On the role of Planck’s oscillator in the construction of Heisenberg’s mechanics

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In the present note we elucidate how physical considerations based on Planck’s oscillator led to the construction of Heisenberg’s mechanics.

I read with the greatest interest the recent paper\textsuperscript{1} by Aitchison, Mac-Manus and Snyder on the epoch-making July 1925 paper\textsuperscript{2} of Heisenberg. They say that this paper of Heisenberg is widely regarded as being difficult to understand mainly because he gave remarkably few details of the calculations he performed; and in particular, he simply wrote down the quantum version:

$$\cos \omega(n, n - 0)t : \omega_0^2 a_0(n) + \frac{1}{4} \left[ a^2(n + 1, n) + a^2(n, n - 1) \right] = 0,$$

(1a)

$$\cos \omega(n, n - 1)t : -\omega^2(n, n - 1) + \omega_0^2 = 0,$$

(1b)

$$\cos \omega(n, n - 2)t : \left[ -\omega^2(n, n - 2) + \omega_0^2 \right] a(n, n - 2) +$$

$$\frac{1}{2} \left[ a(n, n - 1)a(n - 1, n - 2) \right] = 0,$$

(1c)

$$\cos \omega(n, n - 3)t : \left[ -\omega^2(n, n - 3) + \omega_0^2 \right] a(n, n - 3) +$$

$$\frac{1}{2} \left[ a(n, n - 1)a(n - 1, n - 3) + a(n, n - 2)a(n - 2, n - 3) \right] = 0,$$

(1d)

of the classical equations:

$$\cos 0 \omega t : \omega_0^2 a_0(n) + \frac{1}{2} a_1^2(n) = 0,$$

(2a)

$$\cos 1 \omega t : (-\omega^2 + \omega_0^2) = 0,$$

(2b)
\[
\cos 2\omega t : \left( -(2\omega)^2 + \omega_0^2 \right) a_2 + \frac{1}{2} a_1^2 = 0, \quad (2c) \\
\cos 3\omega t : \left( -(3\omega)^2 + \omega_0^2 \right) a_3 + a_1 a_2 = 0, \quad (2d)
\]

which are obtained by substituting

\[
x = \lambda a_0 + a_1 \cos \omega t + \lambda a_2 \cos 2\omega t + \lambda^2 a_3 \cos 3\omega t + \cdots + \lambda^{r-1} a_r \cos r\omega t \quad (3)
\]
in the classical equation of motion

\[
\ddot{x} + \omega_0^2 x + \lambda x^2 = 0 \quad (4)
\]

for the anharmonic oscillator, the left-hand sides (LHS) of Eqs. (2) being, respectively, the coefficients of \( \cos 0 \omega t \) (= 1), \( \cos 1 \omega t \), \( \cos 2 \omega t \) and \( \cos 3 \omega t \), which we have expressly indicated. (The reason for doing so will become evident soon.)

How did Heisenberg arrive at Eqs. (1)?, ask Aitchison et al\(^4\). And then they go on to show how Eqs. (1) are derived from the recursion relation, Eq. (22) of their paper\(^1\), and conjecture that this is what Heisenberg\(^2\) did, calling their approach to be Heisenberg’s.

In the present note we elucidate how physical considerations\(^4\) based on the quantum oscillator of Planck enabled Heisenberg to simply write down Eqs. (1) – with no details necessary.

The energies of the stationary states of the Planck oscillator are given by

\[
W(n) = \frac{n \hbar \omega}{2\pi}, \quad (5)
\]

with \( \omega \) the frequency of the oscillator \( x = a_1 \cos \omega t \). Let us label these [Planck stationary states] as

\[
\cdots, n-3, n-2, n-1, n, n+1, n+2, n+3, \cdots
\]

