MAXIMAL SINGULAR LOCI OF SCHUBERT VARIETIES
IN $SL(n)/B$

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Abstract. Schubert varieties in the flag manifold $SL(n)/B$ play a key role in our understanding of projective varieties. One important problem is to determine the locus of singular points in a variety. In 1990, Lakshmibai and Sandhya showed that the Schubert variety $X_w$ is nonsingular if and only if $w$ avoids the patterns 4231 and 3412. In this paper we give an explicit combinatorial description of the irreducible components of the singular locus of the Schubert variety $X_w$ for any element $w \in S_n$. These irreducible components are indexed by permutations which differ from $w$ by a cycle depending naturally on a 4231 or 3412 pattern in $w$. Our description of the irreducible components is computationally more efficient ($O(n^6)$) than the previously best known algorithms, which were all exponential in time. Furthermore, we give simple formulas for calculating the Kazhdan-Lusztig polynomials at the maximum singular points.

1. Introduction

Schubert varieties play an essential role in the study of the homogeneous spaces $G/B$ for any semisimple group $G$ and Borel subgroup $B$; every closed subvariety in $G/B$ can be written as the union of Schubert varieties, the classes of Schubert varieties form a basis for the cohomology ring of $G/B$ and the Schubert varieties correspond to the lower order ideals of a partial order associated to $G/B$. Specifically, this Bruhat order is an order on the $T$-fixed points in $G/B$ where $T$ is the maximal torus in $B$. The $T$-fixed points, $e_w$, correspond bijectively with elements in the Weyl Group $W = N(T)/T$ of $G$ and $T$. A tremendous amount of information about a Schubert variety can be obtained by examining the corresponding Weyl group element. Our main theorem gives a simple and efficient method for giving the irreducible components of the singular locus of a Schubert variety $^1$.

In the late 1950’s, Chevalley [Che94] showed that all Schubert varieties in $G/B$ are nonsingular in codimension one. Since that time, many beautiful results on determining singular points of Schubert varieties have surfaced (see [BL00]). By definition, the Schubert variety $X_w$ is the closure of the

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$^1$While in the process of preparing this submission, the authors have learned that Manivel [Man01] has recently independently proved a theorem equivalent to Theorem $^1$. 
$B$-orbit of $e_w$. Therefore any point $p \in X_w$ is singular if and only if all points in the orbit $Bp$ are singular. Since the singular locus of a variety is closed, the singular locus of $X_w$ is a union of Schubert varieties indexed by the maximal elements $v < w$ such that $e_v$ is singular in $X_w$.

Let $\text{maxsing}(X_w)$ denote the maximal set of Weyl group elements corresponding to singular points in $X_w$ in Bruhat order, i.e., $v$ is an irreducible component of the singular locus of $X_w$ if and only if $v \in \text{maxsing}(X_w)$. The goal of this paper is to give an explicit algorithm for finding $\text{maxsing}(X_w)$ in the case where $G$ is $SL_n(\mathbb{C})$, $B$ is the set of invertible upper triangular matrices, $T$ is the set of invertible diagonal matrices, and $W$ is the symmetric group $\mathfrak{S}_n$. The algorithm we present is very efficient, $O(n^6)$, and removes the need to search through all nonsingular $T$-fixed points (as is the case with previously known techniques).

In type $A$ (i.e., $G = SL(n)$), smoothness is equivalent to rational smoothness ([Deo85], see also [CK99] in the case of $ADE$) so the maximal singular locus of $X_w$ also determines the maximal permutations $x \leq w$ for which the corresponding Kazhdan-Lusztig polynomial is different from 1. We use the explicit form of $\text{maxsing}(X_w)$ to compute all Kazhdan-Lusztig polynomials at maximal singular points (msp’s); they are either $1 + q + \cdots + q^k$ or $1 + q^k$ depending on whether the corresponding bad pattern is $4231$ or $3412$ (respectively).

2. Main results

In 1990, Lakshmibai and Sandhya [LS90] showed that the Schubert variety $X_w \subset SL(n)/B$ is smooth at every point if and only if the permutation matrix for $w$ does not contain any $4 \times 4$ submatrix equal to $3412$ or $4231$. We use these two permutation patterns to produce the maximal permutations below $w$ which correspond to points in the singular locus. This verifies the conjecture stated in [LS90]. (Gasharov, using a map similar to the one we introduce in Section 3 shows in [Gas00] that the points constructed in [LS90] are singular. His result proves one direction of this conjecture.) In fact, our proof starts from an arbitrary maximal singular $T$-fixed point $e_x$ in $X_w$, and shows that $w$ must contain a $4231$ or $3412$ pattern and $x$ must contain a $2143$ or $1324$ pattern (respectively).

The main theorem below shows that elements of $\text{maxsing}(X_w)$ are obtained by acting on $w$ by certain cycles. These cycles, described in the following theorem, are best absorbed graphically in terms of the permutation matrices $\text{mat}(x)$ and $\text{mat}(w)$. Examples are shown in Figures 1 and 2.

**Theorem 1.** $X_x$ is an irreducible component of the singular locus of $X_w$ if and only if

$$x = w \circ (\alpha_1, \ldots, \alpha_m, \beta_k, \ldots, \beta_1)$$
for disjoint sequences
\[ 1 \leq \alpha_1 < \cdots < \alpha_m \leq n, \text{ with } w(\alpha_1) > \cdots > w(\alpha_m), \text{ and } \]
\[ 1 \leq \beta_1 < \cdots < \beta_k \leq n, \text{ with } w(\beta_1) > \cdots > w(\beta_k), \]
the interiors of the shaded regions in Figures 1 and 2 do contain any other 1’s in the permutation matrix of \( w \), and one of the following cases holds:

1. **4231 Case:**
   \[ k, m \geq 2 \text{ and } \alpha_1 < \beta_1, \ldots, \beta_{k-1} < \alpha_2, \ldots, \alpha_m < \beta_k \text{ and } w(\alpha_m) > w(\beta_1). \]

2. **3412 Case:**
   \[ k, m \geq 2 \text{ and } \beta_1, \ldots, \beta_{k-1} < \alpha_1 < \beta_k < \alpha_2, \ldots, \alpha_k \text{ and } w(\alpha_{m-1}) > w(\beta_1) > w(\alpha_m) > w(\beta_2). \]

3. **45312 Case:**
   \[ k = m = 2 \text{ and } \beta_1 < \alpha_1 < \beta_2 < \alpha_2 \text{ and } w(\alpha_1) > w(\beta_1) > w(\alpha_2) > w(\beta_2) \text{ and } \]
   entries of \( \text{mat}(w) \) in region A of Figure 2 are in decreasing order.

![Figure 1. Example of Case 1 of Theorem 1. ⊗’s denote 1’s in \( \text{mat}(w) \), •’s denote 1’s in \( \text{mat}(x) \).](image)

After introducing basic notation in Section 3, we then introduce in Section 4 the pictorial characterization of the Bruhat order we rely on. In sections 5 and 6, we discuss the Lakshmibai-Seshadri basis for the tangent space of a Schubert variety indexed by transpositions and the set
Figure 2. Examples for Cases 2 and 3 of Theorem 1. For clarity in stating the theorem, region A in (2) is not shaded as it would be in the remainder of the paper. The ⊙'s represent points where mat(x) and mat(w) both have entries.

\[ R(x, w) = \{ t : x < xt \leq w \} \]

We also define a set of maps that allows us to relate \( R(x, w) \) and \( R(y, w) \) when \( x \) and \( y \) differ by a transposition. These maps will then allow us to investigate not only whether a point \( e_x \) is singular, but whether it is maximally singular. To describe those permutations \( x \in \text{maxsing}(X_w) \), we show that related permutations \( \overline{x} \) must, among other qualities, avoid the patterns 231, 312 and 1234. We complete the description of \( \text{maxsing}(X_w) \) in sections 8, 9 and 10.

The remaining sections contain applications arising from our description of \( \text{maxsing}(X_w) \). In Section 11, we prove the conjecture of Lakshmibai and Sandhya on the composition of \( \text{maxsing}(X_w) \). Using the tools we have developed, in Section 12 we calculate the values of the Kazhdan-Lusztig polynomials at maximal singular points. In Section 13, we give some example calculations pertaining to the composition of \( \text{maxsing}(X_w) \). Finally, in Section 14, we state a simple method for determining the number of elements in \( \text{maxsing}(X_w) \) in terms of pattern avoidance and containment.

3. Preliminaries

We begin by introducing our basic notation and terminology. Let \( S_n \) denote the symmetric group on \( n \) letters. We will view elements of \( S_n \) as permutations on \([1, \ldots, n]\). To this end, we identify \( s_i \) with the transposition \((i, i+1)\). Let \( w(i) \) be the image of \( i \) under the permutation \( w \). We have a one-line notation for a permutation \( w \) given by writing the image of \([1, \ldots, n]\) under the action of \( w \): \([w(1), w(2), \ldots, w(n)]\). We will also often utilize the permutation matrix for \( w \) (denoted mat(\( w \))).

We use the standard presentation

\[ S_n = \langle s_1, \ldots, s_{n-1} : s_i^2 = 1, \]

\[ s_is_j = s_js_i \text{ for } |i - j| > 1, \text{ and} \]

\[ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \].
Corollary 4. If \( x \leq y \leq w \), then \( d_{x,w} - d_{y,w} \) is everywhere non-negative.
The Bruhat graph of \( w \) is the graph with vertices labeled by \( \{ v \leq w \} \) and \( v_1 \) is joined to \( v_2 \) by a directed edge if \( v_1 = v_2 t \) for some \( t \in T \) and \( v_1 < v_2 \) in Bruhat order. This graph plays a central role in the study of Schubert varieties. For example, Lakshmibai and Seshadri have shown that in \( SL(n)/B \), the tangent space to \( X_w \) at \( e_x \) has a basis indexed by \( \{ t \in T : xt \leq w \} \), i.e. the edges of the Bruhat graph adjacent to \( x \). This fact forms the main criterion we will use in Section \( 3 \) for smoothness at a point. In fact, since \( xt < x \) implies \( xt < w \) we will just need to consider the edges “going up” from \( x \) in the Bruhat graph of \( w \). As mentioned in the introduction, this set will be denoted by

\[
R(x, w) := \{ t \in T : x < xt \leq w \}.
\]

Over the last few years, it has become apparent that properties of the Bruhat order can often be efficiently characterized by “pattern avoidance” [BL98, Ber92, BW, LS85, Ste96]. We say that \( w = [w(1), \ldots, w(n)] \) avoids the pattern \( v = [v(1), \ldots, v(k)] \) for \( k \leq n \) if we cannot find \( 1 \leq i_1 < \cdots < i_k \leq n \) with \( w(i_1), \ldots, w(i_k) \) in the same relative order as \( v(1), \ldots, v(k) \) - i.e., no submatrix of \( \text{mat}(w) \) on rows \( i_1, \ldots, i_k \) and columns \( w(i_1), \ldots, w(i_k) \) is the permutation matrix of \( v \). Our characterization of the maximal singular locus is yet another example of the utility of this notion.

More generally, we can define pattern avoidance or containment in terms of the following flattening function. For any set \( Z = \{ z_1 < z_2 < \cdots < z_k \} \subseteq [1, \ldots, n] \), and \( x \in S_n \), define \( \text{fl}_Z(x) \) to be the “flattened” permutation on \( [1, \ldots, k] \) whose elements are in the same relative order as \( [x(z_1), \ldots, x(z_k)] \). When the set \( Z \) is clear from context, we will abbreviate \( \text{fl}_Z(x) \) by \( \hat{x} \). We will also write \( \text{fl}(i, j, \ldots, k) \) for the flattened permutation on the sequence \( i, j, \ldots, k \) and write \( x^i \) for \( \text{fl}_{[1, \ldots, n] \setminus \{i\}}(x) \).

It will also be useful to have notation for an “unflattening” operator. Given a permutation \( x \in S_n \), a set \( Z \subseteq [1, \ldots, n] \), and a permutation \( u \in S_k \), we can define a new permutation \( \text{unfl}_Z^x(u) \in S_n \) by requiring that

1. \( \text{fl}_Z(\text{unfl}_Z^x(u)) = u \), and
2. \( x(a) = (\text{unfl}_Z^x(u))(a) \) if \( a \in [1, \ldots, n] \setminus Z \).

When \( x \) and \( Z \) are clear from context, we abbreviate \( \text{unfl}_Z^x(u) \) by \( \hat{u} \).

**Example 5.** For \( x = [5, 2, 4, 1, 6, 3] \) and \( Z = [3, 5, 6] \), we have \( \text{fl}_Z(x) = [2, 3, 1] \) and \( \text{unfl}_Z^x([3, 1, 2]) = [5, 2, 6, 1, 3, 4] \). Note that \( x = \text{fl}_Z(x) \).

4. **Bruhat pictures**

Our main theorem is concerned not only with determining which points in a given Schubert variety are singular, but which are maximally singular. The function \( d_{x, w} \) affords us a graphical view of the Bruhat order. Most importantly, it lets us see the set \( R(x, w) \). We will now introduce the graphical notation utilized in the remainder of the paper that allows us to do this. A diagram displaying the notation we are about to describe is offered in Figure 3.
Figure 3. We see (among other facts) that $d_{x,w}(\triangle) \geq 1$, $d_{y,w}(\Delta) = d_{x,w}(\Delta) - 1 \geq 0$, $pt_x(c) = pt_w(c)$ and $t_{\alpha,\beta} \in \mathcal{R}(x,w)$.

First, we plot, as black disks, all or some of the positions containing 1’s in the permutation matrix $\text{mat}(x)$ of $x$. We will sometimes overlay $\text{mat}(x)$ and $\text{mat}(w)$. In these cases, 1’s in $\text{mat}(w)$ will be marked by open circles. Points that are simultaneously in both diagrams will consist of a black disk and a larger concentric circle. Let $[a, b] \times [c, d]$ denote the set of all points $(p, q) \in \mathbb{R}^2$ such that $a \leq p \leq b$ and $c \leq q \leq d$. The following notation will be handy:

**Definition 6.** For $t_{p,q} \in \mathcal{R}(x,w)$, set

1. $A_{p,q} := A_{p,q}(x) = [p+1, q-1] \times [x(p)+1, x(q)-1]$,
2. $\overline{A}_{p,q} := \overline{A}_{p,q}(x) = [p, q] \times [x(p), x(q)]$,
3. $pt_x(c) := (c, x(c))$ for $c \in [1, \ldots, n]$.

Along with the points of $\text{mat}(x)$, we will often shade parts of our diagram in order to specify that $d_{x,w}$ satisfies a particular inequality on a given region. Light shading on a region signifies that $d_{x,w} \geq 1$ on that region. Dark shading signifies $d_{x,w} \geq 2$. No shading places no restrictions on the values $d_{x,w}$. A region with a black border is one where $d_{x,w}$ achieves the minimum possible value allowed by the shading on that region. Dotted borders are used to demarcate regions we wish to discuss in the text.

