Quenched, Minisuperspace, Bosonic $p$-brane Propagator

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Abstract

We borrow the \textit{minisuperspace approximation} from Quantum Cosmology and the \textit{quenching approximation} from \textit{QCD} in order to derive a new form of the bosonic $p$–brane propagator. In this new approximation we obtain an \textit{exact} description of both the \textit{collective mode} deformation of the brane and the center of mass dynamics in the target spacetime. The collective mode dynamics is a generalization of string dynamics in terms of area variables. The final result is that the evolution of a $p$–brane in the quenched–minisuperspace

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approximation is formally equivalent to the effective motion of a particle in a spacetime where \textit{points as well as hypersurfaces} are considered on the same footing as fundamental geometrical objects. This geometric equivalence leads us to define a new \textit{tension–shell condition} that is a direct extension of the Klein–Gordon condition for material particles to the case of a physical \textit{p–brane}. 
I. INTRODUCTION

The purpose of this paper is to introduce a new approximation scheme to the quantum dynamics of extended objects. Our approach differs from the more conventional ones, such as the normal modes expansion or higher dimensional gravity, in that it is inspired by two different quantization schemes: one is the minisuperspace approach to Quantum Cosmology (QC), the other is the quenching approximation to QCD. Even though these schemes apply to different field theories, they have a common rationale, that is, the idea of quantizing only a finite number of degrees of freedom while freezing, or “quenching,” all the others. In QC this idea amounts, in practice, to quantizing a single scale factor (thereby selecting a class of cosmological models, for instance, the Friedman–Robertson–Walker spacetime) while neglecting the quantum fluctuations of the full metric. The effect is to turn the exact, but intractable, Wheeler–DeWitt functional equation \[1\] into an ordinary quantum mechanical wave equation \[2\]. As a matter of fact, the various forms of the “wave function of the universe” that attempt to describe the quantum birth of the cosmos are obtained through this kind of approximation \[3\], or modern refinements of it \[4\]. On the other hand, in QCD, the dynamics of quarks and gluons cannot be solved perturbatively outside the small coupling constant domain. The strong coupling regime is usually dealt with by studying the theory on a lattice. However, even in that case, the computation of the fermionic determinant by Monte Carlo simulation is actually impossible. Thus, in the “quenched approximation,” the quark determinant is set equal to unity, which amounts to neglecting the effect of virtual quark loops. In other words, this extreme approximation in terms of heavy–quarks with a vanishing number of flavors assumes that gauge fields affect quarks while quarks have no dynamical effect on gauge fields \[5\].

Let us compare the above two situations with the basic problem that one faces when dealing with the dynamics of a relativistic extended object, or \[p\]-brane, for short. Ideally, one would like to account for all local deformations of the object configuration, i.e., those deformations that may take place at a generic point on the \[p\]-brane. However, any attempt to provide a local description of this shape–shifting process leads to a functional differential equation similar to the Wheeler–DeWitt equation in QC. The conventional way of handling that functional equation, or the equivalent infinite set of ordinary differential equations \[6\], is through perturbation theory. There, the idea is to quantize the small oscillations about a classical configuration and assign to them the role of “particle states” \[7\]. Alternatively to this approach is the quantization of a brane of preassigned geometry. This (minisuperspace) approach, pioneered in Ref.(\[8\]), was used in Refs.(\[9\], \[10\]) in order to estimate the nucleation rate of a spherical membrane. Presently, the minisuperspace approximation is introduced for the purpose of providing an exact algorithm for computing two specific components of the general dynamics of a \[p\]-brane: one is the brane collective mode of oscillation in terms of global volume variations, the other is the evolution of the brane center of mass.

In broad terms, this paper is divided into two parts: Section II deals with classical dynamics; Section III deals with quantum dynamics in terms of the path–integral, or “sum over histories.”

Since the action for a classical \[p\]-brane is not unique, we start our discussion by providing
the necessary background with the intent of justifying our choice of action. We then take
the first step in our approximation scheme in order to separate the center of mass motion
from the bulk and boundary dynamics. In subsections IIC and IID we derive an effective
action for the bulk and boundary evolution, while in subsection IIE we discuss the meaning
of the “quenching approximation” at the classical level.
An approximation scheme for a dynamical problem is truly meaningful and useful only when
the full theory is precisely defined, so that the technical and logical steps leading to the ap-
proximate theory are clearly identified. Thus, in Section III we first tackle the problem of
computing the general quantum amplitude for a \( p \)-brane to evolve from an initial configu-
ration to a final one. The full quantum propagator is obtained as a sum over all possible
histories of the world–manifold of the relativistic extended object. In subsection IIIA, we
show in detail what that “sum over histories” really means, both mathematically and phys-
ically, in order to explain why the bulk quantum dynamics cannot be solved exactly. What
can be calculated, namely, the boundary and center of mass propagator, is discussed in
subsections IIIB and IIIC. The final expression for the quantum propagator and the gen-
eralized “tension–shell” condition in the quenched–minisuperspace approximation is given
in subsection IIDD. Finally, subsection IIE checks the self–consistency of the result against
some special cases of physical interest.

II. CLASSICAL DYNAMICS

A. Background

The action of a classical \( p \)-brane is not unique. The first (mem)brane action, dates back
to 1962 and was introduced by Dirac in an attempt to resolve the electron–muon puzzle
\( \text{[1]} \). The Dirac action was reconsidered in Ref.( \( \text{[8]} \) ) and quantized following the pioneering
path traced by Nambu–Goto in the lower dimensional string case. The Dirac–Nambu–Goto
action represents the world volume of the membrane trajectory in spacetime. Thus, it can
be generalized to higher dimensional \( p \)-branes as follows

\[
S_{\text{DNG}} [Y] = -m_{p+1} \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{-\gamma} \quad , \quad \gamma \equiv \det ( \partial_m Y^\mu \partial_n Y_\mu ) , \quad (2.1)
\]

where \( m_{p+1} \) represents the “\( p \)-tension” ( we denote with “\( p \)” the spatial dimensionality of
the brane ) and the coordinates \( \sigma^m, m = 0,1,\ldots,p \), span the \( (p + 1) \)-dimensional world–
manifold \( \Sigma \) in parameter space. On the other hand, the embedding functions \( Y^\mu(\sigma) \), \( \mu =
0,1,\ldots,D-1 \), represent the brane coordinates in the target spacetime.
An alternative description that preserves the reparametrization invariance of the world–
manifold is achieved by introducing an auxiliary metric \( g_{mn}(\sigma) \) in parameter space together
with a “cosmological constant” on the world–manifold \( \text{[12], [13]} \)

\[
S_{\text{HTP}} [Y, g] = -\frac{m_{p+1}}{2} \int_{\Sigma} d^{p+1} \sigma \sqrt{-g} \left[ g^{mn} \partial_m Y^\mu \partial_n Y_\mu - (p - 1) \right] , \quad (2.2)
\]

where \( g \equiv \det g_{mn} \). The two actions \( (2.1) \) and \( (2.2) \) are classically equivalent in the sense
that the “field equations” \( \delta S/\delta g^{mn}(\sigma) = 0 \) require the auxiliary world metric to match the
induced metric, i.e., \( g_{mn} = \gamma_{mn} = \partial_m Y^\mu \partial_n Y_\mu \). The two actions are also complementary: \( S_{\text{DNG}} \) provides an “extrinsic” geometrical description in terms of the embedding functions \( Y^\mu(\sigma) \) and the induced metric \( \gamma_{mn} \), while \( S_{\text{HTP}} \) assigns an “intrinsic” geometry to the world manifold \( \Sigma \) in terms of the metric \( g_{mn} \) and the “cosmological constant” \( m_{p+1} \), with the \( Y^\mu(\sigma) \) functions interpreted as a “multiplet of scalar fields” that propagate on a curved \((p+1)\)-dimensional manifold.