Eq. (5) can be rewritten in the form

\[
\omega = \frac{2\pi}{\hbar} \left[ W(n) - W(n-1) \right] = \frac{2\pi}{\hbar} \left[ W(n+1) - W(n) \right], \quad (6)
\]

\[
e_{n \rightarrow n-1}^{\text{emission}} \quad a_{n+1-n}^{\text{absorption}}
\]

which says that frequency of the emitted or absorbed Planck quantum (light) is the same\(^5\) as that of the oscillator \( (a_1 \cos \omega t) \) and that the Planck oscillator makes transitions only to neighbouring stationary states by emitting or
absorbing one quantum of energy \( \frac{\hbar \omega}{2\pi} \) or of frequency \( \omega \). Furthermore, if the Planck oscillator is to go from the state \( n \) to the state \( n - 2 \), it got to emit two quanta of frequency \( \omega \), one to make the transition from the state \( n \) to the state \( n - 1 \), and another to make the transition from \( n - 1 \) to \( n - 2 \). Now suppose there is another Planck oscillator but of frequency \( 2\omega \) \((a_2 \cos 2\omega t)\) which acts independently of the one with frequency \( \omega \). Then the transition from \( n \) to \( n - 2 \) can also be achieved by the emission of a single quantum of frequency \( 2\omega \). In the same vein, a transition from \( n \) to \( n - 3 \) can occur by the emission of a single quantum of frequency \( 3\omega \) \((a_3 \cos 3\omega t)\) or by the emission of two quanta, one of frequency \( 2\omega \) from \( n \) to \( n - 2 \) and the other of frequency \( \omega \) from \( n - 2 \) to \( n - 3 \), or by the emission of three quanta each of frequency \( \omega \); and a transition from \( n \) to \( n - 2 \) can occur by the absorption of a single quantum of frequency \( \omega \) to go from \( n \) to \( n + 1 \) and then the emission of a single quantum of frequency \( 3\omega \) to go from \( n + 1 \) to \( n - 2 \); etc.

Now in his general theory of line spectra Bohr’s second assumption states that the radiation absorbed or emitted during a transition between two stationary states \( E' \) and \( E'' \) possesses a frequency \( \nu \) \((= \frac{\omega}{2\pi})\), given by the relation \(^6\)

\[
\omega = \frac{2\pi}{\hbar} (E' - E'').
\]  

\( (7) \)

Note that implicit in this assumption is the notion of the Planck oscillator. If one analogously labels the Bohr stationary states as

\[
\cdots, n - 3, n - 2, n - 1, n, n + 1, n + 2, n + 3, \cdots
\]

(with \( n \) the reference state) then Eq. (7) can be rewritten as

\[
\omega(n, n - 1) = \frac{2\pi}{\hbar} \left[ W(n) - W(n - 1) \right];
\]

\( n \rightarrow n - 1 \) emission \( (8a) \)

or

\[
\omega(n, n + 2) = \frac{2\pi}{\hbar} \left[ W(n) - W(n + 2) \right];
\]

\( n \rightarrow n + 2 \) absorption \( (8b) \)

or

3
\[ \omega(n+4, n-1) = \frac{2\pi}{h} [W(n+4) - W(n-1)]; \]

\[ n+4\rightarrow n-1 \text{ emission} \]

etc., the frequencies \( \omega(n, n-1), \omega(n, n+2), \omega(n+4, n-1), \) etc. being different from one another, represent independent Planck oscillators. In other words, the Planck oscillator of frequency \( \omega(n, n-1) \) cannot make a transition from \( n \) to \( n-2 \) or from \( n \) to \( n-3 \), etc. Thus if one wants to make a quantum mechanics of the spectral lines and their intensities, one needs to make quantum mechanics of the Planck harmonic oscillator as Bohr’s jumps are the jumps (transitions) of the Planck oscillators. Therefore, first and foremost, Heisenberg tried to do just that – with his expertise in Bohr-Sommerfeld theory, dispersion theory, and his knowledge of Born’s prescription to translate classical quantities into quantum mechanical ones. And he succeeded.

