THE RAMSEY THEORY OF HENSON GRAPHS

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Abstract. For \( k \geq 3 \), the Henson graph \( H_k \) is the analogue of the Rado graph in which \( k \)-cliques are forbidden. Building on the author’s result for \( H_3 \) in [4], we prove that for each \( k \geq 4 \), \( H_k \) has finite big Ramsey degrees: To each finite \( k \)-clique-free graph \( G \), there corresponds an integer \( T(G, H_k) \) such that for any coloring of the copies of \( G \) in \( H_k \) into finitely many colors, there is a subgraph of \( H_k \), again isomorphic to \( H_k \), in which the coloring takes no more than \( T(G, H_k) \) colors.

Prior to this article, the Ramsey theory of \( H_k \) for \( k \geq 4 \) had only been resolved for vertex colorings by El-Zahar and Sauer in [7]. We develop a unified framework for coding copies of \( H_k \) into a new class of trees, called strong \( H_k \)-coding trees, and prove Ramsey theorems for these trees, forming a family of Halpern-Läuchli and Milliken-style theorems which are applied to deduce finite big Ramsey degrees. The approach here streamlines the one in [4] for \( H_3 \) and provides a general methodology opening further study of big Ramsey degrees for homogeneous structures with forbidden configurations. The results have bearing on topological dynamics via work of Kechris, Pestov, and Todorcevic [18] and Zucker [41].

Overview

A central program of the theory of infinite structures is to find which structures have partition properties resembling Ramsey’s Theorem. In this context, one colors the copies of a finite structure \( A \) inside the infinite structure \( S \) into finitely many colors and looks for an infinite substructure \( S' \), isomorphic to \( S \), in which the copies of \( A \) have the same color. A wide collection of infinite structures the Ramsey property for colorings of singletons. However, even the rationals as a linearly ordered structure do not have the Ramsey property for colorings of pairs, as seen by Sierpiński’s example of a two-coloring of pairs of rationals so that each subcopy of the rationals retains both colors on its pairsets. This leads to the following question: Given an infinite structure \( S \) and a finite substructure \( A \), is there a positive integer \( T(A, S) \) such that for any coloring of all copies of \( A \) in \( S \) into finitely many colors, there is a substructure \( S' \) of \( S \), isomorphic to \( S \), in which all copies of \( A \) take no more than \( T(A, S) \) colors? This number, when it exists, is called the big Ramsey degree of \( A \) in \( S \). Research in this area has gained recent momentum, as it was highlighted by Kechris, Pestov, and Todorcevic in [18]. Big Ramsey degrees have implications for topological dynamics, as shown in [18] and further developed in Zucker’s work [41].

The development of Ramsey theory for infinite structures has progressed quite slowly. After Sierpiński’s coloring for pairs of rationals, work of Laver and Devlin (see [2]) established the exact big Ramsey degrees for finite sets of rationals by 1979.

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In the mid 1970's, Erdős, Hajnal, and Posá began work on the big Ramsey degrees of the Rado graph, establishing an analogue of Sierpiński’s coloring for edges in [8]. Building on work in [32], the full Ramsey theory of the Rado graph for colorings of copies of any finite graph was finally established in 2006 in the two papers [35] by Sauer and [21] by Laflamme, Sauer, and Vuksanovic. Around that time, due to the interest in big Ramsey degrees generated by [18], the Ramsey theory of other ultrahomogeneous structures was established in [29] and [20]. A principal component in the work in [2] and [35] is a Ramsey theorem for strong trees due to Milliken [25], while [20] depended on the authors’ development of a colored version of this theorem. The lack of similar means of coding infinite structures and Ramsey theorems for such coded structures have been the largest obstacles in the further development of this area, especially for ultrahomogeneous structures with forbidden configurations. As stated by Nguyen Van Thé in [31], “so far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties.”

In this paper, we prove that for each \( k \geq 4 \), the Henson graph \( H_k \) has finite big Ramsey degrees. Given \( k \geq 3 \), the Henson graph \( H_k \) is the universal ultrahomogeneous \( K_k \)-free graph; that is, the \( k \)-clique-free analogue of the Rado graph. The only prior work on the big Ramsey degrees of \( H_k \) for \( k \geq 4 \) was work of El-Zahar and Sauer in [7] for vertex colorings in 1989. In [4], we proved that the triangle-free Henson graph has finite big Ramsey degrees. The work in this paper follows the general outline in [4], but the extension of Ramsey theory to all Henson graphs required expanded ideas, a better understanding of the nature of coding structures with forbidden configurations, and many new lemmas. This article presents a unified framework for the Ramsey theory of Henson graphs. We develop new techniques for coding copies of \( H_k \) via strong \( H_k \)-coding trees and prove Ramsey theorems for these trees, forming a family of Milliken-style theorems. The approach here streamlines the one in [4] for \( H_3 \) and provides a general methodology opening further study of big Ramsey degrees for homogeneous structures with forbidden configurations.

1. Introduction

The field of Ramsey theory was established by the following celebrated result.

**Theorem 1.1** (Infinite Ramsey Theorem, [33]). Given positive integers \( m \) and \( j \), suppose the collection of all \( m \)-element subsets of \( \mathbb{N} \) into \( j \) colors. Then there is an infinite set \( N \) of natural numbers such that all \( m \)-element subsets of \( N \) have the same color.

From this, Ramsey deduced the following finite version, which also can be proved directly.

**Theorem 1.2** (Finite Ramsey Theorem, [33]). Given positive integers \( m, n, j \) with \( m \leq n \), there is an integer \( r > n \) such that for any coloring of the \( m \)-element subsets of \( r \) into \( j \) colors, there is a subset \( N \subseteq r \) of cardinality \( n \) such that all \( m \)-element subsets of \( N \) have the same color.

In both cases, we say that the coloring is monochromatic on \( N \), and that the set \( N \) is homogeneous for the coloring. Interestingly, Theorem 1.2 was motivated by Hilbert’s Entscheidungsproblem: to find a decision procedure deciding which
formulas in first order logic are valid. Ramsey applied Theorem 1.2 to prove that the validity, or lack of it, for certain types of formulas in first order logic (those with no existential quantifiers) can be ascertained algorithmically. Later, Church and Turing each showed that a general solution to Hilbert’s problem is impossible, so Ramsey’s success for the class of existential formulas is remarkable. Ever since the inception of Ramsey theory, its connections with logic have continually spurred progress in both fields. This phenomenon occurs once again in Sections 6 and 7, where methods of logic are used to deduce Ramsey theorems.

Structural Ramsey theory investigates which structures satisfy versions of Ramsey’s Theorem. In this setting, one tries to find a substructure isomorphic to some fixed structure on which the coloring is monochromatic. Given structures A and B, we write \( A \leq B \) if and only if there is an embedding of A into B. A substructure \( A' \) of B is called a copy of A if and only if \( A' \) is the image of some embedding of A into B. The collection of all copies of A in B is denoted by \( (B)_A \). Given structures A, B, C with \( A \leq B \leq C \) and an integer \( j \geq 1 \), we write

\[
(1) \quad C \rightarrow (B)_A^j
\]

to mean that for each \( c : (C)_A \rightarrow j \), there is a \( B' \in (B)_A \) for which \( c \) takes only one color on \( (B')_A \). A class \( \mathcal{K} \) of finite structures is said to have the Ramsey property if given \( A, B \in \mathcal{K} \) with \( A \leq B \), for any integer \( j \geq 1 \), there is some \( C \in \mathcal{K} \) for which \( B \leq C \) and \( C \rightarrow (B)_A^j \).

Some classic examples of classes of structures with the Ramsey property include finite Boolean algebras (Graham and Rothschild [14]), finite vector spaces over a finite field (Graham, Leeb, and Rothschild [12] and [13]), finite ordered relational structures (independently, Abramson and Harrington, [1] and Nešetřil and Rödl, [27], [28]), in particular, the class of finite ordered graphs. The papers [27] and [28] further proved that all set-systems of finite ordered relational structures omitting some irreducible substructure have the Ramsey property. This includes the classes of finite ordered graphs omitting \( k \)-cliques, denoted \( \mathcal{G}_k^\mathcal{C} \), for each \( k \geq 3 \). Fraissé classes are natural objects for structural Ramsey theory investigations, for as shown by Nešetřil, any class with the Ramsey property must satisfy the amalgamation property. Since Fraissé theory is not central to the proofs in this article, we refer the interested reader to [11] and Section 2 of the more recent [18] for background; the properties of the specific examples contained in this article will be clear.

In contrast, most classes of finite unordered structures do not have the Ramsey property. However, if equipping the class with an additional linear order produces the Ramsey property, then some remnant of it remains in the unordered reduct. This is the idea behind small Ramsey degrees. Following notation in [18], given any Fraissé class \( \mathcal{K} \) of finite structures, for \( A \in \mathcal{K} \), \( t(A, \mathcal{K}) \) denotes the smallest number \( t \), if it exists, such that for each \( B \in \mathcal{K} \) with \( A \leq B \) and for each \( j \geq 2 \), there is some \( C \in \mathcal{K} \) into which \( B \) embeds such that for any coloring \( c : (C)_A \rightarrow j \), there is a \( B' \in (B)_A \) such that the restriction of \( c \) to \( (B')_A \) takes no more than \( t \) colors. In the arrow notation, this is written as

\[
(2) \quad C \rightarrow (B)^j_{A, t(A, \mathcal{K})}.
\]

A class \( \mathcal{K} \) has finite (small) Ramsey degrees if for each \( A \in \mathcal{K} \) the number \( t(A, \mathcal{K}) \) exists. The number \( t(A, \mathcal{K}) \) is called the Ramsey degree of \( A \) in \( \mathcal{K} \) [10]. Notice that \( \mathcal{K} \) has the Ramsey property if and only if \( t(A, \mathcal{K}) = 1 \) for each \( A \in \mathcal{K} \).
The connection between Fraïssé classes with finite Ramsey degrees and ordered expansions is made explicit in Section 10 of [18], where it is shown that if an ordered expansion $K^<\mathcal{K}$ of a Fraïssé class $\mathcal{K}$ has the Ramsey property, then $\mathcal{K}$ has finite small Ramsey degrees. Furthermore, the degree of $A \in \mathcal{K}$ can be computed from the number of non-isomorphic order expansions it has in $\mathcal{K}^<\mathcal{K}$. Nguyen Van Thé has extended this to the more general notion of pre-compact expansions (see [31]). In particular, the classes of finite (unordered) graphs and finite (unordered) graphs omitting $k$-cliques have finite small Ramsey degrees.

Continuing this expansion of Ramsey theory leads to investigations of which infinite structures have properties similar to Theorem 1.1. Notice that the infinite homogeneous subset $N \subseteq \mathbb{N}$ in Theorem 1.1 is actually isomorphic to $\mathbb{N}$ as a linearly ordered structure. Ramsey theory on infinite structures is concerned with finding substructures isomorphic to the original infinite structure in which a given coloring is as simple as possible. Many infinite structures have been proved to be indivisible: given a coloring of its single-element substructures into finitely many colors, there is an infinite substructure isomorphic to the original structure in which all single-element substructures have the same color. The natural numbers and the rational numbers as linearly ordered structures are indivisible, the proofs being straightforward. Similarly, it is folklore that the Rado graph is indivisible, the proof following naturally from the definitive properties of this graph. In contrast, it took much more effort to prove the indivisibility of the Henson graphs, and this was done for the triangle-free Henson graphs in [19] and all other Henson graphs in [7]. When one considers colorings of structures of two or more elements, more complexity begins to emerge. Even for the simple structure of the rationals, there is a coloring of pairsets into two colors such that each subset isomorphic to the rationals has pairsets in both colors. This is the infamous example of Sierpiński, and it immediately leads to the notion of big Ramsey degree. We take the definition from [18], slightly changing some notation.

**Definition 1.3** ([18]). Given an infinite structure $\mathcal{S}$ and a finite substructure $A \leq \mathcal{S}$, let $T(A, \mathcal{S})$ denote the least integer $T \geq 1$, if it exists, such that given any coloring of $\binom{\mathcal{S}}{A}$ into finitely many colors, there is a substructure $\mathcal{S}'$ of $\mathcal{S}$, isomorphic to $\mathcal{S}$, such that $\binom{\mathcal{S}'}{A}$ takes no more than $T$ colors. This may be written succinctly as

\[
\forall j \geq 1, \quad \mathcal{S} \rightarrow (\binom{\mathcal{S}}{A})^j_{T(A, \mathcal{S})}.
\]

We say that $\mathcal{S}$ has *finite big Ramsey degrees* if for each finite substructure $A \leq \mathcal{S}$, there is an integer $T(A, \mathcal{S}) \geq 1$ such that (3) holds.

Infinite structures which have been investigated in this light include the rationals ([2]), the Rado graph ([8], [32], [35], [21]), ultrametric spaces ([29]), the rationals with a fixed finite number of equivalence relations, and the tournaments $\mathcal{S}(2)$ and $\mathcal{S}(3)$ ([29]), and recently, the triangle-free graph Henson graph ([4]). These results will be discussed below. See [51] for an overview of results on big Ramsey degrees obtained prior to 2013. Each of these structures is ultrahomogeneous: any isomorphism between two finitely generated substructures can be extended to an automorphism of the infinite structure. Recently, Mašulović has widened the investigation of big Ramsey degrees to universal structures, regardless of ultrahomogeneity, and proved transfer principles in [24] from which big Ramsey degrees for one structure may be transferred to other categorically related structures. More background
on the development of Ramsey theory on infinite structures will be given below, but first, we present some recent motivation from topological dynamics for further exploration of big Ramsey degrees.

Connections between topological dynamics and Ramsey theory have been known for some time. The work of Kechris, Pestov, and Todorcevic in [18] subsumed the previously known phenomena by proving several general correspondences between Ramsey theory and topological dynamics. A Fraïssé class which has at least one relation which is a linear order is called a Fraïssé order class. One of the main theorems in [18] (Theorem 4.7) shows that the extremely amenable closed subgroups of the infinite symmetric group $S_\infty$ are exactly those of the form $\text{Aut}(F)$, where $F$ is the Fraïssé limit (and hence an ultrahomogeneous structure) of some Fraïssé order class satisfying the Ramsey property. Another significant theorem (Theorem 10.8) provides a way to compute the universal minimal flow of topological groups which arise as the automorphism groups of Fraïssé limits of Fraïssé classes with the Ramsey property and the ordering property. That the ordering property can be relaxed to the expansion property was proved by Nguyen Van Thé in [30].

Connections between Ramsey theory of ultrahomogeneous structures and topological dynamics have been established by Kechris, Pestov, and Todorcevic. By a Fraïssé structure, we mean the Fraïssé limit of a Fraïssé class. These are exactly ultrahomogeneous structures. In [18], they demonstrated how big Ramsey degrees for Fraïssé structures $F$ are related to big oscillation degrees for their automorphism groups, $\text{Aut}(F)$. Recently, Zucker proved in [41] that if a Fraïssé structure $F$ has finite big Ramsey degrees and moreover, $F$ admits a big Ramsey structure, then any big Ramsey flow of $\text{Aut}(F)$ is a universal completion flow, and further, any two universal completion flows are isomorphic.

1.1. Overview of big Ramsey degrees and main obstacles to its development. Returning to the history of big Ramsey degrees, in contrast to the robust development for finite structures, results on the Ramsey theory of infinite structures have been meager and the development quite slow. Motivated by Sierpiński’s coloring for pairs of rationals which admits no isomorphic copy in one color, Laver investigated the more general problem of finding whether or not there are bounds for colorings of $m$-sized subsets of rationals, for any positive integer $m$. In the 1970’s, Laver showed that the rationals have finite big Ramsey degrees, finding good upper bounds. Guided by Laver’s results and methods, Devlin found the exact bounds in [2]. Interestingly, these numbers turn out to be coefficients of the Taylor series for the tangent function. Around the same time, Erdős, Hajnal, and Pösí initiated investigations of the Rado graph. Recall that the Rado graph is the universal ultrahomogeneous graph on countably many vertices, and can be constructed as the Fraïssé limit of the class of all finite graphs. In 1975, they proved in [8] that there is a coloring of edges into two colors in which each subcopy of the Rado graph has edges in both colors. That the upper bound for the big Ramsey degree of edges in the Rado graph is exactly two was proved much later (1996) by Pouzet and Sauer in [32]. The full Ramsey theory of the Rado graph was finally established a decade later by Sauer in [35] and by Laflamme, Sauer, and Vuksanovic in [21]. Together, these two papers gave a full description of the big Ramsey degrees of the Rado graph in terms of types of certain trees. A recursive procedure for computing these numbers was given by Larson in [22] soon after. It is notable that while the big
Ramsey degrees for the rationals are computed by a closed formula, there is no closed formula producing the big Ramsey degrees for the Rado graph.

The successful completion of the Ramsey theory of the Rado graph and the contemporaneous work of Kechris, Pestov, and Todorcevic stimulated more interest in Ramsey theory of infinite structures, especially ultrahomogeneous structures, which are obtained as limits of Fraïssé classes. In 2008, Nguyen Van Thé investigated big Ramsey degrees for ultrahomogeneous ultrametric spaces. Given $S$ a set of positive real numbers, $U_S$ denotes the class of all finite ultrametric spaces with strictly positive distances in $S$. Its Fraïssé limit, denoted $Q_S$, is called the Urysohn space associated with $U_S$ and is a homogeneous ultrametric space. In [29], Nguyen Van Thé proved that $Q_S$ has finite big Ramsey degrees whenever $S$ is finite. Moreover, if $S$ is infinite, then any member of $U_S$ of size greater than or equal to 2 does not have a big Ramsey degree. Soon after this, Laflamme, Nguyen Van Thé, and Sauer proved in [20] that enriched structures of the rationals, and two related directed graphs, have finite big Ramsey degrees. For each $n \geq 1$, $Q_n$ denotes the structure $(Q, Q_1, \ldots, Q_n, <)$, where $Q_1, \ldots, Q_n$ are disjoint dense subsets of $Q$ whose union is $Q$. This is the Fraïssé limit of the class $P_n$ of all finite linear orders equipped with an equivalence relation with $n$ many equivalence classes. Laflamme, Nguyen Van Thé, and Sauer proved that each member of $P_n$ has a finite big Ramsey degree in $Q_n$. Further, using the bi-definability between $Q_n$ and the circular directed graphs $S(n)$, for $n = 2, 3$, they proved that $S(2)$ and $S(3)$ have finite big Ramsey degrees. Central to these results is a colored version of Milliken’s theorem which they proved in order to deduce the big Ramsey degrees. For more details, we recommend the paper [31] containing a good overview of these results.

A common theme emerges when one looks at the proofs in [2], [35], and [20]. The first two rely in an essential way on Milliken’s Theorem, (see Theorem 2.7 in Section 2). The third proves a new colored version of Milliken’s Theorem and uses it to deduce the results. The results in [20] use Ramsey’s theorem. This would lead one to conclude or at least conjecture that, aside from Ramsey’s Theorem itself, Milliken’s Theorem contains the core combinatorial content of big Ramsey degree results, at least for binary relational structures. The lack of useful representations and Milliken-style theorems for infinite structures in general pose the two main obstacles to broader investigations of big Ramsey degrees. Upon the author’s initial interest in the Ramsey theory of the triangle-free Henson graph, these barriers were pointed out to the author as the main obstacles by Todorcevic in 2012 and by Sauer in 2013; this idea is also expressed in [31], quoted in the Overview.

The work in this paper overcomes these obstacles for the Henson graphs. We present a unified development of new types of trees which code Henson graphs and prove new Milliken-style theorems for these classes of trees which are applied to determine upper bounds for the big Ramsey degrees.

For $k \geq 3$, the Henson graph $H_k$ is the universal ultrahomogeneous $k$-clique free graph. These graphs were first constructed by Henson in 1971 in [17]. It was later noticed that $H_k$ is isomorphic to the Fraïssé limit of the Fraïssé class of finite $k$-clique free graphs, $G_k$. Henson proved in [17] that these graphs are weakly indivisible: given a coloring of the vertices into two colors, either there is a subgraph isomorphic to $H_k$ in which all vertices have the first color, or else every finite $k$-clique free graph has a copy whose vertices all have the second color. However, the indivisibility of $H_k$ took longer to prove. In 1986, Komjáth and Rödl proved in
that given a coloring of the vertices of $H_3$ into finitely many colors, there is
an induced subgraph isomorphic to $H_3$ in which all vertices have the same color.
A few years later, El-Zahar and Sauer proved more generally that $H_k$ is indivisible
for each $k \geq 4$ in [7]. Prior to the author’s work in [4], the only further progress on
big Ramsey degrees for Henson graphs was for edge relations on the triangle-free
Henson graph. In 1998, Sauer proved in [34] that the big Ramsey degree for edges
in $H_3$ is two. There, progress stalled for lack of techniques and general methodology
until the author’s recent work in [4] proving that $H_3$ has finite big Ramsey degrees.

In this paper, we provide a unified approach to the Ramsey theory of the uni-
versal ultrahomogeneous $k$-clique graph, $H_k$, for each $k \geq 3$. This presentation
encompasses and streamlines work in [4] for $H_3$. The outline of the proof is similar
to that in [4], which built on ideas from Sauer’s proof in [35] that the Rado graph
has finite big Ramsey degrees. However, new obstacles were present for $k \geq 4$.
These and their solutions are discussed as the sections of the paper are delineated
now.

1.2. Outline of paper. Section 2 provides basic definitions and notation. A review
of strong trees and the Halpern-Läuchli and Milliken Theorems (Theorems 2.4 and
2.7) is included to provide key ideas behind the approach taken in this paper for
the Ramsey theory of the Henson graphs. An outline of Sauer’s work in [35] for
the Rado graph can be found in the Introduction of the author’s work for $H_3$
in [4]. The reader interested in more details behind the approach taken in this paper
is referred there.

The article consists of three main phases. The first of these occurs in Sections
3 through 5 where we define the tree structures and prove extension lemmas. In
Section 3 we introduce the notion of strong $K_k$-free trees, analogues of Milliken’s
strong trees capable of coding $k$-clique free graphs. These trees contain certain
distinguished nodes, called coding nodes, which code the vertices of a given graph.
These trees branch maximally, subject to the constraint of the coding nodes not
coding any $k$-cliques, and thus are the analogues of strong trees for the $K_k$-free
setting. Although it is not possible to fully develop Ramsey theory on strong $K_k$-
free trees, they have the main structural aspects of the trees for which we will prove
analogues of Halpern-Läuchli and Milliken Theorems, defined in Section 4. Section
3 is given for the sole purpose of building the reader’s understanding of strong
$H_k$-coding trees.

Our approach in this paper simplifies the one given in [4] and unifies the classes
of trees coding Henson graphs. Section 4 presents a streamlined definition of strong
$H_k$-coding trees as subtrees of a given tree $T_k$ which are stably isomorphic (Defini-
tion 4.9) to $T_k$. The class of these trees is denoted $T_k$, and these trees are best
thought of as skew versions of the trees presented in Section 3. Secondarily, an
internal description of the trees in $T_k$ is given. An important property which these
trees have is the Witnessing Property (Definition 4.12). This means that certain
configurations which can give rise to codings of pre-cliques (Definition 4.6) are wit-
tnessed by coding nodes. The effect is a type of book-keeping to guarantee when
finite trees can be extended within a given tree $T \in T_k$ to another tree in $T_k$.

Section 5 holds Extension Lemmas, guaranteeing when a given finite tree can be
extended to a desired configuration. For $k \geq 4$, some new difficulties arise which did
not exist for $k = 3$. The lemmas in this section extend work in [4], while addressing
new complexities. Further, this section includes some new extension lemmas not
in [4]. These have the added benefit of streamlining proofs in Section 6 in which analogues of the Halpern-Läuchli Theorem are proved.

In the second phase of the paper, Sections 6 and 7, we prove a Ramsey theorem for colorings of certain types of finite trees, namely those with the Strict Witnessing Property (see Definition 7.1). First, we prove analogues of the Halpern-Läuchli Theorem for strong $H_k$-coding trees in Theorem 6.2. The proof builds on ideas from Harrington’s forcing proof of the Halpern-Läuchli Theorem, and uses distinct forcings for the separate cases that the level set being colored has a coding node versus a splitting node. These forcings are not Cohen forcings, as the tree structure and Witnessing Property are centrally tied to the efficacy of the forcings. The main new ingredient for $k \geq 4$ is that all pre-cliques need to be considered and witnessed, not just pre-$k$-cliques. It is important to note that although the technique of forcing is used, the proof of this theorem is established using only the axioms of ZFC.

In Section 7, we apply a third forcing to obtain a true analogue the Halpern-Läuchli Theorem for colorings of level sets containing a coding node. Then, after several more lemmas using induction and fusion, we obtain our first Ramsey Theorem for colorings of finite trees.

**Theorem 7.3.** Let $k \geq 3$ be given and let $T \in T_k$ be a strong $H_k$-coding tree and let $A$ be a finite subtree of $T$ satisfying the Strict Witnessing Property. Then for any coloring of the copies of $A$ in $T$ into finitely many colors, there is a strong $H_k$-coding tree $S \leq T$ such that all copies of $A$ in $S$ have the same color.

In the third phase of the article, Sections 8 and 9, we prove a Ramsey theorem for finite antichains of coding nodes (Theorem 7.3), which is then applied to deduce that each Henson graph has finite big Ramsey degrees. To do this, we must first develop a way to transform antichains of coding nodes into finite trees with the Strict Witnessing Property. This is accomplished in Subsections 8.1 and 8.2 where we develop the notions of incremental new pre-cliques and envelopes. Given any finite $K_k$-free graph $G$, there are only finitely many strict similarity types (Definition 8.4) of antichains coding $G$. Given a coloring $c$ of all copies of $G$ in $T_k$ into finitely many colors, we transfer the coloring to the envelopes of copies of $G$ in a given strong coding tree $T$. Then we apply the results in previous sections to obtain a strong $H_k$-coding tree $T' \leq T$ in which all envelopes encompassing the same strict similarity type have the same color. Upon thinning to an incremental strong subtree $S \leq T'$ while simultaneously choosing a set $W \subseteq T'$ of witnessing coding nodes, each finite antichain $X$ of nodes in $S$ is incremental and has an envelope comprised of nodes from $W$ satisfying the Strict Witnessing Property. Applying Theorem 7.3 finitely many times, once for each strict similarity type, we obtain our second Ramsey theorem for strong $H_k$-coding trees, extending the first one.

**Theorem 8.9.** (Ramsey Theorem for Strict Similarity Types). Fix $k \geq 3$. Let $Z$ be a finite antichain of coding nodes in a strong $H_k$-coding tree $T$, and let $h$ be a coloring of all subsets of $T$ which are strictly similar to $Z$ into finitely many colors. Then there is an incremental strong $H_k$-coding tree $S \leq T$ such that all subsets of $S$ strictly similar to $Z$ have the same $h$ color.

Upon taking an antichain of coding nodes $D \subseteq S$ coding $H_k$, the only sets of coding nodes in $D$ coding a given finite $K_k$-free graph $G$ are automatically antichains which are incremental. Applying Theorem 8.9 to the finitely many strict similarity types of antichains coding $G$, we arrive at the main theorem.
Theorem 9.2. The universal homogeneous k-clique free graph has finite big Ramsey degrees.

For each $G \in K_k$, the number $T(G, K_k)$ is bounded by the number of strict similarity types of antichains of coding nodes coding $G$. It is presently open to see whether this number is in fact the lower bound. If so, then recent work of Zucker in [41] would provide an interesting connection with topological dynamics, as the bad colorings obtainable from our structures cohere in the manner necessary to apply Zucker’s work.

2. Background: Coding graphs, Halpern-Läuchli and Milliken

Theorems, and outline of Sauer’s proof for the Rado graph

2.1. Coding vertices in graphs via finite binary sequences. The following notation, standard in mathematical logic, shall be used throughout. The set of all natural numbers $\{0, 1, 2, \ldots\}$ is denoted by $\omega$. Each natural number $k \in \omega$ is equated with the set of all natural numbers strictly less than $k$; thus, $k = \{0, \ldots, k-1\}$ and in particular, 0 denotes the emptyset. For each natural number $k$, $2^k$ denotes the set of all functions from $\{0, \ldots, k-1\}$ into $\{0, 1\}$, in other words, a binary sequence of length $k$. Given $k \in \omega$ and $s \in 2^k$, we may write $s$ as $(s(0), \ldots, s(k-1))$. For each $i < k$, $s(i)$ denotes the $i$-th value or entry of the sequence $s$. The length of $s$, denoted $|s|$, is the domain of $s$.

We shall use $2^{<\omega}$ to denote $\bigcup_{k \in \omega} 2^k$, the collection of all finite binary sequences. For nodes $s, t \in 2^{<\omega}$, we write $s \subseteq t$ if and only if $|s| \leq |t|$ and for each $i < |s|$, $s(i) = t(i)$. In this case, we say that $s$ is an initial segment of $t$, or that $t$ extends $s$. If $s$ is an initial segment of $t$ and $|s| < |t|$, then we write $s \subset t$ and say that $s$ is a proper initial segment of $t$. For $i < \omega$, we let $s \upharpoonright i$ denote the function $s$ restricted to domain $i$. Thus, if $i < |s|$, then $s \upharpoonright i$ is the proper initial segment of $s$ of length $i$, $s \upharpoonright i = (s(0), \ldots, s(i-1))$; if $i \geq |s|$, then $s \upharpoonright i$ equals $s$.

In [8], Erdős, Hajnal and Pósa used the edge relation on a given graph to induce a natural lexicographic order on the vertices, which they employed to solve problems regarding strong embeddings of graphs. With this lexicographic order, vertices in a given graph can be viewed as nodes within the binary tree of finite sequences of 0’s and 1’s, a view made explicit in [34] which we review below.

Definition 2.1. Given nodes $s, t \in 2^{<\omega}$, if $|s| < |t|$, we say that the passing number of $t$ at $s$ is $t(|s|)$. Let $v, w$ be vertices in some graph. Two nodes $s, t \in 2^{<\omega}$ with $|s| < |t|$ represent $v$ and $w$, respectively, if

$$v E w \iff t(|s|) = 1.$$  

Thus, if $t$ has passing number 1 at $s$, then $s$ and $t$ code an edge between $v$ and $w$; and if $t$ has passing number 0 at $s$, then $s$ and $t$ code a non-edge between $v$ and $w$.

Using this correspondence between the edge relation and passing numbers, any graph can be coded by nodes in a binary tree as follows. Let $G$ be a graph with $N$ vertices, where $N \leq \omega$, and let $(v_n : n < N)$ be any enumeration of the vertices of $G$. Choose any node $t_0 \in 2^{<\omega}$ to represent the vertex $v_0$. For $n > 0$, given nodes $t_0, \ldots, t_{n-1}$ in $2^{<\omega}$ coding the vertices $v_0, \ldots, v_{n-1}$, take $t_n$ to be any node in $2^{<\omega}$ such that $|t_n| > |t_{n-1}|$ and for all $i < n$, $v_n$ and $v_i$ have an edge between them if and only if $t_n(|t_i|) = 1$. Then the set of nodes $(t_n : n < N)$ codes the graph $G$. For the purposes of developing the Ramsey theory of Henson graphs, we make
the convention that the nodes in a tree used to code the vertices in a graph have different lengths. Figure 1. shows a set of nodes \( \{t_0, t_1, t_2, t_3\} \) from \( 2^{<\omega} \) coding the four-cycle \( \{v_0, v_1, v_2, v_3\} \).

2.2. Trees. In this paper, we use definitions which are standard for Ramsey theory on trees, which differ slightly from the routine definitions. The meet of two nodes \( s \) and \( t \) in \( 2^{<\omega} \), denoted \( s \land t \), is the longest member \( u \in 2^{<\omega} \) which is an initial segment of both \( s \) and \( t \). Thus, \( u = s \land t \) if and only if \( u = s \upharpoonright |u| = t \upharpoonright |u| \) and \( s \upharpoonright (|u| + 1) \neq t \upharpoonright (|u| + 1) \). In particular, if \( s \subseteq t \) then \( s \land t = s \). A set of nodes \( A \subseteq 2^{<\omega} \) is closed under meets if \( s \land t \) is in \( A \), for each pair \( s, t \in A \).

**Definition 2.2.** A subset \( T \subseteq 2^{<\omega} \) is a tree if \( T \) is closed under meets and for each pair \( s, t \in T \) with \( |s| \leq |t| \), \( t \upharpoonright |s| \) is also in \( T \).

Thus, in this article, a tree is not necessarily closed under initial segments in \( 2^{<\omega} \). Given a tree \( T \subseteq 2^{<\omega} \), let \( \widehat{T} \) denote the set of all \( s \in 2^{<\omega} \) such that \( s = t \upharpoonright n \) for some \( t \in T \) and \( n \leq |t| \). Notice that \( \widehat{T} \) is a tree in the more commonly defined sense, since it contains all members of \( 2^{<\omega} \) which are initial segments of members of \( T \).

Given \( n < \omega \) and a set of nodes \( A \subseteq 2^{<\omega} \), define

\[
A(n) = \{ t \in A : |t| = n \}.
\]

A set \( X \subseteq A \) is a level set if \( X \subseteq A(n) \) for some \( n < \omega \). Note that a tree \( T \) does not have to contain all initial segments of its members, but for each \( s \in T \), the level set \( T(|s|) \) must equal \( \{ t \upharpoonright |s| : t \in T \text{ and } |t| \geq |s| \} \).

2.3. The Halpern-Läuchli and Milliken Theorems. The theorem of Halpern and Läuchli below was established as a technical lemma containing core combinatorial content of the proof that the Boolean Prime Ideal Theorem (the statement that any filter can be extended to an ultrafilter) is strictly weaker than the Axiom of Choice, assuming the Zermelo-Fraenkel axioms of set theory. (See [16].) The Halpern-Läuchli Theorem forms the basis for a Ramsey theorem on strong trees due to Milliken, which in turn forms the backbone of all previously found finite big Ramsey degrees, except where Ramsey’s Theorem itself suffices. An in-depth presentation of the various versions of the Halpern-Läuchli Theorem as well as Milliken’s Theorem can be found in [37]. An account focused solely on the theorems relevant to the present work can be found in [3]. Here, we merely give an overview sufficient for this article, and shall restrict to subtrees of \( 2^{<\omega} \), though the results hold more generally for finitely branching trees.
Figure 2. A strong subtree of $2^{<\omega}$ of height 3

Definition 2.3 (Strong tree). Let $T \subseteq 2^{<\omega}$ be a tree and let $L = \{|s| : s \in T\}$. When $L$ is infinite, then $T$ is a strong tree if and only if every node in $T$ splits in $T$; that is, for each $t \in T$, there are $u, v \in T$ such that $u$ and $v$ properly extend $t$, and $u(|t|) = 0$ and $v(|t|) = 1$. When $L$ is finite, then $T$ is a strong tree if and only if each node $t \in T$ with $|t| < \max(L)$ $t$ splits in $T$. A finite strong tree subtree of $2^{<\omega}$ with $k$ many levels is called a strong tree of height $k$.

Note that each finite strong subtree of $2^{<\omega}$ is isomorphic as a tree to some binary tree of height $k$, where the isomorphism preserves relative lengths of nodes. In particular, a strong tree of height 1 is simply a node in $2^{<\omega}$. See Figure 2. for an example of a strong tree of height 3.

The following is the strong tree version of the Halpern-Läuchli Theorem. It is a Ramsey theorem for colorings of products of level sets of finitely many trees. Here, we restrict to the case of binary trees, since that is sufficient for the exposition in this paper.

Theorem 2.4 (Halpern-Läuchli, [15]). Let $T_i = 2^{<\omega}$ for each $i < d$, where $d$ is any positive integer, and let

$$c : \bigcup_{n < \omega \land i < d} T_i(n) \rightarrow k$$

be a given coloring, where $k$ is any positive integer. Then there is an infinite set of levels $L \subseteq \omega$ and infinite strong subtrees $S_i \subseteq T_i$, each with nodes exactly at the levels in $L$, such that $c$ is monochromatic on

$$\bigcup_{n \in L \land i < d} S_i(n).$$

This theorem of Halpern and Läuchli was applied by Laver in [23] to prove that given $k \geq 2$ and given any coloring of the product of $k$ many copies of the rationals $Q^k$ into finitely many colors, there are subsets $X_i$ of the rationals which again are dense linear orders without endpoints such that $X_0 \times \cdots \times X_{k-1}$ has at most $k!$ colors. Laver further proved that $k!$ is the lower bound. Thus, the big Ramsey degree for the simplest object (single $k$-length sequences) in the Fraïssé class of products of finite linear orders has been found. The full result for all big Ramsey degrees for $\text{Age}(Q^k)$ would involve applications of the extension of
Milliken’s theorem to products of finitely many copies of $2^{<\omega}$; such an extension has been proved by Vlitas in [39].

Harrington produced an innovative method of proof of the Halpern-Läuchli Theorem which uses the set-theoretic technique of forcing, but which takes place entirely in the standard axioms of set theory, ZFC. No new external model is actually built, but rather, finite bits of information, guaranteed by the existence of a generic filter for the forcing, are used to build the subtrees satisfying the Halpern-Läuchli Theorem. This proof is said to provide the clearest intuition into the theorem (see [37]). Harrington’s proof was never published, though the ideas were well-known in certain circles. A version close to his original proof appears in [3], where a proof was reconstructed based on an outline provided to the author by Laver in 2011. This proof formed the starting point for our proofs in Sections 6 and 7 of Halpern-Läuchli style theorems for strong $\mathcal{H}_k$-coding trees.

Harrington’s proof for $d$ many trees uses the forcing which adds $\kappa$ many Cohen subsets of the product of level sets of $d$ many copies of $2^{<\omega}$, where $\kappa$ satisfies a certain partition relation, depending on $d$. For any set $X$ and cardinal $\mu$, $[X]^\mu$ denotes the collection of all subsets of $X$ of cardinality $\mu$.

