Solution of the simplified fluid dynamic equation for transient processes in a gas centrifuge

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Abstract. The equation describing a transient process in a gas centrifuge is a partial differential equation and has to be solved by using a numerical method. The Crank-Nicolson scheme and a central difference scheme are employed, respectively, for time discretization and space discretization. Under the condition of full circulation flow, the solution of the equation coincides with the result of the linear theory, verifying the correctness of numerical solution. The transient processes of a centrifuge are simulated with two withdrawal models to reveal the variations of the axial velocity with time in the processes. The results shows that for a given rotor peripheral speed, the radial distribution of the axial velocity depends mainly on the wall pressure and the withdrawal strength, but the influence of the withdrawals is much weaker than the wall pressure. The results also demonstrate that the partial differential equations describing the fluid dynamics in a transient process does exhibit the dynamic variations, and can be further applied to the analysis of separation performance.

1. Introduction
During the operation of a gas centrifuge, the process always exists that its hydraulics and separation varies with time from one state to another. This non-equilibrium process is referred to as a transient process and is unavoidable. For example, the start-up period of a cascade is a transient process, in which the flow and separation in every centrifuge of the cascade are experiencing dramatic changes. The switch of operation state, such as adjusting the feed or withdrawals, would also cause transient processes. On one hand, it is desirable that a transient process should be as short as possible, so that a centrifuge can reach the set-up working state as quickly as possible [1]. On the other hand, it is also desirable that a transient process can be utilized to create special separation effects that are otherwise unable to be produced by an ordinary steady state operation [2,3,4], so that components of small abundance can be separated efficiently. No matter what situation it is, a better understanding of the hydraulics as well as the law of separation during transient processes would allow us to have better control and use of transient processes.

The first step in the understanding of transient processes is the understanding of the fluid dynamics in a gas centrifuge, and is the basis for analyzing the separation effects during transient processes. In [5], based on the time-dependent Navier-Stokes equations with sources and sinks, an equation describing the flow field in a transient process in a gas centrifuge was obtained by introducing the isothermal rigid-body rotation of variable density as well as using the purely axial flow assumption. The equation, unlike the equation describing the flows at steady state, is time-dependent, which makes it unable to be solved by analytically and has to be solved numerically.
The purpose of this paper is to present a numerical approach to the solution of the equation.

2. The equation describing transient process
A schematic illustration of a gas centrifuge is given in figure 1. Now only the separation chamber is considered. The radius and the height of the rotor rotating at an angular velocity \( \Omega \) are, respectively, \( a \) and \( Z \). The cylindrical coordinate system is fixed on the rotor and rotates with the same angular velocity as the rotor. The feed enters the rotor at the axial position \( F \). Without loss of generality, it is assumed that the product \( P \) and waste \( W \) are withdrawn at the bottom and top, respectively. In the following, we go through very briefly the process of derivation of the equation describing the flow field during transient process. For the details of the derivation, see [5]. For simplicity, we refer to this equation as the transient equation of flow field.

Consider the following model of the isothermal rigid body rotation of variable density:

\[
\begin{align*}
\rho &= \rho_0, \quad V = V_0, \quad p = p_0, \quad T = T_0, \\
0 \rho_r &= \rho_0, \\
0 p_r &= 0, \\
0 p_r &= 0, \\
0 T_r &= 0. 
\end{align*}
\]

Here, \( \rho \) is the gas density, \( p \) the pressure, \( V \) the velocity, and \( T \) the temperature. \( \rho_0 \), \( p_0 \), \( V_0(=0) \), and \( T_0 \) define the isothermal rigid body rotation of variable density, which is just the well-known rigid body rotation with the density \( \rho_0(t) \) that is now variable with time \( t \):

\[
\rho_0(r,t) = \rho_{00}(t) \exp(-x),
\]

where \( \rho_{00} \) is the density at the rotor wall, and \( x = A^2[1-(r/a)^2] \) is the scale height, with \( A = \Omega a \sqrt{2RT_0} \). The gas constant \( R = R_0/M \), with \( R_0 \) being the universal gas constant and \( M \) the molar weight of the gas. For a perfect gas,

\[
p = \rho RT.
\]

