Correlations and beam splitters for quantum Hall anyons

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We study the system of two localized anyons in the lowest Landau level and show how anyonic signatures extrapolating between anti-bunching tendencies of fermions and bunching tendencies of bosons become manifest in the two-particle correlations. Towards probing these correlations, we discuss the influence of a saddle potential on these anyons; we exploit analogies from quantum optics to analyze the time-evolution of such a system. We show that the saddle potential can act as a beam splitter akin to those in bosonic and fermionic systems and can provide a means of measuring the derived anyonic signatures.

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The quantum statistics of bosons and fermions plays a fundamental role over a vast range of length scales and diverse physics such as the spatial distribution and energetics of electrons in atoms, certain constraints on scattering cross sections in nuclear physics, the existence of superfluids and the stability of neutron stars. Statistical signatures key to these phenomena have been explored over the past decades via analyses of two-particle correlations including in seminal studies of bosonic bunching properties by Hanbury Brown and Twiss \cite{1} and Hong et al. \cite{2}. The latter, which also has its fermionic counterpart \cite{3, 4, 5}, performs time-resolved coincidence measurements of pairs of photons incident on a beam splitter from two uncorrelated sources collected at two detectors. While these analyses have established the allowed quantum nature of particles in three dimensions, the past decade has drawn attention to the study of two-dimensional “anyons”, quasiparticles which obey fractional statistics interpolating between those of fermions and bosons \cite{6, 7}. Given the current rapid experimental progress in two-dimensional systems and the keen quest for topologically ordered states which can be ascertained by the detection of anyons, a fundamental understanding of these entities analogous to that of fermions and bosons is much called for.

Here, we explore the (anti)bunching properties of a system of two non-interacting Abelian anyons by analyzing the behavior of specific observables and propose a means of realizing a beam splitter wherein these anyonic properties become manifest. The common wave function for these anyons by definition picks up a phase of \( e^{i\pi \alpha} \) \((e^{-i\pi \alpha})\) upon an anticlockwise (clockwise) exchange of the particles\cite{6}. The parameter \( \alpha \) lies in the range \( 0 < \alpha < 1; \alpha = 0 \) and 1 correspond to bosons and fermions respectively. Of direct relevance to bulk quasihole excitations in the quantum Hall system\cite{8, 9, 10, 11} – a paradigm for anyonic statistics – we study a two-dimensional system of two anyons in a magnetic field projected onto the lowest Landau level(LLL). (In particular, for Laughlin states \cite{9}, quasiholes have fractional charge \( q = -e/m \) and statistics \( \alpha = 1/m \), where \( m \) is an odd integer \[12, 13\].) We find that while anyonic signatures in the LLL are subtle, they clearly extrapolate between their bosonic and fermionic counterparts. We show that the presence of a saddle potential offers a means for LLL anyons to approach one another along two incoming limbs and then propagate away along two out-going limbs, akin to the photonic beam splitter settings, and that analogous coincidence measurements made along the limbs can reflect our predicted anyonic signatures.

The Hamiltonian for two anyons in a perpendicular magnetic field \( B = \hbar k \) has the decoupled form

\[
\hat{H} = \frac{1}{4\mu} \left( \hat{P}_x + qBc\hat{Y}_c \right)^2 + \frac{1}{4\mu} \left( \hat{P}_y - qBc\hat{X}_c \right)^2 + \frac{1}{\mu} \left( \hat{P}_x + qBc\hat{y}_r \right)^2 + \frac{1}{\mu} \left( \hat{P}_y - qBc\hat{x}_r \right)^2 ,
\]

