Quantizing Poisson Manifolds

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Abstract. This paper extends Kontsevich’s ideas on quantizing Poisson manifolds. A new differential is added to the Hodge decomposition of the Hochschild complex, so that it becomes a bicomplex, even more similar to the classical Hodge theory for complex manifolds.

These notes grew out of the author’s attempt to understand Kontsevich’s ideas [Kon95a] on quantizing Poisson manifolds. We introduce a new differential on the Hochschild complex, so that it becomes a bicomplex, see Theorem 2.1. This differential respects the Hodge decomposition of the Hochschild complex of a commutative algebra discovered by Gerstenhaber-Schack [GS87]. Thus, the Hochschild complex becomes similar to the $\partial-\bar{\partial}$-complex in complex geometry. Hopefully, Hodge-theoretic ideas “a la” Deligne-Griffiths-Morgan-Sullivan [DGMS75, Sul77] will eventually result in proving Kontsevich’s Formality Conjecture, which implies local quantization of an arbitrary Poisson manifold, a hard problem that has been around for almost twenty years [BFF+78], see [Wei95] for the most state-of-the-art survey of this subject.

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1. Kontsevich’s Formality Conjecture

1.1. Some formalities. Let $A = C^\infty(X)$ be the algebra of smooth functions on a smooth real manifold $X$. Let $C^\bullet(A, A)$ be the (local) Hochschild complex of the algebra $A$ over $X$, i.e., $C^n(A, A) = \{ \phi \in \text{Hom}(A^\otimes n, A) | \phi(f_1, \ldots, f_n) \text{ is a differential operator in each entry } f_1, \ldots, f_n \}$. The Hochschild-Kostant-Rosenberg Theorem [HKR62] provides the computation of the corresponding Hochschild cohomology

$$H^\bullet(A, A) = \Lambda^\bullet TX,$$

which is nothing but the smooth multivector fields on $X$.

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Notice that both the Hochschild complex and its cohomology (more precisely, the suspensions thereof) are differential graded Lie algebras (DGLa's). The complex $C^\bullet = C^\bullet(A, A)[1]$, where $K^n[1] = K^{n+1}$, $n \in \mathbb{Z}$, of a complex $K^\bullet$, carries a Gerstenhaber bracket, which may be defined naturally, see [Sta93], by observing that the Hochschild cochains are exactly the coderivations of the tensor coalgebra $T(A) = \bigoplus_{n \geq 0} A^\otimes n$; then the Gerstenhaber bracket is just the bracket of coderivations. This bracket defines a DGLA structure on $C^\bullet$. The (suspended) $\mathbb{Z}$-graded vector space $H^\bullet = \Lambda^\bullet T X[1]$ of multivector fields is a DGLA with respect to the trivial differential $d = 0$ and the Schouten-Nijenhuis bracket of multivector fields:

\[
[v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_n] = \sum_{1 \leq i \leq m} (-1)^{m+i+j-1} [v_i, w_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_m \wedge w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_n.
\]

Every DGLA $L^\bullet$ induces an obvious DGLA structure on its cohomology $H^\bullet(L^\bullet)$ with the trivial differential. In this sense the Hochschild-Kostant-Rosenberg Theorem may be refined by saying that the cohomology DGLA of the Hochschild complex is isomorphic to the DGLA of multivector fields, see [GS88]. Kontsevich’s Formality Conjecture [Kon95a] suggests a further, profound refinement of the Hochschild-Kostant-Rosenberg Theorem.

**Conjecture 1.1** (Kontsevich’s Formality Conjecture). The Hochschild complex $C^\bullet$ is quasi-isomorphic as a DGLA to its cohomology $H^\bullet$.

We recall that two DGLA’s $L$ and $L'$ are quasi-isomorphic, if there is a chain $L = L_1 \to L_2 \leftarrow L_3 \to \cdots \leftarrow L_n = L'$ of DGLA homomorphisms all of which induce isomorphisms of cohomology. Perhaps, in this conjecture one should consider a weaker notion of quasi-isomorphism, where the intermediate steps $L_2, L_3, \ldots, L_{n-1}$ are $L_\infty$-algebras rather than DG Lie.

