Gauge Theories on ALE Space
and
Super Liouville Correlation Functions

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Abstract

We present a relation between $\mathcal{N} = 2$ quiver gauge theories on the ALE space $\mathcal{O}_{\mathbb{P}^1}(-2)$ and correlators of $\mathcal{N} = 1$ super Liouville conformal field theory, providing checks in the case of punctured spheres and tori. We derive a blow-up formula for the full Nekrasov partition function and show that, up to a $U(1)$ factor, the $\mathcal{N} = 2^*$ instanton partition function is given by the product of the character of $\hat{SU}(2)_2$ times the super Virasoro conformal block on the torus with one puncture. Moreover, we match the perturbative gauge theory contribution with super Liouville three-point functions.

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1 Introduction

A relation between $\mathcal{N} = 2$ superconformal quiver gauge theories on $\mathbb{C}^2$ and Liouville conformal field theory correlators was pointed out in [1] building on M-theory geometrization of non-perturbative dualities [2]. In this paper we show that an analogous relation holds between gauge theories on an ALE space and $\mathcal{N} = 1$ super Liouville conformal field theory. In particular we analyze $\mathcal{N} = 2$ gauge theories on the cotangent bundle of the two-sphere $\mathcal{O}_\mathbb{P}^1(-2)$, namely the minimal resolution of the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold, and show that its perturbative and instanton sectors provide respectively the three-point functions and conformal blocks of $\mathcal{N} = 1$ super Liouville. This latter relation concerning the conformal blocks of super Virasoro algebra was recently discussed in [3, 4, 5] in some particular cases.

An M-theory perspective on such a correspondence was elaborated in [6] by suggesting that $N$ M5-branes on $\mathbb{C}^2/\mathbb{Z}_m$ should give rise to a two-dimensional system with $U(1), \hat{SU}(m)_N$ and $m$-th para-$W_N$ symmetries. This proposal was checked by computing the central charge arising from the anomaly polynomial of the multiple M5-brane system following the approach of [7, 8].

We present a detailed analysis of $\mathcal{N} = 2$ quiver gauge theories and find a blow-up equation expressing the full partition function on the ALE space in terms of $\mathbb{C}^2$ partition functions. We then compare with super Liouville correlators on punctured spheres and tori. A byproduct of our analysis is that the partition function of $\mathcal{N} = 2^*$ theory, corresponding to the torus with one puncture, reproduces the character of $\hat{SU}(2)_2$ times the super Virasoro conformal block and also the contribution from $U(1)$, confirming the general arguments of [6].

In Section 2 we discuss the details of the full Nekrasov function on $\mathcal{O}_\mathbb{P}^1(-2)$ for quiver gauge theories. In Section 3 and 4 we work out the relation with the four and five points super Liouville correlators on the sphere respectively. In Section 5 we study the same relation for the torus with one puncture and highlight the relation with the character of $\hat{SU}(2)_2$. Finally in Section 6 we present our conclusions and discussions on further directions, and collect some useful identities in the Appendices.

2 Gauge theories on ALE space

In this section we consider gauge theories on the $\mathcal{O}_\mathbb{P}^1(-2)$ complex surface, which is the minimal resolution of the $A_1$ singularity $\mathbb{C}^2/\mathbb{Z}_2$. In subsection 2.1 and 2.2 we will consider the instanton and perturbative parts of the partition function respectively.
2.1 Instanton partition function

The compactification of the moduli space $\tilde{\mathcal{M}}(r, k, n)$ of rank $r$ instantons on $\mathcal{O}_{\mathbb{P}^1}(-2)$ can be described in terms of the moduli space of framed torsion free sheaves $(\mathcal{E}, \Phi)$, where $\Phi$ is the framing on a suitable divisor, on the global quotient $\mathbb{P}^2/\mathbb{Z}_2$ with minimal resolution of the singularity at the origin [9]. The resulting variety corresponds to a “stacky” compactification of $\mathcal{O}_{\mathbb{P}^1}(-2)$ obtained by adding a divisor $\tilde{C}_\infty \simeq \mathbb{P}^1/\mathbb{Z}_2$ over which the framing is defined [10]. This variety is an algebraic Deligne-Mumford stack $X_2$ whose coarse space is the second Hirzebruch surface $\mathbb{F}_2$. The crucial point that lead to consider this compactification is that on $X_2$ one has line bundles with half-integer first Chern class supported on the exceptional divisor. From the physics viewpoint, this allows to include the contribution of anti-self-dual gauge connections with nontrivial holonomy at infinity, see [10] for details. The moduli space $\tilde{\mathcal{M}}(r, k, n)$ is characterized by the rank $r$, the first Chern class of the $\mathcal{E}$ sheaf $c_1(\mathcal{E}) = kC$, where $C$ is the exceptional divisor resolving the singularity at the origin, and the discriminant

$$n = \int \left( c_2(\mathcal{E}) - \frac{r-1}{2r} c_1^2(\mathcal{E}) \right).$$

(2.1)

Since the exceptional divisor squares to $-2$, we have $\int c_1(\mathcal{E}) \wedge c_1(\mathcal{E}) = -2k^2$. Then the instanton action, given by the integral of the second Chern character, reads

$$S_{\text{inst}} = \int \left( c_2(\mathcal{E}) - \frac{1}{2} c_1(\mathcal{E})^2 \right) = n + \frac{k^2}{r}.$$ 

(2.2)

We underline that $k$ is in general half integer due to the “stacky” compactification of the ALE space [10]. Indeed, half-integer classes take into account anti-self-dual connections which asymptote flat connections with nontrivial holonomy at infinity. These will play an important rôle in the correspondence with super Liouville theory, being related to the Neveu-Schwarz (NS) and Ramond (R) sectors respectively. In the following we will concentrate on $k \in \mathbb{Z}$ which corresponds to the NS sector.

The evaluation of the instanton partition function can be obtained by using the localization techniques developed in [11, 12, 13, 14, 15]. The torus action $T$ on the instanton moduli space is given by the Cartan gauge rotations parametrized in terms of the vevs of the vector multiplet scalars $a_{\alpha}$ ($\alpha = 1, \ldots, r$) and the space rotations with angles $\epsilon_{1,2}$. This acts on $\mathcal{O}_C(-2)$ as $T: [z : w] \rightarrow [t_1 z : t_2 w]$ on the exceptional divisor $C$ and as $(z_1, z_2) \rightarrow (t_1^2 z_1, t_2^2 z_2)$ on the fibers over it, where $t_{1,2} = e^{\epsilon_{1,2}}$. On the exceptional divisor the fixed points of the torus action are given by $w = 0$ and $z = 0$.

We are interested in the fixed points of the instanton moduli space under the above torus action. As discussed in [16, 10], these are given by ideal sheaves $I_\alpha$ twisted by $\mathcal{O}(k_\alpha C)$ line
bundles, \( \alpha = 1, \ldots, r \), and are specified in terms of \( \vec{k} = (k_1, \ldots, k_r) \) and a pair of Young diagrams \( \vec{Y}_1, \vec{Y}_2 \), where \( \vec{Y}_1 \) (resp. \( \vec{Y}_2 \)) parametrizes the contribution from the fixed point at \( w = 0 \) (resp. \( z = 0 \)). In the following we will use the compact notation \( I_\alpha (k_\alpha C) \). The fixed points data have to satisfy the following relations

\[
n = |\vec{Y}_1| + |\vec{Y}_2| + \frac{1}{r} \sum_{\alpha < \beta} (k_{\alpha\beta})^2, \quad k = \sum_\alpha k_\alpha,
\]

where \( k_{\alpha\beta} = k_\alpha - k_\beta \).

