Scarcity of periodic orbits in outer billiards

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To the memory of my dear teacher Gennadi Henkin

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Abstract. We give a simple proof of the result of [12] that the set of period 4 orbits in planar outer billiard with piecewise smooth convex boundary has empty interior, provided that no four corners of the boundary form a parallelogram. We also obtain results on period 5 and 6 orbits.

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1 Introduction

In this paper we study the set of periodic orbits in planar outer billiard.

Let $\Gamma$ be a piecewise smooth convex curve in the plane $\mathbb{R}^2$. The dynamics of the outer (or dual) billiard is defined in the exterior of $\Gamma$ as follows. The outer billiard map sends a point $z_1$ to the point $z_2$, so that the line between the two points is a supporting line for $\Gamma$ and meets $\Gamma$ at the midpoint of the segment $[z_1, z_2]$.

The outer billiard was originally introduced by Bernhard Neumann (1960-s) as a model for a stability problem. Moser [7] attracted much attention to the question whether all orbits in an outer billiard are bounded. Schwartz [9] did extensive work on the question for polygonal outer billiards. See [11] for more discussion and related results.

A relevant question on the outer billiard is whether the set of periodic points (orbits) must have measure zero, or in a milder version – empty interior. The interest to this question

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comes from the classical Birkhoff billiard, for which the question is related to asymptotic
distribution of large eigenvalues of the Laplacian \[5\].

The question has turned out to be surprisingly hard and remains open. For the classical
planar billiard, Rychlik \[8\] showed that the set of period 3 points has measure zero. The proof
involved symbolic calculations that were later removed by Stojanov \[10\]. Wojtkowski \[14\]
gave a simpler proof relying on Jacobi’s fields. Vorobets \[13\] extended this result to higher
dimensional billiards. Glutsyuk and Kudryashov \[3\] showed that the set of period 4 points
in a planar billiard has measure zero. For the planar outer billiard, Genin and Tabachnikov
\[2\] proved that the set of period 3 points has empty interior. The author and Zharnitsky
\[12\] proved that the set of period 4 points has empty interior unless there are four corners
of \(\Gamma\) that form a parallelogram. The proof in \[12\] involves computer aided computations.

In this paper we give a simple proof of the result of \[12\] avoiding heavy computations.
In addition to the result about period 4, we obtain results about period 5 and 6 orbits.

Following \[12\], we use an approach based on exterior differential systems (EDS). This
approach was introduced by Baryshnikov, Landsberg, and Zharnitsky, see \[1, 6\]. An EDS
on a smooth manifold \(X\) is a subbundle \(D\) of the tangent bundle \(T(X)\). We call a smooth
manifold \(M \subset X\) an integral manifold for \(D\) if the tangent space of \(M\) is contained in \(D\).
There is an explicitly defined EDS \(D\) in \(\mathbb{R}^{2n}\) so that 2-dimensional integral manifolds of \(D\)
correspond to outer billiards with open sets of period \(n\) points (see Proposition 2.1). Thus,
the question whether there is an outer billiard with an open set of periodic points reduces
to the question whether there is a 2-dimensional integral manifold for \(D\).

There is an algorithm by E. Cartan and Kähler (see, e.g. \[4\]) that answers the question
whether an EDS has integral manifolds of certain dimension, however, the algorithm often
leads to calculations that are difficult to accomplish even on a computer. The first step in
the algorithm consists of describing integral elements, which are subspaces of \(D\) of given
dimension (in our case, of dimension 2) that can be candidates for tangent spaces to \(M\).
They are subspaces on which the defining 1-forms for \(D\) vanish together with their exterior
differentials. In this paper we do not go beyond this first step. Our main idea is based
on an observation that the convexity of the billiard curve \(\Gamma\) can be formulated in terms of
integral elements. It turns out that in some cases that we describe here, there are no integral
elements that arise from a convex curve, therefore there does not exist an integral manifold
that corresponds to a convex outer billiard.

We point out, however, that for the conventional, inner billiards, our approach does not
work. That is, the convexity of the billiard curve does not affect the existence of integral
elements for the corresponding EDS.

