Factorisation of Greedoid Polynomials of Rooted Digraphs

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Received: 27 July 2019 / Revised: 7 February 2021 / Accepted: 31 May 2021 / Published online: 21 June 2021
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Abstract

Gordon and McMahon defined a two-variable greedoid polynomial $f(G; t, z)$ for any greedoid $G$. They studied greedoid polynomials for greedoids associated with rooted graphs and rooted digraphs. They proved that greedoid polynomials of rooted digraphs have the multiplicative direct sum property. In addition, these polynomials are divisible by $1 + z$ under certain conditions. We compute the greedoid polynomials for all rooted digraphs up to order six. A polynomial is said to factorise if it has a non-constant factor of lower degree. We study the factorability of greedoid polynomials of rooted digraphs, particularly those that are not divisible by $1 + z$. We give some examples and an infinite family of rooted digraphs that are not direct sums but their greedoid polynomials factorise.

Keywords Factorisation · Greedoid polynomial · Greedoid · Directed branching greedoid · Rooted digraph · Arborescence

Kai Siong Yow: (the research, and most of the writing, was done at Monash University Australia as part of the author’s PhD research)

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Greedoids were introduced by Korte and Lovász as collections of sets that generalise matroids [9]. Korte and Lovász observed that the optimality of some “greedy” algorithms including breadth-first search could be traced back to an underlying combinatorial structure that satisfies the greedoid, but not the matroid, framework. Björner and Ziegler [1] used two algorithmic constructions of a minimum spanning tree of a connected graph, i.e., Kruskal’s and Prim’s algorithms, to distinguish between greedoids and matroids. For each step in both algorithms, an edge with the minimum weight is added into the minimum spanning tree. The edge sets of the trees/forests that are obtained in each step form the feasible sets of a greedoid. Feasible sets obtained via Kruskal’s algorithm remain feasible when removing any edge from the sets. However, this is not always true for feasible sets that are obtained via Prim’s algorithm. Therefore, the greedoid that is obtained by using Kruskal’s algorithm (but not Prim’s algorithm) is in fact a matroid.

There are two equivalent ways to define greedoids, using set systems or hereditary languages [11, 12]. We define greedoids based on set systems. A greedoid over a finite ground set \( E \) is a pair \( (E, F) \) where \( F \subseteq 2^E \) is a collection of subsets of \( E \) (called the feasible sets) satisfying:

\[
\begin{align*}
\text{(G1)} & \quad \text{For every non-empty } X \subseteq E, \text{ there is an element } x \in X \text{ such that } X - \{x\} \in F. \\
\text{(G2)} & \quad \text{For } X, Y \subseteq E \text{ with } |X| < |Y|, \text{ there is an element } y \in Y - X \text{ such that } X \cup \{y\} \in F.
\end{align*}
\]

The rank \( r(A) \) of a subset \( A \subseteq E \) in a greedoid \( (E, F) \) is defined as \( r(A) = \max\{|X| : X \subseteq A, X \in F\} \). Any greedoid is uniquely determined by its rank function.

**Theorem 1.1** [10] A function \( r : 2^E \to \mathbb{N} \cup \{0\} \) is the rank function of a greedoid \( (E, F) \) if and only if for all \( X, Y \subseteq E \) and for all \( x, y \in E \) the following conditions hold:

\[
\begin{align*}
\text{(R1)} & \quad r(X) \leq |X|. \\
\text{(R2)} & \quad \text{If } X \subseteq Y, \text{ then } r(X) \leq r(Y). \\
\text{(R3)} & \quad \text{If } r(X) = r(X \cup \{x\}) = r(X \cup \{y\}), \text{ then } r(X) = r(X \cup \{x\} \cup \{y\}).
\end{align*}
\]

Important classes of greedoids are those associated with rooted graphs and rooted digraphs. These are called branching greedoids and directed branching greedoids, respectively. We focus on directed branching greedoids. Hence, all our digraphs are rooted.

An arborescence [19] is a directed tree rooted at a vertex \( v \) such that every edge that is incident with \( v \) is an outgoing edge, and exactly one edge is directed into each of the other vertices. For every non-root vertex in an arborescence, there exists a unique directed path in the arborescence that leads from the root vertex to the non-root vertex. Occasionally, to highlight this property, people describe the root vertex...
as Rome\(^1\) [19]. Some authors define arborescences by reversing the direction of each edge in our definition, giving a set of arborescences that is different to ours. In this scenario, each unique directed path in the arborescence directs into rather than away from the root vertex. In both definitions, the number of arborescences rooted at each vertex is identical. To change from one definition to the other, simply reverse the direction for all the edges.

Let \(D\) be a rooted digraph. A subdigraph \(F\) of \(D\) is feasible if \(F\) is an arborescence rooted at the root of \(D\). We call the edge set of \(F\) a feasible set. If the edge set of a feasible \(F\) is maximal, then it is a basis. A spanning arborescence of \(D\) is a subdigraph of \(D\) that is an arborescence which includes every vertex of \(D\).

