Riemann-Hilbert Approach to the Helmholtz Equation in a quarter-plane. Revisited

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Abstract

We revisit the Helmholtz equation in a quarter-plane in the framework of the Riemann-Hilbert approach to linear boundary value problems suggested in late 90s by A. Fokas. We show the role of the Sommerfeld radiation condition in Fokas’s scheme.

1 Introduction

This paper is a complement to our previous paper [10] as well as the second author’s paper [9], where, following the general ideas of Fokas’ method [3]-[7], we started to develop the Riemann-Hilbert scheme for solving the elastodynamic equation in the quarter-plane. In [10], we show that the problem can be reduced to the solution of a certain matrix Riemann-Hilbert problem with a shift posed on a torus. A detail analysis of this problem is our ultimate goal. The modest objective of this paper is to reveal the Riemann-Hilbert interpretation of the Sommerfeld radiation condition in Fokas’ scheme by considering the more simple case of the Helmholtz equation in a quarter-plane.

2 RH approach for Helmholtz equation in a quarter-space

The classical boundary value problem for the Helmholtz equation in the quarter-plane \((x, z), x \geq 0, z \geq 0\) is given as follows:

\[ u_{xx} + u_{zz} + \hbar^2 u = 0, \quad u_x(z, 0) = u_1, \quad u_z(0, x) = u_2 \]  

(2.1)

It’s Lax pair in terms of the spectral parameter \(\zeta\) has the following form:

\[ \phi_z - \frac{i\hbar}{2}(\zeta + \frac{1}{\zeta})\phi = Q \]  

(2.2)

\[ \phi_x + \frac{\hbar}{2}(\zeta - \frac{1}{\zeta})\phi = \tilde{Q} \]  

(2.3)
where

\[ Q = \frac{1}{2} \tau - \frac{i h}{2 \zeta} u, \quad \tilde{Q} = \frac{i}{2} \tau - \frac{h}{2 \zeta} u \quad (2.4) \]

where \( \tau = u_z - iu_x \). The spectral function \( \phi \) is limited and decaying as

\[ \phi = u + O(\zeta) \quad \text{as} \quad \zeta \to 0 \quad (2.5) \]

and

\[ \phi = \frac{i \tau}{h \zeta} \quad \text{as} \quad \zeta \to \infty \quad (2.6) \]

Integrating along the three rays which are discussed in details in [3] - [7], one obtains three solutions:

\[ \phi_1(\zeta, x, z) = \int_{-\infty}^{\zeta} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) (z - z')} Q(\zeta, z', x) dz' \quad (2.7) \]

\[ \phi_2(\zeta, x, z) = \int_{0}^{x} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) (\frac{z}{x} - x')} Q(\zeta, 0, x') dx' + \int_{0}^{\zeta} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) (z - z')} Q(\zeta, z', x) dz' \quad (2.8) \]

\[ \phi_3(\zeta, x, z) = \int_{\infty}^{\zeta} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) (x' - x)} Q(\zeta, z, x') dx' \quad (2.9) \]

and the jump functions \( \rho_{ij}, i, j = 1, 2, 3 \)

\[ \rho_{13}(\zeta) = - \int_{0}^{\infty} e^{-\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z'} Q(\zeta, z', 0) dz' + \int_{0}^{\infty} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z'} \tilde{Q}(\zeta, 0, x') dx', \quad \rho_{31} = - \rho_{13} \quad (2.10) \]

\[ \rho_{21}(\zeta) = - \phi_1(0, 0) = \int_{0}^{\infty} e^{-\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z'} Q(\zeta, z', 0) dz' \quad (2.11) \]

\[ \rho_{32}(\zeta) = \phi_3(\zeta, 0, 0) = - \int_{0}^{\infty} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) x'} \tilde{Q}(\zeta, 0, x') dx' \quad (2.12) \]