**Remark 1.2.** There exists a natural embedding $H^\bullet \to C^\bullet$, “a multivector field is considered as a multid derivation of the algebra $A$ of functions”, which induces an isomorphism of cohomology. This embedding does not satisfy the conditions of the conjecture, because it does not respect the brackets. It is not hard to come up with a counterexample. In rational homotopy theory, there is a similar discouragement: the mapping $H^\bullet(X) \to \Omega^\bullet(X)$ which takes a cohomology class to its harmonic representative is a quasi-isomorphism, but the product of two harmonic forms is not harmonic in general. Nevertheless, the two differential graded associative algebras $H^\bullet(X)$ and $\Omega^\bullet(X)$ are quasi-isomorphic for a compact Kähler $X$, see Section 1.4.

We will take the “physical” point of view and discuss evidence for the Formality Conjecture after seeing what implications it has.

**1.2. Deformation quantization of Poisson manifolds.** Recall that a deformation quantization [BFF78] of a Poisson manifold $X$, whose algebra of smooth functions will be denoted by $A$, as above, is a formal deformation of $A$ in the direction of the Poisson bracket. More precisely, it is a multiplication $a \ast b$ on $A[[\hbar]] = A \otimes \mathbb{R}[[\hbar]]$ making it an associative $\mathbb{R}[[\hbar]]$-algebra, such that for $a, b \in A$

\[
a \ast b = ab + \{a, b\} \hbar + B(a, b) \hbar^2 + \ldots,
\]
where $ab$ is the usual, undeformed multiplication and $\{a, b\}$ is the Poisson bracket. When the Poisson bracket is nondegenerate, i.e., coming from a symplectic structure, the existence of deformation quantization was proven by De Wilde and Lecomte [DWL83] and Fedosov [Fed85]. When the Poisson bracket is arbitrary, the existence of deformation quantization (even locally, for $\mathbb{R}^n$) is an open problem.

The remarkable fact noticed by Kontsevich is that if you assume the Formality Conjecture, the problem of quantization will be solved.

**Theorem 1.3 (Kontsevich).** The Formality Conjecture for a manifold $X$ implies deformation quantization of any Poisson structure on $X$.

**Proof.** We will only sketch the idea of the proof; (some) details may be found in Kontsevich’s Berkeley lectures [Kon95b].

According to Deligne-Schlessinger-Stasheff-Goldman-Millson’s approach to deformation theory, see [GM90, Mil91, SS85], with each DGLA $L^\bullet$ one can associate the formal moduli space $M = \{ \gamma \in L^1 \mid d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \}/\text{exp}(L^0)$, where $\text{exp}(L^0)$ is the Lie group corresponding to the Lie algebra $L^0$, and the defining equation $d\gamma + \frac{1}{2}[\gamma, \gamma] = 0$ is called the Maurer-Cartan equation. If $L^\bullet_1 \rightarrow L^\bullet_2$ is a quasi-isomorphism of DGLA’s, then the corresponding formal moduli spaces can be identified. This is done using the standard machinery of minimal models.

Formal deformations are usually formal paths in the formal moduli spaces. Consider the cases of the above two DGLA’s associated to a manifold $X$. The formal moduli space associated to the DGLA $C^\bullet$ is $M_Q = \{ \gamma \in C^2(A, A) \mid d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \}/\text{GL}(A)$. Since the Hochschild differential $d$ is equal to the bracket $[m, ]$ with the multiplication two-cocycle $m \in C^2(A, A)$, the Maurer-Cartan equation may be rewritten as $[m + \gamma, m + \gamma] = 0$, which is equivalent to the associativity of $m + \gamma \in \text{Hom}(A \otimes A, A)$ understood as a new multiplication. A deformation quantization (in an arbitrary direction) is then a formal path originating at $\gamma = 0$ in the formal moduli space $M_Q$. If this deformation quantizes a Poisson bracket, the tangent vector to the formal path should coincide with the Poisson bracket.

The formal moduli space associated to the DGLA $H^\bullet$ of multivector fields on $X$ is $M_P = \{ \gamma \in \Lambda^2TX \mid [\gamma, \gamma] = 0 \}/\text{exp}(\text{Vect} X)$. The Maurer-Cartan equation in this case is equivalent to the Jacobi identity for the skew bracket $\{f, g\} = \gamma(df, dg)$ of functions on $X$. Thus a solution of the Maurer-Cartan equation is a Poisson structure on $X$.

