Abstract. This paper studies closed 3-manifolds which are the attractors of a system of finitely many affine contractions that tile $\mathbb{R}^3$. Such attractors are called self-affine tiles. Effective characterization and recognition theorems for these 3-manifolds as well as theoretical generalizations of these results to higher dimensions are established. The methods developed build a bridge linking geometric topology with iterated function systems and their attractors.

A method to model self-affine tiles by simple iterative systems is developed in order to study their topology. The model is functorial in the sense that there is an easily computable map that induces isomorphisms between the natural subdivisions of the attractor of the model and the self-affine tile. It has many beneficial qualities including ease of computation allowing one to determine topological properties of the attractor of the model such as connectedness and whether it is a manifold. The induced map between the attractor of the model and the self-affine tile is a quotient map and can be checked in certain cases to be monotone or cell-like. Deep theorems from geometric topology are applied to characterize and develop algorithms to recognize when a self-affine tile is a topological or generalized manifold in all dimensions. These new tools are used to check that several self-affine tiles in the literature are 3-balls. An example of a wild 3-dimensional self-affine tile is given whose boundary is a topological 2-sphere but which is not itself a 3-ball. The paper describes how any 3-dimensional handlebody can be given the structure of a self-affine 3-manifold. It is conjectured that every self-affine tile which is a manifold is a handlebody.



1. Introduction

A great deal of work in the literature has concentrated on tilings of $\mathbb{R}^n$ whose tiles are defined by a finite collection of contractions. One of the most prevalent examples are tilings by self-affine tiles where the contractions are affine translates of a single linear contraction. A long-standing open question is whether there exists a closed 3-manifold which is a nontrivial self-affine tile, a so-called self-affine 3-manifold. To settle this question in the affirmative the current paper effectively characterizes and recognizes self-affine 3-manifolds and gives theoretical generalizations of these results to higher dimensions. The methods developed in this paper build a bridge linking geometric topology with iterated function systems and their attractors.

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1. Introduction

A great deal of work in the literature has concentrated on tilings of $\mathbb{R}^n$ whose tiles are defined by a finite collection of contractions. One of the most prevalent examples are tilings by self-affine tiles where the contractions are affine translates of a single linear contraction. A long-standing open question is whether there exists a closed 3-manifold which is a nontrivial self-affine tile, a so-called self-affine 3-manifold. To settle this question in the affirmative the current paper effectively characterizes and recognizes self-affine 3-manifolds and gives theoretical generalizations of these results to higher dimensions. The methods developed in this paper build a bridge linking geometric topology with iterated function systems and their attractors.
two previously unrelated areas of mathematics: geometric topology on the one side and iterated function systems and their attractors on the other side.

Much work is devoted to how a subset of the Euclidean plane can admit a tiling by self-affine tiles. In the planar case, the topology of these tiles has been studied thoroughly. Much less is known about the topology of self-affine tiles of Euclidean 3-space. In particular it has been an open question as to which (if any) 3-manifolds admit a nontrivial self-affine tiling of $\mathbb{R}^3$. A number of examples have appeared in the literature which were conjectured to be self-affine tilings of $\mathbb{R}^3$ by 3-balls. In the current paper we address these questions by describing an often effective method of determining that a given 3-dimensional self-affine tile is a tamely embedded 3-manifold. The method gives affirmative answers for the previously conjectured examples, and is also used to give examples of 3-dimensional self-affine tiles which are handlebodies of higher genus. Examples are also given of self-affine tiles in $\mathbb{R}^3$ whose boundaries are wildly embedded surfaces and thus are not 3-manifolds. Our method also has potential to allow effective computations in higher dimensions. The proofs of our results require a careful formulation of the problem in terms of a certain type of algebro-geometric complexes which are used to approximate the tile and allow for arbitrarily fine computations due to their recursive structure. Deep tools of geometric topology developed by Cannon and Edwards are then used to determine the homeomorphism type of the boundary and check if it is a tamely embedded surface in the case of dimension 3. We offer Conjectures 1 and 2 stating that every self-affine manifold is homeomorphic to a handlebody.

The study of self-affine tiles and their tiling properties goes back to the work of Thurston [50] and Kenyon [30]. In the 1990s Lagarias and Wang [34, 35, 36] proved fundamental properties of self-affine tiles. Wang [52] surveys these early results on tiles like tiling properties, Hausdorff dimension of the boundary and relations to wavelets. By now there exists a vast literature on self-affine tiles. The topics of research include their geometric, topological, and fractal properties, characterization problems, relations to number systems, and wavelet theory (see e.g. [1, 6, 12, 23, 26, 37, 48]). The topology of (mainly planar) self-affine tiles is the topic of considerable study (cf. for instance [3, 25, 27, 31, 35]). In particular, the case where the self-affine tile is a 2-manifold (i.e., a closed disk) has been well understood from early on (cf. [3, 5, 38, 39, 40]). The question of determining the topology of higher dimensional self-affine tiles arose naturally: in particular, if they could be manifolds in a nontrivial way and if there is a method to recognize whether a self-affine tile (or its boundary) is a manifold (see Gelbrich [24] who first raised this question in 1996, and more recently [2, 4, 17, 38]). We will call a 3-dimensional self-affine tile which is a topological 3-manifold with boundary a self-affine 3-manifold. Numerous specific examples of nontrivial self-affine tiles were conjectured in the literature to be topological 3-dimensional balls (see for instance [4, 24]). However, until this point, no example of a nontrivial self-affine 3-manifold has been exhibited (in [41] self-affine tiles that are $n$-dimensional parallelepipeds are characterized).

In the current article we give conditions that characterize self-affine 3-manifolds. We go on to give an effective algorithm to decide whether a 3-dimensional self-affine tile is a manifold with boundary. Finally we apply our algorithm to conjectured examples and show that they are topological 3-dimensional balls. During the course of our inquiry we show that many of our results generalize to the $n$-dimensional case.

1.1. Classical conjectures and solutions. We start with the exact definition of the fundamental objects studied in the present paper.

**Definition 1.1** (Self-affine tile). Let $A$ be an expanding $n \times n$ integer matrix, that is, a matrix each of whose eigenvalues has modulus strictly greater than one. Let $D \subset \mathbb{Z}^n$ be a complete set of residue class representatives of $\mathbb{Z}^n/\mathbb{Z}^n$, called the digit set. We define the self-affine tile $T = T(A, D)$ as the unique nonempty compact set satisfying

\[
AT = T + D.
\]

If the self-affine tile $T$ tiles $\mathbb{R}^n$ with respect to the lattice $\mathbb{Z}^n$ we say that $T$ induces a self-affine tiling and call $T$ a self-affine $\mathbb{Z}^n$-tile.

The self-affine tile $T$ is well-defined because it is the unique solution of the iterated function system $\{\varphi_d \mid d \in D\}$ with $\varphi_d(x) = A^{-1}(x + d)$ (cf. Hutchinson [29] and note that there is a norm
|| · || on $\mathbb{R}^n$ that makes $A^{-1}$ a contraction; see (3.1) below). In view of the results in [36] the tiling property in Definition 1.1 is not a strong restriction and can easily be checked algorithmically (see for instance [51]). Moreover, without loss of generality, we will assume that $0 \in \mathcal{D}$.

Until now it has not been known whether a nontrivial self-affine $\mathbb{Z}^n$-tile, $n > 2$, could be homeomorphic to an $n$-manifold with boundary. For instance, Gelbrich [24] as well as Bandt and Mesing [4] give examples of self-affine $\mathbb{Z}^3$-tiles which are conjectured to be homeomorphic to 3-dimensional balls. In this paper we give a method of checking that a self-affine tile is a manifold. We check for the presence of an ideal tile (Definition 2.22) and use Theorems 2.18, and 2.23, as well as Corollary 7.23 to check that classically conjectured self-affine 3-manifolds are indeed 3-manifolds. We are unaware of conjectural self-affine tiles which are manifolds in dimensions higher than 3.

1.2. The characterization and recognition problems. One of the hallmarks of a complete theory of a class of examples in mathematics is the solution of the characterization and recognition problem. In other words, when can one formally characterize a class of examples and furthermore effectively recognize those examples given only simple data that defines them? The methods of the present paper go considerably beyond checking examples and allow one to characterize and recognize tame self-affine 3-manifolds.

1.2.1. The 3-dimensional case. In the case of 3-dimensional tame tiles (e.g. those with connected, 1-ULC interior) we give a characterization of those which are manifolds (monotone models are defined in Definition 2.9):

**Theorem 2.18.** Let $T$ be a self-affine $\mathbb{Z}^3$-tile whose interior is connected and 1-ULC. Then $T$ is a self-affine 3-manifold if and only if it admits a monotone model with a boundary that is a closed 2-manifold.

Corollary 7.23 allows to algorithmically recognize which 3-dimensional tile is a tame 3-ball (for the in-out graph see Definition 7.14):

**Corollary 7.23.** Let $T$ be a self-affine $\mathbb{Z}^3$-tile. If $\partial T$ is a 2-sphere in $\mathbb{R}^3$ and each loop in the in-out graph (which is always finite) $\mathcal{I}$ contains a node $N$ such that

(i) $\partial N \cong S^2$,

(ii) $\partial N \cap \partial T$ is connected.

Then $\partial T$ is locally spherical and thus tame. Consequently $T$ is homeomorphic to $D^3$.

This leaves the problem of checking that the boundary of a 3-dimensional tile is a 3-sphere (see (2.3) for the quotient map $Q$):

**Corollary 2.15.** Let $T$ be a self-affine $\mathbb{Z}^3$-tile with connected interior which admits a monotone model $M$ with 2-sphere boundary. Then $\partial T$ is homeomorphic to the 2-sphere and point preimages of the quotient map $Q|_{\partial M}$ do not path separate.

Given the above, the fact that 2-dimensional self-affine manifolds are 2-disks and that Proposition 2.29 (see also the paragraph after it) shows that every 3-dimensional handlebody is homeomorphic to a self-affine manifold, we offer the following conjecture:

**Conjecture 1.** Every self-affine 3-manifold is homeomorphic to a handlebody (a regular neighborhood of a graph in $\mathbb{R}^3$).

1.2.2. The general case. In higher dimensional cases we can recognize self-affine generalized manifolds (see Definition 4.6 for boundary stars):

**Theorem 5.12.** Let $T$ be a self-affine $\mathbb{Z}^n$-tile which admits a monotone model $M$. Assume that almost all boundary stars of $M$ are cellular and $\partial M$ is a manifold. Then $\partial T$ is a generalized $(n - 1)$-manifold with $Q|_{\partial M} : \partial M \to \partial T$ a cellular quotient map from the manifold $\partial M$. In other words, $\partial M$ is a cell-like resolution of the generalized manifold $\partial T$.

In dimensions higher than $n = 3$, one needs extra hypotheses to guarantee that a cell-like quotient of a manifold is a manifold. In dimensions $n = 6$ and higher, the appropriate notion is the disjoint disk property (DDP) made famous by Cannon and Edwards [9, 19].
Theorem 5.15. For \( n \geq 6 \) let \( T \) be a self-affine \( \mathbb{Z}^n \)-tile which admits a monotone model \( M \). Assume that almost all boundary stars of \( M \) are cellular and \( \partial M \) is a manifold. If \( T \) satisfies the disjoint disks property then \( \partial T \) is an \((n-1)\)-manifold.

Theorem 7.22 allows us to algorithmically recognize which \( n \)-dimensional tiles are tame \( n \)-balls.

Theorem 7.22. Let \( T \) be a self-affine \( \mathbb{Z}^n \)-tile. If \( \partial T \) is an \((n-1)\)-sphere in \( \mathbb{R}^n \) and each loop in the in-out graph (which is always finite) \( T \) contains a node \( N \) such that

1. \( \partial N \cong S^{n-1} \),
2. \( \partial N \setminus \partial T \) is simply connected,

then \( \partial T \) is locally spherical and thus tame. Consequently \( T \) is homeomorphic to \( \mathbb{D}^n \).

Given that a self-affine manifold tiles itself by arbitrarily small copies of itself, it seems that its topology cannot be very complicated. We offer the following stronger conjecture:

Conjecture 2. Every self-affine \( n \)-manifold is homeomorphic to an \( n \)-dimensional handlebody.

2. Main results: models, the tiling complex, and self-affine manifolds

In this section we present our main results. The underlying idea behind our theory is to “model” a given self-affine \( \mathbb{Z}^n \)-tile \( T \) by a set \( M \subset \mathbb{R}^n \) that tiles \( \mathbb{R}^n \) by \( \mathbb{Z}^n \)-translates and retains as many topological properties of \( T \) as possible. To understand the topology of \( T \) it is then sufficient to study \( M \). We give ways that allow to construct \( M \) as a finite simplicial complex. This strategy will enable us to derive various new results on topological properties of self-affine \( \mathbb{Z}^n \)-tiles.

We will need the following notations. Let \( M \subset \mathbb{R}^n \) be a compact set that is the closure of its interior and whose boundary has \( \mu(\partial M) = 0 \), where \( \mu \) denotes the \( n \)-dimensional Lebesgue measure. If the \( \mathbb{Z}^n \)-translates of \( M \) cover \( \mathbb{R}^n \) with disjoint interiors we say that \( M \) is a \( \mathbb{Z}^n \)-tile. Note that a \( \mathbb{Z}^n \)-tile always has \( \mu(M) = 1 \). As a self-affine \( \mathbb{Z}^n \)-tile \( T \) is equal to the closure of its interior and \( \mu(\partial T) = 0 \) (cf. [52, Theorem 2.1]), \( T \) is a \( \mathbb{Z}^n \)-tile.

Recall that a function \( f : \mathbb{R}^n \to \mathbb{R}^n \) is called \( \mathbb{Z}^n \)-equivariant if \( f(x + z) = f(x) + z \) holds for each \( x \in \mathbb{R}^n \), \( z \in \mathbb{Z}^n \). If, in addition, we have \( f(0) = 0 \) then \( f(z) = z \) holds for each \( z \in \mathbb{Z}^n \).

2.1. Models for self-affine \( \mathbb{Z}^n \)-tiles. We start with a fundamental definition.

Definition 2.1 (Model). A model \((M,F)\) for the self-affine \( \mathbb{Z}^n \)-tile \( T = T(A,D) \) is a \( \mathbb{Z}^n \)-tile \( M \) equipped with a homeomorphism \( F : \mathbb{R}^n \to \mathbb{R}^n \) satisfying

\[
FM = M + D
\]

such that \( A^{-1}F \) is \( \mathbb{Z}^n \)-equivariant and \( F(0) = 0 \).

If \( F \) is clear from the context, we will write \( M \) instead of \((M,F)\) for a model. Note that \( F \) is not assumed to be contracting in Definition 2.1. Thus in general, \( M \) is not uniquely defined by the set equation (2.2).

Let \((M,F)\) be a model for the self-affine tile \( T = T(A,D) \). To transfer topological information from \( M \) to \( T \) we will make extensive use of the canonical quotient map \( Q : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
Q = \lim_{k \to \infty} A^{-k}F^{(k)}
\]

(\( F^{(k)} \) denotes the \( k \)-th iterate of \( F \)). The following result will be proved in Theorem 3.8.

Theorem 2.4. If \((M,F)\) is a model for the self-affine \( \mathbb{Z}^n \)-tile \( T = T(A,D) \) then \( Q \) is continuous, \( T = Q(M) \), and \( \partial T = Q(\partial M) \).

Note that that any subspace \( X \subset M \) has its canonical quotient \( Q(X) \) under this map. In particular, Theorem 2.4 shows that \( T \) and \( \partial T \) are the canonical quotients of \( M \) and \( \partial M \), respectively.
2.2. Neighbors, cells, and the tiling complex. Let \( M \) be a \( \mathbb{Z}^n \)-tile. In all what follows, the intersection of \( M \) with subsets of its \( \mathbb{Z}^n \)-translates will play an important role. We consider sets of \( n \)-tuples of integers to be coordinates for unoriented “cells” in a “tiling complex” defined below and accordingly introduce the notation

\[
\langle S \rangle_M = \bigcap_{s \in S} (M + s) \quad (S \subset \mathbb{Z}^n).
\]

We extend the range of \( \langle \cdot \rangle_M \) to complexes, i.e., to sets \( C \) of sets in \( \mathbb{Z}^n \) by setting

\[
\langle C \rangle_M = \bigcup_{S \in C} \langle S \rangle_M.
\]

If \( M \) is the self-affine \( \mathbb{Z}^n \)-tile \( T \), we often omit the subscript \( T \) and write \( \langle S \rangle \) instead of \( \langle S \rangle_T \).

