Engineering non-Abelian topological memories from Abelian lattice models

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We propose a novel fault-tolerant scheme for quantum computation based on the topological phase of a spin lattice model. Abelian anyons are used to simulate non-Abelian properties relevant to topological quantum computation. Universality is realized by using anyon braiding supplemented with non-topological operations, namely single spin measurements. Along with the topological nature of the encoding and a finite energy gap, logical states are protected from thermal excitations by a novel method that enforces the required degeneracies. A proof of principle scheme is proposed that can be realized in Josephson junctions with state of the art technology.

Introduction:—It has been theoretically demonstrated that quantum computation can perform tasks that are virtually impossible with classical computers. Physical realizations are presently hindered by environmental and control errors that do not allow demonstration of quantum computation beyond the classical limit [1]. Schemes based on topologically ordered systems promise to drastically resolve this problem by employing physical principles as well as algorithmic constructions. Their aim is to encode and manipulate information in a way that is intrinsically resilient to errors, thus allowing fault-tolerant quantum computation.

Topological quantum computation employs anyons, the quasiparticles of topological models [2, 3]. Non-Abelian anyons have complex behavior well suited to encoding and processing quantum information, but the simpler Abelian anyons are more experimentally accessible. Here we bridge the gap, by using well established control techniques to engineer non-Abelian properties from Abelian lattice models [4]. Our method utilizes symmetries of the Hamiltonian, which may be enforced by single spin interactions. These enable us to interpret certain states of Abelian anyons in terms of new quasiparticles. These states exhibit non-Abelian like behavior, while not constituting a full mapping to non-Abelian anyons. Using these new quasiparticles, quantum information can be stored non-locally in the analogues of fusion spaces, and manipulated using single spin measurements and adiabatic techniques. The addition of non-topological operations, namely measurements of single spins in the underlying lattice, leads to universal quantum computation [2, 3, 5, 6, 7, 8, 9]. State purification is employed to efficiently eliminate errors introduced by these manipulations. All employed operations, including measurements, always produce eigenstates of the Hamiltonian, allowing the energy gap to remain intact at all times and ensuring fault tolerance [10]. Though our method is general, we focus on the quantum double models [3], analytically tractable topological lattice models described by the stabilizer formalism [11]. Specifically we consider the $D(Z_6)$ model, which an be realized with Josephson junctions [12]. An experiment is proposed to demonstrate the quantum memory of our scheme, as well as our method to enforce symmetries.

The $D(Z_6)$ model.—The $D(Z_6)$ anyon model is defined on an oriented two-dimensional square lattice [3]. On each edge there resides a six-level spin spanned by the states $|g⟩_i$ where $g = 0, ..., 5$ labels an element of $Z_6$, the cyclic group of six elements. The generalized Pauli operators for these spins are $σ^x_i = ∑_g∈Z_6 |g⟩⟨g| + 1,i$ and $σ^z_i = ∑_g∈Z_6 e^{-iπg/3} |g⟩⟨g|$. We choose the orientations of our lattice to point upwards for each vertical link and right for each horizontal link. We can now define the following operators,

$$A(v) = σ^x_jσ^x_kσ^z_lσ^z_m, \quad B(p) = σ^z_jσ^x_kσ^x_lσ^z_m.$$  \hspace{1cm} (1)

The edges $j, k, l, m$ are those sharing the plaquette $p$ or vertex $v$. We take $j$ to be the edge to the top of the plaquette or vertex and the rest to proceed clockwise from this. The Hamiltonian of the model is given by,

$$H = -Δ \left( ∑_v P_1(v) + ∑_p P_1(p) \right),$$  \hspace{1cm} (2)

where $Δ > 0$. A state within the ground state space, $|gs⟩$, is then defined by the following projectors,

$$P_1(v) = \frac{1}{6} ∑_{g∈Z_6} A(v)^g, \quad P_1(p) = \frac{1}{6} ∑_{g∈Z_6} B(p)^g,$$