According to the decomposition \( \mathcal{E} = \oplus_\alpha I_\alpha (k_\alpha C) \), the tangent space at the fixed points 
\( T_{(\xi, \phi)} \mathcal{M}(r, k, n) = \text{Ext}^1 \left( \mathcal{E}, \mathcal{E}(\tilde{C}_\infty) \right) \) is decomposed as \( \oplus_{\alpha, \beta} \text{Ext}^1 \left( I_\alpha (k_\alpha C), I_\beta (k_\beta C - \tilde{C}_\infty) \right) \). It can be shown (see [10] for details) that the non-vanishing contributions to the above are given by

\[
\text{Ext}^1 \left( \mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \tilde{C}_\infty) \right), \quad t_1^{2k_{\alpha\beta}} \text{Ext}^1 \left( I_\alpha^1, I_\beta^1 \right) \quad \text{and} \quad t_2^{2k_{\alpha\beta}} \text{Ext}^1 \left( I_\alpha^2, I_\beta^2 \right).
\]

The corresponding T-module structure of the tangent space at the fixed points, corresponding to the vector multiplet contribution, is the following

\[
\chi^{\text{vector}}(\vec{a}) = \sum_{\alpha, \beta=1}^{r} \left( L_{\alpha, \beta}(t_1, t_2) + t_1^{2k_{\alpha\beta}} N_{\alpha, \beta}(t_1^2, t_2/t_1) + t_2^{2k_{\alpha\beta}} N_{\alpha, \beta}(t_1/t_2, t_2^2) \right),
\]

where \( L_{\alpha, \beta}(t_1, t_2) \) is given by

\[
L_{\alpha, \beta}(t_1, t_2) = c_\beta c_\alpha^{-1} \times \begin{cases} \sum_{i,j \geq 0, i+j \leq 2(k_{\alpha\beta}-1)} t_1^{-i} t_2^{-j} & k_{\alpha\beta} > 0 \\ \sum_{i,j \geq 0, i+j \leq -2k_{\alpha\beta}} t_1^{i} t_2^{j+1} & k_{\alpha\beta} < 0 \\ 0 & k_{\alpha\beta} = 0 \end{cases}
\]

and

\[
N_{\alpha, \beta}(t_1, t_2) = c_\beta c_\alpha^{-1} \times \left\{ \sum_{s \in Y_\alpha} \left( t_1^{-l_\alpha(s)} t_2^{l_\alpha(s)+a_\alpha(s)} \right) + \sum_{s \in Y_\beta} \left( t_1^{l_\alpha(s)} t_2^{a_\beta(s)} \right) \right\},
\]

which is the character of the tangent space corresponding to the vector multiplet on \( \mathbb{C}^2 \). The three terms in [2.4] correspond to the three nonvanishing components of the \( \text{Ext}^1 \) respectively. In eq. [2.6] and in the rest of the paper we use the standard notation \( a_\gamma(s) \) for the (relative) arm and \( l_\gamma(s) \) for the (relative) leg lengths. Note that we choose a convention \( c_\alpha = e^{-a_\alpha} \) such that the above character agrees with the usual one in the literature [13] [1].

Let us turn to the contribution of bifundamental hypermultiplets in the \( (r, \bar{r}) \) representation of \( U(r) \times U(\bar{r}) \). The fixed points are specified in terms of the arrays \( \vec{\bar{k}} \) and \( \bar{k} \) and two sets of
Young diagrams $\vec{Y}_{1,2}$ and $\vec{W}_{1,2}$. The contribution of one massive bifundamental hypermultiplet of mass $m$ can be obtained by generalizing the above procedure and is given by

$$
\chi^{\text{bifund}}(\vec{a}, \vec{a}'; m) = - \sum_{\alpha=1}^{r} \sum_{\beta'=1}^{r} \left( L_{\alpha, \beta'}(t_1, t_2) + t_1^{2k_{\alpha, \beta'}} N_{\alpha, \beta'}^{\vec{Y}_{1,2}}(t_2, t_2/t_1) + t_2^{2k_{\alpha, \beta'}} N_{\alpha, \beta'}^{\vec{W}_{1,2}}(t_1, t_1/t_2) \right) e^{-m},
$$

(2.7)

where $a_{\alpha}$ and $\tilde{a}_{\beta'}$ are the vevs of the vector multiplet scalars of the two gauge groups and $L_{\alpha, \beta'}$ can be obtained from (2.5) by replacing $e_{\beta}e_{\alpha}^{-1}$ with $e_{\beta'}e_{\alpha}^{-1}$ and $k_{\alpha, \beta'} \equiv k_{\alpha} - \tilde{k}_{\beta'}$ respectively, with $e_{\beta'} = e^{\tilde{a}_{\beta'}}$. Moreover, $N_{\alpha, \beta'}^{\vec{Y}, \vec{W}}$ is given by

$$
N_{\alpha, \beta'}^{\vec{Y}, \vec{W}}(t_1, t_2) = e_{\beta}e_{\alpha}^{-1} \times \left\{ \sum_{s \in Y_{\alpha}} \left( t_1^{1+t_{W_{\beta'}}(s)} t_2^{1+a_{Y_{\alpha}}(s)} \right) + \sum_{s \in W_{\beta'}} \left( t_1^{1+t_{Y_{\alpha}}(s)} t_2^{-a_{W_{\beta'}}(s)} \right) \right\}.
$$

(2.8)

The contribution of adjoint hypermultiplets can be obtained by setting $\vec{a} = \vec{a}$ and $\vec{W} = \vec{Y}$. This extends the results of [17] to the “stacky” compactification according to the rules stated in [10]. The character of the fundamental hypermultiplet of mass $m$ is given by an analogous extensions of the formula in [17] obtained by setting $\vec{W} = \emptyset$, $\tilde{k}_{\beta'} = 0$ and $\tilde{a}_{\beta'} = 0$ in (2.7)

$$
\chi^{\text{fund}}(\vec{a}, m) = - \sum_{\alpha=1}^{r} \left( L_{\alpha}(t_1, t_2) + t_1^{2k_{\alpha}} N_{\alpha}^{\vec{Y}}(t_2, t_2/t_1) + t_2^{2k_{\alpha}} N_{\alpha}^{\vec{W}}(t_1, t_1/t_2) \right) e^{-m+\epsilon_+},
$$

(2.9)

where $\epsilon_+ = \epsilon_1 + \epsilon_2$,

$$
L_{\alpha}(t_1, t_2) = e_{\alpha} \times \left\{ \sum_{i, j \geq 0, \ i+j \leq 2(k_{\alpha}-1)} t_1^{i+1} t_2^{j+1}, \ k_{\alpha} > 0 \right\}
$$

$$
+ \left\{ \sum_{i, j \geq 0, \ i+j \geq 2(k_{\alpha}-1)} t_1^{-i} t_2^{-j}, \ k_{\alpha} < 0 \right\}
$$

$$
+ \left\{ 0, \ k_{\alpha} = 0 \right\}
$$

(2.10)

and

$$
N_{\alpha}^{\vec{Y}}(t_1, t_2) = e_{\alpha} \sum_{s \in Y_{\alpha}} t_1^{-(i(s)-1)} t_2^{-(j(s)-1)}
$$

(2.11)

Finally, the contribution of anti-fundamental hypermultiplets is given by

$$
\chi^{\text{anti-fund}}(\vec{a}, m) = \chi^{\text{fund}}(\vec{a}, \epsilon_+ - m).
$$

(2.12)

*We note that this formula coincides with the one of $\mathbb{C}^2$, up to an overall sign due to a different notation w.r.t. [1].
The above discussion can be easily generalized to quiver gauge theories. In this case, the fixed points are described in terms of vectors \( \vec{k}^s \) and pairs of Young diagrams \( \vec{Y}^s_1, \vec{Y}^s_2 \) where \( s = 1, \ldots, n \) labels the nodes of the quiver. Let \( r_s \) and \( \vec{a}^s \) be the rank and the Cartan parameters of the \( s \)-th gauge group. The relations (2.3) have to be satisfied for each \( \vec{k}^s \) and \( \vec{Y}^s_1, \vec{Y}^s_2 \). The contributions of vector and matter multiplets for each node are given by the same formulae above, written in terms of the \( s \) fixed points.