A directed branching greedoid over a finite set \(E\) of directed edges of a rooted digraph is a pair \((E, F)\) where \(F\) is the set of feasible subsets of \(E\). This was defined and shown to be a greedoid by Korte and Lovász [10].

Let \(G\) be a greedoid. Gordon and McMahon [7] defined a two-variable greedoid polynomial of \(G\)

\[
f(G; t, z) = \sum_{A \subseteq E(G)} t^{r(G) - r(A)} z^{|A| - r(A)}
\]

which generalises the one-variable greedoid polynomial \(\lambda(G; t)\) given by Björner and Ziegler in [1]. We call the two-variable greedoid polynomial \(f(G; t, z)\) the greedoid polynomial. The greedoid polynomial is motivated by the Tutte polynomial of a matroid [18], and is an analogue of the Whitney rank generating function [20]. This polynomial is one of the digraph polynomials that is analogous of the Tutte polynomial. A survey of such polynomials for directed graphs can be found in [4].

Gordon and McMahon studied greedoid polynomials for branching greedoids and directed branching greedoids. They showed that \(f(D; t, z)\) can be used to determine if a rooted digraph \(D\) is a rooted arborescence [7]. However, this result does not hold when \(D\) is an unrooted tree [5].

Suppose \(D, D_1\) and \(D_2\) are rooted digraphs, and \(E(D_1), E(D_2) \subseteq E(D)\). The digraph \(D\) is the direct sum of \(D_1\) and \(D_2\), if \(E(D_1) \cup E(D_2) = E(D), E(D_1) \cap E(D_2) = \emptyset\) and the feasible sets of \(D\) are precisely the unions of feasible sets of \(D_1\) and \(D_2\). Gordon and McMahon proved that the greedoid polynomials of rooted digraphs have the multiplicative direct sum property, that is, if \(D\) is the direct sum of \(D_1\) and \(D_2\), then \(f(D; t, z) = f(D_1; t, z) \cdot f(D_2; t, z)\). This raises the question of whether this is the only circumstance in which this polynomial can be factorised. The Tutte polynomial of a graph \(G\) factorises if and only if \(G\) is a direct sum [14], but the situation for the chromatic polynomial is more complex [15].

Gordon and McMahon showed that the greedoid polynomial of a rooted digraph that is not necessarily a direct sum has \(1 + z\) among its factors under certain conditions (see Theorems 1.3 and 1.4). We address more general types of factorisation in this article.

\(^1\) From the proverb: All roads lead to Rome.
Gordon and McMahon gave a recurrence formula to compute $f(D; t, z)$ where $D$ is a rooted digraph. The following proposition gives the formula, which involves the usual deletion-contraction operations.

**Proposition 1.2** [7] Let $D$ be a digraph rooted at a vertex $v$, and $e$ be an outgoing edge of $v$. Then

$$f(D; t, z) = f(D/e; t, z) + t^{r(D) - r(D/e)}f(D \setminus e; t, z).$$

A greedoid loop [13] in a rooted graph, or a rooted digraph, is an edge that is in no feasible set. It is either an ordinary (directed) loop, or an edge that belongs to no (directed) path from the root vertex.

**Theorem 1.3** [13] Let $D$ be a rooted digraph that has no greedoid loops. Then $D$ has a directed cycle if and only if $1 + z$ divides $f(D)$.

Let $G$ be a greedoid. A subset $S \subseteq E(G)$ is spanning if $S$ contains a basis. Gordon and McMahon gave a graph-theoretic interpretation for the highest power of $1 + z$ which divides $f(G)$ in the following theorem.

**Theorem 1.4** [8] Let $G$ be the directed branching greedoid associated with a rooted digraph $D$ with no greedoid loops or isolated vertices. If $f(G; t, z) = (1 + z)^k h(t, z)$, where $1 + z$ does not divide $h(t, z)$, then $k$ is the minimum number of edges that need to be removed from $D$ to leave a spanning acyclic directed graph.

Tedford [17] defined a three-variable greedoid polynomial $f(G; t, p, q)$ for any finite rooted graph $G$, which generalises the two-variable greedoid polynomial. He showed that $f(G; t, p, q)$ obeys a recursive formula. He also proved that $f(G; t, p, q)$ determines the number of greedoid loops in any rooted graph $G$. His main result shows that $f(G; t, p, q)$ distinguishes connected rooted graphs $G$ that are loopless and have at most one cycle. He extended $f(G; t, p, q)$ from rooted graphs to general greedoids, and proved that the polynomial determines the number of loops for a larger class of greedoids.

In this article, we compute the greedoid polynomials for all rooted digraphs (up to isomorphism unless otherwise stated) up to order six. All the labelled rooted digraphs (without loops and multiple edges, but cycles of size two are allowed) up to order six were provided by Brendan McKay [2] on 28 March 2018 (personal communication from McKay to Farr). We then study the factorability of these polynomials, particularly those that are not divisible by $1 + z$.

Two rooted digraphs are GM-equivalent if they both have the same greedoid polynomial. If a rooted digraph is a direct sum, then it is separable. Otherwise, it is non-separable.

