Haldane’s Fractional Statistics and the Riemann-Roch Theorem

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Abstract
The new definition of fractional statistics given by Haldane can be understood in some special cases in terms of the Riemann-Roch theorem.

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1 Introduction

Statistics plays an important role in the physics of quantum many-body systems. Bosonic and fermionic statistics have been known to us for quite a long time and were believed to be unique. However, a few years ago, it was found that, in two-dimensional systems, fractional statistics, neither bosonic or fermionic [1], can exist. The particles obeying fractional statistics have been called anyons.

It is now established that quasiparticles in the fractional quantum Hall effect (FQHE) obey fractional statistics. Indeed, in ref. [2], Halperin conjectured that the fractional quantum Hall quasiparticles are anyons; he also suggested that condensation of such quasiparticles in the Laughlin states (at the Landau filling \( \nu = 1/m \)) gives rise to hierarchical states (\( \nu \neq 1/m \)) (see also ref. [3]). The theory of hierarchical states developed by Halperin and Haldane is called the standard hierarchical theory (for a review on the quantum Hall effect, see Refs. [4, 5]). The Halperin conjecture about anyonic quasiparticles was soon proved by Arovas, Schrieffer and Wilczek [6]. Other possible applications of anyon physics, as anyonic high temperature superconductivity, can be found in ref. [7].

Recently, Haldane proposed [8] a new definition of fractional statistics (NDFS), based on Hilbert space counting argument. The NDFS has arose a lot of interest and has been used for the study of the FQHE hierarchical states [4, 9, 10]. It can be viewed as a generalization of the Pauli exclusion principle in the case of systems with a finite Hilbert space or subspace. Precisely, the generalization of the Pauli exclusion principle to fractional statistics has been investigated in ref. [12] in the case of anyons in a strong magnetic field, confined to the infinite Hilbert space of the lowest Landau level (LLL). In this situation, it was found that at most \( \pi/|\theta| \) anyons can occupy a given quantum LLL state (\( \theta = 0 \), Bose statistics, \( |\theta| = \pi \), Fermi statistics, anyonic statistics \( |\theta| < \pi \); notations and conventions will be introduced later), and that at the critical filling, the magnetic field is entirely screened by the flux tubes carried by the anyons. Very recently, Wu has shown [13] how Haldane’s Hilbert space counting arguments in a mean field approach can lead to similar conclusions. However, Haldane’s NDFS can extend \( |\theta| \) to all possible values, as we shall see later.

The NDFS can also be used to calculate the size of the full Hilbert space of many-particle states. In ref. [4, 11], the emphasis was put on the Hilbert space
of the low-energy sector of fractional quantum Hall states in the presence of quasielectron (QE) or quasihole (QH) excitations. On the other hand, the energy spectra of a few electrons can be calculated numerically. A low-energy sector was found which is well separated from the groundstate and corresponds to Hall states with QE or QH excitations. The dimension of the Hilbert space of such states predicted in ref. \[8\] is indeed in agreement with the number of states in the low-energy sector. Furthermore, the low-energy sector has been investigated in Jain’s theory \[14\] of hierarchical states, the so called composite fermion theory of the FQHE. The number of low-energy states can also be obtained in this framework \[13\], and was found to be identical to the one predicted in the standard hierarchical theory using the NDFS concept. This might give a further evidence of the equivalence of the two hierarchical theories.

Note finally that the NDFS can in principle help to define statistics in any dimension and thus suggest a possible generalization of the notion of fractional statistics to space of dimension other than two.

However, it is clear that a complete and convincing picture of NDFS is still lacking. In view of the several points mentioned above, it is important to improve our understanding about it. We will show that in one case, conventional anyons interacting with a strong magnetic field, the new definition is equivalent to the old one. The Riemann-Roch theorem is used to demonstrate this equivalence. Several examples (including the FQHE) will be worked out in detail to illustrate our claim.

## 2 Anyons in a strong magnetic field

We start by a simple example based on ref. \[16\]. Consider on the sphere $N$ anyons with hard core boundary conditions interacting with a magnetic field and use projective coordinates on the plane, to get the Landau Hamiltonian \[17\] (see also ref. \[18\])

$$H = \sum_{i=1}^{N} H_i = \frac{2}{M} \sum_{i=1}^{N} (1 + z_i \bar{z}_i)^2 (P_{z_i} - A_{z_i})(P_{\bar{z}_i} - A_{\bar{z}_i}).$$

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\[2\] by strong magnetic field we mean that one concentrates on the groundstate, meaning that thermal excitations are negligible compared with the cyclotron gap.

\[3\] The Landau problem for anyons on the plane has been originally discussed in ref. \[17\]
with
\[ A_{zi} = i \frac{\theta}{2\pi} \sum_{j \neq i} \left[ \frac{1}{z_i - z_j} \right] - i \phi \frac{z_i}{2(1 + z_i \bar{z}_i)}. \] (2)

and
\[ P_{z_i} = -i \partial_{z_i}, \quad P_{\bar{z}_i} = -i \partial_{\bar{z}_i}. \] (3)

