LANGLANDS’ LAMBDA FUNCTION FOR QUADRATIC TAMELY RAMIFIED EXTENSIONS

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Abstract. Let $K/F$ be a quadratic tamely ramified extension of a non-Archimedean local field $F$ of characteristic zero. In this paper, we give an explicit formula for Langlands’ lambda function $\lambda_{K/F}$.

1. Introduction

Let $K/F$ be a finite subextension (but need not be Galois) in $\overline{F}/F$, where $\overline{F}$ is an algebraic closure of a non-Archimedean local field $F$ of characteristic zero. Let $\psi$ be a nontrivial additive character of $F$. Then Langlands’s lambda function (or simply $\lambda$-function) (cf. [5]) of the extension $K/F$ is:

$$
\lambda_{K/F}(\psi) := W(\text{Ind}_{G_K}^{G_F}(1_K), \psi),
$$

where $1_K$ is the trivial representation of $G_K := \text{Gal}(\overline{F}/K)$. Here $W$ denotes for local constant (or epsilon factor) (cf. [10]). We also can define the $\lambda$-function via Deligne’s constant $c(\rho) := \frac{W(\rho)}{W(\det(\rho))}$, where $\rho$ is a finite dimensional representation of $G_F$ and $\det(\rho)$ is the determinant of $\rho$.

Langlands has shown (cf. Theorem 1 on p. 105 of [10]) that the local constants are weakly extendible functions. Therefore, to compute the local constant of any induced local Galois representation, we have to compute the $\lambda$-function explicitly. Since the local Langlands correspondence preserves local constants, the explicit computation of local constants is an important part of the Langlands program.

In [8], Saito has computed the $\lambda$-function for an arbitrary extension assuming the residual characteristic of the base field is not equal to 2 (cf. Theorem on p. 508 of [8]). In Theorem II 2B on p. 508 of [8], when ramification index is even, Saito has computed the lambda functions for even degree extensions via the Legendre symbol and Hilbert symbol.

In this paper, we also compute this $\lambda$-functions for quadratic tamely ramified extensions. In our computation, we use the classical quadratic Gauss sums and these computations are different from the Saito’s result and explicit. The main idea for tamely ramified quadratic extension case is to reduce the $\lambda$-functions computation to the classical quadratic Gauss sums computations.

If $K/F$ is a tamely ramified quadratic extension, by using classical Gauss sum we have the following explicit formula for $\lambda_{K/F}$.

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Theorem 1.1. Let $K$ be a tamely ramified quadratic extension of $F/\mathbb{Q}_p$ with $q_F = p^r$. Let $\psi_F$ be the canonical additive character of $F$. Let $c \in F^\times$ with $-1 = \nu_F(c) + d_F/\mathbb{Q}_p$, and $c' = \frac{c}{\sqrt[p]{F/F_0}(pc)}$, where $F_0/\mathbb{Q}_p$ is the maximal unramified extension in $F/\mathbb{Q}_p$. Let $\psi_1 = c' \cdot \psi_F$, then
\[
\lambda_{K/F}(\psi_F) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_{-1}),
\]
where
\[
\lambda_{K/F}(\psi_{-1}) = \begin{cases}
(-1)^{s-1} & \text{if } p \equiv 1 \pmod{4} \\
(-1)^{s-1}i^s & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]
If we take $c = \pi_F^{-1-d_F/\mathbb{Q}_p}$, where $\pi_F$ is a norm for $K/F$, then
\[
\Delta_{K/F}(c') = \begin{cases}
1 & \text{if } \sqrt[p]{F/F_0}(pc) \in k_F^s = k_F^P \text{ is a square}, \\
-1 & \text{if } \sqrt[p]{F/F_0}(pc) \in k_F^s = k_F^\times \text{ is not a square}.
\end{cases}
\]

Remark 1.2. But in general, computation of $\lambda_{K/F}$, where $K/F$ is a wildly ramified quadratic extension, seems subtle. When $F = \overline{\mathbb{Q}}_2$, in Example 3.4.14, pp. 60-63 of [1], we have explicit computation for $\lambda_{K/\mathbb{Q}_2}$. In [7], one also can find some particular cases (cf. on pp. 252-255 of [7]). But if $F/\mathbb{Q}_2$ is an arbitrary finite extension and $K/F$ is quadratic extension, then computation of $\lambda_{K/F}$ is still open. And if $K$ is an abelian extension with $N_{K/F}(K^\times) = F^{\times 2}$, we have the following theorem.

