COADJOINT ORBITS OF THE GENERALISED
$Sl(2)$, $Sl(3)$ KdV HIERARCHIES

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ABSTRACT

In this paper we develop two coadjoint orbit constructions for the phase spaces
of the generalised $Sl(2)$ and $Sl(3)$ KdV hierarchies. This involves the construction
of two group actions in terms of Yang Baxter operators, and an Hamiltonian reduc-
tion of the coadjoint orbits. The Poisson brackets are reproduced by the Kirillov
construction. From this construction we obtain a ‘natural’ gauge fixing procedure
for the generalised hierarchies.
1. Introduction

This paper analyses the initial steps in a coadjoint orbit construction for the generalised KdV hierarchies, [5,7]. The analysis proceeds principally through the medium of illustration, using the KdV hierarchies constructed on the Kac Moody algebras \( \hat{sl}(2) \) and \( \hat{sl}(3) \). In this paper we prove that for these theories there exist two orbit constructions for the phase space, reproducing via the Kirillov construction, [13], the two Poisson brackets constructed in [5].

The Coadjoint Orbit Method (also known as the Adler-Kostant-Symes formalism, or AKS formalism) is a construction that uses Lie algebras to define integrable models, [3,9]. The essential ingredients are a Lie algebra \( g \), and an endomorphism \( R : g \rightarrow g \) satisfying the modified Yang Baxter Equation, mYBE. The mYBE implies that the bracket \([X,Y]_R = [RX,Y] + [X,RY]\) satisfies the Jacobi identity, and thus defines a second Lie bracket on the Lie algebra \( g \). The Kirillov construction for Poisson brackets on the dual Lie algebra \( g^* \), [13], defines two Poisson brackets \( \{ , \}_R \) and \( \{ , \}_\sigma \) on \( g^* \) induced from \([ , \]_R\) and \([ , \]_\sigma\) respectively. The fact that the Poisson bracket \( \{ , \}_R \) is constructed from a group action allows the symplectic leaves of \( \{ , \}_R \) to be constructed, each symplectic leaf, \( \mathcal{O} \), furnishing a phase space for a dynamical system with the symplectic structure \( \{ , \}_R|_{\mathcal{O}} \). Further, a set of commuting Hamiltonians can be constructed, these being the \( \text{Ad}^* \)-invariant functions. Thus there exist a set of Hamiltonians \( \{ H_i \} \) satisfying \( \{ H_i, H_j \}_R = 0 \) which generate, under Poisson brackets, a set of commuting time flows. The fact that these theories are integrable follows from the existence of sufficient commuting Hamiltonians.

The aspect of this method that concerns us in this paper is the construction of the symplectic leaves of \( \{ , \}_R \). These are the coadjoint orbits of the group \( G_R \) on \( g^* \), where \( G_R \) denotes the exponential of the Lie algebra \( g \) with commutator \([ , \]_R\). If there exists an inner product on \( g \), then the dual algebra can be identified with the Lie algebra. Thus the coadjoint orbits are identified with subspaces of \( g \). In the case of a current algebra with the Schwinger central extension, the theory is of Lax type, with a Lax operator of the form \( \mathcal{L} = \partial_x + \kappa \), where \( \kappa \in \mathcal{O} \subset g \). The equations of motion take the form of the zero curvature condition, \( [\partial_t + M_i, \partial_x + \kappa] = 0 \), where \( M_i \) is related to the functional derivative of the Hamiltonian \( H_i \) generating the time coordinate \( t_i \), \( M_i = R(d \kappa H_i) \).

We observe that the KdV hierarchies of [7,8] have many features reminiscent of this coadjoint orbit construction. In particular, the theories are constructed by exploiting the structure of a Lie algebra, \textit{i.e.} the current algebra on a loop algebra \( \hat{g} \); the Poisson structures are expressed in terms of Lie brackets involving R-operators, [5]; and the equations of motion take the form of the zero curvature equations. In addition, there exist coadjoint orbit formulations for the Toda Chain, [3], and Non-linear Schrödinger equation, [9], both special cases of the generalised KdV-hierarchies, [7]. These are special cases because they have no gauge group, and only possess a single Poisson structure. The KdV hierarchies are in general bi-Hamiltonian, [5], and it is for this reason that a coadjoint orbit construction may be inappropriate in describing these theories; there appears to be no known method to extend the AKS procedure to create bi-Hamiltonian systems. In this paper, we attempt to initiate a bi-Hamiltonian construction from AKS systems. The idea is to construct two Lie algebra commutators on the current algebra \( C^\infty(S^1, \hat{g}) \), denoted \([ , , ]_R \) and \([ , , ]_\sigma \), such that the Poisson brackets of the KdV hierarchies, [5], are reproduced by the Kirillov
bracket construction: $\{ , \}_\sigma = \{ , \}_1, \{ , \}_R = \{ , \}_2$. We denote the ‘exponential’ of these Lie algebras by $G_R, G_\sigma$ respectively. If we perform the AKS process, we would obtain two integrable systems with phase spaces $O_R$ and $O_\sigma$, coadjoint orbits of the groups $G_R$ and $G_\sigma$ respectively, and a set of commuting Hamiltonians that are identical for both theories. The idea is that the gauge symmetry of the KdV hierarchy is the additional ingredient that equates the two theories dynamically. We perform an Hamiltonian reduction on the two orbits such that the reduced phases spaces become identified. Further, the symplectic structures are inequivalent, leading to a bi-Hamiltonian structure.

This is in fact a simplification of the process. Our final conclusion is that the reduced phase space of $O_\sigma$ is identical to the phase space of a generalised KdV hierarchy, for an appropriate choice of orbit and gauge group. However, the reduced orbits of the group $G_R$ foliate this phase space, i.e. under the Poisson bracket $\{ , \}_R$, the phase space of the generalised KdV hierarchy is not symplectic, and breaks it into symplectic leaves, [21], leaves that can be reproduced as Hamiltonian reductions of $G_R$-orbits. The flows of the hierarchy are such that this foliation is preserved, and thus there is no inconsistency. This foliation induces a partition of the potentials of the hierarchy into ‘types’, part of this partition reproducing the distinction between mKdV type, and ‘true’ KdV type potentials.

The coadjoint orbit structure proposed here should be contrasted with the construction for the $sl(2)$-KdV hierarchy as a coadjoint orbit of a central extension of $Diff^+(S^1)$, [14,17,18]. This differs from the orbit structures considered in this paper, because we do not consider reparametrisations of $S^1$. This structure is specific to the case of $sl(2)$, not generalising to more general hierarchies. We further comment that a coadjoint orbit construction of the traditional $A_n$-KdV hierarchies exists within the framework of Pseudo-Differential Operators, [2]. Orbits of a group of formal pseudo-differential symbols are constructed, the second Poisson bracket $\{ , \}_2$ of the hierarchy being reproduced as the orbit symplectic structure. Since the generalised hierarchies do not appear to possess a description in terms of pseudo-differential operators, this orbit structure cannot be generalised to these cases.

This paper is organised as follows. In section 2 we review the theory of the momentum map and the theory of Hamiltonian reduction. We specialise this discussion to the case of the Hamiltonian reduction of a coadjoint orbit. In section 3 we review the content of [5] and the definition of the two Poisson brackets of the generalised KdV hierarchies. We propose, in section 4, a construction of the KdV hierarchies as the Hamiltonian reduction of a coadjoint orbit of the group $G_\sigma$. We further propose that an Hamiltonian reduction of the coadjoint orbits of the $G_R$-action are capable of describing the dynamics, providing an explanation for the existence of the two Poisson structures. In section 5 we construct the momentum maps for the two gauge groups, $H_R, H_\sigma$, the symmetry groups to be used in the Hamiltonian reduction of the orbits $O_R, O_\sigma$ of the traditional $A_n$-hierarchies. In section 6, we discuss the $G_R$-action, and the reduction by the gauge group $C^\infty(S^1,\mathbb{N}^-)$. In the following two sections we analyse the traditional $Sl(2), Sl(3)$ KdV hierarchies as coadjoint orbit systems. In section 9 we extend the momentum map analysis to include the case of the fractional KdV hierarchies, [4], and generalised hierarchies of [7]. In section 10, we discuss the reduction of the $G_R$-orbits for these theories, the symmetry group of $O_R$ being a subgroup of $C^\infty(S^1,\mathbb{N}^-)$ in general, and not $C^\infty(S^1,\mathbb{N}^-)$ itself. We use these results.
in our two further examples involving the following non-traditional choices for \( \Lambda \),

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
z & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
z & 0 & 0 \\
0 & z & 0
\end{bmatrix}.
\]

Throughout this paper we shall not treat the difficulties involved in infinite dimensional phase spaces, assuming that the finite results generalise without difficulty.

2. HAMILTONIAN REDUCTION: GENERAL THEORY

In this section we summarise the salient features of momentum mappings and Hamiltonian reduction. This exposition follows that of [1].

Consider a symplectic manifold \((P, \omega)\) with group action \(\Phi : G \times P \rightarrow P\), such that \(\Phi_g\) is a symplectomorphism for all \(g \in G\). The momentum map is defined as follows:

**Definition 1.** The momentum mapping is a map \(J : P \rightarrow g^*\) such that the Hamiltonian functions \(\hat{J}_X\) defined by

\[
\hat{J}_X(x) = \langle J(x), X \rangle, \quad \forall X \in g, x \in P,
\]

(2.1)

generate the action \(\Phi\) under Poisson brackets, i.e.

\[
d\hat{J}_X = i_X \omega, \quad \text{where} \quad (X, i_Y \omega) = (X \wedge Y, \omega), \quad \forall X, Y \in TP.
\]

(2.2)

Note that here we are using the embedding \(g \rightarrow TP\) induced from the infinitesimal action of \(G\) on \(P\), i.e. we use the same symbol \(X\) to denote \(X \in g\) and the vector field \(Xf(x) = \frac{d}{d\epsilon}\big|_{\epsilon=0} f(\Phi(e^{\epsilon X}, x))\). Equation (2.2) corresponds to using the isomorphism \(T_xP \cong T^*_xP\) induced by \(\omega\) to map the vector \(X\) to the corresponding 1-form.

Given a symplectic action \(\Phi : G \times P \rightarrow P\) there may not exist a momentum mapping. The obstruction lies in solving the equation (2.2) globally. If a solution \(\hat{J}_X\) exists for all \(X \in g\) then a momentum mapping exists, and the group action is generated by Poisson brackets \(\delta_X \phi(x) = \frac{d}{d\epsilon}\big|_{\epsilon=0} \phi(\Phi(e^{\epsilon X}, x)) = \{\hat{J}_X, \phi\}(x)\), using (2.2). Under Poisson brackets the Lie algebra \(g\) may be centrally extended. This depends on the equivariance relation of the momentum map.

**Definition 2.** The equivariance relation of the momentum map \(J\) is the commutation relation:

\[
J(\Phi_g(x)) = \Psi_g \cdot J(x), \quad \forall x \in P, g \in G,
\]

where \(\Psi\) is a group action of \(G\) on \(g^*\) defined by a cohomology class \(\sigma \in H^1(G, g^*)\)

\[
\Psi : (g, l) \rightarrow \text{Ad}^*(g) \cdot l + \sigma(g), \forall l \in g^*, g \in G.
\]

The momentum map of an action \(G \times P \rightarrow P\) is classified by the cohomology class \(\sigma \in H^1(G, g^*)\).
Note. Given a symplectic action $\Phi$, the momentum map can be calculated by solving for the Hamiltonian functions $\hat{J}_X$ from equation (2.2). If a momentum map exists globally, we can calculate the equivariance relation from the formula $\sigma(g) = J(\Phi_g(x)) - \text{Ad}^*(g) \cdot J(x)$ for any point $x \in P$.

The importance of the cocycle $\sigma$ is that the Poisson bracket algebra is centrally extended:

**Theorem 1.** Given a momentum map $J : P \to g^*$ classified by the cohomology class $\sigma \in H^1(G, g^*)$, the Lie algebra $g$ is centrally extended under Poisson brackets

$$\{\hat{J}_X, \hat{J}_Y\} = \hat{J}_{[X,Y]} + \Sigma(X,Y), \text{ where } \Sigma(X,Y) = \left. \frac{d}{dc} \right|_{c=0} \langle \sigma(\exp(cX)), Y \rangle.$$

We now consider the Hamiltonian reduction by a symmetry $\Phi : G \times P \to P$. Assume that there exists a momentum map $J : P \to g^*$ with an equivariance relation involving a cohomology class $\sigma \in H^1(G, g^*)$. Then the Hamiltonian reduction involves two processes. First we restrict to a submanifold $J^{-1}(l_0) \subset P$, where $l_0 \in g^*$ is a regular value of $J$. These submanifolds are the level sets of the momentum map $J$, and correspond to fixing the values of the constants of motion $\hat{J}_X, \forall X \in g$. Then we take equivalence classes under the little group of the point $l_0$: $G(l_0) = \{ g \in G \mid \Psi(g) \cdot l_0 = l_0 \}$. The reduced phase space is thus $P_{\text{red}} = J^{-1}(l_0) / G(l_0)$.

General theory implies that there is a symplectic structure on $P_{\text{red}}$ induced by this process, the corresponding 2-form being denoted $\omega_{l_0}$. The 2-form $\omega_{l_0}$ is related to the original 2-form $\omega$ by the relation, [16], $\pi^*_0 \omega_{l_0} = i_{i_{l_0}}^* \omega$, where $\pi_{l_0}$ is the projection $\pi_{l_0} : J^{-1}(l_0) \to P_{\text{red}}$, and $i_{l_0}$ is the inclusion $i_{l_0} : J^{-1}(l_0) \to P$. This relation also holds in the infinite dimensional case under certain assumptions. The construction of the reduced manifold $P_{\text{red}}$ can be considered as a method for constructing a symplectic leaf of the Poisson manifold $P/G$, the space of $\Phi$-orbits of $G$ in $P$. Since the action $\Phi$ is symplectic, the space $P/G$ inherits a Poisson structure, i.e. if $f, l$ are functions on $P/G$ and $\pi : P \to P/G$, then the Poisson structure satisfies $\{\pi^* f, \pi^* l\} = \pi^* \{ f, l \}$ which implies that it is well defined on $G$-invariant functions. The momentum defined by $l_0$ selects a symplectic leaf in $P/G$. We observe that in this framework, the fact that $P$ is symplectic is not necessary, i.e. we can reduce a Poisson manifold by a symmetry, selecting a symplectic leaf as the phase space.

Comparing to a more familiar reduction process, we observe that for a system with translation invariance in a direction $z$, restricting to the level set corresponds to fixing the $p_z$ momentum, while taking equivalence classes under the little group corresponds to eliminating the $z$ coordinate.

We observe that the image of the moment map $J(P) \subset g^*$ is necessarily a symplectic submanifold of $g^*$, endowed with the Kirillov bracket constructed from the extended group action $\Psi$. This implies that the image is in fact an orbit of the $\Psi$ action of $G$ on $g^*$.

