On synthetic and transference properties of group homomorphisms

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Abstract

We study Borel homomorphisms \( \theta : G \to H \) for arbitrary locally compact second countable groups \( G \) and \( H \) for which the measure

\[
\theta_* (\mu) (\alpha) = \mu (\theta^{-1} (\alpha)) \quad \text{for} \quad \alpha \subseteq H \quad \text{a Borel set}
\]

is absolutely continuous with respect to \( \nu \), where \( \mu \) (respectively, \( \nu \)) is a Haar measure for \( G \), (respectively, \( H \)). We define a natural mapping \( \theta \) from the class of maximal abelian selfadjoint algebra bimodules (masa bimodules) in \( B (L^2 (H)) \) into the class of masa bimodules in \( B (L^2 (G)) \) and we use it to prove that if \( k \subseteq G \times G \) is a set of operator synthesis, then \( (\theta \times \theta)^{-1} (k) \) is also a set of operator synthesis and if \( E \subseteq H \) is a set of local synthesis for the Fourier algebra \( A (H) \), then \( \theta^{-1} (E) \subseteq G \) is a set of local synthesis for \( A (G) \). We also prove that if \( \theta^{-1} (E) \) is an \( M \)-set (respectively, \( M_1 \)-set), then \( E \) is an \( M \)-set (respectively, \( M_1 \)-set) and if \( \text{Bim} (I^1) \) is the masa bimodule generated by the annihilator of the ideal \( I \) in \( V N (G) \), then there exists an ideal \( J \) such that \( G (\text{Bim} (J)) = \text{Bim} (J^1) \). If this ideal \( J \) is an ideal of multiplicity, then \( I \) is an ideal of multiplicity. In case \( \theta_* (\mu) \) is a Haar measure for \( \theta (G) \), we show that \( J \) is equal to the ideal \( \rho_* (I) \) generated by \( \rho (I) \), where \( \rho (u) = u \circ \theta, \forall u \in I \).

1. Introduction

Arveson discovered the connection between spectral synthesis and operator synthesis, [2]. Froelich found the precise connection for separable abelian groups, [8], and Spronk and Turowska for separable compact groups, [16]. Ludwig and Turowska generalized the previous results in the case of locally compact second countable groups, G. They proved that if \( E \subseteq G \) is a closed set and \( E^* = \{(s, t) \in G \times G : ts^{-1} \in E \} \), then \( E \) is a set of local synthesis if and only if \( E^* \) is a set of operator synthesis, [11]. Anoussis, Katavolos and Todorov stated in [1] that given a closed ideal \( I \) of the Fourier algebra \( A (G) \), where \( G \) is a locally compact second countable group, there are two natural ways to construct a \( \nu^* \)-closed maximal abelian selfadjoint (masa) bimodule.

(i) Let \( I^\perp \) be the annihilator of \( I \) in \( V N (G) \) and then take the masa bimodule \( \text{Bim} (I^\perp) \) in the space of bounded operators acting on \( L^2 (G), B (L^2 (G)) \), generated by \( I^\perp \).

(ii) Consider the space \( \text{Sat} (I) = \text{span} \{ N (I) T (G) \} / _{\parallel T (G)} \) where \( N (u) (s, t) = u (ts^{-1}) \) for all \( u \in I \) and \( T (G) \) is the projective tensor product \( L^2 (G) \hat{\otimes} L^2 (G) \) and then take its annihilator \( \text{Sat} (I)^\perp \) in \( B (L^2 (G)) \).

One of their main results is that \( \text{Bim} (I^\perp) = \text{Sat} (I)^\perp \). They used this in order to prove that if \( A (G) \) possesses an approximate identity, then \( E \subseteq G \) is a set of spectral synthesis if and only if \( E^* \) is a set of operator synthesis.

The transference of results from Harmonic Analysis to Operator Theory and vice versa is not limited to the case of synthesis. In [13], Shulman, Todorov and Turowska proved that if \( G \) is a
locally compact second countable group and \( E \subseteq G \), then \( E \) is an \( M \)-set (respectively, \( M_1 \)-set) if and only if \( E^* \) is an \( M \)-set (respectively, \( M_1 \)-set). Subsequently, Todorov and Turowska in [17] proved that an ideal \( J \subseteq A(G) \) is an ideal of multiplicity if and only if \( \text{Bim}(J^\perp) \) is an operator space of multiplicity.

In Section 2, we consider arbitrary locally compact second countable groups \( G \) and \( H \), Borel homomorphisms \( \theta : G \to H \) for which

\[
\theta_*(\mu)(\alpha) = \mu(\theta^{-1}(\alpha)) \quad \text{for } \alpha \subseteq H \text{ a Borel set}
\]
is absolutely continuous with respect to \( \nu \), \( (\theta_*(\mu) \ll \nu) \), where \( \mu \) (respectively, \( \nu \)) is a Haar measure for \( G \) (respectively, \( H \)). Recall that Borel measurable homomorphisms between locally compact groups are automatically continuous, [10, 12]. We define a natural mapping \( G \) from the class of masa bimodules in \( B(L^2(H)) \) to the class of masa bimodules in \( B(L^2(G)) \). We prove that

\[
G(M_{\max}(k)) = M_{\max}((\theta \times \theta)^{-1}(k)), \quad G(M_{\min}(k)) = M_{\min}((\theta \times \theta)^{-1}(k)),
\]

where \( M_{\max}(k) \) is the biggest masa bimodule supported on the \( \omega \)-closed \( k \), see the definition below, and \( M_{\min}(k) \) is the smallest. Therefore if \( M_{\max}(k) \) is a synthetic operator space, then \( M_{\max}((\theta \times \theta)^{-1}(k)) \) is operator synthetic. This implication can also be deduced from [14, Theorem 4.7] or from [6, Theorem 5.2]. Here we present a different proof. We also prove that if \( E \subseteq H \) is a set of local synthesis, then \( \theta^{-1}(E) \) is a set of local synthesis and if \( U \) is a \( w^* \)-closed masa bimodule for which \( G(U) \) contains a non-zero compact operator (or a non-zero finite rank operator or a rank one operator), so does \( U \). We use this result to prove that if \( \theta^{-1}(E) \) is an \( M \)-set (respectively, \( M_1 \)-set), then \( E \) is an \( M \)-set (respectively, \( M_1 \)-set). If \( I \) is an ideal of \( A(H) \), we prove that there exists an ideal \( J \subseteq A(G) \) such that

\[
G(\text{Bim}(I^\perp)) = \text{Bim}(J^\perp), \quad \text{Sat}(J) = \text{span}\{N(\rho(I)(s,t)) : \|T(G)\| = 1\},
\]

where \( \rho(u) = u \circ \theta \), \( \forall u \in I, \quad N(\rho(u))(s,t) = \rho(u)(ts^{-1}). \)

We use equalities (1.1) to prove that if \( J \) is an ideal of multiplicity, then \( I \) is an ideal of multiplicity. In Section 3, we assume that the measure \( \theta_*(\mu) \) is a Haar measure for the group \( \theta(G) \). We prove that if \( I \) is a closed ideal of \( A(H) \), then \( \rho(I) \subseteq A(G) \) and so we can choose in (1.1) as \( J \) the ideal \( \rho_*(I) \) generated by \( \rho(I) \). We also prove that if \( A(G) \) possesses an approximate identity and \( E \) is an ultra strong Ditkin set, then \( \theta^{-1}(E) \) is also an ultra strong Ditkin set.

