TIME-DEPENDENT ASYMPTOTIC BEHAVIOR OF THE SOLUTION FOR PLATE EQUATIONS WITH LINEAR MEMORY

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ABSTRACT. In this article, we consider the long-time behavior of solutions for the plate equation with linear memory. Within the theory of process on time-dependent spaces, we investigate the existence of the time-dependent attractor by using the operator decomposition technique and compactness of translation theorem and more detailed estimates. Furthermore, the asymptotic structure of time-dependent attractor, which converges to the attractor of fourth order parabolic equation with memory, is proved. Besides, we obtain a further regular result.

1. Introduction. Let Ω be an open bounded set of \( \mathbb{R}^n (n \geq 5) \) with smooth boundary \( \partial \Omega \). We consider the following equations

\[
\begin{align*}
\varepsilon(t)u_{tt} + \alpha u_t + \Delta^2 u + \int_0^{+\infty} \mu(s)\Delta^2(u(t) - u(t-s))ds + f(u) &= g(x), \\
\text{in } \Omega \times (\tau, \infty), \\
u(x, t) = \frac{\partial u(x, t)}{\partial n} &= 0, \quad x \in \partial \Omega, \ t \in \mathbb{R}, \\
u(x, t) = u_0(x, t), u_t(x, t) = \partial_t u_0(x, t), \ x \in \Omega, \ t \leq \tau,
\end{align*}
\]

(1.1)

where \( u_0 : \Omega \times (-\infty, \tau] \rightarrow \mathbb{R} \) is the given past history of \( u \), \( g(\cdot) \in L^2(\Omega) \), \( \varepsilon = \varepsilon(t) \) is a decreasing bounded function and

\[
\lim_{t \to +\infty} \varepsilon(t) = 0.
\]

(1.2)

In particular, there exists \( L > 0 \), such that

\[
\sup_{t \in \mathbb{R}} |\varepsilon(t)| + |\varepsilon'(t)| \leq L.
\]

(1.3)

The nonlinear term \( f \in C^2(\mathbb{R}) \), \( f(0) = 0 \), and for some \( C \geq 0 \) satisfies

\[
|f'(s)| \leq C(1 + |s|^{\frac{4}{n-4}}), \ \forall \ s \in \mathbb{R}, \ n \geq 5,
\]

(1.4)

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along with the dissipation condition
\[
\liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1, \quad \forall \ s \in \mathbb{R},
\]
where \(\lambda_1\) is the first eigenvalue of the strictly positive operator \(A = \Delta^2\).

With respect to the memory component, we assume that
\[
\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \int_0^\infty \mu(s)ds = m_0 \geq 0,
\]
(1.6)

\[
\mu'(s) \leq -\rho \mu(s) \leq 0, \quad \forall s \geq 0,
\]
(1.7)

where \(\rho\) is a positive constant.

The problem (1.1) stems from the elastic equation established by Woinowsky-Krieger([21]). However, the strictly mathematical analysis and the survey of the global solutions as well as the asymptotic behavior for the linear plate equations should start with Ball who studied the stable property of the linear elastic beam equation in 1973 ([2, 3]).

When \(\varepsilon(t)\) is a positive constant independent of time \(t\), the problem (1.1) has been extensively studied in many literatures. For instance, when \(\mu(s) \equiv 0\), Yang and Zhong in [24, 25] achieved the existence of global attractors for the plate equation on a bounded domain, including not only the autonomous system with nonlinear damping but also the non-autonomous system with localized damping and critical nonlinearity. Asymptotic dynamics of plate equations on the unbounded domain were investigated by several authors such as Khanmamedov [9, 10] and Xiao [22]. Ma and Yang obtained the existence of exponential attractors for the plate equation with strong damping in [13]. Recently, the random attractors of stochastic plate equation with strong damping and additive noise were considered in [14].

In the case when \(\varepsilon(t)\) is a positive decreasing function which vanish at infinity, the problem (1.1) becomes more complex and interesting; the reason is that the corresponding dynamical system is still understood under the non-autonomous framework even the forcing term in the equation is independent of time \(t\). In order to deal with these problems, in [5], Conti, Pata and Temam presented a notion of time-dependent attractor exploiting the minimality with respect to the pull-back attraction property, and gave a sufficient condition proving the existence of time-dependent attractor based on the theory established by Plinio, Duane and Temam ([8]). Besides, they applied the new methods into the following weak damped wave equations with time-dependent speed of propagation
\[
\varepsilon(t)u_{tt} + \alpha u_t - \Delta u + f(u) = g(x).
\]
(1.8)

Also, they proved that the time-dependent global attractor of (1.8) converged in a suitable sense to the attractor of the parabolic equation \(\alpha u_t - \Delta u + f(u) = g(x)\) when \(\varepsilon(t) \to 0\) as \(t \to +\infty\) ([6]). Successively, in [7], they continued to show the asymptotic structure of time-dependent global attractor to the following specific one-dimensional wave equation
\[
\varepsilon(t)u_{tt} - u_{xx} + [1 + \varepsilon f'(u)]u_t + f(u) = h.
\]
(1.9)

Recently, Meng et al. investigated the long-time behavior of the solution for the wave equation with nonlinear damping \(g(u_t)\) on the time-dependent space, in which they found a new technical method verifying compactness of the process via defining the contractive functions, see [16]. In [17], Meng and Liu gave the necessary and
sufficient conditions of the existence of time-dependent global attractor borrowed from the ideas in [15].

To the best of our knowledge, in [11, 12] Liu and Ma have studied the existence of time-dependent attractors for the plate equations without linear memory. Just for the problem (1.1), the presence of the memory make it impossible to utilize \((I - P_m)u\) as the test function to capture the asymptotic compactness of the solution process, so the methods in [17] is out of action to our problem. For our purpose, we firstly construct a relatively complicated triple solution space by introducing a new variable. Secondly, we capitalize the decomposition techniques and compactness transitivity theorem to conquer the barriers induced by the critical nonlinearity and the history memory.

It is worth mentioning that the dissipative condition (1.5) is weaker than one in [16], because the authors made use of the dissipative condition, i.e., \(\liminf_{|s|\to\infty} f'(s) > -\lambda_1\).

For convenience, hereafter, \(C\) (or \(c\)) denotes an arbitrary positive constant which may be different from line to line even in the same line.

The rest of this article consists of five Sections. In the next Section, we define some functions sets and iterate some useful lemmas. In Section 3, the existence of the time-dependent global attractor is obtained. In Section 4, the existence of the attractor is proved, and then we prove the upper-semicontinuous convergence of non-autonomous attractors of (1.1) to the global attractor of (4.1). Finally, in Section 5, we consider the further regularity of attractors.

2. Preliminaries.

2.1. Mathematical setting. As in ([4, 18]), we introduce a new variable \(\eta\) as follows

\[
\eta = \eta^t(x, s) = u(x, t) - u(x, t - s), (x, s) \in \Omega \times \mathbb{R}^+ = [0, +\infty), t \geq \tau,
\]

(2.1)

then we can rewritten (1.1) as

\[
\begin{align*}
\varepsilon(t)u_{tt} + \alpha u_t + \Delta^2 u + \int_0^\infty \mu(s)\Delta^2 \eta^t(s)ds + f(u) &= g(x), \\
\eta^t_t &= -\eta^t_s + u_t,
\end{align*}
\]

(2.2)

with initial and boundary conditions

\[
\begin{align*}
\begin{cases}
  u(x, t) = \frac{\partial u_0(x, t)}{\partial n} = 0, & x \in \partial \Omega, \ t \geq \tau \\
  \eta^t(x, s) = \frac{\partial \eta_0(x, s)}{\partial n} = 0, & (x, s) \in \partial \Omega \times \mathbb{R}^+, \ t \geq \tau \\
  u(x, \tau) = u_0(x), \ u_t(x, \tau) = u_1(x), & x \in \Omega, \\
  \eta^\tau(x, s) = \eta^0(x, s), & (x, s) \in \Omega \times \mathbb{R}^+, \\
\end{cases}
\end{align*}
\]

(2.3)

where

\[
\begin{align*}
\begin{cases}
  u_0(x) = u_0(x, \tau), \ u_1(x) = \frac{\partial u_0(x, t)}{\partial t}|_{t=\tau}, \\
  \eta^0 = \eta^0(x, s) = u_0(x, \tau) - u_0(x, \tau - s).
\end{cases}
\end{align*}
\]

Without loss of generality, set \(H = L^2(\Omega)\), endowed with the inner product \(\langle \cdot, \cdot \rangle\) and norm \(\| \cdot \|\), respectively. For \(s \in \mathbb{R}^+\), we define the hierarchy of (compactly) nested Hilbert spaces

\[
H^s = D(A^{\frac{s}{2}}), \ \langle w, v \rangle_s = \langle A^{\frac{s}{2}}w, A^{\frac{s}{2}}v \rangle, \ |w|_s = \| A^{\frac{s}{2}}w \|;
\]

especially, we have the embeddings \(H^{s+1} \hookrightarrow H^s\).
For \( s \in \mathbb{R}^+ \), let \( L_{\mu}^2(\mathbb{R}^+; H^*) \) be the family of Hilbert spaces of functions \( \varphi : \mathbb{R}^+ \to H^* \), equipped with the inner product and norm, respectively,
\[
\langle \varphi_1, \varphi_2 \rangle_{\mu, s} = \langle \varphi_1, \varphi_2 \rangle_{\mu, H^*} = \int_0^\infty \mu(s) \langle \varphi_1(s), \varphi_2(s) \rangle_{H^*} ds,
\]
\[
\| \varphi \|_{\mu, s}^2 = \| \varphi \|_{\mu, H^*}^2 = \int_0^\infty \mu(s) \| \varphi(s) \|_{H^*}^2 ds.
\]
Now, for \( t \in \mathbb{R} \) and \( s \in \mathbb{R}^+ \), write the following symbols,
\[
\mathcal{H}_t^s = H^{s+2} \times H^s \times L_{\mu}^2(\mathbb{R}^+; H^{s+2}),
\]
with the norm
\[
\| z \|_{\mathcal{H}_t^s}^2 = \| (u, u_t, \eta^t) \|_{\mathcal{H}_t^s}^2 = \| u \|_{s+2}^2 + \varepsilon(t) \| u_t \|_s^2 + \| \eta^t \|_{\mu,s+2}^2.
\]
The letter \( s \) is always omitted whenever zero. Especially, we consider the time-dependent phase space
\[
\mathcal{H}_t = H^2 \times H \times L_{\mu}^2(\mathbb{R}^+; H^2),
\]
with the norm
\[
\| z \|_{\mathcal{H}_t}^2 = \| (u, u_t, \eta^t) \|_{\mathcal{H}_t}^2 = \| u \|_2^2 + \varepsilon(t) \| u_t \|_2^2 + \| \eta^t \|_{\mu,2}^2.
\]
We will use sometimes also the space
\[
\mathcal{H}_t^1(\mathbb{R}^+; H^*) = \{ \varphi : \varphi(t), \partial_t \varphi(t) \in L_{\mu}^2(\mathbb{R}^+; H^*) \}.
\]

2.2. Notations and concepts. In the following, we review briefly the notations, some definitions and abstract results about process on time-dependent spaces, see ([5, 6]) for more details.

