Optimal resource states for local state discrimination

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Abstract
We study the problem of locally distinguishing pure quantum states using shared entanglement as a resource. For a given set of locally indistinguishable states we define a resource state to be useful if it can enhance local distinguishability and optimal if it can distinguish the states as well as global measurements and is also minimal with respect to a partial ordering defined by entanglement and dimension. We present examples of useful resources and show that an entangled state need not be useful for distinguishing a given set of states. We obtain optimal resources with explicit local protocols to distinguish multipartite GHZ and Graph states; and also show that a maximally entangled state is an optimal resource under one-way LOCC to distinguish any bipartite orthonormal basis which contains at least one entangled state of full Schmidt rank.

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I. INTRODUCTION

The paradigm of Local Operations and Classical Communication (LOCC) [1] is of central importance in quantum information theory. In a LOCC protocol, two or more distant parties perform arbitrary quantum operations on local subsystems and communicate classically, but are not allowed to exchange quantum information (qubits). Fundamental questions on quantum nonlocality, and properties of entangled states (see [2] for a review), especially those related to the notion of entanglement as a resource, are generally explored within the framework of LOCC.

LOCC protocols have limitations in that they cannot implement all quantum operations on a composite system, parts of which are spatially separated. For example, it is impossible, by LOCC, to entangle two or more quantum systems, even with nonzero probability. Shared entanglement, however, can help to overcome such limitations. The local protocols which use shared entanglement as a resource define the class of LOCCE, short for Local Operations, Classical Communication and Entanglement. These protocols, using appropriate entangled states, can enable local implementation of any quantum operation on the whole system. It is in this sense, we say that entanglement is a resource for quantum operations, e.g. quantum teleportation [3], superdense coding [4], entanglement catalysis [5], entangling measurements and unitaries [6–10]. The present paper considers a problem along these lines, namely quantum state discrimination by LOCCE. This problem has been previously explored, primarily in specific instances of bipartite systems [8, 9, 11–15], while a recent work [16] initiated a more general treatment of both bipartite and multipartite systems.

In a local state discrimination problem [11, 14, 17–36], the goal is to learn about the state of a multipartite quantum system, prepared in one of a known set of states, by LOCC measurements. In some cases, LOCC can indeed perform this task optimally, i.e. as well as global measurements. For example, any two pure states can be optimally distinguished by LOCC regardless of their dimensions, entanglement and multipartite structure [18, 19]. On the other hand, there exist states which cannot be optimally distinguished by LOCC, and such states are said to be locally indistinguishable (LI); e.g. three Bell states [20], a complete orthogonal basis where not all states are product [20, 22, 24–26], the orthogonal product bases exhibiting “nonlocality without entanglement” [11], and unextendible product bases [23]. LI states are said to exhibit a new kind of nonlocality as emphasized by many authors [11, 23, 24, 32] and imply that global information encoded in multipartite systems may not be completely accessible by local means [37, 38]. The latter has found useful applications in data hiding [39–42] and secret sharing [43].

The existence of locally indistinguishable (LI) states imply that auxiliary entanglement, shared between the parties, may be necessary for optimal discrimination of such states by LOCC. Indeed, entanglement
is necessary to perfectly distinguish any bipartite or multipartite orthonormal basis containing entangled states \[9, 20, 22, 24\]. On the other hand, with sufficiently many entangled states any set of LI states can be optimally distinguished. For example, the teleportation protocol \[8, 12, 16\] can optimally distinguish any set of LI states in \((\mathbb{C}^d)^{\otimes N}\) while consuming \((N - 1) \log d\) ebits. However, from a resource perspective the teleportation protocol in general is not optimal. For example, Cohen \[12\] presented protocols which use entanglement more efficiently than teleportation to perfectly distinguish certain classes of unextendible product bases.

The purpose of the present work is to better understand the role of entanglement, as a resource, in local state discrimination problems. We therefore focus on the characterization of resource states and also present results on multipartite state discrimination that are optimal under LOCCE.

Local fidelity \[44\], which quantifies how well a set of states can be distinguished by LOCC, plays a central role in our analysis. This is briefly reviewed in Section II vis-a-vis the problem of local state discrimination. In Section III we give a sufficiently general formulation of the problem of state discrimination under LOCCE. Although we concern ourselves only with distinguishing pure states, the formulation and much of the subsequent analysis can be easily extended to mixed states.

In Section IV we define the resource states to be useful or optimal for a given set \(S\) of locally indistinguishable states and illustrate these definitions with some general results and examples. We say that a resource state is useful if and only if it can enhance local distinguishability of the states in \(S\). In bipartite systems, we show that any pure entangled state of Schmidt rank \(r \geq 2\) is useful for distinguishing the elements of any \(S \subset \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}, 2 \leq d_1 \leq d_2\) provided \(r \geq d_1\). We also show that a resource state can be useful when \(2 \leq r < d_1\): in particular, \(m\) copies of a Bell state are shown useful to distinguish an orthonormal basis of Lattice states in \((\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n}\) for any \(1 \leq m \leq n\).

Here one is tempted to ask: For a given set \(S\), is every pure entangled state useful as a resource? We answer this question in negative. As an example, we prove that any pure state with only bipartite entanglement is not useful for distinguishing a three-qubit GHZ basis. Similar arguments show that any \(N'\)-partite state, no matter how entangled, cannot be useful for distinguishing a \(N\)-qubit GHZ basis whenever \(N' \leq N - 1\).

Next we consider the question of optimality of resource states. Let \(\mathcal{R}\) be the set of all resource states that optimally distinguishes the states in \(S\) under LOCCE. Since most states in \(\mathcal{R}\) are not optimal from a resource point of view, we give two additional conditions that an optimal resource \(|\Psi\rangle \in \mathcal{R}\) must satisfy. The first condition is that the amount of entanglement consumed is no more than what is both necessary and sufficient. This can be duly satisfied by requiring that \(E(\Psi) \leq E(\Psi')\) for any \(|\Psi'\rangle \in \mathcal{R}\) where \(E\) is a well-defined measure of entanglement. The second condition requires that the dimension
of the optimal resource space must be the smallest; i.e. \( \dim \mathcal{H}_\Psi \leq \dim \mathcal{H}_{\Psi''} \) for any \( |\Psi''\rangle \in \mathcal{R} \) satisfying \( E(\Psi'') = E(\Psi) \).

As examples, we obtain optimal resources for distinguishing GHZ and Graph states in Section V. We show that a \( m \)-qubit \( m \)-partite GHZ state is an optimal resource for distinguishing a \( N \)-qubit \( m \)-partite orthonormal GHZ basis with \( N \geq m \geq 2 \) for any partitioning of the \( N \) qubits among \( m \) parties. This result is generalized to Graph states where an optimal resource is obtained for distinguishing a basis defined for any graph \( G \) on \( N \) vertices with each party holding a qubit. Section VI considers optimal resources in one-way LOCCE for bipartite systems. Here we show that a maximally entangled state is an optimal resource for distinguishing any bipartite orthonormal basis containing an entangled state of full Schmidt rank. We conclude in Section VII with a discussion on some of the open problems.