As mentioned above, the great utility of these diagrams arises from being able to visualize $\mathcal{R}(x,w)$ along with the information on the Bruhat order. To see how we do this, suppose we have some reflection $t_{a,b} \in \mathcal{R}(x,w)$ (which implies $x < xt_{a,b} \leq w$). Now compare the shading (with respect to $w$) in $\text{mat}(x)$ and $\text{mat}(xt_{a,b})$. We see (as in Figure 3), that in the region $A_{a,b}(x)$, $d_{xt_{a,b},w} = d_{x,w} - 1$. Hence, by Lemma 3 we can state the following:

**Fact 7.** Let $t_{a,b} \in \mathcal{T}$ with $x < xt_{a,b}$. The transposition $t_{a,b}$ is in $\mathcal{R}(x,w)$ if and only if it corresponds to a region in $\text{mat}(x)$ that is entirely shaded (i.e., $d_{x,w}|_{A_{a,b}} \geq 1$). An example is given in Figure 4.
Note that the values of $d_{x,w}$ on the region $A_{a,b} \setminus A_{a,b}$ are not considered in determining the membership of $t_{a,b}$ in $R(x,w)$.

In order to highlight reflections that we are particularly interested in, we will often draw an arc in our diagram. A solid or dotted curve connecting two points in $\text{mat}(x)$ will denote an element of $R(x, w)$. A dotted curve will be used to designate $t$ when we are particularly interested in $yt' = xt$. A dashed curve will be used when we wish to mark a reflection $t' \in R(y, w)$. Of course, if $tt' \neq t' t$, and our picture is of $\text{mat}(x)$, then only one of the endpoints of our dashed curve will correspond to a point of $\text{mat}(x)$. There are numerous instances when an arc corresponds to an element of $R(x, w) \cap R(y, w)$. In this case, whether we use a solid or dashed arc depends on context.

The following lemma will be used several times in future sections. It allows us to infer the presence of points in $\text{mat}(x)$ in a region based on a particular common pattern of shading.

**Lemma 8.** Let $x < w$ and suppose $p, p', q, q' \in \mathbb{Z}$ such that

1. $p < p'$, $q < q'$,
2. $d_{x,w}(p, q) = 0$,
3. $d_{x,w}(p, q) = \alpha$, $d_{x,w}(p', q') = \beta$, $d_{x,w}(p', q) = \gamma$.

Then there exist at least $\alpha + \beta - \gamma$ values $m$ such that $pt_{x}(m) \in [p + 1, p' - 1] \times [q + 1, q' - 1]$ with $x(m) \neq w(m)$.

**Proof.** Let's define four regions as follows:

- $A = [1, p] \times [q, q' - 1]$,
- $B = [p + 1, p'] \times [q', n]$,
- $C = [p + 1, p'] \times [q, q' - 1]$,
- $D = [1, p] \times [q', n]$.

For every subset $R \subset [1, n] \times [1, n]$, define

$$\Theta_{x,w}(R) = \# \{(p, q) \in R : q = w(p)\} - \# \{(p, q) \in R : q = x(p)\}. \tag{4.4}$$

Then

- $d_{x,w}(p, q') = 0$ implies that $\Theta_{x,w}(D) = 0$,
- $d_{x,w}(p', q') = \beta$ implies that $\Theta_{x,w}(B) = \beta$, and
- $d_{x,w}(p, q') = \alpha$ implies that $\Theta_{x,w}(A) = \alpha$. 
Figure 5. We have indicated certain values of \( d_{x,w} \) at the lower left corner of each region.

Now,

\[
(4.5) \quad d_{x,w}(p', q) = \Theta_{x,w}(A) + \Theta_{x,w}(B) + \Theta_{x,w}(C) + \Theta_{x,w}(D),
\]

so

\[
(4.6) \quad \gamma = \alpha + \beta + \Theta_{x,w}(C) + 0.
\]

So \( \Theta_{x,w}(C) = -(\alpha + \beta - \gamma) \) and there are exactly \( \alpha + \beta - \gamma \) more 1's of \( \text{mat}(x) \) than 1's of \( \text{mat}(w) \) in region \( C \). This finishes the proof.

5. A Criterion for Maximal Smoothness

To prove Theorem 1, we start from the fact that (by definition) \( X_w \) is smooth at \( e_x \) if and only if the dimension of the Zariski tangent space at that point is equal to \( l(w) = \dim(X_w) \). Lakshmibai and Seshadri, [LS84], describe the dimension of this tangent space in terms of the root system. Using the fact that \( \#\{t \in T : xt < x \leq w\} = l(x) \), we can paraphrase their result as:

**Theorem 9.** [LS84] The Schubert variety \( X_w \in SL(n)/B \) is smooth at \( e_x \) if and only if \( \#R(x,w) := \#\{t \in T : x < xt \leq w\} = l(w) - l(x) \).

This yields the following characterization of the permutations in \( \text{maxsing}(X_w) \):

**Fact 10.** \( x \in \text{maxsing}(X_w) \) if and only if

1. \( \#R(x,w) > l(w) - l(x) \) and
2. for all \( t \in R(x,w) \), \( \#R(xt,w) = l(w) - l(xt) \).

As may be ascertained from Theorem 1, the criteria for \( x \) to be an element of \( \text{maxsing}(X_w) \) are local in nature. This implies that we may concentrate on only certain indices in our permutation \( w \) in order to determine \( \text{maxsing}(X_w) \). We now describe these indices explicitly.

**Definition 11.** Let

\[
(5.1) \quad \Delta(x,w) = \{i, 1 \leq i \leq n : \exists j, 1 \leq j \leq n, \text{ with } t_{i,j} \in R(x,w)\}.
\]
For $\Delta(x, w) = \{d_1 < d_2 < \cdots < d_k\}$, set
\[
\bar{x} = \text{fl}([x(d_1), x(d_2), \ldots, x(d_k)]) \quad \text{and}
\]
\[
\bar{w} = \text{fl}([w(d_1), w(d_2), \ldots, w(d_k)]).
\]

Note that $\bar{x}$ and $\bar{w}$ are permutations in $S_k$.

We state here for reference the following useful characterization of $\Delta(x, w)$.

**Corollary 12.** Let $x \leq w$. Then $d_{x,w}(pt_x(p)+(-1,0)) = 0$ and $d_{x,w}(pt_x(p)+(0,1)) = 0$ if and only if $p \notin \Delta(x, w)$.

We now give a sufficient condition for an index $b$ to be in $\Delta(x, w)$.

**Proposition 13.** Suppose $x < w$ and $x(b) \neq w(b)$ with $1 \leq b \leq n$.

1. If $w(b) < x(b)$, then $\exists a < b$ with $t_{a,b} \in R(x, w)$ and $x(a) \neq w(a)$.
2. If $w(b) > x(b)$, then $\exists c > b$ with $t_{b,c} \in R(x, w)$ and $x(c) \neq w(c)$.

**Proof.** First we prove the case of $w(b) < x(b)$. Note that $d_{x,w}(b-1, x(b)) = 1 + d_{x,w}(b, x(b)) \geq 1$ since $w(b) < x(b)$. Let $p' = b - 1$. Choose $q$ as large as possible such that $q < x(b)$ and $d_{x,w}(p', q) = 0$ (see Figure 6). Such a $q$ must exist since $d_{x,w}(-1, 0) = 0$. Now choose $p$ as small as possible such that $p < p'$ and $d_{x,w}(g, h) \geq 1$ for all $g, h$ with $(g, h) \in [p+1, p'] \times [q+1, x(b)]$. Then there exists a $q'$, $q < q' \leq x(b)$ such that $d_{x,w}(p, q') = 0$. By construction,
\[
d_{x,w}(p, q') = 0, \quad d_{x,w}(p', q) = 0, \quad d_{x,w}(p, q) \geq 0 \quad \text{and} \quad d_{x,w}(p', q') \geq 1.
\]
That is (in the notation of Lemma 8), $\alpha \geq 0$, $\beta \geq 1$ and $\gamma = 0$. So by this lemma, there exists an $a$ such that $pt_x(a) \in [p+1, p'] \times [q, q'-1]$ and $x(a) \neq w(a)$. Then $d_{x,w}|_{A_{a,b}} \geq 1$, so by Fact 7, $t_{a,b} \in R(x, w)$. This proves our claim.

To prove the case of $w(b) > x(b)$, it is easiest to use dual rank and difference functions:
\[
r_w'(p, q) := \#\{i \geq p : w(i) \leq q\},
\]
\[
d_{x,w}' := r_w' - r_x'.
\]
One can check that $x \leq w$ if and only if $d'_{x,w} \geq 0$ and then argue as above using this new rank function. (Note, to define $X_w$ using $r'_{w}$, we have to modify our fixed flag $F$ and (3.4).)

**Corollary 14.** If $x \leq w$ and $d_{x,w}(pt_x(b)) > 0$, then there exists $b' < b$ with $t_{b',b} \in \mathcal{R}(x,w)$.

Proposition 13 tells us that if $x(i) \neq w(i)$ then $i \in \Delta(x,w)$. It turns out that the question of whether or not $x \in \text{maxsing}(X_w)$ depends only on the pair $\bar{x}, \bar{w}$. This is borne out by the following simple facts. They will be used without comment in the remainder of the paper.

**Lemma 15.** We have the following:

1. If $x(i) = w(i)$, then $x^i \leq w^i \iff x \leq w$.
2. $\bar{x} \leq \bar{w} \iff x \leq w$.

**Proof.** The first equivalence follows from Lemma 3 by comparing $d_{x,w}$ and $d_{\bar{x},\bar{w}}$. The second follows from the first by noting that $i \in \Delta(x,w)$ whenever $x(i) \neq w(i)$.

**Proposition 16.** We have the following:

1. $l(w) - l(x) = l(\bar{w}) - l(\bar{x})$.
2. There exists a bijection $\mathcal{R}(\bar{x}, \bar{w}) \sim \mathcal{R}(x,w)$.
3. $x \in \text{maxsing}(X_w)$ if and only if $\bar{x} \in \text{maxsing}(X_{\bar{w}})$.

**Proof.** Pick some $i \notin \Delta(x,w)$. Now,

\[
(5.7) \quad l(w) - l(x) - (l(w^i) - l(x^i)) = d_{x,w}(pt_x(i)) + d'_{x,w}(pt_x(i)).
\]

We know by Corollary 14 that $d_{x,w}(pt_x(i)) = 0$. Applying Lemma 8 with $p = i$, $p' = n$, $q = 1$ and $q' = x(i)$, we see that $d'_{x,w}(pt_x(i))$ is also 0. This proves Part 1. Part 2 follows immediately from Fact 5 and the definition of $\Delta(x,w)$ by comparing $d_{x,w}$ and $d_{\bar{x},\bar{w}}$. Part 3 follows from the first two parts along with Corollary 24 (stated below).

6. The Map $\phi_t$

In Fact 10 we claimed that $\text{maxsing}(X_w)$ can be identified in terms of $\mathcal{R}(x,w)$ for $x \leq w$. To do this, we will need to relate $\mathcal{R}(x,w)$ to $\mathcal{R}(y,w)$ when $x,y$ differ by an element of $\mathcal{T}$. So, for every triple $yt < y \leq w$ with $t \in \mathcal{T}$, we will define a map $\phi^{yt,w}_t : \mathcal{R}(y,w) \rightarrow \mathcal{T}$. In Theorem 21 we will show that the image is actually contained in $\mathcal{R}(yt,w)$. The values of $y,w$ are usually clear from context and we will often abbreviate $\phi^{yt,w}_t$ as $\phi_t$.

A similar map has been defined by Gasharov [Gas00] for the purpose of showing that certain elements constructed by Lakshmibai and Sandhya in [LS90] are, in fact, singular points. See Section 11 for details.
Definition 17. Fix $yt < y \leq w$. Given some $t' \in \mathcal{R}(y, w)$, if $t$ and $t'$ commute, we define $\phi_t(t') = t'$. Otherwise, we can find $a < b < c$ such that $d \notin \{a, b, c\}$ implies $y(d) = yt(d) = yt'(d)$. Then we define $\phi_t^{y,w}(t')$ according to Table 1.

| Case | $\mathcal{R}$ | $t$ | $t'$ | $t_{a,c} \in \mathcal{R}(y, w)$ | $\phi_t(t')$ | $yt$ | $yt\phi_t^{y,w}(t')$ |
|------|----------------|-----|------|-------------------------------|---------------|------|---------------------|
| A.i) | $\mathbb{R}$  | 213 | $t_{a,b}$, $t_{a,c}$ | ✓ | $t_{b,c}$ | 231 | 132 |
| ii)  | $\mathbb{R}$  | 213 | $t_{a,b}$, $t_{b,c}$ | ✓ | $t_{a,c}$ | 231 | 321 |
| iii) | $\mathbb{R}$  | 213 | $t_{a,b}$, $t_{b,c}$ | ✓ | $t_{a,c}$ | 231 | 321 |
| B.i) | $\mathbb{R}$  | 132 | $t_{b,c}$, $t_{a,c}$ | ✓ | $t_{a,b}$ | 231 | 213 |
| ii)  | $\mathbb{R}$  | 132 | $t_{b,c}$, $t_{a,b}$ | ✓ | $t_{a,b}$ | 312 | 213 |
| iii) | $\mathbb{R}$  | 132 | $t_{b,c}$, $t_{a,b}$ | ✓ | $t_{a,c}$ | 312 | 321 |
| C.i) | $\mathbb{R}$  | 312 | $t_{a,b}$, $t_{b,c}$ | ✓ | $t_{a,c}$ | 321 | 231 |
| ii)  | $\mathbb{R}$  | 312 | $t_{a,c}$, $t_{b,c}$ | ✓ | $t_{b,c}$ | 321 | 231 |
| D.i) | $\mathbb{R}$  | 231 | $t_{b,c}$, $t_{a,b}$ | ✓ | $t_{a,c}$ | 321 | 312 |
| ii)  | $\mathbb{R}$  | 231 | $t_{a,c}$, $t_{a,b}$ | ✓ | $t_{a,b}$ | 321 | 312 |

### Table 1. Definition of map $\phi_t^{y,w}$. We have split into cases indexed by $t_{a,b,c}(y)$ and whether $t' = t_{a,b}$, $t' = t_{a,c}$ or $t' = t_{b,c}$. Note that the matter of inclusion of $t_{a,c}$ in $\mathcal{R}(y, w)$ is determined by the first three columns in Cases A.i, B.i, C and D. The final two columns are used in proving that $\phi_t$ maps $\mathcal{R}(y, w)$ into $\mathcal{R}(yt, w)$.

**Figure 7.** Graphical depiction of Case A from Definition 17. The dashed (resp. dotted, solid) arcs represent $t'$ (resp. $t$, $\phi_t(t')$).
Maximal Singular Loci of Schubert Varieties in $SL(n)/B$

Figure 8. Graphical depiction of Case C from Definition 17. The dashed (resp. dotted, solid) arcs represent $t'$ (resp. $t, \phi_t(t')$).

Remark 18. It is not sufficient to define $\phi_t(t_{b,c}) = t_{\{t(b),t(c)\}}$. For example, in the situation of Case C.ii, where $t = t_{a,c}$ and $t' = t_{b,c}$, we have $t_{\{t(b),t(c)\}} = t_{a,b} \not\in R(yt, w)$.

Example 19. In order to elucidate the definition, we give here several example of the map $\phi_t$. Let $w = [2,4,5,3,1]$ and $x_0 = [2,1,5,4,3]$. We see that $R(x_0, w) = \{t_{2,4}, t_{2,5}\}$ (see Figure 9.1). Let $x_1 = x_0 t_{1,2} < x_0$. We see that

1. $\phi_{t_{1,2}}(t_{2,4}) = t_{2,4}$ (Case A.ii).
2. $\phi_{t_{1,2}}(t_{2,5}) = t_{2,5}$ (Case A.ii).

