A Cornucopia of AdS\(_5\) Vacua

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Abstract

We report on a systematic search for AdS\(_5\) vacua corresponding to critical points of the potential in the five-dimensional \(\mathcal{N} = 8\) SO(6) gauged supergravity. By employing Google’s TensorFlow Machine Learning library, we find the total of 31 critical points including 5 previously known ones. All 26 new critical points are non-supersymmetric. We compute the mass spectra of scalar fluctuations for all points and find that the non-supersymmetric AdS\(_5\) vacua are perturbatively unstable. Many of the new critical points can be found analytically within consistent truncations of the \(\mathcal{N} = 8\) supergravity with respect to discrete subgroups of the \(\text{SO}(6) \times \text{GL}(2,\mathbb{R})\) symmetry of the potential. In particular, we discuss in detail a \(\mathbb{Z}_2^3\)-invariant truncation with 10 scalar fields and 15 critical points. We also compute explicitly the scalar potential in a \(\mathbb{Z}_2^3\)-invariant extension of that truncation to 18 scalar fields and reproduce 17 of the 31 critical points from the numerical search.
1 Introduction

The AdS/CFT correspondence is deeply rooted in string theory and its low-energy supergravity limits. Therefore, it is important to understand fully the landscape of consistent AdS backgrounds in string theory. A fruitful strategy has been to identify a consistent Kaluza-Klein (KK) truncation of a ten- or eleven-dimensional supergravity to lower $d$ dimensions and to study critical points of the scalar potential in the resulting gauged supergravity. Each critical point with a negative value of the potential leads to an AdS$_d$ solution and thus candidate AdS background of string theory.

Our goal in this paper is to use a mixture of old analytic and modern numerical methods to search systematically for critical points of the scalar potential in $\mathcal{N} = 8$ SO(6) gauged supergrav-
ity in five dimensions [1–3]. This is interesting for several reasons. First, there is now a complete, constructive proof that this five-dimensional supergravity is a consistent KK truncation of type IIB string theory on $S^5$ [4–8]. In particular, this means that all AdS$_5$ vacua corresponding to critical points of the supergravity potential can be uplifted to AdS solutions of string theory. Secondly, the problem should be amenable to similar computational techniques based on Machine Learning that were successfully applied in [9] to find hundreds of new critical points of the scalar potential in the de Wit-Nicolai SO(8) gauged supergravity in four dimensions [10]. Finally, by extrapolating the results in [11], it is natural to expect that a large fraction of the critical points might be accessible analytically, or semi-analytically, within a suitable truncation with respect to a discrete subgroup of the full symmetry group of the theory.

Through holography, the SO(6) gauged supergravity has been an indispensable tool for studying the $\mathcal{N} = 4$ SYM theory and its deformations. Indeed, AdS$_5$ vacua are dual to conformal fixed points obtained by deforming $\mathcal{N} = 4$ SYM and domain wall solutions between these critical points are dual to RG flows between the CFTs [12–14]. From this perspective, one would also like to determine the stability of those vacua. If an AdS$_5$ solution is supersymmetric, it is necessarily stable [15] and the dual CFT is unitary. However, if there are scalar fluctuations with negative masses violating the Breitenlohner-Freedman (BF) bound [16], the dual operators have complex dimensions and the dual CFT is not unitary. In fact, it has been argued in [17] using the Weak Gravity Conjecture [18] that all non-supersymmetric vacua in string theory should be unstable. The violation of the BF bound for a given AdS solution is then the simplest sign of that instability.

It is perhaps surprising that not much progress has been made in classifying AdS$_5$ vacua of the SO(6) gauged supergravity since the initial discovery in 1998 of five critical points listed in Table 1.1$^1$ in an SU(2)-invariant sector of the theory [4]. One reason might be that the Leigh-Strassler analysis [20] of $\mathcal{N} = 1$ deformations of $\mathcal{N} = 4$ SYM suggests that there should be no other supersymmetric critical points beyond the $\mathcal{N} = 8$ point, T0750000, and the $\mathcal{N} = 2$ point, T0839947, already found in [4]. The other three points in Table 1.1 are non-supersymmetric and perturbatively unstable as discussed further in Appendix D. It is then reasonable to expect that any missing point is non-supersymmetric and thus perturbatively unstable as well. Note, however, that the latter need not be necessarily true given that there is a perturbatively stable yet non-supersymmetric SO(3) × SO(3)-invariant AdS$_4$ solution in four dimensions [21, 22],$^2$ and there are multiple examples of perturbatively stable AdS$_3$ vacua in three-dimensional supergrav-

$^1$Following the convention for labelling critical points in four-dimensional supergravity [19], we propose to denote the points in five dimensions according to the value of the first seven digits in their cosmological constant by $T_{n_1 n_2 n_3 n_4 n_5 n_6 n_7}$.

$^2$However, it has been shown recently that this solution is unstable in string theory due to brane-jet instability [23] and higher KK-modes violating the BF bound [24].
Point Symmetry $P_*$ $\mathcal{N}$ SUSY BF Stability
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T0750000 SO(6) $-\frac{3}{4}$ $\mathcal{N} = 8$ S
T0780031 SO(5) $-\frac{3^{2/3}}{8}$ $-$ U
T0839947 SU(2) × U(1) $-\frac{2^{4/3}}{3}$ $\mathcal{N} = 2$ S
T0843750 SU(3) $-\frac{27}{32}$ $-$ U
T0870297 SU(2) × U(1)$^2$ $-\frac{3}{8} \left(\frac{25}{2}\right)^{1/3}$ $-$ U

Table 1.1: The SU(2)-invariant extrema [4].

Given the large number of known critical points of the scalar potentials in maximal gauged supergravities in three [25, 26] and four dimensions [21, 27, 28, 22, 19, 29, 9, 11], it is to be expected that there are comparably many AdS$_5$ vacua of the five-dimensional SO(6) gauged supergravity beyond the ones in Table 1.1. It is the lower symmetry (less than SU(2)) of these vacua that makes looking for them a challenging problem.

Recall that the potential of the $\mathcal{N} = 8$ $d = 5$ supergravity is a function on the 42-dimensional scalar manifold, which is a coset of the maximally noncompact group $E_{6(6)}$ modded by its compact subgroup, USp(8). In the conventions of [2], the potential can be written as

$$P = -\frac{1}{32} g^2 \left[ 2(W_{ab})^2 - (W_{abcd})^2 \right],$$

which looks deceptively simple until fully unpacked. Indeed, the $W_{ab}$ and $W_{abcd}$ tensors are quadratic in the components of the scalar 27-bein, $\mathcal{V} = (\mathcal{V}^{IJ}_{ab}, \mathcal{V}^{\alpha}_{Iab})$, which, modulo a linear transformation, is a group element of $E_{6(6)}$ obtained by exponentiating non-compact elements, $\Phi = \sum \phi_A T_A$, of the Lie algebra $\mathfrak{e}_{6(6)}$, where $T_A$ are some fixed generators and $\phi_A$ are the 42 scalars fields. It follows from the construction of the $W$-tensors that the potential is manifestly invariant under the SO(6) gauge symmetry acting on the $I, J = 1, \ldots, 6$ indices as well as the axion-dilaton SL(2, $\mathbb{R}$) that acts on the $\alpha = 7, 8$ index of the 27-bein. This reduces the number of independent degrees of freedom in (1.1) to $42 - 15 - 3 = 24$. In fact, by including discrete symmetries one can show that the actual symmetry is $S(O(6) \times GL(2, \mathbb{R}))$ [30, 31], which we will exploit in Section 2. When viewed as a function on $E_{6(6)}$, the potential (1.1) is also invariant under local USp(8) transformations acting on the $a, b = 1, \ldots, 8$ indices of the 27-bein, but that symmetry is already fixed by the USp(8) gauge choice in $\Phi$.

See also Appendix A.
The problem now is to compute the potential, \( P(\phi_A) \), as an explicit function of the scalar fields and then find its critical points. Analytically, neither is possible in full generality. A time-tested method, first used by Warner [21, 27] in four dimensions, is to truncate the potential of interest to a smaller number of fields that are invariant under some subgroup, \( G \), of the full symmetry group of the theory. The critical points of the truncated potential are then automatically critical points of the full potential. For a judicious choice of the subgroup, \( G \), one may end up with an analytically tractable problem leading to a potential with new critical points. As we discuss briefly in Section 2, this method has not been too successful thus far in five dimensions beyond the original analysis in [4]. The scalar potentials in various truncations considered over the years in the literature either did not include new critical points or were deemed too complicated to attempt extremization.

Another way to make progress is to attack the problem numerically. This has been initiated about ten years ago by one of the authors and resulted in around 40 new AdS\(_4\) vacua [26, 28, 32, 22, 19] in the de Wit-Nicolai SO(8) gauged supergravity for the total of 50 critical points known in 2013.\(^4\) Recently, a more powerful numerical code using Machine Learning (ML) and Google’s TensorFlow libraries [33] was developed in [9] and led to the total of 194 points that include 2 additional ones found in the follow up analytic work [11]. It is rather straightforward to port the ML code included with [9] from four to five dimensions and, in fact, considerably simplify it using the new publicly available TensorFlow2 libraries\(^5\) as well as by exploiting symmetries of the potential.

By performing a systematic, numerical search using the new ML code, we find the total of 31 AdS\(_5\) vacua in \( \mathcal{N} = 8 \) d = 5 SO(6) gauged supergravity. Those include the 5 classic ones in Table 1.1. We also compute the gravitini and scalar spectra at each point, which are needed to determine unbroken supersymmetries and the BF (in)stability. We find that all 26 new points are non-supersymmetric, which is compatible with the expectation that the dual \( \mathcal{N} = 4 \) SYM theory does not admit relevant deformations, apart from the one in [20], which lead to interacting supersymmetric CFTs. All new points have BF unstable scalar modes, which is perhaps disappointing, but not unexpected. Hence our results further support the instability conjecture for non-supersymmetric AdS vacua in string theory [17].

It turns out that many of the new AdS\(_5\) vacua can also be found using more analytic methods. We generalize here an observation in [11] about the existence of a special truncation in four dimensions in which the scalar manifold is a product of mutually commuting Poincaré disks. That truncation arises from the subalgebra \( \mathfrak{su}(1, 1)^7 \subset \mathfrak{e}_7(7) \) and can be obtained by imposing a discrete \( \mathbb{Z}_3^2 \) symmetry on the scalar fields. This truncation is quite remarkable in that it is very

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\(^4\)Those include the 7 original points found in the “classic period” [21, 27] and one further point in [29].

\(^5\)Cf. [https://blog.tensorflow.org/2019/09/tensorflow-20-is-now-available.html](https://blog.tensorflow.org/2019/09/tensorflow-20-is-now-available.html)
easy to analyze analytically and yet its potential captures 25% of the 194 known critical points.

A natural question is whether there exists a similarly marvelous truncation for the $\mathcal{N} = 8$ supergravity in five dimensions. We find that indeed it does and corresponds to the embedding $\mathfrak{o}(1, 1)^2 \oplus \mathfrak{su}(1, 1) \subset \mathfrak{e}_{6(6)}$ for which the scalar coset is a product of 6 simple commuting factors,

$$\mathcal{M}_{(10)} \equiv O(1, 1)^2 \times \left( \frac{SU(1, 1)}{U(1)} \right)^4,$$

that is 2 half-lines and 4 Poincaré disks. In fact, there exist two different consistent truncations for which the full scalar potential has been already worked out in the literature. Both use a $Z_2^3$ symmetry and have the same looking coset, but preserve different amount of the $SO(6) \times SL(2, \mathbb{R})$ symmetry. The first one, found 20 years ago in [34], has an $U(1)^4$ unbroken symmetry so that the truncated potential can be reduced to 6 scalar fields. The second truncation, found quite recently in [31], breaks all continuous symmetries and the potential is a function of all 10 fields. For that reason we will refer to them as the 6-scalar and the 10-scalar model, respectively. As we show in Section 2, all critical points in the 6-scalar model lie within the 10-scalar model. We find that the latter has 15 AdS$_4$ vacua, with many of the new critical points computable exactly and a few remaining ones easily accessible to standard numerical routines for example in Mathematica.

Finally, we note that the 6-scalar and the 10-scalar models arise by imposing different $Z_2$ symmetry on the same intermediate truncation of the $\mathcal{N} = 8$ supergravity with respect to $Z_2^2 \subset SO(6)$. That truncation has the scalar coset,

$$\mathcal{M}_{(18)} \equiv O(1, 1)^2 \times \frac{SO(4, 4)}{SO(4) \times O(4)},$$

and preserves $U(1)^4$ continuous symmetry. This means that the potential in this model depends on $18 - 4 = 14$ scalar fields. By carefully choosing the parametrization of both factors, we are able to compute the potential in a closed analytic form and determine, once more using a simple numerical routine, that it has in total 17 critical points. This shows that more than 50% of all critical points are accessible analytically either exactly or by using a high level numerical routines to solve a system of explicit equations for the extrema of the potential. This provides an independent validation of the numerical results obtained with TensorFlow.

We begin in Section 2 with a detailed discussion of the 10-scalar model and compute analytically whenever possible its 15 critical points. In Section 3 we describe the numerical search performed with TensorFlow and elucidate relevant differences between the computational strategies in the $d = 5$ search here and the $d = 4$ search in [9]. We conclude with some open questions in Section 4. A lot of technical details can be found in the appendices. Throughout the paper we use the same conventions as in [2]. However, to avoid any ambiguities, in Appendix A we
summarize the details of our parametrization of the scalar coset of the SO(6) supergravity. Appendix B has a detailed discussion of the consistent truncation to 18 scalars in (1.3). We present a careful derivation of the full potential in this 14-scalars model and find its critical points. The results of the full numerical search can be found in Appendix C. We give a list of all 31 critical points together with their locations (partially canonicalized) and the mass spectra. Finally, in Appendix D, we collect some old results for the scalar mass spectra around the critical points in Table 1.1, most of which where known to many but never published.

Note added: The results in this paper were reported in seminars and at a conference [35, 36]. While we were preparing this manuscript, we became aware of the recent work [37], which finds 32 critical points of the scalar potential in $\mathcal{N} = 8$ $d = 5$ gauged supergravity using TensorFlow. The authors of [37] also calculate the gravitini spectra and find no new supersymmetric points. Apart from solution #26 in [37], which is missing from our list, we find a complete match between their values of the potential at critical points and the ones in our search. This provides yet another validation of the two numerical searches.

2 The 10-scalar model

As explained in the Introduction, one way to deal with the complexity of the five-dimensional SO(6) gauged supergravity is to look for consistent truncations of the theory by imposing invariance under a subgroup, $G$, of the full symmetry group. In most examples $G \subset$ SO(6), or SO(6) × SL(2, $\mathbb{R}$), but the most interesting truncation discussed in this section is when $G \subset$ S(O(6) × GL(2, $\mathbb{R}$)). Several such consistent truncations have been studied in the literature in various contexts, see for example [12, 13, 38, 6, 30, 39–42], but no new critical points have been found after a systematic search within an SU(2) invariant truncation in [4]. Other SU(2)-invariant truncations listed in Table A.1 in [14] merely reproduce a subset of critical points in [4]. The same is true for an U(1)-invariant truncation in [42]. There are, however, two truncations with respect to discrete symmetries obtained in [34] and [31], respectively, in which “holomorphic” superpotentials, and thus the full scalar potentials, are known explicitly. It appears that those potentials have never been fully analyzed. Therefore we begin our search by carefully examining these two models.