Note that in both functionals (2.1) and (2.2), the brane tension \( m_{p+1} \) is a pre-assigned parameter. More recently, new action functionals have been proposed that bridge the gap between relativistic extended objects and gauge fields [14], [15], [16]. The brane tension itself, or world–manifold cosmological constant, has been lifted from an \( a \text{ priori} \) assigned parameter to a dynamically generated quantity that may attain both positive and vanishing values. Either a Kaluza–Klein type mechanism [17] or a modified integration measure have been proposed as dynamical processes for producing tension at the classical [18] and semi-classical level [19].

For our present purposes, the form (2.2) of the \( p \)-brane action is the more appropriate starting point. There are essentially two reasons for this choice:

1. Unlike the Nambu–Goto–Dirac action, or the Schild action [20], Eq. (2.2) is quadratic in the variables \( \partial_m X^\mu \). As we shall see in the following subsections, this property, together with the choice of an appropriate coordinate system on \( \Sigma_{p+1} \), facilitates the factorization of the center of mass motion from the deformations of the brane.

2. Equation (2.2) can be interpreted as a scalar field theory in curved spacetime. From this point of view, the minisuperspace quantization approach is equivalent to a quantum field theory in a fixed background geometry, at least as far as the auxiliary metric is concerned.

B. Center of Mass Dynamics

The dynamics of an extended body can be formulated in general as the composition of the center of mass motion and the motion relative to the center of mass. A \( p \)-brane is by definition a spatially extended object. Thus we expect to be able to separate the motion of its center of mass from the shape–shifting about the center of mass. However, given the point like nature of the center of mass, its spacetime coordinates depend on one parameter only, say, the proper time \( \tau \). Thus, the factorization of the center of mass motion automatically breaks the general covariance of the action in parameter space since it breaks the symmetry between the temporal parameter \( \tau \) and the spatial coordinates \( s^i \). We can turn “needs into virtue” by choosing a coordinate mesh on \( \Sigma_{p+1} \) that reflects the breakdown of general covariance in parameter space. Indeed, we can choose the model manifold \( \Sigma_{p+1} \) of the form

\[
\Sigma_{p+1} = I \otimes \Sigma_p \quad , \quad \partial I = \{ P_0, P \} \quad , \quad \partial \Sigma_p = \emptyset ,
\]

where \( I \) is an open interval of the real axis, which has two points, say \( P_0 \) and \( P \), as its boundary and \( \Sigma_p \) is a finite volume, \( p \)-dimensional manifold, without boundary. Thus,
\[ \partial \Sigma_{p+1} = P_0 \otimes \Sigma_p \cup P \otimes \Sigma_p \] and the spacetime image of \( \partial \Sigma_{p+1} \) under the embedding \( Y \) represents the initial and final brane configuration in target spacetime.

In terms of coordinates, the above factorization of \( \Sigma_{p+1} \) amounts to defining \( \tau \) as the center of mass proper time and the \( s_i \)'s as spatial coordinates of \( \Sigma_p \). Accordingly, the invariant line element reads:

\[
dl^2 = \mathcal{g}_{mn} \, d\sigma^m \, d\sigma^n = -e^2(\tau) \, d\tau^2 + h_{ij}(\vec{s}) \, ds^i \, ds^j \tag{2.4}\]

where \( \tau \) plays the role of “cosmological time”, that is, all clocks on \( \Sigma_p \) are synchronized with the center of mass clock.

Now, we are in a position to introduce the center of mass coordinates \( x^\mu(\tau) \) and the relative coordinates \( Y^\mu(\tau, s^i) \):

\[
X^\mu(\tau, \vec{s}) \equiv x^\mu(\tau) + \frac{1}{\sqrt{m_{p+1}}} \, Y^\mu(\tau, \vec{s}) , \tag{2.5}
\]

\[
x^\mu(\tau) \equiv \frac{1}{V_p} \int_{S_p} d^p s \sqrt{h(\vec{s})} \, X^\mu(\tau, \vec{s}) \tag{2.6}
\]

\[
V_p \equiv \int_{S_p} d^p s \sqrt{h(\vec{s})}, \quad h(\vec{s}) \equiv \det(h_{ij}) . \tag{2.7}
\]

Using the above definitions in the action (2.2) and replacing \( g_{mn} \) with \( \mathcal{g}_{mn} \) as indicated in Eq.(2.4), we find

\[
S = -\frac{1}{2} \int_{\Sigma} d^{p+1} \sigma \sqrt{\mathcal{g}} \left[ m_{p+1} \mathcal{g}^{00} \dot{x}^\mu(\tau) \dot{x}_\mu(\tau) + (\mathcal{g}^{mn} \, \partial_m Y^\mu \, \partial_n Y_\mu - m_{p+1} (p-1)) \right] , \quad p \geq 1
\]

\[
= -\frac{1}{2} m_{p+1} V_p \int_0^T d\tau \left[ -\frac{\dot{x}^\mu(\tau) \dot{x}_\mu(\tau)}{e(\tau)} + e(\tau) \right] - \frac{1}{2} \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{\mathcal{g}} \left[ \mathcal{g}^{mn} \, \partial_m Y^\mu \, \partial_n Y_\mu - m_{p+1} p \right] . \tag{2.8}
\]

The first term describes the free motion of the bulk center of mass. The absence of a mixed term, one that would couple the center of mass to the bulk oscillation modes, is due to the vanishing of the metric component \( \mathcal{g}^{0i} \) in the adopted coordinate system (2.4). The last term represents the usual bulk modes free action for a covariant “scalar field theory” in parameter space.

Finally, if we define the brane volume mass, \( M_0 \equiv V_p m_{p+1} \), representing the brane inertia under volume variation, then, from the above expression, we can read off the center of mass action and the corresponding Lagrangian

\[
S_{cm} = -\frac{M_0}{2} \int_0^T d\tau \left[ -\frac{\dot{x}^\mu(\tau) \dot{x}_\mu(\tau)}{e(\tau)} + e(\tau) \right] \equiv \int_0^T d\tau L_{cm}(\dot{x}^\mu ; e(\tau)) , \tag{2.9}
\]

where the \textit{einbein} \( e(\tau) \) ensures \( \tau \) reparametrization invariance along the center of mass world–line.

Summarizing, the final result of this subsection is that, in the adopted coordinate frame where the center of mass motion is separated from the bulk and boundary dynamics, we can write the total action as the sum of two terms

\[
S = S_{cm} - \frac{1}{2} \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{\mathcal{g}} \left[ \mathcal{g}^{mn} \, \partial_m Y^\mu \, \partial_n Y_\mu - m_{p+1} p \right] . \tag{2.10}
\]
We emphasize that, in order to derive the expression (2.10), it was necessary to break the full invariance under general coordinate transformations of the initial theory, preserving only the more restricted symmetry under independent time and spatial coordinate reparametrizations.

C. Induced Bulk and Boundary Actions

In this subsection we wish to discuss those features of the brane classical dynamics which are instrumental for the subsequent evaluation of the quantum path–integral. In agreement with the restricted reparametrization invariance of the action (2.10), as discussed in the previous subsection, we first set up a “canonical formulation” which preserves that same symmetry through all computational steps. This means that all the world indices \( m, n, \ldots \) are raised, lowered and contracted by means of the center of mass metric \( \bar{g}_{mn} \). From the brane action (2.10) we extract the brane relative momentum \( P^m_\mu \) and the corresponding Hamiltonian \( H \):

\[
P^m_\mu \equiv \frac{\partial L}{\partial \partial_{m} Y^\mu(\sigma)} = -\bar{g}^{mn} \partial_n Y^\mu(\sigma)
\]

\[
H \equiv P^m_\mu \partial_m Y^\mu - L = -\frac{1}{2} \left[ \bar{g}_{mn} P^m_\mu P^n_\mu + m_{p+1} p \right].
\]  

(2.11)

Thus, we can write the action in the following canonical form

\[
S = \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{\bar{g}} \left( P^m_\mu \partial_m Y^\mu - H \right)
\]

\[
= \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{\bar{g}} \left[ P^m_\mu \partial_m Y^\mu + \frac{1}{2} \left( \bar{g}_{ab} P^a_\mu P^b_\mu + p m_{p+1} \right) \right].
\]  

