Cosmological evolution in exponential gravity

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Abstract

We explore the cosmological evolution in the exponential gravity $f(R) = R + c_1 (1 - e^{-c_2 R})$ ($c_{1,2} = \text{constant}$). We summarize various viability conditions and explicitly demonstrate that the late-time cosmic acceleration following the matter-dominated stage can be realized. We also study the equation of state for dark energy and confirm that the crossing of the phantom divide from the phantom phase to the non-phantom (quintessence) one can occur. Furthermore, we illustrate that the cosmological horizon entropy globally increases with time.

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I. INTRODUCTION

There exist two representative approaches to account for the current accelerated expansion of the universe. One is to introduce “dark energy” in the framework of general relativity. The other is to consider a modified gravitational theory, such as $f(R)$ gravity.

Several viable theories of $f(R)$ gravity have been constructed; e.g., power-law, Nojiri-Odintsov, Hu-Sawicki, Starobinsky’s, Appleby-Battle, and Tsubakawa’s models (for more detailed references, see a recent review on $f(R)$ gravity). It is known that these models can satisfy the following conditions for the viability: (i) positivity of the effective gravitational coupling, (ii) stability of cosmological perturbations, (iii) asymptotic behavior to the standard Λ-Cold-Dark-Matter (ΛCDM) model in the large curvature regime, (iv) stability of the late-time de Sitter point, (v) constraints from the equivalence principle, and (vi) solar-system constraints.

Recently, an interesting model of $f(R) = R + c_1 (1 - e^{-c_2 R})$, called “exponential gravity”, has been proposed in Refs. with $c_{1,2}$ being constants. The important feature of the exponential gravity is that it has only one more parameter than the ΛCDM model. The constraints from the violation of the equivalence principle and cosmological observations on the exponential gravity have been examined. The exponential gravity in the framework of $f(R)$ gravity has been extended to a gravitational theory in terms of the torsion scalar (for a related work on torsion gravity, see ). We note that the cosmological dynamics in the gravitational theory consisting only of the exponential term without the Einstein-Hilbert one has also been studied in Ref.

In this paper, we explicitly investigate the cosmological evolution in the exponential gravity model given by Cognola et al. and Linder in more detail by using the analysis method in Ref. We also check the above six viability conditions for the model. In particular, we demonstrate that after the matter-dominated stage, the current accelerated expansion of the universe and the crossing of the phantom divide from the phantom phase to the non-phantom (quintessence) one can be realized. It is interesting to note that the crossing of the phantom divide is implied by the cosmological observational data, while the exponential gravity is a ghost free theory. In addition, we illustrate that the cosmological horizon entropy globally increases with time. We use units of $k_B = c = \hbar = 1$ and denote
the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}$ GeV.

The paper is organized as follows. In Sec. II, we review the model of the exponential gravity in Refs. [25, 26] and summarize its viability conditions. In Sec. III, we explore the cosmological evolution of the model. We examine the horizon entropy in Sec. IV. Finally, conclusions are given in Sec. V.

II. EXPONENTIAL GRAVITY

A. The model

The action of $f(R)$ gravity with matter is given by

$$I = \int d^4x \sqrt{-g} \frac{f(R)}{2\kappa^2} + I_{\text{matter}}(g_{\mu\nu}, \Upsilon_{\text{matter}}),$$

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$, $I_{\text{matter}}$ is the action of matter which is assumed to be minimally coupled to gravity, i.e., the action $I$ is written in the Jordan frame, and $\Upsilon_{\text{matter}}$ denotes matter fields. Here, we use the standard metric formalism.

Taking the variation of the action in Eq. (2.1) with respect to $g_{\mu\nu}$, one obtains

$$F G_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(\text{matter})} - \frac{1}{2} g_{\mu\nu} (FR - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F,$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R$ is the Einstein tensor, $F(R) \equiv df(R)/dR$, $\nabla_\mu$ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d’Alembertian for a scalar field, and $T_{\mu\nu}^{(\text{matter})}$ is the contribution to the energy-momentum tensor from all perfect fluids of generic matter.

