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ON SOME NEW DOUBLE SEQUENCE SPACES OF INVARIANT MEANS DEFINED BY ORLICZ FUNCTIONS

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Abstract. The sequence space $BV_\sigma$ was introduced and studied by Musaleen[14]. In this paper we extend $BV_\sigma$ to $2BV_\sigma(p,r,s)$ and study some properties and inclusion relations on this space.

1. Introduction

Let $l_\infty$, and $c$ denote the Banach spaces of bounded and convergent sequences $x = (x_i)$, with complex terms, respectively, normed by $\|x\|_\infty = \sup_i |x_i|$, where $i \in \mathbb{N}$. Let $\sigma$ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits that is to say, if and only if, for all $i = 0$, $j = 0$, $\sigma(j) \neq i$ and $T$ be the operator defined on $l_\infty$ by $(T(x_i))_{i=1}^\infty = (x_{\sigma(i)})_{i=1}^\infty$.

A continuous linear functional $\phi$ on $l_\infty$ is said to be an invariant mean or $\sigma$-mean if and only if

1. $\phi(x) \geq 0$, when the sequence $x = (x_i)$ has $x_i \geq 0$ for all $i$,
2. $\phi(e) = 1$, where $e = \{1,1,1,\ldots\}$ and
3. $\phi(x_{\sigma(i)}) = \phi(x)$ for all $x \in l_\infty$.

If $x = (x_i)$ write $T_x = (Tx_i) = (x_{\sigma(i)})$. It can be shown that

$$V_\sigma = \left\{ x = (x_i) : \sum_{m=1}^\infty t_{m,i}(x) = L \text{ uniformly in } i, L = \sigma - \lim x \right\} \quad (1)$$

where $m \geq a, i > 0$.
where $\sigma^m(i)$ denote the mth iterate of $\sigma(i)$ at i. In the case $\sigma(i) = i + 1$ is often called a Banach limit and $V_{\sigma}$, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequence. Subsequently invariant means have been studied by Ahmad and Mursaleen[1], Mursaleen[12,13], Raimi[15] and many others.

The concept of paranorm is closely related to linear metric spaces. It is generalization of that of absolute value. Let $X$ be a linear space. A Paranorm is a function $g : X \to \mathbb{R}$ which satisfies the following axioms: for any $x, y, x_0 \in X$, $\lambda, \lambda_0 \in \mathbb{C}$,

(i) $g(\theta) = 0$;
(ii) $g(x) = g(-x)$;
(iii) $g(x + y) \leq g(x) + g(y)$
(iv) the scalar multiplication is continuous, that is $\lambda \to \lambda_0, x \to x_0$ imply $\lambda x \to \lambda_0 x_0$.

Any function $g$ which satisfies all the condition (i)-(iv) together with the condition

(v) $g(x) = 0$ if only if $x = \theta$,

is called a Total Paranorm on $X$ and the pair $(X, g)$ is called Total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (cf.[18],Theorm 10.42,p183)

An Orlicz Function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If convexity of $M$ is replaced by $M(x + y) \leq M(x) + M(y)$ then it is called Modulus function.

An Orlicz function $M$ satisfies the $\Delta_2 -$ condition ($M \in \Delta_2$ for short) if there exist constant $k \geq 2$ and $u_0 > 0$ such that

$M(2u) \leq KM(u)$

whenever $|u| \leq u_0$.

