reasonable results for non-central collisions.

### 7.1 Calculating \( v_2 \)

This section outlines how the calculation of \( v_2 \) is carried out. A set of coupled differential equations must be solved numerically to obtain the wavefunctions \( \psi_p^{(-)} \), and then a five-dimensional integral must be performed (two of which can be done analytically with suitable approximations.)

#### 7.1.1 The Wavefunctions \( \psi_p^{(-)}(x) \)

\( \psi_p^{(-)} \) satisfies [Equation 6.19]. Since \( U(b) \) is independent of \( t \) and \( z \), we can write

\[
\psi_p^{(-)}(x) \equiv e^{-i\omega_p t} e^{ip_z z} \psi_p^{(-)}(b), \tag{7.7}
\]

and [Equation 6.19] becomes

\[
\left( \nabla_\perp^2 - U(b) + p_\perp^2 \right) \psi_p^{(-)}(b) = 0, \tag{7.8}
\]

or

\[
\left( \frac{\partial^2}{\partial b^2} + \frac{1}{b} \frac{\partial}{\partial b} + \frac{1}{b^2} \frac{\partial^2}{\partial \phi^2} - U(b) + p_\perp^2 \right) \psi_p^{(-)}(b) = 0. \tag{7.9}
\]

Decomposing \( \psi_p^{(-)} \) and \( U(b) \) into angular moments

\[
\psi_p^{(-)}(b) = \sum_{m=-\infty}^{\infty} f_m(p,b)(-i)^m e^{im(\phi_b - \phi_p)}, \tag{7.10}
\]

\[
U(b) = \sum_{n} U_n(b) e^{in\phi}, \tag{7.11}
\]
results in

\[
\sum_m \left[ \left( \frac{\partial^2}{\partial b^2} + \frac{1}{b} \frac{\partial}{\partial b} - \frac{m^2}{b^2} + p_\perp^2 \right) f_m - \sum_n U_{m-n} i^n e^{im\phi_v} \right] e^{im\phi} e^{-im\phi_v} = 0. \tag{7.12}
\]

So the term in brackets vanishes identically for each \(m\), and we must solve a set of coupled differential equations. In practice, every \(f_m\) above a certain \(m_{\max}\) is set to zero, and a finite set of coupled equations is solved numerically.

The boundary conditions are the same as for the cylindrically symmetric case—far outside the medium one should have a canonically normalized plane wave plus an outgoing wave, i.e.

\[
f_m(b \gg R_{WS}) = J_m(p b) + T_m H_m^{(1)}(p b) \tag{7.13}
\]

with \(J_m\) and \(H_m^{(1)}\) Bessel functions and Hankel functions of the first kind, respectively.

Details of this calculation can be found in appendix A.1. The program used to calculate the wavefunctions was tested in part by comparing to a semi-analytic solution described in appendix A.2.

### 7.1.2 Integration

Once the wavefunctions are found, the integrals must be performed:

\[
v_2 \equiv \langle \cos(2\phi_p) \rangle = \frac{\int d\phi_p \cos(2\phi_p) S(p)}{\int d\phi_p S(p)}. \tag{7.14}
\]

with

\[
S(p) = \int d^4 x d^3 x' \frac{d^3 p'}{(2\pi)^3} e^{-ip' \cdot x'} \psi_p(-)(x + x'/2) \psi_p(-)^*(x - x'/2) S_0(x, p')
\]

\[
= \int \tau d\tau d\eta db d\phi d^3 x \frac{d^3 p'}{(2\pi)^3} e^{-ip' \cdot x'} \psi_p(-)(x + x'/2) \psi_p(-)^*(x - x'/2) S_0(x, p') \tag{7.15}
\]

Several approximations can make this more numerically tractable. If one assumes the the optical potential is approximately independent of the beam direction \(z\) as well as time, the \(\tau\) integral can be done analytically

\[
\int \tau d\tau e^{-\frac{(\tau - \tau_0)^2}{2\Delta^2}} = \sqrt{2\pi} \tau_0 \Delta. \tag{7.16}
\]
The $\eta$ integral can also be done analytically with the following approximations (as in Ref. [142])

$$e^{-\frac{\eta^2}{2\Delta^2}} \approx e^{-\frac{\cosh \eta}{\Delta^2}}$$

(7.17)

$$\frac{1}{e^{(p\cdot u - \mu_\pi)/T} - 1} \approx \sum_{j=1}^{j_{\text{max}}} e^{(-p\cdot u + \mu_\pi)j/T},$$

(7.18)

where the Bose-Einstein distribution is approximated by a sum over Boltzmann distributions truncated at some $j_{\text{max}}$, and so

$$\int d\eta \ c_2h \ c^{1/2} \ c^{\eta} \ e^{-\cosh \eta (\frac{1}{\Delta^2} + \frac{M_\perp}{T} \ c^{\eta})} = 2K_1 \left( \frac{1}{\Delta \eta^2} + \frac{j}{T} M_\perp \ c^{\eta} \right).$$

(7.19)

Finally, we use the large source approximation [142]

$$\psi^{(-)}_{p_i}(b + b'/2)\psi^{(-)*}_{p_j}(b - b'/2) g(b^2) \approx \psi^{(-)}_{p_i}(b)\psi^{(-)*}_{p_j}(b) g(b^2) \exp(iK_\perp \cdot b'),$$

(7.20)

with

$$g(b^2) = 2 \int d^2 K_\perp \ M_\perp \ \exp \left[ -\frac{M_\perp \ c^{\eta}}{T} \right] \ \exp \left[ -i K_\perp \cdot b' \right].$$

(7.21)

After implementing all these approximations, for the numerator we have

$$\int d\phi_p \ \cos(2\phi_p) \ \int d^4 x \ S(p, x)$$

$$= \frac{2 \tau_0 M_\perp}{(2\pi)^3} e^{1/\Delta^2} \sum_{m,n,j} i^{n-m} e^{\frac{j \ c^{\eta}}{T}}$$

$$\times \int d^2 b \ \rho(b) f_m(p, b) f_n^*(p, b) e^{i(m-n)\phi} K_1 \left( \frac{1}{\Delta \eta^2} + \frac{j}{T} M_\perp \ c^{\eta} \right)$$

$$\times \int d\phi_p \ \cos(2\phi_p) e^{-i(m-n)\phi_p} e^{\frac{j \ c^{\eta}}{T} \sinh(\eta) \ c^{\phi}} \ c^{\phi_p},$$

(7.22)

and similarly for the denominator. The final three integrals are done numerically.

More details of this part of the calculation can also be found in appendix A.

### 7.2 Results

We would like to determine the effect of adding final state interactions to hydrodynamic fits. To gain insight into this, we consider an emission function with parameter values taken from Refs. [4, 142], which give the best description of the single particle data in general,
Table 7.1: Best fit parameter sets. The top line (Fit 1) is a general fit \cite{142} while the bottom line (Fit 2) is from a fit where $Im(w_2)$ is held at 0.0001 \cite{4}.

|   | $T$   | $\eta_f$ | $\Delta \tau$ | $R_{WS}$ | $a_{WS}$ | $w_0$ | $w_2$ | $\tau_0$ | $\Delta \eta$ |
|---|-------|-----------|--------------|---------|---------|-------|-------|---------|-------------|
| Fit 1: | 156.58 | 1.310 | 2.0731 | 11.867 | 1.277 | 0.0693 | 0.856+i0.116 | 9.04 | 1.047 |
| Fit 2: | 121 | 1.05 | 0 | 11.7 | 1.11 | 0.495 | 0.762+i0.0001 | 9.20 | 70.7 |

and also with the imaginary part of the optical potential held at zero (see Table 7.1. Also note that in both fits the chemical potential was fixed at the pion mass).

