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On niche graphs of bipartite tournaments

(이분 토너먼트의 니치 그래프에 대하여)

2018년 2월

서울대학교 대학원
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이 논문을 교육학석사 학위논문으로 제출함

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On niche graphs of bipartite tournaments

A dissertation
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by

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Abstract

Let $D$ be a digraph. The niche graph of $D$ has the same set of vertices as $D$ and an edge between vertices $u$ and $v$ if and only if there exists a common in-neighbor or a common out-neighbor of $u$ and $v$ in $D$. Kim et al. [The competition graphs of oriented complete bipartite graphs, Discrete Applied Mathematics 201 (2016) 182–190] studied the competition graphs of bipartite tournaments. In this thesis, we study the niche graphs of bipartite tournaments to extend their results. We characterize graphs that can be represented as the niche graphs of bipartite tournaments. Then we present forbidden induced subgraphs for niche graphs of bipartite tournaments. We also study niche graphs of strongly connected bipartite tournaments. Finally, we consider the extremal cases of niche graphs of bipartite tournaments each of which has the maximum number of edges or the minimum number of edges.

Key words: competition graph, bipartite tournament, niche graph, niche-realizable, strongly niche-realizable

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Chapter 1

Introduction

1.1 Basic notions in graph theory

In this section, we introduce some basic notions in graph theory. For undefined terms, readers may refer to [2, 4].

Let $G$ be a graph. Two vertices $u$ and $v$ in $G$ are called *adjacent* if there is an edge $e$ in $G$ which connects $u$ and $v$. Then we say $u$ and $v$ are the *end vertices* of $e$. Two distinct edges are also called *adjacent* if they have a common end vertex.

Two graphs $G$ and $H$ are said to be *isomorphic* if there exist bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that for every edge $e \in E(G)$, $e$ connects vertices $u$ and $v$ in $G$ if and only if $\phi(e)$ connects vertices $\theta(u)$ and $\theta(v)$ in $H$. If $G$ and $H$ are isomorphic, then we write $G \cong H$ and call $(\theta, \phi)$ a *graph isomorphism* from $G$ to $H$.

Let $G$ be a graph. A graph $H$ is a *subgraph* of $G$ if $V(H) \subset V(G)$, $E(H) \subset E(G)$, and we write $H \subset G$. If $V(H) = V(G)$, then $H$ is a *spanning subgraph* of $G$. A maximally connected subgraph of $G$ is called a *component* of $G$. For a nonempty subset $X$ of $V(G)$, the subgraph of $G$ whose vertex set is $X$ and whose edge set consists of all edges of $G$ which have both ends in $X$ is called the *subgraph of $G$ induced by $X$* and is denoted by $G[X]$. The subgraph
induced by $V(G) \setminus X$ is denoted by $G - X$. For notational convenience, we write notion $G - v$ instead of $G - \{v\}$ for a vertex $v$ in $G$.

Given a simple graph $G$, the complement $\overline{G}$ of $G$ is the simple graph whose vertex set is $V$ and whose edges are the pairs of nonadjacent vertices of $G$.

A walk in a graph $G$ is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_l \in V(G)$ such that $v_{i-1}v_i \in E(G)$ for $i = 2, \ldots, l$ and is denoted by $v_1v_2\ldots v_l$. If the vertices in a walk are distinct, then the walk is called a path. A cycle in $G$ is a path $v_1v_2\ldots v_k$ together with the edge $v_kv_1$ where $k \geq 3$. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a chord of that cycle. A chordal graph is a simple graph in which every cycle of length greater than three has a chord.

A complete graph $K_n$ is a simple graph with $n$ vertices in which every pair of distinct vertices is joined by exactly one edge. A clique of a graph is a set of mutually adjacent vertices.

A graph is bipartite if its vertex set can be partitioned into two subsets $V_1$ and $V_2$ so that every edge has one end in $V_1$ and the other end in $V_2$; such a partition $(V_1, V_2)$ is called a bipartition of the graph, and $V_1$ and $V_2$ are called its parts. If a bipartite graph is simple and every vertex in one part is joined to every vertex in the other part, then the graph is called a complete bipartite graph. We denote by $K_{m,n}$ a complete bipartite graph with bipartition $(V_1, V_2)$ if $|V_1| = m$ and $|V_2| = n$.

A digraph (directed graph) $D$ is an ordered pair $(V(D), A(D))$ consisting of a set $V(D)$ of vertices and a set $A(D)$ of arcs. If $(u,v)$ is an arc of $D$, we say that $u$ is an in-neighbor of $v$ and $v$ is an out-neighbor of $u$. Given a digraph $D$, we denote by $N_D^+(u)$ (resp. $N_D^-(u)$) the set of out-neighbors (resp. in-neighbors) of a vertex $u$ in $D$. The outdegree (resp. indegree) of $u$ in $D$ is defined to be $|N_D^+(u)|$ (resp. $|N_D^-(u)|$).

A directed walk in a digraph $D$ is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_l \in V(D)$ such that $v_{i-1}v_i \in A(D)$ for $i = 2, \ldots, l$ and is
denoted by \( v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_l \). If the vertices in a walk are distinct, then the walk is called a directed path.

For a digraph \( D \), the underlying graph of \( D \) is the graph \( G \) such that \( V(G) = V(D) \) and \( E(G) = \{ uv \mid (u, v) \in A(D) \} \). An orientation of a graph \( G \) is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is \( G \). An oriented graph is a graph with an orientation. A tournament is an oriented complete graph.

Two vertices \( u \) and \( v \) are called strongly connected provided there are directed walks from \( u \) to \( v \) and from \( v \) to \( u \) in a digraph \( D \). We say a digraph \( D \) is strongly connected if, for arbitrary vertices \( u \) and \( v \) of \( D \), there exists a directed path from \( u \) to \( v \) in \( D \). It is well-known that a digraph \( D \) is strongly connected if and only if there is a closed directed walk which contains each vertex (at least once). Given a strongly connected digraph \( D \), the index of imprimitivity of \( D \), denoted by \( k(D) \), is the greatest common divisor of the lengths of the closed directed walks of \( D \).

### 1.2 Competition graphs and its variants

Given a digraph \( D \), the competition graph \( C(D) \) of \( D \) has the same vertex set as \( D \) and has an edge \( uv \) if for some vertex \( x \in V(D) \) the arcs \((u, x)\) and \((v, x)\) are in \( D \). Competition graphs were introduced by Cohen [7] in 1968 in connection with a problem in ecology. Competition graphs also have applications in communications, coding, radio and television transmission, and large modeling problems by considering generalized competition graphs. (See [24] and [25] for a summary of these applications.) Since Cohen introduced the notion of competition graph, a variety of generalizations of the notion of competition graph have also been introduced.

One of its variants, the notion of niche graph of a digraph was introduced by Cable et al. [5]. The niche graph, denoted by \( \mathcal{N}(D) \), of a digraph \( D \) is the graph which has the same vertex set as \( D \) and has an edge between vertices
and $v$ if and only if there exists a common in-neighbor or a common out-neighbor of $u$ and $v$ in $D$. There are results on the niche number of a graph (see [3, 5, 9]), and niche graphs of various types of graphs also have been actively studied (see [15, 23]). Another variant, the common enemy graph is related to the niche graph. Lundgren and Maybee [20] introduced the notion of common enemy graph of a digraph. The common enemy graph, denoted by $CE(D)$, of a digraph $D$ is the graph which has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a common in-neighbor of $u$ and $v$ in $D$. By the definitions of the competition graph and the common enemy graph, we can see that the niche graph is the union of the competition graph and the common enemy graph. Their study led Scott [26] to introduce the competition-common enemy graph of a digraph as one of its variants. The competition-common enemy graph (CCE graph), denoted by $CCE(D)$, of a digraph $D$ is the graph which has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exist both a common in-neighbor and a common out-neighbor of $u$ and $v$ in $D$. This graph is essentially the intersection of the competition graph and the common enemy graph. From the definition of these graphs, we know the relationship between these graphs: $CCE(D) \subset C(D) \subset N(D)$ and $CCE(D) \subset CE(D) \subset N(D)$. (See also [13, 14, 19, 27] for study on CCE graphs.) Among the other variants, the notion of $p$-competition graph was introduced by Kim et al. [17] as a generalization of the competition graph. The $p$-competition graph, denoted by $C_p(D)$, of a digraph $D$ is the graph which the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exist $p$ common out-neighbors of $u$ and $v$ in $D$ for a positive integer $p$. If $p = 1$, then $C_1(D)$ is the ordinary competition graph. For more results on $p$-competition graphs, readers may refer to [1, 18]. Also, the notion of $m$-step competition graph was introduced by Cho et al. [6] such a generalization. The $m$-step competition graph of a digraph $D$, denoted by $C^m(D)$, is defined to be the graph having the same vertex set as $D$ and
having an edge $xy$ if and only if there exists an $m$-step common prey of $x$ and $y$ in $D$. A vertex $y$ is called an $m$-step prey of a vertex $x$ if there is a directed walk of length $m$ from $x$ to $y$ in $D$. If $m = 1$, then $C^1(D)$ is the ordinary competition graph of $D$. For more recent work on $m$-step competition graphs, the reader may refer to [12, 22]. The competition graphs of tournaments have been actively studied (see [8, 10]). Recently, Kim et al. [16] studied the competition graphs of oriented complete bipartite graphs. In this vein, it seems to be a natural shift to take a look at the niche graph of oriented complete bipartite graphs.

1.3 A preview of thesis

In Chapter 2, we see some fundamental structures of niche graphs of bipartite tournaments and show that a niche graph of a bipartite tournament has no induced path of length greater or equal to 3 and has no $K_{1,3}$ as an induced subgraph. Based on these structures, we discuss niche-realizable pairs.

In Chapter 3, we consider strongly connected bipartite tournaments. We study strongly niche-realizable pairs and show that a relationship between lengths of directed cycles in a bipartite tournament $D$ and the connectedness of the subgraph of $\mathcal{N}(D)$ induced by one partite set of $D$.

In Chapter 4, we consider the extremal cases of niche graphs of bipartite tournaments in aspect of number of edges.
Chapter 2

Niche graphs of bipartite tournaments

2.1 Fundamental structures of niche graphs of bipartite tournaments

An orientation of a complete bipartite graph is sometimes called a bipartite tournament and we use whichever of the two terms is more suitable for a given situation throughout this paper. In addition, we denote \( \{1, 2, \ldots, n\} \) by \([n]\) for some positive integer \(n\).

Let \(D\) be a bipartite tournament with bipartition \((U, V)\). For a subset \(W\) of \(V\), we denote by \(D_W\) the subdigraph of \(D\) induced by \(U \cup W\). For simplicity, we use notation \(D_v\) instead of \(D_{\{v\}}\) for a vertex \(v \in V\).

