We present a first example of an integrable (3+1)-dimensional dispersionless system with nonisospectral Lax pair involving algebraic, rather than rational, dependence on the spectral parameter, thus showing that the class of integrable (3+1)-dimensional dispersionless systems with nonisospectral Lax pairs is significantly more diverse than it appeared before. The Lax pair in question is of the type recently introduced in [A. Sergeyev, Lett. Math. Phys. 108 (2018), no. 2, 359-376, arXiv:1401.2122].

Introduction

Integrable systems are well known to play an important role in modern mathematics and physics, cf. e.g. [1] [6] [8] [9] [10] [11] [12] [13] [14] [16] [17] [18] [19] [21] [23] [24] [25] [26]. According to Einstein’s general relativity our spacetime is four-dimensional, so the search for (3+1)-dimensional integrable systems, that is, integrable partial differential systems in four independent variables, is quite naturally of particular significance, see e.g. [9] [16] [19] [24]; also cf. e.g. [22] on peculiarities of 4-manifolds. As discussed e.g. in [19], most of integrable (3+1)-dimensional systems known to date are dispersionless, which, roughly speaking, means that they can be written as first-order quasilinear homogeneous systems.

While a fairly large number of (3+1)-dimensional integrable systems is presently known, see e.g. [16] [18] [19] and references therein, the breadth of their class yet remains to be fully understood. In 2+1 dimensions there are many examples of dispersionless integrable systems whose nonisospectral Lax pairs involve highly sophisticated dependence on the spectral parameter, including, for example, Weierstrass $\wp$-functions, cf. e.g. [11], but, to the best of our knowledge, for all previously known examples of (3+1)-dimensional integrable dispersionless systems in finitely many dependent variables with nonisospectral Lax pairs the latter are polynomial or rational in the spectral parameter, which begets the question of whether more sophisticated nonisospectral Lax pairs could exist in 3+1 dimensions.

We answer this question in the affirmative by presenting in Section 2 below system (5) which is, as far as the present author is aware, the first example of an integrable (3+1)-dimensional dispersionless system in finitely many dependent variables with a nonisospectral Lax pair being algebraic, rather than merely rational, in the spectral parameter, which shows that nonisospectral dispersionless Lax pairs in 3+1 dimensions are significantly more diverse than it appeared before. The example in question is found within the framework of a new systematic construction for (3+1)-dimensional integrable systems related to contact geometry, see [19], the follow-up papers [4] [20], and a brief review in Section 1 below.


# 1 Preliminaries

Recall that a *dispersionless* or *hydrodynamic-type* system in four independent variables \(x, y, z, t\) is, cf. e.g. [10, 11, 19, 21] and references therein, a system that can be written in the form

\[
A_0(u)u_t + A_1(u)u_x + A_2(u)u_y + A_3(u)u_z = 0,
\]

where \(u = (u_1, \ldots, u_N)^T\) is an \(N\)-component vector of unknown functions of \(x, y, z, t\), \(A_i\) are \(M \times N\)-matrix-valued functions of \(u\), \(M\) and \(N\) are nonzero nonnegative integers such that \(M \geq N\), and the superscript \(T\) indicates the transposed matrix.

Here and below all functions are assumed sufficiently smooth for all computations to make sense. This can be readily formalized using the language of differential algebra, cf. e.g. [7, 19], and references therein.

There exist [19] infinitely many *integrable* dispersionless systems of the general form (1) that admit, for suitable Lax functions \(f = f(p, u)\) and \(g = g(p, u)\), nonisospectral Lax pairs of the form introduced in [19] and intimately related to contact geometry,

\[
\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),
\]

where \(\chi = \chi(x, y, z, t, p)\), and \(p\) is the so-called variable spectral parameter (note that \(u_p \equiv 0\)).

For any \(h = h(p, u)\) the operator \(X_h\) is defined as

\[
X_h = h_p \partial_x + (ph_z - h_x) \partial_p + (h - ph_p) \partial_z
\]

and formally looks exactly like the contact vector field with a contact hamiltonian \(h\) on a contact 3-manifold with local coordinates \(x, z, p\) and contact one-form \(dz + pdx\); cf. e.g. [3, 5] and references therein for more details on contact geometry.