\((A_{\bar{z}_i} \) is the complex conjugate of \( A_{z_i} \)). The Hamiltonian with Laplace-Beltrami ordering is

\[ H_{L-B} = \frac{1}{M} \sum_{i=1}^{N} (1 + z_i \bar{z}_i)^2 [(P_{z_i} - A_{z_i})(P_{\bar{z}_i} - A_{\bar{z}_i}) + (P_{\bar{z}_i} - A_{\bar{z}_i})(P_{z_i} - A_{z_i})] \] (4)

with
\[ H_{L-B} - H = \sum_{i} (1 + z_i \bar{z}_i)^2 (\frac{\phi}{1 + z_i \bar{z}_i})^2 - \theta \sum_{i \neq j} \delta^2(z_i - z_j). \] (5)

So, to the exception of a constant term and \( \delta \) functions which can be omitted because of hard core boundary conditions which exclude the diagonal of the configuration space (i.e. wavefunctions have to vanish when any 2 anyons coincide), \( H \) and \( H_{L-B} \) are equivalent.

In (2), the first term of \( A_{z_i} \) encodes the fractional statistics interaction with statistical parameter \( \theta/\pi \). The many-body wavefunction has to be symmetric under coordinate exchange since by convention the particles obey bosonic statistics. If a singular gauge transformation is performed to eliminate this term, the many-body wavefunction explicitly obeys fractional statistics: the statistical phase factor is \( e^{i\theta} \), thus the statistical phase is periodic with period \( 2\pi \). It will become soon clear that if \( \theta \) is shifted by \( 2\pi \), many physical quantities will be affected.

The second term in (2) describes the anyons coupling with the magnetic field. \( \phi \) is related to \( q\Phi \), where \( q \) is the charge of the particle and \( \Phi \) is the magnetic flux out of the surface. In the case of FQHE quasiparticles, \( \phi - q\Phi \) is a finite constant related to the spin of the particle (see ref. [19] and the following sections).

Now let us fix the coordinates of \( z_i, \quad i \neq 1 \). The Hamiltonian of particle 1 is simply \( H_1 \). The Dirac quantization condition for \( H_1 \) is

\[ \phi - (N - 1)\frac{\theta}{\pi} = n \] (6)
with \( n \) an integer. In what follows, all parameters, in particular \( \phi \) and \( n \), are assumed to be positive for convenience. Other situations, for example the QE case, will be discussed afterwards. The Dirac quantization condition can be also presented as follows: the flux felt by a given particle should be an integer expressed in magnetic quantum flux unit.

Since \( H_1 \) is positive definite, if one finds \( \psi(z_1) \) such that \( H_1 \psi(z_1) = 0 \), it is the groundstate. At low temperature and in a strong magnetic field (\( \phi/M \) is big compared with the thermal energy), the system will be confined to the groundstate, i.e. the LLL, defined by

\[
(P_{z_1} - A_{z_1})\psi(z_1) = 0.
\]

(7)

Solutions are found to be

\[
\psi(z_1) = (z_1)^k \prod_{i \neq 1} \left( \frac{(z_1 - z_i)(\bar{z}_1 - \bar{z}_i)}{(1 + z_1\bar{z}_1)} \right)^{\frac{\phi}{2\pi}}(1 + z_1\bar{z}_1)^{-n/2},
\]

(8)

with \( k = 0, 1, \ldots, n \). The dimension \( d \) of this Hilbert subspace (groundstate) is thus equal to \( n + 1 \).

Before applying Haldane’s NDFS to this particular finite dimensional Hilbert space, let us first recall its general definition. Consider a \( N \)-body system, and fix \( N - 1 \) particles among them. Analyze the Hamiltonian of the remaining particle (for example particle 1) assuming that the dimension \( d \) of the Hilbert space for this particle (or subspace of the Hilbert space) is finite and independent of which particle has been chosen. The NDFS statistical parameter \( g \) is by definition

\[
\Delta d = -g\Delta N.
\]

(9)

If one considers varying number of particles in order that a thermodynamic limit can be properly defined, \( \Delta d = -g\Delta N \) is an integer, so is \( \Delta N \), thus \( g \) must be rational.

The size of the Hilbert space of the many-body system (bosonic) at fixed \( N \) is then

\[
\frac{(d + N - 1)!}{(N)!(d - 1)!}.
\]

(10)

