Shielding at a distance due to anomalous resonance

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Abstract

A cylindrical plasmonic structure with a concentric core exhibits an anomalous localized resonance which results in cloaking effects. Here we show that if the structure has an eccentric core, a new kind of shielding effect can happen. In contrast to conventional shielding devices, our proposed structure can block the effect of external electrical sources, even in a region which is not enclosed by any conducting materials. In fact, the shielded region is located at a distance from the device. We analytically investigate this phenomenon by using the Möbius transformation, through which an eccentric annulus is transformed into a concentric one. We also present several numerical examples.

1. Introduction

A cylindrical superlens with a shell that has a relative permittivity of $\varepsilon_0 = -1 + i\delta$, exhibits anomalous resonant behavior as the loss parameter $\delta$ tends to zero [6, 7]. The electric field distribution generated by this structure diverges in magnitude throughout the region localized within a specific distance from the superlens, while it converges to a smooth field farther away from the superlens. Furthermore, the localized region, which changes depending on the source location, has a boundary that does not coincide with any discontinuity in the permittivity distribution. This so-called anomalous localized resonance was first discovered by Nicorovici, McPhedran and Milton [7] and is responsible for the subwavelength resolving power of the superlens [10]; see also [9, 11, 12]. The superlens acts as a cloaking device for certain sources since the resonance cancels the effect of those sources [8].

Extensive work has been produced on cloaking by an anomalous localized resonance, for example in [1–7]. In this paper, we show that a cylindrical superlens can also act as a new kind of quasistatic shielding device if the core is eccentric to the shell. The historical root of the shielding effect goes back to 1896, when Michael Faraday discovered that a region coated with a conducting material is not affected by external electric fields. While such a conventional method shields the region enclosed by the device, a superlens with an eccentric core can shield a non-coated region which is located outside the device. We call this phenomenon shielding at a distance. The aim of this paper is to investigate the conditions required for shielding at a distance, as well as the geometric features such as the location and size of the shielded region. The key element to study in the eccentric case is the Möbius transformation, through which a concentric annulus is transformed into an eccentric one. The quasistatic properties of the eccentric superlens can be derived in a straightforward way from those of the concentric case, since the Möbius transformation maps conformally.

The anomalous resonance in the eccentric case can be very different from that of the concentric case. Interestingly, it is possible to have an undisturbed region of the electric field inside a resonance region. Also, the resonance can happen even for electric sources at infinity. Our proposed shielding device is based on these two features of the anomalous resonance. We emphasize that this unusual resonance might have more important applications besides shielding.

Throughout this paper, the superlens is assumed to be much smaller than the operating wavelength, and thus, the quasistatic approximation is valid. While we assume the smallness for the superlens, we do not restrict...
the range of location for the external sources. Although the concentric superlens only causes cloaking for sources well located within a wavelength, the eccentric one can induce shielding even for sources located far away from the structure. This feature will be explained in a later section.

The paper is organized as follows. In section 2, we review the cloaking due to the anomalous localized resonance caused by the concentric superlens. In section 3, we introduce the Möbius transformation and discuss its geometrical properties. In section 4, we consider how the electric potential is transformed by the Möbius transformation. In section 5, we explain how a new shielding effect can happen and derive the explicit conditions required for its occurrence. Finally, we illustrate shielding at a distance by presenting several numerical examples in section 6.

2. Anomalous localized resonance caused by the concentric superlens

In this section, we review the anomalous resonance caused by the concentric superlens whose geometry is described in figure 1.

We first fix some notations to explicitly state the phenomenon. We let $\Omega_i$ and $\Omega_e$ denote circular disks centered at the origin with the radii $\rho_i$ and $\rho_e$, respectively, satisfying $0 < \rho_i < \rho_e < 1$. Identifying $\mathbb{R}^2$ as $\mathbb{C}$, they can be represented as

$$\Omega_i = \{z \in \mathbb{C} : |z| < \rho_i\} \quad \text{and} \quad \Omega_e = \{z \in \mathbb{C} : |z| < \rho_e\}.$$ 

