Coefficient Estimates for a General Subclass of Bi-univalent Functions

Khosrow Hosseinzadeh

ABSTRACT: In this paper, we introduce and investigate an interesting subclass \( S_{h,p}^{A,B,C,\lambda} \) of bi-univalent functions in the open unit disk \( \mathbb{U} \). Furthermore, we find estimates on the \( |a_2| \) and \( |a_3| \) coefficients for functions in this subclass. The coefficient bounds presented here generalize some recent works of several earlier authors.

Key Words: Analytic functions, Univalent functions, Bi-univalent functions, Coefficient estimates.

Contents

1 Introduction 1
2 Coefficient bounds for the function class \( S_{h,p}^{A,B,C,\lambda} \) 3
3 Corollaries and Consequences 5

1. Introduction

Let \( A \) denote the class of analytic functions in the unit disk

\[
\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \},
\]

that have the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\] (1.1)

Further, we shall denote by \( S \) the class of functions in \( A \) which are univalent in \( \mathbb{U} \) (for details see [1, 3, 5]). Since univalent functions are one-to-one, they are invertible and inverse functions need not be defined on the entire unit disk \( \mathbb{U} \). The Koebe one-quarter theorem [5] ensures that the image of \( \mathbb{U} \) under every univalent function \( f \in S \) contains a disk of radius \( \frac{1}{4} \). So every function \( f \in S \) has an inverse \( f^{-1} \), which is defined by

\[
f^{-1}(f(z)) = z \ (z \in \mathbb{U})
\]

and

\[
f(f^{-1}(w)) = w \left( |w| < r_0(f), \ r_0(f) \geq \frac{1}{4} \right).
\]

In fact, the inverse function \( f^{-1} \) is given by

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.
\] (1.2)

A function \( f \in A \) is said to be bi-univalent in \( \mathbb{U} \), if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{U} \) (see[10]). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1.1). The class of bi-univalent functions was first introduced and studied by Lewin [6], where it was proved that \( |a_2| \leq 1.51 \).

Brannan and Taha [1](see also [2]), also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \). For a brief history and interesting examples of functions in the class \( \Sigma \), see [10].

Netanyahu [8], showed that \( \max_{f \in \Sigma} |a_2| = \frac{4}{3} \). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients \( |a_n| \) for \( n = 3, 4, \ldots \) is presumably still an open problem.
Two of the most famous subclasses of univalent functions are the class \( S^*(\beta) \) of starlike functions of order \( \beta (0 \leq \beta < 1) \) and the class \( K(\beta) \) of convex functions of order \( \beta (0 \leq \beta < 1) \). By definition, we have

\[
S^*(\beta) = \left\{ f \in S : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta, \ z \in \mathbb{U} \right\}
\]

and

\[
K(\beta) = \left\{ f \in S : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \ z \in \mathbb{U} \right\}.
\]

For \( \beta (0 \leq \beta < 1) \), a function \( f \in \Sigma \) is in the class \( S^*(\beta) \) of strongly bi-starlike functions of order \( \beta (0 \leq \beta < 1) \), or \( K_2(\beta) \) of strongly bi-convex functions of order \( \beta (0 \leq \beta < 1) \), if both \( f \) and its inverse map \( f^{-1} \) are, respectively, starlike or convex of order \( \beta (0 \leq \beta < 1) \).

The object of the present paper is to introduce a new subclass of the function class \( \Sigma \) and find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in this new subclass of the functions class \( \Sigma \) employing the techniques used earlier by Srivastava et al. (see [9]).

**Definition 1.1** ([7]). A function \( f(z) \) given by (1.1) is said to be in the \( S_\Sigma(\alpha, \lambda) \) \( (0 < \alpha \leq 1, 0 \leq \lambda \leq 1) \), if the following conditions are satisfied:

\[
f \in \Sigma, \quad \left| \arg \left( \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \right) \right| < \frac{\alpha \pi}{2} (0 < \alpha \leq 1, 0 \leq \lambda \leq 1, z \in \mathbb{U})
\]

and

\[
\left| \arg \left( \frac{w^2g'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \right) \right| < \frac{\alpha \pi}{2} (0 < \alpha \leq 1, 0 \leq \lambda \leq 1, w \in \mathbb{U}),
\]

where \( g \) is the extension of \( f^{-1} \) to \( \mathbb{U} \).

