What is mass in desitterian physics?

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In the present paper we discuss the relevance for de Sitter fields of the mass and spin interpretation of the parameters appearing in the theory. We show that these apparently conceptual interrogations have important consequences concerning the field theories. Among these, it appeared that several authors were using masses which they thought to be different, but which corresponded to a common unitary irreducible representation (UIR), hence to identical physicals systems. This could actually happen because of the arbitrariness of their mass definition in the de Sitter (dS) space. The profound cause of confusion however is to be found in the lack of connexion between the group theoretical approach on the one hand, and the usual field equation (in local coordinates) approach on the other hand. This connexion will be established in the present paper and by doing so we will get rid of any ambiguity by giving a consistent and univocal definition of a “mass” term uniquely defined with respect to a specific UIR of the de Sitter group.

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I. INTRODUCTION

In the present paper we would like to give an unambiguous method to identify a field in dS space. This appeared to be necessary after we noticed that various authors were describing fields belonging to the same UIR although the involved masses were different. This could only be due to different ways of introducing the mass parameters into the field equations. We feel that the question of a preferred mass definitions can only be answered in reference to the flat case where mass and spin are well defined. More precisely, we would like to show that the specificities of the fields are best recognized according to their membership of the carrier space of an UIR of the dS group. Indeed, contrary to the mass values which after all (see section IV) seem to be arbitrary, one finds that the concerned fields all belong to a characteristic family of UIR’s in the group representation classification. Starting with a given UIR, we then will be able to follow it in the flat limit (H = 0), owing to the group representation contraction procedure \[ \mathbb{Z} \]. This provides an efficient method to define a mass in dS space.

Before we actually more precisely point out why one should be careful when working with the mass and spin in dS space, let us first remind how one usually introduces these labels. Following a minkowskian tradition, in almost every work one tends to discriminate the various dS fields according to their mass and spin values. As stated in \[ \mathbb{Z} \] for scalar fields, “…field quantization in curved spacetime proceeds in close analogy to the minkowskian case. We start with the Lagrangian density

\[ \mathcal{L}(x) = \frac{1}{2} (-g)^{1/2} \left\{ (g^{\mu\nu} \nabla_\mu \nabla_\nu \phi) - \left[ m^2 + \xi R \right] \phi^2 \right\}, \]

where \( \phi \) is the scalar field and \( m \) the mass of the field quanta. The coupling between the scalar field and the gravitational field represented by the term \( \xi R \phi^2 \) where \( \xi \) is a numerical factor and \( R \) is the Ricci scalar…”.

A generalized Lagrangian density is defined for fields of arbitrary spin in curved space times by working on the flat Lagrangian density (\( \partial \) replaced by \( \nabla \)) but without changing the mass parameter (see \[ \mathbb{Z} \]). This however raises the question of knowing to which extent the two imported entities (the spin and \( m^2 \)) are adapted to the dS space.

Concerning the mass parameter, there are several facts discussed in the following and which should make us at least suspicious towards a minkowskian interpretation of \( m \). First of all, it appears that the mass parameter for higher spin fields features remarkable properties first observed in \[ \mathbb{Z} \] in the general framework of constant curvature spaces. It has been pointed out that for specific mass values a new gauge invariance allows to reduce the degrees of freedom of the associated field (partially massless field). These partially massless fields are found at border values which separate unitary from non unitary regions of the field. A consequence of the non unitary regions is that the mass range admits holes corresponding to these non unitary representation of the dS group. In Ref. \[ \mathbb{Z} \] this forbidden mass range phenomenon has been discussed for spin-2 fields, and more recently \[ \mathbb{Z} \] the systematic appearance of these forbidden mass values has been addressed in the case of higher spin fields. In short, it turned out that very specific mass values yielded very specific properties of the involved field \[ \mathbb{Z} \]. We will show on several examples that the mass values for these fields do not tell anything in particular whereas they all belong to the same family of UIR.

Secondly, it is found that the use of a mass parameter is sometimes misleading because it actually happens that negative values possibly correspond to some dS unitary irreducible representation \[ \mathbb{Z} \]. Contrary to the Poincaré group, the dS group Casimir eigenvalues can take negative values. This observation is the starting point of this paper. Indeed, the common approach to dS field theory where a positive constant -the mass- is introduced into the Lagrangian, fails to describe all the UIR’s of the dS group. In the same way as for instance a minkowskian mass somehow restricted to strictly positive values would fail to describe the massless case, the introduced parameter \( m^2 \) might set a lower bound (the “massless” case, \( m^2 = 0 \)) even though a negative value of this parameter still corresponds to an UIR of the dS group. These negative values are then simply ruled out or one is forced to find an explanation for a phenomenon which in our view is a simple consequence of a wrong choice of mass parameter. **We therefore claim that a mass parameter is improper if one cannot be ensured that it covers the entire list of UIR’s.** We will illustrate this discussion with an example concerning the mass of the graviton and the question of its value debated in \[ \mathbb{Z} \].

Another puzzling aspect of the mass interpretation of \( m \) in constant curvature spaces is that the fields associated to the value \( m = 0 \) is not trivially linked to conformal invariance or light cone propagation \[ \mathbb{Z} \]. In fact, whereas gauge invariance, helicities ±s, light cone propagation or masslessness are essentially synonymous in flat space, the situation is rather more complicated in (A)dS spaces. In \[ \mathbb{Z} \] it is for instance shown that while gauge invariant, the spin \( \frac{3}{2} \), 2 fields do not propagate only on the null cone for the AdS space. Moreover, as it is stressed in \[ \mathbb{Z} \], “A scalar field propagating according to \( \nabla_\mu \nabla^\mu \phi = 0 \) (no mass term ) scatters from the
background, thereby propagating both on and inside the local null cones. In this sense, the field appears to be “massive”. Again we see that a mass interpretation is really confusing.

On the other hand, and since at large radius the dS space is close to the flat space, we want to be able to indicate when the parameter \( m \) can be viewed as a physical mass for the minkowskian observer. It turns out that the appropriate tool providing a satisfying resolution of all the questions we have raised is the systematic use of the group representation approach. Indeed, the group representations contractions will enable us to compare the dS representations parameters to those of the Poincaré group whenever this makes sense. It is found \([12]\) that not every UIR of the dS group admits a Poincaré group representation in the limit \( H=0 \). Fortunately, there are UIR’s which contract toward the minkowskian massless fields. These UIR’s will be unambiguously associated to the dS massless fields and will set a lower bound for our dS mass definition. Moreover as we have already said, the mentioned partially massless fields seem to all belong to a specific family of UIR’s. Thus we are able to characterize these new gauge fields and possibly predict their occurrence.

Although the best would be not to use the term of mass for desitterian fields, we nevertheless will introduce the entity \( m_H^2 \geq 0 \) (as it is usually done in most of the papers concerned with desitterian physics) supposed to depend on the curvature \( H \) and which will be uniquely and explicitly determined through the involved UIR. We will show that a reasonable mass definition reads

\[
m_H^2 = [(p - q)(p + q - 1)] H^2 \quad \in \mathbb{R},
\]

where \( 2p \in \mathbb{N} \) and \( q \in \mathbb{C} \) label the various dS UIR’s. We will show later on that \( m_H^2 \) is real for every value of \( q \). This parameter will be considered as a mass only for those UIR’s which have a minkowskian interpretation in the \( H=0 \) limit or which admit a unique extension to the conformal group \( SO(2,4) \). Note also that when the limit \( H=0 \) can be defined, the parameter \( p \) remains constant whereas \( q \sim im/H \) where \( m \) is the minkowskian mass. For these representations, the representation parameters are connected to the familiar Poincaré group representation parameters which are the mass and the spin. Consequently, it is only for these representations that we will speak of mass and spin for the involved parameters. Our point of view provides a nomenclature for desitterian fields, which are divided into purely desitterian (no minkowskian interpretation) and more familiar fields. This approach may seem a little anthropomorphic but it is motivated since at large radius \( (1/H) \) at least locally, the measurable entities should match the minkowskian ones.

Because our mass definition may be different from those given in the literature, we present a systematic procedure for identifying dS fields starting with the field equation. Hence we will be able to compare the various involved parameters. Throughout this work, the “identity” of a field will be given by specifying the associated unitary irreducible representation. This will be done within the framework of ambient space formalism, most adapted to group theoretical matters because Casimir operators are simple to express. We systematically will reduce the field equation to the Casimir eigenvalue equation (with the Casimir operators \( \langle Q^{(1)} \rangle \) ) given by

\[
\left( Q^{(1)} - \langle Q^{(1)} \rangle \right) K(x) = 0.
\]

We thus construct dS elementary systems (in Wigner’s sense) in perfect analogy with the minkowskian case. Recall that in the minkowskian case, the field equations are the Casimir equation with eigenvalues \( m^2 \) and \( s \) (the mass and the spin). However, we insist that neither the mass nor the spin serve to classify the dS UIR’s, or to label the field. A further advantage of our dS nomenclature is that its graphical representation in terms of the values of \( p \) and \( q \) will allow to identify the various fields in a straightforward way. Notably we will characterize the new fields recently addressed in the series of paper due to S. Deser and A. Waldron \([3, 4, 6, 7, 8, 9]\) and possibly predict their appearance for higher spins.

