MONOTONE VERSIONS OF $\delta$-NORMALITY

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Abstract. We continue the study of properties related to monotone countable paracompactness, investigating various monotone versions of $\delta$-normality. We factorize monotone normality and stratifiability in terms of these weaker properties.

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1. Introduction

Dowker [1] proves that the product of a space $X$ and the closed unit interval $[0, 1]$ is normal iff $X$ is both normal and countably paracompact. Mack [11] proves that a space $X$ is countably paracompact iff $X \times [0, 1]$ is $\delta$-normal and that every countably paracompact space is $\delta$-normal (see below for definitions).

In [6] and its sequels [3, 5], the first author et al. introduce and study a monotone version of countable paracompactness (MCP) closely related to stratifiability. In [4], the current authors consider various other possible monotone versions of countable paracompactness and the notion of $\text{m}\delta\text{n}$ (monotone $\delta$-normality) arises naturally in this study. It turns out that MCP and $\text{m}\delta\text{n}$ are distinct properties and that, if $X \times [0, 1]$ is $\text{m}\delta\text{n}$, then $X$ (and hence $X \times [0, 1]$) is MCP.

In this paper we take a closer look at monotone versions of $\delta$-normality.

Our notation and terminology are standard as found in [2] or [8]. All spaces are assumed to be $T_1$ and regular.

2. Monotone versions of $\delta$-normality

Definition 1. Let $X$ be a space. A subset $D$ of $X$ is said to be a regular $G_\delta$-set iff there exist open sets $U_n$, $n \in \omega$, such that $D \subseteq U_n$ for each $n$ and $D = \bigcap_{n \in \omega} \overline{U_n}$.
Clearly, a set $D$ is a regular $G_\delta$-set iff there exist open sets $U_n$, $n \in \omega$, such that $D = \bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} \overline{\bigcup_{n \in \omega} U_n}$.

**Definition 2.** $X$ is said to be $\delta$-normal [11] iff any two disjoint closed sets, one of which is a regular $G_\delta$-set, can be separated by open sets.

$X$ is said to be weakly $\delta$-normal [10] iff any two disjoint regular $G_\delta$-sets can be separated by open sets.

We note in passing the following facts about regular $G_\delta$-sets. Finite unions and countable intersections of regular $G_\delta$-sets are again regular $G_\delta$. If $X$ is $T_3$, for every $x \in X$ and every open neighbourhood $V$ of $x$ there exists a regular $G_\delta$-set $K$ such that $x \in K \subseteq V$. In any space $X$, the zero-sets are regular $G_\delta$-sets and so in a normal space $X$, if $C$ is a closed set contained in an open set $U$, then there exists an open set $W$ such that $W$ is the complement of a regular $G_\delta$-set and $C \subseteq W \subseteq \overline{W} \subseteq U$. If $E$ is a regular $G_\delta$-set in $X$, then $E \times \{\alpha\}$ is a regular $G_\delta$-set in $X \times M$ for any infinite compact metrizable space $M$ and $\alpha \in M$. If $Y$ is any compact space, since the projection map is both closed and open, then the projection of a regular $G_\delta$-set in $X \times Y$ is itself a regular $G_\delta$-set in $X$. On the other hand, a regular $G_\delta$-subset of a regular $G_\delta$-subset of $X$ is not necessarily a regular $G_\delta$-set in $X$: for example, the x-axis, $A$, is a regular $G_\delta$-subset of the Moore plane and every subset of $A$ is a regular $G_\delta$-subset in $A$.

Let us make the following definition.

**Definition 3.** Let $X$ be a space and $\mathcal{C}$ be a collection of pairs of disjoint closed sets. We shall say that $H$ is a $\mathcal{C}$-mn operator on $X$ iff $H$ assigns to each pair $(C,D) \in \mathcal{C}$ an open set $H(C,D)$ such that

1. $C \subseteq H(C,D) \subseteq \overline{H(C,D)} \subseteq X \setminus D$,
2. if $C \subseteq C'$ and $D' \subseteq D$, then $H(C,D) \subseteq H(C',D')$.

**Definition 4.** Let $H$ be a $\mathcal{C}$-mn operator on $X$.

1. If $\mathcal{C}$ is the collection of pairs of disjoint closed subsets of $X$, then $X$ is monotonically normal.
2. If $\mathcal{C}$ is the collection of disjoint closed subsets $(C,D)$ such that $C$ is a regular $G_\delta$-set, then $X$ is left monotonically $\delta$-normal or lm$\delta$n.
3. If $\mathcal{C}$ is the collection of pairs of disjoint closed subsets of $X$ at least one of which is a regular $G_\delta$-set, then $X$ is monotonically $\delta$-normal or m$\delta$n.
4. If $\mathcal{C}$ is the collection of pairs of disjoint regular $G_\delta$-subsets of $X$, then $X$ is m$\delta\delta$n.

It can easily be shown that right monotone $\delta$-normality (where $D$, rather than $C$, is assumed to be a regular $G_\delta$-set) is equivalent to lm$\delta$n.
Note that, replacing $H(C, D)$ with $\overline{H(C, D)} \setminus H(D, C)$ if necessary, we may assume that $H(C, D) \cap H(D, C) = \emptyset$ whenever $H$ is an mn, mδn or mδδn operator.

There are a number of characterizations of monotone normality, amongst them the equivalence of conditions (1) and (2) in Theorem 5 (see [7]) (the proof of the extension stated here is routine). Mimicking the proof of this characterization, we obtain the hierarchy of monotone versions of $\delta$-normality listed in Theorem 8.