in terms of center of mass (c.o.m.) and relative variables. Here the anyons are assumed to have mass \( \mu \) (which is immaterial when states are projected to the LLL) and charge \( q \). We have chosen the symmetric gauge \( \mathbf{A}_s = (B/2)(-y, i - x, j), \gamma = 1, 2 \) for each particle. The c.o.m. coordinate and momentum are given by \( \mathbf{R}_c = (r_1 + r_2)/2 \) and \( \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 \), while the relative coordinate and momentum are given by \( \mathbf{r} = r_1 - r_2 \) and \( \mathbf{p} = (p_1 - p_2)/2 \). In both the c.o.m. and relative coordinate sectors, the LLL is spanned by energy degenerate angular momentum eigenstates. The c.o.m Hilbert space is identical to that of a single particle; angular momentum states \( | n \rangle \) are eigenstates of angular momentum \( h\mathbf{A}\mathbf{A}^\dagger \) having eigenvalues \( nh \), where \( n \) is an integer \textsuperscript{14}. Here, the usual commutation rules \([\mathbf{A}, \mathbf{A}^\dagger] = 1\) are satisfied and the components of the guiding centers have the form \( \hat{X} = l(\hat{A} + \hat{A}^\dagger)/2 \) and \( \hat{Y} = il(\hat{A} - \hat{A}^\dagger)/2 \), where \( l = \sqrt{\hbar c/qB} \) is the single-particle magnetic length\textsuperscript{22}. In the relative coordinate sector, the anyon boundary condition is not respected by the guiding center coordinates, \( \hat{x}, \hat{y} \), but it is by their quadratic combinations,
\[ \hat{a} \equiv (\hat{x}^2 + \hat{y}^2)/8l^2, \quad \hat{b} \equiv (\hat{x}^2 - \hat{y}^2)/8l^2, \quad \hat{c} \equiv (\hat{x}\hat{y} - \hat{y}\hat{x})/8l^2. \]

These operators respect a \( sp(1, R) \) algebra \[10\]. The relative coordinate Hilbert space consists of irreducible representations of this algebra \([k, \alpha]_r\), where \( k \) is an integer, and correspond to eigenstates of the angular momentum \( \hat{L} = \hbar(2\hat{a} - 1/2) \) having eigenvalues \((2k + \alpha)\hbar\) \[10\].

Localized anyons can be composed as linear combinations of the degenerate LLL angular momentum states; Ref. \[11\] provides a careful derivation of the form of such anyon-coordinate states that most closely describe localized LLL quasiholes, leading to the expected exchange statistics. These states can be decomposed into product states of localized states centered at the dimensionless c.o.m. coordinates \( Z = (z_1 + z_2)/2 \) and relative coordinates \( z = z_1 - z_2 \), where the individual anyons are centered at \( z_\gamma = (x_\gamma + iy_\gamma)/l, \gamma = 1, 2 \). These localized states have the form \[11\, 24]\]

\[ Z_c = e^{-|z|^2/2} \sum_{n=0}^\infty \frac{(Z^*)^n}{\sqrt{n!}} |n\rangle_c, \]

\[ |z\rangle = N_{\alpha, z} \sum_{k=0}^\infty \frac{(z^*/2)^{2k+\alpha}}{\Gamma(2k + \alpha + 1)} |k, \alpha\rangle_r. \]

Here, we express the normalization \( N_{\alpha, z} \), which in itself contains information on statistics, in terms of a sum of two confluent hypergeometric functions as \( 2|N_{\alpha, z}|^2 = \frac{2}{\Gamma(1 + \alpha)} = (|z|^2/2)^{2\alpha} |M(1, 1 + \alpha, |z|^2/4) + M(1, 1 + \alpha, -|z|^2/4)|^2 \). The convention chosen for the relative co-ordinate localized state explicitly respects the anyonic boundary condition in picking up the desired phase under the exchange action \( z \rightarrow ze^{i\pi} \).

The localized states defined above are consistent with the expected forms for fermions and bosons. For these cases, (anti)symmetrization is brought about by the construction

\[ |z\rangle_1 = e^{-|z|^2/8} N_{1/\alpha, z} |z\rangle_1 \mp |\mp - z\rangle_1, \]

where \( |z\rangle_1 \) refers to the localized state form for distinguishable particles, analogous to Eq.\[2\] but with \( Z \rightarrow z/2 \) in view of the change in effective magnetic length\[23\]. In fact, for localized states, the probability density must be symmetrically peaked close to both relative coordinates \( z \) and \(-z\) for all indistinguishable particles; this can be ascertained by analyzing the anyon state of Eq. \[3\]. The construction \[4\] explicitly shows that the fermion/boson boundary conditions allow only odd/even angular momentum states in the relative co-ordinate localized state decomposition. By evaluating the overlap between \( |z\rangle_1 \) and \( |\mp - z\rangle_1 \), it can be shown that the normalization constant has the simple limiting forms \( |N_{1, z}|^{-2} = \sinh(|z|^2/4) \) and \( |N_{0, z}|^{-2} = \cosh(|z|^2/4) \), respectively.

