Abstract

Generalizing the octahedral configuration of six congruent cylinders touching the unit sphere, we exhibit configurations of congruent cylinders associated to a pair of dual Platonic bodies.
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1 Introduction

In this paper we study compounds of cylinders. By a compound $\kappa$ of cylinders we mean a configuration of infinite open nonintersecting right circular congruent cylinders in $\mathbb{R}^3$, touching the unit sphere $S^2$, such that some of the cylinders in $\kappa$ touch each other. We denote by $r(\kappa)$ the common radius of cylinders forming the compound.

Let $M^k$ be the manifold of $k$-tuples of straight lines

$$m = \{u_1, ..., u_k\}$$

tangent to the sphere $S^2$. The manifold $M^k$ has dimension $3k$. The space of compounds of $k$ cylinders can be conveniently identified with the manifold $M^k$: the straight line corresponding to a cylinder is its unique ruling touching the unit sphere. Therefore the space of compounds of $k$ cylinders is $3k$-dimensional.

For a configuration $m$ we denote by $d(m)$ the minimal distance between the lines of the configuration.

If $\kappa$ is a compound with tangent rulings $m$, then

$$r(\kappa) = \frac{d(m)}{2 - d(m)}.$$  \hspace{1cm} (1)

By a slight abuse of notation we allow the lines $u_i$ to intersect; then $r(\kappa) = d(m) = 0$.

For every $k \geq 3$ let us define the number

$$R_k = \max_{\kappa \in M^k} r(\kappa).$$

By definition, $R_k$ is the maximal possible radius of $k$ congruent nonintersecting cylinders touching $S^2$. It is easy to see that $R_3 = 3 + 2\sqrt{3}$, and a natural (open) conjecture is that $R_4 = 1 + \sqrt{2}$.

The compound $C_6$ of six unit parallel cylinders has appeared in one question of Kuperberg [K] in 1990. For quite some time the common belief was that $R_6 = r(C_6) = 1$, until M. Firsching [F] in 2016 has found by numerical exploration of the manifold $M^6$ a compound $\kappa_F \in M^6$ with $r(\kappa_F) \approx 1.049659$. That finding has triggered our interest in the problem,
and in the paper [OS] we have discovered a curve $C_6(t)$ in the compound space $M^6$, $C_6(0) = C_6$, along which the function $r(C_6(t))$ increases to its maximal value on this curve, $r_m = \frac{1}{8} \left(3 + \sqrt{33}\right) \approx 1.093070331$, taken at the compound $C_m$.

The compounds $C_6$ and $C_m$ are shown on Figure 1 (the green unit ball is in the center).

![Figure 1: Two compounds of cylinders: the compound $C_6$ of six parallel cylinders of radius 1 (on the left) and the compound $C_m$ of six cylinders of radius $\approx 1.0931$ (on the right)](image)

We have shown in [OS-C6] that in fact the compound $C_m$ is a point of local maximum (modulo global rotations) of the function $r$: for every compound $\mathcal{X}$ close to $C_m$ we have $r(\mathcal{X}) < r_m$.

We believe that in fact

$$R_6 = r_m = \frac{1}{8} \left(3 + \sqrt{33}\right).$$

For values of $k$ larger than 3 the quantities $R_k$ are unknown; at the end of this paper we present some lower bounds for few values of $k$.

In his paper Kuperberg has presented yet another compound, $O_6 \in M^6$ with the same radius $r(O_6) = 1$. 

The compound $O_6$ is shown on Figure 2. the central unit ball is again shown in green.

Contrary to $C_6$, the compound $O_6$ is indeed a point of local maximum (modulo global rotations) of the function $r$ : for every compound $\kappa$ close to $O_6$ one has $r(\kappa) < 1$. This was proved in our paper [OS-O6].

The configuration $O_6$ of tangent lines is shown on Figure 3.
Sometimes the configuration \(O_6\) is called ‘octahedral’ because, probably, the tangency points lie at the vertices of the regular octahedron. We present an interpretation of the configuration \(O_6\) which relates it to the configuration of rotated edges of the regular tetrahedron. Following this interpretation we introduce configurations of tangent lines for each pair of dual Platonic bodies, that is, for the tetrahedron/tetrahedron, octahedron/cube and icosahedron/dodecahedron; for the pair tetrahedron/tetrahedron this is precisely the locally maximal configuration \(O_6\). The number of lines in our configurations is equal to the number of edges of either of Platonic bodies in the pair, that is, twelve for the pair octahedron/cube and thirty for the pair icosahedron/dodecahedron.

