Quantum correlation measure in arbitrary bipartite systems

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Abstract – Quantum correlation with a novel definition is presented for an arbitrary bipartite quantum state in terms of the skew information of the complete set of rank-one orthogonal projectors. This definition not only inherits the good properties of skew information including the contractivity, but also shows a powerful analytic computability for a large range of states. In addition, the measure for a general state can be easily numerically obtained by the well-developed technique of the approximate joint diagonalization. As a comparison, we give both the analytic and the numerical quantum correlation for many high-dimensional states. The relation between our measure and quantum metrology is also analyzed.

Introduction. – When quantum correlation is mentioned, one might be immediately tempted to think of quantum entanglement which results from the merging of the superposition principle of states and the tensor structure of the composite quantum state space. This is not strange because as one of the most intriguing feature of quantum mechanics that distinguishes the quantum world from the classical one, quantum entanglement has been employed in most of the quantum information processing tasks (QIPTs) and has been paid close attention to in wide field [1]. However, quantum entanglement cannot cover all the quantumness of correlations in a composite quantum system. It has been shown that some QIPTs without any quantum entanglement might still demonstrate quantum advantage if such QIPTs own quantum discord [2–9] which was first introduced as the discrepancy between the generalizations of two classically equivalent mutual information [10]. This could be one of the potential reasons why quantum discord has attracted so many interests in the past few years (see refs. [11–23] and the references therein).

As an important branch of the researches on quantum correlation, the quantification of quantum correlation is a hot topic [10,24–41]. The information theoretic definition of quantum discord is found to be only analytically calculated for some special states [10,24,25,28,29]. The geometric quantum discord can be calculated analytically for all \((2 \otimes d)\)-dimensional systems [26,27], but it confronts some contradictions [37,38]. For example: 1) it could be increased by the nonunitary evolution on the subsystem without measurements; 2) it could be reduced by an extra product state. In order to avoid these unexpected properties, some new measures based on the trace norm [32–36] are proposed at the cost of losing the computability for a general state compared with the geometric discord. In addition, some variational attempts [37,39] only cover 2). Even though a recent progress [36] according to the skew information [42] has effectively covered both the aspects, it is ineffective for the high-dimensional mixed states and even pure states. It is much like the entanglement [1] that the effective quantification of correlation is still restricted to some particular states [41] due to the complex optimization for the high-dimensional case. So far it has still been an open question how to provide a good quantification or even an effective and easily comprehensible algorithm for the quantum correlation in a high-dimensional quantum system.

In this paper, we give a definition of quantum correlation pertaining to arbitrary bipartite quantum systems based on the skew information of the complete set of rank-one projectors. This definition automatically inherits the good properties such as the contractivity, so it is guaranteed to be a good measure. It is also found to have an interesting relation with quantum metrology. Our measure has two obvious advantages: 1) it can be analytically
calculated for a large range of quantum states; 2) the optimization question covered in our measure can be naturally converted to an existing and easy optimization question (approximate joint diagonalization (AJD)) that can be steadily and effectively solved by the well-developed technique. As a demonstration, we give the analytic expressions for an arbitrary pure state, the \( (2 \otimes 2) \)-dimensional states, a type of positive partial transpose (PPT) states, the high-dimensional Werner states and the Isotropic states, whilst as a comparison, we also provide the numerical expressions for these states by employing the Jacobi algorithm for AJD. The results show, on the one hand, the power of our definition of quantum correlation and, on the other hand, the effectiveness of the proposed numerical method. The paper is organized as follows. In the following section, we first present our definition for quantum correlation and elaborate the good properties including the analytic result for an arbitrary pure state. In the third section, we convert our measure into the standard AJD question and briefly discuss the effectiveness. In the fourth section, the relation between our measure and quantum metrology is analyzed. In the fifth section, \( \{ 1 \} \) has been employed in many fields [43–45], and has many good properties [36]. For example, it vanishes if and only if \( \rho \) and \( O \) commute and it is positive in other cases; it does not increase under classical mixing; in particular, if we select an observable \( O = K_A \otimes 1_B \) with \( K_A \) some observable on subsystem \( A \). \( I(\rho_{AB}, O) \) is contractive under completely positive and trace-preserving maps \( \Phi \) on \( B \), that is, \( I((1_A \otimes \Phi)\rho_{AB}, K_A \otimes 1_B) \leq I(\rho_{AB}, K_A \otimes 1_B) \).

