Comparison Theorems of Kolmogorov Type for Classes Defined by Cyclic Variation Diminishing Operators and Their Application

Fang Gensun Li Xuehua
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

Abstract

Using present a unified approach, we establish a Kolmogorov type comparison theorem for the classes of $2\pi$-periodic functions defined by a special class of operators having certain oscillation properties, which includes the classical Sobolev class of functions with $2\pi$-periodic, the Achiesser class, and the Hardy-Sobolev class as its special examples. Then, using these results, we prove a Taikov type inequality, and calculate the exact values of Kolmogorov, Gel’fand, linear and information $n$–widths of this class of functions in some space $L_q$, which is the classical Lebesgue integral space of $2\pi$–periodic with the usual norm.

Key words and phrases: Comparison theorems of Kolmogorov type, $n$-widths, oscillation properties

AMS classification: 41A46, 30D55

1 Introduction

Osipenko [13] noticed that many extremal problems of approximation theory, such as optimal recovery, optimal quadrature formulae, and $n$-widths etc., can be solved for the classes of smooth periodic functions which can be represented as convolutions with cyclic variation diminishing kernels. However, the functions in some classes of analytic functions do not seem to be representable as convolutions with cyclic variation diminishing kernels. Nevertheless, results that have been known for these classes are very close to those obtained in the smooth case above (see [9, 11, 12, 2, 3]). The idea of constructing general theory, which is covering both of the smooth and analytic cases, has been repeatedly expressed by Tikhomirov (see [13]). Osipenko [13] proposed a unified approach to the problem of the exact calculation of $n$-widths on the classes of functions, defined by a special class of (in general, nonlinear) operators that have the property of cyclic variation diminishing. In this paper, we continue the work of Osipenko [13] to study comparison theorems of Kolmogorov type, inequalities of Taikov type, and determine the precise values $n$–widths on such class of functions.

First we recall some definitions. Let $W$ be a subset of a linear space $X$. We consider the problem of the optimal recovery of a linear functional $L$ on this subset from the values of linear functionals $l_1,\ldots,l_n$. For $x \in W$, we set

$$Ix := (l_1 x,\ldots,l_n x).$$

*Project supported by the National Natural Science Foundation of China(Grant No. 10371009) and Research Fund for the Doctoral Program Higher Education(Grant No. 20050027007).
**Corresponding author.
E-mail address: fanggs@bnu.edu.cn
The operator $I: W \rightarrow K^n$, where $K = \mathbb{R}$ or $\mathbb{C}$, depending on whether $X$ is a real or a complex space, is called the information operator. The quantity

$$e(L, W, I) := \inf_{S: K^n \rightarrow K} \sup_{x \in W} |Lx - S(Ix)|$$

is called the error of the optimal recovery of the functional $L$ on the set $W$. Each method $S_0$ such that

$$\sup_{x \in W} |Lx - S_0(Ix)| = e(L, W, I)$$

is said to be an optimal method of recovery. Smolyak [17] proved that in the real case, for a convex centrally symmetric set $W$, there exists among optimal methods of recovery a linear method and

$$e(L, W, I) := \sup_{x \in W, Ix = 0} |Lx|, \quad (1.1)$$

Each element $x_0 \in W$ such that $Ix_0 = 0$ and

$$|Lx_0| = \sup_{x \in W, Ix = 0} |Lx|,$$

is said to be extremal. The problem of finding an extremal element often turns out to be simpler than that of finding an optimal recovery method.

Let $X$ be a normed linear space with norm $\| \cdot \|$, and $X_n$ be an $n$–dimensional subspace of $X$. For each $x \in X$, $E(x; X_n)$ denotes the distance of the $n$–dimensional subspace $X_n$ from $x$, defined by

$$E(x; X_n) := \inf_{y \in X_n} \|x - y\|, \quad (1.2)$$

and the quantity

$$E(A, X_n) := \sup_{x \in A} \inf_{y \in X_n} \|x - y\| \quad (1.3)$$

is said to be the deviation of $A$ from $X_n$. Thus $E(A, X_n)$ measures how well the “worst element” of $A$ can be approximated from $X_n$.

Given a subset $A$ of $X$, one might ask how well one can approximate $A$ by $n$–dimensional subspaces of $X$. Thus, we consider the possibility of allowing the $n$–dimensional subspaces $X_n$ to vary within $X$. This idea, introduced by Kolmogorov in 1936, is now referred to as the Kolmogorov $n$–width of $A$ in $X$. It is defined by

$$d_n(A, X) := \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|, \quad (1.4)$$

where $X_n$ runs over all $n$–dimensional subspaces of $X$.

The Kolmogorov $n$–width $d_n(A, X)$ describes the minimum error of $A$ approximated by any $n$–dimensional subspace $X_n$ in $X$. In addition to the Kolmogorov $n$–width, there are three other related concepts that will be studied in this paper. The linear $n$–width of $A$ in $X$ is defined by

$$\lambda_n(A, X) := \inf_{P_n} \sup_{x \in A} \|x - P_n x\|, \quad (1.5)$$

where the infimum is taken over all bounded linear operators mapping $X$ into itself whose range has dimension at most $n$. 
The Gel’fand $n$–width of $A$ in $X$ is given by
\[
d^0(A, X) := \inf_{X^n \in A \setminus X^n} \sup_{x \in X^n} \|x\|, \tag{1.6}\]
where $X^n$ runs over all subspaces of $X$ of codimension $n$ (here we assume that $0 \in A$), and the information $n$–width is the quantity
\[
i_n(A, X) := \inf_{l_1, \ldots, l_n \in \mathbb{Z}^n \to X} \inf_{x \in A} \sup_{m \in \mathbb{N}} \|x - m(l_1 x, \ldots, l_n x)\|, \tag{1.7}\]
where $l_1, \ldots, l_n$ run over all continuous linear functionals on $A$ and $m$ is taken over all maps of $\mathbb{Z}^n$ into $X$ ($Z = \mathbb{R}$ or $\mathbb{C}$ depending on whether $A$ is a set of real–valued or complex–valued functions).

More detailed information about the $n$–widths, we refer to monographs of Pinkus [16], and Lorentz, Golischek and Makovoz [7].

Now we recall some notions of sign changes of vectors and functions, and the definitions of Property $B$ and NCVD.

**Definition 1.1.** (see [16], pp. 45, 59) Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$ be a real non–trivial vectors.

(i) $S^-(x)$ indicates the number of sign changes in the sequence $x_1, \ldots, x_n$ with zero terms discarded. The number $S^-_c(x)$ of cyclic variations of sign of $x$ is given by
\[
S^-_c(x) := \max_i S^-(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_i) = S^-(x_k, \ldots, x_n, x_1, \ldots, x_k),
\]
where $k$ is any integer for which $x_k \neq 0$. Obviously $S^-_c(x)$ is invariant under cyclic permutations, and $S^-_c(x)$ is always an even number.

(ii) $S^+(x)$ counts the maximum number of sign changes in the sequence $x_1, \ldots, x_n$ where zero terms are arbitrarily assigned values $+1$ or $-1$. The number $S^+_c(x)$ of maximum cyclic variations of sign of $x$ is defined by
\[
S^+_c(x) := \max_i S^+(x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_i).
\]

Let $f$ be a piecewise continuous, $2\pi$–periodic function. We assume that $f(x) = [f(x^+) + f(x^-)]/2$ for all $x$ and
\[
S_c(f) := \sup S^-_c ((f(x_1), \ldots, f(x_m))), \tag{1.8}\]
where the supremum is taken over all $x_1 < \cdots < x_m < x_1 + 2\pi$ and all $m \in \mathbb{N}$.

Suppose that $f$ is a continuous function of periodic $2\pi$. We define
\[
\tilde{Z}_c(f) := \sup S^+_c ((f(x_1), \ldots, f(x_m))), \tag{1.9}\]
where the supremum runs over all $x_1 < \cdots < x_m < x_1 + 2\pi$ and all $m \in \mathbb{N}$. Clearly,
\[
S_c(f) \leq \tilde{Z}_c(f),
\]
where $S_c(f)$ denotes the number of sign changes of $f$ on a period, $\tilde{Z}_c(f)$ denotes the number of zeros of $f$ on a period, and sign changes are counted once and zeros which are not sign changes are counted twice.

**Definition 1.2.** (see [16], p.129) A real, $2\pi$–periodic, continuous function $G$ satisfies Property $B$ if for every choice of $0 \leq t_1 < \cdots < t_m < 2\pi$ and each $m \in \mathbb{N}$, the subspace
\[
X_m := \{b + \sum_{j=1}^{m} b_j G(-t_j) : \sum_{j=1}^{m} b_j = 0\}
\]
is of dimension $m$, and is a weak Tchebycheff ($WT$–) system (see [16], p.39) for all $m$ odd.
Let $\phi$ be a piecewise continuous $2\pi$–periodic function satisfying $\phi \perp 1$ and set $\psi(x) := a + (G * \phi)(x)$. If $G$ satisfies Property $B$, then $S_c(\psi) \leq S_c(\phi)$ (see [16, p.129]).

**Definition 1.3.** (see [16], p.60 and p.126) We say that a real, continuous, $2\pi$–periodic function $k$ is non–degenerate cyclic variation diminishing ($NCVD$) if $S_c(k * h) \leq S_c(h)$ for each real, $2\pi$–periodic, piecewise continuous function $h$, and

$$\dim \text{span}\{k(x_1 - \cdot), \ldots, k(x_n - \cdot)\} = n$$

for every choice of $0 \leq x_1 < \cdots < x_n < 2\pi$ and all $n \in \mathbb{N}$.