(2.12)

The first term in Eq. (2.12) can be rewritten as follows

\[
\int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{\bar{g}} P^m_\mu \partial_m Y^\mu = \int_{\Sigma_{p+1}} d^{p+1}\sigma \left[ \partial_m \left( \sqrt{\bar{g}} P^m_\mu Y^\mu \right) - Y^\mu \partial_m \left( \sqrt{\bar{g}} P^m_\mu \right) \right].
\]  

(2.13)

According to Eq. (2.13) we can write the total action as the sum of a boundary term plus a bulk term

\[
S = S_B[\partial \Sigma_{p+1}] + S_J[\Sigma_{p+1}]
\]

\[
= \int_{\Sigma_{p}} d^{p}s \sqrt{h} N_n p^n_\mu g^\mu - \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{\bar{g}} Y^\mu(\sigma) \nabla_m P^m_\mu - \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{\bar{g}} H,
\]  

(2.14)

where \( p^n_\mu \) and \( g^\mu \) are the momentum and coordinate of the boundary, \( d^{p}s \sqrt{h} N_n \) represents the oriented surface element of the boundary, and \( \nabla_m \) stands for the covariant derivative with respect to the metric \( \bar{g}_{mn} \). The distinctive feature of this rearrangement is that the bulk coordinates \( Y^\mu(\sigma) \) enter the action as Lagrange multipliers enforcing the classical equation of motion:

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\[
\frac{\delta S}{\delta Y^\mu(\sigma)} = 0 \quad \Rightarrow \quad \partial_m \left( \sqrt{g} \, P^m_\mu \right) = 0 . \tag{2.15}
\]

The general solution of Eq. (2.15) may be expressed as follows
\[
P^m_\mu = \bar{g}^{mn} \partial_n \phi_\mu + \frac{1}{p!} \epsilon^{mm_2\ldots m_{p+1}} \left( P^{00}_\mu \partial_{m_2} Y^{m_2} \ldots \partial_{m_{p+1}} Y^{m_{p+1}} + \partial_{[m_2} A_{m_3\ldots m_{p+1}]} \right) = \bar{g}^{mn} \partial_n \phi_\mu + \frac{1}{p!} \epsilon^{mm_2\ldots m_{p+1}} \left( P^{00}_\mu + F(A)_{m_2\ldots m_{p+1}} \right) \tag{2.16}
\]
with
\[
\Box \phi^\mu(\sigma) = 0 \tag{2.17}
\]
\[
\partial_m P^{00}_\mu = 0 . \tag{2.18}
\]

Here, the components \( \phi_\mu \) represent local harmonic modes on the bulk, and \( \epsilon^{mm_2\ldots m_{p+1}} \equiv (\bar{g})^{-1/2} \delta^{[m_1 m_2\ldots m_{p+1}] \mu} \) stands for the totally antisymmetric tensor. Moreover, the constant antisymmetric tensor \( P^{00}_\mu \) represents the volume momentum zero-mode, or collective-mode, that describes the global volume variation of the brane.

In order to be able to treat \( \phi, P^{00} \) and \( A \) as independent oscillation modes, we demand that the following orthogonality relations are satisfied:
\[
\mathcal{C}_1 \equiv \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{g} \epsilon^{mm_2\ldots m_{p+1}} \partial_m \phi^\mu \partial_{m_2} Y^{m_2} \ldots \partial_{m_{p+1}} Y^{m_{p+1}} = 0 , \tag{2.19}
\]
\[
\mathcal{C}_{II} \equiv \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{g} \epsilon^{mm_2\ldots m_{p+1}} \partial_m \phi^\mu \partial_{[m_2} F^{00}_{m_3\ldots m_{p+1}]} = 0 , \tag{2.20}
\]
\[
\mathcal{C}_{III} \equiv \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{g} \epsilon^{mm_2\ldots m_{p+1}} \partial_m \phi^\mu \partial_{[m_2} A_{m_3\ldots m_{p+1}]} = 0 . \tag{2.21}
\]
The three orthogonality constraints on the bulk determine the field behavior on the boundary through Stokes’ theorem. In particular \( \mathcal{C}_1 \) gives
\[
\mathcal{C}_1 = \int_{\Sigma_p} \phi^\mu dy^{m_2} \wedge \ldots \wedge dy^{m_{p+1}} = 0 \quad \Rightarrow \quad \phi^\mu(\vec{s}) = 0 , \tag{2.22}
\]
which is a Dirichlet boundary condition, whereas in \( \mathcal{C}_{III} \) two integrations remain:
\[
\mathcal{C}_{III} = \int_{\Sigma_p} d^p s F_{\mu m_2\ldots m_{p+1}} \partial_m y^{m_2} \ldots \partial_{m_{p+1}} y^{m_{p+1}}
- \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{g} \left( \partial_{m_2} F_{\mu m_3\ldots m_{p+1}} Y^{m_3} \ldots \partial_{m_{p+1}} Y^{m_{p+1}} \right) . \tag{2.23}
\]
Since the two integrations are carried over the boundary and the bulk respectively, the orthogonality condition can be satisfied only if each integral is identically vanishing,
\[
\mathcal{C}_{III} = 0 \quad \Rightarrow \quad A^\mu_{m_3\ldots m_{p+1}}(\vec{s}) = \partial_{[m_3} A^\mu_{m_4\ldots m_{p+1}]} ,
\text{ and } \partial_{m_2} F_{\mu m_3\ldots m_{p+1}} = 0 . \tag{2.24}
\]
Thus, \( A \) must solve free Maxwell-type equations on the bulk and reduce to a pure gauge configuration on the boundary. Note that under these conditions for \( \phi \) and \( A \), \( \mathcal{C}_{II} \) is satisfied as well.

In summary, the classical solution for the brane momentum reduces the original field content of the model to:
• a multiplet $\phi^\mu$ of world, harmonic, scalar fields (target spacetime vector) which satisfy Dirichlet boundary conditions;

• a multiplet $A^{\mu m_3...m_{p+1}}$ of world, Kalb–Ramond fields (target spacetime vector) which reduce to a pure gauge configuration on the boundary;

• a world–manifold (cosmological) constant $P^{0)\mu_2...\mu_{p+1}}$ (target spacetime constant tensor) corresponding to a constant energy background along the brane world–manifold.

D. Effective Bulk and Boundary Actions

By inserting Eq. (2.18) into the action (2.12), and taking into account the conditions (2.19), (2.20), (2.21), (2.22), (2.23), we can write an effective classical action for the three types of oscillation modes,

$$S_{eff} \equiv S_B + S_J$$

$$S_B = \frac{1}{(p+1)!} P^{0)\mu_2...\mu_{p+1}} \int_{\partial S} d\sigma^{\mu_2...\mu_{p+1}}(\vec{s}) + \int_{\Sigma_p} d^p s y^\mu N^m(\vec{s}) \partial_m \phi_\mu$$

$$S_J = -\frac{1}{2m_{p+1}} \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{g} \left( y^{mn} \partial_m \phi^\mu \partial_n \phi_\mu - \frac{1}{p!} F^\mu_{m_2...m_{p+1}} F^m_{m_2...m_{p+1}} \right) + \frac{1}{2m_{p+1}} \left[ \frac{1}{(p+1)!} P^{0)\mu m_2...m_{p+1}} P^{0)\mu m_2...m_{p+1} - m_{p+1}^2 p \right] \Omega_{p+1},$$

where $N^m(\vec{s})$ represents the normal to the boundary and

$$\sigma^{\mu_2...\mu_{p+1}} = \int_{\partial S} y^\mu d\gamma^2 \wedge \ldots \wedge d\gamma^{p+1} \equiv \int_{\partial S} d\sigma^{\mu_2...\mu_{p+1}}(\vec{s}), \quad p \geq 1$$

stands for the volume tensor of the brane in target spacetime, while $d\sigma^{\mu_2...\mu_{p+1}}(\vec{s})$ represents the oriented volume element attached to the original $p$–brane at the contact point $x^\mu = y^\mu(\vec{s})$. Finally, by definition, we set

$$\Omega_{p+1} \equiv \int_{\Sigma_{p+1}} d^{p+1} \sigma \sqrt{g} = \int_0^T d\tau e(\tau) \int_{\Sigma_p} d^p \sigma \sqrt{h} \equiv V_p \int_0^T d\tau e(\tau).$$