We must make alterations to this central collision model to approximate a more peripheral collision. The results for a central collision do not unambiguously imply what a peripheral collision will look like without appealing to a particular model for the dynamics of the system. We therefore choose reasonable parameters to approximately represent a collision with impact parameter $\sim 7$ fm, and then see how the resulting $v_2$ depends on the strength of the optical potential. In principle one could vary all the parameters and do a separate fit of all the relevant experimental data (multiplicity, HBT radii, $v_2$, etc.) for each of various collision centralities. However, the computing time to do so would be prohibitive, and here we are most interested in investigating only the effect of the interactions, so we proceed as follows.

First, as in Ref. \cite{142}, we scale down $R_{WS}$, $a_{WS}$, and $\tau_0$ by the number of participants to the $1/3$ power, with $N_{part}$ taken from the Glauber model (with the same parameters used in Ref. \cite{1}) for an impact parameter of 0 and 7 fm ($N_{part} = 377.5$ and 171.544). Specifically $R_{WS} \rightarrow 0.7688 R_{WS}$. Then we adjust the ratio $\frac{R_y}{R_x}$ such that the spatial eccentricity

$$
\epsilon \equiv \frac{\langle y^2 \rangle - \langle x^2 \rangle}{\langle y^2 \rangle + \langle x^2 \rangle} = \frac{R_y^2 - R_x^2}{R_y^2 + R_x^2}
$$

has a value of 0.035. This is a reasonable value corresponding to the spatial eccentricity at freezeout of hydrodynamic fits of peripheral collisions with impact parameter $\sim 7$ fm. Note that the brackets in Equation 7.23 indicate a spatial average with weight given by Equation 7.1 while the spatial eccentricity in hydrodynamic simulations are typically
Figure 7.1: Calculated $v_2$ as a function of momentum with $a_2 = 0$ (a) and $a_2 = 0.11, 0.10$ (Fit 1, 2) (b). Points with error bars are experimental data for pions at 20–30% centrality from the STAR Collaboration [153].

given with respect to, e.g., energy density. We nevertheless keep the eccentricity from Equation 7.23 fixed at this value with an understanding that it is only a rough but still realistic guide to the shape.

Lastly we must specify how much elliptic fluid flow is built up in earlier stages of the collision, represented by the value of $a_2$ (recall Equation 7.4). First we set $a_2 = 0$ and see what $v_2$ is generated by interactions with the optical potential in the absence of significant elliptic fluid flow (Figure 7.1(a)). The calculated elliptic flow coefficient $v_2$ is plotted as a function of momentum, along with the relevant experimental data. (Note that $p$ in our calculation is the momentum of an asymptotically free pion detected far outside the medium, not the momentum of a particle as it is emitted inside the medium, and can therefore be compared directly to experiment.) Although we are only able to calculate up to a limited momentum, it is clear that final state interactions alone do not generate an appreciable value for $v_2$ for either the general best-fit parameters (Fit 1) or those with a vanishing imaginary part of the optical potential (Fit 2).

Next we increase $a_2$ such that the experimental value for $v_2$ is roughly obtained (Figure 7.1(b)). A value of $a_2 = 0.11$ was required for the parameters from Fit 1, while $a_2 = 0.10$ was sufficient to bring the emission function from Fit 2 into the physical regime. One can see that
the optical potential has a small but non-negligible effect—it decreases $v_2$ on the order of $10$–$25\%$ of its zero-interaction value with a slightly smaller effect as momentum increases.

### 7.3 Conclusion of DWEF $v_2$ Calculation

Final state interactions in the DWEF model were found to have a small, though not entirely insignificant effect on the elliptic flow coefficient $v_2$. This is in addition to the indirect effect of adding final state interactions. For example, adding an optical potential changes other observables such as the multiplicity, which would alter parameters in a hydrodynamic fit such as freezeout temperature, which would then in turn have an effect on the calculated value of $v_2$.

The precise size of these effects in general can only be determined with a better understanding of the model fits (e.g. Fit 1 versus Fit 2) in addition to a more detailed analysis—a full parameter search using all the relevant experimental data, or perhaps even by adding final state interactions directly into hydrodynamic simulations (i.e. a hydrodynamic afterburner in the vein of, e.g., Refs. [83, 84, 85, 86, 154]). It is reasonable, however, to conclude that final state interactions can affect the calculated value of $v_2$ by as much as $\sim 20\%$ (in agreement with other investigations of final state interactions, e.g., Ref. [84]), and so must be properly taken into account to have confidence in the quantitative predictions of hydrodynamic simulations at that level of precision.
Appendix 1

DETAILS OF DWEF $v_2$ CALCULATION

A.1 Numerical Implementation

A program was written in C++, making use of the GNU Scientific Library (GSL) version 1.9, to do the calculation of $v_2$, as detailed here.

The integral over the azimuthal angle of the pion momentum, $\phi_p$, is done as a sum using a simple trapezoid rule. This is because for each different value of $\phi_p$, a new set of differential equations must be solved. This also allows for the numerator and denominator of Equation 7.14 to be solved simultaneously, with just a factor of $\cos(2\phi_p)$ multiplied to the numerator when adding terms to the sum.

For each term in the sum, then, first the wavefunctions $\psi_p^{(-)}$ are obtained. They obey a set of coupled differential equations of the form

$$\left(\frac{\partial^2}{\partial b^2} + \frac{1}{b} \frac{\partial}{\partial b} - \frac{m^2}{b^2} + p_{\perp}^2\right)f_m - \sum_n U_{m-n} f_{m-n} \cdot e^{i n \phi_p} = 0 \quad (A.1)$$

for all integers $m$. This set is truncated, since large $m$ moments ($f_m$ for $m > p_{\perp} R_{ws}$) contribute little to the wavefunction. Therefore, all $f_m$ for $m$ greater than some $m_{\text{max}}$ are set to zero, leaving a finite $(2m_{\text{max}} + 1)$ number of coupled ordinary differential equations. These are solved by calling a GSL solver. Using an embedded Runge-Kutta-Fehlberg method seemed to give the best performance. For these solutions, Equation 7.1 is integrated numerically to find the moments $U_n$. This is done with the GSL adaptive integration routine for oscillatory functions.

To match to the proper boundary conditions, one must find $(2m_{\text{max}} + 1)$ linearly independent solutions to this set of equations and take the correct linear combination of these solutions that matches the desired boundary conditions. The straightforward choice for these linearly independent solutions is to sequentially solve for the case where only one of
the partial waves is non-zero near the origin. For example, for the n’th solution let:
\[
\begin{align*}
    f_m(b = b_{\text{min}} << \frac{1}{p}) &= \delta_{m,n} \\
    f'_m(b_{\text{min}}) &= \frac{m}{b} \delta_{m,n}
\end{align*}
\]  
(A.2)

and then solve the set of differential equations up to some arbitrarily large \( b_{\text{max}} \) far outside the potential. We can then match each partial wave in this \( n^{th} \) solution to the form:
\[
f_{m,n}(b_{\text{max}}) = A_{m,n} J_m(p b) + B_{m,n} H_m^{(1)}(p b).
\]  
(A.3)

The final wavefunction is then given by the linear combination of these solutions that matches the form of Equation 7.13 at \( b_{\text{max}} \):
\[
f_m(b) = \sum_n C_n f_{m,n}(b).
\]  
(A.4)

This part of the program was tested with the trivial case of zero optical potential, in addition to comparing to a separately written program that calculates only the cylindrically symmetric case, as well as to the results of the semi-analytical test case described in appendix A.2.

Once these wavefunctions are obtained and stored in memory, the integral over \( b \) and \( \phi \) in Equation 7.22 can be performed in addition to the sum over Boltzmann factors. The integrations are done with two GSL adaptive integration routines, one embedded in the other. The sum is done inside the argument of the integrals.