Let \( X_t \) be a family of normed spaces, we introduce the \( R \)-ball of \( X_t \)
\[
B_t(R) = \{ z \in X_t : \| z \|_{X_t} \leq R \}.
\]
We denote the Hausdorff semidistance of two sets \( B, C \subset X_t \) by:
\[
\delta_t(B, C) = \sup_{x \in B} \text{dist}_{X_t}(x, C) = \sup_{x \in B} \inf_{y \in C} \| x - y \|_{X_t}.
\]

Definition 2.1. Let \( X_t \) be a family of normed spaces. A process is a two-parameter family of mappings \( \{ U(t, \tau) : X_\tau \to X_t, \ t \geq \tau \in \mathbb{R} \} \) with properties
(i) \( U(\tau, \tau) = Id \) is the identity on \( X_\tau \), \( \tau \in \mathbb{R} \);
(ii) \( U(t, s)U(s, \tau) = U(t, \tau), \ \forall \ t \geq s \geq \tau \).

Definition 2.2. A family \( \mathcal{C} = \{ C_t \}_{t \in \mathbb{R}} \) of bounded sets \( C_t \subset X_t \) is called uniformly bounded if there exist a constant \( R > 0 \) such that \( C_t \subset B_t(R), \ \forall \ t \in \mathbb{R} \).

Definition 2.3. A time-dependent absorbing set for \( U(t, \tau) \) is a uniformly bounded family \( \mathcal{B} = \{ B_t \}_{t \in \mathbb{R}} \) with the following property: for every \( R > 0 \) there exists a \( t_0 \) such that
\[
\tau \leq t - t_0 \Rightarrow U(t, \tau)B_t(R) \subset B_t.
\]

Definition 2.4. The time-dependent global attractor for \( U(t, \tau) \) is the smallest family \( \mathcal{A} = \{ A_t \}_{t \in \mathbb{R}} \) with the following properties:
(i) Each \( A_t \) is compact in \( X_t \);
(ii) \( \mathcal{A} \) is pullback attracting, namely, it is uniformly bounded and the limit
\[
\lim_{\tau \to -\infty} \delta_t(U(t, \tau)C_\tau, A_t) = 0,
\]
holds for every uniformly bounded family \( \mathcal{C} = \{ C_t \}_{t \in \mathbb{R}} \) and every \( t \in \mathbb{R} \).
Theorem 2.5. The time-dependent attractor $\mathcal{A}$ exists and it is unique if and only if the process $U(t, \tau)$ is asymptotically compact, namely, the set

$$\mathbb{K} = \{ \mathcal{R} = \{ K_t \}_{t \in \mathbb{R}} : K_t \subset X_t \text{ compact, } \mathcal{R} \text{ pullback attracting} \}$$

is not empty.

Definition 2.6. A function $t \to Z(t) \in X_t$ is a complete bounded trajectories (CBT) of $U(t, \tau)$ if and only if

1. $\sup_{t \in \mathbb{R}} \| Z(t) \|_{X_t} < \infty$;
2. $Z(t) = U(t, \tau)Z(\tau), \forall t \geq \tau, \tau \in \mathbb{R}$.

Definition 2.7. We say $\mathcal{A} = \{ A_t \}_{t \in \mathbb{R}}$ is invariant if $U(t, \tau)A_t = A_t, \forall t \geq \tau$.

Theorem 2.8. When the attractor $\mathcal{A} = \{ A_t \}_{t \in \mathbb{R}}$ is invariant, it coincides with the sets of all CBT of the process $U(t, \tau)$, namely

$$\mathcal{A} = \{ Z : t \to Z(t) = (u(t), u_0(t)) \in \mathcal{H}_t \text{ with } Z \text{ CBT of } U(t, \tau) \}.$$

2.3. Some lemmas. Let

$$F(u) = \int_0^u f(y)dy.$$

Lemma 2.9. ([24, 25]) From (1.5), we can get for $0 < \nu < 1$ and $c_i > 0$ ($i = 1, 2$), there hold

$$2(F(u), 1) \geq -(1 - \nu)\| u \|_2^2 - c_i,$$

$$\langle f(u), u \rangle \geq -(1 - \nu)\| u \|_2^2 - c_2, \forall u \in H^2.$$  (2.5)

Lemma 2.10. ([4, 18]) Assume that $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ is a nonnegative function, and satisfies the following: if there exists $s_0 \in \mathbb{R}^+$ such that $\mu(s_0) = 0$, then $\mu(s) = 0$, for all $s \geq s_0$ holds. Moreover, let $B_0, B_1, B_2$ be Banach spaces, where $B_0, B_2$ are reflexive and satisfies $B_0 \hookrightarrow B_1 \hookrightarrow B_2$, where the embedding $B_0 \hookrightarrow B_1$ is compact. Let $C \subset L^2_{\mu}(\mathbb{R}^+, B_1)$ satisfy:

1. $C$ is bounded in $L^2_{\mu}(\mathbb{R}^+, B_0) \cap H^1_{\mu}(\mathbb{R}^+, B_2)$;
2. $\sup_{c \in C} \| \eta(s) \|_{H^1_{\mu}(\mathbb{R}^+, B_1)} \leq h(s), \forall s \in \mathbb{R}^+, h(s) \in L^1_{\mu}(\mathbb{R}^+)$;
then $C$ is relatively compact in $L^2_{\mu}(\mathbb{R}^+, B_1)$.

Lemma 2.11. ([5]) (Gronwall-type lemma) Let $Y(t) : [\tau, \infty) \to \mathbb{R}^+$ be an absolutely continuous function satisfying the inequality

$$\frac{d}{dt}Y(t) + 2\varepsilon Y(t) \leq h(t)Y(t) + k,$$

for some $\varepsilon > 0$, $k \geq 0$ and where $h : [\tau, \infty) \to \mathbb{R}^+$ fulfills

$$\int_{\tau}^{\infty} h(s)ds \leq m,$$

with $m \geq 0$. Then,

$$Y(t) \leq Y(\tau)e^{\varepsilon(t-\tau)} + ke^{-\varepsilon}e^m.$$
3. Existence of the time-dependent global attractor.

3.1. A priori estimates. Global existence of solution \( u \) to (2.2)-(2.3) is classical, by using the standard Galerkin approximation method, see ([4], [1]): that is, if (1.2)-(1.7) hold, then the problem (2.2)-(2.3) has a unique weak solution \( z(t) = (u(t), u_t(t), \eta^t(\cdot)) \). Thus combining with Lemma 3.1, we can define:

\[
U(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_\tau \text{ acting as } U(t, \tau)z(\tau) = \{u(t), u_t(t), \eta^t(\cdot)\},
\]

with the initial data \( z_\tau = z(\tau) = \{u_0, u_1, \eta^0\} \in \mathcal{H}_\tau \).

Lemma 3.1. Under the assumptions (1.2)-(1.7), for every pair of initial data \( z_\tau \in \mathcal{H}_\tau \), there exists a unique solution \( z(t) = (u(t), u_t(t), \eta^t(\cdot)) \) of problem (2.2)-(2.3) in space \( \mathcal{H}_t \) and for \( \tau \in \mathbb{R}, t \geq \tau \) satisfy

\[
 u \in C([\tau, t]; H^2), \quad u_t \in C([\tau, t]; H), \quad \eta^t \in C([\tau, t]; L^2_\mu(\mathbb{R}^+; H^2)).
\]

Furthermore, let \( z_i(\tau) \in \mathcal{H}_\tau \) be the initial data such that \( \|z_i(\tau)\|_{\mathcal{H}_\tau} \leq R \) \( (i = 1, 2) \), and \( z_i(t) \) be the solution of problem (2.2)-(2.3). Then there exists \( C = C(R) > 0 \), such that

\[
\|z_1(t) - z_2(t)\|_{\mathcal{H}_\tau} \leq C(t-\tau)\|z_1(\tau) - z_2(\tau)\|_{\mathcal{H}_\tau}, \quad \forall \ t \geq \tau. \tag{3.1}
\]

To prove Lemma 3.1, we first need the following estimate:

Lemma 3.2. Under the assumptions (1.2)-(1.7), for any initial data \( z(\tau) \in B_\tau(R) \subset \mathcal{H}_\tau \), there exists \( R_0 > 0 \), such that

\[
\|U(t, \tau)z(\tau)\|_{\mathcal{H}_\tau} \leq R_0, \quad \forall \ \tau \leq t.
\]

Proof. Denote

\[
E(t) = \|u\|_2^2 + \varepsilon(t)\|u_t\|^2 + \delta\|u\|^2 + 2\delta\varepsilon(t)\|u_t\|_2^2 + 2\delta\|\eta^t\|_{H^2} + 2\langle F(u), 1 \rangle - 2\langle g, u \rangle.
\]

Multiplying (2.2) by \( 2u_t + 2\delta u \) in \( L^2 \) we achieve

\[
\frac{d}{dt}E(t) + [2\alpha - \varepsilon'(t) - 2\delta\varepsilon(t)]\|u_t\|^2 + 2\delta\|u\|_2^2 + 2\delta\langle \eta^t(s), \eta^t(s) \rangle_{H^2} + 2\delta\langle f(u), u \rangle - 2\delta\langle g, u \rangle = 2\delta\varepsilon'(t)\langle u_t, u \rangle \tag{3.2}
\]

By virtue of Hölder, Young and Poincaré inequalities, and (1.2)-(1.3), there hold

\[
2\delta\varepsilon'(t)\langle u_t, u \rangle \leq 2\delta L\|u_t\||u||u\| \leq \frac{\alpha}{2}\|u_t\|^2 + \frac{2\delta^2 L^2}{\alpha}\|u\|^2 \leq \frac{\alpha}{2}\|u_t\|^2 + \frac{2\delta^2 L^2}{\lambda_1\alpha}\|u\|^2,
\]

where \( \|u\|_2 \geq \lambda_1\|u\|^2, \forall u \in H^2(\Omega) \).

