SECOND ORDER ROTATABLE DESIGNS OF SECOND TYPE USING BALANCED INCOMPLETE BLOCK DESIGNS

P. CHIRANJEEVI*, B. RE. VICTOR BABU

Department of Statistics, Acharya Nagarjuna University, Guntur-522510, A.P., India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract: In this paper second order rotatable designs of second type using balanced incomplete block designs is suggested. This design is compared with second order rotatable designs of first type using balanced incomplete block designs (cf. Das and Narasimham [6]) on the basis of efficiency.

Keywords: response surface methodology; balanced incomplete block design, rotatability; orthogonality, efficiency.

2010 AMS Subject Classification: 62K05.

1. INTRODUCTION

Response surface designs is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable or response. Box and Hunter [1] introduced designs having spherical variance function are called rotatable designs. Das and Narasimham [6] constructed rotatable designs using balanced incomplete block designs (BIBD). Raghavarao [21] constructed second order rotatable designs (SORD) using incomplete block designs. Draper and Guttman [7] suggested an index of

*Corresponding author
E-mail address: chirupunugupati@email.com
Received January 20, 2021

2341
rotatability. Khuri [14] introduced a measure of rotatability for response surface designs. Draper and Pukelshein [8] developed another look at rotatability. Park et al. [16] introduced new measure of rotatability for second order response surface designs. Das et al. [5] developed modified response surface designs. Kim [13] introduced extended central composite designs (CCD) with the axial points are indicated by two numbers. Victorbabu and Vasundharadevi [25] suggested modified second order response surface designs using BIBD. Victorbabu [23] constructed modified SORD and second order slope rotatable designs using a pair of BIBD. Victorbabu et al. [26] studied modified second order response surface designs using pairwise balanced designs (PBD). Victorbabu [22] suggested a review on SORD. Victorbabu et al. [27] suggested modified second order response surface designs using CCD. Victorbabu and Vasundharadevi [28] studied second order response surface designs using SUBA with two unequal block sizes. Victorbabu [24] constructed modified SORD using a pair of SUBA with two unequal block sizes. Park and Park [17] suggested the extension of CCD for second order response surface models. Victorbabu and Surekha [29] suggested measure of rotatability for second order response surface designs using incomplete block designs. Victorbabu and Surekha [30] developed measure of rotatability for second order response surface designs using BIBD. Victorbabu et al. [31-32] studied measure of rotatability for second order response surface designs using a pair of SUBA with two unequal block sizes and a pair of BIBD. Jyostna et al. [9] suggested measure of rotatability for second degree polynomial using CCD. Jyostna and Victorbabu [10-12] studied measure of modified rotatability for second degree polynomials using BIBD, PBD and SUBA with two unequal block sizes. Chiranjeevi et al. [2] extended the work of Kim [13] and suggested SORD of second type using CCD for $9 \leq v \leq 17$ (v: number of factors). Chiranjeevi and Victorbabu [3-4] studied SORD of second type using SUBA with two unequal block sizes and PBD.

In this paper, second order rotatable designs of second type using balanced incomplete block designs is suggested. This design is compared with second order rotatable designs of first type using balanced incomplete block designs (Das and Narasimham [6]) on the basis of efficiency.
2. PRELIMINARIES

Stipulations and formulas for second order rotatable designs

Suppose we want use the second order polynomial response surface design $D = ((x_{iu}))$ to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_i x_{iu}^2 + \sum_{i<j} \sum b_{ij} x_{iu} x_{ju} + \varphi_u \quad (2.1)$$

where $x_{iu}$ represents the level of $i^{th}$ factor ($i=1,2,\ldots,v$) in the $u^{th}$ run ($u=1,2,\ldots,N$) of the experiment and $\varphi_u$ are uncorrelated random error with mean zero and variance $\sigma^2$. Then ‘D’ is said to be SORD if the variance of $Y_u(x_1, x_2, \ldots, x_v)$ with respect to each of independent variable ($x_i$) is only a function of the distance $d^2 = \sum_{i=1}^{v} x_i^2$ of the point $(x_1, x_2, \ldots, x_v)$ from the origin (center) of the design such a spherical variance function for estimation of responses in the second order polynomial model is achieved if the design points satisfy the following the following conditions (cf. Box and Hunter [1]).

All odd order moments are must be zero. In their words when at least one odd power $x$’s equal to zero.

1. $\sum x_{iu} = 0$, $\sum x_{iu}^2 = 0$, $\sum x_{iu} x_{ju} = 0$, $\sum x_{iu} x_{ju}^2 = 0$, $\sum x_{iu} x_{ju} x_{ku} = 0$, $\sum x_{iu}^3 = 0$, $\sum x_{iu} x_{ju}^3 = 0$, $\sum x_{iu} x_{ju} x_{ku} x_{lu} = 0$.

   for $i \neq j \neq k \neq l$; \hspace{1cm} (2.2)

2. (i) $\sum x_{iu}^2 = \text{constant} = N\mu_2$ \hspace{1cm} (ii) $\sum x_{iu}^4 = \text{constant} = cN\mu_4$ for all $i$ \hspace{1cm} (2.3)

3. $\sum x_{iu}^2 x_{ju}^2 = \text{constant} = N\mu_4$; for all $i \neq j$ \hspace{1cm} (2.4)

4. $\frac{\mu_4}{\mu_2^2} > \frac{v}{(c+v-1)}$ \hspace{1cm} (2.5)

5. $\sum x_{iu}^4 = c \sum x_{iu}^2 x_{ju}^2$ \hspace{1cm} (2.6)
where \( \mu_4 \) and \( \mu_2 \) are constants.