The flow in the separation chamber is treated as the superposition of disturbances of the density, pressure, velocity and temperature on the rigid-body rotation:

\[
\rho = \rho_0 + \rho', \quad V = V_0 + V', \quad p = p_0 + p', \quad T = T_0 + T',
\]

where the quantities with a prime are the disturbances. Substituting (4) into the Navier-Stokes equations, neglecting the terms of being second order small, and applying the assumption of the purely axial flow to simplify the equations:

\[
u' = v' = 0, \quad \partial \nu'/\partial z = 0,
\]

Figure 1. The rotor of a gas centrifuge.
where \( u', v', \) and \( w' \) are, respectively, the components of the velocity disturbance \( \mathbf{V}' \) in the radial, azimuthal, and axial directions, we have the following non-dimensionalized transient equation for the axial velocity [5]:

\[
\frac{\partial w}{\partial t} = -\frac{\Omega}{2A^2} \frac{\partial \theta}{\partial z} x + 4A^4\Omega \frac{1}{\rho_0} \frac{\partial}{\partial x} \left( 1 - \frac{x}{A^2} \right) \frac{\partial w}{\partial x} + C.
\]  

(6)

Here \( E = \mu / \rho_o \alpha^2 \) is the Ekman number, and \( C \) is a function of time. The dimensional counterparts of the dimensionless quantities in the above equation are:

\[
z \to z/a, \quad w \to w'/\Omega a, \quad \rho_0 \to \rho_0/\rho_{\omega_0}, \quad \theta \to T'/T_0.
\]  

(7)

Note that, because \( \rho_{\omega_0} \) is a function of time, the Ekman number \( E \) is time-dependent.

The temperature field \( \theta \) is solely determined by the temperature on the rotor wall and is already known. Under the assumption of the purely axial flow, the temperature distribution on the rotor wall should be linear:

\[
\theta = a_\theta z + b_\theta.
\]  

(8)

It is worth pointing out that either or both of the parameters \( a_\theta \) and \( b_\theta \) may be a function of time.

The boundary conditions for the equation are:

on the rotor wall, no-slip condition is satisfied:

\[
w = 0 \quad (x = 0);
\]  

(9)

at the rotor axis, the natural condition is imposed:

\[
\frac{\partial w}{\partial x} = 0 \quad (x = A^2);
\]  

(10)

in the enriching section and stripping section, mass conservation is required:

\[
\int_0^z \rho_0 w dx = -\frac{A^2}{\pi a^2 \Omega \rho_{\omega_0}} P \quad (0 < z < Z_f/a);
\]  

(11)

\[
\int_0^z \rho_0 w dx = \frac{A^2}{\pi a^2 \Omega \rho_{\omega_0}} W \quad (Z_f/a < z < Z/a).
\]  

(12)

The initial condition is as follows:

\[
w = w^{(0)}, \quad \theta = \theta^{(0)}, \quad p_{\omega_0} = p_{\omega_0}^{(0)},
\]  

(13)

where \( w^{(0)}, \theta^{(0)}, \) and \( p_{\omega_0}^{(0)} \) are the specified initial state of the axial velocity, temperature and pressure.

Following the radial averaging approach of Cohen [6], the separation effect can be analyzed when the axial velocity is known. However, it is hard to obtain an analytical solution to Equation (6) as in the case of steady situation.

3. Numerical solution of the transient equation

3.1. The approach to solution

The time discretization of Equation (6) employs the Crank-Nicolson scheme:

\[
\frac{w^{(n+1)} - w^{(n)}}{\Delta t} = \frac{1}{2} \left[ -\frac{\Omega}{2A^2} \frac{\partial \theta}{\partial z} x + 4A^4\Omega \frac{1}{\rho_0} \frac{\partial}{\partial x} \left( 1 - \frac{x}{A^2} \right) \frac{\partial w}{\partial x} + C \right]^{(n+1)}
\]

\[
+ \frac{1}{2} \left[ -\frac{\Omega}{2A^2} \frac{\partial \theta}{\partial z} x + 4A^4\Omega \frac{1}{\rho_0} \frac{\partial}{\partial x} \left( 1 - \frac{x}{A^2} \right) \frac{\partial w}{\partial x} + C \right]^{(n)},
\]  