2.1 The consistent truncation

Motivated by this we start our discussion by studying the critical points of the consistent truncation in [31] which is invariant under a $\mathbb{Z}^3$ subgroup of S(O(6) × GL(2, $\mathbb{R}$)) and contains 10 out of the 42 scalar fields of the maximal theory. The procedure to obtain this truncation is outlined
in detail in [31] and so we will be brief. Consider the O(6) matrices

\[ P_1 = \text{diag}(-1, -1, 1, 1, 1, 1), \]
\[ P_2 = \text{diag}(1, 1, -1, -1, 1, 1), \]
\[ P_3 = \text{diag}(1, -1, 1, -1, 1, -1), \]

and the following GL(2) matrices

\[ Q = \text{diag}(-1, 1), \quad Q' = \text{diag}(-1, 1). \]

The truncation consists of the five-dimensional metric in addition to all fields that are invariant under the action of \( P_1 Q, P_2 Q, P_3 Q' \). Even though the third matrix \( P_3 Q' \) is not inside \( \text{SL}(6, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \), it is still a valid discrete symmetry to impose as explained in [30, 31]. In this paper we focus on AdS\(_5\) vacua of the theory and so are only concerned with scalar fields that are invariant. Form fields must be set to zero. Imposing these symmetries leaves ten scalars parametrizing the scalar manifold

\[ \mathcal{M}_{(10)} = \text{O}(1, 1) \times \text{O}(1, 1) \times \left( \frac{\text{SU}(1, 1)}{\text{U}(1)} \right)^4. \]

These consist of five \( 20' \) scalars, four \( 10 \oplus \overline{10} \) scalars, and the dilaton. Two of the \( 20' \) scalars are singled out as they parametrize the two \( \text{O}(1, 1) \)-factors in (2.3). An explicit parametrization of the coset (2.3) is given by specifying the generators of \( \mathfrak{e}_{6(6)} \) that are invariant with respect to our choice of discrete symmetries. As explained in Appendix A we use an \( \text{SL}(6, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) basis to define our generators. In particular the two \( \text{O}(1, 1) \)-factors correspond to the generators \( g_\alpha \) and \( g_\beta \) defined by

\[ g_\alpha = \hat{\Lambda}_1^1 + \hat{\Lambda}_2^2 - \hat{\Lambda}_3^3 - \hat{\Lambda}_4^4, \]
\[ g_\beta = \hat{\Lambda}_1^1 + \hat{\Lambda}_2^2 + \hat{\Lambda}_3^3 + \hat{\Lambda}_4^4 - 2\hat{\Lambda}_5^5 - 2\hat{\Lambda}_6^6, \]

using the notation in Appendix A. The remaining scalars are best parametrized in terms of one of the non-compact generators of \( \mathfrak{su}(1, 1) \) together with the compact one. The remaining non-compact generator can be obtained as the commutator of the other two. Using the notation in Appendix A the four compact generators \( \mathfrak{r} \) and the non-compact generators \( \mathfrak{t} \) are specified by

\[ t_1 = \frac{1}{\sqrt{2}}(\hat{\Sigma}_{1357} - \hat{\Sigma}_{2468}), \quad r_1 = \frac{1}{\sqrt{2}}(\hat{\Sigma}_{1357} + \hat{\Sigma}_{2468}), \]
\[ t_2 = \frac{1}{\sqrt{2}}(\hat{\Sigma}_{2367} - \hat{\Sigma}_{1458}), \quad r_2 = -\frac{1}{\sqrt{2}}(\hat{\Sigma}_{2367} + \hat{\Sigma}_{1458}), \]
\[ t_3 = \frac{1}{\sqrt{2}}(\hat{\Sigma}_{2457} - \hat{\Sigma}_{1368}), \quad r_3 = -\frac{1}{\sqrt{2}}(\hat{\Sigma}_{2457} + \hat{\Sigma}_{1368}), \]
\[ t_4 = \frac{1}{\sqrt{2}}(\hat{\Sigma}_{1467} - \hat{\Sigma}_{2358}), \quad r_4 = \frac{1}{\sqrt{2}}(\hat{\Sigma}_{1467} + \hat{\Sigma}_{2358}). \]
The full E$_{6(6)}$ group element is constructed as follows

$$V = \exp(\alpha g_\alpha) \cdot \exp(\beta g_\beta) \cdot \prod_{i=1}^{4} \exp(-\omega_i t_i) \cdot \exp(\rho_i \mathbf{t}_i) \cdot \exp(\omega_i \mathbf{t}_i) \ .$$  (2.6)

Notice that the commutator of the SU(1, 1) generators $t_i$ and $\mathbf{t}_i$ gives a linear combination of 20' generators $\hat{\Lambda}_j$ in addition to the dilaton generator $g_{\text{dilaton}} = \hat{\Lambda}_7 - \hat{\Lambda}_8$. It thus follows that one of the ten scalars is the dilaton. Since the scalar potential of the full SO(6) gauged theory does not depend on the dilaton, the same will be true in the truncated 10-scalar model. The way we have parametrized the manifold, $\mathcal{M}_{(10)}$, in (2.6), the dilaton is mixed with all the other SU(1, 1) scalars and isolating it is difficult. The action of SL(2, $\mathbb{R}$) on $V$ is given by the transformation

$$V \mapsto V \cdot \exp(t g_{\text{dilaton}}) \ ,$$  (2.7)

which leaves the potential invariant. Even though in principle it should be possible to translate what this action implies for the scalars $\rho_i$ and $\omega_i$, in practice the transformation is a complicated simultaneous action on all eight fields.

The scalar potential of this truncation can be compactly written as

$$\mathcal{P} = \frac{1}{32} e^\kappa \left( \frac{1}{6} |\partial_\alpha \mathcal{W}|^2 + \frac{1}{2} |\partial_\beta \mathcal{W}|^2 + \mathcal{K}^{ij} D_i \mathcal{W} D_j \overline{\mathcal{W}} - \frac{8}{3} |\mathcal{W}|^2 \right) \ ,$$  (2.8)

where the Kähler covariant derivative is $D_i \mathcal{F} \equiv \partial_i \mathcal{F} + \mathcal{F} \partial_i \kappa$, the Kähler potential is

$$\kappa = -\sum_{i=1}^{4} \log(1 - |z_i|^2) \ ,$$  (2.9)

and determines the kinetic terms through the Kähler metric $\kappa_{ij} \equiv \frac{\partial \kappa}{\partial z_i \partial \overline{z}_j}$ and its inverse $\kappa^{ij}$. The superpotential is [31]

$$\mathcal{W} = e^{-4\alpha}(1 + z_1 z_2 - z_1 z_3 - z_1 z_4 - z_2 z_3 - z_2 z_4 + z_3 z_4 + z_1 z_2 z_3 z_4) + e^{2\alpha + 2\beta}(1 + z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 + z_1 z_2 z_3 z_4) + e^{2\alpha - 2\beta}(1 - z_1 z_2 + z_1 z_3 - z_1 z_4 - z_2 z_3 + z_2 z_4 - z_3 z_4 + z_1 z_2 z_3 z_4) \ .$$  (2.10)

The complex scalars $z_i$ are related to the $\rho_i$ and $\omega_i$ in (2.6) as follows:

$$z_j = i \tanh \frac{\rho_j}{2} e^{-i\omega_j} \ .$$  (2.11)

The 10-scalar model exhibits a number of discrete symmetries, some of which were identified in [31]. For example $z_i \mapsto \pm \overline{z}_i$ and $z_i \mapsto -z_i$. Here we would like to point out a rather large group of symmetries that leaves the superpotential invariant. It can be specified by

$$e_1 : \alpha \mapsto -\frac{\alpha + \beta}{2} \ , \ \beta \mapsto -\frac{3\alpha + \beta}{2} \ , \ z_1 \mapsto -z_2 \mapsto z_1 \ .$$

$$e_2 : \alpha \mapsto -\frac{\alpha + \beta}{2} \ , \ \beta \mapsto \frac{3\alpha - \beta}{2} \ , \ z_2 \mapsto z_3 \mapsto -z_4 \mapsto z_2 \ .$$

$$e_3 : \alpha \mapsto -\frac{\alpha + \beta}{2} \ , \ \beta \mapsto \frac{3\alpha + \beta}{2} \ , \ z_1 \mapsto -z_2 \mapsto z_4 \mapsto -z_3 \mapsto z_1 \ .$$  (2.12)
These satisfy $e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = 1$ and therefore generate the group $S_4$. As we show in the next section this model has 15 critical points including all five of $[4]$. Furthermore, by computing the masses of the 10 scalar fields we have checked that all non-supersymmetric critical points are perturbatively unstable within the 10-scalar model.

We note that a simpler six-scalar model can be obtained by setting the real parts of all $z_i$ scalars to zero. The potential then reduces to that of $[34]$, see also $[39]$. In $[34]$ a 10-scalar truncation was considered which is different from the one we have been discussing here. This latter truncation has a potential which only depends on six fields. Explicitly, the potential for this 6-scalar model can be obtained from the one in (2.8) by setting:

$$
\begin{align*}
  z_1 &= i \tanh \frac{1}{2} (\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4), \\
  z_2 &= i \tanh \frac{1}{2} (\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4), \\
  z_3 &= i \tanh \frac{1}{2} (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4), \\
  z_4 &= i \tanh \frac{1}{2} (\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4),
\end{align*}
$$

The potential can also be written in terms of a superpotential,

$$
\mathcal{P} = \frac{1}{8} \left( \frac{1}{6} (\partial_\alpha W)^2 + \frac{1}{2} (\partial_\beta W)^2 + (\partial_i W)^2 - \frac{8}{3} W^2 \right),
$$

where $\partial_i$ denotes a derivative with respect to $\varphi_i$ and $[34]

$$
\begin{align*}
  W &= \frac{1}{4} e^{-4\alpha} (+ \cosh 2\varphi_1 - \cosh 2\varphi_2 - \cosh 2\varphi_3 - \cosh 2\varphi_4) \\
  &\quad + \frac{1}{4} e^{2\alpha+2\beta} (- \cosh 2\varphi_1 - \cosh 2\varphi_2 + \cosh 2\varphi_3 - \cosh 2\varphi_4) \\
  &\quad + \frac{1}{4} e^{2\alpha-2\beta} (- \cosh 2\varphi_1 + \cosh 2\varphi_2 - \cosh 2\varphi_3 - \cosh 2\varphi_4).
\end{align*}
$$

Four of the points in Table 1.1 are critical points of the 6-scalar potential in (2.14). Only the SO(5) invariant point, T0780031, lies outside it. There are in total eight critical points in the 6-scalar model.

### 2.2 Critical points

Here we provide a list of the 15 critical points of the 10-scalar model potential in (2.8). Some of these can be obtained analytically. For the others we have used the `FindRoot` routine in Mathematica.
\begin{align}
z_a &= 0, \quad \alpha = \beta = 0. \\
\mathcal{P} &= -\frac{3}{4} = -0.750000.
\end{align}
\tag{2.16}

Symmetry: SO(6), \quad \mathcal{N} = 8. 
Comment: Critical point of the 6-scalar model.

\begin{align}
z_2 &= z_3 = 0, \quad z_1 = z_4 = 2 - \sqrt{3}, \quad \beta = 3\alpha = \frac{\log 3}{8}. \\
\mathcal{P} &= -\frac{3 \times 3^{2/3}}{8}.
\end{align}
\tag{2.17}

Symmetry: SO(5), \quad \mathcal{N} = 0

\begin{align}
z_1 &= i(\sqrt{3} - 2), \quad z_2 = z_3 = -z_4 = -z_1, \quad 3\alpha = \beta = \frac{1}{4} \log 2. \\
\mathcal{P} &= -\frac{2 \times 2^{1/3}}{3} \approx -0.839947.
\end{align}
\tag{2.18}

Symmetry: SU(2) \times U(1), \quad \mathcal{N} = 2. 
Comment: Critical point of the 6-scalar model.

\begin{align}
z_1 &= -i\sqrt{5 - 2\sqrt{6}}, \quad z_2 = z_4 = -z_3 = -z_1, \quad \alpha = \beta = 0. \\
\mathcal{P} &= -\frac{27}{32} = -0.843750.
\end{align}
\tag{2.19}

Symmetry: SU(3), \quad \mathcal{N} = 0

Comment: Critical point of the 6-scalar model.

\begin{align}
z_1 &= z_2 = 0, \quad z_3 = -z_4 = i\frac{2\sqrt{10} + \sqrt{15} - 5}{2\sqrt{10} + \sqrt{15} + 5}, \quad \beta = -3\alpha = -\frac{1}{8} \log 10. \\
\mathcal{P} &= -\frac{3 \times 5^{2/3}}{8 \times 2^{1/3}}.
\end{align}
\tag{2.20}

Symmetry: SU(2) \times U(1), \quad \mathcal{N} = 0

Comment: Critical point of the 6-scalar model. In Table 1.1, the symmetry of this point is listed as SU(2) \times U(1)^2. The second U(1) factor is the compact generator of SL(2,\mathbb{R}) which lies outside of the SO(6) gauge algebra.
This point has $\mathcal{N} = 0$ and is a critical point of the 6-scalar model found by setting
\[ z_1 = z_2 = 0, \quad z_3 = -z_4 = i \sqrt{-X^2 - (1 - Y)^4 Y^4 + 2XY^2(1 + 6Y + Y^2)} \]
\[ X + (1 - Y)^2 Y^2 + 2\sqrt{XY}(1 + Y), \] (2.26)
with
\[ \alpha = \frac{1}{24} \log X, \quad \beta = \frac{1}{2} \log Y - \frac{1}{8} \log X. \] (2.27)
The potential is then
\[ \mathcal{P} = -\frac{X^2 + Y^4(1 - Y^2)^2 - 2XY^2(3 + 4Y + 3Y^2)}{16X^{1/3} Y^2 (X + Y^2(1 - Y)^2)}. \] (2.28)
This potential has two critical points, T0878939 and T1001482 which are correlated as follows. First one takes one of the four real roots of the equation
\[ 8 - 44Y + 33Y^2 + 74Y^3 + 33Y^4 - 44Y^5 + 8Y^6 = 0. \] (2.29)
Note that the equation is self-reciprocal or palindromic and therefore the solutions come in inverse pairs which lead to the same cosmological constant. Use a given solution for $Y$ to find $X$ as a solution of the equation
\[ 5X^2 + (1 - Y)^4 Y^4 + 2XY^2(1 - 10Y + Y^2) = 0. \] (2.30)
where the solution for $X$ must be correlated with the solution of $Y$. That is the choice of sign in the above second order equation for $X$ is correlated with which of the two different solution we start with for $Y$. For T0878939 we then find the approximate values
\[ X = 0.006865, \quad Y = 0.283702, \quad \mathcal{P} = -0.878939. \] (2.31)
The value of $\mathcal{P}$ can be obtained as a root of the polynomial
\[ 729 + 5723136 \mathcal{P}^3 + 14123008 \mathcal{P}^6 + 8388608 \mathcal{P}^9. \] (2.32)
Note that the T0870298 is another critical point of (2.28) with $X = 10, Y = 1$.