As a result one can express the spectral function \( \phi \) as the following Cauchy integral over the oriented contour \( K \) presented on Figure 11

\[
\phi(x, z, \zeta) = \frac{1}{2\pi i} \int_{C_1} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{21}(\zeta') d\zeta' + \frac{1}{2\pi i} \int_{i\infty}^{i\infty} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{32}(\zeta') d\zeta' \\
+ \frac{1}{2\pi i} \int_{0}^{-i} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{32}(\zeta') d\zeta' + \frac{1}{2\pi i} \int_{0}^{\infty} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{21}(\zeta') d\zeta' \\
+ \frac{1}{2\pi i} \int_{C_1} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{32}(\zeta') d\zeta' + \frac{1}{2\pi i} \int_{C_2} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{31}(\zeta') d\zeta' \\
+ \frac{1}{2\pi i} \int_{C_3} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{32}(\zeta') d\zeta' + \frac{1}{2\pi i} \int_{C_4} e^{\frac{ih}{2}(\zeta + \frac{1}{\zeta}) z - \frac{h}{2}(\zeta' + \frac{1}{\zeta'}) x} \rho_{31}(\zeta') d\zeta' 
\]

where \( C_1 - C_4 \) are pieces of the circular part of \( K \) in the first, second, third and forth quadrants respectfully. Or, taking (2.5) into account ( and changing from \( \zeta' \) to \( \zeta \))
Substituting the boundary conditions \((2.1)\) into the jump functions \((2.10-2.12)\) we obtain

\[
\rho_{31} = \frac{ih}{4}(\zeta - \frac{1}{\xi}) \int_{0}^{\infty} e^{-\frac{ih}{2}(\zeta + \frac{1}{\xi})z} u(z, 0) dz - \frac{i}{2} \int_{0}^{\infty} e^{-\frac{ih}{2}(\zeta + \frac{1}{\xi})z} u_{1}(z) dz \\
\rho_{21} = \frac{ih}{4}(\zeta - \frac{1}{\xi}) \int_{0}^{\infty} e^{-\frac{ih}{2}(\zeta + \frac{1}{\xi})z} u(z, 0) dz - \frac{i}{2} \int_{0}^{\infty} e^{-\frac{ih}{2}(\zeta + \frac{1}{\xi})z} u_{1}(z) dz - \frac{1}{2} u(0, 0)
\]

Figure 1: Oriented contour \(K\) for \(H\) equation
\[ \rho_{32} = \frac{\hbar}{4}(\zeta + \frac{1}{\zeta}) \int_0^\infty e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})^2} u(0, x) dx - \frac{i}{2} \int_0^\infty e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})^2} u_2(x) dz + \frac{1}{2} u(0, 0) \]  

(2.17)

The jump functions are not completely defined by the known \( u_1 \) and \( u_2 \) functions, so one has to use the global relationship

\[ \rho_{21} + \rho_{32} = \rho_{31} \equiv 0. \]  

(2.18)

to find the integrals of the unknown \( u(z, 0) \) and \( u(0, x) \). Keeping for these integrals the same notations \( \Phi_1 \) and \( \Phi_3 \) as in [10] we can write for

\[ \text{if} \quad |\zeta| \leq 1, \quad 0 \leq \text{arg} \, \zeta \leq \frac{\pi}{2}, \quad \text{and} \quad |\zeta| \geq 1, \quad \pi \leq \text{arg} \, \zeta \leq \frac{3\pi}{2}. \]

the following global relationship

\[ \frac{i\hbar}{4}(\zeta - \frac{1}{\zeta})\Phi_1(\zeta) - \frac{h}{4}(\zeta + \frac{1}{\zeta})\Phi_3(\zeta) = F(\zeta) \]  

(2.19)

where

\[ \Phi_1(\zeta) = \int_0^\infty e^{\frac{h}{2}(\zeta + \frac{1}{\zeta})^2} u(0, z) dz \]  