One of the most direct measures of observing quantum statistics and related (anti)bunching behavior is the average guiding center separation squared, \( \langle \hat{r}^2 \rangle \equiv \langle \hat{x}^2 + \hat{y}^2 \rangle \).

It is well known that for any generic system of spinless fermions/bosons, (anti)symmetrization leads to this average separation being measurably greater/smaller than the value for distinguishable particles\[12\, 13\]. Generally, this statistical effect becomes most pronounced at smaller separation while at larger separation, statistical correlations decay out in a manner characteristic to the particular system and the average separation for distinguishable and indistinguishable particles coincide. Here, to quantify this statistical effect, we define a bunching parameter

\[ \chi(|z|, \alpha) \equiv \frac{1}{4l^2} \left[ \langle |z| \hat{r}^2 |z\rangle_\alpha - \langle z | \hat{r}^2 |z\rangle_\alpha \right], \]

where the factor of \( 4l^2 \) is a matter of convention. A (positive)negative value of \( \chi \) implies (anti)bunching in comparison with distinguishable particles.

We now evaluate the bunching parameter for localized LLL anyons. For distinguishable particles, we have the expected form \( \langle z | \hat{r}^2 |z\rangle_\alpha \equiv (|z|^2 + 2)l^2 \), where the non-zero minimum value reflects the finite width associated with the minimum uncertainty in guiding center positions \( x \) and \( y \), characteristic of states in the LLL. For fermions and bosons, \( \langle \hat{r}^2 \rangle \) can be directly evaluated using the definition in Eq\[4\]. The bunching parameter takes the forms \( \chi(z, 1) = ((|z|^2/4)[\coth((|z|^2/4) - 1]) \) for fermions and \( \chi(z, 0) = (|z|^2/4)[\tanh((|z|^2/4) - 1]) \) for bosons. In keeping with expectations, the bunching parameter is always positive/negative for fermions/bosons and decays exponentially towards zero for large \( |z| \).

For anyons, the desired expectation values can be evaluated by using \( \hat{r}^2 = 8l^2 \hat{a} \), and the eigenstate property \( \hat{a}|k, \alpha\rangle_r = (k + \alpha/2 + 1/4)|k, \alpha\rangle_r \), and hence

\[ \langle |z| \hat{a} \rangle_\alpha = \sum_{k=0}^\infty \frac{(8k + 4\alpha)(|z|^2/4)^{2k+\alpha}}{\Gamma(2k + \alpha + 1)} \]

\[ = 4\alpha \left[ M(1, \alpha, |z|^2/4) + M(1, 1 + \alpha, -|z|^2/4) \right]/\left[ M(1, 1 + \alpha, -|z|^2/4) + M(1, 1 + \alpha, -|z|^2/4) \right]. \]

Fig\[14\] shows the trend exhibited by the bunching parameter obtained from Eq\[6\]. Quite remarkably, the value of \( \chi \) at \( |z|^2 = 0 \) (which is not physically accessible in quantum Hall samples) directly reflects the statistical parameter: \( \chi(0, \alpha) = \alpha \). The limiting case of the fermion, as a function of \( |z| \), \( \chi(|z|, 1) \) begins at a value of unity and then decays to zero in a monotonic fashion. For the bosonic case, \( \chi(|z|, 0) \) always remains negative, beginning at zero decreasing to a minimum value and then rising to taper towards zero. The intermediate anyonic values of \( \alpha \) interpolate between these two limiting behaviors. For all anyons, \( \chi(|z|, \alpha) \) begins at \( \alpha \), decreases below zero, reaches a minimum and finally tapers towards the zero. Hence, the bunching parameter shows that all anyons exhibit anti-bunching at short length scales and bunching.
at long scales and that the trend evolves continuously as a function of $\alpha$. This result is surprising in that one might expect $\alpha < 1/2$ to be boson-like and $\alpha > 1/2$ to be fermion-like. The behavior of $\chi(|z|, \alpha)$ shown in Fig. 1 and its connection to fractional statistics forms the heart of our results.