The core $\Omega_i$ and the background $\mathbb{R}^2 \setminus \overline{\Omega_e}$ are assumed to be occupied by the isotropic material of permittivity 1, and the shell $\Omega_i \setminus \overline{\Omega_e}$ by the plasmonic material of permittivity $-1 + i\delta$ with a given loss parameter $\delta > 0$, i.e., the permittivity distribution $\varepsilon_\delta$ is given by

$$\varepsilon_\delta = \begin{cases} 
1 & \text{in the core}, \\
-1 + i\delta & \text{in the shell}, \\
1 & \text{in the background}.
\end{cases} \quad (2.1)$$

We also assume the annulus structure to be small compared to the operating wavelength so that we can adopt the quasistatic approximation. Then the (quasistatic) electric potential $V$ satisfies

$$\nabla \cdot \varepsilon_\delta \nabla V = f \quad \text{in} \quad \mathbb{C}, \quad (2.2)$$

where $f$ represents an electrical source. We assume that $f$ is a point multipole source of order $n$ located at $z_0 \in \mathbb{R}^2 \setminus \overline{\Omega_e}$. Then the potential $F$ generated by the source $f$ can be represented as

$$F(z) = \sum_{k=1}^{n} \text{Re} \{c_k(z - z_0)^{-k}\}, \quad z \in \mathbb{C},$$

with the complex coefficients $c_k$. When $n = 1$, the source $f$ (or the potential $F$) means a point dipole source.

Now we discuss the anomalous localized resonance. Its rigorous mathematical description was given in [1] as follows:

(i) The dissipation energy $W_\delta$ diverges as the loss parameter $\delta$ goes to zero, if and only if a point source $f$ is located inside the region $\Omega_\delta := \{|z| < \rho_\delta\}$, where $\rho_\delta := \sqrt{\rho_i^2 / \rho_e}$ and $W_\delta$ is given by

Figure 1. Cloaking due to the anomalous localized resonance: (a) shows the structure of the superlens with a concentric core; (b) illustrates the cloaking effect.

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Figure 1. Cloaking due to the anomalous localized resonance: (a) shows the structure of the superlens with a concentric core; (b) illustrates the cloaking effect.
\[ W_\delta := \text{Im} \int_{\mathbb{R}^2} \varepsilon_0 |\nabla V_\delta|^2 \, dx = \delta \int_{\Omega_\delta \setminus \Omega} |\nabla V|^2. \]  

Let us call \( \Omega_\delta \) (or \( \rho_\delta \)) the critical region (or the critical radius), respectively.

(ii) The electric field \(-\nabla V_\delta\) stays bounded outside some circular region regardless of \(\delta\). More precisely, we have

\[ |\nabla V_\delta(z)| \leq C, \quad z \in \Omega_\delta := \{ |z| > \rho_\delta^2 / \rho_\delta \}, \]

for some constant \(C\) independent of \(\delta\). Here, the subscript ‘b’ in \(\Omega_\delta\) indicates the boundedness of the electric field. Let us call \(\Omega_\delta\) the calm region.

Now we explain why the above statements represent the cloaking effect. Suppose that the point source \(f\) is located inside the critical region \(\Omega_\delta\). Then, by fact (i), the energy \(W_\delta\) goes to infinity as \(\delta \to 0\). Since the blow-up of the energy \(W_\delta\) is unphysical, we have to consider the normalized potential \(V_\delta / \sqrt{W_\delta}\) instead of \(V_\delta\). The corresponding energy will remain bounded regardless of \(\delta\). Then, by fact (ii), the normalized electric field \(-\nabla V_\delta / \sqrt{W_\delta}\) goes to zero as \(\delta \to 0\) in the region \(\Omega_\delta\). In other words, assuming the smallness of \(\delta\), the field generated by the point source \(f\) is negligible, which means that the point source \(f\) is cloaked; see figure 1. In contrast, if the source \(f\) is located outside the critical region \(\Omega_\delta\), then the energy dissipation does not blow up. We can detect the presence of the source by observing the potential away from the superlens since the electric field is not negligible on \(\Omega_\delta\) everywhere.

3. Möbius transformation

In this section, we will show that the concentric annulus can be transformed into an eccentric one by applying the Möbius transformation \(\Phi\) defined as

\[ \zeta = \Phi(z) := \frac{az + 1}{z - 1} \]  

with a given positive number \(a\). We shall also discuss how the critical region is transformed depending on the critical parameter \(\rho_\delta\).