**Definition 2.5.** Given cardinals $r, \sigma, \kappa, \lambda$,

\[
\lambda \rightarrow (\kappa)^r_\sigma,
\]

means that for each coloring of $[\lambda]^r$ into $\sigma$ many colors, there is a subset $X$ of $\lambda$ such that $|X| = \kappa$ and all members of $[X]^r$ have the same color.

The following ZFC result guarantees cardinals large enough to have the Ramsey property for colorings into infinitely many colors.

**Theorem 2.6 (Erdős-Rado, [9]).** For $r < \omega$ and $\mu$ an infinite cardinal,

\[
\beth_r(\mu)^+ \rightarrow (\mu^+)^{r+1}_\mu.
\]

For $d$ many trees, letting $\kappa = \beth_{2d-1}(\aleph_0)^+$ suffices for Harrington’s proof. A modified version of Harrington’s proof appears in [38], where the assumption on $\kappa$ is weaker, only $\beth_{d-1}(\aleph_0)^+$, but the construction is more complex. This proof informed the approach in [5] to reduce the large cardinal assumption for obtaining the consistency of the Halpern-Läuchli Theorem at a measurable cardinal. Building on this and ideas from [36] and [6], Zhang proved the consistency of Laver’s result for the $\kappa$-rationals, for $\kappa$ measurable, in [40].

The Halpern-Läuchli Theorem forms the essence of the next Theorem; the proof follows by induction on $k$, applying Theorem 2.4 to $k$ many infinite strong trees.

**Theorem 2.7 (Milliken, [25]).** Let $k \geq 1$ be given and let all strong subtrees of $2^{<\omega}$ of height $k$ be colored by finitely many colors. Then there is an infinite strong subtree $T$ of $2^{<\omega}$ such that all strong subtrees of $T$ of height $k$ have the same color.

2.4. Outline of Sauer’s proof of upper bound for big Ramsey degrees of the Rado graph. Sauer’s proof in [35] that the Rado graph has finite big Ramsey degrees provided a strategy for our proof in [4] for $\mathcal{H}_3$ and for the extended work to all Henson graphs in this paper. An outline of Sauer’s proof is as follows: Graphs can be coded by collections of finite binary sequences. In particular, the graph coded by all the nodes in $2^{<\omega}$, where nodes of the same length code no edge between their represented vertices, is bi-embeddable with the Rado graph. This important aspect
of the Rado graph was one of the keys to Sauer’s work. No analogue of this exists for the Henson graphs, this being one of the main reasons big Ramsey degrees for Henson graphs had not been proved earlier.

Similarly to Devlin’s work for the rationals in [2], Sauer showed that only certain types of antichains of binary sequences need to be considered when handling tree codings of a given finite graph $G$. These antichains induce trees which are skew and have further properties Sauer called strongly diagonal. The second key to Sauer’s proof is that any finite strongly diagonal set can be enveloped into a finite strong tree.

The third important point is that any coloring on the copies of a finite graph $G$ in the Rado graph can be extended to color the strong tree envelopes. Applying Milliken’s Theorem for strong trees finitely many times produces an infinite strong subtree $S$ of $2^{<\omega}$ in which for all diagonal antichains coding $G$ with the same strong similarity type have the same color. To finish, Sauer takes a strongly diagonal $D$ subset of $S$ which codes the Rado graph, so that all codings of $G$ in $D$ must be antichains which are strongly diagonal. Since there are only finitely many similarity types of strongly diagonal antichains coding $G$, this yields the upper bound for the big Ramsey degree of $G$ in the Rado graph.

A more detailed outline of the work in [35] appears in Section 3 of [3], which surveys some recent work regarding Halpern-Läuchli and Milliken Theorems and variants. Chapter 6 of [37] provides a solid foundation for understanding how Milliken’s theorem is used to attain big Ramsey degrees for both Devlin’s result on the rationals and Sauer’s result on the Rado graph. Of course, we recommend foremost Sauer’s original article [35].

We point out that Milliken’s Theorem has been shown to consistently hold at a measurable cardinal by Shelah in [36], using ideas from Harrington’s proof. An enriched version was proved by Džamonja, Larson, and Mitchell in [6] and applied to obtain the consistency of finite big Ramsey degrees for colorings of finite subsets of the $\kappa$-rationals, where $\kappa$ is a measurable cardinal. They obtained the consistency of finite big Ramsey degrees for colorings of finite subgraphs of the $\kappa$-Rado graph for $\kappa$ measurable in [6]. The uncountable height of the tree $2^{<\kappa}$ coding the $\kappa$-rationals and the $\kappa$-Rado graph renders the notion of strong similarity type more complex than for the countable cases.

There is another theorem stronger than Theorem 2.7 also due to Milliken in [26], which shows that the collection of all infinite strong subtrees of $2^{<\omega}$ forms a topological Ramsey space, meaning that it satisfies an infinite-dimensional Ramsey theorem for Baire sets when equipped with its version of the Ellentuck topology (see [37]). This fact informed some of our intuition when approaching the present work.

3. Trees coding $\mathcal{H}_k$, $k \geq 3$: A first approach

This section introduces a unified approach for coding the Henson graphs via trees with special distinguished nodes. These trees are called strong $K_k$-free trees (Definition 3.10), since they branch as fully as possible without coding $k$-cliques. The constructions build on and extend ideas behind the strong triangle-free trees in [4] which code the triangle-free Henson graph. While it is not possible to fully develop Ramsey theory on strong $K_k$-free trees, as shown by Example 3.18 of [4], the trees in this section provide the essential structure behind the strong coding
trees in Section 3. This section is intended to build the reader’s understanding of the structure responsible for coding \( \mathcal{H}_k \), so that the structure of strong coding trees, on which we will develop Ramsey theory throughout this paper, will be more clear.

### 3.1. Henson’s Criterion

Recall that \( K_k \) denotes a complete graph on \( k \) vertices, also called a \( k \)-clique. In [17], for each \( k \geq 3 \), Henson constructed a homogeneous \( K_k \)-free graph which is universal for all \( K_k \)-free graphs on countably many vertices. We denote these graphs by \( \mathcal{H}_k \). It was later seen that \( \mathcal{H}_k \) is isomorphic to Fraïssé limit of the Fraïssé class of finite \( K_k \)-free graphs. Given a graph \( H \) and a subset \( V_0 \) of the vertices of \( H \), the notation \( H|V_0 \) denotes the induced subgraph of \( H \) on the vertices in \( V_0 \). In [17], Henson proved that a countable graph \( H \) is universal for \( K_k \)-free graphs if and only if \( H \) satisfies the following property.

\[
(A_k) \quad \text{(i)} \quad H \text{ does not admit any } k\text{-cliques,}
\]

\[
\text{(ii)} \quad \text{If } V_0, V_1 \text{ are disjoint finite sets of vertices of } H \text{ and } H|V_0 \text{ does not admit any } (k-1)\text{-cliques, then there is another vertex which is connected in } H \text{ to every member of } V_0 \text{ and to no member of } V_1.
\]

The following modification will be useful for our constructions.

\[
(A_k)' \quad \text{(i)} \quad H \text{ does not admit any } k\text{-cliques.}
\]

\[
\text{(ii)} \quad \text{Let } \langle v_n : n < \omega \rangle \text{ enumerate the vertices of } H \text{, and let } \langle F_i : i < \omega \rangle \text{ be any enumeration of the finite subsets of } \omega \text{ such that for each } i < \omega, \max(F_i) < i \text{ and each finite set appears infinitely many times in the enumeration. Then there is a strictly increasing sequence } \langle n_i : i < \omega \rangle \text{ such that for each } i < \omega, \text{ if } H|\{v_m : m \in F_i\} \text{ has no } (k-1)\text{-cliques, then for all } m < i, v_n, E v_m \iff m \in F_i.
\]

It is routine to check that any countably infinite graph \( H \) is universal for \( K_k \) if and only if \( (A_k)' \) holds.

### 3.2. Trees with coding nodes and strong \( K_k \)-free trees

As seen for the case of triangle-free graphs in [1], enriching trees with a collection of distinguished nodes allows for coding graphs with forbidden configurations into trees which have properties similar to strong trees.

**Definition 3.1** ([3]). A tree with coding nodes is a structure \( (T, N; \subseteq, <, c) \) in the language \( L = \{\subseteq, <, c\} \), where \( \subseteq \) and \( < \) are binary relation symbols and \( c \) is a unary function symbol, satisfying the following: \( T \) is a subset of \( 2^{<\omega} \) satisfying that \( (T, \subseteq) \) is a tree (recall Definition 2.2), \( N \leq \omega \) and \( < \) is the usual linear order on \( N \), and \( c : N \to T \) is an injective function such that \( m < n < N \) implies \( |c(m)| < |c(n)| \).

The \( n \)-th coding node in \( T \), \( c(n) \), will usually be denoted as \( c_n^T \).

The length \( |c_n^T| \) of the \( n \)-th coding node in \( T \) shall be denoted by \( l_n^T \). Whenever no ambiguity arises, we shall drop the superscript \( T \).

**Definition 3.2** ([3]). A graph \( G \) with vertices enumerated as \( \langle v_n : n < N \rangle \) is represented by a tree \( T \) with coding nodes \( \langle c_n : n < N \rangle \) if and only if for each pair \( i < n < N \), \( v_n E v_i \iff c_n(l_i) = 1 \). We will often simply say that \( T \) codes \( G \).

The following observation shows exactly how cliques are coded.

**Observation 3.3.** For \( a \geq 2 \), given an index set \( I \) of size \( a \), a collection of coding nodes \( \{ c_i : i \in I \} \) in \( T \) codes an \( a \)-clique if and only if for each pair \( i < j \) in \( I \), \( c_j(l_i) = 1 \).
We now present a criterion which will ensure that a tree with coding nodes does not code a $k$-clique.

**Definition 3.4** ($K_k$-Free Criterion). Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n : n < N \rangle$, where $N \leq \omega$. We say that $T$ satisfies the $K_k$-Free Criterion if the following holds: For each $n \geq k - 2$, for all increasing sequences $i_0 < i_1 < \cdots < i_{k-2} = n$ such that $\{c_{i_j} : j < k - 1\}$ codes a $(k - 1)$-clique, for each $t \in T$ such that $|t| > l_n$,

$$\forall j < k - 2 \quad t(i_j) = 1 \implies t(l_n) = 0.$$  

Thus, a tree $T$ with coding nodes $\langle c_n : n < N \rangle$ satisfies the $K_k$-Free Criterion if for each $n < N$, whenever a node $t$ in $T$ has the same length as the coding node $c_n$, and $t$ and $c_n$ both code edges with some collection of $k - 2$ many coding nodes which themselves code a $(k - 2)$-clique, then $t$ does not split in $T$; its only allowable extension in $T$ is $t \smallfrown 0$.

The next lemma characterizes tree representations of $K_k$-free graphs. We say that the coding nodes in $T$ are dense in $T$, if for each $t \in T$, there is some coding node $c_n \in T$ such that $t \subseteq c_n$. Note that a finite tree $T$ in which the coding nodes are dense will necessarily have coding nodes (of differing lengths) as its maximal nodes.

**Lemma 3.5.** Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n : n < N \rangle$ coding a countable graph $G$ with vertices $\langle v_n : n < N \rangle$, where $N \leq \omega$. Assume that the coding nodes in $T$ are dense in $T$. Then $G$ is $K_k$-free if and only if $T$ satisfies the $K_k$-Free Criterion.

**Proof.** If $T$ does not satisfy the $K_k$-Free Criterion, then there are $i_0 < \cdots < i_{k-2} < N$ and $t \in T$ with $|t| > l_{i_{k-2}}$ such that $\{c_{i_j} : j < k - 1\}$ codes a $(k - 1)$-clique and $t(i_j) = 1$ for all $j < k - 1$. Since the coding nodes are dense in $T$, there is an $n > i_{k-2}$ such that $c_n \supseteq t$. Then $\{c_{i_j} : j < k - 1\} \cup \{c_n\}$ codes a $k$-clique. On the other hand, if $G$ contains a $k$-clique, then there are $i_0 < \cdots < i_{k-1}$ such that the coding nodes $\{c_{i_j} : j < k\}$ in $T$ code a $k$-clique, and these coding nodes witness the failure of the $K_k$-Free Criterion in $T$. \qed

The next criterion ensures maximal branching, subject to never coding a $k$-clique.

**Definition 3.6** ($K_k$-Free Branching Criterion). A tree $T$ with coding nodes $\langle c_n : n < N \rangle$ satisfies the $K_k$-Free Branching Criterion ($k$-FBC) if for each non-maximal node $t \in T$, $t \smallfrown 0$ is always in $T$, and $t \smallfrown 1$ is in $T$ if and only if adding $t \smallfrown 1$ as a coding node to $T$ would not code a $k$-clique with coding nodes in $T$ of shorter length.

Thus, a tree $T$ satisfies the $K_k$-Free Branching Criterion if and only if $T$ is maximally splitting subject to satisfying the $K_k$-Free Criterion.

As we move toward defining strong $K_k$-free coding trees in Definition 8.10, we recall that the modified Henson criterion $(A_k)'$ is satisfied by an infinite $K_k$-free graph if and only if it is homogeneous and universal for all countable $K_k$-free graphs. The following reformulation translates $(A_k)'$ in terms of trees with coding nodes. We say that a tree $T \subseteq 2^{<\omega}$ with coding nodes $\langle c_n : n < \omega \rangle$ satisfies property $(A_k)_\text{tree}$ if the following hold:

$(A_k)_\text{tree}$ (i) $T$ satisfies the $K_k$-Free Criterion.
(ii) Let \( \langle F_i : i < \omega \rangle \) be any enumeration of finite subsets of \( \omega \) such that for each \( i < \omega \), \( \max(F_i) < i - 1 \), and each finite subset of \( \omega \) appears as \( F_i \) for infinitely many indices \( i \). Given \( i < \omega \), if for each subset \( J \subseteq F_i \) of size \( k - 1 \), \( \{c_j : j \in J\} \) does not code a \((k - 1)\)-clique, then there is some \( n \geq i \) such that for all \( j < i \), \( c_n(l_j) = 1 \) if and only if \( j \in F_i \).

**Observation 3.7.** If \( T \) satisfies \((A_k)^{\text{tree}}\), then the coding nodes in \( T \) code \( \mathcal{H}_k \).

To see this, suppose that \( T \) satisfies \((A_k)^{\text{tree}}\), and let \( \mathcal{H} \) be the graph with vertices \( \langle v_n : n < \omega \rangle \) where for \( m < n \), \( v_n \triangleleft v_m \) if and only if \( c_n(l_m) = 1 \). Then \( \mathcal{H} \) satisfies Henson’s property \((A_k)\), and hence is homogeneous and universal for countable \( k\)-clique-free graphs.

The next lemma shows that any finite tree with coding nodes satisfying the \( k \)-FBC preserves all types, and hence can be extended to a tree satisfying \((A)^{\text{tree}}\). This is a step toward proving Theorem 3.9. Any tree with no maximal nodes, a dense set of coding nodes, and satisfying the \( k \)-FBC codes \( \mathcal{H}_k \).

**Lemma 3.8.** Let \( T \) be a finite tree with coding nodes \( \langle c_n : n < N \rangle \), where \( N < \omega \), with all maximal nodes of length \( |N| - 1 \) and satisfying the \( K_k \)-Free Branching Criterion. Given any \( F \subseteq N - 1 \) for which the set \( \{c_n : n \in F\} \) codes no \((k - 1)\)-cliques, there is a maximal node \( t \in T \) such that for all \( n < N - 1 \),

\[
(10) \quad t(l_n) = 1 \iff n \in F.
\]

**Proof.** The proof is by induction on \( N < \omega \) over all such trees with \( N \) coding nodes. For \( N = 1 \), there is only one coding node in \( T \), \( c_0 \) with length \( l_0 = 1 \). Since the only subset of \( N - 1 = 0 \) is the emptyset, the lemma trivially holds.

Now suppose \( N \geq 2 \) and suppose the lemma holds for all trees with less than \( N \) coding nodes. Let \( T \) be a tree with coding nodes \( \langle c_n : n < N \rangle \) satisfying the \( k \)-FBC. Let \( F \) be a subset of \( N - 1 \) such that \( \{c_n : n \in F\} \) codes no \((k - 1)\)-cliques. By the induction hypothesis, \( T \restriction l_{N-2} \) satisfies the lemma, recalling that \( T \restriction l_{N-2} \) denotes \( \{t \in T : |t| \leq l_{N-2}\} \). So there is a node \( t \in T \) of length \( l_{N-2} \) such that for all \( n < N - 2 \), \( t(l_n) = 1 \) if and only if \( n \in F \setminus \{N - 2\} \). If \( N - 2 \notin F \), then as \( t \rhd 0 \) is guaranteed to be in \( T \) by the \( k \)-FBC, the node \( t \rhd 0 \) satisfies the lemma.

Now suppose \( N - 2 \in F \). We claim that \( t \rhd 1 \) is in \( T \). By the \( k \)-FBC, if \( t \rhd 1 \) is not in \( T \), it is because there is some sequence \( i_0 < \cdots < i_{k-2} = N - 2 \) such that \( \{c_{i_j} : j < k - 1 \} \) codes a \((k - 1)\)-clique and \( t(l_{i_j}) = 1 \) for each \( j < k - 2 \). Since for all \( i < N - 2 \), \( t(l_i) = 1 \) if and only if \( i \in F \setminus \{N - 2\} \), it follows that \( \{i_j : j < k - 2 \} \subseteq F \). But then \( F \supseteq \{i_j : j < k - 2 \} \), which contradicts that \( F \) codes no \((k - 1)\)-cliques. Therefore, \( t \rhd 1 \) is in \( T \), and this node satisfies the lemma.

**Theorem 3.9.** Let \( T \) be a tree with no maximal nodes and coding nodes dense in \( T \), and satisfying the \( K_k \)-Free Branching Criterion. Then \( T \) satisfies \((A_k)^{\text{tree}}\), and hence codes \( \mathcal{H}_k \).

**Proof.** Since \( T \) satisfies the \( k \)-FBC, it automatically satisfies (i) of \((A_k)^{\text{tree}}\). Let \( \langle F_i : i < \omega \rangle \) be any enumeration of finite subsets of \( \omega \) as in (ii) of \((A_k)^{\text{tree}}\). For \( i = 0 \), \( F_i \) is the emptyset, so every coding node in \( T \) fulfills (ii) of \((A_k)^{\text{tree}}\). Let \( 1 \leq i < \omega \) be given and suppose that for each subset \( J \subseteq F_i \) of size \( k - 1 \), \( \{c_j : j \in J\} \) does not code a \((k - 1)\)-clique. By Lemma 3.8, there is some node \( t \in T \) of length \( l_{i-1} \) such that for all \( n < i - 1 \), \( t(l_n) = 1 \) if and only if \( n \in F_i \). Since the coding nodes are dense in \( T \), there is some \( j > i \) such that \( c_j \) extends \( t \). This coding node \( c_j \) fulfills (ii) of \((A_k)^{\text{tree}}\).
In this section, when defining strong $K_k$-free trees, we will add two requirements in addition to satisfying the $K_k$-Free Branching Criterion and having a dense set of coding nodes. We shall use ghost coding nodes for the first $k - 3$ levels. Coding nodes will start at length $k - 2$, and all coding nodes of length $k - 2$ or more will end in a sequence of $(k - 2)$ many 1’s. The effect is that coding nodes will only be extendible by 0; coding nodes will never split. This will serve to reduce the upper bound on the big Ramsey degrees for $\mathcal{H}_k$.

**Definition 3.10 (Strong $K_k$-Free Tree).** A strong $K_k$-free tree is a tree with coding nodes, $(T, \omega; \subseteq, <, c)$ satisfying the following:

1. $T$ has no maximal nodes, the coding nodes are dense in $T$, and no coding node splits in $T$.
2. The first $k - 2$ levels of $T$ are exactly $2^{\leq k - 2}$, and the least coding node $c_0$ is exactly $1^{(k - 2)}$.
3. For each $n < \omega$, the $n$-th coding node $c_n$ has length $n + k - 2$, and has final segment a sequence of $k - 2$ many 1’s.
4. $T$ satisfies the $K_k$-free Branching Criterion.

Moreover, $T$ has ghost coding nodes $c_{-k + 2}, \ldots, c_{-1}$ defined by $c_n = 1^{(k + n - 2)}$ for $n \in [-k + 2, -1]$, where $1^{(0)}$ denotes the empty sequence. A finite strong $K_k$-free tree is the restriction of a strong $K_k$-free tree to some finite level.

By Theorem [8.10] each strong $K_k$-free tree codes $\mathcal{H}_k$.

**Remark 3.11.** The ghost coding nodes mimic what would happen if the current tree was taken as a subtree of some larger tree $U$ coding $\mathcal{H}_k$ consisting of nodes above lengths $k$ in $U$. We point out that for any leftmost node $t$ in $T$ of length at least $k - 2$, the cone in $T$ above $t$ will code $\mathcal{H}_k$. Further, the structure of its first $k - 2$ levels above $|t|$ are tree isomorphic to $2^{\leq k - 2}$, and no coding node will split. The ghost coding nodes provide the correct structure which subtrees will automatically inherit, enabling us to build the collection of all subtrees of a given tree $T$ isomorphic to $T$, in a strong way to be made precise in the next section.

The presentation of strong $K_k$-free trees in this paper is minimal. The aim of this section is simply to build the reader’s understanding of their structural properties which will be inherited by their skewed versions introduced in the next section. We now present a method for constructing strong $K_k$-free trees. This construction method is simpler than the one given in [2] and accomplishes the same goals.

**Example 3.12 (Construction of a Strong $K_k$-Free Tree, $S_k$).** Let $\{u_i : i < \omega\}$ enumerate the nodes in $2^{<\omega}$ in such a manner that $i < j$ implies $|u_i| \leq |u_j|$. We will build a strong $K_k$-free tree $S_k \subseteq 2^{<\omega}$ with the $n$-th coding node $c_n$ of length $l_n = n + k - 2$ satisfying the following:

(i) For $i < \omega$, if $u_i$ is in $S_k \upharpoonright (\leq l_{i(k - 1)})$, then $u_i$ is extended by the coding node $c_{i+1(k-1)}$.

(ii) For $n = i(k - 1) + j$, where $i < \omega$ and $1 \leq j < k - 1$, $c_n = 0^{(n)}\upharpoonright 1^{(k - 2)}$.

The first $k - 2$ levels of $S_k$ are exactly $2^{\leq k - 2}$. The ghost coding nodes are defined as in Definition [8.10] with $c_{-k + 2}$ being the empty sequence and the longest ghost coding node being $c_{-1} = 1^{(k - 3)}$. The shortest coding node is $c_0 = 1^{(k - 2)}$. For each $0 \leq n < k - 1$, having defined $S_k \upharpoonright l_n$ and the coding node $c_n$, let every node in $S_k \upharpoonright l_n$ except for $c_n$ split to form the next level, $S_k \upharpoonright l_{n+1}$. Define $c_{n+1}$ to be the...
node $0^{(n+1)-1}(k-2)$. This constructs $S_k \upharpoonright (\leq l_{k-1})$ satisfying (i) and (ii), since $c_{k-1}$ extends $u_0$, which is the empty sequence.

Now suppose $n = i(k-1)$ for some $i \geq 1$, and $S_k \upharpoonright (\leq l_i)$ has been constructed satisfying (i) and (ii). Given $S_k \upharpoonright (\leq l_{i+j})$ where $j < k-2$, let the nodes in $S_k \upharpoonright l_{i+j}$ split according to the $K_k$-Free Branching Criterion and declare $c_{n+j+1}$ to be $0^{(n+j+1)-1}(k-2)$. This builds $S_k \upharpoonright l_{(i+1)(k-1)-1}$ satisfying (i) and (ii). Let $p = (i+1)(k-1)$. And let the nodes in $S_k \upharpoontright l_{p-1}$ split according to the $k$-FBC. The task now is to choose $c_p$ so that (i) will be satisfied. If $u_i$ is not in $S_k \upharpoontright (\leq l_{i(k-1)})$, then let $c_p = 0^{(p)-1}(k-2)$. If $u_i$ is in $S_k \upharpoontright (\leq l_{i(k-1)})$, then let $p - |u_i|$ and define $u' = u \cdot 0(p \cdot 1)(k-2)$. We claim that $u'$ is in $S_k \upharpoontright l_p$.

We shall show that given any collection of $k-1$ many coding nodes in $S_k \upharpoontright (\leq l_{p-1})$ coding a $(k-1)$-clique, there must be some node in that collection for which $u'$ has passing number 0. It will then follow that $u'$ is in $S_k \upharpoontright l_p$, since $u'$ satisfies the $K_k$-Free Criterion and $S_k$ is being built to be maximally branching subject to satisfying the $k$-FBC. Let $l_0 < \cdots < l_{k-2} \leq p-1$ be given such that $\{c_{i_j} : j \leq k-2\}$ codes a $(k-1)$-clique. Suppose that $i_{k-2} = i(k-1)+m$, where $1 \leq m \leq p-2$. By our construction, $c_{i(k-1)+m}$ codes edges with only $k-2$ many coding nodes with smaller indices, and these are exactly $c_{i(k-1)+m-j}$ for $1 \leq j \leq k-2$. If $m \geq 1$, then $c_{i(k-1)+m}$ codes only $k-2$ many edges with coding nodes of smaller indices; hence $\{c_{i_j} : j \leq k-2\}$ cannot code a $(k-1)$-clique. Thus, it must be the case that $i_{k-2} < i(k-1)$. If $|u_i| \leq l_{i_{k-2}}$, then $u'$ has passing number 0 at $c_{i(k-1)}$. If $|u_i| > l_{i_{k-2}}$, then $u_i$ must have passing number 0 at $c_{i_j}$ for some $j < k-1$ since $S_k \upharpoontright (\leq l_{i(k-1)})$ satisfies the $k$-FBC. Thus, $u'$ is a member of $S_k \upharpoontright l_p$. Declare $c_p = u'$, and note that $S \upharpoontright (\leq l_{i(k-1)})$ satisfies (i) and (ii).

This inductive process constructs a tree $S_k = \bigcup_{n<\omega} S_k \upharpoontright l_n$ which is a strong $K_k$-free tree.

Remark 3.13. For $k = 3$, the previous construction of a strong triangle-free tree produces a strong triangle-free tree in the sense of [4], albeit in a more streamlined fashion.

Example 3.14 (A Strong Triangle-Free Tree). In keeping with the description of strong $K_k$-free trees above, we present the first six levels of the construction of a strong $K_3$-free tree. Let $u_0$ denote the empty sequence, and suppose $u_1 = \langle 1 \rangle$ and $u_2 = \langle 0 \rangle$. Here, $k - 2 = 1$, and the first two levels of the tree are simply $2^{\leq 1}$. The ghost coding node is $c_{-1} = \langle \rangle$, the empty sequence. The first coding node is $c_0 = \langle 1 \rangle$; split according to the $K_3$-Free Branching Criterion to construct $S_3 \upharpoontright (\leq 2)$. Since $u_0$ is in $S_3 \upharpoontright (\leq 0)$, the next coding node $c_1$ should end in a one and extend $u_0$. The node $\langle 0, 1 \rangle$ is in $S_3 \upharpoontright 2$ and satisfies these requirements, so let $c_1 = \langle 0, 1 \rangle$, and split according to the 3-FBC to construct $S_3 \upharpoontright (\leq 3)$. Let $c_2 = \langle 0, 0, 1 \rangle$, which is in $S_3 \upharpoontright 3$, and extend to the next level according to the 3-FBC to obtain $S_3 \upharpoontright (\leq 4)$. The node $u_1$ is in $S_3 \upharpoontright (\leq 2)$. Let $c_3 = \langle 1, 0, 0, 1 \rangle$, since this node is in $S_3 \upharpoontright 4$, extends $u_1$, and ends with a one. Split according to the 3-FBC to construct $S_3 \upharpoontright (\leq 5)$. Let $c_4 = \langle 0, 0, 0, 1 \rangle$, and split again by the 3-FBC to construct $S_3 \upharpoontright (\leq 6)$. The node $u_2$ is in $S_3 \upharpoontright (\leq 4)$, and we can take $c_5$ to be $\langle 0, 1, 0, 1, 0, 1 \rangle$, since this node is in $S_3 \upharpoontright 6$ and extends $u_2$. In this manner, one constructs the tree in Figure 3.

Example 3.15 (A Strong $K_4$-Free Tree). The following tree $S_4$ in Figure 4, is an example of a strong $K_4$-free coding tree. The ghost coding nodes $c_{-2} = \langle \rangle$ and
$c_{-1} = \{1\}$ represent that $S_4$ is to be thought of as being a subtree of some larger tree which already has coding nodes below $c_0$, each of which has $\langle 1,1 \rangle$ as its final segment. Suppose that $u_0 = \emptyset$ and $u_1 = \{1\}$. According to the construction in Example 3.12, the first three coding nodes of $S_4$ are $c_0 = \langle 1,1 \rangle$, which extends $u_0$, $c_1 = \langle 0,1,1 \rangle$, and $c_2 = \langle 0,0,1,1 \rangle$, each time splitting according to the $K_4$-Free Branching Criterion to construct a tree $S_4 \upharpoonright (\leq l_2)$. Since $u_1$ is in the tree constructed so far, we split the nodes in $S_4 \upharpoonright l_2$ according to the 4-FBC and take $c_3$ in this new level to extend $u_1$; letting $c_3 = \langle 1,1,0,1,1 \rangle$ works. Then split according to the 4-FBC and take $c_4 = \langle 0,0,0,1,1 \rangle$ and continue the construction according to Example 3.12.

As in the case of $H_3$ in [4], the purpose of not allowing coding nodes to split is to reduce the number of different types of trees coding a given finite $K_k$-free graph. Having the coding nodes be dense in the tree enables the development of Ramsey theory. The same example of a bad coloring as given in Example 3.18 of [4] provides a bad coloring for any strong $K_k$-free tree, for any $k \geq 3$. So we immediately turn to the next section where we develop the skewed version of these trees so that the relevant Ramsey theory can be developed.
4. Strong $\mathcal{H}_k$-coding trees

In order to prove that $\mathcal{H}_k$ has finite big Ramsey degrees, we develop structures called strong $\mathcal{H}_k$-coding trees (Definition 4.1). These trees are skewed versions of the strong $K_k$-free trees constructed in the previous section, the skewing being necessary to avoid bad colorings. The space of strong $\mathcal{H}_k$-coding trees, defined in Subsection 4.2, provides the main tool for developing Ramsey theory applicable to $\mathcal{H}_k$. By the end of Section 7 these spaces of strong $\mathcal{H}_k$-coding trees will be shown to have many similarities to the Milliken space of strong trees [25]. The difficulty is that they involve several new structural properties which must be handled with precision. This section extends results of Section 4 in [4] to $\mathcal{H}_k$ for all $k \geq 3$, while providing a new, more streamlined approach.

4.1. Definitions, notation, and maximal strong $\mathcal{H}_k$-coding trees

The following terminology and notation will be used throughout. A subset $X \subseteq 2^{<\omega}$ is a level set if all nodes in $X$ have the same length. By a tree, we mean exactly a subset $T \subseteq 2^{<\omega}$ which is closed under meets and is a union of level sets; that is, if $s, t \in T$ and $|t| \geq |s|$ then $t^{|s|}$ is also a member of $T$. Let $T \subseteq 2^{<\omega}$ be a tree with coding nodes $\langle c_n^T : n < N \rangle$, where $N \leq \omega$, and let $\hat{I}_n^T$ denote $|c_n^T|$ (recall Definition 3.1). The collection of all initial segments of nodes in $T$ is denoted by $\hat{T}$; thus, $\hat{T} = \{ t \mid n : t \in T \text{ and } n \leq |t| \}$. A node $s \in T$ is called a splitting node if both $s \upharpoonright 0$ and $s \upharpoonright 1$ are in $\hat{T}$; equivalently, $s$ is a splitting node in $T$ if there are nodes $s_0, s_1 \in T$ such that $s_0 \supseteq s \upharpoonright 0$ and $s_1 \supseteq s \upharpoonright 1$. Given $t$ in a tree $T$, the level of $T$ of length $|t|$ is the set of all $s \in T$ such that $|s| = |t|$. This is denoted at $T \upharpoonright |t|$. And is the set of $s \upharpoonright |t|$ such that $s \in T$ and $|s| \geq |t|$. $T$ is skew if each level of $T$ has exactly one of either a coding node or a splitting node. A skew tree $T$ is strongly skew if additionally for each splitting node $s \in T$, every $t \in T$ such that $|t| > |s|$ and $t \not\supset s$ also satisfies $t(|s|) = 0$; that is, the passing number of any node passing by, but not extending, a splitting node is 0. The set of levels of a skew tree $T \subseteq 2^{<\omega}$, denoted $\ell T$, is the set of those $l \leq \omega$ such that $T$ has either a splitting or a coding node of length $l$.

Given a skew tree $T$ with coding nodes $\langle c_n^T : n < N \rangle$, the enumeration of all coding and splitting nodes of $T$ in increasing order of length is denoted as $\langle d_m^T : m < M \rangle$. The nodes $d_m^T$ will be called the critical nodes of $T$. Applying the standard notation for strong trees, for each $m < M$, the $m$-th level of $T$ is

\begin{equation}
T(m) = \{ s \in \hat{T} : |s| = |d_m^T| \}.
\end{equation}

Then for any strongly skew tree $T$,

\begin{equation}
T = \bigcup_{m < M} T(m).
\end{equation}

For $m < M$, the $m$-th approximation of $T$ is defined to be

\begin{equation}
r_m(T) = \bigcup_{j < m} T(j).
\end{equation}

Let $m_n$ denote the integer such that $c_n^T \in T(m_n)$. Then $d_m^T = c_n^T$. Note that a critical node $d_m^T$ is a splitting node if and only if $m \neq m_n$ for any $n$. For each $0 < n < N$, the $n$-th interval of $T$ is $\bigcup \{ T(m) : m_{n-1} < m \leq m_n \}$. The 0-th interval of $T$ is defined to be $\bigcup_{m \leq m_0} T(m)$. Thus, the 0-th interval of $T$ is the set of those nodes in $T$ with lengths in $[0, \hat{I}^T_0]$, and for $0 < n < N$, the $n$-th interval of $T$ is the
set of those nodes in $T$ with lengths in $([l^T_{n-1}, l^T_n])$. A skew tree $T$ is regular if for each $n < N$, the lengths of the splitting nodes in the $n$-th interval of $T$ increase as their lexicographic order decreases.

In what follows, for any given $k \geq 3$, the only ghost coding node of a tree $T$ will be $c^{T}_{T - 1} = \text{stem}(T)$. The structure of $S_k$ in the previous section will be used essentially, but since we now work with skew trees, we can preserve the coding structure of $S_k$ while still requiring all coding nodes not to split. The ghost coding node $c^{T}_{T - 1}$ will function as the ghost coding node $c^{S}_{k - 1}$ did in the strong $H_k$-free tree $S_k$ in the previous section. The coding nodes $c^{T}_{0}, \ldots, c^{T}_{k - 3}$ will take the place of the ghost coding nodes $c^{S}_{k - k + 2}, \ldots, c^{S}_{k - 1}$ code in $S_k$.

**Definition 4.1 (K$_k$-Free Branching Criterion for Skew Trees).** Let $T$ be a skew tree with coding nodes $\langle c^T_n : n < N \rangle$, $N \leq \omega$. Let the ghost coding node $c^T_{T - 1}$ be the stem of $T$. Then $T$ satisfies the $K_k$-Free Branching Criterion ($k$-FBC) if the following holds: For each $n \in [-1, N - 2]$ and for each non-coding node $t$ in $T \mid ([l^T_n, l^T_{n+1}]$, $t$ splits in $T$ before reaching the level of $c^T_{n+1}$ if and only if, letting $u = c^T_{n+1} \mid ([l^T_n, l^T_{n+1}]$, for each subset $I \subseteq [-1, n]$ of size $k - 2$ for which $C = \{c^T_i : i \in I\}$ codes a $(k - 2)$-clique and $u$ has passing number 1 at each $c \in C$, there is some $c \in C$ at which $t$ has passing number 0.

In words, $t$ splits in $T$ if and only if extending $t$ to a coding node will not code a $k$-clique with $c^T_{n+1}$ and $k - 2$ many coding nodes in $T$ below $c^T_{n+1}$. Notice that if a skew tree $T$ satisfies Definition 4.1 then counting the ghost coding node $c^T_{T - 1}$ as among the coding nodes of a skew tree $T$, $T$ satisfies Definition 3.6 the definition of the $K_k$-Free Branching Criterion in the previous section. Thus, Theorem 3.9 applies, so any skew tree satisfying the $K_k$-Free Branching Criterion in which the coding nodes are dense codes a copy of $H_k$.

In contrast to our approach in [4] where we defined strong $H_3$-coding trees via several structural properties, in this paper we shall construct a particular strong
\[ d_0 = c_{-1} \]

\[ \mathcal{H}_k \text{-coding tree } T_k \]

and then define a subtree to be a strong \( \mathcal{H}_k \)-coding tree if it is isomorphic to \( T_k \) in a strong sense (see Definition 4.11 in the next section).

**Theorem 4.2** (The Strong \( \mathcal{H}_k \)-Coding Tree \( T_k \)). For each \( k \geq 3 \), there is a tree \( T_k \) with coding nodes \( \langle c^k_n : n < \omega \rangle \) which is strongly skew and regular, and satisfies the \( K_k \)-Free Branching Criterion for Skew Trees. Furthermore, the coding nodes in are dense in \( T_k \) and code \( \mathcal{H}_k \).