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An Orlicz function $M$ can always be represented in the integral form
\[ M(x) = \int_0^x q(t)dt, \]
where $q$ known as the kernel of $M$, is right differentiable for $t \geq 0$, $q(t) > 0$ for $t > 0, q$ is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Note that an Orlicz function satisfies the inequality
\[ M(\lambda x) \leq \lambda M(x) \]
for all $\lambda$ with $0 < \lambda < 1,$ since $M$ is convex and $M(0) = 0.$

W. Orlicz used the idea of Orlicz function to construct the space $(L^M)$. Lindenstrauss and Tzafriri [9] used the idea of Orlicz sequence space;
\[ l_M := \left\{ x \in \mathbb{R}^\infty : \sum_{k=1}^\infty M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\} \]

which is Banach space with the norm the norm
\[ \|x\|_M = \inf\left\{ \rho > 0 : \sum_{k=1}^\infty M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. \]
The space $l_M$ is closely related to the space $l_p$, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty.$

Throughout a double sequence is denoted by $x = (x_{ij}). A$ double sequence is a double infinite array of elements $x_{ij} \in \mathbb{R}$ for all $i, j \in \mathbb{N}.$ Let $2l_\infty$ and $2c$ denote the Banach spaces of bounded and convergent double sequence $x = (x_{ij})$ respectively. Double sequence spaces have been studied by Moricz and Rhoads[11], E.Savas and R.F.Patterson[16], V.A.Khan[4] and many others.

Let $\sigma$ be an injection having no finite orbits and $T$ be the operator defined on $2l_\infty$ by
\[ T((x_{ij})_{i,j=1}^\infty) = (x_{\sigma(i,j)})_{i,j}^\infty. \]
The idea of $\sigma$-convergence for double sequences has recently been introduced in [2] and further studied by Mursaleen and Mohiuddine [12].

For double sequences,
\[ 2V_\sigma = \left\{ x = (x_{i,j}) : \sum_{m=1}^\infty \sum_{n=1}^\infty t_{mn,pq}(x) = L \text{ uniformly in } p, q, L = \sigma - \lim x \right\}, \]
\[ t_{mn,pq}(x) = \frac{1}{(m+1)(n+1)} \sum_{i=1}^\infty \sum_{j=1}^\infty x_{\sigma^i(p), \sigma^j(q)}, \quad p, q = 0, 1, 2, \ldots \]
\[ t_{0,0,p,q}(x) = x_{pq}, t_{-1,0,p,q}(x) = x_{p-1,q}(x), t_{0,-1,p,q}(x) = x_{p,q-1}, \]

and \( x_{\sigma^i(p),\sigma^j(q)} = 0 \) for all \( i \) or \( j \) or both negative.

A double sequence space \( E \) is said to be solid if \( (\alpha_{i,j}x_{i,j}) \in E \), whenever \( (x_{i,j}) \in E \), for all double sequences \( (\alpha_{i,j}) \) of scalars with \( |\alpha_{i,j}| \leq 1 \), for all \( i,j \in \mathbb{N} \).

Let

\[ K = \{(n_i,k_j) : i,j \in \mathbb{N}; n_1 < n_2 < n_3 < ... \text{ and } k_1 < k_2 < k_3 < ... \} \subseteq \mathbb{N} \otimes \mathbb{N} \]

and \( E \) be a double sequence space. A \( K \)-step space of \( E \) is a sequence space

\[ \lambda^E_K = \{(\alpha_{i,j}x_{i,j}) : (x_{i,j}) \in E \}. \]

A canonical pre-image of a sequence \( (x_{n_i,k_j}) \in E \) is a sequence \( (b_{n,k}) \in E \) defined as follows:

\[ b_{nk} = \begin{cases} a_{nk} & \text{if } (n,k) \in K, \\ 0 & \text{otherwise}. \end{cases} \]

A canonical pre-image of step space \( \lambda^E_K \) is a set of canonical pre-images of all elements in \( \lambda^E_K \).

A double sequence space \( E \) is said to be monotone if it contains the canonical pre-images of all its step spaces.

A double sequence space \( E \) is said to be symmetric if \( (x_{i,j}) \in E \) implies \( (x_{\pi(i),\pi(j)}) \in E \), where \( \pi \) is a permutation of \( \mathbb{N} \).