Now, $R(x_1, w) = \{t_{1,2}, t_{2,4}, t_{2,5}\}$ (see Figure 9.2). Let $x_2 = [1,2,5,3,4] = x_1 t_{4,5} < x_1$. We see that

1. $\phi_{t_{4,5}}(t_{1,2}) = t_{1,2}$ (Because $t_{4,5} t_{1,2} = t_{1,2} t_{4,5}$).
2. $\phi_{t_{4,5}}(t_{2,4}) = t_{2,5}$ (Case B.iii).
3. $\phi_{t_{4,5}}(t_{2,5}) = t_{2,4}$ (Case B.i).

Now $R(x_2, w) = \{t_{1,2}, t_{2,4}, t_{2,5}, t_{4,5}\}$. Let $x_3 = [1,2,3,5,4] = x_2 t_{3,4} < x_2$ (see Figure 9.3). We see that

1. $\phi_{t_{3,4}}(t_{1,2}) = t_{1,2}$ (Because $t_{3,4} t_{1,2} = t_{1,2} t_{3,4}$).
2. \( \phi_{t_{3,4}}(t_{2,4}) = t_{2,3} \) (Case B.i).
3. \( \phi_{t_{3,4}}(t_{2,5}) = t_{2,5} \) (Because \( t_{3,4}t_{2,5} = t_{2,5}t_{3,4} \)).
4. \( \phi_{t_{3,4}}(t_{4,5}) = t_{3,5} \) (Case C.i).

**Remark 20.** It is possible for \( \# R(yt, w) > \# R(y, w) + 1 \) for \( yt < y \leq w \) and \( t \in T \). For example, let \( w = [4, 2, 3, 1], y = [2, 4, 1, 3] \) and \( x = yt_{2,3} = [2, 1, 4, 3] \). Then \( R(y, w) = \{t_{1,2}, t_{3,4}\} \) and \( R(yt, w) = \{t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}\} \).

It is clear from the definition that \( \phi_t(t') \) is always a reflection and that \( yt < yt \cdot \phi_t(t') \). But for \( \phi_t \) to be useful, we will need the following property.

**Theorem 21.** Fix \( yt < y \leq w \). The map \( \phi_t(R(y, w)) \hookrightarrow R(yt, w) \setminus \{t\} \) is injective.

For the proof of this theorem, we’ll need the following simple lemma.

**Lemma 22.** Let \( u, w \in S_n \) and suppose \( 1 \leq i < j < k \leq n \) such that \( f_{ijk}(u) = 123 \). If both

\[
\begin{align*}
(6.1) & \quad w \geq x = u \circ (k, j, i) \quad (i.e., \overline{x} = 312) \quad \text{and} \\
(6.2) & \quad w \geq y = u \circ (i, j, k) \quad (i.e., \overline{y} = 231)
\end{align*}
\]

then \( w \geq z = u \circ (i, k) \) (i.e., \( \overline{z} = 321 \)).

**Proof.** Notice that \( d_{z,w}(p, \cdot) = d_{x,w}(p, \cdot) \) for \( p < j \) and \( d_{z,w}(p, \cdot) = d_{y,w}(p, \cdot) \) for \( p \geq j \). By Lemma 3, \( v \leq w \) if and only if \( d_{v,w} \geq 0 \). Since \( x, y \leq w, d_{x,w}, d_{y,w} \geq 0 \). Combining this with our first observation implies that \( z \leq w \).

Now we are ready to prove Theorem 21.

**Proof.** First we show that \( \phi_t(R(y, w)) \subset R(yt, w) \) — i.e., \( yt < yt \cdot \phi_t(t') \leq w \) for all \( t' \in R(y, w) \). It is clear from the definition that \( yt \leq yt \cdot \phi_t(t') \). So this amounts to showing that one of the two dotted arrows in Figure 10 (corresponding to relation under the Bruhat order) exists.

\[
\begin{aligned}
&\text{Figure 10.} \\
\end{aligned}
\]

First suppose \( t't = tt' \) — hence \( \phi_t(t') = t' \). We wish to show that \( yt' > yt\phi_t(t') = ytt' = yt't \). Letting \( t = t_{a,b} \) and \( t' = t_{c,d} \), this reduces to showing
that \( yt'(a) > yt'(b) \). Now \( t't = tt' \) implies that \( \{a, b\} \cap \{c, d\} = \emptyset \). So \( yt'(a) > yt'(b) \) if \( y(a) > y(b) \). But this last inequality holds by choice of \( t \).

Now suppose that \( t't \neq tt' \). In all of the cases in Table \( \mathcal{I} \) except A.iii and B.iii, \( y\phi(t'(t')) \leq y\phi(t) \). Since \( y\phi(t'(t')) \) and \( y\phi(t) \) agree at all indices except \( a, b, c \), we can use Lemma \( \mathcal{I} \) to conclude that \( y\phi(t'(t')) \leq y\phi(t) \leq w \). So \( \phi(t'(t')) \in \mathcal{R}(yt, w) \) for all cases except possibly A.iii and B.iii.

In Case A.iii, we know that both \( t_{a,c}, t_{b,c} \in \mathcal{R}(y, w) \) — i.e., \( y < y\phi(t_{a,c}, y\phi(t_{b,c}) \leq w \). Since \( y\phi(t_{a,c}) = 312 \) and \( y\phi(t_{b,c}) = 231 \), we can therefore invoke Lemma \( \mathcal{I} \) to conclude that \( w \geq 321 = y\phi(t') \). Case B.iii is similar.

It is clear from Table \( \mathcal{I} \) that \( \phi(t'(t')) \) can share at most one index with \( t \). As we have already shown the inclusion \( \phi(t'(t')) \subseteq \mathcal{R}(yt, w) \), we conclude that \( \phi(t'(t')) \subseteq \mathcal{R}(yt, w) \setminus \{t\} \).

Now we show that \( \phi(t') \) is an injection. Suppose \( t = t_{i,j} \), \( t' = t_{i,j,k} \in \mathcal{R}(y, w) \). No matter which case of Table \( \mathcal{I} \) we are in, we see that \( \phi(t') = t_{i,j,k} \) or \( t_{i,j,k} \). In other words, the index \( t' \) doesn’t share with \( t \) must be an index of \( \phi(t') \). In particular, if \( t', t'' \in \mathcal{R}(y, w) \) such that \( \phi(t') = \phi(t'') \), the index they don’t share with \( t \) must be the same. It is then easy to check by inspection of Table \( \mathcal{I} \) that \( \phi(t') = \phi(t'') \) implies that \( t' = t'' \). Since \( \phi(t') = t' \) whenever \( t \) and \( t' \) don’t share any indices, we conclude that \( \phi(t') \), in fact, an injection.

The above theorem has as a simple corollary a special case of Deodhar’s conjecture \[\text{Deo85}].

**Corollary 23.** \#\( \mathcal{R}(y, w) \geq l(w) - l(y) \).

Various forms and generalizations of the preceding corollary have been proven by Dyer \[\text{Dye93} \] (arbitrary Coxeter systems), Deodhar \[\text{Deo85} \] (type A), Polo \[\text{Pol94} \] (finite Weyl groups) and Carrell-Peterson \[\text{Car94} \] (crystallographic groups).

Recall from Definition \[\mathcal{I} \] that

\[(6.3) \quad \Delta(x, w) = \{i, 1 \leq i \leq n : \exists j, 1 \leq j \leq n, \text{ with } t_{i,j} \in \mathcal{R}(x, w)\}. \]

From the proof of Theorem \[\mathcal{I} \], we obtain the following:

**Corollary 24.** For \( t \in \mathcal{T} \), \( yt < y \leq w \) implies that \( \Delta(y, w) \subseteq \Delta(yt, w) \).

Next, we see that when \( t \) is an adjacent transposition with \( wt < w \), \( \phi_{t}^{b,w} \) surjects onto \( \mathcal{R}(yt, w) \setminus \{t\} \). It would be interesting to classify all such \( t \) for which this happens.

The following fact about the Bruhat order will in useful in the proof of the next proposition and later in the paper. An analogous left-handed version exists.

**Lemma 25** \[\text{Hum90}, \text{Lemma} 7.4\]. If \( s \in \mathcal{S} \) and \( ws < w \), then \( xs \leq w \Leftrightarrow x < w \).
Proposition 26. Let \( s_i \in S \) (i.e., \( s_i \) is an adjacent transposition) and suppose \( y, w \) satisfy the relations \( y s_i < y \leq w \) with \( w s_i < w \). Then \( R(y s_i, w) = \phi_{s_i}(R(y, w)) \cup \{ s_i \} \) and \( e_y \) is smooth in \( X_w \) if and only if \( e_{y s_i} \) is smooth in \( X_w \).

Proof. The last statement follows immediately from the decomposition of \( R(y s_i, w) \). So, taking advantage of Theorem \( \text{(21)} \), we need only show that if \( t_{j,k} \in R(y s_i, w) \setminus \{ s_i \} \), then \( \phi_{s_i}^{-1}(t') \neq \emptyset \).

First consider the case where \( \{ i, i+1 \} \cap \{ j, k \} = \emptyset \). Then \( y t_{j,k} = y s_i t_{j,k} = y s_i t_{j,k} s_i \). As \( t_{j,k} \in R(y s_i, w) \), we can apply Lemma \( \text{(25)} \) to conclude that \( t_{j,k} \in R(y, w) \). Finally, since \( \{ i, i+1 \} \cap \{ j, k \} = \emptyset \), \( \phi_{s_i}(t_{j,k}) = t_{j,k} \). So \( \phi_{s_i}^{-1}(t_{j,k}) \neq \emptyset \) in this case.

Now, as in the proof of injectivity, we can restrict to the cases where \( \{ i, i+1 \} \cap \{ j, k \} \neq \emptyset \). We group into cases according to \( \overline{ys} \), \( \overline{ys} \) and \( t' \). Cases I, III, V have \( i \in \{ j, k \} \); Cases II, IV, VI have \( i + 1 \in \{ j, k \} \). In each case we show that \( t' \in \text{Im} \phi_{s_i} \).

| Case | \( \overline{ys} \) | \( \overline{ys} \) | \( t' \) | \( \overline{ys}t' \) | \( \phi_{s_i}^{-1}(t') \) |
|------|-----------------|-----------------|--------|-----------------|-----------------|
| I    | 123             | 213             | \( t_{a,c} \) | 321             | Since \( w \geq 321 \), \( w \geq 312 = yt_{a,c} \) and \( w \geq 231 = yt_{b,c} \). So we are in Case A.iii and we see that \( t_{a,c} = \phi_{s_i}(t_{b,c}) \). |
| II   | 123             | 132             | \( t_{a,c} \) | 321             | Case B.iii — analogous to I. |
| III  | 123             | 213             | \( t_{b,c} \) | 132             | By Lemma \( \text{(25)} \), \( 312 \leq w \) if and only if \( 132 \leq w \). The latter inequality is true since \( ys_i t' = 132 \). So \( t_{a,c} \in R(y, w) \). We’re in Case A.i and \( t_{b,c} = \phi_{s_i}(t_{a,c}) \). |
| IV   | 123             | 132             | \( t_{a,b} \) | 213             | Case B.i — analogous to III. |
| V    | 132             | 312             | \( t_{a,c} \) | 231             | \( w \geq y = 312, w \geq y s_i t' = 321 \). Hence, by Lemma \( \text{(22)} \), \( w \geq 321 \). So \( t_{b,c} \in R(y, w) \). We’re in Case C.i and \( t_{a,c} = \phi_{s_i}(t_{b,c}) \). |
| VI   | 213             | 231             | \( t_{a,c} \) | 312             | Case D.i — analogous to V. |

Cases I-VI are the only ones possible since \( \overline{ys} = 231, 312, 321 \) cannot have \( s, t' \) as hypothesized.

Proposition 27. Let \( s_i \in S \) (i.e., \( s_i \) is an adjacent transposition) and suppose \( y, w \) satisfy the relations \( s_i y < y \leq w \) with \( s_i w < w \). Then \( R(s_i y, w) = y^{-1}s_i \phi_{s_i}(R(y^{-1}, w^{-1}))s_i y \cup \{ y^{-1}s_i y \} \).

Proof. By Proposition \( \text{(26)} \), \( R(y^{-1}s_i, w^{-1}) = \phi_{s_i}(R(y^{-1}, w^{-1})) \cup \{ s_i \} \). The result follows from the identity \( R(x, w) = x^{-1}R(x^{-1}, w^{-1})x \).

Corollary 28. If \( x \leq w \) is an MSP, \( s, s' \) any simple reflections, then \( sw < w \) or \( ws' < w \) imply, respectively, that \( sx < x \) or \( xs' < x \).
The preceding corollary is well-known [BL00, 8.2.10]. However, Proposition 29 gives a different proof.

There is one more fundamental property of \( \phi_t \) that we will need to know for the rest of the paper. Namely, for a pair of reflections \( t, t' \) where \( t \in \text{Im} \phi_t \), it will be useful to know what we can say about the membership of \( t' \) in \( \text{Im} \phi_t \).

**Proposition 29** (Reciprocity). If \( t, t' \in \mathcal{R}(x, w) \), \( t \neq t' \), with \( l(xt) = l(xt') = l(x) + 1 \), then \( t' \in \text{Im} \phi_{xt, w} \iff t \in \text{Im} \phi_{x't', w} \).

**Remark 30.** Reciprocity does not necessarily hold if \( l(xt') > l(x) + 1 \). For example, take \( x = [1, 2, 3], w = [3, 2, 1], t = t_{1,2} \) and \( t' = t_{1,3} \). Then \( t' \in \text{Im} \phi_{xt, w} \) but \( t \not\in \text{Im} \phi_{x't', w} \).

**Proof.** Suppose \( t \in \text{Im} \phi_t \). We will show that \( t' \in \text{Im} \phi_t \).

First, consider the case where \( t't' = t't \). From the definition of \( \phi_t \), we see that \( \phi_t^{-1}(t) = t \). So \( w \geq xt't = xt' \). This implies that \( t' \in \mathcal{R}(x, w) \) and therefore \( \phi_t(t') = t' \).

Now we suppose \( t't' \neq t't \). So \( a < b < c \) are determined such that \( d \not\in \{a, b, c\} \) implies \( x(d) = xt(d) = xt'(d) \). Let \( \varpi = \pi_{a,b,c}(x) \). Note that:

1. By hypothesis, \( l(xt) = l(xt') = l(x) + 1 \).
2. If \( \varpi \in \{231, 312, 321\} \) then at most one of \( t_{a,b}, t_{a,c}, t_{b,c} \in \mathcal{R}(x, w) \); not two.

Hence, the cases below are the only ones we need consider.