Recall that a polynomial is said to factorise if it has a non-constant factor of lower degree. We say a rooted digraph $D$ factorises if its greedoid polynomial factorises. Every rooted digraph that is a direct sum has a factorisation.

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[2] More combinatorial data can be found at https://users.cecs.anu.edu.au/~bdm/data/.
An irreducible factor of a greedoid polynomial is *basic* if the factor is either \(1 + t\) or \(1 + z\). Otherwise, the irreducible factor is *nonbasic*. We are most interested in nonbasic factors. Some irreducible nonbasic factors are known to be greedoid polynomials, while it is not known if all irreducible nonbasic factors are greedoid polynomials. For \(k \geq 1\), a non-separable digraph is \(k\)-*nonbasic* if its greedoid polynomial has \(k\) nonbasic factors. A non-separable digraph is *totally \(k\)-nonbasic* if it is \(k\)-nonbasic and contains no basic factors.

Our results show that there exist non-separable digraphs that factorise and their polynomials have neither \(1 + t\) nor \(1 + z\) as factors. In some cases (but not all), these non-separable digraphs of order \(n\) are GM-equivalent to a separable digraph of order at most \(n\). We give the numbers of polynomials of this type of non-separable digraph. For rooted digraphs up to order six and \(k \geq 3\), we found that there exist no \(k\)-nonbasic digraphs. We also provide the numbers of 2-nonbasic digraphs and totally 2-nonbasic digraphs. We then give the first examples of totally 2-nonbasic digraphs. Lastly, we give an infinite family of non-separable digraphs where their greedoid polynomials factorise into at least two non-basic factors.

## 2 Results

The greedoid polynomials of all rooted digraphs up to order six were computed based on the deletion-contraction recurrence in Proposition 1.2. We simplified and factorised all these greedoid polynomials using Wolfram Mathematica.

Throughout, rooted digraphs are up to isomorphism unless stated otherwise.

### 2.1 Separability and Non-separability

For each order, we determined the numbers of rooted digraphs, separable digraphs, non-separable digraphs, and non-separable digraphs of order \(n\) that are GM-equivalent to some separable digraph of order at most \(n\) (see Table 2, and the list of abbreviations in Table 1).

Note that the sequences of numbers of labelled rooted digraphs (\(T\)) and rooted digraphs (\(T\)-ISO) are not listed in The On-Line Encyclopedia of Integer Sequences (OEIS) [16].

| Abbreviation | Description |
|--------------|-------------|
| \(T\)        | Number of labelled rooted digraphs |
| \(T\)-ISO    | Number of rooted digraphs |
| \(S\)        | Number of separable digraphs |
| \(NS\)       | Number of non-separable digraphs |
| \(NSE\)      | Number of non-separable digraphs of order \(n\) that are GM-equivalent to some separable digraph of order at most \(n\) |
We observe that the ratio of $T$-ISO to $T$ shows an increasing trend. The ratio of $NS$ to $T$-ISO is also increasing (for $n/C_2^3$), as expected.

For each order, we also provide the number $PU$ of unique greedoid polynomials and the ratio of $PU$ to $T$-ISO, in Table 3.

Bollobás, Pembody and Riordan conjectured that almost all graphs are determined by their chromatic or Tutte polynomials [2]. However, this conjecture does not hold for matroids. The ratio of the number of unique Tutte polynomials of matroids to the number of non-isomorphic matroids approaches 0 as the cardinality of matroids increases, which can be shown using the bounds given in Exercise 6.9 in [3]. We believe that greedoid polynomials of rooted digraphs behave in a similar manner to matroids. According to our findings, the ratio of $PU$ to $T$-ISO shows a decreasing trend. We expect that as $n$ increases, this ratio continues to decrease. The question is, does this ratio eventually approach 0 or is it bounded away from 0? Further computation may give more insight on this question.

### 2.2 Factorability

For $n \in \{1, \ldots, 5\}$, we identified the numbers of greedoid polynomials that factorise for rooted digraphs of order $n$. Details are given in Table 5 (see Table 4 for the list of abbreviations and Figure 1 for the corresponding Venn diagram).

We found that the ratio of $PF$ to $PU$ shows an upward trend, and the ratio stands at 0.9785 when $n = 5$. It seems that most likely as $n$ increases, the ratio will either approach 1 in which case almost all greedoid polynomials of rooted digraphs

| Table 2 | Numbers of various types of rooted digraphs (up to order six) |
|---------|----------------------------------------------------------|
| $n$    | $T$ | $T$-ISO | $S$ | $NS$ | $NSE$ |
| 1      | 1   | 1      | 0   | 1    | 0     |
| 2      | 6   | 4      | 0   | 4    | 0     |
| 3      | 48  | 36     | 6   | 30   | 7     |
| 4      | 872 | 752    | 88  | 664  | 200   |
| 5      | 48040 | 45960  | 2404 | 43556 | 10641 |
| 6      | 9245664 | 9133760 | 150066 | 8983694 | 1453437 |

