Sampling and periodization of generators of Heisenberg modules

Franz Luef

Noncommutativity in the North
Gothenborg

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This is based on joint work with **Mads S. Jakobsen**.

- **Sampling and periodization of generators of Heisenberg modules**, International Journal of Mathematics, Vol. 30, No. 10, 1950051 (2019).
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**Main theme**
Study of Heisenberg modules from the perspective of frame theory. Use results on Gabor frames in the continuous and discrete setting to approximate Heisenberg modules by finite-dimensional Heisenberg modules.
**Definitions**

Let $E$ be an $A$-$B$-equivalence bimodule. For $g \in E$ we define the **analysis operator** by

$$\Phi g : E \to A \quad f \mapsto \langle f, g \rangle,$$

and the **synthesis operator**:

$$\Psi g : A \to E \quad a \mapsto a \cdot g.$$
Frames for Hilbert $C^*$-modules

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\quad \quad f \mapsto \langle f, g \rangle,
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$$\Theta_{g,h} : E \rightarrow E$$

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Let $E$ be an $A$-$B$-equivalence bimodule. We define the frame-like operator $\Theta_{g,h}$ to be

$$\Theta_{g,h} : E \rightarrow E$$

$$f \mapsto \langle f, g \rangle \cdot h.$$ 

$$\Theta_{g,h} = \Psi_h \Phi_g = \Phi_h^* \Phi_g.$$ 

The frame operator of $g$ is the operator

$$\Theta_g := \Theta_{g,g} = \Phi_g^* \Phi_g.$$
Module frames

Suppose we have \( g_1, \ldots, g_k \in E \), such that \( \sum_{i=1}^{k} \Theta_{g_i} \) is invertible \( E \). Then we call \( \{g_1, \ldots, g_k\} \) a **module frame** for \( E \).
Module frames

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This is equivalent to existence of constants \( C, D > 0 \) such that

\[
C \langle f, f \rangle \leq \sum_{i=1}^{k} \langle f, g_i \rangle \langle g_i, f \rangle \leq D \langle f, f \rangle
\]

for all \( f \in E \).
Frames for Hilbert $C^*$-modules

Module frames

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for all $f \in E$.

Frame condition

Let $E$ be an $A$-$B$-equivalence bimodule. Then $f = \sum_{i=1}^{k} \Theta g_i, h_i f$ for all $f \in E$ if and only if $B$ is unital and $\sum_{i=1}^{k} \langle g_i, h_i \rangle \bf{=} 1_B$. 

Lemma

Let $E$ be an $A$-$B$-equivalence bimodule. Suppose $g, h \in E$ satisfy $\langle f, h \rangle g = f$ for all $f \in E$. Then

$$f = h \langle g, f \rangle \quad \text{for all } f \in h \cdot B.$$

Definition

Let $E$ be an $A$-$B$-equivalence bimodule. If $g \in E$ is such that $\Theta_g$ is invertible on $E$, then $h = \Theta_g^{-1} g$ is called the \textbf{canonical dual atom} of $g$. 
Heisenberg modules

Modulation operator and translation operator

$$E_{\beta}f(t) = e^{2\pi i \beta t}f(t), \ T_\alpha f(t) = f(t - \alpha), \ \alpha, \beta \in \mathbb{R}\{0\}, \ f \in L^2(\mathbb{R}),$$
Heisenberg modules

Modulation operator and translation operator

\[ E_{\beta} f(t) = e^{2\pi i \beta t} f(t), \quad T_{\alpha} f(t) = f(t - \alpha), \quad \alpha, \beta \in \mathbb{R} \setminus \{0\}, \quad f \in L^2(\mathbb{R}), \]

\[ \mathcal{A} = \left\{ a \in B(L^2(\mathbb{R})) : a = \sum_{m,n \in \mathbb{Z}} a(m, n) E_{m\beta} T_{n\alpha}, \quad a \in \ell^1(\mathbb{Z}^2) \right\}. \]
Heisenberg modules

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\( a \rightarrow a \) is a faithful representation of the twisted group algebra \( \ell^1(\alpha \mathbb{Z} \times \beta \mathbb{Z}, c) \):
Heisenberg modules

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\( a \to a \) is a faithful representation of the twisted group algebra \( \ell^1(\alpha \mathbb{Z} \times \beta \mathbb{Z}, c) \):

\[ a_1 \sharp a_2(m, n) = \sum_{m', n' \in \mathbb{Z}} a_1(m', n') a_2(m - m', n - n') e^{2\pi i \beta \alpha (m-m')n'}, \quad (1) \]

\[ a^*(m, n) = e^{2\pi i \alpha \beta mn} \overline{a(-m, -n)}. \quad (2) \]
Heisenberg modules