**Proof.** For each \( k \geq 3 \) and any enumeration \( \langle u_i : i \in \omega \rangle \) of the nodes in \( 2^{<\omega} \) so that \( i < j \) implies \( |u_i| \leq |u_j| \), we will construct a tree \( T_k \subseteq 2^{<\omega} \) with ghost coding node \( c^k_{-1} = \langle \rangle \) and coding nodes \( c^k_n, n \geq 0 \), letting \( l^k_n \) denote \( |c^k_n| \). Letting \( \langle d^k_m : m < \omega \rangle \) denote the critical nodes (splitting and coding nodes) of \( T_k \) in order of increasing length, let \( m_n \) be the index such that \( d^k_{m_n} = c^k_n \). We shall construct \( T_k \) satisfying the following:

1. For each \( n \geq k - 3 \), there is a node \( w^k_n \in T_k(m_n + 1) \) such that for all \( j \in [-1, n] \), \( w^k_n(l^k_j) = 1 \) if and only if \( j \in [n - k + 3, n] \).
2. For \( i \in \omega \) and \( n = (i + 1)(k - 1) \), if \( u_i \) is in \( r_{m_i(k-1)}(T_k) \), then the coding node \( c^k_n \) extends \( u_i \); furthermore, \( c^k_n(l^k_j) = 1 \) for all \( j \in [n - k + 2, n - 1] \).
3. For \( n = i(k - 1) + m \), where \( i \in \omega \) and \( 1 \leq m < k - 1 \), for all \( j \in [-1, n] \), \( c^k_n(l^k_j) = 1 \) if and only if \( j \in [n - k + 2, n - 1] \).
4. \( T_k \) satisfies the \( K_k \)-Free Branching Criterion.

We shall concretely construct \( T_3(m_0 + 1) \) and \( T_4(m_1 + 1) \) and then provide the general construction method for \( T_k(m_{k-3} + 1) \) for \( k \geq 5 \). For each \( k \geq 3 \), we will give a general construction satisfying (i) - (iv) above the length \( l^k_{k-3} \), similarly to the construction of \( S_k \) in the previous section, except that here we will be constructing a strongly skew, regular tree.
For \( k = 3 \), let the ghost coding node \( c^3_{-1} \) equal the least splitting node \( d^3_0 \), which is the empty sequence. The next splitting node is \( d^3_1 = \langle 0 \rangle \), and the next coding node \( c^3_0 = \langle 1, 0 \rangle \). Then \( l^3_0 = 2 \), and we let \( T_3(m_0) = \{ \langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle \} \). To construct \( T_3(m_0 + 1) \), extend each node according to its last digit; thus, \( T_3(m_0 + 1) \) consists of the nodes \( s_{00} \sim 0 = \langle 0, 0, 0 \rangle, s_{01} \sim 0 = \langle 0, 1, 1 \rangle \), and \( s_{10} \sim 0 = \langle 1, 0, 0 \rangle \).

For \( k = 4 \), let the ghost coding node \( c^4_{-1} \) equal the least splitting node \( d^4_0 \), which is the empty sequence. Let the next splitting nodes be \( d^4_1 = \langle 1 \rangle \), and \( d^4_2 = \langle 0, 0 \rangle \). Let \( l^4_0 = 3 \) and let \( T_4(m_0) = \{ s_z : z \in 2^3 \} \), where \( s_{00} = \langle 0, 0, 0 \rangle, s_{01} = \langle 0, 0, 1 \rangle, s_{10} = \langle 1, 0, 0 \rangle, s_{11} = \langle 1, 1, 0 \rangle \). Let the coding node \( c^4_0 = \langle 1, 0, 0 \rangle \). Let \( T_4(m_0 + 1) \) consist of their extensions \( s_z \sim z_1 \), for each \( z = \langle z_0, z_1 \rangle \in 2^2 \), so that

\[
T_4(m_0 + 1) = \{ \langle 0, 0, 0, 0 \rangle, \langle 0, 0, 1, 1 \rangle, \langle 1, 0, 0, 0 \rangle, \langle 1, 1, 0, 1 \rangle \}.
\]

Extend the nodes in \( (T_4(m_0 + 1)) \setminus \{ \langle 1, 1, 1 \rangle \} \) in reverse lexicographic order to the next set of splitting nodes as follows: Let the splitting nodes with lengths between \( l^4_0 \) and \( l^4_1 \) be \( \langle 1, 0, 0, 0 \rangle, \langle 0, 0, 1, 1, 0 \rangle, \) and \( \langle 0, 0, 0, 0, 0 \rangle \). Since the longest of these has length 6, the length of the next coding node will be 7. Extend each of these splitting nodes both right and left and then extend by 0’s to length 7. Extend \( \langle 1, 1, 1 \rangle \) leftmost to \( \langle 1, 1, 0, 1, 0, 0, 0 \rangle \) and let \( c^4_1 \) be this node. Thus,

\[
T_4(m_1) = \{ \langle 0, 0, 0, 0, 0, 0, 0 \rangle, \langle 0, 0, 0, 0, 0, 0, 1 \rangle, \langle 0, 0, 1, 1, 0, 0, 0 \rangle, \\
\langle 0, 0, 1, 1, 0, 1, 0, 0 \rangle, \langle 1, 0, 0, 0, 0, 0, 0, 0 \rangle, \langle 1, 0, 0, 0, 0, 1, 0, 0 \rangle, \\
\langle 1, 1, 0, 0, 1, 0, 0, 0 \rangle \}.
\]

Label these nodes according to their lexicographic order as \( s_z \), where \( z \in 2^3 \setminus \{ \langle 1, 1, 1 \rangle \} \). Extend them according to the last digit in their index to form \( T_4(m_1 + 1) \), so that

\[
T_4(m_1 + 1) = \{ \langle 0, 0, 0, 0, 0, 0, 0, 0 \rangle, \langle 0, 0, 0, 0, 0, 0, 0, 1 \rangle, \langle 0, 0, 1, 1, 0, 0, 0, 0 \rangle, \\
\langle 0, 0, 1, 1, 0, 1, 0, 0 \rangle, \langle 1, 0, 0, 0, 0, 0, 0, 0 \rangle, \langle 1, 0, 0, 0, 1, 0, 0, 0 \rangle, \\
\langle 1, 1, 0, 1, 0, 0, 0, 0 \rangle \}.
\]

The general method for constructing \( r_{m_k + 3}(T_k) \) for \( k \geq 5 \) is as follows. Let the ghost coding node \( c^k_{-1} \) equal the least splitting node \( d^k_0 \), which is the empty sequence; hence, \( l^k_{-1} = 0 \). Let the next splitting nodes be \( d^k_1 = \langle 1 \rangle \), and \( d^k_2 = \langle 0, 0 \rangle \). Let \( l^k_0 = 3 \) and let \( T_k(m_0) = \{ s_z : z \in 2^2 \} \), where \( s_{00} = \langle 0, 0, 0 \rangle, s_{01} = \langle 0, 0, 1 \rangle, s_{10} = \langle 1, 0, 0 \rangle, s_{11} = \langle 1, 1, 0 \rangle \). Let the coding node \( c^k_0 = \langle 1, 0, 0 \rangle \). Let \( T_k(m_0 + 1) \) consist of their extensions \( s_z \sim z_1 \), for each \( z = \langle z_0, z_1 \rangle \in 2^2 \). Thus,

\[
T_k(m_0 + 1) = \{ \langle 0, 0, 0, 0 \rangle, \langle 0, 0, 1, 1 \rangle, \langle 1, 0, 0, 0 \rangle, \langle 1, 1, 0, 1 \rangle \}.
\]

Label these \( t_z \) for \( z \in 2^2 \), where the labeling is according to their lexicographic order so that \( t_z \supset s_z \).

Suppose we have constructed \( r_{m_j + 2}(T_k) \) satisfying (i) - (iv), where \( 0 \leq j < k - 3 \). For \( j < k - 4 \), the level set \( T_k(m_j + 1) \) has \( 2^{j+2} \) many nodes. Each node in \( T_k(m_j + 1) \) will split before reaching the level of the next coding node, \( c^k_{j+1} \), so the length of the next coding node will be \( l^k_{j+1} = l^k_j + 2^{j+2} \). Enumerate the nodes in \( T_k(m_j + 1) \) in reverse lexicographic order as \( \langle t_i : i < 2^{j+2} \rangle \). Let the next splitting nodes be \( t_i \sim 0^{(i)} \), for each \( i < 2^{j-1} \), where \( 0^{(0)} \) denotes the empty sequence. Extend each splitting
node both left and right, and then extend with 0’s to reach length $l^k_{j+1}$. Thus, for each $i < 2^{j+2}$, the nodes $t_i \sim 0^{2(i+2)}$ and $t_i \sim 0^{i+1} \sim 0^q$, where $q = 2^{j+3} - i - 1$, are the two nodes in $T_k(m_{j+1})$ extending the node $t_i$. This constructs $T_k(m_{j+1})$.

Label the nodes in $T_k(m_{j+1})$ according to their lexicographic order as $s_z$, $z \in 2^{j+3}$, and let $t_z = s_z \sim z_{j-1}$. This forms $T_k(m_{j+1} + 1)$.

For $j = k-4$, every node except for the rightmost node in $T_k(m_{k-4}+1)$ will split before reaching the level of the coding node $c^k_{k-3}$. Let $t_i$, $i < 2^{k-2}$, enumerate the members of $T_k(m_{k-4}+1)$ in reverse lexicographic order. Let the coding node $c^k_{k-3}$ be $t_0 \sim 0^{2^{k-2}-1}$. Let the next splitting nodes be $t_i \sim 0^{(i-1)}$, for each $1 \leq i < 2^{k-2}$, where $0^{(0)}$ denotes the empty sequence. Extend each splitting node both left and right, and then extend with 0’s until it attains length $l^k_{j+1}$. Thus, for each $1 \leq i < 2^{k-2}$, the nodes $t_i \sim 0^{2^{j+2}-1}$ and $t_i \sim 0^{i-1} \sim 0^q$, where $q = 2^{k-2} - i - 1$, are the two extensions of $t_i \sim 0^{(i-1)}$ at the level of $c^k_{k-3}$. This forms $T_k(m_{k-3})$. Label the members of $T_k(m_{k-3})$ according to their lexicographic order as $s_z$, $z \in 2^{k-1} \{1^{(k-1)}\}$. Let $T_k(m_{k-3}+1)$ consist of the nodes $s_z \sim z_{k-2}$, for each $z = (z_0, \ldots, z_{k-2}) \in 2^{k-1} \setminus \{1^{(k-1)}\}$.

Notice that for each $−1 \leq j < k−4$, every node in $T_k(m_j)$ extends to a splitting node before reaching $T_k(m_{j+1})$. Moreover, (i) - (iv) hold for $r_{m_{k-3}+2}(T_k)$. This concludes the construction of $r_{m_{k-3}+2}(T_k)$ for each $k \geq 3$.

Now let $k \geq 3$ be fixed and suppose $r_{m_{n-1}+2}(T_k)$ has been constructed, where $n \geq k-2$, so that (i) - (iv) hold.

**Claim 1.** There is a node $v^k_n \in T_k(m_{n-1}+1)$ which we can extend to the coding node $c^k_n$ so that (i) - (iv) will hold in $r_{m_{n-2}+2}(T_k)$.

**Proof.** Case 1. Either $n = i(k-1) + j$ for some $i \in \omega$ and $1 \leq j < k-1$, or else $n = (i+1)(k-1)$ for some $i \in \omega$ and $u_i \notin r_{m_{i(k-1)+1}}(T_k)$. Then let $w^k_{n-1}$ denote the node in $T_k(m_{n-1}+1)$ such that for all $p \in [-1,n-1]$,

$$w^k_{n-1}(l^k_p) = 1 \iff p \in [n-k+2,n-1].$$

This $w^k_{n-1}$ is a node in $T_k(m_{n-1}+1)$, since we are assuming (i) holds for $r_{m_{n-1}+2}(T_k)$. Let $v^k_n = w^k_{n-1}$.

Case 2. $n = (i+1)(k-1)$ for some $i \in \omega$, and $u_i \in r_{m_{i(k-1)+1}}(T_k)$. Let $\hat{m}$ be minimal such that $l^k_{\hat{m}} \geq |u_i|$, and let $v^k_n$ be the node in $2^{\omega \setminus i}$ of length $l^k_{\hat{m}} + 1$ which extends $u_i$ so that for all $j \in [\hat{m}+1, i(k-1)]$, $v^k_n(l^k_j) = 0$ and for all $j \in [i(k-1) + 1, n-1]$, $v^k_n(l^k_j) = 1$. We claim that this node $v^k_n$ is a member of $T_k(m_{n-1}+1)$. To prove this, we shall show that given any collection of $k-1$ many coding nodes in $r_{m_{n-1}+1}(T_k)$ coding a $(k-1)$- clique, there must be some node in that collection for which $v^k_n$ has passing number 0. It will then follow that $v^k_n$ is in $r_{m_{n-1}+2}(T_k)$, since this tree is maximally splitting, subject to satisfying the $k$-FBC. Let $i_0 < \cdots < i_{k-2} \leq n-1$ be a fixed sequence for which $\{c^k_j : j \leq k-2\}$ codes a $(k-1)$-clique.

Suppose first that $i_{k-2} = i(k-1) + p$, for some $1 \leq p \leq k-2$. Then $v^k_{i_{k-2}}$ codes edges with only $k-2$ many coding nodes with smaller indices, and these are exactly $c^k_{i(k-1)+p-j}$ for $1 \leq j \leq k-2$. It must follow that these nodes are exactly the nodes in $\{c^k_j : j \leq k-2\}$. Notice that the least index $i_0$ is no greater than $i(k-1)$, so it must be the case that $i_0 = i(k-1)$. For if not, then $\{c^k_j : j \leq k-2\}$ cannot code
a \((k-1)\)-clique since in that case, \(v^k_{i_{k-2}}(l^k_{i_{k-1}})\) would be 0, a contradiction. Thus, \(i_0 = i(k-1)\) and it follows that \(v^k_{i_{k-1}}(l^k_{i_{k-1}}) = v^k_{i_{k-1}}(l^k_{i_{k-1}+1}) = 0\).

Now suppose that \(i_{k-2} = i(k-1)\). Recall that \(m \leq i(k-1)\), since \(|u_i| \leq l^k_{i_{i(k-1)}}\) and \(m\) is minimal such that \(|u_i| \leq l^k_{i_{i(k-1)}}\). If \(m < i_{k-2} \leq i(k-1)\), then \(v^k_m\) has passing number 0 at \(c^k_{i_{i(k-1)}}\). Otherwise, \(i_{k-2} \leq m\). Since \(r_m(i_{i(k-1)}+1)\) satisfies the k-FBC, it follows that there is some \(j \leq k-2\) such that \(v^k_n\) has passing number 0 at \(c^k_j\).

Thus, in all cases, \(v^k_n\) has passing number 0 at least one member of the sequence \(\{c^k_j : j \leq k-2\}\). Since this was an arbitrary sequence coding a \((k-1)\)-clique in \(r_{m_n-1+2}(T_k)\), it follows by the k-FBC that \(v^k_n\) is a member of \(r_{m_n+1}(T_k)\). □

Given \(v^k_n\) as in Claim 1, define \(c^k_n\) to be the lefmost extension of \(v^k_n\) to length \(l^k_n\), this length being determined as follows. Let \(NSpl(T_k, n)\) denote the set of those nodes in \(r_{m_n+1}(T_k)\) which will not split before reaching the level of \(c^k_n\); they will extend by 0’s until they reach length \(l^k_n\). \(NSpl(T_k, n)\) is exactly the set of those nodes \(t \in r_{m_n+1}(T_k)\) for which there exists some \(I \subseteq [n, n-1]\) of size \(k-2\) such that \(\{c^k_j : j \in I\}\) codes a \((k-2)\)-clique, and for each \(i \in I\), \(t(i) = v^k_n(l_i) = 1\). This is the only reason that a node may not extend past \(c^k_n\) with passing number 1. The complement of these nodes,

\[
\text{Spl}(T_k, n) = T_k(m_n + 1) \setminus NSpl(T_k, n),
\]

is the set of nodes in \(T_k(m_n + 1)\) which will extend to splitting nodes before reaching the level of \(c^k_n\).

Let \(l^k_n = l^k_n + 1 + S^k_n\), where \(S^k_n\) is the number of nodes in \(\text{Spl}(T_k, n)\). Enumerate the nodes in \(\text{Spl}(T_k, n)\) in reverse lexicographic order as \(\langle t_j : j < S^k_n \rangle\). The splitting nodes in the interval between \(l^k_n + 1\) and \(l^k_n\) are \(t_j \sim 0(j)\), for \(j < S^k_n\), where \(0^{(j)}\) denotes the empty sequence. Extend each of these splitting nodes both right and left and then extend with 0’s to length \(l^k_n\). For each \(j < S^k_n\), let \(s_{j,0} = t_j \sim 0^{(p_j)}\), where \(p_j = l^k_n - (l^k_n - 1)\); and let \(s_{j,1} = t_j \sim 0^{(j)} \sim 0^{(p_j)}\), where \(p_j = l^k_n - (l^k_n - 1) + 2\). For each \(w \in NSpl(T_k, n)\), let \(s_w = w \sim 0^{(p_j)}\). Then \(T_k(m_n)\) consists of the nodes \(\{s_{j,i} : j < S^k_n, i < 2\}\) \(\cup\) \(\{s_w : w \in NSpl(T_k, n)\}\). Designate \(c^k_n = v^k_n(0^{(p_j)}).\) To extend the nodes in \(\text{Spl}(T_k(n))\) to length \(l^k_n\), let

\[
T_k(m_n + 1) = \{s_{j,1} \sim 1 : j < S^k_n\} \cup \{s_{j,0} \sim 0 : j < S^k_n\} \cup \{s_w \sim 0 : w \in NSpl(T_k, n)\}\]

This inductive process constructs a strong \(H_k\)-coding tree \(T_k\) which satisfies (i) - (iv). Thus, \(T_k\) splits according to the \(K_k\)-Free Branching Criterion and the coding nodes in \(T_k\) are dense in \(T_k\). Therefore, the coding nodes in \(T_k\) code the Henson graph \(H_k\).

\[
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\]

4.2. The space of strong \(H_k\)-coding trees. Let \(k \geq 3\) be given, and fix a strong \(H_k\)-coding tree \(T_k\), constructed as in Theorem 4.2. In preparation for defining the space of strong \(H_k\)-coding trees contained in \(T_k\), we provide the following definitions.
A subset $A$ of $T_k$ is an antichain if $s \subseteq t$ implies $s = t$, for all $s, t \in A$. A subset $X$ of $T_k$ is called a level set if all members of $X$ have the same length. Thus, each level set is an antichain. Given a subset $S \subseteq T_k$, recall that the meet closure of $S$, denoted $S^\wedge$, is the set of all meets of pairs of nodes in $S$. In this definition $s$ and $t$ may be equal, so $S^\wedge$ contains $S$. We say that $S$ is meet-closed if $S = S^\wedge$. The lexicographic order on $2^{\omega^*}$ between two nodes $s, t \in 2^{\omega^*}$, with neither extending the other, is defined by $s <_{\text{lex}} t$ if and only if $s \supseteq (s \wedge t)^\sim 0$ and $t \supseteq (s \wedge t)^\sim 1$. Given a subset $S \subseteq T_k$ and any $l < \omega$, we shall usually abuse notation and simply write $S | l$ to denote $S \upharpoonright l$, whether or not $S$ has nodes of length $l$. For $u \in S$, if there is a unique immediate extension of $u$ in $S | (|u| + 1)$, then $u^+$ denotes this extension of $u$. Notice that for any non-splitting node $u$ in $S$, $u^+$ is uniquely determined by $T_k$, regardless of $S$. If $u$ is the length of a coding node $c$ in $T_k$, then the passing number of $u^+$ at $c$ is uniquely determined by $T_k$. Thus, we say that $i$ is the passing number of $u$ at $c$ exactly when $u^+(|c|) = i$.

If $X$ is a level set, let $l_X$ denote the length of the members of $X$, and let $X^+$ denote the set $\{s^+: s \in X\}$. The following is Definition 4.9 in [4].

**Definition 4.3** (Strong Similarity Map). Let $k \geq 3$ be given and let $S, T \subseteq T_k$ be meet-closed subsets. A function $f : S \to T$ is a strong similarity of $S$ to $T$ if for all nodes $s, t, u, v \in S$, the following hold:

1. $f$ is a bijection.
2. $f$ preserves lexicographic order: $s <_{\text{lex}} t$ if and only if $f(s) <_{\text{lex}} f(t)$.
3. $f$ preserves meets, and hence splitting nodes: $f(s \wedge t) = f(s) \wedge f(t)$.
4. $f$ preserves relative lengths: $|s \wedge t| < |u \wedge v|$ if and only if $|f(s) \wedge f(t)| < |f(u) \wedge f(v)|$.
5. $f$ preserves initial segments: $s \wedge t \subseteq u \wedge v$ if and only if $f(s) \wedge f(t) \subseteq f(u) \wedge f(v)$.
6. $f$ preserves coding nodes: $f$ maps the set of coding nodes in $S$ onto the set of coding nodes in $T$.
7. $f$ preserves passing numbers at images of coding nodes: If $c$ is a coding node in $S$ and $u$ is a node in $S$ with $|u| = |c|$, then $(f(u))^+(|f(c)|) = u^+(|c|)$.

In words, the passing number of the immediate successor of $f(u)$ at $f(c)$ equals the passing number of the immediate successor of $u$ at $c$.

We say that $S$ and $T$ are strongly similar, and write $S \sim T$, exactly when there is a strong similarity map between $S$ and $T$.

It follows from (3) that $s \in S$ is a splitting node in $S$ if and only if $f(s)$ is a splitting node in $T$. In all cases above, it may be that $s = t$ and $u = v$, so in particular, that (5) implies $s \subseteq u$ if and only if $f(s) \subseteq f(u)$. Notice that strong similarity is an equivalence relation, since the inverse of a strong similarity map is a strong similarity map, and the composition of two strong similarity maps is a strong similarity map. If $T' \subseteq T$ and $f$ is a strong similarity of $S$ to $T'$, then we say that $f$ is a strong similarity embedding of $S$ into $T$, and call $T'$ a strong similarity copy of $S$ in $T$.

Our goal in this section is to define a space of subtrees of $T_k$ for which a development of Ramsey theory is possible. Necessary for this is a way to extend a given finite subtree of $T_k$ to an infinite subtree of $T_k$ of an a priori fixed strong similarity type. However, this is not always possible: There are finite subtrees of $T_k$ which are strongly similar to a finite initial subtree of $T_k$ which cannot be extended within
$T_k$ to a subtree strongly similar to $T_k$. In this subsection we make precise what the obstructions are, and then determine criteria which guarantee that a subtree has no obstructions.

For $k > 3$, the next definitions are new to $k$-clique-free graphs and necessary for the work in this paper. When $k = 3$, the rest of this section simply reproduces the concepts of sets of parallel 1’s and the Parallel 1’s Criterion used throughout [4], though in a new and more streamlined manner. Fix $k \geq 3$ throughout the rest of Section [4].

**Definition 4.4** (Mutual Pre-$a$-Clique). Let $k \geq 3$ be fixed, and let $a \in [3, k]$. A level subset $X$ of $T_k$ of size at least two has a mutual pre-$a$-clique (or simply pre-$a$-clique) if there is an index set $I \subseteq \omega$ of size $a - 2$ such that, letting $i_* = \max(I)$ and $l_* = |c^{k}_{i_*}|$, the following hold:

1. $l_*$ is less than or equal to $l_X$, the length of the nodes in $X$, and there are exactly the same number of nodes in the level set $X \upharpoonright l_*$ as in $X$;
2. The set $\{c_i^k : i \in I\}$ codes a $(a - 2)$-clique: For each pair $i < j$ in $I$, the coding node $c_i^k$ has passing number 1 at $c_j^k$;
3. Each node in $X^+$ has passing number 1 at $c_i^k$, for each $i \in I$.

We say that $X$ has a mutual pre-$a$-clique at $l_*$, and that $X \upharpoonright l_*$ is a mutual pre-$a$-clique. The set of coding nodes $\{c_i^k : i \in I\}$ is said to witness that $X$ has a mutual pre-$a$-clique at $l_*$. Notice that in the case that $l_X$ is the length of some splitting node in $T_k$, $X^+$ will have at most one member of length $l_X + 1$ with passing number 1 at $l_X$ and hence is irrelevant in (3) of Definition [4.4]. That is, the extension of $X$ to $X^+$ is relevant only in the case that $l_X$ is the length of some coding node in $T_k$.

To simplify terminology, from now on we omit the word mutual and simply refer to such sets as pre-$a$-cliques. We write $P_a(X)$ exactly when $X$ has a pre-$a$-clique. If $T$ is a subset of $T_k$ and there is a set of coding nodes in $T$ witnessing that $X$ has a pre-$a$-clique, then we write $\text{WP}_a(X; T)$.

**Remark 4.5.** Whenever a level set $X$ has a pre-$k$-clique, then for any set of coding nodes $C$ witnessing that $P_k(X)$, the set $C$ codes a $(k - 1)$-clique, and each $x \in X$ has passing number 1 at each $c \in C$. It follows that for any coding node $c \in T_k$ extending some $x \in X$, every other node in $X \setminus \{x\}$ cannot extend to have passing number 1 at $c$, as that would code a $k$-clique.

Thus, the nodes in a pre-$k$-clique are ‘entangled’: The splitting possibilities in the cone above one of these nodes depends on the cones above the other nodes. If $X$ is contained in some finite subtree $A$ of $T_k$ and $P_k(X)$ is not witnessed by coding nodes in $A$, then the graph coded by $A$ has no knowledge that the cones above $X$ in $T_k$ are entangled - some types are missing. Then no extension of $A$ into $T_k$ can be strongly similar to $T_k$.

In the set-up to the space of strong coding trees, we are considering pre-$a$-cliques for all $a \in [3, k]$, because these will be necessary to witness in order to prove the extension lemmas in Subsection [5.2].

**Definition 4.6.** Let $a \in [3, k]$. We say that a level set $X \subseteq T_k$ of size at least two has a new pre-$a$-clique at $l$ if $X \upharpoonright l$ is a pre-$a$-clique and for each $l' < l$ for which $X \upharpoonright l$ and $X \upharpoonright l'$ have the same number of nodes, $X \upharpoonright l'$ is not a pre-$a$-clique.

Given $T$ a subset of $T_k$, a set of coding nodes $\{c_i^k : i \in I\}$ in $T$, $|I| = a - 2$, witnesses in $T$ that $X$ has a new pre-$a$-clique at $l$ if, letting $i_* = \max(I)$,
(1) \{c^T_i : i \in I\} codes an \((a - 2)\)-clique;
(2) \(|c^T_{i-1}| \geq l\) and \(T\) has no critical nodes in the interval \([l, |c^T_i|]\); and
(3) For each \(x \in X\), the node \(y_x\) in \(T \upharpoonright |c^T_i|\) extending \(x \upharpoonright l\) has passing number 1 at \(c^T_i\), for each \(i \in I\).

Note that the set \(\{y_x : x \in X\}\) in (3) above is a well-defined, since \(T\) has no critical nodes (in particular no splitting nodes) in the interval \([l, |c^T_i|]\). Further, recall that the ‘passing number’ of \(y_x\) at \(c^T_i\) is uniquely determined by \(\mathbb{T}_k\) as the passing number of \((y_x)^+\) at \(c^T_i\).

We also point out that the set of coding nodes \(\{c^T_i : i \in I\}\) is not necessarily the least set in \(\mathbb{T}_k\) witnessing \(P_a(X)\). That is, there may be other sets of coding nodes in \(\mathbb{T}_k\) witnessing that \(X\) has a pre-a-clique at some length less than \(|c^T_i|\). However, \(|c^T_i|\) is the least length of the longest node in any set of coding nodes in \(T\) witnessing \(P_a(X)\); any collection of \(a - 2\) many coding nodes in \(T\) such that the longest node has length less than \(|c^T_i|\) does not witness \(P_a(X)\).

The next concept will be used throughout the rest of this article.

**Definition 4.7.** Let \(X \subseteq \mathbb{T}_k\) be a level set of length \(l\) and let \(a \in [3, k]\). We say that a level set \(Y\) of length \(l' > l\) end-extending \(X\) contains no new pre-a-cliques over \(X\) if for each \(j \in (l, l']\) and each \(Z \subseteq Y\), if \(Z \upharpoonright j\) is a pre-a-clique, then \(Z \upharpoonright l\) already has a pre-a-clique. We say that \(Y\) has no new pre-a-cliques over \(X\) if \(Y\) has no new pre-a-cliques over \(X\) for any \(a \in [3, k]\).

The next definition gives precise conditions under which a new pre-a-clique at \(l\) in a subtree \(T\) of \(\mathbb{T}_k\) is maximal in the interval of \(T\) containing \(l\).

**Definition 4.8 (Maximal New Pre-a-Clique).** Let \(T\) be a subtree of \(\mathbb{T}_k\) and let \(a \in [3, k]\). We say that a level set \(X \subseteq T\) has a maximal new pre-a-clique in \(T\) at \(l\) if \(X \upharpoonright l\) is a new pre-a-clique which is also maximal in \(T\) in the following sense: Let \(d\) denote the critical node in \(T\) of maximum length satisfying \(|d| < l\). If \(m\) is the index so that \(d = d^T_m\), let \(e\) denote \(d^T_{m+1}\) and note that \(l \leq |e|\). Then for any \(l' \in (l, |e|]\) and any new pre-a-clique \(Y \subseteq T \upharpoonright l'\), if \(Y \upharpoonright l\) contains \(X \upharpoonright l\) then these sets are equal; hence \(l' = l\), since \(T\) has no splitting nodes in the interval \([|d|, |e|]\).

We write \(\text{MP}_a(X; T)\) if \(X\) has a maximal new pre-a-clique in \(T\) in the interval of \(T\) containing the length of the nodes in \(X\). Thus, if \(l_X = |d^T_m|\), then \(\text{MP}_a(X; T)\) means that for some \(l \in (l^T_{m-1}, l^T_m]\), \(X\) has a maximal new pre-a-clique at \(l\). If the maximal new pre-a-clique \(X \upharpoonright l\) is witnessed by a set of coding nodes in \(T\), we write \(\text{WMP}_a(X; T)\).

In Definition 4.8 for any level set \(Z\) end-extending \(X\), we say that \(Z\) has a maximal new pre-a-clique in \(T\) at \(l\). We will say that a set \(Y \subseteq T\) contains a maximal new pre-a-clique at \(l\) if \(\text{MP}_a(X; T)\) for some subset \(X \subseteq Y \upharpoonright l\).

**Definition 4.9 (Stable Map).** Let \(S\) and \(T\) be strongly similar subtrees of \(\mathbb{T}_k\) with \(M \leq \omega\) many critical nodes; if \(M\) is finite, we assume that the maximal levels of \(S\) and \(T\) contain a critical node. The strong similarity map \(f : T \rightarrow S\) is stable if for each \(m\) such that \(m \in [1, M]\), the following holds: For each \(a \in [3, k]\), a level subset \(X \subseteq T \upharpoonright |d^S_m|\) has a maximal new pre-a-clique in \(T\) in the interval \([|d^S_{m-1}|, |d^S_m]|\) if and only if \(f(X)\) has a maximal new pre-a-clique in \(S\) in the interval \([|d^S_{m-1}|, |d^S_m]|\).

When there is a stable map between \(S\) and \(T\), we say that \(S\) and \(T\) are stably isomorphic, or simply isomorphic, and write \(S \cong T\).
Notice that any strong similarity map \( f : T \to S \) maps each set of coding nodes in \( T \) witnessing a maximal new \( \alpha \)-clique \( X \) in \( T \) to some set of coding nodes in \( S \) witnessing that \( f[X] \) is a maximal new \( \alpha \)-clique in \( S \), since strong similarity maps preserve coding nodes and passing numbers. Recall that any strong similarity map takes the \( i \)-th coding node in \( T \) to the \( i \)-th coding node in \( S \): \( f(c_i^T) = c_i^S \).

Thus,

**Observation 4.10.** If \( f : T \to S \) is a stable map, then for each \( a \in [3, k] \) and each level set \( X \subseteq T \) satisfying \( \text{MP}_a(X; T) \), a set of coding nodes \( \{c_i^T : i \in I\} \) (\( |I| = a-2\)) witnesses \( \text{MP}_a(X; T) \) if and only if \( \{c_i^S : i \in I\} \) witnesses \( \text{MP}_a(f[X]; S) \). Hence, \( \text{WMP}_a(X; T) \) if and only if \( \text{WMP}_a(f[X]; S) \). Furthermore, \( \cong \) is an equivalence relation, since the inverse of a stable map is stable and composition of two stable maps is stable.

Stable maps preserve all relevant structure regarding the shape of the tree, coding nodes and passing numbers, and maximal new \( \alpha \)-cliques and their witnesses. They provide the essential structure of the members of our space of strong \( H_k \)-coding trees.

**Definition 4.11** (The space \((T_k, \leq, r)\)). A tree \( T \subseteq T_k \) is a member of \( T_k \) if and only if there is a stable map from \( T_k \) onto \( T \), which we denote as \( f_T \). Thus, \( T_k \) consists of all subtrees of \( T_k \) which are stably isomorphic to \( T_k \). Call the members of \( T_k \) strong \( H_k \)-coding trees, or just strong coding trees when \( k \) is clear. The partial ordering \( \leq \) on \( T_k \) is simply inclusion: For \( S, T \in T_k, S \leq T \) if and only if \( S \) is a subtree of \( T \).

For \( T \in T_k \), let \( \langle c_n^T : n < \omega \rangle \) and \( \langle d_m^T : m < \omega \rangle \) enumerate the coding nodes and critical nodes, respectively, of \( T \) in order of increasing length. Since \( f_T \) is a strong similarity map, \( c_n^T = f_T(c_n^k) \) and \( d_m^T = f_T(d_m^k) \). The finite approximations to \( T \) are defined as

\[
(21) \quad r_m(T) = \{t \in T : |t| < |d_m^T|\},
\]

for \( m < \omega \). Thus for \( m < n \), \( r_n(T) \) end-extends \( r_m(T) \), and \( T = \bigcup_{m<\omega} r_m(T) \).

For each \( m < \omega \), define

\[
(22) \quad \mathcal{AT}_m^k = \{r_m(T) : T \in T_k\},
\]

and let

\[
(23) \quad \mathcal{AT}_k^k = \bigcup_{m<\omega} \mathcal{AT}_m^k.
\]

Given \( A \in \mathcal{AT}_k^k \) and \( T \in T_k \), define

\[
(24) \quad [A, T] = \{S \in T_k : \exists m (r_m(S) = A) \text{ and } S \leq T\}.
\]

Given \( j < m < \omega \), \( A \in \mathcal{AT}_j^k \) and \( T \in T_k \), define

\[
(25) \quad r_m[A, T] = \{r_m(S) : S \in [A, T]\}.
\]

For \( A \in \mathcal{AT}_k^k \) and \( B \in \mathcal{AT}_k^k \cup T_k \), if for some \( m \), \( r_m(B) = A \), then we write \( A \sqsubseteq B \) and say that \( A \) is an initial segment of \( B \). If \( A \sqsubseteq B \) and \( A \neq B \), then we write \( A \sqsubset B \) and say that \( A \) is a proper initial segment of \( B \).

If a subset \( A \subseteq T_k \) does not contain sequences of 0’s of unbounded length, there is an \( n \geq 0 \) such that each node in \( A \) has passing number 1 at \( c_i^k \), for some \( i \in [-1, n] \). Such an \( A \) cannot satisfy property \((A_k)_{\text{tree}}\) so it does not code \( H_k \);
hence it is not strongly similar to $T_k$. Thus, the leftmost path through any member of $T_k$ is the infinite sequence of 0’s. It follows that for $T \in T_k$, the stable map $f_T : T_k \to T$ must take each splitting node in $T_k$ consisting only of 0’s to a splitting node in $T$ consisting only of 0’s. In particular, $f_T$ takes $c_n^T$ to the stem of $T$, which is a splitting node in $T$ consisting of a finite sequence of 0’s. For the next definition, recall Definition 4.6 of witnessing a new pre-a-clique and Definition 4.8 of a maximal new pre-a-clique.

**Definition 4.12 (Witnessing Property).** A subtree $T$ of $T_k$ has the Witnessing Property (WP) if for each $a \in [3, k]$, each new pre-a-clique in $T$ takes place in some interval in $T$ of the form $([d_{m-1}^T, |c_n^T|]$ and is witnessed by a set of coding nodes in $T$.

Notice that the coding node $c_n^T$ in Definition 4.12 is obliged (by Definition 4.6) to be among the set of coding nodes witnessing $P_a(X; T)$. Further, in order to satisfy Definition 4.12 it suffices that the maximal new pre-$k$-cliques are witnessed in $T$, as this automatically guarantees that every new pre-$k$-clique is witnessed in $T$.

**Lemma 4.13.** If $A$ is a finite subtree of $T_k$ which has the Witnessing Property and $A \cong B$, then $B$ has the Witnessing Property.

*Proof.* Given the hypotheses, let $f : B \to A$ be a stable map from $B$ to $A$. Suppose $X \subseteq B$ is a level set which has a new pre-a-clique, for some $a \in [3, k]$. Let $m$ be the index such that the new pre-a-clique in $X$ takes place in the interval $([d_{m-1}^B, |d_m^B|]$. Without loss of generality, assume that $X$ has a maximal new pre-a-clique in $B$ in this interval. Since $f$ is stable, $f[X]$ has a maximal new pre-a-clique in $A$ in the interval $([d_{m-1}^A, |d_m^A|]$. Since $A$ has the WP, $d_m^A$ must be a coding node in $A$, and this coding node must be among the set of coding nodes in $A$ witnessing that $f[X]$ has a new pre-a-clique. Therefore, $d_m^B$ is a coding node which is among the set of coding nodes witnessing that $X$ has a new pre-a-clique, since $f$ being a strong similarity map implies $f$ preserves coding nodes and passing numbers. Thus, each new pre-a-clique in $B$ takes place in an interval at or just below a coding node in $B$ and is witnessed in $B$. Hence, $B$ has the WP. □

**Lemma 4.14.** Suppose $A, B$ are subtrees of $T_k$ and that $A$ has the Witnessing Property. Then $A \cong B$ if and only if $A \sim B$ and $B$ also has the Witnessing Property.