In this paper, we concentrate on the exponential gravity in Refs. [25, 26], given by

$$f(R) = R - \beta R_s \left(1 - e^{-R/R_s}\right),$$

where $c_1 = -\beta R_s$ and $c_2 = R_s^{-1}$. Note that $R_s$ corresponds to the characteristic curvature modification scale.

B. Viability conditions on exponential gravity

For the model of the exponential gravity in Eq. (2.3), it is straightforward to show that the conditions for the viability can be satisfied, which are summarized as follows:
(i) Positivity of the effective gravitational coupling

When $\beta < e^{R/R_s}$, $F(R) = 1 - \beta e^{-R/R_s} > 0$. This is required for the positivity of the effective gravitational coupling $G_{\text{eff}} \equiv G/F(R) > 0$ to avoid anti-gravity. In the sense of the quantum theory, the graviton is not a ghost.

(ii) Stability of cosmological perturbations

When $\beta > 0$ and $R_s > 0$, $f''(R) = F'(R) = (\beta/R_s) e^{-R/R_s} > 0$, where the prime denotes differentiation with respect to $R$. This is required for the stability of cosmological perturbations [18–20]. In the sense of the quantum theory, the scalaron, which is a new scalar degree of freedom in $f(R)$ gravity, is not a tachyon [14].

(iii) Asymptotic behavior to the ΛCDM model in the large curvature regime

Since $f(R) - R \rightarrow -\beta R_s = \text{constant for } R/R_s \gg 1$, this model is reduced to the ΛCDM model in the large curvature regime $R/R_s \gg 1$. Such a behavior is necessary for the presence of the matter-dominated stage.

(iv) Stability of the late-time de Sitter point

When $\beta > 1$, $0 < m(R = R_d) < 1$ [27], where $m \equiv R f''(R)/f'(R) = R F'(R)/F(R)$ and $R_d = 2 f(R_d)/F(R_d)$ is the value of the scalar curvature at the de Sitter point. This condition is required for the stability of the late-time de Sitter point [10, 21, 22]. The quantity $m$ characterizes the deviation from the ΛCDM model because $m = 0$ for the ΛCDM model. In the exponential gravity, by making the following conformal transformation [35]: $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Xi^2 g_{\mu\nu}$, the action of $f(R)$ gravity in Eq. (2.1) can be rewritten in the Einstein frame, where $\Xi^2 \equiv F = e^{\sqrt{2/3\kappa\phi}}$ with the scalar field $\phi$. In what follows, a tilde represents the quantity in the Einstein frame. We consider a spherically symmetric body with radius $\tilde{r}_c$ in the Minkowski space-time. Here, $\tilde{r}$ is the distance of the center of the body, and $\rho^* = e^{-\sqrt{2/3\kappa\phi}} \rho$ is a conserved matter density in the Einstein frame with $\rho$ the energy density of matter in the Jordan frame. We assume that a spherically symmetric body has constant densities of $\rho^* = \rho_{\text{in}}$ and $\rho_{\text{out}} (\ll \rho_{\text{in}})$ inside
\( \tilde{r} < \tilde{r}_c \) and outside \( \tilde{r} > \tilde{r}_c \), respectively. In this case, the effective potential has two minima at the field values \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) satisfying the conditions \( dV_{\text{eff}}(\phi_{\text{in}})/d\phi = 0 \) and \( dV_{\text{eff}}(\phi_{\text{out}})/d\phi = 0 \) with a heavier mass squared \( m^2_{\text{in}} \equiv d^2V_{\text{eff}}(\phi_{\text{in}})/d\phi^2 \) and a lighter mass squared \( m^2_{\text{out}} \equiv d^2V_{\text{eff}}(\phi_{\text{out}})/d\phi^2 \), respectively. The thin-shell parameter is defined as \( \epsilon_{\text{th}} \equiv -\kappa (\phi_{\text{out}} - \phi_{\text{in}})/\sqrt{\Phi_c} \) [34], where \( \Phi_c = GM_c/\tilde{r}_c \) is the gravitational potential at the surface of the body and \( M_c = (4\pi/3) \tilde{r}_c^3 \rho_{\text{in}} \).

