Chevalley’s restriction theorem for reductive symmetric superpairs

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Abstract

Let \((\mathfrak{g}, \mathfrak{k})\) be a reductive symmetric superpair of even type, i.e. so that there exists an even Cartan subspace \(\mathfrak{a} \subset \mathfrak{p}\). The restriction map \(S(\mathfrak{p}^*)^k \to S(\mathfrak{a}^*)^W\) where \(W = W(\mathfrak{g}_0 : \mathfrak{a})\) is the Weyl group, is injective. We determine its image explicitly.

In particular, our theorem applies to the case of a symmetric superpair of group type, i.e. \((\mathfrak{k} \oplus \mathfrak{k}, \mathfrak{k})\) with the flip involution where \(\mathfrak{k}\) is a classical Lie superalgebra with a non-degenerate invariant even form (equivalently, a finite-dimensional contragredient Lie superalgebra). Thus, we obtain a new proof of the generalisation of Chevalley’s restriction theorem due to Sergeev and Kac, Gorelik.

For general symmetric superpairs, the invariants exhibit a new and surprising behaviour. We illustrate this phenomenon by a detailed discussion in the example \(\mathfrak{g} = \mathfrak{C}(q + 1) = \mathfrak{osp}(2|2q, \mathbb{C})\), endowed with a special involution. Here, the invariant algebra defines a singular algebraic curve.

1 Introduction

The physical motivation for the development of supermanifolds stems from quantum field theory in its functional integral formulation, which describes fermionic particles by anticommuting fields. In the 1970s, pioneering work by Berezin strongly suggested that commuting and anticommuting variables should be treated on equal footing. Several theories of supermanifolds have been advocated, among which the definition of Berezin, Kostant, and Leites is one of the most commonly used in mathematics.

Our motivation for the study of supermanifolds comes from the study of certain nonlinear \(\sigma\)-models with supersymmetry. Indeed, it is known
from the work of the third named author [Zir96] that Riemannian symmetric superspaces occur naturally in the large \( N \) limit of certain random matrix ensembles, which correspond to Cartan’s ten infinite series of symmetric spaces. In spite of their importance in physics, the mathematical theory of these superspaces is virtually non-existent. (But compare [DP07, LSZ08, Goe08].) We intend to initiate the systematic study of Riemannian symmetric superspaces, in order to obtain a good understanding of, in particular, the invariant differential operators, the spherical functions, and the related harmonic analysis. The present work lays an important foundation for this endeavour: the generalisation of Chevalley’s restriction theorem to the super setting.

To describe our results in detail, let us make our assumptions more precise. Let \( g \) be a complex Lie superalgebra with even centre such that \( g_0 \) is reductive in \( g \) and \( g \) carries an even invariant supersymmetric form. Let \( \theta \) be an involutive automorphism of \( g \), and denote by \( g = \mathfrak{k} \oplus \mathfrak{p} \) the decomposition into \( \theta \)-eigenspaces. We say that \((g, \mathfrak{k})\) is a reductive superpair, and it is of even type if there exists an even Cartan subspace \( \mathfrak{a} \subset \mathfrak{p}_0 \).

Assume that \((g, \mathfrak{k})\) is a reductive symmetric superpair of even type. Let \( \bar{\Sigma}^+_1 \) denote the set of positive roots of \( g_1 : \mathfrak{a} \) such that \( \lambda, 2\lambda \) are no roots of \( g_0 : \mathfrak{a} \). To each \( \lambda \in \bar{\Sigma}^+_1 \), one associates a set \( \mathcal{R}_\lambda \) of differential operators with rational coefficients on \( \mathfrak{a} \).

Our main results are as follows.

**Theorem (A).** Let \( I(\mathfrak{a}^*) \) be the image of the restriction map \( S(\mathfrak{p}^*)^\mathfrak{f} \to S(\mathfrak{a}^*) \) (which is injective). Then \( I(\mathfrak{a}^*) \) is the set of \( W \)-invariant polynomials on \( \mathfrak{a} \) which lie in the common domain of all operators in \( \mathcal{R}_\lambda, \lambda \in \bar{\Sigma}^+_1 \). Here, \( W \) is the Weyl group of \( g_0 : \mathfrak{a} \).

For \( \lambda \in \bar{\Sigma}^+_1 \), let \( A_\lambda \in \mathfrak{a} \) be the corresponding coroot, and denote by \( \partial(A_\lambda) \) the directional derivative operator in the direction of \( A_\lambda \). Then the image \( I(\mathfrak{a}^*) \) can be characterised in more explicit terms, as follows.

**Theorem (B).** We have \( I(\mathfrak{a}^*) = \bigcap_{\lambda \in \bar{\Sigma}^+_1} S(\mathfrak{a}^*)^W \cap I_\lambda \) where

\[
I_\lambda = \bigcap_{j=1}^{2m_{1,\lambda}} \text{dom} \lambda^{-j} \partial(A_\lambda)^j \quad \text{if} \quad \lambda(A_\lambda) = 0,
\]

and if \( \lambda(A_\lambda) \neq 0 \), then \( I_\lambda \) consists of those \( p \in \mathbb{C}[\mathfrak{a}] \) such that

\[
\partial(A_\lambda)^k p|_{\ker \lambda} = 0 \quad \text{for all odd integers} \quad k, \quad 1 \leq k \leq m_{1,\lambda} - 1.
\]

Here, \( m_{1,\lambda} \) denotes the multiplicity of \( \lambda \) in \( g_1 \) (and is an even integer).
If the symmetric pair \((g, \mathfrak{k})\) is of group type, i.e., \(g = \mathfrak{k} \oplus \mathfrak{k}\) with the flip involution, then for all \(\lambda \in \Sigma^+_1\), \(\lambda(A_\lambda) = 0\), and the multiplicity \(m_{1,\lambda} = 2\). In this case, Theorem (B) reduces to \(I(a^*) = \bigcap_{\lambda \in \Sigma^+_1} S(a^*)^W \cap \text{dom} \lambda^{-1} \partial(A_\lambda)\).

The situation where \(\lambda(A_\lambda) \neq 0\) for some \(\lambda \in \Sigma^+_1\) occurs if and only if \(g\) contains symmetric subalgebras \(s \cong C(2) = \mathfrak{osp}(2|2)\) where \(s_0 \cap \mathfrak{k} = \mathfrak{sl}(2, \mathbb{C})\). This is case for \(g = C(q + 1)\) with a special involution, and in this case, the invariant algebra \(I(a^*)\) defines the singular curve \(z^{2q+1} = w^2\) (Corollary 4.6).

Let us place our result in the context of the literature. The Theorems (A) and (B) apply to the case of classical Lie superalgebras with non-degenerate invariant even form (equivalently, finite-dimensional contragredient Lie superalgebras), considered as symmetric superspaces of group type. In this case, the result is due to Sergeev [Ser99], Kac [Kac84], and Gorelik [Gor04], and we simply furnish a new (and elementary) proof. (The results of Sergeev are also valid for basic Lie superalgebras which are not contragredient.) For some particular cases, there are earlier results by Berezin [Ber87].

Sergeev’s original proof involves case-by-case calculations. The proof by Gorelik—which carries out in detail ideas due to Kac in the context of Kac–Moody algebras—is classification-free, and uses so-called Shapovalov determinants. Moreover, the result of Kac and Gorelik actually characterises the image of the Harish-Chandra homomorphism rather than the image of the restriction map on the symmetric algebra, and is therefore more fundamental than our result.

Still in the case of symmetric superpairs of group type, Kac [Kac77a] and Santos [San99] describe the image of the restriction morphism in terms of supercharacters of certain (cohomologically) induced modules (instead of a characterisation in terms of a system of differential equations). This approach cannot carry over to the case of symmetric pairs, as is known in the even case from the work of Helgason [Hel64].

Our result also applies in the context of Riemannian symmetric superspaces, where one has an even non-degenerate \(G\)-invariant supersymmetric form on \(G/K\) whose restriction to the base \(G/K\) is Riemannian. In this setting, it is to our knowledge completely new and not covered by earlier results. We point out that a particular case was proved in the PhD thesis of Fuchs [Fuc95], in the framework of the ‘supermatrix model’, using a technique due to Berezin.

In the context of harmonic analysis of even Riemannian symmetric spaces \(G/K\), Chevalley’s restriction theorem enters crucially, since it determines the image of the Harish-Chandra homomorphism, and thereby, the spectrum of the algebra \(\mathcal{D}(G/K)\) of \(G\)-invariant differential operators on \(G/K\). It is an
important ingredient in the proof of Harish-Chandra’s integral formula for the spherical functions. In a series of forthcoming papers, we will apply our generalisation of Chevalley’s restriction theorem to obtain analogous results in the context of Riemannian symmetric superspaces.

Let us give a brief overview of the contents of our paper. We review some basic facts on root decompositions in sections 2.1-2.2. In section 2.3, we introduce our main tool in the proof of Theorem (A), a certain twisted action $u_z$ on the supersymmetric algebra $S(p)$. In section 3.1, we define the ‘radial component’ map $\gamma_z$ via the twisted action $u_z$. The proofs of Theorems (A) and (B) are contained in sections 3.2 and 3.3, respectively. The former comes down to a study of the singularities of $\gamma_z$ as a function of the semi-simple $z \in p_0$, whereas the latter consists in an elementary and explicit discussion of the radial components of certain differential operators. In sections 4.1 and 4.2, we discuss the generality of the ‘even type’ condition, and study an extreme example in some detail.

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2 Some basic facts and definitions

In this section, we mostly collect some basic facts concerning (restricted) root decompositions of Lie superalgebras, and the (super-) symmetric algebra, along with some definitions which we find useful to formulate our main results. As general references for matters super, we refer the reader to [Kos77, DM99, Kac77b, Sch79].
2.1 Roots of a basic quadratic Lie superalgebra

Definition 2.1. Let $g = g_0 \oplus g_1$ be a Lie superalgebra over $\mathbb{C}$ and $b$ a bilinear form $b$. Recall that $b$ is supersymmetric if $b(u, v) = (-1)^{|u||v|}b(v, u)$ for all homogeneous $u, v$. We shall call $(g, b)$ quadratic if $b$ is a non-degenerate, $g$-invariant, even and supersymmetric form on $g$. We shall say that $g$ is basic if $g_0$ is reductive in $g$ (i.e. $g$ is a semi-simple $g_0$-module) and $z(g) \subset g_0$ where $z(g)$ denotes the centre of $g$.

2.2. Let $(g, b)$ be a basic quadratic Lie superalgebra, and $b$ be a Cartan subalgebra of $g_0$.

As usual [Sch79, Chapter II, § 4.6], we define

$$V^\alpha = \{ x \in V \mid \exists n \in \mathbb{N} : (h - \alpha(h))^n(x) = 0 \text{ for all } h \in b \} , \quad \alpha \in b^*$$

for any $b$-module $V$. Further, the sets of even resp. odd roots for $b$ are

$$\Delta_0(g : b) = \{ \alpha \in b^* \setminus 0 \mid g_0^\alpha \neq 0 \} \quad \text{and} \quad \Delta_1(g : b) = \{ \alpha \in b^* \mid g_1^\alpha \neq 0 \} .$$

We also write $\Delta_j = \Delta_j(g : b)$. Let $\Delta = \Delta(g : b) = \Delta_0 \cup \Delta_1$. The elements of $\Delta$ are called roots. We have

$$g = b \oplus \bigoplus_{\alpha \in \Delta} g^\alpha = b \oplus \bigoplus_{\alpha \in \Delta_0} g_0^\alpha \oplus \bigoplus_{\alpha \in \Delta_1} g_1^\alpha .$$

It is obvious that $\Delta_0 = \Delta(g_0 : b)$, so in particular, it is a reduced abstract root system in its real linear span. Also, since $g_0$ is reductive in $g$, the root spaces $g_\alpha^\alpha$ are the joint eigenspaces of ad $h$, $h \in b$ (and not only generalised ones).

We collect the basic statements about $b$-roots. The results are known (e.g. [Sch79, Ben00]), so we omit their proofs.

Proposition 2.3. Let $g$ be a basic quadratic Lie superalgebra with invariant form $b$, and $b$ a Cartan subalgebra of $g_0$.

(i). For $\alpha, \beta \in \Delta \cup 0$, we have $b(g_\alpha^\alpha, g_\beta^\beta) = 0$ unless $j = k$ and $\alpha = -\beta$.

(ii). The form $b$ induces a non-degenerate pairing $g_\alpha^\alpha \times g_\beta^{-\beta} \to \mathbb{C}$. In particular, we have $\dim g_\alpha^\alpha = \dim g_\beta^{-\beta}$ and $\Delta_j = -\Delta_j$ for $j \in \mathbb{Z}/2\mathbb{Z}$.

(iii). The form $b$ is non-degenerate on $b$, so for any $\lambda \in b^*$, there exists a unique $h_\lambda \in b$ such that $b(h_\lambda, h) = \lambda(h)$ for all $h \in b$.

(iv). If $\alpha(h_\alpha) \neq 0$, $\alpha \in \Delta_1$, then $2\alpha \in \Delta_0$. In particular, $\Delta_0 \cap \Delta_1 = \emptyset$.

(v). We have $g_0^0 = z_1(g) = \{ x \in g_1 \mid [x, g] = 0 \} = 0$, so $0 \notin \Delta_1$.

(vi). All root spaces $g_\alpha^\alpha$, $\alpha \in \Delta$, $\alpha(h_\alpha) \neq 0$, are one-dimensional.
Section 2.2 Restricted roots of a reductive symmetric superpair

**Definition 2.4.** Let \((g, b)\) be a complex quadratic Lie superalgebra, and \(\theta : g \rightarrow g\) an involutive automorphism leaving the form \(b\) invariant. If \(g = \mathfrak{k} \oplus \mathfrak{p}\) is the \(\theta\)-eigenspace decomposition, then we shall call \((g, \mathfrak{k})\) a **symmetric superpair**. We shall say that \((g, \mathfrak{k})\) is **reductive** if, moreover, \(g\) is basic.

Note that for any symmetric superpair \((g, \mathfrak{k})\), \(\mathfrak{k}\) and \(\mathfrak{p}\) are \(b\)-orthogonal and non-degenerate. It is also useful to consider the form \(b_{\theta}(x, y) = b(x, \theta y)\) which is even, supersymmetric, non-degenerate and \(k\)-invariant.

