A Characterization Result for Non-Distributive Logics

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Abstract

Recent published work has addressed the Shalqvist correspondence problem for non-distributive logics. The natural question that arises is to identify the fragment of first-order logic that corresponds to logics without distribution, lifting van Benthem’s characterization result for modal logic to this new setting. Carrying out this project is the contribution of the present article.

The article is intended as a demonstration and application of a project of reduction of non-distributive logics to (sorted) residuated modal logics. The reduction is an application of recent representation results by this author for normal lattice expansions and a generalization of a canonical and fully abstract translation of the language of substructural logics into the language of their companion sorted, residuated modal logics. The reduction of non-distributive logics to sorted modal logics makes the proof of a van Benthem characterization of non-distributive logics nearly effortless, by adapting and reusing existing results, demonstrating the usefulness and suitability of this approach in studying logics that may lack distribution.

1 Introduction

Results on the model theory of non-distributive logics are quite recent. They include published results of their Shalqvist theory [6,34], studies of a Goldblatt-Thomason theorem [17] for the logics of normal lattice expansions, as well as modal translation semantics [21, 22] for non-distributive logics.

We consider here the logics of normal lattice expansions, whose relational interpretation is over sorted frames (with sorts 1,∂) \( F = (Z_1, Z_∂, I, \ldots) \), \( I \subseteq Z_1 \times Z_∂ \), and we show that the fragment of first-order formulae in the sorted first-order language of the structures \( F \) equivalent to a translation of a sentence of the language of our propositional logic consists of formulae \( \Phi(u) \) that are stable, meaning that \( \Phi(u) \equiv \forall \partial v \ (I(u,v) \rightarrow \exists^1 z \ (I(z,v) \land \Phi(z))) \), and invariant under sorted bisimulations. The result is a demonstration and application of
a project of reduction of non-distributive logics to (sorted) residuated modal logics. The reduction is an application of recent representation results [19, 23] for normal lattice expansions, by this author, and a generalization of a canonical and fully abstract translation [22] of the language of substructural logics into the language of their companion sorted, residuated modal logics.

2 Logics of Normal Lattice Expansions

By a distribution type we mean an element \( \delta \) of the set \( \{1, \partial\}^{n+1} \), for some \( n \geq 0 \), typically to be written as \( \delta = (i_1, \ldots, i_n; i_{n+1}) \) and where \( \delta(i_{n+1}) = i_{n+1} \in \{1, \partial\} \) will be referred to as the output type of \( \delta \). A similarity type \( \tau \) is then defined as a finite sequence of distribution types, \( \tau = (\delta_1, \ldots, \delta_k) \).

An \( n \)-ary lattice operator \( f : \mathcal{L}^n \rightarrow \mathcal{L} \) is called additive if it distributes over finite joins of \( \mathcal{L} \) in each argument place. If \( \mathcal{L}_1, \ldots, \mathcal{L}_n, \mathcal{L} \) are bounded lattices, then a function \( f : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \rightarrow \mathcal{L} \) is additive, if for each \( i, f \) distributes over finite joins of \( \mathcal{L}_i \).

We write \( \mathcal{L} \) for \( \mathcal{L}^1 \) and \( \mathcal{L}^0 \) for its opposite lattice (where order is reversed, often designated as \( \mathcal{L}^{op} \)).

An \( n \)-ary operator \( f \) on a lattice \( \mathcal{L} \) is normal if it is an additive function \( f : \mathcal{L}^i \times \cdots \times \mathcal{L}^n \rightarrow \mathcal{L}^{i+1} \), where each \( i_j \), for \( j = 1, \ldots, n, n + 1 \), is in the set \( \{1, \partial\} \), i.e. \( \mathcal{L}^i \) is either \( \mathcal{L} \), or \( \mathcal{L}^0 \). For a normal operator \( f \) on \( \mathcal{L} \), its distribution type is the \( (n + 1) \)-tuple \( \delta(f) = (i_1, \ldots, i_n; i_{n+1}) \).

Definition 2.1 (Lattice Expansions). A normal lattice expansion is a structure \( \mathcal{L} = (\mathcal{L}, \wedge, \vee, 0, 1, (f_i)_{i \in k}) \) where \( k > 0 \) is a natural number and for each \( i \in k \), \( f_i \) is a normal operator on \( \mathcal{L} \) of some specified arity \( \alpha(f_i) \in \mathbb{N}^+ \) and distribution type \( \delta(i) \). The similarity type of \( \mathcal{L} \) is the \( k \)-tuple \( \tau(\mathcal{L}) = (\delta(0), \ldots, \delta(k-1)) \).

Example 2.1. A bounded lattice with a box and a diamond operator \( \mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, 0, 1, \Box, \Diamond, \wedge) \) is a normal lattice expansion of similarity type \( \tau \), where \( \tau = ((1, 1), (\partial, \partial)) \) where \( \delta(\wedge) = (1, 1) \), i.e. \( \Box : \mathcal{L} \rightarrow \mathcal{L} \) distributes over joins of \( \mathcal{L} \), while \( \delta(\Box) = (\partial; \partial) \), i.e. \( \Box : \mathcal{L}^0 \rightarrow \mathcal{L}^0 \) distributes over “joins” of \( \mathcal{L}^0 \) (i.e. meets of \( \mathcal{L} \)), delivering “joins” of \( \mathcal{L}^0 \) (i.e. meets of \( \mathcal{L} \)).

Similarly for an implicative lattice, of similarity type \( \tau' = ((1, \partial; \partial)) \) and where \( (1, \partial; \partial) = \delta(\rightarrow) \) is the distribution type of the implication operator, regarded as a map \( \rightarrow : \mathcal{L} \times \mathcal{L}^0 \rightarrow \mathcal{L}^0 \) distributing over “joins” in each argument place, i.e. co-distributing over joins in the first place, turning them to meets, and distributing over meets (joins of \( \mathcal{L}^0 \)) in the second place, delivering “joins” of \( \mathcal{L}^0 \), i.e. meets of \( \mathcal{L} \).

An FL-algebra (Full Lambek algebra [30]) is a normal lattice expansion with similarity type \( \tau'' = ((1, 1; 1), (\partial; \partial), (\partial, 1; \partial)) \). In other words, it is a residuated lattice \( \mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, 0, 1, \rightarrow, \leftarrow, \bowtie) \), with \( \delta(\leftarrow) = (\partial, 1; \partial), \delta(\rightarrow) = (1, 1; 1) \) and \( \delta(\bowtie) = (1, \partial; \partial) \).

Let \( \mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, 0, 1, \bowtie) \) be a lattice expansion, where \( (\mathcal{L}, \leq, \wedge, \vee, 0, 1) \) is a bounded lattice which may not be distributive, each of \( \bowtie \) is normal and
with respective output types 1 and ∂. Let \( \tau = ((i_1, \ldots, i_n;1), (i'_1, \ldots, i'_n;\partial)) \) be the similarity type of \( \mathcal{L} \). The language \( \Lambda_\tau \) of the lattice expansion is displayed below.

\[
\Lambda_\tau \ni \varphi \equiv p_i \ (i \in \mathbb{N}) \mid \top \mid \bot \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \Diamond (\Box) \mid \Box (\Diamond)
\]

The minimal axiomatization of the logic adds to axioms and rules for Positive Lattice Logic (the logic of bounded lattices) the normality axioms (distribution axioms) determined by the distribution types of the operators.

Normal lattice expansions are the algebraic models of non-distributive logics. Residuated (and co-residuated) lattices, in particular, have been extensively investigated, as the algebraic models of substructural logics. Consult [11] for a comprehensive presentation and literature review.

The relational (Kripke) semantics for the logics of normal lattice expansions use frames \( (A,I,B,\ldots) \) where \( A,B \) are sets and \( I \subseteq A \times B \). Sorted relational semantic frameworks for substructural and non-distributive, more generally, logics have been proposed by Suzuki [33,35] and Gehrke and co-workers [5,8,13]. In both cases the semantics is based on sorted representation theorems for lattices [25,27]. Single-sorted approaches have been also studied [7,9,31], based on a different representation [32,36]. More recently, a representation and Stone type duality result for normal lattice expansions was presented by this author [23] (an improvement over the duality of [19] by the same author), extending the lattice duality result for normal lattice expansions was presented by this author [23] (an improvement over the duality of [19] by the same author), extending the lattice representation of [25], with applications to specific cases of interest in [20,20]. Structures \( \mathcal{K} = (A,\perp,B), \perp \subseteq A \times B \) have been introduced (named ‘polarities’) and studied by Birkhoff [2] and subsequently formed the basic structures of Formal Concept Analysis (FCA) [12] where they are called ‘formal contexts’. The dual structure \( \mathcal{K}^\perp \) of a formal context \( \mathcal{K} \) is its ‘formal concept lattice’, a complete lattice of ‘formal concepts’ \((C,D)\) where \( C \in A \) with \( C = \perp(C\perp) \) (a Galois stable set) and \( D = C\perp \) (a Galois co-stable set), hence also \( C = \perp D \), and where \( \perp : \mathcal{P}(A) \cong \mathcal{P}(B) : \perp \) is the Galois connection generated by the relation \( \perp \).

\[
\begin{align*}
C\perp &= \{d \in D \mid \forall c \in C \ c \perp d\} = \{d \in D \mid C \perp d\} \\
\perp D &= \{c \in C \mid \forall d \in D c \perp d\} = \{c \in C \mid c \perp D\}
\end{align*}
\]

We let \( \mathcal{G}(X), \mathcal{G}(Y) \) designate the complete lattices of Galois stable and co-stable sets, respectively.