such that $P_1(v)|gs⟩ = P_1(p)|gs⟩ = |gs⟩$ for all $v$ and $p$. All $A(v)$ and $B(p)$ operators mutually commute, and so may be described as stabilizers of a stabilizer code [11]. The ground state space is then identified with the stabilizer space. The elementary excitations of $D(Z_6)$ are the anyons $e_g$ and $m_g$ for $g = 1, ..., 5$, and correspond to violations of the vertex and plaquette stabilizers, respectively. The absence of an anyon is referred to as the vacuum. The projectors for the anyon states are given by,

$$P_{e_g}(v) = \frac{1}{6} ∑_{h∈Z_6} e^{-iπgh/3} (A(v))^h,$$

$$P_{m_g}(p) = \frac{1}{6} ∑_{h∈Z_6} e^{-iπgh/3} (B(p))^h.$$  \hspace{1cm} (3)
When the system is in the state $|\psi\rangle$, a quasiparticle of type $a$ at vertex $v$ (plaquette $p$) is defined by $P_a(v) |\psi\rangle = |\psi\rangle$ ($P_a(p) |\psi\rangle = |\psi\rangle$). An important feature of the Hamiltonian (2) is that it assigns the same energy, $\Delta$, to all $e_g$ anyons, and also to all $m_g$ anyons. This symmetry is not a necessary for a valid $D(Z_6)$ Hamiltonian, but it is necessary for the scheme we propose. Later we present a simple method to protect this symmetry.

To encode quantum information in a protected way, we define new types of quasiparticle. We focus on the vertex excitations, while the plaquette excitations are similarly defined. Let us introduce the projectors,

$$
\begin{align*}
P_\lambda(v) &= P_{e_3}(v), \quad P_\phi(v) = P_{e_1}(v) + P_{e_4}(v) \\
P_{\bar{\phi}}(v) &= P_{e_2}(v) + P_{e_5}(v).
\end{align*}
$$

In terms of the anyons of $D(Z_6)$, the quasiparticle $\phi$ is an $e_1$ or $e_4$ and its antiparticle $\bar{\phi}$ is an $e_2$ or $e_5$. The quasiparticle $\lambda$ is directly identified with $e_3$. The fact that the Hamiltonian assigns the same energy to each of these anyons allows arbitrary states of these quasiparticles to be eigenstates. Measuring the stabilizer code with the projectors in (4) rather than (3) extracts less information about the type of anyon present. This is equivalent to 'holes' in the code, where stabilizers are not enforced, except that the effective holes are now carried by each $(\phi, \bar{\phi})$ pair. Our scheme uses these to store quantum information.

States with $(\phi, \bar{\phi})$ or $\lambda$ pairs on vertices connected by a single edge, $i$, can be created by acting on the ground state with the operators:

$$
W_i^\phi = \frac{1}{2} \sigma_i^z \left( \mathbb{1} + (\sigma_i^z)^3 \right), \quad W_i^\lambda = (\sigma_i^z)^3,
$$

respectively. The projection $(\mathbb{1} + (\sigma_i^z)^3)/2$ present in $W_i^\phi$ may be performed deterministically by measuring the observable $(\sigma_i^z)^3$ and applying $(A(v))^3$ to a neighboring vertex if the $-1$ eigenvalue is obtained. By considering the action of the creation operators on two edges sharing a vertex, one obtains the fusion rules,

$$
\begin{align*}
\phi \times \phi &= 1 + \lambda, \quad \phi \times \lambda = \phi, \\
\phi \times \bar{\phi} &= \phi, \quad \phi \times \phi &= \phi, \\
\bar{\phi} \times \phi &= \phi, \quad \bar{\phi} \times \bar{\phi} &= \lambda, \\
\bar{\phi} \times \bar{\phi} &= 1 + \lambda.
\end{align*}
$$

In other words, if both a $\phi$ and $\bar{\phi}$ are moved into the same vertex $v$, their state can be stabilized by either $P_1(v)$ or $P_3(v)$. Note that $W_i^\phi W_i^\lambda = W_i^\phi$, ensuring that the fusion of a $\lambda$ with a $\phi$ cannot be locally distinguished from a single $\phi$. This provides a non-trivial fusion space where information can be encoded in a non-local way. The $\lambda$ particles can be transported using chains of $W_i^\lambda$, whereas $\phi$ and $\bar{\phi}$ require controlled operations $[4, 13]$ such as,

$$
C_i = \sum_{g,h \in Z_6} P_{e_1+h}(v)P_{e_3}(v') (\sigma_i^z)^g,
$$

which moves the particle coherently from vertex $v$ to the neighboring vertex $v'$ through the edge $i$. Alternatively, our quasiparticles are well suited to be moved adiabatically using local potentials [9], without affecting the degeneracy of the logical states.

