On a subtle point of sum rules calculations: toy model.

A.A. Penin

and

A.A. Pivovarov

Institute for Nuclear Research of the Russian Academy of Sciences,
Moscow 117312, Russia

Abstract

We consider a two-point correlator in massless $\phi^3$ model within the ladder approximation. The spectral density of the correlator is known explicitly and does not contain any resonances. Meanwhile making use of the standard sum rules technique with a simple "resonance + continuum" model of the spectrum allows to predict parameters of the "resonance" very accurately in the sense that all necessary criteria of stability are perfectly satisfied.

PACS number(s): 11.50.Li
QCD sum rules have revealed themselves as a powerful method to study hadron properties [1-4]. However certain effects were discovered that disprove efficiency of the method. Thus in the very beginning of QCD sum rules it has been found that instantons lead to the appearance of so-called exponential terms which are omitted within standard operator product expansion (OPE) and induce the strong dependence of sum rules results on detailed structure of QCD vacuum [5]. Recently it has been pointed out that sum rules can also suffer from infrared renormalons that cannot be associated with the expectation value of local operators [6]. Fortunately, in most of the practically interesting cases these undesirable effects can be separated. Though some important questions still remain unclear. The use of sum rules implies hadron properties to be mainly determined by several leading terms of asymptotic expansion of the correlator of relevant interpolating currents in deep Euclidean domain that cannot be guaranteed by itself. This problem was studied within two-dimensional electrodynamic (the Schwinger model) [7] where the exact solution is known and sum rules calculations can be directly checked. In these papers the stability of the result with respect to inclusion of higher order corrections was proposed as an intrinsic criterion of sum rules applicability. In the present paper we demonstrate within another exactly soluble model an opposite example when truncation of the asymptotic expansion of a correlator leads to missing the main properties of its spectral density while the formal stability requirement for corresponding sum rules is completely satisfied . This situation reflects some general property of sum rules and can be realized in QCD as well.

Let us describe the model. We consider a ladder approximation of massless $\phi^3$ model. In the $\phi^3$ model the expansion parameter is dimensionless ratio of the dimensionful coupling constant and the energy of the process in question, so the interaction dies out in the limit of infinitely large energies. On the other hand at low energies the parameter of the expansion becomes large and the model requires a non-perturbative analysis. Thus it would be instructive to investigate the model along the line of ordinary QCD methods. Not to be taken quite seriously it gives nevertheless a way to go beyond the perturbation theory because here the explicit expressions for diagrams in any order of loop expansions are known [8-10]. The model is attractive also because other attempts to break the bound of perturbation theory tend to be in dimensions different from 4.

We modify the usual $\phi^3$ model slightly to make it more convenient for our purpose. The Lagrangian of our model reads

$$L = L_0 + e_1\phi^2A + e_2\phi^2A$$

where $L_0$ is a free kinetic term for all fields and we choose $e_2 = -e_1 = e$. All questions of stability of the model (the existence of a stable ground state, for example) remain beyond the scope of our toy consideration.

We study a correlator of two composite operators $j$ = $\varphi\phi$ in the ladder approximation. The correlator has the form [8]

$$\Pi(q) = i \int \langle 0|Tj(x)j(0)|0\rangle e^{iqx}dx, \quad \Pi(Q^2) = \frac{1}{16\pi^2} \ln \left( \frac{\mu^2}{Q^2} \right) + \Delta \Pi(Q^2)$$

where

$$\Delta \Pi(Q^2) = \frac{1}{16\pi^2} \sum_{L=2}^{\infty} \left( -\frac{e^2}{16\pi^2Q^2} \right)^{L-1} \left( \frac{2L}{L} \right) \zeta(2L-1),$$

(3)
$Q^2 = -q^2$ and $\zeta(z)$ is the Riemann’s $\zeta$-function.

Since the coupling constant $e$ is dimensionful the expansion (3) simulates power corrections or OPE of ordinary $QCD$. Setting $e^2/16\pi^2 = 1$ we get

$$p(Q^2) \equiv 16\pi^2 \Pi(Q^2) = \ln \left( \frac{\mu^2}{Q^2} \right) + \Delta p(Q^2)$$

and

$$\Delta p(Q^2) = \sum_{L=2}^{\infty} \left( -\frac{1}{Q^2} \right)^{L-1} \left( \frac{2L}{L} \right) \zeta(2L-1)$$

(4)

in analogy with $QCD$ where the scale is given and all condensates are expressed through $\Lambda_{QCD}$.

What we see first is the alternating character of the series (4) in Euclidean direction ($Q^2 > 0$) due to the special choice of interaction (1). This was actually the reason to choose all decorations for the simple $\phi^3$ model, in which the series has the same sign for each term for Euclidean $q$.