A.2 Semi-Analytic Test Case

To test the numerics, the case of a pion moving through an elliptically-shaped step-function potential was solved (semi-)analytically making use of elliptic coordinates. This can be compared to the case of \( a_{ws} \to 0 \) (see chapter 7).

We want to solve Equation 6.19 with \( U(b) \) an elliptically shaped step function—a finite potential inside an ellipse in the transverse plane, with zero potential outside.

It is useful to change to elliptic (cylindrical) coordinates, denoted \( u \) and \( v \). Think of \( u \) as a 'radial' coordinate that runs from 0 to \( \infty \) and \( v \) as an 'angular' coordinate that runs
from 0 to $2\pi$

$$x = a \cosh(u) \cos(v)$$
$$y = a \sinh(u) \sin(v).$$

(A.5)

Note the major and minor axes of the resulting confocal ellipses are reversed from the shape of the density used in the main calculation (which is larger in the $y$ direction). This is to maintain consistency with the conventional definition of elliptic coordinates. At the end one can simply take $\phi_p \rightarrow (\phi_p + \pi)$ to match the usual convention in RHIC papers.

Consider the case

$$U(b) = U(u) = U_0 \Theta(u_0 - u).$$

(A.6)

The sharp boundary at $u = u_0$ is an ellipse with major and minor axes

$$R_x = a \cosh(u_0)$$
$$R_y = a \sinh(u_0).$$

(A.7)

In this coordinate system the Laplacian is

$$\nabla^2_{\bot} = \frac{1}{a^2(\sinh^2(u) + \sin^2(v))} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)$$

(A.8)

and so Equation 6.19 becomes

$$\left[ \frac{1}{a^2(\sinh^2(u) + \sin^2(v))} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) - U(u) + p^2 \right] \psi_p(b) = 0$$

(A.9)

or equivalently

$$\left[ \frac{\partial^2}{\partial u^2} + 2q(u) \cosh(2u) + \frac{\partial^2}{\partial v^2} - 2q(u) \cos(2v) \right] \psi_p(b) = 0$$

(A.10)

with

$$q(u) = \frac{a^2}{4} \left( p^2 - U(u) \right).$$

(A.11)

On the inside of the potential and on the outside separately, $q(u)$ does not depend on $u$ and these cases can be solved with separation of variables and the solutions patched together at $u = u_0$. Let

$$q_{\text{in}} = \frac{a^2}{4} \left( p^2 - U_0 \right)$$
$$q_{\text{out}} = \frac{a^2}{4} p^2.$$
Start by expanding $\psi_p(b)$ in terms of so-called elliptic sines and cosines of the ‘angular’ variable $v$. They are solutions of ‘Mathieu’s equation’ \[ A.13 \):

$$
\left(-\frac{\partial^2}{\partial v^2} + 2q \cos(2v)\right) C(\alpha, q, v) = \alpha C(\alpha, q, v).
$$

The general solutions are called ‘Mathieu functions,’ usually denoted $C(\alpha, q, v)$ for solutions even in the coordinate $v$ and $S(\alpha, q, v)$ for odd. Demanding periodicity of the variable $v$ allows only certain discrete eigenvalues $\alpha$ (denoted here $\alpha_n$ for the even functions and $\beta_n$ for the odd functions). This (complete) set of periodic solutions is commonly called elliptic sines and elliptic cosines:

$$
C(\alpha_n, q, v) \equiv ce_n(v, q)
$$
$$
S(\beta_n, q, v) \equiv se_n(v, q).
$$

The general solution of \[ A.10 \] can be written in terms of these elliptic sines and cosines:

$$
\psi_p(b) \equiv \sum_{n=0}^{\infty} \left[ f_{cn}(u)ce_n(v, q) + f_{sn}(u)se_n(v, q) \right]. \tag{A.15}
$$

Plugging this in to \[ A.10 \] gives

$$
\left[ \frac{\partial^2}{\partial u^2} + 2q \cosh(2u) - \alpha_n \right] f_{cn}(u) = 0 \tag{A.16}
$$
$$
\left[ \frac{\partial^2}{\partial u^2} + 2q \cosh(2u) - \beta_n \right] f_{sn}(u) = 0. \tag{A.17}
$$

This is called the modified Mathieu equation, which can be obtained from \[ A.13 \] by replacing $v \rightarrow (i \, u)$. Note that the eigenvalues are different for the functions corresponding to $ce_n$ and $se_n$ ($f_{cn}$ and $f_{sn}$ above, respectively). The general solution is then the same as for the original Mathieu equation, analytically continued with $v \rightarrow (i \, u)$, though typically they are organized by boundary conditions analogous to Bessel and Neumann functions (denoted $Je_n(u, q)$, $Ne_n(u, q)$, etc.) \[ A.13 \] :
Note that there are many different sets of so-called Mathieu functions, each being a complete orthogonal basis. Replacing \( q_{in} \) with \( q_{out} \) results in a different basis, and there are separate sets of modified Mathieu functions corresponding to the eigenvalues of the elliptic sines and elliptic cosines (\( \alpha_n \) and \( \beta_n \)).

By requiring continuity at the \( u = 0 \) line segment one finds that the general solution inside the potential is:

\[
\psi_{p}^{in}(u, v) = \sum_n \left[ C e_{in}^n J e_n(u, q_{in}) c e_n(v, q_{in}) + C o_{in}^n J o_n(u, q_{in}) s e_n(v, q_{in}) \right] \tag{A.20}
\]

with undetermined coefficients \( C e_{in}^n, C o_{in}^n \).

Outside, we write the solution as the sum of a plane wave and an outgoing wave

\[
\psi_{p}^{out}(u, v) = \sum_n \left( \frac{1}{p_n} J e_n(u, q_{out}) + C e_{out}^n H e_n^{(1)}(u, q_{out}) \right) c e_n(v, q_{out}) c e_n(\phi_p, q_{out}) \\
+ \left( \frac{1}{s_n} J o_n(u, q_{out}) + C o_{out}^n H o_n^{(1)}(u, q_{out}) \right) s e_n(v, q_{out}) s e_n(\phi_p, q_{out}) \right], \tag{A.21}
\]

where the H’s are analogous to Hankel functions

\[
He_n^{(1)}(u, q) \equiv J e_n(u, q) + i N e_n(u, q) \tag{A.22}
\]

\[
Ho_n^{(1)}(u, q) \equiv J o_n(u, q) + i N o_n(u, q) \tag{A.23}
\]

and the plane wave coefficients \( p_n \) and \( s_n \) are

\[
\frac{1}{p_n} = \frac{1}{\pi} \int_0^{2\pi} dv \ e^{ip \cdot x} c e_n(v, q_{out}) \tag{A.24}
\]

\[
\frac{1}{s_n} = \frac{1}{\pi} \int_0^{2\pi} dv \ e^{i s \cdot x} s e_n(v, q_{out}) \tag{A.25}
\]

The coefficients \( C e_{out}^n \) and \( C o_{out}^n \), along with the analogous 'inside' coefficients are determined by matching boundary conditions.

