Some results about the Tight Span of spheres

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Abstract

The smallest hyperconvex metric space containing a given metric space \( X \) is called the tight span of \( X \). It is known that tight spans have many nice geometric and topological properties, and they are gradually becoming a target of research of both the metric geometry community and the topological/geometric data analysis community. In this paper, we study the tight span of \( n \)-spheres (with either geodesic metric or \( \ell_\infty \)-metric).
Nomenclature

\( \bar{x} \) Antipodal point of \( x \) in an antipodal metric space \( X \).

\( d_{\text{GH}} \) Gromov-Hausdorff distance.

\( d_H \) Hausdorff distance.

\( \text{diam}(X) \) Diameter of the metric space \( X \).

\( \ell^2 \) Hilbert space of infinite real sequences with finite square sums.

\( \mathcal{S} \) Hilbert cube.

\( \mathbb{R}^n_\infty \) \( \mathbb{R}^n \) with \( \ell^\infty \)-norm.

\( \mathbb{S}^n \) \( n \)-dimensional sphere with its geodesic metric.

\( \mathbb{S}^n_\infty \) \( n \)-dimensional sphere with \( \ell^\infty \)-norm (coming from usual embedding into \( \mathbb{R}^{n+1} \)).

\( E(X) \) Tight span of the metric space \( X \).

\( B_r(X, E) \) Open \( r \)-thickening of \( X \) in \( E \), for \( X \) a sub-metric space of \( E \).

\( L^\infty(X) \) Banach space of real valued functions on \( X \) with \( \ell^\infty \)-norm.

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1 Introduction

A metric space \((E, d_E)\) is called hyperconvex if it satisfies the following property:

For every family \((x_i, r_i)_{i \in I}\) of \(x_i\) in \(E\) and \(r_i \geq 0\) such that \(d_E(x_i, x_j) \leq r_i + r_j\) for every \(i, j\) in \(I\), there exists a point \(x \in E\) such that \(d_E(x, x_i) \leq r_i\) for every \(i\) in \(I\).

Hyperconvex spaces were first studied by Aronszajn and Panitchpakdi in [AP56], and the authors proved that a metric space \(E\) is hyperconvex if and only if \(E\) satisfies the following “injectivity” condition:

For each 1-Lipschitz map \(f : X \to E\) and an isometric embedding of \(X\) into \(\tilde{X}\), there exists a 1-Lipschitz map \(\tilde{f} : \tilde{X} \to E\) extending \(f\):

\[\begin{array}{ccc}
X & \xrightarrow{f} & \tilde{X} \\
\downarrow & & \downarrow \tilde{f} \\
E & & E
\end{array}\]

This notion of injectivity of metric spaces was rediscovered in [Isb64] and the theory was developed further in [Dre84, Lan13, LPZ13]. One important fact is that any arbitrary metric space \((X, d_X)\) can be isometrically embedded into an injective metric space. This can be easily shown by using the Kuratowski embedding \(x \mapsto d_X(x, \cdot)\) and the fact that \(L_\infty(X)\), the space of all real-valued maps on \(X\) with the uniform norm, is hyperconvex.

In the first place, we became interested in hyperconvex metric spaces because of their application to Topological Data Analysis. In [LMO20], the authors proved that the filtration obtained through increasing thickenings of \(X\) inside any hyperconvex metric space is homotopy equivalent to the Vietoris-Rips filtration. Also, the recent papers [JJ19, JJ20] introduced a new notion of curvature for metric spaces through a quantification of their deviation from hyperconvexity, and suggested applications of this notion of curvature for topological/geometric data analysis.

For an arbitrary metric space \(X\), \(E(X)\), the tight span of \(X\) is defined to be the smallest (up to isometric embedding) hyperconvex space containing \(X\). It is known that tight spans have many nice geometric properties. For example, \(E(X)\) is always a contractible geodesic metric space, and it inherits the compactness, diameter, and \(\delta\)-hyperbolicity from its inducing space \(X\) [Isb64, Lan13].

It is known that every complete metric tree \(T\) is isometric to its own tight span \(E(T)\) (hence it is itself hyperconvex) [Dre84, Theorem 8]. A natural next step is to understand the tight span of metric graphs: one motivation for this is that, firstly, metric graphs can approximate arbitrary geodesic metric spaces [BBI01, MOW18] in the Gromov-Hausdorff sense and, secondly, the tight span is itself Gromov-Hausdorff stable [LPZ13, Theorem 3.1]. Since the simplest example of a non-tree metric graph is \(S^1\) (with geodesic metric), this motivates one of the main goals of this paper which is to characterize \(E(S^1)\). Some of the arguments we use also help us obtain a better understanding of \(E(S^n)\), the tight span of \(n\)-spheres for \(n \geq 2\).
Another reason why precise knowledge about the structure of the tight span of spheres is important comes from applied algebraic topology (AAT) [Car09], where the Vietoris-Rips complex (and the filtration it induces) plays a fundamental role [AA17, AAF18]. Being natural “model spaces”, it is of interest to fully characterize the persistent homology induced by the Vietoris-Rips filtrations of spheres (endowed with their geodesic metric). Since it is known [LMO20, Theorem 5] that for any compact metric space \( X \), its Vietoris-Rips filtration and the filtration \( B_r(X, \mathcal{E}(X)) \) (arising from thickening \( X \) inside its tight span) are naturally isomorphic, we hope that by better understanding \( \mathcal{E}(S^n) \) we will be able to eventually characterize the successive homotopy types of Vietoris-Rips complexes of \( S^n \). For partial results in this direction, see §3.3.

Contributions and organization. In §2, we review preliminary notions and theorems which will be required throughout this paper. We give a succinct proof of the equivalence between hyperconvexity and injectivity (Proposition 2.2). Also, with the aid of the tight span, we classify the successive homotopy types of the Vietoris-Rips filtration of tree-like metric spaces (Corollary 2.13).

In §3, first, motivated by the cases of finite antipodal metric spaces, we prove that for the circle \( S^1 \) with geodesic metric, \( \mathcal{E}(S^1) \) is homeomorphic to the Hilbert cube. Next, we figure out the explicit form of those functions in \( \mathcal{E}(S^1) \) which play the role of “vertices” of the Hilbert cube. Finally, with this understanding, we prove that the homotopy type of \( B_r(S^1, \mathcal{E}(S^1)) \) is that of \( S^1 \) for \( r \in \left(0, \frac{\pi}{3}\right) \) and that of a point for \( r = \frac{\pi}{2} \).

In §4, we review the notion of mountain range function introduced by Katz in [Kat91], and with that we generate some examples of functions belonging to \( \mathcal{E}(S^n) \) for arbitrary \( n > 2 \).

In §5, we move our focus to \( S^n_\infty \), which is the \( n \)-sphere equipped with the \( \ell_\infty \)-metric instead of the geodesic metric. To understand \( \mathcal{E}(S^n_\infty) \), we introduce the notions of \( X \)-surrounding points and \( X \)-minimal points associated to a metric space \( X \). In [KK16], the authors prove that, for the circle \( S^1_\infty \), \( \mathcal{E}(S^1_\infty) \) is isometric to \( D^2_\infty \). This result suggests the question whether for all \( n \geq 1 \) it holds that \( \mathcal{E}(S^n_\infty) \) is isometric to \( D^{n+1}_\infty \). We answer this question to the negative through an application of the notions of \( X \)-surrounding points and \( X \)-minimal points. We prove that \( \mathcal{E}(S^n_\infty) \) is not isometric to \( D^{n+1}_\infty \), \( D^{n+1}_\infty \) is not hyperconvex for \( n \geq 2 \) (See Theorem 5.8), and also find an alternative description of \( \mathcal{E}(S^2_\infty) \) (See Theorem 5.15).

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2 Preliminaries

Let us introduce some notations that will be used throughout this paper.

Between two metric spaces \( X \) and \( Y \), \( X \cong Y \) denotes that \( X \) is isometric to \( Y \), and \( X \preceq Y \) (resp. \( X \prec Y \)) denotes that \( X \) can be isometrically embedded in \( Y \) (resp. \( X \) can be isometrically and properly embedded in \( Y \)).
Suppose that $X$ is a subspace of a metric space $(E, d_E)$. For every $r > 0$, let $B_r(X, E) := \{ z \in E : \exists x \in X \text{ with } d_E(z, x) < r \}$ denote the open $r$-thickening of $X$ in $E$. Respectively, the closed $r$-thickening of $X$ in $E$, denoted by $\overline{B}_r(X, E)$ is defined in the analogous way. In particular, if $X = \{ x \}$ for some $x \in E$, it is just denoted by $B_r(x, E)$ (resp. $\overline{B}_r(x, E)$), the usual open (resp. closed) $r$-ball around $x$ in $E$.

As one more convention, whenever there is an isometric embedding $\iota : X \hookrightarrow E$, we will use the notation $B_r(X, E)$ instead of $B_r(\iota(X), E)$.

Now we will review the concepts of injective and hyperconvex metric spaces. The main references for this subsection are [Dre84, DHK+12, Lan13].

**Definition 1** (Injective metric space). A metric space $E$ is called injective if for every 1-Lipschitz map $f : X \to E$ and isometric embedding of $X$ into $\tilde{X}$, there exists a 1-Lipschitz map $\tilde{f} : \tilde{X} \to E$ extending $f$:

$$X \hookrightarrow \tilde{X} \xrightarrow{\tilde{f}} E$$

**Definition 2** (Hyperconvex space). A metric space $E$ is called hyperconvex if for every family $(x_i, r_i)_{i \in I}$ of $x_i$ in $E$ and $r_i \geq 0$ such that $d_E(x_i, x_j) \leq r_i + r_j$ for every $i, j$ in $I$, there exists a point $x \in E$ such that $d_E(x_i, x) \leq r_i$ for every $i$ in $I$.

The following lemma is easy to deduce from the definition of hyperconvex space.

**Lemma 2.1.** Any nonempty intersection of closed balls in a hyperconvex space is hyperconvex.

For a proof of the following proposition, see [AP56] or [Lan13, Proposition 2.3].

**Proposition 2.2.** A metric space is injective if and only if it is hyperconvex.

Soon, we will provide an elegant proof of Proposition 2.2 as one of the applications of tight span.

Moreover, every injective metric space is a contractible geodesic metric space, as one will see in Lemma 2.3 and Corollary 2.4.

**Definition 3** (Geodesic bicombing). By a geodesic bicombing $\gamma$ on a metric space $(X, d_X)$, we mean a continuous map $\gamma : X \times X \times [0, 1] \to X$ such that, for every pair $(x, y) \in X \times X$, $\gamma(x, y, \cdot)$ is a geodesic from $x$ to $y$ with constant speed. In other words, $\gamma$ satisfies the following:

1. $\gamma(x, y, 0) = x$ and $\gamma(x, y, 1) = y$.
2. $d_X(\gamma(x, y, s), \gamma(x, y, t)) = (t - s) \cdot d_X(x, y)$ for all $0 \leq s \leq t \leq 1$.

**Lemma 2.3** ([Lan13, Proposition 3.8]). Every injective metric space $(E, d_E)$ admits a geodesic bicombing $\gamma$ such that, for all $x, y, x', y' \in E$ and $t \in [0, 1]$, it satisfies the following conditions:

1. **Conical:** $d_E(\gamma(x, y, t), \gamma(x', y', t)) \leq (1 - t) d_E(x, x') + t d_E(y, y')$.
2. **Reversible:** $\gamma(x, y, t) = \gamma(y, x, 1 - t)$. 

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Remark 2.6. Here are some remarks. All of them are either straightforward, or can be found in

Corollary 2.4. Every injective metric space $E$ is contractible.

Proof. By Lemma 2.3, there is a geodesic bicombing $γ$ on $E$. Fix arbitrary point $x_0 ∈ E$. Then, restricting $γ$ to $E \times \{x_0\} \times [0, 1]$ gives a deformation retraction of $E$ onto $x_0$. Hence, $E$ is contractible. □

Example 2.5. For every set $S$, the Banach space $L^∞(S)$ consisting of all the bounded real-valued functions on $S$ with the metric induced by $ℓ^∞$-norm is injective. $R^n_∞$ is one of such spaces.

Definition 4. For a compact metric space $(X, d_X)$, the map $κ_X : X \to L^∞(X), x \mapsto d_X(x, ·)$ is an isometric embedding, and it is called the Kuratowski embedding. Hence, every compact metric space can be isometrically embedded into an injective metric space.

There are other injective metric spaces associated to $(X, d_X)$.

Definition 5. Let $(X, d_X)$ be a metric space. Consider the following spaces associated to $X$:

\[
\begin{align*}
\Delta(X) &:= \{ f ∈ L^∞(X) : f(x) + f(x') ≥ d_X(x, x') \text{ for all } x, x' ∈ X \}, \\
E(X) &:= \{ f ∈ \Delta(X) : \text{if } g ∈ \Delta(X) \text{ and } g ≤ f, \text{ then } g = f \text{ (i.e., } f \text{ is minimal)} \}, \\
\Delta_1(X) &:= \Delta(X) \cap \text{Lip}_1(X),
\end{align*}
\]

where $\text{Lip}_1(X)$ denotes the set of all 1-Lipschitz functions $f : X \to R$. We endow these spaces with the metric induced by $ℓ^∞$-norm. Then, it is known that all the three spaces above are injective metric spaces (cf. [Lan13, Section 3]).

In particular, $E(X)$ is said to be the tight span of $X$ [Dre84, Isb64] and it is an especially interesting space: $E(X)$ is the smallest injective metric space into which $X$ can be embedded, and it is unique up to isometry.

Remark 2.6. Here are some remarks. All of them are either straightforward, or can be found in [Lan13]:

1. By choosing $x = x'$ in the definition of $f ∈ \Delta(X)$, we see that any such $f$ is nonnegative [Lan13, §3].

2. $f ∈ \Delta_1(X)$ if and only if $∥f - d_X(x, ·)∥_∞ = f(x)$ for all $x ∈ X$ [Lan13, (3.2)].

3. For every $r > 0$ and $E = \Delta_1(X)$ or $E(X)$, one has that $B_r(X, E) = \{ f ∈ E : \exists x ∈ X \text{ such that } f(x) < r \}$.

4. If $f ∈ E(X)$, then $f(x) = \sup_{x' ∈ X} (d_X(x, x') - f(x'))$ [Lan13, (3.1)]. In particular, this implies: $f(x) ∈ [0, \text{diam}(X)]$ for all $x ∈ X$. With this and the fact that $X$ is isometrically embedded in $E(X)$, one can conclude $\text{diam}(X) = \text{diam}(E(X))$.

5. If $f ∈ E(X)$ and $X$ is compact, then for all $x ∈ X$ there exists $x' ∈ X$ such that $d_X(x, x') = f(x) + f(x')$.

6. There exists a “projection” map $p_X : \Delta(X) \to E(X)$ such that for all $f ∈ \Delta(X)$, $p_X(f)(x) ≤ f(x)$ for all $x ∈ X$ [Lan13, Proposition 3.1].
Lemma 2.7 ([Lan13, Proposition 3.5]). If $X$ is isometrically embedded in $Y$, then there exists an isometric embedding $h_{X,Y}: \mathbb{E}(X) \hookrightarrow \mathbb{E}(Y)$ such that for all $f \in \mathbb{E}(X)$, $h_{X,Y}(f)_{|X} = f$.

At this step, we provide a proof of Proposition 2.2 as promised. Our proof has a similar structure to [Her92, Proposition, P182], but by directly alluding to the tight span, our proof is significantly more concise.

Proof of Proposition 2.2. Since a metric space $X$ is injective if and only if $X = \mathbb{E}(X)$, it is enough to prove that $X$ is hyperconvex if and only if $X$ and $\mathbb{E}(X)$ are isometric.