| Table 3 | Numbers PU of unique greedoid polynomials of rooted digraphs (up to order six) and the ratio of $PU$ to $T$-ISO |
|---------|----------------------------------------------------------|
| $n$    | $T$-ISO | $PU$ | PU/T-ISO |
| 1      | 1       | 1    | 1.0000 |
| 2      | 4       | 4    | 1.0000 |
| 3      | 36      | 22   | 0.6111 |
| 4      | 752     | 201  | 0.2673 |
| 5      | 45960   | 6136 | 0.1335 |
| 6      | 9133760 | 849430 | 0.0930 |
Table 4  Abbreviations for Fig. 1 and Table 5

| Abbreviation | Description |
|--------------|-------------|
| PNF          | ...cannot be factorised |
| PF           | ...can be factorised |
| PFS          | ...can be factorised and the digraph is separable |
| PFNS         | ...can be factorised and the digraph is non-separable |
| PF PFS       | PFS ∪ PFNS |
| COMM         | PFS ∩ PFNS |
| PFSU         | PFS − COMM |
| PFNSU        | PFNS − COMM |

Table 5  Factorability of greedoid polynomials of rooted digraphs (up to order five)

| n  | PNF | PF  | PFS | PFNS | COMM | PFSU | PFNSU |
|----|-----|-----|-----|------|------|------|-------|
| 1  | 1   | 0   | 0   | 0    | 0    | 0    | 0     |
| 2  | 3   | 1   | 0   | 1    | 0    | 0    | 1     |
| 3  | 5   | 17  | 6   | 14   | 3    | 3    | 11    |
| 4  | 16  | 185 | 41  | 166  | 22   | 19   | 144   |
| 5  | 132 | 6004| 444 | 5779 | 219  | 225  | 5560  |

Fig. 1  Venn diagram that represents the factorability of greedoid polynomials of rooted digraphs where $PU = PF \cup PNF$ and $PF = PFS \cup PFNS$

Fig. 2  Digraphs that have the same greedoid polynomial where $a$ is non-separable and $b$ is separable

Fig. 3  The non-separable digraph of order two that factorises
factorise, or the ratio will approach a fixed $z$ where $0.9785 \leq z < 1$. We ask, what is the limiting proportion of greedoid polynomials of rooted digraphs that factorise, as $n \to \infty$?

We categorised these polynomials into two classes, according to whether they are polynomials of separable or non-separable digraphs. Some of these polynomials are polynomials of both separable and non-separable digraphs. The number of such polynomials is given in column 6 (COMM) in Table 5. One such example for digraphs of order three is shown in Fig. 2, where the two digraphs have the same greedoid polynomial $(1 + t)(1 + z)$.

We are interested in non-separable digraphs that can be factorised, especially those digraphs that have greedoid polynomials that are not the same as the greedoid polynomial of any separable digraph. The numbers of greedoid polynomials of these digraphs are given in column PFNSU in Table 5, and examples of such rooted digraphs of order two and three are given in Figs. 3 and 4, respectively. It is easy to verify that the greedoid polynomial of the rooted digraph in Fig. 3 is $(1 + t)(1 + z)$.

**Table 6** Greedoid polynomials of non-separable digraphs of order three that factorise and these polynomials are not the same as polynomials of any separable digraph of order three, and the numbers of associated non-separable digraphs (making 16 non-separable rooted digraphs altogether)

| Greedoid polynomials                          | Number of non-separable rooted digraphs of order three |
|-----------------------------------------------|--------------------------------------------------------|
| 1. $(1 + z)^3$                                | 2                                                      |
| 2. $(1 + z)(1 + t + t^2 + t^2z)$              | 3                                                      |
| 3. $(1 + z)(2 + 2t + t^2 + z + tz + t^2z)$   | 2                                                      |
| 4. $(1 + z)^4$                                | 1                                                      |
| 5. $(1 + z)^2(1 + t + t^2 + t^2z)$            | 3                                                      |
| 6. $(1 + t)(1 + z)^3$                         | 1                                                      |
| 7. $(1 + z)^3(2 + 2t + t^2 + z + tz + t^2z)$ | 1                                                      |
| 8. $(1 + z)^3(3 + 2t + t^2 + z + t^2z)$      | 1                                                      |
| 9. $(1 + z)^3(1 + t + t^2 + t^2z)$            | 1                                                      |
| 10. $(1 + z)^3(3 + 2t + t^2 + z + t^2z)$     | 1                                                      |
The greedoid polynomials of rooted digraphs in Fig. 4 are (from left to right starting from the first row) given in Table 6.

2.3 2-Nonbasic Digraphs

We investigate greedoid polynomials that contain nonbasic factors, for rooted digraphs up to order six. We classify (totally) 2-nonbasic digraphs into types I and II.

**Fig. 5** Venn diagram that represents four types of 2-nonbasic digraphs in Table 8 where \( U \) is the set of digraphs (up to order six) that can be factorised.

**Fig. 6** Three separable digraphs of order five that have two nonbasic factors

The greedoid polynomials of rooted digraphs in Fig. 4 are (from left to right starting from the first row) given in Table 6.