To match at the \( u = u_0 \) boundary, project the 'inside' angular functions (e.g. \( c e_n(v, q_{in}) \)) in terms of the 'outside' ones (e.g. \( c e_n(v, q_{out}) \)).

\[
ce_j(v, q_{in}) = \sum_{n=0}^{\infty} B_{jn}^c c e_n(v, q_{out}) \tag{A.26}
\]

\[
se_j(v, q_{in}) = \sum_{n=0}^{\infty} B_{jn}^s s e_n(v, q_{out}) \tag{A.27}
\]
with
\[ B_{jn}^c = \frac{1}{\pi} \int_0^{2\pi} dv c_j(v, q_{in}) c_n(v, q_{out}) \] (A.28)
\[ B_{jn}^s = \frac{1}{\pi} \int_0^{2\pi} dv s_j(v, q_{in}) s_n(v, q_{out}) \] (A.29)

Then the 'inside' wave functions are
\[ \psi_p^{in} = \sum_{j,n} [C_{e_j}^{in} J_{e_j}(u, q_{in}) B_{jn}^c c_n(v, q_{out}) + C_{o_j}^{in} J_{o_j}(u, q_{in}) B_{jn}^s s_n(v, q_{out})] . \] (A.30)

The coefficients \((C_{e_n}^{in}, C_{o_n}^{in}, C_{e_n}^{out}, C_{o_n}^{out})\) can then be determined by demanding that \(\psi\) and its gradient be continuous at \(u = u_0\), which gives the following relations:

\[ \sum_j C_{e_j}^{in} J_{e_j}(u_0, q_{in}) B_{jn}^c = \frac{1}{p_n} J_{e_n}(u_0, q_{out}) c_n(\phi_p, q_{out}) + C_{e_n}^{out} H^{(1)}(u_0, q_{out}) c_n(\phi_p, q_{out}) \] (A.31)
\[ \sum_j C_{o_j}^{in} J_{o_j}(u_0, q_{in}) B_{jn}^s = \frac{1}{s_n} J_{o_n}(u_0, q_{out}) s_n(\phi_p, q_{out}) + C_{o_n}^{out} H^{(1)}(u_0, q_{out}) s_n(\phi_p, q_{out}) \] (A.32)
\[ \sum_j C_{e_j}^{in} J_{e_j}'(u_0, q_{in}) B_{jn}^c = \frac{1}{p_n} J_{e_n}'(u_0, q_{out}) c_n(\phi_p, q_{out}) + C_{e_n}^{out} H^{''(1)}(u_0, q_{out}) c_n(\phi_p, q_{out}) \] (A.33)
\[ \sum_j C_{o_j}^{in} J_{o_j}'(u_0, q_{in}) B_{jn}^s = \frac{1}{s_n} J_{o_n}'(u_0, q_{out}) s_n(\phi_p, q_{out}) + C_{o_n}^{out} H^{''(1)}(u_0, q_{out}) s_n(\phi_p, q_{out}) \] (A.34)

The plane wave coefficients \((p_n, s_n)\) as well as the coefficients from the projection \((B_{jn}^c, B_{jn}^s)\) must be solved numerically. In addition, to compare to the \(f_m\) in the main calculation, the resulting wavefunctions are integrated to project out the usual angular moments. Hence the description as a “semi-analytical” test case. In fact, this implementation (done in Mathematica) saves no time over the original numerical version, but it does provide an independent check.
Appendix 2

NOTATION, CONVENTIONS, AND DEFINITIONS

All notational definitions are defined when first introduced, but frequently used notation is collected here for easy reference (or at least notation that is used in well-separated parts of the manuscript).

- All quantities are reported using a system of units such that $c = \hbar = k_B = 1$ ("natural units"). I.e., all velocities are measured as fractions of the speed of light $c$, etc.

- The space-time metric in flat space is taken as $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, such that timelike 4-vectors have positive norm and spacelike vectors negative.

- Projectors:
  \begin{equation}
  \Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu ,
  \end{equation}
  \begin{equation}
  P^{\mu\nu}_{\alpha\beta} \equiv \Delta^{\mu}_{\alpha} \Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta} \Delta^{\nu}_{\alpha} - \frac{2}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} ,
  \end{equation}
  such that $u_\mu \Delta^{\mu\nu} = u_\mu P^{\mu\nu}_{\alpha\beta} = g_{\mu\nu} P^{\mu\nu}_{\alpha\beta} = 0$. Projecting with $\Delta^{\mu\nu}$ makes a quantity transverse to a fluid velocity $u^\mu$, while $P^{\mu\nu}_{\alpha\beta}$ makes it transverse, traceless, and symmetric under interchange of indices.

- Derivatives:
  \begin{equation}
  D \equiv u^\mu \partial_\mu ,
  \end{equation}
  \begin{equation}
  \nabla_\mu \equiv \Delta^\alpha_{\mu} \partial_\alpha ,
  \end{equation}
  so that $\partial_\mu = u_\mu D + \nabla_\mu$. In the fluid rest frame, these are the time derivative and spatial gradient, respectively. I.e., in the non-relativistic limit
  \begin{equation}
  D \approx \partial_t + \vec{v} \cdot \vec{\partial} + \mathcal{O} (|\vec{v}|^2) ,
  \end{equation}
  \begin{equation}
  \vec{\nabla} \approx -\vec{\partial} + \mathcal{O} (|\vec{v}|) .
  \end{equation}
Brackets:

\[
A^{(\alpha B^\beta)} \equiv \frac{1}{2} \left( A^\alpha B^\beta + A^\beta B^\alpha \right), \quad (B.7)
\]

\[
A^{[\alpha B^\beta]} \equiv \frac{1}{2} \left( A^\alpha B^\beta - A^\beta B^\alpha \right), \quad (B.8)
\]

\[
A^{(\alpha B^\beta)} \equiv P_{\mu \nu}^{\alpha \beta} A^\mu B^\nu, \quad (B.9)
\]

which are used to define \( \sigma^{\mu \nu} \equiv \nabla^{(\mu} u^{\nu)} \) and the fluid vorticity \( \omega^{\mu \nu} \equiv -\nabla^{[\mu} u^{\nu]} \).
BIBLIOGRAPHY

[1] M. Luzum and P. Romatschke, “Conformal Relativistic Viscous Hydrodynamics: Applications to RHIC results at $\sqrt{s_{NN}} = 200$ GeV,” *Phys. Rev. C78* (2008) 034915, arXiv:0804.4015 [nucl-th].

[2] M. Luzum and P. Romatschke, “Erratum: Conformal Relativistic Viscous Hydrodynamics: Applications to RHIC results at $\sqrt{s_{NN}} = 200$ GeV [1],” *Phys. Rev. C79* (2009) 039903(E), arXiv:0804.4015 [nucl-th].

[3] M. Luzum and P. Romatschke, “Viscous Hydrodynamic Predictions for Nuclear Collisions at the LHC,” arXiv:0901.4588 [nucl-th].

[4] M. Luzum, J. G. Cramer, and G. A. Miller, “Understanding the Optical Potential in HBT Interferometry,” *Phys. Rev. C78* (2008) 054905, arXiv:0809.0520 [nucl-th].

[5] M. Luzum and G. A. Miller, “Elliptic flow from final state interactions in the DWEF model,” *Phys. Rev. C79* (2009) 024902, arXiv:0810.1550 [nucl-th].

[6] U. W. Heinz, “From SPS to RHIC: Maurice and the CERN heavy-ion programme,” *Phys. Scripta 78* (2008) 028005, arXiv:0805.4572 [nucl-th].

[7] U. W. Heinz and M. Jacob, “Evidence for a new state of matter: An assessment of the results from the CERN lead beam programme,” arXiv:nucl-th/0002042.

[8] Figure 2.1 obtained from the Brookhaven National Laboratory Flickr web page under Creative Commons Attribution-Noncommercial-No Derivative Works 2.0 Generic License. http://www.flickr.com/photos/brookhavenlab/sets/72157613690851651/
[9] **BRAHMS** Collaboration, I. Arsene et al., “Quark gluon plasma and color glass condensate at RHIC? The perspective from the BRAHMS experiment,” *Nucl. Phys. A757* (2005) 1–27, arXiv:nucl-ex/0410020.

[10] **PHOBOS** Collaboration, B. B. Back et al., “The PHOBOS perspective on discoveries at RHIC,” *Nucl. Phys. A757* (2005) 28–101, arXiv:nucl-ex/0410022.

[11] **STAR** Collaboration, J. Adams et al., “Experimental and theoretical challenges in the search for the quark gluon plasma: The STAR collaboration’s critical assessment of the evidence from RHIC collisions,” *Nucl. Phys. A757* (2005) 102–183, arXiv:nucl-ex/0501009.

