Simulating Quantum Algorithms Using Fidelity and Coherence Time as Principle Models for Error

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As various quantum computing technologies continue to compete for quantum supremacy, several parameters have emerged as benchmarks for the quality of qubits. These include fidelity, coherence times, connectivity, and a few others. In this paper, we aim to study the importance of these parameters and their impact on quantum algorithms. We propose a realistic connectivity geometry and form quantum circuits for the Bernstein-Vazirani, QFT, and Grover Algorithms based on the limitations of the chosen geometry. We then simulate these algorithms using error models to study the impact of gate fidelity and coherence times on success of the algorithms. We report on the findings of our simulations and note the various benchmarking values which produce reliably successful results.

I. INTRODUCTION

In the years to come, the race for bigger and better quantum computers will yield a new plethora of NISQ (noisy intermediate-scale quantum) devices, nearing the milestone of 100 qubits. With major commercial players such as IBM, Rigetti, Google, Microsoft, and several others competing to drive the technological limitations of these machines, the need for classifying and benchmarking meaningful strides in the field has risen. In particular, many have recognized the shift in importance from “more qubits” to “better qubits”.

Among the various studies and reporting sources for the quality of qubits \cite{1,2}, several parameters have emerged as the benchmarks for competing hardwares. Analogous to the way in which classical computers are categorized by meaningful criteria: CPU, GPU, RAM, etc., the field of quantum computing has naturally gravitated towards the quantities: fidelity ($f$), coherence time ($T_1$, $T_2$), and connectivity. In this paper, we set out to study these parameters and develop deeper understandings as to their importance in quantum computations.

In classical computing an error with a single bit is discrete, where a bit unintentional flips between a 0 and a 1. By contrast, qubits which make up quantum computers possess the ability to have errors in several forms, some of which are not discrete. Qubit errors pertaining to amplitudes, phases, and superpositions are continuous, and their effects can propagate throughout an entire algorithm via the nature of qubit entanglement.

There have been many studies on the sources of errors on qubits, with models describing the sensitive interactions between qubits and the environment \cite{3-12}. These studies provide important foundations on which advances in hardware can be made. Simultaneously, others used these models, as well as propose new ones, in order to simulate quantum systems at both the level of physical interactions \cite{13,14} as well as quantum gates and computations \cite{17,19}. We consider this study be to in the latter camp, whereby all of our simulations are at the quantum circuit level.

In this paper, our primary motivation is to understand how errors related to gate fidelities and coherence times directly inhibit the success of quantum algorithms. Other studies have shown means for estimating, compensating, or minimizing errors related to gate fidelities \cite{20-23}, but here we propose our own general formalism for gate errors and implement them into our simulations. Similarly, we discuss how to incorporate decoherence errors, studying the degree to which spontaneous decoherent collapses can ruin quantum algorithms. For these decoherence errors, we use the parameters $T_1$ and $T_2$ to determine probabilities of collapses, based on values found experimentally \cite{5,6}.

Lastly, it is important to note that the merit in studying these quantum algorithm simulations is not in any exact values, but rather in the orders of magnitude. Specifically, the quantum circuit proposed in this paper are by no means optimal, but are designed with specific interests in mind. Different technologies, and even different quantum computers within the same technologies will differ in the parameters $f$, $T_1$, $T_2$. Our interest in this paper is not in the viability of any specific combination of such parameters, but rather the role of each parameter in improving the success of quantum algorithms.

A. Layout

In section 2 we outline the specifics of the quantum systems we simulate, namely connectivity constraints and circuit diagrams for the algorithms studied \cite{24,25}. In section 3, we outline our model for coherent noisy gates. We provide full mathematical descriptions for our implementation of these noisy gates and their relation to the parameter $f$. Section 4 contains the results of simulating the various quantum algorithms using our noisy gates model. In section 5, we outline our methodology for implementing decoherent errors into the simulations, whereby qubits probabilistically collapse according to the types of errors associated with $T_1$ and $T_2$. Section 6 discusses the results of these decoherent errors and their im-
pact on algorithm success when they are the only source of error. In section 7 we combine both of the previously studied error models, showcasing how each algorithm performs under realistic constraints. Section 8 is a concluding summary of the results found throughout the paper as well as a discussion of potential future work.

II. QUBIT GEOMETRIES AND CIRCUITS

A. Connectivity

As mentioned in the introduction, everything that will go into the simulations throughout this paper is constructed with current hardware metrics and limitations in mind. Thus, we will begin by discussing a limitation on current quantum computers that indirectly affects quantum algorithm success, qubit connectivity. Different quantum computing technologies offer various qubit connectivity, some better than others. In this paper we will be basing our simulations with superconducting qubits in mind, which typically have qubit connections around the 3-5 range.

We will propose a limited qubit geometry here that is on par with current hardware, shown below in figure 1. Consequently, for the quantum algorithms in this study, we adapt the idealized versions of these algorithms to run on this chosen geometry. Doing so requires the use of additional gates in order to carry out 2-qubit operations between qubits that do not share a direct connection.

FIG. 1: Top row of qubits marked by “Q” (dark red): computational qubits. Lower qubits marked by “a” (black): ancilla qubits. The computational qubits represent the main quantum system where each algorithm will take place, while the role of the supporting ancilla qubits is to act as intermediates for multi-qubit gate operations between computational qubits which do not have a direct connection.

The motivation for the geometry shown in figure 1 is twofold: 1) The connections shown can be mapped to several current hardware designs (for example, IBM’s 20-qubit chip “Tokyo”), requiring qubits only have at most four nearest neighbor connections. 2) This geometry is scalable up to any size for producing $2^N$ computational qubits, requiring $2^N-1$ ancilla qubits. Most importantly, higher orders of $N$ do not require more connectivity, only more qubits. The computational qubits require a connectivity of 2 (top row of qubits in figure 1), while the ancilla qubits require 4 (except for the very bottom-most ancilla qubit). For another example, a geometry for $N = 3$ is shown below in figure 2.

![Figure 2](image-url)

**FIG. 2:** $N = 3$ qubit geometry. The top layer consists of $2^3$ computational qubits, requiring 7 ancilla qubits.

The tradeoff for this scalable geometry comes in two forms: 1) 2 or 3-qubit gate operations between distant computational qubits require increasingly more intermediate quantum gates, resulting in potentially more errors from imperfect gate operations and overall longer quantum circuits. 2) Working with $2^N$ computational qubits requires a total qubit geometry of nearly double size, making the overall algorithms twice as sensitive to coherence errors.