II. PRELIMINARIES: QUANTUM STATE DISCRIMINATION AND FIDELITY

In a quantum state discrimination problem, we wish to quantify how much can be learned about a quantum system, prepared in one of a known set \( S = \{p_i, |\psi_i\rangle\} \) of pure quantum states \( |\psi_i\rangle \) occurring with probabilities \( p_i \). The average fidelity is one measure calculated with respect to a particular physical protocol and a decoding scheme, defined initially in [45]. Thus, for fixed set \( S = \{p_i, |\psi_i\rangle\} \), a measurement (POVM) \( M = \{M_a\} \), and a guessing strategy \( G: a \rightarrow |\phi_a\rangle \), the average fidelity is given by [44, 45],

\[
F(S|M, G) = \sum_{i,a} p_i \langle \psi_i | M_a | \psi_i \rangle |\langle \psi_i | \phi_a \rangle|^2
\]

This measures our ability to prepare a new quantum system in a state which is close to the original state \( |\psi_i\rangle \). It may be noted that \( 0 \leq F(S|M, G) \leq 1 \), and \( F(S|M, G) = 1 \) if and only if the procedure \( (M, G) \) identifies the given state of our system perfectly which is possible only if the states \( |\psi_i\rangle \) are orthogonal. The optimal fidelity is defined as

\[
F_{\text{opt}}(S) = \sup_{M \in \text{ALL}, G} F(S|M, G),
\]

where the optimization is over all quantum measurements and all guessing strategies. Note that the problem of finding an optimal measurement strategy is difficult in general but that for any fixed measurement \( M \), it is a straightforward calculation to calculate the optimal guessing strategy \( G \).

In LOCC state discrimination we suppose that the states \( |\psi_i\rangle \in S \) belong to a \( N \)-partite quantum system \( \mathcal{H}_S = \otimes_{i=1}^N \mathbb{C}^{d_i} \), \( N \geq 2 \) with the allowed measurements belonging to the LOCC class. If we label the parties as \( A_1, A_2, \ldots, A_N \), then the LOCC measurements are realized with respect to the partitioning \( A_1|A_2|\cdots|A_N \) unless stated otherwise. To simplify the notation, a LOCC protocol \( P \) will denote the
associated LOCC measurement $M$ and the corresponding guessing strategy $G$. Accordingly, the optimal local fidelity $F_{\text{local}}(S)$ is defined as

$$F_{\text{local}}(S) = \sup_{P \in \text{LOCC}} F(S|P) \leq F_{\text{opt}}(S),$$

(3)

where the optimization is over all LOCC protocols. We say that the states $|\psi_i\rangle$ are locally indistinguishable if and only if $F_{\text{local}}(S) < F_{\text{opt}}(S)$.

In multipartite systems with $N \geq 3$ we can often determine whether a given set of states is locally indistinguishable or not by examining local distinguishability across various bipartitions. Let $A|B$ be a bipartition where $A$ and $B$ hold $m$ and $(N - m)$ subsystems respectively. The optimal local fidelity across this bipartition is given by

$$F_{\text{local}}(S_{A|B}) = \sup_{P \in \text{LOCC}} F(S_{A|B}|P),$$

(4)

where the optimization is now over all LOCC protocols $P$ with respect to the bipartition $A|B$. Since local fidelity cannot increase by further partitioning of the subsystems, the following inequality holds:

$$F_{\text{local}}(S) \leq \min_{\{A|B\}} F_{\text{local}}(S_{A|B}),$$

(5)

where the minimum is obtained over all bipartitions. It is clear from the above inequality that the states must be locally indistinguishable if they are locally indistinguishable across at least one bipartition. However, it should be noted that a set of states can be locally indistinguishable even though they can be optimally distinguished by LOCC across every bipartition [23].

### III. Quantum State Discrimination by LOCCE

We now suppose that the states $|\psi_i\rangle \in S$ are locally indistinguishable, and furthermore that the parties share a resource state $|\Psi\rangle$ in addition to an unknown element of $S$. Since in multipartite systems it may not be always necessary that the resource state is shared by all, we suppose that $|\Psi\rangle$ belongs to a $N'$-partite quantum system $\mathcal{H}_\Psi = \otimes_{i=1}^{N'} \mathbb{C}^{d'_i}$, where $2 \leq N' \leq N$. Note that we do not make any assumption on the structure of the resource state; it can be either genuine multipartite entangled, e.g. a GHZ state or a tensor product of entangled states.

Observe that the task of local discrimination of the states $|\psi_i\rangle$ using a resource state $|\Psi\rangle$ is, in fact, equivalent to the task of local discrimination of the states $|\Psi\rangle \otimes |\psi_i\rangle$ where the states $|\Psi\rangle \otimes |\psi_i\rangle$ now belong to an enlarged joint Hilbert space $\mathcal{H} = \mathcal{H}_\Psi \otimes \mathcal{H}_S$. In this setting, LOCC is understood as follows: Relabel the parties who share the resource state as $A_1, A_2, \ldots, A_{N'}$, $2 \leq N' \leq N$. With this, the joint
Hilbert space can be expressed as $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$ where $\mathcal{H}_i = \mathbb{C}^{d'_i} \otimes \mathbb{C}^{d_i}$ for $i = 1, \ldots, N'$ and $\mathcal{H}_i = \mathbb{C}^{d_i}$ for $i = (N' + 1), \ldots, N$. This means that for $i = 1, \ldots, N'$ each party $A_i$ holds two quantum systems: the first system of dimension $d'_i$ is part of the shared resource state, and the second system, of dimension $d_i$ is part of the shared unknown state, and therefore, joint quantum operations are allowed on these two systems.

As before, local distinguishability of the states $|\Psi\rangle \otimes |\psi_i\rangle$ can be duly quantified by the local fidelity $F(\Psi \otimes S | P)$ for some LOCC protocol $P$. The optimal local fidelity is defined in (3) as

$$F_{\text{local}}(\Psi \otimes S) = \sup_{P \in \text{LOCC}} F(\Psi \otimes S | P) \leq F_{\text{opt}}(S).$$

Computing the local fidelity is sufficient to ensure how well a given set of states can be locally distinguished using a given resource state. We use this fact in the next section for our definition of useful resources. On the other hand, local fidelity doesn’t tell us anything about “efficient” use of entanglement in the process of local state discrimination. As one may recall, it is easy to achieve the global optimum via the teleportation protocol with $|\Psi\rangle$ being a tensor product of many bipartite maximally entangled states. In the next section, we therefore lay down the criteria an optimal resource must satisfy besides achieving the global optimum.

IV. CHARACTERIZATION OF THE RESOURCE STATES: USEFUL AND OPTIMAL RESOURCES

In this section we discuss useful and optimal resources in a state discrimination problem under LOCCE.

A. Useful resource states

In quantum information theory, an entanglement state is considered useful for a task if and only if it helps to perform the task better than LOCC alone. In the same spirit we define useful resources in local state discrimination.