Our interest is in the Hamiltonian reduction of a coadjoint orbit of a Lie group $G$. Consider a coadjoint orbit $\mathcal{O} = \text{Ad}^*_R(G) \cdot \Lambda \subset g^*$, $\Lambda \in g^*$ of a Lie group $G$ on the dual Lie algebra $g^*$, with the group action defined in terms of a classical Yang Baxter operator $R$, [3]. Suppose we are reducing with respect to a Lie group $H$, with Lie algebra $h$, action $\Phi$ and momentum map
Then equation (2.2) can be rewritten as

$$\text{ad}_R^*(d_\kappa \hat{J}_Y) \cdot \kappa = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Phi(\exp_\epsilon Y, \kappa), \forall Y \in \mathcal{H}, \quad \text{and} \quad \kappa \in \mathcal{O}. \quad (2.3)$$

The fact that the orbit $\mathcal{O}$ is symplectic implies that this equation is consistent locally, i.e. the only problem in solving this equation is the exactness of the $1$-form $d_\kappa \hat{J}_Y$ as before.

The tangent space of the orbit $\mathcal{O}$ at a point $\kappa \in \mathcal{O}$ is generated by the Lie algebra $\mathfrak{g}$ through $\text{ad}_R^*$ action, i.e. $T_\kappa \mathcal{O} = \text{span}_{X \in \mathfrak{g}}(\text{ad}_R^*(X) \cdot \kappa)$. Correspondingly, the tangent space of the level set of $J$ is given by restricting $\mathfrak{g}$ to a subspace $S_\kappa \subset \mathfrak{g}$ given by

$$S_\kappa = \{ X \in \mathfrak{g} \mid \langle \text{ad}_R^*(X) \cdot \kappa, d\hat{J}_Y \rangle = 0 \quad \forall Y \in \mathcal{H} \}. \quad (2.4)$$

This is simply the requirement that the Hamiltonians $\hat{J}_Y$ are constant on the level sets. The tangent space of the level set is given by $\text{span}_X S_\kappa (\text{ad}_R^*(X) \cdot \kappa)$. If we define $\mathcal{V}_\kappa$ as the set of directions that are generated infinitesimally by $\Phi$ through the point $\kappa \in \mathcal{O}$, i.e.

$$\mathcal{V}_\kappa = \left\{ r \in T_\kappa \mathcal{O} \left| \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Phi(\exp_\epsilon Y, \kappa) = \kappa + \epsilon r + O(\epsilon^2), \text{ for some } Y \in \mathcal{H} \right. \right\},$$

then the subset $S_\kappa$ can be rewritten as

$$S_\kappa = \{ X \in \mathfrak{g} \mid \langle X, \mathcal{V}_\kappa \rangle = 0 \} \quad (2.5)$$

This follows from a rearrangement of (2.4),

$$\langle \text{ad}_R^*(d\hat{J}_Y) \cdot \kappa, X \rangle \equiv \left. \left( \frac{d}{d\epsilon} \right) \right|_{\epsilon=0} \Phi(\exp_\epsilon Y, \kappa), X \rangle = 0.$$

This means that the set $S_\kappa$ is orthogonal to the directions generated by the infinitesimal action of $\mathcal{H}$. Under the equivalence relation with the little group $G(l_0), l_0 = J(\kappa)$, the tangent space at the equivalence class $\kappa/G(l_0)$ is represented by the set

$$\text{ad}_R^*(X) \cdot \kappa \mod \mathcal{V}_\kappa, \forall X \in S_\kappa. \quad (2.6)$$

Observe that the restriction to the subspace $S_\kappa$, (2.4), is very similar to the gauge invariance constraint on the functional derivatives in the theory of the KdV hierarchy, [5, 8]. Further, the equivalence in (2.6) is very similar to the gauge equivalence also employed.
3. REVIEW

In this section we review the content of [5]. In [5], two Poisson structures are constructed for the generalised KdV hierarchies. This construction was restricted to those hierarchies constructed on untwisted Kac Moody algebras.

The central object in the construction of the hierarchies is a Kac-Moody algebra \( \hat{g} \), realized as the loop algebra \( \hat{g} = g \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}d \), where \( g \) is a finite Lie algebra with Lie group \( G \). The derivation \( d \) is chosen to induce the homogeneous gradation, so that \( [d, a \otimes z^n] = n a \otimes z^n \forall a \in g \).

We use the following notation: \( \{ e_i \} \) \( i=1 \) to \( \text{rank}(g) \) are the raising operators associated to the simple roots of \( g \) in a Cartan-Weyl basis of \( g \); \( \psi \) is the highest root, \( e_{-\psi} \) the corresponding lowering operator; \( N_\pm, B_\pm \) are the Borel subgroups of \( G \); \( k_i \) are the Kac Labels of \( g \), and \( [w] \) denotes the conjugacy class of the Weyl reflection \( w \) of the root space of \( g \). There are other gradations on \( \hat{g} \) given by,

**Definition 3.** A gradation of type \( s \), is defined via the derivation \( d_s \) which satisfies

\[
[d_s, e_i \otimes z^n] = (N + s_i) e_i \otimes z^n,
\]

where \( N = \sum_{i=0}^{\text{rank}(g)} k_i s_i \) and \( s = (s_0, s_1, \ldots, s_{\text{rank}(g)}) \) is a vector of \( \text{rank}(g) + 1 \) non-negative integers.

Under a gradation of type \( s \), \( \hat{g} \) is a \( \mathbb{Z} \)-graded algebra:

\[
\hat{g} = \bigoplus_{i \in \mathbb{Z}} \hat{g}_i(s).
\]

The homogeneous gradation corresponds to \( s_{\text{hom}} \equiv (1, 0, \ldots, 0) \). A special class of gradations are those constructed from conjugacy classes of the Weyl group of \( g \), [10]. Associated to a conjugacy class \( [w] \), there exists a gradation denoted \( s[w] \). The conjugacy classes of the Weyl group also classify the Heisenberg subalgebras, [11], the Heisenberg subalgebra corresponding to \( [w] \) being denoted \( H[w] \). The Coxeter class, \( [w_{\text{co}}] \), defines the familiar principal gradation, and corresponds to \( s = (1, 1, \ldots, 1) \). There exists a partial ordering on the set of gradations of \( \hat{g} \), [7], given by : \( s \preceq s' \) if \( s_i \neq 0 \) whenever \( s'_i \neq 0 \). The construction of the hierarchies involves the use of two gradations, one of which is induced from a conjugacy class \( [w] \). These are denoted \( s \) and \( s[w] \).

To reduce the complexity of the notation, we employ the following notation to refer to them. We use subscripts to denote \( s \)-grade, and superscripts to denote \( s[w] \)-grades, i.e. \( \hat{g}_j \equiv \hat{g}_j(s) \) and \( \hat{g}^j \equiv \hat{g}_j(s[w]) \). In this paper our interest is principally in the KdV-type hierarchies, constructed from the homogeneous gradation \( s = s_{\text{hom}} \).

The KdV hierarchies are constructed from the data \( (\Lambda, s, [w]) \), where \( s \preceq s[w] \) and \( \Lambda \) is a (constant) element of \( \hat{g} \) with \( s[w] \)-grade \( i > 0 \). From this data, one defines the Lax operator

\[
\mathcal{L} = \partial_x + q + \Lambda.
\]

The potential \( q \) is defined to be an element of \( C^\infty(S^1, \hat{g}_{\geq 0} \cap \hat{g}_{<i}) \). The potentials are taken to be periodic functions, this avoiding technical complications [8]. The function \( q(x) \) plays the rôle
of the phase space coordinate in this system. However, there exist symmetries in the system corresponding to the gauge transformation \( \mathcal{L} \to S\mathcal{L}S^{-1} \), with \( S \) generated by \( x \) dependent functions on the subalgebra \( \hat{g}_0 \cap \hat{g}^{<0} \). The phase space of the system \( \mathcal{M} \) is the set of gauge equivalence classes of operators of the form \( \mathcal{L} = \partial_x + q + \Lambda \). The space of functions \( \mathcal{F} \) on \( \mathcal{M} \) is the set of gauge invariant functionals of \( q \) of the form

\[
\varphi[q] = \int_{\mathbb{R}/\mathbb{Z}} dx f \left( x, q(x), q'(x), \ldots, q^{(n)}(x), \ldots \right).
\]

It is straightforward to find a basis for \( \mathcal{F} \), the gauge invariant functionals. One simply performs a non-singular gauge transformation to take \( q \) to some canonical form \( q^{\text{can}} \). The components of \( q^{\text{can}} \) and their derivatives then provide the desired basis.

The outcome of applying the procedure of Drinfel’d and Sokolov, [8], to (3.1) is that there exists an infinite number of commuting flows on the gauge equivalence classes of \( \mathcal{L} \), [7]. In [5] it is further proved that these flows are bi-Hamiltonian, i.e. there exist two coordinated Poisson brackets on the phase space \( \mathcal{M} \) such that the flows are generated by commuting Hamiltonians. These Hamiltonians are labeled by elements of the Heisenberg algebra \( \mathcal{H}[w] \), [7], i.e. \( H_b \) denotes the Hamiltonian defined by the constant element \( b \in \mathcal{H}[w] \) with \( \text{s}\{w\}\text{-grade} > 0 \). It is important that the hierarchy is constructed with a gradation induced from a conjugacy class \([w]\) because the existence of the Heisenberg subalgebra \( \mathcal{H}[w] \) is essential to the construction. We extract the following theorem from [5]

**Theorem 2.** There is a one parameter family of Hamiltonian structures on the gauge equivalence classes of the generalized KdV hierarchy given by

\[
\{\varphi, \psi\}_\mu = \left( q + \Lambda, [d_q \varphi, d_q \psi]_{R_\mu} \right) - \left( d_q \varphi, (d_q \psi)' \right),
\]

where \([, ]_{R_\mu}\) is the Lie algebra commutator constructed from \( R_\mu = (P_{\geq 0} - P_{<0})/2 - \mu/z \). Expanding in powers of \( \mu \), \( \{, \}_\mu = \mu \{, \}_1 + \{, \}_2 \), we obtain the two coordinated Hamiltonian structures on \( \mathcal{M} \)

\[
\{\varphi, \psi\}_1 = - (d_q \varphi, z^{-1} [d_q \psi, \mathcal{L}]) ,
\]

\[
\{\varphi, \psi\}_2 = (q + \Lambda, [d_q \varphi, d_q \psi]_{R}) - (d_q \varphi, (d_q \psi)' ) ,
\]

where \( R = (P_{\geq 0} - P_{<0})/2 \). Under time evolution in the coordinate \( t_b \), the following recursion relation holds:

\[
\frac{\partial \varphi}{\partial t_b} = \{\varphi, H_{zb}\}_1 = \{\varphi, H_b\}_2.
\]
4. THE ORBITS

In this section we outline a program to explain the previous results in terms of a coadjoint orbit construction on the current algebra of $\hat{g}$. This construction will be further explained in another publication. The two Poisson brackets are interpreted as Kirillov brackets by the construction of two Lie algebra commutators on $C^\infty(S^1, \hat{g})$ defined in terms of two Yang Baxter operators, $R, R^\sigma$. The bi-Hamiltonian structure then follows if certain conditions are satisfied, in particular the orbits being dynamically equivalent under Hamiltonian reduction.

Define the two Yang Baxter operators $R = P_{>0} - P_{<0}$ and $R^\sigma = P_{>0} - P_{<0}$ on the Kac Moody algebra $\hat{g}$. These define Lie algebra commutators on $\hat{g}$, denoted by $\{,\}$, and $\{,\}_\sigma = \frac{1}{Z} \{,\}_{R^\sigma}$. On the current algebra, the commutator $\{,\}_R$ is centrally extended with the two form

$$\omega(X,Y) = \int_{S^1} dx \langle P_0(X(x)), Y(x)' \rangle.$$ (4.1)

The Kirillov construction of Poisson brackets on the dual of a Lie algebra, [13], defines two Poisson brackets on $C^\infty(S^1, \hat{g}^*)$, which are identical to those of the hierarchy, [5], $\{,\}_1 = \{,\}_\sigma$, $\{,\}_2 = \{,\}_R$. As in the traditional analysis of the coadjoint orbit method, we induce the coadjoint actions $\text{Ad}^*_{R}$ and $\text{Ad}^*_\sigma$ on the dual $C^\infty(S^1, \hat{g}^*)$. The coadjoint orbits of $\text{Ad}^*_R/\sigma$ define the symplectic leaves of the Poisson brackets $\{,\}_R$, $\{,\}_\sigma$ respectively. We know that on each orbit we can construct an integrable system. The proposal in this paper is that these orbits can be modified (by Hamiltonian reduction) such that the generalised KdV hierarchies are reproduced as dynamical systems, the hierarchies constructed from $(s_{\text{hom}}, s[w])$ having a description in terms of both $\text{Ad}^*$ actions, thus reproducing the two Poisson brackets of [5, 8].

Before discussing the $\text{Ad}^*_R/\sigma$-orbits, we analyse the relationship between the two Poisson structures on the phase space. For a bi–Hamiltonian system, the dynamics are generated through either Poisson bracket, with the bi-Hamiltonian constraint

$$\{\phi, H_b\}_2 = \{\phi, H_{zh}\}_1$$

for all Hamiltonians $H_b$, and functionals $\phi \in \mathcal{F}$. It is only through this relation that the Poisson brackets are related, thus the requirement that the theories are dynamically equivalent does not in fact imply that the phase spaces are identical. The condition for dynamical equivalence is that the phase space of the theory is symplectic under the Poisson bracket $\{,\}_1$, and the Poisson bracket $\{,\}_2$ produces a foliation of the phase space, this foliation being preserved under all the flows. Since the flows of the hierarchy are generated by the Poisson bracket $\{,\}_2$, the requirement that the flows preserve the foliation is trivially satisfied. Thus, in attempting to construct the generalised KdV hierarchies from the Coadjoint Orbit Method, we should choose an orbit of $\text{Ad}^*_\sigma$ as the phase space and prove that this phase space is preserved by the $\text{Ad}^*_R$-action, thus proving the foliation requirement. However, this is an over simplification of the construction, and will in
fact fail. This is because the phase spaces of the KdV hierarchies are not \( \text{Ad}^* \) orbits. Thus we must modify our proposal. This modification can be deduced from an analysis of the orbits, and involves an Hamiltonian reduction of the orbits.

Identifying the dual with the original Lie algebra through the inner product, the \( \text{Ad}^* \) orbits are identified as subspaces of \( C^\infty(S^1, \hat{g}) \) and have the following structure

The \( G_R \) action

\[
\kappa \mapsto P_{\leq 0} \left( g_{\geq 0} \mathcal{L} g_{\geq 0}^{-1} \right) + P_{> 0} \left( g_{< 0}^{-1} \mathcal{L} g_{< 0} \right) \tag{4.2}
\]

This action preserves the decomposition \( \kappa = \{ \kappa_{> 0}, \kappa_{\leq 0} \} \).