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2 Outer billiard and EDS

Let $G \subset \mathbb{R}^2$ be a convex bounded domain with piecewise $C^2$ boundary $\Gamma$. We recall the definition of the outer billiard map $F$. Let $z_1 \in \mathbb{R}^2 \setminus \bar{G}$. We put $F(z_1) = z_2$ if the oriented line $L$ from $z_1$ to $z_2$ is a supporting line for $G$ that meets $\Gamma$ at the midpoint $(z_1 + z_2)/2$ and for definiteness $G$ lies on the left of $L$. The line $L$ is either tangent to $\Gamma$ or passes through the corner.

The map $F$ is well defined in the exterior of $G$ except on supporting lines $L$ for which the intersection $\Gamma \cap L$ is not a single point. These lines are countable, and their union $E$ is nowhere dense. The map $F$ is $C^1$ smooth at $z_1$ unless $\Gamma$ has zero curvature at the midpoint $(z_1 + z_2)/2$ or $L$ is one of the two tangent lines at a corner. The union $E_0$ of tangent lines $L$ through points of zero curvature is a closed nowhere dense set, and $E \subset E_0$.

The orbit $O(z)$ of the point $z$ is the set $O(z) = \{F_j(z) : j \in \mathbb{N}\}$, here $F^j$ is the $j$-th iterate of $F$. We call $z$ a period $n$ point if $F^n(z) = z$. In this case we also call $O(z)$ a period $n$ orbit.

We are concerned with the question whether the set of periodic points can have nonempty interior. Following [12] we reduce the problem to the existence of integral manifolds of a certain exterior differential system (EDS), which we now introduce.

We view the set of period $n$ orbits as a subset in

$$\mathbb{R}^{2n} = \{(z_1, \ldots, z_n) : z_i = (x_i, y_i) \in \mathbb{R}^2, i \in \mathbb{Z}/n\mathbb{Z}\}.$$  

Define the differential forms

$$\theta_i = \det(d\bar{r}_i, r_i), \quad r_i = \frac{z_i - z_{i+1}}{2}, \quad \bar{r}_i = \frac{z_i + z_{i+1}}{2}, \quad i \in \mathbb{Z}/n\mathbb{Z}. \tag{1}$$

Here $\det$ stands for the determinant. We will see (Proposition 3.1) that if no three consecutive points $z_i, z_{i+1}, z_{i+2}$ are collinear, then the forms $\theta_i$ are linearly independent, and their common zero set forms a rank $n$ subbundle $D$ of the tangent bundle of $\mathbb{R}^{2n}$, an EDS. We call a smooth manifold $M \subset \mathbb{R}^{2n}$ an integral manifold for $D$ if the tangent space of $M$ is contained in $D$, that is, $\theta_i|_M = 0$ for all $i$.

**Proposition 2.1.** [12] Let the set of period $n$ points have non-empty interior. Then there exists $M$, an integral manifold for $D$ such that $\dim M = 2$ and $dx_1 \wedge dy_1|_M$ is non-vanishing.

The converse is also true, but we do not need it here. For completeness, we give a proof.

**Proof.** Let the set of period $n$ points have non-empty interior. Then there exists an open set $U \subset \mathbb{R}^2$ such that all iterates $F^j, 1 \leq j \leq n - 1$, are $C^1$ smooth in $U$, and for every $z_1 \in U$, we have $F^n(z_1) = z_1$. Define

$$M = \{(z_1, \ldots, z_n) : z_1 \in U, z_{i+1} = F(z_i), i \in \mathbb{Z}/n\mathbb{Z}\}. \tag{2}$$
Clearly \( \dim M = 2 \) and \( dx_1 \wedge dy_1 |_M \) is non-vanishing. We claim that \( M \) is an integral manifold for \( D \). Let \( z(t) = (z_1(t), \ldots, z_n(t)) \) be a curve in \( M \). Then \( \bar{r}_i(t) \in \Gamma \). If for some \( t \), the midpoint \( \bar{r}_i(t) \) is a corner, then \( \bar{r}_i'(t) = 0 \), hence \( \theta_i(z'(t)) = 0 \). Otherwise, \( \bar{r}_i'(t) \) is tangent to \( \Gamma \) at \( \bar{r}_i(t) \). Then the vectors \( \bar{r}_i'(t) \) and \( r_i(t) \) are collinear, and again \( \theta_i(z'(t)) = 0 \) as desired.