We recall the definition of a critical configurations from [OS-O6]. We call a cylinder configuration critical if for each small deformation \(t\) that keeps the radii of the cylinders, either

(T1) some cylinders start to intersect, or else

(T2) the distances between all of them increase, but by no more than

\[
\sim \|t\|^2,
\]

or stay zero, for some.

Here \(\|t\|\) is the natural norm, see details in [OS-O6].

A critical configuration is called a locally maximal configuration if all its deformations are of the first type. Any other critical configuration is called a saddle configuration, and the deformations of type (T2) are then called the unlocking deformations.

For the pair octahedron/cube we find two possible candidates for the critical points; For the pair icosahedron/dodecahedron we find four possible candidates for the critical points.

The paper is organized as follows. In Section 2 we present an interpretation of the configuration \(O_6\) relating it to the edges of the tetrahedron. In Sections 3, 4 and 5 we describe configurations of tangent lines associated to a pair of dual Platonic bodies. We conclude the article by several conjectures concerning local maxima and critical points for the configurations related to the pairs octahedron/cube and icosahedron/dodecahedron.
2 Interpretation of the configuration $O_6$: $A_4$-symmetric configurations

We shall now give an interpretation of the configuration $O_6$ which shows that it rather deserves to be called the tetrahedral configuration.

There is a family of configurations possessing the $A_4$ symmetry ($A_4$ is the group of proper symmetries of the regular tetrahedron, alternating group on four letters). We start with the configuration of the tangent to the unit sphere lines which are continuations of the edges of the regular tetrahedron. The points of the sphere at which tangent lines pass are the edge middles of the regular tetrahedron. The initial position of the edges of the tetrahedron are shown in blue on Figure 4.

Then each edge is rotated around the diameter of the unit sphere, passing through the middle of the edge, by an angle $\delta$. On Figure 4 the point $A$ (in green) is the middle of the edge $UV$. The line, passing through the point $A$ and rotated by the angle $\delta$, is shown in red. The lines passing through other middles of edges are rotated according to $A_4$ symmetry. The positions of other lines are well defined because the stability subgroup of the edge $UV$ of the regular tetrahedron is generated by the rotation by angle $\pi$ around the diameter of the sphere passing through the point $A$ and, clearly, the red line on Figure 4 is invariant under this rotation.

The minimum of the squares of the distances between the lines is given
by the formula

\[ d^2 = \frac{16T^2}{(3T^2 + 1)(T^2 + 3)} \equiv -\frac{4\sin^2(2\delta)}{(\cos(2\delta) - 2)(\cos(2\delta) + 2)}. \quad (2) \]

Here \( T = \tan(\delta) \). When \( \delta \) changes from 0 to \( \pi/2 \), the value of \( d^2 \) increases, achieves the maximal value 1 for \( \delta_0 = \pi/4 \), and afterwards decreases to 0, see Figure 5.

![Figure 5: Graph of \( d^2(T) \)](image)

For \( \delta \neq 0, \pi/4, \pi/2 \) the symmetry of the configuration is \( A_4 \). For \( \delta = 0 \) the symmetry group is \( S_4 \), the full symmetry group of the regular tetrahedron. For \( \delta = \pi/4 \) the configuration is centrally symmetric, its symmetry group is \( A_4 \times C_2 \), where the cyclic group \( C_2 \) is generated by the central reflection \( I \). For \( \delta = \pi/4 \) this is exactly the configuration \( O_6 \). For \( \delta = \pi/2 \) the tangent lines become again the continuations of the edges of the regular tetrahedron.

The convex span of the edge middles of the regular tetrahedron is the regular octahedron. Our above observation boils down to the fact that rotating the edges in the plane tangent to the mid-sphere (the sphere which touches the middle of each edge of \( P \)) we obtain the configuration \( O_6 \) when the angle of rotation is equal to \( \pi/4 \).

The following pictures of models made with the help of a tensegrity kit (Figure 6) and its Martian version (Figure 7) might be helpful for understanding.
Figure 6: Tensegrity models. The rotating edges of the tetrahedron (left). The position of edges at angle $\pi/4$ is the configuration $O_6$ (right).

Figure 7: Martian version (located in Woking, the hometown of H. G. Wells) of the rotating upper three edges of the tetrahedron, view from the center of the unit sphere.
3 Generalizations of the configuration $O_6$: 
\[ \delta \)-rotation process

We denote by $T$, $O$ and $I$, respectively, the dual pairs tetrahedron/tetrahedron, octahedron/cube and icosahedron/dodecahedron of Platonic bodies. Let $N_T = 6$, $N_O = 12$ and $N_I = 30$ be the number of edges of a Platonic body in the corresponding pair. Finally let $G_O \cong S_4$ and $G_I \cong A_5$ be the proper symmetry groups of the dual pairs octahedron/cube and icosahedron/dodecahedron.