[12] **PHENIX** Collaboration, K. Adcox et al., “Formation of dense partonic matter in relativistic nucleus nucleus collisions at RHIC: Experimental evaluation by the PHENIX collaboration,” *Nucl. Phys. A757* (2005) 184–283, arXiv:nucl-ex/0410003.

[13] P. F. Kolb, U. W. Heinz, P. Huovinen, K. J. Eskola, and K. Tuominen, “Centrality dependence of multiplicity, transverse energy, and elliptic flow from hydrodynamics,” *Nucl. Phys. A696* (2001) 197–215, arXiv:hep-ph/0103234.

[14] R. Hanbury Brown and R. Q. Twiss, “A Test of a new type of stellar interferometer on Sirius,” *Nature* 178 (1956) 1046–1048.

[15] R. Hanbury Brown and R. Q. Twiss, “A New type of interferometer for use in radio astronomy,” *Phil. Mag.* 45 (1954) 663–682.

[16] G. Goldhaber, S. Goldhaber, W.-Y. Lee, and A. Pais, “Influence of Bose-Einstein statistics on the antiproton proton annihilation process,” *Phys. Rev.* 120 (1960) 300–312.

[17] M. A. Lisa, S. Pratt, R. Soltz, and U. Wiedemann, “Femtoscopy in Relativistic Heavy Ion Collisions: Two Decades of Progress,” *Ann. Rev. Nucl. Part. Sci.* 55 (2005) 357–402, arXiv:nucl-ex/0505014.
[18] P. Romatschke, “New Developments in Relativistic Viscous Hydrodynamics,”

\[\text{arXiv:0902.3663 [hep-ph]}\]

[19] L. Euler, “Principes généraux du mouvement des fluides,” Mém. Acad. Sci. Berlin 11 (1755 [printed 1757]) .

[20] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, vol. 6 of Course of Theoretical Physics. Butterworth-Heinemann, 2nd ed., Jan., 1987.

[21] C. Navier, “Mémoire sur les lois du mouvement des fluides,” Mém. Acad. Sci. Inst. France 6 (1822) 389–440.

[22] W. A. Hiscock and L. Lindblom, “Generic instabilities in first-order dissipative relativistic fluid theories,” Phys. Rev. D31 (1985) 725–733.

[23] P. Romatschke, “Relativistic Viscous Fluid Dynamics and Non-Equilibrium Entropy,” arXiv:0906.4787 [hep-th].

[24] I. Shih Liu, I. Müller, and T. Ruggeri, “Relativistic thermodynamics of gases,” Ann. Phys. 169 (1986) 191–219.

[25] I. Müller, “Zum Paradoxon der Wärmeleitungstheorie,” Z. Phys. 198 (1967) 329.

[26] W. Israel, “Nonstationary irreversible thermodynamics: A causal relativistic theory,” Ann. Phys. 100 (1976) 310–331.

[27] W. Israel and J. M. Stewart, “Thermodynamics of nonstationary and transient effects in a relativistic gas,” Phys. Lett. A58 (1976) 213–215.

[28] W. Israel and J. M. Stewart, “Transient relativistic thermodynamics and kinetic theory,” Ann. Phys. 118 (1979) 341–372.

[29] A. Muronga, “Causal Theories of Dissipative Relativistic Fluid Dynamics for Nuclear Collisions,” Phys. Rev. C69 (2004) 034903, arXiv:nucl-th/0309055.

[30] R. Loganayagam, “Entropy Current in Conformal Hydrodynamics,” JHEP 05 (2008) 087, arXiv:0801.3701 [hep-th].
[31] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, “Relativistic viscous hydrodynamics, conformal invariance, and holography,” *JHEP* **04** (2008) 100, arXiv:0712.2451 [hep-th].

[32] D. Teaney, J. Lauret, and E. V. Shuryak, “Flow at the SPS and RHIC as a quark gluon plasma signature,” *Phys. Rev. Lett.* **86** (2001) 4783–4786, arXiv:nucl-th/0011058.

[33] P. Huovinen, P. F. Kolb, U. W. Heinz, P. V. Ruuskanen, and S. A. Voloshin, “Radial and elliptic flow at RHIC: Further predictions,” *Phys. Lett.* **B503** (2001) 58–64, arXiv:hep-ph/0101136.

[34] T. Hirano and K. Tsuda, “Collective flow and two pion correlations from a relativistic hydrodynamic model with early chemical freeze out,” *Phys. Rev.* **C66** (2002) 054905, arXiv:nucl-th/0205043.

[35] P. F. Kolb and R. Rapp, “Transverse flow and hadro-chemistry in Au + Au collisions at s(NN)**(1/2) = 200-GeV,” *Phys. Rev.* **C67** (2003) 044903, arXiv:hep-ph/0210222.

[36] G. Policastro, D. T. Son, and A. O. Starinets, “The shear viscosity of strongly coupled N = 4 supersymmetric Yang-Mills plasma,” *Phys. Rev. Lett.* **87** (2001) 081601, arXiv:hep-th/0104066.

[37] P. Arnold, G. D. Moore, and L. G. Yaffe, “Transport coefficients in high temperature gauge theories. II: Beyond leading log,” *JHEP* **05** (2003) 051, arXiv:hep-ph/0302165.

[38] A. Nakamura and S. Sakai, “Transport Coefficients of Gluon Plasma,” *Phys. Rev. Lett.* **94** (2005) 072305, arXiv:hep-lat/0406009.

[39] P. Arnold, C. Dogan, and G. D. Moore, “The bulk viscosity of high-temperature QCD,” *Phys. Rev.* **D74** (2006) 085021, arXiv:hep-ph/0608012.
[40] S. C. Huot, S. Jeon, and G. D. Moore, “Shear viscosity in weakly coupled $\mathcal{N}=4$ Super Yang-Mills theory compared to QCD,” *Phys. Rev. Lett.* **98** (2007) 172303, arXiv:hep-ph/0608062.

[41] J.-S. Gagnon and S. Jeon, “Leading order calculation of electric conductivity in hot quantum electrodynamics from diagrammatic methods,” *Phys. Rev.* **D75** (2007) 025014, arXiv:hep-ph/0610235.

[42] G. Aarts, C. Allton, J. Foley, S. Hands, and S. Kim, “Spectral functions at small energies and the electrical conductivity in hot, quenched lattice QCD,” *Phys. Rev. Lett.* **99** (2007) 022002, arXiv:hep-lat/0703008.

[43] H. B. Meyer, “A calculation of the shear viscosity in SU(3) gluodynamics,” *Phys. Rev.* **D76** (2007) 101701, arXiv:0704.1801 [hep-lat].

[44] A. Buchel, “Bulk viscosity of gauge theory plasma at strong coupling,” *Phys. Lett.* **B663** (2008) 286–289, arXiv:0708.3459 [hep-th].

[45] S. Caron-Huot and G. D. Moore, “Heavy quark diffusion in perturbative QCD at next-to-leading order,” *Phys. Rev. Lett.* **100** (2008) 052301, arXiv:0708.4232 [hep-ph].

[46] D. Teaney, “Effect of shear viscosity on spectra, elliptic flow, and Hanbury Brown-Twiss radii,” *Phys. Rev.* **C68** (2003) 034913, arXiv:nucl-th/0301099.

[47] P. Romatschke, “Causal viscous hydrodynamics for central heavy-ion collisions. II: Meson spectra and HBT radii,” *Eur. Phys. J.* **C52** (2007) 203–209, arXiv:nucl-th/0701032.

[48] P. Romatschke and U. Romatschke, “Viscosity Information from Relativistic Nuclear Collisions: How Perfect is the Fluid Observed at RHIC?,” *Phys. Rev. Lett.* **99** (2007) 172301, arXiv:0706.1522 [nucl-th].