\[ \square \]

### 3 Integral elements

We say that a 2-subspace \( \sigma \subset T_z \mathbb{R}^{2n}, z \in \mathbb{R}^{2n}, \) is an integral element for \( D \) if \( \theta_i|_\sigma = 0 \) and \( d\theta_i|_\sigma = 0 \) for all \( i \). Clearly, all tangent spaces to \( M \) defined by (2) are integral elements. To describe integral elements, we include the forms \( \theta_i \) in a coframe. Following [12], we put

\[ \omega_i = \det(dr_i, r_i). \]

**Proposition 3.1.** [12] The forms \( \theta_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z} \), are linearly independent provided that no three consecutive points are collinear.

For completeness and subsequent use we include a proof.

**Proof.** It suffices to express the standard frame \( dz_i \) in terms of the forms \( \theta_i, \omega_i \). We first recall how to solve the system

\[ \det(x, a) = \alpha, \quad \det(x, b) = \beta \]

for given \( a, b \in \mathbb{R}^2 \) and \( \alpha, \beta \in \mathbb{R} \). One verifies if \( \Delta = \det(a, b) \neq 0 \), then the system has a unique solution

\[ x = \frac{\beta a - \alpha b}{\Delta}. \quad (3) \]

We introduce

\[ \Delta_{ij} = \det(r_i, r_j), \quad \Delta_i = \Delta_{i-1,i}. \]

Geometrically, \( \Delta_i \) is the half of the area of the triangle with vertices \( z_{i-1}, z_i, z_{i+1} \). These points are collinear exactly when \( \Delta_i = 0 \). We have

\[ z_i = \bar{r}_i + r_i, \quad z_{i+1} = \bar{r}_i - r_i. \]

Accordingly, we get

\[ \det(dz_i, r_i) = \theta_i + \omega_i, \quad \det(dz_{i+1}, r_i) = \theta_i - \omega_i. \]
Solving the system
\[ \det(dz_i, r_{i-1}) = \theta_{i-1} - \omega_{i-1}, \quad \det(dz_i, r_i) = \theta_i + \omega_i \]
using (3) yields
\[ dz_i = \frac{(\omega_i + \theta_i)r_{i-1} + (\omega_{i-1} - \theta_{i-1})r_i}{\Delta_i}, \quad (4) \]
as desired.

For a period \( n \) orbit in outer billiard, no three consecutive points \( z_{i-1}, z_i, z_{i+1} \) are collinear, hence \( \Delta_i > 0 \), and \( \theta_i, \omega_i, i \in \mathbb{Z}/n\mathbb{Z} \) form a coframe, which we assume from now on.

**Proposition 3.2.** \cite{12} Let \( \sigma \subset T_z \mathbb{R}^{2n} \) be a 2-subspace such that \( \theta_i|_\sigma = 0 \) for all \( i \). Then \( \sigma \) is an integral element for \( D \) if and only if for all \( i, j \in \mathbb{Z}/n\mathbb{Z} \) we have
\[ \Delta_i^{-1} \omega_{i-1} \wedge \omega_i|_\sigma = \Delta_j^{-1} \omega_{j-1} \wedge \omega_j|_\sigma \quad (5) \]

For completeness we include a proof.

**Proof.** Differentiating (11), we get
\[ d\theta_i = -\det(d\bar{r}_i, dr_i) = -\frac{1}{4}(\det(dz_i, dz_i) - \det(dz_{i+1}, dz_{i+1})), \quad (6) \]
where the determinants are calculated using the wedge product. Note that the determinant of two 1-forms is commutative. Since \( \theta_i|_\sigma = 0 \), the equation (11) turns into
\[ dz_i|_\sigma = \Delta_i^{-1}(\omega_{i-1}r_{i-1} + \omega_{i-1}r_i). \]
Then we obtain
\[ \det(dz_i, dz_i) = \Delta_i^{-2} \det(\omega_{i-1}r_{i-1} + \omega_{i-1}r_{i-1}, \omega_{i-1}r_{i-1} + \omega_{i-1}r_i) = -2\Delta_i^{-1}\omega_{i-1} \wedge \omega_i. \quad (7) \]
The equation (5) now follows by (6) and (7).