Kim et al. [16] show that the competition graph of a bipartite tournament has no edges between the vertices in one partite set and the vertices in the other partite set. It is a common phenomenon for the niche graph of a bipartite tournament.

Proposition 2.1. Let \(D\) be an orientation of a complete bipartite graph \(K_{m,n}\) with bipartition \((U, V)\), where \(|U| = m\) and \(|V| = n\). Then, the niche graph
of $D$ has no edges between the vertices in $U$ and the vertices in $V$.

**Proof.** Take any vertex $u$ in $U$ and any vertex $v$ in $V$. Then $N_D^+(u) \cup N_D^-(u) = V$ and $N_D^+(v) \cup N_D^-(v) = U$ and therefore $N_D^+(u) \cap N_D^+(v) \subseteq V \cap U$ and $N_D^-(u) \cap N_D^-(v) \subseteq V \cap U$. Since $U \cap V = \emptyset$, $N_D^+(u) \cap N_D^+(v) = \emptyset$ and $N_D^-(u) \cap N_D^-(v) = \emptyset$. Thus $u$ and $v$ are not adjacent in $\mathcal{N}(D)$. \qed

By Proposition 2.1, it is sufficient to study the structure of the subgraph $\mathcal{N}(D)[U]$ of $\mathcal{N}(D)$ induced by $U$ in order to characterize $\mathcal{N}(D)$ for a bipartite tournament $D$ with bipartition $(U, V)$.

**Proposition 2.2.** Let $D$ be a bipartite tournament with bipartition $(U, V)$ where $|U| = m$. Then, for each $v \in V$, the subgraph of $\mathcal{N}(D_v)$ induced by $U$ is isomorphic to either $K_m$ or $m \geq 2$ and $K_i \cup K_{m-i}$ for some $i \in \left[\frac{m}{2}\right]$.

**Proof.** Take $v \in V$. If either $U = N_D^-(v)$ or $U = N_D^+(v)$, then $\mathcal{N}(D_v)[U] \cong K_m$. Suppose $U \neq N_D^-(v)$ and $U \neq N_D^+(v)$. Then $m \geq 2$, and $|N_D^-(v)| = i$ or $|N_D^+(v)| = i$ for some $i \in \left[\frac{m}{2}\right]$, and so the subgraph of $\mathcal{N}(D_v)$ induced by $N_D^-(v)$ or the subgraph of $\mathcal{N}(D_v)$ induced by $N_D^+(v)$ is isomorphic to $K_i$. Now $U \setminus N_D^-(v)$ (resp. $U \setminus N_D^+(v)$) forms the out-neighborhood (resp. in-neighborhood) of $v$ with size $m-i$, and so the subgraph of $\mathcal{N}(D_v)$ induced by $U \setminus N_D^-(v)$ or $U \setminus N_D^+(v)$ is isomorphic to $K_{m-i}$. Hence $\mathcal{N}(D_v)[U] \cong K_i \cup K_{m-i}$ for some $i \in \left[\frac{m}{2}\right]$. \qed

**Corollary 2.3.** Let $D$ be a bipartite tournament with bipartition $(U, V)$ where $|U| = m$ and $H_v$ be the subgraph of $\mathcal{N}(D_v)$ induced by $U$ for each $v \in V$. Then $\overline{H_v}$ is isomorphic to an edgeless graph with $m$ vertices or $m \geq 2$ and $K_{i, m-i}$ for some $i \in \left[\frac{m}{2}\right]$.

**Proof.** It immediately follows from Proposition 2.2. \qed

**Proposition 2.4.** Let $D$ be a bipartite tournament with bipartition $(U, V)$ where $|U| = m$. Then the edge set of the subgraph $H$ of $\mathcal{N}(D)$ induced by $U$
is in the form of
\[ \bigcup_{j=1}^{V} E_j \]
where \( E_j \) is either \( E(K_m) \) or \( m \geq 2 \) and \( E(K_i \cup K_{m-i}) \) for some \( i \in \lfloor \frac{m}{2} \rfloor \).

**Proof.** Obviously
\[ E(H) = \bigcup_{v \in V} E(\mathcal{N}(D_v)). \]  
By Proposition 2.2, \( \mathcal{N}(D_v)[U] \) is isomorphic to either \( K_m \) or \( m \geq 2 \) and \( K_i \cup K_{m-i} \) for some \( i \in \lfloor \frac{m}{2} \rfloor \) for each \( v \in V \).

**Corollary 2.5.** Let \( D \) be a bipartite tournament with bipartition \( (U, V) \) where \( |U| = m \) and \( H \) be the subgraph of \( \mathcal{N}(D) \) induced by \( U \). Then the edge set of \( \overline{H} \) is in the form of
\[ \bigcap_{j=1}^{V} E_j \]
where either \( E_j = \emptyset \) or \( m \geq 2 \) and \( E(K_{i,m-i}) \) for some \( i \in \lfloor \frac{m}{2} \rfloor \).

**Proof.** By (2.1), \( E(H) = \bigcap_{v \in V} E(\overline{\mathcal{N}(D_v)}) \). By Corollary 2.3, \( \mathcal{N}(D_v)[U] \) is isomorphic to either edgeless graph with \( m \) vertices or \( m \geq 2 \) and \( K_{i,m-i} \) for some \( i \in \lfloor \frac{m}{2} \rfloor \) for each \( v \in V \).

**Corollary 2.6.** Let \( D \) be a bipartite tournament with bipartition \( (U, V) \). Then the subgraph \( H \) of \( \mathcal{N}(D) \) induced by \( U \) is disconnected if and only if \( H \) consists of only two disjoint cliques.

**Proof.** The “if” part is obvious. We show the “only if” part. Let \( |U| = m \). By the hypothesis, \( m \geq 2 \). By Proposition 2.2, \( \mathcal{N}(D_v)[U] \) is isomorphic to either \( K_m \) or \( m \geq 2 \) and \( K_i \cup K_{m-i} \) for some \( i \in \lfloor \frac{m}{2} \rfloor \) for each \( v \in V \). By the hypothesis, only the latter case occurs. Let \( U_v \) be the subset of \( U \) corresponding to the vertex set of \( K_i \) for each \( v \in V \). Suppose to the contrary that \( U_v \neq U_w \) and \( U_v \neq U \setminus U_w \) for some distinct vertices \( v \) and \( w \) in \( V \). Then
there exist two vertices \( x \) and \( y \) in \( U \) satisfying \( \{x, y\} \subset U_v \) or \( \{x, y\} \subset U \setminus U_v \); \( x \in U_w \) and \( y \in U \setminus U_w \). Since the subgraph of \( \mathcal{N}(D_v) \) induced by \( U \) or \( U \setminus U_v \) is complete, \( x \) and \( y \) are adjacent in \( H \). Since \( \mathcal{N}(D_w)[U] \) is a spanning subgraph of \( H \) and the edge \( xy \) connects \( K_i \) and \( K_m - i \) for some \( i \in \left[ \frac{m}{2} \right] \), we may conclude that \( H \) is connected, which contradicts the hypothesis. Thus \( U_v = U_w \) or \( U_v = U \setminus U_w \) for any vertices \( v \) and \( w \) in \( V \). Hence, by Proposition 2.4, \( H \) is isomorphic to \( K_i \cup K_m - i \) for some \( i \in \left[ \frac{m}{2} \right] \).

Proposition 2.7. Let \( D \) be a bipartite tournament with bipartition \((U, V)\). If \( |U| \geq 5 \), then the subgraph \( H \) of \( \mathcal{N}(D) \) induced by \( U \) contains a triangle.

Proof. Let \( |U| = m \geq 5 \). By (2.1), \( E(\mathcal{N}(D_v)) \subseteq E(H) \) for some \( v \in V \). By Proposition 2.2, \( \mathcal{N}(D_v)[U] \) is isomorphic to either \( K_m \) or \( m \geq 2 \) and \( K_i \cup K_m - i \) for some \( i \in \left[ \frac{m}{2} \right] \). If \( \mathcal{N}(D_v)[U] \cong K_m \), then \( H \) clearly contains a triangle. Suppose that \( m \geq 2 \) and \( \mathcal{N}(D_v)[U] \cong K_i \cup K_m - i \) for some \( i \in \left[ \frac{m}{2} \right] \). If \( i \geq 3 \), then \( K_i \) contains a triangle. If \( i \leq 2 \), then \( K_m - i \) contains a triangle by the hypothesis that \( m \geq 5 \). Thus \( H \) contains a triangle.

2.2 Forbidden induced subgraphs for niche graphs of bipartite tournaments

Theorem 2.8. Let \( D \) be a bipartite tournament with bipartition \((U, V)\). Then the niche graph \( \mathcal{N}(D) \) of \( D \) has no induced path of length greater than or equal to 3.

Proof. By Proposition 2.1, it is sufficient to consider \( \mathcal{N}(D)[U] \) by symmetry. Let \( U = \{u_1, u_2, \ldots, u_m\} \). If \( 1 \leq m \leq 3 \), then it clearly holds. Assume \( m \geq 4 \). Suppose to the contrary that \( \mathcal{N}(D)[U] \) has an induced path \( P \) of length greater than or equal to 3. By relabeling vertices of \( U \), we may assume that \( P = u_1u_2 \ldots u_j \) for some \( j \geq 4 \). Since \( u_2 \) and \( u_3 \) are adjacent, they have either a common in-neighbor or a common out-neighbor. Without loss of generality
we may assume that $u_2$ and $u_3$ have a common out-neighbor, say $v$. By the definition of niche graph, $N_D^-(v)$ forms a clique $K^*$ in $N(D)$. Since $P$ has no chord, any pair of edges in $P$ belongs to distinct cliques, and so any of $u_1, u_4, \ldots, u_j$ does not belong to $V(K^*)$. Thus $u_1, u_4, \ldots, u_j \in N_D^+(v)$. Since $N_D^+(v)$ forms a clique containing $u_1$ and $u_4$, $u_1u_4$ is a chord of the induced path $P$, which is a contradiction. Thus $N(D)[U]$ has no induced path of length greater than or equal to 3, so does $N(D)$.

**Corollary 2.9.** Let $D$ be a bipartite tournament with bipartition $(U, V)$. Then the niche graph of $D$ has no induced cycle of length greater or equal to 5.

**Corollary 2.10.** Let $D$ be a bipartite tournament with bipartition $(U, V)$. If there is a vertex $v \in V$ such that either outdegree or indegree of $v$ is 1, then the subgraph of $N(D)$ induced by $U$ is chordal.

