TRIANGLE-ROUNDEDNESS IN MATROIDS

JOÃO PAULO COSTALONGA\textsuperscript{1} AND XIANQIANG ZHOU\textsuperscript{2}

Abstract. A matroid $N$ is said to be triangle-rounded in a class of matroids $\mathcal{M}$ if each 3-connected matroid $M \in \mathcal{M}$ with a triangle $T$ and an $N$-minor has an $N$-minor with $T$ as triangle. Reid gave a result useful to identify such matroids as stated next: suppose that $M > N$ are binary 3-connected matroids, $T$ is a triangle of $M$ and $e \in T \cap E(N)$; then $M$ has a 3-connected minor $M'$ with an $N$-minor such that $T$ is a triangle of $M'$ and $|E(M')| \leq |E(N)| + 2$. We strengthen this result by dropping the condition of the existence of such element $e$ and proving that there is a 3-connected minor $M'$ of $M$ with an $N$-minor $N'$ such that $T$ is a triangle of $M'$ and $E(M') - E(N') \subseteq T$. This result is extended to the non-binary case and, as an application, we prove that $M(K_5)$ is triangle-rounded in the class of the regular matroids.

Key words: matroid minors; roundedness; matroid connectivity.

1. Introduction

Let $\mathcal{M}$ be a class of matroids closed for minors and isomorphisms and $\mathcal{F}$ a family of matroids. An $\mathcal{F}$-minor of a matroid $M$ is a minor of $M$ isomorphic to a member of $\mathcal{F}$. We say that $\mathcal{F}$ is $(k, t)$-rounded in $\mathcal{M}$ if each element of $\mathcal{F}$ is $k$-connected and, for each $k$-connected matroid $M \in \mathcal{M}$ and each $t$-subset $T \subseteq E(M)$, $M$ has an $\mathcal{F}$-minor using $T$. We define $\mathcal{F}$ to be $t$-rounded in $\mathcal{M}$ if it is $(t + 1, t)$-rounded in $\mathcal{M}$. A matroid $N$ is said $(k, t)$-rounded (resp. $t$-rounded) in $\mathcal{M}$ if so is $|N|$. When we simply say that a matroid or family of matroids is $(k, t)$-rounded or $t$-rounded with no mention to a specific class of matroids, we are referring to the class of all matroids.

Bixby \cite{1} proved that $U_{2,4}$ is 1-rounded. Seymour \cite{14} established a method to find a minimal 1-rounded family containing a given family of matroids; in that work it is established that $\{U_{2,4}, M(K_4)\}$, $\{U_{2,4}, F_7, F_7^*\}$, $\{U_{2,4}, F_7, F_7^*, M^*(K_{3,3}), M^*(K_5), M^*(K_5')\}$ and $\{U_{2,5}, U_{3,5}, F_7, F_7^*\}$ are 1-rounded.

Seymour \cite{16} proved that $U_{2,4}$ is also 2-rounded and, later, in \cite{17}, established a method to find a minimal 2-rounded family containing a given family of matroids.

Khan \cite{6} and Coullard \cite{5} proved independently that $U_{2,4}$ is not 3-rounded. To the best of our knowledge, there is no known criterion to check $(k, t)$-roundedness for $k \geq 4$. For $t \geq 3$, Oxley \cite{9} proved that $\{U_{2,4}, \mathcal{W}^3\}$ is $(3,3)$-rounded. Moss \cite{7} proved that $\{\mathcal{W}^2, \mathcal{W}^3, \mathcal{W}^4, M(\mathcal{W}_3), M(\mathcal{W}_4), Q_6\}$ is $(3,4)$-rounded and $\{M(\mathcal{W}_3), M(\mathcal{W}_4), M(\mathcal{W}_5), M(K_5 \setminus e), M(K_5 \setminus e), M(K_{1,2,3}), M^*(K_{1,2,3}), S_9\}$ is $(3,5)$-rounded in the class of the binary matroids.

\textsuperscript{1}Departamento de Matemática, Universidade Federal do Espírito Santo. Av. Fernando Ferrari, 514; Campus de Goiabeiras, 29075-910, Vitória, ES, Brazil. e-mail: joacostalonga@gmail.com (corresponding author).

\textsuperscript{2}Department of Mathematics and Statistics, Wright State University, Dayton, OH, 45435, USA and School of Mathematical Sciences, Huaqiao University, Fujian, China. e-mail: xiangqian.zhou@wright.edu.
There are works on classification of small \( r \)-rounded families of matroids for \( t = 1, 2 \). Oxley \[8\] proved that for \( |E(N)| \geq 4 \), \( N \) is 1-rounded if and only if \( N \cong U_{2,4}, P(U_{1,3}, U_{1,1}) \) or \( Q_6 \) and 2-rounded if and only if \( N \cong U_{2,4} \). Reid and Oxley \[12\] proved that, up to isomorphisms, the unique 2-rounded matroids with more than three members in the class of \( GF(q) \)-representable matroids are \( M(\mathcal{W}_3) \) and \( \mathcal{M}(\mathcal{W}_4) \) for \( q = 2, U_{2,4} \) and \( W^3 \) for \( q = 3 \) and \( U_{2,4} \) for \( q \geq 4 \).