We now present the definitions and notation that will be used in this paper. If \( S \) is a subset of a linear space, we denote by \([S]\) its linear span. If \( H \) and \( K \) are Hilbert spaces, \( B(H,K) \) is the set of bounded operators from \( H \) to \( K \). We write \( B(H) \) for \( B(H,H) \). If \( \mathcal{X} \subseteq B(H,K) \) is a subspace, we write \( \text{Ref}(\mathcal{X}) \) for the reflexive hull of \( \mathcal{X} \), that is,

\[
\text{Ref}(\mathcal{X}) = \{T \in B(H,K) : T\xi \in \overline{\mathcal{X}\xi}, \quad \forall \xi \in H\}.
\]

Let \( G \) be a locally compact group with Haar measure \( \mu \), and \( T(G) \) the projective tensor product \( L^2(G) \hat{\otimes} L^2(G) \). Every element \( h \in T(G) \) is an absolutely convergent series,

\[
h = \sum_i f_i \otimes g_i, \quad f_i, g_i \in L^2(G), \quad i \in \mathbb{N},
\]

where \( \sum_i \|f_i\|_2 \|g_i\|_2 < +\infty \). Such an element may be considered as a function \( h : G \times G \to C \), defined by \( h(s,t) = \sum_i f_i(s)g_i(t) \). The norm in \( T(G) \) is given by

\[
\|h\|_t = \inf \left\{ \sum_i \|f_i\|_2 \|g_i\|_2 : h = \sum_i f_i \otimes g_i \right\}.
\]
The space $T(G)$ is predual to $B(L^2(G))$. The duality is given by
\[(T, h)_t = \sum_i (T(f_i), \gamma_i),\]
where $h$ is as above and $(\cdot, \cdot)$ is the inner product of $L^2(G)$.

A subset $F \subseteq G \times G$ is called marginally null if $F \subseteq (\alpha \times G) \cup (G \times \beta)$, where $\alpha$ and $\beta$ are Borel sets such that $\mu(\alpha) = \mu(\beta) = 0$. In this case, we write $F \simeq 0$. If $F_1$ and $F_2$ are subsets of $G \times G$, we write $F_1 \simeq F_2$ if the symmetric difference $F_1 \Delta F_2$ is marginally null. If $F \subseteq G \times G$, we denote by $M_{\max}(F)$ the subspace of $B(L^2(G))$ consisting of all those operators $T$ satisfying
\[(\alpha \times \beta) \cap F \simeq 0 \Rightarrow P(\beta)TP(\alpha) = 0.\]

Here $P(\beta)$, and similarly $P(\alpha)$ is the projection onto $L^2(\beta, \mu)$. We usually identify the algebra $L^\infty(G, \mu)$ with the algebra of operators
\[M_f : L^2(G) \to L^2(G), g \to fg,\]
where $f \in L^\infty(G, \mu)$. This algebra is a maximal abelian selfadjoint algebra, referred to as ‘masa’ in what follows. If $F \subseteq G \times G$, then $M_{\max}(F)$ is an $L^\infty(G)$-bimodule. An $L^\infty(G)$-bimodule will be referred to as a ‘masa bimodule’. The space $M_{\max}(F)$ is reflexive. Also, there exists a $w^*$-closed masa bimodule $U_0$ with the property that it is the smallest $w^*$-closed masa bimodule $\mathcal{U}$ such that $\text{Ref}(\mathcal{U}) = M_{\max}(F)$. We write $U_0 = M_{\min}(F)$. Given a reflexive masa bimodule $\mathcal{V}$, there exists a set $k \subseteq G \times G$ which is marginally equal to $(\cup_{n \in \mathbb{N}} \alpha_n \times \beta_n)\complement$, where $\alpha_n, \beta_n$ are Borel subsets of $G$ such that $\mathcal{V} = M_{\max}(k)$. An operator $T$ belongs to $\mathcal{V}$ if and only if $P(\beta_n)TP(\alpha_n) = 0$, $\forall \, n$. A set $k$ that is marginally equal to a complement of a countable union of Borel rectangles is called an $\omega$-closed set.

An $\omega$-closed set $k$ is called operator synthetic if $M_{\max}(k) = M_{\min}(k)$.

If $s \in G$, we denote by $\lambda_s$ the operator given by
\[\lambda_s(f)(t) = f(s^{-1}t), \quad \forall \, f \in L^2(G).\]
The homomorphism $G \to B(L^2(G)) : s \to \lambda_s$ is called the left regular representation. We denote by $A(G)$ the set of maps $u : G \to \mathbb{C}$ given by $u(s) = (\lambda_s(\xi), \eta)$ for $\xi, \eta \in L^2(G)$. For any $u \in A(G)$, we write
\[\|u\|_{A(G)} = \inf\{\|\xi\|_2, \|\eta\|_2 : u(s) = (\lambda_s(\xi), \eta) \forall \, s\}.\]
The set $A(G)$ is an algebra under the usual multiplication, and $\| \cdot \|_{A(G)}$ is a norm making $A(G)$ a commutative regular semisimple Banach algebra. We call this algebra a Fourier algebra. We denote by $VN(G)$ the following von Neumann subalgebra of $B(L^2(G))$:
\[VN(G) = |\lambda_s : s \in G|^{-w^*}.\]
This algebra is the dual of the Fourier algebra $A(G)$. The duality is given by $(\lambda_s, u)_\alpha = u(s)$ for all $u \in A(G)$ and $s \in G$.

If $E \subseteq G$ is a closed set, we write
\[I(E) = \{u \in A(G) : u|_E = 0\},\]
\[J_0(E) = \{u \in A(G) : \exists \, \Omega \text{ open set, } E \subseteq \Omega, \, u|_\Omega = 0\},\]
and $J(E)$ for the closure of $J_0(E)$ in $A(G)$. The spaces $I(E)$ and $J(E)$ are closed ideals of $A(G)$ and $J(E) \subseteq I(E)$. The set $E$ is called a set of spectral synthesis if $J(E) = I(E)$. Let $I^c(E)$ be the set of all compactly supported functions $f \in I(E)$. We say that $E$ is a set of local spectral synthesis if $I^c(E) \subseteq J(E)$.

If $u : G \to L$ is an arbitrary map, we write $N(u)$ for the map
\[N(u) : G \times G \to L, \quad (s, t) \to u(ts^{-1}).\]
If \( u \in A(G) \), the map \( N(u) \) can be written as
\[
N(u) = \sum_{i \in \mathbb{N}} \phi_i \otimes \psi_i,
\]
where \( \phi_i, \psi_i : G \to \mathbb{C} \) are Borel maps satisfying
\[
\left\langle \sum_{i \in \mathbb{N}} |\phi_i|^2 \right\rangle_{\infty} < +\infty, \quad \left\langle \sum_{i \in \mathbb{N}} |\psi_i|^2 \right\rangle_{\infty} < +\infty.
\]
The map \( N(u) \) satisfies \( N(u)T(G) \subseteq T(G) \). See in [15] for more details.

