Abstract

Research has shown that widely used deep neural networks are vulnerable to carefully crafted adversarial perturbations. Moreover, these adversarial perturbations often transfer across models. We hypothesize that adversarial weakness is composed of three sources of bias: architecture, dataset, and random initialization. We show that one can decompose adversarial examples into an architecture-dependent component, data-dependent component, and noise-dependent component and that these components behave intuitively. For example, noise-dependent components transfer poorly to all other models, while architecture-dependent components transfer better to retrained models with the same architecture. In addition, we demonstrate that these components can be recombined to improve transferability without sacrificing efficacy on the original model.

1 Introduction

Due to the recent successes of Neural Networks on a wide variety of tasks, they are now being widely applied in the real-world. However, despite their major successes, recent works [9, 1] have shown that in the presence of adversarially perturbed input, they fail catastrophically. Moreover, [9, 1] showed that inputs adversarially generated for one model often cause other models to misclassify images as well, a phenomenon commonly called transferability.

Our understanding of the causes of transferability is fairly limited. [11] analyzes local similarity of decision boundaries to define a local decision boundary metric that determines how transferable adversarial examples between two models are likely to be. However, many questions are still open. A recent work [12] hypothesized that adversarial perturbations could be decomposed into initialization-specific and data-dependent components. It is also hypothesized that the data-dependent component is what primarily contributes to transfer. However, the paper provides neither theoretical nor empirical evidence to justify this hypothesis.

Our work aims to examine this hypothesis in greater detail. We first augment the previous hypothesis to provide decomposition into three parts: architecture-dependent, data-dependent, and noise-dependent components. In addition, we propose a method for decomposing adversarial perturbations into noise-dependent and noise-reduced components, as well as decomposing the noise-reduced

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component into architecture-dependent and data-dependent components. We also conduct ablation studies to show the significance of the choices made in our methodology.

2 Motivation and Approach

Motivated by the reviewers’ comments on [12], we seek to provide further evidence that an adversarial example can be decomposed into model-dependent and data-dependent portions. First, we augment our hypothesis to claim that an adversarial perturbation can be decomposed into architecture-dependent, data-dependent, and noise-dependent components. We note that it is clear that these are the only things that could contribute in some way to the adversarial example. An intuition behind why noise-dependent components exist and wouldn’t transfer despite working on the original dataset is shown in Figure 2. Not drawn above is the architecture-dependent component. As neural networks induce biases in the decision boundary, and specific network architectures induce specific biases, we would expect that an adversarial example could exploit these biases across all models with the same architecture.

2.1 Notation

We denote \( A = \{A_0, A_1, \ldots, A_k\} \) to be the set of model architectures. Let \( \mathcal{M}_i = \{\mathcal{M}_{ij}\} \) to be a set of fully trained models of architecture \( A_i \) initialized with random noise. The superscript will be omitted when architecture is clear.

We define an attack \( A(x, y, \mathcal{M}_j, L, \Delta x_0) = \Delta x \), where \( x \) is an image, \( y \) its corresponding label, \( \mathcal{M}_j \) is a neural network model as defined above, \( L \) is a loss function, \( \Delta x_0 \) the initial perturbation of \( x \), and \( \Delta x \) a perturbation of \( x \) such that \( L(\mathcal{M}_j(x + \Delta x), y) \) is maximal.

For fixed architecture \( A_i \), model \( \mathcal{M}_j \), and attack \( A \), we denote \( \Delta x_{\text{noise}}, \Delta x_{\text{arch}}, \Delta x_{\text{data}} \) to be the three components of \( \Delta x \) introduced in previous sections. Let \( \Delta x_{\text{noise reduced}} = \Delta x_{\text{arch}} + \Delta x_{\text{data}} \); we will use the short hand \( \Delta x_{\text{nr}} \).

Let \( P_{x_1}(x_2) \) denotes the projection of vector \( x_2 \) onto vector \( x_1 \). Let \( \hat{x} = \frac{x}{\|x\|} \) be the unit vector with same direction as \( x \).