Heisenberg’s construction is reproduced on page 26 of Ref. 3. We recast it as follows:

For the simple harmonic oscillator (\( \omega = 1\omega \))

\[ x = a_1 \cos \omega t \] (9)

the action variable

\[ J = \oint p\, dx = \int_0^{2\pi/\omega} m\ddot{x} \, dt = \pi ma_1^2 \omega; \] (10)

or

\[ 1 = \pi m \frac{\partial}{\partial J}(a_1^2 \omega). \] (11)

Then using Born’s prescription, (11) becomes

\[ h = \pi m \left[ a^2(n+1, n)\omega(n+1, n) - a^2(n, n-1)\omega(n, n-1) \right] \] (12)

as the Planck oscillator is characterised by only two distinct amplitudes, \( a(n, n+1) \) for absorption of a Planck quantum of frequency \( \omega(n, n+1) = \omega \), and \( a(n, n-1) \) for emission of a quantum of the same frequency \( \omega = \omega(n, n-1) \). Eq. (12) is then rewritten as

\[ h = \pi m \omega \left[ a^2(n+1, n) - a^2(n, n-1) \right], \] (13)
which has the solution
\[ a^2(n, n - 1) = \frac{nh}{m\pi\omega}, \] (14)
with the stipulation \( a(0, -1) = 0 \), i.e. there is a lowest bound for the Planck energy levels, namely \( n = 0 \), and there can be no emission of a quantum from it.

Now one can calculate the energy of the \( n \)-th Planck state of the oscillator by substituting (9) in
\[ W = \frac{1}{2}m\left(x^2 + \omega^2x^2\right) \] (15)
and obtaining
\[ W = \frac{1}{2}m\omega^2a_1^2. \] (16)
But as \( a_2^2 \) has only two distinct possible values, \( a^2(n + 1, n) \) and \( a^2(n, n - 1) \), Eq. (16) becomes
\[ W(n) = \frac{1}{2}m\omega^2a_1^2 \left[a^2(n + 1, n) + a^2(n, n - 1)\right], \] (17)
or, using (14),
\[ W(n) = \left(n + \frac{1}{2}\right)\omega\frac{h}{2\pi}. \] (18)
Thus Heisenberg made the quantum mechanics of the harmonic oscillator\textsuperscript{12,13}.

But this was not enough. In order to solve the problem of atomic spectral lines and their intensities, one needed to know the relation between different transition amplitudes, say, for example, between \( a(n, n - 2) \) [the transition amplitude from state \( n \) to state \( n - 2 \) of frequency \( \omega(n, n - 2) \)] and \( a(n, n - 1) \) [the transition amplitude from state \( n \) to \( n - 1 \) of frequency \( \omega(n, n - 1) \)], etc. So Heisenberg went back to his favourite\textsuperscript{14} anharmonic oscillator. And there it [the relation he sought for] was in Eq. (2c). The amplitude \( a_2 \), which represents in Planck’s oscillator language, the transition amplitude from state \( n \) to \( n - 2 \) of frequency \( \omega(n, n - 2) \) as well as from state \( n \) to \( n + 2 \) of frequency \( \omega(n, n + 2) \), was expressible as the product of \( a_1 \) and \( a_1 \) (and not as the sum of \( a_1 \) and \( a_1 \)), \( a_1 \) being the transition amplitude from state \( n \) to \( n - 1 \) or from \( n - 1 \) to \( n - 2 \), etc.\textsuperscript{15} In view of the discussion given earlier, Eq. (2c) has the obvious translation in terms of Bohr labeling: \( 2\omega \rightarrow \omega(n, n - 2) \); \( a_2 \rightarrow a(n, n - 2) \); \( a_2^2 \rightarrow a(n, n - 1)a(n - 1, n - 2) \), and is simply written as Eq. (1c). Similarly in Eq. (2d), \( 3\omega \rightarrow \omega(n, n - 3) \); \( a_3 \rightarrow a(n, n - 3) \); and
\[ a_1 a_2 \rightarrow \frac{1}{2} [a(n, n-1)a(n-1, n-3) + a(n, n-2)a(n-2, n-3)] \]