The \( G_\sigma \) action

\[
\kappa \mapsto P_{< 0} \left( g_{\geq 0} \mathcal{L} g_{\geq 0}^{-1} \right) + P_{\geq 0} \left( g_{< 0}^{-1} \mathcal{L} g_{< 0} \right) \tag{4.3}
\]

This action preserves the decomposition \( \kappa = \{ \kappa_{> 0}, \kappa_{< 0} \} \).

Here \( g_{\geq 0} \) denotes the formal exponential of an element of the Lie algebra \( C^\infty(S^1, \hat{g}_{\geq 0}) \), and similarly for \( g_{< 0} \). These actions can be made more precise through the use of a representation, or the universal enveloping algebra.

Taking the hint from the KdV hierarchy, we simplify the group actions by generating the orbits from a point in \( \hat{g}_{\geq 0} \). Both group actions preserve the space \( \hat{g}_{\geq 0} \), which implies that the orbits also lie in \( \hat{g}_{\geq 0} \). These orbits take the form

The \( G_R \)-orbit

\[
\kappa \mapsto \left( g_0 \mathcal{L}_0 g_0^{-1} \right) + P_{> 0} \left( g_{< 0}^{-1} \mathcal{L} g_{< 0} \right) \tag{4.4}
\]

Note that the second term only contributes if \( \mathcal{L}_{> 2} \neq 0 \), i.e. if the Lax operator possesses terms of homogeneous degree \( > 1 \). In this paper we restrict to the case \( \mathcal{L}_2 = 0 \), and hence the second term reduces to \( \mathcal{L}_1, \forall g_{< 0} \).

The \( G_\sigma \)-orbit

\[
\kappa \mapsto P_{\geq 0} \left( g_{< 0}^{-1} \mathcal{L} g_{< 0} \right) \tag{4.5}
\]

Observe that both group actions treat \( \hat{g}_{> 0} \) identically. However they differ in their treatment of the space \( \hat{g}_{0} \). Hence we require a mechanism that equates the dynamical degrees of freedom in \( \hat{g}_{0} \). This process must preserve the symplectic nature of the orbits, which suggests that we perform an Hamiltonian reduction with respect to a symmetry group of each orbit. Thus for each orbit, \( O_\sigma, O_R \), we require a group of symplectomorphisms, \( H_\sigma, H_R \) respectively, with which to perform an Hamiltonian reduction. One class of symplectomorphisms that are easily identified are those induced from an algebra homomorphism \( \chi : C^\infty(S^1, \hat{g}) \to C^\infty(S^1, \hat{g}) \), \( \chi \) being an algebra homomorphism with respect to both Lie algebra structures \([ , ]_R \) and \([ , ]_\sigma \). Since these Lie brackets depend on the homogeneous gradation, adjoint action by any element of \( G \) will
necessarily be an algebra homomorphism. The symplectic submanifolds $\mathcal{O}_\sigma$ and $\mathcal{O}_R$ are in fact sufficiently similar that an Hamiltonian reduction by subgroups $H_\sigma,H_R \subset G$ is sufficient, where $H_\sigma,H_R$ act by Ad$^*$-action. Note that one of the constraints on $H_\sigma,H_R$ is that under Ad$^*$-action the orbits are preserved, i.e. we require Ad$^*$($H_\sigma$)$\cdot \mathcal{O}_\sigma \subset \mathcal{O}_\sigma$, Ad$^*$($H_R$)$\cdot \mathcal{O}_R \subset \mathcal{O}_R$. Assume for the moment that these actions possess moment maps $J_\sigma : \mathcal{O}_\sigma \to h^*_\sigma$, $J_R : \mathcal{O}_R \to h^*_R$, where $h_\sigma,h_R$ are the Lie algebras of $H_\sigma,H_R$ respectively, and $h^*_\sigma,h^*_R$ denote the corresponding duals. Then the Hamiltonian reduction is performed by restricting to the inverse image $J_{\sigma}^{-1}(n_\sigma)$, and taking equivalence classes under the little group $G_\sigma(n_\sigma)$, $n_\sigma \in h^*_\sigma$. The condition that the reduced phase spaces are equivalent dynamically now reduces to the requirement that $J_{\sigma}^{-1}(n_\sigma)/G_\sigma(n_\sigma)$ is foliated by spaces of the form $J_{R}^{-1}(n_R)/G_R(n_R)$. Further, we expect for appropriate choices of orbit $\mathcal{O}_\sigma$, symmetry group $H_\sigma$ and momentum $n_\sigma$, that the reduced phase space is identical to that of a generalised KdV hierarchy of $[7]$.

The preceding discussion leads us to propose the following conjecture for the integrable models constructed in $[7]$.

**Conjecture.** The $(s_{\text{hom}},s_{[w]})$ KdV Hierarchies have phase spaces that are Hamiltonian reductions of $G_\sigma$-orbits, by a symmetry group $H_\sigma \subset C^\infty(S^1,N_-)$. The leaves of the foliation induced by the second Poisson structure, $\{J_\sigma,J_R\}$, are Hamiltonian reductions of $G_R$-orbits, with a symmetry group $H_R \subset C^\infty(S^1,N_-)$. This foliation induces a partition of the potentials that is finer than the separation of the potentials into modified KdV, partially modified KdV and KdV type potentials.

We observe that prior to the introduction of the gauge group $H_\sigma$, the orbit $\mathcal{O}_\sigma$ is effectively finite since there is no central extension in the $G_\sigma$-action. It is only through the gauge group that the $G_\sigma$-orbit acquires a complexity capable of describing theories such as the generalised KdV hierarchies.

The first test of this conjecture is whether it is able to reproduce the traditional $A_\kappa$-KdV hierarchies, $[8]$. In these theories, the element $\Lambda$ has $[w_{\text{co}}]$-grade equal to 1, and the homogeneous decomposition $\Lambda = I + z e_\psi$, where $I = \sum_{i=1}^{\text{rank}(\Lambda)} e_i$. The orbits are constructed such that they pass through the point $\kappa = \kappa_0 + z e_\psi \in C^\infty(S^1,\hat{g}_{\geq 0})$, i.e. initially we only fix the component of homogeneous degree 1. The orbits can now be written down explicitly:

\[ \mathcal{O}_R = \{ z e_\psi + g^{-1} (\partial_x + \kappa_0) g \mid \forall g \in C^\infty(S^1,G) \}, \]
\[ \mathcal{O}_\sigma = \{ \kappa + [e_\psi,Y] \mid \forall Y \in C^\infty(S^1,g \mod \text{Ker}(e_\psi)) \}, \]

where we note that the $\mathcal{O}_\sigma$ orbit is parametrised by the space $C^\infty(S^1,g \mod \text{Ker}(e_\psi))$, with $\text{Ker}(X) = \{ Y \in g \mid [Y,X] = 0 \}$. To simplify the comparison of the orbits, we write down the tangent spaces of the orbits at the point $\kappa = \Lambda_{z \to z+\mu}$,

\[ T_{\Lambda_{z \to z+\mu}} \mathcal{O}_R = \text{span}_{X \in C^\infty(S^1,g)} (\mu [e_\psi,X] + [\partial_x + I,X]) , \]
\[ T_{\Lambda_{z \to z+\mu}} \mathcal{O}_\sigma = \text{span}_{X \in C^\infty(S^1,g)} ([e_\psi,X]) . \]  

(4.6)

All elements of the tangent space of $\mathcal{O}_\sigma$ are orthogonal to $\text{Ker}(e_\psi)$.
Observe that the two tangent spaces are similar, the choice $\kappa = \Lambda + \mu e_{-\psi}$ in (4.6) ecentrating the similarity. However the tangent spaces $T_cO_R$ and $T_cO_\sigma$ of the two orbits are not equivalent, in particular the $G_R$-orbit has directions that are not orthogonal to $\text{Ker}(e_{-\psi})$. Further, there are additional degrees of freedom over and above those in $g \mod \text{Ker}(e_{-\psi})$, corresponding to the terms $[\partial_x + I, X]$. Thus the $O_\sigma$ orbit is more restricted than the $O_R$ orbit as a model for a phase space. Hamiltonian reduction is able to reduce this discrepancy between the tangent spaces, i.e. we could enforce $T_cO_R$ to be orthogonal to $\text{Ker}(e_{-\psi}) \subset g$ through the introduction of a symmetry. However, this is too strong a condition, since the introduction of a symmetry also introduces an equivalence relation on directions in the tangent space, (2.6). Thus we only impose a symmetry by a subgroup of the Lie group generated by $C^\infty(S^1, \text{Ker}(e_{-\psi}))$. Since the tangent space $T_cO_\sigma$ can never have a direction in the upper triangular subalgebra $n_+$ for any point in the orbit, the gauge algebra $h_R$ must at least include $C^\infty(S^1, n_-)$ as a subalgebra. From the known structure of the KdV hierarchy, this should in fact be sufficient.

5. THE GAUGE GROUP MOMENTUM MAPS AND HAMILTONIAN REDUCTION

In this section, for the case when $\Lambda$ has $s|\text{w}_{\text{co}}|$-grade 1, we construct the two momentum maps $J_R, J_\sigma$ for the $\text{Ad}^*$-action of $H_R, H_\sigma \subset C^\infty(S^1, G)$ on the orbits $O_\sigma$ and $O_R$. The moment map $J_R$ is a simple quotient, while the moment map $J_\sigma$ requires a rather complex calculation. After calculating the equivariance relation of $J_\sigma$, we find that under Poisson brackets the Lie algebra $h_\sigma$ is centrally extended.

Given a symmetry $\Phi : G \times P \to P$, the momentum map is calculated by solving equation (2.2), or more specifically (2.3) in the case of a coadjoint orbit. This corresponds to infinitesimally generating the action $\Phi$ under Poisson brackets. For the gauge group actions, $H_R, H_\sigma \subset C^\infty(S^1, G)$, this equation reads

\[
\begin{align*}
\text{ad}_R^*(d_\kappa J_{R,X}) \cdot \kappa &= \text{ad}^*(X) \cdot \kappa, \forall X \in h_R, \\
\text{ad}_\sigma^*(d_\kappa J_{\sigma,X}) \cdot \kappa &= \text{ad}^*(X) \cdot \kappa, \forall X \in h_\sigma.
\end{align*}
\]

Observe that this relates the gauge transformation in terms of the $\text{Ad}^*$-action to the two $\text{Ad}^*$-actions $\text{Ad}_R^*$ and $\text{Ad}_\sigma^*$ employed in the Poisson brackets.

Consider the $O_R$ orbit. We need the group $H_R$ to preserve the orbit, in particular this implies that $H_R$ stabilises $e_{-\psi}$ such that the term $L_1 = ze_{-\psi}$ is preserved. Equation (5.1) has the form $[L, d_\kappa J_{R,X}] = [L, X]$. This implies that $d_\kappa J_{R,X} = X$, which integrates to $J_{R,X} = \kappa, X)$. Comparing to equation (2.1), we obtain the momentum map

\[ J_R : \hat{g} \to \hat{g}/\text{Ann}(h_R). \]  

The dual map $J_R^* : C^\infty(S^1, h_R) \to C^\infty(S^1, \hat{g})$ is the inclusion map. The trivial calculation:

\[ \langle J_R(\text{Ad}^*(g) \cdot \kappa), X \rangle = \langle \kappa, \text{Ad}(g^{-1}) \cdot X \rangle \equiv \langle \text{Ad}^*(g) \cdot J_R(\kappa), X \rangle \]

proves that $J_R$ is $\text{Ad}^*$-equivariant. Thus we can reduce the orbit $O_R$ by any subgroup $H_R$ that
stabilises $e_{-\psi}$. This momentum map has previously been constructed in relation to the \( A_n \)-KdV hierarchies, [8,18], these hierarchies being constructed as Hamiltonian reductions of the orbits. Thus from the point of view of the \( G_R \)-orbits this description of the KdV hierarchies is well known.

For the orbit \( \mathcal{O}_\sigma \), the requirement that \( H_\sigma \) preserves the orbit is highly restrictive. Define the subgroup \( \tilde{N}_- \subset \mathcal{C}^\infty(S^1,G) \) as the subgroup that preserves the orbit and \( \tilde{n}_- \) as the corresponding Lie algebra. Thus for all \( \tilde{X} \in \tilde{n}_- \) there exists an \( X \in \mathcal{C}^\infty(S^1,g \text{ mod Ker}(e_{-\psi})) \) satisfying \( \text{ad}^*_\psi(X) \cdot \kappa = \text{ad}^*(\tilde{X}) \cdot \kappa, \forall \kappa \in \mathcal{O}_\sigma \), which implies that \( \tilde{n}_- \) not only annihilates \( e_{-\psi} \), but lies in \( \mathcal{C}^\infty(S^1,n_-) \). More specifically, \( \tilde{n}_- \) consists of matrices of the form

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
A & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
B & A & 0 & \cdots & 0 & 0 & 0 & 0 \\
C & B - A' & A & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & A & 0 & 0 & 0 \\
\vdots & B - (n - 4)A' & A & \cdots & 0 & 0 & 0 \\
\vdots & C - (n - 4)B' & B - (n - 3)A' & A & 0 & 0 & 0 \\
\end{pmatrix}
\tag{5.3}
\]

where \( A, B, \ldots \) are \( \mathcal{C}^\infty(S^1) \) functions, and \( n \) is the dimension of the matrix. The dual \( \tilde{n}_-^\ast \) is the strictly upper triangular analogue of (5.3). Observe that the derivative terms are only present if the representation has dimension \( n \geq 4 \).