If \( I \) is a closed ideal of \( A(G) \), \( I^\perp \) is its annihilator in \( VN(G) \):
\[
I^\perp = \{ T \in VN(G) : (T, u)_\alpha = 0, \ \forall u \in I \}.
\]
We also write
\[
\text{Sat}(I) = [N(I)T(G)]^{-\|\cdot\|} \subseteq T(G).
\]
If \( \mathcal{X} \) is a subspace of \( VN(G) \), we write \( \text{Bim}(\mathcal{X}) \) for the following subspace of \( B(L^2(G)) \):
\[
\text{Bim}(\mathcal{X}) = [M_\phi XM_\psi : X \in \mathcal{X}, \ \phi, \psi \in L^\infty(G)]^{-w^*}.
\]

2. Synthetic and transference properties of group homomorphisms

In this section, we assume that \( G \) and \( H \) are locally compact second countable groups with Haar measures \( \mu \) and \( \nu \), respectively, \( \theta : G \to H \) is a continuous homomorphism and \( \theta_\ast(\mu) << \nu \). We conclude that the map
\[
\hat{\theta} : L^\infty(H) \to L^\infty(G), \quad \hat{\theta}(f) = f \circ \theta
\]
is a weak*-continuous homomorphism. If \( \alpha \subseteq H \), (respectively, \( \beta \subseteq G \)) is a Borel set, we denote by \( P(\alpha) \), (respectively, \( Q(\beta) \)) the projection onto \( L^2(\alpha, \nu) \) (respectively, \( L^2(\beta, \mu) \)). If \( \phi \in L^\infty(H) \) (respectively, \( \psi \in L^\infty(G) \)), we denote by \( M_\phi \) (respectively, \( M_\psi \)) the operator \( L^2(H) \to L^2(H) : f \to f\phi \) (respectively, \( L^2(G) \to L^2(G) : g \to g\psi \)). We define the following ternary ring of operators (TRO):
\[
\mathcal{N} = \{ X : XP(\alpha) = Q(\theta^{-1}(\alpha))X, \ \text{for} \ \alpha \subseteq H, \ \text{a Borel set} \}.
\]
(For the definition and properties of TROs, see [3]). Observe that for every \( \phi \in L^\infty(H) \) and \( X \in \mathcal{N} \), we have \( XM_\phi = M_{\phi \circ \theta}X \). Suppose that \( Ker(\hat{\theta}) = L^\infty(\alpha_0^\circ) \), for some Borel set \( \alpha_0^\circ \subseteq H \). Then the map
\[
L^\infty(\alpha_0) \to L^\infty(G), \quad \hat{\theta}(f|_{\alpha_0}) = f \circ \theta, \quad f \in L^\infty(H)
\]
is a one-to-one \( * \)-homomorphism. We now define the following TRO:
\[
\mathcal{M} = \{ X : XP(\alpha) = Q(\theta^{-1}(\alpha))X, \ \alpha \subseteq \alpha_0, \ \text{Borel} \} \subseteq B(L^2(\alpha_0), L^2(G)).
\]
If \( R \) is the projection onto \( L^2(\alpha_0) \), we can easily see that \( \mathcal{N} = \mathcal{M}R \).

Let \( A \subseteq B(L^2(G)) \) be the commutant of the algebra \( \{ M_{\phi \circ \theta} : \phi \in L^\infty(H) \} \). By [5, Theorem 3.2],
\[
[M^*M]^{-w^*} = L^\infty(\alpha_0), \quad [MM^*]^{-w^*} = A.
\]
For every masa bimodule \( U \subseteq B(L^2(H)) \), we define
\[
\mathcal{G}(U) = [NU^\ast]^{-w^*}
\]
and for every masa bimodule \( U \subseteq B(L^2(\alpha_0)) \), we define
\[
\mathcal{F}(U) = [MU^\ast]^{-w^*}.
\]
By [5, Proposition 2.11], the map $\mathcal{F}$ is a bijection from the masa bimodules acting on $L^2(\alpha_0)$ onto the $\mathcal{A}$-bimodules acting on $L^2(G)$. The inverse of $\mathcal{F}$ is given by

$$
\mathcal{F}^{-1}(\mathcal{V}) = [\mathcal{M^*\mathcal{V}\mathcal{M}]}^{-w^*}.
$$

We can easily see that

$$
\mathcal{G}(\mathcal{U}) = \mathcal{F}(\mathcal{R}\mathcal{U}\mathcal{R}).
$$

**Remark 2.1.** We inform the reader of the following.

(i) The spaces $\mathcal{U}$ and $\mathcal{F}(\mathcal{U})$ are stably isomorphic in the sense that there exists a Hilbert space $\mathcal{H}$ such that the spaces $\mathcal{U} \otimes B(\mathcal{H})$ and $\mathcal{F}(\mathcal{U}) \otimes B(\mathcal{H})$ are isomorphic as dual operator spaces, where $\otimes$ is the normal spatial tensor product, [7].

(ii) The spaces $\mathcal{U}, \mathcal{F}(\mathcal{U})$ are spatially Morita equivalent in the sense of [6].

**Lemma 2.2.** Suppose $k \subseteq \alpha_0 \times \alpha_0$ is an $\omega$-closed set. Suppose further that $\mathcal{U} = M_{\text{max}}(k)$ and $\mathcal{V} = M_{\text{max}}(\sigma)$, where $\sigma = (\theta^{-1}(\alpha_n))^{-1}(k)$. Then $\mathcal{F}(\mathcal{U}) = \mathcal{V}$.

**Proof.** Suppose that $\alpha_n \subseteq \alpha_0$ and $\beta_n \subseteq \alpha_0, n \in \mathbb{N}$ are Borel sets such that $k = (\bigcup_n (\alpha_n \times \beta_n))^c$. Then $\sigma = (\bigcup_n (\theta^{-1}(\alpha_n) \times \theta^{-1}(\beta_n)))^c$. If $Z \in \mathcal{U}, X, Y \in \mathcal{M}$, then

$$
Q(\theta^{-1}(\beta_n))XZY^*Q(\theta^{-1}(\alpha_n)) = XP(\beta_n)ZP(\alpha_n)Y^* = X0Y^* = 0, \; \forall n.
$$

Therefore $\mathcal{M^*\mathcal{U}\mathcal{M}^*} \subseteq \mathcal{V}$. Similarly, we can prove $\mathcal{M^*\mathcal{V}\mathcal{M}} \subseteq \mathcal{U}$. The above relations imply

$$
\mathcal{M^*\mathcal{V}\mathcal{M}^*} \subseteq \mathcal{M^*\mathcal{U}\mathcal{M}^*} \subseteq \mathcal{V}.
$$

Since $[\mathcal{M^*\mathcal{U}\mathcal{M}^*}]^{-w^*}$ is an unital algebra,

$$
\mathcal{V} = [\mathcal{M^*\mathcal{U}\mathcal{M}^*}]^{-w^*} = \mathcal{F}(\mathcal{U}).
$$

**Lemma 2.3.** Let $\mathcal{U}$, be as in Lemma 2.2. Then $\mathcal{F}(\mathcal{U}_{\text{min}}) = \mathcal{F}(\mathcal{U})_{\text{min}}$.

