Growth Kinetics in a Phase Field Model with Continuous Symmetry.

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We discuss the static and kinetic properties of a Ginzburg-Landau spherically symmetric $O(N)$ model recently introduced (Phys. Rev. Lett. 75, 2176, (1995)) in order to generalize the so called Phase field model of Langer. The Hamiltonian contains two $O(N)$ invariant fields $\phi$ and $U$ bilinearly coupled. The order parameter field $\phi$ evolves according to a non conserved dynamics, whereas the diffusive field $U$ follows a conserved dynamics. In the limit $N \to \infty$ we obtain an exact solution, which displays an interesting kinetic behavior characterized by three different growth regimes. In the early regime the system displays normal scaling and the average domain size grows as $t^{1/2}$, in the intermediate regime one observes a finite wavevector instability, which is related to the Mullins-Sekerka instability; finally, in the late stage the structure function has a multiscaling behavior, while the domain size grows as $t^{1/4}$.

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I. INTRODUCTION

When a system described by an order parameter, initially placed into a high temperature single phase region of its phase diagram, is brought to a point inside the coexistence curve by a sudden change of temperature it becomes thermodynamically unstable and phase separates as a result of the existence of many competing ground states. After the quench the system can order kinetically through either nucleation or spinodal decomposition. In the latter process a microscopic long-wavelength fluctuation initially present is amplified and determines the formation and evolution of various patterns characterized by the presence of a universal length scale $L(t)$, associated with the typical domain-size and separation among topological defects. As the system orders, $L(t)$ grows in time in a power-law fashion $t^{1/2}$ and the time-dependent structure factor $C(k, t)$ displays dynamical scaling.

A successful approach to the study of these phenomena is represented by the time dependent Ginzburg-Landau equation. Many years ago Hohenberg and Halperin provided a useful classification of the various models, which comprises a vast class of dynamic critical phenomena, in terms of few parameters. Within their scheme two models have received a great deal of attention: model A, where a single field evolves towards equilibrium with non conserved order parameter dynamics (NCOP) and model B where the order parameter is conserved (COP).

The NCOP dynamics is aimed to describe an ordering process similar to that of the Ising model with conventional Monte Carlo spin flip dynamics, while COP accounts for the approach to equilibrium of an alloy. In this case, one observes that the system orders by growing droplets larger than a critical size at the expenses of smaller droplets, while keeping the total amount of material fixed. The effect of the conservation law is to slow down the phase separation process, because the material has to be transported via diffusion through the system before being added to a growing region. The value of the dynamical exponent $z$ is $z = 2$ for NCOP, whereas for conserved dynamics is $z = 3$ for a scalar order parameter and $z = 4$ for vector order parameter, indicating that conservation laws play an important role in the dynamical process.

While COP and NCOP dynamics has been widely studied, the Phase field model which describes the coupling of a NCOP system with a diffusive COP field such as temperature or concentration, seems not to have been completely explored, in spite of the fact that it displays a variety of interesting peculiar features. Only to mention the most striking of these, we recall that the Phase field model accounts for the regularity of the shapes observed during the growth of crystals into a supercooled melt. According to the Phase field model a planar solid front growing in the supercooled liquid undergoes the so called Mullins-Sekerka instability. This phenomenon can be understood as follows. The latent heat, released when the liquid freezes, is diffused into the colder liquid and thus promotes the freezing of more material. The larger is the temperature gradient the faster is the advancement of the front. Now, imagine to slightly perturb the isochemical flat solid-liquid interface in a slowly varying fashion. As a result of the deformation, the temperature gradient will be larger on the bulges of such a boundary and so the heat flux. This fact makes a solid tip to grow faster than a flat portion of interface and
provides a mechanism by which a perturbation of finite wavelength is amplified, as it was discovered by Mullins and Sekerka [4,5].

In a recent letter [6] we have introduced an N-component version of the Phase field model in order to study the evolution of a non conserved order parameter $\phi$ bilinearly coupled to a conserved field $U$. In the limit $N \to \infty$ we were able to obtain some analytical results on the non-equilibrium relaxation behaviour of the model. Here we expand these results.

The model, inspired by the model C [1], displays several interesting features: the evolution of the vector field $\phi$ is non conserved in the early regime, then after a crossover time it develops an instability at finite wavelength due to the coupling with the conserved field $U$. In the very late regime the COP behaviour becomes eventually dominant and $\phi$ shows a genuine COP evolution, including multiscaling. A similar mechanism was reported by Somozza and Sagui [3] in a numerical study of the model C where they observed that notwithstanding the non conserved field evolves faster than the conserved field. For late times the growth is driven by diffusion of the conserved variable and the order parameter becomes slaved by the diffusive field.

The paper is organized as follows. In section II we give a brief physical motivation of the model. The equation defining the model are given in section III and its equilibrium and dynamical properties are discussed in section IV and section V, respectively. Finally section VI contains a brief summary of the results and discussions.

II. MOTIVATION OF THE MODEL

The model we discuss in the present paper belongs to the family of the $O(N)$ spherical models and it has been introduced with the aim of studying exactly the coupling of a NCOP field with a diffusive COP field. The $O(N)$ generalization proves to be fruitful because, while retaining the salient features of the phenomena occurring during the diffusion limited growth it allows for some analytical results in the limit $N \to \infty$.

Historically, models containing couplings quadratic with respect to the material NCOP field $\phi$ and linear with respect to the COP diffusive thermal field $u$ were introduced as early as the seventies in the framework of the dynamical critical phenomena and named models C. Few years later, Langer, in order to study first order phase transitions accompanied by latent heat of fusion, put forward the so called Phase field model [3], in which the coupling was assumed to be bilinear with respect to the two fields.