A construction, similar to the one described in Section 2, works for each pair of dual Platonic bodies. Namely, let a unit sphere touch the edge middles of a Platonic body $P$. Rotate the edges of $P$ around the axes passing through the center of the sphere and tangency points. By the same reason as for the tetrahedron, it can be done preserving the proper symmetry group of $P$; spelled differently, we rotate each edge in the same direction, say, counterclockwise, viewed from the tip of the axis, passing through the center of the sphere and the tangency point. When the angle $\delta$ of the rotation reaches $\pi/2$ the edges form the configuration of edges of the dual to $P$ Platonic body.

In the subsequent Sections we investigate the details of this ‘$\delta$-rotation’ process for the remaining pairs $O$ and $I$.

3.1 Neighboring tangent lines

For a Platonic body $P$, assume that the symmetry axis $a_v$ passing through the vertex $v$ of $P$ is $k$-fold. By ‘neighboring’ edges of $P$ we mean edges which share a vertex $v$ and can be obtained from one another by a rotation through the angle $2\pi/k$ around the axis $a_v$. We keep the name ‘neighboring’ for the tangent lines obtained from the neighboring edges in the $\delta$-rotation process. Note that this notion does not depend on a concrete Platonic body in a pair: neighboring tangent lines for a Platonic body $P$ remain neighboring for the dual of $P$.

For some value of $\delta$ the distance between the neighboring tangent lines takes the maximal value.

We now present the maximal values $d_O$ and $d_I$ between the neighboring tangent lines for the pairs $O$ and $I$. 

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3.1.1 Pair $O$

For the pair octahedron/cube, let the octahedron correspond to $\delta = 0$.

**Theorem 1** The maximum distance in the $\delta$-rotation process for the pair $O$ is achieved at the angle $\delta_O$ such that

$$\tan(\delta_O) = \frac{3^{1/4}}{\sqrt{2}}$$

(approximately, $\delta_O \simeq 0.23856\pi \approx 0.74946$). The square of the maximal distance between the tangent lines is the value of the function

$$-\frac{4\sin^2(2\delta)}{(\cos(2\delta) - 3)(\cos(2\delta) + 5)}$$

at $\delta = \delta_O$. It is equal to

$$d_O^2 = 2 - \sqrt{3} = \frac{(-1 + \sqrt{3})^2}{2} \approx 0.26795.$$  \hspace{1cm} (5)

**Proof.** The square of the distance between the rotated neighboring lines is given by the formula (10) which reduces to (4) for the octahedron.

A straightforward but more lengthy verification shows that in the configuration obtained by the $\delta$-rotation through the angle $\delta_O$ all other (non-neighboring) pairwise distances between the tangent lines are bigger than $d_O$ so the other pairs of corresponding cylinders do not intersect. See Section 4.1 for details. ■

The corresponding radius of touching cylinders, see formula (1), is

$$r_O = \frac{\sqrt{3} - 1}{1 + 2\sqrt{2} - \sqrt{3}} = 7 - \sqrt{2} - 4\sqrt{3} + 3\sqrt{6} \approx 0.3492.$$  \hspace{1cm} (6)

The resulting compound, at the angle $\delta_O$, of the twelve cylinders of radius $r_O$ is shown on Figures 8 and 9.

**Remark.** Let $C_{12}$ be the configuration of twelve vertical equidistant lines tangent to the unit sphere. We denote by the same symbol $C_{12}$ the corresponding configuration of touching vertical cylinders. In [OS] we have proved that the configuration $C_{12}$ is unlockable along a curve, which keeps the $D_6$ symmetry of the configuration, in the moduli space. It is interesting to know if the maximal point on this curve beats the radius (6) of cylinders.
Figure 8: Octahedron/cube maximal configuration of cylinders, view from a vertex of the cube

Figure 9: Octahedron/cube maximal configuration of cylinders, view from a vertex of the octahedron
3.1.2 Pair I

For the pair icosahedron/dodecahedron, let the icosahedron correspond to \( \delta = 0 \). Then the maximum of the square of distances between the neighboring edges is achieved at the angle \( \delta_I \) such that

\[
\tan(\delta_I) = \left( \frac{6}{5 + \sqrt{5}} \right)^{1/4}
\]

