On finite groups with all simple modules of low
dimension in characteristic $p$

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Abstract

We offer a short and reasonably elementary proof that if $G$ is a finite
group, $F$ is an algebraically closed field of prime characteristic $p$, and
all simple $FG$-modules have dimension less than $p$, then $G$ has a normal
Sylow $p$-subgroup.

1 Introduction

Let $G$ be a finite group, $p$ be a prime, and let $F$ be an algebraically closed field of
characteristic $p$. We recall that $O_p(G)$ acts trivially on each simple $FG$-module. We
will prove that if each simple $FG$-module has dimension less than $p$, then $G$ has a normal Sylow $p$-subgroup. This may be compared with an analogous result for ordinary irreducible representations proved in [1] by Isaacs and Passman, where the resulting normal Sylow $p$-subgroup is necessarily Abelian, which need not be the case in the present modular context.

We will make use of the following reciprocity theorem, which appears in [2] with a proof using properties of the Reynolds ideal of $Z(FG)$, but which may also be proved using projective homomorphisms: if $H$ is a subgroup of $G$ and $S$ is simple $FG$-module, $T$ is a simple $FH$-module, then the multiplicity of the projective cover of $S$ as a summand of $\text{Ind}_G^H(T)$ is equal to the multiplicity of the projective cover of $T$ as a summand of $\text{Res}_H^G(S)$.

Now we prove our main:

Theorem: Let $G$ be a finite group and $F$ be an algebraically closed field of
prime characteristic $p$. Suppose that every simple $FG$-module has dimension
less than $p$. Then $G$ has a normal Sylow $p$-subgroup.

Proof: We proceed by induction on $|G|$. Since $O_p(G)$ acts trivially on every
simple $FG$-module, we may suppose that $O_p(G) = 1$, but that $p$ divides $|G|$.

Since the hypotheses imply (on consideration of the composition factors of
the respective regular modules) that for any section $K$ of $G$, each simple $FK$-
module has dimension less than $p$, we may suppose by induction that every
proper section $K$ of $G$ has a normal Sylow $p$-subgroup.
We claim that $P \cap P^g = 1$ for all $g \in G \setminus N_G(P)$. It suffices to prove that $N_G(V) \leq N_G(P)$ whenever $1 \neq V \leq P$. Since $O_p(G) = 1$, we have $N_G(V) < G$ for all such $V$. Hence, by induction, we know that $N_G(V)$ has a normal (and hence unique) Sylow $p$-subgroup for each such $V$. It follows that $N_G(V)$ is contained in the normalizer of its unique Sylow $p$-subgroup, so, using induction on $[P : V]$ (and the fact that $N_P(V) > V$ whenever $V < P$), the claim follows.

Now $N_G(P) < G$ and we have $\text{Ind}_{N_G(P)}^G(F) = F \oplus M$ where $M$ is a (non-zero) projective $FG$-module, using Mackey decomposition, for example. Let $Q$ be a projective indecomposable summand of $M$, and let $S$ be the socle of $Q$ (which is also isomorphic to the head of $Q$). Then the multiplicity of the projective cover of the trivial $FN_G(P)$-module $F$ as a summand of $\text{Res}_{N_G(P)}^G(S)$ is strictly positive by the above reciprocity theorem (applied with the trivial $FN_G(P)$-module $F$ in place of $T$). Hence $\dim_F(S) \geq |P|$, contrary to our assumption that each simple $FG$-module has dimension less than $p$. This contradiction shows that $|G|$ is not divisible by $p$, and so completes the proof of the Theorem.

2 References

[1] Isaacs, I.M and Passman, D.S.; A characterization of groups in terms of the degree of their characters, Pacific J. Math., 15, 3, (1965), 877-903.

[2] Robinson, G.R.; On projective summands of induced modules, Journal of Algebra, 122, (1989), 106-111.