**Proof.** If $\mathcal{W} = M_{\text{max}}(\Omega)$ is a reflexive masa bimodule, we write $\mathcal{W}_{\text{min}} = M_{\text{min}}(\Omega)$. From Lemma 2.2, $\mathcal{F}(\mathcal{U}) = M_{\text{max}}(\sigma)$. Therefore $\mathcal{F}(\mathcal{U})_{\text{min}} = M_{\text{min}}(\sigma)$. Since $\mathcal{A}$ contains the masa $L^\infty(G)$ and $\mathcal{F}(\mathcal{U})$ is a reflexive $\mathcal{A}$–bimodule the space

$$
\Pi = \begin{pmatrix} \mathcal{A} & \mathcal{F}(\mathcal{U}) \\ 0 & \mathcal{A} \end{pmatrix}
$$

is a CSL algebra. From the proof of [5, Proposition 4.7], we have

$$
\begin{pmatrix} 0 & \mathcal{F}(\mathcal{U})_{\text{min}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{F}(\mathcal{U}) \\ 0 & 0 \end{pmatrix}_{\text{min}} \subseteq \Pi_{\text{min}}.
$$

Also the diagonal of $\Pi$, $(\mathcal{A} \ 0 \ 0 \ \mathcal{A})$, belongs to $\Pi_{\text{min}}$. Thus,

$$
\begin{pmatrix} \mathcal{A} & \mathcal{F}(\mathcal{U})_{\text{min}} \\ 0 & \mathcal{A} \end{pmatrix} \subseteq \Pi_{\text{min}}.
$$

But

$$
\text{Ref} \left( \begin{pmatrix} \mathcal{A} & \mathcal{F}(\mathcal{U})_{\text{min}} \\ 0 & \mathcal{A} \end{pmatrix} \right) = \Pi.
$$
Therefore,
\[ \Pi_{\text{min}} \subseteq \begin{pmatrix} A & F(U)_{\text{min}} \\ 0 & A \end{pmatrix}. \]
We conclude that
\[ \Pi_{\text{min}} = \begin{pmatrix} A & F(U)_{\text{min}} \\ 0 & A \end{pmatrix}. \]
Since \( \Pi \) is an algebra by [4, Theorem 22.19], \( \Pi_{\text{min}} \) is also an algebra, which implies \( AF(U)_{\text{min}}A \subseteq F(U)_{\text{min}}. \)

Observe that
\[ MU_{\text{min}}M^* \subseteq MM^* \subseteq F(U). \]
Since by Lemma 2.2 \( F(U) \) is a reflexive space,
\[ \text{Ref}(MU_{\text{min}}M^*) \subseteq F(U). \]
Let \( Z \in F(U) \) and assume that \( Z \) does not belong to \( \text{Ref}(MU_{\text{min}}M^*) \). Thus, there exists a \( \xi \in L^2(G) \) such that \( Z\xi \) does not belong to \( \text{Ref}(MU_{\text{min}}M^* \xi) \). Thus, there exists an \( \omega \in L^2(G) \) such that \( (XSY^*\xi, \omega) = 0, \forall X, Y \in M, S \in U_{\text{min}} \) and \( (Z\xi, \omega) \neq 0 \). We have \( (SY^*\xi, X^*\omega) = 0, \forall X, Y \in M, S \in U_{\text{min}} \).

Since \( F(U) = [MM^*]^{-w*} \),
\[ (T\xi, \omega) = 0, \forall T \in F(U). \]
Therefore \( (Z\xi, \omega) = 0. \) This contradiction shows that
\[ F(U) \subseteq \text{Ref}(MU_{\text{min}}M^*) \Rightarrow F(U) = \text{Ref}(MU_{\text{min}}M^*). \]
Since \( [MU_{\text{min}}M^*]^{-w*} \) is a masa bimodule, \( F(U)_{\text{min}} \subseteq [MU_{\text{min}}M^*]^{-w*}. \) By symmetry, we have \( U_{\text{min}} \subseteq [M^*F(U)_{\text{min}}M]^{-w*}. \) Thus,
\[ F(U)_{\text{min}} \subseteq [MU_{\text{min}}M^*]^{-w*} \subseteq [MM^*F(U)_{\text{min}}M]^{-w*} \subseteq [A^*F(U)_{\text{min}}A]^{-w*} \subseteq F(U)_{\text{min}}. \]

Therefore \( F(U)_{\text{min}} = F(U)_{\text{min}}. \)

**Theorem 2.4.** Let \( k \subseteq H \times H \) be an \( \omega \)-closed set. Then
(i) \( G(M_{\text{max}}(k)) = M_{\text{max}}((\theta \times \theta)^{-1}(k)); \)
(ii) \( G(M_{\text{min}}(k)) = M_{\text{min}}((\theta \times \theta)^{-1}(k)). \)

**Proof.** (i) By Lemma 2.2,
\[ F(M_{\text{max}}(k \cap (\alpha_0 \times \alpha_0))) = M_{\text{max}}((\theta \times \theta)^{-1}(k \cap (\alpha_0 \times \alpha_0))). \]
We can easily see that if \( k_1, k_2 \) are \( \omega \)-closed sets, then
\[ M_{\text{max}}(k_1 \cap k_2) = M_{\text{max}}(k_1) \cap M_{\text{max}}(k_2), \]
thus
\[ M_{\text{max}}((\theta \times \theta)^{-1}(k \cap (\alpha_0 \times \alpha_0))) = M_{\text{max}}((\theta \times \theta)^{-1}(k)) \cap M_{\text{max}}((\theta \times \theta)^{-1}(\alpha_0 \times \alpha_0)). \]
Since \( \theta^{-1}(\alpha_0) = G \) up to measure zero, the sets \( G \times G, (\theta \times \theta)^{-1}(\alpha_0 \times \alpha_0) \) are marginally equal, thus \( M_{\text{max}}((\theta \times \theta)^{-1}(\alpha_0 \times \alpha_0)) = B(L^2(G)). \) Therefore
\[ G(M_{\text{max}}(k)) = F(M_{\text{max}}(k \cap (\alpha_0 \times \alpha_0))) = M_{\text{max}}((\theta \times \theta)^{-1}(k)). \]
(ii) By Lemma 2.3,
\[ \mathcal{F}(M_{\min}(k \cap (\alpha_0 \times \alpha_0))) = M_{\min}((\theta \times \theta)^{-1}(k \cap (\alpha_0 \times \alpha_0))) = M_{\min}((\theta \times \theta)^{-1}(k \cap (\theta \times \theta)^{-1}(\alpha_0 \times \alpha_0))). \]

Since the sets \( G \times G, (\theta \times \theta)^{-1}(\alpha_0 \times \alpha_0) \) are marginally equal, we conclude that
\[ \mathcal{G}(M_{\min}(k)) = \mathcal{F}(M_{\min}(k \cap (\alpha_0 \times \alpha_0))) = M_{\min}((\theta \times \theta)^{-1}(k)). \]

\[ \square \]

**Corollary 2.5.** If \( \mathcal{U} = M_{\max}(k) \) is a synthetic masa bimodule acting on \( L^2(H) \), then \( \mathcal{G}(\mathcal{U}) = M_{\max}((\theta \times \theta)^{-1}(k)) \) is also synthetic.

**Remark 2.6.** The implication of the previous corollary was first proved in [14, Theorem 4.7]. In the present paper, we have given a different proof.