The material is characterized by an order parameter $\phi$ which assumes a positive value in the solid phase and a negative value in the liquid phase. The local temperature of the system is treated as an additional dynamical field obeying a heat diffusion equation in the presence of sources represented by the amount of material changing phase. The solidification takes place adiabatically so that no heat can flow to the outside. One defines the dimensionless temperature field as $u(x,t) = c_p(T(x,t) - T_m)/L$, where $L$ is the latent heat of fusion per mole, $c_p$ the specific molar heat at constant pressure and $T_m$ is the bulk melting temperature. The spatial average of $u$ at the initial time is the so called undercooling parameter $\Delta$ and is a negative quantity.

The thermal field diffuses according to the modified Fourier equation:

$$\frac{\partial u(x,t)}{\partial t} = D \nabla^2 u(x,t) + \frac{\partial \phi(x,t)}{\partial t}$$  \hspace{1cm} (1)

where $D$ is the thermal diffusivity and the last term on the r.h.s. is the amount of material which crystallizes per unit time and thus proportional to the heat released during the first order transition. Both $c_p$ and $D$ are assumed to be equal in the two phases. The evolution of the order parameter is determined by the non linear time-dependent equation of the Ginzburg-Landau type [8,9,10]:

$$\frac{\partial \phi(x,t)}{\partial t} = -\Gamma \frac{\delta}{\delta \phi(x,t)} F[\phi,u]$$

$$= -\Gamma [-\nabla^2 \phi + r \phi + g \phi^3 + \alpha u]$$  \hspace{1cm} (2)

To describe a two phase system the form of $F$ is chosen to be a double well for $r < 0$. The coupling to the thermal field $u$ can create an unbalance, in such a way that for negative values of $\alpha u$ the liquid phase ($\phi \sim -\sqrt{-r/g}$) is metastable with respect to the solid ($\phi \sim +\sqrt{-r/g}$). In the absence of coupling to the temperature field, i.e. $\alpha = 0$, eq. (2) represents the familiar Cahn-Hilliard equation, also called Model A.

The process contains two stages: during the first stage the solid grows at the expenses of the liquid, while in the second stage the total amount of solid is nearly constant and the growth is limited by diffusion of the thermal field.

Interestingly, the two dynamical equations (1) and (2) can be derived from a Lyapounov functional $\mathcal{F}$, which plays the role of the time dependent Ginzburg-Landau potential in the present problem. If one performs the transformation $U = u - \phi$ and eliminates $\phi$ in favor of the new field $U$ one can write eqs. (1)-(2) as

$$\frac{\partial \phi(x,t)}{\partial t} = -\Gamma \frac{\delta F}{\delta \phi(x,t)} \bigg|_{\phi}$$  \hspace{1cm} (3)

$$\frac{\partial U(x,t)}{\partial t} = D \frac{\delta F}{\delta U(x,t)} \bigg|_{\phi}$$  \hspace{1cm} (4)

with the Lyapounov functional

$$\mathcal{F}[\phi,U] = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{r}{2} \phi^2 + \frac{g}{4} \phi^4 + \frac{\alpha}{2} (U + \phi)^2\right].$$  \hspace{1cm} (5)
The functional $\mathcal{F}$ has two equal minima when the temperature field vanishes, i.e., for $U = -\phi$ and generates a complex dynamical behavior which has been the object of some studies. However, its global properties are not so well known. This fact, lead us to formulate an $O(N)$ invariant vectorial generalization of the above model. This kind of models, in fact, lend themselves to nearly analytical solutions thus providing useful insights on the properties of the scalar order parameter solutions.

### III. THE $O(N)$ MODEL

We shall consider a system described by two coupled $N$-component vector fields $\phi = (\phi_1(x, t), \ldots, \phi_N(x, t))$ and $U = (U_1(x, t), \ldots, U_N(x, t))$, whose Hamiltonian can be represented by

\[ H[\phi(x), U(x, t)] = \int d^d x \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{r}{2} \phi^2 + \frac{g}{4N} (\phi^2)^2 + \frac{w}{2} \phi^2 + \mu U \phi \right] \]

where $r$ and $g$, with $g > 0$ and $w > 0$, are the standard quadratic and quartic couplings of the Ginzburg-Landau model and the last term represent a bilinear coupling between the field $\phi$ and $U$. The first three terms in eq. (6) constitute the familiar Ginzburg-Landau-Wilson Hamiltonian describing an $O(N)$ $\phi^4$ model, whereas the last two terms represent the interaction between the order parameter field and an external fluctuating field $U$.

We assume that the field $\phi$ evolves according to NCOP dynamics:

\[ \frac{\partial \phi_\alpha(x, t)}{\partial t} = -\Gamma_\phi \frac{\delta}{\delta \phi_\alpha(x, t)} H[\phi, U] + \eta_\alpha(x, t) \tag{7} \]

whereas the field $U$ is conserved and relaxes according to

\[ \frac{\partial U_\alpha(x, t)}{\partial t} = \Gamma_U \nabla^2 \frac{\delta}{\delta U_\alpha(x, t)} H[\phi, U] + \xi_\alpha(x, t) \tag{8} \]

The noises appearing on the right hand sides of eqs. (7)- (8) have zero average and two-point correlations:

\[ \langle \eta_\alpha(x, t) \eta_\beta(x', t') \rangle = 2T_J \Gamma_\phi \delta_{\alpha, \beta} \delta(x - x') \delta(t - t') \tag{9} \]

\[ \langle \xi_\alpha(x, t) \xi_\beta(x', t') \rangle = -2T_J \Gamma_U \delta_{\alpha, \beta} \nabla^2 \delta(x - x') \delta(t - t') \tag{10} \]

\[ \langle \eta_\alpha(x, t) \xi_\beta(x', t') \rangle = 0 \tag{11} \]

where $T_J$ is the temperature of the final equilibrium state whereas $\Gamma_U$ and $\Gamma_\phi$ are the kinetic coefficients.