Suppose $X$ is hyperconvex. Let $f \in \mathbb{E}(X)$. Then, for all $x, x' \in X$, $f(x) + f(x') \geq d_X(x, x')$. By the hyperconvexity of $X$, $\cap_{x \in X} B_{f(x)}(x, X) \neq \emptyset$. Let $x_0 \in \cap_{x \in X} B_{f(x)}(x, X)$. Then, $d_X(x_0, x) \leq f(x)$ for all $x \in X$. This implies that $f \geq d_X(x_0, \cdot)$. By the minimality of $\mathbb{E}(X)$, we have $f = d_X(x_0, \cdot)$. This shows that $\mathbb{E}(X) = X$.

Conversely, suppose $X = \mathbb{E}(X)$. Consider any index set $I$ and let $X_I := \{x_i\}_{i \in I} \subseteq X$ and $\{r_i\}_{i \in I}$ be such that $r_i + r_j \geq d_X(x_i, x_j)$. Then, the function $f_I : X_I \to \mathbb{R}$ taking $x_i$ to $r_i$ belongs to $\Delta(X_I)$. Consider the projection $p_I : \Delta(X_I) \to \mathbb{E}(X_I)$ and the embedding $h_I : \mathbb{E}(X_I) \hookrightarrow \mathbb{E}(X)$ from Lemma 2.7. Then, $f := h_I(p_I(f_I)) \in \mathbb{E}(X) = X$. Hence, there exists $x_0 \in X$ such that $d_X(x_0, \cdot) = f$. Then,

$$d_X(x_0, x_i) = f(x_i) = p_I(f_I)(x_i) \leq f_I(x_i) = r_i$$

and thus $x_0 \in \cap_{i \in I} B_{r_i}(x_i, X)$. This implies that $X$ is hyperconvex. \qed

Finally, it is known that some topological/geometric properties of $X$ are reflected in $\mathbb{E}(X)$.

Theorem 2.8 ([Isb64, 2.11]). If $X$ is compact, then $\mathbb{E}(X)$ is also compact.

Definition 6 ($\delta$-hyperbolic space). A metric space $(X, d_X)$ is called $\delta$-hyperbolic, for some constant $\delta \geq 0$, if

$$d_X(x, w) + d_X(y, z) \leq \max\{d_X(w, y) + d_X(x, z), d_X(x, y) + d_X(w, z)\} + \delta$$

for all quadruples of points $x, y, z, w \in X$.

Proposition 2.9 ([Lan13, Proposition 1.3]). If $X$ is a $\delta$-hyperbolic geodesic metric space for some $\delta \geq 0$, then its tight span $\mathbb{E}(X)$ is also $\delta$-hyperbolic.

Moreover,

$$B_r(X, \mathbb{E}(X)) = \mathbb{E}(X)$$

for each $r > \delta$.

Stability of the tight span. It is known that the map $X \mapsto \mathbb{E}(X)$ is stable with respect to the Gromov-Hausdorff distance.

Theorem 2.10 ([LPZ13, Theorem 3.1]). For any two metric spaces $A$ and $B$,

$$d_{\text{GH}}(\mathbb{E}(A), \mathbb{E}(B)) \leq 2 d_{\text{GH}}(A, B).$$
2.1 Homotopy types of Vietoris-Rips complexes of tree-like metric spaces

A 0-hyperbolic metric space $X$ is called, alternatively, tree-like metric space. i.e., a metric space $X$ is tree-like if it satisfies

$$d_X(w, x) + d_X(y, z) \leq \max\{d_X(w, y) + d_X(x, z), d_X(x, y) + d_X(w, z)\}.$$  

for every $x, y, z, w \in X$.

In this section, we show the homotopy types of Vietoris-Rips complexes of tree-like metric spaces can be easily determined using the knowledge of tight span. More precisely, we show that for a tree-like metric space $X$ and for any scale $r > 0$, each connected component of the neighborhood $B_r(X, E(X))$ is contractible. Then we obtain the result for Vietoris–Rips complex $\text{VR}_{2r}(X)$ by the identification of homotopy type between $\text{VR}_{2r}(X)$ and $B_r(X, E(X))$ in [LMO20].

Proposition 2.11 ([LMO20, Proposition 2.3]). Let $X$ be a subspace of an injective metric space $(E, d_E)$. Then, for every $r > 0$, the Vietoris-Rips complex $\text{VR}_{2r}(X)$ is homotopy equivalent to $B_r(X, E(X))$.

The tight span approach that we adopt in this section both generalizes the claim and offers an alternative technique to the one followed in the case of finite tree-like metric spaces [CCR13, Supp. Material].

Proposition 2.12. Let $X$ be a tree-like metric space. Then for any $r > 0$, the $r$-neighborhood of $X$ inside its tight span $E(X)$ is homotopy equivalent to a disjoint union of points with cardinality equal the number of connected components of $B_r(X, E(X))$.

Corollary 2.13. Let $X$ be a tree-like metric space. Then for any $r > 0$, $\text{VR}_{2r}(X)$ is homotopy equivalent to a disjoint union of points with cardinality equal to the number of connected components of $B_r(X, E(X))$.

Proof. Apply Proposition 2.11 and Proposition 2.12. □

Our proof of the above proposition relies on the characterization of the tight span of a tree-like metric space as an $\mathbb{R}$-tree [Dre84]. We now recall the definition of an $\mathbb{R}$-tree.

Definition 7. Let $X$ be a metric space, $X$ is called an $\mathbb{R}$-tree if the following two conditions are satisfied:

- For every $x, x' \in X$ there exist a unique isometric embedding $\gamma : [0, d_X(x, x')] \to X$ such that $\gamma(0) = x$ and $\gamma(d_X(x, x')) = x'$.
- For every injective continuous map $\gamma : [0, 1] \to X$, for each $t \in [0, 1]$, one has $d_X(\gamma(0), \gamma(t)) + d_X(\gamma(t), \gamma(1)) = d_X(\gamma(0), \gamma(1))$.

In [Dre84], Dress shows the tight span of a tree-like metric space is an $\mathbb{R}$-tree.

Theorem 2.14 ([Dre84, Theorem 8]). Let $X$ be a tree-like metric space, then the tight span of $X$ is isometric to an $\mathbb{R}$-tree.

Now we are ready to prove Proposition 2.12.

Proof of Proposition 2.12. By Theorem 2.14, $E(X)$ is an $\mathbb{R}$-tree. Fix $r > 0$. Let $C$ be a connected component of $B_r(X, E(X))$. As $\mathbb{R}$-trees are locally path connected, the connected component $C$ is path-connected. Therefore, $C$ is also an $\mathbb{R}$-tree and hence contractible. □
2.2 Tight span of antipodal metric spaces

In this section, we recall known results on the characterization of the tight span of antipodal metric spaces. These results are then used to establish a simpler description of $\mathbb{E}(S^n)$ which in turn leads to a simpler function space $F(S^1)$ which is isometric to $\mathbb{E}(S^1)$ (see Definition 10) in §3.1. Also, this simpler description will be implicitly used in §4.

**Definition 8.** A metric space $X$ is called antipodal if for every $x \in X$, there exists $\bar{x} \in X$ (called an antipodal point of $x$) such that $d_X(x, \bar{x}) = d_X(x, y) + d_X(y, \bar{x})$ holds for every $y \in X$.

**Lemma 2.15 ([GM00, Lemma 7.7]).** If $X$ is an antipodal compact metric space, then a function $f \in \mathbb{E}(X)$ if and only if $f$ is 1-Lipschitz and $f(x) + f(\bar{x}) = \text{diam}(X)$, $\forall x \in X$, where $\bar{x}$ is the antipodal point of $x$.

It is then easy to see that for any $f \in \mathbb{E}(X)$, we have $0 \leq f \leq \text{diam}(X)$. This equation easily allows us to characterize the center and the radius of $\mathbb{E}(X)$.

**Definition 9.** For a bounded metric space $(X, d_X)$, we call $\text{rad}(X) := \inf_{x \in X} \sup_{y \in X} d_X(x, y)$ the radius of $X$. If $x_0 \in X$ satisfies that $\text{rad}(X) = \sup_{y \in X} d_X(x_0, y)$, we say that $x_0$ is a center of $X$.

**Proposition 2.16.** Let $X$ be a compact antipodal metric space. Then, $\text{rad}(\mathbb{E}(X)) = \frac{\text{diam}(X)}{2}$ and the constant function $f_0 \equiv \frac{\text{diam}(X)}{2}$ is the unique center of $\mathbb{E}(X)$.

**Proof.** For every $f \in \mathbb{E}(X)$ and every $x \in X$, we have that
\[
\|f - d_X(x, \cdot)\|_{\infty} = f(x) \text{ and } \|f - d_X(\bar{x}, \cdot)\|_{\infty} = f(\bar{x}) = \text{diam}(X) - f(x).
\]

Then,
\[
\sup_{g \in \mathbb{E}(X)} \|f - g\|_{\infty} \geq \max\{\|f - d_X(x, \cdot)\|_{\infty} ; \|f - d_X(\bar{x}, \cdot)\|_{\infty}\} = \max\{f(x), \text{diam}(X) - f(x)\} \geq \frac{\text{diam}(X)}{2}.
\]

Moreover, if for a given $f \in \mathbb{E}(X)$ there exists $x \in X$ such that $f(x) \neq \frac{\text{diam}(X)}{2}$, we must have
\[
\sup_{g \in \mathbb{E}(X)} \|f - g\|_{\infty} \geq \max\{f(x), \text{diam}(X) - f(x)\} > \frac{\text{diam}(X)}{2}.
\]

Now, consider $f_0 \equiv \frac{\text{diam}(X)}{2} \in \mathbb{E}(X)$. For each $f \in \mathbb{E}(X)$ and each $x \in X$, since $0 \leq f(x) \leq \text{diam}(X)$, we have $\left|f(x) - \frac{\text{diam}(X)}{2}\right| \leq \frac{\text{diam}(X)}{2}$. This implies that $\|f - f_0\|_{\infty} \leq \frac{\text{diam}(X)}{2}$. Also, it is obvious that $\|d_X(x, \cdot) - f_0\|_{\infty} = \frac{\text{diam}(X)}{2}$. Hence,
\[
\sup_{f \in \mathbb{E}(X)} \|f_0 - f\|_{\infty} = \|f_0 - d_X(x, \cdot)\|_{\infty} = \frac{\text{diam}(X)}{2}.
\]

Therefore, $\text{rad}(\mathbb{E}(X)) = \frac{\text{diam}(X)}{2}$ and $f_0$ is the unique center of $\mathbb{E}(X)$. \qed
We conclude this section with an interesting result about the convexity of the tight span of antipodal spaces.

**Proposition 2.17.** Let $X$ be a compact metric space. Then, $X$ is antipodal if and only if $E(X)$ is convex as a subset of $C(X)$, the space of all continuous functions $f : X \to \mathbb{R}$.

**Proof.** We first assume that $X$ is antipodal. Let $f, g \in E(X)$ and let $\lambda \in (0, 1)$. Consider $h := \lambda f + (1 - \lambda) g \in C(X)$. Since both $f$ and $g$ are 1-Lipschitz, it is obvious that $h$ is 1-Lipschitz.

Now, for every $x \in X$, we have that
\[
\begin{align*}
  h(x) + h(\bar{x}) &= \lambda (f(x) + f(\bar{x})) + (1 - \lambda) (g(x) + g(\bar{x})) \\
  &= \lambda \text{diam}(X) + (1 - \lambda) \text{diam}(X) \\
  &= \text{diam}(X).
\end{align*}
\]

Then, by Lemma 2.15, we have that $h \in E(X)$. Therefore, $E(X)$ is convex.

Now, we assume that $E(X)$ is a convex subspace of $C(X)$. Recall that $\kappa_X(x) = d_X(x, \cdot) \in E(X)$ for every $x \in X$. Since $X$ is compact, there exist $x_1, x_2 \in X$ such that $d_X(x_1, x_2) = \text{diam}(X)$. We will prove that $x_2 = \bar{x}_1$. By convexity of $E(X)$, $f := \frac{1}{2}(d_X(x_1, \cdot) + d_X(x_2, \cdot)) \in E(X)$. Now, pick an arbitrary $y \in X$. Then, there exists $z \in X$ such that
\[
  f(y) = d_X(y, z) - f(z).
\]

by item (4) of Remark 2.6. This implies that
\[
2d_X(y, z) = d_X(x_1, y) + d_X(x_2, y) + d_X(x_1, z) + d_X(x_2, z) \geq d_X(x_1, x_2) + d_X(x_1, x_2) = 2\text{diam}(X).
\]

Since $d_X(y, z) \leq \text{diam}(X)$, we have that all equalities in the above formula holds. Therefore, $d_X(y, z) = \text{diam}(X)$ and $d_X(x_1, x_2) = d_X(x_1, y) + d_X(y, x_2)$.

Since $y$ is arbitrary, we have that $x_2 = \bar{x}_1$. Moreover, we have proved that for every $y \in X$, there exists $z \in X$ such that $d_X(y, z) = \text{diam}(X)$. Then, we can apply the same argument as above to deduce that $z = \bar{y}$, which concludes the proof. \qed

## 3 Characterization of $E(S^1)$ and ancillary results

Given any positive integer $k$, let $C_{2k}$ denote the circular graph as shown in Fig. 1 with $2k$ vertices. We regard $C_{2k}$ as a metric space by equipping it with the shortest path distance. Then, [GM00] established the following exact description of the tight span of $C_{2k}$.

**Proposition 3.1 ([GM00, Section 9]).** $E(C_{2k})$ is a $k$-dimensional hypercube living in $\mathbb{R}^{2k}$. More precisely, $E(C_{2k})$ is the convex combination of the $2^k$ vertices described as follows: for every $\sigma = (\sigma_i)_{i=1}^k \in \{\pm 1\}^k$, consider $h_\sigma \in E(C_{2k})$ defined by
\[
  h_\sigma = \frac{k}{2} + \sum_{i=1}^k \sigma_i \cdot g_i
\]

where $g_i$ is the function mapping vertices $i, i+1, \ldots, i+k-1$ to $\frac{1}{2}$, and the remaining vertices to $-\frac{1}{2}$. Then, the set of vertices of $E(C_{2k})$ is given by all the points $(h_\sigma(1), \ldots, h_\sigma(2k)) \in \mathbb{R}^{2k}$ for $\sigma \in \{\pm 1\}^k$. 

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Up to this point, the proposition above, and the fact that tight spans are Gromov-Hausdorff stable, suggests that the tight span of $S^1$ should be, in a certain sense, an infinite dimensional hypercube. For any metric space $(X, d_X)$ and any $\lambda > 0$, we denote by $\lambda X$ the metric space $(X, \lambda \cdot d_X)$. Then, it is obvious that,

$$\lim_{k \to \infty} d_{GH}(S^1, \pi \frac{\pi}{k} C_{2k}) = 0.$$  

In other words, $S^1$ is the Gromov-Hausdorff limit of the sequence $\{ \frac{\pi}{k} C_{2k} \}_{k=1}^{\infty}$. By Theorem 2.10, we conclude that $E(S^1)$ is in fact the Gromov-Hausdorff limit of the sequence $\{ E(\pi \frac{\pi}{k} C_{2k}) \}_{k=1}^{\infty}$. For each $k = 1, 2, \ldots$, note that $E(\pi \frac{\pi}{k} C_{2k}) = \frac{\pi}{k} E(C_{2k})$ is a $k$-dimensional hypercube. Therefore, this suggests that $E(S^1)$ should be an infinite-dimensional hypercube (in a sense to be determined).

In this section, we provide a precise description of $E(S^1)$ and prove that $E(S^1)$ is homeomorphic to the Hilbert cube, a proper notion of an infinite-dimensional hypercube. Furthermore, we will clarify which functions in $E(S^1)$ play the role of “vertices” of the Hilbert cube and characterize the homotopy types of some open thickenings of $S^1$ in $E(S^1)$.