In particular, for any natural \(m\) and \(n\) the pairs of Lax functions

\[
f = p^{n+1} + \sum_{j=0}^{n} v_j p^j, \quad g = p^{m+1} + \frac{m}{n} v_n p^m + \sum_{k=0}^{m-1} w_k p^k
\]

and

\[
f = \sum_{j=1}^{m} a_j / (v_i - p), \quad g = \sum_{k=1}^{n} b_k / (w_k - p)
\]

yield [19, 20] (3+1)-dimensional integrable systems of the general form (1) with \(M = N\) for \(u = (v_0, \ldots, v_n, w_0, \ldots, w_{m-1})^T\) and \(u = (a_1, \ldots, a_m, b_1, \ldots, b_n, v_1, \ldots, v_m, w_1, \ldots, w_n)^T\) respectively.

Notice [19] that if \(u_z = 0\) and \(\chi_z = 0\) then the Lax pairs (2) boil down to well-known (2+1)-dimensional Lax pairs involving Hamiltonian (rather than contact) vector fields, cf. e.g. [11, 26] and references therein for the associated (2+1)-dimensional integrable systems.

**Proposition 1** ([19]). A system (1) admits a linear Lax pair of the form (2) if and only if it admits a nonlinear Lax pair for \(\psi = \psi(x, y, z, t)\) of the form

\[
\psi_y = \psi_z f(\psi_x / \psi_z, u), \quad \psi_t = \psi_z g(\psi_x / \psi_z, u)
\]

with the same functions \(f\) and \(g\) as in (2).

In closing note that systems of the form (1) belong to a broader class of multitime Hamilton–Jacobi systems, cf. e.g. [15] for more details on the latter.
2 New integrable system with algebraic Lax pair

Consider the following (3+1)-dimensional evolutionary system for \( \mathbf{u} = (a, b, r, s, u, v, w)^T \):

\[
\begin{align*}
a_t &= \frac{1}{r^2 - 2rsa + 2s^2b} \left( (4w(ra - sb) - vr)a_x + ra_y \\
&\quad + (2w(2a(ra - sb) - rb) - ur)a_z \\
&\quad + (vs - 2wr)b_x - sb_y + (2w(sb - ra) + us)b_z \\
&\quad + (r - sa)u_x + (ra - 2sb)u_z + (2s - ra)v_x \\
&\quad + 2(a(sb - ra) + rb)v_z \\
&\quad + 2(2a^2(ar - sb) - 3rab + 2s^2b^2)w_z \right),
\end{align*}
\]

\[
\begin{align*}
b_t &= \frac{1}{r^2 - 2rsa + 2s^2b} \left( (2wr - vs)ba_x + 2sba_y \\
&\quad + 2(2w(ra - sb) - us)ba_z \\
&\quad + (2s(2a - wb) - wr)b_x + (r - 2sa)b_y \\
&\quad + (2s(b - a^2) + ra)u_x + (2r - sa)bu_z \\
&\quad - 2(r - sa)bu_x - 2(ra - 2sb)bu_z \\
&\quad + 2(ra - 2sb)bw_x + 2a(ra - sb - rb)bw_z \right),
\end{align*}
\]

\[
\begin{align*}
r_t &= \frac{1}{r^2 - 2rsa + 2s^2b} \left( (vs - 2wr)ra_x - rsa_y \\
&\quad - (2w(ra - sb) - us)ra_z \\
&\quad + (2wr - vs)sb_x + s^2b_y + (wr^2 - us^2)b_z \\
&\quad + (sa - r)su_x + (2s - ra)su_z + (r - sa)rv_x + (ra - 2sb)rv_z \\
&\quad + (2s - ra)rw_x - 2(a(ra - sb) - rb)rw_z \right),
\end{align*}
\]

\[
\begin{align*}
s_t &= w_x + aw_z + wa_z, \\
u_t &= ar_x + 2br_z - sb_z, \\
v_t &= r_x + ar_z + as_x + 2bs_z - sa_x + sb_z, \\
w_t &= s_x + as_z + sa_z.
\end{align*}
\]