**Definition 1.** For a given set $S$ of locally indistinguishable states, a resource state $|\Psi\rangle$ is **useful** iff there exists a LOCC protocol $P$ that $F(\Psi \otimes S | P) > F_{\text{local}}(S)$. That is: $F_{\text{local}}(\Psi \otimes S) > F_{\text{local}}(S)$

Thus, $|\Psi\rangle$ is useful iff it can enhance the distinguishability of the set $S$ under LOCC. For a fixed $S$ it seems difficult to ascertain whether a given resource state is useful or not, but nonetheless, we present some general results and examples that answer some of the closely related questions.

Consider a set $S$ of LI states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $d_1 \leq d_2$. Proposition shows that any pure entangled state of Schmidt rank $r$ is useful if $r \geq d_1$. We also show that useful resource states with $r < d_1$ exist. In Example
A resource state of the form $|\Phi|^\otimes m$ where $|\Phi\rangle$ is a Bell state, is shown to be useful in distinguishing an orthonormal basis of Lattice states in $(\mathbb{C}^2)^\otimes n \otimes (\mathbb{C}^2)^\otimes n$ for any $1 \leq m \leq n$.

In Example 2 we show that any bipartite pure entangled state is not a useful resource for distinguishing a three-qubit $GHZ$ basis. Generalizing this, we point out that a $N'$-partite pure entangled state, no matter how entangled, cannot be considered useful for distinguishing a $N$-qubit $GHZ$ basis provided $N' \leq N - 1$.

**Proposition 1.** Let $S$ be a set of locally indistinguishable states in $\mathcal{H}_S = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $2 \leq d_1 \leq d_2$. Then any bipartite pure state $|\Psi\rangle \in \mathcal{H}_\Psi$ of Schmidt rank $r \geq d_1$ is a useful resource.

**Proof.** We prove the proposition by giving an explicit local protocol. Since $r \geq d_1$ we assume that $|\Psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, where $d_1 \leq d'_1 \leq d_2$. In the first step of the protocol we attempt to convert $|\Psi\rangle$ to a maximally entangled state $|\Psi'\rangle$ of Schmidt rank $d_1$ by LOCC. From Vidal’s theorem [46] we know that this local conversion succeeds with probability $p > 0$, and when it does, we use $|\Psi'\rangle$ to optimally distinguish the states by the teleportation protocol. The local conversion, however, fails with probability $(1 - p)$ in which case we resort to the optimal LOCC measurement to distinguish the states. Thus with respect to this local protocol, say $P$, the fidelity is given by

$$F(\Psi \otimes S | P) = pF_{\text{opt}}(S) + (1 - p)F_{\text{local}}(S)$$

since $p > 0$ and $F_{\text{opt}}(S) > F_{\text{local}}(S)$. Hence, $|\Psi\rangle$ is useful for locally distinguishing $S$. \[\square\]

**Example 1.** Let $|\Phi_i\rangle$, $i = 1, \ldots, 4$ be the states in the Bell basis $B \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ where $|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\Phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and $|\Phi_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Now consider a local state discrimination problem where two parties hold $n \geq 2$ unknown Bell states. This means that the unknown state belongs to the maximally entangled basis

$$B' = \left\{ \bigotimes_{j=1}^{n} |\Phi_{i_j}\rangle \middle| i_j \in \{1, 2, 3, 4\} \right\} \subset (\mathbb{C}^2)^\otimes n \otimes (\mathbb{C}^2)^\otimes n$$

The states in $B'$ are known as Lattice States in the literature (see for example [35]). It was shown in [35] that for $n \geq 3$, there are small subsets $S \subset B'$ with $|S| < 2^n$ such that $S$ is locally indistinguishable. Here we restrict our attention to distinguishing a complete basis.

Assuming that the states in $B'$ are all equiprobable, we will show that the resource state $|\Psi_m\rangle = |\Phi_1|^\otimes m$ is useful for any $1 \leq m \leq n$, i.e.

$$F_{\text{local}}(\Psi_m \otimes B') > F_{\text{local}}(B')$$.
Note that, $\Psi_m \otimes B'$ is a set of $2^{2n}$ maximally entangled states in $(\mathbb{C}^2)^{(n+m)} \otimes (\mathbb{C}^2)^{(n+m)}$.

We proceed as follows. We first bound $F_{\text{local}} (B')$ using a result from [34] which states that the local fidelity when distinguishing a set of $k$ $\mathbb{C}^d \otimes \mathbb{C}^d$ maximally entangled states is bounded above by $\frac{d}{k}$. For $B', d = 2^n$ and $k = 2^{2n}$. Hence

$$F_{\text{local}} (B') \leq \frac{2^n}{2^{2n}} = \frac{1}{2^n}$$

On the other hand, using the resource state resource state $|\Psi_m\rangle$ we can teleport $m$ of our $n$ systems from one party to the other, allowing us to identify the value of $(i_1, i_2, \ldots, i_m)$. This reduces the original problem to that of distinguishing $(n - m)$ unknown Bell states for which the optimal local fidelity is exactly $\frac{1}{2^{n-m}}$ (since the bound from [34] is saturated by measuring in the computational basis). Thus, $F_{\text{local}} (\Psi_m \otimes B') = \frac{1}{2^{n-m}} > \frac{1}{2^n} \geq F_{\text{local}} (B')$ for $1 \leq m \leq n$. Note that $B'$ become perfectly distinguishable using this protocol if and only if $m = n$.

The following example shows that not every pure entangled state can be useful for a fixed set $S$ of LI states.

**Example 2.** Consider the problem of local discrimination of the three-qubit GHZ basis $\mathcal{G} = \{|\Phi_i\rangle\}$, $i = 1, \ldots, 8$:

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \quad |\Phi_5\rangle = \frac{1}{\sqrt{2}} (|010\rangle + |101\rangle)$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}} (|000\rangle - |111\rangle) \quad |\Phi_6\rangle = \frac{1}{\sqrt{2}} (|010\rangle - |101\rangle)$$
$$|\Phi_3\rangle = \frac{1}{\sqrt{2}} (|001\rangle + |110\rangle) \quad |\Phi_7\rangle = \frac{1}{\sqrt{2}} (|011\rangle + |100\rangle)$$
$$|\Phi_4\rangle = \frac{1}{\sqrt{2}} (|001\rangle - |110\rangle) \quad |\Phi_8\rangle = \frac{1}{\sqrt{2}} (|011\rangle - |100\rangle)$$

We assume that the states are all equiprobable. Let the qubits be labeled as $A$, $B$ and $C$. Let $F_{\text{local}}^{(i|jk)} (\mathcal{G})$ denote the optimal fidelity across a bipartition $(i|jk)$, where $i \neq j \neq k \in \{A, B, C\}$. We show that local distinguishability of the above states cannot be enhanced by any bipartite pure state shared between any two parties.