2.2 \( \Delta x_{\text{noise}} \) and \( \Delta x_{\text{nr}} \) Decomposition

Description: We fix our architecture \( A_0 \) and have \( \{\mathcal{M}_1, \ldots, \mathcal{M}_n\} \) as our set of trained models. Set \( L \) to be the cross-entropy loss and let \( \Delta x = A(x, y, \mathcal{M}_1, L, 0) \) be the generated adversarial perturbation for \( \mathcal{M}_1 \).

Proposition: \( \Delta x \) can be decomposed into \( \Delta x_{\text{noise}} + \Delta x_{\text{nr}} \) such that the attack \( \Delta x_{\text{noise}} \) is effective on \( \mathcal{M}_1 \) but transfers poorly to \( \mathcal{M}_2, \ldots, \mathcal{M}_n \), while \( \Delta x_{\text{nr}} \) transfers well on all models.
We note that we start off with a set of 10 retrained ResNet18 [2] models. We empirically verify the approaches given in the motivation above and show that the isolated noise achieves a far lower transfer rate than either \( \Delta x_{nr} \) or \( \Delta x \) while still maintaining relatively high error rate on the original model, providing evidence for the success of this decomposition. To the best of our knowledge, this is the first methodology that is able to construct adversarial examples with especially low transferability. Although this is of low practical use, this is theoretically interesting. We note that although we attempt to generate \( \Delta x_{nr} \) by multi-fooling across 9 retrained models, reducing noise in high dimensions is difficult, so we are unable to achieve a perfect decomposition of \( \Delta x_{noise} \). Ablation studies in Appendix B suggest that we may be able to achieve a better decomposition with a larger set of retrained models.

### 3.1 Recombining components

As the components \( \Delta x_{noise} \) and \( \Delta x_{nr} \) are linearly independent unit vectors, and by definition, \( \Delta x \) is in the span of these vectors, we can find unique scalars \( a \) and \( b \) such that \( a \cdot \Delta x_{noise} + b \cdot \Delta x_{nr} = \Delta x \). Experimentally, we find that under our setting, \( a \approx 1.319 \) and \( b \approx 0.386 \). We note that for our original perturbation, this is perhaps an undue amount of focus paid to the noise-specific perturbation. We can now try setting \( a \) and \( b \) to different ratios, which correspond to how much we wish to emphasize attacking the original model vs. transferability. As we are now able to set an arbitrarily high \( a \) and \( b \), allowing us to saturate the epsilon constraints, we sign maximize (ie: \( \text{sign}(x) \cdot \epsilon \)), as motivated in [11] to level the playing field. The results in Table 3 show the results of performing these experiments.
Table 1: $\Delta x_{\text{noise}}$ decomposition (ResNet18)

| $\Delta x$       | $M_{\text{orig}}$ | $\{M_{\text{avg}}\}$ | $\{M_{\text{test}}\}$ |
|------------------|-------------------|------------------------|------------------------|
| $\Delta x$       | 68.3%             | 45.6%                  | 46.7%                  |
| $\Delta x_{\text{nr}}$ | 63.7%             | 61.9%                  | 59.5%                  |
| $\Delta x_{\text{noise}}$ | 60.2%             | 19.8%                  | 20.3%                  |

Table 2: $\Delta x_{\text{noise}}$ decomposition (DenseNet121)

| $\Delta x$       | $M_{\text{orig}}$ | $\{M_{\text{avg}}\}$ | $\{M_{\text{test}}\}$ |
|------------------|-------------------|------------------------|------------------------|
| $\Delta x$       | 70.0%             | 47.1%                  | 49.8%                  |
| $\Delta x_{\text{nr}}$ | 64.3%             | 65.3%                  | 66.6%                  |
| $\Delta x_{\text{noise}}$ | 64.9%             | 27.1%                  | 29.6%                  |

All numbers reported are fooling ratios. Observe that $\Delta x_{\text{noise}}$ exhibits exceptionally low transferability.

on ResNet18. We find that we are able to generate perturbations that perform equivalently with $\Delta x$ on $M_{\text{orig}}$, while performing substantially better when transferring to $\{M_{\text{avg}}\}$ and $M_{\text{test}}$.