### 3.1 A precise description of $E(S^1)$

We will use $S^n$ to denote the $n$-dimensional unit sphere with the geodesic (round) metric. Also, for $n = 1$ case, we will view $S^1$ as $\mathbb{R}/2\pi$. As an abuse of notation, $\theta \in \mathbb{R}$ will also denote the equivalence class of $\theta$ in $\mathbb{R}/2\pi = S^1$.

By Lemma 2.15, we have that $E(S^1) = \{ f \in \text{Lip}_1(S^1) | f(\theta) + f(\bar{\theta}) = \pi, \forall \theta \in S^1 \}$. 

In the sequel, we provide a representation of $E(S^1)$ simpler than Equation (2), and then determine the homeomorphism type of $E(S^1)$.

**Definition 10** (Reduced tight span of $S^1$). Let

$$F(S^1) := \{ f \in \text{Lip}_1([0, \pi]) | 0 \leq f \leq \pi, f(0) + f(\pi) = \pi \}.$$  

Endow $F(S^1)$ with the sup norm. Then, we call $F(S^1)$ the reduced tight span of $S^1$. 

![Figure 1: Examples of $C_{2k}$ when $k = 2$ and $k = 3$.](image)
Remark 3.2. Actually, \( F(S^1) = \{ f \in \text{Lip}_1([0, \pi]) | f(0) + f(\pi) = \pi \} \). The condition that \( 0 \leq f \leq \pi \) is implied by \( f(0) + f(\pi) = \pi \) and the 1-Lipschitz condition.

Remark 3.3. It is easy to check that \( F(S^1) \) is a compact convex subset of the space \( C([0, \pi]) \) consisting of all continuous maps from \([0, \pi]\) to \( \mathbb{R} \).

Proposition 3.4. \( F(S^1) \) is isometric to \( E(S^1) \).

To prove the theorem, we explicitly construct a map from \( F(S^1) \) to \( E(S^1) \) as follows: for every \( f \in F(S^1) \), we define \( \tilde{f} : S^1 \to \mathbb{R} \) in the following way:

\[
\tilde{f} : S^1 \longrightarrow \mathbb{R} \\
\theta \longmapsto \begin{cases} 
  f(\theta) & \text{if } \theta \in [0, \pi] \\
  \pi - f(\theta - \pi) & \text{if } \theta \in (\pi, 2\pi] 
\end{cases}
\]

Lemma 3.5. If \( f \in F(S^1) \), then \( \tilde{f} \) belongs to \( E(S^1) \).

Proof. It suffices to show \( \tilde{f}(\theta) + \tilde{f}(\bar{\theta}) = \pi \) for all \( \theta \in S^1 \) and \( \tilde{f} \) is 1-Lipschitz.

Fix arbitrary \( \theta \in S^1 \). If \( \theta \in [0, \pi] \),

\[
\tilde{f}(\theta) + \tilde{f}(\bar{\theta}) = f(\theta) + f(\bar{\theta}) = f(\theta) + (\pi - f(\theta)) = \pi.
\]

If \( \theta \in (\pi, 2\pi] \),

\[
\tilde{f}(\theta) + \tilde{f}(\bar{\theta}) = (\pi - f(\theta - \pi)) + (\pi - f(\theta - \pi)) = f(\theta - \pi) + f(\theta - \pi) = \pi.
\]

Hence, the first condition is satisfied.

Now, let’s check that \( \tilde{f} \) is 1-Lipschitz. Fix arbitrary \( \theta, \theta' \in S^1 \). If \( \theta, \theta' \in [0, \pi] \),

\[
|\tilde{f}(\theta) - \tilde{f}(\theta')| = |f(\theta) - f(\theta')| \leq |\theta - \theta'| = d_{S^1}(\theta, \theta').
\]

If \( \theta, \theta' \in [\pi, 2\pi] \),

\[
|\tilde{f}(\theta) - \tilde{f}(\theta')| = |(\pi - f(\theta - \pi)) - (\pi - f(\theta' - \pi))| = |f(\theta - \pi) - f(\theta' - \pi)| \leq |\theta - \theta'| = d_{S^1}(\theta, \theta').
\]

Finally, if \( \theta \in [0, \pi] \) and \( \theta' \in [\pi, 2\pi] \), we need a more subtle case-by-case analysis.

1. If \( \pi \leq \theta' \leq \theta + \pi \),

\[
|\tilde{f}(\theta) - \tilde{f}(\theta')| = |f(\theta) - (\pi - f(\theta' - \pi))| \\
\leq |f(\theta) + f(0) - \pi| + |f(\theta' - \pi) - f(0)| \\
= |f(\theta) - f(\pi)| + |f(\theta' - \pi) - f(0)| \\
\leq |\theta - \pi| + |\theta' - \pi| \\
= |\theta - \theta'| = d_{S^1}(\theta, \theta').
\]
2. If $\theta + \pi \leq \theta' \leq 2\pi$,
\[
|\tilde{f}(\theta) - \tilde{f}(\theta')| = |f(\theta) - (\pi - f(\theta' - \pi))|
\leq |f(\theta' - \pi) + f(0) - \pi| + |f(\theta) - f(0)|
= |f(\theta' - \pi) - f(\pi)| + |f(\theta) - f(0)|
\leq |\theta' - 2\pi| + \theta
= 2\pi - \theta' + \theta = d_{S^1}(\theta, \theta').
\]

This concludes the proof.

\textbf{Proof of Proposition 3.4.} By Lemma 3.5, $f \mapsto \tilde{f}$ is a well-defined map from $F(S^1)$ to $E(S^1)$. To establish the claim, it is enough to show this map is surjective and preserves distances.

Let’s prove the first claim. Fix arbitrary $g \in E(S^1)$. Then, one can view $g$ as a map from $\mathbb{R}$ to $\mathbb{R}$ such that (1) $g(\theta + 2\pi) = g(\theta) \forall \theta \in \mathbb{R}$, (2) $g(\theta + \pi) = -g(\theta) \forall \theta \in \mathbb{R}$, and (3) $g$ is 1-Lipschitz. Let $f := g|_{[0, \pi]}$. Then one can easily check $f \in F(S^1)$ and $\tilde{f} = g$.

Next, consider two arbitrary functions $f, g \in F(S^1)$. Fix an arbitrary $\theta \in S^1$. If $\theta \in [0, \pi]$, then $|\tilde{f}(\theta) - \tilde{g}(\theta)| = |f(\theta) - g(\theta)|$. If $\theta \in [\pi, 2\pi]$, then $|\tilde{f}(\theta) - \tilde{g}(\theta)| = |(\pi - f(\theta - \pi)) - (\pi - g(\theta - \pi))| = |f(\theta - \pi) - g(\theta - \pi)|$. Hence, $\|\tilde{f} - \tilde{g}\|_{\infty} = \|f - g\|_{\infty}$. This concludes the proof.

Already suggested by Proposition 3.1, via Proposition 3.4 we next show that $E(S^1)$ is an infinite-dimensional cube, namely, the Hilbert cube:

\textbf{Definition 11.} Let $\ell^2$ denote the Hilbert space consisting of all infinite real sequences with finite square sums. The subspace
\[
\mathcal{H} := \prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right] \subseteq \ell^2
\]
is called the \textit{Hilbert cube}.

Recall that in a linear space $E$, the affine hull of any subset $S \subseteq E$ is defined as
\[
\text{aff}(S) := \left\{ \sum_{i=1}^{k} t_i x_i : k > 0, t_i \in \mathbb{R} \text{ and } \sum_{i=1}^{k} t_i = 1 \right\}.
\]
For any convex $S$, the dimension of $S$ is defined to be the dimension of $\text{aff}(S)$. The following result characterizes a large family of spaces homeomorphic to $\mathcal{H}$.

\textbf{Lemma 3.6 ([Kle55]).} Any infinite-dimensional compact convex subset of a separable normed linear space is homeomorphic to the Hilbert cube.

\textbf{Lemma 3.7.} $F(S^1) \subseteq C([0, \pi])$ is infinite dimensional.

\textbf{Proof.} It is easy to see that the affine hull $\text{aff}(F(S^1))$ can be characterized as follows
\[
\text{aff}(F(S^1)) = \{ f \in \text{Lip}([0, \pi]) | f(0) + f(\pi) = \pi \},
\]
where $\text{Lip}([0, \pi])$ is the space of all Lipschitz functions (not necessarily 1-Lipschitz). $\text{Lip}([0, \pi])$ is dense in $C([0, \pi])$ and thus infinite dimensional. Therefore $\text{aff}(F(S^1))$ is a codimension-1 subspace of $\text{Lip}([0, \pi])$ and thus of infinite dimension.
Combining Remark 3.3, Lemma 3.6, and Lemma 3.7, we obtain the following topological characterization of $F(S^1)$.

**Theorem 3.8.** $F(S^1)$ is homeomorphic to the Hilbert cube $\mathcal{H}$.

As a direct consequence of this theorem, we have the following result.

**Theorem 3.9.** For every positive integer $n > 0$, $E(S^n)$ is homeomorphic to the Hilbert cube $H$.

**Proof.** By Proposition 3.4 and Theorem 3.8, we have that $E(S^1)$ is homeomorphic to $\mathcal{H}$.

Now, for any $n > 1$, notice that $E(S^n)$ is a compact convex subset of the separable normed linear space $C(S^n)$ (cf. Theorem 2.8 and Proposition 2.17). By Lemma 3.6, we then only need to show that $\dim(\operatorname{aff}(E(S^n))) = \infty$.

Let $S^1$ be any great circle in $S^n$. By Lemma 2.7, there exists an isometry $h_n : E(S^1) \to E(S^n)$ such that for every $f \in E(S^1)$, $h_n(f)|_{S^1} = f$. Then, $\operatorname{aff}(h_n(E(S^1))) \subseteq \operatorname{aff}(E(S^n))$. Consider the restriction map $\operatorname{res} : C(S^n) \to C(S^1)$ sending any $f \in C(S^n)$ to $f|_{S^1}$. The map $\operatorname{res}$ is obviously a linear map and it is easy to see that $\operatorname{res}(\operatorname{aff}(h(E(S^n)))) = \operatorname{aff}(E(S^1))$.

By Proposition 3.4 and Lemma 3.7, we have that $\dim(\operatorname{aff}(E(S^1))) = \infty$. This implies that $\infty = \dim(\operatorname{aff}(h(E(S^1)))) \leq \dim(\operatorname{aff}(E(S^n)))$,

which concludes the proof.

The tight span of a metric space $X$ can be viewed as a “hyperspace” of $X$, i.e., a metric space containing $X$ as a subspace. The Hausdorff hyperspace of $X$, which consists of nonempty closed subsets of $X$ and is endowed with the Hausdorff distance, is another type of hyperspace. It is known that the Hausdorff hyperspaces of many types of compact metric spaces are homeomorphic to $\mathcal{H}$ [SW74, CS74, SW75]. This fact suggests us to also look into the tight span of more general compact metric spaces than spheres to determine whether an analogue of Theorem 3.8 still holds. We leave this for future work.

### 3.2 The vertex set of $E(S^1)$

From Proposition 3.1 we know that $E(C_{2k})$ has the combinatorial structure of a hypercube, and we have an explicit description of its vertex set. In the previous section, we established that $E(S^1)$ is an infinite-dimensional cube, i.e., the Hilbert cube. In this section, therefore, we provide an explicit description of the vertex set of $E(S^1)$.

**Definition 12.** For every $\theta \in \mathbb{R}$, define $g_\theta : S^1 \to \mathbb{R}$ as follows

$$
g_\theta(\varphi) := \begin{cases} 
\frac{1}{2} & \varphi \in [\theta, \theta + \pi) \\
-\frac{1}{2} & \text{otherwise}
\end{cases}
$$

(3)
As a generalization of Equation (1), we identify the following family of functions defined on $S^1$. Given any Lebesgue measurable $A \subseteq [0, \pi]$, define $h_A : S^1 \to \mathbb{R}$ as follows

$$h_A(\varphi) := \frac{\pi}{2} + \int_0^\pi g_\theta(\varphi) \left( 1_A(\theta) - 1_{[0, \pi] \setminus A}(\theta) \right) d\theta.$$ 

**Proposition 3.10.** Given any Lebesgue measurable $A \subseteq [0, \pi]$, the function $h_A : S^1 \to \mathbb{R}$ defined above belongs to $E(S^1)$.

**Proof.** Note that $h_A$ is 1-Lipschitz, since

$$|h_A(\alpha) - h_A(\beta)| \leq \int_0^\pi |g_\theta(\alpha) - g_\theta(\beta)| d\theta = d_{S^1}(\alpha, \beta)$$

for every $\alpha, \beta \in S^1$.

Now, for every $\theta \in [0, \pi]$ and $\varphi \in S^1$, we have that $g_\theta(\varphi) + g_\theta(\bar{\varphi}) = 0$. This implies that $h_A(\varphi) + h_A(\bar{\varphi}) = \pi$. Then, by Lemma 2.15, $h_A \in E(S^1)$. \hfill $\square$

**Lemma 3.11.** Given two measurable subsets $A, B \subseteq [0, \pi]$ such that $\mu(A \Delta B) \neq 0$, we have that $h_A \neq h_B$.

**Example 3.12.** For any $\alpha \in [0, \pi]$, let $A = [0, \alpha]$, then $h_A = d_{S^1}(e^{i(\pi+\alpha)}, \cdot)$ and $h_{[0,\pi] \setminus A} = d_{S^1}(\alpha, \cdot)$.

Let $\mu$ denote the Lebesgue measure on $[0, \pi]$. Then, it is easy to observe the following explicit formula for $h_A$:

$$h_A(\varphi) = \begin{cases} 
\frac{\pi}{2} - \frac{1}{2} \mu(A \cap [\varphi, \pi]) + \frac{1}{2} \mu(A \cap [0, \varphi]) + \frac{1}{2} \mu([\varphi, \pi] \setminus A) - \frac{1}{2} \mu([0, \varphi] \setminus A), & \varphi \in [0, \pi] \\
\frac{\pi}{2} + \frac{1}{2} \mu(A \cap [\bar{\varphi}, \pi]) - \frac{1}{2} \mu(A \cap [0, \varphi]) - \frac{1}{2} \mu([\varphi, \pi] \setminus A) + \frac{1}{2} \mu([0, \varphi] \setminus A), & \varphi \in [\pi, 2\pi]
\end{cases}$$

where $\bar{\varphi} = \varphi - \pi$ denotes the antipodal point of $\varphi$ in $S^1$.

For every $\varphi \in (\pi, 2\pi]$, $h_A(\varphi) = \pi - h_A(\varphi - \pi)$. So we focus on $\varphi \in [0, \pi]$. Then, the above formula can be further simplified as follows

$$h_A(\varphi) = h_A(0) + 2\mu(A \cap [0, \varphi]) - \varphi,$$

and $h_A(0) = \pi - \mu(A)$.

**Definition 13.** In a convex set $S$, a point $x \in S$ is called an extreme point/vertex point, if it is not the midpoint of any pair of distinct points $y, z \in S$.

It is clear that a function $f \in E(S^1)$ is an extreme point if and only if $f|_{[0, \pi]}$ is an extreme point in $F(S^1)$. Below, we analyze extreme points in $F(S^1)$. First of all, by Rademacher’s theorem, $h_A|_{[0, \pi]} : [0, \pi] \to \mathbb{R}$ is differentiable a.e., and moreover by Lebesgue differentiation theorem, we have the following observation.

**Lemma 3.13.** Given any Lebesgue measurable $A \subseteq [0, \pi]$, consider the derivative $(h_A|_{[0, \pi]})'$. Then, we have that $(h_A|_{[0, \pi]})'(x) = 1$ when $x \in A$ and $(h_A|_{[0, \pi]})'(x) = -1$ when $x \in [0, \pi] \setminus A$ a.e.

