Discriminating Equivalent Algorithms via Relative Performance

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Abstract—In scientific computing, it is common that a mathematical expression can be computed by many different algorithms (sometimes over hundreds), each identifying a specific sequence of library calls. Although mathematically equivalent, those algorithms might exhibit significant differences in terms of performance. However in practice, due to fluctuations, there is not one algorithm that consistently performs noticeably better than the rest. For this reason, with this work we aim to identify not the one best algorithm, but the subset of algorithms that are reliably faster than the rest. To this end, instead of using the usual approach of quantifying the performance of an algorithm in absolute terms, we present a measurement-based clustering approach to sort the algorithms into equivalence (or performance) classes using pair-wise comparisons. We show that this approach, based on relative performance, leads to robust identification of the fastest algorithms even under noisy system conditions. Furthermore, it enables the development of practical machine learning models for automatic algorithm selection.

Index Terms—performance analysis, algorithm ranking, benchmarking, sampling

I. INTRODUCTION

Given a set $\mathcal{A}$ of mathematically equivalent algorithms (i.e., algorithms that in exact arithmetic would all return the same output), we aim to identify the subset $\mathcal{F} \subseteq \mathcal{A}$ that contains all those algorithms that are “equivalently” fast to one another, and “noticeably” faster than the algorithms in the subset $\mathcal{A} \setminus \mathcal{F}$. We will clarify the meaning of “equivalent” and “noticeable” shortly; for now, we simply state that in order to identify $\mathcal{F}$, we develop a measurement-based approach that assigns a higher score to the algorithms in $\mathcal{F}$ compared to those in $\mathcal{A} \setminus \mathcal{F}$. In order to capture the performance of an algorithm, we compute a relative score that compares the current algorithm with respect to the fastest algorithm(s) in $\mathcal{A}$. We refer to such scores as “relative performance estimates”.

It is well known that execution times are influenced by many factors, and that repeated measurements, even with same input data and compute environment, often result in different execution times [1]–[3]. Therefore, comparing the performance of any two algorithms involves comparing two sets of measurements (or “distributions”). In common practice, time distributions are summarized into few statistical estimates (such as minimum or median execution time, possibly in combination with standard deviations or quantiles), which are then used to compare and rank algorithms [2], [4], [5]. However, when system noise begins to have a significant impact on program execution, the common statistical quantities cannot reliably capture the profile of the time distribution [6]; as a consequence, when time measurements are repeated, the ranking of algorithms would most likely change at every repetition and this makes the development of reliable machine learning models for automatic algorithm comparisons difficult. The lack of consistency in ranking stems from not considering the possibility that two algorithms can be equivalent when comparing their performance. In order for one algorithm to be faster (or slower) than the other, there should be “noticeable” difference in their time distributions; for instance, comparing the blue and red time distributions in Fig. 1a, the blue algorithm could be considered reliably faster than the red one. On the other hand, the performance of algorithms is comparable if their distributions are “equivalent” or have significant overlap (Figure 1b). Therefore, the comparison of two algorithms will yield one of the three outcomes: faster, slower, or equivalent.

In this paper, we describe a methodology to use this three-way comparison to sort the set of algorithms $\mathcal{A}$ into performance classes by merging the ranks of algorithms whose distributions are significantly overlapping with one another.

The algorithms in $\mathcal{A}$ represent different, alternative ways of computing the same mathematical expression. For instance, consider the following expression which appears in an image restoration application [7]:

$$y := H^T y + (I_n - H^T H)x;$$

where $H$ is a square matrix in $\mathbb{R}^{n \times n}$, $x$ and $y$ are vectors in $\mathbb{R}^n$, and $I_n$ is an identity matrix. If the product $H^T H$ is computed explicitly, the code would perform a $O(n^3)$ matrix-matrix multiplication; by contrast, by applying distributivity, one can rewrite this assignment as

$$y := H^T(y - Hx) + x,$$

obtaining an alternate algorithm which computes the same expression by using only matrix-vector multiplications, for a cost of $O(n^2)$. In this example, the two algorithms differ in the order of magnitude of floating point operations (FLOPs), hence noticeable differences in terms of execution times are naturally expected. However, with increasing parallelism or efficient

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1In general, distributivity does not always lead to lower FLOP count.
memory access patterns, it is possible that an algorithm with higher FLOP counts result in faster executions [8].

In practice, one has to consider and compare more than just two algorithms. For instance, for the Generalized Least Square problem

$$y := (X^T S^{-1} X)^{-1} X^T S^{-1} z,$$

where $X \in \mathbb{R}^{n \times m}$, $S \in \mathbb{R}^{n \times n}$ and $z \in \mathbb{R}^n$, it is possible to find more than 100 different algorithms that compute the same solution and that differ not more than 1.4x in terms of FLOP count. All these different algorithms arise by changes in the sequence of BLAS calls due to properties of input matrices (such as symmetric positive definite, lower or upper triangular, etc.), different parenthesizations for matrix chains, identifying common sub-expressions, etc. [9]. The number of equivalent algorithms increases exponentially when considering the possible splits of computations among different resources (such as CPU and GPUs). In this work, we consider linear algebra expressions which occur at the heart of many scientific computing applications. We consider solution algorithms generated by the Linnea framework [8], which accepts linear algebra expressions as input, and generates a family of algorithms (in the form of sequential Julia code [10]) consisting of (mostly, but not limited to) sequences of BLAS and LAPACK calls.

In this paper, we develop a measurement-based methodology for reliable discrimination (or clustering) of algorithms. This methodology can also be extended for use in the (more challenging) scenario in which it is not feasible to time all algorithms. In that case, in order to get closer in identifying the set of fastest algorithms $\mathcal{F}$ within a given time budget (for instance, 6 hours), one should make as many “intelligent” measurements as possible. To this end, the clustering is done on a sample of algorithms and further measurements are sampled in such a way that the probability of being in the set $\mathcal{F}$ is higher. The details of this intelligent sampling, which can be achieved by using our methodology, is beyond the scope of this article.

**Contributions:** The efficient computation of mathematical expressions is critical both for complex simulations and for time-critical computations done on resource-constrained hardware [11] (e.g., instantaneous object detection for autonomous driving applications [12]). A framework that identifies the best algorithm should take into consideration at least the systematic noise factors, such as regular interrupts, impacts of other applications that are simultaneously sharing the resources, etc. Moreover, we observed that multiple algorithms can yield similar performance profiles. Therefore, we develop a methodology for robust identification of not one, but a set of fast algorithms for a given operational setting. From the resulting set of fast algorithms, one may seek to select an algorithm based on additional performance metrics such as energy [13]. To this end, we use the bootstrapping technique [14] to rank the set of equivalent algorithms $\mathcal{A}$ into clusters, where more than one algorithm can obtain the same rank. The robust rankings produced by our methodology can also be used as ground truth to train realistic performance models that can predict the ranks without executing all the algorithms.