Table 3: Linear Combinations of $\Delta x_{\text{noise}}$ and $\Delta x_{\text{nr}}$

| $b : a$ | $M_{\text{orig}}$ | $\{M_{\text{avg}}\}$ | $M_{\text{test}}$ |
|---------|-------------------|------------------------|------------------|
| $\Delta x_{\text{nr}}$ | 65.8%             | 63.6%                  | 65.1%             |
| 2:1     | 68.5%             | 63.7%                  | 65.2%             |
| 1.5:1   | 69.4%             | 61.2%                  | 62.8%             |
| 1:1     | 69.8%             | 56.0%                  | 56.4%             |
| 1:2     | 70.0%             | 53.1%                  | 53.5%             |
| $\Delta x$ | 69.8%             | 51.0%                  | 51.0%             |

All numbers reported are fooling ratios. Observe that as you increase the ratio $b : a$ we obtain better transferability with lowered effectiveness on $M_{\text{orig}}$. Also note that we are able to construct perturbations that are strictly superior to either $\Delta x$ or $\Delta x_{\text{nr}}$.

3.2 $\Delta x_{\text{arch}}$ and $\Delta x_{\text{data}}$ Decomposition

To evaluate decomposition into architecture and data, we consider the 4 models ResNet18 [2], GoogLeNet [10], DenseNet121 [4], and SENet18 [3]. In each experiment we first fix a source architecture $A_i$ and generate $\Delta x_{\text{nr}}$ by attacking 4 retrained copies of $A_i$, denoted as $\{M_{\text{source}}\}$. We then generate $\Delta x_{\text{data}}$ by attacking four copies of each $A_j \neq i$ for twelve models total. We then test on another 4 retrained copies of $A_i$ called $\{M'_{\text{source}}\}$ as well as $\{M'_{\text{other}}\}$, consisting of four copies of each of the other three architectures $\{A_j \neq i\}$. We see that for all four models, $\Delta x_{\text{arch}}$ obtains significantly higher error rate on $\{M'_{\text{source}}\}$ than on $\{M'_{\text{other}}\}$. In addition, the relative error between $\{M'_{\text{source}}\}$ and $\{M'_{\text{other}}\}$ for $\Delta x_{\text{arch}}$ are close to the relative error between $\{M_{\text{source}}\}$ and $\{M_{\text{other}}\}$ for $\Delta x_{\text{nr}}$ when averaged across models, supporting the success of our decomposition.

3.3 Ablation

Orthogonality We assume that $\Delta x_{\text{noise}}$, $\Delta x_{\text{arch}}$, and $\Delta x_{\text{data}}$ terms are orthogonal. We note that if these vectors had no relation to each other, then due to the properties of high dimensional space, they are approximately orthogonal with very high probability.

We vary orthogonality by modifying the method in 2.2 to generate $\Delta x_{\text{noise}}$ with $\Delta x - \alpha P_{\Delta x_{\text{nr}}} (\Delta x)$. When $\alpha = 1$, we recover the original algorithm, and when $\alpha = 0$, $\Delta x_{\text{noise}} = \Delta x$. We also experimentally vary the orthogonality of $\Delta x_{\text{noise}}$ and $\Delta x_{\text{nr}}$ in Table 5 and note that we achieve the greatest difference in efficacy between the original model and transferred models when they are near-orthogonal, suggesting that the assumption we made is reasonable.

However, it is not true that orthogonal components achieve the best isolation. This suggests that our current method of decomposition may simply be an approximation for the true components, and that a more nuanced method may be necessary for better isolation.

