GEOMETRIC STRUCTURES IN FIELD THEORY

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Abstract

This review paper is concerned with the generalizations to field theory of the tangent and cotangent structures and bundles that play fundamental roles in the Lagrangian and Hamiltonian formulations of classical mechanics. The paper reviews, compares and constrasts the various generalizations in order to bring some unity to the field of study. The generalizations seem to fall into two categories. In one direction some have generalized the geometric structures of the bundles, arriving at the various axiomatic systems such as $k$-symplectic and $k$-tangent structures. The other direction was to fundamentally extend the bundles themselves and to then explore the natural geometry of the extensions. This latter direction gives us the multisymplectic geometry on jet and cojet bundles and $n$-symplectic geometry on frame bundles.
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1 Introduction

This review paper is inspired by the geometric formulations of the Lagrangian and Hamiltonian descriptions of classical mechanics. The mathematical arenas of these well-known formulations are respectively, the tangent and cotangent bundles of the configuration space. Over the years many have sought to study classical field theory in analogous ways, using various generalizations or extensions of the tangent and cotangent bundles and/or their structures. No one has yet achieved a perfect formalism, but there are beautiful and useful results in many arenas.

The generalizations seem to fall into two categories. In one direction, some have generalized the geometric structures of these bundles, sometimes arriving at a formalism pertinent to field theory. Another direction was to fundamentally extend the bundles themselves and then explore the natural geometry of the extensions. The former gives us the various axiomatic systems such as $k$-symplectic and $k$-tangent structures. The latter gives us the multisymplectic geometry on jet and cojet bundles and $n$-symplectic geometry on frame bundles.

1.1 Cotangent-like structures

The first step in this direction of generalization was the development of symplectic geometry [1]. Later, around 1960, Bruckheimer [2] introduced the notion of almost cotangent structures. These were further investigated by Clark and Goel [3] in 1974. In both cases the canonical 2-form became the model from which axioms were designed.

Between 1987 and 1991, several independent and closely related generalizations were developed. Polysymplectic geometry [4], almost $k$-cotangent structures [5, 6], and $k$-symplectic geometry [7, 8] were based around the natural structure of the $k$-cotangent bundle. This bundle, which can be thought of as the fiberwise product of the cotangent bundle $k$ times, has a $k$-tuple of 1-forms with which one works. Also, the development of the $n$-symplectic geometry of the frame bundle and its $\mathbb{R}^n$-valued soldering 1-form $\theta$ began during this time period [9, 10, 11, 12]. While the development of $k$-tangent structures and $k$-symplectic geometry had purely geometric motivations, polysymplectic geometry was created to study
field theory and $m$-symplectic geometry sought to generalize Hamiltonian mechanics.

1.2 Tangent-like structures

Around 1960, the theory of almost tangent structures was developed by Clark and Bruckheimer [13] and Eliopoulos [14] separately. Almost tangent structures are generalizations of the tangent bundle. The canonical vector valued one-form $J$, viewed as the object of central interest, was axiomatized.

Almost $k$-tangent structures [15, 16] arose around 1988 as a generalization of the geometry of the $k$-tangent bundle. This bundle is, among other interpretations, the fiberwise product of the tangent bundle with itself $k$ times. A section of this bundle is equivalent to a $k$-tuple of vector fields. The central geometric object becomes a $k$-tuple of $J$’s.

Another version of the tangent structure arises on the jet bundle (see [17]). This is a very broad level of generalization since the idea of the jet of a section generalizes and incorporates the notions of tangent vectors, cotangent vectors, $k$-tangent vectors, and $k$-cotangent vectors. Such geometry has clear importance to field theory since one can envision any type of field as a section of a fiber bundle.

We present here also a new tangent-like structure, namely a canonically defined set of tensor fields $J^i, i = 1...n$ on the bundle of frames $LM$ of a manifold $M$. This tangent-like structure will be shown to induce the tangent structure on $TM$.

1.3 Interconnections, and plan of the paper

In this review paper our goal is to identify and clarify important connections between the various structures mentioned above. We also will consider relationships between some of the formalisms built on top of these structures.

The $k$-cotangent, $k$-symplectic, and polysymplectic structures are nested generalizations with $k$-cotangent being the most specific. The $n$-symplectic geometry of the frame bundle is also an example of a polysymplectic structure. Later in the paper, we will draw some interesting connections between the frame bundle and the $k$-cotangent bundle.

The frame bundle is an interesting case since in addition to having a cotangent-like
structure, it also has a tangent-like structure. Exploiting the natural correlation of frames and co-frames, we can define an \( n \)-tuple of \( Js \) in addition to the \( m \)-tuple of \( \theta \)s mentioned earlier. These objects acquire additional properties and relationships on the frame bundle.

The vector valued one-form \( S_\alpha \) on the jet bundle is later shown to be directly related to the other tangent structures in the special cases where they are comparable. Additionally, using new results regarding the adapted frame bundle we show a similar relationship between the \( k \)-tangent structure there and the \( S_\alpha \) on the jet bundle.

Venturing into the realm of multi-symplectic geometry, we show how the canonical multi-symplectic form on the cojet bundle is tied to the canonical \( k \)-symplectic structures we discuss. Moreover we show how the Cartan-Hamilton-Poincaré \( n \)-form on \( J^1\pi \) is induced from the \( m \)-symplectic structure on \( L_\pi E \).

What we strive to do in this paper is to unify perspectives. We show similarities and differences among the approaches and draw strong correlations. Since no one geometry has emerged as dominant, it is important that everyone be aware of the options. We hope this work may serve as a guidebook and translation table for those desiring to explore other formalisms.

All the manifolds are supposed to be smooth. The differential of a mapping \( F : M \rightarrow N \) at a point \( x \in M \) will be denoted by \( F_*(x) \) or \( TF(x) \). The induced tangent mapping will be denoted as \( TF : TM \rightarrow TN \).

The names of the various theories are different, yet two names are so similar that we feel it necessary to introduce the following convention that will be followed throughout the paper.

- We use the term \textit{k-symplectic geometry} to refer to the works of Awane and the works of de León, Salgado, et. al.

- We use the terms \textit{n-symplectic geometry and/or m-symplectic geometry} to refer to the works of Norris et. al.
2 Spaces with tangent-like structures

In this section we first recall the definitions and main properties of almost tangent and almost $k$-tangent structures. We describe the canonical $n$-tangent structure of the frame bundle $LM$ of an $n$-dimensional manifold $M$ in terms of the soldering form.

Secondly we recall Saunders’s construction of the vector valued 1-form $S_{\alpha}$. This 1-form is a generalization to field theories defined on jet bundles of fibered manifolds, of the almost tangent structure.

2.1 Almost tangent structures and $TM$

An almost tangent structure $J$ on a $2n$-dimensional manifold $M$ is tensor field of type $(1, 1)$ of constant rank $n$ such that $J^2 = 0$. The manifold $M$ is then called an almost tangent manifold. Almost tangent structures were introduced by Clark and Bruckheimer \cite{13} and Eliopoulos \cite{14} around 1960 and have been studied by numerous authors (see \cite{15, 19, 20, 21, 22, 23, 24, 25, 26}).

The canonical model of these structures is the tangent bundle $\tau_M : TM \to M$ of an arbitrary manifold $M$. Recall that for a vector $X_x$ at a point $x \in M$ its vertical lift is the vector on $TM$ given by

$$X^V_x(v_x) = \frac{d}{dt}(v_x + tX_x)|_{t=0} \in T_{v_x}(TM)$$

for all points $v_x \in TM$.

The canonical tangent structure $J$ on $TM$ is defined by

$$J_{v_x}(Z_{v_x}) = ((\tau_M)_*(v_x)Z_{v_x})^V_{v_x}$$

for all vectors $Z_{v_x} \in T_{v_x}(TM)$, and it is locally given by

$$J = \frac{\partial}{\partial u^j} \otimes dx^i$$

with respect the bundle coordinates on $TM$. This tensor $J$ can be regarded as the vertical lift of the identity tensor on $M$ to $TM$ \cite{27}.

The integrability of these structures, which means the existence of local coordinates such that the tensor field $J$ is locally given like as in \cite{27}, is characterized as follows.
**Proposition 2.1** An almost tangent structure \( J \) on \( M \) is integrable if and only if the Nijenhuis tensor \( N_J \) of \( J \) vanishes. ■

Crampin and Thompson [20] proved that an integrable almost tangent manifold \( M \) satisfying some natural global hypotheses is essentially the tangent bundle of some differentiable manifold.

### 2.2 Almost \( k \)-tangent structures and \( T^1_kM \)

The almost \( k \)-tangent structures were introduced as generalization of the almost tangent structures [15, 16].

**Definition 2.2** An almost \( k \)-tangent structure \( J \) on a manifold \( M \) of dimension \( n + kn \) is a family \((J^1, \ldots, J^k)\) of tensor fields of type \((1, 1)\) such that

\[
J^A \circ J^B = J^B \circ J^A = 0, \quad \text{rank} \ J^A = n, \quad \text{Im} \ J^A \cap (\bigoplus_{B \neq A} \text{Im} \ J^B) = 0, \tag{2}
\]

for \( 1 \leq A, B \leq k \). In this case the manifold \( M \) is then called an almost \( k \)-tangent manifold.

The canonical model of these structures is the \( k \)-tangent vector bundle \( T^1_kM = J^1_0(\mathbb{R}^k, M) \) of an arbitrary manifold \( M \), that is the vector bundle with total space the manifold of 1-jets of maps with source at \( 0 \in \mathbb{R}^k \) and with projection map \( \tau(j^1_0\sigma) = \sigma(0) \). This bundle is also known as the tangent bundle of \( k^1 \)-velocities of \( M \) [27].

The manifold \( T^1_kM \) can be canonically identified with the Whitney sum of \( k \) copies of \( TM \), that is

\[
T^1_kM \equiv TM \oplus \cdots \oplus TM, \quad j^1_0\sigma \equiv (j^1_0\sigma_1 = v_1, \ldots, j^1_0\sigma_k = v_k)
\]

where \( \sigma_A = \sigma(0, \ldots, t, \ldots, 0) \) with \( t \in \mathbb{R} \) at position \( A \) and \( v_A = (\sigma_A)_*(0)(\frac{d}{dt}\big|_0) \).

If \((x^i)\) are local coordinates on \( U \subseteq M \) then the induced local coordinates \((x^i, v^i_A), 1 \leq i \leq n, 1 \leq A \leq k, \) on \( \tau^{-1}(U) \equiv T^1_kU \) are given by

\[
x^i(j^1_0\sigma) = x^i(\sigma(0)), \quad v^i_A(j^1_0\sigma) = \frac{d}{dt}(x^i \circ \sigma_A)|_{t=0} = v_A(x^i).
\]
Definition 2.3 For a vector $X_x$ at $M$ we define its vertical $A$-lift $(X_x)^A$ as the vector on $T^1_kM$ given by

$$(X_x)^A(j^1_0\sigma) = \frac{d}{dt}((v_1)_x, \ldots, (v_{A-1})_x, (v_A)_x + tX_x, (v_{A+1})_x \ldots, (v_k)_x)|_{t=0} \in T_{j^1_0\sigma}(T^1_kM)$$

for all points $j^1_0\sigma \equiv ((v_1)_x, \ldots, (v_k)_x) \in T^1_kM$.

In local coordinates we have

$$(X_x)^A = \sum_{i=1}^n a^i \frac{\partial}{\partial v^i_A} (3)$$

for a vector $X_x = a^i \partial/\partial x^i$.

The canonical vertical vector fields on $T^1_kM$ are defined by

$$C^A_B(x, X_1, X_2, \ldots, X_k) = (X_B)^A (4)$$

and are locally given by $C^A_B = v^i_B \frac{\partial}{\partial v^i_A}$. The canonical $k$-tangent structure $(J^1, \ldots, J^k)$ on $T^1_kM$ is defined by

$$J^A(Z_{j^1_0\sigma}) = (\tau_*(Z_{j^1_0\sigma}))^A$$

for all vectors $Z_{j^1_0\sigma} \in T_{j^1_0\sigma}(T^1_kM)$. In local coordinates we have

$$J^A = \frac{\partial}{\partial v^i_A} \otimes dx^i (5)$$

The tensors $J^A$ can be regarded as the $(0, \ldots, 1_A, \ldots, 0)$-lift of the identity tensor on $M$ to $T^1_kM$ defined in [27].

We remark that an almost 1-tangent structure is an almost tangent structure.

In [15, 16] the almost $k$-tangent structures are described as $G$-structures, and the integrability of these structures, which is defined as the existence of local coordinates such that the tensor fields $J^A$ are locally given as in (5), is characterized by following proposition.

Proposition 2.4 An almost $k$-tangent structure $(J^1, \ldots, J^k)$ on $M$ is integrable if and only if $\{J^A, J^B\} = 0$ for all $1 \leq A, B \leq k$, where

$$\{J^A, J^B\}(X, Y) = [J^AX, J^BY] + J^A J^B[X, Y] - J^A[X, J^B Y] - J^B[J^A X, Y],$$

for any vector fields $X$, $Y$ on $M$. ■

In [15, 16] it is proved (in a way analogous to [20]) that an integrable almost tangent manifold $M$ satisfying some natural global hypotheses is essentially the $k$-tangent bundle of some differentiable manifold.
2.3 The canonical \( n \)-tangent structure of \( LM \)

We shall show that \( LM \) has an intrinsic \( n \)-tangent structure described in terms of the soldering form and fundamental vertical vectors fields.

Let \( M \) be a \( n \)-dimensional manifold and \( \lambda_M : LM \to M \) the principal fiber bundle of linear frames of \( M \). A point \( u \) of \( LM \) will be denoted by the pair \((x,e_i)\) where \( x \in M \) and \((e_1,e_2,\ldots,e_n)_x\) denotes a linear frame at \( x \). The projection map \( \lambda_M : LM \to M \) is defined by \( \lambda_M(x,e_i) = x \).

If \((U,x^i)\) is a chart on \( M \) then we can introduce two different coordinates on \( \lambda_M^{-1}(U) \). First consider the **coframe** or \( n \)-symplectic momentum coordinates \((x^i,\pi^i_j)\) on \( \lambda_M^{-1}(U) \) defined by

\[
x^i(u) = x^i(x), \quad \pi^i_j(u) = e^i(\frac{\partial}{\partial x^j}),
\]

where \((e^1,\ldots,e^n)_x\) is the dual frame to \( u = (e_1,\ldots,e_n)_x \).

Secondly consider the **frame** or \( n \)-symplectic velocity coordinates \((x^i,v^i_j)\) on \( \lambda_M^{-1}(U) \) defined by

\[
x^i(u) = x^i(x), \quad v^i_j(u) = e_j(x^i),
\]

The relationship between the two coordinates systems on \( LM \) is given by

\[
v^i_j(u)\pi^i_k(u) = \delta^i_k, \quad v^i_j(u)\pi^i_k(u) = \delta^k_j,
\]

for all \( u \) in the domain of the \( \pi^i_j \) momentum coordinates.

Denoting the standard basis of \( \text{gl}(n,\mathbb{R}) \) by \( \{E^i_j\} \), the corresponding fundamental vertical vector fields \( E^*_j^i \) on \( LM \) are given in momentum coordinates by

\[
E^*_j^i = -\pi^i_k \frac{\partial}{\partial \pi^i_k}.
\]

The bundle of linear frames \( LM \) is an open and dense submanifold of the \( n \)-tangent bundle \( T^1_n M \), where \( n = \text{dim} \, M \). The general linear group \( GL(n,\mathbb{R}) \) acts naturally on both \( LM \) and \( T^1_n M \). However, since each point in \( LM \) is a linear frame, the action of \( Gl(n,\mathbb{R}) \) is free.
on \( LM \) but not on \( T_n^1 M \). This reflects the fact that \( LM \) has more intrinsic structure than \( T_n^1 M \).

On \( LM \) we have an \( \mathbb{R}^n \)-valued one-form, the \textit{soldering one-form} \( \hat{\theta} = \theta^i \hat{r}_i \). Here \( \hat{r}_i \) denotes the standard basis of \( \mathbb{R}^n \). In momentum coordinates, \( \theta^i \) has the local expression

\[
\theta^i = \pi^i_j dx^j. \tag{10}
\]

\( \hat{\theta} \) is the \( n \)-symplectic potential on \( LM \).

Now the restriction of the \( n \)-tangent structure on \( T_n^1 M \) to \( LM \) will yield an \( n \)-tangent structure on \( LM \). It is not difficult to show that the restriction of \( (\mathbb{R}^n) \) to \( LM \) has, in \( n \)-symplectic momentum coordinates, the form

\[
J^i = -\pi^i_a \pi^j_b \frac{\partial}{\partial \pi^a_j} \otimes dx^b, \tag{11}
\]

We will present now an alternative derivation of this \( n \)-tangent structure on \( LM \) that is reminiscent of the geometric origins of other tangent-like structures. We recall the formula

\[
\xi^*(u) = \frac{d}{dt} (u \cdot \exp(t\xi))|_{t=0} \tag{12}
\]

for the value of the associated fundamental vertical vector field \( \xi^* \) on \( LM \) defined at \( u = (x, e_i) \) for each \( \xi \in gl(n, \mathbb{R}) \). These vector fields are smooth. We define the vector-valued 1-forms \( J^i \) by

\[
(J^i)_u(X) = (E^i_j \theta^j_u(X))^*(u) \quad \forall \ X \in T_u(LM) \tag{13}
\]

This definition uses the group action on \( LM \) in a manner that parallels the definition of the tangent structure on \( TM \) and mixes in the canonical soldering 1-forms in a fundamental way. The difference is that the action of \( GL(n, \mathbb{R}) \) on \( LM \) is global, while the definition of \( J \) on \( TM \) uses the fiberwise action of \( T_n M \) on \( T_n M \).

The mapping \( \xi \to \xi^* \) is a linear mapping from the Lie algebra \( gl(n, \mathbb{R}) \) to the Lie algebra of fundamental vertical vector fields on \( LM \). Hence

\[
(J^i)_u(X) = \theta^i_u(X)(E^i_j)^*(u) \quad \forall \ X \in T_u(LM)
\]
so that
\[ J^i = E^*_{j} \otimes \theta^j \quad (14) \]
Substituting (9) and (10) into this formula yields the local expression (11). This formula tells us that the canonical \( n \)-tangent structure on \( T^1_n M \) is in fact another representation of the soldering 1-form \( \hat{\theta} \). To see this explicitly we note that the mapping
\[ \hat{r}_i \rightarrow E^j_i \otimes \hat{r}_j \rightarrow E^*_j \otimes \hat{r}_i \]
is a linear representation of the basis vectors \( \hat{r}_i \) of \( \mathbb{R}^n \) in the space of \( gl(n, \mathbb{R}) \otimes \mathbb{R}^n \). Extending this representation to \( \hat{\theta} = \theta^i \otimes \hat{r}_i \) we obtain the \( n \)-tangent structure \( \hat{J} \):
\[ \hat{\theta} = \theta^i \otimes \hat{r}_i \rightarrow (E^*_j \otimes \theta^j) \otimes \hat{r}_i = \hat{J}. \]

2.4 The vector-valued one-form \( S_\alpha \) on \( J^1 \pi \)

We now turn our attention to 1-jets and review the tangent-like structure present on \( J^1 \pi \).

Let \( \pi : E \rightarrow M \) be a fiber bundle where \( M \) is \( n \)-dimensional and \( E \) is \( m = (n + k) \)-dimensional. Let \( \tau_E|_{V_\pi} : V_\pi \rightarrow E \) be the vertical tangent bundle to \( \pi \). We shall denote by \( \pi_{1,0} : J^1 \pi \rightarrow E \) the canonical projection and by \( V_{\pi_{1,0}} \) the vertical distribution defined by \( \pi_{1,0} \).

Throughout this paper if \( (x^i, y^A) \) are local fiber coordinates on \( E \) we take standard jet coordinates \( (x^i, y^A, y^A_i) \), \( 1 \leq i \leq n, 1 \leq A \leq k \), on the first jet bundle \( J^1 \pi \) the manifold of 1-jets of sections of \( \pi \).

**Definition 2.5** Let \( \phi : M \rightarrow E \) be a section of \( \pi \), \( x \in M \) and \( y = \phi(x) \). The vertical differential of the section \( \phi \) at the point \( y \in E \) is the map
\[
d^V_y \phi : T_y E \longrightarrow V_y \pi
\]
\[ u \mapsto u - (\phi \circ \pi)_x u \]
As \( d^V_y \phi \) depends only on \( j^1_x \phi \), the vertical differential can be lifted to \( J^1 \pi \) in the following way.
Definition 2.6 The canonical contact 1-form $\omega^1$ on $J^1\pi$ is the $V\pi$-valued 1-form defined by

$$\omega^1(j^1_x\phi) : T_{j^1_x\phi}(J^1\pi) \rightarrow V_{\pi(\phi(x))}$$

$$\tilde{X}_{j^1_x\phi} \mapsto (d\nu^\phi)((\pi_1)_*(\tilde{X}_{j^1_x\phi}))$$

In coordinates,

$$\omega^1 = (dy^B - y^B_j dx^j) \otimes \frac{\partial}{\partial y^B}.$$  \hfill (15)

Next let us recall the definition of the vector-valued 1-form $S_\alpha$ on $J^1\pi$ where $\alpha$ is a 1-form on $M$. Given a point $j^1_x\phi \in J^1\pi$, a cotangent vector $\eta_x \in T^*_{\pi(\phi(x))}M$ and a tangent vector $\xi \in V_{\phi(x)\pi}$, there exists a well defined vector $\eta_x \otimes_{j^1_x\phi} \xi \in V_{\phi(x)\pi}$ called the vertical lift of $\xi$ to $V_{\pi(\phi(x))}$ by $\eta$. This vector is locally given by

$$\eta_x \otimes_{j^1_x\phi} \xi(\phi(x)) = \eta_i \xi^A \frac{\partial}{\partial v_i^A}(j^1_x\phi).$$  \hfill (16)

Definition 2.7 Let $\alpha \in \Lambda^1 M$ be any 1-form on $M$. The vector-valued 1-form $S_\alpha$ on $J^1\pi$ is defined by

$$S_\alpha(j^1_x\phi) : T_{j^1_x\phi}(J^1\pi) \rightarrow (V_{\pi(\phi(x))})_{j^1_x\phi},$$

$$\tilde{X}_{j^1_x\phi} \mapsto S_\alpha(j^1_x\phi)(\tilde{X}_{j^1_x\phi}) = \alpha_x \otimes_{j^1_x\phi} \omega^1(\tilde{X}_{j^1_x\phi}).$$

From (16) and (15) we have that in coordinates

$$S_\alpha = \alpha_j (dy^A - y^A_i dx^i) \otimes \frac{\partial}{\partial v_i^A}. $$  \hfill (17)

$S_\alpha$ can be considered a more general version of the canonical tangent and $k$-tangent structures. This relationship is explored in section 3.2. Note also that $S_\alpha$ plays an important role in the construction of the Cartan-Hamilton-Poincaré $n$-form (see section 7.1).

2.5 The adapted frame bundle $L_\pi E$

An adapted frame at $e \in E$, $\pi : E \rightarrow M$, is a frame where the last $k$ basis vectors are vertical with respect to $\pi$. The adapted frame bundle of $\pi$ \cite{28, 29}, denoted by $L_\pi E$, consists of all adapted frames for $E$,

$$L_\pi E = \{(e_i, e_A)_e : e \in E, \{e_i, e_A\} \text{ is a basis for } T_e E, \text{ and } \pi_*(e)(e_A) = 0\}$$
The canonical projection, $\lambda : L_\pi E \to E$, is defined by $\lambda(e_i, e_A)_e = e$.