Let \((g, \mathfrak{k})\) be a reductive symmetric superpair. For arbitrary subspaces \(c, d \subset g\), let \(z_g(c) = \{ d \in d | [d, c] = 0 \}\) denote the centraliser of \(c\) in \(d\). Any linear subspace \(a = z_p(a) \subset p_0\) consisting of semi-simple elements of \(g_0\) is called an **even Cartan subspace**. If an even Cartan subspace exists, then we say that \((g, \mathfrak{k})\) is of **even type**.

We state some generalities on even Cartan subspaces. These are known and straightforward to deduce from standard texts such as [Dix77, Bor98].

**Lemma 2.5.** Let \(a \subset g\) be an even Cartan subspace.

(i). \(a\) is reductive in \(g\), i.e. \(g\) is a semi-simple \(a\)-module.

(ii). \(z_{g_0}(a)\) and \(z_{g_1}(a)\) are \(b\)-non-degenerate.

(iii). \(z_{g_0}(a) = m_0 \oplus a\) and \(z_{g_1}(a) = m_1\) where \(m_i = z_{\mathfrak{g}_i}(a)\), and the sum is \(b\)-orthogonal.

(iv). \(m_0, m_1,\) and \(a\) are \(b\)-non-degenerate.

(v). There exists a \(\theta\)-stable Cartan subalgebra \(b\) of \(g_0\) containing \(a\).

2.6. Let \(\mathfrak{k}\) be a classical Lie superalgebra with a non-degenerate invariant even form \(B\) [Kac78]. Then \(\mathfrak{k}_0\) is reductive in \(\mathfrak{k}\), and \(\mathfrak{g}(\mathfrak{k})\) is even. We may define \(g = \mathfrak{k} \oplus \mathfrak{k}\), and \(b(x, y, x', y') = B(x, x') + B(y, y')\). Then \((g, b)\) is basic quadratic. The flip involution \(\theta(x, y) = (y, x)\) turns \((g, \mathfrak{k})\) into a reductive symmetric superpair (where \(\mathfrak{k}\) is, as is customary, identified with the diagonal in \(g\)). We call such a pair of **group type**.

Moreover, any Cartan subalgebra \(a\) of \(\mathfrak{k}_0\) yields an even Cartan subspace for the superpair \((g, \mathfrak{k})\). Indeed, \(p = \{(x, -x) | x \in \mathfrak{k}\}\), and the assertion follows from Proposition 2.3 (v).
2.7. In what follows, let \((\mathfrak{g}, \mathfrak{k})\) be a reductive symmetric superpair of even type, \(\mathfrak{a} \subset \mathfrak{p}\) an even Cartan subspace, and \(\mathfrak{b} \subset \mathfrak{g}_0\) a \(\theta\)-stable Cartan subalgebra containing \(\mathfrak{a}\). The involution \(\theta\) acts on \(\mathfrak{b}^*\) by \(\theta\alpha = \alpha \circ \theta\) for all \(\alpha \in \mathfrak{b}^*\). Let \(\alpha_\pm = \frac{1}{2}(1 \pm \theta)\alpha\) for all \(\alpha \in \mathfrak{b}^*\), and set

\[
\Sigma_j = \Sigma_j(\mathfrak{g} : \mathfrak{a}) = \{ \alpha_- \mid \alpha \in \Delta_j, \alpha \neq \theta\alpha \}, \quad \Sigma = \Sigma(\mathfrak{g} : \mathfrak{a}) = \Sigma_0 \cup \Sigma_1.
\]

(The union might not be disjoint.) Identifying \(\mathfrak{a}^*\) with the annihilator of \(\mathfrak{b} \cap \mathfrak{k}\) in \(\mathfrak{b}^*\), these may be considered as subsets of \(\mathfrak{a}^*\). The elements of \(\Sigma_0\), \(\Sigma_1\), and \(\Sigma\) are called even restricted roots, odd restricted roots, and restricted roots, respectively. For \(\lambda \in \Sigma\), let

\[
\Sigma_j(\lambda) = \{ \alpha \in \Delta_j \mid \lambda = \alpha_- \}, \quad \Sigma(\lambda) = \Sigma_0(\lambda) \cup \Sigma_1(\lambda).
\]

In the following lemma, observe that \(\lambda \in \Sigma_j(\lambda)\) means that \(\lambda \in \Delta_j\). We omit the simple proof, which is exactly the same as in the even case [War72, Chapter 1.1, Appendix 2, Lemma 1].

**Lemma 2.8.** Let \(\lambda \in \Sigma_j, j = 0, 1\). The map \(\alpha \mapsto -\theta\alpha\) is a fixed point free involution of \(\Sigma_j(\lambda) \setminus \lambda\). In particular, the cardinality of this set is even.

2.9. For \(\lambda \in \Sigma\), let

\[
\mathfrak{g}_{j,a}^\lambda = \{ x \in \mathfrak{g}_j \mid \forall h \in \mathfrak{a} : [h, x] = \lambda(h) \cdot x \}, \quad \mathfrak{g}_a^\lambda = \mathfrak{g}_{0,a}^\lambda \oplus \mathfrak{g}_{1,a}^\lambda,
\]

and \(m_{j,\lambda} = \dim_{\mathbb{C}} \mathfrak{g}_{j,a}^\lambda\), the even or odd multiplicity of \(\lambda\), according to whether \(j = 0\) or \(j = 1\). It is clear that

\[
\mathfrak{g}_{j,a}^\lambda = \bigoplus_{\alpha \in \Sigma_j(\lambda)} \mathfrak{g}_j^\alpha, \quad m_{j,\lambda} = \sum_{\alpha \in \Sigma_j(\lambda)} \dim_{\mathbb{C}} \mathfrak{g}_j^\alpha, \quad \text{and} \quad \mathfrak{g} = \mathfrak{z}(\mathfrak{a}) \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_a^\lambda.
\]

The following facts are certainly well-known. Lacking a reference, we give the short proof.

**Proposition 2.10.** Let \(\alpha, \beta \in \Delta, \lambda \in \Sigma, \text{ and } j, k \in \{0, 1\}\).

(i). The form \(b^\theta\) is zero on \(\mathfrak{g}_j^\alpha \times \mathfrak{g}_k^\beta\), unless \(j = k\) and \(\alpha = -\theta\beta\), in which case it gives a non-degenerate pairing.

(ii). There exists a unique \(A_\lambda \in \mathfrak{a}\) such that \(b(A_\lambda, h) = \lambda(h)\) for all \(h \in \mathfrak{a}\).

(iii). We have \(\dim_{\mathbb{C}} \mathfrak{g}_j^\alpha = \dim_{\mathbb{C}} \mathfrak{g}_j^{-\theta\alpha}\).

(iv). The subspace \(\mathfrak{g}_j(\lambda) = \mathfrak{g}_{j,a}^\lambda \oplus \mathfrak{g}_{j,a}^{-\lambda}\) is \(\theta\)-invariant and decomposes into \(\theta\)-eigenspaces as \(\mathfrak{g}_j(\lambda) = \mathfrak{t}_j^\lambda \oplus \mathfrak{p}_j^\lambda\).
(v). The odd multiplicity $m_{1,\lambda}$ is even, and $b^\theta$ defines a symplectic form on both $\mathfrak{t}_1^\lambda$ and $\mathfrak{p}_1^\lambda$.

**Proof.** The form $b^\theta$ is even, so $b^\theta(\mathfrak{g}_0, \mathfrak{g}_1) = 0$. For $x \in \mathfrak{g}_0^\alpha$, $y \in \mathfrak{g}_1^\beta$, we compute, for all $h \in \mathfrak{b}$,

$$(\alpha + \theta \beta)(h)b^\theta(x, y) = b^\theta([h, x], y) + b^\theta(x, [\theta h, y]) = b^\theta([h, x] + [x, h], y) = 0.$$ 

Hence, $b^\theta(x, y) = 0$ if $\alpha \neq -\theta \beta$. Since $b^\theta$ is non-degenerate and $\mathfrak{g}/\mathfrak{b}$ is the sum of root spaces, $b^\theta$ induces a non-degenerate pairing of $\mathfrak{g}_j^\alpha$ and $\mathfrak{g}_j^{-\theta \alpha}$. We also know already that $\mathfrak{a}$ is non-degenerate for $b^\theta$, and (i)-(iii) follow. Statement (iv) is immediate.

We have

$$\mathfrak{g}_1^\lambda / \mathfrak{g}_1^\lambda \cong \bigoplus_{\alpha \in \Sigma_j(\lambda) \setminus \lambda} \mathfrak{g}_1^\alpha.$$ 

By (iii) and Lemma 2.8 this space is even-dimensional. But $\lambda$ is a root if and only if $\lambda = -\theta \lambda$. Then $b^\theta$ defines a symplectic form on $\mathfrak{g}_1^\lambda$ by (i), and this space is even-dimensional. Thus, $m_{1,\lambda}$ is even, and again by (i), $\mathfrak{g}_{1,a}^\lambda$ is $b^\theta$-non-degenerate. It is clear that $\mathfrak{t}_1^\lambda$ and $\mathfrak{p}_1^\lambda$ are $b^\theta$-non-degenerate because $\mathfrak{g}_{1,a}^\lambda$ and $\mathfrak{g}_{1,a}^{-\lambda}$ are. Hence, we obtain assertion (v). □

**Remark 2.11.** Unlike the case of unrestricted roots, there may exist $\lambda \in \Sigma_1$ such that $2\lambda \not\in \Sigma$ but $\lambda$ is still anisotropic, i.e. $\lambda(A_\lambda) \neq 0$. Indeed, consider $\mathfrak{g} = \mathfrak{osp}(2|2, \mathbb{C}) \cong \mathfrak{sl}(2|1, \mathbb{C})$. Then $\mathfrak{g}_0 = \mathfrak{o}(2, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C}) = \mathfrak{gl}(2, \mathbb{C})$ and $\mathfrak{g}_1$ is the sum of the fundamental representation of $\mathfrak{g}_0$ and its dual.

Define the involution $\theta$ to be conjugation by the element $(\begin{smallmatrix} \sigma & 0 \\ 0 & 1 \end{smallmatrix})$ where $\sigma = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. One finds that $\mathfrak{t}_0 = \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{p}_0 = \mathfrak{a} = \mathfrak{z}(\mathfrak{g}_0)$ which is one-dimensional and non-degenerate for the supertrace form $b$. On the other hand, $\mathfrak{g}_1 = \mathfrak{g}_1(\lambda)$ is the sum of the root spaces for certain odd roots $\pm \alpha$, $\pm \theta \alpha$ which restrict to $\pm \lambda$. Clearly, there are no even roots, so $2\lambda$ is not a restricted root. Since $A_\lambda$ generates $\mathfrak{a}$, it is a $b$-anisotropic vector. We discuss this issue at some length in section 4.2.

We point out that it is also not hard to prove that any such root $\lambda$ occurs in this setup. I.e., given a reductive symmetric superpair $(\mathfrak{g}, \mathfrak{t})$, for any $\lambda \in \Sigma_1$, $2\lambda \not\in \Sigma$, $\lambda(A_\lambda) \neq 0$, there exists a $b$-non-degenerate $\theta$-invariant subalgebra $\mathfrak{s} \cong \mathfrak{osp}(2|2, \mathbb{C})$ such that $\mathfrak{p} \cap \mathfrak{s}_0 = \mathbb{C} A_\lambda = \mathfrak{z}(\mathfrak{s}_0)$ (the centre of $\mathfrak{s}_0$), and $\dim \mathfrak{s} \cap \mathfrak{g}_1(\lambda) = 4$.

This phenomenon, of course, cannot occur if the symmetric superpair $(\mathfrak{g}, \mathfrak{t})$ is of group type. This reflects the fact that the conditions characterising the invariant algebra may be different in the general case than one might
expect from the knowledge of the group case (i.e. the theorems of Sergeev and Kac, Gorelik).

2.3 The twisted action on the supersymmetric algebra

2.12. Let $V = V_0 \oplus V_1$ be a finite-dimensional super-vector space over $\mathbb{C}$. We define the supersymmetric algebra $S(V) = S(V_0) \otimes \Lambda(V_1)$. It is $\mathbb{Z}$-graded by total degree, as follows: $S^{k, \text{tot}}(V) = \bigoplus_{p+q=k} S^p(V_0) \otimes \Lambda^q(V_1)$. This grading is not compatible with the $\mathbb{Z}_2$-grading, but will of be of use to us nonetheless.

Let $U$ be another finite-dimensional super-vector space, and moreover, let $b : U \times V \to \mathbb{C}$ be a bilinear form. Then $b$ extends to a bilinear form $S(U) \times S(V) \to \mathbb{C}$: It is defined on linear generators by

$$b(x_1 \cdots x_m, y_1 \cdots y_n) = \delta_{mn} \sum_{\sigma \in S_n} \alpha^\sigma_{x_1, \ldots, x_n} \cdot b(x_{\sigma(1)}, y_1) \cdots b(x_{\sigma(n)}, y_n)$$

for all $x_1, \ldots, x_m \in U$, $y_1, \ldots, y_n \in V$ where $\alpha = \alpha^\sigma_{x_1, \ldots, x_n} = \pm 1$ is determined by the requirement that $\alpha \cdot x_{\sigma(1)} \cdots x_{\sigma(n)} = x_1 \cdots x_n$ in $S(V)$. If $b$ is even (resp. odd, resp. non-degenerate), then so is its extension. Here, recall that a bilinear form has degree $i$ if $b(V_j, V_k) = 0$ whenever $i + j + k \equiv 1 \, (2)$.

In particular, the natural pairing of $V$ and $V^*$ extends to a non-degenerate even pairing $\langle \cdot, \cdot \rangle$ of $S(V)$ and $S(V^*)$. By this token, $S(V)$ embeds injectively as a subsuperspace in $\hat{S}(V) = S(V^*)$. Its image coincides with the graded dual $S(V^*)^{\text{tr}}$ whose elements are the linear forms vanishing on $S^{k, \text{tot}}(V^*)$ for $k \gg 1$.

We define a superalgebra homomorphism $\partial : S(V) \to \text{End}(\hat{S}(V^*))$ by

$$\langle p, \partial(q) \pi \rangle = \langle pq, \pi \rangle$$

for all $p, q \in S(V)$, $\pi \in S(V)^*$

where $\hat{S}(V^*) = S(V^*)$. Clearly, $\partial(q)$ leaves $S(V^*)$ invariant.