(approximately, \( \delta_I \approx 0.24255 \pi \)). The square of the maximal distance between the neighboring tangent lines is the value of the function

\[
d^2 = -\frac{4 \sin^2(2\delta)}{(\cos(2\delta) - (4 + \sqrt{5})) \cdot (\cos(2\delta) - (1 - 2\sqrt{5})\tau^3)}
\]

at \( \delta = \delta_I \). Here \( \tau \) is the golden ratio,

\[
\tau = \frac{1 + \sqrt{5}}{2}
\]

**Remark.** Compare the golden integers, appearing in the denominator of (7) with the numbers \( \beta_{11} \) and \( \gamma_{-19} \), see Section 4.1 in [OS-C6], which enter the formulas for the differentials of distances around the configuration \( C_6 (\varphi_m, \delta_m, \kappa_m) \). This coincidence hints at some hidden relation between the configuration \( C_6 (\varphi_m, \delta_m, \kappa_m) \) and the exceptional Platonic bodies (the icosahedron and dodecahedron).

The value of the function (7) at \( \delta = \delta_I \) is

\[
d_I^2 = \frac{2\sqrt{6(5 + \sqrt{5})}}{75 + 33\sqrt{5} + 12\sqrt{6(5 + \sqrt{5})} + 5\sqrt{30(5 + \sqrt{5})}} = \frac{9 - \sqrt{5} - \sqrt{6(5 + \sqrt{5})}}{4} = \left( \frac{5^{1/4}\sqrt{3\tau} - \tau}{2\tau} \right)^2 \approx 0.0437 .
\]

The corresponding radius of touching neighboring cylinders is

\[
r_I = 11 - 5\sqrt{5} + \sqrt{3\left(85 - 38\sqrt{5}\right)} \approx 0.1167 .
\]
This compound of cylinders is shown on Figure 10.

In contrast to the pair $\mathcal{O}$, there are non-neighboring tangent lines at a smaller (than for the neighboring tangent lines) distance; see the overlapping blue cylinders on Figure 10. The final result concerning the maximal point for the pair $\mathcal{I}$ is contained in Subsection 4.2.

![Figure 10: Configuration of cylinders of radius $r_I$ at $\delta = \delta_I$](image)

The square of the minimal distance at the angle $\delta_I$ is approximately $0.00291762$ which corresponds to a much smaller cylinder radius, equal approximately to 0.0277571, compare with (8) and (9).

A more detailed analysis, given in Subsection 4.2 of the $\delta$-rotation pro-
cess, shows that the actual radius of cylinders at the maximal point is not so much smaller than the one given by the expression (9); the maximal radius is approximately 0.115558.

Remark. It is interesting to note that the values $d^2_1$ and $r_I$, see formulas (8) and (9), define the same extension of the field of rational numbers, $\mathbb{Q}[d^2_1] \cong \mathbb{Q}[r_I]$. Indeed,

$$r_I = 12(d^2_1)^3 - 62(d^2_1)^2 + 74d^2_1 - 3.$$ 

It is not so for the pair $O$, see formulas (5) and (6).

### 3.2 Comment about calculations in Subsection 3.1

To find the values $\delta_O$ and $\delta_I$, we place a vertex of a Platonic body $P$ at the point $(0, 0, h)$, $h > 1$, on the $z$-axis. We draw the line $\iota$ in the plane $xz$, passing through the point $(0, 0, h)$ and touching the unit sphere at the point with a positive $x$-coordinate. Then the tangency point is

$$p := \left(\frac{\sqrt{h^2 - 1}}{h}, 0, \frac{1}{h}\right).$$

Let $\ell_1$ be the line, tangent to the unit sphere at the point $p$ and forming the angle $\delta$ with the line $\iota$. Let $\ell_2$ be the line obtained from $\ell_1$ by a rotation around the $z$-axis by the angle $2\pi/k$ where $k$ is the number of faces of $P$ sharing a common vertex. The distance to be explored is the distance between lines $\ell_1$ and $\ell_2$. The tangency points of $\ell_1$ and $\ell_2$ are at the same latitude

$$\varphi = \arccos\left(\frac{\sqrt{h^2 - 1}}{h}\right).$$

Therefore the square of the distance between the lines, obtained from the lines $\ell_1$ and $\ell_2$ in the $\delta$-rotation process, given by the formula (28) from Section 6.1 in [OS], is

$$d^2 = \frac{4\sin(\alpha)^2(1 - S^2)^2T^2}{(S^2 + T^2)(1 + \sin(\alpha)^2S^2 + \cos(\alpha)^2T^2)}. \quad (10)$$

Here $S = \sin(\varphi) = 1/h$, $\alpha = \pi/k$ and $T = \tan(\delta)$.