The tightest experimental bound on \( \epsilon_{\text{th}} \) obtained from the violation of the equivalence principle for the accelerations of the Earth and the moon toward the Sun is given by \( \epsilon_{\text{th}, \oplus} < 2.2 \times 10^{-6} \) [37, 38]. This is the thin-shell parameter for the Earth. By using the value of the gravitational potential for the Earth \( \Phi_{c, \oplus} = 7.0 \times 10^{-10} \) and \( |\phi_{\text{out, \oplus}}| > |\phi_{\text{in, \oplus}}| \), the condition on \( \epsilon_{\text{th, \oplus}} \) is reduced to \( |\kappa \phi_{\text{out, \oplus}}| < 3.7 \times 10^{-15} \) [36]. The field value \( \phi_{\text{out, \oplus}} \) can be found by solving \( dV_{\text{eff}}(\phi_{\text{out}})/d\phi = 0 \) with \( \rho^* = \rho_{\text{out}} \), which gives \( R \simeq \kappa^2 \rho_{\text{out}} \).

For the exponential gravity, \( \kappa \phi_{\text{out}} \simeq -\sqrt{3/2} e^{-\kappa^2 \rho_{\text{out}}/R_0} \) [27] and \( \beta R_s/R_0 \approx \Omega_{m(0)} \), where \( R_0 \approx 12H_0^2 \) is the current scalar curvature, \( H_0 \) is the current Hubble parameter, \( \Omega_{m(0)} \equiv \rho_{m(0)}/\rho_{\text{crit}(0)} \) is the current density parameter of non-relativistic matter (cold dark matter and baryon), \( \rho_{m(0)} \) is the energy density of non-relativistic matter at the present time, and \( \rho_{\text{crit}(0)} = 3H_0^2/\kappa^2 \) is the critical density. As a consequence, by using \( \rho_{\text{crit}(0)} \approx 10^{-28} \text{g/cm}^3 \) and the homogeneous baryon/dark matter density \( \rho_{\text{out}} \approx 10^{-24} \text{g/cm}^3 \), we find \( \kappa \phi_{\text{out}} \approx -\beta \exp(-10^5 \beta) \) [27]. When \( \beta > 1 \), which is the stability condition for the late-time de Sitter point in the exponential gravity, the above constraint on \( |\kappa \phi_{\text{out, \oplus}}| \) is satisfied very well. For example, if \( \beta = 1.1 \), \( |\kappa \phi_{\text{out}}| = O(10^{-50000}) \). In what follows, the superscript (0) denotes the present value.

(vi) Solar-system constraints

The bound on the thin-shell parameter coming from the solar-system constraint \( \epsilon_{\text{th, \oplus}} < 2.3 \times 10^{-5} \) [7] is weaker than that from the violation of the equivalence principle \( \epsilon_{\text{th, \oplus}} < 2.2 \times 10^{-6} \) shown above.

III. COSMOLOGICAL EVOLUTION

We assume the flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with the metric,

\[
 ds^2 = -dt^2 + a^2(t)dx^2, \tag{3.1}
\]
where \( a(t) \) is the scale factor. From Eq. (2.2), we obtain the following gravitational field equations:

\[
3F H^2 = \kappa^2 \rho_M + \frac{1}{2} (FR - f) - 3HF,
\]

\[
-2F \dot{H} = \kappa^2 (\rho_M + P_M) + \ddot{F} - H\dot{F},
\]

where \( H = \dot{a}/a \) is the Hubble parameter, the dot denotes the time derivative of \( \partial/\partial t \), and \( \rho_M \) and \( P_M \) are the energy density and pressure of all perfect fluids of generic matter, respectively.