Modulation operator and translation operator

\[ E_\beta f(t) = e^{2\pi i \beta t} f(t), \quad T_\alpha f(t) = f(t - \alpha), \quad \alpha, \beta \in \mathbb{R} \setminus \{0\}, \quad f \in L^2(\mathbb{R}), \]

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The enveloping \( C^* \)-algebra is the twisted group \( C^* \)-algebra \( C^*(\mathbb{Z}^2, c) \).
Heisenberg modules

The left-action that $a \in \mathcal{A}$ has on functions $f \in L^2(\mathbb{R})$ is given by

$$a \cdot f = \sum_{m,n \in \mathbb{Z}} a(m, n) E_{m\beta} T_{n\alpha} f.$$
Heisenberg modules

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For functions in $S_0(\mathbb{R})$ we define an $\mathcal{A}$-valued inner-product in the following way:  

$$\langle \cdot, \cdot \rangle: S_0(\mathbb{R}) \times S_0(\mathbb{R}) \to \mathcal{A},$$

$$\langle f, g \rangle = \sum_{m,n \in \mathbb{Z}} \langle f, E_{m\beta} T_{n\alpha} g \rangle E_{m\beta} T_{n\alpha}.$$
Heisenberg modules

The left-action that \( a \in A \) has on functions \( f \in L^2(\mathbb{R}) \) is given by

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\langle f, g \rangle = \sum_{m,n \in \mathbb{Z}} \langle f, E_{m\beta} T_{n\alpha} g \rangle E_{m\beta} T_{n\alpha}.
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Here \( S_0(\mathbb{R}) \) is Feichtinger’s algebra: a suitable Banach space of test-functions, which is widely used in time-frequency analysis. Let \( g \) be a Gaussian function.
Heisenberg modules

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Here \( S_0(\mathbb{R}) \) is Feichtinger's algebra: a suitable Banach space of test-functions, which is widely used in time-frequency analysis. Let \( g \) be a Gaussian function. Then \( f \in L^2(\mathbb{R}) \) is in \( S_0(\mathbb{R}) \) if and only if
\[
\| f \|_{S_0} = \iint_{\mathbb{R}^2} |\langle f, E_{\omega} T_x g \rangle| \, dx \, d\omega < \infty,
\]
where \( V_g f(x, \omega) = \langle f, E_{\omega} T_x g \rangle \) is the short-time Fourier transform.
Let $\mathcal{B}$ be the twisted group algebra $\ell^1(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$ and $C^*(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$ its enveloping $C^*$-algebra.
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We define a $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle : S_0(\mathbb{R}) \times S_0(\mathbb{R}) \to \mathcal{B} :$

$$\langle f, g \rangle = \frac{1}{|\alpha\beta|} \sum_{m,n \in \mathbb{Z}} \langle g, (E_{m/\alpha} T_{n/\beta})^* f \rangle (E_{m/\alpha} T_{n/\beta})^*$$
Let $\mathcal{B}$ be the twisted group algebra $\ell^1(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$ and $C^*(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$ its enveloping $C^*$-algebra.

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that has a right-action on functions $g \in L^2(\mathbb{R})$ given by

$$g \cdot b = \frac{1}{|\alpha \beta|} \sum_{m,n \in \mathbb{Z}} b(m, n) \left( E_{m/\alpha} T_{n/\beta} \right)^* g.$$
Let $B$ be the twisted group algebra $\ell^1\left(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \bar{c}\right)$ and $C^*\left(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \bar{c}\right)$ its enveloping $C^*$-algebra.

We define a $B$-valued inner product $\langle \cdot, \cdot \rangle : S_0(\mathbb{R}) \times S_0(\mathbb{R}) \to B : \langle f, g \rangle = \frac{1}{|\alpha\beta|} \sum_{m,n \in \mathbb{Z}} \langle g, (E_{m/\alpha} T_{n/\beta})^* f \rangle (E_{m/\alpha} T_{n/\beta})^*$ that has a right-action on functions $g \in L^2(\mathbb{R})$ given by

$$g \cdot b = \frac{1}{|\alpha\beta|} \sum_{m,n \in \mathbb{Z}} b(m, n) (E_{m/\alpha} T_{n/\beta})^* g.$$

Associativity condition:

$$\langle f, g \rangle \cdot h = f \cdot \langle g, h \rangle$$ for all $f, g, h \in S_0(\mathbb{R})$. 

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Let $\mathcal{B}$ be the twisted group algebra $\ell^1(\mathbb{Z}/\beta \mathbb{Z} \times \mathbb{Z}/\alpha \mathbb{Z}, \bar{c})$ and $C^*(\mathbb{Z}/\beta \mathbb{Z} \times \mathbb{Z}/\alpha \mathbb{Z}, \bar{c})$ its enveloping $C^*$-algebra.