**Theorem 1.1** ([7]). Let the function \( f(z) \) given by (1.1) be in the \( S_\Sigma(\alpha, \lambda) \) \( (0 < \alpha \leq 1, 0 \leq \lambda \leq 1) \). Then

\[
|a_2| \leq \frac{2\alpha}{\sqrt{\alpha(1 - 2\lambda + 25\lambda^2 - 44\lambda^3 + 20\lambda^4) + (1 + 3\lambda - 2\lambda^2)^2}}
\]

and

\[
|a_3| \leq \frac{\alpha}{1 + 2\lambda^2} + \frac{4\alpha^2}{(1 + 3\lambda - 2\lambda^2)^2}.
\]

**Definition 1.2** ([7]). A function \( f(z) \) given by (1.1) is said to be in the \( S_\Sigma(\beta, \lambda) \) \( (0 \leq \beta < 1, 0 \leq \lambda \leq 1) \), if the following conditions are satisfied:

\[
f \in \Sigma, \quad \text{Re} \left( \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \right) > \beta (0 \leq \beta < 1, 0 \leq \lambda \leq 1, z \in \mathbb{U})
\]

and

\[
\text{Re} \left( \frac{w^2g'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \right) > \beta (0 \leq \beta < 1, 0 \leq \lambda \leq 1, w \in \mathbb{U}),
\]

where \( g \) is the extension of \( f^{-1} \) to \( \mathbb{U} \).

It is stated that in Theorem 3.1 in [7], the calculations done by Magesh for the bound \( |a_3| \) are inaccurate. To remove this remarkable mistake, we’ve revised the calculations appropriately (see Theorem 1.2).

**Theorem 1.2** ([7]). Let the function \( f(z) \) given by (1.1) be in the \( S_\Sigma(\beta, \lambda) \) \( (0 \leq \beta < 1, 0 \leq \lambda \leq 1) \). Then

\[
|a_2| \leq \sqrt{\frac{2(1 - \beta)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}}
\]

and

\[
|a_3| \leq \frac{2(1 - \beta)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1} + \frac{1 - \beta}{2\lambda^2 + 1}.
\]
2. Coefficient bounds for the function class $S_{\Sigma}^{h,p}(A, B, C, \lambda)$

In this section, we introduce the subclass $S_{\Sigma}^{h,p}(A, B, C, \lambda)$ ($0 \leq \lambda \leq 1$) and find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this subclass.

**Definition 2.1.** Let the functions $h, p : U \to \mathbb{C}$ be analytic functions so that

$$\min \{ |\Re((h(z))|, |\Re(p(z))| \} > 0 \ (z \in \mathbb{U}) \text{ and } h(0) = p(0) = 1.$$  

Also, let the continuous functions $A, B, C : [0, 1] \to \mathbb{R}$ be so constrained that

$$A(\lambda) + B(\lambda) + C(\lambda) = 1, \ C(\lambda) \neq 2 \text{ and } 3 + 3A(\lambda) - C(\lambda) \neq 0; \ \lambda \in [0, 1].$$

A function $f(z) \in A$ given by (1.1) is said to be in the class $S_{\Sigma}^{h,p}(A, B, C, \lambda)$ ($0 \leq \lambda \leq 1$), if the following conditions are satisfied:

$$f \in \Sigma, \quad \frac{zf'(z) + A(\lambda)z^2f''(z)}{B(\lambda)z + A(\lambda)zf'(z) + C(\lambda)f(z)} \in h(\mathbb{U}) \ (z \in \mathbb{U}) \quad (2.1)$$

and

$$\frac{wg'(w) + A(\lambda)w^2g''(w)}{B(\lambda)w + A(\lambda)wg'(w) + C(\lambda)g(w)} \in p(\mathbb{U}) \ (w \in \mathbb{U}), \quad (2.2)$$

where $g$ is the extension of $f^{-1}$ to $U$.

