The condition of the resonant break-up of a gas bubble subjected to an acoustic wave in liquid

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Abstract. The problem of a gas bubble break-up in liquid is considered in the conditions of the frequencies resonance of the radial and nth axially symmetric deformational mode 2:1. The nonlinear energy transfer between the modes is described using an efficient Krylov-Bogolyubov averaging technique. It is shown that the deformational mode magnitude can be some orders larger than the radial mode magnitude which is damped by the thermal, viscous and acoustic dissipation. The estimative criterion of bubble break-up is obtained in the cases of slow and fast acoustic wave start. The obtained pressure magnitudes in the wave for break-up are very small and the mechanism can have strong medical and technical applications.

1. Introduction

The mechanisms of break-up and the subharmonics’ appearance are not yet fully understood and have gained much interest in present science. The bubble break-up problem has applications in medicine, oceanology and in all fields related to cavitation and sonoluminescence. For example one of the most promising applications of bubble break-up is the brain-blood-barrier opening by forcing break-up near barrier boundary with the focused ultrasound [1] and minimization of the ultrasound magnitude is very important for safeness.

There is much evidence that the bubbles in acoustic field emit subharmonic sound at frequency being the integer part of the driving frequency [2]. This fact can be attributed to the existence of nonspherical bubble harmonics resonant to driving frequency. There exist many experimental works that prove the possibility of bubble break-up because of its nonspherical shape oscillations [3]. Theory on this topic usually deals with the nonlinear interaction between oscillation modes and there are many works predicting the amplified energy transfer at frequency resonance 2:1 [4]. The possibility of break-up was mentioned in the latest work on this topic [5] that dealt with the problem of resonant energy transfer from radial oscillations mode to any deformation mode in case of free oscillations. In the latter the period of energy transfer and asymptotic trajectories were obtained and the enormous growth of deformation mode was attributed as a sign of break-up possibility. The main disadvantage of this work is that the damping was not considered at all. In this work a bubble in an acoustic wave is considered and for the purpose of this paper the case of axially symmetric deformation oscillations are considered (as shown in [5] the axially symmetric mode is the most dangerous for the break-up phenomenon). The problem is solved in a Hamiltonian approach by an effective Krylov-Bogolyubov
considered to be small coefficient radial and deformational oscillations, respectively. The two last terms in the second equation can be here \( n \) is the Legendre polynomial index, \( \rho \) fluid density, \( s \) sound speed in fluid; \( S_t=2a_0/\rho_\infty \) – the ratio of the Laplace pressure to the external pressure, \( \sigma \) – surface tension; \( Vis=4\mu\Omega/\rho_\infty \) – the ratio of the viscous pressure to the external pressure, \( \mu \) – fluid viscosity; \( Pe=\rho_0^2\Omega/k \) – Peclet number, \( k \) – gas thermal diffusivity. In figure 1 the resonant curves for the bubbles of different radii and their enveloping curve (by dash line) are shown.

\[
\text{Resp}(\Omega) = \frac{A(\Omega)}{F} = \left( \frac{Dy + St - i Vi s - \frac{3}{1 + (1 + i \gamma)}G}{\Omega} \right)^{-1}, G = 1 - 3 \frac{\sqrt{\text{Pe}} \cosh \sqrt{\text{Pe}} - 1}{\text{Pe}}
\]

(1)

Here we have introduced dimensionless parameters: \( D y = \rho a_0^2 \Omega^2/\rho_\infty \) – the ratio of the dynamic pressure to the external pressure, \( \rho \) – fluid density; \( M a = a_0/\rho_\infty c \) – Mach number, \( c \) – sound speed in fluid; \( S_t=2a_0/\rho_\infty \) – the ratio of the Laplace pressure to the external pressure, \( \sigma \) – surface tension; \( Vi s=4\mu\Omega/\rho_\infty \) – the ratio of the viscous pressure to the external pressure, \( \mu \) – fluid viscosity; \( Pe=\rho_0^2\Omega/k \) – Peclet number, \( k \) – gas thermal diffusivity. In figure 1 the resonant curves for the bubbles of different radii and their enveloping curve (by dash line) are shown.

**Figure 1.** The resonant curves for the air bubbles of different radii in water

Following [5] the axially symmetric bubble with two degrees of freedom is considered and its surface is described by \( r(\theta,r)=a_0(1+x(r)+\xi(r))P_n(\cos \theta) \). The bubble Hamiltonian obtained in [5] is used. The terms responsible for dissipation of the radial mode and driving force are added to the Hamilton equations:

\[
\begin{align*}
\dot{x} - u &= -(n+3)yv, \quad \dot{u} + 4x = (3/2)v^2 - 4(n+1)y^2 - \alpha \cos 2t - \beta u\dot{y} - v = -(n+3)yv - 3xy, \\
\dot{y} + v &= (n+3)u\dot{v} - 8(n+1)xy, \quad v = \xi/\sqrt{(n+1)(2n+1)}
\end{align*}
\]

(2)

