EXAMPLES OF NON-SYMMETRIC KÄHLER-EINSTEIN TORIC FANO MANIFOLDS

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Abstract. In this note we report on examples of 7- and 8-dimensional toric Fano manifolds that are not symmetric and still admit a Kähler-Einstein metric. This answers a question first posed by V.V. Batyrev and E. Selivanova. The examples were found in the classification of $\leq 8$-dimensional toric Fano manifolds obtained by M. Øbro. We also discuss related open questions and conjectures.

1. Introduction

Let us first recall our setting. In the toric case, there is a correspondence between $n$-dimensional nonsingular Fano varieties and $n$-dimensional Fano polytopes, where the Fano varieties are biregular isomorphic if and only if the corresponding Fano polytopes are unimodularly equivalent. Here, given a lattice $N$ of rank $n$, a Fano polytope $Q \subseteq N_\mathbb{Z} := N \otimes \mathbb{Z} \mathbb{R}$ is given as a lattice polytope containing the origin strictly in its interior such that the vertices of any facet of $Q$ form a lattice basis of $M$. In this case, when we denote the dual lattice by $M$, the dual polytope is given as

$$P := Q^* := \{ y \in M_\mathbb{R} : \langle y, x \rangle \geq -1 \ \forall \ x \in Q \}.$$ 

Since $Q$ is a Fano polytope, $P$ is also a lattice polytope. In particular, $Q$ and $P$ are reflexive polytopes.

In 2003 X. Wang and X. Zhu clarified completely which nonsingular toric Fano varieties admit a Kähler–Einstein metric [20]:

**Theorem 1.1** (Wang/Zhu). Let $X$ be a nonsingular toric Fano variety with associated reflexive polytope $P$. Then $X$ admits a Kähler–Einstein metric if and only if the barycenter $b_P$ of $P$ is zero.

Here, the barycenter of $P$ equals the Futaki character of the holomorphic vector field of $X$, [11]. It is also known that the existence of a Kähler–Einstein metric implies that the automorphism group of $X$ is reductive. The converse does not hold (for related combinatorial questions see also [13]).
Prior to the previous theorem, in 1999 V.V. Batyrev and E. Selivanova had already proved a sufficient condition. For this, let us define by $W(P)$ the group of lattice automorphisms of $M$ that map $P$ onto itself. Now, $P$ is called symmetric, if the origin is the only lattice point of $M$ fixed by all elements of $W(P)$. Note that $P$ is symmetric if and only if $Q$ is symmetric (e.g., Proposition 5.4.2 in [12]).

**Theorem 1.2** (Batyrev/Selivanova). Let $X$ be a nonsingular toric Fano variety with associated reflexive polytope $P$. If $P$ is symmetric, then $X$ admits a Kähler–Einstein metric.

**Question 1.3** (Batyrev/Selivanova). Does the converse also hold?

This question was also posed by J. Song (remark after Proposition 4.3 of [17]), by K. Chan and N.C. Leung (Remark 4.1 of [3]), and by A. Futaki, H. Ono, and Y. Sano (introduction of version v1 of [9] and Remark 1.4 of [16]). The hope was that several technical assumptions may be omitted, if the answer would be positive. Unfortunately, this is not true in higher dimensions.

**Proposition 1.4.** The answer to Question 1.3 is negative, if $n \geq 7$.

This observation is explained in the next section. Related open questions and conjectures are discussed in the last section of the paper.

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2. THE EXAMPLES

M. Øbro described in [14] an efficient algorithm to classify Fano polytopes, that he used to compute complete lists of all isomorphism classes of $n$-dimensional Fano polytopes (and their duals) for $n \leq 8$. Now, a simple computer search in Øbro’s database found the examples we were interested in. For this, let us denote by $v_Q$ the sum of all the vertices of a lattice polytope $Q$, and by $b_Q$ the barycenter of $Q$.

**Proposition 2.1.** Let $n \leq 8$, and $Q$ be an $n$-dimensional Fano polytope with dual polytope $P$ such that $b_P = 0$. Then $v_Q = 0$, except if $Q$ is one of the following Fano polytopes $Q_1, Q_2, Q_3$:

1. $Q_1$ is 7-dimensional and has 12 vertices:
The associated nonsingular toric Fano variety \( X_1 \) is a \( \mathbb{P}^1 \)-bundle over \( (\mathbb{P}^1)^3 \times \mathbb{P}^3 \).