In order to quantify the quantum correlation, we would like to restrict ourselves to the \emph{rank-one} local orthogonal projectors \( K_A \) operated on subsystem \( A \). Let \( \rho_{AB} \) be an \( (m \otimes n) \)-dimensional density matrix and suppose

\[
K_k = |k\rangle \langle k| \otimes 1_n
\]

with \( |k\rangle \) in the arbitrary orthonormal set \( S \) of subsystem \( A \), we will be able to find a useful quantity \( Q(\rho_{AB}) \) as follows.

**Definition 1.** \( Q(\rho_{AB}) \), which collects minimal total contributions of the skew information induced by a group of orthonormal projectors, is defined by

\[
Q(\rho_{AB}) := -\frac{1}{2} \min_k \sum_{k=0}^{m-1} \text{Tr}[\sqrt{\rho_{AB}, K_k}]^2 ,
\]

with \( K_k \) defined by eq. (2).

With this definition, we would like to claim the following theorem.

**Theorem 1.** \( Q(\rho_{AB}) \) vanishes if and only if \( \rho_{AB} \) is a classical-quantum state.

**Proof.** To begin with, we should note that the set of classical-quantum (CQ) states includes the set of classical-classical (CC) states. Let \( \rho_{AB} = \sum \lambda_k |\tilde{k}\rangle \langle \tilde{k}| \otimes \rho_k \) which is obviously a CQ state, one can always find such a \( \tilde{k} \) that \( [\rho_{AB}, K_{\tilde{k}}] = 0 \) holds for all \( \tilde{k} \). On the contrary, given an arbitrary \( \rho_{AB} \), if \( [\rho_{AB}, K_{k_1}] = 0 \) for some particular \( k_1 \), we can write

\[
\rho_{AB} = \lambda_{k_1} |k_1\rangle \langle k_1| \otimes \rho_1 + \rho_{k_1}, \quad (4)
\]

where \( \rho_{k_1} \) means that it can be completely expanded in the orthogonal space of \( |k_1\rangle \langle k_1| \) and \( \lambda_{k_1} \geq 0 \). If \( \rho_{AB} \) given in eq. (4) continues to commuting with \( K_{k_2} \) with \( |k_1| \langle k_2| = 0 \), one can further write \( \rho_{AB} \) as

\[
\rho_{AB} = \lambda_{k_1} |k_1\rangle \langle k_1| \otimes \rho_1 + \lambda_{k_2} |k_2\rangle \langle k_2| \otimes \rho_2 + \rho_{k_1} \cap k_2. \quad (5)
\]

If \( [\rho_{AB}, K_{k_j}] = 0 \) holds for all \( k_j \) such that \( \sum_j |k_j| \langle k_j| = 1_m \), one will draw the conclusion that

\[
\rho_{AB} = \sum_j \lambda_{k_j} |k_j\rangle \langle k_j| \otimes \rho_j, \quad (6)
\]

which is obviously a CQ state. The proof is completed.

Since \( Q(\rho_{AB}) \) can effectively evaluate whether a state is quantum correlated or not, now, we will show that actually \( Q(\rho_{AB}) \) is also a good measure of quantum correlation, \( i.e. \), it satisfies all the good properties that a measure should satisfy.

i) \( Q(\rho_{AB}) \) is invariant under local unitary operations. It is apparent that \( I(\rho_{AB}, O) \) is invariant under local unitary operations, which is analogous to the proof in ref. [36].

ii) \( Q(\rho_{AB}) \) is contractive under completely positive and trace-preserving maps \( \Phi \) on \( B \). Since \( I((1_m \otimes \Phi)\rho_{AB}, K_m \otimes 1_B) \) given in eq. (1) is contractive for the observable \( K_m \), the same property is inherited by \( \sum_k I(\rho_{AB}, K_k) \). For an optimal set \( \{ K_k \} \), \( Q(\rho_{AB}) \geq \sum_k I((1_m \otimes \Phi)\rho_{AB}, K_k) \geq Q((1_m \otimes \Phi)\rho_{AB}) \).