*Proof.* For the forward direction, note that $A \cong B$ implies $A \sim B$, by the definition of stably isomorphic. If moreover, $A$ has the WP then Lemma 4.13 implies $B$ also has the WP.

Now suppose that $A \sim B$ and both $A$ and $B$ have the WP. Let $f : A \to B$ be the strong similarity map. Suppose $X$ is a level set in $A$ which has a maximal new pre-a-clique, for some $a \in [3, k]$. Since $A$ has the WP, there is a set of coding nodes $C \subseteq A$ witnessing that $X$ has a new pre-a-clique. Furthermore, $t_X$ must be the length of some coding node in the set $C$. Since $f$ preserves coding nodes and passing numbers, it follows that $f[C]$ is a set of coding nodes in $B$ witnessing that $f[X]$ has a pre-a-clique. It remains to show that $f[X]$ is new and maximal in $B$.

If $f[X]$ is not a new pre-a-clique in $B$, then there is some critical node $d$ in $B$ below $f(c)$ such that $f[X] \upharpoonright |d|$ has a new pre-a-clique in $B$, where $c$ is the longest coding node in $C$. Since $B$ satisfies the WP, this new pre-a-clique in $f[X]$...
appears at some coding node in $B$ below $d$. Further, $f[X]$ must be witnessed by some set of coding nodes $D$ in $B$. But then $f^{-1}[D]$ is a set of coding nodes in $A$ witnessing a pre-$a$-clique in $X$. Since the longest length of a coding node in $f^{-1}[D]$ is shorter than $|c|$, the pre-$a$-clique in $X$ occurs first at some coding node below $c$, a contradiction to $X$ having a new pre-$a$-clique. Therefore, $f[X]$ is a new pre-$a$-clique in $B$.

If $f[X]$ is not maximal in $B$, then there is some level set $Z$ of nodes of length $l_X$ properly containing $f[X]$ which has a new pre-$a$-clique in $B$. Since $B$ has the WP, there is some set of coding nodes $D \subseteq B$ witnessing $Z$. Then $f^{-1}[D]$ witnesses that $f^{-1}[Z]$ is a pre-$a$-clique in $A$ properly containing $X$, contradicting the maximality of $X$ in $A$.

Therefore, $f$ preserves maximal new pre-$a$-cliques, and hence is a stable map. Hence, $A \cong B$. □

Lemma 4.15. (1) If $T \subseteq \mathbb{T}_k$ is strongly similar to $\mathbb{T}_k$, then $T$ satisfies the $K_k$-Free Branching Criterion.

(2) If $T \subseteq \mathbb{T}_k$ is strongly similar to $\mathbb{T}_k$ and has the Witnessing Property, then the strong similarity map from $\mathbb{T}_k$ to $T$ is stable, and hence $T$ is a member of $\mathbb{T}_k$.

Proof. (1) follows in a straightforward manner from the definitions of $k$-FBC and strong similarity map, along with the structure of $\mathbb{T}_k$, as we now show. Suppose $T \subseteq \mathbb{T}_k$ is strongly similar to $T$, and let $f : \mathbb{T}_k \to T$ be the strong similarity map. Note that for each $n \in [-1, \omega)$, $c^T_n = f(c^k_n)$. Fix $n \in [-1, \omega)$ and a node $t \in T \upharpoonright l^T_n$ which does not extend to $c^T_{n+1}$. Then $s := f^{-1}(t)$ is in $\mathbb{T}_k \upharpoonright l^k_n$. Since $f$ is a strong similarity map, $s$ does not extend to the coding node $c^k_{n+1}$ in $\mathbb{T}_k$. Since $T_k$ satisfies the $k$-FBC, $s$ splits in $\mathbb{T}_k$ before reaching the level of $c^k_{n+1}$ if and only if, letting $u = c^k_{n+1} \upharpoonright (l^k_n + 1)$, for each subset $I \subseteq [-1, n]$ of size $k - 2$ such that $C = \{c^k_i : i \in I\}$ codes a $(k-2)$-clique and $u$ has passing number 1 at each $c \in C$, there is some $c \in C$ at which $s^+$ has passing number 0. Since $t = f(s)$ and $f$ is a strong similarity map, $t$ splits in $T$ before reaching the level of $c^T_{n+1}$ if and only if, letting $v = c^T_{n+1} \upharpoonright (l^T_n + 1)$, for each subset $I \subseteq [-1, n]$ of size $k - 2$ for which $D = \{c^T_i : i \in I\}$ codes a $(k-2)$-clique and $v$ has passing number 1 at each $c \in D$, there is some $c \in D$ at which $t^+$ has passing number 0.

For (2), if $T \subseteq \mathbb{T}_k$ is strongly similar to $\mathbb{T}_k$ and has the Witnessing Property, then it follows from Lemma 4.14 that $T \cong \mathbb{T}_k$ since $\mathbb{T}_k$ has the Witnessing Property. □

Lemma 4.16. Every $T \in \mathcal{T}_k$ has the following properties:

(1) $T \cong \mathcal{T}_k$.

(2) $T$ satisfies the $K_k$-Free Branching Criterion.

(3) $T$ has the Witnessing Property.

Proof. (1) is immediate from the definition of $\mathcal{T}_k$. (2) follows from Lemma 4.15 part (1). (3) follows from (1) and Lemma 4.14 □

5. Extension Lemmas

Unlike Milliken’s strong trees, not every finite subtree of a strong $\mathcal{H}_k$-coding tree can be extended within that ambient tree to another member of $\mathcal{T}_k$, nor necessarily even to another tree of a desired configuration. This section provides structural properties of finite subtrees which are necessary and sufficient to extend to larger
tree of a particular type. The first subsection lays the groundwork for these properties and the second subsection proves extension lemmas which are fundamental to developing Ramsey theory on strong $\mathcal{H}_k$-coding trees. The extension lemmas extend and streamline similar lemmas in [4], taking care of new issues that arise when $k \geq 4$. Furthermore, these lemmas lay new groundwork for general extension principles, with the benefit of a simpler proof of Theorem 5.2 than the proof of its instance for $\mathcal{H}_3$ in [4].

5.1. Free level sets and finite valid subtrees. In this subsection, we provide criteria which will aid in the extension lemmas in Subsection 5.2. These requirements will guarantee that a finite subtree of a strong coding tree $T$ can be extended within $T$ to another strong coding tree.

**Definition 5.1.** Let $T \in \mathcal{T}_k$ be fixed. We say that a level set $X \subseteq \hat{T}$ with length $l$ is free in $T$ if given $n$ least such that $l_X^n \geq l$, letting $Y$ consist of the leftmost extensions of members of $X$ in $T \upharpoonright l_X^n$, then $Y$ has no new pre-$a$-cliques over $X$, for any $a \in [3, k]$.

In particular, any level set $X \subseteq T$ with length that of some coding node in $T$ is free in $T$.

**Remark 5.2.** For $k = 3$, this is equivalent to the concept of “$X$ has no pre-determined new parallel 1’s in $T$” in [4].

**Terminology 5.3.** For a level set $Y$ end-extending a level set $X$, we say that $Y$ has no new pre-cliques over $X$ if $Y$ has no new pre-$a$-cliques over $X$, for any $a \in [3, k]$.

**Lemma 5.4.** Let $T \in \mathcal{T}_k$ be fixed, $X \subseteq \hat{T}$ be a level set which is free in $T$. Let $n$ be least such that $l^n_X \geq l$. Then for all $m \geq n$, the set of leftmost extensions of the nodes in $X$ to $T \upharpoonright l^n_X$ contains no new pre-$a$-cliques over $X$. Furthermore, the leftmost extensions of $X$ in $T \upharpoonright l^n_p$ have passing numbers 0 at $c_p^n$, for each $p > n$.

**Proof.** This follows from the fact that $T \cong \mathbb{T}_k$. To see this, let $f : \mathbb{T}_k \rightarrow T$ be the stable map witnessing that $T \in \mathcal{T}_k$, and let $n$ be least such that $l^n_X \geq l$. Let $m \geq n$ and $a \in [3, k]$ be given, and let $Y$ be the end-extension of $X$ in $T \upharpoonright l^n_m$ consisting of the nodes which are leftmost extensions in $T$ of the nodes in $X$. Since $X$ is free in $T$, $X \upharpoonright l^n_m$ has no new pre-$a$-cliques over $X$. Since $f^{-1}$ is a strong similarity map, $f^{-1}[Y]$ is the collection of leftmost extensions in $\mathbb{T}_k \upharpoonright l^n_m$ of the level set $f^{-1}[X]$. In particular, $f^{-1}[Y]$ has no new pre-$a$-cliques in the interval $(l^n_m, l^n_m]$. In particular, the passing numbers of members of $f^{-1}[Y]$ in this interval $(l^n_m, l^n_m]$ are all 0. Since $f$ is stable, $Y = f \circ f^{-1}[Y]$ has no new pre-$a$-cliques over $Y \upharpoonright l^n_m$, and all passing numbers of the leftmost extensions of $X$ in $T$ are 0.

An important property of $\mathcal{T}_k$ is that all of its members contain unbounded sequences of 0’s.

**Lemma 5.5.** Suppose $T \in \mathcal{T}_k$ and $s$ is a node in the leftmost branch of $T$. Then $s$ is a sequence of 0’s.

**Proof.** Suppose there is a $T \in \mathcal{T}_k$ such that for some $n$, no node of $T$ extends $0^{(n)}$. Then there is a finite set $C$ of coding nodes in $\mathcal{T}_k$ such that each node in $T$ has passing number 1 by at least one member of $C$.

Let $l_C$ be the longest length of the coding nodes in $C$. If for some $l$, for each $c \in C$ there is some coding node $e_c \in T$ so that $\{ t \in T \upharpoonright l : t(e_c) = 1 \} = \{ t \in T \upharpoonright l : 0 \}$. Then there is a level set $X \subseteq \hat{T}$ with $l_X \geq l_C$ such that $X$ is free in $T$. Since $f$ is stable, $Y = f \circ f^{-1}[Y]$ has no new pre-$a$-cliques over $Y \upharpoonright l^n_m$, and all passing numbers of the leftmost extensions of $X$ in $T$ are 0.

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**Proof.** Suppose there is a $T \in \mathcal{T}_k$ such that for some $n$, no node of $T$ extends $0^{(n)}$. Then there is a finite set $C$ of coding nodes in $\mathcal{T}_k$ such that each node in $T$ has passing number 1 by at least one member of $C$.

Let $l_C$ be the longest length of the coding nodes in $C$. If for some $l$, for each $c \in C$ there is some coding node $e_c \in T$ so that $\{ t \in T \upharpoonright l : t(e_c) = 1 \} = \{ t \in T \upharpoonright l : 0 \}$. Then there is a level set $X \subseteq \hat{T}$ with $l_X \geq l_C$ such that $X$ is free in $T$. Since $f$ is stable, $Y = f \circ f^{-1}[Y]$ has no new pre-$a$-cliques over $Y \upharpoonright l^n_m$, and all passing numbers of the leftmost extensions of $X$ in $T$ are 0.

An important property of $\mathcal{T}_k$ is that all of its members contain unbounded sequences of 0’s.
Lemma 5.10. Suppose \( Z \) is the set of leftmost extensions of \( T \), for some \( i \in \omega \). Fix any subset \( X' \subseteq X \). Given \( n \) such that \( l^T_n > l^T_i \), suppose \( Y' \subseteq T \mid l^T_n \) end-extends \( X' \), and \( Y' \) has no new pre-cliques over \( X' \). Let \( Y'' \) denote the set of leftmost extensions of \( X \setminus X' \) in \( T \mid l^T_i \). Then \( Y = Y' \cup Y'' \) is free in \( T \) and contains no new pre-cliques over \( X \).

Proof. It is trivial that \( Y \) is free in \( T \), since these nodes have the length of a coding node in \( T \). Suppose not. Then there is some \( a \in [3, k] \) and some \( Z \subseteq Y \) with at least two nodes such that \( Z \) has a new pre-a-clique in the interval \((l^T_i, l^T_n)\). Note that \( Z \cap Y' \) and \( Z \cap Y'' \) must both be non-empty, since by hypothesis, \( Y' \) has no

Notation 5.6. Let \( A \) be a finite subtree of \( T_k \). We let \( l_A \) denote the maximum of the lengths of the nodes in \( A \), and let

\[
(26) \quad \max(A) = \{ t \in A : |t| = l_A \}.
\]

The next notion of valid subtree is central to the constructions in this paper. In particular, we will start with valid subtrees of a given strong coding tree \( T \) and extend them to larger valid subtrees of \( T \).

Definition 5.7. Suppose \( T \in T_k \) and let \( A \) be a finite subtree of \( T \). We say that \( A \) is valid in \( T \) if and only if \( A \) has the Witnessing Property and \( \max(A) \) is free in \( T \).

Definition 5.8 (Finite strong coding tree). A finite subtree \( A \subseteq T_k \) is a finite strong coding tree if and only if \( A \in \mathcal{A}T^k_{m+1} \) for \( m \) such that either \( d^k_m \) is a coding node or else \( m = 0 \).

Lemma 5.9. Given \( T \in T_k \), each finite strong coding tree \( A \) contained in \( T \) is valid in \( T \).

Proof. Fix \( T \in T_k \) and let \( A \) be a finite strong coding tree contained in \( T \). If \( A \in \mathcal{A}T^k_0 \), then \( A \) is the empty set vacuously is valid in \( T \). Otherwise, by Definition 5.8, \( \max(A) \) contains a coding node, so \( \max(A) \) is free in \( T \). Further, \( A \cong r_{m+1}(T_k) \) for some \( m \) such that \( d^k_m \) is a coding node. Since \( r_{m+1}(T_k) \) has the WP, it follows from Lemma 4.14 that \( A \) also has the WP. Therefore, \( A \) is valid in \( T \). □

Valid subtrees are safe to work with: They can always be extended within the ambient strong coding tree to any desired structure, as will be shown in the next subsection.

5.2. Extension Lemmas. The next series of lemmas will be used extensively throughout the rest of the paper. In particular, they ensure that every tree in \( T_k \) contains infinitely many subtrees which are also members of \( T_k \), and that for any \( A \in \mathcal{A}T^k_{m+1} \) for some \( m < \omega \) which is valid in some \( T \in T_k \), the set \( r_j[A, T] \) defined in [21] of Definition 4.11 is infinite, for each \( j > m \).

Lemma 5.10. Suppose \( T \in T_k \) and \( X \) is a level set of two or more nodes in \( T \) of length \( l^T_i \), for some \( i \in \omega \). Fix any subset \( X' \subseteq X \). Given \( n \) such that \( l^T_n > l^T_i \), suppose \( Y' \subseteq T \mid l^T_n \) end-extends \( X' \), and \( Y' \) has no new pre-cliques over \( X' \). Let \( Y'' \) denote the set of leftmost extensions of \( X \setminus X' \) in \( T \mid l^T_i \). Then \( Y = Y' \cup Y'' \) is free in \( T \) and contains no new pre-cliques over \( X \).

Proof. It is trivial that \( Y \) is free in \( T \), since these nodes have the length of a coding node in \( T \). Suppose not. Then there is some \( a \in [3, k] \) and some \( Z \subseteq Y \) with at least two nodes such that \( Z \) has a new pre-a-clique in the interval \((l^T_i, l^T_n)\). Note that \( Z \cap Y' \) and \( Z \cap Y'' \) must both be non-empty, since by hypothesis, \( Y' \) has no
Lemma 5.12. Suppose \( k \) code a pre-X. Let \( \{ t \} \subseteq X \) be any nonempty subset of \( X \). Choose a node \( n \geq |X| \), and let \( s' \) be the minimal length at which this new pre-clique occurs. Let \( m \) be the least integer such that \( |T_m| < l \leq |T_{m+1}| \). Since \( T \) has the Witnessing Property (recall Lemma 4.10), \( T_{m+1} \) must be a coding node in \( T \), say \( c^T_j \). Since the new pre-a-clique \( Z \setminus l \) must be witnessed at the level of \( c^T_j \), it follows that all nodes in \( Z \) have passing number 1 at \( c^T_j \). Let \( f \) be the stable map from \( T \) to \( T'_k \) and let \( V \) denote \( f[Z] \). Then \( V \) is a level set in \( T'_k \) of length \( t^k \). Since \( f \) is stable, it preserves passing numbers. Hence, all nodes in \( V \) have passing number 1 at \( c^T_j \) in \( T'_k \).

However, letting \( Y'' = X \setminus X' \), since \( Y'' \) consists of the leftmost extensions in \( T \) of the nodes in \( X'' \), it follows that \( f[Y''] \) is the set of leftmost extensions in \( T'_k \) of \( f[X'] \). Thus, each member of \( f[K[Y'']] \) has passing number 0 at \( c^T_j \). Since \( Z \cap Y'' \neq \emptyset \), also \( V \cap f[Y''] \neq \emptyset \), so \( V \) has at least one node with passing number 1 at \( c^T_j \), contradicting the previous paragraph. Thus, \( Y \) has no new pre-cliques over \( X \).

Lemma 5.11. Suppose \( s \) is a node in a strong coding tree \( T \in T_k \). If \( n \) is least such that \( |s| \leq t^k_n \), then there is splitting node \( t \in T \) extending \( s \) such that \( |t| \leq t^k_{n+1} \). In particular, every strong coding tree is perfect.

Proof. It suffices to work with \( T_k \), since each member of \( T_k \) is strongly similar to \( T \). We make use here of the particular construction of \( T_k \) from Theorem 4.22.

Let \( s \) be a node in \( T_k \), and let \( n \) be least such that \( t^k_n \geq |s| \). Let \( p > n \) be least such that \( p = (i+1)(k-1) \) for some \( i \), and let \( s' \) be the leftmost extension of \( s \) in \( T_k \). Note that \( p < n+k \) and that \( s' \) has passing number 0 at \( c^k_p \). By the construction of \( T_k \), the next coding node \( c^k_{p+1} \) in \( T_k \) will have passing number 1 at precisely the coding nodes \( c^k_{p-k+1}, \ldots, c^k_p \), and at no others. Let \( v \) denote the truncation of \( c^k_{p+1} \) to length \( t^k_p + 1 \). The number of coding nodes in \( T_k \) at which both \( s' \) and \( v \) have passing number 1 is at most \( k-3 \). Therefore, \( s' \) and \( v \) do not code a pre-k-clique. So by the \( K_k \)-Free Splitting Criterion, \( s' \) extends to a splitting node \( t \in T_k \) before reaching the level of \( c^k_{p+1} \).

Given a set of nodes \( Z \subseteq T_k \), by the tree induced by \( Z \) we mean the set of nodes \( \{ t \ | \ |v| : t \in T_k, v \in Z \} \).

Lemma 5.12. Suppose \( A \) is a finite valid subtree of some strong coding tree \( T \in T_k \). Let \( X \) be any nonempty subset of \( \text{max}(A) \), and let \( Z \) be any subset of \( \text{max}(A) \setminus X \). Let \( \{ s_i : i < i \} \) be any enumeration of \( X \) and suppose \( l \geq l_A \) is given. Then there exist \( l > l \) and extensions \( t^0_i, t^1_i \supseteq s_i \) \( (i < i) \) and \( t_z \supseteq z \) \( (z \in Z) \), each in \( T \setminus l_A \), such that letting

\[
Y = \{ t^i_j : i < i, j < 2 \} \cup \{ t_z : z \in Z \},
\]

and letting \( B \) denote the tree induced by \( A \cup Y \), the following hold:

1. The splitting in \( B \) above level \( l_A \) occurs in the order of the enumeration of \( X \). Thus, for \( i < i' < i \), \( |t^0_i \cap t^1_i| < |t^0_{i'} \setminus t^1_{i'}| \).
2. \( Y \) has no new pre-cliques over \( \text{max}(A) \) and is free in \( T \).

Proof. If \( l_A \) is not the level of some coding node in \( T \), begin by extending each member of \( X \) leftmost in \( T \) to the level of the very next coding node in \( T \). In this case, abuse notation and let \( X \) denote this set of extensions. Since \( A \) is valid in \( T \), this adds no new pre-cliques over \( \text{max}(A) \).
By Lemma 5.14 every node in X extends to a splitting node in T. Let \( s_i^1 \) be the splitting node of least length in T extending \( s_0 \), and let \( c_n^1 \) be the coding node in T of least length above \( |s_n^0| \). Extend all nodes in \( \{ s_i : 1 \leq i < \tilde{i} \} \leftarrow \) leftmost in T to length \( l_n^T \) and label their extensions \( \{ s_i^1 : 1 \leq i < \tilde{i} \} \). Given \( 1 \leq p < \tilde{i} \) and the nodes \( \{ s_i^p : p \leq i < \tilde{i} \} \), let \( s_p^\prime \) be the splitting node of least length in T extending \( s_p^p \), and let \( c_p^\prime \) be the coding node in T of least length above \( |s_p^\prime| \). If \( p < \tilde{i} - 1 \), then extend all nodes in \( \{ s_i^p : p + 1 \leq i < \tilde{i} \} \leftarrow \) leftmost in T to length \( l_n^T \), and label these \( \{ s_i^p : p + 1 \leq i < \tilde{i} \} \).

When \( p = \tilde{i} - 1 \), let \( n = n_{\tilde{i} - 1} \) and for each \( i < \tilde{i} \) and \( j < 2 \), let \( t_i^j \) be the leftmost extension in T of \( s_i^\tilde{i} \leftarrow j \) to length \( l_n^T \). For each \( z \in Z \), let \( t_z \) be the leftmost extension in T of \( z \) to length \( l_n^T \). This collection of nodes composes the desired set \( Y \). By Lemma 5.10 \( Y \) has no new pre-cliques over \( \max(A) \). \( Y \) is free in T since the nodes in T have the length of a coding node in T.

**Convention 5.13.** Recall that when working within a strong coding tree \( T \in \mathcal{T}_k \), the passing numbers at coding nodes in T are completely determined by \( T \); in fact, they are determined by \( \mathcal{T}_k \). For a finite subset \( A \) of T such that \( l_A = l_n^T \) for some \( n < \omega \), we shall say that \( A \) has the Witnessing Property if and only if the extension \( A \cup \{ s^+ : s \in \max(A) \} \) has the Witnessing Property.

The next Lemma 5.14 shows that given a valid subtree of a strong coding tree \( T \), any of its maximal nodes can be extended to some coding node \( c_n^T \) in T while the rest of the maximal nodes can be extended to length \( l_n^T \) so that their passing numbers are anything desired, subject only to the \( K_k \)-Free Criterion.

**Lemma 5.14 (Passing Number Choice).** Fix \( T \in \mathcal{T}_k \) and a finite valid subtree \( A \) of \( T \). Let \( \{ s_i : i < \tilde{i} \} \) be any enumeration of \( \max(A) \), and fix some \( d < i \). To each \( i \in \tilde{i} \leftarrow \{ d \} \) associate an \( \varepsilon_i \in \{ 0, 1 \} \), with the stipulation that \( \varepsilon_i \) must equal 0 if \( \{ s_i, s_{\tilde{i}} \} \) has a pre-\( k \)-clique. In particular, \( \varepsilon_d = 0 \).

Then given any \( j < \omega \), there is an \( n \geq j \) such that the coding node \( c_n^T \) extends \( s_d \), and there are extensions \( u_i \geq s_i, i \in \tilde{i} \leftarrow \{ d \} \), in \( T \leftarrow I_n^T \) such that each \( u_i \) has passing number \( \varepsilon_i \) at \( c_n^T \). Letting \( u_d = c_n^T \), any new pre-cliques among \( \{ u_i : i < \tilde{i} \} \) have their first instances occurring in the interval \( (d_{m_n - 1}, l_n^T) \). In particular, \( A \cup \{ u_i : i < \tilde{i} \} \) is valid in T.

**Proof.** Assume the hypotheses of the lemma. Let \( m \) be least such that \( l_m^T \geq l_A \), and for each \( i < \tilde{i} \), let \( s_i^\prime \) be the leftmost extension of \( s_i \) in T of length \( l_m^T \). The level set \( \max(A) \) is free in T since \( A \) is valid in T, so \( \{ s_i^\prime : i < \tilde{i} \} \) has no new pre-cliques over \( A \). Given \( j < \omega \), take \( n \geq j \) minimal above \( \max(j, m + 1) \) such that \( c_n^T \geq c_i^T \), and let \( u_d = c_n^T \). Such an \( n \) exists, as the coding nodes in any strong coding tree are dense in that tree, by its strong similarity to \( \mathcal{T}_k \). For \( i \neq d \), extend \( s_i^\prime \) via its leftmost extension in T to length \( l_n^T \), and label it \( t_i \). By Lemma 5.10 \( \{ t_i : i \in \tilde{i} \leftarrow \{ d \} \} \cup \{ u_d \leftarrow l_n^T \} \) has no new pre-cliques over \( \{ s_i^\prime : i < \tilde{i} \} \), and hence no new pre-cliques over \( A \).

For \( i < \tilde{i} \) with \( \varepsilon_i = 0 \), let \( u_i \) be the leftmost extension of \( t_i \) of length \( l_n^T \). For \( i < \tilde{i} \) with \( \varepsilon_i = 1 \), note that \( \{ t_i, u_d \leftarrow l_n^T \} \) has no \( k \)-pre-cliques, since \( \varepsilon_i = 1 \) implies \( \{ s_i, s_{\tilde{i}} \} \) has no \( k \)-pre-cliques, and their extensions to length \( l_n^T \) have no new pre-cliques by Lemma 5.10. By the \( k \)-Free Branching Criterion of \( T \), \( t_i \) splits in T before reaching the level of \( c_n^T \). Let \( u_i \) be the rightmost extension of \( t_i \) to
length $l_T^u$. Note that for each $i < \tilde{i}$, the passing number of $u_i$ at $u_d = \varepsilon_i$. Let $X = \{s_i : i < \tilde{i}\}$ and $Y = \{u_i : i < \tilde{i}\}$. Since the nodes in $Y$ have the length of the coding node $u_d$, $Y$ is free in $T$. By construction, any new pre-cliques in $Y$ over $A$ occur in the interval $(l_{n-1}, l_n^T)$. Since $T$ has the WP, any new pre-cliques of $Y$ in this interval must actually occur in the interval $(|d_{m_n-1}|, |l_n^T|)$. Thus it remains to show that any new pre-cliques in $Y$ over $A$ in the interval $(|d_{m_n-1}|, l_n^T)$ are witnessed by coding nodes in $A \cup Y$.

Suppose $I \subseteq i$ has size at least two and $\{u_i : i \in I\}$ has a new pre-$a$-clique over $A$, for some $a \in [3, k]$. Let $Z$ denote $\{u_i : i \in I\}$ and let $l$ be least such that $Z \upharpoonright l$ is a pre-$a$-clique, and note that $l$ must be in the interval $(|d_{m_n-1}|, |l_n^T|)$. Since $T$ has the WP, there is some subset of nodes in $C$ in $T$ witnessing $Z \upharpoonright l$. As $c_n^T$ is the least coding node in $T$ above $Z \upharpoonright l$, $c_n^T$ must be in $C$, again by the WP of $T$. It follows that each node in $Z$ must have passing number 1 at $c_n^T$. Note that $c_n^T$ is not in $Z$, since its passing number at $c_n^T$ is 0.

If $a = 3$, then the set $\{c_n^T\}$ is contained in $Y$ and witnesses the pre-$3$-clique in $Z$. Now suppose that $a \geq 4$. Then $C \setminus \{c_n^T\}$ witnesses that $Z' = Z \sqcup \{c_n^T\}$ has a pre-$(a - 1)$-clique. The $l'$ at which $Z' \upharpoonright l'$ is a new pre-$(a - 1)$-clique must be below $|d_{m_n-1}|$, since $T$ cannot witness it at the level of $c_n^T$. Since $Y$ has no new pre-cliques over $A$ in the levels between $l_A$ and $|d_{m_n-1}|$, it must be that $l' \leq l_A$. Since $Z' \upharpoonright l_A$ is contained in $A$ and $A$ has the WP, there is a set of coding nodes $C'$ contained in $A$ witnessing the pre-$(a - 1)$-clique $Z \upharpoonright l'$. Then $C' \sqcup \{c_n^T\}$ is contained in $A \cup Y$ and witnesses the pre-$a$-clique $Z$. It follows that $A \cup Y$ has the WP, and hence is valid in $T$.

The next lemma shows that any valid subtree of a strong coding tree can be extended to another valid subtree with any prescribed strong similarity type. This will be central to the constructions involved in proving the Ramsey theorems for strong coding trees as well as in further sections.

**Lemma 5.15.** Suppose $A$ is a valid subtree of a strong coding tree $T \in \mathcal{T}_k$ and fix any member $u \in \max(A)$. Let $X$ be any subset of $\max(A)$ such that for each $s \in X$, the pair $\{s, u\}$ has no pre-$k$-cliques, and let $Z$ denote $\max(A) \setminus (X \cup \{u\})$. Let $l \geq l_A$ be given.

Then there is an $l_s > l$ and there are extensions $u_s \supseteq u$, $s^0_s, s^1_s \supseteq s$ for all $s \in X$, and $s_s \supseteq s$ for all $s \in Z$, each of length $l_s$, such that, letting

\[ Y = \{u_s\} \cup \{s^i_s : s \in X, \ i \leq 1\} \cup \{s_s : s \in Z\}, \]

and letting $B$ be the tree induced by $A \cup Y$, the following hold:

1. $u_s$ is a coding node.
2. For each $s \in X$ and $i \leq 1$, the passing number of $s^i_s$ at $u_s$ is $i$.
3. For each $s \in Z$, the passing number of $s_s$ at $u_s$ is 0.
4. Splitting among the extensions of the $s \in X$ occurs in reverse lexicographic order: For $s$ and $t$ in $X$, $|s^0_s \wedge s^1_s| < |t^0_t \wedge t^1_t|$ if and only if $s_s > \text{lex} t_s$.
5. There are no new pre-cliques among the nodes in $X$ below the length of the longest splitting node in $B$ below $u_s$.

In particular, $B$ is valid in $T$.

**Proof.** Since $A$ is valid in $T$, apply Lemma 5.12 to extend $\max(A)$ to have splitting nodes in the desired order without adding any new pre-cliques and so that this
extension is free in $T$. Then apply Lemma 5.13 to extend to a level with a coding node and passing numbers as prescribed, with the extension being valid in $T$. □

These lemmas lead to the main extension theorem of this section.

**Theorem 5.16.** Suppose $T \in \mathcal{T}_k$, $m < \omega$, and $A \in \mathcal{AT}_m$ is a valid subtree of $T$. Then the set $r_{m+1}[A,T]$ is infinite. In particular, for each $l < \omega$, there is a member $B \in r_{m+1}[A,T]$ with $B$ is valid in $T$ and $l_B \geq l$. Furthermore, $[A,T]$ is infinite, and for each strictly increasing sequence of integers $(l_j)_{j \geq m}$, there is a member $S \in [A,T]$ such that $[d^S]_j \geq l_j$ and $r_j(S)$ is valid in $T$, for each $j > m$.

**Proof.** This follows immediately from Lemma 5.15. □

The final lemmas of this section set up for constructions in the main theorem of Section 3.

**Lemma 5.17.** Suppose $T \in \mathcal{T}_k$, and $X = X' \cup X''$ is a level set with length of some coding node $c^T_j$, which is a member of $X'$. Suppose further that $j < n$ and $Y' \subseteq T \upharpoonright l^T_{n-1}$ end-extends $X'$ with the following properties: $Y'$ has no new pre-cliques over $X'$, and each node in $Y'$ has the same passing number at $c^T_n$ as it does at $c^T_j$. Then there is a level set $Y'' \subseteq T \upharpoonright l^T_{n-1}$ end-extending $X''$ such that each node in $Y''$ has the same passing number at $c^T_n$ as it does at $c^T_j$, and $Y = Y' \cup Y''$ has no new pre-cliques over $X$.

**Proof.** If $n > j + 1$, first extend the nodes in $X''$ leftmost in $T$ to length $l^T_{n-1}$, and label this set of nodes $Y'' \upharpoonright l^T_{n-1}$. By Lemma 5.10 $Y'' \upharpoonright l^T_{n-1} \cup Y' \upharpoonright l^T_{n-1}$ has no new pre-cliques over $X$. Apply Lemma 5.14 to extend the nodes in $Y'' \upharpoonright l^T_{n-1}$ to $Y'' \subseteq T \upharpoonright l^T_{n-1}$ such that for each node $t \in Y''$, $t$ has the same passing number at $c^T_n$ as it does at $c^T_j$. Let $Y = Y' \cup Y''$.

Suppose towards a contradiction that for some $a \in [3, k]$, there is a new pre-a-clique $Z \subseteq Y$ above $X$. If $a = 3$, then $c^T_n$ witnesses this pre-a-clique. Since each node in $Y$ has the same passing number at $c^T_n$ as it does at $c^T_j$, it follows that $Z \upharpoonright l^T_j$ has a pre-a-clique which is witnessed by $c^T_j$. Thus, $Z$ was not new over $X$.

Now suppose that $a \geq 4$. Then $Z \cup \{c^T_n\}$ has a pre-b-clique, where $b = a - 1$. Since $Z$ is a new pre-a-clique and $T \cong \mathcal{T}_k$, it must be that the the level where the pre-a-clique in $Z \cup \{c^T_n\}$ is new must be at some $l \leq l^T_{n-1}$. Since $Y$ has no new pre-a-cliques in the interval $(l^T_j, l^T_{n-1}]$, this $l$ cannot be greater than $l^T_j$. Since the passing numbers of members of $Y$ are the same at $c^T_n$ as they are at $c^T_j$, it follows that $Z \cup \{c^T_j\}$ has a pre-b-clique. This pre-b-clique must occur at some level strictly below $l^T_j$, since the passing number of the coding node $c^T_n$ at itself is 0. Hence, $Z \upharpoonright l^T_j \cup \{c^T_j\}$ is a pre-a-clique. Therefore, $Z$ is not new over $X$, a contradiction. □

**Lemma 5.18.** Suppose $T \in \mathcal{T}_k$, $m < \omega$, and $B \in r_{m+1}[0,T]$ with $B$ valid in $T$. Let $x_*$ be the critical node of $\max(B)$, let $X \subseteq \max(B)$ with $x_* \in X$, and let $X' = \max(B) \setminus X$. Suppose that $Y$ end-extends $X$ into $T$ so that $Y$ has no new pre-cliques over $X$, $Y$ is free in $T$, and the critical node $x_*$ is extended to the same type of critical node $y_*$ in $Y$. If $x_*$ is a coding node, assume that for each $y \in Y$, the passing number of $y$ at $y_*$ is the same as the passing number of $y$ at $x_*$. Then there is a level set $Y' \upharpoonright l^T_T$ end-extending $X'$ in $T$ to length $l_Y$ such that $r_{m+1}[B \cup (Y \cup Y')]$ is a member of $r_{m+1}[r_{m+1}(B), T]$. □
Proof. Suppose first that $x_*$ is a splitting node. Let $Y'$ consist of the leftmost extensions in $T$ of members of $X'$ to length $l_Y$. In this case, $y_*$ is a splitting node extending $x_*$, so it suffices to show that $Y \cup Y'$ has no new pre-cliques over $r_m(B)$. Since $B$ has the WP and $\max(B)$ has a splitting node, it follows that $X'$ has no new pre-cliques over $r_m(B)$. Since $B$ being valid in $T$ implies that $\max(B)$ is free in $T$, it follows that $Y'$ has no new pre-cliques over $r_m(B)$ and is free in $T$, by Lemma 5.10.

The following argument is similar to the proof of Lemma 5.10. Suppose toward a contradiction that for some $a \in [3,k]$ there is a subset $Z \subseteq Y \cup Y'$ with at least two nodes such that $Z$ has a new pre-$a$-clique in the interval $(l_X,l_Y]$. Let $l$ be the minimal length where this new pre-$a$-clique occurs, and let $m$ be the least integer such that $|d^l_m| < l \leq |d^l_{m+1}|$. Since $T$ has the WP, $d^l_{m+1}$ is a coding node in $T$, say $c^T_j$. As the new pre-$a$-clique $Z \upharpoonleft l$ must be witnessed in $T$ at the level of $c^T_j$, it follows that all nodes in $Z$ have passing number 1 at $c^T_j$. Let $f$ be the stable map from $T$ to $T_k$ and let $V$ denote $f[Z]$. Then $V$ is a level set in $T_k$ of length $l^T_k$, and all nodes in $V$ have passing number 1 at $c^T_j$ in $T_k$. However, since $Y'$ consists of the leftmost extensions in $T$ of $X'$ which is free in $T$, all nodes in $f[Y']$ have passing number 0 at $c^T_j$. Since $Y$ has no new pre-cliques over $X$, the set $Z \cap Y'$ must contain at least one node. Hence, at least one member of $V$ has both passing number 0 and passing number 1 at $c^T_j$, a contradiction.