$L_\pi E$ is a reduced subbundle of $LE$, the frame bundle of $E$. As such it is a principal fiber bundle over $E$. Its structure group is $G_v$ the nonsingular block lower triangular matrices

$$G_v = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A \in \text{Gl}(n, \mathbb{R}), B \in \text{Gl}(k, \mathbb{R}), C \in \mathbb{R}^{kn} \right\} \quad (18)$$

$G_v$ acts on $L_\pi E$ on the right by

$$(e_i, e_A)_e \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = (A^i e_i + C^j_A e_A, B^A_B e_A)_e. \quad (19)$$

If $(x^i, y^A)$ are adapted coordinates on an open set $U \subseteq E$, then one may induce several different coordinates on $\lambda^{-1}(U)$. First consider the coframe or m-symplectic momentum coordinates $(x^i, y^A, \pi^i_j, \pi^A_j, \pi^A_B)$ on $\lambda^{-1}(U)$ defined in (6). Let us observe that $\pi^i_A = 0$ on $L_\pi E$.

We have as is customary retained the same symbols for the induced horizontal coordinates.

Secondly consider the frame or m-symplectic velocity coordinates $(x^i, y^A, v^i_j, v^A_j, v^A_B)$ on $\lambda^{-1}(U)$ defined in (7). Let us observe that $v^A_j = 0$ on $L_\pi E$.

The $v$ coordinates, viewed together as a block triangular matrix, form the inverse of the $\pi$ coordinates defined above. The blocks have the following relations:

$$v^i_j \pi^j_s = \delta^i_s \quad v^i_j \pi^j_s + v^A_j \pi^B_s = 0 \quad v^A_B \pi^B_s = \delta^A_C$$

Lastly consider the following coordinates which are constructed from the previous two.

Define $(x^i, y^A, u^i_j, u^A_j, u^A_B)$ on $\lambda^{-1}(U)$ by

$$x^i((e_i, e_A)_e) = x^i(e) \quad u^i_j = \pi^i_j \quad u^A_j = v^A_j \pi^i_j = -v^A_B \pi^B_j$$

$$y^A((e_i, e_A)_e) = y^A(e) \quad u^A_B = \pi^A_B$$

In Section 3.3 we discuss the fact that $L_\pi E$ is an $H = \text{Gl}(n) \times \text{Gl}(k)$ principal bundle $\rho : L_\pi E \to J^1 \pi$. It will turn out that the $u^A_j$ coordinates are pull-ups under $\rho$ of the standard jet coordinates on $J^1 \pi$. As such, we refer to these coordinates as Lagrangian coordinates.

The fundamental vertical vector fields $E^*_j$, $E^*_B$, and $E^*_A$, on $L_\pi E$ are given, in Lagrangian coordinates, by

$$E^*_j = -u^i_s \frac{\partial}{\partial u^A_s} \quad E^*_B = -u^A_C \frac{\partial}{\partial u^B_C} \quad E^*_A = u^i_s v^B_A \frac{\partial}{\partial u^B_s} \quad (20)$$
On $L\pi E$ we have also a $\mathbb{R}^{m+k}$-valued 1-form, the **soldering one-form** $\hat{\theta} = \theta^i \hat{\gamma}_i + \theta^A \hat{\gamma}_A$, which is the restriction of the canonical soldering 1-form on $LE$ to $L\pi E$. Here $\hat{\gamma}_i, \hat{\gamma}_A$ denotes the standard basis of $\mathbb{R}^{n+k}$. In Lagrangian coordinates, $\theta^i, \theta^A$ have the local expression

\[
\theta^i = u^i_j dx^j, \quad \theta^A = u^A_B (dy^B - u^B_j dx^j).
\] (21)

From (14) we have that the $(n+k)$-tangent structure on $LE$ is given by

\[
J^i = E^{*i}_j \otimes \theta^j + E^{*i}_B \otimes \theta^B, \quad J^A = E^{*A}_j \otimes \theta^j + E^{*A}_B \otimes \theta^B
\]

Now considering its restriction to the principal fiber bundle $L\pi E$ we have

\[
(J^i)|_{L\pi E} \equiv J^i, \quad 1 \leq i \leq n,
\]

\[
J^A|_{L\pi E} \equiv E^{*A}_j |_{L\pi E} \otimes \theta^j + E^{*A}_B \otimes \theta^B \quad 1 \leq A \leq k.
\]

## 3 Relationships among the tangent-like structures

In this section we show how the tangent, $k$-tangent, and similar structures on various spaces are related. We have already remarked in Section 2.3 that the $n$-tangent structure on $LM$ and the one on $T^n_1 M$ ($n = \dim M$) induce each other. Now we complete the circle by showing that the tangent structure on $TM$ induces the $k$-tangent structure on $T^1_k M$ and that the $n$-tangent structure on $LM$ induces the tangent structure on $TM$.

Secondly, we show that in the special cases where comparison makes sense the vector valued one-form on $J^1\pi$ is directly related to the $k$-tangent structure on $T^1_k M$. Furthermore, using recent results relating the jet bundle and adapted frame bundle, we show a similar relationship with the $(n+k)$-tangent structure on $L\pi E$.

### 3.1 Relationships among $TM$, $T^1_k M$, and $LM$

The $k$-tangent structure on $T^1_k M$ in terms of the tangent structure on $TM$

One can induce $J^A$ on $T^1_k M$ from $J$ on $TM$. We make use of the **inclusion maps**

\[
i_A : TM \to T^1_k M \quad 1 \leq A \leq k
\]

\[
v_x \to (0, \ldots, 0, v_x, 0, \ldots, 0)
\]
From (1), (5) we obtain

**Proposition 3.1**

\[ J^A(u) = i_{A*}(\phi(u)) \circ J_{\phi(u)} \circ \phi^*(u) \]

for all \( u \in T^1_kM \), where \( \phi: T^1_kM \rightarrow TM \) is any \( C^1 \) bundle morphism over the identity on \( M \) (one of the \( k \) projections for example).

The tangent structure on \( TM \) viewed from \( LM \)

**Lemma 3.2** Let \((J^1, \ldots, J^n)\) be the canonical \( n \)-tangent structure of \( LM \). For all vector fields \( X \) on \( LM \) we have

\[ J^i \circ R_{g*}(X) = (g^{-1})^i_a R_{g*} \circ J^a(X) \]

where \( R_g \) denotes the right translation with respect to \( g \in GL(n, \mathbb{R}) \).

**Proof** It follows from (11) and the identities

\[ R_g^* (\pi_j^i) = (g^{-1})^j_a \pi^a_k, \quad R_g^* (d \pi_j^i) = (g^{-1})^j_a d \pi^a_k, \quad R_g^* (dx^i) = dx^i, \]

\[ R_g^* \left( \frac{\partial}{\partial \pi_j^i} \right) = (g^{-1})^a_j \frac{\partial}{\partial \pi^a_k}, \quad R_g^* \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i}. \]

where \( g \) is any element of \( GL(n, \mathbb{R}) \).

Let \( \tilde{TM} \) denotes the manifold obtained from the tangent bundle \( TM \) by deleting the zero section. For a fixed, non-zero element \( \xi \in \mathbb{R}^n \) let \( \psi_{\xi} \) denote the mapping from \( LM \) to \( \tilde{TM} \) defined as follows. For each \( u \in LM \) let

\[ \psi_{\xi}(u) = [u, \xi] \]

where we are identifying the tangent bundle \( TM \) with the bundle associated to \( LM \) and the standard action of \( GL(n, \mathbb{R}) \) on \( \mathbb{R}^n \). The following lemma is easily verified for this map.

**Lemma 3.3**

\[ (\psi_{\xi})_* (E^{*b}_c(u)) = \xi^b \psi^i_c(u) \frac{\partial}{\partial y^i} \mid_{[u, \xi]} \]
Remark In this case Let $h = h_{ij} dx^i \otimes dx^j$ be any metric tensor field on the manifold $M$. Then its associated covariant tensorial function on $LM$ is the $\mathbb{R}^{n \times n}$ valued function with components

$$\hat{h}_{ij} = (h_{ab} \circ \lambda) v^a_i v^b_j$$

(see [30]). For simplicity we will drop the $\circ \lambda$ notation and write simply $\hat{h}_{ij} = h_{ab} v^a_i v^b_j$. Moreover, we know that $(\hat{h}_{ab})$ obeys the transformation law

$$\hat{h}_{ab} (u \cdot g) = g_{am} g_{bn} \hat{h}_{mn} (u)$$

for all $g \in GL(n)$.

**Definition 3.4** Let $h$ be a fixed positive definite metric tensor field on $M$. The associated covariant $m$-tangent structure $(J_i^{(h)})$ based on $h$ is

$$J_i^{(h)} = \sum_j h_{ij} J^j$$

**Lemma 3.5**

$$J_a^{(h)}(u \cdot g)(R_{g*}(X)) = g^b_a g_{g*} (J_b^{(h)}(u)(X))$$

**Proof** The proof follows easily from (22) and (26).

**Theorem 3.6** Let $h$ be an arbitrary positive definite metric tensor field on the manifold $M$, and let $(J_i^{(h)})$ denote the covariant $m$-tangent structure on $LM$ defined by $h$. For each point $[u, \xi] \in \tilde{T}M$ (note: $\xi = (\xi^i)$ is by assumption non-zero) let $\psi_\xi : LM \to \tilde{T}M$ be the map defined in (23) above. Then the vector-valued 1-form $\mathcal{J}$ on $\tilde{T}M$ defined pointwise by

$$X \mapsto \mathcal{J}([u, \xi])(X) = \frac{\psi_\xi^* \left( \xi^i J_i^{(h)}(u)(\tilde{X}) \right)}{h_{ab}(u) \xi^a \xi^b}, \quad \forall X \in T_{[u, \xi]}(TM)$$

is the canonical tangent structure on $\tilde{T}M$ given in local coordinates by

$$\mathcal{J} = \frac{\partial}{\partial y^i} \otimes dx^i$$

In equation (29) $\tilde{X}$ is any tangent vector at $u \in LM$ that projects to the same vector at $\lambda_{LM}(u)$ as does the vector $X \in T_{[u, \xi]}(TM)$; i.e. $d\lambda_{LM}(\tilde{X}) = d\tau(X)$. 

Proof. We first show that the tangent vector \( J([u, \xi]) \) is well-defined. Since \([u, \xi] = [u \cdot g, g \cdot \xi] \) we need to show that the right-hand side of formula (29) remains unchanged if we make the substitutions \( u \to u \cdot g \) and \( \xi \to g \cdot \xi = ((g^{-1})^i_j \xi^j) \). Making the substitutions we have

\[
J([u \cdot g, g \cdot \xi])(X) = \frac{\psi(g \xi)^* \left((g \xi)^i J_i^{(h)}(u \cdot g)(R_g, \tilde{X})\right)}{\hat{h}_{ab}(u \cdot g)(g \cdot \xi)^a(g \cdot \xi)^b}
\]  

(31)

Using \( (g \xi^i) = (g^{-1})^i_m \xi^m \) and (28) the numerator in this equation can be reduced as follows:

\[
\psi(g \xi)^* \left((g \xi)^i J_i^{(h)}(u \cdot g)(R_g, \tilde{X})\right) = \psi(g \xi)^* \left((g^{-1})^i_m \xi^m g_i^b R_g^* \left( J_b^{(h)}(u)(X)\right)\right)
\]

\[
= \psi(g \xi)^* \left( R_g^* (\xi^i J_i^{(h)}(u)(\tilde{X})\right)
\]

\[
= (\psi(g \xi) \circ R_g)^* \left( \xi^i J_i^{(h)}(u)(\tilde{X})\right)
\]

\[
= \psi_\xi^* \left( \xi^i J_i^{(h)}(u)(\tilde{X})\right)
\]

where the last equality follows from the fact that \( \psi \circ R_g = \psi_\xi \).

Similarly, using (27) the denominator in equation (31) can be reduced as follows:

\[
\hat{h}_{ab}(u \cdot g)(g \cdot \xi)^a(g \cdot \xi)^b = \hat{h}_{ab}(u) \xi^a \xi^b
\]

Substituting the last two results into (31) we obtain

\[
\frac{\psi(g \xi)^* \left((g \xi)^i J_i^{(h)}(u \cdot g)(R_g, \tilde{X})\right)}{\hat{h}_{ab}(u \cdot g)(g \cdot \xi)^a(g \cdot \xi)^b} = \frac{\psi_\xi^* \left( \xi^i J_i^{(h)}(u)(\tilde{X})\right)}{\hat{h}_{ab}(u) \xi^a \xi^b}
\]

which proves that the mapping given in (29) above is well-defined.

We now calculate the numerator on the right-hand-side of the above identity. From (24), (27), we obtain

\[
\psi_\xi^* \left( \xi^i J_i^{(h)}(u)(\tilde{X})\right) = \left( \xi^i \hat{h}_{ij}(u) \theta^b(u)(\tilde{X})\right) \psi_\xi^* \left( E_k^{\xi_j}(u)\right)
\]

\[
= \left( \xi^i \hat{h}_{ij}(u) \pi^k_i(u)dx^i(\tilde{X})\right) \left( \xi^j v_{ki}^a(u) \frac{\partial}{\partial y^l}([u, \xi])\right)
\]

\[
= \left( \hat{h}_{ij}(u) \xi^i \xi^j\right) \left( \partial_{[u, \xi]} dx^a(X)\right)
\]

Since the metric \( h \) is definite, the coefficient \( \hat{h}_{ij}(u) \xi^i \xi^j \) is non-zero for all \( u \in LM \). Hence we may divide both sides of the last equation by this term and use linearity of the mapping \( \psi_\xi \) to obtain the desired result. \( \blacksquare \)
3.2 The relationship between the vertical endomorphism on $J^1\pi$ and the canonical $k$-tangent structures

Now we shall describe the relationship between the vertical endomorphism on $J^1\pi$ and the canonical $k$-tangent structure on $T^1_kM$ when $E$ is the trivial bundle $E = \mathbb{R}^k \times M \to \mathbb{R}^k$. In this case $J^1\pi$ is diffeomorphic to $\mathbb{R}^k \times T^1_kM$ via the diffeomorphism given by $j^1_t\phi \equiv (t, j^0_t\phi_t)$ where $\phi_t(s) = \phi(t + s)$. In this case, (see (17)), the vector valued 1-form $S_\alpha$ is locally given by

$$S_\alpha = \frac{\partial}{\partial v^B_i} \otimes (\alpha_B (dx^i - v^i_A dt^A))$$

with respect the coordinates $(t^A, x^i, v^i_A)$ on $\mathbb{R}^k \times T^1_kM$.

In the case $k = 1$, we consider $\omega = dt$ and thus

$$S_{dt} = \frac{\partial}{\partial v^i} \otimes (dx^i - v^i dt) = \frac{\partial}{\partial v^i} \otimes dx^i - v^i \frac{\partial}{\partial v^i} \otimes dt$$

where $(t, x^i, v^i)$ are the coordinates in $\mathbb{R}^n \times TM$. Then we have

$$S_{dt} = J - C \otimes dt$$

where $C$ denotes the canonical or Liouville vector field on $TM$ and $J$ is the canonical tangent structure $J$ on $TM$.

In the general case, with $k$ arbitrary, if we fix $B$, $1 \leq B \leq k$, we have

$$S_{dtB} = \frac{\partial}{\partial v^i_B} \otimes (dx^i - v^i_A dt^A) = \frac{\partial}{\partial v^i_B} \otimes dx^i - v^i_A \frac{\partial}{\partial v^i_B} \otimes dt^A = J^B - C^B_A \otimes dt^B$$

where $J^B$ is the canonical $k$-tangent structure on $T^1_kM$, and the $C^B_A$ are the canonical vertical vector fields defined in equation (4).

**Proposition 3.7** The relationship between the canonical $k$-tangent structure on $T^1_kM$ and the vertical endomorphism $S_{dtB}$, up to some obvious identifications, is given by

$$J^B = S_{dtB} + C^B_A \otimes dt^A$$
3.3 Strong relationships between $J^1\pi$ and $L_\pi E$

We shall consider two ways of describing 1-jets, each with its own charm:

1. Equivalence classes of local sections of $\pi$.

\[ J^1\pi = \{j^1_x\phi : x \in M, \phi \in \Gamma_x(\pi)\} \]

where $\Gamma_x(\pi)$ denotes the set of sections of $\pi$ defined in a neighborhood of $x$.

2. Linear right-inverses to $\pi_*(e)$.

\[ J^1\pi = \{\tau_e : T_{\pi(e)}M \to T_eE : \pi_*(e) \circ \tau_e = id_{T_{\pi(e)}M}\} \]

We will use either description of $J^1\pi$ when it is convenient.

Let $H$ be the subgroup of $G_v$ isomorphic to $Gl(n) \times Gl(k)$ (defined in (18)) given by

\[ H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in Gl(n, \mathbb{R}), B \in Gl(k, \mathbb{R}) \right\}, \]

and let $J$ be the following subgroup of $G_v$

\[ J = \left\{ \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} : \xi \in \mathbb{R}^{kn} \right\}. \]

Although $H$ is a closed Lie subgroup of $G_v$, it is not normal. As such $G_v/H$ does not have a natural group structure; it is a manifold with a left $G_v$-action. For each coset $gH \in G_v/H$, we select the unique representative in $J$.

\[ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \]

By choosing these representatives, we identify $G_v/H$ with $J$ and hence $\mathbb{R}^{kn}$. These identifications are diffeomorphisms.

Consider how the left $G_v$-action looks for our selected representatives.

\[ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ C + B\xi & B \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ CA^{-1} + B\xi A^{-1} & I \end{pmatrix} \tag{32} \]

So the $G_v$-action appears affine when $G_v/H$ is identified with $\mathbb{R}^{kn}$. Therefore it is prudent to use this identification to define an affine structure on $G_v/H$ modeled on $\mathbb{R}^{kn}$. This $G_v$-invariant structure will pass to the fibers of the associated bundle discussed below, making it an affine bundle.
Theorem 3.8 \( L\pi E \times_{G_v} (G_v/H) \cong J^1\pi \)

**Proof:** The affine bundle isomorphism maps each equivalence class \([(e_i, e_A)_e, (\xi_i^A)]\) to the linear map \( \phi : T_{\pi(e)}M \to T_eE \) defined by \( \phi(\hat{e}_i) = e_i + \xi_i^A e_A \), where we use the basis \( \{\hat{e}_i\} \) where \( \hat{e}_i = \pi^* (e)(e_i) \). The inverse isomorphism is given by

\[
j^1_\phi \mapsto \left[ \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} \right)_{\phi(x)} \right]
\]

The following corollary, whose simple proof is made possible by the preceding development, is a fundamental tool in lifting Lagrangian field theory to the adapted frame bundle.

**Corollary 3.9** \( L\pi E \) is a principal fiber bundle over \( J^1\pi \) with structure group \( H \).

**Proof:** This fact follows directly from proposition 5.5 in reference [30].

We will denote the projection from \( L\pi E \) to \( J^1\pi \) by \( \rho \). It is given by

\[
\rho : \quad L\pi E \quad \longrightarrow \quad J^1\pi
\]

\[
(e_i, e_A)_e \quad \longmapsto \quad \tau_e : \quad T_{\pi(e)}M \quad \rightarrow \quad T_eE
\]

\[
\pi^*_e (e_i) \quad \longmapsto \quad e_i
\]

We now show that the \( u^A_j \)-coordinates defined in Section 2.5 are the pull-ups of the jet coordinates. If \( (x^i, y^A) \) are adapted coordinates on an open set \( U \subseteq E \) and \( u = (e_i, e_A)_e \in \lambda^{-1}(U) \) then

\[
y^A_i \circ \rho(u) = y^A_i (\tau_e) = (dy^A)e_i \left( \tau_e \left( \frac{\partial}{\partial x^i} \right) \right) = (dy^A)(\hat{e}_i)(\frac{\partial}{\partial x^i})_{\pi(e)} e_j
\]

\[
= e_j(\frac{\partial}{\partial x^i})_{\pi(e)} (dy^A)_e (e_j) = \pi^*_i(u) v^A_j(u) = u^A_j(u)
\]

3.4 The vector-valued 1-form \( S_\alpha \) on \( J^1\pi \) viewed from \( L\pi E \).

\( L\pi E \) is a principal bundle over \( J^1\pi \), we shall establish in this subsection the relationship between the vertical endomorphism \( S_\alpha \) on \( J^1\pi \) and the restriction of the \((n+k)\)-tangent structure of \( LE \) to the vertical adapted bundle \( L\pi E \). To be more precise, we show that \( S_\alpha \) corresponds to the tensors on \( L\pi E \):

\[
E^s_i \otimes \theta^B = J^i - E^s_i \otimes \theta^j, \quad 1 \leq i \leq n.
\]

Note the similarity to proposition [3.7].
Proposition 3.10 Let \( u = (e_i, e_A)_e \) be a frame on \( L_\pi E \) and let us denote by \( u \cdot \xi \) the frame

\[
\begin{align*}
    u \cdot \xi &= (e_i, e_A)_e \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} = (e_i + \xi^B e_B, e_A)_e
\end{align*}
\]

Let \( \alpha \) be any 1-form on \( M \) and \([u, \xi] = [(e_i, e_A)_e, (\xi^A_i)] \) an element of \( J^1 \pi \). Then the relationship between \( S_\alpha \) and the tensor fields \( E_B^\alpha \otimes \theta^B \) is given by

\[
S_\alpha([u, \xi]) (X_{[u, \xi]}) = \rho_*(u \cdot \xi) \left( (\pi^* \alpha)_e(e_i) (E_B^\alpha \otimes \theta^B)(u \cdot \xi)(\hat{X}_{u \cdot \xi}) \right)
\]

(33)

where

\[
X_{[u, \xi]} \in T_{[u, \xi]}(J^1 \pi), \quad \hat{X}_{u \cdot \xi} \in T_{u \cdot \xi}(L_\pi E)
\]

are vectors that project onto the same vector on \( E \).

**Proof**: First let us observe that, from the definition of \( \rho \), we have

\[
\rho(u \cdot \xi) = \rho((e_i + \xi^B e_B, e_A)_e) = [(e_i, e_A)_e, (\xi^A_i)] = [u, \xi]
\]

Now we shall prove that the right side of (33) does not depend on the choice of the representative of the equivalence class \([(e_i, e_A)_e, (\xi^A_i)]\). If

\[
[u, \xi] = [(e_i, e_A)_e, (\xi^A_i)] = [(\bar{e}_i, \bar{e}_A)_e, (\bar{\xi}^A_i)] = [\bar{u}, \bar{\xi}]
\]

we must prove that

\[
\rho_*(u \cdot \xi) \left( (\pi^* \alpha)_e(e_i) (E_B^\alpha \otimes \theta^B)(u \cdot \xi)(\bar{X}_{u \cdot \xi}) \right) = \rho_*((\bar{u} \cdot \bar{\xi}) \left( (\pi^* \alpha)_e(\bar{e}_j) (E_C^\alpha \otimes \theta^C)((\bar{u} \cdot \bar{\xi})(\bar{X}_{\bar{u} \cdot \bar{\xi}}))
\]

for any vectors \( X_{u \cdot \xi} \in T_{u \cdot \xi}(L_\pi E), \bar{X}_{\bar{u} \cdot \bar{\xi}} \in T_{\bar{u} \cdot \bar{\xi}}(L_\pi E) \) that project at the same vector on \( E \).