While the correlations discussed above bear distinct signatures of statistics, they are static in nature due to the projection to the LLL. In practice, probing these correlations requires endowing them with dynamics via the application of appropriate potentials that lift the LLL degeneracy. Here we propose the application of a saddle potential whose effect on each particle can be described by $H_s = \sum_{\gamma=1,2} U \hat{x}_\gamma \hat{y}_\gamma$, $U > 0$, where $U P$ is much smaller than the Landau level spacing thus retaining the LLL projection. In terms of LLL eigenstate solutions for a single particle\cite{4}, it has been shown that the saddle potential acts as a beam splitter in that particles approaching the origin along the $x$-axis tend to scatter either along the positive or the negative $y$-axis. Moreover, for two particles, the potential has the advantage of being separable in terms of the relative and center of mass motion\cite{19}, hence preserving the decoupling of these degrees of freedom, and of respecting the anyon boundary conditions. The saddle potential, when projected to the LLL\cite{11, 14}, can be expressed as

$$\hat{H}_s^P = \frac{1}{2} i U t^2 [\hat{A}^2 - (\hat{A}^\dagger)^2] + 2 U t^2 \hat{c}$$

It can be shown that, as for the single particle case, eigenstates of this Hamiltonian correspond to scattering states in the relative and center of mass sectors and that anyonic statistics translates to scattering phase shifts in the relative sector\cite{13}. Here we show that for pairs of localized particles traveling along opposite limbs of the saddle on the $x$-axis, the choice of propagation along the $y$-axis directly reflects the correlations similar to those shown in Fig. 1.

To gain an insight on the propagation of localized states along saddle potentials, we present an analysis of single particle physics exploiting analogies in quantum optics\cite{14} which can also be applied for the c.o.m. behavior. The associated localized c.o.m. state in Eq. 8 has the coherent state form $|Z\rangle_c = \exp(Z \hat{A}^\dagger - Z^* \hat{A}) |0\rangle_c \equiv \mathcal{D}(Z) |0\rangle_c$. In the Schrödinger picture, we can consider the time-evolution of the coherent state due to the saddle potential: $|Z(t)\rangle_c = e^{-i \mathcal{H}_s t / \hbar} |Z\rangle_c$. In the language of quantum optics, the time-evolution operator has the form of the squeeze operator $\hat{S}(\xi) = \exp[\xi (\hat{A}^\dagger)^2 / 2 - \xi^* \hat{A}^2 / 2]$, where for the c.o.m. sector we have $\xi = -U t t^2 / \hbar$. The squeeze parameter $\xi \equiv r e^{i \phi}$ corresponds to squeezing along the direction $\phi$ with associated aspect ratio $r$; here the squeeze is of magnitude $U t t^2 / \hbar$ along the $x$-axis. We can now invoke the identity $\hat{S}(\xi) \hat{D}(Z) = \hat{D}(Z \cosh r + Z^* e^{i \phi} \sinh r) \hat{S}(\xi)$ and the fact that $\hat{D}(\beta) \hat{S}(\xi) |0\rangle$ represents a squeezed state having squeeze parameter $\xi$ centered at $\beta$\cite{19}. Hence, the time-evolved coherent state flattens along the $y$-axis and its center follows the trajectory $(X e^{-U t t^2 / \hbar}, Y e^{U t t^2 / \hbar})$, where $(X, Y)$ is the initial position of the coherent state. Consistent with semiclassical dynamics along equipotentials of a saddle potential, the center obeys $X(t) Y(t)$ being a constant. Furthermore, any state having the initial condition $Y(t) = 0$ evolves asymptotically towards $Y(t \to \infty) \to +\infty$ and likewise for the lower quadrant.