Now taking \( \delta \) small enough, such that \( 1 - \frac{2\delta L^2}{\lambda_1\alpha} > 0 \), and set

\[
I(t) = [2 - \frac{2\delta L^2}{\lambda_1\alpha}\|u\|_2^2 + \varepsilon(t)\|u_t\|^2 + \frac{2}{\delta}\|\eta^t(s), \eta^t(s)\|_{H^2} + 2\|u\|_2^2 + 2\|\eta^t(s), \eta^t(s)\|_{H^2} + 2\delta\|f(u), u \rangle - 2\|g, u \rangle,
\]

so we conclude that

\[
\frac{d}{dt}E(t) + \delta I(t) \leq 0,
\]

then

\[
E(t) \leq \delta \int_\tau^t I(s)ds + E(\tau). \tag{3.3}
\]
Together with (2.4) and Hölder, Young, Poincaré inequalities, it follows that

\[ E(t) \geq \nu \|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \delta \alpha \|u\|^2 + 2\delta \varepsilon(t) (u_t, u) + \|\eta^t\|_{p,2}^2 - 2\langle g, u \rangle - c_1 \]

\[ \geq \nu \|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t\|_{p,2}^2 - \varepsilon(t) \|u_t\|^2 - \frac{2\delta L}{\lambda_1} \|u\|_2^2 - \frac{\delta}{2} \|u\|^2 - \frac{2}{\delta \lambda_1} \|g\|^2 - c_1 \]

\[ \geq (\nu - \frac{2\delta L}{\lambda_1} - \frac{\delta}{2}) \|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t\|_{p,2}^2 - \frac{2}{\delta \lambda_1} \|g\|^2 - c_1 \]

\[ \geq \frac{\nu}{8} (\|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t\|_{p,2}^2) - \left( \frac{2}{\lambda_1 \nu} \right) \|g\|^2 + c_1, \]  

(3.4)

where, we use \( \nu - \frac{2\delta L}{\lambda_1} - \frac{\delta}{2} > \frac{\nu}{2} \) for \( \delta < \nu \) small enough.

In addition, by Hölder, Young inequalities, and together with (1.6)-(1.7) we obtain

\[ \langle \eta^t, \eta^t \rangle_{\mu,2} = \frac{1}{2} \int_0^\infty \mu(s) \frac{d}{ds} \|\eta^t(s)\|_{\mu}^2 ds = -\frac{1}{2} \int_0^\infty \mu'(s) \|\eta^t(s)\|_{\mu}^2 ds \geq \frac{\rho}{2} \int_0^\infty \mu(s) \|\eta^t(s)\|_{\mu}^2 ds = \frac{\rho}{2} \|\eta^t(\cdot)\|_{\mu,2}^2, \]

\[ 2 \int_0^\infty \mu(s) \langle \Delta \eta^t(s), \Delta u(t) \rangle ds \leq 2 \int_0^\infty \mu(s) \|\Delta \eta^t(s)\| \cdot \|\Delta u(t)\| ds \leq \frac{\rho}{2\delta} \|\eta^t(\cdot)\|_{\mu,2}^2 + \frac{2\delta m_0}{\rho} \|u\|_2^2. \]  

(3.5)

Combining with the above estimates (2.5), Hölder, Young inequalities, it leads to

\[ I(t) \geq 2(\nu - \frac{\delta L^2}{\lambda_1 \alpha}) \|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \frac{\rho}{2} \|\eta^t(\cdot)\|_{\mu,2}^2 - \frac{2\delta m_0}{\rho} \|u\|_2^2 \]

\[ \geq (2\nu - \frac{2\delta m_0}{\rho} - \frac{2\delta L^2}{\lambda_1 \alpha}) \|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \frac{\rho}{2\delta} \|\eta^t(\cdot)\|_{\mu,2}^2 - \frac{2}{\delta \lambda_1} \|g\|^2 - 2c_2 \]

\[ \geq C_1 (\|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t(\cdot)\|_{\mu,2}^2) \]  

(3.6)

where \( C_1 = \min \{ 2\nu - \frac{2\delta m_0}{\rho} - \frac{2\delta L^2}{\lambda_1 \alpha}, 1, \frac{\nu}{2\delta} \} \), for \( \delta < \nu \) small enough such that \( 2\nu - \frac{2\delta m_0}{\rho} - \frac{2\delta L^2}{\lambda_1 \alpha} > \frac{\nu}{2} > 0 \).

Then by (3.3), (3.4)-(3.6) we get

\[ \frac{\nu}{8} (\|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t(\cdot)\|_{\mu,2}^2) - m_1 \]

\[ \leq -\delta \int_\tau^t \left[ C_1 (\|u\|_2^2 + \varepsilon(t) \|u_t\|^2 + \|\eta^t(\cdot)\|_{\mu,2}^2) - m_2 \right] dt + E(t), \]

where \( m_1 = c_1 + \frac{2}{\lambda_1 \alpha} \|g\|^2 \), \( m_2 = \frac{2}{\lambda_1 \alpha} \|g\|^2 + 2c_2 \). So, for any \( K_0 > \frac{m_2}{c_1} \), there exists \( t_0 > \tau \) such that

\[ \|u(t_0)\|_2^2 + \varepsilon(t_0) \|u_t(t_0)\|^2 + \|\eta^{t_0}\|_{\mu,2}^2 \leq K_0. \]
As a result, let \( B_t = \bigcup_{t \geq \tau} U(t, \tau)B_0 \), where

\[
B_0 = \{(u_0, u_1, \eta^0) \in H_T : \|u_0\|_2^2 + \|u_1\|_2^2 + \|\eta^0(s)\|_{\mu, 2}^2 \leq K_0\},
\]

then \( B_t \) is a bounded absorbing set for process \{U(t, \tau)\}.

On the other hand, from the above discussion, there exist a positive constant \( R_0 \) such that

\[
\|u\|_2^2 + \varepsilon(t)\|u_t\|_2^2 + \|\eta^t\|_{\mu, 2}^2 \leq R_0, \quad \forall \, t \geq t_0 > \tau.
\]

\[\square\]

**Proof of Lemma 3.1.** Let \( z_1(\tau), \ z_2(\tau) \in H_T \) such that \( \|z_i(\tau)\|_{H_T} \leq R, \ i = 1, 2, \) and denote by \( C \) a generic positive constant depending on \( R \) but independent of \( z_i \). We first observe that the energy estimate in Lemma 3.2 above ensures:

\[
\|U(t, \tau)z_i(\tau)\|_{H_T} \leq C. \quad (3.7)
\]

We set \( \{u_i(t), \partial_t u_i(t), \eta^i(\cdot)\} = U(t, \tau)z_i(\tau) \) and denote \( \tilde{z}(t) = \{\tilde{u}(t), \tilde{u}_t(t), \tilde{\eta}^t(\cdot)\} = U(t, \tau)z_1(\tau) - U(t, \tau)z_2(\tau) \). Then the difference between the two solutions with initial data \( \tilde{z}(\tau) = z_1(\tau) - z_2(\tau) \) satisfies

\[
\varepsilon(t)\tilde{u}_t + \alpha\tilde{u} + \Delta^2\tilde{u} + \int_0^\infty \mu(s)\Delta^2\tilde{\eta}(s)ds + f(u_1) - f(u_2) = 0.
\]

Multiplying the above equation by \( 2\tilde{u}_t \) and integrating over \( \Omega \), we obtain

\[
\frac{d}{dt}\|\tilde{z}\|_{H_t}^2 + [2\alpha - \varepsilon'(t)]\|\tilde{u}_t\|^2 + 2\langle\tilde{\eta}, \eta^t\rangle_{\mu, 2} = -2(f(u_1) - f(u_2), \tilde{u}_t).
\]

By exploiting (1.4), (3.7), and Hölder, Young inequality, and the embedding \( H^2 \hookrightarrow L^{\frac{4}{n-2}}(n \geq 5) \), it yields

\[
-2(f(u_1) - f(u_2), \tilde{u}_t) \leq C\int_\Omega (1 + |u_1|^{\frac{4}{n-2}} + |u_2|^{\frac{4}{n-2}}) \cdot \tilde{u} \cdot \tilde{u}_t \, dx
\]

\[
\leq C[1 + \|u_1\|_{L^{\frac{4}{n-2}}}^{\frac{4}{n-2}} + \|u_2\|_{L^{\frac{4}{n-2}}}^{\frac{4}{n-2}}] \cdot \|\tilde{u}\|_2 \cdot \|\tilde{u}_t\|
\]

\[
\leq C\|\tilde{u}\|_2 \cdot \|\tilde{u}_t\|
\]

\[
\leq 2\alpha \|\tilde{u}_t\|^2 + C\|\tilde{u}\|_2^2,
\]

from (3.5) yields

\[
\langle\tilde{\eta}, \eta^t\rangle_{\mu, 2} \geq \frac{D}{2}\|\tilde{\eta}\|_{\mu, 2}^2.
\]

Combining with the above estimates, we have

\[
\frac{d}{dt}\|\tilde{z}(t)\|_{H_t}^2 \leq C\|\tilde{z}(t)\|_{H_t}^2,
\]

then applying the Gronwall lemma on \( [\tau, t] \), we obtain

\[
\|\tilde{z}(t)\|_{H_t}^2 \leq \|\tilde{z}(\tau)\|_{H_t}^2 \cdot e^{C(t-\tau)} = \|z_1(\tau) - z_2(\tau)\|_{H_T}^2 \cdot e^{C(t-\tau)},
\]

where \( C > 0 \) is a constant dependent on \( R \). \[\square\]

From Lemma 3.1 and Lemma 3.2, we can get the following conclusion:
Lemma 3.3. Under the assumptions (1.2)-(1.7), there exists $R_0 > 0$, such that the $\mathcal{B} = \{ \mathbb{B}_t(R_0) \}_{t \in \mathbb{R}}$ is a time-dependent absorbing sets for the process $\{U(t, \tau)\}$ associated with (1.1), and for $M_0 \geq R_0$, there holds
\begin{equation}
\sup_{z \in \mathbb{B}_s(R_0)} \{ \| U(t, \tau) z(\tau) \|_{\mathcal{H}_t} + \int_\tau^\infty \| u_t(y) \| dy \} \leq M_0, \ \forall \ \tau \in \mathbb{R}. \tag{3.8}
\end{equation}

Proof. Let $\delta \equiv 0$ in equality (3.2) and integrating on $[\tau, t]$, we get $\int_\tau^t \| u_t(y) \| dy \leq M_0 > 0$. Then together with Lemma 3.2 we conclude that (3.8) is true. \hfill \Box

3.2. Existence of the time-dependent global attractor.

3.2.1. The decomposition. We write $f = f_0 + f_1$, where $f_0, \ f_1 \in C^2(\mathbb{R})$ fulfill respectively,
\begin{equation}
|f_1'(u)| \leq C(1 + |u|^\gamma), \ 0 < \gamma < \frac{4}{n - 4}, \ \forall \ u \in \mathbb{R}, \tag{3.9}
\end{equation}
\begin{equation}
|f_0'(u)| \leq C(1 + |u|^\frac{4}{n-4}), \ \forall \ u \in \mathbb{R}, \tag{3.10}
\end{equation}
\begin{equation}
\liminf_{|u| \to \infty} \frac{f_1(u)}{u} > -\lambda_1, \ \forall \ u \in \mathbb{R}, \tag{3.11}
\end{equation}
\begin{equation}
f_0(0) = f_0'(0) = 0, \ f_0(u)u \geq 0, \ \forall \ u \in \mathbb{R}. \tag{3.12}
\end{equation}

Denote
\begin{equation}
0 < \sigma = \min\{\frac{1}{4}, 2 - \frac{\gamma(n-4)}{2}\}, \tag{3.13}
\end{equation}
where $\gamma$ is defined in (3.9).