We now derive parametric equations of integral elements. We introduce a \( n \times n \) matrix
\[ C = \begin{pmatrix} c_1 & \Delta_1 & 0 & \ldots & 0 & \Delta_2 \\ \Delta_3 & c_2 & \Delta_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \Delta_n & 0 & 0 & \ldots & \Delta_1 & c_n \end{pmatrix} \quad (8) \]
Proposition 3.3. Let \( \sigma \subset T_z \mathbb{R}^{2n} \) be a 2-subspace such that \( \theta_i|_{\sigma} = 0 \) for all \( i \). Then \( \sigma \) is an integral element if and only if there exist real parameters \( c_i, 1 \leq i \leq n \), such that \( \text{rank} \, C = n - 2 \), and \( C \omega|_{\sigma} = 0 \). Here \( \omega \) is the vector with components \( \omega_i \).

Proof. Suppose \( \sigma \) is an integral element. Then for every \( i \) the forms \( \omega_i \) and \( \omega_{i+1} \) are linearly independent on \( \sigma \). Otherwise, if for some \( i \) they are linearly dependent, then by (5) all the forms \( \omega_i \) will be multiples of one another on \( \sigma \), which is absurd because \( \dim \sigma = 2 \). Since every three forms \( \omega_i \) are linearly dependent on \( \sigma \), for some \( a_i, b_i \) we have

\[
\omega_{i+2} = a_i \omega_i + b_{i+1} \omega_{i+1}.
\]  

(9)

Wedge-multiplying both parts by \( \omega_{i+1} \) and using (5) we obtain

\[
a_i = -\Delta^{-1}_{i+1} \Delta_{i+2}.
\]

We put \( c_i = -b_i \Delta_i \). Then (9) turns into

\[
\Delta_{i+2} \omega_i + c_{i+1} \omega_{i+1} + \Delta_{i+1} \omega_{i+2} = 0.
\]  

(10)

We put these scalar equations in a matrix-vector form \( C \omega = 0 \). Note that \( \text{rank} \, C \geq n - 2 \) because \( C \) has a nonzero minor of order \( n - 2 \) in the left lower corner. Since \( \dim \sigma = 2 \), we have exactly \( \text{rank} \, C = n - 2 \), hence the conclusion. Clearly, the converse is also true.

For every polygon with vertices \( z_i, 1 \leq i \leq n \), we introduce a special integral element. We put

\[
d_i = \Delta_{i-1,i+1} = \det(r_{i-1}, r_{i+1}).
\]

Proposition 3.4. Putting \( c_i = -d_i \) defines an integral element.

Proof. We claim that if \( c_i = -d_i \), then \( \text{rank} \, C = n - 2 \). To see that, we check that \( Cr = 0 \), where \( r \) is a formal vector with components \( r_i \). Since the vectors \( r_i \) are not all collinear, we will have two linearly independent vectors in the null-space of \( C \), hence \( \text{rank} \, C = n - 2 \).

We check that

\[
\Delta_{i+2} r_i - d_{i+1} r_{i+1} + \Delta_{i+1} r_{i+2} = 0.
\]  

(11)

Without loss of generality, for simplicity of notation, we put \( i = 0 \). The equation (11) turns into

\[
f(r_0, r_1, r_2) := r_0 \det(r_1, r_2) + r_1 \det(r_2, r_0) + r_2 \det(r_0, r_1) = 0.
\]

We note that \( f \) is multilinear and alternating. Since \( r_0, r_1, r_2 \in \mathbb{R}^2 \) are linearly dependent, we have \( f(r_0, r_1, r_2) = 0 \).

Remark 3.5. If \( n \) is even, then putting \( c_i = d_i \) also defines an integral element. In this case the vector with components \( (-1)^i r_i \) is in the null-space of \( C \).
4 Convexity

We determine what integral elements correspond to convex billiards.

Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^2$. We put $s_i = |r_i|$.

**Lemma 4.1.** $\Delta_2 \langle r_0, r_1 \rangle + \Delta_1 \langle r_1, r_2 \rangle = s_2^2 d_1$.

**Proof.** Immediate by inner multiplying (11) by $r_{i+1}$. \hfill $\Box$

Let $\kappa_i$ be the curvature of $\Gamma$ at the midpoint $\bar{r}_i$. We assume $\kappa_i > 0$ because the tangent lines at the points with zero curvature form a nowhere dense set. We put $\kappa_i = \infty$ if $\Gamma$ has a corner at $\bar{r}_i$.