**Organization:** In Sec. II, we highlight the challenges and survey the state of art. In Secs. III and IV we introduce the idea behind relative performance, and describe a methodology to obtain relative scores, respectively; the methodology is then evaluated in Sec. V. Finally, in Sec. VI we draw conclusions and discuss the applications.

II. RELATED WORKS

The problem of mapping one target linear algebra expression to a sequence of library calls is known as the Linear Algebra Mapping Problem (LAMP) [9] and it is mainly applied in auto-tuning High-Performance Computing applications [15]; typical problem instances have many mathematically equivalent solutions, and high-level languages and
environments such as Matlab [16], Julia [10] etc., ideally should select the best one. However, it has been shown that most of these languages choose algorithms that are suboptimal in terms of performance [8], [9]. A general approach to identify the best candidate is by ranking the solution algorithms according to their predicted performance. For linear algebra computations, the most common performance metric to be used as performance predictor is the FLOP count, even though it is known that the number of FLOPs is not always a direct indicator of the fastest code [8], especially when the computation is bandwidth-bound or executed in parallel. For selected bandwidth-bound operations, Jakymchuk et al. developed analytical performance models based on memory access patterns [17], [18]; while those models capture the program execution accurately, their construction requests not only a deep understanding of the processor, but also of the details of the implementation.

Performance metrics that are a summary of execution times (such as minimum, median etc.) lack in consistency when the measurements of the programs are repeated; this is due to system noise [2], cache effects [1], behavior of collective communications [19] etc., and it is not realistic to eliminate the performance variations entirely [20]. The distribution of execution times obtained by repeated measurements of a program is known to lack in textbook statistical behaviors and hence specialized methods to quantify performance have been developed [21]–[24]. However, approximating statistical distributions require executing the algorithms several times.

The performance of an algorithm can be predicted using regression or machine learning based methods; this requires careful formulation of an underlying problem. A wide body of significant work has been done in this direction for more than a decade [25]–[28]. Peise et al in [25] creates a prediction model for individual BLAS calls and estimates the execution time for an algorithm by composing the predictions from several BLAS calls. In [27] Barve et al predicts performance to optimize resource allocation in multi-tenant systems. Barnes et al in [29] predicts scalability of parallel algorithms. In all these approaches, measurements are designed to estimate the parameters of certain performance “model”; once these parameters are estimated, the model can automatically predict the performance for a specific use-case and a well-defined compute environment. Porting the prediction model for a different compute setting requires running the measurements and re-parametrizing the model again.

Online performance prediction can be used to tackle the sensitivities to compute environments; the dynamic nature of performance is encoded through real-time update of the model parameters via algorithm comparisons [29]–[31]. In [29], Reiji Suda selects the best algorithm by solving a “multi-armed bandit problem”. Here, an application program calls a library function iteratively and there are several equivalent algorithms for the library routine. Candidate algorithms are compared among each other during the course of the iteration and a Bayesian model is updated to minimize the total execution time.

Performance metrics based on algorithm comparisons (or discrimination) are central to developing a class of performance models based on reinforcement learning. The multi-armed bandit problem is a reinforcement learning problem. In such approaches, the model parameters are updated using a reward function, which is formulated based on algorithm discriminations; faster algorithms gets higher rewards and the slower algorithms are penalized. Therefore, in this paper, we develop and analyse a metric based on relative performance, which is estimated by clustering the algorithms into performance classes. This metric can be used to define rewards and facilitate development of a class of practical performance models for real-time applications.

In [13], we discuss the application of relative performance in an edge computing environment, where scientific computations are split among various devices. Our relative performance metric, based on clustering methodology, can be used to select algorithms with respect to more than one criteria; for instance, from a subset of equivalently fast algorithm, one can then select an algorithm that consumes the least energy.

III. TOWARDS RELATIVE PERFORMANCE

Fundamentally, for every algorithm in \( \mathcal{A} \) we want to compute a score that indicates its chance of being the fastest. As an example, consider the distributions of execution times shown in Fig. 1 for three different algorithms \( \mathcal{A} := \{\text{“Red”}, \text{“Blue”}, \text{“Yellow”}\} \) that solve the Ordinary Least Square problem \( (X^TX)^{-1}X^Ty \) where \( X \in \mathbb{R}^{m \times n} \) and \( y \in \mathbb{R}^n \). The distributions in the example were obtained by measuring the execution time of every algorithm \( \text{alg}_j \in \mathcal{A} \) 500 times [5] let \( t_j \in \mathbb{R}^{500} \) be the distribution of execution times for algorithm \( \text{alg}_j \). In order to ensure that the time measurements are unbiased of system noise, we adopted the following procedure: Let the set of all executions of the three algorithms be \( \mathcal{E} = e_1 \oplus e_2 \oplus e_3 \), where \( e_j \in \{ \text{alg}_{j,1}, \ldots, \text{alg}_{j,500} \} \) is the set of 500 executions of \( \text{alg}_j \) and \( \oplus \) is a concatenation operation. The set \( \mathcal{E} \) was shuffled before the executions were timed and the measurements \( t_j \) were obtained.

A “straightforward” approach to relatively score algorithms consists in finding the minimum (or median, or mean) execution time of every algorithm \( \text{alg}_j \in \mathcal{A} \) from \( N = 500 \) measurements and then use that to construct a ranking [3]. By doing this, every algorithm would obtain a unique rank. If we choose to assign the algorithm with rank 1 to the set of fastest algorithm \( \mathcal{F} \), then only one algorithm (Yellow or Blue) could be assigned to \( \mathcal{F} \), although the distributions of Yellow and Blue are nearly identical. This will lead to inconsistent rankings when all the executions \( \mathcal{E} \) are repeated, unless there is a methodology to assign rank 1 to both Yellow and Blue algorithms. Reproducibility of rankings is essential in order to derive mathematically sound performance predictions. In order

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4The pseudocode of solution algorithms are shown in Appendix A. For the sake of explanation, we only consider three solution algorithms; for this problem. Linnea generates many more than three [6].

5In Sec. VII-B, we show that our methodology produces reliable results also for smaller values of \( N \).
to ensure reproducibility of relative performance estimates, both Yellow and Blue should be assigned to \( F \). To this end, one could choose the \( k \)-best algorithms \( \{2\} \) with \( k = 2 \), and assign both Yellow and Blue to \( F \). However, the fixed value \( k = 2 \) might not be suitable for other scenarios in which either only one algorithm or more than two algorithms are superior to the rest.