Theorem 1.3. Let $F$ be an extension of $\mathbb{Q}_2$. Let $K$ be the abelian extension for which $N_{K/F}(K^\times) = F^{\times 2}$. Then $\lambda_{K/F} = 1$.

2. Notations and Preliminaries

Let $F$ be a non-Archimedean local field of characteristic zero, i.e., a finite extension of the field $\mathbb{Q}_p$ (field of $p$-adic numbers), where $p$ is a prime. Let $O_F$ be the ring of integers in the local field $F$ and $P_F = \pi_F O_F$ is the unique prime ideal in $O_F$ and $\pi_F$ is a uniformizer, i.e., an element in $P_F$ whose valuation is one, i.e., $\nu_F(\pi_F) = 1$. Let $q_F$ be the cardinality of the residue field $k_F$ of $F$. Let $U_F = O_F - P_F$ be the group of units in $O_F$. Let $P_F^i = \{x \in F : \nu_F(x) \geq i\}$ and for $i \geq 0$ define $U_F^i = 1 + P_F^i$ (with proviso $U_F^0 = U_F = O_F^\times$). We also consider that $a(\chi)$ is the conductor of nontrivial character $\chi : F^\times \rightarrow \mathbb{C}^\times$, i.e., $a(\chi)$ is the smallest integer $m \geq 0$ such that $\chi$ is trivial on $U_F^m$. We say $\chi$ is unramified if the conductor of $\chi$ is zero and otherwise ramified. Throughout the paper, when $K/F$ is unramified we choose uniformizers $\pi_K = \pi_F$. And when $K/F$ is ramified (both tame and wild) we choose uniformizers $\pi_K = N_{K/F}(\pi_K)$, where $N_{K/F}$ is the norm map from $K^\times$ to $F^\times$. In this paper $\Delta_{K/F} := \det(\text{Ind}_{K/F}(1))$.

The conductor of any nontrivial additive character $\psi$ of the field $F$ is an integer $n(\psi)$ if $\psi$ is trivial on $P_F^r$, but nontrivial on $P_F^{n(\psi)-1}$.
2.1. Local constant formula for character. For a nontrivial multiplicative character $\chi$ of $F^\times$ and nontrivial additive character $\psi$ of $F$, we have (cf. [10], p. 94):

\begin{equation}
W(\chi, \psi) = \chi(c) q_F^{-(a(\chi)/2 - a(\psi)/2)} \sum_{x \in \frac{U_F}{U^\times}} \chi^{-1}(x) \psi(x/c),
\end{equation}

where $c = \pi_F^{\alpha(\chi) + n(\psi)}$.

2.2. Classical Gauss sums. Let $k_q$ be a finite field. Let $p$ be the characteristic of $k_q$; then the prime field contained in $k_q$ is $k_p$. The structure of the canonical additive character $\psi_q$ of $k_q$ is the same as the structure of the canonical (cf. [10], p. 92) character $\psi_F$, namely it comes by trace from the canonical character of the base field, i.e.,

$$\psi_q = \psi_p \circ \text{Tr}_{k_q/k_p},$$

where

$$\psi_p(x) := e^{2 \pi i x/p} \text{ for all } x \in k_p.$$

**Gauss sums:** Let $\chi, \psi$ be a multiplicative and an additive character respectively of $k_q$. Then the Gauss sum $G(\chi, \psi)$ is defined by

\begin{equation}
G(\chi, \psi) = \sum_{x \in k_q^\times} \chi(x) \psi(x).
\end{equation}

For computation of $\lambda_{K/F}$, where $K/F$ is a tamely ramified quadratic extension, we will use the following theorem.