In this work, we focus on a different type of roundedness. A family of matroids \( \mathcal{F} \) is said to be \textit{triangle-rounded} in \( \mathcal{M} \) if all members of \( \mathcal{F} \) are 3-connected and, for each matroid \( M \in \mathcal{M} \) with an \( \mathcal{F} \)-minor and each triangle \( T \) of \( M \), there is an \( \mathcal{F} \)-minor of \( M \) with \( T \) as triangle. We say that a matroid \( N \) is \textit{triangle-rounded} in \( \mathcal{M} \) if \( \{N\} \) is \( \{N\} \). Some examples of triangle-rounded matroids and families are \( U_{2,4} \) in the class of all matroids, \( F_7 \) in the class of binary matroids and \( M^*(K_{3,3}) \) in the class of regular matroids (Asano, Nishizeki and Seymour \[2\]), \( M(K_5 \setminus e) \) in the class of regular matroids and \( \{S_8, J_{10}\} \) in the class of binary matroids (Reid \[13\]). The proofs for the triangle-roundedness of the later two rely on the following criterion:

\textbf{Theorem 1.} (Reid \[13\] Theorem 1.1) \( \{e, f, g\} \) be a triangle of a 3-connected binary matroid \( M \) and \( N \) be a 3-connected minor of \( M \) with \( e \in E(N) \). Then, there exists a 3-connected minor \( M' \) of \( M \) using \( \{e, f, g\} \) such that \( M' \) has a minor which is isomorphic to \( N \) and \( E(M') \) has at most \( |E(N)| + 2 \) elements.

Here we establish a stronger result for binary matroids:

\textbf{Theorem 2.} If \( M \) is a 3-connected binary matroid with a 3-connected minor \( N \) with \( |E(N)| \geq 4 \) and \( T \) is a triangle of \( M \), then \( M \) has a 3-connected minor \( M' \) with an \( N \)-minor \( N' \) such that \( E(M') = E(N') \subseteq T \).

Theorem 2 follows from our more general result:

\textbf{Theorem 3.} Let \( M \) be a 3-connected matroid with a 3-connected minor \( N \) satisfying \( |E(N)| \geq 4 \). Suppose that \( T \) is a triangle of \( M \) and \( M \) is minor-minimal with the property that \( M \) has no \( N \)-minor using \( T \). Then \( r(M) - r(N) \leq 2 \) and for some \( N \)-minor \( N' \) of \( M \), \( |E(M) - (E(N') \cup T)| \leq 1 \). Moreover, if \( E(M) - E(N') \nsubseteq T \), then one of the following assertions holds:

(a) \( r(M) - r(N) = 1, r^*(M) - r^*(N) \in \{2, 3\} \) and \( M \) has an element \( x \) such that \( E(M) - E(N') \subseteq T \cup x \) and \( x \) is the unique element of \( M \) such that \( \text{si}(M/x) \) is 3-connected with an \( N \)-minor; or

(b) \( r^*(M) - r^*(N) = 2, r(M) - r(N) \in \{1, 2\} \) and \( M \) has an element \( y \) such that \( E(M) - E(N') \subseteq T \cup y \) and \( y \) is a 4-circuit of \( M \) such that \( M/y \) has no \( N \)-minor and \( M \setminus A \) has no \( N \)-minor for each subset \( A \) of \( T \).

The possible cases described in this theorem indeed occur, we give examples in Section 3. We say that a graph \( G \) is \textit{triangle-rounded} if so is \( M(G) \) in the class of graphic matroids. Using Theorem 2 we establish that \( K_5 \) is triangle-rounded, in other words:

\textbf{Theorem 4.} If \( G \) is a 3-connected graph with a triangle \( T \) and a \( K_5 \)-minor, then \( G \) has a \( K_5 \)-minor with \( E(T) \) as edge-set of a triangle.

\textbf{Remark: } \( K_3 \) and \( K_4 \) are triangle-rounded, but no larger complete graph than \( K_5 \) is triangle-rounded. Indeed, consider, for disjoint sets \( X, Y \) and \( z \) satisfying \( |X|, |Y| \geq 2 \), a complete
graph $K$ on vertex set $X \cup Y \cup \{z\}$. Consider also a graph $G$ extending $K$ by two vertices $x$ and $y$ with $E(G) - E(K) = \{xy, xz, yz, xx', yy' : x' \in X \text{ and } y' \in Y\}$. Note that $G \setminus xy \cong K_n$. But no $K_n$-minor of $G$ uses $\{xy, xz, yz\}$ because contracting any other edge than $xy$ in $G$ results in a graph with more than one parallel pair of edges.

The next result allows us to derive triangle-roundedness in the class of regular matroids from triangle-roundedness in the classes of graphic and cographic matroids.

**Theorem 5.** If a family $\mathcal{F}$ of internally 4-connected matroids with no triads is triangle-rounded both in the class of graphic and cographic matroids, then $\mathcal{F}$ is triangle-rounded in the class of regular matroids not isomorphic to $R_{10}$.

As $R_{10}$ has no $M(K_5)$-minor and $M(K_5)$ is internally 4-connected with no triads and trivially triangle-rounded in the class of cographic matroids, it follows from Theorems 4 and 5 that:

**Corollary 6.** $M(K_5)$ is triangle-rounded in the class of regular matroids.

All proofs are in the next section.

## 2. Proofs

In this section we prove the theorems. Next we state some results used in the proofs.

**Lemma 7.** (Whittle, [18, Lemma 3.6]) Let $M$ be a 3-connected matroid with elements $x$ and $p$ such that $si(M/x)$ and $si(M/x,p)$ are 3-connected, but $si(M/p)$ is not 3-connected. Then, $r(M) \geq 4$ and there is a rank-3 cocircuit $C^*$ of $M$ containing $x$ such that $p \in cl_M(C^*) - C^*$.

**Lemma 8.** (Whittle [18, Lemma 3.7]) Let $C^*$ be a rank-3 cocircuit of a 3-connected matroid $M$ such that $p \in cl_M(C^*) - C^*$.

(a) If $z_1, z_2 \in C^*$, then $si(M/p, z_1) \equiv si(M/p, z_2)$.

(b) If $N$ is a matroid and for some $x \in C^*$, $si(N/x, p)$ is 3-connected with an $N$-minor, then $si(N/z, p)$ is 3-connected with an $N$-minor for each $z \in C^*$.