Introducing the Fourier components of the fields one can write the evolution equation as,

\[ \frac{\partial \phi_\alpha(k, t)}{\partial t} = F_\phi^k(k) + \eta_\alpha(k, t) \tag{12} \]

\[ \frac{\partial U_\alpha(k, t)}{\partial t} = F_U^k(k) + \xi_\alpha(k, t) \tag{13} \]

where $F_\phi, U$ are the Fourier transforms of the first term on the r.h.s of eqs. (7) and (8).

In the limit $N \to \infty$ the cubic term entering into $F_\phi$ can be decoupled and we have:

\[ F_\phi^k(k) = M_{\phi\phi}(k, t) \phi_\alpha(k, t) + M_{\phi U}(k, t) U_\alpha(k, t) \tag{14} \]

\[ F_U^k(k) = M_{\phi\phi}(k, t) \phi_\alpha(k, t) + M_{UU}(k, t) U_\alpha(k, t) \tag{15} \]

where the matrix elements are given by

\[ M_{\phi\phi}(k, t) = -\Gamma_\phi [k^2 + r + gS(t)], \]

\[ M_{\phi U}(k, t) = -\Gamma_\mu \mu^2, \]

\[ M_{UU}(k, t) = -\Gamma_U w^2. \tag{16} \]

The quantity $S(t)$ is the integrated $\phi$-structure function

\[ S(t) = \frac{1}{N} \sum_\alpha \langle \phi_\alpha(x, t) \phi_\alpha(x, t) \rangle \]

\[ = \int \frac{d^d k}{(2\pi)^d} \langle \phi_\alpha(k, r) \phi_\alpha(-k, t) \rangle \tag{17} \]

and the integral contains a phenomenological momentum cutoff $\Lambda$. The average is over the external noises $\eta$ and $\xi$ and initial conditions.

To study the behaviour at finite temperature $T_J$ it is useful to introduce the equations of motion for the three equal-time real space connected correlation functions $C_{\phi\phi}(r, t) = \langle \phi_\alpha(R + r, t) \phi_\alpha(R, t) \rangle$ $C_{\phi U}(r, t) = \langle \phi_\alpha(R + r, t) U_\alpha(R, t) \rangle$ and $C_{UU}(r, t) = \langle U_\alpha(R + r, t) U_\alpha(R, t) \rangle$, whose Fourier transforms are the structure functions. These correlations are independent of the index $\alpha$ due to the internal symmetry. In the $N \to \infty$ limit the structure functions evolve according to the following set of equations:

\[ \frac{1}{2} \frac{\partial}{\partial t} C_{\phi\phi}(k, t) = M_{\phi\phi}(k, t)C_{\phi\phi}(k, t) \]

\[ + M_{\phi U}(k, t)C_{\phi U}(k, t) + \Gamma_\phi T_J \tag{18} \]

\[ \frac{\partial}{\partial t} C_{\phi U}(k, t) = M_{\phi\phi}(k, t)C_{\phi U}(k, t) \]

\[ + (M_{UU}(k, t) + M_{\phi\phi}(k, t))C_{UU}(k, t) \]

\[ + M_{\phi U}(k, t)C_{UU}(k, t) \tag{19} \]

\[ \frac{1}{2} \frac{\partial}{\partial t} C_{UU}(k, t) = M_{\phi\phi}(k, t)C_{UU}(k, t) \]

\[ + M_{UU}(k, t)C_{UU}(k, t) + \Gamma_U T_J k^2. \tag{20} \]

In what follows we shall be more interested into the behaviour of the field $\phi$, since it is the relevant order parameter of the system.
IV. EQUILIBRIUM PROPERTIES

In this section we investigate the equilibrium properties of the model \([\underline{12}, \underline{13}]\). It can be shown that the random process characterized by the Langevin equations \([\underline{9}]-[\underline{13}]\) obeys detailed balance since the following “potential conditions”, analogous of the Onsager relations, are fulfilled \([\underline{12}]\):

\[
\frac{\delta}{\delta \phi_\alpha(k)} F_\phi^{\beta}(-k') = \frac{\delta}{\delta \phi_\beta(k')} F_\phi^{\alpha}(-k), \tag{21}
\]

\[
\frac{\delta}{\delta \phi_\alpha(k)} F_U^{\beta}(-k') = \frac{\Gamma_U}{\Gamma_\phi} \frac{\delta}{\delta U_\beta(k')} F_\phi^{\alpha}(-k), \tag{22}
\]

\[
\frac{\delta}{\delta U_\alpha(k)} F_U^{\beta}(-k') = \frac{k'^2}{k^2} \frac{\delta}{\delta U_\beta(k')} F_U^{\alpha}(-k). \tag{23}
\]

If detailed balance holds, the stationary probability density reads

\[
P_{st}[\phi, U] = N \exp\left(-\frac{1}{T_f} H[\phi, U]\right) \tag{24}
\]

where \(N\) is a normalization constant.