This point has $\mathcal{N} = 0$ and is a critical point of the 6-scalar model located at
\[ z_2 = -z_3 = i \frac{1 - \sqrt{Y + \sqrt{Y^2 - 1}}}{1 + \sqrt{Y + \sqrt{Y^2 - 1}}}, \quad \beta = 3\alpha = \frac{1}{8} \log \left( \frac{20 + 4\sqrt{34}}{3} \right), \]
\[ z_1 = i \frac{X + \sqrt{X^2 - 1} - \sqrt{Y + \sqrt{Y^2 - 1}}}{X + \sqrt{X^2 - 1} + \sqrt{Y + \sqrt{Y^2 - 1}}}, \]
\[ z_4 = i \frac{(X + \sqrt{X^2 - 1}) \sqrt{Y + \sqrt{Y^2 - 1} - 1}}{(X + \sqrt{X^2 - 1}) \sqrt{Y + \sqrt{Y^2 - 1} + 1}}, \] (2.33)
where the potential reduces to
\[
\mathcal{P} = \frac{(1 + Y)^{1/3} \left( 2(1 + Y)(Y^2 - 3) - X^2(7 + 3Y) \right)}{16 \times 2^{2/3} X^{4/3}},
\]
(2.34)
and
\[
X^2 = \frac{1}{243} (88 + 40\sqrt{34}) , \quad Y = \frac{1}{9}(-1 + 2\sqrt{34}).
\]
(2.35)
The value of the potential is
\[
\mathcal{P} = -\frac{(196079 + 33524\sqrt{34})^{1/3}}{3^{7/3} \times 2^{8/3}} \approx -0.887636,
\]
(2.36)
which is the smaller of the two real roots of the polynomial
\[
107811 + 100392448\mathcal{P}^3 + 143327232\mathcal{P}^6.
\]
(2.37)

\[\text{T0892913}\]
\[
z_3 = \frac{1}{\sqrt{9}}, \quad z_1 = z_2 = z_4 = 0, \quad 3\alpha = \beta = \frac{1}{2} \log 2.
\]
(2.38)
\[
\mathcal{P} = -\frac{9}{8 \times 2^{1/3}}.
\]
(2.39)
Symmetry: \(\mathcal{N} = 0\)
Comment: Critical point of the 6-scalar model.

\[\text{T0964525}\]
\[
\alpha = -0.0262713, \quad \beta = 0.254756, \quad z_1 = 0.224701 - i0.487424,
\]
\[
z_2 = 0.0709794, \quad z_3 = -0.256605, \quad z_4 = 0.0116728 - i0.507927.
\]
(2.40)
\[
\mathcal{P} \approx -0.9645259.
\]
(2.41)
Symmetry: \(\mathcal{N} = 0\)

\[\text{T0982778}\]
\[
z_1 = -z_4 = \frac{\sqrt{19 - 4\sqrt{22}}}{\sqrt{3}}, \quad z_2 = z_3 = 0.280116 + i0.485175, \quad \beta = 3\alpha = \frac{1}{8} \log(3/2).
\]
(2.42)
\[
\mathcal{P} = -\frac{3 \times 3^{2/3}}{4 \times 2^{2/3}}.
\]
(2.43)
Symmetry: \(\mathcal{N} = 0\).
Comment: The values of \(z_1\) and \(z_2\) can be obtained as roots of the polynomials \(3 - 38X^2 + 3X^4\) and \(4 + 14Y^2 + 45Y^4 + 14Y^6 + 4Y^8\), respectively.
This point has $\mathcal{N} = 0$ and is a critical point of the 6-scalar model obtained in the same way as $T0878939$ with the following approximate values for the roots of the polynomials in (2.29) and (2.30)

\begin{align}
X &= 0.097733, \quad Y = 0.337328, \quad \mathcal{P} = -1.001482. \quad (2.44)
\end{align}

Note that $\mathcal{P}$ is a root of the polynomial in (2.37).

This point has $\mathcal{N} = 0$ and is located at

\begin{align}
z_1 &= -z_3 = -\sqrt{1 + 2Y - Y^2 + iY}, \quad z_2 = -\bar{z}_4 = \frac{1 - X - z_1(1 + X)}{1 + X - z_1(1 - X)}, \\
\beta &= \frac{1}{2} \log X, \quad \alpha = 0,
\end{align}

which gives the potential

\begin{align}
\mathcal{P} &= -\frac{9}{8}. \quad (2.46)
\end{align}

The value of $X$ is a root of the polynomial $1 - 5X^2 + X^4$. The value of $Y$ is unfixed. This is due to the fact that the five-dimensional dilaton is a flat direction in the potential. Therefore we can fix $Y$ to any convenient value by an SL(2, $\mathbb{R}$) symmetry transformation. The only constraint when fixing $Y$ is that one has to ensure that all four scalars $z_a$ lie inside the unit disk.

\begin{align}
\alpha &= 0.0713344, \quad \beta = 0.214003, \\
z_1 &= 0.340985 - i 0.385628, \quad z_2 = 0.109181 + i 0.698203, \\
z_3 &= 0.0805304 - i 0.315369, \quad z_4 = -0.481872 - i 0.341603.
\end{align}

\begin{align}
\mathcal{P} &\approx -1.304606. \quad (2.48)
\end{align}

Symmetry: $\mathcal{N} = 0$

\begin{align}
z_1 &= -z_4 = i \frac{\sqrt{(9 - 4\sqrt{2})(1 + i 4\sqrt{3})}}{7}, \quad z_2 = z_3 = i(\sqrt{2} - 1), \quad \beta = 3\alpha = \frac{1}{4} \log 2. \quad (2.49)
\end{align}

\begin{align}
\mathcal{P} &= -\frac{9}{4 \times 2^{2/3}}. \quad (2.50)
\end{align}

Symmetry: $\mathcal{N} = 0$
\[ \alpha = 0.0766018, \quad \beta = 0.0519887, \]
\[ z_1 = -0.214941 + i 0.285334, \quad z_2 = -0.0554356 + i 0.297182, \]
\[ z_3 = 0.483533 + i 0.610042, \quad z_4 = 0.293764 - i 0.686. \]  

(2.51)

\[ \mathcal{P} \approx -1.501862. \]  

(2.52)

Symmetry: \( \mathcal{N} = 0 \)

3 Critical points with TensorFlow

Following the basic strategy explained in [9], the numerical search for critical points was performed with TensorFlow. In this section, we want to elucidate relevant differences between the computational strategies used for \( d = 4 \) in this earlier publication and \( d = 5 \) supergravity in this article. These mainly come from two sources, differences in physics, and also advances in the software ecosystem.

3.1 TensorFlow and other options

The commonly used conventions for de Wit-Nicolai maximal gauged \( d = 4 \) supergravity use complex \( E_7(7) \) generator matrices. When employing numerical minimization with backpropagation as an effective strategy to search for vacuum solutions of the equations of motion, the stationarity condition is a smooth \( \mathbb{R}^{70} \to \mathbb{R} \) function. Using Machine Learning terminology, one would regard this as the ‘Loss Function’. If we want to keep the code in close alignment with the formulae from the published literature, we hence need a framework for reverse-mode automatic differentiation (AD) that supports Einstein summation, taking (ideally also higher) derivatives of matrix exponentiation, complex matrix exponentiation, and, importantly, taking gradients of \( \mathbb{R}^n \to \mathbb{R} \) computations even if intermediate steps involve complex quantities and holomorphic functions.

It is especially this last point that is slightly subtle and apparently not widely appreciated in the Machine Learning world, which makes TensorFlow at the time of this writing (to the best of the authors’ knowledge) the only AD framework with which the \( d = 4 \) calculation could be done using the established conventions. We want to briefly explain why.

For loss functions that involve complex intermediate quantities, it is not sufficient for a computational framework to simply support complex derivatives: it must be able to in particular correctly handle the case that a real-valued result is the magnitude-square of a complex intermediate result, schematically: 

\[ y = f_j(z_k(x_m)) \cdot \overline{f_j(z_k(x_m))}, \]

with the \( f_j \) being holomorphic functions of the intermediate complex quantities \( z_k \) that in turn are functions of the real input parameters \( x_k \). When backpropagating such an expression, the AD framework repeatedly answers the
question by how much the final result would change, relative to \( \varepsilon \), if one interrupted the calculation right after the currently-in-focus intermediate quantity \( q_n \) was obtained and changed it \( q_n \to q_n + \varepsilon \). This answer, i.e. the sensitivity of the end result on \( q_n \), is found by referring, in every step, to the already-known sensitivities for later intermediate quantities \( q_{n+k} \). Starting with the sensitivity of the end result on the end result, which is 1, we proceed through the entire computation a second time, in reverse, to ultimately obtain the sensitivities of the end result on the input parameters, i.e. the gradient. For a product of the above schematic form, the sensitivity of the end result \( y \) on the intermediate quantity \( z_k(x_m) \) is \( f_j(z_k(x_m)) \cdot \partial_k f_j(z_k(x_m)) \), and the sensitivity of \( y \) on the intermediate quantity \( z_k(x_m) \) is the complex-conjugate of this value. Clearly, a reverse mode automatic differentiation framework that only knows about holomorphic derivatives and not this subtlety involving complex conjugation will not be able to produce the expected gradients. TensorFlow uses a modified definition of a ‘complex gradient’ that is not the holomorphic derivative, but also involves complex conjugation in precisely the way that is needed to make this case work.\(^6\)

While the 56-dimensional fundamental representation of \( E_{7(-133)} \) is pseudoreal (i.e. does not permit all-real generator matrices), this is not the case for \( E_{7(7)} \), closely paralleling the familiar situation for \( SU(2) \) and \( SL(2,\mathbb{R}) \). It is indeed possible to translate de Wit-Nicolai supergravity from the ‘\( SU(8) \)-aligned’ basis that makes fermion couplings look simple to a ‘\( SL(8,\mathbb{R}) \)-aligned’ basis with all-real \( E_{7(7)} \) generator matrices, and this alternative description has been used e.g. in [43] to great effect. In maximal gauged five-dimensional supergravity, the commonly used conventions employ a real basis for the corresponding \( E_{6(6)} \) generator matrices of size \( 27 \times 27 \), and so there would be the option to also base the computation on some other reverse-mode AD numerical framework, such as perhaps the – in comparison to TensorFlow – much more lightweight ‘JAX’ library [44].

For this work, we nevertheless decided to stay with TensorFlow, partly out of the desire to develop further software tools for supergravity research that are generally applicable also in situations where complex derivatives occur.

### 3.2 The \( d = 5 \) calculation

As in maximal four-dimensional supergravity, critical points of the equations of motion are saddle points, except for the maximum at the origin with unbroken SO(6) symmetry. For this work, we did not use a stationarity condition that is expressed in terms of the gradient of the potential with respect to an infinitesimal frame change that multiplies the Vielbein matrix from one side, as in (2.8) and (2.9) of [9]. Rather, we took as stationarity condition the length-squared of the gradient of the potential, and let TensorFlow work out the gradient of this (scalar) stationarity condition.

\(^6\)For technical details, cf. [https://github.com/tensorflow/tensorflow/issues/3348](https://github.com/tensorflow/tensorflow/issues/3348)
The theory of Automated Differentiation tells us that the computational effort for obtaining the gradient of a scalar function is no more than six times the effort to compute the function (ignoring the effect of caches), and so computing the gradient of the stationarity-condition here is no more than $6^2 \times$ the effort of evaluating the potential, which is quite affordable with only 42 parameters.

For the de Wit-Nicolai theory, spin(8) symmetry can be employed to rotate a solution in such a way that one of the two symmetric traceless matrices $M_{\alpha\beta}, M_{\dot{\alpha}\dot{\beta}}$ that describe the location of a critical point (cf. (D.3) in [9]) gets diagonalized. For five-dimensional maximal supergravity, we first performed a scan in the full 42-dimensional parameter space, starting from 100,000 seeded pseudorandom starting locations, and then checked that we could indeed re-identify all solutions found in this way by performing another (similarly large) scan using a reduced coordinate-parametrization that set the non-diagonal entries of the $\Lambda^I_J$ and also the two SL(2, R)/SO(2) axion-dilaton parameters to zero. As the volume of the SO(6) orbit of a solution is a function of the distance from the origin, one would naturally expect these two different scanning methods to produce any given solution with very different probability, and so using only the latter, reduced, parametrization, might have increased the risk of overlooking solutions. Also, the conjecture that one can indeed always set the axion-dilaton parameters to zero seems to be currently unproven.

As for the $d = 4$ calculation, we employed residual unbroken SO(6) symmetry that is associated with degenerate entries on the diagonal of $\Lambda^I_J$ to further reduce the number of non-zero $\Sigma_{ijk;\alpha}$-coefficients, but there is no guarantee in our tables that the number of parameters found in each case is indeed minimal.

Given that TensorFlow currently is limited to performing calculations with at most IEEE 754 64-bit float precision, and also the inherent problems of solving nonlinear equation systems via minimization to good accuracy, we found it effective to further increase the accuracy of a solution-candidate as obtained from minimization via a modified multi-dimensional Newton method. Here, one has to be careful due to the presence of flat (“Goldstone mode”) directions in the potential and hence also stationarity condition.

### 3.3 Modern TensorFlow

In this work, TensorFlow mostly serves as a “fast numerical gradients” library for high-dimensional numerical minimization. While it is useful to adopt Machine Learning terminology for easier communication with other (mostly Machine Learning) users of TensorFlow, this is not strictly necessary. Due to the public release of TensorFlow2 in September 2019,\(^7\) which moves away from the explicit meta-programming paradigm, much of the scaffolding that was used on the

\(^7\)Cf. [https://blog.tensorflow.org/2019/09/tensorflow-20-is-now-available.html](https://blog.tensorflow.org/2019/09/tensorflow-20-is-now-available.html)
example Colab notebook\(^8\) published alongside [9] can be eliminated. In particular, the need for continuation-passing techniques (such as provided by: `call_with_critical_point_scanner()`) in order to evaluate a function “in session context” is now gone.

There broadly are two major approaches to reverse mode Automatic Differentiation (AD), program-transformation based AD and tape-based AD. TensorFlow\(^1\) was based on program transformation, where the ‘program’ is a description of a calculation in terms of a (tensor-)arithmetic graph that can be evaluated on general purpose CPUs or alternatively also hardware that is more specialized towards parallel numerics, i.e. GPUs or Google’s Tensor Processing Units\(^9\) (TPUs). The Python programming language is here used as a ‘Meta-Language’ to manipulate ‘graph’ objects that represent computations.