**Definition 2.6** (Neighbor structure and tiling complex). The neighbor structure of a \( \mathbb{Z}^n \)-tile \( M \) is the set

\[
\mathcal{K}(M) = \{ S \subset \mathbb{Z}^n \mid S \neq \emptyset \text{ and } \langle S \rangle_M \neq \emptyset \},
\]

which reflects the underlying intersection structure of the tiling induced by \( M \) and will also be considered a formal simplicial complex. The faces of \( S \in \mathcal{K}(M) \) are the elements of

\[
\mathcal{F}_M(S) = \{ S' \in \mathcal{K}(M) \mid S \subset S' \}.
\]

A face of \( S \in \mathcal{K}(M) \) is proper if it is not equal to \( S \). We consider the set \( \mathcal{K}(M) \) of cells along with the notion of faces induced by \( \mathcal{F}_M(S) \) to be the tiling complex for \( M \). Moreover, we set

\[
\mathcal{K}(M)^i := \{ S \in \mathcal{K}(M) \mid |S| = i \}
\]

for the set of all \( i \)-cells of the tiling complex \( \mathcal{K}(M) \).

Given a simplex \( S \in \mathcal{K}(M) \) we define the operator

\[
\delta S = \{ S \cup \{ x \} \mid x \in \mathbb{Z}^n \setminus S \} \cap \mathcal{K}(M),
\]

that is, the set of maximal proper faces of \( S \) in \( \mathcal{K}(M) \). We think of \( \langle \delta S \rangle_M \) as a form of simplicial boundary of \( \langle S \rangle_M \). The operator \( \delta \) can be extended to a collection \( C \) of \( i \)-cells by setting

\[
\delta C = \{ S \cup \{ x \} \mid S \in C, x \in \mathbb{Z}^n \setminus \bigcup_{S \in C} S \} \cap \mathcal{K}(M).
\]

We intuit the sets \( S \) as formal unoriented simplices and \( \langle S \rangle_M \) as their (dual) geometric realizations. Note that it can happen that \( S \) is a proper face of \( S' \) but that \( \langle S \rangle_M = \langle S' \rangle_M \).

2.3. Monotone models. We refine the notion of model so that it preserves the neighbor structure of the modeled self-affine \( \mathbb{Z}^n \)-tile \( T \). Proposition 4.1 shows that for a model \( M \) having the same neighbor structure\(^1\) as \( T \), the canonical quotient map \( Q \) preserves intersections: \( Q(S)_M = \langle S \rangle \) for each \( S \subset \mathbb{Z}^n \). The following notion of monotone model enables us to say more about the point preimages of the canonical quotient map \( Q \).

**Definition 2.9** (Monotone model). A model \( M \) is a monotone model for the self-affine \( \mathbb{Z}^n \)-tile \( T \) if \( \mathcal{K}(T) = \mathcal{K}(M) \) and each \( \langle S \rangle_M \) is connected.

Recall that a quotient map is monotone if point preimages are connected. We state the following result, which follows immediately from Proposition 4.3 (i).

**Theorem 2.10.** If \( M \) is a monotone model for the self-affine \( \mathbb{Z}^n \)-tile \( T \) then the restriction

\[
Q \mid_{\langle S \rangle_M} : \langle S \rangle_M \to \langle S \rangle
\]

is a monotone quotient map for each \( S \in \mathcal{K}(T) \).

\(^1\)Compare this with the definition of respecting intersection given in [45, Definition 1.2.5].
We saw in Theorem 2.4 that \( Q\partial M = \partial T \) holds for a model \( M \) of the self-affine \( \mathbb{Z}^n \)-tile \( T \). If \( M \) is even a monotone model, this follows from Theorem 2.10 because \( \partial M = \partial \{0\}^M = \{\delta\{0\}\} \) and \( \partial T = \partial \{0\} \). Under certain conditions \( Q \) behaves nicely also for boundaries of cells \( \langle S \rangle \) with \( |S| \geq 2 \). To make this precise let \( M \) be a \( \mathbb{Z}^n \)-tile and set

\[
K(M)_0 := \{ S \in K(M) \mid 0 \in S \}
\]

for the set of all \( i \)-cells of \( K(M) \) containing \( 0 \). Assume that the (dual) geometric realization \( (K(M)_0^i)_M \) of \( K(M)_0^i \) carries the subspace topology inherited from \( \mathbb{R}^n \) and, for \( i \geq 2 \), denote by \( \partial_i = \partial_i^M \) (we omit the index \( M \) as it is clear from the context) the boundary relative to \( (K(M)_0^i)_M \). The following definition contains a property of \( K(M)_0^i \) that allows us to derive results on the behaviour of \( Q \) with respect to these boundary operators.

**Definition 2.11** (Combinatorial tile). Let \( M \) be a \( \mathbb{Z}^n \)-tile. We say that \( M \) is combinatorial if \( \partial_i(S)_M = \{\delta S\}_M \) for all \( S \in K(M)_0^i \) and all \( i \geq 2 \).

We can show the following result which is proved at the end of Section 4.1.

**Theorem 2.12.** Let \( M \) be a monotone model for a self-affine \( \mathbb{Z}^n \)-tile \( T \), \( i \geq 2 \), and \( S \in K(T)_0^i \).

(i) If \( M \) is combinatorial, then \( \partial_i(S) \subset Q\partial_i(S)_M \).

(ii) If \( T \) is combinatorial, then \( Q\partial_i(S)_M \subset \partial_i(S) \).

Thus, if both, \( M \) and \( T \) are combinatorial we have that \( Q\partial_i(S)_M = \partial_iQ(S)_M \).

2.4 Self-affine tiles and manifolds. In Section 5.1 we will state a criterion for a self-affine \( \mathbb{Z}^2 \)-tile to be homeomorphic to a closed disk. In view of the Jordan Curve Theorem this is equivalent to \( \partial T \) being homeomorphic to a circle. Criteria of that kind have been published before (see e.g. [5, 39]). We shall prove analogs of these results for boundaries of self-affine \( \mathbb{Z}^3 \)-tiles that are homeomorphic to a closed surface. To treat surfaces of positive genus we need to exclude pathological point preimages of the restriction of \( Q \) to \( \partial M \). We thus define the following property of point preimages.

**Definition 2.13** (Semi-contractible). Let \( M \) be a monotone model for a self-affine \( \mathbb{Z}^3 \)-tile with \( \partial M \) being a closed surface and let \( Q \) be the associated quotient map. We say that \( M \) is semi-contractible if each point preimage of \( Q|\partial M \) is contained in a contractible neighborhood (e.g. a disk).

We note that a model with spherical boundary is semi-contractible.

**Theorem 2.14.** Let \( T \) be a self-affine \( \mathbb{Z}^2 \)-tile with connected interior which admits a semi-contractible monotone model \( M \) whose boundary is the closed surface \( S \). Then the following assertions hold.

(i) \( \partial T \) is homeomorphic to \( S \).

(ii) Under the restriction \( Q|\partial M \) preimages of points do not path separate \( \partial M \).

We immediately obtain the following consequence.

**Corollary 2.15.** Let \( T \) be a self-affine \( \mathbb{Z}^3 \)-tile with connected interior which admits a monotone model \( M \) with 2-sphere boundary. Then \( \partial T \) is homeomorphic to the 2-sphere and point preimages of the quotient map \( Q|\partial M \) do not path separate.

Theorem 2.14 can be extended to finite unions of \( \mathbb{Z}^3 \)-translates of \( T \) (see Proposition 5.9).

The next theorem shows that under certain conditions \( \partial T = \bigcup_{s \in \mathbb{Z}^n \setminus \{0\}} \langle \{0, s\} \rangle \) admits a natural CW-structure defined by the intersections \( \langle S \rangle \). Recall that a set is degenerate if it contains fewer than 2 points.

**Theorem 2.16.** Let \( T \) be a self-affine \( \mathbb{Z}^3 \)-tile with connected interior which admits a semi-contractible combinatorial monotone model \( M \) whose boundary is the closed surface \( S \).

Let \( S \in K(T) \) be nondegenerate. If \( \langle S \rangle_\partial M \) is a closed topological manifold or a ball then its canonical quotient \( \langle S \rangle_\partial M \) is either homeomorphic to \( \langle S \rangle_M \) or degenerate.
The proofs of Theorems 2.14 and 2.16 are contained in Section 5.2 where we also state an easy criterion for checking whether the interior of a self-affine $\mathbb{Z}^n$-tile is connected (see Lemma 5.10). It is important to note that the homeomorphisms asserted in these theorems are usually not $Q$ since $Q$ is not necessarily injective. However, one can use $Q$ to construct this homeomorphism with the help of Moore’s decomposition theorem (see Proposition 5.2 and [43, 44]). Theorem 2.16 requires $S$ to contain at least 2 elements. Under the given conditions we cannot expect $T = \langle \{0\} \rangle$ to be homeomorphic to a 3-ball even if the same is true for $M$. Indeed, in Section 8.2 we give an example of a monotone model which is homeomorphic to a 3-ball but whose underlying self-affine $\mathbb{Z}^3$-tile is wild and indeed not even simply connected, even though its boundary can be shown to be a sphere by applying Corollary 2.15. An image of this example is depicted on the right hand side of Figure 1. It is the self-affine $\mathbb{Z}^3$-tile $T = T(A, D)$ with $A = \text{diag}(9, 9, 9)$ whose digits set is visualized on the left hand side of Figure 1 (see Section 8.2 for an exact definition). In studying this example we shall prove the following result.

**Theorem 2.17.** There exists a self-affine $\mathbb{Z}^3$-tile whose boundary is a 2-sphere, but which is not homeomorphic to a 3-ball (a self-affine wild crumpled cube).

We are also able to prove generalizations of Theorem 2.14 to higher dimensions and give criteria for boundaries of self-affine $\mathbb{Z}^n$-tiles to be $(n - 1)$-manifolds. Since their statements require more notation we postpone it to Section 5.3.

In Section 7 we deal with self-affine $\mathbb{Z}^n$-tiles that are homeomorphic to closed balls. For $n = 3$ a combination of Theorem 2.14 and a result of Bing [7] yields the following characterization result.

**Theorem 2.18.** Let $T$ be a self-affine $\mathbb{Z}^3$-tile whose interior is connected and 1-ULC. Then $T$ is a self-affine 3-manifold if and only if it admits a monotone model with a boundary that is a closed 2-manifold.

Using a result of Cannon [8] we will give an algorithmic criterion that allows to check whether a self-affine $\mathbb{Z}^n$-tile with spherical boundary is homeomorphic to a closed ball. The exact statement of this result, which requires the definition of the in-out graph, can be found in Theorem 7.22. Together with Theorem 2.14 we shall use this result to give nontrivial examples of $\mathbb{Z}^3$-tiles that are homeomorphic to closed 3-balls and other 3-manifolds (see Section 2.7).
2.5. **Subdivision.** Let \( T = T(A, D) \) be a self-affine \( \mathbb{Z}^n \)-tile and \((M,F)\) a model for \( T \) satisfying \( K(T) = K(M) \). We now describe a generalization of the set equations (1.2) and (2.2) to the cells \((S)\) and \((S)_M\) for \( S \in K(T) \), respectively. We start with the following definition.

**Definition 2.19** (Subdivision operator). Let \( T = T(A, D) \) be a self-affine \( \mathbb{Z}^n \)-tile. Then the subdivision operator \( P \) is given by

\[
P(S) = \{ (p + A)(S) \in K(T) \mid p \in D^S \} \quad (S \in K(T)),
\]

where \( D^S \) denotes the set of functions from \( S \) to \( D \).

It is unnecessary to define a similar operator for the model \( M \) since \( K(T) = K(M) \) and \( F(z) = Az \) holds for each \( z \in \mathbb{Z}^n \) which implies that \( P(S) = \{ (p + F)(S) \in K(M) \mid p \in D^S \} \).

As \( P(\{0\}) = \{ \{d\} \mid d \in D \} \), using the operator \( P \) the set equations in (1.2) and (2.2) become

\[
A(\{0\}) = \langle P(\{0\}) \rangle \quad \text{and} \quad F(\{0\})_M = \langle P(\{0\}) \rangle_M,
\]

respectively. The terminology **subdivision operator** is further testified by the set equations

\[
A(S) = \langle P(S) \rangle \quad \text{and} \quad A(S)_M = \langle P(S) \rangle_M,
\]

which we shall prove in Theorems 3.13 and 3.21. In Lemma 6.1 we will see that \( P \) behaves nicely with respect to the “simplicial boundary operator” \( \delta \) defined in (2.7) and (2.8), in particular,

\[
\delta P(S) = P(\delta S).
\]

2.6. **Ideal tiles.** Up to this point it is unclear how one could construct nontrivial models for a given self-affine \( \mathbb{Z}^n \)-tile. We now give necessary conditions that define an **ideal tile** for a self-affine \( \mathbb{Z}^n \)-tile which allows to construct a monotone model for that tile.

**Definition 2.22** (Ideal tile). A \( \mathbb{Z}^n \)-tile \( Z \) is an ideal tile of \( T(A, D) \), if \( Z \) has connected interior and the following conditions hold.

1. \( K(Z) = K(T) \).
2. For each \( S \in K(Z) \) the set \( \langle S \rangle_Z \) is connected and homeomorphic to \( \langle P(S) \rangle_Z \).
3. For each \( S \in K(Z) \) each homeomorphism from \( \langle \delta S \rangle_Z \) to \( \langle \delta P(S) \rangle_Z \) extends to a homeomorphism from \( \langle S \rangle_Z \) to \( \langle P(S) \rangle_Z \).

Note that since \( Z \) is a \( \mathbb{Z}^n \)-tile one only needs to check the above hypotheses for all \( S \) containing 0. Condition (i) can be checked algorithmically by using so-called boundary and vertex graphs (cf. e.g. [46] and see Remark 3.15). In most cases ideal tiles are chosen to be polyhedra. Then Conditions (ii) and (iii) can be checked by direct inspection for \( n = 3 \). For higher dimensions, in Section 6.2 we show how to check (ii) by techniques from algebraic topology in certain cases. Lemma 6.2 shows that (iii) is true for large classes of topological spaces.

**Theorem 2.23.** Let \( Z \) be an ideal tile for a self-affine \( \mathbb{Z}^n \)-tile \( T \) and \( u \in \text{int}(Z) \) then there is a homeomorphism \( F \) such that \((Z - u, F)\) is a monotone model for \( T \).

Examples of ideal tiles are given in Figures 3 and 5.

2.7. **Examples.** We now illustrate our theory by examples of self-affine \( \mathbb{Z}^3 \)-tiles. In Section 2.7.1 we give a first example of a tile that is homeomorphic to a 3-ball. Section 2.7.2 is devoted to a tile that was already studied in 1996 by Gelbrich [24]. We can now show that it is homeomorphic to a 3-ball. Finally, in Section 2.7.3 we state an existence result for a self-affine \( \mathbb{Z}^3 \)-tile whose boundary is a surface of genus \( g \) for each \( g \in \mathbb{N} \).

2.7.1. **A self-affine \( \mathbb{Z}^3 \)-tile that is homeomorphic to a 3-ball.** Let

\[
A = \begin{pmatrix} 0 & 0 & -4 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\},
\]

and let \( T \subset \mathbb{R}^3 \) be the unique nonempty compact set satisfying \( AT = T + D \). As \( A \) is expanding and \( D \) is a complete set of residue class representatives of \( \mathbb{Z}^3/\mathbb{Z}^3 \) the set \( T \) is a self-affine tile. Moreover, [36, Corollary 6.2] yields that \( T \) tiles \( \mathbb{R}^3 \) by \( \mathbb{Z}^3 \)-translates making \( T \) a self-affine \( \mathbb{Z}^3 \)-tile. An image of \( T \) is depicted in Figure 2. We shall prove the following result.
Figure 2. A self-affine $\mathbb{Z}^n$-tile that is homeomorphic to a closed ball.

**Theorem 2.25.** Let $T$ be the self-affine $\mathbb{Z}^3$-tile defined by the set equation $AT = T + D$ with $A$ and $D$ given as in (2.24). $T$ is homeomorphic to a closed 3-dimensional ball.

Moreover, we are able to establish the following topological characterization result for the cells of $T$ (the finite graphs $\Gamma_2$, $\Gamma_3$, and $\Gamma_4$ are defined in Remark 3.15 and constructed explicitly in Section 8).

**Proposition 2.26.** Let $T$ be the self-affine $\mathbb{Z}^3$-tile defined by $AT = T + D$ with $A$ and $D$ defined as in (2.24), and let $S \subset \mathbb{Z}^3$ with $0 \in S$ be given.

- If $|S| = 2$ then $\langle S \rangle \cong \mathbb{D}^2$ if $S$ is a node of the graph $\Gamma_2$ and $\langle S \rangle = \emptyset$ otherwise.
- If $|S| = 3$ then $\langle S \rangle \cong [0, 1]$ if $S$ is a node of the graph $\Gamma_3$ and $\langle S \rangle = \emptyset$ otherwise.
- If $|S| = 4$ then $\langle S \rangle \cong \{p\}$ if $S$ is a node of the graph $\Gamma_4$ and $\langle S \rangle = \emptyset$ otherwise.
- If $|S| \geq 5$ then $\langle S \rangle = \emptyset$.

In Section 8.1 will give detailed proofs of these results.