**Bootstrapping:** Bootstrapping \([14]\) is a common procedure to summarize statistics from an arbitrary distribution. Here, the “straightforward” approach is repeated \( \text{Rep} \) times, and for each iteration, \( K < N \) measurements are sampled from the original \( N \) measurements and rankings are constructed. If an algorithm \( \text{alg}_j \) obtains rank 1 in at least one out of \( \text{Rep} \) iterations, it is assigned to \( F \), and will receive the score of \( c_j / \text{Rep} \), where \( c \leq \text{Rep} \) is the number of times \( \text{alg}_j \) obtains rank 1. The steps for this approach are shown in Procedure 1.

For a given set of algorithms \( \text{alg}_1, \text{alg}_2, \ldots, \text{alg}_p \in \mathcal{A} \) for which the execution times \( t_1, t_2, \ldots, t_p \in \mathbb{R}^N \) are measured \( N \) times each, Procedure 1 returns the subset \( F \subseteq \mathcal{A} \) of fastest algorithms \( f_1, f_2, \ldots, f_q \in F \), each with an associated score \( c_1, c_2, \ldots, c_q \). As an example, when this procedure is applied to the distributions in Fig. 1 (with \( \text{Rep} = 100 \) and a sample size \( K = 5 \)), both Yellow and Blue are assigned to \( F \), with scores 0.55 and 0.45 respectively.

**Procedure 1**

**Input:** \( \text{alg}_1, \text{alg}_2, \ldots, \text{alg}_p \in \mathcal{A} \)

**Output:** \( (f_1, c_1), (f_2, c_2), \ldots, (f_q, c_q) \in F \times \mathbb{R}^+ \)

1: \( p = |\mathcal{A}| \)
2: for \( i = 1, \ldots, p \) do
3: \( t_i = \text{get_measurements}(\text{alg}_i) \)
4: \( a \leftarrow \emptyset \)
5: \( f \leftarrow \emptyset \)
6: for \( i = 1, \ldots, \text{Rep} \) do
7: \( f_j \leftarrow \text{sample } K \text{ values from } t_j \)
8: \( a_i \leftarrow \text{algorithm} \left( \text{argmin} \left( \text{min} \left( \tilde{t}_j \right) \right) \right) \)
9: \( f \leftarrow \text{select unique algorithms in } a \)
10: \( q = |f| \)
11: for \( i = 1, \ldots, q \) do
12: \( c_i \leftarrow \text{number of occurrences of } f_i \text{ in } a \)
13: \( c_i \leftarrow c_i / \text{Rep} \)
14: return \( (f_1, c_1), (f_2, c_2), \ldots, (f_q, c_q) \)

However, in practical applications, the execution times of algorithms are typically measured only a few times (likely much fewer than 500 times, possibly just once), and therefore, only a snapshot of the distributions shown in Fig. 1 will be available. Procedure 1 does not account for the uncertainty in measurement data in capturing the true distributions. The recommended approach is to use tests for significant differences (such as Kruskal-Wallis ANOVA \([6]\)) between pairs of algorithms, and to merge the ranks of algorithms if there is not enough evidence of one algorithm dominating over the other.

In this paper, we incorporate the test for significant differences in Procedure 1 by introducing a three-way comparison function.

**IV. METHODOLOGY**

For each algorithm \( \text{alg}_1, \ldots, \text{alg}_p \in \mathcal{A} \), its execution time is measured \( N \) times using the measurement strategy described in Section III. Let \( t_j \in \mathbb{R}^N \) be the array of time measurements for algorithm \( \text{alg}_j \). We do not make any assumptions about the nature of the distributions of timings. In order to compute the relative score for an algorithm, we first cluster all the algorithms in \( \mathcal{A} \) into performance classes. Let \( r_1, r_2, \ldots, r_w \in \mathbb{Z}^+ \) be the ranks of \( w \) performance classes. The number of performance classes \( w \) (or ranks) \( w \) is lesser than or equal to the total number of algorithms \( p \) (or \( w \leq p \), as more than one algorithm can be assigned with the same rank. For illustration, let us consider an example of clustering four algorithms \( \text{alg}_1, \text{alg}_2, \text{alg}_3, \text{alg}_4 \) \((p = 4)\) into performance classes. Significant overlap in the distributions are observed between algorithms \( \text{alg}_2, \text{alg}_4 \) and \( \text{alg}_1, \text{alg}_3 \); thus, the algorithms should be ranked into two performance classes \((w = 2)\). We represent the outcome of this procedure as a sequence set with tuples \((\text{alg}_j, r_j)\), where \( r_j \) is the rank assigned to \( \text{alg}_j \) and the sequence set for this illustration should result as \( \{(\text{alg}_2, 1), (\text{alg}_4, 1), (\text{alg}_1, 2), (\text{alg}_3, 2)\} \); where an algorithm with a lesser rank is better. All the algorithms with rank 1 are assigned to the set of fastest algorithms \( F \). To this end, we first sort the algorithms in \( \mathcal{A} \) using a three-way comparison function.

The comparison of the distributions of execution times of any two algorithms \( t_i, t_j \in \mathbb{R}^N \) is indicated in Procedure 2.

The comparison yields one of three outcomes:

- Outcome A: algorithm \( \text{alg}_i \) is faster than \( \text{alg}_j \) \((<)\)
- Outcome B: algorithm \( \text{alg}_i \) is as good as \( \text{alg}_j \) \((\sim)\)
- Outcome C: algorithm \( \text{alg}_i \) is slower than \( \text{alg}_j \) \((>)\)

Outcomes A and C imply that there are noticeable differences between the distributions; outcome B implies that the distributions are performance-wise equivalent. In Procedure 2 the distributions \( t_i \) and \( t_j \) are sampled \((t_i \subset t_i \text{ and } t_j \subset t_j)\) with sample size \( K < N \), and the minimum execution time from the respective samples are compared. This comparison procedure is repeated \( M \geq 1 \) times and the counter \( e \) is updated based on the result of comparison in each iteration. In theory, one could also use other metrics such as mean or median execution time (instead of the minimum statistic in line 6 and 8 of Procedure 2), or even a combination of multiple statistics across different comparisons. Peise et al. in \([33]\) show that minimum and median statistics are less sensitive to fluctuations in execution time for linear algebra computations.