**Lemma 9.** (Whittle [18, Lemma 3.8]) Let $C^*$ be a rank-3 cocircuit of a 3-connected matroid $M$. If $x \in C^*$ has the property that $cl_M(C^*) - x$ contains a triangle of $M/x$, then $si(M/x)$ is 3-connected.

**Lemma 10.** (Wu, [Lemma 3.15][19]) If $I^*$ is a coindependent set in a matroid $M$ and $M \setminus I^*$ is vertically 3-connected, then so is $M$.

Using Seymour’s Splitter Theorem (as stated in [11, Corollary 12.2.1]) and proceeding by induction on $i$ using Lemma 10, we may conclude:

**Corollary 11.** Let $N < M$ be 3-connected matroids such that $M$ has no larger wheel or whirl-minor than $N$ in case $N$ is a wheel or whirl respectively. Then, there is a chain of 3-connected matroids $N \cong M_n < \cdots < M_1 < M_0 = M$ such that for each $i = 1, \ldots, n$ there is $x_i \in E(M_i)$ satisfying $M_i = M_{i-1} / x_i$ or $M_i = M_{i-1} \setminus x_i$. Moreover, for $I := \{x_i : M_{i-1} = M_i / x_i\}$ and $I^* := \{x_i : M_{i-1} = M_i / x_i\}$,

(a) $I$ is an independent set and $I^*$ is a coindependent set of $M$. 


(b) for each $1 \leq i \leq n$, $M/(I \cap \{x_1, \ldots, x_i\})$ and $(M\setminus\{I^* \cap \{x_1, \ldots, x_i\}\})^*$ are vertically $3$-connected.

**Theorem 12.** (Whittle, Corollary 3.3) Let $N$ be a $3$-connected minor of the $3$-connected matroid $M$. If $r(M) \geq r(N) + 3$, then for each element $x$ such that $si(M\setminus x)$ is $3$-connected with an $N$-minor, there exists $y \in E(M)$ such that $si(M\setminus y)$ and $si(M\setminus x, y)$ are $3$-connected with $N$-minors.

**Theorem 13.** (Whittle Lemma 3.4 and Theorem 3.1 and Costalonga Theorem 1.3) Let $k \in \{1,2,3\}$ and let $M$ be a $3$-connected matroid with a $3$-connected minor $N$ such that $r(M) - r(N) \geq k$. Then $M$ has a $k$-independent set $J$ such that $si(M\setminus x)$ is $3$-connected with an $N$-minor for all $x \in J$.

**Lemma 14.** (Costalonga Corollary 4) Suppose that $M > N$ are $3$-connected matroids with $r^*(M) - r^*(N) \geq 4$ and $N$ is cossimple. Then:

(a) $M$ has a coinddependent set $S$ of size $4$ such that $co(M\setminus e)$ is $3$-connected with an $N$-minor for all $e \in S$; or

(b) $M$ has distinct elements $a_1, a_2, b_1, b_2, b_3$ such that, $T_S := \{a_s, b_t, b_3\}$ is a triangle for $(s, t) = \{1,2\}$, $T^* := \{b_1, b_2, b_3\}$ is a triad of $M$ and $co(M\setminus T^*)$ is $3$-connected with an $N$-minor.

**Proof of Theorem 13:** Suppose that the result does not hold. This is, for each $N'$-minor of $M$, $E(M) - E(N') \not\subseteq T$ and items (a) and (b) of the theorem do not hold. It is already known that $U_{2,4}$ is triangle-rounded [2], so, we may assume that $|E(N)| \geq 5$. The proof will be based on a series of assertions. First, note that it follows from the minimality of $M$ that:

(I). If, for $x \in E(M)$, $si(M\setminus x)$ is $3$-connected with an $N$-minor, then $x \in cl_M(T)$.

(II). If $T^*$ is a triad and $T$ is a triangle of $M$ such that $T^* \setminus T = \{x\}$, then $M\setminus x$ has no $N$-minor.

**Subproof:** Suppose the contrary. Let $T^* \setminus T = \{a, b\}$. As $N$ is simple and cosimple, $M\setminus x$ is $3$-connected with an $N$-minor. But $M\setminus x/a \setminus b$ has an $N$-minor. By Lemma 9, $si(M\setminus x)$ is $3$-connected. By [1] $x \in cl_M(T)$. As $x \notin T$, then $T^*$ meets a $4$-segment of $M$. This implies that $M \cong U_{2,4}$, a contradiction.

(III). If, for $x \in E(M)$, $co(M\setminus x)$ is $3$-connected with an $N$-minor, then $x \in T$.

**Subproof:** Suppose the contrary. Then $T \not\subseteq E(co(M\setminus x))$ and therefore, there is a triad $T^*$ meeting $x$ and $T$. A contradiction to [11].

(IV). If $si(M\setminus x)$ and $si(M\setminus x, y)$ are $3$-connected with $N$-minors, then so is $si(M\setminus y)$ and $x, y \in T$.

**Subproof:** First we prove that $si(M\setminus y)$ is $3$-connected. Suppose the contrary. By Lemma 7 there is a rank-3 cocircuit $C^*$ such that $x \in C^*$ and $y \in cl(C^*) - C^*$. By [11] $x \in cl(T)$. By orthogonality, $T \subseteq cl(C^*)$. As $r(C^*) = 3$, there is $z \in C^* - cl(T)$ and $T$ is a triangle of $M$ containing in $cl_{M}(z)$ in $C^*$. By Lemma 9, $si(M\setminus z)$ is $3$-connected and, by Lemma 8, $M\setminus z$ is an $N$-minor. A contradiction to [11]. So, $si(M\setminus y)$ is $3$-connected.