In the present case, \( d = 1 + \phi - (N - 1)\theta/\pi \), one finds \( g = \theta/\pi \): the NDFS statistical parameter is thus identical to the fractional statistical parameter.
Now one can find the wavefunctions of the many-body state
\[
\Psi(z_1, \cdots, z_i, \cdots, z_N) = f(z_i) \prod_{i<j} \left[ \frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)} \right]^{\frac{\theta}{2\pi}} \times \prod_i (1 + z_i \bar{z}_i)^{-n/2},
\]
(11)
where \( f(z_i) \) is a symmetric analytic function of \( z_i \), with monomial in \( z_i \) of power smaller or equal to \( n \). Symmetric polynomials \( \sigma_i \) are generated by the function
\[
P(z_i) = \prod_i (z - z_i) = \sum_{i=0}^N (-1)^i \sigma_i z^{N-i},
\]
(12)
Thus
\[
f(z_i) = \prod_i \sigma_i^{s_i}
\]
(13)
with \( \sum_i s_i \leq n \). Due to this restriction, the number of \( f(z_i) \)’s is finite (so is the dimension of the groundstate) and is equal to
\[
\frac{(N+n)!}{N!n!},
\]
(14)
in agreement with Haldane’s equation (10) (in this case, \( d = n + 1 \)). It can be rewritten as in the mean field approach of ref. [13]
\[
\frac{(G + N - 1 - (N - 1)g)!}{N!(G - 1 - (N - 1)g)!},
\]
(15)
with \( G = \phi + 1 \) (\( g = \theta/\pi \)). \( G \) is the the number of degenerate Landau states a particle can occupy \( (d = G - (N - 1)\theta/\pi \), thus \( G = d \) when \( N = 1 \) as it should). But \( G \) can be non-integer (however \( G - (N - 1)g = d \) must be an integer).

If one increases the number of particles, \( d \) decreases to 1. \( d = 1 \) is critical since below that point one has not enough room to put all the particles in the LLL. In this critical situation, \( \phi - (N - 1)\theta/\pi = 0 \), the magnetic field is entirely screened by the flux tubes carried by the anyons. The critical filling is (in the thermodynamic limit \( N \to \infty \)) [12, 13]
\[
\nu = \frac{N}{\phi} = \frac{1}{g},
\]
(16)
The non degenerate groundstate wavefunction at the critical filling is simply

\[ \Psi_{\text{crit}}(z_1, \ldots, z_i, \ldots, z_N) = \prod_{i<j} \left[ \frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)} \right]^{\frac{\theta}{2\pi}}. \]  

(17)

One can generalize the above results to the case of multispecies anyons (labelled by an index \( l \)) with mutual statistics [20]. Now the Hamiltonian is

\[ H = \sum H_{i,l} = \sum \frac{2}{M_l} (1 + z_{i,l} \bar{z}_{i,l})^2 (P_{z_{i,l}} - A_{z_{i,l}})(P_{\bar{z}_{i,l}} - A_{\bar{z}_{i,l}}), \]  

(18)

where \( z_{i,l} \) is the coordinate of the \( i \)th anyon of species \( l \). The gauge field \( A_{z_{i,l}} \) is

\[ A_{z_{i,l}} = i \sum_{j \neq i} \frac{\theta_{l,l}}{2\pi} \frac{1}{z_{i,l} - z_{j,l}} + i \sum_{j,k \neq l} \frac{\theta_{l,k}}{2\pi} \frac{1}{z_{i,l} - z_{j,k}} - i \frac{\phi_{l}}{2} \frac{\bar{z}_{i,l}}{1 + z_{i,l} \bar{z}_{i,l}}. \]  

(19)

The Dirac quantization condition for species \( l \) reads

\[ \phi_{l} - (N_l - 1) \frac{\theta_{l,l}}{\pi} - \sum_{k \neq l} N_k \frac{\theta_{l,k}}{\pi} = n_l \]  

(20)

where \( N_k \) is the number of anyons of species \( l \) and

\[ d_l = n_l + 1. \]  

(21)

One defines

\[ \Delta d_l = - \sum_k g_{l,k} \Delta N_k. \]  

(22)

Then

\[ g_{l,k} = \frac{\theta_{l,k}}{\pi}. \]  

(23)

The many-body groundstate wavefunctions are

\[ \Psi = f(z_{i,l}) \prod_{i,j,l,k} \left[ \frac{(z_{i,l} - z_{j,k})(\bar{z}_{i,l} - \bar{z}_{j,k})}{(1 + z_{i,l} \bar{z}_{i,l})(1 + z_{j,k} \bar{z}_{j,k})} \right]^{\frac{\theta_{l,k}}{2\pi}} \prod_{i,l}(1 + z_{i,l} \bar{z}_{i,l})^{-n_l/2} \]  

(24)

where if \( l = k \), then \( i < j \). \( f(z_{i,l}) \) is an holomorphic function generated by

\[ f(z_{i,l}) = \prod_{i,l} \sigma_i(l)^{s_{i,l}} \]  

(25)
where the $\sigma_i(l)$’s are symmetric polynomials in the coordinates $z_{j,l}$. One has also the restrictions
\[
\sum_i s_{i,l} \leq n_l. \tag{26}
\]
The number of solutions is
\[
\prod_l \frac{(N_l + n_l)!}{N_l! n_l!}, \tag{27}
\]
or
\[
\prod_l \frac{[G_l + N_l - 1 - \sum_k g_{l,k}(N_k - \delta_{l,k})]!}{N_l! [G_l - 1 - \sum_k g_{l,k}(N_k - \delta_{l,k})]!}, \tag{28}
\]
with $G_l = \phi_l + 1$.

At the critical filling, the $d_l$’s are all equal to 1 and the groundstate is non degenerate.