The variances and covariances of the estimated parameters are

\[
V(\hat{b}_o) = \frac{\mu_4 (c+v-1)\sigma^2}{N[\mu_4 (c+v-1)-v\mu_2^2]},
\]

\[
V(\hat{b}_i) = \frac{\sigma^2}{N\mu_2},
\]

\[
V(\hat{b}_{ij}) = \frac{\sigma^2}{N\mu_4},
\]

\[
V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\mu_4}\left[\frac{\mu_4 (c+v-2)-(v-1)\mu^2_2}{\mu_4 (c+v-1)-v\mu^2_2}\right],
\]

\[
\text{Cov}(\hat{b}_o, \hat{b}_n) = -\frac{\mu_2 \sigma^2}{N[\mu_4 (c+v-1)-v\mu^2_2]},
\]

\[
\text{Cov}(\hat{b}_{ii}, \hat{b}_{ij}) = \frac{(\mu^2_2 - \mu_4)\sigma^2}{(c-1)N\mu_4[\mu_4 (c+v-1)-v\mu^2_2]} \quad \text{and other covariances vanish.} \quad (2.7)
\]

The variance of the estimated response at the point \((x_{10}, x_{20}, \ldots, x_{v0})\) is

\[
V(\hat{Y}_o) = V(\hat{b}_o) + \left[ V(\hat{b}_i) + 2 \text{ Cov}(\hat{b}_o, \hat{b}_{ii}) \right] d^2 + V(\hat{b}_{ii}) d^4 + \sum x_{i0}^2 x_{j0}^2 \left[ V(\hat{b}_{ij}) + 2 \text{ Cov}(\hat{b}_{ii}, \hat{b}_{jj}) - 2 V(\hat{b}_{ii}) \right] \quad (2.8)
\]

The coefficient of \( \sum x_{i0}^2 x_{j0}^2 \) in the above equation (2.8) is simplified to \((c-3)\sigma^2/(c-1)N\mu_4\).

A second order response surface design D is said to be rotatable, if in this design \(c=3\) and all the other conditions (2.2) to (2.7) are satisfied.

3. Main Results

3.1. SORD of first type using balanced incomplete block designs (cf. Das and Narasimham [6])

**Balanced incomplete block design:**

The parameters of BIBD denote by \((v, b, r, k, \lambda)\) is an arrangement of \(v\) treatments in \(b\) blocks
each block contains \( k \) \((<v)\) treatments, if

(i) Every treatment occurs at most once in each block

(ii) Each treatment occurs in exactly \( r \) blocks and

(iii) Every pair of treatments occurs together in \( \lambda \) times.

**Incidence matrix:**

Let \((v, b, r, k, \lambda)\) denote parameters of a BIBD. Associated with any design \( D \) is the incidence matrix \( N_{v \times b} = ((\eta_{ij})), (i=1,2,...,v; j=1,2,...,b) \) where \( \eta_{ij} \) denote the number of times the \( j^{th} \) treatment occurs in the \( i^{th} \) block.

Where

\[ \eta_{ij} = 1 \text{ if the } j^{th} \text{ treatment occurs in the } i^{th} \text{ block} \]

\[ = 0, \text{ otherwise} \]

For example the plan and incidence matrix of BIBD is given as following design \((v=3, b=3, r=2, k=2, \lambda=1)\).

**Plan of BIBD**

|   | 1 | 2 |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 |
| 1 | 3 |

**Incidence matrix of BIBD**

| Treatments | 1 | 2 | 3 |
|------------|---|---|---|
| I          | 1 | 1 | 0 |
| Blocks II  | 0 | 1 | 1 |
| III        | 1 | 0 | 1 |
Generation of design points

\[
\begin{array}{ccc}
  x_1 & x_2 & x_3 \\
  1 & 1 & 1 & 0 \\
  2 & 1 & -1 & 0 \\
  3 & -1 & 1 & 0 \\
  4 & -1 & -1 & 0 \\
  5 & 0 & 1 & 1 \\
  6 & 0 & 1 & -1 \\
  7 & 0 & -1 & 1 \\
  8 & 0 & -1 & -1 \\
  9 & 1 & 0 & 1 \\
 10 & 1 & 0 & -1 \\
11 & -1 & 0 & 1 \\
12 & -1 & 0 & -1 \\
13 & a & 0 & 0 \\
14 & -a & 0 & 0 \\
15 & 0 & a & 0 \\
16 & 0 & -a & 0 \\
17 & 0 & 0 & a \\
18 & 0 & 0 & -a \\
19 & 0 & 0 & 0 \\
\end{array}
\]

Here the total number of experimental points \(N=b2^{(k)}+2v+n_0\).

Let \((v, b, r, k, \lambda)\) denote parameters of BIBD, \(2^{(k)}\) denote a fractional replicate of \(2^k\) in +1 or -1 levels in which no interaction with less than five factors are confounded. \([1-(v, b, r, k, \lambda)]\) denote the design points generated from transpose of the incidence matrix of BIBD. \([1-(v,b,r,k,\lambda)]2^{(k)}\) are the \(b2^{(k)}\) design points generated from BIBD by “multiplication” (cf. Raghavarao [18]), pp 298-300). We use the additional set of points like \((\pm a,0,\ldots,0),(0,\pm a,0,\ldots,0),\ldots,(0,0,\ldots,\pm a)\) etc. Here \((a,0,0,\ldots,0)2^{l}\) denote the \(2v\) axial points generated from \((a,0,0,\ldots,0)\) point set. Let \(U\) denote the union of the design points generated from different sets of points, \((n_0)\) denote the number of central points. The method of SORD of first type using BIBD (cf. Das and Narasimham [6]) is given in the following result.

**Result:** The design points, \([1-(v,b,r,k,\lambda)]2^{(k)}U(a,0,\ldots,0)2^{l}U(n_0)\) will give a
v-dimensional SORD of first type using BIBD in \( N = b 2^{(k)} + 2v + n_0 \) design points, with

\[
a^4 = \frac{2^{(k)}(3 \lambda - r)}{2}.
\]

The condition for the design becomes an orthogonal design.