**Proposition 4.2.** Let $\{c_i\}$ define an integral element arising from a convex billiard curve $\Gamma$. Then we have

$$c_i = d_i - \frac{2\Delta_i \Delta_{i+1}}{\kappa_i s_i^3}.$$ 

In particular, $c_i \leq d_i$, where the equality occurs if $\Gamma$ has a corner at $\bar{r}_i$.

**Proof.** Let $\{z_i(t)\}$ be a curve in $M$ defined by (2) through the point where $t = 0$. Then the quantities $r_i, \bar{r}_i, \text{etc.}$, will depend on $t$. For simplicity of notation, we derive the needed formula for $i = 1$. By a prime we denote the derivative with respect to $t$ at $t = 0$. We first assume $\kappa_1 < \infty$ and $\bar{r}_1' \neq 0$. By the definition of the outer billiard, we put $r_1 = \lambda \bar{r}_1'$. Then $r_1' = \lambda r_1'' + \lambda v$. Since $\bar{r}_1 \in \Gamma$, the normal component of the acceleration $\bar{r}_1''$ has the form $a_N = \kappa_1 v^2$, here $v = |\bar{r}_1'|$. The inner unit normal vector to $\Gamma$ at $\bar{r}_1$ has the form $N_1 = -s_1^{-1} J r_1$, here $J$ is the counterclockwise rotation by $\pi/2$. Then we have

$$\omega_1(z') = \det(r_1', r_1) = \det(\lambda r_1'' + \lambda v, r_1) = \det(\lambda \kappa_1 v^2 N_1, \lambda r_1') = \frac{\lambda^2 v^2 \kappa_1}{s_1} \det(-J r_1, \bar{r}_1').$$

Note $\lambda^2 v^2 = s_1^2$, and $\det(-J x, y) = \langle x, y \rangle$. Then $\omega_1(z') = \kappa_1 s_1 \langle \bar{r}_1', r_1 \rangle$, and on the integral element we have

$$\omega_1 = \kappa_1 s_1 \langle d\bar{r}_1, r_1 \rangle.$$ (12)

We have obtained (12) assuming that $\bar{r}_1' \neq 0$. However, if $\bar{r}_1' = 0$, then (12) still holds because in this case $\omega_1(z') = 0$. Using (4), we write

$$2d\bar{r}_1 = dz_1 + dz_2 = \Delta_1^{-1}(\omega_0 r_0 + \omega_0 r_1) + \Delta_2^{-1}(\omega_2 r_1 + \omega_1 r_2),$$

$$2\Delta_1 \Delta_2 \langle d\bar{r}_1, r_1 \rangle = \omega_1(\Delta_2 \langle r_0, r_1 \rangle + \Delta_1 \langle r_1, r_2 \rangle) + (\Delta_2 \omega_0 + \Delta_1 \omega_2) s_1^2.$$
Using Lemma 4.1 we get
\[ 2s_1^{-2} \Delta_1 \Delta_2 (d\bar{r}_1, r_1) = d_1 \omega_1 + \Delta_2 \omega_0 + \Delta_1 \omega_2, \]
and using (12) we get
\[ \frac{2\Delta_1 \Delta_2}{\kappa_1 s_1^2} \omega_1 = d_1 \omega_1 + \Delta_2 \omega_0 + \Delta_1 \omega_2. \]  
We have obtained (14) assuming \( \kappa_1 < \infty \). However, if \( \kappa_1 = \infty \), then (14) still holds because in this case \( d\bar{r}_1 = 0 \), and (13) implies (14). Now by comparing (14) with (10) we obtain the desired equation for \( c_1 \).

We call an integral element convex if \( c_i \leq d_i \). Our results on the empty interior of the set of periodic orbits follow from the absence of convex integral elements.

5 The cases \( n = 3 \) and \( n = 4 \)

We first consider the simple cases \( n = 3 \) and \( n = 4 \).

**Theorem 5.1.** For every triangle \( \{z_1, z_2, z_3\} \), there is only one integral element \( c_i = \Delta \), here \( \Delta \) is one-half the area of the triangle. This integral element is not convex. Hence, the set of period 3 orbits has empty interior.