[49] A. K. Chaudhuri, “Saturation of elliptic flow and shear viscosity,” arXiv:0708.1252 [nucl-th].
[50] A. Muronga and D. H. Rischke, “Evolution of hot, dissipative quark matter in relativistic nuclear collisions,” arXiv:nucl-th/0407114.

[51] A. K. Chaudhuri and U. W. Heinz, “Hydrodynamical evolution of dissipative QGP fluid,” J. Phys. Conf. Ser. 50 (2006) 251–258, arXiv:nucl-th/0504022.

[52] A. Muronga, “Viscous hydrodynamics,” J. Phys. G31 (2005) S1035–S1039, arXiv:0710.3277 [nucl-th].

[53] A. K. Chaudhuri, “Dissipative hydrodynamics in 2+1 dimension,” Phys. Rev. C74 (2006) 044904, arXiv:nucl-th/0604014.

[54] P. Mota, G. S. Denicol, T. Koide, and T. Kodama, “New formulation of causal dissipative hydrodynamics: Shock wave propagation,” J. Phys. G34 (2007) S1011, arXiv:hep-ph/0701162.

[55] H. Song and U. W. Heinz, “Suppression of elliptic flow in a minimally viscous quark-gluon plasma,” Phys. Lett. B658 (2008) 279–283, arXiv:0709.0742 [nucl-th].

[56] K. Dusling and D. Teaney, “Simulating elliptic flow with viscous hydrodynamics,” Phys. Rev. C77 (2008) 034905, arXiv:0710.5932 [nucl-th].

[57] H. Song and U. W. Heinz, “Causal viscous hydrodynamics in 2+1 dimensions for relativistic heavy-ion collisions,” Phys. Rev. C77 (2008) 064901, arXiv:0712.3715 [nucl-th].

[58] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231–252, arXiv:hep-th/9711200.

[59] D. Kharzeev and K. Tuchin, “Bulk viscosity of QCD matter near the critical temperature,” JHEP 09 (2008) 093, arXiv:0705.4280 [hep-ph].

[60] S. Sakai and A. Nakamura, “Lattice calculation of the QGP viscosities - Present results and next project -,” PoS LAT2007 (2007) 221, arXiv:0710.3625 [hep-lat].
[61] H. B. Meyer, “A calculation of the bulk viscosity in SU(3) gluodynamics,” *Phys. Rev. Lett.* **100** (2008) 162001, [arXiv:0710.3717 [hep-lat]].

[62] F. Karsch, D. Kharzeev, and K. Tuchin, “Universal properties of bulk viscosity near the QCD phase transition,” *Phys. Lett.* **B663** (2008) 217–221, [arXiv:0711.0914 [hep-ph]].

[63] A. Muronga, “Second order dissipative fluid dynamics for ultra-relativistic nuclear collisions,” *Phys. Rev. Lett.* **88** (2002) 062302, [arXiv:nucl-th/0104064].

[64] R. Baier, P. Romatschke, and U. A. Wiedemann, “Dissipative hydrodynamics and heavy ion collisions,” *Phys. Rev.* **C73** (2006) 064903, [arXiv:hep-ph/0602249].

[65] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” *JHEP* **02** (2008) 045, [arXiv:0712.2456 [hep-th]].

[66] M. Natsuume and T. Okamura, “Causal hydrodynamics of gauge theory plasmas from AdS/CFT duality,” *Phys. Rev.* **D77** (2008) 066014, [arXiv:0712.2916 [hep-th]].

[67] R. Baier and P. Romatschke, “Causal viscous hydrodynamics for central heavy-ion collisions,” *Eur. Phys. J.* **C51** (2007) 677–687, [arXiv:nucl-th/0610108].

[68] P. Huovinen and P. V. Ruuskanen, “Hydrodynamic Models for Heavy Ion Collisions,” *Ann. Rev. Nucl. Part. Sci.* **56** (2006) 163–206, [arXiv:nucl-th/0605008].

[69] C++ versions of the relativistic viscous hydrodynamic codes with and without radial symmetry may be obtained from [http://hep.itp.tuwien.ac.at/~paulrom/](http://hep.itp.tuwien.ac.at/~paulrom/).

[70] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, 2nd ed., Oct., 1992.

[71] A. Dumitru, E. Molnar, and Y. Nara, “Entropy production in high-energy heavy-ion collisions and the correlation of shear viscosity and thermalization time,” *Phys. Rev.* **C76** (2007) 024910, [arXiv:0706.2203 [nucl-th]].
[72] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, “The order of the quantum chromodynamics transition predicted by the standard model of particle physics,” *Nature* **443** (2006) 675–678, [arXiv:hep-lat/0611014](http://arxiv.org/abs/hep-lat/0611014).

[73] M. Laine and Y. Schroder, “Quark mass thresholds in QCD thermodynamics,” *Phys. Rev.* **D73** (2006) 085009, [arXiv:hep-ph/0603048](http://arxiv.org/abs/hep-ph/0603048).

[74] C. M. Hung and E. V. Shuryak, “Equation of state, radial flow and freeze-out in high energy heavy ion collisions,” *Phys. Rev.* **C57** (1998) 1891–1906, [arXiv:hep-ph/9709264](http://arxiv.org/abs/hep-ph/9709264).

[75] F. Cooper and G. Frye, “Comment on the Single Particle Distribution in the Hydrodynamic and Statistical Thermodynamic Models of Multiparticle Production,” *Phys. Rev.* **D10** (1974) 186.

[76] P. V. Ruuskanen, “Transverse Hydrodynamics With a First Order Phase Transition In Very High-Energy Nuclear Collisions,” *Acta Phys. Polon.* **B18** (1987) 551.

[77] D. H. Rischke and M. Gyulassy, “The time-delay signature of quark-gluon plasma formation in relativistic nuclear collisions,” *Nucl. Phys.* **A608** (1996) 479–512, [arXiv:nucl-th/9606039](http://arxiv.org/abs/nucl-th/9606039).

[78] S. R. de Groot, W. van Leeuwen, and C. G. van Weert, *Relativistic kinetic theory, Principles and applications*. North-Holland, Amsterdam, Jan., 1980.

[79] P. F. Kolb and U. W. Heinz, “Hydrodynamic description of ultrarelativistic heavy-ion collisions,” [arXiv:nucl-th/0305084](http://arxiv.org/abs/nucl-th/0305084).

[80] J. Sollfrank, P. Koch, and U. W. Heinz, “The Influence of resonance decays on the P(t) spectra from heavy ion collisions,” *Phys. Lett.* **B252** (1990) 256–264.

[81] J. Sollfrank, P. Koch, and U. W. Heinz, “Is there a low p(T) ‘anomaly’ in the pion momentum spectra from relativistic nuclear collisions?,” *Z. Phys.* **C52** (1991) 593–610.

[82] Azhydro version 0.2, available from [http://karman.physics.purdue.edu/OSCAR/](http://karman.physics.purdue.edu/OSCAR/).
[83] S. A. Bass and A. Dumitru, “Dynamics of hot bulk QCD matter: From the quark-gluon plasma to hadronic freeze-out,” *Phys. Rev. C61* (2000) 064909, arXiv:nucl-th/0001033.

[84] D. Teaney, J. Lauret, and E. V. Shuryak, “A hydrodynamic description of heavy ion collisions at the SPS and RHIC,” arXiv:nucl-th/0110037.

[85] T. Hirano, U. W. Heinz, D. Kharzeev, R. Lacey, and Y. Nara, “Hadronic dissipative effects on elliptic flow in ultrarelativistic heavy-ion collisions,” *Phys. Lett. B636* (2006) 299–304, arXiv:nucl-th/0511046.

[86] C. Nonaka and S. A. Bass, “Space-time evolution of bulk QCD matter,” *Phys. Rev. C75* (2007) 014902, arXiv:nucl-th/0607018.