This paper is organized as follows. First we shall recall a complete classification of the UIR’s of the dS group in terms of the eigenvalues of the Casimir operators (Section II). This classification is the sum of the works found in \([13, 14]\), see related references therein. Since our goal is to connect our approach to the field equations traditionally given in literature, we must indicate how to relate the Casimir operators to the covariant derivatives or other intrinsic entities (Section III) and finally show how to consistently introduce a mass parameter which includes every UIR (Section IV). We then give examples of recent debates where our approach may contribute to clarify the situation.
II. CLASSIFICATION OF THE UNITARY IRREDUCIBLE REPRESENTATIONS OF THE DE SITTER GROUP $SO_0(1, 4)$.

Let us first recall that the de Sitter is conveniently seen as a hyperboloid embedded in a five-dimensional Minkowski space $M^5$

$$X_H = \{ x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2} = \frac{3}{\Lambda} \},$$

where $\eta_{\alpha\beta} = \text{diag}(1,-1,-1,-1,-1)$, and with the minkowskian induced metric $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = g_{\mu\nu}^2 dX^\mu dX^\nu$, $\mu = 0, 1, 2, 3$. The $X^\mu$'s are 4 space-time intrinsic coordinates of the dS hyperboloid and $\Lambda$ is the cosmological constant. A tensor field $K_{\eta_1...\eta_r}(x)$ on $X_H$ can be viewed as an homogeneous function $K(x)$ on $M^5$ with an arbitrary degree of homogeneity $\lambda$. In order to guarantee that $K(x)$ lies in the dS tangent space, one must require the transversality condition

$$x \cdot K(x) = 0.$$

On the dS space we define the tangential (or transverse) derivative $\partial_h$ in the following way

$$\partial_h = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x \cdot \partial,$$

verifying $x \cdot \partial = 0$. (2.1)

The tensor with components $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is called the transverse projection operator. It satisfies $\theta_{\alpha\beta} x^\alpha = \theta_{\alpha\beta} x^\beta = 0$.

The kinematical group of the de Sitter space is the 10-parameter group $SO_0(1, 4)$ (connected component of the identity in $SO(1,4)$), which is one of the two possible deformations of the Poincaré group (the other one being $SO_0(2,3)$). The unitary irreducible representations (UIR) of $SO_0(1,4)$ are characterized by the eigenvalues of the two Casimir operators $Q^{(1)}$ and $Q^{(2)}$. This is because these operators commute with the action of the group generators and therefore are constant in each unitary irreducible representation. They read

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^\alpha_\beta, \quad Q^{(2)} = -W_\alpha W^\alpha,$$  (2.2)

where

$$W_\alpha = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} L_\beta^\gamma L_\delta^\eta, \quad \text{with 10 infinitesimal generators} \quad L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}. \quad (2.3)$$

The orbital part $M_{\alpha\beta}$ reads

$$M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) \quad \text{with} \quad Q_0^{(1)} = -\frac{1}{2} M_{\alpha\beta} M^\alpha_\beta,$$

where the operator $Q_0^{(1)}$ represents the pure scalar part in the action of $Q^{(1)}$. In order to precise the action of the spinorial part $S_{\alpha\beta}$ on a field $K$, one must treat separately the integer and half-integer cases. Integer spin fields can be represented by tensors of rank $r$ and the spinorial action reads

$$S_{\alpha\beta} K_{\eta_1...\eta_r} = -i \sum_i \left( \eta_{\alpha\eta_i} K_{\eta_1...\eta_i+\eta_r} - \eta_{\beta\eta_i} K_{\eta_1...\eta_i+\eta_r} \right).$$  (2.4)

Half-integer spin fields with spin $s = r + \frac{1}{2}$ are represented by four component spinor-tensor $K^i_{\eta_1...\eta_r}$ with $i = 1, 2, 3, 4$. The spinorial action is then divided in two different parts

$$S^{(a)}_{\alpha\beta} = S_{\alpha\beta} + S_{\alpha\beta}^{(\frac{1}{2})} \quad \text{with} \quad S_{\alpha\beta}^{(\frac{1}{2})} = -\frac{i}{4} [\gamma_{\alpha}, \gamma_{\beta}],$$

where the five matrices $\gamma_{\alpha}$ are determined by the relations

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^\alpha_\beta \quad \gamma^{\alpha\dagger} = \gamma^0 \gamma^\alpha \gamma^0.$$

The operator $S_{\alpha\beta}$ defined by acts upon the tensors indexes $\eta_1...\eta_r$ and $S_{\alpha\beta}^{(\frac{1}{2})}$ acts upon the spinor indexes given by $i$. The symbol $\epsilon_{\alpha\beta\gamma\delta}$ holds for the usual antisymmetrical tensor. In fact the UIR’s may be labelled
by a pair of parameters $\Delta = (p, q)$ with $2p \in \mathbb{N}$ and $q \in \mathbb{C}$, in terms of which the eigenvalues of $Q^{(1)}$ and $Q^{(2)}$ are expressed as follows [13, 14]:

$$Q^{(1)} = [-p(p + 1) - (q + 1)(q - 2)]\text{Id}, \quad Q^{(2)} = [-p(p + 1)q(q - 1)]\text{Id}.$$ 

According to the possible values for $p$ and $q$, three series of inequivalent unitary representations may be distinguished: the principal, complementary and discrete series.

The **Principal series of representations**

Also called “massive” representations, they are denoted by $U_{p,\nu}$, and labelled with $\Delta = (p, q) = (p, \frac{1}{2} + i\nu)$ where

$$p = 0, 1, 2, \ldots \quad \text{and} \quad \nu \geq 0 \quad \text{or},$$

$$p = \frac{1}{2}, \frac{3}{2}, \ldots \quad \text{and} \quad \nu > 0.$$

The operators $Q^{(1)}$ and $Q^{(2)}$ take respectively the following forms:

$$Q^{(1)} = \left[\left(\frac{9}{4} + \nu^2\right) - p(p + 1)\right]\text{Id}, \quad Q^{(2)} = \left[\left(\frac{1}{4} + \nu^2\right)p(p + 1)\right]\text{Id}.$$

The **complementary series representations**

The complementary series is denoted by $V_{p,\nu}$ with $\Delta = (p, q) = (p, \frac{1}{2} + \nu)$ and

$$p = 0 \quad \text{and} \quad \nu \in \mathbb{R}, \quad 0 < |\nu| < \frac{3}{2} \quad \text{or},$$

$$p = 1, 2, 3, \ldots \quad \text{and} \quad \nu \in \mathbb{R}, \quad 0 < |\nu| < \frac{1}{2}.$$

The operators $Q^{(1)}$ and $Q^{(2)}$ assume the following values

$$Q^{(1)} = \left[\left(\frac{9}{4} - \nu^2\right) - p(p + 1)\right]\text{Id}, \quad Q^{(2)} = \left[\left(\frac{1}{4} - \nu^2\right)p(p + 1)\right]\text{Id}.$$

The **discrete series of representations**

The elements of the discrete series of representations are denoted by $\Pi_{p,0}$ and $\Pi_{p,q}^{\pm}$ where the signs $\pm$ stand for the helicity. The relevant values for the couple $\Delta = (p, q)$ are

$$p = 1, 2, 3, \ldots \quad \text{and} \quad q = p, p - 1, \ldots, 1, 0 \quad \text{or},$$

$$p = \frac{1}{2}, \frac{3}{2}, \ldots \quad \text{and} \quad q = p, p - 1, \ldots, 1, \frac{1}{2}.$$

Let us add a few precisions concerning the UIR’s which extend to the conformal group $\text{SO}_0(2,4)$. First recall that in our view, these UIR’s will correspond to the massless fields in de Sitter space. Masslessness will in fact be synonymous to conformal invariance throughout this paper. In Ref. [20], the reduction of the $\text{SO}_0(2,4)$ unitary irreducible representations to the de Sitter subgroup $\text{SO}_0(1,4)$ UIR’s are examined. It is found that the $\text{SO}_0(1,4)$ UIR’s which can be extended to UIR’s of the conformal group are the following:

- The scalar representation with $p = 0$, $q = 1$ and $\langle Q^{(1)} \rangle = 2$, which, in the above classification, belongs to the **complementary series** of UIR. In that case, the $\text{SO}_0(2,4)$ representation remains irreducible when restricted to the $\text{SO}_0(1,4)$ subgroup.

- The UIR’s characterized by $p = q = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, which correspond to some terms of the **discrete series** of UIR. For any values such that $p = q$, there are two inequivalent unitary irreducible representations of $\text{SO}_0(2,4)$ and both remain irreducible when restricted to $\text{SO}_0(1,4)$. These two UIR’s denoted $\Pi_{p,p}^{\pm}$ differ in the sign of the parameter $k_0 = \pm p$ connected to a subgroup $\text{SO}(3)$ and there is no operator in $\text{SO}_0(2,4)$ which changes the value of that sign. Therefore these two UIR’s are distinguished by an entity which we are allowed to name the helicity.
We have pictured these representations (up to \( p = 3 \)) in terms of \( p \) and \( q \) on Figure 1. The symbols \( \circ \) and \( \square \) stand for the discrete series with half-integer and integer values of \( p \) respectively. The complementary series is represented in the same diagram by bold lines. The principal series is represented in the \( \text{Re}(q) = 1/2 \) plane by dashed lines. We have superposed the three discrete series of representation with values \( p = 1/2, 3/2, 5/2 \), \( \text{Re}(q) = 1/2 \) and \( \text{Im}(q) = 0 \) to the principal series in order to show how these two diagrams fit together. Note that the substitution \( q \rightarrow (1 - q) \) does not alter the eigenvalues; the representations with labels \( \Delta = (p, q) \) and \( \Delta = (p, 1 - q) \) are said to be “Weyl equivalent”. The weyl equivalent points can be localized on figure 6 starting from the points \( q = \frac{1}{2} \) and \( p = 0, 1, 2, \ldots \): the bold lines (complementary series here) on the right hand side of these points are weyl equivalent to the bold lines on the left hand side, including the limiting points belonging to the discrete series in the case \( p > 0 \).