Every complete lattice \( \mathcal{C} \) can be represented as the formal concept lattice of the context \( (C,\leq,\mathcal{C}) \) and it was further shown in [27] (following the FCA approach and building on Urquhart’s [30]) and in [25] (building on Goldblatt’s representation of ortholattices [14]) that every lattice can be represented as a sublattice of the formal concept lattice of a suitable formal context. This was generalized in [19,23] to the case of normal lattice expansions, using sorted frames with additional relations \( \mathcal{K} = (A,I,B,(R_t)_{t \in T}) \). By the similarity type of a sorted frame \( \mathcal{K} \) we shall mean the tuple \( \langle \sigma(R_k) \rangle_{k \in K} \).

The relational semantics based on [19,23] associates to every distribution type \( \delta = (i_1,\ldots,i_n;i_{n+1}) \) a sorted relation \( R \subseteq Z^{i_{n+1}} \times \prod_{j=1}^{i_{n}} Z^{i_j} \) on the frame \((A,I,B)\), of sorting type \( \sigma(R) = (i_{n+1};i_1\cdots i_n) \), where \( Z^{i_j} \) is \( A \), if \( i_j = 1 \) and it
is $B$ when $i_j = \emptyset$. Hence, to an algebra of similarity type $\tau$, a frame of the same similarity type is associated for the interpretation of the language $\Lambda_\tau$. Set operators are then canonically extracted from the relation (cf. [19] [20] [23] [26] for details). The representation is uniform and all cases reduce to the cases of relations of sorting types $(1;i_1\ldots i_n)$ and $(\emptyset;i'_1\ldots i'_n)$, corresponding to normal lattice operators that take their values in the lattice $\mathcal{L}$, or in its opposite lattice $\mathcal{L}^\emptyset$. Hence we will be only considering sorted structures $(A, I, B, R, S)$ with relations $I \subseteq A \times B$, $R \subseteq A \times \prod_{i=1}^{j_1} Z^{i_j}$ and $S \subseteq B \times \prod_{i=1}^{j_2} Z^{i'_j}$, nothing depending on having more than one relation of each sorting type, or on having relations of different arities. The Galois dual relations $R', S'$ of $R, S$ are defined by setting $R'u_1\ldots u_n = (Ru_1\ldots u_n)^\perp$ and, similarly, $S'v_1\ldots v_n = (Sv_1\ldots v_n)$. By a section of an $(n+1)$-ary relation we mean the set obtained by leaving one argument place unfilled.

**Proposition 2.2.** Let $\alpha_R, \alpha_S$ be the classical (but sorted) image operators generated by the relations $R, S$

\[
\alpha_R(W_1, \ldots, W_n) = \{ u \mid \exists w_1\ldots w_n(uRw_1\ldots w_n \land \bigwedge_j (w_j \in W_j)) \} = \bigcup_{w_j \in W_j} Ru_1\ldots w_n
\]

and similarly for $\alpha_S$ and let $\overline{\alpha}_R, \overline{\alpha}_S$ be the Galois closure of the restriction of $\alpha_R, \alpha_S$ to Galois stable, or co-stable sets, according to the sort type of the relations. If every section of the Galois dual relations $R', S'$ of $R, S$ is a Galois stable (or co-stable, according to the sort type) set, then $\overline{\alpha}_R, \overline{\alpha}_S$ distribute over arbitrary joins in each argument place.

**Proof.** Section stability was first invoked by Gehrke in [13] in modeling the implication-fusion fragment of the Lambek calculus, where operators were generated by relations which are in effect the Galois dual relations $R', S'$ of $R, S$. The argument was generalized by Goldblatt [17] to the case of operators that either distribute in each argument place over arbitrary joins of Galois stable sets, returning a join, or they distribute over arbitrary meets of Galois stable sets, returning a meet. The argument can be generalized to that of operators of an arbitrary distribution type. The proof was given in [24], to which we refer the reader for details.

The operator $\overline{\alpha}_R$ (similarly for $\overline{\alpha}_S$) is sorted and its sorting is inherited from the sort type of $R$. For example, if $\sigma(R) = (\emptyset;11)$, $\alpha_R : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$, hence $\overline{\alpha}_R : \mathcal{G}(X) \times \mathcal{G}(X) \rightarrow \mathcal{G}(Y)$. Single sorted operations

\[
\overline{\alpha}^1_R : \mathcal{G}(X) \times \mathcal{G}(X) \rightarrow \mathcal{G}(X) \text{ and } \overline{\alpha}^2_R : \mathcal{G}(Y) \times \mathcal{G}(Y) \rightarrow \mathcal{G}(Y)
\]

can be then extracted by composing appropriately with the Galois connection: $\overline{\alpha}^1_R(F, C) = (\overline{\alpha}_R(F, C))^\perp$ (where $F, C \in \mathcal{G}(X)$) and, similarly, $\overline{\alpha}^2_R(G, D) = \overline{\alpha}_R(G', D')$ (where $G, D \in \mathcal{G}(Y)$). Similarly for the $n$-ary case and for an arbitrary distribution type.
Remark 2.3. A lattice operator $\phi$ of distribution type $\delta(\phi) = (i_1, \ldots, i_n; i_{n+1})$ is canonically represented in $\mathfrak{M}$ as the operator $\sigma^i_R$, where the relation $R$, of sort type $\sigma(R) = (i_{n+1}; i_1 \cdots i_n)$, is defined classically by the condition

$$uRw_1 \cdots w_n \text{ iff } \forall a_1 \cdots a_n \left( \bigwedge_{j=1}^{i_{n+1}} (a_j \in w_j) \rightarrow \phi(a_1, \ldots, a_n) \in u \right)$$

and it can be shown (cf. [23][24]) that the section stability requirement of Proposition 2.2 holds in the canonical frame construction.

Frames $\mathfrak{F} = (A, I, B, R, S)$ of similarity type $\tau$ are considered for the relational semantics of the language $\Lambda_\tau$ and models $\mathfrak{M} = (\mathfrak{F}, V)$ are equipped with an interpretation that assigns a Galois-stable set $V(p_i)$ to each propositional variable. A co-interpretation $V^\perp$ is also defined, by setting $V^\perp(p_i) = V(p_i)^\perp$. Interpretation and co-interpretation are extended to all sentences and we write $\llbracket \varphi \rrbracket \subseteq A, \llbracket \varphi \rrbracket \subseteq B$, respectively, so that $\llbracket \varphi \rrbracket = \llbracket \varphi \rrbracket^\perp$. Equivalently, we may say that each sentence $\varphi$ is interpreted as a formal concept $(\llbracket \varphi \rrbracket, \llbracket \varphi \rrbracket^\perp)$ in the formal concept lattice of the frame. The satisfaction and co-satisfaction (refutation) relations $\models \in A \times \Lambda, \models^\perp \in B \times \Lambda$ are defined as usual, $a \models \varphi$ iff $a \in \llbracket \varphi \rrbracket$, $b \models^\perp \varphi$ iff $b \in \llbracket \varphi \rrbracket$. Since interpretation and co-interpretation determine each other, it suffices to provide for each logical operator the clause for either $\models$, or $\models^\perp$, as we do in Table 1. The relations $R', S'$ in Table 1 are the Galois dual relations of $R, S$ and we recall that they are defined by setting $R'w_1 \cdots w_n = (Rw_1 \cdots w_n)^\perp$ and, similarly, $S'w_1 \cdots w_n = \perp(Sw_1 \cdots w_n)$.

Table 1: (Co)Satisfaction relations

| $a \models p_i$ | $a \in V(p_i)$ |
| $a \models \top$ | $a = a$ |
| $b \models^\perp \bot$ | $b = b$ |
| $a \models \varphi \land \psi$ | $a \models \varphi$ and $a \models \psi$ |
| $b \models^\perp \varphi \lor \psi$ | $b \models^\perp \varphi$ and $b \models^\perp \psi$ |
| $b \models^\perp \land \varphi_1, \ldots, \varphi_n$ | $\forall v_1, \ldots, v_n \left( \Lambda_{ij}=1(u_j \models \varphi_j) \land \Lambda_{ir}=1(u_r \models^\perp \varphi_r) \rightarrow bR'v_1 \cdots v_n \right)$ |
| $a \models \lor \varphi_1, \ldots, \varphi_n$ | $\forall v_1, \ldots, v_n \left( \Lambda_{ij}=1(v_j \models \varphi_j) \land \Lambda_{ir}=1(v_r \models^\perp \varphi_r) \rightarrow aS'v_1 \cdots v_n \right)$ |

Soundness of the logics of normal lattice expansions is proven in the class of frames where the relations $R, S$ satisfy the section stability requirement of Proposition 2.2. Completeness is shown by applying the representation arguments of [19][13][23], see Remark 2.3.

Example 2.2. If $\oplus = \circ$ (the fusion (cotenability) operator) and $\Rightarrow = \rightarrow$ (implication), of respective distribution types $(1, 1; 1), (1, \partial; \partial)$, the semantic clauses run as follows

5
\[ b \vdash_{\beta} \varphi \circ \psi \text{ iff } \forall a, c \in A \ (c \vdash \varphi \land a \vdash \psi \rightarrow bR'ca) \]
\[ a \vdash \varphi \rightarrow \psi \text{ iff } \forall c \in A \ \forall b \in B \ (c \vdash \varphi \land b \vdash_{\beta} \psi \rightarrow bS'ca) \]

where suitable conditions on \( R, S \) ensure residuation of the operators (cf [20] for details).