iii) \( Q(\rho_{AB}) \) is reduced to entanglement for pure states. Because \( Q(\rho_{AB}) \) is not changed by the local unitary operations, we can safely consider the pure state in the form of a Schmidt decomposition which is given by \( |\chi\rangle_{AB} = \sum_{i=0}^{r-1} \mu_i |ii\rangle_{AB} \) with \( \mu_i \) the Schmidt coefficients and \( r = \min\{m, n\} \). Substitute \( |\chi\rangle_{AB} \) into eq. (3), one will
easily find that
\[
Q(\rho_{AB}) = 1 - \max_S \left[ \sum_{k=0}^{n-1} \left| \sum_{i,j=0}^{r-1} \mu_i \mu_j \langle \langle k | \otimes | l_n \rangle | jj \rangle_{AB} \right|^2 \right] \\
= 1 - \max_S \left\{ \sum_{k=0}^{n-1} \left| \sum_{i=0}^{r-1} \mu_i^2 | i \rangle_{A} \langle i | k \rangle \right|^2 \right\} \\
\geq 1 - \max_S \sum_{k=0}^{n-1} \mu_k^4 = 1 - \operatorname{Tr} \rho_{\theta_0}^2, \tag{7}
\]
where \(\rho_{\theta_0}\) is the reduced density matrix of \(\rho_{AB}\) and the "\(\rho_{\theta_0}\)" in eq. (7) can always be satisfied if the optimized set \(S = \{|i\}\). It is obvious that eq. (7) is actually equivalent to the remarkable entanglement measure, (a half) squared concurrence for pure states [1].

iv) The maximum of \(Q\) is \(\frac{1}{r}\) in \((m \otimes n)\)-dimensional Hilbert space. To see this, we have to turn to eq. (7). For any \(k\) and the pure state \(|\chi\rangle_{AB}\), one can have \(Q_k = 1 - \sum_{k=0}^{n-1} |\langle k | \sum_{i=0}^{r-1} \mu_i^2 | i \rangle_{A} \langle i | k \rangle|^2 \leq 1 - \sum_{k=0}^{n-1} \mu_i | \langle k | \mu_i \rangle |^4 \leq 1 - \frac{1}{r}\). The first inequality is based on the convexity of the "\(\rho_{\theta_0}\)" is saturated if all \(\mu_i\) are the same, i.e., \(\mu_i = \frac{1}{\sqrt{r}}\). In other words, it is saturated by the maximally entangled states. Since any density matrix can be written as a convex sum of the pure states, due to the convexity of the skew information, one can easily find that the maximum of \(Q\) is \(\frac{1}{r}\).

The effective numerical approach for \(Q(\rho_{AB})\). The previous results show that \(Q(\rho_{AB})\) is a good measure of quantum correlation. Next, we will convert the complex optimization problem into an existing and well-done question, by which we will give the almost analytic expression for \(Q(\rho_{AB})\). To do so, we would like to first introduce the approximate joint diagonalization (AJD).

Consider \(n\) \(d\)-dimensional matrices \(M_1, M_2, \ldots, M_n\). As usual, \(M_i\) cannot be diagonalized simultaneously, since \(M_i\) could neither be Hermitian nor commute with each other. However, one can always try to find a unitary matrix \(U\) such that the optimization problem \(J(U) = \max_U \sum_{j=1}^{n} \left| U M_j U^\dagger \right|_{kk}^2\) is achieved. Such an optimization problem is a standard expression of AJD of the series of matrices \(M_i\) [42,46]. This AJD problem is widely met in the Blind Source Separation and Independent Component Analysis (See ref. [46] and the references therein) and is well solved numerically by many effective algorithms. A very remarkable algorithm is the Jacobi algorithm [42,43] which can be solved as steadily, reliably, fast and perfectly as the diagonalization of a single matrix [42,43].

The spirit of the Jacobi algorithm can be easily understood (for detailed understanding given in refs. [42,43,46]). The optimization \(J(U)\) is obviously equivalent to find the unitary matrix \(U\) such that the sum of off-diagonal contributions of \(M_i\) are minimized. The Jacobi algorithm deals with such a question by decomposing \(U\) in \(J(U)\) into a series of unitary matrices (called Givens rotations) only operated on \((2 \times 2)\) subspace of the \(n M_i\). In other words, each Givens rotation only simultaneously deals with \(n\) \((2 \times 2)\) block matrices. It happened that for such a series of \((2 \times 2)\) block matrices, the Givens rotation is uniquely and analytically determined such that the off-diagonal contribution in each \(M_i\) is transferred to the corresponding diagonal parts to the maximal extent, whilst \(M_i\) is updated. When all the subspace is done with once, it is called one sweep. The sweep is not done until no contribution subject to any preconditioned precision could be transferred. Since there are \(mn(m-1)\) off-diagonal entries for one \(M_i\), it is obvious that the AJD of \(n M_i\) needs \(\frac{d(d-1)}{2}\) Givens rotations for one sweep [42] for all \(M_i\) which is like the diagonalization of \(n\) \(M_i\) [42]. In particular, it was said that about \(\log(m)\) sweeps were needed for a single matrix \(M_i\) to an acceptable precision (Examples show around \(10^{-10}\) in [43]).