---

1Here $A \Delta B := A \setminus B \cup B \setminus A$ denotes the symmetric difference between two sets.
It turns out that this phenomenon holds for every extreme point in $F(S^1)$:

**Proposition 3.14.** $h \in F(S^1)$ is an extreme point if and only if $|h'| \equiv 1$ on $[0, \pi]$ a.e.

*Proof.* Suppose first that $|h'| \equiv 1$ a.e. Assume that there exist $f, g \in F(S^1)$ such that $\frac{f + g}{2} = h$. Let $A := \{x \in [0, \pi] : h'(x) = 1\}$. Then, $h'(x) = 1$ for $x \in A$ and $h'(x) = -1$ for $x \in [0, \pi] \setminus A$ almost everywhere. Since both $f$ and $g$ are 1-Lipschitz, they are differentiable a.e. and $|f'|, |g'| \leq 1$. Since $\frac{f + g}{2} = h$, we have that $f'(x) = g'(x) = 1$ for $x \in A$ a.e. and $f'(x) = g'(x) = -1$ for $x \in [0, \pi] \setminus A$ a.e. In other words, $f' = g'$ a.e. on $[0, \pi]$. Since $f(0) + f(\pi) = g(0) + g(\pi) = \pi$, by the fundamental theorem of calculus with respect to absolute continuous functions, we must have that $f = g = h$. Therefore, $h$ is an extreme point.

Conversely, assume that $h$ is an extreme point. Let $\text{Lip}_1([0, \pi], 0)$ denote the set of 1-Lipschitz functions $f : [0, \pi] \to \mathbb{R}$ such that $f(0) = 0$. Then, $f \in \text{Lip}_1([0, \pi], 0)$ is an extreme point of $\text{Lip}_1([0, \pi], 0)$ if and only if $|f'| = 1$ a.e. (cf. [Rol84, Proposition 6]). Now, it suffices to prove that $h_0 := h - h(0)$ is an extreme point in $\text{Lip}_1([0, \pi], 0)$ which implies that $|h' = (h - h(0))'| = 1$ a.e. Assume that there exist $f_0, g_0 \in \text{Lip}_1([0, \pi], 0)$ such that $h_0 = \frac{f_0 + g_0}{2}$. Define $f := f_0 + \frac{\pi - f_0(\pi)}{2}$ and $g := g_0 + \frac{\pi - g_0(\pi)}{2}$. It is easy to check that $f, g \in F(S^1)$ and that $h = \frac{f + g}{2}$. Since $h$ is an extreme point, we have that $f = g = h$ and thus $f_0 = g_0 = h_0$ which implies that $h_0$ is an extreme point as we required.

**Corollary 3.15.** $h_A \in E(S^1)$ is an extreme point for every measurable $A \subseteq [0, \pi]$.

**Lemma 3.16.** If $f \in E(S^1)$ is an extreme point, then there is $A \subseteq [0, \pi]$ such that $f = h_A$.

*Proof.* Let $A := \{x \in [0, \pi] : f'(x) = 1\}$. Then, $f'(x) = 1$ for $x \in A$ and $f'(x) = -1$ for $x \in [0, \pi] \setminus A$ a.e. Then, $(f|_{[0, \pi]})' = (h_A|_{[0, \pi]})'$ a.e. By the fundamental theorem of calculus for Lebesgue measure, we have that $f|_{[0, \pi]} = h_A|_{[0, \pi]}$ and thus $f = h_A$.

Therefore, the set $\{h_A\}_{A \subseteq [0, \pi]}$ (where $A$ is required to be Lebesgue measurable) coincides with the set of all extreme points in $E(S^1)$. Then, by the Krein-Milman theorem, we deduce the following theorem which, in analogy with Proposition 3.1, permits interpreting $\{h_A\}_{A \subseteq [0, \pi]}$ as the “set of vertices” of the infinite dimensional cube $E(S^1)$ (cf. Theorem 3.8).

**Theorem 3.17.** The closure of the convex hull of $\{h_A\}_{A \subseteq [0, \pi]}$ coincides with $E(S^1)$.

### 3.3 Homotopy types of open $r$-thickenings of $S^1$ within $E(S^1)$

Recall that, given any $r > 0$, $B_r(S^1, E(S^1))$ denotes the open $r$-thickening of $S^1$ in $E(S^1)$. When $r > \frac{\pi}{2}$, we have that $B_r(S^1, E(S^1)) = E(S^1)$ which is contractible. When $r \in (0, \frac{\pi}{2})$, the homotopy type of $B_r(S^1, E(S^1)) \subseteq E(S^1)$ can be characterized as follows. By Proposition 2.11 we have that $B_r(S^1, E(S^1))$ is homotopy equivalent to the Vietoris-Rips complex $VR_{2r}(S^1)$. Then, through the characterization of $VR_{2r}(S^1)$ in [AA17], we have that for any non-negative integer $n$, when $r \in \left(\frac{n\pi}{2n+1}, \frac{(n+1)\pi}{2n+3}\right)$, $B_r(S^1, E(S^1))$ is homotopy equivalent to $S^{2n+1}$.

Note that the case when $r = \frac{\pi}{2}$ is not covered in the above argument. In this section, (1) we consider the case $r = \frac{\pi}{2}$, i.e., we prove that $B_{\frac{\pi}{2}}(S^1, E(S^1))$ is contractible, and (2) we provide a geometric proof, which does not rely on the Vietoris-Rips complex structure, of the fact that $B_r(S^1, E(S^1))$ is homotopy equivalent to $S^1$ when $0 < r \leq \frac{\pi}{2}$. 

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The case $r = \frac{\pi}{2}$. We establish the following proposition.

**Proposition 3.18.** $B_{\frac{\pi}{2}}(S^1, E(S^1))$ is contractible.

To prove this result, we first observe that the $r$-neighborhood of $S^1$ in $E(S^1)$ is in fact the complement of the $(\frac{\pi}{2} - r)$-neighborhood of the center $f_0 \equiv \frac{\pi}{2}$ (cf. Proposition 2.16) in $E(S^1)$:

**Lemma 3.19.** For every $r \in [0, \frac{\pi}{2}]$, $B_r(S^1, E(S^1)) = E(S^1) \setminus \overline{B}_{\frac{\pi}{2} - r}(f_0, E(S^1))$, where $f_0 \equiv \frac{\pi}{2}$.

**Proof.** If $f \in \overline{B}_{\frac{\pi}{2} - r}(f_0, E(S^1))$, then $|f(\theta) - \frac{\pi}{2}| \leq \frac{\pi}{2} - r$ for every $\theta \in S^1$, which implies that $r \leq f(\theta) \leq \pi - r$ for every $\theta \in S^1$. Then, $\|f - d_{S^1}(\theta, \cdot)\| = f(\theta) \geq r$ (cf. item 2 in Remark 2.6) for every $\theta \in S^1$. Therefore, $f \notin B_r(S^1, E(S^1))$.

On the other hand, if $f \notin \overline{B}_{\frac{\pi}{2} - r}(f_0, E(S^1))$, then $\exists \theta \in S^1$ such that $|f(\theta) - \frac{\pi}{2}| > \frac{\pi}{2} - r$, which implies that $f(\theta) < r$ or $f(\theta) > \pi - r$. Then,

$$\|f - d_{S^1}(\theta, \cdot)\|_{\infty} = f(\theta) < r \quad \text{or} \quad \|f - d_{S^1}(\theta, \cdot)\|_{\infty} = f(\theta) = \pi - f(\theta) < r.$$ 

Therefore, $f \in B_r(S^1, E(S^1))$. □

**Proof of Proposition 3.18.** By Lemma 3.19, $B_{\frac{\pi}{2}}(S^1, E(S^1)) = E(S^1) \setminus \{f_0\}$. Therefore, $B_{\frac{\pi}{2}}(S^1, E(S^1))$ is the whole space $E(S^1)$ minus one point. By Proposition 3.4 and Theorem 3.8, $B_{\frac{\pi}{2}}(S^1, E(S^1))$ is homeomorphic to the Hilbert cube $\mathcal{S}$ minus one point, which is of the same homotopy type as the Hilbert cube itself (cf. [Cha72, page 330]) and is therefore contractible . □

The proof of Lemma 3.19 above entirely relies on the facts that $S^1$ is antipodal and $\text{diam}(S^1) = \pi$. Therefore, for every $n \in \mathbb{N}$, the proof (and thus the statement) of Lemma 3.19 can be easily generalized to the case of $S^n$ for $n \geq 2$. In this way, by Theorem 3.9 and using a argument similar to the one used for proving Proposition 3.18, we conclude that:

**Proposition 3.20.** $B_{\frac{\pi}{2}}(S^n, E(S^n))$ is contractible for every $n \in \mathbb{N}$.

The case $r \in (0, \frac{\pi}{2}]$. We now provide a direct proof\(^2\) of the fact that $B_r(S^1, E(S^1))$ is homotopy equivalent to $S^1$ when $0 < r \leq \frac{\pi}{2}$. This proof can be seen as being in the same spirit as but more direct than the one given by Katz in [Kat91, Section 3]. In fact, at the beginning of [Kat91, Section 3] the author comments:

"It is tempting to retract to the (suitably weighted) center of mass of the sublevel set $f \leq r$ which is contained in a semicircle. However, more work has to be done."

and then follows a different approach. Our proof departs from Katz’s in that the strategy we follow is exactly that of considering such a weighted center of mass map (cf. equations (4) and (5) below) and then directly checking that it gives us the required deformation retraction (cf. the proof of Theorem 3.25).

Recall that for every $f \in E(S^1)$, we have that $f(\theta) = \|f - d_{S^1}(\theta, \cdot)\|_{\infty}$ for any $\theta \in S^2$ (cf. item 2 in Remark 2.6). Then, $f^{-1}([0, r]) = \{\theta \in S^1 : \|f - d_{S^1}(\theta, \cdot)\|_{\infty} < r\}$.

We will also use the following version of Jung’s theorem:

\(^2\)Here “direct” means a proof which does not invoke the result by Adams and Adamszek described above.
Theorem 3.21 (Jung’s Theorem [Kat83, Lemma 2]). Every subset of $S^n$ with diameter less than or equal to $\arccos \left(-\frac{1}{n+1}\right)$ either coincides with the set of vertices of some inscribed regular $(n + 1)$-simplex, or is contained in some ball of radius $\pi - \arccos \left(-\frac{1}{n+1}\right)$.

Lemma 3.22. If $r \in (0, \frac{\pi}{3}]$ and $f \in E(S^1)$, $f^{-1}([0, r))$ is contained in an arc with length no longer than $2r$.

Proof. For every $\theta, \theta' \in f^{-1}([0, r))$, one has that $\|f - d_{\mathbb{S}^1}(\theta, \cdot)\|_\infty, \|f - d_{\mathbb{S}^1}(\theta', \cdot)\|_\infty < r$. Then,

$$d_{\mathbb{S}^1}(\theta, \theta') = \|d_{\mathbb{S}^1}(\theta, \cdot) - d_{\mathbb{S}^1}(\theta', \cdot)\|_\infty \leq \|f - d_{\mathbb{S}^1}(x, \cdot)\|_\infty + \|f - d_{\mathbb{S}^1}(x', \cdot)\|_\infty < 2r.$$  

This implies that $\text{diam}(f^{-1}([0, r))) \leq 2r \leq \frac{2\pi}{3}$. By Jung’s theorem (cf. Theorem 3.21), $f^{-1}([0, r))$ is contained in a semicircle and thus inside an arc with length not exceeding $2r$.

Let $s : \mathbb{R} \to \mathbb{R}/2\pi = S^1$ be the canonical quotient map. Given $r \in (0, \frac{\pi}{3}]$, let $f$ be in $B_r(S^1, E(S^1))$, then $f^{-1}([0, r))$ is contained in a semicircle (cf. Lemma 3.22). Then, there exists $\theta_f \in [0, 2\pi)$ such that $f^{-1}([0, r)) \subseteq s([\theta_f, \theta_f + \pi])$. Pick any such $\theta_f$ for every $f \in B_r(S^1, E(S^1))$.

Consider the following weighted center of mass:

$$n^r_f := \frac{\int_{\theta_f}^{\theta_f + \pi} x(r - f \circ s(x))1_{f^{-1}([0, r))}(s(x)) \, dx}{\int_{\theta_f}^{\theta_f + \pi} (r - f \circ s(x))1_{f^{-1}([0, r))}(s(x)) \, dx} \in \mathbb{R}$$

and let

$$m^r_f := s(n^r_f) \in S^1.$$  

(5)

See Figure 2 for an illustration of $n^r_f$ when $\theta_f = 0$. Note that since $f \in B_r(S^1, E(S^1))$, the denominator in Equation (4) is positive and thus $n^r_f$ is well-defined. Recall that $\theta_f \in [0, 2\pi)$ is arbitrarily chosen such that $f^{-1}([0, r)) \subseteq s([\theta_f, \theta_f + \pi])$. It is also not hard to see that $n^r_f$ does not depend on the choice of $\theta_f$.

Moreover, we have

$$m^r_{d_{\mathbb{S}^1}(\theta, \cdot)} = \theta \text{ for every } \theta \in S^1.$$  

(6)

Lemma 3.23. If $r \in (0, \frac{\pi}{3}]$, the map $m^r : B_r(S^1, E(S^1)) \to S^1$ taking $f$ to $m^r_f$ is a retraction.

Proof. We only need to prove the continuity of $m^r$. Fix an arbitrary $f \in B_r(S^1, E(S^1))$ and $\varepsilon > 0$. Next, choose an arbitrary $g \in B_r(S^1, E(S^1))$ such that $\|f - g\|_\infty < \varepsilon$. Let $B_f := f^{-1}([0, r))$, $B_g := g^{-1}([0, r))$ and $B := B_f \cap B_g$. Consider any $x \in s^{-1}(B_f \setminus B)$. Then, $g \circ s(x) \geq r$ and $g \circ s(x) - f \circ s(x) < \varepsilon$. Thus, $r - f \circ s(x) < \varepsilon$. Similarly, for every $x \in s^{-1}(B_g \setminus B)$, $r - g \circ s(x) < \varepsilon$.

Now, we show that $d_{\mathbb{S}^1}(m^r_f, m^r_g) \leq C_f \varepsilon$ where $C_f > 0$ is a constant depending on $f$. This will establish the continuity of $m^r$. By the requirement that $\theta_f, \theta_g \in [0, 2\pi)$, there are only two possible cases:

1. $s^{-1}(B) \cap [\theta_f, \theta_f + \pi] = s^{-1}(B) \cap [\theta_g, \theta_g + \pi].$
Figure 2: The construction of $n_r'$. The real value $n_r'$ is the projection of the barycenter of the red region onto the the x-axis. See equations (4) and (5) below.

2. $s^{-1}(B) \cap [\theta_f, \theta_f + \pi] = s^{-1}(B) \cap [\theta_g, \theta_g + \pi] \pm 2\pi$.