**Remark 2.1.** There are many choices of the functions $h(z)$ and $p(z)$ which would provide interesting subclasses of the analytic function class $A$. For example, if we get

$$h(z) = p(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha \ (0 < \alpha \leq 1, \ z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f(z) \in S_{\Sigma}^{h,p}(A, B, C, \lambda)$, $A(\lambda) = 2\lambda^2 - \lambda, \ B(\lambda) = 4(\lambda - \lambda^2)$ and $C(\lambda) = 2\lambda^2 - 3\lambda + 1$ then

$$f \in \Sigma, \quad \arg \left( \frac{zf'(z) + (2\lambda^2 - \lambda)z^2f''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \right) < \frac{\alpha \pi}{2} \ (0 < \alpha \leq 1, \ 0 \leq \lambda \leq 1, \ z \in \mathbb{U})$$

and

$$\left| \arg \left( \frac{wg'(w) + (2\lambda^2 - \lambda)w^2g''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \right) \right| < \frac{\alpha \pi}{2} \ (0 < \alpha \leq 1, \ 0 \leq \lambda \leq 1, \ w \in \mathbb{U}).$$

In this case, the function $f$ is said to be in the class $S_{\Sigma}(\alpha, \lambda)$ introduced and studied by Magesh and Yamini [7].

By putting $\lambda = 0$ ($A(\lambda) = B(\lambda) = 0$ and $C(\lambda) = 1$), the class $S_{\Sigma}(\alpha, \lambda)$ reduces to the class of strongly bi-starlike functions of order $\alpha(0 < \alpha \leq 1)$ and denoted by $S_1^\alpha$.

By putting $\lambda = \frac{1}{4}$ ($A(\lambda) = C(\lambda) = 0$ and $B(\lambda) = 1$), the class $S_{\Sigma}(\alpha, \lambda)$ reduces to the class $H_1^\alpha$ introduced and studied by Srivastava et al. [10] and for $\lambda = 1$ ($B(\lambda) = C(\lambda) = 0$ and $A(\lambda) = 1$), the class $S_{\Sigma}(\alpha, \lambda)$ reduces to the class of strongly bi-convex functions of order $\alpha(0 < \alpha \leq 1)$ and denoted by $K_1^\alpha$.

If we get

$$h(z) = p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \ (0 \leq \beta < 1, \ z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. If $f(z) \in S_{\Sigma}^{h,p}(A, B, C, \lambda)$, $A(\lambda) = 2\lambda^2 - \lambda, \ B(\lambda) = 4(\lambda - \lambda^2)$ and $C(\lambda) = 2\lambda^2 - 3\lambda + 1$ then
A function

\[
f \in \Sigma, \quad Re \left( \frac{zf'(z) + (2\lambda^2 - \lambda)zf''(z)}{4(\lambda - \lambda^2)z + (2\lambda^2 - \lambda)zf'(z) + (2\lambda^2 - 3\lambda + 1)f(z)} \right) > \beta(0 \leq \beta < 1, 0 \leq \lambda \leq 1, z \in \mathbb{U})
\]

and

\[
Re \left( \frac{wg'(w) + (2\lambda^2 - \lambda)wg''(w)}{4(\lambda - \lambda^2)w + (2\lambda^2 - \lambda)wg'(w) + (2\lambda^2 - 3\lambda + 1)g(w)} \right) > \beta(0 \leq \beta < 1, 0 \leq \lambda \leq 1, w \in \mathbb{U}).
\]

In this case, the function \( f \) is said to be in the class \( S_{\Sigma}(\beta, \lambda) \) introduced and studied by Magesh and Yamini [7].

By putting \( \lambda = 0(A(\lambda) = B(\lambda) = 0 \) and \( C(\lambda) = 1) \), the class \( S_{\Sigma}(\beta, \lambda) \) reduces to the class of strongly bi-starlike functions of order \( \beta(0 \leq \beta < 1) \) and denoted by \( S_{\Sigma}^*(\beta) \).

By putting \( \lambda = \frac{1}{2}(A(\lambda) = C(\lambda) = 0 \) and \( B(\lambda) = 1) \), the class \( S_{\Sigma}(\beta, \lambda) \) reduces to the class \( \mathcal{S}_{\Sigma}(\beta) \) introduced and studied by Srivastava et al. [10] and for \( \lambda = 1(B(\lambda) = C(\lambda) = 0 \) and \( A(\lambda) = 1) \), the class \( S_{\Sigma}(\beta, \lambda) \) reduces to the class of strongly bi-convex functions of order \( \beta(0 \leq \beta < 1) \) and denoted by \( \mathcal{K}_{\Sigma}(\beta) \).

Note: Let \( A := A(\lambda), B := B(\lambda) \) and \( C := C(\lambda) \).

