ON LINEAR SYSTEMS OF CURVES ON RATIONAL SCROLLS

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Abstract. In this paper we prove a conjecture on the dimension of linear systems, with base points of multiplicity 2 and 3, on an Hirzebruch surface.

1. Introduction

Consider \( r \) points in general position on an algebraic surface \( S \), to each one of these points \( p_i \) we associate a natural number \( m_i \) called the multiplicity of the point. Let \( r_j \) be the number of \( p_i \) with multiplicity \( m_i \) and let \( \mathcal{L} \) be the complete linear system associated to the line bundle \( L \in \text{Pic}(S) \). By \( \mathcal{L}(m_1^{r_1}, \cdots, m_k^{r_k}) \) we mean the linear system of curves in \( \mathcal{L} \) with \( r_j \) general base points of multiplicity at least \( m_j \) for \( j = 1 \cdots k \). Now, define the effective dimension of the system to be:

\[
l(m_1^{r_1}, \cdots, m_k^{r_k}) = \dim \mathcal{L}(m_1^{r_1}, \cdots, m_k^{r_k}),
\]

and the virtual dimension to be:

\[
v(m_1^{r_1}, \cdots, m_k^{r_k}) = \dim \mathcal{L} - \sum r_i \frac{m_i(m_i + 1)}{2}.
\]

Observe that it may happen that \( v < -1 \), in this case \( v \) does not represent the dimension of a linear system and instead of it we may define the expected dimension \( e = \max\{v, -1\} \). From the previous definitions it follows immediately that for a given system we have:

\[
1 \leq e \leq l.
\]

Let \( Z \) be the 0-dimensional scheme defined by the multiple points \( p_i \) and consider the exact sequence of sheaves:

\[
0 \to \mathcal{I}(Z) \to \mathcal{O}_S \to \mathcal{O}_Z \to 0
\]
where $\mathcal{I}(Z)$ is the ideal sheaf of $Z$. Tensoring with $L$ and taking cohomology we obtain:

$$
0 \rightarrow H^0(L \otimes \mathcal{I}(Z)) \rightarrow H^0(L) \rightarrow H^0(L_Z) \rightarrow \\
\rightarrow H^1(L \otimes \mathcal{I}(Z)) \rightarrow H^1(L) \rightarrow 0.
$$

In this way we see that

$$
h^1(L \otimes \mathcal{I}(Z)) - h^1(L) = l(m_1^{r_1}, \ldots, m_k^{r_k}) - v(m_1^{r_1}, \ldots, m_k^{r_k})
$$

Observe that the second inequality of (2) may be strict since the conditions imposed by the points may fail to be independent. In this case we say that the system is special. The aim of this paper is to give a characterization of special systems on rational scrolls and a complete classification will be given in the case of homogeneous systems of multiplicity $\leq 3$.

The paper is organized as follows:

In section 2 we give some preliminaries on rational scrolls and linear systems on them. Then we give an example of special systems $L$ whose base locus contains a particular kind of multiple curves which we will call $(-1)$-curves. This will allow us to make a general conjecture on the structure of special linear systems. In section 3 we restrict our attention to linear systems with point of equal multiplicity, which we call homogeneous. Then we give a classification of homogeneous $(-1)$-curves of multiplicity 1. In section 4 we study a class of birational transformations from $\mathbb{F}_n \rightarrow \mathbb{F}_{n-1}$ which are a natural generalization of quadratic transformations on $\mathbb{P}^2$. In section 5 we give the complete list of homogeneous $(-1)$-special systems with multiplicity $m \leq 3$. In section 6 we describe the fundamental technique, which consists in a degeneration of the $\mathbb{F}_n$ to a suitable reducible surface. In section 7 we prove the conjecture for all homogeneous systems systems with base points of multiplicity $\leq 3$.

2. Preliminaries

Consider an $\mathbb{F}_n$ surface, i.e. $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ with $n \in \mathbb{N}$ and let $F, H$ be the two generators of $\text{Pic}(\mathbb{F}_n)$ such that $F^2 = 0$, $H^2 = n$, $F \cdot H = 1$. With $\Gamma_n$ we indicate the $(-n)$-curve of $\mathbb{F}_n$ (i.e. the rational curve $\Gamma_n \in |H - nF|$ of self intersection $-n$). Let $L = aF + bH$ be a divisor, with $B_s \mid L \mid$ we mean the base locus of the linear system $|L|$. 
Proposition 2.1. On an $\mathbb{F}_n$ surface, the ample divisors coincide with the very ample ones and are of the form $aF + bH$ with $a, b > 0$.

Proof. [8, Cap. V, 2.18]: □

Proposition 2.2. Let $L = aF + bH$ be an effective divisor on an $\mathbb{F}_n$ surface, then it is linearly equivalent to a divisor of the form: $q\Gamma_n + a_1F + b_1H$, with $B_s \mid L = q\Gamma_n$ and $a_1, b_1 \geq 0$.

Proof. Consider the product $\cdot \cdot \cdot \rightarrow \mathcal{O}_{\mathbb{F}_n}(-F) \rightarrow \mathcal{O}_{\mathbb{F}_n} \rightarrow \mathcal{O}_F \rightarrow 0$,
tensoring with $\mathcal{O}_{\mathbb{F}_n}(aF + bH)$ we obtain surjective maps for each $a \in \mathbb{Z}$:
$$\cdots \rightarrow H^1((a-1)F + bH) \rightarrow H^1(aF + bH) \rightarrow 0.$$ 
Again from the sequence:
$$0 \rightarrow \mathcal{O}_{\mathbb{F}_n}(-H) \rightarrow \mathcal{O}_{\mathbb{F}_n} \rightarrow \mathcal{O}_H \rightarrow 0,$$
tensoring with $\mathcal{O}_{\mathbb{F}_n}(bH)$, with $b \geq 0$, we obtain:
$$\cdots \rightarrow H^1((b-1)H) \rightarrow H^1(bH) \rightarrow 0.$$ 
Now $h^1(-H) = 0$ since $H^1(\mathcal{O}_{\mathbb{F}_n}) = 0$ and the map $H^0(\mathcal{O}_{\mathbb{F}_n}) \rightarrow H^0(\mathcal{O}_H)$
is an isomorphism, hence $h^1(aF + bH) = 0$.
By Riemann-Roch and the preceding discussion, it follows that $h^0(aF + bH) = ((aF + bH)^2 - (aF + bH) \cdot K)/2 + 1 = (b+1)(2a + 2 + nb)/2$. □
Let \( \mathcal{L}_n(a, b, m_1 r_1, \ldots, m_k r_k) \) denote the linear system \(| aF + bH |\) on the \( \mathbb{F}_n \) surface with \( r_i \) base points of multiplicity \( m_i \). By a generality assumption, we may suppose that no one of these points lies on the \((-n)\)-curve \( \Gamma_n \), hence, by [2,2] we are lead to consider only systems with \( a, b \geq 0 \).

In what follows we will assume that our \( r \) points on \( \mathbb{F}_n \) will be in general position; this means that we choose the points in such a way that each linear systems with these simple base points in non-special. The scheme that parameterize such points is a Zariski open set of the scheme that parameterize \( r \) points on an \( \mathbb{F}_n \) surface (this is why we use the word "generic").

**Proposition 2.4.** A linear system \( \mathcal{L} \) on an \( \mathbb{F}_n \) surface, with base points of multiplicity one and in general position, is non-special.