The moment map of the \( G_\sigma \)-orbit is less trivial than that of the \( G_R \)-orbit treated previously. We are required to solve the equation, (5.1)

\[
[e_{-\psi}, d_\kappa J_{\sigma,X}] = [\mathcal{L}, X], \forall X \in \tilde{n}_-.
\]

This splits into the following two parts. Since an element of the orbit has the form \( \kappa = \Lambda + [e_{-\psi}, Y], Y \in \mathcal{C}^\infty(S^1,g \text{ mod Ker}(e_{-\psi})) \), we have

\[
[e_{-\psi}, d_\kappa J_{\sigma,X}] = [\partial_x + I, X] + [e_{-\psi}, [Y, X]], \forall X \in \tilde{n}_-.
\]

Observe that there is a constant term, and a term linear in the variable \( Y \), implying that the Hamiltonian function \( J_{\sigma,X} \) is quadratic in \( Y \). It takes the form

\[
J_{\sigma,X}(\kappa(Y)) = -\langle Y, [\partial_x + I, X] \rangle + \frac{1}{2} \langle [e_{-\psi}, Y], [Y, X] \rangle, \tag{5.4}
\]

in terms of the variable \( Y \) parametrising the orbit \( \mathcal{O}_\sigma \). This equation is well defined given that \( Y \in \mathcal{C}^\infty(S^1,g \text{ mod Ker}(e_{-\psi})) \), because of the presence of \( e_{-\psi} \) in the second term, and the fact
that for \( X \in \tilde{n}_- \) there exists an \( s_X \in C^\infty(S^1, g) \) satisfying \([\partial_x + I, X] = [e_{-\psi}, s_X]\). Calculation of the functional derivative \( d_\epsilon J_{\sigma,X} \) verifies that this is the desired Hamiltonian function. This is accomplished by using the observation that all tangent vectors of \( O_\sigma \) have a form \( r = [e_{-\psi}, u] \), \( u \in C^\infty(S^1, g \mod \text{Ker}(e_{-\psi})) \), and thus

\[
\langle d_\kappa J_{\sigma,X}, r \rangle = -\langle [e_{-\psi}, d_\kappa J_{\sigma,X}], u \rangle = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} J_{\sigma,X}(\kappa(Y + \epsilon u)).
\]

From (5.4) we obtain the momentum map \( J_\sigma : \hat{\tilde{n}}^* \rightarrow \tilde{n}_-^* \),

\[
J_\sigma(\kappa(Y)) = [\partial_x + I, Y] + \frac{1}{2} [[e_{-\psi}, Y], Y] \mod \text{Ann}(\tilde{n}_-).
\] (5.5)

To perform the Hamiltonian reduction with respect to \( J_\sigma \), it is necessary to pull back a point \( n_\sigma \in C^\infty(S^1, \tilde{n}_-^*) \) and reduce by the little group of \( n_\sigma \). To calculate the little group of \( n_\sigma \) we need to know the equivariance relation of \( J_\sigma \). We require the action \( \Psi \) of \( \tilde{N}_- \) on \( \tilde{n}_-^* \) such that \( J_\sigma(\text{Ad}^*(g) \cdot \kappa) = \Psi(g) \cdot J_\sigma(\kappa), \forall g \in \tilde{N}_- \). We perform an infinitesimal calculation. Define the function \( s : \tilde{n}_- \rightarrow C^\infty(S^1, g \mod \text{Ker}(e_{-\psi})) \) by \([\partial_x + I, X] = [e_{-\psi}, s_X]\). Then we have

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} J_\sigma(\kappa(Y + \epsilon([Y, X] + s_X))) = [\partial_x + I, [Y, X] + s_X] + \frac{1}{2} [[[e_{-\psi}, [Y, X] + s_X], Y]
+ \frac{1}{2} [[e_{-\psi}, [Y, X] + s_X]] \mod \text{Ann}(\tilde{n}_-).
\]

This rearranges as follows

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} J_\sigma(\text{Ad}^*(e^{-\epsilon X}) \cdot \kappa(Y)) = [J_\sigma(\kappa(Y)), X] + [\partial_x + I, s_X] \mod \text{Ann}(\tilde{n}_-).
\] (5.6)

From this formulae it appears that the action on \( \tilde{n}_- \) is extended. Thus there exists a 1-cocycle \( \sigma : \tilde{N}_- \rightarrow \tilde{n}_-^* \) such that the momentum map \( J_\sigma \) is equivariant with respect to the action \( \Psi : (g, \eta) \rightarrow \text{Ad}^*(g^{-1}) \cdot \eta + \sigma(g) \), where \( \sigma(g) = J(\text{Ad}^*(g) \cdot \kappa) - \text{Ad}^*(g) \cdot J(\kappa) \). The 1-cocycle \( \sigma \) is the ‘exponential’ of the additional term in (5.6), and takes the form

\[
\sigma(g^{-1}) = [\partial_x + I, S_g] + \frac{1}{2} [[[e_{-\psi}, S_g], S_g] \mod \text{Ann}(n_-),
\] (5.7)

where \( S \) is a map \( S : \tilde{N}_- \rightarrow C^\infty(S^1, g \mod \text{Ker}(e_{-\psi})) \) defined by

\[
g^{-1}(\partial_x + I) g = I + [e_{-\psi}, S_g], \forall g \in \tilde{N}_-.
\] (5.8)

The proof is omitted, being a specific case of theorem 3, section 9. The fact that \( \sigma \) is a 1-cocycle follows from general Hamiltonian reduction theory. By theorem 1, the gauge algebra will
be centrally extended under Poisson brackets, with the cocycle (using (5.6))

\[ \Sigma(X,Y) = -\langle e^{-\psi}, [s_X, s_Y]\rangle, \forall X, Y \in \tilde{n}_-, \]

which gives the following Poisson bracket algebra

\[ \{\hat{J}_\sigma X, \hat{J}_\sigma Y\} = \hat{J}_\sigma [X,Y] - \langle e^{-\psi}, [s_X, s_Y]\rangle, \]

a central extension of \( \tilde{n}_- \).

6. THE GAUGE REDUCTION OF THE \( G_R \)-ORBIT

In this section we consider the \( G_R \)-action on \( C^\infty(S^1, \hat{g}^*) \) and analyse the observation that the orbit through \( \Lambda \) is only able to reproduce mKdV type potentials. The only remaining degrees of freedom in the construction are the \( s[w]- \)grade \( \leq 0 \) components of the original point, \( i.e. \kappa^0_0 \). Through the choice of \( \kappa^0_0 \) we propose that the \( G_R \)-orbits split the KdV potentials, \( \{u_k(x)\} \), into various functional forms that are preserved under the flows of the hierarchy.

Consider the traditional KdV hierarchy with \( \Lambda \) of \( s[w_{co}]- \)grade 1. The \( G_R \)-orbit has the form

\[ n_-^{-1}b_+^{-1}(\partial_x + \kappa_0) b_+ n_- + ze^{-\psi}, \quad (6.1) \]

where we have decomposed the action into two parts \( n_-(x) \in N_-, b_+(x) \in B_+ \). This orbit contains dynamical degrees of freedom that cannot be reproduced by a \( G_\sigma \)-orbit, (4.5). In particular the upper triangular dynamical components must be removed, which is accomplished by an Hamiltonian reduction with respect to \( C^\infty(S^1, N_-) \). The appropriate value of the momentum map is \( n_R = I \in C^\infty(S^1, n^*_+) \), which gives the level set constraint as

\[ P_0^{>0} (b_+^{-1}(\partial_x + \kappa_0) b_+) = I. \quad (6.2) \]

The action by \( C^\infty(S^1, N_-) \) preserves this constraint, and is thus the little group. We further observe that the choice \( n_- = 1 \) is a valid gauge choice, and hence if the orbit (6.1) passes through \( \Lambda, (\kappa_0 = I) \), the theory is only capable of reproducing mKdV type potentials. Since the orbits never intersect, we further deduce that under the time evolution of the hierarchy the potential always remains as an mKdV potential, a familiar observation. To reproduce general KdV type potentials, we propose that the components \( \kappa^0_0 \) of the original point are varied, \( i.e. \) we consider the set of orbits that foliate the space \( ze^{-\psi} + C^\infty(S^1, \hat{g}_0 \cap \hat{g}^{\leq 1}) \). Because the little group is identical for each orbit, the Hamiltonian reduction of the \( G_R \)-orbits induces a map \( C^\infty(S^1, \hat{g}_0) \to \mathcal{M} \) that preserves the property of foliation, \( i.e. \) leaves are mapped to leaves. This gives a construction for the leaves of the foliation induced from the poisson bracket \( \{ , \} _R, [21] \), as Hamiltonian reductions.
of $G_R$-orbits. The fact that we are considering a foliation by reduced $G_R$-orbits means that there is nothing more to prove, i.e. we only need to verify that a symmetry reduction exists such that the level sets are submanifolds of $C^\infty(S^1, \hat{g}_{>0} \cap \hat{g}^{\leq 1})$. This implies that from the point of view of the Poisson bracket $\{ \cdot, \cdot \}_R$, the phase space is not $\mathcal{M}$, but a submanifold of $\mathcal{M}$ given by the gauge equivalence classes of one of the level sets.

An interesting question is the number of group orbits that foliate the phase space $\mathcal{M}$. This is of interest because for each distinct orbit, we obtain a certain parametrisation of the potentials $\{u_k(x)\}$. For example, as indicated above we obtain a distinction between mKdV type potentials and true KdV type potentials depending on the type of $G_R$-orbit from which the potential originates. Thus, each choice of orbit $O_R$ containing a point $\kappa = \Lambda + \kappa^{<0}_0$ gives a set of potentials $\{u_k\}$ that cannot be described by any other $G_R$-orbit. In addition, since the flows of the hierarchies are generated by the Poisson bracket $\{ \cdot, \cdot \}_R$, this partitioning of the functional form of the potentials $\{u_k\}$ is also preserved by the flows of the hierarchy. The Muira maps $\{M_s, \{ \cdot, \cdot \}_s\} \rightarrow \{M_{co}, \{ \cdot, \cdot \}_2\}$ are a weakened form of this conclusion, e.g. at the crudest level this gives the decomposition into mKdV type (passing through $\Lambda$), and non mKdV type solution (from orbits that do not pass through $\Lambda$). The exact nature of this partition is unknown. In this paper we do not attempt to analyse this foliation of the phase space $\mathcal{M}$ further.

We know from [5] that for gauge invariant functionals the Poisson bracket $\{ \cdot, \cdot \}_R$ is expressible in terms of the alternative gradation endomorphism $R[w]$, $R[w] = P^{\geq 0} - P^{< 0}$. This suggests that the phase space can be analysed in terms of a $G_{R[w]}$-action. Using the constraint (6.2), we observe that the orbit (6.1) takes the form

$$n^{-1}_- b_+^{-1} (\partial_x + \kappa_0) b_+ n_- + \varepsilon e^{-\psi} = P^{\leq 0} (n^{-1}_- b_+^{-1} (\partial_x + \kappa_0) b_+ n_-) + \Lambda.$$

The first term can be rewritten as

$$n^{-1}_- P^{\leq 0} (b_+^{-1} (\partial_x + \kappa_0) b_) n_- + P^{\leq 0} (n^{-1}_- I n_-),$$

using the constraint (6.2), and the decomposition $\hat{g}_0 = \hat{g}_0^{>0} + \hat{g}_0^{\leq 0}$. Hence, the level set can be rewritten in terms of a $G_{R[w]}$-action

$$n^{-1}_- \left( P^{\leq 0} \left( b_+^{-1} \left( \partial_x + \kappa_0^{<0} \right) b_+ \right) + \Lambda \right) n_- \quad (6.3)$$

where $b_+$ satisfies the differential equation (6.2). By fixing the $n_-$ component of the group action, equation (6.3) implies that each level set (with momentum $J_R(\kappa) = I$), can be gauge fixed to be a subspace of a $G_{R[w]}$-orbit. Thus, a $G_{R[w]}$-orbit of the form $P^{\leq 0} (b_+^{-1} I b_+) + \Lambda$ is a partially gauge fixed Lax operator corresponding to a collection of level sets, and thus after gauge fixing will correspond to a collection of leaves of the $\{ \cdot, \cdot \}_R$-foliation. This implies that the $G_{R[w]}$-orbits induce a partition of the potential type that is coarser than that induced by the $G_R$-action, but still finer (or equal) to that into mKdV, pmKdV and true KdV type.
7. THE CASE OF $Sl(2)$

In this example, we explicitly construct the orbits in $C^\infty(S^1, \hat{sl}(2)^*)$ under the two group actions $G_R$ and $G_\sigma$, exploiting the fundamental representation of $Sl(2)$. We perform the Hamiltonian reductions with respect to the symmetry groups $H_{R/\sigma} = C^\infty(S^1, N_-)$, and verify that for a certain choice of inverse images $J_{R/\sigma}^{-1}$ the phase spaces are dynamically identical.

The $G_R$-orbit.

We split the action of $G_R$ into an action by $N_-$ and $B_+$. We are interested in orbits that intersect the space $\kappa = ze^{-\psi} + X, X \in C^\infty(S^1, \hat{\gamma}_0)$, with the upper triangular component non-zero for all $x \in S^1$. For these elements, we can use the lower Borel subgroup, $B_-$, to express them in the form $\Lambda + \mu e^{-\psi}, \mu \in C^\infty(S^1)$. Thus the orbits of interest have the form:

$$\kappa = ze^{-\psi} + b_+^{-1} \left( \partial_x + I + \mu e^{-\psi} \right) b_+$$

(7.1)

with the additional $C^\infty(S^1, N_-)$ component of the group action suppressed. The momentum map $J_R: O_R \rightarrow n_+^* \cong n_+$ is

$$J_R(\kappa) = \alpha^{-2} ((\alpha \beta)' + 1) - \mu \beta^2$$

(7.2)

Since $n_+$ is one dimensional we are suppressing the basis element in this, and all following formulae. To perform the Hamiltonian reduction, we restrict $O_R$ to the inverse image of a point $n_R \in C^\infty(S^1, n_+^*)$, and reduce by the little group of $n_R$. For all choices of $n_R$ the little group is $C^\infty(S^1, n_-)$, as follows from the abelian nature of $n_-$ and the fact that $J_R$ is $Ad^*$-equivariant. This Hamiltonian reduction has been considered before in [8,18].

The $G_\sigma$-orbit.

Under the action of $G_\sigma$ the orbit through $\Lambda$ takes the form:

$$\kappa = \Lambda + [e^{-\psi}, Y] = ze^{-\psi} + \begin{pmatrix} -B & 1 \\ 2A & B \end{pmatrix}$$

for $Y = \begin{pmatrix} A & B \\ * & -A \end{pmatrix} \in C^\infty(S^1, \hat{sl}(2))$. (7.3)

The momentum map $J_\sigma$ of the $Ad^*$-action of $C^\infty(S^1, N_-)$ on this orbit takes the form, (5.5)

$$J_\sigma(\kappa(Y)) = B' - B^2 - 2A.$$ (7.4)

To carry out the Hamiltonian reduction by $C^\infty(S^1, N_-)$, we must evaluate the function $s_X$ defined
by \([\partial_z + I, X] = [e^{-\psi}, s_X]\), \(X \in C^\infty(S^1, n_-)\). It has the form

\[
s_X = \begin{pmatrix}
\frac{1}{2} C' & -C \\
* & -\frac{1}{2} C'
\end{pmatrix}, \quad \text{for } X = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.
\]

This gives the following central extension to the Poisson bracket algebra

\[
\Sigma(X, Y) = \int dx \langle \{\partial_z + I, X\}, \psi \rangle = \int dx \text{tr} \left( \begin{pmatrix} D & 0 \\ D' & -D \end{pmatrix} \begin{pmatrix} \frac{1}{2} C' & -C \\ * & -\frac{1}{2} C' \end{pmatrix} \right) = 2 \int dx DC',
\]

for \(X = De^{-\psi}, Y = Ce^{-\psi} \in C^\infty(S^1, n_-)\). Thus the Poisson bracket algebra is

\[
\{\hat{J}_{\sigma}, X\} = 2 \int dx DC',
\]

since \(n_-\) is abelian. The appropriate action of \(C^\infty(S^1, N_-)\) on \(C^\infty(S^1, n_-)\) is given by

\[
\Psi(g) \cdot n_{\sigma} = n_{\sigma} + \sigma(g), \forall g \in N_-,
\]

where the cocycle \(\sigma\) is defined in (5.7). The function \(S_g\) has the form

\[
S_g = \begin{pmatrix}
\frac{1}{2} (\gamma' - \gamma^2) & -\gamma \\
* & -\frac{1}{2} (\gamma' - \gamma^2)
\end{pmatrix}, \quad \text{for } g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix},
\]

giving the cocycle \(\sigma(g) = 2\gamma'\). Hence the little group of a point \(n_{\sigma} \in C^\infty(S^1, n_-)\) only consists of constant elements of the gauge group, \(i.e. G_{\sigma}(n_{\sigma}) = N_-\).