Number of Models We find that the higher the number of models are used to approximate $\Delta x_{\text{nr}}$, the more successfully we are able to isolate $\Delta x_{\text{noise}}$. Check Appendix B for full results.
Table 4: $\Delta x_{arch}$ decomposition

| source     | $\Delta x_{nr}$ | $\{M_{source}\}$ | $\{M_{other}\}$ | $\{M'_{source}\}$ | $\{M'_{other}\}$ |
|------------|-----------------|-------------------|------------------|-------------------|-------------------|
| ResNet18   | 60.9%           | 50.7%             | 59.4%            | 50.7%             |
|           | 54.6%           | 61.4%             | 54.8%            | 60.8%             |
|           | 52.4%           | 26.7%             | 36.9%            | 30.3%             |
| DenseNet121| 62.8%           | 46.2%             | 62.9%            | 47.1%             |
|           | 58.4%           | 58.3%             | 57.2%            | 55.7%             |
|           | 54.1%           | 24.4%             | 43.1%            | 26.0%             |
| GoogLeNet  | 65.3%           | 41.9%             | 65.7%            | 41.9%             |
|           | 59.5%           | 59.2%             | 59.5%            | 58.3%             |
|           | 57.9%           | 22.8%             | 44.8%            | 26.2%             |
| SENet18    | 53.8%           | 48.4%             | 53.2%            | 49.0%             |
|           | 55.7%           | 64.5%             | 54.8%            | 63.8%             |
|           | 47.1%           | 28.1%             | 38.6%            | 29.8%             |

All numbers reported are fooling ratios. Note that for all architectures, the adversarial decomposition holds. Namely, $\Delta x_{arch}$ is more transferable to its specific architecture than to others, whereas $\Delta x_{data}$ is equally transferable across architectures.

Table 5: Varying $\alpha$

| $\alpha$ | $M_{orig}$ | $\{M_{avg}\}$ | Difference |
|----------|------------|----------------|------------|
| $\Delta x$ | 68.3%      | 45.6%          | 22.7       |
| $\Delta x_{nr}$ | 63.7%      | 61.9%          | 1.8        |
| 0.1      | 66.7%      | 42.2%          | 24.5       |
| 0.5      | 63.4%      | 29.1%          | 34.3       |
| 0.8      | 59.7%      | 21.4%          | 38.3       |
| 1.0      | 52.7%      | 16.6%          | 36.1       |
| 1.2      | 51.9%      | 11.3%          | 40.6       |
| 1.5      | 42.9%      | 9.4%           | 33.5       |
| 2.0      | 33.5%      | 7.2%           | 26.3       |

All percentages reported are fooling ratios. Note that the $\alpha = 1.2$ setting is what produces maximal difference, which is slightly different from the assumed orthogonality ($\alpha = 1.0$).

4 Conclusion

We demonstrate that it is possible to decompose adversarial perturbations into noise-dependent and data-dependent components, a hypothesis reviewers thought was interesting but unsupported in [12]. We go further beyond by decomposing an adversarial perturbation into model related, data related, and noise related perturbations. A major contribution here is a new method of analyzing adversarial examples; this creates many potential future directions for research. One interesting direction would be extending these decompositions to universal perturbations [7, 8] and thus removing the dependence on individual data points. Another avenue to explore is analyzing various attacks and defenses and how they interplay with these various components.
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Appendix

A. Different attack settings

To show that our decomposition is effective across a variety of attack settings, we perform the experiment of Section 3.1 with three different iFGSM settings corresponding to $\epsilon = 0.01, 0.03, 0.06$. Results are shown in Table 6.

| $\epsilon$ | $\Delta x$ | $\{M_{orig}\}$ | $\{M_{avg}\}$ | $\{M_{test}\}$ |
|------------|------------|------------------|----------------|------------|
| $0.01$     | $\Delta x$ | 39.0%            | 16.4%          | 14.4%      |
|            | $\Delta x_{nr}$ | 25.1%            | 28.4%          | 22.2%      |
|            | $\Delta x_{noise}$ | 26.6%            | 06.2%          | 05.3%      |
| $0.03$     | $\Delta x$ | 68.3%            | 45.6%          | 46.7%      |
|            | $\Delta x_{nr}$ | 63.7%            | 61.9%          | 59.5%      |
|            | $\Delta x_{noise}$ | 60.2%            | 19.8%          | 20.3%      |
| $0.06$     | $\Delta x$ | 81.2%            | 69.7%          | 73.6%      |
|            | $\Delta x_{nr}$ | 81.1%            | 80.5%          | 85.8%      |
|            | $\Delta x_{noise}$ | 77.7%            | 39.4%          | 40.0%      |