[87] R. S. Bhalerao and S. Gupta, “Aspects of causal viscous hydrodynamics,” *Phys. Rev. C77* (2008) 014902, arXiv:0706.3428 [nucl-th].

[88] G. Baym, B. L. Friman, J. P. Blaizot, M. Soyeur, and W. Czyz, “Hydrodynamics of Ultrarelativistic Heavy Ion Collisions,” *Nucl. Phys. A407* (1983) 541–570.

[89] M. Chojnacki and W. Florkowski, “Characteristic form of boost-invariant and cylindrically non-symmetric hydrodynamic equations,” *Phys. Rev. C74* (2006) 034905, arXiv:nucl-th/0603065.

[90] M. I. Nagy, T. Csorgo, and M. Csanad, “Detailed description of accelerating, simple solutions of relativistic perfect fluid hydrodynamics,” *Phys. Rev. C77* (2008) 024908, arXiv:0709.3677 [nucl-th].

[91] D. Kharzeev, E. Levin, and M. Nardi, “QCD saturation and deuteron nucleus collisions,” *Nucl. Phys. A730* (2004) 448–459, arXiv:hep-ph/0212316.

[92] H.-J. Drescher, A. Dumitru, A. Hayashigaki, and Y. Nara, “The eccentricity in heavy-ion collisions from color glass condensate initial conditions,” *Phys. Rev. C74* (2006) 044905, arXiv:nucl-th/0605012.
[93] L. D. McLerran and R. Venugopalan, “Computing quark and gluon distribution functions for very large nuclei,” *Phys. Rev. D49* (1994) 2233–2241, arXiv:hep-ph/9309289.

[94] L. D. McLerran and R. Venugopalan, “Gluon distribution functions for very large nuclei at small transverse momentum,” *Phys. Rev. D49* (1994) 3352–3355, arXiv:hep-ph/9311205.

[95] T. Lappi and R. Venugopalan, “Universality of the saturation scale and the initial eccentricity in heavy ion collisions,” *Phys. Rev. C74* (2006) 054905, arXiv:nucl-th/0609021.

[96] P. F. Kolb, J. Sollfrank, and U. W. Heinz, “Elliptic and hexadecupole flow from AGS to LHC energies,” *Phys. Lett. B459* (1999) 667–673, arXiv:nucl-th/9906003.

[97] J.-Y. Ollitrault, “Anisotropy as a signature of transverse collective flow,” *Phys. Rev. D46* (1992) 229–245.

[98] P. F. Kolb, J. Sollfrank, and U. W. Heinz, “Anisotropic transverse flow and the quark-hadron phase transition,” *Phys. Rev. C62* (2000) 054909, arXiv:hep-ph/0006129.

[99] U. W. Heinz, “Thermalization at RHIC,” *AIP Conf. Proc. 739* (2005) 163–180, arXiv:nucl-th/0407067.

[100] M. Lublinsky and E. Shuryak, “How much entropy is produced in strongly coupled quark-gluon plasma (sQGP) by dissipative effects?,” *Phys. Rev. C76* (2007) 021901, arXiv:0704.1647 [hep-ph].

[101] PHENIX Collaboration, S. S. Adler et al., “Identified charged particle spectra and yields in Au + Au collisions at $s_{[NN]}^{**}(1/2) = 200$-GeV,” *Phys. Rev. C69* (2004) 034909, arXiv:nucl-ex/0307022.

[102] PHOBOS Collaboration, B. Alver et al., “Elliptic flow fluctuations in Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV,” arXiv:nucl-ex/0702036.
[103] **STAR** Collaboration, B. I. Abelev *et al.*, “Centrality dependence of charged hadron and strange hadron elliptic flow from $\sqrt{s_{NN}} = 200$ GeV Au+Au collisions,” *Phys. Rev. C77* (2008) 054901, [arXiv:0801.3466 [nucl-ex]].

[104] A. Poskanzer. Private communication.

[105] J.-Y. Ollitrault, “Collective flow from azimuthal correlations,” *Nucl. Phys. A590* (1995) 561c–564c.

[106] P. Kovtun, D. T. Son, and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” *Phys. Rev. Lett. 94* (2005) 111601, [arXiv:hep-th/0405231].

[107] P. Huovinen. Talk given at RHIC Winter Workshop, INT, Seattle, 2002.

[108] W. Broniowski, M. Chojnacki, W. Florkowski, and A. Kisiel, “Uniform Description of Soft Observables in Heavy-Ion Collisions at $\sqrt{s_{NN}} = 200$ GeV,” *Phys. Rev. Lett. 101* (2008) 022301, [arXiv:0801.4361 [nucl-th]].

[109] T. Hirano and M. Gyulassy, “Perfect Fluidity of the Quark Gluon Plasma Core as Seen through its Dissipative Hadronic Corona,” *Nucl. Phys. A769* (2006) 71–94, [arXiv:nucl-th/0506049].

[110] T. Schafer, “The Shear Viscosity to Entropy Density Ratio of Trapped Fermions in the Unitarity Limit,” *Phys. Rev. A76* (2007) 063618, [arXiv:cond-mat/0701251].

[111] S. Gavin and M. Abdel-Aziz, “Measuring Shear Viscosity Using Transverse Momentum Correlations in Relativistic Nuclear Collisions,” *Phys. Rev. Lett. 97* (2006) 162302, [arXiv:nucl-th/0606061].

[112] **PHENIX** Collaboration, A. Adare *et al.*, “Energy Loss and Flow of Heavy Quarks in Au+Au Collisions at $\sqrt{s_{NN}} = 200$ GeV,” *Phys. Rev. Lett. 98* (2007) 172301, [arXiv:nucl-ex/0611018].
[113] H.-J. Drescher, A. Dumitru, C. Gombeaud, and J.-Y. Ollitrault, “The centrality dependence of elliptic flow, the hydrodynamic limit, and the viscosity of hot QCD,” *Phys. Rev. C76* (2007) 024905, arXiv:0704.3553 [nucl-th].

[114] STAR Collaboration, J. Adams *et al.*, “Transverse-momentum p(t) correlations on (eta,Phi) from mean-p(t) fluctuations in Au - Au collisions at s(NN)**(1/2) = 200-GeV,” *J. Phys. G32* (2006) L37–L48, arXiv:nucl-ex/0509030.

[115] STAR Collaboration, P. Sorensen, “Elliptic flow fluctuations in Au + Au collisions at s(NN)**(1/2) = 200-GeV,” *J. Phys. G34* (2007) S897, arXiv:nucl-ex/0612021.

[116] S. Vogel, G. Torrieri, and M. Bleicher, “Elliptic flow fluctuations in heavy ion collisions at RHIC and the perfect liquid hypothesis,” arXiv:nucl-th/0703031.

[117] H.-J. Drescher and Y. Nara, “Eccentricity fluctuations from the Color Glass Condensate at RHIC and LHC,” *Phys. Rev. C76* (2007) 041903, arXiv:0707.0249 [nucl-th].

[118] T. A. Trainor, “The RHIC azimuth quadrupole: ‘perfect liquid’ or gluonic radiation?,” *Mod. Phys. Lett. A23* (2008) 569–589, arXiv:0708.0792 [hep-ph].

[119] Y. Hama *et al.*, “NeXSPheRIO Results on Elliptic-Flow Fluctuations at RHIC,” *Phys. Atom. Nucl. 71* (2008) 1558–1564, arXiv:0711.4544 [hep-ph].

[120] P. Romatschke, “Fluid turbulence and eddy viscosity in relativistic heavy-ion collisions,” *Prog. Theor. Phys. Suppl. 174* (2008) 137–144, arXiv:0710.0016 [nucl-th].

[121] U. W. Heinz, H. Song, and A. K. Chaudhuri, “Dissipative hydrodynamics for viscous relativistic fluids,” *Phys. Rev. C73* (2006) 034904, arXiv:nucl-th/0510014.