**Proof.** By Corollary 2.9, it is sufficient to show that $N(D)[U]$ has no $C_4$ as an induced subgraph. If $|U| \leq 3$, the statement is immediately true. Suppose $|U| =: m \geq 4$. By the hypothesis, $N(D_v)[U]$ is $K_1 \cup K_{m-1}$ and so $N(D_v)[U] = K_{1,m-1}$. By Corollary 2.5, $\overline{N(D)}[U]$ is a subgraph of $K_{1,m-1}$. Therefore any two edges in $\overline{N(D)}[U]$ are adjacent. If $N(D)$ has $C_4$ as an induced subgraph, then $\overline{N(D)}[U]$ contains nonadjacent two edges, which is impossible. Thus $N(D)[U]$ cannot have an induced $C_4$. 

The niche graph of a bipartite tournament satisfying the condition given in Corollary 2.9 may have a 4-cycle as an induced subgraph. For example, consider the digraph $D$ defined as follows.

$V(D) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3, v_4, v_5\}$;

$A(D) = \{(u_1, v_1), (u_1, v_4), (u_1, v_5), (u_2, v_1), (u_2, v_5), (u_3, v_3), (u_3, v_4), (u_3, v_5),$

$(v_1, u_3), (v_2, u_1), (v_2, u_2), (v_2, u_3), (v_3, u_1), (v_3, u_2), (v_4, u_2)\}.$

Then the niche graph of $D$ has an induced 4-cycle (see Figure 2.1).
Corollary 2.11. Let $D$ be a bipartite tournament with bipartition $(U, V)$ and $H$ be the subgraph of $\mathcal{N}(D)$ induced by $U$. Then $\overline{H}$ has no induced path of length greater than or equal to 3.

Proof. If $1 \leq |U| \leq 3$, then it clearly holds. Assume $|U| \geq 4$. Suppose to the contrary that $\overline{H}$ has an induced path $P := x_1x_2\ldots x_j$ for some $j \geq 4$. Then it is easy to check that $x_2x_4x_1x_3$ is an induced path with length 3, which is a contradiction by Theorem 2.8. Thus $\overline{H}$ has no induced path of length greater than or equal to 3. \hfill $\Box$

Corollary 2.12. Let $D$ be a bipartite tournament with bipartition $(U, V)$ and $H$ be the subgraph of $\mathcal{N}(D)$ induced by $U$. Then $\overline{H}$ contains no odd cycle. Furthermore, if $\overline{H}$ contains an induced cycle $C$, then $C$ is a 4-cycle.

Proof. Let $C$ be a cycle of $\overline{H}$. By Corollary 2.5, $E(C) \subseteq \bigcap_{j=1}^{l} E_j$ for some $l \in [|V|]$ where $E_j = E(K_{i,m-i})$ for some $i \in \left[ \left\lfloor \frac{m}{2} \right\rfloor \right]$. Thus $E(C)$ is contained in $E(K_{i,m-i})$ for some $i \in \left[ \left\lfloor \frac{m}{2} \right\rfloor \right]$. Since a bipartite graph cannot contain an odd cycle, $C$ has an even length. If $C$ is an induced cycle, then there is no induced cycle of length greater than or equal to 6 in $\overline{H}$ by Corollary 2.11, and thus $C$ is a 4-cycle. \hfill $\Box$

Theorem 2.13. Let $D$ be a bipartite tournament with bipartition $(U, V)$. Then the niche graph of $D$ contains no $K_{1,3}$ as an induced subgraph.

Proof. Suppose $\mathcal{N}(D)$ contains a $K_{1,3}$. Without loss of generality, we may assume that $\mathcal{N}(D)[U]$ contains a $K_{1,3}$ by Proposition 2.1. Then $\overline{\mathcal{N}(D)[U]}$ contains a triangle, which contradicts Corollary 2.12. \hfill $\Box$
Theorem 2.14. Let $D$ be a bipartite tournament with bipartition $(U,V)$ where $|U| = m$. A component of $\overline{\mathcal{N}(D)}[U]$ with at least two vertices is isomorphic to $K_{p,q}$ where $1 \leq p \leq \lfloor \frac{m}{2} \rfloor$, and $p + q \leq m$.

Proof. Let $X$ be a component of $\overline{\mathcal{N}(D)}[U]$ with at least two vertices. Suppose $X$ contains no cycle. By Corollary 2.11, the longest induced path of $X$ is of the length at most 2. Then $X$ should be $K_{1,q}$ for some $q \in [m - 1]$. Suppose $X$ contains a cycle $C$. By Corollary 2.12, $C$ has an even length and so $X$ contains an induced 4-cycle. By Corollary 2.5, $X$ is a subgraph of $K_{i,m-i}$ for some $i \in [\lfloor \frac{m}{2} \rfloor]$ and so, by Corollary 2.11 and the previous argument, $X$ should be isomorphic to $K_{p,q}$ for some $p, q$, $2 \leq p \leq \lfloor \frac{m}{2} \rfloor$, $p + q \leq m$. \hfill \Box

2.3 Niche-realizable pairs

Definition 2.15. Let $G_1$ and $G_2$ be graphs with $m$ vertices and $n$ vertices, respectively. The pair $(G_1, G_2)$ is said to be niche-realizable through $K_{m,n}$ (in this paper, we only consider orientations of $K_{m,n}$ and so we omit “through $K_{m,n}$”) if the disjoint union of $G_1$ and $G_2$ is the niche graph of an orientation of the complete bipartite graph $K_{m,n}$ with bipartition $(V(G_1), V(G_2))$.

Proposition 2.16. Let $D$ be a bipartite tournament with bipartition $(U,V)$. If $|U| \geq 4$, then the subgraph $H$ of $\mathcal{N}(D)$ induced by $U$ is not a tree.

Proof. Suppose to the contrary, $H$ is a tree. Let $H^*$ be a subtree of $H$ induced by four vertices. Let $P$ be a longest one among the induced paths of $H^*$. By Theorem 2.8, $P$ is of length at most 2. If $P$ is of length 1, then four vertices on $H^*$ form $K_4$ which is impossible by Theorem 2.13. Thus $P$ is of length 2. Then the graphs possibly isomorphic to $H^*$ are given in Figure 2.2. Since $H^*$ contains no cycle, $K_{1,3}$ is the only graph isomorphic to $H^*$, which is impossible. Hence $H$ is not a tree. \hfill \Box

Given a digraph $D$ and vertex sets $S$ and $T$ of $D$, we denote the set of arcs from $S$ to $T$ by $[S,T]$, that is, $[S,T] = \{(x,y) \in A(D) \mid x \in S, y \in T\}$. 
Figure 2.2: The graphs possibly isomorphic to $H^*$. 

**Proposition 2.17.** The pair $(K_m, K_n)$ is niche-realizable for positive integers $m$ and $n$. 

*Proof.* Let $D$ be a bipartite tournament with bipartition $(U, V)$ and the arc set $[U, V]$ where $|U| = m$, $|V| = n$. Then the niche graph of $D$ is $K_m \cup K_n$. Thus the pair $(K_m, K_n)$ is niche-realizable. \qed 

**Proposition 2.18.** For integers $m$ and $n$ with $m \geq 2$ and $n \geq 1$, the pair $(K_i \cup K_{m-i}, K_n)$ is niche-realizable for any $i \in \left[ \left\lfloor \frac{m}{2} \right\rfloor \right]$. 

*Proof.* Fix $i \in \left[ \left\lfloor \frac{m}{2} \right\rfloor \right]$. We define a bipartite tournament $D$ with bipartition $(U, V)$ satisfying $|U| = m$, $|V| = n$ in the following. Let $S$ be a subset of $U$ of size $i$. The arc set of $D$ is $[S, V] \cup [V, U \setminus S]$. Then the niche graph of $D$ is $(K_i \cup K_{m-i}) \cup K_n$. Thus the pair $(K_i \cup K_{m-i}, K_n)$ is niche-realizable. \qed 

**Proposition 2.19.** For integers $m$ and $n$ both of which are at least $2$, the pair $(K_i \cup K_{m-i}, K_j \cup K_{n-j})$ is niche-realizable for any $i \in \left[ \left\lfloor \frac{m}{2} \right\rfloor \right]$ and any $j \in \left[ \left\lfloor \frac{n}{2} \right\rfloor \right]$. 

*Proof.* We define a bipartite tournament $D$ with bipartition $(U, V)$, where $|U| = m \geq 2$ and $|V| = n \geq 2$ as follows. Fix $i \in \left[ \left\lfloor \frac{m}{2} \right\rfloor \right]$ and $j \in \left[ \left\lfloor \frac{n}{2} \right\rfloor \right]$. Let $S$ and $T$ be subsets of $U$ and $V$ respectively where $|S| = i$, $|T| = j$. The arc set of $D$ is 

$$[S, T] \cup [T, U \setminus S] \cup [U \setminus S, V \setminus T] \cup [V \setminus T, S].$$ 

Then it is easy to check that the niche graph of $D$ is $(K_i \cup K_{m-i}) \cup (K_j \cup K_{n-j})$. Thus the pair $(K_i \cup K_{m-i}, K_j \cup K_{n-j})$ is niche-realizable. \qed
Figure 2.3: The digraphs $D_1, D_2,$ and $D_3$.

**Proposition 2.20.** Let $m$ and $n$ be integers such that $m \geq n \geq 3$. Then the pair $(C_m, C_n)$ is niche-realizable if and only if $(m, n) = (3, 3), (4, 3), (4, 4)$.

**Proof.** First, we show the “only if” part. Let $(C_m, C_n)$ be a niche-realizable pair. By Theorem 2.8, $m, n \leq 4$. Thus we have $(m, n) = (3, 3), (4, 3), (4, 4)$.

Now we show the “if” part. Consider the digraphs $D_1, D_2$ and $D_3$ defined as follows (see Figure 2.3).

$V(D_1) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$;
$A(D_1) = \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_2, v_1), (u_2, v_2), (u_2, v_3), (u_3, v_1), (u_3, v_2), (u_3, v_3)\}$;

$V(D_2) = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3\}$;
$A(D_2) = \{(u_1, v_1), (u_1, v_3), (u_2, v_1), (u_2, v_2), (u_2, v_3), (u_3, v_2), (u_3, v_3), (u_3, v_4), (u_4, v_1), (u_4, v_2), (u_4, v_3), (v_1, u_3), (v_2, u_3), (v_3, u_3), (v_3, u_4)\}$;

$V(D_3) = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3, v_4\}$;
$A(D_3) = \{(u_1, v_1), (u_1, v_4), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_3), (u_4, v_3), (u_4, v_4), (v_1, u_3), (v_1, u_4), (v_2, u_1), (v_2, u_4), (v_3, u_1), (v_3, u_2), (v_4, u_2), (v_4, u_3)\}$.

It is easy to check that the niche graphs of $D_1, D_2$ and $D_3$ are $C_3 \cup C_3$, $C_4 \cup C_3$, and $C_4 \cup C_4$, respectively. Therefore $(C_3, C_3), (C_4, C_3), (C_4, C_4)$ are niche-realizable. \qed
**Proposition 2.21.** Let $m$ and $n$ be positive integers such that $m \geq n$. Then the pair $(P_m, P_n)$ is niche-realizable if and only if $(m, n) = (1, 1), (2, 1), (2, 2), (3, 2), (3, 3)$.