as the transition from state \( n \) to state \( n - 3 \) can occur in two possible ways: by a transition from \( n \rightarrow n - 1 \) by the action of \( a_1 \) followed by the transition from \( n - 1 \) to \( n - 3 \) by the action of \( a_2 \); or by a transition from \( n \rightarrow n - 2 \) by the action of \( a_2 \) followed by the transition from \( n - 2 \rightarrow n - 3 \) by the action of \( a_1 \); and it is simply written as Eq. (1d). Eq. (2b) obviously translates into Eq. (1b) which simply verifies that the transition frequency \( \omega(n, n - 1) \) of the Planck oscillator is the same as the frequency of the classical oscillator. Now the LHS of Eq. (2a) is the constant term, i.e. it is multiplied by \( \cos 0 \omega t \), and thus represents no transition from the state \( n \). Hence \( a_0(n) \rightarrow a_0(n, n - 0) \equiv a_0(n); \) since \( a_1^2 \) can act on state \( n \) in two ways: \( a(n, n+1)a(n+1, n) \) and \( a(n, n-1)a(n-1, n), a_1^2 \rightarrow \frac{1}{2} [a^2(n+1, n) + a^2(n, n-1)] \) with \( a(n, n+1) = a(n+1, n) \) and \( a(n-1, n) = a(n, n-1) \). And Eq. (2a) translates into Eq. (1a).16 Aitchison et al.1 add the terms of order \( \lambda^2 \) to Eqs. (2) (see Eqs. (27) of Ref. 1), the term added to (2d) being

\[ \cos 3 \omega t : \lambda^2(a_1 a_4 + 2a_0 a_3) + \cdots. \]  

Let us see how it translates into quantum theoretical labeling. Since it multiplies \( \cos 3 \omega t \), it represents the transition from the state \( n \) to \( n - 3 \) of frequency \( \omega(n, n - 3) \). Now by the action of \( a_1 a_4 \) this can only be achieved by \( a(n, n+1)a(n+1, n-3) \) or by \( a(n, n-4)a(n-4, n-3) \), i.e. \( a_1 a_4 \rightarrow \frac{1}{2} [a(n, n+1)a(n+1, n-3) + a(n, n-4)a(n-4, n-3)] \); and by the action of \( a_0 a_3 \) by \( \frac{1}{2} [a_0(n) a(n, n-3) + a(n, n-3) a_0(n-3, n-3)] \). So the expression (19) translates into

\[ \lambda^2 \left[ \frac{1}{2} \{a(n, n+1)a(n+1, n-3) + a(n, n-4)a(n-4, n-3)\} + \right. \]

\[ \left. \{a_0(n) a(n, n-3) + a(n, n-3) a_0(n-3, n-3)\} \right], \]

which is the same that appears in Eq. (46) of their paper1.

Put succinctly, one can indeed write down the quantum version of Eqs. (2) [or Eqs. (27) of Aitchison et al.1] simply by considerations based on the Planck oscillator.

One may note that the energies of the stationary states given by (18) for the simple harmonic oscillator come from (16) – the constant term (no periodic terms) – strongly suggesting that the energies of the anharmonic
oscillator should likewise be given by constant terms which result after substituting (3) in
\[ W = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2x^2 + \frac{1}{3}m\lambda x^3. \] (21)
To first order in \( \lambda \) the only constant term that results (after using trigonometric identities) is \( \frac{1}{2}m\omega_0^2a_1^2 \) giving \( W(n) \) by Eq. (18). We emphasize that all periodic terms vanish — a fact that has been verified by Aitchison et al.\(^1\), showing thereby that the quantum theoretical labeling is indeed internally consistent.