According to Lemma 3.3 we know that $\mathcal{B} = \{ \mathbb{B}_t(R_0) \}_{t \in \mathbb{R}}$ is a time-dependent absorbing set. Then, for any $z(\tau) \in \mathbb{B}_s(R_0)$, we split $U(t, \tau) z(\tau)$ into the sum
\begin{equation*}
U(t, \tau) z(\tau) = \{u(t), u_t(t), \eta^t(\cdot)\} = D(t, \tau) z(\tau) + K(t, \tau) z(\tau),
\end{equation*}
where
\begin{equation}
D(t, \tau) z(\tau) = \{v(t), v_t(t), \zeta^t(\cdot)\}, \ K(t, \tau) z(\tau) = \{w(t), w_t(t), \xi^t(\cdot)\}
\end{equation}
solve the systems, respectively,
\begin{equation}
\begin{cases}
\varepsilon(t) v_t + \alpha v_t + Av + \int_0^{\infty} \mu(s) A\zeta^t(s) ds + f_0(v) = 0, \\
\zeta^t = -\zeta^t + v_t, \\
v|_{\partial \Omega} = \frac{\partial}{\partial n}|_{\partial \Omega} = 0, \ v(x, \tau) = u_0(x), \ v_t(x, \tau) = u_1(x), \\
\xi^t|_{\partial \Omega} = \frac{\partial}{\partial n}|_{\partial \Omega} = 0, \ \zeta^0(x, s) = u_0(x) - u_0(x, \tau - s)
\end{cases}
\tag{3.14}
\end{equation}
and
\begin{equation}
\begin{cases}
\varepsilon(t) w_t + \alpha w_t + Aw + \int_0^{\infty} \mu(s) A\xi^t(s) ds + f(u) - f_0(v) = g, \\
\xi^t = -\xi^t + w_t, \\
w|_{\partial \Omega} = \frac{\partial}{\partial n}|_{\partial \Omega} = 0, \ w(x, \tau) = 0, \ w_t(x, \tau) = 0, \\
\xi^t|_{\partial \Omega} = \frac{\partial}{\partial n}|_{\partial \Omega} = 0, \ \zeta^0(x, s) = 0.
\end{cases}
\tag{3.15}
\end{equation}

In what follows, the generic constant $C > 0$ depends only on $\mathcal{B}$.

Lemma 3.4. There exists $\delta = \delta(\mathcal{B}) > 0$ such that
\begin{equation*}
\| D(t, \tau) z(\tau) \|_{\mathcal{H}_t} \leq C e^{-\delta(t-\tau)}, \ \forall \ t \geq \tau.
\end{equation*}
Proof. First, we have
\[
\|D(t, \tau)z(\tau)\|_{H_t} \leq C. \tag{3.16}
\]
Then, denoting:
\[
E_0(t) = \|D(t, \tau)z(\tau)\|_{H_t}^2 + \delta \alpha \|v\|^2 + 2\delta \epsilon \langle v, v \rangle + 2\langle F_0(v), 1 \rangle,
\]
where
\[
F_0(s) = \int_0^s f_0(y)dy.
\]
From (3.10), (3.16) yields
\[
\frac{1}{2} \|D(t, \tau)z(\tau)\|_{H_t}^2 \leq E_0(t) \leq C \|D(t, \tau)z(\tau)\|_{H_t}^2. \tag{3.17}
\]
Multiplying the first equation of (3.14) by \(2v_t + 2\delta v\) in \(L^2\) we get
\[
\frac{d}{dt} E_0(t) + [2\alpha - \epsilon(t) - 2\delta \epsilon(t)] \|v_t\|^2 + 2\delta \|v\|^2 + 2\langle \zeta^t, \zeta^t \rangle_{\mu,2} + 2\delta \langle f_0(v), v \rangle + 2\delta \int_0^t \mu(s) \langle \Delta \zeta^s(s), \Delta v(t) \rangle ds = 2\delta \epsilon(t) \langle v_t, v \rangle.
\]
By Hölder, Young inequalities we get
\[
2\delta \epsilon(t) \langle v_t, v \rangle \leq 2\delta \|D(t, \tau)z(\tau)\|_{H_t}^2 \leq C \|D(t, \tau)z(\tau)\|_{H_t}^2.
\]
Substituting the above estimate into (3.18) and by (3.12), then for \(\delta\) small enough, we conclude that
\[
\frac{d}{dt} E_0(t) + \delta \|D(t, \tau)z(\tau)\|_{H_t}^2 \leq 0.
\]
Combining with (3.17) and applying the Gronwall’s lemma on \([\tau, t]\), we can get the result.

Summing up, the following uniform bound holds
\[
\sup_{t \geq \tau} \|U(t, \tau)z(\tau)\|_{H_t} + \|D(t, \tau)z(\tau)\|_{H_t} + \|K(t, \tau)z(\tau)\|_{H_t} \leq C. \tag{3.19}
\]

Lemma 3.5. For \(\sigma\) defined in (3.13), there exists \(M = M(\mathcal{B}) > 0\) such that
\[
\|K(t, \tau)z(\tau)\|_{H_t} \leq M, \ orall \ t \geq \tau.
\]

Proof. Let
\[
\Lambda(t) = \|K(t, \tau)z\|_{H_t}^2 + \delta \alpha \|w\|_\sigma^2 + 2\delta \epsilon \langle w, A^{\sigma/2}w \rangle + 2\|f(u) - f_0(v) - g, A^{\sigma/2}w\| + C.
\]
By (3.19) and the growth condition of \(f\) and the embedding equality, we have
\[
2\|f(u) - f_0(v), A^{\sigma/2}w\| \leq 2\|f(u) - f_0(v)\|_{A^{\sigma/2}} \|w\|_{A^{\sigma/2}} \leq C \|w\|_{A^{\sigma/2}} \leq C \|w\|_{A^{\sigma/2}}^2 + C.
\]
Then for \(\delta\) small enough, in line with Hölder, Young inequalities, it follows that
\[
2\delta \epsilon \langle w_t, A^{\sigma/2}w \rangle \leq \frac{\epsilon(t)}{2} \|w_t\|_{A^{\sigma/2}}^2 + \frac{\delta \alpha}{2} \|w\|_{A^{\sigma/2}}^2.
\]
Therefore
\[ \frac{1}{2} \| K(t, \tau)z(\tau) \|_{L^q_t}^2 \leq \Lambda(t) \leq 2 \| K(t, \tau)z(\tau) \|_{L^q_t}^2 + 2C. \tag{3.20} \]

Multiplying the first equation of (3.15) by \( 2A^{\sigma/2}w_t + 2\delta A^{\sigma/2}w \) in \( L^2 \) we get
\[
\frac{d}{dt} \Lambda(t) + [2\alpha - \varepsilon'(t) - 2\delta \varepsilon(t)] \| w_t \|_\sigma^2 + 2\delta \| w \|_{\sigma+2}^2 + 2\langle \xi^t, \xi^s \rangle_{\mu,\sigma+2} 
+ 2\delta \int_0^\infty \mu(s) \langle A\xi(s), A^{\sigma/2}w(t) \rangle ds + 2\delta (f(u) - f_0(v) - g, A^{\sigma/2}w) \tag{3.21}
= 2\delta \varepsilon'(w_t, A^{\sigma/2}w) + I_1 + I_2 + I_3,
\]

where
\[
I_1 = 2\langle f_0'(u) - f_0'(v) | u_t, A^{\sigma/2}w \rangle,
I_2 = 2\langle f_0'(v) | u_t, A^{\sigma/2}w \rangle,
I_3 = 2\langle f_1'(u) | u_t, A^{\sigma/2}w \rangle.
\]

By exploiting conditions (3.10),(3.19)-(3.20) and the embedding \( H^{2+\sigma} \hookrightarrow L^{\frac{2p}{2p-2\sigma}}, H^{-\sigma} \hookrightarrow L^{\frac{2p}{2p-2\sigma}}, \) as well as the continuous embedding \( H^{(5p-10)/2p} \hookrightarrow L^p(\Omega)(p > 2), \) we have
\[
I_1 \leq C \int_\Omega (1 + |u|^{\frac{8-n}{4}} + |v|^{\frac{8-n}{4}}) \cdot |w| \cdot |u_t| \cdot |A^{\sigma/2}w| dx 
\leq C(1 + \| u \|_{L^{2n/(n-4)}}^{\frac{8-n}{4}} + \| v \|_{H^{0,0,0}}^{\frac{8-n}{4}}) \cdot \| w \|_{L^{\frac{2p}{2p-2\sigma}}} \cdot \| u_t \| \cdot \| A^{\sigma/2}w \|_{L^{\frac{2p}{2p-2\sigma}}}
\leq C\tilde{C}(1 + \| u \|_{L^{2n/(n-4)}}^{\frac{8-n}{4}} + \| v \|_{H^{0,0,0}}^{\frac{8-n}{4}}) \cdot \| w \|_{\sigma+2}^2 \cdot \| u_t \|
\leq C\| u_t \| \| w \|_{\sigma+2}^2 \leq \frac{\delta}{4} \| w \|_{\sigma+2}^2 + \frac{C^2}{\delta} \| u_t \| \cdot \| w \|_{\sigma+2}^2
\leq \frac{\delta}{2} \Lambda + C \| u_t \| \| w \|_{\sigma+2}^2;
\]
\[
I_2 \leq C(1 + \| v \|_{L^{2n/(n-4)}}^{\frac{8-n}{4}}) \cdot \| u_t \|_{L^{\frac{2p}{2p-2\sigma}}} \cdot \| A^{\sigma/2}w \|_{L^{\frac{2p}{2p-2\sigma}}}
\leq C(1 + \| v \|_{H^{0,0,0}}^{\frac{8-n}{4}}) \cdot \| u_t \| \cdot \| A^{\sigma/2}w \|_{L^{\frac{2p}{2p-2\sigma}}}
\leq C\| u_t \| \cdot \| w \|_{\sigma+2}
\leq \frac{\alpha}{2} \| u_t \|_{\sigma+2}^2 + C \| v \|_{H^{0,0,0}}^{\frac{8-n}{4}} \cdot \| w \|_{\sigma+2}^2.
\]