Let \( e_i \) and \( e_j \) be the minimum execution time of samples \( \tilde{t}_i \) and \( \tilde{t}_j \) respectively. The probability that \( e_i \) is less than \( e_j \) is approximated from the results of \( M \) comparisons. That is, if \( e_i \) is less than \( e_j \) in \( c \) out of \( M \) comparisons, then the

\[ c \text{ of } M \]
empirical probability $P[e_i \leq e_j]$ is $c/M$. The outcome of the CompareAlgs function (Procedure 2) is determined by comparing the empirical probability against certain threshold (see lines 11 - 16 in Procedure 3). The outcome of CompareAlgs function is not entirely deterministic: If $c/M \rightarrow threshold$, then $alg_i$ approaches the “state of being faster” than $alg_j$ and the outcome can tip over A ($<$) or B ($>$) when the CompareAlgs function is repeated (even for the same input distributions). As a consequence, the comparisons need not be transitive: $alg_1 < alg_2$ and $alg_2 < alg_3$ does not imply $alg_1 < alg_3$.

Effect of threshold: Valid values of threshold for Procedure 3 lie in the range $[0.5, 1]$. When threshold $\rightarrow 0.5$, the outcome B ($>$) becomes less and less likely (and impossible for threshold $= 0.5$), while for threshold $\rightarrow 1$, the conditions for outcomes A ($<$) and C ($\sim$) becomes stricter and outcome B become more and more likely.

Effect of K: As $K \rightarrow N$, the minimum of the samples $(e_i, e_j)$ approximates the minimum execution time $(m_i, m_j)$ of the distributions $t_i$ and $t_j$ respectively, and the result of comparison $e_i \leq e_j$ (line 9 of Procedure 2) becomes more and more deterministic. As a consequence, $c/M$ tends to either 0 or 1 and outcome B ($alg_1 \sim alg_j$) becomes less and less likely. When $K = N$, outcome B become impossible and this invalidates the advantages of bootstrapping. When $K \rightarrow 1$, the minimum estimates of samples $(e_i, e_j)$ point to an instance of execution time $t_i, t_j \in R$ from the distribution. Then, outcome B becomes more likely even for marginal overlap of distributions (as in Fig 1).

Effect of M: When $M = 1$, the threshold does not affect the outcome of comparison and the outcome B becomes impossible. As $M \rightarrow N$, the accuracy of the empirical probability $P[e_i \leq e_j]$ for a given $K$ increases.

Sorting procedure: The CompareAlgs function (Procedure 2) is used in Procedure 3 to sort the algorithms $alg_1, alg_2, \ldots, alg_p \in A$ and assign them to one of the performance classes $r_1, r_2, \ldots, r_w$. Initially, the number of performance classes $w$ is the same as the number of algorithms $p$. The outcome of Procedure 3 is represented as a sequence set $\langle (alg_{s[1]}, r_1), (alg_{s[2]}, r_2), \ldots, (alg_{s[p]}, r_p) \rangle$; for instance, if $\langle (alg_2, 1), (alg_4, 1), (alg_1, 2), (alg_3, 2) \rangle$ is the returned sequence, $s_1$ (or $s[1]$) is the index of the first (and best) algorithm, which is 2, and this algorithm obtains rank 1. $s_2$ is the second index, which is 4 with rank 1 and so on. To this end, a swapping procedure that adapts bubble-sort [34] with three-way comparisons is used (the bubble-sort algorithm is used only to demonstrate the idea behind the methodology; therefore it is not optimized for performance). Procedure 3 first creates an initial sequence set with all the algorithms in $A$ assigned with distinct consecutive ranks $\langle (alg_1, 1), \ldots, (alg_p, p) \rangle$. Starting from the first element of the initial sequence, bubble sort compares adjacent algorithms and swaps indices if an algorithm occurring later in the sequence $alg_{s[j+1]}$ is faster (according to Procedure 2) than the previous algorithm $alg_{s[j]}$, and then ranks are updated. When the comparison of two algorithms results to be as good as each other, both are assigned with the same rank, but their indices are not swapped. In order to illustrate the rank update rules in detail, consider the example in Fig 2 which shows the intermediate steps while sorting the distributions of

Procedure 2 CompareAlgs ($alg_i, alg_j, threshold, M, K$)

**Inp:** $alg_i, alg_j \in A$ $threshold \in [0.5, 1]$ $M, K \in Z^+$

**Out:** \{ “$alg_i < alg_j$”, “$alg_i > alg_j$”, “$alg_i \sim alg_j$” \}

1: $t_i = get\_measurements(alg_i)$ $\triangleright t_i \in R^N$
2: $t_j = get\_measurements(alg_j)$ $\triangleright t_j \in R^N$
3: $p \leftarrow 0$
4: for $m = 1, \ldots, M$ do
5: $t_i \leftarrow $ sample $K$ values from $t_i$
6: $e_i = min(t_i)$
7: $t_j \leftarrow $ sample $K$ values from $t_j$
8: $e_j = min(t_j)$
9: if $e_i \leq e_j$ then
10: $c = c + 1$
11: if $\frac{c}{M} \geq threshold$ then
12: return “$alg_i < alg_j$”
13: else if $\frac{c}{M} < 1 - threshold$ then
14: return “$alg_i > alg_j$”
15: else
16: return “$alg_i \sim alg_j$”

Procedure 3 SortAlgs ($A, threshold, M, K$)

**Input:** $alg_{s[1]}, alg_{s[2]}, \ldots, alg_{s[p]} \in A$ $threshold \in [0.5, 1]$ $M, K \in Z^+$

**Output:** $\langle (alg_{s[1]}, r_1), (alg_{s[2]}, r_2), \ldots, (alg_{s[p]}, r_p) \rangle$

1: Initialize $w \leftarrow p$
2: for $i = 1, \ldots, p$ do
3: Initialize $r_i \leftarrow i$ $\triangleright$ Initialize Alg rank
4: Initialize $s_i \leftarrow i$ $\triangleright$ Initialize Alg Index
5: for $i = 1, \ldots, p$ do
6: for $j = 0, \ldots, p-1$ do
7: ret = CompareAlgs($alg_{s[j]}, alg_{s[j+1]}, threshold, M, K$)
8: if $alg_{s[j+1]}$ is faster than $alg_{s[j]}$ then
9: Swap values of $s_j$ and $s_{j+1}$
10: if $r_{j+1} = r_j$ then
11: if $r_{j-1} \neq r_j$ or $j = 0$ then
12: Increment ranks $r_{j+1}, \ldots, r_p$ by 1
13: else
14: if $r_{j-1} = r_j$ and $j \neq 0$ then
15: Decrement ranks $r_{j+1}, \ldots, r_p$ by 1
16: else if $alg_{s[j+1]}$ is as good as $alg_{s[j]}$ then
17: if $r_{j+1} \neq r_j$ then
18: Decrement ranks $r_{j+1}, \ldots, r_p$ by 1
19: else $alg_{s[j]}$ is faster than $alg_{s[j+1]}$ then
20: Leave the ranks as they are
21: return $\langle (alg_{s[1]}, r_1), \ldots, (alg_{s[p]}, r_p) \rangle$
four algorithms \(\text{alg}_1, \text{alg}_2, \text{alg}_3, \text{alg}_4\) (initialized with ranks 1, 2, 3, 4 respectively). All possible update rules that one might encounter appear in one of the intermediate steps of this example.