![FIG. 1: SO(1, 4) unitary irreducible representation diagrams. Note that the right hand side diagram corresponds to the \( \text{Re}(q) = 1/2 \) plane.](image)

### III. MINKOWSKIAN OBSERVER POINT OF VIEW

First of all, let us recall that the Minkowski spacetime is obtained from the de Sitter spacetime in the \( H = 0 \) limit. One easily shows that the dS metric tends toward the Minkowskian metric in that limit \([17]\). A minkowskian interpretation of the parameters \( p, q \) is made possible when the group contraction method allows to connect them to the Poincaré group parameters \( s, m \). Recall that the group contraction allows to follow a dS UIR in the limit \( H = 0 \) \([1, 18]\). More precisely, one considers a family of representations \( U^H \) of a group \( G \) into some spaces \( \mathcal{H}_H \) and a representation \( U \) of a group \( G' \) into a space \( \mathcal{H} \). The contraction procedure consists in giving a precise meaning to the assertion \( U^H \rightarrow U \) for \( H \rightarrow 0 \) (one says that the representations \( U^H \) contract toward \( U \)).

For instance, a way of performing the contraction of a dS UIR toward a Poincaré group UIR denoted \( \mathcal{P}(s, \pm m) \) (where \( \pm \) stands for the positive and negative energies), will be to balance the vanishing \( H \) with the parameter \( q \). Thus, a contraction by this method will only be possible if \( q \) has no upper bound which in view of our classification corresponds to the principal series of UIR. Let us now indicate more generally for which unitary irreducible representation of the dS group such a procedure can be implemented.

**de Sitter massive fields:**

- They correspond to the **principal series** of unitary representation (also called the massive representations of the dS group). In this case, the contraction procedure can be done by having \( p \) fixed, and the hole family of UIR’s with \( 0 < q < +\infty \) contracting toward the massive Poincaré group UIR. This is achieved with \( q \) such that in the limit \( H = 0 \), one gets \( q \sim \text{im}/H \) \([1, 18]\). In the limit \( H = 0 \), one gets

\[
U_{s, \frac{1}{2} + \frac{\text{im}}{H}} \rightarrow \mathcal{P}(s, \pm m)
\]  

(3.1)
In the massive case, the contraction is performed with respect to the subgroup $\text{SO}_0(1,3)$ which is identified as the Lorentz subgroup in both relativities, and the concerned de Sitter representations form the principal series. They are precisely those ones which are induced by the minimal parabolic subgroup $\text{SO}(3) \times \text{SO}(1,1) \times (\text{a certain nilpotent subgroup})$, where $\text{SO}(3)$ is the space rotation subgroup of the Lorentz subgroup in both cases. This fully clarifies the concept of spin in de Sitter since it is issued from the same $\text{SO}(3)$. Thus the principal series UIR’s contract toward the massive spin $s$ representations of the Poincaré group. The relevant tensor field equation is given by

$$\left[ Q^{(1)} - \left( \frac{9}{4} + \nu^2 \right) + s(s+1) \right] \mathcal{K} = 0.$$  \hspace{1cm} (3.2)

**de Sitter massless fields:**

In that case we select those representations having a natural extension to the conformal group $\text{SO}_0(2,4)$ and which are equivalent to the massless spin $s$ UIR of the conformal extension of the Poincaré group [20, 21].

- For the scalar case, this UIR is found in the **complementary series** of unitary representation with the values $\Delta = (0,1)$. It is also called conformally coupled massless case since it corresponds to the conformally invariant field equation

$$\left( Q^{(1)} - 2 \right) \phi = 0, \quad \Rightarrow \quad (\Box_H + 2H^2) \phi = 0.$$  \hspace{1cm} (3.3)

In the limit $H = 0$ they correspond to the massless scalar Poincaré group UIR $V_{0,1} \to \mathcal{P}(0,0)$.

- When $p \neq 0$, the only physical representations in the sense of Poincaré limit are those with $p = q = s > 0$ which lie at the lower end of the **discrete series**. They are called the massless representations of the dS group. In the limit $H = 0$ they correspond to the massless spinorial Poincaré group UIR $\Pi_{s,s}^\pm \to \mathcal{P}^\pm(s,0)$.

The corresponding $r$-rank tensor field equations read

$$\left[ Q^{(1)} + 2(s^2 - 1) \right] \mathcal{K} = 0.$$  \hspace{1cm} (3.6)

To summarize, we could say that in our point of view, the dS massive fields with arbitrary spin $s$ correspond to the principal series of unitary representations, and that the massless fields correspond to the discrete series of unitary rep. with $p = q$ except for the scalar case which belongs to the complementary series with the values $p = 0, q = 1$. These are the fields for which it is justified to use the terms of mass and spin. On the contrary, an example of a purely desitterian field is given by the so-called “massless” minimally coupled scalar field. This field corresponds to the lowest term in the discrete series of unitary representations (with $p = 1, q = 0$) and obeys the scalar field equation $Q_0^{(1)} \phi = 0$. This representation admits no minkowskian interpretation.

**IV. INTRINSIC FIELD EQUATIONS AND UNITARY IRREDUCIBLE REPRESENTATIONS**

We now would like to adapt the above classification to the more familiar language of the wave equations in local coordinates on the manifold $X_H$ (intrinsic coordinates). This is actually a simple way to identify every field given in literature on dS space. As we have said, the procedure consists in reducing the intrinsic wave equation to the eigenvalue equation

$$\left( Q^{(1)} - \langle Q^{(1)} \rangle \right) \mathcal{K}(x) = 0,$$  \hspace{1cm} (4.1)

where for integer spin fields, $\mathcal{K}(x)$ is a transverse, symmetric tensor of rank $r = s$. For half integer spin fields the carrier space is made of transverse tensor spinor fields of rank $r = s - \frac{1}{2}$. 
The first step will be to rewrite the intrinsic wave equations in terms of ambient space notations. These will help to make explicit the group theoretical content, since they allow to easily write the equation in terms of the operator $Q^{(1)}$.

The link between the ambient and the intrinsic notations is provided by

$$h_{\mu_1...\mu_r}(X) = \frac{\partial x^{\alpha_1}}{\partial X^\mu_1}...\frac{\partial x^{\alpha_r}}{\partial X^\mu_r}K_{\alpha_1...\alpha_r}(x),$$  \hspace{1cm} (4.2)

and the covariant derivatives acting on a $r$-rank tensor are transformed according to

$$\nabla_\mu\nabla_\nu...\nabla_\rho\hat{h}_{\lambda_1...\lambda_r} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}...\frac{\partial x^\gamma}{\partial X^\rho}\frac{\partial x^\eta}{\partial X^\lambda_1}...\frac{\partial x^\eta_r}{\partial X^\lambda_r}\text{Trpr\hat{\partial}_\gamma}\text{Trpr\hat{\partial}_\beta}...\text{Trpr\hat{\partial}_\lambda_1}\hat{\partial}_{\eta_r},$$ \hspace{1cm} (4.3)

where the transverse projection operator is defined by

$$(\text{Trpr\hat{K}})_{\lambda_1...\lambda_r} = \theta^{\eta_1}_{\lambda_1}...\theta^{\eta_r}_{\lambda_r}\hat{K}_{\eta_1...\eta_r}.$$  

Actually, our main task will be to transcribe the action of the covariant derivatives $\nabla_\mu$ in particular the resulting expression for the intrinsic d’Alembertian operator (in local coordinates) $\Box_H = \nabla_\mu\nabla^\mu$ will be very useful since the d’Alembertian appears in every field equation. The transcription of the action of $\Box_H$ upon ambient space fields depends on the rank of the tensor used to represent the field. For a scalar field $\phi$, the situation is simple since $\Box_H$ is linked to the scalar part $Q_0^{(1)}$ of the Casimir operator $Q^{(1)}$ through

$$\Box_H \phi(X) = \partial^2 \phi(x) = -H^2 Q_0^{(1)} \phi(x).$$

For a $r$-rank tensor field, the corresponding expression depends upon $r$ and contains the operator $Q_0^{(1)}$. In Appendix A we prove that

$$\Box_H h_{\mu_1...\mu_r}(X) = \frac{\partial x^{\beta_1}}{\partial X^\mu_1}...\frac{\partial x^{\beta_r}}{\partial X^\mu_r} [ -H^2 \left(Q_0^{(1)} + r\right) K_{\beta_1...\beta_r} + 2H^4 \sum_{j=1}^{r} x_{\beta_j} \sum_{i<j} x_{\beta_i} K'_{\beta_1...\beta_i...\beta_j...\beta_r}$$

$$- 2H^2 \sum_{i=1}^{r} x_{\beta_i} \left( \hat{\partial} \cdot K_{\beta_i...\beta_i...\beta_r} - H^2 x \cdot K_{\beta_1...\beta_i...\beta_r} \right),$$ \hspace{1cm} (4.4)

where the symbol $\hat{\beta}_i$ indicates that this index should be removed, and where $K'$ is the trace of $K$ defined by

$$K' \equiv K_{\alpha_1...\alpha_{r-2}} = \eta^{\alpha_{r-1}\alpha_r}K_{\alpha_1...\alpha_r}.$$ \hspace{1cm} (4.5)

Note that the trace of a tensor of rank $r$ is a tensor of rank $r-2$.