By using the above results one can readily compute the instanton part of the Nekrasov partition function of quiver gauge theories on \( \mathcal{O}_{p_1}(-2) \). We resum the contributions by weighting them in terms of the instanton topological action \( q^{S_{\text{inst}}} = q^{n + \frac{k^2}{2}} \). Then we consider the operator insertion \( Z_{\text{ALE}}^{\text{inst}} \equiv \langle e^{-\sum_{s=1}^{n} v_s f c_1(E_s) \wedge c_1(C)} \rangle \) where \( C \) is the exceptional divisor and \( z_s \equiv e^{2v_s} \)

\[
Z_{\text{inst}}^{\text{ALE}}(\epsilon_1, \epsilon_2, \vec{a}; q_s, z_s) = \sum_{k^s \in \mathbb{Z}} \left( \prod_{s=1}^{n} \frac{(k^s)^2}{q_s^{r_s}} (z_s)^{k^s} \right) Z_{\text{inst}}^{\{k^s\}}(\epsilon_1, \epsilon_2, \vec{a}^s, m_i; q_s), \tag{2.13}
\]

where

\[
Z_{\text{inst}}^{\{k^s\}}(\epsilon_1, \epsilon_2, \vec{a}^s, m_i; q_s) = \sum_{\sum k^s = k^s} \left( \prod_{s=1}^{n} \frac{(k^s)^2}{q_s^{r_s}} \right) \ell(\epsilon_1, \epsilon_2, \vec{a}^s, \vec{k}^s, m_i) \tag{2.14}
\]

\[
\times Z_{\text{inst}}^{C^2} \left( 2\epsilon_1, \epsilon_2 - \epsilon_1, a^s - 2\epsilon_1 \vec{k}^s, q_s \right) Z_{\text{inst}}^{C^2} \left( \epsilon_1 - \epsilon_2, 2\epsilon_2, \vec{a}^s - 2\epsilon_2 \vec{k}^s, q_s \right).
\]

Here \( Z_{\text{inst}}^{C^2}(\epsilon_1, \epsilon_2, \vec{a}, q) \) is the instanton part of the Nekrasov partition function of the same quiver gauge theory on \( \mathbb{C}^2 \).

The factor \( \ell \) in (2.14), depending on the matter content of the gauge theory, is obtained from the characters (2.4), (2.7), (2.9) and (2.12). Let us spell it out in detail. First of all, let us define

\[
\ell_{\alpha\beta'}(x, k_\alpha; \vec{x}, \vec{k}_{\beta'}; m) = \begin{cases} 
\prod_{i,j \geq 0, i+j \leq 2(2k_{\alpha\beta'}-1)} (x - \vec{x} - i\epsilon_1 - j\epsilon_2 - m), & k_{\alpha\beta'} > 0 \\
\prod_{i,j \geq 0, i+j \leq -2(2k_{\alpha\beta'}+1)} (x - \vec{x} + (i+1)\epsilon_1 + (j+1)\epsilon_2 - m), & k_{\alpha\beta'} < 0 \\
1, & k_{\alpha\beta'} = 0
\end{cases} \tag{2.15}
\]
Similarly, we also define

\[
\ell_{\alpha}(x, k_{\alpha}) = \begin{cases} 
\prod_{i,j \geq 0, \ i+j \leq 2(k_{\alpha} - 1) \mod 2} (-x + (i + 1)e_1 + (j + 1)e_2), & k_{\alpha} > 0 \\
\prod_{i,j \geq 0, \ i+j \leq -2(k_{\alpha} + 1) \mod 2} (-x - i\epsilon_1 - j\epsilon_2), & k_{\alpha} < 0 \\
1, & k_{\alpha} = 0 
\end{cases} \tag{2.16}
\]

By using these definitions, the $\ell$ factors for the bifundamental and (anti-)fundamental hyper-multiplets are given by

\[
\ell_{\text{bifund}}(\vec{a}, \vec{k}; \vec{\tilde{a}}, \vec{\tilde{k}}; m) = \prod_{\alpha=1}^{r} \ell_{\alpha\beta'}(a_{\alpha}, k_{\alpha}; \tilde{a}_{\beta'}, \tilde{k}_{\beta'}; m), \tag{2.17}
\]

\[
\ell_{\text{fund}}(\vec{a}; m) = \prod_{\alpha} \ell_{\alpha}(a_{\alpha} + \epsilon_+ - m, k_{\alpha}), \quad \ell_{\text{anti-fund}}(\vec{a}; m) = \prod_{\alpha} \ell_{\alpha}(a_{\alpha} + m, k_{\alpha}),
\]

while the adjoint and the vector contributions are

\[
\ell_{\text{adj}}(\vec{a}; m) = \ell_{\text{bifund}}(\vec{a}, \vec{k}; \vec{\tilde{a}}, \vec{\tilde{k}}; m), \quad \ell_{\text{vector}}(\vec{a}) = \ell_{\text{adj}}(\vec{a}; 0)^{-1}. \tag{2.18}
\]

Analogously to what discussed in [1] for the flat space case, the bifundamental contribution satisfies

\[
\ell_{\text{bifund}}(\vec{a}, \vec{k}; \vec{\tilde{a}}, \vec{\tilde{k}}; m) = \ell_{\text{bifund}}(\vec{\tilde{a}}, \vec{\tilde{k}}; \vec{\tilde{a}}, \vec{\tilde{k}}; \epsilon_+ - m), \tag{2.19}
\]

corresponding to the exchange of the two gauge factors. Moreover, the (anti-)fundamental contribution is obtained from the bifundamental one as

\[
\ell_{\text{bifund}}(\vec{a}, \vec{k}; \vec{\bar{\mu}}, \vec{0}; m) = \prod_{f=1}^{n} \ell_{\text{fund}}(\vec{a}; m + \mu_f),
\]

\[
\ell_{\text{bifund}}(\vec{\bar{\mu}}, \vec{0}; \vec{a}, \vec{k}; m) = \prod_{f=1}^{n} \ell_{\text{anti-fund}}(\vec{a}; m - \mu_f), \tag{2.20}
\]

where $\vec{\bar{\mu}} = (\mu_1, \ldots, \mu_n)$.

### 2.2 Classical and perturbative parts of the partition function

In this subsection, we consider the perturbative part of the partition function. As we will show in the following, this is directly related to the three-point functions of super Liouville theory. The inclusion of the perturbative contribution will allow us to derive a blow-up formula for the full partition function on the ALE space. Although our result are valid for generic rank, we will
focus in this section on the $U(2)$ gauge theory, in which we fix $k = k_1 + k_2 = 0$ and decouple the abelian Coulomb parameter. This is the simplest case for a comparison with super Liouville theory.

The perturbative part for the vector and the adjoint hypermultiplet contributions of $SU(2)$ gauge theory is

$$Z_{\text{pert}}^{\text{vector}}(a) = \exp \left[ -\gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(2a) - \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(2a + \epsilon_+ + \epsilon_-) - (\epsilon_1 \leftrightarrow \epsilon_2) \right],$$

$$Z_{\text{pert}}^{\text{adj}}(a; m) = \exp \left[ \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(2a + \epsilon_+ - m) + \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(-2a + \epsilon_+ - m) + (\epsilon_1 \leftrightarrow \epsilon_2) \right],$$

where the definition of the gamma function is presented in Appendix A. These results are obtained by adapting the approach of [17] to our case. Since $k_1$ can be integer or half-integer, the instanton partition function can also be divided into an “even” sector with integer $k_1$ and an “odd” sector with half-integer $k_1$. Since matter in the (anti-)fundamental representation couples to the $\mathcal{O}(k_\alpha C)$ line bundles, its perturbative contribution is different in the two different sectors, giving

$$Z_{\text{pert,even}}^{\text{fund}}(a, m) = \prod_{\alpha=1,2} \exp \left[ \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(a_\alpha + \epsilon_+ - m) + (\epsilon_1 \leftrightarrow \epsilon_2) \right],$$

$$Z_{\text{pert,even}}^{\text{anti-fund}}(a, m) = \prod_{\alpha=1,2} \exp \left[ \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(a_\alpha + m) + (\epsilon_1 \leftrightarrow \epsilon_2) \right]$$

(2.22)

with $\tilde{a} = (a, -a)$, for the even sector, and

$$Z_{\text{pert,odd}}^{\text{fund}}(a, m) = \prod_{\alpha=1,2} \exp \left[ \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(a_\alpha + \epsilon_+ - m + \epsilon_1) + (\epsilon_1 \leftrightarrow \epsilon_2) \right],$$

$$Z_{\text{pert,odd}}^{\text{anti-fund}}(a, m) = \prod_{\alpha=1,2} \exp \left[ \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(a_\alpha + m + \epsilon_1) + (\epsilon_1 \leftrightarrow \epsilon_2) \right]$$

(2.23)

for the odd sector. The contribution of the bifundamental hypermultiplet can also be written as

$$Z_{\text{pert,even}}^{\text{bifund}}(a, \tilde{a}; m) = \prod_{\alpha=1,2} \prod_{\beta'=1,2} \exp \left[ \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m) + (\epsilon_1 \leftrightarrow \epsilon_2) \right],$$

$$Z_{\text{pert,odd}}^{\text{bifund}}(a, \tilde{a}; m) = \prod_{\alpha=1,2} \prod_{\beta'=1,2} \exp \left[ \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m + \epsilon_1) + (\epsilon_1 \leftrightarrow \epsilon_2) \right],$$

(2.24)

(2.25)

with $\tilde{a} = (a, -a)$ and $\tilde{\tilde{a}} = (\tilde{a}, -\tilde{a})$. Note that the instanton partition function including bifundamental hypermultiplets depends on $\tilde{k}$ and $\tilde{k}$. The even (odd) sector of the bifundamental corresponds to the case where $k_{1,l}(= k_1 - \tilde{k}_1)$ is (half-)integer, because we fixed $k = \tilde{k} = 0$.