Besides, in view of (3.9),(3.13), (3.20) it leads to
\[
I_3 \leq C \int_\Omega (1 + |u|^{\gamma}) \cdot |u_t| \cdot |A^{\sigma/2}w| dx \leq \| u \|_{L^{2n/(n-4)}}^\gamma \cdot \| u_t \| \cdot \| A^{\sigma/2}w \|_{L^{\frac{2p}{2p-2\sigma}}}
\leq \| u_t \| \cdot \| w \|_{\sigma+2}^2 + C,
\]

and from (3.5) we know that
\[
2\langle \xi^t, \xi^s \rangle_{\mu,\sigma+2} \geq \rho\| \xi^t(s) \|_{\mu,\sigma+2}^2,
\]
\[
2\delta \int_0^\infty \mu(s) \langle A\xi(s), A^{\sigma/2}w(t) \rangle ds \leq \frac{\rho}{2} \| \xi^t(s) \|_{\mu,\sigma+2}^2 + \frac{2\delta^2 m_0}{\rho} \| w \|_{\sigma+2}^2.
\]
As a consequence
\[
\frac{d}{dt} \Lambda(t) + \delta \Lambda(t) + (\alpha - \varepsilon'(t) - 3\delta \varepsilon(t))\|u_t\|_\sigma^2 + (\delta - \frac{2\delta^2 L^2}{\lambda_1 \alpha} - \frac{2\delta^2 m_0}{\rho}) \|w\|_{\sigma+2}^2 \\
+ (\frac{\rho}{2} - \delta) \|\xi^t\|_{\mu, \sigma+2}^2 - \frac{2\delta^2 \varepsilon(t)}{\rho} \|w_t\|_\sigma^2 - 2 \delta^2 \varepsilon(t) \langle w_t, L^\sigma w \rangle 
\leq \frac{\delta}{2} \Lambda(t) + C \|w\|_{\sigma+2}^2 \|u_t\|_\sigma^2 + C \|v\|_{\sigma+2}^2 \|w\|_{\sigma+2}^2 + C,
\]
thus, according to (3.20) and taking \( \delta < \rho \) small enough, such that
\[
\frac{d}{dt} \Lambda(t) + \frac{\delta}{2} \Lambda(t) \leq q(t) \Lambda(t) + C,
\]
where \( q(t) = C(\|u_t\|_\sigma^2 + \|v\|_{\sigma+2}^2) \). Lemma 3.3, 3.4 implies that
\[
\int^{\infty}_\tau q(y)dy \leq C,
\]
so applying Lemma 2.11, there holds
\[
\Lambda(t) \leq CA(\tau)e^{-\frac{\delta}{2}(t-\tau)} + C \leq C.
\]
Combining with (3.20), we can get the boundedness of \( K(t, \tau)z(\tau) \) in \( \mathcal{H}_t^q \).

3.2.2. Existence of the invariant attractor.

Theorem 3.6. Under the conditions (1.2)-(1.7), the process \( U(t, \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_t \) generated by problem (1.1) has an invariant time-dependent global attractor \( \mathfrak{A} = \{A_t\}_{t \in \mathbb{R}} \).

For proving this theorem, we need first to verify the compactness of the memory term.

We know that, for any \( \xi^0 \in L^2_{\mu}((R^+, H^2)) \), the Cauchy problem
\[
\begin{cases}
\partial_t \xi^t = -\partial_s \xi^t + w_t, t > \tau, \\
\xi^\tau = \xi^0,
\end{cases}
\] (3.22)
has a unique solution \( \xi^t \in C([\tau, +\infty); L^2_{\mu}(R^+, H^2)) \) and the explicit expression
\[
\xi^t(s) = \begin{cases}
w(t) - w(t-s), \tau < s \leq t, \\
w(t), s > t.
\end{cases}
\] (3.23)

Let \( B_t = B_t(R_0) \) be the bounded absorbing set from Lemma 3.3, we can prove the following result:

Lemma 3.7. Assume that (1.4)-(1.5) hold, \( g \in L^2(\Omega) \). For every given \( T > \tau \) and any \( \epsilon > 0 \), setting
\[
K_T := PK(T, \tau)B_\tau,
\]
there is a positive constant \( N_1 = N_1(\|B_T\|_{\mathcal{H}_T}) \) such that
(1) \( K_T \) is bounded in \( L^2_{\mu}(R^+, H^{\sigma+2}) \cap H^1_{\mu}(R^+, H^2) \);
(2) \( \sup_{\xi \in K_T} \|\xi(s)\|_{H^2}^2 \leq N_1 \),
where \( \sigma = \min\{\frac{1}{4}, 2 - \frac{\gamma(n-4)}{2}\} \), \( \{K(t, \tau)\}_{t \geq \tau} \) is the solution operator of (3.15), \( P : H^2 \times H \times L^2_{\mu}(R^+, H^2) \rightarrow L^2_{\mu}(R^+, H^2) \) is a projection operator.

Proof. According to (3.23), we have
\[
\|\xi^T(x, s)\|_{H^2} \leq \begin{cases}
\|w(T) - w(T-s)\|_{H^2} \leq 2\|w(T)\|_{H^2}, \tau < s \leq T, \\
\|w(T)\|_{H^2}, s > T.
\end{cases}
\] (3.24)
By virtue of Lemma 3.5, we know (1),(2) hold. \( \square \)
Thus, combining with Lemma 2.10, it is easy to know that $K_T$ is relatively compact in $L^2_t(\mathbb{R}^+,H^2)$. Furthermore, using the compact embedding $H^{r+2} \times H^r \hookrightarrow H^2 \times H$ we get:

**Lemma 3.8.** Under the conditions of Lemma 3.7, for any $T > \tau, K(T,\tau) B_\tau$ is relatively compact in $\mathcal{H}_T$.

**Proof of Theorem 3.6.** According to Lemma 3.5, 3.7, we consider the family $\mathcal{R} = \{K_t\}_{t \in \mathbb{R}}$, where

$$K_t = \{ z \in \mathcal{H}_t^T : \|z\|_{\mathcal{H}_t^T} \leq M \}.$$ 

Applying the compact embedding $\mathcal{H}_T^r \hookrightarrow \mathcal{H}_T$ and together with Lemma 3.8, we know that $K_t$ is compact; since the injection constant $M$ is independent of $t$, the set $\mathcal{R}$ is uniformly bounded.

Besides, from Lemma 3.2, 3.3, 3.4, it’s easy to know that $\mathcal{R}$ is pullback attracting, indeed,

$$\delta_t(U(t,\tau) B_\tau(R_0), K_t) \leq C e^{-\delta(t-\tau)}, \forall t \geq \tau.$$ 

Hence the process $\{U(t,\tau)\}$ is asymptotically compact, which allows the application of Theorem 2.5 and achieve the existence of the unique time-dependent global attractor $\mathfrak{A} = \{A_t\}_{t \in \mathbb{R}}$. Due to Lemma 3.1, we know the process $\{U(t,\tau)\}$ is strongly continuous, so $\mathfrak{A}$ is invariant([11, Theorem 5.6]).

3.2.3. Regularity of the attractor.

**Theorem 3.9.** Under the assumptions (1.2)-(1.7), $\{A_t\}_{t \in \mathbb{R}}$ is bounded in $\mathcal{H}_T^1$, and the bound is independent of $t$.

To prove Theorem 3.9, we fix $\tau \in \mathbb{R}$, and for $z(\tau) \in A_\tau$, we split the solution $U(t,\tau)z(\tau)$ into the sum $D_1(t,\tau)z(\tau) + K_1(t,\tau)z(\tau)$, where $D_1(t,\tau)z(\tau) = \{v(t), v(t,\tau), \xi(t)\}$ and $K_1(t,\tau)z(\tau) = \{w(t), w(t,\tau), \xi(t)\}$, solve respectively

\[
\begin{aligned}
\varepsilon(t) v_{tt} + \alpha v_t + A v + \int_0^\infty \mu(s) \Delta^2 \xi_t(s) ds &= 0, \\
\xi_t &= -\xi_t + v_t, \\
v|_{\partial \Omega} = \frac{\partial v}{\partial n}|_{\partial \Omega} = 0, v(x,\tau) = u_0(x), \quad v_t(x,\tau) = u_1(x), \\
\xi_t|_{\partial \Omega} = \frac{\partial \xi}{\partial n}|_{\partial \Omega} = 0, \quad \xi(0,x) = u_0(x,\tau) - u_0(x,\tau - s) 
\end{aligned}
\tag{3.25}
\]

and

\[
\begin{aligned}
\varepsilon(t) w_{tt} + \alpha w_t + A w + \int_0^\infty \mu(s) \Delta^2 \xi_t(s) ds + f(u) &= g, \\
\xi_t &= -\xi_t + w_t, \\
w|_{\partial \Omega} = \frac{\partial w}{\partial n}|_{\partial \Omega} = 0, \quad w(x,\tau) = 0, \quad w_t(x,\tau) = 0, \\
\xi_t|_{\partial \Omega} = \frac{\partial \xi}{\partial n}|_{\partial \Omega} = 0, \quad \xi(0,x) = 0. 
\end{aligned}
\tag{3.26}
\]

As a particular case of Lemma 3.4, we learn that

$$\|D_1(t,\tau)z(\tau)\|_{\mathcal{H}_T^1} \leq C e^{-\delta(t-\tau)}, \forall t \geq \tau. \tag{3.27}$$