Expression (2.29) allows us to establish a relation between functional derivatives in $p$–loop space,

$$\frac{\delta}{\delta y^\mu(\vec{s})} = y^{\mu_2...\mu_{p+1}}(\vec{s}) \frac{\delta}{\delta \sigma^{\mu_2...\mu_{p+1}}(\vec{s})}, \quad y^{\mu_2...\mu_{p+1}}(\vec{s}) \equiv \epsilon^{m_2...m_{p+1}} \partial_{m_2} y^{\mu_2} \ldots \partial_{m_{p+1}} y^{\mu_{p+1}}.$$
effective action $S_B$ leads us to define the boundary momentum density as the dynamical variable canonically conjugated to the boundary coordinate $y^\mu(\vec{s})$:

$$\frac{\delta S_B}{\delta y^\mu(\vec{s})} = P_\mu(\vec{s}) = \frac{1}{p!} \epsilon^{\mu_2...\mu_{p+1}} P_{\mu_2...\mu_{p+1}} \partial_{\mu_2} y^{\mu_2} \ldots \partial_{\mu_{p+1}} y^{\mu_{p+1}} + N^m \partial_m \phi_\mu . \quad (2.32)$$

Here, $P_\mu(\vec{s})$ describes the overall response of the $p$–brane boundary to local volume deformations encoded into $d\sigma^{\mu_1...\mu_{p+1}}(\vec{s})$, as well as to induced harmonic deformations, orthogonal to the boundary, described by the normal derivative of $\phi_\mu$. In a similar way, we can define the energy density of the system as the dynamical variable canonically conjugated to the $p$–brane history volume variation

$$\frac{\delta S_J}{\partial \Omega_{p+1}} = \frac{1}{2m_{p+1}(p+1)!} P_{\mu_2...\mu_{p+1}} P^{0)_{\mu_2...\mu_{p+1}} - \frac{m_{p+1}}{2} p . \quad (2.33)$$

Finally, from the anti-symmetry of $P^{0)_{\mu_2...\mu_{p+1}}}$ under index permutations we deduce the following identity

$$P_\mu P^\mu \equiv \frac{\hbar}{(p+1)!} P^{0)_{\mu_2...\mu_{p+1}} P^{0)_{\mu_2...\mu_{p+1}}} . \quad (2.34)$$

Thus, we arrive at the main result of the classical formulation in the form of a reparametrization invariant, relativistic, effective Jacobi equation

$$\frac{1}{2m_{p+1}V_p} \int_{\Sigma_p} \sqrt{h} \left( \frac{\delta S^{\text{eff}}}{\delta y_\mu(\vec{s})} - N^m(\vec{s}) \partial_m \phi_\mu \right) \left( \frac{\delta S^{\text{eff}}}{\delta y^\mu(\vec{s})} - N^j(\vec{s}) \partial_j \phi_\mu \right) - \frac{m_{p+1}}{2} p = \frac{\partial S^{\text{eff}}}{\partial \Omega_{p+1}} . \quad (2.35)$$

This Jacobi equation encodes the boundary dynamics of the $p$–brane with respect to an evolution parameter represented by the world–volume of the $p$-brane history. This is a generalization of the areal string dynamics originally introduced by Eguchi [22] via reparametrization of the Schild action [20], [23].

E. “Classical Quenching” → Volume Dynamics

The Jacobi equation derived in the previous subsection takes into account both the intrinsic fluctuations $\delta y^\mu(\vec{s})$ and the normal boundary deformations $dN^m \partial_m \phi^\mu$ induced by the bulk field $\phi^\mu$. The problem is that, even neglecting the boundary fluctuations induced by the bulk harmonic mode, the $p$–loop space Jacobi equation is difficult to handle [24], [25], [26]. In order to make some progress, it is necessary to forgo the local fluctuations of the brane in favor of the simpler, global description in terms of hyper-volume variations, without reference to any specific point on the $p$-brane where a local fluctuation may actually occur. Thus, in our formulation of $p$–brane dynamics, classical quenching means having to relinquish the idea of describing the local deformations $\phi^\mu(\sigma)$ of the brane, and to focus instead on the collective mode of oscillation. In turn, by “collective dynamics,” we mean volume variations with no reference to the local fluctuations which cause the volume to vary. In
this approximation we can write a “global”, i.e., non-functional, \( p \)-brane wave equation.

The effective action that encodes the volume dynamics, say \( S_0 \), is obtained from \( S_J + S_B \) by “freezing” both the harmonic and Kalb–Ramond bulk modes. The simplification is that the general action reduces to the following form

\[
S_J + S_B \rightarrow S_0 = \frac{1}{(p+1)!} \sigma^{\mu_2\ldots\mu_{p+1}} P^{(0)}_{\mu_2\ldots\mu_{p+1}} + \Omega_{p+1} \left[ \frac{1}{2m_{p+1} (p+1)!} P^{(0)}_{\mu_2\ldots\mu_{p+1}} P^{(0)\mu_2\ldots\mu_{p+1}} - \frac{m_{p+1}}{2} p \right]
\]

so that the functional equation reduces to a partial differential equation

\[
\frac{1}{2m_{p+1}} \frac{\partial S_0}{\partial \sigma_{\mu_1\ldots\mu_{p+1}}} \frac{\partial S_0}{\partial \sigma^{\mu_1\ldots\mu_{p+1}}} - \frac{m_{p+1}}{2} p = \frac{\partial S_0}{\partial \Omega_{p+1}}.
\]

The collective–mode dynamics is much simpler to handle. In fact, the functional derivatives that describe the shape variation of the brane have been replaced by “ordinary” partial derivatives that take into account only hyper–volume variations, rather than local distortions. In other words, while the original equation \((2.35)\) describes the shape dynamics, the global equation \((2.37)\) accounts for the collective dynamics of the brane. The advantage of the Jacobi equation \((2.37)\) is that the partial derivative is taken with respect to a matrix coordinate \( \sigma_{\mu_1\ldots\mu_{p+1}} \) instead of the usual position four–vector.

The similarity with the point particle case \((p=0)\) suggests the following ansatz for \( S_0 \)

\[
S_0 (\sigma; \Omega) = \frac{B}{2\Omega_{p+1} (p+1)!} \left( \sigma^{\mu_1\ldots\mu_{p+1}} - \sigma^{\mu_1\ldots\mu_{p+1}}_{0} \right)^2 - \frac{m_{p+1}}{2} p V_{p+1},
\]

where \( \sigma^{\mu_1\ldots\mu_{p+1}}_{0} \) represents a constant ( matrix ) of integration to be determined by the “initial conditions”, while the value of the \( B \) factor is fixed by the equation \((2.37)\). Indeed

\[
\frac{\partial S_0}{\partial \sigma^{\mu_1\ldots\mu_{p+1}}} = \frac{B}{\Omega_{p+1}} \left( \sigma^{\mu_1\ldots\mu_{p+1}} - \sigma_{0}^{\mu_1\ldots\mu_{p+1}} \right)
\]

and

\[
\frac{\partial S_0}{\partial \Omega_{p+1}} = - \frac{B}{2\Omega_{p+1} (p+1)!} \left( \sigma^{\mu_1\ldots\mu_{p+1}} - \sigma_{0}^{\mu_1\ldots\mu_{p+1}} \right)^2 - \frac{m_{p+1}}{2} p \Omega_{p+1}
\]

so that

\[
B = -m_{p+1},
\]

\[
S_0 (\sigma; V) = - \frac{m_{p+1}}{2V_{p+1} (p+1)!} \left( \sigma^{\mu_1\ldots\mu_{p+1}} - \sigma^{\mu_1\ldots\mu_{p+1}}_{0} \right)^2 - \frac{m_{p+1}}{2} p V_{p+1}.
\]