B. Varying number of models/iterations

We investigate the effectiveness of the Section 3.1 decomposition as we vary hyper-parameters. Results for increasing iFGSM iterations in Table 7 and results for increasing the results for increasing the number of models are given in Table 8.

| # of iters | $\Delta x$ | $\{M_{orig}\}$ | $\{M_{avg}\}$ |
|------------|------------|------------------|----------------|
| $5$        | $\Delta x$ | 65.2%            | 43.4%          |
|            | $\Delta x_{nr}$ | 58.8%            | 58.8%          |
|            | $\Delta x_{noise}$ | 55.5%            | 20.6%          |
| $10$       | $\Delta x$ | 68.3%            | 46.7%          |
|            | $\Delta x_{nr}$ | 63.7%            | 61.9%          |
|            | $\Delta x_{noise}$ | 60.2%            | 19.8%          |
| $100$      | $\Delta x$ | 72.9%            | 48.6%          |
|            | $\Delta x_{nr}$ | 67.3%            | 65.2%          |
|            | $\Delta x_{noise}$ | 60.3%            | 18.7%          |

C. Justification of Equations

Justification of Equations in 3.1

Recall that the equations are given by

$$\Delta x_{nr} \approx A(x, y, \frac{1}{n-1} \sum_{j=2}^{n} M_j, \mathcal{L}, 0), \quad \Delta x_{noise} \approx \Delta x - P_{\Delta x_{nr}}(\Delta x)$$

We assume that the expected value of our noise term $\Delta x_{noise}$ is 0 over all random noise. This is motivated because the random noise $i$ at initialization is a Gaussian distribution centered at 0, and it is reasonable to assume that the model distribution and the noise distribution follows a similar pattern.
Letting $\Delta x^j = A(x, y, M_j, L, 0)$ over all random initialization $i$, we claim that $E_j[\Delta x^j] = \Delta x_{arch} + \Delta x_{data}$. Since $\Delta x_{arch}$ and $\Delta x_{data}$ are noise independent, which means that

$$
\Delta x^j = \Delta x_{noise}^j + \Delta x_{arch} + \Delta x_{data}
$$

where $\Delta x_{noise}^j$ is the noise component corresponding with the noise of model $M_j$. Therefore, it follows that

$$
E_j[\Delta x^j] = \Delta x_{arch} + \Delta x_{data} + E_j[\Delta x_{noise}^j] = \Delta x_{arch} + \Delta x_{data} = \Delta x_{nr}
$$

By the law of large numbers, it follows that $\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \Delta x^j = \Delta x_{nr}$. Therefore, we note that, for sufficiently large $n$, it follows that

$$
\frac{1}{n} \sum_{j=1}^{n} \Delta x^j \approx \Delta x_{nr}
$$

We see that, since the cross entropy loss $L$ is additive and the attack $A$ that we examine are first order differentiation methods, we have

$$
L(\frac{1}{n-1} \sum_{j=2}^{n} M_j(x), y) = \frac{1}{n-1} \sum_{j=2}^{n} L(M_j(x), y)
$$

$$
\implies A(x, y, \frac{1}{n-1} \sum_{j=2}^{n} M_j, L, 0) = \frac{1}{n-1} \sum_{j=2}^{n} A(x, y, M_j, L, 0) \approx \Delta x_{nr}
$$

To prove the other claim, we have already shown through empirical results and an intuition that $\Delta x_{noise}$ and $\Delta x_{nr}$ are linearly independent that $\Delta x_{noise}$ and $\Delta x_{nr}$ are very close to orthogonal and compose $\Delta x$. Therefore, it follows that we can take the use the projection of $\Delta x_{nr}$ implies that

$$
\Delta x_{noise} + P_{\Delta x_{nr}}(\Delta x) \approx \Delta x \implies \Delta x_{noise} \approx \Delta x - P_{\Delta x_{nr}}(\Delta x)
$$

up to a scaling constant.