**Theorem 2.1** ([6], p. 199, Theorem 5.15). Let $k_q$ be a finite field with $q = p^s$, where $p$ is an odd prime and $s \in \mathbb{N}$. Let $\chi$ be the quadratic character of $k_q$ and let $\psi$ be the canonical additive character of $k_q$. Then

\begin{equation}
G(\chi, \psi) = \begin{cases} 
(-1)^{s-1}q^{\frac{1}{2}} & \text{if } p \equiv 1 \pmod{4}, \\
(-1)^{s-1-i^*q^{\frac{1}{2}}} & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\end{equation}

3. Explicit computation of $\lambda_{K/F}$, where $K/F$ is a quadratic extension

Let $K/F$ be a quadratic extension of the field $F/\mathbb{Q}_p$. Let $G = \text{Gal}(K/F)$ be the Galois group of the extension $K/F$. Let $t$ be the ramification break or jump (cf. [9]) of the Galois group $G$ (or of the extension $K/F$). Then it can be proved that the conductor of $\omega_{K/F}$ (a quadratic character of $F^\times$ associated to $K$ by class field theory) is $t + 1$. When $K/F$ is unramified we have $t = -1$, therefore the conductor of a quadratic character $\omega_{K/F}$ of $F^\times$ is zero, i.e., $\omega_{K/F}$ is unramified. And when $K/F$ is tamely ramified we have $t = 0$, then $a(\omega_{K/F}) = 1$. In the wildly ramified case (which occurs if $p = 2$) it can be proved that $a(\omega_{K/F}) = t + 1$ is, **up to the exceptional case** $t = 2 \cdot e_{F/\mathbb{Q}_2}$, always an **even number** which can be seen by the filtration of $F^\times$ (cf. p. 50 of [1]).
3.1. Computation of $\lambda_{K/F}$, where $K/F$ is a tamely ramified quadratic extension. The existence of a tamely ramified quadratic character (which is not unramified) of a local field $F$ implies $p \neq 2$ for the residue characteristic. Then

$$F^\times / F^{\times 2} \cong V$$

is isomorphic to Klein’s 4-group. So we have only 3 nontrivial quadratic characters in that case, corresponding to 3 quadratic extensions $K/F$. One is unramified and other two are ramified. The unramified case is well settled. The two ramified quadratic characters determine two different quadratic ramified extensions of $F$.

In the ramified case we have $a(\chi) = 1$ because it is tame, and we take $\psi$ of conductor $-1$. Then we have $a(\chi) + n(\psi) = 0$ and therefore in the formula of $W(\chi, \psi)$ (cf. equation (2.1)) we can take $c = 1$. So we obtain:

$$W(\chi, \psi) = q_F^{-\frac{1}{2}} \sum_{x \in U_F / U_1^F} \chi^{-1}(x) \psi(x) = q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} \chi^{-1}(x) \overline{\psi}(x),$$

where $\chi$ is the quadratic character of the residue field $k_F^\times$, and $\overline{\psi}$ is an additive character of $k_F$. When $n(\psi) = -1$, we observe that both the ramified characters $\chi$ give the same $W(\chi, \psi)$, because one is different from other by a quadratic unramified character twist. To compute an explicit formula for $\lambda_{K/F}(\psi_{-1})$, where $K/F$ is a tamely ramified quadratic extension and $\psi_{-1}$ is an additive character of $F$ with conductor $-1$, we need to use classical quadratic Gauss sums.