Quasiparticles $\chi, \lambda$ and $\mu$ on plaquettes can be defined equivalently to $\phi, \bar{\phi}$ and $\lambda$, respectively. The corresponding projectors $P_\chi(p)$, $P_\lambda(p)$ and $P_\mu(p)$ and the creation operators $W_i^\chi$ and $W_i^\mu$ are obtained from (4), (5) and (7) using the substitutions $A(v) \rightarrow B(p)$ and $\sigma_i^z \rightarrow \sigma_i^z$. The braiding of the quasiparticles can be determined from the constituent $e_g$ and $m_g$ anyons. For example, a $\mu$ around a $\lambda$ gives the statistical phase $e^{i \pi}$ due to their identification with $m_3$ and $e_3$, respectively.

**Quantum computation:** Employing the new quasiparticle states, we define a non-local encoding equivalent to quantum computational schemes with non-Abelian anyons [2, 3]. Consider the following operations on the ground state (see Fig.1(a) for enumeration): (i) The application of $W_i^\phi W_i^\phi$. (ii) The application of $W_i^\lambda$. Applying operation (i) creates two $(\phi, \bar{\phi})$ pairs at vertices $v_2$, $v_4$, and $v_1$, $v_3$. The fusion of either pair would result in the vacuum. Applying both (i) and (ii) also creates two $(\phi, \bar{\phi})$ pairs, but in this case they would both fuse to a $\lambda$. These states can be used to encode a $v$-type logical qubit, with basis states $|0_v\rangle$ and $|1_v\rangle$ associated with the vacuum and $\lambda$ fusion channels, respectively. In terms of spin operators acting on the ground state they can be written as,

$$
\begin{align*}
|0_v\rangle &= \frac{1}{4} \sigma_3^x \sigma_3^z \left[ \mathbb{1} + (\sigma_3^z)^3 \right] \left[ \mathbb{1} + (\sigma_3^z)^3 \right] \left[ \mathbb{1} + (\sigma_3^z)^3 \right] \left[ \mathbb{1} + (\sigma_3^z)^3 \right] |gs\rangle, \\
|1_v\rangle &= \frac{1}{4} (\sigma_3^z)^3 \sigma_3^x \sigma_3^z \left[ \mathbb{1} + (\sigma_3^z)^3 \right] \left[ \mathbb{1} + (\sigma_3^z)^3 \right] |gs\rangle.
\end{align*}
$$

No local operator (4-local such as $A(v)$ or smaller) can distinguish between the two states. The fusion channel of a $(\phi, \bar{\phi})$ pair is a non-local property, and can only be detected by observables acting on both particles of a pair. The logical qubit operations are given by $X = \ldots$
(\sigma_1^z)^3 or (\sigma_2^z)^3, and \( Z = A^3(v_1)A^3(v_3) \) or \( A^3(v_2)A^3(v_4) \). Consider keeping the \( \phi \) and \( \phi' \) of each pair together, but using \( 7 \) to move the two pairs away from each other. The \( X \) operations become products of \((\sigma_1^z)^3\)’s along strings connecting the targeted vertices, while the \( Z \) operation remains the same. The two possible forms of both \( X \) and \( Z \) allow two possible means to measure in each basis. These should both give the same result, with any difference being a signature of errors. The fusion channels of \( \chi \)'s and \( \chi' \)'s may similarly be used to encode \( p \)-type qubits on corresponding plaquettes \( p_1 \ldots p_4 \) with logical states \(|0_p\rangle \) and \(|1_p\rangle \). The \( X \) and \( Z \) rotations for the \( p \)-qubits are obtained using the substitutions above. The eigenstates of the \( X \) basis may be denoted as \(|\pm_{v/p}\rangle \).