By using formulae (A.7), (A.8) and (A.10), it is straightforward to see that

$$\exp \left[ -\gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(2a - 2k_{12}\epsilon_1) - \gamma_{2\epsilon_1,\epsilon_2-\epsilon_1}(2a + \epsilon_+ - 2k_{12}\epsilon_1) - (\epsilon_1 \leftrightarrow \epsilon_2) \right]$$

$$= (-\Lambda^2)^{(k_{12})^2} \ell_{\text{vector}}(a) Z_{\text{pert}}^{\text{vector}}(a, m)$$

(2.26)
for the vector part,

\[
\exp \left[ \gamma_{\epsilon_1, \epsilon_2 - \epsilon_1} (2a + \epsilon_+ - m - 2k_1 \epsilon_1) + \gamma_{\epsilon_1, \epsilon_2 - \epsilon_1} (-2a + \epsilon_+ - m + 2k_1 \epsilon_1) + (\epsilon_1 \leftrightarrow \epsilon_2) \right] \\
= \Lambda^{-2(k_1^2)} \ell_{\text{adj}}(a, m) Z_{\text{pert}}^{\text{adj}}(a, m) \tag{2.27}
\]

for the adjoint,

\[
\prod_{\alpha=1,2} \prod_{\beta'=1,2} \exp \left[ \gamma_{\epsilon_1, \epsilon_2 - \epsilon_1} (a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m - 2k_1 \beta' \epsilon_1) + (\epsilon_1 \leftrightarrow \epsilon_2) \right] \\
= \prod_{\alpha=1,2} \prod_{\beta'=1,2} \left( \Lambda^{-2(k_1^2)} \right)^2 \ell_{\text{fund}}(a, \bar{a}; m) \times \begin{cases} Z_{\text{pert}}^{\text{bifund, even}}(a, \bar{a}; m), & k_{11'} \in \mathbb{Z} \\ Z_{\text{pert}}^{\text{bifund, odd}}(a, \bar{a}; m), & k_{11'} \in \mathbb{Z} + \frac{1}{2} \end{cases} \tag{2.28}
\]

for the bifundamental, and

\[
\prod_{\alpha=1,2} \exp \left[ \gamma_{\epsilon_1, \epsilon_2 - \epsilon_1} (a_\alpha + \epsilon_+ - m - 2k_1 \epsilon_1) + (\epsilon_1 \leftrightarrow \epsilon_2) \right] \\
= \Lambda^{-(k_1^2-(k_2^2)} \ell_{\text{fund}}(a, m) \times \begin{cases} (-1)^{(k_1^2+(k_2^2)} Z_{\text{pert}}^{\text{fund, even}}(a, \bar{a}; m), & k_1 \in \mathbb{Z} \\ (-1)^{(k_1^2+(k_2^2)-1/2) Z_{\text{pert}}^{\text{fund, odd}}(a, \bar{a}; m), & k_1 \in \mathbb{Z} + \frac{1}{2} \tag{2.29}
\end{cases}
\]

for the fundamental part.

For theories with vanishing beta function, we set \( \Lambda = 1 \). In this case, we have to multiply further by the classical part of the partition function:

\[
Z_{\text{cl}} = \prod_{s=1}^n q_s^{-\frac{(a^s)^2}{2k_1^2}}. \tag{2.30}
\]

By combining all together, namely the classical, the perturbative and the instanton parts, we obtain the blow-up formula

\[
Z_{\text{full}}^{k_2^2} (\epsilon_1, \epsilon_2, \bar{a}^s, q_s) = \sum_{\tilde{k}^+ | k^+ = 0} (-1)^{(k_1^2)} Z_{\text{full}}^{\text{C}^2} (2\epsilon_1, \epsilon_2 - \epsilon_1, \bar{a}^s - 2\epsilon_1 \tilde{k}^s, q_s) Z_{\text{full}}^{\text{C}^2} (\epsilon_1 - \epsilon_2, 2\epsilon_2, \bar{a}^s - 2\epsilon_2 \tilde{k}^s, q_s), \tag{2.31}
\]

where

\[
Z_{\text{full}}^{\text{C}^2} (\epsilon_1, \epsilon_2, \bar{a}^s, q_s) = Z_{\text{cl}}^{\text{C}^2} (\epsilon_1, \epsilon_2, \bar{a}^s, q_s) Z_{\text{pert}}^{\text{C}^2} (\epsilon_1, \epsilon_2, \bar{a}^s, q_s) Z_{\text{inst}}^{\text{C}^2} (\epsilon_1, \epsilon_2, \bar{a}^s, q_s). \tag{2.32}
\]

We have defined the classical and the perturbative parts of the partition function on \( \mathbb{C}^2 \) as follows:

\[
Z_{\text{cl}}^{\text{C}^2} (\epsilon_1, \epsilon_2, \bar{a}^s, q_s) = \prod_{s=1}^n q_s^{-\frac{(a^s)^2}{2k_1^2}}, \tag{2.33}
\]

\[
Z_{\text{pert}}^{\text{C}^2} (\epsilon_1, \epsilon_2, \bar{a}^s, q_s) = 
\]
and the perturbative part is the same as the ones in (2.21), (2.22) and (2.24). Notice that the factors $q^{\frac{k_1 k_2^2}{2}}$ and $\ell$ in the instanton part (2.14) were absorbed, respectively, by the inclusions of the classical and the perturbative parts with shifted arguments: $\epsilon_1 \rightarrow 2\epsilon_1$ and $\epsilon_2 \rightarrow \epsilon_2 - \epsilon_1$ and $\epsilon_{1,2}$ exchanged.

In the case of the asymptotically free theory with one $SU(2)$ gauge group, $q = \Lambda^{b_0}$ where $b_0 = 4-N_f$, and there is no classical part. Then, the $\Lambda$ dependence in (2.26), (2.28) and (2.29) leads to $\Lambda^{\frac{N_f}{2}k^2 - \frac{b_0}{2}k_{1,2}}$. The last factor cancels the $q^{\frac{k^2}{2}}$ in the instanton part (2.14). Therefore, the final form of the full partition function (2.32) is universal for the theory with $b_0 \geq 0$ up to a sign.

The blow-up formula (2.31) is the analog of the one found in [15] in the $O_{p1}(-1)$ case. The existence of blow-up formulae for $N=2$ gauge theories on toric manifolds was suggested in [18].

### 3 Four-point super Liouville correlator on the sphere

We are now ready to compare the gauge theory partition functions on the $O_{p1}(-2)$ surface with correlators of the super Liouville conformal field theory. As we saw in the previous section, the instanton moduli spaces are classified by $k$. The (half-)integer classes correspond to flat connections with (non)trivial holonomy at infinity. It is then natural that the integer and half-integer classes are on different footings in the super Liouville theory: we expect that they correspond to the Neveu-Schwarz (NS) and Ramond (R) sectors respectively. In this paper we focus on the former sector and see the relation with the gauge theory. In the present section, we consider the four-point correlation function on the sphere and show that this is related to the full partition function of $SU(2)$ gauge theory with four flavors, by concentrating on the case with $k = 0$, which is the simplest sector to check.