2.7.2. Gelbrich’s twin dragon. In 1996 Gelbrich [24] studied the twin dragon $T = T(A, D)$ with

$$(2.27) \quad A = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$  

Again it is easy to check that $T$, which is depicted in Figure 3, is a self-affine $\mathbb{Z}^3$-tile. In his paper, Gelbrich asked whether $T$ is homeomorphic to a closed 3-dimensional ball. We are now able to answer his question in the affirmative.

Figure 3. Gelbrich’s twin dragon $T$ (left) and its ideal tile $\mathbb{Z}$ (right).

**Theorem 2.28.** Let $T$ be Gelbrich’s twin dragon which is defined by the set equation $AT = T + D$ with $A$ and $D$ given as in (2.27). $T$ is homeomorphic to a closed 3-dimensional ball.

The proof of this result is contained in Section 8.3.
2.7.3. Self-affine \( \mathbb{Z}^3 \)-tiles whose boundaries are surfaces of positive genus. We give the following result on boundary surfaces of positive genera (see also Section 8.4).

**Proposition 2.29.** For each genus \( g \in \mathbb{N} \) there is a self-affine \( \mathbb{Z}^3 \)-tile \( T \) whose boundary is a surface of genus \( g \).

Although we do not want to go into details, we mention that it is possible to show that \( T \) is a self-similar 3-manifold by adapting the ball-checking algorithm provided in Section 7 (see in particular Remark 7.24). The fundamental neighborhoods can be chosen to be cubes here. Thus each 3-dimensional handlebody is homeomorphic to a \( \mathbb{Z}^3 \)-tile.

3. Self-affine \( \mathbb{Z}^n \)-tiles and their models

In this section we extend standard notations and results on self-affine \( \mathbb{Z}^n \)-tiles and show how much of these results remains true for models. Before we start we equip \( \mathbb{R}^n \) with a norm \( \| \cdot \| \) such that the associated operator norm (also denoted by \( \| \cdot \| \)) satisfies

\[
\| A^{-1} \| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\| A^{-1} x \|}{\| x \|} < 1.
\]

As \( A \) is expanding such a choice is possible (cf. e.g. [35, Section 3]).

3.1. Basic properties of the canonical quotient map. Let \( T = T(A, D) \) be a self-affine \( \mathbb{Z}^n \)-tile and \((M, F)\) a model for it. We now give some auxiliary results on models and study properties of the canonical quotient map \( Q : \mathbb{R}^n \to \mathbb{R}^n \) defined in (2.3). Our first aim is to associate with a given model \( M \) a sequence of models which converges to the tile \( T \). For this reason define the set

\[
M_k = q_k M \quad \text{with} \quad q_k = A^{-k} F^{(k)}
\]

and let \( F_k = A^{-k} F A^k \) for each \( k \in \mathbb{N} \).

**Lemma 3.3.** For each \( k \in \mathbb{N} \) the functions \( A^{-1} F_k \) and \( q_k = A^{-k} F^{(k)} \) are \( \mathbb{Z}^n \)-equivariant and fix each element of \( \mathbb{Z}^n \).

**Proof.** Let \( k \in \mathbb{N} \), \( x \in \mathbb{R}^n \), and \( z \in \mathbb{Z}^n \). The \( \mathbb{Z}^n \)-equivariance of \( A^{-1} F \) implies that

\[
A^{-1} F_k(x + z) = A^{-k} A^{-1} F(A^k x + A^k z) = A^{-k}(A^{-1} F A^k x + A^{-1} F A^k z) = A^{-1} F_k(x) + z
\]
and $A^{-1}F_k$ is $\mathbb{Z}^n$-equivariant. Because $F(0) = 0$ we also have $A^{-1}F_k(0) = 0$ and, hence, by $\mathbb{Z}^n$-equivariance, $A^{-1}F_k$ fixes each element of $\mathbb{Z}^n$. Finally, as $q_k = A^{-1}F_{k-1} \circ \cdots \circ A^{-1}F_0$, also $q_k$ has the required properties. □

We now are in a position to prove the following convergence result for models. The limit of a sequence of models in its statement is taken with respect to the product of the Hausdorff metric and the metric of uniform convergence.

**Proposition 3.4.** Let $(M, F)$ be a model for the self-affine $\mathbb{Z}^n$-tile $T = T(A, D)$. Then $(M_k, F_k)$ is a model for $T$ for each $k \in \mathbb{N}$ and $\lim_{k \to \infty} (M_k, F_k) = (T, A)$.

**Proof.** We first prove that $(M_k, F_k)$ is a model for each $k \in \mathbb{N}$. We begin by showing that $M_k$ is a $\mathbb{Z}^n$-tile. The properties of $q_k$ asserted in Lemma 3.3 yield that $M_k = A^{-1}q_{k-1}F(M) = A^{-1}(q_{k-1}M + D) = A^{-1}(M_{k-1} + D)$. Iterating this for $k$ times ($q_0$ is the identity) and setting

$$D_k = D + AD + \cdots + A^{k-1}D$$

we gain

$$M_k = A^{-k}(M + D_k) \quad (k \in \mathbb{N}).$$

Since $D$ is a complete set of residue class representatives of $\mathbb{Z}^n/\mathbb{Z}^n$, $D_k$ is a complete set of residue class representatives of $\mathbb{Z}^n/A^k\mathbb{Z}^n$. Thus, as $M$ is a $\mathbb{Z}^n$-tile, equation (3.6) implies that also $M_k$ is a $\mathbb{Z}^n$-tile. Next we show that $(M_k, F_k)$ satisfies the set equation (2.2). Indeed, as $F_kq_k = q_kF$ holds by the definition of $q_k$ and $F_k$, we have

$$F_kM_k = F_kq_kM = q_kFM = q_k(M + D) = q_kM + D = M_k + D.$$

As Lemma 3.3 implies that $A^{-1}F_k$ is $\mathbb{Z}^n$-equivariant and $F_k(0) = 0$, we have obtained that $(M_k, F_k)$ is a model for $T$ for each $k \in \mathbb{N}$.

To prove the convergence result let $D(\cdot, \cdot)$ be the Hausdorff metric. For each $\varepsilon > 0$ we may choose $k \in \mathbb{N}$ in a way that $A^{-k}T$ and $A^{-k}M$ are contained in a ball of diameter $\varepsilon$ around the origin. Using (3.6) and the $k$-th iterate of the set equation (1.2), we gain

$$D(M_k, T) = D(A^{-k}(M + D_k), A^{-k}(T + D_k))$$

$$\leq \max_{d \in D_k} \{D(A^{-k}(M + d), A^{-k}(T + d))\}$$

$$< \varepsilon.$$

As $||A|| > 1$ and $A^{-1}F$ is continuous and $\mathbb{Z}^n$-equivariant, the maps $A^{-k}(A^{-1}F)A^k$ uniformly converge to the identity. Thus $F_k \to A$ uniformly for $k \to \infty$ and the proof is finished. □

We mention that the sequence $(M_k)$ of models yields approximations not only of the tile $T$, but also of the dynamical system defined by its set equation (see e.g. Theorem 3.21).

The following result contains Theorem 2.4 as well as some properties of $Q$.

**Theorem 3.8.** Let $(M, F)$ be a model for the self-affine $\mathbb{Z}^n$-tile $T$. Then the canonical quotient map $Q = \lim_{k \to \infty} q_k$ is continuous, $\mathbb{Z}^n$-equivariant, and satisfies the following properties.

(i) $Q(M) = T$, i.e., $T$ is a quotient space of $M$.

(ii) $Q(\partial M) = \partial T$, i.e., $\partial T$ is a quotient space of $\partial M$.

**Proof.** We first show that $(q_k(x))_{k \geq 0}$ is a Cauchy sequence for each $x \in \mathbb{R}^n$. Observe that by continuity and $\mathbb{Z}^n$-equivariance of $A^{-1}F$ there is an absolute constant $c > 0$ such that

$$||q_kx - q_{k-1}x|| = ||A^{-k}F(k)x - A^{-k+1}F(k-1)x||$$

$$= ||A^{-k+1}A^{-1}F(k)x - A^{-k+1}F(k-1)x||$$

$$\leq ||A^{-1}||^{k-1}||A^{-1}F(F(k)x) - (F(k)x)||$$

$$\leq c||A^{-1}||^{k-1} \quad \forall x \in \mathbb{R}^n, \ k \in \mathbb{N}.$$
By (3.1) we have $||A^{-1}|| < 1$ and, by the triangle inequality and a geometric series consideration
\begin{equation}
||q_k x - q x|| < ||A^{-1}||^k \left( \frac{c}{1 - ||A^{-1}||} \right) \quad \forall x \in \mathbb{R}^n, \; k \geq 1 \in \mathbb{Z}.
\end{equation}

Thus $(q_k(x))_{k \geq 0}$ is Cauchy and, hence, converges for each $x \in \mathbb{R}^n$. This defines the function $Q$ for each $x \in \mathbb{R}^n$. As the convergence in (3.9) is uniform in $x$ we conclude that $Q$ is continuous.

$\mathbb{Z}^n$-equivariance of $Q$ follows from $\mathbb{Z}^n$-equivariance of $q_k$ (see Lemma 3.3), and (i) is an immediate consequence of the convergence statement in Proposition 3.4.

To prove (ii) we have to show that for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that
\begin{equation}
D(\partial T, \partial M_k) < \varepsilon
\end{equation}
holds for each $k \geq k_0$. To prove this let $\varepsilon > 0$ be arbitrary and choose $k_0$ in a way that (3.7) holds for $k \geq k_0$.

First we note that, since $M_k$ and $T$ are $\mathbb{Z}^n$-tiles, we have
\begin{equation}
\partial T = \bigcup_{s \in \mathbb{Z}^n \setminus \{0\}} \{0, s\} = \langle \delta \{0\} \rangle \quad \text{and} \quad \partial M_k = \bigcup_{s \in \mathbb{Z}^n \setminus \{0\}} \{0, s\}_M = \langle \delta \{0\} \rangle_{M_k}.
\end{equation}

Thus for each $x \in \partial T$ there is $s \in \mathbb{Z}^n \setminus \{0\}$ s.t. $x \in T$ and $x \in T + s$. By (3.7) there exist $y_1 \in M_k$, $y_2 \in M_k + s$ with $||x - y_i|| < \varepsilon$ ($i = 1, 2$). As $M_k$ is a $\mathbb{Z}^n$-tile this implies that there is an element $y \in \partial M_k$ with $||x - y|| < \varepsilon$. By analogous reasoning, for each $x \in \partial M_k$ there exists $y \in \partial T$ with $||x - y|| < \varepsilon$. This proves (3.10) and, hence, also (ii).

The following lemma will be needed in some computations.

**Lemma 3.12.** If $(M, F)$ is a model for the self-affine $\mathbb{Z}^n$-tile $T$ then $AQ = QF$ and $Q^{-1}A = FQ^{-1}$.

**Proof.** The first identity follows immediately from the definition of $Q$. The second one follows as
\begin{align*}
FQ^{-1}x &= \{Fy \mid Qy = x\} = \{Fy \mid AQy = Ax\} = \{Fy \mid QFy = Ax\} \\
&= \{z \mid Qz = Ax\} = Q^{-1}Ax.
\end{align*}

\hfill \square

3.2. The generalized set equation. Let $T = T(A, D)$ be a self-affine $\mathbb{Z}^n$-tile. Using the subdivision operator $P$ from Definition 2.19 we will now extend the standard notion of set equation (1.2) to intersections $\langle S \rangle$ (see also [46] where this is done by using so-called boundary graphs). To this end we need “powers” of $P$ which we define inductively by
\begin{equation}
P^{(k)}(S) = \{S'' \in P(S') \mid S' \in P^{(k-1)}(S)\} \quad (k \geq 2).
\end{equation}

**Theorem 3.13** (The generalized set equation). Let $T$ be a self-affine $\mathbb{Z}^n$-tile and $S \in \mathcal{K}(T)$. Then
\begin{equation}
\langle S \rangle = A^{-k} \langle P^{(k)}(S) \rangle \quad (k \in \mathbb{N}).
\end{equation}

As $(T, A)$ trivially is a model of $T$ having the same neighboring structure as $T$, Theorem 3.13 is a special case of the first equality in Theorem 3.21. Thus we refrain from proving Theorem 3.13 here.

**Remark 3.15.** For $S \in \mathcal{K}(T)$ with $0 \in S$ equation (3.14) yields
\begin{equation}
\langle S \rangle = \bigcup_{S' \in P(S)} A^{-1}(S').
\end{equation}

and each shifted set $S' = S + s'$ is an element of $\mathcal{K}(T)$ containing 0. To each $S \in \mathcal{K}(T)$ with $0 \in S$ we associate an indeterminate $X_S$ whose range of values is the space of nonempty compact subsets of $\mathbb{R}^n$. Using (3.16) we define the (finite) collection
\begin{equation}
X_S = \bigcup_{S' \in P(S)} A^{-1}(X_{S' - s'} + s') \quad (S \in \mathcal{K}(T), \; 0 \in S)
\end{equation}
of set equations which defines a graph directed iterated function system whose unique solution is given by $X_S = \langle S \rangle$ for each $S \in \mathcal{K}(T)$ with $0 \in S$. Note that $|S| = |S'|$ holds for each $S' \in P(S)$. Thus, following [46], for each $i \geq 1$ we define the graph $\Gamma_i$ as follows. The set of nodes of $\Gamma_i$ is
The algorithmic construction of the graphs $\Gamma_i$ way. Indeed, it induces a sequence of subdivisions $(\text{Walks})$. For a set of walks $w$, the sequence $(C_w)$ for an “ancestor” function $Z$ of cosets of $\{\cdots\} = \{\cdots\}$, so that $S_k = 1$ implies $P(S) \cap P(S') = \emptyset$. For this reason there is again an “ancestor” function $R$ from $P(S)$ to the set of finite subsets of $\mathbb{Z}^n$ such that $P = R^{-1}$, i.e., $R(S') = S$ if $S' \in P(S)$. This motivates the following precise definition of walks.

**Definition 3.18 (Walks).** For $S \in K(T)$, the set $W(S)$ of walks in $S$ is the inverse limit of the sequence $(P(k)(S))$ with bonding map $R : P(k)(S) \to P(k-1)(S)$. Let $\pi_k : W(S) \to P(k)(S)$ be the canonical projection and define

$$w_k = A^{-k}(\pi_k(w))$$

for a walk $w$ and

$$C_k = A^{-k}(\pi_k(C))$$

for a set of walks $C$.

Walks are coordinates for a nested collection of sets. The set $W(S)$ can also be described as the set of infinite walks in a finite graph (see e.g., [46]); this motivates the terminology “walk”.

Recall that the image of $P(k)$ is contained in $K(T)$ for each $k$ by definition. Thus for a walk $w$, the sequence $(w_k)$ is a nested sequence of nonempty compact sets. The diameter of $w_k$ approaches zero (since $A^{-1}$ is a contraction) and consequently defines a unique limit point.

**Definition 3.19 (Limit points of walks).** Let $S \in K(T)$ and $w \in W(S)$ be a walk. The mapping

$$w \mapsto w_T = \bigcap_{k \geq 1} w_k$$

is a continuous map of the Cantor set $W(S)$ to $\langle S \rangle$. If $C$ is a set of walks, we will use the notation

$$C_T = \bigcap_{k \geq 1} C_k.$$

In particular, set $W = W(\{0\})$ and $\partial W = \bigcup_{s \in \mathbb{Z}^n \setminus \{0\}} W(\{0, s\})$. Theorem 3.13 implies

$$T = W_T, \quad \partial T = \partial W_T, \quad \text{and} \quad \langle S \rangle = W(S)_T.$$
3.4. The generalized set equation and walks in a model. There are many similarities between a self-affine $\mathbb{Z}^n$-tile and its model, particularly if they have the same neighbor structure.

We obtain the following analog of Theorem 3.13 for models. Recall that $M_k$ is defined in (3.2).