TensorFlow\(^2\) tries to hide much of this meta-programming complexity by making the graph invisible to the user and mostly following the ‘tape-based’ paradigm. Here, the idea is that the sequence of computational steps in a calculation for which we want to have a fast and accurate gradient are recorded on a ‘tape’. Once the calculation is done, the tape is ‘played in reverse’, in each step updating sensitivities of the final result on intermediate quantities, in their natural latest-to-earliest order. Pragmatically, this means that a TensorFlow\(^2\) ‘Tensor’ object can be seen as an envelope around a NumPy array that can be tracked on a tape, but otherwise is passed around and manipulated mostly like an array of numbers. This in particular means that with TensorFlow\(^2\), interfacing with optimizers such as `scipy.optimize.fmin_bfgs()` no longer requires a TensorFlow-provided wrapper such as `ScipyOptimizerInterface()`, or initiating numerical evaluation through an explicitly managed ‘session’, but instead can be done by simply wrapping up numpy-arrays in TensorFlow tensors for gradient computations, roughly along these lines:

```python
def tf_minimize(tf_func, x0):
    """Minimizes a TensorFlow tf.Tensor -> tf.Tensor function.""
    def f_opt(xs):
        return tf_func(tf.constant(xs, dtype=tf.float64)).numpy()
    def fprime_opt(xs):
        t_xs = tf.constant(xs, dtype=tf.float64)
        tape = tf.GradientTape()
        with tape:
            tape.watch(t_xs)
            t_val = tf_func(t_xs)
        return tape.gradient(t_val, t_xs).numpy()
    opt = scipy.optimize.fmin_bfgs(
        f_opt, numpy.array(x0), fprime=fprime_opt, disp=0)
    return f_opt(opt), opt
```

\(^{8}\)https://research.google.com/seedbank/seed/so_supergravity_extrema
\(^{9}\)Cf. https://tinyurl.com/y6gmmwes
4 Conclusions

In this paper we presented a numerical exploration of the AdS\(_5\) vacua corresponding to critical points of the scalar potential of the SO(6) maximal gauged supergravity. Out of the 31 critical points, we find that there are only 2 that are supersymmetric and perturbatively stable. Usually one would dismiss the 29 unstable AdS\(_5\) solutions as physically irrelevant. Nevertheless, the existence of these critical points may point towards some interesting dynamics in the supersymmetry broken phases of the planar \(\mathcal{N} = 4\) SYM theory. Perhaps some of these vacua admit an interpretation as holographic duals to complex CFTs \([45, 46]\) or can serve as lampposts for other type of approximately conformal QFT dynamics similar to the ones studied in \([47]\). To understand this question better one can study holographic RG flows represented by domain wall solutions connecting our new vacua. This can be done most explicitly for the 10-scalar and 6-scalar consistent truncations. For example, if there are supersymmetric RG flows that closely approach some of the unstable AdS\(_5\) vacua this may suggest an approximately conformal supersymmetric phase of \(\mathcal{N} = 4\) SYM. It should also be noted that the 10- and 14-scalar consistent truncations have wider applications in the context of holography. As emphasized in \([31, 48]\) they can be used to study the holographic dual description of the \(\mathcal{N} = 1^*\) mass deformation of \(\mathcal{N} = 4\) SYM on \(\mathbb{R}^4\) and \(S^4\) for general values of the complex mass parameters.

All of the AdS\(_5\) vacua we constructed can be uplifted to solutions of type IIB supergravity using the explicit formulae in \([8]\). This will result in ten-dimensional AdS\(_5\) solutions with non-trivial fluxes on \(S^5\). Given that the new critical points are perturbatively unstable, they can be used as a test ground for exploring the general mechanisms responsible for instabilities in non-supersymmetric flux compactifications. In addition, using the ten-dimensional uplift may allow for the possibility of stabilizing some of the AdS\(_5\) vacua by projecting out the unstable modes using an appropriate orbifold action in type IIB string theory \([49]\).

Finally, we note that there are other gaugings that lead to maximal supergravity theories in five dimensions, see \([2, 50]\). It will be interesting to apply similar numerical and analytical tools to study the critical points of these theories.

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A Conventions

Throughout this paper we use the same conventions as in [2], which the reader should consult for details. Here we summarize an explicit parametrization of the scalar manifold

$$\mathcal{M}_{(42)} \equiv \frac{E_{6(6)}}{USp(8)},$$

of the $N = 8$ $d = 5$ supergravity as needed for the truncations in Section 2 and Appendix B, an explicit construction of the potential in Section 3, and specifying the location of its critical points in Appendix C.

The most straightforward description of the $e_{6(6)}$ generators in the so-called SL$(6, \mathbb{R}) \times$ SL$(2, \mathbb{R})$ basis is through their action on 27-dimensional vectors with components $(z_{IJ}, z^I\alpha)$, $z_{IJ} = -z_{JI}$, \footnote{For the corresponding $27 \times 27$ matrix, see (A.36) in [2].}

$$\delta z_{IJ} = -\Lambda^K_{IJ}z_{KJ} - \Lambda^K_{KJ}z_{IK} + \Sigma_{IJK\beta}z^{K\beta},$$

$$\delta z^I\alpha = \Lambda^I_Kz^{K\alpha} + \Lambda^\alpha_\beta z^I\beta + \Sigma^{KLI\alpha}z_{KL},$$

(A.2)

where $(\Lambda^I_J)$ and $(\Lambda^\alpha_\beta)$ are real matrices in $\mathfrak{sl}(6, \mathbb{R})$ and $\mathfrak{sl}(2, \mathbb{R})$, respectively, and $\Sigma_{IJK\alpha} = \Sigma_{[IJ]K\alpha}$ is real with

$$\Sigma_{IJK\alpha} = \frac{1}{6}\epsilon^{IJKLMN}\epsilon^{\alpha\beta}\Sigma_{LMN\beta}.$$\footnote{Note that unlike [2] we use the range $\alpha, \beta = 7, 8$ for the SL$(2, \mathbb{R})$ indices.}

(A.3)

Note that the transformation (A.2) can be extended to arbitrary $(\Lambda^I_J) \in \mathfrak{gl}(6, \mathbb{R})$ and $(\Lambda^\alpha_\beta) \in \mathfrak{gl}(2, \mathbb{R})$. This can be used to introduce a convenient basis of generators $(\hat{\Lambda}^I_J, \hat{\Lambda}^\alpha_\beta, \hat{\Sigma}_{IJK\alpha})$ in $e_{6(6)} \oplus \mathbb{R}^2$ defined by the following nonvanishing parameters in (A.2) for each generator: \footnote{For the corresponding $27 \times 27$ matrix, see (A.36) in [2].}

$$\hat{\Lambda}^I_J : \quad \Lambda^I_J = 1, \quad I, J = 1, \ldots, 6,$$

$$\hat{\Lambda}^\alpha_\beta : \quad \Lambda^\alpha_\beta = 1, \quad \alpha, \beta = 7, 8,$$

$$\hat{\Sigma}_{IJK\alpha} : \quad \Sigma_{IJK\alpha} = \Sigma_{KIJ\alpha} = \ldots = -\Sigma_{KJI\alpha} = 1, \quad I < J < K.$$ (A.4)

The coset, $E_{6(6)}/USp(8)$, has a trivial topology of $\mathbb{R}^{42}$ and, via the exponential map, is isomorphic to the corresponding quotient of the Lie algebras, $e_{6(6)}/usp(8)$. The usual choice of the coset representatives is then given by the noncompact generators for which

$$\Lambda^I_J = \Lambda^J_I, \quad \Lambda^\alpha_\beta = \Lambda^\beta_\alpha, \quad \Sigma_{IJK\alpha} = \Sigma^{IJK\alpha}.$$ (A.5)

An ordered set of the 20+2+20 independent parameters in (A.5) provides then global coordinates on the scalar manifold, $\mathcal{M}_{(42)}$. 

B 14-scalar model

In this appendix we present a truncation of the potential to a 14-scalar model that arises as an intermediate step in the construction of the 6-scalar model in [34] and/or the 10-scalar model [31] discussed in Section 2. The main result is an explicit, albeit rather complicated, form of the scalar potential in this sector. It yields a subset of 17 extrema of the full potential.

B.1 $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant truncations

There are two equivalent methods to obtain the 14-scalar model we are interested in. The first one is to truncate with respect to a $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \text{SO}(6)$ symmetry generated by

$$g_1 = \text{diag}(-1, -1, -1, -1, 1, 1) \quad \text{and} \quad g_2 = \text{diag}(1, 1, -1, -1, -1, -1).$$

(B.1)

The second method is to use $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \text{SO}(6) \times \text{GL}(2, \mathbb{R})$ generated by $P_1Q$ and $P_2Q$ [31], where

$$P_1 = \text{diag}(-1, -1, 1, 1, 1, 1), \quad P_2 = \text{diag}(1, 1, -1, -1, 1, 1), \quad Q = \text{diag}(-1, -1).$$

(B.2)

are the same as in (2.1) and (2.2). The truncations with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ in (B.1) or (B.2), respectively, yield the same set of invariant generators of $\mathfrak{o}(1, 1)^2 \times \mathfrak{so}(4, 4) \subset \mathfrak{e}_6(6)$, with the resulting scalar coset

$$\mathcal{M}_{\mathbb{O}(1,1)^2} \times \mathcal{M}_{\mathbb{SO}(4,4)} \equiv \text{SO}(1, 1)^2 \times \frac{\text{SO}(4, 4)}{\text{SO}(4) \times \text{SO}(4)}.$$  

(B.3)

To compute the potential, we need a workable parametrization of the second factor.

B.2 Polar parametrization of the coset

In the vector representation of $\text{SO}(4, 4)$, the compact $\text{SO}(4) \times \text{O}(4)$ subgroup is given by block matrices

$$O = \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix}, \quad O_1, O_2 \in \text{O}(4), \quad O_1O_2 \in \text{SO}(4).$$

(B.4)

The non-compact generators are of the form

$$X = \begin{pmatrix} 0 & M \\ MT & 0 \end{pmatrix},$$

(B.5)

and the $4 \times 4$ matrices, $M$, provide global coordinates on the coset. Now, note that

$$OXO^T = \begin{pmatrix} 0 & O_1MO_2^T \\ O_2MO_1^T & 0 \end{pmatrix},$$

(B.6)
and use the fact that any generic real matrix can be diagonalized by two orthogonal matrices, that is
\[ M = O_1 \Lambda O_2^T. \]

The diagonal matrix, \( \Lambda \), consists of 4 commuting, noncompact generators. Furthermore, any 4 such generators are conjugate under the action of the compact subgroup. The idea now is to parametrize the \( \mathcal{M}_{SO(4,4)} \) coset in terms of Euler angles for \( O_1 \) and \( O_2 \) and the four parameters in \( \Lambda \).

To this end, we first decompose the compact generators of \( \mathfrak{so}(4,4) \subset \mathfrak{e}_{6(6)} \) into generators of 4 mutually commuting \( \mathfrak{su}(2) \)'s, which are labelled by \( \alpha, \beta, \gamma, \delta \).\(^{12}\) Inside the \( \mathfrak{e}_{6(6)} \), one can choose those generators as follows:

\[
\begin{align*}
\mathbf{r}_1^{(\alpha)} &= \frac{1}{\sqrt{2}} \left( -\hat{\Sigma}_{1357} + \hat{\Sigma}_{1368} + \hat{\Sigma}_{1458} + \hat{\Sigma}_{1467} + \hat{\Sigma}_{2358} + \hat{\Sigma}_{2367} + \hat{\Sigma}_{2457} - \hat{\Sigma}_{2468} \right), \\
\mathbf{r}_2^{(\alpha)} &= \frac{1}{\sqrt{2}} \left( -\hat{\Sigma}_{1358} - \hat{\Sigma}_{1367} - \hat{\Sigma}_{1457} + \hat{\Sigma}_{1468} - \hat{\Sigma}_{2357} + \hat{\Sigma}_{2368} + \hat{\Sigma}_{2458} + \hat{\Sigma}_{2467} \right), \\
\mathbf{r}_3^{(\alpha)} &= \hat{A}^{12} + \hat{A}^{34} + \hat{A}^{56} + \hat{A}^{78}, \\
\mathbf{r}_1^{(\beta)} &= \frac{1}{\sqrt{2}} \left( -\hat{\Sigma}_{1357} + \hat{\Sigma}_{1368} - \hat{\Sigma}_{1458} - \hat{\Sigma}_{1467} - \hat{\Sigma}_{2358} - \hat{\Sigma}_{2367} + \hat{\Sigma}_{2457} - \hat{\Sigma}_{2468} \right), \\
\mathbf{r}_2^{(\beta)} &= \frac{1}{\sqrt{2}} \left( \hat{\Sigma}_{1358} + \hat{\Sigma}_{1367} - \hat{\Sigma}_{1457} + \hat{\Sigma}_{1468} - \hat{\Sigma}_{2357} + \hat{\Sigma}_{2368} - \hat{\Sigma}_{2458} - \hat{\Sigma}_{2467} \right), \\
\mathbf{r}_3^{(\beta)} &= \hat{A}^{12} + \hat{A}^{34} - \hat{A}^{56} - \hat{A}^{78}, \\
\mathbf{r}_1^{(\gamma)} &= \frac{1}{\sqrt{2}} \left( -\hat{\Sigma}_{1357} - \hat{\Sigma}_{1368} + \hat{\Sigma}_{1458} - \hat{\Sigma}_{1467} - \hat{\Sigma}_{2358} + \hat{\Sigma}_{2367} - \hat{\Sigma}_{2457} - \hat{\Sigma}_{2468} \right), \\
\mathbf{r}_2^{(\gamma)} &= \frac{1}{\sqrt{2}} \left( \hat{\Sigma}_{1358} - \hat{\Sigma}_{1367} + \hat{\Sigma}_{1457} - \hat{\Sigma}_{1468} + \hat{\Sigma}_{2357} - \hat{\Sigma}_{2368} + \hat{\Sigma}_{2458} - \hat{\Sigma}_{2467} \right), \\
\mathbf{r}_3^{(\gamma)} &= \hat{A}^{12} - \hat{A}^{34} + \hat{A}^{56} - \hat{A}^{78}, \\
\mathbf{r}_1^{(\delta)} &= \frac{1}{\sqrt{2}} \left( -\hat{\Sigma}_{1357} - \hat{\Sigma}_{1368} - \hat{\Sigma}_{1458} + \hat{\Sigma}_{1467} + \hat{\Sigma}_{2358} - \hat{\Sigma}_{2367} - \hat{\Sigma}_{2457} - \hat{\Sigma}_{2468} \right), \\
\mathbf{r}_2^{(\delta)} &= \frac{1}{\sqrt{2}} \left( \hat{\Sigma}_{1358} - \hat{\Sigma}_{1367} - \hat{\Sigma}_{1457} + \hat{\Sigma}_{1468} + \hat{\Sigma}_{2357} + \hat{\Sigma}_{2368} + \hat{\Sigma}_{2458} - \hat{\Sigma}_{2467} \right), \\
\mathbf{r}_3^{(\delta)} &= -\hat{A}^{12} + \hat{A}^{34} + \hat{A}^{56} - \hat{A}^{78},
\end{align*}
\]

where \( \hat{A}^{IJ} = \hat{A}^{I}_{J} - \hat{A}^{J}_{I} \). They satisfy,

\[
[\mathbf{r}_i, \mathbf{r}_j] + 4 \epsilon_{ijk} \mathbf{r}_k = 0, \quad \text{Tr} \mathbf{r}_i \mathbf{r}_j = -48 \delta_{ij},
\]

within each \( \mathfrak{su}(2) \) subalgebra.