Using figure 2 as an example, one can see that the number of connections separating some of the computational qubits is as high as five. This means that a 2-qubit gate between such qubits would require five times as many operations (often more), drastically increasing the chance of errors impacting the algorithm. Simultaneously, these longer operations require more time and qubits to achieve, opening up more possibilities for both the computational and ancilla qubits to decohere. Nevertheless, we have chosen these qubit geometries, with all their faults, such that we may study the way in which these connectivity restraints impact quantum algorithm success.

B. Algorithm Circuits

Because of limited connectivity, we must adapt the idealized versions of each quantum algorithm to fit our particular geometry choice. In general, these adapted versions follow all of the same steps as the idealized algorithms, but require additional control gates to and from the ancilla qubits. The quantum circuits for the Bernstein-Vazirani, Grover, and QFT algorithms are shown below in figures 3-6, for the case of an $N = 2$ geometry ($2^2$ computational qubits).

All of the circuits shown are the exact instructions used in our simulations. Obeying the geometry laid out in figure 1 in conjunction with the gates $X$, $H$, $T$, $R_\phi$, and $\text{CNOT}$, the circuits presented here are all in principle realizable on any available quantum computing hardware that can support the required connectivity. Thus, all of the simulation results obtained in the following sections are comparable with potential experimental results.
FIG. 3: Quantum circuit for the Bernstein-Vazirani Algorithm, shown for the case where \( a = [1, 0, 1, 0] \) (\( a \) is the hidden bit-string). Note that some of the CNOT gates in this circuit can be parallelized in order to shorten the circuit depth, but our simulations do not do so.

FIG. 4: Quantum Circuit for a Quantum Fourier Transformation. When studying this circuit throughout the paper, only the gate operations shown here are subject to fidelity and coherence errors. Additionally, we do not include the standard SWAP gates at the end of the circuit.

III. COHERENT NOISY GATES

When evaluating different quantum computing technologies in terms of quality, often times gate fidelity is the first metric people gravitate towards. Justifiably so, quantum algorithms require precise gate operations in order to maximize the advantages that superposition states allow for [27]. Thus, identifying “how good” a quantum computer’s gates are is a natural first benchmark. The parameter fidelity (\( f \)) is most often used to classify this metric, defined in several closely related ways [28–30], but generally always interpreted as the “closeness” between two quantum states. In this study we will associate the parameter \( f \) with each quantum gate, denoting how close a particular gate operation transforms a qubit(s) to the intended final state:

\[
U|\Psi\rangle = |\Phi\rangle \\
\tilde{U}|\Psi\rangle = |\phi\rangle \\
f = |\langle \Phi | \phi \rangle|^2 \tag{1}
\]

Equation (1) above shows the definition of fidelity between the two pure states \( |\Phi\rangle \) and \( |\phi\rangle \), where \( U \) is some theoretical gate operation and \( \tilde{U} \) represents an imperfect version of the same gate. \( \tilde{U} \) carries an error with it, achieving some final state differing from \( |\Phi\rangle \), which we will define as a coherent error for this paper. Specifically, a coherent error is one where an imperfect gate operation can be modeled by a unitary operator. These coherent errors result in pure states that are skewed in some way, such that their overlap with the intended output state defines the gate’s fidelity.

Supposing we would like to determine \( f \) for some unitary gate \( U \) experimentally, one simple way is to apply \( UU^\dagger \) to a qubit(s) and then make a measurement. In principle, applying such an operation should always return a qubit back to its original state, \( |0\rangle \) in most cases. However, experimentally one may occasionally find the state \( |1\rangle \), implying that the operation \( UU^\dagger \) did not transform the qubit’s state as intended (assuming one can rule out other sources of error). Repeating this process many times, one can determine an average fidelity \( \langle f \rangle \), which is the value most often reported [1–4, 7]. It is important to note that on any given individual application of some \( U \), we cannot say for certain if the operation was successful or not, thus we must most often discuss fidelities in terms of averages.
A. Coherent Amplitude Error

In the experiment just described, there are several contributing factors as to why one might measure the $|1\rangle$ state (when expecting to find $|0\rangle$). The issue is that it is very difficult to pin down quantum errors to a single source. The interactions that a qubit has with gates, other qubits, and the environment are all very delicate and intertwined. Thus, the aim of our study here is to simulate each quantum algorithm using models that isolated isolate each source of error.

To begin, our first error model focuses solely on the quantity of fidelity and its relationship to imperfect gate operations. Specifically, we will study a model for imperfect gates whereby the error occurs on the amplitudes of operations. Specifically, we will study a model for imper-

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Instead forced to work with averages. Additionally, we should assume that repeated uses of the same gate do not result in the exact same amplitude error, so we cannot assign a single $\epsilon$ for each gate.

Because average fidelities are the standard quantity often reported for quantum computers [1,4,7], we will incorporate them into our model here. In particular, our simulations will assign each individual application of a gate a randomly chosen fidelity, some better than others, ultimately averaging out to $\langle f \rangle$ for a given gate. We incorporate this randomness through the parameter $\epsilon$, whereby the simulation selects random $\epsilon$ values from some probability distribution, $P(\epsilon)$. The only requirement on this probability distribution is that it must give rise to the expected macroscopic value for $\langle f \rangle$ through random sampling. Rewriting $\mathbb{E}$ and $\mathbb{V}$ in terms of averages, we get equations which set the constraints on choosing probability distributions:

$$
\langle f_{\text{1-qubit}} \rangle = 1 - \langle \epsilon^2 \rangle 
$$

$$
\langle f_{\text{2-qubit}} \rangle = 1 - \langle \epsilon_1^2 \rangle - \langle \epsilon_2^2 \rangle + \langle \epsilon_1^2 \epsilon_2^2 \rangle
$$

So long as one samples $\epsilon$’s from a probability distribution that satisfies equations [3] and [4], the coherent error gates will reflect any value chosen for $\langle f \rangle$. Ideally then, we would like to sample from a probability distribution that accurately reflects the underlying nature of each gate’s error tendencies. However, it is difficult to say what the nature of such a distribution might be, especially when considering the same quantum gate achieved through various technologies.