Suppose such an optimization question is done with some satisfying precision. Let \(U_o\) be the optimal unitary matrix such that the optimal value of \(J(U)\) can be attained, then \(U_o\) is called as the joint diagonalizer of all the \(M_i\) and \(\lambda_k = (U_o M_i U_o^\dagger)_{kk}\) is called as the \(k\)-th joint eigenvalue of \(M_i\). With this knowledge, we can give our Theorem 2.

**Theorem 2.** Let \(|i\rangle\) and \(|j\rangle\) belong to a set of orthonormal bases of the subspace \(B\) of the \((m \otimes n)\)-dimensional state \(\rho_{AB}\), and the matrices \(A_{ij} = (1_m \otimes |i\rangle \langle j|)_{AB}(1_n \otimes |j\rangle \langle i|)\), the quantum correlation \(Q(\rho_{AB})\) of \(\rho_{AB}\) defined in eq. (3) can be explicitly given by

\[
Q(\rho_{AB}) = 1 - \sum_{i,j,k}^{n-1} m-1 \sum_{k=0}^{n} |\lambda_{ij}^k|^2 ,
\]

where \(\lambda_{ij}^k\) is the \(k\)-th joint eigenvalue of \(A_{ij}\) which is defined by \((U_o A_{ij} U_o^\dagger)_{kk}\) with \(U_o\) the joint diagonalizer of all the \(A_{ij}\).

**Proof.** From eq. (3), one will directly arrive at

\[
Q(\rho_{AB}) = \min_S \sum_{k=0}^{m-1} \left[ \operatorname{Tr} \rho_{AB} K_k^2 - \operatorname{Tr} \sqrt{\rho_{AB} K_k \sqrt{\rho_{AB} K_k}} \right] \\
= 1 - \max_S \sum_{k=0}^{m-1} \operatorname{Tr} \sqrt{\rho_{AB} K_k \sqrt{\rho_{AB} K_k}}. \tag{9}
\]

By substituting \(K_k = |k\rangle \langle k| \otimes 1_n\) and any orthonormal bases of subsystem \(B\) into eq. (9), it follows that

\[
Q(\rho_{AB}) = 1 - \max_S \sum_{i,j,k}^{n-1} \sum_{k=0}^{m-1} \operatorname{Tr} \sqrt{\rho_{AB} \langle k | \langle k | \otimes | i \rangle \langle i | \rangle} \\
\times \sqrt{\rho_{AB} (|k\rangle \langle k| \otimes | j \rangle \langle j|)} \\
= 1 - \max_S \sum_{i,j,k}^{n-1} \sum_{k=0}^{m-1} |\langle k | A_{ij} | k \rangle|^2 = 1 - \tilde{J}(U), \tag{10}
\]
with $A_{ij} = (1_m \otimes \langle i |) \sqrt{\rho_{AB}} (1_m \otimes | j \rangle)$ and

$$
\tilde{J}(U) = \max_U \sum_{i,j=0}^{n-1} \sum_{k=0}^{m-1} |U A_{ij} U^\dagger|_{kk}^2.
$$

(11)

Thus, our calculation of the quantum correlation is directly changed into the optimization problem of eq. (11) which is the standard AJD introduced previously. In particular, we emphasize that $A_{ij}$ is not necessarily Hermitian. Let $U_\alpha$ be the joint diagonalizer of all the $A_{ij}$ and assume $X^\alpha_{ij} = (U_\alpha A_{ij} U_\alpha^\dagger)_{kk}$, one will easily find that the final expression of $Q(\rho_{AB})$ can be written as eq. (8). It is obvious that if $[A_{ij}, A_{kl}] = 0$ holds for all $A$, the question can be exactly solved. Of course, that these $A_{ij}$ commute with each other is just a sufficient condition for the exact solution of eq. (8), which will be seen from our latter examples. In addition, the number of the matrices that need to be AJD can be reduced further based on ref. [44].