\(^{12}\)Note that \( \alpha \) and \( \beta \) have different meaning in the main text than in this appendix.
The group elements of these four commuting SU(2)’s inside $E_{6(6)}$ are now parametrized by the Euler angles $\alpha_1, \ldots, \delta_3$ defined by

$$g(\varphi_1, \varphi_2, \varphi_3) = \exp(\varphi_1 r_1^{(\varphi)}) \cdot \exp(\varphi_2 r_2^{(\varphi)}) \cdot \exp(\varphi_3 r_3^{(\varphi)}), \quad \varphi = \alpha, \beta, \gamma, \delta.$$  \hspace{1cm} (B.13)

By simultaneously diagonalizing the four Casimir operators, one can bring the group element of the compact subgroup into a block diagonal form corresponding to the branching

$$27 \rightarrow 3 \times (1,1,1,1) + (2,2,1,1) + (2,1,2,1) + (1,2,2,1) + (2,1,1,2) + (1,2,1,2) + (1,1,2,2),$$  \hspace{1cm} (B.14)

where the $4 \times 4$ blocks of each SU(2) are of the form

$$\begin{pmatrix}
  s_1 & s_2 & \sqrt{2} s_3 & \sqrt{2} s_4 \\
  -s_2 & s_1 & -\sqrt{2} s_4 & \sqrt{2} s_3 \\
  -s_3/\sqrt{2} & s_4/\sqrt{2} & s_1 & -s_2 \\
  -s_4/\sqrt{2} & \sqrt{3}/\sqrt{2} & s_2 & s_1
\end{pmatrix},$$  \hspace{1cm} (B.15)

modulo a permutation of signs between some terms that make the two SU(2)’s in each $(2,2)$ block commute. The $s_i$’s above are

$$s_1 = \cos 2\varphi_2 \cos 2(\varphi_1 + \varphi_3), \quad s_2 = \cos 2\varphi_2 \sin 2(\varphi_1 + \varphi_3),$$
$$s_3 = \sin 2\varphi_2 \cos 2(\varphi_1 - \varphi_3), \quad s_4 = \sin 2\varphi_2 \sin 2(\varphi_1 - \varphi_3),$$  \hspace{1cm} (B.16)

for each of the angles $\varphi = \alpha, \beta, \gamma, \delta$. Note that

$$s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1,$$  \hspace{1cm} (B.17)

so that each block is simply a unit quaternion. Then the $24 \times 24$ block of the $E_{6(6)}$ matrix corresponding to the second line in (B.14) is a diagonal matrix parametrized by 4 commuting quaternions, $q_\alpha, \ldots, q_\delta$.

Next we choose 4 commuting noncompact generators, cf. (2.5),

$$g_1 = \frac{1}{\sqrt{2}} \left( \Sigma_{1357} - \Sigma_{2468} \right), \quad g_2 = \frac{1}{\sqrt{2}} \left( \Sigma_{1467} - \Sigma_{2358} \right),$$
$$g_3 = \frac{1}{\sqrt{2}} \left( \Sigma_{2367} - \Sigma_{1458} \right), \quad g_4 = \frac{1}{\sqrt{2}} \left( \Sigma_{2457} - \Sigma_{1368} \right).$$  \hspace{1cm} (B.18)

Then the scalar 27-bein

$$\mathcal{V}_{SO(4,4)}(\alpha, \beta, \gamma, \delta; \rho) = g(\alpha) \ldots g(\delta) \exp \left( \sum_i \rho_i g_i \right) g(\delta)^{-1} \ldots g(\alpha)^{-1} \in E_{6(6)},$$  \hspace{1cm} (B.19)
parametrizes the coset $\mathcal{M}_{SO(4,4)}$. The matrix $V_{SO(4,4)}$ is somewhat sparse with $195$ out of $27^2 = 729$ nonzero entries. In the following, it will be useful to work with the corresponding matrix obtained by replacing the nonvanishing entries in $V_{SO(4,4)}$ with symbolic entries, say $m_{ij}$.

Adding the $O(1,1)$ factor does not change much. We choose generators, cf. (2.4),

$$
\tilde{g}_1 = \frac{3}{2} (\hat{\Lambda}_1 + \hat{\Lambda}_2^2 - \hat{\Lambda}_3^3 - \hat{\Lambda}_4^4),
$$

$$
\tilde{g}_2 = \frac{5}{3} (\hat{\Lambda}_1 + \hat{\Lambda}_2^2 + \hat{\Lambda}_3^3 + \hat{\Lambda}_4^4 - 2\hat{\Lambda}_5^5 - 2\hat{\Lambda}_6^6),
$$

with the corresponding group element

$$
V_{O(1,1)^2}(\xi_1, \xi_2) = \exp(\xi_1 \tilde{g}_1 + \xi_2 \tilde{g}_2).
$$

This matrix is diagonal and simply “decorates” the $m_{ij}$’s in (B.19) by exponential factors. Finally, the full scalar 27-bein is

$$
V(\xi; \alpha, \beta, \gamma, \delta; \rho) = V_{O(1,1)^2}(\xi) \cdot V_{SO(4,4)}(\alpha, \beta, \gamma, \delta; \rho).
$$

### B.3 Computation of the potential

Using symbolic representation of $V$, the potential is a sum of $2784$ terms quartic in $m_{ij}$’s, which fall into $6$ different groups depending on the $O(1,1)^2$ prefactors,

$$
e^{-3\xi_1 - \frac{40}{3} \xi_2}, \quad e^{3\xi_1 - \frac{40}{3} \xi_2}, \quad e^{-\frac{40}{3} \xi_1}, \quad e^{\frac{40}{3} \xi_1}, \quad e^{-\frac{60}{3} \xi_1 + \frac{20}{3} \xi_2}, \quad e^{\frac{60}{3} \xi_1 + \frac{20}{3} \xi_2}.
$$

After substituting for $m_{ij}$’s, we find the prefactors in (B.23) are multiplied by $48$, $48$, $18$, $48$, $18$, $18$ different quartic products of $\cosh \rho_{ij}$ and $\sinh \rho_{ij}$, with $\rho_{ij} = \rho_i - \rho_j$, respectively, for the total of $198$ terms. In turn, each of those terms is multiplied by a homogenous polynomial of order $16$ in $16$ different $s_i$’s (B.15) for the $12$ Euler angles. A typical number of terms in those trigonometric polynomials is on the order of $40,000$. That number is drastically reduced upon repeated use of (B.17), usually to less than a $100$. Finally, the substitution of explicit $s_i$’s in terms of the angles further collapses each group to a relatively small number of terms.

In the last stage, all dependence of the potential on the four angles $\alpha_1, \beta_1, \gamma_1$ and $\delta_1$ disappears and one is left with the potential that depends on $8$ Euler angles and $6$ noncompact fields. This is a nice consistency check for this long calculation. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ truncation preserves $U(1)^4 \subset SO(6) \times SL(2, \mathbb{R})$, generated by $v_3^{(\psi)}$’s, which is a symmetry of the potential. Hence the latter should be a function of $18 - 4 = 14$ independent scalar fields, as indeed it is.

Even a simplified expression for the potential is too long to be written down in a reasonable amount of space here. Instead, it is made available as a Mathematica input file, see Section B.5.
B.4 The critical points

We have found 17 critical points of the scalar potential in this $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant sector using the `FindRoot[... ]` routine in Mathematica starting at random locations on the scalar manifold. Those points are listed in Table B.1. As expected, they include all critical points found in the 10-scalar model in Section 2, with only two additional ones, $T1054687$ and $T1547778$, whose positions in the polar parametrization used here are given in Table B.2.\footnote{Note that \[ -\frac{35}{2^{13/3}} \cdot 5 = -1.5477783979193562580662234151917585735219771770242937517061887 \ldots, \] which agrees with the value of the potential for $T1547778$ to the numerical accuracy we tested it.}

A major inconvenience when working with the polar-type coordinates, as compared to the ones used in Sections 2 and 3, is the presence of coordinate singularities in the parametrization of the scalar coset. As a result the search routine yields a large fraction of “fake critical points”. Those are then eliminated by an explicit check of criticality, that is by evaluating the potential to the first order in $\epsilon$ on the scalar vielbein

\[(1 + \epsilon \sum_A \psi_A T_A) V_*, \tag{B.24}\]

where $V_*$ is the presumed critical point. The sum in (B.24) runs over all 78 generators of $\mathfrak{e}_{6(6)}$ and to eliminate a point it is sufficient to verify that (B.24) does not vanish for some random values of the parameters $\psi_A$.

The numerical search for critical points in this sector appears to be quite efficient, so we believe that there should be no missing critical points from our search. It is then quite remarkable that the 17 points found here constitute more than 50% of all critical points found by the TensorFlow search in Section 3.

B.5 Ancillary files

A text file with a Mathematica input for the full potential in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant sector in this section is available for download as an ancillary file with this arXiv submission. The potential

\[
\begin{array}{cccccccc}
T0750000 & T0780031 & T0839947 & T0843750 & T0870297 & T0878939 \\
T0887636 & T0892913 & T0964525 & T0982778 & T1001482 & T1054687 \\
T1125000 & T1304606 & T1417411 & T1501862 & T1547778 & \\
\end{array}
\]

Table B.1: The critical points in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant sector.
depends on 14 scalar fields, which are denoted by the same symbols as in the text above. The locations of the critical points can be found in a Mathematica input file which is available for download as an ancillary file with this arXiv submission.

## C Critical points and mass spectra

In this appendix, we list numerical data obtained from TensorFlow on the locations, gravitino and scalar mass spectra, cosmological constant, as well as residual gauge symmetry and supersymmetry. Gravitino masses are normalized relative to the AdS radius such that for every unbroken supersymmetry, there is a $m^2/m_0^2[\psi] = 1$ gravitino, and the BF bound is $m^2/m_0^2[\phi] \geq -4$. The (symmetric-traceless) $\Lambda_{IJ}$ parameters have been diagonalized. The diagonal $\Lambda_{II}$ entries listed sum to zero as expected, hiding a linear constraint on the numerical data.

This list of solutions, produced by starting numerical optimization from $10^5$ random points, is likely to be mostly complete. Notably, an independent second deep scan that used a modified ‘loss function’ to guide the search towards supersymmetric solutions did not find any supersymmetric critical points beyond the two already known ones.

In the list of solutions we give the location on the scalar manifold in terms of the generators in Appendix A. The $\xi_{\hat{\mathfrak{g}(6)}}$ element is constructed as a linear combination of the generators $\hat{\Lambda}^\alpha_\beta$, $\hat{\Lambda}^I_J$, and $\hat{\Sigma}_{IJKa}$. The coefficient of $\hat{\Lambda}^\alpha_\beta$ is set to zero for all solutions as explained in Section 3. The coefficients of $\hat{\Lambda}^I_J$ are denoted by $\Lambda^I_J$, and the coefficients multiplying $\hat{\Sigma}_{IJK7}$ are $\pm \sqrt{2} \Sigma^{\pm(IJK;1+...;2)}$. Only nonzero coefficients of the $\hat{\Sigma}_{IJK\alpha}$-generators are displayed. This accounts for all non-compact generators. The group element is obtained by exponentiating the linear combination just described.

| Point      | $\xi_1$ | $\xi_2$ | $\rho_1$ | $\rho_2$ | $\rho_3$ | $\rho_4$ | $\alpha_{2/3}$ | $\beta_{2/3}$ | $\gamma_{2/3}$ | $\delta_{2/3}$ |
|------------|---------|---------|----------|----------|----------|----------|----------------|---------------|---------------|---------------|
| T1054687   | 0       | 0       | 0.59672  | -0.00571 | 0.60944  | 0.59486  | 0.05047        | 0.38645       | 0.52874       | 0.28433       |
|            |         |         |          |          |          |          | 0.46797        | 1.17929       | 1.64474       | 0.85757       |
| T1547778   | 0.14931 | 0.04479 | 0.34047  | 1.36783  | 0.56949  | 0.58893  | 0.58938        | 1.58999       | 1.38975       |
|            |         |         |          |          |          |          | 0.23497        | 0.41012       | 1.56041       | 1.36012       |

Table B.2: Positions of T1054687 and T1547778 in the polar coordinates.
\[ \mathcal{P} / g^2 = -0.75000000, \, \mathcal{N} = 8, \, \mathfrak{so}(6) \to \mathfrak{so}(6) \] (C.1)

\[ m^2 / m_0^2[\psi] : 1.000_{\times 8} \]

\[ m^2 / m_0^2[\phi] : -4.000_{\times 20}, -3.000_{\times 20}, 0.000_{\times 2} \]

\[ \Lambda_1^1 = \Lambda_2^2 = \Lambda_3^3 = \Lambda_4^4 = \Lambda_5^5 = \Lambda_6^6 = 0.00000 \]

\[ \mathcal{P} / g^2 = -0.78003143, \, \mathfrak{so}(6) \to \mathfrak{so}(5) \] (C.2)

\[ m^2 / m_0^2[\psi] : 1.185_{\times 8} \]

\[ m^2 / m_0^2[\phi] : -5.333_{\times 14}, -2.000_{\times 20}, 0.000_{\times 7}, 8.000 \]

\[ \Lambda_1^1 = \Lambda_2^2 = \Lambda_3^3 = \Lambda_4^4 = \Lambda_5^5 \approx -0.09155, \, \Lambda_6^6 \approx 0.45776 \]

\[ \mathcal{P} / g^2 = -0.83994737, \, \mathcal{N} = 2, \, \mathfrak{so}(6) \to \mathfrak{su}(2) + \mathfrak{u}(1) \] (C.3)

\[ m^2 / m_0^2[\psi] : 1.000_{\times 2}, 1.361_{\times 4}, 1.778_{\times 2} \]

\[ m^2 / m_0^2[\phi] : -4.000_{\times 3}, -3.750_{\times 12}, -3.437_{\times 4}, -3.000_{\times 2}, -2.438_{\times 4}, -1.292, 0.000_{\times 13}, 3.000_{\times 2}, 9.292 \]

\[ \Lambda_1^1 = \Lambda_2^2 = \Lambda_3^3 = \Lambda_4^4 \approx -0.11552, \, \Lambda_5^5 = \Lambda_6^6 \approx 0.23105 \]

\[ \Sigma_{+125;1+346;2}, 1_{+126;2+345;1} = \Sigma_{+135;2+246;1} = \Sigma_{+136;1+245;2} \approx 0.27465 \]

\[ \mathcal{P} / g^2 = -0.84375000, \, \mathfrak{so}(6) \to \mathfrak{su}(3) \] (C.4)

\[ m^2 / m_0^2[\psi] : 1.210_{\times 6}, 2.000_{\times 2} \]

\[ m^2 / m_0^2[\phi] : -4.444_{\times 12}, -1.778_{\times 12}, 0.000_{\times 17}, 8.000 \]
\[ \Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_4 = \Lambda_5 = \Lambda_6 = 0.00000 \]

