A note on non-vanishing divergence of the stress-energy tensor in theories of gravity

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Abstract

In this paper we investigate three theories characterised by non-vanishing divergence of the stress-energy tensor, namely \( f(R, \mathcal{L}_m) \), \( f(R, T) \), and Rastall theory. We show that it is not possible to obtain the third from the first two, unless in some very specific case. Nonetheless, we show that in the framework of cosmology in the \( f(R, T) \) theory, a result similar to that found in the Rastall one is reproduced, namely that the dynamics of the \( \Lambda \)CDM model of standard cosmology can be exactly mimicked, even though the dark energy component is able to cluster.

1 Introduction

In the general relativity (GR) theory the conservation of the stress-energy tensor (SET), \( T^\mu\nu \), in a curved spacetime is a straightforward generalization of its conservation in Minkowski space through the equivalence principle. This is a kind of minimalist (even if apparently quite natural) approach to implement the behaviour of matter fields in a curved space. In this case, the non-trivial geometry intervenes in the conservation law by means of the covariant derivatives and the metric field. This approach can be easily incorporate in a Lagrangian formalism through the Einstein-Hilbert Lagrangian.

The simplest way of generalizing the conservation law, while still maintaining the validity of the equivalence principle, is to make the divergence of the SET proportional to the gradient of the curvature scalar \( R \), i.e.

\[
T^{\mu\nu}_{\;\;\;\;\;\;\mu} \propto R^\nu ,
\]

with \( \partial^\nu \) denoting the covariant derivative. This proposal was first made by P. Rastall [1]. The field equations compatible with Eq. (1) can be obtained by combining the expressions written originally in Ref. [1] and are given by

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G (T_{\mu\nu} - \frac{\gamma-1}{2}g_{\mu\nu}T) ,
\]

where \( \gamma \) is the free parameter of the theory (setting \( \gamma = 1 \) recovers GR). Using these field equations together with the Bianchi identities, the conservation of the SET can be recast in the following form

\[
T^{\mu\nu}_{\;\;\;\;\;\;\;\nu} = \frac{\gamma-1}{2}T^{\nu\nu} .
\]

Rastall gravity has received some criticism. For example, Visser stated in Ref. [2] that Rastall gravity is nothing more than GR somehow disguised, a claim which has been criticized in Ref. [3]. In some sense, this controversy does not touch the fundamental original approach by Rastall, stated as a modification of the conservation laws due to space-time curvature which is, in our opinion, the more appealing and physically sound aspect of Rastall gravity. It is also not evident how Visser’s argument applies to multi-fields models, since the redefinition in the SET are less evident in this case. We remember, concerning this point, that our universe contains many different components which can only approximately be represented by a single fluid approach.

Another difficulty that Rastall’s proposal encounters is that Eq. (2) was assumed ad-hoc, that is, without any Lagrangian formulation, due to the non-conservative nature of the theory. A Lagrangian formulation for Rastall theory was first proposed in Ref. [4], but with the serious drawback, recognized by the author himself, that the Lagrangian function found was not a scalar density.

More recently, motivated by several different modifications of the Einstein-Hilbert Lagrangian, theories of the type \( f(R, \mathcal{L}_m) \) [5] and \( f(R, T) \) [6], where \( \mathcal{L}_m \) is the usual matter Lagrangian and \( T \) is the trace of the SET, were shown to inherently possess \( T^{\mu\nu}_{\;\;\;\;\;\;\;\mu} \) different from zero. This led some authors [7, 8, 9] to attempt to find a connection between Rastall theory and the aforementioned models. In these references, the authors propose a similar Rastall Lagrangian of the type \( f(R, T) = R + \alpha T \), however, it was not thoroughly investigated how the addition of matter affects the relationship with Rastall’s proposal. Specifically in Ref. [7] it is also claimed that Rastall gravity may be recast using the following prescription \( f(R, \mathcal{L}_m) = \alpha R + \mathcal{G}(\mathcal{L}_m) \), where \( \mathcal{G}(\mathcal{L}_m) \) is a general function of \( \mathcal{L}_m \).