The $N=1$ superconformal symmetry of Liouville is generated by the holomorphic currents $T(z)$, $G(z)$ and their anti-holomorphic counterparts. The algebra is

\[ [L_m, L_n] = (m-n)L_{m+n} + \frac{\hat{c}}{8}(m^3 - m)\delta_{m+n,0}, \]
\[ [L_n, G_k] = \left(\frac{n}{2} - k\right)G_{n+k}, \]
\[ \{G_k, G_l\} = 2L_{k+l} + \frac{\hat{c}}{2}\left(k^2 - \frac{1}{4}\right)\delta_{k+l,0}, \quad (3.1) \]

and the same for the right part $\bar{L}_n$ and $\bar{G}_k$. In our notation the central charge is $\hat{c} = 1 + 2Q^2$ with $Q = b + 1/b$. The NS sector which we focus on corresponds to the case with $k, l$ are half-integers.
The primary field $V_\alpha(x)$ corresponds to the highest weight vector satisfying

$$L_n V_\alpha = 0, \quad G_k V_\alpha = 0, \quad \text{for } k, n > 0 \quad \text{and} \quad L_0 V_\alpha = \Delta V_\alpha,$$  

(3.2)

where the conformal dimension of the primary is

$$\Delta = \frac{Q^2}{8} - \frac{\alpha^2}{2},$$  

(3.3)

and similar for the anti-holomorphic part. The super-Verma module $\mathcal{V}$ is formed by the descendants $V_{KM} = L_{-M} G_{-K} V$ which are obtained by acting with the raising operators $L_{-M} = L_{-m_1} L_{-m_2} L_{-m_3} \cdots$, where $\{m_i\}$ are positive integers with $m_1 < m_2 < \cdots$, and $G_{-K} = G_{-k_1} G_{-k_2} G_{-k_3} \cdots$, where $\{k_i\}$ are positive half-integers with $k_1 < k_2 < \cdots$.

The four-point correlation function is

$$\langle V_1(\infty) V_2(1) V_3(q) V_4(0) \rangle = \int dP \left[ C_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |q^{\Delta - \Delta_3 - \Delta_4} B_{0,4}^{even}(\Delta_i, q)|^2 - \tilde{C}_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} |q^{\Delta - \Delta_3 - \Delta_4} B_{0,4}^{odd}(\Delta_i, q)|^2 \right],$$  

(3.4)

up to an irrelevant constant. The modulus $q$ will be identified with the gauge coupling constant $q = e^{2\pi i \tau}$. The expressions of the three-point functions

$$C_{\alpha_1, \alpha_2, \alpha_3} = C_{\alpha_1 + Q/2, \alpha_2 + Q/2, \alpha_3 + Q/2}, \quad \tilde{C}_{\alpha_1, \alpha_2, \alpha_3} = -\tilde{C}_{\alpha_1 + Q/2, \alpha_2 + Q/2, \alpha_3 + Q/2}$$  

(3.5)

are given in Appendix B. We divide the conformal block $B_{0,4}$ into the even and the odd sectors which are expanded in $q$ as

$$B_{0,4}^{even} = \sum_{n \in \mathbb{N}} B_n q^n, \quad B_{0,4}^{odd} = \sum_{n \in \mathbb{N} + \frac{1}{2}} B_n q^n.$$

(3.6)

The lower order terms are computed as [19, 20, 21]

$$B_{\frac{1}{2}}^{\text{even}} = \frac{1}{2\Delta}, \quad B_1 = \frac{(\Delta + \Delta_2 - \Delta_1)(\Delta + \Delta_3 - \Delta_4)}{2\Delta},$$  

$$B_{\frac{3}{2}}^{\text{even}} = \frac{(1 + 2\Delta + 2\Delta_2 - 2\Delta_1)(1 + 2\Delta + 2\Delta_3 - 2\Delta_4)}{8\Delta(1 + 2\Delta)} + \frac{4(\Delta_2 - \Delta_1)(\Delta_3 - \Delta_4)}{\hat{\epsilon} + 2(-3 + \hat{\epsilon}) \Delta + 4\Delta^2(1 + 2\Delta)},$$

(3.7)

and so on.

Let us first compare the conformal blocks with the Nekrasov instanton partition function of the gauge theory with two fundamental fields with masses $\mu_{3,4}$ and two anti-fundamental fields with masses $\mu_{1,2}$. We first redefine the mass parameters to the ones associated with the $SU(2)^4$ flavor symmetries:

$$\mu_1 = n_0 + m_0 + \frac{\epsilon_+}{2}, \quad \mu_2 = n_0 - m_0 + \frac{\epsilon_+}{2},$$
$$\mu_3 = n_1 + m_1 + \frac{\epsilon_+}{2}, \quad \mu_4 = n_1 - m_1 + \frac{\epsilon_+}{2}$$  

(3.8)
Then the instanton partition function is written as
\[
Z_{\text{inst}}^{\text{ALE}} = \sum_{k \in \mathbb{Z}} q^{\frac{k^2}{2}} z^k Z_{\text{inst}}^k
\] (3.9)
where \(Z_{\text{inst}}^k\) is given by (2.14), whose \(\ell\) factor includes the contributions for the vector, the two fundamentals and the two anti-fundamentals. Note that \(\ell\) in this case depends on both the sum \(k\) and the difference \(k_{12}\). We calculate \(Z_{\text{inst}}^k\) with \(k = 0\) here.

The conformal block above agrees with the instanton partition function by the following relation [5]:
\[
Z_{\text{inst}}^{k=0} = (1 - q)^{\frac{\epsilon_i}{2} + n_0}\left(\frac{\epsilon_i}{2} - n_1\right) \left(B_{0,4}^{\text{even}} + \frac{1}{2} B_{0,4}^{\text{odd}}\right)
\] (3.10)
under the identification of the parameters:
\[
\epsilon_1 = b, \quad \epsilon_1 = 1/b, \quad iP = a,
\]
\[
\alpha_1 = m_0, \quad \alpha_2 = n_0, \quad \alpha_3 = n_1, \quad \alpha_4 = m_1.
\] (3.11)
where \(\alpha_i\) are the external momenta. The factor in front of the conformal block in (3.10) is understood as the \(U(1)\) factor as introduced in [1]. Indeed, the conformal block is invariant under the transformation corresponding to the Weyl reflection of the flavor symmetry: \(n_i \to -n_i\) and \(m_i \to -m_i\) (\(i = 0, 1\)).

Let us then compare the remaining part of the correlation function (3.4). It is easy to check that
\[
C_{\alpha_1,\alpha_2,iP} C_{\alpha_3,\alpha_4,-iP} = C_0 \left| Z_{\text{pert}}(a) \prod_{i=1,2} Z_{\text{pert}}^\text{anti-fund,even}(a, \mu_i) \prod_{i=3,4} Z_{\text{pert}}^\text{fund,even}(a, \mu_i) \right|^2
\] (3.12)
for the even sector, and
\[
\tilde{C}_{\alpha_1,\alpha_2,iP} \tilde{C}_{\alpha_3,\alpha_4,-iP} = -4C_0 \left| Z_{\text{pert}}(a) \prod_{i=1,2} Z_{\text{pert}}^\text{anti-fund,odd}(a, \mu_i) \prod_{i=3,4} Z_{\text{pert}}^\text{fund,odd}(a, \mu_i) \right|^2
\] (3.13)
for the odd sector, where \(C_0\) is a factor which does not depend on the internal momentum (or the vev \(a\)). Notice that the difference between the even and the odd sectors in CFT is related with the difference between (2.22) and (2.23) of the gauge theory, which correspond to the sectors where \(k_1\) is integer and half-integer, respectively.

Also, the factor in front of the conformal block in (3.1) is identified with the classical part of the partition function (2.30): \(q^{\Delta - \Delta_3 - \Delta_4} \sim q^{-\frac{a^2}{2}}\), up to an \(a\) independent factor. Therefore, by combining all together, we can see that the four-point correlation function of the super Liouville theory is written as the integral over the internal momentum \(P\) with integrand \(|\mathcal{Z}_{\text{full}}^{k=0}(a; q)|^2\).

