Synchronization in nonlinear oscillators with conjugate coupling

Wenchen Han,† Mei, Zhang,‡ and Junzhong Yang

†School of Science, Beijing University of Posts and Telecommunications, Beijing, 100876, People’s Republic of China
‡Department of Physics, Beijing Normal University, Beijing, 100875, People’s Republic of China

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In this work, we investigate the synchronization in oscillators with conjugate coupling in which oscillators interact via dissimilar variables. The synchronous dynamics and its stability are investigated theoretically and numerically. We find that the synchronous dynamics and its stability are dependent on both coupling scheme and the coupling constant. We also find that the synchronization may be independent of the number of oscillators. Numerical demonstrations with Lorenz oscillators are provided.

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The study of synchronization phenomena in coupled periodic oscillators has been active since the early days of physics [1, 2]. Chaos implies sensitive dependence on initial conditions, with nearby trajectories diverging exponentially, and the synchronization among chaotic oscillators has become a topic of great interest since 1990 [3, 4]. The general theories on complete synchronization in which the distance between states of interacting identical chaotic units approaches zero for \( t \rightarrow \infty \) have been well framed [5, 6]. In these theories, chaotic oscillators interact with each other through the same (nonconjugate) variables of different oscillators. However, coupling via dissimilar (conjugate) variables is also natural in real situations [7, 8]. One example is the coupled-semiconductor-laser experiments by Kim and Roy [9], where the photon intensity fluctuation from one laser is used to modulate the injection current of the other, and vice versa. In the nonconjugate coupling case, the interaction term vanishes with the buildup of complete synchronization and the synchronous state is a solution of isolated system. In contrast, the interaction term in conjugate coupling case may stay nonzero even when the units are synchronized.

The dynamical system with conjugate coupling has been used to realize the amplitude death [10, 11] in coupled identical units, the phenomenon in which unstable equilibrium in isolated unit becomes stable with the assistance of the coupling, in several recent works [12–14]. Interestingly, the realized amplitude death has indeed been shown to be possible but the synchronous state is not necessarily a stable solution of isolated units. Then questions arise: Can synchronization in chaotic oscillators with conjugate coupling be realized? What is the synchronous state in chaotic oscillators with conjugate coupling and how about its stability?

The main goal in this work is to theoretically investigate the synchronous dynamics in a ring of identical chaotic oscillators with conjugate coupling and its stability by following the methods in Ref. [5, 6]. The statements are demonstrated through numerical simulations with Lorenz oscillators. We also show that the statements are valid for regular random networks in which each oscillator has the same number of neighbors.

The model we consider takes the general form

\[
\dot{x}_i = f(x_i) + \epsilon(D_2 x_{i+1} - D_1 x_i) + \epsilon(D_2 x_{i-1} - D_1 x_i)
\]

where \( x_i \in \mathbb{R}^n (i = 1, 2, \ldots, N) \), \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is nonlinear and capable of exhibiting rich dynamics such as chaos. The periodic boundary conditions are imposed on Eq. (1). The parameter \( \epsilon \) is a scalar coupling constant. \( D_1 \) and \( D_2 \) are constant matrices describing coupling schemes. When \( D_1 = D_2 \), the interaction terms become \( D_1 (x_{i+1} + x_{i-1} - 2x_i) \) and the ordinary non-conjugate coupled oscillators are recovered in which oscillators interact with each other through the same variables.

Now we are interested in synchronous states; the states reside on a synchronization manifold defined by

\[
M = \{ (x_1, \ldots, x_N) : x_i = s(t) \}
\]

where \( s(t) \) satisfies the equation of motion

\[
\dot{s} = f(s) + 2\epsilon(D_2 - D_1) s.
\]

To be noted that the synchronous state is not the solution of the isolated oscillator any more and its dynamics depends on the coupling constant and the matrices \( D_1 \) and \( D_2 \) (or the coupling scheme). The stability of the synchronous state can be investigated by letting \( x_i = s + \xi_i \) and linearizing Eq. (1) about \( s(t) \). This leads to

\[
\frac{d}{dt} \xi = I \otimes (DF(s) - 2\epsilon D_1) \xi + \epsilon C \otimes D_2 \xi
\]