For another example, consider the case of modal operators \( \Theta = \Box, \Diamond = \Diamond, \) of respective distribution types \((\partial; \delta)\) and \((1; 1)\). The semantic clauses run as follows, after some logical manipulation of the respective clauses in Table 1, with \( \Phi \) dually interpreted as necessity (cf [18, 26] for details),

\[ b \vdash_{\beta} \Phi \varphi \text{ iff } \forall d \in B \ (bR''d \rightarrow d \vdash_{\beta} \varphi) \]
\[ a \vdash \Box \varphi \text{ iff } \forall c \in A \ (aS''c \rightarrow c \vdash \varphi) \]

where we define \( bR'' = \frac{1}{2}(bR') \) (and recall that \( R' \) is defined from \( R \subseteq A \times A \) by setting \( R'c = (Rc)^{\downarrow} \)) and similarly \( S'' \) is defined from \( S \subseteq B \times B \) by first letting \( S'b = \frac{1}{2}(Sb) \), then defining \( aS'' = \frac{1}{2}(aS') \). □

A translation of the language \( \Lambda_{\tau} \) into a (necessarily sorted) first-order language needs to take into account the fact that propositional variables are interpreted as Galois stable sets \( C = \frac{1}{2}(C_{\downarrow}) \), hence merely introducing a unary predicate \( P \) for each propositional variable \( p_i \) falls short of the goal. The obstacle can be sidestepped by observing that the complement \( I \) of the relation \( \frac{1}{2} \) generates a pair of residuated operators \( \Diamond : \mathcal{P}(A) \rightleftharpoons \mathcal{P}(B) : \Box \) such that the generated (by composition) closure operators coincide with those generated by the Galois connection, i.e. \( \frac{1}{2}(U_{\downarrow}) = \Box \Diamond U \) and \( (\downarrow V)_{\downarrow} = \Box \Diamond V \), for \( U \subseteq A, V \subseteq B \) and where \( \Box = -\Diamond - \) and \( \Diamond = -\Box - \). This leads to considering the sorted modal logic of polarities with relations, a project initiated in [21, 22].

## 3 Sorted Modal Logics of Polarities with Relations

### 3.1 Sorted Residuated Modal Logic

Fix any \( \tau \)-structure (a structure of similarity type \( \tau \)) \( \mathfrak{S} = (A, B, I, R, S) \), where \( I \subseteq A \times B, \) and \( R, S \) are \( (n + 1) \)-ary relations of respective sorting types \( \sigma(R) = (1; i_1, \ldots, i_n) \) and \( \sigma(S) = (\partial; i'_1, \ldots, i'_n) \), i.e. \( R \subseteq A \times \prod_{i=1}^{i_n} Z_{i_j} \) and \( S \subseteq B \times \prod_{j=1}^{j_m} Z'_{i_j} \), where \( Z_1 = A \) and \( Z_2 = B \). The relation \( I \) generates residuated operators \( \Diamond : \mathcal{P}(A) \rightleftharpoons \mathcal{P}(B) : \Box \)

\[ \Diamond U = \{ b \in B \mid \exists a \in A \ (aIb \land a \in U) \} \]
\[ \Box V = \{ a \in A \mid \forall b \in B \ (aIb \rightarrow b \in V) \} \] (1)

and each of \( R, S \) generates a sorted image operator in the sense of [28, 29]

\[ \Diamond (U_1, \ldots, U_n) = \{ a \in A \mid \exists u_1 \ldots u_n \ (aRu_1 \ldots u_n \land \bigwedge_j (u_j \in U_j)) \} \] (2)
\[ \Box (V_1, \ldots, V_n) = \{ b \in B \mid \exists v_1 \ldots v_n \ (bSv_1 \ldots v_n \land \bigwedge_j (v_j \in V_j)) \} \] (3)
where for each \( j = 1, \ldots, n \), \( U_j \subseteq \mathbb{Z}_{i_j} \in \{ A, B \} \) and \( V_j \subseteq \mathbb{Z}_{i'_j} \in \{ A, B \} \).

To the structure (frame) \( \mathfrak{F} = (A, B, I, R, S) \) with sorting types of \( R, S \) as above, we may associate a residuated sorted modal logic with residuated modal operators \( \Diamond, \mathbf{B} \) and sorted polyadic diamonds \( \Phi, \varnothing \) of sorting types determined by the sorting types of the relations. The language \( L = (L_1, L_\partial) \) of sorted, residuated modal logic, given countable, nonempty and disjoint sets of propositional variables, is defined as follows

\[
L_1 \ni \alpha, \zeta, \eta : = P_i (i \in \mathbb{N}) | \neg \alpha | \alpha \rightarrow \alpha | \mathbf{B} \beta | \Phi (\overline{\theta})
\]

\[
L_\partial \ni \beta, \delta, \xi : = Q_i (i \in \mathbb{N}) | \neg \beta | \beta \rightarrow \beta | \Box \alpha | \varnothing (\overline{\theta})
\]

where \( \overline{\theta} = (\theta_1, \ldots, \theta_n) \), the sorting type of \( \Phi \) is \( \sigma(\Phi) = (i_1, \ldots, i_n; 1) \) and if \( \sigma(j) = 1 \), then \( \theta_j \in L_1 \), else \( \theta_j \in L_\partial \). Similarly for \( \varnothing \), of sorting type \( \sigma' = (i'_1, \ldots, i'_n; \partial) \). Nothing of significance for our purposes is obtained by proliferating diamonds (and frame relations) and considering indexed families \( (\Phi_j)_{j \in J}, (\varnothing_k)_{k \in K} \) of each. Note that \( \sigma(\mathbf{B}) = (\partial; 1) \) and \( \sigma(\Box) = (1; \partial) \). The same symbols are used for negation and implication in the two sorts and we rely on context to disambiguate. Diamond operators \( \Diamond = \neg \Box \neg \) and \( \varnothing = \neg \mathbf{B} \neg \) are defined as usual, except that the two occurrences of negation in each definition are of different sort. Sorted box operators \( \mathbf{B}, \Box \) are defined accordingly from \( \Phi, \varnothing \) and negation. Disjunction and conjunction for each sort is defined in the classical way. We let \( \tau, \iota \in L_1 \) and \( \tau, \iota \in L_\partial \) designate the (definable) true and false constants for each sort. The operators \( \iota^{\uparrow} (\cdot), (\cdot)^{\uparrow} \) are defined by \( \iota^{\uparrow} \beta = \mathbf{B} (\neg \beta) \) and \( \alpha^{\uparrow} = \Box (\neg \alpha) \).

**Remark 3.1.** The sorted residuated companion modal logic of the logic of a normal lattice expansion is determined by the similarity type of the expansion. As an example, consider \( \mathbf{FL} \), the associative Lambek calculus, with algebraic semantics in residuated lattices \( \mathcal{L} = (L, \leq, \wedge, \vee, 0, 1, \leftarrow, \circ, \rightarrow) \), of similarity type \( \tau = (1, 1, (1; \partial, \partial), (1, 1; \partial)) \), where \( \delta(\circ) = (1; \partial, \partial) \). Its companion modal logic includes three diamond operators \( \Diamond, \varnothing, \varnothing^* \) of respective sorting type \( (\partial, 1; \partial) \), \( (1, 1; 1) \) and \( (1, 1; \partial) \). An implication operator of the first sort is defined by setting \( \alpha \rightarrow \eta = \iota(\alpha \varnothing^* \eta) \) (reminiscent of the classical definition of implication as \( \varphi \rightarrow \psi = \neg (\varphi \land \neg \psi) \)) and similarly for \( \leftarrow \). The two languages are interpreted over the same class of frames, determined by the similarity type at hand. Frames \( \mathfrak{F} = (A, I, B, L, F, R) \) include ternary relations of respective sorting types \( \sigma(L) = (\partial; \partial 1) \), \( \sigma(F) = (1; 11) \) and \( \sigma(R) = (\partial; 1 \partial) \), in other words \( L \subseteq B \times (B \times A) \), \( F \subseteq A \times (A \times A) \), while \( R \subseteq A \times (A \times B) \). Operators are generated on both arbitrary (as classical image operators) and (co)stable subsets (as closures of suitable compositions of the image operators with the Galois connection of the frame), as detailed in [19]. Appropriate frame conditions ensure that residuation obtains, as detailed in [20][22]. In this particular case, two of the relations can be dispensed with, as they are definable in terms of the third and the Galois connection (cf [20][22] for details).

Given a sorted frame \( \mathfrak{F} = (A, I, B, R, S) \) as above, a model \( \mathfrak{M} = (\mathfrak{F}, V) \) on the frame \( \mathfrak{F} \) is equipped with a sorted valuation function \( V \) such that \( V(P) \subseteq A \)
and $V(Q_i) \subseteq B$. The sorted modal language is interpreted in the expected way, as in Table 2 where we use $\models \subseteq A \times L_1$ and $\models \subseteq B \times L_0$ for the two satisfaction relations and, to simplify notation, we let $\models$ be either $\models$ or $\models$, as appropriate for the sorting type at hand. Furthermore, we let $[[\alpha]]_{\mathcal{M}} \subseteq A$ and $[[\beta]]_{\mathcal{M}} \subseteq B$ be the generated interpretations of the sorted modal formulae of the first and second sort, respectively.