**Theorem 2.1.** A function \( f(z) \) given by (1.1) is said to be in the \( S_{\Sigma}^{\mathcal{L},\mathcal{P}}(A, B, C, \lambda) \) \( (0 \leq \lambda \leq 1) \). Then

\[
|a_2| \leq \min \left\{ \frac{\sqrt{|h''(0)|^2 + |p''(0)|^2}}{4(3 + 3A - C) + (C - 2)(2A + C) \neq 0} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{|h''(0)| + |p''(0)|}{4(3 + 3A - C) + (C - 2)(2A + C) \neq 0}, \right\}
\]

**Proof.** First of all, we write the argument inequalities in (2.1) and (2.2) in their equivalent forms as follows:

\[
\frac{zf'(z) + Az^2f''(z)}{Bz + Azf'(z) + Cf(z)} = h(z) \quad (z \in \mathbb{U}) \tag{2.3}
\]

and

\[
\frac{wg'(w) + Aw^2g''(w)}{Bw + Awg'(w) + Cg(w)} = p(w) \quad (w \in \mathbb{U}), \tag{2.4}
\]

respectively, where functions \( h(z) \) and \( p(w) \) satisfy the conditions of Defintion 2.1. Furthermore, the functions \( h(z) \) and \( p(w) \) have the following Taylor-Maclaurin series expensions:

\[
h(z) = 1 + h_1z + h_2z^2 + h_3z^3 \ldots \tag{2.5}
\]

and

\[
p(w) = 1 + p_1w + p_2w^2 + p_3w^3 \ldots \tag{2.6}
\]

respectively. Now, upon substituting from (2.5) and (2.6) into (2.3) and (2.4), respectively, and equating the coefficients, we get

\[
(2 - C)a_2 = h_1, \tag{2.7}
\]

\[
(3 + 3A - C)a_3 + (C - 2)(2A + C)a_2 = h_2, \tag{2.8}
\]

\[
-(2 - C)a_2 = p_1 \tag{2.9}
\]
and
\[-(3 + 3A - C)a_3 + \{2(3 + 3A - C) + (C - 2)(2A + C)\}a_2^2 = p_2.\]  
(2.10)

From (2.7) and (2.9), we obtain
\[p_1 = -h_1,\]  
(2.11)
\[a_2^2 = \frac{h_1^2 + p_1^2}{2(2 - C)^2}.\]  
(2.12)

If \((3 + 3A - C) + (C - 2)(2A + C) \neq 0\), then by adding (2.8) and (2.10), we get
\[a_2^2 = \frac{h_2 + p_2}{2[(3 + 3A - C) + (C - 2)(2A + C)]}.\]  
(2.13)

Therefore, we find from the equations (2.12) and (2.13) that
\[|a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2(C - 2)^2}\]
and
\[|a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{4[(3 + 3A - C) + (C - 2)(2A + C)]^2}\]
respectively. So we get the desired estimate on the coefficient \(|a_2|\) asserted. Next, in order to find the bound on the coefficient \(|a_3|\), we subtract (2.10) from (2.8). We thus get
\[2(3 + 3A - C)a_3 - 2(3 + 3A - C)a_2^2 = h_2 - p_2.\]  
(2.14)

Upon substituting the value of \(a_2^2\) from (2.12) into (2.14), it follows that
\[a_3 = \frac{h_2 - p_2}{2(3 + 3A - C)} + \frac{h_1^2 + p_1^2}{2(2 - C)^2}.\]  
(2.15)

We thus find that
\[|a_3| \leq \frac{|h''(0)| + |p''(0)|}{4[3 + 3A - C]^2} + \frac{|h'(0)|^2 + |p'(0)|^2}{2(C - 2)^2}\]
If \((3 + 3A - C) + (C - 2)(2A + C) \neq 0\), then by substituting the value of \(a_2^2\) from (2.13) into (2.14), it follows that
\[a_3 = \frac{h_2 - p_2}{2(3 + 3A - C)} + \frac{h_2 + p_2}{2[(3 + 3A - C) + (C - 2)(2A + C)]}.\]  
(2.16)

Consequently, we have
\[|a_3| \leq \frac{|h''(0)| + |p''(0)|}{4[3 + 3A - C]} + \frac{|h''(0)| + |p''(0)|}{4[(3 + 3A - C) + (C - 2)(2A + C)]}.\]

\[\square\]