The energies of the stationary states of the anharmonic oscillator
\[ \ddot{x} + \omega_0^2x + \lambda x^3 = 0 \] (22)
are likewise calculable by substituting
\[ x = a_1 \cos \omega t + \lambda a_3 \cos 3\omega t + \lambda^2 a_5 \cos 5\omega t + \cdots \] (23)
in the expression
\[ W = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2x^2 + \frac{1}{4}m\lambda x^4 \] (24)
and finding the constant terms after using trigonometric identities. They turn out to be (to first order in \( \lambda \))
\[ W = \frac{1}{2}m\omega_0^2a_1^2 + \frac{1}{4}m\lambda \frac{3}{8}(a_1^2)^2, \] (25)
or
\[ W(n) = \left( n + \frac{1}{2} \right) \omega_0 \frac{h}{2\pi} + \lambda \frac{3}{8} \frac{(n^2 + n + \frac{1}{2})h^2}{4\pi^2\omega_0^2m}, \] (26)
after the use of (14). One may note that the anharmonic oscillator (22) represents the Planck oscillators of transition frequencies
\[ \omega(n, n - 1) = \frac{2\pi}{h} \left[ W(n) - W(n - 1) \right] \]
\[ = \omega_0 + \lambda \frac{3}{8} \frac{hn}{\pi \omega_0^2 m}, \] (27)
and transition amplitudes\(^1^7\)
\[ a(n, n - 1) = \sqrt{\frac{nh}{m\pi \omega(n, n - 1)}} = \sqrt{\frac{nh}{m\pi \omega_0}} \left[ 1 - \lambda \frac{3}{16} \frac{hn}{\pi \omega_0^3 m} \right]. \] (28)
Before closing we emphasize that the quantum method given by Heisenberg could be applied to solving the real problem of atomic spectral lines and their intensities only in its reformulated form by Born and Jordan, as demonstrated by Pauli in his solution of the Hydrogen spectrum.

Last, we may remark that the paper of Aitchison et al. and the present note teach unequivocally the lesson – which every student must learn – that creative work comes about via many routes, including (i) mathematical expertise and sophistication, and (ii) physical insight and considerations.

ACKNOWLEDGEMENTS

I thank Heike Moore and Matthias Burkardt for making translations from German to English for me, and R.S. Bhalerao for a critical reading of the manuscript.

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2. W. Heisenberg, “Quantum-theoretical re-interpretation of kinematic and mechanical relations”, Z. Phys. 33, 879-893 (1925); English translation in Ref. 3, pp. 261-276.

3. Sources of Quantum Mechanics, edited by B.L. van der Waerden (Dover Publications, New York, 1968).

4. In tune with the character of Heisenberg’s physics; in this connection, see Pauli’s reaction, Ref. 3, p. 37.

5. Ref. 3, p. 22.

6. Ref. 3, p. 5.

7. In this connection see also Ref. 3, p. 189.
8. The method of trial and error is indispensable to research; see, for example, B. Ram, “Presenting the Planck’s relation $E = n\hbar\nu$”, Am. J. Phys. 42, 1092-1094 (1974).

9. Ref. 3, pp. 21-22.

10. H.A. Kramers and W. Heisenberg, “On the dispersion of radiation by atoms”, Z. Phys. 31, 681-708 (1925); English translation in Ref. 3, pp. 223-252.

11. M. Born, “Quantum Mechanics”, Z. Phys. 26, 379-395 (1924); English translation in Ref. 3, pp. 181-198; see footnote 3 on p. 182.

12. As envisioned by Planck; see I. Duck and E.C.G. Sudarshan, 100 Years of Planck’s Quantum (World Scientific, Singapore, 2000), pp. 51 and 73.

13. One may note that the extra term $\frac{1}{2}\omega\hbar$ had already appeared in the work of Planck; see M. Planck, The Theory of Heat Radiation (Dover Publications, New York, 1991) [originally published in 1914], p. 142.

14. See Ref. 3, pp. 23-24.

15. Remembering that the Planck oscillator makes transitions only to neighboring states.

16. Here we may comment that actually $a^2$ in Eq. (16) should be written as $\frac{1}{2}[a(n, n + 1)a(n + 1, n) + a(n, n - 1)a(n - 1, n)]$, which is equivalent to the expression on the RHS of Eq. (17).

17. This verifies that the energies of stationary states must be given by constant terms.

18. M. Born and P. Jordan, “On quantum mechanics”, Z. Phys. 34, 858-888 (1925); English translation in Ref. 3, pp. 277-306.

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