*Proof.* First, we show the “only if” part. Let $(P_m, P_n)$ be a niche-realizable pair. By Theorem 2.8, $m, n \leq 3$. Thus we have $(m, n) = (1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)$. When $(m, n) = (3, 1)$, the niche graph of a bipartite tournament is either $K_3 \cup K_1$ or $(K_2 \cup K_1) \cup K_1$ by Proposition 2.2. Hence the possible pairs of $(m, n)$ are $(1, 1), (2, 1), (2, 2), (3, 2), (3, 3)$.

Now we show the “if” part. Since $P_1 \cong K_1$ and $P_2 \cong K_2$, the pairs $(P_1, P_1)$, $(P_2, P_1)$, and $(P_2, P_2)$ are niche-realizable by Proposition 2.17. Consider the digraphs $D_1$ and $D_2$ defined as follows (see Figure 2.4).

$$V(D_1) = \{u_1, u_2, u_3\} \cup \{v_1, v_2\};$$

$$A(D_1) = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (v_1, u_3), (v_2, u_2), (v_2, u_3)\};$$

$$V(D_2) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\};$$

$$A(D_2) = \{(u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (u_3, v_3), (v_1, u_3), (v_2, u_1), (v_3, u_1), (v_3, u_2)\}.$$  

It is easy to check that the niche graphs of $D_1$ and $D_2$ are $P_3 \cup P_2$ and $P_3 \cup P_3$, respectively. Therefore $(P_3, P_2)$, $(P_3, P_3)$ are niche-realizable. \qed

An orientation $D$ of a complete bipartite graph with bipartition $(U, V)$ is said to give *niche-connectedness by $U$* if the subgraph of $\mathcal{N}(D)$ induced by $U$ is a connected non-complete graph.
Lemma 2.22. Let $D$ be an orientation of a complete bipartite graph with
bipartition $(U, V)$, where $|U| \geq 3$ and $|V| \geq 2$. If $D$ gives niche-connectedness
by $U$, then there is a vertex in $U$ each of whose indegree and outdegree is at
least 1.

Proof. Suppose that $\mathcal{N}(D)[U]$ is not complete but connected. Suppose to
the contrary that each vertex in $U$ has either outdegree 0 or indegree 0.
If every vertex in $U$ has outdegree 0 or every vertex in $U$ has indegree 0,
then $\mathcal{N}(D)[U]$ becomes a complete graph, which is impossible. Therefore $U$
is partitioned into two parts $A$ and $U \setminus A$ where $A$ is the set of vertices with
indegree 0. By our assumption, $U \setminus A$ is the set of vertices with outdegree
0. Thus the subgraph of $\mathcal{N}(D_v)$ induced by $A$ (resp. $U \setminus A$) is a clique $K_{|A|}$
(resp. $K_{|U| - |A|}$) of $\mathcal{N}(D)$ for each $v \in V$. By (2.1), $\mathcal{N}(D)[U] \cong K_{|A|} \cup K_{|U| - |A|}$,
which is impossible as $\mathcal{N}(D)[U]$ is connected. Hence there is a vertex in $U$
each of whose indegree and outdegree is at least 1.

Theorem 2.23. Let $D$ be an orientation of a complete bipartite graph with
bipartition $(U, V)$, where $|U| \geq 3$ and $|V| \geq 2$. If $D$ gives niche-connectedness
by $U$, then the subgraph of $\mathcal{N}(D)$ induced by $V$ is connected.

Proof. We prove by induction on $|V| \geq 2$ where $V$ is a partite set of an
orientation $D$ of a complete bipartite graph which gives niche-connectedness
by the other partite set $U$ with $|U| \geq 3$. Assume $|V| = 2$. Then $\mathcal{N}(D)[U]$ is
a connected non-complete graph by the hypothesis. Let $V = \{v_1, v_2\}$. Since
$\mathcal{N}(D)[U]$ is not a complete graph, there are two vertices $u_1$ and $u_2$ in $H$
which are not adjacent. Since $\mathcal{N}(D)[U]$ is connected, there is a path between
$u_1$ and $u_2$. By Theorem 2.8, a shortest path $P$ between $u_1$ and $u_2$ is of length
2. Then $P = u_1w u_2$ for a vertex $w \in U$. Since $u_1$ and $w$ are adjacent, they
have a common out-neighbor or a common in-neighbor in $V$. Without loss of
generality, we may assume that they have a common out-neighbor $v_1$. Since
$u_1$ and $u_2$ are not adjacent, $u_2$ is an out-neighbor of $v_1$. Since $w$ and $u_2$
are adjacent, $w$ and $u_2$ have a common out-neighbor or a common in-neighbor,
say $v_2$, in $V$. If $v_2$ is a common out-neighbor of $w$ and $u_2$, $w$ is a common in-neighbor of $v_1$ and $v_2$, and so $v_1$ and $v_2$ are connected in $\mathcal{N}(D)[V]$. If $v_2$ is a common in-neighbor of $w$ and $u_2$, $u_2$ is a common out-neighbor of $v_1$ and $v_2$, and so $v_1$ and $v_2$ are connected in $\mathcal{N}(D)[V]$. Thus the result holds for an orientation $D$ of a complete bipartite graph with bipartition $(U, V)$ such that $|V| = 2$ and $D$ gives niche-connectedness by $U$.

Suppose that the statement is true for any orientation $D$ of a complete bipartite graph with bipartition $(U, V)$ such that $|U| = m \geq 3$, $|V| = n - 1$ for $n \geq 3$ and $D$ gives niche-connectedness by $U$. Now take an orientation $D^*$ of a complete bipartite graph with bipartition $(U, V)$ such that $|U| \geq 3$, $|V| = n$ and $D^*$ gives niche-connectedness by $U$. Suppose to the contrary that $\mathcal{N}(D^* - v)[U]$ is disconnected for every vertex $v \in V$.

For $v \in V$, we denote the two clique components of $\mathcal{N}(D_{v})[U]$ by $K_{i_{v}}^{(v)}$ and $K_{|U|-i_{v}}^{(v)}$ for some $i \in \left[\left\lfloor \frac{|U|}{2} \right\rfloor \right]$. Then,

\[
(*) \text{ for any pair of vertices } v \text{ and } w \text{ in } V, V(K_{i_{v}}^{(v)}), \text{ the out-neighborhood (resp. in-neighborhood) of } w \text{ and } V(K_{m-i_{v}}^{(v)}), \text{ the in-neighborhood (resp. out-neighborhood) of } w.
\]

If \( \{V(K_{i_{v}}^{(v)}), V(K_{m-i_{v}}^{(v)})\} \neq \{V(K_{i_{w}}^{(w)}), V(K_{m-i_{w}}^{(w)})\} \) for any vertices $v, w$ in $V$, then $\mathcal{N}(D^*)[U]$ is disconnected and we reach a contradiction. Therefore there exist two vertices, say $v_1$ and $v_2$, for which \( \{V(K_{i_{v_{1}}}^{(v_{1})}), V(K_{m-i_{v_{1}}}^{(v_{1})})\} \neq \{V(K_{i_{v_{2}}}^{(v_{2})}), V(K_{m-i_{v_{2}}}^{(v_{2})})\} \). For notational convenience, we use $K_{i_{j}}^{(j)}$ and $K_{m-i_{j}}^{(j)}$ for $K_{i_{v_{1}}}^{(v_{1})}$ and $K_{m-i_{v_{1}}}^{(v_{1})}$, respectively, for each $j = 1, 2$. Since $D^*$ gives niche-connectedness by $U$, there are vertices $u_1 \in V(K_{i_{1}}^{(1)})$ and $u_2 \in V(K_{m-i_{1}}^{(1)})$ which have $v_2$ as a common out-neighbor or a common in-neighbor. Then both of $u_1$ and $u_2$ belong to either $V(K_{i_{2}}^{(2)})$ or $V(K_{m-i_{2}}^{(2)})$, for otherwise $K_{i_{2}}^{(2)}$ and $K_{m-i_{2}}^{(2)}$ would be connected by the edge $u_1u_2$. Since $n \geq 3$, there exists a vertex $v_3$ in $V \setminus \{v_1, v_2\}$. By $(*)$ for $v = v_2$, $v_3$ is a a common out-neighbor or a common in-neighbor of $u_1$ and $u_2$. However, since $u_1 \in V(K_{i_{1}}^{(1)})$ and $u_2 \in V(K_{m-i_{1}}^{(1)})$, $v_3$ cannot be a common out-neighbor or a common in-neighbor of
and $u_2$ by $(\ast)$ for $v = v_1$. Thus we have reached a contradiction. Hence we may conclude that there exist a vertex $v \in V$ such that $\mathcal{N}(D^* - v)[U]$ is connected. Moreover, $\mathcal{N}(D^* - v)[U]$ is not a complete graph as it is a spanning subgraph of $\mathcal{N}(D^*)[U]$. Now let $D = D^* - v$. By the induction hypothesis, $\mathcal{N}(D)[V]$ is connected. Since $|V \setminus \{v\}| \geq 2$ and $\mathcal{N}(D)[U]$ is not complete but connected, there is a vertex $u$ in $U$ whose each of indegree and outdegree is at least 1 by Lemma 2.22. Let $v'$ be an in-neighbor of $u$ and $v''$ be an out-neighbor of $u$ for some distinct $v', v'' \in V \setminus \{v\}$. If $v$ is an in-neighbor of $u$, then $v$ is adjacent to $v'$, and otherwise, $v$ is adjacent to $v''$ in $\mathcal{N}(D^*)[V]$. Thus the statement is true for orientation $D^*$ of a complete bipartite graph with bipartition $(U, V)$ such that $|U| \geq 3$, $|V| = n$ and $D^*$ gives niche-connectedness by $U$. Hence this completes the proof.

**Corollary 2.24.** Let $(G_1, G_2)$ be niche-realizable pair of an orientation of a complete bipartite graph with $|V(G_1)| \geq 3, |V(G_2)| \geq 3$. If none of $G_1$ and $G_2$ is complete, then $G_1$ and $G_2$ both are either connected or disconnected.

**Proof.** It immediately follows from Theorem 2.23.

**Proposition 2.25.** Let $U$ be a partite set of complete bipartite graph $K_{m,n}$ with size $m$ and a graph $G$ be isomorphic to the subgraph of $\mathcal{N}(D)$ induced by $U$ for some orientation $D$ of $K_{m,n}$. Then $(G, K_n)$ is niche-realizable.