From equation 2 (i) of (2.3) and 3 of (2.4), we have

\[
\sum x_{iu}^2 = r 2^{(k)} + 2a^2 = N \mu_2
\]

\[
\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{(k)} = N \mu_4
\]

For the convenience \( N \) is replaced by \( M \)

By using the orthogonality condition we have

\[
\mu_2^2 = \mu_4
\]

\[
\left( \frac{r 2^{(k)} + 2a^2}{M} \right)^2 = \frac{\lambda 2^{(k)}}{M}
\]

then we can obtain

\[
a^2 = \left( \frac{\sqrt{\lambda 2^{(k)} M - r 2^{(k)}}}{2} \right) \text{(for orthogonality)}
\]

and the condition for the design becomes rotatability.

From equation 2 (ii) of (2.3) and 3 of (2.4), we have

\[
\sum x_{iu}^4 = r 2^{(k)} + 2a^4 = 3N \mu_4
\]

\[
\sum x_{iu}^2 x_{ju}^2 = \lambda 2^{(k)} = N \mu_4
\]

For the convenience \( N \) is replace by \( M \)

Then the rotatability condition equation (2.6), we have

\[
\sum x_{iu}^4 = c \sum x_{iu}^2 x_{ju}^2
\]

\[
\Rightarrow r 2^{(k)} + 2a^4 = 3(\lambda 2^{(k)})
\]

\[
\Rightarrow a^4 = \frac{2^{(k)}(3 \lambda - r)}{2} \text{(for rotatability)}
\]
3.2. Proposed method of SORD of second type using balanced incomplete block design

Following methods construction of Das and Narasimham [6] and Kim [13], Here we studied SORD of second type using BIBD is given below.

In the same design \( (v=3, b=3, r=2, k=2, \lambda=1) \) generation of design points in BIBD

\[
\begin{array}{ccc}
X_1 & X_2 & X_3 \\
1 & 1 & 1 & 0 \\
2 & 1 & -1 & 0 \\
3 & -1 & 1 & 0 \\
4 & -1 & -1 & 0 \\
5 & 0 & 1 & 1 \\
6 & 0 & 1 & -1 \\
7 & 0 & -1 & 1 \\
8 & 0 & -1 & -1 \\
9 & 1 & 0 & 1 \\
10 & 1 & 0 & -1 \\
11 & -1 & 0 & 1 \\
12 & -1 & 0 & -1 \\
13 & a_1 & 0 & 0 \\
14 & -a_1 & 0 & 0 \\
15 & 0 & a_1 & 0 \\
16 & 0 & -a_1 & 0 \\
17 & 0 & 0 & a_1 \\
18 & 0 & 0 & -a_1 \\
19 & a_2 & 0 & 0 \\
20 & -a_2 & 0 & 0 \\
21 & 0 & a_2 & 0 \\
22 & 0 & -a_2 & 0 \\
23 & 0 & 0 & a_2 \\
24 & 0 & 0 & -a_2 \\
25 & 0 & 0 & 0 \\
\end{array}
\]

Here number of central points is an integer are greater than or equal to 1 In BIBD the axial points are indicated by two numbers \( a_1 \) and \( a_2 \) and \( a_2 \geq a_1 > 0 \), the total number of experimental points are \( N=b^2+4v+n_0 \).

Let \( (v, b, r, k, \lambda) \) denote parameters of BIBD, \( 2^{f(k)} \) denote a fractional replicate of \( 2^k \) in +1 or -1 levels in which no interaction with less than five factors are confounded. [1- \( (v, b, r, k, \lambda) \)]
denote the design points generated from transpose of the incidence matrix of BIBD. Let 
$[1-(v, b, r, k, \lambda)]^{2t(k)}$ are the $b^{2t(k)}$ design points generated from BIBD by “multiplication” (cf. 
Raghavarao [18], pp 298-300). We use the additional set of points like 
$(\pm a_1, 0, \ldots, 0), (0, \pm a_1, 0, \ldots, 0), \ldots, (0, 0, \ldots, \pm a_1)$ and $(\pm a_2, 0, \ldots, 0), (0, \pm a_2, 0, \ldots, 0), \ldots, (0, 0, \ldots, \pm a_2)$ are 
two sets of axial points. Here $(a_1, 0, 0, \ldots, 0)^T U(a_2, 0, 0, \ldots, 0)^T$ denote the $4v$ axial points generated 
from $(a_1, 0, 0, \ldots, 0)$ and $(a_2, 0, 0, \ldots, 0)$ point sets. Let $U$ denote the union of the design points 
generated from different sets of points, and $(n_0)$ denote the number of central points. The method 
of construction of SORD of second type (cf. Kim [13]) using BIBD is given in the following 
theorem.

**Theorem (1):** The design points, 

$$[1-(v, b, r, k, \lambda)]^{2t(k)} U(a_1, 0, \ldots, 0)^T U(a_2, 0, \ldots, 0)^T U(n_0)$$

will give a $v$-dimensional SORD of second type using BIBD in $N=b^{2t(k)}+4v+n_0$ design points, with

$$a_1^4 + a_2^4 = \frac{2^{t(k)}(3\lambda-r)}{2} \quad \text{(for rotatability)}$$

**Proof:** For the design points generated from SORD of second type using BIBD, simple 
symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows:

\[
\sum x_{iu}^2 = r^{2t(k)} + 2(a_1^2 + a_2^2) = N\mu_2 \tag{3.3}
\]

\[
\sum x_{iu}^4 = r^{4t(k)} + 2(a_1^4 + a_2^4) = cN\mu_4 \tag{3.4}
\]

\[
\sum x_{iu}^2 x_{ju}^2 = \lambda^2 r^{2t(k)} = N\mu_4 \tag{3.5}
\]

Solving the equations (3.4) and (3.5), we get

$$a_1^4 + a_2^4 = \frac{2^{t(k)}(3\lambda-r)}{2} \quad \text{(for rotatability)}$$