Table 2: Sorted Interpretation

| $a \models P_i$       | iff $a \in V(P_i)$ | $b \models Q_i$ | iff $b \in V(Q_i)$ |
|-----------------------|--------------------|-----------------|--------------------|
| $a \models \neg \alpha$ | iff $a \not\models \alpha$ | $b \models \neg \beta$ | iff $b \not\models \beta$ |
| $a \models \alpha \land \eta$ | iff $a \models \alpha$ and $a \models \eta$ | $b \models \beta \land \delta$ | iff $b \models \beta$ and $b \models \delta$ |
| $a \models \Box \beta$ | iff $\forall b \ (aIb \ implies \ b \models \beta)$ | $b \models \Box \alpha$ | iff $\forall a \ (aIb \ implies \ a \models \alpha)$ |

Let $\mathcal{F}$ be a class of sorted frames and $\mathcal{M}$ the class of models $\mathcal{M} = (\mathcal{F}, V)$ over frames $\mathcal{F} \in \mathcal{F}$. The following notions have a standard definition just as in the single sorted case. Consult [3] for details.

- $\alpha$ (or $\beta$) is *locally true*, or *satisfiable* in $\mathcal{F} = (A, I, B, R, S)$
- $\alpha$ (or $\beta$) is *globally true* in $\mathcal{F}$
- $\alpha$ (or $\beta$) is *valid in the class* $\mathcal{F}$ of frames
- A set $\Sigma$ of sentences (perhaps of both sorts) *defines* the class $\mathcal{F}$ of frames.

The weakest sorted normal modal logic is an extension of a KB type of system, where the B-axioms (one for each sort) are $\alpha \rightarrow \Box \alpha$ and, for the second sort, $\beta \rightarrow \Box \beta$. In addition, it includes normality axioms for the polyadic sorted diamonds $\Diamond, \Diamond \Diamond$. Note that in the K-axiom (one for each sort) implication on both sorts is involved, witness $\Box(\alpha \rightarrow \eta) \rightarrow (\Box \alpha \rightarrow \Box \eta)$. We shall refer to it as the logic $2KB$. For frames satisfying the seriality conditions

$$\forall a \in A \ \exists b \in B \ aIb \ \ \ \ \ \forall b \in B \ \exists a \in A \ aIb$$

the D-axioms $\Box \beta \rightarrow \Diamond \beta$ and $\Box \alpha \rightarrow \Diamond \alpha$ are valid. We shall refer to the corresponding system as $2KDB$.

### 3.2 Translation Semantics for Logics of Normal Lattice Expansions

In [22], a translation of the language of substructural logics (in the language on the signature $\{\land, \lor, \top, \bot, \rightarrow, \circ, \rightarrow\}$) was introduced, proven to be full and faithful.
(fully abstract). To prove a characterization theorem, generalizing the van Benthem result for modal logic, we define a family of translations, parameterized on permutations of natural numbers, given an enumeration of modal sentences of the second sort.

Let \( \beta_0, \beta_1, \ldots \) be an enumeration of modal sentences of the second sort and \( \pi : \omega \to \omega \) a permutation of natural numbers. The translation \( T^\pi_\varphi \) and co-translation \( T^\varphi_\pi \) of sentences of the language \( \Lambda_\pi \) are defined as in Table 3.

| Table 3: Modal translation and co-translation |
|-----------------------------------------------|
| \( T^\pi_\varphi(p_i) \) = \( \blacksquare \beta_\pi(i) \) | \( T^\varphi_\pi(p_i) \) = \( \boxtimes \neg \beta_\pi(i) \) |
| \( T^\pi(\top) \) = \( \top \) | \( T^\varphi(\top) \) = \( \bot \) |
| \( T^\pi(\bot) \) = \( \bot \) | \( T^\varphi(\bot) \) = \( \top \) |
| \( T^\pi(\varphi \land \psi) = T^\pi_\varphi(\varphi) \land T^\pi_\psi(\psi) \) | \( T^\varphi(\varphi \land \psi) = \boxtimes (\boxtimes T^\pi_\varphi(\varphi) \land \boxtimes T^\pi_\psi(\psi)) \) |
| \( T^\pi(\varphi \lor \psi) = \boxtimes (\boxtimes T^\pi_\varphi(\varphi) \lor \boxtimes T^\pi_\psi(\psi)) \) | \( T^\varphi(\varphi \lor \psi) = T^\pi_\varphi(\varphi \lor \psi) \) |
| \( T^\pi(\diamondsuit(\varphi_1, \ldots, \varphi_n)) = \boxtimes \diamondsuit(\ldots, T^\pi_\varphi(\varphi_j), \ldots, T^\pi_\varphi(\varphi_i), \ldots) \) | \( T^\varphi(\diamondsuit(\varphi_1, \ldots, \varphi_n)) = \blacksquare \boxtimes (\ldots, T^\varphi_\varphi(\varphi_j), \ldots, T^\varphi_\varphi(\varphi_i), \ldots) \) |
| \( T^\varphi(\diamondsuit(\varphi_1, \ldots, \varphi_n)) = \blacksquare \boxtimes (\ldots, T^\varphi_\varphi(\varphi_j), \ldots, T^\varphi_\varphi(\varphi_i), \ldots) \) | \( T^\varphi(\diamondsuit(\varphi_1, \ldots, \varphi_n)) = \blacksquare \boxtimes (\ldots, T^\varphi_\varphi(\varphi_j), \ldots, T^\varphi_\varphi(\varphi_i), \ldots) \) |

**Theorem 3.2.** Let \( \mathcal{F} = (X, I, Y, R, S) \) be a frame (a sorted structure) and \( \mathfrak{M} = (\mathcal{F}, V) \) a model of the sorted modal language. For any enumeration \( \beta_0, \beta_1, \ldots \) of modal sentences of the second sort and any permutation \( \pi : \omega \to \omega \) of the natural numbers define a model \( \mathfrak{M}_\pi \) on \( \mathcal{F} \) for the language \( \Lambda_\pi \) by setting \( V^\pi_{\mathfrak{M}}(p_i) = (\beta_\pi(i))_{\varphi_\mathfrak{M}} \). Then for any sentences \( \varphi, \psi \) of \( \Lambda_\pi \)

1. \( (\lceil \varphi \rceil_{\mathfrak{M}} = (\lceil T^\pi_\varphi(\varphi) \rceil_{\varphi_\mathfrak{M}} = (\lceil \boxtimes T^\pi_\varphi(\varphi) \rceil_{\varphi_\mathfrak{M}} = (\lceil \boxtimes T^\pi_\varphi(\varphi) \rceil_{\varphi_\mathfrak{M}}) \)

2. \( (\lceil \psi \rceil_{\mathfrak{M}} = (\lceil T^\varphi_\psi(\psi) \rceil_{\psi_\mathfrak{M}} = (\lceil \boxtimes T^\varphi_\psi(\psi) \rceil_{\psi_\mathfrak{M}} = (\lceil \boxtimes T^\varphi_\psi(\psi) \rceil_{\psi_\mathfrak{M}}) \)

3. \( \varphi \vdash \psi \) if \( T^\pi_\varphi(\varphi) \models T^\varphi_\psi(\psi) \) if \( T^\varphi_\psi(\psi) \models T^\pi_\varphi(\varphi) \)

**Proof.** The proof is a modification of the proof given in [22]. In [22] we gave a translation of the language of substructural logics into the language of sorted modal logic (more precisely, into the language of \( 2 \mathbf{KB} \mathbf{D} \)) and proved it to be fully abstract ([22], Theorem 4.1, Corollary 4.9). The difference with the current translation is that instead of fixing the translation of propositional variables we define a family of translations, parameterized by a permutation \( \pi \) on the natural numbers. This is needed in the proof of a van Benthem type correspondence result for propositional logics without distribution (Theorem 5.7). The second difference is that we treat here arbitrary normal operators, rather than the operators \( \leftrightarrow, \circ, \rightarrow \) of the language of a substructural logic.
Claim 3) is an immediate consequence of the first two, which we prove simultaneously by structural induction. Note that, for 2), the identities \((\Box □\neg T^*(\varphi))_{3n} = (\Box □\neg T^*(\varphi))_{3n}\) are easily seen to hold for any \(\varphi\), given the proof of claim 1), since

\[
(\varphi)_{3n} = ([\varphi]_{3n} = \Box (\neg T^*(\varphi))_{3n} = (\Box □\neg T^*(\varphi))_{3n}
\]

where \(\Box\) is the set operator interpreting the modal operator \(□\) and similarly for the other cases, differentiating set operators from logical operators by a larger font size for the first.

For the induction proof, we separate cases.

**Case** \(p_i\) \([p_i]_{3n} = V_r(p_i) = [\Box\beta_{\pi(i)}] = [\Box T^*(p_i)]\), by definitions. The other two equalities of 1) are a result of residuation and of the fact that, by residuation again, every boxed formula is stable, i.e. \(\Box\beta \equiv \Box\Box\beta\). Similarly for 2), using definitions and residuation.