Under the assumption of general position we consider a particular class of special system, obtained in the following way:

Consider the blow up \( S \) of \( \mathbb{F}_n \) at \( p_1, \ldots, p_r \) and denote by \( \mathcal{L} \) the strict transform of the system \( \mathcal{L}_n(a, b, m_1 r_1, \ldots, m_k r_k) \). Define the virtual and the expected dimension of \( \mathcal{L} \) as those of \( \mathcal{L}_n(a, b, m_1 r_1, \ldots, m_k r_k) \). By Riemann-Roch, the virtual dimension of \( \mathcal{L} \) may be given in the following way:

\[
v(\mathcal{L}) = \frac{\mathcal{L}^2 - \mathcal{L} \cdot K_S}{2},
\]

where \( K_S \) denote the canonical bundle of \( S \). We recall that \( E \) is a \((-1)\)-curve on \( S \) if \( E \) is irreducible and \( E^2 = E \cdot K_S = -1 \). Suppose that \( \mathcal{L} \cdot E = -t < 0 \), then \( tE \) is contained in the fixed part of \( \mathcal{L} \). Let \( \mathcal{M} = \mathcal{L} - tE \) be the residual system, then:

\[
v(\mathcal{L}) = \frac{\mathcal{L}^2 - \mathcal{L} \cdot K_S}{2} = \frac{(\mathcal{M} + tE)^2 - (\mathcal{M} + tE) \cdot K_S}{2} = v(\mathcal{M}) + \frac{t - t^2}{2}
\]

So if \( t \geq 2 \) and \( \mathcal{L} \) is not empty, then \( l(\mathcal{L}) = l(\mathcal{M}) \geq v(\mathcal{M}) > v(\mathcal{L}) \), which means that the system \( \mathcal{L} \) is special.

If \( t = 1 \), then \( v(\mathcal{L}) = v(\mathcal{M}) \), but it may happen that \( \mathcal{M} \cdot \Gamma_n < 0 \). In this case the \((-n)\)-curve \( \Gamma_n \) is contained in the fixed part of \( \mathcal{L} \) and the system may again be special. For example consider the system \( \mathcal{L}_6(0, 4, 3^{11}) \), this has expected dimension equal to \(-1\). Consider the \((-1)\)-curve \( E = \mathcal{L}_6(2, 1, 1^{11}) \), then \( \mathcal{L} \cdot E = -1 \). The system \( \mathcal{L} - E = \)
\( \mathcal{L}_6(-2,3,2^{11}) \) has again a fixed part, since it has negative product with the \((-n)\)-curve. The residual system is \( \mathcal{L} - E - \Gamma_n = \mathcal{L}_6(4,2,2^{11}) \), and this is exactly \( 2E \), hence \( \mathcal{L} = \Gamma_n + 3E \) and the initial system is not empty.

Given a linear system \( \mathcal{L} \), consider the following procedure:

1) if it does exist a \((-1)\)-curve \( E \) such that \( t := \mathcal{L} \cdot E < 0 \), then substitute \( \mathcal{L} \) with \( \mathcal{L} - tE \) and goto step 1), else goto step 2).

2) if \( \mathcal{L} \cdot \Gamma_n < 0 \) then substitute \( \mathcal{L} \) with \( \mathcal{L} - \Gamma_n \) and goto step 1), else finish.

After this procedure, that must obviously ends in a finite number of steps, we have a new linear system \( \mathcal{M} \):

**Definition 2.5.** A linear system \( \mathcal{L} \) is \((-1)\)-special if \( v(\mathcal{L}) > v(\mathcal{M}) \)

For linear systems on \( \mathbb{P}^2 \) a conjecture due to Hirschowitz says that the only special systems are the \((-1)\)-special ones. Reformulating this conjecture on \( \mathbb{F}_n \) surfaces we have:

**Conjecture 2.6.** A linear system \( \mathcal{L}_n(a,b,m_1^{r_1},\cdots,m_k^{r_k}) \) is special if and only if it is \((-1)\)-special.

If \( m_1 = \cdots m_k = m \) the system is called homogeneous of multiplicity \( m \). The aim of this paper is to prove the conjecture for homogeneous systems with \( m \leq 3 \). This conjecture may be easily proved in the following case:

**Proposition 2.7.** The conjecture is true on an \( \mathbb{F}_n \) surface if the number \( r \) of points is less then or equal to \( n + 3 \).

*Proof.* It is enough to prove the conjecture when \( r = n + 3 \), so from now on all the sums are intended to be over the \( n + 3 \) points.

Let \( \pi : S \to \mathbb{F}_n \) be the blow-up map and observe that \( -K_S = N + \Delta \),

where \( N := \pi^*(\frac{2}{n}\Gamma_n + H + 2F) - \sum_i E_i \) and \( \Delta := \pi^*(\frac{n-2}{n}\Gamma_n) \). Moreover \( N \) is a nef and big divisor.

Given a non-empty linear system \( \mathcal{L} \) which is not \((-1)\)-special, we may always assume that it does not have negative product with any \((-1)\)-curve and with \( \pi^*\Gamma_n \). What we want to prove in this case is that \( h^1(L) = 0 \).

Let \( L \) be the line bundle associated to the proper transform of the system \( \mathcal{L} \) on the blow-up surface \( S \). If \( C \) is an irreducible curve such that \( L \cdot C < 0 \), then \( C \) is contained in the fixed part of \( \mathcal{L} \) and \( C^2 < 0 \).
This means that $2g(C) - 2 = C^2 + C \cdot K_S < 0$, which implies that $g(C) = 0$ and $C^2$ equal $-1$ or $-2$. By hypothesis $C^2$ cannot be a ($-1$)-curve so that it has to be a ($-2$)-curve. From $0 = C \cdot (N + \Delta)$ and the fact that $N$ is nef we deduce that $C \cdot \Delta = 0$ so that $C \cdot (\pi^*(H + 2F) - \sum E_i) = 0$. The last equality implies that $C \cdot (\pi^*(H + F) - \sum E_i) < 0$ which is a contradiction since $\pi^*(H + F) - \sum E_i$ is a ($-1$)-curve, so that it is irreducible and distinct from $C$.

We proved that $L$ is nef so that $N + L$ is nef and big. We conclude by observing that $L = K_S + (L - K_S) = K_S + (L + N) + \Delta$, so that $h^1(L) = 0$, by Kawamata-Viehweg vanishing theorem (see [10, Theorem 9.1.18]).

3. ($-1$)-Curves

In what follow, with abuse of language, we call ($-1$)-curve also the curves of $\mathbb{F}_n$ whose strict transform is a ($-1$)-curve on the blow up surface $S$. In order for $E \in \mathcal{L}_n(a, b, m^r)$ to be a ($-1$)-curve, we must have:

\begin{equation}
\begin{cases}
E^2 = -1 \\
E \cdot K = -1
\end{cases}
\end{equation}

which leads to the system:

\begin{equation}
\begin{cases}
2ab + nb^2 - rm^2 + 1 = 0 \\
2a + nb + 2b - rm - 1 = 0
\end{cases}
\end{equation}

By eliminating $a$, we obtain

\begin{equation}
2b^2 - rmb - b + rm^2 - 1 = 0.
\end{equation}

This is the equation of an irreducible conic in the coordinates $b, r$ for $m > 1$. For $m > 1$, observe that $r = 0$, $b = 1$ is an integral solution of (5), hence we obtain a rational parameterization which gives us all the rational solutions of (5). Let $b = rt + 1$ where $t = p/q$ with $p, q \in \mathbb{Z}$ and gcd$(p, q) = 1$, substituting in (5) we obtain:

\[ b = \frac{m^2p + q}{mp - 2q} \quad \text{and} \quad r = \frac{p}{q} \frac{p(m^2 - m) + 3q}{pm - 2q}. \]

Now we look for integral solutions, from the expression of $r$ it follows that $q \mid m(m - 1)$ because $p$ and $q$ are coprime and $q \mid p(m^2 - m) + 3q$. This allows us to calculate all the possible values of $q$ and also all the integral solutions of (5) for a given $m$.