2. Main Results

**Lemma 1** A sequence space \( E \) is solid implies \( E \) is monotone.

Mursaleen[14] defined the sequence space

\[ BV_\sigma = \{ x \in l_\infty : \sum_{m} |\phi_{m,i}(x)| < \infty, \text{ uniformly in } i \}, \]

where \( \phi_{m,i}(x) = t_{m,i}(x) - t_{m-1,i}(x) \)

assuming that \( t_{m,i}(x) = 0 \) for \( m = -1 \).

A straightforward calculation shows that
\[
\phi_{m,n}(x) = \begin{cases} 
\frac{1}{m(m+1)} \sum_{n=1}^{m} n[x^n_{\sigma}(i) - x^{n-1}_{\sigma}(i)] & (m \geq 1) \\
x_i & (m = 0).
\end{cases}
\] (6)

We define
\[
2BV_{\sigma} = \{x \in 2l_{\infty} : \sum_{m,n} |\phi_{mnpq}(x)| < \infty, \text{ uniformly in } p \text{ and } q\},
\] (7)

where
\[
\phi_{mnpq}(x) = \begin{cases} 
\frac{1}{m(m+1)n(n+1)} \sum_{i,j=1}^{m,n} ij [x_{\sigma^{(p)},\sigma^{(q)}}(i) - x_{\sigma^{(p)}(i),\sigma^{(q)}}(j) - x_{\sigma^{(p)}(i),\sigma^{(q)}}(j) + x_{\sigma^{(p)}(i),\sigma^{(q)}}(j)] & (m,n \geq 1) \\
x_{ij} & m \text{ or } n \text{ or both zero}.
\end{cases}
\] (8)

Let \(M\) be an Orlicz function, \(p = (p_i)\) be any sequence of strictly positive real numbers and \(r \geq 0\). V.A.Khan[5] defined the following sequence space:
\[
BV_{\sigma}(M,p,r) = \{x = (x_i) : \sum_{m=1}^{\infty} \frac{1}{m^r} \left[M\left(\frac{\phi_{m,n}(x)}{\rho}\right)\right]^{p_i} < \infty, \\
\text{uniformly in } i \text{ and for some } \rho > 0\}.
\]

Let \(p = (p_{ij})\) be any double sequence of strictly positive real numbers and \(r, s \geq 0\). We define the following double sequence spaces as:
\[
2BV_{\sigma}(M,p,r,s) = \{x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[M\left(\frac{\phi_{mnpq}(x)}{\rho}\right)\right]^{p_{ij}} < \infty, \\
\text{uniformly in } p, q \text{ and for some } \rho > 0\}.
\]

For \(M(x) = x\), we get
\[
2BV_{\sigma}(p,r,s) = \{x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} |\phi_{mnpq}(x)|^{p_{ij}} < \infty, \text{ uniformly in } p, q\}.
\]

For \(p_{ij} = 1\) for all \(i,j\) we get
\[
2BV_{\sigma}(M,r,s) = \{x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[M\left(\frac{\phi_{mnpq}(x)}{\rho}\right)\right] < \infty, \\
\text{uniformly in } p, q \text{ and for some } \rho > 0\}.
\]
For $r, s = 0$, we get
\[ 2BV_\sigma(M, p) = \left\{ x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M \left( \frac{|\phi_{mnpq}(x)|}{\rho} \right)^{p_{ij}} < \infty, \right\} \]
uniformly in $p, q$ and for some $\rho > 0$.

For $M(x) = x$ and $r, s = 0$, we get
\[ 2BV_\sigma(p) = \left\{ x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\phi_{mnpq}(x)|^{p_{ij}} < \infty, \text{ uniformly in } p, q \right\}. \]

For $p_{i,j} = 1$ for all $i, j$ and $r, s = 0$, we get
\[ 2BV_\sigma(M) = \left\{ x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M \left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) < \infty, \text{ uniformly in } p, q \right\}, \]
and for some $\rho > 0$.