Now suppose that the Formality Conjecture is true. Then the two moduli spaces $M_Q$ and $M_P$ are identified. Given a Poisson structure on $X$, we can connect it by a straight line with the origin in the moduli space of Poisson structures. Consider this line as a formal path. Using the isomorphism of the moduli spaces, we have a formal path in the moduli space of quantizations, which is a deformation quantization we were looking for.

**1.3. Evidence for the Formality Conjecture.** It is known that every nondegenerate Poisson structure can be quantized [DWL83, Fed85]. Moreover, the following analogue of the conjecture related to the nondegenerate case is true. Consider the Hochschild DGLA of the function algebra with respect to the deformed multiplication on a symplectic manifold $X$. The other DGLA will be the multivector fields on $X$ with the differential being the Schouten-Nijenhuis bracket with the canonical Poisson tensor on $X$. Then the two DGLA’s are quasi-isomorphic. One uses Fedosov’s connection to prove this fact. [Kon95a].
Different evidence comes from quantizing an arbitrary Poisson structure in the Lie-theoretic context. The recent theorem of P. Etingof and D. Kazhdan [EK96] solves the conjecture of Drinfeld asserting that every Poisson Lie group has a canonical quantization.

Mirror Symmetry predicts that for a Calabi-Yau manifold \( Y \), the corresponding holomorphic version \( \mathcal{R} \Gamma(Y, C^\bullet) \) of the Hochschild DGLA for the sheaf of holomorphic functions on \( Y \) gives rise to a smooth formal moduli space, which may be interpreted as the moduli space of “noncommutative Calabi-Yau manifolds”. On the other hand, one can show that the holomorphic multivector field DGLA \( \mathcal{R} \Gamma(Y, H^\bullet) \) produces a smooth formal moduli space. Thus, if \( C^\bullet \) and \( H^\bullet \) were known to be quasi-isomorphic, it would prove the smoothness of the first moduli space as confirmed by Mirror Symmetry.

1.4. Formality in rational homotopy theory. Kontsevich’s Formality Conjecture has a very close analogy with the Deligne-Griffiths-Morgan-Sullivan Formality Theorem [DGMS75]: the Sullivan model of a compact Kähler manifold \( X \) is formal. The Sullivan model of \( X \) may be represented by the differential graded commutative algebra (DGA) \( \Omega^\bullet(X) \) of smooth differential forms on \( X \). Formality means that \( \Omega^\bullet(X) \) is quasi-isomorphic to its cohomology DGA \( H^\bullet(X) \). A simple way to prove this is using Hodge theory, see [Sul77]: decompose the de Rham differential into the holomorphic and antiholomorphic parts: \( d = \partial + \bar{\partial} \). Standard Hodge-theoretic arguments (the \( \partial-\bar{\partial} \)-Lemma of [DGMS75]) imply that \((\text{Ker} \partial, d) \subset (\Omega^\bullet(X), d)\) is an embedding of DGA’s, which is a quasi-isomorphism. On the other hand, the natural morphism \((\text{Ker} \partial, d) \to (\text{Ker} \partial/\text{Im} \partial, d) = (H^\bullet(X), 0)\) of DGA’s is also a quasi-isomorphism for the same reasons.

2. Hodge theory for the Hochschild complex

In this section, we are going to develop Hodge theory in the Hochschild context. The construction of Hodge decomposition of the Hochschild complex of a commutative algebra \( A \) over a field of characteristic zero goes back to Gerstenhaber and Schack [GS87], who decomposed the Hochschild complex \( C^\bullet(A, A) \) into the direct sum of \( C^{p,q}(A, A) \), with the Hochschild differential \( d \) acting like \( \partial \) in the Dolbeault complex: \( d : C^{p,q}(A, A) \to C^{p-1,q+1}(A, A) \). Here we add a new ingredient to Gerstenhaber-Schack’s Hodge theory: we define an extra, \( \partial \)-like differential \( d' : C^{p,q}(A, A) \to C^{p-1,q}(A, A) \) on the Hochschild complex, so that it becomes a bicomplex. This bicomplex is similar to the \( \partial-\bar{\partial} \)-complex of a compact Kähler manifold: the total cohomology of the bicomplex is equal to the cohomology of one of the differentials. Our new differential is also similar to the differential \( B \) of the cyclic cohomology complex. Together with the Hochschild differential, the differential \( B \) provides the cyclic cohomology complex with the structure of a bicomplex and, moreover, respects the Hodge decomposition of the cyclic cohomology complex in a similar way, see J.-L. Loday [Lod89]. Another similarity between the cyclic \( B \) and our differential is that the cohomology of both vanish.