For $m = 1$, $2b^2 - rb - b + r - 1 = (b - 1)(2b + 1 - r)$. $b = 1$ gives the solution: $\mathcal{L}_n(e, 1, l^{2e+n+1})$. If $r = 2b + 1$ substituting in the second
equation of \( \mathcal{L} \) we obtain \( a = (2 - nb)/2 \), which is negative unless one has either \( b \leq 2 \) or \( n = 0 \). If \( n = 0 \) then \( a = 1 \) and the system is \( \mathcal{L}_0(1, b, 1^{2b + 1}) \) which is the same as \( \mathcal{L}_0(b, 1, 1^{2b + 1}) \) since on \( \mathbb{F}_0 \), \( F \) and \( H \) have the same properties. so this system is already contained in \( \mathcal{L}_n(e, 1, 1^{2e + n + 1}) \). If \( n > 0 \) the systems are: \( \mathcal{L}_n(1, 0, 1), \mathcal{L}_2(0, 1, 1^3), \mathcal{L}_1(0, 2, 1^5) \). Also \( \mathcal{L}_2(0, 1, 1^3) \) is contained in \( \mathcal{L}_n(e, 1, 1^{2e + n + 1}) \). These are all the homogeneous \((-1\))-curves with multiplicity one:

\[
\begin{array}{ccc}
\mathcal{L}_1(0, 2, 1^5) & \mathcal{L}_n(1, 0, 1) & \mathcal{L}_n(e, 1, 1^{2e + n + 1}) \\
\end{array}
\]

4. Elementary Transformations

In this section we consider a well known class of birational transformations from an \( \mathbb{F}_n \) surface to an \( \mathbb{F}_{n-1} \): the elementary transformation. These transformations have two good properties for our scope:

- The virtual dimension of linear systems are preserved through any such transformation.
- The \((-1\))-curves goes into \((-1\))-curves.

Sometimes one has also the following:

- The effective dimension of linear systems is preserved.

These transformations, allow us in some cases, to translate the problem of homogeneous linear systems on \( \mathbb{F}_n \) to a problem of quasi-homogeneous linear systems on \( \mathbb{P}^2 \).

In order to understand the action of such transformations on linear systems, consider a surface \( \mathbb{F}_n \) and a point \( p \) not on the \((-n\))-curve. Blow up the fiber through \( p \) and then blow down the strict transform of the fiber. Consider the system: \( \mathcal{L}_n(a, b, m) \) with only one singular point \( p \). Let \( \pi_1 \) be the blow up of \( p \) with exceptional divisor \( E \). Let \( H_n, F_n, H_{n-1}, F_{n-1} \) be the generators of Pic(\( \mathbb{F}_n \)) and Pic(\( \mathbb{F}_{n-1} \)) respectively, with \( H_i^2 = i, H_i \cdot F_i = 1, F_i^2 = 0 \) with \( i \in \{n-1, n\} \). Let \( F \) be the strict transform of the fiber \( F_n \) through the point \( p \) and let \( H = \pi_1^* H_n \). Now let \( \pi_2 \) be the blow down of \( F \), we obtain:

\[
\pi_1^* H_n = H, \pi_1^* F_n = E + F, \pi_2^* H_{n-1} = H - E, \pi_2^* F_{n-1} = E + F.
\]

The strict transform of a curve belonging to the system \( \mathcal{L}_n(a, b, m) \) is

\[
a \pi_1^* F_n + b \pi_1^* H_n - m E = a(F + E) + bH - mE = (a - m + b)(E + F) + b(H - E) - (b - m) F = (a - m + b)\pi_2^* F_{n-1} + b\pi_2^* H_{n-1} - (b - m) F.
\]

After blowing down it becomes \( \mathcal{L}_{n-1}(a - m + b, b, b - m) \). Observe that if
$b > m$ then in the last system the point of multiplicity $b - m$ belongs to the $1 - n$ curve. This gives the following transformation for $k < n$ points:

\begin{equation}
L_n(a, b, m^r) \rightarrow L_{n-k}(a + k(b - m), b, m^{n-k}, (b - m)^{k})
\end{equation}

It is a simple calculation to verify that the virtual dimension of such systems is the same. If $C$ is a $(-1)$-curve of $\mathbb{P}_n$ through $r$ points, after an elementary transformation (with center in one of the $r$ points), the self-intersection and the genus of the new curve through $r - 1$ points are the same, hence these transformations preserve $(-1)$-curves.

Now we are able to prove the conjecture in the following cases:

**Proposition 4.1.** The system $L_n(a, b, m^r)$ with $b \leq m + 1$ is special if and only if it is $(-1)$-special.

**Proof.** By [2,7] we need to consider only systems with more than $n + 1$ points. We distinguish three cases:

- **$b < m$.** In this case, since $\mathcal{L}_n(a, b, m^r) \cdot \mathcal{L}_n(1, 0, 1) = b - m < 0$, then the fibers $F$ through the singular points are fixed components of $\mathcal{L}_n(a, b, m^r)$. Let $k = m - b$, the system consists of a fixed part and of the residual system $\mathcal{L}_n(a - kr, m - k, (m - k)^r)$. If $b \leq m - 2$ and the residual system is not empty, then the system is $(-1)$-special. If $b = m - 1$ then the residual system is $\mathcal{L}_n(a - r, m - 1, (m - 1)^r)$. In the last system we have $b = m - k$, so we need only to study the case $b = m$.

- **$b = m$.** By making a sequence of $n - 1$ elementary transformations we obtain the system $\mathcal{L}_1(a, m, m^{r-n+1})$. Blowing down the $(-1)$-curve of $\mathbb{P}_1$ we obtain the quasi-homogeneous system on $\mathbb{P}^2$: $\mathcal{L}_{\mathbb{P}^2}(a + m, -a, m^{r-n+1})$, which is special if and only if it is $(-1)$-special (see [2, Proposition 6.2]).

- **$b = m + 1$.** Consider the system $\mathcal{L}_n(a, m + 1, m^r)$, after $n - 1$ elementary transformations, the system becomes $\mathcal{L}_1(a + n - 1, m + 1, m^{r-n+1}, 1^{n-1})$, with the $n - 1$ points belonging to the $(-1)$-curve. This system may be again transformed by blowing up two points outside the $(-1)$-curve and blowing down the two fibers through these points. In this way we obtain the system $\mathcal{L}_1(a + n - m, m + 1, m^{r-n+1}, 1^{n+1})$ with the $n + 1$ points belonging to a section of self intersection 1. Blowing down the $(-1)$-curve we obtain the plane system $\mathcal{L}_{\mathbb{P}^2}(a + n + n + 1, m + 1, m^{r-n+1}, 1^{n+1})$. 


1, a + n - m, m^{r-n-1}, 1^{n+1}) where the n + 1 points belong to a line R. Now we distinguish two cases, according to the fact that the system \( S = L_{P^2}(a + n + 1, a + n - m, m^{r-n-1}) \) is special or not.

If \( S \) is special then, as shown in [2, Proposition 6.4], it is \((-1)\)-special. This means that the initial system is \((-1)\)-special.

If \( S \) is non-special, but \( L_{P^2}(a + n + 1, a + n - m, m^{r-n-1}, 1^{n+1}) \) is special, this means that this last system contains the line \( R \) through the \( q_i \)'s points, otherwise these (generic) points of the line, will impose independent conditions on the curves of \( S \). This means than \( h^0(S - \sum_{i=0}^n q_i) = h^0(S) \), hence we have the following exact sequence:

\[
0 \to H^0(S_R) \to H^1(S - R) \to H^1(S) \to H^1(S|_R) \to \cdots
\]

If \( h^1(S - R) \neq 0 \), then \( S - R \) is special and hence \((-1)\)-special (by [2, Proposition 6.4]). This imply that also \( S \) is \((-1)\)-special.

If \( h^1(S - R) = 0 \) then \( h^0(S|_R) = 0 \) and this happen iff deg \( S|_R < 0 \), which means that \( n + 1 > \deg S \). This can not happen since \( \deg S = a + n + 1 \).

\[\square\]

5. \((-1)\)-Special Systems

A \((-1)\)-special homogeneous system \( L \) may contain a \((-1)\)-curve \( A \) which is not homogeneous. In this case it must contain also all the \((-1)\)-curves \( A_i \) obtained from \( A \) by a permutation of the base points.