1) **Both indices and ranks are swapped**: In the first pass of bubble sort, pair-wise comparison of adjacent algorithms are done starting from the first element in the sequence. Currently, the sequence is \(\{\text{alg}_1, \text{alg}_2, \text{alg}_3, \text{alg}_4\}\). As a first step, algorithms \(\text{alg}_1\) and \(\text{alg}_2\) are compared, and \(\text{alg}_2\) ends up being faster. As the slower algorithm should be shifted towards the end of the sequence, \(\text{alg}_1\) and \(\text{alg}_2\) swap positions (line 9 in Procedure 3). Since all the algorithms still have unique ranks, \(\text{alg}_2\) and \(\text{alg}_1\) also exchange their ranks, and no special rules for updating ranks are applied. So, \(\text{alg}_2\) and \(\text{alg}_1\) receive rank 1 and 2, respectively.

2) **Indices are not swapped and the ranks are merged**: Next, algorithm \(\text{alg}_1\) is compared with its successor \(\text{alg}_3\); since they are just as good as one another, no swap takes place. Now, the rank of \(\text{alg}_3\) should also indicate that it is as good as \(\text{alg}_1\); so \(\text{alg}_3\) is given the same rank as \(\text{alg}_1\) and the rank of \(\text{alg}_4\) is corrected by decrementing by 1. (line 16-18 in Procedure 3). Hence \(\text{alg}_1\) and \(\text{alg}_3\) have rank 2, and \(\text{alg}_4\) is corrected to rank 3.

3) **Indices are swapped and the ranks are merged**: In the last comparison of the first sweep of bubble sort, Algorithm \(\text{alg}_4\) results to be faster than \(\text{alg}_3\), so their positions are swapped. However, ranks cannot be exchanged the same way it happened in step 1, as \(\text{alg}_3\) would receive rank 3, even though it was evaluated to be as good as \(\text{alg}_1\), which has rank 2 (recall from step 2: \(\text{alg}_3 \sim \text{alg}_1\)). So instead of changing the rank of \(\text{alg}_3\), the rank of \(\text{alg}_4\) is improved (line 14-15 in Procedure 3). Thus \(\text{alg}_1\), \(\text{alg}_2\), and \(\text{alg}_3\) are given rank 2, but \(\text{alg}_4\) is now earlier in the sequence than \(\text{alg}_3\). This indicates that there is an inherent ordering even when algorithms share the same rank. Even though \(\text{alg}_4\) gets the same rank as \(\text{alg}_2\), despite resulting to be faster than the latter, it still has chances to be assigned with better rank as the sort procedure progress. Every pass of bubble sort pushes the slowest algorithm to the end of the sequence. At this point, the sequence is \(\{\text{alg}_2, \text{alg}_1, \text{alg}_4, \text{alg}_3\}\).

4) **Swapping indices with algorithms having same rank**: In the second pass of bubble sort, the pair-wise comparison of adjacent algorithms, except the “right-most” algorithm in sequence, is evaluated (note that the right-most algorithm can still have its rank updated depending upon the results of comparisons of algorithms occurring earlier in the sequence). The first two algorithms \(\text{alg}_2\) and \(\text{alg}_1\) were already compared in Step 1. So now, the next comparison is \(\text{alg}_1\) vs. \(\text{alg}_4\). Algorithm \(\text{alg}_4\) results to be faster than \(\text{alg}_1\), so the algorithms are swapped as usual. However, the algorithms currently share the same rank. Since \(\text{alg}_4\) reached the top of its performance class—having defeated all the other algorithms with the same rank—it should now get a higher rank than the other algorithms in its class. Therefore, the rank of \(\text{alg}_1\) and \(\text{alg}_3\) is incremented by 1 (line 10-12 in Procedure 3). \(\text{alg}_1\) stays at rank 2, and \(\text{alg}_3\) and \(\text{alg}_4\) receive rank 3. This completes the second pass of bubble sort and the two slowest algorithms have been pushed to the right.

5) **Indices are not swapped and the ranks are merged**: (This is the same rule applied in Step 2). In the third and final pass, we again start from the first element on the left of the sequence and continue the pair-wise comparisons until the third last element. This leaves only one comparison to be done, \(\text{alg}_4\) vs. \(\text{alg}_2\). Algorithm \(\text{alg}_4\) is evaluated to be as good as \(\text{alg}_2\), so both are given the same rank and the positions are not swapped. The ranks of algorithms occurring later than \(\text{alg}_4\) in the sequence are decremented by 1. Thus, the final sequence is \(\{\text{alg}_2, \text{alg}_4, \text{alg}_1, \text{alg}_3\}\). Algorithms \(\text{alg}_2\) and \(\text{alg}_4\) obtain rank 1, and \(\text{alg}_1\) and \(\text{alg}_3\) obtain rank 2.

All the algorithms with rank 1 are assigned to the set of fastest algorithms \(\mathcal{F}\). From the above illustration, \(\mathcal{F} \leftarrow \{\text{alg}_2, \text{alg}_4\}\). The results of the Compare Function (Procedure 3) are not entirely deterministic due to the non-transitivity of comparisons. As a consequence, the final ranks obtained by Procedure 3 are also not entirely deterministic. For instance, in step 5, algorithm \(\text{alg}_4\) could be right at the threshold of being better than \(\text{alg}_2\); that is, if \(\text{alg}_4\) were estimated to be better than \(\text{alg}_2\) once in every two runs of Procedure 3 then fifty percent of the times \(\text{alg}_2\) would be pushed to rank 2 and would not be assigned to \(\mathcal{F}\). To address this, the SortAlgs function is repeated \(Rep\) times and in each iteration, all the
algorithms that received rank 1 are accumulated (in the list ‘a’ in Procedure 4). Then, we follow the steps similar to lines 10-14 in Procedure 1 to compute the relative score. If an algorithm \( \text{alg}_2 \) was assigned rank 1 in \( c \) out of \( \text{Rep} \) iterations, then \( \text{alg}_1 \) would appear \( c \) times in the list ‘a’. Then, the relative confidence (or relative score) for \( \text{alg}_2 \) is \( c/\text{Rep} \). The modified version of Procedure 1 is shown in Procedure 4 which returns a set of fastest algorithms \( F \) (all the unique occurrences in \( a \)) and their corresponding relative scores \( \text{score} \). Since we are only interested in the fastest algorithms, all the other candidates that were not assigned rank 1 even once are given a relative score of 0. For the illustration in Figure 1, if \( \text{alg}_4 < \text{alg}_2 \) in approximately one out of two executions of SortAlgs function, \( \text{alg}_2 \) would get a relative score of 0.5 and \( \text{alg}_4 \) would get 1.0. \( \text{alg}_1 \) and \( \text{alg}_3 \) would be given relative scores 0. A higher relative score implies a greater confidence of being one of the fastest algorithms; that is, the result can be reproduced with a higher likelihood when all the time measurements are repeated, irrespective of fluctuations in system noise, but for a given operation setting (computing architecture, operating system and run time settings).