In light of this, this work’s main goal is to analyze if
these models are consistent with Rastall gravity, and if so, under which conditions. In order to achieve this goal, we first analyze, in Sec. 2, the \( f(R, L_m) \) model and show how it cannot recover Rastall gravity; in the subsequent section, Sec. 3, we explicitly calculate how \( f(R, T) \) gravity behaves for several different choices of SETs, and demonstrate that it is possible to obtain a similar structure as that given by Rastall gravity but only for perfect fluids; we fully develop this correspondence in Sec. 4, and show a direct relation with AC
dM model; we conclude this work, discussing our main results, in Sec. 5.

2 Rastall theory as a \( f(R, L_m) \) theory

As an extension of \( f(R) \) models \([10]\), a specific non-minimal coupling of the matter Lagrangian was proposed as a way to investigate non-geodesic motion of massive test particles \([11]\). The complete generalization of this model, now perceived as an \( f(R, L_m) \) theory, possesses the following Lagrangian formulation

\[
L = \frac{1}{16\pi G} \sqrt{-g} f(R, L_m) + \sqrt{-g} L_m .
\]

The correspondent field equations for the metric field are given by

\[
f_R R_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R - \frac{1}{2} [f(R, L_m) - f(L_m)] g_{\mu\nu} = 8\pi G f(L_m) T_{\mu\nu} ,
\]

where we define \( f_R = \frac{\partial f}{\partial R} \), \( f(L_m) = \frac{\partial f}{\partial L_m} \) and adopt the usual definition of the SET as

\[
T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu\nu}} = g_{\mu\nu} L_m - \frac{\delta L_m}{\delta g^{\mu\nu}} .
\]

In Ref. \([7]\) it is claimed that choosing \( f(R, L_m) = \alpha R + G(L_m) \), where \( G(L_m) \) is a general function of the matter Lagrangian, Rastall gravity is recovered for any choice of \( G \). The field equations for this case are

\[
G_{\mu\nu} = \frac{8\pi G}{\alpha} [G' T_{\mu\nu} + (G - G' L_m) g_{\mu\nu}] ,
\]

and using the Bianchi identities, the conservation equations read

\[
T^{\mu\nu} = \frac{(G' L_m - G)_{\mu\nu} - T_{\mu\nu} G'}{G'} ,
\]

where the prime in \( G \) denotes the derivative with respect to \( L_m \).

Through inspection of Eq. (7), one sees that the only way to recover a structure similar to Eq. (2), is to set the function multiplying \( T_{\mu\nu} \) to a constant, that is, \( G' = \text{cst} \). With this choice we are left with \( G = c_1 \times L_m + c_2 \), \( c_{1,2} \) constants, which recovers GR. Non-linear dependences of \( G \) on \( L_m \) lead to strong departures from GR but also from Rastall’s gravity.

We check that a general choice of \( G(L_m) \) cannot recover Rastall gravity. For instance, considering scalar fields such that \( L_m = \phi, \phi^2 \) reduces the above formulation to the k-essence class of theories \([12]\) instead of Rastall’s theory.

3 Rastall theory as a \( f(R, T) \) theory

An approach which further extends GR is to assume that the cosmological constant, responsible for the accelerated expansion of the universe, might be a dynamical term (see Ref. \([13]\) for a review in several different proposals). One possible formulation of this idea is to assume that the cosmological term depends on the trace of the SET, i.e. \( T \). This is now dubbed as \( \Lambda(T) \) gravity \([14]\), and such model is an instance of the general class of \( f(R, T) \) models.

Although this dependence on \( T \) might be motivated due to some exotic fluid, or some quantum effect, there is one general conceptual important aspect for these theories: they use in the Lagrangian a quantity, the trace of the SET, which is defined by a variational principle of the same Lagrangian. This drawback has been addressed by Ref. \([8]\) considering \( T \) as the trace of a general, arbitrary, tensor at first. But in order to recover a structure similar to Rastall gravity, it is necessary to identify this tensor with the energy-momentum tensor and we essentially come back to the \( f(R, T) \) approach. In what follows we ignore this (very important) conceptual aspect and we focus only on its consequences.