(2.20)

\[ \Phi_3(\zeta) = \int_0^\infty e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})^2} u(x, 0) dx \]  

(2.21)

and

\[ F(\zeta) = \frac{i}{2} \int_0^\infty e^{\frac{h}{2}(\zeta - \frac{1}{\zeta})^2} u_2(x) dx + \frac{i}{2} \int_0^\infty e^{\frac{h}{2}(\zeta + \frac{1}{\zeta})^2} u_1(z) dz \]  

(2.22)

Using the symmetry one can extend (2.19) to the complex plane \( \zeta \) as follows:

\[ \zeta \to -\frac{1}{\zeta} : \quad \frac{i\hbar}{4}(\zeta - \frac{1}{\zeta})\Phi_1(-\zeta) + \frac{h}{4}(\zeta + \frac{1}{\zeta})\Phi_3(-\zeta) = F(-\frac{1}{\zeta}) \]  

(2.23)

\[ \zeta \to \frac{1}{\zeta} : \quad -\frac{i\hbar}{4}(\zeta - \frac{1}{\zeta})\Phi_1(\zeta) + \frac{h}{4}(\zeta + \frac{1}{\zeta})\Phi_3(\zeta) = F(\frac{1}{\zeta}) \]  

(2.24)

\[ \zeta \to -\zeta : \quad \frac{i\hbar}{4}(\zeta - \frac{1}{\zeta})\Phi_1(-\zeta) - \frac{h}{4}(\zeta + \frac{1}{\zeta})\Phi_3(\frac{1}{\zeta}) = -F(-\zeta) \]  

(2.25)

The distribution of the relevant functions in the \( \zeta \)-plane is given in Figure 2. Introducing a new function \( \Omega \) in terms of \( \Phi_1 \) as it is shown in Figure 2 we see that it has jumps on the real line and on the circumference. Using (2.19, 2.25) we can find its jumps on the real line in terms of the known \( F(\zeta) \) function on the intervals I-IV as follows

\[ I : \quad \Omega_+ - \Omega_- = -\Phi_1(-\zeta) - \Phi_1(\zeta) = -\frac{4\zeta}{i\hbar(\zeta^2 - 1)}(F(-\frac{1}{\zeta}) + F(\zeta)) \]  

(2.26)

\[ II : \quad \Omega_+ - \Omega_- = \Phi_1(-\zeta) + \Phi_1(\zeta) = -\frac{4\zeta}{i\hbar(\zeta^2 - 1)}(F(-\zeta) + F(\frac{1}{\zeta})) \]  

(2.26)

\[ III : \quad \Omega_+ - \Omega_- = \Phi_1(\zeta) + \Phi_1(-\zeta) = \frac{4\zeta}{i\hbar(\zeta^2 - 1)}(F(\zeta) + F(-\frac{1}{\zeta})) \]
Figure 2: Regions of analyticity of $\Phi$ and $\Omega$ for the second Riemann-Hilbert problem for $\Pi$ equation

$IV : \quad \Omega_+ - \Omega_- = -\Phi_1(\zeta) - \Phi_1(-\zeta) = \frac{4\zeta}{i\hbar(\zeta^2 - 1)}(F(\frac{1}{\zeta}) + F(-\zeta))$

However we cannot do the same on the circumference. In a similar way introducing a new function $\tilde{\Omega}$ in terms of $\Phi_3$ we can find jump functions on the imaginary axis, but not on the circle. Therefore to set up Riemann-Hilbert problem for any of these functions we have to supplement the boundary conditions with some physical, for example Sommerfeld’s radiation conditions $[16]$. That means that we have to estimate the asymptotic value of $u$ (2.14) for big $R$, where $R^2 = x^2 + z^2$ ($x = R \cos \theta$, $z = R \sin \theta$) we will use the steepest descent method. Let’s introduce

$$E(\zeta) = \frac{i\hbar}{2}((\zeta + i)\sin \theta + i(\zeta - i)\cos \theta) \tag{2.27}$$