**Procedure 4** Get\( F(\mathcal{A}, \text{Rep}, \text{thresh}, M, K) \)

| Input: | \( \text{alg}_1, \text{alg}_2, \ldots, \text{alg}_p \in \mathcal{A} \)  
| thresh \( \in [0.5, 1] \), M, K, Reps \( \in \mathbb{Z}^+ \) |
| Output: | \((f_1, c_1), (f_2, c_2), \ldots, (f_q, c_q) \in \mathcal{F} \times \mathbb{R} \) |
| 1: | \( a \leftarrow [ ] \)  
| 2: | \( f \leftarrow [ ] \)  
| 3: | for \( i = 1, \ldots, \text{Rep} \) do |
| 4: | SortAlgs(\( \mathcal{A}, \text{thresh}, M, K \)) |
| 5: | \( \tilde{a} \leftarrow \text{select algorithms with rank 1} \) |
| 6: | append \( \tilde{a} \) to the list \( a \) |
| 7: | \( f \leftarrow \text{select unique algorithms in } a \)  
| | \( \triangleright |f| \leq |a| \) |
| 8: | \( q = |f| \) |
| 9: | for \( i = 1, \ldots, q \) do |
| 10: | \( c_i \leftarrow \text{number of occurrences of } f_i \text{ in } a \) |
| 11: | \( c_i \leftarrow c_i/\text{Rep} \) |
| 12: | return \((f_1, c_1), (f_2, c_2), \ldots, (f_q, c_q) \) |

V. Experiments

The relative score can serve as a metric to discriminate an algorithm from the faster algorithms in \( \mathcal{A} \). In this section, we validate the correctness of relative scores calculated using our methodology. For our experiments, we consider four solution algorithms \( \text{alg}_0, \text{alg}_1, \text{alg}_2, \text{alg}_3 \in \mathcal{A} \) for the Ordinary Least Square problem \((X^T X)^{-1} X^T y\) where \( X \in \mathbb{R}^{m \times n} \) and \( y \in \mathbb{R}^n \) (pseudocode in Appendix A) coded in the Julia language [10]. The distribution of every algorithm consists of 50 time measurements: \( t_0, t_1, t_2, t_3 \in \mathbb{R}^{50} \). In order to validate the consistency of the algorithms rankings to fluctuations in system noise, measurements were repeated under two different settings. In the first setting, the number of threads was fixed to 24; in the second setting, the number of threads for each repetition of an algorithm was randomly chosen between 20 and 24 in order to simulate noticeable fluctuations. Fig. 3 presents the distributions of time measurements for both the settings; the distribution of \( \text{alg}_0 \) is compared against the distributions of the other three algorithms. The distributions of \( \text{alg}_0, \text{alg}_1, \text{alg}_2 \) are largely overlapping; as these algorithms perform the same number of floating point operations but differ in the order in which they execute matrix operations. (Note that, despite similar FLOP counts, the order of computations can still have an effect on the execution time, due to caching effects between sequence of calls within the algorithm [11]).

For all intents and purposes, as indicated by the observations from Fig. 3, \( \text{alg}_0, \text{alg}_1, \text{alg}_2 \) are performance equivalent and we require a statistically sound metric which indicates that choosing any of these three algorithm would not make much difference. Such a metric is important for developing machine-learning based performance prediction models. Recall that the relative scores returned by Procedure 4 indicate the chance of an algorithm to be assigned to the set of fast algorithms \( F \).

In this example, one would naturally expect high scores for \( \text{alg}_0, \text{alg}_1, \text{alg}_2 \). We show that the assignments made to \( F \) are robust and invariant to system noise or fluctuation and also when the number of time measurements is reduced (from 50).

A. Robustness to Fluctuations

If algorithms are ranked according to a single performance number, such as minimum or mean execution time, then the resulting ranking is generally highly sensitive to small changes to the measurement data. In other words, if new measurements were acquired, a different ranking would arise with a high likelihood. We illustrate the impact of such fluctuations in the ranking of algorithms by measuring the execution times under the two different settings described earlier in this section. Table 3 shows the distribution statistics of the algorithms \( \text{alg}_0, \text{alg}_1, \text{alg}_2, \text{alg}_3 \) for setting 1 (number of threads fixed to 24) and setting 2 (number of threads randomly chosen between 20 - 24). Under setting 1, when considering the minimum execution time, \( \text{alg}_1 \) emerges as the best algorithm. However, under setting 2, \( \text{alg}_2 \) emerges as the best algorithm and \( \text{alg}_1 \) is significantly behind \( \text{alg}_0 \) and \( \text{alg}_2 \). If mean execution time is considered instead of minimum, then under setting 1, \( \text{alg}_0 \) becomes the best algorithm instead of \( \text{alg}_1 \). Hence, relying on a single performance number does not lead to robust ranking.

One could argue that, despite the inconsistency, the best algorithm that is identified with a single statistic is indeed always one of the fastest algorithms \( \{\text{alg}_0, \text{alg}_1 \text{ or } \text{alg}_2\} \). However, recall that we are interested in not just one, but all algorithms that are equivalently fast. In Fig 3 we notice that the distributions of \( \text{alg}_0, \text{alg}_1 \) and \( \text{alg}_2 \) are largely overlapping for time measurements under both the settings.
(a) Setting 1: Number of threads is set to 24

(b) Setting 2: Threads are randomly chosen between 20 - 24

Fig. 3: Fifty measurements of algorithms 1, 2, 3 compared against algorithm 0. Algorithm 0: Blue; Algorithm 1: Orange; Algorithm 2: Yellow; Algorithm 3: Red.