For the first case, we have that

$$\left| \int_{\theta_f}^{\theta_f + \pi} (r - f \circ s(x))1_{B_f}(s(x)) \, dx - \int_{\theta_g}^{\theta_g + \pi} (r - g \circ s(x))1_{B_g}(s(x)) \, dx \right|$$

$$= \left| \left( \int_{[\theta_f, \theta_f + \pi] \cap s^{-1}(B)} + \int_{[\theta_f, \theta_f + \pi] \cap s^{-1}(B_f \setminus B)} \right) (r - f \circ s(x)) \, dx \right.$$

$$- \left. \left( \int_{[\theta_g, \theta_g + \pi] \cap s^{-1}(B)} + \int_{[\theta_g, \theta_g + \pi] \cap s^{-1}(B_g \setminus B)} \right) (r - g \circ s(x)) \, dx \right|$$

$$\leq \int_{[\theta_f, \theta_f + \pi] \cap s^{-1}(B)} |g \circ s(x) - f \circ s(x)| \, dx + \int_{[\theta_f, \theta_f + \pi] \cap s^{-1}(B_f \setminus B)} |r - f \circ s(x)| \, dx$$

$$+ \int_{[\theta_g, \theta_g + \pi] \cap s^{-1}(B_g \setminus B)} |r - g \circ s(x)| \, dx$$

$$\leq 3\pi \varepsilon,$$

Similarly,

$$\left| \int_{\theta_f}^{\theta_f + \pi} x(r - f \circ s(x))1_{B_f}(s(x)) \, dx - \int_{\theta_g}^{\theta_g + \pi} x(r - g \circ s(x))1_{B_g}(s(x)) \, dx \right| \leq 9\pi^2 \varepsilon.$$
For notational simplicity, we let
\[ S_f := \int_{\theta_f}^{\theta_f + \pi} (r - f \circ s(x))1_{f^{-1}(\{0,r\}))(s(x)) \ dx \]
and
\[ E_f := \int_{\theta_f}^{\theta_f + \pi} x(r - f \circ s(x))1_{f^{-1}(\{0,r\}))(s(x)) \ dx. \]

We also define \( S_g \) and \( E_g \) in a similar manner. Then, \( |S_f - S_g| \leq 3\varepsilon \) and \( |E_f - E_g| \leq 9\pi\varepsilon^2 \). Since \( S_f > 0 \), we can choose \( \varepsilon \) small enough so that \( S_g \geq S_f - 3\varepsilon > \frac{S_f}{2} \). With this choice of \( \varepsilon \), we then have
\[
|n_f^r - n_g^r| = \left| \frac{E_f - E_g}{S_f} \right| = \left| \frac{E_f S_g - E_g S_f}{S_f S_g} \right| \\
\leq 2 \left| \frac{E_f (S_g - S_f)}{S_f^2} \right| + 2 \left| \frac{(E_f - E_g) S_f}{S_f^2} \right| \\
\leq 2(3\pi E_f + 9\pi^2 S_f) \varepsilon.
\]

Hence, by the continuity of the quotient map \( s \), we have that \( d_{S^1}(m_f^r, m_g^r) \leq C_f \varepsilon \), where the constant \( C_f := \frac{2(3\pi E_f + 9\pi^2 S_f)}{S_f^2} > 0 \) depends only on \( f \).

Now for the second case, without loss of generality one can assume \( s^{-1}(B) \cap [\theta_f, \theta_f + \pi] = s^{-1}(B) \cap [\theta_g, \theta_g + \pi] - 2\pi \). By the periodicity of \( g \circ s \) and \( s \), one has
\[
\int_{\theta_g}^{\theta_g + \pi} x(r - g \circ s(x))1_{B_g}(s(x)) \ dx = \int_{\theta_g - 2\pi}^{\theta_g + \pi - 2\pi} x(r - g \circ s(x))1_{B_g}(s(x)) \ dx \\
= \int_{\theta_g - 2\pi}^{\theta_g + \pi - 2\pi} (r - g \circ s(x))1_{B_g}(s(x)) \ dx + 2\pi.
\]

Then \( m_g^r = s(n_g^r) = s(n_f^r) \). Take \( \theta_g' = \theta_g - 2\pi \). Then \( s^{-1}(B) \cap [\theta_f, \theta_f + \pi] = s^{-1}(B) \cap [\theta_g', \theta_g' + \pi] \)
and thus the argument for the first case applies here to conclude that \( d_{S^1}(m_f^r, m_g^r) \leq C_f' \varepsilon \), where \( C_f' > 0 \) is some constant depending only on \( f \).

In conclusion, \( m_f^r : B_r(S^1, E(S^1)) \to S^1 \) is continuous and, in view of (6), it is therefore a retraction. \( \square \)

**Lemma 3.24.** For every \( r \in (0, \frac{\pi}{3}] \) and \( f \in B_r(S^1, E(S^1)) \), there exists \( \theta \in S^1 \) such that \( d_{S^1}(\theta, m_f^r) < r \) and \( \|f - d_{S^1}(\theta, \cdot)\|_{\infty} < r \).

**Proof.** In fact, this follows directly from Lemma 3.22. Since \( f^{-1}(\{0,r\}) \) is contained in an arc with length no longer than \( 2r \), \( m_f^r \) also lies in this arc. More precisely, \( m_f^r \) belongs to the geodesic convex hull of \( f^{-1}(\{0,r\}) \). Then, there exists \( \theta \in f^{-1}(\{0,r\}) \) such that \( d_{S^1}(\theta, m_f^r) < r \). Since \( \theta \in f^{-1}(\{0,r\}) \), one has that \( \|f - d_{S^1}(\theta, \cdot)\|_{\infty} = f(\theta) < r \) \( \square \)

**Theorem 3.25.** For every \( r \in (0, \frac{\pi}{3}] \), \( B_r(S^1, E(S^1)) \) is homotopy equivalent to \( S^1 \).

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Proof. Recall that \( \kappa_{S^1} : S^1 \rightarrow B_r(S^1, E(S^1)) \) denotes the Kuratowski embedding. We build a homotopy
\[
H : B_r(S^1, E(S^1)) \times [0, 1] \rightarrow B_r(S^1, E(S^1))
\]
between \( \kappa_{S^1} \circ m^r : B_r(S^1, E(S^1)) \rightarrow B_r(S^1, E(S^1)) \) and \( \text{id} : B_r(S^1, E(S^1)) \rightarrow B_r(S^1, E(S^1)) \) as follows. For every \( (f, t) \in B_r(S^1, E(S^1)) \times [0, 1] \), let \( H(f, t) := tf + (1 - t)d_{S^1}(m_f^r, \cdot) \). We need to show \( H \) is well-defined, i.e., the image of \( H \) lies in \( B_r(S^1, E(S^1)) \). In fact, by Lemma 3.24, there is \( x \in S^1 \) such that \( \| d_{S^1}(x, \cdot) - f \|_\infty < r \) and \( \| d_{S^1}(x, \cdot) - d_{S^1}(m_f^r, \cdot) \|_\infty < r \). Hence
\[
\| tf + (1 - t)d_{S^1}(m_f^r, \cdot) - d_{S^1}(x, \cdot) \|_\infty \leq t \| f - d_x \|_\infty + (1 - t) \| d_{S^1}(x, \cdot) - d_{S^1}(m_f^r, \cdot) \|_\infty < r.
\]
Thus \( H(f, t) \in B_r(S^1, E(S^1)) \). It is then easy to check that \( H \) is continuous and thus \( S^1 \) is a deformation retraction of \( B_r(S^1, E(S^1)) \).

\[ \square \]

4 Mountain range functions and the tight span

In this section, we first review the notion of \textit{mountain range function} that was introduced by Katz in [Kat91]. We then study the problem of determining sufficient conditions for a mountain range function on a sphere to be an element in the tight span of sphere.

\textbf{Definition 14.} Let \( X \) be a metric space and \( P = \{p_1, \cdots, p_n\} \subseteq X \) be a finite subset along with a set of real numbers \( V = \{v_1, \cdots, v_n\} \). We define the \textit{mountain range function} \( \text{MR}_X(P, V) : X \rightarrow \mathbb{R} \) as
\[
\text{MR}_X(P, V)(x) := \min_{i=1,\cdots,n} (d_X(x, p_i) + v_i).
\]

\textbf{Remark 4.1.} For a mountain range function \( \text{MR}_X(P, V) \), if all \( v_i \) are equal to the same number \( v \), obtain that \( \text{MR}_X(P, V)(x) \) equals \( d_X(x, P) + v \) where \( d_X(x, P) \) denotes the distance function to the set \( P \).

Now, we give conditions under which a mountain range function on \( X \) is an element in \( \Delta_1(X) \) (cf. Definition 5).

\textbf{Proposition 4.2.} Let \( X \) be a compact metric space and let \( P \) be a finite subset equipped with the induced metric. For every function \( f \in \Delta(P) \), we associate it with a mountain range function \( \text{MR}_X(P, f) \) in the following way,
\[
\text{MR}_X(P, f)(x) := \min_{p_i \in P} (d_X(x, p_i) + f(p_i))
\]
Then \( \text{MR}_X(P, f) \in \Delta_1(X) \). Moreover, if \( f \in \Delta_1(P) \), then \( \text{MR}_X(P, f) \) is an extension of \( f \) onto \( X \).

\textbf{Proof.} Let \( f \) be a function in \( \Delta(P) \), then for every \( p_i, p_j \in P \), we have \( f(p_i) + f(p_j) \geq d_X(p_i, p_j) \). We first show that \( \text{MR}_X(P, f) \in \Delta(X) \). For any \( x, x' \in X \), there are some \( i, j \) such that \( \text{MR}_X(P, f)(x) = d_X(x, p_i) + f(p_i) \) and \( \text{MR}_X(P, f)(x') = d_X(x', p_j) + f(p_j) \). Therefore,
\[
\text{MR}_X(P, f)(x) + \text{MR}_X(P, f)(x') = d_X(x, p_i) + f(p_i) + d_X(x', p_j) + f(p_j)
\geq d_X(x, p_i) + d_X(p_i, p_j) + d_X(p_j, x')
\geq d_X(x, x').
\]
As $X$ is compact, $\text{MR}_X(P, f)$ is a bounded function and hence, $\text{MR}_X(P, f) \in \Delta(X)$.

Next, we need to show $\text{MR}_X(P, f)$ is a 1-Lipschitz function. For any two points $x, x'$ in $X$, let $p_j \in P$ be such that $\text{MR}_X(P, f)(x') = d_X(x, p_j) + f(p_j)$, as before. Then, we have

$$\text{MR}_X(P, f)(x) - \text{MR}_X(P, f)(x') \leq (d_X(x, p_j) + f(p_j)) - (d_X(x', p_j) - f(p_j))$$

$$= d_X(x, p_j) - d_X(x', p_j)$$

$$\leq d_X(x, x').$$

Similarly, we have $\text{MR}_X(P, f)(x') - \text{MR}_X(P, f)(x) \leq d_X(x, x')$. This shows that $\text{MR}_X(P, f)$ is a 1-Lipschitz function and $\text{MR}_X(P, f)$ belongs $\Delta_1(X)$.

If $f$ is inside $\Delta_1(X)$, then $f$ is furthermore 1-Lipschitz. For fixed $p_i \in P$ and any other $p_j \in P$, we have

$$d_X(p_i, p_i) + f(p_i) \leq f(p_j) + d_X(p_i, p_j).$$

Therefore, $\text{MR}_X(P, f)(p_i) = f(p_i)$ for every $p_i \in P$, that is, $\text{MR}_X(P, f)$ is an extension of $f$. □

**Remark 4.3.** The above construction gives the following commutative diagram:

$$
\begin{array}{ccc}
P & \xleftarrow{\text{MR}} & X \\
\kappa_P & \downarrow & \kappa_X \\
\Delta_1(P) & \xleftarrow{\text{MR}} & \Delta_1(X)
\end{array}
$$

where the vertical maps are Kuratowski embeddings.

Now we focus on the case where $X$ is a sphere and $f$ is a constant function. It turns out that for $\text{MR}_{\mathbb{S}^n}(P, f)$ to be an element in the tight span $\mathcal{E}(\mathbb{S}^n)$, $P$ must be a special configuration that we will now introduce. For any two points $x, x' \in P$, we say that $x$ and $x'$ are comaximal if $d_X(x, x') = \text{diam}(P)$. We use the notation $\text{comax}_P(x)$ to denote the set of points $x'$ in $P$ that is comaximal with $x$.

**Definition 15** ([Kat89]). Let $P$ be a subset of $\mathbb{S}^n$ containing no antipodal pairs. We say that a point $p \in P$ is held by $P$ if for every tangent vector $v \in T_p\mathbb{S}^n$, there exists a point $p' \in \text{comax}_P(p)$ such that the inner product $\langle v, \exp_p^{-1}(p') \rangle_{T_p\mathbb{S}^n} \geq 0$, where $\exp_p^{-1}(p')$ is the unit tangent vector at $p$ that is tangent to the unique shortest geodesic connecting $p$ to $p'$.

We say that $P$ is pointwise extremal if every point $p \in P$ is held by $P$.

**Remark 4.4.** In $\mathbb{S}^1$, a finite subset is pointwise extremal if and only if it is the vertex set of an inscribed regular odd $n$-gon, see [Kat91, Lemma 4.3].

**Proposition 4.5.** If $P$ is a finite subset of $\mathbb{S}^n$ with no antipodal pairs. Then $P$ is pointwise extremal if and only if for every $p_i \in P$, the function $d_{\mathbb{S}^n}(\cdot, P)$ attains a local maximum at $\bar{p}_i$, the antipodal of $p_i$.

**Proof.** Let $p \in P$ be a fixed point in $P$. If $p$ is held by $P$, then for every $v \in T_p\mathbb{S}^n$, there exists a point $p_i \in \text{comax}_P(p)$ such that the inner product $\langle -v, \exp_p^{-1}(p_i) \rangle \geq 0$, that is $\langle v, \exp_p^{-1}(p_i) \rangle \leq 0$. This is equivalent to the function

$$\Phi_p(x) = \max_{p_i \in \text{comax}_P(p)} d_{\mathbb{S}^n}(p_i, x)$$
obtaining a local minimum at \( p \). Note that, as \( P \) is a finite set, for every \( x \) in a small neighborhood around \( \bar{p} \), we have
\[
d_{S^n}(x, P) = \min_{p_i \in \text{comax}_{P}(p)} d_{S^n}(x, p_i) = \min_{p_i \in \text{comax}_{P}(p)} (\pi - d_{S^n}(\bar{x}, p_i)) = \pi - \Phi_p(\bar{x}).
\]
Therefore, \( d_{S^n}(\cdot, P) \) attaining a local maximum at \( \bar{p} \) is equivalent to \( \Phi_p(\cdot) \) attaining a local minimum at \( p \) which is equivalent to \( p \) being held by \( P \). \hfill \Box

**Proposition 4.6.** Let \( P \) be a finite subset of \( S^n \) without antipodal pairs. If the function
\[
\text{MR}_{S^n}(P, a)(x) = d_{S^n}(x, P) + a
\]
is inside \( E(S^n) \), then a equals \( \frac{1}{2} \text{diam}(P) \) and \( P \) is a pointwise extremal set.

**Proof.** Fix a \( p \) in \( P \) such that there is some \( p' \in P \) with \( d_{S^n}(p, p') = \text{diam}(P) \). Then we have \( d_{S^n}(\bar{p}, P) = \pi - \text{diam}(P) \). As \( \text{MR}_{S^n}(P, a)(x) \in E(S^n) \), we have
\[
\pi = \text{MR}_{S^n}(P, a)(p) + \text{MR}_{S^n}(P, a)(\bar{p}) = a + d_{S^n}(\bar{p}, P) + a
\]
This shows \( d_{S^n}(\bar{p}, P) = \pi - 2a \), that is, \( a = \frac{1}{2} (\pi - d_{S^n}(\bar{p}, P)) = \frac{1}{2} \text{diam}(P) \). For every \( p_i \in P \), note that, the function \( \text{MR}_{S^n}(P, a)(x) \) obtains minimum at \( p_i \). Furthermore, according to the equality
\[
\pi = \text{MR}_{S^n}(P, a)(p_i) + \text{MR}_{S^n}(P, a)(\bar{p}_i),
\]
we deduce that \( \text{MR}_{S^n}(P, a)(x) \) obtains maximum at \( \bar{p}_i \). Therefore, the function \( d_{S^n}(\cdot, P) \) obtains local maximum at \( \bar{p}_i \) as well. By Proposition 4.5, we know that \( p_i \) is held by \( P \). This shows that \( P \) is pointwise extremal. \hfill \Box

On the other hand, for every pointwise extremal configuration on \( S^1 \), we get a function inside the \( E(S^1) \) from the mountain range function construction, see Remark 4.7 below. However, it is not the case for higher-dimensional spheres.