We will recall Hodge decomposition of the Hochschild complex, following the modification of M. Ronco, A. B. Sletsjøe, and H. L. Wolfgang, see [BW93] for more detail. Let \( r \) and \( s \) be positive integers and \( n = r + s \). The shuffle product of tensors \( a_1 \otimes \cdots \otimes a_r \in A^{\otimes r} \) and \( a_{r+1} \otimes \cdots \otimes a_n \in A^{\otimes s} \) is the element

\[
\sum \text{sgn}(\sigma) a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} \in A^{\otimes n},
\]
where the summation runs over those \( \sigma \in S_n \) for which \( \sigma(1) < \sigma(2) < \cdots < \sigma(r) \) and \( \sigma(r+1) < \cdots < \sigma(n) \). Let \( \text{Sh}_k \) denote the image of shuffle products of \( k \) elements in the tensor algebra \( T(A) = \bigoplus_{n \geq 0} A^\otimes n \). By definition \( \text{Sh}^0 = T(A) \) and \( \text{Sh}^1 = \bigoplus_{n \geq 0} A^\otimes n \). We have a filtration of the tensor algebra

\[
T(A) = \text{Sh}^0 \supset \text{Sh}^1 \supset \text{Sh}^2 \supset \text{Sh}^3 \ldots
\]

Define

\[
C^{p,q} = \text{Hom}(\text{Sh}^p \cap A^\otimes p+q / \text{Sh}^{p+1} \cap A^\otimes p+q, A),
\]

where \( p,q \geq 0 \). Of course, one can describe \( C^{p,q} \) as \( A \)-valued functionals \( \phi \) on the subspace of \( A^\otimes p+q \) generated by the shuffle products \( \text{Sh}^p \) of \( p \) elements, such that \( \phi \) vanishes on the shuffle products of \( p+1 \) elements. One can check that the Hochschild differential induces a mapping \( d : C^{p,q} \to C^{p,q+1}, d^2 = 0 \), and \( C^n(A,A) \cong \bigoplus_{p+q=n} C^{p,q} \).

This gives the **Hodge decomposition** of the Hochschild complex into the direct sum of complexes:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & C^{2,0} & \xrightarrow{d} & C^{2,1} & \xrightarrow{d} & C^{2,2} & \xrightarrow{d} & \ldots \\
0 & \longrightarrow & C^{1,0} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & C^{1,2} & \xrightarrow{d} & \ldots \\
0 & \longrightarrow & C^{0,0} & \xrightarrow{d} & 0 & \longrightarrow & 0 & \longrightarrow & \ldots 
\end{array}
\]

The second-to-last row is known as the **Harrison complex**\(^1\). The Hodge decomposition of the Hochschild complex induces one on the Hochschild cohomology: \( H^n(A,A) \cong \bigoplus_{p+q=n} H^{p,q}(A,A) \).

**Theorem 2.1.** 1. There exists a differential \( d' : C^{p,q} \to C^{p-1,q} \) which is a derivation of the Gerstenhaber bracket and defines the structure of a bicomplex on the Hochschild complex:

\[
\begin{array}{ccccccc}
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \longrightarrow & C^{2,0} & \xrightarrow{d} & C^{2,1} & \xrightarrow{d} & C^{2,2} & \xrightarrow{d} & \ldots \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \longrightarrow & C^{1,0} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & C^{1,2} & \xrightarrow{d} & \ldots \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \longrightarrow & C^{0,0} & \xrightarrow{d} & 0 & \longrightarrow & 0 & \longrightarrow & \ldots \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots
\end{array}
\]

2. The spectral sequence associated to the first filtration \( \mathcal{F}^p = \bigoplus_{i \leq p} C^{i,j} \) is convergent: \( E_\infty = H^*(C^{*,*}, d+d') \). Moreover, \( E_1 \) is equal to the Hochschild cohomology \( H^*(A,A) \).

\(^1\)The last row does not have a name yet, but will hopefully acquire one soon.
3. The cohomology of the differential $d'$ vanishes. The spectral sequence associated to the second filtration $^{"}F_q = \bigoplus_{j \geq q} C^{i,j}$ collapses at $^{"}E_1$, which is equal to 0.