By [11] $x, y \in cl(T)$. If, for some $(a, b) = (x, y)$, $a \notin T$, then, as $M\setminus b$ has an $N$-minor and $a$ is in a parallel pair of $M\setminus b$, it follows that $M\setminus a$ has an $N$-minor. Moreover, in this case, $T \cup a$ is $4$-segment of $M$ and $M\setminus a$ is $3$-connected with an $N$-minor, contradicting the minimality of $M$. Thus, $x, y \in T$. 
(V). \( r(M) - r(N) \leq 2 \) and \( r^*(M) - r^*(N) \leq 3. \)

**Subproof:** If \( r(M) - r(N) \geq 3 \), then, by Theorem \([13]\) there is an independent set \( J \) of size 3 such that \( \text{si}(M/x) \) is 3-connected with an \( N \)-minor for all \( x \in J \). So, there is \( x \in J - \text{cl}_M(T) \), a contradiction to \([11]\). Thus, \( r(M) - r(N) \leq 2. \)

If \( r^*(M) - r^*(N) \geq 4 \), then Lemma \([14]\) applies. If item (a) of that Lemma holds, then we have an element \( x \in E(M) - T \) such that \( \text{co}(M\backslash x) \) is 3-connected with an \( N \)-minor, contradicting \([III]\). So, consider the elements given by item (b) of Lemma \([14]\). Let \( s \in \{1, 2\} \). Note that \( M\backslash T^* \) is isomorphic to a minor of \( M\backslash a_s \). By the dual version of Lemma \([9]\) on \( T_s \) and \( a_s \), it follows that \( \text{co}(M\backslash a_s) \) is 3-connected with an \( N \)-minor and, therefore, \( a_s \in T \) by \([III]\). So, \( a_1, a_2 \in T \). By orthogonality between \( T \) and \( T^* \), \( b_3 \notin T \). As \( M\backslash b_3 \) has an \( N \)-minor, then, by \([III]\) \( \text{co}(M\backslash b_3) \) is not 3-connected. By Bixby’s Lemma, \( \text{si}(M/ b_3) \) is 3-connected. As \( M\backslash T^* \) has an \( N \)-minor, then, so has \( M\backslash b_3 \), and, therefore, \( \text{si}(M/ b_3) \) is 3-connected with an \( N \)-minor, contradicting the fact that they are the elements of a set with rank at least 3 in \( W \).

Thus, \( r(M) = r(W) + 1 \). Now, by Theorem \([13]\) there is an element \( x \) such that \( \text{si}(M/x) \) is 3-connected with an \( W \)-minor \( W' \). By Lemma \([10]\) \( M, x \) is vertically 3-connected with an \( N \)-minor for each non-spoke \( y \) of \( W \). By \([IV]\) all non-spokes of \( W' \) are in \( T \), a contradiction again.

Now, we assume that there is no wheel or whirl \( W \) such that \( N < W \leq M \) and the hypotheses of Seymour’s Splitter Theorem now hold for \( M \) and \( N \).

(VII). If \( x \in E(M) \) and \( \text{si}(M/x) \) is 3-connected with an \( N \)-minor, then \( x \in T \).

**Subproof:** Suppose the contrary. By \([11]\) \( x \in \text{cl}_M(T) \), which is a line with more than 3 points. As \( M\backslash x \) is 3-connected for all \( z \in \text{cl}_M(T) - T \), then \( M\backslash z \) has no \( N \)-minor if \( z \in \text{cl}_M(T) - T \). As \( M/x \) has an \( N \)-minor, \( \text{cl}_M(T) = T \cup x \) and \( M\backslash x \) has no \( N \)-minor. This implies that \( r^*(M) - r^*(N) \geq 2 \) as \( T \) is in a parallel class of \( M/x \).

Let us check that for each \( z \in E(M) - x \), \( \text{si}(M/z) \) is not 3-connected with an \( N \)-minor. Suppose the contrary, by \([11]\) \( z \in \text{cl}_M(T) \), this implies that \( M\backslash x \) has an \( N \)-minor, a contradiction. So \( x \) is the unique element of \( M \) such that \( \text{si}(M/x) \) is 3-connected with an \( N \)-minor. By Theorem \([13]\) \( r(M) - r(N) = 1 \).

Consider the structures defined as in Corollary \([11]\). By what we proved, for all choices of \( M_1, \ldots, M_n \), we have \( I = \{x\} \) and \( n = 3 \) or 4. As \( M\backslash x \) has a parallel class with 3 elements, then \( x = x_3 \) or \( x = x_4 \), so, we have two cases to consider:
Case 1. We may pick $M_1, \ldots, M_n$ with $x = x_n$: For all $y \in E(M) - x$ such that $M\setminus y$ is 3-connected with an $N$-minor, we have $y \in T$ by (III). In particular this holds for each $y \in I^* \cup \cl_M(T) - x$. So, $I^* \cup (\cl_M(T) - x) \subseteq T$. This implies the validity of the theorem and, in particular, of item (a).

Case 2. Otherwise: Now, necessarily, $n = 4$ and $x = x_3$. Moreover, $M' := M\setminus x_1, x_2, x_4$ is not 3-connected, but so is $M'/x$. This implies that $x$ is in a cocircuit with size at most two in $M'$. As $M_2 = M\setminus x_1, x_2$ is 3-connected, then $x$ is in a serial pair of $M'$ with an element $z$. This implies that $M\setminus I^*/z \cong N$. By Lemma (III), $\si(M/z)$ is 3-connected with an $N$-minor. A contradiction to the uniqueness of $x$ established before.

(VIII). If $\co(M\setminus x)$ and $\co(M\setminus x, y)$ are 3-connected with $N$-minors, then $\co(M\setminus y)$ is 3-connected and $x, y \in T$.

Subproof: Suppose the contrary. By (III), $\co(M\setminus y)$ is not 3-connected. By the dual of Lemma (2) there is a corank-3 circuit $C$ containing $x$ with $y \in \cl_*(C) - C$.