Before proceeding, we would like to emphasize that the complexity of the computation of our measure is at the same level of the diagonalization of $\rho_{AB}$ by the same method. Since the density matrix $\rho_{AB}$ is $(m \otimes n)$-dimensional, one will generally have $n^2 m$-dimensional $A_{ij}$. Thus, based on the Jacobi algorithm, the diagonalization of $\rho_{AB}$ needs $\frac{m(m-1)}{2}$ Givens rotations for one sweep, but AJD for $A_{ij}$ needs $\frac{m(m-1)n^2}{2}$ Givens rotations for one sweep. In this sense, we say that calculating $Q(\rho_{AB})$ is as the same level as diagonalizing $\rho_{AB}$.

**Relation with quantum metrology.** Here we will show that our quantum correlation in Definition 1 connects some quantum metrology scheme in an interesting way. Let $\{\{k\}\}$ denote the group of the optimal orthonormal set of projectors that achieves the exact value of $Q(\rho_{AB})$. Assume the state $\rho_{AB}$ is an $(m \otimes n)$-dimensional probing state with subsystem $A$ undergoing one (or series of) unitary transformation which endows some unknown phases $\phi_k$ on $\rho_{AB}$ by $\rho_2 = e^{-i H(\phi_k)} \rho_{AB} e^{i H(\phi_k)}$ with $H(\phi) = \sum_{k=1}^N \phi_k |k\rangle \langle k | \otimes 1_n$. We aim to estimate $\phi_k$ one by one by $N$ runs of detection with high precision quantified by the uncertainty of the estimated phase $\phi_k$ defined by $(\delta \phi_k)^2 = \left(\frac{\partial \phi_k}{\partial \sigma_{ij}}\right)^2$ (45). This variance $\delta \phi_k$, for an unbiased estimator, is bounded by the quantum Cramér-Rao bound $(\delta \phi_k)^2 \geq \frac{1}{N F_{QK}}$ that can be attained asymptotically by the projective measurements in the basis of the symmetric logarithmic derivative operator and the maximum likelihood estimation, where $F_{QK}$ given by $F_{QK} = -\text{Tr} [\sqrt{\rho_{AB}} K_k^2]$ is the quantum Fisher information subject to the phase $\phi_k$ [44,45]. So one can easily find that

$$
\sum_k \frac{1}{N} (\delta \phi_k)^2 \leq \frac{1}{N} F_{QK} = 2Q(\rho_{AB}).
$$

(12)

Note that the skew information is invariant under local unitary operations, so when one estimates one phase, the other phases are equivalent to local unitary operations. This should be distinguished from the standard multivariable estimation [47]. In the current setting, the “=” in eq. (12) cannot be satisfied since the other $\phi_j$ are unknown, when we are estimating one $\phi_k$. That is, our quantum correlation measure provides an upper bound that characterizes the contributions of all the inverse variances of the estimated phases. In this probing experiment, it is implied that all the phases are imposed before we estimate any phase. However, in a different setting, we can require that only a single $\phi_k$ can be endowed by the similar unitary transformation and the phase should be estimated before another one was imposed. Thus when we repeat the probing experiment $m$ times with all $|k\rangle$ considered, it will be found that each time the minimal value of $(\delta \phi_k)^2$ can be achieved if the proper measurements chosen. Thus, the “=” in eq. (12) can hold. In this case, we have actually finished $m$ single-parameter estimations. However, the $m$ single-parameter estimations are not simply equivalent to $m$-independent single-parameter estimations. In particular, in practical scenarios we do not know the optimal basis $|k\rangle$, so to find the optimal one, we have to consider the sum of $F_{QK}$, i.e., the minimal $\sum_k F_{QK}$, generated in the $m$ experiments. As a comparison, the local quantum uncertainty [36] is only related to the variance of a single-phase estimation.