No two of the \( A_i \) can meet, because if two \((-1)\)-curves on a rational surface meet, their union moves in a linear system, and so the union cannot be part of the fixed divisor of \( L \). Let \( p_1, \cdots, p_r \) be the base points of the linear system, since the Picard group of \( S \) has rank \( r + 2 \) there can be at most \( r + 1 \) of these disjoint \((-1)\)-curves. On the other hand let \( k_i = \#\{p_i \mid \mu_{p_i}(A) = h_i\} \), where \( \mu_{p_i}(A) \) is the multiplicity of \( A \) at the point \( p_i \) which may be also 0. In this way we have \( \sum_{i=1}^s k_i = r \)

and the number of distinct \( A_i \) through the \( r \) points is: \( r!/(k_1! \cdots k_s!) \).

**Lemma 5.1.** Let \( r, s, k_i \in \mathbb{N} \) such that \( \sum_{i=1}^s k_i = r \), then \( k_1! \cdots k_s! \leq (r - s + 1)! \)

**Proof.** By hypothesis one has: \( r - s + 1 = 1 + \sum_{i=1}^s (k_i - 1) \). Define \( s_0 = 1 \) and \( s_t = 1 + \sum_{i=1}^t (k_i - 1) \) for \( t \geq 1 \), observe that

\[
\prod_{j=1+s_t}^{s_{t+1}} j \geq k_{t+1}!
\]
because on the left side of the inequality there are \( k_{t+1} - 1 \) terms which are greater than or equal to those on the right side (on this side the terms are \( k_{t+1} \) but the first is 1 and does not give contribution in the product). Taking products on \( t \) we obtain the thesis. \( \square \)

From the preceding lemma we obtain the following:

\[
(7) \quad \frac{r!}{k_1! \cdots k_s!} \geq \frac{r!}{(r-s+1)!}.
\]

The right side is a polynomial \( P_s(r) \) in \( r \) of degree \( s - 1 \), it is easy to prove that \( P_s(r) > r + 1 \) for \( r \geq s \geq 3 \). If \( s = 1 \) we have \( P_1(r) = 1 \) and the system \( A \) is homogeneous through the \( r \) points. If \( s = 2 \) then \( P_2(r) = r \). In this case, the left hand of (7) is a binomial coefficient: \( \binom{r}{k_1} \).

A simple calculation shows that \( \binom{r}{k_1} \leq r + 1 \) unless \( r \leq 3 \), but in this case \( k_2 \leq 1 \). In the other cases we have \( \binom{r}{k_1} \geq \binom{r}{2} \) for \( 2 \leq k_1 \leq \lfloor r/2 \rfloor \). So we may assume that \( k_1 = 1 \) and \( k_2 = r - 1 \). In this case the \((-1)\)-curves are of type \( \mathcal{L}_n(a,b,m_1,m_2^{r-1}) \). Consider two of them with multiplicity \( m_1 \) at two different points, \( p_i \) and \( p_k \) and call them \( A_i \) and \( A_k \) respectively.

The conditions \( A_i \cdot A_k = 0 \) if \( i \neq k \) and \( A_i^2 = -1 \) give us \( m_1 = m_2 \pm 1 \). For our purposes we need only to consider those with \( m_i \leq 1 \) (since they must be contained twice in the base locus of the linear system to give a \((-1)\) special system). So we have two possibilities:

- \( m_1 = 0 \) and \( m_2 = 1 \), in this case the system must have multiplicity at least \( 2r \), since it contains all these curves as fixed components twice.

  From \( m \geq 2r \), if \( m \leq 3 \) we obtain \( r = 1 \).

- \( m_1 = 1 \) and \( m_2 = 0 \), in this case there are no restriction on the multiplicity of the system. We call such systems compound.

**Proposition 5.2.** All homogeneous \((-1)\)-special systems with multiplicity \( \leq 3 \) on \( \mathbb{F}_n \) are listed in the following table:

**Proof.** We consider homogeneous linear systems \( \mathcal{L}(a,b,m^r) \) which have negative products with \((-1)\)-curves. The first product is:

\[
\mathcal{L}_1(a,b,m^r) \cdot \mathcal{L}_1(0,2,1^5) < 0,
\]

in this case the system is of the form \( \mathcal{L}_1(a,b,m^5) \), with \( m = 2, 3 \). If \( m = 2 \), then \( 2a + 2b - 10 < 0 \). The left hand of the preceding inequality is an even number which can not be less of \(-2 \) (because the linear system \( \mathcal{L}_1(a,b,2^5) \) can not contain the system \( \mathcal{L}_1(0,2,1^5) \) more
than twice). This implies that $a + b = 4$ and that $\mathcal{L}_1(a, b, m^5) = 2\mathcal{L}_1(0, 2, 1^5) + \mathcal{L}_1(a, b - 4)$. Hence, in order for the system to be not empty, it must be $b \geq 4$. This gives $b = 4$ and $a = 0$, so the initial system is $\mathcal{L}_1(0, 4, 2^5)$.

If $m = 3$, then $\mathcal{L}_1(a, b, 3^5) \cdot \mathcal{L}_1(0, 2, 1^5) \leq -1$ gives: $2(a + b) - 15 \leq -1$. If the product is equal to $-1$, then $a + b = 7$ and the residual system is $\mathcal{L}_1(a, b - 2, 2^5) = \mathcal{L}_1(a, 5 - a, 2^5)$. This system is empty if $a \geq 3$ and non-special if $a \geq 2$. If the product is equal to $-3$, then $a + b = 6$. The residual system is $\mathcal{L}_1(a, b - 6)$, hence $b \geq 6$, which implies that $b = 6$ and $a = 0$ and the system is $\mathcal{L}_1(0, 6, 3^5)$.

The second product is:

$$\mathcal{L}_n(a, b, m^r) \cdot \mathcal{L}_n(1, 0, 1) < 0.$$ 

If $m = 2$ then, since the product is equal to $b - 2$, $b$ may be equal to 0 or 1.

- If $b = 0$, then the residual system $\mathcal{L}_n(a - 2r, 0)$ is not empty if and only if $a \geq 2r$. In this case the initial system is $(-1)$-special or empty.
- If $b = 1$, then after removing the fixed part, the residual system $\mathcal{L}_n(a - r, 1, 1^r)$ is non-special, since it has base points of multiplicity one.

In this case the initial system is non-special, since the $(-1)$-curves have product equal to $-1$ with the initial system.
If \( m = 3 \) then, since the product is equal to \( b - 3 \), \( b \) belong to the set \{0, 1, 2\}.

- If \( b = 0 \), then the residual system \( \mathcal{L}_n(a - 3r, 0) \) is not empty if and only if \( a \geq 3r \). In this case the initial system is \((-1)\)-special or empty.
- If \( b = 1 \), then the residual system \( \mathcal{L}_n(a - 2r, 1) \) is non-special of virtual dimension \( 2a + n - 5r + 1 \). Hence the initial system is \((-1)\)-special or empty.
- If \( b = 2 \), then the residual system is \( \mathcal{L}_n(a - r, 2) \). This system may be \((-1)\)-special, only if it has negative product with some of the other \((-1)\)-curves of the \( \mathbb{F}_n \), hence it is already studied in the other cases of the present classification.

The third product is:

\[
\mathcal{L}_n(a, b, m^{2e + n + 1}) \cdot \mathcal{L}_n(e, 1, 1^{2e + n + 1}) < 0
\]

Here we consider only systems with \( b \geq m \), since the other systems where already considered in the preceding case. We begin by considering the case \( m = 2 \).

In this case, the product may be equal to \(-1\) or to \(-2\). In the first case one has \( a = e(4 - b) + n(2 - b) + 1 \) and the residual system is \( \mathcal{L}_n(a - e, b - 1, 1^{2e + n + 1}) \).