**Case** \(\top,\bot,\wedge,\vee\) See \([21]\), or \([22]\), Theorem 4.1.

**(Case 3)** To prove this case, let \(\Box\) be the sorted image operator generated by the relation \(R\)

\[
\Box (\ldots, U_j, \ldots, U_r, \ldots) = \{a \in A \mid \exists u_1 \cdots u_s (aRu_1 \cdots u_s \wedge \bigwedge_{s=1}^{s=n} U_s \in U_s)\}
\]

let also \(\Box\) be the operator on \(P(A)\) resulting by composition with the Galois connection and defined on \(W_1, \ldots, W_n \in A\) by

\[
\Box (W_1, \ldots, W_n) = \Box (\ldots, W_j, \ldots, \Box (\neg W_r), \ldots)
\]

and let \(\Box\) be obtained as the closure of the restriction of \(\Box\) on stable subsets \(C_s = \Box\Box C_s \subseteq A\), for \(s = 1, \ldots, n\), i.e.

\[
\Box (C_1, \ldots, C_n) = \Box\Box (\ldots, C_j, \ldots, \Box (\neg C_r), \ldots)
\]

A dual operator \(\Box^\partial\) on co-stable subsets \(D_s = \Box\Box D_s \subseteq B\) is defined by composition with the Galois connection

\[
\Box^\partial (D_1, \ldots, D_n) = \Box (\neg \Box (\neg D_1), \ldots, \Box (\neg D_n))
\]
In particular, if $C_s = [[\varphi_s]]_{\Lambda_1}$ and $D_s = (\varphi_s)_{\Omega_1}$ we obtain
\[
\mathcal{D}(\varphi_1, \ldots, \varphi_n) = \square \Diamond (\varphi_1, \ldots, \varphi_n)
\]
\[
\mathcal{D}(\varphi_1, \ldots, \varphi_n) = \square \Diamond (\varphi_1, \ldots, \varphi_n)
\]
In particular, if $C_s = [[\varphi_s]]_{\Omega_1}$ and $D_s = (\varphi_s)_{\Omega_1}$ we obtain
\[
\mathcal{D}(\varphi_1, \ldots, \varphi_n) = \square \Diamond (\varphi_1, \ldots, \varphi_n)
\]
\[
\mathcal{D}(\varphi_1, \ldots, \varphi_n) = \square \Diamond (\varphi_1, \ldots, \varphi_n)
\]
Computing membership in the sets $D(C_1, \ldots, C_n)$ and $D(D_s, \ldots, D_n)$, see \cite{19}, Lemma 3.6, for the particular case $C_s = [[\varphi_s]]_{\Omega_1}$ and $D_s = (\varphi_s)_{\Omega_1}$ we obtain the interpretation of Table\cite{19} i.e. $\mathcal{D}(\varphi_1, \ldots, \varphi_n) = \mathcal{D}(\varphi_1, \ldots, \varphi_n) = \mathcal{D}(\varphi_s, \ldots, \varphi_n)$. Computing with the translation, which was defined in line with the representation results of \cite{19} by
\[
T_\pi(\mathcal{D}(\varphi_1, \ldots, \varphi_n)) = \square \Diamond (\varphi_1, \ldots, T_\pi(\varphi_1), \ldots, T_\pi(\varphi_n))
\]
and using the induction hypothesis both claims 1) and 2) follow.

The case for $\Theta$ is similar to the case for $\mathcal{D}$.

\begin{corollary}
A modal formula $\alpha \in L_\tau$ is equivalent to a translation $T_\pi(\varphi)$, for some permutation $\pi : \omega \rightarrow \omega$, of a formula $\varphi$ in the language $\Lambda_\tau$ of normal lattice expansions of similarity type $\tau$ if there is a modal formula $\beta \in L_\tau$ such that $\alpha \equiv \Box \Diamond \alpha$.

Similarly for a formula $\beta \in L_\tau$ and a translation $T_\pi(\varphi)$, in which case $\beta \equiv T_\pi(\varphi)$ iff $\beta \equiv \Box \Diamond \beta$.
\end{corollary}

\begin{proof}
The direction left-to-right follows from Theorem \ref{3.2} Conversely, every formula $\Box \Diamond \alpha$ is in the range of a translation $T_\pi$ for some permutation $\pi$.
\end{proof}

\begin{remark}
Call a modal formula $\alpha$ stable if it is equivalent to $\Box \Diamond \alpha$, and analogously for co-stable. The stable fragment (analogously for the co-stable fragment) of the sorted modal logic is the fragment of modal formulae that are stable in the above sense. The translation $T_\pi$ maps a sentence of the non-distributive logic into the stable fragment of its companion modal logic. Analogously for the co-translation. Note that in the statement and proof of Theorem \ref{3.2} we did not need to place any restrictions on the frame relations $R, S$ and the proof remains valid when restricting to the class of frames where the relations $R, S$ satisfy the section stability requirement of Proposition \ref{2.2}.
\end{remark}
4 First-Order Languages and Structures

A sorted subset \( S = (S_1, S_\partial) \subseteq Z = (Z_1, Z_\partial) \) is finite (more generally, of cardinality \( \kappa \)) if both \( S_1, S_\partial \) are finite (resp. of cardinality \( \kappa \)). The sorted membership relation \( a \in_1 Z, b \in_\partial Z \) means that \( a \in Z_1 = A \) and, respectively, \( b \in Z_\partial = B \). For a pair \( c = (a, b) \), with \( a \in A, b \in B \), the statement \( c \in Z \) has the obvious intended meaning. A sorted function \( h : Z \rightarrow Z' \) is a pair of functions \( h_1 : Z_1 \rightarrow Z'_1, h_\partial : Z_\partial \rightarrow Z'_\partial \). A sorted \((n + 1)\)-ary relation is a subset \( R \subseteq Z_{i_{n+1}} \times \prod_{j=1}^{n} Z_{i_j} \), where for each \( j, i_j \in \{1, \partial\} \). The tuple \( \sigma = (i_{n+1}; i_1; \ldots; i_n) \) is referred to as the sorting type of \( R \) and \( i_{n+1} \in \{1, \partial\} \) as its output type. For an \((n + 1)\)-ary relation we typically use the notation \( uRv_1; \ldots; v_n \) and sometimes, for notational transparency, \( uR(v_1, \ldots, v_n) \).

Consider a structure \( \mathfrak{F} = (A, B, R) \), where \( Z_1 = A, Z_\partial = B \) are the sort sets and \( R \) is a relation of some sorting type \( \sigma = (i_{n+1}; i_1; \ldots; i_n) \). Nothing of significance for our current purposes changes if we consider expansions \((A, B, (R_s)_{s \in S})\) with a tuple of relations \( R_s \), with \( s \) in some index set \( S \), each of some sorting type \( \sigma_s \).

The sorted first-order language with equality \( L^1_s(V_1, V_\partial, R, =_1, =_\partial) \) of a structure \( \mathfrak{F} = (A, B, R) \), for some \((n + 1)\)-ary sorted relation, is built on a countable sorted set \( (V_1, V_\partial) \) of individual variables \( v_1^0, v_1^1, \ldots \) and \( v_\partial^0, v_\partial^1, \ldots \), respectively, and an \((n + 1)\)-ary sorted predicate \( R \) of some sorting type \( \sigma = (i_{n+1}; i_1; \ldots; i_n) \). Well-formed (meaning also well-sorted) formulae are built from atomic formulae \( v_1^1, v_\partial^1 =_1 v_\partial^0 =_\partial v_m^\partial \) and \( R(v_1^{i_{n+1}}; v_1^{i_1}, \ldots; v_1^{i_n}) \) using negation, conjunction and sorted quantification \( \forall v_1^1 \Phi, \exists \partial v_2^\partial \Psi \). We typically simplify notation and write \( \forall v_1 \Phi, \exists \partial v_2 \Psi \) etc. with an understanding and assumption of well-sortedness. We assume the usual definition of other logical operators (\( \land, \neg, \exists \partial, \exists^1, \exists^\partial \)) and of free and bound (occurrences) of a variable, as well as that of a closed formula (sentence), and we follow the usual convention about the meaning of displaying variables in a formula, as in \( \Phi(v_1^0; v_\partial^0) \).

Given a sorted valuation \( V \) of individual variables, \( \mathfrak{F} \models_s \Phi[V] \) is defined exactly as in the case of unsorted FOL. When \( V(u_1^0) = a \in A = Z_1 \), we may also display the assignment in writing \( \mathfrak{F} \models_s \Phi(u_1^0)[u_1^0 := a] \) and similarly for more variables occurring free in \( \Phi \). A formula \( \Phi \) in \( n \) free variables is also referred to as an \( n \)-ary type. A valuation \( V \) realizes the type \( \Phi \) in the structure \( \mathfrak{F} \) iff \( V \) satisfies \( \Phi, \mathfrak{F} \models_s \Phi[V] \). A structure \( \mathfrak{F} \) realizes \( \Phi \) iff some valuation \( V \) does (iff \( \Phi \) is satisfiable in \( \mathfrak{F} \)), otherwise \( \mathfrak{F} \) omits the type. Similarly for a set \( \Sigma \) of \( n \)-ary types, which will itself, too, be referred to as an \( n \)-ary type.

An \( L^1_s \)-theory \( T \) is a set of \( L^1_s \)-sentences and a complete theory is a theory whose set of consequences \( \{ \Phi \mid T \models_s \Phi \} \) is maximal consistent. The (complete) \( L^1_s \)-theory of a structure is designated by \( \text{Th}_s(\mathfrak{F}) \). If \( C = (C_1, C_\partial) \subseteq (A, B) \) is a sorted subset, the expansion \( L^1_s[C] \) of the language includes sorted constants \( c^1 \in C_1, c^\partial \in C_\partial \), for each member of \( C_1, C_\partial \). We sometimes simplify notation writing \( c_{a}, c_{b} \) for the constants naming the elements \( a \in A, b \in B \). It is assumed, as usual, that a constant is interpreted as the element that it names. The extended structure interpreting the expanded signature of the language is designated by
Proof. Straightforward.