For the sake of simplicity, one may choose the integration constant to vanish, i.e., one may set \( \sigma^{\mu_1\ldots\mu_{p+1}}_{0} = 0 \). Thus, in the quenching approximation, we have obtained the classical Jacobi action for the hyper-volume dynamics of a free \( p \)-brane.
III. THE FULL QUANTUM PROPAGATOR

A. “Momentum Space” Propagator

Eventually, one is interested in computing the quantum amplitude for the brane to evolve from an initial (vacuum) state to a final, finite volume, state. In general, the $p$–brane “two–point” Green function represents the correlation function between an initial brane configuration $y_0(\vec{s})$ and a final configuration $y(\vec{s})$. In the quantum theory of $p$–branes the Green function is obtained as a sum over all possible histories of the world–manifold $\Sigma_{p+1}$ in the corresponding phase space. For the sake of simplicity, one may “squeeze” the initial boundary of the brane history to a single point. In other words, the physical process that we have in mind represents the quantum nucleation of a $p$–brane so that the propagator that we wish to determine connects an initial brane of zero size to a final object of proper volume $V_p$. The corresponding amplitude $G$ is represented by the path–integral:

$$G = \int [\mathcal{D}g_{mn}] \int_{x_0}^{x} [\mathcal{D}x(\tau)] \int [\mathcal{D}p(\tau)] \int^{y} [\mathcal{D}Y^\mu] [\mathcal{D}P^m_\mu] \times \exp \left\{ \frac{i}{p+1} \sigma \sqrt{g} \left[ p_\mu \dot{x}^\mu - g^{00} \frac{1}{2m_{p+1}} p_\mu p^\mu + \frac{m_{p+1}}{2} \right] \right\} \times \delta \left[ \nabla_m P^m_\mu \right] \times \delta \left[ g_{mn} - h_{ij}(\vec{s}) \right].$$

(3.1)

Summing over “all” the brane histories in phase space means summing over all the dynamical variables, that is, the shapes $Y^\mu(\sigma)$ of the world–manifold $\Sigma_{p+1}$, over the rates of shape change, $P^m_\mu(\sigma)$, and over the bulk intrinsic geometries, $g_{mn}(\sigma)$, with the overall condition that the shape of the boundary is described by $x^\mu = y^\mu(\vec{s})$ and its intrinsic geometry by $h_{ij}(\vec{s})$. Thus,

$$[\mathcal{D}g_{mn}] \equiv \left[ \mathcal{D}g_{mn} \right] \delta \left[ g_{mn} - \bar{g}_{mn} \right] = \left[ \mathcal{D}g_{00} \right] \left[ \mathcal{D}e \right] \delta \left[ g_{00} - e^2(\tau) \right] \left[ \mathcal{D}g_{ik} \right] \delta \left[ g_{ik} - h_{ik}(\vec{s}) \right],$$

(3.2)

where the only non-trivial integration is over the unconstrained field $e(\tau)$. Carrying out all three functional integrations would give us a boundary effective theory encoding all the information about bulk quantum dynamics, in agreement with the Holographic Principle. However, there are several technical difficulties that need to be overcome before reaching that goal.

Let us begin with the shape variables $Y^\mu(\sigma)$; they appear in the path–integral as “Fourier integration variables” linearly conjugated to the classical equation of motion. Hence, the $Y$ integration gives a (functional) Dirac–delta that confines $P^m_\mu$ on-shell:

$$\int [\mathcal{D}Y^\mu] \exp \left\{ -i \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{g} Y^\mu(\sigma) \nabla_m P^m_\mu \right\} \times \delta \left[ \nabla_m P^m_\mu \right].$$

(3.3)

The proportionality constant in front of the Dirac–delta is physically irrelevant and can be set equal to unity.
Integrating out the brane coordinates is equivalent to “shifting from configuration space to momentum space” in a functional sense. However, the momentum integration is not free, but is restricted by Eq. (3.3) to the family of classical trajectories that are solutions of equation (2.13). Then, we can write the two–points Green function as follows

$$G = N \int [dP^0] \int [Dg_{mn}] \int [D\phi][DA] \exp \left( iS_{\text{eff}} \right), \quad (3.4)$$

where $N$ is a normalization factor to be determined at the end of the calculations, $S_{\text{eff}}$ is the effective action (2.25) and we integrate over the zero mode components in the ordinary sense, that is, we integrate over numbers and not over functions. It may be worth emphasizing that we have traded the original set of scalar fields $Y^\mu(\sigma)$ with the “Fourier conjugated” modes $\phi, A, P^0$. Then

$$G = N \int [Dg] \exp \left\{ -i \frac{m_{p+1}}{2} p \int_{\Sigma} d^{p+1} \sigma \sqrt{- \det g_{mn}} \right\} K[\sigma ; g] Z_{\phi,A}[g]. \quad (3.5)$$

We recall that

$$V_{p+1} \equiv \int_{\Sigma} d^{p+1} \sigma \sqrt{- \det g_{mn}} = \int_{\Sigma} d^p s \sqrt{- \det h_{ij}} \int_0^T d\tau e(\tau) \equiv V_p \int_0^T d\tau e(\tau) \quad (3.6)$$

and

$$K[\sigma ; e] = N \int [dP^0_{\mu_1 \ldots \mu_{p+1}}] \exp \left\{ \frac{i}{(p+1)!} P^0_{\mu_1 \ldots \mu_{p+1}} \sigma^{\mu_2 \ldots \mu_{p+1}} \right\} \times \exp \left\{ i V_p \frac{2m_{p+1}}{(p+1)!} \int_0^T d\tau e(\tau) P^0_{\mu_1 \ldots \mu_{p+1}} P^0_{\mu_2 \ldots \mu_{p+1}} \right\}. \quad (3.7)$$

At this point, we would like to factor out of the whole path–integral the boundary dynamics, i.e. we would like to write $K = K(\text{boundary}) \times K(\text{bulk})$. In order to achieve this splitting between bulk and boundary dynamics, we need to remove the dependence on the 00–component of the bulk metric $g_{mn}$ in $K[\sigma ; e$. In other words, we are looking for a propagator where $\sqrt{|\sigma|}$ plays the role of “euclidean distance” between the initial and final configuration. In support of this interpretation, we also need a suitable parameter that plays the role of “proper time” along the history of the branes connecting the initial and final configurations. The obvious candidate for that role is the proper time lapse $\int_0^T d\tau e(\tau)$. However, the $e$–field is subject to quantum fluctuations, so that the proper time lapse is a quantum variable itself. Accordingly, a $c$–number $\Omega_{p+1}$ can be defined only as a quantum average of the proper world volume operator

$$\Omega_{p+1} = V_p \langle \int_0^T d\tau e(\tau) \rangle. \quad (3.8)$$

Thus, we replace the quantum proper volume $V_{p+1}[g]$ with the quantum average $\Omega_{p+1}$ in $K[\sigma ; e]$ and write the boundary propagator in the form
\[ K [\sigma ; \Omega_{p+1}] = N \int \left[ dP^0 \right] \exp \left\{ \frac{i}{(p+1)!} P^0_{\mu_2...\mu_{p+1}} \sigma^{\mu_2...\mu_{p+1}} + \right. \]
\[ + i \frac{\Omega_{p+1}}{2m_{p+1}(p+1)!} P^0_{\mu_2...\mu_{p+1}} P^0_{\mu_2...\mu_{p+1}} \right\}, \quad (3.9) \]

while the amplitude becomes
\[ \overline{G} = N \int_0^\infty d\Omega_{p+1} \exp \left\{ -i \frac{m_{p+1}}{2} \Omega_{p+1} \right\} K [\sigma ; \Omega_{p+1}] \]
\[ \times \int [dg] \delta \left[ \Omega_{p+1} - V_p \left( \int_0^T d\tau e(\tau) \right) \right] Z_{\phi,A} [g]. \quad (3.10) \]