3. Corollaries and Consequences

By putting
\[A(\lambda) = 2\lambda^2 - \lambda, \quad B(\lambda) = 4(\lambda - \lambda^2), \quad C(\lambda) = 2\lambda^2 - 3\lambda + 1\]
and
\[h(z) = p(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha \quad (0 < \alpha \leq 1, \ z \in \mathbb{U})\]
in Theorem 2.1, we obtain the following result.
Corollary 3.1. Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_{\Sigma}(\alpha, \lambda)$ ($0 < \alpha \leq 1$, $0 \leq \lambda \leq 1$). Then

$$|a_2| \leq \min \left\{ \frac{2\alpha}{1 + 3\lambda - 2\lambda^2}, \alpha \sqrt{\frac{2}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}} \right\} = \alpha \sqrt{\frac{2}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}}$$

and

$$|a_3| \leq \min \left\{ \frac{\alpha^2}{2\lambda^2 + 1} + \frac{4\alpha^2}{(1 + 3\lambda - 2\lambda^2)^2} \frac{\alpha^2}{2\lambda^2 + 1} + \frac{2\alpha^2}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1} \right\}$$

$$= \frac{\alpha^2}{2\lambda^2 + 1} + \frac{2\alpha^2}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}$$

Remark 3.1. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.1 are better than those given in Theorem 1.1. Because

$$2\alpha \leq \frac{\alpha^2}{\sqrt{\alpha(1 - 2\lambda + 25\lambda^2 - 44\lambda^3 + 20\lambda^4)}} + (1 + 3\lambda - 2\lambda^2)^2 \leq \frac{\alpha^2}{\alpha - \lambda} \leq \frac{4\alpha^2}{(1 + 3\lambda - 2\lambda^2)^2}$$

By putting $\lambda = \frac{1}{2}$ in Corollary 3.1, we conclude the following corollary.

Corollary 3.2. Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $H_{\Sigma}^2$ ($0 < \alpha \leq 1$). Then

$$|a_2| \leq \min \left\{ \alpha, \sqrt{\frac{2}{3\alpha}} \right\} = \sqrt{\frac{2}{3\alpha}}$$

and

$$|a_3| \leq \min \left\{ \frac{5}{3}, \frac{4}{3}, \frac{4\alpha^2}{3\alpha^2} \right\} = \frac{4}{3}\alpha^2.$$

Remark 3.2. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.2 are better than those given by Srivastava [10, Theorem 1].

By putting $\lambda = 1$ in Corollary 3.1, we conclude the following corollary.

Corollary 3.3. Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $K_{\Sigma}(\alpha)$ ($0 < \alpha \leq 1$). Then

$$|a_2| \leq \alpha \quad \text{and} \quad |a_3| \leq \frac{4}{3}\alpha^2.$$
Remark 3.4. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.4 are better than those given by Çağlar [4, Corollary 2.5]. Because

\[
\sqrt{2\alpha} \leq \frac{2\alpha}{\sqrt{\alpha + 1}}
\]

and

\[
3\alpha^2 \leq 4\alpha^2 + \alpha.
\]

By putting

\[
\begin{align*}
A(\lambda) &= 2\lambda^2 - \lambda, \quad B(\lambda) = 4(\lambda - \lambda^2), \quad C(\lambda) = 2\lambda^2 - 3\lambda + 1
\end{align*}
\]

and

\[
h(z) = p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1, \ z \in \mathbb{U})
\]

in Theorem 2.1, we obtain the following result.

Corollary 3.5. Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S_\Sigma(\beta, \lambda)$ ($0 \leq \beta < 1, \ 0 \leq \lambda \leq 1$). Then

\[
|a_2| \leq \min \left\{ \frac{2(1 - \beta)}{1 + 3\lambda - 2\lambda^2}, \frac{2(1 - \beta)}{\sqrt{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1}} \right\}
\]

and

\[
|a_3| \leq \min \left\{ \frac{1 - \beta}{2\lambda^2 + 1} + \frac{4(1 - \beta)^2}{(1 + 3\lambda - 2\lambda^2)^2}, \frac{1 - \beta}{2\lambda^2 + 1} + \frac{2(1 - \beta)}{12\lambda^4 - 28\lambda^3 + 15\lambda^2 + 2\lambda + 1} \right\}.
\]

By setting $\lambda = \frac{1}{2}$ in Corollary 3.5, we conclude the following corollary.