The function \(\Phi\) is a conformal mapping from \(\mathbb{C} \setminus \{1\}\) to \(\mathbb{C} \setminus \{a\}\), which maps the point \(z = 1\) to infinity, infinity to \(\zeta = a\), and \(z = 0\) to \(\zeta = -a\). It also maps a circle centered at the origin, say \(S_\rho := \{ z \in \mathbb{C} : |z| = \rho \}\), to the circle given by

\[ \Phi(S_\rho) = \{ z \in \mathbb{C} : |z - c| = r \}, \quad \text{where} \quad c = a\rho^2 + 1 - 2a \rho^2 + 2a / (\rho^2 - 1). \]

So, the concentric circles \(S_\rho\) with \(\rho \neq 1\) are transformed into eccentric ones in the \(\zeta\)-plane; see figure 2.

Let us discuss how the concentric superlens described in section 2 is geometrically transformed by the mapping \(\Phi\). Note that for \(0 < \rho < 1\), the transformed circle \(\Phi(S_\rho)\) always lies in the left half plane of \(\mathbb{C}\). Since we assume that \(0 < \rho_\delta < \rho_\delta < 1\), the concentric annulus in the \(z\)-plane is changed to an eccentric one contained in the left half \(\zeta\)-plane. We let \(\Omega_\delta\) (or \(\Omega_\delta\)) denote the transformed disk of \(\Omega_\delta\) (or \(\Omega_\delta\)), respectively.

Now we consider the critical region \(\Omega_{\delta'e} = \{ |z| < \rho_{\delta'} \}\) and the calm region \(\Omega_{\delta\delta'}\). Let us denote the transformed critical region (or calm region) by \(\tilde{\Omega}_{\delta'e}\) (or \(\tilde{\Omega}_{\delta\delta'}\), respectively. The shape of \(\tilde{\Omega}_{\delta'e}\) can be very different depending on the value of \(\rho_{\delta'}\). Suppose \(0 < \rho_{\delta'} < 1\) for a moment. Then the region \(\tilde{\Omega}_{\delta'e}\) is a circular disk contained in the left half \(\zeta\)-plane. Next, assume that \(\rho_{\delta'} > 1\). In this case, \(\tilde{\Omega}_{\delta'e}\) becomes the region outside a disk which is disjointed from the eccentric annulus. Contrary to the case when \(\rho_{\delta'} < 1\), the region \(\tilde{\Omega}_{\delta'e}\) is now unbounded. Similarly, the shape of \(\tilde{\Omega}_{\delta\delta'}\) depends on the parameter \(\rho_{\delta\delta'} := \rho_{\delta'}^2 / \rho_{\delta}\). If \(0 < \rho_{\delta\delta'} < 1\), \(\tilde{\Omega}_{\delta\delta'}\) is a region outside the circle. But, if \(\rho_{\delta\delta'} > 1\), \(\tilde{\Omega}_{\delta\delta'}\) becomes a bounded circular region which does not intersect with the eccentric superlens. This unbounded (or bounded) feature of the shape of \(\tilde{\Omega}_{\delta'e}\) (or \(\tilde{\Omega}_{\delta\delta'}\)) will essentially be used to design a new shielding device.

4. Potential in the transformed space

Here, we will transform the potential \(V_\delta\) via the Möbius map \(\Phi\) and then show that the resulting potential describes the physics of the eccentric superlens. Let us define the transformed potential \(V_\delta'\) by

\[ V_\delta'(\zeta) := V_\delta \circ \Phi^{-1}(\zeta). \]  

Since the Möbius transformation \(\Phi\) is a conformal mapping, it preserves the harmonicity of the potential and interface conditions. It can be easily shown that the transformed potential \(V_\delta'\) satisfies
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\[ e \ddot{V} \approx \dddot{d} \cdot ( ) \]
where \( f_1 \) and \( f_2 \) are given by
\[ f(z) = \frac{1}{|\Phi'|} (f \circ \Phi^{-1})(\zeta) \]
and the permittivity \( e_\ddot{V} \) is given by
\[ e_\ddot{V}(\zeta) = \begin{cases} 1 & \text{in } \Omega_0, \\ -1 + i\delta & \text{in } \Omega_0 \setminus \Omega_1, \\ 1 & \text{in the background.} \end{cases} \]

Therefore, the transformed potential \( \ddot{V} \) represents the quasistatic electrical potential of the eccentric superlens (4.2) induced by the source \( \ddot{f} \).

