NEW RATES FOR EXPONENTIAL APPROXIMATION AND THE THEOREMS OF RÉNYI AND YAGLOM

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Abstract

We introduce two abstract theorems that reduce a variety of complex exponential distributional approximation problems to the construction of couplings. These are applied to obtain rates of convergence with respect to the Wasserstein and Kolmogorov metrics for the theorem of Rényi on random sums and generalizations of it, hitting times for Markov chains, and to obtain a new rate for the classical theorem of Yaglom on the exponential asymptotic behavior of a critical Galton-Watson process conditioned on non-extinction. The primary tools are an adaptation of Stein’s method, Stein couplings, as well as the equilibrium distributional transformation from renewal theory.

1 INTRODUCTION

The exponential distribution arises as an asymptotic limit in a wide variety of settings involving rare events, extremes, waiting times, and quasi-stationary distributions. As discussed in the preface of Aldous (1989), the tremendous difficulty in obtaining explicit bounds on the error of the exponential approximation in more than the most elementary of settings apparently has left a gap in the literature. The classical theorem of Yaglom (1947) describing the asymptotic exponential behavior of a critical Galton-Watson process conditioned on non-extinction, for example, has a large literature of extensions and embellishments (see Lalley and Zheng (in press) for example) but the complex dependencies between offspring have apparently not previously allowed for obtaining explicit error bounds. Stein’s method, introduced in Stein (1972), is now a well established method for obtaining explicit bounds in distributional approximation problems in settings with dependencies (see Ross and Peköz (2007) for an introduction). Results for the normal and Poisson approximation, in particular, are extensive but also are currently very actively being further developed; see e.g. Chatterjee (2008) and Chen and Röllin (2009).
While many discrete distributions on the real line have been tackled using Stein’s method, such as the binomial, geometric, compound Poisson distributions in Ehm (1991), Peköz (1996), Barbour, Chen, and Loh (1992a) and many other articles, results for continuous distributions besides the normal are far less developed. Diaconis and Zabell (1991) and Schoutens (2001) give theoretical material on general classes of probability distribution on the real line and make connections with orthogonal polynomials. Luk (1994) and Pickett (2004) give some applications for $\Gamma$ and $\chi^2$ approximations, but the methodology is very specific for their applications. Nourdin and Peccati (2009) give general theorems for $\Gamma$ approximation of functionals of Gaussian fields.

There have been a few attempts to apply Stein’s method to exponential approximation. Weinberg (2005) sketches a few potential applications but only tackles simple examples thoroughly, and Bon (2006) only considers geometric convolutions. Chatterjee, Fulman, and Röllin (2006) breaks new ground by applying the method to a challenging problem in spectral graph theory using exchangeable pairs, but the calculations involved are application-specific and far from elementary. In this article, in contrast, we develop a general framework that more conveniently reduces a broad variety of complex exponential distributional approximation problems to the construction of couplings. We provide evidence that our approach can be fruitfully applied to non-trivial applications and in settings with dependence—settings where Stein’s method typically is expected to shine.

The article is organized as follows. In the Section 2 we present two abstract theorems formulated in terms of couplings. We introduce a distributional transformation (the ‘equilibrium distribution’ from renewal theory) which has not yet been extensively explored using Stein’s method. We also make use of Stein couplings similar to those introduced in Chen and Röllin (2009). This is followed by some more concrete coupling constructions that can be used along with the theorems. In Section 3 we give applications using these couplings to obtain exponential approximation rates for the theorem of Rényi on random sums and generalizations of it, hitting times for Markov chains, and to the classical theorem of Yaglom on the exponential asymptotic behavior of a critical Galton-Watson process conditioned on non-extinction; this is the first place this latter result has appeared in the literature. In Section 4 we then give the postponed proofs for the main theorems.

2 MAIN RESULTS

In this section we present the framework in abstract form that will subsequently be used in concrete applications in Section 3. This framework is comprised of two approaches that we will describe here and then prove in Section 4.

Let us first define the probability metrics used in this article. Define the
sets of test functions

\[ F_K = \{ \lfloor z \rfloor \mid z \in \mathbb{R} \}, \]
\[ F_W = \{ h : \mathbb{R} \to \mathbb{R} \mid h \text{ is Lipschitz, } \|h'\| \leq 1 \}, \]
\[ F_{BW} = \{ h : \mathbb{R} \to \mathbb{R} \mid h \text{ is Lipschitz, } \|h\| \leq 1 \text{ and } \|h'\| \leq 1 \}. \]

Then, for a general set of test functions \( F \), the distance between two probability measures \( P \) and \( Q \) with respect to \( F \) is defined as

\[ d_{F}(P, Q) := \sup_{f \in F} \left| \int_{\mathbb{R}} f \, dP - \int_{\mathbb{R}} f \, dQ \right| \]  \hspace{1cm} (2.1) \{1\}

if the corresponding integrals are well-defined. Denote by \( d_K \), \( d_W \) and \( d_{BW} \) the respective distances induced by the sets \( F_K \), \( F_W \) and \( F_{BW} \). The subscripts respectively denote the Kolmogorov, Wasserstein, and bounded Wasserstein distances. We have

\[ d_{BW} \leq d_{W}, \quad d_K(P, \text{Exp}(1)) \leq 1.74 \sqrt{d_W(P, \text{Exp}(1))}. \]  \hspace{1cm} (2.2) \{2\}

The first relation is clear as \( F_{BW} \subset F_W \). We refer to \cite{GibbsSu.2002} for the second relation.

Our first approach is related to the zero-bias coupling introduced in \cite{GoldsteinReinert.1997} in the context of normal approximation. It was also (independently of the current work) previously used by \cite{Bon.2006, Lemma 6}) to study geometric convolutions. Whereas zero-bias couplings are difficult to construct in general (see e.g. \cite{Goldstein.2005, Goldstein.2007} and \cite{Ghosh.2009}), it turns out that there is a convenient way to obtain this type of coupling in our case; see Section 2.1.1.

**Definition 2.1.** Let \( X \) be a non-negative random variable with finite mean. We say that a random variable \( X^e \) has the *equilibrium distribution w.r.t. \( X \)* if for all Lipschitz \( f \)

\[ \mathbb{E} f(X) - f(0) = \mathbb{E} X \mathbb{E} f'(X^e). \]  \hspace{1cm} (2.3) \{3\}

We use the term ‘equilibrium distribution’ due to its common use for this transformation in renewal theory (see \cite{RossPekoz.2007}): for a stationary renewal process with inter-event times having distribution \( X \), the time until the next renewal starting from an arbitrary time point has the equilibrium distribution \( X^e \). Though there is some connection, we choose not to refer to the distribution of \( X^e \) as ‘exponential zero-biased’ since there are other Stein operators in the context of exponential approximation (as used by \cite{Chatterjeeetal.2006}) that may lead to distributional characterizations more similar to the actual zero-bias transformation of \cite{GoldsteinReinert.1997} than to (2.3). The connection between this transformation and the ‘size-biased’ transformation is discussed in Section 2.1.1 below.

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For nonnegative \( X \) having finite first moment, define the distribution function
\[
F^e(x) = \frac{1}{\mathbb{E}X} \int_0^x \mathbb{P}[X > y]dy
\]
(2.4) \{4\}
on \( x \geq 0 \) and \( F^e(x) = 0 \) for \( x < 0 \). Then
\[
\mathbb{E} f(X) - f(0) = \mathbb{E} \int_0^X f'(s)ds = \mathbb{E} \int_0^\infty f'(s)\mathbb{P}[X > s]ds = \int_0^\infty f'(s)\mathbb{P}[X > s]ds,
\]
so that \( F^e \) is the distribution function of \( X^e \) and our definition via (2.3) is consistent with that from renewal theory. It is clear from (2.4) that \( F^e \) always exists if \( X \) has finite expectation and that \( F^e \) is absolutely continuous with respect to the Lebesgue measure. It is also clear from (2.3) that conditions on the moments of \( L(X^e) \) go along with corresponding conditions on the moments of \( L(X) \): finite \( m \)-th moment of \( X^e \), \( m > 0 \), requires finite \( (m + 1) \)-th moment of \( X \).

For a random variable \( W \) it can be seen that \( L(W) = L(W^e) \) if and only if \( W \) has an exponential distribution; when \( W \) is exponential it is clear that \( L(W) = L(W^e) \) and we show the converse below. Our first result here can be thought of as formalizing the notion that when \( L(W) \) and \( L(W^e) \) approximately equal then \( W \) has approximately an exponential distribution. We not only give bounds for \( W \), but also for \( W^e \) as this quantity itself may be of interest; see Section 3.2 for such an example.

**Theorem 2.1.** Let \( W \) be a non-negative random variable with \( \mathbb{E}W = 1 \) and let \( W^e \) have the equilibrium distribution w.r.t. \( W \). Then, for any \( \beta > 0 \),
\[
d_K(L(W), \text{Exp}(1)) \leq 12\beta + 2\mathbb{P}[\lvert W^e - W \rvert > \beta] \tag{2.5} \{5\}
\]
and
\[
d_K(L(W^e), \text{Exp}(1)) \leq \beta + \mathbb{P}[\lvert W^e - W \rvert > \beta]. \tag{2.6} \{6\}
\]
If in addition \( W \) has finite second moment, then
\[
d_W(L(W), \text{Exp}(1)) \leq 2\mathbb{E}[W^e - W] \tag{2.7} \{7\}
\]
and
\[
d_K(L(W^e), \text{Exp}(1)) \leq \mathbb{E}|W^e - W|; \tag{2.8} \{8\}
\]
bound (2.8) also holds for \( d_W(L(W^e), \text{Exp}(1)) \).

The key to our second approach is a coupling of three random variables \((W, W', G)\); this approach was recently introduced by Chen and Rölpin (2009). Here, \( W \) is the random variable of interest, \( W' \) is a ‘small perturbation’ of \( W \), and \( G \) is an auxiliary random variable which in some sense brings the coupling into the domain of the Stein operator that is used for the approximation.