Let $r_m$ be the midradius (the radius of the sphere which touches the middle of each edge of $P$) and $r_c$ the radius of a circumscribed sphere (the one that
passes through all vertices of $\mathcal{P}$). We want the sphere touching the middles of edges of $\mathcal{P}$ to be unit. Then

$$h = r_c/r_m.$$ 

The well-known values of $r_c$ and $r_m$ for the Platonic bodies of edge length $a$ are collected in the following table; the last column gives the value of $h$.

|            | $r_c$    | $r_m$    | $h$  |
|------------|----------|----------|------|
| tetrahedron| $\frac{\sqrt{3}}{\sqrt{8}} a$ | $\frac{1}{\sqrt{8}} a$ | $\sqrt{3}$ |
| octahedron | $\frac{1}{\sqrt{2}} a$         | $\frac{1}{2} a$        | $\sqrt{2}$  |
| cube       | $\frac{\sqrt{3}}{2} a$         | $\frac{1}{\sqrt{2}} a$ | $\sqrt{3}$ |
| icosahedron| $\frac{5^{1/4} \tau^{1/2}}{2} a$ | $\frac{\tau}{2} a$       | $5^{1/4} \tau^{1/2}$ |
| dodecahedron| $\frac{\sqrt{3} \tau}{2} a$   | $\frac{\tau^2}{2} a$   | $\sqrt{3} \tau$ |

**Remark.** The construction of this section associates (see denominators of the formulas (2), (4) and (7)) pairs of numbers to each exceptional finite subgroup of $SO(3)$ (the proper symmetry group of either of dual Platonic bodies), or, via the McKay correspondence, to exceptional Lie groups $E_6$, $E_7$ and $E_8$. These pairs are:

- 2 and -2 for $E_6$;
- 3 and -5 for $E_7$;

- two golden integers, of norms 11 and -19, for $E_8$.

It would be interesting to have a uniform formula giving these values in terms of some Lie groups data.
4  Platonic compounds: global picture

We will analyze different pairwise distances between the tangent lines in the \( \delta \)-rotation process. This question is equivalent to the following one. Let \( \mathcal{P} \) be a Platonic body. The proper symmetry group \( G_{\mathcal{P}} \) acts on the set of unordered pairs of distinct edges of \( \mathcal{P} \). The set of pairwise distances is described by the orbits of this action.

The further analysis of the \( \delta \)-rotation process for the pairs \( \mathcal{O} \) and \( \mathcal{I} \) is found in the next two subsections.

4.1  Octahedron/Cube

We recall that the octahedron corresponds to \( \delta = 0 \) and the cube corresponds to \( \delta = \pi/2 \); during the \( \delta \)-rotation process the edges are rotated counterclockwise.

We fix an edge of the cube; it is shown in purple on Figure 11. The set of orbits of the action of the group \( G_{\mathcal{O}} \) on the set of unordered pairs of distinct edges has cardinality five, shown in red, green, blue, black and yellow on Figure 11.

![Figure 11: Pairwise distances for the pair \( \mathcal{O} \)](image)
The graphs of the squares of four distances during the $\delta$-rotation process are shown on Figure 12; $\delta$ varies from 0 to $\pi/2$. The colours of the graphs correspond to the colours of the rotated red, green, blue and black edges. The remaining fifth distance is the distance between the purple and the yellow edges; it keeps the constant value 2 in the $\delta$-rotation process.

The graphs on Figure 12 show that only two out of five distances are relevant for the minimum; these are the graphs shown in red and green on Figure 12. The red graph is the graph of the function $f_1$; the green graph is the graph of the function $f_2$.

\[
\frac{(4 \cos(2\delta) + \sqrt{2} \sin(2\delta))^2}{6 \cos^4(\delta) + 8\sqrt{2} \cos(\delta) \sin(\delta) + 8 \sin^4(\delta)}.
\]

Figure 12: Graphs of squares of four distances for the pair $O$

The graph of the minimum of the squares of the distances in the $\delta$-rotation process is shown on Figure 13. The maximum of the graph corresponds to the value $\delta_0$ given by (3).
The first corner point on Figure 13 occurs at
\[ \delta \simeq 0.800811 . \]
The value of the square of the minimal distance at this point is
\[ \simeq 0.26534 , \]
compare to the value (5).

### 4.2 Icosahedron/Dodecahedron

In the \( \delta \)-rotation process we let the icosahedron correspond to \( \delta = 0 \) and the dodecahedron correspond to \( \delta = \pi/2 \); during the \( \delta \)-rotation process the edges are rotated counterclockwise.

The set of orbits of the action of the group \( G_T \) on the set of unordered pairs of distinct edges has cardinality eleven. We illustrate the orbits on Figure 14. We fix an edge of the dodecahedron. It is marked by a red disk on the dodecahedron net on Figure 14. The eleven different distances during
the \( \delta \)-rotation process correspond to the edges on the dodecahedron net with the marks from 1 to 11.

