Symmetric Union Presentations for 2-Bridge Ribbon Knots

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ABSTRACT
We prove that all 2-bridge ribbon knots are symmetric unions.

1. Overview and definitions
Symmetric unions have been defined as generalisations of Kinoshita-Terasaka’s construction in 1957 ([7], [9]). They are given by diagrams which look like the connected sum of a knot and its mirror image with additional twist tangles inserted near the symmetry axis. Because all symmetric unions are ribbon knots, we can ask how big a subfamily of ribbon knots they form. It is, for instance, known that all 21 (non-trivial, prime) ribbon knots with crossing numbers \( \leq 10 \) are symmetric unions. In this article we prove that all 2-bridge ribbon knots are symmetric unions.

Fig. 1. Examples of disks with ribbon singularities
This article is based on a talk at the Joint Meeting of AMS and DMV in June 2005 at Mainz University (Germany). We include a postscriptum containing comments and later developments. Note that in 2005 it was not yet known that the three Casson-Gordon families of 2-bridge ribbon knots are a complete list of all 2-bridge ribbon knots. This was proved by Paolo Lisca in 2006.

**Definition 1.1.** A knot $K$ in $S^3$ is a *ribbon knot* if it bounds an immersed disk in $S^3$ with only ribbon singularities.

Recall that $K$ is a *slice knot* if it bounds a smoothly embedded 2-disk $D^2$ in $B^4$. Every ribbon knot is slice. A notorious question in knot theory is whether the converse is true.

There are 21 non-trivial prime ribbon knots with crossing numbers $\leq 10$. A list is shown in Appendix F.5 in the book "A survey of knot theory" [5]. An example can be seen in Figure 2.

![Fig. 2. A ribbon presentation for the knot 10_{87}](image)

Symmetric unions yield another way of showing that knots are ribbon. As a general reference we refer to [9]. Before we give the definition of symmetric unions we mention that all 21 non-trivial prime ribbon knots with crossing number less or equal to 10 are symmetric unions. As an example we show in Figure 3 a symmetric union for 10_{87} a case which was missing from my list in 1998 [9].

**Definition 1.2.** Let $D$ be an unoriented knot diagram and $D^*$ the diagram $D$ reflected at an axis in the plane. If, as in Figure 4 we insert the tangle $\rightleftharpoons$ and twist tangles $n_i$, we call the result a *symmetric union* of $D$ and $D^*$ with twist parameters $n_i$, $i = 1, \ldots, k$. The *partial knot* $\hat{K}$ of a symmetric union of $D$ and $D^*$ is the knot given by the diagram $D$. 
Fig. 3. The knot $10_{87}$ as a knot diagram (left), as a symmetric ribbon with twists (center) and the partial knot diagram (right).

Fig. 4. Definition of symmetric unions

Examples:
1.) As an example we consider again the diagram in Figure 3. Here the partial knot is the knot $6_{1}$, shown in the right of the figure.
2.) If all $n_{i} = 0$, then we get the well-known symmetric ribbon for $\hat{K}^{*} - \hat{K}$ (we use the notation $-K$ for the mirror image of $K$, with reversed orientation).

2. Properties of symmetric unions

The following Theorem 2.1 was already known to Kinoshita and Terasaka in the case that only one twist tangle is inserted, see [7]. The theorem is proved for instance by using the matrix definition of the Alexander polynomial and the Goeritz matrix [9].

Theorem 2.1. The Alexander polynomial of a symmetric union of $D$ and $D^{*}$
depends only on the parity of the twist numbers \( n_i \). The determinant of a symmetric union of \( D \) and \( D^\ast \) is independent of the twist numbers and equals the square of the determinant of the partial knot.

In this article we are interested mainly in the ribbon property of symmetric unions:

**Theorem 2.2.** Symmetric unions are ribbon knots.

The proof uses the same construction as for \( K^\#_2 - K \) with additional twists in the ribbon, see Figure 5 for an example. The articles [9], [14] and [16] contain the proof and more details on symmetric unions.