Now suppose that $x_*$ is a coding node. Let $n$ be the integer such that $y_* = c^T_n$. Then $l_X \leq l^T_{n-1}$. Let $W'$ denote the leftmost extensions of the nodes in $X'$ in $T \upharpoonleft l^T_{n-1}$. Since $x_*$ is a coding node, $\max(B)$ is free in $T$. By the same argument as above, when $x_*$ is a splitting node, the set $W' \cup (Y \upharpoonleft l^T_{n-1})$ has no new pre-cliques over $B$. For $i \in 2$, let $W'_i$ be the set of those $w \in W'$ which have passing number $i$ at $x_*$. Note that for each $w \in W'_1$, the set $\{w \upharpoonleft l_X,x_*\}$ has no pre-$k$-clique, and since no new pre-cliques are added between the levels of $l_X$ and $l^T_{n-1}$, the set $\{w,y_* \upharpoonleft l^T_{n-1}\}$ has no pre-$k$-clique. Since $T$ satisfies the $K_k$-Free Branching Criterion, each $w \in W'_1$ can be extended to a node $y \in T \upharpoonleft l_Y$ with passing number 1 at $y_*$. Extend each node in $W'_0 \upharpoonleft l_Y$ with length $l_Y$. Let $Y' = W'_0 \cup W'_1$. Then $Y'$ end-extends $W'$ which end-extends $X'$, and each $y \in Y'$ has the same passing number at $y_*$ as it does at $x_*$. We claim that $Y \cup Y'$ has no new pre-cliques over $B$. Suppose towards a contradiction that $Z \subseteq Y \cup Y'$ is a new pre-$a$-clique above $B$, for some $a \in [3,k]$. Since $Z \upharpoonleft l^T_{n-1}$ has no new pre-cliques over $B$, this new pre-$a$-clique must take place at some level $l \in ([d],l_Y]$. Since $T$ has the WP, $l$ must be in the interval $(|d|,l_Y]$, where $d$ is the longest splitting node in $T$ of length less than $l_Y$. If $a = 3$, then $y_*$ witnesses the pre-3-clique $Z$. But then $Z \upharpoonleft l_X$ must also be a pre-3-clique, since the passing numbers at $y_*$ are the same as at $x_*$, and $x_*$ witnesses the pre-3-clique $Z \upharpoonleft l_X$. Hence, $Z$ is not new over $B$. Now suppose that $a \geq 4$. Then $Z \cup \{x_*\}$ has a pre-$(a-1)$-clique at some level $l' < l$. Since $T$ has the WP, $Z \cup \{x_*\}$ can have at most one new pre-clique in the interval $(|d|,l_Y]$, and $T$ has no new pre-cliques between $(l^T_{n-1},|d|]$. Thus, it must be that $l' \leq l_X$. Therefore, the minimal level of a pre-$(a-1)$-clique in $Z \cup \{x_*\}$ at some level in $B$. Since $B$ has the WP, this is witnessed in $B$. Since $y_* \geq x_*$ and each $z \in Z$ has the same passing number at $y_*$ as at $x_*$, $x_*$ cannot be a witness of the pre-$(a-1)$-clique in $Z \cup \{x_*\}$. Therefore, $Z \cup \{x_*\}$ must be witnessed in $r_m(B)$, say by coding nodes $\{c^B_j : j < a-3\}$, where
Now we will show that $Y \cup Y'$ has no new pre-cliques over $r_m(B)$. Suppose $Z \subseteq Y \cup Y'$ has a pre-a-clique, for some $a \in [3, k]$. Since this pre-a-clique is not new over $B$, there is some $l \leq l_X$ where $Z \upharpoonright l$ is a new pre-a-clique in $B$. Since $B$ is valid, it has the WP, so there are some coding nodes $c_{i_0}^B, \ldots, c_{i_{a-3}}^B$ in $B$ witnessing $Z \upharpoonright l$. If $i_{a-3} < m$, then these witnesses are in $r_m(B)$. Now suppose that $i_{a-3} = m$. Note that $y_s \supseteq x_s = c_m^B$. Thus, $\{y_s\} \cup \{c_{i_j}^B : j < a - 3\}$ forms a pre-$(a - 1)$-clique which witnesses $Z$. Therefore, $r_m(B) \cup Y' \cup Y$ has the WP. Since it is strongly similar to $B$, $r_m(B) \cup Y' \cup Y$ is a member of $r_{m+1}[r_m(B), T]$ by Lemma 4.14. □

Remark 5.19. As was remarked for $T_3$ in [3], each space $(T_k, \leq, r)$, $k \geq 3$, satisfies Axioms A.1, A.2, and A.3(1) of Todorcevic’s axioms in Chapter 5 of [38] guaranteeing a topological Ramsey space, and it is routine to check this. However, Axiom A.3(2) does not hold. The pigeonhole principle, Axiom A.4, holds exactly when the finite subtree is valid inside the given strong coding tree; this will follow from Theorems in Section 6 and 7.

6. HALPERN-LÄUCHLI-STYLE THEOREMS FOR STRONG CODING TREES

The Ramsey theory content for strong coding trees begins in this section. The ultimate goal is to obtain a Ramsey theorem for colorings of strictly similar (Definition 8.3) copies of any given finite antichain of coding nodes, as these are the structures which will code finite triangle-free graphs. This is accomplished in Theorem 8.9. In Section 7 we will prove Milliken-style theorems for finite trees satisfying a strict version of the Witnessing Property. Just as the Halpern-Läuchli Theorem forms the core content of Milliken’s Theorem in the setting of strong trees, so too in the setting of strong coding trees, Halpern-Läuchli-style theorems are proved first and then applied to obtain Milliken-style theorems.

Theorem 6.2 encompasses colorings of two different types of level set extensions of a fixed finite tree: The level set either contains a splitting node (Case (a)) or a coding node (Case (b)). In Case (a), we obtain a direct analogue of the Halpern-Läuchli Theorem. In Case (b), we obtain a weaker version of the Halpern-Läuchli Theorem, which is later strengthened to the direct analogue in Lemma 7.8. The proof given here basically follows the outline of the proof in [4], but is more streamlined, treating the two cases simultaneously most of the time. This is due to having proved more general extension lemmas in Section 4 needed here.

Let $k \geq 3$ be fixed, and fix the following terminology and notation. Given subtrees $U, V$ of $T_k$ with $U$ finite, we write $U \supseteq V$ if and only if $U = \{v \in V : |v| \leq l_U\}$; in this case we say that $V$ extends $U$, or that $U$ is an initial subtree of $V$. We write $U \subset V$ if $U$ is a proper initial subtree of $V$. Recall the following notation from Definition 4.11 of the space $(T_k, \leq, r)$: $S \preceq T$ means that $S$ and $T$ are members of $T_k$ and $S$ is a subtree of $T$. Given $A \in \mathcal{AR}_m$ for some $m < \omega$, $[A, T]$ denotes the set of all $S \preceq T$ such that $S$ extends $A$. We now begin setting up for the two possible cases before stating the theorem.

**The Set-up for Theorem 6.2** Let $T \in T_k$ be given, and let $A$ be a finite valid subtree of $T$ with the Witnessing Property. $A$ is allowed to have terminal nodes at different levels. In order to simplify notation in the proof, without loss of generality, we assume that $0^{(l_A)}$ is in $A$. Let $A^+$ denote the set of immediate extensions in $T$. But then $x_s \cup \{c_j^B : j < a - 3\}$ witnesses the pre-a-clique $Z$. Hence, $Z$ is not new over $B$.

Now we will show that $Y \cup Y'$ has no new pre-cliques over $r_m(B)$. Suppose $Z \subseteq Y \cup Y'$ has a pre-a-clique, for some $a \in [3, k]$. Since this pre-a-clique is not new over $B$, there is some $l \leq l_X$ where $Z \upharpoonright l$ is a new pre-a-clique in $B$. Since $B$ is valid, it has the WP, so there are some coding nodes $c_{i_0}^B, \ldots, c_{i_{a-3}}^B$ in $B$ witnessing $Z \upharpoonright l$. If $i_{a-3} < m$, then these witnesses are in $r_m(B)$. Now suppose that $i_{a-3} = m$. Note that $y_s \supseteq x_s = c_m^B$. Thus, $\{y_s\} \cup \{c_{i_j}^B : j < a - 3\}$ forms a pre-$(a - 1)$-clique which witnesses $Z$. Therefore, $r_m(B) \cup Y' \cup Y$ has the WP. Since it is strongly similar to $B$, $r_m(B) \cup Y' \cup Y$ is a member of $r_{m+1}[r_m(B), T]$ by Lemma 4.14. □

Remark 5.19. As was remarked for $T_3$ in [4], each space $(T_k, \leq, r)$, $k \geq 3$, satisfies Axioms A.1, A.2, and A.3(1) of Todorcevic’s axioms in Chapter 5 of [38] guaranteeing a topological Ramsey space, and it is routine to check this. However, Axiom A.3(2) does not hold. The pigeonhole principle, Axiom A.4, holds exactly when the finite subtree is valid inside the given strong coding tree; this will follow from Theorems in Section 6 and 7.

6. HALPERN-LÄUCHLI-STYLE THEOREMS FOR STRONG CODING TREES

The Ramsey theory content for strong coding trees begins in this section. The ultimate goal is to obtain a Ramsey theorem for colorings of strictly similar (Definition 8.3) copies of any given finite antichain of coding nodes, as these are the structures which will code finite triangle-free graphs. This is accomplished in Theorem 8.9. In Section 7 we will prove Milliken-style theorems for finite trees satisfying a strict version of the Witnessing Property. Just as the Halpern-Läuchli Theorem forms the core content of Milliken’s Theorem in the setting of strong trees, so too in the setting of strong coding trees, Halpern-Läuchli-style theorems are proved first and then applied to obtain Milliken-style theorems.

Theorem 6.2 encompasses colorings of two different types of level set extensions of a fixed finite tree: The level set either contains a splitting node (Case (a)) or a coding node (Case (b)). In Case (a), we obtain a direct analogue of the Halpern-Läuchli Theorem. In Case (b), we obtain a weaker version of the Halpern-Läuchli Theorem, which is later strengthened to the direct analogue in Lemma 7.8. The proof given here basically follows the outline of the proof in [4], but is more streamlined, treating the two cases simultaneously most of the time. This is due to having proved more general extension lemmas in Section 4 needed here.

Let $k \geq 3$ be fixed, and fix the following terminology and notation. Given subtrees $U, V$ of $T_k$ with $U$ finite, we write $U \supseteq V$ if and only if $U = \{v \in V : |v| \leq l_U\}$; in this case we say that $V$ extends $U$, or that $U$ is an initial subtree of $V$. We write $U \subset V$ if $U$ is a proper initial subtree of $V$. Recall the following notation from Definition 4.11 of the space $(T_k, \leq, r)$: $S \preceq T$ means that $S$ and $T$ are members of $T_k$ and $S$ is a subtree of $T$. Given $A \in \mathcal{AR}_m$ for some $m < \omega$, $[A, T]$ denotes the set of all $S \preceq T$ such that $S$ extends $A$. We now begin setting up for the two possible cases before stating the theorem.

**The Set-up for Theorem 6.2** Let $T \in T_k$ be given, and let $A$ be a finite valid subtree of $T$ with the Witnessing Property. $A$ is allowed to have terminal nodes at different levels. In order to simplify notation in the proof, without loss of generality, we assume that $0^{(l_A)}$ is in $A$. Let $A^+$ denote the set of immediate extensions in $T$. But then $x_s \cup \{c_j^B : j < a - 3\}$ witnesses the pre-a-clique $Z$. Hence, $Z$ is not new over $B$.

Now we will show that $Y \cup Y'$ has no new pre-cliques over $r_m(B)$. Suppose $Z \subseteq Y \cup Y'$ has a pre-a-clique, for some $a \in [3, k]$. Since this pre-a-clique is not new over $B$, there is some $l \leq l_X$ where $Z \upharpoonright l$ is a new pre-a-clique in $B$. Since $B$ is valid, it has the WP, so there are some coding nodes $c_{i_0}^B, \ldots, c_{i_{a-3}}^B$ in $B$ witnessing $Z \upharpoonright l$. If $i_{a-3} < m$, then these witnesses are in $r_m(B)$. Now suppose that $i_{a-3} = m$. Note that $y_s \supseteq x_s = c_m^B$. Thus, $\{y_s\} \cup \{c_{i_j}^B : j < a - 3\}$ forms a pre-$(a - 1)$-clique which witnesses $Z$. Therefore, $r_m(B) \cup Y' \cup Y$ has the WP. Since it is strongly similar to $B$, $r_m(B) \cup Y' \cup Y$ is a member of $r_{m+1}[r_m(B), T]$ by Lemma 4.14. □

Remark 5.19. As was remarked for $T_3$ in [4], each space $(T_k, \leq, r)$, $k \geq 3$, satisfies Axioms A.1, A.2, and A.3(1) of Todorcevic’s axioms in Chapter 5 of [38] guaranteeing a topological Ramsey space, and it is routine to check this. However, Axiom A.3(2) does not hold. The pigeonhole principle, Axiom A.4, holds exactly when the finite subtree is valid inside the given strong coding tree; this will follow from Theorems in Section 6 and 7.
of the members of $\max(A)$; thus,
\begin{equation}
A^+ = \{ s \upharpoonright i : s \in \max(A), \ i \in \{0,1\}, \ \text{and} \ s \upharpoonright i \in \hat{T} \}.
\end{equation}
Note that $A^+$ is a level set of nodes of length $l_A + 1$. Let $A_e$ be a subset of $A^+$ containing $0^{(l_A+1)}$ and of size at least two. (If $A^+$ has only one member, then $A$ consists of one non-splitting node of the form $0^{(l)}$ for some $l$, and the theorem in this section does not apply.) Suppose that $X$ is a level set of nodes in $T$ extending $A_e$ so that $A \cup X$ is a finite valid subtree of $T$ satisfying the Witnessing Property. Assume moreover that $0^{(l_X)}$ is a member of $X$, so that the node $0^{(l_A)}$ in $A_e$ is extended by $0^{(l_X)}$ in $X$. There are two possibilities:

**Case (a).** $X$ contains a splitting node.

**Case (b).** $X$ contains a coding node.

In both cases, define
\begin{equation}
\Ext_T(A, \hat{X}) = \{ X \subseteq T : X \supseteq \hat{X} \text{ is a level set, } A \cup X \cong A \cup \hat{X}, \text{ and } A \cup X \text{ is valid in } T \}.
\end{equation}

The next lemma shows that seemingly weaker properties suffice to guarantee that a level set is in $\Ext_T(A, \hat{X})$.

**Lemma 6.1.** Let $X$ be a level set in $T$ extending $\hat{X}$. Then $X \in \Ext_T(A, \hat{X})$ if and only if $X$ is free in $T$, $A \cup X$ is strongly similar to $A \cup \hat{X}$, and $X$ has no new pre-cliques over $A \cup \hat{X}$.

**Proof.** The forward direction follows from the definition of $\Ext_T(A, \hat{X})$: $A \cup X \cong A \cup \hat{X}$ implies that these trees are strongly similar. In Case (a), $A \cup X \cong A \cup \hat{X}$ and $A \cup X$ has the WP implies that $X$ is free in $T$ and has no new pre-cliques over $A$, and hence over $A \cup \hat{X}$. In Case (b), since $X$ contains a coding node, $X$ is automatically free in $T$. Since $A \cup X$ has the WP, if $Z \subseteq X$ has a new pre-a-clique at some level $l$ above $l_A$, then $Z$ must be witnessed by a set of coding nodes $C \subseteq A$ along with the coding node $c_X$ in $X$. It follows that the passing numbers of nodes in $Z$ must be 1 at $l_X$. Since $A \cup X \cong A \cup \hat{X}$, the passing numbers of nodes in $Z$ are 1 at $l_X$; so $C$ along with the coding node in $\hat{X}$ witnesses $Z$. Thus, $Z$ extends a pre-a-clique already occurring in $A \cup \hat{X}$.

Now suppose that $X \supseteq \hat{X}$ as in the second part of the statement. Then $X$ has no new pre-cliques over $A \cup \hat{X}$, and $A \cup X$ is strongly similar to $A \cup \hat{X}$. In Case (a), this implies that $A \cup X$ has the WP, since $\hat{X}$ has no new pre-cliques over $A$. In Case (b), if $Z \subseteq X$ has a new pre-a-clique over $A$, then $Z \upharpoonright l_X$ has a new a pre-a-clique over $A$, since we are assuming $X$ has no new pre-cliques over $A \cup \hat{X}$. Since $A \cup \hat{X}$ has the WP, the $Z \upharpoonright l_X$ is witnessed by some coding nodes $C \subseteq Z$ along with the coding node in $\hat{X}$. By strong similarity of $A \cup X$ with $A \cup \hat{X}$, the coding node in $X$ along with $C$ witnesses the new pre-a-clique. Thus, $A \cup X$ has the WP. By Lemma 4.13, $A \cup X$ being strongly similar to $A \cup \hat{X}$ and both having the WP implies that $A \cup X \cong A \cup \hat{X}$. Since $X$ is free in $T$, $A \cup X$ is valid in $T$. Therefore, $X$ is a member of $\Ext_T(A, \hat{X})$. 

In the following, for a finite subtree $A$ of some $T \in \mathcal{T}_k$, recall that $\max(A)$ denotes the set $\{ t \in A : |t| = l_A \}$, the set of all nodes in $A$ of the maximum length, and $A^+$ denotes the set of all immediate successors of $\max(A)$ in $\hat{T}$. We now prove the analogue of the Halpern-Läuchli Theorem for strong coding trees.
Theorem 6.2. Fix $T \in T_k$ and $B$ a finite valid subtree of $T$ such that $B \in \mathcal{AT}_m$, for some $m \geq 1$. Let $A$ be a subtree of $B$ with $l_A = l_B$ and $0^{(l_A)} \in A$ such that $A$ has the Witnessing Property and is valid in $T$. Let $A_e$ be a subset of $A^+$ of size at least two such that $0^{(l_A+1)}$ is in $A_e$. Let $\tilde{X}$ be a level set in $T$ end-extending $A_e$ with at least two members, one of which is the node $0^{(l_A)}$ such that $A \cup \tilde{X}$ is a finite valid subtree of $T$ with the Witnessing Property.

Given any coloring $h : \text{Ext}_T(A, \tilde{X}) \to 2$, there is a strong coding tree $S \in [B, T]$ such that $h$ is monochromatic on $\text{Ext}_S(A, \tilde{X})$. If $\tilde{X}$ has a coding node, then the strong coding tree $S$ is, moreover, taken to be in $[r_{m_0-1}(B'), T]$, where $m_0$ is the integer for which there is a $B' \in r_{m_0}[B, T]$ with $\tilde{X} \subseteq \text{max}(B')$.

Proof. Let $T, A, A_e, B, \tilde{X}$ be given satisfying the hypotheses, and let $h : \text{Ext}_T(A, \tilde{X}) \to 2$ be a given coloring. Fix the following notation: Let $d+1$ equal the number of nodes in $\tilde{X}$, and enumerate the nodes in $\tilde{X}$ as $s_0, \ldots, s_d$ so that $s_d$ is the critical node in $\tilde{X}$. Let $i_0$ denote the integer such that $s_{i_0}$ is the node which is a sequence of $0$’s. Notice that $i_0$ can equal $d$ only if we are in Case (a) and the splitting node in $\tilde{X}$ is a sequence of $0$’s. In Case (b), let $I_0$ denote the set of all $i < d$ such that $s_i(l_{\tilde{X}}) = 0$ and let $I_1$ denote the set of all $i < d$ such that $s_i(l_{\tilde{X}}) = 1$.

Let $L$ denote the collection of all $l < \omega$ such that there is a member of $\text{Ext}_T(A, \tilde{X})$ with nodes of length $l$. In Case (a), since $B$ is valid in $T$, $L$ consists of those $l < \omega$ for which there is a splitting node of length $l$ extending $s_d$, is infinite by Lemma 5.10. In Case (b), since $\tilde{X}$ contains a coding node, it follows from the proof of Lemma 5.14 that $L$ is exactly the set of all $l < \omega$ for which there is a coding node of length $l$ extending $s_d$.

For each $i \in (d+1) \setminus \{i_0\}$, let $T_i = \{t \in T : t \supseteq s_i\}$; and let $T_{i_0} = \{t \in T : t \supseteq s_{i_0}$ and $t \in 0^{<\omega}\}$, the collection of all leftmost nodes in $T$ extending $s_{i_0}$. Let $\kappa = \beth_{2d}$. The following forcing notion $\mathbb{P}$ adds $\kappa$ many paths through $T_i$, for each $i \in d \setminus \{i_0\}$, and one path through $T_d$. If $i_0 \neq d$, then $\mathbb{P}$ will add one path through $T_{i_0}$, but with $\kappa$ many ordinals labeling this path. We allow this in order to simplify notation.

$\mathbb{P}$ is the set of conditions $p$ such that $p$ is a function of the form

$$p : (d \times \delta_p) \cup \{d\} \to T \upharpoonright l_p,$$

where $\delta_p \in [\kappa]^{<\omega}$, $l_p \in L$, $\{p(i, \delta) : \delta \in \delta_p\} \subseteq T_i \upharpoonright l_p$ for each $i < d$, and the following hold:

Case (a). (i) $p(d)$ is the splitting node extending $s_d$ of length $l_p$;

(ii) $\{p(i, \delta) : (i, \delta) \in d \times \delta_p\} \cup \{p(d)\}$ is free in $T$.

Case (b). (i) $p(d)$ is the coding node extending $s_d$ of length $l_p$;

(ii) For each $\delta \in \delta_p$, $j \in \{0, 1\}$, and $i \in I_j$, the passing number of $p(i, \delta)$ at $p(d)$ is $j$.

Given $p \in \mathbb{P}$, the range of $p$ is defined as

$$\text{ran}(p) = \{p(i, \delta) : (i, \delta) \in d \times \delta_p\} \cup \{p(d)\}.$$ 

If also $q \in \mathbb{P}$ and $\delta_q \subseteq \delta_p$, then we let $\text{ran}(q \upharpoonright \delta_q)$ denote $\{q(i, \delta) : (i, \delta) \in d \times \delta_q\} \cup \{q(d)\}$. In both Cases (a) and (b), the partial ordering on $\mathbb{P}$ is defined as follows: $q \leq p$ if and only if $l_q \geq l_p$, $\delta_q \supseteq \delta_p$, and the following hold:
Then \( q(d) \supseteq p(d) \), and \( q(i, \delta) \supseteq p(i, \delta) \) for each \( (i, \delta) \in d \times \delta_p \), and
\( \text{ran}(q \upharpoonright \vec{\delta}_p) \) has no new pre-cliques over \( \text{ran}(p) \).

Since all conditions in \( \mathbb{P} \) have ranges which are free in \( T \), we shall say that \( q \) is valid over \( p \) to mean that (ii) holds.

The theorem will be proved in two main parts. In Part I, we check that \( \mathbb{P} \) is an atomless partial order and then prove the main Lemma 6.6. In Part II, we apply Lemma 6.6 to build the tree \( S \) such that \( h \) is monochromatic on \( \text{Ext}_S(A, \bar{X}) \).

\textbf{Part I.}

\textbf{Lemma 6.3.} \( (\mathbb{P}, \leq) \) is an atomless partial ordering.

\textit{Proof.} The order \( \leq \) on \( \mathbb{P} \) is clearly reflexive and antisymmetric. Transitivity follows from the fact that the requirement (ii) in the definition of the partial order on \( \mathbb{P} \) is a transitive property. To see this, suppose that \( p \geq q \) and \( q \geq r \). Then \( \vec{\delta}_p \subseteq \vec{\delta}_q \subseteq \vec{\delta}_r \), \( l_p \leq l_q \leq l_r \), \( r \) is valid over \( q \), and \( q \) is valid over \( p \). Since \( \text{ran}(r \upharpoonright \vec{\delta}_p) \) is contained in \( \text{ran}(r \upharpoonright \vec{\delta}_q) \) which has no new pre-cliques over \( \text{ran}(q) \), it follows that \( \text{ran}(r \upharpoonright \vec{\delta}_p) \) has no new pre-cliques over \( \text{ran}(q \upharpoonright \vec{\delta}_p) \). Since \( \text{ran}(q \upharpoonright \vec{\delta}_p) \) has no new pre-cliques over \( \text{ran}(p) \), it follows that \( \text{ran}(r \upharpoonright \vec{\delta}_p) \) has no new pre-cliques over \( \text{ran}(p) \). Therefore, \( r \) is valid over \( p \), so \( p \geq r \).

\textbf{Claim 2.} \textit{For each} \( p \in \mathbb{P} \) \textit{and} \( l > l_p \), \textit{there are} \( q, r \in \mathbb{P} \) \textit{with} \( l_q, l_r > l \) \textit{such that} \( q, r < p \) \textit{and} \( q \) \textit{and} \( r \) \textit{are incompatible.}

\textit{Proof.} Let \( p \in \mathbb{P} \) and \( l > l_p \) be given, and let \( \vec{\delta} \) denote \( \vec{\delta}_p \) \textit{and let} \( \vec{\delta}_r = \vec{\delta}_q = \vec{\delta} \). In Case (a), take \( q(d) \) \textit{and} \( r(d) \) \textit{to be incomparable splitting nodes in} \( T \textit{ extending} p(d) \) \textit{to some lengths greater than} \( l \). Such splitting nodes exist by Lemma 5.11 showing that strong coding trees are perfect. Let \( l_q = |q(d)| \) \textit{and} \( l_r = |r(d)| \). For each \( (i, \delta) \in d \times \vec{\delta}_r \), \textit{let} \( q(i, \delta) \) \textit{be the leftmost extension in} \( T \) \textit{ of} \( p(i, \delta) \) \textit{to length} \( l_q \), \textit{and let} \( r(i, \delta) \) \textit{be the leftmost extension of} \( p(i, \delta) \) \textit{to length} \( l_r \). Then \( q \) \textit{and} \( r \) \textit{are members of} \( \mathbb{P} \). Since \( \text{ran}(p) \) \textit{is free in} \( T \), both \( \text{ran}(q) \) \textit{and} \( \text{ran}(r) \) \textit{are free in} \( T \) \textit{and} \( \text{ran}(q \upharpoonright \vec{\delta}_p) \) \textit{and} \( \text{ran}(r \upharpoonright \vec{\delta}_p) \) \textit{have no new pre-cliques over} \( \text{ran}(p) \), by Lemma 5.10. \textit{It follows that} \( q \) \textit{and} \( r \) \textit{are both valid over} \( p \). Since neither of \( q(d) \) \textit{and} \( r(d) \) \textit{extends the other}, \( q \) \textit{and} \( r \) \textit{are incompatible.}

In Case (b), let \( s \) be a splitting node in \( T \) \textit{ of length greater than} \( l \) \textit{ extending} \( p(d) \). Let \( k \) be minimal such that \( |c_k^T| \geq |s| \). \textit{Let} \( u, v \) \textit{extend} \( s^0, s^1 \), \textit{respectively, leftmost in} \( T \upharpoonright l_k^T \). \textit{For each} \( (i, \delta) \in d \times \vec{\delta}_p \), \textit{let} \( p'(i, \delta) \) \textit{be the leftmost extension of} \( p(i, \delta) \) \textit{to length} \( l_k \). By Lemma 5.14, there are \( q(d) \supseteq u \) \textit{and} \( q(i, \delta) \supseteq p'(i, \delta) \), \textit{(i, \delta) \in d \times \vec{\delta}_p}, \textit{such that}
\begin{itemize}
  \item (1) \( q(d) \) \textit{is a coding node;}
  \item (2) \( q \) \textit{is valid over} \( p \);
  \item (3) \textit{For each} \( j < 2, i \in I_j \) \textit{if and only if the immediate extension of} \( q(i, \delta) \) \textit{is} \( j \).
\end{itemize}
\textit{Then} \( q \in \mathbb{P} \) \textit{and} \( q \leq p \). \textit{Likewise by Lemma 5.14 there is a condition} \( r \in \mathbb{P} \) \textit{which extends} \( \{p'(i, \delta) : (i, \delta) \in d \times \vec{\delta}_p\} \cup \{v\} \) \textit{such that} \( r \leq p \). \textit{Since the coding nodes} \( q(d) \) \textit{and} \( r(d) \) \textit{are incomparable,} \( q \) \textit{and} \( r \) \textit{are incompatible conditions in} \( \mathbb{P} \). \( \square \)

\textit{It follows from Claim 2 that} \( \mathbb{P} \) \textit{is atomless.} \( \square \)

\textit{From now on, whenever ambiguity will not arise by doing so, for a condition} \( p \in \mathbb{P} \), \textit{we will use the terminology critical node of} \( p \) \textit{to refer to} \( p(d) \), \textit{which is a}
splitting node in Case (a) and a coding node in Case (b). Let $\dot{b}_d$ be a $\mathbb{P}$-name for the generic path through $T_d$; that is, $\dot{b}_d = \{ (p(d), p) : p \in \mathbb{P} \}$. Note that for each $p \in \mathbb{P}$, $p$ forces that $\dot{b}_d \upharpoonright l_p = p(d)$. By Claim 2, it is dense to force a critical node in $\dot{b}_d$ above any given level in $T$, so $\mathbb{P}$ forces that the set of levels of critical nodes in $\dot{b}_d$ is infinite. Thus, given any generic filter $G$ for $\mathbb{P}$, $\dot{b}_d^G = \{ p(d) : p \in G \}$ is a cofinal path of critical nodes in $T_d$. Let $\dot{L}_d$ be a $\mathbb{P}$-name for the set of lengths of critical nodes in $\dot{b}_d$. Note that $\mathbb{P} \Vdash \dot{L}_d \subseteq L$. Let $\dot{U}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}_d$. For $i < d$ and $\alpha < \kappa$, let $\dot{b}_{i,\alpha}$ be a $\mathbb{P}$-name for the $\alpha$-th generic branch through $T_i$; that is, $\dot{b}_{i,\alpha} = \{ (p(i, \alpha), p) : p \in \mathbb{P} \}$ and $\alpha \in \dot{\delta}_p$. Then for any $p \in \mathbb{P}$,

$$p \Vdash (\forall i < d \, \forall \alpha \in \dot{\delta}_p (\dot{b}_{i,\alpha} \upharpoonright l_p = p(i, \alpha))) \land (\dot{b}_d \upharpoonright l_p = p(d)).$$

We shall write sets $\{ \alpha_i : i < d \}$ in $[\kappa]^d$ as vectors $\vec{\alpha} = \langle \alpha_0, \ldots, \alpha_{d-1} \rangle$ in strictly increasing order. For $\vec{\alpha} \in [\kappa]^d$, we use the following abbreviation:

$$\dot{b}_{\vec{\alpha}} \text{ denotes } \{ \dot{b}_{0,0,\alpha_0}, \ldots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d \}.$$ 

Since the branch $\dot{b}_d$ is unique, this abbreviation introduces no ambiguity. For any $l < \omega$,

$$\text{let } \dot{b}_{\vec{\alpha}} \upharpoonright l \text{ denote } \{ \dot{b}_{0,0,\alpha_0} \upharpoonright l, \ldots, \dot{b}_{d-1,\alpha_{d-1}} \upharpoonright l, \dot{b}_d \upharpoonright l \}.$$ 

Using the abbreviations just defined, $h$ is a coloring on sets of nodes of the form $\dot{b}_{\vec{\alpha}} \upharpoonright l$ whenever this is forced to be a member of $\text{Ext}_T(A, \dot{X})$. Given $\vec{\alpha} \in [\kappa]^d$ and a condition $p \in \mathbb{P}$ with $\alpha \subseteq \dot{\delta}_p$, let

$$X(p, \vec{\alpha}) = \{ p(i, \alpha_i) : i < d \} \cup \{ p(d) \}.$$ 

We now set up to prove Lemma 6.3. For each $\vec{\alpha} \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ such that

1. $\vec{\alpha} \subseteq \dot{\delta}_{p_{\vec{\alpha}}}$,
2. $X(p_{\vec{\alpha}}, \vec{\alpha}) \in \text{Ext}_T(A, \dot{X})$,
3. There is an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash “h(\dot{b}_{\vec{\alpha}} \upharpoonright l) = \varepsilon_{\vec{\alpha}}$ for $\dot{U}$ many $l$ in $\dot{L}_d$’’,
4. $h(X(p_{\vec{\alpha}}, \vec{\alpha})) = \varepsilon_{\vec{\alpha}}$.

Properties (1) - (4) can be guaranteed as follows. Recall that $\{ s_i : i \leq d \}$ enumerates $\dot{X}$ and that $s_d$ is the critical node in $\dot{X}$. For each $\vec{\alpha} \in [\kappa]^d$, define

$$p_{\vec{\alpha}}^0 = \{ ((i, \delta), t_i) : i < d, \delta \in \vec{\alpha} \} \cup \{ (d, t_d) \}.$$ 

Then $p_{\vec{\alpha}}^0$ is a condition in $\mathbb{P}$ with ran($p_{\vec{\alpha}}^0$) = $\dot{X}$, and $\dot{\delta}_{p_{\vec{\alpha}}^0} = \vec{\alpha}$ which implies (1) holds for any $p \leq p_{\vec{\alpha}}^0$. The following fact will be used many times.

Claim 3. Given $\vec{\alpha} \in [\kappa]^d$, for any $p \leq p_{\vec{\alpha}}^0$, the set of nodes $X(p, \vec{\alpha})$ is a member of $\text{Ext}_T(A, \dot{X})$.

Proof. Suppose $p \leq p_{\vec{\alpha}}^0$. Then $p$ is valid over $p_{\vec{\alpha}}^0$, so $X(p, \vec{\alpha})$ has no new pre-cliques over $\dot{X}$. Since $p$ is a condition of $\mathbb{P}$, $X(p, \vec{\alpha})$ is free in $T$ and $A \cup X(p, \vec{\alpha})$ is strongly similar to $A \cup \dot{X}$. It follows from Lemma 6.4 that $X(p, \vec{\alpha})$ is in $\text{Ext}_T(A, \dot{X})$. \[\square\]

Thus, (2) holds for any $p \leq p_{\vec{\alpha}}^0$. Take an extension $p_{\vec{\alpha}}^1 \leq p_{\vec{\alpha}}^0$ which forces $h(\dot{b}_{\vec{\alpha}} \upharpoonright l)$ to be the same value for $\dot{U}$ many $l \in \dot{L}_d$. Since $\mathbb{P}$ is a forcing notion, there is a $p_{\vec{\alpha}}^2 \leq p_{\vec{\alpha}}^1$ deciding a value $\varepsilon_{\vec{\alpha}}$ for which $p_{\vec{\alpha}}^2$ forces that $h(\dot{b}_{\vec{\alpha}} \upharpoonright l) = \varepsilon_{\vec{\alpha}}$ for $\dot{U}$ many $l \in \dot{L}_d$. Then (3) holds for any $p \leq p_{\vec{\alpha}}^2$. If $p_{\vec{\alpha}}^2$ satisfies (4), then let
\( p_\alpha = p_\alpha^2 \). Otherwise, take some \( p_\alpha^3 \leq p_\alpha^2 \) which decides some \( l \in \hat{L} \) such that \( l_{p_\alpha^3} < l_{p_\alpha^2} < l \leq l_{p_\alpha^3} \) for some \( n \), and \( p_\alpha^3 \) forces \( h(\bar{b}_\alpha \upharpoonright l) = \varepsilon_\alpha \). Since \( p_\alpha^3 \) forces "\( \bar{a}_\alpha \upharpoonright l = \{ p_\alpha^3 \upharpoonright (i, \alpha_i) \mid l \leq i < d \} \cup \{ p_\alpha^3 \upharpoonright l \} \)" and \( h \) is defined in the ground model, this means that \( p_\alpha^3 (d) \upharpoonright l \) is a splitting node in Case (a) and a coding node in Case (b), and

\[
(35) \\
h(X(p_\alpha^3, \bar{\alpha}) \upharpoonright l) = \varepsilon_\alpha,
\]

where \( X(p_\alpha^3, \bar{\alpha}) \upharpoonright l \) denotes \( \{ p_\alpha^3 \upharpoonright (i, \alpha_i) \mid l \leq i \} \) \( \cup \{ p_\alpha^3 \upharpoonright l \} \). If \( l = l_{p_\alpha^3} \), let \( p_\alpha = p_\alpha^3 \), and note that \( p_\alpha \) satisfies (1) - (4).

Otherwise, \( l < l_{p_\alpha^3} \). In Case (a), let \( p_\alpha \) be defined as follows: Let \( \bar{\delta}_\alpha = \bar{\delta}_\alpha \) and

\[
(36) \\
\forall (i, \delta) \in d \times \bar{\delta}_\alpha, \text{ let } p_\alpha(i, \delta) = p_\alpha^3(i, \delta) \upharpoonright l \text{ and let } p_\alpha(d) = p_\alpha^3(d) \upharpoonright l.
\]

Since \( p_\alpha^3 \) is a condition in \( \mathbb{P} \), \( \text{run}(p_\alpha^3) \) is free in \( T \). Furthermore, \( p_\alpha^3 \leq p_\alpha^2 \) implies that \( \text{run}(p_\alpha^3 \upharpoonright \delta p_\alpha^2) \) has no new pre-cliques over \( \text{run}(p_\alpha^2) \). Therefore, least extensions of \( \text{run}(p_\alpha) \) have no new pre-cliques, so \( \text{run}(p_\alpha) \) is free in \( T \). Therefore, \( p_\alpha \) is a condition in \( \mathbb{P} \) and \( p_\alpha \leq p_\alpha^2 \). Thus, \( p_\alpha \) satisfies (1) - (3), and (4) holds by equation (35)

In Case (b), we construct \( p_\alpha \leq p_\alpha^2 \) as follows: As in Case (a), let \( \bar{\delta}_\alpha = \bar{\delta}_\alpha \).