Denote by $L_q := L_q[0, 2\pi]$ the classical Lebesgue integral space of $2\pi$–periodic real–valued functions on $\mathbb{R}$ with the usual norm $\| \cdot \|_q$, $1 \leq q \leq \infty$. We now introduce the classes of functions to be studied here. Let $\varphi$ be a differentiable, odd and strictly increasing function for which $\varphi'$ is continuous on $[-1, 1]$. It is clear that the functions

$$\varphi_0(z) = \tan(\pi z/4) \quad (1.10)$$

and

$$\varphi_1(z) = z \quad (1.11)$$

satisfy the above conditions. Let $f * g$ denote the convolution of the functions $f$ and $g$, i.e.,

$$(f * g)(x) := \frac{1}{2\pi} \int_0^{2\pi} f(x - t)g(t) \, dt,$$

and let $G$ be a kernel satisfying Property $B$ or $NCVD$. Then $G$ has some variation-diminishing properties, we are interested in the class of $2\pi$–periodic functions representable as follows:

$$\widetilde{K}^G,\varphi^r_{\infty,\beta} = \{f : f = a + G * \varphi(K_\beta * u), a \in \Theta, \varphi(K_\beta * u) \perp \Theta, \|u\|_\infty \leq 1\}, \quad (1.12)$$

where $\Theta = \mathbb{R}$ if $G$ satisfies Property $B$ or $\Theta = \emptyset$ if $G$ is a $NCVD$ kernel, and

$$K_\beta(z) := 1 + 2 \sum_{k=1}^{\infty} \frac{\cos(kz)}{\cosh(k\beta)}, \quad \beta > 0. \quad (1.13)$$

For conciseness, we will regard $\varphi(K_\beta * u)$ as $u$ if $\beta = 0$. It can be seen that $\widetilde{K}^G,\varphi^0_{\infty,\beta}$ and $\widetilde{K}^G,\varphi^1_{\infty,\beta}$ are convex. Set

$$D_r(t) := 2 \sum_{k=1}^{\infty} \frac{\cos(kt - (\pi r/2))}{k^r}, \quad r = 1, 2, \ldots. \quad (1.14)$$

It is known that for each $r \geq 2$, $D_r$ satisfies Property $B$ ($D_1$ also satisfies all the conditions of Property $B$ except that it has a jump discontinuity at $x = 0$) (see [16], p.133). Then from [13] we have

$$\widetilde{K}^G,\varphi^r_{\infty,\beta} := \begin{cases} \widetilde{W}^r_{\infty}, & G = D_r, \beta = 0, \\ \widetilde{W}^r_{\infty,\beta}, & G = D_r, \varphi = \varphi_0, \beta > 0, \\ \widetilde{W}^r_{\infty,\beta}, & G = D_r, \varphi = \varphi_1, \beta > 0, \end{cases} \quad (1.15)$$
where \( \tilde{W}_\infty^r \) is the classical Sobolev class of real, \( 2\pi \)-periodic functions \( f \) whose \((r - 1)\)th derivative is absolutely continuous and whose \( r \)th derivative satisfies the condition \( \| f^{(r)} \|_\infty \leq 1 \); \( \tilde{h}_{\infty, \beta}^r \) is the Ahieser class \[1\] p. 214, and p. 219 which are all real–valued on \( \mathbb{R} \) with \( 2\pi \)–periodic, and can be continued analytically in the strip \( S_\beta := \{ z \in \mathbb{C} : |\text{Im}z| < \beta \} \), satisfying the restriction condition \( |\text{Re} f^{(r)}(z)| \leq 1 \) in this strip; and \( \tilde{H}_{\infty, \beta}^r \) is the class of \( 2\pi \)–periodic, real–valued functions on \( \mathbb{R} \) which are analytic in the strip \( S_\beta \) and satisfy the condition \( |f^{(r)}(z)| \leq 1, z \in S_\beta \). Thus the class \( \tilde{K}_{\infty, \beta}^{G, \varphi} \) includes the three famous classes as its special cases.

The exact values of \( n \)–widths were investigated for the classical Sobolev class \( \tilde{W}_p^r \) (see \[3\]) in \( L_q \) space, by the efforts of many mathematicians some similar results were also built for the classes \( \tilde{h}_{\infty, \beta}^r \) and \( \tilde{H}_{\infty, \beta}^r \). The Kolmogorov \( n \)–widths of the class \( \tilde{h}_{\infty, \beta}^r \) in the space \( L_\infty \) were obtained by Tikhomirov \[20\], Forst \[5\] for \( r = 0 \) and by Sun \[13\], and Osipenko \[12\] for all \( r \in \mathbb{N} \). After this, the exact values of the even \( n \)–widths of the class \( \tilde{h}_{\infty, \beta}^r \) in \( L_q \), \( 1 \leq q < \infty \) were calculated by Pinkus \[16\] for \( r = 0 \) and by Osipenko \[11\] for all \( r \in \mathbb{N} \). The exact estimates of the even \( n \)–widths of the class \( \tilde{H}_{\infty, \beta}^r \) in the space \( L_q \) were determined by Osipenko \[9\] for \( r = 0 \), \( 1 \leq q < \infty \) and by Osipenko \[12\]–\[13\], Osipenko and Wilderotter \[14\] for all \( r \in \mathbb{N} \), \( q = \infty \). The exact values of Gel’fand \( n \)–width, Kolmogorov \( 2n \)–width, linear \( 2n \)–width and information \( 2n \)–width of \( \tilde{H}_{\infty, \beta}^r \) in the space \( L_q \) were obtained by us \[2\] for all \( r \in \mathbb{N} \), \( 1 \leq q < \infty \).

Now we outline the rest of this paper. In Section \[2\] we establish a comparison theorem of Kolmogorov type on \( \tilde{K}_{\infty, \beta}^{G, \varphi} \) by using the property of the kernel \( G \). In Section \[3\] using the results of Section \[2\] we get a Taikov type inequality which will be used as the upper estimates of the Gel’fand \( n \)–width of the class \( \tilde{K}_{\infty, \beta}^{G, \varphi} \) in \( L_q \) for \( G \) satisfying Property \( B \) and \( 1 \leq q < \infty \). In Section \[4\] following the method of \[2\] and \[13\], we solve two minimum norm questions on analogue of the classical polynomial perfect splines, and then we use them to prove the lower estimates of \( n \)–widths, which together with some results of Section \[4\] determine the precise values of \( n \)–widths of \( \tilde{K}_{\infty, \beta}^{G, \varphi} \) in the space \( L_\infty \) for \( \varphi \) being such that \( \tilde{K}_{\infty, \beta}^{G, \varphi} \) is convex, the exact values of the Gel’fand \( n \)–width, and the lower estimate of the Kolmogorov \( 2n \)–width of \( \tilde{K}_{\infty, \beta}^{G, \varphi} \) in \( L_q \) for \( G \) satisfying Property \( B \) and \( 1 \leq q < \infty \).

2 Comparison Theorem of Kolmogorov Type

The Kolmogorov comparison theorem (see \[6\]), which concerns the comparison of derivatives of differentiable functions defined on the real line, plays an important role in establishing some sharp estimates of extremal problems in approximation theory. In this section, we will prove a comparison theorem of Kolmogorov type on the class \( \tilde{K}_{\infty, \beta}^{G, \varphi} \). Before advancing our discuss further, we introduce the following notions. Let

\[
\Lambda_{2n} = \{ \xi : \xi = (\xi_1, \ldots, \xi_{2n}), 0 \leq \xi_1 < \cdots < \xi_{2n} < 2\pi \}, \quad n \in \mathbb{N}. \quad (2.1)
\]

The closure of \( \Lambda_{2n}, \overline{\Lambda}_{2n} \) is given by

\[
\overline{\Lambda}_{2n} := \{ \xi : \xi = (\xi_1, \ldots, \xi_{2m}), 0 \leq \xi_1 < \cdots < \xi_{2m} < 2\pi, \quad m \leq n \}. \quad (2.2)
\]

For each \( \xi \in \Lambda_{2n} \), we define

\[
h_\xi(t) := (-1)^j, \quad t \in [\xi_{j-1}, \xi_j), \quad j = 1, \ldots, 2n + 1, \quad (2.3)
\]
where $\xi_0 = 0$ and $\xi_{2n+1} = 2\pi$. There is one particularly important function of this form, say $h_n$, which corresponds to the choice $\xi_j = (j - 1)\pi/n$, $j = 1, \ldots, 2n$.

We introduce the standard function $\Phi_{n,\beta}^{G,\varphi}$ of the comparison theorem on the class $\tilde{K}_{n,\beta}^{G,\varphi}$ and discuss some of its properties. Let

$$\Phi_{n,\beta}^{G,\varphi}(x) := (G * \varphi(K_\beta * h_n))(x).$$

Then it is easily seen that $\Phi_{n,\beta}^{G,\varphi}(x + \pi/n) = -\Phi_{n,\beta}^{G,\varphi}(x)$ and $\Phi_{n,\beta}^{G,\varphi}(x)$ is continuous with respect to $x$. So $\Phi_{n,\beta}^{G,\varphi}$ is $2\pi/n$-periodic and there exist at least $2n$ points $0 \leq t_1 < \cdots < t_{2n} < 2\pi$ such that

$$\Phi_{n,\beta}^{G,\varphi}(t_j) := \epsilon(-1)^j \|\Phi_{n,\beta}^{G,\varphi}\|_{\infty}, \quad j = 1, \ldots, 2n,$$

where $\epsilon = 1$ or $-1$.

Now we are in position to prove the comparison theorem on the class $\tilde{K}_{n,\beta}^{G,\varphi}$.

**Theorem 2.1 (Comparison theorem of Kolmogorov type).** Let $f \in \tilde{K}_{n,\beta}^{G,\varphi}$ be such that

$$\|f\|_{\infty} \leq \|\Phi_{n,\beta}^{G,\varphi}\|_{\infty}$$

for some positive integer $n$, and $f(\alpha) = \Phi_{n,\beta}^{G,\varphi}(\gamma)$, $f'(\alpha)\Phi_{n,\beta}^{G,\varphi'}(\gamma) \geq 0$ for some $\alpha, \gamma \in \mathbb{R}$. Then

$$|f'(\alpha)| \leq |\Phi_{n,\beta}^{G,\varphi'}(\gamma)|.$$

**Proof.** Without loss of generality we may assume $\alpha = \gamma$. Assume that for some $f = a + G * \varphi(K_\beta * u) \in \tilde{K}_{n,\beta}^{G,\varphi}$, $a \in \Theta$, $\varphi(K_\beta * u) \perp \Theta$, $\|u\|_{\infty} \leq 1$, positive integer $n$ and $\alpha \in \mathbb{R}$, we have

$$\|f\|_{\infty} \leq \|\Phi_{n,\beta}^{G,\varphi}\|_{\infty}, f(\alpha) = \Phi_{n,\beta}^{G,\varphi}(\alpha), f'(\alpha)\Phi_{n,\beta}^{G,\varphi'}(\alpha) \geq 0 \text{ and } |f'(\alpha)| \leq |\Phi_{n,\beta}^{G,\varphi'}(\alpha)|.$$

It follows from the definitions that $f'$ and $\Phi_{n,\beta}^{G,\varphi'}$ are continuous. If $f(\alpha) \neq 0$, the continuity of them ensures the existence of $0 < \rho < 1$ and $\alpha_0 \in \mathbb{R}$ such that

$$\rho f(\alpha_0) = \Phi_{n,\beta}^{G,\varphi}(\alpha_0), \quad \rho |f'(\alpha_0)| > |\Phi_{n,\beta}^{G,\varphi'}(\alpha_0)|.$$

If $f(\alpha) = \Phi_{n,\beta}^{G,\varphi}(\alpha) = 0$, we can choose some $\rho$ satisfying $1 > \rho > \frac{|\Phi_{n,\beta}^{G,\varphi'}(\alpha)|}{|f'(\alpha)|}$ and let $\alpha_0 = \alpha$. Then (2.7) also holds. Put $f := \rho f$, then $f \in \tilde{K}_{n,\beta}^{G,\varphi}$ and $\|f\|_{\infty} < \|\Phi_{n,\beta}^{G,\varphi}\|_{\infty}$. It is enough to consider one possible case $f(\alpha_0) = \Phi_{n,\beta}^{G,\varphi}(\alpha_0) \geq 0$, $f'(\alpha_0) > \Phi_{n,\beta}^{G,\varphi'}(\alpha_0) > 0$ (the other cases can be treated by the same way). From the property of $\Phi_{n,\beta}^{G,\varphi}$, it has at least $2n$ monotone intervals on every interval whose length is $2\pi$. Choose a interval whose length is $2\pi$ and whose endpoints are extremal points of $\Phi_{n,\beta}^{G,\varphi}$, in which $\alpha_0$ is contained, written as $\Delta_{\alpha_0}$. By geometrical consideration we see that in the monotone interval, which is in $\Delta_{\alpha_0}$ and contains $\alpha_0$, the graphs of $f$ and $\Phi_{n,\beta}^{G,\varphi}$ intersect at least three times, while on each of the other monotone intervals in $\Delta_{\alpha_0}$ these graphs intersect at least once. Hence for $F(x) := \Phi_{n,\beta}^{G,\varphi}(x) - \rho f(x)$, we have $\tilde{Z}_c(F) \geq 2n + 2$.