**Proof.** Let $V$ be the other partite set of $K_{m,n}$. Then $|V| = n$. In addition, let $G := \mathcal{N}(D)[U]$ for an orientation $D$ of $K_{m,n}$. If $G$ is complete, then $(G, K_n)$ is niche-realizable by Proposition 2.17. If $G$ is disconnected, then $(G, K_n)$ is niche-realizable by Proposition 2.18. Suppose $G$ is connected and non-complete. Then $|V(G)| \geq 3$ and $|V| \geq 2$. By Theorem 2.23, $\mathcal{N}(D)[V]$ is connected. Since $|V| \geq 2$, $\mathcal{N}(D)[V]$ contains at least one edge, say $v_1v_2$. Then there is a vertex $u \in U$ such that $u$ is either a common in-neighbor or a common out-neighbor of $v_1$ and $v_2$. Without loss of generality, we may assume that $u$ is a common out-neighbor of $v_1$ and $v_2.$
Now we construct an orientation $D^*$ of $K_{m,n}$ as in the following. Take a vertex $v \in V$. Then, by Proposition 2.2, $\mathcal{N}(D_v)[U]$ is isomorphic to either $K_m$ or $m \geq 2$ and $K_i \cup K_{m-i}$ for some $i \in \left[\left\lfloor \frac{m}{2} \right\rfloor \right]$. By the case assumption, the latter holds. Then $u$ belongs to one of the two components, say $K_v$, of $\mathcal{N}(D_v)[U]$. In $D$, one of the following is true:

(i) There are arcs from $v$ to each vertex in $K_v$ and the arcs from each vertex in $U \setminus V(K_v)$ to $v$.

(ii) There are arcs from each vertex in $K_v$ to $v$ and the arcs from $v$ to each vertex in $U \setminus V(K_v)$.

If (i) holds, then we add those arcs to $D^*$. If (ii) holds, then we reverse the directions of those arcs to add the resulting arcs to $D^*$. It is easy to check that $\mathcal{N}(D^*)[V]$ is isomorphic to $K_n$ and $\mathcal{N}(D^*)[U]$ is still $G$. \qed

**Theorem 2.26.** The pair $(F, F')$ is niche-realizable for forests $F$ and $F'$ with $|V(F)| \geq |V(F')|$ if and only if $(F, F')$ is one of the following:

(i) $(K_1, K_1)$;

(ii) $(K_2, K_1), (K_2, K_2), (K_2, K_1 \cup K_1), (K_1 \cup K_1, K_1), (K_1 \cup K_1, K_2), (K_1 \cup K_1, K_1 \cup K_1)$;

(iii) $(P_3, K_2), (P_3, P_3), (K_2 \cup K_1, K_1), (K_2 \cup K_1, K_2), (K_2 \cup K_1, K_1 \cup K_1), (K_2 \cup K_1, K_2 \cup K_1)$;

(iv) $(K_2 \cup K_2, K_1), (K_2 \cup K_2, K_2), (K_2 \cup K_2, K_1 \cup K_1), (K_2 \cup K_2, K_2 \cup K_1), (K_2 \cup K_2, K_2 \cup K_2)$

**Proof.** Let $(F, F')$ be a niche-realizable forest pair with $|V(F)| = m$ and $|V(F')| = n$. By Proposition 2.7, $m \leq 4$ and $n \leq 4$.

If $m = 1$, then $(K_1, K_1)$ is the only candidate for niche-realizable pair, by Proposition 2.17, $(K_1, K_1)$ is niche-realizable.
Suppose $m = 2$. Then $F$ is either $K_2$ or $K_1 \cup K_1$. If $F = K_2$, then $F'$ is one of $K_1$, $K_2$, and $K_1 \cup K_1$, and by Propositions 2.17 and 2.18, the first three pairs in the item (ii) are niche-realizable pairs. If $F = K_1 \cup K_1$, then $F'$ is one of $K_1$, $K_2$, and $K_1 \cup K_1$, and by Propositions 2.18 and 2.19, the last three pairs in the item (ii) are niche-realizable pairs.

Suppose $m = 3$. Since $F$ cannot be a triangle, $F$ is either $P_3$ or $K_2 \cup K_1$. Then $F'$ is one of $K_1$, $K_2$, $K_1 \cup K_1$, $P_3$, and $K_2 \cup K_1$ as $F'$ does not contain a triangle. Suppose $F = P_3$. Then, by Proposition 2.2 and Theorem 2.23, $F'$ cannot be any of $K_1$, $K_1 \cup K_1$, $K_2 \cup K_1$. Thus the first two pairs listed in the item (iii) are the only candidates for niche-realizable pairs. Suppose $F = K_2 \cup K_1$. By Theorem 2.23, $F' \neq P_3$. Thus the last four pairs listed in the item (iii) are the only candidates for niche-realizable pairs. By Propositions 2.18, 2.19, and 2.21, all the pairs listed in the item (iii) are niche-realizable.

Suppose $m = 4$. Since $F$ contains no triangle, $F = K_2 \cup K_2$. Then $F'$ is one of $K_1$, $K_2$, $K_1 \cup K_1$, $P_3$, $K_2 \cup K_1$, and $K_2 \cup K_2$ as $F'$ does not contain a triangle. By the way, $P_3$ should be excluded by Theorem 2.23. Thus all the pairs listed in the item (iv) are the only candidates for niche-realizable pairs. By Propositions 2.18 and 2.19, all the pairs listed in the item (iv) are niche-realizable. 

\[\square\]
Chapter 3

Niche graphs of strongly connected bipartite tournaments

There are some results on the competition graph of strongly connected digraphs (see [11, 21]). In this vein, we study niche graphs of strongly connected bipartite tournaments throughout this chapter.

3.1 Strongly niche-realizable pairs

Proposition 3.1. Let $G_1 \cup G_2$ be the niche graph of an orientation $D$ of a complete bipartite graph. If $G_1$ and $G_2$ both are either disconnected graphs or non-complete connected graphs, then $D$ is strongly connected.

Proof. Let $(U, V)$ be the bipartition of the complete bipartite graph whose orientation is $D$. We first suppose that $G_1$ and $G_2$ are disconnected. Then by Corollary 2.6, each of $G_1$ and $G_2$ consists of two clique components. Let $S$ and $T$ be the vertex sets of a clique component of $G_1$ and a clique component of $G_2$, respectively. Then the out-neighborhood of each vertex in $S$ is either $T$ or
Without loss of generality, we may assume that the out-neighborhood of each vertex in $S$ is $T$. Then,

$$\text{(§)} \text{ the arc set of } D = [S, T] \cup [T, U \setminus S] \cup [U \setminus S, V \setminus T] \cup [V \setminus T, S].$$

Take two vertices $u$ and $v$ in $D$. If $u$ and $v$ belong to distinct sets among $S, T, U \setminus S$, and $V \setminus T$, then they are obviously on a directed 4-cycle. If $u$ and $v$ belong to the same set among $S, T, U \setminus S$, and $V \setminus T$, then it is easy to check that they are on a closed directed walk of length 8. Therefore $D$ is strongly connected.

Now suppose $G_1$ and $G_2$ are non-complete connected graphs. Since $G_1$ and $G_2$ are non-complete, $|V(G_1)| \geq 3, |V(G_2)| \geq 3$, and there are two nonadjacent vertex pairs, say $\{u_1, u_2\}$ and $\{v_1, v_2\}$, of $G_1$ and $G_2$, respectively. Since $v_1$ and $v_2$ (resp. $u_1$ and $u_2$) are not adjacent, one of $v_1$ and $v_2$ (resp. $u_1$ and $u_2$) is an out-neighbor of $u$ (resp. $v$) and the other is an in-neighbor of $u$ (resp. $v$) for any $u \in U$ (resp. $v \in V$). Without loss of generality, we may assume that $v_1$ is an out-neighbor of $u_1$. Then $(u_1, v_1), (v_1, u_2), (u_2, v_2)$ and $(v_2, u_1)$ are arcs which form a directed 4-cycle in $D$. Take two vertices $x$ and $y$. Suppose that $x$ and $y$ both belong to same partite set. Without loss of generality, we may assume $x$ and $y$ both are in $V(G_1)$ and that $(x, v_1)$ and $(v_2, x)$ are arcs of $D$. If $(y, v_1)$ and $(v_2, y)$ are arcs of $D$, then $xv_1u_2v_2y$ and $yv_1u_2v_2x$ are directed walks from $x$ to $y$ and from $y$ to $x$, respectively. If $(v_1, y)$ and $(y, v_2)$ are arcs of $D$, then $xv_1y$ and $yv_2x$ are directed walks from $x$ to $y$ and from $y$ to $x$, respectively. Suppose $x$ and $y$ belong to different partite sets. Without loss of generality, we may assume $x \in V(G_1)$ and $y \in V(G_2)$ and $(x, v_1)$ and $(v_2, x)$ are arcs of $D$. If $(y, u_1)$ and $(u_2, y)$ are arcs of $D$, then $xv_1u_2y$ and $yu_1v_1u_2v_2x$ are directed walks from $x$ to $y$ and from $y$ to $x$, respectively. If $(u_1, y)$ and $(y, u_2)$ are arcs of $D$, then $xv_1u_2v_2u_1y$ and $yu_2v_2x$ are directed walks from $x$ to $y$ and from $y$ to $x$, respectively. Thus $D$ is strongly connected.

The graph $G$ given in Proposition 2.25 may be disconnected. However,
\((G, K_n)\) is not strongly niche-realizable if \(G\) is disconnected by the following Proposition.

**Proposition 3.2.** If \(G \cup K_n\) is the niche graph of some orientation \(D\) of \(K_{m,n}\) for \(m \geq 2, n \geq 2\) such that each vertex of \(D\) has both indegree and outdegree at least 1, then \(G\) is connected.