Here we have used that the gauge theory partition function is rewritten as (2.32).
4 Five-point super Liouville correlator on the sphere

We consider in this section five-point correlation function on the sphere and compare it with the full partition function of $SU(2)^2$ quiver gauge theory with two fundamentals in the first $SU(2)$, two anti-fundamentals in the second $SU(2)$ and one bifundamental.

The correlation function is written as

$$\left\langle \prod_{i=1}^{5} V_{a_i}(z_i) \right\rangle = \int dP_1 dP_2 |(q_1 q_2)^{\Delta - \Delta_4 - \Delta_5} q_1^{\Delta - \Delta - \Delta_3}|^2$$

$$\left[ C_{\alpha_1, \alpha_2, iP_1} C_{-iP_2, \alpha_3, iP_3} C_{-iP_2, \alpha_4, \alpha_5} |B_{0,5}^{e, \alpha}|^2 - C_{\alpha_1, \alpha_2, iP_1} \tilde{C}_{-iP_2, \alpha_3, iP_3} \tilde{C}_{-iP_2, \alpha_4, \alpha_5} |B_{0,5}^{o, \alpha}|^2 \right]$$

where $P_{1,2}$ are the internal momenta and we have chosen the coordinates of the insertions as $z_1 = \infty$, $z_2 = 1$, $z_3 = q_1$, $z_4 = q_1 q_2$ and $z_5 = 0$. The moduli of the sphere $q_i$ will be identified with the gauge coupling constants of the gauge theory $q_i = e^{2 \pi i \alpha_i}$. The indices, $e$ and $o$, of the conformal block $B_{0,5}$ denote the even and the odd sectors. Namely the first index $e$ (or $o$) means that the conformal block includes (half-)integer powers in $q_1$ and the second in $q_2$. We can calculate their coefficients as

$$B_{1,0} = \frac{(\Delta + \Delta_2 - \Delta_1)(\Delta + \Delta_3 - \tilde{\Delta})}{2\Delta}, \quad B_{0,1} = \frac{(\tilde{\Delta} + \Delta_3 - \Delta)(\tilde{\Delta} + \Delta_4 - \Delta_5)}{2\Delta},$$

$$B_{1,0} = \frac{1}{2\Delta}, \quad B_{1,1} = \frac{(\tilde{\Delta} + \Delta_3 - \Delta - \frac{1}{2})(\tilde{\Delta} + \Delta_4 - \Delta_5)}{4\Delta \tilde{\Delta}},$$

$$B_{0,0} = \frac{1}{2\Delta}, \quad B_{1,1} = \frac{(\Delta + \Delta_2 - \Delta_1)(\Delta + \Delta_3 - \tilde{\Delta} - \frac{1}{2})}{4\Delta \tilde{\Delta}}, \quad B_{1,1} = \frac{\Delta - \Delta_3 + \tilde{\Delta}}{4\Delta \tilde{\Delta}},$$

and so on.

We now first compare these conformal blocks with the instanton partition function of $SU(2)^2$ gauge theory. We denote the masses of the anti-fundamentals, the fundamentals and the bifundamental by $\mu_{1,2}$, $\mu_{4,5}$ and $\mu_3$ respectively. The vevs of the vector multiplet scalars of two $SU(2)$ are $a$ and $\tilde{a}$. The partition function is

$$Z_{\text{inst}}^{\text{ALE}} = \sum_{k \in \mathbb{Z}} \sum_{\tilde{k} \in \mathbb{Z}} q_1^k q_2^{\frac{k^2}{2}} z^{k \tilde{k}} z^{k \tilde{k}} Z_{\text{inst}}^{k, \tilde{k}}.$$

\footnote{In this section and in the following we normalize the three-point functions in the odd-odd sector up to factors $(\Delta_1 - \Delta_2 + \Delta_3) = \frac{\langle G_{-1/2} V_1 V_2 G_{-1/3} V_3 \rangle}{\langle V_1 V_2 V_3 \rangle}$ which get included in the conformal block. This normalization is more natural for the comparison with the gauge theory.}
We propose that the instanton partition function with fixed first Chern classes $k = 0$ and $\tilde{k} = 0$ is related to the five-point conformal block as

$$Z_{\text{inst}}^{k=0,\tilde{k}=0} = Z_{U(1)} \left( B_{0,5}^{c,e} + \frac{1}{2}(B_{0,5}^{c,o} - B_{0,5}^{a,o}) \right),$$

(4.4)

where the $U(1)$ factor is

$$Z_{U(1)} = (1 - q_1^{(\frac{1}{2} + n_0)})(1 - q_1 q_2^{(\frac{1}{2} + m_3)})(1 - q_1 q_2^{(\frac{1}{2} - n_1)})(1 - q_2^{(\frac{1}{2} + n_0)}).$$

(4.5)

The identification of the parameters are similar to (3.11):

$$iP_1 = a, \quad iP_2 = \tilde{a},$$

$$\alpha_1 = m_0, \quad \alpha_2 = n_0, \quad \alpha_3 = m_3, \quad \alpha_4 = n_1, \quad \alpha_5 = m_1.$$  

(4.6)

We have checked this relation in lower orders in $q_1$ and $q_2$.

Next we consider the three-point function. We can show that for the even-even sector

$$\mathbb{C}_{\alpha_1,\alpha_2,iP_1,iP_3,\alpha_3,iP_2} \mathbb{C}_{\alpha_4,\alpha_5,-iP_2} = \tilde{C}_0 \left| Z_{\text{pert}}^{\text{vector}}(a) Z_{\text{pert}}^{\text{vector}}(b) \prod_{i=1,2} Z_{\text{pert}}^{\text{anti-fund,even}}(a, \mu_i) Z_{\text{pert}}^{\text{bifund,even}}(a, \tilde{a}; \mu_3) \prod_{i=4,5} Z_{\text{pert}}^{\text{fund,even}}(a, \mu_i) \right|^2.$$  

(4.7)

To obtain the analog of (4.7) in the other three sectors, one has to keep into account the numerical $(-4)$ factor appearing in (3.13) to match the correct normalization as dictated by the blow-up formula (2.31).

Finally, the $q_{1,2}$ factor in the first line in (4.1) can be seen as the classical part of the gauge theory $q_1^{\Delta_3 - \Delta_4 - \Delta_5} q_2^{\tilde{d}_3 - \Delta_4 - \Delta_5} \sim q_1^2 q_2^{-\frac{a}{2}}$. Therefore, we conclude that the integrand of the five-point correlation function can be written as $|Z_{\text{full}}^{k=0,\tilde{k}=0}|^2$.

The arguments presented in this section should generalize to the $n$-point functions on the sphere and the corresponding linear quiver gauge theories. The identification among moduli and gauge couplings should proceed along the same lines as in [1] as well as the one among momenta and Coulomb or mass parameters. This should apply to the $U(1)$ factors too, but with different shifted exponents as in (4.5).

5 One-point super Liouville correlator on the torus

In this subsection we consider the one-point correlation function on a torus. This can be written as

$$\langle V_{\alpha_1} \rangle_{\text{torus}} = \int dP \mathbb{C}_{iP,\alpha_1,-iP} \left| q^{\Delta - \frac{a}{2}} B_{1,1}(q) \right|^2,$$  

(5.1)

$^5$We have checked this relation up to terms of order $q_1^a q_2^b$ with $a + b = 2$.  

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again up to an irrelevant constant. In (5.1) the one-point conformal block on the torus is denoted by $B_{1,1}(q)$ and expands as $\sum_{n \in \mathbb{N}/2} B_n q^n$ with coefficients

$$B_{1,1} = -\frac{\Delta_1 + 2\Delta}{2\Delta}, \quad B_1 = \frac{\Delta_1^2 - \Delta_1 + 2\Delta}{2\Delta},$$

$$B_3 = 2 + \frac{-(\hat{c} + 20\Delta^2 + 4\Delta\hat{c})\Delta_1 + (4\Delta^2 + 2\Delta\hat{c} + 10\Delta + \hat{c})\Delta_1^2 - (2\Delta + \hat{c})\Delta_1^3}{2\Delta(4\Delta^2 + 2\Delta\hat{c} - 6\Delta + \hat{c})}$$

and so on.