2.13. If $U$ is an even finite-dimensional vector space over $\mathbb{C}$, then we have the well-known isomorphism $S(U^*) \cong \mathbb{C}[U]$ as algebras, where $\mathbb{C}[U]$ is the set of polynomial mappings $U \to \mathbb{C}$. We recall that the isomorphism can be written down as follows.

The pairing $\langle \cdot, \cdot \rangle$ of $S(U)$ and $S(U^*)$ extends to $\hat{S}(U) \times S(U^*)$. For any $d \in S(U)$, the exponential $e^d = \sum_{n=0}^{\infty} \frac{d^n}{n!}$ makes sense as an element of the algebra $\hat{S}(U) = \prod_{n=0}^{\infty} S^n(U)$. Now, define a map $S(U^*) \to \mathbb{C}[U] : p \mapsto P$ by

$$P(z) = \langle e^z, p \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle z^n, p \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle 1, \partial(z)^n p \rangle$$

Observe

$$\frac{d}{dt} P(z_0 + tz) \big|_{t=0} = \frac{d}{dt} \langle e^{tz} e^{z_0}, p \rangle \big|_{t=0} = \langle ze^{z_0}, p \rangle.$$
Iterating this formula, we obtain $\langle z_1 \cdots z_n, p \rangle$ for any $z_j \in U$ as a repeated directional derivative of $P$, and the map is injective. Since it preserves the grading by total degree, it is bijective because of identities of dimension in every degree.

2.14. Let $V = V_0 \oplus V_1$ be a finite-dimensional super-vector space. We apply the above to define an isomorphism $\phi : S(V^*) \to \text{Hom}_{S(V_0)}(S(V), \mathbb{C}[V_0])$. Here, $S(V_0)$ acts on $S(V)$ by left multiplication, and it acts on $\mathbb{C}[V_0]$ by natural extension of the action of $V_0$ by directional derivatives:

$$(\partial_z P)(z_0) = \frac{d}{dt} P(z_0 + tz)\bigg|_{t=0} \text{ for all } P \in \mathbb{C}[V_0], \ z, z_0 \in V_0.$$ 

The isomorphism $\phi$ is given by the following prescription for $P = \phi(p)$:

$$P(d; z) = (-1)^{|d||p|} (e^z, \partial(d)p) \text{ for all } p \in S(V^*), \ z \in V_0, \ d \in S(V).$$

Here, note that $\bar{S}(V_0) \subset \bar{S}(V)$ since $S(V_0^*)$ is a direct summand of $S(V^*)$, $S(V^*) = S(V_0^*) \oplus S(V_0^*) \wedge^+ (V_1^*)$, where $\wedge^+ = \bigoplus_{k \geq 1} \wedge^k$. Hence, $e^z$ may be considered as an element of $\bar{S}(V)$.

The map $\phi$ is an isomorphism as the composition of the isomorphisms

$$\text{Hom}_{S(V_0)}(S(V), \mathbb{C}[V_0]) \cong \text{Hom}_{S(V_0)}(S(V_0) \otimes \wedge V_1, S(V_0^*)) \cong S(V_0^*) \otimes \wedge V_1^* \cong S(V^*).$$

**Definition 2.15.** Let $(\mathfrak{g}, \mathfrak{k})$ be a reductive symmetric superpair of even type, and $\mathfrak{a} \subset \mathfrak{p}$ an even Cartan subspace. We apply the isomorphism $\phi$ for $V = \mathfrak{p}$ to define natural *restriction homomorphisms*

$$S(\mathfrak{p}^*) \to S(\mathfrak{p}_0^*): p \mapsto \bar{p} \quad \text{and} \quad S(\mathfrak{p}^*) \to S(\mathfrak{a}^*): p \mapsto \bar{p}.$$ 

Here, $\bar{p} \in S(\mathfrak{p}_0^*)$ (resp. $\bar{p} \in S(\mathfrak{a}^*)$) is defined via its associated polynomial $P \in \mathbb{C}[\mathfrak{p}_0]$ (resp. $P \in \mathbb{C}[\mathfrak{a}]$) where

$$\bar{P}(z) = P(1; z) \quad \text{and} \quad P = \phi(p).$$

This is a convention we will adhere to in all that follows.

Since $\mathfrak{p}_0$ is complemented by $\mathfrak{p}_1$ in $\mathfrak{p}$, and $\mathfrak{a}$ is complemented in $\mathfrak{p}_0$ by $\bigoplus_{\lambda \in \Sigma_0} \mathfrak{p}_0^\lambda$, we will in the sequel consider $\mathfrak{p}_0^* \subset \mathfrak{p}^*$ and $\mathfrak{a}^* \subset \mathfrak{p}_0^*$.

2.16. Let $K$ be a connected Lie group with Lie algebra $\mathfrak{k}_0$ such that the restricted adjoint representation $\text{ad} : \mathfrak{k}_0 \to \text{End}(\mathfrak{g})$ lifts to a homomorphism $\text{Ad} : K \to \text{GL}(\mathfrak{g})$. (For instance, one might take $K$ simply connected.) Then $\mathfrak{k}$ (resp. $\mathfrak{K}$) acts on $S(\mathfrak{p})$, $S(\mathfrak{p}^*)$, $\bar{S}(\mathfrak{p})$, $\bar{S}(\mathfrak{p}^*)$ by suitable extensions of $\text{ad}$ and
ad* (resp. Ad and Ad*) which we denote by the same symbols. Here, the sign convention for ad* is
\[ \langle y, \text{ad}^*(x)\rangle = \langle [y, x], \eta \rangle = -(-1)^{|x||y|}\langle \text{ad}(x)(y), \eta \rangle \]
for all \( x, y \in \mathfrak{g} \), \( \eta \in \mathfrak{g}^* \).

Let \( z \in \mathfrak{p}_0 \). We have \( e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \in \hat{S}(\mathfrak{p}) \), and this element is invertible with inverse \( e^{-z} \). Define
\[ u_z(x)d = \text{ad}(x)(de^z)e^{-z} \quad \text{for all } x \in \mathfrak{k}, \ d \in \hat{S}(\mathfrak{p}) . \]

Observe that
\[ \text{ad}(x)(e^z) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{ad}(x)(z^n) = \sum_{n=1}^{\infty} \frac{n}{n!}[x, z]z^{n-1} = [x, z]e^z , \]
because \( z \) is even. Hence,
\[ u_z(x)d = \text{ad}(x)(de^z)e^{-z} = [x, z]d + \text{ad}(x)(d) . \]

In particular, \( u_z(x) \) leaves \( S(\mathfrak{p}) \subset \hat{S}(\mathfrak{p}) \) invariant.

**Lemma 2.17.** Let \( z \in \mathfrak{p}_0 \). Then \( u_z \) defines a \( \mathfrak{k} \)-module structure on \( S(\mathfrak{p}) \), and for all \( x \in \mathfrak{k}, \ k \in K \), we have
\[ \text{Ad}(k) \circ u_z(x) = u_{\text{Ad}(k)(z)}(\text{Ad}(k)(x)) \circ \text{Ad}(k) . \]

**Proof.** We clearly have
\[ u_z(x)u_z(y)d = (\text{ad}(x)\text{ad}(y))(de^z)e^{-z} . \]

Now \( u_z \) is a \( \mathfrak{k} \)-action because ad is a homomorphism. Similarly,
\[ \text{Ad}(k)(u_z(x)d) = \text{ad}(\text{Ad}(k)(x))(\text{Ad}(k)(d))e^{\text{Ad}(k)(z)}e^{-\text{Ad}(k)(z)} = u_{\text{Ad}(k)(z)}(\text{Ad}(k)(x))\text{Ad}(k)(d) , \]

which manifestly gives the second assertion. \( \square \)

2.18. Let \( u_z \) also denote the natural extension of \( u_z \) to \( \mathfrak{U}(\mathfrak{k}) \). Then we may define an action \( \ell \) of \( \mathfrak{U}(\mathfrak{k}) \) on \( \text{Hom}_{S(\mathfrak{p}_0)}(S(\mathfrak{p}), \mathbb{C}[\mathfrak{p}_0]) \) via
\[ (\ell_vP)(d; z) = (-1)^{|v||P|}P(u_z(S(v))d; z) \]
for all \( P \in \text{Hom}_{S(\mathfrak{p}_0)}(S(\mathfrak{p}), \mathbb{C}[\mathfrak{p}_0]), \ v \in \mathfrak{U}(\mathfrak{k}), \ d \in S(\mathfrak{p}), \ z \in \mathfrak{p}_0 \). Here, we denote by \( S : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \) the unique linear map such that \( S(1) = 1 \),
\(S(x) = -x\) for all \(x \in g\), and \(S(uv) = (-1)^{|u||v|} S(v)S(u)\) for all homogeneous \(u, v \in \mathfrak{U}(g)\) (i.e. the principal anti-automorphism). Compare \([\text{Kos83}]\) for a similar definition in the context of the action of a supergroup on its algebra of superfunctions.

We also define
\[
\ell_k P)(d; z) = P(\text{Ad}(k^{-1})(d); \text{Ad}(k^{-1})(z))
\]
for all \(P \in \text{Hom}_{S(p_0)}(S(p), \mathbb{C}[p_0])\), \(k \in K\), \(d \in S(p)\), \(z \in p_0\).

**Lemma 2.19.** The map \(\ell\) (resp. \(L\)) defines on \(\text{Hom}_{S(p_0)}(S(p), \mathbb{C}[p_0])\) the structure of a module over \(k\) (resp. \(K\)) making the isomorphism \(\phi\) equivariant for \(k\) (resp. \(K\)).

**Proof.** Let \(P = \phi(p)\). Then
\[
(\ell_x P)(d; z) = -(-1)^{|x||p|} P(u_x(x)d; z) = -(-1)^{|d||p|} \langle \text{ad}(x)(e^zd), p \rangle
\]
\[
= (-1)^{|d||x+|p||} \langle e^zd, \text{ad}^*(x)(p) \rangle = \phi(\text{ad}^*(x)(p))(d; z).
\]

Similarly, we check that
\[
(L_k P)(d; z) = P(\text{Ad}(k^{-1})(d); \text{Ad}(k^{-1})(z))
\]
\[
= (-1)^{|d||p|} \langle e^{\text{Ad}(k^{-1})(z)} \text{Ad}(k^{-1})(d), p \rangle
\]
\[
= (-1)^{|d||p|} \langle \text{Ad}(k^{-1})(e^zd), p \rangle = \phi(\text{Ad}^*(k)(p))(z; d).
\]
This proves our assertion.

\[\square\]

## 3 Chevalley’s restriction theorem

### 3.1 The map \(\gamma_z\)

From now on, let \((g, \mathfrak{k})\) be a reductive symmetric superpair of even type, and let \(\mathfrak{a} \subset p_0\) be an even Cartan subspace.

**Definition 3.1.** An element \(z \in p_0\) is called **oddly regular** whenever the map \(\text{ad}(z) : \mathfrak{k}_1 \to \mathfrak{p}_1\) is surjective. Recall that \(z \in p_0\) is called **regular** if \(\dim_{\mathfrak{k}_0}(z) = \dim_{\mathfrak{a}_0}(\mathfrak{a})\). We shall call \(z\) **super-regular** if it is both regular and oddly regular.

Fix an even Cartan subspace \(\mathfrak{a}\), and let \(\Sigma\) be the set of (both odd and even) restricted roots. Let \(\Sigma^+ \subset \Sigma\) be any subset such that \(\Sigma\) is the disjoint union of \(\pm \Sigma^+\). Define \(\Sigma^\pm_j = \Sigma_j \cap \Sigma^\pm\) for \(j \in \mathbb{Z}/2\mathbb{Z}\). Let \(\bar{\Sigma}_1\) be the set of
\[ \lambda \in \Sigma_1 \text{ such that } m \lambda \notin \Sigma_0 \text{ for } m = 1, 2. \text{ Denote } \Sigma_1^+ = \Sigma_1 \cap \Sigma^+. \text{ Note that } \Pi_1 \in S(a^*)^W \text{ where } \Pi_1(h) = \prod_{\lambda \in \Sigma_1} \lambda(h), \text{ and } W \text{ is the Weyl group of } \Sigma_0. \]

By Chevalley’s restriction theorem, restriction \( S(p_0^0)^{tot} \to S(a^*)^W \) is a bijective map. Let \( \Pi_1 \) also denote the unique extension to \( S(p_0^0)^{tot} \) of \( \Pi_1 \).

**Remark 3.2.** The space \( p_0 \) contains non-semi-simple elements, and the definitions we have given above work in this generality. However, it will suffice for our purposes to consider the set of semi-simple super-regular elements in \( p_0 \), by the following reasoning.

First, the set of semi-simple elements in \( p_0 \) is Zariski dense (a linear endomorphism is semi-simple if and only if its minimal polynomial has only simple zeros). Second, the set of semi-simple elements in \( p_0 \) equals \( \text{Ad}(K)(a) \) [Hel84, Chapter III, Proposition 4.16]. Thus, given any semi-simple \( z \in p_0 \), \( z \) is oddly regular (super-regular) if and only if \( \lambda(\text{Ad}(k)(z)) \neq 0 \) for all \( \lambda \in \Sigma_1 \) (\( \lambda \in \Sigma \)), and for some (any) \( k \in K \) such that \( \text{Ad}(k)(z) \in a \). In particular, the set of super-regular elements of \( a \) is the complement of a finite union of hyperplanes. Hence, the set of semi-simple super-regular elements of \( p_0 \) is non-void and therefore Zariski dense; in particular, this holds for the set of semi-simple oddly regular elements.

**Lemma 3.3.** If \( z \in p_0 \) is semi-simple, then \( \mathfrak{t}_i = \mathfrak{z}_i(z) \oplus [z, \mathfrak{p}_i] \), and the subspaces \( \mathfrak{z}_i(z) \) and \( [z, \mathfrak{p}_i] \) are \( b \)-non-degenerate.

**Proof.** Since \( ad_z \) is a semi-simple endomorphism of \( \mathfrak{g} \) (\( \mathfrak{g} \) is a semi-simple \( g_0 \)-module and \( z \) is semi-simple), we have \( \mathfrak{g}_i = \mathfrak{z}_i(z) \oplus [z, \mathfrak{g}_i] \). Taking \( \theta \)-fixed parts, we deduce \( \mathfrak{t}_i = \mathfrak{z}_i(z) \oplus [z, \mathfrak{p}_i] \). The summands, being \( b \)-orthogonal, are non-degenerate. \( \square \)

3.4. Let \( z \in p_0 \) be semi-simple and oddly regular. Let \( \beta : S(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g}) \) be the supersymmetrisation map. Let

\[ \Gamma_z : \Lambda(p_1) \otimes S(p_0) \to S(p) : q \otimes p \mapsto u_z(\beta([z, q])) p \]

on elementary tensors and extend linearly.