**Theorem 5.** The following are equivalent for a space $X$:

1. $X$ is monotonically normal.
2. There is an operator $\psi$ assigning to each open set $U$ in $X$ and $x \in U$, an open set $\psi(x, U)$ such that
   a. $x \in \psi(x, U)$,
   b. if $\psi(x, U) \cap \psi(y, V) \neq \emptyset$, then either $x \in V$ or $y \in U$.
3. There is an operator $\psi$ as in (2) such that, in addition, $\psi(x, U) \subseteq U$.
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In Theorem 5 monotone normality is characterized in terms of an operator assigning an open set to each point $x$ and open neighbourhood $U$ of $x$. We define several new properties, analogous to these characterizations, by considering an operator acting on a regular $G_\delta$-set $L$ and an open set containing $L$.

**Definition 6.** A space $X$ is weakly coherently $\delta$-normal (wc$\delta$n) iff there is an operator $\varphi$ assigning to each regular $G_\delta$-set $L$ and open set $U$ containing $L$, an open set $\varphi(L, U)$ such that

1. $L \subseteq \varphi(L, U)$,
2. if $\varphi(L, U) \cap \varphi(K, V) \neq \emptyset$ then either $L \cap V \neq \emptyset$ or $K \cap U \neq \emptyset$.

$X$ is coherently $\delta$-normal (c$\delta$n) if in addition,

3. $L \subseteq \varphi(L, U) \subseteq \overline{\varphi(L, U)} \subseteq U$.

$X$ is monotonically coherently $\delta$-normal (mc$\delta$n) if in addition,

4. if $L \subseteq L'$ and $U \subseteq U'$ then $\varphi(L, U) \subseteq \varphi(L', U')$.

If $\varphi$ is an operator witnessing that $X$ is wc$\delta$n, there is no assumption that $\varphi(L, U)$ is monotone in $L$ or $U$ nor that it is a subset of $U$. We have the following proposition.

**Proposition 7.** Suppose that $X$ is wc$\delta$n. Then there is a wc$\delta$n operator $\varphi$ on $X$ such that:
(1) \( L \subseteq \varphi(L, U) \subseteq U \) and
(2) if \( L \subseteq L' \) and \( U \subseteq U' \), then \( \varphi(L, U) \subseteq \varphi(L', U') \).

Proof. Suppose \( \psi \) is a wc\( \delta \)n operator on \( X \) and let \( L \) be a regular \( G_\delta \)-set contained in an open set \( U \). Define

\[
\varphi(L, U) = U \cap \bigcup \{ \psi(J, W) : J \subseteq L, J \text{ is regular } G_\delta, W \text{ is open, } J \subseteq W \subseteq U \}.
\]

Then \( \varphi(L, U) \) is open and \( L \subseteq \varphi(L, U) \subseteq U \) and clearly \( \varphi(L, U) \subseteq \varphi(L', U') \) whenever \( L \subseteq L' \) and \( U \subseteq U' \).

It remains to verify that \( \varphi \) is, indeed, a wc\( \delta \)n operator. So suppose that \( \varphi(L, U) \cap \varphi(K, V) \neq \emptyset \). Then for some regular \( G_\delta \)-sets \( L' \) and \( K' \), and open sets \( U' \) and \( V' \), such that \( L' \subseteq L \), \( K' \subseteq K \), \( L' \subseteq U' \subseteq U \) and \( K' \subseteq V' \subseteq V \), we have \( \psi(L', U') \cap \psi(K', V') \neq \emptyset \). Hence either \( \emptyset \neq L' \cap V' \subseteq L \cap V \) or \( \emptyset \neq K' \cap U' \subseteq K \cap U \), as required.

On the other hand, it is not clear whether c\( \delta \)n implies m\( \delta \)n.

In light of Theorem 5, we might expect there to be a relationship between m\( \delta \)n, wc\( \delta \)n and c\( \delta \)n. Indeed, we have the following theorem.

Theorem 8. Each of the following properties of a space \( X \) implies the next:

1. Monotonically normal,
2. m\( \delta \)n,
3. mc\( \delta \)n,
4. c\( \delta \)n,
5. wc\( \delta \)n,
6. m\( \delta \delta \)n.

Moreover, every mc\( \delta \)n space is lm\( \delta \)n and every lm\( \delta \)n space is m\( \delta \delta \)n.

Proof. The proofs of (1) \( \rightarrow \) (2), (3) \( \rightarrow \) (4), (4) \( \rightarrow \) (5) and the fact that lm\( \delta \)n implies m\( \delta \delta \)n are trivial.

(2) \( \rightarrow \) (3): We modify the proof of Theorem 5. Suppose \( H \) is a m\( \delta \)n operator for \( X \) with \( H(L, K) \cap H(K, L) = \emptyset \). Let \( L \) be a regular \( G_\delta \)-set and \( U \) an open set such that \( L \subseteq U \) and define \( \psi(L, U) = H(L, X \setminus U) \).