But, in this case, we have from (19) and (32)

\[
\bar{u} = (\bar{e}_j, \bar{e}_B)_e = (A^i_j e_i + C^A_j e_A, B^A_B e_A), \quad \bar{\xi}^B_j = -(B^{-1})^B_C C^C_j + (B^{-1})^B_C \xi^C_A A^A_j \quad .
\]

(34)

Let us consider the frames

\[
\tilde{u} = (\tilde{e}_i, \tilde{e}_A)_e = u \cdot \xi = (e_i + \xi^B e_B, e_A)_e
\]

\[
\hat{u} = (\hat{e}_j, \hat{e}_B)_e = \tilde{u} \cdot \tilde{\xi} = (\tilde{e}_j + \tilde{\xi}^C_i \tilde{e}_C, \hat{e}_B)_e = (A^i_j \tilde{e}_i, B^A_B \tilde{e}_A)_e
\]
where the last identity comes from (34). Then we deduce that the relationship between the coordinates of \( \bar{u} \) and \( \hat{u} \) are

\[
\hat{v}_j^i = A^i_j \bar{v}_i^j, \quad \hat{v}_j^C = A^i_j \bar{v}_i^C, \quad \hat{v}_B^C = B^B_A \bar{v}_A^C, \quad \hat{u}_j^i = (A^{-1})^i_j \bar{u}_j^i, \quad \hat{u}_i^A = \bar{u}_i^A. \tag{35}
\]

On the other hand, the tensor fields \( E_B^i \otimes \theta^B \) are locally given by

\[
E_B^i \otimes \theta^B = u_j^i (dy^B - u_i^A dx^t) \otimes \frac{\partial}{\partial u_j^B}. \tag{36}
\]

From (37), and (35) we obtain

\[
(E_C^e \otimes \theta^C)(\hat{u}) = (A^{-1})^j_i \bar{u}_i^j ((dy^A)_\bar{u} - \bar{u}_i^A (dx^t)_\bar{u}) \otimes \frac{\partial}{\partial u_j^A}(\hat{u}) \tag{37}
\]

Since \( (\pi^* \alpha)_e(\bar{e}_j) = A^j_i (\pi^* \alpha)_e(e_i) \) we deduce that

\[
(\pi^* \alpha)_e(\bar{e}_j) (E_C^e \otimes \theta^C)(\hat{u}) = (\pi^* \alpha)_e(e_i) \bar{u}_i^j ((dy^A)_\bar{u} - \bar{u}_i^A (dx^t)_\bar{u}) \otimes \frac{\partial}{\partial u_j^A}(\hat{u}) \tag{38}
\]

and

\[
(\pi^* \alpha)_e(e_i) (E_B^i \otimes \theta^B)(\hat{u}) = (\pi^* \alpha)_e(e_i) \bar{u}_i^j ((dy^A)_\bar{u} - \bar{u}_i^A (dx^t)_\bar{u}) \otimes \frac{\partial}{\partial u_j^A}(\hat{u}) \tag{39}
\]

If the vectors \( X_u \xi \in T_u \xi (L_s E) \), \( \bar{X}_\bar{u} \bar{\xi} \in T_{\bar{u}} \bar{\xi} (L_s E) \) project onto the same vector on \( E \) then its components with respect the coordinates \( x^i \) and \( y^A \) are equal and from (38) and (39) we obtain that

\[
\rho_*(\bar{u}) ((\pi^* \alpha)_e(e_i) (E_B^i \otimes \theta^B)(\bar{u})(X_{\bar{u}})) = \rho_*(\hat{u}) ((\pi^* \alpha)_e(\bar{e}_j) (E_C^e \otimes \theta^C)(\hat{u})(\bar{X}_{\hat{u}}))
\]

because \( \rho(\bar{u}) = [u, \xi] = [\bar{u}, \bar{\xi}] = \rho(\hat{u}) \).

Now we shall prove the identity (33) using the Theorem 3.8. If \( j_1^A \phi \equiv [(e_i, e_A)_e, (\xi_i^A)] \) and

\[
e_i = v_i^j \frac{\partial}{\partial x^j}(e) + v_i^B \frac{\partial}{\partial y^B}(e), \quad e_A = v_A^B \frac{\partial}{\partial y^B}(e) \tag{40}
\]

then

\[
[(e_i, e_A)_e, (\xi_i^A)] = [(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^B})_e, \left( \begin{array}{cc} v_i^j & 0 \\ v_i^B & v_A^B \end{array} \right) |e(\xi_i^A)]
\]
From this identity and (32) we deduce that the coordinates of the 1-jet \( j_{x}^{1} \phi \), defined by the class \([u, \xi] = [(e_{i}, e_{A})_{e}, (\xi^{A})_{e}]\), are

\[
\frac{\partial \phi^{B}}{\partial x^{s}}(\pi(e)) = u_{r}^{s} (v_{r}^{B} + v_{A}^{B} \xi_{r}^{A})|_{e} \tag{41}
\]

and therefore from (17) we have

\[
S_{\alpha}([u, \xi]) = \alpha_{j} (\pi(e)) ((dy^{A})_{[u, \xi]} - u_{r}^{s} (v_{r}^{A} + v_{B}^{A} \xi_{r}^{B})|_{e} (dx^{i})_{[u, \xi]} \otimes \frac{\partial}{\partial u_{j}^{i}}([u, \xi]) \tag{42}
\]

The coordinates of the frame \( \tilde{u} \) satisfy the identities

\[
\tilde{u}_{j}^{i} = u_{j}^{i}, \quad \tilde{u}_{j}^{A} = \tilde{u}_{i}^{i} \tilde{u}_{j}^{A} = u_{i}^{i} (v_{i}^{A} + v_{B}^{A} \xi_{B}^{i}) \nonumber
\]

Therefore, from (36) we have

\[
(E_{B}^{*i} \otimes \theta^{B})(\tilde{u}) = u_{j}^{i}|_{e} ((dy^{A})_{\tilde{u}} - u_{i}^{l} (v_{l}^{A} + v_{B}^{A} \xi_{l}^{B})|_{e} (dx^{i})_{\tilde{u}} \otimes \frac{\partial}{\partial u_{j}^{i}}(\tilde{u}) \tag{43}
\]

If \( \alpha = \alpha_{r} dx^{r} \), then \( (\pi^{*} \alpha)_{e}(e_{i}) = \alpha_{r}(\pi(e)) v_{i}^{r} \) and from (13) we obtain that

\[
(\pi^{*} \alpha)_{e}(e_{i}) (E_{B}^{*i} \otimes \theta^{B})(\tilde{u}) = \alpha_{j} ((dy^{A})_{\tilde{u}} - u_{i}^{l} (v_{l}^{A} + v_{B}^{A} \xi_{l}^{B})|_{e} (dx^{i})_{\tilde{u}} \otimes \frac{\partial}{\partial u_{j}^{i}}(\tilde{u}))
\]

Now, since \( \rho(\tilde{u}) = [u, \xi] \), from this last identity and (12) we get the identity (B3) taking into account that \( \rho^{*} y_{j}^{A} = u_{j}^{A} \). 

\section{Spaces with cotangent-like structures}

In this section we shall define and give the main properties of the almost cotangent structure and its generalizations.

\subsection{Almost cotangent structures and \( T^{*} M \)}

Almost cotangent structures were introduced by Bruckheimer \[2\]. An almost cotangent structure on a 2m-dimensional manifold \( M \) consists of a pair \((\omega, V)\) where \( \omega \) is a symplectic form and \( V \) is a distribution such that

\[
(i) \quad \omega|_{V \times V} = 0, \quad (ii) \quad \ker \omega = \{0\}
\]
The canonical model of this structure is the cotangent bundle $\tau_M^*: T^*M \to M$ of an arbitrary manifold $M$, where $\omega$ is the canonical symplectic form $\omega_0 = -d\theta_0$ on $T^*M$ and $V$ is the vertical distribution. Let us recall the definition of the Liouville form $\theta_0$ in $T^*M$:

$$\theta_0(\alpha)(\tilde{X}_\alpha) = \alpha((\tau_M^*)_*(\alpha)(\tilde{X}_\alpha)),$$

for all vectors $\tilde{X}_\alpha \in T_\alpha(T^*M)$. In local coordinates $(x^i, p_i)$ on $T^*M$

$$\omega_0 = dx^i \wedge dp_i, \quad V = \langle \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_k} \rangle. \quad (44)$$

Clark and Goel [3] also investigated these structures, defining them as a certain type of $G$-structure. They proved that the integrability of these structures, that is the existence of coordinates on the manifold such that $\omega_0$ and $V$ have the form of (44), is characterized by

**Proposition 4.1** An almost cotangent structure $(\omega, V)$ on $M$ is integrable if and only if $\omega$ is closed and the distribution $V$ is involutive.

Thompson [26, 31] proved that an integrable almost cotangent manifold $M$ satisfying some natural global hypotheses is essentially the cotangent bundle of some differentiable manifold.

### 4.2 $k$-symplectic structures and $(T^1_k)^*M$

**Definition 4.2** [7, 8] A $k$-symplectic structure on a manifold $M$ of dimension $N = n + kn$ is a family $(\omega_A, V; 1 \leq A \leq k)$, where each $\omega_A$ is a closed 2-form and $V$ is an $nk$-dimensional distribution on $M$ such that

$$(i) \quad \omega_A|_{V \times V} = 0, \quad (ii) \quad \cap_{A=1}^k \ker \omega_A = \{0\}.$$  

In this case $(M, \omega_A, V)$ is called a $k$-symplectic manifold.

The canonical model of this structure is the $k$-cotangent bundle $(T^1_k)^*M = J^1(M, \mathbb{R}^k)_0$ of an arbitrary manifold $M$, that is the vector bundle with total space the manifold of 1-jets of maps with target at $0 \in \mathbb{R}^k$, and projection $\tau^*(j^1_{x,0}\sigma) = x$. 
The manifold \((T^*_k)^*M\) can be canonically identified with the Whitney sum of \(k\) copies of \(T^*M\), say
\[
(T^*_k)^*M \equiv T^*M \oplus \cdots \oplus T^*M, \quad j_{x,0}\sigma \equiv (j^1_{x,0}\sigma^1, \ldots, j^k_{x,0}\sigma^k)
\]
where \(\sigma^A = \pi_A \circ \sigma : M \to \mathbb{R}\) is the \(A\)-th component of \(\sigma\).

The canonical \(k\)-symplectic structure \((\omega_A, V; 1 \leq A \leq k)\), on \((T^*_k)^*M\) is defined by
\[
\omega_A = (\tau^*_A)^*(\omega_0)
\]
\[
V(j^1_{x,0}\sigma) = \ker(\tau^*_A)(j^1_{x,0}\sigma)
\]
where \(\tau^*_A = (T^*_k)^*M \to T^*M\) is the projection on the \(A\)-th copy \(T^*M\) of \((T^*_k)^*M\), and \(\omega_0\) is the canonical symplectic structure of \(T^*M\).

One can also define the 2-forms \(\omega_A\) by \(\omega_A = -d\theta_A\) where \(\theta_A\) is the 1-form defined as follows
\[
\theta_A(j^1_{x,0}\sigma)(\bar{X}^1_{j^1_{x,0}\sigma}) = \sigma^*(x)((\tau^*_A)(j^1_{x,0}\sigma))\bar{X}^1_{j^1_{x,0}\sigma})
\]
for all vectors \(\bar{X}^1_{j^1_{x,0}\sigma} \in T^1_{j^1_{x,0}\sigma}(T^*_k)^*M\).

If \((x^i)\) are local coordinates on \(U \subseteq M\) then the induced local coordinates \((x^i, p^A_i), 1 \leq i \leq n, 1 \leq A \leq k\) on \((T^*_k)^*U = (\tau^*)^{-1}(U)\) are given by
\[
x^i(j^1_{x,0}\sigma) = x^i(x), \quad p^A_i(j^1_{x,0}\sigma) = d_x \sigma^A(\frac{\partial}{\partial x^i} \bigg|_x).
\]
Then the canonical \(k\)-symplectic structure is locally given by
\[
\omega_A = \sum_{i=1}^{n} dx^i \wedge dp^A_i, \quad V = \langle \frac{\partial}{\partial p^1_i}, \ldots, \frac{\partial}{\partial p^k_i} \rangle \quad 1 \leq A \leq k.
\]

**Theorem 4.3** [7] Let \((\omega_A, V; 1 \leq A \leq k)\) be a \(k\)-symplectic structure on \(M\). About every point of \(M\) we can find a local coordinate system \((x^i, p^A_i), 1 \leq i \leq n, 1 \leq A \leq k\) such that
\[
\omega_A = \sum_{i=1}^{n} dx^i \wedge dp^A_i, \quad 1 \leq A \leq k \quad (45)
\]

In [4] Günther introduces the following definitions.
Definition 4.4 A closed non-degenerate $\mathbb{R}^n$-valued 2-form

$$\bar{\omega} = \sum_{A=1}^{n} \omega_A \hat{r}_A$$

on a manifold $M$ of dimension $N$ is called a polysymplectic form. The pair $(M, \bar{\omega})$ is a polysymplectic manifold.

A polysymplectic form $\bar{\omega}$ on a manifold $M$ is called standard iff for every point of $M$ there exists a local coordinate system such that $\omega_A$ is written locally as in (43).

From Theorem 4.3 it now follows that the $k$-symplectic manifold structures coincide with the standard polysymplectic structures.

$\bar{\omega}$ is called by Norris [32] a general $n$-symplectic structure. The difference in the formalism is that there exist natural definitions of Poisson brackets in the $n$-symplectic theory of Norris. See Section 9 for a discussion of $n$-symplectic Poisson brackets in the general case.

4.3 Almost $k$-cotangent structures and $(T^1_k)^*M$

In [5] the almost $k$-cotangent structures were defined and described as $G$-structures.

Definition 4.5 An almost $k$-cotangent structure is a family $(\omega_A, V_A; 1 \leq A \leq k)$, where each $\omega_A$ is a 2-form of constant rank $2n$ and $V_A$ is a $n$-dimensional distribution on $M$, such that

(i) $V_A \cap (\oplus_{B \neq A} V_B) = 0$,  \quad (ii) $\ker \omega_A = \oplus_{B \neq A} V_B$,  \quad (iii) $\omega_A|_{V_A \times V_A} = 0$

for all $1 \leq A \leq k$.

The canonical model of this structure is $(T^1_k)^*M$ with the 2-forms $\omega_A$, and $V_A = \ker T\rho_A$ where $\rho_A : (T^1_k)^*M \to (T^1_{k-1})^*M$ is the projection given by

$$\rho_A(\alpha_1, \ldots, \alpha_k) = (\alpha_1, \ldots, \alpha_{A-1}, \alpha_{A+1}, \ldots, \alpha_k).$$

The integrability of these structures is characterized by

Proposition 4.6 An almost $k$-cotangent structure $(\omega_A, V_A; 1 \leq A \leq k)$ on $M$ is integrable if and only if the 2-forms $\omega_A$ are closed and all distributions $V_A \oplus \cdots \oplus V_A$ are involutive.

Remark It can be proved that an integrable almost $k$-cotangent structure on a manifold $M$ is a $k$-symplectic structure on $M$ setting $V = \oplus_{A=1}^{k} V_A$. 
4.4 The $n$-symplectic structure of $LM$

The frame bundle $LM$ has a canonical $n$-symplectic structure given by $\omega_i = -d\theta^i$, $V = \ker \lambda_M$ where $\theta^i$ are the components of the soldering one-form and $V$ is the vertical distribution. This structure was first introduced in [9, 10] under the name generalized symplectic geometry on $LM$, and later referred to as $n$-symplectic geometry in [11]. $n$-symplectic geometry is the generalized geometry that one obtains on $LM$ when $d\hat{\theta} = d\theta^i \hat{r}_i$ is taken as a generalized symplectic 2-form. The structure is rich enough to allow the definition of generalized Poisson brackets and generalized Hamiltonian vector fields. The ideas are "generalized" in the sense that the observables of the theory are vector-valued on $LM$ rather than $\mathbb{R}$-valued. Moreover the generalized Hamiltonian vector fields are equivalence classes of vector-valued vector fields. The details of this geometry in the more general case of a general $n$-symplectic manifold are given in Section 9 of this paper.

The relationship between $n$-symplectic geometry on the bundle of linear frames $LM$ and canonical symplectic geometry on the cotangent bundle $T^*M$ has been developed in [11], showing that the ordinary symplectic geometry of $T^*M$ can be induced from the $n$-symplectic geometry of $LM$ using the associated bundle construction. This relationship will be discussed further in Section 5.1.

In [28] it is shown that $m$-symplectic geometry on frame bundles can be viewed as a "covering theory" for the Hamiltonian formulation of field theory (multisymplectic manifolds). This relationship will be discussed in Section 7.4.

Also in [12] it is shown that the Schouten-Nijenhuis brackets of both symmetric and antisymmetric contravariant tensor fields have a natural geometrical interpretation in terms of $n$-symplectic geometry on the bundle of linear frames $LM$. Specifically, the restriction of the $n$-symplectic Poisson bracket to the subspace of $GL(n)$-tensorial functions is in fact the lift to $LM$ of the Schouten-Nijenhuis brackets. See Section 9.4.

4.5 $k$-cosymplectic structures and $\mathbb{R}^k \times (T^*_k)^*M$ 

Let us begin by recalling that a cosymplectic manifold is a triple $(M, \theta, \omega)$ consisting of a smooth $(2n + 1)$-dimensional manifold $M$ with a closed 1-form $\theta$ and a closed 2-form $\omega$,
such that \( \theta \wedge \omega^n \neq 0 \). The standard example of a cosymplectic manifold is provided by
\[
(J^1(\mathbb{R}, N) \equiv \mathbb{R} \times T^*N, dt, \pi^*\omega_0),
\]
with \( t : \mathbb{R} \times T^*N \to \mathbb{R} \) and \( \pi : \mathbb{R} \times T^*N \to T^*N \) the canonical projections and \( \omega_0 \) the canonical symplectic form on \( T^*N \).

**Definition 4.7** Let \( M \) be a differentiable manifold of dimension \((k + 1)n + k\). A family \((\eta_A, \omega_A, V; 1 \leq A \leq k)\), where each \( \eta_A \) is a closed 1-form, each \( \omega_A \) is a closed 2-form and \( V \) is an \( nk \)-dimensional integrable distribution on \( M \), such that

1. \( \eta_1 \wedge \cdots \wedge \eta_k \neq 0, \quad \eta_{A_1V} = 0, \quad \omega_{A_1V} = 0, \)
2. \((\cap^k_{A=1} \ker \eta_A) \cap (\cap^k_{A=1} \ker \omega_A) = \{0\}, \quad \dim(\cap^k_{A=1} \ker \omega_A) = k, \)

is called a \( k \)-cosymplectic structure and the manifold \( M \) a \( k \)-cosymplectic manifold.

The canonical model for these geometrical structures is \( \mathbb{R}^k \times (T^1_k)^*M = J^1(M, \mathbb{R}^k) \). Let \( J^1(M, \mathbb{R}^k) \) be the \((k + (k + 1)n)\)-dimensional manifold of one jets from \( M \) to \( \mathbb{R}^k \), with elements denoted by \( j^1_{x,t}\sigma \). We recall that one jets of mappings from \( M \) to \( \mathbb{R}^k \) can be identified with the manifold \( J^1\pi \) of one jets of sections of the trivial bundle \( \pi : \mathbb{R}^k \times M \to M \).

\( J^1\pi \) is diffeomorphic to \( \mathbb{R}^k \times (T^1_k)^*M \) via the diffeomorphism given by
\[
j^1_x\sigma \in J^1\pi \to (\sigma(x), j^1_{x,0}\sigma_x) \in \mathbb{R}^k \times (T^1_k)^*M,
\]
where \( \sigma_x(\tilde{x}) = \sigma(\tilde{x}) - \sigma(x) \) and \( \tilde{x} \) denotes an arbitrary point in \( M \).

Let \( \tau^* : \mathbb{R}^k \times (T^1_k)^*M \to M \) denote the canonical projection. If \( (x^i) \) are local coordinates on \( U \subseteq M \) then the induced local coordinates \((t^A, x^i, p^A_i), 1 \leq i \leq n, 1 \leq A \leq k \), on \((\tau^*)^{-1}(U) \equiv \mathbb{R}^k \times (T^1_k)^*U \) are given by
\[
t^A(j^1_x\sigma) = t^A, \quad x^i(j^1_x\sigma) = x^i(x), \quad p^A_i(j^1_x\sigma) = d(\sigma^A_x)(x)(\frac{\partial}{\partial x^i}|_x)
\]
where \( \sigma^A_x = \pi_A \circ \sigma_x \).

An \( \mathbb{R}^k \)-valued 1-form \( \eta_0 \) and an \( \mathbb{R}^k \)-valued 2-form \( \omega_0 \) on \( \mathbb{R}^k \times (T^1_k)^*M \) are defined by
\[
\eta_0 = \sum_{A=1}^m (\eta_0)_A \hat{r}_A = \sum_{A=1}^k ((\pi^1_A)^*dt) \hat{r}_A, \quad \omega_0 = \sum_{A=1}^k (\omega_0)_A \hat{r}_A = \sum_{A=1}^m (\pi^2_A)^*(\omega_M) \hat{r}_A
\]
(46)
where \( \pi^1_A : \mathbb{R}^k \times (T^1_k)^*M \to \mathbb{R} \) and \( \pi^2_A : \mathbb{R}^k \times (T^1_k)^*M \to T^*M \) are the projections defined by
\[
\pi^1_A((t^B), (p^B)) = t^A, \quad \pi^2_A((t^B), (p^B)) = p^A,
\]
and $\omega_M$ is the canonical symplectic form on $T^*M$. In local coordinates we have

$$(\eta_0)_A = dt^A, \quad (\omega_0)_A = \sum_{i=1}^m dx^i \wedge dp_i^A \quad 1 \leq A \leq k \quad (47)$$

Moreover, let $V = \ker T\mu^*$, where $\mu^* : \mathbb{R}^k \times (T^*_k)^* M \to \mathbb{R}^k \times M$. Then locally

$$V = \left\langle \frac{\partial}{\partial p_i^1}, \ldots, \frac{\partial}{\partial p_i^k} \right\rangle \quad 1 \leq A \leq k,$$

and the canonical $k$-cosymplectic structure on $\mathbb{R}^k \times (T^*_k)^* M$ is $((\eta_0)_A, (\omega_0)_A, V)$. Indeed a simple computation in local coordinates shows that the forms $((\eta_0)_A, (\omega_0)_A, V)$ satisfy the conditions of Definition [4,7].

For any $k$-cosymplectic structure $(\eta_A, \omega_A, V)$ on $M$, there exists a family of $k$ vector fields $(\xi_1, \ldots, \xi_k)$ characterized by the conditions

$$\eta_A(\xi_B) = \delta_{AB}, \quad \iota_{\xi_B} \omega_A = 0,$$

for all $1 \leq A, B \leq k$. These vector fields are called the Reeb vector fields associated to the $k$-cosymplectic structure.

**Theorem 4.8** [4,7] Let $(\eta_A, \omega_A, V, 1 \leq A \leq k)$ be a $k$-cosymplectic structure on $M$. About every point of $M$ we can find a local coordinate system $(t^A, x^i, p_i^A)$ such that

$$(\eta_0)_A = dt^A, \quad (\omega_0)_A = \sum_{i=1}^n dx^i \wedge dp_i^A, \quad V = \left\langle \frac{\partial}{\partial p_i^1}, \ldots, \frac{\partial}{\partial p_i^k} \right\rangle \quad 1 \leq A \leq k,$$

and the Reeb vector fields are given by $\xi_A = \frac{\partial}{\partial t^A}$.

**4.6 Multisymplectic structures**

In $k$-symplectic geometry the model is the Whitney sum of $k$-copies of the cotangent bundle of a manifold $M$. In multisymplectic geometry [34, 35, 36, 37] one uses a completely different model.

Let $E$ be an $m$-dimensional differentiable manifold and denote by $\Lambda^k E$ the bundle of exterior $k$-forms on $E$ with canonical projection $\rho_k : \Lambda^k E \to E$. Notice that $\Lambda^1 E = T^*E$. On $\Lambda^k E$ there exists a canonical $k$-form $\Theta_E$ defined by

$$(\Theta_E)_\alpha(v_1, \ldots, v_k) = \alpha(T\rho_k(v_1), \ldots, T\rho_k(v_k))$$
for \( \alpha \in \bigwedge^k E \) and \( v_1, \ldots, v_k \in T_\alpha(\bigwedge^k E) \). This is a direct extension of the construction of the canonical Liouville 1-form on a cotangent bundle.