The proof is by contradiction. Suppose that a bipartite pure state $|\Psi_{jk}\rangle$ shared between two parties $j$ and $k$ can enhance the local distinguishability of the states in $\mathcal{G}$. This means that there must exist a local protocol $\mathbf{P}$ such that the inequalities

$$F_{\text{local}} (\mathcal{G}) < F (|\Psi_{jk} \otimes \mathcal{G}| \mathbf{P}) \leq F_{\text{local}}^{(i|jk)} (\mathcal{G}) \quad (7)$$

must hold. Since across every bipartition (for example, $A|BC$) each state has a maximum Schmidt coefficient of $\frac{1}{2}$, we can apply the result [47] to obtain

$$F_{\text{local}}^{(i|jk)} (\mathcal{G}) \leq F_{\text{sep}}^{(i|jk)} (\mathcal{G}) \leq \frac{1}{2} : i \neq j \neq k \in \{A, B, C\}$$
As the separable fidelity in a multipartite setting is bounded by the minimum separable fidelity across all bi-partitions,

\[ F_{\text{sep}}(G) \leq \min \left\{ F_{\text{sep}}(A|BC) (G), F_{\text{sep}}(B|AC) (G), F_{\text{sep}}(C|AB) (G) \right\} \leq \frac{1}{2} \]

This upper bound is attained by LOCC simply by measuring in the computational basis and decoding with one of the two possible inputs as in Example 1. Thus we have proved that \( F_{\text{local}} (G) = F_{\text{local}}^{(i|jk)} (G) = \frac{1}{2} \) which is in contradiction with (7).

The above example can be immediately extended to the problem of distinguishing a \( N \)-qubit GHZ basis using a \( N' \)-partite resource state; and a similar argument exploiting the symmetry properties of a GHZ basis shows that a \( N' \)-partite state, no matter how entangled, cannot be useful as a resource if \( N' \leq N - 1 \).

B. Optimal resource states

We now come to the question: When is a resource state optimal for local discrimination of a given set \( S \) of locally indistinguishable states? We have the following definition.

Definition 2. For a set \( S \) of locally indistinguishable states, let \( \mathcal{R} \) be the set of all resource states \(|\Phi\rangle\) such that

\[ F_{\text{local}} (\Phi \otimes S) = F_{\text{opt}} (S) \]

A resource state \(|\Psi\rangle \in \mathcal{H}_\Psi\) is optimal if \(|\Psi\rangle \in \mathcal{R}\) and if there exists a well-defined entanglement measure \( E \) such that

- \( E (\Psi) \leq E (\Psi') \) for any \(|\Psi'\rangle \in \mathcal{R}\).
- \( \dim \mathcal{H}_\Psi \leq \dim \mathcal{H}_{\Psi''} \), for any \(|\Psi''\rangle \in \mathcal{R}\) such that \( E (\Psi'') = E (\Psi) \).

In bipartite systems \( E \) can be chosen to be entanglement entropy which is exactly computable for pure states and has a clear physical interpretation. For multipartite systems one may consider a measure that is most suitable to the specific problem under consideration. For the optimality results presented later in this paper we have chosen the Schmidt measure of entanglement.

For a given set \( S \) of LI states what necessary conditions must an optimal resource state satisfy? In general, this is difficult to answer, but some conditions can still be had when \( S \) is an orthonormal basis.
Let Example 3. what follows, we give couple of examples to illustrate the idea of optimal resources.

\[ S \] enables perfect discrimination of \( \dim \) which in turn implies that \( 1 \) cannot be perfectly distinguished by LOCC across the bipartition \( A|B \). Assume that the states are equally likely. The above states are locally indistinguishable because they have \( F \) with \( \{ \psi_1 \} \) and \( \{ \psi_2 \} \). Clearly \( F \) is sufficient; it is also known to be necessary: see [13] for \( S_1 \) and [14] for \( S_2 \). Note that even though \( S_2 \) is locally more distinguishable than \( S_1 \), \( F_{\text{local}} (S_2) = \frac{3}{4} \) while \( F_{\text{local}} (S_1) = \frac{2}{3} \) [34], both the sets require the same amount of entanglement for optimal discrimination.

\[ \text{Example 3.} \] Let \( S \) be a maximally entangled orthonormal basis in \( \mathbb{C}^d \otimes \mathbb{C}^d \), \( d \geq 2 \). If a resource state \( \ket{\Phi} \) enables perfect discrimination of \( S \) by LOCC then from Proposition [2] we see that \( E (\Phi) \geq \log d \) ebits which in turn implies that \( \dim \mathcal{H}_\Phi \geq d^2 \). Since for any maximally entangled state \( \ket{\Psi} \in \mathbb{C}^d \otimes \mathbb{C}^d \) we have \( F_{\text{local}} (\Psi \otimes S) = 1 \), \( E (\Psi) = \log d \) ebits and \( \dim \mathcal{H}_\Psi = d^2 \), \( \ket{\Psi} \) is therefore optimal.

Using similar arguments it’s also easy to see that a maximally entangled state \( \ket{\Psi} \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) is optimal for distinguishing the set \( S_1 = \{ \text{any three Bell states} \} \) as well as the set \( S_2 = \{ \ket{\Phi_1}, \ket{\Phi_2}, \ket{01}, \ket{10} \} \). Clearly \( \ket{\Psi} \) is sufficient; it is also known to be necessary: see [13] for \( S_1 \) and [14] for \( S_2 \). Note that even though \( S_2 \) is locally more distinguishable than \( S_1 \), \( F_{\text{local}} (S_2) = \frac{3}{4} \) while \( F_{\text{local}} (S_1) = \frac{2}{3} \) [34], both the sets require the same amount of entanglement for optimal discrimination.

The next example shows that an optimal resource may not be shared among all the parties in a multipartite system.

\[ \text{Example 4.} \] Consider the problem of locally distinguishing three-qubit GHZ states \( \mathcal{G}' = \{ \ket{\Phi_i} \}, i = 1, \ldots, 4 \), where the states are given by

\[
\begin{align*}
\ket{\Phi_1}_{ABC} & = \frac{1}{\sqrt{2}} (\ket{000}_{ABC} + \ket{111}_{ABC}) \\
\ket{\Phi_2}_{ABC} & = \frac{1}{\sqrt{2}} (\ket{000}_{ABC} - \ket{111}_{ABC}) \\
\ket{\Phi_3}_{ABC} & = \frac{1}{\sqrt{2}} (\ket{001}_{ABC} + \ket{110}_{ABC}) \\
\ket{\Phi_4}_{ABC} & = \frac{1}{\sqrt{2}} (\ket{001}_{ABC} - \ket{110}_{ABC})
\end{align*}
\]

Assume that the states are equally likely. The above states are locally indistinguishable because they cannot be perfectly distinguished by LOCC across the bipartition \( C|AB \). This follows from the observation
that across the bipartition $C|AB$ the states are locally equivalent to the Bell basis. On the other hand, the states can be perfectly distinguished by LOCC across the two other bipartitions $A|BC$ and $B|CA$.