Now, in the following, we present two important relations which make explicit the link between the operators $Q_0^{(1)}$ and $Q^{(1)}$ for integer and half-integer spin fields. Thus, with the help of equation (4.4), any intrinsic field equation which contains a d’Alembertian operator and covariant derivatives will be expressed in terms of $Q^{(1)}$. In this way, we will be able to recast any intrinsic field equation in a form comparable to Eq. (4.4).

### A. Integer spin case

The Casimir operator are simple to manipulate in ambient space notations. In particular it is easy to show that for a $r$-rank tensor $K_{\alpha_1...\eta_r}(x)$ one has

$$Q^{(1)} K(x) = \left(Q_0^{(1)} - r(r+1)\right) K(x) + 2\eta K' + 2S x \partial \cdot K(x) - 2S \partial x \cdot K(x),$$ \hspace{1cm} (4.6)

where $K'$ is the trace of the $r$-rank tensor $K(x)$ viewed as a homogeneous function of the variables $x^\alpha$ and where $S$ is the non normalized symmetrization operator defined for two vectors $\xi_\alpha$ and $\omega_\beta$ by $S(\xi_\alpha\omega_\beta) = \xi_\alpha\omega_\beta + \xi_\beta\omega_\alpha$.

This indeed follows from (4.2),

$$Q^{(1)} K(x) = -\frac{1}{2} L^{(r)} \omega L^{\alpha\beta(r)} K(x) = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta(r)} K(x) - \frac{1}{2} S^{\alpha\beta(r)} S^{\alpha\beta(r)} K(x) - M_{\alpha\beta} S^{\alpha\beta(r)} K(x),$$ \hspace{1cm} (4.7)
and
\[ \frac{1}{2} S^{(r)}_{\alpha\beta} S^\alpha S^\beta K(x) = r(r+3)K(x) - 2\eta K' , \quad M_{\alpha\beta} S^{\alpha\beta} K(x) = 2S \partial x \cdot K(x) - 2S x \partial \cdot K(x) - 2r K(x) . \] (4.8)

As we have said, Eq. (4.6) will be useful to express \( Q^{(1)}_0 \) in terms of \( Q^{(1)} \) which given (4.1) will allow to write every intrinsic field equation in the form of (4.1). One can distinguish among the fields between those which satisfy the properties of tracelessness and divergenceless and the others.

- For the massive fields the tensor can be chosen to be traceless and divergenceless. These conditions which must be consistent with the field equation (25) allow to constrain the number of propagating degrees of freedom (dof) to the \( 2s + 1 \) degrees of a massive field. In our view, these fields should be associated to the principal series of unitary representation because of their minkowskian behaviour in the flat limit. For such fields, the eigenvalue equation becomes

\[ \left( Q^{(1)}_0 - r(r+1) - \langle Q^{(1)} \rangle \right) K(x) = 0 \quad \text{with} \quad \langle Q^{(1)} \rangle \quad \text{corresponding to the princ. series of UIR} . \]

Since in that case (traceless, divergenceless), one has

\[ \square_H h_{\mu_1..\mu_s}(X) = -\frac{\partial x^{r_1}}{\partial X_{\mu_1}} \frac{\partial x^{r_2}}{\partial X_{\mu_2}} \left( H^2 Q^{(1)}_0 + H^2 r \right) K_{\alpha r..\alpha r}(x) , \] (4.9)

it follows that the field equation in local coordinates writes

\[ \left( \square_H + H^2 r(r+2) + H^2 \langle Q^{(1)} \rangle \right) h_{\mu_1..\mu_s}(X) = 0 . \] (4.10)

This expression explicitly conveys the information contained in \( \langle Q^{(1)} \rangle \) to the traceless and divergenceless field in intrinsic notations. This equation really connects (in the massive case) two approaches largely developed in the framework of field theory, but sometimes a little independently: field equations in local coordinates and group representation theory.

Actually, any massive (in the minkowskian sense) spin \( s = r \) field equation on dS space can be written under the above form. Equation (4.10) therefore allows to unambiguously identify any massive field equation on ds space. Note that we will come across fields satisfying Eq. (4.10) but with \( \langle Q^{(1)} \rangle \) different from the principal series. For instance this will happen for the members of the complementary series and for the partially massless fields belonging to the discrete series. But as we have said we do not consider these fields as true massive fields since they do not admit a massive minkowskian interpretation in the \( H = 0 \) limit. Moreover, we will see that although the partially massless fields can be taken traceless and divergenceless they have additional properties which aren’t expected for a massive field (see below). Therefore we really distinguish among the fields satisfying (4.10), the subspace characterized by \( \langle Q^{(1)} \rangle \) in the principal series as the true massive fields.

- For the UIR’s different from the principal series, it is not always possible to impose the divergenceless or traceless conditions (see for instance the massless spin-1 and spin-2 fields). One can still rewrite the corresponding field equation in terms of \( Q^{(1)} \) and search for the relevant physical subspace corresponding to a massless UIR of the dS group. We have seen that these UIR’s correspond to the lowest end of the discrete series with \( p = q = s \) with Casimir eigenvalue \( \langle Q^{(1)} \rangle = -2(s^2 - 1) \). Therefore, the field equation will be of the form

\[ \left( \square_H + H^2 r(r+2) - 2H^2(s^2 - 1) \right) h_{\mu_1..\mu_s}(X) + G(x) = 0 , \] (4.11)

where \( G(x) \) depends on divergencies and traces of \( h \). Examples are given by the massless spin-1 and spin-2 fields where the physical subspaces respectively correspond to the field equations (26, 27)
B. Half-integer spin case

In that case one can represent the field with a four component symmetric tensor spinor $K(x) = K^{i_{1}..i_{p}}(x)$ with $i = 1,..,4$. The difference with the integer case is due to the action of the spinorial part $S_{\alpha\beta}$ which now reads

$$S_{\alpha\beta}^{(s)} = S_{\alpha\beta}^{(r)} + S_{\alpha\beta}^{(\frac{1}{2})}$$

with $S_{\alpha\beta}^{(\frac{1}{2})} = -\frac{i}{4}[\gamma_{\alpha}, \gamma_{\beta}]$ and $s = r + \frac{1}{2}$.

and with the Dirac gamma matrices

$$\gamma^{0} = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}, \gamma^{1} = \begin{pmatrix} 0 & i\sigma^{1} \\ i\sigma^{1} & 0 \end{pmatrix}, \gamma^{2} = \begin{pmatrix} 0 & -i\sigma^{2} \\ i\sigma^{2} & 0 \end{pmatrix}, \gamma^{3} = \begin{pmatrix} 0 & i\sigma^{3} \\ i\sigma^{3} & 0 \end{pmatrix}, \gamma^{4} = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}.$$

Note that $S_{\alpha\beta}^{(\frac{1}{2})}$ acts only upon the index $i$. Now using \[23\] we have

$$S_{\alpha\beta}^{(\frac{1}{2})}S_{\gamma\delta}^{(r)}K(x) = rK(x) - S_{\gamma}(\gamma \cdot K(x)),$$

and the above relations \[10\] and \[11\] one finally obtains

$$Q^{(1)}K(x) = (Q_{0}^{(1)} - r(r + 2))K(x) + 2\eta K' + 2\mathcal{S} \partial K(x) - 2\mathcal{S} \partial x \cdot K(x) + \left(\frac{i}{2} \gamma_{\alpha} \gamma_{\beta} M^{\alpha\beta} - \frac{5}{2}\right)K + S_{\gamma\gamma} \gamma \cdot K(x).$$

Again it is possible with help of this relation to express the intrinsic wave equation in terms of the Casimir operator $Q^{(1)}$.

V. THE MASS RELATION

We would now like to introduce a parameter $m_{H}^{2}$ for dS fields. Since after all it labels the UIR’s, we at least expect it to be complete (every UIR labelled). Thus, the massless cases must correspond to the lowest values of the Casimir operators for a given spin (the discrete series of representations because of their minkowskian massless interpretation). Moreover, we do not want to systematically interpret any parameter as a mass since we have seen that not every UIR contracts toward the well known Poincaré parameters. In order to understand this difference we must be able to relate the mass parameter to these UIR’s. In short, given that we have taken care not to introduce in the field equation any arbitrary parameter, we do not want to do it at this stage! This is precisely what we will do: the mass parameter will be unambiguously defined with reference to the UIR’s. First it should be noted that in our classification all the UIR’s which admit a minkowskian interpretation have in common that the spin is given by the value of $p$. From that, we would like to fix the zero of the mass parameter for a given $p$. A consistent way of introducing a positive mass parameter $m_{H}^{2}$ for a given spin, is to connect the mass parameter to the UIR’s of the dS group in such a way that the UIR which corresponds to the strictly massless field yields the value zero for the parameter $m_{H}^{2}$. Of course the masslessness of these UIR’s must be understood in a minkowskian sense, which means that these UIR’s are precisely those which admit a clear minkowskian masslessness and spin $s$ interpretation. Our strategy is now clear: for each value of $p$ we search for the UIR corresponding to the minkowskian massless field and set the zero of $m_{H}^{2}$ in reference to that UIR.