### 3 The Riemann-Roch theorem and the NDFS

The results of the preceding section can be easily understood in terms of the Riemann-Roch theorem. The Riemann-Roch theorem and other recent developments in Algebraic Geometry have been used in ref. [21] to investigate Landau quantum mechanics on various Riemann surfaces.

Following ref. [21], let us define the metric $ds^2 = g_{z\bar{z}} dz d\bar{z}$ where $z$ is a complex coordinate on a given Riemann surface. The volume form is $dv = [ig_{z\bar{z}}/2] dz \wedge d\bar{z} = g_{z\bar{z}} dx \wedge dy$. Take a constant magnetic field applied perpendicularly to the Riemann surface, $F = Bdv = (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) dz \wedge d\bar{z}$ implying $\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = ig_{z\bar{z}} B/2$. The flux of the magnetic field is $2\pi \Phi = \int F = BV$, where $V$ is the area of the surface and $B > 0$ has been assumed ($\Phi > 0$; $\Phi$ should be an integer because of the Dirac quantization condition). The Landau Hamiltonian reads
\[
H_{L-B} = \left[1/2M \sqrt{g}(P_\mu - A_\mu)g^{\mu\nu} \sqrt{g}(P_\nu - A_\nu) \right.
\]
\[
= [g^{z\bar{z}}/M][P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) + (P_{\bar{z}} - A_{\bar{z}})(P_z - A_z)]
\]
\[
= [2g^{z\bar{z}}/M](P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) + B/2M \tag{29}
\]
where $g^{z\bar{z}} = [1/g_{z\bar{z}}]$ and $P_z = -i\partial_z$, $P_{\bar{z}} = -i\partial_{\bar{z}}$. The inner product is defined as $<\psi_1|\psi_2> = \int dv \bar{\psi}_1 \times \psi_2$. $H_{L-B} - B/2M$ is a positive definite hermitian
operator thus \((P_z - A_z)\psi = 0\) has for solutions the groundstate wavefunctions of the Hamiltonian \(H_{L-B}\), i.e. the LLL. The existence of such solutions is guaranteed by the Riemann-Roch theorem. They belong to the holomorphic line bundle under the gauge field. The Riemann-Roch theorem states that \(h^0(L) - h^1(L) = \deg(L) - h + 1\), where \(h^0(L)\) is the dimension of the holomorphic line bundle or the degeneracy of the groundstate of \(H_{L-B}\), \(h^1(L)\) is the dimension of the holomorphic line bundle \(K \cdot L^{-1}\) where \(K\) is the canonical bundle, and \(h\) is the genus of the surface. \(\deg(L)\) is the degree of the line bundle which is equal to the first Chern number of the gauge field, or the magnetic flux through the surface, \(\Phi\). If \(\deg(L) > 2h - 2\) (case of a strong magnetic field as in the FQHE), \(h^1(L) = 0\) and \(h^0(L) = \Phi - h + 1\).

Now, consider on the surface \(N\) anyons with hard-core boundary conditions coupled to a constant magnetic field. The Hamiltonian becomes

\[
H_{L-B} = \sum_i [g^{z_i\bar{z}_i}/M][(P_{z_i} - A_{z_i})(P_{\bar{z}_i} - A_{\bar{z}_i}) + (P_{\bar{z}_i} - A_{\bar{z}_i})(P_{z_i} - A_{z_i})].
\tag{30}
\]

The vector potential \(A_{z_i}\) is a sum of two terms \(A^1_{z_i} + A^2_{z_i}\). \(A^1_{z_i}\) encodes the fractional statistics interaction between the anyons and \(A^2_{z_i}\) describes the coupling of the anyons with the constant external magnetic field. The flux is \(n = \phi - (N - 1)\theta/\pi\), where \(\phi\) is the contribution of \(A^2_{z_i}\) and \(-(N - 1)\theta/\pi\) is those of \(A^1_{z_i}\). \(n\) must be an integer, which is assumed to be positive, as well as \(\phi\) and \(\theta\) which are also positive.

Define

\[
H = \sum_i [2g^{z_i\bar{z}_i}/M][(P_{z_i} - A_{z_i})(P_{\bar{z}_i} - A_{\bar{z}_i})].
\tag{31}
\]

\(H_{L-B} - H\) contains only \(\delta\) interactions and constant terms as for the case of the sphere. Because of hard core boundary conditions these two Hamiltonians are in fact equivalent. After fixing the coordinates \(z_i, i \neq 1\), one analyzes the Hamiltonian of particle 1. If some solutions of the equation \((P_{z_i} - A_{z_i})\psi(z_1) = 0\) exist, they are the groundstate wavefunctions of \(H_1\). If \(n > 2h - 2\), and following the previous discussions, the number of solutions \(\psi(z_1)\) is \(d = n - h + 1\) (in the case of the sphere, \(h = 0\), then \(d = n + 1\) as it should). One obtains \(g = \theta/\pi\) as in the previous section. Because the many-body wavefunctions are bosonic in the gauge chosen, the size of the Hilbert space is again given by \((N + d - 1)!/[N!(d-1)!]\). Also, note that the Riemann-Roch theorem actually proves that the groundstate wavefunctions found in the last section are complete.
The above discussion applies to other various situations. In the case of a strong magnetic field and at low temperature, the system will be confined in the groundstate \( (P_{\bar{z}_i} - A_{\bar{z}_i})\psi(z_1) = 0 \). The Riemann-Roch theorem does not depend on details of the interactions, as for example the exact extension or profile of the vortex described by the gauge potential \( A_{\bar{z}_i}^1 \) (here we have specialized to point-like flux tubes), but only on the flux \( n \) stemming from them.