**2.3 2-Nonbasic Digraphs**

We investigate greedoid polynomials that contain nonbasic factors, for rooted digraphs up to order six. We classify (totally) 2-nonbasic digraphs into types I and II.
(to be indicated in parentheses) according as the digraph is not known (type I), or known (type II), to be a greedoid polynomial of any rooted digraph of smaller order. Details are given in Table 8 (see Table 7 for the list of abbreviations and Fig. 5 for the corresponding Venn diagram).

All rooted digraphs up to order four either have one nonbasic factor or only basic factors in their polynomials. There are 120 rooted digraphs of order five that have greedoid polynomials with at least two nonbasic factors. The number of distinct greedoid polynomials of these 120 rooted digraphs is 34. Further examination showed that the number of nonbasic factors in these polynomials is exactly two. Nonetheless, 117 of the 120 rooted digraphs have greedoid polynomials that contain at least one basic factor, and the remaining three are separable digraphs (as shown in Fig. 6). Hence, there exist no totally 2-nonbasic digraphs of order five. In addition, none of the polynomials of these 120 rooted digraphs contains a factor that is not known to be a greedoid polynomial. This implies that none of the rooted digraphs up to order five are 2-nonbasic digraphs (I). Each of the factors of greedoid polynomials of rooted digraphs of order five is either basic, or is a factor of some greedoid polynomials of rooted digraphs of smaller order.

There are 12348 rooted digraphs of order six that have greedoid polynomials with at least two nonbasic factors. The number of distinct greedoid polynomials of these 12348 rooted digraphs is 837. A quick search showed that all these digraphs are 2-nonbasic. Note that 11096 of them are 2-nonbasic digraphs (II). We found that 6 of these 2-nonbasic digraphs (II) are totally 2-nonbasic. One of the totally 2-nonbasic digraphs (II) \( D_1 \) of order six is shown in Fig. 7 and its greedoid polynomial is as follows:

\[
f(D_1) = (1 + t + t^2 + t^2 z)(2 + 2t + t^2 + t^2 + z + tz + t^2 z + 3t^3 z + 3t^3 z^2 + t^3 z^3).
\]

Both of the nonbasic factors of \( f(D_1) \) are greedoid polynomials of rooted digraphs \( G \) and \( H \) that have order three and four, respectively (see Fig. 8). We have \( f(G) = 1 + t + t^2 + t^2 z \) and \( f(H) = 2 + 2t + t^2 + t^3 + z + tz + t^2 z + 3t^3 z + 3t^3 z^2 + t^3 z^3 \). However, \( D_1 \) is a non-separable digraph and hence not the direct sum of \( G \) and \( H \).

There are also 1252 2-nonbasic digraphs (I) of order six, and all these digraphs are non-separable. However, only nine of them are totally 2-nonbasic digraphs (I). One of the 2-nonbasic digraphs (I) \( D_2 \) of order six is shown in Fig. 9 and it has the following greedoid polynomial:

\[
f(D_2) = 1 + t + t^2 + t^2 z + 2t^2 + t^3 + z + tz + t^2 z + 3t^3 z + 3t^3 z^2 + t^3 z^3.
\]
$$f(D_2) = (1 + t + t^2 + t^2z)(4 + 3t + t^2 + t^3 + 4z + 2tz + t^2z + 4t^2z + z^2 + 6t^3z^2 + 4t^3z^3 + t^3z^4)$$.

Note that the first factor $1 + t + t^2 + t^2z$ is the greedoid polynomial of the rooted digraph $G$ in Fig. 8. The second factor $4 + 3t + t^2 + t^3 + 4z + 2tz + t^2z + 4t^2z + z^2 + 6t^3z^2 + 4t^3z^3 + t^3z^4$ in $f(D_2)$ does not appear as a factor of any greedoid polynomial of rooted digraph of order less than 6. Note that this factor itself is not a greedoid polynomial of any rooted digraph of order 6. It can however be seen in
greedoid polynomials of some other rooted digraphs of order 6. One such example is shown in Fig. 10, and the digraph $D_3$ has the following greedoid polynomial:

$$f(D_3) = (1 + z)(1 + t + t^2 + t^2 z)(4 + 3t + t^2 + r^3 + 4z + 2tz + r^2 z + 4r^3 z + z^2 + 6r^3 z^2 + 4r^3 z^3 + r^3 z^4).$$

The digraph $D_3$ is not a totally 2-nonbasic digraph because of the basic factor $1 + z$. Our data shows that not all the greedoid polynomials of rooted digraphs of order 6 that contain the factor $4 + 3t + t^2 + r^3 + 4z + 2tz + r^2 z + 4r^3 z + z^2 + 6r^3 z^2 + 4r^3 z^3 + r^3 z^4$ can be obtained from $f(D_2)$ by just multiplying the basic factor $1 + z$. For example, the digraph $D_4$ (shown in Fig. 11) has greedoid polynomial

$$f(D_4) = (2 + 2t + t^2 + z + tz + t^2 z)(4 + 3t + t^2 + r^3 + 4z + 2tz + r^2 z + 4r^3 z + z^2 + 6r^3 z^2 + 4r^3 z^3 + r^3 z^4).$$

The fact that a greedoid polynomial of a rooted digraph is not divisible by $1 + z$ implies that the associated rooted digraph has neither a directed cycle nor a greedoid loop. Our results show that there exist some non-separable digraphs (of order six) that factorise into only nonbasic factors. This implies that the multiplicative direct sum property, and the existence of greedoid loops and directed cycles, are not the only characteristics that determine if greedoid polynomials of rooted digraphs factorise.