Then \( L \subseteq \psi(L, U) \subseteq \psi(L, U) \subseteq U \). Assume \( L \cap V = \emptyset \) and \( K \cap U = \emptyset \) where \( K \) is a regular \( G_\delta \)-set contained in an open set \( V \). Then \( L \subseteq X \setminus V \) and \( K \subseteq X \setminus U \). So by monotonicity, \( \psi(L, U) \subseteq H(L, K) \). Similarly, \( \psi(K, V) \subseteq H(K, L) \). Therefore \( \psi(L, U) \cap \psi(K, V) = \emptyset \). Monotonicity of the operator \( \psi \) follows from the monotonicity of \( H \), hence \( \psi \) is a mc\( \delta \)n operator for \( X \).
Again we modify the proof of Theorem 5. Suppose \( \psi \) is a wc\( \delta \)-\( \omega_n \) operator for \( X \) and let \( L \) and \( K \) be disjoint regular \( G_\delta \)-sets in \( X \). Define

\[
H(L,K) = \bigcup \{ \psi(J,U) : J \subseteq L \cap U, J \text{ is regular } G_\delta, U \text{ is open}, U \cap K = \emptyset \}.
\]

Then \( H(L,K) \) is open with \( L \subseteq H(L,K) \). We show that \( \overline{H(L,K)} \subseteq X \setminus K \). Since \( X \) is wc\( \delta \)-\( \omega_n \), if \( U \) is open with \( U \cap K = \emptyset \) and \( J \) is any regular \( G_\delta \)-set contained in \( L \cap U \), then \( \psi(K,X \setminus L) \cap \psi(J,U) = \emptyset \). Hence \( \psi(K,X \setminus L) \cap H(L,K) = \emptyset \) and so \( K \cap \overline{H(L,K)} = \emptyset \). It is routine to show that the operator \( H \) is monotone.

To see that mc\( \delta \)-\( \omega_n \) implies lm\( \delta \)-\( \omega_n \), assume \( \psi \) is a mc\( \delta \)-\( \omega_n \) operator for \( X \). Let \( C \) and \( D \) be disjoint closed sets, \( C \) a regular \( G_\delta \)-set. Define \( H(C,D) = \psi(C,X \setminus D) \). Then \( C \subseteq H(C,D) \subseteq \overline{H(C,D)} \subseteq X \setminus D \). Suppose \( C \subseteq C' \) and \( D' \subseteq D \). Then \( X \setminus D \subseteq X \setminus D' \), hence \( H(C,D) \subseteq H(C',D') \).

The proof of the following is routine.

**Proposition 9.** Let \( M \) be a compact metrizable space. If \( X \times M \) satisfies any of the properties listed in Theorem 8, then so does \( X \).

### 3. Factorizations of monotone normality

Kohli and Singh [10] factorize normality in terms of various weak normality properties. They define a space to be \( \Sigma \)-normal if for each closed set \( C \) contained in an open set \( U \), there exists a set \( W \) that is the complement of a regular \( G_\delta \)-set such that \( C \subseteq W \subseteq U \) and show that a space is normal iff it is both weakly \( \delta \)-normal and \( \Sigma \)-normal. There is an obvious monotone version of this result that factorizes monotone normality into monotone \( \Sigma \)-normality and m\( \delta \)-\( \omega_n \). However, it turns out that we can do better than this in the monotone case.

**Definition 10.** A space \( X \) is monotonically \( \Sigma \)-normal, or m\( \Sigma \)-\( n \), iff there is an operator \( W \) assigning to each closed set \( C \) and each open set \( U \) containing \( C \), an open set \( W(C,U) \) such that

1. \( X \setminus W(C,U) \) is a regular \( G_\delta \)-set,
2. \( C \subseteq W(C,U) \subseteq U \) and
3. if \( C \subseteq C' \) and \( U \subseteq U' \), then \( W(C,U) \subseteq W(C',U') \).

**Proposition 11.** \( X \) is m\( \Sigma \)-\( n \) iff there are operators \( D \) and \( W \) assigning to each closed set \( C \) and each open set \( U \) containing \( C \), sets \( D(C,U) \) and \( W(C,U) \) such that

1. \( D(C,U) \) and \( X \setminus W(C,U) \) are regular \( G_\delta \)-sets
2. \( C \subseteq D(C,U) \subseteq W(C,U) \subseteq U \),
(3) \( D(C, U) \cap W(X \setminus U, X \setminus C) = \emptyset \),
(4) if \( C \subseteq C' \) and \( U \subseteq U' \), and then \( D(C, U) \subseteq D(C', U') \) and \( W(C, U) \subseteq W(C', U') \).

Proof. Suppose the conditions of the theorem hold, then clearly \( X \) is \( m\Sigma n \). Conversely, suppose \( V \) is a \( m\Sigma n \) operator for \( X \) and that \( C \subseteq U \). Define \( D'(C, U) = X \setminus V(X \setminus U, X \setminus C) \), so that \( C \subseteq D'(C, U) \subseteq U \) and \( D'(C, U) \) is a regular \( G_\delta \), and define \( W(C, U) = V(D'(C, U), U) \). It is routine to check conditions (1), (2) and (4). Now define \( D(C, U) = D'(C, U) \setminus W(X \setminus U, X \setminus C) \), which is the intersection of two regular \( G_\delta \)-sets. Since \( W(X \setminus U, X \setminus C) = V(X \setminus V(C, U), X \setminus C) \) and \( C \cap W(X \setminus U, X \setminus C) = \emptyset \), we have operators \( D \) and \( W \) satisfying all four conditions. \( \square \)

Proposition 12. Every monotonically normal space and every perfectly normal space is \( m\Sigma n \).

Proof. To show that every monotonically normal space is \( m\Sigma n \), we extend the proof that every normal space is \( \Sigma \)-normal [10] and use the monotone version of Urysohn’s lemma [12].

Suppose \( X \) is perfectly normal. Then every open set is the complement of a regular \( G_\delta \)-set and defining \( W(C, U) = U \) shows that \( X \) is \( m\Sigma n \). \( \square \)

It turns out that a weaker property (that might be termed monotone \( \Sigma \) Hausdorff) is all that is needed to factorize monotone normality in terms of \( m\diamond n \).