Equation (3.2) can be rewritten to

\[
H^2 - (F - 1) \left( H \frac{dH}{d\ln a} + H^2 \right) + \frac{1}{6} (f - R) + H^2 F' \frac{dR}{d\ln a} = \frac{\kappa^2 \rho_M}{3},
\]

while the scalar curvature \( R \) is expressed as

\[
R = 12H^2 + 6H \frac{dH}{d\ln a}.
\]

To solve Eqs. (3.4) and (3.5), we introduce the following variables [13]:

\[
y_H \equiv \frac{\rho_{DE}}{\rho_m^{(0)}} = \frac{H^2}{\bar{m}^2} - a^{-3} - \chi a^{-4},
\]

\[
y_R = \frac{R}{\bar{m}^2} - 3a^{-3},
\]

with

\[
\bar{m}^2 \equiv \frac{\kappa^2 \rho_m^{(0)}}{3},
\]

\[
\chi \equiv \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4},
\]

where \( \rho_{DE} \) is the energy density of dark energy and \( \rho_r^{(0)} \) is the energy density of radiation at the present time. In our analysis, the contribution from radiation is also taken into consideration. Equations (3.4) and (3.5) are reduced to a coupled set of ordinary differential equations

\[
\frac{dy_H}{d\ln a} = \frac{y_R}{3} - 4y_H,
\]

\[
\frac{dy_R}{d\ln a} = 9a^{-3} - \frac{1}{y_H + a^{-3} + \chi a^{-4}} \frac{1}{\bar{m}^2 F'}
\times \left[ y_H - (F - 1) \left( \frac{1}{6} y_R - y_H - \frac{1}{2} a^{-3} - \chi a^{-4} \right) + \frac{1}{6} \frac{f - R}{\bar{m}^2} \right].
\]
The equation of state for dark energy $w_{\text{DE}} \equiv P_{\text{DE}}/\rho_{\text{DE}}$ is given by
\[ w_{\text{DE}} = -1 - \frac{1}{3} \frac{1}{y_H} \frac{d y_H}{d \ln a}, \] (3.12)
derived by the continuity equation
\[ \dot{\rho}_{\text{DE}} + 3H (1 + w_{\text{DE}}) \rho_{\text{DE}} = 0. \] (3.13)
on the other hand, the effective equation of state $w_{\text{eff}}$ is defined as
\[ w_{\text{eff}} \equiv -\frac{1}{3} \frac{\dot{H} H}{H^2} = \frac{P_{\text{tot}}}{\rho_{\text{tot}}}, \] (3.14)
where $\rho_{\text{tot}} \equiv \rho_{\text{DE}} + \rho_{m} + \rho_{r}$ and $P_{\text{tot}} \equiv P_{\text{DE}} + P_{m} + P_{r}$ are the total energy density and pressure of the universe, respectively. Here, $P_{\text{DE}}, P_{m} (= 0)$ and $P_{r}$ are the pressure of dark energy, non-relativistic matter and radiation, respectively.

Combining Eqs. (3.10) and (3.11), we obtain
\[ \frac{d^2 y_H}{d (\ln a)^2} + J_1 \frac{d y_H}{d \ln a} + J_2 y_H + J_3 = 0, \] (3.15)
where
\[ J_1 = 4 + \frac{1}{y_H + a^{-3} + \chi a^{-4} 6 \bar{m}^2 F'}, \] (3.16)
\[ J_2 = \frac{2 - F}{y_H + a^{-3} + \chi a^{-4} 3 \bar{m}^2 F'}, \] (3.17)
\[ J_3 = -3a^{-3} - \frac{(1 - F)(a^{-3} + 2\chi a^{-4}) + (R - f) / (3 \bar{m}^2)}{y_H + a^{-3} + \chi a^{-4}} \frac{1}{6 \bar{m}^2 F'}. \] (3.18)