**Remark 4.7.** Let \( P \) be the vertex set of an inscribed regular odd \( n \)-gon in \( S^1 \). Then it is clear that the function
\[
\text{MR}_{S^1} \left( P, \frac{1}{2} \text{diam}(P) \right)(\theta) = d_{S^1}(\theta, P) + \frac{1}{2} \text{diam}(P)
\]
is \( 1 \)-Lipschitz and satisfies the property that the sum of function values on any antipodal pair is \( \pi \) and therefore by Lemma 2.15 this function is in \( E(S^1) \).

**Remark 4.8.** The above mountain range function construction for a pointwise extremal configurations on higher dimensional spheres does not necessarily produce a function in the tight span. In fact, let \( P \) be a subset of \( S^n \) such that the mountain range function
\[
\text{MR}_{S^n} \left( P, \frac{1}{2} \text{diam}(P) \right)(x) = d_{S^n}(x, P) + \frac{1}{2} \text{diam}(P)
\]
is inside \( E(S^n) \). If \( n \geq 2 \) then \( P \) must contain infinitely many points. First we observe that if \( \text{diam}(P) = \pi \) then \( \text{MR}_{S^n}(P, \frac{1}{2} \text{diam}(P))(x) \geq \frac{\pi}{2} \) for every \( x \) and by Lemma 2.15, we have \( d_{S^n}(x, P) = 0 \) for every \( x \) and hence, \( P \) is the whole sphere \( S^n \). Therefore, for other cases, we can assume \( P \) does not contain a pair of antipodal points. Fix a point \( p_i \in P \) and let \( \bar{p}_i \) be its antipodal point. If \( P \) is finite, then there exists some \( \epsilon > 0 \) such that for every point \( x \in B_i(p_i, S^n) \), \( d_{S^n}(x, P) = d_{S^n}(x, p_i) \). Note that, for \( \text{MR}_{S^n}(P, \frac{1}{2} \text{diam}(P)) \) to be a function in the tight span \( E(S^n) \), it must satisfy
• MR_{S^n}(P, \frac{1}{2}\text{diam}(P))(p_i) + MR_{S^n}(P, \frac{1}{2}\text{diam}(P))(\bar{p}_i) = \pi
• MR_{S^n}(P, \frac{1}{2}\text{diam}(P))(x) + MR_{S^n}(P, \frac{1}{2}\text{diam}(P))(\bar{x}) = \pi

From our assumption on \(x\), we have

\[
MR_{S^n} \left( P, \frac{1}{2}\text{diam}(P) \right) (x) = MR_{S^n} \left( P, \frac{1}{2}\text{diam}(P) \right) (p_i) + d_{S^n}(x, p_i).
\]

Therefore,

\[
d_{S^n}(\bar{x}, P) = MR_{S^n} \left( P, \frac{1}{2}\text{diam}(P) \right) (\bar{x}) - \frac{1}{2}\text{diam}(P)
= MR_{S^n} \left( P, \frac{1}{2}\text{diam}(P) \right) (\bar{p}_i) - d_{S^n}(x, p_i) - \frac{1}{2}\text{diam}(P)
= MR_{S^n} \left( P, \frac{1}{2}\text{diam}(P) \right) (\bar{p}_i) - d_{S^n}(\bar{x}, \bar{p}_i) - \frac{1}{2}\text{diam}(P)
= d_{S^n}(\bar{p}_i, P) - d_{S^n}(\bar{x}, \bar{p}_i)
\]

This implies that there must be a point \(p_\nu\) in \(\text{comax}_P(p_i)\) such that \(\bar{x}, \bar{p}_i, p_\nu\) lie on a geodesic of \(S^n\). When \(n \geq 2\), there are infinitely many geodesic circles based at \(\bar{p}_i\) such that pairwise intersections are exactly \(\{p_i, \bar{p}_i\}\). Therefore, by varying \(\bar{x} \in B_1(P)\), there must be infinitely many different \(p_\nu\) for different choices of \(x\) and thus \(P\) consists of infinitely many points.

Nevertheless, the following rotated odd \(n\)-gon on \(S^n\) gives a function in the tight span through the mountain range function construction. Let us introduce a convenient notion for subsets of \(S^n\) that give rise to an element in \(E(S^n)\) through the mountain range function construction.

**Definition 16.** Let \(P\) be a subset of \(S^n\), if the mountain range function

\[
MR_{S^n} \left( P, \frac{1}{2}\text{diam}(P) \right) (x) = d_{S^n}(x, P) + \frac{1}{2}\text{diam}(P)
\]

is inside \(E(S^n)\), we then say \(P\) is an admissible subset in \(S^n\).

The following example shows that we can get admissible subsets of \(S^n\) by iteratively revolving odd \(n\)-gon on \(S^1\).

**Example 4.9 (Examples of Mountain range functions inside \(E(S^n)\)).** For every positive integer \(m\), the regular \((2m+1)\)-gon \(P^1_m\) and the constant \(\frac{1}{2}\text{diam}(P^1_m) = \frac{m\pi}{2m+1}\) give rise to a mountain range function \(\text{MR}(P^1_m, \frac{m\pi}{2m+1}) \in E(S^1)\). Based on \(P^1_m\), we will introduce \(P^n_m \subseteq S^n\) through rotations for every integer \(n \geq 1\) such that \(\text{MR}(P^n_m, \frac{m\pi}{2m+1}) \in E(S^n)\). Consider the spherical coordinate for \(S^n \subseteq \mathbb{R}^{n+1}\) as follows

\[
x_1 = \cos(\varphi_1) \\
x_2 = \sin(\varphi_1) \cos(\varphi_2) \\
x_3 = \sin(\varphi_1) \sin(\varphi_2) \cos(\varphi_3) \\
\vdots \\
x_n = \sin(\varphi_1) \sin(\varphi_2) \cdots \sin(\varphi_{n-1}) \cos(\varphi_n) \\
x_{n+1} = \sin(\varphi_1) \sin(\varphi_2) \cdots \sin(\varphi_{n-1}) \sin(\varphi_n)
\]
where $\varphi_1 \in [0, \pi]$ and $\varphi_k \in [0, 2\pi)$ for all $k = 2, \ldots, n$. Assume that $P^1_m$ is represented by polar coordinates $\left\{ \frac{2k\pi}{2m+1} | k = 0, 1, \ldots, 2m+1 \right\}$. We let

$$Q_m := \left\{ \frac{2k\pi}{2m+1} | k = 0, 1, \ldots, 2m+1 \right\} \cap [0, \pi] = \left\{ \frac{2k\pi}{2m+1} | k = 0, 1, \ldots, m+1 \right\}.$$ 

Then, we let $P^n_m := Q_m \times [0, 2\pi]^{n-1}$ denote a set of points in $S^n$ in terms of spherical coordinate. For every point $x \in S^n$ represented by spherical coordinate $(\psi_1, \ldots, \psi_n)$, consider the great circle $S^1_x$ containing $x$ and $(0, \ldots, 0)$ represented by the following set of equations: $(\varphi_1, \ldots, \varphi_n) \in S^1_x$ iff $\varphi_k = \psi_k$ for all $k = 2, \ldots, n$. Of course, $(0, \varphi_2, \ldots, \varphi_n)$ and $(0, \ldots, 0)$ represent the same point in $S^n$. Then, $S^1_x \cap P^n_m = \left\{ \left( \frac{2k\pi}{2m+1}, \psi_2, \ldots, \psi_n \right) | k = 0, 1, \ldots, m+1 \right\}$ is itself a regular $(2m+1)$-gon, denoted by $P^1_{m,x}$. Since the antipodal point $\bar{x} \in S^1_x$, we have that

$$\text{MR} \left( P^n_m, \frac{m\pi}{2m+1} \right) (x) + \text{MR} \left( P^n_m, \frac{m\pi}{2m+1} \right) (\bar{x})$$

$$= \text{MR} \left( P^1_{m,x}, \frac{m\pi}{2m+1} \right) (x) + \text{MR} \left( P^1_{m,x}, \frac{m\pi}{2m+1} \right) (\bar{x})$$

$$= \pi,$$

where the last equality follows from $\text{MR} \left( P^1_{m,x}, \frac{m\pi}{2m+1} \right) \in \text{E}(S^1_x)$ and Lemma 2.15. It is obvious that $\text{MR} \left( P^n_m, \frac{m\pi}{2m+1} \right)$ is 1-Lipschitz. Then, by Lemma 2.15 again, we have that $\text{MR} \left( P^n_m, \frac{m\pi}{2m+1} \right) \in \text{E}(S^n)$.

In general, we have the following result regarding revolved admissible sets.

**Proposition 4.10.** Let $P_{n+1}$ be a subset of $S^{n+1}$ such that there exists a point $a \in P_{n+1}$ with the following properties,

1. any rotation at the point $a$ preserves $P_{n+1}$,

2. for any $S^n$ that contains both $a$ and $\bar{a}$, that is a slice under the rotation at $a$, $P_{n+1} \cap S^n$ is admissible in $S^n$. 

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Then, $P_{n+1}$ is admissible in $\mathbb{S}^{n+1}$.

**Proof.** Let $x$ be a point in $\mathbb{S}^{n+1}$, then there exists a $\mathbb{S}^n$ slice that contains $x$, $a$ and $\bar{a}$. For any point $p \in P_{n+1}$, the orbit of $p$ under the rotation intersects perpendicularly with the above chosen $\mathbb{S}^n$ at a point $p'$. Therefore, $d_{\mathbb{S}^n}(x, p') = d_{\mathbb{S}^{n+1}}(x, p') \leq d_{\mathbb{S}^{n+1}}(x, p)$ which further implies

$$d_{\mathbb{S}^{n+1}}(x, P_{n+1}) = d_{\mathbb{S}^n}(x, P_{n+1} \cap \mathbb{S}^n).$$

This allows us to show $\text{diam}(P_{n+1}) = \text{diam}(P_{n+1} \cap \mathbb{S}^n)$ where $\mathbb{S}^n$ is a slice under rotation. Note that, as rotations are isometries and $P_{n+1}$ is preserved under rotations, the quantity $\text{diam}(P_{n+1} \cap \mathbb{S}^n)$ is independent of the choice of slice. For any point $p$ in $P_{n+1}$, the diameter of $P_{n+1}$ at $p$ is

$$\max_{p' \in P_{n+1}} d_{\mathbb{S}^{n+1}}(p', p) = \pi - d_{\mathbb{S}^{n+1}}(\bar{p}, P_{n+1}) = \pi - d_{\mathbb{S}^n}(\bar{p}, P_{n+1} \cap \mathbb{S}^n) = \max_{p' \in P_{n+1} \cap \mathbb{S}^n} d_{\mathbb{S}^n}(p', p),$$

where $\mathbb{S}^n$ is a slice containing $p$ and hence, $\bar{p}$. As $p$ is arbitrary, we get $\text{diam}(P_{n+1}) = \text{diam}(P_{n+1} \cap \mathbb{S}^n)$. We then have the following calculation.

$$\text{MR}_{\mathbb{S}^n} \left( P_{n+1}, \frac{1}{2} \text{diam}(P_{n+1}) \right)(x) + \text{MR}_{\mathbb{S}^n} \left( P_{n+1}, \frac{1}{2} \text{diam}(P_{n+1}) \right)(\bar{x})$$

$$= d_{\mathbb{S}^n}(x, P_{n+1})(x) + \frac{1}{2} \text{diam}(P_{n+1}) + d_{\mathbb{S}^n}(x, P_{n+1})(\bar{x}) + \frac{1}{2} \text{diam}(P_{n+1})$$

$$= d_{\mathbb{S}^n}(x, P_{n+1} \cap \mathbb{S}^n)(x) + \frac{1}{2} \text{diam}(P_{n+1} \cap \mathbb{S}^n) + d_{\mathbb{S}^n}(x, P_{n+1} \cap \mathbb{S}^n)(\bar{x}) + \frac{1}{2} \text{diam}(P_{n+1} \cap \mathbb{S}^n)$$

$$= \pi$$

The last line comes from the fact that $P_{n+1} \cap \mathbb{S}^n$ is admissible and applying Lemma 2.15. As $\text{MR}_{\mathbb{S}^n} \left( P_{n+1}, \frac{1}{2} \text{diam}(P_{n+1}) \right)$ is a 1-Lipschitz function by Proposition 4.2, the result then comes from the characterization of functions in the tight span as in Lemma 2.15.

## 5 The case of $\mathbb{S}^n$ with $\ell^\infty$-metric

Let $n$ be a positive integer. For simplicity of notation, let $d_{\infty}$ denote the metric induced by $\ell^\infty$-norm on $\mathbb{R}^n$, and we write $\mathbb{R}_n^\infty := (\mathbb{R}^n, \ell^\infty)$. We denote by $\mathbb{S}_n^\infty := \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ the $n$-dimensional sphere equipped with $\ell^\infty$-metric, and similarly $\mathbb{D}_n^{n+1} := \{x \in \mathbb{R}^{n+1} : \|x\|_2 \leq 1\}$ the $(n + 1)$-dimensional ball equipped with $\ell^\infty$-metric. In this section, we study the tight span of $\mathbb{S}^\infty_n$.

### The tight span of $\mathbb{S}^\infty_n$. We start with the case of $\mathbb{S}^\infty_n \subset \mathbb{R}^2_\infty$. Notice that $\mathbb{R}^2_\infty$ and the Manhattan plane $(\mathbb{R}^2, \ell^1)$ are isometric, but for $n > 2$, $\mathbb{R}^n_\infty$ and $(\mathbb{R}^n, \ell^1)$ are not isometric. In particular, while $\mathbb{R}^n_\infty$ is hyperconvex for every $n$, $(\mathbb{R}^n, \ell^1)$ is hyperconvex only when $n \leq 2$. On the other hand, the $\ell^\infty$ and $\ell^1$ metrics on real spaces are closely related: [Her92, Theorem 5] proves that the space $\mathbb{R}^{n+1}_\infty$ is isometric to the tight span of $(\mathbb{R}^n, \ell^1)$.

The tight span of a subset of $\mathbb{R}^2$ equipped with either the $\ell^\infty$ or the $\ell^1$-norm has been completely studied in [Epp11, KK15, KK16], in which case the tight span is the same as the orthogonal convex hull. In addition, in [KKÖ21], the authors develop a procedure to compute the tight span of finite subsets of $(\mathbb{R}^2, \ell^1)$. 

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In $\mathbb{R}^2_\infty$, a subspace $Y$ is said to be geodesically convex if any two points in $Y$ are connected by a geodesic that is fully contained in $Y$. By [KK16, Theorem 2], the tight span of $X \subseteq \mathbb{R}^2_\infty$ is isometric to any closed geodesically convex subspace $Y$ containing $X$, which is minimal (w.r.t. inclusion) with these properties. From this, we have:

**Corollary 5.1.** The tight span of $\mathbb{S}^1_\infty$ is isometric to $\mathbb{D}^2_\infty$.

Unlike the case of $\mathbb{R}^2_\infty$, when $n \geq 3$ the tight span of subsets of $\mathbb{R}^n_\infty$ appear difficult to describe. Some related results in the literature include studies of the injectivity of subsets of $\mathbb{R}^n_\infty$, such as [Pav16] for the injectivity of convex polyhedrons in $\mathbb{R}^n_\infty$ and [DP17] for the injectivity of more general subsets of $\mathbb{R}^n_\infty$. In this section, we study the tight span of $\mathbb{S}^n_\infty$, as well as the injectivity of $\mathbb{D}^{n+1}_\infty$.