One has only discussed compact surfaces so far, but real samples do have boundaries. Then the Riemann-Roch theorem with boundaries [22] (or more generally, the index theorem with boundaries) should be relevant. One can show that, at the condition that the boundary conditions are unchanged when particles are added, \( \Delta d \) is still equal to \(-\theta/\pi \Delta N\) and thus \( g \) is equal to \( \theta/\pi \).

4 the FQHE and Haldane’s statistics

Following the original motivation of Haldane [8], let us use the NDFS concept for the FQHE quasiparticles excitations.

We remind that the Hamiltonian of an electron on a sphere coupled to a magnetic field is

\[
H = \frac{2}{M_e} (1 + z\bar{z})^2 (P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}),
\]

with

\[
eA_z = -\frac{i\Phi}{2} \frac{\bar{z}}{1 + z\bar{z}},
\]

where \( \Phi \) is the magnetic flux out of the surface. Define

\[
d_{ij} = \frac{z_i - z_j}{(1 + z_i\bar{z}_i)^{\frac{1}{2}}(1 + z_j\bar{z}_j)^{\frac{1}{2}}}.
\]

The Laughlin wavefunctions at filling \( \nu = 1/m \) [23, 3, 16, 19] then read

\[
\prod_{i < j} d_{ij}^{n_i}
\]
where $N_e$ is the number of electrons. The Hamiltonian for the quasiparticles is \[19\]

$$H_{qh} = \frac{2}{M_{qh}} \sum_{i=1}^{N_{qh}} (1 + z_i\bar{z}_i)^2 (P_{z_i} - A_{qh}(\bar{z}_i))(P_{z_i} - A_{qh}(z_i)),$$

$$H_{qe} = \frac{2}{M_{qe}} \sum_{i=1}^{N_{qe}} (1 + z_i\bar{z}_i)^2 (P_{z_i} - A_{qe}(z_i))(P_{\bar{z}_i} - A_{qe}(\bar{z}_i)) \tag{36}$$

where $N_{qh}$ ($N_{qe}$) is the number of QH (QE). Different normal orderings for the QH and QE Hamiltonians will be justified later. However, because of the hard core boundary conditions on the quasiparticles \[9\], Hamiltonians with different normal orderings are equivalent. One has

$$A_{qh}(z_i) = \frac{-i}{2m} \sum_{j \neq i} \frac{1}{z_i - z_j} - \frac{i}{2} \left( \frac{1}{m} - 1 \right) \frac{\bar{z}_i}{1 + z_i\bar{z}_i} + \frac{i\Phi}{2m} \frac{\bar{z}_i}{1 + z_i\bar{z}_i},$$

$$A_{qe}(z_i) = \frac{-i}{2m} \sum_{j \neq i} \frac{1}{z_i - z_j} - \frac{i}{2} \left( \frac{1}{m} + 1 \right) \frac{\bar{z}_i}{1 + z_i\bar{z}_i} - \frac{i\Phi}{2m} \frac{\bar{z}_i}{1 + z_i\bar{z}_i} \tag{37}$$

$A_{qh}(z_i)$ was obtained in ref. \[19\], whereas $A_{qe}(z_i)$ can be obtained in the same way by calculating the Berry phase of quasiparticles on the sphere \[6\]. The “guiding center” coordinates do not commute and such systems are described by the quasiparticles Hamiltonian projected on the LLL. The QH and QE statistical phases are both equal to $e^{-i\pi/m}$. The QH charge is $1/m$ (assuming that the charge of an electron is $-1$) and thus the QE charge is $-1/m$. The first term in $A_{qh}(z_i)$ or $A_{qh}(z_i)$ is due to fractional statistics. The last term in $A_{qh}(z_i)$ or $A_{qh}(z_i)$ describes the coupling of the quasiparticles to the magnetic field. The remaining term in $A_{qh}(z_i)$ or $A_{qh}(z_i)$ is related to the intrinsic spin of the quasiparticles \[19\]. For completeness, the effective Coulomb interactions between quasiparticles should be included, and they actually are taken into account via the hard core boundary conditions on the quasiparticles (we will comment on this later).