We will show that a two-qubit Bell state $|\Psi\rangle_{BC} = \frac{1}{\sqrt{2}} (|00\rangle_{BC} + |11\rangle_{BC})$ shared between $B$ and $C$ is an optimal resource. The LOCC protocol is as follows. In the first step $A$ performs a measurement in the $|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$ basis on the qubit he/she holds as part of the unknown state and informs the outcome to $B$. If the outcome is $+$, $B$ does nothing. If the outcome is $-$, $B$ applies $\sigma_z$ on the qubit that he/she holds as part of the unknown state. This results in an unknown Bell state between $B$ and $C$ with the following mapping:

$$
\begin{align*}
|\Phi_1\rangle_{ABC} &\rightarrow |\Phi^+\rangle_{BC} \\
|\Phi_2\rangle_{ABC} &\rightarrow |\Phi^-\rangle_{BC} \\
|\Phi_3\rangle_{ABC} &\rightarrow |\Psi^+\rangle_{BC} \\
|\Phi_4\rangle_{ABC} &\rightarrow |\Psi^-\rangle_{BC}
\end{align*}
$$

The protocol is completed by $B$ and $C$ who distinguish the unknown Bell state using $|\Psi\rangle_{BC}$ following the teleportation protocol. Note that the necessary conditions in Proposition 2 and Corollary 1 are not violated because $G'$ is not a complete basis.

V. OPTIMAL RESOURCES FOR GHZ AND GRAPH STATES

A. Discrimination of a GHZ basis by LOCCE

Let $G^N_m$ denote a $N$-qubit $m$-partite GHZ basis for $2 \leq m \leq N$. Assuming that the $i^{th}$ party holds $n_i$ qubits, $1 \leq n_i \leq (N - m + 1)$ the basis states are given by a collection of $2^{N-1}$ conjugate pairs

$$
|\Phi_{\alpha}^\pm\rangle_{N,m} = \frac{1}{\sqrt{2}} (|k_1^\alpha\rangle |k_2^\alpha\rangle \cdots |k_m^\alpha\rangle \pm |\overline{k_1^\alpha}\rangle |\overline{k_2^\alpha}\rangle \cdots |\overline{k_m^\alpha}\rangle), \alpha = 1, \ldots, 2^{N-1}
$$

where for every $j$, $j = 1, \ldots, m$ $k_j$ is a $n_j$-bit binary string and $\overline{k_j}$ is its bit-wise orthogonal complement and $k_1^\alpha k_2^\alpha \cdots k_m^\alpha \neq k_1^\beta k_2^\beta \cdots k_m^\beta$ whenever $\alpha \neq \beta$. The state space of the states with respect to this partitioning is given by $\mathcal{H}_S = \otimes_{i=1}^{m} \mathbb{C}^{2^{n_i}}$.

The set $G^N_m$ of states is locally indistinguishable for any partitioning of the $N$ qubits among $m$ parties. This is because across any bipartition the states remain entangled (by inspection), and we know that any bipartite orthonormal basis containing entangled states is locally indistinguishable [24].

We wish to obtain an optimal resource for distinguishing the elements of $G^N_m$. Let us suppose that a resource state $|\Psi\rangle$ is optimal. Since the average entanglement of the states in $G^N_m$ is one ebit across every bipartition, Corollary 1 tells us that $|\Psi\rangle$ must be $m$-partite; that is, $\mathcal{H}_\Psi = \otimes_{i=1}^{m} \mathbb{C}^{d_i}$, $d_i \geq 2$ and therefore, $\dim \mathcal{H}_\Psi \geq 2^m$. We now state the main result:
Theorem 1. A $m$-qubit GHZ state $|\Phi\rangle$ is an optimal resource for distinguishing the states in $G_N^m$ using LOCC for any $2 \leq m \leq N$ and any partitioning of the $N$ qubits among $m$ parties.

Without loss of generality let $|\Phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes m + |1\rangle \otimes m)$. To establish optimality it suffices to show that $F_{local}(\Phi \otimes G_N^m) = 1$. This is because the other conditions for optimality are satisfied as follows: First, $E_{SM}(\Phi) = 1$, where $E_{SM}$ is the Schmidt measure of entanglement. Since for any multipartite pure state $|\phi\rangle$, $E_{SM}(\phi) \geq 1$, entanglement of $|\Phi\rangle$ thus achieves the minimum. Next, the dimension of the resource space being $2^m$ also achieves the minimum dimension required by any $m$-partite state. What remains to show is that, in fact, $F_{local}(\Phi \otimes G_N^m) = 1$. We begin with the case $m = N$:

Lemma 1. For any $N \geq 2$, a GHZ basis $G_N^N$ and a resource state $|\Phi'\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes N + |1\rangle \otimes N)$, we have $F_{local}(\Phi' \otimes G_N^N) = 1$.

The proof is given in appendix A. The LOCC protocol, however, is simple. It consists of a sequence of Bell measurements by each party $A_i$, $i = 1, \ldots, N$ followed by appropriate Pauli corrections. While any sequence works, in the proof we assume that the sequence $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_N$ is followed. The first Bell measurement by $A_1$ entangles the resource state and the shared unknown state but does not eliminate any state. However, in each subsequent measurement performed in the order $A_2 \rightarrow \cdots \rightarrow A_{N-1}$, the outcome maps exactly half of the candidate states (remaining in that round) onto a new set of orthonormal states and eliminates the rest. In the end $A_N$ who is left with the task of distinguishing four Bell states completes the protocol by performing a Bell measurement.

We now extend the theorem to the case $m < N$ with the following lemma which shows that local discrimination of the states in $\Phi \otimes G_N^m$ cannot be harder than the states in $\Phi' \otimes G_N^N$.

Lemma 2. For any $N \geq m \geq 2$,

$$F_{local}(\Phi \otimes G_N^m) = F_{local}(\Phi'_m \otimes G_N^m) \geq F_{local}(\Phi' \otimes G_N^N)$$

where $|\Phi'_m\rangle$ is the $N$-qubit GHZ state shared by $m$ parties with the same partitioning of the qubits as in $G_N^m$.

The first equality is easy to see, since $|\Phi\rangle \leftrightarrow |\Phi'_m\rangle$ by LOCC with unit probability. If we start with $|\Phi\rangle$ then we can locally transform $|\Phi\rangle$ into $|\Phi'_m\rangle$ by having each party append $(n_i - 1)$ additional qubits and then perform a control-NOT on each to entangle them, mapping $|0\rangle$ to $|0\rangle \otimes m$ and $|1\rangle$ to $|1\rangle \otimes m$. Conversely, if we start with $|\Phi'_m\rangle$, each party simply performs the inverse unitary which is again a control-NOT to disentangle the additional qubits to arrive at $|\Phi\rangle$.

The proof of the second inequality comes from the observation that the sets $\Phi'_m \otimes G_N^N$ and $\Phi' \otimes G_N^N$ contain the same states but in the latter the states are partitioned further. Thus, the problem of locally
distinguishing the elements of the former cannot be harder than locally distinguishing the elements of the latter. Hence the lemma.

The proof the theorem immediately follows since \( F_{\text{opt}}(G_N^N) = F_{\text{local}}(\Phi' \otimes G_N^N) = 1 \) by Lemma 1.

