Akizuki’s counterexample

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Abstract

Following [Akizuki], I construct a Noetherian local integral domain $C_M$ whose normalisation (integral closure) is not finite over $C_M$. My proof follows closely Akizuki’s ingenious calculations.

Let $A$ be a DVR with local parameter $t$ and residue field $k = A/(t)$, and $\hat{A}$ the completion of $A$. There are no restrictions on the characteristic of $A$ or $k$, but I assume that $\hat{A}$ contains a transcendental element over $A$. (For DVRs of interest, $\hat{A}$ usually has infinite transcendence degree over $A$.) The rings $B, C$ constructed below, and their localisations, are intermediate subrings between $A$ and $\hat{A}$.

The construction depends on a power series

$$z = z_0 = a_0 + a_1 t^{n_1} + a_2 t^{n_2} + \cdots \in \hat{A},$$

(not just on the element $z$). Assume:

(2) Each $a_i \in A$ is a unit.

(3) $n_r \geq 2n_{r-1} + 2$ for every $r \geq 1$, where I set $n_0 = 0$; for example, the smallest possible choice is $n_r = 2(2^r - 1) = 0, 2, 6, 14, 30, \ldots$.

(4) $z$ is transcendental over $A$, so that $A \subset A[Z] \subset \hat{A}$ is a polynomial extension.

Akizuki’s construction is as follows: for $r \geq 0$, let

$$z_r = \frac{z_0 \text{ - first } r \text{ terms}}{t^{m_r}} = a_r + a_{r+1} t^{m_{r+1}} + \cdots,$$

where

$$m_r = n_r - n_{r-1} \text{ so that (3) gives } 2m_r \geq n_r + 2.$$

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Then the \( z_r \) satisfy the identities
\[
\begin{align*}
z_r - a_r &= t^{m_r+1} z_{r+1}, \\
t^{m_r} z_r &= z_0 - \sum_{i=0}^{r-1} a_i t^{m_i}, \quad \text{with} \sum_{i=0}^{r-1} a_i t^{m_i} \in A.
\end{align*}
\] (6) (7)

Then set \( B = A[z_0, z_1, \ldots] = A[(z_0 - a_0), (z_1 - a_1), \ldots] \). The properties of \( B \) are easy (compare, for example, [UCA], Ex. 8.5).

**Theorem** The principal ideal \( m = tB \subset B \) is maximal, with \( B/m = k = A/(t) \), and the localisation \( B_m \) is a DVR with the same parameter \( t \).

**Proof** Consider the natural “evaluation” homomorphism \( B \to k = A/(t) \) defined by \( t \mapsto 0, \ z_r \mapsto a_r \). This is obviously surjective, and by (6), the kernel is the principal ideal \( m = tB \). The localisation \( B_m \) is a local ring; its maximal ideal \( mB_m = tB_m \) is principal; and \( \bigcap (t^n) = 0 \) in \( B_m \), because \( B_m \subset A \), and the same holds there. This proves that \( B \) is a DVR (see, for example, [UCA], Proposition 8.4) with local parameter \( t \), residue field \( B/tB = A/(t) = k \) and \( \text{Frac} B = \text{Frac} A(z) \). Q.E.D.

Now the big one: set \( C = A[t(z_0 - a_0), \{(z_i - a_i)^2\}_{i=0}^\infty] \subset B \).

**Theorem** \( M = (t, t(z_0 - a_0)) \subset C \) is a maximal ideal with \( C/M = k = A/(t) \), and the localisation \( C_M \) has the following properties:

(i) \( B \) and \( C \) have the same field of fractions:
\[
\text{Frac} B = \text{Frac} C = \text{Frac} A(z).
\]

(ii) \( B_m \) is integral over \( C_M \), so that \( B_m = \widehat{C_M} \) (the normalisation).

(iii) \( C_M \) is a 1-dimensional Noetherian local ring.

(iv) \( B_m \) is not finite as a \( C_M \)-module.

Statements (i) and (ii) are immediate. The surprise, of course, is that \( C \) is Noetherian.

**Proof** Manipulating the identities (6), (7) gives two standard tricks. First, by (7), the difference between \( t(z_0 - a_0) \) and \( t^{r+1}(z_r - a_r) \) is an element of \( A \) for any \( r \geq 0 \). This allows me to replace \( t^{n+1}(z_i - a_i) \) wherever it appears by an element of \( A \) plus \( t^{n+1}(z_j - a_j) \) with \( j > i \).