**Theorem 1.** The (3+1)-dimensional seven-component evolutionary system [5] is integrable since it admits a Lax pair [2] with algebraic Lax functions \( f \) and \( g \) given by

\[
\begin{align*}
f &= u + vp + wp^2 + (r + sp)\sqrt{p^2 + 2ap + 2b}, \\
g &= \sqrt{p^2 + 2ap + 2b},
\end{align*}
\]
that is,

\[
\chi_y = \frac{1}{g} \left( (2sp^2 + (r + 3sa + 2wg)p + ra + vg + 2sb)\chi_x \\
+ (-sp^3 - (wg + sa)p^2 + pra + 2rb + ug)\chi_z \\
+ (s_zp^4 + (2as_x + sa_x + r_x + gw_x - s_x)p^3 \\
+ ((v_x - w_x)g + 2bs_x + ra_x + sb_x - 2as_x - sa_x - r_x + 2ar_x)p^2 \\
+ ((u_x - v_x)g + rb_x - ra_x - sb_x - 2bs_x - 2ra_x + 2br_x)p \\
- rb_x - gu_x - 2br_x\chi_p) \right),
\]

\[
\chi_t = \frac{1}{g} \left( (p + a)\chi_x + (ap + 2b)\chi_z + (a_zp^2 + p(b_z - a_x) - b_x)\chi_p) \right),
\]

and a nonlinear Lax pair of the form (4) with \( f \) and \( g \) given by (6), that is,

\[
\psi_y = uv\psi_z + v\psi_x + w\psi_x^2/\psi_z + (r\psi_x + s\psi_x)\sqrt{(\psi_x/\psi_z)^2 + 2a\psi_x/\psi_z + 2b},
\]

\[
\psi_t = \psi_z\sqrt{(\psi_x/\psi_z)^2 + 2a\psi_x/\psi_z + 2b};
\]

both of the Lax pairs (7) and (8) are expressed in terms of algebraic, rather than rational, functions.

Proof. By definition, proving that (5) admits (7) and (8) amounts to proving that both (7) and (8) are compatible by virtue of (5).

First of all, by Proposition 1 compatibility of (7) by virtue of (5) implies the same for (8).

Next, by Proposition 1 of [19], in order to prove that (7) is compatible by virtue of (5), it suffices to show that for \( f \) and \( g \) given by (6) the system (5) implies the equation

\[
f_t - g_y + \{f, g\} = 0
\]

where the contact bracket \( \{,\} \) is defined [19] as

\[
\{f, g\} = f_g - f_x g_p - p(f_g - g_{pz}) + f_{gz} - g_{f_z}.
\]

Substituting (6) into (9) we readily see that this is indeed the case, and the result follows.

3 Conclusions

We have presented above a new integrable (3+1)-dimensional dispersionless evolutionary system (5) whose linear Lax pair (7) and nonlinear Lax pair (8) are expressed in terms of algebraic rather than rational functions.

To the best of our knowledge, this is the first example of an integrable (3+1)-dimensional dispersionless system in finitely many dependent variables admitting a nonisospectral Lax pair which is not rational in the variable spectral parameter. Indeed, as far as the present author is able to tell, for all previously known examples of integrable dispersionless (3+1)-dimensional systems in finitely many dependent variables with nonisospectral Lax pairs, including e.g. the equations for (anti-)self-dual four-dimensional conformal structures [10], the Dunajski equation, and systems (15), (17), (38), and (40) from [19], their Lax pairs have at most rational dependence on the variable spectral parameter.

We conjecture that (5) has no nontrivial linear or nonlinear Lax pairs written in terms of rational (rather than algebraic) functions.

In closing note that it could be of interest to study symmetries, conservation laws, Hamiltonian operators, and other related structures for (5) in spirit of [14][17].
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References