Switching to $R$ it can be rewritten as

$$E(\zeta) = \frac{iRh}{2}((\zeta + i)\sin \theta + i(\zeta - i)\cos \theta) \tag{2.28}$$

Then from the equation $E'(\zeta) = 0$ one obtains two stationary phase points

$$\zeta_1 = \sin \theta - i \cos \theta, \quad \zeta_2 = -\zeta_1 \tag{2.29}$$

Taking into account that we are considering a quarter-space $x, z \geq 0$, which means $0 \leq \theta \leq \pi/2$, one can see that $\zeta_1$ is located on $C_1$ and $\zeta_2$ is located on $C_2$. It means that only these two integrals in (2.14) will contribute in the asymptotic value of $u$. One can easily obtain that

$$E(\zeta_1) = iRh, \quad E''(\zeta_1) = \frac{iRh}{(\sin \theta - i \cos \theta)^2} = -iRe^{-2i\theta} \tag{2.30}$$
This in turn implies that in the neighborhood of the point \( \zeta_1 \) the exponent \( E(\zeta) \) takes the form,

\[
E(\zeta) \sim E(\zeta_1) + \frac{1}{2} E''(\zeta_1)(\zeta - \zeta_1)^2 = iRh - \frac{1}{2} iRhe^{-2i\theta}(\zeta - \zeta_1)^2, \tag{2.31}
\]

where \( \zeta \) lies on the line tangent to the arc \( C_4 \) at the point \( \zeta_1 \). Accordingly, the integral over \( C_4 \) can be estimated as

\[
I_{C_4} = \frac{1}{2\pi i} \int_{C_4} e^{\frac{ih}{4}(\zeta + \frac{1}{\zeta})} \frac{1}{\zeta} \rho_{31}(\zeta) d\zeta \sim \frac{1}{2\pi i} \frac{\rho_{31}(\zeta_1)}{\zeta_1} \int_{\mathbb{R}} e^{iRh - \frac{1}{2} iRhe^{-2i\theta}(\zeta - \zeta_1)^2} d\zeta \tag{2.32}
\]

Changing variables as \( \zeta' = e^{-i\theta}(\zeta - \zeta_1) \), \( d\zeta' = e^{-i\theta} d\zeta \) and using again \( \zeta \) for \( \zeta' \) one obtains

\[
I_{C_4} = \frac{1}{2\pi i} \frac{\rho_{31}(\zeta_1)}{\zeta_1} e^{iRh} e^{i\theta} \int_{\mathbb{R}} e^{-\frac{i}{2} iRh\zeta'^2} d\zeta' \tag{2.33}
\]

Finally introducing \( X = \sqrt{\frac{Rh}{2}} \zeta \) we can finish the estimate as

\[
I_{C_4} = \frac{1}{2\pi i} \frac{\rho_{31}(\zeta_1)}{\zeta_1} e^{iRh} e^{i\theta} \sqrt{\frac{2}{Rh}} \int_{\mathbb{R}} e^{-iX^2} dX = \frac{1}{2\pi i} \frac{\rho_{31}(\zeta_1)}{\zeta_1} e^{iRh} e^{i\theta} \sqrt{\frac{2}{Rh}} e^{-i\pi/4} \sqrt{\pi} \tag{2.34}
\]

Similarly,

\[
E(\zeta_2) = -iRh, \quad E''(\zeta_1) = -\frac{iRh}{(-\sin \theta + i \cos \theta)^2} = iRhe^{-2i\theta}, \tag{2.35}
\]

and the integral over \( C_2 \) satisfies the asymptotic relation

\[
I_{C_2} = \frac{1}{2\pi i} \frac{\rho_{13}(\zeta_1)}{\zeta_1} e^{-iRh} e^{i\theta} \sqrt{\frac{2}{Rh}} \int_{\mathbb{R}} e^{iX^2} dX = \frac{1}{2\pi i} \frac{\rho_{13}(\zeta_1)}{\zeta_1} e^{-iRh} e^{i\theta} \sqrt{\frac{2}{Rh}} e^{i\pi/4} \sqrt{\pi} \tag{2.36}
\]