**Proposition 3.5.** Let \( z \) be oddly regular and semi-simple. Then \( \Gamma_z \) is bijective, and \( \gamma_z = (\varepsilon \otimes 1) \circ \Gamma_z^{-1} : S(p) \to S(p_0) \) satisfies

\[ \gamma_{\text{Ad}(k)(z)} \circ \text{Ad}(k) = \text{Ad}(k) \circ \gamma_z \text{ for all } k \in K. \]

Here \( \varepsilon : \Lambda(p_1) \to \mathbb{C} \) is the unique unital algebra homomorphism.

Moreover, on \( S^{m,tot}(p) \), \( \Pi_1(z)^m \gamma_z \) is polynomial in \( z \), i.e. it extends to an element \( \Pi_1(\cdot)^m \gamma_z \) of the space \( \mathbb{C}[p_0] \otimes \text{Hom}(S^{m,tot}(p), S(p_0)) \).
Proof. By the assumption on $z$, $\text{ad } z : p_1 \to [z, p_1]$ is bijective. Moreover, $\Gamma_z$ respects the filtrations by total degree, and the degrees of these filtrations are equidimensional by the assumption. Hence, $\Gamma_z$ will be bijective once it is surjective. In degree zero, $\Gamma_z$ is the identity. We proceed to prove the surjectivity in higher degrees by induction.

By assumption, $\text{ad } z : [z, p_1] \to p_1$ is also bijective (since its kernel is $\mathfrak{z}_1(z) \cap [z, p_1]$, which is $0$ by Lemma 3.3). Let $y_1, \ldots, y_m \in p_1, y'_1, \ldots, y'_n \in p_0$. Let $x_j \in p_1$ such that $[[z, x_j], z] = y_j$. We find

$$\Gamma_z(x_1 \cdots x_m \otimes y'_1 \cdots y'_n) = y_1 \cdots y_m y'_1 \cdots y'_n \left( \bigoplus_{k < m + n} S^{k, \text{tot}}(p) \right),$$

so the first assertion follows by induction.

As to the covariance property, observe first that we have the identity $\text{Ad}(k)([z, p_1]) = [\text{Ad}(k)(z), p_1]$. Moreover,

$$(\text{Ad}(k) \circ \gamma_z)(\Gamma_z(v \otimes d)) = \varepsilon(v) \text{Ad}(k)(d) = \varepsilon(\text{Ad}(k)(v)) \text{Ad}(k)(d)$$

$$= \gamma_{\text{Ad}(k)(z)} (\Gamma_{\text{Ad}(k)(z)}(\text{Ad}(k)(v) \otimes \text{Ad}(k)(d)))$$

$$= \gamma_{\text{Ad}(k)(z)} (\text{Ad}(k)(\beta([z, v])) \text{Ad}(k)(d))$$

$$= \gamma_{\text{Ad}(k)(z)} (\text{Ad}(k)(\beta([z, v]))(d))$$

$$= (\gamma_{\text{Ad}(k)(z)} \circ \text{Ad}(k))(\Gamma_z(v \otimes d))$$

for all $v \in \wedge(p_1)$ and $d \in S(p_0)$, by Lemma 2.17.

To show that $\Pi_1(z)^m \gamma_z : S^{m, \text{tot}}(p) \to S(p_0)$ is given by the restriction of a polynomial function, we remark that its domain of definition—the set $U$ of semi-simple oddly regular elements in $p_0$—is Zariski dense. We need only prove that $f : U \to \text{Hom}(p_1, \mathfrak{t}_1)$, $f(z) = \Pi_1(z)(\text{ad } z)^{-1}$, is polynomial in $z$, where we consider $(\text{ad } z)^{-1} : p_1 \to [z, p_1]$ as a linear map $p_1 \to \mathfrak{t}_1$.

Thus, let $z \in p_0$ be semi-simple and oddly regular. It is contained in some even Cartan subspace $\mathfrak{a}$ (say). We have $\mathfrak{z}_1(\mathfrak{a}) = m_1 = \mathfrak{t}_1 \cap [z, p_1]$ by Lemma 3.3 and $(\mathfrak{t}_1 \cap m_1^\perp) \oplus p_1 = \bigoplus_{\lambda \in \Sigma^+_1} g^{\lambda}_{1,a}$. If $x = u + v \in g^{\lambda}_{1,a}$, and $u \in \mathfrak{t}_1, v \in p_1$, then $[z, u] = \lambda(z)v$. It follows that $\Pi_1(z)(\text{ad } z)^{-1}$ depends polynomially on $z$, proving our claim. \hfill \Box

Proposition 3.6. Let $p \in S(p^*)^\ell$. Then $P(d; z) = P(\gamma_z(d); z)$ for all oddly regular and semi-simple $z \in p_0$ and $d \in S(p)$.

Proof. Fix an oddly regular $z \in p_0$, and let $x_1, \ldots, x_n \in p_1$. By Lemma 2.19 we find for $n > 0$

$$P(\Gamma_z(x_1 \cdots x_n \otimes q); z) = (\ell(\beta([z, x_1 \cdots x_n])); z) = 0.$$
Since \( d - \gamma_z(d) \in \Gamma_z(\bigwedge^+(p_1) \otimes S(p_0)) \), where \( \bigwedge^+(p_1) \) denotes the kernel of \( \varepsilon : \bigwedge(p_1) \to \mathbb{C} \) (i.e., the set of elements without constant term), the assertion follows immediately.

**Corollary 3.7.** Let \((g, \mathfrak{t})\) be a reductive symmetric superpair of even type. The algebra homomorphism \( p \mapsto \bar{p} : I(p^*) = S(p^*)_\mathfrak{t} \to S(p_0^*)\) is injective. In particular, \( I(p^*)\) is commutative and purely even.

**Proof.** Let \( p \in I(p^*) \). Assume that \( \bar{p} = 0 \). Let \( d \in S(p) \). For all \( z \in p_0 \) which are oddly regular and semi-simple,

\[
P(d; z) = P(\gamma_z(d); z) = [\partial_{\gamma_z(d)} \bar{P}](z) = 0 ,
\]

by Proposition 3.6. It follows that \( P(d; -) = 0 \) on \( p_0 \), since it is a polynomial. Since \( d \) was arbitrary, we have established our contention.

**Remark 3.8.** The statement of the Corollary can, of course, be deduced by applying the inverse function theorem for supermanifolds, as in [Ser99, Proposition 1.1]. Nonetheless, we find it instructive to give the above proof based on the map \( \gamma_z \), as it illustrates the approach we will take to determine the image of the restriction map.

### 3.2 Proof of Theorem (A)

3.9. Let \((g, \mathfrak{t})\) be a reductive symmetric superpair of even type, and let \( \mathfrak{a} \) be an even Cartan subspace. We denote by \( \mathfrak{a}' \) the set of super-regular elements of \( \mathfrak{a} \). Let \( \mathcal{R} \) be the algebra of differential operators on \( \mathfrak{a} \) with rational coefficients which are non-singular on \( \mathfrak{a}' \). For any \( z \in \mathfrak{a}' \) and any \( D \in \mathcal{R} \), let \( D(z) \) be the local expression of \( D \) at \( z \). This is defined by the requirement that \( D(z) \) be a differential operator with constant coefficients, and

\[
(Df)(z) = (D(z)f)(z) \quad \text{for all} \quad z \in \mathfrak{a}' ,
\]

and all regular functions \( f \).

We associate to \( \Sigma \subset \mathfrak{a}^* \), the restricted root system of \( g : \mathfrak{a} \), the subset \( \mathcal{R}_\Sigma = \bigcup_{\lambda \in \Sigma^+_1} \mathcal{R}_\lambda \subset \mathcal{R} \) where

\[
\mathcal{R}_\lambda = \{ D \in \mathcal{R} \mid \exists d \in S(p_1^\lambda) : D(z) = \gamma_z(d) \text{ for all } z \in \mathfrak{a}' \} .
\]

I.e., \( \mathcal{R}_\Sigma \) consists of those differential operators which are given as radial parts of operators with constant coefficients on the \( \mathfrak{p} \)-projections \( p_1^\lambda \) of the restricted root spaces for the \( \lambda \in \Sigma_1^+ \). For any \( D \in \mathcal{R} \), let the domain \( \text{dom} D \) be the set of all \( p \in \mathbb{C}[\mathfrak{a}] \) such that \( Dp \in \mathbb{C}[\mathfrak{a}] \).
As we shall see, the image of the restriction map is the set of \(W\)-invariant polynomials in the common domain of \(\mathcal{R}_\Sigma\). We will subsequently determine \(\mathcal{R}_\Sigma\) in order to describe this common domain in more explicit terms.

**Theorem 3.10.** The restriction homomorphism \(I(p^*) \to S(a^*)\) from Definition 2.15 is a bijection onto the subspace \(I(a^*)= S(a^*)^W \cap \bigcap_{D\in \mathcal{R}_\Sigma} \text{dom } D\).

The proof of the Theorem requires a little preparation.

**Lemma 3.11.** Let \(q \in S(p_0^*)^K\), \(Q = \phi(q)\), and \(z \in p_0\) be super-regular and semi-simple. For all \(x \in \mathfrak{t}\), and \(w \in S(p)\), we have
\[
Q(\gamma_z(u_z(x)w); z) = 0 .
\]

**Proof.** There is no restriction to generality in supposing \(z \in a'\), so that \(\mathfrak{z}(z) = \mathfrak{z}(a) = m\) and \(\mathfrak{z}(z) = \mathfrak{z}(a) = m_0\). We define linear maps
\[
\gamma_z' : S(p_0) \to S(a) \quad \text{and} \quad \gamma_z'' : S(p) \to S(a)
\]
by the requirements that \(v - \gamma_z'(v) \in u_z(m_0^\perp \cap \mathfrak{k}_0)(S(p_0))\) for all \(v \in S(p_0)\) and \(w - \gamma_z''(w) \in u_z(m^\perp \cap \mathfrak{t})(S(p))\) for all \(w \in S(p)\). (That such maps exist and are uniquely defined by these properties follows in exactly the same way as for Proposition 3.5.) We remark that \([z, \mathfrak{p}_1] = \mathfrak{t} \cap m_1^\perp\) by Lemma 3.3.

Then
\[
w - \gamma_z'(\gamma_z(w)) = w - \gamma_z(w) + \gamma_z(w) - \gamma_z'(\gamma_z(w))
\]
\[
\in u_z(m_1^\perp \cap \mathfrak{t})(S(p)) + u_z(m_0^\perp \cap \mathfrak{k}_0)(S(p_0)) \subset u_z(m^\perp \cap \mathfrak{t})(S(p))
\]
for all \(w \in S(p)\), where \(m_1 = \mathfrak{z}(a)\). This shows that \(\gamma_z'' = \gamma_z' \circ \gamma_z\).

Moreover, by the \(K\)-invariance of \(q\), we have \(Q(v; z) = Q(\gamma_z'(v); z)\) for all \(v \in S(p_0)\). We infer
\[
Q(\gamma_z(u_z(x)w); z) = Q(\gamma_z'(u_z(x)w); z) = 0 \quad \text{for all } x \in m^\perp \cap \mathfrak{t}, w \in S(p)
\]
since \(u_z(x)w \in u_z(m^\perp \cap \mathfrak{t})(S(p))\) belongs to \(\ker \gamma_z''\).

Next, we need to consider the case of \(x \in m\). Then \(\text{ad}(x) : S(p) \to S(p)\) annihilates the subspace \(S(a)\), and moreover, \(\text{ad}(x)(e^z) = 0\). From this we find for all \(y \in m^\perp \cap \mathfrak{t}, d \in S(p)\)
\[
\text{ad}(x)(u_z(y)(d)) = (\text{ad}(x) \text{ad}(y)(de^z))e^{-z}
\]
\[
= (\text{ad}([x, y])(de^z))e^{-z} + (-1)^{|x||y|} \text{ad}(y) \text{ad}(x)(d)e^z e^{-z}
\]
\[
= u_z([x, y])d + (-1)^{|x||y|} u_z(y) \text{ad}(x)(d) .
\]
Since $\mathfrak{m}$ is a subalgebra and $b$ is $\mathfrak{t}$-invariant, $\mathfrak{m}^\perp \cap \mathfrak{t}$ is $\mathfrak{m}$-invariant. Hence, the above formula shows that $\ker \gamma''_z = u_z(\mathfrak{m}^\perp \cap k)(S(\mathfrak{p}))$ is $\mathrm{ad}(x)$-invariant.

By the definition of $\gamma''_z$, we find that

$$\gamma''_z(\mathrm{ad}(x)d) = \mathrm{ad}(x)\gamma''_z(d) = 0 \quad \text{for all } x \in \mathfrak{m}, \ d \in S(\mathfrak{p}) .$$

Reasoning as above, we see that

$$Q(\gamma_z(u_z(x)d); z) = Q(\gamma_z(\mathrm{ad}(x)d); z) = 0 \quad \text{for all } x \in \mathfrak{m}, \ d \in S(\mathfrak{p}) .$$

Since $\mathfrak{t} = \mathfrak{m} \oplus (\mathfrak{m}^\perp \cap \mathfrak{t})$, this proves the lemma.

Let $\mathfrak{p}'_0$ be the set of semi-simple super-regular elements in $\mathfrak{p}_0$. Recall the polynomial $\Pi_1$, and consider the localisation $\mathbb{C}[\mathfrak{p}_0][\Pi_1]$. Let $q \in S(\mathfrak{p}'_0)^K$, $Q = \phi(q)$, and define

$$P(v; z) = Q(\gamma_z(v); z) \quad \text{for all } v \in S(\mathfrak{p}) , \ z \in \mathfrak{p}'_0 .$$

By Proposition 3.5, $P \in \mathrm{Hom}(S(\mathfrak{p}), \mathbb{C}[\mathfrak{p}_0][\Pi_1])$. We remark that the $\mathfrak{t}$-action $\ell$ defined in 2.18 extends to $\mathrm{Hom}(S(\mathfrak{p}), \mathbb{C}[\mathfrak{p}_0][\Pi_1])$, by the same formula.