Let us first consider the one-body QH Hamiltonian. The flux quantization is

$$-\frac{\Phi}{m} + \left( \frac{1}{m} - 1 \right) + \frac{(N_{qh} - 1)}{m} = -N_e \tag{38}$$
where the relation \( m(N_e - 1) + N_{qh} = \Phi \) has been used. Because of the negative flux \(-N_e\), the groundstate wavefunctions satisfy \((P_{z_i} - A_{z_i})\Psi = 0\), and not \((P_{z_i} - A_{z_i})\Psi = 0\). As a special normal ordering as been used in the QH Hamiltonian, the QH groundstate energy is 0 (the same reasoning also applies to QE). If we would include Coulomb interactions, the energies of those eigenstates would be different from each other and form a band.

\[ d \] is now equal to \( N_e + 1 \). The QH wavefunctions can be constructed following the discussions of section 2

\[
\Psi(z_1, \ldots, z_i, \ldots, z_{N_{qh}}) = f(\bar{z}_i) \prod_{i<j}^{N_{qh}} \left( \frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{(1 + z_i\bar{z}_i)(1 + z_j\bar{z}_j)} \right)^{1/2m} \\
\times \prod_{i}^{N_{qh}} (1 + z_i\bar{z}_i)^{-N_e/2},
\]

with degeneracy equal to

\[
\frac{(d - 1 + N_{qh})!}{(d - 1)!N_{qh}!},
\]

The NDFS parameter \( g \) is equal to \( 1/m \), according to the previous discussion in section 2. The \( f(\bar{z}_i) \) are again symmetric polynomials. At the critical filling, \( d \) is required to be 1. But it means that \( N_e = 0 \), which is not physically interesting. When the QH condense and the QH wavefunction is a Laughlin wavefunction type, the FQHE state is a hierarchical state. The QH wavefunction at \( d = 1 \) is

\[
\prod_{i<j}^{N_{qh}} |d_{ij}|^{1/m} (\bar{d}_{ij})^p,
\]

where \( p \) is a positive even integer. As in the case of Laughlin states formed by electrons, Coulomb interactions between quasiparticles are needed to form the incompressible state \([\Pi]\). Because the charge of the quasiparticle is \( 1/m \), the Coulomb interaction between quasiparticles is weaker than the Coulomb interaction between electrons. Then one should expect that such states are difficult to obtain. However, we will show that in the QE case, fractional statistics interactions play a significant role to produce the Laughlin states. In fact, it will come out that QE Laughlin states are easier to form than QH Laughlin states. This conclusion was used in ref. [9] to explain the QH and QE states asymmetry.
The flux quantization condition for the one-body QE Hamiltonian is
\[ \frac{\Phi}{m} + \left( \frac{1}{m} + 1 \right) + \left( \frac{N_{qe} - 1}{m} \right) = N_e \] (42)
where the relation \( m(N_e - 1) - N_{qe} = \Phi \) has been used. The one-body QE groundstate (the coordinates of the other QE are fixed) is determined by
\[ (P_{z_1} - A_{qe}(\bar{z}_1))\psi(z_1) = 0, \] (43)
The solutions are
\[ \psi(z_1) = f(z_1) \prod_{i \neq 1}^{N_{qe}} \left[ \frac{(z_1 - z_i)(\bar{z}_1 - \bar{z}_i)}{(1 + z_1 \bar{z}_1)} \right]^{-\frac{1}{2m}} (1 + z_1 \bar{z}_1)^{-N_e/2}. \] (44)
with \( f(z_1) = z_k^k, k = 0, 1, \cdots, N_e \). It appears that the number of solutions is \( d = N_e + 1 \), in agreement with the Riemann-Roch theorem. However, due to the hard core boundary conditions imposed on the quasiparticles, only a subset of the above solutions can be retained (note that hard core boundary conditions are automatically satisfied for QH wavefunctions). Indeed, \( f(z_1) \) should have zeros at \( z_1 = z_i, \ i \neq 1 \). But as the many-body wavefunctions should obey bosonic statistics, the order of zeros at \( z_1 = z_i, \ i \neq 1 \) must be equal or greater than two. \( f(z_1) \) should thus take the form \( f(z_1) = \prod_{j \neq 1} (z_1 - z_j)^2 f'(z_1), \) with \( f'(z_1) = z_k^k, k = 1, 2, \cdots, N_e - 2(N_{qe} - 1) \). We conclude that the number of wavefunctions with hard core boundary conditions is \( d' = N_e - 2(N_{qe} - 1) + 1 \). This result can also be understood by the Riemann-Roch theorem. The number of zeros of the wavefunctions is equal to the flux out of the surface, or the Chern number (in this case the number of zeros or the Chern number is \( N_e \)). Now, zeros are at \( \bar{z}_1 = z_i, \ i \neq 1 \) with at least order two (the total order of zeros at these points is at least \( 2(N_{qe} - 1) \)). The number of such linear independent functions is \( N_e - 2(N_{qe} - 1) + h + 1 \), by the Riemann-Roch theorem.
Because of the hard core boundary conditions, the dimension of the one-body QE Hilbert space is \( d' \). Since \( d' = \frac{\Phi}{m} + \left( \frac{1}{m} + 1 \right) - (N_{qe} - 1)(2 - \frac{1}{m}) \), the NDFS parameter is found to be \( g = 2 - \frac{1}{m} \) (this result was also numerically obtained in ref. [11]). Thus the critical filling at \( d' = 1 \) is \( 1/(2 - 1/m) \).
The dimension of the many-body QE Hilbert space is
\[ \frac{(d' - 1 + N_{qe})!}{(d' - 1)!N_{qe}!}, \] (45)
and the construction of the QE many-body wavefunctions proceed as in the case of QH:

$$
\Psi(z_1, \cdots, z_i, \cdots, z_{N_{qe}}) = f'(z_i) \prod_{i<j}^{N_{qe}} \frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)}^{1/2m} \times \left(\frac{z_i - z_j}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)}\right)^2 \times \prod_{i}^{N_{qe} - 2(N_{qe} - 1)} (1 + z_i \bar{z}_i)^{-N_e + 2(N_{qe} - 1)/2} \tag{46}
$$

where the $f'(\bar{z}_i)$'s stand for symmetric polynomials (their construction has been discussed in section 2) and where the highest power of any coordinate is $N_e - 2(N_{qe} - 1)$. The number of such polynomials is indeed equal to (45). The critical filling occurs at $d'_c = 1$, which implies $N_e = 2(N_{qe} - 1)$ and corresponds to the hierarchical states with the electron filling at $1/(m - 1/2)$.

The QE wavefunction at $d'_c = 1$ is then

$$
\prod_{i<j}^{N_{qe}} |d_{i,j}|^{-1/m} d_{i,j}^2. \tag{47}
$$

Contrary to the QH Laughlin state (41), the statistical interaction is largely responsible for the QE Laughlin state (47), at the critical filling. Coulomb interactions do not need to be very strong to form such a QE Laughlin state. Thus the QE Laughlin state of (47) should be observed relatively easily. This explains why there are more FQHE hierarchical states due to QE condensation than to QH condensation. By no means, one should conclude that Coulomb interactions are not necessary to form QE Laughlin states. The physical origin of hard core boundary conditions for quasiparticles can actually be found in the Coulomb short-range repulsion between quasiparticles [9].

The above discussion also shows that the statistical phase does not uniquely determine the NDFS parameter. The QE and QH statistical phases are the same, but the NDFS parameters for QH and QE are different.

In the hierarchical theory of Haldane and Halperin, the filling of the $n$-
level hierarchical states is
\[ \nu = \frac{1}{p_1 - \frac{1}{p_2 - \frac{1}{\cdots - \frac{1}{p_n}}}}, \]  
(48)
where \( p_1 \) is a positive integer, and the \( p_i, \ i \neq 1 \) are even integers. The standard FQHE hierarchical theory suggests that at this filling the FQHE state is obtained by condensation of quasiparticles of the previous levels.

The fractional fillings observed to date occur in the following sequence (filling \( \nu < 1 \)) [24]:

\[ \nu = \frac{n}{2n + 1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \cdots, \frac{9}{19}, \cdots, \]
\[ \nu = 1 - \frac{n}{2n + 1} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \cdots, \frac{9}{19}, \cdots, \]
\[ \nu = \frac{n}{4n + 1} = \frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \cdots, \]
\[ \nu = 1 - \frac{n}{4n + 1} = \frac{4}{5}, \frac{7}{9}, \cdots, \]
\[ \nu = \frac{n}{4n - 1} = \frac{2}{7}, \frac{3}{11}, \frac{4}{15}, \cdots, \]
\[ \nu = 1 - \frac{n}{4n - 1} = \frac{5}{7}, \cdots. \]  
(49)
Consider in eq. (48) the sequence of fillings \( \nu = (n/2n + 1), n/(4n + 1), n/(4n - 1) \), where the other fillings correspond to the conjugate states. The filling \( \nu = n/(2n + 1) \) can be written in the form (48) with \( p_1 = 3 \) and \( p_i = 2, \ i \neq 1 \). Thus states at filling \( \nu = n/(2n + 1) \) are due to the hierarchical QE condensations of the \( \nu = 1/3 \) Laughlin state (in the way QE condense mainly because of statistical interactions, as we discussed previously). FQHE states at such fillings are thus expected to be more easily observed, and it is actually so.

For \( \nu = n/(4n + 1), p_1 = 5, \) and \( p_i = 2, \ i \neq 1 \). Thus states at filling \( n/(4n + 1) \) are due to the hierarchical QE condensations of the \( \nu = 1/5 \) Laughlin state. Since the \( \nu = 1/5 \) filling of the parent state is more difficult to
observe than the $\nu = 1/3$ filling, so are states in the sequence $\nu = n/(4n+1)$ compared with states in the sequence $\nu = n/(2n+1)$.

$\nu = n/(4n - 1)$ can be written as

$$
\nu = \frac{1}{p_1 + \frac{1}{1 + \frac{1}{p_2 - \frac{1}{\ldots - \frac{1}{p_n}}}}},
$$

with $p_1 = 3$, $p_i = 2$, $i \neq 1$. When $n = 2$, the state is an hierarchical state due to the QH condensation of the Laughlin state at filling $1/3$ (the QH wavefunction is given by eq. (41)). When $n > 2$, the states are due to the hierarchical QE condensation of the hierarchical state at $n = 2$. It is found experimentally that this sequence is weaker than the sequence $\nu = n/(2n + 1)$, as it should from the preceding discussion (sequences involving QH condensations should be weaker than sequences which do not).