**Corollary 2.7.** Let \( E \subseteq H \) be a closed set. Then

(i) \( \mathcal{G}(M_{\max}(E^*)) = M_{\max}(\theta^{-1}(E^*)) \) and \( \mathcal{G}(M_{\min}(E^*)) = M_{\min}(\theta^{-1}(E^*)) \);

(ii) if \( E \) is a set of local synthesis, then \( \theta^{-1}(E) \) is a set of local synthesis;

(iii) if \( E \) is a set of local synthesis and \( A(G) \) possess an approximate identity, then \( \theta^{-1}(E) \) is a set of spectral synthesis.

**Proof.** (i) By Theorem 2.4,
\[ \mathcal{G}(M_{\max}(E^*)) = M_{\max}((\theta \times \theta)^{-1}(E^*)) = M_{\max}(\theta^{-1}(E^*)). \]

Similarly,
\[ \mathcal{G}(M_{\min}(E^*)) = M_{\min}((\theta \times \theta)^{-1}(E^*)) = M_{\min}(\theta^{-1}(E^*)). \]

(ii) If \( E \) is a set of local synthesis, then \( M_{\max}(E^*) \) is a masa bimodule of operator synthesis. By Corollary 2.5 and (i), \( M_{\max}(\theta^{-1}(E^*)) \) is also a masa bimodule of operator synthesis. Thus, by [11], \( \theta^{-1}(E) \) is a set of local synthesis.

(iii) If \( A(G) \) possess an approximate identity, then \( \theta^{-1}(E) \) is a set of local synthesis if and only if \( \theta^{-1}(E) \) is a set of spectral synthesis. Now use (ii).

\[ \square \]

**Theorem 2.8.** Let \( \mathcal{U} \subseteq B(L^2(H)) \) be a masa bimodule. If \( \mathcal{G}(\mathcal{U}) \) contains a non-zero compact operator, then so does \( \mathcal{U} \). The same holds replacing compact by finite rank or by rank one operator.

**Proof.** We have \( \mathcal{G}(\mathcal{U}) = \mathcal{F}(RUR) \) and
\[ RUR = \mathcal{F}^{-1}(\mathcal{G}(\mathcal{U})) = [\mathcal{M}^* \mathcal{G}(\mathcal{U}) \mathcal{M}]^{-w^*}. \]

If \( K \in \mathcal{G}(\mathcal{U}) \) is a non-zero compact operator, then
\[ \mathcal{M}^* K \mathcal{M} \subseteq RUR \subseteq \mathcal{U}. \]

It suffices to prove that \( \mathcal{M}^* K \mathcal{M} \neq 0 \).

Suppose \( \mathcal{M}^* K \mathcal{M} = 0 \). Then \( [\mathcal{M} \mathcal{M}^*]^{-w^*} K [\mathcal{M} \mathcal{M}^*]^{-w^*} = 0 \). Since \( [\mathcal{M} \mathcal{M}^*]^{-w^*} = \mathcal{A} \) is an unital algebra, \( K = 0 \). This contradiction shows that \( \mathcal{U} \) contains a non-zero compact operator. The remaining cases are proved similarly.

\[ \square \]

**Corollary 2.9.** Let \( k \subseteq H \times H \) be an \( \omega \)-closed set and assume that \( M_{\max}((\theta \times \theta)^{-1}(k)) \) (respectively, \( M_{\min}((\theta \times \theta)^{-1}(k)) \)) contains a non-zero compact operator, then \( M_{\max}(k) \),
(respectively, $M_{\min}(k)$), also contains a non-zero compact operator. The same holds replacing compact by finite rank or by rank one operator.

Remark 2.10. The implication that if $G(M_{\max}(k))$ contains a non-zero compact operator, then $M_{\max}(k)$ also contains a non-zero compact operator, was first proved in [13, Corollary 4.8] for some special cases of $\theta$.

Theorem 2.11. Let $I$ be a closed ideal of $A(H)$ and $U = \text{Bim}(I^\perp)$. Then there exists a closed ideal $J$ of $A(G)$ such that $G(U) = \text{Bim}(J^\perp)$.

Proof. Let $\rho^G : G \to B(L^2(G))$, $t \to \rho^G_t$, be the right regular representation of $G$ on $L^2(G)$, that is, the representation 
$$\rho^G_t(f)(x) = \Delta^G(x)^\frac{1}{2} f(xt), \quad t, x \in G, \quad f \in L^2(G),$$
where $\Delta^G$ is the modular function of $G$. Similarly, we define the right regular representation $\rho^H : H \to B(L^2(H))$ of the group $H$. By [1, Theorem 4.3], it suffices to prove
$$\rho^G_t \mathcal{G}(U) \rho^G_{t^{-1}} \subseteq \mathcal{G}(U), \quad \forall \ t \in G.$$
If $P \in L^\infty(H, \nu)$ is a projection, there exists a Borel set $\alpha$ such that $P = P(\alpha) \equiv L^2(\alpha, \nu)$. If $s \in H$, we denote by $P_s$ the projection onto $L^2(\alpha s)$. We can easily see that $P_s^H P_s^H = P_s$. Let $\alpha \subseteq H$ be a Borel set and $t \in G$. Then 
$$\hat{\theta}(P_{\theta(t)}) = \hat{\theta}(P(\alpha \theta(t))) = Q(\theta^{-1}(\alpha \theta(t))) = Q(\theta^{-1}(\alpha) t) = Q(\theta^{-1}(\alpha)) \equiv \hat{\theta}(P(\alpha)) \equiv \hat{\theta}(P),$$
where $Q(\beta) \equiv L^2(\beta, \mu)$.

Thus if $X \in \mathcal{N}, P \in L^\infty(H)$ and $t \in G$,
$$XP_{\theta(t)} = \hat{\theta}(P)_t X.$$ 
Therefore,
$$XP_{\theta(t)} P_{\theta(t)^{-1}} = XP_{\theta(t)^{-1}} = \hat{\theta}(P)_{t^{-1}} X = \rho^G_t \hat{\theta}(P) \rho^G_{t^{-1}} X.$$ 
Also,
$$\rho^G_{t^{-1}} XP_{\theta(t)} P_{\theta(t)^{-1}} \rho^G_t = \hat{\theta}(P) \rho^G_{t^{-1}} X P^H_{\theta(t)},$$
for all $t \in G$ and $P \in L^\infty(H)$. We conclude that
$$\rho^G_{t^{-1}} N P^H_{\theta(t)} \rho^G_t \subseteq \mathcal{N}.$$ 
Now take $X,Y \in \mathcal{N}, t \in G$ and $Z \in \mathcal{U}$. There exist $X_1,Y_1 \in \mathcal{N}$ such that
$$\rho^G_t X = X_1 P^H_{\theta(t)} \text{ and } \rho^G_t Y = Y_1 P^H_{\theta(t)}.$$ 
Therefore
$$\rho^G_t XZY^* \rho^G_{t^{-1}} = X_1 P^H_{\theta(t)} Z P^H_{\theta(t)^{-1}} Y_1^*.$$ 
By [1, Theorem 4.3], $P^H_{\theta(t)} Z P^H_{\theta(t)^{-1}} \in \mathcal{U}$. Thus $\rho^G_t XZY^* \rho^G_{t^{-1}} \in \mathcal{N} \mathcal{U} \mathcal{N}^*$. We have proven
$$\rho^G_t \mathcal{G}(U) \rho^G_{t^{-1}} \subseteq \mathcal{G}(U),$$
which implies
$$\rho^G_t \mathcal{G}(U) \rho^G_{t^{-1}} \subseteq \mathcal{G}(U).$$
Remark 2.12. If \( u \in A(H) \), we denote by \( \rho(u) \) the function \( u \circ \theta \). There exist cases of \( G, H, \theta \) such that \( \rho(A(H)) \cap A(G) = \{0\} \). For example, if \( G \) is a non-compact group, \( \theta: G \to H \) is the trivial homomorphism and \( u(e_H) \neq 0 \), then \( \rho(u) \) is a non-zero constant map and therefore does not belong to \( A(G) \). Therefore in case \( \rho(A(H)) \cap A(G) = \{0\} \) if \( I \) is a closed ideal of \( A(H) \), then \( \rho(I) \) is not contained in \( A(G) \). Nevertheless, by Theorem 2.11, if \( U = \text{Bim}(I^\perp) \), there is a closed ideal \( J \subseteq A(G) \) such that \( \mathcal{G}(U) = \text{Bim}(J^\perp) \). We are going to prove that \( \text{Sat}(J) = [N(\rho(I))T(G)]^{-\|\| \cdot} \).