**Lemma 3.10.** For $M_1 = M_1(\mathfrak{A}) > 0$, there holds

$$\sup_{t \geq \tau} \|K_1(t,\tau)z(\tau)\|_{\mathcal{H}_T^1} \leq M_1.$$ 

**Proof.** Let

$$E_1(t) = \|K_1(t,\tau)z\|_{\mathcal{H}_T^1}^2 + \delta \alpha \|w\|_{-1}^2 + 2\delta \varepsilon(w_1, A^{1/2}w) - 2\langle g, A^{1/2}w \rangle + C.$$
For $\delta > 0$ small and some $C > 0$ (depending on $|g|$) large enough such that
\[
\frac{1}{4}\|K_1(t, \tau)z\|_{H^1_t}^2 \leq E_1(t) \leq 2\|K_1(t, \tau)z\|_{H^1_t}^2 + 2C. \tag{3.28}
\]
Multiplying (3.26) by $2A^{1/2}w_t + 2\delta A^{1/2}w$ in $L^2$ we find
\[
\frac{d}{dt}E_1(t) + (2\alpha - \varepsilon'(t) - 2\delta \varepsilon(t))\|w_t\|_2^2 + 2\|w\|_2^2 + 2(\xi^t, \xi^\alpha)_{\mu, \lambda}
\]
\[
+ 2\delta \int_0^\infty \mu(s)(A\xi^t(s), A^{1/2}w(t))ds - 2\delta(g, A^{1/2}w)
\]
\[
= 2\delta \varepsilon'(t)(w_t, A^{1/2}w) - 2(f(u), A^{1/2}w_t) - 2\delta(f(u), A^{1/2}w),
\]
where
\[
2(\xi^t, \xi^\alpha)_{\mu, \lambda} \geq \rho\|\xi^t(s)\|_{\mu, \lambda}^2,
\]
\[
2\delta \int_0^\infty \mu(s)(A\xi^t(s), A^{1/2}w(t))ds \leq \frac{\rho}{2}\|\xi^t(s)\|_{\mu, \lambda}^2 + \frac{2\delta^2g_0}{\rho}\|w\|_2^2.
\]
Taking $\delta < \rho$ small enough, we get
\[
\frac{d}{dt}E_1(t) + \delta E_1(t) \leq -2(f(u), A^{1/2}w_t) - 2\delta(f(u), A^{1/2}w) + \delta C.
\]
Then, exploiting $\sigma = \frac{\delta}{4}$ in Lemma 3.5, (3.19), and the embeddings $H^{(5p-10)/2p} \hookrightarrow L^p$ ($p \geq 2$), $H^1 \hookrightarrow L^{\frac{4}{3}}$, $H^2 \hookrightarrow L^20$ there holds
\[
\|f(u)\|_1 \leq C\left(\int_\Omega (1 + |u|^{\frac{n+2}{n-2}})^2 \cdot |A^{1/4}u|^{2}dx \right)^{\frac{1}{2}} \leq C(1 + \|u\|_{L^{\frac{4}{3}}}^{\frac{n}{n-2}}) \cdot \|A^{1/4}u\|_{L^4} \leq C\|u\|_{L^{\frac{4}{3}}} \|u\|_2 \leq C.
\]
So
\[
-2(f(u), A^{1/2}w_t) - 2\delta(f(u), A^{1/2}w) \leq 2\|f(u)\|_1(\|w_t\|_1 + \|w\|_1) \leq \frac{\delta}{2}E_1 + C,
\]
where $C$ depends on $\delta$, $L$. We finally end up with
\[
\frac{d}{dt}E_1(t) + \frac{\delta}{2}E_1(t) \leq C,
\]
and an application of the Gronwall lemma, recalling (3.28) we can get the uniform boundedness of $\|K_1(t, \tau)z(\tau)\|_{H^1_t}$.

\textbf{Proof of Theorem 3.9.} Let
\[
K^1_t = \{z \in H^1_t : \|z\|_{H^1_t} \leq M_1\}.
\]
From (3.27) and Lemma 3.10, for all $t \in \mathbb{R}$, it yields
\[
\lim_{\tau \to -\infty} \delta_t(U(t, \tau)A_t, K^1_t) = 0.
\]
Since $\mathfrak{A}$ is invariant, this means
\[
\delta_t(A_t, K^1_t) = 0.
\]
Hence, $A_t \subset K^1_t = K^1_t$, proving that $A_t$ is bounded in $H^1_t$ with a bound independent of $t \in \mathbb{R}$.

Now from Lemma 3.3 and Theorem 3.9, the following conclusion is obtained immediately:
Lemma 3.11. For any $\tau \in \mathbb{R}, u \in A_t$, there exists a positive constant $C$ such that
\[ \sup_{t \geq \tau} \|u(t)\|_3^2 + \varepsilon(t) \|u_t\|_1^2 + \|\eta^t\|_{\mu,3}^2 + \int_\tau^\infty \|u_t(y)\|_2^2 \, dy \leq C. \] (3.29)

4. Asymptotic structure of the time-dependent attractor. We now investigate the relationship between the time-dependent global attractor of $U(t, \tau)$ and the global attractor of the limit equation (4.1) formally corresponding to (1.1) when $t \to +\infty$.

4.1. Attractor of fourth order parabolic equation with linear memory.
If $\varepsilon(t) \equiv 0$ in (2.2) we obtain the following system
\[ \begin{cases} \alpha u_t + \Delta^2 u + \int_0^\infty \mu(s) \Delta^2 \eta^t(s) \, ds + f(u) = g(x), & x \in \Omega, \ t \geq 0, \\ \eta^t_t + \eta^t_s = u_t, \end{cases} \] (4.1)
where $\Omega$ is an open bounded set of $\mathbb{R}^n(n \geq 5)$ with smooth boundary $\partial \Omega$, and with boundary and initial conditions
\[ \begin{cases} u|_{\partial \Omega} = \frac{\partial u}{\partial n}|_{\partial \Omega} = 0, \\ \eta^t|_{\partial \Omega} = \frac{\partial \eta}{\partial n}|_{\partial \Omega} = 0, \\ u(x, \tau) = u_0(x), \ \eta^t(x, s) = \eta^0(x, s). \end{cases} \]

Then under the conditions (1.4)-(1.7), and applying the standard semigroup theory [19, 23] and a priori estimates (4.2), we can obtain that the problem (4.1) has a unique solution $(u, \eta^t)$. Therefore, we can define the semigroup $\{S(t)\}_{t \geq 0}$ acting on the space $H^2 \times L^2_\mu(\mathbb{R}^+, H^2)$ associated with the problem (4.1), such that $(u(t), \eta^t(\cdot)) = S(t)(u_0, \eta^0)$, where $(u_0, \eta^0)$ is the initial data of (4.1).

Theorem 4.1. If (1.4)-(1.7) hold, then the semigroup $\{S(t)\}_{t \geq 0}$ associated with problem (4.1) is asymptotically compact in $H^2 \times L^2_\mu(\mathbb{R}^+, H^2)$. As a result, the semigroup $\{S(t)\}_{t \geq 0}$ has a global attractor $A_\infty$ in $H^2 \times L^2_\mu(\mathbb{R}^+, H^2)$.

To get the results of Theorem 4.1, we need the following estimates:

Lemma 4.2. For any $z_0 = \{u_0, \eta^0\} \in H^2 \times L^2_\mu(\mathbb{R}^+, H^2)$, there exists a positive constant $C$ depending on $z_0$ such that
\[ \sup_{t \geq 0} \|S(t)z_0\|_{H^2 \times L^2_\mu(\mathbb{R}^+, H^2)} \leq C(\|z_0\|_{H^2 \times L^2_\mu(\mathbb{R}^+, H^2)}). \] (4.2)

Proof. Taking the inner product of the first equation of (4.1) with $u_t$ in $L^2(\Omega)$, we get
\[ \frac{d}{dt} \mathcal{L}(u(t), \eta^t) + 2\alpha \|u_t\|^2 + 2 \langle \eta^t(s), \eta^t_s(s) \rangle_{\mu,2} = 0, \] (4.3)
where
\[ \mathcal{L}(u(t), \eta^t) = \|\Delta u\|^2 + \|\eta^t\|^2_{\mu,2} + 2 \langle F(u, 1), u \rangle. \]

From (2.4) and Hölder, Young inequalities, for $c_1', c_2'$, we have
\[ \frac{\nu}{2} \|S(t)z_0\|_{H^2 \times L^2_\mu(\mathbb{R}^+, H^2)} \leq \mathcal{L}(u(t), \eta^t) \leq c_1' \|S(t)z_0\|^2_{H^2 \times L^2_\mu(\mathbb{R}^+, H^2)} + c_2'. \] (4.4)

Integrating (4.3) on $[0, t]$ and combining with (4.4), we conclude
\[ \|S(t)z_0\|_{H^2 \times L^2_\mu(\mathbb{R}^+, H^2)} + \frac{1}{\varepsilon} \int_0^\infty \|u_t(y)\|^2 \, dy \leq C(\|z_0\|_{H^2 \times L^2_\mu(\mathbb{R}^+, H^2)}), \]
where $\bar{c}$ depends on $\alpha, \nu$. \qed
Now we prove the following results from borrowing the method introduced in [23].

**Lemma 4.3.** Let \( \{z_m\}_{m=1}^{\infty} \) be weakly convergent to \( z_0 \) in \( H^2 \times L^2_{\mu}(\mathbb{R}^+, H^2) \). Then

\[
\lim_{T \to +\infty} \lim_{m \to +\infty} \|S(T)z_m - S(T)z_0\|_{H^2 \times L^2_{\mu}(\mathbb{R}^+, H^2)} = 0.
\]

**Proof.** There exists bounded subset \( B \subset H^2 \times L^2_{\mu}(\mathbb{R}^+, H^2) \), such that for any \( m \), \( \{z_m\} \subset B \). Let \( S(t)z_m = \{u^m(t), (\eta^m)^t\} \) be the solution of problem (4.1) with initial data \( z_m = \{u_0^m, \eta_0^m\} \) and \( S(t)z_0 = \{u(t), \eta^s\} \) be the solution of initial data \( z_0 = \{u_0, \eta_0\} \). From Lemma 4.2 we have

\[
\sup_{t \geq 0, m \geq 0} \|S(t)z_m\|_{H^2 \times L^2_{\mu}(\mathbb{R}^+, H^2)} \leq M_2,
\]

for some constant \( M_2 \) depending on \( B \). Taking the inner product of (4.1) with \( u_t + \delta u \), we obtain

\[
\frac{d}{dt}[\mathcal{L}(u(t), \eta^t) + \delta \alpha \|u\|^2] + 2(\delta - \frac{\delta^2 m_0}{\rho}) \|\Delta u\|^2 + \frac{\rho}{2} \|\eta^t\|^2_{\mu, 2} + 2\alpha \|u_t\|^2 + 2\delta (f(u), u) - 2\delta (g, u) \leq 0.
\]

Take \( \delta \) small enough, and define

\[
\mathcal{L}_0(u(t), \eta^t) = \|\Delta u\|^2 + \|\eta^t\|^2_{\mu, 2}.
\]

Integrating (4.5) over \([0, T]\), combining with (4.4) yields

\[
\left| \int_0^T [\mathcal{L}_0(u(t), \eta^t) + 2(f(u), u) - 2(g, u)] dt \right| \leq \frac{C_0^0}{\delta},
\]

where \( C_0^0 \) depends on \( \|z_0\|_{H^2 \times L^2_{\mu}(\mathbb{R}^+, H^2)} \). Similarly to (4.6), we also have

\[
\left| \int_0^T [\mathcal{L}_0(u^m(t), (\eta^m)^t) + 2(f(u^m), u^m) - 2(g, u^m)] dt \right| \leq \frac{C_1^1}{\delta},
\]

where \( C_1^1 \) depends on \( \|z_0\|_{H^2 \times L^2_{\mu}(\mathbb{R}^+, H^2)} \).