where \( DF(s) \) is the Jacobian of \( f \) on \( s \), \( I \) is the \( N \times N \) unit matrix, the coupling matrix \( C \) is an \( N \times N \) matrix with zero elements except that \( c_{i,i+1} = c_{i-1,i} = 1 \), which describes the interaction among oscillators. The eigenvalues and eigenvectors of \( C \) satisfy \( C \phi_i = \lambda_i \phi_i \). By
expanding $\xi$ into the eigenvectors of $C$, $\xi = \sum_{i=1}^{N} \eta_{i} \phi_{i}$ where $\eta_{i}$ is the coefficient and is dependent on time, the linear stability equations is diagonalized and gives

$$
\dot{\eta}_{i} = [Df(x^{*}) - 2\epsilon D_{1} + \epsilon \lambda_{i} D_{2}] \eta_{i},
$$

where $\lambda_{i} = 2 \cos \frac{2\pi}{N}$ is the eigenvalue of $C$. It can be demonstrated that the synchronous manifold coincides with the subspace spanned by the eigenvector of $C$ with eigenvalue $\lambda_{N} = 2$. The Lyapunov exponents given by the linear stability equation with $\lambda_{N} = 2$ determine the dynamics of the synchronous state while the modes characterized by all other eigenvalues govern the motion transversal to the synchronous manifold. Suppose that the mode with $\lambda_{i}$ gives Lyapunov exponents $\Lambda_{1}^{(1)} \geq \Lambda_{2}^{(1)} \geq \cdots \geq \Lambda_{N}^{(1)}$. Then the stability of the synchronous state requires $\Lambda_{1}^{(i)} < 0$ for all $i$ ranging from 1 to $N - 1$.

To make above analysis clear, we take some specific systems as examples. We begin with a pair of oscillators coupled together which is a special case of a ring. For a pair of oscillators where $N = 2$, there is only one coupling term in Eq.(1), therefore the motion equation of the synchronous state Eq.(2) and the linear stability equation Eq.(4) should be modified by replacing the factor $2\epsilon$ with $\epsilon$. Correspondingly, the eigenvalues are changed to $\lambda_{i} = \pm 1, (i = 1, 2)$ in which the mode with $\lambda_{2} = 1$ accounts for the synchronous motion. Firstly, we consider two identical Lorenz oscillators coupled conjugately in which $D_{1} = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right)$ and $D_{2} = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right)$. The motion of a Lorenz oscillators follows

$$
\begin{align*}
\dot{x} &= \sigma (y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - \beta z.
\end{align*}
$$

Eq. (5) has a chaotic attractor for parameters $\sigma = 10$, $r = 28$, and $\beta = 1$. Figure 1(a) and (b) show the bifurcation diagrams of coupled oscillators and the synchronous motion against the coupling constant, respectively. With the variance of the coupling strength, rich dynamics is found. Especially, the synchronous motion displays various periodic windows in which the regular motions transit to chaotic ones. The resemblance between the diagrams of coupled oscillators and the synchronous motion indicates that there is no other stable attractors other than the synchronous solution even if the synchronous motion is unstable. Figure 1(c) shows the synchronization error $\Delta = \langle \sqrt{(x_{1} - x_{2})^{2} + (y_{1} - y_{2})^{2} + (z_{1} - z_{2})^{2}} \rangle$ in which $\langle \cdot \rangle$ means the average over a long time interval after transient. $\Delta$ is a measure on desynchronization and the two oscillators get synchronized when $\Delta = 0$. The dependence of $\Delta$ on $\epsilon$ in Fig.1(c) reveals an interesting feature that there exists one regime of $\epsilon$ in which the synchro-
nization and desynchronization alternate strongly, which is not quite common. Figure 1(d) shows the first two Lyapunov exponents \( \Lambda_{1,2} \) of the synchronous motion and the largest Lyapunov exponent \( \Lambda_{1}^{(1)} \) of the transversal mode against the coupling strength \( \epsilon \). The first two Lyapunov exponents of the synchronous motion show plenty of periodic windows in which the period-doubling bifurcation sequence can be found. \( \Lambda_{1}^{(1)} \) stays at zero for \( \epsilon > 0.41 \) and fluctuates around zero in the range of \( \epsilon \in (0.26, 0.41) \). As analyzed above, the negative largest Lyapunov exponent of the transversal model indicates the synchronization between oscillators. The stability regime of the synchronous motion indicated by negative \( \Lambda_{1}^{(1)} \) is in agreement with that shown by \( \Delta \) in Fig.1(c). To be addressed, the synchronization between two oscillators does not rely on whether the synchronous dynamics is periodic or chaotic as shown in Fig.1.