- If \( b = 2 \) then \( a = 2e + 1 \) and the residual system is \( \mathcal{L}_n(e + 1, 1^{2e + n + 1}) \). This system is non-special by proposition 2.4.
- If \( b = 3 \) then \( a = e - n + 1 \) and the residual system is \( \mathcal{L}_n(-n + 1, 1, 1^{2e + n + 1}) \). If \( n \leq 1 \), then by proposition 2.4 the system is non-special. If \( n \geq 2 \), then the residual system has negative product with the \(-n\)-curve \( \Gamma_n \) of \( \mathbb{F}_n \), hence it is equal to \( \Gamma_n + \mathcal{L}_n(1, 1^{2e + n + 1}) \). By proposition 2.4, this system is non-special of dimension \( 2 - 2e \). This means that \( e \leq 1 \). Since \( a = e - n + 1 \geq 0 \), the only possibility with \( n \geq 2 \) is \( e = 1 \) and \( n = 2 \). In this case the initial system is \( \mathcal{L}_2(0, 3, 2^5) \), it has virtual dimension 0, hence it is non-special.
- If \( b = 4 \) then \( a = -2n + 1 \) is negative unless \( n = 0 \) and in this case the residual system is empty.
- If \( b \geq 5 \) then \( a \) is always negative and the system is empty.

If the product is equal to \(-2\), then \( a = e(4 - b) + n(2 - b) \) and the residual system is \( \mathcal{L}_n(a - 2e, b - 2, 0) \).
Now we consider the case $m - a_1^2 \cdot 2 + \cdots$.

If $b = 2$ then $a = 2e$ and the residual system $\mathcal{L}_n(0, 0, 0)$ is not empty and the system $\mathcal{L}_n(2e, 2, 2^{2e+n+1})$ is $(-1)$-special.

If $b = 3$ then $a = e - n$ and the residual system $\mathcal{L}_n(-e - n, 1, 0)$ is empty.

If $b \geq 4$ then the system is always empty.

Now we consider the case $m = 3$.

In this case the product may be equal to $-1, -2, -3$. In the first case $a = e(6 - b) + n(3 - b) + 2$ and the residual system is $\mathcal{L}_n(a - e, b - 1, 2^{2e+n+1})$.

- If $b = 3$ then $a = 3e + 2$ and the residual system is $\mathcal{L}_n(2e + 2, 2, 2^{2e+n+1})$. By Proposition 2.4 this system is non-special.
- If $b = 4$ then $a = 2e - n + 2$ and the residual system is $\mathcal{L}_n(e - n + 2, 3, 2^{2e+n+1})$. If $e - n + 2 \geq 0$ then by Proposition 2.4 the system is non-special. If $e - n + 2 < 0$ then the system has negative product with the $-n$ curve $\Gamma_n$. So $\mathcal{L}_n(e - n + 2, 3, 2^{2e+n+1}) = \Gamma_n + \mathcal{L}_n(e + 2, 2, 2^{2e+n+1})$. Since $\mathcal{L}_n(e + 2, 2, 2^{2e+n+1}) \cdot \mathcal{L}_n(e, 1, 1^{2e+n+1}) = -e$, then $e \leq 2$ otherwise the system is empty.

If $e = 0$ then $a = -n + 2 < 0$ because we suppose that $e - n + 2 < 0$, hence there is no such a possibility.

If $e = 1$ then $a = 4 - n$ and $n \geq 4$. This means that $n = 4$ and $a = 0$. This gives the system $\mathcal{L}_4(0, 4, 3^7)$. This system is equal to $\mathcal{L}_4(1, 1, 1^7) + \mathcal{L}_4(-1, 3, 2^7) = \mathcal{L}_4(1, 1, 1^7) + \mathcal{L}_4(-4, 1, 0) + \mathcal{L}_4(3, 2, 2^7)$ since $\mathcal{L}_4(3, 2, 2^7) \cdot \mathcal{L}_4(1, 1, 1^7) = -1$, the initial system is equal to $2\mathcal{L}_4(1, 1, 1^7) + \mathcal{L}_4(-4, 1, 0) + \mathcal{L}_4(2, 1, 1^7)$. By Proposition 2.4 the last residual system is non-special of dimension 2. Since the virtual dimension of the initial system is 2, the system is non-special.

If $e = 2$ then $a = 6 - n$ and $n \geq 5$. This means that $n$ may be equal to 5 or 6. If $n = 5$ then $\mathcal{L}_5(1, 4, 3^{10}) = \mathcal{L}_5(2, 1, 1^{10}) + \mathcal{L}_5(-1, 3, 2^{10})$, this is equal to $\mathcal{L}_5(2, 1, 1^{10}) + \mathcal{L}_5(-5, 1, 0) + \mathcal{L}_5(4, 2, 2^{10})$. Since $\mathcal{L}_5(4, 2, 2^{10}) \cdot \mathcal{L}_5(2, 1, 1^{10}) = -2$, we have that the initial system is equal to $3\mathcal{L}_5(2, 1, 1^{10}) + \mathcal{L}_5(-5, 1, 0)$. The virtual dimension of the initial system is $-1$, hence the system is $(-1)$-special. If $n = 6$ then $\mathcal{L}_6(0, 4, 3^{11}) = \mathcal{L}_6(2, 1, 1^{11}) + \mathcal{L}_6(-2, 3, 2^{11})$, this is equal to $\mathcal{L}_6(2, 1, 1^{11}) + \mathcal{L}_6(-6, 1, 0) + \mathcal{L}_6(4, 2, 2^{11})$. Since $\mathcal{L}_6(4, 2, 2^{11}) \cdot \mathcal{L}_6(2, 1, 1^{11}) = -2$, we have that the initial system is equal to $3\mathcal{L}_6(2, 1, 1^{11}) + \mathcal{L}_6(-6, 1, 0)$.
The virtual dimension of the initial system is \(-1\), hence the system is \((-1)\)-special.

- If \(b = 5\) then \(a = e - 2n + 2\) and the residual system is \(L_n(-2n + 2, 4, 2^{2e+n+1})\). This system has negative product with the \(-n\)-curve only if \(n \geq 2\), otherwise the system is non-special, since it does not have negative product with any other \((-1)\)-curve. If \(n = 2\) then the residual system \(L_2(-2, 4, 2^{2e+3})\) is equal to \(L_2(-2, 1, 0) + L_2(0, 3, 2^{2e+3})\). The last residual system is not \((-1)\)-special of virtual dimension \(6 - 6e\) and since \(a = e - 2 \geq 0\), this dimension is negative. Hence the initial system is not \((-1)\)-special.

- If \(b = 6\) then \(a = -3n + 2\) and the system is empty.

If the product is equal to \(-2\), then \(a = e(6 - b) + n(3 - b) + 1\) and the residual system is \(L_n(a - 2e, b - 2, 1^{2e+n+1})\).

- If \(b = 3\) then \(a = 3e + 1\)
- If \(b = 4\) then \(a = 2e - n + 1\) and the residual system is \(L_n(-n + 1, 2, 1^{2e+n+1})\). This is equal to \(\Gamma_n + L_n(1, 1, 1^{2e+n+1})\). The last system has dimension \(2 - 2e\), hence it is not empty only if \(e = 0, 1\).

- If \(b = 5\) then \(a = e - 2n + 1\) and the residual system is \(L_n(-e - 2n + 1, 2, 1^{2e+n+1})\).

If the product is equal to \(-3\), then \(a = e(6 - b) + n(3 - b)\) and the residual system is \(L_n(a - 3e, b - 3, 0)\).

- If \(b = 3\) then \(a = 3e\) and the system \(L_n(3e, 3, 3^{2e+n+1}) = 3L_n(e, 1, 1^{2e+n+1})\) is \((-1)\)-special.
- If \(b = 4\) then \(a = 2e - n\) and the residual system \(L_n(-e - n, 1, 0)\) is empty.
- If \(b = 5\) then \(a = e - 2n\) and the residual system \(L_n(-2e - 2n, 2, 0)\) is empty.
- If \(b = 6\) then \(a = -3n\) is negative and there are no such systems.