In the sequel, we fix a closed ideal \( I \subseteq A(H) \), and write \( U = \text{Bim}(I^\perp) \) and \( \Xi = [N(\rho(I))T(G)]^{-\|\| \cdot} \). Let \( J \subseteq A(G) \) be a closed ideal such that \( \mathcal{G}(U) = \text{Bim}(J^\perp) \).

Lemma 2.13. The space \( \Xi^\perp \) is a \( \mathcal{A} \)-bimodule.

Proof. Let \( V_1, V_2 \in \mathcal{N}, X \in \Xi^\perp, u \in I \) and \( f, g \in L^2(G) \). If \( N(u) = \sum_i \phi_i \otimes \psi_i \), we have
\[
(V_1 V_2^* X V_3 V_4^*, N(u \circ \theta)(f \otimes g))_t
\]
\[
= \sum_i (V_1 V_2^* X V_3 V_4^*, ((\phi_i \circ \theta)f) \otimes ((\psi_i \circ \theta)g))_t
\]
\[
= \sum_i (V_2 V_4^*(X V_3, V_4^*(\phi_i \circ \theta(f)) \otimes V_1^*(\psi_i \circ \theta(g)))_t
\]
\[
= \sum_i (V_2^* X V_3 V_4^*(\phi_{\psi_i}(f)) \otimes V_1^*(\psi_{\phi_i}(g)))_t = \sum_i (V_2^* X V_3 M_{\psi_{\phi_i}} V_1^*(g))_t
\]
Since \( N(u \circ \theta)(V_1 V_2^* X V_3 V_4^*, N(u \circ \theta)(f \otimes g))_t = 0. \) Thus \( V_1 V_2^* X V_3 V_4^* \in \Xi^\perp \). The algebra \( \mathcal{A} \) is equal to \( [\mathcal{N}, \mathcal{N}^*]^{-\|\| \cdot} \), therefore \( \mathcal{A} \Xi^\perp \mathcal{A} \subseteq \Xi^\perp \).

Theorem 2.14. The spaces \( \Xi \) and \( \text{Sat}(J) \) are equal.

Proof. First we are going to prove that
\[
\mathcal{N}\text{Bim}(I^\perp) \mathcal{N}^* \subseteq \Xi^\perp.
\]
Let \( V_1, V_2 \in \mathcal{N}, X \in \Xi^\perp, u \in I \) and \( f, g \in L^2(G) \). If \( N(u) = \sum_i \phi_i \otimes \psi_i \), we have
\[
(V_2 X V_1^*, N(u \circ \theta)(f \otimes g))_t = \sum_i (V_2 X V_1^*, ((\phi_i \circ \theta)f) \otimes ((\psi_i \circ \theta)g))_t
\]
\[
= \sum_i (X, V_1^* M_{\phi_i \circ \theta} V_2^*(f) \otimes M_{\psi_i \circ \theta} V_2^*(g))_t = \sum_i (X, M_{\phi_i \circ \theta} V_1^*(f) \otimes M_{\psi_i \circ \theta} V_1^*(g))_t
\]
Thus
\[
\mathcal{N}\text{Bim}(I^\perp) \mathcal{N}^* \subseteq \Xi^\perp = \text{Bim}(J^\perp) \subseteq \Xi^\perp \Rightarrow \Xi \subseteq \text{Sat}(J).\]
If $X \in \Xi^+$ and $V_1, V_2, V_3, V_4 \in \mathcal{N}$, then Lemma 2.13 implies

$$V_1 V_2^* X V_3 V_4^* \in \Xi^+.$$  

Thus for all $u \in I, f, g \in L^2(G)$, we have

$$0 = (V_1 V_2^* X V_3 V_4^*, N(u \circ \theta)(f \otimes g))_t = (V_2^* X V_3, N(u)(V_1^*(f) \otimes V_4^*(g)))_t.$$  

Since

$$R(L^2(H)) = [\mathcal{M}^c(L^2(G))],$$

we conclude that

$$0 = (V_2^* X V_3, N(u|_{\alpha_0 \times \alpha_0}(f \otimes g)), \forall f, g \in L^2(\alpha_0), u \in I.$$  

Since

$$R\text{Bim}(I^\perp) R = [N(u|_{\alpha_0 \times \alpha_0}(f \otimes g) : u \in I, \ f, g \in L^2(\alpha_0)]^\perp,$$

we have that $V_2^* X V_3 \in R\text{Bim}(I^\perp) R$. Therefore

$$\mathcal{N}^\perp \mathcal{N}^* \subset R\text{Bim}(I^\perp) R,$$

which implies

$$\mathcal{N}^\perp \mathcal{N}^* \subset \mathcal{F}(R\text{Bim}(I^\perp) R) = \mathcal{G}(\text{Bim}(I^\perp)) = \text{Bim}(J^\perp).$$

The space $\mathcal{A} = [\mathcal{N}^\perp]^\perp$ is an unital algebra, thus

$$\Xi^\perp \subset \text{Bim}(J^\perp) \Rightarrow \text{Bim}(J^\perp)^\perp \subset \Xi \Rightarrow \text{Sat}(J) \subset \Xi.$$

Since we have already shown $\Xi \subset \text{Sat}(J)$, we obtain the required equality. \hfill \Box

For the following theorem, we recall from [17] that a closed ideal $J \subset A(G)$ is an ideal of multiplicity if $J^\perp \cap C^*_r(G) \neq \{0\}$, where $C^*_r(G)$ is the reduced $C^*$-algebra of $G$.

**Theorem 2.15.** Let $I$ be a closed ideal of $A(H)$. By Theorems 2.11 and 2.14, there exists a closed ideal $J \subset A(G)$ such that

$$\mathcal{G}(\text{Bim}(I^\perp)) = \text{Bim}(J^\perp) \text{ and Sat}(J) = [N(I)(T(\mathcal{G}))]^\perp.$$  

If $J$ is an ideal of multiplicity, then $I$ is also an ideal of multiplicity.