**Theorem 3.21.** Let $(M, F)$ be a model for the self-affine $\mathbb{Z}^n$-tile $T$ satisfying $\mathcal{K}(T) = \mathcal{K}(M)$. Then for every $S \in \mathcal{K}(T)$ and every $k \in \mathbb{N}$,

\begin{equation}
F^{(k)}\langle S \rangle_M = \langle F^{(k)}(S) \rangle_M = A^k\langle S \rangle_{M_k}, \tag{3.22}
\end{equation}

**Proof.** As $A^{-1}F$ is $\mathbb{Z}^n$-equivariant and fixes $\mathbb{Z}^n$ point wise, we have $F(x + z) = F(x) + Az$ for each $x \in \mathbb{R}^n$, $z \in \mathbb{Z}^n$. Together with the set equation (2.2) for $M$ this yields

\[
F\langle S \rangle_M = \bigcap_{s \in S} F(M + s) = \bigcap_{s \in S} (F(M) + As)
= \bigcap_{s \in S} (M + D + As) = \bigcap_{s \in S} \bigcup_{d \in D} (M + d + As)
= \bigcup_{p \in D^S} \bigcap_{s \in S} (M + p(s) + As) = \bigcup_{p \in D^S} \langle (p + A)(S) \rangle_M.
\]

Since $\mathcal{K}(T) = \mathcal{K}(M)$, this subdivision is again governed by the function $P$ and we arrive at

\[
F\langle S \rangle_M = \bigcup_{S' \in P(S)} \langle S' \rangle_M.
\]

Iterating this for $k$ times proves the first equality. To prove the second one recall that $q_k = A^{-k}F^{(k)}$ and $M_k = q_k M$. By Lemma 3.3 we have

\begin{equation}
q_k\langle S \rangle_M = \bigcap_{s \in S} q_k(M + s) = \bigcap_{s \in S} (q_k(M) + s) = \bigcap_{s \in S} (M_k + s) = \langle S \rangle_{M_k}, \tag{3.23}
\end{equation}

and, hence, $F^{(k)}\langle S \rangle_M = A^k\langle S \rangle_{M_k}$. \hfill \Box

Let $(M, F)$ be a model of a self-affine $\mathbb{Z}^n$-tile $T$ satisfying $\mathcal{K}(M) = \mathcal{K}(T)$. For $S \in \mathcal{K}(M)$ to a walk $w \in W(S)$ we associate the nested collection

\[
w_{M,k} = (F^{-1})^{(k)}\langle \pi_k(w) \rangle_M
\]

and, for a set $C \subset W(S)$, we define

\[
C_{M,k} = (F^{-1})^{(k)}\langle \pi_k(C) \rangle_M.
\]

We denote

\[
w_M = \bigcap_{k \geq 1} w_{M,k}
\]

which may contain more than one point as $F^{-1}$ is not necessarily a contraction. This is an important difference between a self-affine $\mathbb{Z}^n$-tile and its model. However, as $\mathcal{K}(M) = \mathcal{K}(T)$, the definition of a walk assures that $w_M$ cannot be empty. Again, this definition extends to sets $C$ of walks by setting $C_M = \bigcap_{k \geq 1} C_{M,k}$. Using this notation we obtain

\begin{equation}
M = W_M, \quad \partial M = \partial W_M, \quad \text{and} \quad \langle S \rangle_M = W(S)_M. \tag{3.24}
\end{equation}

4. Properties of monotone models

In this section we investigate mapping properties of the canonical quotient map $Q$ under the condition that $(M, F)$ is a monotone model for a self-affine $\mathbb{Z}^n$-tile $T$. 
4.1. Quotients of cells. The following proposition gives results on images of certain sets under $Q$. Recall that the model $M_k$ is defined in (3.2).

**Proposition 4.1.** If $(M, F)$ is a model for the self-affine $\mathbb{Z}^n$-tile $T$ with $\mathcal{K}(M) = \mathcal{K}(T)$ then the following assertions hold.

(i) The sequence $(S)_{M_k}$ converges to $(S)$ in the Hausdorff metric $D$ for each $S \in \mathcal{K}(T)$.

(ii) $Q(S)_{M_k} = (S)$ holds for each $S \in \mathcal{K}(T)$.

(iii) If $S \in \mathcal{K}(T)$ then for each $w \in W(S)$ we have $Q(w_{M_k}) = w_T$.

**Proof.** To prove (i) let $\varepsilon > 0$ be arbitrary. Since $A$ is expanding and both $T$ and $M$ are compact we may choose $k_0 \in \mathbb{N}$ in a way that $A^{-k}T$ and $A^{-k}M$ are contained in a ball of diameter $\varepsilon$ around the origin for each $k \geq k_0$. Using Theorems 3.13 and 3.21 we get

$$D((S), (S)_{M_k}) = D \left( \bigcup_{S' \in P(k)(S)} A^{-k}(S'), \bigcup_{S' \in P(k)(S)} A^{-k}(S'_{M_k}) \right) \leq \max \{ D(A^{-k}(S'), A^{-k}(S'_{M_k})) \mid S' \in P(k)(S) \}.$$ 

Note that $A^{-k}(S') \subset A^{-k}(T + s)$ and $A^{-k}(S'_{M_k}) \subset A^{-k}(T + s)$ for each $S' \subset \mathbb{Z}^n$ and each $s \in S'$. Thus $A^{-k}(S')$ as well as $A^{-k}(S'_{M_k})$ is contained in a ball of diameter $\varepsilon$ around $A^{-k}s$ implying that $D(A^{-k}(S'), A^{-k}(S'_{M_k})) < \varepsilon$ for each $S' \subset \mathbb{Z}^n$. Thus $D((S), (S)_{M_k}) < \varepsilon$ and (i) is proved.

By (3.23) we have $q_k(S)_{M_k} = (S)_{M_k}$. Thus, as $Q = \lim_{k \to \infty} q_k$, Assertion (ii) follows from (i).

To prove (iii), using Lemma 3.12 and (ii) we derive

$$Q(w_{M_k}) = Q((F^{-1}(k))(\pi_k(w)))_{M_k} = A^{-k}Q((\pi_k(w))_{M_k}) = A^{-k}Q(w_k) = w_k.$$ 

Now (iii) follows from

$$Q(w_M) = Q \left( \lim_{k \to \infty} w_{M_k} \right) = \lim_{k \to \infty} Q(w_{M_k}) = \lim_{k \to \infty} w_k = w_T. \quad \square$$

We will now use the sets $w_M$ to study preimages of $Q$.

**Lemma 4.2.** If $M$ is a model for $T$ with $\mathcal{K}(M) = \mathcal{K}(T)$, for any nonempty set of walks $C$

$$\bigcap_{w \in C} w_M = \emptyset \iff \bigcap_{w \in C} w_T = \emptyset \iff |C_T| > 1.$$ 

**Proof.** As the sets $w_M$ and $w_T$, $w \in C$, are compact it suffices to check the first equivalence for each finite subset $C'$ of $C$ by the finite intersection property for compact sets. Assume that $\bigcap_{w \in C'} w_M \neq \emptyset$. As $(\bigcap_{w \in C'} w_{M_k})_{k \in \mathbb{N}}$ is a nested sequence of compact sets this is equivalent to $\bigcap_{w \in C'} w_k \neq \emptyset$ for each $k \in \mathbb{N}$. Because $\mathcal{K}(M) = \mathcal{K}(T)$, this is in turn equivalent to $\bigcap_{w \in C'} w_{M_k} \neq \emptyset$ for each $k \in \mathbb{N}$. As the sequence $(\bigcap_{w \in C'} w_{M_k})_{k \in \mathbb{N}}$ is a nested sequence of compact sets, this is finally equivalent to $\bigcap_{w \in C'} w_M \neq \emptyset$, proving the first equivalence. As for the second one note that the limit point $w_T$ is a singleton for each $w \in C_T$. Thus $C_T$ contains more than one element if and only if there exist $w, w' \in C$ having disjoint limit points which is equivalent to $\bigcap_{w \in C'} w_M \neq \emptyset$. \quad \square

Recall that a quotient is monotone if point preimages are connected.

**Proposition 4.3.** Suppose that $M$ is a monotone model for the self-affine $\mathbb{Z}^n$-tile $T$.

(i) $Q(S)_{M_k} = (S)$ is a monotone quotient of $(S)_{M_k}$ for each $S \in \mathcal{K}(T)$.

(ii) $Q(S)_{M_k} = (S)$ is a monotone quotient of $(\delta S)_{M_k}$ for each $S \in \mathcal{K}(T)$.

(iii) $Q\partial M = \partial T$ is a monotone quotient of $\partial M$.

**Proof.** By Proposition 4.1 (ii) we have $Q(S)_{M_k} = (S)$. To show (i) we thus have to prove that $(Q|_{(S)_{M_k}})^{-1}(x)$ is connected for each $x \in (S)$. Let $x \in (S)$ be fixed and choose $w \in W(S)$ with $x = w_T$. Proposition 4.1 (iii) implies that

$$(Q|_{(S)_{M_k}})^{-1}(x) = \bigcup_{w' \in W(S), w' = w_T} w'_{M_k}.$$
By Lemma 4.2 all the sets $w'_M$ in the union on the right share a common point. It therefore remains to prove that $w'_M$ is connected for each $w' \in W(S)$. However, since $M$ is a monotone model for $T$, the cell $(S')_M$ is connected for each $S' \subset \mathbb{Z}^n$. Thus the set $w'_M$ is a nested union of the connected sets $w'_{M,k}$ and therefore itself connected.

To prove (ii) first observe that
\[ Q(\delta S)_M = \bigcup_{S' \in \delta S} Q(S')_M = \bigcup_{s \in \mathbb{Z}^n \setminus S} Q(S \cup \{s\})_M = \bigcup_{s \in \mathbb{Z}^n \setminus S} \langle S \cup \{s\} \rangle = \langle \delta S \rangle. \]

Choose $x \in \langle \delta S \rangle$. Then, by Proposition 4.1 (iii) we gain
\[ (Q|_{\langle \delta S \rangle})^{-1}(x) = \bigcup_{s \in \mathbb{Z}^n \setminus S} Q(\langle S \cup \{s\} \rangle)_M^{-1}(x) = \bigcup_{s \in \mathbb{Z}^n \setminus S} \bigcup_{w' \in W(S \cup \{s\})} w'_M \]
and everything runs exactly as in (i). In view of (3.11), (iii) is an immediate consequence of (ii). \qed

Theorem 2.10 is a consequence of Proposition 4.3 (i). Just note that all decompositions are accomplished by the restrictions $Q_{\langle S \rangle}_M$ of $Q$.

We now turn to the proof of Theorem 2.12 which clarifies the effect of the canonical quotient map $Q$ on boundaries of cells.

**Proof of Theorem 2.12.** Let $S \in \mathcal{K}(T)_{T_0}$, $i \geq 2$, be arbitrary but fixed. To prove (i) let $x \in \partial_i(S)$ be given. Then, since $\mathcal{K}(T)_{T_0}$ is a finite complex and cells are closed in $\langle \mathcal{K}(T)_{T_0} \rangle$, there exists $S' \in \mathcal{K}(T)_{T_0}, S' \neq S$ such that $x \in \langle S' \rangle$. As $x \in \langle S \rangle \cap \langle S' \rangle$ there exists $s \in S' \setminus S$ such that $x \in \langle S \cup \{s\} \rangle$. Proposition 4.1 (ii) implies that $Q(S \cup \{s\})_M = \langle S \cup \{s\} \rangle$. Thus, because $M$ is combinatorial, we obtain that $x \in Q(S \cup \{s\})_M \subset Q(\delta S)_M = Q(\delta S)_M$.

To prove (ii) let $x \in Q\partial_i(S)_M$ be given. By the same argument as in (i) there is $s \in \mathbb{Z}^n \setminus S$ with $x \in Q\langle S \cup \{s\} \rangle_M$. Thus, as $T$ is combinatorial, $x \in Q\langle S \cup \{s\} \rangle_M = \langle S \cup \{s\} \rangle \subset \langle \delta S \rangle = \partial_i(S)$. \qed

### 4.2. Cell-like maps.

In order to prove our higher dimensional results in Section 5.3 we need to make sure that $Q$ is a cellular mapping in the following sense.

**Definition 4.4** (Cellular and cell-like). A compact subset $K$ of an $n$-manifold $M$ is cellular in $M$ if $K$ is the intersection of a properly nested decreasing sequence of closed $n$-cells in $M$, i.e., if there is a sequence $(C_i)_{i \geq 1}$ of $n$-cells such that $C_{i+1} \subset \text{int}(C_i)$ and $K = \bigcap_{i \geq 1} C_i$. A space $X$ is cell-like if there is an embedding $i$ of $X$ in a manifold $M$ such that $i(X)$ is cellular in $M$. A mapping is cellular or cell-like if its point preimages are cellular or cell-like, respectively.

A simple diagonalization argument gives the following lemma.

**Lemma 4.5.** A set is cellular if it is the intersection of a properly nested decreasing sequence of cellular sets.

To formulate the result on the cellularity of $Q$ we need one more definition.

**Definition 4.6** (Boundary star). Let $M$ be a monotone model for a self-affine $\mathbb{Z}^n$-tile $T$. For $y \in \partial M$ and $k \in \mathbb{N}$ define the boundary star for $y$ of level $k$ by
\[
\partial\text{-}\text{star}_k(y) = \bigcup_{w \in \partial W} w_{M,k}.\]

One checks that $\partial\text{-}\text{star}_k(y)$ is the closed star of $y$ in the complex $\partial M$ induced by the sets $A^{-k}(S)_M$. The following lemma contains a more convenient representation for boundary stars.

**Lemma 4.7.** If $(M,F)$ is a monotone model for the self-affine $\mathbb{Z}^n$-tile $T$ satisfying $K(M) = K(T)$ and $y \in \partial M$ then
\[
\partial\text{-}\text{star}_k(y) = \bigcup_{w \in \partial W \atop Qy = w_T} w_{M,k}.\]

(4.8)
Theorem 5.1. Thus for each we have $y$ with $H \subseteq G$ be a monotone upper semi-continuous decomposition of $M$. By the generalized set equation for models in (3.22), we may choose $S_j \in P(S_{j-1})$ such that $y \in (F^{-1}(i)|S_j)^M$. Then $w' = (S_i)$ satisfies $\pi_k(w') = \pi_k(w)$ and, hence, $w'_{M,k} = w_{M,k}$. Moreover, we have $y \in w'_{M}$ by the definition of $w'_{M}$ and $Q_y \in w'_T \cap P$ by Proposition 4.1 (iii).

Proposition 4.9. Let $(M, F)$ be a monotone model for the self-affine $\mathbb{Z}^n$-tile $T$. Assume that almost all boundary stars of $M$ are cellular and that $\partial M$ is an $(n-1)$-manifold. Then $Q_{\partial M}$ is a cellular map.

Proof. We have to show that $(Q)(\partial M)^{-1}(x)$ is cellular for $x \in \partial T$. Let $P = \{w \in \partial W \mid w_T = x\}$ and thus by Proposition 4.1 (iii) we have $P_M = (Q)(\partial M)^{-1}(x)$. Now, $P_M = \cap_{k \geq 1} P_{M,k}$, with $P_{M,k} = \bigcup_{w \in P} (F^{-1}(n)(\pi_k(w)))_M$, is the intersection of a nested sequence.

Suppose that $P_M \cap \partial P_{M,k} \neq \emptyset$ for some $j \geq 0$. Since $(P_{M,k})$ is a nested sequence containing $P_M$, we have that $P_M \cap \partial P_{M,k} \neq \emptyset$ for each $k \geq j$. As $\partial W_{M,k}$ covers $\partial M$ for each fixed $k$ by the set equation for $\langle S \rangle$ in (3.22), there is a walk $w \in \partial W$ such that $Qw \neq x$ but $P_M \cap w_{M,k} \neq \emptyset$ holds for each $k \geq j$. Thus $P_M \cap w_{M,k} \neq \emptyset$, which implies that there is some $w \in P$ satisfying $w_{M,k} \neq \emptyset$. However, by Lemma 4.2 this yields that $Qw_M = w_T = w_T = x$, a contradiction. Thus for each $k \geq j$ we have $P_M \cap \partial Qw_{M,k}$ (where the interior is taken relative to $\partial M$) and we may choose a properly nested subsequence of $(P_{M,k})$.

In view of Lemma 4.5 it remains to prove that $P_{M,k}$ is cellular for large $k$. To this end let $y \in (Q)(\partial M)^{-1}(x)$. Then Lemma 4.7 implies that

$$P_{M,k} = \bigcap_{w \in P} w_{M,k} = \bigcup_{w \in W} w_{M,k} \cap \partial T \cap \star_k(y)$$

and $P_{M,k}$ is cellular for large $k$ by the assumption that almost all boundary stars are cellular. □

5. **Self-affine $\mathbb{Z}^n$-tiles whose boundaries are manifolds**

5.1. **The planar case.** We start this section with an easy criterion for a self-affine $\mathbb{Z}^2$-tile $T \subset \mathbb{R}^2$ to be homeomorphic to a closed disk $D^2$.

**Theorem 5.1.** A self-affine $\mathbb{Z}^2$-tile $T$ admits a monotone model which is homeomorphic to $D^2$ if and only if $T$ is homeomorphic to $D^2$.

Proof. If $T$ admits a monotone model which is homeomorphic to $D^2$, Proposition 4.3 (iii) implies that $\partial T$ is a monotone quotient of $S^1$ and thus is either a singleton or homeomorphic to $S^1$. As $T$ is the closure of its interior, $\partial T$ cannot be a singleton and Jordan’s Curve Theorem implies that $T \cong D^2$. For the converse just observe that $(T, A)$ is a monotone model for $T$ (it is not hard to check that $\langle T \rangle$ is always connected for a self-affine tile $T$ which is homeomorphic to $D^2$). □

It is known that a $\mathbb{Z}^2$-tile $M$ which is a closed disk has either 6 or 8 “neighbors” (i.e., $K(M)^2_0$ has either 6 or 8 elements; see e.g. [3, Lemma 5.1]). This makes it easy to find a suitable ideal tile $M$ to apply Theorem 5.1. For a similar criterion we refer to [5, Theorems 2.1 and 2.2].