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Definition 2.2. A coupling \((W, W', G)\) is called a \textit{constant Stein coupling} if
\[
\mathbb{E}\{Gf(W') - Gf(W)\} = \mathbb{E} f(W) \tag{2.9} \{9\}
\]
for all \(f\) with \(f(0) = 0\) and for which the expectations exist.

This contrasts with the \textit{linear Stein coupling} introduced in Chen and Röllin (2009), where the right hand side of (2.9) is replaced by \(\mathbb{E}\{Wf(W)\}\) and is used for normal approximation. We emphasize that the equality in (2.9) need not be exactly satisfied for our next theorem to be applied. But in order to obtain useful results, (2.9) should be approximately true. To this end we define
\[
r_1(\mathcal{F}) = \sup_{f \in \mathcal{F}, f(0)=0} |\mathbb{E}\{Gf(W') - Gf(W) - f(W)\}|.
\]
This term measures the extent to which (2.9) holds with respect to the class of functions from \(\mathcal{F}\) with \(f(0) = 0\). Another primary error term that will appear in the error bound is
\[
r_2 = \mathbb{E}|\mathbb{E}^{W''}(GD) - 1|,
\]
where here and in the rest of the article \(D := W' - W\). The random variable \(W''\) is defined on the same probability space as \((W, W', G)\) and can be used to simplify the bounds (it is typically chosen so that \(r_2 = 0\)); let \(D' := W'' - W\). At first reading one may simply set \(W'' = W\) (in which case typically \(r_2 \neq 0\)); we refer to Chen and Röllin (2009) for a more detailed discussion of Stein couplings.

Theorem 2.2. Let \(W, W', W''\) and \(G\) be random variables with finite first moments such that also \(\mathbb{E}|GD| < \infty\) and \(\mathbb{E}|GD'| < \infty\). Then with the above definitions,
\[
d(W, \mathcal{L}(W), \text{Exp}(1)) \leq r_1(\mathcal{F}_W) + r_2 + 2r_3 + 2r_3' + 2r_4 + 2r_4', \tag{2.10} \{10\}
\]
where
\[
\begin{align*}
r_3 &= \mathbb{E}|GD[I|D| > 1]|, & r_3' &= \mathbb{E}|(GD - 1)I[D'| > 1]|, \\
r_4 &= \mathbb{E}|G(D^2 \wedge 1)|, & r_4' &= \mathbb{E}|(GD - 1)(D'| \wedge 1)|.
\end{align*}
\]
The same bound holds for \(d_{BW}\) with \(r_1(\mathcal{F}_W)\) replaced by \(r_1(\mathcal{F}_{BW})\). Furthermore, for any \(\alpha, \beta\) and \(\beta'\),
\[
d(K, \mathcal{L}(W), \text{Exp}(1)) \leq 2r_1(\mathcal{F}_{BW}) + 2r_2 + 2r_5 + 2r_5' + 22(\alpha\beta + 1)\beta' + 12\alpha\beta^2, \tag{2.11} \{11\}
\]
where
\[
\begin{align*}
r_5 &= \mathbb{E}|GD[I|G| > \alpha \text{ or } |D| > \beta]|, \\
r_5' &= \mathbb{E}|(1 - GD)I[G| > \alpha \text{ or } |D| > \beta \text{ or } |D'| > \beta']|.
\end{align*}
\]
2.1 Couplings

In this section we present a way to construct the \( W^e \) distribution more explicitly and also a few constant Stein couplings. We note that the main goal of the coupling \((W, W', G)\) is to achieve \((2.9)\), so that Theorem 2.2 can be applied with \( r_1 \) being small. We will not discuss how to obtain \( W'' \) as this depends very much on the concrete application; see Section 3.1 for an example.

2.1.1. Equilibrium distribution via size biasing. Assume that \( EW = 1 \) and let \( W^s \) have the size bias distribution of \( W \), i.e. \( f(W) = f(W^s) \) for all \( f \) for which the expectation exist. Then, if \( U \) has the uniform distribution on \([0,1]\) independent of all else, \( W := U W^s \) has the equilibrium distribution w.r.t. \( W \). Indeed, for any Lipschitz \( f \) with \( f(0) = 0 \) we have

\[
\mathbb{E} f(W) = \mathbb{E} f(W) - f(0) = \mathbb{E} f'(U W) = \mathbb{E} f'(U W^s) = \mathbb{E} f'(W^e).
\]

We note that this construction was also considered by Goldstein (2009). It has been observed by Pakes and Khattree (1992) that for a non-negative random variable \( W \) with \( EW < \infty \), we have that \( \mathcal{L}(W) = \mathcal{L}(U W^s) \) if and only if \( W \) has exponential distribution.

It is clear here that we do not intend to couple \((W, W^s)\) as closely as possible, but \((W, U W^s)\), so that, unfortunately, we cannot utilize the large literature available for size-bias coupling, such as Barbour, Holst, and Janson (1992b), Goldstein and Penrose (2008) and others.

2.1.2. Exchangeable pairs. Let \((W, W')\) be an exchangeable pair. Assume that

\[
\mathbb{E}^W (W' - W) = -\lambda + \lambda R \quad \text{on} \quad \{W > 0\}.
\]

Then, if we set \( G = (W' - W)/(2\lambda) \), we have \( r_1(\mathcal{F}_{BW}) \leq \mathbb{E}|R| \) and \( r_1(\mathcal{F}_W) \leq \mathbb{E}|RW| \).

This coupling was used by Chatterjee et al. (2006) to obtain an exponential approximation for the spectrum of the Bernoulli-Laplace Markov chain. In order to obtain optimal rates, Chatterjee et al. (2006) develop more application specific theorems than ours.

2.1.3. Conditional distribution of \( W \) given \( E^c \). Let \( E \) be an event and let \( p = \mathbb{P}[E] \), where \( p \) is thought to be small. Assume that \( W' \) and \( Y \) are defined on the same probability space and that \( \mathcal{L}(W') = \mathcal{L}(W|E^c) \) and \( \mathcal{L}(Y) = \mathcal{L}(W|E) \). Then, for any Lipschitz \( f \) with \( f(0) = 0 \), and with
\( G = (1 - p)/p, \)
\[
\mathbb{E} \{ Gf(W') - Gf(W) \} = \frac{1 - p}{p} \mathbb{E} f(W') - \frac{1 - p}{p} \mathbb{E} f(W) + \mathbb{E} f(W)
\]
\[
= \frac{1 - p}{p} \mathbb{E} f(W') - \frac{1 - p}{p} \mathbb{E} (f(W)|E^c) - \mathbb{E} (f(W)|E) + \mathbb{E} f(W)
\]
\[
= \frac{1 - p}{p} \mathbb{E} f(W') - \frac{1 - p}{p} \mathbb{E} f(Y) + \mathbb{E} f(W)
\]
\[
= \mathbb{E} f(W) - \mathbb{E} \{ Yf'(UY) \},
\]
so that \( r_1(\mathcal{F}_W) \leq \mathbb{E} Y. \) This coupling is mainly used by Peköz (1996) for geometric approximation in total variation. The Stein operator used there is a discrete version of the Stein operator used in this article. Clearly, one will typically aim for an event \( E \supset \{ W = 0 \} \) in order to have \( Y = 0. \)

2.1.4. Conditional distribution of \( W' \) given \( E^c. \) The roles of \( W \) and \( W' \) from the previous coupling can be reversed. Let \( E \) and \( p \) be as before. However, assume now that \( \mathcal{L}(W) = \mathcal{L}(W'|E^c) \) and \( \mathcal{L}(Y) = \mathcal{L}(W'|E). \) Then, for any Lipschitz \( f \) with \( f(0) = 0, \) and with \( G = -1/p \)
\[
\mathbb{E} \{ Gf(W') - Gf(W) \} = \frac{1 - p}{p} \mathbb{E} f(W) - \frac{1}{p} \mathbb{E} f(W)
\]
\[
= \frac{1 - p}{p} \mathbb{E} f(W) - \frac{1}{p} \mathbb{E} f(W) + \frac{1 - p}{p} \mathbb{E} f(W)
\]
\[
= \mathbb{E} f(W) - \mathbb{E} \{ Yf'(UY) \},
\]
so that \( r_1(\mathcal{F}_W) \leq \mathbb{E} Y. \)

3 APPLICATIONS

3.1 Random sums

It is well-known that a geometric distribution divided by its mean, when the mean is large, is close to the standard exponential distribution. If \( N \sim \text{Ge}(p) \) (starting at 1) then we can write \( N = \sum_{i=1}^{N} 1 \) and one may ask whether the 1’s in this sum can be replaced by random variables \( X_1, X_2, \ldots \) having mean of order 1, so that we still obtain an exponential limit. A classical result by Rényi (1957) states that this is true if the sequence consists of i.i.d. random variables under the surprisingly mild condition that \( X_1 \) has finite first moment (c.f. Sugakova (1990)). A more general result by Brown (1990, pp. 1400/1) states that \( \sum_{i=1}^{N} X_i/N \to 1 \) almost surely is a sufficient condition for an exponential limit—even when the variables are not i.i.d. or not independent of \( N. \)
Quite a few (mostly Soviet) authors have considered the uniform rate of convergence in the 80s and early 90s; Kalashnikov (1997) is the standard reference here. Although convergence results for the non i.i.d. case have been considered in previous articles, to the best of our knowledge the only article that goes beyond the above i.i.d. case for uniform bounds is Sugakova (1995) who considers independent, non-identically distributed random variables with equal mean, again under the assumption that $N$ has a geometric distribution. Our aim is to relax the assumptions of independence of the $X_i$, equal mean, and the geometric distribution of $N$. Let us first handle the case of some martingale-like dependence structure using the equilibrium distribution approach. For a random variable $X$ denote by $F_X$ its distribution function and by $F_X^{-1}$ its generalized inverse. We adopt the standard convention that $\sum_{a}^{b} = 0$ if $b < a$.