The “bulk” quantum physics is encoded now into the path–integral
\[ Z_{\phi,A} [g] = \int [D\phi][DA] \exp \left\{ \frac{i}{2m_{p+1}(p+1)!} \right. \]
\[ \times \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{g} \left( g^{mn} \partial_m \phi^\mu \partial_n \phi^\mu - \frac{1}{p!} F^\mu_{m_1...m_p}(A) F_{m_1...m_p}(A) \right) \left\} \right. \]. \quad (3.11) \]

Equation (3.10) presents a new problem: the constraint over the metric integration, which allowed us to factor out the boundary dynamics, is highly non-linear as it depends on the vacuum average of the quantum volume. However, we can get around this difficulty by replacing the Dirac delta with an exponential weight factor
\[ \delta \left[ \Omega_{p+1} - V_p \left( \int_0^T d\tau e(\tau) \right) \right] \rightarrow \exp \left\{ -i \Lambda \left( \Omega_{p+1} - V_p \int_0^T d\tau e(\tau) \right) \right\}, \quad (3.12) \]

where \( \Lambda \) is a constant Lagrange multiplier. Thus, we first perform all calculations with \( \Omega_{p+1} \) as an arbitrary evolution parameter and only at the end we impose the condition
\[ \frac{\partial}{\partial \Lambda} \overline{G} = 0 \quad \Rightarrow \quad \Omega_{p+1} = V_p \left( \int_0^T d\tau e(\tau) \right). \quad (3.13) \]

In this way, we can write the \( \Lambda \) dependent amplitude in the form
\[ \overline{G} = N \int_0^\infty d\Omega_{p+1} \exp \left\{ -i \Omega_{p+1} \left( \Lambda + \frac{m_{p+1}}{2} p \right) \right\} K [\sigma ; \Omega_{p+1}] \times \int [dg] Z_{\phi,A} [g ; \Lambda], \quad (3.14) \]

where the bulk quantum physics is encoded into the path–integral
\[ Z_{\phi,A} [g ; \Lambda] = \exp \left\{ i \Lambda V_p \int_0^T d\tau e(\tau) \right\} \int [D\phi][DA] \exp \left\{ \frac{i}{2m_{p+1}(p+1)!} \right. \]
\[ \times \int_{\Sigma_{p+1}} d^{p+1}\sigma \sqrt{g} \left( g^{mn} \partial_m \phi^\mu \partial_n \phi^\mu - \frac{1}{p!} F^\mu_{m_1...m_p}(A) F_{m_1...m_p}(A) \right) \left\} \right. \]. \quad (3.15) \]
B. The Boundary Propagator

The main point of the whole discussion in the previous subsection is this: even though the bulk quantum dynamics cannot be solved exactly, since there is no way to compute the bulk fluctuations in closed form, the boundary propagator can be evaluated exactly. This is because the integral (3.4) is gaussian in $P^0$:

$$K[\sigma ; \Omega_{p+1}] = \left[ \frac{m_{p+1}}{i\pi \Omega_{p+1}} \right]^{\frac{D(p)}{2}} \exp \left\{ \frac{im_{p+1}}{2(p+1)!\Omega_{p+1}} \sigma^{\mu_1...\mu_{p+1}} \right\} . \quad (3.16)$$

Moreover, one can check through an explicit calculation that the kernel $K$ solves the (matrix) Schroedinger equation

$$\left[ \frac{1}{2m_{p+1}(p+1)!} \frac{\partial^2}{\partial \sigma^{\mu_1...\mu_{p+1}} \partial \sigma_{\mu_1...\mu_{p+1}}} \right] K[\sigma - \sigma_0 ; \Omega_{p+1}] = -i \frac{\partial}{\partial \Omega_{p+1}} K[\sigma - \sigma_0 ; \Omega_{p+1}] \quad (3.17)$$

with the boundary condition

$$\lim_{\Omega \to 0} K[\sigma - \sigma_0 ; \Omega_{p+1}] = \delta[|\sigma - \sigma_0|] . \quad (3.18)$$

Notice that the average proper volume $\Omega_{p+1}$ enters the expression of the kernel $K$ only through the combination $m_{p+1}/\Omega_{p+1}$. Thus, the limit (3.18) is physically equivalent to the infinite tension limit where $\Omega_{p+1}$ is kept fixed and $m_{p+1} \to \infty$:

$$\lim_{m_{p+1} \to \infty} K[\sigma - \sigma_0 ; \Omega_{p+1}] = \delta[|\sigma - \sigma_0|] . \quad (3.19)$$

In the limit (3.19) the infinite tension shrinks the brane to a pointlike object. From the above discussion we infer that the quantum dynamics of the collective mode can be described either by the zero mode propagator (3.16), or by the wavelike equation

$$\Psi_0[\sigma ; \Omega_{p+1}] = \int [d\sigma_0] K[\sigma - \sigma_0 ; \Omega_{p+1}] \phi[\sigma_0 ; 0] , \quad (3.20)$$

where $\phi[\sigma_0 ; 0]$ represents the initial state wave function. Comparing the “Schroedinger equation” (3.17) with the Jacobi equation (2.35) suggests the following Correspondence Principle among classical variables and quantum operators:

$$P^0_{\mu_1...\mu_{p+1}} \to i \frac{\partial}{\partial \sigma^{\mu_1...\mu_{p+1}}} , \quad (p \geq 1) \quad (3.21)$$

$$E \to -i \frac{\partial}{\partial \Omega_{p+1}} . \quad (3.22)$$

In summary, the main result of this section is that the general form of the quantum propagator for a closed bosonic $p$-brane can be written in the following form.
\[ G = N \int_0^\infty d\Omega_{p+1} \exp \left\{ i\Omega_{p+1} \left[ \Lambda + \frac{m_{p+1}}{2} p \right] \right\} \left[ \frac{m_{p+1}}{i\pi\Omega_{p+1}} \right]^{\frac{D}{2}(p+1)} \times \exp \left\{ \frac{im_{p+1}}{2(p+1)!}\Omega_{p+1} \right\} \int [De] \ K_{cm} [x - x_0; e(\tau)] Z [e; \Lambda]. \] (3.23)

IV. “MINISUPERSPACE–QUENCHED PROPAGATOR”

At this stage in our discussion, we can explicitly define our approximation scheme. It consists of three main steps.

The first essential step, discussed in subsection IIB, consists in splitting the metric of the world–manifold according to Eq.(2.4). In a broad sense, this is a “minisuperspace approximation” to the extent that it restricts the general covariance of the action in parameter space. In other words, separating the center of mass proper time from the spatial coordinates on the world manifold \( \Sigma_{p+1} \) effectively breaks the full invariance under general coordinate transformations into two symmetry groups:

\[ \text{General Diffs} \rightarrow (\text{time})\text{Rep} \otimes (\text{spatial})\text{Diffs}, \] (4.1)

so that the metric (2.4) shows a residual symmetry under independent time reparametrizations and spatial diffeomorphisms. By virtue of this operation, we were able to separate the center of mass motion from the bulk and boundary dynamics. This is encoded in the split form (2.10) of the total action for the \( p \)-brane.

The second and more strict interpretation of the minisuperspace approximation is that we now “freeze” the world metric into a background configuration \( g_{mn} \) where the space is a \( p \)-sphere, while \( g_{00} \), or its square root \( e(\tau) \), is free to fluctuate. In other words, we work in a minisuperspace of all possible world geometries.