Corollary 3.6. Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $H_\Sigma(\beta)$ ($0 \leq \beta < 1$). Then

\[
|a_2| \leq \left\{ \begin{array}{ll}
\sqrt{\frac{4}{3}}(1 - \beta) & ; 0 \leq \beta \leq \frac{1}{3} \\
(1 - \beta) & ; \frac{1}{3} \leq \beta < 1
\end{array} \right.
\]

and

\[
|a_3| \leq \left\{ \begin{array}{ll}
\frac{4}{3}(1 - \beta) & ; 0 \leq \beta \leq \frac{1}{3} \\
\frac{(1 - \beta)(5 - 3\beta)}{3} & ; \frac{1}{3} \leq \beta < 1.
\end{array} \right.
\]

Remark 3.5. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.6 are better than those given by Srivastava [10, Theorem 2].

By putting $\lambda = 1$ in Corollary 3.5, we conclude the following corollary.

Corollary 3.7. Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $K_\Sigma(\beta)$ ($0 \leq \beta < 1$). Then

\[
|a_2| \leq \min \left\{ (1 - \beta), \sqrt{1 - \beta} \right\} = (1 - \beta)
\]

and

\[
|a_3| \leq \min \left\{ \frac{4}{3}(1 - \beta), \frac{1}{3}(1 - \beta) + (1 - \beta)^2 \right\} = \frac{1}{3}(1 - \beta) + (1 - \beta)^2.
\]

Remark 3.6. The bound on $|a_2|$ given in Corollary 3.7 is better than that given by Xiao-Fei-li [11, Theorem 3.2], when $\lambda = 1$.

By putting $\lambda = 0$ in Corollary 3.5, we conclude the following corollary.
Corollary 3.8. Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $S^*_2(\beta)$ ($0 \leq \beta < 1$). Then

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)} ; & 0 \leq \beta \leq \frac{1}{2} \\ 2(1-\beta) ; & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} 3(1-\beta) ; & 0 \leq \beta \leq \frac{1}{2} \\ (1-\beta)(5-4\beta) ; & \frac{1}{2} \leq \beta < 1 \end{cases}$$

Remark 3.7. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.8 are better than those given by Çağlar [4, Corollary 3.5].

Acknowledgments

I would like to thank the referees by your suggestions.

References

1. Brannan, D. A., Taha, T. S., *On some classes of bi-univalent functions*, Studia Universitatis Babes-Bolyai, Series Mathematica 31, 70-77, (1986).
2. Brannan, D. A., Clunie, J., *Aspect of contemporary complex analysis*, Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham, Academic Press, New York and London, 1980.
3. Breaz, D., Breaz, N., Srivastava, H. M., *An extension of the univalent conditions for a family of integral operators*, Appl. Math. Lett. 22, 41-44, (2009).
4. Çağlar, M., Orhan, H., Yağmur, N., *Coefficient bounds for new subclasses of bi-univalent functions*, Filomat. 27, 1165-1171, (2013).
5. Duren, P. L., *Univalent Functions*, Springer-Verlag, New York, Berlin, 1983.
6. Lewin, M., *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. 18, 63-68, (1967).
7. Magesh, N., Yamini, J., *Coefficient bounds for a certain subclass of bi-univalent functions*, International Mathematical Forum. 8, 1337-1344, (2013).
8. Netanyahu, E., *The minimal distance of the image boundary from the origin and second coefficient of a univalent functions in $|z| < 1$*, Arch. Rational Mech. Anal. 32, 100-112, (1969).
9. Srivastava, H. M., Bulut, S., Çağlar, M., Yağmur, N., *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat. 27, 831-842, (2013).
10. Srivastava, H. M., Mishra, A. K., Gochhayat, P., *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. 23, 1188-1192, (2010).
11. Xiao-Fei-li, An-Ping Wang, *Two new subclasses of bi-univalent functions*, International Mathematical Forum. 7, 1495-1504, (2012).
12. Zireh, A., Analouei Adegani, E., Bulut, S., *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions defined by subordination*, Bull. Belg. Math. Soc. Simon Stevin., 23, 487-504, (2016).
13. Zireh, A., Analouei Adegani, E., Bidkham, M., *Faber polynomial coefficient estimates for subclass of bi-univalent functions defined by quasi-subordinate*, Math. Slovaca, 68, 369-378, (2018).

Khosrow Hosseinzadeh,
Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Iran.
E-mail address: zadeh.khosrow@gmail.com, kh.hosseinzadeh@shahroodut.ac.ir