Next, we define a \((k + 1)\)-form \( \Omega_E \) by

\[
\Omega_E = -d\Theta_E.
\]

Taking bundle coordinates \((x^i, p_{i_1 \ldots i_k})\), \(1 \leq i \leq m\), \(1 \leq i_1 < \cdots < i_k \leq m\), on \( \bigwedge^k E \), we have

\[
\Theta_E = p_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \Omega_E = -dp_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
\]

Assume that \( E \) itself is fibered over some manifold \( M \), with projection \( \pi : E \to M \). For any \( r \), with \( 0 \leq r \leq k \), let \( \bigwedge^r E \) denote the bundle over \( E \) consisting of those exterior \( k \)-forms on \( E \) which vanish whenever \( r \) of its arguments are vertical tangent vectors with respect to \( \pi \). Obviously, \( \bigwedge^r E \) is a vector subbundle of \( \bigwedge^k E \), and we will denote by \( i_{k,r} : \bigwedge^k E \to \bigwedge^k E \) the natural inclusion.

The restriction of \( \Theta_E \) and \( \Omega_E \) to \( \bigwedge^r E \) will be denoted by \( \Theta^r_E \) and \( \Omega^r_E \), respectively; that is

\[
\Theta^r_E = i_{k,r}^* \Theta_E, \quad \Omega^r_E = i_{k,r}^* \Omega_E,
\]

and, clearly, \( \Omega^r_E = -d\Theta^r_E \).

Based on the properties of the \((k + 1)\)-forms \( \Omega_E \) and \( \Omega^r_E \), we introduce the following definition.

**Definition 4.9** A closed \((k + 1)\)-form \( \alpha \) on a manifold \( N \) is called multisymplectic if it is non-degenerate in the sense that for a tangent vector \( X \) on \( N \), \( X \lrcorner \alpha = 0 \) if and only if \( X = 0 \). The pair \((N, \alpha)\) will then called a multisymplectic manifold.

Of course the manifolds \((\bigwedge^k E, \Omega_E)\) and \((\bigwedge^r E, \Omega^r_E)\), \(0 \leq r \leq k\), are multisymplectic.

To develop the multisymplectic formalism of field theory we will use the canonical multisymplyctic manifold \((\bigwedge^2 E, \Omega^2_E)\) and the manifold \((\bigwedge^1 E, \Omega^1_E)\). If \( M \) is oriented with volume form \( \omega \) we can consider coordinates \((x^i, y^A)\) on \( E \) such that \( \omega = d^n x = dx^1 \wedge \cdots \wedge dx^n \).

Elements of \( \bigwedge^2 E \) and \( \bigwedge^1 E \) can be written, respectively, as follows

\[
p d^n x, \quad p d^n x + p_A^i dy^A \wedge d^{n-1} x_i
\]
where \(d^{n-1} x_i = \frac{\partial}{\partial x^i} \int d^n x\). Then we take local coordinates \((x^i, y^A, p)\) on \(\bigwedge^n_1 \mathcal{E}\) and \((x^i, y^A, p, p^i_A)\) on \(\bigwedge^n_2 \mathcal{E}\). Therefore the canonical multisymplectic \((n+1)\)-form \(\Omega^2_\mathcal{E}\) on \(\bigwedge^n_2 \mathcal{E}\) is locally given by

\[
\Omega^2_\mathcal{E} = -dp \wedge d^n x - dp^i_A \wedge dy^A \wedge d^{n-1} x_i
\]

and \(\Theta^2_\mathcal{E} = p d^n x + p^i_A dy^A \wedge d^{n-1} x_i\).

**Remark** In [38] the authors have developed a geometrical study of multisymplectic manifolds, exhibiting the complexity of a classification. A characterization of multisymplectic manifolds which are exterior bundles can be found in [39].

## 5 Relationships among the cotangent-like structures

Here we show how the symplectic, \(k\)-symplectic, \(m\)-symplectic and similar structures are related. We also venture further into the realm of multisymplectic geometry by showing how the canonical \(k\)-symplectic structure is induced from a special case of the multisymplectic structure on \(J^1\pi^*\). We use here the definition of \(J^1\pi^*\) given in [40] rather than the affine dual definition of \(J^1\pi^*\) given in [41].

### 5.1 Relationships among \(T^*M\), \((T^*_1k)^*M\), and \(LM\)

In Section 4.2 we have already seen the relationship between the \(k\)-symplectic structure on \((T^*_1k)^*M\) and the symplectic structure on \(T^*M\). The relationship between the canonical symplectic structure on \(T^*M\) and the soldering form on \(LM\) can be found in [11]: if \(\theta_0\) is the Liouville 1-form on \(T^*M\) and \(\theta\) the soldering 1-form on \(LM\) then

\[
(\theta_0)[u,\alpha](\bar{X}_{[u,\alpha]}) = \alpha(\theta_u(X_u)) , \quad [u, \alpha] \in T^*M \equiv LM \times_{GL(n,\mathbb{R})} (\mathbb{R}^n)^*.
\]

In this equation \(u\) is a point in \(LM\), \([u, \alpha]\) denotes a point (equivalence class) in \(T^*M\) thought of as the associated bundle \(LM \times_{GL(m,\mathbb{R})} (\mathbb{R}^n)^*\) and

\[
\bar{X}_{[u,\alpha]} \in T_{[u,\alpha]}T^*M , \quad X_u \in T_u(LM)
\]

are vectors that project to the same vector on \(M\), and \(\alpha \in \mathbb{R}^n^*\) is non-zero.
5.2 The multisymplectic form and the canonical $k$-symplectic structure

Now we shall describe the relationship between the canonical multisymplectic form $\Omega^2_k E$ on $\Lambda^k_2 E$ and the canonical $k$-symplectic structure on $(T^*_1)^* M$ when $E$ is the trivial bundle $E = \mathbb{R}^k \times M \to \mathbb{R}^k$. In this case $\Lambda^k_2 E$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{R} \times (T^*_1)^* M$. Let us recall that $\Lambda^k_2 E$ is the vector bundle $
abla_{\pi} \Lambda^k_2 (\mathbb{R}^k \times M) = \{ \alpha((t,x)) \in \Lambda^k_2 (\mathbb{R}^k \times M) : \langle v \rangle \alpha((t,x)) = 0 \forall v, w \in (V\pi)_{(t,x)} \}$ where $V\pi$ is the vertical fiber bundle corresponding to $\pi$.

We define

$$\Psi : \Lambda^k_2 (\mathbb{R}^k \times M) \to \mathbb{R}^k \times \mathbb{R} \times (T^*_1)^* M$$

$$\alpha((t,x)) \mapsto (t, r, (\alpha^1_x, \ldots, (\alpha^k_x))$$

where

$$r = \alpha((t,x)) \left( \frac{\partial}{\partial t^1}(t,x), \ldots, \frac{\partial}{\partial t^k}(t,x) \right)$$

and

$$(\alpha^B)_x (-) = i^*_t (\alpha((t,x)) \left( \frac{\partial}{\partial t^1}(t,x), \ldots, \frac{\partial}{\partial t^{B-1}}(t,x), -\frac{\partial}{\partial t^{B+1}}(t,x), \ldots, \frac{\partial}{\partial t^k}(t,x) \right) 1 \leq B \leq k,$$

where $i_t : M \to R^k \times M$ denotes the inclusion $x \to (t,x)$.

The inverse of $\Psi$

$$\Psi^{-1} : \mathbb{R}^k \times \mathbb{R} \times (T^*_1)^* M \to \Lambda^k_2 (\mathbb{R}^k \times M)$$

$$(t, r, (\alpha^1_x, \ldots, (\alpha^k_x)) \mapsto \alpha((t,x))$$

is given by

$$\alpha((t,x)) = r (d^k t)_{(t,x)} + (pr^*_2)_{(t,x)} ((\alpha^B)_x) \wedge (d^{k-1} t^B)_{(t,x)}$$

where

$$d^k t = dt^1 \wedge \cdots \wedge dt^k, \quad d^{k-1} t^B = \frac{\partial}{\partial t^B} \int dt^k$$

and $pr_2 : \mathbb{R}^k \times M \to M$ is the canonical projection.

Elements of $\Lambda^k_2 E$ can be written uniquely as

$$p_i^B dx^i \wedge d^{k-1} t^B + pd^k t$$
where \((x^i)\) are coordinates on \(M\). Let us denote by \((t_B, p, x^i, p_i^A)\) the corresponding coordinates on \(\bigwedge^k E \equiv \mathbb{R}^k \times \mathbb{R} \times (T^*_k)^*M\). Locally \(\Psi\) is written as the identity.

The canonical \(k\)-form on \(\bigwedge^k E \equiv \mathbb{R}^k \times \mathbb{R} \times (T^*_k)^*M\) is locally given in this case by

\[
\Theta^k_E = p_i^B \, dx^i \wedge d^{k-1}t^B + p \, dt^k
\]

and the corresponding canonical multisymplectic \((k+1)\)-form \(\Omega^2_E = -d\Theta^k_E\) is locally given by

\[
\Omega^2_E = dx^i \wedge dp_i^B \wedge d^{k-1}t^B - dp \wedge dt^k
\]

Let \(i : (T^*_k)^*M \rightarrow \mathbb{R}^k \times \mathbb{R} \times (T^*_k)^*M\) be the natural inclusion. We define on \((T^*_k)^*M\) the 1-forms \(\lambda_B, 1 \leq B \leq k\), by

\[
\lambda_B(-) = i^*(\Theta^2_E(-), \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^{B-1}}, -, \frac{\partial}{\partial t^{B+1}}, \ldots, \frac{\partial}{\partial t^k}),
\]

and from (49) we deduce \(\lambda_B = p_i^B \, dx^i\). Hence \(\lambda_B\) is the Liouville form on the \(B\)-th copy \(T^*M\) of \((T^*_k)^*M\). To get this local expression apply \(\lambda_B\) to the partials \(\partial/\partial x^i\) and \(\partial/\partial p_i^B\).

Therefore the 2-forms

\[
\omega_B = -d\lambda_B = dx^i \wedge dp_i^B, \quad 1 \leq B \leq k
\]

define the canonical \(k\)-symplectic structure on \((T^*_k)^*M\), and \(\omega_B\) can also be defined as follows

\[
\omega_B(-, -) = i^*(\Omega^2_E(-), \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^{B-1}}, -, \frac{\partial}{\partial t^{B+1}}, \ldots, \frac{\partial}{\partial t^k}),
\]

The case \(k = 1\) gives us the canonical symplectic structure of \(T^*M\).

**Proposition 5.1** The relationship between the 2-forms of the canonical \(k\)-symplectic structure on \((T^*_k)^*M\) and the canonical multisymplectic form \(\Omega^2_E\) is given by (50). □

### 6 Field Theory on \(k\)-symplectic and \(k\)-cosymplectic manifolds

Here we discuss the polysymplectic formalism [4] for Hamiltonian and Lagrangian field theory using \(k\)-symplectic manifolds. We discuss the Günther’s formalism (autonomous case) using the \(k\)-symplectic structures and the \(k\)-tangent structures. The non autonomous case will be developed using the \(k\)-cosymplectic structures and the stable \(k\)-tangent structures [33, 41].
6.1 $k$-vector fields

Let $M$ be an arbitrary manifold and $\tau : T^1_k M \to M$ its $k$-tangent bundle.

**Definition 6.1** A section $X : M \to T^1_k M$ of the projection $\tau$ will be called a $k$-vector field on $M$.

Since $T^1_k M$ can be canonically identified with the Whitney sum $T^1_k M \equiv TM \oplus \cdots \oplus TM$ of $k$ copies of $TM$, we deduce that a $k$-vector field $X$ defines a family of vector fields $X_1, \ldots, X_k$ on $M$.

**Definition 6.2** An integral section of the $k$-vector field $X$ on $M$ is a map $\phi : U \subset \mathbb{R}^k \to M$, where $U$ is an open subset of $\mathbb{R}^k$ such that

$$\phi_*(t)\left( \frac{\partial}{\partial t^A} \right) = X_A(\phi(t)) \quad \forall t \in U, \quad 1 \leq A \leq k,$$

or equivalently, $\phi$ satisfies

$$X \circ \phi = \phi^{(1)}, \quad \text{(51)}$$

where $\phi^{(1)}$ is the first prolongation of $\phi$ defined by

$$\phi^{(1)} : U \subset \mathbb{R}^k \quad \to \quad T^1_k M$$

$$t \quad \mapsto \quad \phi^{(1)}(t) = \sum_{i=0}^{k-1} \phi_{it}$$

where $\phi_t(s) = \phi(s+t)$ for all $t, s \in \mathbb{R}$. If $X$ has an integral section, $X$ is said to be integrable.

**Remark** Let us consider the trivial bundle $\pi : E = R^k \times M \to R^k$. A jet field $\gamma$ on $\pi$ (see [17]) is a section of the projection $\pi_{1,0} : J^1 \pi \equiv \mathbb{R}^k \times T^1_k M \to E \equiv \mathbb{R}^k \times M$. If we identify each $k$-vector field $X$ on $M$ with the jet field $\gamma = (id_{\mathbb{R}^k}, X)$, that is $\gamma(t, x) = (t, X_1(x), \ldots, X_k(x))$, then the integral sections of the jet field $\gamma$ correspond, as defined by Günther, to the solutions of the $k$-vector field $X$.

We remark that if $\phi$ is an integral section of a $k$-vector field $(X_1, \ldots, X_k)$ then each curve on $M$ defined by $\phi_A = \phi \circ h_A$, where $h_A : \mathbb{R}^n \to \mathbb{R}^k$ is the natural inclusion $h_A(t) = (0, \ldots, t, \ldots, 0)$, is an integral curve of the vector field $X_A$ on $M$, with $1 \leq A \leq k$. We refer to [12, 43] for a discussion on the existence of integral sections.
6.2 Hamiltonian formalism and $k$-symplectic structures

In this section, following the ideas of Günther [4], we will describe the Hamilton equations, for an autonomous Hamiltonian, in terms of the geometry of $k$-symplectic structures, showing that the role played by symplectic manifolds in classical mechanics is here played by the $k$-symplectic manifolds.

Let $(M,\omega_A, V; 1 \leq A \leq k)$ be a $k$–symplectic manifold. Since $M$ is a polysymplectic manifold let us consider the vector bundle morphism defined by Günther:

\[
\Omega^k : T^1_k M \longrightarrow T^* M
\]

\[
(X_1, \ldots, X_k) \longrightarrow \Omega^k(X_1, \ldots, X_k) = \sum_{A=1}^k X_A \omega_A.
\]  

(52)

Definition 6.3 Let $H : M \longrightarrow \mathbb{R}$ be a function on $M$. Any $k$-vector field $(X_1, \ldots, X_k)$ on $M$ such that

\[
\Omega^k(X_1, \ldots, X_k) = dH
\]

will be called an evolution $k$-vector field on $M$ associated with the Hamiltonian function $H$.

It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [11] that there always exists an evolution $k$-vector field associated with a Hamiltonian function $H$.

Let $(x^i, p^A_i)$ be a local coordinate system on $M$. Then we have

Proposition 6.4 If $(X_1, \ldots, X_k)$ is an integrable evolution $k$-vector field associated to $H$ then its integral sections

\[
\sigma : \mathbb{R}^k \longrightarrow M
\]

\[
(t^B) \longrightarrow (\sigma^i(t^B), \sigma^A_i(t^B)),
\]

are solutions of the classical local Hamilton equations associated with a regular multiple integral variational problem [44]:

\[
\frac{\partial H}{\partial x^i} = -\sum_{A=1}^k \frac{\partial \sigma^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \sigma^i}{\partial t^A}, \quad 1 \leq i \leq n, 1 \leq A \leq k.
\]
6.3 Hamiltonian formalism and $k$-cosymplectic structures

In this section we will describe the Hamilton equations for a non-autonomous Hamiltonian in terms of the geometry of $k$-cosymplectic structures, showing that the role played by cosymplectic manifolds in classical mechanics (see [45, 46, 47]) is here played by the $k$-cosymplectic manifolds.

Let $(M, \eta_A, \omega_A, V; 1 \leq A \leq k)$ be a $k$–cosymplectic manifold. Let us consider the vector bundle morphism defined by:
\[
\Omega^k : T^1_k M \longrightarrow T^* M \quad (X_1, \ldots, X_k) \longrightarrow \Omega^k(X_1, \ldots, X_k) = \sum_{A=1}^{k} X_A \mathcal{J} \omega_A + \eta_A(X_A) \eta_A.
\]
(53)

Let $\xi_A$ the Reeb vector fields associated to the $k$-cosymplectic structure $(\eta_A, \omega_A, V)$. Notice here that the hamiltonian $H(t_A, x_i, p^A_i)$ is non-autonomous.

Definition 6.5 Let $H : M \longrightarrow \mathbb{R}$ be a function on $M$. Any $k$-vector field $(X_1, \ldots, X_k)$ on $M$ such that
\[
\eta_A(X_B) = \delta^A_B, \quad \Omega^k(X_1, \ldots, X_k) = dH + \sum_{i=1}^{k} (1 - \xi_A(H)) \eta_A
\]
will be called an evolution $k$-vector field on $M$ associated with the Hamiltonian function $H$ for all $1 \leq A, B \leq k$.

It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [41] that there always exists an evolution $k$-vector field associated with a Hamiltonian function $H$.

Let $(t^A, x^i, p^A_i)$ be a local coordinate system on $M$. Then we have

Proposition 6.6 If $(X_1, \ldots, X_k)$ is an integrable evolution $k$-vector field associated to $H$ then its integral sections
\[
\sigma : \mathbb{R}^k \longrightarrow M \quad (t^B) \longrightarrow (\sigma^A(t^B), \sigma^i(t^B), \sigma^A_i(t^B)),
\]
satisfy $\sigma^A(t^1, \ldots, t^k) = t^A$ and are solutions of the classical local Hamilton equations associated with a regular multiple integral variational problem [44]:
\[
\frac{\partial H}{\partial x^i} = -\sum_{A=1}^{k} \frac{\partial \sigma^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \sigma^i}{\partial t^A}, \quad 1 \leq i \leq n, 1 \leq A \leq k.
\]
6.4 Second Order Partial Differential Equations on $T^1_k M$

The idea of this subsection is to characterize the integrable $k$-vector fields on $T^1_k M$ such that their integral sections are canonical prolongations of maps from $\mathbb{R}^k$ to $M$.

**Definition 6.7** A $k$-vector field on $T^1_k M$, that is, a section $\xi \equiv (\xi_1, \ldots, \xi_k) : T^1_k M \to T^1_k T^1_k M$ of the projection $\tau_{T^1_k M} : T^1_k (T^1_k M) \to T^1_k M$, is a Second Order Partial Differential Equation (SOPDE) if and only if it is also a section of the vector bundle $T^1_k (\tau_{M}) : T^1_k (T^1_k M) \to T^1_k M$, where $T^1_k (\tau_{M})$ is defined by $T^1_k (\tau_{M}) (j^1_0 \sigma) = j^1_0 (\tau_{M} \circ \sigma)$.

Let $(x^i)$ be a coordinate system on $M$ and $(x^i, v^i_A)$ the induced coordinate system on $T^1_k M$. From the definition we deduce that the local expression of a SOPDE $\xi$ is

$$\xi_A(x^i, v^i_A) = v^i_A \frac{\partial}{\partial x^i} + (\xi_A)_B \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k. \quad (54)$$

We recall that the first prolongation $\phi^{(1)}$ of $\phi : U \subset \mathbb{R}^k \to M$ is defined by

$$\phi^{(1)} : U \subset \mathbb{R}^k \quad \xrightarrow{t} \quad T^1_k M \quad \xrightarrow{\phi^{(1)}}(t) = j^1_0 \phi_t$$

where $\phi_t(s) = \phi(s + t)$ for all $t, s \in \mathbb{R}$. In local coordinates:

$$\phi^{(1)}(t^1, \ldots, t^k) = (\phi^{(1)}(t^1, \ldots, t^k), \frac{\partial \phi^{(1)}}{\partial t^A}(t^1, \ldots, t^k)), \quad 1 \leq A \leq k, \quad 1 \leq i \leq n. \quad (55)$$

**Proposition 6.8** Let $\xi$ an integrable $k$-vector field on $T^1_k M$. The necessary and sufficient condition for $\xi$ to be a Second Order Partial Differential Equation (SOPDE) is that its integral sections are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbb{R}^k \to M$. That is

$$\xi_A(\phi^{(1)}(t)) = \phi^{(1)}_A(t) (\frac{\partial}{\partial t_A})(t)$$

for all $A = 1, \ldots, k$. These maps $\phi$ will be called solutions of the SOPDE $\xi$.

From (55) and (54) we have

**Proposition 6.9** $\phi : \mathbb{R}^k \to M$ is a solution of the SOPDE $\xi = (\xi_1, \ldots, \xi_k)$, locally given by $(54)$, if and only if

$$\frac{\partial \phi^{(1)}}{\partial t_A}(t) = v^i_A(\phi^{(1)}(t)), \quad \frac{\partial^2 \phi^{(1)}}{\partial t^A \partial t^B}(t) = (\xi_A)_B(\phi^{(1)}(t)).$$
If $\xi : T_k^1 M \to T_k^1 T_k^1 M$ is an integrable SOPDE then for all integral sections $\sigma : U \subset \mathbb{R}^k \to T_k^1 M$ we have $(\tau_M \circ \sigma)(1) = \sigma$ where $\tau_M : T_k^1 M \to M$ is the canonical projection.

Now we show how to characterize the SOPDEs using the canonical $k$-tangent structure of $T_k^1 M$.

**Definition 6.10** The canonical vector field $C$ on $T_k^1 M$ is the infinitesimal generator of the one parameter group

$$
\mathbb{R} \times (T_k^1 M) \to T_k^1 M, \\
(s, (x^i, v^i_B)) \mapsto (x^i, e^s v^i_B).
$$

Thus $C$ is locally expressed as follows:

$$
C = \sum_B C_B = \sum_{i,B} v^i_B \frac{\partial}{\partial v^i_B},
$$

where each $C_B$ corresponds with the canonical vector field on the $B$-th copy of $TM$ on $T_k^1 M$.

Let us remark that each vector field $C_A$ on $T_k^1 M$ can also be defined using the $A$-lifts of vectors as follows: $C_A((v_1)_q, \ldots, (v_k)_q) = ((v_A)_q)^A(v)$.

From (5), (54) and (56) we deduce the following

**Proposition 6.11** A $k$-vector field $\xi = (\xi_1, \ldots, \xi_k)$ on $T_k^1 M$ is a SOPDE if and only if

$$
J_A^A(\xi_A) = C_A, \quad \forall 1 \leq A \leq k,
$$

where $(J^1, \ldots, J^k)$ is the canonical $k$-tangent structure on $T_k^1 M$.

### 6.5 Lagrangian formalism and $k$-tangent structures

Given a Lagrangian function of the form $L = L(x^i, v^i_A)$ one obtains, by using a variational principle, the *generalized Euler-Lagrange equations* for $L$:

$$
\sum_{A=1}^k \frac{d}{dt^A}(\frac{\partial L}{\partial v^i_A}) - \frac{\partial L}{\partial x^i} = 0, \quad v^i_A = \frac{\partial x^i}{\partial t^A}.
$$

In this section, following the ideas of Günther [4], we will describe the above equations (57) in terms of the geometry of $k$-tangent structures. In classical mechanics the symplectic structure of Hamiltonian theory and the tangent structure of Lagrangian theory play
complementary roles \cite{21,22,23,24,25}. Our purpose in this section is to show that the \( k \)-symplectic structures and the \( k \)-tangent structures play similarly complementary roles.

First of all, we note that such a \( L \) can be considered as a function \( L : T^1_k M \to \mathbb{R} \) with \( M \) a manifold with local coordinates \((x^i)\). Next, we construct a \( k \)-symplectic structure on the manifold \( T^1_k M \), using its canonical \( k \)-tangent structure for each \( 1 \leq A \leq k \). We consider:

- the vertical derivation \( \iota_{J^A} \) of \( \iota_* \) defined by \( J^A \), which is defined by
  \[
  \iota_{J^A} f = 0
  \]
  \[
  (\iota_{J^A} \alpha)(X_1, \ldots, X_p) = \sum_{j=1}^{p} \alpha(X_1, \ldots, J^A X_j, \ldots, X_p),
  \]
  for any function \( f \) and any \( p \)-form \( \alpha \) on \( T^1_k M \);

- the vertical differentiation \( d_{J^A} \) of forms on \( T^1_k M \) defined by
  \[
  d_{J^A} = [\iota_{J^A}, d] = \iota_{J^A} \circ d - d \circ \iota_{J^A},
  \]
  where \( d \) denotes the usual exterior differentiation.