**Theorem 3.1.** Let $X = (X_1, X_2, \ldots)$ be a sequence of square integrable, non-negative random variables, independent of all else, such that, for all $i \geq 1$,

$$E(X_i|X_1, \ldots, X_{i-1}) = \mu_i < \infty \quad \text{almost surely.} \quad (3.1)$$

Let $N$ be a positive and integer valued random variable with $E N < \infty$ and let $M$ be another random variable with distribution defined by

$$P(M = m) = \mu_m P(N \geq m)/\mu, \quad m = 1, 2, \ldots \quad (3.2)$$

with

$$\mu = \sum_{i=1}^{N} X_i = \sum_{m \geq 1} \mu_m P(N \geq m).$$

Then, with $W = \mu^{-1} \sum_{i=1}^{N} X_i$, we have

$$d_W(\mathcal{L}(W), \text{Exp}(1)) \leq 2\mu^{-1} (E|X_M - X_M^e| + \sup_{i \geq 1} \mu |N - M|), \quad (3.3)$$

where each $X_i^e$ is a random variables having the equilibrium distribution w.r.t. $X_i$ given $X_1, \ldots, X_{i-1}$. If, in addition, $X_i \leq C$ for all $i$ and $|N - M| \leq K$, then

$$d_K(\mathcal{L}(W), \text{Exp}(1)) \leq 12\mu^{-1} \left\{ \sup_{i \geq 1} \|F_{X_i^e}^{-1} - F_{X_i}^{-1}\| + CK \right\}; \quad (3.4)$$

if $K = 0$, the same bound also holds for unbounded $X_i$.

**Proof.** First, let us prove that $W^e := \mu^{-1} (\sum_{i=1}^{M-1} X_i + X_M^e)$ has the equilibrium distribution w.r.t. $W$. Note first from (2.3), (3.1) and the assumptions on $X_i^e$ that, for Lipschitz $f$ and every $m$,

$$E f'\left(\mu^{-1} \sum_{i=1}^{m-1} X_i + \mu^{-1} X_m^e\right) = \frac{\mu}{\mu_m} E \left[ f\left(\mu^{-1} \sum_{i=1}^{m} X_i\right) - f\left(\mu^{-1} \sum_{i=1}^{m-1} X_i\right) \right].$$

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Note also that, for any Lipschitz function $g$ with $g(0) = 0$ we have

$$\mu \mathbb{E} \left[ \frac{g(M)}{\mu_M} - \frac{g(M - 1)}{\mu_M} \right] = \sum_{m \geq 0} P(N \geq m) (g(m) - g(m - 1)) = \mathbb{E} g(N).$$

We may now assume that $f(0) = 0$. Hence, using the above two facts and independence between $M$ and the sequence $X_1, X_2, \ldots$, we have

$$\mathbb{E} f'(W^e) = \mathbb{E} f' \left( \mu^{-1} \sum_{i=1}^{M-1} X_i + \mu^{-1} X^e_M \right)$$

$$= \mu \mathbb{E} \left[ f \left( \mu^{-1} \sum_{i=1}^{M} X_i \right) / \mu_M - f \left( \mu^{-1} \sum_{i=1}^{M-1} X_i \right) / \mu_M \right]$$

$$= \mathbb{E} f \left( \mu^{-1} \sum_{i=1}^{N} X_i \right) = \mathbb{E} f(W).$$

Now,

$$W^e - W = \mu^{-1} \left\{ (X^e_M - X_M) + \text{sgn}(M - N) \sum_{i=(M \land N)+1}^{N \lor M} X_i \right\}. \tag{3.5} \{16\}$$

Due to Strassen’s Theorem, we can always couple two random variables $X$ and $Y$ in such a way that $|X - Y| \leq \| F_X^{-1} - F_Y^{-1} \|$. Hence, choosing

$$\beta = \mu^{-1} \left\{ \sup_{i \geq 1} \| F_{X_i}^{-1} - F_{Y_i}^{-1} \| + CK \right\},$$

we have $\mathbb{P} [|W^e - W| > \beta] = 0$, and thus, from (2.5), (3.4) follows; the remark after (3.4) follows similarly. Using (3.5) we can also easily deduce (3.3) from (2.7). \qed

Note that the idea of replacing a single summand by a distributional transform is omnipresent in the literature related to Stein’s method; see e.g. Goldstein and Reinert (1997) in connection with the zero-bias distribution for normal approximation.

**Remark 3.1.** Let $N \sim \text{Ge}(p)$ (starting at 1) and assume that the $\mu_i$ are bounded from above and bounded away from 0. This implies in particular that $1/\mu \asymp p$ as $p \to 0$. Now, from the Kantorovic-Rubinstein Theorem (see Kantorovič and Rubinštejn (1958) and Vallender (1973)) we have that

$$d_W(\mathcal{L}(N), \mathcal{L}(M)) = \inf_{(N,M)} \mathbb{E}|N - M|, \tag{3.6} \{17\}$$

where the infimum ranges over all possible couplings of $N$ and $M$. As (3.3) holds for any coupling $(N, M)$ we can replace $\mathbb{E}|N - M|$ in (3.3) by the left
hand side of (3.6). To bound this quantity note first that from (3.2) we deduce

$$\mathbb{E} h(M) = \mathbb{E} \left\{ \frac{\mu N}{\mu p} h(N) \right\}$$  \hspace{1cm} (3.7)  \hspace{1cm} \{18\}

for every function $h$ for which the expectations exist. Note also that

$$\mathbb{E}(\mu N) = \mathbb{E} \sum_{k=1}^{\infty} \mu_k p [N = k] = \mathbb{E} \sum_{k=1}^{\infty} \mu_k p [N \geq k] = \mu p.  \hspace{1cm} (3.8)  \hspace{1cm} \{19\}$$

Let $h$ now be Lipschitz with Lipschitz constant 1 and assume without loss of generality that $h(0) = 0$, so that $|h(N)| \leq N$. Then

$$\left| \mathbb{E} \left\{ h(M) - h(N) \right\} \right| = \left| \mathbb{E} \left\{ \left( \frac{\mu N}{\mu p} - 1 \right) h(N) \right\} \right| \leq \mathbb{E} \left| \left( \frac{\mu N}{\mu p} - 1 \right) N \right| \leq \sqrt{\text{Var}(\mu N)} \frac{\sqrt{N}}{\mu p} \leq \sqrt{2 \text{Var}(\mu N)} \frac{\sqrt{N}}{\mu p^2}.$$

Hence, under the assumptions of this remark, for the second term in (3.3) to converge to zero it is sufficient that $\text{Var}(\mu N) \to 0$ as $p \to 0$. The following remark gives a sufficient condition for this.

**Remark 3.2.** We say that a sequence $a_0, a_1, a_2, \ldots$ is Abel-summable with limit $A$ if for all $|x| < 1$ the sum $\sum_{k \geq 0} a_k x^k$ exists and if

$$\sum_{k \geq 0} a_k x^k \to A \quad \text{as } x \to 1.$$

Now, if $N \sim \text{Ge}(p)$ and $s_k = a_0 + a_1 + \cdots + a_k$ the partial sums, we have

$$\mathbb{E}(s_{N-1}) = p \sum_{k \geq 0} s_k (1-p)^k = \sum_{k \geq 0} a_k (1-p)^k.$$

Hence, we have $\text{Var}(\mu N) = \mathbb{E}(\mu^2 N) - (\mathbb{E} \mu N)^2 \to 0$ as $p \to 0$ if and only if the sequences $\mu_1^2, \mu_2^2 - \mu_1^2, \mu_3^2 - \mu_2^2, \ldots$ and $\mu_1, \mu_2 - \mu_1, \mu_3 - \mu_2, \ldots$ are Abel-summable with limits $A^2$ and $A$ respectively. This is certainly the case if $\mu_k \to A$ as $k \to \infty$, because Abel-summability is consistent with regular summability; see Korevaar (2004, p. 4).

Next is a corollary, and first we need a definition.

**Definition 3.3.** A nonnegative random variable $X$ with finite mean is said to be NBUE (new better than used in expectation) or NWUE (new worse than used in expectation) if we respectively have either $\mathbb{E}[X - s|X > s] \leq \mathbb{E}[X]$ for all $s > 0$ or $\mathbb{E}[X - s|X > s] \geq \mathbb{E}[X]$ for all $s > 0$. If $X$ is nonnegative integer-valued then we say it is discrete NBUE (discrete NWUE) if we $\mathbb{E}[X - s|X > s] \leq (\geq)\mathbb{E}[X]$ for all $s = 1, 2, \ldots$.\hspace{1cm}  \{rem2\} \hspace{1cm} {20}
It can be shown (see Shaked and Shanthikumar, 2007, Theorem 1.4.A31) that if \( X \) is NBUE then \( X^e \leq_{st} X \) and if \( X \) is NWUE then \( X^e \geq_{st} X \). If \( N \) is integer valued and discrete NBUE (discrete NWUE) then it is also true (see Sengupta, Chatterjee, and Chakraborty, 1995, p. 1477) that for the corresponding \( M \) in (3.2) with \( \mu_1 = \mu_2 = \cdots \) we have \( M \leq_{st} N \). In this case we can couple \( X \) and \( X^e \) so either \( X^e \leq X \) or \( X^e \geq X \) holds (and \( N \) and \( M \) so either \( N \leq M \) or \( N \geq M \) holds) and we then have

\[
\mathbb{E}|X^e - X| = \mathbb{E}X^e - \mathbb{E}X = \frac{\mathbb{E}X^2}{2 \mathbb{E}X} - \mathbb{E}X
\]  

(3.9) \{21\}

and also

\[
\mathbb{E}|M - N| = \mathbb{E}M - \mathbb{E}N = \left| \frac{\mathbb{E}N^2}{2 \mathbb{E}N} + \frac{1}{2} - \mathbb{E}N \right|
\]  

(3.10) \{22\}

and thus we immediately obtain the following corollary.