[122] T. Koide, G. S. Denicol, P. Mota, and T. Kodama, “Relativistic dissipative hydrodynamics: A Minimal causal theory,” *Phys. Rev. C75* (2007) 034909, arXiv:hep-ph/0609117.
[123] H. Song and U. W. Heinz, “Multiplicity scaling in ideal and viscous hydrodynamics,” *Phys. Rev. C78* (2008) 024902, arXiv:0805.1756 [nucl-th].

[124] P. Huovinen and D. Molnar, “The applicability of causal dissipative hydrodynamics to relativistic heavy ion collisions,” *Phys. Rev. C79* (2009) 014906, arXiv:0808.0953 [nucl-th].

[125] **NA49** Collaboration, C. Alt *et al.*, “Directed and elliptic flow of charged pions and protons in Pb + Pb collisions at 40-A-GeV and 158-A-GeV,” *Phys. Rev. C68* (2003) 034903, arXiv:nucl-ex/0303001.

[126] M. Prakash, M. Prakash, R. Venugopalan, and G. Welke, “Nonequilibrium properties of hadronic mixtures,” *Phys. Rept. 227* (1993) 321–366.

[127] L. Ramello. Private communication.

[128] J. D. Bjorken, “Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region,” *Phys. Rev. D27* (1983) 140–151.

[129] D. Kharzeev, E. Levin, and M. Nardi, “Color glass condensate at the LHC: Hadron multiplicities in p p, p A and A A collisions,” *Nucl. Phys. A747* (2005) 609–629, arXiv:hep-ph/0408050.

[130] N. Armesto, “Predictions for the LHC: an Overview,” *J. Phys. G35* (2008) 104042, arXiv:0804.4158 [hep-ph].

[131] N. Armesto (ed) *et al.*, “Heavy Ion Collisions at the LHC - Last Call for Predictions,” *J. Phys. G35* (2008) 054001, arXiv:0711.0974 [hep-ph].

[132] W. Busza, “Trends in multiparticle production and some ‘predictions’ for pp and PbPb collisions at LHC,” *J. Phys. G35* (2008) 044040, arXiv:0710.2293 [nucl-ex].

[133] M. A. York and G. D. Moore, “Second order hydrodynamic coefficients from kinetic theory,” arXiv:0811.0729 [hep-ph].
[134] Y. Aoki et al., “The QCD transition temperature: results with physical masses in the continuum limit II,” arXiv:0903.4155 [hep-lat].

[135] A. Bazavov et al., “Equation of state and QCD transition at finite temperature,” arXiv:0903.4379 [hep-lat].

[136] M. Gyulassy and T. Matsui, “Quark Gluon Plasma Evolution In Scaling Hydrodynamics,” Phys. Rev. D29 (1984) 419–425.

[137] A. K. Chaudhuri, “Multiplicity, mean $p_T$, $p_T$-spectra and elliptic flow of identified particles in Pb+Pb collisions at LHC,” Phys. Lett. B672 (2009) 126–131, arXiv:0803.0643 [nucl-th].

[138] PHOBOS Collaboration, B. Alver et al., “System size, energy, pseudorapidity, and centrality dependence of elliptic flow,” Phys. Rev. Lett. 98 (2007) 242302, arXiv:nucl-ex/0610037.

[139] H. Niemi, K. J. Eskola, and P. V. Ruuskanen, “Elliptic flow in nuclear collisions at the Large Hadron Collider,” Phys. Rev. C79 (2009) 024903, arXiv:0806.1116 [hep-ph].

[140] G. A. Miller, “Charge Density of the Neutron,” Phys. Rev. Lett. 99 (2007) 112001, arXiv:0705.2409 [nucl-th].

[141] U. W. Heinz and P. F. Kolb, “Two RHIC puzzles: Early thermalization and the HBT problem,” arXiv:hep-ph/0204061.

[142] G. A. Miller and J. G. Cramer, “Polishing the Lens: I Pionic Final State Interactions and HBT Correlations- Distorted Wave Emission Function (DWEF) Formalism and Examples,” J. Phys. G34 (2007) 703–740, arXiv:nucl-th/0507004.

[143] J. G. Cramer, G. A. Miller, J. M. S. Wu, and J.-H. Yoon, “Quantum opacity, the RHIC HBT puzzle, and the chiral phase transition,” Phys. Rev. Lett. 94 (2005) 102302, arXiv:nucl-th/0411031.
[144] S. Pratt, “Resolving the HBT Puzzle in Relativistic Heavy Ion Collision,”
\url{arXiv:0811.3363 [nucl-th]}.

[145] S. Pratt, “The Long Slow Death of the HBT Puzzle,” Acta Phys. Polon. B40 (2009)
1249–1256, \url{arXiv:0812.4714 [nucl-th]}.

[146] S. Pratt, “Pion interferometry for exploding sources,” Phys. Rev. Lett. 53 (1984)
1219–1221.

[147] M. Gyulassy, S. K. Kauffmann, and L. W. Wilson, “Pion interferometry of nuclear
collisions. I. Theory,” Phys. Rev. C20 (1979) 2267–2292.

[148] P. Danielewicz and P. Schuck, “Formulation of particle correlation and cluster
production in heavy-ion-induced reactions,” Phys. Lett. B274 (1992) 268–274.

[149] S. Pratt, “Interpreting scattering wave functions in the presence of energy-dependent
interactions,” Phys. Rev. C77 (2008) 014609 \url{arXiv:0710.5739 [nucl-th]}.

[150] F. Retiere and M. A. Lisa, “Observable implications of geometrical and dynamical
aspects of freeze-out in heavy ion collisions,” Phys. Rev. C70 (2004) 044907.
\url{arXiv:nucl-th/0312024}.

[151] STAR Collaboration, J. Adams et al., “Identified particle distributions in p p and
Au + Au collisions at s**(1/2) = 200-GeV,” Phys. Rev. Lett. 92 (2004) 112301.
\url{arXiv:nucl-ex/0310004}.

[152] STAR Collaboration, J. Adams et al., “Pion interferometry in Au + Au collisions
at s(NN)**(1/2) = 200-GeV,” Phys. Rev. C71 (2005) 044906.
\url{arXiv:nucl-ex/0411036}.

[153] STAR Collaboration, J. Adams et al., “Azimuthal anisotropy in Au + Au collisions
at s(NN)**(1/2) = 200-GeV,” Phys. Rev. C72 (2005) 014904.
\url{arXiv:nucl-ex/0409033}.
[154] H. Petersen, J. Steinheimer, G. Burau, M. Bleicher, and H. Stocker, “A Fully Integrated Transport Approach to Heavy Ion Reactions with an Intermediate Hydrodynamic Stage,” [arXiv:0806.1695 [nucl-th]]

[155] I. S. Gradshteyn, I. M. Ryzhik, A. Jeffrey, and D. Zwillinger, *Table of Integrals, Series, and Products.* Academic Press, 6th ed., July, 2000.

[156] J. C. Gutiérrez-Vega, R. M. Rodríguez-Domingo, M. A. Meneses-Nava, and S. Chávez-Cerda, “Mathieu functions, a visual approach,” *Am. J. Phys.* 71 (2003) 233.

[157] N. W. McLachlan, *Theory and Application of Mathieu Functions.* Oxford University Press, Oxford, 1947.
VITA

Matt was born in Benson, Minnesota in 1980. He graduated from Benson High School in 1999 and then attended Saint John’s University in Collegeville, Minnesota, where in 2003 he obtained a Bachelor of Arts degree with a major in Physics and minors in Mathematics and Chemistry. He earned a Master of Science in 2004 and a Doctor of Philosophy in 2009, both in physics from the University of Washington. He then moved on to a postdoctoral research position at the Institut de Physique Théorique in Saclay, France.