Let us consider the 1–forms \((\beta_L)_A = d_{J^A} L, 1 \leq A \leq k \). In a local coordinate system \((x^i, v^i_A)\) we have
\[
(\beta_L)_A = \frac{\partial L}{\partial v^i_A} dx^i, \ 1 \leq A \leq k. \tag{58}
\]

**Definition 6.12** A Lagrangian \( L \) is called regular if and only if
\[
\det(\frac{\partial^2 L}{\partial v^i_A \partial v^B_j}) \neq 0, \quad 1 \leq i, j, \leq n, \ 1 \leq A, B \leq k. \tag{59}
\]

By introducing the following 2–forms \((\omega_L)_A = -d(\beta_L)_A, 1 \leq A \leq k \), one can easily prove the following.

**Proposition 6.13** \( L : T^1_k M \to \mathbb{R} \) is a regular Lagrangian if and only if \((\omega_L)_A, \ldots, (\omega_L)_k, V)\) is a \( k \)-symplectic structure on \( T^1_k M \), where \( V \) denotes the vertical distribution of \( \tau : T^1_k M \to M \). \( \blacksquare \)
Let $L : T^1_k M \longrightarrow \mathbb{R}$ be a regular Lagrangian and let us consider the $k$–symplectic structure $((\omega_L)_1, \ldots, (\omega_L)_k, V)$ on $T^1_k M$ defined by $L$. Let $\Omega^L_k$ be the morphism defined by this $k$–symplectic structure

$$\Omega^L_k : T^1_k (T^1_k M) \longrightarrow T^*(T^1_k M).$$

Thus, we can set the following equation:

$$\Omega^L_k (X_1, \ldots, X_k) = dE_L,$$

where $E_L = C(L) - L$, and where $C$ is the canonical vector field of the vector bundle $\tau : T^1_k M \rightarrow M$.

**Proposition 6.14** Let $L$ be a regular Lagrangian. If $\xi = (\xi_1, \cdots, \xi_k)$ is a solution of (60) then it is a SOPDE. In addition if $\xi$ is integrable then the solutions of $\xi$ are solutions of the Euler-Lagrange equations (57).

**Proof** It is a direct computation in local coordinates using (54), (56), (58) and (59).

**Remark** The Legendre map defined by Günther [4]

$$FL : T^1_k M \longrightarrow (T^1_k)^* M$$

can be described here as follows: if $v_x = (v_1, \ldots, v_k)_x \in (T^1_k M)_q$ with $q \in M$ and $v_A \in T_q M$, then $FL(v_x) = (\tilde{v}^1, \ldots, \tilde{v}^k) \in (T^1_k M)^*_x$, where $\tilde{v}^A \in T^*_x M$ is given by

$$\tilde{v}^A(z) = (\beta_L)_A(\bar{z}), \quad 1 \leq A \leq k,$$

for any $z \in T_q M$, where $\bar{z} \in T_{v_x}(T^1_k M)$ with $\tau_*(\bar{z}) = z$.

From (58) we deduce that $FL$ is locally given by

$$(x^i, v^A) \longrightarrow (x^i, \frac{\partial L}{\partial v^A}).$$

(61)

and from (58) and (61) we deduce the following

**Lemma 6.15** For every $1 \leq A \leq k$, we have $(\omega_L)_A = FL^* \omega_A$, where $\omega_1, \ldots, \omega_k$ are the 2-forms of the canonical $k$–symplectic structure of $(T^1_k)^* M$. ■
Then from \((65)\) we get that

**Proposition 6.16** Let \(L\) be a Lagrangian. The following conditions are equivalent:

1) \(L\) is regular.
2) \(FL\) is a local diffeomorphism.
3) \(((ω_L)_1, \ldots, (ω_L)_k, V)\) is a \(k\)-symplectic structure on \(T^1_kM\).

### 6.6 Second order partial differential equations on \(\mathbb{R}^k \times T^1_kM\)

The idea of this subsection is to characterize the integrable \(k\)-vector fields on \(\mathbb{R}^k \times T^1_kM\) such that their integral sections are canonical prolongations of maps from \(\mathbb{R}^k\) to \(M\).

**Definition 6.17** A \(k\)-vector field on \(\mathbb{R}^k \times T^1_kM\), that is, a section \(ξ \equiv (ξ_1, \ldots, ξ_k) : \mathbb{R}^k \times T^1_kM \rightarrow T^1_k(\mathbb{R}^k \times T^1_kM)\) of the projection \(τ_{\mathbb{R}^k \times T^1_kM} : T^1_k(\mathbb{R}^k \times T^1_kM) \rightarrow \mathbb{R}^k \times T^1_kM\), is a Second Order Partial Differential Equation (SOPDE) if and only if:

1) \(dt^A (ξ_B) = δ^A_B\)
2) \(Tpr_2 \circ ξ_B \circ i_t\) is a SOPDE on \(T^1_kM\), \(∀ t \in \mathbb{R}^k\), where \(pr_2 : \mathbb{R}^k \times T^1_kM \rightarrow T^1_kM\) is the canonical projection and \(i_t : T^1_kM \rightarrow \mathbb{R}^k \times T^1_kM\) is the canonical inclusion.

Let \((x^i)\) be a coordinate system on \(M\) and \((t^A, x^i, v^i_A)\) the induced coordinate system on \(\mathbb{R}^k \times T^1_kM\). From \((63)\) we deduce that the local expression of a SOPDE \(ξ\) is

\[
ξ_A(t^1, \ldots, t^k, x^i, v^1_A) = \frac{∂}{∂t_A} + v^i_A \frac{∂}{∂x^i} + (ξ_A)_i^i B \frac{∂}{∂v^i_B}, \quad 1 \leq A \leq k
\]  

(62)

where \((ξ_A)_B^i\) are functions on \(\mathbb{R}^k \times T^1_kM\).

**Definition 6.18** For \(φ : \mathbb{R}^k \rightarrow M\) a map, we define the first prolongation \(φ^{(1)}\) of \(φ\) as the map

\[
φ^{(1)} : \mathbb{R}^k \rightarrow J^1π \equiv \mathbb{R}^k \times T^1_kM,
\]

\[
t \mapsto j^1_t φ \equiv (t, j^1_0 φ_t).
\]

In local coordinates:

\[
φ^{(1)}(t^1, \ldots, t^k) = (t^1, \ldots, t^k, φ_i(t^1, \ldots, t^k), \frac{∂φ^i}{∂t_A}(t^1, \ldots, t^k)), \quad 1 \leq A \leq k, \ 1 \leq i \leq n.
\]  

(63)
Proposition 6.19 Let $\xi$ an integrable $k$-vector field on $\mathbb{R}^k \times T^1_k M$. The necessary and sufficient condition for $\xi$ to be a Second Order Partial Differential Equation (SOPDE) is that its integral sections are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbb{R}^k \to M$. That is

$$\xi_A(\phi^{(1)}(t)) = \phi^{(1)}_*(t)(\frac{\partial}{\partial t_A})(t)$$

for all $A = 1, \ldots, k$.

These maps $\phi$ will be called solutions of the SOPDE $\xi$.

¿From (63) and (62) we have

Proposition 6.20 $\phi : \mathbb{R}^k \to M$ is a solution of the SOPDE $\xi$, locally given by (62), if and only if

$$\frac{\partial \phi^i}{\partial t_A}(t) = v^i_A(\phi^{(1)}(t)), \quad \frac{\partial^2 \phi^i}{\partial t_A \partial t_B}(t) = (\zeta_A)^i_B(\phi^{(1)}(t)).$$

If $\xi$ is an integrable SOPDE then for all integral sections $\sigma : U \subset \mathbb{R}^k \to \mathbb{R}^k \times T^1_k M$ we have $(\tau_M \circ \sigma)^{(1)} = \sigma$ where $\tau_M : \mathbb{R}^k \times T^1_k M \to M$ is the canonical projection. Now we show how to characterize the SOPDEs on $\mathbb{R}^k \times T^1_k M$ using the canonical $k$-tangent structure of $T^1_k M$. Let us consider on $\mathbb{R}^k \times T^1_k M$ the tensor fields $\hat{J}^1, \ldots, \hat{J}^k$ of type $(1, 1)$, defined as follows:

$$\hat{J}^A = J^A - C_A \otimes dt^A, \quad 1 \leq A \leq k,$$

where we have transported the canonical $k$-tangent structure $(J^1, \ldots, J^k)$ of $T^1_k M$ to $\mathbb{R}^k \times T^1_k M$.

Proposition 6.21 A $k$-vector field $\xi = (\xi_1, \ldots, \xi_k)$ on $\mathbb{R}^k \times T^1_k M$ is a SOPDE if and only if

$$\hat{J}^A(\xi_A) = 0, \quad \bar{\eta}_A(\xi_B) = \delta_{AB},$$

for all $1 \leq A, B \leq k$.

Remark: Let us consider the trivial bundles $\pi : E = \mathbb{R}^k \times M \to \mathbb{R}^k$ and $\pi_1 : \mathbb{R}^k \times T^1_k M \to \mathbb{R}^k$. We identify each SOPDE $(\xi_1, \ldots, \xi_k)$ with the following semi-holonomic second order jet field

$$J^1 \pi \equiv \mathbb{R}^k \times T^1_k M \to J^1 \pi_1 \equiv \mathbb{R}^k \times T^1_k(T^1_k M)$$

$$(t^A, q^i, v^i_A) \to (t^A, q^i, v^i_A, v^i_A, (\xi_A)^i_B)$$
If the SOPDE $\xi$ on $\mathbb{R}^k \times T^1_k M$ is integrable, then its integral sections are canonical prolongations of maps from $\mathbb{R}^k$ to $M$ and then $\xi$ defines a second-order jet field $\Gamma$ on $\pi$ whose coordinate representation of the corresponding connection $\tilde{\Gamma}$ is

$$\tilde{\Gamma} = dt^A \otimes \left( \frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (\xi_A)_B^i \frac{\partial}{\partial v_B^i} \right),$$

since $(\xi_A)_B^i = (\xi_B)_A^i$ (see [17]).

The integrability of the SOPDE is equivalent to the condition given by $R = 0$, where $R$ is the curvature tensor of the above connection (see [12] and [17]).

6.7 Lagrangian formalism and stable $k$-tangent structures

Given a nonautonomous Lagrangian $\mathcal{L} = \mathcal{L}(t^A, q^i, v_A^i)$ one realizes that such an $\mathcal{L}$ can be considered as a function $\mathcal{L} : \mathbb{R}^k \times T^1_k M \to \mathbb{R}$.

In this section we shall give a geometrical description of Euler Lagrange equations (57) using a $k$-cosymplectic structure on $\mathbb{R}^k \times T^1_k M$ associated to the regular Lagrangian $\mathcal{L}$. This $k$-cosymplectic structure shall be constructed using the canonical tensor fields $\tilde{J}^A$, $1 \leq A \leq k$ of type $(1,1)$ on $\mathbb{R}^k \times T^1_k M$ defined by

$$\tilde{J}^A = \frac{\partial}{\partial t^A} \otimes dt^A + J^A = \frac{\partial}{\partial t^A} \otimes dt^A + \sum_{i=1}^{n} \frac{\partial}{\partial v_A^i} \otimes dq^i, \quad 1 \leq A \leq k,$$

where we have transported the canonical $k$-tangent structure $(J^1, \ldots, J^k)$ of $T^1_k M$ to $\mathbb{R}^k \times T^1_k M$. The family $(\tilde{J}^A, dt^A, \frac{\partial}{\partial t^A})$ is called the canonical stable $k$-tangent structure on $\mathbb{R}^k \times T^1_k M$.

For each $1 \leq A \leq k$, we define:

- the vertical derivation $\iota_{\tilde{J}^A}$ of forms on $\mathbb{R}^k \times T^1_k M$ by
  $$\iota_{\tilde{J}^A} f = 0, \quad (\iota_{\tilde{J}^A} \alpha)(X_1, \ldots, X_p) = \sum_{j=1}^{p} \alpha(X_1, \ldots, \tilde{J}^A X_j, \ldots, X_p),$$
  for any function $f$ and any $p$-form $\alpha$ on $\mathbb{R}^k \times T^1_k M$;

- the vertical differentiation $d_{\tilde{J}^A}$ of forms on $\mathbb{R}^k \times T^1_k M$ by
  $$d_{\tilde{J}^A} = [\iota_{\tilde{J}^A}, d] = \iota_{\tilde{J}^A} \circ d - d \circ \iota_{\tilde{J}^A},$$
  where $d$ denotes the usual exterior differentiation.
Let us consider the 1–forms
\[(\beta_L)_A = d_jA \mathcal{L} - \xi_A(\mathcal{L}) dt^A, \quad 1 \leq A \leq k.\]

In bundle coordinates \((t^A, q^i, v^i_A)\) we have
\[(\beta_L)_A = \sum_{i=1}^n \frac{\partial \mathcal{L}}{\partial v^i_A} dq^i, \quad 1 \leq A \leq k.\] (64)

**Definition 6.22** A Lagrangian \(\mathcal{L}\) is called regular if and only if the Hessian matrix
\[
\left( \frac{\partial^2 \mathcal{L}}{\partial v^i_A \partial v^j_B} \right)
\]
is non–singular.

Now, we introduce the following 2–forms
\[(\omega_L)_A = -d(\beta_L)_A, \quad 1 \leq A \leq k.\]

Using local coordinates one can easily prove the following proposition.

**Proposition 6.23** Let \(\mathcal{L} : \mathbb{R}^k \times T^1_k M \to \mathbb{R}\) be a regular Lagrangian, and \(V_{1,0}\) the vertical distribution of the bundle \(\pi_{1,0} : \mathbb{R}^k \times T^1_k M \to \mathbb{R}^k \times M\). Then, \(\mathcal{L}\) is regular if and only if \((\mathbb{R}^k \times T^1_k M, \bar{\eta}_A, (\omega_L)_A, V_{1,0})\) is a \(k\)-cosymplectic manifold. ■

Let \(\mathcal{L} : \mathbb{R}^k \times T^1_k M \to \mathbb{R}\) be a regular Lagrangian and \((dt^A, (\omega_L)_A, V_{1,0})\) the associated \(k\)-cosymplectic structure on \(\mathbb{R}^k \times T^1_k M\). The equations
\[dt^A((\xi_L)_B) = \delta^A_B, \quad (\xi_L)_A \dashv (\omega_L)_B = 0, \quad 1 \leq A, B \leq k.\] (66)

define the Reeb vector fields \\{((\xi_L)_1, \ldots, (\xi_L)_k)\} on \(\mathbb{R}^k \times T^1_k M\) which are locally given by
\[(\xi_L)_A = \frac{\partial}{\partial t^A} + ((\xi_L)_A)_B \frac{\partial}{\partial v^i_B},\] (67)

where the functions \(((\xi_L)_A)_B^i\) satisfy
\[
\frac{\partial^2 \mathcal{L}}{\partial t^A \partial v^j_C} + \frac{\partial^2 \mathcal{L}}{\partial v^i_B \partial v^j_C}((\xi_L)_A)_B^i = 0,
\] (68)

for all \(1 \leq A, B, C \leq k\) and \(1 \leq i, j \leq n\).
Since $\mathcal{L}$ is regular, from the local conditions we can define, in a neighbourhood of each point of $\mathbb{R}^k \times T^1_k M$, a $k$–vector field that satisfies (66). Next one can construct a global $k$–vector field $\xi_L$, which is a solution of (66), by using a partition of unity.

Let $\mathcal{L}$ be a regular Lagrangian and let $\Omega^k_\mathcal{L}$ be the $\sharp$-morphism defined by the $k$-cosymplectic structure $(dt^A, (\omega_\mathcal{L})_A, V_{1,0})$, as in (52):

$$
\Omega^k_\mathcal{L} : T^1_k(\mathbb{R}^k \times T^1_k M) \longrightarrow T^*(\mathbb{R}^k \times T^1_k M)
$$

$$(X_1, \ldots, X_k) \mapsto \Omega^k_\mathcal{L}(X_1, \ldots, X_k) = \sum_{A=1}^k X_A \cdot (\omega_\mathcal{L})_A + dt^A(X_A)dt^A.
$$

A direct computation in local coordinates proves the following Proposition.

**Proposition 6.24** Let $\mathcal{L}$ be a regular Lagrangian and let $X = (X_1, \ldots, X_k)$ be a $k$-vector field such that

$$
dt^A(X_B) = \delta_{AB}, \quad 1 \leq A, B \leq k
$$

$$
\Omega^k_\mathcal{L}(X_1, \ldots, X_k) = dE_L + \sum_{A=1}^k (1 - (\xi_L)_A(E_L))dt^A
$$

where $E_L = C(\mathcal{L}) - \mathcal{L}$. Then $X = (X_1, \ldots, X_k)$ is a SOPDE. In addition, if $X = (X_1, \ldots, X_k)$ is integrable then its solutions satisfy the Euler-Lagrange equations (57).

In conclusion, we can consider Eqs. (70) as a geometric version of the Euler-Lagrange field equations for a regular Lagrangian.

**Remark** We have given a geometric version of the Euler-Lagrange equations for a non autonomous Lagrangian by constructing a $k$-cosymplectic structure on $\mathbb{R}^k \times T^1_k M$ defined from the Lagrangian and the canonical stable $k$-tangent structure on $\mathbb{R}^k \times T^1_k M$. We can also construct this $k$-cosymplectic structure using the Legendre transformation $\mathcal{F}\mathcal{L}$ of $\mathcal{L}$ which is the map

$$
\mathcal{F}\mathcal{L} : \mathbb{R}^k \times T^1_k M \longrightarrow \mathbb{R}^k \times (T^1_k)^* M
$$

defined as follows:

If $(t, v) = (t^1, \ldots, t^k, v_1, \ldots, v_k) \in \mathbb{R}^k \times (T^1_k M)_x$ with $x \in M$ and $v_A \in T_x M$, then

$$
\mathcal{F}\mathcal{L}(t, y) = (t^1, \ldots, t^k, p^1, \ldots, p^k) \in \mathbb{R}^k \times (T^1_k M)_x^*, \quad p^A \in T^*_x M
$$
is given by

\[ p^A(v_x) = (\beta_{x})_A(\bar{v}_x), \quad 1 \leq A \leq k, \]

for any \( v_x \in T_xM \), where \( \bar{v}_x \in T_v(T^1_kM) \) is any tangent vector such that \( d\tau_M(v)(\bar{v}_x) = v_x \), with \( \tau_M : T^1_kM \to M \) the canonical projection. In induced coordinates we have

\[ \mathcal{FL} : (t^A, q^i, v^i_A) \to (t^A, q^i, \frac{\partial L}{\partial v^i_A}). \quad (71) \]

Now, from (64) and (71) we deduce the following.

**Lemma 6.25** \((\omega_L)_A = \mathcal{FL}^*((\omega_0)_A), \quad dt^A = \mathcal{FL}^*((\eta_0)_A), \) for all \( A \).

Then we have

**Proposition 6.26** The following conditions are equivalent:

1) \( \mathcal{L} \) is regular.

2) \( \mathcal{FL} \) is a local diffeomorphism.

3) \((dt^A, (\omega_\mathcal{L})_A, V_{1,0}) \) is a \( k \)-cosymplectic structure on \( \mathbb{R}^k \times T^1_kM \).

7 The Cartan-Hamilton-Poincaré Form on \( J^1\pi \) and \( L_\pi E \)

In this section we further explore relationships between \( n \)-symplectic geometry on frame bundles and multisymplectic geometry. Since \( m = n + k = \text{dim}(E) \) we will refer to the \( n \)-symplectic geometry on \( LE \) as \( m \)-symplectic geometry, and base the discussion on the \( n \)-form on \( J^1\pi \) considered by Cartan, Hamilton and Poincaré. This form has various names in the literature; here we will use the name Cartan-Hamilton-Poincaré (CHP) form. Although this \( n \)-form on \( J^1\pi \) has been in the literature for many years, its definition on \( L_\pi E \) is relatively recent. It appeared first in [18], where the \( n \)-form was defined in terms of newly defined Cartan-Hamilton-Poincaré 1-forms. These Cartan-Hamilton-Poincaré 1-forms play the role of an \( m \)-symplectic potential on \( L_\pi E \) and are discussed in Section 7.5. In Section 7.4 we give a new geometrical definition of \( \Theta_L \) on \( J^1\pi \). See also Section 9.4 where the Cartan-Hamilton-Poincaré 1-forms are defined using an \( m \)-symplectic Legendre transformation.
7.1 The Cartan-Hamilton-Poincaré Form on $J^{1\pi}$

One method used to construct the Cartan-Hamilton-Poincaré Form on $J^{1\pi}$ is to first construct a vector valued $m$-form $S_\omega$ on $J^{1\pi}$ associated with a volume form $\omega$ on $M$, as follows: For each 1-form $\sigma$ on $J^{1\pi}$ the vector valued 1-form $S\sigma$ along $\pi_1: J^{1\pi} \to M$ is defined by

$$\alpha((S\sigma)(X)) = \sigma(S_\alpha(X))$$

for any vector field $X$ on $J^{1\pi}$ and any 1-form $\alpha$ on $M$. Recall $S_\alpha$ was defined in Section 2.4.

Now $S_\omega$ is defined according to the rule

$$S_\omega \downarrow \sigma = \iota_{S\sigma} \omega$$

where $\iota_{S\sigma}$ is the derivation of type $\iota_*$ corresponding to $S\sigma$, that is

$$\sigma(S_\omega(X_1, \ldots, X_m)) = (\iota_{S\sigma} \omega)(X_1, \ldots, X_m) = \sum_{i=1}^{n} \omega((\pi_1)_*X_1, \ldots, S\sigma(X_i), \ldots, (\pi_1)_*X_m)$$

for any vector fields $X_1, \ldots, X_m$ on $J^{1\pi}$. In coordinates

$$S_\omega = (dy^A - y^A_j dx^j) \wedge \left( \frac{\partial}{\partial x^i} \downarrow \omega \right) \otimes \frac{\partial}{\partial v^A_i} \quad (72)$$

If $L_\pi: J^{1\pi} \to \Lambda^n M$ is a Lagrangian density, then $L_\pi = L \omega$ where $L: J^{1\pi} \to \mathbb{R}$. The Cartan-Hamilton-Poincaré $n$-form of $L$ is defined by

$$\Theta_L = L \omega + S_\omega \ast dL = L \omega + dL \circ S_\omega. \quad (73)$$

In coordinates

$$\Theta_L = L \omega + \frac{\partial L}{\partial y^A_i}(dy^A - y^A_j dx^j) \wedge \left( \frac{\partial}{\partial x^i} \downarrow \omega \right)$$

$$\Theta_L = L \omega + \frac{\partial L}{\partial x^i} \wedge \left( \frac{\partial}{\partial v^A_i} \downarrow \omega \right) \quad (74)$$

7.2 The tensors $S_\alpha$ and $S_\omega$ on $J^{1\pi}$ viewed from $L_\pi E$

For each 1-form $\alpha$ on $M$, we shall define on $L_\pi E$ a tensor field $\tilde{S}_\alpha$, of type $(1, 1)$ that projects on the tensor $S_\alpha$ on $J^{1\pi}$. Let $(B_i = B(\hat{r}_i), B_A = B(\hat{r}_A))$ be the standard vector fields of any torsion free linear connection on $\lambda: L_\pi E \to E$. In local coordinates we have

$$B_i = v_i^s \frac{\partial}{\partial x^s} + v_i^C \frac{\partial}{\partial y^C} + V_i, \quad B_A = v_A^C \frac{\partial}{\partial y^C} + V_A \quad (75)$$
where $V_i, V_A$ are vertical with respect to $\lambda$.