The equilibrium probability density is quadratic in the field \(U\), therefore as far as the static properties of \(\phi\) are involved, the field \(U\) can be traced out. One is then left with an effective Hamiltonian for the field \(\phi\):

\[
H_{\text{eff}}[\phi] = \int d^d x \left[ \frac{1}{2} \overline{(\nabla \phi)^2} + \frac{r_{\text{eff}}}{2} (\phi_1)^2 + \frac{g}{4 N} (\phi^2)^2 \right] \tag{25}
\]

where \(r_{\text{eff}} = r - \mu^2/w\) is the “renormalized mass”. The importance of the field \(U\) can be fully appreciated only in the non-equilibrium dynamics of the system, as will be discussed in the next section.

Before considering the dynamics, we briefly discuss the static properties of this model. Unlike the case where the \(U\) is quenched \([\underline{13}]\) the system displays for space dimensions \(d > 2\) an order-disorder transition when \(T_f\) is lower than the critical temperature \(T_c\). In order to locate the critical surface \(T_f = T_c(r, g, \mu, w)\) one considers the long-range behaviour of the structure functions \(C_{\phi\phi}(k)\), which can be computed from \([\underline{23}]\). The fourth order term makes the calculation difficult for finite \(N\). However for \(N \to \infty\) we can use the Hartree approximation, exact in this limit, and we readily obtain for \(T > T_c\):

\[
C_{\phi\phi}(k) = \frac{T_f}{k^2 + r + g S_\infty - \mu^2/w}. \tag{26}
\]

\[
S_\infty = \int_{|k| < \Lambda} \frac{d^d k}{(2\pi)^d} C_{\phi\phi}(k). \tag{27}
\]

For \(T_f \leq T_c\) the structure function diverges at small \(k\) because the full mass term \(r + g S_\infty - \mu^2/w\) vanishes, signaling the appearance of the ordered phase. In fact, the model for \(r < \mu^2/w\) and \(g > 0\) displays a high temperature paramagnetic phase and a low temperature ordered phase. The critical temperature is given by the usual form of the \(\phi^4\) theory \(T_c = (\mu^2/w - \nu)(d-2)/(g \Lambda^{d-2} k_d)\) with \(1/k_d = 2\pi^{d/2} \Gamma(d/2)\), where \(\Gamma(x)\) is the Gamma function.

For temperatures \(T_f\) below \(T_c\) there exists a non vanishing order parameter \(M = \langle \phi_1 \rangle\), which can be assumed to be directed along the \(\alpha = 1\) direction without loss of generality. The \((N - 1)\) components of the correlation function orthogonal to the order parameter direction diverge at small \(k\), reflecting the existence of Nambu-Goldstone modes, i.e., excitations of vanishing energy cost in the long wavelength limit. The real space two point correlation function takes the form

\[
\langle \phi_\alpha(r) \phi_\alpha(r) \rangle = M^2 \delta_{\alpha 1} + S_\infty(t) \tag{28}
\]

\[
M^2 = -\frac{1}{g} \left(r - \frac{\mu^2}{w}\right) \left(1 - \frac{T_f}{T_c}\right) \tag{29}
\]

where \(S_\infty\) defined in equation \([27]\) comes from the transverse components only.

The other equilibrium correlation functions can also be obtained from the stationary equilibrium distribution and read for \(T > T_c\):

\[
C_{UU}(k) = \frac{T_f}{w - \mu^2(k^2 + r + g S_\infty)^{-1}} \tag{30}
\]

\[
C_{\phi U}(k) = -\left(\frac{\mu}{w}\right) \frac{T_f}{k^2 + r + g S_\infty - \mu^2/w} \tag{31}
\]

Note that both \(C_{\phi U}(k)\) and \(C_{UU}(k)\) are singular for \(k \to 0\).

We conclude by noting that to obtain the static structure functions from the dynamical equations one has to supplement the requirement that the right hand sides of eqs. \([\underline{13}]-[\underline{20}]\) vanish with the following stronger condition:

\[
\lim_{t \to \infty} \left[M_{UU}(k, t) C_{\phi\phi}(k, t) + M_{UU}(k, t) C_{\phi U}(k, t)\right] = 0, \tag{32}
\]

\[
\lim_{t \to \infty} \left[M_{\phi\phi}(k, t) C_{\phi U}(k, t) + M_{\phi U}(k, t) C_{UU}(k, t)\right] = 0 \tag{33}
\]

to ensure that the equilibrium properties of the model are independent on the kinetic coefficients \(\Gamma_\phi\) and \(\Gamma_U\). The conditions \([\underline{12}]-[\underline{13}]\) can also be deduced from the the equilibrium properties of the model (see Appendix).
V. DYNAMICAL PROPERTIES

Since the behavior at \( T_f = 0 \) is representative of the entire dynamics in the ordered phase when \( T_j < T_c \), we shall neglect the noise terms in the following analysis.

For general initial conditions the two fields are not in equilibrium, and we may expect that the relaxation of \( \phi \) is only slightly modified by the external field \( U \). Since the dynamics of \( U \) is sufficiently slow compared with that of \( \phi \), the presence of \( U \) does not modify qualitatively the NCOP behaviour of \( \phi \). In particular, the size of the domains of \( \phi \) should grow with a characteristic length \( L(t) \sim t^{1/2} \), while the maximum of the structure factor is located at \( k = 0 \) and should increase in time with the power \( t^{d/2} \).

This kind of behavior persists until the domain size reaches the typical length associated with the field \( U \), and given by the maximum of the structure function of \( U \). At this stage the dynamics of \( \phi \) slows down because the coupling with the conserved field \( U \) introduces an additional constraint on the dynamics of \( \phi \). For longer times the two fields equilibrate and the COP behaviour eventually becomes dominant.