1. \( \varpi = 132 \).
   Then \( \{t, t'\} = \{t_{a,b}, t_{a,c}\} \) and \( w \geq \overline{312}, \overline{321} \). By Lemma 22, \( w \geq \overline{321} \).
   So \( t_{a,b} \in \mathcal{R}(xt_{a,c}, w) \) and, as we are in Case D.ii of Definition 17, \( t_{a,b} = \phi_{t_{a,c}, t_{a,b}} \).
   Similarly, \( t_{b,c} \in \mathcal{R}(xt_{a,b}, w) \) and, as we are in Case C.i of Definition 17, \( t_{a,c} = \phi_{t_{a,b}, t_{b,c}} \).
2. \( \varpi = 213 \).
   The argument is parallel to that in the previous case.
3. \( \varpi = 123 \).
   Here \( \{t, t'\} = \{t_{a,b}, t_{b,c}\} \). (Note that \( t_{a,c} \) is not considered since \( l(xt_{a,c}) > l(x) + 1 \)) Suppose \( t_{a,b} \in \text{Im} \phi_{t_{a,b}} \). From Case B of Definition 17, this implies that

\[
\begin{cases}
w \geq \overline{312} \\
w \geq \overline{231}
\end{cases}
\Rightarrow
\begin{cases}
t_{a,c} \in \mathcal{R}(xt_{a,b}, w) \\
t_{b,c} \in \mathcal{R}(xt_{a,b}, w)
\end{cases}
\]

Then, from Case A of Definition 17, we see that \( t_{b,c} \in \text{Im} \phi_{t_{a,b}} \). The argument is analogous if we instead assume \( t_{b,c} \in \text{Im} \phi_{t_{a,b}} \).
7. Preparatory lemmas

Let $xt < x \leq w$. We make the following observation (see Theorem 9): If
\begin{equation}
\#\phi_t(\mathcal{R}(x, w)) < \#\mathcal{R}(xt, w) - l(xt) + l(x),
\end{equation}
then $e_{xt}$ is a singular point of $X_w$.

The above fact is most conveniently expressed in terms of the following notation:

**Definition 31.** For $x < w$ and $t \in \mathcal{R}(x, w)$, let
\begin{equation}
E_t(x, w) = \mathcal{R}(x, w) \setminus \{t\} \cup \phi_t(\mathcal{R}(xt, w))
\end{equation}
denote the set of “extra” reflections corresponding to $x$ and $t$. We often write $E_{a,b}(x, w)$ for $E_{t_{a,b}}(x, w)$.

If $t' \in E_t(x, w)$, then we say that $t$ and $t'$ are incompatible edges (in the Bruhat graph). The elements of $E_t(x, w)$ are “extra” edges in the sense that they correspond to an increase in the dimension of the Zariski tangent space.

The utility of $E_t(x, w)$ is embodied in the following two facts.

**Fact 32.** If $t, t' \in \mathcal{R}(x, w)$ with $t' \in E_t(x, w)$ and $l(xt) = l(x) + 1$, then $x < w$ is singular.

**Fact 33.** $x$ is an MSP for $w$ if and only if, for every $t \in \mathcal{R}(x, w)$ with $l(xt) = l(x) + 1$, $E_t(x, w) \neq \emptyset$.

Note, however, that if $x$ is a singular point, but not an MSP, then it is possible that $E_t(x, w) = \emptyset$. An example is afforded by $w = [4, 2, 3, 1]$, $x = [1, 2, 3, 4]$ and $t = t_{1, 2}$. Conversely, if $l(xt) > l(x) + 1$, then we may have $E_t(x, w) \neq \emptyset$ even if $X_w$ is entirely smooth. Take, for example, $w = [3, 2, 1]$, $x = [1, 2, 3]$, and $t = t_{1, 3}$.

There will be numerous instances in the remainder of the paper where we do the following:

1. Assume we have an MSP $x$ for $X_w$.
2. Construct some $y = xt'' > x$.
3. Conclude that $y < w$ from the fact that $A_w(x)$ is shaded.
4. Find incompatible edges $t, t'$ as in Fact 32 to conclude that $y$ is also a singular point of $X_w$.
5. Obtain a contradiction with our first assumption.

The previous technique will allow us to significantly pare down the possibilities for what $\bar{x}$ looks like for $x$ an MSP. The following lemma is the first example of this strategy.

**Lemma 34 (Ell Lemma).** Let $x \leq w$ and $1 \leq i < j < k \leq n$.

1. If $f_{ijk}(x) = 213$ and $t_{i,k}, t_{j,k} \in \mathcal{R}(x, w)$, then $t_{i,k} \in \text{Im} \phi_{t_{i,k}}^{x_{t_{i,k},w}}$ and $t_{j,k} \in \text{Im} \phi_{t_{j,k}}^{x_{t_{i,k},w}}$. 

---

**Note:** The above text contains mathematical expressions and notation that are typical for a research paper in mathematics, particularly in the fields of algebraic geometry or group theory. The lemmas and facts presented are foundational to understanding the subsequent arguments and proofs in the document. The use of specific notation, such as $\mathcal{R}$ for a reflection set and $\phi_t$ for a reflection map, indicates a formal and rigorous approach to the discussion.
2. If $f_{ijk}(x) = 132$ and $t_{i,j}, t_{i,k} \in \mathcal{R}(x,w)$, then $t_{i,j} \in \text{Im} \phi_{t_{i,k}}^{x_{t_{i,k}}}w$ and $t_{i,k} \in \text{Im} \phi_{t_{i,j}}^{x_{t_{i,j}}}w$.

**Proof.** We only prove 1 as the proof for 2 is entirely analogous. Diagrams for $x$, $xt_{i,k}$ and $xt_{j,k}$ are given in Figure 11.

![Diagrams](image)

**Figure 11.**

We see that $t_{i,k} \in \mathcal{R}(x,w)$ implies $w \geq \hat{312}$ and $t_{j,k} \in \mathcal{R}(x,w)$ implies $w \geq \hat{231}$. So, by Lemma 22, $xt_{i,k}t_{j,k} = xt_{j,k}t_{i,j} = 321 \leq w$. Equivalently, $t_{j,k} \in \mathcal{R}(xt_{i,k},w)$ and $t_{i,j} \in \mathcal{R}(xt_{j,k},w)$. So, (Case C.ii of Definition 17) $\phi_{xt_{i,k}}^{x_{t_{i,k}}}w(t_{j,k}) = t_{j,k}$ and (Case D.i of Definition 17) $\phi_{xt_{j,k}}^{x_{t_{j,k}}}w(t_{i,j}) = t_{i,k}$. □

The next lemma is used frequently. It gives us criteria for determining when two reflections are, in fact, incompatible.

**Lemma 35.** Let $t_{a,b} \in \mathcal{R}(x,w)$.

1. **Patch Incompatibility.**
   
   If $t_{c,d} \in \mathcal{R}(x,w)$ with $\{a,b\} \cap \{c,d\} = \emptyset$, then $t_{a,b} \in \mathcal{E}_{c,d}(x,w)$ if and only if $\min(d_{x,w}|A) = 1$ (with region $A$ as in Figure 12.1).

2. **Link Incompatibility.**
   
   If $t_{b,c} \in \mathcal{R}(x,w)$, then
   
   $$t_{a,b} \in \mathcal{E}_{b,c}(x,w) \iff t_{b,c} \in \mathcal{E}_{a,b}(x,w)$$
   
   $$\iff \min(d_{x,w}|B) = \min(d_{x,w}|C) = 0,$$
   
   (7.3)
   
   (where regions $B$ and $C$ are as in Figure 12.2).

![Diagrams](image)

**Figure 12.** In 1, we display only one possible configuration where $t_{a,b}$ and $t_{c,d}$ are patch incompatible.

**Proof.** The proof of Patch Incompatibility is clear. To prove Link Incompatibility, it suffices to consider Cases A and B of Definition 17. □
In Fact 32 we give a sufficient condition for $x$ to be a singular point of $X_w$ that is expressed in terms of the map $\phi_t$. Namely, $x$ is singular if $E_t(x, w)$ is non-empty for some $t \in R(x, w)$ with $l(xt) = l(x) + 1$. As the lemma below shows, many elements of $R(x, w)$ aren’t even candidates to be elements of $E_t(x, w)$.

**Lemma 36.** Let $t_{a,b}, t_{c,d} \in R(x, w)$. If $A_{a,b} \cap A_{c,d} = \emptyset$, then $t_{a,b} \notin E_{c,d}(x, w)$ (i.e. $t_{a,b} \in \text{Im } \phi_{t_{c,d}}$).

**Proof.** For any point $\circ \in A_{a,b}(x)$, $d_{xt_{a,b},w}(\circ) = d_{x,w}(\circ) - 1$. Similarly for the pair $t_{c,d}$ and $A_{c,d}(x)$. Now, $A_{a,b} \cap A_{c,d} = \emptyset$ implies that $A_{a,b}(x) = A_{a,b}(xt_{c,d})$ and $d_{xt_{c,d},w} \geq 1$ on $A_{a,b}(xt_{c,d})$. This implies that $t_{a,b} \in R(xt_{c,d}, w)$ and $t_{a,b} \in \text{Im } \phi_{t_{c,d}}$.

The following lemma is technically useful for Proposition 38.

**Lemma 37.** Let $x < w$, $t_{a,b} \in R(x, w)$, $l(xt_{a,b}) = l(x) + 1$, and $t_{c,d} \in E_{a,b}(x, w)$. If $pt_x(a) \in A_{c,d}(x)$ or $pt_x(b) \in A_{c,d}(x)$, then $x$ is not an MSP for $w$.

**Proof.** First consider the case where both $pt_x(a), pt_x(b) \in A_{c,d}(x)$. Suppose there is a point $pt_x(f)$ in region A of Figure 13.1. Choose such an $f$ as small as possible. Then we see that $t_{c,b}$ and $t_{a,f}$ are patch incompatible reflections for $x' = xt_{b,d}t_{f,d} \leq w$ and $l(x't_{a,f}) = l(x') + 1$. By Fact 32, $x'$ is then singular. This contradicts the fact that $x$ is an MSP for $w$.

Now suppose region A of Figure 13.1 is empty — this is shown in Figure 13.2. Then $t_{c,b}$ and $t_{a,d}$ are incompatible reflections for $xt_{b,d} \leq w$ and $l(xt_{b,d}t_{a,d}) = l(xt_{b,d}) + 1$. Since $x < xt_{b,d}$, this contradicts the fact that $x$ is an MSP for $w$.

We now argue the case of $pt_x(b) \in A_{c,d}(x) \neq pt_x(a)$. (The arguments for $pt_x(b) \notin A_{c,d}(x) \ni pt_x(a)$ are parallel.)

Clearly $d > b$ and $x(d) > x(b)$. There are four possibilities with regard to the position of $pt_x(c)$.

1. $c = a$.

We are in Case A.iii of the definition of $\phi$. Hence, $t_{c,d} \in \text{Im } \phi_{t_{a,b}}$, which is a contradiction. So this case cannot occur.

2. $c > a$, $x(c) > x(a)$.

This case cannot occur as it violates $l(xt_{a,b}) = l(x) + 1$. 

![Figure 13.1 and 13.2](image-url)
3. $c > a$, $x(c) < x(a)$.
   This case is depicted in Figure 14.1. Suppose $l(xt_{b,d}t_{a,d}) = l(xt_{b,d}) + 1$. Then $t_{a,d}$ and $t_{c,b}$ are patch incompatible for $xt_{b,d} \leq w$. This contradicts the fact that $x$ is an MSP for $w$. If $l(xt_{b,d}t_{a,d}) > l(xt_{b,d}) + 1$, then we can argue as in Figure 13.1 to obtain our contradiction.

4. $c < a$, $x(c) > x(a)$.
   See Figure 14.2. This is analogous to the previous case.

Proposition 38 below gives us our first non-trivial restriction regarding the composition of $R(x, w)$. This proposition will greatly reduce the amount of work we need to do later on to determine possibilities for $\tilde{x}$.

**Proposition 38.** Let $x < w$ be an MSP. If $t \in R(x, w)$ then $l(xt) = l(x) + 1$.

**Proof.** Suppose that $t \in R(x, w)$ and $l(xt) > l(x) + 1$. We will obtain a contradiction.

Let $t = t_{a,c}$. Choose $b$ as large as possible such that $pt_{c}(b) \in A_{a,c}(x)$. Note that $t_{a,b}, t_{b,c} \in R(x, w)$ and $l(xt_{b,c}) = l(x) + 1$. Since $x$ is an MSP, we can invoke Fact 33 to find a $t_{e,f} \in E_{b,c}(x, w)$.

Suppose $A_{a,b}(x) \cap A_{e,f}(x) = \emptyset$. Since $t_{a,c} \in R(x, w)$, $A_{a,c}$ is shaded so $t_{b,c} \in R(xt_{a,b}, w)$. Hence $t_{b,c}$ and $t_{e,f}$ are incompatible for $xt_{a,b} \leq w$ and $l(xt_{a,b}t_{b,c}) = l(xt_{a,b}) + 1$. This contradicts $x \in \text{maxsing}(X_w)$.

Otherwise, $A_{e,f}$ overlaps both $A_{a,b}$ and $A_{b,c}$, so, by Lemma 37, we are in one of the following two scenarios.

1. $e = b$.
   By choice of $b$, $f \not\in A_{b,c}(x)$. (Note that $f \neq c$.) So either $f > c$, $x(f) < x(c)$ or $f < c$, $x(f) > x(c)$ (the latter case is shown in Figure 13.1). In either case, we can apply the Ell Lemma 34 to conclude that $t_{e,f} \in \text{Im} \phi_{b,c}$. This contradicts the choice of $t_{e,f}$.

2. $f = b$.
   Since $t_{a,c} \in R(x, w)$, for $t_{e,f}$ to be an element of $E_{b,c}(x, w)$, we need $e < a$, $x(e) < x(a)$ and $d_{x,w} = 0$ for some point in each of regions A and B in Figure 13.2. But then $t_{e,b}$ and $t_{b,c}$ are link incompatible for $xt_{a,b} \leq w$. Furthermore, by having chosen $b$ as large as possible, we ensure that $l(xt_{a,b}t_{b,c}) = l(xt_{a,b}) + 1$. This contradicts $x \in \text{maxsing}(X_w)$.
We are now able to give a graphical description of all possible pairs $t_{a,b}, t_{c,d} \in \mathcal{R}(x, w)$ such that $t_{c,d} \in \mathcal{E}_{a,b}(x, w)$ when $x \in \maxsing(X_w)$. The result follows immediately from Lemma 36 and Proposition 38 along with the Ell Lemma 34. Note that by Proposition 29, we can assume, without loss of generality, that $a < c$.

**Corollary 39.** If $t_{a,b} \in \mathcal{R}(x, w)$ and $t_{c,d} \in \mathcal{E}_{a,b}(x, w)$, then the relative positions of $A_{a,b}$ and $A_{c,d}$ are one of the ones shown in Figure 16.

This greatly simplifies our future investigations. We now use Corollary 39 and Lemma 8 to prove the following crucial lemma.

**Lemma 40 (Cross Lemma).** Let $x < w$ be an MSP and suppose $1 \leq i < j < k < l \leq n$ such that $\mathsf{fl}_{ijkl}(x) = 2143$. If $t_{j,k} \in \mathcal{R}(x, w)$ and $t_{i,l} \in \mathcal{E}_{j,k}(x, w)$, then $t_{i,k}, t_{j,l} \in \mathcal{R}(x, w)$.

**Proof.** We can visualize the situation as in Figure 17. Since $t_{i,l} \in \mathcal{E}_{j,k}(x, w)$,
there is necessarily a point $\triangle$ in region A for which $d_{x,w}(\triangle) = 1$. Suppose $t_{i,k} \notin R(x,w)$. Then there is a point $\square$ in region B such that $d_{x,w}(\square) = 0$. Then we can apply Lemma 8 (with $\alpha, \beta \geq 1, \gamma = 1$) and Proposition 38 to conclude that there is a point $pt_x(p)$ of mat$(x)$ in region $C$ (see Figure 17.2). If we choose $\square$ to be as low as possible in our diagram, then $d_{x,w}|D \geq 1$ (see Figure 17.3). But then $t_{i,l}$ and $t_{j,k}$ are patch incompatible for $xt_{p,k} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. Therefore $t_{i,k} \in R(x,w)$ and we can shade the entire region B.