**Example 1:** We illustrate the theorem (1) to obtain a SORD of second type using BIBD with 
parameters $(v=3, b=3, r=2, k=2, \lambda=1)$. The design points 
$[1-(3,3,2,2,1)]^{2t(k)} U(a_1, 0, 0, \ldots, 0)^T U(a_2, 0, 0, \ldots, 0)^T U(n_0 = 1)$ will give $v$-dimensional SORD of
second type using BIBD in N=25 design points with one central point. From (3.3), (3.4) and (3.5) we have

\[
\sum x_{iu}^2 = 8 + 2(a_1^2 + a_2^2) = N\mu_2 \quad (3.6)
\]

\[
\sum x_{iu}^4 = 8 + 2(a_1^4 + a_2^4) = cN\mu_4 \quad (3.7)
\]

\[
\sum x_{iu}^2 x_{ju}^2 = 4 = N\mu_4 \quad (3.8)
\]

From (3.7) and (3.8), we can obtain the rotatability value \(a_1^4 + a_2^4 = 2\), here we assume an arbitrary value \(a_1 = 1\), then we get \(a_2 = 1\) and \(c = 3\) (for rotatability).

The Non singularity condition (2.5), we have

\[
\frac{0.16}{0.16} > \frac{3}{3+3-1} \Rightarrow 1.0000 > 0.6
\]

Hence the non singularity condition is also satisfied.

The variance and covariance of the estimated parameters are given as fallows

\[
V(\hat{b}_0) = 0.1\sigma^2
\]

\[
V(\hat{b}_i) = 0.1\sigma^2
\]

\[
V(\hat{b}_j) = 0.25\sigma^2
\]

\[
V(\hat{b}_\mu) = 0.125\sigma^2
\]

\[
\text{Cov}(\hat{b}_0, \hat{b}_\mu) = -0.05\sigma^2
\]

\[
\text{Cov}(\hat{b}_i, \hat{b}_\mu) = 0
\]

(3.9)

The variance of the estimated response at the point \((x_{i0}, x_{20}, \ldots, x_{v0})\) is

\[
V(\hat{Y}) = 0.1539\sigma^2 + d^2(0.2433\sigma^2) + d^4(0.0403\sigma^2)
\]

(3.10)

Table 1 gives the values of the variance of the estimated responses for different factors of SORD of second type using BIBD.
Table 1: The variance of estimated response for different factors for $3 \leq v \leq 15$ (by using theorem 1)

| $(v, b, r, k, \lambda)$ | $n_0$ | $N$ | $a_1^4 + a_2^4$ | $a_1$ | $a_2$ | $V(\hat{Y})$ |
|-------------------------|-------|-----|-----------------|------|------|----------------|
| $(3,3,2,2,1)$           | 1     | 25  | 2               | 1    | 1    | $0.1\sigma^2 + d^4(0.125\sigma^2)$ |
| $(4,4,3,3,2)$           | 1     | 49  | 12              | 1    | 1.8212 | $0.0612\sigma^2 + d^4 (0.0313 \sigma^2)$ |
| $(5,5,4,4,3)$           | 1     | 101 | 40              | 1    | 2.4989 | $0.0346\sigma^2 + d^4(0.0104\sigma^2)$ |
| $(6,10,5,3,2)$          | 1     | 105 | 4               | 1    | 1.3161 | $0.0381\sigma^2 + d^4 (0.0312 \sigma^2)$ |
| $(7,7,4,4,2)$           | 1     | 141 | 16              | 1    | 1.9679 | $0.0319\sigma^2 + d^4 (0.0156 \sigma^2)$ |
| $(8,14,7,4,3)$          | 6     | 262 | 16              | 1    | 1.9679 | $0.0194\sigma^2 + d^2 (-0.0001 \sigma^2) + d^4 (0.0104 \sigma^2)$ |
| $(9,18,8,4,3)$          | 18    | 342 | 8               | 1    | 1.6265 | $0.0160\sigma^2 + d^4 (0.0104 \sigma^2)$ |
| $(10,18,9,5,4)$         | 1     | 329 | 24              | 1    | 2.1899 | $0.0183\sigma^2 + d^2 (-0.0001 \sigma^2) + d^4 (0.0078 \sigma^2)$ |
| $(11,11,5,5,2)$         | 1     | 221 | 8               | 1    | 1.6265 | $0.0294\sigma^2 + d^2 (0.0001 \sigma^2) + d^4 (0.0156 \sigma^2)$ |
| $(12,22,11,6,5)$        | 23    | 775 | 64              | 1    | 2.8173 | $0.0090\sigma^2 + d^4 (0.0031 \sigma^2)$ |
| $(13,26,12,6,5)$        | 40    | 924 | 48              | 1    | 2.5607 | $0.0081\sigma^2 + d^4 (0.0031 \sigma^2)$ |
| $(15,15,7,7,3)$         | 26    | 1046| 96              | 1    | 3.1219 | $0.0081\sigma^2 + d^4 (0.0026 \sigma^2)$ |

4. Study of Orthogonality in SORD of Second Type Using BIBD

An orthogonal design is one in which the terms in the fitted model are uncorrelated with one another and thus the parameter estimates are uncorrelated. In this case, the variance of the predicted response at any point $x$ in the experimental region, is expressible as a weighted sum of the variance of the parameter estimates in the model. For second order moments $\sum x_m^2$ and $\sum x_m^2 x_j^2$ is difficult to obtain. This is because the moments $\sum x_m^2$ and $\sum x_m^2 x_j^2$ are necessarily positive. Hence, we consider the model with the pure quadratic terms correlated for their means. In regard to orthogonality, this model is often used for the sake of simplicity in the
calculation. A design is said to be orthogonal we shall investigate the restriction
\[
\left( \sum x_{iu}^2 \right)^2 = N \sum x_{iu}^2 x_{ju} \text{ i.e. } \left( N \mu_2 \right)^2 = N \left( N \mu_4 \right) \text{ i.e. } \mu_2^2 = \mu_4
\]
to get SORD of second type using BIBD.

\[
a_1^2 + a_2^2 = \frac{\sqrt{N(\lambda 2^{(t)}) - r_2^{(t)}}}{2}
\]

(4.1)

It must be established the equation (4.1) makes SORD of second type using BIBD an orthogonal system. However \( N = b^2 + 4v + n_0 \), the value of (4.1) depends on factors \( (v), n_0 \) and the design points of SORD of second type using BIBD. The following table 2 gives the values of orthogonality of second order response surface methodology using various parameters of SORD of second type using BIBD and \( n_0 \), the value of \( 'a_1^2 + a_2^2' \) makes orthogonal second order response surface designs by using SORD of second type using BIBD.