For the relative motion of anyons the analysis presented above cannot be directly applied as the associated $sp(1, R)$ algebra is rather involved. However, we surmise a few common features and explicitly derive time-evolved expectation values of relevant observables. Given that the initial state probability density for the relative coordinate is peaked at $z$ and $-z$, over time, we expect it to asymptotically be distributed in the upper and lower quadrants in a manner which depends on the statistics of the particles. The functioning of the saddle potential as a beam splitter is best seen when two localized state anyons are placed along or close to the $x$-axis, diametrically across one another with respect to the saddle point origin. As a function of time, the particles approach one another and then get deflected along the $y$-axis. Whether or not they travel in the same direction (along either the positive or negative $y$-direction) or in opposite directions depends on the magnitude of $\langle Y^2 \rangle$ compared to that of $\langle \hat{y}^2 \rangle$. In fact, the quantity analogous to those measured in photonic and electronic beam splitters is $\langle \hat{y}_1 \hat{y}_2 \rangle = \langle \hat{Y}^2 - \hat{y}^2 / 2 \rangle$; a positive/negative value of $\langle \hat{y}_1 \hat{y}_2 \rangle$ indicates that the anyons traveled out along the same/opposite limbs, thus exhibiting bunching/anti-bunching behavior. These correlations are analogous to those between reflected and transmitted currents in electronic beam splitters\cite{4, 2}.

The desired time-evolved expectation values are most

![FIG. 1: (Color online)The bunching parameter as a function of the dimensionless distance between particles $|z|$ for different values of anyonic phase $\alpha$. Curves from the top-most along the $y$-axis correspond to values $\alpha = 1$, $\alpha = 3/5$, $\alpha = 1/3$ and $\alpha = 0$.](image-url)
easily evaluated in the Heisenberg representation. From the commutation relations of \( \hat{a}, \hat{b}, \hat{c} \), one finds
\[
\frac{d\hat{a}}{dt} = -2U\hat{b}\hat{b}\hat{a}/dt = -2U\hat{a}\hat{b}\hat{a}.
\]
The solutions of these equations yield \( \hat{x}^2(t) = e^{-2Ut^2/\hbar}\hat{x}^2(0) \) and \( \hat{y}^2(t) = e^{2Ut^2/\hbar}\hat{y}^2(0) \) for the relative co-ordinates. Similarly, and consistent with the above discussion, one finds \( \hat{X}^2(t) = e^{-2U\hat{t}^2/\hbar}\hat{X}^2(0) \) and \( \hat{Y}^2(t) = e^{2U\hat{t}^2/\hbar}\hat{Y}^2(0) \) for the c.o.m. coordinates. By straightforward but lengthy evaluations of the expectation values of these operators on the initial state described by \( Z \) and \( z \), we find that the correlator for measuring the relative motion of particles along the \( y \)-axis has the form
\[
\langle \hat{y}_1\hat{y}_2 \rangle = l^2 e^{2U\hat{t}^2/\hbar}\left[ \text{Im}[Z]^2 - \frac{1}{4}\text{Im}[z]^2 - \frac{1}{2}\chi + \delta \right]
\]
where \( \chi([z], \alpha) \) is the bunching parameter introduced in Eq [10]. The function \( \delta(z, \alpha) \) is a small correction, which takes a maximum value of 0.018, vanishes for \( z = 0 \) and is present due to the deviation of localized states from coherent states. Hence, for anyons placed on the \( x \)-axis, the sign of \( \langle \hat{y}_1\hat{y}_2 \rangle \), or equivalently, whether the particles went into the same limb or opposite limbs, is entirely determined by the statistics and the bunching parameter, which in turn depends on initial conditions. Given the exponential dependence of \( \langle \hat{y}_1\hat{y}_2 \rangle \) on time, the saddle potential acts as a beam splitter whose read out amplifies initial uncertainties in initial positions. Average separations can be expected for a stream of particle pairs and associated uncertainties in initial positions. Average separations can be varied by changing the field, and thus the magnetic length, within a quantum Hall state. Alternatively, one can study correlations between dilute beams of quasiparticles conveyed along quantum Hall edge states and brought together via pinching [21, 22]; correlations described here would require the application of a saddle potential at the pinched region. While a complete analysis of such a system would require connecting edge physics to the bulk, we expect some signatures predicted here to be robust.

In conclusion, we have presented fundamental correlations characterizing LLL anyons and distinguishing them from their fermionic and bosonic counterparts. We have proposed the application of a saddle potential as a means of realizing a quantum Hall beam splitter that can display these correlations and associated direct signatures of fractional statistics.

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mass and guiding center co-ordinates are \( l_c = l / \sqrt{2} \) and \( l_c = \sqrt{2} l \), where \( l \) is the single particle magnetic length. [24] For the relative coordinates, these states are not equivalent to coherent states except for the fermionic and bosonic cases but they do exhibit the same asymptotic behavior.