*Proof.* Note \( \Delta_i = \Delta > 0 \). Likewise, say \( d_1 = \Delta_{02} = -\Delta < 0 \). Since rank \( C = 1 \), we have \( c_i = \Delta > 0 \). Then the convexity means that \( 0 < c_1 \leq d_1 < 0 \), which is absurd. \( \square \)

**Theorem 5.2.** For every convex quadrilateral \( \{z_i\} \), there is only one convex integral element \( c_i = d_i \), that is, the midpoints \( \bar{r}_i \) are the corners of \( \Gamma \). Hence, the set of period 4 orbits has empty interior unless there are 4 corners of \( \Gamma \) forming a parallelogram.

*Proof.* Let \( \{c_i\} \) define a convex integral element. Then rank \( C = 2 \). In particular the 3-minor in the rows 1,2,3 and columns 1,3,4 vanishes. This condition yields \( c_1 + c_3 = 0 \). By Proposition 3.4 the numbers \( \{-d_i\} \) always define an integral element, hence \( d_1 + d_3 = 0 \). By the convexity condition, \( c_i \leq d_i \). Then \( 0 = c_1 + c_3 \leq d_1 + d_3 = 0 \). Then we must have \( c_i = d_i \) for \( i = 1, 3 \). Similarly, it holds for the remaining values \( i = 2, 4 \). \( \square \)

For completeness, we note that for \( n = 4 \), all integral elements are defined by the equations
\[ c_1 + c_3 = 0, \quad c_2 + c_4 = 0, \quad c_1 c_2 + \Delta_2 \Delta_4 = \Delta_1 \Delta_3. \]
They form a smooth curve, a hyperbola, unless \( \Delta_2 \Delta_4 = \Delta_1 \Delta_3 \), which means the quadrilateral is a trapezoid.
6 The case \( n = 5 \)

We describe integral elements for \( n = 5 \).

**Proposition 6.1.** For \( n = 5 \), integral elements are defined by the equation

\[ c_1 c_2 = c_4 \Delta_2 + \Delta_1 \Delta_3 \]  

(15)

and the equations obtained from (15) by shifting indices.

**Proof.** The first three columns of the matrix \( C \) given by (8) are linearly independent. Then 
\( \text{rank } C = n - 2 = 3 \) if and only if the columns 4 and 5 are linear combinations of columns 1, 2, and 3. We include calculations for column 4 and leave the rest to the reader. Denote the coefficients of the linear combination by \( a_1, a_2, a_3 \). Then we have

\[ a_1 c_1 + a_2 \Delta_1 = 0, \quad a_1 \Delta_3 + a_2 c_2 + a_3 \Delta_2 = 0, \quad a_2 \Delta_4 + a_3 c_3 = \Delta_3, \quad a_3 \Delta_5 = c_4, \quad a_1 \Delta_5 = \Delta_1. \]

We find \( a_1 = \Delta_1 / \Delta_5, \quad a_3 = c_4 / \Delta_5, \quad a_2 = -c_1 / \Delta_5 \). Plugging them in the remaining equations, we obtain

\[ \Delta_1 \Delta_3 - c_1 c_2 + c_4 \Delta_2 = 0, \quad \Delta_3 \Delta_5 - c_3 c_4 + c_1 \Delta_4 = 0 \]

as desired. \( \square \)

For a periodic orbit in outer billiard, we can introduce a winding number if we regard the polygon as a path. Denote the interior angles by \( \alpha_i \). Then the angle from \( r_{i-1} \) to \( r_i \), that is, the exterior angle at \( z_i \) will be \( \delta_i = \pi - \alpha_i \). Then the winding number will be

\[ m = (2\pi)^{-1} \sum \delta_i. \]

A \((n, m)\)-orbit is a period \( n \) orbit with winding number \( m \). We have 
\( \sum \alpha_i = \pi(n - 2m) \), hence \( 0 < 2m < n \). In particular, for \( n = 5 \), there may be \((5, 1)\) and \((5, 2)\)-orbits.