We first consider the relation between the conformal block and the gauge theory partition function of $\mathcal{N} = 2^*$ gauge theory. We note that differently from to the $\mathcal{N} = 4$ case, the $\ell$ factor in this case depends only on the difference $k_{12}$. Thus, the instanton partition function factorizes as

$$Z_{\text{inst}}^{\text{ALE}} = \left(\sum_{k \in \mathbb{Z}} q^{k^2} z^k\right) Z_{\text{inst}}^k = \vartheta_3(q; z) Z_{\text{inst}}^{k=0},$$

where $Z_{\text{inst}}^{k=0}$ is calculated from (2.14).

We find that this partition function coincides with the conformal block on the torus with one insertion by the following relation\footnote{We have checked this relation up to terms of order $q^2$.}

$$Z_{\text{inst}}^{\text{ALE}}(q; z) = \vartheta_3(q; z)\eta(q)^{-2+m(Q-m)}\chi(q)B_{1,1}(q)$$

where $\eta$ and $\chi$ are

$$\eta(q) = \prod_{n>0} (1 - q^n), \quad \chi(q) = \prod_{n>0} (1 + q^{n-1/2}).$$

The identification of the parameters is

$$iP = a, \quad \alpha_1 = m - \frac{\epsilon_+}{2}.$$\hspace{1cm} (5.6)

Note that since the factors $\vartheta_3$ and $\chi$ include half-integer powers of $q$, this non-trivially mix the even and the odd parts of the conformal block to give the instanton partition function.

We note that the prefactor in (5.4) can be written in terms of the character of the affine $SU(2)_2$:

$$Z_{\text{inst}}^{\text{ALE}}(q; z) = q^{\frac{1}{12}} \left( \chi_{[2,0]}^{SU(2)_2}(q; z) + \chi_{[0,2]}^{SU(2)_2}(q; z) \right) \eta(q)^{-1-m(Q-m)} B_{1,1}(q)$$

where $[2,0]$ and $[0,2]$ represent the integrable highest weight representations. This result nicely agrees with that of [6] where the sum of the central charges of super Virasoro, affine $\widehat{SU(2)}_2$ and free boson was derived from the M-theory considerations.
Let us briefly comment on this result. First of all one can wonder why the \( \chi_{[1,1]}^{\hat{SU}(2)_{2}}(q; z) \) term does not appear in the sum (5.7). The point is that we restricted our computation to \( k \) integer and this corresponds in the conformal field theory to restrict to the NS sector. We expect the missing character to arise when including the contribution of instantons with \( k \) half-integer, which should be related to the R sector. Secondly, the appearance of the character of the \( \hat{SU}(2)_{2} \) algebra indicates that the fixed point sector of the instanton moduli space on the \( \mathbb{Z}_2 \) orbifold provides also a realization of this algebra. The result we got in equation (5.7) suggests that the vertex operator of the entire algebra is non-trivially represented in the super Liouville sector only. In other words, in the punctured torus example we do not see conformal blocks of WZW model appearing in the instanton computation, at least when restricting to the \( k \) integer case, namely to the NS sector.

Finally, let us consider the full correlator on the torus (5.1). The three-point function in (5.1) can be written as

\[
\mathbb{C}_{iP,\alpha_{1},-iP} \left| q^{\Delta-\frac{\ell}{6}} \right|^{2} = \tilde{C}_{0} \left| q^{-\frac{\ell}{2}} \mathcal{Z}_{pert}(a) \mathcal{Z}_{adj}(a; m) \right|^{2}, \tag{5.8}
\]

where again \( \tilde{C}_{0} \) is an irrelevant constant which does not depend on \( a \). Therefore, the one-point correlation function on the torus leads to the integral over the vev \( a \) of \( \mathcal{Z}_{full}^{\text{ALE}} \).

\( \mathcal{N} = 4 \) partition function

The \( \mathcal{N} = 4 \) partition functions on ALE spaces have been computed and considered from different points of view in relation with affine Lie algebras in the past literature [22]. Let us briefly consider the \( m = 0 \) limit of the \( \mathcal{N} = 2^{*} \) theory, where \( \mathcal{N} = 4 \) supersymmetry is recovered. We can easily compute the full instanton partition function \( \mathcal{Z}_{\text{inst}}^{\text{ALE}} \) in this limit as

\[
\mathcal{Z}_{\text{inst}}^{\text{ALE}} = \left( \sum_{k \in \mathbb{Z}} q^{\frac{k^{2}}{2}} z^{k} \right) \left( \sum_{k_{12} \in \mathbb{Z}} q^{\frac{k_{12}^{2}}{2}} \right) \eta(q)^{-4} = \vartheta_{3}(q; z) \eta(q)^{-3} \chi(q)^{2} \tag{5.9}
\]

where we have used that the instanton partition function on \( \mathbb{C}^{2} \) reduces in this limit to \( \eta(q)^{-2} \) and that the \( \ell \) factor in (2.14) equals 1 in the massless limit, so the sum over \( k_{12} \) gets factorized. In the last equality, we have used the fact that \( \vartheta_{3}(q; z = 0) = \eta(q) \chi(q)^{2} \). The expression (5.9) was obtained in [23, 24, 25]. As argued in [5], the character of the super Virasoro is

\[
\chi_{\text{superVirasoro}}(q) = \eta(q)^{-1} \chi(q) \tag{5.10}
\]

which gives a partial check of our result (5.4).
6 Conclusions

In this paper we presented a relation between gauge theories on the ALE space $\mathcal{O}_{\mathbb{P}^1}(-2)$ and $\mathcal{N} = 1$ super Liouville conformal field theory. We found a blow up formula relating the full partition function of the gauge theory on the ALE space to a convolution of partition functions on the flat space, see Eq.(2.31). This simple formula begs for an interpretation in two-dimensional superconformal field theory, possibly relating its correlation functions to the ones of bosonic Liouville theory. Moreover, Eq.(2.31) could be used to get information on the existence of special geometry relations for gauge theories on the ALE space by using similar arguments to [26]. We remark that in this paper we considered the correspondence between the integer $k$ sector of the gauge theory and the NS sector in the CFT. We expect that the half integer $k$ sector will be related to the R sector. Further investigations are needed in this direction.

On the other side, we found that the super Liouville correlators can be written in terms of four-dimensional gauge theory building blocks. Our results then point to a direct interpretation of the correlation functions in super Liouville theory, analogous to the one pointed out in [1], as partition functions of the corresponding gauge theories on $S^4/\mathbb{Z}_2$. It would be interesting to pursue this direction more precisely on the gauge theory side along the lines of [27], in particular carefully analyzing the peculiarities of the gluing conditions in the orbifold case.

We checked the relations we are proposing at lower orders in the expansion in the instanton counting parameters. It would be clearly desirable to have an exact, that is to all orders, proof of them by using recursion relations in the super Virasoro conformal block structure [19, 20, 28, 29], possibly generalizing the approach of [30, 31].

Our results open the way to generalize AGT correspondence to more general ADE quotients of $\mathbb{C}^2$ by comparing with para Liouville/Toda conformal field theories as suggested in [6].

It would be very useful in our opinion to find a topological string theory engineering of gauge theories on ALE spaces, possibly implementing a $\Gamma$-equivariant refined BPS counting of M-theory states à la Gopakumar-Vafa [32, 33, 34]. Further investigations are also needed on the geometry of M-theory compactifications for this class of theories and its relation to (quantum) Hitchin systems [7, 35, 36, 37, 38, 39].