### IV. FIDELITY GATES ANALYSIS

#### A. Role of Probability Distributions

In principle, although two $P(\epsilon)$ distributions may result in the same $\langle f \rangle$, the way in which they represent errors could impact a quantum algorithm differently. So then, in order to understand the role that an underlying $P(\epsilon)$ distribution may have, we will study two probability distributions that possess distinct differences. Each distribution satisfies equations [3] and [4] as well as $\langle \epsilon \rangle = 0$, implying that the average $\epsilon$ value has no bias (the errors are symmetric in the way they deviate from the intended final state). Both distributions are modeled as gaussians, but the major distinction between them lies in where the most probable $\epsilon$’s occur.

$$
P_1(\epsilon) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{\epsilon^2}{2\sigma_1^2}}
$$

$$
P_2(\epsilon) = \frac{1}{2\sqrt{2\pi}\sigma_2} e^{-\frac{(\epsilon-\bar{\epsilon})^2}{2\sigma_1^2}} e^{-\frac{(\epsilon+\bar{\epsilon})^2}{2\sigma_2^2}}
$$

$P_1(\epsilon)$ corresponds to an underlying error model where $\epsilon = 0$ is the most probable value, representing the situation where gate operations are most often close to a fidelity of 1, but the chance for large $\epsilon$ errors are still non-negligible. Conversely, $P_2(\epsilon)$ reflects the case where the most probable $\epsilon$ values are centered around the peaks $\pm \bar{\epsilon}$, which become closer to $\epsilon = 0$ as fidelity approaches 1. This distribution represents the situation where large $\epsilon$ errors are much less common, but so too are values close to $\epsilon = 0$. Plotted in figure [9] below are $P_1(\epsilon)$ and $P_2(\epsilon)$ for various fidelity values.

![Figure 9: Plots for $P_1(\epsilon)$ and $P_2(\epsilon)$ with fidelity values of 0.9 and 0.99. Both distributions extend out to the infinity, but for our simulations we only use values where $|\epsilon| < 1$.](image)

Looking at figure [9] the important difference to note is where the majority of each probability distribution is concentrated. For completeness, the values for $\sigma_1$, $\sigma_2$, and $\bar{\epsilon}$ for equations [7] and [8] are given below.

$$
\sigma_1 = \sqrt{1 - \langle f \rangle}
$$

$$
\sigma_2 = \bar{\epsilon}/4
$$

$$
\bar{\epsilon} = \sqrt{\frac{16}{17}(1 - \langle f \rangle)}
$$

Note that while $P_1(\epsilon)$ and $P_2(\epsilon)$ are true probability distributions, in our simulations we do not allow for values of $|\epsilon| > 1$. This constraint on $\epsilon$ is required by our error gate models in order to stay unitary. Incidents of $|\epsilon| > 1$ in our random sampling are thrown out and a new $\epsilon$ value is simulated. For reference, at the lower bound of our simulations of $\langle f \rangle = 0.9$, the probabilities of picking an $|\epsilon| > 1$ are $10^{-3}$ and $10^{-6}$ for $P_1(\epsilon)$ and $P_2(\epsilon)$ respectively. By $\langle f \rangle = 0.99$, these probabilities become smaller than $10^{-10}$.

For the 2-qubit gates, we again use the probability distributions [7] and [8], where we will assume the errors on each qubit are independent but still determined solely by a single average fidelity (equation [6]). Specifically, we have:
\[ P_n(\epsilon_1, \epsilon_2) = P_n(\epsilon_1)P_n(\epsilon_2) \quad n \in [1, 2] \] (12)

Equation 12 reflects that in our model the error for each qubit is independent of the other, but both contribute to the overall fidelity of the gate. The subscript \( n \) in the equation refers to the two probability distributions \( P_1 \) and \( P_2 \) Substituting \( P_1 \) and \( P_2 \) into this equation and integrating gives us the following average fidelities for \( P_1(\epsilon_1, \epsilon_2) \) and \( P_2(\epsilon_1, \epsilon_2) \):

\[ P_1(\epsilon_1, \epsilon_2) : \quad \langle f \rangle = (1 - \sigma_1^2)(1 - \sigma_2^2) \] (13)
\[ P_2(\epsilon_1, \epsilon_2) : \quad \langle f \rangle = \left(1 - \frac{17}{16}\right)^2(1 - \frac{17}{16}\bar{\epsilon}^2) \] (14)

Here, the subscripts 1 and 2 on the \( \sigma \)'s and \( \bar{\epsilon} \)'s refer to qubits 1 and 2 (1 for the control qubit, 2 for the target). Equations 13 and 14 are general, allowing for different \( \sigma \) and \( \bar{\epsilon} \) values for each of the qubits, which could be motivated experimentally. Here, we will assume that the inherent probabilities for error on the control and target qubits are equal. Setting these quantities to be equal results in the values given below.

\[ \sigma_1 = \sigma_2 = \sqrt{1 - \langle f \rangle^2} \] (15)
\[ \bar{\epsilon}_1 = \bar{\epsilon}_2 = \sqrt{\frac{16}{17}(1 - \langle f \rangle^2)} \] (16)

By substituting the values in 15 and 16 into \( P_1(\epsilon_1, \epsilon_2) \) and \( P_2(\epsilon_1, \epsilon_2) \), we now how our complete formalism for simulating coherent noisy gates with the two different underlying probability distributions for \( \epsilon \). Plotted in figure 10 are the findings of our simulations for the Bernstein-Vazirani algorithm. The figure shows the influence of \( P_1(\epsilon) \) and \( P_2(\epsilon) \) for various fidelity values, as well as the difference between them.

As figure 10 suggests, the two differing underlying probability distributions seem to have no overall impact on the success of the algorithm (we discuss our metric for determining algorithm success at the start of the next section). In the region where \( 0.9 < \langle f \rangle < 0.99 \), the difference between average successes is at most 0.008 (0.8\% success probability). These differences become negligible by the point \( \langle f \rangle = 0.99 \) and beyond. This result suggests that our fidelity model is strongly governed by \( \langle f \rangle \), and not any particular \( P_i(\epsilon) \).

As a final note, the results from figure 10 seem to suggest that any probability distribution that satisfies the condition \( \langle \epsilon \rangle = 0 \) will lead to the same average success. However, working with an underlying \( P(\epsilon) \) that does not meet this condition \( \epsilon \) values are bias towards either positive or negative values) may very likely lead to differing results. We leave this as an open question, one possibly experimentally motivated, to see the impact of physical gates that may tend to produce errors with a bias.