To shade the lower left corner, apply the preceding argument to $x^{-1}$ and $w^{-1}$.

8. Restrictions on $\bar{x}$

Recall that $\bar{x}$ and $\bar{w}$ are the restrictions of $x$ and $w$ to those positions in $\Delta(x,w)$ (see Definition 11). In order to determine the structure of $\text{maxsing}(X_w)$, we first prove the following necessary conditions on $\bar{x}$ for any MSP $x$ for $w$.

**Theorem 41.** If $x < w$ is an MSP, then $\bar{x}$ is 231- and 312-avoiding.

**Theorem 42.** If $x < w$ be an MSP, then $\bar{x}$ is 1234-avoiding.

These two theorems are, in fact, almost enough to describe $\bar{x}$ for any MSP $x$.

8.1. Technical lemmas regarding $\bar{x}$. In order to streamline the proof of Theorem 41, we first present two technical lemmas. These lemmas simply show that mat$(\bar{x})$, for $x$ an MSP, must avoid certain patterns of points and shading.

**Lemma 43.** If $x < w$ is an MSP, then mat$(\bar{x})$ does not contain either of the configurations in Figure 18 (regardless of whether or not these reflections are incompatible).

![Figure 18](image)

**Proof.** We only prove that mat$(\bar{x})$ must avoid the configuration in Figure 18.1. The proof of the other case is parallel.

First suppose that $\min(d_{x,w}|A) = 1$ (see Figure 18.1). Then by the Cross Lemma 40, $t_{a,\gamma} \in R(x,w)$. But this contradicts Proposition 38 as $l(xt_{a,\gamma}) >$
l(x) + 1. We get a similar contradiction if min(d_{x,w}|B) = 1. So we can henceforce assume that our configuration is actually as in Figure 19 — i.e., min(d_{x,w}|A), min(d_{x,w}|B) ≥ 2.

Figure 19. Recall that dark shading denotes regions on which d_{x,w} ≥ 2.

Since x is an MSP, we must have some reflection t_{d,δ} ∈ E_{c,γ}(x, w). Suppose

(8.1) \[ A_{d,δ} \cap A_{a,α} = \emptyset \]
(8.2) \[ A_{d,δ} \cap A_{b,β} = \emptyset \]

In the case of (8.1), t_{c,γ} and t_{d,δ} are incompatible for xt_{a,α} ≤ w. This contradicts x ∈ maxsing(X_w). (Similarly for (8.2).) So these intersections must be non-empty. It is clear from Figure 19 that for these intersections to be non-empty, we need t_{d,δ} to be patch (rather than link) incompatible with each of t_{c,γ}, t_{a,α} and t_{b,β}. By Proposition 38 and because the intersections in (8.1) and (8.2) must be non-empty, it is readily seen that we require δ < α and d > b. Hence, there are only four possible ways in which A_{d,δ} may overlap A_{a,α}, A_{b,β} and A_{c,γ}. These are shown in Figure 20.

1. d > c, δ > γ.

If min(d_{x,w}|C) ≥ 2 then t_{c,γ} and t_{d,δ} are patch incompatible for xt_{a,α} ≤ w. This contradicts x ∈ maxsing(X_w). Otherwise, we can apply the Cross Lemma 40 to t_{d,δ} and t_{a,α} to conclude that t_{a,δ} ∈ R(x, w). This contradicts Proposition 38.

2. d < c, δ < γ.

The argument is parallel to the previous case.

3. d < c, δ > γ.

If min(d_{x,w}|C) = 1 or min(d_{x,w}|D) = 1 then we can apply the Cross Lemma 40 to t_{d,δ} and t_{a,α} to conclude that t_{a,δ} ∈ R(x, w). This contradicts Proposition 38. The only alternative is that d_{x,w} ≥ 2 on regions C and D. But then t_{d,δ} and t_{c,γ} are patch incompatible for xt_{a,α} ≤ w. This contradicts x ∈ maxsing(X_w).

4. d > c, δ < γ.

Here t_{c,γ} and t_{d,δ} are patch incompatible for xt_{a,α} ≤ w. This contradicts x ∈ maxsing(X_w).
Lemma 44. If $x < w$ is an MSP, then $\mat(\x)$ does not contain the configuration in Figure 21.

Proof. Since $x$ is an MSP for $w$, there exists some $t_{d,\delta} \in \mathcal{E}_{c,\gamma}(x, w)$. Clearly, if

\begin{equation}
\mathcal{A}_{d,\delta} \cap \mathcal{A}_{c,\alpha} = \emptyset,
\end{equation}

then $t_{c,\gamma}$ and $t_{d,\delta}$ are (patch or link) incompatible reflections for $xt_{c,\gamma} \leq w$. This would contradict $x \in \maxsing(X_w)$. 
So, to ensure that (8.3) does not hold, we need $p_\alpha(d)$ in region A of Figure 22.1 and $p_\gamma(\delta)$ in region B. Here we are including the possibilities that $d = a$ or $\delta = c$. Note that (as is shown in Figure 22.1) Proposition 38 requires that $x(\delta) < x(b)$ and $x(d) > x(\beta)$. Clearly if $d = a$ then $\delta \neq c$ and vice versa. Hence, by symmetry, we can treat only the cases where $\delta \neq c$. These two cases are illustrated in Figures 22.2 and 22.3.

Figure 22.

For both cases, we can apply the Cross Lemma 40 to $t_{d,\delta}$ and $t_{c,\gamma}$ to conclude that $t_{d,\gamma} \in R(x, w)$. Then $l(xt_{d,\delta}) > l(x) + 1$, which contradicts Proposition 38.

8.2. Proofs of Theorems 41 and 42.

Theorem 41. By passing to inverses, it is enough to prove that $\tilde{x}$ is 231-avoiding. So choose $a, b, c \in \Delta(x, w)$ with $1 \leq a < b < c \leq n$ such that $f_{abc} = 231$.

Figure 23. $t_{a,b} \in R(x, w)$.

1. Assume $t_{a,b} \in R(x, w)$.

We have the situation of Figure 23. By definition of $\Delta(x, w)$, there exists a $d \in \Delta(x, w)$ with $t_{c,d} \in R(x, w)$. We’ll assume that $c < d$ as all cases where $d < c$ are analogous to one of the cases we cover by transposing over the antidiagonal. Clearly $A_{a,b} \cap A_{c,d} = \emptyset$.

Since $x$ is an MSP, there exists a $t_{\alpha,\beta} \in E_{a,b}(x, w)$. We claim that

\[ A_{\alpha,\beta} \cap A_{c,d} \neq \emptyset. \]
Suppose the intersection is empty. Then \( t_{a,b} \) and \( t_{\alpha,\beta} \) are (patch or link) incompatible for \( xt_{c,d} \leq w \). This contradicts \( x \in \text{maxsing}(X_w) \). So we may assume that this intersection is non-empty. There are three cases according to whether \( t_{a,b} \) and \( t_{\alpha,\beta} \) are

(a) link incompatible
(b) patch incompatible with \( \alpha < a \)
(c) patch incompatible with \( \alpha > a \).

We only describe the arguments explicitly in the case of link incompatibility — the arguments are similar in the latter two cases. We will argue only \( b = \alpha \) as the case of \( \beta = a \) is analogous.

By Proposition \( \text{38} \), there are three possibilities for the relative positions of \( pt_x(d) \) and \( pt_x(\beta) \). They are displayed in Figure 24.

(a) \( d > \beta \) (i.e., \( x(d) < x(\beta) \)).
We have that \( t_{a,b} \) and \( t_{\alpha,c} \) are link incompatible for \( xt_{c,d} \leq w \). This contradicts \( x \in \text{maxsing}(X_w) \).
(b) \( d = \beta \) (i.e., \( x(d) = x(\beta) \)).
The argument is the same as in the previous case.
(c) \( d < \beta \) (i.e., \( x(d) > x(\beta) \)).
If \( \min(d_{x,w}|c) \geq 2 \), then \( t_{a,b} \) and \( t_{\alpha,\beta} \) are link incompatible for \( xt_{c,d} \leq w \). This contradicts \( x \in \text{maxsing}(X_w) \). Otherwise, we can apply the Cross Lemma \( \text{10} \) to \( t_{c,d} \) and \( t_{\alpha,\beta} \) to conclude that \( t_{c,\beta} \in \mathcal{R}(x,w) \). Then, as in Figure 24.3, \( t_{a,b} \) and \( t_{b,c} \) are link incompatible for \( xt_{c,\beta} \leq w \). This contradicts \( x \in \text{maxsing}(X_w) \).

2. Assume \( t_{a,b} \not\in \mathcal{R}(x,w) \).
Since \( a, b, c \in \Delta(x,w) \), we can find \( \alpha, \beta, \gamma \) such that \( t_{\{a,\alpha\}}, t_{\{b,\beta\}}, t_{\{c,\gamma\}} \in \mathcal{R}(x,w) \). If

(a) \( pt_x(\alpha) \) is in region A or A’ of Figure 25.1 or
(b) \( pt_x(\beta) \) is in region C or C’ of Figure 25.2 or
(c) \( pt_x(\gamma) \) is in region E or E’ of Figure 25.3,
then we can reduce to the previous case (of \( t_{a,b} \in \mathcal{R}(x,w) \)) or we violate Proposition \( \text{38} \). So we must have

(a) \( pt_x(\alpha) \in B \cup B’ \)
(b) \( pt_x(\beta) \in D \cup D’ \)
(c) \( pt_x(\gamma) \in F \cup F’ \).
As the argument of $\text{pt}_x(\gamma) \in F$ is analogous, we will assume $\text{pt}_x(\gamma) \in F'$.

We now argue the four cases we have left according to whether $\text{pt}_x(\alpha) \in B$ and whether $\text{pt}_x(\beta) \in D$.

(a) $\text{pt}_x(\alpha) \in B$, $\text{pt}_x(\beta) \in D$.

See Figure 26.1. Note that we can reduce to Case 1 if $x(\alpha) > x(\beta)$.
So we assume $x(\alpha) < x(\beta)$. We have already shown in Lemma 44 that this configuration contradicts $x \in \text{maxsing}(X_w)$.

(b) $\text{pt}_x(\alpha) \in B$, $\text{pt}_x(\beta) \in D'$.

Since $x$ is an msp, there exists some $t_{d,\delta} \in \mathcal{E}_{b,\beta}(x, w)$. As $\min(d_{x,w}|A) = 0$, one can see in Figure 26.2 that $\overline{A}_{d,\delta} \cap \overline{A}_{\alpha,a} = \emptyset$. Hence, $t_{d,\delta}$ and $t_{b,\beta}$ are (patch or link) incompatible for $xt_{\alpha,a} \leq w$.

(c) $\text{pt}_x(\alpha) \in B'$, $\text{pt}_x(\beta) \in D$.

See Figure 26.3. We have already shown in Lemma 43 that this case contradicts $x \in \text{maxsing}(X_w)$. 

Figure 25.

Figure 26.
MAXIMAL SINGULAR LOCI OF SCHUBERT VARIETIES IN $SL(n)/B$

(d) $pt_x(\alpha) \in B'$, $pt_x(\beta) \in D'$.

See Figure 24. Note that if $\alpha > \beta$, then in this case we can reduce to Case 4 with $\text{fl}_{b\gamma\alpha}(x) = 231$. So we assume $\alpha < \beta$.

We have already shown in Lemma 13 that this case contradicts $x \in \text{maxsing}(X_w)$.

This completes the proof that $\bar{x}$ is 231- and 312-avoiding.

**Theorem 42.** Suppose we have $a < b < c < d$ with $a, b, c, d \in \Delta(x, w)$ and $\text{fl}_{abcd}(x) = 1234$. We will obtain a contradiction.

By Theorem 41, no points of $\text{mat}(\bar{x})$ may occur in regions I or II of Figure 27.1. Since $a, d \in \Delta(x, w)$, there exist $b', c'$ such that $t_{a, b'}, t_{c', d} \in I \cup II$.

As we have now constructed $a < b' < c' < d$ with $t_{a, b'}, t_{b', c'}, t_{c', d} \in \mathcal{R}(x, w)$, we will assume that $b$ and $c$ were chosen initially such that $t_{a, b}, t_{b, c}, t_{c, d} \in \mathcal{R}(x, w)$. Note that by the above construction, we can assume that $t_{a, b} \in \mathcal{E}_{b, c}(x, w)$. Suppose $t_{b, c} \not\in \mathcal{E}_{b, c}(x, w)$. Then $t_{a, b}$ and $(t_{b, c} \text{ or } t_{b, d})$ are link incompatible for $xt_{c, d} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$.

So our diagram looks like that pictured in Figure 28.1 and we have $t_{a, b} \in \mathcal{E}_{b, c}(x, w), t_{b, c} \in \mathcal{E}_{c, d}(x, w)$.

Therefore, we can find a point in each of the regions U & V such that $d_{x, w} = 0$. Choose the point in region U to be as low as possible. Choose the point in region V to be as far right as possible. Such points are shown in Figure 28.2. Apply Lemma 8 to the rectangle determined by these two points with $\alpha, \beta \geq 1$ and $\gamma = 0$. This, along with Proposition 88, implies that there is another point $pt_x(p)$ in either region P or Q. Without loss of generality,
Figure 28.

assume it is in region P. By having chosen the point in region U as low as possible, we find that \( t_{p,d} \in \mathcal{R}(x,w) \) (see Figure 28.3). Hence, \( t_{a,b} \) and \( t_{b,c} \) are link incompatible for \( xt_{p,d} \leq w \). This contradicts \( x \in \text{maxsing}(X_w) \).

9. Restrictions on \( \bar{w} \)

Combining Theorem 41 and Theorem 42, we see that if \( x \in \text{maxsing}(X_w) \), then

\[
\bar{x} = [k, \ldots, 1, k + l, \ldots, k + 1, k + l + m, \ldots, k + l + 1]
\]

for some \( k, l, m \geq 0 \). If two out of three of \( k, l, m \) are 0, then \( \bar{x} \) is strictly decreasing, so \( x \leq w \) implies that \( x = w \). But then \( x \) cannot possibly be an MSP. So now we determine the possible values of \( k, m \) in Proposition 45 when \( l = 0 \) and the possible values of \( k, l, m \) in Proposition 46 when \( k, l, m > 0 \). In each proposition, we also determine what \( \bar{w} \) must be to allow \( \bar{x} \) to be singular.

We know from Proposition 34 that \( x \in \text{maxsing}(X_w) \) if and only if \( \bar{x} \in \text{maxsing}(X_{\bar{w}}) \). Hence, for the remainder of this section, we will only consider the case where \( \bar{x} = x \) and \( \bar{w} = w \).

9.1. Two decreasing sequences in \( \bar{x} \).

Proposition 45. Let \( x \in \text{maxsing}(X_w) \) with \( \bar{x} = x \) and \( \bar{w} = w \). Suppose that \( x \) consists of exactly two decreasing sequences:

\[
x = [k, \ldots, 1, k + m, \ldots, k + 1],
\]

for some \( k, m \geq 1 \). Then

1. \( k, m \geq 2 \) and
2. \( w = [k + m, k, \ldots, 2, k + m - 1, \ldots, k + 1, 1] \) (shown in Figure 28.3).