Figure 14: Pairwise distances for the pair \( I \)

The graphs of the ten distances during the \( \delta \)-rotation process, are shown on Figure 15, \( \delta \) varies from 0 to \( \pi/2 \). The remaining eleventh distance is between opposite edges; it keeps the constant value 2 during the \( \delta \)-rotation process.

These graphs on Figure 15 show that only four out of eleven distances are relevant for the minimum.
Figure 15: Graphs of squares of ten distances for the pair $I$

The graphs of relevant distances are shown on Figure 16.

Figure 16: Relevant distances for the pair $I$
We have explicitly calculated the distances during the $\delta$-rotation process with the help of Mathematica [W].

The red graph corresponds to the distance marked by 1 on Figure 14. This function is given by the formula (7).

The blue graph corresponds to the distance marked by 5 on Figure 14. This function is

$$8 \left( \sqrt{5} \sin(2\delta) - 2 \cos(2\delta) \right)^2 \frac{1}{21 + 4\sqrt{5} + 4\sqrt{5} \cos(2\delta) - \cos(4\delta) + 8 \sin(2\delta) - 4\sqrt{5} \sin(4\delta)}.$$  \hfill (11)

The orange line corresponds to the distance marked by 3 on Figure 14. This function is

$$4 \cos^2(2\delta) \frac{3 + \cos^2(2\delta)}{3 + \cos^2(2\delta)}.$$ \hfill (12)

The brown graph corresponds to the distance marked by 8 on Figure 14. This function is

$$8(2 \cos(2\delta) + \sin(2\delta))^2 \frac{1}{25 + 8\sqrt{5} + 4\tau^3(2 \cos(2\delta) - \sin(2\delta)) + 3 \cos(4\delta) + 4 \sin(4\delta)}.$$ \hfill (13)

The distance marked by 9 is also important in the analysis. Its graph is dashed on Figure 15. This function is

$$8(2 \cos(2\delta) + \sin(2\delta))^2 \frac{1}{\tau^3(25 - 8\sqrt{5}) + 4(2 \cos(2\delta) - \sin(2\delta)) + \tau^3(3 \cos(4\delta) + 4 \sin(4\delta))}.$$ \hfill (14)

The graph of this function goes above the graph of the min function (the minimum of the squares of the distances) everywhere except the point

$$\frac{\pi - \arctan(2)}{2}.$$ 

At this point the functions (13) and (14) vanish simultaneously and this complicates the corresponding minimal configuration, see details in Subsection 5.2.

The graph of the minimum of the squares of the distances in the $\delta$-rotation process is shown on Figure 17. At $\delta_x$ ($\delta_x$ is slightly greater than $\delta_{\text{max}}$) the min function is given by (12) while the value (8) is the value of the function (7). For this reason the analysis of the pair $\mathcal{I}$ has to be continued.
Figure 17: Graph of the square of the minimal distance for the pair $\mathcal{I}$

The seemingly peak, for $\delta$ between 0.874 and 0.876, on Figure 17, is illusive. The fine structure of the graph around these values of $\delta$ is shown on Figure 18.

Figure 18: Fine structure

The resulting maximal compound, at the angle $\delta_{\text{max}}$, of thirty cylinders is shown on Figures 19 and 20.
Figure 19: Maximal configuration, view from the tip of a 5-fold axis

Figure 20: Maximal configuration, view from the tip of a 3-fold axis
We summarize the results about the maximal distance in the $\delta$-rotation process for the pair $I$.

**Theorem 2** The maximum distance in the $\delta$-rotation process for the pair $I$ is achieved at the angle

$$\delta_{\text{max}} = \arctan (\sqrt{t_0}) ,$$

where

$$t_0 \simeq 0.694356$$

is a root of the polynomial

$$5t^6 - 80t^5 + 190t^3 - 4t^2 - 84t + 9 ;$$

all roots of this polynomial are real. Approximately,

$$\delta_{\text{max}} \simeq 0.694707 .$$

The value of the square of the minimal distance at $\delta_{\text{max}}$ is approximately

$$d_{\text{max}} \simeq 0.0429216 ,$$

compare with (8). The corresponding radius of cylinders is approximately

$$r_{\text{max}} \simeq 0.115558 ,$$

compare with (9).

## 5 Minima

For completeness, we describe here the critical points corresponding to the local minima during the $\delta$-rotation process. There are two ‘trivial’ local minima corresponding to the angles $\delta = 0$ and $\delta = \pi/2$; these are Platonic bodies themselves. Other local minima happen when continuations of certain non-neighboring edges start to intersect.