**FIG. 10:** (top) Average success of the Bernstein-Vazirani Algorithm for \( \langle f \rangle \) values ranging from 0.9 to 0.9999. Each data point represents the average from 10000 simulations. (bottom) The difference in success between \( P_1(\epsilon) \) and \( P_2(\epsilon) \) for each data point in the top plot.

### B. Smaller Scale Algorithms

Based on the findings from the previous section, it is clear that the influences from \( P_1(\epsilon) \) versus \( P_2(\epsilon) \) are negligible towards the overall success of the algorithms. To confirm this fact, both the Grover and QFT circuits were tested as well, showing similar results. Consequently, we will choose to have all of the remaining results from this point forward be generated using only \( P_1(\epsilon) \). The choice for using \( P_1(\epsilon) \) is motivated by simplicity reasons, electing to work with a single gaussian model versus a double.

Having settled on \( P_1(\epsilon) \) as the probability distribution for our simulations, let us now discuss the impact of these coherent error gates on the algorithms outlined in figures 3 - 6. We shall start by presenting the results for the Bernstein-Vazirani, QFT, and CCNOT circuits, shown in figure 11. While the Bernstein-Vazirani algorithm is perhaps of little practical importance, the same cannot be said about the QFT and CCNOT circuits. Several of the quantum algorithms currently thought to be contenders for true quantum supremacy [31-34] rely critically on quantum subroutines which require QFT and CCNOT. Thus, benchmarking their gate fidelity dependence is an important step towards realizing grander quantum algorithms.

For completeness, we must specify the way in which the simulations determine the success of each algorithm (which includes the results shown in figure 10). Starting with Bernstein-Vazirani, all qubits are initialized in the \( |0\rangle \) state, and the success of the algorithm is based on the probability of measuring the state \( |1010\rangle \). Specifically, let \( |\Psi_f\rangle \) be the final state at the end of the circuit, which has absorbed all of the errors from the imperfect gate operations. Then, the success of the algorithm is the quantity \( |\langle 1010|\Psi_f\rangle|^2 \).
For the CNOT circuit, we define the measure of success as the quantity $|\langle 1100|\langle 100|\Psi\rangle|^2$, where the control qubits are $Q_1$ and $Q_2$, and the target is $a_3$ (this state follows the structure $|Q_1Q_2Q_3Q_4\rangle|a_1a_2a_3\rangle$), see figure [1]. To produce this desired final state, we initialize qubits $Q_1$ and $Q_2$ in the state $|1\rangle$, and all other qubits in $|0\rangle$. These initialized qubits are done so perfectly in our simulation, ensuring that the only sources of error come from the gates outlined in [5].

For the QFT circuit, the measure of success is slightly different from the previous two. Because the QFT is often used in larger algorithms for the way in which it uniquely handles phases on each state, we have chosen to include phase into our model the success of our QFT simulations. To do this, we initialize the computational qubits in a specific superposition state $|\Psi\rangle$, which has a desired output state that contains no repeating phases:

$$|\Psi\rangle_i = \frac{1}{2} \left( |0011\rangle + |0111\rangle + |1011\rangle + |1111\rangle \right)$$

$$|\Phi\rangle = QFT|\Psi\rangle_i = \frac{1}{2} \left( |0000\rangle - i|1000\rangle - |0100\rangle + i|0001\rangle \right) (17)$$

The state shown in equation [17] is used as our metric of success for the QFT circuit, $|\langle \Phi|\Psi\rangle|^2$. The initialization of $|\Psi\rangle_i$ is done using perfect gates, isolating the QFT circuit as shown in figure [4] as the only source of error. Additionally, because the QFT circuit does not end with a measurement, and in principle may be followed by further quantum operations, we also impose a strict condition on the ancilla qubits. Specifically, because of the way in which the quantum circuit is designed, we only consider final states where the ancilla qubits are returned to the state $|00\rangle$.

In all three circuits, the quantities of interest are represented by inner products squared, which can be observed experimentally (with the exception of the phases from the QFT). However, because we are simulating these quantum systems classically, we have the advantage of being able to observe wavefunctions and amplitudes directly. Consequently, we can use the amplitudes of the desired final states to directly calculate the average probabilities of success, rather than simulating measurements. Figure [11] shows the results of our simulations, showing the average success rates for each circuit.

The algorithm results shown in figure [11] were chosen due to their similarity in fidelity dependence. In particular, all three circuits show the largest increase in success in the region $0.9 \geq \langle f \rangle \geq 0.99$, becoming dependably successful by 0.999. In terms of current NISQ hardware, 99.9% fidelity is certainly within the realm of feasibility [2] [3] [7], with perhaps the exception of the CNOT gate. However, while the averages shown above may look promising, they do not tell the whole story. Figure [12] shows the standard deviations accompanying the results in [11] revealing that individual runs of these noisy circuits can vary drastically. Even for $\langle f \rangle$ values between 0.99 and 0.999, our simulations showed frequent individual trials with successes below 50%, despite the averages being 85 - 98%.

When comparing the results in figure [12] to [11] we can see that the CCNOT and Bernstein-Vazirani circuits have nearly mirror results. Intuitively, one might expect the CCNOT circuit to have smaller standard deviations due to its higher average success, but the data in figure [11] is actually revealing a critical feature.

Because the Bernstein-Vazirani algorithm uses Hadamard gates on five out of the seven qubits, nearly all of the algorithm’s amplitude is spread evenly in superposition, only collapsing down to the $|1010\rangle$ state at the very end. During this superposition, the effects of the noisy gates appear to be distributed more evenly, leading to a smaller variance in final amplitudes. Conversely, because the CCNOT circuit deals with just three qubits, only one of which is in a superposition, the
effects of the noisy gates tend to be more pronounced.

Lastly, the results of the QFT circuit seem to follow trends distinct from the other two, largely responsible by the increased size of the algorithm. The data from figure 11 clearly shows that the increased number of gate operations impedes the algorithm’s success. Simultaneously, the notably higher standard deviations indicates that the circuit’s complexity leads to consistently varying final states. The exception to this being the region where \( \langle f \rangle < 0.99 \), where we can attribute the small standard deviations to the algorithm’s overall low average success.