**The applications.** From the following examples, one will find that $Q(\rho_{AB})$ can be analytically solved, whilst these examples will demonstrate the effectiveness of the AJD method in the calculation for high-dimensional systems and illustrate the perfect consistency between the strictly analytic solutions and the almost analytic ones obtained by the AJD method.

a) **Qubit-qudit states.** As a comparison with the previous jobs, we will first consider the quantum correlation of a $(2 \otimes d)$-dimensional state. For such a state $\rho_{AB}$, eq. (3) can be rewritten as

$$
Q(\rho_{AB}) = -\frac{1}{2} \min_S \sum_{k=0}^{n-1} \text{Tr} [\sqrt{\rho_{AB}} K_k] \langle K_k \rangle^2
$$

$$
= -\frac{1}{2} \min_S \text{Tr} [\sqrt{\rho_{AB}} K_0] \langle K_0 \rangle^2.
$$

(13)

Since any pure state can be expanded in the Bloch representation, one can always write $K_0$ as

$$
K_0 = \frac{1}{2} (I_d + \tilde{n} \cdot \tilde{\sigma}) \otimes 1_d
$$

(14)

with $\tilde{n}_i^2 = 1$. Substitute eq. (14) into eq. (13), one will arrive at

$$
Q(\rho_{AB}) = \frac{1}{2} - \frac{1}{2} \max \sum_{i,j} \text{Tr} n_i T_{ij} n_j
$$

$$
= \frac{1}{2} (1 - v_{\text{max}}),
$$

(15)

where $v_{\text{max}}$ is the maximal eigenvalue of the matrix $T$ with

$$
T_{ij} = \text{Tr} [\sqrt{\rho_{AB}} (\sigma_i \otimes 1_n) \sqrt{\rho_{AB}} (\sigma_j \otimes 1_n)].
$$

(16)
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Equation (15) happened to be the half of that in ref. [36].

b) \((3 \otimes 3)\)-dimensional PPT states. Let us consider such a PPT state given by [48]

\[
\rho_{PPT} = \frac{2}{3} |\Phi\rangle \langle \Phi| + \frac{\alpha}{3} \rho_+ + \frac{5 - \alpha}{3} \rho_-, \quad \alpha \in [2, 4],
\]

where \(|\Phi\rangle_m = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |kk\rangle\) and \(\rho_+ = \frac{1}{3} \sum_{k=0}^{2} |k, k+1\rangle \langle k, k+1|\) and \(\rho_- = \frac{1}{3} \sum_{k=0}^{2} |k \oplus 1, k\rangle \langle k \oplus 1, k|\) with \(\oplus\) the modulo-3 addition. Note that, only when \(\alpha \in (3, 4)\), PPTPPT is entangled. If \(\alpha \leq 3\), PPTPPT is separable. But if \(4 < \alpha \leq 5\), PPTPPT is not a PPT state, but a free entangled state. For integrity, we also consider this type of free entangled states here. It is interesting that \(Q(\rho_{PPT})\) can be analytically solved for \(\alpha \in [2, 5]\), which by

\[
Q(\rho_{PPT}) = \begin{cases}
\frac{21 - \sqrt{6(5-\alpha) - \sqrt{65 - 3\sqrt{\alpha(5-\alpha)}}}}{31.5}, & 2 \leq \alpha \leq N_T, \\
\frac{4}{\pi}, & N_T < \alpha \leq 5
\end{cases}
\]

with \(N_T = \frac{15 + \sqrt{136 \cdot \sqrt{136} - 1307}}{3} = 3.066885\). The numerical results based on our Theorem 2 are plotted in fig. 1, which shows the perfect consistency between our Theorem 2 and the strict analytic expression. In particular, we can analytically find the sudden change point of quantum correlation near the critical point of the separable state and the bound entangled state.

c) Werner states and Isotropic states in \((m \otimes m)\) dimension. Besides the above examples, our quantum correlation measure \(Q(\cdot)\) for both the \((m \otimes m)\)-dimensional Isotropic states and Werner states [48] can be analytically calculated. Thus they can serve as important examples that show the effectiveness of \(Q(\cdot)\) for larger systems. The Werner state can be written as

\[
\rho_W = \frac{m-x}{m^2-m} |1, m^2-I\rangle \langle 1, m^2-I| + \frac{x}{m^2-m} |V, x\rangle \langle V, x|,
\]

with \(V = \sum_{k=0}^{m-1} |k\rangle \langle k|\) the swap operator. This state has no quantum correlation if and only if \(x = \frac{1}{m}\). Through a simple algebra, one can have the analytic expression of the quantum correlation as follows:

\[
Q(\rho_W) = \frac{m-x-\sqrt{m^2-I\sqrt{1-x^2}}}{2(1+m)}.
\]