On the other hand, since $\varphi$ is a differentiable odd and strictly increasing function, and $K_\beta$ is NCVD, if $G$ satisfies Property B, by [165] Chapt. IV, Prop. 6.4] we can get

$$\tilde{Z}_c(F) \leq S_c(\varphi(K_\beta * h_n) - \rho \varphi(K_\beta * u)) \leq S_c(h_n(\cdot) - u^*(\cdot)) \leq 2n,$$

where $u^*$ is defined by the equality $\rho \varphi(K_\beta * u) = \varphi(K_\beta * u^*)$, satisfying $\varphi(K_\beta * u^*) \perp \Theta$ and $\|u^*\|_{\infty} < 1$. If $G$ is a NCVD kernel, by [163] Chapt. III, Section 3], we shall assume that $G$ is extended CVD by means of “smoothing”. The property of $G$ being extended CVD implies that

$$\tilde{Z}_c(F) \leq S_c(\varphi(K_\beta * h_n) - \rho \varphi(K_\beta * u)) \leq S_c(h_n(\cdot) - u^*(\cdot)) \leq 2n,$$

where $u^*$ is the same as that above. Thus we get a contradiction. Theorem 2.1 is proved. \square
Remark 2.2. Among others, Sun [18] established a comparison theorem of Kolmogorov type for the class $\tilde{h}_{\infty,\beta}$. Moreover, Fisher [4] demonstrated an inequality of Landau–Kolmogorov type for the class of non–periodic analytic functions on the open unit disk whose $r$th derivatives are bounded by 1 and which are real–valued on the interval $(-1,1)$. Osipenko [11] established Kolmogorov inequalities for the class of functions analytic in the strip $S_{\beta}$ real on the real axis $\mathbb{R}$ and satisfying the condition $|f^{(r)}(z)| \leq 1$, $z \in S_{\beta}$, which includes the class $H_{\infty,\beta}$ as a subset with $2\pi$-periodic. Using a method different from that of Osipenko [11] we established a comparison theorem of Kolmogorov type on the class $H_{\infty,\beta}$ (see [2]).

Our next aim is to establish a Landau–Kolmogorov type inequality and some other corollaries of the comparison theorem.

Corollary 2.3 (Landau–Kolmogorov type inequality). Let $f \in \tilde{K}_{\infty,\beta}^{G\varphi}$ be such that $\|f\|_{\infty} \leq \|\Phi_{n,\beta}^{G\varphi}\|_{\infty}$ for some positive integer $n$. Then

$$\|f'\|_{\infty} \leq \|\Phi_{n,\beta}^{G\varphi'}\|_{\infty}. \quad (2.8)$$

Proof. Assume that there exists a $\xi \in \mathbb{R}$ for which $|f'(\xi)| > \|\Phi_{n,\beta}^{G\varphi}\|_{\infty}$. Since $\Phi_{n,\beta}^{G\varphi}(t)$ is a continuous function of $t$ and

$$\|f\|_{\infty} \leq \|\Phi_{n,\beta}^{G\varphi}\|_{\infty}, \quad -\|\Phi_{n,\beta}^{G\varphi}\|_{\infty} \leq \Phi_{n,\beta}^{G\varphi}(t) \leq \|\Phi_{n,\beta}^{G\varphi}\|_{\infty}, \quad t \in \mathbb{R},$$

there exists an $\eta \in \mathbb{R}$ such that $f(\xi) = \Phi_{n,\beta}^{G\varphi}(\eta)$ and $|f'(\xi)| > |\Phi_{n,\beta}^{G\varphi'}(\eta)|$, which contradicts Theorem 2.1. Corollary 2.3 is proved. \hfill \Box

Let $C(\mathbb{R})$ be the set of all continuous functions on the real line $\mathbb{R}$. A function $\psi \in C(\mathbb{R})$ is said to be regular (see [3], p. 107) if it has a period $2l$ and if on some interval $(a, a+2l)$ ($a$ is a point of an absolute extremum of $\psi$) there is a point $c$ such that $\psi$ is strictly monotone on $(a,c)$ and $(c, a+2l)$. In order to emphasize the length of the period $2l$, sometimes we shall speak about $\psi$ as being $2l$–regular.

We say that a function $f \in C(\mathbb{R})$ possesses a $\mu$–property with respect to a regular function $\psi$ if for every $\alpha \in \mathbb{R}$ and on every interval of monotonicity of $\psi$ the difference $\psi(t) - f(t+\alpha)$ either does not change sign or changes sign exactly once–from $+\to -$ if $\psi$ decreases or from $-\to +$ if $\psi$ increases. It is clear that $f$ will possess the $\mu$–property with respect to $\psi(t+\beta)$, $\beta \in \mathbb{R}$ if it possesses the $\mu$–property with respect to $\psi$.

Corollary 2.4. Let $G$ satisfy Property B. Let $f \in \tilde{K}_{\infty,\beta}^{G\varphi}$ be such that $\|f\|_{\infty} \leq \|\Phi_{n,\beta}^{G\varphi}\|_{\infty}$ for some positive integer $n$. Then the function $f$ possesses a $\mu$–property with respect to $\Phi_{n,\beta}^{G\varphi}$.

Proof. Let $G$ satisfy Property B. First we will prove that $\Phi_{n,\beta}^{G\varphi}$ is $2\pi/n$–regular. Assume that $\Phi_{n,\beta}^{G\varphi}$ is not $2\pi/n$–regular. Since $\Phi_{n,\beta}^{G\varphi}(x + \pi/n) = -\Phi_{n,\beta}^{G\varphi}(x)$ and $\Phi_{n,\beta}^{G\varphi}(x)$ is continuous with respect to $x$, there exists a constant $c \in \mathbb{R}$ such that $\Phi_{n,\beta}^{G\varphi}(-c)$ has at least $2n + 2$ zeros, i.e.,

$$\overline{Z}_{c}(\Phi_{n,\beta}^{G\varphi}(-c)) \geq 2n + 2.$$

On the other hand, since $\varphi$ is a differentiable, odd and strictly increasing function, and $K_{\beta}$ is NCVD, the condition that $G$ satisfies Property B implies that

$$\overline{Z}_{c}(\Phi_{n,\beta}^{G\varphi}(-c)) \leq S_{c}(\varphi(K_{\beta} \ast h_n)) \leq S_{c}(h_n) \leq 2n.$$
by [16] Chapt. IV, Prop. 6.4. This contradiction shows that \( \Phi_{n,\beta}^{G,\varphi} \) is \( 2\pi/n \)-regular. Therefore there exist exactly 2n points \( 0 \leq t_{n,\beta,1}^{G,\varphi} < \cdots < t_{n,\beta,2n}^{G,\varphi} < 2\pi \) such that

\[
\Phi_{n,\beta}^{G,\varphi}(t_{n,\beta,j}^{G,\varphi}) := \epsilon (1)^j \| \Phi_{n,\beta}^{G,\varphi} \|_{\infty}, \quad t_{n,\beta,j}^{G,\varphi} - t_{n,\beta,j-1}^{G,\varphi} = \frac{\pi}{n}, \quad j = 1, \ldots, 2n,
\]

where \( \epsilon = 1 \) or \(-1\), and \( t_{n,\beta,2n+1}^{G,\varphi} = t_{n,\beta,1}^{G,\varphi} + 2\pi \). So put

\[
\Delta_{n,\beta}^{G,\varphi}(j) = [t_{n,\beta,j}^{G,\varphi}, t_{n,\beta,j+1}^{G,\varphi}), \quad j = 1, \ldots, 2n.
\]

Then \( \Phi_{n,\beta}^{G,\varphi} \) is monotonic on each interval \( \Delta_{n,\beta}^{G,\varphi}(j), j = 1, \ldots, 2n \).

Suppose that \( f \) does not possess a \( \mu \)-property with respect to \( \Phi_{n,\beta}^{G,\varphi} \). Then for some \( \alpha \in \mathbb{R} \), the difference \( \Phi_{n,\beta}^{G,\varphi}(t) - f(t + \alpha) \) changes sign on a monotone interval \( \Delta_{n,\beta}^{G,\varphi}(k) \) \((k \in \{1, \ldots, 2n\})\) of the standard function \( \Phi_{n,\beta}^{G,\varphi} \) from + to − if \( \Phi_{n,\beta}^{G,\varphi} \) increases and from − to + if \( \Phi_{n,\beta}^{G,\varphi} \) decreases. By continuity arguments this fact will also hold for the difference

\[
F(t) := \Phi_{n,\beta}^{G,\varphi}(t) - (1 - \epsilon)f(t + \alpha)
\]

for \( 0 < \epsilon < 1 \) sufficiently small. Since \((1 - \epsilon) \| f(\cdot + \alpha) \|_{\infty} < \| \Phi_{n,\beta}^{G,\varphi} \|_{\infty}, F \) changes sign at least three times on the interval \( \Delta_{n,\beta}^{G,\varphi}(k) \) and at least once on each of the other intervals \( \Delta_{n,\beta}^{G,\varphi}(j) \) \((j \in \{1, \ldots, 2n\} \setminus \{k\})\). Hence \( S_{\epsilon}(F) \geq 2n + 2 \).

On the other hand, we can prove that \( S_{\epsilon}(F) \leq 2n \) by the same method as that in the proof of Theorem 2.1. This contradiction proves the corollary.