4. Suppose that $A$ is the algebra of smooth functions on a manifold or regular functions on a nonsingular affine scheme. Then the first spectral sequence collapses at $'E_1$, which is equal to $H^{\bullet,0}(A, A)$, the space of global multivector fields. This coincides with the total cohomology of the bicomplex.

Remark 2.2. The differential $d'$, being a derivation of the Gerstenhaber bracket of degree $-1$, defines the structure of a DGLA on the Hochschild complex $(C^{\bullet}(A, A)[1], d', d'')$. However, the total complex $(C^{\bullet}(A, A)[1], d + d'')$ is only a differential $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra: the degree of the total differential $d + d''$ is equal to one modulo two.

Proof. 1. Define $d'$ as the inner derivation

$$d' \phi = [1, \phi],$$

where $[,]$ is the Gerstenhaber bracket (see Section 1.1) and $1 \in A = C^0(A, A)$ is the unit element of $A$. In other words,

$$(d' \phi)(a_1, \ldots, a_{n-1}) = \phi(a_1, \ldots, a_{n-1}, 1) - \phi(a_1, \ldots, a_{n-2}, 1, a_{n-1}) + \cdots + (-1)^{n-1} \phi(1, a_1, \ldots, a_{n-1}).$$

Then $(d')^2 = 0$, because $[1, 1] = 0$, and $d'd + dd' = 0$, because $d = [m, ]$, where $m \in C^2(A, A)$ is the multiplication cocycle, and $[m, 1] = 0$: $[m, 1](a) = m(1, a) - m(a, 1) = 1 \cdot a - a \cdot 1 = 0$ for any $a \in A$. It is also clear that $d' : C^n(A, A) \rightarrow C^{n-1}(A, A)$. Let us verify that moreover $d' : C^{p,q} \rightarrow C^{p-1,q}$. Indeed, if $\phi$ is an $A$-valued functional defined on the $p$-shuffles $Sh^p$, then $d'\phi$ is obviously defined on $p - 1$-shuffles via the natural mapping $Sh^{p-1} \rightarrow Sh^p$, the shuffle product with $1 \in A$. For the same reason, if $\phi$ vanishes on $Sh^{p+1}$, then $d'\phi$ will vanish on $Sh^p$. This proves the first statement of the theorem.

2. The first spectral sequence is convergent, because the first filtration is regular: moreover, $'F^{-1} = 0$. Since $'^{FP} / '^{FP+1} = C^{p,\bullet}$ with the differential $d$, $'E'_1 = H^{\bullet}( '^{FP} / '^{FP+1}, d) = H^{\bullet,\bullet}(A, A)$.

3. We will define for each $n \geq 0$ a null-homotopy $k : C^n(A, A) \rightarrow C^{n+1}(A, A)$ on the complex $C^{\bullet}(A, A)$ with respect to the differential $d'$:

$$(k\phi)(a_1, \ldots, a_{n+1}) = \frac{1}{n+1} \sum_{i=0}^{n+1} (-1)^{i-1} a_i \phi(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1}).$$

If $\phi = 0$ on $Sh'$, then $k\phi = 0$ on $Sh'^{+1}$, therefore $k$ is well-defined on $C^{p,q}$ and maps it to $C^{p+1,q}$. A straightforward computation shows that $kd' + d'k = id$. Thus, the cohomology of $d'$ vanishes. Since $'^{"}F_q / '^{"}F_{q+1} = C^{\bullet,q}$ with the differential $d'$, $'^{"}E'_1 = H^{\bullet}('^{"}F_q / '^{"}F_{q+1}, d') = 0$, and the second spectral sequence collapses.

4. If $A$ is a regular algebra of functions, its Hochschild cohomology $H^{\bullet}(A, A)$ is equal to the space of multivector fields, see [HKR62]. The multivector fields are skew multiderivations of $A$ and therefore project bijectively on $H^{\bullet,0}(A, A)$. In this case, the differential $d'$ vanishes on all Hochschild cocycles, because derivations of $A$ vanish on constants. Therefore $'E_1 = 'E_2 = \cdots = 'E_\infty = H^{\bullet,0}(A, A)$. The computation of the total cohomology then follows from Part 2. \qed
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