5.2. **Surface boundaries of self-affine $\mathbb{Z}^3$-tiles.** In this section we will prove our results on self-affine $\mathbb{Z}^3$-tiles with surface boundary stated in Theorems 2.14 and 2.16. To this end we will use the following result on monotone upper semi-continuous decompositions of 2-manifolds that was proved by Roberts and Steenrod [44] generalizing a theorem of Moore [43].

**Proposition 5.2** (cf. [44, Theorem 1]). Let $S$ be a compact 2-manifold without boundary and let $G$ be a monotone upper semi-continuous decomposition of $S$. If $G$ contains at least 2 elements and $H_1(g; \mathbb{Z}_2) = 0$ for each $g \in G$, then $G$ is homeomorphic to $S$.

The following easy lemma, which will be needed on several occasions, starts a list of preparatory results.
Lemma 5.3. Let $M \subset \mathbb{R}^n$ be compact and equal to the closure of its interior. Assume that $U$ is a nonempty bounded component of $\mathbb{R}^n \setminus M$. Then $\text{int}(M + a) \not\subset U$ holds for each $a \in \mathbb{R}^n$.

Proof. As $\text{int}(M) \neq \emptyset$ the result is true for $a = 0$, and we may assume that $a \neq 0$. If the assertion was wrong we had $\partial(U + a) \subset M + a \subset U$ which is absurd for a bounded set $U$. □

Lemma 5.4. Let $M$ be a $\mathbb{Z}^n$-tile with $\text{int}(M)$ connected. Then $\mathbb{R}^n \setminus M$ is connected.

Proof. If this is wrong, $M$ separates $\mathbb{R}^n$ and we may choose a bounded complementary component $U$ of $M$. As $\text{int}(M)$ is connected and $M$ tiles $\mathbb{R}^n$ with $\mathbb{Z}^n$-translates, there exists $s \in \mathbb{Z}^n \setminus \{0\}$ such that $\text{int}(M + s) \subset U$. This contradicts Lemma 5.3. □

The next two propositions are of interest in their own right.

Proposition 5.5. Let $M \subset \mathbb{R}^n$ be a $\mathbb{Z}^n$-tile. If $\text{int}(M)$ is connected then no proper subset of $\partial M$ separates $\mathbb{R}^n$.

Proof. By Lemma 5.4 $\partial M$ is the common boundary of the components $\text{int}(M)$ and $\mathbb{R}^n \setminus M$ of its complement. By a classical result this implies that $\partial M$ is an irreducible separator, i.e., that none of its subsets separates $\mathbb{R}^n$ (see e.g. [32, §46, VII, Theorem 4]). □

Proposition 5.6. Let $M \subset \mathbb{R}^n$ be a $\mathbb{Z}^n$-tile. If $\text{int}(M)$ is connected then $\partial M$ has no cut point.

Proof. For $n = 1$ the result is trivial, for $n = 2$ it follows from [39, Theorem 1.1 (iii)]. To prove the result for $n \geq 3$ assume on the contrary that there is a singleton $\{x\}$ that separates $\partial M$. Then there are nonempty compact proper subsets $A, B \subset \partial M$ satisfying $\partial M = A \cup B$ with $A \cap B = \{x\}$. We will now use the Mayer-Vietoris-sequence for Čech cohomology groups (see [42, p. 67]) in order to prove our result. Since $A, B$ are closed subsets of $\mathbb{R}^n$ a part of the Mayer-Vietoris sequence reads

\[
\cdots \rightarrow H^{n-2}(A \cap B) \rightarrow H^{n-1}(A \cup B) \rightarrow H^{n-1}(A) \oplus H^{n-1}(B) \rightarrow \cdots.
\]

As $A$ and $B$ satisfy the conditions of [42, Theorem 3.13] we conclude that the sequence in (5.7) is exact. We apply Alexander’s Duality Theorem (see e.g. Dold [18, Chapter VIII, 8.15]) together with standard results from singular homology to derive that

\[
\begin{align*}
H^{n-2}(A \cap B) &= H_1(\mathbb{R}^n \setminus (A \cap B)) = 0 \quad \text{(as } A \cap B = \{x\}, \text{ a singleton)},
H^{n-1}(A \cup B) &= H_0(\mathbb{R}^n \setminus (A \cup B)) = \mathbb{Z} \quad \text{(as } \mathbb{R}^n \setminus (A \cup B) = \mathbb{R}^n \setminus \partial M \text{ has 2 components by Lemma 5.4),}
H^{n-1}(A) &= H_0(\mathbb{R}^n \setminus A) = 0 \quad \text{(as } \mathbb{R}^n \setminus A \text{ has one component by Proposition 5.5),}
H^{n-1}(B) &= H_0(\mathbb{R}^n \setminus B) = 0 \quad \text{(as } \mathbb{R}^n \setminus B \text{ has one component by Proposition 5.5).}
\end{align*}
\]

Inserting this in (5.7) yields that the sequence $\cdots \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \cdots$ is exact, which is absurd. This yields the desired contradiction and the result is proved. □

We are now in a position to give the proof of Theorems 2.14 and 2.16.

Proof of Theorem 2.14. As $M$ is a monotone model for $T$, by Theorem 3.8 the canonical quotient map $Q$ in (2.3) is well-defined and maps $M$ onto $T$. In view of Proposition 5.2 to prove (i) it suffices to prove that for each $x \in \partial T$ the preimage $(Q|_{\partial M})^{-1}(x)$ is connected and satisfies $H_1((Q|_{\partial M})^{-1}(x); \mathbb{Z}_2) = 0$ (note that $\partial T$ is not a singleton as $T$ is the closure of its interior). Connectivity of $(Q|_{\partial M})^{-1}(x)$ is a consequence of Proposition 4.3 (iii). Since $M$ is semi-contractible, to prove $H_1((Q|_{\partial M})^{-1}(x); \mathbb{Z}_2) = 0$ it is sufficient to show (ii).

To prove (ii) assume on the contrary that $(Q|_{\partial M})^{-1}(x)$ path separates $\partial M$ between two elements $u, v \in \partial M$. We first observe that being the continuous image of the locally connected continuum $\partial M$ (by Theorem 3.8 (ii)), the set $\partial T$ is a locally connected continuum. Moreover, as $\text{int}(T)$ is connected, Proposition 5.6 implies that $\partial T$ has no cut point. Thus $\partial T \setminus \{x\}$ is arcwise connected (see e.g. [32, §52, II, Theorem 16]) and we may connect $Q(u)$ and $Q(v)$ by a path $\ell$ in $\partial T \setminus \{x\}$. Since $(Q|_{\partial M})^{-1}(y)$ is connected for each $y \in \partial T$ it is easy to see that the preimage of a continuum is a continuum. In particular, $(Q|_{\partial M})^{-1}(\ell)$ is a continuum which contains $u$ and $v$ and is disjoint from $(Q|_{\partial M})^{-1}(x)$. Thus $u$ and $v$ are $\varepsilon$-chain connected in $(Q|_{\partial M})^{-1}(\ell)$ with an $\varepsilon$ that is so small that (in view of the fact that $\partial M$ is a locally arcwise connected continuum) we can construct an arc that connects $u$ and $v$ in $\partial M$ avoiding $(Q|_{\partial M})^{-1}(x)$. This contradicts our assumption. □
As mentioned in the introduction, there is a version of Theorem 2.14 for finite unions of \( \mathbb{Z}^3 \)-translates of \( T \). Before we make this precise, for a \( \mathbb{Z}^n \)-tile \( M \) we set
\[
[S]_M = \bigcup_{s \in S} (M + s) \quad (S \subset M).
\]
Again, we write \([S]\) instead of \([S]_T\) if \( T \) is a self-affine \( \mathbb{Z}^n \)-tile.

**Proposition 5.9.** Let \( T \) be a self-affine \( \mathbb{Z}^3 \)-tile which admits a semi-contractible monotone model \( M \) and let \( S \subset \mathbb{Z}^n \) be nonempty and finite. Assume that \( \text{int}([S]) \) and \( \mathbb{R}^3 \setminus [S] \) are connected. If \( \partial [S]_M \cong S \) for a closed surface \( S \) then also \( \partial [S] \cong S \).

**Proof.** As \( \partial [S]_M \) is the boundary of the components \( \text{int}([S]) \) and \( \mathbb{R}^3 \setminus [S] \), no proper subset of \( \partial [S]_M \) separates \( \mathbb{R}^n \) (see [32, §46, VII, Theorem 4]). Imitating the proof of Proposition 5.6 we see that this implies that \( \partial [S] \) has no cut point. Thus the result follows by the same proof as the one of Theorem 2.14 (just note that Proposition 4.3 can be extended immediately to show that \( \partial [S] \) is a monotone quotient of \( \partial [S]_M \)). \( \Box \)

**Proof of Theorem 2.16.** Let \( S \in \mathcal{K}(T) \) be nondegenerate and assume, without loss of generality, that \( 0 \in S \). Assume that \( (S)_M \) is a closed manifold or a ball inside of the surface \( \partial M \cong S \). Thus \( (S)_M \) is either a closed surface, a closed disk, a circle, an arc, or a point. We have to show that \( (S)_M \) has the required properties. To this end we will use the fact that \( (S)_M = Q(S)_M \) is a monotone quotient of \( (S)_M \) by Proposition 4.3 (i).

We first dispose off the easy cases. If \( (S)_M \) is a point, observing that the monotone quotient of a point is a point, we gain that \( (S)_M = Q(S)_M \) is a point. Similarly, if \( (S)_M \) is an arc, the fact that the monotone quotient of an arc is either an arc or a point yields that \( (S)_M \) is an arc or a point. The case where \( (S)_M \) is a circle is also settled because it is well-known that the monotone quotient of a circle is a circle or a point. Finally, if \( (S)_M \) is a closed surface, we have \( (S)_M = S \) as a closed surface has no proper subsurface. Thus Theorem 2.14 (i) yields that \( (S)_M \cong S \).

It remains to consider the case where \( (S)_M \) is a closed disk. Since \( M \) is combinatorial, this implies that \( |S| = 2 \) and \( \partial_2(S)_M = \{\delta S\}_M \). Thus, as \( (\mathcal{K}(M))_{\partial_2} \cong \partial M \cong S \), the boundary \( \partial_2(S)_M \) is a circle and by Proposition 4.3 (ii) the canonical quotient \( Q\partial_2(S)_M \) of \( \partial_2(S)_M \) is monotone. Therefore, \( Q\partial_2(S)_M \) is either a point or a circle. We treat these alternatives separately.

Assume first that \( Q\partial_2(S)_M \) is a point \( x \). By Theorem 2.14 (ii), the preimage \( (Q|_{\partial M})^{-1}(x) \) cannot path separate \( M \). As this preimage contains the (path separating) circle \( \partial_2(S)_M \), it therefore has to contain one of the two complementary components of \( \partial_2(S)_M \). Consequently, either \( (S)_M \subset (Q|_{\partial M})^{-1}(x) \) or \( \partial M \setminus (S)_M \subset (Q|_{\partial M})^{-1}(x) \). If \( (S)_M \subset (Q|_{\partial M})^{-1}(x) \) we have \( Q(S)_M = (S)_M = \{x\} \) and we are done. If, on the other hand, \( \partial M \setminus (S)_M \subset (Q|_{\partial M})^{-1}(x) \) we gain \( (S) = Q((S)_M) = \{\delta M\} = \partial T \). As \( S = \{0, s\} \) for some \( s \in \mathbb{Z}^3 \setminus \{0\} \) this implies that \( \partial T = (S) \subset \partial(T + s) \), thus \( \text{int}(T) \) is a bounded component of \( \mathbb{R}^n \setminus \partial(T + s) \). This contradicts Lemma 5.3, this situation does not come up.

Now assume that \( Q\partial_2(S)_M \) is a circle. As the monotone image of a disk cannot be a circle (see e.g. [13, I, §4, Exercise 9]), observe that a disk is unicoherent but a circle is not) \( Q(S)_M \setminus Q\partial_2(S)_M \neq \emptyset \). Since \( M \) is combinatorial, Theorem 2.12 (i) implies that \( \partial_2(S)_M \subset Q\partial_2(S)_M \), hence, \( (S)_M \setminus \partial_2(S)_M \neq \emptyset \). As \( (S)_M \) is a proper subset of \( \partial T \) (otherwise we get a contradiction to Lemma 5.3 as above) we conclude that \( \partial_2(S)_M \) is a subset of the circle \( Q\partial_2(S)_M \) which separates \( \partial T \cong S \). This implies that \( \partial_2(S)_M = Q\partial_2(S)_M \). Recall that a circle in a closed surface is nulhomotopic if and only if it bounds two components, at least one of which is a disk. Since \( \partial_2(S)_M \) bounds \( (S)_M \), \( \partial_2(S)_M \) is nulhomotopic, and thus its canonical image \( \partial_2(S) \) is a nulhomotopic circle in the surface \( \partial T \) which must itself bound a disk. Any component of the complement of \( \partial_2(S)_M \) which is a disk must map to a complementary component of \( \partial_2(S) \) which is a disk under \( Q \). Thus, as \( (S)_M \) is a disk, also \( (S)_M = Q(S)_M \) is a disk. \( \Box \)

To check that \( \text{int}(T) \) is connected, the following easy criterion is often applicable (cf. [2, Proposition 13.1]).

**Lemma 5.10.** Let \( T = T(A, D) \) be a self-affine \( \mathbb{Z}^n \)-tile. If there is a connected set \( E \subset \text{int}(T) \) such that \( E \cap A^{-1}(E + d) \neq \emptyset \) holds for each \( d \in D \) then \( \text{int}(T) \) is connected.
Proof. We note that \( \text{int}(T) = \bigcup_{k \in \mathbb{N}} \left( \bigcup_{d \in \mathcal{D}_k} A^{-1}(E + d) \right) \) with \( \mathcal{D}_k \) as in (3.5) is a nested union of open sets each of which is, by induction, connected. \( \square \)

5.3. Manifolds and the disjoint disks property. In the present section we present generalizations of Theorem 2.14 to higher dimensions. In this setting we are able to give a checkable criterion for the boundary of a self-affine \( \mathbb{Z}^n \)-tile \( T \) to be a generalized manifold. To make sure that \( \partial T \) is actually a manifold (for dimension \( n \geq 6 \)), according to the work of Cannon [10], one has to assume the disjoint disks property.

Definition 5.11 (Generalized \( n \)-manifold; see e.g. [10]). A space \( X \) is a generalized \( n \)-manifold if it has the following properties.

- \( X \) is a Euclidean neighborhood retract (ENR), i.e., for some integer \( n \) it embeds in \( \mathbb{R}^n \) as a retract of an open subset of \( \mathbb{R}^n \).
- \( X \) is a homology \( n \)-manifold, i.e., \( H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \) for each \( x \in X \).

A generalized \( n \)-manifold is called resolvable if it is a proper cell-like upper semi-continuous decomposition of an \( n \)-manifold.

Theorem 5.12. Let \( T \) be a self-affine \( \mathbb{Z}^n \)-tile which admits a monotone model \( M \). Assume that almost all boundary stars of \( M \) are cellular and \( \partial M \) is a manifold. Then \( \partial T \) is a generalized \((n-1)\)-manifold with \( Q|_{\partial M} : \partial M \to \partial T \) a cellular quotient map from the manifold \( \partial M \). In other words, \( \partial M \) is a cell-like resolution of the generalized manifold \( \partial T \).

Proof. By assumption, \( M \) satisfies the conditions of Proposition 4.9 and thus \( Q|_{\partial M} : \partial M \to \partial T \) is a cellular quotient map. Now recall that Lacher [33, (11.2) Corollary] implies that a cell-like image of a compact manifold is an ENR, and [16, Proposition 8.5.1] states that every \( n \)-dimensional resolvable space is an \( n \)-dimensional homology manifold. This implies the result. \( \square \)

Remark 5.13. A priori there are infinitely many boundary stars to check for cellularity. However, as will be explained in Remark 7.21, it suffices to check only one representative of finitely many “equivalence classes” of boundary stars.

To make the step from a generalized manifold to a topological manifold, we need the well-known disjoint disks property (cf. [9]).

Definition 5.14 (Disjoint disks property). A metric space \((X,d)\) has the disjoint disks property if for every pair of maps \( g_1, g_2 : D^2 \to X \) and every \( \varepsilon > 0 \) there exist maps \( g'_1, g'_2 : D^2 \to X \) such that \( \max\{d(g_1, g'_1), d(g_2, g'_2)\} < \varepsilon \) and \( g'_1(D^2) \cap g'_2(D^2) = \emptyset \).

Generalizing Theorem 2.14 this allows us to state a result on self-affine \( \mathbb{Z}^n \)-tiles of dimension \( \geq 6 \) whose boundary is a manifold.