**Corollary 3.2.** Consider the situation from Theorem 3.1 and assume in addition that the \( X_i \) are independent with \( \mathbb{E}X_1 = 1 \) and, for each \( i \), we have that \( X_i \) is either NBUE or NWUE. If \( N \) is integer-valued and either discrete NBUE or discrete NWUE then

\[
d_W(\mathcal{L}(W), \text{Exp}(1)) \leq 2\mu^{-1} \sup_{i \geq 1} \left| \frac{1}{2} \mathbb{E}X_i^2 - 1 \right| + 2 \left| \frac{\mathbb{E}N^2}{2 \mathbb{E}N} + \frac{1}{2} - \mathbb{E}N \right|. 
\]  

(3.11) \{23\}

**Example 3.4** (Geometric convolution of i.i.d. random variables). Assume that \( N \sim \text{Ge}(p) \) and that \( \mathbb{E}X_1 = 1 \). Then, it is straightforward that \( \mathcal{L}(M) = \mathcal{L}(N) \), hence we can set \( M = N \). Denote by \( \delta(F) \) the distance between \( \mathcal{L}(X_1) \) and \( \text{Exp}(1) \) as defined in (2.1) with respect to the set of test functions \( \mathcal{F} \); define \( \delta^e(F) \) analogously but between \( \mathcal{L}(X) \) and \( \mathcal{L}(X^e) \).

In this case the estimates of Theorem 3.1 reduce to

\[
d_W(\mathcal{L}(W), \text{Exp}(1)) \leq 2\mu^{-1} \sup_{i \geq 1} \left| \frac{1}{2} \mathbb{E}X_i^2 - 1 \right| + 2 \left| \frac{\mathbb{E}N^2}{2 \mathbb{E}N} + \frac{1}{2} - \mathbb{E}N \right|. 
\]  

(3.12) \{25\}

\[
d_K(\mathcal{L}(W), \text{Exp}(1)) \leq 12p\|F_{X_1}^{-1} - F_{X_i}^{-1}\|.
\]  

(3.13) \{26\}

Inequality (3.12) follows again from the Kantorovich-Rubinstein Theorem. Note that these two bounds are small not only if \( p \) is small but also if the \( X_i \) are close to exponential.

Let us compare this with analogous results from the literature. From Kalashnikov, 1997, Theorem 3.1 for \( s = 2 \), page 151) we have the (slightly simplified) bound

\[
d_W(\mathcal{L}(W), \text{Exp}(1)) \leq p\delta(\mathcal{F}_W) + 2p\delta(\mathcal{F}_2),
\]  

(3.14) \{26\}

where

\[\mathcal{F}_2 = \{f \in C^1(\mathbb{R}) \mid f' \in \mathcal{F}_W\}.\]
Known results from the literature typically involve a direct measure of closeness between $X_1$ and $\text{Exp}(1)$, whereas our results (3.12)-(3.13) incorporate a measure of closeness between $L(X_1)$ to its equilibrium distribution. By means of (2.7) and again the Kantorovich-Rubinstein Theorem we have the relation

$$\delta(F_W) \leq 2\delta^e(F_W).$$

Let $Z \sim \text{Exp}(1)$ and let $h$ be a differentiable function with $h(0) = 0$. Then, recalling that $\mathcal{L}(Z^e) = \mathcal{L}(Z)$, we have from (2.3) that $\mathbb{E}h(Z) = \mathbb{E}h'(Z)$ and, using again (2.3) for $X$ and $X^e$,

$$\mathbb{E}h(X) - \mathbb{E}h(Z) = \mathbb{E}h'(X^e) - \mathbb{E}h'(Z).$$

This implies

$$\delta(F_2) = d_W\left(\mathcal{L}(X^e_1), \text{Exp}(1)\right)$$

and hence, from (2.8), we have $\delta(F_2) \leq \delta^e(F_W)$, so that (3.14) gives a bound which is not as good as our (3.12) if the bound is to be expressed in terms of $\delta^e(F_W)$.

On the other hand, from (3.16) and the triangle inequality,

$$\delta^e(F_W) \leq \delta(F_W) + d_W\left(\mathcal{L}(X^e_1), \text{Exp}(1)\right) = \delta(F_W) + \delta(F_2).$$

Hence, our bound is not as good as (3.14) if the bound is to be expressed in terms of closeness of $\mathcal{L}(X_1)$ to $\text{Exp}(1)$. It seems therefore that, although much broader in applicability, our theorems are able to yield comparable results for the case of geometric convolutions, at least with respect to the Wasserstein metric.

In the case where $\mathcal{L}(X_1)$ has a density, we have from Kalashnikov (1997, Theorem 4.1 for $s = 2$, page 152) the (again slightly simplified) bound

$$d_K\left(\mathcal{L}(W), \text{Exp}(1)\right) \leq p\delta(F_K) + (1 + \bar{\rho})p\delta(F_W) + p(1 \wedge 2\bar{\rho})\delta(F_2),$$

where $\bar{\rho}$ is the supremum of the density of $\mathcal{L}(W)$. Then, according to Kalashnikov and Vsekhsvyat-skii (1989, p. 99), $\bar{\rho}$ can be bounded by

$$\bar{\rho} \leq 1 + \frac{\delta^e(F_W) + 2\sup_x |F'_{X_1}(x) - (1 - F_{X_1}(x))|}{(2 - \delta^e(F_W))},$$

where $F_{X_1}$ is the distribution function of $\mathcal{L}(X_1)$ with derivative $F'_{X_1}$. Kalashnikov and Vsekhsvyat-skii (1989) also obtain a more complicated bound for general, non-continuous $X_1$; we refer to their paper for that result. It seems difficult to directly compare this bound with our result (3.13) as there is no obvious correspondence between the right hand side quantity in (3.13) and the quantities appearing in (3.17).
If for each $i$ the variable $X_i$ is either NBUE or NWUE, (3.12) in combination with (2.2) gives us

$$d_K(\mathcal{L}(W), \text{Exp}(1)) \leq 2.47 \left( p \sup_{i \geq 1} \left| \frac{X_i^2}{2} - 1 \right| \right)^{1/2}. \quad (3.18)$$

Although (3.18) is non-optimal in rate, it is purely formulated in terms of moments of the involved random variables. Results similar to (3.18) have appeared elsewhere: Theorem 6.1 in Brown and Ge (1984) has the same result with a larger constant, but Brown (1990) and Daley (1988) subsequently derived significant improvements of this result.

Let us look now at a more flexible, but less precise approach. In particular, we do not assume a martingale-like dependence like (3.1). This comes at the cost of a non-vanishing bound in the case where the summands are i.i.d. exponential and the number of summands is geometrically distributed. Note that, in the context of this approach, it is more convenient to think of $N$ as starting from 0. If it starts at 1, then the modifications are minor, in particular, one needs to make use of the random variable $Y$ from Section 2.1.3.

**Theorem 3.3.** Let $X = (X_1, X_2, \ldots)$ be a sequence of random variables with $\mathbb{E}X_i = \mu_i$ and $\mathbb{E}X_i^2 < \infty$. Let $N$, $N'$ and $N''$ be non-negative, square integrable, integer valued random variables independent of the sequence $X$. Assume that

$$p := \mathbb{P}[N = 0] > 0, \quad \mathcal{L}(N') = \mathcal{L}(N|N > 0), \quad N'' \leq N \leq N'.$$

Define $S(k, l) := X_{k+1} + \cdots + X_l$ for $k < l$ and $X(k, l) = 0$ for $k \geq l$. Let $\mu = \mathbb{E}S(0, N)$ and $W = S(0, N)/\mu$. Then

$$d_W(\mathcal{L}(W), \text{Exp}(1)) \leq \frac{qs}{p\mu} + \frac{4q\mathbb{E}\{S(N, N')(1 + S(N'', N))\}}{p\mu^2} + \frac{4\mathbb{E}S(N'', N)}{\mu},$$

where $s^2 = \text{Var} \mathbb{E}(S(N, N')|\mathcal{F}_{N''})$ and $\mathcal{F}_k := \sigma(X_1, \ldots, X_k)$. If, in addition,

$$X_i \leq C, \quad N' - N \leq K_1, \quad N - N'' \leq K_2, \quad (3.19)$$

for positive constants $C$, $K_1$ and $K_2$, then

$$d_K(\mathcal{L}(W), \text{Exp}(1)) \leq \frac{qs}{p\mu} + \frac{2CK_2}{\mu} + \frac{2C^2K_1(11K_2 + 6K_1)}{p\mu^2}. \quad (3.20)$$

**Proof.** We make use of the coupling construction from Section 2.1.3. Let $E = \{N = 0\}$, let $Y = 0$, let $W' = \mu^{-1} \sum_{i=1}^{N'} X_i$ and likewise $W'' = \mu^{-1} \sum_{i=1}^{N''} X_i$. Then the conditions of Section 2.1.3 are satisfied with $G =$
(1 − p)/p and we can apply Theorem 2.2 in particular (2.11). We have $r_1(F_{BW}) = 0$ as proved in Section 2.1.3. Note now that $D = S(N, N')$ and $D' = S(N'', N)$. Hence $r_2 = E[1 - q(pµ)^{-1}E(S(N, N')|F_{N''})]$. As (2.9) implies that $E(GD) = EW = 1$ the variance bound of $r_2$ follows. The $d_W$-bound follows from (2.10), using the rough estimates $r_3 + r_4 \leq 2E(D') + 2E(GDD')$ as we assume bounded second moments.