The second example is still the Lorenz oscillator but with \( D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and \( D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). The transition to the synchronization at \( \epsilon = 2.5 \) can be found either from \( \Delta \) in Fig.2(a) or from the largest \( \Lambda_{1} \) of the transversal mode in Fig.2(b). Figure 2(b) shows that the stable synchronous motion could be chaotic or time-independent. It can be found that there are two stable time-independent synchronous solutions which satisfy \( x = y \neq 0 \) and are symmetrical under transformation of \( (x,y,z) \rightarrow (-x,-y,z) \). The properties of the stable time-independent synchronous solutions indicate that they are actually the pair of unstable fixed points in the isolated Lorenz oscillator. In terminology, amplitude death refers to a situation where individual oscillators cease to oscillate when coupled and settle down to their unstable equilibrium. Previous studies have shown that systems with different chaotic oscillators non-conjugately coupled can give rise to time-independent solution either. However, these realized solutions do not set the systems onto unstable equilibria of their own and such phenomena are always named as phase death [13][17] or quenched death. It is an interesting finding that using conjugate coupling may generate amplitude death in coupled chaotic oscillators in its original sense. The state of amplitude death in Fig.2 loses its stability with the decrease of the coupling constant \( \epsilon \) through a subcritical Hopf bifurcation, which gives rise to a synchronous chaotic motion.

There is only one nonzero element in the matrices \( D_1 \) and \( D_2 \) in the above two examples. However, the coupling schemes in Eq.(1) may be more complicated than them. For example, \( D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) in Fig. 3(a), and \( D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \) and \( D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \) in Fig. 3(b). In these two examples, there are two and three non-zero elements in the matrices \( D_1 \) and \( D_2 \), respectively. In each case, we find that coupling strength may adjust the system from desynchronization to synchronization. It is worth mentioning that the oscillators with conjugate coupling may provide plenty of coupling schemes. For example, in coupled Lorenz oscillators, there are eighteen coupling schemes even if there is only one non-zero elements in \( D_1 \) and \( D_2 \). Large number of coupling schemes may give rise to rich synchronous dynamics and de-synchronous dynamics, which may render the oscillators with conjugate coupling a platform for investigating exotic nonlinear dynamics and pattern formation.

Now we consider a ring of Lorenz oscillators in which \( N > 2 \). In a ring structure, the eigenvalues of the coupling matrix \( C \) take the form of \( 2 \cos \frac{k \pi}{N} \) and distribute between -2 and 2. Larger \( N \), more denser the eigenvalues of \( C \). The stability of the synchronous dynamics can be treated in a two-step procedure. In the first step, the largest transversal Lyapunov exponent for each \( \lambda \) is calculated using Eq. (4) and Eq. (2). Then the region in the complex plane of \( \lambda \) in which the largest transversal Lyapunov exponent is positive, is obtained. In the second step, the eigenvalues of the matrix \( C \) are calculated. If there is any eigenvalue except for the one for synchronous dynamics falling onto the region, the synchronous dynamics is unstable. Otherwise, the synchronous dynamics is stable. Following the procedure, the stability of the synchronous dynamics for a ring of oscillators with the size \( N \) can be determined. A specific example is given in Fig. 3(a) where the dependence