**Proof.** By [17, Corollary 1.5], if $J$ is an ideal of multiplicity, then $\text{Bim}(J^\perp)$ contains a non-zero compact operator. By Theorem 2.8, $\text{Bim}(I^\perp)$ contains a non-zero compact operator. Thus, again by [17, Corollary 1.5], $I$ is an ideal of multiplicity. \hfill \Box

A closed set $E \subset H$ is called an $M$-set (respectively, an $M_1$-set) if the ideal $J_H(E)$ (respectively, $J_H(E)$) is an ideal of multiplicity. Corollaries 2.7(i) and 2.9 together with [17, Corollary 3.6] imply the following:

**Corollary 2.16.** If $E \subset H$ is a closed set such that $\theta^{-1}(E)$ is an $M$-set (respectively, an $M_1$-set), then $E$ is an $M$-set (respectively, an $M_1$-set).

**Remark 2.17.** The previous corollary was proven in [13] for some special cases of $\theta$.

3. *The case when $\nu_\ast(\mu)$ is a Haar measure for $\theta(G)$*

Let $G$ and $H$ be locally compact, second countable groups with Haar measures $\mu$ and $\nu$, respectively. Suppose that $\theta : G \to H$ is a continuous homomorphism, and assume
\[ m = \theta_\ast(\mu) \ll \nu. \] Since \( G \) is a \( \sigma \)-compact set and \( \theta \) is a continuous map, then \( \theta(G) \) is also a \( \sigma \)-compact set and hence a Borel set. Also \( \theta_\ast(\mu) \ll \nu \) implies \( \nu(\theta(G)) > 0 \). By Steinhaus theorem, the group \( \theta(G) \) contains an open set. We conclude that \( \theta(G) \) is an open set. We note that the open subgroups of a locally compact group are closed. Using these facts we can easily see that \( \nu|_{H_0} \) is a Haar measure of \( H_0 = \theta(G) \). In some cases, \( m \ll \nu \) implies that \( m \) is a Haar measure for \( H_0 \). Thus there exists \( c > 0 \) such that \( m|_{H_0} = cv|_{H_0} \). In this section, we investigate this equality. We can replace the Haar measure \( \nu \) with \( cv \) and thus we may assume that \( m(\alpha) = \nu(\alpha) \) for all Borel sets \( \alpha \subseteq H_0 \).

For every \( u \in A(H) \), we define \( \rho(u) = u \circ \theta \). We are going to prove that \( \rho : A(H) \to A(G) \) is a continuous homomorphism and that if \( \mathcal{I} \) is a closed ideal of \( A(H) \) and \( \mathcal{U} = \text{Bim}(\mathcal{I}^\perp) \), then \( \mathcal{G}(\mathcal{U}) = \text{Bim}(\rho_\ast(\mathcal{I}^\perp)) \), where \( \rho_\ast(\mathcal{I}) \) is the closed ideal of \( A(G) \) generated by \( \rho(\mathcal{I}) \) and \( \mathcal{G} \) is the map defined in Section 2. For every \( u \in A(H) \), we denote by \( \pi(u) \) the function \( u|_{H_0} \). By [9, Theorem 2.6.4], \( \pi(u) \in A(H_0) \) for all \( u \in A(H) \). Thus if \( u \in A(H) \), there exist \( f, g \in L^2(H_0) \) such that

\[ u(t) = \pi(u)(t) = (\lambda^H_t f, g), \; \forall \; t \in H_0. \]

By [9, Corollary 2.6.5], the map \( \pi : A(H) \to A(H_0) \) is contractive onto homomorphism, thus \( \pi(\mathcal{I}) \) is an ideal of \( A(H_0) \). Also, observe that the map \( A : L^2(H_0) \to L^2(G) \) given by \( A(f) = f \circ \theta \) is an isometry.

**Lemma 3.1.** Let \( u \in A(H) \). Then \( \rho(u) \in A(G) \). Actually, if

\[ u(t) = \pi(u)(t) = (\lambda^H_t f, g), \; \forall \; t \in H_0, \]

then

\[ \rho(u)(s) = (\lambda^G_s Af, Ag), \; \forall \; s \in G. \]

**Proof.** For every \( s \in G \), we have

\[ (\lambda^G_s Af, Ag) = \int_G Af(s^{-1}t)Ag(t)d\mu(t) = \int_G (f \circ \theta)(s^{-1}t)(g \circ \theta)(t)d\mu(t) \]

\[ = \int_G (f_{\theta(s^{-1})} \circ \theta)(t)(g \circ \theta)(t)d\mu(t) = \int_{H_0} f_{\theta(s^{-1})}(t)g(t)dm(t) \]

\[ = \int_{H_0} f_{\theta(s^{-1})}(t)g(t)dm(t) = (\lambda^H_{\theta(s)} f, g) = u(\theta(s)). \]

**Theorem 3.2.** The map \( \rho : A(H) \to A(G) \) is a continuous homomorphism.

**Proof.** If \( u_1, u_2 \in A(H) \), then

\[ \rho(u_1u_2) = (u_1u_2) \circ \theta = u_1 \circ \theta \cdot u_2 \circ \theta = \rho(u_1)\rho(u_2). \]

Let \( u \in A(H) \). We assume

\[ f, g \in L^2(H_0), \; \pi(u)(t) = (\lambda^H_t f, g), \; \forall \; t \in H_0. \]

By Lemma 3.1,

\[ \rho(u)(s) = (\lambda^G_s Af, Ag), \; \forall \; s \in G. \]

Thus

\[ \|\rho(u)\|_{A(G)} \leq \|Af\|_2\|Ag\|_2 = \|f\|_2\|g\|_2 \]
for all $f$ and $g$ such that $\pi(u)(t) = (\lambda^H_t f, g)$. Thus
\[ \|\rho(u)\|_{A(G)} \leq \|\pi(u)\|_{A(H^0)}. \]

Since $\pi$ is a contraction,
\[ \|\rho(u)\|_{A(G)} \leq \|u\|_{A(H)}. \]

\[ \square \]

**Lemma 3.3.** Let $I \subseteq A(H)$ be an ideal and $X \in B(L^2(G))$ such that
\[ (X, N(\rho(u))h)_t = 0, \ \forall \ h \in T(G), \ u \in I. \]
Then
\[ (X, N(v)h)_t = 0, \ \forall \ h \in T(G), \ v \in \rho_*(I). \]

**Proof.** Let
\[ K = \{ v \in \rho_*(I) : (X, N(v)h)_t = 0, \ \forall \ h \in T(G) \}. \]
Clearly $K$ is a closed subset of $A(G)$ and $\rho(I) \subseteq K \subseteq \rho_*(I)$. If $v_1 \in K$ and $v_2 \in A(G)$, we have
\[ (X, N(v_1v_2)h)_t = (X, N(v_1)(N(v_2)(h))_t. \]

Since $N(v_2)(h) \in T(G), \ v_1 \in K$, we have $(X, N(v_1v_2)h)_t = 0$. Thus $v_1v_2 \in K$. Therefore $K$ is an ideal. We conclude that $K = \rho_*(I)$.