\(\square\)

6. Degeneration of \(\mathbb{F}_n\) Surfaces

Let \(\Delta\) be: \(\{z \in \mathbb{C} \mid |z| < 1\}\), consider the product with \(\mathbb{F}_n\) and the two projections maps \(\pi_i\). Now blowing up the rational normal curve \(H\) contained in \(\pi^{-1}_1(0)\) we obtain a threefold \(X\) and two maps:
Let $X_t = p^{-1}_1(t)$, which for $t \neq 0$ is $\mathbb{F}_n$, while $X_0$ is the union of the proper transforms $\tilde{\mathbb{F}}$ of $\mathbb{F}_n$ and of the exceptional divisor $\mathbb{F}$ which is $\mathbb{P}(N_{\mathbb{F}_n, \Delta})$. But $N_{\mathbb{F}_n, \Delta} = \mathcal{O}_{\mathbb{F}^1} (n) \oplus \mathcal{O}_{\mathbb{F}^1}$. Hence the exceptional divisor is also an $\mathbb{F}_n$ that intersects the proper transform in a curve $R_{\tilde{\mathbb{F}}}$ whose class is $H$. Let us denote by $R_{\tilde{\mathbb{F}}}$ the curve $R$ when we consider it as a divisor on $\mathbb{F}$. To understand the class of $R_{\tilde{\mathbb{F}}}$ in Pic($\mathbb{F}$) observe that $(\mathbb{F} + \tilde{\mathbb{F}}) \cdot \mathbb{F} = X_0 \cdot \mathbb{F} = X_t \cdot \mathbb{F} = 0$. Hence $\mathbb{F} \cdot \mathbb{F} = -\tilde{\mathbb{F}} \cdot \mathbb{F} = -R_{\tilde{\mathbb{F}}}$. Now consider the line bundle $\mathbb{F}$ and the two restrictions: $\mathbb{F}|_{\tilde{\mathbb{F}}} = -R_{\tilde{\mathbb{F}}}$ and $\mathbb{F}_{\tilde{\mathbb{F}}} = R_{\tilde{\mathbb{F}}}$. If we consider the restrictions of these systems to $R$ they must agree, since they come from the same line bundle. So is $-R_{\tilde{\mathbb{F}}}^2 = R_{\tilde{\mathbb{F}}}^2 = n$, which tells us that $R_{\tilde{\mathbb{F}}}$ is the $(-n)$-curve of $\mathbb{F}$.

Now consider a linear system $|aF + bH|$ on the generic fiber $X_t$, we have: $p_2^* (aF + bH)|_{\tilde{\mathbb{F}}} = aF + bH$ and $p_2^* (aF + bH)|_{\mathbb{F}} = (a + nb)F$. So this is not sufficient to generate Pic($\mathbb{F}$). If we consider the linear system $\mathcal{X}(a, b, k) = |p_2^* (aF + bH) + k\mathbb{F}|$, this restricted to $X_t$, gives again $|aF + bH|$, while restricted to $X_0$ gives $|aF + (b - k)H|$ on $\tilde{\mathbb{F}}$ and $|(a + nb - nk)F + kH|$ on $\mathbb{F}$. Now consider an homogeneous linear system $\mathcal{L} = \mathcal{L}(a, b, m)$. We define a $(k, s)$-degeneration by sending $s$ of the $r$ points to $\tilde{\mathbb{F}}$ and considering the system $\mathcal{X}(a, b, k)|_{\mathbb{F}}$ with $s$ base points on $\tilde{\mathbb{F}}$ and $r - s$ base points on $\mathbb{F}$. Now let $\mathcal{X} = p_2^* \mathcal{L}$, since the map $p_1$ is a flat morphism, by semi-continuity is:

$$h^0(\mathcal{L}) = h^0(X_t, \pi_2^* \mathcal{L}) = h^0(X_t, \mathcal{X}) \leq h^0(X_0, \mathcal{X}).$$

We call $\mathcal{L}_0 = \mathcal{X}|_{X_0}$ and let $l_0 = \dim \mathcal{L}_0$. This is the key of the method: in order to prove the conjecture for a supposed non special system, one proves that $v(\mathcal{L}) = l_0$ and this implies that also $v(\mathcal{L}) = l(\mathcal{L})$.

In order to give a linear system on $X_0$, one has to give two linear system on $\mathbb{F}$ and $\tilde{\mathbb{F}}$ which agree on $R = \mathbb{F} \cap \tilde{\mathbb{F}}$. If we call $\mathcal{R}_{\tilde{\mathbb{F}}}$ (respectively $\mathcal{R}_{\mathbb{F}}$)
the restrictions of $\mathcal{L}$ (respectively $\mathcal{L}$) to $R$, then we have the following exact sequences:

$$0 \to \hat{\mathcal{L}} \to \mathcal{L} \to \mathcal{R} \to 0$$

$$0 \to \hat{\mathcal{L}} \to \mathcal{L} \to \mathcal{R} \to 0.$$

Where $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}$ are the kernels of the restrictions to $R$. So we obtain the four systems:

| System | Formula |
|--------|---------|
| $\mathcal{L}$ | $\mathcal{L}_n(a, b - t, m^{r-s})$ |
| $\mathcal{L}$ | $\mathcal{L}_n(a + n(b - t), t, m^s)$ |
| $\hat{\mathcal{L}}$ | $\hat{\mathcal{L}}_n(a, b - t - 1, m^{r-s})$ |
| $\hat{\mathcal{L}}$ | $\hat{\mathcal{L}}_n(a + n(b - t + 1), t - 1, m^s)$ |

Now, in order to evaluate $h^0(X_0, \mathcal{L})$ we must consider the fibered product:

$$H^0(X_0, \mathcal{L}) = H^0(\mathbb{F}, \mathcal{L}) \otimes H^0(R, \mathcal{L}) H^0(\mathbb{F}, \mathcal{L})$$

which is obtained by taking the sections of $H^0(\mathbb{F}, \mathcal{L})$ and $H^0(\mathbb{F}, \mathcal{L})$ whose restrictions to $R$ give the same element of $H^0(R, \mathcal{L})$. Hence one has:

$$l_0 = \dim(\mathcal{R} \cap \mathcal{R}) + l + \hat{l} + 2.$$

In order to evaluate $\dim(\mathcal{R} \cap \mathcal{R})$, we recall the following lemma proved in [2, proposition (6.1)].

**Lemma 6.1.** Let $L \in \text{Pic}(\mathbb{P}^1)$, given two vector subspaces: $V_1, V_2 \subset H^0(\mathbb{P}^1, L)$ which are not transverse, there exists an isomorphism $\psi$ of $\mathbb{P}^1$ such that $\psi^*V_1$ is transverse with $V_2$.

Now, observe that such an isomorphism $\psi$ of $\mathbb{P}^1$ may be extended to an isomorphism $\tau : \mathbb{F} \to \mathbb{F}$ such that $\pi \circ \tau = \psi \circ \pi$ where $\pi : \mathbb{F} \to \mathbb{P}^1$ is the projection map. To see this, it is sufficient to consider the isomorphism $\psi$ as defined between two rational section of the $\mathbb{F}$ and to extend it with the identity map on the fibers.