Let \( (v, b, r, k, \lambda) \) denote parameters of BIBD, \( 2^{(t \_ k)} \) denote a fractional replicate of \( 2^k \) in \(+1\) or \(-1\) levels in which no interaction with less than five factors are confounded. \([1-(v, b, r, k, \lambda)]\) denote the design points generated from transpose of the incidence matrix of BIBD. Let \([1-(v, b, r, k, \lambda)]2^{(t \_ k)}\) are the \( b^2 \) design points generated from BIBD by “multiplication” (cf. Raghavarao [18], pp 298-300). We use the additional set of points like \((\pm a_1,0,\ldots,0),(0,\pm a_1,0,\ldots,0),\ldots,(0,0,\ldots,\pm a_1)\) and \((\pm a_2,0,\ldots,0),(0,\pm a_2,0,\ldots,0),\ldots,(0,0,\ldots,\pm a_2)\) are two sets of axial points. Here \( (a_1,0,0,\ldots,0)2^1U(a_2,0,0,\ldots,0)2^1 \) denote the \( 4v \) axial points generated from \( (a_1,0,0,\ldots,0) \) and \( (a_2,0,0,\ldots,0) \) point sets. Let \( U \) denote the union of the design points generated from different sets of points, and \( (n_0) \) denote the number of central points. The method of study of orthogonality of a SORD of second type using BIBD is given in the following theorem.

Theorem(2): The design points,

\[
[1-(v,b,r,k,\lambda)]2^{(t \_ k)}U(a_1,0,\ldots,0)2^1U(a_2,0,\ldots,0)2^1U(n_0)
\]

will give a \( v \)-dimensional SORD of second type using BIBD in \( N = b^2 + 4v + n_0 \) design points, with

\[
a_1^2 + a_2^2 = \frac{\sqrt{N(\lambda 2^{(t \_ k)}) - r_2^{(t \_ k)}}}{2} \text{ (for orthogonality)}.
\]
Proof: For the design points generated from second order rotatable designs of second type using BIBD, simple symmetry conditions (2.2) are true. Further, conditions (2.3) and (2.4) are true as follows

\[ \sum x_i^2 = r^{2n(k)} + 2(a_1^2 + a_2^2) = N \mu \]  

\[ \sum x_i^4 = r^{4n(k)} + 2(a_1^4 + a_2^4) = cN \mu \]  

\[ \sum x_i^2 x_j^2 = \lambda 2^{n(k)} = N \mu \]  

Solving equations (4.2) and (4.4) using \( \mu_2 = \mu_4 \),

\[ \left( \frac{r^{2n(k)} + 2(a_1^2 + a_2^2)}{N} \right)^2 = \frac{\lambda 2^{n(k)}}{N} \]

then we can obtain

\[ a_1^2 + a_2^2 = \frac{\sqrt{N(\lambda 2^{n(k)}) - r^{2n(k)}}}{2} \] (for orthogonality).

Example 2. We illustrate the theorem 2 second order rotatable designs of second type using BIBD with parameters \( (v=3, b=3, r=2, k=2, \lambda=1) \). The design points \[ 1-(3,3,2,2,1)] 2 \( t(k) \) \( U(a_1,0,0,0,...,0) 2 \( \lambda \) \( U(a_2,0,0,0,...,0) 2 \( n_0=1 \) \) will give a \( v \)-dimensional SORD of second type using BIBD in \( N=25 \) design points with one central point. From (3.3), (3.4) and (3.5) we have

\[ \sum x_i^2 = 8 + 2(a_1^2 + a_2^2) = N \mu \]  

\[ \sum x_i^4 = 8 + 2(a_1^4 + a_2^4) = cN \mu \]  

\[ \sum x_i^2 x_j^2 = 4 = N \mu \]

From (4.5) and (4.7), using \( \mu_2 = \mu_4 \), we can obtain the orthogonality value

\[ a_1^2 + a_2^2 = \frac{\sqrt{(25)^2 - 8}}{2} \]

\[ = 1. \]
Table 2: Values of orthogonality of SORD of second type using BIBD (using theorem 2)

| (3,3,2,2,1) | (4,4,3,3,2) | (5,5,4,4,3) | (6,10,5,3,2) |
|-------------|-------------|-------------|-------------|
| (n₀=1)      | N a₁² + a₂² | (n₀=1)      | N a₁² + a₂² | (n₀=1)      | N a₁² + a₂² | (n₀=1)      | N a₁² + a₂² |
| n₀          | 25 1        | n₀          | 49 2        | n₀          | 101 2.8138  | n₀          | 105 0.4939  |
| n₀+1        | 26 1.099    | n₀+1        | 50 2.1421   | n₀+1        | 102 2.9857  | n₀+1        | 106 0.5913  |
| n₀+2        | 27 1.1962   | n₀+2        | 51 2.2829   | n₀+2        | 103 3.1568  | n₀+2        | 107 0.6882  |
| n₀+3        | 28 1.2915   | n₀+3        | 52 2.4222   | n₀+3        | 104 3.327   | n₀+3        | 108 0.7846  |
| n₀+4        | 29 1.3852   | n₀+4        | 53 2.5602   | n₀+4        | 105 3.4965  | n₀+4        | 109 0.8806  |