The sum (4) has the following closed form [10]

$$-\sum_{n=1}^{\infty} nQ^2 \left[ \left( 1 + \frac{4}{n^2 Q^2} \right)^{-\frac{1}{2}} - 1 + \frac{2}{n^2 Q^2} \right]$$

and does not contain any resonances.

The spectral density for the function $\Delta p(Q^2)$ reads

$$\Delta \rho(s) = -\sum_{n=1}^{\infty} \frac{ns\sqrt{s}}{\sqrt{\frac{4}{n^2} - s}} \theta \left( \frac{4}{n^2} - s \right) \theta(s).$$

(5)

Unfortunately the whole spectral density of the correlator, $\rho(s) = 1 + \Delta \rho(s)$, has negative sign in some domains. In case $e_1e_2 > 0$ it would have a correct positive sign but the cut would be situated in the wrong place of the complex plane (negative semiaxis). So, any choice of the interaction sign leads to unphysical spectral density and the set of ladder diagrams is hardly representative for the exact correlator $\langle 0|Tj(x)j(0)|0 \rangle$. The same situation is realized in $QCD$. If one takes seriously the leading order correlator for vector currents, for example, one finds that the spectral density contains an unphysical pole at $Q^2 = \Lambda^2_{QCD}$ and does not satisfy the spectrality condition being negative at $s < \Lambda^2_{QCD}$.

We omit these delicate points and proceed as in $QCD$. We pose a question: does the asymptotic behavior of the correlator (2) contradict to the presence of a resonance in the considered channel? Namely, whether the expansion (4) can be described successfully with the simple formula

$$\rho^{test}(s) = F\delta(s - m^2) + \theta(s - s_0)$$

(6)

for the spectral density $\rho(s)$.

An explicit expansion for the correlator reads

$$\Delta p(Q^2) = -\frac{6\zeta(3)}{Q^2} + \frac{20\zeta(5)}{Q^4} - \frac{70\zeta(7)}{Q^6} + \frac{252\zeta(9)}{Q^8} + \ldots$$

(7)
while the "test" form of the correlator is

\[ p_{\text{test}}(Q^2) = \ln \left( \frac{\mu^2}{Q^2 + s_0} \right) + \frac{F}{Q^2 + m^2}. \]  

(8)

We intend to connect the expressions (7) and (8) by means of sum rules. First we use finite energy sum rules \( (F\text{ESR}) \) [11]

\[ \int_0^{s_0} s^k \rho(s) ds = \int_0^{s_0} s^k \rho_{\text{test}}(s) ds \]  

(9)

where \( k = 0, 1, 2 \) because the test spectral density has three parameters to be determined. Eq. (9) has an explicit form

\[ F = s_0 - 6\zeta(3), \quad Fm^2 = \frac{1}{2}s_0^2 - 20\zeta(5), \quad Fm^4 = \frac{1}{3}s_0^3 - 70\zeta(7) \]  

(10)

that leads to the equation for determination of the duality threshold \( s_0 \)

\[ \frac{1}{12}s_0^4 - 2\zeta(3)s_0^3 + 20\zeta(5)s_0^2 - 70\zeta(7)s_0 + 420\zeta(7)s_0^3 - 400\zeta(5)^2 = 0. \]  

(11)

It has solutions \( s_0 = 16.9 \) and \( s'_0 = 2.24 \), and for corresponding parameters \( F = 9.67 \) and \( m^2 = 12.6 \) whilst \( F' = -4.97 \) and \( m'^2 = 3.67 \). We will study the first solution because the second one gives an unnatural relation between "resonance mass" and "duality interval" \( s'_0 < m'^2 \).

Now we check corresponding expressions in the Borel sum rules approach [\text{I}]. The Borel transformation \( p(M^2) \) of the function \( p(Q^2) \) is

\[ p(M^2) = 1 - \frac{6\zeta(3)}{M^2} + \frac{20\zeta(5)}{M^4} - \frac{35\zeta(7)}{M^6} + \frac{42\zeta(9)}{M^8} + \ldots, \]  

(12)

continuum gives

\[ c(M^2) = \exp \left( -\frac{s_0}{M^2} \right) \]  

(13)

and the resonance contribution reads

\[ r(M^2) = \frac{F}{M^2}\exp \left( -\frac{m^2}{M^2} \right). \]  

(14)