The controlled-\( Z \) gate can be applied between a \( v \)-type and a \( p \)-type qubit by moving the \((\phi, \tilde{\phi})\) pair on \( v_1, v_3 \) around the \((\chi, \tilde{\chi})\) pair on \( p_1, p_3 \). This is due to the statistical phase \( e^{i\pi} \) obtained when moving a \( \lambda \) around a \( \mu \). Arbitrary single qubits rotations can be performed by employing suitable logical ancillary states. To do this we initially prepare the logical state \( |0_v\rangle \). The projector \( \Pi_0^v = \frac{1}{2} |1\rangle \langle 1| + \cos \theta (\sigma_1^z)^2 + i \sin \theta (\sigma_1^z)^3(\sigma_1^z)^3 \) is then applied to the spin on edge 1. This can be performed probabilistically, by a measurement. In general, \( \Pi_0^v \) creates a superposition of a pair of \( \lambda \)'s and \( \mu \)'s on the vertices and plaquettes sharing the spin on edge 1. The \( \lambda \)'s will fuse with the \( \phi \)'s, and the \( \mu \)'s may be measured. If a pair of \( \mu \)'s is detected, they can be annihilated by applying \((\sigma_1^z)^3\). This leaves the system in the logical ancillary state \(|a^v_0\rangle = (\cos \theta |0_v\rangle - i \sin \theta |1_v\rangle)/\sqrt{2} \). If no \( \mu \)'s are detected, the protocol can be repeated until successful. The ancillary plaquette state \(|a^p_0\rangle \) can be prepared similarly. By utilizing these states, arbitrary single qubit rotations and controlled-\( X \) gates can be performed using the circuits shown in Fig. 2. During the preparation stage, the ancilla states are stored on neighboring plaquettes or vertices, making them vulnerable to \( X \) errors and affecting the fault-tolerance of our scheme. However, note that all single qubit rotations can be constructed from \( e^{i(\pi/8)Z} \) and \( e^{i(\pi/8)X} \) \[14\], whose implementation requires only preparation of \(|a^{\pi/8}_v\rangle, |a^{\pi/8}_p\rangle \) and \(|\pm_{p/v}\rangle \).

Fault-tolerance:- We consider errors that do not excite the system, requiring a temperature low enough for topological order to be stable \[16, 17\]. The errors can then be considered as perturbations in the Hamiltonian, due to imprecise tuning of the system or coupling with the environment.

The Hamiltonian \[2\] can be expressed as an equally weighted sum of the stabilizers \( A(v) \) and \( B(p) \). However, physical systems will likely produce perturbed Hamiltonians, lifting the degeneracy of the anyons and breaking the symmetries our scheme requires. This is a problem that not only affects realizations of \( D(Z_0) \), but all quantum double models, Abelian or non-Abelian \[2, 5, 13\]. Here we present a method to enforce the symmetries in \( D(Z_0) \), but the principle applies in general. For the states of \((\phi, \bar{\phi})\) pairs to be eigenstates of the Hamiltonian we require both the \( e_1 \) and \( e_4 \) anyon states and the \( e_2 \) and \( e_5 \) states to be degenerate. To ensure this, note that the operator \( W_0^v \) involves a projection \( (1 + (\sigma_1^z)^3)/2 \). This causes the state of the pair to become an eigenstate of the operator \((\sigma_1^z)^3\). Since this operation creates \( e_3 \) anyons, and so performs the mapping \( e_1 \rightarrow e_4 \) and \( e_2 \rightarrow e_5 \), it does not commute with any perturbation that lifts the required degeneracies. Application of a single spin magnetic field term \( B(\sigma_1^z)^3 \) then suppresses a perturbation of strength \( \delta \), such as \( \delta A^v(v) \), by \((\delta/B)^2 \). Moving the quasiparticles apart means that the magnetic field is no longer a single \((\sigma_1^z)^3\) but a product along a chain between them. An equivalent method for the \((\chi, \tilde{\chi})\) pairs can be obtained with \( B(\sigma_1^z)^3 \) terms.