For each \( i < d \), define \( p_\alpha(i, \alpha_i) = p_\alpha^3(i, \alpha_i) \upharpoonright l \), and let \( p_\alpha(d) = p_\alpha^3(d) \upharpoonright l \). Then \( X(p_\alpha, \bar{\alpha}) = \{ p_\alpha^3 \upharpoonright (i, \alpha_i) \mid l \leq i \} \) \( \cup \{ p_\alpha^3 \upharpoonright l \} \), so \( h(X(p_\alpha, \bar{\alpha})) = \varepsilon_\alpha \). Let \( U \) denote \( X(p_\alpha^2, \bar{\alpha}) \) and let \( U' = \text{run}(p_\alpha^2) \setminus X \). Let \( X \) denote \( X(p_\alpha, \bar{\alpha}) \) and note that \( X \) end-extends \( U \), and \( X \) is free in \( T \) and has no new pre-cliques over \( U \). By Lemma 5.17 there is an \( X' \) end-extending \( U' \) to nodes in \( T \upharpoonright l \) so that the following hold: \( X \cup X' \) is free in \( T \) and has no new pre-cliques over \( U \cup U' \); furthermore, each node in \( X' \) has the same passing number at \( l \) as it does at \( l_{p_\alpha^2} \). Let \( \text{run}(p_\alpha) \) be this set of nodes \( X \cup X' \), where for each \( i < d \) and \( (i, \delta) \in d \times \bar{\delta}_p \) with \( \delta \neq \alpha_i \), we let \( p_\alpha(i, \delta) \) be the node in \( Y' \) extending \( p_\alpha^3(i, \delta) \). This defines a condition \( p_\alpha \leq p_\alpha^2 \) satisfying (1) - (4).

The rest of Part I follows by arguments in [4] for the case \( k = 3 \), with no modifications. It is included here for the reader’s convenience. We are assuming \( \kappa = \beth_{2d} \Rightarrow (\kappa_1)_{\beth_0} \) by the Erdős-Rado Theorem (Theorem 2.6). Given two sets of ordinals \( J, K \) we shall write \( J < K \) if every member of \( J \) is less than every member of \( K \). Let \( D = \{ 0, 2, \ldots, 2d - 2 \} \) and \( D_o = \{ 1, 3, \ldots, 2d - 1 \} \), the sets of even and odd integers less than \( 2d \), respectively. Let \( I \) denote the collection of all functions \( \iota : 2d \to 2d \) such that \( \iota \upharpoonright D \) and \( \iota \upharpoonright D_o \) are strictly increasing sequences and \( \{ \iota(0), \iota(1) \} < \{ \iota(2), \iota(3) \} < \cdots < \{ \iota(2d - 2), \iota(2d - 1) \} \). Thus, each \( \iota \) codes two strictly increasing sequences \( \iota \upharpoonright D \) and \( \iota \upharpoonright D_o \), each of length \( d \). For \( \bar{\theta} \in [\kappa]^{2d} \), \( \iota(\bar{\theta}) \) determines the pair of sequences of ordinals \( (\theta_{(0)}, \theta_{(2)}, \ldots, \theta_{(2d - 2)}), (\theta_{(1)}, \theta_{(3)}, \ldots, \theta_{(2d - 1)}) \), both of which are members of \( [\kappa]^{d} \). Denote these as \( \iota_c(\bar{\theta}) \) and \( \iota_o(\bar{\theta}) \), respectively. To ease notation, let \( \bar{\delta}_\alpha \) denote \( \bar{\delta}_p \), \( k_\alpha \) denote \( \bar{\delta}_\alpha \), and let \( l_\alpha \) denote \( l_{p_\alpha} \). Let \( \delta_\alpha(j) : j < k_\alpha \) denote the enumeration of \( \bar{\delta}_\alpha \) in increasing order.

Define a coloring \( f \) on \([\kappa]^{2d}\) into countably many colors as follows: Given \( \bar{\theta} \in [\kappa]^{2d} \) and \( \iota \in I \), to reduce the number of subscripts, letting \( \bar{\alpha} \) denote \( \iota_c(\bar{\theta}) \) and \( \bar{\beta} \) denote
Let \( f \) be the sequence \( \langle f(\bar{\theta}) : \bar{\theta} \in \mathcal{I} \rangle \), where \( \mathcal{I} \) is given a fixed some ordering. Since the range of \( f \) is countable, apply the Erdős-Rado Theorem to obtain a subset \( K \subseteq \kappa \) of cardinality \( \kappa_1 \) which is homogeneous for \( f \). Take \( K' \subseteq K \) such that between each two members of \( K' \) there is a member of \( K \) and \( \min(K') > \min(K) \).

Take subsets \( K_i \subseteq K' \) such that \( K_0 < \cdots < K_{d-1} \) and each \( |K_i| = n_0 \).

**Lemma 6.4.** There are \( \varepsilon^* \in 2, k^* \in \omega, t_d \), and \( \langle t_{i,j} : j < k^* \rangle \), such that for all \( \bar{\alpha} \in \prod_{i<d} K_i \) and each \( i < d \), \( \varepsilon_{\bar{\alpha}} = \varepsilon^*, k_{\bar{\alpha}} = k^* \), \( p_{\alpha}(d) = t_d \), and \( \langle p_{\alpha}(i, \delta_{\bar{\alpha}}(j)) : j < k_{\bar{\alpha}} \rangle = \langle t_{i,j} : j < k^* \rangle \).

**Proof.** Let \( \bar{\theta} \) be the member in \( \mathcal{I} \) which is the identity function on \( 2d \). For any pair \( \bar{\alpha}, \bar{\beta} \in \prod_{i<d} K_i \), there are \( \bar{\alpha}', \bar{\beta}' \in [K]^{2d} \) such that \( \bar{\alpha} = \iota_{\alpha}(\bar{\theta}) \) and \( \bar{\beta} = \iota_{\beta}(\bar{\theta}) \).

Since \( f(\bar{\alpha}) = f(\bar{\alpha}') = f(\bar{\beta}) = f(\bar{\beta}') \), it follows that \( \varepsilon_{\bar{\alpha}} = \varepsilon_{\bar{\beta}} \), \( k_{\bar{\alpha}} = k_{\bar{\beta}} \), \( p_{\alpha}(d) = p_{\beta}(d) \), and \( \langle p_{\alpha}(i, \delta_{\bar{\alpha}}(j)) : j < k_{\bar{\alpha}} \rangle = \langle p_{\beta}(i, \delta_{\bar{\beta}}(j)) : j < k_{\bar{\beta}} \rangle : i < d \rangle \) to be \( \varepsilon_{\bar{\alpha}} \), \( k_{\bar{\alpha}} \), \( p_{\alpha}(d) \), \( \langle p_{\alpha}(i, \delta_{\bar{\alpha}}(j)) : j < k_{\bar{\alpha}} \rangle : i < d \rangle \) for any \( \bar{\alpha} \in \prod_{i<d} K_i \).

Let \( l^* \) denote the length of \( t_d \). Then all the nodes \( t_{i,j}, i < d, j < k^* \), also have length \( l^* \).

**Lemma 6.5.** Given any \( \bar{\alpha}, \bar{\beta} \in \prod_{i<d} K_i \), if \( j, k \in k^* \) and \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \), then \( j = k \).

**Proof.** Let \( \bar{\alpha}, \bar{\beta} \) be members of \( \prod_{i<d} K_i \) and suppose that \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \) for some \( j, k < k^* \). For each \( i < d \), let \( \rho_i \) be the relation from among \( \{<,=,>\} \) such that \( \alpha_i \rho_i \beta_i \). Let \( \bar{\theta} \) be the member of \( \mathcal{I} \) such that for each \( \bar{\gamma} \in [K]^{2d} \) and each \( i < d, \theta_i(\beta_{i+1}) = \theta_{i+1}(\beta_i) \). Then there is a \( \bar{\theta} \in [K]^{2d} \) such that \( \iota_{\alpha}(\bar{\theta}) = \bar{\alpha} \) and \( \iota_{\beta}(\bar{\theta}) = \bar{\beta} \). Since between any two members of \( K' \) there is a member of \( K \), there is a \( \bar{\gamma} \in [K]^{2d} \) such that for each \( i < d, \alpha_i \gamma_i \beta_i \) and \( \gamma_i \rho_i \beta_i \), and furthermore, for each \( i < d - 1, \{\alpha_i, \beta_i, \gamma_i\} \subseteq \{\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}\} \). Given that \( \alpha_i \rho_i \gamma_i \) and \( \gamma_i \rho_i \beta_i \) for each \( i < d \), there are \( \bar{\mu}, \bar{\nu} \in [K]^{2d} \) such that \( \iota_{\gamma}(\bar{\mu}) = \bar{\alpha}, \iota_{\gamma}(\bar{\mu}) = \bar{\beta}, \iota_{\gamma}(\bar{\nu}) = \bar{\gamma}, \) and \( \iota_{\gamma}(\bar{\mu}) = \bar{\beta} \). Since \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \), the pair \langle \bar{\gamma}, \bar{\nu} \rangle \) is the last sequence in \( f(\bar{\alpha}, \bar{\beta}) \).

Since \( f(\bar{\gamma}, \bar{\nu}) = f(\bar{\gamma}, \bar{\nu}) = f(\bar{\gamma}, \bar{\nu}) \), also \( \langle j, k \rangle \) is in the last sequence in \( f(\bar{\gamma}, \bar{\nu}) \) and \( f(\bar{\gamma}, \bar{\nu}) \). It follows that \( \delta_{\bar{\gamma}}(j) = \delta_{\bar{\beta}}(k) \) and \( \delta_{\bar{\gamma}}(j) = \delta_{\bar{\beta}}(k) \). Hence, \( \delta_{\bar{\alpha}}(j) = \delta_{\bar{\beta}}(k) \), and therefore \( j \) must equal \( k \).

For any \( \bar{\alpha} \in \prod_{i<d} K_i \) and any \( \bar{\theta} \in [K]^{2d} \) such that \( \bar{\alpha} = \iota_{\alpha}(\bar{\theta}) \). By homogeneity of \( f \) and by the first sequence in the second line of equation (37), there is a strictly increasing sequence \( \langle j_i : i < d \rangle \) of members of \( k^* \) such that for each \( \bar{\alpha} \in \prod_{i<d} K_i \), \( \delta_{\bar{\alpha}}(j_i) = \alpha_i \). For each \( i < d \), let \( t_i^* \) denote \( t_{i,j_i} \). Then for each \( i < d \) and each \( \bar{\alpha} \in \prod_{i<d} K_i \),
\[
p_{\bar{\alpha}}(i, \alpha_i) = p_{\bar{\alpha}}(i, \delta_{\bar{\alpha}}(j_i)) = t_{i,j_i} = t_i^*.
\]

Let \( t_d^* \) denote \( t_d \).
Lemma 6.6. For any finite subset $\vec{J} \subseteq \prod_{i<d} K_i$, the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is compatible. Moreover, $p_{\vec{J}} := \bigcup \{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a member of $P$ which is below each $p_{\vec{\alpha}}$, $\vec{\alpha} \in \vec{J}$.

Proof. For any $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, whenever $j, k < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, then $j = k$, by Lemma 6.5. It then follows from Lemma 6.4 that for each $i < d$, 
$$p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j} = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(k)).$$
Thus, for each $\vec{\alpha}, \vec{\beta} \in \vec{J}$ and each $\delta \in \delta_{\vec{\alpha}} \cap \delta_{\vec{\beta}}$, for all $i < d$,
$$p_{\vec{\alpha}}(i, \delta) = p_{\vec{\beta}}(i, \delta).$$
Thus, $p_{\vec{J}} := \bigcup \{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a function. Let $\vec{J}_f = \bigcup \{\vec{J} : \vec{\alpha} \in \vec{J}\}$. For each $\delta \in \vec{J}_f$ and $i < d$, $p_{\vec{J}}(i, \delta)$ is defined, and it is exactly $p_{\vec{\alpha}}(i, \delta)$, for any $\vec{\alpha} \in \vec{J}$ such that $\delta \in \delta_{\vec{\alpha}}$. Thus, $p_{\vec{J}}$ is a member of $P$, and $p_{\vec{J}} \leq p_{\vec{\alpha}}$ for each $\vec{\alpha} \in \vec{J}$. \qed

The final lemma of Part I will be used in the next section.

Lemma 6.7. If $\beta \in \prod_{i<d} K_i$, $\vec{\alpha} \in \prod_{i<d} K_i$, and $\beta \notin \vec{\alpha}$, then $\beta$ is not a member of $\delta_{\vec{\alpha}}$.

Proof. Suppose toward a contradiction that $\beta \in \delta_{\vec{\alpha}}$. Then there is a $j < k^*$ such that $\beta = \delta_{\vec{\alpha}}(j)$. Let $i$ be such that $\beta \in K_i$. Since $\beta \neq \alpha_i = \delta_{\vec{\alpha}}(j_i)$, it must be that $j \neq j_i$. However, letting $\vec{\beta}$ be any member of $\prod_{i<d} K_i$ with $\beta_i = \beta$, then $\beta = \delta_{\vec{\beta}}(j_i) = \delta_{\vec{\alpha}}(j)$, so Lemma 6.3 implies that $j_i = j$, a contradiction. \qed

Part II. In this last part of the proof, we build a strong coding tree $S$ valid in $T$ on which the coloring $h$ is homogeneous. Cases (a) and (b) must be handled separately.

Part II Case (a). Recall that $\{s_i : i \leq d\}$ enumerates the members of $A_e$, which is a subset of $B^+$. Let $m'$ be the integer such that $B \in \mathcal{AT}_{m'}$. Let $M = \{m_j : j < \omega\}$ be the strictly increasing enumeration of those $m > m'$ such that the splitting node in $\max(r_m(T))$ extends $s_d$. We will find $U_{m_0} \in r_{m_0}[B, T]$ and in general, $U_{m_{j+1}} \in r_{m_{j+1}}[U_{m_j}, T]$ so that for each $j < \omega$, $h$ takes color $\varepsilon^*$ on $\Ext_{U_{m_j}}(A, \vec{\alpha})$. Then setting $S = \bigcup_{j<\omega} U_{m_j}$ will yield $S$ to be a member of $[B, T]$ for which $\Ext_S(A, \vec{\alpha})$ is homogeneous for $h$, with color $\varepsilon^*$.

First extend each node in $B^+$ to level $l^*$ as follows. The set $\{t^*_i : i \leq d\}$ end-extends $A_e$, has no new pre-cliques over $A_e$, and is free in $T$. For each node $u$ in $B^+ \setminus A_e$, let $u^*$ denote its leftmost extension in $T \upharpoonright l^*$. Then the set
$$U^* = \{t^*_i : i \leq d\} \cup \{u^* : u \in B^+ \setminus A_e\}$$
end-extends $B^+$, is free in $T$, and has no new pre-cliques over $B$, by Lemma 5.10. Thus, $U^*$ is free in $T$, and $B \cup U^*$ satisfies the Witnessing Property so is valid in $T$. If $m_0 = m'^{+} + 1$, then $B \cup U^*$ is a member of $r_{m_0}[B, T]$, by Lemma 5.18. In this case, let $U_{m_0'} = B \cup U^*$ and extend $U_{m_0'}$ to a member $U_{m_0'} \in r_{m_0'}[U_{m_0'}, T]$, using Theorem 5.10. If $m_0 > m'^{+} + 1$, apply Lemma 5.18 and Theorem 5.10 to extend above $U^*$ to construct a member $U_{m_0} \in r_{m_0}[B, T]$ which is valid in $T$. In this case, note that $\max(r_{m_0}([U_{m_0}]))$ is not $U^*$, but rather $\max(r_{m_0}([U_{m_0}]))$ end-extends $U^*$. 


Assume \( j < \omega \) and we have constructed \( U_{m_j-1} \), valid in \( T \), so that every member of \( \text{Ext}_{U_{m_j-1}}(A, \bar{X}) \) is colored \( \varepsilon^* \) by \( h \). Fix some \( V \in r_{m_j}[U_{m_j-1}, T] \) with \( V \) valid in \( T \), and let \( Z = \max(Y_j) \). The nodes in \( Z \) will not be in the tree \( S \) we are constructing; rather, we will extend the nodes in \( Z \) to construct \( U_{m_j} \in r_{m_j}[U_{m_j-1}, T] \).

We now start to construct a condition \( q \) which will satisfy Lemma 6.8 below. Let \( q(d) \) denote the splitting node in \( Z \) and let \( l_q = |q(d)| \). For each \( i < d \), let \( Z_i \) denote the set of those \( z \in T_i \cap Z \) such that \( z \in X \) for some \( X \in \text{Ext}_T(A, \bar{X}) \).

For each \( i < d \), take a set \( J_i \subseteq K_i \) of cardinality \( |Z_i| \) and label the members of \( Z_i \) as \( \{z_\alpha : \alpha \in J_i\} \). Notice that each member of \( \text{Ext}_T(A, \bar{X}) \) above \( Z \) extends some set \( \{z_\alpha : i < d\} \cup \{q(d)\} \), where each \( \alpha \in J_i \). Let \( J \) denote the set of those \( \langle \alpha_0, \ldots, \alpha_{d-1} \rangle \in \prod_{i<d} J_i \) such that the set \( \{z_\alpha : i < d\} \cup \{q(d)\} \) is in \( \text{Ext}_T(A, \bar{X}) \).

Then for each \( i < d \), \( J_i = \{\alpha_i : \alpha_i \in \bar{\alpha} \in \bar{J}\} \). It follows from Lemma 6.6 that the set \( \{p_\alpha : \alpha \in \bar{J}\} \) is compatible. The fact that \( p_J \) is a condition in \( \mathbb{P} \) will be used to make the construction of \( q \) very precise.

Let \( \bar{\delta}_q = \bigcup \{\delta_\alpha : \alpha \in \bar{J}\} \). For each \( i < d \) and \( \alpha \in J_i \), define \( q(i, \alpha) = z_\alpha \). Notice that for each \( \bar{\alpha} \in \bar{J} \) and \( i < d \),

\[
q(i, \alpha_i) \supseteq t_i^* = p_\alpha(i, \alpha_i) = p_J(i, \alpha_i),
\]
and

\[
q(d) \supseteq t_d^* = p_\alpha(d) = p_J(d).
\]

For each \( i < d \) and \( \gamma \in \bar{\delta}_q \setminus J_i \), there is at least one \( \bar{\alpha} \in \bar{J} \) and some \( k < k^* \) such that \( \delta_\alpha(k) = \gamma \). Let \( q(i, \gamma) \) be the leftmost extension of \( p_J(i, \gamma) \) in \( T \) of length \( l_q \).

Define

\[
q = \{q(d)\} \cup \{(i, \delta), q(i, \delta) : i < d, \delta \in \bar{\delta}_q\}.
\]

Since \( C \) is valid in \( T \), \( Z \) is free in \( T \). Since \( \text{ran}(q) \) consists of \( Z \) along with leftmost extensions of nodes in \( \text{ran}(p_J(i, \gamma)) \), all of which are free, \( \text{ran}(q) \) is free. Therefore, \( q \) is a condition in \( \mathbb{P} \).

**Lemma 6.8.** For all \( \bar{\alpha} \in \bar{J} \), \( q \leq p_\alpha \).

**Proof.** Given \( \bar{\alpha} \in \bar{J} \), it follows from the definition of \( q \) that \( \bar{\delta}_q \supseteq \bar{\delta}_\alpha, q(d) \supseteq p_\alpha(d) \), and for each pair \( (i, \gamma) \in d \times \bar{\delta}_\alpha, q(i, \gamma) \supseteq p_\alpha(i, \gamma) \). So it only remains to show that \( q \) is valid over \( p_\alpha \). It follows from Lemma 6.7 that \( \bar{\delta}_\alpha \cap \bigcup_{i<d} K_i = \bar{\alpha} \); so for each \( i < d \) and \( \gamma \in \bar{\delta}_\alpha \setminus \{\alpha_i\}, q(i, \gamma) \) is the leftmost extension of \( p_\alpha(i, \gamma) \). Since \( \bar{\alpha} \) is in \( \bar{J} \), \( X(q, \bar{\alpha}) \) is in \( \text{Ext}_T(A, \bar{X}) \). This implies that \( X(q, \bar{\alpha}) \) has no new pre-cliques over \( A \), and hence, none over \( X(p_\alpha, \bar{\alpha}) \). It follows that \( \text{ran}(q \upharpoonright \bar{\delta}_\alpha) \) is valid over \( \text{ran}(p_\alpha) \), by Lemma 5.10. Therefore, \( q \leq p_\alpha \).

**Remark 6.9.** Notice that we did not prove that \( q \leq p_J \); in fact that is generally false.

To construct \( U_{m_j} \), take an \( r \leq q \) in \( \mathbb{P} \) which decides some \( l_j \) in \( \bar{L}_q \) for which \( h(b_\alpha \upharpoonright l_j) = \varepsilon^* \), for all \( \bar{\alpha} \in \bar{J} \). This is possible since for all \( \bar{\alpha} \in \bar{J} \), \( p_\alpha \) forces \( h(b_\alpha \upharpoonright l) = \varepsilon^* \) for \( \bar{U} \) many \( l \) in \( \bar{L}_q \). By the same argument as in creating the conditions \( p_\alpha \leq p_\alpha^*_\alpha \) to satisfy (4) in Part I, we may assume that the nodes in the image of \( r \) have length \( l_j \). Since \( r \) forces \( h_\alpha \upharpoonright l_j = X(r, \bar{\alpha}) \) for each \( \bar{\alpha} \in \bar{J} \), and since the coloring \( h \) is defined in the ground model, it follows that \( h(X(r, \bar{\alpha})) = \varepsilon^* \) for
each $\vec{\alpha} \in \vec{J}$. Extend the splitting node $q(d)$ in $Z$ to $r(d)$. For each $i < d$ and $\alpha_i \in J_i$, extend $q(i, \alpha_i)$ to $r(i, \alpha_i)$. Let $Z'$ denote $Z \setminus \{q(i, \alpha_i) : i < d, \alpha_i \in J_i\} \cup \{q(d)\}$. For each node $v$ in $Z'$, let $v^*$ be the leftmost extension of $v$ in $T \setminus I_j$. Let

$$(45) \quad U_{m_j} = U_{m_j-1} \cup \{r(d)\} \cup \{r(i, \alpha_i) : i < d, \alpha_i \in J_i\} \cup \{v^* : v \in Z'\}.$$ 

**Lemma 6.10.** $U_{m_j} \in r_m[S_{m_j}, T]$ and is valid in $T$, and every $X \in \text{Ext}_{U_{m_j}}(A, \vec{X})$ satisfies $h(X) = \varepsilon^*$. 

**Proof.** Recall that $C \in r_{m_j}[U_{m_j-1}, T]$, and both $U_{m_j-1}$ and $C$ are valid in $T$. Since $r \leq q$, it follows that $\text{ran}(r \upharpoonright \delta_x)$ has no new pre-cliques over $\text{ran}(q)$, which is a subset of $Z$. All other nodes in $\max(U_{m_j})$ are leftmost extensions of nodes in $Z$. Thus, by Lemma 5.15 $\max(U_{m_j})$ is free in $T$ and has no new pre-cliques over $Z$; hence $U_{m_j}$ is valid in $T$. Since $U_{m_j} \cong r_m(T_k)$, it follows that $U_{m_j} \in r_m[U_{m_j-1}, T]$. 

For each $X \in \text{Ext}_{U_{m_j}}(A, \vec{X})$ with $X \subseteq \max(U_{m_j})$, the truncation $X \upharpoonright l_q$ is a member of $\text{Ext}_Z(A, \vec{X})$. There corresponds a sequence $\vec{\alpha} \in \vec{J}$ such that $X \upharpoonright l_q = X(q, \vec{\alpha})$. Then $X = X(r, \vec{\alpha})$, which has $h$-color $\varepsilon^*$. By the induction hypothesis on $U_{m_j-1}$, the lemma holds. □ 

Let $S = \bigcup_{j < \omega} U_{m_j}$. Then for each $X \in \text{Ext}_S(A, \vec{X})$, there corresponds a $j < \omega$ such that $X \in \text{Ext}_{U_{m_j}}(A, \vec{X})$. By Lemma 6.10 $h(X) = \varepsilon^*$. Thus, $S \in [B, T]$ and satisfies the theorem. This concludes the proof of the theorem for Case (a).

**Part II Case (b).** Let $m_0$ be the integer such that there is a $B' \in r_{m_0}[B, T]$ with $\vec{X} \subseteq \max(B')$. Let $U_{m_0-1}$ denote $r_{m_0-1}(B')$. Since $\vec{X} \subseteq \max(B')$, it follows that $l^* \geq l_{B'}$. Let $V = \{t^*_i : i \leq d\}$, and recall that this set has no new pre-cliques over $\vec{X}$. By Lemma 5.15 there is a set of nodes $V'$ end-extending $\max(B') \setminus V$ such that $U_{m_0-1} \cup V \cup V'$ is a member of $r_{m_0}[U_{m_0-1}, T]$; label this $U_{m_0}$. Since $\max(U_{m_0})$ is at the level of the coding node $t^*_0$, $\max(U_{m_0})$ is free in $T$. Since $U_{m_0} \in r_{m_0}[U_{m_0-1}, T]$, it satisfies the WP. Therefore, $U_{m_0}$ is valid in $T$. Notice that $\{t^*_i : i \leq d\}$ is the only member of $\text{Ext}_{U_{m_0}}(A, \vec{X})$, and it has $h$-color $\varepsilon^*$.

Let $M = \{m_j : j < \omega\}$ enumerate the set of $m \geq m_0$ such that the coding node $c^*_m \geq r_{m_0}$. Assume that $1 \leq j < \omega$ and we have constructed $U_{m_j-1} \in A T_{m_j-1}$ valid in $T$ so that every member of $\text{Ext}_{U_{m_j-1}}(A, \vec{X})$ is colored $\varepsilon^*$ by $h$. By Theorem 5.10 we may fix some $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$ which is valid in $T$. Take some $C \in r_{m_0}[U_{m_j-1}, T]$, and let $Z$ denote $\max(C)$. The nodes in $Z$ will not be in the tree $S$ we are constructing; rather, we will construct $U_{m_j} \in r_{m_j}[U_{m_j-1}, T]$ so that $\max(U_{m_j})$ extends $Z$. Let $q(d)$ denote the coding node in $Z$ and let $l_q = |q(d)|$. Recall that for $k \in \{0, 1\}$, $I_k$ denotes the set of $i < d$ for which $t^*_i$ has passing number $k$ at $t^*_g$. For each $k \in \{0, 1\}$ and each $i \in I_k$, let $Z_i$ be the set of nodes $z$ in $T_i \cap Z$ such that $z$ has passing number $k$ at $q(d)$.

We now construct a condition $q$ similarly to, but not exactly as in, Case (a). For each $i < d$, let $J_i$ be a subset of $K_i$ with the same size as $Z_i$. For each $i < d$, label the nodes in $Z_i$ as $\{z_\alpha : \alpha \in J_i\}$. Let $\vec{J}$ denote the set of those $\langle \alpha_0, \ldots, \alpha_{d-1}\rangle \in \prod_{i < d} J_i$ such that the set $\{z_\alpha : i < d\} \cup \{q(d)\}$ is in $\text{Ext}_T(A, \vec{X})$. Notice that for each $i < d$ and $\vec{\alpha} \in \vec{J}$, $z_\alpha \geq t^*_i = p_2(i, \alpha_i)$, and $q(d) \geq t^*_q = p_2(d)$. Furthermore, for each $i < d$ and $\delta \in J_i$, there is an $\vec{\alpha} \in \vec{J}$ such that $\alpha_i = \delta$. Let $\delta_q = \bigcup \{\delta_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For each pair $(i, \gamma) \in d \times \delta_q$ with $\gamma \in J_i$, define $q(i, \gamma) = z_\gamma$. 


Let $\mathcal{J} = \{(i, \gamma) \in d \times \vec{\delta}_q : i < d \text{ and } \gamma \in \vec{\delta}_q \setminus J_i\}$. For each pair $(i, \gamma) \in \mathcal{J}$, there is at least one $\vec{\alpha} \in \vec{\mathcal{J}}$ and some $k < k^*$ such that $\delta_{\vec{\alpha}}(k) = \gamma$. By Lemma 6.6, $p_\beta(i, \gamma) = p_{\vec{\alpha}}(i, \gamma) = t_{i,k}^*$, for any $\vec{\beta} \in \vec{\mathcal{J}}$ for which $\gamma \in \vec{\delta}_\vec{\beta}$. For each $i \in I_0$ and $(i, \gamma) \in \mathcal{J}$, $q(i, \gamma)$ is the leftmost extension of $t_{i,k}^*$ in $T \upharpoonright l_q$. For $i \in I_1$ and $(i, \gamma) \in \mathcal{J}$, let $q(i, \gamma)$ be the node which extends $t_{i,k}^*$ leftmost until length of the longest coding node in $T$ strictly below $q(d)$, and then takes the rightmost path to length $l_q$. Note that for $i \in I_k$, $q(i, \gamma)$ has passing number $k$ at $q(d)$. By the arguments of Lemma 5.14 the set $\{q(i, \gamma) : (i, \gamma) \in \mathcal{J}\}$ has no new pre-cliques over $\{t_i^* : i \leq d\}$.

Define

\[(46) \quad q = \{q(d)\} \cup \{(i, \delta) : i < d, \delta \in \vec{\delta}_q\}.\]

By the construction, $q$ is a member of $\mathcal{P}$.

**Claim 4.** For each $\vec{\alpha} \in \vec{\mathcal{J}}$, $q \leq p_{\vec{\alpha}}$.

**Proof.** It suffices to show that for each $\vec{\alpha} \in \vec{\mathcal{J}}$, $\text{ran}(q \upharpoonright \vec{\delta}_q)$ has no new pre-cliques over $\text{ran}(p_{\vec{\alpha}})$, since by construction, we have that $q(i, \delta) \supseteq p_{\vec{\alpha}}(i, \delta)$ for all $(i, \delta) \in d \times \vec{\delta}_q$.

Let $\vec{\alpha} \in \vec{\mathcal{J}}$ be given. Then

\[(47) \quad \text{ran}(q \upharpoonright \vec{\delta}_q) \subseteq \{q(i, \gamma) : (i, \gamma) \in \mathcal{J}\} \cup X(q, \vec{\alpha}),\]

recalling that $X(q, \vec{\alpha}) = \{q(i, \alpha_i) : i < d\} \cup \{q(d)\}$. By definition of $\vec{\mathcal{J}}$, $\vec{\alpha} \in \vec{\mathcal{J}}$ implies that $X(q, \vec{\alpha})$ is a member of $\text{Ext}_T(A, \vec{\delta}_q)$. Thus, $X(q, \vec{\alpha})$ has no new pre-cliques over $A \cup \vec{\delta}_q$, by Lemma 5.11. Since $\{t_i^* : i \leq d\}$ end-extends $\vec{\delta}_q$, it follows that $X(q, \vec{\alpha})$ has no new pre-cliques over $\{t_i^* : i \leq d\}$. It follows from the proof of Lemma 5.17 that $\text{ran}(q \upharpoonright \vec{\delta}_q)$ has no new pre-cliques over $\text{ran}(p_{\vec{\alpha}})$. Therefore, $q \leq p_{\vec{\alpha}}$. \qed

To construct $U_{m_j}$, take an $r \leq q$ in $\mathcal{P}$ which decides $l_r \in \hat{L}_d$ such that $h(b_{\vec{\alpha}} \upharpoonright l_r) = \varepsilon^*$ for all $\vec{\alpha} \in \vec{\mathcal{J}}$, using the same ideas as in the construction of $p_{\vec{\alpha}}$. Let $\mathcal{Y} = \bigcup \{X(r, \vec{\alpha}) : \vec{\alpha} \in \vec{\mathcal{J}}\}$, and let $\mathcal{Z}^* = \{r(d)\} \cup \bigcup_{i < d} \mathcal{Z}_i$. Since $\text{ran}(r \upharpoonright \vec{\delta}_q)$ has no new pre-cliques over $\text{ran}(q)$, it follows that $\mathcal{Y}$ has no new pre-cliques over $\mathcal{Z}^*$. By Lemma 5.17, extend the nodes in $\mathcal{Z} \setminus \mathcal{Z}^*$ to a set $\mathcal{Y}' \subseteq T \upharpoonright l_r$ so that each node in $\mathcal{Y}'$ has the same passing number at $r(d)$ as it does at $q(d)$, and such that $\mathcal{Y} \cup \mathcal{Y}'$ has no new pre-cliques over $\mathcal{Z}$. Then $U_{m_j-1} \cup \mathcal{Y} \cup \mathcal{Y}'$ is a member of $r_{m_j}[U_{m_j-1}, T]$ which is valid in $T$.

To finish the proof of the theorem for Case (b), Define $S = \bigcup_{j \in \omega} U_{m_j}$. Then $S \in [B', T]$, and for each $Z \in \text{Ext}_S(A, \vec{\delta}_q)$, there is a $j < \omega$ such that $Z \in \text{Ext}_{U_{m_j}}(A, \vec{\delta}_q)$, so $h(Z) = \varepsilon^*$.

This concludes the proof of the theorem. \qed

7. **Ramsey Theorem for finite trees with the Strict Witnessing Property**

This section contains the Ramsey theorem for colorings of finite trees with the following strong version of the Witnessing Property.

**Definition 7.1** (Strict Witnessing Property). A subtree $A$ of a strong coding tree satisfies the **Strict Witnessing Property (SWP)** if $A$ satisfies the Witnessing Property and for each interval $\{|d_m^A|, |d_{m+1}^A|\}$ in $A$ the following hold:
Let \( \tilde{T} \) coding tree \( d \) \( X \) then follows by induction and an application of Ramsey’s Theorem.

Assumption 7.4. \( 6.2, \) will be used in much of this section.

Lemma 7.2. If \( A \subseteq \mathbb{T}_k \) has the Strict Witnessing Property and \( B \cong A \), then \( B \) also has the Strict Witnessing Property.

Proof. If \( B \cong A \) and \( A \) has the WP, then \( B \) also has the WP by Lemma \( 6.14 \). Let \( f : A \rightarrow B \) be the stable map between them. Since new pre-cliques can only occur in intervals of \( A \) where the upper critical node is a coding node, \( f \) ensures this holds for \( B \) as well. If \( d_{m+1}^B \) is a coding node in \( B \) and \( Y \) is has new pre-a-clique in \( B \) in the interval \( (|d_m^B|, |d_{m+1}^B|] ) \) for some \( a \in [3, k] \), then this is witnessed by some set of coding nodes \( N \subseteq B \) which includes \( d_{m+1}^B \).

If \( f^{-1}[Y] \) has a new pre-a-clique in the interval \( (|d_m^A|, |d_{m+1}^A|] \) and this is witnessed by \( f^{-1}[N] \). Since \( f \) is stable, \( f \) preserves properties (2) and (3) of Definition \( 7.1 \) from \( A \) to \( B \).

Given a finite tree \( A \) with the SWP, we say that \( B \) is a copy of \( A \) if \( A \cong B \). The main theorem of this section, Theorem \( 7.3 \), will guarantee a Ramsey Theorem for colorings of copies of a finite tree with the SWP inside a strong coding tree.

Theorem 7.3. Let \( T \in \mathcal{T}_k \) be a strong coding tree and let \( A \) be a finite subtree of \( T \) satisfying the Strict Witnessing Property. Then for any coloring of the copies of \( A \) in \( T \) into finitely many colors, there is a strong coding subtree \( S \leq T \) such that all copies of \( A \) in \( S \) have the same color.

Theorem \( 7.3 \) will be proved via four lemmas and an induction argument. The main difficulty is that Case (b) of Theorem \( 6.2 \) provides homogeneity for \( \text{Ext}_S(A, \bar{X}) \) for some strong coding tree \( S \); in particular, homogeneity only holds for level sets \( X \) end-extending \( \bar{X} \). We need a strong coding tree in which every \( X \) satisfying \( A \cup X \cong A \cup \bar{X} \) has the same color. This will be addressed by the following: Lemma \( 7.4 \) will build a fusion sequence to obtain an \( S \leq T \) which is homogeneous on \( \text{Ext}_S(A, Y) \) for each minimal level set \( Y \) extending \( A_+ \) such that \( A \cup Y \cong A \cup \bar{X} \). Lemma \( 7.5 \) will use a new forcing and arguments from the proof of Theorem \( 6.2 \) to obtain a strong coding tree \( S \in [B, T] \) in which every \( X \) satisfying \( A \cup X \cong A \cup \bar{X} \) has the same color. The last two lemmas involve fusion to construct a strong coding subtree which is homogeneous for the induced color on copies of \( A \). The theorem then follows by induction and an application of Ramsey’s Theorem.

The following basic assumption, similar to but stricter than Case (b) of Theorem \( 6.2 \), will be used in much of this section.

Assumption 7.4. Let \( A \) and \( C \) be fixed non-empty finite valid subtrees of a strong coding tree \( T \in \mathcal{T}_k \) such that

1. \( A \) and \( C \) both satisfy the Strict Witnessing Property; and
2. \( C \setminus A \) is a level set containing both a coding node and the sequence \( 0^{(c)} \).

Let \( \bar{X} \) denote \( C \setminus A \), and let \( A_+ \) be the subset of \( A^+ \) which is extended to \( \bar{X} \). Let \( d \) be the number of nodes in \( X \). List the nodes in \( A_+ \) as \( \langle s_i : i \leq d \rangle \) and the nodes of \( \bar{X} \) as \( \langle t_i : i \leq d \rangle \) so that each \( t_i \) extends \( s_i \) and \( t_d \) is the coding node in \( \bar{X} \). For \( j \in \{0, 1\} \), let \( I_j \) denote the set of \( i \leq d \) such that \( t_i \) has passing number \( j \) at \( t_d \). If \( \bar{X} \) has a new pre-clique over \( A \), let \( I_* \) denote the set of \( i \in I \) such that
\{t_i : i \in I_*\} is the new pre-clique in \(\tilde{X}\) over \(A\). Note that \(I_* \subseteq I_1\) and \(t_d\) must be among the coding nodes in \(C\) witnessing this new pre-clique.

For any \(X\) such that \(A \cup X \cong C\), let \(\text{Ext}_T(A, X)\) be defined as in equation (30) of Section 8. Thus, \(\text{Ext}_T(A, X)\) is the collection of level sets \(Y \subseteq T\) such that \(Y\) end-extends \(X\) and \(A \cup Y \cong A \cup X\), (equivalently, \(A \cup Y \cong C\)), and \(A \cup Y\) is valid in \(T\). Recall that, since \(\tilde{X}\) contains a coding node, \(A \cup X \cong A \cup \tilde{X}\) implicitly includes that the stable map from \(A \cup \tilde{X}\) to \(A \cup X\) preserves passing numbers between \(\tilde{X}^+\) and \(X^+\). We hold to the convention that given \(Y\) such that \(A \cup Y \cong C\), the nodes in \(Y\) are labeled \(y_i, i \leq d\), where each \(y_i \supseteq s_i\). In particular, \(y_d\) is the coding node in \(Y\).