To a structure \( \mathfrak{F} = (A, B, R) \) we may also associate an unsorted (single-sorted) first-order language with equality \( \mathcal{L}^1(V', U_1, U_\partial, R, =) \) where the interpretation of \( U_1, U_\partial \) is, respectively, \( Z_1 = A, Z_\partial = B \) and \( V' = V_1 \cup V_\partial \). Assuming the sorting type of \( R \) is \( \sigma = (i_{n+1}; i_1 \ldots i_n) \), the structure validates all sentences pertaining to sorting constraints, which are of the following form, with \( i_j, \in \{1, \partial\} \), for each \( r \).

\[
\forall v_1 \ldots \forall v_{n+1} (R(v_{n+1}, v_1, \ldots, v_n) \rightarrow \bigwedge_{r=1}^{n+1} U_{i_r}(v_r)) \tag{5}
\]

\[
\forall v_1 \forall v_2 (v_1 = v_2 \rightarrow ((U_1(v_1) \land U_1(v_2)) \lor (U_\partial(v_1) \land U_\partial(v_2)))) \tag{6}
\]

In particular, (6) implies the sentence \( \forall v (U_1(v) \lor U_\partial(v)) \). The (unsorted) \( \mathcal{L}^1 \)-theory of \( \mathfrak{F} \) will be designated by \( \text{Th}(\mathfrak{F}) \).

By sort-reduction (for details cf. [10], ch. 4), the language \( \mathcal{L}^1_s \) can be translated into \( \mathcal{L}^1 \), by relativising quantifiers (where \( i_r \in \{1, \partial\} \))

\[
\Psi = \forall^{i_r} u^{i_r}_k \Phi \implies \Psi^* = \forall u^{i_r}_k (U_{i_r}(u^{i_r}_k) \rightarrow \Phi^*)
\]

and replacing \( =_1, =_\partial \) by a single equality predicate =. For later use we list the following result.

**Theorem 4.1** (Enderton [10], ch. 4.3).

1. (Sort-reduction) If \( \Phi^* \) is the sort-reduct of \( \Phi \) and \( V \) a valuation of variables, then \( \mathfrak{F} =_s \Phi[V] \iff \mathfrak{F} = \Phi^*[V] \).

2. (Compactness) If every finite subset of a set \( \Sigma \) of many-sorted sentences in \( \mathcal{L}^1_s \) has a model, then \( \Sigma \) has a model. \( \square \)

### 4.1 Standard Translation of Sorted Modal Logic

The standard translation of sorted modal logic into sorted FOL is exactly as in the single-sorted case, except for the relativization to two sorts, displayed in Table 4.1, where \( \text{ST}_u() \), \( \text{ST}_v() \) are defined by mutual recursion and \( u, v \) are individual variables of sort 1, \( \partial \), respectively.

**Proposition 4.2.** For any sorted modal formulae \( \alpha, \beta \) (of sort 1, \( \partial \), respectively), for any model \( \mathfrak{M} = ((A, I, B, R, S), V) \) and for any \( a \in A, b \in B, \mathfrak{M}, a = \alpha \) iff \( \mathfrak{M} = \text{ST}_u(\alpha)[u := a] \) and \( \mathfrak{M}, b \equiv \beta \) iff \( \mathfrak{M} = \text{ST}_v(\beta)[v := b] \).

**Proof.** Straightforward. \( \square \)

We next review and adapt to the sorted case the basics on ultraproducts and ultrapowers that will be needed in the sequel. Consult [11][4] for details.
4.2 Sorted Ultraproducts

Let \( \mathfrak{F}_J = (A_j, B_j, R_j)_{j \in J} \), with \( J \) some index set, be a family of structures with sorted relations \( R_j \) of some fixed sorting type \( \sigma \).

An ultraproduct over \( J \) is an ultraproduct (maximal filter) \( U \) of the powerset Boolean algebra \( \mathcal{P}(J) \). Let \( \Pi_U A_j, \Pi_U B_j \) be the ultraproducts of the families of sets \( (A_j)_{j \in J} \) over the ultrafilter \( U \). Members of \( \Pi_U A_j \) are equivalence classes \( f_U \) of functions \( f \in \Pi_j A_j \) (i.e. functions \( f : J \to \bigcup_j X_j \) such that for all \( j \in J \), \( f(j) \in A_j \)) under the equivalence relation \( f \sim_U g \iff \{ j \in J \mid f(j) = g(j) \} \in U \).

Goldblatt [15–17] introduced ultraproducts for polarities (sorted structures with a binary relation), slightly generalizing the classical construction. We review the definition, adapting to the case of an arbitrary \((n + 1)\)-ary relation.

Definition 4.3 (Ultraproducts of Sorted Structures). Given a family \( \mathfrak{F}_J = (A_j, B_j, R_j)_{j \in J} \) of structures (models) with \( J \) some index set, their ultraproduct is the sorted structure \( \Pi_U \mathfrak{F}_J = (\Pi_U A_j, \Pi_U B_j, R_U) \) where

1. \( \Pi_U A_j, \Pi_U B_j \) are the ultraproducts over \( U \) of the families of sets \( (A_j)_{j \in J} \), \( (B_j)_{j \in J} \).
2. Where the sorting type of \( R_j \) for all \( j \in J \) is \( \sigma = (i_{n+1}; i_1, \ldots, i_n) \) and for each \( r \in \{ 1, \ldots, n + 1 \} \) we have \( h_{r,U} \in \Pi_U A_j \), if \( i_r = 1 \), and \( h_{r,U} \in \Pi_U B_j \) if \( i_r = \partial \), the relation \( R_U \), of sorting type \( \sigma \) is defined by setting

\[
h_{n+1,U} R_U (h_1, U, \ldots, h_n, U) \iff \{ j \in J \mid h_{n+1}(j)R_j(h_1(j), \ldots, h_n(j)) \} \in U
\]

If for all \( j \in J \), \( \mathfrak{F}_j = \mathfrak{F} \), then the ultraproduct is referred to as the ultrapower \( \Pi_U \mathfrak{F} \) of \( \mathfrak{F} \) over the ultrafilter \( U \).
Considering the structures \( \mathfrak{F}_j \) as \( \mathcal{L}^1 \)-structures, by the fundamental theorem of ultraproducts (Los’s theorem) we have

\[
\prod_U \mathfrak{F}_j = \Phi[f_1,u, \ldots, f_t,u] \quad \text{iff} \quad \{ j \in J \mid \mathfrak{F}_j = \Phi[f_1(j), \ldots, f_t(j)] \} \in U \tag{8}
\]

By sort reduction, Los’s theorem holds when the \( \mathfrak{F}_j \) are regarded as models of the sorted language (as \( \mathcal{L}^s \)-structures), as well. Indeed

\[
\prod_U \mathfrak{F}_j = \Phi^*[f_1,u, \ldots, f_t,u] \quad \text{iff} \quad \{ j \in J \mid \mathfrak{F}_j = \Phi^*[f_1(j), \ldots, f_t(j)] \} \in U
\]

We use the standard notation \( \mathfrak{F} \models \mathfrak{G} \) for \textit{elementarily equivalent structures} (satisfying the same set of sentences) and \( \mathfrak{F} \prec \mathfrak{G} \) to designate the fact that \( \mathfrak{G} \) is an \textit{elementary extension} of \( \mathfrak{F} \), meaning that \( \mathfrak{F} \subseteq \mathfrak{G} \) (\( \mathfrak{F} \) is a substructure of \( \mathfrak{G} \)) and for any \( n \)-ary type \( \Phi(w_{i_1}, \ldots, w_{i_n}) \) of some sort \( (i_1, \ldots, i_n) \in \{1, d\}^n \) and any valuation \( V \) for \( \mathfrak{F} \) we have \( \mathfrak{F} \models \Phi[V] \) iff \( \mathfrak{G} \models \Phi[V] \). Finally, we recall that a map \( h : \mathfrak{F} \models \mathfrak{G} \) is an \textit{elementary embedding} iff for any \( n \)-ary type \( \Phi \) as above we have \( \mathfrak{F} \models \Phi(\tau_{\mathfrak{G}})(V) \) iff \( \mathfrak{G} \models \Phi(\tau_{\mathfrak{G}})(h \circ V) \).

The same argument as above, appealing to sort-reduction, applies to lift to the sorted case well-known consequences of Los’s theorem (in particular, Corollary 4.1.13 of [4], restated for the sorted case below).

**Corollary 4.4.** If \( \mathfrak{F} \) is an \( \mathcal{L}^1 \)-structure, \( J \) an index set and \( U \) an ultrafilter over \( J \), then \( \mathfrak{F} \) and the ultrapower \( \prod_U \mathfrak{F} \) are elementarily equivalent, \( \mathfrak{F} \models \prod_U \mathfrak{F} \). Furthermore, the embedding \( e = (e_1 : Z_1 \to \prod_U Z_1, e_\partial : Z_\partial \to \prod_U Z_\partial) \) sending elements \( a \in Z_1 = A, b \in Z_\partial = B \) to the respective equivalence classes \( e(a) = e_1(a) = f_{a,U}, e(b) = e_\partial(b) = f_{b,U} \) of the constant functions \( f_a(j) = a, f_b(j) = b, \) for all \( j \in J \), is an elementary embedding \( e : \mathfrak{F} \models \mathfrak{G} \).