C. Larger Scale Algorithms

Let us now turn our attention to the Grover Algorithm, which is considerably longer than the previous circuits. In the coming results, we will be examining the success of the Grover Algorithm at the point of each iteration. Much like the Bernstein-Vazirani and CCNOT circuits, the metric for success will be in the probability of measuring a single desired state, which by the design of the circuit will be the state \( |0101\rangle \).

The only difference between the success metric here and the ones previously studied is that the theoretical desired final state does not have a probability of 1. In particular, the theoretical probabilities of measuring the desired state are roughly 0.473, 0.908, and 0.961 after one, two, and three Grover iterations respectively. As a result, we must adjust the success metric accordingly:

| Grover Iterations | Success Metric |
|-------------------|----------------|
| 1                 | \( \frac{1}{0.473} |\langle 0101 | \Psi \rangle_f |^2 \) |
| 2                 | \( \frac{1}{0.908} |\langle 0101 | \Psi \rangle_f |^2 \) |
| 3                 | \( \frac{1}{0.961} |\langle 0101 | \Psi \rangle_f |^2 \) |

Using the adjusted success metrics shown in equation 18 (in the actual simulations we compare using values of higher decimal accuracy), we can track the success of the Grover Algorithm after each iteration. Plotted in figure 13 are the results found from our simulations.

In contrast to the plots for the smaller algorithms, the success of the Grover Algorithm is noticeably worse. This result is perhaps unsurprising, considering that the Grover Algorithm is several times larger in gate count, even containing several CCNOT gates within it. If we compare the success of this algorithm at the average fidelity point 99.9%, we can see that even optimistic quantum computing hardware would produce unreliable results.

When we compare the trends shown in figures 11 and 13, one way to look at the respective successes is in terms of the order of magnitude where we see the largest growth. While the smaller algorithms see the largest gain in success in the average fidelity region [0.9 - 0.99], the Grover Algorithm achieves similar growth in the [0.99 - 0.999] region. Thus, we can say that the success of the Grover Algorithm requires an additional order of magnitude in gate fidelity. To confirm this result once more, figure 14 shows the accompanying standard deviations to the Grover plots.

![FIG. 13](image13.png) FIG. 13: Plotted are the average values of success for the Grover algorithm after one (black circle), two (red +), and three (blue triangle) iterations. Each data point reflects the average success of the algorithms for a particular \( \langle f \rangle \) value, generated from 2000, 1500, and 1000 simulations per value for one, two, and three iterations respectively.

![FIG. 14](image14.png) FIG. 14: Plotted are the standard deviations for the three Grover iterations. Each data point corresponds to the same data used to generate the plots in figure 13, but without the factors for the success metric. Thus, the standard deviation values shown here represent the variance in amplitude found on the \( |0101\rangle \) state at the end of each iteration.

The trends found in figure 14 are in agreement with the trends shown in 13. The data shown in this figure represents the standard deviations for the unadjusted probabilities of measuring the \( |0101\rangle \) state (the same quantities in equation 18 but without the prefactors). We can see that each Grover iteration undergoes the same peak in standard deviation in the region where \( 0.99 < \langle f \rangle < 0.999 \). However, unlike the trends seen with the smaller algorithms, the Grover Algorithm doesn’t find reliable results until the gate fidelities are beyond 99.99%.
V. MODELING DECOHERENCE ERRORS

It is important to remember that imperfect gates are just one source of error that plague NISQ computers. The second major source can be categorized as decoherent errors. In contrast to the coherent errors studied up to this point, these are errors which occur spontaneously and often times cannot be described by unitary operators. There are several well documented models for the source of these decoherence errors \[5, 6, 9–12\], but our interest in this study will be a more general model and its impact on algorithm success.

When a quantum system experiences a decoherence error, what we will mean is that a qubit (or multiple qubits) has undergone a collapse in some way, a process which irreversibly disturbs the system. Specifically, there are two standard models for decoherence errors, named \(T_1\) and \(T_2\). For this study, a \(T_1\) decoherence error will represent a qubit in the \(|1\rangle\) state (excited state) collapsing to \(|0\rangle\) (ground state), while a \(T_2\) error will represent the case of a qubit in some superposition state collapsing to either \(|0\rangle\) or \(|1\rangle\).

While the names “\(T_1\)” and “\(T_2\)” are often interchangeably used to describe the errors they represent, strictly speaking \(T_1\) and \(T_2\) are lengths of time (referred to as coherence times). The standard means for determining \(T_1\) and \(T_2\) are to prepare a qubit in either an excited state (for \(T_1\)) or superposition state (for \(T_2\)), and wait for various amounts of time to determine how long a qubit can be expected to probabilistically hold onto its coherence. In this study we will model each kind of collapse as the standard probabilistic exponential decay function:

\[
P(\Delta t) = e^{-\frac{\Delta t}{T_f}}
\]  

(19)

According to this probabilistic function, the values \(T_1\) and \(T_2\) correspond to the lengths of time where one expects that a given qubit has collapsed with a probability of \(1 - 1/e\) (roughly 63%). The equation tells us that the chances of a decoherence error occurring decreases with either shorter algorithm times (\(\Delta t\)) or longer \(T_1\) and \(T_2\) decay times. While better decay rates will certainly be realized as technology continues to improve, the same cannot be said for algorithm times. Algorithms can always be optimized to try and minimize \(\Delta t\), but in principle we should expect that future advanced algorithms will inherently require larger \(\Delta t\)’s. Thus, much like the importance of gate fidelities in influencing the success of quantum algorithms, so too are the coherence times on the qubits.

A. Simulating \(T_1\) and \(T_2\)

Just like the simulations of average fidelity from earlier, the motivation for studying decoherence errors is to understand their impact on algorithm success. In particular, we will again be working with the standard metrics for benchmarking, \(T_1\) and \(T_2\), and construct models whereby they are the only parameters. It is important to mention that there are more advanced metrics for coherence times, specifically \(T_2\), but they are usually technology specific and oftentimes not publically reported. Incorporating such advanced decoherence metrics could certainly lead to more accurate results, but for this study we will choose to keep things more general, working only with the standard \(T_2\).