Theorem 5.15. For \( n \geq 6 \) let \( T \) be a self-affine \( \mathbb{Z}^n \)-tile which admits a monotone model \( M \). Assume that almost all boundary stars of \( M \) are cellular and \( \partial M \) is a manifold. If \( T \) satisfies the disjoint disks property then \( \partial T \) is an \((n-1)\)-manifold.

Proof. Theorem 5.12 yields that \( \partial T \) is a resolvable generalized \((n-1)\)-manifold. As we assume that \( \partial T \) satisfies the disjoint disks property, Edwards’ Cell-like Approximation Theorem (cf. [19]) implies that \( \partial T \) is a manifold (see also [13] for \( n-1 > 5 \) and [14] for \( n-1 = 5 \)). \( \square \)

For dimensions less than 5 the disjoint disks property is not suited to detect manifolds, so we cannot use it for boundaries of self-affine tiles of dimension \( n < 6 \). In Daverman and Repovš [15] alternatives for the disjoint disks property for 3-manifolds are proposed; for 4-manifolds no such alternatives seem to be known so far.

6. Ideal tiles

6.1. Ideal tiles and monotone models. The first aim of this section is to prove Theorem 2.23 which states that each ideal tile is a monotone model up to translation. To this end we need to prove (2.21) which is contained in the following lemma.
Lemma 6.1. Let $T = T(A, D)$ be a self-affine $\mathbb{Z}^n$-tile. For each $S \in \mathcal{K}(T)$ we have $\delta P(S) = P(\delta S)$.

Proof. Since $D$ is a complete set of residue classes of $\mathbb{Z}^n/A\mathbb{Z}^n$ we have

$$P(\delta S) = \{(p + A)(\delta S) \in \mathcal{K}(T) \mid p \in D^{\delta S}\}$$

$$= \bigcup_{s \in \mathbb{Z}^n \setminus S} \{(p + A)(S \cup \{s\}) \in \mathcal{K}(T) \mid p \in D^{S \cup \{s\}}\}$$

$$= \bigcup_{s \in \mathbb{Z}^n \setminus S} \bigcup_{d \in D} \{(p + A)(S \cup \{As + d\}) \in \mathcal{K}(T) \mid p \in D^{S}\}$$

$$= \bigcup_{s' \in \mathbb{Z}^n \setminus (AS + D)} \{(p + A)(S \cup \{s'\}) \in \mathcal{K}(T) \mid p \in D^{S}\}$$

$$= \delta(P(S)).$$

We proceed with the proof of the theorem.

Proof of Theorem 2.23. By Definition 2.22 each translate of $Z$ is again an ideal tile of $T$, hence, $M = Z - u$ is an ideal tile of $T$. To prove the theorem we have to construct a $\mathbb{Z}^n$-equivariant homeomorphism $f$ fixing 0 so that $F = Af$ satisfies $FM = M + D$. Let

$$C_{-1} = \{S \subset \mathbb{Z}^n \mid \langle S \rangle_M = \emptyset\},$$

and inductively set

$$C_{\ell} = \{S \in \mathcal{K}(M) \mid \delta S \subset C_{\ell-1}\} \quad (\ell \geq 0).$$

Note that by compactness of $M$, $S \in C_{\ell}$ for $\ell \geq 0$ implies that $S$ is finite.

Now assume that the homeomorphism $f$ has been defined $\mathbb{Z}^n$-equivariantly on $\langle C_{\ell-1} \rangle_M$, that is, $f(\langle S \rangle_M + z) = f(\langle S \rangle_M) + z$ for $z \in \mathbb{Z}^n$, and $S \in C_{\ell-1}$ such that $f(\langle S \rangle_M) = A^{-1}(P(S))_M$.

Choose a maximal set $E_\ell$ of pairwise inequivalent elements of $C_\ell \setminus C_{\ell-1}$, that is, $E_\ell$ contains precisely one element of $S + \mathbb{Z}^n$ for each $S \in C_\ell \setminus C_{\ell-1}$.

Let $S \in E_\ell$. By definition, $\delta S$ is a union of cells of $C_{\ell-1}$. The induction hypothesis yields that each cell in $\langle \delta S \rangle_M$ is mapped homeomorphically to the according cell in $A^{-1}(P(\delta S))_M$. Thus $f(\langle \delta S \rangle_M)$ is a $\mathbb{Z}^n$-equivariant homeomorphism onto $A^{-1}(P(\delta S))_M = A^{-1}(P(S))_M$ (the equality follows from Lemma 6.1). Now $f$ is defined from $\langle \delta S \rangle_M$ to $A^{-1}(P(S))_M$ in an appropriate way.

By Condition (iii) of the definition of the ideal tile $M$ we may extend the domain of $f$ to $\langle S \rangle_M$ in a way that $f$ maps $\langle S \rangle_M$ homeomorphically to $A^{-1}(P(S))_M$.

One now easily checks that $f$ has the desired properties on $\langle S \rangle_M$. Doing this for each of the finitely many elements of $E_\ell$ and extending $f$ equivariantly to $\langle C_\ell \rangle_M$ concludes the induction step.

At the end of this construction we arrive at $\langle S \rangle_M = \{\emptyset\}_M = M$ and $fM = A^{-1}(M + D)$ which implies that $FM = AFM = M + D$. In this case we need to make sure that 0 is fixed by $f$. However, as $0 \in \text{int}(M)$ and, as $0 \in D$, therefore also $0 \in \text{int}(A^{-1}(M + D)) = \text{int}(\langle P(S) \rangle_M)$, this can be achieved by the connectivity of $\text{int}(M)$.

The following lemma is useful because it can be used to check Property (iii) of the definition of ideal tile (which is Definition 2.22) in nonpathological examples.

Lemma 6.2. Let $X$ be a contractible union of cones over spheres which pairwise intersect only at single points on their boundaries. The boundary of $X$ is the union of the bases of the cones which comprise it. Then $X$ has the property that any self-homeomorphism of its boundary can be extended to a self-homeomorphism of $X$.

Proof. Let $C_i$ be the cones which comprise $X$. Suppose $\psi : \partial X \to \partial X$ is a homeomorphism. Then, by considering cut points in the boundary, we see that $\psi|_{\partial C_i}$ is a homeomorphism onto $\partial C_i$ for some unique choice of $i'$. Coning the map $\psi|_{\partial C_i}$, we obtain an extension homeomorphism $\tilde{\psi}|_{C_i} : C_i \to C_i$ satisfying $\tilde{\psi}|_{\partial C_i} = \psi|_{\partial C_i}$. By again considering cut points on the boundary, we see that the map $i \to i'$ is a permutation. Since the $C_i$ only meet at single boundary points, the maps $\tilde{\psi}|_{\text{int}(C_i)}$ have disjoint images. Thus the homeomorphisms $\tilde{\psi}|_{C_i}$ patch together to form the desired homeomorphism $\psi$. 

\square
6.2. Checking whether a simplicial complex is a ball. Let $T$ be a self-affine $\mathbb{Z}^n$-tile. In Theorem 2.23 we require that $Z$ is an ideal tile for $T$. Particularly for low dimensions $n$, Condition (ii) of Definition 2.22 can be checked easily by direct inspection. However, in higher dimensions this may be tricky and we are forced to use a systematic approach based on methods from classical algebraic topology. Such an approach is discussed in the present section. Indeed, for many instances, the ideal tile $Z$ can be chosen to be a triangulable complex such that each nonempty set $\langle S \rangle_Z$ is a closed ball that is the closure of a single cell of this complex (see e.g. the examples discussed in Sections 8.1 and 8.2). We shall discuss how one can check Property (ii) of Definition 2.22 in this case.

In particular, we have to check that $\langle P(S) \rangle_Z$ is a ball of the same dimension as $\langle S \rangle_Z$. By translation invariance it suffices to check this for all $S \in \mathcal{K}(Z)$ containing 0. The intersections $\langle S \rangle_Z$ as well as $\langle P(S) \rangle_Z$ can be nonempty only for finitely many of these sets since $Z$ as well as $\langle P(\{0\}) \rangle_Z$ is compact. Thus there are only finitely many instances to check. By definition, $\langle P(S) \rangle_Z$ is a triangulable complex made up of finitely many triangulable complexes of the form $\langle S' \rangle_Z$. We have to check that this simplicial complex is the underlying set of a ball.

As we will need this later on for different complexes, we now switch to a general setting and describe how to check whether a given simplicial complex is the underlying set of a ball. In particular, let $K$ be a simplicial complex and $\sigma$ be one of its cells. Recall that the star of $\sigma$ in $K$, denoted by $\text{st}(\sigma)$, is given by the set of all simplices in $K$ that have $\sigma$ as a face. Moreover, the link of $\sigma$ in $K$, denoted by $\text{lk}(\sigma)$, is given by

$$\text{lk}(\sigma) = \text{cl}(\text{st}(\sigma)) \setminus \text{st}(\text{cl}(\sigma)),$$

where $\text{cl}(X)$ is the smallest subcomplex of $K$ that contains each simplex in $X$.

As we first have to check that $K$ is a manifold, we need the following result (cf. [16, Theorem 8.10.2]; note that this is a consequence of Cannon’s Double Suspension Theorem, see [9]).

**Lemma 6.3.** A simplicial complex $K$ is the underlying set of a topological n-manifold if and only if, for each k-simplex $\sigma \in K$, $\text{lk}(\sigma)$ has the homology of $S^{n-k-1}$ (i.e., $H_i(\text{lk}(\sigma); \mathbb{Z}) = 0$ for $i \neq n-k-1$ and $H_{n-k-1}(\text{lk}(\sigma); \mathbb{Z}) = \mathbb{Z}$) and, for each vertex $v \in K$, $\text{lk}(v)$ is simply connected.

We now give an outline on how to check that $K$ is a ball. For all classical theorems from algebraic topology we are using here, we refer the reader e.g. to Hatcher [28]: (i) we need to check that the simplicial complex $K$ is a manifold. By Lemma 6.3 it suffices to calculate homology groups and the fundamental group of certain links. This can be done by using the Mayer-Vietoris-Sequence and the Seifert-van Kampen Theorem, respectively. (ii) We check that $K$ has trivial homology groups and trivial fundamental group. Again we use the Mayer-Vietoris-Sequence and the Seifert-van Kampen Theorem. (iii) The Theorems of Hurewicz and Whitehead now imply that $K$ is homotopy equivalent to a ball. (iv) The generalized Poincaré Theorem proved by Smale [47], Freedman [21], and Perelman (see [11]) yields that $K$ is a ball.

Summing up, checking that $K$ is a ball of appropriate dimension is achieved by calculating homology groups and fundamental groups of simplicial complexes.

7. Self-affine $\mathbb{Z}^n$-tiles that are homeomorphic to a ball

Let $T$ be a self-affine $\mathbb{Z}^n$-tile. In Theorems 2.14 and 5.15 we gave criteria for $\partial T$ to be homeomorphic to a manifold. In the present section we shall assume that $\partial T$ is homeomorphic to $\mathbb{S}^{n-1}$ and give criteria under which this implies that the tile $T$ itself is homeomorphic to the closed $n$-dimensional disk $\mathbb{D}^n$, i.e., that $\partial T$ is tamely embedded in $\mathbb{R}^n$.

7.1. Cannon’s criterion and fundamental neighborhoods. We start with some terminology.

**Definition 7.1 (1-LCC).** A set $X \subseteq \mathbb{R}^n$ is said to be 1-LCC if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each loop of diameter less than $\delta$ in $\mathbb{R}^n \setminus X$ can be contracted to a point in a subset of $\mathbb{R}^n \setminus X$ of diameter less than $\varepsilon$.

**Definition 7.2 (Locally spherical).** An $(n-1)$-sphere $S \subseteq \mathbb{R}^n$ is said to be locally spherical if each $p \in S$ has a neighborhood basis $\{U_m \mid m \in \mathbb{N}\}$ such that $\partial U_m \cong S^{n-1}$ and $\partial U_m \setminus S$ is simply connected.
If \( n = 3 \) the simple connectivity of \( \partial U_1 \setminus S \) is equivalent to the fact that \( \partial U_1 \cap S \) is connected. Cannon’s criterion now reads as follows.

**Proposition 7.3** (cf. [8, 5.1 Theorem]). If \( S \) is an \((n - 1)\)-sphere in \( \mathbb{R}^n \) that is locally spherical then \( S \) is 1-LCC.

We combine this criterion with the following result.

**Proposition 7.4.** If an \((n - 1)\)-sphere \( S \) in \( \mathbb{R}^n \) is 1-LCC then \( S \) is tamely embedded.

Bing [7] proved this result for \( n = 3 \), for \( n = 4 \) it is proved by Freedman and Quinn [22], and for \( n \geq 5 \) it is due to Daverman (see [16, Theorem 7.6.5]).

We will use the next corollary, which is just a combination of Propositions 7.3 and 7.4.

**Corollary 7.5.** If \( S \) is an \((n - 1)\)-sphere in \( \mathbb{R}^n \) that is locally spherical then \( S \) is tamely embedded in \( \mathbb{R}^n \).

The self-affine structure of \( T \) produces natural candidates for neighborhood bases. To make this more precise we first recall from (5.8) that \( [S] = \bigcup_{s \in S}(T + s) \) for \( S \subset \mathbb{Z}^n \).

**Definition 7.6** (Fundamental neighborhood). Let \( T \) be a self-affine \( \mathbb{Z}^n \)-tile and let \( S \subset \mathbb{Z}^n \) be given in a way that

\[(7.7) \quad \langle S \rangle \neq \emptyset \quad \text{and} \quad \langle S \cup \{s\} \rangle = \emptyset\]

holds for each \( s \in \mathbb{Z}^n \setminus S \). In this case we call the union \([S]\) the fundamental neighborhood of \( \langle S \rangle \).

The following lemma is an immediate consequence of this definition.

**Lemma 7.8.** Let \( T \) be a self-affine \( \mathbb{Z}^n \)-tile. The set \( \mathcal{A} = \{ A^{-k}L \mid L \text{ a fundamental neighborhood} \} \) forms a basis for the topology of \( \mathbb{R}^n \). In particular, each \( x \in \partial T \) admits a neighborhood basis made up of elements of \( \mathcal{A} \).

If \( N \in \mathcal{A} \) is given in a way that \( A^{k_i}N \) is a fundamental neighborhood we say that \( N \) is of level \( k \) and write \( \text{level}(N) = k \). Lemma 7.8 implies that the set

\[\mathcal{B} = \{ N \in \mathcal{A} \mid \text{int}(N) \cap \partial T \neq \emptyset \}\]

contains a neighborhood basis for each \( x \in \partial T \). Our aim is to provide an algorithm that allows to check whether this neighborhood basis can always be chosen in a way that it meets the conditions of Corollary 7.5. To this matter we define the following equivalence relation on \( \mathcal{B} \).

**Definition 7.9** (Equivalent neighborhoods). Let \( T \) be a self-affine \( \mathbb{Z}^n \)-tile, let \( N_1, N_2 \in \mathcal{B} \) be given and set \( k_i = \text{level}(N_i) \) for \( i \in \{1, 2\} \). If there exists \( u \in \mathbb{Z}^n \) such that

\[A^{k_1}N_1 = A^{k_2}N_2 + u \quad \text{and} \quad A^{k_1}(N_1 \cap \partial T) = A^{k_2}(N_2 \cap \partial T) + u\]

we say that \( N_1 \) is equivalent to \( N_2 \). In this case we write \( N_1 \sim N_2 \).

If \( N \in \mathcal{B} \) with \( \text{level}(N) \geq 1 \) is given, we often need a larger neighborhood in \( \mathcal{B} \) that contains \( N \). To this matter we define

\[\text{Parents}(N) = \{ N' \in \mathcal{B} \mid N \subset N' \text{ and } \text{level}(N') = \text{level}(N) - 1 \}\]

We need the following result.

**Lemma 7.10.** If \( N \in \mathcal{B} \) with \( \text{level}(N) \geq 1 \) then \( \text{Parents}(N) \neq \emptyset \).

**Proof.** Let \( k = \text{level}(N) \) and choose \( S \) in a way that \( N = A^{-k}[S] \). For each \( s \in \mathbb{Z}^d \) there is a unique \( s'(s) \in \mathbb{Z}^d \) such that \( A^{-k}(T + s) \) is contained in \( A^{-(k-1)}(T + s') \) (see Section 3.3). Let \( S' = \{ s'(s) \mid s \in S \} \). If \( S' \) satisfies the maximality condition in (7.7) we are done, if not, successively add elements of \( \mathbb{Z}^d \) to \( S' \) until it satisfies this condition. Since \( \mathbb{Z}^n \) is discrete and \( T \) is compact at most finitely many elements can be added.

\[\square\]
7.2. The in-out graph. Using the relation “Parents” we can define an infinite directed graph $\mathcal{I}$ whose nodes are the elements of $\mathcal{B}$ and whose edges are defined by

$$N \to N' \iff N' \in \text{Parents}(N).$$

The following lemma shows that the Parent relation depends only on the equivalence class of the edges $N$ with respect to the equivalence relation defined in Definition 7.9.