Second, consider the identity
\[
(z_{i-1} - a_{i-1})^2 = (t^{m_i} z_i)^2 = t^{2m_i}((z_i - a_i)^2 - a_i^2) + 2a_i t^{2m_i} z_i.
\]

(8)
It’s easy to check that both terms on the right are in $tC$: the second, because of (7) and the assumption $2m_r \geq n_r + 2$ (see (3) and (5)). A first consequence is that the kernel of the map $C \to k$ defined by the evaluation $t \mapsto 0$ and $z_i \mapsto a_i$ is the maximal ideal $M = (t, t(z_0 - a_0))$.

The second trick allows me to replace $(z_{r-1} - a_{r-1})^2$ wherever it appears by

$$t^{2m_r}(z_i - a_i)^2 + \text{multiple of } t^{n_r+2}(z_i - a_i) + \text{element of } A.$$ 

Performing these two tricks repeatedly gives that, for any specified $r \geq 0$ and $N > 0$, any element $f \in C$ can be written

$$f = X + Yt^{n_r+1}(z_r - a_r) + t^NZ, \quad \text{with } X, Y \in A \text{ and } Z \in C. \quad \text{(9)}$$

**Main Claim** For $0 \neq f \in M$, the principal ideal $fC_M$ contains a power of $t$.

**Proof of Claim** There exists $N$ such that $f \notin t^NA$. Choose $r$ with $n_r \geq N - 1$, and consider the expression (10). Then necessarily $X = t^nu$ with $n < N$ and $u$ a unit of $A$. Dividing through by $u$, I assume that $X = t^n$, and

$$f = t^n(1 + t^{N-n}Z) + Yt^{n_r+1}(z_r - a_r).$$

To prove the claim, multiply $f$ by $g = t^n(1 + t^{N-n}Z) - Yt^{n_r+1}(z_r - a_r)$:

$$fg = t^{2n}(1 + t^{N-n}Z)^2 - Y^2t^{2n+2}(z_r - a_r)^2.$$ 

This is obviously of the form $t^{2n}$ times an element of $C \setminus M$. Q.E.D.

I prove that the local ring $C_M$ is Noetherian and 1-dimensional. It is clear from (10) that $C_M/t^NC_M$ is generated over $A/(t^N)$ by 1 and $t^{n_r+1}z_r$, and therefore is a Noetherian $A$-module. Now any nonzero ideal $I \subset C_M$ contains $t^N$ for some $N$, and then the quotient ring $C_M/I$ is also Noetherian. Therefore bigger ideals $I \subset J \subset C_M$ have the a.c.c. A nonzero prime ideal of $C_M$ contains some $t^N$, and therefore also $t$ and $t(z_0 - a_0)$, so that Spec $C_M = \{0, MC_M\}$.

Under the assumption that $z$ is transcendental, I now prove that $B_m$ is not finite over $C_M$, arguing by contradiction. Since $C_M$ is Noetherian, if $B_m$ were finite, it would be a Noetherian $C_M$-module. Consider the ascending chain of submodules generated by $\{(z_i - a_i)\}_{i \leq r'}$; for some $r$, I get a relation

$$z_r - a_r = \sum_{i=0}^{r-1} g_i(z_i - a_i) \quad \text{with } g_i \in C_M. \quad \text{(10)}$$

Writing $g_i = f_i/f_r \in C_M$ gives

$$f_r(z_r - a_r) = \sum_{i=0}^{r-1} f_i(z_i - a_i) \quad \text{with } f_0, \ldots, f_r \in C \text{ and } f_r \notin M. \quad \text{(11)}$$
Now multiplying (11) by $t^{n_r}$ and using (7) gives

$$f_r(z - \sum_{j=0}^{r+1} a_j t^{n_j}) = \sum_{i=0}^{r-1} f_i t^{n_r-n_i}(z - \sum_{j=0}^{i+1} a_j t^{n_j}).$$

(12)

Now all the $f_i \in C$ are polynomials in $z$ with coefficients in $A$, and the left-hand side is a unit times $z$ (because $f_r \notin M$), whereas every coefficient on the right-hand side is divisible by $t$. Therefore (12) is a nontrivial polynomial relation $F(z) = 0$ with coefficients in $A$. This contradiction completes the proof of the theorem. Q.E.D.