We now consider perturbations that do not come from fine tuning, such as those acting on spins forming strings across the lattice. If one end of a string connects with a \( \phi \) or \( \bar{\phi} \), it may move the quasiparticle. The simplest examples are single spin perturbations \((\sigma_1^z)^{\alpha} \) acting on a spin surrounding a \( \phi \) or \( \bar{\phi} \). Lowering the coupling of the vortex operators where these quasiparticles are placed from \( \Delta \rightarrow \Delta - \alpha \) will suppress this process by an effective gap of size \( \alpha \). If string-like perturbation stretches between the two \((\phi, \bar{\phi})\) pairs, it can distinguish the logical states of the \( X \) basis and so lift the degeneracy. If the environment can produce string-like \( k \)-local perturbations in the Hamiltonian then the \((\phi, \bar{\phi})\) pairs should be moved \( k + 1 \) spins apart, to protect the encoded qubit. Perturbations may also act on spins that form loops. If these loop around a single \( \phi \) or \( \bar{\phi} \), they may cause errors on

![FIG. 2: The circuits that implement (a) \( e^{\pm i\theta Z} \) rotations and (b) \( e^{\pm i\theta X} \) rotations. The sign depends on the outcome of the measurements and can be corrected by subsequent rotations. In (c) the controlled-\( X \) gates are depicted on \( v \)-type qubits. The \( X \) measurements here may be realized by single spin measurements \((\sigma_1^z)^3 \) for \( v \)-type qubits and \((\sigma_1^z)^3 \) for \( p \)-type.](image-url)
the results of $X$ measurements. These will be suppressed by the magnetic field $(\sigma^z)^3$ as long as they do not loop around both the $\phi$ and $\bar{\phi}$ of a pair. Such errors become highly correlated, and so increasingly difficult for nature to produce, as these particles are moved apart. Such errors may also be dealt with by encoding each qubit on $2N$ pairs rather than just $2$. Only perturbations acting on spins forming loops around $N$ quasiparticles can then cause uncorrectable errors.

Josephson junction realization:- We now present a means to experimentally demonstrate the quantum memory of our scheme, including its resilience to perturbations breaking the required symmetries. Consider the Josephson junction element in Fig. 3(a). A flux $2\pi/6$ passes through each loop, creating six degenerate ground states that can be used as a six-level spin [12]. Constructing a lattice of such elements, as in Fig. 3(b), imposes the vacuum state on all plaquettes, and gives the following Hamiltonian for the vertices,

$$H' = -r \sum_v (A(v) + A^\dagger(v)).$$

Here $2r$ is the energy gap resulting from the tunnelling processes within the junctions. Using a semi-classical approximation we find the coupling to be $r \approx E_J^4/E_C^2 \exp(-S_0)$ with $S_0 \approx 0.651 \sqrt{E_C}$. Here $E_J$ and $E_C$ are the Josephson and charging energies, respectively. To realize a single qubit memory, the states [3] must be prepared. This can be done by pumping charge between vertices to implement the $\sigma^z$ operations on the connecting link [15]. The degeneracy of these states must then be confirmed. Since the Hamiltonian is a perturbed version of [2], it does not assign equal energies to the $e_g$ anyons. Hence, the terms $B(\sigma^z)^3$ and $B(\sigma^3)^3$ must be applied to each pair to ensure the degeneracies. These can be simply implemented by passing a flux of $2\pi/3$ through the elements on links 2 and 3, rather than $2\pi/6$. The experimental verification of such memories, even without the application of any quantum gates, would form a major breakthrough in the realization of anyonic quantum memories.

Conclusions:- We have demonstrated non-Abelian like encoding of quantum information using the more experimentally accessible Abelian anyon models. Since the scheme requires a fine tuned Hamiltonian, we have also introduced a method to enforce symmetries that can be applied to other models, both Abelian and non-Abelian.

Further, we have proposed an experiment to demonstrate the quantum memory and enforcing of symmetries with cutting edge technology. It would be interesting to study whether the enhanced fault-tolerance provided matches that of non-Abelian schemes requiring non-topological operations for universality [11].

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