A. Non zero spin fields

In that case, the UIR’s which admit a minkowskian massless interpretation are located at the bottom of the discrete series (with $p = q = s$) for $s > 0$. It can be checked that for a given $s > 0$ these UIR’s also correspond to the lowest values of the Casimir operator! We are now in position to define a consistent “mass” parameter with respect to the UIR’s characterized by the values $p = q = s$ (and where $Q_{p=q}^{(1)}$ is the corresponding value of the Casimir) by

$$m_{H}^{2} = H^{2} \left(\langle Q^{(1)} \rangle - \langle Q_{p=q}^{(1)} \rangle \right) = \langle Q^{(1)} \rangle H^{2} + 2(p^{2} - 1)H^{2} = [(p - q)(p + q - 1)]H^{2}. \quad (5.1)$$
Since we have set the zero of our mass parameter according to the lowest value of the Casimir operator, this ensures that every UIR for that spin are labelled by $m_H^2$ and with $m_H^2 \geq 0$.

We insist, that this parameter is a true mass only when $⟨Q^{(1)}⟩$ belongs to the principal series of unitary representation or to the mentioned massless UIR’s. In these cases the parameter $p$ also corresponds to the spin and can therefore be replaced by $s$. For example, the field equation for massive integer spin fields (i.e. principal series of UIR) given by (4.11) finally takes the form
\[
(D_H + [2 - s(s - 2)]H^2 + m_H^2) h_{\mu_1\ldots\mu_s}(X) = 0.
\] (5.2)
Whenever $⟨Q^{(1)}⟩$ does not belong to a UIR with possible minkowskian interpretation, we can still use $m_H^2$ but without referring to a minkowskian mass meaning.

**B. The scalar fields**

We have seen previously that in the scalar case, the massless UIR (with the conformal invariance) corresponds to the complementary series with the values $p = 0$, $q = 1$. But this UIR is weyl equivalent (same eigenvalue for the Casimir operator) to the UIR with values $\Delta = (p, 1 - q)$, in that case $\Delta = (0, 0)$. Hence we can also define the scalar mass relation with respect to the UIR characterized by $p = q$ and denoted by $⟨Q^{(1)}⟩$. The mass relation given by Eq. (5.1) is therefore valid also in the scalar case.

Finally let us mention some peculiarities of the scalar case.

- Note that by setting the massless case in reference to the conformally invariant UIR, we implicitly have assumed the coupling $\xi = 1/6$ between the field and the background in the usual scalar wave equation
\[
(D_H + m_H^2 + \xi R) \phi = 0,
\] (5.3)
where $\xi$ is the coupling constant and $R = 12H^2$ the Ricci scalar.

- Note that our mass relation given by
\[
m_H^2 = [(p - q)(p + q - 1)]H^2,
\] (5.4)
also describes the massless minimally coupled scalar field. This field corresponds to the values $m_H^2 = \xi = 0$ in the wave equation (5.3). The involved UIR belongs to the discrete series of UIR $Π_{1,0}$ with the values $p = 1$, $q = 0$ and the eigenvalue $⟨Q^{(1)}⟩ = 0$. It is straightforward to verify that $m_H^2 = 0$ for $p = 1, q = 0$ in the mass definition (5.4). This field is said to be massless but since the UIR $Π_{1,0}$ cannot be linked to the Poincaré group, $m_H$ does not represent a mass and the value $p = 1$ does not represent the spin.

- In the scalar case, the UIR $⟨Q^{(1)}⟩ = 2$ is not the lowest possible eigenvalue for $p = 0$. In fact the eigenvalues of the Casimir operator with $p = 0$ and $-1 < q < 0$ (also their weyl equivalent UIR’s with $1 < q < 2$) are smaller. This also authorizes imaginary values of $m_H$. This is not a serious inconvenient since on the one hand the corresponding UIR’s have no minkowskian counterpart and on the other hand these are the only UIR’s which do not yield $m_H^2 \in \mathbb{R}^+$.

**C. Discussion**

First let us state that one can use the mass relation (5.1) even if the involved UIR are not linked to the Poincaré group through group representation contraction. This corresponds to the fact that we do not expect $m_H^2$ to be interpretable as a minkowskian mass for every couple $(p, q)$.

It is easy to check that $m_H^2$ defined through the equation (5.1), is a real number for every UIR listed in the above classification. This actually is a consequence of the fact that the Casimir eigenvalue are real valued. Indeed, the only case where a complex number appears in $⟨Q^{(1)}⟩$ concerns the principal series with $q = 1/2 + i\nu$ and $\nu \in \mathbb{R}$. Since it occurs under the form $(q - 1)(q - 2)$ it also yields a real $m_H^2$. In fact the value of $m_H^2$ is a positive real number (it is constructed in that way) except for the UIR’s with $p = 0$ and $-1 < q < 0$, $1 < q < 2$ but which as we have said do not correspond to any minkowskian UIR in the $H = 0$ limit.
Let us finally examine the vanishing curvature limit \((H = 0)\). One distinguishes essentially two cases. If the mass parameter is defined with \((Q^{(1)})\) belonging to the principal series of UIR, then its limit is well defined. In this case \(p\) remains constant, and the hole family of UIR’s with \(0 < q < +\infty\) contract toward the massive Poincaré group UIR. The parameter \(q\) is then given by \(q \sim -im/H\) such that \(m^2_H \sim m\) where \(m\) is the minkowskian mass. On the other hand, for all the representations different from the principal series and which yield a non zero mass parameter, the limit gives \(m^2_H = 0\). This is because in the complementary and discrete series of representations a fixed value of \(p\) entails a bounded value of \(q\). Therefore there is no term in \(m^2_H\) to balance the vanishing \(H\). Note that this vanishing mass parameter should not be interpreted as a minkowskian massless field since at the level of group representation the limit is not well defined. Although the group contraction procedure allows to give an insight into the desitterian physics, the status of the purely desitterian field is not really understood. Because these fields can still be considered at very large radius (locally almost flat) we do not know how a minkowskian observer would perceive them. Certainly not in terms of mass and spin.

With the help of examples we now show how our properly defined mass relations really simplifies the issues of recent debates. These are first the new gauge states known as “partially massless” fields largely discussed in a recent series of papers of S. Deser and A. Waldron \([4, 6, 7, 8]\). Secondly we address the question of the mass of the graviton in (A)dS space which has recently been debated by M. Novello and R. P. Neves in \([10]\).

VI. EXAMPLES AND APPLICATIONS

We can now use our properly defined mass definition to label the fields in dS space according to \((m^2_H, p = s)\). This will be useful since a large amount of authors actually use a mass parameter in their field equation. Our main advantage is that not only we will be able to trace back the involved UIR for a given value of \(m^2_H\) (and thus check if the corresponding mass deserves the mass denomination by considering the UIR contraction), but we are also sure that every UIR of the above classification can be described by \(m^2_H\). We will illustrate the efficiency of our approach with two examples.

A. Gauge invariant fields and group representation

First of all, we examine the mass values of a given type of fields characterized by gauge invariance. In fact, we will call (strictly) massive a field that has \(2s + 1\) degrees of freedom (dof) with all the \(2s + 1\) helicities and no gauge invariance. Starting from there one distinguishes essentially two classes of gauge invariant fields:

1. Strictly massless fields where the gauge invariance allows to reduce the degrees of freedom to two (helicities \(\pm s\)). The corresponding field equation are invariant under conformal transformation.

2. Intermediate fields where the gauge invariance allows to remove subsets of lowest helicity modes. Such fields which correspond to a novel gauge invariance, were first observed by S. Deser in \([8]\) in the spin 2 case. Following S. Deser and A. Waldron \([6, 7, 8]\) we will designate these fields as partially massless fields. This designation is due to their light cone propagation properties \([8]\). The new gauge invariance permits “intermediate sets of higher helicities ” between the \(2s + 1\) massive degrees of freedom (dof) and the \(2\) strictly massless helicities.

Of course, all the mentioned fields can be completely characterized by their \((m^2_H, s)\) values. Given a value of \(s > 1\), S. Deser and A. Waldron \([6, 7, 8]\) have shown (for de Sitter or anti-de Sitter) that the plane defined by a mass parameter \(m^2_H\) and the cosmological constant \(\Lambda = \pm 3H^2\) was divided into different phases which correspond to unitary or non-unitary regions. The non unitary regions correspond to forbidden mass ranges. In the spin-2 case for instance, this mass range has been discussed by Higuchi in \([2]\). These regions are separated by lines of the gauge invariant fields described above.

In the following we examine the mass values of the gauge invariant fields in the de Sitter case \((\Lambda = 3H^2)\). Beyond the mere value of \(m^2_H\), our mass definition linked to the UIR’s of the de Sitter group enables us to characterize the partially massless fields following their group representation content. Actually, we would like to show how the partially massless fields with spin \(s\) are linked to very specific representations of the dS group and how the forbidden mass ranges correspond to the gaps between two unitary representations in the classification. This is easily done if we compare for a given spin, the mass relations given by \([5, 1]\) with the \((m^2_{DW}, \Lambda)\) pictures found in \([2]\) where \(m_{DW}\) designates the mass used by S. Deser and A. Waldron. Since the
authors of [9] consider both dS and AdS geometries, their figures include negative values of the cosmological constant. But as our mass relation is given in terms of the UIR’s of the dS group, we restrict ourself to the unitary regions where $\Lambda > 0$.