Let $\psi_{-1}$ be an additive character of $F/\mathbb{Q}_p$ of conductor $-1$, i.e., $\psi_{-1} : F/P_F \rightarrow \mathbb{C}^\times$. Now restrict $\psi_{-1}$ to $O_F$, it will be one of the characters $a \cdot \psi_{q_F}$, for some $a \in k_F^\times$ and usually it will not be $\psi_{q_F}$ itself. Therefore, choosing $\psi_{-1}$ is very important and we have to choose $\psi_{-1}$ such a way that its restriction to $O_F$ is exactly $\psi_{q_F}$. Then we will be able to use the quadratic classical Gauss sum in the $\lambda$-function computation. We also know that there exists an element $c \in F^\times$ such that

$$\psi_{-1} = c \cdot \psi_F$$

induces the canonical character $\psi_{q_F}$ on the residue field $k_F$.

Now question is: Finding proper $c \in F^\times$ for which $\psi_{-1}|_{O_F} = c \cdot \psi_F|_{O_F} = \psi_{q_F}$, i.e., the canonical character of the residue field $k_F$.

From the definition of conductor of the additive character $\psi_{-1}$ of $F$, we obtain from the construction (3.2)

$$-1 = \nu_F(c) + n(\psi_F) = \nu_F(c) + d_{F/\mathbb{Q}_p},$$

where $d_{F/\mathbb{Q}_p}$ is the exponent of the different $D_{F/\mathbb{Q}_p}$. In the next two lemmas we choose the proper $c$ for our requirement.

**Lemma 3.1.** Let $F/\mathbb{Q}_p$ be a local field and let $\psi_{-1}$ be an additive character of $F$ of conductor $-1$. Let $\psi_F$ be the canonical character of $F$. Let $c \in F^\times$ be any element such that $-1 = \nu_F(c) + d_{F/\mathbb{Q}_p}$, and

$$Tr_{F/F_0}(c) = \frac{1}{p},$$

where
where $F_0/Q_p$ is the maximal unramified subextension in $F/Q_p$. Then the restriction of $\psi_{-1} = c \cdot \psi_F$ to $O_F$ is the canonical character $\psi_{q_p}$ of the residue field $k_F$ of $F$.

**Proof.** Since $F_0/Q_p$ is the maximal unramified subextension in $F/Q_p$, we have $\pi_{F_0} = p$, and the residue fields of $F$ and $F_0$ are isomorphic, i.e., $k_{F_0} \cong k_F$, because $F/F_0$ is totally ramified extension. Then every element of $O_F/P_F$ can be considered as an element of $O_{F_0}/P_{F_0}$. Moreover, since $F_0/Q_p$ is the maximal unramified extension, then from Proposition 2 of [11] on p. 140, for $x \in O_{F_0}$ we have

$$\rho_p(\text{Tr}_{F_0/Q_p}(x)) = \text{Tr}_{k_{F_0}/k_{Q_p}}(\rho_0(x)),$$

where $\rho_0, \rho_p$ are the canonical homomorphisms of $O_{F_0}$ onto $k_{F_0}$, and of $O_{Q_p}$ onto $k_{Q_p}$, respectively. Then for $x \in k_{F_0}$ we can write

$$\text{Tr}_{F_0/Q_p}(x) = \text{Tr}_{k_{F_0}/k_{Q_p}}(x).$$

Furthermore, since $F/F_0$ is totally ramified, we have $k_F = k_{F_0}$, then the trace map for the tower of the residue fields $k_F/k_{F_0}/k_{Q_p}$ is:

$$\text{Tr}_{k_F/k_{Q_p}}(x) = \text{Tr}_{k_{F_0}/k_{Q_p}} \circ \text{Tr}_{k_F/k_{F_0}}(x) = \text{Tr}_{k_{F_0}/k_{Q_p}}(x),$$

for all $x \in k_F$. Then from the equations (3.5) and (3.6) we obtain

$$\text{Tr}_{F_0/Q_p}(x) = \text{Tr}_{k_F/k_{Q_p}}(x)$$

for all $x \in k_F$.