**A Comparison of the orbits.**

Comparing the two orbits in (7.1) and (7.3), we observe that they can never be identical because of the existence of a dynamical degree of freedom in the upper triangular component of the \(G_R\)-orbit. We conclude that an Hamiltonian reduction of the orbits is necessary. Recall that the first step in the Hamiltonian reduction is to specify the constants of motion, \(i.e.\) choosing the points \(n_R, n_{\sigma} \in C^\infty(S^1, n^\pm)\). Comparing the two orbits (7.1) and (7.3), we observe that the appropriate choice is \(n_R = 1\) with the associated level set consisting of matrices that satisfy

\[
(\alpha \beta)' + \mu (\alpha \beta)^2 + 1 = \alpha^2.
\]

This is a differential equation for the combination \(\delta = \alpha \beta\), given \(\alpha\). Reversing the logic, this equation gives \(\alpha\) in terms of \(\delta\). Thus the orbit has one degree of freedom, \(\delta\). The Lax operator
now takes the form
\[ z e_{-\psi} + \left( \frac{\alpha^{-1} \alpha' - \mu \delta}{\alpha'} \begin{pmatrix} 1 \\ \mu \alpha^2 \\ -\alpha^{-1} \alpha' + \mu \delta \end{pmatrix} \right), \] (7.6)

where \( \alpha^2 \) is related to \( \delta \) through (7.5). The gauge group introduces the following equivalence relation
\[ \begin{pmatrix} F & 1 \\ G & -F \end{pmatrix} \sim \begin{pmatrix} F + \gamma & 1 \\ G + \gamma' - \gamma^2 - 2\gamma F & -F - \gamma \end{pmatrix}, \] (7.7)

the matrix expansion of the formula \( \mathcal{L} \sim g^{-1} \mathcal{L} g, \forall g \in C^\infty(S^1, N_-) \). As is well known, we can choose a gauge slice with no diagonal component, which implies that there is only one dynamical field. By the choice \( \gamma = -F = -\alpha^{-1} \alpha' + \mu \delta \), we obtain the traditional gauge slice with \( u(x) = -F' + F^2 + \mu \alpha^2 \). Thus the potential \( u(x) \) has a rather complex functional form in terms of the dynamical variable \( \delta \). Recall that under the flows of the hierarchy, this functional form is preserved, since the \( \mathcal{G}_R \)-orbit is preserved by the flow. Due to the foliation of \( C^\infty(S^1, \hat{g}^*) \) by the orbits of \( \mathcal{G}_R \), the full space \( \Lambda + C^\infty(S^1, \hat{g_0} \cap \hat{g}^{<1}) \) is covered by the level sets, and hence the full phase space \( \mathcal{M} \) is reproduced, i.e. generic \( u(x) \) can be reproduced through changes in \( \mu(x) \). Unfortunately, the number of orbits required is unknown, and the corresponding classification of these orbits by choices for the function \( \mu \in C^\infty(S^1) \) is also unknown.

We performed an Hamiltonian reduction on the \( \mathcal{G}_R \)-orbits because they contained additional degrees of freedom over and above those contained in the \( \mathcal{G}_\sigma \)-orbit. This produced a system with only 1 dynamical field. In contrast, the orbit \( \mathcal{O}_\sigma \) has 2 dynamical fields, i.e. \( A \) and \( B \), (7.3). Thus we must reduce the degrees of freedom by 1. This is accomplished through an Hamiltonian reduction by \( J_\sigma \). Imposing the level set constraint \( J_\sigma(\kappa) = n_\sigma \), (7.4), we obtain the level set as
\[ \kappa = z e_{-\psi} + \begin{pmatrix} -B \\ B' - B^2 - n_\sigma \\ B \end{pmatrix}, \] (7.8)

which only passes through \( \Lambda \) if \( n_\sigma = 0 \). Under equivalence with the little group \( G_\sigma(n_\sigma) = N_- \), we obtain the equivalence relation
\[ \begin{pmatrix} -B \\ B' - B^2 - n_\sigma \\ B \end{pmatrix} \sim \begin{pmatrix} \gamma - B \\ (B + \gamma)' - (B - \gamma)^2 - n_\sigma \\ B - \gamma \end{pmatrix}, \] (7.9)

where \( \gamma \) is a constant. This implies that the constant component of \( B \) is gauge, and thus non-dynamical. We deduce that (7.8) should be a gauge slice for the \( sl(2) \)-KdV hierarchy with the gauge fixing constraint \( \int B(x) dx = 0 \). This is proved by using the full gauge group \( C^\infty(S^1, N_-) \), i.e. \( \gamma \) arbitrary, to map (7.8) into the traditional gauge slice with no Cartan subalgebra component. We derive the relationship between the traditional potential \( u(x) \) and the dynamical field \( B(x) \) as
\[ u(x) = 2B' - n_\sigma, \] a relationship that supports the conclusion that the constant
component of $B$ is non-dynamical. This parametrisation of $u(x)$ also implies that the constant component $\int u(x)dx$ is not dynamical, i.e. a constant of motion. This is in fact observed in the traditional analysis of the KdV hierarchy, since $\int u(x)dx$ is the Hamiltonian that generates the chiral flow, and is thus constant under all the flows. We conclude that (7.8) is a good gauge slice for the theory, a gauge slice that was in fact used in [15], in the derivation of the $sl(2)$-KdV hierarchy from the Self Dual Yang-Mills Equations. We shall find that in our other examples, we also obtain a gauge fixing that is similar to this gauge slice, and refer to them as Mason-Sparling gauge slices.

We deduce that a coadjoint orbit construction with group action $G_\sigma$ is able to reproduce the phase space of the $sl(2)$-KdV hierarchy. This provides an underlying group theoretic description for the first Poisson bracket $\{\ , \}_1$. In addition, the leaves of the foliation of the phase space $\mathcal{M}$ defined by the second poisson bracket $\{\ , \}_R$ are gauge group reductions of $G_R$-orbits.

8. THE CASE OF $Sl(3)$

In [5, 7], three theories were constructed from the Kac Moody algebra $sl(3)$, with the element $\Lambda$ taking the various regular forms

$$
\Lambda_{co} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
z & 0 & 0
\end{pmatrix}, \quad \Lambda_{co}^2 = \begin{pmatrix}
0 & 0 & 1 \\
z & 0 & 0 \\
0 & z & 0
\end{pmatrix}, \quad \Lambda_\alpha = \begin{pmatrix}
0 & 0 & 1 \\
z & 0 & 0
\end{pmatrix}
$$

Recall that the $G_\sigma$-orbit only depends on the components of $\mathcal{L}$ with homogeneous degree greater than zero, i.e. $\mathcal{L}_{>0}$. Thus the $G_\sigma$-orbits of $\Lambda = \Lambda_{co}$ and $\Lambda = \Lambda_\alpha$ are essentially identical, i.e. the orbits are linear translates of each other. The difference in these theories lies in their different gauge group actions, leading to different dynamical degrees of freedom. In this section we perform an analysis of the case $\Lambda = \Lambda_{co}$, the traditional $sl(3)$-KdV hierarchy. The remaining two theories are examined in later sections, and serve as useful comparisons. The traditional hierarchy, $\Lambda = \Lambda_{co}$, is constructed from the Lax operator

$$
\mathcal{L} = \partial_x + z e_{-\psi} + \begin{pmatrix}
U^+ & 1 & 0 \\
G^+ & -U^+ - U^- & 1 \\
T & G^- & U^-
\end{pmatrix}, \quad (8.1)
$$

with an equivalence relation $\mathcal{L} \cong n^{-1} \mathcal{L} n_-, \forall n_- \in C^\infty(S^1, N_-)$. This implies that (8.1) is equiv-
alent to

\[ n_{-1} \mathcal{L}_0 n_- = \]

\[
\begin{pmatrix}
U^+ + \alpha & 1 & 0 \\
G^+ + \alpha' + \gamma - \alpha (2U^+ + U^- + \alpha) & -U^+ - U^- - \alpha + \beta & 1 \\
\hat{T} & G^- + \beta' - \gamma + \beta (U^+ + 2U^- + \alpha - \beta) & U^- - \beta
\end{pmatrix},
\]

with \( \hat{T} = T + \gamma' - \beta\alpha' + G^- \alpha - G^+ \beta + \alpha \beta (2U^+ + U^-) \), \( n_- = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \gamma & \beta & 1 \end{pmatrix} \).

The above Lax operator can be reproduced from an \( O_\sigma \)-orbit. The orbit \( O_\sigma \) takes the form

\[
\kappa = \hat{\kappa} + [\epsilon_{-\psi}, Y] = \hat{\kappa} + \begin{pmatrix} -B & 0 & 0 \\ -C & 0 & 0 \\ 2D & A & B \end{pmatrix}, \quad \text{for} \ Y = \begin{pmatrix} * & * & C \\ * & * & -D \end{pmatrix} \in C^\infty(S^1, sl(3)), \quad (8.3)
\]

where \( \hat{\kappa} \in C^\infty(S^1, \mathfrak{g}_{\geq 0} \cap \mathfrak{g}_{\leq 1}) \) is the original point of the orbit, satisfying the restriction \( P^{\geq 1} (\hat{\kappa}) = \Lambda \). This orbit has four degrees of freedom, too many to be equivalent to the \( sl(3) \)-KdV hierarchy. We reduce the number of degrees of freedom by two by an Hamiltonian reduction with \( \bar{N}_- \), the subgroup of \( C^\infty(S^1, N_-) \) that preserves the orbit. With \( \Lambda = \Lambda_{co} \), the momentum map \( J_\sigma \) for the \( \text{Ad}^*(\bar{N}_-) \)-action reads

\[
J_\sigma(\kappa(Y)) = [\partial_x + I, Y] + \frac{1}{2} [[\epsilon_{-\psi}, Y], Y] = \begin{pmatrix} * & \hat{A}' - D - B\hat{A} & B' - A + C - B^2 \\ * & * & \hat{A}' - D - B\hat{A} \\ * & * & * \end{pmatrix},
\]

where we realise \( C^\infty(S^1, n^*_-) \) as the matrix analogue of (5.3), and define \( \hat{A} = \frac{1}{2}(A + C) \). Thus the level set with momentum \( J_\sigma(\kappa) = -\frac{1}{2} \mu I - \nu \epsilon_{\psi} \) has the form

\[
\kappa = \Lambda_{co} + \begin{pmatrix} -B & 0 & 0 \\ -\hat{A} + \frac{1}{2} (B' - B^2 + \nu) & 0 & 0 \\ \mu + 2\hat{A}' - 2B\hat{A} & \hat{A} + \frac{1}{2} (B' - B^2 + \nu) & B \end{pmatrix}, \quad (8.4)
\]

and passes through the point \( \Lambda + \mu \epsilon_{-\psi} + \frac{1}{2} \nu (e_{-1} + e_{-2}) \).
As in the case of $\text{sl}(2)$, the little group of a point $n_\sigma \in C^\infty(S^1, \tilde{N}_-)$ is not the full gauge group, $G_\sigma(n_\sigma) = \{g \in C^\infty(S^1, \tilde{N}_-) \mid \sigma(g) = 0\}$ since $\tilde{N}_-$ is abelian. Using definition (5.8), we find that

$$
S_g = \begin{pmatrix}
\frac{1}{2} (\gamma' - \alpha \left(2\gamma + \alpha' - \alpha^2\right)) & -\gamma + \alpha' & -\alpha \\
* & * & -\gamma - \alpha + \alpha^2 \\
* & * & -\frac{1}{2} \left(\gamma' - \alpha \left(2\gamma + \alpha' - \alpha^2\right)\right)
\end{pmatrix},
$$

which implies that the cocycle has the form

$$
\sigma(g^{-1}) = \begin{pmatrix}
* & -\frac{3}{2} (\gamma' - \alpha \alpha') & -3\alpha' \\
* & * & -\frac{3}{2} (\gamma' - \alpha \alpha') \\
* & * & *
\end{pmatrix}, \text{ where } g = \begin{pmatrix}
1 & 0 & 0 \\
\alpha & 1 & 0 \\
\gamma & \alpha & 1
\end{pmatrix}.
$$

The little group is given by elements $g \in \tilde{N}_-$ satisfying the constraint $\sigma(g) = 0$, i.e. $\alpha' = \gamma' = 0$. Thus the little group is the constant subgroup of $\tilde{N}_-$. Under equivalence by $\text{Ad}^*(G_\sigma(n_\sigma))$ we deduce that the constant components of $\hat{A}$ and $B$ are non-dynamical, the little group defining the equivalence $\hat{A} \sim \hat{A} - \gamma + \frac{1}{2} \alpha^2, B \sim B - \alpha, \alpha, \gamma$ arbitrary constants, (8.2). This equivalence can be fixed by imposing $\int \hat{A} dx = \int B dx = 0$. With these constraints, (8.4) should be a gauge slice of the $\text{sl}(3)$-KdV hierarchy, a gauge slice that is the $\text{sl}(3)$ analogue of the Mason-Sparling gauge slice, [15]. This is proved by showing that the gauge slice (8.4) is equivalent to that traditionally employed. This involves the construction of a gauge transformation that transforms (8.4) into the traditional gauge slice, such that the the parametrisations are in 1:1 correspondence. Using full gauge equivalence, we obtain the relations $u_1(x) = \mu + 3\hat{A}' - \frac{3}{2} B''$, $u_2(x) = 3B' + \nu$. These expressions confirm the conclusion that the constant components of $\hat{A}, B$ are non-dynamical. Thus the gauge slice (8.4) is equivalent to the traditional slice of [8] under full gauge equivalence since generic potentials $u_1(x), u_2(x)$ can be reproduced. The only restriction is that the Fourier components $\int u_1 dx, \int u_2 dx$ are non-dynamical. These are Hamiltonians as in the case of $\text{sl}(2)$, confirming that they are constants of motion.

We have proved that the traditional $\text{sl}(3)$-KdV hierarchy can be constructed as an Hamiltonian reduction of a $G_\sigma$-orbit. We now wish to prove that the $G_R$-orbits are able to reproduce the dynamics of this system, i.e. that the orbits $O_R$ have an Hamiltonian reduction such that the level sets possess a gauge fixing to the traditional gauge slice, or equivalently to (8.4). The $G_R$-orbits have the form $g^{-1}\mathcal{L}_0 g + \mathcal{L}_1$, and thus foliate the space $C^\infty(S^1, \tilde{g}_0) + \mathcal{L}_1$. Since $C^\infty(S^1, N_-)$ stabilises $\mathcal{L}_1 = \{c\} \in \mathcal{M}$, we can Hamiltonian reduce with respect to $C^\infty(S^1, N_-)$. Comparing to the $G_\sigma$-orbit, the appropriate momentum is $J_R(\kappa) = I$. The corresponding little group is $C^\infty(S^1, N_-)$, the full gauge group. Thus the level sets are of the form (8.1) with a gauge equivalence under $C^\infty(S^1, N_-)$. Hence the reduced phase spaces are the symplectic leaves of $\mathcal{M}$ as required, proving that the $\text{sl}(3)$-KdV hierarchy can be described dynamically as reduced $G_R$-orbits. This foliation appears to be finer than the split of the potentials into mKdV, pmKdV,
and true KdV type potentials. For example, orbits that possess a point with the form

\[
\begin{pmatrix}
0 & 1 & 0 \\
\nu & 0 & 1 \\
0 & \nu & 0
\end{pmatrix}
\]

(8.5)
can only produced pmKdV (or mKdV) type potentials, since only through an action by the gauge group \( C_\infty(S^1, N_-) \) can a term proportional to \( e_{-\psi} \) be reproduced. By definition, this is the pmKdV type potential, \([7]\). We note however that the pmKdV type potentials are probably not covered by only one \( G_R \)-orbit, since there exist other initial points than the choice (8.5) that do not lie on the \( G_R \)-orbit of (8.5).