First assume that $C \notin T$. If $M$ has a 4-cocircuit $D^*$ contained in $C \cup y$, then, as $|D^* \cap C| \geq 3$ and $T \not\subseteq C$, there is $z \in (D^* \cap C) - T$. So, $D^* - z$ is a triad of $M\setminus z$ contained in $\cl_{M\setminus z}((C - z)$ and, by the dual of Lemmas (3) and (4), $\co(M/z)$ is 3-connected with an $N$-minor. But this contradicts (III). Thus, $C \cup y$ contains no 4-cocircuit of $M$. But $r^*(C \cup y) = 3$ and $y \in \cl_*(C) - C$, so $C \cup y$ is the disjoint union of a singleton set $\{e\}$ and a non-trivial coline $L^*$ containing $y$. By the dual of Lemmas (3) and (4), $\co(M\setminus e)$ is 3-connected with an $N$-minor. By (III), $e \in T$. By the dual of Lemma (3), for some $f \in T^*$, $\co(M\setminus x, y) \cong \co(M\setminus f, y) \cong \co(M\setminus T^*)$ has an $N$-minor. Thus, $M\setminus y$ has an $N$-minor and, by Bixby's Lemma, $\si(M\setminus y)$ is 3-connected with an $N$-minor. By (VII), $y \in T$. Since $T$ meets $L^*$, it follows that $L^*$ is a triad and, as a consequence, $|C| = 3$. By orthogonality, there is $g \in (T^* \cap T) - y \subseteq C$. Since $e \in (C \cap T^*) - T^*$, then $C \cap T$ is a 4-segment of $M$ meeting a triad, a contradiction. Therefore, $C = T$.

Let $C^*$ be a cocircuit such that $y \in C^* \subseteq T \cup y$. If $C^*$ is a triad, we have a contradiction to (II) since $y \in C^* - T$ and $M\setminus y$ has an $N$-minor. So, $C^*$ is a 4-cocircuit and $C^* = T \cup y$. By Bixby's Lemma, $\si(M\setminus y)$ is 3-connected. Since $y \notin T$, then $M\setminus y$ has no $N$-minor by (VII). If $r(M) = r(N)$, then $N \cong M\setminus x, y$ and $M\setminus y$ is 3-connected, therefore $r(M) - r(N) \in \{1, 2\}$.

For all 2-subsets $A$ of $T$, $M\setminus A$ has no $N$-minor because, otherwise, $y$ would be in the serial pair $C^* - A$ of $M\setminus A$ and a $M\setminus y$ would have an $N$-minor.

As $\co(M\setminus x, y)$ has an $N$-minor, hence $r^*(M) - r^*(N) \geq 2$. If $r^*(M) - r^*(N) \geq 3$, then, by Theorem (12) there is $z \in E(M)$ such that $\co(M\setminus z)$ and $\co(M\setminus x, z)$ are 3-connected with $N$-minors. By (III), $z \in T$. So, for $A := \{x, z\} \subseteq T$, $M\setminus A$ has an $N$-minor, a contradiction. Therefore, $r^*(M) - r^*(N) = 2$. To prove the theorem and item (b), we have to find an $N$-minor $N'$ of $M$ with $E(M) - E(N') \subseteq T \cup y$. Consider a chain of matroids, sets and elements as in Corollary (11). Let $a$ and $b$, in this order, be the elements deleted from $M$ in order to get $M_n$ from $M$ as in the chain (recall that $r^*(M) - r^*(N) = 2$). By Lemma (10), $\co(M\setminus a)$ is 3-connected with an $N$-minor, hence, by (III), $a \in T$. It follows from Lemma (10) and (IV), (V) and (VII) that $I \subseteq T$. We just have to prove now that $b = y$. Suppose the contrary. If $b \notin T$, then, for $A := \{a, b\} \subseteq T$, $M\setminus A$ has an $N$-minor, a contradiction, as we saw before. Thus, $b \notin T$ and, by (III), $\co(M\setminus b)$ is not 3-connected. So, $a$ and $b$ play similar roles as $x$ and $y$ and applying the same steps for $a$ and $b$.
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as we did for \( x \) and \( y \), we conclude that \( D^* := T \cup b \) is a cocircuit of \( M \). By circuit elimination on \( C^* := T \cup y, D^* \) and any element \( e \) of \( T \), it follows from the cosimplicity of \( M \) and from the orthogonality with \( T \) that \( (T - e) \cup \{y, b\} \) is a cocircuit of \( M \). Therefore, \( y \) is in a series class of \( M \setminus a, b \), which has an \( N \)-minor. But this implies that \( M / y \) has an \( N \)-minor, a contradiction. So, \( b = y \) and (b) holds.

Now, consider the structures as given by Corollary I[11]. It follows from \([IV], [V]\) and \([VII]\) that \( I \subseteq T \). If \( r^*(M) - r^*(N) \leq 2 \), it follows from \([III]\) and \([VIII]\) that \( I^* \subseteq T \). This implies that \( T \subseteq E(M) - E(N') \) for \( N' \leq N \), and the theorem holds in this case. So, we may assume that \( r^*(M) - r^*(N) = 3 \). If \( |I| = 0 \), then \( N' = M \setminus I^* \) and \( M \setminus e \) is 3-connected with an \( N \)-minor for all \( e \in I^* \) and by \([III]\), \( I^* \subseteq T \). So, \( |I| \) \geq 1. By Lemma \([10]\) and \([VIII]\), the elements of \( I^* \) with the two least indices in \( x_1, \ldots, x_n \) are in \( T \). So, \( |T \cap I^*| \leq 1 \) and as \( I \subseteq T \), \( |I| = 1 \). Therefore, \( n = 4 \). Since \( I \subseteq T \), \( x_1 \in I \) by the simplicity of \( M_1 \). If \( I = \{x_1\} \), then \( M \setminus e \) is 3-connected for all \( e \in I^* \), a contradiction, as before. Therefore, \( I = \{x_3\} \text{ or } \{x_2\} \). This implies that \( T = \{x_1, x_2, x_3\} \).