B. Discrimination of Graph states by LOCCE

Let \( \mathcal{P}_N \) be the set of \( N \)-fold tensor products of Pauli operators: \( \mathcal{P}_N = \{ \otimes_{k=1}^N \sigma_k \} \). For any graph \( G \) on \( N \) vertices, we can define an associated set of \( N \)-partite graph state in which each party holds a qubit. Following the definitions in, e.g. [50, 51], each vertex \( a \in V \) is associated with a unitary \( K_G^{(a)} = \otimes_{k=1}^N \sigma_{k_i} \in \mathcal{P}_N \), where \( \sigma_{i_k} = X; \sigma_{k_i} = Z \) if \( v_k \) and \( v_i \) are neighbors in \( G \); and \( \sigma_{k_i} = I \) otherwise.

The set of operators \( \{ K_G^{(a)} : a \in V \} \) commute and (except in degenerate cases) define a unique basis of common eigenvectors, \( S \subset (\mathbb{C}^2)^\otimes N \). We call these the graph states corresponding to the graph \( G \). We identify the state \( |\Psi_G\rangle \in S \) as the unique state which is simultaneously an eigenvector of each \( K_G^{(a)} \) with eigenvalue one, and we propose that \( |\Psi_G^*\rangle \) is an optimal resource to distinguish the elements of \( S \) (where * denotes the entrywise complex conjugate). We start with the following elementary observation:

**Lemma 3.** The set of graph states \( S \) contains the orbit of \( |\Psi_G\rangle \) under the action of \( \mathcal{P}_N \).

The proof is straightforward: For any \((i_1, i_2, \ldots, i_N) \in \{0, 1, 2, 3\}^N\), define

\[
|\Psi\rangle = (\otimes_{k=1}^N \sigma_{i_k}) |\Psi_G\rangle
\]

Since \( K_G^{(a)} \in \mathcal{P}_N \), we can use standard commutation properties to get that

\[
K_G^{(a)} (\otimes_{k=1}^N \sigma_{i_k}) = \pm (\otimes_{k=1}^N \sigma_{i_k}) K_G^{(a)}
\]

This means that

\[
K_G^{(a)} |\Psi\rangle = K_G^{(a)} (\otimes_{k=1}^N \sigma_{i_k}) |\Psi_G\rangle = \pm (\otimes_{k=1}^N \sigma_{i_k}) K_G^{(a)} |\Psi_G\rangle
\]

\[
= \pm (\otimes_{k=1}^N \sigma_{i_k}) |\Psi_G\rangle = \pm |\Psi\rangle
\]

Hence, for each \( a \in V \), \( |\Psi\rangle \) is an eigenvector of \( K_G^{(a)} \), which implies that \( |\Psi\rangle \in S \).

This allows us to state our result:

**Theorem 2.** For any graph \( G \) on \( N \) vertices that uniquely defines states \( S \) and \( |\Psi_G\rangle \) as above: The state \( |\Psi_G^*\rangle \) is an optimal resource to distinguish the elements of \( S \) under LOCCE.
The proof is a consequence of the lemma. Suppose we start with the state $|\Psi_G^*\rangle \otimes |\Psi_x\rangle$ for $|\Psi_x\rangle \in \mathcal{S}$. Each party can measure their two-qubit system in the Bell basis. This is equivalent to performing the global measurement

$$M = \{(I_{2N} \otimes \sigma_N) |\Psi\rangle \langle \Phi| (I_{2N} \otimes \sigma_N) : \sigma_N \in \mathcal{P}\}$$

where $|\Phi\rangle$ is the canonical maximally entangled state. (Note that the elements of $M$ project onto the Lattice States discussed in Example 1.)

Given the initial state $|\Psi_G^*\rangle \otimes |\Psi_x\rangle$, the probability of getting the outcome $\sigma_N$ is given by

$$|\langle \Psi_G^* \otimes \Psi_x |(I_{2N} \otimes \sigma_N)|\Phi\rangle|^2 = \frac{1}{2^N} |\langle \Psi_x |\sigma_N |\Psi_G\rangle|^2 = \frac{1}{2^N} |\langle \Psi_x |\Psi_y\rangle|^2$$

for some $|\Psi_y\rangle \in \mathcal{S}$ by the lemma. Since the elements of $\mathcal{S}$ are mutually orthogonal, the probability of getting this outcome is zero unless $y = x$. It also confirms that each $|\Psi_y\rangle$ corresponds to $2^N$ measurement outcomes. Since there are $4^N$ possible outcomes, this implies that the orbit of $\mathcal{P}_N$ in fact reaches all $2^N$ elements of $\mathcal{S}$ and that the set $|\Psi_G^*\rangle \otimes \mathcal{S}$ can be perfectly distinguished with LOCC.

On the other hand, the result in [16] asserts that for any optimal resource $|\Psi\rangle$, $|\Psi^*\rangle$ must be locally transformable into $|\Psi_G\rangle$, which implies that for every entanglement measure, $E(\Psi) \geq E(\Psi_G) = E(\Psi_G^*)$ and every local space must have dimension at least two. Hence, $|\Psi_G^*\rangle$ is an optimal resource state for $\mathcal{S}$.

Note that the $(m, m)$-GHZ states are locally equivalent to the graph states corresponding to the complete graph $K_m$ on $m$ vertices; hence Theorem 2 is a direct generalization of Theorem 1.

VI. OPTIMAL RESOURCES FOR ONE-WAY LOCC IN BIPARTITE SYSTEMS

Optimal resource states can be defined with respect to any restricted set of measurements. One familiar restriction on LOCC in bipartite systems is that of one-way communication, in which Alice can communicate her measurement results to Bob but Bob cannot communicate back to Alice. We denote the optimal fidelity with respect to this restriction $F_{\text{local}}^\text{–1}$.

Given a set of bipartite states $\{|\psi_i\rangle\} \subset \mathbb{C}^d \otimes \mathbb{C}^d$, we can identify each state with a $d \times d$ matrix in the standard way

$$|\psi_i\rangle = (I \otimes M_i) |\Phi\rangle$$

where $|\Phi\rangle$ is the standard maximally-entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$. It was noted in [15, 31] that a necessary condition for one-way LOCC discrimination is the existence of a state $|\varphi\rangle$ such that the $\langle \varphi | M_i^* M_j | \varphi \rangle = 0$ whenever $i \neq j$. We can use this condition to state the following:
**Proposition 3.** Let $S = \{ |\psi_i\rangle \}$ be a complete orthogonal basis of $\mathbb{C}^d \otimes \mathbb{C}^d$, and let $|\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ be the standard maximally-entangled state. If $S$ contains at least one state with full Schmidt rank $d$, then $|\Phi\rangle$ is an optimal resource for the problem of one-way LOCC discrimination.

**Proof.** It is clear that $F_{\text{local}}(\Phi \otimes S) = 1$, since we can use one-way LOCC to teleport one half of our states to the other subsystem. What is less clear is that this is optimal, which we show next.