From eq. (20), one will also find that \(Q(\rho_W) = 0\) for \(x = \frac{1}{m}\). Analogously, we also plot \(Q(\rho_W)\) based on eq. (19) and eq. (8), respectively, in fig. 2(a) which shows a perfect consistency. The isotropic state can be given by

\[
\rho_I = \frac{1-x}{m^2-I} |1, m^2-I\rangle \langle 1, m^2-I| + \frac{m^2-I}{m^2-I} |\Phi\rangle \langle \Phi|, \quad x \in [0, 1],
\]

with \(|\Phi\rangle = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |kk\rangle\). Based on our definition, we can analytically obtain

\[
Q(\rho_I) = \frac{1-2\sqrt{m^2-I\sqrt{x(1-x)}} + (m^2-2)x}{m(1+m)}.
\]

It is obvious that \(x = \frac{1}{m}\) will lead to \(Q(\rho_I) = 0\), which is consistent to ref. [24]. As a comparison, we plot \(Q(\rho_I)\) given by eq. (8) and eq. (22), respectively, in fig. 2(b) which shows a perfect consistency again.

Conclusions and discussion. – We have presented a new definition of quantum correlation for any bipartite quantum system with some good properties. In particular, this measure can be used to obtain the analytic expression for a large number of states. In addition, this measure can be easily calculated by a well-developed numerical way.
This measure has been effectively related to the quantum metrology. As applications, we have found that the quantum correlations of many quantum states can be strictly analytically solved, which also provides a direct support for the effectiveness of our Theorem 2.

Finally, we would like to emphasize that the AJD technique plays an important role in the numerical process. Whether it can induce other contributions to the relevant researches such as quantum correlation of other forms, quantum entanglement measure, quantum nonlocality etc. is worthy of our forthcoming efforts.