Definition 13. A space \( X \) has property (\( \ast \)) iff there are operators \( D \) and \( E \) assigning to every \( x \in X \) and open set \( U \) containing \( x \), disjoint sets \( D(x, U) \) and \( E(x, U) \) such that

1. \( D(x, U) \) and \( E(x, U) \) are regular \( G_\delta \)-sets,
2. \( x \in D(x, U) \subseteq U \) and
3. for every open set \( V \) and \( y \in V \), if \( x \notin V \) and \( y \notin U \), then \( D(y, V) \subseteq E(x, U) \).

Of course, if \( X \) is a regular space we can, without loss of generality, drop the assumption that \( D(x, U) \subseteq U \).

Proposition 14. A space \( X \) has property (\( \ast \)) iff there are operators \( D \) and \( W \) assigning to each \( x \in X \) and each open \( U \) containing \( x \), sets \( D(x, U) \) and \( W(x, U) \) such that

1. \( D(x, U) \) and \( X \setminus W(x, U) \) are regular \( G_\delta \)-sets,
2. \( x \in D(x, U) \subseteq W(x, U) \subseteq U \) and
Proposition 11. Suppose that $U$ is an unbounded set. Let $X$ and $D$ be the topology on $X$ and $\mathbb{R}$ such that, whenever $x, y \in X \setminus U$, then $D(x, U) \cap W(x, U) = \emptyset$.

Proof. If $D$ and $E$ witness that $X$ has property $(\star)$, define $W(x, U) = X \setminus E(x, U)$ for each $x \in U$. If $z \notin U$ and $\hat{V} = X \setminus D(z, U)$, then $x \notin \hat{V}$ and so $z \in D(z, \hat{V}) \subseteq E(x, U)$. Hence $X \setminus U \subseteq E(x, U)$ and so $W(x, U) \subseteq U$. Since $D(x, U)$ and $E(x, U)$ are disjoint, $D(x, U) \subseteq W(x, U)$ and condition (3) is clear. The converse follows just as easily. \qed

Property $(\star)$ is relatively easy to achieve.

Theorem 15. Every $m\Sigma n$ space and every Tychonoff space with $G_\delta$ points has property $(\star)$.

Hence every monotonically normal space, every perfectly normal space, every first countable Tychonoff space and every Tychonoff space with a $G_\delta$-diagonal has property $(\star)$.

Proof. Suppose that $X$ is $m\Sigma n$. Let $D$ and $W$ satisfy the conditions of Proposition 11. Suppose that $U$ and $V$ are open sets and that $x \in U \setminus V$ and $y \in V \setminus U$. By (4), $D(\{y\}, V) \cap W(\{x\}, U) \subseteq D(\{y\}, X \setminus \{x\} \cap W(\{x\}, X \setminus \{y\})$, which is empty by (3). Hence $D(\{x\}, U)$ and $W(\{x\}, U)$ define operators satisfying property $(\star)$.

Suppose now that $X$ is Tychonoff and has $G_\delta$ points. Let $x \in U$. Since $\{x\}$ is a $G_\delta$-set, regularity implies that it is a regular $G_\delta$-set. Since $X$ is Tychonoff, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) = 0$. Define $D(x, U) = \{x\}$ and $E(x, U) = f^{-1}(0)$. Then $D(x, U)$ and $E(x, U)$ are disjoint regular $G_\delta$-sets such that $x \in D(x, U) \subseteq U$ and $X \setminus U \subseteq E(x, U)$, so that $D(y, V) \subseteq E(x, U)$, whenever $y \in V \setminus U$. \qed

Example 16. Assuming $\clubsuit^*$, there is a space with property $(\star)$ that is not $m\Sigma n$.

Proof. $\clubsuit^*$ asserts the existence of a sequence $R_\alpha = \{\beta_{\alpha,n} : n \in \omega\}$ for every limit ordinal $\alpha \in \omega_1$ that is cofinal in $\alpha$ such that, whenever $X$ is an uncountable subset of $\omega_1$, $\{\alpha \in \omega_1 : X \cap R_\alpha$ is cofinal in $\alpha\}$ contains a closed unbounded set. $\clubsuit^*$ holds, for example, in any model of $V = L$.

Let $X = \omega_1 \times 2$. For each limit $\alpha$ and $n \in \omega$, define $B(\alpha, n) = \{(\alpha, 1)\} \cup \{(\beta_{\alpha,k}, 0) : n \leq k\}$. Let $T$ be the topology on $X$ generated by the collection

$$\{(\alpha, i) : \alpha \text{ is a successor or } i = 0\} \cup \{B(\alpha, n) : \alpha \text{ is a limit, } n \in \omega\}.$$  

With this topology, $X$ is zero-dimensional, hence Tychonoff, and first countable, so has property $(\star)$.  

If $U$ is an open set containing an uncountable subset of $\omega_1 \times \{0\}$, for closed unboundedly many $\alpha$, $R_\alpha \cap \{\beta : (\beta, 0) \in U\}$ is cofinal in $\alpha$, so that $\{\alpha : (\alpha, 1) \in U\}$ contains a closed unbounded subset. Since the intersection of countably many closed unbounded subsets of $\omega_1$ is again, closed and unbounded, it follows that every uncountable regular $G_\delta$-set in $X$ contains a closed unbounded subset of $\omega_1 \times \{1\}$. Hence, if $C$ is any uncountable, co-uncountable subset of $\omega_1 \times \{1\}$, $U = C \cup (\omega_1 \times \{0\})$ and $D$ is any regular $G_\delta$-set containing $C$, then $C \subseteq U$, $U$ is open but $D \not\subseteq U$. Hence $X$ is not $m\Sigma n$. □

Interestingly, property ($\star$) is enough to push $m\delta\delta n$ up to monotone normality.