In order to properly implement decoherence errors into our algorithm simulations, we must be careful in the way we model collapses. In particular, we will use the parameters \(T_1\) and \(T_2\) to determine error rates according to equation [19] which will then be used to probabilistically simulate error occurrences. For the instances when a decoherence error occurs, we update the system according to the error type and the amplitudes on the states. Consider the example below, which shows how a quantum system would collapse under \(T_1\) and \(T_2\) errors in our model:

\[
|\Psi\rangle_i = \alpha|010\rangle + \beta|110\rangle + \gamma|011\rangle
\]  

(20)

\[
T_1: \text{ qubit}_2 \rightarrow |0\rangle
|\Psi\rangle_f = \alpha|000\rangle + \beta|100\rangle + \gamma|001\rangle
\]  

(21)

\[
T_2: \text{ qubit}_1 \rightarrow |0\rangle
|\Psi\rangle_f = \frac{1}{\sqrt{\alpha^2 + \gamma^2}}(\alpha|010\rangle + \gamma|011\rangle)
\]  

(22)

Beginning with the \(T_1\) error, equation [21] shows the result of qubit 2 collapsing from the \(|1\rangle\) state down to \(|0\rangle\). In our model, this type of error is only applicable when a qubit is solely in the \(|1\rangle\) state (no superposition). This type of error will be far less common in our simulations, as most algorithms require nearly all of the qubits to be in superpositions for the majority of a run. Conversely, \(T_2\) errors will make up the majority of the decoherence errors in this study, causing states and amplitudes to change as shown in equation [22]. In essence, this type of error is equivalent to a measurement, where the probability of collapsing into either \(|0\rangle\) or \(|1\rangle\) is determined by the amplitudes prior to the collapse, and the total state is normalized based on the result.

While the mathematics describing a \(T_2\) collapse are equivalent to a measurement, the importance here is that these collapses happen unbeknownst to the experimenter. When a single qubit loses its superposition spontaneously, the effects ripples throughout the rest of the algorithm and can impact future gate operations in unanticipated ways. Because we have the luxury of simulating these quantum systems classically, we can record when and where these \(T_2\) collapses happen, leading to insights otherwise unavailable through experimental means (see section VLC).
In determining when and where decoherence errors occur, our simulations work through each algorithm in “moments” to determine the $\Delta t$ for equation 19. A moment is defined as any grouping of gates that can happen in parallel, represented pictorially by gates in a vertical stack in figures 3-6 (with the exception of a few CCNOT gates in the Grover Circuit that are drawn in separate moments for display purposes). Because these gates would be physically occurring at the same time, our simulations treat them in the same way. Specifically, the $\Delta t$ for all of the qubits in a given moment is determined by the longest gate in the moment. Figure 15 shows an example of this using gates with varying $\Delta t$’s.

![Example Circuit Showing Resulting Times for Each Moment](image)

**FIG. 15:** Example circuit showing the resulting times for each moment, based on different pairings of gates. The operation with the longest gate time in a given moment determines the total time for that moment.

For our simulations, we implement occurrences of decoherence errors after all of the gates in a given moment. Since it is unclear what the model for a decoherence error during a gate implementation would be, we will elect to let gate operations happen independent of decoherence errors. Thus, after a grouping of gates have been applied in parallel, our simulation works through each qubit and randomly samples $P(\Delta t)$ based on the $\Delta t$ for that moment. Figure 15 shows an example of this using gates with varying $\Delta t$’s.

![Gates and Delta t](image)

**FIG. 15:** Example circuit showing the resulting times for each moment, based on different pairings of gates. The operation with the longest gate time in a given moment determines the total time for that moment.

VI. DECOHERENCE ANALYSIS

We will now present the results of our decoherence simulations here, focusing on noteworthy trends in the data. In all of the coming simulation results, the only sources of error for each quantum system are $T_1$ and $T_2$ collapses. All of the gate operations for this section assume perfect fidelity as to isolate the impact of the decoherence errors. In addition, for all reported $T_2$ times we set the value of $T_1 = 2T_2$, a result commonly reported for superconducting qubits. Lastly, we apply the same values for $T_1$ and $T_2$ to each qubit, assuming all qubits are of equal quality.

A. Algorithm Success

Analogous to the results shown in figures 11 and 13, our first quantity of interest will be the relation between $T_2$ and algorithm success. Because these decoherence errors are spontaneous, one can expect to have entire runs with no errors, and some with multiple. Thus, our first result will be to run each algorithm numerous times and derive trends in average algorithm success.

Plotted in figure 17 are the average success rates for the Bernstein-Vazirani and QFT circuits as a function of $T_2$. Just like the fidelity simulations from earlier, the metric for success of each algorithm is defined as $|\langle \Phi | \Psi_f \rangle |^2$. As before, $|\Phi\rangle$ is the desired final state for each algorithm and $|\Psi_f\rangle$ is the final state of the system prone to errors.

As one might expect, the simulations show that the Bernstein-Vazirani circuit is less prone to coherence errors as a result of having a shorter total circuit time. However, unlike the data from the fidelities errors, we do not see the two plots converging to 1 quite as quickly. Despite being the smallest two algorithms, their success rates only reach 98.7% and 97.3% for the maximum studied value of $T_2 = 500\mu s$.

![Circuit Times and Gates](image)

**FIG. 16:** (top) Gate times for the 1 and 2-qubit gates used in the simulations. (bottom) Breakdown of each algorithm’s total time as well as the number of 1 and 2-qubit gates. In determining the total length of time for a circuit, one must consider both the total number of moments as well as types of gate in each moment.

Using equation 23 along with the circuit diagrams, we can calculate the total times required for each algorithm. Figure 16 shows the times that we have chosen for the 1 and 2-qubit gates, as well as the resulting total times for each algorithm. These times are based on average results found from reports for 1 and 2-qubit gates on superconducting qubits.
FIG. 17: The average success rates of the Bernstein-Vazirani (black circle) and QFT (red +) as a function of coherence time $T_2$, with $T_1 = 2T_2$. Alongside the data are plots for $P(\Delta t)$, using the total $\Delta t$ for each circuit from 19. These curves represent the probability of having zero decoherence errors as a function of increasing coherence times.

probability of no error occurring for each circuit as a function of $T_2$. The reason these additional plots are of interest is because they represent the scenario in which a single decoherence error results in a 0% success probability for an algorithm. So then, the large discrepancy between these curves and the data points is indicative that these decoherence errors do not completely kill an algorithm. In the next section we will explore this topic in further detail, but first we will continue our preliminary analysis by looking at data for the Grover circuit(s), shown in figure 18.