The general \( f(R, T) \) Lagrangian is given by

\[
L = \frac{1}{16\pi G} \sqrt{-g} f(R, T) + \sqrt{-g} L_m .
\]

Using the variational principle, and the standard definition of the SET as in Eq. (6), we obtain the field equations

\[
f_R R_{\mu\nu} - \frac{1}{2} f(R, T) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R = 8\pi G T_{\mu\nu} - f_T(R, T) T_{\mu\nu} + \Theta_{\mu\nu} ,
\]

where,

\[
\Theta_{\mu\nu} = g^{\sigma\rho} \delta T_{\mu\rho} / \delta g^{\mu\nu} .
\]

Let us choose

\[
f(R, T) = \alpha R + 8\pi G \beta T .
\]

Hence, the field equations become:

\[
G_{\mu\nu} = 8\pi G \left[ \frac{1 - \beta}{\alpha} T_{\mu\nu} + \frac{\beta}{2\alpha} (g_{\mu\nu} T - 2 \Theta_{\mu\nu}) \right] ,
\]

and the SET conservation:

\[
T^{\mu\nu} = \frac{\beta}{2(\beta - 1)} (T_{\mu\nu} - 2 \Theta_{\mu\nu}) ,
\]

with both \( \alpha \) and \( \beta \) being constants.

Following Ref. \([7]\), Rastall theory is obtained if the term \( \Theta_{\mu\nu} \) is ignored and if

\[
\alpha = 1 - \beta , \quad \beta = \gamma - 1 / \gamma - 2 .
\]

As we show in the following subsections, ignoring the contribution of \( \Theta_{\mu\nu} \) has plenty of theoretical consequences.

Let us consider now the explicit form of \( \Theta_{\mu\nu} \) for some relevant cases. In the subsequent calculations we use the following relations:

\[
\frac{\delta g^{\mu\nu}}{\delta g^{\mu\nu}} = \delta g^{\mu\nu} ,
\]

\[
\frac{\delta g_{\mu\nu}}{\delta g^{\mu\nu}} = - g_{\alpha\nu} g_{\beta\mu} \delta^{\lambda}_\alpha \delta^\beta_\mu = - g_{\alpha\mu} g_{\beta\nu} .
\]
3.1 Electromagnetic case

In Ref. [6] the Lagrangian for the electromagnetism in vacuum, i.e. \( L_m = -F_{\mu\nu}F^{\mu\nu} \), is considered, yielding

\[
\Theta_{\mu\nu} = -T_{\mu\nu} .
\]  

(18)

Using this formula, and the fact that the electromagnetic SET is traceless, the field equation, Eq. (13) becomes

\[
G_{\mu\nu} = \frac{8\pi G}{\alpha} T_{\mu\nu} ,
\]

(19)

and thus, the conservation of the SET is

\[
T^{\mu\nu,\mu} = 0 .
\]

(20)

That is, this is essentially GR with a trivial redefinition of the electromagnetic field. This result had to be expected from the beginning due to the traceless nature of the electromagnetic field.

Therefore, until now we have found no surprising result, since for the same case Rastall gravity also reduces to GR.

3.2 Perfect-fluid case

The SET of a perfect-fluid is

\[
T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu} ,
\]

(21)

with the use of the definition of the tensor \( \Theta_{\mu\nu} \) in Eq. (11), we obtain

\[
\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu} .
\]

(22)

With this expression, the field equations and the conservation of the SET read

\[
G_{\mu\nu} = \frac{8\pi G}{\alpha} \left[ \frac{1 + \beta}{\alpha} T_{\mu\nu} + \frac{\beta}{2\alpha} g_{\mu\nu} (T + 2p) \right] ,
\]

(23)

\[
T^{\mu\nu,\mu} = \frac{\beta}{2(1 + \beta)} (T^{\mu\nu} + 2p^{\mu}) .
\]

(24)

The modified Einstein equations cannot recover GR, nor Rastall theory. Nonetheless, this set of equations will be necessary for our hydrodinamical correspondence in the sections to come.