\[ \Sigma_{+123;1+456;2} \approx 0.29343, \Sigma_{+123;2-456;1} \approx 0.02565, \Sigma_{+124;1-356;2} \approx 0.00727, \Sigma_{+124;2+356;1} \approx 0.02437, \]
\[ \Sigma_{+125;1+346;2} \approx 0.01975, \Sigma_{+125;2-346;1} \approx -0.30327, \Sigma_{+126;1-345;2} \approx -0.08422, \]
\[ \Sigma_{+126;2+345;1} \approx 0.02034, \Sigma_{+134;1+256;2} \approx 0.01948, \Sigma_{+134;2-256;1} \approx -0.31733, \Sigma_{+135;1-246;2} \approx 0.00577, \]
\[ \Sigma_{+135;2+246;1} \approx 0.00693, \Sigma_{+136;1+245;2} \approx 0.12256, \Sigma_{+136;2-245;1} \approx -0.01130, \Sigma_{+145;1+236;2} \approx 0.32741, \]
\[ \Sigma_{+145;2-236;1} \approx 0.01353, \Sigma_{+146;1-235;2} \approx -0.01950, \Sigma_{+146;2+235;1} \approx -0.08083, \]
\[ \Sigma_{+156;1+234;2} \approx -0.01220, \Sigma_{+156;2-234;1} \approx -0.12477 \]

\[ T0870297 : \mathcal{P}/g^2 = -0.87029791, \text{so}(6) \rightarrow \text{su}(2) + u(1) \quad (C.5) \]
\[ m^2/m_0^2[\psi] : 1.440_{\times 4}, 1.600_{\times 4} \]
\[ m^2/m_0^2[\phi] : -5.440_{\times 6}, -4.000_{\times 4}, -2.560_{\times 8}, -2.400_{\times 6}, 0.000_{\times 13}, 3.360_{\times 2}, 9.600, 10.400_{\times 2} \]
\[ \Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_4 \approx -0.19188, \Lambda_5 = \Lambda_6 \approx 0.38376 \]
\[ \Sigma_{+135;2+246;1} = \Sigma_{+136;1+245;2} \approx -0.35635 \]

\[ T0878939 : \mathcal{P}/g^2 = -0.87893974, \text{so}(6) \rightarrow u(1) \quad (C.6) \]
\[ m^2/m_0^2[\psi] : 1.358_{\times 4}, 1.802_{\times 4} \]
\[ m^2/m_0^2[\phi] : -5.827, -5.221_{\times 4}, -4.990, -4.937_{\times 2}, -4.780_{\times 2}, -4.475, -2.522_{\times 2}, -2.288_{\times 4}, \]
\[ -1.532_{\times 4}, 0.000_{\times 16}, 0.820, 4.437, 9.803, 12.622_{\times 2} \]
\[ \Lambda_1 = \Lambda_2 \approx -0.21480, \Lambda_3 = \Lambda_4 \approx -0.20031, \Lambda_5 = \Lambda_6 \approx 0.41511 \]
\[ \Sigma_{+136;2-245;1} = \Sigma_{+146;1-235;2} \approx 0.34860 \]

\[ T0887636 : \mathcal{P}/g^2 = -0.88763615, \text{so}(6) \rightarrow u(1) \quad (C.7) \]
\[ m^2/m_0^2[\psi] : 1.483_{\times 2}, 1.604_{\times 4}, 1.838_{\times 2} \]
\[ m^2/m_0^2[\phi] : \begin{align*}
&-5.907x_2, -5.489x_4, -5.208, -4.731, -4.581x_2, -4.054x_2, -2.293x_4, -1.182x_2, \\
&-0.253x_2, 0.000x_16, 2.070x_2, 3.344, 9.651, 14.863x_2
\end{align*} \]

\[ \Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_4 \approx -0.22251, \Lambda_5 = \Lambda_6 \approx 0.44502 \]

\[ \Sigma_{+125;1+346;2} \approx 0.16666, \Sigma_{+126;2+345;1} \approx 0.03828, \Sigma_{+135;2+246;1} \approx -0.14968, \Sigma_{+136;1+245;2} \approx 0.14968, \Sigma_{+145;1+236;2} \approx -0.15351, \Sigma_{+146;2+235;1} \approx 0.37172 \]

---

T0892913 : \( \mathcal{P}/g^2 = -0.89291309, \ so(6) \rightarrow u(1) + u(1) \) \hfill (C.8)

\[ m^2/m_0^2[\psi] : 1.667 \times 8 \]

\[ m^2/m_0^2[\phi] : \begin{align*}
&-6.000x_4, -5.572, -5.000x_4, -4.520, -3.500x_4, -0.833x_4, 0.000x_16, 1.667x_4, 2.667, \\
&9.572, 16.000, 16.520
\end{align*} \]

\[ \Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_4 \approx -0.23105, \Lambda_5 = \Lambda_6 \approx 0.46210 \]

\[ \Sigma_{+125;1+346;2} \approx -0.00001, \Sigma_{+125;2+345;1} \approx 0.00001, \Sigma_{+126;1+345;2} \approx -0.00198, \]

\[ \Sigma_{+126;2+345;1} \approx -0.00217, \Sigma_{+135;1+246;2} \approx 0.35483, \Sigma_{+135;2+246;1} \approx -0.32504 \]

---

T0963952 : \( \mathcal{P}/g^2 = -0.96395224, \ so(6) \rightarrow \emptyset \) \hfill (C.9)

\[ m^2/m_0^2[\psi] : 1.739x_4, 1.924x_2, 1.942x_2 \]

\[ m^2/m_0^2[\phi] : \begin{align*}
&-5.874, -5.026x_2, -4.449x_2, -4.076x_2, -4.017x_2, -4.010x_2, -3.485x_2, -2.623, \\
&-0.499, -0.156x_2, 0.000x_{17}, 6.042x_2, 7.106x_2, 9.730, 11.012x_2, 12.757
\end{align*} \]

\[ \Lambda_1 \approx -0.32137, \Lambda_2 = \Lambda_3 \approx -0.24369, \Lambda_4 = \Lambda_5 \approx 0.24012, \Lambda_6 \approx 0.32852 \]

\[ \Sigma_{+126;2+345;1} = \Sigma_{+136;1+245;2} \approx 0.17139, \Sigma_{+146;1+235;2} = \Sigma_{+156;2+234;1} \approx 0.52343 \]

---

T0964097 : \( \mathcal{P}/g^2 = -0.96409727, \ so(6) \rightarrow \emptyset \) \hfill (C.10)

\[ m^2/m_0^2[\psi] : 1.734x_4, 1.947x_4 \]
\[ m^2/m_0^2[\phi] : -5.922, -5.120, -5.100, -4.711, -4.163, -4.146, -4.097, -4.056, -4.040, -3.996, -3.943, -3.455, -3.450, -2.553, -0.456, 0.000_{18}, 0.137, 5.735, 5.821, 7.182, 7.379, 9.682, 10.417, 11.561, 13.019 \]

\[ \Lambda_1 \approx -0.32305, \Lambda_2 \approx -0.24370, \Lambda_3 \approx -0.23916, \Lambda_4 \approx 0.22698, \Lambda_5 \approx 0.24757, \Lambda_6 \approx 0.33137 \]

\[ \Sigma_{+136;2-245;1} \approx 0.21467, \Sigma_{+146;2+235;1} \approx -0.51776, \Sigma_{+156;1+234;2} \approx 0.54357 \]

---

**T0964525 :** \( \mathcal{P}/g^2 = -0.96452592, \ so(6) \rightarrow u(1) \) (C.11)

\[ m^2/m_0^2[\psi] : 1.798_{x4}, 1.899_{x4} \]

\[ m^2/m_0^2[\phi] : -5.981, -5.350_{x2}, -4.441_{x2}, -4.119_{x4}, -3.992_{x2}, -3.453_{x2}, -2.393, 0.000_{x16}, 0.275_{x4}, 5.108_{x2}, 7.699_{x2}, 9.546, 11.108_{x2}, 13.615 \]

\[ \Lambda_1 \approx -0.32814, \Lambda_2 = \Lambda_3 \approx -0.23391, \Lambda_4 = \Lambda_5 \approx 0.22848, \Lambda_6 \approx 0.33900 \]

\[ \Sigma_{+146;2+235;1} = \Sigma_{+156;1+234;2} \approx -0.55645 \]

---

**T0982778 :** \( \mathcal{P}/g^2 = -0.98277802, \ so(6) \rightarrow u(1) \) (C.12)

\[ m^2/m_0^2[\psi] : 1.630_{x4}, 2.222_{x4} \]

\[ m^2/m_0^2[\phi] : -6.371, -5.431, -5.333_{x2}, -4.114_{x2}, -4.000_{x2}, -3.277, -3.033_{x2}, -2.846, -1.903, -1.127_{x2}, 0.000_{x17}, 2.194_{x2}, 6.611, 7.033_{x2}, 9.333, 9.799, 11.714_{x2}, 14.085 \]

\[ \Lambda_1 = \Lambda_2 \approx -0.32349, \Lambda_3 \approx -0.19823, \Lambda_4 = \Lambda_5 \approx 0.25591, \Lambda_6 \approx 0.33339 \]

\[ \Sigma_{+135;2+246;1} = \Sigma_{+156;1+234;2} \approx 0.56869 \]

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**T1001482 :** \( \mathcal{P}/g^2 = -1.00148265, \ so(6) \rightarrow u(1) \) (C.13)

\[ m^2/m_0^2[\psi] : 1.744_{x4}, 2.188_{x4} \]
\[
m^2/m_0^2[\phi] : -6.436, -5.586_{\times 4}, -4.410_{\times 2}, -2.824, -2.719_{\times 2}, -2.306_{\times 4}, 0.000_{\times 16}, 3.171_{\times 2}, 5.354_{\times 2}, 7.693_{\times 4}, 8.788, 9.401, 11.392, 11.598
\]
\[
\Lambda_1 = \Lambda_2 \approx -0.34956, \Lambda_3 = \Lambda_4 \approx 0.15576, \Lambda_5 = \Lambda_6 \approx 0.19379
\]
\[
\Sigma +136;1+246;1 = \Sigma +146;1+235;2 \approx -0.63157
\]

\section*{C.14 \textbf{T1054687} :} \[ P/g^2 = -1.05468750, \ so(6) \rightarrow \emptyset \]
\[
m^2/m_0^2[\psi] : 1.733_{\times 2}, 2.326_{\times 6}
\]
\[
m^2/m_0^2[\phi] : -6.512, -5.251_{\times 2}, -4.161_{\times 3}, -4.062_{\times 2}, -2.925_{\times 2}, -1.560_{\times 3}, 0.000_{\times 18}, 4.490_{\times 2}, 6.697_{\times 3}, 10.600, 15.735_{\times 3}, 16.638_{\times 2}
\]
\[
\Lambda_1 = \Lambda_2 = \Lambda_3 \approx -0.37177, \Lambda_4 = \Lambda_5 = \Lambda_6 \approx 0.37177
\]
\[
\Sigma +124;1+356;2 \approx -0.40776, \Sigma +124;2+356;1 \approx -0.09457, \Sigma +125;2+346;1 \approx -0.20112,
\]
\[
\Sigma +126;1+345;2 \approx -0.07976, \Sigma +126;2+345;1 \approx 0.00066, \Sigma +134;1+256;2 \approx 0.04717, \Sigma +134;2+256;1 \approx 0.21114,
\]
\[
\Sigma +135;1+246;2 \approx 0.34833, \Sigma +135;2+246;1 \approx 0.16791, \Sigma +136;1+245;2 \approx -0.06930,
\]
\[
\Sigma +136;2+245;1 \approx -0.14455, \Sigma +145;1+236;2 \approx 0.43585, \Sigma +146;2+235;1 \approx -0.17902,
\]
\[
\Sigma +156;1+234;2 \approx -0.00305
\]

\section*{C.15 \textbf{T1073529} :} \[ P/g^2 = -1.07352975, \ so(6) \rightarrow \emptyset \]
\[
m^2/m_0^2[\psi] : 1.877_{\times 2}, 2.011_{\times 2}, 2.433_{\times 2}, 2.720_{\times 2}
\]
\[
m^2/m_0^2[\phi] : -6.536, -4.882_{\times 2}, -4.631_{\times 2}, -4.068_{\times 2}, -3.714_{\times 2}, -3.354, -2.637_{\times 2}, 0.000_{\times 17}, 1.273_{\times 2}, 5.470_{\times 2}, 8.041, 8.130_{\times 2}, 10.762, 14.391_{\times 2}, 19.457_{\times 2}, 21.617
\]
\[
\Lambda_1 = \Lambda_2 \approx -0.39487, \Lambda_3 \approx -0.35452, \Lambda_4 = \Lambda_5 \approx 0.34969, \Lambda_6 \approx 0.44490
\]
\[
\Sigma +124;2+356;1 \approx 0.47010, \Sigma +125;1+346;2 \approx -0.47010,
\]
\[
\Sigma +134;1+256;2 = \Sigma +146;2+235;1 = \Sigma +156;1+234;2 \approx -0.25201, \Sigma +135;2+246;1 \approx 0.25201
\]
\textbf{T1125000} : \( \mathcal{P}/g^2 = -1.12500000, \ \mathfrak{so}(6) \rightarrow \emptyset \) \hfill (C.16)

\( m^2/m_0^2[\psi] : 2.444_{\times 8} \)

\( m^2/m_0^2[\phi] : -7.325, -6.206, -4.769, -4.748_{\times 2}, -4.427, -3.550_{\times 2}, -3.280, -1.276, 0.000_{\times 18}, \\
4.806_{\times 2}, 4.883_{\times 2}, 5.333, 6.504, 7.861_{\times 2}, 10.829, 16.719, 18.634, 22.748_{\times 2}, 25.265 \)

\( \Lambda_1 \approx -0.48597, \Lambda_2 = \Lambda_3 \approx -0.39170, \Lambda_4 = \Lambda_5 \approx 0.39170, \Lambda_6 \approx 0.48597 \)

\( \Sigma_{+125;1+346;2} \approx 0.60475, \Sigma_{+134;2-256;1} \approx -0.60475 \)

\textbf{T1297247} : \( \mathcal{P}/g^2 = -1.29724786, \ \mathfrak{so}(6) \rightarrow \emptyset \) \hfill (C.17)

\( m^2/m_0^2[\psi] : 1.943_{\times 2}, 2.738_{\times 4}, 3.733_{\times 2} \)

\( m^2/m_0^2[\phi] : -5.445, -4.874, -4.060, -3.966, -3.942, -3.204, -3.186, -2.907, -1.014, -0.934, \\
0.000_{\times 18}, 6.983, 7.652, 11.889, 12.146, 12.543, 13.041, 13.946, 14.490, 18.175, 18.187, \\
21.001, 21.574, 22.590, 22.900 \)

\( \Lambda_1 = \Lambda_2 \approx -0.59541, \Lambda_3 = \Lambda_4 \approx 0.23362, \Lambda_5 = \Lambda_6 \approx 0.36178 \)