**Sketch of Proof.** By appealing to sort-reduction (cf Theorem 4.3). In fact, a direct argument for the sorted case is literally the same as in the unsorted case, as seen by consulting for example the proof in [1], Lemma 2.3. □

Note, in particular, that for a unary type \( \Phi(u^1) \in \mathcal{L}^1 \) and any element say \( a \in A \) (i.e., a valuation \( V \) such that \( V(u^1) = a \in A \)) we have (dropping the sorting superscript on the variable) the following

**Corollary 4.5.** For \( u \in V_1, \mathfrak{F} \models \Phi(u)[u := a] \) if \( \prod_U \mathfrak{F} \models \Phi(u)[u := a] \).

The same holds for a type with a free variable \( v \in V_\partial \).

**Proof.**

\[
\prod_U \mathfrak{F} \models \Phi(u)[u := a] \quad \text{iff} \quad \prod_U \mathfrak{F} \models \Phi^*(u)[u := a] \quad \text{(by sort-reduction)}
\]

\[
\quad \text{iff} \quad \{ j \mid \mathfrak{F} \models \Phi^*(u)[u := a(j)] \} \in U \quad \text{(by Los’s theorem)}
\]

\[
\quad \text{iff} \quad \{ j \mid \mathfrak{F} \models \Phi^*(u)[u := a] \} \in U \quad \text{(by sort-reduction)}
\]

\[
\mathfrak{G} \text{ is a filter, so } \{ j \mid \mathfrak{F} \models \Phi^*(u)[u := a] \} \neq \emptyset
\]

and this proves the claim. □
\textbf{Definition 4.6} (Ultrapowers of Models). If $\mathfrak{M} = (\mathfrak{A}, V)$ is a model and $U$ is an ultrafilter over an index set $J$, the ultrapower of $\mathfrak{M}$ is defined by $\prod_U \mathfrak{M} = (\prod_U \mathfrak{A}, V_U)$ where $V_U(u) = f_{n,U}$ if $V(u) = a$.

4.3 Saturated Structures

Let $\mathfrak{A} = (A, B, R)$ be an $\mathcal{L}_1$-structure. The structure $\mathfrak{A}$ is called $\omega$-saturated iff for any finite subset $C \subseteq A \cup B$, every unary type $\Sigma$ of the expanded language $\mathcal{L}_1(C)$ that is consistent with the theory $\text{Th}_\omega(\mathfrak{A}, \bar{c})_{c \in C}$ is realized in $(\mathfrak{A}, \bar{c})_{c \in C}$. If reference to sorting is disregarded, this is precisely the meaning of $\omega$-saturated structures for (unsorted) first-order languages. The definition generalizes to $\kappa$-saturated structures, for any cardinal $\kappa$, but we shall only have use of $\omega$-saturated structures in the sequel.

$\omega$-saturated first-order (unsorted) structures can be constructed as unions of elementary chains, or as ultrapowers. Consult Bell and Slomson \cite{1}, Theorem 1.7 and Theorem 2.1, or Chang and Keisler \cite{4}, ch. 5, for details. Any two elementarily equivalent $\kappa$-saturated structures are isomorphic (\cite{4}, Theorem 5.1.13, \cite{1}, Theorem 3.1), so we only discuss ultrapowers. With some necessary adaptation, the original arguments for the unsorted case (for the existence of $\omega$-saturated extensions) can be reproduced for the sorted case. It is easier, however, to derive the result for the sorted case by reducing the problem to the unsorted case, using sort-reduction, as we do below.

\textbf{Theorem 4.7.} Every $\mathcal{L}_1$-structure $\mathfrak{A}$ has an elementary $\omega$-saturated extension $h : \mathfrak{A} \leq \mathfrak{B}$. \footnote{As a countably incomplete ultrafilter we may take an ultrafilter over the set of natural numbers that does not contain any singletons (cf \cite{3}, Example 2.72).}

\textbf{Proof.} By standard model-theoretic results (\cite{4}, Proposition 5.1.1, Theorem 6.1.1), for every first-order structure ($\mathcal{L}_1$-structure) $\mathfrak{A}$ and any countably incomplete ultrafilter $U$ over some index set $J$, its ultrapower $\prod_U \mathfrak{A}$ is an elementary $\omega$-saturated extension of $\mathfrak{A}$, $e : \mathfrak{A} \not\leq \prod_U \mathfrak{A}$, by the embedding of (the unsorted version of) Corollary 4.1.13 (see \cite{4}, Corollary 4.1.13).

Let $\mathfrak{A} = (A, B, R)$ be a sorted first-order structure, $C \subseteq A \cup B$ and $\Sigma(v)$, with $v \in V_1 \cup V_2$, a unary type in the expanded language $\mathcal{L}_1(C)$ consistent with $\text{Th}_\omega(\mathfrak{A}, \bar{c})$. We claim that the sort reducible $\Sigma^* = \{ \Phi^*(v) \mid \Phi(v) \in \Sigma \}$ is consistent with the (unsorted) theory $\text{Th}(\mathfrak{A}, \bar{c})$. Assuming for the moment that the claim is proved, by $\omega$-saturation of the ultrapower of the $\mathcal{L}_1$-structure $\mathfrak{A}$, $\prod_U \mathfrak{A}\Gamma \equiv \Sigma^*[S]$ and then by sort reducibility $\prod_U \mathfrak{A}\Gamma \equiv \Sigma^*[S]$, i.e. the type $\Sigma$ in the sorted language $\mathcal{L}_1(C)$ is realized in $\prod_U \mathfrak{A}\Gamma$ by some valuation $S$. Hence $\prod_U \mathfrak{A}$, regarded as an $\mathcal{L}_1$-structure, is an elementary $\omega$-saturated extension of $\mathfrak{A}$.

To prove the claim we made in course of the above argument, recall that we assume that $\Sigma(v)$ is consistent with $\text{Th}_\omega(\mathfrak{A}, \bar{c})$, so that a structure $\mathfrak{M}$ and a valuation $V_N$ exist such that $V_N$ satisfies in $\mathfrak{M}$ every formula in $\Sigma(v)$ and sentence in $\text{Th}_\omega(\mathfrak{A}, \bar{c})$.

If $\Sigma^*$ is not consistent with the theory $\text{Th}(\mathfrak{A}, \bar{c})$, let $\Phi \in \mathcal{L}_1(C)$ be such that both $\Phi$ and $\neg \Phi$ are derivable from $\Sigma(v) \cup \text{Th}(\mathfrak{A}, \bar{c})$. By compactness,
5 Bisimulations and van Benthem Characterization

5.1 Bisimulations

**Definition 5.1** (Bisimulation on Sorted Structures). Let $\mathfrak{F} = (A, I, B, R, S)$, $\mathfrak{F}' = (A', I', B', R', S')$ be frames, where recall that the sorting types of $R, S$ are $(1; i_1, \ldots, i_m)$ and $(\beta; i'_1, \ldots, i'_n)$, respectively, and assume that $z \leq Z \times Z'$ (where $Z = A \cup B$ and $Z' = A' \cup B'$) is a well-sorted relation (i.e. for $a \in A$, $b \in B$, the set $a \leq z = \{b \mid a \leq b\}$ is a subset of $A'$ and similarly $b \leq z \in B'$). Then the relation $z$ is a simulation iff

1. If $a \leq a'$ then
   - if $aIb$, then $b \leq b'$ for some $b' \in B'$ such that $a'I'b'$
   - if $aRu_1 \ldots u_m$, then $a'R'u'_1 \ldots u'_m$ for some $u'_j$ such that $u_j \leq u'_j$

2. If $b \leq b'$ then
   - if $aIb$, then $a \leq a'$ for some $a' \in A'$ such that $a'I'b'$
   - if $bSv_1 \ldots v_m$, then $b'S'v'_1 \ldots v'_m$ for some $v'_j$ such that $v_j \leq v'_j$

A relation $z$ is a bisimulation if both $z$ and its inverse $z^{-1}$ are simulations. We use the notation $\sim$ for bisimulations. If $\sim$ is a bisimulation for the frames, we write $\mathfrak{F} \sim \mathfrak{F}'$.

A relation $\sim$ is a bisimulation of models $\mathcal{M} = (\mathfrak{F}, V), \mathcal{M}' = (\mathfrak{F}', V')$ iff

1. $\mathfrak{F} \sim \mathfrak{F}'$ and
2. for any propositional variable $P_i$ of the first sort, if $a \in V(P_i)$ and $a \sim a'$, then $a' \in V'(P_i)$
3. for any propositional variable $Q_i$ of the second sort, if $b \in V(Q_i)$ and $b \sim b'$, then $b' \in V'(Q_i)$

If $\sim$ is a bisimulation for the models, we write $\mathcal{M} \sim \mathcal{M}'$ and we use $\mathcal{M}, w \sim \mathcal{M}', w'$ when $w \sim w'$ are points of either (but the same) sort.

**Proposition 5.2.** Sorted modal formulas are invariant under bisimulation. In other words, if $\mathcal{M}, a \sim \mathcal{M}', a'$ (resp. $\mathcal{M}, b \sim \mathcal{M}', b'$), then $\mathcal{M} \vDash \alpha(u)[v := a]$ iff $\mathcal{M}' \vDash \alpha(u)[v := a']$ (resp. $\mathcal{M} \vDash \beta(v)[v := b]$ iff $\mathcal{M}' \vDash \beta(v)[v := b']$).
Proof. By structural induction, observing that the argument for the base case is built into the definition of bisimulations, while for negations and implications it reduces to that for the subsentences and the induction hypothesis is used, while for modal operators the corresponding clauses in the definition of bisimulations allow directly the use of the inductive hypothesis on subsentences.