[1] M. Ablowitz, H. Segur, Solitons and the inverse scattering transform, SIAM, Phil., PA, 1981.
[2] H. Baran, M. Marvan, Jets. A software for differential calculus on jet spaces and diffieties, available online at http://jets.math.slu.cz
[3] D.E. Blair, Riemannian geometry of contact and symplectic manifolds. 2nd ed. Birkhäuser, Boston, MA, 2010.
[4] M. Błaszak, A. Sergyeyev, Contact Lax pairs and associated (3+1)-dimensional integrable systems, in Nonlinear Systems and Their Remarkable Mathematical Structures, vol. 2, CRC Press, submitted, [arXiv:1901.05181]
[5] A. Bravetti, Contact Hamiltonian dynamics: the concept and its use, Entropy 19 (2017), no. 10, art. 535, 12 pp.
[6] F. Calogero, Why are certain nonlinear PDEs both widely applicable and integrable?, in What is integrability?, Springer, Berlin, 1991, 1–62.
[7] A. De Sole, V.G. Kac, D. Valeri, A new scheme of integrability for (bi)Hamiltonian PDE, Comm. Math. Phys. 347 (2016), no. 2, 449–488, [arXiv:1508.02549]
[8] S. Dimas, I.L. Freire, Study of a fifth order PDE using symmetries, Appl. Math. Lett. 69 (2017), 121–125.
[9] B. Dorizzi, B. Grammaticos, A. Ramani, P. Winternitz, Are all the equations of the Kadomtsev–Petviashvili hierarchy integrable? J. Math. Phys. 27 (1986), no. 12, 2848–2852.
[10] M. Dunajski, E.V. Ferapontov, B. Kruglikov, On the Einstein-Weyl and conformal self-duality equations. J. Math. Phys. 56 (2015), no. 8, 083501, 10 pp.
[11] E.V. Ferapontov, A. Moro, V.V. Sokolov, Hamiltonian systems of hydrodynamic type in 2 + 1 dimensions. Comm. Math. Phys. 285 (2009), no. 1, 31–65.
[12] O.Ye. Hentosh, B.Yu. Kyshakhevych, D. Blackmore, A.K. Prykarpatski, New fractional nonlinear integrable Hamiltonian systems, Appl. Math. Lett. 88 (2019) 41–49.
[13] Y. Kodama, Dispersionless integrable systems and their solutions, in Integrability: the Seiberg–Witten and Whitham equations (Edinburgh, 1998), 199–212, Gordon & Breach, Amsterdam, 2000.
[14] J. Krasil’shchik, A.M. Verbovetsky, R. Vitolo, The symbolic computation of integrability structures for partial differential equations, Springer, Cham, 2017.
[15] P.-L. Lions, J.-C. Rochet, Hopf formula and multitime Hamilton-Jacobi equations, Proc. Amer. Math. Soc. 96 (1986), no. 1, 79–84.
[16] L.J. Mason, N.M.J. Woodhouse, Integrability, self-duality, and twistor theory, Clarendon & OUP, N.Y., 1996.
[17] P.J. Olver, Applications of Lie groups to differential equations, 2nd ed., Springer, N.Y., 1993.
[18] A. Sergyeyev, A simple construction of recursion operators for multidimensional dispersionless integrable systems. J. Math. Anal. Appl. 454 (2017), no. 2, 468–480, [arXiv:1501.01955]
[19] A. Sergyeyev, New integrable (3+1)-dimensional systems and contact geometry, Lett. Math. Phys. 108 (2018), no. 2, 359–376, [arXiv:1401.2122]
[20] A. Sergyeyev, Integrable (3+1)-dimensional systems with rational Lax pairs, Nonlin. Dynamics 91 (2018), no. 3, 1677–1680, [arXiv:1711.07395]
[21] D. Serre, Systèmes hyperboliques riches de lois de conservation, in Nonlinear partial differential equations and their applications. Collège de France Seminar, vol. XI (Paris, 1989–1991), Longman, Harlow, 1994, 248–281.
[22] C. Taubes, What we know and what we don’t know about 4-dimensions, in Introduction to modern mathematics, Int. Press, Somerville, MA, 2015, 391–408.
[23] O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Equivalence transformations in the study of integrability, Phys. Scr. 89 (2014), 038003, [arXiv:1308.5126]
[24] E. Witten, Searching for integrability, J. Geom. Phys. 8 (1992), no. 1-4, 327–334.
[25] J. Yang, Nonlinear waves in integrable and nonintegrable systems. SIAM, Phil., PA, 2010.
[26] V.E. Zakharov, Dispersionless limit of integrable systems in 2+1 dimensions, in Singular Limits of Dispersive Waves, ed. by N.M. Ercolani et al., pp. 165–174, Plenum Press, N.Y., 1994.