3.3 Scalar field case

The Lagrangian for a self-interacting scalar field \( \phi \), subject to a generic potential \( V(\phi) \) is

\[
L_m = -\frac{1}{2} \dot{\phi}^2 + V(\phi) ,
\]

(25)

and its corresponding tensor \( \Theta_{\mu\nu} \) is

\[
\Theta_{\mu\nu} = -2T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} T - g_{\mu\nu} V .
\]

(26)

In Ref. [6] a factor 2 is missing multiplying the first term of the right-hand-side, and it is probably a misprint. This factor appears in another approach, through the fluid representation, as we will see later. However, it does not change the main aspect of the analysis for our present purposes.

The field equations and the conservation of the SET, for the present case, are then given by

\[
G_{\mu\nu} = \frac{8\pi G}{\alpha} \left[ (1 + \beta) T_{\mu\nu} - \beta g_{\mu\nu} T + \beta g_{\mu\nu} V \right] ,
\]

(27)

\[
T^{\mu\nu,\mu} = \frac{\beta}{\beta - 1} V^{,\nu} .
\]

(28)

This result can be verified by inserting Eq. (25) into the action, as given in Eq. (9), with the particular choice of Eq. (12).

We rewrite these two equations in terms of the scalar field \( \phi \) in order to better visualize the differences with respect to GR:

\[
G_{\mu\nu} = \frac{8\pi G}{\alpha} \left[ (1 + \beta) \left( \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\lambda} \phi_{,\lambda} \right) + (1 + 2\beta) g_{\mu\nu} V \right] ,
\]

(29)

\[
\Box \phi = -\frac{1 + 2\beta}{1 + \beta} V_{,\phi} ,
\]

(30)

with \( V_{,\phi} \) representing the derivative with respect to \( \phi \). From these equations, one sees that GR can be recovered by a trivial redefinition of the scalar field and the potential as

\[
\sqrt{\frac{1 + \beta}{\alpha}} \phi \rightarrow \phi , \quad \frac{1 + 2\beta}{\alpha} V \rightarrow V .
\]

(31)

One aspect, however, must be remarked: depending on the sign of \((1 + \beta)/\alpha\), an ordinary scalar field can become phantom and an attractive potential can become repulsive. Even so, the general structure is not the same as found in the corresponding case of Rastall gravity.

3.4 Self-interacting scalar field and fluid correspondence

In Ref. [15] it was shown that when we consider a self-interacting scalar field given by

\[
L = \sqrt{-g} L(X, \phi) ,
\]

(32)

where \( L(X, \phi) \) is an arbitrary function of the field \( \phi \) and the kinetic term \( X \), defined as

\[
X = \frac{1}{2} \phi_{,\lambda} \phi^{,\lambda} ,
\]

(33)

we can make a correspondence with thermodynamic quantities. If we assume a canonical scalar field, that is, \( L = X - V(\phi) \), the pressure, density and four-velocity of a fluid are respectively related to the scalar field as

\[
p = L , \quad \rho = 2X L_X - L , \quad u_\mu = \phi_{,\mu}/\sqrt{2X} .
\]

(34) 

(35) 

(36)

With this, we can rewrite both the pressure and the density in the following form:

\[
\rho = X + V , \quad p = X - V .
\]

(37)

From now on we denote the fluid components as “\( f_\)” and the scalar field ones as “\( \phi \)”. Using the relations above, the SET of the fluid and the scalar field are related by

\[
T_{\mu\nu}^{f_\} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu} ,
\]

(38)

\[
= \phi_{,\mu} u_\mu - \frac{1}{2} g_{\mu\nu} \phi_{,\rho} \phi^{,\rho} + g_{\mu\nu} V(\phi) = T_{\mu\nu}^{\phi} .
\]

(39)
Similarly, the function $\Theta_{\mu\nu}$ of the fluid may be recast in terms of the scalar field SET as

$$\Theta_{\mu\nu}' = -2T_{\mu\nu}' - pg_{\mu\nu} = -2T_{\mu\nu} - (X - V)g_{\mu\nu}, \quad (39)$$

$$= -2T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}T_{\phi} - g_{\mu\nu}V. \quad (40)$$

If we use the hydrodynamical representation, the field equations take the form

$$G_{\mu\nu} = 8\pi G \left[ \frac{1 + \alpha}{1 - \alpha} T_{\mu\nu} + \frac{\alpha}{1 - \alpha} g_{\mu\nu} \left( \rho - p \right) \right]. \quad (41)$$

This expression reduces to the standard scalar field case when the correspondance fluid/scalar field described above is applied. In particular, a very simple result is obtained when setting $p = \rho$, which, as it can be seen from the relations (37), represents a free scalar field.