To obtain the $d_K$-bound choose $\alpha = G = (1 - p)/p$, $\beta = CK_1/µ$ and $\beta' = CK_2/µ$; then $r_5 = r_5' = 0$. Hence (2.11) yields

$$d_K(\mathcal{L}(W), \text{Exp}(1)) \leq r_2 + 22(\alpha \beta + 1)\beta' + 12\alpha \beta^2.$$ 

Plugging in the value for $r_2$ and the constants, the theorem is proved.

Example 3.5 (Geometric convolution under local dependence). If $N + 1 \sim \text{Ge}(p)$ (that is, $N$ is a geometric distribution starting at 0) we can choose $N' = N + 1$, as $\mathcal{L}(N|N > 0) = \mathcal{L}(N + 1)$ due to the well-known lack-of-memory property; hence $K_1 = 1$. Assume now there is a non-negative integer $m$ such that, for each $i$, $(X_1, \ldots, X_i)$ is independent of $(X_{i+m+1}, X_{i+m+2}, \ldots)$. We can set $N'' = \max(N - m, 0)$, hence $s^2 = \text{Var}(\mu_{N+1})$, where $\mu_i := EX_i$. Assume also that $\mu_i \geq \mu_0$ for some $\mu_0 > 0$, so that $\mu \geq \mu_0/p$. Hence Theorem 3.3 yields

$$d_K(\mathcal{L}(W), \text{Exp}(1)) \leq r_2 + 22(\alpha \beta + 1)\beta' + 12\alpha \beta^2.$$ 

Hence, convergence is obtained only if $\text{Var}(\mu_{N+1}) \to 0$ as $p \to 0$; c.f. Remarks 3.1 and 3.2.

3.2 First passage times

It is well-known that the time until the occurrence of a rare event can often be well approximated by an exponential distribution; Aldous (1989) gives a wide survey of the settings where this phenomenon occurs, and Aldous and Fill (Preprint) summarizes many results in the setting of Markov chain hitting times. The articles by Aldous and Brown (1992) and Aldous and Brown (1993) are other good entry points to the large literature on approximately exponential passage times in Markov chains.

Let $X_0, X_1, \ldots$ be an ergodic and stationary Markov chain taking values in a denumerable space $\mathcal{X}$ with transition probability matrix $P = (P_{i,j})_{i,j \in \mathcal{X}}$ and stationary distribution $\pi = (\pi_i)_{i \in \mathcal{X}}$ and let

$$T_{\pi,i} = \inf\{t \geq 0 : X_t = i\}$$

be the time of the first visit to state $i$ when started at time 0 according to the stationary distribution $\pi$ and let

$$T_{i,j} = \inf\{t > 0 : X_t = j\}$$

starting with $X_0 = i$. 

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be the first time the Markov chain started in state $i$ next visits state $j$. The Markov chains in the two definitions may be separate copies coupled together on the same probability space.

**Corollary 3.4.** With the above definitions we have

$$ d_K(\mathcal{L}(\pi_i T_{\pi,i}), \text{Exp}(1)) \leq 1.5 \pi_i + \pi_i \mathbb{E}[T_{\pi,i} - T_{i,i}] \tag{3.21} \quad \{32\} $$

and

$$ d_K(\mathcal{L}(\pi_i T_{\pi,i}), \text{Exp}(1)) \leq 2 \pi_i + \mathbb{P}(T_{\pi,i} \neq T_{i,i}) \tag{3.22} \quad \{33\} $$

**Proof.** We first claim that, with $U$ a uniform $[0, 1]$ random variable independent of all else, $\mathcal{L}(T_{i,i}) = \mathcal{L}(T_{i,i} + U)$. To see this, first note that

$$ \mathbb{P}(T_{\pi,i} = k) = \pi_i \mathbb{P}(T_{i,i} > k) $$

follows because visits to state $i$ constitute a renewal process and a cycle (having mean length $\mathbb{E}T_{i,i} = 1/\pi_i$) will have precisely one time $t$ when the excess

$$ Y_t = \inf\{s \geq t : X_s = i\} - t $$

is exactly equal to $k$ if and only if the cycle length is greater than $k$. We then apply the renewal reward theorem; see Ross and Peköz (2007).

So with $f(0) = 0$ and using (2.3), we have

$$ \mathbb{E} f'(T_{\pi,i} + U) = \mathbb{E} f(T_{\pi,i} + 1) - f(T_{\pi,i}) $$

$$ = \pi_i \sum_k \mathbb{P}(T_{i,i} > k)(f(k + 1) - f(k)) $$

$$ = \pi_i \sum_k \sum_{j > k} \mathbb{P}(T_{i,i} = j)(f(k + 1) - f(k)) $$

$$ = \pi_i \sum_{j} \sum_{0 \leq k < j} \mathbb{P}(T_{i,i} = j)(f(k + 1) - f(k)) $$

$$ = \pi_i \sum_{j} \mathbb{P}(T_{i,i} = j)f(j) $$

$$ = \pi_i \mathbb{E} f(T_{i,i}) $$

and the claim follows from (2.3). We then have

$$ d_K(\mathcal{L}(\pi_i T_{\pi,i}), \text{Exp}(1)) \leq \pi_i + d_K(\mathcal{L}(\pi_i T_{\pi,i} + U), \text{Exp}(1)) $$

$$ = \pi_i + d_K(\mathcal{L}(\pi_i T_{i,i}^c), \text{Exp}(1)) \tag{3.23} \quad \{34\} $$

where we use $d_K(\mathcal{L}(T_{\pi,i}), \mathcal{L}(T_{\pi,i} + U)) \leq \pi_i$ in the first line and $\mathcal{L}(T_{i,i}^c) = \mathcal{L}(T_{\pi,i} + U)$ in the second line. We obtain inequality (3.21) from (3.23) and (2.8) and then using

$$ \mathbb{E}|T_{\pi,i} + U - T_{i,i}| \leq \mathbb{E}U + \mathbb{E}|T_{\pi,i} - T_{i,i}| \leq 0.5 + \mathbb{E}|T_{\pi,i} - T_{i,i}| $$

and we obtain (3.22) from (3.23) and (2.6) using $\beta = \pi_i$ (since $\{|T_{\pi,i} + U - T_{i,i}| > 1\}$ implies $\{T_{\pi,i} \neq T_{i,i}\}$). \(\square\)
We say that a stopping time \( T_{i,\pi} \) is a stationary time starting from state \( i \) if \( \mathcal{L}(X_{T_{i,\pi}} \mid X_0 = i) = \pi \). Also, whenever \( T_{i,i} \) and \( T_{i,\pi} \) are used together in an expression, it is understood that they are both based on the same copy of the Markov chain started from state \( i \).

**Corollary 3.5.** With the above definitions and \( \rho = \mathbb{P}[T_{i,i} < T_{i,\pi}] \),

\[
d_K(\mathcal{L}(\pi_i T_{\pi,i}), \text{Exp}(1)) \leq \pi_i (1.5 + \mathbb{E}T_{i,\pi} + \rho \sup_j \mathbb{E}T_{j,i}) \tag{3.24} \quad \{35\}
\]

and

\[
d_K(\mathcal{L}(\pi_i T_{\pi,i}), \text{Exp}(1)) \leq 2\pi_i + \sum_{n=1}^{\infty} |P_{i,i}^{(n)} - \pi_i| \tag{3.25} \quad \{36\}
\]

**Proof.** Letting \( X_0 = i \),

\[
T_{\pi,i} = \inf\{t \geq 0 : X_{T_{i,\pi} + t} = i\}
\]

and

\[
T_{i,i} = \inf\{t > 0 : X_t = i\}
\]

and \( A = \{T_{i,i} < T_{i,\pi}\} \) we have

\[
|T_{\pi,i} - T_{i,i}| \leq (T_{\pi,i} + T_{i,\pi}) I_A + T_{i,\pi} I_{A^c}
\]

and (3.24) follows from (3.21) after noting \( \mathbb{E}[T_{\pi,i} \mid A] \leq \sup_j \mathbb{E}T_{j,i} \).

For (3.25), let \( X_1, X_2, \ldots \) be the stationary Markov chain and let \( Y_0, Y_1, \ldots \) be a coupled copy of the Markov chain started in state \( i \) at time 0, but let \( Y_1, Y_2, \ldots \) be coupled with \( X_1, X_2, \ldots \) according to the maximal coupling of Griffeath (1974/75) so that we have \( \mathbb{P}(X_n = Y_n = i) = \pi_i \wedge P_{i,i}^{(n)} \). Let \( T_{\pi,i} \) and \( T_{i,i} \) be hitting times respectively defined on these two Markov chains. Then

\[
\mathbb{P}(T_{\pi,i} \neq T_{i,i}) \leq \sum_n \mathbb{P}(X_n = i, Y_n \neq i) + \mathbb{P}(Y_n = i, X_n \neq i)
\]

and since

\[
\mathbb{P}(X_n = i, Y_n \neq i) = \pi_i - \mathbb{P}(X_n = i, Y_n = i)
\]

\[
= \pi_i - \pi_i \wedge P_{i,i}^{(n)}
\]

\[
= [\pi_i - P_{i,i}^{(n)}]^+
\]

and a similar calculation yields

\[
\mathbb{P}(Y_n = i, X_n \neq i) = [P_{i,i}^{(n)} - \pi_i]^+
\]