**Lemma 3.12.** Retain the above assumptions. Then $P$ is $S(\mathfrak{p}_0)$-linear and $\mathfrak{t}$-invariant, i.e. $P \in \mathrm{Hom}_{S(\mathfrak{p}_0)}(S(\mathfrak{p}), \mathbb{C}[\mathfrak{p}_0][\Pi_1])^{\mathfrak{t}}$.

**Proof.** By Lemma 3.11 $P$ is $\mathfrak{t}$-invariant. It remains to prove that $P$ is $S(\mathfrak{p}_0)$-linear. To that end, we first establish that $P$ is $K$-equivariant as linear map $S(\mathfrak{p}) \to \mathbb{C}[\mathfrak{p}_0][\Pi_1]$. Since $q$ is $K$-invariant,

$$P(\mathrm{Ad}(k)(v); \mathrm{Ad}(k)(z)) = Q(\gamma_{\mathrm{Ad}(k)(z)}(\mathrm{Ad}(k)(v)); \mathrm{Ad}(k)(z))$$

$$= Q(\mathrm{Ad}(k)(\gamma_z(v)); \mathrm{Ad}(k)(z))$$

$$= Q(\gamma_z(v); z) = P(v; z) .$$

Next, fix $z \in \mathfrak{p}'_0$. Then $S(\mathfrak{p}) = S(\mathfrak{p}_0) \oplus u_z(\mathfrak{z}_1^\perp \cap \mathfrak{t}_1)(S(\mathfrak{p}))$ where the second summand equals $\ker \gamma_z$. We may check the $S(\mathfrak{p}_0)$-linearity on each summand separately.

For $v \in S(\mathfrak{p}_0)$, we have $P(v; z) = Q(v; z)$, so for any $y \in \mathfrak{p}_0$

$$[\partial_y P(v; -)](z) = [\partial_y Q(v; -)](z) = Q(yv; z) = P(yv; z) .$$

We are reduced to considering $v = u_z(x)v'$ where $x \in \mathfrak{z}_1^\perp \cap \mathfrak{t}_1$ and $v' \in S(\mathfrak{p})$. We may assume w.l.o.g. $z \in \mathfrak{a}$ (since $z$ is semi-simple), so that $\mathfrak{z}_1(z) = \mathfrak{z}_1(a) = \mathfrak{m}_1$. By our assumption on $z$, $\mathfrak{p}_0 = \mathfrak{a} \oplus [\mathfrak{t}_0, z]$, and we may consider $y$ in each of the two summands separately.

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Let \( y \in \mathfrak{a} \). For sufficiently small \( t \), we have \( z + ty \in \mathfrak{a}' = \mathfrak{a} \cap \mathfrak{p}'_0 \), so that \( \mathfrak{z}_1(z + ty) = \mathfrak{m}_1 = \mathfrak{z}_1(z) \). Hence, \( \gamma_{z+ty}(u_{z+ty}(x)v') = 0 \). By the chain rule,

\[
0 = \frac{d}{dt}\gamma_{z+ty}(u_{z+ty}(x)v') \bigg|_{t=0} = d\gamma_z(y) + \gamma_z(\frac{d}{dt}u_{z+ty}(x)v' \bigg|_{t=0}) ,
\]

Since \( \frac{d}{dt}u_{z+ty}(x)v' \big|_{t=0} = [x,y]v' \), we have

\[
d\gamma_z(y) = -\gamma_z(\frac{d}{dt}u_{z+ty}(x)v' \big|_{t=0}) = \gamma_z([x,x]v') .
\]

Moreover, as operators on \( S(\mathfrak{p}) \),

\[
[y, u_z(x)] = y[x, z] + y \text{ad}(x) - [x, z]y - \text{ad}(x)y = [y, x],
\]

and thus \( yv = yu_z(x)v' \equiv [y, x]v' \) modulo \( \text{ker} \gamma_z \). We conclude

\[
d\gamma_z(v) z(y) = \gamma_z([y, x]v') = \gamma_z(yv) - y\gamma_z(v) - \gamma_z(v) = 0. \]

This shows that the image lies in \( I \mathfrak{z} \mathfrak{z}_1 \mathfrak{z}_1(x) \). Hence,

\[
[\partial_y P(v; -)](z) = Q(d\gamma_z(y) + y\gamma_z(v); z) = Q(\gamma_z(yv); z) = P(yv; z) .
\]

Now let \( y = [u, z] \) where \( u \in \mathfrak{k}_0 \). We may assume that \( u \perp \mathfrak{z}_0(z) \). Define \( k_t = \exp tu \). Then by the \( K \)-invariance of \( P \),

\[
[\partial_y P(v; -)](z) = \frac{d}{dt}P(v; \text{Ad}(k_t(z))) \bigg|_{t=0} = \frac{d}{dt}P(\text{Ad}(k_t^{-1})(v); z) \bigg|_{t=0} = -P(\text{ad}(u)(v); z) = P(yv; z) - P(u_z(v); z) = P(yv; z)
\]

where in the last step, we have used Lemma 3.11

Proof of Theorem 3.11 The restriction map is injective by Corollary 3.7 and Chevalley’s restriction theorem for \( \mathfrak{g}_0 \). By the latter, the image lies in the set of \( W \)-invariants. Let \( \bar{p} \in S(\mathfrak{a}^*) \) be the restriction of \( p \in I(\mathfrak{p}^*) \), and \( P = \phi(p) \). For any \( d \in S(\mathfrak{p}) \), and \( D \in \mathcal{R}_\Sigma \) given by \( D(z) = \gamma_z(d) \), we have by Proposition 3.6

\[
(D\bar{p})(z) = (\partial_{\gamma_z(d)} P)(z) = P(\gamma_z(d); z) = P(d; z) \quad \text{for all } z \in \mathfrak{a}' .
\]

The result is clearly polynomial in \( z \), so \( \bar{p} \in \text{dom } D \). This shows that the image of the restriction map lies in \( I(\mathfrak{a}^*) \).

Let \( r \in I(\mathfrak{a}^*) \). By Chevalley’s restriction theorem, there exists a unique \( q \in I(\mathfrak{p}_0^*) = S(\mathfrak{p}_0^*)^K \) such that \( Q(h) = R(h) \) for all \( h \in \mathfrak{a} \).

Next, recall that for \( d \in S(\mathfrak{p}) \) and \( z \in \mathfrak{p}_0' \):

\[
P(d; z) = Q(\gamma_z(d); z) .
\]
By Lemma 3.12, \( P \in \text{Hom}_{S(p_0)}(S(p), \mathbb{C}[p_0]_{\Pi_1})^\sharp \). Hence, \( P \) will define an element \( p \in I(\mathfrak{p}^*) \) by virtue of the isomorphism \( \phi \), as soon as it is clear that, as a linear map \( S(p) \to \mathbb{C}[p_0]_{\Pi_1} \), it takes its values in \( \mathbb{C}[p_0] \).

We only have to consider \( z \) in the Zariski dense set \( p'_0 \). The function \( \Pi_1(z)^k \cdot P(d; z) \) depends polynomially on \( z \), where we assume \( d \in S^{\leq k, \text{tot}}(\mathfrak{p}) \).

To prove that \( P \) has polynomial values, it will suffice (by the removable singularity theorem and the conjugacy of Cartan subspaces) to prove that \( P(d; h) \) is bounded as \( h \in \mathfrak{a}' = \mathfrak{a} \cap p'_0 \) approaches one of the hyperplanes \( \lambda^{-1}(0) \) where \( \lambda \in \Sigma^+ \) is arbitrary. Since \( r \) is \( W \)-invariant, \( r - r_0 \) (where \( r_0 \) is the constant term of \( r \)) vanishes on \( \lambda^{-1}(0) \) if a multiple of \( \lambda \) belongs to \( \Sigma^+_0 \). Such a multiple could only be \( \pm \lambda, \pm 2 \lambda \). Hence, it will suffice to consider \( \lambda \in \Sigma^+_1 \). By definition, \( 2\lambda \notin \Sigma \).

Consider \( P(d; h) \) as a map linear in \( d \), and let \( N_h = \ker P(-; h) \). Let \( d \in S^{\leq k, \text{tot}}(\mathfrak{p}) \). Assume that \( d = zd' \) where \( z \) is defined by \( x = y + z \), \( y \in \mathfrak{t} \), \( z \in \mathfrak{p} \), for some \( x \in g^0_\mathfrak{p} \) and \( \mu \in \Sigma^+ \), \( \mu \neq \lambda \). Then, modulo \( N_h \),

\[
d = zd' \equiv zd' + \frac{u_h(y)d'}{\mu(h)} = zd' + \frac{[y, h]d'}{\mu(h)} + \frac{\text{ad}(y)(d')}{\mu(h)} = \frac{\text{ad}(y)(d')}{\mu(h)}.
\]

The root \( \mu \) is not proportional to \( \lambda \) and the total degree of \( \text{ad}(y)(d') \) is strictly less than that of \( d \). By induction, modulo \( N_h \),

\[
d \equiv \frac{\tilde{d}}{\prod_{\mu \in \Sigma^+ \setminus \lambda} \mu(h)^k}
\]

for some \( \tilde{d} \) which lies in the subalgebra of \( S(\mathfrak{p}) \) generated by \( \mathfrak{a} \oplus p_1^{\lambda} \), and depends polynomially on \( h \) and linearly on \( d \in S^{\leq k, \text{tot}}(\mathfrak{p}) \). Hence, the problem of showing that \( P(d; h) \) remains bounded as \( h \) approaches \( \lambda^{-1}(0) \) is reduced to the case of \( d \in S(\mathfrak{a} \oplus p_1^{\lambda}) \). For \( d \in S(p_1^{\lambda}) \), the polynomiality of \( P(d; -) \) immediately follows from the assumption on \( r \). If \( d = d'd'' \) where \( d' \in S(\mathfrak{a}) \) and \( d'' \in S(p_1^{\lambda}) \), then \( P(d; z) = [\partial(d')P(d''; -)](z) \) since \( P \) is \( S(p_0) \)-linear. But \( P(d''; -) \in \mathbb{C}[p_0] \) and this space is \( S(p_0) \)-invariant, so \( P(d; -) \in \mathbb{C}[p_0] \).

Therefore, there exists \( p \in I(\mathfrak{p}^*) \) such that \( P = \phi(p) \). By its definition, it is clear that \( p \) restricts to \( r \), so we have proved the theorem. \( \square \)

### 3.3 Proof of Theorem (B)

3.13. In order to give a complete description of the image of the restriction map, we need to compute the radial parts \( \gamma_h(d) \) for \( d \in S(p_1^{\lambda}) \) and \( h \in \mathfrak{a}' \) explicitly. First, let us choose bases of the spaces \( S(p_1^{\lambda}) \).
Let $\lambda \in \Sigma_1^+$. By Proposition 2.10(v) we may choose $b^\theta$-symplectic bases $y_i, \tilde{y}_i \in \mathfrak{p}_1^\lambda$, $z_i, \tilde{z}_i \in \mathfrak{p}_1^\lambda$, $i = 1, \ldots, \frac{1}{2}m_{1,\lambda}$, $m_{1,\lambda} = \dim \mathfrak{g}_{1,a}^\lambda$. I.e.,

$$b(y_i, \tilde{y}_j) = b(\tilde{z}_j, z_i) = \delta_{ij}, b(y_i, y_j) = b(\tilde{y}_i, \tilde{y}_j) = b(z_i, z_j) = b(\tilde{z}_i, \tilde{z}_j) = 0.$$  

We may impose the conditions $x_i = y_i + z_i, \tilde{x}_i = \tilde{y}_i + \tilde{z}_i \in \mathfrak{g}_{1,a}^\lambda$, so that

$$[h, y_i] = \lambda(h)z_i, [h, \tilde{y}_i] = \lambda(h)\tilde{z}_i, [h, z_i] = \lambda(h)y_i, [h, \tilde{z}_i] = \lambda(h)\tilde{y}_i$$

for all $h \in \mathfrak{a}$. (Compare Proposition 2.10(iv).)

Given partitions $I = (i_1 < \cdots < i_k), J = (j_1 < \cdots < j_{i})$, we define monomials $z_I \tilde{z}_J = z_{i_1} \cdot \cdots \cdot z_{i_k} \tilde{z}_{j_1} \cdot \cdots \cdot \tilde{z}_{j_{i}}$ in $S(\mathfrak{p}_1^\lambda) = \bigwedge(\mathfrak{p}_1^\lambda)$. They form a basis of $S(\mathfrak{p}_1^\lambda)$.

**Lemma 3.14.** Fix $\lambda \in \Sigma_1^+$. Let $h \in \mathfrak{a}$ be oddly regular, $I, J$ be multi-indices where $k = |I|, \ell = |J|$, and let $m$ be a non-negative integer. Modulo $\ker \gamma_h$,

$$z_I \tilde{z}_J \lambda^m \equiv \begin{cases} 0 & I \neq J, \\
A^m & I = J = \emptyset, \\
(-1)^k z_I \tilde{z}_J \sum_{j=0}^{m} (-1)^j \frac{\lambda(A)^j}{\lambda(h)^{j+1}} (m)_j A^{m+1-j} & I = J = (i < I'), \end{cases}$$

where $(m)_j$ is the falling factorial $m(m-1) \cdots (m-j+1)$, and $(m)_0 = 1$.