\( \Sigma_{+123;1+456;2} \approx 0.48716, \Sigma_{+124;2+356;1} \approx -0.48716, \Sigma_{+125;1+346;2} = \Sigma_{+126;2+345;1} \approx 0.04462, \\
\Sigma_{+134;1+256;2} = \Sigma_{+156;1+234;2} \approx -0.37352, \Sigma_{+135;2+246;1} \approx 0.28567, \Sigma_{+136;1+245;2} \approx 0.33709, \\
\Sigma_{+145;1+236;2} \approx -0.47047, \Sigma_{+146;2+235;1} \approx 0.03521 \)

\textbf{T1302912} : \( \mathcal{P}/g^2 = -1.30291232, \ \mathfrak{so}(6) \rightarrow \emptyset \) \hfill (C.18)

\( m^2/m_0^2[\psi] : 1.878_{\times 2}, 2.924_{\times 2}, 2.982_{\times 2}, 3.694_{\times 2} \)

\( m^2/m_0^2[\phi] : -5.530, -5.163, -4.601, -4.088, -3.938, -3.876, -2.975, -1.241, -0.535, 0.000_{\times 18}, \\
1.055, 10.323, 11.248, 11.304, 11.795, 12.471, 12.633, 15.614, 15.958, 16.392, 16.582, \\
22.633, 22.985, 23.633, 24.258 \)
\[ \Lambda_1^1 \approx -0.62113, \Lambda_2^2 \approx -0.58860, \Lambda_3^3 \approx 0.24575, \Lambda_4^4 \approx 0.29746, \Lambda_5^5 \approx 0.32510, \Lambda_6^6 \approx 0.34142 \]

\[ \Sigma_{+123;2-456;1} \approx 0.46285, \Sigma_{+124;1-356;2} \approx 0.38576, \Sigma_{+126;2+345;1} \approx -0.06801, \]

\[ \Sigma_{+135;1-246;2} \approx -0.73256, \Sigma_{+145;2-236;1} \approx -0.31760, \Sigma_{+156;1+234;2} \approx -0.44500 \]

| T1304606 : \[ \mathcal{P}/g^2 = -1.30460644, \mathfrak{so}(6) \rightarrow \emptyset \] | (C.19) |
|---|---|
| \( m^2/m_0^2[\psi] \) | 1.863_{x2}, 3.045_{x4}, 3.623_{x2} |
| \( m^2/m_0^2[\phi] \) | -5.709, -5.072, -4.820, -4.308, -4.051, -3.851, -2.894, -0.409, 0.000_{x18}, 0.705_{x2}, 10.707, 11.268, 12.495, 12.520_{x2}, 13.495, 14.491, 15.601_{x2}, 16.679, 22.729, 23.752_{x2}, 24.989 |
| \( \Lambda_1^1 = \Lambda_2^2 \approx -0.60876, \Lambda_3^3 = \Lambda_4^4 \approx 0.28534, \Lambda_5^5 = \Lambda_6^6 \approx 0.32343 \) |
| \( \Sigma_{+123;2-456;1} = \Sigma_{+124;1-356;2} \approx -0.41059, \Sigma_{+135;1-246;2} \approx 0.32857, \Sigma_{+145;2-236;1} \approx 0.87485 \) |

| T1319179 : \[ \mathcal{P}/g^2 = -1.31917968, \mathfrak{so}(6) \rightarrow \emptyset \] | (C.20) |
|---|---|
| \( m^2/m_0^2[\psi] \) | 2.007_{x2}, 2.444_{x2}, 3.209_{x2}, 3.479_{x2} |
| \( m^2/m_0^2[\phi] \) | -6.083_{x2}, -4.130, -3.789_{x2}, -3.689, -3.654_{x2}, -0.971, 0.000_{x17}, 0.946_{x2}, 8.035, 9.713, 10.759_{x2}, 11.497, 12.191, 12.714, 17.658_{x2}, 17.814, 21.132_{x2}, 23.045, 23.374 |
| \( \Lambda_1^1 = \Lambda_2^2 \approx -0.60188, \Lambda_3^3 = \Lambda_4^4 \approx 0.22066, \Lambda_5^5 = \Lambda_6^6 \approx 0.38122 \) |
| \( \Sigma_{+123;2-456;1} = \Sigma_{+124;1-356;2} \approx 0.59938, \Sigma_{+135;1-246;2} \approx 0.31540, \Sigma_{+136;2-245;1} \approx -0.00424, \Sigma_{+145;2-236;1} \approx 0.24695, \Sigma_{+146;1-235;2} \approx 0.55810 \) |

| T1382251 : \[ \mathcal{P}/g^2 = -1.38225189, \mathfrak{so}(6) \rightarrow \emptyset \] | (C.21) |
|---|---|
| \( m^2/m_0^2[\psi] \) | 2.392_{x4}, 3.822_{x2}, 4.380_{x2} |
| \( m^2/m_0^2[\phi] \) | -6.955, -5.615, -4.845, -3.442, -3.429, -3.048, -1.888, -0.873, 0.000_{x18}, 2.191, 2.392_{x4}, 3.822_{x2}, 4.380_{x2} |
\[\Lambda_1 = \Lambda_2 \approx -0.61986, \Lambda_3 = \Lambda_4 \approx 0.12499, \Lambda_5 \approx 0.49487\]

\[\Sigma_{+123;1+456;2} = \Sigma_{+124;2+356;1} \approx -0.57846, \Sigma_{+125;2-346;1} = \Sigma_{+126;1-345;2} \approx 0.13899,\]
\[\Sigma_{+134;1+256;2} = \Sigma_{+156;1+234;2} \approx -0.44722, \Sigma_{+135;1-246;2} \approx -0.37639, \Sigma_{+136;2-245;1} \approx 0.00529,\]
\[\Sigma_{+145;2-236;1} \approx 0.14221, \Sigma_{+146;1-235;2} \approx -0.12738\]

**T1391035 :** \[\mathcal{P}/g^2 = -1.39103566, \mathfrak{so}(6) \rightarrow \emptyset\] \hspace{1cm} (C.22)

\[m^2/m_0^2[\psi] : 2.615_{\times 4}, 3.670_{\times 4}\]

\[m^2/m_0^2[\phi] : -7.422, -5.873, -5.133, -3.439, -3.258, -3.039, -1.551, 0.000_{\times 18}, 0.990, 3.614,\]
\[5.111, 8.912, 9.520, 9.929, 11.031, 12.107, 15.840, 16.038, 16.254, 19.051, 20.981, 22.866,\]
\[23.627, 40.460, 40.498\]

\[\Lambda_1 \approx -0.65089, \Lambda_2 \approx -0.60760, \Lambda_3 \approx 0.08094, \Lambda_4 \approx 0.20776, \Lambda_5 \approx 0.48464, \Lambda_6 \approx 0.48515\]

\[\Sigma_{+123;1+456;2} \approx 0.61508, \Sigma_{+124;2+356;1} \approx -0.57701, \Sigma_{+126;1-345;2} \approx -0.20028,\]
\[\Sigma_{+134;2-256;1} \approx -0.56777, \Sigma_{+136;1+245;2} \approx 0.17976, \Sigma_{+146;2+235;1} \approx 0.41246\]

**T1416746 :** \[\mathcal{P}/g^2 = -1.41674647, \mathfrak{so}(6) \rightarrow \emptyset\] \hspace{1cm} (C.23)

\[m^2/m_0^2[\psi] : 2.545_{\times 4}, 4.675_{\times 4}\]

\[m^2/m_0^2[\phi] : -7.588, -6.331, -4.664, -3.637, -2.868, -2.381, -0.500, 0.000_{\times 18}, 2.018, 3.674,\]
\[5.078, 10.577, 11.395, 14.830, 15.208, 15.285, 17.454, 19.273, 19.969, 27.930, 27.963,\]
\[28.305, 28.585, 55.234, 55.252\]

\[\Lambda_1 \approx -0.73037, \Lambda_2 \approx -0.46700, \Lambda_3 \approx -0.03604, \Lambda_4 \approx 0.22540, \Lambda_5 \approx 0.50269, \Lambda_6 \approx 0.50532\]

\[\Sigma_{+123;1+456;2} \approx -0.63728, \Sigma_{+126;1-345;2} \approx -0.22177, \Sigma_{+134;2-256;1} \approx 0.88862,\]
\[\Sigma_{+135;2+246;1} \approx -0.10297, \Sigma_{+146;2+235;1} \approx 0.35261, \Sigma_{+156;2-234;1} \approx -0.20665\]
\textbf{T1417411}: \mathcal{P}/g^2 = -1.41741118, \mathfrak{so}(6) \to \emptyset \quad \text{(C.24)}

\begin{align*}
m^2/m_{0}[\psi] & : 2.667_{x4}, 4.444_{x4} \\
m^2/m_{0}[\phi] & : -7.719, -6.560, -4.692, -3.535, -2.851, -2.492, 0.000_{\times 18}, 0.276, 1.895, 4.301, 5.567, \\
 & \quad \quad 10.235, 11.353, 14.431, 15.458, 15.502, 17.694, 20.814, 22.838, 24.048, 25.549, 25.724, \\
 & \quad \quad 25.766, 53.186, 53.213
\end{align*}

\begin{align*}
\Lambda_1 & \approx -0.73052, \Lambda_2 \approx -0.47160, \Lambda_3 \approx -0.04027, \Lambda_4 \approx 0.24055, \Lambda_5 \approx 0.49948, \Lambda_6 \approx 0.50237 \\
\Sigma_{+123;2-456;1} & \approx -0.64026, \Sigma_{+126;2+345;1} \approx 0.23639, \Sigma_{+134;1+256;2} \approx -0.90336, \\
\Sigma_{+146;1-235;2} & \approx 0.36681
\end{align*}

\textbf{T1460654}: \mathcal{P}/g^2 = -1.46065435, \mathfrak{so}(6) \to \emptyset \quad \text{(C.25)}

\begin{align*}
m^2/m_{0}[\psi] & : 2.701_{x2}, 2.724_{x2}, 3.531_{x2}, 4.726_{x2} \\
m^2/m_{0}[\phi] & : -7.782, -7.781, -5.074, -3.077_{x2}, -2.429, -0.156, 0.000_{\times 18}, 5.812_{x2}, 8.206, 8.928, \\
 & \quad \quad 9.049, 11.748, 15.607_{x2}, 18.569, 18.574, 18.983, 23.132, 23.187, 26.681, 26.684, 44.562_{x2}
\end{align*}

\begin{align*}
\Lambda_1 & \approx -0.80626, \Lambda_2 \approx -0.17618, \Lambda_3 \approx 0.04003, \Lambda_4 \approx 0.04284, \Lambda_5 \approx 0.44978, \Lambda_6 \approx 0.44979 \\
\Sigma_{+123;2-456;1} & \approx -0.83799, \Sigma_{+124;1-356;2} \approx 0.83732, \Sigma_{+125;2-346;1} \approx -0.03678, \\
\Sigma_{+126;1-345;2} & \approx 0.03663, \Sigma_{+134;2-256;1} \approx -0.00494, \Sigma_{+135;1-246;2} \approx -0.22033, \\
\Sigma_{+136;2-245;1} & \approx -0.22065, \Sigma_{+145;2-236;1} \approx -0.21718, \Sigma_{+146;1-235;2} \approx 0.21663, \\
\Sigma_{+156;2-234;1} & \approx -0.13058
\end{align*}

\textbf{T1497042}: \mathcal{P}/g^2 = -1.49704248, \mathfrak{so}(6) \to \emptyset \quad \text{(C.26)}

\begin{align*}
m^2/m_{0}[\psi] & : 2.565_{x2}, 2.752_{x2}, 4.909_{x2}, 4.985_{x2} \\
m^2/m_{0}[\phi] & : -7.876, -7.552, -6.086, -2.529, -1.033, -0.827, 0.000_{\times 18}, 3.290, 4.630, 5.745, 7.840, \\
 & \quad \quad 8.291, 9.757, 11.599, 15.632, 18.415, 18.640, 21.703, 21.866, 28.340, 28.422, 33.168,
\end{align*}

34
33.169, 60.129, 60.135

\[ \Lambda_1 \approx -0.82463, \Lambda_2 \approx -0.19891, \Lambda_3^3 \approx 0.00718, \Lambda_4^4 \approx 0.00932, \Lambda_5^5 \approx 0.50354, \Lambda_6^6 \approx 0.50351 \]

\[ \Sigma_{+123;2-456;1} \approx 0.82383, \Sigma_{+124;1-356;2} \approx 0.81305, \Sigma_{+125;1+346;2} \approx -0.13392, \]
\[ \Sigma_{+136;1+245;2} \approx -0.28798, \Sigma_{+146;2+235;1} \approx -0.25309, \Sigma_{+156;2-234;1} \approx 0.38869 \]

\[
\text{T1499666 : } \mathcal{P}/g^2 = -1.49966681, \text{ so}(6) \rightarrow \emptyset \quad (C.27)
\]

\[ m^2/m_0^2[\psi] : 2.800_{\times4}, 4.665_{\times4} \]

\[ m^2/m_0^2[\phi] : -8.146, -7.859, -6.341, -2.381, -0.858, 0.000_{\times18}, 0.587, 3.588, 4.575, 6.215, 7.213, \\
8.309, 8.389, 11.542, 16.667, 18.400, 18.880, 22.379, 22.570, 25.467, 25.509, 31.179, \\
31.185, 57.469, 57.478 \]

\[ \Lambda_1 \approx -0.81651, \Lambda_2^2 \approx -0.22484, \Lambda_3^3 \approx -0.00212, \Lambda_4^4 \approx 0.03391, \Lambda_5^5 \approx 0.50465, \Lambda_6^6 \approx 0.50490 \]

\[ \Sigma_{+123;2-456;1} \approx 0.83606, \Sigma_{+124;1-356;2} \approx 0.83978, \Sigma_{+126;2+345;1} \approx -0.09844, \]
\[ \Sigma_{+135;2+246;1} \approx -0.29598, \Sigma_{+145;1+236;2} \approx 0.27814, \Sigma_{+156;2-234;1} \approx -0.26863 \]

\[
\text{T1501862 : } \mathcal{P}/g^2 = -1.50186250, \text{ so}(6) \rightarrow \emptyset \quad (C.28)
\]

\[ m^2/m_0^2[\psi] : 2.958_{\times4}, 4.361_{\times4} \]

\[ m^2/m_0^2[\phi] : -8.390, -8.149, -6.547, -2.243, 0.000_{\times18}, 0.534_{\times2}, 3.715, 4.659, 6.331, 6.591, 8.036_{\times2}, \\
11.530, 17.908, 19.043, 19.546, 22.488_{\times2}, 22.827_{\times2}, 28.910, 28.950, 54.551, 54.564 \]

\[ \Lambda_1 \approx -0.81048, \Lambda_2^2 \approx -0.24631, \Lambda_3^3 = \Lambda_4^4 \approx 0.02461, \Lambda_5^5 \approx 0.50349, \Lambda_6^6 \approx 0.50407 \]

\[ \Sigma_{+123;1+456;2} \approx 0.85496, \Sigma_{+124;2+356;1} \approx -0.85496, \Sigma_{+135;1-246;2} = \Sigma_{+145;2-236;1} \approx 0.30382 \]

\[
\text{T1510900 : } \mathcal{P}/g^2 = -1.51090053, \text{ so}(6) \rightarrow \emptyset \quad (C.29)
\]

\[ m^2/m_0^2[\psi] : 2.341_{\times2}, 2.536_{\times2}, 5.210_{\times2}, 5.535_{\times2} \]
\[ m^2/m_0^2[\phi] : \begin{align*} &-7.096, -6.092, -5.664, -3.912, -3.831, -1.977, 0.000_{18}, 4.992, 5.327, 5.471, 8.192, \\ &10.978, 11.323, 12.154, 15.682, 18.760, 19.789, 23.119, 24.250, 32.619, 32.622, 38.729, \\ &38.866, 54.809, 54.833 \end{align*} \]

\[ \Lambda_1 \approx -0.80362, \Lambda_2 \approx -0.54087, \Lambda_3^3 \approx 0.11891, \Lambda_4^4 \approx 0.28645, \Lambda_5^5 \approx 0.46652, \Lambda_6^6 \approx 0.47261 \]

\[ \Sigma_{+123;2-456;1} \approx -0.65751, \Sigma_{+126;1-345;2} \approx -0.15110, \Sigma_{+134;1+256;2} \approx -0.55854, \]

\[ \Sigma_{+135;2+246;1} \approx 0.32344, \Sigma_{+146;2+235;1} \approx 0.35501, \Sigma_{+156;1+234;2} \approx -0.74695 \]