**Remark 6.2.** Let $v$ be the virtual dimension of the system $\mathcal{L}_n(a, b, m^r)$. Let $l_v = l - l_v - 1$ and $r_v = l_v + 1$ be the dimensions of the restrictions (to $R$) of the linear systems $\mathcal{L}$ and $\mathcal{L}$ respectively. Let $v, v, \hat{v}, \hat{v}$ be the virtual dimensions of the systems $\mathcal{L}$, $\mathcal{L}$, $\hat{\mathcal{L}}$, $\hat{\mathcal{L}}$. An easy calculation shows that the following identities hold:

1. $v + v = v + a + n(b - k)$
2. $\hat{v} + v = \hat{v} + v = v - 1$
3. If $r_v + r_v \leq a + n(b - k) - 1$, then $l_0 = l_v + l_v + 1$.
4. If $r_v + r_v \geq a + n(b - k) - 1$, then $l_0 = l_v + l_v - a - n(b - k)$. 
7. The Main Theorem

Now we are ready to prove the main conjecture in the case \( m = 2, 3 \).
We need first a lemma:

Lemma 7.1. For each \( r \leq n + 1 \), the linear system \( \mathcal{L}_n(a, b, m^r) \) is special if and only if it is \((-1)\)-special.

Proof. This is due to the fact that \( K_S \) is nef. \( \square \)

Theorem 7.2. Every special homogeneous system of multiplicity \( \leq 3 \) on an \( \mathbb{P}_n \) surface, is a \((-1)\)-special system.

Proof. Multiplicity 2

The case \( n = 1 \) is the same as the case of quasi homogeneous (i.e. systems with all except one points of the same multiplicity) systems on \( \mathbb{P}^2 \), which is completely classified in [2] for multiplicity \( \leq 3 \). For the remaining \( n \), we proceed by induction on the number \( r \) of points. For systems with only one point we proved the thesis in 7.1. By lemma 4.1 we may consider the linear system \( \mathcal{L} = \mathcal{L}_n(a, b, 2^r) \) with \( b \geq 4 \). First consider the case \( v(\mathcal{L}_n(a, b, 2^r)) < 0 \). Performing a \((1, s)\)-degeneration, we obtain the four systems:

\[
\hat{\mathcal{L}} = \mathcal{L}_n(a, b - 1, 2^{r-s}), \quad \mathcal{L} = \mathcal{L}_n(a + n(b - 1), 1, 2^s)
\]

\[
\hat{\mathcal{L}} = \mathcal{L}_n(a, b - 2, 2^{r-s}), \quad \mathcal{L} = \mathcal{L}_n(a + nb, 0, 2^s).
\]

Step I. We want to find an \( s \) such that

\[
\hat{\mathcal{L}} = \hat{\mathcal{L}} = \emptyset.
\]

Observe that \( \hat{\mathcal{L}} \) is a \((-1)\)-special system of dimension \( a + nb - 2s \), so it is empty if \( s > (a + nb)/2 \). There always exists such an \( s \) because, from the inequalities \( v(\mathcal{L}) < 0 \) and \( b \geq 3 \) it follows that \( r > (a + nb)/2 \). A sufficient condition to have \( \hat{\mathcal{L}} = \emptyset \) is that it is non special and \( v(\hat{\mathcal{L}}) \leq v(\mathcal{L}) \). This gives \( s \leq 2(a + nb - n + 1)/3 \). So \( s \) must satisfy:

\[
\frac{a + nb}{2} < s \leq \frac{2}{3}(a + nb - n + 1)
\]

A sufficient condition for \( s \) to exist is that \( 2(a + nb - n + 1)/3 - (a + nb)/2 \geq 1 \). Solving the inequality we obtain: \( n(b - 4) \geq 2 - a \) which is always true if \( b \geq 5 \) or if \( b = 4 \) and \( a \geq 2 \). In the remaining case, \( b = 4, a = 1 \) it is again possible to find \( s = 2n + 1 \). \( \hat{\mathcal{L}} \) may be \((-1)\)-special in only one case (see Proposition 5.2) with \( b \geq 4 \): \( \hat{\mathcal{L}} = \mathcal{L}_n(2e, 2, 2^{2e+n+1}) \). For \( e \geq 4 \) there are two values of \( s \) that satisfy
so we may choose the value for which \( \hat{L}_{\tilde{X}} \) is non special. In the remaining cases \( e \leq 3 \) \( v(\mathcal{L}) \geq 0 \). Now we have chosen \( s \) such that \( \hat{L}_{\tilde{X}} \) and \( \hat{L}_{\tilde{X}} \) are empty and we finish step I.

*Step II.* We want to prove that \( r_{\tilde{X}} + r_{\tilde{X}} \leq a + n(b - 1) - 1 \) which by [6.2 iii] ends the proof. Observe that (Proposition 5.2) \( \tilde{L}_{\tilde{X}} \) is non special \( L_{\tilde{X}} \) is non special for \( b \geq 4 \). Then by [6.2 i], \( l_{\tilde{X}} + l_{\tilde{X}} = a + n(b - 1) + v \) and since \( v \leq -1 \), by [6.2 iii] we finish.

Suppose now that \( v(\mathcal{L}_n(a, b, 2^r)) \geq 0 \) and consider again a \((1, s)\)-degeneration. In this case we need the regularity of \( \mathcal{L}_{\tilde{X}} \) and \( \mathcal{L}_{\tilde{X}} \) and the inequality:

\[
r_{\tilde{X}} + r_{\tilde{X}} \geq a + n(b - 1) - 1.
\]

So applying [6.2 i] and [6.2 iv] we prove that \( l_0 = v \) and we finish. Choosing \( s \leq (a + nb)/2 \), then both \( \mathcal{L}_{\tilde{X}} \) and \( \hat{L}_{\tilde{X}} \) are not empty and by proposition [5.2] for \( b \geq 4 \), \( \mathcal{L}_{\tilde{X}} \) and \( \hat{L}_{\tilde{X}} \) are non special. \( \hat{L}_{\tilde{X}} \) is \((-1)\)-special of dimension \( a + nb - 2s \) and \( l_{\tilde{X}} = 2(a + bn) - n - 3s + 1 \) and This gives \( r_{\tilde{X}} = l_{\tilde{X}} - \hat{l}_{\tilde{X}} = a + bn - n - s \). So we have only to check that \( r_{\tilde{X}} \geq s - 1 \).

If \( \hat{L}_{\tilde{X}} \) is non special the inequality is true for \( b \geq 2 \). If \( \hat{L}_{\tilde{X}} \) is \((-1)\)-special the only possibility with \( b \geq 4 \) is: \( \mathcal{L}_{\tilde{X}}(a, b - 2, 2^{r-s}) = \mathcal{L}_{\tilde{X}}(2e, 2, 2e+n+1) \) which has dimension 0. In this case \( \mathcal{L}_{\tilde{X}} \) is non special of dimension \( 2e + 3n \). So \( r_{\tilde{X}} \geq n + s - 1 \) iff \( s \leq 2e + 2n \) which is always true.

**Multiplicity 3**

The technique is the same as in the multiplicity 2 case, so we proceed again by induction assuming that \( b \geq 5 \) by lemma 4.1. First suppose that \( v(\mathcal{L}_n(a, b, 3^r)) < 0 \). In this case, we make a \((2, s)\)-degeneration, obtaining the four systems:

\[
\mathcal{L}_{\tilde{X}} = \mathcal{L}_n(a, b - 2, 3^{r-s}), \quad \mathcal{L}_{\tilde{X}} = \mathcal{L}_n(a + n(b - 2), 2, 3^s)
\]

\[
\hat{L}_{\tilde{X}} = \mathcal{L}_n(a, b - 3, 3^{r-s}), \quad \hat{L}_{\tilde{X}} = \mathcal{L}_n(a + n(b - 1), 1, 3^s)
\]

*Step I.* We need to find an \( s \) that satisfies (8). Observe that by [5.2] \( \hat{L}_{\tilde{X}} \) is a \((-1)\)-special system of dimension \( 2(a + nb) - n - 5s + 1 \). So choosing \( s > (2(a + nb) - n + 1)/5 \), we obtain: \( \hat{L}_{\tilde{X}} = \emptyset \). It is possible to chose such an \( s \) since \( v < 0 \) and \( b \geq 5 \) imply that \( r > (2a + nb + 2)/2 - 1/6 \). Now, solving the inequality \( v(\hat{L}_{\tilde{X}}) \leq v(\mathcal{L}) \) we obtain \( s \leq (a + bn - n + 1)/2 \). So we want:

\[
\frac{2(a + nb) - n + 1}{5} < s \leq \frac{a + bn - n + 1}{2},
\]
A sufficient condition for \( s \) to exist is that \((a+bn-n+1)/2-(2(a+nb)-n+1)/5 \geq 1\), which is equivalent to \((b-3)n+a-7 \geq 0\). This inequality is not satisfied only if \( b = 5\), \( n = 2\) and \( a \leq 2\), but in these cases \( s = 5 \) works. If \( \hat{L} \) is \((-1)\)-special, then by \([5.2]\) it is one of the following: \( \mathcal{L}_n(4e+n+1,2,3^{2e+n+1})\), \( \mathcal{L}_n(3e+1,3,3^{2e+n+1})\), \( \mathcal{L}_n(3e,3,3^{2e+n+1})\).