and then we obtain the result from (3.22). \( \square \)
Example 3.6. With the above definitions and further assuming $X_n$ is an $m$-dependent Markov chain, we can let $T_{i,\pi} = m$ and we thus have
\[
d_K\left(\mathscr{L}(\pi_i T_{\pi,i}), \text{Exp}(1)\right) \leq \pi_i (m + 1.5 + \mathbb{P}(T_{i,i} < m) \sup_j \mathbb{E}T_{j,i})
\]
and
\[
d_K\left(\mathscr{L}(\pi_i T_{\pi,i}), \text{Exp}(1)\right) \leq 2\pi_i + \sum_{n=1}^{m-1} |P_{i,i}^{(n)} - \pi_i|
\]
If we consider flipping a biased coin repeatedly, let $T$ be the number of flips required until the beginning of a given pattern of heads and tails of length $k$ first appearing as a run. The current run of $k$ flips can be encoded in the state space of a $k$-dependent Markov chain. Suppose the Markov chain at time $n$ is in state $i$ if some given desired pattern that can not overlap with itself appears as a run starting with flip $n$. This means $\mathbb{P}(T_{i,i} < m) = 0$ and $P_{i,j}^{(n)} = 0$, $n < m$. By applying the second result above we then obtain
\[
d_K\left(\mathscr{L}(\pi_i T_{\pi,i}), \text{Exp}(1)\right) \leq \pi_i (k + 1)
\]
If we are instead interested in the time until a run of $k$ heads first appears, we can use the “de-clumping” trick of waiting for the (non-overlapping) pattern of tails followed by $k$ heads in row and notice that this differs from the first appearance time of $k$ heads in a row only if the first $k$ flips are heads. If we let $T$ be the number of flips required prior to the start of a run of $k$ heads in row, we have
\[
d_K\left(\mathscr{L}(q_p k T_{\pi,i}), \text{Exp}(1)\right) \leq (k + 2)p^k
\]
where $p = 1 - q$ is the probability of heads. This result is nearly a factor of 2 improvement over the result from Barbour et al. (1992), Page 164, where a Poisson approximation is used for the number of times the pattern appears to estimate the error bound on the exponential tail probability. This type of bound also appears in the context of geometric approximation in Peköz (1996).

Our next results (and improvements of them) have previously appeared in the literature in Brown and Ge (1984), Brown (1990), and Daley (1988). Recall the definitions of NBUE and NWUE from Definition 3.3. The following is an immediate consequence of Theorem 2.1, (3.9) and (2.2).

Corollary 3.6. If $W$ is either NBUE or NWUE with $\mathbb{E}W = 1$, finite second moment and letting
\[
\rho = |\frac{1}{2} \mathbb{E}W^2 - 1|
\]
we have
\[
d_W\left(\mathscr{L}(W), \text{Exp}(1)\right) \leq 2\rho, \quad d_K\left(\mathscr{L}(W), \text{Exp}(1)\right) \leq 2.47\rho^{1/2}, \quad (3.26)
\]
and
\[ d_W(\mathcal{L}(W^e), \text{Exp}(1)) \leq \rho, \quad d_K(\mathcal{L}(W^e), \text{Exp}(1)) \leq \rho. \] (3.27)

Remark 3.7. The class of stochastic processes in applications where first passage times are either NBUE or NWUE is quite large; see Karasu and Özekici (1989) or Lam (1990) for a survey and some examples. The class of mixtures of exponential distributions, also called completely monotone distributions, appears prominently in the setting of reversible Markov chain hitting times; see Aldous and Fill (Preprint). It is straightforward to verify that the class of completely monotone distributions is a subset of the class of NWUE distributions.

3.3 Critical Galton-Watson branching process

Let \( Z_0 = 1, Z_1, Z_2, \ldots \) be a Galton-Watson branching process with offspring distribution \( \nu = \mathcal{L}(Z_1) \). A theorem due to Yaglom (1947) states that, if \( \mathbb{E}Z_1 = 1 \) and \( \text{Var}Z_1 = \sigma^2 < \infty \), then \( \mathcal{L}(n^{-1}Z_n|Z_n > 0) \) converges to an exponential distribution with mean \( \sigma^2/2 \). We give a rate of convergence for this asymptotic under finite third moment of the offspring distribution using the idea from Section 2.1.1. Though exponential limits in this context are an active area of research (see, for example, Lalley and Zheng (in press)), the question of rates does not appear to have been previously studied in the literature. To this end, we make use of the construction from Lyons, Pemantle, and Peres (1995); we refer to that article for more details on the construction and only present what is needed for our purpose.

Theorem 3.7. For a critical Galton-Watson branching process with offspring distribution \( \nu = \mathcal{L}(Z_1) \) such that \( \mathbb{E}Z_1^3 < \infty \) we have
\[ d_W(\mathcal{L}(2Z_n/\sigma^2 n|Z_n > 0), \text{Exp}(1)) = O\left(\frac{\log n}{n}\right). \]

Proof. First we construct a size-biased branching tree as in Lyons et al. (1995). We assume that this tree is labeled and ordered, in the sense that, if \( w \) and \( v \) are vertices in the tree from the same generation and \( w \) is to the left of \( v \), then the offspring of \( w \) is to the left of the offspring of \( v \), too. Start in generation 0 with one vertex \( v_0 \) and let it have a number of offspring distributed according to the size-bias distribution of \( \nu \). Pick one of the offspring of \( v_0 \) uniformly at random and call it \( v_1 \). To each of the siblings of \( v_1 \) attach an independent Galton-Watson branching process with offspring distribution \( \nu \). For \( v_1 \) proceed as for \( v_0 \), i.e., give it a size-biased number of offspring, pick one at uniformly at random, call it \( v_2 \), attach independent Galton-Watson branching process to the siblings of \( v_2 \) and so on. It is clear that this will always give an infinite tree as the “spine” \( v_0, v_1, v_2, \ldots \) of the tree will never die out.
Let us have some notation now. Denote by $S_n$ the total number of particles in generation $n$. Denote by $L_n$ and $R_n$, respectively, the number of particles to the left (exclusive $v_n$) and to the right (inclusive $v_n$), respectively, of vertex $v_n$, so that $S_n = L_n + R_n$. We can describe these particles in more detail, according to the generation at which they split off from the spine. Denote by $S_{n,j}$ the number of particles in generation $n$ that stem from any of the siblings of $v_j$ (but not $v_j$ itself). Clearly, $S_n = 1 + \sum_{j=1}^{n} S_{n,j}$, where the summands are independent. Likewise, let $L_{n,j}$ and $R_{n,j}$, respectively, be the number of particles in generation $n$ that stem from the siblings to the left and right, respectively, of $v_j$ (note that $L_{n,n}$ and $R_{n,n}$ are just the number of siblings of $v_n$ to the left and to the right, respectively). We have the relations $L_n = \sum_{j=1}^{n} L_{n,j}$ and $R_n = 1 + \sum_{j=1}^{n} R_{n,j}$. Note that, for fixed $j$, $L_{n,j}$ and $R_{n,j}$ are in general not independent, as they are linked through the offspring size of $v_{j-1}$.

Let now $R_{n,j}'$ be independent random variables such that $L(R_{n,j}') = L(R_{n,j} \mid L_{n,j} = 0)$.

Define also $R_n^* = 1 + \sum_{j=1}^{n} R_{n,j}$. Let us collect a few facts which we will then use to give the proof of the theorem:

(i) for any non-negative random variable $X$ the size-biased distribution of $L(X)$ is the same as the size-biased distribution of $L(X \mid X > 0)$;

(ii) $S_n$ has the size-biased distribution of $Z_n$;

(iii) given $S_n$, the vertex $v_n$ is uniformly distributed among the particles of the $n$th generation;

(iv) $\mathcal{L}(R_n^*) = \mathcal{L}(Z_n \mid Z_n > 0)$;

(v) $\mathbb{E}\{R_{n,j}I_{A_{n,j}}\} \leq \sigma^2 \mathbb{P}[A^c_{n,j}]$;

(vi) $\mathbb{E}\{R_{n,j}I_{A_{n,j}}\} \leq \gamma \mathbb{P}[A^c_{n,j}]$, where $\gamma = \mathbb{E}Z^3$;

(vii) $\mathbb{P}[A_{n,j}^c] \leq \sigma^2 \mathbb{P}[Z_{n-j} > 0] \leq C(\nu)/(n-j+1)$ for some absolute constant $C(\nu)$.

Statement (i) is easy to verify, (ii) follows from Lyons et al. (1995, Eq. (2.2)), (iii) follows from Lyons et al. (1995, comment after (2.2)), (iv) follows from Lyons et al. (1995, Proof of Theorem C(ii)). Using independence, 

$$\mathbb{E}\{R_{n,j}I_{A_{n,j}}\} = \mathbb{E}R_{n,j} \mathbb{P}[A^c_{n,j}] \leq \sigma^2 \mathbb{P}[A^c_{n,j}],$$

where the second inequality is due to Lyons et al. (1995, Proof of Theorem C(ii)), which proves (v). If $X_j$ denotes the number of siblings of $v_j$, having
the size bias distribution of $Z_1$ minus 1, we have

$$
\mathbb{E}\{R_{n,j}I_{A_{n,j}^c}\} \leq \mathbb{E}\{X_jI_{A_{n,j}^c}\} \leq \sum_k k\mathbb{P}[X_j = k, A_{n,j}^c] \\
\leq \sum_k k\mathbb{P}[X_j = k]\mathbb{P}[A_{n,j}^c]\mathbb{P}[X_j = k] \\
\leq \sum_k k^2\mathbb{P}[X_j = k]\mathbb{P}[A_{n,j}^c] \leq \gamma\mathbb{P}[A_{n,j}^c],
$$

hence (vi). Finally,

$$
\mathbb{P}[A_{n,j}^c] = \mathbb{E}\{\mathbb{P}[A_{n,j}^c|X_j]\} \leq \mathbb{E}\{X_j\mathbb{P}[Z_{n-j} > 0]\} \leq \sigma^2\mathbb{P}[Z_{n-j} > 0].
$$

Using Kolmogorov’s estimate (see Lyons et al. (1995, Theorem C(i))) we have

$$
\lim_{n \to \infty} n\mathbb{P}[Z_n > 0] = 2/\sigma^2
$$

which implies (vii).