For $M(x) = x, p_{i,j} = 1$ and $r, s = 0$, we get
\[ 2BV_\sigma = \left\{ x = (x_{ij}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\phi_{mnpq}(x)| < \infty, \text{ uniformly in } p, q \right\}. \]

**Theorem 1** The sequence space $2BV_\sigma(M, p, r, s)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

**Proof** Let $x = (x_{i,j})$ and $y = (y_{i,j}) \in 2BV_\sigma(M, p, r, s)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers $\rho_1$ and $\rho_2$ such that
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^n n^s} M \left( \frac{|\phi_{mnpq}(x)|}{\rho_1} \right)^{p_{ij}} < \infty \]
and
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^n n^s} M \left( \frac{|\phi_{mnpq}(y)|}{\rho_2} \right)^{p_{ij}} < \infty \]
uniformly in $p$ and $q$ and $r, s \geq 0$.

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M$ is non-decreasing and convex we have,
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^n n^s} M \left( \frac{|\alpha\phi_{mnpq}(x) + \beta\phi_{mnpq}(y)|}{\rho_3} \right)^{p_{ij}} < \infty \]
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^n n^s} M \left( \frac{|\alpha\phi_{mnpq}(x)|}{\rho_3} + \frac{|\beta\phi_{mnpq}(y)|}{\rho_3} \right)^{p_{ij}} < \infty \]
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\begin{align*}
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[ M \left( \frac{\phi_{mnpq}(x)}{\rho_1} \right) + M \left( \frac{\phi_{mnpq}(y)}{\rho_2} \right) \right] < \infty
\end{align*}

uniformly in \( p \) and \( q \) and \( r, s \geq 0 \).

This proves that \( 2BV_\sigma(M, p, r, s) \) is a linear space over the field \( \mathbb{C} \) of complex numbers.

**Theorem 2** For any Orlicz function \( M \) and a bounded sequence \( p = (p_{i,j}) \) of strictly positive real numbers, \( 2BV_\sigma(M, p, r, s) \) is a paranormed space with paranorm

\[
g((x_{ij})) = \sup_i |x_{i,1}| + \sup_j |x_{1,j}| + \inf \left\{ \rho^{\frac{p_{i,j}}{P}} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M \left( \frac{\phi_{mnpq}(x)}{\rho} \right) \right)^{p_{i,j}} \right\} 
\]

where \( H = \max(1, \sup_{i,j} p_{i,j}) \).

**Proof** Clearly \( g(0) = 0, g(-x_{ij}) = g((x_{i,j})) \).

Using Theorem[1], for \( \alpha = \beta = 1 \), we get

\[
g(x + y) \leq g(x) + g(y).
\]

For continuity of scalar multiplication let \( \eta \neq 0 \) be any complex number. Then by definition we have

\[
g(\eta(x_{ij})) = \sup_i |\eta x_{i,1}| + \sup_j |\eta x_{1,j}| + \inf \left\{ \rho^{\frac{p_{i,j}}{P}} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M \left( \frac{\phi_{mnpq}(\eta x)}{\rho} \right) \right)^{p_{i,j}} \right\} 
\]

\[
= \sup_i |\eta||x_{i,1}| + \sup_j |\eta||x_{1,j}| + \inf \left\{ \rho^{\frac{p_{i,j}}{P}} \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M \left( \frac{\phi_{mnpq}(x)}{r} \right) \right)^{p_{i,j}} \right\} 
\]

uniformly in \( p \) and \( q \).
where \( \frac{1}{r} = \frac{|\eta|}{p} = \max(1, |\eta|^H g((x_{ij})) \) and therefore \( g(\eta(x_{ij})) \) converges to zero when \( g((x_{ij})) \) converges to zero in \( 2BV_\sigma(M, p, r, s) \).

Now let \( x \) be a fixed element in \( 2BV_\sigma(M, p, r, s) \). There exist \( \rho > 0 \) such that

\[
g((x_{ij})) = \sup_i |x_{i,1}| + \sup_j |x_{1,j}| + \inf \left\{ \rho^{\frac{1}{p}} : \left( \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{m^n} \left[ M\left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) \right]^{p_{ij}} \right)^{\frac{1}{p}} \leq 1 \right\}
\]

uniformly in \( p \) and \( q \).