For integer spins our mass definition (5.1) agrees with theirs, in particular the strictly massless case (which we have used to set the zero of $m_H^2$) yields the value $m_{DW}^2 = 0$. Therefore we set $m_{DW}^2 = m_H^2$ in the following for the integer case.

The situation is a little different for half integer spins. In [7] it is argued that the partially massless fields correspond to AdS fields (in order to have a positive mass, see Figure 3) because otherwise it would entail negative values of $m_{DW}^2$. On the contrary we believe that these fields actually can be found in de Sitter space and that the negative mass values are merely a consequence of a bad choice of mass relation. For instance, we will see that with the choice of mass $m_{DW}^2$, the strictly massless fields have a mass $m_{DW}^2 \neq 0$ whereas they would yield $m_H^2 = 0$ if defined according to (5.1).

Finally for integer and half integer spin fields, we will see that all the gauge invariant fields correspond to the family of the discrete series of representation of the de Sitter group.

Integer spins examples

As announced, the mass definition (5.1) agrees with that given in [7]. In the following we examine the $(m_H^2, \Lambda = 3H^2)$ figures for spins up to $s=3$ which are given in [7]. We would like to see how the gauge fields can be characterized.

- For the scalar and vector cases, there are no positive forbidden mass ranges (no new gauge invariance occurs). This can be seen on Figure 1 since for $p = 0$ and $p = 1$ one covers the unitary region continuously. The corresponding mass relations are shown in Figure 2 (the discrete series members are represented by the symbol □).

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
Spin 0 & Spin 1 \\
\hline
Compl. S. & Princ. S. & $m_H^2$ & Compl. S. & Princ. S. & $m_H^2$ \\
0 & $H^2/4$ & & 0 & $H^2/4$ & \\
\end{tabular}
\caption{Scalar and vector ($p = 0$ and $p = 1$) mass relations.}
\end{figure}

- According to [7] the spin-2 partial massless gauge field is given by the value $m_H^2 = 2\Lambda/3 = 2H^2$ and the strictly massless field corresponds to $m_{DW}^2 = 0$. One notices with the help of definition (5.1) that both belong to the discrete series of unitary irreducible representations corresponding to the values $p = 2$ with $q = 1$ ($m_H^2 = 2H^2$) or $q = 2$ ($m_H^2 = 0$). Both cases are characterized by a certain gauge invariance which allows to reduce the number of the degrees of freedom of the corresponding fields. One gets two helicities in the case $m_H^2 = 0$ and four degrees of freedom for the field with $m_H^2 = 2H^2$. The strictly massless case $m_H^2 = 0$ with $p = q = 2$ corresponds to the spin-2 linearized gravity (see Figure 3).

- For the spin-3 (see Figure 4) case the lines of new gauge invariance are given by the values $m_H^2 = 4\Lambda/3 = 4H^2$, $m_H^2 = 2\Lambda = 6H^2$ and the strictly massless field corresponds to $m_{DW}^2 = 0$. According to (5.1) they belong to the discrete series of unitary irreducible representations with the values $p = 3$ with $q = 1$ ($m_H^2 = 6H^2$), $q = 2$ ($m_H^2 = 4H^2$) and $q = 3$ ($m_H^2 = 0$).

The situation is quite clear, the partially massless and the strictly massless fields (all characterized by some gauge invariance) correspond to the members of the discrete series of representations! They are represented in figure 9 (the squares up to spin 3) where dashed lines indicate the degrees of freedom shared by the members of the same diagonal. Such a figure provides an efficient tool for identifying the partially massless lines and given the mass relation (5.1) allows to locate the regions which correspond to forbidden mass values.

Half integer spins examples

In the half integer case, for $s = \frac{1}{2}$, there is no forbidden region (same as spin-0 or spin-1). More interesting are the examples $s = \frac{3}{2}$ and $s = \frac{5}{2}$ because in that case we disagree with the mass definition given in [9]. Let us present both point of views.
• Following the figures (see Figure 3) found in [9], it seems that the “mass” relations are defined relatively to the first terms of the discrete series of representation (with $q = 1/2$ rather than $q = p$) for a given spin. In other words, this means that the relation (5.1) should be replaced by

$$m_{DW}^2 = H^2 \left( \langle Q^{(1)} \rangle - \langle Q^{(1)}_{p,q=1/2} \rangle \right) = \frac{\Lambda}{3} \left( \langle Q^{(1)} \rangle - \langle Q^{(1)}_{p,q=1/2} \rangle \right) = \frac{\Lambda}{3} \left[ -q(q - 1) - \frac{1}{4} \right]. \quad (6.1)$$

Indeed, following this mass relation, one can check that the partially massless lines are disposed for half integer spin fields as shown in Figure 5 (therefore in agreement with [9]). The strictly massless fields (2 helicities only) in that case still correspond to the lowest values of the discrete series of representation namely $\langle Q^{(1)} \rangle$ with $p = q$ ($m_{DW}^2 = -\Lambda/3$ for the spin $3/2$ and $m_{DW}^2 = -4\Lambda/3$ for the spin $5/2$). Note also that the values with $q = 1/2$ ($m_{DW}^2 = 0$) are not partially massless gauge fields. In [9] it is claimed that the gauge lines correspond to AdS fields (since $m_{DW}^2 < 0$ if $\Lambda > 0$), thus there are no strictly massless fields in dS space for these spin values.

• We would like to give some arguments to show why we believe that this absence of strictly massless fields in dS space is not a physical fact but rather that it is due to an erroneous choice of mass parameter. **First note that the strictly massless fields for (6.1) do not yield the value $m_{DW}^2 = 0$.** Of course one could say that this is of no importance since the zero can be chosen arbitrarily. But having made this choice one should still take into account the negative values which might occur! In fact the consequence of the mass definition (6.1) is that the members of the discrete series with $p$ fixed and $q > 1/2$ assume negative
Discussion

It appears simple and consistent to us to always use definition (5.1) which leads to partially massless fields in dS space for integer as well as semi-integer spins. These are represented together on Figure 8. We can characterize the partially and strictly massless lines by remarking that they all correspond to members of the discrete series of representation (except for the value $q = 1/2$ which is contiguous to the principal series). Thus there seems to be something very special about the discrete series in relation with gauge invariance.

Although this comparison has only been done for spins up to 3, it is clear that the UIR’s figure will behave in the same way for $p > 3$ and one can expect that also in these cases the members of discrete series of representations will correspond to partially massless lines. Since the mass relation (5.1) can be written

\[
m_H^2 = \left| p(p - 1) - q(q - 1) \right| H^2 = \left| (p - q)(p + q - 1) \right| H^2 \quad \text{for} \ p > 0,
\]

one could conjecture that the partially massless fields will assume the values given by (6.2) with $q = p - 1$, ..., 1 or 3/2. Of course, this still must be worked out properly. In Figure 9, we give a complete picture of the various fields up to spin 3 with the corresponding degrees of freedom.
Let us at last compute the mass values for the borderlines of the unitary regions in order to view how these regions evolve with increasing $p$. The ultimate discrete series representation values before the unitary regions (which starts with the complementary series in the integer spin cases and directly with the principal series in the half integer cases) correspond to $q = 1$ and $q = 1/2$ for the integer and half integer cases respectively. Thus, according to (6.2) and using $\Lambda = 3H^2$ one gets

$$\Lambda = \frac{3m_H^2}{p(p - 1)} \quad \text{for the integer case,} \quad \Lambda = \frac{3m_H^2}{(p - \frac{1}{2})^2} \quad \text{for the half integer case.}$$

It is clear that the allowed unitary region for both cases approaches the region $\Lambda = 0$ for increasing $p$ (spin!). This phenomenon has already been pointed out in [7] with the difference that the fermion higher spin field unitary region approaches the region around $\Lambda = 0$ from below $\Lambda < 0$, since these fields correspond to AdS fields. Apart from this, we agree with S. Deser and A. Waldron on the fact that as higher spin fields are included, the overlap between the allowed unitary region narrows to the value of $\Lambda = 0$. This “provides a sample of how particle kinematics can be affected by cosmological backgrounds allowing only certain partial gauge theories and a restricted range of $m_H^2$ for a given $\Lambda$”.

B. The mass of the graviton

An interesting approach to the spin-2 field theory is given in [28] where the use of the so-called Fierz representation allows to deal with the consistency problems inherent to spin-2 coupling which originate in the non
commutative nature of the covariant derivatives. The ambiguities arise naturally since ones deals with second order derivatives quantities. The difficulties encountered for general spin-2 coupling (to gravity or other fields) consist in finding enough conditions (analogous to the divergencelessness and tracelessness condition in the flat case) in order to reduce the degrees of freedom of the field to pure spin-2 (for a clear review see [25]). These conditions which must be consistent with the equation of motion can be derived at least in the massive case. In the massless case, consistency is related to the presence of a local gauge invariance.