In Section 5.1, for a given compact set $X \subseteq \mathbb{R}^n_\infty$ we introduce a set $\mathcal{F}(X) \subseteq \mathbb{R}^n_\infty$ that is a sub-metric space of the tight span $\mathcal{E}(X)$, and use it to show that $\mathbb{D}^{n+1}_\infty$ is not injective when $n \geq 2$. In Section 5.2, we identify an injective set $\mathcal{E}(X) \subseteq \mathbb{R}^n_\infty$ containing $\mathcal{E}(X)$, and prove that $\mathcal{E}(\mathbb{S}^2_\infty)$ (in this case, equal to $\mathcal{F}(\mathbb{S}^2_\infty)$) is isometric to $\mathcal{E}(\mathbb{S}^2_\infty)$.

### 5.1 The set of $X$-surrounding points $\mathcal{F}(X)$

In this section, we study a particular subspace $\mathcal{F}(X)$ of the tight span $\mathcal{E}(X)$, for a given compact set $X \subseteq \mathbb{R}^n_\infty$. We show that when $n \geq 2$, the tight span of $\mathbb{S}^n_\infty$ (or $\mathbb{D}^{n+1}_\infty$) properly contains $\mathbb{D}^{n+1}_\infty$ up to isometry.

As in [DP17], we divide $\mathbb{R}^n_\infty$ into the following cone-shaped regions. For each $1 \leq i \leq n$, let $
abla_i := \{x \in \mathbb{R}^n_\infty : x_i = \|x\|_\infty\}$. For each $x \in \mathbb{R}^n_\infty$, $\xi \in \{\pm 1\}$ and $i = 1, \ldots, n$, let

$$x + \xi \nabla_i := \{x + \xi z : z \in \nabla_i\} = \{z \in \mathbb{R}^n_\infty : z_i - x_i = \xi d_\infty(z, x)\}.$$ 

Clearly, $\mathbb{R}^n_\infty = \bigcup_{\xi \in \{\pm 1\}, i \in \{1, \ldots, n\}} (x + \xi \nabla_i)$. See Figure 4 for illustrations of $\xi \nabla_i$ in $\mathbb{R}^2_\infty$ and $\mathbb{R}^3_\infty$.

**Figure 4:** The regions $\xi \nabla_i$ in $\mathbb{R}^2_\infty$ and $\mathbb{R}^3_\infty$.

**Proposition 5.2.** Let $x, y \in \mathbb{R}^n_\infty$, for every $i \in \{1, \ldots, n\}$ and $\xi \in \{\pm 1\}$,
Proof. Part (1) is trivial. Part (2) is true, because for \( y \in x + \xi \Lambda_i \) and any \( z \in y + \xi \Lambda_i \), we have
\[
d_{\infty}(z, x) \geq \xi(z_i - x_i) = \xi(z_i - y_i) + \xi(y_i - x_i) = d_{\infty}(z, y) + d_{\infty}(y, x) \geq d_{\infty}(z, x)
\]
which implies \( \xi(z_i - x_i) = d_{\infty}(z, x) \), i.e. \( y + \xi \Lambda_i \subseteq x + \xi \Lambda_i \). The other inclusion is proved in a similar way. \( \square \)

**Definition 17** (X-surrounding points). For a compact subset \( X \subseteq \mathbb{R}^n_{\infty} \), we call a point \( p \in \mathbb{R}^n_{\infty} \) a **X-surrounding point** if \( (p + \xi \Lambda_i) \cap X \neq \emptyset \) for every \( i \in \{1, \ldots, n\} \) and \( \xi \in \{\pm 1\} \).

Denote \( \mathcal{F}(X) := \{p \in \mathbb{R}^n_{\infty}: p \text{ is a } X\text{-surrounding point}\} \), and call it the **set of X-surrounding points**.

Clearly, \( X \subseteq \mathcal{F}(X) \), and \( \mathcal{F}(X) \subseteq \mathcal{F}(Y) \) whenever \( X \subseteq Y \subseteq \mathbb{R}^n_{\infty} \).

For each \( z \in \mathbb{R}^n_{\infty} \), let \( d^X_{\infty}(z, \cdot) := d_{\infty}(z, \cdot)|_{X} \in L^\infty(X) \). We then have the following map:
\[
\iota_X : \mathbb{R}^n_{\infty} \longrightarrow L^\infty(X)
\]
\[
z \longmapsto d^X_{\infty}(z, \cdot)
\]
By definition, the set \( \iota_X^{-1}(E(X)) \) consists of all \( z \in \mathbb{R}^n_{\infty} \) such that the function \( d^X_{\infty}(z, \cdot) \) is minimal (see Definition 5).

**Proposition 5.3.** Let \( X \) be a compact subset of \( \mathbb{R}^n_{\infty} \). Then, for every X-surrounding point \( p \), \( d^X_{\infty}(p, \cdot) \) is minimal, i.e., \( \iota_X(\mathcal{F}(X)) \subseteq E(X) \). In addition, the map \( \iota_X : \mathcal{F}(X) \rightarrow E(X) \) preserves distances. Thus, \( \mathcal{F}(X) \subseteq E(X) \).

**Proof.** Fix an arbitrary \( p \in \mathcal{F}(X) \) and arbitrary \( x \in X \). Then, \( x \in p + \xi \Lambda_i \) for some \( i \in \{1, \ldots, n\} \) and \( \xi \in \{\pm 1\} \). Take \( y \in (p - \xi \Lambda_i) \cap X \neq \emptyset \) (because \( p \in \mathcal{F}(X) \)). Notice that
\[
d_{\infty}(x, y) \leq d^X_{\infty}(p, x) + d^X_{\infty}(p, y) = \xi(x_i - p_i) + (-\xi)(y_i - p_i) = \xi(x_i - y_i) \leq d_{\infty}(x, y).
\]
Hence, \( d^X_{\infty}(p, x) + d^X_{\infty}(p, y) = d_{\infty}(x, y) \) so that \( d^X_{\infty}(p, \cdot) \) is minimal. i.e., \( d^X_{\infty}(p, \cdot) \in E(X) \).

We now prove that \( \iota_X : \mathcal{F}(X) \rightarrow E(X) \) preserves distances. Fix arbitrary \( p, q \in \mathcal{F}(X) \). One can assume that \( q \in p + \xi \Lambda_i \) for some \( i \in \{1, \ldots, n\} \) and \( \xi \in \{\pm 1\} \). Because \( q \in \mathcal{F}(X) \), \( (q + \xi \Lambda_i) \cap X \neq \emptyset \). Take \( x \in (q + \xi \Lambda_i) \cap X \subseteq (p + \xi \Lambda_i) \cap X \) where the last inclusion holds by the item (2) of Proposition 5.2. Therefore,
\[
d^X_{\infty}(p, x) - d^X_{\infty}(q, x) = \xi(x_i - p_i) - \xi(x_i - q_i) = \xi(q_i - p_i) = d_{\infty}(p, q).
\]
Thus, \( \|d^X_{\infty}(p, \cdot) - d^X_{\infty}(q, \cdot)\|_{\infty} \geq d_{\infty}(p, q) \). The reverse inequality \( \|d^X_{\infty}(p, \cdot) - d^X_{\infty}(q, \cdot)\|_{\infty} \leq d_{\infty}(p, q) \) follows directly from the triangle inequality. Therefore, \( \iota_X : \mathcal{F}(X) \rightarrow E(X) \) preserves distances, implying that \( \mathcal{F}(X) \subseteq E(X) \). \( \square \)
Example 5.4. Notice that $\mathcal{F}(X)$ is not necessarily injective. Consider the example $X := (0, 3) \cup \left\{0 \right\} \times [-1, 1] \subseteq \mathbb{R}^2$, for which we have $X \subseteq \mathcal{F}(X) \prec \mathbf{E}(X)$. See Figure 5.

Remark 5.5. The set $\mathcal{F}(X)$ is injective if and only if $\mathcal{F}(X) \cong \mathbf{E}(X)$.

Proposition 5.6. If $X \subseteq \mathbb{R}_\infty^n$ is compact, then $\mathcal{F}(X)$ is compact.

Proof. Because $X$ is bounded, we can choose a large enough bounded $n$-dimensional cube $C = [-a, a]^n \supseteq X$ for some positive number $a$. For any $p \notin C$, there exists some $1 \leq i \leq n$ such that $|p_i| > a$. Without loss of generality, assume $p_i$ is positive. Any point $x \in p + \Lambda_i$ satisfies $x_i > p_i > a$, and thus cannot be in $C$. Therefore, $(p + \Lambda_i) \cap C = \emptyset$, implying that $p \notin \mathcal{F}(X)$. Thus, $\mathcal{F}(X)$ is contained in $C$ and is bounded.

Let $p \in \mathbb{R}_\infty^n \setminus \mathcal{F}(X)$ and assume that $\xi \in \{\pm 1\}$ and $1 \leq i \leq n$ are such that $(p + \xi \Lambda_i) \cap X = \emptyset$. Since $X$ and $p + \xi \Lambda_i$ are both closed, $r := d_\infty(X, p + \xi \Lambda_i) > 0$. For any $q \in B_r(p, \mathbb{R}_\infty^n)$, we claim that $q + \xi \Lambda_i \subseteq B_r(p + \xi \Lambda_i, \mathbb{R}_\infty^n)$. Indeed, for every $z \in q + \xi \Lambda_i$, we have $z + p - q \in (q + p - q) + \xi \Lambda_i = p + \xi \Lambda_i$, and it follows from

$$d_\infty(z, p + \xi \Lambda_i) \leq d_\infty(z, z + p - q) = d_\infty(p, q) < r$$

that $z \in B_r(p + \xi \Lambda_i, \mathbb{R}_\infty^n)$. Therefore, $B_r(p, \mathbb{R}_\infty^n) \subseteq \mathbb{R}_\infty^n \setminus \mathcal{F}(X)$, and thus $\mathcal{F}(X)$ is closed. Hence, $\mathcal{F}(X)$ is compact. \qed

Proposition 5.7. If $X \subseteq \mathbb{R}_\infty^n$ is convex, then $\mathcal{F}(X)$ is convex.

Proof. For every two points $p, q \in \mathcal{F}(X)$ and each $\lambda \in [0, 1]$, let $w := \lambda p + (1 - \lambda)q$. By the definition of $\mathcal{F}(X)$, there exist $x \in (p + \Lambda_1) \cap X$ and $y \in (q + \Lambda_1) \cap X$. Because $X$ is convex, we have $z := \lambda x + (1 - \lambda)y \in X$. It follows from

$$d_\infty(w, z) \leq \lambda d_\infty(p, x) + (1 - \lambda)d_\infty(q, y) = \lambda(x_1 - p_1) + (1 - \lambda)(y_1 - q_1) = z_1 - w_1 \leq d_\infty(w, z)$$

that $z \in w + \Lambda_1$. Therefore, $(w + \Lambda_1) \cap X \neq \emptyset$, and by a similar argument we have $(w + \xi \Lambda_i) \cap X \neq \emptyset$ for every $\xi \in \{\pm 1\}$ and $1 \leq i \leq n$. Thus, $\mathcal{F}(X)$ is convex. \qed

Next, we show that the tight span of the $n$-sphere $S_\infty^n$ (or the $(n + 1)$-ball $\mathbb{D}_\infty^{n+1}$) with $\ell^\infty$ metric properly contains the $(n + 1)$-ball $\mathbb{D}_\infty^{n+1}$ for $n \geq 2$.

Theorem 5.8. Let $n \geq 2$. For the tight spans of $S_\infty^n$ and $\mathbb{D}_\infty^{n+1}$, we have

- $\mathbf{E}(S_\infty^n) \supseteq \mathcal{F}(S_\infty^n) \supseteq \mathbb{D}_\infty^{n+1}$;
• $E(D_{\infty}^{n+1}) \supseteq F(D_{\infty}^{n+1}) \supseteq F(S_{\infty}^{n}) \supseteq D_{\infty}^{n+1}$.

As a consequence, $D_{\infty}^{n+1}$ is not injective.

**Proof.** To prove $F(S_{\infty}^{n}) \supseteq D_{\infty}^{n+1}$ it is enough to find a point $p \in F(S_{\infty}^{n}) - D_{\infty}^{n+1}$.

Let $x = \frac{1}{\sqrt{n+1}}(1, \ldots, 1) \in \mathbb{R}_{\infty}^{n+1}$. Let $p = (1 + \lambda)x$, where $\lambda$ is chosen to be a positive number such that $\lambda^2 + 2\lambda \leq \frac{(n-1)^2}{4n}$. It is clear that $p \notin D_{\infty}^{n+1}$. See Figure 6 for the case of $n = 2$.

![Figure 6](image)

**Remark 5.9.** Let us see why the above proof of Theorem 5.8 does not go through when $n = 1$. We use the same notation as the theorem. When $\lambda > 0$, $\|p - tv^i\|^2 - 1 = t^2 + (\lambda^2 + 2\lambda) = 0$ does not have a solution for $t$. Thus, $p = \frac{1+\lambda}{\sqrt{2}}(1, 1) \notin F(S_{\infty}^1)$ for every $\lambda > 0$, unlike the case when $n \geq 2$.

### 5.2 The set of $X$-minimal points $E(X)$

In this section, we identify an injective set $E(X)$ containing a given set $X \subseteq \mathbb{R}_{\infty}^n$, which thus also contains the tight span of $X$ up to isometry.

For every $x, y \in \mathbb{R}^n$, the closed metric interval (see [DD09, page 13]) between $x$ and $y$ is defined as the set

$I_{xy} := \{z \in \mathbb{R}^n : d_{\infty}(x, z) + d_{\infty}(z, y) = d_{\infty}(x, y)\}$.

See Figure 7 for illustrations of $I_{xy}$ in both $\mathbb{R}^2$ and $\mathbb{R}^3$, and below for some property of $I_{xy}$.
**Proposition 5.10.** Let \( x, y \in \mathbb{R}^n_{\infty} \).

1. Suppose that \( y \in x + \xi \Lambda_i \). Then, \( I_{xy} = (x + \xi \Lambda_i) \cap (y - \xi \Lambda_i) \).
2. For every \( z \in I_{xy} \), \( I_{xz} \subseteq I_{xy} \).

**Proof.** For Part (1), notice that for any \( z \in (x + \xi \Lambda_i) \cap (y - \xi \Lambda_i) \),

\[
d_\infty(x, z) + d_\infty(z, y) = \xi(z_i - x_i) + \xi(y_i - z_i) = \xi(y_i - x_i) = d_\infty(x, y).
\]

Hence, \( z \in I_{xy} \) so that \( (x + \xi \Lambda_i) \cap (y - \xi \Lambda_i) \subseteq I_{xy} \).

Next, for any \( z \in I_{xy} \), we have

\[
d_\infty(x, z) + d_\infty(z, y) = d_\infty(x, y) = \xi(y_i - x_i) = \xi(y_i - z_i) + \xi(z_i - x_i) \leq d_\infty(x, z) + d_\infty(z, y)
\]

implying \( \xi(y_i - z_i) = d_\infty(z, y) \) and \( \xi(z_i - x_i) = d_\infty(x, z) \). Hence, \( z \in (x + \xi \Lambda_i) \cap (y - \xi \Lambda_i) \) so that \( I_{xy} \subseteq (x + \xi \Lambda_i) \cap (y - \xi \Lambda_i) \).

For Part (2), without loss of generality one can assume that \( y \in x + \xi \Lambda_i \) for some \( \xi \) and \( i \). Then for every \( z \in I_{xy} = (x + \xi \Lambda_i) \cap (y - \xi \Lambda_i) \) (by Part (1)), we have

\[
I_{xz} = (x + \xi \Lambda_i) \cap (z - \xi \Lambda_i), \quad \text{(because } z \in x + \xi \Lambda_i \text{})
\]

\[
\subseteq (x + \xi \Lambda_i) \cap (y - \xi \Lambda_i), \quad \text{(because } z \in y - \xi \Lambda_i \text{ and Proposition 5.2)}
\]

\[
= I_{xy}.
\]

\( \square \)

**Definition 18 (X-minimal points).** For a compact subset \( X \subseteq \mathbb{R}^n_{\infty} \), we say that \( z \in \mathbb{R}^n_{\infty} \) is an \( X \)-minimal point (in \( \mathbb{R}^n_{\infty} \)), if the distance function \( d_\infty(z, \cdot) : X \to \mathbb{R}_{\geq 0} \) is minimal, i.e., for every \( x \in X \), there exists \( y \in X \) such that \( d_\infty(x, z) + d_\infty(z, y) = d_\infty(x, y) \).