Suppose that $|\Psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ is an optimal resource to distinguish our basis $S$ with one-way LOCC. We write

$$|\Psi\rangle = \left( I \otimes \Lambda^{1/2} \right) |\Phi\rangle \quad \Lambda = \sum_{i=1}^{d} \lambda_i |i\rangle \langle i|$$

If we can distinguish the set $\Psi \otimes S$ with one-way LOCC, then there exist positive constants $\{a_k\}$ and states $\{|\varphi_k\rangle\} \subset \mathbb{C}^{d^2}$ such that $\sum_k a_k |\varphi_k\rangle \langle \varphi_k| = I_{d^2}$ and $\langle \varphi_k|(\Lambda \otimes M_i^* M_j)|\varphi_k\rangle = 0$ for $i \neq j$ [15]. If we write $|\varphi_k\rangle = (I \otimes R_k)|\Phi\rangle$, then whenever $i \neq j$,

$$\langle \varphi_k|(\Lambda \otimes M_i^* M_j)|\varphi_k\rangle = \frac{1}{d} \text{Tr}(R_k \Lambda R_k^*) M_i^* M_j = 0 \quad (9)$$

Since the elements of $S$ are linearly independent, so are the matrices $\{M_i\}$. By assumption, at least one of the states in $S$ (say $|\psi_1\rangle$) has Schmidt rank $d$, which means that the corresponding matrix $M_1$ is invertible. This implies that the matrices $\{M_i^* M_j\}_{j=2}^{d^2}$ are linearly independent; and since they are all traceless, the orthogonal complement of $\{M_i^* M_j\}_{j=2}^{d^2}$ is simply the multiples of the identity matrix $I$. Setting $i = 1$ in (9), we get that for any $k$, $R_k \Lambda R_k^* = t_k I_d$ is a multiple of the identity. This implies that each $R_k$ is full rank and that, in fact, for each $k$, there exists a unitary $U_k$ such that

$$R_k = \sqrt{t_k} U_k \Lambda^{-1/2}$$

Since $|\varphi_k\rangle$ is a normalized pure state, $\text{Tr} R_k^* R_k = d$, which implies that $t_k = t$ does not depend on $k$. We can now rewrite our decomposition of the identity to get

$$I_{d^2} = \sum_k a_k |\varphi_k\rangle \langle \varphi_k|$$

$$= \sum_k a_k t \left( I \otimes U_k \Lambda^{-1/2} \right) |\Phi\rangle \langle \Phi| \left( I \otimes \Lambda^{-1/2} U_k^* \right)$$

$$= \sum_k a_k t \left( \Lambda^{-1/2} \otimes U_k \right) |\Phi\rangle \langle \Phi| \left( \Lambda^{-1/2} \otimes U_k^* \right)$$

$$\Lambda \otimes I_d = \sum_k a_k t \left( I \otimes U_k \right) |\Phi\rangle \langle \Phi| \left( I \otimes U_k^* \right)$$

In the last line, all of the states on the right side are maximally-entangled, so if we trace out the second system, we get the maximally mixed state, which implies that

$$\Lambda = \frac{t}{d} \left( \sum_k a_k \right) I_d = t d I_d$$
Since $\text{Tr}\Lambda = d$, we get that $\Lambda = I_d$.

Conclusion: If the state $|\Psi\rangle = (I \otimes \Lambda^{1/2}) |\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ can be used as a resource to locally distinguish a complete basis of $\mathbb{C}^d \otimes \mathbb{C}^d$ containing a full-rank state, then $\Psi$ must be maximally-entangled.  

We note that this same result can be shown using the operator system methods in the recent work of Kribs, et al. [52].

VII. CONCLUSION AND OPEN PROBLEMS

The notion of entanglement as a resource stems from the fact that shared entanglement can help us to realize nonlocal quantum operations on composite systems by LOCC. In this paper, we have considered the task of quantum state discrimination within the framework of LOCCE, short of Local Operations, Classical Communication and Entanglement. To better understand the role of entanglement as a resource, we focused on the characterization of resource states and defined useful and optimal resource states for any given local state discrimination problem. These definitions were further illustrated with results and examples in both bipartite and multipartite systems.

Some interesting questions emerge from the notion of useful resources. For example, let $\mathcal{S}$ be a set of LI states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $3 \leq d_1 \leq d_2$. Is there a pure entangled state $|\Phi\rangle$ of Schmidt rank $r$ such that $r < d_1$ which is useful for distinguishing $\mathcal{S}$? From the example given in this paper we know that such a pair $(\mathcal{S}, \Phi)$ can be found but a general answer is wanting. More generally, for a fixed set $\mathcal{S}$, how can we characterize the set of states $|\Phi\rangle$ such that $|\Phi\rangle$ is useful for distinguishing $\mathcal{S}$?

Some other open problems which may also be of interest are discussed below.

Consider, for example, the following orthonormal basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$:

\[
|\psi_1\rangle = \alpha |00\rangle + \beta |11\rangle \quad |\psi_2\rangle = \beta |00\rangle - \alpha |11\rangle \\
|\psi_3\rangle = \gamma |01\rangle + \delta |10\rangle \quad |\psi_4\rangle = \delta |01\rangle - \gamma |10\rangle
\]

where $\alpha, \beta, \gamma, \delta$ with $\alpha \geq \beta \geq 0$ and $\gamma \geq \delta \geq 0$ are real numbers satisfying $\alpha^2 + \beta^2 = 1$ and $\gamma^2 + \delta^2 = 1$. The local fidelity of the above set of states can be shown to be $F_{\text{local}}(\mathcal{S}) = \frac{1}{2} (\alpha^2 + \gamma^2)$. The states are locally indistinguishable except for the case $\alpha = \gamma = 1$. i.e. when the set reduces to the computational basis. Clearly, the states can be perfectly distinguished using a Bell state as resource; and if $\max(\alpha, \gamma) = 1$, this is necessarily optimal [16]. However, we do not know whether this is optimal in other cases, and it would be useful to understand how this depends on $\alpha$ and $\gamma$.

Another problem worth considering is motivated by the no-go results on local distinguishability of maximally entangled states [20, 23, 26, 33, 35, 36, 53]. Suppose $\mathcal{S}$ is a set of orthonormal maximally en-
tangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$. If the states form a basis then we know that any maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$ is an optimal resource. On the other hand, if the states do not form a basis, that is, $|S| < d^2$, they can still be locally indistinguishable [20, 26, 33, 35, 36] and in these cases, except when $d = 2$, we do not know the optimal resources.

In multipartite systems, questions related to optimal resources may pose different kinds of challenges, especially because of the complex structure of the states, computability of entanglement measures and existence of multiple SLOCC equivalence classes [54, 55]. In fact, the existence of multiple SLOCC classes led to a recent no-go result [16] which states that for a given multipartite system, a universal resource (a state which can optimally distinguish any set of locally indistinguishable states) almost always does not exist in the same state space. For example, one cannot find a three-qubit pure entangled state that can perfectly distinguish any three-qubit orthonormal basis by LOCC. This in turn implies that any universal resource for a three-qubit system must belong to higher dimensions. In view of this, finding optimal resources in multipartite systems could be challenging. In this paper, we were able to make partial progress by solving for GHZ and Graph states; however, optimal resources for distinguishing any other orthonormal basis with states chosen from other SLOCC classes are not yet known.

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[47] For a set of $N$ equally likely orthogonal states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle$ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, we have

$$F_{\text{sep}} \leq \frac{\lambda_{\text{max}} d_1 d_2}{N}$$

(10)

where $\lambda_{\text{max}} = \max_i \lambda_i$, $\sqrt{\lambda_i}$ being the largest Schmidt coefficient of the state $|\psi_i\rangle$.