We define a $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle_\bullet : S_0(\mathbb{R}) \times S_0(\mathbb{R}) \rightarrow \mathcal{B}$:

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The completion of $S_0(\mathbb{R})$ with respect to $\| f \|_E = \| \langle f, f \rangle_\bullet \|^{1/2}$ (or $\| \langle f, f \rangle_\bullet \|^{1/2}$) is the Heisenberg module, $E$. 
Heisenberg modules

Bessel family

Denote by $B_{\alpha\beta}$ the subspace of $L^2(\mathbb{R})$ consisting of those $g \in L^2(\mathbb{R})$ such that

$$\sum_{k,l \in \mathbb{Z}} |\langle f, E_{\beta l} T_{\alpha k} g \rangle|^2 \leq \infty,$$

for all $f \in L^2(\mathbb{R})$. 
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$\{E_{\beta l} T_{\alpha k} g\}_{k,l \in \mathbb{Z}}$ is a Gabor frame if there exists an $h \in L^2(\mathbb{R})$ such that

$$f = \langle f, g \rangle h = \sum_{m,n \in \mathbb{Z}} \langle f, E_{\beta n} T_{\alpha m} g \rangle E_{\beta n} T_{\alpha m} h$$
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Austad-Enstad

If $g \in E$ is an element of the Heisenberg module, then it is a Bessel vector of the Gabor system $\{E_{\beta l} T_{\alpha k} g\}_{k,l}$. 

For $g \in S^0(\mathbb{R})$ this was shown in (Luef, JFA 2009).
Heisenberg modules

Austad-Enstad

- If $g \in E$ is an element of the Heisenberg module, then it is a Bessel vector of the Gabor system $\{ E_{\beta l} T_{\alpha k} g \}_{k,l \in \mathbb{Z}}$.
- The set $\{ g_1, \ldots, g_k \}$ generates $E$ as a left $C^*(\alpha \mathbb{Z} \times \beta \mathbb{Z}, c)$-module if and only if $\{ E_{\beta n} T_{\alpha m} g_i : i = 1, \ldots, k \}_{m,n \in \mathbb{Z}}$ is a (multi-window) frame for $L^2(\mathbb{R})$, i.e. there exist constant $A, B > 0$ such that

$$A \| f \|_2^2 \leq \sum_{i=1}^{k} \sum_{m,n \in \mathbb{Z}} |\langle f, E_{\beta n} T_{\alpha m} g_i \rangle|^2 \leq B \| f \|_2^2$$

for all $f \in L^2(\mathbb{R})$. For $g \in S_0(\mathbb{R})$ this was shown in (Luef, JFA 2009).
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$$A \|f\|_2 \leq \sum_{i=1}^k \sum_{m, n \in \mathbb{Z}} |\langle f, E_{\beta m}T_{\alpha n}g_i \rangle|^2 \leq B \|f\|_2 \text{ for all } f \in L^2(\mathbb{R}).$$

For $g \in \mathcal{S}_0(\mathbb{R})$ this was shown in (Luef, JFA 2009).
Setting

We assume that $\theta$ is such that $\theta = a/M = b/N$ for some $a, b, M, N \in \mathbb{N}$ and take $d = aN$. Then the noncommutative torus $\mathcal{A}_\theta$ has faithful representations in terms of the Schrödinger representations of the Heisenberg group that act on $L^2(\mathbb{R})$, $\ell^2(a^{-1}\mathbb{Z})$ and $\ell^2(\mathbb{Z}_d) \cong \mathbb{C}^d$, where $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z} \cong \{0, 1, \ldots, d - 1\}$.

Noncommutative tori

For convenience, we denote the respective realization by $\mathcal{A}_\theta^\mathbb{R}$, $\mathcal{A}_\theta^{a^{-1}\mathbb{Z}}$ and $\mathcal{A}_\theta^{\mathbb{Z}_d}$ and the canonical generators:

$$U_{\mathbb{R}} f(t) = e^{2\pi i \theta t} f(t), \quad V_{\mathbb{R}} f(t) = f(t - 1), \quad f \in L^2(\mathbb{R}), \ t \in \mathbb{R},$$
Noncommutative tori – ctd

\[ U_{a^{-1}\mathbb{Z}} f(t) = e^{2\pi i \theta t} f(t), \quad V_{a^{-1}\mathbb{Z}} f(t) = f(t - 1), \quad f \in \ell^2(a^{-1}\mathbb{Z}), \quad t \in a^{-1}\mathbb{Z}, \]

\[ U_{\mathbb{Z}_d} f(t) = e^{2\pi i b t/d} f(t), \quad V_{\mathbb{Z}_d} f(t) = f(t - a), \quad f \in \ell^2(\mathbb{Z}/d\mathbb{Z}), \quad t \in \{0, 1, \ldots, d - 1\}. \]