Proof. For brevity in the following, we use the convention that \( \alpha, a, a' \in [1, \ldots, k] \) and \( \beta, b, b' \in [k + 1, \ldots, k + m] \).

Condition 1: If \( k = 1 \) or \( m = 1 \), then by Lemma 34, \( \mathcal{E}_{1,k+1}(x,w) = \emptyset \). This contradicts Fact 33.

Condition 2: We split this proof into proving the following facts:

1. \( t_{1,k+1}, t_{k,k+m} \in \mathcal{R}(x,w) \).
2. \( t_{a,b} \in \mathcal{R}(x,w) \) for all \( 1 \leq a \leq k \) and \( k + 1 \leq b \leq k + m \).
3. \( d_{x,w} \leq 1 \).
4. \( w \) is as in the statement of Condition 2.

We now prove these claims.

Step 1. \( t_{1,k+1}, t_{k,k+m} \in \mathcal{R}(x,w) \).

Assume \( t_{1,k+1} \notin \mathcal{R}(x,w) \). We will obtain a contradiction.

By Proposition 13, we can find \( \alpha, \beta \) such that
\[ t_{1,\beta}, t_{\alpha,k+1} \in \mathcal{R}(x,w) \]
(see Figure 29). Choose \( \alpha \) as large, \( \beta \) as small as possible subject to this restriction.

If \( t_{1,\beta} \in \mathcal{E}_{\alpha,k+1}(x,w) \), then an application of the Cross Lemma 40 would offer the desired contradiction. So assume that this is not the case (i.e., assume \( d_{x,w} \geq 2 \) on region \( R \) of Figure 29).

Since \( x \) is an MSP, by Fact 33, we can find some \( t_{a,b} \in \mathcal{E}_{1,\beta}(x,w) \). Recall that we chose \( \alpha \) as large as possible such that \( t_{a,k+1} \in \mathcal{R}(x,w) \). It follows then that \( a \leq \alpha \). Similarly, our choice of \( \beta \) as small as possible such that \( t_{1,\beta} \in \mathcal{R}(x,w) \), in conjunction with the Cross Lemma 40 and Ell Lemma 34, implies that \( b > \beta \). Suppose \( a = \alpha \). This is depicted in Figure 30.1. We see that \( t_{k+1,\beta} \) and \( t_{1,b} \) are patch incompatible for \( xt_{a,k+1} \leq w \). This contradicts \( x \in \text{maxsing}(X_w) \). So we may assume \( a < \alpha \) as in Figure 30.2.

Suppose that \( d_{x,w} \geq 2 \) on region \( A \). Then \( t_{a,b} \) and \( t_{1,\beta} \) are patch incompatible for \( xt_{a,k+1} \leq w \). This contradicts \( x \in \text{maxsing}(X_w) \). So there is at least one point in region \( A \) for which \( d_{x,w} \) has value 1. Now we can apply the Cross Lemma 40 to the patch incompatible pair
Figure 31.

$t_{\alpha,k+1}, t_{a,b}$ to conclude that $d_{x,w} \geq 1$ on regions B and C. We display this knowledge in Figure 31.1.

Now suppose that there is a point in region D for which $d_{x,w} = 1$. Then $t_{k+1}, \beta$ and $t_{1,b}$ are patch incompatible reflections for $x t_{\alpha,k+1} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. Since, by construction, $t_{a,b} \in E_{1,\beta}(x, w)$, the only possibility left is that $\min(d_{x,w}|_{E_1}) = 1$ (as in Figure 31.2). We can now apply the Cross Lemma 40 to $t_{1,b}$ and $t_{a,k+1}$ to conclude that $d_{x,w} \geq 1$ on region F. Hence $t_{1,k+1} \in \mathcal{R}(x, w)$ as claimed.

The proof that $t_{k,k+m} \in \mathcal{R}(x, w)$ is entirely analogous when one uses $d'_{x,w}$ from (5.6).

Step 2. $t_{a,b} \in \mathcal{R}(x, w)$ for all $1 \leq a \leq k$ and $k+1 \leq b \leq k+m$.

By the previous step, we know that we can shade regions I and II in Figure 31.3. For every $a, b$, by the definition of $\Delta(x, w)$, we can shade the corresponding regions U and V, respectively. This completes the claim.

Step 3. $d_{x,w} \leq 1$.

Suppose, on the contrary, that $d_{x,w} \geq 2$ for some point on region A in Figure 32.1.

Figure 32. We have displayed the case of $a < k$, but the argument holds for $a = k$ too.

Since $d_{x,w}$ is non-decreasing as we move down or left in region A, we can assume that $d_{x,w}(\Delta) \geq 2$ for $\Delta = (k, k+1)$. But then there must be some $a, b$ with $1 < a \leq k$ and $k+1 < b \leq k+m$ with either
$a < k$ or $b < k + m$ and $t_{a,b} \in E_{1,k+1}(x, w)$ (see Figure 32.2). Note that 
$\min(d_{x,w}|_{B\cup C\cup D}) = 1$ by choice of $t_{a,b}$. If $\min(d_{x,w}|_{B}) = 1$ then $t_{k,k+1}$ and $t_{1,b}$ are patch incompatible for $x t_{1,k+m} \leq w$ (see Figure 32.3). This contradicts $x \in \text{maxsing}(X_w)$. So we can assume $d_{x,w}|_{B} \geq 2$. Since 
$\min(d_{x,w}|_{B\cup C\cup D}) = 1$, and $d_{x,w}$ is non-decreasing in region A as move down or left, we can now assume that 
$\min(d_{x,w}|_{C}) = 1$. But then $t_{1,b}$ and $t_{a,k}$ are patching incompatible for $x t_{k,k+1} \leq w$ (see Figure 33.1). 
This contradicts $x \in \text{maxsing}(X_w)$. So $d_{x,w}|_{A} \leq 1$ as claimed.

FIGURE 33.

Step 4. By the previous step, there is at most one point of mat($w$) in region 
A. But as $w > x$, $\tilde{x} = x$ and $\tilde{w} = w$, this fixes all the remaining points 
and we see that 

$$w = [k + m, k, \ldots , 2, k + m - 1, \ldots , k + 1, 1],$$

as claimed. This is displayed in Figure 33.2.

9.2. Three decreasing sequences in $\tilde{x}$. We repeat the task of the previous section when $\tilde{x}$ consists of three decreasing sequences rather than just two.

**Proposition 46.** Let $x \in \text{maxsing}(X_w)$ with $\tilde{x} = x$ and $\tilde{w} = w$. Suppose 
that $x$ consists of exactly three decreasing subsequences:

$$x = [k, \ldots , 1, k + l, \ldots , k + 1, k + l + m, \ldots , k + l + 1],$$

for some $k, l, m \geq 1$. Then

1. $l \geq 2$,
2. $l = 2$ if $k > 1$ or $m > 1$,
3. 

$$w = [k + l, k, \ldots , 2, k + l + m, k + l - 1, \ldots , k + 2, 1, k + l + m - 1, \ldots , k + l + 1, k + 1].$$

(Shown in Figure 44.)
Proof. Again for purposes of brevity, we’ll assume throughout this proof that $a, a', \alpha \in [1, \ldots, k]$, $b, b', \beta \in [k + 1, \ldots, k + l]$ and $c, \gamma \in [k + l + 1, \ldots, k + l + m]$. We now prove a series of claims elucidating the structure of $R(x, w)$.

As the chain of reasoning has several steps, we summarize them here before beginning.

1. There exist $a, b, c$ such that $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$.
2. $t_{a,b}$ and $t_{b,c}$ are link incompatible.
3. Such $a$ and $c$ exist for any $b$.
4. $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$ for any triple $a, b, c$.
5. Conclude that we can shade the diagram as in Figure 40.1.
6. $d_{x,w} \leq 1$ in Figure 40.1.
7. Condition 1 holds.
8. $w$ is as claimed in Condition 3.
9. Condition 2 holds.

Step 1. There exist $a, b, c$ such that $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$.

Suppose there is no such triple of indices. Then by the definition of $\Delta(x, w)$, for given $a, c$ there exist $b, b'$ ($b \neq b'$) such that $t_{a,b}, t_{b',c} \in \mathcal{R}(x, w)$. By the Ell Lemma 34, along with the assumptions that $x$ is an MSP and that such triples do not exist, we can find $\alpha \neq a$ and $\beta \neq b'$ such that $t_{\alpha,\beta} \in \mathcal{E}_{a,b}(x, w)$ (illustrated in Figure 34).

![Figure 34](image)

Figure 34. We have displayed the case of $b' < \beta < b$, $\alpha < a$, but there are other possibilities.

But then $t_{a,b}$ and $t_{\alpha,\beta}$ are patch incompatible for $xt_{b',c} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. So we must be able to find a triple as claimed.

Step 2. If $t_{a,b}, t_{b,c} \in \mathcal{R}(x, w)$ then $t_{a,b} \in \mathcal{E}_{b,c}(x, w)$.

Suppose, on the contrary, that $t_{a,b} \in \text{Im} \phi_{t_{b,c}}$. Then $d_{x,w} \geq 1$ on either all of region A or all of region B in Figure 40.1.

Assume that $d_{x,w} \geq 1$ on region A. Now, since $x$ is an MSP, by Fact 33 there exists $t_{p,q} \in \mathcal{E}_{b,c}(x, w)$. We now consider the two possibilities for the relative positions of $t_{p,q}$ and $t_{b,c}$.

Suppose that $t_{p,q}$ and $t_{b,c}$ are link incompatible — i.e., we have $q = b$ (Figure 35.2). For $t_{p,q}$ to be link incompatible with $t_{b,c}$, we need $p > a$ (as depicted in Figure 35.2) since we are assuming $\min(d_{x,w}|A) \geq 1$. Additionally, as $t_{p,q} \in \mathcal{E}_{b,c}(x, w)$, $d_{x,w}$ must have value 0 for at least one point on each of regions C and D. Thus $t_{p,b}$ and $t_{b,c}$ are link incompatible for $xt_{a,b} \leq w$. This contradicts $x \in \text{maxsing}(X_w)$. 
On the other hand, $t_{p,q}$ and $t_{b,c}$ may be patch incompatible. Then there are four possibilities for the relative positions of $\mathbf{p}_x(p)$, $\mathbf{p}_x(b)$, $\mathbf{p}_x(q)$ and $\mathbf{p}_x(c)$ depending on whether $p < b$ and whether $q < c$ (see Figure 36). In each situation, $t_{p,q}$ and $t_{b,c}$ are patch incompatible for $x \tau_{a,b} \leq w$. This contradicts $x \in \operatorname{maxsing}(X_w)$.

We have obtained a contradiction for every scenario in which $d_{x,w} \geq 1$ on region A. Arguing similarly if $d_{x,w} \geq 1$ on region B, we conclude that $t_{a,b} \in E_{b,c}(x,w)$.

**Step 3.** Given $\beta$, there exist $\alpha, \gamma$ such that $t_{\alpha,\beta}, t_{\beta,\gamma} \in \mathcal{R}(x,w)$.

By Step 1, there exist $a, b, c$ such that $t_{a,b}, t_{b,c} \in \mathcal{R}(x,w)$. If $b = \beta$ then we are done — so assume not. We can at least find a $q$ with $t_{\{\beta,q\}} \in \mathcal{R}(x,w)$. Without loss of generality, assume $q = \gamma$ for some $\gamma > \beta$. We split into cases according to whether $\gamma < c$, $\gamma = c$ or $\gamma > c$. These are depicted in Figure 37. Note that by the previous step,
$t_{a,b} \in \mathcal{E}_{b,c}(x,w)$, so $\min(d_{x,w}|_{B}) = 0$. In addition, if $\min(d_{x,w}|_{A}) \geq 1$, then $t_{a,b} \in \mathcal{R}(x,w)$ as desired. So in the following arguments (and Figure 37), we assume $\min(d_{x,w}|_{A}) = 0$ and derive a contradiction.

Assume $\gamma > c$. If $\min(d_{x,w}|_{C}) \geq 2$, then $t_{a,b}, t_{b,c}$ are link incompatible for $xt_{2,\gamma} \leq w$. This contradicts $x \in \maxsing(X_{w})$. Otherwise, by the Cross Lemma 40, $t_{\beta,c} \in \mathcal{R}(x,w)$. Then $t_{a,b}, t_{b,c}$ are link incompatible for $xt_{\beta,c} \leq w$. This contradicts $x \in \maxsing(X_{w})$.

If $\gamma \leq c$, then $t_{\beta,c} \in \mathcal{R}(x,w)$ and we get a contradiction as above.

Step 4. For every $\alpha, \beta, \gamma$, we have $t_{\alpha,\beta} \in \mathcal{E}_{a,b}(x,w)$.
Suppose $t_{\alpha,\beta} \notin \mathcal{R}(x,w)$. By the definition of $\Delta(x,w)$ and the fact that $\bar{x} = x$, we know that there exists $b$ such that $t_{a,b} \in \mathcal{R}(x,w)$. Now we can apply the previous step to obtain a $c$ such that $t_{b,c} \in \mathcal{R}(x,w)$. Note that by Step 2, $t_{a,b}$ and $t_{b,c}$ are link incompatible. So our situation is as depicted as in Figure 38.

Using the logic of the previous step, we see that $\min(d_{x,w}|_{A}) = 0$ contradicts $x \in \maxsing(X_{w})$. Hence $t_{\alpha,\beta} \in \mathcal{R}(x,w)$ as desired.

The argument for showing $t_{\beta,\gamma} \in \mathcal{R}(x,w)$ is analogous.

Step 5. We can shade our diagram as in Figure 40.1. This follows immediately from the previous four steps.

Step 6. $d_{x,w} \leq 1$ in Figure 40.1.
We start by showing that if $a \neq \alpha$ and $b \neq \beta$, then $t_{a,b}, t_{\alpha,\beta} \in \mathcal{R}(x,w)$ implies $t_{\alpha,\beta} \in \mathcal{E}_{a,b}(x,w)$.

By Steps 2 and 3, $t_{b,c} \in \mathcal{E}_{a,b}(x,w)$ for some $c$ (see Figure 33). Suppose $t_{\alpha,\beta} \notin \mathcal{E}_{a,b}(x,w)$ — i.e., $d_{x,w} \geq 2$ on region $A$. Then $t_{a,b}$ and $t_{b,c}$ are link incompatible for $xt_{\alpha,\beta} \leq w$. This contradicts $x \in \maxsing(X_{w})$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure37.png}
\caption{We have displayed the case of $\beta < b$, but the proof of Step 3 also holds for $\beta > b$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure38.png}
\caption{We have displayed the case of $b > \beta$.}
\end{figure}
A similar argument can be used to show that if \( b \neq \beta \) and \( c \neq \gamma \), then \( t_{b,c} t_{\beta,\gamma} \in \mathcal{R}(x, w) \) implies \( t_{\beta,\gamma} \in \mathcal{E}_{b,c}(x, w) \).

The claim of \( d_{x,w} \leq 1 \) then follows by inspection from these two facts.

Step 7. Condition 1 in Proposition 46 holds, namely \( l \geq 2 \).

Recall that in Step 4 we showed that \( t_{a,b}, t_{b,c} \in \mathcal{R}(x, w) \) for any \( a, b, c \). This implies that we can shade \( \text{mat}(\tilde{x}) \) as in Figure 40.1.