We are now in the position to prove the theorem using (2.5) of Theorem 2.1. Let $c = 2/\sigma^2$. Due to (iv) we can set $W = cR_n/n$. Due to (i) and (ii), $S_n$ has the size bias distribution of $R_n^e$. Let $U$ be an independent and uniform random variable on $[0,1]$. Now, $R_n - U$ is a continuous random variable taking values on $[0,S_n]$ and, due to (iii), has distribution $\mathcal{L}(US_n)$; hence we can set $W^e = c(R_n^e - U)/n$. It remains to bound $\mathbb{E}\|W - W^e\|_1$.

From (3.28) and using (v)–(vii) we have

$$
nc^{-1}\mathbb{E}\|W - W^e\|_1 \leq \mathbb{E}U + \mathbb{E}|R_n^e - R_n| \leq 1 + \sum_{j=1}^n \mathbb{E}\{R_{n,j}^eI_{A_{n,j}^c} + R_{n,j}I_{A_{n,j}^c}\} \\
\leq 1 + C(\nu)\sum_{j=1}^n \frac{\sigma^2 + \gamma}{n - j + 1} \leq 1 + C(\nu)(\sigma^2 + \gamma)(1 + \log n).
$$

Hence, for a possibly different constant $C(\nu)$,

$$
\mathbb{E}\|W - W^e\|_1 \leq \frac{C(\nu)\log n}{n}.
$$

Plugging this into (2.7) yields the final bound.

\begin{remark}
Note that, from (2.2), Theorem 3.7 implies a rate of convergence of $O((\log(n)/n)^{1/2})$ for the Kolmogorov metric. However, if the number of offspring has a geometric distribution (started at 0) with mean 1 then standard generating function arguments (e.g. Athreya and Ney (1972)) show that the number of particles at time $n$ conditioned on non-extinction has geometric distribution Ge$(1/(n+1))$ (started at 1). Hence, in this case, the actual order of convergence is $n^{-1}$. We conjecture that this is also the correct rate of convergence for more general offspring distributions.
\end{remark}
Our main results are based on the Stein operator

\[ Af(x) = f'(x) - f(x) \quad (4.1) \]

previously studied (independently of each other and, in the case of the first two, independent of the present work) by Weinberg (2005), Bon (2006) and Chatterjee et al. (2006).

Let us first make a connection between this Stein operator and survival analysis. Assume that \( F \) is a differentiable distribution function of a non-negative random variable \( X \) having finite first moment such that the derivative \( F' > 0 \). Let \( \bar{F} = 1 - F \) be the survival function associated with \( X \). Then, for every function \( f \in \mathcal{F}_0 \) we have

\[ \mathbb{E} f(X) = -\int_0^\infty f(x)\bar{F}'(x)dx = \int_0^\infty f'(x)\bar{F}(x)dx \]

\[ = \int_0^\infty f'(x) \frac{\bar{F}(x)}{F'(x)}dF(x) = \mathbb{E}\{h(X)f'(X)\}, \]

where \( h(x) = \frac{\bar{F}(x)}{F'(x)} \) is the inverse hazard rate function of \( F \). This approach seems to be new and contrast approaches such as the density approach by Stein in Reinert (2005), where \( f'(x) \) is held fixed and the coefficient in front of \( f(x) \) is chosen accordingly, or Nourdin and Peccati (2009), where \( xf(x) \) is held fixed and the coefficient in front of \( f'(x) \) is altered. Now as is well known, the hazard rate function of the standard exponential distribution is 1, so that this approach yields the Stein operator \((4.1)\) and the Stein equation hence becomes

\[ f'(w) - f(w) = h(w) - \mathbb{E}h(Z), \quad w \geq 0. \quad (4.2) \]

Note that the density approach by Stein in Reinert (2005) yields the same operator but with no apparent probabilistic interpretation. The solution \( f \) to \((4.2)\) can be explicitly written as

\[ f(w) = -e^w \int_w^\infty (h(x) - \mathbb{E}h(Z))e^{-x}dx \quad (4.3) \]

To apply Stein’s method we need to study the properties of the solution \((4.3)\). Some preliminary results can be found in Weinberg (2005), Bon (2006), Chatterjee et al. (2006) and Daly (2008). We give self-contained proofs of the following bounds.

**Lemma 4.1** (Properties of the solution to the Stein equation). Let \( f \) be the solution to \((4.2)\). If \( h \) is bounded we have

\[ \|f\| \leq \|h\|, \quad \|f'\| \leq 2\|h\|. \quad (4.4) \]
If $h$ is Lipschitz we have

$$|f(w)| \leq (1 + w)\|h'||, \quad \|f'|| \leq \|h'||, \quad \|f''|| \leq 2\|h'||. \quad (4.5) \quad \{43\}$$

For any $a > 0$ and any $\varepsilon > 0$ let

$$h_{a,\varepsilon}(x) := \varepsilon^{-1} \int_0^\varepsilon [x + s \leq a] \, ds. \quad (4.6) \quad \{44\}$$

Define $f_{a,\varepsilon}$ as in $\{4.3\}$ with respect to $h_{a,\varepsilon}$. Define $h_{a,0}(x) = [x \leq a]$ and $f_{a,0}$ accordingly. Then, for all $\varepsilon > 0$,

$$\|f_{a,\varepsilon}\| \leq 1, \quad \|f_{a,\varepsilon}'\| \leq 1, \quad (4.7) \quad \{45\}$$

$$|f_{a,\varepsilon}(w + t) - f_{a,\varepsilon}(w)| \leq 1, \quad |f_{a,\varepsilon}'(w + t) - f_{a,\varepsilon}'(w)| \leq 1, \quad (4.8) \quad \{45b\}$$

$$|f_{a,\varepsilon}'(w + t) - f_{a,\varepsilon}'(w)| \leq (|t| \wedge 1) + \varepsilon^{-1} \int_{t \wedge 0}^{t \vee 0} [a - \varepsilon \leq w + u \leq a] \, du; \quad (4.9) \quad \{46\}$$

$(4.7)$ and $(4.8)$ also hold for $\varepsilon = 0$.

Proof. Write $\tilde{h}(w) = h(w) - \mathbb{E}h(Z)$. Assume now that $h$ is bounded. Then

$$|f(w)| \leq e^w \int_w^\infty |\tilde{h}(x)| e^{-x} \, dx \leq \|h||.$$  

Rearranging $(4.2)$ we have $f'(w) = f(w) + \tilde{h}(w)$, hence

$$|f'(w)| \leq |f(w)| + |\tilde{h}(w)| \leq 2\|h||.$$  

This proves $(4.4)$. Assume now that $h$ is Lipschitz. We can further assume without loss of generality that $h(0) = 0$ as $f$ will not change under shift; hence we may assume that $|h(x)| \leq x\|h'||$. Thus,

$$|f(w)| \leq e^w \int_w^\infty x\|h'|| e^{-x} \, dx = (1 + w)\|h'||,$$

which is the first bound of $(4.5)$. Now, differentiate both sides of $(4.2)$ to obtain

$$f''(w) - f'(w) = h'(w), \quad (4.10) \quad \{47\}$$

hence, analogous to $(4.3)$, we have

$$f'(w) = -e^w \int_w^\infty h'(x) e^{-x} \, dx.$$  

The same arguments as before lead to the second and third bound of $(4.5)$.

Let us now look at the properties of $f_{a,\varepsilon}$. It is easy to check that

$$f_{a,0}(x) = (e^{x-a} \wedge 1) - e^{-a}, \quad f_{a,0}'(x) = e^{x-a} [x \leq a] \quad (4.11) \quad \{47b\}$$
is the explicit solution to \((4.10)\) with respect to \(h_{a,0}\). Now, it is not difficult to see that, for \(\varepsilon > 0\), we can write
\[
f_{a,\varepsilon}(x) = \varepsilon^{-1} \int_0^\varepsilon f_{a,0}(x + s)ds
\]
and this \(f_{a,\varepsilon}\) satisfies \((4.12)\). These representations immediately lead to the bounds \((4.7)\) and \((4.8)\) for \(\varepsilon \geq 0\) from the explicit formulas \((4.11)\). Let now \(\varepsilon > 0\); observe that, from \((4.10)\),
\[
f'(x + t) - f'(x) = (f(x + t) - f(x)) + (h(x + t) - h(x)).
\]
Again from \((4.11)\), we deduce that
\[
|f_{a,\varepsilon}(x + t) - f_{a,\varepsilon}(x)| \leq (|t| \wedge 1),
\]
and this proves the second part of the bound \((4.9)\) for \(t > 0\); a similar argument yields the same bound for \(t < 0\).

The following lemmas are straightforward and hence given without proof.

**Lemma 4.2** (Smoothing lemma). For any \(\varepsilon > 0\)
\[
d_K(\mathcal{L}(W), \mathcal{L}(Z)) \leq \varepsilon + \sup_{a > 0} |\mathbb{E} h_{a,\varepsilon}(W) - \mathbb{E} h_{a,\varepsilon}(Z)|,
\]
where \(h_{a,\varepsilon}\) are defined as in Lemma 4.1.

**Lemma 4.3** (Concentration inequality). For any random variable \(V\),
\[
\mathbb{P}[a \leq V \leq b] \leq (b - a) + 2d_K(\mathcal{L}(V), \text{Exp}(1)).
\]

For the rest of the article write \(\kappa = d_K(\mathcal{L}(W), \text{Exp}(1))\).