By comparison to the results from the Bernstein-Vazirani and QFT circuits, the data in figure 18 shows a significantly worse trend. Whereas the two smaller circuits reach the 90% success mark for $T_2$ coherence times as short as 60$\mu$s and 132$\mu$s, the Grover iterations do not achieve such success until 850$\mu$s, 1490$\mu$s, and 1930$\mu$s. Just like the case of the coherent amplitude errors, we find that the increased size of the Grover circuit causes the algorithm to find success roughly an entire order of magnitude later than the smaller algorithms.

B. Total Decoherence Time

To better understand the plots shown in figures 17 and 18 it is helpful to not only consider the total time for the entire circuit, but also the total individual times for which each qubit may undergo a decoherence collapse. For example, table 16 shows that the difference in total time between the Bernstein-Vazirani and QFT circuits is roughly 5$\mu$s, but when we sum up the total amount of time in the circuits for which each individual qubit must sustain a superposition, we find the difference between the two circuits to be nearly 22$\mu$s. Table 19 shows the total times for which each circuit must endure spontaneous $T_2$ decoherence errors.

FIG. 19: The sum of the total amount of time in each circuit for which a qubit is prone to a $T_2$ decoherence error. Instances where $T_1$ collapses may occur were purposely excluded in order to show the total amount of time in which each algorithm must sustain superpositions.

Based on the way in which we have chosen to model spontaneous decoherence errors, the numbers shown in figure 19 represent the primary governing factor for the likelihood of an error in each algorithm. That is to say, substituting the times from this table into equation 19 and plotting as a function of $T_2$ will reveal curves that are nearly identical to those plotted in figure 17 (the curves for no error). By comparison, our simulations found results which are also very close in shape to exponential curves, despite being averages that incorporate many different trials spanning various combinations of errors.

In terms of general quantum algorithm analysis, knowing the total decoherence time for a quantum circuit is an important quantity to track when considering real qubits. Unlike our simulations however, the values for $T_1$ and $T_2$ typically vary from qubit to qubit on real devices. Consequently, one must consider the total coherence times on each individual qubit in order to maximize algorithm success.
C. Impact of Single Decoherence Errors

While the graphs from the previous section are good indicators into the relation between \( T \) and algorithm success, here we will delve a bit deeper into the exact nature of what these types of errors may do to the quantum systems. Often times a single decoherence error is assumed to be the death of an algorithm, but this is not necessarily always the case. Certain algorithms could in principle be designed such that a decoherence error on particular qubits has a tolerable impact on the overall success of the algorithm.

For example, consider the way in which we use ancilla qubits in the circuits for this study, often only serving a temporary purpose in the form of CNOT gates. After successfully delivering the effect of a CNOT gate between two distant computational qubits, a decoherence error on them may have little to no impact on the overall success of the algorithm.

To demonstrate that not all decoherence errors are fatal to an algorithm, for example, consider the way in which we use ancilla qubits in the circuits for this study, often only serving a temporary purpose in the form of CNOT gates. After successfully delivering the effect of a CNOT gate between two distant computational qubits, a decoherence error on them may have little to no impact on the overall success of the algorithm.

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It is clear that the biggest cost in algorithm success comes from the first decoherent collapse a quantum system experiences, after which each successive error has a diminishing effect. This turns out to be especially true for the QFT circuit, which can be seen as the least resilient algorithm to a single decoherence error. Ultimately, given enough decoherence errors in a single run, we can see

![Figure 20: Success rates for the Grover Algorithm (1 Iteration) for the situation where exactly one \( T_2 \) collapse has occurred, chosen for two distinct moments in the circuit (here we see examples of which CCNOT gates happen in parallel). The numbers in each box represent the overall success of the algorithm, defined as in equation 18. The left value in each box is the overall success of the algorithm, corresponding to the overall 10th moment (and again the sub moments in the CCNOT circuit), we now find that the success rates of the algorithm vary between 0.219 and 0.5 depending on the location and value of the collapse. These numbers tell us two interesting things: 1) The overall success of this particular algorithm is more resilient if a single \( T_2 \) error were to occur in this later moment. 2) There are select instances where a decoherence error actually results in a better final state.

To understand this second point, recall that 1 iteration of the Grover Algorithm results in a probability of measuring the desired state of about 47.3% (equation 18). This probability comes from a final state where the \( |01010⟩ \) state is most probable, and all other 15 states in the system share the remaining probability. So then, if now we imagine that a \( T_2 \) error were to happen on a qubit at the end of the circuit and its resulting collapse were to be in the desired state, this would in turn remove a portion of the non-desired states from the system and consequently boost the overall probability of the desired state.

Decoherence errors collapsing in favorable ways is certainly a rarity, and in general only applicable to certain algorithms. For example, for example, there is no single collapse which can boost an algorithm with a desired final state probability of 1 such as Bernstein-Vazirani. Nevertheless, these rare collapses in the Grover Algorithm further the claim that not all decoherence errors are fatal to an algorithm.

D. Success By Error Count

Having just seen some examples of algorithm success for cases of exactly one \( T_2 \) error, we will now turn our attention to the impact of numerous decoherence errors. Figure 21 shows the average success of each algorithm as a function of the total number of decoherence errors \( (T_1 \text{ and } T_2) \). These plots were generated from the same results used in average date trends in figures 17 and 18, but now separated by instances of various error counts.

The bar plots in figure 21 show the rate at which numerous decoherence errors impede the overall success of each algorithm. For all three of the circuits studied, it is clear that the biggest cost in algorithm success comes from the first decoherent collapse a quantum system experiences, after which each successive error has a diminishing effect. This turns out to be especially true for the QFT circuit, which can be seen as the least resilient algorithm to a single decoherence error. Ultimately, given enough decoherence errors in a single run, we can see
FIG. 21: Plotted are the average success rates for the Bernstein-Vazirani (top left), QFT (top right), and Grover (bottom) circuits as a function of total number of decoherence errors. For the Grover Algorithm, each iteration is plotted as its own color, highlighting the resilience of the algorithm for the various lengths. All three plots show that the largest drop in success occurs from the first error.

that the quantum systems reach a point where the effect of each algorithm is completely washed out and we are left with probabilities nearing an equal distribution of all states.