**Theorem 17.** A space is monotonically normal iff it has property ($\star$) and is $m\delta\delta n$.

**Proof.** Suppose $H$ is an $m\delta\delta n$ operator for $X$ such that $H(E, F) \cap H(F, E) = \emptyset$. Let $U$ be an open set with $x \in U$. By property ($\star$), there exist disjoint regular $G_\delta$-sets $D(x, U)$ and $E(x, U)$ such that $x \in D(x, U) \subseteq U$ and for any open set $V$ with $x \notin V$, if $y \in V \setminus U$ then $D(y, V) \subseteq E(x, U)$.

Define $\psi(x, U) = H(D(x, U), E(x, U))$. Then $D(x, U) \subseteq \psi(x, U)$, so $x \in \psi(x, U)$. Suppose $x \notin V$ and $y \in V \setminus U$. Then by monotonicity of $H$, $H(D(y, V), E(y, V)) \subseteq H(E(x, U), D(x, U))$. It follows that $H(D(y, V), E(y, V)) \cap H(D(x, U), E(x, U)) = \emptyset$. Hence $\psi(y, V) \cap \psi(x, U) = \emptyset$. By Theorem 5, $X$ is monotonically normal.

The converse is trivial given Theorems 8 and 15. □

Hence, in any space with property ($\star$), for example in a first countable Tychonoff space, each of the properties listed in Theorem 8 is equivalent to monotone normality.

**Theorem 18.**

1. If every point of $X$ is a regular $G_\delta$-set, then $X$ is monotonically normal iff it is $wc\delta n$.
2. $X$ is $c\delta n$ iff it is $wc\delta n$ and $\delta$-normal.
3. If $X$ is normal, then $X$ is $c\delta n$ iff it is $m\delta\delta n$.

**Proof.** In each case one implication follows from Theorem 8 and from the fact that a $c\delta n$ space is obviously $\delta$-normal.

To complete (1) and (2), suppose that $\psi$ satisfies conditions (1) and (2) of Definition 6. If every $x \in X$ is a regular $G_\delta$, then $\varphi(x, U) = \psi(\{x\}, U)$ satisfies conditions (2) of Theorem 5 and $X$ is monotonically normal. If $X$ is $\delta$-normal and $L$ is a regular $G_\delta$-subset of the open set $U$, then there is an
open set \( \varphi(L,U) \) such that \( L \subseteq \varphi(L,U) \subseteq \varphi(L,U) \subseteq \psi(L,U) \subseteq U \). It is trivial to check that, in this case, \( \varphi \) is a \( \text{c\&\&} \) operator.

To complete (3), suppose \( H \) is an m\( \text{\&\&} \) operator for \( X \) with \( H(L,K) \cap H(K,L) = \emptyset \). Let \( L \) be a regular \( G_\delta \)-set and \( U \) an open set such that \( L \subseteq U \). Since \( X \) is normal, there exists an open set \( W_L \) such that \( W_L \) is the complement of a regular \( G_\delta \)-set and \( L \subseteq W_L \subseteq U \). Define \( \psi(L,U) = H(L,X \setminus W_L) \), then \( L \subseteq \psi(L,U) \subseteq \psi(L,U) \subseteq W_L \subseteq U \). Now suppose \( L \cap V = \emptyset \) and \( K \cap U = \emptyset \) where \( K \) is a regular \( G_\delta \)-set contained in an open set \( V \). Then \( L \subseteq X \setminus W_K \) and \( K \subseteq X \setminus W_L \). By monotonicity, \( \psi(L,U) \subseteq H(L,K) \) and \( \psi(K,V) \subseteq H(K,L) \), hence \( \psi(L,U) \cap \psi(K,V) = \emptyset \). Therefore \( \psi \) is a \( \text{c\&\&} \) operator for \( X \).

\[ \square \]

4. PRODUCTS WITH COMPACT METRIZABLE SPACES AND STRATIFIABILITY

A space \( X \) is semi-stratifiable if there is an operator \( U \) assigning to each \( n \in \omega \) and closed set \( D \) an open set \( U(n,D) \) containing \( D \) such that \( \bigcap_{n \in \omega} U(n,D) = D \) and \( U(n,D') \subseteq U(n,D) \) whenever \( D' \subseteq D \). If, in addition, \( \bigcap_{n \in \omega} \overline{U(n,D)} = D \), then \( X \) is said to be stratifiable. A space \( X \) is stratifiable iff \( X \times M \) is monotonically normal for any (or all) infinite compact metrizable \( M \) iff \( X \) is both semi-stratifiable and monotonically normal (see [9]).

**Definition 19.** A space \( X \) is \( \delta \)-semi-stratifiable iff there is an operator \( U \) assigning to each \( n \in \omega \) and regular \( G_\delta \)-set \( D \) in \( X \), an open set \( U(n,D) \) containing \( D \) such that

1. if \( E \subseteq D \), then \( U(n,E) \subseteq U(n,D) \) for each \( n \in \omega \) and
2. \( D = \bigcap_{n \in \omega} U(n,D) \).

If in addition,

3. \( D = \bigcap_{n \in \omega} \overline{U(n,D)} \),

then \( X \) is \( \delta \)-stratifiable.

Just as for stratifiability, we may assume that the operator \( U \) is also monotonic with respect to \( n \), so that \( U(n+1,D) \subseteq U(n,D) \) for each \( n \) and regular \( G_\delta \)-set \( D \).

The proof of the following is essentially the same as the proof of the corresponding results for stratifiability and monotone normality.
Theorem 20. \hspace{1em} (1) If $X$ is $\delta$-stratifiable, then $X$ is $\delta$-semi-stratifiable and $m\delta\omega$.