Again, integrating (4.3) over \([s, T]\) we conclude

\[
\mathcal{L}(u(T), \eta^T) + 2\alpha \int_s^T \|u_t(t)\|^2 dt + 2\int_s^T \langle \eta^t(\xi), \eta^t(\xi) \rangle_{\mu, 2} dt = \mathcal{L}(u(s), \eta^s(\xi)),
\]

integrating the above equality over \([0, T]\) it leads to

\[
T \mathcal{L}(u(T), \eta^T) + 2\alpha \int_0^T \|u_t(t)\|^2 dt ds + 2\int_0^T \int_s^T \langle \eta^t(\xi), \eta^t(\xi) \rangle_{\mu, 2} dt ds = \int_0^T \mathcal{L}(u(s), \eta^s(\xi)) ds.
\]
Thus combining with (4.6), there holds
\[
\mathcal{L}_0(u(T), \eta^T) + 2\langle F(u(T)), 1 \rangle - 2\langle g, u(T) \rangle + \frac{2\alpha}{T} \int_0^T \int_s^T \|u_t\|^2 dt ds
\]
\[
+ \frac{2}{T} \int_0^T \int_s^T \langle \eta_t^2(\xi), \eta_t^2(\xi) \rangle_{\mu, 2} dt ds
\]
\[
= \frac{1}{T} \int_0^T [\mathcal{L}_0(u(s), \eta^s(\xi)) + 2\langle F(u(s)), 1 \rangle - 2\langle g, u(s) \rangle] ds
\]
\[
\geq \frac{1}{T} \int_0^T [-2\langle f(u(s)), u(s) \rangle + 2\langle F(u(s)), 1 \rangle] ds - \frac{C_0'}{\delta T}.
\]

The same way for \{u^m, \eta^m\} implies
\[
\mathcal{L}_0(u^m(T), (\eta^m)^T) + 2\langle F(u^m(T)), 1 \rangle - 2\langle g, u^m(T) \rangle + \frac{2\alpha}{T} \int_0^T \int_s^T \|u_t^m\|^2 dt ds
\]
\[
+ \frac{2}{T} \int_0^T \int_s^T \langle ((\eta^m)^t(\xi), (\eta^m)^t(\xi)) \rangle_{\mu, 2} dt ds
\]
\[
= \frac{1}{T} \int_0^T [\mathcal{L}_0(u^m(s), (\eta^m)^s(\xi)) + 2\langle F(u^m(s)), 1 \rangle - 2\langle g, u^m(s) \rangle] ds
\]
\[
\leq \frac{1}{T} \int_0^T [-2\langle f(u^m(s)), u^m(s) \rangle + 2\langle F(u^m(s)), 1 \rangle] ds + \frac{C_0'}{\delta T}.
\]

From (4.7)–(4.8), it follows that
\[
\mathcal{L}_0(u^m(T), (\eta^m)^T) + 2\langle F(u^m(T)), 1 \rangle - 2\langle g, u^m(T) \rangle + \frac{2\alpha}{T} \int_0^T \int_s^T \|u_t^m\|^2 dt ds
\]
\[
+ \frac{2}{T} \int_0^T \int_s^T \langle ((\eta^m)^t(\xi), (\eta^m)^t(\xi)) \rangle_{\mu, 2} dt ds
\]
\[
\leq \frac{1}{T} \int_0^T \{2\langle f(u), u \rangle - \langle f(u^m), u^m \rangle \} - 2\langle (F(u(s)), 1) - (F(u^m(s)), 1) \rangle \} \}
\]
\[
+ \mathcal{L}_0(u(T), \eta^T) + 2\langle F(u(T)), 1 \rangle - 2\langle g, u(T) \rangle + \frac{2\alpha}{T} \int_0^T \int_s^T \|u_t\|^2 dt ds
\]
\[
+ \frac{2}{T} \int_0^T \int_s^T \langle \eta_t^2(\xi), \eta_t^2(\xi) \rangle_{\mu, 2} dt ds + \frac{C_0' + C_1'}{\delta T}.
\]

For every \(2 \leq q < \frac{2n}{n-1}\), we know that \(H^2 \hookrightarrow L^q(\Omega)\). Thanks to \(u^m \rightarrow u(t)\) weakly in \(H^2(\Omega)\) ([4, Lemma 1]), we have
\[
\lim_{m \to +\infty} \int_{\Omega} |u(T) - u^m(T)|^q dx = 0.
\]

Further, it follows from (2.4)–(2.5), (4.10), and \(g \in L^2(\Omega)\), Lemma 4.2, we infer that
\[
\lim_{m \to +\infty} \frac{1}{T} \int_0^T \langle F(u^m(t)), 1 \rangle dt = \frac{1}{T} \int_0^T \langle F(u(t)), 1 \rangle dt,
\]
\[
\lim_{m \to +\infty} \frac{1}{T} \int_0^T \langle f(u^m), u^m \rangle dt = \frac{1}{T} \int_0^T \langle f(u), u \rangle dt,
\]
\[
\lim_{m \to +\infty} \frac{1}{T} \int_{\Omega} \langle g(x), u^m(T) \rangle dx = \frac{1}{T} \int_{\Omega} \langle g(x), u(T) \rangle dx.
\]
Also, we know that \( u^n \) is convergent weakly to \( u_t(t) \) in \( L^2 \) for a.e. \( t \in [0, T] \), see [20, Theorem 2.1], and \( \eta^I \in C([0,T]; L^2_{\mu}(\mathbb{R}^+, \mathbb{H}^2)) \), so due to Fatou Lemma and weakly lower-semicontinuity of norm, it yield

\[
\liminf_{m \to +\infty} \frac{1}{T} \int_0^T \int_s^T \|u^n(t)\|^2 dt ds \geq \frac{1}{T} \int_0^T \int_s^T \|u(t)\|^2 dt ds. \tag{4.14}
\]

\[
\liminf_{m \to +\infty} \frac{2}{T} \int_0^T \int_s^T ((\eta^m)^I(\xi) - \eta^I(\xi), (\eta^m)^I(\xi) - \eta^I(\xi)) d\mu d\xi dt ds \geq \frac{\rho}{T} \int_0^T \int_s^T \|\eta^m - \eta^I\|^2 d\mu d\xi dt ds. \tag{4.15}
\]

Taking (4.10)-(4.15) into account and passing the limit in (4.9), we obtain for arbitrary \( \epsilon > 0 \),

\[
\limsup_{m \to +\infty} \mathcal{L}_0(u^m(T), (\eta^m)^I) \leq \mathcal{L}_0(u(T), \eta^I) + \epsilon. \tag{4.16}
\]

From Lemma 4.2 and [4, Lemma 1], we achieve

\[
\limsup_{m \to +\infty} \|S(T)z_m - S(T)z_0\|_{L^2} = \limsup_{m \to +\infty} \mathcal{L}_0(u^m(T) - u(T), (\eta^m)^I - \eta^I) = \limsup_{m \to +\infty} \left[ \int_{\Omega} |\Delta u^m(T)|^2 dx - 2 \int_{\Omega} \Delta u^m(T) \Delta u(T) dx + \int_{\Omega} |\Delta u(T)|^2 dx \right. \]

\[
\left. + \| (\eta^m)^I \|_{H^2}^2 - 2 ( (\eta^m)^I, \eta^I )_{H^2} + \| \eta^I \|_{H^2}^2 \right] = \limsup_{m \to +\infty} \mathcal{L}_0(u^m(T), (\eta^m)^I) - \mathcal{L}_0(u(T), \eta^I), \tag{4.17}
\]

which together with (4.16) the proof is finished. \( \square \)

**Proof of Theorem 4.1.** Similar to Lemma 3.7, 3.8, the compactness of the memory term in (4.1) is checked, then combining with Lemma 4.3, we know that \( S(t) \) is asymptotically compact ([9, 23]), so the semigroup \( S(t) \) possesses a global attractor \( A_{\infty} \) in \( H^2 \times L^2_{\mu}(\mathbb{R}^+, \mathbb{H}^2) \).

\( \square \)

### 4.2. Asymptotic structure.

From Theorem 4.1, we know that the semigroup \( S(t) \) associated with (4.1) has a unique attractor \( A_{\infty} \) in \( H^2 \times L^2_{\mu}(\mathbb{R}^+, \mathbb{H}^2) \), moreover, according to Theorem 2.8, for all \( s \in \mathbb{R} \), we have

\[
A_{\infty} = \{ w(s) : w \ is \ the \ CBT \ of \ S(t) \}.
\]

The main result of this part establishes the asymptotic closeness of the time-dependent global attractor \( \mathfrak{A} = \{ A_t \}_{t \in \mathbb{R}} \) of the process generated by (2.2)-(2.3) and the global attractor \( A_{\infty} \) of the semigroup \( S(t) \) generated by (4.1).

**Theorem 4.4.** The following limit holds:

\[
\lim_{t \to \infty} \delta_{H^2 \times L^2_{\mu}(\mathbb{R}^+, \mathbb{H}^2)}(\Pi_t A_t, A_{\infty}) = 0, \tag{4.18}
\]

where \( \Pi_t A_t \) denotes the projection of \( A_t \) into it’s first component, that is,

\[
\Pi_t A_t = \{ \zeta \in H^2 : (\zeta, v) \in A_t \}.
\]

To prove (4.18), we need to verify the following result:

**Lemma 4.5.** For any sequence \( z_n = (u_n, \partial_t u_n, \eta^n_{\mu}) \) of CBT of the process \( U(t, \tau) \) and any \( t_n \to \infty \), there exists a CBT \( y = (w, \xi^I) \) of the semigroup \( S(t) \) such that, for every \( T > 0 \),

\[
\sup_{t \in [-T, T]} \|u_n(t + t_n) - w(t)\|_2 \to 0. \tag{4.19}
\]
and
\[
\sup_{t \in [-T, T]} \| \eta_n^t(s + t_n) - \xi^t(s) \|_{\mu_2} \to 0, \quad (4.20)
\]
as \( n \to \infty \), up to a subsequence.

**Proof.** Let \( z_n = (u_n, \partial_t u_n, \eta_n^t) \) be a CBT of \( U(t, \tau) \) and \( t_n \to +\infty \) be given, owing to (3.29), for every \( T > 0 \),

\[
u(\cdot + t_n) \text{ is bounded in } L^\infty(-T, T; H^3) \cap W_2^{2,2}(-T, T; H).
\]

Then a direct application of [22, corollary 4] shows that \( u_n(\cdot + t_n) \) is relatively compact in \( C([-T, T], H^2) \), for every \( T > 0 \). By Lemma 3.7, the sequence \( \eta_n^t(\cdot + t_n) \) in the space

\[L^\infty(-T, T; L^2_\mu(\mathbb{R}^+, H^{\sigma+2}) \cap H^1_\mu(\mathbb{R}^+, H^2)),\]

is bounded, so \( \eta_n^t(\cdot + t_n) \) is relatively compact in \( C([-T, T], L^2_\mu(\mathbb{R}^+, H^2)) \). Hence there exists a function

\[y : H^2 \times L^2_\mu(\mathbb{R}^+, H^2) \to H^2 \times L^2_\mu(\mathbb{R}^+, H^2)\]
such that for sequence \( u_n, \eta_n^t \), the convergence

\[u_n(\cdot + t_n) \to w(\cdot), \quad \eta_n^t(\cdot + t_n) \to \xi^t(s)\]
hold, and \( y \in H^2 \times L^2_\mu(\mathbb{R}^+, H^2) \). Besides, recalling (3.29),

\[
\sup_{t \in \mathbb{R}} \| y(t) \|_{H^2 \times L^2_\mu(\mathbb{R}^+, H^2)} \leq c. \quad (4.21)
\]