A simple analysis of the equation of motion for \( N \to \infty \) gives the scaling of the crossover time with the coupling constant \( \mu \). Indeed it is simple to see that making the rescalings

\[
\begin{align*}
t \mu^2 & \to t; \quad k/\mu \to k; \\
U/\mu & \to U; \quad r/\mu \to r; \\
g/\mu^{d-1} & \to g; \quad \Lambda/\mu \to \Lambda; \\
\end{align*}
\]

the parameter \( \mu \) disappears from equations of motion for \( \phi \equiv \phi_\alpha \) and \( U \equiv U_\alpha \):

\[
\begin{align*}
\frac{\partial}{\partial t} \phi(k, t) &= -\Gamma_\phi [k^2 + r + gS(t)] \phi(k, t) \\
& \quad - \Gamma_\phi \mu U(k, t)
\end{align*}
\]

\[
\frac{\partial}{\partial t} U(k, t) = -\Gamma_U \mu k^2 \phi(k, t) - \Gamma_U wk^2 U(k, t).
\]

As a consequence the crossover time scales as \( 1/\mu^2 \). From this analysis it follows that if the dynamics of \( U \) is sufficiently slow then for \( 1 \ll t < 1/\mu^2 \) the field \( \phi \) exhibits a NCOP behaviour while for \( t \gg 1/\mu^2 \) a COP behaviour. If the dynamics of \( U \) becomes too fast the first NCOP behaviour shrinks and becomes hardly observable.

This scenario can be confirmed by solving the equation of motion (\ref{eq:phi}) and (\ref{eq:U}) in a quasilinear approximation. To this end we assume that

\[
R(t) = r + gS(t)
\]

is slowly varying in time, so that it can be considered constant over successive intervals of time. In other words, we make a piecewise linearization of the equation of motion along the trajectory. In spite of that, the approximation is sufficient to identify the different regimes of the relaxation process.

If we neglect the time dependence of \( R(t) \) and assume it to be nearly constant eqs. (\ref{eq:phi}) and (\ref{eq:U}) become a linear system whose solution has the form:

\[
\begin{align*}
\phi(k, t) &= c^+_\phi(k) e^{\omega_+(k) t} + c^-_\phi(k) e^{\omega_-(k) t} \\
U(k, t) &= c^+_U(k) e^{\omega_+(k) t} + c^-_U(k) e^{\omega_-(k) t}
\end{align*}
\]

where \( \omega_+(k) \) and \( \omega_-(k) \) are the eigenvalues of the \( M \) matrix.

\[
\omega_{\pm}(k) = \frac{1}{2} \left[ -\Gamma_\phi(k^2 + R) - \Gamma_U wk^2 \\
\pm \sqrt{\left[ \Gamma_\phi(k^2 + R) + \Gamma_U wk^2 \right]^2 + 4 \Gamma_\phi \Gamma_U \mu^2 k^2} \right]
\]

For time \( t \gg 1 \) the dynamical behavior of the solution is determined by the larger eigenvalue \( \omega_+(k) \). For large \( k^2 \), the eigenvalue \( \omega_+(k) \) decreases proportionally to \(-k^2 \) and hence large momenta are exponentially damped. Moreover we see that \( \omega_+(k) \) is a function of \( k^2 \) which either has an extremum at \( k = 0 \) or a single maximum for \( k \neq 0 \), as one can verify by inspecting the small-\( k \) behaviour of \( \omega_+(k) \). The behavior of \( \omega+(k) \) is shown in Fig. 1 for \( t < \tau_f \) and \( t > \tau_f \). The crossover time \( \tau_f \) is defined as the time when the fastest growing mode moves from \( k = 0 \) to \( k \neq 0 \). Other definitions of \( \tau_f \) are possible, e.g., the time when the peak at \( k \neq 0 \) becomes higher than the \( k = 0 \) one. However, all definitions lead to similar results.

Below the critical temperature \( T_c \) and in the early stage of the ordering process the value of \( gS(t) \) is small compared with \( r \), i.e. \( R < 0 \), and the larger eigenvalue is well approximated by:

\[
\omega_+(k) = \Gamma_\phi |R| - \left( \Gamma_\phi - \Gamma_U \frac{\mu^2}{|R|} \right) k^2 + O(k^4)
\]

A brief calculation reveals that

\[
\begin{align*}
c^+_{\phi} &= \phi(k, 0) + \frac{\mu}{R} U(k, 0) + O(k^2) \\
c^+_U &= \frac{\Gamma_U \mu}{\Gamma_\phi R} \left[ \phi(k, 0) + \frac{\mu}{R} U(k, 0) \right] k^2 + O(k^4).
\end{align*}
\]

Therefore, assuming \( \phi(k, 0) + (\mu/R) U(k, 0) = O(1) \) for \( k \to 0 \), the coefficient \( c^+_U(k) \) is finite for \( k \to 0 \) while \( c^+_\phi(k) \) vanishes, indicating that the amplitudes of the longest wavelength components of \( U \) are decreased due to the conservation law.

As a consequence, below \( T_c \) the structure factor \( C_{\phi\phi}(k, t) \) develops a peak centered at \( k = 0 \), growing in time as a power of \( t \). The structure functions \( C_{\phi U}(k, t) \) and \( C_{UU}(k, t) \) also develop a peak, centered at a finite value of \( k \), say \( k_f \), as a result of the competing effect between the \( k \) dependence of the exponential factor \( e^{\omega_+(k) t} \) and the amplitude \( c^+_\phi(k) \), see eq (\ref{eq:amplitude}).
This mechanism selects a set of exponentially growing \( U \)-modes with wavevectors in a certain range centered around \( k_f \), whose dependence on the coupling \( \mu \) is shown in Fig. 3. Such modes represent an inhomogeneity of the \( U \) field which in turn affects the spatial properties of the \( \phi \) subsystem. One witnesses a strong feedback process between the two fields and the outcome is the slaving of the NCOP dynamics of the field \( \phi \) to the COP dynamics of the \( U \)-field.