### 2.4 An Infinite Family

Lastly, we show that there exists an infinite family of digraphs where their greedoid polynomials factorise into at least two nonbasic factors. We first characterise greedoid polynomials of two classes of rooted digraphs.

Let $P_{m,v_0}$ be a directed path $v_0v_1\ldots v_m$ of size $m \geq 0$ rooted at $v_0$, and $C_{m,v_0}$ be a directed cycle $v_0v_1\ldots v_{m-1}v_0$ of size $m \geq 1$ rooted at $v_0$. For convenience, we usually write $P_m$ for $P_{m,v_0}$ and $C_m$ for $C_{m,v_0}$.

**Lemma 2.1**

$$f(P_m; t, z) = 1 + \frac{t(1 - (t(1 + z))^m)}{1 - t(1 + z)}.$$  

**Proof** By induction on the number of edges. 

Let $G$ be a rooted undirected graph and $X \subseteq E(G)$. The *rank* $r(X)$ of $X$ is defined as $r(X) = \max \{|A| : A \subseteq X, A$ is a rooted subtree}. Let $F$ be the set of subtrees of $G$ containing the root vertex. Korte and Lovász [10] showed that $(G, F)$ is a greedoid called the *branching greedoid* of $G$. 

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Suppose \( Q_m \) is an undirected path \( v_0v_1\ldots v_m \) of size \( m \geq 0 \) rooted at either \( v_0 \) or \( v_m \). Then \( f(P_m; t, z) = f(Q_m; t, z) \), since there is a rank-preserving bijection between \( 2^{E(P_m)} \) and \( 2^{E(Q_m)} \).

**Lemma 2.2**

\[
f(C_m; t, z) = (1 + z)f(P_{m-1}; t, z).
\]

**Proof** By induction on the number of edges.

Gordon gave a formula for the greedoid polynomials of rooted undirected cycles in [6]. Those polynomials are different to the ones given by Lemma 2.2.

We now give an infinite family of digraphs where their greedoid polynomials factorise into at least two nonbasic factors, extending the example in Fig. 7.

**Lemma 2.3** There exists an infinite family of non-separable digraphs \( D \) that have at least two nonbasic factors, where

\[
f(D) = f(P_{k+1})\left(f(C_{k+1}) + f(P_{k+1}) + t^{k+2}(1 + z)^{k+2}\right), \text{ for } k \geq 1.
\]
Proof Let $D$ be the non-separable digraph rooted at vertex $v_0$ shown in Fig. 12, where $a_0, \ldots, a_k$ and $b_0, \ldots, b_k$ are two directed paths in $D$ of length $k \geq 1$ starting at $a_0$ and $b_0$, respectively. To compute the greedoid polynomial of $D$ by using Proposition 1.2, we first choose the edge $e = v_0v_1$. By deleting and contracting $e$, we obtain the digraphs $D_1 = D/e$ and $D_2 = D \setminus e$ as shown in Fig. 13.

Note that $D_1$ is a separable digraph rooted at $v_0$. Let $R = \{v_0, a_0, \ldots, a_k\} \subset V(D_1)$, $S = \{v_0, b_0, \ldots, b_k\} \subset V(D_1)$ and $T = \{v_0, a_0, \ldots, a_k\} \subset V(D_2)$. Suppose $A = D_1[R]$ and $B = D_1[S]$ are the subdigraphs of $D_1$ induced by $R$ and $S$ respectively, and $C = D_2[T]$ is the subdigraph of $D_2$ induced by $T$. Clearly, $B \cong C \cong P_{k+1}$. Hence we have $f(B) = f(C) = f(P_{k+1})$. Note that every edge $g \in E(D_2) \setminus E(C)$ is a greedoid loop, and $|E(D_2) \setminus E(C)| = k + 2$. By using the recurrence formula, we have
\[ f(D) = f(D/e) + r^{r(D)-r(D/e)}f(D\setminus e) \]
\[ = f(A) \cdot f(B) + t^{(2k+3)-(k+1)}f(C) \cdot (1 + z)^{k+2} \]
\[ = f(P_{k+1}) \left( f(A) + t^{k+2}(1 + z)^{k+2} \right) \quad \text{(since } f(B) = f(C) = f(P_{k+1}) \text{)} \]