The alternative theory with \( \Lambda = \Lambda_0 \) can also be analysed in terms of the orbit structures. It contrasts very nicely with the theory associated with \( \Lambda = \Lambda_{co} \), showing that a different choice of \( \Lambda_0 \) exist such that the reduced orbits describe bi-Hamiltonian systems. This theory is pursued further in section 11 after we have extended the analysis of sections 5 and 6 to include this more general \( \Lambda \).

9. THE GAUGE GROUP MOMENTUM MAPS AND HAMILTONIAN REDUCTION II

In this section we extend the momentum map analysis of section 5 to include theories constructed with \( \Lambda \) of \( s[w] \)-grade greater than 1, i.e. with a general form \( \Lambda = zI_- + I_+ \).

Define the homogeneous decomposition of \( \Lambda \) as \( \Lambda = zI_- + I_+ \), where \( I_{\pm} \) only have components of \( s[w] \)-grade \( \leq r_{\pm} < N \) respectively, \( r_+ = N + r_- = i > 0 \). We have not given \( \Lambda \) a definite \( s[w] \)-gradation as suggested in \([7]\). However this structure will be reproduced after a study of the dynamics, when we separate the components in \( \Lambda \) into those with \( s[w] \)-grade \( < r_+ \) as constants of motion (or absorbed into dynamical fields), and identify the components with \( s[w] \)-grade \( r_+ \) as the element of the Heisenberg algebra \( H[w] \) called \( \Lambda \) in \([7]\). The \( G_\sigma \)-orbit takes the form

\[
\kappa = \Lambda + [I_-, Y], \quad Y \in \hat{g}_0 \mod \ker(I_-).
\]

(9.1)

Upper and lower subscripts denote principal and homogeneous gradation respectively. As in section 5, the space \( C_\infty(S^1, \hat{g}_0 \mod \ker(I_-)) \) is used to parametrise the orbit \( O_\sigma \).

Define the gauge group \( \tilde{N}_- \) as the subgroup of \( C_\infty(S^1, N_-) \) that preserves the orbit (9.1), and \( \tilde{n}_- \) the corresponding Lie algebra. This implies that \( \tilde{n}_- \subset \text{Ann}(I_-) \), since if \( [I_-, X] \neq 0 \) the gauge group action would not preserve the homogeneous grade 1 term, i.e. \( zI_- \). We make the following definition
Definition 4. Define the map $s: \tilde{n}_- \to C^\infty(\mathbb{S}^1, \hat{g}_0 \mod \text{Ker}(I_-))$ as

$$[\partial_x + I_+, X] = [I_-, s_X],$$

such that under infinitesimal gauge transformations we have the relation

$$[\partial_x + \kappa, X] = [I_-, s_X + [Y, X]], \forall X \in \tilde{n}_-.$$

Given a symmetry $\Phi : G \times P \to P$, the momentum map is calculated by solving equation (2.3), which takes the form

$$\text{ad}^*_\sigma(d\kappa J_\sigma : X) \cdot \kappa = \text{ad}(X) \cdot \kappa, \forall X \in \tilde{n}_-.$$

This has the solution

$$J_{\sigma,X}(\kappa(Y)) = -\langle Y, [\partial_x + I_+, X]\rangle + \frac{1}{2}[\langle I_-, Y \rangle, [Y, X]],$$

(9.2)

in terms of the variable $Y \in C^\infty(\mathbb{S}^1, g \mod \text{Ker}(I_-))$, that parametrises the orbit $O_\sigma$. This is verified through a calculation of the functional derivative $d\kappa \hat{J}_\sigma : X$, accomplished by using the observation that all tangent vectors of $O_\sigma$ have a form $r = [I_-, u], u \in C^\infty(\mathbb{S}^1, g \mod \text{Ker}(I_-))$. Thus

$$\langle d\kappa J_{\sigma,X}, r \rangle = -\langle [I_-, d\kappa J_{\sigma,X}], u \rangle \bigg|_{\epsilon=0} J_{\sigma,X}(\kappa(Y + \epsilon u)).$$

From (9.2) we obtain the momentum map $J_\sigma : \hat{g}^* \to \hat{\tilde{n}}_-$,

$$J_\sigma(\kappa(Y)) = [\partial_x + I_+, Y] + \frac{1}{2}[\langle I_-, Y \rangle, Y] \mod \text{Ann}(\tilde{n}_-).$$

(9.3)

The Hamiltonian reduction with respect to $J_\sigma$ is accomplished by restricting to the pull back of a point $n_\sigma \in \hat{\tilde{n}}_-$ and reducing by the little group of $n_\sigma$. To calculate the little group of $n_\sigma$ we need to know the equivariance relation of $J_\sigma$, i.e. we require the action $\Psi$ of $\tilde{N}_-$ on $\hat{\tilde{n}}_-$ such that $J_\sigma(\text{Ad}(g) \cdot \kappa) = \Psi(g) \cdot J_\sigma(\kappa), \forall g \in \tilde{N}_-$. Following the calculation in section 5, we define the ‘exponential’ of the map $s$
Definition 5. Define the map $S : \tilde{N} \rightarrow C^\infty(S^1, g \mod \ker(I_-))$ by

$$g^{-1}(\partial_x + I_+) g = I_+ + [I_-, S_g], \forall g \in \tilde{N}.$$  

Theorem 3. The equivariance relation of the gauge group momentum map is with respect to the action $\Psi : (g, \eta) \rightarrow \text{Ad}^*(g) \cdot \eta + \sigma(g)$, where $\sigma(g)$ is a 1-cocycle of $\tilde{N}$ given by

$$\sigma(g^{-1}) = [\partial_x + I_+, S_g] + \frac{1}{2} [[I_-, S_g], S_g] \mod \text{Ann}(\tilde{n}-). \tag{9.4}$$  

Proof. This is proved by calculating $J_\sigma (\text{Ad}^*(g^{-1}) \cdot \kappa)$ and extracting the Ad$^*$ action by $g \in G$. Using definition 5, we obtain

$$J_\sigma (\text{Ad}^*(g^{-1}) \cdot \kappa(Y)) \equiv J_\sigma (\kappa (\text{Ad}(g^{-1}) \cdot Y + S_g)) = \left[ \partial_x + I_+, \text{Ad}(g^{-1}) \cdot Y + S_g \right] + \frac{1}{2} \left[ [I_-, \text{Ad}(g^{-1}) \cdot Y + S_g], \text{Ad}(g^{-1}) \cdot Y + S_g \right]$$  

We now rearrange, the second term taking the form

$$\frac{1}{2} \text{Ad}(g^{-1}) \cdot [[I_-, Y], Y] + \frac{1}{2} [[I_-, S_g], S_g] + [[I_-, S_g], \text{Ad}(g^{-1}) \cdot Y],$$  

where terms of the form $[I_-, ]$ are zero after taking the quotient with Ann$(I_-)$. The cocycle (9.4) now follows. The fact that this is a cocycle can be proved by using the relation $S_{gh} = h^{-1}S_p h + S_h$.

By theorem 1, the gauge algebra will be centrally extended under Poisson brackets,

$$\{\tilde{J}_\sigma; \cdot X, \tilde{J}_\sigma; \cdot Y \}_\sigma = \tilde{J}_\sigma; \cdot [X, Y] - \langle I_-, [s_X, s_Y] \rangle,$$

a central extension of $\tilde{n}_-$. 

25
10. THE GAUGE REDUCTION OF THE $G_R$-ORBIT II

In this section we generalise the analysis of section 6 to include the case of a more general $\Lambda$, $\Lambda = I_+ + z I_-$. We assume that there exists a gradation $s'$ such that the space $\oplus_{k=1}^{n-1} \hat{g}_k(s') \cap \hat{g}_{\geq 0}(s)$ lies in the image of $\text{Ad}(P_-) \cdot \Lambda - \Lambda$, where $P_-$ is the Lie group generated by $\hat{g}_0(s) \cap \hat{g}_{<0}(s')$. We further assume that the gradation $s'$ is maximal with respect to $\preceq$ for all gradations satisfying this property, and that $s'_0 \neq 0$. In [7], the gradation $s'$ is required to be induced from an equivalence class $[w]$ of the Weyl group. This will also be assumed, $s' = s[w]$, but takes no part in the following calculations.

With $\Lambda = I_+ + z I_-$, $I_\pm$ having components of $s[w]$-grade $\leq r_\pm$ respectively, the $G_R$-orbit takes the form

$$\kappa = n_-^{-1} b_+^{-1} (\partial_x + I_+) b_+ n_- + z I_-$$  \hspace{1cm} (10.1)

where we choose $\kappa = I_+$ for the original point. The fact that $I_+$ has negative $s[w]$-grade components implies that (10.1) describes all orbits of this type. As in section 9, the fact that $\Lambda$ has a well defined $s[w]$-grade will be reproduced in the final analysis. Note that only degrees of freedom generated through the action of $G_R$ are dynamical. In particular this means $L_1$ is non-dynamical. Thus we must allow $I_+$ to contain components with $s[w]$-grade less than $r_- = r_+ - N$ in order to achieve exact equality with the KdV hierarchies of [7].

Recall that the generalised KdV hierarchies are defined on the phase space $\mathcal{M}$, gauge equivalence classes of $C^\infty(S^1, \hat{g}_{\geq 0} \cap \hat{g}^{<0})$ with the gauge group generated by $C^\infty(S^1, \hat{g}_{0} \cap \hat{g}^{<0})$. This suggests that we attempt to Hamiltonian reduce the $G_R$-orbits with respect to the group $C^\infty(S^1, P_-)$. However, under $\text{Ad}^*$-action this group does not preserve the orbits (10.1) for general $I_-$. We now have two courses of action, use the smaller gauge group preserving the orbits, or increase the space we wish to reduce to the image of the $C^\infty(S^1, P_-)$ action. The second option removes us from the theory of Hamiltonian reduction, the image only having a Poisson structure. Pursuing the first option, define the symmetry group $\tilde{N}_- \subset C^\infty(S^1, P_-)$ as the subgroup that stabilises $I_-$ under $\text{Ad}$-action, i.e. $\text{Ad}(\tilde{N}_-) \cdot I_- = I_-$. Under $\text{Ad}^*$-action, $\tilde{N}_-$ preserves the orbits (10.1), and has a momentum map $J_R : \mathcal{O}_R \to C^\infty(S^1, g^* \mod \text{Ann}(\tilde{n}_-))$. Performing an Hamiltonian reduction with respect to $\tilde{N}_-$, we choose the level set constraint

$$P^{>0} \left( n_+^{-1} b_+^{-1} (\partial_x + I_+) b_+ n_- \right) = I_+ \mod \text{Ann}(\tilde{n}_-)$$  \hspace{1cm} (10.2)

where we have assumed with no loss of generality that $I_+$ is the value of the momentum map, i.e. we generate the orbit from a point on the level set of interest. The assumption that $s'_0 \neq 0$ implies that all lowering operators of $s[w]$-gradation $\leq -r_+$ are members of the Lie algebra $\tilde{n}_-$. This implies that all the components of the Lax operator with $s[w]$-grade $\geq r_+$ are totally specified by (10.2).

The little group $G(I_+)$ is not in general the full symmetry group $\tilde{N}_-$. However it is identical
for each orbit. Thus the Lax operator has a form

\[ \mathcal{L} = \partial_x + \Lambda + q(x), \quad q(x) \in \text{Ann}(\tilde{n}_-) \cap \hat{g}_0, \]  

(10.3)

with a gauge equivalence under the group \( G(I_+) \subset \tilde{N}_- \). The fact that the little group is not \( \tilde{N}_- \) implies that the Lax operator is partially gauge fixed, i.e. we can increase the equivalence relation to an equivalence under \( \tilde{N}_- \) invoking the penalty of altering the momentum. Only the momentum with \( s[w]-\text{grade} < r_+ \) can be altered because \( \tilde{n}_- \in C^\infty(\mathbb{S}^1, \hat{g}_0 \cap \hat{g}^{<0}) \), and the assumption regarding the image of \( \text{Ad}(P_-) \cdot \Lambda \) implies that all momenta of this type are altered. We deduce that the space of Lax operators of the form (10.3) is equivalent to the space of Lax operators with the form

\[ \mathcal{L} = \partial_x + \Lambda + q(x), \quad q(x) \in \hat{g}_0 \cap \hat{g}^{<i}, \]

with gauge group \( \tilde{N}_- \). This is similar to the structure in [7], except for the treatment of the homogeneous grade 1 components, and smaller gauge group. The question arises as to whether this is a partial gauge fixed form of a generalised KdV Lax operator, the partial gauge fixing corresponding to the reduction \( C^\infty(\mathbb{S}^1, P_-) \rightarrow \tilde{N}_- \). Since the only constraint on \( \tilde{N}_- \) is that it stabilises \( I_- \), we can increase the gauge group to \( P_- \) if we treat the components of \( s[w]-\text{grade} < r_- \) in \( I_- \) as dynamical. Thus we extract the fact that \( \Lambda \) has well defined \( s[w]-\text{grade} \) through an increase of the symmetry group from \( G(I_+) \) to \( C^\infty(\mathbb{S}^1, P_-) \).

Since we have enlarged the equivalence relation in (10.3) to an equivalence under \( \tilde{N}_- \), the momentum constraint becomes

\[ P^{\geq i} \left( b_+^{-1} (\partial_x + \kappa_0) b_+ \right) = P^{\geq i} I_+. \]

(10.4)

Thus the level set of the orbit (10.1) can be rewritten as

\[ \kappa = n_-^{\leq 0} P^{\leq 0} \left( b_+^{-1} (\partial_x + \kappa_0) b_+ \right) n_- + n_-^{\leq 1} I_+ n_- + z I_- . \]

From the definition of \( \tilde{N}_- \), we have \( n_-^{\leq 1} I_- n_- = I_- \), and hence the level set is gauge equivalent to a subspace of a \( G_{\mathbb{R}[w]} \)-orbit. Thus we again obtain the conclusion that the \( G_{\mathbb{R}[w]} \)-orbits induce a partition of the potentials that is coarser than that defined by the \( G_R \)-orbits.