**Justification of Equations in 3.2**

Recall that the equations are, given $\Delta_{nr}$ generated on $A_0$,

| # of models | $\Delta$ | $\{M_{orig}\}$ | $\{M_{avg}\}$ | $\{M_{test}\}$ |
|-------------|----------|----------------|----------------|----------------|
| 3           | $\Delta x$ | 69.4% | 46.6% | 45.6% | $\Delta x_{nr}$ | 57.6% | 62.1% | 51.9% | $\Delta x_{noise}$ | 60.1% | 24.9% | 29.2% |
| 5           | $\Delta x$ | 68.3% | 45.6% | 46.7% | $\Delta x_{nr}$ | 60.1% | 62.0% | 55.2% | $\Delta x_{noise}$ | 57.5% | 22.4% | 24.6% |
| 10          | $\Delta x$ | 63.7% | 61.9% | 59.5% | $\Delta x_{nr}$ | 60.2% | 19.8% | 20.3% | $\Delta x_{noise}$ | 60.2% | 19.8% | 20.3% |
\[ \Delta x_{data} \approx \mathbb{E}_{A_i \in A} \left[ A(x, y, \frac{1}{n-1} \sum_{j=2}^{n} \mathcal{M}_j, \mathcal{L}, \Delta x_{nr}) \right] , \quad \Delta x_{arch} \approx \Delta x_{nr} - P_{\Delta x_{data}}(\Delta x_{nr}) \]

We make two core assumptions:

- The value of \( \Delta \mathbb{E}_A[x_{arch}] = 0 \). This is a reasonable assumption since our generated architectures \( A \) should produce roughly symmetric error vectors \( x_{arch} \).

- \( A(x, y, \frac{1}{n-1} \sum_{j=2}^{n} \mathcal{M}, \mathcal{L}, \Delta x') \) is equivalent \( A(x, y, \mathcal{M}, \mathcal{L}, 0) \) in the sense that the former produces a noised reduce gradient closer to \( \Delta x' \). This is reasonable because the space of there are many adversarial perturbations (different directions) and changing our start location won’t cripple our search space. Furthermore, we use this to generate a \( \Delta x_{nr} \) close to \( \Delta x' \).

We claim that \( \mathbb{E}_A[\Delta x_{nr}] = \Delta x_{data} \) where we take \( \Delta x_{nr} \) over architecture \( A \). To see this, we note that

\[
\mathbb{E}_A[\Delta x_{nr}] = \mathbb{E}_A[\Delta x_{arch}] + \mathbb{E}_A[\Delta x_{data}] = \mathbb{E}_A[\Delta x_{data}]
\]

and so again we can approximate it with

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \Delta x_{nr}^i = \Delta x_{data} \] 

where \( \Delta x_{nr}^i \) is the \( \Delta x_{nr} \) component generated for model \( A_i \). For sufficiently large \( n \), it follows that

\[
\frac{1}{n} \sum_{i=1}^{n} \Delta x_{nr}^i = \Delta x_{data}
\]

Therefore we have

\[
\Delta x_{data} = \mathbb{E}_{A_i \in A} [\Delta x_{nr}^i] = \mathbb{E}_{A_i \in A} \mathbb{E}_{j} [(x, y, \mathcal{M}_j^i, \mathcal{L}, 0)] \approx \mathbb{E}_{A_i \in A} \left[ A(x, y, \frac{1}{n-1} \sum_{j=2}^{n} \mathcal{M}_j^i, \mathcal{L}, 0) \right]
\]

and by our assumption this is roughly equivalent to

\[
\mathbb{E}_{A_i \in A} \left[ A(x, y, \frac{1}{n-1} \sum_{j=2}^{n} \mathcal{M}_j^i, \mathcal{L}, \Delta x_{nr}) \right]
\]

as desired. To prove the other claim, we use an analogous argument to the one above as we have shown that \( \Delta x_{arch} \) and \( \Delta x_{data} \) are orthogonal and applying the same projection technique yields

\[
\Delta x_{data} + P_{\Delta x_{data}}(\Delta x_{nr}) \approx \Delta x_{nr} \implies \Delta x_{arch} \approx \Delta x - P_{\Delta x_{data}}(\Delta x_{nr})
\]

up to a scaling constant.