| (7,7,4,4,2) | (8,14,7,4,3) | (9,18,8,4,3) | (10,18,9,5,4) |
|-------------|-------------|-------------|-------------|
| (n₀=1)      | N a₁² + a₂² | (n₀=16)     | N a₁² + a₂² | (n₀=18)     | N a₁² + a₂² | (n₀=1)      | N a₁² + a₂² |
| n₀          | 141 1.5857  | n₀          | 262 0.1714  | n₀          | 342 0.0625  | n₀          | 329 0.5534  |
| n₀+1        | 142 1.7046  | n₀+1        | 263 0.1783  | n₀+1        | 343 0.1561  | n₀+1        | 330 0.6636  |
| n₀+2        | 143 1.8231  | n₀+2        | 264 0.285   | n₀+2        | 344 0.2495  | n₀+2        | 331 0.7736  |
| n₀+3        | 143 1.9411  | n₀+3        | 265 0.3915  | n₀+3        | 345 0.3428  | n₀+3        | 332 0.8835  |
| n₀+4        | 144 2.0588  | n₀+4        | 266 0.4978  | n₀+4        | 346 0.436   | n₀+4        | 333 0.9932  |

| (11,11,5,5,2) | (12,22,11,6,5) | (13,26,12,6,5) | (15,15,7,7,3) |
|---------------|---------------|---------------|---------------|
| (n₀=1)        | N a₁² + a₂²   | (n₀=2)        | N a₁² + a₂²   | (n₀=4)        | N a₁² + a₂²   | (n₀=1)        | N a₁² + a₂²   |
| n₀            | 221 2.0476    | n₀            | 775 0.0682    | n₀            | 924 0.2498    | n₀            | 1046 0.0714   |
| n₀+1          | 222 2.1426    | n₀+1          | 776 0.1817    | n₀+1          | 925 0.3538    | n₀+1          | 1047 0.1785   |
| n₀+2          | 223 2.2374    | n₀+2          | 777 0.2952    | n₀+2          | 926 0.4578    | n₀+2          | 1048 0.2855   |
| n₀+3          | 224 2.332     | n₀+3          | 778 0.4086    | n₀+3          | 927 0.5617    | n₀+3          | 1049 0.3925   |
| n₀+4          | 225 2.4264    | n₀+4          | 779 0.522     | n₀+4          | 928 0.6655    | n₀+4          | 1050 0.4994   |
5. Efficiency Comparison for SORD of Second Type Using BIBD with SORD of First Type Using BIBD

In this section, SORD of second type using BIBD is used as the basis for estimating specific coefficient in the response surface model. SORD of second type using BIBD is compared with SORD of first type using BIBD. This comparison criterion is based on the precision at which the coefficient is estimated. It is consider that the numbers of experimental plots are required at same way.

For example in terms of estimating mixed quadratic coefficient $b_{ij}$ ($i \neq j$), two experimental designs, let's try to compare $D_1$ and $D_2$. The number of experimental plots required in $D_1$ and $D_2$ are $M$ and $N$ respectively. The relative efficiency of $D_1$ and $D_2$ is given by the following equation (see Myers [15], section 7.2).

$$E(D_1/D_2) = \frac{\left\{\text{Var}(b_{ij})\text{in } D_2\right\} N}{\left\{\text{Var}(b_{ij})\text{in } D_1\right\} M}$$

(5.1)

Where $N=b^{2(k)}+4v+n_0$ (Design points in SORD of second type using BIBD)

$M=b^{2(k)}+2v+m_0$ (Design points in SORD of first type using BIBD)

In this case, in order to compare fairly, the experimental system should make the second product equal to value of $\sum_{i=1}^{N}x_{\mu}^2$. It must be scaled and for this the following scaling criteria is used.

(i) $\frac{1}{N} \sum_{i=1}^{N}x_{\mu} = 0$

(ii) $\frac{1}{N} \sum_{i=1}^{N}x_{\mu}^2 = 1$, (i=1,2,...,v) (taking $\mu_2 = 1$)

(5.2)

5.1 Comparison in mixed quadratic coefficient $b_{ij}$ ($i \neq j$)

According to equation (2.7) $V(\hat{b}_{ij})=\frac{\sigma^2}{N\mu_4}$, but this before scaling equation (5.2) will be write in SORD of second type using BIBD. From (ii) of (5.2) $\frac{(r2^{2(k)}+2\mu^2+2\mu_2^2)}{N}$ each time making equation (5.2) will be is equal to 1, i.e. (ii) =1, then the scaling factor $g$
\[ g = \left( \frac{b^{2(\text{ii})} + 4v + n_0}{r^{2(\text{ii})} + 2a^2 + 2a^2} \right)^{\frac{1}{2}} \]  

(5.3)

However, the \( V(b_{ij}) \) is multiplied by the scaling factor ‘\( g \)’ than the \( V(b_{ij}) = \frac{\sigma^2}{N\mu_4} \cdot \frac{1}{g^2} \) that is

\[ V(b_{ij}) = \frac{\sigma^2}{N\mu_4} \left( \frac{r^{2(\text{ii})} + 2a^2 + 2a^2}{b^{2(\text{ii})} + 4v + n_0} \right)^2 \]

According to equation (5.1) the relative efficiency SORD of second type using BIBD versus SORD of first type using BIBD in the mixed quadratic coefficient of \( b_{ij} \) is obtained as follows

\[ E \left( \frac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}} \right) = \frac{\frac{\sigma^2}{N\mu_4} \left( \frac{r^{2(\text{ii})} + 2a^2}{b^{2(\text{ii})} + 2v + m_0} \right)^2 (b^{2(\text{ii})} + 2v + m_0)}{\frac{\sigma^2}{N\mu_4} \left( \frac{r^{2(\text{ii})} + 2a^2 + 2a^2}{b^{2(\text{ii})} + 4v + n_0} \right)^2 (b^{2(\text{ii})} + 4v + n_0)} \]

\[ = \frac{(r^{2(\text{ii})} + 2a^2)^2 (b^{2(\text{ii})} + 4v + n_0)}{(r^{2(\text{ii})} + 2a^2 + 2a^2)^2 (b^{2(\text{ii})} + 2v + m_0)} \]  