Finally in this section, we reexamine the reduction of the \( G_R \)-orbits by \( C^\infty(\mathbb{S}^1, P_-) \). Since the \( G_R \)-orbits are not preserved under the \( \text{Ad}^* \)-action by \( C^\infty(\mathbb{S}^1, P_-) \), we consider the space

\[ \{ g^{-1} L_0 g + z I_- + \hat{u} \mid g \in G, \hat{u} \in C^\infty(\mathbb{S}^1, \hat{g}_1 \cap \hat{g}^{<i}) \}, \]

(10.5)

where \( L_0 = \partial_x + I_+ + q \), with \( q \) a point in \( C^\infty(\mathbb{S}^1, \hat{g}_0 \cap \hat{g}^{<i}) \). We take \( I_+, I_- \) as having well defined \( s[w]-\text{grade} \), such that \( \Lambda = I_+ + z I_- \) has \( s[w]-\text{grade} i \). This is a Poisson manifold, Poisson
bracket $\{,\}_R$, and kernel generated by the coordinate $\hat{u}$. To perform an analogue of Hamiltonian reduction, we consider the space of equivalence classes of (10.5) under $\text{Ad}^*(C^\infty(S^1, P_-))$, and construct a leaf of the foliation induced by the Poisson bracket $\{,\}_R$, [21]. Using the corresponding Hamiltonian reduction of $O_R$ as a guide, we construct the gauge invariant functions $\hat{J}_X$, $X \in C^\infty(S^1, \hat{g}_0 \cap \hat{g}^{\leq -i})$. These functions are elements of the kernel $\ker(\{,\}_R)$, and thus are constants of motion. Fixing these constants of motion reproduces a Lax operator with the form $L = \partial_x + \Lambda + q(x), q(x) \in C^\infty(S^1, \hat{g}_{\geq 0} \cap \hat{g}^{\leq i})$ and a gauge equivalence under $C^\infty(S^1, P_-)$, i.e. the generalised hierarchy of [7]. The remaining constants of motion are those that are constructed from the equivalence classes of $\hat{u}$. Choosing these constants of motion selects a leaf of the foliation induced by the second Poisson bracket.

11. ANOTHER VERY INTERESTING EXAMPLE

In this section we complete the analysis of the $Sl(3)$ theory associated with the element $\Lambda = \Lambda_\alpha$. We prove that under gauge reduction of the $G_\sigma$-orbit we can reproduce the KdV hierarchy as defined in [7].

In section 8, we constructed the $G_\sigma$-orbit, equation (8.3). The fact that we have altered $I_+$ means that the gauge group is different from that of section 8. In particular the symmetry group $\hat{N}_-$ is reduced from that in section 8 to the Lie group generated by $C^\infty(S^1, e^{-\psi})$. The momentum map takes the form

$$J_\sigma(\kappa(Y)) = B' - B^2 - 2D,$$

in terms of the parametrisation (8.3) of $C^\infty(S^1, \hat{g}_0 \text{mod Ker}(I_-))$. Thus we obtain the level set constraint, momentum= $-2\mu$

$$D = \frac{1}{2} (B' - B^2) + \mu,$$

with the level set

$$\kappa = \Lambda_\alpha + \begin{pmatrix} -B & 0 & 0 \\ -C & 0 & 0 \\ (B' - B^2) + \mu & A & B \end{pmatrix} \quad (11.1)$$

Anticipating that the little group is again trivial, this form for $\kappa$ will be a gauge slice for the theory. It is similar to the Mason-Sparling gauge slice in [15].

The little group of this level set is calculated by solving for the map $S$ of definition 5,
$S: \tilde{N}_- \to C^\infty(S^1, g \text{ mod } \text{Ker}(I_-))$. The calculation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & 0 & 1 \end{pmatrix} (\partial_x + I_+) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 1 \\ 0 & 0 & 0 \\ \alpha' - \alpha^2 & 0 & -\alpha \end{pmatrix}$$

implies that

$$S(e^{\alpha e^{-\psi}}) = \begin{pmatrix} \frac{1}{2}(\alpha' - \alpha^2) & 0 & -\alpha \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(\alpha' - \alpha^2) \end{pmatrix}$$

Thus the cocycle $\sigma(g)$ is given by $\sigma(e^{\alpha e^{-\psi}}) = 2\alpha'$, which implies that the little group is the constant subgroup of $\tilde{N}_-$. Under $\text{Ad}^*$-action the little group gauges away the constant component of $B$, implying that this component is non-dynamical.

In order to prove that the phase space, as parametrised in (11.1), reproduces the KdV hierarchy defined in [7], it is necessary to prove that (11.1) is a gauge slice for the theory. The Lax operator of the hierarchy has an homogeneous grade zero component with the general form, (before gauge fixing), [7]

$$L_0 = \partial_x + \begin{pmatrix} U^+ & J^+ & 1 \\ G^+ & -U^+ - U^- & J^- \\ T & G^- & U^- \end{pmatrix}$$

(11.2)

with a gauge equivalence under $C^\infty(S^1, N_-)$. On the homogeneous grade zero component, gauge equivalence reads

$$n_0^{-1}L_0n_- =$$

$$\begin{pmatrix} U^+ + \alpha J^+ + \gamma & J^+ + \beta & 1 \\ \hat{G}^+ & -U^+ - U^- - \alpha J^+ + \beta J^- - \alpha \beta & J^- - \alpha \\ \hat{T} & \hat{G}^- & U^- - \gamma - \beta J^- + \alpha \beta \end{pmatrix}$$

(11.3)

where $n_-$ has the form given in (8.2), and

$$\hat{T} = T + \gamma' - \beta \alpha' + \alpha G^- - \beta G^+ + \gamma(U^- - U^+) + \alpha \beta(2U^+ + U^-) + \alpha^2 \beta J^+ - \gamma(\alpha J^+ + \beta J^-) - \gamma^2 + \alpha \beta \gamma$$

$$\hat{G}^+ = G^+ + \alpha' - \alpha (2U^+ + U^-) + \gamma J^- - \alpha \gamma - \alpha^2 J^+$$

$$\hat{G}^- = G^- + \beta' + \beta (U^+ + 2U^-) - \gamma J^+ - \beta \gamma - \beta^2 J^- + \alpha \beta J^+ + \alpha \beta^2.$$

Thus, through the gauge degrees of freedom we can remove the variables $J^\pm$ and the component proportional to $H_1 + H_2$, i.e. let $\alpha = J^-, \beta = -J^+, \gamma = \frac{1}{2}(U^- - U^+) - J^+ J^-$. This gives a
gauge slice

\[ \kappa = z e^{-\psi} + \begin{pmatrix} \frac{1}{2} U & 0 & 1 \\ \frac{1}{2} U & -U & 0 \\ T & G^{-} & \frac{1}{2} U \end{pmatrix} \]  

(11.5)

similar to the type used in the other examples of [7]. We must prove that (11.1) is also a good gauge slice. This follows by performing a gauge transformation on (11.1) such that it attains the form (11.5). Performing a gauge transformation from (11.1) to (11.5), we find that \( T = 2B' + \mu \). However, no components in the Cartan subalgebra are reproduced, i.e. we appear to have a discrepancy in the treatment of \( H_1 - H_2 \). However, \( H_1 - H_2 \in \mathcal{H}[w] \) and hence the field \( U \) is non-dynamical, a field that can be reproduced in (11.1) by its inclusion in \( \Lambda \). Thus, provided \( \int T(x)dx \) is non-dynamical, the phase space of the KdV hierarchy is reproduced as the Hamiltonian reduction of a \( G_\sigma \)-orbit. Calculating the first Hamiltonian of the theory through the Drinfel’d-Sokolov procedure, [8], we find that \( H_\Lambda = \int T(x)dx \), confirming that this Fourier component of \( T \) is a constant under all the flows.

Consider the \( G_R \)-orbit of \( \Lambda_\alpha \)

\[ \kappa = z e^{-\psi} + g^{-1} (\partial_x + \kappa_0) g, \ g \in G. \]

In order that this is dynamically equivalent (8.3), it is necessary to perform a gauge reduction as before, since this orbit contains dynamical degrees of freedom that are not present in (8.3). The momentum map is projection onto \( n_+ \) as before, (5.2). However the appropriate value of the momentum is \( P^0_\alpha (\Lambda_\alpha) \), which has a little group generated by \( e^{-\psi} \). Fixing this gauge freedom, the Lax operator takes the form (11.5), with the dynamical variables \( G^\pm, T \) parametrised by \( g \in G \) satisfying the level set constraints. Thus the phase space is foliated by Hamiltonian reductions of \( G_R \)-orbits, reproducing the hierarchy as a dynamical system.

12. A FRACTIONAL KdV EXAMPLE: Sl(3)

In this section we consider the case of a ‘fractional’ hierarchy with \( \Lambda \) of \( s[w] \)-grade 2. This example has previously been studied in [4, 5, 7]. The fractional KdV hierarchy is defined by the Lax operator \( \mathcal{L} = \partial_x + \Lambda + q(x) \), where

\[ \Lambda = \begin{pmatrix} 0 & 0 & 1 \\ z & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \]

and \( q(x) \in \hat{g}_{\geq 0} \cap \hat{g}^{\leq 1} \), [7]. The gauge group is generated by \( C^\infty (S^1, \hat{g}_0 \cap \hat{g}^{<0}) \). In this section, we prove that the two orbit constructions in section 4 reproduce the \( Sl(3) \) fractional KdV hierarchy as a dynamical system.
From our previous examples, we know that the $G_\sigma$-orbit is more restrictive as a model for the phase space, and gives a gauge fixing procedure. We observe from [7] that the components of the phase space with homogeneous degree zero are identical to the example considered in the previous section, i.e. the Lax operator has a homogeneous degree zero form identical to (11.2) before gauge fixing. Thus, we use a realisation for $g \mod \text{Ker}(I_-)$ that produces a parametrisation of the $G_\sigma$-orbit similar to (11.2),

$$Y = \begin{pmatrix} \frac{1}{2} (2G^+ + G^-) & -U^+ & -J \\ \frac{1}{2} T & -\frac{1}{3} (G^+ - G^-) & U^- \\ 0 & -\frac{1}{2} T & -\frac{1}{3} (G^+ + 2G^-) \end{pmatrix}$$

The orbit takes the form, equation (9.1)

$$\kappa = \Lambda + \begin{pmatrix} U^+ & J & 0 \\ G^+ & -(U^+ + U^-) & -J \\ T & G^- & U^- \end{pmatrix}, \quad (12.1)$$

where we have not included a (non-dynamical) component proportional to $ze^{-\psi}$ for simplicity. To reproduce the most general form of the fractional KdV hierarchy, i.e. reproducing all constants of motion, we should include this component. We observe that this orbit gives a Lax operator that is very similar to the Lax operator constructed in [7], before introducing gauge equivalence. They differ principally in the symmetry of the components $J^\pm$, and the fact that no component proportional to $ze^{-\psi}$ is created through the $G_\sigma$-action. Introduction of the gauge group in [7] reduces the number of dynamical fields to four, a suitable gauge slice taking the form,

$$\kappa = \begin{pmatrix} 0 & 0 & 1 \\ z & 0 & 0 \\ 0 & z & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} U & 0 & 0 \\ G^+ & -U & 0 \\ T & G^- & \frac{1}{2} U \end{pmatrix} + \phi \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z & 0 & 0 \end{pmatrix} \quad (12.2)$$

The field $\phi$ is not dynamical, and is in fact set to zero in [4]. In order to obtain equivalence between the integrable system constructed from the coadjoint orbit (12.1) and the fractional KdV hierarchy defined by the Lax operator (12.2), we must reduce the degrees of freedom by 2. This is achieved through the Hamiltonian reduction of the orbit by the gauge group $\tilde{N}_- \subset C^\infty(S^1, \tilde{g}_0 \cap \tilde{g}^{<0})$, the subgroup that preserves the orbit. This requires that $[X, I_-] = 0, \forall X \in \tilde{N}_-$ such that the homogeneous degree 1 components of the Lax operator are preserved. Thus $\tilde{N}_-$ consists of matrices of the form (5.3). From section 9, the momentum map of the gauge group $\tilde{N}_-$ is

$$J_\sigma(\kappa(Y)) = [\partial_x + I_+, Y] + \frac{1}{2} [[I_-, Y], Y] \mod \text{Ann}(\tilde{n}_-)$$

31
where $Y$ parametrises the orbit (12.1). This takes the form

$$J_\sigma(\kappa(Y)) =$$

$$\begin{pmatrix}
* & U'^+ + \frac{1}{2} T + U'^2 + \frac{1}{2} U^+ U^- & J' + J (U^+ - U^-) \\
& + \frac{1}{2} J (G^+ - G^-) & + (G^+ + G^-) \\
& - U' - \frac{1}{2} T + U'^2 + \frac{1}{2} U^+ U^- & + \frac{1}{2} J (G^+ - G^-) \\
& * & * \\
& * & *
\end{pmatrix}$$

which gives the level set constraints

$$J' + J (U^+ - U^-) + (G^+ + G^-) = 0,$$

$$T + (U^+ - U^-)' + U'^2 + U'^2 + U^+ U^- + J (G^+ - G^-) = 0,$$

(12.3)

where we have chosen the momentum map to have the value zero for the convenience of discussion.

We anticipate that, as in previous examples, the little group is effectively trivial, so that the Hamiltonian reduction by $\tilde{N}_-$ reduces the degrees of freedom by 2, through the 2 constraints in (12.3). This is the correct number for the theory to be equivalent to the fractional KdV hierarchy. The level set takes the form

$$\kappa = \Lambda^+$$

$$\begin{pmatrix}
U^+ \\
G - \frac{1}{2} (J' + J (U^+ - U^-)) \\
(U^- - U^+)' - U'^2 - U'^2 - U^+ U^- - 2 J G \\
- G - \frac{1}{2} (J' + J (U^+ - U^-)) \\
\end{pmatrix}$$

where we have defined $G = \frac{1}{2} (G^+ - G^-)$. This level set possesses 4 dynamical degrees of freedom: $U^+, U^-, G \equiv (G^+ - G^-)$ and $J$. We propose that this parametrisation of the level set corresponds to a gauge slice of the fractional KdV hierarchy of [7]. To verify this, we must prove that the little group $G_\sigma(n_\sigma)$ is ‘trivial’, and that under a full $C^\infty(S^1, N_-)$ gauge transformation, the two slices (12.2) and (12.4) are equivalent. The gauge equivalence relation for the homogeneous degree zero component is identical to that of the previous example, and takes the form (11.3). The homogeneous grade 1 components are invariant because $[X, I_-] = 0, \forall X \in \tilde{N}_-$. We choose

$$\alpha = \beta = -J, \ \gamma = \frac{1}{2} (U^- - U^+) + \frac{1}{2} J^2,$$

in a gauge group action on (12.4), which produces a gauge slice similar to (12.2), i.e. removing the $s[u]$-grade 1 components, and the Cartan subalgebra component proportional to $H_1 + H_2$. 