**Exercises**

1. Use (8) to prove that $M^2 = t M$.

2. Prove that $t^{n_r}(z_r - a_r) \notin C$ for any $r \geq 0$. [Hint: following the method of (12), use $t^{n_r}(z_r - a_r) \in C$ to derive an algebraic dependence relation for $z$ over $A$.]

**History**

My treatment follows Akizuki in all essentials. Clearly under the influence of the papers of Krull and his followers, Akizuki only considers the case where $\hat{A} = \mathbb{Z}_p$ is the ring of $p$-adic integers. His proof that $C$ is infinite over $B$ is indirect. He argues by contradiction, based on the notion of “analytically unramified” (in later terminology): the element $x_r = t^{n_r+1}(z_r - a_r) \notin t C$ (by Ex. 2 above), but $x_r^2 \in t^{2n_r+2} C$. Thus $x = \lim_{r \to \infty} x_r$ is a nilpotent element of the $t$-adic completion of $C$.

As discussed in [UCA], 9.4, the real point of this counterexample, and of those of Nagata (see the appendix to [Nagata]) is that there is really no hope of making everything that works for geometric rings go through for Noetherian rings. At some time you have to make assumptions of a concrete nature, for example that your ring is finitely generated over $k$ or $\mathbb{Z}$.

**Geometric interpretation of B**

If $A = \mathbb{C}[t](0)$ and the power series $z$ has positive radius of convergence, I can consider the analytic arc $\Gamma \subset \mathbb{C}^2$ defined by $(z = z(t))$. There is an obvious sense in which $B_m$ is the ring of regular functions on $\Gamma$ that are restrictions of rational functions on $\Gamma$ that are restrictions of rational functions of $t, z$.

More algebraically, for each $r$, I can view $\mathbb{A}_r = \text{Spec } A[z_r]$ as the “affine plane” with coordinates $t, z_r$, or its germ at $(t = 0, z = a_r)$. The inclusion of rings $\text{Spec } A[z_{r-1}] \subset \text{Spec } A[z_r]$ corresponds to the “blow-up” $\mathbb{A}_r \to \mathbb{A}_{r-1}$ defined by $(t, z_r) \mapsto (t, a_r + t^{n_r} z_{r-1})$. The limit $\text{Spec } B$ is the surface in infinite dimensional space defined by the relations (11). The projection to each $\mathbb{A}_r$ can be viewed as an infinitely thin cusp-shaped region around the analytic arc $z = z(t)$.

Rings like $B$ are interesting because of their proclivity to dimensional ambiguity, arising from the question as to whether or not (1) is a functional dependence relation $z = z(t)$. This ambiguity is the starting point for Nagata's
examples of noncatenary rings, see \cite{Nagata}, Example 2, p. 203 or \cite{UCA}, Example 9.4, (2). As we have seen, $B$ becomes a DVR when localised at $(t)$, because modulo $t^N$ the identities (6)–(7) imply the “obvious” relation $z = z(t)$. On the other hand, if I delocalise $A$ (taking, say, $A = k[t]$), I can pass to the ring of fractions $B[1/t]$. Then the identities (6) give all the $z_i$ as functions of $z = z_0$, so that, assuming $z$ is transcendental over $A$, $B[1/t] = A[1/t][z]$ is clearly 2-dimensional (for example, $B = k[t,z][1/t]$ is just polynomial functions on the $z,t$ plane).

**Geometric interpretation of $C$** Even after the event, I don’t know how to motivate Akizuki’s example to make it completely natural, and it’s hard to imagine how he discovered his incredibly ingenious construction.

For what it’s worth, I have in mind the following geometric picture, by analogy with the above picture of $B$: the ring $C \subset B$ is the union over $r$ of subrings

$$A[t^{n_r+1}z_r, (z_r - a_r)^2] \subset A[z_r].$$

In other words, the monomials that are missing are $t^i z_r$ for $i = 1, \ldots, n_r$. This can be interpreted as creasing the $z,t$ plane $A_r$ along the analytic arc $\Gamma$ to have a cusp for $n_r + 1$ infinitesimal steps. Of course, it’s hard to predict on the basis of the geometric picture why such a weird procedure should lead to a Noetherian ring.

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**Ad** This note is a free sample of my forthcoming book \cite{UCA}. Place your order soon to avoid disappointment.

**References**

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