(5.4)

From equation (5.4) the condition that \( E \left( \frac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}} \right) > 1 \), than the SORD of second type using BIBD is more efficient than SORD of first type using BIBD

\[ a_i^2 + a_2^2 < \frac{1}{2} \left( \frac{(r^{2(\text{ii})} + 2a^2) \sqrt{b^{2(\text{ii})} + 4v + n_0 - r^{2(\text{ii})}}}{b^{2(\text{ii})} + 2v + m_0} \right) \]  

(5.5)

From the values of (3.1) and (4.1) substitute in (5.4) and then we get the value of greater than 1. From this orthogonal SORD of second type using BIBD has the same degree of efficiency as orthogonal SORD of first type using BIBD, and consider the efficiency of SORD of second type using BIBD is giving the better efficiency than SORD of first type using BIBD. Now, the efficiency comparison of SORD of second type using BIBD versus SORD of first type using BIBD with rotatability. Substituting the values of (3.2) into (5.5) and we evaluated that the SORD of second type using BIBD will be more efficient than the SORD of first type using BIBD with rotatability.
For example, in SORD of first type using BIBD when \((v=3,b=3,r=2,k=2,\lambda=1)\) and \(m_0=1\) then we get \(M = 19\) then \(a=1.1892\) and \(n_0=1\), then the equation (5.6)

\[
a_1^2 + a_2^2 < 3.7216
\]

(5.7)

Among the rotatability of SORD of second type using BIBD, it is easy to find an experimental plan that satisfies the equation (5.7). For example in SORD of second type using BIBD with \(a_1=1\), \(a_2=1\) satisfy the rotatability property equation (3.7), and \(a_1^2 + a_2^2 = 2\) as it satisfies the equation (5.7) as well, it is more efficient than the rotatability SORD of first type using BIBD. Then the relative efficiency of SORD of second type using BIBD versus SORD of first type using BIBD equation (5.4) is as follows.

\[
\left(\frac{8+2(1.4142)}{8+2(2)}\right)^2 \frac{(12+12+1)}{(12+6+1)} = 1.0714
\]

5.2 Comparison in the pure quadratic coefficient \(b_{ii}\)

Now this time in terms of estimating the pure quadratic coefficient \(b_{ii}\), the efficiency of SORD of second type using BIBD is comparing with SORD of first type using BIBD, here the scaling factor the equation (5.2) is applied. The relative efficiency SORD of second type using BIBD versus SORD of first type using BIBD is as follows based on the equation (5.1)

\[
E\left(\frac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}}\right) = \frac{\sigma^2 e_1 \left(\frac{r_2^{(k)} + 2a_i^2}{b_2^{(k)} + 2v + m_0}\right)^2 \left(b_2^{(k)} + 2v + m_0\right)}{\sigma^2 e_2 \left(\frac{r_2^{(k)} + 2a_i^2 + 2a_j^2}{b_2^{(k)} + 4v + n_0}\right)^2 \left(b_2^{(k)} + 4v + n_0\right)}
\]

\[
= \frac{e_1 \left(\frac{r_2^{(k)} + 2a_i^2}{b_2^{(k)} + 4v + n_0}\right)^2 \left(b_2^{(k)} + 4v + n_0\right)}{e_2 \left(\frac{r_2^{(k)} + 2a_i^2 + 2a_j^2}{b_2^{(k)} + 2v + m_0}\right)^2 \left(b_2^{(k)} + 2v + m_0\right)}
\]

(5.8)

Where,

\(e_1 = \nu(b_{ii})\) in SORD of first type using BIBD \hspace{1em} \text{and} \hspace{1em} \(e_2 = \nu(b_{ii})\) in SORD of second type using BIBD.
For example, in SORD of second type using BIBD of design \((v=3,b=3,r=2,k=2,\lambda=1)\), \(n_0=1\), \(a_1=1\), \(a_2=1\), \(e_2=0.125\) and in SORD of first type using BIBD \(m_0=1\), \(a=1.1892\), \(e_1=0.1947\), lets us compare the relative efficiency of SORD of second type using BIBD versus SORD of first type using BIBD, if you get the equation \((5.8)\) as \(1.6688\), then we conclude that the SORD of second type using BIBD is more efficient than SORD of first type using BIBD.

### 5.3 Comparison in terms estimating the first order coefficient \(b_i\)

It can be developed in the same process of the \(V(b_i)\) by multiplying the scaling factor then
\[
V(b_i) = \frac{\sigma^2}{(r^2+2a_1^2+2a_2^2)}
\]
is to be multiplied by \(1/g^2\) then we get,
\[
V(b_i) = \frac{\sigma^2}{(r^2+2v+n_0)}
\]simply

the \(V(b_i)\) is multiplied by scaling factor in SORD of first type using BIBD the we obtained
\[
V(b_i) = \frac{\sigma^2}{(r^2+2v+n_0)}
\]
so finally we compare the relative efficiency of
\[
E\left( \frac{\text{SORD of second type using BIBD}}{\text{SORD of first type using BIBD}} \right)
\]
and it obtained 1, then the efficiency of SORD of second type using BIBD is more efficient than SORD of first type using BIBD.

### 6. Conclusion

In this paper, SORD of second type using BIBD is developed. The variance covariance of the estimated parameters are studied and we evaluated for the SORD of second type using BIBD is most orthogonal for second order response surface designs and the results of the orthogonality are also provided in the paper.

The comparison between the SORD of second type using BIBD versus SORD of first type using BIBD for different coefficients are studied then we conclude that the SORD of second type using BIBD is more efficient than SORD of first type using BIBD. It is convenient to use the practical situations and give the more efficiency when compared to SORD of first type using BIBD.
CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] G. E. P. Box, J. S. Hunter, Multifactor experimental designs for exploring response surfaces, Ann. Math. Stat. 28 (1957), 195-241.