In Figs. 1, 2 and 3, we depict the cosmological evolutions of the density parameters of dark energy $\Omega_{\text{DE}} \equiv \rho_{\text{DE}}/\rho_{\text{crit}}^{(0)}$, non-relativistic matter $\Omega_{m} \equiv \rho_{m}/\rho_{\text{crit}}^{(0)}$ and radiation $\Omega_{r} \equiv \rho_{r}/\rho_{\text{crit}}^{(0)}$ as functions of the redshift $z \equiv 1/a - 1$ for $\beta = 1.1$, $\beta = 1.8$ and $\beta = 2.5$, respectively. In the high $z$ regime ($z \gtrsim 3.0$), the universe is at the matter-dominated stage ($\Omega_{m} > \Omega_{\text{DE}}, \Omega_{m} \gg \Omega_{r}$). As $z$ decreases, the dark energy becomes dominant over matter for $z < z_{\text{DE}}$, where $z_{\text{DE}}$ is the crossover point in which $\Omega_{\text{DE}} = \Omega_{m}$. Explicitly, we have $z_{\text{DE}} = 0.55, 0.47$ and $0.45$ for $\beta = 1.1, 1.8$ and $2.5$, respectively. The values of $z_{\text{DE}}$ become smaller for the larger values of $\beta$. At the present time ($z = 0$), $(\Omega_{\text{DE}}^{(0)}, \Omega_{m}^{(0)}, \Omega_{r}^{(0)}) = (0.77, 0.23, 7.0 \times 10^{-5}), (0.76, 0.24, 7.3 \times 10^{-5})$ and $(0.75, 0.25, 7.3 \times 10^{-5})$ for $\beta = 1.1, 1.8$ and $2.5$, respectively.

In Fig. 4, we also show the cosmological evolution of $\Omega_{r}$ for $\beta = 1.8$. The qualitative behaviors of $\Omega_{r}$ for $\beta = 1.1$ and $2.5$ are similar to that for $\beta = 1.8$. Thus, the current
accelerated expansion of the universe following the matter-dominated stage can be realized in the exponential gravity.

We note that in solving Eq. (3.18) numerically, we have taken the initial conditions at $z = z_1$ as $y_H (z = z_1) = 3.0$ and $dy_H/d\ln a (z = z_1) = 0$, where $z_1 = 4.0, 3.5$ and $3.0$ for $\beta = 1.1$, $\beta = 1.8$ and $\beta = 2.5$, respectively. The values of $z_1$ have been chosen so that $RF'(z = z_1) \sim 10^{-13}$, i.e., the exponential gravity at $z = z_1$ can be very close to the $\Lambda$CDM model, in which $RF' = 0$. Since $R/R_s \gg 1$ in the high $z$ regime ($z \simeq z_1$), $\beta R_s/\bar{m}^2 \simeq 6y_H$. Consequently, the value of the combination $\beta R_s$ is set as $\beta R_s \simeq 18H_0^2\Omega_m^{(0)}$. Therefore, we have only one free parameter $\beta$ in the exponential gravity in Eq. (2.3). Furthermore, from Eq. (3.10) we see that $y_R = 12y_H$ at $z = z_1$ and it follows from Eq. (3.12) that $w_{DE} = -1$ at
$z = z_i$. All numerical calculations have been executed for $\Omega_m(0) = 0.26^{[2]}$.

The cosmological evolution of the equation of state for dark energy $w_{DE}$ in Eq. (3.12) is shown in Fig. 5. From the figure, we see that $w_{DE}$ starts at the phase of a cosmological constant $w_{DE} = -1$ and evolves from the phantom phase ($w_{DE} < -1$) to the non-phantom (quintessence) phase ($w_{DE} > -1$). The crossing of the phantom divide occurs at $z = z_{cross}$, where $z_{cross} = 0.78, 0.57$ and $0.46$ for $\beta = 1.1, 1.8$ and $2.5$, respectively. The values of $z_{cross}$ become smaller for the larger values of $\beta$. Moreover, the present values of $w_{DE}$ are $w_{DE}(z = 0) = -0.85, -0.93$ and $-0.97$ for $\beta = 1.1, 1.8$ and $2.5$, respectively. Since $\beta R_s$ is a constant, the larger $\beta$ is, the closer the exponential gravity is to the $\Lambda$CDM model. The results on $w_{DE}$ are qualitatively the same as the analysis in Refs. [26, 28]. Thus, the crossing of the phantom divide from the phantom phase to the non-phantom one can be
FIG. 3: Legend is the same as Fig. 1 but for $\beta = 2.5$.

realized in the exponential gravity. We remark that the similar behaviors can occur in Hu-Sawicki [13, 42], Appleby-Battye [43], and Starobinsky’s [44] models as well.