By Step 3, \( t_{k,k+1} \) and \( t_{k+1,k+l+m} \) are link incompatible. This implies that \( \min(d_{x,w}|_{A}) = 0 \). Similarly, \( \min(d_{x,w}|_{B}) = 0 \). It then follows from our explicit description of \( x \) that the values of \( w(i) \) for \( i = 1, k + 1 \) are as shown in Figure 40.2. Arguing with \( d'_{x,w} \) (see 5.6) and region B, we see that \( w(i) \) for \( i = k + l, k + l + m \) is as shown in the same figure. But this means that \( w^{-1}(k + l) = 1 \) and \( w^{-1}(k + 1) = k + l + m \). This can only happen if \( l > 1 \).

Step 8. \( w \) is as stated in Condition 3.

Step 4 tells us that we can conclude that \( d_{x,w} = 1 \) on all shaded areas of Figure 40. Therefore, \( w(i) = k + 2 - i \) for \( 2 \leq i \leq k \). A similar argument to that in Step 4 shows that \( w(i) = 2(k + l) + m - i \) for \( k + l + 1 \leq i < k + l + m \). So we need only investigate the values of \( w(i) \) for \( k + 1 < i < k + l \). To do this, assume that \( w(i) = x(i) \) for \( k + 1 < i \leq j \) for some \( j \) with \( k + 1 \leq j < k + l - 1 \). Then, as in Figure 40.1, we see that \( d_{x,w} = 0 \) on region B.

Therefore, \( t_{k,k+1} \) and \( t_{k+1,j+2} \) are link incompatible for \( x t_{j+2,k+l+m} \leq w \). This contradicts \( x \in \maxsing(X_w) \). Hence \( w(i) = x(i) \) for all \( i \) with
$k + 1 < i < k + l$. So

$$w = [k + l, k, \ldots, 2, k + l + m, k + l - 1, \ldots, k + 2, 1, k + l + m - 1, \ldots, k + l + 1, k + 1].$$

as desired.

Step 9. Condition 2 in Proposition 46 holds.

We need to show that if $l > 2$ then $k, m = 1$. So assume $k > 1$. By Steps 4 and 6, $t_{1,k+1}$ and $t_{2,k+2}$ are patch incompatible reflections for $xt_{k+l,k+l+m} \leq w$ (see Figure 41.2). This contradicts $x \in \text{maxsing}(X_w)$.

The argument showing that $m = 1$ is analogous.

This completes the proof of Proposition 46.

This completes the proof of Proposition 46.

\[\Box\]

10. Maximal singularity of candidates

We now finish the proof of Theorem 1 by showing that the restrictions we have discovered for $\tilde{x}$ in Propositions 15 and 16 are sufficient to show that these points correspond to MSP’s in the appropriate Schubert variety. This task consists of two steps:

1. Show that the points $x$ are singular points.
2. Show that any cover of $x$ that is still below $w$ is a smooth point.

So that we can describe maxsing($X_w$) succinctly, we introduce the following notation:

Definition 47. For $k, m \geq 2$, define

\begin{align*}
\text{(10.1)} & \quad x_{k,m} = [k, \ldots, 1, k + m, \ldots, k + 1], \\
\text{(10.2)} & \quad w_{k,m} = [k + m, k, \ldots, 2, k + m - 1, \ldots, k + 1].
\end{align*}
For $k, m \geq 1$ and $l \geq 2$, define
\begin{align*}
(10.3) \quad x_{k,l,m} &= [k, \ldots, 1, k+l, \ldots, k+1, k+l+m, \ldots, k+1+l], \\
(10.4) \quad w_{k,l,m} &= [k+l, k, \ldots, 2, k+l+m, k+l-1, \ldots, k+2, 1, k+l+m-1, \ldots, k+1].
\end{align*}

**Theorem 48** (Rephrasing of Theorem 1). $x$ is an MSP of $X_w$ if and only if
\begin{enumerate}
\item $t \in \mathcal{R}(x, w)$ implies $l(xt) = l(x) + 1$.
\item (a) For some $k, m \geq 2$, we have $\tilde{x} = x_{k,m}$ and $\tilde{w} = w_{k,m}$ or
       (b) For some $k, m \geq 1, l = 2$ or $k = m = 1, l \geq 2$, we have $\tilde{x} = x_{k,l,m}$ and $\tilde{w} = w_{k,l,m}$.
\end{enumerate}

**Proof.** Proposition 38 tells us that Condition 1 is necessary. Propositions 15 and 16 tell us that Conditions 2a and 2b are necessary. So all we need to show is sufficiency.

Let $t$ be a reflection such that $x < y = xt \leq w$. As $\phi_t$ is injective, to calculate $\#\mathcal{R}(y, w)$ from $\mathcal{R}(x, w)$ we need only count how many reflections in $\mathcal{R}(x, w)$ are not in the image of $\phi_t$. Note by Proposition 16 that $\#\mathcal{R}(x, w) = \#\mathcal{R}(\tilde{x}, \tilde{w})$ and $l(w) - l(x) = l(\tilde{w}) - l(\tilde{x})$.

Consider first the case shown in Figure 42 of two decreasing sequences for $\tilde{x}$. Note that
\begin{align*}
(10.5) \quad &l(w_{k,m}) = \binom{k}{2} + \binom{m}{2} + k + m - 1, \\
(10.6) \quad &l(x_{k,m}) = \binom{k}{2} + \binom{m}{2} \quad \text{and} \\
(10.7) \quad &\#\mathcal{R}(x, w) = k \cdot m.
\end{align*}
Hence,
\begin{align*}
(10.8) \quad l(w) - l(x) - \#\mathcal{R}(x, w) &= k + m - km - 1.
\end{align*}
Since $k, m \geq 2$, (10.8) is negative. So by Theorem 8, $e_x$ is a singular point of $X_w$. 
To prove that it is a maximal singular point, we consider some \( t_{a,b} \in \mathcal{R}(x,w) \) and let \( y = xt_{a,b} \). Then, viewing Figure 43, it is easily seen that

![Figure 43. \( y = xt_{a,b} \).](image)

\[ \#\mathcal{R}(y,w) = (k - 1) + (m - 1) = k + m - 2. \] Since \( l(y) = l(x) + 1 \), by Theorem 9 and (10.8), \( y \) is a smooth point of \( X_w \). Since \( y \) was chosen as an arbitrary cover of \( x \), \( x \) is an MSP for \( w \).

![Figure 44.](image)

Now we prove the case shown in Figure 44 of three decreasing sequences for \( \bar{x} \). Note that

\[
(10.9) \quad l(w_{k,l,m}) = \binom{k}{2} + \binom{l}{2} + \binom{m}{2} + k + m + 2(l - 2) + 1,
\]

\[
(10.10) \quad l(x_{k,l,m}) = \binom{k}{2} + \binom{l}{2} + \binom{m}{2} \quad \text{and}
\]

\[
(10.11) \quad \#\mathcal{R}(x,w) = l(k + m).
\]
Hence,
\begin{equation}
  l(w) - l(x) - \#R(x, w) = (l - 1) \left( 1 + \frac{l - 2}{l - 1} - k - m \right).
\end{equation}

Since \(k, m \geq 1\) and \(l \geq 2\), (10.12) is negative. So by Theorem 9, \(x\) is a singular point of \(X_w\). To prove it is an MSP for \(w\), as above we consider some \(t_{a,b} \in R(x, w)\) and let \(y = xt_{a,b}\). We have \(l(w) - l(y) = k + m + 2(l - 2)\).

Viewing Figure 45, it is clear that \(\#R(y, w) = (k - 1) + (l - 1) + m(l - 1)\). If \(l = 2\), then \(\#R(y, w) = k + m = l(w) - l(y)\). If \(l > 2\), then by 2 of Proposition 46, we have that \(k = m = 1\), and \(\#R(y, w) = 2(l - 1) = l(w) - l(y)\). So, in either case, \(y\) is a smooth point of \(X_w\).

So in both cases, \(x\) is an MSP of \(X_w\) as claimed. \(\square\)

This completes the proof of Theorem 48. It is easy to check that the above formulation is equivalent Theorem 1. (Note, however, that the values of \(k, l, m\) in the statement of Theorem 48 differ from those used in Theorem 1.)

11. Lakshmibai-Sandhya Conjecture

Let \(w = [w(1), \ldots, w(n)] \in S_n\). Define \(E_w\) to be the set of all \(x = [x(1), \ldots, x(n)]\) satisfying the following conditions:

1. There exist \(i < j < k < l\) and \(i' < j' < k' < l'\) such that (as sets) \(\{w(i), w(j), w(k), w(l)\} = \{x(i'), x(j'), x(k'), x(l')\}\).

2. One of the following holds:
   (a) \(f_{ijkl}(w) = 3412\) and \(f_{i'j'k'l'}(x) = 1324\).
   (b) \(f_{ijkl}(w) = 4231\) and \(f_{i'j'k'l'}(x) = 2143\).

3. Using the notation of Section 3.
(a) If 2a holds, then set 
\[ w = \text{unfl}_{ijkl}^{(1324)} \] and 
\[ \hat{x} = \text{unfl}_{j'k'l'}^{(3412)}. \]

(b) If 2b holds, then set 
\[ w = \text{unfl}_{ijkl}^{(2143)} \] and 
\[ \hat{x} = \text{unfl}_{j'k'l'}^{(4231)}. \]

Then, 
\[ w \leq x \leq \hat{x} \leq w. \]

Theorem 49 (Conjecture in [LS90]). For \( w \in \mathcal{S}_n \), the singular locus of \( X_w \) is equal to \( \cup_x X_x \), where \( x \) runs over the maximal elements of \( E_w \) in Bruhat order.

Proof. We only give the argument for singular points of the type 4231 (i.e., those described in Case 1 of Theorem 1). The argument for singular points of type 3412 and 45312 is analogous.

We start by proving that \( \text{maxsing}(X_w) \subseteq \cup_x X_x \). To do this, fix some \( x \in \text{maxsing}(X_w) \) (of type 4231). We will choose indices \( i, j, k, l \) and \( i', j', k', l' \) as described in the definition of \( E_w \), and show that (11.1) is satisfied for our choice of indices. So, using the notation of Theorem 1, let

\[ \alpha_1 < \beta_1 < \beta_2 < \cdots < \beta_{k-1} < \alpha_2 < \alpha_3 < \cdots < \alpha_m < \beta_k \]

(11.2)

(11.3)

(11.4)

Now, recall from Lemma 3 that \( u \leq v \) if and only if \( d_{u,v} \) is everywhere non-negative. But then (11.1) follows from Figure 16 along with the observation that \( d_{u,v} \leq 1 \) in each of these diagrams.

Now we need to show that any \( x \) satisfying (11.1) for some \( w \) and some set of indices is a singular point of \( X_w \). Since \( x \leq \hat{x} \leq w \) by hypothesis, we see that \( i, j, k, l \in \Delta(x, w) \). Combining this with Corollary 12 and the fact that \( w \leq x \), we conclude that

\[ i \leq i', l' \leq l, \quad j \leq j' \leq l, \quad i \leq k' \leq k. \]

(11.5)

By hypothesis, we also know that \( i' \leq j' \leq k' \leq l' \). Finally, recall that

\[ \{w(i), w(j), w(k), w(l)\} = \{x(i'), x(j'), x(k'), x(l')\}. \]

(11.6)

One possible configuration of the points of (11.6) is shown in Figure 47.1. Note that (11.1) implies that \( d_{x,w} \equiv 1 \) on region \( A \cup B \cup C \). Hence, \( t_{i',k'}, t_{j',l'} \) are patch incompatible for \( x \leq w \). If \( l(x_{j',l'}) = l(x) + 1 \), then we can conclude that \( x \) is a singular point of \( X_w \) by Fact 32. Otherwise, consider Figure 47.2. Pick the index \( p \) as small as possible such that \( pt_x(p) \in D \).
Then $t_{i',k'}, t_{j',p}$ are patch incompatible for $xt_{p,l'} \leq w$. But then $x$ is a singular point $X_w$ as $l(xt_{p,l'}t_{j',p}) = l(xt_{p,l'}) + 1$ by construction.

Carrying out the analogous arguments for the 3412 and 45312 type singularities (using link incompatible reflections) completes the proof of Conjecture 49.

12. **Kazhdan-Lusztig polynomials at elements of $\text{maxsing}(X_w)$**

The determination of $\text{maxsing}(X_w)$ has applications to the study of multiplicities in Verma modules through the Kazhdan-Lusztig polynomials. These polynomials, in the type A case, are indexed by two permutations $x, w$ (for properties of these polynomials, see [Hum90]). A result of Kazhdan and Lusztig [KL79] is that $P_{x,w} = 1$ if and only if $e_x$ is a smooth point of $X_w \subseteq SL(n)/B$. Furthermore, by a conjecture of Kazhdan and Lusztig ([KL79]), proved independently by Beilinson-Bernstein [BBS] and Brylinski-Kashiwara.
PK81, \( P_{x,w}(1) \) gives the multiplicity of an irreducible module associated to \( w \) in the Verma module associated to \( x \).

Theorem 1 gives explicit conditions for this Verma module multiplicity to be greater than 1. In this section, we calculate the \( P_{x,w} \) for \( x \in \text{maxsing}(X_w) \). Setting \( q = 1 \) therefore yields the exact multiplicities in these cases.

There are some explicit descriptions of the \( P_{x,w} \) in special cases. Lascoux and Schützenberger [LS81] and Zelevinsky [Zel83] combine to give a small resolution of \( X_w \) and corresponding formulas for \( P_{x,w} \) when \( w \) is a Grassmannian permutation. Lascoux [Las95] has extended this result to vexilary permutations. Brenti [Bre94, Bre97, Bre98] has given several beautiful, general, alternating sum formulas. Finally, formulas have been calculated in several specific classes (e.g., [BS98b, BW, Pol99]). In particular, Theorem 55 is proved in [BS98a] and Theorem 54 is proved in [LS81], but both are only proved in the case where \( \tilde{x} = x \) and \( \tilde{w} = w \).

A result of Polo, [Pol99], states that every polynomial in \( \mathbb{N}[q] \) with constant term 1 can be realized as a Kazhdan-Lusztig polynomial in \( S_n \) for some \( n \). However, as we will see in the below three theorems, the Kazhdan-Lusztig polynomials at elements of \( \text{maxsing}(X_w) \) are of very limited forms.

For pairs of permutations \( x, w \in S_n \), we can define the Kazhdan-Lusztig polynomials by the following properties:

1. \( P_{x,w} = 0 \) if \( x \not\leq w \).
2. \( P_{x,w} = 1 \) if \( x \leq w \) and \( l(w) - l(x) \leq 2 \).
3. \( \deg(P_{x,w}) \leq 1 \).
4. If \( s \in S \) such that \( ws < w \) then

\[
(12.1) \quad P_{x,w} = q^c P_{x,ws} + q^{1-c} P_{xs,ws} - \sum_{x \leq z < ws, zs < z} \mu(z, ws)q^{(l(w)-l(z))/2} P_{x,z},
\]

where \( \mu(z, ws) \) is the coefficient of \( q^{(l(ws)-l(z)-1)/2} \) in \( P_{z,ws} \) and \( c = 1 \) if \( xs < x \), \( c = 0 \) if \( xs > x \).

Lemma 50. If \( i \notin \Delta(x, w) \), then \( P_{x,w} = P_{x^i,w^i} \).

Proof. Fix \( x < w \) and pick some \( i \notin \Delta(x, w) \). We know by Proposition 13 that \( x(i) = w(i) \). By Corollary 24, this implies that if \( x \leq z \leq w \) for some \( z \), then \( z(i) = x(i) = w(i) \).