**Proof of Theorem 2.1**. Let \(\Delta := W - W^e\). Define \(I_1 := I[|\Delta| \leq \beta]\); note that \(W^e\) may not have finite first moment. With \(f\) as in \((4.2)\) with respect to \((4.6)\), the quantity \(\mathbb{E} f'(W^e)\) is well defined as \(\|f'\| < \infty\), and we have
\[
\mathbb{E} \{f'(W) - f(W)\} = \mathbb{E} \{f'(W) - f'(W^e)\} + \mathbb{E} \{(1 - I_1)(f'(W) - f'(W^e))\} =: J_1 + J_2.
\]
Using \((4.17)\), \(|J_2| \leq \mathbb{P}[|\Delta| > \beta]\). Now, using \((4.10)\) and in the last step Lemma 4.3
\[
J_1 = \mathbb{E} \left\{ I_1 \int_0^\Delta f''(W + t)dt \right\}
= \mathbb{E} \left\{ I_1 \int_0^\Delta f'(W + t) - \varepsilon^{-1}I[a - \varepsilon \leq W + t \leq a]dt \right\}
\leq \mathbb{E} |I_1|\Delta + \int_{-\beta}^0 \mathbb{P}[a - \varepsilon \leq W + t \leq a]dt \leq 2\beta + 2\beta\varepsilon^{-1}\kappa.
\]
Similarly,
\[ J_1 \geq -\mathbb{E}|I_1\Delta| - \int_0^\beta \mathbb{P}[a - \varepsilon \leq W + t \leq a]dt \geq -2\beta - 2\beta\varepsilon^{-1}\kappa, \]
hence \( |J_1| \leq 2\beta + 2\beta\varepsilon^{-1}\kappa. \) Using Lemma 4.2 and choosing \( \varepsilon = 4\beta, \)
\[ \kappa \leq \varepsilon + \mathbb{P}[|\Delta| > \beta] + 2\beta + 2\beta\varepsilon^{-1}\kappa \leq \mathbb{P}[|\Delta| > \beta] + 6\beta + 0.5\kappa. \]
Solving for \( \kappa \) proves (2.5).

To obtain (2.6), write
\[
\mathbb{E}\left\{ f'(W^e) - f(W^e) \right\} = \mathbb{E}\left\{ f'(W) - f(W^e) \right\} \\
= \mathbb{E}\left\{ I_1 (f(W) - f(W^e)) \right\} + \mathbb{E}\left\{ (1 - I_1) (f(W) - f(W^e)) \right\}
\]
Hence, using Taylor’s expansion along with the bounds (4.7) for \( \varepsilon = 0, \)
\[ |\mathbb{E}f'(W^e) - f(W^e)| \leq \|f''\|\mathbb{E}|I_1\Delta| + \mathbb{P}[|\Delta| > \beta] \leq \mathbb{P}[|\Delta| > \beta], \]
which gives (2.6).

Assume now in addition that \( W \) has finite variance so that \( W^e \) has finite mean. Then
\[ |\mathbb{E}\{ f'(W) - f(W) \}| = |\mathbb{E}\{ f'(W) - f'(W^e) \}| \leq \|f''\|\mathbb{E}|\Delta| \]
From the bound (4.5), (2.7) follows. Also,
\[ |\mathbb{E}\{ f'(W^e) - f(W^e) \}| \leq \|f''\|\mathbb{E}|\Delta| \]
which yields (2.5) from (4.7) with \( \varepsilon = 0; \) the remark after (2.8) follows from (4.5).

**Proof Theorem 2.2.** Let \( f \) be the solution (4.2) to (4.3), hence \( f(0) = 0, \)
and assume that \( f \) is Lipschitz. From the fundamental theorem of calculus we have
\[ f(W') - f(W) = \int_0^D f'(W + t)dt. \]
Multiplying both sides by \( G \) and comparing it with the left hand side of (4.2) we have
\[
f'(W) - f(W) = Gf(W') - Gf(W) - f(W) \\
+ (1 - GD)f'(W') \\
+ (1 - GD)(f'(W) - f'(W')) \\
- G \int_0^D (f'(W + t) - f'(W))dt.
\]

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Note that we can take expectation component-wise due to the moment assumptions. Hence,

$$
\mathbb{E} h(W) - \mathbb{E} h(Z) = R_1(f) + R_2(f) + R_3(f) - R_4(f)
$$

where

$$
R_1(f) = \mathbb{E}\{Gf(W') - Gf(W) - f(W)\},
$$

$$
R_2(f) = \mathbb{E}\{(1 - GD)f'(W'')\},
$$

$$
R_3(f) = \mathbb{E}\{(1 - GD)(f'(W) - f'(W''))\},
$$

$$
R_4(f) = \mathbb{E}\left\{G \int_0^D (f'(W + t) - f'(W))dt\right\}.
$$

Assume now that $h \in \mathcal{F}_{BW}$ and $f$ the solution to (4.2). Then from (4.4) and (4.5)

$$
\|f\| \leq 1, \quad \|f'\| \leq 1, \quad \|f''\| \leq 2.
$$

Hence, $f \in \mathcal{F}_{BW}$ and

$$
|R_1(f)| \leq r_1(\mathcal{F}_{BW}), \quad |R_2(f)| \leq r_2.
$$

Furthermore,

$$
|R_3(f)| \leq \mathbb{E}\left|(1 - GD)(f'(W'') - f'(W))\right|
$$

$$
\leq 2\mathbb{E}\{|1 - GD| |D'| > 1\} + 2\mathbb{E}\{|1 - GD| (|D'| \wedge 1)\}
$$

$$
= 2r'_3 + 2r'_4,
$$

and

$$
|R_4(f)| \leq \mathbb{E}\left|G \int_0^D (f'(W + t) - f'(W))dt\right|
$$

$$
\leq 2\mathbb{E}|GD| |D'| > 1\} + 2\mathbb{E}|G(D^2 \wedge 1)|
$$

$$
= 2r_3 + 2r_4.
$$

This yields the $d_{BW}$ results. Let now $h \in \mathcal{F}_W$ and $f$ the solution to (4.2). Then, from (4.7) and (4.5),

$$
|f(x)| \leq (1 + x), \quad \|f'\| \leq 1, \quad \|f''\| \leq 2,
$$

hence the bounds on $R_2(f)$, $R_3(f)$ and $R_4(f)$ remain, whereas now $f \in \mathcal{F}_W$ and, thus, $|R_1(f)| \leq r_1(\mathcal{F}_W)$. This proves the $d_W$ estimate.

Let now $f$ be the solution to (4.2) with respect to $h_{a,\varepsilon}$ as in (4.6). Then, from (4.7)

$$
\|f\| \leq 1, \quad \|f'\| \leq 1,
$$

hence the bounds on $R_2(f)$, $R_3(f)$ and $R_4(f)$ remain, whereas now $f \in \mathcal{F}_W$ and, thus, $|R_1(f)| \leq r_1(\mathcal{F}_W)$. This proves the $d_W$ estimate.
hence $f \in \mathcal{F}_{BW}$, $|R_1(f)| \leq r_1(\mathcal{F}_{BW})$ and $|R_2(f)| \leq r_2$. Let $I_1 = I[|G| \leq \alpha, |D| \leq \beta', |D'| \leq \beta']$. Write

$$R_3(f) = \mathbb{E}\{(1 - I_1)(1 - GD)(f'(W'') - f'(W))\}
+ \mathbb{E}\{I_1(1 - GD)(f'(W'') - f'(W))\} =: J_1 + J_2.$$

Using (4.7), $|J_1| \leq r_5$ is immediate. Using (4.9) and Lemma 4.3

$$|J_2| \leq \mathbb{E}|(GD - 1)I_1(f'(W'') - f'(W))|$$

$$\leq (\alpha\beta + 1)\beta' + (\alpha\beta + 1)\varepsilon^{-1}\int_{-\beta'}^{\beta'} \mathbb{P}[a - \varepsilon \leq W + u \leq a]du$$

$$\leq (\alpha\beta + 1)\beta' + (\alpha\beta + 1)\varepsilon^{-1}\int_{-\beta'}^{\beta'} (\varepsilon + 2\kappa)du$$

$$= 3(\alpha\beta + 1)\beta' + 4(\alpha\beta + 1)\beta'\varepsilon^{-1}\kappa.$$

Similarly, let $I_2 = I[|G| \leq \alpha, |D| \leq \beta]$ and write

$$R_4(f) = \mathbb{E}\left\{G(1 - I_2) \int_0^D (f'(W + t) - f'(W))dt\right\}
+ \mathbb{E}\left\{GI_2 \int_0^D (f'(W + t) - f'(W))dt\right\} =: J_3 + J_4.$$

By (4.7), $|J_3| \leq r_5$. Using again (4.9) and Lemma 4.3

$$|J_4| \leq \mathbb{E}\left\{GI_2 \int_{D \wedge 0}^{D \wedge 0} |f'(W + t) - f'(W)|dt\right\}$$

$$\leq \alpha\mathbb{E}\left\{\int_{-\beta}^{\beta} |(t| + 1) + \varepsilon^{-1}\int_{t \wedge 0}^{t \vee 0} [a - \varepsilon \leq W + u \leq a]du\right\} dt\right\}$$

$$\leq \alpha\beta^2 + \alpha\varepsilon^{-1}\mathbb{E}\left\{\int_{-\beta}^{\beta} \int_{t \wedge 0}^{t \vee 0} (\varepsilon + 2\kappa)du\right\}dt = 2\alpha\beta^2 + 2\alpha\beta^2\varepsilon^{-1}\kappa.$$

Using Lemma 4.12 and collecting the bounds above we obtain

$$\kappa \leq \varepsilon + r_1(\mathcal{F}_{BW}) + r_2 + |J_1| + |J_2| + |J_3| + |J_4|$$

$$\leq \varepsilon + r_1(\mathcal{F}_{BW}) + r_2 + r_5 + r_5' + 3(\alpha\beta + 1)\beta' + 2\alpha\beta^2$$

$$+ (4(\alpha\beta + 1)\beta' + 2\alpha\beta^2)\varepsilon^{-1}\kappa.$$

so that, setting $\varepsilon = 8(\alpha\beta + 1)\beta' + 4\alpha\beta^2$,

$$\kappa \leq \varepsilon + r_1(\mathcal{F}_{BW}) + r_2 + r_5 + r_5' + 11(\alpha\beta + 1)\beta' + 6\alpha\beta^2 + 0.5\kappa.$$

Solving for $\kappa$ yields the final bound.
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