**Proof.** For $k = \ell = 0$, there is nothing to prove. We assume that $k > 0$ or $\ell > 0$, and write $I = (i < I')$ if $k > 0$, $J = (j < J')$ if $\ell > 0$. We claim that modulo $\ker \gamma_h$,

$$z_I \tilde{z}_J \lambda^m \equiv \begin{cases} 0 & k \neq \ell \text{ or } i \neq j, \\
(-1)^k z_I \tilde{z}_J \sum_{n=0}^{m} (-1)^n \frac{\lambda(A)^m}{\lambda(h)} (m)_n A^{m+1-n} & i = j. \end{cases}$$

We argue by induction on $\max(k, \ell)$. There will also be a sub-induction on the integer $m$. First, we assume that $k > 0$, and compute

$$z_I \tilde{z}_J \lambda^m \equiv z_I z_J \tilde{z}_J \lambda^m + \frac{1}{\lambda(h)} u_h(y_i)(z_I \tilde{z}_J \lambda^m) = \frac{1}{\lambda(h)} \text{ad}(y_i)(z_I \tilde{z}_J \lambda^m).$$

For any $q$, we have

$$b([y_i, z_q], h') = -\lambda(h')b(y_i, y_q) = 0 \quad \text{for all } h' \in \mathfrak{a},$$

so $b([y_i, z_q], a) = 0$, and $[y_i, z_q] \in \mathfrak{p}_0$. Hence $[y_i, z_q] \in \mathfrak{g}_{0,\lambda}^2 \oplus \mathfrak{g}_{0,\lambda}^{-2\lambda} = 0$. Similarly, for $i \neq q$, we have $[y_i, \tilde{z}_q] = 0$. Now, assume that $i \leq J$. Then

$$z_I \tilde{z}_J \lambda^m \equiv (-1)^{k-1} \frac{1}{\lambda(h)} z_I \text{ad}(y_i)(\tilde{z}_J \lambda^m)$$

$$= (-1)^{k-1} \frac{1}{\lambda(h)}[y_i, \tilde{z}_J] \tilde{z}_J \lambda^m = -m \frac{\lambda(A)^m}{\lambda(h)} z_I \tilde{z}_J \lambda^{m+1} \quad (\ast)$$

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by standard arguments, we find, by adding equations (*), we obtain
\[ z_I \tilde{z}_J A^m_\lambda \equiv (-1)^m m! \frac{\lambda(A_\lambda)^m}{\lambda(h)^m} z_I \tilde{z}_J = (-1)^{m+k-1} m! \frac{\lambda(A_\lambda)^m}{\lambda(h)^m} [y_i, z_J] z_I \tilde{z}_J = 0. \]

Virtually the same reasoning goes through for \( \ell \). In particular, whenever \( \gamma_h (z_I \tilde{z}_J A^m_\lambda) \neq 0 \) and \( k > 0 \), then \( i < J \) implies \( \ell > 0 \) and \( i = j \).

If \( \ell > 0 \) and \( j < I \), then we observe that \( z_I \tilde{z}_J = (-1)^k \tilde{z}_I z_I \). Formally exchanging the letters \( z_s \) and \( \tilde{z}_s \) in the above equations, and reordering all terms in the appropriate fashion, we obtain
\[ z_I \tilde{z}_J A^m_\lambda \equiv (-1)^k \frac{1}{\lambda(h)} [\tilde{y}_j, z_I] z_I \tilde{z}_J A^m_\lambda - m \frac{\lambda(A_\lambda)}{\lambda(h)} z_I \tilde{z}_J A^m_\lambda - 1, \quad (**) \]

because \( k \ell + \ell - 1 + (k - 1)(\ell - 1) = k(2\ell - 1) \equiv k(2) \). Arguing as above, the right hand side of equation (**) is equivalent to 0 modulo ker \( \gamma_h \) if \( k = 0 \) or \( j < I \). Therefore, \( \gamma_h (z_I \tilde{z}_J A^m_\lambda) \) vanishes unless \( k, \ell > 0 \) and \( i = j \).

We consider the case of \( k, \ell > 0 \) and \( i = j \). Since \([y_i, \tilde{z}_J] - [\tilde{y}_i, z_I] = -2A_\lambda \)
by standard arguments, we find, by adding equations (*) and (**),
\[ z_I \tilde{z}_J A^m_\lambda \equiv (-1)^k \frac{1}{\lambda(h)} [\tilde{y}_j, z_I] z_I \tilde{z}_J A^m_\lambda + m \frac{\lambda(A_\lambda)}{\lambda(h)} z_I \tilde{z}_J A^m_\lambda - 1. \]

We may now apply this formula recursively to the second summand, to conclude
\[ z_I \tilde{z}_J A^m_\lambda \equiv (-1)^k z_I \tilde{z}_J A^m_\lambda + m \frac{\lambda(A_\lambda)}{\lambda(h)} z_I \tilde{z}_J A^m_\lambda - 1. \]

By induction on \( \max(k, \ell) \), the right hand side belongs to ker \( \gamma_h \) unless \( k = \ell \). We have proved our claim, and thus, we arrive at the assertion of the lemma.

3.15. Fix \( \lambda \in \Sigma_1^+ \) and \( h \in a' \). Let \( I = (i_1 < \cdots < i_k) \) and \( 1 < \ell < k \). Set \( I' = (i_{\ell+1} < \cdots < i_k) \). Let
\[ e^k_{\ell} = (-1)^{\sum_{j=\ell}^k \delta} = (-1)^{\sum_{j=\ell}^k \delta}. \]

We claim that there are \( b_{s\ell} \in \mathbb{N} \), \( s < \ell \), \( b_{01} = 1 \), such that, modulo ker \( \gamma_h \),
\[ z_I \tilde{z}_I \equiv e^k_{\ell} z_I \tilde{z}_I \sum_{j=0}^{\ell-1} b_{s\ell} (-\lambda(A_\lambda))^{j} A^j_{\lambda}. \quad (***). \]
The case \( \ell = 1 \) has already been established. To prove the inductive step, let \( I'' = (i_{\ell}, \ldots, i_k) = (i_{\ell} < I') \), and \( J = (i_0 < I) \). We compute
\[
\begin{align*}
  z_J \tilde{z}_I &
  \equiv \varepsilon_\ell^{k+1} z_{I''} \tilde{z}_{I''} \sum_{j=0}^{(-1)} b_j t_j \frac{(-\lambda(A_\lambda))}{\lambda(h)^{j+1}} A_\lambda^{\ell-j} \\
  &
  \equiv (-1)^{k-\ell+1} \varepsilon_\ell^{\ell+1} z_{I''} \tilde{z}_{I''} \sum_{j=0}^{\ell} \sum_{s=0}^{\min(s,\ell-1)} (\ell - j) s_j b_j t_j \frac{(-\lambda(A_\lambda))}{\lambda(h)^{j+1+s}} A_\lambda^{\ell+1-s},
\end{align*}
\]
so
\[
b_{s,\ell+1} = \sum_{j=0}^{\min(s,\ell-1)} (\ell - j) s_j b_j t_j = \frac{1}{(\ell-s)!} \sum_{j=0}^{\min(s,\ell-1)} (\ell - j) s_j b_j t_j.
\]
This proves our claim, where the constants \( b_{s,\ell} \) obey the recursion relation set out above.

To solve this recursion, we claim that
\[
b_{s,\ell} = \frac{(\ell - 1 + s)!}{2^s(\ell - 1 - s)!} \quad \text{for all} \quad 0 \leq s < \ell.
\]
This is certainly the case for \( \ell = 1 \). By induction, for all \( 0 \leq s \leq \ell, \ell \geq 1 \),
\[
b_{s,\ell+1} = \frac{1}{(\ell-s)!} \sum_{j=0}^{\min(\ell, s-1)} (\ell - j) \frac{(-1+\ell)^s}{2^j}.
\]
As is easy to show by induction, \( \sum_{j=0}^{N}(\ell - j) \frac{(-1+\ell)^s}{2^j} = \frac{(\ell+N)!}{2^N N!} \). Hence,
\[
b_{s,\ell+1} = \left\{ \begin{array}{ll}
  \frac{(\ell-s)!}{2^s(\ell-s)!} & 0 \leq s < \ell, \\
  \frac{2^s \ell^s}{(2\ell)^s} (2\ell-1)! & s = \ell
\end{array} \right.
\]
which establishes the claim.

Setting \( \ell = k = |I| \) in (***), we obtain the following lemma.

**Lemma 3.16.** Fix \( \lambda \in \Sigma_1^+ \). Let \( h \in a \) be oddly regular, \( I \) be a multi-index where \( k = |I| \). Then
\[
\gamma_h(z_I \tilde{z}_I) = (-1)^{\frac{k(k+1)}{2}} \sum_{j=0}^{k-1} \frac{(k-1+j)!}{2^j (k-1-j)!} \frac{(-\lambda(A_\lambda))}{\lambda(h)^{k+j}} A_\lambda^{k-j}.
\]

**Remark 3.17.** In passing, note that \( b_{k-2k} = b_{k-1k} = \frac{(2k-2)!}{2^k (k-1)!} \). We remark also that \( \theta_n(z) = \sum_{j=0}^{n} b_{j,n+1} z^{n-j} \) are so-called Bessel polynomials [Gro78, Slc09, A001498].

3.18. Let \( \lambda \in \Sigma_1^+, \lambda(A_\lambda) = 0 \). By Lemma 3.16, we find for all \( I, |I| = k \), that \( \gamma_h(z_I \tilde{z}_I) = (-1)^{\frac{k(k+1)}{2}} \lambda(h)^{-k} A_\lambda^k \). Hence,
\[
\bigcap_{D \in \mathcal{R}_\lambda} \text{dom } D = \bigcap_{k=1}^{m_{1,\lambda}} \text{dom } \lambda^{-k} o(A_\lambda)^k.
\]

The situation in the case \( \lambda(A_\lambda) \neq 0 \) is different and requires a more detailed study.
3.19. Let $\lambda \in \Sigma_1^+$, $\lambda(A_\lambda) \neq 0$. Then $\mathbb{C}[\lambda] \cong R[\lambda]$ where $R = \mathbb{C}[\text{ker } \lambda]$. This isomorphism is equivariant for $S(\mathbb{C}A_\lambda)$ if we define an action $\partial$ on $R[\lambda]$ by requiring that $\partial(A_\lambda)$ be the unique $R$-derivation for which $\partial(A_\lambda)\lambda = \lambda(A_\lambda)$.

Now, let $R$ be an arbitrary commutative unital $\mathbb{C}$-algebra. We define an action $\partial$ of $S(\mathbb{C}A_\lambda)$ on $R[\lambda, \lambda^{-1}]$ by requiring that $\partial(A_\lambda)$ be the unique $R$-derivation such that $\partial(A_\lambda) = \lambda(A_\lambda)$ and $\partial(A_\lambda)\lambda^{-1} = -\lambda(A_\lambda)\lambda^{-2}$. The action $\partial$ is faithful, because $\lambda(A_\lambda) \neq 0$.

Let $D_\lambda$ be the subalgebra of $\text{End}_\mathbb{C}(R[\lambda, \lambda^{-1}])$ generated by $\partial(S(\mathbb{C}A_\lambda))$ and $\mathbb{C}[\lambda, \lambda^{-1}]$. In particular, we may embed $\mathcal{R}_\lambda \subset D_\lambda$. We consider the action of $D \in \mathcal{R}_\lambda$, $D(h) = \gamma_h(z_i z_j)$, $|I| = k$, on $p = \sum_{j=0}^n a_j \lambda_j \in R[\lambda]$, $Dp = (-1)^{k(k+1)}s \sum_{j=1}^n a_j \lambda(A_\lambda)^{k+1} \lambda^{-2k} \sum_{i=(k-j)+}^{k-1} (-1)^i (j)_{k-j} b_{ik} \in R[\lambda, \lambda^{-1}]$.

Since $\lambda(A_\lambda) \neq 0$, we have $Dp \in R[\lambda]$ if and only if 
\[ a_j \sum_{i=(k-j)+}^{k-1} (-1)^i (j)_{k-j} b_{ik} = 0 \quad \text{for all } j = 1, \ldots, 2k - 1. \]

We need to determine when the number 
\[ a_j = \sum_{i=(k-j)+}^{k-1} (-1)^i (j)_{k-j} b_{ik} = \sum_{i=(k-j)+}^{k-1} \left( \frac{-1}{2} \right)^i (j)_{k-j} \frac{(k-1+i)!}{(k-1-i)!i!} \] 
is non-zero.

3.20. Fix $k \geq 1$. For $x \in \mathbb{R}$ and $1 \leq j \leq k$, let 
\[ a_{2k-j,k}(x) = \sum_{i=0}^{k-1} x^i (2k-j)_{k-j} \frac{(k-1+i)!}{(k-1-i)!i!}. \]

We claim that 
\[ a_{2k-j,k}(x) = \frac{(j-1)!(2k-j)!}{(k-1)!} \sum_{\ell=0}^{j-1} \left( \begin{array}{c} k-1 \\ j-1-\ell \end{array} \right) \left( \begin{array}{c} k-1 \\ i \end{array} \right) x^\ell (1+x)^{k-1-\ell}. \] 
To that end, we rewrite 
\[ a_{2k-j,k}(x) = \frac{(j-1)!(2k-j)!}{(k-1)!} \sum_{i=0}^{k-1} \left( \begin{array}{c} k-1 \\ i \end{array} \right) \left( \begin{array}{c} k+i-1 \\ j-1 \end{array} \right) x^i. \]

Then, for fixed $x \in \mathbb{R}$, we form the generating function 
\[ f(z) = \sum_{j=1}^{\infty} z^{j-1} \sum_{i=0}^{k-1} \left( \begin{array}{c} k-1 \\ i \end{array} \right) \left( \begin{array}{c} k+i-1 \\ j-1 \end{array} \right) x^i. \]
It is easy to see
\[ f(z) = \sum_{i=0}^{k-1} \binom{k-1}{i} x^i \sum_{j=1}^{k+i} \binom{k+i-1}{j-1} z^{j-1} \]
\[ = (1 + z)^{2k-2} \sum_{i=0}^{k-1} \binom{k-1}{i} x^i \left( \frac{1}{1 + z} \right)^{k-1-i} \]
\[ = (1 + z)^{k-1}((1 + z)x + 1)^{k-1}. \]

On the other hand, we may form the generating function for the right hand side of (3.2),
\[ g(z) = \sum_{j=1}^{\infty} z^{j-1} \sum_{\ell=0}^{j-1} \binom{k-1}{\ell} \binom{k-1}{j-1-\ell} x^\ell (1 + x)^{k-1-\ell}. \]
Then
\[ g(z) = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} x^\ell (1 + x)^{k-1-\ell} \sum_{j=\ell+1}^{k-1} \binom{k-1}{j} z^j \]
\[ = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (xz)^\ell (1 + x)^{k-1-\ell} \sum_{j=0}^{k-1} \binom{k-1}{j} z^j \]
\[ = (xz + x + 1)^{k-1}(1 + z)^{k-1} = f(z). \]

Since the generating functions coincide, we have proved (3.2).