The power law growth of \( C_{\phi \phi}(k = 0, t) \) can be extracted from the quasilinear approximation by using \( (11) \). In the early regime \( R \) starts from a negative value and grows towards zero due to the growing of \( C_{\phi \phi}(k, t) \) for small \( k \). This in turn implies that \( S(t) \) tends to a finite value for increasing time. By imposing this condition, and making use of \( (11) \), it follows that \( C_{\phi \phi}(k = 0, t) \sim t^{d/2} \), as in the pure NCOP, i.e. the longest wavelength fluctuations grow faster. We note that while the quasilinear approximation leads to the correct scaling of \( C_{\phi \phi}(k = 0, t) \sim t^{d/2} \) and of the domain size \( L(t) \sim t^{1/2} \), it gives for \( R(t) \sim \log(t)/t \) which reveals that the approximation is slightly crude.

These results are valid for \( R/\mu^2 \) not too large, i.e., far from the crossover region where \( R \) changes sign. Unlike the pure NCOP, where \( R(t) \) goes to zero, after a characteristic time \( \tau_f = O(1/\mu^2) \) the value of \( |R| \) becomes \( O(\mu^2) \) and the NCOP behaviour ends. By inspection of eq.\( (11) \), we see that if \( |R|/\mu^2 < \Gamma_U/\Gamma_{\phi} \) the maximum of \( \omega_+(k) \) moves away from \( k = 0 \) and the system loses its NCOP behaviour.

This regime corresponds in our model to the instability which is observed in systems where a non-conserved \( C_{\phi \phi}(k, t) \) is shown in Fig. 3. For times \( t = O(1/\mu^2) \) the quantity \( R \) changes sign becoming positive, and finally for \( t \to \infty \) tends to a finite value \( \mu^2/w \) while the maximum of \( \omega_+(k) \) moves again towards vanishing wavevectors. The dynamics is therefore dominated in the regime \( t \gg 1/\mu^2 \) by long wavelength fluctuations. We can then expand \( \omega_+(k) \) in powers of \( k \) obtaining

\[
\omega_+(k) = \Gamma_U \left( \frac{\mu^2}{r + gS} - w \right) k^2 - c_4 k^4
\]

where \( c_4 \) is a positive coefficient having a finite limit for \( \mu^2/R \to w \). By imposing that \( S(t) \) has a finite non zero, limit for \( t \to \infty \) and making use of eq.\( (43) \) one obtains that in this regime

\[
C_{\phi \phi}(k, t) = \left[ L(t)^2 k_m(t)^{2-\delta} \right] \varphi(k/k_m(t))
\]

where

\[
L(t) \sim t^{1/4}, \quad k_m(t) \sim \left( \frac{d \log t}{4} \right)^{1/4}
\]

a behaviour typical of COP dynamics \( [4] \). The multi-scaling function \( \varphi(x) \) is given by

\[
\varphi(x) = 1 - (x^2 - 1)^2
\]

The COP behaviour is also observed if one considers the structure functions \( C_{\phi \phi}(k, t) \) and \( C_{UU}(k, t) \). Such a multiscaling behavior follows from the competition of two marginally distinct lengths, namely the domain size \( L(t) \) and \( k_m^{-1} \).

Finally we note that the quasilinear approximation in this regime leads to

\[
R(t) - \mu^2/w \sim \left( \frac{\log t}{t} \right)^{1/2}.
\]

From equation \( (23) \) we see that \( R - \mu^2/w \) plays the role of the mass term \( r + gS(t) \) in pure COP dynamics \( [4] \), therefore in spite of the fact that the quasilinear approximation is quite crude, it gives nevertheless the correct scaling behaviour of COP dynamics.

The above theoretical predictions were checked by integrating numerically the system of equations \( (18)-(20) \) by a Simpson rule discretizing the wavevectors in the interval \([0, \Lambda]\). Figure 4 displays the structure function \( C_{\phi \phi}(k, t) \) for various values of the time \( t \). One clearly sees that in the early regime the fastest growing modes are centered about \( k = 0 \), because long-wavelengths fluctuations of the field \( \phi \) increase more rapidly than shorter ones, whereas for \( t > \tau_f \) a finite wavevector peak appears. Moreover, the growth, in this late regime, has a conserved character because its value at \( k = 0 \) remains constant. The evolution of \( C_{\phi \phi}(k, t) \) and \( C_{UU}(k, t) \) is shown in Figs. 3 and 4. Initially the fields \( U \) and \( \phi \) evolve as if they were nearly independent and the \( C_{UU} \) correlations display the usual finite wavevector peak of the conserved dynamics, whereas the \( \phi \) field evolves according to a faster non conserved dynamics. The long wavelength fluctuations of the \( U \) field are hindered by the conservation law and the presence of the \( C_{\phi \phi} \) term has only a small effect on the \( C_{\phi \phi} \). However, as the domain size \( L(t) \) reaches a critical value and becomes comparable with \( \lambda_f \), the typical length of the oscillations of the diffusive field, the two fields strongly interact. Within this late regime the dynamics becomes controlled by the conservation law induced by the \( U \) field. In Fig. 5 we show the behavior of the height of the peak of \( C_{\phi \phi} \) versus time, where one clearly sees the crossover from the early time behavior \( t^{d/2} \) to the late stage slope \( t^{d/4} \). In the crossover regime due to the presence of a double peak the maximum height decreases until the peak at \( k = 0 \) disappears.
Finally we report the numerical result concerning the multiscaling, observed in the late regime. In Fig. 8 we display the shape function $F(x) = k_m^d(t)C_{\phi}(xk_m, t)$ as a function of $x$. In Fig. 9 we show the multiscaling function $\varphi(x)$ obtained from the best fit of $C_{\phi}(k, t)$ as a function of $L(t)^2 k_m(t)^{2-d}$ for fixed values of $x = k/k_m(t)$. Similar curves can be extracted from the other structure functions. We note that while the data follow quite well the theoretical result (46) for $|x - 1|$ not to large, they display a large deviation as $|x - 1|$ increases. This is due to the terms neglected in (43). We remark, however, that these become less and less important as $k_m$ decreases, as a consequence we expect that the range of values of $|x - 1|$ where there is a good agreement with (46) should increase with time. This is indeed observed by using data for increasing time in the best fit. Roughly the range increases as $1/k_m(t)$.