Fig. 5. Symmetric unions are ribbon knots (left: a diagram of \( 8_{20} \) showing the symmetric ribbon, right: a 3D rendering)

Further examples:

- The knot 10_{153} and the Kinoshita-Terasaka knot are symmetric unions of the trivial knot, see Figure 6. Hence they have determinant 1. For the Kinoshita-Terasaka knot the twist parameter is an even number, hence its Alexander polynomial is equal to 1.
- In 2004 Taizo Kanenobu used the chiral knot in the right of Figure 6 in order to present a knot whose chirality is not detected by the Links/Gould invariant, see [3], [4]. This knot is a symmetric union with respect to two different axes, resulting in different partial knots. A similar diagram can be found in Example 3.1 in [14].
3. 2-bridge knots: definition and the Casson-Gordon families

Recall the definition of the bridge number of a knot: if \( v \in \mathbb{S}^2 \) is a unit vector in \( \mathbb{R}^3 \) and \( K \) is a knot, then let \( b_v(K) \) be the number of maxima of the orthogonal projection of \( K \) on the line spanned by \( v \). Then the bridge number of \( K \) is:

\[
b(K) := \min_{K' \sim K} \min_v b_v(K')
\]

Because knots with bridge number 1 are trivial, the simplest cases occur for bridge number 2. They were studied by:

- Bankwitz/Schumann (Viergeflechte, 1934),
- Schubert (Knoten mit 2 Brücken, 1956) and
- Conway (rational knots, 1970).

We mention some properties of 2-bridge knots: Their double branched coverings are lens spaces \( L(p, q) \) with \( p \) equal to the determinant of the knot. In the plait normal form \( C(a_1, \ldots, a_n) \), also called Conway notation, the numbers of half-twists \( a_i \) are related to the parameters \( p \) and \( q \) by the regular (positive) continued fraction \( [a_1, \ldots, a_n] = \frac{p}{q} \) (see [5], Chapter 2.1). We sometimes call \( \frac{p}{q} \) the ‘fraction’ of \( K \). It determines the knot type.

The following theorem gives a necessary condition for 2-bridge ribbon knots.

**Theorem 3.1 (Casson-Gordon, 1974, [1]).**

Let \( K \) be a 2-bridge knot with fraction \( p^2/q \). Denote by \( \Delta(a, b) \), where \( a, b \in \mathbb{R} \), the triangle with vertices \((0, 0), (a, 0)\) and \((a, b)\). If \( K \) is ribbon then we have:

\[
4(\text{Area}\Delta(pr, \frac{qr}{p}) - \text{Int}\Delta(pr, \frac{qr}{p})) = \pm 1, \quad \forall r \in \{1, \ldots, p - 1\},
\]

where \( \text{Int}\Delta(a, b) \) is a weighted count of the integral points lying in \( \Delta(a, b) \) (see below for the precise definition).
The value of \( \text{Int} \Delta(a, b) \) is computed by counting lattice points \((\mathbb{Z}^2 \cap \Delta(a, b))\) similar to Pick’s formula \[15\]: interior points count as 1, boundary points as 1/2 and vertices different from \((0,0)\) as 1/4. The vertex \((0,0)\) is not counted.

In the example of Figure 7 with \(p = 11\) and \(q = 46\),

- for \(r = 1\) the area of the triangle is 23 and the value of \(\text{Int} \Delta(pr, \frac{q}{p})\) is \(23 + \frac{1}{4}\),
- for \(r = 2\) these values are 92 and 91 + \(\frac{3}{4}\).

It can be checked that for this knot (with fraction \(121/46\)) the difference between the area and the lattice point count is \(\pm \frac{1}{4}\) for all \(r \in \{1, \ldots, 10\}\). It therefore satisfies the necessary condition.

More generally, it was found that the following three families of 2-bridge knots satisfy the necessary condition to be ribbon given in Theorem 3.1.

- Family 0: \(C(a, b, \ldots, w, x, x + 2, w, \ldots, b, a)\), with parameters \(> 0\),
- Family 1: \(C(2a, 2, 2b, -2, -2a, 2b)\), with \(a, b \neq 0\),
- Family 2: \(C(2a, 2, 2b, 2a, 2, 2b)\), with \(a, b \neq 0\),

These three families were for a long time the unique known families of 2-bridge ribbon knots satisfying the necessary condition (see [1] and [10]) and families 0 and 2 where shown to be ribbon (the status of family 1 remained unclear, see end of Section 4). It was a conjecture formulated in 1974 that there are no other 2-bridge ribbon (or slice) knots. Paolo Lisca proved this in 2006, see [20].

4. Symmetric union presentations for the 3 families

Our main theorem is:

**Theorem 4.1.** Every knot contained in one of the three families is a symmetric union.

**Proof.** The proof is given by the following knot diagram transformations. For family 0 we have \(C(a, b, \ldots, w, x + 1, x - 1, w, \ldots, b, a) = C(a, b, \ldots, w, x, 1, -x, -w, \ldots, -b, -a)\) which is a symmetric union, see Figure 8.
This family was already considered by Kanenobu in 1986 to construct knots with the same Jones or Homflypt polynomial, see [6].