First we try to solve in each case the inequality: \((a+bn-n+1)/2-(2(a+nb)-n+1)/5 \geq 2\). This allows us to choose \( s \) in two ways and to avoid the \((-1)\)-special case. In the first case we obtain \( 4e+3n \geq 16 \). This is true if \( e \geq 3 \), in the remaining cases, making the explicit calculation, there are two values for \( s \). In the second case we have \( 3e+3n \geq 16 \), which is true if \( e \geq 4 \). If \( e \leq 3 \) it is again possible to find two values for \( s \) except for \( e = 1\), \( n = 2\), \( s = 7 \) but in this case \( v(L) = 4 \). In the third case we have \( 3e+3n \geq 17 \) which is always true for \( e \geq 4 \). In the remaining cases the only exceptions are \( e = 1\), \( n = 3\), \( s = 9 \) and \( e = 2\), \( n = 2\), \( s = 8 \). But in these cases \( v(L) = 0 \). So we may assume that also \( \hat{L} = \emptyset \).

**Step II.** We want to prove that \( r_{\hat{\phi}} + r_F \leq a + n(b - 1) - 1 \) which by \([6.2] iii\) ends the proof. \( \hat{L} \) is \((-1)\)-special only if \( \hat{L} = \mathcal{L}_n(3e,3,3^{2e+n+1}) \) or \( \hat{L} = \mathcal{L}_n(3e+1,3,3^{2e+n+1}) \). In the first case \( l_{\hat{\phi}} = 0 \) and \( L_F = \mathcal{L}_n(3e+n,2,3^s) \). If \( L_F \) is not \((-1)\)-special, then \( v_F = 9e+12n-6s+2 \), hence \( l_{\hat{\phi}} + l_F = v_F \). From (10) we deduce that \( v_F < (9e+6n-2)/5 < 3e+3n = a+n(b-2) \), so the system is empty. If \( L_F = \mathcal{L}_n(3(e+n),2,3^s) \) is \((-1)\)-special then it must have dimension 0, so \( r_{\hat{\phi}} + r_F = l_{\hat{\phi}} + l_F = 0 < a + n(b-2) \) and the system is empty.

If \( \hat{L} = \mathcal{L}_n(3e+1,3,3^{2e+n+1}) \), then \( l_{\hat{\phi}} = 2 \). If \( L_F = \mathcal{L}_n(3(e+n)+1,2,3^s) \) is non special, then \( v_F = 9e+12n-6s+5 \). From \( s > (2(a+nb)-n+1)/5 \) we deduce that \( l_{\hat{\phi}} + l_F = v_F + 2 < (9e+6n+1)/5 + 2 < 3e+3n = a+n(b-2) \) and the system is empty. If \( L_F = \mathcal{L}_n(3(e+n)+1,2,3^s) \) is \((-1)\)-special then it may have dimension 0 or 2 and again \( l_{\hat{\phi}} + l_F < a+n(b-2) \) and the system is empty.

So we may assume that \( \hat{L} \) is non special. If also \( L_F \) is non special, then \( l_{\hat{\phi}} + l_F = v_F = a + n(b-2) + v(L) < a + n(b-2) \) and again the result follows.

\( L_F \) may be \((-1)\)-special only if \( L_F = \mathcal{L}_n(4e+n+1,2,3^{2e+n+1}) \). In this case we have \( a + n(b-2) = 4e + n + 1 \) and \( s = 2e + n + 1 \). It is
easy to prove that we may choose $s$ in two ways in order to avoid the $(-1)$-special case.

Now we study the case $v(L_n(a, b, 3^r)) \geq 0$. We perform a $(3, s)$-degeneration, obtaining the systems:

$$L_{\tilde{g}} = L_n(a, b - 3, 3^{r-s}), \quad L_{\tilde{r}} = L_n(a + n(b - 3), 3, 3^s)$$

$$\hat{L}_{\tilde{g}} = L_n(a, b - 4, 3^{r-s}), \quad \hat{L}_{\tilde{r}} = L_n(a + n(b - 2), 2, 3^s)$$

Now let $\nu \equiv a + n(b - 1) + 1 \pmod{2}$ and let $s = (a + n(b - 1) + 1 + \nu)/2$. With this choice of $s$ the system $L_{\tilde{r}}$ is empty or $(-1)$-special with $\tilde{v}_F = -1$ and $\tilde{l}_F = 0$. The system $L_{\tilde{r}}$ is always non special and $v_F = a + n(b - 3) - 3\nu$. Then $r_F = a + n(b - 3) - 3\nu + \epsilon$ where $\epsilon = 1$ if the system is $(-1)$-special, otherwise is 0.

Observe that $v_{\tilde{F}} - v = 3\nu \geq 0$, hence $L_{\tilde{F}}$ is not empty. If $L_{\tilde{F}}$ is non special and $\hat{L}_{\tilde{F}} = \emptyset$ then $r_{\tilde{F}} = v + 3\nu$ and $r_{\tilde{F}} + r_{\tilde{F}} = a + n(b - 3) + \epsilon$ so $l_0 = l_{\tilde{F}} + l_{\tilde{F}} - a - n(b - 3) = v$. If $\hat{L}_{\tilde{F}} \neq \emptyset$ is non special, then $r_{\tilde{F}} = v_{\tilde{F}} - \tilde{v}_{\tilde{F}} - 1 = a + n(b - 3)$ and again we finish. If $\hat{L}_{\tilde{F}} \neq \emptyset$ is $(-1)$-special then for all the possibilities we have $l_{\tilde{F}} - \tilde{v}_{\tilde{F}} \leq 5$. Hence $r_{\tilde{F}} \geq a + n(b - 3) - 5$ which gives $r_{\tilde{F}} + r_{\tilde{F}} \geq a + n(b - 3) - 1$ which is true if $a + n(b - 3) \geq 8$.

If $L_{\tilde{F}}$ is $(-1)$-special then it may be only $L_{\tilde{F}} = L_n(3e + 1, 3, 3^{2e+n+1})$. In this case choosing $s = (a + n(b - 1) + 1 + \nu)/2 + 1$ we have $\hat{L}_{\tilde{F}} = \emptyset$ and $L_{\tilde{F}} = L_n(3e + 4n + 1, 2, 3^{3+n+1})$ which is non special. Hence $r_{\tilde{F}} = v_{\tilde{F}} = 3(e + n - \nu) - 5$. $\hat{L}_{\tilde{F}} = L_n(3e + n + 1, 2, 3^{2e+n})$ is non special and empty. $L_{\tilde{F}} = L_n(3e + 1, 3, 3^{2e+n})$ is non special and $v_{\tilde{F}} = 7$. Therefore $r_{\tilde{F}} + r_{\tilde{F}} = 3(e + n - \nu) + 2$, hence if $\nu = 0$ we finish. If $\nu = 1$ then $L = L_n(3e + 1, 6, 3^r)$ with $r = (7e + 7n + 5)/2$ and in this case $v = -2$, a contradiction. \qed

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