3.21. We notice that for \( k \geq 1 \) and \( j = 1, \ldots, k, \) \( k - (2k - j) = j - k \leq 0, \)
so \( a_{2k-j,k} = a_{2k-j,k}(-\frac{1}{2}) \) by (3.1). By (3.2), we obtain
\[ a_{2k-j,k} = \frac{(j-1)!(2k-j)!}{2^{k-1}(k-1)!} \sum_{\ell=0}^{j-1} (-1)^\ell \binom{k-1}{\ell} \binom{k-1}{j-1-\ell} \]
For \( j = 1, \) one gets
\[ a_{2k-1,k} = \frac{(2k-1)!}{2^{k-1}(k-1)!} \neq 0. \]
Now, let \( j = 2n \) where \( 1 \leq n \leq \lfloor \frac{k}{2} \rfloor. \) Then \( \ell \mapsto (-1)^\ell \binom{k-1}{\ell} \binom{k-1}{2n-1-\ell} \) is odd under the permutation \( \ell \mapsto 2n - 1 - \ell \) of \( \{0, \ldots, 2n - 1\}, \) so
\[ a_{jk} = 0 \quad \text{for all} \quad j = k, \ldots, 2k - 2, \ j \equiv 0 \mod{2}. \]
3.2. Next, we study the behaviour of \( a_{k-j,k} \) for \( k \geq 1 \) and \( j = 1, \ldots, k-1 \), by a similar scheme. To that end, write

\[
a_{k-j,k} = \sum_{i=j}^{k-1} \frac{(k-j)! (k-1+i)!}{(i-j)! (k-1-i)!} \left( -\frac{1}{2} \right)^i \]

\[
= \frac{(k-1+j)! (k-j)!}{(k-1)!} \sum_{i=j}^{k-1} \binom{k-1}{i} \binom{k-1+i}{k-1+j} \left( -\frac{1}{2} \right)^i .
\]

Observe that we may sum over \( i = 0, \ldots, k-1 \) since the second binomial coefficient vanishes for \( i < j \).

Now, we fix \( x \in \mathbb{R} \) and define \( f(z) = \sum_{j=1}^{k-1} a_{k-j,k}(x) z^{k+j-1} \in \mathbb{C}[z] \) where

\[
a_{k-j,k}(x) = \sum_{i=0}^{k-1} \binom{k-1}{i} \binom{k-1+i}{k-1+j} x^i .
\]

We wish to study the coefficients of the polynomial \( f(z) \). Observe that the lowest power of \( z \) occurring in \( f(z) \) is \( z^k \). Thus, we compute, modulo \( \mathbb{C}[z]_{<k} \),

\[
f(z) = \sum_{i=0}^{k-1} \binom{k-1}{i} x^i \sum_{j=1}^{k} \binom{k-1+i}{k-1+j} z^{k+j-1}
\]

\[
= \sum_{i=0}^{k-1} \binom{k-1}{i} x^i \sum_{j=k}^{k-1} \binom{k-1+i}{j} z^j
\]

\[
\equiv (1+z)^{k-1} \sum_{i=0}^{k-1} \binom{k-1}{i} (x(1+z))^i = (1+z)^{k-1}(1+x(1+z))^{k-1} .
\]

For \( j = k, \ldots, 2k-2 \), \( a_{2k-j-1,k}(x) \) is the coefficient of \( z^j \) in \( f(z) \). Since

\[
(1+z)^{k-1}(1+x(1+z))^{k-1} = \sum_{j=0}^{2k-2} z^j \sum_{i=0}^{j} \binom{k-1}{j-i} \binom{k-1+i}{k-1-i} (1+x)^{k-1-i} x^i ,
\]

we find, for \( j = k, \ldots, 2k-2 \),

\[
a_{2k-j-1,k}(x) = \sum_{i=0}^{j} \binom{k-1}{j-i} \binom{k-1}{i} (1+x)^{k-1-i} x^i
\]

\[
= (1+x)^{k-1} \sum_{i=j-k+1}^{k-1} \binom{k-1}{j-i} \binom{k-1}{i} \left( \frac{x}{1+x} \right)^i .
\]
In particular,
\[ a_{2k-j-1,k}(-\frac{1}{2}) = 2^{1-k} \sum_{i=j-k+1}^{k-1} (-1)^i \binom{k-1}{j-i} \binom{k-1}{i}. \]

Notice that the function \( i \mapsto (-1)^i \binom{k-1}{j-i} \binom{k-1}{i} \) has parity \( j \) with respect to the permutation \( i \mapsto j - i \) of \( \{j-k+1, \ldots, k-1\} \). Since \( 2k-j-1 \) is even and only if \( j \) is odd, this implies
\[ a_{jk} = 0 \quad \text{for all } j = 2, \ldots, k-1, \ j \equiv 0 \ (2). \]

We summarise the above considerations in the following proposition.

**Proposition 3.23.** Let \( R \) be a commutative unital \( \mathbb{C} \)-algebra, and \( \lambda \in \bar{\Sigma}_1^+ \) such that \( \lambda(A_{\lambda}) \neq 0 \). Let \( m \geq 1 \) be an integer, and for \( k = 1, \ldots, m \), define
\[ D_k = (-1)^{\frac{k(k+1)}{2}} \sum_{j=0}^{k-1} \frac{(k-1+j)!}{2^{k-1-j}j!} (-\lambda(A_{\lambda}))^j A_{\lambda}^{k-j} \in D_\lambda. \]

Let \( p = \sum_{j=0}^{N} a_j \lambda^j \in \mathbb{R}[\lambda] \). Then \( D_k p \in \mathbb{R}[\lambda] \) for all \( k = 1, \ldots, m \) if and only \( a_j = 0 \) for all \( j = 1, \ldots, 2m-1, \ j \equiv 1 \ (2) \).

**Proof.** Let \( 1 \leq k \leq m \). We have \( a_{2k-1}a_{2k-1,k} = 0 \) and \( a_{2k-1,k} \neq 0 \), so \( a_{2k-1} = 0 \). Conversely, there are no further conditions, since \( a_{km} = 0 \) for even \( k \), \( 1 < k < 2m \). \( \square \)

3.24. To apply Proposition 3.23 to the determination of the image of the restriction map, let \( \lambda \in \bar{\Sigma}_1^+ \), \( \lambda(A_{\lambda}) \neq 0 \). Note that \( \mathbb{C}[a] = \mathbb{C}[\ker \lambda][\lambda] \). Then for all \( p \in \mathbb{C}[a] \),
\[ p = \sum_{j=0}^{\infty} (j!)^{-1} \partial(A_{\lambda})^j p|_{\ker \lambda}(\frac{\lambda}{\lambda(A_{\lambda})})^j. \]

I.e., if we take \( R = \mathbb{C}[\ker \lambda] \), then \( p = \sum_j a_j \lambda^j \) where the coefficients are given by \( a_j = \frac{1}{\lambda(A_{\lambda})^j} \partial(A_{\lambda})^j p|_{\ker \lambda} \in R. \) Also, \( \partial(A_{\lambda})^j p|_{\ker \lambda} = 0 \) for all \( i = 1, \ldots, j \) if and only if \( p \in \mathbb{C} \oplus \lambda^{j+1}\mathbb{C}[a] \). Together with Theorem 3.10 we immediately obtain our main result, as follows.

**Theorem 3.25.** The restriction homomorphism \( I(p^*) \rightarrow S(a^*) \) is a bijection onto the subspace \( I(a^*) = \bigcap_{\lambda \in \bar{\Sigma}_1^+} S(a^*)^W \cap I_{\lambda} \) where
\[ I_{\lambda} = \bigcap_{j=1}^{\frac{m+1}{2}} \text{dom } \lambda^{-j} \partial(A_{\lambda})^j \quad \text{if } \lambda(A_{\lambda}) = 0 \]
and if \( \lambda(A_{\lambda}) \neq 0 \), then \( I_{\lambda} \) consists of those \( p \in \mathbb{C}[a] \) such that
\[ \partial(A_{\lambda})^k p|_{\ker \lambda} = 0 \quad \text{for all odd integers } k, \ 1 \leq k \leq m+1, \lambda - 1. \]
4 Examples

4.1 Scope of the theory

4.1. As remarked in 2.6, Theorem 3.25 applies to a symmetric superpair of group type where $\mathfrak{k}$ is classical and carries a non-degenerate invariant even form. The assumptions are still fulfilled if we add to $\mathfrak{k}$ an even reductive ideal. Hence, $\mathfrak{k}$ may be a direct sum of a reductive Lie algebra, and copies of any of the following Lie superalgebras [Kac77b]:

- $\mathfrak{gl}(p|q, \mathbb{C})$,
- $\mathfrak{sl}(p|q, \mathbb{C})$ ($p \neq q$),
- $\mathfrak{sl}(p|p, \mathbb{C})$/C,
- $\mathfrak{osp}(p|2q, \mathbb{C})$,
- $D(1,2; \alpha), F(4), G(3)$.

As follows from Proposition 2.3 (iv), in this situation one has $\lambda(A_\lambda) = 0$ for all $\lambda \in \Sigma^+_1$.

4.2. If we take $(\mathfrak{g}, \mathfrak{k})$ to be an arbitrary reductive symmetric superpair, then the assumption of even type amounts to an additional condition.

As an example, we consider $\mathfrak{g} = \mathfrak{gl}(p + q + r + s, \mathbb{C})$, $p, q, r, s \geq 0$, where $\theta$ is given by conjugation with the diagonal matrix whose diagonal entries are the matrix blocks $1_p, -1_q, 1_r, -1_s$. Let $\mathfrak{a} \subset p_0$ be the maximal Abelian subalgebra of all matrices

$$
\begin{pmatrix}
0 & A & 0 & 0 \\
-A^t & 0 & 0 & 0 \\
0 & 0 & 0 & B \\
0 & 0 & -B^t & 0
\end{pmatrix} \in \mathbb{C}^{(p+q+r+s) \times (p+q+r+s)}
$$

where $A = (D, 0)$ or $A = (0 \, D)$ for a diagonal matrix $D \in \mathbb{C}^{\min(p,q) \times \min(p,q)}$, and similarly for $B$. Let $x_j, j = 1, \ldots, \min(p, q)$, and $y_\ell, \ell = 1, \ldots, \min(r, s)$, be the linear forms on $\mathfrak{a}$ given by the entries of the diagonal blocks of $A, B$.

Consider the $\mathfrak{a}$-module $\mathfrak{g}_1$. Then the non-zero weights are

$$
\pm(x_j \pm y_\ell) \ (2), \pm x_j \ (2|r - s|), \pm y_\ell \ (2|p - q|)
$$

with multiplicities given in parentheses [SZ08]. The sum $U \subset \mathfrak{g}_1$ of the non-zero weight spaces therefore has dimension

$$
8 \min(p,q) \min(r,s) + 4|r - s| \min(p,q) + 4|p - q| \min(r,s)
= 2((p + q)(r + s) - |p - q||r - s|).
$$

(The equation follows by applying the formula $2 \min(a, b) = a + b - |a - b|$.)
We have that $U$ is $\theta$-stable, and the action of a generic $h \in \mathfrak{a}$ induces an automorphism of $U$. Hence, we have $\dim U_\theta = \dim U_p = \frac{1}{2} \dim U$ where $U_\theta$ and $U_p$ are the projections of $U$ onto $\mathfrak{t}_1$ and $\mathfrak{p}_1$, respectively. It follows that

$$\dim \mathfrak{p}_1 = 2(ps + rq) = (p + q)(r + s) - (p - q)(r - s).$$

Hence, $\mathfrak{z}_{\mathfrak{p}_1}(\mathfrak{a}) = 0$ if and only if $(p - q)(r - s) \geq 0$, and $(\mathfrak{g}, \mathfrak{k})$ is of even type if and only if this condition holds.

We remark that in this case, the set $\tilde{\Sigma}_1^+$ consists of the weights $x_j \pm y_\ell$ (for a suitably chosen positive system). For each $\lambda \in \tilde{\Sigma}_1^+$, one has $\lambda(A_\lambda) = 0$. 

4.3. A similar example arises by restricting the involution from 4.2 to the subalgebra $\mathfrak{g} = \mathfrak{osp}(p + q|\mathbb{R}, \mathbb{C})$, where we now assume $r$ and $s$ to be even. We realise $\mathfrak{g}$ by taking the direct sum of the standard non-degenerate symmetric forms on $\mathbb{C}^p \oplus \mathbb{C}^q$, and the direct sum of the standard symplectic forms on $\mathbb{C}^r \oplus \mathbb{C}^s$.

For $k$ even, denote by $J_k \in \mathbb{C}^{k \times k}$ the matrix representing the standard symplectic form. Let $\mathfrak{a} \subset \mathfrak{p}_0$ be the maximal Abelian subalgebra of all matrices

$$
\begin{pmatrix}
0 & A & 0 & 0 \\
-A^t & 0 & 0 & 0 \\
0 & 0 & B & 0 \\
0 & 0 & J_\ell B^t J_r & 0
\end{pmatrix} \in \mathbb{C}^{(p+q+r+s) \times (p+q+r+s)}
$$

where $A = (D, 0)$ or $A = \left(\frac{D'}{0}\right)$ for a diagonal matrix $D \in \mathbb{C}^{\min(p,q) \times \min(p,q)}$, and $B = (D', 0)$ or $B = \left(\frac{D''}{0}\right)$ for a diagonal matrix $D' \in \mathbb{C}^{\min(r,s) \times \min(r,s)}$.

By restriction, we obtain the following non-zero $\mathfrak{a}$-weights in $\mathfrak{g}_1$,

$$\pm(x_j \pm y_\ell) (2), \pm x_j \left(|r - s|\right), \pm y_\ell (2|p - q|),$$

where now $j = 1, \ldots, \min(p,q)$, $\ell = 1, \ldots, \frac{1}{2}\min(r,s)$, and the multiplicities are given in parentheses $[SZ08]$.