Lemma 7.11. Let $N'_1, N'_2 \in \mathcal{B}$ with $N'_1 \sim N'_2$. If $N_1 \to N'_1$ is an edge in $\mathcal{I}$ then there is $N_2 \in \mathcal{B}$ such that $N_1 \sim N_2$ and $N_2 \to N'_2$ is an edge in $\mathcal{I}$.

Proof. Let $k_i := \text{level}(N'_i)$. As $N'_1 \sim N'_2$ there is $u \in \mathbb{Z}^d$ such that

\begin{align}
(7.12) & \quad A^{k_1}N'_1 = A^{k_2}N'_2 + u, \\
(7.13) & \quad A^{k_1}(N'_1 \cap \partial T) = A^{k_2}(N'_2 \cap \partial T) + u.
\end{align}

We will show that $N_2 := A^{-k_2}(A^{k_1}N_1 - u)$ satisfies the requirements of our lemma. It is clear that $N_2 \in \mathcal{A}$ with level$(N_2) = k_2 + 1$. Moreover, as $N_1 \subset N'_1$ equation (7.12) implies that $N_2 \subset N'_2$. It remains to show that $N_2 \in \mathcal{B}$ and $N_1 \sim N_2$. To this matter observe that (7.13) yields

$$A^{k_1}(N_1 \cap \partial T) = A^{k_2}(N_1 \cap N'_1 \cap \partial T)$$

$$= A^{k_1}N_1 \cap A^{k_1}(N'_1 \cap \partial T)$$

$$= A^{k_1}N_1 \cap (A^{k_2}(N'_2 \cap \partial T) + u)$$

$$= (A^{k_2}N_2 + u) \cap (A^{k_2}(N'_2 \cap \partial T) + u)$$

$$= A^{k_2}(N_2 \cap N'_2 \cap \partial T) + u$$

$$= A^{k_2}(N_2 \cap \partial T) + u.$$

From this we get the desired properties. \hfill $\Box$

Definition 7.14 (In-out graph). For $N \in \mathcal{B}$ denote by $\overline{N}$ the equivalence class of $N$ with respect to the equivalence relation ‘$\sim$’. The in-out graph is a directed graph $\overline{\mathcal{I}}$ which is defined as follows.

- The nodes of $\overline{\mathcal{I}}$ are the equivalence classes $\{\overline{N} \mid N \in \mathcal{B}\}$.
- There is a directed edge $\overline{N} \to \overline{N'}$ in $\overline{\mathcal{I}}$ if there is an edge $N \to N'$ in $\mathcal{I}$.

Lemma 7.15. The in-out graph $\overline{\mathcal{I}}$ is finite.

Proof. Choose some order on $\mathbb{Z}^n$, set

$$\mathcal{N} = \{s \in \mathbb{Z}^n \mid \{0, s\} \neq \emptyset\},$$

and let $D_k$ be defined as in (3.5). For each finite set $Y \subset \mathbb{Z}^n$ define the functions

$$\alpha(Y) = Y - u, \quad \beta(Y) = (Y + \mathcal{N}) \cap D_k - u, \quad \gamma(Y) = (Y + \mathcal{N}) \cap (\mathbb{Z}^n \setminus D_k) - u,$$

where $u \in Y$ is chosen to be minimal with respect to this order. Let $N = A^{-k}[S]$ be an element of $\mathcal{B}$. As $(\mathcal{S}) \neq \emptyset$ we have $\text{diam}(S) \leq \text{diam}(\{S\}) \leq 2 \text{diam}(T)$. Moreover, as $\{s, s + v\} \neq \emptyset$ holds for each “neighbor” $v \in \mathcal{N}$ we get

$$\text{diam}(S + \mathcal{N}) \leq \text{diam}(\{S + \mathcal{N}\}) \leq 4 \text{diam}(T).$$

Now pick $u \in S$ minimal with respect to the above order of $\mathbb{Z}^d$. Then

$$A^{k}N - u = [S] - u = [\alpha(S)].$$

Moreover, as $A^{k}\partial T = [D_k] \cap [\mathbb{Z}^n \setminus D_k]$ we get $A^{k}(N \cap \partial T) = [S] \cap [D_k] \cap [\mathbb{Z}^n \setminus D_k]$. As $[S] \cap (T + x) = \emptyset$ whenever $x \notin S + \mathcal{N}$, this implies that $A^{k}(N \cap \partial T) = [S] \cap [(S + \mathcal{N}) \cap D_k] \cap (S + \mathcal{N}) \cap (\mathbb{Z}^n \setminus D_k)]$. Subtracting $u$ this yields

$$A^{k}(N \cap \partial T) - u = [\alpha(S)] \cap [\beta(S)] \cap [\gamma(S)].$$

From (7.18) and (7.19) we see that each equivalence class $\overline{N}$ is completely characterized by the sets $\alpha(S), \beta(S), \gamma(S)$. As these sets are contained in the finite set $S + \mathcal{N} - u$, the estimate in (7.17) implies that they are contained in the ball of radius $4 \text{diam}(T)$ around the origin. Thus there are only finitely many choices for these sets. \hfill $\Box$
Using Lemma 7.11 we can provide the following algorithm to calculate $\overline{T}$.

**Proposition 7.20.** The in-out graph $\overline{T}$ can be constructed by the following finite recurrence process.

Recurrence start: The equivalence class $\overline{N}$ of each fundamental neighborhood $N$ contained in $B$ is a node of $\overline{T}$.

Recurrence step: Suppose that $\overline{N}'$ is a node of $\overline{T}$. For all $N$ satisfying $N' \in \text{Parents}(N)$ the node $\overline{N}$ together with the edge $\overline{N} \rightarrow \overline{N}'$ belong to $\overline{T}$.

End of recurrence: Iterate until no new nodes occur in a recurrence step.

**Proof.** Let $R$ be the graph constructed by this recurrence process. Obviously, each node of $R$ is also a node of $\overline{T}$. Suppose that there is a node $\overline{N}$ of $\overline{T}$ that is not a node of $R$. Choose $N \in B$ in a way that level($N$) is minimal with this property. Let $N' \in \text{Parents}(N)$. Then $\overline{N'} \in R$ by the choice of $N$. The recurrence step above now implies together with Lemma 7.11 that $\overline{N}$ is a node of $\overline{T}$, a contradiction. Thus $R$ and $\overline{T}$ have the same set of nodes. Since the edges are defined in the same way, the result follows.

**Remark 7.21.** Let $(M,F)$ be a monotone model for a self-affine $\mathbb{Z}^n$-tile $T = T(A,D)$. Although $(F^{-1})(\mathcal{S}|_M)$, $S \subset \mathbb{Z}^n$, do not necessarily form a basis for the topology of $\mathbb{R}^n$, by the same arguments as above, a finite in-out graph $\overline{T}_M$ can be constructed also for $M$. Since two boundary stars $B_1$ and $B_2$ are homeomorphic (i.e., equivalent) if $A^{k_1}B_1 = A^{k_2}B_2 + u$, it suffices to check cellularity of boundary stars only for one representative of each equivalence class. The finiteness of $\overline{T}_M$ immediately implies that there are only finitely many such equivalence classes to check. In order to check cellularity of a given boundary star, the methods outlined in Section 6.2 can be used.

### 7.3. Results on self-affine balls.

We can now prove the following theorem.

**Theorem 7.22.** Let $T$ be a self-affine $\mathbb{Z}^n$-tile. If $\partial T$ is an $(n-1)$-sphere in $\mathbb{R}^n$ and each loop in the in-out graph $\overline{T}$ contains a node $\overline{N}$ such that

(i) $\partial N \cong \mathbb{S}^{n-1}$,

(ii) $\partial N \setminus \partial T$ is simply connected,

then $\partial T$ is locally spherical and thus tame. Consequently $T$ is homeomorphic to $\mathbb{D}^n$.

**Proof.** Let $|\overline{I}|$ be the number of nodes in $\overline{T}$ and assume that $k > |\overline{I}|$. Let $N \in B$ be a neighborhood of an element $x \in \partial T$ with level($N$) = $k$ and let $N \rightarrow N_{k-1} \rightarrow \cdots \rightarrow N_1 \rightarrow N_0$ be a walk in the graph $\overline{T}$. The associated walk in $\overline{T}$ is $\overline{N} \rightarrow \overline{N}_{k-1} \rightarrow \cdots \rightarrow \overline{N}_1 \rightarrow \overline{N}_0$. As $k > |\overline{I}|$ the first $|\overline{I}|$ edges of this walk contain a loop. Thus, by assumption, there is $\ell \in \{k - |\overline{I}|, \ldots, k\}$ such that $\partial N_\ell \setminus \partial T$ is simply connected and $\partial N_\ell \cong \mathbb{S}^{n-1}$. As $k$ was arbitrary and $\ell \in \{k - |\overline{I}|, \ldots, k\}$, we constructed an arbitrarily small neighborhood $N_\ell$ of $x$ that satisfies the properties of Corollary 7.7. Since $x \in \partial T$ was arbitrary, this proves the result.

For $n = 3$ we can simplify this by using the remark after Definition 7.2.

**Corollary 7.23.** Let $T$ be a self-affine $\mathbb{Z}^3$-tile. If $\partial T$ is a 2-sphere in $\mathbb{R}^3$ and each loop in the in-out graph $\overline{T}$ contains a node $N$ such that

(i) $\partial N \cong \mathbb{S}^2$,

(ii) $\partial N \cap \partial T$ is connected.

Then $\partial T$ is locally spherical and thus tame. Consequently $T$ is homeomorphic to $\mathbb{D}^3$.

**Remark 7.24.** It seems that the neighbor basis $B$ leads to satisfactory results if the self-affine $\mathbb{Z}^n$-tile $T$ has only face-neighbors (i.e., neighbors that intersect $T$ is an $(n-1)$-dimensional set; see Section 8.1). However, as Cannon’s criterion is necessary and sufficient, for tiles that are homeomorphic to balls such neighborhood bases always exist. In Section 8.3 we consider a tile with “degenerate” neighbors. To show that this tile is homeomorphic to a ball, we will change the fundamental neighborhoods slightly.

As being locally spherical is a local property, the results of the present section can be adapted to check whether $T$ is homeomorphic to other manifolds with boundary (see Section 8.4).
8. Proofs for the examples

We now provide the proofs for the examples given in Section 2.7 and construct a self-affine crumpled cube in order to prove Theorem 2.17. In particular, concerning the example in Section 2.7.1 we work out detailed proofs of Theorem 2.25 and Proposition 2.26. After that, in Section 8.2 we give a proof of Theorem 2.17. In Section 8.3 we sketch the proof of Theorem 2.28 and finally, Section 8.4 is devoted to self-affine \( \mathbb{Z}^3 \)-tiles that have a surface of positive genus as boundary.

8.1. A self-affine \( \mathbb{Z}^3 \)-tile that is homeomorphic to a 3-ball. Let \( A \) and \( D \) be given as in (2.24) and consider the tile \( T = T(A,D) \) defined by this data. Our aim is to prove Theorem 2.25 and Proposition 2.26 using Corollary 2.15, Theorem 2.16, and Corollary 7.23.

Sphere checking. To prove that \( \partial T \) is homeomorphic to the sphere \( S^2 \) we have to establish a monotone model that satisfies the conditions of Corollary 2.15. In view of Theorem 2.23 we start with setting up an ideal tile \( Z \) of \( T \). By inspecting the neighbor structure \( K(T) \) of the tile \( T \) it turns out that choosing \( Z \) to be equal to the prism spanned by the vectors \((0,1,0)^t, (1,\frac{5}{2},0)^t, (\frac{3}{2}, \frac{1}{2}, 1)^t\) is a good candidate. The prism \( Z \) and \( Z_1 = \langle P(\{0\}) \rangle_Z = A^{-1}(Z+D) \) are shown in Figure 5. It is easy to see that \( Z + \mathbb{Z}^3 \) forms a \( \mathbb{Z}^3 \)-tile of \( \mathbb{R}^3 \) having connected interior. Thus it suffices to check Items (i), (ii), and (iii) of Definition 2.22 to make sure that \( Z \) is an ideal tile for \( T \).

![Figure 5](image)

**Figure 5.** The ideal tile \( Z \) of \( T \) and \( \langle P(\{0\}) \rangle_Z \).

![Figure 6](image)

**Figure 6.** The directed graph \( \Gamma_2 \) of double intersections of \( T \). The triple \( abc \) stands for the node \( \{(0,0,0)^t, (a,b,c)^t\} \) and \( \bar{a} = -a \). Thus \( abc \) corresponds to the nonempty 2-fold intersection \( T \cap (T + (a,b,c)^t) \).

In order to verify Item (i) we have to check which of the sets \( \langle S \rangle , \langle S \rangle_Z \) are nonempty. By translation invariance, we can confine ourselves to sets \( \langle S \rangle \) and \( \langle S \rangle_Z \) with \( 0 \in S \). To characterize
all nonempty cells \( \langle S \rangle \) with \( 0 \in S \), by Remark 3.15 it suffices to construct the graphs \( \Gamma_i \) \((i \geq 0)\). This can be done by standard algorithms (see e.g. [46]). Indeed, in our example we get the Graphs \( \Gamma_2 \) and \( \Gamma_3 \) depicted in Figures 6 and 7, respectively. Moreover, the nodes of \( \Gamma_4 \) are contained in Table 1 and \( \Gamma_i \) is empty for \( i \geq 5 \). By inspecting the nodes of these graphs we know all sets \( S \subset \mathbb{Z}^3 \) containing \( 0 \) that correspond to a nonempty intersection \( \langle S \rangle \). On the other hand, the nonempty intersections \( \langle S \rangle \) with \( 0 \in S \) can easily be determined as \( Z \) is an explicitly given prism in \( \mathbb{R}^3 \). Comparing the collection of nonempty sets \( \langle S \rangle \) and \( \langle S \rangle \) we obtain that \( K(T) = K(Z) \) and Item (i) is verified.

Figure 7. The directed graph \( \Gamma_3 \) of triple intersections of \( T \) with two other tiles. Here a node \( a_1 b_1 c_1 \) corresponds to the intersection \( T \cap (T + (a_1, b_1, c_1)^T) \cap (T + (a_2, b_2, c_2)^T) \).

To check Item (ii) we need to make sure that \( \langle S \rangle \) and \( \langle P(S) \rangle \) are connected and that \( \langle S \rangle \cong \langle P(S) \rangle \) holds for each \( S \subset \mathbb{Z}^3 \). Since \( Z \) and \( \langle P(\{0\}) \rangle \) are explicitly given polyhedra (see Figure 5) it is a routine calculation to check (ii). Indeed it is easy to see that all the nonempty sets \( \langle S \rangle \) and \( \langle P(S) \rangle \) are balls of dimension \( 4 - |S| \).

To check (iii) we observe that \( Z \) is combinatorial, i.e., for each \( S \subset \mathbb{Z}^3 \) with \( |S| = 1, i \geq 2 \) we have \( \delta_S \cong \delta_i \langle S \rangle \) and \( \delta_P(S) \cong \delta_i \langle P(S) \rangle \). Thus each homeomorphism between the spheres
We have to show that Proposition 8.2.

\[ (T + s_1) \cap (T + s_2) \neq \emptyset. \]

We have to show that \( (T + s_1) \cap (T + s_2) \neq \emptyset. \) We know that (8.1) holds and only if \( s_1 - s_2 \in \mathcal{N} \) it is easy to set up this graph and to verify it is connected. It is now straightforward to show that \( E \cap \varphi_d(E) \neq \emptyset \) for each \( d \in \mathcal{D} \). Applying Lemma 5.10 we conclude that \( \partial T \) is connected.

Summing up, we may invoke Corollary 2.15 to \( T \) and have thus proved the following result.

**Proposition 8.2.** Let \( T \) be the self-affine \( \mathbb{Z}^3 \)-tile defined by \( AT = T + \mathcal{D} \) with \( A \) and \( \mathcal{D} \) as in (2.24). Then \( \partial T \) is homeomorphic to the sphere \( S^2 \).

Recall that each nonempty \( \langle S \rangle_M \) is a ball of dimension \( 4 - |S| \). Moreover, from each node in the graphs \( \Gamma_2 \) and \( \Gamma_3 \) there lead away infinitely many different infinite walks. Thus the sets \( \langle S \rangle \) with \( S \) being a node of these graphs, contain infinitely many points. Moreover, the sets \( \langle S \rangle \) with \( S \) being...
a node of \( \Gamma_4 \) are single points. Thus, since we already saw that \( Z \) and, hence, \( M \), is combinatorial, Theorem 2.16 implies Proposition 2.26.