A function \( F \) is said to be a periodic integral of a real, \( 2\pi \)-periodic, continuous function \( f \) on \( \mathbb{R} \) if \( F'(x) = f(x) \) and \( F(x + 2\pi) = F(x) \) for all \( x \in \mathbb{R} \). Let \( G \) satisfy Property B. Put

\[
\tilde{G}(x) = \int_0^x (G(y) - a_0) \, dy,
\]

where \( a_0 = 1/2\pi \int_0^{2\pi} G(y) \, dy \). Then \( \tilde{G} \) satisfies Property B (see [16] Chapt. IV, Prop. 6.6) and \( \tilde{\Phi}_{n,\beta}^{G,\varphi} \) is a periodic integral of \( \Phi_{n,\beta}^{G,\varphi} \). From Corollary 2.3, we have immediately the following theorem which will be used in the next section.

**Theorem 2.5.** Let \( G \) satisfy Property B and \( \tilde{G} \) be defined by (2.11). Suppose that \( f \in \tilde{K}_{n,\beta}^{G,\varphi} \), and \( F \) is a periodic integral of \( f \) such that \( \| F \|_{\infty} \leq \| \Phi_{n,\beta}^{G,\varphi} \|_{\infty} \) for some positive integer \( n \). Then

\[
\| f \|_{\infty} \leq \| \Phi_{n,\beta}^{G,\varphi} \|_{\infty}.
\]

### 3 Inequalities of Taikov Type

In this section, we establish an inequality of Taikov type, which leads to the upper estimates of the Gel’fand \( n \)-widths of \( \kappa_{n,\beta}^{G,\varphi} \) in \( L_q \) for \( G \) satisfying Property B and \( 1 \leq q < \infty \). To do this, we need some auxiliary lemmas.

**Lemma 3.1.** Let \( n \in \mathbb{N} \), \( G \) satisfy Property B and \( \tilde{G} \) be defined by (2.11). Then

\[
\int_0^{2\pi} |\tilde{\Phi}_{n,\beta}^{G,\varphi}(t)| \, dt = \sqrt{\int_0^{2\pi} \tilde{\Phi}_{n,\beta}^{G,\varphi} \, dt} = 4n \| \Phi_{n,\beta}^{G,\varphi} \|_{\infty}.
\]


Proof. By (2.5) and (2.10), $\Phi_{n,\beta}^{G,\varphi}$ is $2\pi/n$-periodic, strictly monotonic on $\Delta_{n,\beta}^{G,\varphi}(j) (j = 1, \ldots, 2n)$ and $\Phi_{n,\beta}^{G,\varphi}(t + \frac{\pi}{n}) = -\Phi_{n,\beta}^{G,\varphi}(t)$ for all $t \in [0, 2\pi)$. Hence,

$$\int_0^{2\pi} |\Phi_{n,\beta}^{G,\varphi}(t)| dt = \int_{t_{n,\beta,1}}^{t_{n,\beta,1}+2\pi} |\Phi_{n,\beta}^{G,\varphi}(t)| dt = 2n \int_{t_{n,\beta,1}}^{t_{n,\beta,1}+\frac{\pi}{n}} |\Phi_{n,\beta}^{G,\varphi}(t)| dt = 2n \int_{t_{n,\beta,1}}^{t_{n,\beta,1}+\frac{\pi}{n}} \Phi_{n,\beta}^{G,\varphi}(t) dt$$

$$= 2n \left| \Phi_{n,\beta}^{G,\varphi}(t_{n,\beta,1} + \frac{\pi}{n}) - \Phi_{n,\beta}^{G,\varphi}(t_{n,\beta,1}) \right| = 4n \|\Phi_{n,\beta}^{G,\varphi}\|_{\infty} = 2n \int_0^{2\pi} \Phi_{n,\beta}^{G,\varphi} = \int_0^{\pi/n} \Phi_{n,\beta}^{G,\varphi}.$$}

Now let $f$ be a $2\pi$-periodic integrable function, and denote by $r(f, t)$ the non-increasing rearrangement of $|f|$ (see [8], p. 110). With this notation we have

**Lemma 3.2. (see [6], p.112)** Let $f, g \in L_q (1 \leq q < \infty)$ and

$$\int_0^x r(f, t) dt \leq \int_0^x r(g, t) dt, \quad 0 \leq x \leq 2\pi.$$ Then

$$\|f\|_q \leq \|g\|_q.$$}

**Lemma 3.3. (see [6], p. 114)** Let $f \in C(\mathbb{R})$ possess the $\mu$-property with respect to the $2\pi/n$-regular ($n = 1, 2, \ldots$) function $\psi$ and

$$\int_0^{2\pi/n} \psi(t) dt = 0.$$ If

$$\min_u \psi(u) \leq f(t) \leq \max_u \psi(u), \quad \forall t \in \mathbb{R},$$

and

$$\max_{a,b} \left| \int_a^b f(t) dt \right| \leq \frac{1}{2} \int_0^{2\pi/n} |\psi(t)| dt,$$

then

$$\int_0^x r(f, t) dt \leq \int_0^x r(\psi, t) dt, \quad 0 \leq x \leq 2\pi.$$}

**Theorem 3.4.** Let $G$ satisfy Property B, $\tilde{G}$ be defined by (2.11), $f \in \bar{K}_{\infty,\beta}^{G,\varphi}$ and $F$ be a periodic integral of $f$ such that $\|F\|_\infty \leq \|\Phi_{n,\beta}^{G,\varphi}\|_\infty$ for some positive integer $n$. Then

$$\int_0^x r(f, t) dt \leq \int_0^x r(\Phi_{n,\beta}^{G,\varphi}, t) dt, \quad 0 \leq x \leq 2\pi,$$

$$\|f\|_q \leq \|\Phi_{n,\beta}^{G,\varphi}\|_q, \quad 1 \leq q < \infty.$$}

Proof. By virtue of Lemma 3.2, the inequality (3.3) follows from (3.2). So we only need to prove the inequality (3.3). Since $f \in \bar{K}_{\infty,\beta}^{G,\varphi}$ and $F$ is a periodic integral of $f$ such that $\|F\|_\infty \leq \|\Phi_{n,\beta}^{G,\varphi}\|_\infty$ for some positive integer $n$, we conclude from Theorem 2.5 that $\|f\|_\infty \leq \|\Phi_{n,\beta}^{G,\varphi}\|_\infty$, and it follows
By noticing that \( 3.1 \):

Let \( \text{Lemma } 3.5 \).

as the upper estimates of \( n \) the following lemma will be used in establishing Taikov type inequality in the next section as well.

Proof. First, we claim that the system is a Tchebycheff system, it follows from \([16, p.41]\) that from Corollary 3.9 that \( \Phi \) possesses a \( \mu \)-property with respect to the \( 2\pi/n \)-regular function \( \Phi_{n,\beta}^{G,\varphi} \). Hence the inequality (3.2) is true. Theorem 3.4 is proved.

Let \( T_n = \text{span}\{1, \cos t, \sin t, \ldots, \cos((n-1)t), \sin((n-1)t)\} \) be the trigonometric polynomial subspace of order \( n-1 \). Set

\[
a_j(f) := \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(jt) \, dt, \quad j = 0, 1, \ldots,
\]

\[
b_j(f) := \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(jt) \, dt, \quad j = 1, 2, \ldots,
\]

\[
I_{2n-1}(f) := (a_0(f), a_1(f), b_1(f), \ldots, a_{n-1}(f), b_{n-1}(f)).
\]

The following lemma will be used in establishing Taikov type inequality in the next section as well as the upper estimates of \( n \)-widths in section 4.

**Lemma 3.5.** Let \( n \in \mathbb{N} \) and \( \varphi \) be such that \( \tilde{K}_{\infty,\beta}^{G,\varphi} \) is convex. Then there exists a linear operator \( \mathcal{L}: \mathbb{R}^{2n-1} \to T_n \) such that

\[
\sup_{f \in \tilde{K}_{\infty,\beta}^{G,\varphi}} \| f - \mathcal{L}(I_{2n-1}(f)) \|_\infty = \| G * \varphi(K_{\beta} * h_n) \|_\infty = \| \Phi_{n,\beta}^{G,\varphi} \|_\infty. \tag{3.4}
\]

**Proof.** First, we claim that

\[
\sup_{f \in \tilde{K}_{\infty,\beta}^{G,\varphi}, I_{2n-1}(f) = 0} \| f \|_\infty = \| G * \varphi(K_{\beta} * h_n) \|_\infty = \| \Phi_{n,\beta}^{G,\varphi} \|_\infty. \tag{3.5}
\]

Suppose that there exists a function \( f_0 = a_0 + G * \varphi(K_{\beta} * u_0) \in \tilde{K}_{\infty,\beta}^{G,\varphi} \), where \( a_0 \in \Theta \), \( \varphi(K_{\beta} * u_0) \perp \Theta \) and \( \| u_0 \|_\infty \leq 1 \), such that \( I_{2n-1}(f_0) = 0 \) and \( \| f_0 \|_\infty > \| \Phi_{n,\beta}^{G,\varphi} \|_\infty \). Set

\[
\rho = \frac{\| \Phi_{n,\beta}^{G,\varphi} \|_\infty}{\| f_0 \|_\infty}.
\]

Then \( 0 < \rho < 1 \). We choose \( \alpha \in [0, 2\pi) \) such that \( F(\cdot) \equiv \Phi_{n,\beta}^{G,\varphi}(\cdot + \alpha) - \rho f_0(\cdot) \) has a multiple zero. Since \( \Phi_{n,\beta}^{G,\varphi} \) has the period \( 2\pi/n \), \( I_{2n-1}(\Phi_{n,\beta}^{G,\varphi}) = 0 \). Thus, \( I_{2n-1}(F) = 0 \). Since the trigonometric system is a Tchebycheff system, it follows from [16, p.41] that

\[
\tilde{Z}_c(F(\cdot)) > S_c(F(\cdot)) \geq 2n.
\]
On the other hand, since \( \varphi \) is a differentiable odd and strictly increasing function, and \( K_\beta \) is NCVD, if \( G \) satisfies Property B, by [16 Chapt. IV, Prop. 6.4] we can get

\[
\tilde{Z}_c(F(\cdot)) \leq S_c(\varphi(K_\beta \ast h_n)(\cdot + \alpha) - \rho \varphi(K_\beta \ast u_0)(\cdot)) \\
\leq S_c(h_n(\cdot + \alpha) - u_0^*(\cdot)) \leq 2n,
\]

where \( u_0^* \) is defined by the equality \( \rho \varphi(K_\beta \ast u_0) = \varphi(K_\beta \ast u_0^*) \), satisfying \( \varphi(K_\beta \ast u_0^*) \perp \Theta \) and \( \| u_0^* \|_\infty < 1 \). If \( G \) is a NCVD kernel, by [16 Chapt. III, Section 3], we shall assume that \( G \) is extended CVD by means of “smoothing”. The property of \( G \) being extended CVD implies that

\[
\tilde{Z}_c(F(\cdot)) \leq S_c(\varphi(K_\beta \ast h_n)(\cdot + \alpha) - \rho \varphi(K_\beta \ast u_0)(\cdot)) \\
\leq S_c(h_n(\cdot + \alpha) - u_0^*(\cdot)) \leq 2n,
\]

where \( u_0^* \) is the same as that above. Thus we get a contradiction. So (3.5) holds.