Finally, we have explored, different types of conservation laws represented by $\Gamma f(k^{\mu})$ with $0 < \mu < 2$. In all these cases the dynamics selects for intermediate times a peak at finite values of the wavevector $k$.

VI. CONCLUSIONS

To summarize in this paper we have studied the evolution of an $N$-component version of the Phase field model and shown that the coupling between the massless transverse modes and the diffusive field produce an instability at finite wavelengths. Our model, where the low-energy Nambu-Goldstone modes couple to the diffusive modes provides an interesting new scenario and we believe represents a paradigm for the Mullins-Sekerka type of phenomena where the soft modes are represented by the capillary wave spectrum associated with the solid-melt interface and the diffusive field is the heat transport. These two fields concur to destabilize the solid-melt boundary in analogy with our findings. On physical grounds, one expects this kind of instability to occur during phase separation, because small droplets can dissipate heat more efficiently and reach rapidly thermal equilibrium due to the Gibbs-Thomson effect, whereas larger droplets try to dissipate energy faster by creating bulges thus increasing the curvature. As the system cools down upon reaching equilibrium the typical length of the bulges, $\lambda_f$, increases and diverges together with the average domain-size $L(t)$.

We have demonstrated that the presence of $U$ induces non trivial effects on the field $\phi$ because it acts on a time scale longer than the noise field, characterized by a short correlation time.

Finally, the $O(N)$ model analysed presents unusual features since it displays scaling behavior in the early regime ($t < \tau_f$) and multiscaling in the late regime and constitutes an example of multiscaling without COP, a phenomenon which at the best of our knowledge was not observed before.

In summary the present model reveals an unexpectedly rich dynamical behavior which reminds in many respects the solidification kinetics.

VII. APPENDIX: EQUILIBRIUM PROPERTIES

In the present appendix we shall outline the calculations of the equilibrium properties of the model. The partition function associated with the Hamiltonian

$$Z[\{h(x)\}] = \int_{-\infty}^{\infty} \Pi_{\alpha=1}^{N} d\phi_{\alpha} dU_{\alpha} e^{-\beta H[\phi, U] + \beta h \phi}$$  (48)

Where we have included an external field $h(x)$ coupled linearly to $\phi(x)$ and $\beta = (k_B T_f)^{-1}$. The field $U(x)$ can be traced out and one finds a part from uninteresting constants:

$$Z[\{h(x)\}] = \int_{-\infty}^{\infty} \Pi_{\alpha=1}^{N} d\phi_{\alpha} e^{-\beta H_{\text{eff}}[\phi] + \beta h \phi}$$  (49)

where $H_{\text{eff}}$ is defined by Eq. (23). In order to separate the macroscopic component $P$ of the field we employ the following identity

$$1 = N \int_{-\infty}^{\infty} dP_{\phi} \delta(NP^2 - \sum_{\alpha} \phi_{\alpha}^2)$$  (50)

and rewrite $Z$ as

$$Z = N \int_{-\infty}^{\infty} dP_{\phi} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{\infty} \Pi_{\alpha=1}^{N} d\phi_{\alpha} \times e^{-\beta H_{\text{eff}}[\phi_{\alpha}] + \beta h \phi + \lambda (NP^2 - \sum_{\alpha} \phi_{\alpha}^2)}$$  (51)

$$Z = N \int_{-\infty}^{\infty} dP_{\phi} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-\beta N(\frac{\lambda^2}{4} P^2 + \frac{\beta^2}{4} P^4 - \frac{\lambda^2}{8} F^2)}$$

$$\times \int_{-\infty}^{\infty} \Pi_{\alpha=1}^{N} d\phi_{\alpha} \times e^{-\beta / 2 \sum_{\alpha} [-\int d^d x \nabla \phi_{\alpha} \nabla \phi_{\alpha} + 2i \lambda / \beta \phi_{\alpha}^2 + \beta \sum_{\alpha} h_{\alpha} \phi_{\alpha}]$$  (52)

In the case of a uniform external field directed along the component 1 ($h_1 = h$), after eliminating the $\phi_{\alpha}$ fields, $Z$ reads:

$$Z = N c^{\phi \ln(2\pi/\beta)} \int_{-\infty}^{\infty} dP_{\phi} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{\Omega N}$$  (53)

$$\Omega = -\beta \int d^d x \left[ \frac{\lambda^2}{2} P^2 + \frac{\beta^2}{4} P^4 - \frac{i \lambda}{\beta} \phi_{\alpha}^2 \right]$$

$$- \frac{\beta}{2} \int \left( \frac{d^d + 1}{2\pi m} \ln(k^2 + 2i \lambda / \beta) - \frac{\beta^2 h_{\alpha}^2}{2\pi} \right)$$  (54)