** REFERENCES **

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REFERENCES

[1] Horodecki R., Horodecki P., Horodecki M. and Horodecki K., *Rev. Mod. Phys.*, **81** (2009) 865, and the references therein.
[2] Knill E. and Laflamme R., *Phys. Rev. Lett.*, **81** (1998) 5672.
[3] Datta A., Shah A. and Caves C. M., *Phys. Rev. Lett.*, **100** (2008) 050502.
[4] Lanyon B. P., Barbieri M., Almeida M. P. and White A. G., *Phys. Rev. Lett.*, **101** (2008) 200501.
[5] Roa L., Retamal J. C. and Alid-Vaccarezza M., *Phys. Rev. Lett.*, **107** (2011) 080401.
[6] Chuan T. K. et al., *Phys. Rev. Lett.*, **109** (2012) 070501.
[7] Kay A., *Phys. Rev. Lett.*, **109** (2012) 080503.
[8] Yu Chang-shui et al., *Phys. Rev. A*, **87** (2013) 022113.
[9] Datta A. and Vidal G., *Phys. Rev. A*, **75** (2007) 042310.
[10] Henderson L. and Vedral V., *J. Phys. A: Math. Theor.*, **34** (2001) 6899; Ollivier H. and Zurek W. H., *Phys. Rev. Lett.*, **88** (2001) 017901.
[11] Modi K. et al., *Rev. Mod. Phys.*, **84** (2012) 1655.
[12] Werlang T. et al., *Phys. Rev. A*, **80** (2009) 024103; *Phys. Rev. Lett.*, **105** (2010) 095702.
[13] Yu Chang-shui, Zhang Jun and Fan H., *Phys. Rev. A*, **86** (2012) 052317.
[14] Maziero J. et al., *Phys. Rev. A*, **80** (2009) 044102; Mazzola L., Filo J. and Maniscalco S., *Phys. Rev. Lett.*, **104** (2010) 200401.
[15] Sarandy M. S., *Phys. Rev. A*, **80** (2009) 022108.
[16] Chen Y. X. and Li S. W., *Phys. Rev. A*, **81** (2010) 0323120.
[17] Liu B. Q. et al., *Phys. Rev. A*, **83** (2011) 052112.
[18] Luo S. L. and Fu S. S., *Phys. Rev. Lett.*, **106** (2011) 120401.
[19] Lang M. D. and Caves C. M., *Phys. Rev. Lett.*, **105** (2010) 150501.
[20] Streitsof A., Kampermann H. and Bruß D., *Phys. Rev. Lett.*, **108** (2012) 205051.
[21] Shabani A. and Lidar D. A., *Phys. Rev. Lett.*, **102** (2009) 100402.
[22] Silva I. A. et al., *Phys. Rev. Lett.*, **110** (2013) 140501.
[23] Jin Jin-sen, Zhang F.-Y., Yu Chang-shui and Song He-shian, *J. Phys. A: Math. Theor.*, **45** (2012) 115308.
[24] Luo S. L., *Phys. Rev. A*, **77** (2008) 042303; Luo S. L. and Fu S. S., *Phys. Rev. A*, **82** (2010) 034302.
[25] Chudamani E., *Phys. Rev. A*, **86** (2012) 0323110.
[26] Dakić B., Vedral V. and Brukner C., *Phys. Rev. Lett.*, **105** (2010) 190502.
[27] Vinjanampathy S. and Rau A. R. P., *J. Phys. A: Math. Theor.*, **45** (2012) 095303.
[28] Modi K. et al., *Phys. Rev. Lett.*, **104** (2010) 080501.
[29] Ali M., Rau A. R. P. and Alber G., *Phys. Rev. A*, **81** (2010) 042105.
[30] Yu Chang-shui and Zhaoyi Haiqing, *Phys. Rev. A*, **84** (2010) 062312.
[31] Yu Chang-shui, Zhang Y. and Zhaoyi Haiqing, *Quantum Inf. Process.*, **13** (2014) 1437.
[32] Bartkiewicz K., Lemi K. et al., *Phys. Rev. A*, **87** (2013) 062102.
[33] Paula F. M., de Oliveira T. R. and Sarandy M. S., *Phys. Rev. A*, **87** (2013) 064101.
[34] Cincarello F., Tufarelli T. and Giovannetti V., *New J. Phys.*, **16** (2014) 013038.
[35] Aaronson B., Franco R. L., Compagno G. and Adesso G., *New J. Phys.*, **15** (2013) 095022.
[36] Girolami D., Tufarelli T. and Adesso G., *Phys. Rev. Lett.*, **110** (2013) 240402.
[37] Piani M., *Phys. Rev. A*, **86** (2012) 034101.
[38] Hu X. et al., *Phys. Rev. A*, **87** (2013) 032340.
[39] Tufarelli T. et al., *J. Phys. A: Math. Theor.*, **46** (2013) 275308; Chang L. and Luo S., *Phys. Rev. A*, **87** (2013) 062303.
[40] Wigner E. P. and Yanase M. S., *Proc. Natl. Acad. Sci. U.S.A.*, **49** (1963) 910; Luo S., *Phys. Rev. Lett.*, **91** (2003) 180403.
[41] Sen A., Bhar A. and Sarkar D., arXiv:1304.7019v1 [quant-ph]; Wang Shuhao et al., arXiv:1307.0576v2 [quant-ph].
[42] Cardoso J. F. and Souloumiac A., *IEE Proc. F*, **140** (1993) 362; *SIAM J. Mater. Anal. Appl.*, **17** (1996) 161.
[43] Golub G. H. and Van Loan C. F., *Matrix Computations*, 3rd edition (The Johns Hopkins University Press) 1996.
[44] Yu Chang-shui and Song He-shian, *Phys. Rev. A*, **73** (2006) 032322; *Van Loan C. F. and Pitsianis N. P., in Linear Algebra for Large Scale and Real Time Applications*, edited by Moonen M. S. and Golub G. H. (Kluwer, Dordrecht) 1993, pp. 293–314.
[45] Braunstein S. L. and Caves C. M., *Phys. Rev. Lett.*, **72** (1994) 3439; Dorrer U. et al., *Phys. Rev. Lett.*, **102** (2009) 040403.
[46] Zeh H. et al., *J. Mach. Learn. Res.*, **5** (2004) 777.
[47] Helstrom C. W., *J. Stat. Phys.*, **1** (1969) 231; Brody D. C., *J. Phys. A: Math. Theor.*, **44** (2011) 252002.
[48] Alber G., Beth T., Horodecki M. et al., *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments* (Springer-Verlag, Berlin, Heidelberg) 2001.