The third and last step in our procedure is an adaptation of one of the most useful approximations to the exact dynamics of interacting quarks and gluons, namely, “Quenched QCD”. There, the contribution of the determinant of the quark kinetic operator is set equal to one. In the same spirit we “quench” all the bulk oscillations:

\[ Z_{\phi,A} [e; \Lambda] \rightarrow \exp \{-i \Lambda \Omega_{p+1}\}. \] (4.2)

Combining these three steps, we obtain from (3.23) the quenched–minisuperspace propagator:

\[ G [x - x_0, \sigma] = N \int_0^\infty d\Omega_{p+1} \exp \left\{ ip \Omega_{p+1} \frac{m_{p+1}}{2} \right\} \left[ \frac{m_{p+1}}{i\pi\Omega_{p+1}} \right]^{\frac{D}{2}(p+1)} \times \exp \left\{ \frac{im_{p+1}}{2\Omega_{p+1}(p+1)!} \sigma^2 \right\} \int [De] \ K_{cm} [x - x_0; e(\tau)] \delta \left[ \int_0^T d\tau e(\tau) - \frac{\Omega_{p+1}}{V_p} \right]. \] (4.3)

The center of mass propagator \( K_{cm} [x - x_0; e(\tau)] \) can be computed as follows (27).

From the Lagrangian \( L_{cm} \) we can define the center of mass momentum \( p_{cm} \) as follows
\[ p_\mu \equiv \frac{\partial L_{cm}}{\partial \dot{x}^\mu (\tau)} = M_0 \dot{x}_\mu e(\tau). \quad (4.4) \]

By Legendre transforming \( L_{cm} \) we obtain the center of mass Hamiltonian:

\[ H_{cm} \equiv p_\mu \dot{x}^\mu - L_{cm} = \frac{e(\tau)}{2M_0} \left[ p_\mu p^\mu + M_0^2 \right]. \quad (4.5) \]

Moreover, the canonical form of \( S_{cm} \) reads

\[ S_{cm} = \int_0^T d\tau \left[ p_\mu \dot{x}^\mu - \frac{e(\tau)}{2M_0} \left( p_\mu p^\mu + M_0^2 \right) \right] \quad (4.6) \]

so that the quantum dynamics of the bulk center of mass is described by the path–integral

\[ K_{cm} [x - x_0; e(\tau)] \equiv \int [\mathcal{D}x] [\mathcal{D}p] e^{i S_{cm}[x, T; e(\tau)]}. \quad (4.7) \]

This path–integral can be reduced to an ordinary integral over the constant four momentum \( q_\mu \) of a point particle of mass \( M_0 \)

\[ K_{cm} [x - x_0; e(\tau)] = \int \frac{d^D q}{(2\pi)^D} e^{i q_\mu (x^\mu - x_0^\mu)} \exp \left[ -i \int_0^T d\tau \frac{e(\tau)}{2M_0} \left( q_\mu q^\mu + M_0^2 \right) \right]. \quad (4.8) \]

In order to get the explicit form of the center of mass propagator we have to integrate, in the ordinary sense, over \( q_\mu \). However, before that we must integrate, in the functional sense, over the einbein field \( e(\tau) \).

Using the properties of the Dirac–delta distribution, the einbein field can be integrated out:

\[ \int [\mathcal{D}e] \delta \left[ \int_0^T d\tau e(\tau) - \Omega_{p+1}/V_p \right] \exp \left[ -i \int_0^T d\tau \frac{e(\tau)}{2M_0} \left( q_\mu q^\mu + M_0^2 \right) \right] = \exp \left[ -i \frac{\Omega_{p+1}}{2M_0 V_p} \left( q_\mu q^\mu + M_0^2 \right) \right]. \quad (4.9) \]

and

\[ \int \frac{d^D q}{(2\pi)^D} e^{i q_\mu (x^\mu - x_0^\mu)} \exp \left[ -i \frac{\Omega_{p+1}}{2M_0 V_p} q_\mu q^\mu \right] = \left( \frac{\pi M_0 V_p}{\Omega_{p+1}} \right)^{D/2} \exp \left[ -i \frac{M_0 V_p}{2\Omega_{p+1}} (x - x_0)^2 \right]. \quad (4.10) \]

Using the above results, the “QCD–QC combined approximation” leads to the following expression for the propagator

\[ G [x - x_0, \sigma] = N \int_0^\infty d\Omega_{p+1} \exp \left\{ i \Omega_{p+1} \left[ \frac{m_{p+1}}{2} p + \frac{M_0}{2 V_p} \right] \right\} \left[ \frac{m_{p+1}}{i\pi \Omega_{p+1}} \right] \frac{1}{2} \left( \frac{D}{p+1} \right)^{D/2} \exp \left[ -i \frac{M_0 V_p}{2\Omega_{p+1}} (x - x_0)^2 \right]. \quad (4.11) \]
The amplitude (4.11) can be cast in a more familiar form in terms of the Schwinger–Feynman parametrization

\[ s \equiv \frac{\Omega_{p+1}}{4V_p}, \quad d\Omega_{p+1} = 4V_p ds. \]  

(4.12)

Using the above parametrization, the quantum propagator takes its final form in the minisuperspace–quenched approximation

\[ G[x-x_0, \sigma; M_0] = NV_p \int_0^\infty ds \left( \frac{\pi M_0}{is} \right)^{D/2} \exp \left[ \frac{iM_0}{2s} (x - x_0)^2 \right] \times \exp \left\{ i s \frac{M_0}{2} \left( p + 1 \right) \right\} \left( \frac{\sigma}{2V_p^2 s} \right)^{\frac{D}{2(p+1)}(p+1)!} \times \left( \frac{\sigma}{V_p^2 (p+1)!} \right)^{\frac{D}{2}+1} \exp \left[ i \frac{M_0}{2s} (x - x_0)^2 \right] \times \exp \left\{ i \frac{M_0}{2} \left( p + 1 \right) \right\} \right. \]

(4.13)

Here we have set \( N = i/2M_0 V_p \) in order to match the form of the point particle propagator.

The formula (4.13) represents the main result of all previous calculations and holds for any \( p \) in any number of spacetime dimensions.

**A. Green Function Equation → Tension–Shell Condition**

For completeness of exposition, in this subsection we derive the master equation satisfied by the Green function (4.13) in the quenched–minisuperspace approximation. Then, by formally inverting that equation we arrive at an alternative expression for the Green function in momentum space. The advantage of this procedure is that it provides a useful insight into the structure of the \( p \)-brane propagator.

To begin with, it seems useful to remark that the propagation kernel in (4.13) is the product of the center of mass kernel \( K_{cm} (x - x_0 ; s) \) and the volume kernel \( K(\sigma ; s) \); each term carries a weight given by the phase factor \( \exp i M_0 s \) and \( \exp i p M_0 s \), respectively. Finally, we integrate over all the values of the Feynman parameter \( s \):

\[ G[x-x_0; M_0] = \frac{i}{2M_0} \int_0^\infty ds \exp \left\{ \frac{i M_0}{2} \left( p + 1 \right) \right\} K_{cm} (x - x_0 ; s) \ K (\sigma ; s). \]  

(4.14)

As is customary in the Green function technique, we may add an infinitesimal imaginary part to the mass in the exponent, that is, \( M_0/2 \rightarrow (M_0/2) + i \epsilon \), so that the oscillatory phase turns into an exponentially damped factor enforcing convergence at the upper integration limit. The "\( i \epsilon \)" prescription in the exponent allows one to perform an integration by parts leading to the following expression
\[ G [x - x_0 ; M_0] = \frac{1}{M_0^2 (p + 1)} \left[ \exp \left\{ i \left( \frac{M_0}{2} + i \epsilon \right) s (p + 1) \right\} K_{cm} (x - x_0 ; s) K (\sigma ; s) \right]_0^{\infty} \]

\[- \frac{1}{M_0^2 (p + 1)} \int_0^{\infty} ds \exp \left\{ i \left( \frac{M_0}{2} + i \epsilon \right) s (p + 1) \right\} \frac{\partial}{\partial s} \left( K_{cm} (x - x_0 ; s) K (\sigma ; s) \right) . \] (4.15)