Note that the angle from \( r_{i-1} \) to \( r_{i+1} \) is equal to \( \delta_i + \delta_{i+1} = 2\pi - \alpha_i - \alpha_{i+1} \). Recall the notation \( s_i = |r_i| \). Then we have

\[ d_i = \det(r_{i-1}, r_{i+1}) = -s_{i-1}s_{i+1} \sin(\alpha_i + \alpha_{i+1}). \]  

(16)

**Theorem 6.2.** The set of \((5, 2)\)-orbits has empty interior.

**Proof.** We show that for \((5, 2)\)-orbits, there are no convex integral elements. Let \( \{z_i\} \) be a \((5, 2)\)-orbit, and let \( \{c_i\} \) define a convex integral element. By Proposition 3.4 the numbers \( \{-d_i\} \) define an integral element, hence by (15) we obtain

\[ d_1 d_2 = -d_4 \Delta_2 + \Delta_1 \Delta_3. \]
Subtracting the later from (15) we obtain
\[ c_1c_2 - d_1d_2 = (c_4 + d_4)\Delta_2. \] (17)

Since \( \sum \alpha_i = \pi(n - 2m) = \pi \), we have \( \alpha_i + \alpha_{i+1} < \pi \). Then by (16), \( d_i < 0 \). By convexity,
\[ c_i \leq d_i < 0, \quad c_1c_2 - d_1d_2 \geq 0, \quad c_4 + d_4 < 0, \]
contradicting (17).

\[ \square \]

Remark 6.3. For (5,1) orbits our method does not work because convex integral elements do exist.

7 The case \( n = 6 \)

For even \( n \), there are open sets of period \( n \) orbits in which all the midpoints \( \bar{r}_i \) are corners of \( \Gamma \). For instance, if \( \Gamma \) is a triangle, then there is an open set of period 6 orbits.

We first describe integral elements for \( n = 6 \).

Proposition 7.1. For \( n = 6 \), integral elements are defined by the equations
\[ \Delta_5(c_1c_2 - \Delta_1\Delta_3) + \Delta_2(c_4c_5 - \Delta_4\Delta_6) = 0, \quad \Delta_5(c_1\Delta_4 - c_5\Delta_3) + c_3(c_4c_5 - \Delta_4\Delta_6) = 0, \] (18)
and the equations obtained from (18) by shifting indices.

The proof is similar to that of Proposition 6.1 and we leave it to the reader.

For \( n = 6 \), there may be (6,1) and (6,2)-orbits. We call a (6,2)-orbit paradoxical if \( \alpha_{i-1} + \alpha_i > \pi \) and \( \alpha_{i+1} + \alpha_i > \pi \) for some \( i \). The author does not know whether paradoxical orbits exist for a convex curve \( \Gamma \).

Theorem 7.2. The set of non-paradoxical (6,2)-orbits has empty interior unless there is an orbit \( \{z_i\} \) so that the midpoints \( \bar{r}_i \) are the corners of \( \Gamma \).

Proof. We show that for non-paradoxical (6,2)-orbits, there are no convex integral elements except when all the midpoints \( \bar{r}_i \) are the corners of \( \Gamma \). Let \( \{z_i\} \) be such an orbit, and let \( \{c_i\} \) define a convex integral element. We use only the first equation in (18), which we rewrite in the form
\[ \Delta_5c_1c_2 + \Delta_2c_4c_5 = \Delta_1\Delta_3\Delta_5 + \Delta_2\Delta_4\Delta_6. \] (19)

By Proposition 3.4, the numbers \( \{-d_i\} \) also define an integral element, hence by (19) we obtain
\[ \Delta_5d_1d_2 + \Delta_2d_4d_5 = \Delta_1\Delta_3\Delta_5 + \Delta_2\Delta_4\Delta_6. \]
Subtracting the later from (19) we obtain
\[ \Delta_5(c_1c_2 - d_1d_2) + \Delta_2(c_4c_5 - d_4d_5) = 0. \]  
(20)

Note that for a (6,2)-orbit, \( \sum \alpha_i = 2\pi \). By (16), \( d_i > 0 \) exactly when \( \alpha_i + \alpha_{i+1} > \pi \). If two of the numbers \( d_i \) are positive, then they must be consecutive. Since the orbit is non-paradoxical, they cannot be consecutive either. Hence at most one of these numbers may be positive. For definiteness, let it be \( d_6 \).