By Theorem \([13]\) there is a 3-coindependent set \( J^* \) of \( M \) such that \( co(M \setminus e) \) is 3-connected with \( N \)-minor for all \( e \in J^* \). By \([III]\), \( J^* = \{x_1, x_2, x_3\} \). If \( T \) meets a triad \( T^* \), then, for \( f \in T^* - T \) and \( e \in T \cap T^* \) we have that \( M \setminus e \) and, therefore, \( M \setminus f \), have an \( N \)-minor. But in this case, by Lemma \([9]\), \( si(M \setminus f) \) is 3-connected with an \( N \)-minor, a contradiction to \([IV]\). Thus, \( T \) meets no triads of \( M \). Next, we check:

\((IX)\). We may not pick \( M_1, \ldots, M_4 \) in such a way that \( I = \{x_3\} \).

\textbf{Subproof:} Suppose the contrary. Then, we may pick the chain of matroids in such a way that \( M_4 = M \setminus x_1, x_2 / x_3 \setminus x_4 \) with \( I = \{x_3\} \). As \( x_4 \not\in T \), by \([III]\), \( co(M \setminus x_4) \) is not 3-connected and \( M^* \) has a vertical 3-separation \( (A, x_4, B) \). This is, both \( A \) and \( B \) are 3-separating sets of \( M \), \( x_4 \in cl^*(A) \cap cl^*(B) \) and \( r^*(A), r^*(B) \geq 3 \). So, \( (A, B) \) is a 2-separation of \( M \setminus x_4 \), but \( M \setminus x_4, x_1, x_2 / x_3 \) is 3-connected and we may assume, therefore, that \( |A - \{x_1, x_2, x_3\}| = |A - T| \leq 1 \). As \( |A| \geq 3 \), then \( |A \cap T| \geq 2 \) and \( A \) spans \( T \). This implies that \( Y = A \cup T \) is a 3-separating set of \( M \). Moreover, \( Y = T \) or \( Y = T \cup A \) for some \( y \in E(M) - T \).

Let us prove that \( T \cup x_4 \) is a cocircuit of \( M \). Note that \( x_4 \in cl^*(Y) \). If \( Y = T \), then, as \( T \) meets no triads, it follows that \( T \cup x_4 \) is a 4-cocircuit. So, we may assume that \( Y = T \cup y \) for some \( y \in E(M) - T \). If \( r_M(Y) = 2 \), \( Y \) is a 4-segment of \( M \). As \( M \setminus x_3 \) has an \( N \)-minor, then so has \( M \setminus y \). In this case, \( M \setminus y \) is 3-connected, contradicting \([III]\) since \( y \not\in T \). Thus, \( r_M(Y) = 3 \). Since \( |Y| = 4 \) and \( Y \) is 3-separating, it follows that \( r^*(Y) = 3 = r^*(T) \). Now, \( T \) cospans \( y \), and as \( T \cup y \) cospans \( x_4 \), it follows that \( T \) cospans \( x_4 \) and \( T \cup x_4 \) is a cocircuit of \( M \) since \( T \) meets no triads.

As \( T \cup x_4 \) is a cocircuit of \( M \), hence \( T - \{x_3, x_4\} \) is a serial pair of \( M_2 = M \setminus x_1, x_2 \) which is 3-connected with at least 4 elements, a contradiction. Thus, \((IX)\) holds.

Now, by \((IX)\), \( I = \{x_2\} \) for all choices of chains. This implies that there are no pair of elements \( \{a, b\} \subseteq E(M) \) such that \( M \setminus a \) and \( M \setminus a, b \) are 3-connected with an \( N \)-minor. In particular, \( M \setminus x_1, x_3 \) is not 3-connected. But \( M_3 = M \setminus x_1, x_2 / x_3 \) is 3-connected. As \( M \setminus x_1 \) is 3-connected, then \( x_2 \) is in a serial pair \( \{x_2, z\} \) of \( M \setminus x_1, x_3 \). Hence, \( M / z \) has an \( N \)-minor. Since \( M \setminus x_1 \) is 3-connected, it follows that \( \{z, x_2, x_3\} \) is a triad of \( M \setminus x_1 \). But \( T \) meets no triads of \( M \) and, therefore, \( C^* := \{z, x_1, x_2, x_3\} = T \cup z \) is a 4-cocircuit of \( M \). If \( z \in cl(T) \), then \( r^*(C^*) = 2 \) and \( C^* \) is a 2-separating set of \( M \). This implies that \( r(T) = r(C^*) = 2 \), contradicting the fact that
We define $K_{3,3}^{1,1}$ as the graph in Figure 1. The following lemma is a well-known result and is a straightforward consequence of Seymour’s Splitter Theorem.

**Lemma 15.** If $G$ is a 3-connected graph with a $K_5$-minor then, either $G \cong K_5$ or $G$ has a $K_{3,3}^{1,1}$-minor.