Applying the Fierz representation approach to the massive spin-2 field in (A)dS space, M. Novello and R. P. Neves show that the graviton (helicity $\pm 2$) mass in AdS space is related to the cosmological constant by $m_{NN}^2 = -2\Lambda/3$ where $m_{NN}$ is the mass used in [10]. The origin of this non zero value of the mass parameter for the graviton is related to the new form of the field equation given in the Fierz formalism. This equation is given by

$$\hat{G}_{\mu\nu} + \frac{1}{2} m_{NN}^2 (h_{\mu\nu} - g_{\mu\nu} h) = 0,$$

(6.3)

where

$$G_{\mu\nu}^{(a)} = \frac{1}{2} [\square_H h_{\mu\nu} + \nabla_\mu \nabla_\nu h - (\nabla_\mu \nabla^\lambda h_{\nu\lambda} + \nabla_\nu \nabla^\lambda h_{\mu\lambda}) - g_{\mu\nu} (\square_H h - \nabla_\lambda \nabla_\rho h^\lambda_\rho)] ,$$

$$G_{\mu\nu}^{(b)} = \frac{1}{2} [\square_H h_{\mu\nu} + \nabla_\mu \nabla_\nu h - (\nabla_\mu \nabla^\lambda h_{\nu\lambda} + \nabla_\nu \nabla^\lambda h_{\mu\lambda}) - g_{\mu\nu} (\square_H h - \nabla_\lambda \nabla_\rho h^\lambda_\rho)] ,$$

(6.4)

which is symmetric with respect to the ordering of the covariant derivatives. Now since

$$\nabla_\rho \nabla_\lambda h_{\mu\nu} - \nabla_\lambda \nabla_\rho h_{\mu\nu} = -H^2 (g_{\rho\mu} h_{\lambda\nu} + g_{\rho\nu} h_{\lambda\mu} - g_{\lambda\mu} h_{\rho\nu} - g_{\lambda\nu} h_{\rho\mu}) ,$$

(6.5)

which yields

$$\nabla^\rho \nabla_\mu h_{\rho\nu} = \nabla_\mu \nabla^\rho h_{\rho\nu} - 4H^2 h_{\mu\nu} + H^2 g_{\mu\nu} h ,$$

(6.6)

one finally gets the field equation

$$\square_H + 4H^2 h_{\mu\nu} - g_{\mu\nu} (\square_H + H^2) h - (\nabla_\mu \nabla \cdot h_\nu + \nabla_\nu \nabla \cdot h_\mu) + g_{\mu\nu} \nabla^\lambda \nabla^\rho h_{\lambda\rho} + \nabla_\mu \nabla_\nu h + m_{NN}^2 (h_{\mu\nu} - h g_{\mu\nu}) = 0 .$$

(6.7)

The traditional gauge invariant field equation of the graviton (based on $G_{\mu\nu}^{(a)}$) is given by

$$\square_H + 2H^2 h_{\mu\nu} - g_{\mu\nu} (\square_H - H^2) h - (\nabla_\mu \nabla \cdot h_\nu + \nabla_\nu \nabla \cdot h_\mu) + g_{\mu\nu} \nabla^\lambda \nabla^\rho h_{\lambda\rho} + \nabla_\mu \nabla_\nu h = 0 ,$$
and is equivalent to the former when $m_{NN}^2 = -2\Lambda/3$. This equation can be transcribed to ambient space notations using the relations (4.2), (4.3) and (4.4). One finds that it reads

$$\left(Q^{(1)} + 6\right) \mathcal{K}(x) + D_2 \partial_2 \cdot \mathcal{K}(x) = 0,$$

(6.8)

where the operators $D_2$ et $\partial_2$ are respectively the generalized gradient and divergence

$$D_2 \mathcal{K} = H^{-2} S(\partial - H^2 x) \mathcal{K},$$

$$\partial_2 \cdot \mathcal{K} = \partial \cdot \mathcal{K} - H^2 x \mathcal{K}' - \frac{1}{2} \bar{\partial} \mathcal{K}'.$$

(6.9)

with $S$ a symmetrization operator. If we compare Eq. (6.8) with the eigenvalue equation

$$\left(Q^{(1)} - \langle Q^{(1)} \rangle\right) \mathcal{K} = 0,$$

it is found that the relevant physical subspace (which can be associated to an UIR) is made of the solution of

$$\left(Q^{(1)} + 6\right) \mathcal{K}(x) = 0,$$

(6.10)

and thus corresponds to the discrete series of unitary representations with $p = q = 2$ ($\langle Q^{(1)} \rangle = -6$). Thus we see that the field associated to the mass $m_{NN}^2 = -2\Lambda/3$ and verifying Eq. (6.3) corresponds to the usual strictly massless UIR of the dS group. Precisely the one which corresponds to $m_H^2 = 0$ and where the gauge invariance allows to reduce the degrees of freedom to two (helicity $\pm 2$).

Of course the problem is that in dS space, the value of $m_{NN}$ would be imaginary because $\Lambda = 3H^2$. Let us compare $m_{NN}$ with our mass definition $m_H^2$ in order to understand this problem. It is clear (since $m_H^2 = 0$ when $m_{NN}^2$) that $m_{NN}^2$ is related to $m_H^2$ through the formula

$$m_H^2 = m_{NN}^2 + \frac{2\Lambda}{3}.$$  

(6.11)

Equivalently using (5.1) one has

$$m_{NN}^2 = m_H^2 - \frac{2\Lambda}{3} = H^2 \langle Q^{(1)} \rangle + 4H^2.$$  

(6.12)

It is obvious from that relation that $m_{NN}^2 \in \mathbb{R}^+$ fails to describe all the UIR’s of the de Sitter group. In particular, the UIR with $p = q = 2$ and $\langle Q^{(1)} \rangle = -6$ which is the strictly massless UIR is eliminated. We would like to insist on the fact that we agree with the authors when they claim that the field equation (6.8) is the graviton field equation. But whereas in [10] the corresponding field is not a dS field (because of the imaginary mass) on the contrary our mass definition entails that it is a dS field, which seems reasonable since after all $\langle Q^{(1)}_{p=q=2} \rangle$ is a dS unitary representation (not AdS).

### VII. OUTLOOKS

This mass definition $m_H^2$ and the $(p, q)$ diagrams enabled us to locate the fields where gauge invariance appears as members of the discrete series of representation. This has been done up to $p=3$. Although it must still be rigourously shown, it is reasonable to believe that also for values of $p > 3$, these representations will correspond to fields featuring gauge invariance. The challenge therefore is to establish the precise relation between the discrete series of UIR’s and the property of gauge invariance.

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APPENDIX A: D’ALEMBERTIAN OPERATOR IN AMBIENT SPACE NOTATIONS

In the following, we would like to prove that given a symmetric transverse tensor \( h_{\lambda_1, \ldots, \lambda_r}(x) \) of rank \( r \), and linked to the ambient space tensor \( K_{\beta_1, \ldots, \beta_r}(x) \) by

\[
h_{\lambda_1, \ldots, \lambda_r}(X) = \frac{\partial x^{\beta_1}}{\partial X^{\lambda_1}} \cdots \frac{\partial x^{\beta_r}}{\partial X^{\lambda_r}} K_{\beta_1, \ldots, \beta_r}(x),
\]

the action of the d’Alembertian in local coordinates is given by

\[
\Box_H h_{\lambda_1, \ldots, \lambda_r}(X) = \frac{\partial x^{\beta_1}}{\partial X^{\lambda_1}} \frac{\partial x^{\beta_2}}{\partial X^{\lambda_2}} \cdots \frac{\partial x^{\beta_r}}{\partial X^{\lambda_r}} \left[ -H^2 \left( Q_0^{(1)} + r \right) K_{\beta_1, \ldots, \beta_r} + 2H^4 \sum_{j=1}^{r} \sum_{i<j}^{r} x_{\beta_i} \theta_{\beta_j} K'_{\beta_i, \beta_j, \ldots, \beta_r} \right] - 2H^2 \sum_{i=1}^{r} x_{\beta_i} \left( \partial \cdot K_{\beta_1, \ldots, \beta_i, \ldots, \beta_r} - H^2 x \cdot K_{\beta_1, \ldots, \beta_i, \ldots, \beta_r} \right).
\]

For this, we must compute the expression

\[
\nabla_\mu \nabla_\nu h_{\lambda_1, \ldots, \lambda_r} = \frac{\partial x^{\alpha}}{\partial X^\nu} \frac{\partial x^{\gamma}}{\partial X^\mu} \frac{\partial x^{\beta_1}}{\partial X^\lambda_1} \cdots \frac{\partial x^{\beta_r}}{\partial X^\lambda_r} \text{Trpr} \tilde{\alpha}_a \text{Trpr} \tilde{\beta}_a K_{\beta_1, \ldots, \beta_r},
\]

where the transverse projection operator is defined by

\[(\text{Trpr} K)_{\lambda_1, \ldots, \lambda_r} = \theta_{\lambda_1} \cdots \theta_{\lambda_r} K_{\beta_1, \ldots, \beta_r}.
\]

The first step is to prove that

\[
(\text{Trpr} \tilde{\alpha} K)_{\alpha_1, \ldots, \alpha_r} = \tilde{\partial}_a K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} x_{\beta_i} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r},
\]

(1.4)

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \tilde{\partial}_a \tilde{\partial}_b K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} \sum_{j=i+1}^{r} x_{\beta_i} \sum_{1}^{r} x_{\beta_j} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r} + 2H^4 \sum_{j=1}^{r} \sum_{i<j}^{r} x_{\beta_i} \theta_{\beta_j} K'_{\beta_i, \beta_j, \ldots, \beta_r} \right] - 2H^2 \sum_{i=1}^{r} x_{\beta_i} \left( \partial \cdot K_{\beta_1, \ldots, \beta_i, \ldots, \beta_r} - H^2 x \cdot K_{\beta_1, \ldots, \beta_i, \ldots, \beta_r} \right).
\]

(1.2)