In this section, we want to consider all copies of \(C\) extending \(A\). To that end let
\[
\text{Ext}_T(A, C) = \bigcup \{\text{Ext}_T(A, X) : A \cup X \cong C\}.
\]

Now we define the notion of minimal pre-extension, which will be used in the next lemma. For \(x \in T\), define \(\text{splitpred}_T(x)\) to be \(x \upharpoonright l\) where \(l < |x|\) is maximal such that \(x \upharpoonright l\) is a splitting node in \(T\).

**Definition 7.5** (Minimal pre-extension of \(A\) to a copy of \(C\)). Given \(A, \tilde{X}\), and \(C\) as in Assumption 7.4, for \(X = \{x_i : i \leq d\}\) a level set extending \(A\), such that \(x_i \supseteq s_i\) for each \(i \leq d\) and such that \(l_X\) is the length of some coding node in \(T\), we say that \(X\) is a **minimal pre-extension in \(T\) of \(A\) to a copy of \(C\)** if the following hold:

(i) \(\{i \leq d : \text{the passing number of } x_i \text{ at } x_d \text{ is } 1\} = I_1\).
(ii) \(A \cup \text{SP}_T(X)\) satisfies the Strict Witnessing Property, where
\[
\text{SP}_T(X) = \{\text{splitpred}_T(x_i) : i \in I_1\} \cup \{x_i : i \in I_0\}.
\]
(iii) If \(X\) has a new pre-clique over \(A\), then \(X\) has only one new maximal pre-clique over \(A\) which is exactly \(\{x_i : i \in I_*\} \upharpoonright l\), for some \(l \in (l_A, l_X]\).

Notice that since \(\text{SP}_T(X)\) has no coding nodes, for (ii) to hold, \(\text{SP}_T(X)\) must have no new pre-cliques over \(A\). Let \(\text{MPE}_T(A, C)\) denote the set of minimal pre-extensions in \(T\) of \(A\) to a copy of \(C\). When \(A\) and \(C\) are clear, we call members of \(\text{MPE}_T(A, C)\) simply **minimal pre-extensions**. Minimal pre-extensions are exactly the level sets in \(T\) which can be extended to a member of \(\text{Ext}_T(A, \tilde{X})\).

For \(X \in \text{MPE}_T(A, C)\), define
\[
\text{Ext}_T(A, C; X) = \{Y \subseteq T : A \cup Y \cong C \text{ and } Y \text{ end-extends } X\}.
\]

Then
\[
\text{Ext}_T(A, C) = \bigcup \{\text{Ext}_T(A, C; X) : X \in \text{MPE}_T(A, C)\},
\]

**Definition 7.6.** A coloring on \(\text{Ext}_T(A, C)\) is **end-homogeneous** if for each minimal pre-extension \(X\), every member of \(\text{Ext}_T(A, C; X)\) has the same color.

The following lemma is a slightly modified version of Lemma 6.7 in [4].

**Lemma 7.7** (End-homogeneity). Assume [7,4] and let \(k_*\) be the integer such that \(\max(A) \subseteq r_{k_*}(T)\). Then for any coloring \(h\) of \(\text{Ext}_T(A, C)\) into two colors, there is a \(T' \in [r_{k_*}(T), T]\) such that \(h\) is end-homogeneous on \(\text{Ext}_T(A, C)\).
Proof. Let \((k_i)_{i<\omega}\) enumerate those integers greater than \(k_s\) such that there is a minimal pre-extension of \(A\) to a copy of \(C\) from among the maximal nodes in \(r_{k_s}(T)\). Notice that for each \(i < \omega\), \(\max(r_{k_i}(T))\) contains a coding node, although there can be members of \(\text{MPE}_T(A, C)\) contained in \(\max(r_{k_i}(T))\) not containing that coding node.

Let \(T_{i-1}\) denote \(T\). Suppose that \(i < \omega\) and \(T_{i-1}\) is given so that the coloring \(h\) is homogeneous on \(\text{Ext}_{T_{i-1}}(A; C; X)\) for each minimal pre-extension \(X\) in \(r_{k_i}(T_{i-1})\).

Let \(U_{i-1}\) denote \(r_{k_i}(T_{i-1})\). Enumerate the minimal pre-extensions contained in \(\max(r_{k_i}(T_{i-1}))\) as \(X_0, \ldots, X_n\). By induction on \(m \leq n\), we will obtain \(T_i \in [U_{i-1}, T_{i-1}]\) such that \(\max(r_{k_i}(T_i))\) end-extends \(\max(r_{k_i}(T_{i-1}))\) and \(\text{Ext}_{T_i}(A, C; Z)\) is homogeneous for each minimal pre-extension \(Z\) in \(\max(r_{k_i}(T_{i-1}))\).

Suppose \(m \leq n\) and for all \(j < m\), there are strong coding trees \(S_j\) such that \(S_0 \in [U_{i-1}, T_{i-1}]\), and for all \(j' < j < m\), \(S_j \in [U_{i-1}, S_{j'}]\) and \(h\) is homogeneous on \(\text{Ext}_{S_j}(A; C; X_j)\). Let \(l\) denote the length of the nodes in \(\max(r_{k_i}(T_{i-1}))\). Note that \(X_m\) is contained in \(\max(r_{k_i}(S_{m-1}))\) \(\uparrow l\), though \(l\) does not have to be the length of any node in \(S_{m-1}\). The point is that the set of nodes \(Y_m\) in \(\max(r_{k_i}(S_{m-1}))\) end-extending \(X_m\) is again a minimal pre-extension. Extend the nodes in \(Y_m\) to some \(Z_m \in \text{Ext}_{S_{m-1}}(A, C; Y_m)\), and let \(l'\) denote the length of the nodes in \(Z_m\).

Note that \(Z_m\) has no new pre-cliques over \(Y_m\). Let \(W_m\) consist of the nodes in \(Z_m\) along with the leftmost extensions of the nodes in \(\max(r_{k_i}(S_{m-1})) \setminus Y_m\) to the length \(l'\) in \(S_{m-1}\).

Let \(S_{m-1}'\) be a strong coding tree in \([U_{i-1}, S_{m-1}]\) such that \(\max(r_{k_i}(S_{m-1}'))\) extends \(W_m\). Such an \(S_{m-1}'\) exists by Lemmas 5.10 and 5.14 and Theorem 5.16. Apply Case (b) of Theorem 6.2 to obtain a strong coding tree \(S_m \in [U_{i-1}, S_{m-1}']\) such that the coloring on \(\text{Ext}_{S_m}(A, C; Z_m)\) is homogeneous. At the end of this process, let \(T_i = S_n\). Note that for each minimal pre-extension \(Z \subseteq \max(r_{k_i}(T_i))\), there is a unique \(m \leq n\) such that \(Z\) extends \(X_m\), since each node in \(\max(r_{k_i}(T_i))\) is a unique extension of one node in \(\max(r_{k_i}(T_{i-1}))\), and hence \(\text{Ext}_{T_i}(A, C; Z)\) is homogeneous.

Having chosen each \(T_i\) as above, let \(T' = \bigcup_{j<\omega} r_{k_i}(T_i)\). Then \(T'\) is a strong coding tree which is in \([r_{k_i}(T), T]\), and for each minimal pre-extension \(Z\) in \(T'\), \(\text{Ext}_{T'}(A, C; Z)\) is homogeneous for \(h\). Therefore, \(h\) is end-homogeneous on \(\text{Ext}_{T'}(A, C)\).

The next lemma provides a means for uniformizing the end-homogeneity from the previous lemma to obtain one color for all members of \(\text{Ext}_S(A, C)\). The arguments are often similar to those of Case (a) of Theorem 5.24 but sufficiently different to warrant a proof.

Lemma 7.8. Assume [7.4], and suppose that \(B\) is a finite strong coding tree valid in \(T\) and \(A\) is a subtree of \(B\) such that \(\max(A) \subseteq \max(B)\). Suppose that \(h\) is end-homogeneous on \(\text{Ext}_{T}(A, C)\). Then there is an \(S \in [B, T]\) such that \(h\) is homogeneous on \(\text{Ext}_S(A, C)\).

Proof. Given any \(U \in [B, T]\), recall that \(\text{MPE}_U(A, C)\) denotes the set of all minimal pre-extensions of \(A\) to a copy of \(C\) in \(U\). We are under Assumption [7.4]. Let \(i_0 \leq d\) be such that \(i_0 = 0^{(d)}\), and note that \(i_0\) is a member of \(I_0\). Each member \(Y\) of \(\text{MPE}_T(A, C)\) will be enumerated as \(\{y_i : i \leq d\}\) so that \(y_i \supseteq s_i\) for each \(i \leq d\). Recall notation [49] of \(\text{SP}_T(Y)\).
Since $C$ satisfies the SWP, $X$ is in $\text{MPE}_T(A, C)$. Let $P$ denote $\text{SP}_T(X)$. Since $X$ is contained in an interval of $T$ above the interval containing $\text{max}(A)$, each node of $P$ extends exactly one node of $A_c$. For any $U \in [B, T]$, define

$$X \in \text{Ext}_U(A, P) \iff X = \text{SP}_U(Y) \text{ for some } Y \in \text{MPE}_U(A, C).$$

By assumption, the coloring $h$ on $\text{Ext}_T(A, C)$ is end-homogeneous. This induces a coloring $h$ on $\text{MPE}_T(A, C)$ by defining, for $Y \in \text{MPE}_T(A, C)$, $h(Y)$ to be the $h$-color that all members of $\text{Ext}_T(A, C; Y)$ have. This further induces a coloring $h'$ on $\text{Ext}_T(A, P)$ as follows: For $Q \in \text{Ext}_T(A, P)$, for the $Y \in \text{MPE}_T(A, C)$ such that $\text{SP}_T(Y) = Q$, let $h'(Q) = h(Y)$. Given $Q \in \text{Ext}_T(A, P)$, the extensions of the $q_i \in Q$ such that $i \in I_1$ to the level of next coding node in $T$, with passing number 1 at that coding node recovers $Q$. Thus, $h'$ is well-defined.

Let $L$ denote the collection of all $l < \omega$ such that there is a member of $\text{Ext}_T(A, P)$ with maximal nodes of length $l$. For each $i \in (d + 1) \setminus \{i_0\}$, let $T_i = \{t \in T : t \supseteq s_i\}$. Let $T_{i_0}$ be the collection of all leftmost nodes in $T$ extending $s_{i_0}$. Let $\kappa = 2^{\beth_{2d+2}}$. The following forcing notion $\mathbb{Q}$ will add $\kappa$ many paths through each $T_i$, $i \in (d + 1) \setminus \{i_0\}$ and one path through $T_{i_0}$, though with $\kappa$ many labels. The present case is handled similarly to Case (a) of Theorem 6.2.

Let $\mathbb{Q}$ be the set of conditions $p$ such that $p$ is a function of the form

$$p : (d + 1) \times \delta_p \to T,$$

where $\delta_p \in [\kappa]^{<\omega}$, $l_p \in L$, $(\{p(i, \delta) : \delta \in \delta_p\} \subseteq T_i)$ for each $i < d$, and

- There is some some coding node $\nu_{n(p)}^T$ in $T$ such that $l_{n(p)}^T = l_p$, and
- $\nu_{n(p)}^T < |p(i, \delta)| \leq l_p$ for each $(i, \delta) \in (d + 1) \times \delta_p$.

Let

$$\mathbb{Q} \cap \mathbb{P} = \{p \in \mathbb{Q} : p \text{ is a } \mathbb{P}\text{-name for a non-principal ultrafilter on } L\}.$$

Then

$$\{p(i, \delta) : (i, \delta) \in (d + 1) \times \delta_p\} \subseteq T_i$$

for each $i < d$. For any condition $p \in \mathbb{Q}$, ran$(p)$ is free in $T$: leastmost extensions add no new pre-cliques. Furthermore, all nodes in ran$(p)$ are contained in the $n(p)$-th interval of $T$. We point out that ran$(p)$ may or may not contain a coding node. If it does, then that coding node must appear as $p(i, \delta)$ for some $i \in I_0$; this $i$ may or may not equal $d$.

The partial ordering on $\mathbb{Q}$ is defined as follows: $q \leq p$ if and only if $l_q \geq l_p$, $\delta_q \supseteq \delta_p$,

- $q(i, \delta) \supseteq p(i, \delta)$ for each $(i, \delta) \in (d + 1) \times \delta_p$; and
- ran$(q \upharpoonright \delta_p) := \{q(i, \delta) : (i, \delta) \in (d + 1) \times \delta_p\}$ has no new pre-cliques over ran$(p)$.

By arguments similar to those in the proof of Theorem 6.2, $(\mathbb{Q}, \leq)$ is an atomless partial order, and any condition in $\mathbb{Q}$ can be extended by two incompatible conditions of length greater than any given $l < \omega$.

Let $\mathbb{U}$ be a $\mathbb{Q}$-name for a non-principal ultrafilter on $L$. For each $i \leq d$ and $\alpha < \kappa$, let $\dot{b}_{i, \alpha} = \{p(i, \alpha), p) : p \in \mathbb{Q} \text{ and } \alpha \in \delta_p\}$, a $\mathbb{Q}$-name for the $\alpha$-th generic branch through $T_i$. For any condition $p \in \mathbb{Q}$, for $(i, \alpha) \in I_0 \times \delta_p$, $p$ forces that $\dot{b}_{i, \alpha} \upharpoonright l_p = p(i, \alpha)$. For $(i, \alpha) \in I_1 \times \delta_p$, $p$ forces that splitpred$_T(\dot{b}_{i, \alpha} \upharpoonright l_p) = p(i, \alpha)$. For $\dot{\alpha} = (\alpha_0, \ldots, \alpha_d) \in [\kappa]^{d+1}$,

$$\dot{b}_{\dot{\alpha}} \text{ denote } \langle \dot{b}_{0, \alpha_0}, \ldots, \dot{b}_{d, \alpha_d} \rangle.$$
For $l \in L$, we shall use the abbreviation

$$\hat{b}_\delta \upharpoonright l$$

which is exactly $\{b_{i,\alpha} \upharpoonright l : i \in I_0 \} \cup \{\text{splitpred}_T(\hat{b}_{i,\alpha} \upharpoonright l) : i \in I_1\}$.

Similarly to the proof of Theorem 6.2, we will find infinite pairwise disjoint sets $K_i \subseteq \kappa$, $i \leq d$, such that $K_0 < K_1 < \cdots < K_d$, and conditions $p_\delta$, $\vec{\alpha} \in \prod_{i<d} K_i$, such that these conditions are pairwise compatible, have the same images in $\mathcal{T}$, and force the same color $\varepsilon^*$ for $h'(\hat{b}_\delta \upharpoonright l)$ for $\mathcal{U}$ many levels $l$ in $L$. Moreover, the nodes $\{t_i^* : i \leq d\}$ obtained from the application of the Erdős-Rado Theorem for this setting will extend $\{s_i : i \leq d\}$ and form a member of $\text{Ext}_T(A, P)$. The arguments are quite similar to those in Theorem 6.2, so we only fill in the details for arguments which are necessarily different.

**Part I.** Given $p \in \mathcal{Q}$ and $\vec{\alpha} \in [\vec{\delta}_p]^{d+1}$, let

$$P(p, \vec{\alpha}) = \{p_\delta(i, \alpha_i) : i \leq d\}.$$ 

For each $\vec{\alpha} \in [\kappa]^{d+1}$, choose a condition $p_\delta \in \mathcal{Q}$ such that

1. $\vec{\alpha} \subseteq \vec{\delta}_{p_\delta}$.
2. $P(p, \vec{\alpha}) \in \text{Ext}_T(A, P)$.
3. There is a $\varepsilon_\delta \in 2$ such that $p_\delta \upharpoonright \varepsilon_\delta$ for $\mathcal{U}$ many $l \in \hat{L}_d$.
4. $h'(P(p_\delta, \vec{\alpha})) = \varepsilon_\delta$.

Properties (1) - (4) can be guaranteed as follows. For each $i \leq d$, let $u_i$ denote the member of $P$ which extends $s_i$. For each $\vec{\alpha} \in [\kappa]^{d+1}$, let

$$p_\alpha^0 = \{(i, \delta, u_i) : i \leq d, \delta \in \vec{\alpha}\}.$$ 

Then $p_\alpha^0$ is a condition in $\mathcal{P}$ and $\vec{\delta}_{p_\alpha^0} = \vec{\alpha}$, so (1) holds for every $p \leq p_\alpha^0$. Further, $\text{ran}(p_\alpha^0)$ is a member of $\text{Ext}_T(A, P)$ since it equals $P$. For any $p \leq p_\alpha^0$, (ii) of the definition of the partial ordering on $\mathcal{Q}$ guarantees that $P(p, \vec{\alpha})$ has no new precliques over $\text{ran}(p)$, and hence is also a member of $\text{Ext}_T(A, P)$. Thus, (2) holds for any $p \leq p_\alpha^0$. Take an extension $p_\alpha^1 \leq p_\alpha^0$ which forces $h'(\hat{b}_\delta \upharpoonright l)$ to be the same value for $\mathcal{U}$ many $l \in \hat{L}_d$, and which decides that value, denoted by $\varepsilon_\delta$. Then any $p \leq p_\alpha^1$ satisfies (3).

Take $p_\alpha^2 \leq p_\alpha^1$ which decides $h'(\hat{b}_\delta \upharpoonright l) = \varepsilon_\delta$, for some $l$ such that $l \upharpoonright \varepsilon_\delta < l \upharpoonright \varepsilon_\delta$. If $l = l_{p_\delta}^\varepsilon$, let $p_\delta = p_\delta^2$. Otherwise, let $\vec{\delta}_\delta = \vec{\delta}_{p_\delta}$ and define $p_\delta$ as follows: For each $i \in I_0$, for $\delta \in \vec{\delta}_\delta$, let $p_\delta(i, \delta) = p_\delta^2(i, \delta) \upharpoonright l$. For each $i \in I_1$, for $\delta \in \vec{\delta}_\delta$, let $p_\delta(i, \delta) = \text{splitpred}_T(p_\delta^2(i, \delta) \upharpoonright l)$. Then $p_\delta$ is a condition in $\mathcal{Q}$, and $p_\delta \leq p_\delta^2$, so it satisfies (1) - (3). Furthermore, $h'(P(p_\delta, \vec{\alpha})) = \varepsilon_\delta$, so $p_\delta$ satisfies (4).

We are assuming $\kappa = \aleph_{2d+2}$. Let $D_e = \{0, 2, \ldots, 2d\}$ and $D_o = \{1, 3, \ldots, 2d+1\}$, the sets of even and odd integers less than $2d + 2$, respectively. Let $\mathcal{I}$ denote the collection of all functions $\iota : (2d + 2) \to (2d + 2)$ such that $\iota \upharpoonright D_e$ and $\iota \upharpoonright D_o$ are strictly increasing sequences and $\{\iota(0), \iota(1)\} < \{\iota(2), \iota(3)\} < \cdots < \{\iota(2d), \iota(2d + 1)\}$. For $\vec{\theta} \in [\kappa]^{2d+2}$, $\iota(\vec{\theta})$ determines the pair of sequences of ordinals $(\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d + 1)}), (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d + 1)})$, both of which are members of $[\kappa]^{d+1}$. Denote these as $\iota_\varepsilon(\vec{\theta})$ and $\iota_\delta(\vec{\theta})$, respectively. Let $\vec{\delta}_{\alpha}$ denote $\vec{\delta}_{p_\alpha}$, $k_{\alpha}$ denote $k_{p_\alpha}$, and let $l_{\alpha}$ denote $l_{p_\alpha}$. Let $\langle \delta_{\alpha}(j) : j < k_{\alpha} \rangle$ denote the enumeration of $\vec{\delta}_{\alpha}$ in increasing order. Define a coloring $f$ on $[\kappa]^{2d+2}$ into countably many colors as
follows: Given $\vec{\theta} \in [\kappa]^{2d+2}$ and $i \in I$, to reduce the number of subscripts, letting $\vec{\alpha}$ denote $\iota_\omega(\vec{\theta})$ and $\vec{\beta}$ denote $\iota_\omega(\vec{\theta})$, define

$$f(i, \vec{\theta}) = \langle i, \epsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, \langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle : i \leq d \rangle,$$

(56) \( \langle (i, j) : i \leq d, j < k_{\vec{\alpha}}, \text{ and } \delta_{\vec{\alpha}}(j) = \alpha_i \rangle, \langle (j, k) : j < k_{\vec{\alpha}}, k < k_{\vec{\beta}}, \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k) \rangle. \)

Let $f(\vec{\theta})$ be the sequence $\langle f(i, \vec{\theta}) : i \in I \rangle$, where $I$ is given some fixed ordering. By the Erdős-Rado Theorem, there is a subset $K \subseteq \kappa$ of cardinality $\aleph_1$ which is homogeneous for $f$.

Take $K' \subseteq K$ such that between each two members of $K'$ there is a member of $K$ and $\min(K') > \min(K)$. Then take subsets $K_i \subseteq K'$ such that $K_0 < \cdots < K_d$ and each $|K_i| = \aleph_0$. The following four lemmas are direct analogues of Lemmas 6.3, 6.4, 6.6, and 6.7. Their proofs follow by simply making the correct notational substitutions, and so are omitted.

**Lemma 7.9.** There are $\varepsilon^* \in 2$, $k^* \in \omega$, and $\langle t_{i,j} : j < k^* \rangle$, $i \leq d$, such that for all $\vec{\alpha} \in \prod_{i \leq d} K_i$ and each $i \leq d$, $\epsilon_{\vec{\alpha}} = \varepsilon^*$, $k_{\vec{\alpha}} = k^*$, and $\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}} \rangle = \langle t_{i,j} : j < k^* \rangle$.

Let $l^* = |t_{i0}|$. Then for each $i \in I_0$, the nodes $t_{i,j}$, $j < k^*$, have length $l^*$; and for each $i \in I_1$, the nodes $t_{i,j}$, $j < k^*$, have length in the interval $(l_{n-1}^*, l_n^*)$, where $n$ is the index of the coding node in $T$ of length $l^*$.

**Lemma 7.10.** Given any $\vec{\alpha}, \vec{\beta} \in \prod_{i \leq d} K_i$, if $j, k < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, then $j = k$.

For any $\vec{\alpha} \in \prod_{i \leq d} K_i$, and any $i \in I$, there is a $\vec{\theta} \in [K]^{2d+2}$ such that $\vec{\alpha} = \iota_\omega(\vec{\theta})$. By homogeneity of $f$, there is a strictly increasing sequence $\langle j_i : i \leq d \rangle$ of members of $k^*$ such that for each $\vec{\alpha} \in \prod_{i \leq d} K_i$, $\delta_{\vec{\alpha}}(j_i) = \alpha_i$. For each $i \leq d$, let $t^*_i$ denote $t_{i,j_i}$. Then for each $i \leq d$ and each $\vec{\alpha} \in \prod_{i \leq d} K_i$,

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t^*_{i,j_i} = t^*_i.$$

**Lemma 7.11.** For any finite subset $\vec{J} \subseteq \prod_{i \leq d} K_i$, the set of conditions $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is compatible. Moreover, $p_{\vec{J}} := \bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a member of $\mathcal{P}$ which is below each $p_{\vec{\alpha}}, \vec{\alpha} \in \vec{J}$.

**Lemma 7.12.** If $\beta \in \bigcup_{i \leq d} K_i$, $\vec{\alpha} \in \prod_{i \leq d} K_i$, and $\beta \notin \vec{\alpha}$, then $\beta$ is not a member of $\delta_{\vec{\alpha}}$.

**Part II.** Let $(n_i)_{i < \omega}$ denote the set of indices for which there is an $X \in \text{MPE}_\mathcal{F}(A, C)$ with $X = \max(V)$ for some $V$ of $r_{n_i}[B, T]$. For $i \in I_0$, let $u^*_i = t^*_i$. For $i \in I_1$, let $u^*_i$ be the leftmost extension of $t^*_i$ in $T \upharpoonright l^*$. Note that $\{u^*_i : i \leq d\}$ has no new pre-cliques over $A_\varepsilon$, since lefmost extensions of splitting nodes with no new pre-cliques add no new pre-cliques; this follows from the WP of $T$. Extend each node $u$ in $B^+ \setminus A_\varepsilon$ to its lefmost extension in $T \upharpoonright l^*$ and label that extension $u^*$. Let

$$U^* = \{u^*_i : i \leq d\} \cup \{u^* : u \in B^+ \setminus A_\varepsilon\}.$$

Then $U^*$ extends $B^+$, and $U^*$ has no new pre-cliques over $B$. Let $k_B$ be the integer such that $B = r_{k_B}(B)$. Take $S_0 \in r_{n_0}[B, T]$ such that the nodes in $\max(r_{k_B+1}(S_0))$ extend the nodes in $U^*$. This is possible by Lemma 5.18.
Suppose that \( j < \omega \) and for all \( i < j \), we have chosen \( S_i \in r_{n_i}[B,T] \) such that \( i < i' < j \) implies \( S_i \subseteq S_{i'} \), and \( h' \) is constant of value \( \varepsilon^* \) on \( \operatorname{Ext}_{S_i}(A,P) \). Take \( V_j \in r_{n_j}[S_{j-1},T] \), and let \( X \) denote \( \max(V_j) \). Notice that each member of \( \operatorname{Ext}_X(A,P) \) extends the nodes in \( U^* \). By the definition of \( n_j \), the set of nodes \( X \) contains a coding node. For each \( i \in I_0 \), let \( Y_i \) denote the set of all \( t \in T_i \cap X \) which have immediate extension \( 0 \) in \( T \). Let \( r \) be such that \( l_X = l_T^r \). For each \( i \in I_1 \), let \( Y_i \) denote the set of all splitting predecessors of nodes in \( T_i \cap X \) which split in the interval \( (l_T^{r-1},l_T^r) \) of \( T \). For each \( i \leq d \), let \( J_i \) be a subset of \( K_i \) of size \( |Y_i| \), and enumerate the members of \( Y_i \) as \( q(i,\delta) \), \( \delta \in J_i \). Let \( J \) denote the set of all splitting predecessors of nodes in \( T_i \cap X \) which have length \( \omega \) and \( \delta \) constant of value \( \varepsilon^* \) on \( \operatorname{Ext}_S(A,P) \). Without loss of generality, we may assume that the maximal nodes \( \vec{\gamma} \in \bigcup_{i \leq d} J_i \) such that the set \( \{q(i,\alpha_i) : i \leq d \} \) has no new pre-cliques over \( A \). Thus, the collection of sets \( \{q(i,\alpha_i) : i \leq d \}, \vec{\alpha} \in J \), is exactly the collection of sets of nodes in the interval \( (l_T^{r-1},l_T^r) \) of \( T \) which are members of \( \operatorname{Ext}_T(A,P) \). Moreover, for \( \vec{\alpha} \in J \) and \( i \leq d \),

\[
q(i,\alpha_i) \supseteq t^*_i = p_{\vec{\alpha}}(i,\alpha_i).
\]

To complete the construction of the desired \( q \in \mathbb{Q} \) for which \( q \leq p_{\vec{\alpha}} \) for all \( \vec{\alpha} \in J \), let \( \delta_{\vec{\alpha}} = \bigcup_{i \leq d} \delta_{\vec{\alpha}} = \bigcup_{i \leq d} \delta(\vec{\alpha},i) \). For each pair \((\vec{\alpha},\gamma)\) with \( \gamma \in \delta_{\vec{\alpha}} \setminus J_i \), there is at least one \( \vec{\alpha} \in J \) and some \( j < k^* \) such that \( \gamma = \delta_{\vec{\alpha}}(j) \). As in Case (a) of Theorem 6.2, for any other \( \vec{\beta} \in J \) for which \( \gamma \in \delta_{\vec{\beta}} \), it follows that \( p_{\vec{\beta}}(i,\gamma) = p_{\vec{\alpha}}(i,\gamma) = t^*_i, r \) and \( \delta_{\vec{\beta}}(j) = \gamma \). If \( i \in I_0 \), let \( q(i,\gamma) \) be the leftmost extension of \( t^*_i,r \) in \( T \setminus l_T^r \). If \( i \in I_1 \), let \( q(i,\gamma) \) be the leftmost extension of \( t^*_i,r \) to a splitting node in \( T \) in the interval \( (l_{n_{j-1}},l_{n_j}) \). Such a splitting node must exist, because the coding node in \( X \) must have no pre-cliques with \( t^*_i \) (since \( i \in I_1 \)). Thus, by Lemma 5.10 the leftmost extension of \( t^*_i \) in \( T \) to length \( l_X \) has no pre-cliques with the coding node in \( X \), so it has a splitting predecessor in the interval \( (l_{n_{j-1}},l_{n_j}) \). Define

\[
q = \bigcup_{i \leq d} \{\langle(i,\alpha),q(i,\alpha)\rangle : \alpha \in \delta(q)\}.
\]

By a proof similar to that of Claim 4, it follows that \( q \leq p_{\vec{\alpha}} \), for each \( \vec{\alpha} \in J \).

Take an \( r \leq q \in \mathbb{P} \) which decides some \( l_j \) in \( L \), and such that for all \( \vec{\alpha} \in J \), \( h'(b_{\vec{\alpha}} \upharpoonright l_j) = \varepsilon^* \). Without loss of generality, we may assume that the maximal nodes \( r \) have length \( l_j \). If \( q(i',\alpha') \) is a coding node for some \( i' \in I_0 \) and \( \alpha' \in J_{i'} \), then let \( c_r \) denote \( r(i',\alpha') \). Otherwise, let \( c_r \) denote the leftmost extension in \( T \) of length \( l_j \) of the coding node in \( X \).

Let \( Z_0 \) denote those nodes in the splitting predecessors of \( X \) which are not in \( Y_0 \) and which have length equal to \( c_X \). For each \( z \in Z_0 \), let \( s_z \) denote the leftmost extension of \( z \) in \( T \) to length \( l_j \). Let \( Z_1 \) denote the set of all splitting predecessors of nodes in \( X \) which are not in \( Y_1 \). For each \( z \in Z_1 \), let \( s_z \) denote the splitting predecessor of the leftmost extension of \( z \) in \( T \) to length \( l_j \). This splitting predecessor exists in \( T \) for the following reason: If \( z \) is a splitting predecessor of a node in \( X \), then \( z \) has no pre-cliques with \( c_X \), so the leftmost extension of \( z \) to any length has no pre-cliques with any extension of \( c_X \). In particular, the set \( \{s_z : z \in Z_0 \cup Z_1\} \) has no new pre-cliques over the splitting predecessors in \( X \).

Let

\[
Z^- = \{q(i,\alpha) : i \leq d, \alpha \in J_i\} \cup \{s_z : z \in Z_0 \cup Z_1\}.
\]
Let $Z^*$ denote the extensions in $T$ of all members of $Z^-$ to length $l_j$. Let $j^-$ denote the index such that the maximal coding node in $V_j$ below $c^X$ is $v^j_{n^-}$. Note that $Z^*$ has no new pre-cliques over $splitpred_T(X)$; furthermore, the tree induced by $r_{n^-}((V_j) \cup Z^*)$ is strongly similar to $(V_j)$, except that the coding node might possibly be in the wrong place. Using Lemma 5.18 there is an $S_j \in r_{n^-}(V_j), T$ with max$(S_j)$ extending $Z^*$. Then every member of Ext$_{S_j}(A, P)$ has the same $h'$ color $\varepsilon^*$, by the choice of $r$, since each minimal pre-extension in $MPE_S(A, C)$ extends some member of Ext$_{S_j}(A, P)$ which extends members in ran$(r)$ and so have $h'$-color $\varepsilon^*$.

Let $S = \bigcup_{j < \omega} S_j$. Then $S$ is a strong coding tree in $[B, T]$. Given any $Y \in$ Ext$_S(A, C)$, there is some $X \in MPE_S(A, C)$ such that $Y$ extends $X$. Since $SP_S(X)$ is in Ext$_{S_j}(A, P)$ for some $j < \omega$, $SP_S(X)$ has $h'$ color $\varepsilon^*$. Thus, $Y$ has $h$-color $\varepsilon^*$.

**Lemma 7.13.** Assume $\mathcal{L}_2$. Then there is a strong coding subtree $S \subseteq T$ such that for each copy $A'$ of $A$ in $S$, $h$ is homogeneous on Ext$_S(A', C)$.

**Proof.** Let $(k_i)_{i < \omega}$ be the sequence of integers such that $r_{k_i}(T)$ contains a copy of $A$ which is valid in $r_{k_i}(T)$ and such that max$(A) \subseteq$ max$(r_{k_i}(T))$. Let $k_{-1} = 0$, $T_{-1} = T$, and $U_{-1} = r_0(T)$.

Suppose $i < \omega$ and $U_{i-1} \equiv r_{k_{i-1}}(T)$ and $T_{i-1}$ are given satisfying that for each copy $A'$ of $A$ valid in $U_{i-1}$ with max$(A) \subseteq$ max$(U_{i-1})$, $h$ is homogeneous on Ext$_{U_{i-1}}(A', C)$. Let $U_i$ be in $r_{k_i}[U_{i-1}, T_{i-1}]$. Enumerate all copies $A'$ of $A$ which are valid in $U_i$ and have max$(A') \subseteq$ max$(U_i)$ as $\langle A_0, \ldots, A_n \rangle$. Apply Lemma 7.7 to obtain $R_0 \in [U_i, T_{i-1}]$ which is end-homogeneous for Ext$_{R_0}(A_0, C)$. Then apply Lemma 7.8 to obtain $R'_0 \in [U_i, R_0]$ such that Ext$_{R'_0}(A_0, C)$ is homogeneous for $h$. Given $R'_j$ for $j < n$, apply Lemma 7.4 to obtain a $R_{j+1} \in [U_i, R'_j]$ which is end-homogeneous for Ext$_{R_{j+1}}(A_{j+1}, C)$. Then apply Lemma 7.8 to obtain $R'_{j+1} \in [U_i, R_{j+1}]$ such that Ext$_{R'_{j+1}}(A_{j+1}, C)$ is homogeneous for $c$. Let $T_i = R_{j+1}$.

Let $U = \bigcup_{i < \omega} U_i$. Then $U \leq T$ and $h$ has the same color on Ext$_U(A', C)$ for each copy $A'$ of $A$ which is valid in $U$. Finally, take $S \leq U$ such that for each $m < \omega$, $r_m(S)$ is valid in $U$. Then each copy $A'$ of $A$ in $S$ is valid in $U$. Hence, $h$ is homogeneous on Ext$_S(A', C)$, for each copy $A'$ of $A$ in $S$.

For the setting of Case (a) in Theorem 6.2 a similar lemma holds. The proof is omitted, as it is almost identical, making the obvious changes.

**Lemma 7.14.** Let $T$ be a strong coding tree and let $A, C, h$ be as in Case (a) of Theorem 6.2. Then there is a strong coding tree $S \subseteq T$ such that for each $A' \subseteq S$ with $A' \equiv A$, Ext$_S(A', C)$ is homogeneous for $h$.

**Proof of Theorem 7.3** The proof is by induction on the number of critical nodes. Suppose first that $A$ consists of a single node. Then such a node must be a splitting node in $T$ on the leftmost branch of $T$, so the copies of $A$ are exactly the splitting nodes in $T$ which are sequences of 0’s. Let $h$ be any finite coloring on the splitting nodes in the leftmost branch of $T$. By Ramsey’s Theorem, infinitely many splitting nodes in the leftmost branch of $T$ must have the same $h$ color. By the Extension Lemmas in Section 4 there is a subtree $S \subseteq T$ in which all splitting nodes in the leftmost branch of $S$ have the same $h$ color.

Now assume that $n \geq 1$ and the theorem holds for each finite tree $B$ with $n$ or less critical nodes such that $B$ satisfies the SWP and max$(B)$ contains a node which
is a sequence of all 0’s. Let $C$ be a finite tree with $n + 1$ critical nodes containing a maximal node in $0^{<\omega}$, and suppose $h$ maps the copies of $C$ in $T$ into finitely many colors. Let $d$ denote the maximal critical node in $C$ and let $B = \{t \in C : |t| < |d|\}$. Apply Lemma 8.13 or Lemma 8.14 depending on whether $d$ is a coding or splitting node, to obtain $T' \leq T$ so that for each copy $V$ of $B$ in $T'$, the set $\text{Ext}_{T'}(V, C)$ is homogeneous for $h$. Define $g$ on the copies of $B$ in $T'$ by letting $g(V)$ be the value of $h$ on $V \cup X$ for any $X \in \text{Ext}_{T'}(V, C)$. By the induction hypothesis, there is an $S \leq T'$ such that $g$ is homogeneous on all copies of $B$ in $S$. It follows that $h$ is homogeneous on the copies of $C$ in $S$.

To finish, let $A$ be any tree satisfying the SWP. If $\text{max}(A)$ does not contain a member of $0^{<\omega}$, let $l_A$ denote the longest length of nodes in $A$, and let $\hat{A}$ be the tree induced by $A \cup \{0^{l_A}\}$. Otherwise, let $\hat{A} = A$. Let $g$ be a finite coloring of the copies of $A$ in $T$. To each copy $B$ of $\hat{A}$ in $T$ there corresponds a unique copy of $A$ in $T$, denoted $\varphi(B)$: If $A = \hat{A}$, then $\varphi(B) = B$; if $A \neq \hat{A}$, then $\varphi(B)$ is $B$ with the leftmost node in $\text{max}(B)$ removed. For each copy $B$ of $\hat{A}$, define $h(B) = g(\varphi(B))$. Take $S \leq T$ homogeneous for $h$. Then $S$ is homogeneous for $g$ on the copies of $A$ in $S$. \qed

8. Main Ramsey Theorem for strong $\mathcal{H}_k$-coding trees

The third phase of this article takes place in this section. Subsection 8.1 develops the notion of incremental trees, which set the stage for envelopes for incremental antichains. These envelopes allow applications of Theorem 7.3 to deduce the main Ramsey theorem for colorings of finite antichains coding $K_k$-free graphs, namely Theorem 8.9.