We are left to show that \( y \) solves (4.1). We define

\[v_n(t) = u_n(t + t_n), \quad \varepsilon_n(t) = \varepsilon(t + t_n), \quad \xi_n^t(t) = \eta_n^t(t + t_n),\]

then we write Eq. (2.2) for \( u_n \) in the form

\[\alpha \partial_t v_n = -\varepsilon_n(t) \partial_t v_n - \Delta^2 v_n - \int_0^\infty \mu(s) \Delta^2 \xi_n^t \, ds - f(v_n) + g.\]

We first prove that the sequence \( \varepsilon_n(t) \partial_t v_n \) converges to zero in the distributional sense. Indeed, for every fixed \( T > 0 \) and every smooth \( H \)-valued function \( \varphi \), supported on \((-T, T)\), we have

\[
\int_{-T}^T \varepsilon_n(t) (\partial_t v_n, \varphi(t)) \, dt = -\int_{-T}^T \varepsilon'_n(t) (\partial_t v_n, \varphi(t)) \, dt - \int_{-T}^T \varepsilon_n(t) (\partial_t v_n(t), \partial_t \varphi(t)) \, dt.
\]

Exploiting again (3.29), we find

\[
\begin{align*}
| \int_{-T}^T \varepsilon_n(t) (\partial_t v_n, \varphi(t)) \, dt | & \\
& \leq c \int_{-T}^T \sqrt{\varepsilon_n(t)} \sqrt{\varepsilon_n(t)} \| \partial_t v_n \|_1 dt + c \int_{-T}^T \frac{| \varepsilon'_n(t) |}{\sqrt{\varepsilon_n(t)}} \sqrt{\varepsilon_n(t)} \| \partial_t v_n \|_1 dt \\
& \leq c \int_{-T}^T \varepsilon_n(t) dt + c \int_{-T}^T \frac{| \varepsilon'_n(t) |}{\sqrt{\varepsilon_n(t)}} dt \\
& \leq cT \sup_{t \in [-T, T]} \sqrt{\varepsilon_n(t)} + c(\sqrt{\varepsilon_n(T)} - \sqrt{\varepsilon_n(-T)}),
\end{align*}
\]

where the generic constant \( c > 0 \) depends also on \( \varphi \). Since

\[
\lim_{n \to \infty} \left[ \sup_{t \in [-T, T]} \varepsilon_n(t) \right] = 0,
\]

we have

\[
\lim_{n \to \infty} \left[ \sup_{t \in [-T, T]} \varepsilon'_n(t) \right] = 0.
\]
we reach the desired conclusion
\[
\lim_{n \to \infty} \int_{-T}^{T} \varepsilon_n(t) \langle \partial_{tt} v_n, \varphi(t) \rangle dt = 0.
\]
Now, taking into account (1.4), for every \( T > 0 \), we have the convergence
\[-\Delta^2 v_n - f(v_n) \to -\Delta^2 w - f(w),\]
in the topology of \( L^\infty(-T, T; \mathcal{H}^{-2}) \). At the same time, the convergence
\[\partial_{tt} v_n(t) \to w_1(t), \quad \partial_t \xi_n \to \partial_t \xi^t,\]
hold in the distributional sense. Therefore, we end up with the equality
\[\alpha w_t + \Delta^2 w + f(w) + \int_0^\infty \mu(s) \Delta^2 \xi^t(s)ds = g(x),\]
which together with (4.21), proves that \( y(t) \) is a CBT of the semigroup \( S(t) \).  

5. **Further regularity.** In this section, we discuss further regularity of the attractor, within the additional assumption
\[
\beta := \inf_{t \in \mathbb{R}} [2\alpha + \varepsilon'(t)] > 0. \tag{5.1}
\]

Then, we prove that for \( \{u, u_\epsilon, \eta_\epsilon\} \in \mathfrak{A} \), \( u_t \) is bounded in \( \mathcal{H}^2 \) and the bound is independent of \( \varepsilon(t) \).

**Theorem 5.1.** Let \( \mathfrak{A} = \{A_t\}_{t \in \mathbb{R}} \) be the invariant global attractor. Then within the assumption of (5.1), for \( c = c(\mathfrak{A}) > 0 \), we have the following uniform estimate:
\[
\sup_{t \in \mathbb{R}} \sup_{\{u, u_\epsilon, \eta_\epsilon\} \in \mathfrak{A}} \|u_t(t)\|^2 + \varepsilon(t)\|u_{tt}(t)\|^2 + \|\eta_\epsilon\|_{\mathcal{H}^2}^2 \leq C.
\]

The proof of Theorem 5.1 is based on the following lemma.

**Lemma 5.2.** Let \( \{u, u_\epsilon, \eta_\epsilon\} \in \mathfrak{A} \). Then for every \( t \geq \tau \), the uniform bound
\[\|u_t(t)\|^2 + \varepsilon(t)\|u_{tt}(t)\|^2 + \|\eta_\epsilon\|_{\mathcal{H}^2}^2 \leq \frac{c}{\varepsilon^2(\tau)} e^{-\kappa(t-\tau)} + C\]
hold for some \( \kappa > 0 \), both depending only on \( \mathfrak{A} \).

**Proof.** Differentiating (2.2) in time, let \( q = u_t, \ \xi_\epsilon = \eta_\epsilon \), then we conclude that the following equation
\[\varepsilon(t)q_{tt} + (\alpha + \varepsilon'(t))q_t + \Delta^2 q + \int_0^\infty \mu(s) \Delta^2 \xi_\epsilon^t = -f'(u)u_t, \quad x \in \Omega, \ t \geq \tau, \tag{5.2}\]
where \( \xi_\epsilon^t = -\xi_\epsilon^s + q_t \), and with initial-boundary conditions
\[
\begin{cases}
q|_{\partial \Omega} = 0, \quad \frac{\partial q}{\partial n}|_{\partial \Omega} = 0, \quad \xi_\epsilon^t|_{\partial \Omega} = 0, \quad \frac{\partial \xi_\epsilon^t}{\partial n}|_{\partial \Omega} = 0, \\
q(\tau) = u_t(x), \quad q_t(\tau) = \frac{1}{\varepsilon(\tau)} [g - f(u_0) - \alpha u_t - \Delta \xi_\epsilon^t - \int_0^\infty \mu(s) \Delta^2 \xi_\epsilon^t ds], \\
\xi_\epsilon^\tau(x, s) = u_t(x, \tau) - u_1(x, \tau - s).
\end{cases}
\]

Since \( \{u_0, u_1, \eta_\epsilon^0\} \in A_\tau \), we have
\[\|q(\tau), q_t(\tau), \xi_\epsilon^\tau(s)\|_{\mathcal{H}^2} \leq \frac{c}{\varepsilon(\tau)}. \tag{5.3}\]

Besides, from (3.29) we know
\[\|f'(u)\|^2 \leq c \int_{\Omega} (1 + |u|^{4/3})^2 dx \leq c(1 + \|u\|_2^{3/2}) \leq c. \tag{5.4}\]
\[
\int_\tau^\infty \|f'(u)u_t\|^2 \, dy \leq \int_\tau^\infty \|f'(u)\|^2 \|u_t(y)\|^2 \, dy \leq c \int_\tau^\infty \|u_t(y)\|^2 \, dy \leq C.
\]

Multiplying (5.2) by \(2q_t + 2\delta q\) in \(L^2(\Omega)\), it leads to
\[
\frac{d}{dt} \Lambda'(t) + (2\alpha + \epsilon'(t) - \delta \epsilon(t))\|q_t\|^2 + 2\delta \|q\|^2 + 2\langle \xi^t, \xi_t^t \rangle_{\mu,2} + 2\delta \epsilon(t) q_t \|q\|_2^2,
\]
where
\[
\Lambda'(t) = \|q\|^2_2 + \epsilon(t)\|q_t\|^2 + \|\xi^t\|^2_{\mu,2} + 2\delta \epsilon(t) q_t \|q\|_2^2.
\]
In line with Hölder and Young inequality, there holds
\[
2\delta \epsilon(t) q_t \|q\| \leq \frac{\epsilon(t)}{2} \|q_t\|^2 + 2\delta^2 L \|q\|^2.
\]
Choosing \(\delta\) small enough, such that
\[
\frac{1}{2} \|(q(t), q_t(t), \xi^t(t))\|^2_{H_1} \leq \Lambda'(t) \leq \|(q(t), q_t(t), \xi^t(t))\|^2_{H_1} + 2C. \tag{5.5}
\]
From (3.5) we get
\[
\langle \xi^t, \xi_t^t \rangle_{\mu,2} \geq \frac{\rho}{2} \int_0^\infty \mu(s) \|\xi^t(s)\|^2 \, ds = \frac{\rho}{2} \|\xi^t(s)\|^2_{\mu,2},
\]
\[
2\|\xi^t(s)\|_{\mu,2} = 2 \int_0^\infty \mu(s) \|\Delta \xi^t(s), \Delta q(t)\| \, ds \leq 2 \int_0^\infty \mu(s) \|\Delta \xi^t(s)\| \|\Delta q(t)\| \, ds \leq \frac{\rho}{2\delta} \|\xi^t(s)\|^2_{\mu,2} + \frac{2\delta \mu_0}{\rho} \|q\|^2_2.
\]
Together with (5.4) again, we achieve
\[
\langle -f'(u)u_t, 2q_t + 2\delta q \rangle \leq 2\|f'(u)u_t\| (\|q_t\| + \delta \|q\|) \leq C\|f'(u)u_t\|^2 + \frac{\alpha}{2} \|q_t\|^2 + \delta \|q\|^2_2.
\]
Then taking \(\delta\) small enough, by (5.1), there exist \(\kappa < \delta\) such that
\[
\frac{d}{dt} \Lambda'(t) + \kappa \Lambda'(t) \leq C\|f'(u)u_t\|^2.
\]
Thus, exploiting the Gronwall’s lemma on \([\tau, t]\), it follows that
\[
\Lambda'(t) \leq \Lambda'(\tau)e^{-\kappa(t-\tau)} + C \int_\tau^t \|f'(u)u_t\|^2 \, dy, \quad \forall \ t \geq \tau,
\]
then together with (5.3) and (5.5) we get
\[
\|q\|^2_2 + \epsilon(t)\|q_t\|^2 + \|\xi^t\|^2_{\mu,2} \leq \frac{c}{\epsilon(t)^2} e^{-\kappa(t-\tau)} + C.
\]

The proof is completed. \(\square\)

**Proof of Theorem 5.1.** Let \(\{u, u_t, \eta^t\} \in \mathfrak{A}\). From Lemma 5.2 we know
\[
\|u_t\|^2_2 + \epsilon(t)\|u_t\|^2 + \|\eta_t^t\|^2_{\mu,2} \leq \frac{c}{\epsilon(t)^2} e^{-\kappa(t-\tau)} + C, \quad \forall \ t \geq \tau.
\]
Since \(\epsilon\) is bounded at \(-\infty\), the required estimate follows by taking the limit \(\tau \to -\infty\). \(\square\)

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