(14)
where $\Delta t$ is the time step, and the superscript in the braces indicates the time level. The space discretization needs to consider the strong pressure gradient in the radial direction, which requires non-uniform grid space to obtain sufficient accuracy and stability. Let

$$\xi = \frac{1}{2} (\eta + \eta^{'\prime}) \quad (\eta_i \leq \eta \leq 1),$$

where $\eta = r/a$, that is, the dimensionless radius, $\eta_i = r_i / a$, with $\eta_i$ corresponding to $r = r_i$, the so-called inner boundary. The appropriate value of $N_{\xi}$ depends on the peripheral speed of the rotor, and here $N_{\xi} = 20$. The area beyond $r = r_i$, where the gas is very rarefied, is excluded from consideration, and so the computation is performed for the area $r \leq r_i$. Let $\xi_i = (\eta_i + \eta_{i+1})/2$, and discretize $\xi$ uniformly into $N_{\xi}$ points in the area $[\xi_i, 1]$. Accordingly, $x$ is discretized into $N_{\xi}$ points $(x_1, x_2, \ldots, x_{N_{\xi}})$ in the area $[0, x_i]$, as illustrated in figure 2.

![Figure 2. Illustration of the discrete $x$-coordinate](image)

The discrete values are defined at these discrete coordinates. The second term in the brackets on the right hand side of equation (14) becomes:

$$\frac{1}{\rho_0} \frac{\partial}{\partial x} \left( \frac{1 - x}{A^2} \right) \frac{\partial w}{\partial x} = \frac{1}{\rho_0} \frac{\partial}{\partial x} \left( \eta^2 \frac{d \xi}{dx} \right) \frac{d \xi}{dx}.$$

The space discretization uses a central difference scheme $(i = 2, 3, \ldots, N_{\xi} - 1)$:

$$\frac{1}{\rho_0} \frac{\partial}{\partial x} \left( \eta^2 \frac{d \xi}{dx} \right) \approx \frac{1}{\Delta \xi^2} \frac{1}{\rho_0} \frac{\partial}{\partial x} \left( \frac{d \xi}{dx} \right) \times \left\{ \eta^2 \frac{d \xi}{dx}_{i-1} - \left( \eta^2 \frac{d \xi}{dx}_{i+1} + \eta^2 \frac{d \xi}{dx}_{i-1} \right) \right\}.$$

Here, the subscript indicates the location where the discrete value is evaluated. For example, if $i$ denotes that the value is evaluated at $\xi_i$, then $i+\frac{1}{2}$ tells that the value is evaluated at $(\xi_{i+1} + \xi_i)/2$. For the internal points of the area $[\xi_{i-1}, \xi_i]$ , the discretized form of equation (14) can be written as:

$$c_i w_{i-1}^{(n+1)} - d_i w_i^{(n+1)} + e_i w_{i+1}^{(n+1)} + f_i C^{(n+1)} = b_i,$$

where

$$c_i = -2A^4 \Omega \frac{dt}{\Delta \xi^2} E^{(n+1)} \left( \frac{1}{\rho_0} \frac{d \xi}{dx} \right) \left( \eta^2 \frac{d \xi}{dx} \right)_{i-1},$$

$$d_i = 2A^4 \Omega \frac{dt}{\Delta \xi^2} E^{(n+1)} \left( \frac{1}{\rho_0} \frac{d \xi}{dx} \right) \left( \eta^2 \frac{d \xi}{dx} \right)_{i-1} + 1,$$

$$e_i = -2A^4 \Omega \frac{dt}{\Delta \xi^2} E^{(n+1)} \left( \frac{1}{\rho_0} \frac{d \xi}{dx} \right) \left( \eta^2 \frac{d \xi}{dx} \right)_{i+1},$$

$$f_i = \frac{dt}{2},$$

$$b_i = 2A^4 \Omega \frac{dt}{\Delta \xi^2} E^{(n+1)} \left( \frac{1}{\rho_0} \frac{d \xi}{dx} \right) x_i.$$
At the boundary $x = 0$, equation (9) gives:

$$w_1^{(n+1)} = b_1,$$

(20)