For each minimum we present a picture of intersecting tangent lines. On the plane projection the line which goes below looses its color at the intersection, so the color of the intersection corresponds to the line which goes above.
5.1 Octahedron/Cube

For the pair $O$ we have only one non-trivial minimum, see Figure 13. The minimum happens at the angle

$$\delta = \arctan \left( \sqrt{2} \right).$$

At this angle every tangent line $\ell$ starts to intersect with two other tangent lines, see Figure 11. Therefore the twelve edges split into four triangles.

The resulting figure, shown on Figure 21, is formed by four triangles whose edge length is $2\sqrt{3}$. The triangles are pairwise simply linked.

One can describe this figure as follows. Choose a three-fold axis of the proper symmetry group $G_O$ of the octahedron/cube. Draw an equilateral triangle $T$ in the plane orthogonal to the axis and passing through the origin. Then the figure is obtained by application of the rotations from $G_O$ to the triangle $T$.

![Octahedron/cube minimum](image)

Figure 21: Octahedron/cube minimum

The convex hull of the four triangles is the cuboctahedron, of edge length 2, shown on Figure 22.
5.2 Icosahedron/Dodecahedron

As it can be seen on Figure [17], there are three non-trivial values of $\delta$ for which certain tangent lines start to intersect.

1. The first minimum happens at the angle

$$\delta = \frac{1}{2} \arctan \left( \frac{2}{\sqrt{5}} \right).$$

The intersections correspond to distances marked by 5 on Figure [14]. Since for every tangent line $\ell$ there are two other tangent lines at the distance marked by 5 from $\ell$, the thirty edges have to split into ten triangles.

The resulting figure is a union of ten triangles of edge length $2\sqrt{3}$. It is shown on Figure [23]. The triangles are pairwise simply linked.

One can describe this figure as follows. Choose a three-fold axis of the proper symmetry group $G_T$ of the icosahedron/dodecahedron. Draw an equilateral triangle $T$ in the plane orthogonal to the axis and passing through the origin. Then the figure is obtained by application of the rotations from $G_T$ to the triangle $T$. 
The convex hull of the figure is the icosidodecahedron, of edge length $\sqrt{5} - 1$, shown on Figure 24.
2. The second minimum happens at the angle

\[ \delta = \frac{\pi}{4} . \]

Here the intersections correspond to distances marked by 3 on Figure 14. Since each tangent line enters into four pairs of type marked by 3, the thirty edges have to split into five one-skeletons of the tetrahedron.

The resulting figure is the compound of one-skeletons, of edge length \(2\sqrt{2}\), of the five tetrahedra inscribed in the dodecahedron. It is shown on Figure 25.

Figure 25: Second minimum
The pairwise linking of the one-skeletons is shown on Figure 26.

![Figure 26: Linking of tetrahedra](image)

3. The third minimum happens at the angle

\[ \delta = \arctan(\tau) \equiv \frac{\pi - \arctan(2)}{2}, \]  

(16)

where \( \tau \) is the golden ratio.

At this value of \( \delta \) two distances (13) and (14) vanish simultaneously. These are distances marked by 8 and 9 on Figure 14. For each tangent line \( \ell \) there are two other lines of type marked by 8 and two more lines marked by 9. So each tangent line intersects four other lines, and the intersections are of two different combinatoric types.

It turns out that at the angle (16) the edges form pentagonal stars, so the resulting figure is the compound of six pentagonal stars. It is shown on Figure 27.
One can describe this figure as follows. Choose a five-fold axis of the proper symmetry group $G_I$ of the icosahedron/dodecahedron. Draw a regular pentagonal star $P$ in the plane orthogonal to the axis and passing through the origin. Then the figure is obtained by application of the rotations from $G_I$ to the star $P$.

The inner pentagons of the stars form a compound shown on Figure 28. The pentagons are pairwise simply linked.
The convex hull of either ten stars or ten pentagons is again an icosidodecahedron.

6 Concluding remarks

For the pair $\mathcal{T}$, the square of the minimal distance in the $\delta$-rotation process is a smooth function with a single local maximum. Theorem 1 in [OS-O6] asserts that this maximum is a local maximum in the space of all possible configurations of six tangent lines mod $SO(3)$.