Now we consider the problem of the optimal recovery of the value \( f(0) \) in the class \( \widetilde{K}_{\infty, \beta}^{G, \varphi} \) from the information \( I_{2n-1}(f) \). Since \( \varphi \) is such that \( \widetilde{K}_{\infty, \beta}^{G, \varphi} \) is convex, by general results concerning the problems of the optimal recovery [17], there exists a linear optimal method of recovery, that is, there exist numbers \( a_0, a_1, a_2, \ldots, a_{n-1}, b_{n-1} \) such that

\[
\sup_{f \in \widetilde{K}_{\infty, \beta}^{G, \varphi}} \| f(0) - a_0 a_0(f) - \sum_{j=1}^{n-1} (\alpha_j a_j(f) + \beta_j b_j(f)) \| \infty \leq \sup_{f \in \widetilde{K}_{\infty, \beta}^{G, \varphi}} \| f \| \infty = \| \Phi_n^{G, \varphi} \| \infty. \tag{3.6}
\]

Let \( g \) be an arbitrary function in \( \widetilde{K}_{\infty, \beta}^{G, \varphi} \). For \( t \in [0, 2\pi) \), we set \( f_t(\tau) = g(t + \tau) \). We have \( f_t \in \widetilde{K}_{\infty, \beta}^{G, \varphi} \), \( a_0(f_t) = a_0(g) \), and

\[
a_j(f_t) = a_j(g) \cos(jt) + b_j(g) \sin(jt), \quad j = 1, 2, \ldots; \tag{3.7}
\]

\[
b_j(f_t) = -a_j(g) \sin(jt) + b_j(g) \cos(jt),
\]

therefore setting

\[
\mathcal{L}(I_{2n-1}(g)) := a_0 a_0(g) + \sum_{j=1}^{n-1} ((\alpha_j \cos(jt) - \beta_j \sin(jt)) a_j(g) + (\alpha_j \sin(jt) + \beta_j \cos(jt)) b_j(g)),
\]

from (3.6) with \( \tau = 0 \) we obtain

\[
| g(t) - \mathcal{L}(I_{2n-1}(g)) | \leq \| \Phi_n^{G, \varphi} \| \infty.
\]

Since \( t \in [0, 2\pi) \) can be chosen arbitrarily, it follows that

\[
\| g - \mathcal{L}(I_{2n-1}(g)) \| \infty \leq \| \Phi_n^{G, \varphi} \| \infty.
\]

For \( g = \Phi_n^{G, \varphi} \) the last inequality turns into equality, which proves Lemma 3.5
Let $G$ satisfy Property B and $\tilde{G}$ be defined by (2.11). Consider a subset of $\tilde{K}^{G,\varphi}_{\infty,\beta}$ defined by

$$\tilde{K}^{G,\varphi}_{\infty,\beta} \cap T_n^\perp := \left\{ f \in \tilde{K}^{G,\varphi}_{\infty,\beta} : I_{2n-1}(f) = 0 \right\},$$

(3.8)

and let $F$ be the periodic integral of $f \in \tilde{K}^{G,\varphi}_{\infty,\beta} \cap T_n^\perp$ and satisfy the condition $\int_0^{2\pi} F(t)dt = 0$. Then we have $I_{2n-1}(F) = 0$. By (3.7) we know that

$$\|F\|_\infty \leq \|\Phi_{n,\beta}^{G,\varphi}\|_\infty,$$

(3.9)

which together with Theorem 3.4 gives an inequality of Taikov type as follows.

**Theorem 3.6 (Inequalities of Taikov type).** Let $n \in \mathbb{N}$, $G$ satisfy Property B and $f \in \tilde{K}^{G,\varphi}_{\infty,\beta} \cap T_n^\perp$. Then

$$\|f\|_q \leq \|\Phi_{n,\beta}^{G,\varphi}\|_q, \quad 1 \leq q < \infty.$$

Since $\Phi_{n,\beta}^{G,\varphi}$ is $2\pi/n$-periodic, $\Phi_{n,\beta}^{G,\varphi} \in \tilde{K}^{G,\varphi}_{\infty,\beta} \cap T_n^\perp$. From Theorem 3.6 and (3.9), we obtain

**Corollary 3.7.** Let $n \in \mathbb{N}$ and $G$ satisfy Property B. Then

$$\sup_{f \in \tilde{K}^{G,\varphi}_{\infty,\beta} \cap T_n^\perp} \|f\|_q = \|\Phi_{n,\beta}^{G,\varphi}\|_q, \quad 1 \leq q < \infty.$$

**Remark 3.8.** Among others, Taikov [19] proved that for all $n \in \mathbb{N}$ and $r = 1,2,\ldots$,

$$\sup_{f \in \tilde{H}^{\infty,\beta} \cap T_n^\perp} \|f\|_q \leq \|D_r \ast h_n\|_q, \quad 1 \leq q < \infty.$$ 

(3.10)

The inequality (3.10) is now referred to as Taikov’s inequality. For the case $q = 1$, (3.10) was also proved by Turovets in another way. For more details see [6, p. 172 and p. 207]. We [2] proved that for all $n \in \mathbb{N}$ and $r = 0,1,2,\ldots$,

$$\sup_{f \in \tilde{H}^{\infty,\beta} \cap T_n^\perp} \|f\|_q = \|D_r \ast \varphi_0 (K_\beta \ast h_n)\|_q, \quad 1 \leq q < \infty.$$ 

(3.11)

**Remark 3.9.** We conjecture that the inequality of Taikov type also holds for $G$ being a NCVD kernel, the main difficulty lies proving the corollary is also true for $G$ being a NCVD kernel.

### 4 Exact Values of $n$–Widths of $\tilde{K}^{G,\varphi}_{\infty,\beta}$

In this section, we will solve a uniform minimum norm question of the classical polynomial perfect splines. Using the result, we determine the exact values of the Kolmogorov, Gel’fand, linear and information $n$–widths of $\tilde{K}^{G,\varphi}_{\infty,\beta}$ in $L_\infty$ for $\varphi$ being such that $\tilde{K}^{G,\varphi}_{\infty,\beta}$ is convex, which have been obtained by Osipenko [13] in a different method. And then we solve a minimum norm question on analogue of the classical polynomial perfect splines in the space of $L_q$, $1 \leq q < \infty$, which together with some results of Section 3 gives the exact values of the Gel’fand $n$–width, and the lower estimate of the Kolmogorov $2n$–width of $\tilde{K}^{G,\varphi}_{\infty,\beta}$ in $L_q$ for $G$ satisfying Property B and $1 \leq q < \infty$.

Let $n \in \mathbb{N}$ and let

$$\tilde{\Lambda}^{\varphi,\Theta}_{2n} := \{ \xi \in \tilde{\Lambda}_{2n} : \varphi(K_\beta \ast h_\xi) \perp \Theta \},$$

$$\tilde{K}_\beta^{G,\varphi}(\tilde{\Lambda}^{\varphi,\Theta}_{2n}) := \{ f : f = a + G \ast \varphi(K_\beta \ast h_\xi), a \in \Theta, \xi \in \tilde{\Lambda}^{\varphi,\Theta}_{2n} \},$$

(4.1)

where $G$, $\varphi$, $K_\beta$ and $\Theta$ are the same as those in (1.12). We will regard $\varphi(K_\beta \ast u)$ as $u$ if $\beta = 0$. 

Lemma 4.1. For $n \in \mathbb{N}$, we have
\[
\inf_{f \in \mathcal{K}^G_n(\mathcal{L}^G)} \|f\|_\infty = \|G \ast \varphi(K_\beta * h_n)\|_\infty = \|\Phi_{n,\beta}^G\|_\infty. \tag{4.2}
\]

Proof. Suppose that there exists an $a \in \Theta$ and a $\xi \in \mathcal{L}^G$ for which
\[
\|a + G \ast \varphi(K_\beta * h_\xi)\|_\infty < \|G \ast \varphi(K_\beta * h_n)\|_\infty.
\]
By virtue of (2.5), we have
\[
2n \leq S_c((G \ast \varphi(K_\beta * h_n))(\cdot + \alpha) \pm a \pm (G \ast \varphi(K_\beta * h_\xi))(\cdot) \tag{4.3}
\]
for every $\alpha \in \mathbb{R}$. It follows from the equality $\varphi(K_\beta * h_n)(x + \pi/n) = -\varphi(K_\beta * h_n)(x)$ that $\varphi(K_\beta * h_n)$ is $2\pi/n$–periodic and $\varphi(K_\beta * h_n) \perp 1$. From the fact that $G$ satisfies Property B or is a NCVD kernel, $a \in \Theta$, $\varphi$ is an odd, continuous and strictly increasing function, $K_\beta$ is a NCVD kernel and $\xi \in \mathcal{L}^G$, it follows that
\[
S_c((G \ast \varphi(K_\beta * h_n))(\cdot + \alpha) \pm a \pm (G \ast \varphi(K_\beta * h_\xi))(\cdot)) \leq S_c(h_n(\cdot + \alpha) \pm h_\xi(\cdot)). \tag{4.4}
\]
Using [16] Chapt. V, Lemma 4.1, we obtain the existence of an $a \in \mathbb{R}$ and $\epsilon = 1$ or $-1$ for which
\[
S_c(h_n(\cdot + \alpha) - \epsilon h_\xi(\cdot)) \leq 2(n - 1).
\]
So from (4.3) and (4.4), we have
\[
2n \leq S_c((G \ast \varphi(K_\beta * h_n))(\cdot + \alpha) - \epsilon a - \epsilon(G \ast \varphi(K_\beta * h_\xi))(\cdot)) \leq 2(n - 1)
\]
for some $\alpha \in \mathbb{R}$ and $\epsilon = 1$ or $-1$. This is a contradiction. Lemma 4.1 is proved.