(2) If $X$ is $\delta$-semi-stratifiable and $lm\delta\omega$, then it is $\delta$-stratifiable.

Theorem 21. Let $M$ be any infinite compact metrizable space. $X$ is $\delta$-stratifiable iff $X \times M$ is $\delta$-stratifiable iff $X \times M$ is $m\delta\omega$.

Proof. Let $\pi: X \times M \to X$ be the projection map. Since $M$ is compact, $\pi$ is both open and closed.

Suppose $X \times M$ is $\delta$-stratifiable with $\delta$-stratifiability operator $W$. By Theorem 20, $X \times M$ is $m\delta\omega$. To see that $X$ is $\delta$-stratifiable, let $D$ be a regular $G_\delta$-subset of $X$. Fix some $r \in M$ and define $U(n, D) = \pi(W(n, D \times \{r\}))$. It is routine to verify that $U$ is a $\delta$-stratifiability operator for $X$.

Now suppose that $X$ is $\delta$-stratifiable with operator $U$ such that $U(n, \emptyset) = \emptyset$ and satisfying $U(n + 1, E) \subseteq U(n, E)$ for each $n$ and regular $G_\delta$-set $E$. Suppose $D$ is a regular $G_\delta$-set in $X \times M$. Then $D = \bigcap_{\begin{small}t \in \omega\end{small}} U_i$ where $D \subseteq U_i$ and $U_i$ is open in $X \times M$ for each $i$. Define $D_r = D \cap (X \times \{r\})$ for each $r \in M$. Then each $D_r$ is a regular $G_\delta$-set since $D_r = \bigcap_{\begin{small}t \in \omega\end{small}} (X \times \{r\})$ and $D_r \subseteq U_i \cap (X \times (B_{1/2^k}(r)))$ for all $i \in \omega$. Clearly $D = \bigcup_{r \in M} D_r$. Moreover $\pi(D_r)$ is a regular $G_\delta$-set in $X$ for each $r \in M$.

For each $n \in \omega$ define

$$H(n, D) = \bigcup_{r \in M} U(n, \pi(D_r)) \times B_{1/2^k}(r).$$

We show that $H$ is a $\delta$-stratifiability operator for $X \times M$. Clearly $H(n, D)$ is open for each regular $G_\delta$-set $D$ and $n \in \omega$. That $H$ is monotone is clear from the monotonicity of $U$. It is easily seen that $D \subseteq H(n, D)$ for each $n \in \omega$, so it remains to prove that $\bigcap_{n \in \omega} H(n, D) \subseteq D$.

Suppose $(x, s) \in \bigcap_{n \in \omega} H(n, D) \smallsetminus D$. Then there exists a basic open set $V \ni x$ and $k \in \omega$ such that $(V \times B_{1/2^k}(s)) \cap D = \emptyset$ and so $(V \times B_{1/2^k}(s)) \cap (\pi(D_r) \times \{r\}) = \emptyset$ for all $r \in B_{1/2^k}(s)$. Since $(x, s) \in H(n, D)$ for each $n \in \omega$, we may consider the following two cases:

Case 1: Assume $(x, s) \in \bigcap_{r \in B_{1/2^k}(s)} U(n, \pi(D_r)) \times B_{1/2^k}(r)$ for all $n \geq k + 1$. Then for all such $n$, $(W \times B_{1/2^m}(s)) \cap \bigcup_{r \in B_{1/2^k}(s)} U(n, \pi(D_r)) \times B_{1/2^m}(r) \neq \emptyset$ for all basic open sets $W \ni x, m \in \omega$. It follows that for some $t \in B_{1/2^k}(s)$, $V \cap U(n, \pi(D_t)) \neq \emptyset$ for each $n \geq k + 1$. Then, since $U$ is monotonic with respect to $n$, $V \cap \bigcap_{n \in \omega} U(n, \pi(D_t)) \neq \emptyset$. Therefore $V \cap \pi(D_t) \neq \emptyset$, a contradiction.
Case 2: Assume \((x, s) \in \bigcup_{r \notin B_{1/2^k}(s)} U(n, \pi(D_r)) \times B_{1/2^n}(r)\) for all \(n \geq k+1\).

Then for some \(p \notin B_{1/2^k}(s)\), \((W \times B_{1/2^m}(s)) \cap (U(n, \pi(D_p)) \times B_{1/2^n}(p)) \neq \emptyset\) for all basic open sets \(W \ni x, m \in \omega\) and \(n \geq k + 1\). Thus, for all such \(m\) and \(n\), \(B_{1/2^m}(s) \cap B_{1/2^n}(p) \neq \emptyset\). However, \(B_{1/2^{k+1}}(s) \cap B_{1/2^n}(p) = \emptyset\) for all \(n \geq k + 1\), a contradiction.

Therefore \(D = \bigcap_{n \in \omega} H(n, D)\) as required.

To complete the proof we wish to show that if \(X \times M\) is \(\delta\)-stratifiable. Note first that we may assume that \(X \times \Omega\) is \(\delta\)-stratifiable, where \(\Omega = \omega + 1\) is the convergent sequence. To see this note that if \(W\) is a subspace of \(M\) that is homeomorphic to \(\Omega\), then any regular \(G_\delta\)-subset of \(X \times W\) is in fact a regular \(G_\delta\)-subset of \(X \times M\), so that \(X \times W\) is also \(\delta\) stratifiable.

The proof is now familiar.