We recognize that there seems to be non-trivial situations in the fluid representation (besides the scalar field one) when $p \neq \rho$. Likewise, we think that this correspondence may be further investigated when the potential is exponential, which coincide to a fluid with $p = \omega \rho$. However, we leave this investigation for a future work.

4 ACDM model

Let us compare the results that we can obtain with the $f(R, T)$ formulation we described so far with those obtained in Ref. [16] for the ACDM model.

In the first part we address the background equations and then we work out the perturbative part separately.

4.1 Background evolution

For simplicity, we set $\alpha = 1 + \beta$ in Eqs. (23). This redefinition is possible since it implies a global factor in the Lagrangian. Given this, we have

$$G_{\mu\nu} = 8\pi G \left[ T_{\mu\nu} + \frac{\beta}{2(1 + \beta)} g_{\mu\nu}(T + 2p) \right], \quad (42)$$

$$T_{\mu\nu}''' = -\frac{\beta}{2(1 + \beta)} \left( T_{\nu}'' + 2p'' \right). \quad (43)$$

Now, let us consider a flat FLRW metric,

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j, \quad (44)$$

and a perfect fluid Lagrangian, Eq. (21), with $\rho$ and $p$ depending only on the time coordinate. The resulting equations are

$$3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi G \left[ \frac{2 + 3\beta}{2(1 + \beta)} \rho - \frac{\beta}{2(1 + \beta)} \rho_p \right], \quad (45)$$

$$\left[ \frac{2 + 3\beta}{2(1 + \beta)} \right] \frac{\dot{\rho}}{a^3} + \frac{\beta}{2(1 + \beta)} \frac{\dot{\rho}}{a^3} + 3H(\rho + p) = 0. \quad (46)$$

We decompose the fluid into two components, a pressureless matter $\rho_m$ ($\dot{\rho}_m = 0$) and a cosmological term $\rho_\Lambda$ ($\dot{\rho}_\Lambda = -\rho_\Lambda$), such that,

$$\rho = \rho_m + \rho_\Lambda, \quad p = p_m + \rho_\Lambda = -\rho_\Lambda. \quad (47)$$

Then, both equations can be rewritten as

$$3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi G \left[ \frac{2 + 3\beta}{2(1 + \beta)} \rho_m + 1 + \frac{2\beta}{1 + \beta} \rho_\Lambda \right], \quad (48)$$

$$\left[ \frac{2 + 3\beta}{2(1 + \beta)} \right] \frac{\dot{\rho}_m}{a^3} + \frac{1 + 2\beta}{1 + \beta} \rho_\Lambda + 3H\rho_m = 0. \quad (49)$$

Since we want the structure formation to be preserved, we choose that the matter component conserves separately. Hence

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (50)$$

and, therefore,

$$\rho_m = \frac{\rho_{m0}}{a^3}, \quad (51)$$

in which we have defined the integration constant $\rho_{m0}$ as the matter density at present time. With this hypothesis, the previous two equations becomes

$$3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi G \left[ \frac{2 + 3\beta}{2(1 + \beta)} \rho_{m0} + 1 + \frac{2\beta}{1 + \beta} \rho_\Lambda \right], \quad (52)$$

$$\left[ \frac{2 + 3\beta}{2(1 + \beta)} \right] \frac{\dot{\rho}_{m0}}{a^3} + \frac{1 + 2\beta}{1 + \beta} \rho_\Lambda = 0. \quad (53)$$

Then, Eq. (53) can be integrated, leading to

$$\rho_\Lambda = -\frac{\beta}{2(1 + 2\beta)} \rho_{m0} + \rho_{\Lambda0}, \quad (54)$$

where $\rho_{\Lambda0}$ as the cosmological constant density at present time. In consequence of this, we finally get to

$$3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi G (\rho_{m0} + \rho_{\Lambda0}), \quad (55)$$

$$\dot{\rho}_{\Lambda0} = 1 + \frac{2\beta}{1 + \beta} \rho_{\Lambda0}. \quad (56)$$

Hence, the ACDM model re-appears. All the background tests are thus equally satisfied, as explained in Ref. [16].