\textbf{Theorem 4.2.} Let $n \in \mathbb{N}$ and $\varphi$ be such that $K_{\infty,\beta}^G$ is convex. Then
\[
d_{2n}(K_{\infty,\beta}^G, L_\infty) = \lambda_{2n}(K_{\infty,\beta}^G, L_\infty) = d_{2n}(K_{\infty,\beta}^G, L_\infty) = i_{2n}(K_{\infty,\beta}^G, L_\infty) \tag{4.5}
\]
\[
= d_{2n-1}(K_{\infty,\beta}^G, L_\infty) = \lambda_{2n-1}(K_{\infty,\beta}^G, L_\infty)
\]
\[
= d_{2n-1}(K_{\infty,\beta}^G, L_\infty) = i_{2n-1}(K_{\infty,\beta}^G, L_\infty) = \|\Phi_{n,\beta}^G\|_\infty.
\]

Proof. First, we will prove the lower bound for the Kolmogorov $2n$–width. Set
\[
S^{2n} := \left\{x = (x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1} : \sum_{k=1}^{2n+1} |x_k| = 2\pi\right\}, \tag{4.6}
\]
\[
\tau_0(x) := 0, \quad \tau_j(x) := \sum_{k=1}^{j} |x_k|, \quad j = 1, \ldots, 2n + 1.
\]
For each $x \in S^{2n}$, put

$$g_x(y) := \text{sgn} x_j, \quad \tau_{j-1}(x) \leq y < \tau_j(x), \quad j = 1, \ldots, 2n + 1,$$

$$f_x(y) := (G * \varphi(K_\beta * g_x))(y).$$

Let $X_{2n}$ be any $2n$-dimensional subspace of $L_q$, $1 < q < \infty$. Suppose that $X_{2n} = \text{span}\{f_1, \ldots, f_{2n}\}$ and let

$$f_x^0 := \sum_{j=1}^{2n} a_j(x)f_j$$

be the unique best approximation element to $f_x$ from the subspace $X_{2n}$. If $\Theta$ is not contained in $X_{2n}$ (this happens only when $G$ satisfies Property $B$), then $E(\tilde{K}_{\infty,\beta}^{G,\varphi}, X_{2n}) = \infty$. We shall now assume that $\Theta \subset X_{2n}$. Then $\dim \Theta = 1$ if $G$ satisfies Property $B$ or $\dim \Theta = 0$ if $G$ is a NCVD kernel. Assume that $f_1$ is a basis of $\Theta$ when $G$ satisfies Property $B$. The mapping

$$A_1(x) := \begin{cases} 
(b(x), a_2(x), \ldots, a_{2n}(x)), & \text{if } G \text{ satisfies Property } B, \\
(a_1(x), a_2(x), \ldots, a_{2n}(x)), & \text{if } G \text{ is a NCVD kernel,}
\end{cases}$$

where

$$b(x) := \int_0^{2\pi} \varphi((K_\beta * g_x)(t)) \, dt,$$

is an odd and continuous map of $S^{2n}$ into $\mathbb{R}^{2n}$. Hence, by Borsuk's theorem (see [4], p. 91) there exists an $x^*_1 \in S^{2n}$ for which $A_1(x^*_1) = 0$. Then $g_{x^*_1} \in \{h_\xi : \xi \in \overline{\Lambda}_{2n}^{\Theta} \}$ and $a_i(x^*_1) = 0$, $i = 2, \ldots, 2n$ if $G$ satisfies Property $B$ or $i = 1, \ldots, 2n$ if $G$ is a NCVD kernel. As $\|g_{x^*_1}\|_\infty \leq 1$, $f_{x^*_1} \in \tilde{K}_{\infty,\beta}^{G,\varphi}$.

Therefore, we have

$$E(\tilde{K}_{\infty,\beta}^{G,\varphi}, X_{2n}) \geq \inf_{g \in X_{2n}} \|f_{x^*_1} - g\|_q \geq \|f_{x^*_1} - f_{x^*_1}^0\|_q$$

$$\geq \inf_{f \in \tilde{K}_{\beta}^{G,\varphi}(\overline{\Lambda}_{2n}^{\Theta})} \|f\|_q, \quad 1 < q < \infty. \quad (4.8)$$

Consequently,

$$d_{2n}(\tilde{K}_{\infty,\beta}^{G,\varphi}, L_q) \geq \inf_{f \in \tilde{K}_{\beta}^{G,\varphi}(\overline{\Lambda}_{2n}^{\Theta})} \|f\|_q, \quad 1 < q < \infty.$$

Passing to the limit $q \to \infty$ and using Lemma 4.1, we obtain

$$d_{2n}(\tilde{K}_{\infty,\beta}^{G,\varphi}, L_\infty) \geq \inf_{f \in \tilde{K}_{\beta}^{G,\varphi}(\overline{\Lambda}_{2n}^{\Theta})} \|f\|_\infty = \|\Phi_{n,\beta}^{G,\varphi}\|_\infty.$$

Now let us find a lower bound for the Gel'fand $2n$-width. Suppose that

$$X_{2n}^2 := \{f \in L_\infty : \langle l_j, f \rangle = 0, l_j \in L_\infty^*, j = 1, \ldots, 2n\}.$$

If $\langle l_j, 1 \rangle = 0, j = 1, \ldots, 2n$, then $a \in \tilde{K}_{\infty,\beta}^{G,\varphi} \cap X_{2n}^2$ for all $a \in \mathbb{R}$ (this happens only when $G$ satisfies Property $B$), which gives

$$\sup_{f \in \tilde{K}_{\infty,\beta}^{G,\varphi} \cap X_{2n}^2} \|f\|_\infty = \infty.$$
If there exists a \( j_0 \in \{1, \ldots, 2n\} \) for which \( \langle l_{j_0}, 1 \rangle \neq 0 \), then without loss of generality we may assume that \( \langle l_1, 1 \rangle \neq 0 \). Set
\[
L_j := l_j - \frac{\langle l_j, 1 \rangle}{\langle l_1, 1 \rangle} l_1, \quad j = 2, \ldots, 2n.
\]
For each \( x \in S^{2n} \), denote by \( A_2 \) the mapping
\[
A_2(x) := \begin{cases}
(b(x), (L_2, f_x), \ldots, (L_{2n}, f_x)), & \text{if } G \text{ satisfies Property } B, \\
(\langle l_1, f_x \rangle, \langle l_2, f_x \rangle, \ldots, \langle l_{2n}, f_x \rangle), & \text{if } G \text{ is a NCVD kernel}.
\end{cases}
\]
Since \( A_2 \) is an odd and continuous map of \( S^{2n} \) into \( \mathbb{R}^{2n} \), by Borsuk’s theorem there exists an \( x^*_2 \in S^{2n} \) for which \( A_2(x^*_2) = 0 \). Then \( g_{x^*_2} \in \{ h_\xi : \xi \in \bar{X}_{2n}^\Theta \} \) and
\[
f^*_2 := f_{x^*_2} - \frac{\langle l_1, f_{x^*_2} \rangle}{\langle l_1, 1 \rangle} \in X^{2n} \cap \bar{K}^{G, \varphi}_{\infty, \beta}.
\]
Consequently,
\[
\sup_{f \in \bar{K}^{G, \varphi}_{\infty, \beta} \cap X^{2n}} \| f \|_\infty \geq \| f^*_2 \|_\infty \geq \inf_{f \in \bar{K}^{G, \varphi}_{\infty, \beta} (\bar{x}_{2n}^\Theta)} \| f \|_\infty = \| \Phi^{G, \varphi}_{n, \beta} \|_\infty.
\]
Therefore,
\[
d_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \geq \| \Phi^{G, \varphi}_{n, \beta} \|_\infty.
\] (4.9)

It follows from \[13\] Lemma 1] and the monotonicity of \( n \)-widths that
\[
d_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq \lambda_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq \lambda_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty),
\]
\[
d_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq d_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq \lambda_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty),
\]
\[
d_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq i_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq \lambda_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq \lambda_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty),
\]
\[
d_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq d_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq i_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \leq \lambda_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty).
\]
So it remains to find a suitable upper bound for \( \lambda_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) \). Such an estimate follows from \[35\] Theorem 1.2 is proved.

**Corollary 4.3.** Let \( n, r \in \mathbb{N} \). Then
\[
d_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) = \lambda_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) = d_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) = i_{2n}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty)
\]
\[
= d_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) = \lambda_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) = d_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty) = i_{2n-1}(\bar{K}^{G, \varphi}_{\infty, \beta}, L_\infty)
\]
\[
= \begin{cases}
\| D_r * h_n \|_\infty, & \bar{K}^{G, \varphi}_{\infty, \beta} = \bar{W}^r_{\infty}, \\
\| D_r * (K_\beta * h_n) \|_\infty, & \bar{K}^{G, \varphi}_{\infty, \beta} = \bar{H}^r_{\infty, \beta}, \\
\| D_r * \varphi_0 (K_\beta * h_n) \|_\infty, & \bar{K}^{G, \varphi}_{\infty, \beta} = \bar{H}^r_{\infty, \beta}.
\end{cases}
\] (4.10)
Now we proceed to solve a minimum norm question on analogue of the polynomial perfect splines in the space of $L_q$, $1 \leq q < \infty$. Following the method of [8, 21, 2], we get the following result which will be used in the lower estimates of the $n$-widths in Theorem 4.5 and 4.7.

**Lemma 4.4.** Let $n, r \in \mathbb{N}$ and $G$ satisfy Property B. Then for $1 \leq q < \infty$,

\[
\inf_{f \in \mathcal{K}^G_{\beta}(\mathbb{R}^{2n})} \|f\|_q = \min_{m \in \mathbb{N}, m \leq n} \|G_{m, \beta}^{G, \varphi}\|_q = \|G_{n, \beta}^{G, \varphi}\|_q
\]

\[
= \begin{cases} 
\|D_r \ast h_n\|_q, & G = D_r, \beta = 0, \\
\|D_r \ast (K_\beta \ast h_n)\|_q, & G = D_r, \varphi = \varphi_1, \beta > 0, \\
\|D_r \ast \varphi_0(K_\beta \ast h_n)\|_q, & G = D_r, \varphi = \varphi_0, \beta > 0.
\end{cases}
\]

(4.11)

**Proof.** For $1 \leq q < \infty$, we follow the approach of Zensybaev [21], Micchelli and Pinkus [8], and Pinkus [15].

A compactness argument shows that the infimum in (4.11) is attained, i.e., there exists an $a^* \in \mathbb{R}$ and a $\xi^* \in \mathbb{R}^{2n}$, $\xi^* = (\xi^*_1, \ldots, \xi^*_m)$, $m \leq n$, such that

\[
\inf_{a \in \mathbb{R}, \xi \in \mathbb{R}^{2n}} \|a + G \ast \varphi(K_\beta \ast h_\xi)\|_q^2 = \|a^* + G \ast \varphi(K_\beta \ast h_{\xi^*})\|_q^2.
\]