Now \( g(\eta(x_{ij})) = \sup_i |\eta x_{i,1}| + \sup_j |\eta x_{1,j}| + \inf \left\{ \rho^{\frac{1}{p}} : \left( \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{m^n} \left[ M\left( \frac{|\phi_{mnpq}(\eta x)|}{\rho} \right) \right]^{p_{ij}} \right)^{\frac{1}{p}} \leq 1 \right\} \rightarrow 0 \) as \( \eta \rightarrow 0 \).

This completes the proof.

**Theorem 3** Suppose that \( 0 < p_{ij} \leq q_{ij} < \infty \) for each \( m \in \mathbb{N} \) and \( r, s \geq 0 \). Then

(i) \( 2BV_\sigma(M, p) \subseteq 2BV_\sigma(M, q) \).

(ii) \( 2BV_\sigma(M) \subseteq 2BV_\sigma(M, r, s) \).

**Proof**

(i) Suppose \( x \in 2BV_\sigma(M, p) \). This implies that

\[
\left[ M\left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) \right]^{p_{ij}} \leq 1
\]

for sufficiently large values \( m, n \) say \( m \geq m_0, \ n \geq n_0 \) for some fixed \( m_0, \ n_0 \in \mathbb{N} \). Since \( M \) is non decreasing, we have

\[
\sum_{m=m_0}^\infty \sum_{n=n_0}^\infty \left[ M\left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) \right]^{q_{ij}} \leq \sum_{m=m_0}^\infty \sum_{n=n_0}^\infty \left[ M\left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) \right]^{p_{ij}} < \infty,
\]

uniformly in \( p, q \). Hence \( x \in 2BV_\sigma(M, q) \).

The second proof is trivial.

The following result is a consequence of the above result.
Corollary 1 If $0 \leq p_{ij} \leq 1$ for each $i$ and $j$, then $2BV_{\sigma}(M, p) \subseteq 2BV_{\sigma}(M)$. If $0 \leq p_{ij} \leq 1$ for all $i, j$ then $2BV_{\sigma}(M) \subseteq 2BV_{\sigma}(M, p)$.

Theorem 4 The sequence space $2BV_{\sigma}(M, p, r, s)$ is solid.

Proof Let $x \in 2BV_{\sigma}(M, p, r, s)$.
This implies $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[ M \left( \frac{\phi_{mnpq}(x)}{\rho} \right) \right]^{p_{ij}} < \infty$.

Let $(\alpha_{ij})$ be sequence of scalars such that $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[ M \left( \frac{|\alpha_{ij}\phi_{mnpq}(x)|}{\rho} \right) \right]^{p_{ij}} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[ M \left( \frac{|\alpha_{ij}\phi_{mnpq}(x)|}{\rho} \right) \right]^{p_{ij}} < \infty.$$  

Hence $\alpha x \in 2BV_{\sigma}(M, p, r, s)$, for all sequences of scalars $(\alpha_{ij})$ with $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$ whenever $x \in 2BV_{\sigma}(M, p, r, s)$.

From Theorem 4 and Lemma we have:

Corollary 2 The sequence space $2BV_{\sigma}(M, p, r, s)$ is monotone.

Theorem 5 Let $M_1, M_2$ be Orlicz functions satisfying $\Delta_2$-condition and $r, r_1, r_2, s, s_1, s_2 \geq 0$. Then we have

(i) if $r, s > 1$ then $2BV_{\sigma}(M, p, r, s) \subseteq 2BV_{\sigma}(M \circ M_1, p, r, s)$,
(ii) $2BV_{\sigma}(M_1, p, r, s) \cap 2BV_{\sigma}(M_2, p, r) \subseteq 2BV_{\sigma}(M_1 + M_2, p, r, s)$,
(iii) if $r_1 \leq r_2$ and $s_1 \leq s_2$ then $2BV_{\sigma}(M, p, r_1, s_1) \subseteq 2BV_{\sigma}(M, p, r_2, s_2)$.