Now if $\alpha$ is an arbitrary 1-form on $M$ and $(\pi \circ \lambda)^* \alpha$ its pull-back to $L_\pi E$, we consider on $L_\pi E$ the functions $((\pi \circ \lambda)^* \alpha)(B_i)$ for each $1 \leq i \leq n$. In coordinates, if $\alpha = \alpha_r dx^r$, then from (75)

$$((\pi \circ \lambda)^* \alpha)(B_i) = \alpha_r dx^r \left( v^s_i \frac{\partial}{\partial x^s} + v^C_i \frac{\partial}{\partial y^C} + V_i \right) = \alpha_r v^r_i .$$

(76)

Taken together the function $\hat{\alpha} = (\alpha_a v^a_i) \hat{r}_i$ is the $(\mathbb{R}^n)^*$-valued tensorial 0-form on $LE$ corresponding to $\alpha$ on $M$.

**Definition 7.1** The vector-valued 1-form $\tilde{S}_\alpha$ on $L_\pi E$ is defined by

$$\tilde{S}_\alpha = ((\pi \circ \lambda)^* \alpha)(B_i) E^s_B \otimes \theta^B .$$

From (36) and (76) we obtain that in local coordinates

$$\tilde{S}_\alpha = \alpha_j (dy^B - u^B_i dx^r) \otimes \frac{\partial}{\partial u^B_j} .$$

(77)

**Proposition 7.2** The relationship between $\tilde{S}_\alpha$ on $L_\pi E$ and $S_\alpha$ on $J^1 \pi$ is given by

$$\tilde{S}_\alpha \rhd \rho_* = \rho_* \dashv S_\alpha$$

that is

$$\rho_* (\tilde{S}_\alpha(\rho(u))(X_u)) = S_\alpha(\rho(u))(\rho_* (u)(X_u))$$

for any $u \in L_\pi E$ and any $X_u \in T_u(L_\pi E)$.

**Proof** : It is an immediate consequence of the local expressions of $\tilde{S}_\alpha$ and $S_\alpha$ taking into account that $\rho^* y^B_i = u^B_i$. ■

Now, proceeding analogously, we construct a tensor field $\tilde{S}_\omega$ of type $(1, n)$ on $L_\pi E$ using the tensor field $\tilde{S}_\omega$ on $L_\pi E$, associated with a volume form $\omega$ on $M$. We then construct the corresponding Cartan-Hamilton-Poincaré form on $L_\pi E$.

For each 1-form $\sigma$ on $L_\pi E$ the vector valued 1-form $\tilde{S} \sigma$ along $\pi \circ \lambda : L_\pi E \to M$ is defined by

$$\alpha((\tilde{S} \sigma)(X)) = \sigma(\tilde{S}_\alpha(X))$$

(78)
for any vector field $X$ on $L_\pi E$ and any 1-form $\alpha$ on $M$. We shall compute the local expression of this 1-form. If we write

$$\sigma = \sigma_i dx^i + \sigma_A dy^A + \sigma_j^i du^B_j + \sigma^i_B du^B_A$$

and we take $\alpha = dx^j$ then from (33) and (78) we obtain

$$dx^j(\tilde{S}\sigma(\frac{\partial}{\partial x^i})) = -\sigma^B_B u^B_i, \quad dx^j(\tilde{S}\sigma(\frac{\partial}{\partial y^A})) = \sigma^A_A,$$
$$dx^j(\tilde{S}\sigma(\frac{\partial}{\partial u^A_i})) = dx^j(\tilde{S}\sigma(\frac{\partial}{\partial u^A_B})) = 0$$

Therefore the local expression of $\tilde{S}\sigma$ is

$$\tilde{S}\sigma = \sigma^B_B(dy^B - u^B_i dx^i) \otimes \frac{\partial}{\partial x^j}. \quad (79)$$

**Definition 7.3** The tensor field $\tilde{S}_\omega$ is defined according to the rule

$$\tilde{S}_\omega \bullet \sigma = \iota_{\tilde{S}\sigma} \Omega$$

where $\iota_{\tilde{S}\sigma}$ is the derivation of type $\iota_*$ corresponding to $\tilde{S}\sigma$, that is

$$\sigma(\tilde{S}_\omega(X_1, \ldots, X_n)) = (\iota_{\tilde{S}\sigma} \omega)(X_1, \ldots, X_n)$$
$$= \sum_{j=1}^n \omega((\pi \circ \lambda)_*X_1, \ldots, \tilde{S}\sigma(X_j), \ldots, (\pi \circ \lambda)_*X_n) \quad (80)$$

for any vector fields $X_1, \ldots, X_n$ on $L_\pi E$ and any 1-form $\sigma$ on $L_\pi E$.

From (79) and (80) we obtain that the local expression of $\tilde{S}_\omega$ is

$$\tilde{S}_\omega = (dy^A - u^A_i dx^i) \wedge \left(\frac{\partial}{\partial x^i} \bullet \omega\right) \otimes \frac{\partial}{\partial u^A_i} \quad (82)$$

**Proposition 7.4** The relationship between $\tilde{S}_\omega$ on $L_\pi E$ and $S_\omega$ on $J^1 \pi$ is given by

$$\tilde{S}_\omega \bullet \rho_* = \rho_* \bullet S_\omega$$

that is

$$\rho_*(u) \left(\tilde{S}_\omega(u)((X_u)_1, \ldots, (X_u)_n)\right) = S_\omega(\rho(u))(\rho_*(u)((X_u)_1), \ldots, \rho_*(u)((X_u)_n))$$

for any $u \in L_\pi E$ and any $(X_u)_1, \ldots, (X_u)_n \in T_u(L_\pi E)$.

**Proof**: It is an immediate consequence of the local expressions (72) and (77) of $\tilde{S}_\omega$ and $S_\omega$ taking into account that $\rho^*y^B_i = u^B_i$. □
7.3 The Cartan-Hamilton-Poincaré form $\Theta_L$ on $J^1\pi$ viewed from $L_\pi E$

Using the tensor field $\tilde{S}_\omega$ we shall construct an $m$-form on $L_\pi E$ that projects to the corresponding Cartan-Hamilton-Poincaré $m$-form on $J^1\pi$.

**Definition 7.5** A Lagrangian on $L_\pi E$ is a function $L : L_\pi E \to \mathbb{R}$.

**Definition 7.6** [48] A Lagrangian on $L_\pi E$ is lifted if it satisfies the auxiliary conditions

$$E^*_j(L) = 0 \quad E^*_B(L) = 0 \quad (83)$$

**Remark** Using [20] these conditions imply that $L$ is constant on the fibers of $\rho : L_\pi E \to J^1\pi$, and thus is the pull up of a function $\mathcal{L}$ on $J^1\pi$, that is $\rho^*\mathcal{L} = L$.

**Definition 7.7** If $L : L_\pi E \to \mathbb{R}$ is a lifted Lagrangian on $L_\pi E$, then we define the Cartan-Hamilton-Poincaré $m$-form of $L$ by

$$\theta_L = L \omega + \tilde{S}_\omega^* dL = L \omega + dL \circ \tilde{S}_\omega.$$ 

If $\omega = d^n x = dx^1 \wedge \cdots \wedge dx^n$ then from (82) we obtain that the local expression of $\theta_L$ is

$$\theta_L = \left( L - u_i^A \frac{\partial L}{\partial u_i^A} \right) d^n x + \frac{\partial L}{\partial u_i^A} dy^A \wedge d^{n-1} x_i. \quad (84)$$

**Proposition 7.8** If $L$ is a lifted Lagrangian then the corresponding $m$-form satisfies $\rho^*\Theta_\mathcal{L} = \theta_L$, where $\Theta_\mathcal{L}$ is the Cartan-Hamilton-Poincaré $n$-form on $J^1\pi$ corresponding to $\mathcal{L}$.

**Proof** It follows from the local expressions taking into account that $\rho^*y_i^A = u_i^A$. ■
7.4 The $m$-symplectic structure on $L_πE$ and the formulation of the Cartan-Hamilton-Poincaré $n$-form

We consider next the definition of the Cartan-Hamilton-Poincaré 1-forms on $L_πE$ introduced in [48, 32]. These 1-forms combine into an $\mathbb{R}^m$-valued 1-form whose exterior derivative plays the role of a general $m$-symplectic structure on $L_πE$.

**Definition 7.9** [48] Let $L : L_πE → \mathbb{R}$ be a lifted Lagrangian on $L_πE$, and $τ(n)$ a positive function of $n = \text{dim } M$. The Cartan-Hamilton-Poincaré 1-forms $θ^i_L$ on $L_πE$ are

\[
\begin{align*}
θ^i_L &= τ(n)θ^i + E^*_A(L)θ^A \\
θ^A_L &= θ^A
\end{align*}
\]  

where $E^*_A$ are defined above in (20), and $θ^i$ and $θ^A$ are the components of the canonical soldering 1-form on $L_πE$.

**Remark** The quantities $E^*_A(L)$, referred to as the "covariant canonical momenta" in [48], are **globally defined** on $L_πE$. In local canonical coordinates $(z^α, π^μ_ν)$, these quantities have the local expressions

\[
E^*_A(L) = π^j_A p^j_B u_B, \quad θ^j_B = \frac{∂L}{∂u^j_B}
\]

and clearly are the frame components of the "canonical field momenta" $p^j_B = \frac{∂L}{∂u^j_B}$. For different values of $τ$ one can obtain the de Donder-Weyl theory [19, 44] and the Caratheodory theory [51, 44] as special cases of the formalism presented in reference [48]. The significance of these CHP 1-forms as regards other geometrical theories was also considered by MacLean and Norris. In [48] it was shown that one may construct the CHP $n$-form on $J^1π$ from the CHP 1-forms on $L_πE$. In this regard see also references [11, 28]. We now recall the construction of the Cartan-Hamilton-Poincaré $n$-form on $J^1π$ from these CHP 1-forms.

**Proposition 7.10** [48] Let $(B_i, B_A)$ denote the standard horizontal vector fields of any torsion free linear connection on $λ : L_πE → E$, and let $\text{vol}$ denote the pull up to $L_πE$ of a fixed
volume $n$-form $\omega$ on $M$. Set $\text{vol}_i = B_i \int \text{vol}$. Then when $\tau(n) = \frac{1}{n}$ the $n$-form

$$\theta_L := \theta_L^i \wedge \text{vol}_i$$

passes to the quotient to define the CHP-$n$-form $\Theta_L$ on $J^1\pi$ associated with $\text{vol} = \omega$.

Next we shall show here that the Cartan-Hamilton-Poincaré 1-forms can be obtained from the canonical $m$-tangent structure $J^i, J^A$ on $L_{\pi}E$. Let $\Lambda = f_1 \wedge \cdots \wedge f_n$ be a fixed contravariant volume on $M$, with $f_i$ locally written as $f_i = \alpha_i^j \frac{\partial}{\partial x^j}$. Thus $\Lambda$ is a nowhere vanishing $n$-vector on $M$, which is the covariant version of a volume form on $M$. In coordinates

$$\lambda = \det(\alpha^i_j) \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$$

Now given an arbitrary point $u = (e_i, e_A)_e$ on $L_{\pi}E$ we can define the $n$-vector

$$[\tilde{e}_i] = \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n$$

where $\tilde{e}_i = (\pi \circ \lambda)_*(u)(e_i)$. $[\tilde{e}_i]$ is a well-defined $n$-vector at $(\pi \circ \lambda)(u) = \pi(e) \in M$ since the vectors $\tilde{e}_i$ are linearly independent. In coordinates

$$[\tilde{e}_i] = \det(v^i_j) \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$$

We can now define a function $\sigma : L_{\pi}E \to \mathbb{R}$ relative to the fixed contravariant volume $\Lambda$ on $M$ by the formula

$$[\tilde{e}_i] = \sigma(u) \lambda(\pi(e))$$

Using the local expressions above it is easy to see that in local coordinates on $L_{\pi}E$ one has

$$\sigma(u) = \frac{\det(v^i_j)(u)}{\det(\alpha^i_j(\pi(e)))} \quad (88)$$

**Proposition 7.11** Let $L$ be a lifted Lagrangian on $L_{\pi}E$ and let $\sigma$ be the function defined on $L_{\pi}E$ relative to a fixed contravariant volume $\Lambda$ on $M$. Then the Cartan-Hamilton-Poincaré 1-forms on $L_{\pi}E$ are given by the formula

$$\theta^i_L = \frac{1}{\sigma} d_J^i(\sigma L)$$
where
\[ \tilde{J}^i = \frac{1}{n} (E^*_{ij} \otimes \theta^j) + E^*_A \otimes \theta^A \]
and \( d_{ji} = [y_{ji}, d] \).

**Proof** From (20) and (88) we obtain that
\[ E^*_j (\sigma) = \sigma \delta^i_j \]
and from (83) we obtain
\[ E^*_j (\sigma L) = \sigma L \delta^i_j \]
and from (84) we obtain
\[ E^*_A (\sigma L) = \sigma E^*_A (L) \].

Now from these last identities we have
\[ \frac{1}{\sigma} d_{ji} (\sigma L) = \frac{1}{\sigma} \left( d(\sigma L) \circ \tilde{J}^i \right) = \frac{1}{\sigma} \left( \frac{1}{n} E^*_j (\sigma L) \theta^j + E^*_A (\sigma L) \theta^A \right) \]
\[ = \frac{1}{\sigma} \left( \frac{1}{n} \sigma L \delta^i_j \theta^j + \sigma E^*_A (L) \theta^A \right) = \frac{1}{n} L \theta^i + E^*_A (L) \theta^A. \]

**Remark** To these three constructions of the Cartan-Hamilton-Poincaré 1-forms on \( L \pi E \) we add a fourth in Section 9.4 where we show that the \( \theta^*_L \) are the pull-backs, under a suitable defined \( m \)-symplectic Legendre transformation, of the canonical \( m \)-symplectic structure on \( LE \).

## 8 Multisymplectic formalism

An alternative way to derive the field equations is to use the so-called multisymplectic formalism, developed by the Tulczyjew school in Warsaw (see [36, 37, 51, 52]), and independently by García and Pérez-Rendón [53, 54] and Goldschmidt and Sternberg [1]. This approach was revised by Martin [55, 56] and Gotay et al [34, 35, 57, 58, 59], and more recently by Cantrijn et al [38, 39].

### 8.1 Lagrangian formalism

Assume a Lagrangian \( \mathcal{L} : J^1 \pi \to \mathbb{R} \) where \( J^1 \pi \) is the 1-jet prolongation of a fibered manifold \( \pi : E \to M \). \( M \) is supposed to be oriented with volume form \( \omega \). We take adapted coordinates \( (x^i, y^A, y^i_\alpha) \) such that \( \omega = dx^1 \wedge \cdots \wedge dx^n = d^n x \).
Denote $\Omega_L = -d\Theta_L$ where $\Theta_L$ is the Cartan-Hamilton-Poincaré $m$-form introduced in 7.1. From (73) we have that in local coordinates

$$\Omega_L = d(y_i^A \frac{\partial L}{\partial y_i^A} - L) \wedge d^n x - d(\frac{\partial L}{\partial y_i^A}) \wedge dy^A \wedge d^{n-1}x^i$$

where $d^{n-1}x^i = \frac{\partial}{\partial x^i} \omega$.

**Definition 8.1** $\Omega_L$ is called the Cartan-Hamilton-Poincaré $(n+1)$-form.

One can use this multisymplectic form to re-express, in an intrinsic way, the Euler-Lagrange equations, which in coordinates take the classical form

$$\sum_{i=1}^{k} \frac{\partial}{\partial x^i} (\frac{\partial L}{\partial y_i^A}(x^i, \phi^B(x)), \frac{\partial \phi^B}{\partial x^i}(x)) - \frac{\partial L}{\partial y_i^A}(x^i, \phi^B(x), \frac{\partial \phi^B}{\partial x^i}(x)) = 0,$$  (89)

for a (local) section $\phi$ of $\pi : E \to M$.

**Theorem 8.2** For a section $\phi$ of $\pi$ the following are equivalent:

(i) the Euler-Lagrange equations (89) hold in coordinates;

(ii) for any vector field $X$ on $J^1\pi$

$$(j^1\phi)^*(X \bigtriangledown \Omega_L) = 0.$$  (90)

The proof can be found in [34].

$\Omega_L$ is a multisymplectic form on $J^1\pi$ provided $L$ is regular, that is, the Hessian matrix

$$\left(\frac{\partial^2 L}{\partial y_i^A \partial y_j^B}\right)$$

is nonsingular.

We can extend equations (90) to sections $\tau$ of $J^1\pi \to M$, that is we consider sections $\tau$ such that

$$\tau^*(X \bigtriangledown \Omega_L) = 0,$$  (91)

for any vector field $X$ on $J^1\pi$. If the Lagrangian $L$ is regular then both problems (90) and (91) are equivalent, that is, such a $\tau$ is automatically a 1-jet prolongation $\tau = j^1\phi$. Equation (91) corresponds to the so called de Donder problem (see Binz et al [60].)
8.2 Hamiltonian formalism

We have an exact sequence of vector bundles over $E$:

\[ 0 \to \bigwedge_1^n E \overset{i}{\to} \bigwedge_2^n E \overset{\mu}{\to} J^1\pi^* \to 0 \]

where $J^1\pi^*$ is the quotient vector bundle

\[ J^1\pi^* = \frac{\bigwedge_2^n E}{\bigwedge_1^n E}, \]

$i$ is the inclusion, and $\mu$ is the projection map.

$J^1\pi^*$ is sometimes defined as the affine dual bundle of $J^1\pi$ (see [17]). We have taken local coordinates $(x^i, y^A, p)$ on $\bigwedge_1^n E$ and $(x^i, y^A, p^i_A)$ on $\bigwedge_2^n E$, and then $(x^i, y^A, p^i_A)$ can be taken as local coordinates in $J^1\pi^*$.

To develop a Hamiltonian theory, we need a Hamiltonian, in this case a section $H : J^1\pi^* \to \bigwedge_2^n E$ of the canonical projection $\mu$. In coordinates, we have

\[ H(x^i, y^A, p^i_A) = (x^i, y^A, -\hat{H}, p^i_A) \]

where $\hat{H} = \hat{H}(x^i, y^A, p^i_A) \in C^\infty(J^1\pi^*, \mathbb{R})$.

Take the pull-back $\Omega_H = H^* \Omega^2_E$ (we also have $\Theta_H = H^* \Theta^2_E$ such that $\Omega_H = -d\Theta_H$), then from (48) we have

\[ \Theta_H = -\hat{H} d^n x + p^i_A dy^A \wedge d^{n-1} x^i, \quad \Omega_H = d\hat{H} \wedge d^n x - dp^i_A \wedge dy^A \wedge d^{n-1} x^i, \]

$\Omega_H$ is again a multisymplectic $(n+1)$-form. Now solutions of the Hamilton equations

\[ \frac{\partial \gamma^A}{\partial x^i} = -\frac{\partial \hat{H}}{\partial p^i_A}, \quad \sum_i \frac{\partial \gamma^i_A}{\partial x^i} = \frac{\partial \hat{H}}{\partial y^A}. \]

are obtained by looking for sections

\[ \gamma : M \to J^1\pi^*, \quad (x^i) \mapsto (x^i, \gamma^A, \gamma^i_A) \]

such that

\[ \gamma^*(Y \lrcorner \Omega_H) = 0 \]

for any vector field $Y$ on $J^1\pi^*$, see [38].
To relate both formalisms, we must use the Legendre transformation. For $\mathcal{L}$, we define a fibered mapping over $E$, $\text{Leg} : J^1\pi \to \bigwedge^1_1 E$, by

$$[\text{Leg}(j^1_x\phi)](X_1, \ldots, X_n) = (\Theta_{\mathcal{L}})_{j^1_x\phi}(\tilde{X}_1, \ldots, \tilde{X}_n)$$

for all $X_1, \ldots, X_n \in T_{\phi(x)}E$, where $\tilde{X}_1, \ldots, \tilde{X}_n \in T_{j^1_x\phi}(J^1\pi)$ are such that they project on $X_1, \ldots, X_n$, respectively.

In local coordinates

$$\text{Leg}(x^i, y^A, y^A_i) = (x^i, y^A, \mathcal{L} - y^A_i \frac{\partial \mathcal{L}}{\partial y^A_i}, \frac{\partial \mathcal{L}}{\partial y^A_i}).$$

If we compose $\text{Leg} : J^1\pi \to \bigwedge^1_1 E$ with $\mu : \bigwedge^1_1 E \to J^1\pi^*$, we obtain the reduced Legendre transformation

$$\text{leg} : J^1\pi \to J^1\pi^*$$

$$\begin{array}{c}
(x^i, y^A, y^A_i) \\
\mapsto \\
(x^i, y^A, \frac{\partial \mathcal{L}}{\partial y^A_i})
\end{array}$$

which extends the usual one in mechanics, and the Legendre map defined by Günther. (see remark in Section 6.5).

A direct computation shows that $\text{leg}^*\Theta^2_E = \Theta_{\mathcal{L}}, \quad \text{leg}^*\Omega^2_E = \Omega_{\mathcal{L}}$.

It is clear that $\text{leg} : J^1\pi \to J^1\pi^*$ is a local diffeomorphism if and only if $\mathcal{L}$ is regular. If $\mathcal{L}$ is regular, then we can define a (local) section $H$ as follows $H = \text{Leg} \circ \text{leg}^{-1}$

$$J^1\pi \to \bigwedge^2_2 E$$

Proposition 8.3 The following assertions are equivalents:

1) $\mathcal{L}$ is regular.

2) $\Omega_{\mathcal{L}}$ is multisymplectic, and

3) $\text{leg} : J^1\pi \to J^1\pi^*$ is a local diffeomorphism.

8.3 Ehresmann connections and the Lagrangian and Hamiltonian formalisms

A different geometric version of the field equations was given recently, based on Ehresmann connection [39].
In mechanics we look for curves and their linear approximations; that is, we look for tangent vectors. In Field Theory, we look for sections, and their linear approximations are just horizontal subspaces of Ehresmann connections in the fibration $\pi_1 : J^1\pi \to M$.

A connection in $\pi_1$ (in the sense of Ehresmann [31, 62]) is defined by a complementary distribution $H$ of $V\pi_1$, i.e., we have the following Withney sum of vector bundles over $E$:

$$T(J^1\pi) = H \oplus V\pi_1.$$ 

As is well-known, we can characterize a connection in $\pi_1$ as a (1,1)-tensor field $\Gamma$ on $J^1\pi$ such that

- $\Gamma^2 = Id$, and
- the eigenspace at the point $z \in J^1\pi$ corresponding to the eigenvalue $-1$ is the vertical subspace $(V\pi_1)_z$.

In other words, $\Gamma$ is an almost product structure on $J^1\pi$ whose eigenvector bundle corresponding to the eigenvalue $-1$ is just the vertical subbundle $V\pi_1$.

We denote by

$$h = \frac{1}{2}(Id + \Gamma), \quad v = \frac{1}{2}(Id - \Gamma),$$

the horizontal and vertical projectors, respectively. Hence, the horizontal distribution is given by $H = Im \ h$ and $Im \ v = V\pi_1$.

We say that $\Gamma$ is flat if the horizontal distribution is integrable. In such a case, from the Frobenius theorem, there exists a horizontal local section $\gamma$ of $\pi_1$ passing through each point of $J^1\pi$. Let us recall that a local section $\gamma$ of $\pi_1 : J^1\pi \to M$ is called horizontal if it is an integral submanifold of the horizontal distribution.