8.1. Incremental trees. The new notions of incremental new pre-cliques and incrementally witnessed pre-cliques are defined now. The main lemma of this subsection, Lemma 8.20, shows that given a strong coding tree $T$, there is an incremental strong coding subtree $S \leq T$ and a set $W \subseteq T$ of coding nodes disjoint from $S$ such that all pre-cliques in $S$ are incrementally witnessed by coding nodes in $W$. This sets the stage for the development of envelopes with the Strict Witnessing Property in the next section, enabling application of Theorem 8.20 to obtain the main Ramsey theorem for strong coding trees, Theorem 8.9.

**Definition 8.1** (Incremental Pre-Cliques). Let $S$ be a subtree of $T_k$, and let $\langle l_j : j < \tilde{j} \rangle$ (\(\tilde{j} \leq \omega\)) list in increasing order the minimal lengths of new pre-cliques in $S$. We say that $S$ has incremental new pre-cliques, or simply $S$ is incremental, if letting

$$S_{l_j,1} := \{t \mid l_j < t \in S, \; |t| > l_j, \; \text{and} \; t(l_j) = 1\},$$

(62) the following hold: For each $j < \tilde{j}$,

1. $S_{l_j,1}$ is a new pre-$a$-clique for exactly one $a \in [3, k]$;
2. If $S_{l_j,1}$ has more than two members, then for each proper subset $X \subseteq S_{l_j,1}$ of size at least 2, then for some $i < j$, $X \upharpoonright l_i = S_{l_i,1}$ and is also a pre-$a$-clique;
3. If $a > 3$, then there are $l_{j-1} < l_3 < \cdots < l^a = l_j$ such that for each $3 \leq b \leq a$, $S_{l_{b,1}} \upharpoonright l^b$ is a pre-$b$-clique. Furthermore, for some $m$, $|d^m| < l^3 < l^a = l_j < |d^{m+1}|$. 


A tree $T \in \mathcal{T}_k$ is called an *incremental strong coding tree* if $T$ is incremental and moreover, in (3) of Definition 8.1, $d_{m+1}^T$ is a coding node in $T$. Note that every subtree of an incremental strong coding tree is incremental, but a strong coding subtree of an incremental strong coding tree need not be an incremental strong coding tree.

**Definition 8.2** (Incrementally Witnessed Pre-Cliques). Let $S, T \in \mathcal{T}_k$ be such that $S$ is incremental and $S \subseteq T$. We say that the pre-cliques in $S$ are *incrementally witnessed* by a set of witnessing coding nodes $W \subseteq T$ if the following hold. Given that $(l_j : j < \omega)$ is the increasing enumeration of the minimal lengths of new pre-cliques in $S$, for each $j < \omega$ the following hold:

1. $|d_{m-1}^S| < l_j < l_n^S$ for some $n < \omega$.
2. If $S_{l_j,1}$ is a new pre-$a_j$-clique, where $a_j \in [3,k]$, then there exist coding nodes $w_3^j, \ldots, w_{a_j}^j$ in $T$ such that, letting $W$ denote $\bigcup_{j < \omega} \{w_3^j, \ldots, w_{a_j}^j\}$, the set of all these witnessing coding nodes,
   a. The set of nodes $\{w_3^j, \ldots, w_{a_j}^j : j < \omega\}$ forms a pre-$(a_j - 1)$-clique which witnesses the pre-$a_j$-clique in $S_{l_j,1}$.
   b. The nodes in $\{w_3^j, \ldots, w_{a_j}^j\}$ do not form pre-cliques with any nodes in $(W \setminus \{w_3^j, \ldots, w_{a_j}^j\}) \cup (S \setminus |w_{a_j}^j| \setminus S_{l_j-1})$ where $S_{l_j-1}$ denotes the set of nodes in $S \setminus |w_{a_j}^j|$ which end-extend $S_{l_j,1}$.
   c. If $Z \subseteq \{w_3^j, \ldots, w_{a_j}^j\} \cup (S \setminus |w_{a_j}^j|)$ forms a pre-clique, then $Z \setminus l_j \cap S$ must be contained in $S_{l_j,1}$. In the terminology of $[4]$, the only nodes in $S$ with which $\{w_3^j, \ldots, w_{a_j}^j\}$ has parallel 1’s (pre-3-cliques) are in $S_{l_j,1}$.

For a node $w \in T$, letting $w^\wedge = w \uparrow l$, where $l$ is least such that $w(l) \neq 0$, we have

1. $|d_{m-1}^S| < |(w_3^j)^\wedge| < \cdots < |(w_{a_j}^j)^\wedge| < |w_3^j| < \cdots < |w_{a_j}^j|$.
2. If $|d_{m-1}^S| < l_{j+1} < l_n^T$, then $\max(l_j, |w_{a_j}^j|) < |(w_{a_j}^j)^\wedge|$.

In what follows, we shall say that a strong coding tree $S$ such that $S \subseteq T$ is valid in $T$ if for each $m < \omega$, $r_m(S)$ is valid in $T$. Since $S$ is a strong coding tree, this is equivalent to max($r_m(S)$) being free in $T$ for each $m < \omega$.

**Lemma 8.3.** Let $T \in \mathcal{T}_k$ be a strong coding tree. Then there is an incremental strong coding tree $S \subseteq T$ and a set of coding nodes $W \subseteq T$ such that each new pre-clique in $S$ is incrementally witnessed in $T$ by coding nodes in $W$.

**Proof.** Recall that for any tree $T \in \mathcal{T}_k$, the sequence $\langle m_n : n < \omega \rangle$ denotes the indices such that $d_{m_n}^T = c_n^T$; that is, the $m_n$-th critical node in $T$ is the $n$-th coding node in $T$. Fix some $U_0 \in r_{m_0+1}[0, T]$ which is valid in $T$. Then $U_0$ has exactly one coding node, $c_{l_0}^U$. If $k = 3$, then $U_0$ has exactly one node with passing number 1 at $c_0^{l_0}$. So there are no pre-cliques to witness; in this case, let $S_0 = U_0$.

If $k > 3$, then $U_0$ has exactly two nodes with passing number 1 at $c_0^{l_0}$. Thus, $U_0$ has exactly one pre-clique, which is a pre-3-clique, and $c_0^{l_0}$ witnesses it. By Lemma 3.11, $T$ is a perfect tree, so there is a splitting node $s \in T$ such that $s$ is a sequence of 0’s and $|s| > l_{l_0}$. Extend all nodes in $\max(U_0)$ leftmost in $T$ to the length $|s|$, and call this set of nodes $X$. Then apply Lemma 3.13 to obtain $Y$ end-extending $X^+$ so that the following hold: The node in $Y$ extending $s^{-1}$ is a coding node, call it $w_0$; the nodes in $Y \setminus \{w_0\}$ have the same passing number at $w_0$ as they do.
at $U'_0$; and the extension of $s^{-1}$ in $Y$ is a sequence of 0’s. Lastly, $Y$ has no new pre-cliques over $U_0$. Then apply Lemma 5.15 to end-extend $Y \setminus \{w_0\}$ to a level set $Z$ in $T$ so that $r_{m_0}(U_0) \cup Z$ is a member of $r_{m_0+1}[0,T]$. Let $S_0 = r_{m_0}(U_0) \cup Z$ and let $W_0 = \{w_0\}$.

Suppose now that $n \geq 1$ and we have chosen $S_{n-1} \in r_{m_{n-1}+1}[0,T]$ valid in $T$ and $W_{n-1} \subseteq T$ so that $S_{n-1}$ is incremental and each new pre-clique in $S_{n-1}$ is incrementally witnessed by some coding nodes in $W_{n-1}$. Take some $U_n \in r_{m_{n}+1}[S_{n-1},T]$ such that $r_m(U_n)$ is valid in $T$. Let $V = \max(r_m(U_n))$.

Let $(X_j : j < \tilde{j})$ enumerate those subsets of $\max(U_n)$ which have new pre-cliques over $r_m(U_n)$ so that for each pair $j < j' < \tilde{j}$,

1. If $X_j$ is a new pre-a-clique, then $X_{j'}$ is a new pre-a'-clique where $a \leq a'$; and
2. $X_j \not\subseteq X_{j'}$.

Note that (1) implies that, for each $a \in [3,k-1]$, all new pre-a-cliques are enumerated before any new pre-$(a+1)$-clique is enumerated. Furthermore, every new pre-clique in $\max(U_n)$ over $r_m(U_n)$ is enumerated in $(X_j : j < \tilde{j})$ whether or not it is maximal. By (2), all new pre-a-cliques composed of two nodes are listed before any new pre-a-clique consisting of three nodes, etc. For each $j < \tilde{j}$, let $Y_j = X_j \mid \{V \}$.

By properties (1) and (2), $S_0$ must be a pre-3-clique consisting of two nodes. (Every new coding node level of a strong coding tree has a new pre-3-clique.) The construction process in this case is similar to the construction above for $S_0$ when $k > 3$. By Lemma 5.11 there is a splitting node $s \in T$ such that $s$ is a sequence of 0’s and $|s| > |v_1 + 1$. Extend all nodes in $V$ leftmost in $T$ to the length $|s|$, and call this set of nodes $Z$. Apply Lemma 5.14 to obtain $V_0$ end-extending $Z$ so that the node in $V_0$ extending $s^{-1}$ is a coding node, call it $w_{n,0}$: the two nodes in $V_0$ extending the nodes in $Y_0$ both have passing number 1 at $w_{n,0}$: all other nodes in $V_0$ are leftmost extensions of the nodes in $V^+ \setminus Y_0$; and the only new pre-clique in $V_0$ is the nodes in $V_0$ extending $Y_0$. Let $W_{n,0} = \{w_{n,0}\}$.

Given $j < \tilde{j} - 1$ and $V_j$, let $Y_{j+1}'$ be the set of those nodes in $V_j$ which extend the nodes in $Y_{j+1}$. Let $a \in [3,k]$ be such that $X_{j+1}$ is a new pre-a-clique. Applying Lemma 5.11 $a - 2$ times, obtain splitting nodes $s_i$, $i < a - 2$, in $T$ which are sequences of 0 such that $|v_i| < |s_0| < \cdots < |s_{a-3}|$. Extend all nodes $s_i^{-1}$, $i < a - 2$, leftmost in $T$ to length $|s_{a-3}| + 1$ and extend the nodes in $V_j$ leftmost in $T$ to length $|s_{a-3}| + 1$ and denote this set of nodes as $Z$. By Lemma 5.10, this adds no new pre-cliques over $V_j$. Next apply Lemma 5.14 $a - 2$ times to obtain $V_{j+1}$ end-extending $Z$ and coding nodes $w_{n,j+1,i} \in T$, $i < a - 2$, such that letting $Y''_{j+1}$ be those nodes in $V_{j+1}$ extending nodes in $Y_{j+1}'$, the following hold:

1. $|w_{n,j+1,0}| < \cdots < |w_{n,j+1,a-3}|$;
2. The nodes in $V_{j+1}$ all have length $|w_{n,j+1,a-3}|$;
3. For each $i < a - 2$, all nodes in $\{w_{n,j+1,i'} : i < i' < a - 2\} \cup Y''_{j+1}$ have passing number 1 at $w_{n,j+1,i}$;
4. All nodes in $V_{j+1} \setminus Y''_{j+1}$ are leftmost extensions of nodes in $V_j \setminus Y'_{j+1}$;
5. The only new pre-clique in $V_{j+1}$ above $V^+$ is the set of nodes in $Y''_{j+1}$.

Let $W_{n,j+1} = \{w_{n,j+1,i} : i < a - 2\}$.

After $V_{j-1}$ has been constructed, take some $S_n \in r_{m_{n}+1}[r_{k_n}(U_n),T]$ such that $\max(S_n)$ end-extends $V_{j-1}$, by Lemma 5.18. Let $W_n = \bigcup_{j < j} W_{n,j}$. 
To finish, let $S = \bigcup_{n<\omega} S_n$ and $W = \bigcup_{n<\omega} W_n$. Then $S \leq T$, $S$ is incremental, and the pre-cliques in $S$ are strongly incrementally witnessed by coding nodes in $W$. \hfill $\square$

8.2. Ramsey theorem for strict similarity types. The main Ramsey theorem of this paper is Theorem 8.9. It says that given a finite coloring of all strictly similar copies (Definition 8.4) of a fixed finite antichain in an incremental strong coding tree, there is a subtree which is again a strong coding tree in which all strictly similar copies of the antichain have the same color. Envelopes of incremental antichains will have the Strict Witnessing Property. Moreover, all envelopes of a fixed incremental antichain of coding nodes will be stably isomorphic to each other. This will allow for an application of Theorem 7.3 to obtain the same color for all copies of a given envelope, in some subtree in $T_k$. From this, we will deduce Theorem 8.9.

Recall that a set of nodes $A$ is an antichain if no node in $A$ extends any other node in $A$. In what follows, we shall call a set of nodes an antichain if it is an antichain of coding nodes. If $Z$ is an antichain, then the tree induced by $Z$ is the set of nodes

\begin{equation}
\{z \mid |u| : z \in Z \text{ and } u \in Z^\omega\}.
\end{equation}

We say that an antichain satisfies the Witnessing Property (Strict Witnessing Property) if and only if the tree it induces satisfies the Witnessing Property (Strict Witnessing Property).

Fix, for the rest of this section, an incremental strong coding tree $T \in T_k$, as in Lemma 8.3. Notice that any strong coding subtree of $T$ will also be incremental. Furthermore, any antichain in $T$ must be incremental.

**Definition 8.4** (Strict similarity type). Suppose $Z \subseteq T$ is a finite antichain of coding nodes. Enumerate the nodes of $Z$ in increasing order of length as $\langle z_i : i < \tilde{i} \rangle$. Enumerate all nodes in $Z^\omega$ as $\langle u_m^Z : m < \tilde{m} \rangle$ in order of increasing length. Thus, each $u_m^Z$ is either a splitting node in $Z^\omega$ or else a coding node in $Z$. List the minimal levels of new pre-cliques in $Z$ in increasing order as $\langle l_j : j < \tilde{j} \rangle$. For each $j < \tilde{j}$, let $I_{l_j}^Z$ denote the set of those $i < \tilde{i}$ such that $\{z_i \mid l_j : i \in I_{l_j}^Z\}$ is the new pre-clique in $Z \upharpoonright l_j$. The sequence

\begin{equation}
\langle (l_j : j < \tilde{j}), (I_{l_j}^Z : j < \tilde{j}), \langle |u|^Z_m : m < \tilde{m} \rangle \rangle
\end{equation}

is the strict similarity sequence of $Z$.

Let $Y$ be another finite antichain in $T$, and let

\begin{equation}
\langle (p_j : j < \tilde{k}), (I_{p_j}^Y : j < \tilde{k}), \langle |u|^Y_m : m < \tilde{q} \rangle \rangle
\end{equation}

be its strict similarity sequence. We say that $Y$ and $Z$ have the same strict similarity type or are strictly similar, and write $Y \overset{ss}{\approx} Z$, if

1. The tree induced by $Y$ is stably isomorphic to the tree induced by $Z$, so in particular, $\tilde{m} = \tilde{q}$;
2. $\tilde{j} = \tilde{k}$;
3. For each $j < \tilde{j}$, $I_{p_j}^Y = I_{l_j}^Z$; and
4. The function $\varphi : \langle p_j : j < \tilde{k} \rangle \cup \{|u|^Y_m : m < \tilde{m} \} \rightarrow \langle l_j : j < \tilde{j} \rangle \cup \{|u|^Z_m : m < \tilde{m} \}$, defined by $\varphi(p_j) = l_j$ and $\varphi(u^Y_m) = u^Z_m$, is an order preserving bijection between these two linearly ordered sets of natural numbers.
Define
\[
\sin_{p_s}(Z) = \{ Y \subseteq T : Y \cong_{p_s} Z \).
\]

Note that if \( Y \cong_{p_s} Z \), then the map \( f : Y \to Z \) by \( f(y_i) = z_i \), for each \( i < \tilde{i} \), induces the strong similarity map from the tree induced by \( Y \) onto the tree induced by \( Z \). Then \( f(w^Y_m) = u^Z_m \), for each \( m < \tilde{m} \). Further, by (3) and (4) of Definition 8.2, this map preserves the order in which new pre-cliques appear, relative to all other new pre-cliques in \( Y \) and \( Z \) and the nodes in \( Y^\wedge \) and \( Z^\wedge \).

The following notion of envelope is defined in terms of structure without regard to an ambient strong coding tree. In any given strong coding tree subtree \( U \subseteq T \), there will certainly be finite subtrees of \( W \) consisting of nodes from \( S \) witnessed by coding nodes in \( W \). The next fact follows immediately from the definitions.

**Fact 8.6.** Let \( Z \) be any antichain in an incremental strong coding tree. Then any envelope of \( Z \) has incrementally witnessed pre-cliques, which implies that \( Z \) has the Strict Witnessing Property.

**Lemma 8.7.** Let \( Y \) and \( Z \) be strictly similar incremental antichains of coding nodes. Then any envelope of \( Y \) is stably isomorphic to any envelope of \( Z \), and both envelopes have the Strict Witnessing Property.

**Proof.** Let \( Y = \{ y_i : i < \tilde{i} \} \) and \( Z = \{ z_i : i < \tilde{i} \} \) be the enumerations of \( Y \) and \( Z \) in order of increasing length, and let
\[
\langle \langle l_j : j < \tilde{j} \rangle, \langle l^Y_{\tilde{j}} : j < \tilde{j} \rangle, \langle |u^Y_m| : m < \tilde{m} \rangle \rangle
\]
and
\[
\langle \langle p_j : j < \tilde{j} \rangle, \langle l^Z_{\tilde{j}} : j < \tilde{j} \rangle, \langle |u^Z_m| : m < \tilde{m} \rangle \rangle
\]
be their strict similarity sequences, respectively. Let \( E = Y \cup V \) and \( F = Z \cup W \) be any envelopes of \( Y \) and \( Z \), respectively. For each \( j < \tilde{j} \), let \( a_j \geq 3 \) be such that \( I^Y_j \) is a new \( a_j \)-clique. Then the members of \( V \) may be labeled as \( \{ v^Y_1, \ldots, v^Y_{\tilde{j}} : j < \tilde{j} \} \) with the property that for each \( j < \tilde{j} \), given the least \( m < \tilde{m} \) such that \( |u^Y_{a_j}| < |u^Y_m| \), we have \( |u^Y_{m-1}| < |(v^Y_j)^{\wedge}| \). This follows from Definition 8.2. Since \( Y \) and \( Z \) have the same strict similarity type, it follows that for each \( j < \tilde{j} \), \( I^Z_{\tilde{j}} \) is also a new \( a_j \)-clique. Furthermore, \( W = \{ w^Z_1, \ldots, w^Z_{\tilde{j}} : j < \tilde{j} \} \), where for each \( j < \tilde{j} \), given the least \( m < \tilde{m} \) such that \( |w^Z_{a_j}| < |w^Z_m| \), we have that \( |w^Z_{m-1}| < |(w^Z_j)^{\wedge}| \). Thus, \( V \) and \( W \) both have the same size, label it \( J \).

Let \( \tilde{n} = \tilde{i} + J \), and let \( \{ e_n : n < \tilde{n} \} \) and \( \{ f_n : n < \tilde{n} \} \) be the enumerations of \( E \) and \( F \) in order of increasing length, respectively. For each \( j < \tilde{j} \), let \( n_j \) be the
index in \( \tilde{n} \) such that \( e_{n} = v_{j} \) and \( f_{n} = w_{j} \). For \( n < \tilde{n} \), let \( E(n) \) denote the tree induced by \( E \) restricted to those nodes of length less than or equal to \(|e_{n}|\); precisely, \( E(n) = \{ e \mid |e| = t \in E^{\wedge}, \text{ and } |t| \leq |e_{n}| \} \). Define \( F(n) \) similarly.

We prove that \( E \cong F \) by induction on \( j \). If \( \tilde{j} = 0 \), then \( E = Y \) and \( F = Z \), so \( E \cong F \) follows from \( Y \cong Z \). Suppose now that \( \tilde{j} \geq 1 \) and that, letting \( j = \tilde{j} - 1 \), the induction hypothesis gives that \( E(n) \cong F(n) \) for the maximal \( n < \tilde{n} \) such that \( e_{n} \in Y^{\wedge} \) and \( |e_{n}| < l_{j} \). Let \( m \) be the least integer below \( \tilde{m} \) such that \(|u_{m}^{\wedge}| > l_{j} \). Then \( e_{n} = u_{m-1}^{Y} \) and the only nodes in \( E^{\wedge} \) in the interval \((|u_{m-1}^{Y}|,|u_{m}^{Y}|)\) are \((a_{1}^{3}),\ldots,(a_{1}^{a_{j}})^{\wedge},(a_{j}^{a_{j}})^{\wedge},(a_{j}^{a_{j}})^{\wedge},\ldots,(a_{j}^{a_{j}})^{\wedge},(a_{j}^{a_{j}})^{\wedge} \). Likewise, the only nodes in \( F^{\wedge} \) in the interval \((|u_{m-1}^{Y}|,|u_{m}^{Y}|)\) are \((a_{3}^{3}),\ldots,(a_{j}^{a_{j}})^{\wedge},(a_{j}^{a_{j}})^{\wedge},\ldots,(a_{j}^{a_{j}})^{\wedge},(a_{j}^{a_{j}})^{\wedge} \).

By the induction hypothesis, there is a stable map \( g : E(n) \to F(n) \). Extend it to a stable map \( g^{*} : E(n') \to F(n') \), where \( n' = \tilde{n} - 1 \) as follows: Define \( g^{*} = g \) on \( E(n) \). For each \( i \in [3,a_{j}] \), \( g^{*}((a_{i}^{3}))^{\wedge} = (a_{i}^{3})^{\wedge} \) and \( g^{*}(a_{j}^{a_{j}}) = a_{j}^{a_{j}} \). Recall that the nodes \((a_{3}^{3}),\ldots,(a_{j}^{a_{j}})^{\wedge} \) form a \(-\wedge(\text{a}_{j}-1)-\)clique and only have mutual pre-cliques with nodes in \( \{y_{i} : i \in I_{Y}^{\wedge}\} \), witnessing this set, and no other members of \( E \). Likewise, for \((a_{3}^{3}),\ldots,(a_{j}^{a_{j}})^{\wedge} \) and \( \{z_{i} : i \in I_{Y}^{\wedge}\} \). Thus, \( g^{*} \) from \( E(n'') \to F(n'') \) is a strict similarity map, where \( n'' < \tilde{n} \) is the index such that \( v_{k}^{a_{j}} = e_{a_{j}} \). If \( n'' < \tilde{n} - 1 \), then \( \{e_{k} : n'' < q < \tilde{n}\} \subseteq Y^{\wedge} \) and \( \{f_{k} : n'' < q < \tilde{n}\} \subseteq Z^{\wedge} \). Since these sets have no new pre-cliques and are strictly similar, the map \( g^{*}(e_{k}) = f_{k} \), \( n'' < q < \tilde{n} \), is a stable map. Thus, we have constructed a stable map \( g^{*} : E \to F \). It follows from the definitions that envelopes satisfy the Strict Witnessing Property.

\[ \text{Lemma 8.8. Suppose } Z \text{ is a finite antichain of coding nodes and } E \text{ is an envelope of } Z \text{ in } T. \text{ Enumerate the nodes in } Z \text{ and } E \text{ in order of increasing length as } \langle z_{i} : i < \tilde{i} \rangle \text{ and } \langle e_{k} : k < \tilde{k} \rangle, \text{ respectively. Given any } F \subseteq T \text{ with } F \cong E, \text{ let } F \upharpoonright Z := \{ f_{k} : i < \tilde{i} \}, \text{ where } \{ f_{k} : k < \tilde{k} \} \text{ enumerates the nodes in } F \text{ in order of increasing length and for each } i < \tilde{i}, k_{i} \text{ is the index such that } e_{k_{i}} = z_{i}. \text{ Then } F \upharpoonright Z \text{ is strictly similar to } Z. \]

\[ \text{Proof.} \text{ Recall that } E \text{ has incrementally witnessed new pre-cliques and } F \cong E \text{ implies that } F \text{ also has this property, and hence has the SWP. Let } \iota_{Z,F} : Z \to F \text{ be the injective map defined via } \iota_{Z,F}(z_{i}) = f_{k_{i}}, i < \tilde{i}, \text{ and let } F \upharpoonright Z \text{ denote } \{ f_{k} : i < \tilde{i} \}, \text{ the image of } \iota_{Z,F}. \text{ Then } F \upharpoonright Z \text{ is a subset of } F \text{ which we claim is strictly similar to } Z. \]

Since \( F \) and \( E \) each have incrementally witnessed new pre-cliques, the strong similarity map \( g : E \to F \) satisfies that for each \( j < \tilde{k} \), the indices of the new pre-cliques at level of the \( j \)-th coding node are the same:

\[ \{ k < \tilde{k} : e_{k}(|e_{j}|) = 1 \} = \{ k < \tilde{k} : g(e_{k})(|g(e_{j})|) = 1 \} = \{ k < \tilde{k} : f_{k}(|f_{j}|) = 1 \}. \]

Since \( \iota_{Z,F} \) is the restriction of \( g \) to \( Z \), \( \iota_{Z,F} \) also takes each new pre-clique in \( Z \) to the corresponding new pre-clique in \( F \upharpoonright Z \), with the same set of indices. Thus, \( \iota_{Z,F} \) witnesses that \( F \upharpoonright Z \) is strictly similar to \( Z \).

\[ \text{\textbf{Theorem 8.9 (Ramsey Theorem for Strict Similarity Types). Let } Z \text{ be a finite antichain of coding nodes in an incremental strong coding tree } T, \text{ and suppose } h \text{ colors of all subsets of } S \text{ which are strictly similar to } Z \text{ into finitely many colors. Then there is an incremental strong coding tree } S \subseteq T \text{ such that all subsets of } S \text{ strictly similar to } Z \text{ have the same } h \text{ color.} } \]
Proof. First, note that there is an envelope $E$ of a copy of $Z$ in $T$: By Lemma 8.3 there is an incremental strong coding tree $U \leq T$ and a set of coding nodes $V \subseteq T$ such that each $Y \subseteq U$ which is strictly similar to $Z$ has an envelope in $T$ by adding nodes from $V$. Since $U$ is strongly similar to $T$, there is subset $Y$ of $U$ which is strictly similar to $Z$. Let $E$ be any envelope of $Y$ in $T$, using witnessing coding nodes from $V$.

By Lemma 8.7 all envelopes of copies of $Z$ are stably isomorphic and have the SWP. For each $F \cong E$, define $h^*(F) = h(F \upharpoonright Z)$, where $F \upharpoonright Z$ is the subset of $F$ provided by Lemma 8.3. The set $F \upharpoonright Z$ is strictly similar to $Z$, so the coloring $h^*$ is well-defined. By Theorem 7.3, there is a strong coding tree $T' \leq T$ such that $h^*$ is monochromatic on all stably isomorphic copies of $E$ in $T'$. Lemma 8.3 implies there is an incremental strong coding tree $S \leq T'$ and a set of coding nodes $W \subseteq T'$ such that each $Y \subseteq S$ which is strictly similar to $Z$ has an envelope $F$ in $T'$, so that $h(Y) = h^*(F)$. Therefore, $h$ takes only one color on all strictly similar copies of $Z$ in $S$. \qed

9. The Henson graphs have finite big Ramsey degrees

From the results from previous sections, we now prove the main theorem of this paper, Theorem 9.2. This result follows from Ramsey Theorem 8.3 for strict similarity types along with Lemma 9.1 below.

For a strong coding tree $T$, let $(T, \subseteq)$ be the reduct of $(T, \omega; \subseteq, <, c)$. Then $(T, \subseteq)$ is simply the tree structure of $T$, disregarding the difference between coding nodes and non-coding nodes. We say that two trees $(T, \subseteq)$ and $(S, \subseteq)$ are strongly similar trees if they satisfy Definition 3.1 in [35]. This is the same as modifying Definition 4.3 by deleting (6) and changing (7) to apply to passing numbers of all nodes in the trees. By saying that two finite trees are strongly similar trees, we are implicitly assuming that their extensions to their immediate successors. Thus, strong similarity implies passing numbers of their immediate extensions are preserved. Given an antichain $D$ of coding nodes from a strong coding tree, let $L_D$ denote the set of all lengths of nodes $t \in D^\omega$ such that $t$ is not the splitting predecessor of any coding node in $D$. Define

\begin{equation}
D^* = \bigcup \{ t \upharpoonright l : t \in D^\omega \setminus D \text{ and } l \in L_D \}.
\end{equation}

Then $(D^*, \subseteq)$ is a tree.

Lemma 9.1. Let $T \in T_k$ be a strong coding tree. Then there is an infinite antichain of coding nodes $D \subseteq T$ which code $H_k$ the same way as $T_k$: $c_k^D(l^D_i) = c_k^T(l^T_i)$, for all $i < n < \omega$. Moreover, $(D^*, \subseteq)$ and $(T_k, \subseteq)$ are strongly similar as trees.

Proof. We will construct a subtree $D \subseteq T_k$ such that $D$ the set of coding nodes in $D$ form an antichain satisfying the lemma. Then, since $T \in T_k$ implies $T \cong T_k$, letting $\varphi : T_k \to T$ be the strong similarity map between $T_k$ and $T$, the image of $\varphi$ on the coding nodes of $D$ will yield an antichain of coding nodes $D \subseteq T$ satisfying the lemma.

We will construct $D$ so that for each $n$, the node of length $l^D_n + 1$ which is going to be extended to the next coding node $c_{n+1}^D$ will split in $D$ before any of the other nodes of length $l^D_{n+1}$ split in $D$. Above that, the splitting will be regular in the interval until the next coding node. Recall that for each $i < \omega$, $T_k$ has either a
coding node or else a splitting node of length $i$. To avoid some superscripts, let $l_n = |c^k_n|$ and $p_n = |d^k_n|$. Let $j_n$ be the index such that $c^k_n = c^l_{j_n}$, so that $p_n$ equals $l_{j_n}$. The set of nodes in $\mathbb{D} \setminus \{d^k_n\}$ of length $p_n$ shall be indexed as $\{d_t : t \in T_k \upharpoonright l_n\}$.

We define inductively on $n \in [-1, \omega)$ trees with coding nodes, $\mathbb{D} \upharpoonright (\leq p_{n-1})$, and strong similarity maps of the trees $\varphi : T_k \upharpoonright (\leq l_n) \to \mathbb{D}^* \upharpoonright (\leq p_n)$, where $l_n = |c^k_n|$ and $p_n = |\varphi(c^k_n)|$. Recall that the node $\langle \rangle$ is the ghost coding node $c^k_{-1}$ in $T_k$. Define $d_\langle \rangle = \varphi(\langle \rangle) = \langle \rangle$. The node $\langle \rangle$ splits in $T_k$, so the node $d_\langle \rangle$ will split in $\mathbb{D}$. Suppose that $n \in \omega$ and we have constructed $\mathbb{D} \upharpoonright (\leq p_{n-1})$ satisfying the lemma. By the induction hypothesis, there is a strong similarity map of the trees $\varphi : T_k \upharpoonright (\leq l_{n-1}) \to \mathbb{D}^* \upharpoonright (\leq p_{n-1})$. For $t \in T_k \upharpoonright l_{n-1}$, let $d_t$ denote $\varphi(t)$.

Let $s$ denote the node in $T_k \upharpoonright l_{n-1}$ which extends to the coding node $c^k_n$. Let $v_s$ be a splitting node in $T_k$ extending $d_s$. Let $u_s = v_s \setminus s$ and extend all nodes $d_s, t \in (T_k \upharpoonright l_{n-1}) \setminus \{s\}$, leftmost to length $|u_s|$ and label these $d'_t$. Extend $v_s \setminus 0$ leftmost to length $|u_s|$ and label it $d'_s$. Let $X = \{d'_t : t \in T_k \upharpoonright l_{n-1} \cup \{u_s\}\}$ and let $\text{Spl}(u_s)$ be the set of all nodes in $X$ which have no pre-$k$-cliques with $u_s$. Apply Lemma 5.15 to obtain a coding node $c^k_n$ extending $u_s$ and nodes $d'_w, w \in T_k \upharpoonright l_n$, so that, letting $p_n = |c^k_n|$ and

$$\mathbb{D} \upharpoonright p_n = \{d_t : t \in T_k \upharpoonright l_n\} \cup \{c^k_n\},$$

the following hold. $\mathbb{D} \upharpoonright (\leq p_n)$ satisfies the Witnessing Property, and $\mathbb{D}^* \upharpoonright (\leq p_n)$ is strongly similar as a tree to $T_k \upharpoonright (\leq l_n)$. Thus, the coding nodes in $\mathbb{D} \upharpoonright (\leq p_n)$ code exactly the same graph as the coding nodes in $T_k \upharpoonright (\leq l_n)$.

Let $\mathbb{D} = \bigcup_{n < \omega} \mathbb{D} \upharpoonright (\leq p_n)$. Then the set of coding nodes in $\mathbb{D}$ forms an antichain of maximal nodes in $\mathbb{D}$. Further, the tree generated by the the meet closure of the set $\{c^k_n : n < \omega\}$ is exactly $\mathbb{D}$, and $\mathbb{D}^*$ and $T_k$ are strongly similar as trees. By the construction, for each pair $i < n < \omega$, $c^k_n(p_i) = c^k_i(l_i)$; hence they code $\mathcal{H}_k$ in the same order.

To finish, let $\psi$ be the stable map from $T_k$ to $T$. Letting $D$ be the $\psi$-image of $\{c^k_n : n < \omega\}$, we see that $D$ is an antichain of coding nodes in $T$ such that $D^*$ and $\mathbb{D}^*$ are strongly similar trees, and hence $D^*$ is strongly similar as a tree to $T_k$. Thus, the antichain of coding nodes $D$ codes $\mathcal{H}_k$ and satisfies the lemma.

Recall that the Henson graph $\mathcal{H}_k$ is, up to isomorphism, the homogeneous $k$-clique-free graph on countably many vertices which is universal for all $k$-clique-free graphs on countably many vertices.

**Main Theorem 9.2.** For each $k \geq 3$, the Henson graph $\mathcal{H}_k$ has finite big Ramsey degrees.

**Proof.** Fix $k \geq 3$ and let $G$ be a finite $K_k$-free graph. Suppose $f$ colors of all the copies of $G$ in $\mathcal{H}_k$ into finitely many colors. By Theorem 5.22 there is a strong coding tree $T_k$ such that the coding nodes in $T_k$ code a $\mathcal{H}_k$. Let $A$ denote the set of all antichains of coding nodes of $T_k$ which code a copy of $G$. For each $Y \in A$, let $h(Y) = f(G')$, where $G'$ is the copy of $G$ coded by the coding nodes in $Y$. Then $h$ is a finite coloring on $A$.

Let $n(G)$ be the number of different strict similarity types of incremental antichains of coding nodes in of $T_k$ coding $G$, and let $\{Z_i : i < n(G)\}$ be a set of one representative from each of these strict similarity types. Successively apply Theorem 5.9 to obtain incremental strong coding trees $T_k \geq T_0 \geq \cdots \geq T_{n(G)-1}$ so that
for each $i < n(G)$, $h$ is takes only one color on the set of incremental antichains of coding nodes $A \subseteq T_i$ such that $A$ is strictly similar to $Z_i$. Let $S = T_{n(G) - 1}$.

By Lemma 9.1 there is an antichain of coding nodes $D \subseteq S$ which codes $H_k$ in the same way as $T_k$. Every set of coding nodes in $D$ coding $G$ is automatically incremental, since $S$ is incremental. Therefore, every copy of $G$ in the copy of $H_k$ coded by the coding nodes in $D$ is coded by an incremental antichain of coding nodes. Thus, the number of strict similarity types of incremental antichains in $T_k$ coding $G$ provides an upper bound for the big Ramsey degree of $G$ in $H_k$. □

10. Future Directions

This article developed a unified approach to proving upper bounds for big Ramsey degrees of all Henson graphs. The main phases of the proof were as follows: I. Find the correct structures to code $H_k$ and prove Extension Lemmas. II. Prove an analogue of Milliken’s Theorem for finite trees with certain structure. In the case of the Henson graphs, this is the Strict Witnessing Property. III. Find a means for turning finite antichains into finite trees with the Strict Witnessing Property so to deduce a Ramsey Theorem for finite antichains from the previous Milliken-style theorem. This general approach should apply to a large class of ultrahomogeneous structures with forbidden configurations. It will be interesting to see where the dividing line is between those structures for which this methodology works and those for which it does not. The author conjectures that similar approaches will work for forbidden configurations which are irreducible in the sense of [27] and [28].

Although we have not yet proved the lower bounds to obtain the precise big Ramsey degrees $T(G, K_n)$, we conjecture that they will be exactly the number of strict similarity types of incremental antichains coding $G$. We further conjecture that once found, the lower bounds will satisfy the conditions needed for Zucker’s work in [41] to apply. If so, then each Henson graph would admit a big Ramsey structure and any big Ramsey flow will be a universal completion flow, and any two universal completion flows will be universal.

Lastly, we point out that by a compactness argument, one can obtain finite versions of the two main Ramsey theorems in this article. In particular, the finite version of Theorem 8.9 may well produce better bounds for the sizes of finite $K_n$-free graphs instantiating that the Fraïssé class $G_n^<\subseteq$ has the Ramsey property.

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