If we now compare the results in figure 21 with the additional plots in figure 17, we can see why the average success rates are higher than those of the zero error curves. Specifically, we can think of the data from 17 as showing the combined average of each success rate from 21 multiplied by the weight of that many errors occurring. As we increase the coherence times of $T_1$ and $T_2$, we not only increase the probability of getting a run with zero errors, but we also decrease the occurrence of multiple errors and correspondingly the lower success rates contributing to the overall average.

VII. COMBINING ERROR MODELS

We have now seen the effects of the two models for error studied in this paper: coherent amplitude errors and collapsing decoherent errors. These two error models were derived to solely incorporate the most commonly reported values for benchmarking quantum computers: $\langle f \rangle$, $T_1$, and $T_2$. In this final section, we will combine both of these error models and study their joint impact on algorithm success.

Based on the results from studying each error model in isolation, we have chosen to study their combined effect in a way which assumes a continuous improvement in both parameters. Specifically, each data point in the coming figure represents a consistent percentile improvement in both average fidelity and coherence times from the previous point. The values for $\langle f \rangle$ and $T_2$ will obey the following trends:

\[
\langle f \rangle_1 = 0.99 \quad T_{2i} = 40 \mu s
\]

\[
\langle f \rangle_k = (0.9) \langle f \rangle_{k-1} + 0.1 \quad (24)
\]

\[
T_{2k} = (1.05) T_{2k-1} \quad (25)
\]

As before, we set the value of $T_1$ to be double that of $T_2$ for all points. The motivation for combining the errors in this way is to simulate what one might expect from technological improvements on a continual basis. These chosen values then represent the scenario in which gate fidelities and coherence times improve at rates of 10% and 5% respectively, which is not unreasonable given past trends in technological improvements.

FIG. 22: Average success rate as a function of both sources of error. Each data point in the plot represents a 10% increase in average fidelity and 5% increase in coherence times from the previous point. The accompanying table shows the points at which each algorithm crosses the 90% and 95% average success threshold, and the corresponding $\langle f \rangle$ and $T_2$ values.

Figure 22 shows that the result of incorporating both error models results in noticeably worse results. Interest-
ingly, if we compare the marked thresholds of success to those in figures 17 and 18, we find that the combined error trends are closer to those of the isolated decoherence errors. This result suggests that between the two types of errors, the decoherence errors seem to outweigh the coherent amplitude errors in impeding algorithm success.

If we now consider where current NISQ devices would fall on the x-axis shown in figure 22, leading quantum computing efforts could be categorized as somewhere in the region between (0.99, 87) and (0.999, 255). If we focus on the various successes found within this region, the results indicate promising results for smaller algorithms such as QFT. This in turn suggests that algorithms which can be composed of 20-30 gate operations or less may find reasonable success in the near future.

VIII. CONCLUSION

The results found from the various simulations in this paper explore the degree to which imperfect gate operations and decoherence errors are detrimental to algorithm success. For the case where the only source of error in the system is imperfect gates, it was found that the necessary fidelities in order to achieve average success rates of greater than 90% ranged from 0.99 \( \geq \langle f \rangle \geq 0.999 \) for the smaller algorithms, and upwards of 0.9999 for the Grover iterations. Similarly, in order to achieve the same levels of average success with only decoherence errors, the simulations found that the required coherence times were of the order 100\( \mu s \geq T_2 \geq 1000\mu s \).

A. Impact By Algorithm

The results from the isolated error cases suggest that the smaller algorithms (Bernstein-Vazirani, CCNOT, and QFT) may find reasonable success on current NISQ devices. However, the results from the combined errors simulations showed that even these smaller circuits may be just barely on the cusp of feasible, requiring a combination of gate fidelities and coherence times around the order of 0.997 and 150\( \mu s \). When we compare these values to that of the latest state-of-the-art quantum devices, which promise \( \langle f \rangle \) and \( T_2 \) values around 0.995 and 50 – 100\( \mu s \), it is difficult to imagine even the smaller algorithms achieving the 90% average success percentile.

In regards to the Grover Algorithm, and its success as a function of iterations, the results from all three studies concluded that such a quantum circuit is beyond the reach of current technology. In particular, in order to run circuits with the same level of depth and gate count as those studied in this paper, our simulations show that the critical quality for improvement is \( T_2 \). This result was also found to be consistent for the smaller algorithms as well, which suggests that technological improvements in coherence times will likely result in the biggest jumps in success for near term devices. Conversely, the results from the isolated fidelity study suggest that NISQ devices may already optimistically be in the region where \( \langle f \rangle \) can produce reliable results. That being said however, our results assumed that CNOT gates could perform on the same order of precision as single qubit gates, which has yet to be demonstrated experimentally.

B. Benchmark Parameters, Models, and Future Work

The two models for error studied in this paper can be interpreted as first-order approaches to understanding the impact of noise on quantum algorithm success. In particular, the average success rates found for the various algorithms are most indicative of circuit depth and gate count. Thus, the results shown in the figures throughout this study represent estimates to the orders of magnitude on \( \langle f \rangle \), \( T_1 \), and \( T_2 \) one might require in order to expect reliable results.

Returning to the original objective of this study, we can now make some statements as to the validity of \( \langle f \rangle \), \( T_1 \), and \( T_2 \) as benchmarks for success. Because the current technologies for quantum computers are so complex and varied, the natural rise of these quantities makes sense as a means of comparing various devices at a quick glance. However, being in the unique time in quantum computing we are, NISQ devices require much more detailed parameterization in order to construct meaningful circuits. Specifically, because noise is such an unavoidable entity with these devices, any hopes of achieving near term quantum advantages will likely require algorithms that directly account for and minimize errors down to each individual qubit.

Going forward, there is room for several areas for improvement in the error models studied in this paper, in order to yield results closer to what one might find on a physical device. The most notable improvement would be to vary the values of \( \langle f \rangle \) and \( T_2 \) for the different gates and qubits (in principle, each qubit could have a complete list of \( \langle f \rangle \) values for every gate operation). Additionally, each error model could in principle be customized further to better represent the mathematical nature of errors for a specific NISQ hardware. For research efforts with specific hardware parameters in mind, it would be interesting to see the accuracy of such advanced models as compared to physical results, where each qubit is customized to match the specifications of a real device.

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