Finally, we observe that relations analogous to the AGT correspondence was discussed in [40, 41, 42, 43, 44, 45, 46] connecting three-dimensional superconformal field theories on (squashed) $S^3$ and $SL(2, \mathbb{C})$ Chern-Simons theory on circle bundles over a Riemann surface. It would be interesting to generalize this correspondence to orbifold spaces.
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Appendix

A \( \gamma \) identities

Let us introduce the function

\[
\gamma_{\epsilon_1, \epsilon_2}(x; \Lambda) = \frac{d}{ds}|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-tx} \frac{e^{-tx}}{(e^{-\epsilon_1 t} - 1)(e^{-\epsilon_2 t} - 1)} \tag{A.1}
\]

By changing the argument \( \epsilon_1 \) and \( \epsilon_2 \) to the ones in section 2, we have

\[
\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(x) + \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(x) = \gamma_{2\epsilon_1, 2\epsilon_2}(x + \epsilon_1 + \epsilon_2),
\tag{A.2}
\]

where we have suppressed the \( \Lambda \) dependence. Indeed, from (A.1) we get that the l.h.s. of the above expression is

\[
\frac{d}{ds}|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-tx} \left[ \frac{1}{(e^{-2t\epsilon_1} - 1)(e^{-t(\epsilon_2 - \epsilon_1)} - 1)} + \frac{1}{(e^{-2t\epsilon_2} - 1)(e^{-t(\epsilon_1 - \epsilon_2)} - 1)} \right]
\]

from which (A.2) follows. One can generalize this identity to the following ones: let \( s \) be an integer. For \( s = \pm 1 \), we obtain

\[
\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(x + s\epsilon_1) + \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(x + s\epsilon_2) = \gamma_{2\epsilon_1, 2\epsilon_2}(x + \epsilon_1) + \gamma_{2\epsilon_1, 2\epsilon_2}(x + \epsilon_2),
\tag{A.4}
\]

for \( s > 1 \)

\[
\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(x + s\epsilon_1) + \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(x + s\epsilon_2) = \sum_{\substack{i, j \geq 0 \atop i + j = s}} - \sum_{\substack{i, j \geq 0 \atop i + j = s + 2}} \gamma_{2\epsilon_1, 2\epsilon_2}(x + i\epsilon_1 + j\epsilon_2),
\tag{A.5}
\]

and for \( s < -1 \)

\[
\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(x + s\epsilon_1) + \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(x + s\epsilon_2) = \sum_{\substack{i, j \geq -1 \atop i + j = -s - 2}} - \sum_{\substack{i, j \geq -1 \atop i + j = -s}} \gamma_{2\epsilon_1, 2\epsilon_2}(x - i\epsilon_1 - j\epsilon_2),
\tag{A.6}
\]
By using these identities and generalizing the argument in the Appendix in [26], we can obtain the following useful formula:

\[
\exp \left[ -\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m - 2k_{\alpha\beta'}\epsilon_1) - \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m - 2k_{\alpha\beta'}\epsilon_2) \right] \tag{A.7}
\]

\[
\lambda^{2(k_{\alpha\beta'})^2} \ell_{\alpha\beta'}(a_\alpha, k_{\alpha\beta'}, \tilde{k}_{\beta'}; m)^{-1} \times \begin{cases} 
\exp \left[ -\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m) - \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m) \right], & k_{\alpha\beta} \in \mathbb{Z} \\
\exp \left[ -\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m + \epsilon_1) - \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(a_\alpha - \tilde{a}_{\beta'} + \epsilon_+ - m + \epsilon_2) \right], & k_{\alpha\beta} \in \mathbb{Z} + \frac{1}{2}
\end{cases}
\]

and

\[
\exp \left[ -\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(a_\alpha - \tilde{a}_{\beta'} - m - 2k_{\alpha\beta'}\epsilon_1) - \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(a_\alpha - \tilde{a}_{\beta'} - m - 2k_{\alpha\beta'}\epsilon_2) \right] \tag{A.8}
\]

\[
\left( -\lambda^{2(k_{\alpha\beta'})^2} \ell_{\alpha\beta'}(a_\alpha, k_{\alpha\beta'}, \tilde{k}_{\beta'}; m)^{-1} \times \begin{cases} 
\exp \left[ -\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(a_\alpha - \tilde{a}_{\beta'} - m) - \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(a_\alpha - \tilde{a}_{\beta'} - m) \right], & k_{\alpha\beta} \in \mathbb{Z} \\
\exp \left[ -\gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(a_\alpha - \tilde{a}_{\beta'} - m + \epsilon_1) - \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(a_\alpha - \tilde{a}_{\beta'} - m + \epsilon_2) \right], & k_{\alpha\beta} \in \mathbb{Z} + \frac{1}{2}
\end{cases}
\]

where \( k_{\alpha\beta'} = k_{\alpha} - \tilde{k}_{\beta'} \), \( \ell_{\alpha\beta} \) was defined in (2.15) and we have used

\[
\gamma_{2\epsilon_1, 2\epsilon_2}(x + 2\epsilon_1) + \gamma_{2\epsilon_1, 2\epsilon_2}(x + 2\epsilon_2) - \gamma_{2\epsilon_1, 2\epsilon_2}(x) + \gamma_{2\epsilon_1, 2\epsilon_2}(x + 2\epsilon_1 + 2\epsilon_2) = \log \left( \frac{x}{\Lambda} \right), \tag{A.9}
\]

by which the \( \Lambda \) dependent factor appeared. Similary, for the (anti-)fundamental matter part, we obtain

\[
\exp \left[ \gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(x - 2k_{\alpha}\epsilon_1) + \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(x - 2k_{\alpha}\epsilon_2) \right] \tag{A.10}
\]

\[
\left( -\lambda^{-1(k_{\alpha})^2} \ell_{\alpha}(x, k_{\alpha}) \times \begin{cases} 
\exp \left[ \gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(x) + \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(x) \right], & k_{\alpha} \in \mathbb{Z} \\
\exp \left[ \gamma_{2\epsilon_1, \epsilon_2 - \epsilon_1}(x + \epsilon_1) + \gamma_{\epsilon_1 - \epsilon_2, 2\epsilon_2}(x + \epsilon_2) \right], & k_{\alpha} \in \mathbb{Z} + \frac{1}{2}
\end{cases}
\]

where \( \ell_{\alpha} \) was defined in (2.16). In the case of theories with vanishing beta function, \( \Lambda \) is set to be 1.

### B Three-point functions

For completeness, we collect the expressions for the three-point functions of super Liouville theory. We are interested in the following types of three-point functions:

\[
\langle V_{a_1}(z_1)V_{a_2}(z_2)V_{a_3}(z_3) \rangle = \frac{C_{a_1,a_2,a_3}}{|z_{12}^{\Delta_{1+2-3}} z_{23}^{\Delta_{2+3-1}} z_{31}^{\Delta_{3+1-2}}|}, \tag{B.1}
\]

\[
\langle W_{a_1}(z_1)V_{a_2}(z_2)V_{a_3}(z_3) \rangle = \frac{\tilde{C}_{a_1,a_2,a_3}}{|z_{12}^{\Delta_{1+2-3+1/2}} z_{23}^{\Delta_{2+3-1-1/2}} z_{31}^{\Delta_{3+1-2+1/2}}|}.
\]
where $\Delta_{1+2-3} = \Delta_1 + \Delta_2 - \Delta_3$. The expressions for $C$ and $\tilde{C}$ were given in \cite{47,48} and also \cite{49}

$$C_{a_1,a_2,a_3} = A \frac{\Upsilon_{NS}(2a_1)\Upsilon_{NS}(2a_2)\Upsilon_{NS}(2a_3)}{\Upsilon_{NS}(a-Q)\Upsilon_{NS}(a_1+2-3)\Upsilon_{NS}(a_2+3-1)\Upsilon_{NS}(a_3+1-2)},$$

$$\tilde{C}_{a_1,a_2,a_3} = 2iA \frac{\Upsilon_{NS}(2a_1)\Upsilon_{NS}(2a_2)\Upsilon_{NS}(2a_3)}{\Upsilon_{R}(a-Q)\Upsilon_{R}(a_1+2-3)\Upsilon_{R}(a_2+3-1)\Upsilon_{R}(a_3+1-2)}, \tag{B.2}$$

where $a = a_1 + a_2 + a_3$, $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and

$$A = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) b^{1-b^2} \right)^{a-Q} \Upsilon_{NS}(0),$$

$$\Upsilon_{NS}(x) = \Upsilon \left( \frac{x}{2} \right) \Upsilon \left( \frac{x+Q}{2} \right), \quad \Upsilon_{R}(x) = \Upsilon \left( \frac{x+b}{2} \right) \Upsilon \left( \frac{x+b^{-1}}{2} \right). \tag{B.3}$$

The $\Upsilon$ function can be written in terms of the Barnes’ double Gamma function $\Gamma_2$ as

$$\Upsilon(x) = \frac{1}{\Gamma_2(x|b,b^{-1})\Gamma_2(Q-x|b,b^{-1})}. \tag{B.4}$$

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