Let $U$ be the sum of all weight spaces for non-zero weights of the $\mathfrak{a}$-module $\mathfrak{g}_1$. Then the dimension of $U$ is

$$4 \min(p,q) \min(r,s) + 2|r - s| \min(p,q) + 2|p - q| \min(r,s)$$

$$= (p + q)(r + s) - |p - q||r - s|. $$

If $U_p$ is the projection of $U$ onto $\mathfrak{p}_1$, then by the same argument as in 4.2

$$\dim U_p = \frac{1}{2} \dim U.$$

We have

$$\dim \mathfrak{p}_1 = pq + rs = \frac{1}{2} \left((p + q)(r + s) - (p - q)(r - s)\right),$$
4.2 An extremal class: $\mathfrak{g} = C(q+1) = \mathfrak{osp}(2|2q, \mathbb{C})$, $\mathfrak{k}_0 = \mathfrak{sp}(2q, \mathbb{C})$

4.4. Consider the Lie superalgebra $\mathfrak{g} = C(q+1) = \mathfrak{osp}(2|2q, \mathbb{C})$ where $q \geq 1$ is arbitrary. Let $I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{C}^{2 \times 2}$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^{2q \times 2q}$. If we realise $\mathfrak{g}$ with respect to the orthosymplectic form $I \oplus J$, it consists of the matrices

$$x = \begin{pmatrix} a & 0 & -w^t & z^t \\ 0 & -a & -w^t & z^t \\ z & z' & A & B \\ w & w' & C & -A^t \end{pmatrix}$$

where $a \in \mathbb{C}$, $z, z', w, w' \in \mathbb{C}^q$, $A, B, C = C^t \in \mathbb{C}^{q \times q}$.

The matrix $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{C}^{(2+2q) \times (2+2q)}$ represents an even automorphism of the super-vector space $\mathbb{C}^{2|2q}$, of order 2. Since $g$ leaves the orthosymplectic form invariant, $\theta(x) = gxg$ defines an involutive automorphism of $\mathfrak{g}$. Moreover, since $g^2 = 1$, the supertrace form $b(x, y) = \text{str}(xy)$ on $\mathfrak{g}$ is $\theta$-invariant. Hence, $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k} = \mathfrak{g}_0$, is a reductive symmetric superpair.

We compute

$$\theta(x) = \begin{pmatrix} -a & 0 & -w^t & z^t \\ 0 & a & -w^t & z^t \\ z' & z & A & B \\ w' & w & C & -A^t \end{pmatrix}$$

when $x \in \mathfrak{g}$ is written as above. Hence, the general elements of $\mathfrak{k}$ and $\mathfrak{p}$ are respectively of the form

$$x = \begin{pmatrix} 0 & 0 & -w^t & z^t \\ 0 & 0 & -w^t & z^t \\ z & z & A & B \\ w & w & C & -A^t \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} a & 0 & w^t & -z^t \\ 0 & -a & -w^t & z^t \\ z & -z & 0 & 0 \\ w & -w & 0 & 0 \end{pmatrix}.$$
implies in particular that $\text{j}_{p_1}(a) = 0$. Hence, $a$ is an even Cartan subspace, and $(g, \mathfrak{t})$ is of even type.

Also, there are only two restricted roots, $\pm \lambda$, where $\lambda$ maps $x \in a$ (as above) to $a$. Necessarily, $\lambda$ is odd, so $2\lambda \notin \Sigma = \{\pm \lambda\}$, and $W = W(\Sigma_0) = 1$. Since $A_\lambda$ is $b$-anisotropic, we have $\lambda(A_\lambda) \neq 0$.

Moreover, we must have $p_1 = p_1^\lambda$, and this space has dimension $2q$, so $m_{1,\lambda} = 2q$. From Theorem 3.25, we obtain the following result.

**Proposition 4.5.** Let $g = \text{osp}(2|2q, \mathbb{C})$, with the involution defined above. The image of the restriction map $S(p^*)^t \to S(a^*) = \mathbb{C}[\lambda]$ is

$$I(a^*) = \{ p = \sum_j a_j \lambda^j \mid a_{2j-1} = 0 \ \forall j = 1, \ldots, q \}.$$

In particular, the algebra $I(a^*)$ is isomorphic to the commutative unital $\mathbb{C}$-algebra defined by the generators $\lambda_2, \lambda_{2q+1}$, and the relation

$$(\lambda_2)^{2q+1} = (\lambda_{2q+1})^2.$$

**Proof.** We only need to prove the presentation of $I(a^*)$. Let $A$ be the unital commutative $\mathbb{C}$-algebra defined by the above generators and relations. It is clear that there is a surjective algebra homomorphism from $\phi : A \to I(a^*)$, defined by $\phi(\lambda_n) = \lambda^n$.

Consider on $I(a^*)$ the grading induced by $\mathbb{C}[\lambda]$. For any multiindex $\alpha = (\alpha_2, \alpha_{2q+1})$, define $\lambda_\alpha = (\lambda_2)^{\alpha_2}(\lambda_{2q+1})^{\alpha_{2q+1}}$ in the free commutative algebra $\mathbb{C}[\lambda_2, \lambda_{2q+1}]$. The latter is graded via $|\lambda_\alpha| = |\alpha| = 2\alpha_2 + (2q + 1)\alpha_{2q+1}$. The relation defining $A$ is homogeneous for this grading, so that $A$ inherits a grading.

By definition, $\phi$ respects the grading, and in fact, it is surjective in each degree of the induced filtration (and hence, in each degree of the grading). The relation of $A$ ensures that the image of $\lambda_\alpha$ in $A$, for any $\alpha$, depends only on $|\alpha|$. Hence, dim $A_j \leq 1$ for all $j$. This proves that $\phi$ is injective. \[\square\]

**Corollary 4.6.** Under the assumptions of Proposition 4.5, the algebra $I(a^*)$ defines the singular curve in $\mathbb{C}^2$ given by the equation $z^{2q+1} = w^2$.

4.7. We substantiate the above by some explicit computations. We have

$$\text{str} \left( \begin{array}{ccc} a & 0 & w^t & -z^t \\ 0 & -a & -w^t & z^t \\ z & -z & 0 & 0 \\ w & -w & 0 & 0 \end{array} \right) \left( \begin{array}{cccc} a' & 0 & w'^t & -z'^t \\ 0 & -a' & -w'^t & z'^t \\ z' & -z' & 0 & 0 \\ w' & -w' & 0 & 0 \end{array} \right) = 2aa' + 4(w^tz' - z^tw')$$
for the trace form $b$ on $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_1^\lambda$. In particular,

$$A_\lambda = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \lambda(A_\lambda) = \frac{1}{2} .$$

Setting

$$z_i = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -e_i^\nu \\ 0 & 0 & 0 & e_i^\nu \\ e_i^\nu & -e_i^\nu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \tilde{z}_i = \frac{1}{2} \begin{pmatrix} 0 & 0 & e_i^\nu & 0 \\ 0 & 0 & -e_i^\nu & 0 \\ 0 & 0 & 0 & 0 \\ e_i^\nu & -e_i^\nu & 0 & 0 \end{pmatrix} ,$$

$$y_i = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & -e_i^\nu \\ 0 & 0 & 0 & e_i^\nu \\ e_i^\nu & -e_i^\nu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \tilde{y}_i = \frac{1}{2} \begin{pmatrix} 0 & 0 & e_i^\nu & 0 \\ 0 & 0 & -e_i^\nu & 0 \\ 0 & 0 & 0 & 0 \\ e_i^\nu & -e_i^\nu & 0 & 0 \end{pmatrix} ,$$

one verifies the conditions from 3.13, namely

$$y_i, \tilde{y}_i \in \mathfrak{k}_i , \quad z_i, \tilde{z}_i \in \mathfrak{p}_1 , \quad y_i + z_i, \tilde{y}_i + \tilde{z}_i \in \mathfrak{g}_1^\lambda , \quad b(y_i, \tilde{y}_j) = b(\tilde{z}_i, z_i) = \delta_{ij} ,$$

$$b(y_i, y_j) = b(\tilde{y}_i, \tilde{y}_j) = b(z_i, z_j) = b(\tilde{z}_i, \tilde{z}_j) = 0 .$$

Then one computes

$$[y_i, z_j] = [\tilde{y}_i, \tilde{z}_j] = 0 , \quad [y_i, \tilde{z}_j] = -\delta_{ij} A_\lambda , \quad [\tilde{y}_i, z_j] = \delta_{ij} A_\lambda ,$$

$$[A_\lambda, y_i] = \frac{1}{2} z_i , \quad [A_\lambda, z_i] = \frac{1}{2} y_i , \quad [A_\lambda, \tilde{y}_i] = \frac{1}{2} \tilde{z}_i , \quad [A_\lambda, \tilde{z}_i] = \frac{1}{2} \tilde{y}_i .$$

Let $\zeta_i, \tilde{\zeta}_i, i = 1, \ldots, q$, be the basis of $\mathfrak{p}_i^*$, dual to $z_i, \tilde{z}_i, i = 1, \ldots, q$, so

$$\langle \tilde{z}_i, \zeta_j \rangle = -\langle z_i, \tilde{\zeta}_j \rangle = \delta_{ij} , \quad \langle z_i, \zeta_j \rangle = \langle \tilde{z}_i, \tilde{\zeta}_j \rangle = 0 .$$

Then $\langle z, \zeta \rangle = b(z, z_i), \quad \langle z, \tilde{\zeta} \rangle = b(z, \tilde{z}_i)$, and one has

$$\text{ad}^*(y_i) \zeta_j = \text{ad}^*(\tilde{y}_i) \tilde{\zeta}_j = 0 , \quad -\text{ad}^*(y_i) \tilde{\zeta}_j = \text{ad}^*(\tilde{y}_i) \zeta_j = \delta_{ij} \lambda ,$$

$$\text{ad}^*(y_i) \lambda = -\frac{1}{2} \tilde{\zeta}_i , \quad \text{ad}^*(\tilde{y}_i) \lambda = -\frac{1}{2} \zeta_i \lambda .$$

Also, we observe $\langle z_I \tilde{z}_J h^\nu, \zeta_K \tilde{\zeta}_L \lambda^\mu \rangle = \delta_{IL} \delta_{JK} \delta_{\nu\mu} (-1)^{|I||J|} \nu! \lambda(h)^\nu$. The preimages $p_2, p_{2q+1}$ of the generators $\lambda^2, \lambda^{2q+1}$ in $S(\mathfrak{p}^*)^\mathfrak{k}$ under the restriction map can be deduced from 3.19 because $p = \mathfrak{a} \oplus \mathfrak{p}_1^\lambda$. Indeed, let $P = \phi(p_N)$ where $N = 2$ or $N = 2q + 1$. By the formulæ from 3.19 for $q \geq |I| = k > 0$ and $h \in a'$,

$$\sum_{\nu=0}^{\infty} \frac{1}{\nu!} \langle z_I \tilde{z}_J h^\nu, p_N \rangle = P(z_I \tilde{z}_J; h) = (\partial_{\nu_h(z_I \tilde{z}_J)} \lambda^N)(h)$$

$$= \delta_{IJ} (-1)^{\frac{1}{2} k(k+1)} 2^{-k} a_{NK} \lambda(h)^{N-2k} .$$
where
\[ a_{Nk} = \sum_{i=(k-N)^+}^{k-1} \left( -\frac{1}{2} \right)^i (N)_{k-i} \frac{(k-1+i)!}{(k-i)!} \cdot \]
Thus,
\[ p_N = \lambda^N + \sum_{k=1}^{\min(N,q)} (-1)^{\frac{1}{2}k(k+3)} 2^{-k} a_{Nk} \lambda^{N-2k} \sum_{|I|=k} \zeta_I \tilde{\zeta}_I \cdot \]
When \( N = 2 \) and \( k \geq 2 \), then \( a_{Nk} = 0 \) by 3.21 and 3.22. On the other hand, \( a_{21} = 2 \). Hence,
\[ p_2 = \lambda^2 + \sum_{i=1}^{q} \zeta_i \tilde{\zeta}_i \]
is the super-Laplacian, and
\[ p_{2q+1} = \lambda^{2q+1} + \sum_{k=1}^{q} (-1)^{\frac{1}{2}k(k+3)} 2^{-k} a_{2q+1,k} \lambda^{2(q-k)+1} \sum_{|I|=k} \zeta_I \tilde{\zeta}_I \cdot \]
These elements are clearly subject to the relation \( p_{2q+1}^2 = p_{2q+1} \cdot \)
One readily checks
\[ \text{ad}^*(y_i)p_2 = -\lambda \zeta_i + \zeta_i \lambda = 0 \quad \text{and} \quad \text{ad}^*(\tilde{y}_i)p_2 = -\lambda \tilde{\zeta}_i + \lambda \tilde{\zeta}_i = 0 \cdot \]
In case \( q = 1 \), one has \( p_3 = \lambda^3 + \frac{3}{2} \lambda \zeta_1 \tilde{\zeta}_1 \), and
\[ \text{ad}^*(y_1)p_3 = -\frac{3}{2} \lambda^2 \zeta_1 - \frac{3}{2} \lambda \zeta_1 \lambda \quad \text{ad}^*(y_1)\tilde{\zeta}_1 = -\frac{3}{2} \lambda^2 \zeta_1 + \frac{3}{2} \lambda \zeta_1 \lambda = 0 \ , \]
\[ \text{ad}^*(\tilde{y}_1)p_3 = -\frac{3}{2} \lambda^2 \tilde{\zeta}_1 + \frac{3}{2} \lambda \text{ad}^*(\tilde{y}_1)(\zeta_1)\tilde{\zeta}_1 = -\frac{3}{2} \lambda^2 \tilde{\zeta}_1 + \frac{3}{2} \lambda \tilde{\zeta}_1 \lambda = 0 \ . \]
To verify the \( \mathfrak{t}_0 \)-invariance, let
\[ x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & C & -A^t \end{pmatrix} \in \mathfrak{t}_0 = \mathfrak{sp}(2q, \mathbb{C}) \ . \]
Then
\[ \text{ad}^*(x)\zeta_i = \sum_{j=1}^{q} (A_{ji} \zeta_j + C_{ji} \tilde{\zeta}_j) \quad \text{and} \quad \text{ad}^*(x)\tilde{\zeta}_i = \sum_{j=1}^{q} (B_{ji} \zeta_j + A_{ji} \tilde{\zeta}_j) \ . \]
This implies
\[ \text{ad}^*(x)(\zeta_i \tilde{\zeta}_i) = \sum_{j \neq i} (C_{ji} \tilde{\zeta}_j \tilde{\zeta}_i - B_{ji} \zeta_j \zeta_i) \ . \]
Since \( B = B^t \), \( C = C^t \), we deduce \( \sum_{i=1}^{q} \text{ad}^*(x)(\zeta_i \tilde{\zeta}_i) = 0 \). Since \( a = 3(\mathfrak{g}_0) \) and thus \( \text{ad}^*(\mathfrak{t}_0)\lambda = 0 \), this implies that \( p_2 \) (for general \( q \)) and \( p_3 \) (for \( q = 1 \)) are \( \mathfrak{t} \)-invariant.
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