**Ball checking.** In order prove that \( T \) is homeomorphic to a ball we want to apply Corollary 7.23. To this matter we have to construct the in-out graph \( \mathcal{I} \) which can be done by the algorithm proposed in Proposition 7.20. In the present example there exist 24 fundamental neighborhoods in \( \mathcal{B} \), one for each node of \( \Gamma_4 \) (see Table 1). As these lie in pairwise different equivalence classes (in the sense of Definition 7.9), the recurrence starts with 24 nodes. After eight recurrence steps we arrive at the in-out graph \( \mathcal{I} \) which has 2888 nodes. We now have to verify Conditions (i) and (ii) of Corollary 7.23 to prove that \( T \) is homeomorphic to a ball.

As \( \Gamma_5 \) is empty and each node of \( \Gamma_3 \) is a subset of a node of \( \Gamma_4 \), each set \( S \subset \mathbb{Z}^3 \) with the property \( \langle S \rangle \neq \emptyset \) and \( \langle S \cup \{s\} \rangle = \emptyset \) for all \( s \in \mathbb{Z}^3 \setminus S \) has exactly 4 elements. Therefore, each fundamental neighborhood can be written as \([S] + u\), with \( S \in \Gamma_4 \) and \( u \in \mathbb{Z}^n \) and, hence, each node \( \mathcal{N} \) of \( \mathcal{I} \) is of the form \( N = A^{-k}([S] + u) \), with \( k \in \mathbb{N} \), \( u \in \mathbb{Z}^n \), and \( S \in \Gamma_4 \). This implies that \( N \) is homeomorphic to \([S]\) for some \( S \in \Gamma_4 \) and checking Condition (i) of Corollary 7.23 amounts to checking whether \( \mathcal{N} \cong \mathbb{S}^2 \) holds for each of the 24 nodes of \( \Gamma_4 \). To prove that \( \mathcal{N} \cong \mathbb{S}^2 \) we may use Proposition 5.9. To check the conditions of this proposition, it remains to check the following items for each \( S \in \Gamma_4 \):

1. \([S] \cong \mathbb{S}^2\).
2. \( \text{int}([S]) \) is connected.
3. \( \mathbb{R}^3 \setminus [S] \) is connected.

As \([S] \cong \mathbb{S}^2\) is a union of four prisms one can check (a) by direct inspection or standard methods (see Section 6.2). To check (b) observe that \( T \) tiles \( \mathbb{R}^3 \) by \( \mathbb{Z}^3 \)-translates. Thus the definition of the fundamental neighborhood implies that the singleton \( \langle S \rangle \) is contained in the interior of \([S]\) and, hence, there is a small open \( B \) ball centered in \( \langle S \rangle \) that is contained in \( \text{int}([S]) \). As \( \text{int}(T) \) is connected and \( B \) contains inner points of \( T + s \) for each \( s \in S \), the interior of \([S]\) is connected. As the tiling \( T + \mathbb{Z}^3 \) is locally finite, also (c) can be checked combinatorially by using the connectivity of \( \text{int}(T) \).

Condition (ii) of Corollary 7.23 has to be checked for each of the 2888 nodes of \( \mathcal{I} \). We explain how this is done for a given node of \( \mathcal{I} \). Let \( \mathcal{N} \) be a node of \( \mathcal{I} \) and consider \( \partial \mathcal{N} \cap \partial T \). Suppose \( N = A^{-k}([S] + u) \), then there exist \( S_1, \ldots, S_m \subset \mathbb{Z}^n \) such that

\[
\partial \mathcal{N} \cap \partial T = A^{-k} \bigcup_{i=1}^{m} \langle S_i \rangle.
\]

As \( \langle S_i \rangle \) is connected for each \( i \in \{1, \ldots, m\} \), this set is connected if \( \{S_1, \ldots, S_m\} \) forms a chain. In other words, define a graph \( C(N) \) whose nodes are the sets \( S_1, \ldots, S_m \) and there is an undirected edge between \( S_i \) and \( S_j \) if and only if \( \langle S_i \rangle \cap \langle S_j \rangle = \langle S_i \cup S_j \rangle \neq \emptyset \). All the information required to construct this graph is contained in Proposition 2.26. The set \( \partial \mathcal{N} \cap \partial T \) is connected if and only if \( C(N) \) is a connected finite graph. We checked connectivity for each node of \( \mathcal{I} \) with the aid of a Mathematica program. It turns out that in each walk of length 2 of \( \mathcal{I} \) there is at least one node satisfying (ii).

Summing up, in each loop of \( \mathcal{I} \) there is at least one node satisfying the conditions of Corollary 7.23. This proves that \( T \) is homeomorphic to a closed ball and Theorem 2.25 is established.

8.2. A self-affine \( \mathbb{Z}^3 \)-tile whose boundary is a wild sphere. Let \( A = \text{diag}(9,9,9) \) be the \( 3 \times 3 \) diagonal matrix with 9s in the main diagonal. We define the set of digits \( D \) as follows. Let

\[
C := \{(x_1, x_2, x_3) \mid 0 \leq x_1, x_2, x_3 \leq 8\}
\]

be the “basic cube”. Starting from \( C \) we construct the digit set by attaching and cutting out “horns”. For the “upper horns” set

\[
H^u_1 := \{(1, 4, x_3) \mid 0 \leq x_3 \leq 6\} \cup \{(x_1, 4, 6) \mid 2 \leq x_1 \leq 7\},
\]

\[
H^u_2 := \{(7, 4, x_3) \mid 0 \leq x_3 \leq 4\}.
\]
The “lower horns” we define by \( H_i^0 := \{(x_2, x_1, 8 - x_3) \mid (x_1, x_2, x_3) \in H_i^3 \} \) for \( i \in \{1, 2\} \).

\[
    D = \left( C \cup (H_1^0 + (9, 0, 0)) \cup (H_2^0 + (9, 0, 0)) \cup (H_1^0 - (9, 0, 0)) \cup (H_2^0 - (9, 0, 0)) \right) \setminus \left( H_i^0 \cup H_i^3 \cup H_i^1 \cup H_i^2 \right).
\]

It is easy to see that \( D \) has \( 9^3 \) elements and is a complete set of coset representatives of \( \mathbb{Z}^3 / 3 \mathbb{Z}^3 \). Moreover, using well-known algorithms (cf. e.g. Vince [51]), one checks that \( T = T(A, D) \) is a \( \mathbb{Z}^3 \)-tile and, hence, a self-affine \( \mathbb{Z}^3 \)-tile. The image of \( \langle P(\{0\}) \rangle \) in Figure 1 gives a good geometric imagination of the digit set \( D \).

**Proposition 8.4.** Let the self-affine \( \mathbb{Z}^3 \)-tile \( AT = T + D \) with \( A = \text{diag}(9, 9, 9) \) and \( D \) defined as in \((8.3)\) be given.

(i) \( \partial T \) is homeomorphic to \( S^2 \).

(ii) \( T \) is not homeomorphic to a ball.

We only sketch the proof and omit routine calculations.

**Sketch of the proof.** To prove (i) we will apply Corollary 2.15. To this matter we first construct a single point intersections. For this reason it is harder to construct an ideal tile \( T \) more complicated than the one studied Section 8.1. It has 18 neighbors, four of which correspond to \( \phi \) and the references given there.

To prove (ii) we proceed as in the classical proof for Alexander’s Horned Sphere and show that \( T \) admits a monotone model \( (M, F) \) with \( \partial M \simeq S^2 \).

It remains to show that \( \text{int}(T) \) is connected. In view of Lemma 5.10 we construct a connected set \( E \subset \text{int}(T) \) with the property that \( E \cap \varphi_d(E) \neq \emptyset \) for each \( d \in D \). It is easy to see that the midpoint of the cube \( \varphi_d(Z) \) is an element of \( \text{int}(T) \). For each face of \( \varphi_d(Z) \) that is also contained in another cube \( \varphi_d(Z) \) connect the midpoint of \( \varphi_d(Z) \) to the midpoint of this face by an arc that is contained in \( \text{int}(T) \). Call the union of all these arcs \( Y_d \) and set \( E = \bigcup_{d \in D} Y_d \). One easily checks that \( E \) has the required properties.

To prove (ii) we proceed as in the classical proof for Alexander’s Horned Sphere and show that the complement \( \mathbb{R}^3 \setminus T \) is not simply connected since we cannot homotope out a loop that surrounds one of the “horns” of \( T \) (see e.g. [28, Example 2B.2, page 170ff]).

This seems to be the first example of a self-affine wild “crumpled cube” that tiles \( \mathbb{R}^3 \) and therefore proves Theorem 2.17. It is of interest in the study of possible embedding types of spheres that admit a tiling of \( \mathbb{R}^3 \). We refer to Tang [49] (particularly to Question 2 on p. 422 of this paper) and the references given there.

### 8.3. Gelbrich’s Twin Dragon

Let \( A \) and \( D \) be given as in \((2.27)\). Then \( T = T(A, D) \) is Gelbrich’s twin dragon. We give a sketch of the proof of Theorem 2.28. This example is more complicated than the one studied Section 8.1. It has 18 neighbors, four of which correspond to single point intersections. For this reason it is harder to construct an ideal tile \( Z \) for \( T \) that makes Corollary 2.15 applicable. We choose \( Z \) as follows. Let \( P \) be the prism spanned by the vectors \((1, 0, 0)^t, (\frac{1}{5}, 1, 0)^t, (-\frac{1}{2}, \frac{1}{5}, 1)^t \) and let

\[
    \Sigma_1 := \text{convexhull} \left\{ \left( -\frac{1}{5}, \frac{8}{25}, \frac{2}{5} \right)^t, \left( -\frac{3}{10}, \frac{12}{25}, \frac{3}{5} \right)^t, \left( -\frac{9}{100}, \frac{3}{5}, \frac{1}{2} \right)^t, \left( -\frac{9}{20}, \frac{2}{5}, \frac{1}{2} \right)^t \right\},
\]

\[
    \Sigma_2 := \text{convexhull} \left\{ \left( \frac{11}{10}, \frac{9}{5}, \frac{1}{5} \right)^t, \left( \frac{13}{10}, \frac{9}{5}, \frac{1}{5} \right)^t, \left( \frac{13}{10}, \frac{41}{25}, \frac{1}{5} \right)^t, \left( \frac{6}{5}, 2, 1 \right)^t \right\}.
\]

Then we set

\[
    Z := P \cup \Sigma_1 \cup \Sigma_2 \setminus (\Sigma_1 + (1, 0, 0)^t) \cup (\Sigma_2 - (1, 1, 0)^t).
\]

A picture of \( Z \) is provided in Figure 3. It can be checked by a lengthy but simple direct (computer aided) calculation that \( Z \) satisfies the conditions Definition 2.22 and is therefore an ideal tile for \( T \);
As for Condition (i) it is again a matter of calculating the graphs $\Gamma_i$ ($i \geq 0$) by known algorithms and compare the results with the intersection structure of the polyhedron $Z$. As for Condition (ii) it turns out that four of the intersections $\{\{0, s\}\}_Z$ are the union of two disks intersecting in a single point so that we are not in the situation covered by Section 6.2. Nevertheless, Condition (ii) can be checked easily by direct inspection and Condition (iii) follows from Lemma 6.2. Thus $Z$ is an ideal tile for $T$ and, hence, Theorem 2.23 implies that there is a monotone model $(M, F)$ for $T$ whose boundary is homeomorphic to $S^2$.

To apply Theorem 2.14 it therefore remains to check that $\text{int}(T)$ is connected. This is again done with the help of Lemma 5.10. Summing up we obtain that Gelbrich’s tile $T$ satisfies $\partial T \cong S^2$.

Also, running the ball-checking algorithm of Section 7 is more tricky in this case (see Remark 7.24). Indeed, we have to define the fundamental neighborhoods in the following way. Let $[S]$ with $0 \in S$, $|S| = 4$, and $\langle S \rangle \neq \emptyset$. Then take the 7-th subdivision

$$[S] = \bigcup_{s \in S} \bigcup_{r \in D_s} (T + r + A^7 s) = \bigcup_{s \in S} \bigcup_{r \in D_s} Q_{r,s}.$$ 

Now choose the union of all point neighbors of $[S]$, i.e.,

$$U_S = \bigcup_{z \in \mathbb{Z}^3 : |S| \cap (T + z)} (T + z),$$

and let $V_S = \bigcup_{s \in S, r \in D_s} Q_{r,s}$ be the union of all $Q_{r,s}$ that touch such a point neighbor. Then each fundamental neighborhood has the form $N = [S] \setminus V_S$. Running the ball checking algorithm with these fundamental neighborhoods yields an in-out graph with 12716 nodes. The fact that the fundamental neighborhoods have spherical boundary can be checked by Proposition 5.9 (note that $N$ is of the form $A^{-7}[S']$ for some $S' \subset \mathbb{Z}^3$; the fact that $\partial [S']_Z \cong S^2$ can be checked by using the methods discussed in Section 6.2), and the connectivity of their intersections with $\partial T$ are treated in the same way as in Section 8.1.

8.4. Self-affine $\mathbb{Z}^3$-tiles whose boundary is a surface of positive genus. We first construct a self-affine $\mathbb{Z}^3$-tile whose boundary is a torus. Let $A = \text{diag}(6, 6, 6)$ and define the digit set as follows. First set $C = \{(x_1, x_2, x_3) \mid 0 \leq x_1, x_2, x_3 \leq 5\}$,

$$C_1 = \{(3, x_2, x_3), (3, 4, 5), (3, 5, 5) \mid 0 \leq x_2 \leq 3, 3 \leq x_3 \leq 5\},$$

and $C_2 = \{(x_1, 5 - x_2, 5 - x_3) \mid (x_1, x_2, x_3) \in C_1\}$. The sets $C_1$ and $C_2$ cut out a “hole” from the “cube” $C$. To make this a digit set that forms a complete set of residue classes of $\mathbb{Z}^3/A\mathbb{Z}^3$ we need to insert $C_1$ and $C_2$ at another place. Indeed, we define the digit set by

$$(8.5) \quad D = (C \cup (C_1 + (0, 6, 0)) \cup (C_2 - (0, 6, 0))) \setminus (C_1 \cup C_2).$$

The self-affine $\mathbb{Z}^3$-tile $T = T(A, D)$ is depicted in Figure 4. From these pictures it is plausible to assume that $T$ is a solid torus. Using Theorem 2.14 we can prove the following result.

**Proposition 8.6.** Let the self-affine $\mathbb{Z}^3$-tile $AT = T + D$ with $A = \text{diag}(6, 6, 6)$ and $D$ defined as in (8.5) be given. Then $\partial T$ is a 2-torus.

To construct this self-affine torus $T$, starting with a $6 \times 6 \times 6$ cube, we cut out a hole and – to compensate for the digits killed by digging this hole – we added a “half” handle on the top and on the bottom of the cube. To construct boundary surfaces of genus $g$, we have to dig $g$ holes and to add $g$ such “half” handles on the top and on the bottom. Since, seeing the case $g = 1$ above, this construction is quite obvious and we omit the details for the proof of Proposition 2.29.

**References**

[1] S. Akiyama and B. Loridant. Boundary parametrization of self-affine tiles. *J. Math. Soc. Japan*, 63(2):525–579, 2011.

[2] C. Bandt. Combinatorial topology of three-dimensional self-affine tiles. preprint, available under http://arxiv.org/pdf/1002.0710.pdf.

[3] C. Bandt and G. Gelbrich. Classification of self-affine lattice tilings. *J. London Math. Soc. (2)*, 50(3):581–593, 1994.

[4] C. Bandt and M. Mesing. Self-affine fractals of finite type. In *Convex and fractal geometry*, volume 84 of *Banach Center Publ.*, pages 131–148. Polish Acad. Sci. Inst. Math., Warsaw, 2009.
[43] R. L. Moore. Concerning upper semi-continuous collections of continua. Trans. Amer. Math. Soc., 27(4):416–428, 1925.
[44] J. H. Roberts and N. E. Steenrod. Monotone transformations of two-dimensional manifolds. Ann. of Math. (2), 39(4):851–862, 1938.
[45] K. Rudnik. Self-similar metric inverse limits of invariant geometric inverse sequences. Topology Appl., 48(1):1–17, 1992.
[46] K. Scheicher and J. M. Thuswaldner. Neighbours of self-affine tiles in lattice tilings. In Fractals in Graz 2001, Trends Math., pages 241–262. Birkhäuser, Basel, 2003.
[47] S. Smale. Generalized Poincaré’s conjecture in dimensions greater than four. Ann. of Math. (2), 74, 1961.
[48] R. S. Strichartz and Y. Wang. Geometry of self-affine tiles. I. Indiana Univ. Math. J., 48(1):1–23, 1999.
[49] T.-M. Tang. Crumpled cube and solid horned sphere space fillers. Discrete Comput. Geom., 31(3):421–433, 2004.
[50] W. Thurston. Groups, tilings and finite state automata. in: AMS Colloquium Lecture, 1989.
[51] A. Vince. Digit tiling of Euclidean space. In Directions in mathematical quasicrystals, volume 13 of CRM Monogr. Ser., pages 329–370. Amer. Math. Soc., Providence, RI, 2000.
[52] Y. Wang. Self-affine tiles. In Advances in wavelets (Hong Kong, 1997), pages 261–282. Springer, Singapore, 1999.

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