where $b_1 = 0$. At $x = x_f$, we have from equation (10):

$$-w_{N,1}^{(n+1)} + w_N^{(n+1)} = b_n,$$

(21)

where $b_n = 0$. Integrating equations (11) and (12) by using the trapezoidal rule yields:

$$\sum_{i=1}^{N} g_i w_i^{(n+1)} = b_{N+1},$$

(22)

where

$$g_i = \left( \rho_0 \frac{dx}{d\xi} \right)_i, \quad g_i = 2 \left( \rho_0 \frac{dx}{d\xi} \right)_i \quad (i = 2, 3, \ldots, N_z - 1),$$

(23)

and

$$W_i = \begin{cases} -P & (0 < z < Z_f/a) \\ W & (Z_f/a < z < Z/a) \end{cases},$$

(24)

Hence, the algebraic system consisting of equations (18), (20), (21) and (22) can be expressed as:

$$Aw = b,$$

(25)

where

$$A = \begin{pmatrix}
1 & 0 & 0 & & \\
& c_2 & d_2 & e_2 & f_2 \\
& & c_3 & d_3 & e_3 & f_3 \\
& & & \ddots & \ddots & \ddots \\
& & & & c_{N,1} & d_{N,1} & e_{N,1} & f_{N,1} \\
& & & & & -1 & 1 & 0 \\
g_1 & g_2 & \cdots & g_{N,2} & g_{N,1} & g_{N,1} & \cdots & g_{N,1} & 0
\end{pmatrix},$$

(26)

$$w = (w_1^{(n+1)} \ w_2^{(n+1)} \ w_3^{(n+1)} \ \cdots \ w_N^{(n+1)} \ w_{N+1}^{(n+1)} \ C^{(n+1)})^T,$$

$$b = (b_1 \ b_2 \ b_3 \ \cdots \ b_{N+1} \ b_N \ b_{N+1})^T.$$

Let:

$$M_{1i} = (a_{ik}) \quad (1 \leq j \leq N_z; 1 \leq k \leq N_z), \quad M_{12} = (a_{ik}) \quad (1 \leq j \leq N_z; k = N_z + 1),$$
\[ M_{21} = (a_{jk}) \quad (j = N_x + 1; 1 \leq k \leq N_y), \quad M_{22} = (a_{jk}) \quad (j = N_x + 1; k = N_y + 1); \]

\[ w_1 = (w_j)^T \quad (1 \leq j \leq N_x), \quad w_2 = (w_j)^T \quad (j = N_x + 1); \]

\[ b_1 = (b_j)^T \quad (1 \leq j \leq N_x), \quad b_2 = (b_j)^T \quad (j = N_x + 1). \]

Clearly, \( M_{11} \) is a triangular matrix, which allows us to obtain easily the solution of equation (25):

\[ w_2 = (M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1}(b_2 - M_{21}M_{11}^{-1}b_1), \quad w_1 = M_{11}^{-1}(b_1 - M_{12}w_2). \]  

### 3.2. The withdrawal model

A withdrawal model is needed to create a transient process in a gas centrifuge. Here we construct two simple models. It should be pointed out that the models are constructed simply for demonstration of the dynamic changes of the centrifuge hydraulics in a transient process, not intending to simulate withdrawals in reality. Because the highlight of this paper is to solve the transient equation, the detailed explanations on the models are not given.

Let the withdrawal pressures for the product and waste be \( P_{Pe} \) and \( P_{We} \), respectively (see figure 3) and the pressures at the inlet of the product and waste scoops be \( P_{Pi} \) and \( P_{Wi} \), respectively.