Table 1: Time statistics for the four algorithms (in ms).

| Setting 1 | Setting 2 |
|-----------|-----------|
|           | a₀ | a₁ | a₂ |           | a₀ | a₁ | a₂ |
| min       | 1.46| 1.76| 0.24| 1.47       | 1.83| 0.24|
| mean      | 1.44| 1.82| 0.25| 1.51       | 1.91| 0.44|
| std       | 1.46| 1.75| 0.28| 1.45       | 1.85| 0.24|
| a₃        | 1.74| 2.61| 0.73| 2.05       | 2.6 | 0.36|

TABLE II: Relative Scores; Rep = 500, K = 10

| Setting 1 | Setting 2 |
|-----------|-----------|
|           | a₀ | a₁ | a₂ |           | a₀ | a₁ | a₂ |
| M=1, thr=N/A | 0.45| 0.11| 0.44| 0.53       | 0.07| 0.40|
| M=30, thr=0.50 | 0.53| 0.0 | 0.47| 0.76       | 0.0 | 0.24|
| M=30, thr=0.80 | 0.96| 0.73| 0.90| 0.97       | 0.12| 0.95|
| M=30, thr=0.85 | 0.97| 0.89| 0.94| 0.95       | 0.22| 0.93|
| M=30, thr=0.90 | 0.99| 0.97| 0.99| 0.95       | 0.66| 0.90|
| M=30, thr=0.95 | 1.0 | 0.99| 0.99| 0.97       | 0.88| 0.97|

therefore, one would expect all the three algorithms to be assigned to the set of fastest algorithms $\mathcal{F}$. We consider the assignments made to $\mathcal{F}$ to be robust only if they are invariant to measurement fluctuations.

In order to further understand the assignments made to $\mathcal{F}$, let us first describe the effect of the hyper-parameters that were introduced in our methodology (Sec. IV). Recall that, we introduced the three-way comparison function, which in addition to the outcomes “faster” and “slower”, also captures the “equivalence” ($\sim$) among the two algorithms being compared. In Procedure 2 when the number of bootstrap iterations $M = 1$, the outcome that represents equivalence of algorithms ($\sim$) is no longer possible; and the outcome can just be either faster or slower. Table II shows the relative scores obtained by Procedure 3 for $M = 1$ (threshold does not affect the outcome) and $M = 30$ with different values of threshold. Relative scores close to 1 (or 0) implies that the algorithm consistently obtained (or did not obtain) rank 1 in all the Rep repetitions of the sorting algorithm in Procedure 4; therefore, as scores tend to 1, we consider the assignment of the algorithm to $\mathcal{F}$ to be more consistent or robust. When $M = 1$, alg₀ received a low relative score, despite significantly overlapping with alg₁ and alg₂ in both settings. When $M = 30$, bootstrapping is introduced; however, with threshold = 0.5, the outcome “∼” in the CompareAlg function is still not possible, but it can be seen that the score for alg₁ is already 0 and can never be assigned to $\mathcal{F}$ (an indication of improved consistency). When the threshold is increased from 0.5, the outcome “∼” in Procedure 3 now becomes possible and as the threshold tends to 1, the conditions for one algorithm to be ranked faster (or slower) than the other become stricter. As the threshold increases, the relative score of alg₁ also increases (see Table II), which means that it was assigned with rank 1 more frequently across different repetitions of the SortAlg function in Procedure 4. Now all the algorithms alg₀, alg₁, alg₂, alg₃ with overlapping distributions of execution time gets relative score close to 1 as the threshold increases, which implies that their assignment to $\mathcal{F}$ becomes more and more probable (robust); alg₃ still obtains a relative score of 0; this is because the difference in the distribution of execution time with the other algorithms is noticeable.

The parameter threshold can be seen as something that controls the “tolerance level” up to which two algorithms can be considered equivalent. That is, a high threshold implies that the tolerance to rank an algorithm better than the other is low and as a result, any two algorithms can result to be equivalent (∼) unless there is enough evidence of one distribution dominating the other. The tolerance level is also impacted by the parameter $K$ in Procedure 2 which is the number of measurements sampled from $N$ time measurements of an algorithm in each of the $M$ bootstrap iterations. Let $t_i, t_j \in \mathbb{R}^N$ be the time measurements of algᵢ and algⱼ respectively. In each of the $M$ iterations, let $(e_i, e_j)$ be the
minimum execution time that is computed from the samples \( \tilde{t}_i \subset t_i \) and \( \tilde{t}_j \subset t_j \) (\( \tilde{t}_1, \tilde{t}_j \in \mathbb{R}^K \)). As \( K \to N \), the minimum of the samples \( (\tilde{t}_i, \tilde{t}_j) \) would tend to the minimum of the distributions \( (t_i, t_j) \). As a consequence, when \( K = N \), the evaluation of \( \epsilon_j \leq \epsilon_j \) in Procedure \( \Box \) becomes deterministic, and “~” would become an impossible outcome. For Setting 1 in Table \( \Box \), alg_1 recorded the lowest execution time; therefore, as \( K \to N (= 50) \), the relative score of alg_1 approaches 1.0, while the scores of alg_0 and alg_2 approach 0 (see Figure \( \Box \)). Thus, higher values of \( K \) invalidate the advantages of bootstrapping as the rankings start depending more and more exclusively on a single statistic. It is ideal to have \( K \) randomly chosen from a set of values (say \( K \in [5, 10] \)).

B. Robustness to number of measurements

We evaluate the assignments made to \( \mathcal{F} \) as the number of time measurements \( N \) of each algorithm is reduced. All algorithms that obtain a relative score greater than 0 are assigned to \( \mathcal{F} \). For a given \( N \), let \( \mathcal{F}_N \) be the set of fastest algorithms that have been selected. As an example, let us consider 100 mathematically equivalent algorithms framework for the Generalized Least Square problem \( \Box \); for \( M = 1 \) (no bootstrapping), the set of fastest algorithms that are identified with \( N = 50 \) is \( \mathcal{F}_{50} = \{a_0, a_2\} \), and for \( N = 20 \) is \( \mathcal{F}_{20} = \{a_0, a_1, a_2, a_3, a_4\} \); more algorithms are assigned to \( \mathcal{F}_N \) when \( N \) is reduced. When \( N \) is decreased to 20, one finds less and less evidence of one algorithm dominating over the others; as a consequence, more algorithms obtain rank 1 in at least one of the Rep iterations of Procedure \( \Box \) and are assigned to \( \mathcal{F}_{20} \). We assume that \( \mathcal{F}_{50} \) is closer to the true solution than \( \mathcal{F}_{N<50} \), as higher \( N \) implies more evidence.