**Proof.** Let \((U, V)\) be the bipartition of \(D\) with \(|U| = m\) and \(|V| = n\). To prove the statement by contradiction, suppose that \(G\) is disconnected. Then, by Corollary 2.6, \(G = K^{(1)} \cup K^{(2)}\) where \(K^{(1)} \cong K_i, K^{(2)} \cong K_{m-i}\) for some \(i \in \left[\lfloor \frac{m}{2} \rfloor \right]\). By (2.1), for each \(v \in V\), the subgraph \(H_v\) of \(N(D_v)\) induced by \(U\) is \(K^{(1)} \cup K^{(2)}\). In addition, by the argument for the proof of Proposition 2.2, \(V(K^{(1)})\) is either the out-neighborhood or the in-neighborhood of \(v\) for each \(v \in V\). If there is a pair of vertices \(v_1, v_2 \in V\) such that \(N_D^-(v_1) = V(K^{(1)}), N_D^+(v_1) = V(K^{(2)})\) (resp. \(N_D^-(v_1) = V(K^{(2)}), N_D^+(v_1) = V(K^{(1)})\)) and \(N_D^-(v_2) = V(K^{(1)}), N_D^+(v_2) = V(K^{(2)})\) (resp. \(N_D^-(v_2) = V(K^{(1)}), N_D^+(v_2) = V(K^{(2)})\)), then \(v_1\) and \(v_2\) are not adjacent in \(N(D)[V]\), which is impossible. Thus either \(V(K^{(1)})\) is the out-neighborhood of \(v\) for each vertex \(v \in V\) or \(V(K^{(1)})\) is the in-neighborhood of \(v\) for each vertex \(v \in V\). Then each vertex in \(K^{(1)}\) has outdegree 0 or each vertex in \(K^{(1)}\) in has indegree 0 in \(D\), which contradicts the hypothesis. Therefore \(G\) is connected. \(\square\)

**Definition 3.3.** Let \(G_1\) and \(G_2\) be graphs with \(m\) vertices and \(n\) vertices, respectively. The pair \((G_1, G_2)\) is said to be **strongly niche-realizable through** \(K_{m,n}\) (in this paper, we only consider orientations of \(K_{m,n}\) and so we omit “through \(K_{m,n}\)” if the disjoint union of \(G_1\) and \(G_2\) is the niche graph of a strongly connected orientation of the complete bipartite graph \(K_{m,n}\) with bipartition \((V(G_1), V(G_2))\).

Since each vertex of a strongly connected digraph has both indegree and outdegree at least 1, the following corollary immediately true.

**Corollary 3.4.** If \((G, K_n)\) is strongly niche-realizable for \(n \geq 2\), then \(G\) is connected.
The converse of Proposition 3.2 is not true. For example, see the digraph $D$ in Figure 3.1. The niche graph of $D$ is $T_{3,1} \cup K_3$ where $T_{3,1}$ is a paw graph, and $u_2$ has outdegree 0.

In the following, we characterize strongly niche-realizable pairs $(G_1, G_2)$ where both $G_1$ and $G_2$ are paths or cycles. The following lemma is obviously true.

**Lemma 3.5.** Let $D$ be a bipartite tournament with bipartition $(U, V)$. For two vertices $u$ and $v$ in a component of $N(D)[V]$ with at least two vertices, $N^-_{D}(u) = N^+_{D}(v)$ and $N^+_{D}(u) = N^-_{D}(v)$ if $u$ and $v$ are adjacent and $N^-_{D}(u) = N^-_{D}(v)$ and $N^+_{D}(u) = N^+_{D}(v)$ if $u$ and $v$ are nonadjacent.

**Lemma 3.6.** Let $D$ be a bipartite tournament with bipartition $(U, V)$ such that each vertex in $V$ has both indegree and outdegree at least 1. If $|V| \leq 2$, then the subgraph of $N(D)$ induced by $U$ cannot be a complete graph.

**Proof.** Let $U = \{u_1, u_2, \cdots, u_m\}$. By the hypothesis that each vertex in $V$ has both indegree and outdegree at least 1, $m \geq 2$. Since each vertex in $V$ has both indegree and outdegree at least 1, the subgraph of $N(D_v)$ induced by $U$ cannot be isomorphic to $K_m$ for any $v \in V$. For each $v \in V$, we denote the two clique components of $N(D_v)[U]$ by $K^{(v)}_{i_v}$ and $K^{(v)}_{|U|-i_v}$ for some $i \in \left\lfloor \frac{|U|}{2} \right\rfloor$. Suppose that the subgraph of $N(D_v)$ induced by $U$ has $K_{m-1}$ as a maximal clique for some $v \in V$. Without loss of generality, we may assume...
that \( \left\{ V(K_{m-1}^{(w)}), V(K_1^{(w)}) \right\} = \{\{u_1, u_2, \ldots, u_{m-1}\}, \{u_m\}\}. \) If \( |V| = 1 \), then \( u_m \) is not adjacent to any vertex in \( U \setminus \{u_m\} \), and so \( \mathcal{N}(D)[U] \) is not complete. Suppose \( |V| = 2 \) and \( w \) be the other vertex in \( V \). Then \( u_m \) belongs to one of \( K_{i_w}^{(w)} \) and \( K_{m-i_w}^{(w)} \) and there exists a vertex \( u_k \) which belongs to the other component in \( \mathcal{N}(D)[U] \). Then \( u_k \) and \( u_m \) are not adjacent in \( \mathcal{N}(D)[U] \), so \( \mathcal{N}(D)[U] \) is not complete. \( \square \)

**Proposition 3.7.** Let \( D \) be a bipartite tournament with bipartition \((U, V)\) such that each vertex has both indegree and outdegree at least 1. If \( \overline{\mathcal{N}(D)[V]} \) consists of two components, then \( \mathcal{N}(D)[U] \) is not a complete graph.

**Proof.** Suppose \( |V| \geq 3 \). Let \( X \) and \( Y \) be two components of \( \overline{\mathcal{N}(D)[V]} \). Then \( |X| \geq 2 \) or \( |Y| \geq 2 \). Without loss of generality, we may assume \( |X| \geq 2 \). By Lemma 3.5, \( \mathcal{N}(D_v)[U] \) is the same as \( \mathcal{N}(D_w)[U] \) or \( \mathcal{N}(D_{\{v,w\}})[U] \) for any \( v, w \in X \). Thus we may treat \( X \) as one vertex. If \( |Y| \geq 2 \), then we may apply the same argument to treat \( Y \) as one vertex. Hence we may assume \( |V| = 2 \). Then, by Lemma 3.6, \( \mathcal{N}(D)[U] \) is not a complete. \( \square \)

**Proposition 3.8.** Let \( m \) and \( n \) be integers such that \( m \geq n \geq 3 \). Then the pair \((C_m, C_n)\) is strongly niche-realizable if and only if \((m, n) = (3, 3), (4, 4)\).

**Proof.** By Proposition 2.20, the pairs \((C_3, C_3)\), \((C_4, C_3)\) and \((C_4, C_4)\) are niche-realizable. By Proposition 3.7, \((C_4, C_3)\) is not strongly niche-realizable, so the “only if” part is true.

Consider the digraphs \( D_1 \) and \( D_2 \) defined as follows (see Figure 3.2).

\[
V(D_1) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\};
\]

\[
A(D_1) = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_3), (u_3, v_2), (u_3, v_3), (v_1, u_3), (v_2, u_2), (v_3, u_1)\};
\]

\[
V(D_2) = \{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, v_3, v_4\};
\]

\[
A(D_2) = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_4), (u_3, v_3), (u_3, v_4), (u_4, v_2), (u_4, v_3), (v_1, u_3), (v_1, u_4), (v_2, u_2), (v_2, u_3), (v_3, u_1), (v_3, u_2), (v_4, u_1), (v_4, u_4)\}.
\]
It is easy to check that $D_1$ and $D_2$ are strongly connected bipartite tournaments and the niche graphs of $D_1$ and $D_2$ are $C_3 \cup C_3$ and $C_4 \cup C_4$, respectively. Therefore $(C_3, C_3)$, $(C_4, C_4)$ are strongly niche-realizable, and so the “if” part is true. \hfill \Box

**Proposition 3.9.** Let $m$ and $n$ be positive integers such that $m \geq n$. Then the pair $(P_m, P_n)$ is strongly niche-realizable if and only if $(m, n) = (3, 3)$.

**Proof.** By Proposition 2.21, the pairs $(P_1, P_1), (P_2, P_1), (P_2, P_2), (P_3, P_2)$ and $(P_3, P_3)$ are niche-realizable. Vacuously, the pairs $(P_1, P_1), (P_2, P_1)$ and $(P_2, P_2)$ are not strongly niche-realizable. By Proposition 3.7, $(P_3, P_2)$ is not strongly niche-realizable, so the “only if” part is true.

Consider the digraph $D$ defined as follows.

$$V(D) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\};$$

$$A(D) = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_3, v_3), (v_1, u_3), (v_2, u_2), (v_2, u_3), (v_3, u_1), (v_3, u_2)\}.$$ 

It is easy to check that $D$ is a strongly connected bipartite tournament and the niche graph of $D$ is $P_3 \cup P_3$. Therefore $(P_3, P_3)$ is strongly niche-realizable, and so the “if” part is true. \hfill \Box

Figure 3.2: The digraphs $D_1$ and $D_2$. 

[Diagram of $D_1$ and $D_2$]
3.2 A relationship between directed cycles in a bipartite tournament $D$ with bipartition $(U, V)$ and the connectedness of $\mathcal{N}(D)[U]$

**Lemma 3.10.** Let $D$ be a bipartite tournament with bipartition $(U, V)$ such that each vertex has both indegree and outdegree at least 1. Then there is a directed 4-cycle in $D$.

**Proof.** By the hypothesis, $|U| \geq 2$ and $|V| \geq 2$. Suppose to the contrary that there is no directed 4-cycle in some bipartite tournament $D$ with bipartition $(U, V)$ such that each vertex has both indegree and outdegree at least 1. Since every vertex has outdegree at least 1, there is a directed cycle, say $C$, in $D$. Since the underlying graph of $D$ is a bipartite graph which has no odd cycle, $C$ has an even length at least 6 by the assumption on $D$.

Let $C = v_1v_2\cdots v_{2j}v_1$ for some $j \geq 3$. Without loss of generality, we may assume that $v_{2k+1} \in U$ and $v_{2k} \in V$ for $k = 1, \cdots, j$ where $v_{2j+1}$ is identified with $v_1$. Since there is no directed 4-cycle, $(v_{j-1}, v_{j+2})$ is an arc of $D$. Then $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{j-1} \rightarrow v_{j+2} \rightarrow \cdots \rightarrow v_{2j} \rightarrow v_1$ is a directed cycle of length $2j - 2$. By applying the same argument to the directed cycle $v_1 \rightarrow \cdots \rightarrow v_{j-1} \rightarrow v_{j+2} \rightarrow \cdots \rightarrow v_{2j} \rightarrow v_1$, we may conclude that $(v_{j-2}, v_{j+3})$ is an arc of $D$. We may continue this process to obtain the arc $(v_2, v_{2j-1})$. Then we obtain a directed 4-cycle $v_1 \rightarrow v_2 \rightarrow v_{2j-1} \rightarrow v_{2j} \rightarrow v_1$ and reach a contradiction. \qed

**Proposition 3.11.** Let $D$ be a strongly connected digraph of a complete bipartite graph with bipartition $(U, V)$ with $|U| \geq 3, |V| \geq 3$. If $D$ contains a directed $(4k + 2)$-cycle for a positive integer $k$, then each of the subgraphs of $\mathcal{N}(D)$ induced by $U$ and $V$ is connected. Otherwise, each of the subgraphs of $\mathcal{N}(D)$ induced by $U$ and $V$ is disconnected.