Let \(H\) be an \(\delta\)-stratifiable operator for \(X \times \Omega\) such that \(H(C, D) \cap H(D, C) = \emptyset\) for any regular \(G_\delta\)-sets \(C\) and \(D\). For each \(n \in \omega\), let \(\Omega_n = (\omega + 1) \setminus \{n\}\) and let \(\pi : X \times \Omega \to X\) be the projection map. If \(E\) is a regular \(G_\delta\)-subset of \(X\) define

\[U(n, E) = \pi(H(E \times \{n\}, X \times \Omega_n)).\]

Clearly \(E \subseteq U(n, E)\) for each \(n\). Suppose that \(z \in \bigcap_{n \in \omega} U(n, E) \setminus E\). Then, as \(E\) is closed, there is a regular \(G_\delta\)-set \(D\) such that \(z \in D \subseteq X \setminus E\). Hence \(K = D \cap \bigcap_{n \in \omega} U(n, E)\) is a regular \(G_\delta\) such that \(z \in K, K \cap E = \emptyset\) and \(K \subseteq \bigcap_{n \in \omega} U(n, E)\), from which it follows that

\[K \times \{w\} \subseteq \bigcup_{n \in \omega} H(E \times \{n\}, X \times \Omega_n) = \bigcup_{n \in \omega} H(E \times \{n\}, X \times \Omega_n).\]

Therefore, for some \(n \in \omega\), we have

\[\emptyset \neq H(K \times \{\omega\}, E \times \Omega) \cap H(E \times \{n\}, X \times \Omega_n),\]

but, by monotonicity, this implies that

\[\emptyset \neq H(K \times \{\omega\}, E \times \Omega) \cap H(E \times \Omega, K \times \{\omega\}),\]

which is a contradiction and it follows that \(\bigcap_{n \in \omega} U(n, E) = E\).

\[\square\]

Clearly property \((\star)\) will have an effect on \(\delta\)-stratifiability although it not clear that it is productive. Obviously, by Theorem \[15\], if \(X\) and \(Y\) are Tychonoff with \(G_\delta\) points, in particular if \(Y\) is a compact metrizable space, then \(X \times Y\) has property \((\star)\). Furthermore, if the product of a space with some compact metrizable space does not have property \((\star)\), then the space is not stratifiable.
Corollary 22. Let $M$ be any infinite compact metrizable space. If $X \times M$ has property $(\star)$, in particular if $X$ is a Tychonoff space with $G_\delta$ points, then $X$ is stratifiable iff $X$ is $\delta$-stratifiable iff $X \times M$ is $m\delta\delta$.

5. Examples

The following lemma gives some simple sufficient conditions on the regular $G_\delta$-subsets of a space for it to be wc$n$ or mc$n$.

Lemma 23. Let $X$ be a space.

(1) If, whenever $L$ and $K$ are disjoint regular $G_\delta$-subsets, at least one of them is clopen, then $X$ is wc$n$.

(2) If every regular $G_\delta$-subset of $X$ is clopen, then $X$ is both mc$n$ and $\delta$-stratifiable.

Proof. (1) For any regular $G_\delta$-set $L$ contained in an open set $U$, define $\psi$ as follows:

$$\psi(L,U) = \begin{cases} L & \text{if } L \text{ is clopen} \\ U & \text{if } L \text{ is not clopen}. \end{cases}$$

Suppose $L$ is clopen. Then $\psi(L,U) = L$ and $\psi(K,V) \subseteq V$, where $K$ is a regular $G_\delta$-set contained in an open set $V$. Hence if $L \cap V = \emptyset$ and $K \cap U = \emptyset$, then $\psi(L,U) \cap \psi(K,V) = \emptyset$.

(2) follows immediately by defining $\varphi(L,U) = L$ and $U(n,L) = L$ for any $n \in \omega$ and regular $G_\delta$-set $L$. \hfill $\square$

Given a cardinal $\kappa$, let $\mathbb{L}_\kappa$ denote the space $\kappa+1$ with the topology generated by isolating each $\alpha \in \kappa$ and declaring basic open neighbourhoods of $\kappa$ to take the form $\mathbb{L}_\kappa \setminus C$, where $C$ is some countable subset of $\kappa$. Note that, if $\kappa$ is uncountable, then any regular $G_\delta$-subset of $\mathbb{L}_\kappa$ containing the point $\kappa$ is clopen and co-countable and that a regular $G_\delta$-set that does not contain $\kappa$ is countable.

Example 24. $\mathbb{L}_{\omega_1}$ is monotonically normal and $\delta$-stratifiable, but not semi-stratifiable. Moreover $\mathbb{L}_{\omega_1} \times (\omega+1)$ is $m\delta\delta$.

Proof. By Lemma 23 (2), $\mathbb{L}_{\omega_1}$ is $\delta$-stratifiable. By Theorem 5, defining $\psi(x,U) = U$, if $x = \omega_1$, and $\psi(x,U) = \{x\}$, otherwise, whenever $x$ is in the open set $U$, we see that $\mathbb{L}_{\omega_1}$ is monotonically normal. However, since $\{\omega_1\}$ is not a $G_\delta$-subset, $\mathbb{L}_{\omega_1}$ is not semi-stratifiable. That $\mathbb{L}_{\omega_1} \times (\omega+1)$ is $m\delta\delta$ follows by Theorem 21. \hfill $\square$

Example 25. Let $\mathcal{S}$ be the Sorgenfrey line. $\mathcal{S}$ is monotonically normal but not $\delta$-stratifiable and $\mathcal{S} \times (\omega+1)$ is not $m\delta\delta$.
Proof. Since \( S \times (\omega + 1) \) is first countable and Tychonoff, it has property (\( *) \). Since \( S \) is not stratifiable, \( S \times (\omega + 1) \) is not monotonically normal and therefore not m\( \delta \)n. \( \square \)

**Example 26.** \( X = \left[ \mathbb{L}_{\omega_1} \times (\omega + 1) \right] \setminus \{(\omega_1, \omega)\} \) is wc\( \delta \)n, but neither c\( \delta \)n nor lm\( \delta \)n.