We use “precision” and “recall” metrics to evaluate consistency of the assignments made to \( \mathcal{F}_N \). When \( \mathcal{F}_{20} \) is compared against \( \mathcal{F}_{50} \), the set \( \mathcal{F}_{20} \) has \( \{a_0, a_2\} \) as true positives \((TP)\), \( \{a_1, a_3, a_4\} \) as false positives \((FP)\), and there are no false negatives \((FN)\). Therefore, the precision and recall for this

![Fig. 4: Relative Score vs K; Rep=500, M=30, threshold=0.9](image)

precision and recall values averaged for 25 different linear algebra expressions (taken from \( \Box \)) arising in practical applications, and each expression consisting of at most 100 mathematically equivalent algorithms. It can be seen that the precision improves considerably when \( M = 30 \) (with bootstrapping), even with zero tolerance \((i.e., \text{threshold} = 0.5)\). Furthermore, both precision and recall improve as the threshold increases. However, the recall decreases as \( N \) is reduced. A lower recall score combined with high precision implies that, although not all the true fast algorithms are selected, the quality of the solutions identified is still preserved, as the number of false positives is less; in order to optimally select algorithms based on additional performance metric (such as energy), the set \( \mathcal{F} \) should contain as little false positives as possible.

However, we mentioned earlier that higher a threshold implies that an algorithm will be ranked better than the other only when there is enough evidence of one distribution dominating the other; as \( N \) decreases, the information about the actual distribution is also reduced, and therefore it is recommended to reduce the threshold in proportion to \( N \).

VI. CONCLUSION AND FUTURE OUTLOOK

For a given mathematical expression, there typically exist many solution algorithms; although mathematically equivalent to one another, those algorithms might exhibit significant differences in performance. We developed and evaluated a metric that quantifies the discrimination of an algorithm from the fastest algorithm. Our methodology is a measurement-based approach to identify not one, but a subset of algorithms that are equivalently fast.\(^\text{10}\) This approach can be used when code optimization is done with respect to multiple factors such as execution time and energy or scalability, where identifying more than one fast algorithm becomes helpful.\(^\text{13}\)

The typical development of scientific code involves a lot of trial and error in choosing the best implementation out of several possible alternatives. In order to offer a high-level of precision = \( TP/\left( TP + FP \right) \), recall = \( TP/(TP + FN) \). Hence, \( 0 \leq \text{precision} \leq 1, \) and \( 0 \leq \text{recall} \leq 1 \).

\(^\text{10}\)\text{Code: https://github.com/as641651/Relative-Performance}
abstraction, languages such as Julia [10], Matlab [16], Tensor-ﬂow [35] etc., were developed to automatically decompose a mathematical expression into sequences of library calls. Compilers like Linnea [8] were developed on top of such high-level languages to identify several alternative algorithms for the same mathematical expression. The selection of the best algorithm can be guided by models that predict relative performance.

Relative Performance Modelling: A natural extension to this work would be Relative Performance modelling, where we aim to automatically discriminate the algorithms without having to execute all the alternatives. To this end, our methodology can be used to develop machine learning models that are effective in doing what comes naturally to humans; that is, to learn through discrimination [36]. If a human were assigned the task of selecting a fast algorithm among hundreds of alternatives, within a stipulated time-frame where he or she cannot execute and measure all those possible alternatives, a typical approach to solution would have been the following: First, sample a small set of algorithms and execute them; then, discriminate the faster algorithms from the slower algorithms by identifying the “features” (such as FLOPs, choice of BLAS calls, number of cores etc.) that resulted in an algorithm to execute faster or slower; finally, sample further algorithms for measurements, that have the appropriate features which could most likely result in those algorithm to execute faster. In order to develop a machine learning model for this task, our clustering methodology can be performed on a sample of algorithms and the relative scores can be mapped with the algorithm features to train a model iteratively, which can then be used to sample further algorithms that have high probabilities of being faster algorithms. We call this procedure as “intelligent sampling,” for which a statistically sound relative performance metric is needed.

APPENDIX

A. Algorithms for the Ordinary Least Square problem

The algorithms are all mathematically equivalent; the FLOP count for Algorithm 3 is twice as high as that for Algorithm 0, 1, 2.

Algorithm 0 Blue

Expression: $(X^TX)^{-1}X^Ty$ $X \in \mathbb{R}^{1000 \times 500}$ $y \in \mathbb{R}^{500}$

1: $T_1 \leftarrow \text{syrk}(X^TX)$ $\triangleright T_1^{-1}X^Ty$
2: $LL^T \leftarrow \text{Cholesky}(T_1)$ $\triangleright L^{-T}L^{-1}X^Ty$
3: $t_2 \leftarrow (X^Ty)^T$ $\triangleright L^{-T}L^{-1}T_1$
4: $t_2 \leftarrow L^{-1}t_2$
5: $z = L^{-T}t_2$

Algorithm 1 Orange

Expression: $(X^TX)^{-1}X^Ty$ $X \in \mathbb{R}^{1000 \times 500}$ $y \in \mathbb{R}^{500}$

1: $t_1 = X^Ty$ $\triangleright (X^TX)^{-1}t_1$
2: $T_2 \leftarrow \text{syrk}(X^TX)$ $\triangleright T_2^{-1}t_1$
3: $LL^T \leftarrow \text{Cholesky}(T_2)$ $\triangleright L^{-T}L^{-1}t_1$
4: $t_1 = L^{-1}t_1$
5: $z = L^{-T}t_1$

Algorithm 2 Yellow

Expression: $(X^TX)^{-1}X^Ty$ $X \in \mathbb{R}^{1000 \times 500}$ $y \in \mathbb{R}^{500}$

1: $T_1 \leftarrow \text{syrk}(X^TX)$ $\triangleright T_1^{-1}X^Ty$
2: $t_2 = X^Ty$ $\triangleright T_1^{-1}t_1$
3: $T_2 \leftarrow \text{Cholesky}(T_1)$ $\triangleright L^{-T}L^{-1}X^Ty$
4: $t_2 = L^{-1}t_2$
5: $z = L^{-T}t_2$

Algorithm 3 Red

Expression: $(X^TX)^{-1}X^Ty$ $X \in \mathbb{R}^{1000 \times 500}$ $y \in \mathbb{R}^{500}$

1: $T_1 \leftarrow \text{gemm}(X^TX)$ $\triangleright T_1^{-1}X^Ty$
2: $LL^T \leftarrow \text{Cholesky}(T_1)$ $\triangleright L^{-T}L^{-1}X^Ty$
3: $t_2 \leftarrow X^Ty$ $\triangleright L^{-T}L^{-1}T_1$
4: $t_2 \leftarrow L^{-1}t_2$
5: $z = L^{-T}t_2$

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