With this we were able to show that from an intrinsically non-conservative modified theory of gravity, $f(R, T)$, the ACDM model was re-obtained. On the other hand, the cosmological constant does not behave as in GR, since the non-conservation of this component changes its evolution.

4.2 Evolution of small matter perturbations

So far, we obtained that the ACDM model for the background evolution is recovered. We now show that this is also the case for small perturbations. In order to carry out this investigation, let us write the equations in the alternative form,

$$R_{\mu\nu} = 8\pi G \left[ T_{\mu\nu} - \frac{(1 + 2\beta)}{2(1 + \beta)} g_{\mu\nu} T - \frac{\beta}{1 + \beta} g_{\mu\nu} p \right], \quad (57)$$

$$T_{\mu\nu}''' = -\frac{\beta}{2(1 + \beta)} \left( T_{\nu}'' + 2p'' \right). \quad (58)$$

We choose a linearly perturbed FLRW metric, with the perturbation variable denoted as $h_{\mu\nu}$, in the synchronous coordinate condition, i.e. $h_{\mu0} = 0$ [17]. Then, we define

$$h = \frac{h_{kk}}{a^2}, \quad (59)$$
and similarly to what we have done in the previous section, we split the SET into
\[ T^{\mu\nu} = T^{\mu\nu}_m + T^{\mu\nu}_\Lambda, \]  
with,
\[ T^{\mu\nu}_m = \rho_m u^\mu u^\nu, \quad T^{\mu\nu}_\Lambda = \rho_\Lambda g^{\mu\nu}. \]

Once again, applying the same reasoning as before, we assume that the matter SET conserves separately
\[ T^{\mu\nu} \mu = 0. \]  
Hence, we can rewrite Eq. (58) as
\[ T^{\mu\nu}_\Lambda \mu = -\frac{\beta}{2(1+\beta)} (T^{\mu\nu}_t + 2p^{\mu\nu}_t) , \]
where the subscript \( t \) stands for the total contribution. The set of the perturbed equations are then
\[ \frac{\ddot{h}}{2} + H\dot{h} = 8\pi G \left( \frac{1}{2(1+\beta)} \delta \rho_m - \frac{1 + 2\beta}{1 + \beta} \delta \rho_\Lambda \right), \]
\[ \delta \rho_m = \frac{\dot{h}}{2}, \]
\[ \delta \dot{\rho}_\Lambda = -\frac{\beta}{2(1+2\beta)} \delta \rho_m. \]
Combining these equations, we obtain
\[ \ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi G \rho_m \delta_m = 0 , \]
with the usual definition for the density contrast for the matter component as
\[ \delta_m = \frac{\dot{\delta} \rho_m}{\rho_m}. \]

Therefore, the evolution of small matter fluctuations in the model under consideration is also identical to the one in the \( \Lambda \)CDM model of GR. Despite the similarities, an important difference is that the cosmological constant introduced in the \( f(R, T) \) theory is not really a constant, but has an evolution dictated by the non-conservative character of the theory. This feature is the same as it was found in Ref. [16] for the standard Rastall gravity.

\section{Discussion and conclusions}

In this paper we analyzed the possibility of conceiving a Lagrangian for Rastall theory. First, we showed that theories of type \( f(R, L_m) \) cannot reproduce Rastall gravity. Even assuming \( f(R, L_m) = \alpha R + \mathcal{G}(L_m) \), this structure either recovers GR, or strongly deviates from Rastall gravity.

Then, we demonstrated that despite the structural similarities with Rastall, the only way to cast this theory in the framework of \( f(R, T) \), was through the hydrodynamical representation of a scalar field.

In a cosmological setting, the latter model, is able to reproduce one of the most intriguing results of Rastall theory, proved in Ref. [16]: the recent cosmic history of the \( \Lambda \)CDM model is reproduced, but with the extra feature of having a clustering of dark energy.

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