**Proof.** Let \( T = \{ (\alpha, \omega) : \alpha \in \omega_1 \} \) and \( R = \{ (\omega_1, k) : k \in \omega \} \)

To see that \( X \) is not c\( \delta \)n, note that \( T \) is a regular \( G_\delta \)-set and that \( U = X \setminus R \) is an open set containing \( T \). If \( \varphi(T, U) \) is any open set such that \( T \subseteq \varphi(T, U) \subseteq X \setminus R \), then, for some \( k \in \omega \), \( \{ (\alpha, k) : (\alpha, k) \in \varphi(T, U) \} \) is uncountable, so that \( (\omega_1, k) \in \varphi(T, U) \), but \( (\omega_1, k) \notin U \). The same argument shows that \( X \) is not lm\( \delta \)n either.

To see that \( X \) is wc\( \delta \)n, let \( L \) be a regular \( G_\delta \)-subset of the open set \( U \). First note that if \( (\omega_1, k) \in L \), then \( L \cap (\mathbb{L}_{\omega_1} \times \{ k \}) \) is a clopen subset of \( X \). For each \( (x, \omega) \in L \), there is a least \( k_x \in \omega \) such that \( \{ (x, j) : k_x \leq j \} \) is a subset of \( U \). Let \( B(x, U) = \{ (x, \omega) \} \cup \{ (x, j) : k_x \leq j \} \). Define

\[
\psi(L, U) = L \cup \bigcup \{ B(x, U) : (x, \omega) \in L \}.
\]

Then \( L \subseteq \psi(L, U) \subseteq U \) and \( \psi(L, U) \) is open.

Suppose that \( L \) and \( K \) are regular \( G_\delta \)-sets, \( U \) and \( V \) are open sets and that \( L \subseteq U \setminus V \) and \( K \subseteq V \setminus U \). Then

\[
\psi(L, U) \cap \psi(K, V) = (L \cup \bigcup \{ B(x, U) : (x, \omega) \in L \}) \cap (K \cup \bigcup \{ B(x, V) : (x, \omega) \in K \}) = \bigcup \{ B(x, U) : (x, \omega) \in L \} \cap \bigcup \{ B(x, V) : (x, \omega) \in K \} = \emptyset,
\]

since otherwise, if \( (x, k) \in \psi(L, U) \cap \psi(K, V) \), then \( (x, \omega) \in L \cap K \). \( \square \)

**Example 27.** \( X = \left[ \mathbb{L}_{\omega_1} \times \mathbb{L}_{\omega_2} \right] \setminus \{(\omega_1, \omega_2)\} \) is mc\( \delta \)n and \( \delta \)-stratifiable, but not m\( \delta \)n.

**Proof.** Let \( L \) be a regular \( G_\delta \)-subset of \( X \) containing \( (\omega_1, \alpha) \) (or \( (\alpha, \omega_2) \)). Then \( L \) contains a clopen neighbourhood of \( (\omega_1, \alpha) \) (or \( (\alpha, \omega_2) \)). Hence every regular \( G_\delta \)-subset of \( X \) is clopen and by Lemma 23, \( X \) is mc\( \delta \)n and \( \delta \)-stratifiable.

To see that \( X \) is not m\( \delta \)n, suppose to the contrary that \( H \) is an m\( \delta \)n operator such that \( H(C, D) \cap H(D, C) = \emptyset \). For each \( \alpha \in \omega_1 \) and \( \beta \in \omega_2 \), let

\[
C_{\alpha} = \{ (\alpha, \omega_2) \}, \quad D_{\alpha} = X \setminus \{ \alpha \} \times \mathbb{L}_{\omega_2},
\]

\[
E_{\beta} = \{ (\omega_1, \beta) \}, \quad F_{\beta} = X \setminus \mathbb{L}_{\omega_1} \times \{ \beta \}.
\]
Notice that $C_\alpha \cap D_\alpha = E_\beta \cap F_\beta = \emptyset$, $C_\alpha \subseteq F_\beta$, $E_\beta \subseteq D_\alpha$, $H(C_\alpha, D_\alpha) \subseteq \{\alpha\} \times \mathbb{L}_{\omega_2}$, and $H(E_\beta, F_\beta) \subseteq \mathbb{L}_{\omega_1} \times \{\beta\}$. Hence $H(C_\alpha, D_\alpha) \subseteq H(F_\beta, E_\beta)$, so that $H(C_\alpha, D_\alpha) \cap H(E_\beta, F_\beta) = \emptyset$.

Now, for each $\beta \in \omega_2$, there are no more than countably $\alpha \in \omega_1$ such that $(\alpha, \beta) \notin H(E_\beta, F_\beta)$. This implies that there is a subset $W$ of $\omega_2$ with cardinality $\omega_2$ and some $\alpha_0 \in \omega_1$ such that $(\alpha_0, \omega_1] \times \{\beta\}$ is a subset of $H(E_\beta, F_\beta)$ for each $\beta \in W$. It follows that for any $\alpha_0 \leq \alpha \in \omega_1$ and any $\beta \in W$, $(\alpha, \beta) \notin H(C_\alpha, D_\alpha)$, so that $H(C_\alpha, D_\alpha)$ is not open, which is the required contradiction. \qed

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