In Fig. 6, we also illustrate the cosmological evolution of the effective equation of state $w_{\text{eff}}$ in Eq. (3.14). The present values of $w_{\text{eff}}$ are $w_{\text{eff}}(z = 0) = -0.65, -0.71$ and $-0.74$ for $\beta = 1.1, 1.8$ and $2.5$, respectively. We remark that $w_{\text{eff}}$ does not cross the line of the phantom divide unlike $w_{\text{DE}}$ due to the null energy condition $\rho_{\text{tot}} + P_{\text{tot}} = \rho_{\text{DE}} + \rho_{\text{m}} + \rho_{\text{r}} + P_{\text{DE}} + P_{\text{m}} + P_{\text{r}} \geq 0$.

Finally, we mention that an $f(R)$ gravity model with realizing a crossing of the phantom divide from the non-phantom phase to the phantom one, which is the opposite transition from the above one, has been reconstructed in Ref. [45]. In addition, the behavior of $f(R)$ gravity with realizing multiple crossings of the phantom divide [46] and that of $f(R)$ gravity
FIG. 4: Cosmological evolution of $\Omega_r$ (solid line) as a function of the redshift $z$ for $\beta = 1.8$.

around a crossing of the phantom divide by taking into account the presence of cold dark matter have also been explored.

IV. HORIZON ENTROPY

In Ref. [48], it is shown that it is possible to obtain a picture of equilibrium thermodynamics on the apparent horizon in the FLRW background for $f(R)$ gravity as well as that of non-equilibrium thermodynamics due to a suitable redefinition of an energy momentum tensor of the “dark” component that respects a local energy conservation. For a recent review on the Black hole entropy on scalar-tensor and $f(R)$ gravity, see Ref. [49].

In general relativity, the Bekenstein-Hawking horizon entropy is expressed as $S = \ldots$
FIG. 5: Cosmological evolution of $w_{DE}$ in Eq. (3.12) as a function of the redshift $z$ for $\beta = 1.1$ (dashed line), $\beta = 1.8$ (thick solid line) and $\beta = 2.5$ (dashed and single-dotted line), where the thin solid line shows $w_{DE} = -1$ (cosmological constant).

$A/(4G)$, where $A$ is the area of the apparent horizon [50-52]. The Bekenstein-Hawking entropy

$$S = \frac{A}{4G}$$  \hspace{1cm} (4.1)

is a global geometric quantity which is proportional to the horizon area $A$ with a constant coefficient $1/(4G)$. This quantity is not directly affected by the difference of gravitational theories. We regard the horizon entropy $S$ in Eq. (4.1) as the one in the equilibrium description [48]. On the other hand, in the context of modified gravity theories including $f(R)$ gravity a horizon entropy $\hat{S}$ associated with a Noether charge has been proposed by Wald [53]. The Wald entropy $\hat{S}$ is a local quantity defined in terms of quantities on the
bifurcate Killing horizon. More specifically, it depends on the variation of the Lagrangian density of gravitational theories with respect to the Riemann tensor. This is equivalent to \( \hat{\mathcal{S}} = A/(4G_{\text{eff}}) \), where \( G_{\text{eff}} = G/F \) is the effective gravitational coupling in \( f(R) \) gravity [54]. Therefore, we use the Wald entropy in the exponential gravity in Eq. (2.3)

\[
\hat{S} = \frac{(1 - \beta e^{-R/R_s}) A}{4G}.
\]

In what follows, a hat denotes the quantity in the non-equilibrium description of thermodynamics.