By convexity, for all \( i \neq 6, c_i \leq d_i \leq 0 \), hence \( c_1c_2 - d_1d_2 \geq 0 \) and \( c_4c_5 - d_4d_5 \geq 0 \). If \( \Gamma \) is smooth, then these inequalities are strict, contradicting (20), and we stop here.

In the general case, we have
\[ c_1c_2 = d_1d_2, \quad c_4c_5 = d_4d_5. \]

We first consider the case in which indeed \( d_6 > 0 \). Then since \( \sum \alpha_i = 2\pi \), we have \( d_2 < 0 \) and \( d_4 < 0 \). Then we have \( c_i = d_i \) for \( i = 1, 2, 4, 5 \). We show that it also holds for \( i = 3, 6 \).

By shifting indices in (19), we obtain
\[ \Delta_6c_2c_3 + \Delta_3c_5c_6 = \Delta_1c_3c_4 + \Delta_4c_6c_1 = \Delta_1\Delta_3\Delta_5 + \Delta_2\Delta_4\Delta_6. \]

Since these equations hold for \( d_i \) in place of \( c_i \), and already \( c_i = d_i \) for \( i = 1, 2, 4, 5 \), we obtain
\[ \Delta_6d_2(c_3 - d_3) + \Delta_3d_5(c_6 - d_6) = \Delta_1d_4(c_3 - d_3) + \Delta_4d_1(c_6 - d_6) = 0, \]
in which all terms are non-negative. Since \( d_2 < 0 \), we have \( c_3 = d_3 \). If \( c_6 \neq d_6 \), then \( d_1 = d_5 = 0 \), which contradicts \( \sum \alpha_i = 2\pi \). Hence \( c_i = d_i \) for all \( i \), as desired.

Finally, we consider the case in which all \( d_i \leq 0 \), including \( d_6 \). Then arguing as above, we have \( c_ic_{i+1} = d_id_{i+1} \) for all \( i \). If some \( d_i = 0 \), then there are at most two of them, and they must be consecutive. Then again \( c_i = d_i \) for all \( i \), as desired. \( \square \)

**Remark 7.3.** For (6,1) orbits our method does not work because there exist convex integral elements with \( c_i \neq d_i \).

**References**

[1] Yu. Baryshnikov and V. Zharnitsky, Sub-Riemannian geometry and periodic orbits in classical billiards. Math. Res. Lett. 13 (2006), 587–598.

[2] Daniel Genin and Serge Tabachnikov, On configuration spaces of plane polygons, sub-Riemannian geometry and periodic orbits of outer billiards. J. Mod. Dyn. 1 (2007), 155–173.
[3] A. Glutsyuk and Yu. Kudryashov, No planar billiard possesses an open set of quadrilateral trajectories. J. Mod. Dyn. 6 (2012), 287–326.

[4] T.A. Ivey and J.M. Landsberg, Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems. Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, 2003.

[5] V. Ja. Ivrii, The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary. Funktsional. Anal. i Prilozhen, 14 (1980), 25–34.

[6] J. M. Landsberg, Exterior differential systems and billiards. Proceeding of the 7th international conference on Geometry, Integrability and Quantization, Varna, Bulgaria, 2005.

[7] J. Moser, Stable and random motions in dynamical systems. Ann. of Math. Stud. 77, Princeton University Press, Princeton, N. J., 1973.

[8] M. Rychlik, Periodic orbits of the billiard ball map in a convex domain. J. Diff. Geom. 30 (1989), 191–205.

[9] R. E. Schwartz, Outer Billiards on Kites. Annals of Mathematics Studies, 171, Princeton University Press, Princeton, NJ, 2009.

[10] L. Stojanov, Note on the periodic points of the billiard, J. Diff. Geom. 34 (1991), 835–837.

[11] S. Tabachnikov, Billiards. Panor. Synth. No. 1 (1995), vi+142 pp.

[12] A. Tumanov and V. Zharnitsky, Periodic orbits in outer billiards. Int. Math. Res. Not. 2006, Art. ID 67089, 17 pp.

[13] Ya. Vorobets, On the measure of the set of periodic points of a billiard. Mat. Notes 55 (1994), 455–460.

[14] M. Wojtkovski, Two applications of Jacobi fields to the billiard ball problem. J. Diff. Geom. 40 (1994), 155–164.