![Figure 3. Illustration of feed and withdrawals.](image)

The withdrawals \( P \) and \( W \) satisfy the following relationships:

\[ P_{Pi}^2 - P_{Pi}^2 = c_l^2 P + c_t^2 P^2, \quad P_{Wi}^2 - P_{Wi}^2 = c_l^2 W + c_t^2 W^2, \]

where \( c_l \) and \( c_t \) are the laminar and turbulent resistance coefficients, respectively, and can be measured experimentally. Some explanation is in place. Take the product withdrawal as an example. The withdrawal pressure \( P_{Pi} \) is known and is measured at the outlet of the scoop and pipe assembly. The pressure at the inlet of the product scoop is calculated according to the flow field determined in the above. Assume there is a normal shock wave in front of the scoop, and the pressure before the shock wave is \( p \). The inlet pressure \( P_{Pi} \) can be obtained through the Rankine-Hugoniot relation:

\[ P_{Pi} = \frac{\gamma}{2} \left( \frac{(\gamma + 1) Ma^2}{4 \gamma Ma^2 - 2(\gamma - 1)} \right)^{\frac{1}{\gamma - 1}}, \]

with \( \gamma \) being the specific heat, and \( Ma \) the Mach number, calculated by

\[ Ma = \varepsilon r_0 / \sqrt{\gamma RT_0}. \]

Here, \( \varepsilon \) is the angular velocity deficit coefficient, taking into account the angular velocity loss.
relative to the angular velocity of the rigid body rotation due to the friction of the scoop, and $r_p$ is the radial location of the inlet. The pressure $p$ is given by:

$$p = p_a(t) \exp(-x_p),$$

(32)

with $x_p = A_p^2[1 - (r_p/a)^2]$, where $A_p^2$ is determined from:

$$A_p^2 = \left( \frac{c_p \omega a}{2RT} \right)^2.$$

(33)

With $p_{p_i}$ known, the withdrawal $P$ can be obtained by using either of the following simple models.

3.2.1. Withdrawal model 1 (WM1). From equation (29), the product withdrawal is calculated by:

$$P = \begin{cases} 
0 & (p_{p_i} \leq p_{p_f}) \\
-\frac{c'_i + \sqrt{c'^2_i + 4c'^2_i(p_{p_f} - p_{p_i})}}{2c'_i} & (p_{p_f} > p_{p_i}). 
\end{cases}$$

(34)

Clearly, the withdrawal starts only when the pressure in the withdrawal pipe just before the outlet is larger than $p_{p_i}$, which is maintained during the withdrawal.

3.2.2. Withdrawal model 2 (WM2). Suppose that an orifice plate is installed at the outlet of the withdrawal pipe, and the pressure downstream is very low. Then the flow through the orifice plate is:

$$P = d_p p_{p_f},$$

(35)

with $d_p$ being a coefficient.

Having obtained $P$ and $W$, by using whichever withdrawal model, mass conservation requires:

$$\frac{\partial H}{\partial t} = F - P - W,$$

(36)

where $H$ is the holdup inside the centrifuge and given by:

$$H = \frac{2\pi RTZ \rho_{a0}(t)}{\Omega^2}(1 - e^{-\theta}).$$

(37)

Therefore,

$$\frac{\partial \rho_{a0}(t)}{\partial t} = \frac{(F - P - W) \Omega^2}{2\pi RTZ(1 - e^{-\theta})}.$$

(38)

This variable density makes the Ekman number time-dependent (see equation (6)).

4. Examples and discussions

The process gas is taken to be UF$_6$, and the centrifuge is the well-known Iguaçu model, whose major parameters are given in Tab.1.

| Table 1. Major parameters of the Iguaçu centrifuge. |
|-----------------|---------|---------|---------|
| $a$ (m)         | $Z$ (m) | $Z_c$ (m) | $T_o$ (K) |
| 0.06            | 0.5     | 0.25     | 300      |

The temperature distribution on the rotor wall is assumed to be linear, and its axial gradient to be $d\theta/dz = 0.002$, which is independent of time for the sake of simplicity of demonstration. The inner boundary is taken to be $\eta = 0.7$, corresponding to $r_r = 0.042$ mm. The friction coefficients of the product and waste withdrawal assemblies are given in Tab.2.
Table 2. The friction coefficients of the product and waste assemblies.

|     | $c'_r$ (Torr$^2$·h/g) | $c'_r$ (Torr$^2$·h$^2$/g$^2$) | $c''_r$ (Torr$^2$·h/g) | $c''_r$ (Torr$^2$·h$^2$/g$^2$) |
|-----|----------------------|-----------------------------|----------------------|-----------------------------|
| 0.4 | 0.4                  | 0.01                        | 0.4                  | 0.01                        |

4.1. The case of total reflux

When a gas centrifuge is operated without a feed and withdrawals, that is, when it is working in a closed operation mode, the net flow flux at any cross section is zero, often referred to as the total reflux case. This case is trivial and used here to check the correctness of the numerical method and its implementation. We would like to obtain the radial distribution of the axial velocity $w$ when $\Omega a$ is 500, 600, and 700m/s, respectively, $p_{w_0}$ is 60, 80, and 100 Torr, respectively. The initial condition is: $w_{r(0)} = 0$, $\theta_{r(0)} = 1/60$, and $p_{w_{w(0)}} = 60$, 80, and 100 Torr.