Here \( n \) is the Legendre polynomial index, \( x, u \) and \( y, v \) the generalized coordinates and moments of radial and deformational oscillations, respectively. The two last terms in the second equation can be linked with the bubble physical properties by (1), and the driving force \( \alpha=(4/3)F \) and the damping coefficient \( \beta=2/3(\text{Resp}(\Omega))^{1/2} \) may be obtained. The driving force and the damping coefficient are considered to be small \( \alpha=\varepsilon^2, \beta=\varepsilon \beta_1 \). The stable asymptotic solution is obtained with the help of the Krylov-Bogolyubov averaging technique (the stability is proven by the second Bogolyubov theorem):
be considered critical for the break-up occurrence. Simulations show that a neck separating parts of the bubble appears around shown in figure 3. All the pictures except the bottom left one are drawn for conditions. Functions part) is compared with the numerical simulation of the initial system (2) (top part) with the same initial $t$.

The initial conditions for simulation were: $x(0)=−0.004$, $u(0)=−0.01$, $y(0)=−0.002$, $v(0)=0.004$. The choice of initial conditions may be arbitrary in order to meet the conditions of smallness of driving force and dissipation. The obtained analytical formulas describe the qualitative behaviour of the system at first moment and are very precise in describing the steady oscillations. These oscillations magnitude practically doesn't change when we slightly modify the driving frequency and damping coefficient. Numerical simulations have shown that the possible frequency mismatch is proportional to $\sqrt{\alpha}$.

$$x(t) = -\alpha \left( \frac{8n(9n+16) + 97}{(4n-1)^2} \cos 2t + \frac{8n + 5}{4(4n-1)} \right) + \tilde{x}(t)e^{-\beta t/4}$$

$$y(t) = \pm \sqrt{\alpha} \left( \frac{2}{\sqrt{4n-1}} \sin t - \frac{10n(8n+15) + 91}{(4n-1)^{3/2}} \beta \cos t \right) + \tilde{y}(t)e^{-\beta t/4}$$

Here $\tilde{x}(t),\tilde{y}(t)$ are quickly oscillating functions. In figure 2 the obtained analytical solution (bottom part) is compared with the numerical simulation of the initial system (2) (top part) with the same initial conditions. Functions $x(t)$ and $y(t)$ are plotted with red and blue colors. The left part is devoted to the $t<900$ and the right part to $4972<t<5003$. The initial conditions for simulation were: $\alpha=0.0001, \beta=0.01, n=19/8$, $x(0)=−0.004$, $u(0)=−0.01$, $y(0)=−0.002$, $v(0)=0.004$. The choice of initial conditions may be arbitrary in order to meet the conditions of smallness of driving force and dissipation. The obtained analytical formulas describe the qualitative behaviour of the system at first moment and are very precise in describing the steady oscillations. These oscillations magnitude practically doesn't change when we slightly modify the driving frequency and damping coefficient. Numerical simulations have shown that the possible frequency mismatch is proportional to $\sqrt{\alpha}$.

![Figure 2. The comparison of the asymptotical solution (bottom part) and the numerical simulation (top part) of (2).](image)

Some bubble oscillations modes associated with different Legendre polynomial indices $n$ are shown in figure 3. All the pictures except the bottom left one are drawn for $\zeta=1$. Numerical simulations show that a neck separating parts of the bubble appears around $\zeta=0.5$ and this value may be considered critical for the break-up occurrence.

![Figure 3. Some modes of bubble oscillations.](image)
As may be observed in figure 2, the maximal value of the deformational mode is obtained during the transitional process and is approximately 1.7 times higher than the stationary magnitude. Thus, we obtain the estimative criterion for break-up in the cases of the fast and slow wave start. The latter is the critical acoustic wave magnitude $F_{c} \approx 1/(5n)$. In figure 4 the break-up conditions for the air bubble in water at the atmospheric pressure is shown. The dashed lined stands for the maximal time of the fast acoustic wave start. Diamonds and circles stand for the critical acoustic wave magnitude in the cases of the fast and slow acoustic wave start for the bubbles satisfying the resonance 2:1 between radial and deformational modes condition:

$$a_0 = \frac{(2(n-1)(n+1)(n+2)+1)}{(3\gamma-1)} \cdot \frac{(2\sigma)}{p_{\infty}}$$

![Figure 4. The break-up conditions and the time of the fast acoustic wave start.](image)

One may observe that the break-up acoustic wave magnitude may be really small and this is possible due to two resonance conditions: resonance of frequencies 2:1 between the radial and the deformational modes and the frequency of the acoustic wave in resonance with the radial frequency. Surely, for proving this mechanism the strongly nonlinear deformations should be taken into account and the experiments should be carried out.

3. Conclusions

The estimative criterion of gas bubble break-up in liquid in case of the resonance of the radial and n-th deformational modes frequencies 2:1 and the resonant frequency of the driving force was obtained in cases of slow and fast acoustic wave start. The needed acoustic wave magnitude is very small and the proposed break-up mechanism may have numerous medical and technical applications.

The work was supported by the Russian Science Foundation, project No 14-19-01633.

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