Now we consider some physical properties in the transformed space. The dissipation energy \( \ddot{W}_V \) in the transformed space turns out to be the same as the original one \( W_0 \) as follows:
\[ \ddot{W}_V = \delta \int_{\partial \Omega_0 \setminus \Omega_1} \ddot{V} \frac{\partial \ddot{V}}{\partial n} \, d\ell = \delta \int_{\partial \Omega_0 \setminus \Omega_1} V_0 - \frac{1}{|\ddot{\Phi}'|} \frac{\partial V_0'}{\partial n} |\ddot{\Phi}'| \, d\ell = W_0. \]

In the derivation, we have used the Green’s identity and the harmonicity of the potentials \( V_0 \) and \( \ddot{V}_0 \).

The point source \( f \) is transformed into another point source at a different location. To see this, we recall that the source \( f \) is located at \( z = z_0 \) in the original space, generating the potential \( F(z) = \sum_{k=1}^n \text{Re} \{ d_k (z - z_0)^{-k} \} \).

By the map \( \Phi \), the potential \( F \) becomes \( \ddot{F} := \ddot{f} \circ \Phi^{-1} \), which is of the following form:
\[ \ddot{F}(\zeta) = \sum_{k=1}^n \text{Re} \{ d_k (\zeta - \zeta_0)^{-k} \}, \]
where \( d_k \) are complex constants and \( \zeta_0 = \Phi(z_0) \). So the transformed source \( \ddot{f} \) is a point multipole source of order \( n \) located at \( \zeta = \zeta_0 \). It is also worth remarking that if the point source \( f \) is located at \( z_0 = 1 \) in the original space, then \( \ddot{f} \) becomes a multipole source at infinity in the transformed space. In fact, its corresponding potential \( \ddot{F} \) is of the following form:
\[ \ddot{F}(\zeta) = \sum_{k=1}^n \text{Re} \{ e_k \zeta^k \} \]
for some complex constants \( e_k \). For example, if \( n = 1 \), then the source \( \ddot{f} \) (or potential \( \ddot{F} \)) represents a uniform incident field.
5. Shielding at a distance due to anomalous resonance

In this section, we analyze the anomalous resonance in the eccentric annulus and explain how a new kind of shielding effect can arise. In view of the previous section, the mathematical description of anomalous resonance in the eccentric case can be directly obtained from that in the concentric case as follows:

(i) The dissipation energy $\tilde{W}_d$ diverges as the loss parameter $\delta$ goes to zero if and only if a point source $\tilde{f}$ is located inside the region $\tilde{W}_d$.

(ii) The electric field $\nabla \tilde{V}_d$ stays bounded in the calm region $\Omega_b$, regardless of $\delta$, i.e.,

$$|\nabla \tilde{V}_d(\zeta)| \leq C, \quad \zeta \in \Omega_b,$$

for some constant $C$ independent of $\delta$.

Now we discuss the new shielding effect. Suppose the parameters $\rho$ and $\rho_l$ satisfy $\rho_l = \sqrt{\rho^2 / \rho} > 1$. Then, as explained in section 3, the calm region $\Omega_b$ becomes a bounded circular region, which does not intersect with the eccentric structure. If a point source is located within the critical region $\Omega_b$, then the anomalous resonance occurs and the normalized electric field $\nabla \tilde{V}_d$ is nearly zero inside the calm region $\Omega_b$. In other words, the shielding effect does occur in $\Omega_b$, but there is a significant difference in this shielding effect compared to the standard one. There is no additional material enclosing region $\Omega_0$; the eccentric structure is located disjointly. So we call this effect ‘shielding at a distance’ and $\Omega_b$ ‘the shielding region’. The condition for its occurrence can be summarized as follows: shielding at a distance happens in $\Omega_b$ if and only if the critical parameter $\rho_b$ and the source location $\zeta_0$ satisfy