In order to evaluate $Z$ we apply the saddle point estimate in the limit $N \to \infty$ imposing the conditions
where following condition must be fulfilled:

\[
\frac{2i\lambda}{\beta} = r_{\text{eff}} + gP^2
\]  

(55)

\[
P^2 = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + 2i\lambda/\beta - \beta^2h^2/4\lambda^2}
\]  

(56)

Eliminating \( \lambda \) with the help of eqs. (55), (56) we find:

\[
P^2 = \frac{h^2}{(r_{\text{eff}} + gP^2)^2 + 1} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + r_{\text{eff}} + gP^2}
\]  

\[= m^2 + S_\infty\]

(57)

The last term equals \( m^2 \), the square of the average magnetization per unit volume \( m \neq 0 \), i.e. the following condition must be fulfilled:

\[
\lim_{h \to 0} [r_{\text{eff}} + gP^2] = 0
\]  

(59)

The equation of state reads

\[
\left[ gT_f \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + r_{\text{eff}} + gS_\infty + gm^2 + r_{\text{eff}} + gm^2} \right] m = h
\]  

(60)

where \( S_\infty \) is given by:

\[
S_\infty = gT_f \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 + r_{\text{eff}} + gS_\infty + gm^2}
\]  

(61)

In order to determine the critical temperature \( T_c \) we require \( m^2 = 0 \) and \( r_{\text{eff}} + gS_\infty = 0 \).

\[
T_c = (\mu^2/(w-r))d/(g\Lambda^2K_d)
\]  

(62)

where \( 1/K_d = 2\pi^{d/2}\Gamma(d/2) \) where \( \Gamma(x) \) is the Gamma function.

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FIG. 1. Dispersion relation \( \omega_t(k) \) in the early regime and in the late regime. Both quantities are plotted in arbitrary units. Notice the different long-wavelength behavior in the two cases. The first is typical of NCOP dynamics, whereas the second characterizes COP dynamics.

FIG. 2. Dependence of the wavevector instability \( k_f \) on the coupling parameter \( \mu \) in log-log scale. Both quantities are plotted in arbitrary units. The squares are obtained from the numerical solution of eqs. (58) - (61) with \( \Gamma_\phi = 1, \Gamma_U = 5, r = -0.5, g = 1, w = 0.05 \) and \( d = 3 \). The broken line has slope 1. The value of \( k_f \) is taken when the peak at \( k_f \) becomes larger than the one at \( k = 0 \). Other definitions, e.g., fastest growing mode, lead to a similar dependence as follows from the rescaling (54).

FIG. 3. Dependence of the crossover time \( \tau_f \) on the coupling parameter \( \mu \) in log-log scale. Both quantities are plotted in arbitrary units. The broken line has slope \(-2\). The squares are obtained from the numerical solution of eqs. (58) - (61) with \( \Gamma_\phi = 1, \Gamma_U = 5, r = -0.5, g = 1, w = 0.05 \) and \( d = 3 \). The crossover time \( \tau_f \) is defined as the time when the peak at \( k \neq 0 \) becomes dominant. Other definitions, e.g., fastest growing mode lead to similar dependence as follows from the rescaling (54).

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FIG. 4. Typical evolution of the structure function $C_{\phi\phi}(k, t)$ for different times as shown in the figure. Both quantities are plotted in arbitrary units. Notice that the two largest times correspond to $t > \tau_f$ and display the characteristic COP peak. The data are from the numerical solution of eqs. (13) - (20) with $\Gamma_\phi = 1$, $\Gamma_U = 5$, $r = -0.5$, $g = 1$, $w = 0.05$ and $d = 3$.

FIG. 5. Typical evolution of the structure function $C_{\phi U}(k, t)$ The data are from the numerical solution of eqs. (13) - (20) with parameters as in Fig. 4. Both quantities are plotted in arbitrary units.

FIG. 6. Time evolution of the structure function $C_{UU}(k, t)$ for different times. Notice the conserved dynamics at all times. The data are from the numerical solution of eqs. (13) - (20) with parameters as in Fig. 4. Both quantities are plotted in arbitrary units.

FIG. 7. Height of the peak of the structure function $C_{\phi\phi}(k_m, t)$ as a function of time. Both quantities are plotted in arbitrary units. The crossover from the NCOP behavior to the COP behavior is evident. The dashed line represents the $t^{d/2}$ behavior whereas the dashed-dot line the $t^{d/4}$ behavior. The data are from the numerical solution of eqs. (13) - (20) with parameters as in Fig. 4.

FIG. 8. Shape function $F(x)$ of the $C_{\phi,\phi}(k, t)$ structure function in the late stage evolution. The absence of scaling is evident in figure. The data are from the numerical solution of eqs. (13) - (20) with parameters as in Fig. 4.

FIG. 9. Multiscaling exponent $\varphi(x)$ defined in the text. The data are from the numerical solution of eqs. (13) - (20) with $\Gamma_\phi = 1$, $\Gamma_U = 5$, $r = -0.5$, $g = 1$, $w = 0.05$ and $d = 3$. The crosses are obtained from $C_{\phi\phi}$ while the triangles from $C_{UU}$. The full line is the theoretical prediction (46).
\[ \omega_+(k) \leq 0 \quad (t < \tau_f) \]

\[ \omega_+(k) > 0 \quad (t > \tau_f) \]
$C_{\phi\phi}(k_m, t)$

- NCOP
- COP

$t$

Log-log scale with axes ranging from $10^2$ to $10^7$.
\[ F(x) = \frac{k}{k_m} \]
\[ \phi(x) \]