Suppose that $h$ is locally expressed in fibered coordinates $(x^i, y^A, y^A_i)$ as follows:

$$h = dx^i \otimes \left[ \frac{\partial}{\partial x^i} + \Gamma^A_i \frac{\partial}{\partial y^A} + \Gamma^{A}_{ji} \frac{\partial}{\partial y^A_j} \right]$$ (92)

A direct computation in local coordinates shows that the equation

$$\iota_h \Omega_\mathcal{C} = (n-1)\Omega_\mathcal{C}$$
may be considered as the geometric version of the field equations, where $h$ is the horizontal projector of the Ehresmann connection in $J^1\pi \to M$. Indeed, from (12) and the local expression of $\Omega_L$ we deduce that $\iota_h\Omega_L = (n-1)\Omega_L$ if and only if
\[
\frac{\partial L}{\partial y^A} - \frac{\partial^2 L}{\partial y_i^A \partial x^i} - \Gamma^B_i \frac{\partial^2 L}{\partial y_i^A \partial y^B} - \Gamma^B_j \frac{\partial^2 L}{\partial y_i^A \partial y^j} + (\Gamma^B_i - y^B_i) \frac{\partial^2 L}{\partial y_i^A \partial y^B} = 0 , \tag{93}
\]
and
\[
(\Gamma^B_j - y^B_j) \frac{\partial^2 L}{\partial y_i^A \partial y^B} = 0 . \tag{94}
\]
If $L$ is regular, (94) implies $\Gamma^B_j = y^B_j$, for all $B,j$, and then (93) becomes
\[
\frac{\partial L}{\partial y^A} - \frac{\partial^2 L}{\partial y_i^A \partial x^i} - y^B_i \frac{\partial^2 L}{\partial y_i^A \partial y^B} - \Gamma^B_j \frac{\partial^2 L}{\partial y_i^A \partial y^j} = 0 , \tag{95}
\]
Hence, if $\Gamma$ is flat and $\gamma : M \to J^1\pi$ is a a horizontal local section locally given by $\gamma(x^i) = (x^i, \gamma_i^A, \gamma^A_i)$, then taking into account that $\gamma_*(T_x M) = H_{\gamma(x)}$ we obtain
\[
\Gamma^A_i = y^A_i = \frac{\partial \gamma^A_i}{\partial x^i} = \gamma^A_i , \quad \Gamma^A_j = \frac{\partial \gamma^A_j}{\partial x^i} = \frac{\partial^2 \gamma^A_j}{\partial x^i \partial x^j} . \tag{96}
\]
This implies that $\gamma$ is a 1-jet prolongation, i.e. $\gamma = j^1 \phi$ and, $\phi$ is a solution of (93), that is, $\phi$ is solution of the Euler-Lagrange equations (89).

Again, we can look for Ehresmann connections in the fibration $J^1\pi^* \to M$. Indeed, if $\tilde{h}$ is the horizontal projector of such a connection, we deduce that
\[
\iota_{\tilde{h}}\Omega_H = (n-1)\Omega_H
\]
if and only if
\[
\bigwedge_A^i = -\frac{\partial H}{\partial p^A_i} , \quad \sum_i \bigwedge_A^i = \frac{\partial H}{\partial y^A} ,
\]
where
\[
\tilde{h} = dx^i \otimes \left[ \frac{\partial}{\partial x^i} + \bigwedge_A^i \frac{\partial}{\partial y^A} + \bigwedge_A^j \frac{\partial}{\partial p^A_j} \right].
\]
Therefore, if $\tilde{h}$ is flat, and $\gamma$ is an integral section of $\tilde{h}$, we deduce that $\gamma$ satisfies the Hamilton equations for $H$. 

8.4 Polysymplectic formalism

An alternative formalism for Classical Field Theories is the so-called polysymplectic approach (see [63, 64, 65, 66, 67, 68, 70, 71, 72, 73]). The geometric ingredients are almost the same as in multisymplectic theory, except that we consider vector-valued Cartan-Hamilton-Poincaré forms.

We start with a fibred bundle $\pi : E \rightarrow M$ as above, and introduce the following spaces

- The Legendre bundle
  $$\Pi = \bigwedge^n M \otimes_E V^* \pi \otimes_E TM$$
  where $V^* \pi$ is the dual vector bundle of the vertical bundle $V \pi$.

- The homogeneous Legendre bundle
  $$Z_E = T^* E \wedge (\bigwedge^{n-1} M).$$

$Z_E$ (resp. $\Pi$) will play the role of $\bigwedge^n E$ (resp. $J^1 \pi^*$) in multisymplectic formalism. Accordingly, we introduce coordinates $(x^i, y^A, p, p^i_A)$ on $Z_E$, and $(x^i, y^A, p^i_A)$ on $\Pi$. Moreover, there exists a canonical embedding $\theta : \Pi \rightarrow \bigwedge^{n+1} E \otimes_E TM$ defined by $\theta = -p^i_A dy^A \wedge \omega \otimes \frac{\partial}{\partial x^i}$.

**Definition 8.4** The polysymplectic form on $\Pi$ is the unique $TM$-valued $(n+2)$-form $\Omega$ such that the relation

$$\iota_\phi \Omega = -d(\phi \llcorner \theta)$$

holds for any 1-form $\phi$ on $M$.

A direct computation shows that $\Omega$ has the following local expression

$$\Omega = dp^i_A \wedge dy^A \wedge \omega \otimes \frac{\partial}{\partial x^i}.$$ 

A covariant Hamiltonian is given by a Hamiltonian form, that is, a section $H$ of the canonical projection $Z_E \rightarrow \Pi$, as in the multisymplectic settings. The field equations are provided by a connection $\gamma$ in the fibration $\Pi \rightarrow M$ such that $\gamma \llcorner \Omega$ is closed, and $\gamma$ is then called a Hamilton connection (see [34] for details).
The Cartan-Hamilton-Poincaré $m$-form $\Theta_L$ defines the Legendre transformation

$$\mathcal{FL} : J^1\pi \longrightarrow Z_E$$

by

$$\mathcal{FL}(x^i, y^A, y_i^A) = (x^i, y^A, L - y_i^A \frac{\partial L}{\partial y_i^A} \frac{\partial L}{\partial y_i^A}).$$

On the other hand, notice that $Z_E$ is canonically embedded into $\bigwedge^n M$, so that it inherits the restriction $\Xi_E$ of the canonical multisymplectic form $\Omega_E$, say

$$\Xi_E = \Omega_E|_{Z_E}.$$

Let $Z_L = \mathcal{FL}(J^1\pi)$ and assume that it is embedded into $Z_E$. Therefore we have an $n$-form $\Xi_L$ on $Z_L$ which is just the restriction of $\Xi_E$. Of course we have

$$\Theta_L = \mathcal{FL}^*(\Xi_L).$$

The Legendre morphism $\mathcal{FL}$ permits then to transport sections from the fibration $J^1\pi \rightarrow M$ to $Z_L \rightarrow M$, and conversely:

$$\begin{array}{ccc}
J^1\pi & \xrightarrow{\mathcal{FL}} & Z_L \\
\downarrow s & \swarrow & \leftarrow \downarrow \bar{s} \\
M & & \mathcal{FL}^{-1}
\end{array}$$

such that, if $s$ is a solution of the equation $s^*(X \llcorner d\Theta_L) = 0$ for all vector fields on $J^1\pi$, then $\mathcal{FL} \circ s$ is a solution of the equation $\gamma^*(\bar{X} \llcorner d\Xi_L) = 0$, for all vector fields $\bar{X}$ on $Z_L$, and conversely (see [64]).

In [64] is also analyzed the case of singular Lagrangians in order to compare the Hamiltonian and Lagrangian formalism.

9 \ n-symplectic geometry

$n$-symplectic geometry on frames bundles was originally developed as a generalization of Hamiltonian mechanics. The theory has, however, turned out to be a covering theory of both symplectic and multisymplectic geometries in the sense that these latter structures
can be derived from \( n \)-symplectic structures on appropriate frames bundles \([11, 28]\). In this section we compare the \( n \)-symplectic geometry to \( k \)-symplectic/polysymplectic geometry and to multisymplectic geometry as well. Moreover we present a recent extension of the algebraic structures on an \( n \)-symplectic manifold to a general \( n \)-symplectic manifold.

9.1 The structure equations of \( n \)-symplectic geometry

The difference between \( n \)-symplectic and \( k \)-symplectic/polysymplectic geometry lies not in the properties of the canonical 2-form – they are essentially the same. Instead the real difference lies in the structure equations, the specification of \( LM \), and the algebraic structures based on the \( m \)-symplectic Poisson bracket.

In \( n \)-symplectic geometry, one works with the soldering form on the frame bundle \( LM \). The differential of the soldering form is a family of 2-forms that, together with the right grouping of the fundamental vertical vector fields, makes \( LM \) a \( m \)-symplectic manifold. However in \( n \)-symplectic geometry we prefer to think of \( d\theta \) as a vector valued 2-form – as a single unit rather than a collection.

Recall the structure equation of \( m \)-symplectic geometry for first order observables:

\[
d \hat{f}^i = -X_f \, d\theta^i
\]

(97)

So we have vector-valued observables \((\hat{f}^i)\) and scalar-valued vector fields \((X_f)\), whereas the polysymplectic formalism has scalar observables and vector-valued vector fields.

In the polysymplectic formalism there exist corresponding vector fields for all functions, but these vector fields are not unique. Contrastingly, in the first order \( m \)-symplectic formalism the vector fields are unique, but only exist for a special class of functions (see Section 9.4). This uniqueness allows for the definition of Poisson brackets, which are not available in the polysymplectic formalism.

The \( m \)-symplectic formalism extends to allow higher order observables. For example, in the second order symmetric case we have:

\[
d \hat{f}^{ij} = -2X_f^{(i} \, d\theta^{j)}
\]

(98)

Now we obtain vector-valued vector fields from an \( \mathbb{R}^n \otimes \mathbb{R}^n \)-valued function. In fact, we remark that the trace \( \sum_{i=j} f^{ij} \) will satisfy the polysymplectic equation with the vector field
$-2X^j_i$. In this second order case the vector fields are no longer unique, but this does not impede the definition of Poisson brackets.

For the $p$-th order case in $m$-symplectic geometry we have

$$df^{i_1...i_p} = -p!X^i_{f} (d^i_1...d^i_p) \quad \text{or} \quad df^{i_1...i_p} = -p!X^i_{f} [d^i_1...d^i_p]$$

for the symmetric and anti-symmetric cases respectively. The Poisson bracket of a $p$-th and a $q$-th order observable is a $(p + q - 1)$-th order observable. The full algebra is developed in [9]. There is nothing in the polysymplectic formalism to compare to this in general.

It has been shown recently that the $n$-symplectic Poisson brackets defined on frame bundles extends to Poisson brackets on a general polysymplectic manifold. We present in the next sections a summary of the general results shown by Norris [32] for a general $n$-symplectic (polysymplectic) manifold.

### 9.2 General $n$-symplectic geometry

Let $P$ be an $N$-dimensional manifold, and let $(\hat{r}_\alpha)$ denote the standard basis of $\mathbb{R}^n$, with $1 \leq n \leq N$. We suppose there exists on $P$ a **general $n$-symplectic structure**, namely an $\mathbb{R}^n$-valued 2-form $\hat{\omega} = \omega^\alpha \otimes \hat{r}_\alpha$ that satisfies the following two conditions:

$$(C - 1) \quad d\omega^\alpha = 0 \quad \forall \quad \alpha = 1, 2, \ldots, n$$

$$(C - 2) \quad X \int \hat{\omega} = 0 \quad \Leftrightarrow \quad X = 0$$

**Definition 9.1** The pair $(P, \hat{\omega})$ is a general $n$-symplectic manifold.

**Remark** In references [1, 11, 12, 13, 14, 15] the term $n$-symplectic structure refers to the two-form that is the exterior derivative of the $\mathbb{R}^n$-valued soldering 1-form on frame bundles or subbundles of frame bundles. As outlined earlier in this paper Günther [4] was perhaps the first to consider a manifold with a non-degenerate $\mathbb{R}^n$-valued 2-form, and he used the terms **polysymplectic structure** and **polysymplectic manifold** for the non-degenerate 2-form and manifold, respectively. In addition, when one adds two extra conditions to conditions
C-1 and C-2 one arrives at a *k*-symplectic manifold. Specifically, if $P$ is required to support an $np$-dimensional distribution $V$ such that

\[(C - 3) \quad N = p(n + 1)\]
\[(C - 4) \quad \hat{\omega}|_{V \times V} = 0\]

then $P$ is a *k*-symplectic manifold as defined by both de Leon, Salgado, et al. [5] and also by Awane [7]. To make this identification one needs to make the notational changes $n \rightarrow k$ and $p \rightarrow n$ in the above discussion. Thus all *k*-symplectic manifolds are *n*-symplectic, but not conversely. The example $(LE, d\theta)$ of an *m*-symplectic manifold introduced in Section 4.4 is also a *k*-symplectic manifold. On the other hand the important example of the adapted frame bundle $L_\pi E$ is *m*-symplectic, but not *k*-symplectic. The problem is that the *k*-symplectic dimensional requirement $N = p(m + 1)$ cannot be satisfied on $L_\pi E$.

We will use the name general *m*-symplectic structure for the structure in definition 9.2 in order to emphasis the geometrical and algebraic developments that the *m*-symplectic approach provides. However, the definition of a general *m*-symplectic structure is identical with the definition of a polysymplectic structure.

### 9.3 Canonical coordinates

Awane [7] has proved a generalized Darboux theorem for *k*-symplectic geometry. Thus in the neighborhood of each point $u \in P$ one can find canonical (or Darboux) coordinates $(\pi_\alpha^a, z^b)$, $\alpha, \beta = 1, 2, \ldots k$ and $a, b = 1, 2, \ldots n$. With respect to such canonical coordinates $\hat{\omega}$ takes the form

\[
\hat{\omega} = (d\pi_\alpha^a \wedge dz^a) \otimes \hat{r}_\alpha
\]

Hence we have the following locally defined equations:

\[
d\pi_\alpha^a = -\frac{\partial}{\partial z^a} \int \omega^\alpha, \quad dz^a = \frac{\partial}{\partial \pi_\alpha^a} \int \omega^\alpha, \quad (\Sigma_\alpha)
\]

**Remark** The *n*-symplectic approach used to characterize algebras of observables requires the existence of such canonical coordinates. From the results in [9] one knows that not
all functions are *allowable* \( n \)-symplectic observables, even in the canonical case of frame bundles. Thus, for example, whether or not there exist pairs \((\hat{f}^{\alpha_1\alpha_2...\alpha_p}, X_f^{\alpha_1\alpha_2...\alpha_{p-1}})\), \( p = 1, 2, \ldots \) that satisfy equation \((105)\) below for a general \( m \)-symplectic manifold is an existence question. The formulas \((103)\) will provide local examples of rank 1 solutions of the \( n \)-symplectic structure equations \((105)\) when either the geometry is specialized to \((k = n)\)-symplectic geometry where a Darboux theorem holds, or when canonical coordinates are simply known to exist. Fortunately in the case of the adapted frame bundle \( L_\pi E \), canonical coordinates are known to exist.

**Example:** On the bundle of linear frames \( \lambda : LE \to E \) one can introduce canonical coordinates in the \((z^\alpha, \pi^\alpha_\beta)\) as in section 2.5. With respect to such a coordinate system on \( LE \) the soldering 1-form \( \hat{\theta} \) has the local coordinate expression

\[
\hat{\theta} = (\pi^\alpha_\beta dz^\beta) \otimes \hat{r}_\alpha
\]

(104)

The \( m \)-symplectic 2-form \( d\hat{\theta} \) clearly has the canonical form \((102)\) in such a coordinate system.

9.4 The Symmetric Poisson Algebra Defined by \( \hat{\omega} \)

In this section we generalize the algebraic structures of \( n \)-symplectic geometry on frame bundles to a general \( n \)-symplectic manifold. Throughout this section we let \((P, \hat{\omega})\) be a general \( n \)-symplectic manifold as defined above. It is convenient to introduce the multi-index notation

\[
\hat{r}^{\alpha_1\alpha_2...\alpha_n}_{\mu} = \hat{r}_{\alpha_1} \otimes_s \hat{r}_{\alpha_2} \otimes_s \cdots \otimes_s \hat{r}_{\alpha_n\mu} \quad , \quad 0 \leq \mu \leq n - 1
\]

In addition round brackets around indices \((\alpha\beta\gamma)\) denote symmetrization over the enclosed indices.

**Definition 9.2** For each \( p \geq 1 \) let \( SHF^p \) denote the set of all \((\otimes_s)^p \mathbb{R}^n\)-valued functions 

\[
\hat{f} = (\hat{f}^{\alpha_1\alpha_2...\alpha_p}) = (\hat{f}(\alpha_1\alpha_2...\alpha_p)) \quad \text{on} \ P \quad \text{that satisfy the equations}
\]

\[
d \hat{f}^{\alpha_1\alpha_2...\alpha_p} = -p! X^{(\alpha_1\alpha_2...\alpha_{p-1}}_f \hat{\omega}_{\alpha_p})
\]

(105)
for some set of vector fields \( X_{f_1}^{\alpha_1\alpha_2 \ldots \alpha_{p-1}} \). We then set

\[
SHF = \bigoplus_{p \geq 1} SHF^p
\]

(106)

\( \hat{f} \in SHF^p \) is a symmetric Hamiltonian function of rank \( p \).

**Example:** The locally defined functions \( \hat{f} \) that satisfy (105) for the canonical \( m \)-symplectic manifold \( (LE, d\hat{\theta}) \) were given in reference [9]. In particular, contrary to the situation in symplectic geometry, not all \( (\otimes_s)^p \mathbb{R}^m \)-valued functions on \( LE \) are compatible with equation (105). The \( p = 1, 2 \) cases will clarify the structure. Let \( ST^p(LE) \) denote the vector space of symmetric \( (\otimes_s)^p \mathbb{R}^m \)-valued GL(m)-tensorial functions on \( LE \) that correspond uniquely to symmetric rank \( p \) contravariant tensor fields on \( E \). Similarly let \( C^\infty(E, (\otimes_s)^p \mathbb{R}^m) \) denote the set of smooth \( (\otimes_s)^p \mathbb{R}^m \)-valued functions on \( LE \) that are constant on fibers of \( LE \). Then

\[
SHF^1 = ST^1(LE) + C^\infty(E, \mathbb{R}^m)
\]

(107)

\[
SHF^2 = ST^2(LE) + T^1(LE) \otimes_s C^\infty(E, \mathbb{R}^m) + C^\infty(E, \mathbb{R}^m \otimes_s \mathbb{R}^m)
\]

(108)

For example, if \( \hat{f} = (\hat{f}^\alpha) \in SHF^1 \) and \( \hat{f} = (\hat{f}^{\alpha\beta}) \in SHF^2 \), then in canonical coordinates \((\pi^\beta, \lambda^\gamma)\) the functions \( \hat{f}^\alpha \) and \( \hat{f}^{\alpha\beta} \) have the general forms

\[
\hat{f}^\alpha = A^a_{\alpha} \pi^a + B^\alpha, \quad \hat{f}^{\alpha\beta} = A^{\mu\nu}_{\alpha} \pi^\mu \pi^\nu + B^{\mu(\alpha} \pi^{\beta)} + C^{\alpha\beta}
\]

(109)

where \( A^a, B^\alpha, A^{\mu\nu} = A^{(\mu\nu)}, B^{\mu\nu} \) and \( C^{\mu\nu} = C^{(\mu\nu)} \) are all constant on the fibers of \( \lambda : LE \to E \) and hence are pull-ups of functions defined on \( E \).

The analogous results for the general \( n \)-symplectic form given in (102) above are straightforward to work out in canonical coordinates. For the \( p = 1 \) and \( p = 2 \) symmetric cases, one finds:

\[
\hat{f}^\alpha = A^a_{\alpha} \pi^a + B^\alpha, \quad \hat{f}^{\alpha\beta} = A^{ab}_{\alpha} \pi^a \pi^b + B^{a(\alpha} \pi^{b)} + C^{\alpha\beta}
\]

(110)

where now all coefficients are functions of the coordinates \( z^a \).

Although \( \hat{\omega} \) is non-degenerate in the sense given in equation (101) above, because of the symmetrization on the right-hand-side in (103) the relationship between \( \hat{f} \) and \( (X_{\hat{f}}^{\alpha_1\alpha_2 \ldots \alpha_{p-1}}) \)
is not unique unless \( p = 1 \). Given a pair \( (f_{\alpha_1 \alpha_2 \ldots \alpha_p}, X_{f_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}}) \) that satisfies (105), one can always add to \( X_{f_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}} \) vector fields \( Y_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}} \) that satisfy the kernel equation

\[
Y^{(\alpha_1 \alpha_2 \ldots \alpha_{p-1})} \omega^{\alpha_p} = 0
\]  

(111)
to obtain a new pair \( (f_{\hat{\alpha_1 \alpha_2 \ldots \alpha_p}}, \hat{X}_{f_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}}) \) that also satisfies (105), where

\[
\hat{X}_{f_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}} = X_{f_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}} + Y_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}
\]

Hence we associate with \( \hat{f} \in SHF^p \) an equivalence class of \((\otimes_s)^{p-1} \mathbb{R}^n\)-valued vector fields, which we denote by \([X_{\hat{f}}]\) = \([X_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}]\]. We will see below that even though we obtain equivalence classes of Hamiltonian vector fields rather than vector fields, the geometry still carries natural algebraic structures.

**Definition 9.3** For each \( p \geq 1 \) let \( SHV^p \) denote the vector space of all equivalence classes of \((\otimes_s)^{p-1} \mathbb{R}^n\)-valued vector fields \([\hat{X}] = [X_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}}] \) on \( P \) that satisfy the equations (102) for some \( \hat{f} = f_{\alpha_1 \alpha_2 \ldots \alpha_p} \hat{r}_{\alpha_1 \alpha_2 \ldots \alpha_{p-1}} \in SHF^p \). We then set

\[
SHV = \bigoplus_{p \geq 1} SHV^p
\]

(112)

\([\hat{X}]\) will be referred to as the generalized rank \( p \) Hamiltonian vector field defined by \( \hat{f} \).

**Example:** The Hamiltonian vector field \( X_f \) for the rank 1 element in (109) is unique, and has the form

\[
X_f = A^\alpha \frac{\partial}{\partial z^\alpha} - \left( \frac{\partial A^\beta}{\partial z^\gamma} \pi^\alpha_\beta + \frac{\partial B^\alpha}{\partial z^\gamma} \pi^\alpha_\gamma \right) \frac{\partial}{\partial \pi^\alpha_\gamma}
\]

(113)
The equivalence class of \( \mathbb{R}^m \)-valued Hamiltonian vector fields corresponding to the rank 2 element in (109) on \( LE \) has representatives of the form

\[
X_f = (A^{\mu \nu} \pi_{\mu}^\alpha + B^{\mu \alpha}) \frac{\partial}{\partial z^\nu} - \frac{1}{2} \left( \frac{\partial A^{\mu \beta}}{\partial z^\gamma} \pi_{\mu}^\alpha \pi_{\beta}^\gamma + \frac{\partial B^{\mu (\alpha}}{\partial z^\gamma} \pi_{\mu}^{\nu)} + \frac{\partial C^{\alpha \nu}}{\partial z^\gamma} \right) \frac{\partial}{\partial \pi_{\gamma}^\nu} + Y_{\gamma}^{(\alpha \beta)} \frac{\partial}{\partial \pi_{\gamma}^\nu}
\]

(114)
where \( Y_{\gamma}^{(\alpha \beta)} \) are functions subject to the constraint

\[
Y_{\gamma}^{(\alpha \beta)} = 0
\]

but are otherwise completely arbitrary. The fact that \( Y_{\gamma}^{\alpha \nu} \frac{\partial}{\partial \pi_{\nu}^\gamma} \) is purely vertical on \( \lambda : LE \to E \) follows from (111).
For the $n$-symplectic rank 2 symmetric observable given above in (10), one can check easily that the local coordinate form of a representative $X_f^\alpha$ of the equivalence class of Hamiltonian vector fields $[\hat{X}_f]^\alpha$ that satisfies (105) has the form

$$X^\alpha = (A^{ab}\pi_a^\alpha + B^{ba})\frac{\partial}{\partial z^b} - \frac{1}{2} \left( \frac{\partial A^{ab}}{\partial z^a} \pi_a^\alpha \pi_b^\alpha + \frac{\partial B^{ba}(a)}{\partial z^d} \pi_a^\alpha + \frac{\partial C^{\alpha\sigma}}{\partial \pi_d^a} \right) \frac{\partial}{\partial \pi_d^a} + Y^\alpha$$  \hspace{1cm} (115)

### 9.4.1 Poisson Brackets

We show that the $n$-symmetric Poisson brackets defined on frame bundles can also be defined in a general $n$-symplectic manifold.