For families 1 and 2 we have the following diagrams:
Surprisingly, it is necessary to insert extra 2a/-2a twists to obtain the symmetric diagram for family 1.

By Theorem 2.2 this shows that the knots in the three families are ribbon knots. For family 1 this seemed, in 2005, to be the first time that their ribbon property was explicitly given. See also the remark in [20] at the end of the first section and section 3 in [15].

We explain the steps leading to the proof of Theorem 4.1. We generated symmetric union diagrams and checked if they represent 2-bridge knots using Knotscape [8] and the list in [2]. Four families of diagrams could be distinguished in this list of symmetric 2-bridge knot diagrams. For each family we found a diagram transformation from the symmetric diagram to a plait normal form (corresponding to the direction from right to left in the above figures) and it was possible to reduce the number of families to three. The proof reverses the direction of the diagram transformations.

5. Open questions and a project

The article of 2005 ended with the following conjectures and a suggested project.

**Conjecture 5.1.** Every 2-bridge ribbon knot is contained in one of the three families of Casson and Gordon.

**Conjecture 5.2.** Every ribbon knot is a symmetric union.

As mentioned, the first conjecture was proved in [20]. The second is open.
Project:

- All ribbon knots and symmetric unions with minimal crossing number $\leq 10$ are known. Extend this list to knots with minimal crossing number 11 (is the Conway knot ribbon?).

The project was tackled by Axel Seeliger [24]. He considered knots with crossing number $\leq 14$. His work and the search for ribbon knots not representable as symmetric unions is described in [19] – taking this information into account we believe that Conjecture 5.2 is wrong. Recently, Lisa Piccirillo proved that the Conway knot is not slice [23].

6. Postscriptum

As this article is originally from 2005 we comment on later developments.

A detailed article on the (still open) number-theoretic question related to 2-bridge ribbon knots is [15]. This question asks whether the condition in Theorem 3.1 is sufficient for 2-bridge knots to be ribbon. Additional related topics are discussed in the recent articles [17] and [21].

We mention a few more articles on the topic of symmetric unions: by Paolo Aceto [11], Carlo Collari and Paolo Lisca [12, 13], Toshifumi Tanaka [25, 26] and Feride Ceren Kose [18]. Maggie Miller used symmetric unions as examples (in particular the diagram for family 2) in Section 6 of [22].

We end this article with two new projects:

(1) We propose to study the following relationship: two knots are related if they are partial knots for the same knot, given as (two different) symmetric unions. Of course, by Theorem 2.1 two related knots share the same determinant. Are there further restrictions? Prove that there are knots with the same determinant which are not related.

(2) The second project proposes to find bounds for the crossing number of a symmetric union in terms of the crossing number of its partial knot. In the upper direction, it is easy to see that there is no such bound. For the lower bound, we look at some examples first:

Figure 3 shows a symmetric union for $10_{87}$ with partial knot $6_1$. Because the crossing number of $6_1\#6_1$ is 12 and three additional crossings are added it is remarkable that the crossing number of the resulting knot is as small as 10. Motivated by this example we define a new knot invariant: for each knot $K$ let $\delta(K)$ be the difference between $2 \cdot c(K)$ and the lowest crossing number of a symmetric union with partial knot $K$. Because $c(K \# K - K) \leq 2 \cdot c(K)$, this invariant is non-negative.

Examples: a) $\delta(3_1) = 0$, because there are no knots with determinant 9 and crossing number less than 6. b) $\delta(5_1) = 2$, because $8_8$ and $4_1\#4_1$ are symmetric
unions with partial knot $\tilde{5}_1$ and there is no knot with determinant 25 and
crossing number less than 8. c) $\delta(6_1) = 2$, because the smallest crossing number
of knots with determinant 81 is 10 and we indeed found the case of 10_{87} (see
Figure 3).

The first unknown value in the table of knots seems to be for $6_3$: there are
no knots with determinant 169 and crossing number less than 11 and according
to [19] the two possibilities with 11 crossings (11a164 and 11a326) have not yet
occurred as symmetric unions with partial knot $6_3$, hence $\delta(6_3)$ is 0 or 1.

Taken over all knots, this invariant is not bounded. To see this, use the
family from Section 6.4 in [16]. As in general $\delta$ seems to increase for knots with
increasing crossing number, we conjecture that there are only finitely many
prime knots with $\delta(K) = 0$, and even stronger that for each $d \geq 0$ there are
only finitely many prime knots with $\delta(K) = d$.

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