Let $F = 0$, and set $p_{w_0}$ and $p_{w_{w_0}}$ to a large enough value such that $P = W = 0$. Figure 4 shows the $w$ distribution corresponding to different values of $\Omega a$ and $p_{w_0}$. It is clearly seen that both $\Omega a$ and $p_{w_0}$ have an influence on the distribution. Looking at $w$ at a radial location, for the same $p_{w_0}$, the larger $\Omega a$, the smaller $w$; whereas for the same $\Omega a$, the larger $p_{w_0}$, the larger $w$. All shapes of these curves are similar, but the locations of their zeros are the same.

Figure 4. $w$ distributions for different values of $\Omega a$ and $p_{w_0}$.

In figure 5, the dimensionless axial velocity $w^*$, defined by

$$w^* = w\sqrt{\frac{16A^2E}{d\theta/dz}}$$

is presented corresponding to these values of $\Omega a$ and $p_{w_0}$. It reveals that for the same $\Omega a$, the curves for different $p_{w_0}$ are identical. However, the results of analytical analysis [7] tell us that the relation of $w^*_r$ against $x$ should be independent of both $\Omega a$ and $p_{w_0}$. In fact, the results based on the purely axial flow assumption correspond to the solution of the zero mode obtained by the eigenfunction analysis [8,9], which is unable to take into account the effect of the rotor curvature. In the analytical solution, the so-called pancake approximation, that is, $(1 - x/A^2) \sim 1$ is applied to simplify equation (6). If this approximation was made in the numerical solution, all curves would be overlapped, as in the case of the analytical solution.

4.2. The case with feed and withdrawals

Now we consider the transient process from the start of filling up the centrifuge to the steady state. Take the angular velocity deficit coefficients $\varepsilon_p = 0.95$ and $\varepsilon_w = 0.9$, and the radial locations of the inlets of the product and waste scoops $r_p = 0.055$ mm and $r_w = 0.054$ mm. The feed $F = 10$ mg/s.
4.2.1. WM1. Set the withdrawal pressures $p_{w} = 30$ Torr and $p_{w} = 20$ Torr. Figure 6 demonstrates the variations with time of the pressure on the wall $p_{w}$ as well as the product and waste withdrawals $P$ and $W$.

The wall pressure $p_{w}$ varies linearly with time before 12.5s. At about 12.5s, the waste withdrawal starts, soon followed by the product withdrawal. The equilibrium between the feed and withdrawals is reached approximately at 14s, afterwards the states of $p_{w}$, $P$ and $W$ no longer change. The time to establish the equilibrium is very short, just a couple of seconds. This result seems not consistent with experimental observations. The reason is that the process of head conduction is not considered between the heat sources (such as the scoops due to friction) and heat absorptions (such as the rotor wall due to heat transfer within the rotor wall and it’s the surrounding environment). This process is complicated and would lead to a variable temperature distribution with time on the wall, which spends a much longer time to reach equilibrium.

The $w$ distributions at different times in the radial direction are presented in figure 7 for the enriching section. The $w$ distributions in the stripping section are identical to those in the enriching section.
Before the start of the waste withdrawal, the increase of $w$ is proportional to the time, which is implied by the equally spaced curves.

The $w$ distributions corresponding to the three time instants 12.7s, 13.1s and 14s, indicated in figure 6 by the symbol “■”, are shown in figure 8. One can see that the three curves have only very small differences, which suggests that the feed and withdrawals have negligible influences on the axial flow circulation.