**Definition 9.4** For $p, q \geq 1$ define a map $\{ \cdot, \cdot \} : \text{SHF}^p \times \text{SHF}^q \to \text{SHF}^{p+q-1}$ as follows. For $\hat{f} = f^{\alpha_1\alpha_2 \ldots \alpha_p} \hat{\alpha}_{\alpha_1\alpha_2 \ldots \alpha_p} \in \text{SHF}^p$ and $\hat{g} = g^{\beta_1\beta_2 \ldots \beta_q} \hat{\beta}_{\beta_1\beta_2 \ldots \beta_q} \in \text{SHF}^q$

$$\{ \hat{f}, \hat{g} \}^{\alpha_1\alpha_2 \ldots \alpha_p+q-1} := p! X_f^{(\alpha_1\alpha_2 \ldots \alpha_p+1)}(\hat{g}^{\alpha_0\alpha_p+1 \ldots \alpha_{p+q-1}})$$  \hspace{1cm} (116)

where $X_f^{\alpha_1\alpha_2 \ldots \alpha_p+1}$ is any set of representatives of the equivalence class $[\hat{X}_f]$. We need to make certain that $\{ \hat{f}, \hat{g} \}$ is well-defined. Suppose we have two representatives $X_f^{\alpha_1\alpha_2 \ldots \alpha_p-1}$ and $\tilde{X}_f^{\alpha_1\alpha_2 \ldots \alpha_p-1} = X_f^{\alpha_1\alpha_2 \ldots \alpha_p-1} + Y^{\alpha_1 \alpha_2 \ldots \alpha_p-1}$ of $[\hat{X}_f]$. Then it follows easily from (111) that

$$\tilde{X}_f^{(\alpha_1\alpha_2 \ldots \alpha_p-1)}(\hat{g}^{\alpha_0\alpha_p+1 \ldots \alpha_{p+q-1}}) = X_f^{(\alpha_1\alpha_2 \ldots \alpha_p-1)}(\hat{g}^{\alpha_0\alpha_p+1 \ldots \alpha_{p+q-1}})$$

Hence the bracket is independent of choice of representatives. That $\{ \hat{f}, \hat{g} \}$ actually is in $\text{SHF}^{p+q-1}$ will follow from Corollary (1.7) below.

**Definition 9.5** Let $[\hat{X}_f] = [X_f^{\alpha_1\alpha_2 \ldots \alpha_p-1} \hat{\alpha}_{\alpha_1\alpha_2 \ldots \alpha_p-1}]$ and $[\hat{X}_g] = [X_g^{\alpha_1\alpha_2 \ldots \alpha_p-1} \hat{\alpha}_{\alpha_1\alpha_2 \ldots \alpha_p-1}]$ denote the equivalence classes of vector-valued vector fields determined by $\hat{f} \in \text{SHF}^p$ and $\hat{g} \in \text{SHF}^q$, respectively. Define a bracket $[\cdot, \cdot] : \text{SHV}^p \times \text{SHV}^q \to \text{SHV}^{p+q-2}$ by

$$[[\hat{X}_f], [\hat{X}_g]] = [[X_f^{(\alpha_1\alpha_2 \ldots \alpha_p-1)}, X_g^{\alpha_p\alpha_{p+1} \ldots \alpha_{p+q-2}}] \hat{\alpha}_{\alpha_1\alpha_2 \ldots \alpha_{p+q-2}}]$$  \hspace{1cm} (117)

where the “inside” bracket on the right-hand side is the ordinary Lie bracket of vector fields calculated using arbitrary representatives. (Notice the symmetrization over all the upper indices in this equation.)
We again need to show that this bracket is well-defined. This is shown in the following lemma, in which we will need the formula

\[ L_{X^J}\omega^\alpha = 0 \]  

(118)

which follows easily from (105) and the formula \( L_X\omega = X \cdot d\omega \). In (118) \( J \) denotes the multiindex \( \alpha_1 \alpha_2 \ldots \alpha_{p-1} \), and \( X^J \) denotes a representative of a rank \( p \) Hamiltonian vector field satisfying equations (105). The next lemma shows that the bracket defined in (117) is (i) independent of choice of representatives, and (ii) close s on the set of equivalence classes of vector-valued Hamiltonian vector fields. The proof of the lemma can be found in [32], which is quite similar to the proof of the analogous result in symplectic geometry.

**Lemma 9.6** Let \( [\hat{X}_f] \) and \( [\hat{X}_g] \) denote the equivalence classes of vector-valued vector fields determined by \( \hat{f} \in SHF^p \) and \( \hat{g} \in SHF^q \), respectively. Then

\[ [[\hat{X}_f], [\hat{X}_g]] = \frac{(p + q - 1)!}{p! q!} [[\hat{X}_{\{f, g\}}]] \]  

(119)

**Corollary 9.7**

\[ \{\hat{f}, \hat{g}\} \in SHF^{p+q-1} \]

**Theorem 9.8** \((SHV, [\ , \])\) is a Lie Algebra.

**Proof** The bracket defined in (117) is clearly anti-symmetric. To check the Jacobi identity we note that we only need check it for arbitrary representatives, and we may use the very definition (117) for the calculation. Since the ”inside” bracket on the right-hand-side in (117) is the ordinary Lie bracket for vector fields, we see that the bracket defined in (117) also must obey the identity of Jacobi.

We can now show that \( SHF \) is a Poisson algebra under the bracket defined in (116).

**Theorem 9.9** \((SHF, \{\ , \})\) is a Poisson algebra over the commutative algebra \((SHF, \otimes_s)\).
**Proof** The symmetrized tensor product $\otimes_s$ makes $\text{SHF}$ into a commutative algebra. If we now consider again elements $\hat{f} \in \text{SHF}^p$, $\hat{g} \in \text{SHF}^q$ and $\hat{h} \in \text{SHF}^r$, then by using definition (116) one may show that

$$\{\hat{f}, \hat{g} \otimes_s \hat{h}\} = \{\hat{f}, \hat{g}\} \otimes_s \hat{h} + \hat{g} \otimes_s \{\hat{f}, \hat{h}\} .$$

(120)

Thus the bracket defined in (116) acts as a derivation on the commutative algebra.

**Example:** In the canonical case $P = LE$ the Poisson brackets just defined have a well-known interpretation. As mentioned above the homogeneous elements in $\text{SHF}^p$ make up the space $\text{ST}^p(LE)$, the symmetric rank $p$ $\text{GL}(m)$-tensorial functions that correspond to symmetric rank $p$ contravariant tensor fields on $E$. Then $\text{ST} = \bigoplus_{p \geq 1} \text{ST}^p \subset \text{SHF}$, and the bracket $\{\ , \ \} : \text{ST}^p \times \text{ST}^q \to \text{ST}^{p+q-1}$ has been shown [12] to be the frame bundle version of the Schouten-Nijenhuis bracket of the corresponding symmetric tensor fields on $E$.

There is also a Schouten-Nijenhuis bracket for anti-symmetric contravariant tensor fields on $E$, and as one might expect this bracket also extends to $LE$. This leads to a graded $m$-symplectic Poisson algebra of anti-symmetric tensor-valued functions on $LE$ [11].

### 9.5 The Legendre Transformation in $m$-symplectic theory on $L_\pi E$

One can define the CHP 1-forms, defined above in Definition 7.9, using a frame bundle version of the Legendre transformation. Given a lifted Lagrangian $L : L_\pi E \to \mathbb{R}$ we obtain a mapping $\phi_L : L_\pi E \to LE$ given by

$$\phi_L(u) = \phi_L(e, e_i, e_A) = \left( e, \frac{1}{\tau L(u)} e_i, e_A - \frac{1}{\tau L(u)} E_A^a (L)(u)e_a \right) .$$

(121)

The condition that this mapping end up in $LE$ is that the Lagrangian be non-zero, and for the rest of this paper we will assume this condition. We refer to this mapping as the $m$-symplectic Legendre transformation. Our goal is to prove Theorem (9.13), namely that $\hat{\theta}_L = \phi_L^* (\hat{\theta})$ where $\hat{\theta}$ is the canonical soldering 1-form on the image $Q_L$ of $\phi_L$.

To clarify the meaning of the Legendre transformation (121) we introduce a new manifold $\tilde{P}$ as follows. Let $J$ denote the subgroup of $\text{GL}(n)$ consisting of matrices of the form

$$\begin{pmatrix} I & \xi \\ 0 & I \end{pmatrix} \quad \xi \in \mathbb{R}^{n \times k}$$
Define \( \tilde{P} \) by

\[
\tilde{P} = L_{\pi}E \cdot J = \{(e_i, e_A + \xi_A^j e_j) \mid (e_i, e_A) \in L_{\pi}E, \xi \in \mathbb{R}^{n \times k}\}
\]  
(122)

We collect together the pertinent results that are proved in [48, 32] and that lead up to Theorem (9.13).

**Lemma 9.10** \( \tilde{P} \) is a open dense submanifold of the bundle of frames \( LE \) of \( E \).

**Lemma 9.11** There is a canonical diffeomorphism from \( \tilde{P} \) to the product manifold \( L_{\pi}E \times \mathbb{R}^{m \times k} \).

Using this fact one can prove the following lemma. We let \( Q_L \) denote the range of the Legendre transformation.

**Lemma 9.12** If the Lagrangian \( L \) is non-zero, then the Legendre transformation \( \phi_L : L_{\pi}E \to Q_L \) is a diffeomorphism.

These facts taken together lead to the following fundamental theorem:

**Theorem 9.13** Let \( L \) be the pull-up to \( L_{\pi}E \) of a non-zero Lagrangian on \( J^1\pi \), and let \( \phi_L \) denote the \( m \)-symplectic Legendre transformation defined above in (121). Then

\[
\hat{\theta}_L = \phi^*_L(\hat{\theta})
\]  
(123)

**Proof** The proof is a direct calculation using the definition (121).

**Remark** This theorem has an obvious analogue in symplectic mechanics, where the symplectic form on the velocity phase space \( TE \) is, for a regular Lagrangian, the pull back under the Legendre transformation of the canonical 1-form on \( T^*M \). There is also a similar theorem in multisymplectic geometry where the CHP \( m \)-form on \( J^1\pi \) is known [34] to be the pull back of the canonical multisymplectic \( m \)-form on \( J^{1*}\pi \).

Now \( Q_L \), being a submanifold of \( LE \), supports the restriction \( \hat{\theta}|_{Q_L} \) of the \( \mathbb{R}^m \)-valued soldering 1-form \( \hat{\theta} \). It is easy to verify that the closed \( \mathbb{R}^m \)-valued 2-form \( d\hat{\theta}|_{Q_L} \) is also non-degenerate, and hence \( (Q_L, d(\hat{\theta}|_{Q_L})) \) is an \( m \)-symplectic manifold.

Using the fact that \( Q_L \) and
$L_πE$ are diffeomorphic under the Legendre transformation, we obtain the following corollary to Theorem 9.13.

**Corollary 9.14** $(L_πE, d\hat{θ}_L)$ is an $m$-symplectic manifold.

To find the *allowable observables* of this theory one can set up the equations of $m$-symplectic reduction to find the subset of $m$-symplectic observables on $LE$ that reduce to the submanifold $Q_L$.

9.6 The Hamilton-Jacobi and Euler-Lagrange equations in $m$-symplectic theory on $L_πE$

Working out the local coordinate form of the CHP-1-forms, given in Definition 7.9, in Lagrangian coordinates one finds

$$θ^i_L = -H^i_j dx^j + P^i_A dy^A$$

(124)

$$θ^A_L = P^A_j dx^j + P^A_B dy^B$$

(125)

where

$$H^i_j = u^i_k p^k_B u^B_j - τ(n) L δ^k_j$$

(126)

$$P^i_B = u^i_k p^k_B$$

(127)

$$P^A_j = -u^A_B u^B_j$$

(128)

$$P^A_B = u^A_B$$

(129)

The $H^i_j$ are the components of the **covariant Hamiltonian**, and the $P^i_B$ are the components of the **covariant canonical momentum** [48]. Defining symbols $h^k_j$ by the formula

$$h^k_j = p^k_B u^B_j - τ(n) L δ^k_j$$

(130)

the covariant Hamiltonian (127) can be expressed as $H^i_j = u^i_k h^k_j$. Setting $τ(n) = 1$ one finds that $h^i_j$ has the form of Carathéodory’s Hamiltonian tensor [44, 50]. Similarly, setting $τ = \frac{1}{n}$ one finds that $h = h^i_i$ yields the Hamiltonian in the de Donder-Weyl theory [44, 49].
9.6.1 The $m$-symplectic Hamilton-Jacobi Equation on $L_\pi E$

The Carathéodory-Rund and de Donder-Weyl Hamilton-Jacobi equations occur as special
cases of a general Hamilton-Jacobi equation that can be set up on $L_\pi E$. Proceeding by
analogy with the time independent Hamilton-Jacobi theory we seek Lagrangian submanifolds
of $L_\pi E$. However, since the dimension of $L_\pi E$ is in general not twice the dimension of $E$,
a new definition is needed. For our purposes here we will consider $m = n + k$ dimensional
submanifolds of $L_\pi E$ that arise as sections of $\lambda$. In particular we consider sections $\sigma : E \to
L_\pi E$ that satisfy

$$\sigma^*(d\theta^\alpha_L) = 0$$

These are the $m$-symplectic Hamilton-Jacobi equations [48].

Since $\sigma^*(d\theta^\alpha_L) = d(\sigma^*(\theta^\alpha_L))$ the condition (131) asserts that the 1-forms $\sigma^*(\theta^\alpha_L)$ are locally
exact, and we express this as

$$\sigma^*(\theta^\alpha_L) = dS^\alpha$$

in terms of $m = n + k$ new functions $S^\alpha$ defined on open subsets of $E$. For convenience we
will denote objects on $L_\pi E$ pulled back to $E$ using $\sigma$ with an over-tilde. Thus, for example,
$\tilde{H}^i_j = H^i_j \circ \sigma$ and $\tilde{P}^i_A = P^i_A \circ \sigma$. Then we get from (126)–(129) and (132)

(a) $\tilde{H}^i_j = -\frac{\partial S^i}{\partial x^j}$,
(b) $\tilde{P}^i_A = \frac{\partial S^i}{\partial y^A}$

(a) $\tilde{u}^B_A \tilde{u}^A_j = -\frac{\partial S^A}{\partial x^j}$,
(b) $\tilde{u}^A_B = \frac{\partial S^A}{\partial y^B}$

Recalling that $H^i_j = P^i_B u^B_j - \tau(n) L u^i_j$ and $P^i_A$ are functions of the coordinates $x^i$, $y^A$, $u^i_j$ and
$u^A_i$, equations (133) can be combined into the single equation

$$H^i_j(x^a, y^B, u^a_B, u^B_a, \frac{\partial S^i}{\partial y^B}) \circ \sigma = -\frac{\partial S^i}{\partial x^j}$$

(135)

Similarly combining equations (134) we obtain

$$\frac{dS^A}{dx^j} = 0$$

We next consider special cases of these $m$-symplectic Hamilton-Jacobi equations.
9.6.2 The Theory of Carathéodory and Rund

We note from (126), (127), and (130) that \( H_j^i = u_k^i h_j^k \) and \( P_{A}^i = u_k^i p_{A}^k \), where the matrix of functions \( (u_j^i) \) is \( \text{GL}(n) \)-valued. Using the notation \( P_j^i = -H_j^i \) and \( \tilde{u}_j^i = u_j^i \circ \sigma \) we may rewrite (126) and (127) in the form

\[
\tilde{P}_j^i = -\tilde{u}_k^i h_j^k, \quad \tilde{P}_{A}^i = \tilde{u}_k^i p_{A}^k
\]  

(136)

If we take \( t(n) = 1 \) then these equations are the equations defining the canonical momenta in Rund’s canonical formalism for Carathéodory’s geodesic field theory (see equations (1.22), page 389 in [44], with the obvious change in notation). In this situation equation (133) can be identified with the Rund’s Hamilton-Jacobi equation for Carathéodory’s theory (see equation (3.29) on page 240 in [44]). We recall [44] that one can derive the Euler-Lagrange field equations from this Hamilton-Jacobi equation.

In (136) we have the result that the arbitrary non-singular matrix-valued functions \( (\tilde{u}_j^i) \) that occur in Rund’s canonical formalism for Carathéodory’s theory have a geometrical interpretation in the present setting. Specifically they correspond to the coordinates for linear frames for \( M \). These defining relations are derived from Rund’s transversality condition, and we now show that this condition has the elegant reformulation as the kernel of \( (\theta_L^i) \).

We will say that a vector \( X \) at \( e \in E \) is transverse to a solution surface through \( e \) that is defined by a given Lagrangian \( L \), if \( X = d\lambda(\hat{X}) \), where \( \hat{X} \in T_u(L \pi E) \) satisfies \( \hat{X} \downarrow \theta_L^i = 0 \), for some \( u \in \lambda^{-1}(e) \). \( \hat{X} \) thus satisfies the equations

\[
0 = -H_j^i X^j + P_{A}^i X^A = u_k^i \left( -h_j^k X^j + p_{A}^k X^A \right)
\]

\[
X^j = \hat{X}(x^j), \quad X^A = \hat{X}(y^A)
\]

from which we infer

\[
0 = -h_j^k X^j + p_{A}^k X^A
\]  

(137)

This is Rund’s transversality condition for the theory of Carathéodory when we take \( \tau(n) = 1 \) (see equation (1.10), page 388 in [44]). The canonical momenta \( P_j^i \) and \( P_{A}^i \) are defined by Rund to be solutions of

\[
0 = P_j^i X^j + P_{A}^i X^A
\]  

(138)
when \((X^j, X^A)\) satisfy \((137)\). Rund’s solutions of these equations are given in \((136)\). Looking at \((136), (137)\) and \((138)\) we see that the introduction of the \(u^i_j\) in \((136)\) amounts to the introduction of the \(\text{GL}(n)\) freedom for linear frames for \(M\).

### 9.6.3 de Donder-Weyl Theory

Returning to \((133)\) let us reduce this equation by making several assumptions. We suppose that \(L\) is regular (in the usual sense on \(J^1\pi\)), that the section \(\sigma\) is such that \(\tilde{u}^i_j = \delta^i_j\), and we make the choice \(\tau(n) = \frac{1}{n}\). Now summing \(i = j\) in \((133)\) we obtain

\[
\tilde{h}(x^i, y^B, \frac{\partial S^i}{\partial y^B}) = -\frac{\partial S^i}{\partial x^i}
\]

where \(\tilde{h} = \tilde{p}_A^i \tilde{u}^A_i - \tilde{L}\). This equation is the Hamilton-Jacobi equation of the de Donder-Weyl theory, as presented by Rund (see equation (2.31) on page 224 in [44]). We recall [44] that one can derive in this case also the Euler-Lagrange field equations from the de Donder-Weyl Hamilton-Jacobi equation.

### 9.7 Hamilton Equations in \(m\)-symplectic geometry

The structure of equations \((124) - (127)\) suggests that one should be able to derive generalized Hamilton equations if the canonical momenta \(p_A^i = \frac{\partial L}{\partial u^i_A}\) can be introduced as part of a local coordinate system on \(L_\pi E\). Part of the original philosophy used in developing \(m\)-symplectic geometry in reference [9] was to switch from scalar equations to tensor equations, motivated by the fact that the soldering 1-form is vector-valued. In particular, the basic structure equation \((97)\) in \(m\)-symplectic geometry is tensor-valued. We show next that

\[
u^*(\eta \int d\theta^i_L) = 0 \tag{139}\]

where \(u : M \rightarrow L_\pi E\) is a section of \(\pi \circ \lambda\), and \(\eta\) is any vector field on \(L_\pi E\), yields generalized canonical equations that contain known canonical equations as special cases. We consider here only \(d\theta^i_L\) since by Proposition \((7.10)\) it alone is needed to construct the CHP-\(m\)-form on \(J^1\pi\).

We need the following definition in order to introduce the canonical momenta as part of a coordinate system on \(L_\pi E\).
Definition 9.15 A Lagrangian $L$ on $L\pi E$ is regular if the $(n + k) \times (n + k)$ matrix

$$
\left( E^s_A \circ E^s_B (L) \right)
$$

is non-singular.

Working out the terms of this matrix in Lagrangian coordinates using (20) we obtain

$$
E^s_A \circ E^s_B (L) = u^i_j u^i_k v^j_B v^k_A \left( \frac{\partial^2 L}{\partial u^a \partial u^b} \right)
$$

It is clear that this definition is equivalent to the standard definition of regularity on $J^1\pi$.

We now consider the transformation of coordinates from the set $(x^i, y^A, u^i_k, u^A_B)$ to the new set $(\bar{x}^i, \bar{y}^A, \bar{u}^i_j, p^j_A, \bar{u}^A_B)$ where

$$
\bar{x}^i = x^i, \quad \bar{y}^A = y^A, \quad \bar{u}^i_j = u^i_j, \quad \bar{u}^A_B = u^A_B, \quad \bar{p}^j_A = \frac{\partial L}{\partial u^A_i}
$$

Computing the Jacobian one finds that the new barred functions will be a proper coordinate system whenever the Lagrangian is regular. For the remainder of this section we shall assume that $L$ has this property, despite the fact that many important examples (see [34, 35]) have non-regular Lagrangians. Moreover, for simplicity we will drop the bars on the new coordinates.

In the generalized canonical equation (139) we now take $\eta = \frac{\partial}{\partial p^A_i}$. We find the result

$$
0 = \left( \frac{\partial H^j_k}{\partial p^A_i} \circ u \right) + \left( u^j_i \circ u \right) \left( \frac{\partial (y^A \circ u)}{\partial x^k} \right)
$$

Using $H^j_k = u^j_i h^i_k$ and the fact that $(u^j_i)$ is a non-singular matrix valued function, this last equation reduces to

$$
\frac{\partial h^j_k}{\partial p^A_i} \circ u = \frac{\partial (y^A \circ u)}{\partial x^k} \delta^j_i
$$

This is our first set of $m$-symplectic Hamilton equations. Notice that by summing $j = k$ in this equation we obtain

$$
\frac{\partial h}{\partial p^A_i} \circ u = \frac{\partial (y^A \circ u)}{\partial x^i}
$$
Upon setting $\tau(n) = \frac{1}{n}$ we obtain half of the de Donder-Weyl canonical equations. Under suitable but complicated conditions these equations, with $\tau(n) = 1$, will also reproduce part of Rund’s canonical equations for the theory of Carathéodory.

In the generalized canonical equation (139) we now take $\eta = \frac{\partial}{\partial y^A}$. We find

$$0 = u^* \left( d(u^i_k p^k_A) + u^i_k \frac{\partial h^k_j}{\partial y^A} dx^j \right)$$

Using an “over bar” notation to denote objects pulled back to $M$ by $u$ we may write this as

$$\frac{\partial}{\partial x^j} (\bar{u}^i_k p^k_A) = -\bar{u}^i_k \left( \frac{\partial h^k_j}{\partial y^A} \right) \circ u$$

This is our second set of $m$-symplectic Hamilton equations.

Notice that what is non-standard in (141) is the appearance of the derivatives of the functions $\bar{u}^i_j = u^i_j \circ u$. If, however, the section $u : M \to L_\pi E$ is such that the $\bar{u}^i_j$ are constants, then these equations reduce to

$$\frac{\partial (\bar{p}^k_A)}{\partial x^j} = -\frac{\partial h^k_j}{\partial y^A} \circ u$$

Setting $\tau(n) = \frac{1}{n}$ and summing $k = j$ in this equation we obtain

$$\frac{\partial (\bar{p}^l_A)}{\partial x^i} = -\frac{\partial h^l_i}{\partial y^A} \circ u$$

These equations, together with equations (140) when $\tau(n) = \frac{1}{n}$, are the complete canonical equations in the de Donder-Weyl theory.

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