![FIG. 3: The dynamics of a pair of Lorenz oscillators. The Lyapunov exponents of the synchronous motion (\( \Lambda_{1}^{(2)} \) in red and \( \Lambda_{2}^{(2)} \) in green) and the largest Lyapunov exponent \( \Lambda_{1}^{(1)} \) of the transversal mode (in blue) are plotted against the coupling constant. The matrices \( D_1 \) and \( D_2 \) in plots (a) and (b) are presented in the text. \( \sigma = 10, r = 28, \beta = 1. \)]
of $\Lambda_1$ on $\lambda$ and $\epsilon$ is presented with $D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The black curve in the plot denotes the set of $\lambda$ and $\epsilon$ at which $\Lambda_1 = 0$. Then we consider $N = 7$. The seven eigenvalues are plotted in Fig. 4(a) in orange lines. To be noted, the eigenvalue $2\cos\frac{2\pi}{N}$ is the same as $2\cos\frac{2(N-1)\pi}{N}$ and there are only four orange lines besides $\lambda_N = 2$ in the plot. According to the condition that the stability of the synchronous motion requires $\Lambda_1 < 0$ for all transversal modes (or in another word, at any $\epsilon$, the seven eigenvalues except for $\lambda = 2$ should fall into the region with $\Lambda < 0$), the figure shows that there exists a large range of $\epsilon$ for stable synchronous motions. As a comparison, we plot the synchronization error against $\epsilon$ for $N = 7$ in the inset, which shows a disconnected regimes of $\epsilon$ for stable synchronous motion and is in agreement with the analysis based on the dependence of $\Lambda_1$ on $\lambda$ and $\epsilon$. In the case of non-conjugate coupled oscillators, there is a size instability for synchronous motions: At given oscillator parameters and coupling constant, the synchronous state can only be realized when the number of oscillators is below a threshold $N$. However, an extraordinary feature in Fig. 4(a) is that the stability of the synchronous motion may be independent of the number of oscillators in the ring in certain ranges of $\epsilon$. For example, $\epsilon > 0.25$ in which $\Lambda_1 < 0$ provided that $\lambda \neq 2$ and, consequently, the requirement of stable synchronization is always satisfied regardless of $N$.

Then, we consider the case with $D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which are the same as those in Fig. 2.

The results in Fig. 4(b) shows that the synchronous motion becomes independent of the number of oscillators when $\epsilon > 12$. For $\epsilon < 12$, the synchronous motion can only be realized for small number of oscillators, for example $N < 4$. For $N = 3$, Fig. 4(b) also shows that stability of the synchronous motion is non-monotonically dependent on $\epsilon$ since the eigenvalues $\lambda_1 = 2 \cos\frac{2\pi}{3}$ and $\lambda_2 = 2 \cos\frac{\pi}{3}$ cross the black curves denoting $\Lambda_1 = 0$ several times. Furthermore, we consider the case with $D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $D_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The synchronous motion realized in this case is a quenched state which is not the solution of isolated oscillator. Figures 4(c) and (d) show the results for increasing $\epsilon$ and decreasing $\epsilon$, respectively. The two figures show that oscillators conjugate coupled in this way exhibit strong hysteresis and there exists a range of $\epsilon$ in which synchronous motion coexists with de-synchronous motions. Another feature in these two figures is that the synchronous motion is independent of the number of oscillators once it is realized.

The theoretical frame proposed here is not limited to ring structures and it may be applicable to the regular random networks in which each oscillator has the same number of neighbors. The synchronous dynamics in a regular random network with degree $d$ follows

$$\dot{s} = f(s) + d\epsilon(D_2 - D_1)s. \quad (6)$$

The stability of the synchronous dynamics on a regular random network can be determined in the method presented Fig. 4. The difference lies in that the range of $\lambda$ to be concerned is dependent of $d$. We consider the system consisting of eight Lorenz oscillators with the same $D_2$ and $D_3$ as those in Fig. 1. We presented the bifurcation diagrams and the synchronization errors for the oscillators on three regular random networks in Fig. 5. Though oscillators are sitting on different networks, the synchronous dynamics in these three cases is the same, which is in agreement with the above analysis. As shown in Fig. 5. The discrepancies on the range of $\epsilon$ for stable synchronous dynamics in these three cases are resulted from the different eigenvalue spectrums realized by the underlying networks. In discussion, we have considered the synchronization in the system of oscillators with conjugate coupling in
FIG. 5: The dynamics of Lorenz oscillators with conjugate coupling on different random networks. \( \sigma = 10 \), \( r = 28 \), and \( \beta = 1 \). The top panel shows the bifurcation diagram of oscillators and the bottom shows the synchronization error. The inset in the plots (A)-(C) in the bottom panel shows the underlying network where each oscillator has three neighbors. \( N = 8 \), \( D_1 \) and \( D_2 \) are the same as those in Fig. 1.

which oscillators interact through the coupling of dissimilar variables. We proposed a general theoretical frame work for the synchronous dynamics and its stability. We found that the synchronous dynamics and its stability are dependent on both coupling scheme and the coupling constant. We found that the stability of synchronous dynamics may be independent of the number of oscillators, which is in contrast to the size instability in the system of oscillators with non-conjugate coupling. We also show that the theoretical analysis in this work is applicable to regular random networks. However, the synchronization among oscillators sitting in an arbitrary complex network and with conjugate coupling pose a question which worths further investigations.

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