**Proof.** Suppose to the contrary that there is a strongly connected digraph $D$ of a complete bipartite graph with bipartition $(U, V)$ with $|U| \geq 3, |V| \geq 3$.
3 such that $D$ contains a directed $(4k + 2)$-cycle for a positive integer $k$ but the subgraph of $\mathcal{N}(D)$ induced by one of $U$ and $V$ is disconnected. Without loss of generality, we may assume that $\mathcal{N}(D)[U]$ is disconnected. Then by Theorem 2.23 and Corollary 3.4, $\mathcal{N}(D)[V]$ is disconnected. Now, by Corollary 2.6, each of $\mathcal{N}(D)[U]$ and $\mathcal{N}(D)[V]$ has two clique components. Let $S$ (resp. $T$) be the vertex set of a clique component of $\mathcal{N}(D)[U]$ (resp. $\mathcal{N}(D)[V]$). By a similar argument for (§) in the proof of Proposition 3.1, we may assume that the arc set of $D$ is $[S, T] \cup [T, U \setminus S] \cup [U \setminus S, V \setminus T] \cup [V \setminus T, S]$. It is easy to check that any directed cycle in $D$ has length $4t$ for some positive integer $t$ which is a contradiction.

Now suppose that $D$ does not contain any directed cycle of length $4k + 2$ for a positive integer $k$. By Lemma 3.10, $D$ contains a directed 4-cycle. Since the underlying graph of $D$ is a bipartite graph, the index of imprimitivity of $D$ is 4. Let $V_1, V_2, V_3$ and $V_4$ be the sets of imprimitivity. Since $D$ is a bipartite tournament with bipartition $(U, V)$, each arc is only between $U$ and $V$. For the same reason, either $V_1 \cup V_3 = U$ and $V_2 \cup V_4 = V$, or $V_1 \cup V_3 = V$ and $V_2 \cup V_4 = U$. Without loss of generality, we may assume $V_1 \cup V_3 = U$ and $V_2 \cup V_4 = V$. Since $D$ is a bipartite tournament, $[V_1, V_2] \cup [V_2, V_3] \cup [V_3, V_4] \cup [V_4, V_1]$ is the arc set of $D$. It is easy to check that the niche graph of $D$ is $(K_{|V_1|} \cup K_{|V_3|}) \cup (K_{|V_2|} \cup K_{|V_4|})$, and so the statement is true. \hfill $\Box$

**Theorem 3.12.** Let $D$ be a strongly connected digraph of a complete bipartite graph with bipartition $(U, V)$ with $|U| \geq 3, |V| \geq 3$. Then any cycle of $D$ has length a multiple of 4 if and only if each of the subgraphs of $\mathcal{N}(D)$ induced by $U$ and $V$ is disconnected.

**Proof.** The “only if” part follows from Proposition 3.11. The “if” part is clear by (§) in the proof of Proposition 3.1. \hfill $\Box$
Chapter 4

Extremal cases

Kim et al. [16] showed the following theorem.

**Theorem 4.1 ([16]).** Let \( m \) and \( n \) be positive integers with \( m \geq n \). Then, either the pair \((K_{\lfloor m^2 \rfloor} \cup K_{\lceil m^2 \rceil}, K_{\lfloor n^2 \rfloor} \cup K_{\lceil n^2 \rceil})\) or the pair \((\overline{K_m}, K_n)\) has the minimum number of edges among the competition-realizable pairs \((G_1, G_2)\) with \(|V(G_1)| = m\) and \(|V(G_2)| = n\).

**Proposition 4.2.** Let \( m \) and \( n \) be positive integers. Then the pair \((K_m, K_n)\) has the maximum number of edges, and if \( m \geq 2, n \geq 2 \), the pair \((K_{\lfloor m^2 \rfloor} \cup K_{\lceil m^2 \rceil}, K_{\lfloor n^2 \rfloor} \cup K_{\lceil n^2 \rceil})\) has the minimum number of edges.

**Proof.** By Proposition 2.17, the pair \((K_m, K_n)\) is niche-realizable and \( K_m \cup K_n \) clearly has the maximum number of edges.

By Proposition 2.19, the pair \((K_{\lfloor m^2 \rfloor} \cup K_{\lceil m^2 \rceil}, K_{\lfloor n^2 \rfloor} \cup K_{\lceil n^2 \rceil})\) is niche-realizable if \( m \geq 2 \) and \( n \geq 2 \). On the other hand, by Proposition 2.4, the pair \((\overline{K_m}, K_n)\) is niche-realizable only when \( m = 2 \). It is easy to check that \((K_1 \cup K_1, K_{\lfloor 4^2 \rfloor} \cup K_{\lceil 4^2 \rceil})\) is less edges than \((\overline{K_2}, K_n)\). Since the edge set of \( \mathcal{N}(D) \) contains that of \( C(D) \) for a digraph \( D \), \((K_{\lfloor m^2 \rfloor} \cup K_{\lceil m^2 \rceil}, K_{\lfloor n^2 \rfloor} \cup K_{\lceil n^2 \rceil})\) has the minimum number of edges by Theorem 4.1. \( \square \)

In the following, we find the gaps in \([a, b]\) for the minimum number \( a \)
of edges and the maximum number $b$ of edges in niche graphs of bipartite tournaments.

Given integers $m$ and $n$ with $n \geq 2$, we consider the subgraph of $\mathcal{N}(D)$ induced by $U$ for a bipartite tournament $D$ with bipartition $(U, V)$ satisfying $|U| = m$ and $|V| = n$. By Proposition 2.4, the subgraph of $\mathcal{N}(D)$ induced by $U$ has the minimum number of edges when it is in the form of $K_i \cup K_{m-i}$ for $i \in \left[\left\lfloor \frac{m}{2} \right\rfloor \right]$. Yet, the number of edges of $K_i \cup K_{m-i}$ is

$$\binom{i}{2} + \binom{m-i}{2} = \frac{i(i-1)}{2} + \frac{(m-i)(m-i-1)}{2} = \frac{i^2-mi}{3} + m^2-m.$$

(4.1)

Therefore the subgraph of $\mathcal{N}(D)$ induced by $U$ has the minimum number of edges when it is isomorphic to $K_{\left\lfloor \frac{m}{2} \right\rfloor} \cup K_{\left\lceil \frac{m}{2} \right\rceil}$. Thus

$$\left(\left\lfloor \frac{m}{2} \right\rfloor \right) + \left(\left\lceil \frac{m}{2} \right\rceil \right)$$

is the minimum number of edges of the subgraph of $\mathcal{N}(D)$ induced by $U$.

Now we claim the following Proposition.

**Proposition 4.3.** Given integers $m$ and $n$ with $n \geq 2$, only when $1 \leq m \leq 4$, the following is true: for each $i$, \(\binom{\left\lfloor \frac{m}{2} \right\rfloor}{2} + \binom{\left\lceil \frac{m}{2} \right\rceil}{2} \leq i \leq \binom{m}{2}\), there exists a bipartite tournament $D$ with bipartition $(U, V)$ satisfying $|U| = m$ and $|V| = n$ such that the subgraph of $\mathcal{N}(D)$ induced by $U$ has $i$ edges.

**Proof.** If $m = 1$, the statement is immediately true. If $m = 2$, then the pairs $(K_2, K_n)$ and $(K_1 \cup K_1, K_n)$ are niche-realizable by Propositions 2.17 and 2.18, respectively, and the statement is true. Suppose $m = 3$. Then the pairs $(K_3, K_n)$ and $(K_1 \cup K_2, K_n)$ are niche-realizable by Propositions 2.17 and 2.18, respectively. To construct a bipartite tournament $D$ such that $\mathcal{N}(D)[U]$ has $2$ edges, label the vertices of $U$ by $u_1, u_2$ and $u_3$ and take a vertex $v \in V$. Let $D$ be a bipartite tournament with bipartition $(U, V)$ such that the arc set of $D$ is $\{(u_1, v), (u_2, v), (v, u_3)\} \cup [U \setminus \{u_1\}, V \setminus \{v\}] \cup [V \setminus \{v\}, \{u_1\}]$. Then it is easy to check that $\mathcal{N}(D)$ is $P_3 \cup K_n$, and so the statement is true for $m = 3$. Suppose $m = 4$. Then the pairs $(K_4, K_n)$, $(K_1 \cup K_3, K_n)$, and $(K_2 \cup K_2, K_n)$
are niche-realizable by Propositions 2.17 and 2.18. To construct a bipartite tournament $D_1$ and $D_2$ such that $\mathcal{N}(D_i)[U]$ has $i + 3$ edges for $i = 1, 2$, label the vertices of $U$ by $u_1, u_2, u_3$ and $u_4$ and take a vertex $v \in V$. Let $D_1$ and $D_2$ be bipartite tournaments with bipartition $(U, V)$ such that the arc set of $D_1$ and $D_2$ are $\{(u_1, v), (u_2, v), (v, u_3), (v, u_4)\} \cup \{(u_2, u_3), V \setminus \{v\}\} \cup \{V \setminus \{v\}, \{u_1, u_4\}\}$, $\{(u_1, v), (u_2, v), (u_3, v), (v, u_4)\} \cup \{U \setminus \{u_2\}, V \setminus \{v\}\} \cup \{U \setminus \{u_2\}, V \setminus \{v\}, \{u_2\}\}$, respectively. Then it is easy to check that $\mathcal{N}(D_1)$ and $\mathcal{N}(D_2)$ are $C_4 \cup K_n$ and $M_4 \cup K_n$, respectively, where $M_4$ is a diamond graph. Thus the statement is true for $m = 4$.