**T1547778**: \( \mathcal{P}/g^2 = -1.54777840, \mathfrak{so}(6) \rightarrow \emptyset \)  \hspace{1cm} (C.30)

\[ m^2/m_0^2[\psi] : 2.919_{x2}, 3.511_{x4}, 4.696_{x2} \]

\[ m^2/m_0^2[\phi] : \begin{align*} &-7.690, -7.121, -5.196, -3.768, -3.229, 0.000_{18}, 2.491, 5.758, 5.882, 7.906_{x2}, \\ &11.876, 15.646, 16.511, 16.875_{x2}, 18.734, 19.897, 21.602_{x2}, 25.828, 34.397, 37.884_{x2}, \\ &40.118 \end{align*} \]

\[ \Lambda_1 = \Lambda_2 \approx -0.71461, \Lambda_3^3 \approx 0.13652, \Lambda_4^4 = \Lambda_5^5 \approx 0.41599, \Lambda_6^6 \approx 0.46074 \]

\[ \Sigma_{+123;1+456;2} \approx 0.70763, \Sigma_{+126;2+345;1} \approx -0.25491, \Sigma_{+134;1+256;2} = \Sigma_{+146;2+235;1} \approx 0.45958, \]

\[ \Sigma_{+135;2+246;1} = \Sigma_{+156;1+234;2} \approx -0.51427 \]

**T1738407**: \( \mathcal{P}/g^2 = -1.73840792, \mathfrak{so}(6) \rightarrow \emptyset \)  \hspace{1cm} (C.31)

\[ m^2/m_0^2[\psi] : 2.461_{x2}, 3.106_{x2}, 5.034_{x2}, 5.322_{x2} \]

\[ m^2/m_0^2[\phi] : \begin{align*} &-6.504_{x2}, -6.250_{x2}, -3.019, 0.000_{17}, 3.819_{x2}, 11.175, 11.214_{x2}, 13.297, 17.275_{x2}, \\ &21.000_{x2}, 25.218_{x2}, 26.699_{x2}, 28.635, 30.545_{x2}, 34.291_{x2}, 35.846 \end{align*} \]

\[ \Lambda_1 \approx -1.11580, \Lambda_2^2 \approx 0.20510, \Lambda_3^3 = \Lambda_4^4 \approx 0.22634, \Lambda_5^5 = \Lambda_6^6 \approx 0.22901 \]

\[ \Sigma_{+125;1+346;2} = \Sigma_{+126;2+345;1} \approx -0.80729, \Sigma_{+135;1-246;2} = \Sigma_{+145;2-236;1} = \Sigma_{+146;1-235;2} \approx 0.42803, \]

\[ \Sigma_{+136;2-245;1} \approx -0.42803 \]
D Scalar mass spectra for the classic vacua

In this appendix we collect results for the spectrum of scalar fluctuations around each of the five classic critical points found in [4].

The spectrum of scalar masses at the SO(6) point, T075000, in Table D.1 follows from $\mathcal{N} = 8$ supersymmetry. At the SU(2) $\times$ U(1) point, T083994, the full spectrum was computed and organized into multiplets of $\mathcal{N} = 2$ supersymmetry in [14] with the scalar masses given in Table D.4. Both points are perturbatively stable. The BF instability of the SU(3) point, T0843750, was established in [51] and subsequently confirmed in [52, 12]. The full scalar spectrum at this point, see also [13], is given in Table D.3. Finally, the scalar spectra at the the SO(5) point, T0780031, and the SU(2) $\times$ U(1)$^2$ point, T0870298, were computed in 1999 [53] and are given in Tables D.2 and D.5.
References

[1] M. Gunaydin, L. J. Romans, and N. P. Warner, *Gauged N=8 Supergravity in Five-Dimensions*, Phys. Lett. **154B** (1985) 268–274.

[2] M. Gunaydin, L. J. Romans, and N. P. Warner, *Compact and Noncompact Gauged Supergravity Theories in Five-Dimensions*, Nucl. Phys. **B272** (1986) 598–646.

[3] M. Pernici, K. Pilch, and P. van Nieuwenhuizen, *Gauged N=8 D=5 Supergravity*, Nucl. Phys. **B259** (1985) 460.

[4] A. Khavaev, K. Pilch, and N. P. Warner, *New vacua of gauged N=8 supergravity in five-dimensions*, Phys. Lett. **B487** (2000) 14–21, [hep-th/9810235].

[5] M. Cvetic, H. Lu, C. N. Pope, A. Sadrzadeh, and T. A. Tran, *Consistent SO(6) reduction of type IIB supergravity on S**5**, Nucl. Phys. **B586** (2000) 275–286, [hep-th/0003103].

[6] K. Pilch and N. P. Warner, *N=2 supersymmetric RG flows and the IIB dilaton*, Nucl. Phys. **B594** (2001) 209–228, [hep-th/0004063].

[7] K. Lee, C. Strickland-Constable, and D. Waldram, *Spheres, generalised parallelisability and consistent truncations*, Fortsch. Phys. **65** (2017), no. 10-11 1700048, [arXiv:1401.3360].

[8] A. Baguet, O. Hohm, and H. Samtleben, *Consistent Type IIB Reductions to Maximal 5D Supergravity*, Phys. Rev. **D92** (2015), no. 6 065004, [arXiv:1506.01385].

[9] I. M. Comsa, M. Firsching, and T. Fischbacher, *SO(8) Supergravity and the Magic of Machine Learning*, arXiv:1906.00207.

[10] B. de Wit and H. Nicolai, *N=8 Supergravity*, Nucl. Phys. **B208** (1982) 323.

[11] N. Bobev, T. Fischbacher, and K. Pilch, *Properties of the new N = 1 AdS4 vacuum of maximal supergravity*, JHEP **01** (2020) 099, [arXiv:1909.10969].

[12] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, *Novel local CFT and exact results on perturbations of N=4 superYang Mills from AdS dynamics*, JHEP **12** (1998) 022, [hep-th/9810126].

[13] J. Distler and F. Zamora, *Nonsupersymmetric conformal field theories from stable anti-de Sitter spaces*, Adv. Theor. Math. Phys. **2** (1999) 1405–1439, [hep-th/9810206].

[14] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, *Renormalization group flows from holography supersymmetry and a c theorem*, Adv. Theor. Math. Phys. **3** (1999) 363–417, [hep-th/9904017].

[15] G. W. Gibbons, C. M. Hull, and N. P. Warner, *The Stability of Gauged Supergravity*, Nucl. Phys. **B218** (1983) 173.

[16] P. Breitenlohner and D. Z. Freedman, *Stability in Gauged Extended Supergravity*, Annals Phys. **144** (1982) 249.

[17] H. Ooguri and C. Vafa, *Non-supersymmetric AdS and the Swampland*, Adv. Theor. Math. Phys. **21** (2017) 1787–1801, [arXiv:1610.01533].
[18] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, JHEP **06** (2007) 060, [hep-th/0601001].

[19] T. Fischbacher, *The Encyclopedic Reference of Critical Points for SO(8)-Gauged N=8 Supergravity. Part 1: Cosmological Constants in the Range -\(\Lambda/g^2\) ∈ [6 : 14.7], arXiv:1109.1424.

[20] R. G. Leigh and M. J. Strassler, *Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory*, Nucl. Phys. **B447** (1995) 95–136, [hep-th/9503121].

[21] N. P. Warner, *Some Properties of the Scalar Potential in Gauged Supergravity Theories*, Nucl. Phys. **B231** (1984) 250–268.

[22] T. Fischbacher, K. Pilch, and N. P. Warner, *New Supersymmetric and Stable, Non-Supersymmetric Phases in Supergravity and Holographic Field Theory*, arXiv:1010.4910.

[23] I. Bena, K. Pilch, and N. P. Warner, *Brane-Jet Instabilities*, arXiv:2003.02851.

[24] E. Malek, H. Nicolai, and H. Samtleben. In preparation, 2020.

[25] T. Fischbacher, H. Nicolai, and H. Samtleben, *Vacua of maximal gauged D = 3 supergravities*, Class. Quant. Grav. **19** (2002) 5297–5334, [hep-th/0207206].

[26] T. Fischbacher, *The Many vacua of gauged extended supergravities*, Gen. Rel. Grav. **41** (2009) 315–411, [arXiv:0811.1915].

[27] N. P. Warner, *Some New Extrema of the Scalar Potential of Gauged N = 8 Supergravity*, Phys. Lett. **128B** (1983) 169–173.

[28] T. Fischbacher, *Fourteen new stationary points in the scalar potential of SO(8)-gauged N=8, D=4 supergravity*, JHEP **09** (2010) 068, [arXiv:0912.1636].

[29] A. Borghese, A. Guarino, and D. Roest, *Triality, Periodicity and Stability of SO(8) Gauged Supergravity*, JHEP **05** (2013) 107, [arXiv:1302.6057].

[30] K. Pilch and N. P. Warner, *N=1 supersymmetric renormalization group flows from IIB supergravity*, Adv. Theor. Math. Phys. **4** (2002) 627–677, [hep-th/0006066].

[31] N. Bobev, H. Elvang, U. Kol, T. Olson, and S. S. Pufu, *Holography for \(\mathcal{N} = 1^3\) on \(S^4\), JHEP **10** (2016) 095, [arXiv:1605.00656].

[32] T. Fischbacher, *Numerical tools to validate stationary points of SO(8)-gauged N=8 D=4 supergravity*, Comput. Phys. Commun. **183** (2012) 780–784, [arXiv:1007.0600].

[33] M. Abadi, P. Barham, J. Chen, Z. Chen, A. Davis, J. Dean, M. Devin, S. Ghemawat, G. Irving, M. Isard, M. Kudlur, J. Levenberg, R. Monga, S. Moore, D. G. Murray, B. Steiner, P. Tucker, V. Vasudevan, P. Warden, M. Wicke, Y. Yu, and X. Zheng, *TensorFlow: A system for large-scale machine learning*, in 12th USENIX Symposium on Operating Systems Design and Implementation (OSDI 16), pp. 265–283, 2016.

[34] A. Khavaev and N. P. Warner, *A Class of N=1 supersymmetric RG flows from*
five-dimensional $N=8$ supergravity, Phys. Lett. B495 (2000) 215–222, [hep-th/0009159].

35. T. Fischbacher, “Studying M-Theory Spontaneous Symmetry Breaking with Machine Learning Tools,” Seminar at ETH, Zurich, November 4, 2019. Seminar at Erlangen University, November 27, 2019.

36. K. Pilch, “AdS vacua of maximal supergravities.” Seminar at IPhT Saclay, February 5, 2020. Lecture at the Southwest Strings Meeting 2020, Utah State University, February 14-15, 2020.

37. C. Krishnan, V. Mohan, and S. Ray, Machine Learning $\mathcal{N} = 8, D = 5$ Gauged Supergravity, arXiv:2002.12927.

38. L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, The Supergravity dual of $N=1$ superYang-Mills theory, Nucl. Phys. B569 (2000) 451–469, [hep-th/9909047].

39. N. Bobev, A. Kundu, K. Pilch, and N. P. Warner, Supersymmetric Charged Clouds in $AdS_5$, JHEP 03 (2011) 070, [arXiv:1005.3552].

40. F. Aprile, D. Roest, and J. G. Russo, Holographic Superconductors from Gauged Supergravity, JHEP 06 (2011) 040, [arXiv:1104.4473].

41. N. Bobev, H. Elvang, D. Z. Freedman, and S. S. Pufu, Holography for $N = 2^*$ on $S^4$, JHEP 07 (2014) 001, [arXiv:1311.1508].

42. N. Bobev, K. Pilch, and O. Vasilakis, $(0, 2)$ SCFTs from the Leigh-Strassler fixed point, JHEP 06 (2014) 094, [arXiv:1403.7131].

43. G. Dall’Agata and G. Inverso, On the Vacua of $N = 8$ Gauged Supergravity in 4 Dimensions, Nucl. Phys. B859 (2012) 70–95, [arXiv:1112.3345].

44. J. Bradbury, R. Frostig, P. Hawkins, M. J. Johnson, C. Leary, D. Maclaurin, and S. Wanderman-Milne, JAX: composable transformations of Python+NumPy programs, v 0.1.55, 2018. http://github.com/google/jax.

45. V. Gorbenko, S. Rychkov, and B. Zan, Walking, Weak first-order transitions, and Complex CFTs, JHEP 10 (2018) 108, [arXiv:1807.11512].

46. A. F. Faedo, C. Hoyos, D. Mateos, and J. G. Subils, Holographic Complex CFTs, arXiv:1909.04008.

47. A. Donos, J. P. Gauntlett, C. Rosen, and O. Sosa-Rodriguez, Boomerang RG flows with intermediate conformal invariance, JHEP 04 (2018) 017, [arXiv:1712.08017].

48. N. Bobev, F. F. Gautason, B. E. Niehoff, and J. van Muiden, A holographic kaleidoscope for $\mathcal{N} = 1^*$, JHEP 10 (2019) 185, [arXiv:1906.09270].

49. N. Bobev, N. Halmagyi, K. Pilch, and N. P. Warner, Supergravity Instabilities of Non-Supersymmetric Quantum Critical Points, Class. Quant. Grav. 27 (2010) 235013, [arXiv:1006.2546].

50. B. de Wit, H. Samtleben, and M. Trigiante, On Lagrangians and gaugings of maximal
supergravities, \textit{Nucl. Phys.} \textbf{B655} (2003) 93–126, [hep-th/0212239].

[51] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, “Private communication with J. Distler and F. Zamora.” Unpublished, 1999.

[52] J. Distler and F. Zamora, \textit{Chiral symmetry breaking in the AdS / CFT correspondence}, \textit{JHEP} \textbf{05} (2000) 005, [hep-th/9911040].

[53] K. Pilch, “\textit{Notes on perturbative instability of the SO(5), SU(2)×U(1)×U(1), and SU(3) AdS}$_5$ \textit{vacua}.” Unpublished, 1999.