Using the method of Lagrange’s multiplier, we find that the optimal $a^*$ and $\xi^* = (\xi^*_1, \ldots, \xi^*_m)$ must satisfy the following system of nonlinear equations:

\[
\int_0^{2\pi} f(x) \, dx = 0,
\]

(4.12)

\[
\int_0^{2\pi} f(x) \left[ \int_0^{2\pi} G(t) \varphi'((K_\beta \ast h_\xi)(x-t)) K_\beta(x-t-\xi_j^*) \, dt \right] dx + \frac{\varphi(-1)^j}{\pi} \int_0^{2\pi} \varphi'(\xi_j^*) K_\beta(t-\xi_j^*) \, dt = 0, \quad j = 1, \ldots, 2m,
\]

(4.13)

where $\theta$ is the Lagrange’s multiplier, and

\[
f(x) := q[a^* + G \ast \varphi((K_\beta \ast h_\xi^*)(x))]^{q-1} \text{sgn}[(a^* + G \ast \varphi((K_\beta \ast h_\xi^*)(x))].
\]

(4.14)

Since $\varphi$ is a continuous odd and strictly increasing function, $G$ satisfies Property B and $K_\beta$ is NCVD, we get

\[
S_c(f) = S_c(a^* + G \ast \varphi(K_\beta \ast h_\xi^*)) \leq S_c(\varphi(K_\beta \ast h_\xi^*)) \leq S_c(K_\beta \ast h_\xi^*) \leq S_c(h_\xi^*) \leq 2n.
\]

(4.15)
Therefore, we have

\[\forall a \in S, b \in \mathbb{R}, \quad \text{sgn}(a + b) = \text{sgn}(|a|^{q-1}\text{sgn} a + |b|^{q-1}\text{sgn} b)\]

for every \(a, b \in \mathbb{R}\) and \(q \in (1, \infty)\), it follows that \(S_c(p + r) \leq 2(m - 1)\). Since

\[S_c((K_\beta * h_{\xi^*})(\cdot) + (K_\beta * h_{\xi^*})(\cdot + \delta)) \leq 2(m - 1)\]

for each \(q \in [1, \infty)\). From (4.16),

\[f(x) = q|a^* + G * \varphi((K_\beta * h_{\xi^*})(x))|^q \text{sgn}[a^* + G * \varphi((K_\beta * h_{\xi^*})(x))]\]

and

\[= q|p(x)|^{q-1}\text{sgn}(p(x)).\]

Therefore, we have

\[S_c(f(\cdot) + f(\cdot + \delta)) \leq S_c(p + r) \leq 2(m - 1).\]

Set

\[P(y) = \frac{1}{2\pi} \int_0^{2\pi} f(x)H(x; y) \, dx + \theta \int_0^{2\pi} \varphi'(((K_\beta * h_{\xi^*})(t))K_\beta(t - y) \, dt\]

and

\[R(y) = \frac{1}{2\pi} \int_0^{2\pi} f(x + \delta)H(x; y) \, dx + \theta \int_0^{2\pi} \varphi'((K_\beta * h_{\xi^*})(t + \delta))K_\beta(t - y) \, dt,\]
where
\[ H(x; y) = \int_0^{2\pi} G(t)\varphi'((K_{\beta} * h_{\xi^*})(x - t))K_\beta(x - t - y) \, dt. \]

By change of scale and Fubini’s theorem, we have
\[
P(y) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \left[ \int_0^{2\pi} G(x - t)\varphi'((K_{\beta} * h_{\xi^*})(t))K_\beta(t - y) \, dt \right] dx
+ \theta \int_0^{2\pi} \varphi'((K_{\beta} * h_{\xi^*})(t))K_\beta(t - y) \, dt
\]
\[= \frac{1}{2\pi} \int_0^{2\pi} \varphi'((K_{\beta} * h_{\xi^*})(t))K_\beta(t - y) \left[ \int_0^{2\pi} G(x - t)f(x) \, dx \right] dt
+ \theta \int_0^{2\pi} \varphi'((K_{\beta} * h_{\xi^*})(t))K_\beta(t - y) \, dt
\]
\[= \frac{1}{2\pi} \int_0^{2\pi} \varphi'((K_{\beta} * h_{\xi^*})(t))f_G(t)K_\beta(t - y) \, dt + \theta \int_0^{2\pi} \varphi'((K_{\beta} * h_{\xi^*})(t))K_\beta(t - y) \, dt
\]
\[= \frac{1}{2\pi} \int_0^{2\pi} \varphi'((K_{\beta} * h_{\xi^*})(t))K_\beta(t - y) (f_G(t) + 2\pi\theta) \, dt
\]
\[= \frac{1}{2\pi} \int_0^{2\pi} K_\beta(y - t)\varphi'((K_{\beta} * h_{\xi^*})(t)) (f_G(t) + 2\pi\theta) \, dt,
\]
where
\[ f_G(t) = \int_0^{2\pi} G(x - t)f(x) \, dx. \]

Since \( \varphi' \geq 0 \), \( \varphi' \) is continuous on \([-1, 1] \), \( G \) satisfies Property B and \( K_{\beta} \) is NCVD, by (4.15), we conclude that
\[
\tilde{Z}_c(P(\cdot) + R(\cdot)) \leq S_c(\varphi'((K_{\beta} * h_{\xi^*})(\cdot))(f_G(\cdot) + f_G(\cdot + \delta) + 4\pi\theta))
\]
\[
\leq S_c(f_G(\cdot) + f_G(\cdot + \delta) + 4\pi\theta) \leq S_c(f(\cdot) + f(\cdot + \delta)) \leq 2(m - 1).
\]

A simple change of variable argument shows that \( R(y) = P(y + \delta) \), from which we obtain
\[
\tilde{Z}_c(P(\cdot) + P(\cdot + \delta)) \leq 2(m - 1).
\]

From (4.13), we have
\[ P(\xi_i^*) = 0, \quad i = 1, \ldots, 2m. \]

By our choice of \( \delta \), \( \xi_i^* + \delta \leq \xi_{i+1}^* \), \( i = 1, \ldots, 2m \), and therefore
\[ S_c^+(P(\xi_1^* + \delta), \ldots, P(\xi_{2m}^* + \delta)) = 2m. \]

Thus
\[ S_c^+(P(\xi_1^*) + P(\xi_1^* + \delta), \ldots, P(\xi_{2m}^*) + P(\xi_{2m}^* + \delta)) = 2m, \]
which implies that
\[ \tilde{Z}_c(P(\cdot) + P(\cdot + \delta)) \geq 2m. \]

This is a contradiction, and therefore \( h_{\xi^*}(\cdot) = -h_{\xi^*}(\cdot + \delta) \), i.e., \( h_{\xi^*} = h_m \).
Now we proceed to show that $a^* = 0$. Let

$$f(x; a) := q[a + G \varphi((K_\beta * h_m)(x))]|^{q-1}_{q} \text{sgn}[a + G \varphi((K_\beta * h_m)(x))].$$

Since the constant term is a free variable, $f(:; a^*) \perp 1$. Because $G \varphi((K_\beta * h_m)(x + \pi/2)) = -G \varphi((K_\beta * h_m)(x))$, it follows that $f(:; 0) \perp 1$. Thus $a^* = 0$. Since $h_m \in \{h_\xi : \xi \in \mathcal{C}_2n\}$ for all $m \in \mathbb{N}$ satisfying $m \leq n$, by Theorem 3.5 and (4.15), we get (4.18).

**Theorem 4.5.** Let $n, r \in \mathbb{N}$ and $G$ satisfy Property B. Then for $1 \leq q < \infty$,

$$d^{2n}(\tilde{K}_{\infty,\beta}^{G,\varphi}, L_q) = d^{2n-1}(\tilde{K}_{\infty,\beta}^{G,\varphi}, L_q) = \|\Phi_{n,\beta}^{G,\varphi}\|_q
\begin{cases}
\|D_r * h_n\|_q, & G = D_r, \beta = 0, \\
\|D_r * (K_\beta * h_n)\|_q, & G = D_r, \varphi = \varphi_1, \beta > 0, \\
\|D_r * \varphi_0(K_\beta * h_n)\|_q, & G = D_r, \varphi = \varphi_0, \beta > 0.
\end{cases}
$$

**Proof.** We begin with the lower estimate. Assume that $S^{2n}, g_x$ and $f_x$ are the same as those in the proof of Theorem 4.2 (see (4.6) and (4.7)). Suppose that $1 \leq q < \infty$, and

$$X^{2n} := \{f \in L_q : \langle l_j, f \rangle = 0, l_j \in L_q^*, j = 1, \ldots, 2n\}.$$

If $\langle l_j, 1 \rangle = 0, j = 1, \ldots, 2n$, then

$$\sup_{f \in \tilde{K}_{\infty,\beta}^{G,\varphi} \cap X^{2n}} \|f\|_q = \infty.$$

Therefore, we only need to consider the subspace $X^{2n}$ such that there exists a $j_0 \in \{1, \ldots, 2n\}$ for which $\langle l_{j_0}, 1 \rangle \neq 0$. Then without loss of generality we may assume that $\langle l_1, 1 \rangle \neq 0$. Set

$$L_j := l_j - \frac{\langle l_1, 1 \rangle}{\langle l_1, l_1 \rangle} l_1, \quad j = 2, \ldots, 2n.$$

For each $x \in S^{2n}$, denote by $A_3$ the mapping

$$A_3(x) := (b(x), \langle L_2, f_x \rangle, \ldots, \langle L_{2n}, f_x \rangle),$$

where

$$b(x) := \int_0^{2\pi} \varphi((K_\beta * g_x)(t))dt.$$

Since $A_3$ is an odd and continuous map of $S^{2n}$ into $\mathbb{R}^{2n}$, by Borsuk’s theorem, there exists an $x_3^* \in S^{2n}$ for which $A_3(x_3^*) = 0$. Then $g_{x_3^*} \in \{h_\xi : \xi \in \mathcal{C}_2n\}$, and $\langle L_i, f_{x_3^*} \rangle = 0, i = 2, \ldots, 2n$. Thus

$$f_{x_3^*} := f_{x_3} - \frac{\langle l_1, f_{x_3} \rangle}{\langle l_1, l_1 \rangle} \in X^{2n} \cap \tilde{K}_{\infty,\beta}^{G,\varphi}.$$

Consequently,

$$\sup_{f \in \tilde{K}_{\infty,\beta}^{G,\varphi} \cap X^{2n}} \|f\|_q \geq \|f_{x_3^*}\|_q \geq \inf_{f \in \tilde{K}_{\infty,\beta}^{G,\varphi}(\mathcal{C}_2n)} \|f\|_q.$$
Therefore,
\[ d^{2n-1}(\tilde{K}^{G,\varphi}_{\infty,\beta}, L_q) \geq d^{2n}(\tilde{K}^{G,\varphi}_{\infty,\beta}, L_q) \geq \inf_{f \in \tilde{K}^{G,\varphi}_{\infty,\beta}} \|f\|_q. \] (4.19)

Now we turn to the upper estimate. It follows from Corollary 3.7 that
\[ d^{2n}(\tilde{K}^{G,\varphi}_{\infty,\beta}, L_q) \leq d^{2n-1}(\tilde{K}^{G,\varphi}_{\infty,\beta}, L_q) \leq \|\Phi^{G,\varphi}_{n,\beta}\|_q, \] (4.20)
which together with (4.11), (4.19) and (1.15) gives the proof of Theorem 4.5.