Convergence of the integral enables us to express the partial derivative \( \partial / \partial s \) by means of the diffusion equations for \( K_{cm} \) and \( K \) and to move the differential operators \( \partial_\mu \partial^\mu, \partial^2 / \partial \sigma^2 \) out of the integral:

\[ G [x - x_0 ; M_0] = \frac{1}{M_0^2 (p + 1)} [\delta (x - x_0) \delta (\sigma ; s)] - \frac{1}{M_0^2 (p + 1)} \int_0^{\infty} ds \exp \left\{ i \frac{M_0}{2} s (p + 1) \right\} ] \times \left[ (K (\sigma ; s) \partial^\mu \partial_\mu K_{cm} (x - x_0 ; s)) \right. \]

\[ \left( K_{cm} (x - x_0 ; s) \frac{V_p}{(p + 1)!} \frac{\partial^2 K (\sigma ; s)}{\partial \sigma_{\mu_1} \cdots \partial \sigma_{\mu_{p+1}}} \right) \]

\[ = - \frac{1}{M_0^2 (p + 1)} \delta (x - x_0) \delta (\sigma ; s) \]

\[ + \frac{1}{M_0^2 (p + 1)} \left( \partial_\mu \partial^\mu + \frac{V_p}{(p + 1)!} \frac{\partial^2}{\partial \sigma_{\mu_1} \cdots \partial \sigma_{\mu_{p+1}}} \right) G (x - x_0 , \sigma ; s) , \] (4.16)

from which we deduce the desired result,

\[ \left[ \partial_\mu \partial^\mu + \frac{V_p}{(p + 1)!} \frac{\partial^2}{\partial \sigma_{\mu_1} \cdots \partial \sigma_{\mu_{p+1}}} \right] G (x - x_0 ; \sigma) = \delta^p (x - x_0) \delta (\sigma) . \]

(4.17)

This is the Green function equation for the non–standard differential operator \( \partial^\mu \partial_\mu + V_p \partial^2 / \partial \sigma^2 \). Finally, we “Fourier transform” the Green function by extending the momentum space to a larger space that includes the volume momentum as well:

\[ G (x - x_0 , \sigma) = \int \frac{d^D q}{(2\pi)^D} \int [dk_{\mu_1} \cdots k_{\mu_{p+1}}] \exp \left( i q_\mu (x - x_0)^\mu + \frac{i}{(p + 1)!} k_{\mu_1} \cdots k_{\mu_{p+1}} \right) \]

\[ \times \frac{1}{q^2 + \frac{V_p^2}{(p + 1)!} k_{\mu_1} \cdots k_{\mu_{p+1}} + (p + 1) M_0^2} . \] (4.18)

The vanishing of the denominator in (4.18) defines a new tension–shell condition:

\[ q^2 + \frac{V_p^2}{(p + 1)!} k_{\mu_1} \cdots k_{\mu_{p+1}} + (p + 1) M_0^2 = 0 . \] (4.19)

Real branes, as opposed to virtual branes, must satisfy the condition (4.19) which links together center of mass and volume momentum squared. Equation (4.19) represents an extension of the familiar Klein–Gordon condition for point–particles to relativistic extended objects. Some physical consequences of the “tension–shell condition”, as well as the mathematical structure of the underlying spacetime geometry, are currently under investigation and will be reported in a forthcoming letter. In the next subsection we limit ourselves to check the consistency of our results against some familiar cases of physical interest.
B. Checks

1. Infinite Tension Limit

When probed at low energy (resolution) an extended object effectively looks like a point–particle. In this case, “low energy” means an energy which is small compared with the energy scale determined by the brane tension. In natural units, the tension of a \( p \)-brane has dimension: \( [T_p] = (\text{energy})^{p+1} \). Thus, when probing the brane at energy \( E \ll (T_p)^{1/p+1} \) one cannot resolve the extended structure of the object. From this perspective, the “point–like limit” of a \( p \)-brane is equivalent to the “infinite tension limit”. In either case, no higher vibration modes are excited and one expects the brane to appear concentrated, or “collapsed”, in its own center of mass. This critical limit can be obtained from the general result (4.13) by setting \( p = 0 \) and performing the limit \( V_p \rightarrow 0 \) using the familiar representation of the Dirac–delta distribution:

\[
\delta(x) \equiv \lim_{\epsilon \to 0} \left( \frac{1}{\pi \epsilon} \right)^{d/2} \exp \left( -x^2/\epsilon \right).
\] (4.20)

In our case: \( x^2 \rightarrow \sigma^2/(p+1)! \), \( \epsilon \rightarrow -iM_0/4sV_p^2 \), \( d \rightarrow \left( \frac{D}{p+1} \right) \), and the whole dependence on the volume coordinates of the brane reduces to a delta function which is different from zero only when \( \sigma = 0 \). In this case, \( G[x-x_0;M_0] \) reduces to the familiar expression for the Feynman propagator for a point particle of mass \( M_0 \),

\[
G[x-x_0;M_0] = \delta \left[ \sigma^2 \right] \frac{i}{2M_0} \int_0^\infty ds \left( \frac{\pi M_0}{is} \right)^{D/2} \exp \left( -i \frac{M_0}{2s} \right) \exp \left\{ i \frac{M_0}{2s} (x-x_0)^2 \right\}.
\] (4.21)

2. Spherical Membrane Wave Function

From the results of the previous subsections we can immediately extract the generalized Klein–Gordon equation for a 2–brane in four spacetime dimensions:

\[
\left[ \partial_\mu \partial^\mu + \frac{V_2}{3!} \frac{\partial^2}{\partial \sigma_{\mu_1 \mu_2 \mu_3}} \partial \sigma^{\mu_1 \mu_2 \mu_3} - (p+1) M_0^2 \right] \Psi(x, \sigma) = 0.
\] (4.22)

In the following we show briefly how this wave equation specializes to the case of a gauge fixed, or spherical, membrane of fixed radius \( R \). Moreover, since it is widely believed that \( p \)-brane physics may be especially relevant at Planckian energy, we assume that our 2–brane is a fundamental object characterized by Planck units of tension and length. Thus, for a spherical 2–brane of radius \( R \) we have:

\[
V_2 = 4\pi l_{P1}^2,
\] (4.23)

\[
\sigma^{\mu_1 \mu_2 \mu_3} = \delta_{x}^{[\mu_1} \delta_{y}^{\mu_2} \delta_{z}^{\mu_3]} \frac{4\pi}{3} R^3,
\] (4.24)
\[
\frac{\partial}{\partial \sigma_{\mu_1 \mu_2 \mu_3}} = \frac{1}{4\pi R^2} \delta_{\mu_1}^x \delta_{\mu_2}^y \delta_{\mu_3}^z \frac{\partial}{\partial R}.
\] (4.26)

Since there is no mixing between ordinary and volume derivative, we can use the method of separation of variables to factorize the dependence of the wave function on \( x \) and \( \sigma \)

\[
\Psi (x, \sigma) = \phi(x) \psi_0 (R) \implies \left[ \partial_\mu \partial^\mu - M_0^2 \right] \phi(x) = 0.
\] (4.27)

Here, \( \phi_0(x) \) represents the center of mass wave function, while the "relative motion wave function" \( \psi(R) \) must satisfy the following equation

\[
\left[ \frac{1}{4\pi R^2} \frac{\partial}{\partial R} \frac{1}{R^2} \frac{\partial}{\partial R} - \frac{pM_0^2}{l_{Pl}^2} \right] \psi(R) = 0.
\] (4.28)

Apart from some ordering ambiguities, equation (4.28) is the wave equation found in Ref. [8], [9], [10], for the zero energy eigenstate, provided we identify the membrane tension \( \rho \) through the expression

\[
\rho^2 = \frac{pM_0^2}{4\pi l_{Pl}^2}.
\] (4.29)

It may be worth emphasizing that our approach preserves time reparametrization invariance throughout all computational steps, while the conventional minisuperspace approximation assumes a gauge choice from the very beginning.

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