We have plotted these functions in Fig. 1. As we see our "test" representation accurately simulates the asymptotic form of the correlator at \( M^2 > 8 \). Equating the functions \( p(M^2) \) and \( r(M^2) + c(M^2) \) we obtain the Borel sum rules to determine the parameters of the resonance. The sum rules look like ordinary \( QC\bar{D} \) sum rules. Fig. 2 shows the dependence of mass \( m^2 \) on the Borel variable \( M^2 \) with the parameters \( s_o \) and \( F \) given by \( F\text{ESR} \) for the different numbers of power corrections included. For other close values of the parameters \( s_o \) and \( F \) the results are not very different from fig. 2. A stable region is reached for \( 8 < M^2 < 18 \) where, on the one hand, higher order power corrections are convergent and, on the other hand, the continuum contribution (13) does not prevail over the resonance one (14). The best stability is obtained at the optimal values \( s_o = 16.6, F = 9.39 \) that shows a consistency of the Borel sum rules and \( F\text{ESR} \). Furthermore, from the curves in fig. 2 we see that the inclusion of the high order power corrections does not destroy the Borel sum rules and even extends the region of stability.
Let us emphasize that the sum rules are perfectly saturated by the artificially introduced resonance and show very good stability though the exact spectral density does not contained any resonance singularities. The reason of this phenomenon is quite transparent: using sum rules we neglect the high order terms of large momentum expansion which do not affect the rough integral characteristics of the spectral density but are essentially responsible for its local behavior. The extreme sensitivity of the local form of the spectral density to the high orders of perturbative expansion can be easily demonstrated within considered model. For example, substituting $\zeta$-functions for $L > 2$ by units in the series (4) one neglects the terms with $n > 2$ in eq. (5) and gets the modified spectral density

$$\Delta \tilde{\rho}(s) = -s\sqrt{s} \theta(s) \left( \frac{\theta(4-s)}{\sqrt{4-s}} + \frac{\theta(1-s)}{\sqrt{1-s}} \right).$$  \hspace{1cm} (15)

As we see the tiny variation of the coefficients of the asymptotic expansion leads to a drastic change of the spectral density at low scale $s < 4/9$. Indeed, the modified spectral density has two singular points at $s = 4$ and 1 while the original expression has an infinite number of singularities at $s_n = 4/n^2$, $n = 1, 2, \ldots$. At the same time this variation does not really affect the sum rules result that changes slightly (less then 10%): $\bar{s}_0 = 17.9$, $\bar{F} = 10.7$, $\bar{\eta}^2 = 13.0$.

The lesson we drew from the above consideration is that applying sum rules technique one can find sum rules to be stable while the leading terms of $OPE$ are hardly related to the hadron parameters, i.e. the non-perturbative information contained in the basic $QCD$ condensates is essentially insufficient to determine the hadron properties or the physical spectral density has unexpectedly complicated form. At the same time a direct method to distinguish and separate this effect is absent now. So sum rules approach faces a subtle problem and another criterion of reliability of sum rules predictions is necessary instead of naive stability requirement.

Acknowledgments
This work is supported in part by Russian Fund for Fundamental Research under Contract No. 93-02-14428, 94-02-06427 and by Soros Foundation.

Figure Captions
Fig. 1. The resonance contribution $r(M^2)$ to the Borel sum rules (eq.(14)) and the function $\tilde{r}(M^2) = p(M^2) - c(M^2)$ (eq.(12,13)).
Fig. 2. The mass $m^2$ plotted as a function of Borel variable $M^2$ with two (a), three (b) and four (c) orders of power corrections included. The arrows mark stability interval of the Borel sum rules.

References
[1] M.A.Shifman, A.I.Vainshtein and V.I.Zakharov, Nucl.Phys. B147(1979)385.
[2] L.J.Reinders, H.R.Rubinstein and S.Yazaky, Phys.Rep. 127(1985)1.
[3] M.A.Shifman, ”QCD sum rules: physical picture and historical survey” (Lectures given at 11th Autumn School of Physics, Lisbon, Portugal, Oct 9-14, 1989), Published in Lisbon School 1989.

[4] S.Narison, ”QCD Spectral Sum Rules” (Lecture Notes in Physics, Vol. 26), World Scientific, Singapore 1989.

[5] V.A.Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl.Phys. B191(1981)301.

[6] I.I.Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, preprint CERN-TH.7171/94.

[7] A.A.Pivovarov, N.N.Tavkhelidze and V.F.Tokarev, Phys.Lett. B132(1983)402; A.A.Pivovarov and V.F.Tokarev, Yad.Fiz. 41(1985)524.

[8] N.I.Ussukina and A.I.Davydychev, Phys.Lett. B305(1993)136.

[9] V.V.Belokurov and N.I.Ussukina, J.Phys. A16(1983)2811.

[10] D.J.Brodhurst, Phys.Lett. B307(1993)132.

[11] N.V.Krasnikov and A.A.Pivovarov, Phys.Lett B112(1982)397; N.V.Krasnikov, A.A.Pivovarov and N.N.Tavkhelidze, Z.Phys. C19(1983)301.