With these facts, the result then follows easily by induction on \( l(w) \) using (12.1). (Note that our base case of \( l(w) = 1 \) is trivial.)

Corollary 51. \( P_{\tilde{x},\tilde{w}} = P_{x,w} \).

As it will be used repeatedly in upcoming arguments, for reference we state the following fact [Hum90, Cor. 7.14]:

Fact 52. For \( s, s' \in S, ws < w, s'w < w \), then \( P_{x,w} = P_{xs,w} = P_{s'x,w} \).
We are now ready to calculate $P_{x,w}$ for $x \in \text{maxsing}(X_w)$. By Theorem 48 and Corollary 51, it is enough to calculate $P_{x,w}$ for the pairs $x_{k,m}$, $w_{k,m}$ and $x_{k,l,m}$, $w_{k,l,m}$.

**Theorem 53** (4231-type singularities). For $k, m \geq 2$,

\begin{equation}
P_{x_{k,m}, w_{k,m}} = 1 + q + \cdots + q^{\min(k-1,m-1)}.
\end{equation}

**Proof.** We apply induction on $k + m$. The case of $k = m = 2$ can be checked from (12.1). We assume then, without loss of generality, that $k \geq 3$. Also, for brevity we will often write $x$ and $w$ in place of $x_{k,m}$ and $w_{k,m}$, respectively.

Consider $s = s_1$. The pairs $x, w$ and $x, w_s$ and $x_s, w_s$ are shown in Figure 48. We claim that

\begin{equation}
P_{x, w_s} = P_{x, w} = 1 + q + \cdots + q^{\min(k-2,m-1)}.
\end{equation}

The second equality follows from Figure 48.2, Corollary 12, Lemma 50 and the induction hypothesis. To obtain the first equality, we notice that $s_{k-1} x_s = x$. Then, since $s_{k-1} w_s < w_s$, (12.3) follows from Fact 52.

Substituting this information into (12.4), we obtain

\begin{equation}
P_{x, w} = (1 + q) \left( 1 + q + \cdots + q^{\min(k-2,m-1)} \right) - \sum_{x \leq z < w_s \atop z < z_s} \mu(z, w) q^\frac{\mu(w) - \mu(z)}{2} P_{z, x}.
\end{equation}

Now we need to investigate the possible terms in the sum of (12.4). We will first determine which $z$ with $l(z) < l(w_s) - 1$ can contribute to (12.4). By Corollary 54, Corollary 12 and Lemma 50, if $x \leq z < w_s$ then $P_{z, w_s} = P_{z^1, w_s} = P_{x_{k-1,m}, w_{k-1,m}}$. To have $\mu(z, w) > 0$ while $l(z) < l(w) - 1$, we need $\deg(P_{z, w}) > 0$. Since $x_{k-1,m}$ is the only MSP for $w_{k-1,m}$, this tells us that $z^1 \leq x_{k-1,m}$. But by Fact 52, $P_{z, w_{k-1,m}} = P_{x_{k-1,m}, w_{k-1,m}}$, so this implies that we must have $z^1 = x_{k-1,m}$ by the degree bound. It is then easy to check from (10.5) and (10.6) that $\mu(x, w) > 0$ if and only if $k - 1 = m$.

We now split into two cases depending on the relative values of $k$ and $m$. 

**Figure 48.**
1. $k - 1 \neq m$.
Since $k - 1 \neq m$, we know by the previous paragraph that the only $z$ that can contribute to the sum in (12.4) are those with $l(z) = l(ws) - 1$. Furthermore, since we are summing only over $z$ for which $zs < z$, the only possibility is $z = wss_2$. In this case, by induction, along with Corollaries 12 and 51, $P_{x,wss_2} = 1 + q + \cdots + q^{\min(k-3,m-1)}$. So

\begin{equation}
P_{x,w} = (1 + q)(1 + q + \cdots + q^{\min(k-2,m-1)}) - q(1 + q + \cdots + q^{\min(k-3,m-1)})
= 1 + q + \cdots + q^{\min(k-1,m-1)}.
\end{equation}

2. $k - 1 = m$.
$z = wss_2$ will contribute as in the previous case. However, from our discussion above, $z = x_{k-1,m}$ will also contribute. For $z = x_{k-1,m}$, we have $P_{x,z} = 1$ and $q^{(l(w) - l(z))/2} = q^{k-1}$. Plugging this term into (12.4), along with the term coming from $z = wss_2$, we get

\begin{equation}
P_{x,w} = (1 + q)(1 + q + \cdots + q^{k-2}) - q(1 + q + \cdots + q^{k-3}) - q^{k-1}
= 1 + q + \cdots + q^{k-2} = 1 + q + \cdots + q^{\min(k-1,m-1)}.
\end{equation}

\begin{proof}
We apply induction on $k + m$. The case of $k = m = 1$ is easy to check using (12.1). For brevity, we abbreviate $x_{k,2,m}$ and $w_{k,2,m}$ by $x$ and $w$, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure49.png}
\caption{Figure 49.}
\end{figure}

Let $s = s_{k+1}$. Now, as seen in Figure 43, $w$ is 3412- and 4231-avoiding, hence smooth ([LS90]). Therefore, $P_{y,ws} = 1$ for all $y \leq ws$. Clearly $x, xs \leq$
Thus the first two terms of (12.1) together contribute $1 + q$. We now show that the sum in (12.1) is empty.

Since $P_{y, ws} = 1$ for all $y \leq ws$, $\mu(z, ws) > 0$ implies that $l(z) = l(ws) - 1$.

But, as seen from Figure 50.2, no $z$ satisfying this length condition can satisfy the additional constraint of $z s_{k+1} < z$. This proves the theorem. □

**Theorem 55 (45312-type singularities).** For $l \geq 2$,

(12.8) $P_{x_{1,l,1}, w_{1,l,1}} = 1 + q^{l-1}$.

**Proof.** We apply induction on $l$. The case of $l = 2$ is covered by Theorem 54.

So we assume $l \geq 3$. For clarity, we abbreviate $x_{1,l,1}$ and $w_{1,l,1}$ by $x$ and $w$, respectively.

In Figure 50, we depict the pairs $x, w$ and $x, ws_2$ and $xs_2, ws_2$. We claim that the first two terms in (12.1) contribute $(1 + q)(1 + q^{l-2})$. First consider the pair $x, ws_2$. Since $ws_2 s_1 < ws_2$, by the induction hypothesis, Corollary 12 and Lemma 50, we see that $P_{x, ws_2} = P_{xs_1, ws_2} = 1 + q^{l-2}$. Now consider the pair $xs_2, ws_2$. Since $s_1 ws_2 < ws_2$ and $ws_2 s_1 < ws_2$, it follows that $P_{xs_2, ws_2} = P_{s_1 xs_2 s_1, ws_2}$. But since $s_1 xs_2 s_1 = xs_1$, we get that $P_{xs_2, ws_2} = 1 + q^{l-2}$ also. Plugging this information into (12.1), we can write

(12.9) $P_{x, w} = 1 + q^{l-2} + q + q^{l-1} - \sum_{x \leq z < ws_2 \leq z} \mu(z, ws_2)q^{\frac{l(ws_2) - l(z)}{2}}P_{x, z}$.

Now we check which $z$ will appear in the sum in (12.9). First note that $xs_1$ is the unique MSP for $ws_2$. By induction, $P_{xs_1, ws_2} = 1 + q^{l-2}$. By Fact 52, $P_{e, ws_2} = P_{x_{s_1}, ws_2}$. Hence, the only $z$ such that $l(z) < l(ws_2) - 1$ and $\deg(P_{x, ws_2})$ is maximized is $z = xs_1$. However, $xs_1 s_2 > xs_1$, so $xs_1$ does not appear in the sum. So the only possible terms in the sum are those with $l(z) = l(ws_2) - 1$. From Figure 50.2, we see that $z = ws_2 s_3$ is the only $z$ satisfying both this length condition and $z s_2 < z$. Using Fact 52, Lemma 54 and the induction hypothesis, one can check that $P_{x, ws_2 s_3} = 1 + q^{l-3}$. Hence, the sum in (12.9) contributes $-q - q^{l-2}$. Simplifying, we see that $P_{x, w} = 1 + q^{l-1}$ as claimed. □
13. Examples Calculating $\text{maxsing}(X_w)$

**Example 56.** Using Theorem 1, in Figure 51 we compute the singular locus

$$\text{maxsing}(X_w) = X_{[48376512]} \cup X_{[64387512]} \cup X_{[46587312]} \cup X_{[68174325]}$$

of $X_w$ where $w = [6,8,4,7,5,3,1,2]$.

![Figure 51.](image)

**Remark 57.** The cardinality of the set $\text{maxsing}(X_w)$ may be $O(n^4)$. This is the case, for example, when

$$w = [k + 1, \ldots, k + l, 1, \ldots, k].$$

Then $\# \text{maxsing}(X_w) = \left(\frac{n}{2}\right)^2$.

**Example 58.** Using a computer it is easy to calculate, for example, that

$$\text{maxsing}(X_w) = [17, 6, 2, 15, 12, 11, 3, 8, 16, 7, 14, 5, 13, 9, 10, 1, 4],$$

$$\# \text{maxsing}(X_w) = 29.$$
14. Patterns indexing maxsing($X_w$)

Which 4231 or 3412 patterns lead to elements in maxsing($X_w$)? We can describe these patterns by taking all 4231 and 3412 patterns in $w$ and removing certain “useless patterns” contained in larger patterns of length 5 or 6. For example, if $w = [52341]$, the pattern 5241 will be useless since the shaded region it defines is not empty. We describe the useless patterns in the following way. For each pattern of length 5 or 6 in the left hand column of (14.1), remove the corresponding pattern in the right hand column.

\[
\begin{align*}
(52341) & \quad (5241) \\
(52431) & \quad (5241) \\
(53241) & \quad (5241) \\
(53421) & \quad (5341) \\
(54231) & \quad (5231) \\
(35412) & \quad (3512) \\
(526413) & \quad (5613) \\
(563412) & \quad (5612) \\
(463152) & \quad (4612) \\
(45132) & \quad (4512) \\
(45213) & \quad (4513) \\
(635241) & \quad (6341) \\
(6512) & \quad (612) \\
(43512) & \quad (4512) \\
\end{align*}
\]

The remaining “useful patterns” all index a unique component in maxsing($X_w$). For example, if $w = [7432651]$ then maxsing($X_w$) has only one element namely $x = [4321765]$ and this element would be indexed by 7251. This example corresponds to the shape in Figure 1.

It would be interesting to know the distribution of the various sizes of maxsing($X_w$) for all $w \in S_n$ for large $n$.

References

[Ba81] A. Beilinson and J. Bernstein, Localization of $g$-modules, C. R. Acad. Sci. Paris Ser. I Math 292 (1981), 15–18.
[BB92] Nantel Bergeron, A Combinatorial Construction of the Schubert Polynomials, J. Comb. Theory, Series A 60 (1992), 168–182.
[BK81] J.-L. Brylinski and M. Kashiwara, Kazhdan-Lusztig conjectures and holonomic systems, Invent. Math. 64 (1981), 387–410.
[BL98] Sara C. Billey and Tao Kai Lam, Vexillary elements in the hyperoctahedral group, J. Alg. Combin 8 (1998), no. 2, 139–152.
[BL00] Sara Billey and V. Lakshmibai, Singular Loci of Schubert Varieties, Progress in Mathematics, no. 182, Birkhäuser, 2000.
[Bre94] Francesco Brenti, A combinatorial formula for Kazhdan-Lusztig polynomials, Invent. Math. 118 (1994), no. 2, 371–394.
[Bre97] Francesco Brenti, Combinatorial expansions of Kazhdan-Lusztig polynomials, J. London Math. Soc. 55 (1997), no. 2, 448–472.
[Bre98] Francesco Brenti, Kazhdan-Lusztig polynomials and R-polynomials from a combinatorial point of view, Discrete Math 193 (1998), no. 1–3, 93–116.
[BS98a] A. Vainshtein B. Shapiro, M. Shapiro, Kazhdan-Lusztig polynomials for certain varieties of incomplete flags, Discrete Math. 180 (1998), 345–355.

[BS98b] Francesco Brenti and Rodica Simion, Enumerative aspects of some Kazhdan-Lusztig polynomials, Preprint (1998).

[BW] Sara Billey and Gregory Warrington, Kazhdan-Lusztig polynomials for 321-hexagon-avoiding permutations, to appear in J. Alg. Comb.

[Car94] J. Carrell, On the smooth points of a Schubert variety, CMS Conference proceedings, vol. 16, June 1994, pp. 15–24.

[Che94] C. Chevalley, Sur les décompositions cellulaires des espaces G/B, Proceedings of Symposia in Pure Mathematics 56 (1994), no. 1.

[CK99] J. B. Carrell and J. Kuttler, On the smooth points of T-stable varieties in G/B and the Peterson map, preprint (1999).

[Deo85] V. Deodhar, Local Poincaré duality and non-singularity of Schubert varieties, Comm. Algebra 13 (1985), 1379–1388.

[Dye93] M. Dyer, The nil-Hecke ring and Deodhar’s conjecture on Bruhat intervals, Invent. Math. 111 (1993).

[Ful97] William Fulton, Young tableaux; with applications to representation theory and geometry, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, New York, 1997.

[Gas00] Vesselin Gasharov, Sufficiency of Lakshmibai-Sandhya singularity conditions for Schubert varieties, In preparation (2000).

[Hum90] James E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, 1990.

[KL79] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Inv. Math. 53 (1979), 165–184.

[Las95] Alain Lascoux, Polynomes de Kazhdan-Lusztig pour les varietes de Schubert vexillaires. (French) [Kazhdan-Lusztig polynomials for vexillary Schubert varieties], C. R. Acad. Sci. Paris Sr. I Math. 321 (1995), no. 6, 667–670.

[LS81] Alain Lascoux and Marcel-Paul Schützenberger, Polynomes de Kazhdan & Lusztig pour les Grassmanniennes. (French) [Kazhdan-Lusztig polynomials for Grassmannians], Astérisque 87–88 (1981), 249–266, Young tableaux and Schur functions in algebra and geometry (Toruń, 1980).

[LS84] V. Lakshmibai and C. S. Seshadri, Singular locus of a Schubert variety, Bull. Amer. Math. Soc. 11 (1984), no. 2, 363–366.

[LS85] A. Lascoux and M.-P. Schützenberger, Schubert polynomials and the Littlewood-Richardson rule, Letters in Math. Physics 10 (1985), 111–124.

[LS90] V. Lakshmibai and B. Sandhya, Criterion for smoothness of Schubert varieties in SL(n)/B, Proc. Indian Acad. Sci. (Math Sci.) 100 (1990), no. 1, 45–52.

[Man01] Laurent Manivel, Le lieu singulier des varietes de Schubert, arXiv:math.AG/0102124 (2001).

[Pol94] P. Polo, On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar, Indag. Math. 5 (1994), 483–493.

[Pol99] Patrick Polo, Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups, Represent. Theory 3 (1999), 90–104, (electronic).

[Ste96] John Stembridge, On the fully commutative elements of Coxeter groups, J. Algebraic Combin. 5 (1996), no. 4, 353–385.

[Zel83] A.V. Zelevinskii, Small resolutions of singularities of Schubert varieties, Functional Anal. Appl. 17 (1983), no. 2, 142–144.

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