[2] P. Chiranjeevi, K. J. Benhurst, B. Re. Victorbabu, Second order rotatable designs of second type using central composite designs, Asian J. Probab. Stat. 11 (2021), 30-41

[3] P. Chiranjeevi, B. Re. Victorbabu, Second order rotatable designs of second type using symmetrical unequal block arrangements with two unequal block sizes, Int. J. Res. Appl. Sci. Eng. Technol. 9(2021a), 515-525.

[4] P. Chiranjeevi, B. Re. Victorbabu, Second order rotatable designs of second type using pairwise balanced designs, submitted. 2021.

[5] M. N. Das, N. C. Giri, Design and Analysis of Experiments, New Age International (P) Limited, New Delhi, India, (1999).

[6] M. N. Das, V. L. Narasimham, Construction of rotatable designs through balanced incomplete block designs, Ann. Math. Stat. 33 (1962), 1421-1439.

[7] N. R. Draper, I. Guttman, An index of rotatability, Technometrics, 30 (1988), 105-112.

[8] N. R. Draper, F. Pukelsheim, Another look at rotatability, Technometrics, 32 (1990), 195-202.

[9] P. Jyostna, B. Sulochana, B. Re. Victorbabu, Measure of modified rotatability for second order response surface designs using central composite designs, J. Math. Compt. Sci. 11 (2021), 494-519.

[10] P. Jyostna, B. Re. Victorbabu, Evaluating measure of modified rotatability for second degree polynomial using balanced incomplete block designs, Asian J. Probab. Stat. 10 (2021), 47-59.

[11] P. Jyostna, B. Re. Victorbabu, Evaluating measure of modified rotatability for second degree polynomial using pairwise balanced designs, Int. J. Res. Appl. Sci. Eng. Technol. 9 (2021b), 506-514.

[12] P. Jyostna, B. Re. Victorbabu, Evaluating measure of modified rotatability for second degree polynomial using symmetrical unequal block arrangements with two unequal block sizes, Int. J. Math. Stat. Invention. 9 (2021c), 19-28.
[13] H. J. Kim, Extended central composite designs with the axial points indicated by two numbers, Commun. Stat. Appl. Meth. 9 (2002), 595-605.

[14] A. I. Khuri, A measure of rotatability for response surface designs, Technometrics, 30 (1988), 95-104.

[15] R. H. Myers, Response surface methodology, Blacksburg, VA, (1976).

[16] S. H. Park, J. H. Lim, Y. Baba, A measure of rotatability for second order response surface designs, Ann. Inst. Stat. Math. 45 (1993), 655-664.

[17] H. J. Park, S. H. Park, Extension of central composite designs for second order response surface model building, Commun. Stat. Theor. Method. 39 (2010), 1202-1211.

[18] D. Raghavarao, Constructions and combinatorial problems in Design of Experiments, John Wiley, New York, (1971).

[19] D. Raghavarao, Construction of second order rotatable designs using using incomplete block designs, J. Indian Stat. Assoc. 1 (1963), 221-225.

[20] Ch. V. V. S. Surekha, B. Re. Victorbabu, Construction of measure of rotatable central composite designs, Int. J. Agric Stat. Sci. 7 (2011), 351-360.

[21] H. N. Tyagi, On the construction of second order and third order rotatable designs through pairwise balanced designs and doubly balanced designs, Calcutta Stat. Assoc. Bull. 13 (1964), 150-162.

[22] B. Re. Victorbabu, On second order rotatable designs – a review, Int. J. Agric. Stat. Sci. 3 (2007), 201-209.

[23] B. Re. Victorbabu, Construction of modified second order rotatable and second order slope rotatable designs using a pair of balanced incomplete block designs, Sri Lankan J. Appl. Stat. 7 (2006), 39-53.

[24] B. Re. Victorbabu, Construction of modified second order rotatable designs using a pair of symmetrical unequal block arrangements with two unequal block sizes, ANU J. Phys. Sci. 1 (2009), 73-80.

[25] B. Re. Victorbabu, V. Vasundharadevi, Modified second order response surface designs, rotatable designs using BIBD, Sri Lankan J. Appl. Stat. 6 (2005), 1-11.

[26] B. Re. Victorbabu, V. Vasundharadevi, B. Viswanadam, Modified second order response surface designs, using pairwise block designs, Adv. Appl. Stat. 6 (2006), 323-334.

[27] B. Re. Victorbabu, V. Vasundharadevi, B. Viswanadam, Modified second order response surface designs, using central composite designs, Can. J. Appl. Sci. 2 (2008), 289-294.
[28] B. Re. Victorbabu, V. Vasundharadevi, Modified second order response surface designs, rotatable designs using symmetrical unequal block arrangements with two unequal block sizes, Pak. J. Stat. 24 (2008), 67-76.

[29] B. Re. Victorbabu, Ch. V. V. S. Surekha, A note on measure of rotatability for second order response surface designs using incomplete block designs, J. Stat.: Adv. Theor. Appl. 1 (2013), 137-151.

[30] B. Re. Victorbabu, Ch. V. V. S. Surekha, A note on a measure of rotatability for second order response surface designs using balanced incomplete block designs, Thail. Stat. 13(2015), 97-110.

[31] B. Re. Victorbabu, P. Jyostna, Ch. V. V. S. Surekha, Measure of rotatability for second order response surface designs using a pair of symmetrical unequal block arrangements with two unequal block sizes, Int. J. Agric. Stat. Sci. 12 (2016), 9-11.

[32] B. Re. Victorbabu, P. Jyostna, Ch. V. V. S. Surekha, Measure of rotatability for second order response surface designs using a pair of balanced incomplete block designs, Thail. Stat. 15, (2017), 27-41.