$$\rho_b > 1 \quad \text{and} \quad \zeta_0 \in \tilde{\Omega}_b.$$

The shielding effect occurs not only for the point source but also for an external field like a uniform incident field $\hat{F}(\zeta) = -\text{Re} \{E_0 \zeta\}$ for a complex constant $E_0$. As mentioned previously, an external field of the form $\text{Re} \{\sum_{k=1}^n \hat{E}_k \zeta^k\}$ can be considered as a point source at $\zeta = \infty$. Since the critical region $\tilde{\Omega}_a$ contains the point at infinity when $\rho_b > 1$, the anomalous resonance will happen, and the circular bounded region $\tilde{\Omega}_b$ will then be shielded. It is worth remarking that unlike in the eccentric case, the anomalous resonance cannot result from any external field with a source at infinity for the concentric case.

We now discuss the range of validity of the new shielding effect. Recall that we assume the eccentric superlens to have a negative permittivity. For this, we may use plasmonic materials (such as gold and silver) whose permittivities have a negative real part in the infrared to visible regime. In these regimes, the operating wavelength lies in the range of several hundreds of nanometers. Then the eccentric structure and the shielded region should be of less than a few tens of nanometers in size, because our analysis is based on the quasistatic approximation. They should also be relatively close to each other. We emphasize, however, that the sources do not have to be located near the eccentric structure. In fact, the shielding can happen for external electromagnetic waves surrounding the structure and the shielded region. Suppose, for instance, that a plane wave is incident. Since the eccentric structure is small compared to the wavelength, the incident field is nearly uniform across the
structure. As already explained, the anomalous resonance can happen for the uniform dipole at infinity. So the shielding at a distance will occur for the plane wave. Similarly, it can happen for general surrounding waves.

6. Numerical illustration

In this section, we illustrate shielding at a distance by showing several examples of the field distribution generated by an eccentric annulus and a point source. To compute the field distribution, we use an analytic solution derived by applying the separation of variables method in the polar coordinates to the concentric case, and then using the Möbius transformation \( \Phi \). Although we omit the details, we refer to \([1, 8]\) for the analytic solution in the concentric case.

For all the examples below, we fix \( \rho_b = 0.7 \) for the concentric shell and \( a = 1 \) for the Möbius transformation. We also fix the loss parameter as \( \delta = 10^{-12} \).

6.1. Cloaking of a dipole source

We first present an eccentric annulus which acts as a cloaking device (figure 4). Since we want to make a cloaking device, we need \( \rho_b \) to satisfy the condition \( \rho_b < 1 \). Setting \( \rho_l = 0.55 \) for this example, we have \( \rho_b = \rho_l^2 / \rho_i = 0.89 < 1 \) (\( \rho_a = \sqrt{\rho_l^2 / \rho_i} = 0.79 \)). Applying then the Möbius transformation \( \Phi \), the concentric annulus is transformed into the following eccentric structure from (3.2): the outer region \( \tilde{\Omega}_b = \Phi(\Omega_b) \) is the circular disk of radius 2.75 centered at \((-2.92, 0)\) and the core \( \tilde{\Omega}_c = \Phi(\Omega_c) \) is of radius 1.58 centered at \((-1.87, 0)\). The boundaries of the physical regions \( \partial\tilde{\Omega}_b \) and \( \partial\tilde{\Omega}_c \) are plotted as solid white curves in figure 4. On the other hand, the critical region’s boundary \( \partial\tilde{\Omega}_a \), which is not a material interface and is a circle of radius 4.08 centered at \((-4.55, 0)\), is plotted as a dashed white circle. We do not plot the calm region’s boundary \( \partial\tilde{\Omega}_b \) in the figure for the sake of simplicity; it is relatively close to \( \partial\tilde{\Omega}_a \). Note that the calm region \( \tilde{\Omega}_b \) is an unbounded region whose boundary is slightly outside \( \partial\tilde{\Omega}_a \).

In figure 4 we assume that a dipole source \( \tilde{F}(\zeta) = \text{Re}\{\tilde{b}(\zeta - \zeta_0)^{-1}\} \) is located at \( \zeta_0 = (-3.4, 8.5) \) with the dipole moment \( \tilde{b} = (3, -3) \). The point source is plotted as a small solid disk in white. It is clearly seen that the field distribution is smooth over the entire region except at the dipole source. That is, the anomalous resonance does not happen. We can detect the dipole source by measuring the perturbation of the electric field.