Assume $m > 4$. Let $U$ and $V$ be sets with $|U| = m$ and $|V| = n$ and let $\mathcal{D}_{m,n}$ be the set of bipartite tournaments with the bipartition $(U, V)$. In addition, let $X = \{|E(\mathcal{N}(D)[U])| \mid D \in \mathcal{D}_{m,n}\}$ and $X_c$ be the subset of $X$ defined by

$$X_c = \{|E(\mathcal{N}(D)[U])| \mid \mathcal{N}(D)[U] \text{ is connected and } D \in \mathcal{D}_{m,n}\}.$$ We first find out the minimum of $X_c$. Suppose that the minimum of $X_c$ is achieved by a digraph $D \in \mathcal{D}_{m,n}$. By the minimality of $|E(\mathcal{N}(D)[U])|$, $\mathcal{N}(D)[U]$ is not complete and so, for any vertex $v \in V$, $\mathcal{N}(D_v)[U] \cong K_{i_v} \cup K_{m-i_v}$ for some $i_v$, $1 \leq i_v \leq \lfloor \frac{m}{2} \rfloor$. Fix $v \in V$. Since $|V| \geq 2$ and $\mathcal{N}(D)[U]$ is connected, there is an edge which connects the two clique components $K_{i_v}$ and $K_{m-i_v}$, that is, there are vertices $u_1 \in V(K_{i_v})$ and $u_2 \in V(K_{m-i_v})$ such that $u_1$ and $u_2$ are either in-neighbors or out-neighbors of $w$ for some $w \in V$. Without loss of generality, we may assume that $u_1$ and $u_2$ are in-neighbors of $w$. Let $k$ and $t$ be the numbers of in-neighbors of $w$ in $V(K_{i_v})$ and in $V(K_{m-i_v})$, respectively. Then $1 \leq k \leq i_v$ and $1 \leq t \leq m - i_v$. Now $\mathcal{N}(D_{v,w})[U]$ has

$$kt + (i_v - k)(m - i_v - t)$$

edges more than $\mathcal{N}(D_v)[U]$. If $k = i_v$, then $kt + (i_v - k)(m - i_v - t) \geq i_v$ as
$t \geq 1$. If $k < i_v$, then $kt + (i_v - k)(m - i_v - t) \geq t + (m - i_v - t) = m - i_v \geq i_v$
asummary{as $k \geq 1$, $i_v - k \geq 1$, and $1 \leq i_v \leq \lceil \frac{m}{2} \rceil$. Thus $\mathcal{N}(D)[U]$ has at least $i_v$ edges more than $\mathcal{N}(D_v)[U]$ if $\mathcal{N}(D_v)[U] \cong K_{i_v} \cup K_{m-i_v}$ for some $v \in V$. Hence, for $i_v, 1 \leq i_v \leq \lceil \frac{m}{2} \rceil$,}

$$|E(\mathcal{N}(D)[U])| \geq \left( \frac{i_v}{2} \right) + \left( \frac{m - i_v}{2} \right) + i_v.$$

$$(\frac{i_v}{2}) + (\frac{m - i_v}{2}) + i_v = \frac{i_v(i_v - 1)}{2} + \frac{(m - i_v)(m - i_v - 1)}{2} + i_v$$

$$= i_v^2 - (m - 1)i_v + \frac{m^2 - m}{2} = \left( i_v - \frac{m - 1}{2} \right)^2 + \frac{m^2 - 1}{4}$$

$$\geq \left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + \left\lfloor \frac{m}{2} \right\rfloor.$$ 

Let $S$ and $T$ be subsets of $U$ where $|S| = \left\lceil \frac{m}{2} \right\rceil$, $|T| = \left\lfloor \frac{m}{2} \right\rfloor$, and $S \cup T = U$. Let $D'$ be a bipartite tournament with bipartition $(U, V)$ with arc set

$$[S, V \setminus \{v\}] \cup [V \setminus \{v\}, T] \cup [S \cup \{u\}, \{v\}] \cup [\{v\}, T \setminus \{u\}]$$

for some $v \in V$ and $u \in T$. Then it is easy to check that $D'$ belongs to $\mathcal{D}_{m,n}$, $\mathcal{N}(D'[U]$ is connected, and $|E(\mathcal{N}(D')[U])| = \left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + \left\lfloor \frac{m}{2} \right\rfloor$. Thus $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + \left\lfloor \frac{m}{2} \right\rfloor$ is the minimum of $X_c$.

By (4.1), the minimum and second smallest element in $X_c$ are $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right)$ and $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lceil \frac{m}{2} \right\rceil \right) + \left\lfloor \frac{m}{2} \right\rfloor$, respectively. It is easy to check that $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + \left\lfloor \frac{m}{2} \right\rfloor \leq \left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right)$. Therefore $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right)$ and $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + \left\lfloor \frac{m}{2} \right\rfloor$ are the minimum and second smallest element, respectively, in $X$. However, the difference between $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rfloor \right)$ and $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rceil \right)$ is 2 if $m$ is odd and 1 if $m$ is even. Hence, $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rceil \right) + 1 \notin X$ if $m$ is odd and $\left( \left\lceil \frac{m}{2} \right\rceil \right) + \left( \left\lfloor \frac{m}{2} \right\rceil \right) + 2 \notin X$ if $m$ is even. \qed
Chapter 5

Concluding remarks

Let $\mathcal{D}_{m,n}$ be the set of bipartite tournaments with the bipartition $(U, V)$ satisfying $|U| = m$ and $|V| = n$. For $k \in \{|E(N(D)[U])| \mid D \in \mathcal{D}_{m,n}\}$, it is interesting to know how many non-isomorphic graphs exist in $\{N(D)[U] \mid D \in \mathcal{D}_{m,n}\}$ having $k$ edges. In the case where each component of $N(D)[U]$ is of the form $K_{1,t}$ for $1 \leq t \leq m - 1$, we conjecture that there may exist at most $r$ non-isomorphic subgraphs in $\{N(D)[U] \mid D \in \mathcal{D}_{m,n}\}$ where $r$ is the number of partitions of an integer $k$. 
Bibliography

[1] CA Anderson, L Langley, JR Lundgren, PA McKenna, and SK Merz. New classes of $p$-competition graphs and $\phi$-tolerance competition graphs. Congressus Numerantium, pages 97–108, 1994.

[2] John Adrian Bondy and Uppaluri Siva Ramachandra Murty. Graph theory, volume 244 of graduate texts in mathematics, 2008.

[3] Stephen Bowser and Charles A Cable. Some recent results on niche graphs. Discrete Applied Mathematics, 30(2-3):101–108, 1991.

[4] Richard A Brualdi and Herbert J Ryser. Combinatorial matrix theory, volume 39. Cambridge University Press, 1991.

[5] Charles Cable, Kathryn F Jones, J Richard Lundgren, and Suzanne Seager. Niche graphs. Discrete Applied Mathematics, 23(3):231–241, 1989.

[6] Han Hyuk Cho, Suh-Ryung Kim, and Yunsun Nam. The $m$-step competition graph of a digraph. Discrete Applied Mathematics, 105(1):115–127, 2000.

[7] Joel E Cohen. Interval graphs and food webs: a finding and a problem. RAND Corporation Document, 17696, 1968.

[8] James D Factor. Domination graphs of extended rotational tournaments: chords and cycles. Ars Combinatoria, 82:69–82, 2007.
[9] Peter C Fishburn and William V Gehrlein. Niche numbers. *Journal of graph theory*, 16(2):131–139, 1992.

[10] David C Fisher, J Richard Lundgren, Sarah K Merz, and K Brooks Reid. The domination and competition graphs of a tournament. *Journal of Graph Theory*, 29(2):103–110, 1998.

[11] Kathryn F Fraughnaugh, J Richard Lundgren, Sarah K Merz, John S Maybee, and Norman J Pullman. Competition graphs of strongly connected and hamiltonian digraphs. *SIAM Journal on Discrete Mathematics*, 8(2):179–185, 1995.

[12] Geir T Helleloid. Connected triangle-free m-step competition graphs. *Discrete Applied Mathematics*, 145(3):376–383, 2005.

[13] Kathryn F Jones, J Richard Lundgren, FS Roberts, and S Seager. Some remarks on the double competition number of a graph. *Congr. Numer.*, 60:17–24, 1987.

[14] Seog-Jin Kim, Suh-Ryung Kim, and Yoomi Rho. On CCE graphs of doubly partial orders. *Discrete Applied Mathematics*, 155(8):971–978, 2007.

[15] Suh-Ryung Kim, Jung Yeun Lee, Boram Park, Won Jin Park, and Yoshio Sano. The niche graphs of doubly partial orders. *arXiv preprint arXiv:0905.3954*, 2009.

[16] Suh-Ryung Kim, Jung Yeun Lee, Boram Park, and Yoshio Sano. The competition graphs of oriented complete bipartite graphs. *Discrete Applied Mathematics*, 201:182–190, 2016.

[17] Suh-ryung Kim, Terry A McKee, FR McMorris, and Fred S Roberts. p-competition graphs. *Linear algebra and its applications*, 217:167–178, 1995.
[18] Suh-Ryung Kim, Terry A McKee, Fred R McMorris, and Fred S Roberts. p-competition numbers. *Discrete applied mathematics*, 46(1):87–92, 1993.

[19] Suh-Ryung Kim, Fred S Roberts, and Suzanne Seager. *On 101-clear (0, 1) matrices and the double competition number of bipartite graphs*. Rutgers University. Rutgers Center for Operations Research [RUTCOR], 1989.

[20] J Richard Lundgren and John S Maybee. Food webs with interval competition graphs. In *Graphs and Applications: Proceedings of the First Colorado Symposium on Graph Theory*. Wiley, New York, 1984.

[21] J Richard Lundgren, Craig W Rasmussen, Larry Langley, Patricia A McKenna, and Sarah K Merz. The p-competition graphs of strongly connected and hamiltonian digraphs. 1995.

[22] Boram Park, Jung Yeun Lee, and Suh-Ryung Kim. The m-step competition graphs of doubly partial orders. *Applied Mathematics Letters*, 24(6):811–816, 2011.

[23] Jeongmi Park and Yoshio Sano. The niche graphs of interval orders. *Discussiones Mathematicae Graph Theory*, 34(2):353–359, 2014.

[24] Arundhati Raychaudhuri and Fred S Roberts. Generalized competition graphs and their applications. *Methods of Operations Research*, 49:295–311, 1985.

[25] FS Roberts. Competition graphs and phylogeny graphs. *Graph Theory and Combinatorial Biology, Bolyai Mathematical Studies*, 7:333–362, 1996.

[26] Debra D Scott. The competition-common enemy graph of a digraph. *Discrete Applied Mathematics*, 17(3):269–280, 1987.
[27] Suzanne M Seager. The double competition number of some triangle-free graphs. *Discrete applied mathematics*, 28(3):265–269, 1990.
국문초록

$D$를 유향그래프라고 하자. $D$의 니치 그래프는 $D$와 같은 꼭짓점 집합을 갖고 두 꼭짓점 $u$와 $v$ 사이에 변이 있기 위한 필요충분조건은 $D$에 $u$와 $v$의 공통의 내이웃 또는 공통의 외이웃이 있는 것이다. Kim 등은 이분 토너먼트의 경쟁 그래프를 연구하였다 [방향 차이진 완전 이분그래프의 경쟁 그래프, Discrete Applied Mathematics 201 (2016) 182–190]. 본 논문에서는 그들의 연구 결과를 확장하기 위하여 이분 토너먼트의 니치 그래프를 연구하였다. 우선 이분 토너먼트의 니치 그래프로 나타나는 그래프들을 특정화하였다. 그 다음에 이분 토너먼트의 니치 그래프의 생성된 부분 그래프로 나타날 수 없는 그래프들을 제시하였다. 또한 강하게 연결된 이분 토너먼트에 대해서도 연구하였다. 마지막으로 니치 그래프가 변을 가장 많이 갖거나 변을 가장 적게 가지는 극한을 탐구하였다.

주요어휘: 경쟁 그래프, 이분 토너먼트, 니치 그래프, 니치-실현가능한, 강하게 니치-실현가능한
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