We present several conjectures concerning two other dual pairs. Let $Z_k$ denote the configuration space, mod $SO(3)$, of $k$ non-overlapping cylinders touching the unit ball. Let $Z_k(\geq R)$ denote the subspace of $Z_k$ consisting of cylinders of common radius greater than or equal to $R$. Let $\mathcal{U}$ be a dual pair of Platonic bodies, either $\mathcal{O}$ or $\mathcal{I}$. We denote by $R_{\mathcal{U}}(\delta)$ the radius of touching cylinders corresponding to the minimal distance in the $\delta$-rotation process related to the pair $\mathcal{U}$. 

Figure 28: Third minimum, inner pentagons
Remark. It is interesting to note that the number of pairwise distances is equal to $3t - 1$ where $t$ takes values 1, 2 and 4 for, respectively $T$, $O$ and $I$. The minimum of the squares of the distances is the minimum of exactly $t$ smooth functions. The number of local maxima of the minimum of the squares of the distances is also equal to $t$.

We believe that our conjectures can be proved (or disproved) using the techniques of our proofs of the local maximality of the configuration $C_m$ in [OS-C6] and of the configuration $O_6$ in [OS-O6].

Pair octahedron/cube

Let $\delta_{O,1}$ and $\delta_{O,2}$ be the values of the angle $\delta$ corresponding to two local maxima of the function $R_O(\delta)$, see Figure 13. Here $\delta_{O,1} = \delta_{O}$. In the $\delta$-rotation process for the pair $O$, there are three distinguished values of $\delta$ for $0 < \delta < \pi_2$: $\delta_{O,1}, \delta_{O,2}$ and one more corner point.

- **Conjecture iO, weak form.** The configurations of twelve cylinders of radius $R_O(\delta_{O,j}), j = 1, 2$, are local maxima in the space $Z_{12}$.

- **Conjecture iO, strong form.** The configurations of twelve cylinders of radius $R_O(\delta_{O,j}), j = 1, 2$, are, moreover, isolated points in the space $Z_{12} (\geq R_O(\delta_{O,j}))$.

Comment about the remaining corner point. Here the left and right derivatives of the minimum of the squares of the distances have the same sign. Therefore the resulting (at this angle) configuration of twelve cylinders of the corresponding radius is not a critical point in the space $Z_{12}$.

Pair icosahedron/dodecahedron

Let $\delta_{I,j}, j = 1, 2, 3, 4$, be the values of the angle $\delta$ corresponding to four local maxima of the function $R_I(\delta)$, see Figures 17 and 18. Besides, there are two corner points of the minimum of the squares of the distances. Altogether, there are six distinguished values of the angle $\delta$ in the $\delta$-rotation process for the pair $I$. As it is seen on Figures 17 and 18 the left and right derivatives of the squares of the distances have the same sign for each corner point which is not a local maximum.
• **Conjecture i**\(_I\), **weak form.** The configurations of thirty cylinders of radius \(R_I(δ_{I,j})\), \(j = 1, 2, 3, 4\), are local maxima in the space \(Z_{30}\).

• **Conjecture i**\(_I\), **strong form.** The configurations of thirty cylinders of radius \(R_I(δ_{I,j})\), \(j = 1, 2, 3, 4\), are, moreover, isolated points in the space \(Z_{30}(≥ R_I(δ_{I,j}))\).

**Minima**

The \(δ\)-rotation process for the pair \(O\) exhibits the unique non-trivial minimum, see Figure 21. The convex span of the vertices on Figure 21 is the cuboctahedron, Figure 22. Its cousin for the pair \(I\) is the icosidodecahedron which appears as the convex span of the vertices of the compounds of rotated edges for the two local minima in the \(δ\)-rotation process for the pair \(I\).

The vertices of the cuboctahedron form the FCC configuration of twelve balls touching the unit central ball. Thus the configuration of the thirty balls with centres in the vertices of the icosidodecahedron is a natural analogue of the FCC configuration.

The balls, touching the central unit ball and centered in the vertices of the icosidodecahedron, touch each other when their common radius is equal to

\[
\frac{1}{\sqrt{5}}.
\]

Let us call \(ID\) this configuration of balls. By analogy with the FCC configuration, we expect that the configuration \(ID\) is critical and is unlockable by a move similar to the Coxeter move, see § 8.4 in [C], for the configuration FCC.

**Acknowledgements.** Part of the work of S. S. has been carried out in the framework of the Labex Archimede (ANR-11-LABX-0033) and of the A*MIDEX project (ANR-11- IDEX-0001-02), funded by the Investissements d’Avenir French Government program managed by the French National Research Agency (ANR). Part of the work of S. S. has been carried out at IITP RAS. The support of Russian Foundation for Sciences (project No. 14-50-00150) is gratefully acknowledged by S. S. The work of O. O. was supported by the Program of Competitive Growth of Kazan Federal University and by the grant RFBR 17-01-00585.
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