![Figure 1. $K \cong K_{3,3}^{1,1}$](image)

**Proof of Theorem 4:** We have to prove that for each 3-connected simple graph $G$ with a $K_5$-minor and for each triangle $T$ of $G$, $G$ has a $K_5$-minor using $E(T)$. Consider a counter-example $G$ with $|E(G)|$ as small as possible. By Theorem 2, we may assume $E(G) - E(K_5) \subseteq E(T)$. As no edges may be added to $K_5$ in order to get a 3-connected simple graph, then $|G| = 6$ or $7$. If $|G| = 7$, then $G$ is obtained from $K_5$ by expanding a vertex into the triangle $T$. In this case, there are two vertices $u, v \in V(T)$ with degree 3 and it is clear that for the edges $e, f \in E(T)$ incident to $u$ and $v$ respectively, we have $si(G/e, f) \cong K_5$. So, we may assume that $|G| = 6$. By Lemma 15, up to labels, $G$ is obtained from $K \cong K_{3,3}^{1,1}$ (the graph in Figure 1), by adding the edges of $E(T) - E(K)$. Since $K/uv \cong K_5$, then $uv \in T$ and we may assume without losing generality that $V(T) = \{u, v, a\}$, so $G = K + va$. Now, it is clear that $G/ba/ub$ is a $K_5$-minor of $G$ using $T$. This proves the theorem. □

The following Lemma has a slightly stronger conclusion than [11, Proposition 9.3.5] (it states beyond that $R$ has a $K$-minor, the way it is obtained), but the proof for [11] Proposition 9.3.5] also holds for the following Lemma.

**Lemma 16.** Let $R = K \oplus_3 L$ be a 3-sum of binary matroids, where $K$ and $L$ are 3-connected and $E(K) \cap E(L) = S$. Then there are $X, Y \subseteq E(L) - S$ such that $R/X\setminus Y$ is obtained from $R$ by relabeling the elements $s_1, s_2$ and $s_3$ of $S$ in $K$ by respective elements $l_1, l_2$ and $l_3$ of $L$.

**Proof of Theorem 5:** Let $R$ be a regular matroid with a triangle $T$ and an $M$-minor for $M \in \mathcal{F}$. Let us check that $R$ has an $F$-minor using $T$ for some $F \in \mathcal{F}$. If $R$ is graphic or cographic it is trivial, so assume the contrary. By Seymour’s Decomposition Theorem for regular matroids, there are matroids $K$ and $L$ with at least 7 elements each intersecting in a common triangle $S$ such that $R = K \oplus_3 L$ with $L$ being 3-connected and $K$ being 3-connected up to parallel
classes of size two meeting $S$. Under these circumstances, we may assume that $|E(K) \cap E(M)| \geq |E(L) \cap E(M)|$.

If $C$ is a cycle of $R$ meeting both $E(K)$ and $E(L)$, then there is $s \in S$ such that $(C \cap E(N)) \cup s$ is a cycle of $N$ for $N = K, L$. As we picked $L$ with no parallel pairs, it follows that $cl_R(E(K) - S) \cap E(L) = \emptyset$.

Let us first check that $K$ has an $M$-minor. Let $M = R/I \setminus I^*$ for some independent set $I$ and co-independent set $I^*$ of $R$. Since $\lambda_M(E(K) \cap E(M)) \leq \lambda_R(E(K) \cap E(M)) = 2$, then as $M$ is internally 4-connected, it follows that $|E(M) \cap E(L)| \leq 3$, and, moreover, $E(M) \cap E(L)$ is not a triad of $M$ because $M$ has no triads. This implies that $E(M) \cap E(L) \subseteq cl_M(E(M) - E(L))$. By the format of the family of circuits of $R$, it follows that $E(M) \cap E(L) \subseteq cl_R(E(K) - S)$, which is empty. So, $E(M) \subseteq E(K)$. By Lemma 11 there is a minor $K'$ of $R$ obtained by relabeling the elements $s_1, s_2$ and $s_3$ of $S$ in $K$ by respective elements $I_1, I_2$ and $I_3$ of $L$. Consider the matroid $K''$ obtained from $K'$ by contracting each $I_l$ for those indices $i \in \{1, 2, 3\}$ such that $s_i \in cl_L(I \cap E(L))$. Now $K''$ is obtained from $R/(I \cap E(L)) \setminus (E(L) \cap I^*)$ by relabaling the remaining elements of $S$. This implies that $K''$ and, therefore, $K$, have $M$-minors.

If $T \subseteq E(K)$, then $K$ has an $\mathcal{F}$-minor using $T$ by the minimality of $R$. But $R$ has an $K$-minor using $T$ by Lemma 11 and this implies the theorem. So, $T$ meets $E(L)$. As $cl_R(E(K) - S) \cap E(L) = \emptyset$, it follows that $X := T \cap E(L)$ has at least two elements. As $L$ is 3-connected and $\lambda_L(S) = \lambda_L(X) = 2$, then $\kappa_L(S, X) = 2$. By Tutte’s Linking Theorem [11, Theorem 8.5.2], there is a minor $N$ of $L$ with $E(N) = S \cup X$ such that $\lambda_N(S) = 2$. Hence:

$$2 = \lambda_N(S) = r_N(S) + r_N(X) - r(N) \leq 4 - r(N).$$

So, $r(N) \leq 2$. But $r(N) \geq r_N(S) \geq \lambda_N(S) = 2$. Also $r_N(X) \geq \lambda_N(X) = 2$. This implies that $S$ spans $N$ and $X$ contains no parallel pairs of $N$. Now, each element of $X$ is in parallel with an element of $S$ in $N$. Therefore, for $N = L/A \setminus B$, we have that $R/A \setminus B$ is obtained from $M$ by relabeling the elements of $S$ by elements of $T$. So, $R/A \setminus B$ is 3-connected with $T$ as triangle and has an $M$-minor. By the minimality of $R$, $R/A \setminus B$ has an $\mathcal{F}$-minor using $T$ and this proves the Lemma.

\[ \square \]

3. Sharpness

First we construct a sharp case for Theorem 3 with $E(M) \subseteq E(N) \cup T$. Consider a complete graph $K$ on $n \geq 14$ vertices. Let $X := \{v_{i,j} : i = 1, 2, 3$ and $j = 1, 2, 3, 4\}$ be a 12-subset of $V(K)$. Consider a triangle $T$ on vertices $u_1, u_2$ and $u_3$, disjoint from $K$. Let $G = K \cup T + \{u_iv_{i,j} : i = 1, 2, 3$ and $j = 1, 2, 3, 4\}$. Define, for disjoint subsets $A$ and $B$ of $E(T)$, $H := G \setminus A/B$, $M := M(G)$. For each $x \in E(G) - E(T)$, $G \setminus x$ has at least 3 parallel pairs and, therefore, no $H$-minor. For $e$ incident to $v \in V(K)$, in order to get a minor of $G \setminus e$ with $|K| - 12$ vertices with degree $|K| - 1$ and 12 vertices with degree $|K|$ as in $H$, it is necessary to contract some edge out of $T$, thus $G \setminus e$ has no $H$-minor either.