**Remark 4.6.** From [16, Chapt. V, Theorem 4.11], it is known that the exact values of the Gel’fand 2n-width of $\tilde{W}^r_\infty$ in $L_q$, $r \in \mathbb{N}$, $1 \leq q \leq \infty$ is determined, moreover, Sun [18] determined the exact values of the Gel’fand 2n-widths

\[ d^{2n}(\tilde{h}_{\infty,\beta}, L_q) = d^{2n-1}(\tilde{h}_{\infty,\beta}, L_q) = \|\phi^n_q\|_q, \quad 1 \leq q \leq \infty, \quad n = 1, 2, \ldots, \]

where
\[ \phi^n_q(x) = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{\cos(2\nu + 1)nx}{(2\nu + 1)\cosh(2\nu + 1)n\beta}. \] (4.21)

Osipenko [12] obtained the exact values of the Gel’fand n-width of $\tilde{h}^r_{\infty,\beta}$ in $L_q$, $r \in \mathbb{N}$, $1 \leq q \leq \infty$, and we [2] determined the exact values of the Gel’fand n-width of $\tilde{H}^r_{\infty,\beta}$ in $L_q$, $r \in \mathbb{N}$, $1 \leq q < \infty$.

Now we turn to prove the lower estimate of the Kolmogorov 2n–Width of $\tilde{K}^{G,\varphi}_{\infty,\beta}$ in $L_q$ for $G$ satisfying Property B and $1 \leq q < \infty$.

**Theorem 4.7.** Let $n \in \mathbb{N}$ and $G$ satisfy Property B. Then for $1 \leq q < \infty$,
\[ d_{2n}(\tilde{K}^{G,\varphi}_{\infty,\beta}, L_q) \geq \|\Phi^{G,\varphi}_{n,\beta}\|_q. \] (4.22)

**Proof.** We will prove the lower bound for the Kolmogorov 2n–width. Let $n \in \mathbb{N}$ and $1 < q < \infty$. Assume that $S^{2n}$, $g_x$ and $f_x$ are the same as those in the proof of Theorem 4.2 (see [16, 4.4]). Since the class of functions $\tilde{K}^{G,\varphi}_{\infty,\beta}$ contains all constants, in order to establish the lower bound of $d_{2n}(\tilde{K}^{G,\varphi}_{\infty,\beta}, L_q)$ we only have to consider the subspace $X_{2n} \subset L_q$ which also contains the constants. Let $X_{2n}$ be any 2n-dimensional subspace of $L_q$, $1 < q < \infty$, such that $1 \in X_{2n}$. Suppose that $X_{2n} = \text{span}\{f_1, \ldots, f_{2n}\}$ and $f_1(t) = 1$. Let $a_1(x), \ldots, a_{2n}(x)$ be the coefficients of $f_1, \ldots, f_{2n}$, respectively, in the unique best approximation to $f_x$ from $X_{2n}$. The mapping
\[ A_1(x) := (b(x), a_2(x), \ldots, a_{2n}(x)), \]
where
\[ b(x) := \int_0^{2\pi} \varphi((K_\beta * g_x)(t)) \, dt, \]
is an odd and continuous map of $S^{2n}$ into $\mathbb{R}^{2n}$. By Borsuk’s theorem there exists an $x_4^* \in S^{2n}$ for which $A_1(x_4^*) = 0$. Then $g_{x_4^*} = \{h_\xi : \xi \in \Lambda_{2n}^{\varphi,\beta}\}$ and $a_i(x_4^*) = 0$, $i = 2, \ldots, 2n$. Then $f_{x_4^*} - a_1(x_4^*) \in \tilde{K}_\beta(\Lambda_{2n}^{\varphi,\beta}) \subset \tilde{K}^{G,\varphi}_{\infty,\beta}$. Therefore, if $1 < q < \infty$, by Lemma 4.3, we have
\[ d_{2n}(\tilde{K}^{G,\varphi}_{\infty,\beta}, L_q) \geq \sup_{f \in \tilde{K}^{G,\varphi}_{\infty,\beta}} \inf_{g \in X_{2n}} \|f - g\|_q \geq \inf_{f \in \tilde{K}^{G,\varphi}_{\infty,\beta}} \|f\|_q = \|\Phi^{G,\varphi}_{n,\beta}\|_q, \quad 1 < q < \infty. \] (4.23)
By passing to the limit $q \to 1$, we obtain the lower estimate of the Kolmogorov $2n$–width $d_{2n}(\tilde{K}_{G,\varphi}^{G,\varphi}, L_q)$, $1 \leq q < \infty$. Theorem 4.7 is proved.

**Remark 4.8.** By [2] [6] [11], and [16], if $n, r \in \mathbb{N}$, then for $\tilde{K}_{G,\varphi}^{G,\varphi} = \tilde{W}_r$, $\tilde{h}_r$, or $\tilde{H}_r$, $d_{2n}(\tilde{K}_{G,\varphi}^{G,\varphi}, L_q) = \lambda_{2n}(\tilde{K}_{G,\varphi}^{G,\varphi}, L_q) = i_{2n}(\tilde{K}_{G,\varphi}^{G,\varphi}, L_q) = \|\Phi_{G,\varphi}^{G,\varphi}\|_q$, (4.24)

where

$$
\|\Phi_{n,\varphi}^{G,\varphi}\|_q = \begin{cases} 
\|D_r * h_n\|_q, & \tilde{K}_{G,\varphi}^{G,\varphi} = \tilde{W}_r, \\
\|D_r * (K_{\varphi} * h_n)\|_q, & \tilde{K}_{G,\varphi}^{G,\varphi} = \tilde{h}_r, \\
\|D_r * \varphi_0 (K_{\varphi} * h_n)\|_q, & \tilde{K}_{G,\varphi}^{G,\varphi} = \tilde{H}_r,
\end{cases}
$$

and $1 \leq q < \infty$. So we conjecture that

$$
d_{2n}(\tilde{K}_{G,\varphi}^{G,\varphi}, L_q) = \lambda_{2n}(\tilde{K}_{G,\varphi}^{G,\varphi}, L_q) = i_{2n}(\tilde{K}_{G,\varphi}^{G,\varphi}, L_q) = \|\Phi_{n,\varphi}^{G,\varphi}\|_q
$$

holds for $n \in \mathbb{N}$, $G$ satisfying Property B, $\varphi$ being such that $\tilde{K}_{G,\varphi}^{G,\varphi}$ is convex and $1 \leq q < \infty$. Moreover, we think that all the results proved in this paper also hold for $G$ being a ENCVD kernel (see Section 3 of Chapter III in [16]).

**References**

[1] N. I. Achieser, *Theory of Approximation*, New York, 1956.

[2] Fang Gensun and Li Xuehua, *Comparison theorems of Kolmogorov type and exact values of n-widths on Hardy-Sobolev classes*, Math. Comp. 75(2006), 241-258.

[3] Fang Gensun and Li Xuehua, *Optimal Quadrature Problem on Hardy-Sobolev classes*, Journal of Complexity, 21 (2005), 722-739.

[4] S. D. Fisher, *Envelopes, widths, and Landau problems for analytic functions*, Constr. Approx. 5 (1989), 171–187.

[5] W. Forst, *Über die Breite von Klassen holomorpher periodischer funktionen*, J. Approx. Theory, 19 (1977), 325–331.

[6] N. Korneichuk, *Exact Constants in Approximation Theory*, Cambridge University Press, Cambridge, 1991.

[7] G. G. Lorentz, M. V. Golischek, Y. Makovoz, *Constructive Approximation*, Springer–Verlag, New York, 1993.

[8] C. A. Micchelli, A. Pinkus, *Some problems in the approximation of functions of two variables and n–widths of integral operators*, J. Approx. Theory 24 (1978), 51–77.

[9] K. Yu. Osipenko, *On n–widths, optimal quadrature formulas, and optimal recovery of functions analytic in a strip*, Izv. Ross. Akad. Nauk, Ser. Mat. 58 (1994), 55–79; English transl., in Russian Acad. Sci. Izv. Math. 45 (1995), 55–78.
[10] K. Yu. Osipenko, *Inequalities for derivatives of functions analytic in a strip*, Mat. Zametki 56 (1994), 114–122; English transl., in Math. Notes 56 (1994), 1069–1074.

[11] K. Yu. Osipenko, *Exact values of $n$–widths and optimal quadratures on classes of bounded analytic and harmonic functions*, J. Approx. Theory 82 (1995), 156–175.

[12] K. Yu. Osipenko, *Exact $n$–widths of Hardy–Sobolev classes*, Constr. Approx. 13 (1997), 17–27.

[13] K. Yu. Osipenko, *On the precise values of $n$–widths for classes defined by cyclic variation diminishing operators*, Sbornik Math. 188 (1997), 1371–1383.

[14] K. Yu. Osipenko and K. Wilderotter, *Optimal information for approximating periodic analytic functions*, Math. Comput. 66 (1997), 1579–1592.

[15] A. Pinkus, *On $n$–widths of periodic functions*, J. Analyse Math. 35 (1979), 209–235.

[16] A. Pinkus, *$n$–Widths in Approximation Theory*, Springer–Verlag, Berlin, 1985.

[17] S.A. Smolyak, *On the Optimal Recovery of Functions and Functionals of Them*, Kandidat thesis, Moscow State University, Moscow, 1965.

[18] Sun Yongsheng, *A remark on Kolmogorov's comparison theorem*, Chin. Ann. of Math. (Ser. B), 7 (1986), 463–467.

[19] L. V. Taikov, *The approximation in the mean of certain classes of periodic functions*, Trudy Math. Inst. Steklov, 88 (1967), 61–70.

[20] V. M. Tikhomirov, *Diameters of sets in function spaces and the theory of best approximations*, Uspekhi Mat. Nauk, 15 (1960), 81–120; English transl. in Russian Math. Surveys, 15 (1960), 75–111.

[21] A. A. Zensykbaev, *On the best quadrature formulas on the class $W^r L_p$*, Dokl. Akad. Nauk SSSR, 227 (1976), 277–279; English transl. in Soviet Math. Dokl, 17 (1976), 377–380.