In the following theorem, we fix a closed ideal $I \subseteq A(H)$, we assume that $U = \text{Bim}(I^\perp)$ and define $G(U) = [\mathcal{U} \mathcal{U}^*]^{-\|\cdot\|}$, where $\mathcal{U}$ is the TRO defined in Section 2. We are going to prove that $G(U) = \text{Bim}(\rho_*(I)^\perp)$.

**Theorem 3.4.** The space $G(U)$ is equal to $\text{Bim}(\rho_*(I)^\perp)$.

**Proof.** By Theorem 2.14, there exists a closed ideal $J$ of $A(G)$ such that $G(U) = \text{Bim}(J^\perp)$ and $\text{Sat}(J) = [N(u \circ \theta)h : u \in I, \ h \in T(G)]^{-\|\cdot\|}$. Clearly
\[ \text{Sat}(J) \subseteq \text{Sat}(\rho_*(I)) \Rightarrow \text{Bim}(\rho_*(I)^\perp) \subseteq \text{Bim}(J^\perp) = G(U). \]

We need to prove
\[ G(U) \subseteq \text{Bim}(\rho_*(I)^\perp). \]

It suffices to prove
\[ \mathcal{U} \mathcal{U}^* \subseteq \text{Bim}(\rho_*(I)^\perp). \]

Let $X \in U, \ V_1, V_2 \in \mathcal{N}, \ u \in I$. Assume $N(u) = \sum_i \phi_i \otimes \psi_i$. For all $f, g \in L^2(G)$, we have
\[ (V_1XV_2^*, N(u \circ \theta)(f \otimes g))_t = \sum_i (V_1XV_2^*, (\phi_i \circ \theta)f \otimes (\psi_i \circ \theta)g)_t \]
\[ = \sum_i (X, V_2^*((\phi_i \circ \theta)f) \otimes V_1^*(\psi_i \circ \theta)g)_t. \]

We have
\[ V_2^*((\phi_i \circ \theta)f) = V_2^* M_{\phi_i \circ \theta}(f) = M_{\phi_i} V_2^*(f). \]

Similarly,
\[ V_1^*((\psi_i \circ \theta)g) = M_{\psi_i} V_1^*(g). \]
Therefore

\[
(V_1XV_2^*, N(u \circ \theta)(f \otimes g))_t = \sum_i (X, M_{\phi_i}V_2^*(f) \otimes M_{\psi_i}V_1^*(g))_t = (X, N(u)(V_2^*(f) \otimes V_1^*(g)))_t = 0,
\]

because \( X \in \text{Bim}(I^\perp) \) and \( u \in I \). Therefore

\[
(V_1XV_2^*, N(u \circ \theta)(h))_t = 0 \Rightarrow (V_1XV_2^*, N(\mu(u))(h))_t = 0, \quad \forall \, h \in T(G), \, u \in I
\]

By Lemma 3.3, we have

\[
(V_1XV_2^*, N(v)(h))_t = 0, \quad \forall \, h \in T(G), \, v \in \rho_*(I).
\]

Thus \( V_1XV_2^* \in \text{Bim}(\rho_*(I)^\perp) \), which implies

\[
\mathcal{NU}^* \subseteq \text{Bim}(\rho_*(I)^\perp).
\]

**Theorem 3.5.** Let \( I \) be a closed ideal of \( A(H) \). If \( \rho_*(I) \) is an ideal of multiplicity, then \( I \) is also an ideal of multiplicity.

The proof of the above theorem is consequence of Theorems 2.15 and 3.4.

In the last part of this section, we will prove that if \( \theta_*(\mu) \) is a Haar measure for \( \theta(G) \) and \( A(G) \) contains a (possibly unbounded) identity and \( E \subseteq H \) is an ultra strong Ditkin set, then \( \theta^{-1}(E) \) is also an ultra strong Ditkin set.

**Definition 3.1.** Let \( E \subseteq H \) be a closed set. We call \( E \) an ultra strong Ditkin set if there exists a bounded net \( (u_\lambda) \subseteq J_0^H(E) \) such that \( u_\lambda u \to u \), for every \( u \in I_H(E) \).

**Lemma 3.6.** Let \( E \subseteq H \) be a closed set. Then

\[
\rho(J_0^H(E)) \subseteq J_0^G(\theta^{-1}(E)).
\]

**Proof.** If \( u \in J_0^H(E) \), there is an open set \( \Omega \subseteq H \) such that \( E \subseteq \Omega \) and \( u|_{\Omega} = 0 \). We consider the open set \( \theta^{-1}(\Omega) \). Since \( u \circ \theta|_{\theta^{-1}(\Omega)} = 0 \) and \( \theta^{-1}(E) \subseteq \theta^{-1}(\Omega) \), we conclude that \( \rho(u) = u \circ \theta \in J_0^G(\theta^{-1}(E)) \). \( \square \)

**Lemma 3.7.** Let \( E \subseteq H \) be a closed set and suppose \( (u_\lambda) \subseteq J_0^H(E) \) is a bounded net such that \( u_\lambda u \to u \) for every \( u \in I_H(E) \). Then \( \rho(u_\lambda)v \to v \) for every \( v \in \rho_*(I_H(E)) \).

**Proof.** Define the space

\[
I = \left\{ v \in \rho_*(I_H(E)) : v = \lim_{\lambda} \rho(u_\lambda)v \right\}.
\]

If \( u \in I_H(E) \), we have \( \rho(u_\lambda)u \to \rho(u) \). Then \( \rho(I_H(E)) \subseteq I \). If \( v_1 \in I, v_2 \in A(G) \), we have

\[
\rho(u_\lambda)v_1 \to v_1 \Rightarrow \rho(u_\lambda)v_1v_2 \to v_1v_2,
\]

thus \( v_1v_2 \in I \). Therefore \( I \) is an ideal. Since \( (\rho(u_\lambda)) \) is a bounded net, we can easily see that if \( (v_i) \subseteq I \) is a sequence such that \( v_i \to v \), then \( \lim_{\lambda} \rho(u_\lambda)v = v \). Thus \( I \) is a closed ideal, which implies \( I = \rho_*(I_H(E)) \). The proof is complete. \( \square \)

**Theorem 3.8.** Let \( E \subseteq H \) be a closed set and assume that \( A(G) \) has a (possibly unbounded) approximate identity. If \( E \) is an ultra strong Ditkin set, then \( \theta^{-1}(E) \) is an ultra strong Ditkin set.
Proof. Theorem 5.3 of [1] implies
\[ M_{\min}(\theta^{-1}(E)^*) = \text{Bim}(I_G(\theta^{-1}(E))^\perp). \]
If \( E \) is an ultra strong Ditkin set, then by Corollary 2.5 the set \( \theta^{-1}(E)^* \) is operator synthetic. Thus
\[ M_{\max}(\theta^{-1}(E)^*) = M_{\min}(\theta^{-1}(E)^*) = \text{Bim}(I_G(\theta^{-1}(E))^\perp). \]
From Theorem 3.4, we have
\[ M_{\max}(\theta^{-1}(E)^*) = \text{Bim}(\rho_*(I_H(E)^\perp)). \]
Lemma 4.5 in [1] implies
\[ I_G(\theta^{-1}(E)) = \rho_*(I_H(E)). \quad (3.1) \]
Now Lemmas 3.6 and 3.7 together with (3.1) imply that \( \theta^{-1}(E) \) is also an ultra strong Ditkin set.

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