32
The dynamical variables now take the form, (11.4)

\[
U = (U^+ + U^-) - J^2,
\]

\[
\dot{G}^+ = -J' + G + \frac{3}{2}J(U^+ + U^-) - J^3,
\]

\[
\dot{G}^- = -J' - G - \frac{3}{2}J(U^+ + U^-) + J^3,
\]

\[
\dot{T} = -\frac{3}{2}(U^+ - U^-)' - \frac{3}{4}U^2.
\]

(12.5)

Thus we have reproduced the gauge slice (12.2) with the constraint

\[
\int G^+ + G^- dx = 0, \int T + \frac{3}{4}U^2 dx = 0.
\]

By including a component \(\phi e^{-\psi}\) into \(I_-\), and altering the value of the momentum map, we are able to reproduce the gauge slice (12.2) in all generality, with the proviso that \(\int G^+ + G^- dx, \int T + \frac{3}{4}U^2 dx\) are non dynamical. A calculation of the Hamiltonians of the theory through the Drinfel’d-Sokolov procedure, [7, 8] proves that this is in fact the case, because \(H_{\Lambda_1} = \int G^+ + G^- dx, H_{\Lambda_2} = \int T + \frac{3}{4}U^2 dx\) where \(\Lambda^1 \equiv \Lambda_{\text{co}}, \Lambda^2 \equiv \Lambda_{\text{co}}^2\) are the elements of the Heisenberg subalgebra \(\mathcal{H}[w]\) of \(s[w]\)-grades 1 and 2 respectively.

The final calculation that must be performed to complete the proof of phase space equivalence between the Hamiltonian reduction of the \(G_{\sigma}\)-orbit, (12.4), and the fractional KdV hierarchy is the calculation of the little group, in particular the cocycle \(\sigma\) of theorem 3. This requires that we solve for the map \(S\), definition 5, given by

\[
g^{-1}( \partial_x + I_+) g = I_+ + [I_-, S_g], \quad \text{for } g = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \gamma & \alpha & 1 \end{pmatrix} \in \tilde{N}_.
\]

Using the parametrisation given in equation (12.1) in terms of the variable \(Y\), we deduce that

\[
S_g = \begin{pmatrix} \alpha' - \alpha \gamma + \frac{1}{3} \alpha^3 & -\gamma & -\alpha \\ \frac{1}{3} (\gamma' - \alpha \alpha' - \gamma^2 + \gamma \alpha^2) & \frac{1}{3} \alpha^3 & -\gamma + \alpha^2 \\ 0 & -\frac{1}{3} (\gamma' - \alpha \alpha' - \gamma^2 + \gamma \alpha^2) & -\alpha' + \alpha \gamma - \frac{2}{3} \alpha^3 \end{pmatrix}
\]

This implies that the cocycle \(\sigma(g^{-1}) = [\partial_x + I_+, S_g] + \frac{1}{2} [[I_-, S_g], S_g]\) is

\[
\sigma(g^{-1}) = \begin{pmatrix} * & \frac{1}{3} (-3 \gamma' + \alpha \alpha' + \alpha^4) - \gamma \alpha^2 & -3\alpha' \\ * & * & \frac{1}{2} (-3 \gamma' + 5 \alpha \alpha' - \alpha^4) + \gamma \alpha^2 \\ * & * & * \end{pmatrix}
\]

Since the gauge group \(\tilde{N}_.\) is abelian, the little group \(G(n_\sigma)\) of all points \(n_\sigma \in \tilde{n}_-\) is given by \(G(n_\sigma) = \{ g \in \tilde{N}_- | \sigma(g) = 0 \}\), i.e. an element of \(G(n_\sigma)\) satisfies the constraints \(\gamma' = \alpha' = 0\). Thus
the little group is the constant subgroup of $\tilde{N}_-$, as in the previous examples. Under Ad$^*$-action with this little group, we obtain the equivalence relations

$$U^+ \sim U^+ + \alpha J + \gamma, \quad U^- \sim U^- + \alpha J + \alpha^2 - \gamma,$$

$$J \sim J + \alpha, \quad G \sim G - \frac{1}{2} \left( 3\alpha \left( U^+ + U^- + \alpha J \right) + \alpha^2 \right), \quad (12.6)$$

on the fields $U^+, U^-, G, J$ of the level set $\mathcal{L}$. Thus the constant components of $J$ and $U^+ - U^-$ can be gauged away. It can be proved that there is no contradiction between this gauge transformation and the parametrisation (12.5), because the fields $U, \hat{G}^{\pm}, \hat{T}$ of (12.5) are independent of the equivalence relation (12.6).

The $G_R$-orbit takes the form $g^{-1}\mathcal{L}_0 g + \mathcal{L}_1$, and hence contains dynamical degrees of freedom in excess of those contained in the fractional KdV theory of [4,7]. Thus we introduce the gauge group and perform an Hamiltonian reduction. However, as discussed in section 10, Ad$^*$-action by $C^\infty(S^1, N_-)$ is not a symmetry of the orbit; Ad$^*$ $(C^\infty(S^1, N_-))$ does not stabilise the homogeneous grade 1 component, i.e. $zL_-$. The symmetry group is thus generated by $\epsilon_{-1}, \epsilon_{-1} + \epsilon_{-2}$. The momentum map $J_R$ takes the value $I_+$ mod Ann($h_R$), with the little group equal to the full symmetry group since the momentum map is equivariant and the symmetry group abelian.

Through action by the little group, we obtain the gauge fixed form

$$\kappa = \begin{pmatrix} 0 & 0 & 1 \\ z & 0 & 0 \\ 0 & z & 0 \end{pmatrix} + \begin{pmatrix} U & 0 & 0 \\ -U & 0 & 0 \\ \frac{1}{2}U & \frac{1}{2}U & 0 \end{pmatrix} + \begin{pmatrix} 0 & \phi & 0 \\ 0 & 0 & \phi \\ \frac{1}{2}\phi & \frac{1}{2}\phi & 0 \end{pmatrix}, \quad (12.7)$$

where $\phi, \hat{\phi}$ are constants of motion, and $U = U^+ + U^-$. There are 4 dynamical degrees of freedom $U, G^{\pm}, T$, which are parametrised by $g \in G$ satisfying the two level set constraints. This gauge fixed form is identical to that of the fractional KdV hierarchy of [7], except for the treatment of the constants of motion $\phi$ and $\hat{\phi}$. The resolution of this can be accomplished in many ways. For example, we can choose the momentum map to have the value $\phi = \hat{\phi}$, or we can introduce the remaining part of the gauge group $C^\infty(S^1, N_-)$ not included in $\tilde{N}_-$. This involves extending the reduction to a Poisson manifold, and removes us from the framework of Hamiltonian reduction. It is well known that the Ad$^*$-action by $C^\infty(S^1, N_-)$ defines a Poisson symmetry of $(C^\infty(S^1, \hat{g}_{\geq 0} \cap \hat{g}^{\leq 1}), \{ , \}_R)$, and hence there is a Poisson structure induced on the gauge invariant functions, i.e. the space of gauge equivalence classes of $C^\infty(S^1, \hat{g}_{\geq 0} \cap \hat{g}^{\leq 1})$ is a Poisson manifold.

We now prove that Lax operators of the form (12.2) are a Poisson submanifold of this space, i.e. the constraint $P^2(\kappa) = I_+$ is consistent with the Poisson structure. This follows since $\hat{J}_{e^\phi}$ is both gauge invariant, and an element of the kernel $\text{Ker}(\{ , \}_R)$. Hence $\hat{J}_{e^\phi}$ is a constant of motion, the momentum $\hat{J}_{e^{-\phi}} = 1$ defines the submanifold consisting of Lax operators of the form (12.2). The further restriction to symplectic leaves can be accomplished by a similar analysis, i.e. the gauge group $C^\infty(S^1, N_-)$ preserves the Poisson manifold $\{ \kappa = g^{-1}\mathcal{L}_0 g + zL_- \quad \forall g \in G, \phi \in C^\infty(S^1) \}$, thus the equivalence classes under Ad$^*$ $(C^\infty(S^1, N_-))$ also form a Poisson manifold, with a kernel generated by the image of $\phi$ and $\hat{J}_{e^{-\phi}}$. 

34
13. Conclusions

In this paper we have proved that there exists a coadjoint orbit structure for the phase spaces of the KdV hierarchies associated to $Sl(2)$ and $Sl(3)$, $i < N$. Two group actions are constructed, $G_R$, $G_\sigma$, such that the Poisson structures of [5] are reproduced under the Kirillov construction, [13]. We reconstruct the phase space $M$ of the generalised hierarchies of [7], $i < N$, as an Hamiltonian reduction of a coadjoint orbit of the $G_\sigma$-action. The gauge group $H_\sigma$ is the maximal subgroup of $C^\infty(S^1, N_-)$ that preserves the orbit. The Hamiltonian reduction by this symmetry is non-trivial, the momentum map being equivariant with respect to an extended $Ad^*$ action. However, we observe that the little group is effectively trivial, hence the calculation of the level set gives a gauge fixing of the theory. This gauge slice is a generalisation of the gauge slice employed in [15]. If this extends to all the hierarchies, this process would solve the problem of how to choose a gauge slice for the generalised hierarchies of [7]. The orbits of the $G_R$-action produce a partition of the potentials by ‘type’, this being finer than the partition into mKdV, pmKdV and ‘true’ KdV type potentials. The fact that the phase space $M$ cannot be reproduced from a single orbit is a result of the fact that the Poisson structure $\{ , \}_2 \equiv \{ , \}_R$ possesses a non-trivial kernel on $M$.

This paper has only considered the problem of reproducing the phase space of the KdV hierarchies from a coadjoint orbit method. In particular we choose the gauge groups such that the theories in [7] are reproduced, and use certain observations about constants of motion to verify equivalence. Ultimately, the coadjoint orbit construction should be an independent and self contained construction. We propose that given $I_-$, the symmetry group $H_\sigma$, (which also defines $I_+$), is specified by the requirement that the reduced space is foliated by Hamiltonian reductions of $G_R$-orbits. The gauge group $H_R$ is identical for each $G_R$-orbit of interest. It then remains to prove that the Poisson brackets are coordinated, and further that the theory is bi-Hamiltonian. It is expected that the regularity of the element $\Lambda$ will be reproduced as a method of classification. From our examples we observe that the symmetry group $H_\sigma$ restricts the possible form of $I_+$ given $I_-$. We hope to consider these questions in a later publication.

This construction of the generalised KdV hierarchies of [7] from a Coadjoint Orbit framework suggests many avenues for further generalisations. We observe that the orbits used in this construction are a special case, suggesting that there may exist a further generalisation involving characters, as in the full AKS theory, [3]. We have also only considered untwisted Kac Moody algebras, and restricted the potentials to be periodic.

Finally we remark that the $G_{R[u]}$-action discussed in sections 6 and 10 is very reminiscent of the generation of solutions to the KdV hierarchies through dressing transformations, [6,12,19]. The construction involves a vertex operator representation of a Kac Moody algebra, and generates tau-functions from the vacuum. The use of the vertex representation is suspected to be analogous to our restriction to level sets of the $G_R$-orbits. The connection between coadjoint orbits and unitary representations, [13], supports the suggestion of a relationship. In this vein, we further observe that the Poisson structure analysis of the dressing transformation in [20] is very suggestive of the $G_{R[u]}$-action discussed in sections 6 and 10, and it’s associated Poisson mapping structure.
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REFERENCES

1. R. Abraham, J.E. Marsden.: Foundations of Mechanics. Second Edition. Benjamin/ Cummings Pub. 1980.

2. M. Adler.: On the Trace Functional for Formal Pseudo-Differential Operators and the Symplectic Structure of the Korteweg-Devries Type Equations. Inv. Math. 50, 219-248 (1979).

3. O. Babelon, C.M. Viallet.: Integrable Models, Yang-Baxter Equation and Quantum Groups. Part 1. Preprint SISSA-54/89/EP May 1989.

4. I. Bakas, D.A. Depireux.: A Fractional KdV Hierarchy, Mod. Phys. Lett. A6, 1561 (1991). (Erratum A6, 2351 (1991)).

5. N. Burroughs, M. De Groot, T. Hollowood, J.L. Miramontes.: Generalised Drinfel’d- Sokolov Hierarchies II: The Hamiltonian Structures. Preprint PUPT-1263, IASSNS-HEP- 91/42.

6. E. Date, M. Jimbo, M. Kashiwara, T. Miwa.: Transformation Groups for Soliton Equations-Euclidean Lie Algebras and Reduction of the KP Hierarchy. Publ. RIMS. Kyoto Univ. 18, 1077-1110 (1982).

7. M. De Groot, T.J. Hollowood, J.L. Miramontes.: Generalized Drinfel’d-Sokolov Hierarchies. IAS and Princeton preprint IASSNS-HEP-91/19, PUPT-1251 March 1991. To appear in CMP.

8. V.G. Drinfel’d, V.V. Sokolov.: Lie Algebras and Equations of the Korteweg-de Vries Type. Jour.Sov.Math. 30 (1985) 1975; Equations of Korteweg-De Vries Type and Simple Lie Algebras. Soviet.Math.Dokl. 23 (1981) 457.

9. L.D. Faddeev, L.A. Takhtajan.: Hamiltonian Methods in the Theory of Solitons. Springer-Verlag, 1986.

10. V.G. Kac.: Infinite Dimensional Lie Algebras, 2nd edition. Cambridge University Press 1985.

11. V.G. Kac, D.H. Peterson.: 112 Constructions of the Basic Representations of the Loop Group of $E_8$. In Symposium on Anomalies, Geometry and Topology. Eds: W. A. Bardeen & A. R. White. Singapore, World Scientific 1985.

12. M. Jimbo, T. Miwa.: Solitons and Infinite Dimensional Lie Algebras. Publ. RIMS. Kyoto Univ. 19, 943-1001 (1983).

13. A.A. Kirillov.: Elements of the Theory of Representations. Springer-Verlag 1976.
14. V.F. Lazutkin, T.F. Pankratova.: Normal Forms and Versal Deformations for Hill’s Equation. Func. Anal. & Appl. 9, 306-311 (1975).

15. L.J. Mason, G.A.J. Sparling.: Nonlinear Schroödinger and Korteweg-de-Vries are reductions of self-dual Yang-Mills. Phys.Lett. A137 (1989) 29.

16. J. Marsden, A. Weinstein.: Reduction of Symplectic Manifolds with Symmetry. Rep. Math. Phys. 5, 121-130, 1974.

17. G. Segal.: Unitary Representations of some infinite dimensional groups. CMP 80, 301-342, (1981).

18. G. Segal.: The Geometry of the KdV equation. Int. Jour. Mod. Phy. A. Vol. 6 No 16, 2859-2869, (1991).

19. G. Segal, G. Wilson.: Loop Groups and Equations of KdV Type. IHES Pub. Math, No 61, 5-65 (1985).

20. M. Semenov-Tian-Shansky.: Dressing Transformations and Poisson Group Actions. Publ. RIMS. Kyoto Univ. 21 (1985) 1237.

21. A. Weinstein.: The Local structure of Poisson Manifolds. Jour. Diff. Geom. 18, 523-557 (1983).