(2) \( Q_2 \) is the 8-dimensional Fano polytope with 14 vertices corresponding to \( X_2 := X_1 \times \mathbb{P}^1 \) (i.e., \( Q_2 \) is the bipyramid over \( Q_1 \)).

(3) \( Q_3 \) is 8-dimensional and has 16 vertices:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\
\end{pmatrix}
\]

The associated nonsingular toric Fano variety \( X_3 \) is a \( S_3 \)-bundle over \( (\mathbb{P}^1)^3 \times \mathbb{P}^3 \), where \( S_3 \) is \( \mathbb{P}^2 \) blown up in three torus-invariant points.

In particular, for each \( n \geq 7 \) we see that \( (\mathbb{P}^1)^{n-7} \times X_1 \) is a nonsymmetric toric Fano \( n \)-fold admitting a Kähler–Einstein metric.

### 3. Related questions and results

#### 3.1. The anticanonical degree.

There is a long-standing open conjecture by E. Ehrhart, see section E13 in [7], that can be seen as a generalization of Minkowski’s first theorem:

**Conjecture 3.1** (Ehrhart). Let \( P \subseteq M_\mathbb{R} \) be an \( n \)-dimensional convex body with the origin as its only interior lattice point and barycenter \( b_P = 0 \). Then \( \text{vol}(P) \leq (n+1)^n/n! \). Moreover, equality should only be obtained for \( Q^* \), where \( Q \) is the (unique) Fano simplex corresponding to \( \mathbb{P}^n \).

We checked this conjecture for duals of Fano polytopes up to dimension eight.
In algebro-geometric terms, Conjecture 3.1 implies the following statement, for which no proof is known, too: Any $n$-dimensional toric Fano manifold $X$ that admits a Kähler–Einstein metric has anticanonical degree $(-K_X)^n \leq (n+1)^n$, with equality only for $\mathbb{P}^n$. It was noted in [10] that Bishop’s obstruction [2] yields the following bound:

\[(3.1)\quad I(X)(-K_X)^n \leq (n+1)^n+1,\]

where $I(X)$ is the Fano index. While this inequality seems slightly weaker, note that it is sharp for $\mathbb{P}^n$. We don’t know of a purely combinatorial proof of this result.

**Remark 3.2.** For a general toric Fano $n$-fold $X$ there is no polynomial bound on $\sqrt{(-K_X)^n}$, as was proven by O. Debarre in [8] on p.139. In particular, also inequality (3.1) does not hold in general, as had been suggested in some recent papers (Conjecture 6.4 in [18], Conjecture 1.8 in [6], and inequality (2.22) in [10]).

### 3.2. The alpha-invariant and the log canonical threshold.

In the case of an $n$-dimensional toric Fano manifold $X$ there is an explicit formula [17] for the *alpha-invariant* introduced by G. Tian [Tia87]. For this, let $P$ be the associated reflexive polytope. We denote by $P_G$ the intersection of $P$ with the subspace of all points that are fixed by each element in the group $G \subseteq \mathcal{W}(P)$. Let us also recall the definition of the *coefficient of asymmetry* $ca(P,0)$ of $P$ about the origin:

$$ca(P,0) := \max_{\|y\|=1} \max(\lambda > 0 : \lambda y \in P).$$

The coefficient of asymmetry plays an important role in finding upper bounds on the volume of lattice polytopes with a fixed number of interior lattice points [15].

**Theorem 3.3** (Song). Let $X$ be an $n$-dimensional toric Fano manifold with associated reflexive polytope $P$. Let $G$ be the subgroup of $\text{Aut}(X)$ generated by $\mathcal{W}(P)$ and $(S^1)^n$. Then $\alpha_G(X) = 1$, if $X$ is symmetric, and $\alpha_G(X) = \frac{1}{1+ca(P_{\mathcal{W}(P)},0)}$, otherwise.