Since the conductor of $\psi_{-1}$ is $-1$, for $x \in O_F/P_F(= O_{F_0}/P_{F_0}$ because $F/F_0$ is totally ramified) we have

$$\psi_{-1}(x) = c \cdot \psi_F(x) = \psi_F(cx) = \psi_{Q_p}(\text{Tr}_{F_0/Q_p}(cx)) = \psi_{Q_p}(\text{Tr}_{F_0/Q_p} \circ \text{Tr}_{F_0}(cx))$$

$$= \psi_{Q_p}(\text{Tr}_{F_0/Q_p}(x \cdot \text{Tr}_{F_0}(c)))$$

$$= \psi_{Q_p}(\text{Tr}_{F_0/Q_p}(1/x)),$$

since $x \in O_F/P_F = O_{F_0}/P_{F_0}$ and $\text{Tr}_{F_0}(c) = 1/p$,

$$= \psi_{Q_p}(\text{Tr}_{F_0/Q_p}(x)),$$

because $1/p \in Q_p$

$$= e^{2\pi i \text{Tr}_{F_0/Q_p}(x)/p},$$

because $\psi_{Q_p}(x) = e^{2\pi i x}$

$$= e^{2\pi i \psi_{q_p}(x)/p},$$

using equation (3.7)

This competes the lemma.

□

The next step is to produce good elements $c$ more explicitly. By using Lemma 3.1 in the next lemma we see more general choices of $c$.

**Lemma 3.2.** Let $F/Q_p$ be a tamely ramified local field and let $\psi_{-1}$ be an additive character of $F$ of conductor $-1$. Let $\psi_F$ be the canonical character of $F$. Let $F_0/Q_p$ be the maximal unramified subextension in $F/Q_p$. Let $c \in F^\times$ be any element such that $-1 = \nu_F(c) + d_{F/Q_p}$, then
\[
\psi = \frac{c}{\text{Tr}_{F/F_0}(pc)},
\]
fulfills conditions (3.3), (3.4), and hence \(\psi_{-1}|_{O_F} = \psi \cdot \psi_F|_{O_F} = \psi_{q_F} \).

**Proof.** By the given condition we have \(\nu_F(c) = -1 - d_{F/Q_p} = -1 - (e_{F/Q_p} - 1) = -e_{F/Q_p} \).

Then we can write \(c = \pi^{-e_{F/Q_p}} u(c) = p^{-1} u(c) \) for some \(u(c) \in U_F \) because \(F/Q_p\) is tamely ramified, hence \(p = \pi^{-e_{F/Q_p}} \). Then we can write

\[
\text{Tr}_{F/F_0}(pc) = p \cdot \text{Tr}_{F/F_0}(c) = p \cdot p^{-1} u(c) = u_0(c) \in U_{F_0} \subset U_F,
\]
where \(u_0(c) = \text{Tr}_{F/F_0}(u(c)) \), hence \(\nu_F(\text{Tr}_{F/F_0}(pc)) = 0 \). Then the valuation of \(c'\) is:

\[
\nu_F(c') = \nu_F(\frac{c}{\text{Tr}_{F/F_0}(pc)}) = \nu_F(c) - \nu_F(\text{Tr}_{F/F_0}(pc)) = \nu_F(c) - 0 = \nu_F(c) = -1 - d_{F/Q_p}.
\]

Since \(\text{Tr}_{F/F_0}(pc) = u_0(c) \in U_{F_0} \), we have

\[
\text{Tr}_{F/F_0}(c') = \text{Tr}_{F/F_0}(\frac{c}{\text{Tr}_{F/F_0}(pc)}) = \frac{1}{\text{Tr}_{F/F_0}(pc)} \cdot \text{Tr}_{F/F_0}(c) = \frac{1}{p \text{Tr}_{F/F_0}(c)} \cdot \text{Tr}_{F/F_0}(c) = \frac{1}{p}
\]

Thus we observe that here \(c' \in F^\times\) satisfies equations (3.3) and (3.4). Therefore, from Lemma 3.1 we can see that \(\psi_{-1}|_{O_F} = c' \cdot \psi_F|_{O_F}\) is the canonical additive character of \(k_F\). \(\square\)

By Lemmas 3.1 and 3.2 we get many good (in the sense that \(\psi_{-1}|_{O_F} = c' \cdot \psi_F|_{O_F} = \psi_{q_F}\)) elements \(c\) which we will use in our next theorem to calculate \(\lambda_{K/F}\), where \(K/F\) is a tamely ramified quadratic extension.