Proof (i) Since $M$ is continuous at 0 from right, for $\epsilon > 0$, there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \epsilon$. If we define

$$I_1 = \left\{ m \in \mathbb{N} : M_1 \left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) \leq \delta \text{ for some } \rho > 0 \right\}.$$

$$I_2 = \left\{ m \in \mathbb{N} : M_1 \left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) > \delta \text{ for some } \rho > 0 \right\}.$$

then, when $M_1 \left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) > \delta$ we get

$$M \left( M_1 \left( \frac{|\phi_{mnpq}(x)|}{\rho} \right) \right) \leq \left\{ \frac{M \left( 1 \right)}{\delta} \right\} M_1 \left( \frac{|\phi_{mnpq}(x)|}{\rho} \right).$$
Hence for $x \in 2BV_\sigma(M, p, r, s)$ and $r, s > 1$

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} \left[ M \circ M_1 \left( \frac{\phi_{mn pq}(x)}{\rho} \right) \right]^{p_{ij}} \\
= \sum_{m \in I_1} \sum_{n \in I_1} \frac{1}{m^r n^s} \left[ M \circ M_1 \left( \frac{\phi_{mn pq}(x)}{\rho} \right) \right]^{p_{ij}} \\
+ \sum_{m \in I_2} \sum_{n \in I_2} \frac{1}{m^r n^s} \left[ M \circ M_1 \left( \frac{\phi_{mn pq}(x)}{\rho} \right) \right]^{p_{ij}} \\
\leq \sum_{m \in I_1} \sum_{n \in I_1} \frac{1}{m^r n^s} \left[ \epsilon^{p_{ij}} \right] + \sum_{m \in I_2} \sum_{n \in I_2} \frac{1}{m^r n^s} \left[ \left( \frac{2}{\delta} \right) M_1 \left( \frac{\phi_{mn pq}(x)}{\rho} \right) \right]^{p_{ij}} \\
\leq \max(h, M) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^r n^s} + \max \left( \left\{ \frac{1}{\delta}, h \right\} \left\{ \frac{1}{\epsilon}, \frac{2}{\delta} \right\} \right)
$$

(where $0 < h = \inf p_{ij} \leq p_{ij} \leq H = \sup p_{ij} < \infty$.)

(ii) The proof follows from the following inequality

$$
\frac{1}{m^r n^s} \left( M_1 + M_2 \right) \left( \frac{\phi_{mn pq}(x)}{\rho} \right)^{p_{ij}} \leq \frac{C}{m^r n^s} \left[ M_1 \left( \frac{\phi_{mn pq}(x)}{\rho} \right) \right]^{p_{ij}} \\
+ \frac{C}{m^r n^s} \left[ M_2 \left( \frac{\phi_{mn pq}(x)}{\rho} \right) \right]^{p_{ij}}
$$

(iii) The proof is trivial.

**Corollary 3** Let $M$ be an Orlicz function satisfying $\Delta_2$-condition. Then we have.

(i) if $r, s > 1$ then $2BV_\sigma(p, r, s) \subseteq 2BV_\sigma(M, p, r, s),$

(ii) $2BV_\sigma(M, p) \subseteq 2BV_\sigma(M, p, r, s),$

(iii) $2BV_\sigma(M) \subseteq 2BV_\sigma(M, r, s).$

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**ÖZET:** $BV_\sigma$ dizisi uzayı, Mursaleen tarafından tanımlanmış ve incelenmiştir. Bu makalede ise $BV_\sigma$ uzayı, $2BV_\sigma(p, r, s)$ uzayına genişletilmiş ve bazı özellikleri ile içeme bağıntıları incelenmiştir.
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