In [4] it was shown that for $X$ smooth and $G$ compact, the alpha-invariant $\alpha_G(X)$ coincides with the *global $G$-invariant log canonical threshold* $\text{lct}(X,G)$, for this notion see Definition 1.13 in [4]. I. Cheltsov and C. Shramov also calculated directly the log canonical threshold without assuming smoothness (see also Remark 1.11 in [5]):

**Lemma 3.4** (Cheltsov/Shramov). Let $X$ be an $n$-dimensional toric $\mathbb{Q}$-factorial Fano variety with the polytope $P$ associated to $-K_X$. Let
$G \subset \mathcal{W}(P)$ be a subgroup. Then
\[
\text{lct}\left(X, G\right) = \frac{1}{1 + \max\left\{ \langle w, v \rangle \mid w \in P_G, \ v \in \mathcal{V}(Q) \right\}},
\]
where $\mathcal{V}(Q)$ are the primitive generators of the fan associated to $X$.

From a combinatorial point of view, it is indeed straightforward to notice that the previous two formulas in Theorem 3.3 and Lemma 3.4 agree for a reflexive polytope $P$ dual to a Fano polytope $Q$. Now, for the interested reader we provide the alpha-invariants of our examples:

**Proposition 3.5.** Each Fano polytope $Q_1, Q_2, Q_3$ (see Proposition 2.1) has a 1-dimensional fixspace. Hence, this also holds for their dual reflexive polytopes. Therefore, $\alpha_G(X_i) = \frac{1}{2}$ for $i = 1, 2, 3$.

3.3. **Chern number inequalities.** In [3] a series of Miyaoka-Yau type inequalities were proposed by K. Chan and N.C. Leung for compact Kähler $n$-folds $X$ with negative $c_1(X)$. In the toric case they also conjectured an analogue for positive $c_1(X)$.

**Conjecture 3.6** (Chan/Leung). Let $X$ be a Kähler-Einstein toric Fano $n$-fold. Then
\[
c_1^2(X)H^{n-2} \leq 3c_2(X)H^{n-2}
\]
for any nef class $H$.

**Remark 3.7.** Here is purely combinatorial consequence that was observed in [3]. Let $Q$ be the Fano polytope corresponding to a Kähler-Einstein toric Fano $n$-fold $X$, and $P$ the dual reflexive polytope. Let us denote the Ehrhart polynomial $k \mapsto |(kP) \cap \mathbb{Z}^n|$ of $P$ by $\sum_{i=0}^n a_i t^i$. Then
\[
c_1^2(X)(-K_X)^{n-2} \leq 3c_2(X)(-K_X)^{n-2}
\]
if and only if
\[
a_{n-2} \leq \frac{1}{3} \text{vol}(P^{(2)}),
\]
where $P^{(2)}$ is the union of all codimension two faces of $P$. Using the database, we checked that Equation (3.2) holds for $n \leq 7$. This provides additional evidence in favour of Conjecture 3.6.

In their paper K. Chan and N.C. Leung proved this conjecture in some particular instances (Theorem 1.1 of [3]):

**Theorem 3.8** (Chan/Leung). **Conjecture 3.6** holds, if
1. $n = 2, 3, 4$, or
(2) each facet of the associated reflexive polytope \( P \) contains a lattice point in its interior.

The proof relied on a purely combinatorial property, which the authors conjectured to hold also without additional assumptions on \( X \) (Conjecture 3.1 of [3]):

**Conjecture 3.9** (Chan/Leung). If \( b_P = 0 \), then for any facet \( F \) there exists a point \( x_F \in \text{aff}(F) \) such that

\[
\langle u_G, x_F \rangle \leq \frac{1}{2}
\]

for any facet \( G \) of \( P \) adjacent to \( F \), which is defined via \( \langle u_G, G \rangle = -1 \) and \( \langle u_G, P \rangle \geq -1 \).

Here is an example that shows that this approach of proving Conjecture 3.6 unfortunately fails in general.

**Proposition 3.10.** Conjecture 3.9 does not hold for the 5-dimensional reflexive polytope \( P \) with \( b_P = 0 \), whose dual \( Q \) is a Fano polytope having the following vertices:

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1
\end{pmatrix}
\]

The criterion of Conjecture 3.9 does not hold for the facets of \( P \) associated to the vertices in column four and five. Note that these are precisely the facets that do not contain interior lattice points, as required by Theorem 3.8.

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