Denote by \( \mathcal{E}(X) \) the set of \( X \)-minimal points (in \( \mathbb{R}^n_{\infty} \)), and equip it with the \( \ell^\infty \) metric. Equivalently,

\[
\mathcal{E}(X) = \bigcap_{x \in X} \bigcup_{y \in X} I_{xy}.
\]
Proposition 5.11. Consider a subset $X$ of $\mathbb{R}_\infty^n$. Then, the following properties hold:

1. $X \subseteq \mathcal{E}(X)$.

2. For every $\tilde{X}$ such that $X \subseteq \tilde{X} \subseteq \mathcal{E}(X)$, we have $\mathcal{E}(\tilde{X}) \subseteq \mathcal{E}(X)$. In particular, $\mathcal{E}(\mathcal{E}(X)) = \mathcal{E}(X)$.

3. If $X$ is compact, then $\mathcal{E}(X)$ is also compact.

4. We have $X \subseteq \mathcal{F}(X) \subseteq \mathcal{E}(X)$, and $\mathcal{F}(X) \preceq \mathcal{E}(X) \preceq \mathcal{E}(X)$.

Proof. Part (1) is straightforward. For Part (2), we claim that for every $x \in X$, $\bigcup_{\tilde{y} \in \tilde{X}} I_{x\tilde{y}} = \bigcup_{y \in X} I_{xy}$. Then it follows from the claim that

$$\mathcal{E}(\tilde{X}) = \bigcap_{\tilde{y} \in \tilde{X}} \bigcup_{x \in X} I_{x\tilde{y}} \subseteq \bigcup_{y \in X} I_{xy} = \bigcap_{x \in X} \bigcup_{y \in X} I_{xy} = \mathcal{E}(X).$$

To prove the claim, it suffices to show $\bigcup_{\tilde{y} \in \tilde{X}} I_{x\tilde{y}} \subseteq \bigcup_{y \in X} I_{xy}$, because the reverse inclusion follows directly from the fact $X \subseteq \tilde{X}$. Fix $x \in X$. For each $y \in \bigcup_{\tilde{y} \in \tilde{X}} I_{x\tilde{y}}$, there exists $\tilde{y} \in \tilde{X} \subseteq \mathcal{E}(X)$ such that $z \in I_{x\tilde{y}}$. Because $\tilde{y} \in \mathcal{E}(X)$ and $x \in X$, there exists $y \in X$ such that $\tilde{y} \in I_{xy}$. It follows from item (2) of Proposition 5.10 that $z \in I_{x\tilde{y}} \subseteq I_{xy}$. Therefore,

$$\bigcup_{\tilde{y} \in \tilde{X}} I_{x\tilde{y}} \subseteq \bigcup_{y \in X} I_{xy},$$

and the claim is proved. Taking $\tilde{X} = \mathcal{E}(X)$, we obtain $\mathcal{E}(\mathcal{E}(X)) \subseteq \mathcal{E}(X)$. Combined with Part (1), we then have $\mathcal{E}(X) = \mathcal{E}(\mathcal{E}(X))$.

For Part (3), we assume that $X$ is compact. By Theorem 2.8 the tight span $\mathcal{E}(X)$ is also compact, i.e. every limit of minimal functions is still minimal. Consider a convergent sequence $\{p_i\}_{i \in \mathbb{N}} \subseteq \mathcal{E}(X)$. Then $d_\infty(\lim p_i, \cdot) = \lim d_\infty(p_i, \cdot)$ is the limit of minimal functions, and thus is also minimal. It follows that $\lim p_i \in \mathcal{E}(X)$.

Finally, item (4) can be proved by using Proposition 5.3, the minimality of $\mathcal{E}(X)$, and the injectivity of $\mathcal{E}(X)$ which will be shown in Proposition 5.13.

Example 5.12. The isometric embeddings in Proposition 5.11 (4) can all be non-surjective. For example, let $X := (0, 3) \cup \{0\} \times [-1, 1] \subseteq \mathbb{R}_\infty^2$ as before. Then we have $X \subsetneq \mathcal{F}(X) \subsetneq \mathcal{E}(X) \preceq \mathcal{E}(X)$, see Figure 8.

![Figure 8](image_url)
Next, we show that the set of $X$-minimal points is injective:

**Proposition 5.13.** For a compact $Q \subseteq \mathbb{R}_n^\infty$, $Q$ is injective if and only if $Q = \mathcal{E}(Q)$. In addition, for any every compact set $X \subseteq \mathbb{R}_n^\infty$, $\mathcal{E}(X)$ is injective.

**Proof.** Assume that $Q$ is injective. To show $Q = \mathcal{E}(Q)$, it suffices to show that $Q \supseteq \mathcal{E}(Q)$. Since $Q$ is injective, $Q$ is isometric to $\mathcal{E}(Q)$. As a consequence, any function $f \in \mathcal{E}(Q)$ can be realized as a distance map $d^Q_\infty(z, \cdot) : Q \to \mathbb{R}$ for some $z \in Q$, and thus has a zero. For any $p \in \mathcal{E}(Q)$, the distance map $d^Q_\infty(p, \cdot) : Q \to \mathbb{R}$ is in the tight span $\mathcal{E}(Q)$. Because $d^Q_\infty(p, \cdot) : Q \to \mathbb{R}$ has a zero, there is some $q \in Q$ such that $d^Q_\infty(p, q) = 0$, i.e. $p = q \in Q$.

Conversely, suppose $Q = \mathcal{E}(Q)$. Consider the map $\iota : \mathbb{R}^n_\infty \to L^\infty(Q)$ with $z \mapsto d^Q_\infty(z, \cdot)$. The set $\iota^{-1}(\mathcal{E}(Q))$ consists of $z \in \mathbb{R}^n_\infty$ such that the function $d^Q_\infty(z, \cdot)$ are minimal, and thus is equal to $\mathcal{E}(Q)$ by Definition 18. Thus, $\iota|Q : Q \to \mathcal{E}(Q)$ is surjective. In addition, because $\iota|Q$ preserves distance, $\iota|Q$ is an isometry. It follows from $Q \cong \mathcal{E}(Q)$ that $Q$ is injective.

Recall from Proposition 5.11 (2) that $\mathcal{E}(X) = \mathcal{E}(\mathcal{E}(X))$, for any every compact set $X \subseteq \mathbb{R}_n^\infty$. Thus, $\mathcal{E}(X)$ is injective. \hfill $\Box$

**Remark 5.14.** For every compact set $X \subseteq \mathbb{R}_n^\infty$, $\mathcal{E}(X)$ is isometric to the smallest compact set $Y \subseteq \mathbb{R}_n^\infty$ such that $X \subseteq Y \subseteq \mathcal{E}(X)$ and $Y = \mathcal{E}(Y)$.

### 5.2.1 The tight span of $\mathbb{S}_\infty^2$

We have already seen in Theorem 5.8 that the tight span of $\mathbb{S}_\infty^n$ (or that of $\mathbb{D}_{\infty}^{n+1}$) properly contains $\mathbb{D}_{\infty}^{n+1}$ for $n \geq 2$, up to isometry. Below, we concentrate to the case when $n = 2$, and use the injectivity of $\mathcal{E}(\mathbb{S}_\infty^2)$ to show that $\mathcal{E}(\mathbb{S}_\infty^2)$ is isometric to $\mathcal{F}( \mathbb{S}_\infty^2 ) = \mathcal{E}(\mathbb{S}_\infty^2)$.

**Theorem 5.15.** Let $\mathbb{S}_\infty^2$ be the unit sphere centered at the origin in $\mathbb{R}_\infty^2$. Then,

1. $\mathcal{F}( \mathbb{S}_\infty^2 ) = \mathcal{E}(\mathbb{S}_\infty^2)$, which is then isometric to the tight span $\mathcal{E}(\mathbb{S}_\infty^2)$ of $\mathbb{S}_\infty^2$.

2. $\mathcal{F}( \mathbb{D}_{\infty}^3 ) = \mathcal{E}(\mathbb{D}_{\infty}^3) = \mathcal{F}( \mathbb{S}_\infty^2 )$, which is then isometric to the tight span $\mathcal{E}(\mathbb{D}_{\infty}^3)$ of $\mathbb{D}_{\infty}^3$. In addition, $\mathcal{E}(\mathbb{D}_{\infty}^3) \cong \mathcal{E}(\mathbb{S}_\infty^2)$.

We prepare the following two lemmas for the proof of Theorem 5.15.

**Lemma 5.16.** Let $X$ be a compact subset of $\mathbb{R}_\infty^n$.

1. If the map $\iota_X : \mathcal{E}(X) \to \mathcal{E}(X)$ with $x \mapsto d^X_\infty(x, \cdot)$ preserves distances, then the tight span $\mathcal{E}(X) \cong \mathcal{E}(X)$.

2. If $\mathcal{E}(X) \subseteq \mathcal{F}(X)$, i.e. $\mathcal{E}(X) = \mathcal{F}(X)$, then the tight span $\mathcal{E}(X) \cong \mathcal{E}(X) = \mathcal{F}(X)$.

**Proof.** If the map $\iota_X : \mathcal{E}(X) \to \mathcal{E}(X)$ with $x \mapsto d^X_\infty(x, \cdot)$ preserves distance, then $\mathcal{E}(X)$ is an injective metric space isometrically embedded into $\mathcal{E}(X)$. By the minimality of $\mathcal{E}(X)$, it must be true that $\mathcal{E}(X) \cong \mathcal{E}(X)$.

Assume now $\mathcal{E}(X) = \mathcal{F}(X)$. Recall from Proposition 5.3 that $\iota_X : \mathcal{F}(X) = \mathcal{E}(X) \to \mathcal{E}(X)$ with $x \mapsto d^X_\infty(x, \cdot)$ preserves distances. Thus, $\mathcal{E}(X) \cong \mathcal{F}(X) = \mathcal{E}(X)$. \hfill $\Box$
Lemma 5.17. Let $X \subseteq \mathbb{R}^n_\infty$ and $p \in \mathcal{E}(X)$. For each fixed $1 \leq i \leq n$ and $\xi \in \{\pm 1\}$, if $\text{Int}(p + \xi \Lambda_i) \cap X \neq \emptyset$, then $(p - \xi \Lambda_i) \cap X \neq \emptyset$.

Proof. Without loss of generality, assume that $i = 1$ and $\xi = 1$. Let $x \in \text{Int}(p + \Lambda_1) \cap X$. Since $p \in \mathcal{E}(X) \subseteq \bigcup_{y \in X} I_{xy}$, there exists $y \in X$ such that $p \in I_{xy}$. Suppose $1 \leq j \leq n$ and $\eta \in \{\pm 1\}$ are such that $y \in x + \eta \Lambda_j$. Then $p \in I_{xy} = (x + \eta \Lambda_j) \cap (y - \eta \Lambda_j) \subseteq (x + \eta \Lambda_j)$ by item (1) of Proposition 5.10. Combined with the fact that $p$ is in $\text{Int}(x - \Lambda_1)$, we must have $\eta = -1$ and $j = 1$. Thus, $p \in y - \eta \Lambda_j = y + \Lambda_1$ and it follows that $y \in (p - \Lambda_1) \cap X$, i.e. $(p - \Lambda_1) \cap X \neq \emptyset$. \hfill \Box

Proof of Theorem 5.15. To prove $\mathcal{F}(S^2_\infty) = \mathcal{E}(S^2_\infty)$, it is enough to show $\mathcal{E}(S^2_\infty) \subseteq \mathcal{F}(S^2_\infty)$. Then, item (1) immediately follows from Proposition 5.16. Because the ball $D^3_\infty$ is clearly contained in $\mathcal{F}(S^2_\infty)$, it suffices to show that $\mathcal{E}(S^2_\infty) - D^3_\infty \subseteq \mathcal{F}(S^2_\infty)$.

Take any $p \in \mathcal{E}(S^2_\infty) - D^3_\infty$. Because $S^2_\infty$ cannot be fully embedded into the union of finitely many 2-dimensional planes $\bigcup_{i=1}^{3} (\partial(p + \Lambda_i) \cup \partial(p - \Lambda_i))$, there must exist some $1 \leq i \leq 3$ and $\xi \in \{\pm 1\}$ such that $\text{Int}(p + \xi \Lambda_i) \cap S^2_\infty \neq \emptyset$. Without loss of generality, we assume that $\xi = 1$ and $i = 1$. By Lemma 5.17, we also have $(p - \Lambda_1) \cap S^2_\infty \neq \emptyset$.

If both $\text{Int}(p + \Lambda_1) \cap S^2_\infty = \emptyset$ and $\text{Int}(p - \Lambda_1) \cap S^2_\infty = \emptyset$ for $i = 2$ and $i = 3$, then for every $x \in S^2_\infty$, $x$ is either in $(p + \Lambda_1) - \{p\}$ or $(p - \Lambda_1) - \{p\}$. Equivalently, we have either $x_1 > p_1$ or $x_1 < p_1$. Therefore,

$$S^2_\infty = \{x \in S^2_\infty : x_1 < p_1\} \cup \{x \in S^2_\infty : x_1 > p_1\}.$$ 

In addition, $\{x \in S^2_\infty : x_1 < p_1\} \cup (p + \Lambda_1) \cap S^2_\infty \neq \emptyset$ is nonempty, so is $\{x \in S^2_\infty : x_1 > p_1\}$. Thus, we have written $S^2_\infty$ as the disjoint union of two nonempty open subsets, contradicting that $S^2_\infty$ is connected. Hence, there must exist $2 \leq i \leq 3$ such that $\text{Int}(p + \Lambda_i) \cap S^2_\infty \neq \emptyset$ or $\text{Int}(p - \Lambda_i) \cap S^2_\infty \neq \emptyset$. Without loss of generality, suppose $\text{Int}(p + \Lambda_2) \cap S^2_\infty \neq \emptyset$. By Lemma 5.17, we also have $(p - \Lambda_2) \cap S^2_\infty \neq \emptyset$.

If both $\text{Int}(p + \Lambda_3) \cap S^2_\infty = \emptyset$ and $\text{Int}(p - \Lambda_3) \cap S^2_\infty = \emptyset$, then

$$S^2_\infty \subseteq (p + \Lambda_1) \cup (p - \Lambda_1) \cup (p + \Lambda_2) \cup (p - \Lambda_2) - \{p\}.$$ 

Denote the regions $\text{Int}(p + \Lambda_1) \cap S^2_\infty$, $\text{Int}(p + \Lambda_2) \cap S^2_\infty$, $(p - \Lambda_1) \cap S^2_\infty$ and $(p - \Lambda_2) \cap S^2_\infty$ by $R^1$, $R^2$, $R^3$ and $R^4$ respectively. For $j = 1, 2, 3, 4$, take $x^j \in R^j$, and let $\gamma_j \subseteq R_j \cup R_{j+1}$ be a path connecting $x^j$ and $x^{j+1}$, where we assume $R_5 = R_1$ and $x^5 = x^1$. Because $S^2_\infty$ is simply-connected, the loop $\gamma := \gamma_1 \circ \gamma_2 \circ \gamma_3 \circ \gamma_4$ is contractible in $S^2$, and thus contractible in $(p + \Lambda_1) \cup (p - \Lambda_1) \cup (p + \Lambda_2) \cup (p - \Lambda_2) - \{p\}$. This gives a contradiction. \hfill \Box

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