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \tilde{\partial}_a \tilde{\partial}_b K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} x_{\beta_i} \sum_{j=i+1}^{r} x_{\beta_j} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r} \right) \cdot
\]

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \tilde{\partial}_a \tilde{\partial}_b K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} x_{\beta_i} \sum_{j=i+1}^{r} x_{\beta_j} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r} \right) \cdot
\]

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \tilde{\partial}_a \tilde{\partial}_b K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} x_{\beta_i} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r} \right) \cdot
\]

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \tilde{\partial}_a \tilde{\partial}_b K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} x_{\beta_i} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r} \right) \cdot
\]

(1.5)

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \tilde{\partial}_a \tilde{\partial}_b K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} x_{\beta_i} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r} \right) \cdot
\]

Let us now prove formula (1.6). First of all, given (1.4) one has

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \text{Trpr} \tilde{\partial} \left( \partial K - H^2 \sum xK \right) = (\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} - H^2 \left( \text{Trpr} \tilde{\partial} \sum xK \right)_{\alpha_1, \beta_1, \ldots, \beta_r},
\]

and since \( \theta \cdot x = 0 \) one gets

\[
(\text{Trpr} \tilde{\beta} \text{Trpr} \tilde{\alpha} \text{Trpr} \tilde{\alpha} K)_{\alpha_1, \beta_1, \ldots, \beta_r} = \theta_{\beta_1} \theta_{\beta_1} \cdots \theta_{\beta_1} \cdots \theta_{\beta_1} \cdots \theta_{\beta_1} \cdots \theta_{\beta_1} K_{\beta_1, \ldots, \beta_r} - H^2 \sum_{i=1}^{r} \theta_{\alpha} K_{\beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_r} \right) \cdot
\]

(1.7)
Moreover one has
\[
\theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \partial_\gamma \partial_\alpha K_{\beta_1^r \ldots \beta_r^r} = \theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \left( \partial_\alpha \partial_\gamma K_{\beta_1^r \ldots \beta_r^r} - H^2 x_\gamma \partial_\alpha K_{\beta_1^r \ldots \beta_r^r} \right)
\]
\[
= \theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \left( \partial_\alpha \partial_\gamma K_{\beta_1^r \ldots \beta_r^r} - H^2 x_\gamma \left( \theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \partial_\alpha K_{\beta_1^r \ldots \beta_r^r} \right) \right)
\]
\[
= \theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \left( \partial_\alpha \partial_\gamma K_{\beta_1^r \ldots \beta_r^r} - H^2 x_\gamma \left( \partial_\alpha K_{\beta_1^r \ldots \beta_r^r} - H^2 \sum_{i=1}^{r} x_{\beta_i} K_{\beta_1^r \ldots \beta_r^r} \right) \right) .
\]  
(1.8)

Finally we must calculate \( \theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \partial_\gamma \partial_\alpha K_{\beta_1^r \ldots \beta_r^r} \). The procedure here is to develop the derivatives of \( \theta^\alpha_{\beta_r^r} \) in expressions like
\[
\partial_\alpha \partial_\gamma K_{\beta_1^r \ldots \beta_r^r} = \partial_\alpha \partial_\gamma \theta^\alpha_{\beta_r^r} K_{\beta_1^r \ldots \beta_r^r} .
\]
In this way, after a tedious but simple computation one gets
\[
\theta^\alpha_{\beta_r^r} \partial_\gamma \partial_\alpha K_{\beta_1^r \ldots \beta_r^r} = \partial_\alpha \partial_\gamma \theta^\alpha_{\beta_r^r} K_{\beta_1^r \ldots \beta_r^r} + H^4 x_\alpha x_\beta K_{\beta_1^r \ldots \beta_r^r - \gamma} - H^2 x_\beta S_{\alpha \beta \partial_\gamma \partial_\alpha K_{\beta_1^r \ldots \beta_r^r - \gamma}} ,
\]
where \( S_{\alpha \gamma} \) is the non normalized symmetrization operator with respect to \( \alpha \) and \( \gamma \). Thus finally one obtains
\[
\theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \partial_\gamma \partial_\alpha K_{\beta_1^r \ldots \beta_r^r} = \theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \left( \partial_\alpha \partial_\gamma K_{\beta_1^r \ldots \beta_r^r - \gamma} + H^4 x_\alpha x_\beta K_{\beta_1^r \ldots \beta_r^r - \gamma} - H^2 x_\beta S_{\alpha \gamma} \partial_\alpha K_{\beta_1^r \ldots \beta_r^r - \gamma} \right)
\]
\[
= \theta^\alpha_{\beta_1^r} \theta^\alpha_{\beta_2^r} \ldots \theta^\alpha_{\beta_r^r} \left( \partial_\alpha \partial_\gamma K_{\beta_1^r \ldots \beta_r^r - \gamma} + H^4 x_\alpha x_\beta K_{\beta_1^r \ldots \beta_r^r - \gamma} - H^2 \sum_{\gamma=1}^{r} x_{\beta_j} K_{\beta_1^r \ldots \beta_j \ldots \beta_{r-1} \gamma} \right)
\]
\[
= \partial_\alpha \partial_\gamma K_{\beta_1^r \ldots \beta_r^r} + H^4 x_\alpha \sum_{\gamma=1}^{r} x_{\beta_j} K_{\beta_1^r \ldots \beta_j \ldots \beta_r \gamma}
\]
\[
- H^2 \sum_{\gamma=1}^{r} x_{\beta_j} S_{\alpha \gamma} \left( \partial_\alpha K_{\beta_1^r \ldots \beta_j \ldots \beta_r \gamma} - H^2 \sum_{i<j} x_{\beta_i} K_{\beta_1^r \ldots \beta_i \ldots \beta_r \gamma} \right) ,
\]  
(1.9)
which completes the proof of formula (1.5). We are now in position to compute the d’Alembertian operator on any transverse tensor. Note that it is easy to show that the metric in local coordinates \( g_{\mu \nu} \) corresponds through (1.1) to the transverse projector \( \theta = \eta + H^2 xx \) and therefore we have
\[
\Box_H h_{\lambda_1 \ldots \lambda_r} = \frac{\partial x_{\beta_1}}{\partial x^{\alpha_1}} \ldots \frac{\partial x_{\beta_r}}{\partial x^{\alpha_r}} \theta^{\alpha_1 \ldots \alpha_r} \left( \text{Trpr} \partial_\alpha \partial_\beta K \right)_{\alpha_1 \ldots \alpha_r} \gamma_{\beta_1 \ldots \beta_r} .
\]  
(1.10)

Formula (1.2) then follows from (1.5).

[1] T. Garidi, E. Huguet, J. Renaud, Phys. Rev. D 67, 124028 (2003).
[2] Birrel N. D., Davies P. C. W., Cambridge Monogaphs on Mathematical Physics, (1984), Quantum fields in curved space.
[3] S. Deser, R.I. Nepomechie, Phys. Lett. B, 132, 321 (1983).
[4] S. Deser, R.I. Nepomechie, Annals Phys. 154, 396 (1984).
[5] A. Higuchi, Nucl. Phys. B 282, 397 (1987).
[6] S. Deser, A. Waldron, Phys. Rev. Lett. 87, 031601 (2001).
[7] S. Deser, A. Waldron, Nucl. Phys. B 607, 577-604 (2001).
[8] S. Deser, A. Waldron, Phys. Lett. B 508, 347-353 (2001).
[9] S. Deser, A. Waldron, Phys. Lett. B 513, 137-141 (2001).
[10] M. Novello, R.P. Neves, Class. Quant. Grav. 20, L67- L73 (2003).
[11] B. S. De Witt, Relativity, Groups and Topology Gordon & Breach, New York (1964).


[12] Gazeau J. P., Renaud J., Takook M. V., Class. Quant. Grav. 17 (2000) L1415–L1434.
[13] J. Dixmier, Bull. Soc. Math. France, 89 9 (1961).
[14] B. Takahashi, Bull. Soc. Math. France 91, 289 (1963).
[15] C. Fronsdal, Phys. Rev. D, 20 848 (1979).
[16] J. P. Gazeau, M. Hans, J. Math. Phys., 29 2533 (1988).
[17] P. A. M., Dirac Ann. Math., 36 657 (1935).
[18] J. Mickelsson, J. Niederle, Comm. Math. Phys. 27 167 (1972).
[19] L. Lipsman, Springer Verlag, Group Representations, Lecture Notes in Mathematics, 388 (1974).
[20] A. O. Barut, A. Böhm , J. Math. Phys. 11 2938 (1970).
[21] E. Angelopoulos, M. Flato, C. Fronsdal, D. Sternheimer, Phys. Rev . D 23, 1278 (1981).
[22] J. P. Gazeau, M. Hans, R. Murenzi, Class. Quantum Grav. 6, 329 (1989).
[23] Lesimple M., Letters Math. Phys., 15 (1988) 143.
[24] T. Garidi, J.P. Gazeau, M.V. Takook, J. Math. Phys., 44, 9 3838 (2003), hep-th/0302022.
[25] A. Hindawi, B. A. Ovrut, D. Waldram, Phys. Rev. D 53, 5583 (1996).
[26] Allen B. Jacobson T. , Commun. Math. Phys., 103 (1986) 669.
[27] I. Antoniadis, J. Iliopoulos, T. N. Tomaras, Nuclear Phys. B 462, 437 (1996).
[28] M. Novello, R.P. Neves, Class. Quant. Grav. 19, 5335-5351 (2002).