Now, let us construct an example satisfying item (a) of Theorem 3. Let us pick $M$ as a restriction of the affine space $\mathbb{R}^3$. Consider a 4-subset $L := \{a, b, c, x\}$ of an line $R$. Let $T := L - x$. Now consider for each $y \in T$ a line $R_y$ meeting $L$ in $y$ in such a way that no three lines among $R, R_a, R_b$ and $R_c$ lay in a same plane. Let $m \geq 6$. For each $y \in T$, pick a $m$-subset $L_y$ of $R_y$.
containing $y$. Let $M$ be the restriction of the affine space to $L \cup L_a \cup L_b \cup L_c$. Let $N = M/x\backslash a, b$ or $N = M/x\backslash T$. Note that it is not possible to get a rank-3 minor of $M$ with $3$ disjoint $(m-1)$-segments by contracting other element than $x$. So all $N$-minors of $M$ are minors of $M/x$ and therefore, deleting at least two elements of $T$ is necessary to get an $N$-minor since $T$ is a parallel class of $M/x$. So, this is the unique way to get an $M/x\backslash a, b$-minor of $M$. Moreover, deleting the element in the intersection of the three $m$-lines is the unique way to get and $M/x\backslash T$ from an $M/x\backslash a, b$-minor of $M$.

Next, we construct an example satisfying item (b) of Theorem 3. We denote by $M + e$ the matroid obtained by adding $e$ freely to $M$. Start with a projective geometry $P$ with $r(P) \geq 6$. Let $F$ be a flat of $P$ with $4 \leq r(F) \leq r(P) - 2$. Consider a copy $U$ of $U_{2,4}$ on ground set $T \cup x := \{x, x_1, x_2, x_3\}$ with $(T \cup x) \cap E(P) = \emptyset$. Let $y$ be an element out of $E(P) \cup T \cup x$. Let $M$ be the matroid obtained by adding $y$ freely to the flat $F \cup T$ of $(P + x) \oplus U$. Note that $E(P)$ is a hyperplane of $M$ and, therefore, $T \cup y$ is a 4-cocircuit of $M$. Define $N_1 = M/y\backslash x_2\backslash x_3$ and $N_2 := N_1/x_1$. Note that $N_1 = P + x_1$ and $N_2$ is the truncation of $P$ with rank $r(P) - 1$.

Let $i \in \{1, 2\}$ and $N = M/X\backslash Y$ be an $N_i$-minor of $M$. Note that $r^*(M) - r^*(N) = 2$ and $r(M) - r(N) \in \{1, 2\}$. For each $p \in E(N)$, $|E(s_i(M/p))| < |E(N_i)|$. Thus, no element $p \in P$ may be contracted in $M$ in order to get an $N_i$-minor. So, $X \subseteq T \cup y$.

Let us check that $T \cup y$ meets no circuit of $M$ with less than six elements other than $T$. Indeed, $M/y$ is a two sum of a 4-point line on $T \cup x$ and a matroid with rank greater than five with $x$ as free element. Thus all circuits of $M/y$ meeting $T$, except for $T$ itself, have more than five elements. Moreover, $M$ is obtained from $M/y$ adding $y$ as a free element to a flat with rank greater than 4 and, therefore, all circuits of $M$ containing $y$ also have more than five elements. Hence, the triangles of $M/X$ are precisely the triangles of $P$. Moreover, those must be the same triangles of $N$ since all triangles of $N$ are triangles of $M/X$ and they occur in the same number. As deleting an element of $P$ from $M/X$ would result in a matroid with less triangles than $N_i$, it follows that $Y \subseteq T \cup y$. Hence, $E(P) \subseteq E(N)$ for each minor $N$ of $M$ isomorphic to $N_1$ or $N_2$.

Let us check that $M/y$ has no $N_2$-minor and, therefore, no $N_1$-minor too. Suppose for a contradiction that $N$ is an $N_2$-minor of $M/y$. We may assume that $N = M/y, x_1 \backslash x_2, x_3$. Note that $x_1$ is a free element of the rank-$r_F$ flat $F \cup x_1$ of $M/y\backslash x_1, x_2$. This implies that $N[F]$ is a truncation of rank $r_F - 1$ of the rank-$r_F$ projective geometry $F$. But, as $N_2$ is the rank-$(r_F - 1)$ truncation of $P$ and $r_F \leq r(P) - 2$, then all rank-$(r_F - 1)$ flats of $N_1$ are projective geometries and so is $F$, a contradiction.

Now, for $i = 1, 2$ each $N_i$-minor of $M$ is the form $M/y, x_i/A$ with $1 \leq i \leq 3$ and $A$ being a $i$-subset of $T - x_i$. Let $A$ be a 2-subset of $T$. As we proved for $A = \{x_2, x_3\}$, it follows that $M/A$ has no $N_2$-minor. Moreover, for $x_k \in T - A$ it is clear that $y$ is not a free element of $M\backslash A/x_k$, which, therefore, is not isomorphic to $N_1$. Thus $M\backslash A$ has no minor isomorphic to $N_1$ and neither $N_2$.

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