[48] Any $N$-partite pure state $|\phi\rangle \in \otimes_{i=1}^{N} \mathbb{C}^{d_i}$ can be written in the form

$$|\phi\rangle = \sum_{i=1}^{R} \alpha_i |\phi^{(i)}_1\rangle \otimes |\phi^{(i)}_2\rangle \otimes \cdots \otimes |\phi^{(i)}_N\rangle,$$

(11)

where $|\phi^{(i)}_j\rangle \in \mathbb{C}^{d_j}$, $j = 1, \ldots, N$, $\alpha_i \in \mathbb{C}$, $i = 1, \ldots, R$ for some $R$. Suppose that $r$ is the minimal number of product terms $R$ in such a decomposition. The Schmidt measure [49] is defined as $E(\phi) = \log_2 r$.

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Appendix

A. Proof of Lemma

A $N$-qubit $N$-partite GHZ basis $G_N^N$ is defined by a collection of $2^{N-1}$ mutually orthogonal conjugate pairs which can be written as:

$$|\Phi^{\pm}_\alpha\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle^\alpha \otimes |k_2^\alpha\rangle \cdots |k_N^\alpha\rangle \pm |1\rangle^\alpha \otimes |\overline{k_2^\alpha}\rangle \cdots |\overline{k_N^\alpha}\rangle \right), \alpha = 1, \ldots, 2^{N-1}$$

(12)

where for every $i = 2, \ldots, N$, $k_i \in \{0, 1\}$ and $\overline{k_i}$ is its complement. We now give a LOCC protocol that perfectly distinguishes the states in $G_N^N$ using the resource $|\Phi^\prime\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle^\otimes N + |1\rangle^\otimes N \right)$. First, we write
the (unnormalized) states in $\Phi' \otimes \mathcal{G}_N^2$ as
\[
|\Phi'\rangle \otimes |\Phi'_\alpha\rangle = |0,0\rangle \otimes |0,0\rangle \cdots |0,0\rangle \pm |1,1\rangle \otimes |1,1\rangle \cdots |1,1\rangle \\pm |0,1\rangle \otimes |1,0\rangle \cdots |1,0\rangle \pm |1,0\rangle \otimes |0,1\rangle \cdots |0,1\rangle ; \alpha = 1, \ldots, 2^{N-1}
\]
where the subscript "r" indicates that the qubit belongs to the resource state. The protocol constitutes a series of sequential Bell measurements by all the parties $A_i$, $i = 1, \ldots, N$. We adopt the following sequence: $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_N$.

1. $A_1$ performs a Bell measurement on the two qubits and informs the outcome to $A_2$ who applies the appropriate Pauli correction following the convention of standard teleportation on the resource qubit he/she holds. This measurement completely disentangles the first two qubits held by $A_1$ and results in a state shared between the rest of the parties. This resulting state belongs to one of the two sets $\Phi$ and $\Psi$ (given below) depending on whether the outcome was in $\{\Phi^+ / \Phi^-\}$ or $\{\Psi^+ / \Psi^-\}$:

\[
\Phi : \{ |0, k_2^\alpha \rangle \cdots |0, k_N^\alpha \rangle \pm |1, k_2^\alpha \rangle \cdots |1, k_N^\alpha \rangle \}; \alpha = 1, \ldots, 2^{N-1}
\]

\[
\Psi : \{ |0, k_2^\alpha \rangle \cdots |0, k_N^\alpha \rangle \pm |1, k_2^\alpha \rangle \cdots |1, k_N^\alpha \rangle \}; \alpha = 1, \ldots, 2^{N-1}
\]

Note that, as of now, the measurement by $A_1$ does not eliminate any state; instead, it entangles the resource state and the unknown state.

2. Let us suppose that the outcome of the measurement by $A_1$ was either $\Phi^+$ or $\Phi^-$. The resulting state, now shared between the parties $A_2, A_3, \ldots, A_N$, therefore, belongs to the set $\Phi$. The task is now to distinguish the elements in $\Phi$. The states in $\Phi$ can be grouped into two disjoint subsets $\Phi_0$ and $\Phi_1$ depending on whether $k_2^\alpha$ takes the value 0 or 1. By an appropriate relabeling of the states, the sets $\Phi_0$ and $\Phi_1$ are given by:

\[
\Phi_0 : \{ |0,0\rangle \cdots |0, k_2^\alpha \rangle \cdots |0, k_N^\alpha \rangle \pm |1, k_2^\alpha \rangle \cdots |1, k_N^\alpha \rangle \}; \alpha = 1, \ldots, 2^{N-2}
\]

\[
\Phi_1 : \{ |0,0\rangle \cdots |0, k_2^\alpha \rangle \cdots |0, k_N^\alpha \rangle \pm |1, k_2^\alpha \rangle \cdots |1, k_N^\alpha \rangle \}; \alpha = 2^{N-2} + 1, \ldots, 2^{N-1}
\]

Each of the sets $\Phi_0$ and $\Phi_1$ contains exactly $2^{N-2}$ conjugate pairs. $A_2$ now performs a Bell measurement on the two qubits he/she holds, and informs the result to $A_3$ who applies the appropriate Pauli correction on the resource qubit. This measurement disentangles the two qubits held by $A_2$ and results in a $(N-2)$-partite state shared between $A_3, A_4, \ldots, A_N$. The resulting state belong to one of the following two sets $\Phi'$ and $\Psi'$ depending on whether the outcome was $\Phi^+ / \Phi^-$ or $\Psi^+ / \Psi^-$:

\[
\Phi' : \{ |0, k_3^\alpha \rangle \cdots |0, k_N^\alpha \rangle \pm |1, k_3^\alpha \rangle \cdots |1, k_N^\alpha \rangle \}; \alpha = 1, \ldots, 2^{N-2}
\]

\[
\Psi' : \{ |0, k_3^\alpha \rangle \cdots |0, k_N^\alpha \rangle \pm |1, k_3^\alpha \rangle \cdots |1, k_N^\alpha \rangle \}; \alpha = 1, \ldots, 2^{N-2}
\]
As the resulting state belong to either $\Phi'$ or $\Psi'$, thus, this measurement eliminates $2^{(N-1)}$ states. One can do a similar analysis had the outcome of $A_1$’s measurement was either $\Psi^+$ or $\Psi^-$.

3. The protocol continues in a similar fashion, each round eliminating exactly half of the states that remained to be distinguished in the previous round; that is, the second round eliminates $2^{(N-1)}$ states (or equivalently $2^{(N-2)}$ conjugate pairs), the third round eliminates $2^{(N-2)}$ states (or $2^{(N-3)}$ conjugate pairs) and so on. It is easy to check that after $(N - 1)$ rounds of measurements (note that state elimination starts only from the second round starting with the measurement by $A_2$), all but four states (or two conjugate pairs) get eliminated. The last party $A_N$ therefore performs a complete orthogonal measurement to distinguish these four states. This completes the protocol.