In figure 4 we change the location of the source to \( \zeta_0 = (-3.4, 3.5) \) so that the source’s location belongs to the critical region \( \tilde{\Omega}_b \). Then the anomalous resonance does happen, as shown in the figure. As a result, the potential outside the dashed white circle becomes nearly constant. In other words, the dipole source is almost cloaked.

Figure 4. Cloaking for a dipole source. We set \( \rho_b = 0.55, \rho_b = 0.7 \) and \( a = 1 \). The dipole source is located at \( \zeta_0 = (-3.4, 8.5) \) in the left figure and at \( \zeta_0 = (-3.4, 3.5) \) in the right figure. (left) A dipole source (the small solid disk in white) is located outside the critical region \( \tilde{\Omega}_a \) (dashed white circle). The field outside \( \tilde{\Omega}_a \) is significantly perturbed by the source. (right) A dipole source is located inside the critical region \( \tilde{\Omega}_a \). The anomalous resonance happens near the superlens but the field outside \( \tilde{\Omega}_b \) becomes nearly zero, and the source becomes almost cloaked. The plot range is from -10 (blue) to 10 (red).
6.2. Shielding at a distance for a dipole source

Next we show that changing the size of the core can allow shielding at a distance to happen for a dipole source (figure 5).

In figure 5 we let $\rho_s = 0.55$ as in section 6.1. We also assume that a dipole source $\vec{F}(\zeta) = \text{Re} \{\vec{b} (\zeta - \zeta_0)^{-1}\}$ is located at $\zeta_0 = (5, 5)$ with the dipole moment $\vec{b} = (3, 3)$. Since the source is located outside the critical region, the anomalous resonance does not happen.

Now let us change the size of the core. To make the shielding happen at a distance, the critical radius $\rho_k$ satisfies the condition $\rho_k > 1$. We set $\rho_s = 0.2$ so that $\rho_k = \sqrt{\rho_s^2 / \rho_i} = 1.31 > 1$. Then, the core $\Omega_k = \Phi(\Omega_i)$ becomes the circular disk of radius 0.42 centered at $(-1.08, 0)$. The critical region $\Omega_k$ becomes the region outside the circle of radius 3.53 centered at $(4.06, 0)$. The resulting eccentric annulus and the critical region are illustrated in figure 5. Note that the source is contained in the new critical region $\Omega_k$ and $\rho_k > 1$. In other words,
the condition (5.2) for shielding at a distance is satisfied. Indeed, inside the dashed white circle, the potential becomes nearly constant while there is an anomalous resonance outside it. Thus, the shielding at a distance does happen.

6.3. Shielding at a distance for a uniform field
Finally, we consider shielding at a distance for a uniform field (figure 6). We keep the parameters \( a, \rho_1 \) and \( \rho_2 \), as in the previous example, but change the dipole source to a uniform field \( \hat{F}(\xi) = -\text{Re} \{ E_0 \xi \} \) with \( E_0 = 1 \). As mentioned previously, an external field can be considered as a point source located at infinity.

In the left figure, the critical region does not contain infinity, so the anomalous resonance does not happen, and the uniform field can be easily detected. In the right figure, we changed the core as in the previous example. Now the critical region (the region outside the dashed white circle) does contain infinity, so the anomalous resonance does happen. Again, the potential becomes nearly constant in the region inside the dashed circle. This means there is shielding at a distance for the uniform field.

7. Conclusion
We considered an eccentric superlens and showed that it exhibits a new kind of shielding effect. In contrast to the conventional shielding device, the new shielding effect does not require any material enclosing the region to be shielded. In this paper, we assumed a quasistatic regime in two-dimensional space. In our approach, the Möbius map \( \Phi \) is essentially used, and it can be represented in terms of the bipolar coordinates. We expect that it will be possible to extend our result to the three-dimensional quasistatic case by using the bispherical coordinates. It would also be interesting to extend it to general electromagnetic waves by considering the full Maxwell equations.

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