4.2.2. WM2. Figure 9 gives the variations of $p_{w0}$, $P$ and $W$ with time. Obviously, compared with the transient process with WM1 (see figure 6), the transient process with MD2 experiences a much longer time. The reason is that the feed and the withdrawals take place simultaneously, leading to a much slower accumulation of the process gas inside the centrifuge and, consequently, a much slower increase of the wall pressure. Therefore, the transient process is naturally extended to a longer period. This observation tells us that to shorten a transient process during the start-up of a cascade, an appropriate feed and withdrawal scheme may need well worked out.

As a comparison with figure 7, figure 10 demonstrates the $w$ distributions at different times and shows a different transient process from figure 7.
Figure 9. Time history of $p_{w0}$, $P$ and $W$ with WD2

Figure 10. $w$ distributions at different times

5. Conclusions

The partial differential equation describing the $w$ velocity during a transient process is solved by numerical solution. The Crank-Nicolson scheme is used to perform the temporal discretization, and an ordinary central difference scheme is employed to do the spatial discretization. The resulting algebraic system can be easily solved. Two example cases, the case of total reflux and the case with feed and withdrawals, are investigated for the Iguaçu centrifuge.

The results show that, in the case of total reflux, the $w$ distribution in the radial direction depends only on the wall pressure and the peripheral speed of the rotor. At the same peripheral speed, the $w$ distribution varies with the wall pressure, but the dimensionless $w$ distribution only depends on the peripheral speed. This observation is consistent with the results obtained from analytical analysis and verifies the correctness of the numerical solution.

Two simple withdrawal models, WD1 and WD2, are designed to demonstrate the transient process during the filling up of the centrifuge. The transient process is significantly influenced by the way in which the withdrawals are carried out. With WD2, the time taken to complete the transient process is much longer than with WD1. This suggests that the transient process during the cascade filling up period can be greatly shortened by application of a specially designed withdrawal scheme.
Both withdrawal models show that the $w$ distribution is related to the wall pressure and withdrawals, but the influence from the withdrawals are much smaller relative to the influence from the wall pressure. Therefore, the influence from the withdrawals can be neglected. Note that, as a result of application of the purely axial flow, the shapes of the $w$ distribution are all similar in case of total reflux to those with withdrawals, but the zeros kept unchanged.

With the $w$ velocity known during a transient process, the separation effect can be investigated, which is the work in a future research.

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References
[1] Vetsko V M, Laguntsov N I, Levin E V, Nikolaev B I and Sulaberidze G A 1987 Transient processes in double cascades for separating multicomponent isotopic mixtures At. Energy 63(3) 692–697
[2] Sosnin L Yu, Tcheltsov A N, Kuchelev A P, Remin G V and Hobotov A A 2002 Centrifugal extraction of highly enriched $^{120}$Te and $^{122}$Te using the non-steady state method of separation Nucl. Instr. Methods Phys. Rese. A 480 36–39
[3] Cao Y F, Zeng S, Lei Z G and Ying C T 2004 Study of a nonstationary separation method with gas centrifuge cascade Sep. Sci. Technol. 39(14) 3405–3429
[4] Orlov A, Ushakov A and Sovach V 2017 Mathematical modeling of nonstationary separation processes in gas centrifuge cascade for separation of multicomponent isotope mixtures MATEC Web of Conferences 72 01106
[5] Zeng S 2020 A method for analyzing the transient hydraulics in a gas centrifuge Proc. 15th Int. Workshop on Separation Phenomena in Liquids and Gases ed S Zeng and M S Zhou (Tsinghua University Press) May 13-17, 2019, Wuxi, Jiangsu Province, China 78–85
[6] Cohen K 1951 The Theory of Isotope Separation as Applied to the Large-scale Production of $^{235}$U McGraw-Hill Book Company, Inc.
[7] Von Halle E 1977 The countercurrent gas centrifuge for the enrichment of U-235. US DOE Rep. K/OA-4058
[8] Brouwers J J H 1976 On the motion of a compressible fluid in a rotating cylinder Ph.D. thesis Twente University of Technology, Enschede, The Netherlands
[9] Wood H G and Morton J B 1980Önsager’s pancake approximation for the fluid dynamics of a gas centrifuge J. Fluid Mech. 101(1) 1–31