**Proof of Theorem 1.1** From [2], p. 190, part (2) of the Proposition, we have

\[
\lambda_{K/F}(\psi) = \lambda_{K/F}(c' \psi_F) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_F).
\]

Since \(\Delta_{K/F} = \Delta_{K/F}^{-1}\), we can write \(\Delta_{K/F} = \Delta_{K/F}^{-1}\). So we obtain

\[
\lambda_{K/F}(\psi) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_{-1}).
\]

Now we have to compute \(\lambda_{K/F}(\psi_{-1})\), and which we do in the following:

Since \([K : F] = 2\), we have \(\text{Ind}_{K/F}(1) = 1_F \oplus \omega_{K/F}\). The conductor of \(\omega_{K/F}\) is 1 because \(K/F\) is a tamely ramified quadratic extension, and hence \(t = 0\), so \(a(\omega_{K/F}) = t + 1 = 1\). Therefore, we can consider \(\omega_{K/F}\) as a character of \(F^\times/U_F^1\). So the restriction of \(\omega_{K/F}\) to \(U_F\), \(\text{res}(\omega_{K/F}) := \omega_{K/F}|_{U_F}\), we may consider as the uniquely determined character of \(k_F^\times\) of order 2. Since \(c'\) satisfies equations (3.3), (3.4), then from Lemma 3.2 we have \(\psi_{-1}|_{O_F} = c' \cdot \psi_F|_{O_F} = \psi_{q_F}\), and this is the canonical character of \(k_F\). Then from equation (3.1) we can write

\[
\lambda_{K/F}(\psi_{-1}) = q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} \text{res}(\omega_{K/F})(x) \psi_{q_F}(x)
\]

Moreover, by Theorem 2.1 we have

\[
G(\text{res}(\omega_{K/F}), \psi_{q_F}) = \begin{cases} (-1)^{s-1} q_F^{\frac{1}{2}} & \text{if } p \equiv 1 \pmod{4} \\ (-1)^{s-1} i q_F^{\frac{1}{2}} & \text{if } p \equiv 3 \pmod{4} \end{cases}
\]

Thus \(\lambda_{K/F}(\psi_{-1}) = q_F^{-\frac{1}{2}} \cdot G(\text{res}(\omega_{K/F}), \psi_{q_F})\).
By using the classical quadratic Gauss sum we obtain

\begin{equation}
\lambda_{K/F}(\psi_{-1}) = \begin{cases} 
(-1)^{s-1} & \text{if } p \equiv 1 \pmod{4} \\
(-1)^{s-1}i^s & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\end{equation}

We also can write \( \Delta_{K/F} = \det(\text{Ind}_{K/F}(1)) = \det(1_F \oplus \omega_{K/F}) = \omega_{K/F} \). So we have

\[ \Delta_{K/F}(\pi_F) = \omega_{K/F}(\pi_F) = 1, \]

because \( \pi_F \in N_{K/F}(K^\times) \).

Under the assumption of the Theorem 1.1 we have \( \pi_F \in N_{K/F}(K^\times) \), \( \Delta_{K/F} = \omega_{K/F} \) and \( c' = \frac{c}{\text{Tr}_{F/F_0}(pc)} \), where \( c \in F^\times \) with \( \nu_F(c) = -1 - d_{F/Q_p} \). Then we can write

\[ \Delta_{K/F}(c') = \omega_{K/F}(c') = \omega_{K/F} \left( \frac{c}{\text{Tr}_{F/F_0}(pc)} \right) \]

\[ = \omega_{K/F} \left( \frac{\pi_F^{-e_{F/Q_p}} u(c)}{u_0(c)} \right), \quad \text{where } c = \pi_F^{-e_{F/Q_p}} u(c), \text{Tr}_{F/F_0}(pc) = u_0(c) \in U_{F_0} \]

\[ = \omega_{K/F}(\pi_F^{-e_{F/Q_p}}) \omega_{K/F}(v), \quad \text{where } v = \frac{u(c)}{u_0(c)} \in U_F \]

\[ = \omega_{K/F}(x) \]

\[ = \begin{cases} 
1 & \text{when } x \text{ is a square element in } k_F^\times \\
-1 & \text{when } x \text{ is not a square element in } k_F^\times,
\end{cases} \]

where \( v = xy \), with \( x = (\omega_{K/F}, c) \in U_F/U_F^1 \), and \( y \in U_F^1 \).