It can be shown that the horizon entropy \( S \) in the equilibrium description has the following

\( \text{FIG. 6: Cosmological evolution of } w_{\text{eff}} \text{ in Eq. (3.14) as a function of the redshift } z \text{ for } \beta = 1.1 \) (dashed line), \( \beta = 1.8 \) (thick solid line) and \( \beta = 2.5 \) (dashed and single-dotted line).
relation with \( \hat{S} \) in the non-equilibrium description \[48\]:

\[
dS = \frac{1}{1 - \beta e^{-R/R_s}} d\hat{S} + \frac{1}{1 - \beta e^{-R/R_s}} \frac{2H^2 + \dot{H}}{4H^2 + H} d_i\hat{S},
\]

where \( d_i\hat{S} \) is the new term which can be interpreted as a term of entropy produced in the non-equilibrium thermodynamics. The difference between \( S \) and \( \hat{S} \) appears in \( f(R) \) gravity due to \( dF \neq 0 \). Note that \( S \) is identical to \( \hat{S} \) in general relativity due to \( F = 1 \). From Eq. (4.3), we see that the change of the horizon entropy \( S \) in the equilibrium framework involves the information of both \( d\hat{S} \) and \( d_i\hat{S} \) in the non-equilibrium framework.

In Figs. 7, 8 and 9, we show the cosmological evolution of the horizon entropy \( \hat{S} \) in Eq. (4.2) in the non-equilibrium description of thermodynamics and \( S \) in Eq. (4.1) in the equilibrium description of thermodynamics for \( \beta = 1.1, 1.8 \) and 2.5, respectively. In these figures, we illustrate the normalized quantities \( \tilde{S} \equiv \hat{S}/S_0 \) and \( \bar{S} \equiv S/S_0 \) with \( S_0 = \pi/(GH_0^2) \) being the present value of the horizon entropy \( S \). Furthermore, we also depict the evolution of \( \bar{H} \equiv H/H_0 \). We note that as \( S \propto H^{-2} \), \( S \) increases with time as long as \( H \) continues to decreases to the de Sitter point, in which \( H \) becomes a constant.

In the high \( z \) regime (\( z \gtrsim 1 \)), since the deviation of the exponential gravity from the \( \Lambda \)CDM model, i.e., general relativity, is very small, the evolution of \( S \) is similar to that of \( \hat{S} \). In other words, for the high \( z \) regime (the higher curvature regime) \( F(R) = 1 - \beta e^{-R/R_s} \approx 1 \) because \( R/R_s \gg 1 \). As \( z \) decreases (and \( R \) also decreases), the deviation of the exponential gravity from the \( \Lambda \)CDM model emerges, i.e., \( F(R) < 1 \) and \( F(R) \) decreases. Hence, there appears a difference between the evolution of \( S \) and that of \( \hat{S} \). Note that \( S > \hat{S} \propto F(R) \).

The present values of \( \tilde{S} \) are \( \tilde{S}(z = 0) = 0.90, 0.96 \) and 0.99 for \( \beta = 1.1, 1.8 \) and 2.5, respectively. It is clear from Figs. 7, 8 and 9 that both \( S \) and \( \hat{S} \) globally increases with time for any values of \( \beta \). This confirms that the second law of thermodynamics on the apparent horizon always holds. The similar behaviors for both \( S \) and \( \hat{S} \) have been obtained in the Starobinsky’s model \[48\]. Furthermore, we see that the larger \( \beta \) is, the closer the evolution of \( \tilde{S} \) is to that of \( \bar{S} \).
FIG. 7: Cosmological evolutions of $\bar{S} = S/S_0$ (solid line), $\tilde{S} = \dot{S}/S_0$ (dashed line) and $\bar{H} = H/H_0$ (dashed and single-dotted line) as functions of the redshift $z$ for $\beta = 1.1$.

V. CONCLUSIONS

In the present paper, we have studied the cosmological evolution in the exponential gravity. We have summarized various viability conditions and explicitly illustrated that the late-time cosmic acceleration can be realized after the matter-dominated stage. We have also shown that the crossing of the phantom divide from the phantom phase to the non-phantom one can occur and the cosmological horizon entropy globally increases with time. Phenomenologically, at least in the light of the background cosmological evolution, the exponential gravity can be regarded as one of the most promising viable modified gravitational theories because (a) it satisfies all conditions for the viability; (b) in substance it
has only one model parameter; and (c) both the current cosmic acceleration following the matter-dominated stage and the crossing of the phantom divide can be realized.

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FIG. 9: Legend is the same as Fig. 7 but for $\beta = 2.5$.

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