It remains to show that \( f(A) \) can be expressed in terms of \( f(P_{k}) \) and \( f(C_{k}) \). By taking \( h = v_0d_k \in E(A) \) (see Fig. 14) as the outgoing edge in the recurrence formula, we have

\[ f(A) = f(A/h) + r^{r(A)-r(A/h)}f(A \setminus h) \]
\[ = f(C_{k+1}) + t^{(k+1)-(k+1)}f(P_{k+1}) \quad \text{(since } A/h \cong C_{k+1} \text{ and } A \setminus h \cong P_{k+1} \text{)} \]
\[ = f(C_{k+1}) + f(P_{k+1}). \]

Therefore,

\[ f(D) = f(P_{k+1}) \left( f(C_{k+1}) + f(P_{k+1}) + t^{k+2}(1 + z)^{k+2} \right). \]

Clearly, both factors of \( f(D) \) are nonbasic factors. Since \( D \) is non-separable and \( k \geq 1 \), the proof is complete. \( \Box \)

We extend the infinite family in Lemma 2.3, and characterise the greedoid polynomials of a new infinite family, as follows.

**Theorem 2.4** There exists an infinite family of non-separable digraphs \( D \) that have at least two nonbasic factors, where

\[ f(D) = f(P_{k+1}) \left( f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1 + z)^{k+2}(1 - (t(1 + z))^\ell)}{1 - t(1 + z)} \right), \]

for \( k, \ell \geq 1 \).

**Proof** Let \( D \) be the non-separable digraph rooted at vertex \( v_0 \) shown in Fig. 15, where \( L = v_0 \ldots v_\ell \) is a directed path in \( D \) of length \( \ell \geq 1 \) starting at \( v_0 \). We proceed by induction on the length \( \ell \) of \( L \).

For the base case, suppose \( \ell = 1 \). By Lemma 2.3, we have

\[ f(D) = f(P_{k+1}) \left( f(C_{k+1}) + f(P_{k+1}) + t^{k+2}(1 + z)^{k+2} \right) \]
\[ = f(P_{k+1}) \left( f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1 + z)^{k+2}(1 - (t(1 + z))\ell)}{1 - t(1 + z)} \right), \]

and the result for \( \ell = 1 \) follows.
Assume that \( \ell > 1 \) and the result holds for every \( r < \ell \).

Let \( e = v_0v_1 \in E(D) \). By applying the deletion-contraction recurrence in Proposition 1.2 on \( e \), we obtain the digraphs \( D_1 = D/e \) and \( D_2 = D \setminus e \) as shown in Fig. 16.

Note that \( D_1 \) is a non-separable digraph rooted at \( v_1 \). Since the directed path \( v_1 \ldots v_\ell \) in \( D_1 \) has length \( \ell - 1 \), we use the inductive hypothesis to obtain \( f(D_1) \).

Let \( R = \{v_0, a_0, \ldots, a_k\} \subset V(D_2) \), and \( A = D_2[R] \) be the subdigraph of \( D_2 \) induced by \( R \). Clearly, \( A \cong P_{k+1} \). Hence, we have \( f(A) = f(P_{k+1}) \). Note that every edge \( g \in E(D_2) \setminus E(A) \) is a greedoid loop, and \( |E(D_2) \setminus E(A)| = k + \ell + 1 \). By using the recurrence formula, we have

\[
f(D) = f(D/e) + r^{(D)-r(D)e}f(D \setminus e)
= f(P_{k+1}) \left( f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1 + z)^{k+2}(1 - (t(1 + z))^{\ell - 1})}{1 - t(1 + z)} \right)
+ t^{(2k+\ell+2)-(k+1)} \left( f(P_{k+1}) \cdot (1 + z)^{k+\ell+1} \right)
= f(P_{k+1}) \left( f(C_{k+1}) + f(P_{k+1}) + \left( \frac{t^{k+2}(1 + z)^{k+2}(1 - (t(1 + z))^{\ell - 1})}{1 - t(1 + z)} \right) \right)
+ t^{k+\ell+1}(1 + z)^{k+\ell+1}
= f(P_{k+1}) \left( f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1 + z)^{k+2}(1 - (t(1 + z))^{\ell - 1})}{1 - t(1 + z)} \right).
\]

\[\Box\]

We observe that if every directed path has length at most one in a digraph \( D \) rooted at a vertex \( v \), the greedoid polynomial of \( D \) is trivial. In this scenario, every vertex in \( D \) is either a sink vertex or a source vertex. If \( v \) is a sink vertex, then every edge in \( D \) is a greedoid loop. If \( v \) is a source vertex, every edge that is not incident with \( v \) is a greedoid loop.

![Fig. 16](image-url) Two minors \( D/e \) and \( D \setminus e \) of \( D \)
In the following theorem, we show that the greedoid polynomial of any digraph that has a directed path of length at least two is a nonbasic factor of the greedoid polynomial of some non-separable digraph. The proof follows similar approaches as in Lemma 2.3 and Theorem 2.4.

**Theorem 2.5** For any digraph $G$ that has a directed path of length at least two, there exists a non-separable digraph $D$ where $f(D)$ has $f(G)$ as a nonbasic factor.