It is interesting to observe that QH condensation only occurs in the second level of the hierarchical states (or QH condensation only occurs in parent Laughlin states at filling $1/p_1$). The reason for this is possibly that in high levels of hierarchical states, QH has a smaller electric charge and thus Coulomb interactions are weaker. The hierarchical states due to QH condensation in high levels are rather difficult to observe. However, since the QE statistical interactions play a significant role in producing the Laughlin states (47), it is still possible to have hierarchical states due to QE condensations in high level hierarchical states despite their rather small charge. In conclusion, the standard hierarchical theory also explains the order of stability of sequences found in experiments (as in Jain’s theory).

Recent works show that the standard hierarchical theory may be equivalent to Jain’s theory (9, 10). However, the construction of explicit trial wavefunctions of electrons at hierarchical fillings is still an open problem in the standard hierarchical theory. In Jain’s theory, such trial wavefunctions can be constructed without involving quasiparticles. Recent attempts have been made to get the electron wavefunctions based on the standard hierarchical theory (25), but much more remain to be done.

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References

[1] J.M. Leinaas and J. Myrheim, Nuovo Cimento B37 (1977) 1; G.A. Goldin, R. Menikoff and D.H. Sharp, J. Math. Phys. 22 (1981) 1664; F. Wilczek, Phys. Rev. Lett. 48 (1982) 1144; 49 (1982) 957; Y.S. Wu, Phys. Rev. Lett. 52 (1984) 2103; 53 (1984) 111.

[2] B.I. Halperin, Phys. Rev. Lett. 52 (1984) 1583.

[3] F.D.M. Haldane, Phys. Rev. Lett. 51 (1983) 605.

[4] R. Prange and S. Girvin, The Quantum Hall Effect (Springer-Verlag, New York, Heidelberg, 1990, 2nd ed); and references therein.

[5] A.H. MacDonald, Perspective on the Quantum Hall Effect (Klewer, Boston, 1989); M. Stone, Quantum Hall Effect (World Scientific, 1992); and references therein.

[6] D. Arovas, J.R. Schrieffer and F. Wilczek, Phys. Rev. Lett. 53 (1984) 722.

[7] F. Wilczek, Fractional Statistics and Anyon Superconductivity (World Scientific, 1990); and references therein.

[8] F.D.M. Haldane, Phys. Rev. Lett. 67 (1991) 937.

[9] S. He, X.C. Xie, and F.C. Zhang, Phys. Rev. Lett. 68 (1992) 3460; M. Ma and F.C. Zhang, Phys. Rev. Lett. 66 (1991) 1769.

[10] J. Yang and W.P. Su, Phys. Rev. Lett. 68 (1992) 2382; Phys. Rev. Lett. 70 (1993) 1163; Phys. Rev. B 47 (1993) 12953.

[11] M.D. Johnson and G.S. Canright, Florida preprint UCF-CM-93-105.

[12] A. Dasnières de Veigy and S. Ouvry, Phys. Rev. Lett. 72 (1994) 600.

[13] Y.S. Wu, “Statistical Distribution for Particles Obeying Fractional Statistics”, Utah preprint.

[14] J.K. Jain, Phys. Rev. Lett. 63 (1989) 199; Phys. Rev. B 41 (1990) 7653; Adv. phys. 41 (1992) 105.
[15] G. Dev and J.K. Jain, Phys. Rev. Lett. 69 (1992) 2843;

[16] D. Li, Nucl. Phys. B 396 (1993) 411 (FS); see also R. Iengo and K. Lechner, Phys. Rep. C213 (1992) 179.

[17] G. V. Dunne, A. Lerda and C. A. Trugenberger, Mod. Phys. Lett. A 6 (1991) 2891; Int. Jour. Mod. Phys. B 5 (1991) 1675; G.V. Dunne, A. Lerda , S. Sciuto and C.A. Trugenberger, Nucl. Phys. B 370 (1992) 601; J. Grundberg, T.H. Hansson, A. Karlhede and E. Westerberg, Stockholm preprint USITP-91-2; A.P. Polychronakos, Phys. Lett. B 264 (1991) 362; C. Chou, Phys. Lett. A 155 (1991) 245; Phys. Rev. D 44 (1991) 2533; A. Govari, Technion preprint Phy.-92.

[18] A. Comtet, J. McCabe and S. Ouvry, Phys. Rev. D. 45 (1992) 709.

[19] D. Li, Phys. Lett. A 169 (1992) 82.

[20] A. Dasnières de Veigy and S. Ouvry, Phys. Lett. B 307 (1993) 91.

[21] R. Iengo and D. Li, Nucl. Phys. B 413 (1994) 735 (FS).

[22] T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rep. 66 (1980) 213.

[23] R.B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395.

[24] J.K. Jain, Comments Cond. Mat. Phys. 16 (1993) 307.

[25] M. Greiter, Princeton preprint IASSNS-HEP-92/78.