In particular, if we choose \( c \) such a way that \( u(c) = 1 \), i.e., \( c = \pi_F^{-1-d_{F/Q_p}} \), then we have \( \Delta_{K/F}(c') = \Delta_{K/F}(\text{Tr}_{F/F_0}(pc)) \). Since \( \text{Tr}_{F/F_0}(pc) \in O_{F_0} \) is a unit and \( \Delta_{K/F} = \omega_{K/F} \) induces the quadratic character of \( k_F^\times = k_{F_0}^\times \), then for this particular choice of \( c \) we obtain

\[ \Delta_{K/F}(c') = \begin{cases} 
1 & \text{if } \text{Tr}_{F/F_0}(pc) \text{ is a square in } k_{F_0}^\times \\
-1 & \text{if } \text{Tr}_{F/F_0}(pc) \text{ is not a square in } k_{F_0}^\times.
\end{cases} \]

3.2. Computation of \( \lambda_{K/F} \), where \( K/F \) is a wildly ramified extension. In the case \( p = 2 \), the square class group of \( F \), i.e., \( F^\times/F^{\times 2} \) can be very large (cf. Theorem 2.29 on p. 165 of [1]), so we can have many quadratic characters but they are wildly ramified, not tame. In Remark 1.2 we mention the current status of this wildly ramified quadratic case.

Proof of Theorem 1.3. Let \( G = \text{Gal}(K/F) \). From Theorem 2.29 on p. 165 of [1], if \( F/Q_2 \), we have \( |F^\times/F^{\times 2}| = 2^m \), \( (m \geq 3) \) and the 2-rank of \( G \) (i.e., the dimension of \( G/G^2 \) as a vector space over \( \mathbb{F}_2 \)) \( \text{rk}_2(G) \neq 1 \) and \( G \) is not metacyclic. Then from Bruno Kahn’s result, the second Stiefel-Whitney class \( s_2(\text{Ind}_{K/F}(1)) = 0 \) (cf. Theorem 1 of [3]). Since \( s_2(\text{Ind}_{K/F}(1)) = 0 \), the Deligne constant \( c(\text{Ind}_{K/F}(1)) = 1 \) (cf. Theorem 3 on p. 129 of [10]). Again since here \( \text{rk}_2(G) \neq 1 \) and \( G \) is not metacyclic, we have \( \Delta_{K/F} \equiv 1 \). Therefore, we can conclude that

\[ \lambda_{K/F}(\psi) = c(\text{Ind}_{K/F}(1)) \cdot W(\Delta_{K/F}, \psi) = 1, \]
where $\psi$ is a nontrivial additive character of $F$. \hfill \square

**Example 3.3 (Computation of $\lambda_{K/Q_2}$, where $K/Q_2$ is a quadratic extension).** In this case, we have (cf. pp. 60-63 of [1]):

\[
\begin{align*}
\lambda_{Q_2(\sqrt{5})/Q_2} &= 1, \lambda_{Q_2(\sqrt{-1})/Q_2} = i, \lambda_{Q_2(\sqrt{-5})/Q_2} = i, \lambda_{Q_2(\sqrt{2})/Q_2} = 1, \\
\lambda_{Q_2(\sqrt{10})/Q_2} &= i, \lambda_{Q_2(\sqrt{-2})/Q_2} = i, \lambda_{Q_2(\sqrt{-10})/Q_2} = -i.
\end{align*}
\]

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