For models \(\mathcal{M}, \mathcal{M}'\), points \(a \in A, a' \in A'\), respectively are **modally equivalent** iff for any \(\alpha \in L_T\), \(\mathcal{M}, a \vDash_s \alpha\) iff \(\mathcal{M}', a' \vDash_s \alpha\). In symbols \(\mathcal{M}, a \equiv \mathcal{M}', a'\). Similarly for points \(b \in B, b' \in B'\).

A kind of converse of Proposition 5.2 is provided below.

**Proposition 5.3.** Assume \(\mathcal{M}, a \equiv \mathcal{M}', a'\) and let \(U\) be a countably incomplete ultrafilter over some index set \(J\). Then \(\prod_D \mathcal{M}, f_{a,U} \equiv \prod_D \mathcal{M}', f_{a',U}\) and the relation of modal equivalence on the ultrapowers is a bisimulation.

**Proof.** By Proposition 4.2 we have \(\mathcal{M}, a \equiv \prod_D \mathcal{M}, f_{a,D}\) and so the first hypothesis implies that \(\prod_D \mathcal{M}, f_{a,U} \equiv \prod_D \mathcal{M}', f_{a',U}\).

From the second hypothesis and Theorem 4.7 it is obtained that \(\prod_D \mathcal{M}, \prod_D \mathcal{M}'\) are \(\omega\)-saturated and the claim is that this implies that modal equivalence is a bisimulation. The proof of this claim is the same as the corresponding proof in the unsorted case (cf. [3], ch. 2, Proposition 2.54 and Theorem 2.65).

If \(\Phi = \Phi(u) \in \mathcal{L}\) has only the displayed variable \(u \in V_1\) free (i.e. it is a unary type) and it holds that \(\Phi \vDash \text{ST}_a(\alpha)\), for some \(\alpha\) (of sort 1) in the sorted modal language, then we say that \(\text{ST}_a(\alpha)\) is a modal \(1\)-consequence of \(\Phi\). Similarly, if \(\Psi(v) \vDash \text{ST}_v(\beta)\) then \(\text{ST}_v(\beta)\) is a modal \(\partial\)-consequence of \(\Psi\). Let \(m_1^{\Phi}(\Phi), m_1^{\Psi}(\Psi)\) be sets of \(1\)- and \(\partial\)-consequences of \(\Phi, \Psi\), respectively.

**Lemma 5.4.** Let \(\Phi(u)\) be a sorted first-order formula in one free variable \(u \in V_1\) and let \(m_1^{\Phi}(\Phi)\) be the set of its modal 1-consequences. If \(\Phi\) is invariant under bisimulation, then \(m_1^{\Phi}(\Phi) \equiv_s \Phi\). Similarly, for a bisimulation invariant formula \(\Psi(v) \in \mathcal{L}\), with \(v \in V_\partial\), \(m_1^{\Psi}(\Psi) \equiv_s \Psi\).

**Proof.** The proof is again similar to that for the unsorted case, see for example the proof in Theorem 2.68 of [3]. We provide some details.

Let \(\mathcal{M} = ((A, I, B, R, S), V)\), \(a \in A\), assume \(\mathcal{M} \equiv_s m_1^{\Phi}(\Phi)[u := a]\) and observe that \(\Phi \cup m_1^{\Phi}(a)\) is consistent. Otherwise, by compactness of sorted FOL [10], we obtain that \(\vDash_s \Phi \rightarrow \neg \land m_0^{\Phi}(a)\), for some finite \(m_0^{\Phi}(a) \subseteq m_1^{\Phi}(a)\). Hence, \(\neg \land m_0^{\Phi}(a) \in m_1^{\Phi}(\Phi)\) which implies that \(\mathcal{M} \equiv_s \neg \land m_0^{\Phi}(a)\). This is in contradiction with the fact that \(m_0^{\Phi}(a) \subseteq m_1^{\Phi}(a)\) and \(\mathcal{M} \equiv_s \text{ST}_u(a)[u := a]\) for all \(\text{ST}_u(a) \in m_1^{\Phi}(a)\).

By consistency of \(\Phi(u) \cup m_1^{\Phi}(a)\), let \(\mathcal{M}' = ((A', I', B', R', S'), V')\) be a model and \(a' \in A'\) such that \(\mathcal{M}' \equiv_s \{\Phi(u)\} \cup m_1^{\Phi}(a)[u := a']\). Then for any sentence \(\alpha\) of the first sort in the sorted modal language, \(\mathcal{M}, a \equiv_s \alpha\) iff \(\mathcal{M}', a' \equiv_s \alpha\). i.e. \(a, a'\) are modally equivalent. This is because if \(\mathcal{M}, a \equiv_s \alpha\), then \(\text{ST}_u(\alpha) \in m_1^{\Phi}(a)\) and therefore by \(\mathcal{M}' \equiv_s \{\Phi(u)\} \cup m_1^{\Phi}(a)[u := a']\) and Proposition 4.2 it follows that
In an incomplete ultrafilter over some index set $J$. Conversely, if $\mathcal{M}', a' \models s \alpha$, then it must be that $\mathcal{M}, a \models s \alpha$ for, if not, then $\mathcal{M}, a \models s \neg \alpha$ and this implies $\mathcal{M}', a' \models s \neg \alpha$ which is a contradiction.

To obtain $\mathcal{M} \models s \Phi(u)[u := a]$ from $\mathcal{M}' \models s \Phi(u)[u := a']$, let $U$ be a countably incomplete ultrafilter over some index set $J$. By Proposition 5.3 and Corollaries 4.4 and 4.5 we obtain a sequence of implications:

$$\mathcal{M}' \models s \Phi(u)[u := a'] \implies \prod_U \mathcal{M}' \models s \Phi(u)[u := f_{a,U}']$$
$$\implies \prod_U \mathcal{M} \models s \Phi(u)[u := f_{a,U}']$$
$$\implies \mathcal{M} \models s \Phi(u)[u := a']$$

This establishes that $m_1(\Phi) \models s \Phi$. The argument for a formula $\Psi(v) \in L^1$, with $v \in V_{\emptyset}$ is similar.

\section{Van Benthem Characterization}

Fix a similarity type $\tau$. Let $\Lambda_\tau$ be the language of normal lattice expansions of type $\tau$, $L_{\tau} = (L_1, L_{\emptyset})$, be the sorted modal language of type $\tau$ and $L_{1,\tau}$ the sorted first-order language of the same type $\tau$. All the necessary work to lift van Benthem’s characterization theorem to sorted modal logic has been presented and we state the result.

\textbf{Theorem 5.5.} Let $\Phi(u) \in L_{1,\tau}$ be a formula in one free variable in the sorted first-order language $L_{1,\tau}$, with $u \in V_1$. Then $\Phi$ is equivalent to the translation $ST_u(\al)$ of a modal formula $\al \in L_{\tau}$ iff $\Phi$ is bisimulation invariant. Similarly for a formula $\Psi(v)$ with $v \in V_{\emptyset}$.

\textit{Proof.} If $\Phi$ is equivalent to the translation $ST_u(\al)$ of a modal formula $\al \in L_{\tau}$, then $\Phi$ is bisimulation invariant by Proposition 5.2. For the converse, by Lemma 5.3 we obtain $m_1(\Phi) \models s \Phi$. By compactness for sorted FOL (Theorem 4.4), let $\mu_1(\Phi) = \{ST_u(\al_1), \ldots, ST_u(\al_n)\} \subseteq m_1(\Phi)$ be a finite subset of $m_1(\Phi)$ such that $\mu_1(\Phi) \models s \Phi$. Then $\models s \Phi \leftrightarrow \land \mu_1(\Phi)$, hence $\models s \Phi \leftrightarrow ST_u(\eta)$, where we set $\eta = \al_1 \land \cdots \land \al_n$. \hfill \Box

It remains to adapt the result to the case of the logics of normal lattice expansions of similarity type $\tau$.

\textbf{Definition 5.6.} $\Phi(u)$, with $u \in V_1$, is \textit{stable} if and only if it is equivalent to the formula $\forall^g v \exists^1 z \ (I(u,v) \rightarrow I(z,v) \land \Phi(z))$.

\textbf{Theorem 5.7} (van Benthem Characterization). Fix a similarity type $\tau$. Let $\Phi(u) \in L_{1,\tau}$ be a formula with one free variable in the sorted first-order language $L_{1,\tau}$, with $u \in V_1$. Then $\Phi$ is equivalent to the translation $ST_u^s(\varphi)$, for some permutation $\pi: \omega \rightarrow \omega$, of a sentence in the language of lattice expansions of similarity type $\tau$ iff $\Phi$ is bisimulation invariant and stable.

\textit{Proof.} The claim of the theorem follows immediately by combining the characterization result for sorted modal logic of similarity type $\tau$ (Theorem 5.5) and Corollary 3.3 (a consequence of Theorem 5.2). \hfill \Box
6 Conclusions

This article is part of a project of employing modal methods and lifting results proved for modal logic to the case of non-distributive propositional logics, i.e. the logics of normal lattice expansions of some similarity type $\tau$. An intermediate step in carrying out the proof has been the lifting of the van Benthem characterization result to the case of sorted modal logic, which is unproblematic, though burdened with the usual technicalities one needs to deal with when moving from unsorted to sorted domains. Its core idea, on which it relies heavily, is the modal representation of normal lattice expansions developed in [19,20,23,26] and the possibility to provide fully abstract modal translations of the languages of logics for normal lattice expansions into sorted modal logic, an idea first explored in [21,22].

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