**Proof** Let $G$ be a digraph that has a directed path $K = a_0a_1\ldots a_k$ of length $k \geq 2$, and $G'$ be a copy of $G$. The copy of $K$ in $G'$ is denoted by $K' = a'_0a'_1\ldots a'_k$.

We construct a non-separable digraph $D_\ell$ using $G$ and $G'$, as follows. We first create a directed path $L = a_0v_1\ldots v_{\ell-1}a'_0$ of length $\ell$. We add a directed edge $a'_0a_k$, and assign $a_0$ as the root vertex of $D_\ell$ (see Fig. 17).

To show that $f(G)$ is a nonbasic factor of $f(D_\ell)$, we proceed by induction on the length $\ell$ of $L$.

For the base case, suppose $\ell = 1$. We apply the deletion-contraction recurrence in Proposition 1.2 on $e = a_0a'_0$. We denote $a_0$ the root vertex of the separable digraph $D_1/e$. We have

$$f(D_1) = f(D_1/e) + t^{r(D_1/e)} f(D_1 \setminus e)$$

$$= f(G + a_0a_k) \cdot f(G) + t^{r(G) - r(G)} f(G) \cdot (1 + z)^{|E(G)|+1}$$

$$= f(G) \left( f(G + a_0a_k) + t^{r(G) + 1} (1 + z)^{|E(G)|+1} \right).$$

Hence, the result for $\ell = 1$ follows.

Assume that $\ell > 1$ and the result holds for every $r < \ell$.

For the inductive steps, we apply the deletion-contraction recurrence on $e = a_0v_1$. We have
Let \( D \) be a non-separable digraph that belongs to the infinite family in Theorem 2.5. By replacing the edge \( a_0a_k \in E(D) \) by any digraph \( R \) such that every edge in \( E(R) \) that is incident with \( a_k \) is an incoming edge of \( a_k \), then \( f(D) \) has \( f(G) \) as a nonbasic factor. 

We now have the following corollary.

**Corollary 2.6** Let \( D \) be a non-separable digraph that belongs to the infinite family in Theorem 2.5. By replacing the edge \( a_0a_k \in E(D) \) by any digraph \( R \) such that every edge in \( E(R) \) that is incident with \( a_k \) is an incoming edge of \( a_k \), then \( f(D) \) has \( f(G) \) as a nonbasic factor. 

### 3 Computational Methods

All labelled rooted digraphs (without loops and multiple edges, but cycles of size two are allowed) up to order six were provided by Brendan McKay on 28 March 2018 (personal communication from McKay to Farr). Each digraph is given as a list of numbers on one line separated by a single space. The first number is the order of the digraph, the second number is the size of the digraph, and each pair of subsequent numbers represent a directed edge of the digraph. For instance, \( 3 2 0 2 1 \) represents a digraph of order 3 and size 2. The directed edges of the digraph are \((2, 0)\) and \((2, 1)\). Details are as follows:

\[
\begin{array}{c@{}c@{}c@{}c@{}c@{}c}
\text{order} & \text{edge} \\
3 & 2 & 0 & 1 \\
\text{size} & \text{edge}
\end{array}
\]

We use the set of numbers \( \{0, 1, \ldots, n - 1\} \) to represent vertices for each digraph of order \( n \), and an edge list to represent the edge set of each digraph, e.g., \([0, 1]\) represents a digraph with a single edge directed from vertex 0 to vertex 1.

We use Python 3, Wolfram Mathematica 11 and Bash Shell (Mac OS Version 10.13.4), in computing results for greedoid polynomials of rooted digraphs up to order six.

Algorithms of our programs and the steps used in obtaining our results are given in [21] and [22].

### 4 Concluding Remarks

In this paper, we presented (i) the results from exhaustive computation of all small rooted digraphs and (ii) the first results of the factorability of greedoid polynomials of rooted digraphs.

We computed the greedoid polynomials for all rooted digraphs up to order six. From Table 3, the ratio of PU to T-ISO shows a decreasing trend. We expect that as
As \( n \) increases, this ratio continues to decrease. Hence, we have the following conjecture.

**Conjecture 4.1** *Most rooted digraphs are not determined by their greedoid polynomials.*

We found that the multiplicative direct sum property, and the existence of greedoid loops and directed cycles, are not the only characteristics that determine if greedoid polynomials of rooted digraphs factorise. We showed that there exists an infinite family of non-separable digraphs where their greedoid polynomials factorise. We also characterised the greedoid polynomials of rooted digraphs that belong to the family.

We now suggest some problems for further research.

1. Investigate the factorability of greedoid polynomials of rooted graphs, or even greedoids in general.

2. Does there exist a graph-theoretic interpretation for the highest power of \( 1 + t \) for greedoid polynomials of rooted digraphs?

For rooted digraphs of order six, there are nine totally 2-nonbasic digraphs (I) and six totally 2-nonbasic digraphs (II).

3. For \( k \geq 2 \), can we characterise greedoid polynomials of totally \( k \)-nonbasic digraphs?

4. Determine necessary and sufficient conditions for greedoid polynomials of rooted digraphs to factorise.

**Acknowledgements** We thank Gary Gordon and the referee for their useful feedback.

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