Now, taking into account the radiation condition, we arrive at the equation,

\[
\rho_{31}(\zeta) = 0, \quad \forall \zeta \in C_2 \tag{2.37}
\]

That means that (2.19) holds in \( C_2 \):

\[
\frac{i}{4h} (\zeta - \frac{1}{\zeta}) \Phi_1(\zeta) - \frac{1}{4h} (\zeta + \frac{1}{\zeta}) \Phi_3(\zeta) = F(\zeta) \tag{2.38}
\]

We could think about this equation as about a jump between \( \Phi_3 \) which is analytic outside of \( C_2 \) and \( \Phi_1 \) which is analytic inside (see Figure 2). However, our goal is to write the jump for the function \( \Omega \) on \( C_2 \) which we need to supplement the auxiliary Riemann-Problem for this function. One can see that \( \Phi_1(-\zeta) \) is also analytic outside \( C_2 \) and is related to \( \Phi_3 \) by (2.23). Therefore, this equation can be used to express \( \Phi_3 \) in terms of \( \Phi_1(-\zeta) \):

\[
-\frac{h}{4}(\zeta + \frac{1}{\zeta}) \Phi_3(\zeta) = -F(-\frac{1}{\zeta}) + \frac{i}{4h} (\zeta - \frac{1}{\zeta}) \Phi_1(-\zeta) \tag{2.39}
\]

Finally, substituting (2.39) into (2.38) we obtain the following jump for the function \( \Omega \) on \( C_2 \):

\[
C_2: \quad \Omega_+ - \Omega_- = -\Phi_1(\zeta) - \Phi_1(-\zeta) = -\frac{4\zeta}{ih(\zeta^2 - 1)} (F(\zeta) + F(-\frac{1}{\zeta}))
\]
Using the symmetry of $\Phi$ functions in the same way as we did before we could obtain the similar relations for $C_3$, $C_4$ and $C_1$ and obtain the jump function $\Omega$ on the whole circle:

$$
C_3, C_4: \quad \Omega_+ - \Omega_- = \frac{4 \zeta}{i h (\zeta^2 - 1)} (F(-\zeta) + F(\frac{1}{\zeta}))
$$

$$
C_1, C_2: \quad \Omega_+ - \Omega_- = -\frac{4 \zeta}{i h (\zeta^2 - 1)} (F(\zeta) + F(-\frac{1}{\zeta})
$$

Our principal message now is the following:

_The global relations together with the Sommerfeld radiation condition provide the complete set of the jump relation for the unknown function $\Omega$. _

The function $\Omega(\zeta)$, can be now written in the form of the Cauchy integral,

$$
\Omega = \frac{1}{2 \pi i} \int_{\Gamma} \frac{r(\zeta)}{\zeta' - \zeta} d\zeta, \tag{2.40}
$$

where the oriented contour $\Gamma$ consists of the real line and the unit circle, and the density function $r(\zeta)$ is given by the equations,

$$
r(\zeta) = -\frac{4 \zeta}{i h (\zeta^2 - 1)} (F(\zeta) + F(-\frac{1}{\zeta}))
$$

on the parts I, $C_1$, $C_2$ and $-III$ of the contour $\Gamma$, and

$$
r(\zeta) = \frac{4 \zeta}{i h (\zeta^2 - 1)} (F(-\zeta) + F(\frac{1}{\zeta}))
$$

on the parts $IV$, $C_3$, $C_4$ and $-II$ of the contour $\Gamma$.

Equation (2.40) completes the solution of the boundary value problem (2.1). It would be very interesting to compare the method of this paper with the alternative approach developed in [17] and [18] for the quarter-plane problem for the same Helmholtz equation.

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