We shall not so much use the operators \( U \) and \( V \) but rather the time-frequency shift operators:

(i) For \((x, \omega) \in \mathbb{R}^2\) and \(f \in L^2(\mathbb{R})\) we define
\[ \pi(x, \omega) f(t) = e^{2\pi i \omega t} f(t - x), \quad t \in \mathbb{R} \text{ and that } \pi(1, \theta) = U_{\mathbb{R}} V_{\mathbb{R}}. \]

(ii) For \((x, \omega) \in a^{-1}\mathbb{Z} \times [0, a)\) and \(f \in \ell^2(a\mathbb{Z})\) we define
\[ \pi(x, \omega) f(t) = e^{2\pi i \omega t} f(t - x), \quad t \in a^{-1}\mathbb{Z} \text{ and that } \pi(1, \theta) = U_{a^{-1}\mathbb{Z}} V_{a^{-1}\mathbb{Z}}. \]

(iii) For \((x, \omega) \in \mathbb{Z}_d \times \mathbb{Z}_d\) and \(f \in \ell^2(\mathbb{Z}_d)\) we define
\[ \pi(x, \omega) f(t) = e^{2\pi i \omega t/d} f(t - x) \text{ and that } \pi(a, b) = U_{\mathbb{Z}_d} V_{\mathbb{Z}_d}. \]
Noncommutative tori – ctd

\[ A^R_\theta = \{ a \in B(L^2(\mathbb{R})) : a = \sum_{n,m \in \mathbb{Z}} a(n, m) \pi(n, \theta m), a \in \ell^1(\mathbb{Z}^2) \}, \]

\[ A^{a^{-1}Z}_\theta = \{ a \in B(\ell^2(a^{-1}\mathbb{Z})) : a = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} a(n, m) \pi(n, \theta m), a \in \ell^1(\mathbb{Z} \times \mathbb{Z}_M) \}, \]

\[ A^{Z^d}_\theta = \{ a \in \mathbb{C}^d : a = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} a(n, m) \pi(na, mb), a \in \ell^1(\mathbb{Z}_N \times \mathbb{Z}_M) \}. \]

The enveloping C*-algebras of \( A^R_\theta \), \( A^{a^{-1}Z}_\theta \) and \( A^{Z^d}_\theta \) are *-representations of the noncommutative torus \( A_\theta \).
For functions in Feichtinger’s algebra $S_0(\mathbb{R})$, sequences in $\ell^1(a^{-1}\mathbb{Z})$, and vectors in $\mathbb{C}^d$ we define a respective $\mathcal{A}_\theta$-valued inner-product in the following ways:

\[
\langle \cdot, \cdot \rangle^\mathbb{R} : S_0(\mathbb{R}) \times S_0(\mathbb{R}) \rightarrow \mathcal{A}_\theta^\mathbb{R},
\]

\[
\langle f, g \rangle^\mathbb{R} = \sum_{n,m \in \mathbb{Z}} \langle f, \pi(n, \theta m)g \rangle \pi(n, \theta m),
\]

\[
\langle \cdot, \cdot \rangle^{a^{-1}\mathbb{Z}} : \ell^1(a^{-1}\mathbb{Z}) \times \ell^1(a^{-1}\mathbb{Z}) \rightarrow \mathcal{A}_\theta^{a^{-1}\mathbb{Z}},
\]

\[
\langle f, g \rangle^{a^{-1}\mathbb{Z}} = \sum_{n \in \mathbb{Z}} \sum_{m=0}^{M-1} \langle f, \pi(n, \theta m)g \rangle \pi(n, \theta m),
\]

\[
\langle \cdot, \cdot \rangle^{\mathbb{Z}_d} : \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathcal{A}_\theta^{\mathbb{Z}_d},
\]

\[
\langle f, g \rangle^{\mathbb{Z}_d} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \langle f, \pi(na, mb)g \rangle \pi(na, mb),
\]
The *module norm* of a function in $S_0(\mathbb{R})$, a sequence in $\ell^1(a^{-1}\mathbb{Z})$ and a vector in $\mathbb{C}^d$ is given, respectively, by

$$\|g\|_{A^R_\theta} = \|\langle g, g \rangle^R\|_{\text{op}}^{1/2}, \quad g \in S_0(\mathbb{R}),$$

$$\|g\|_{A^{a^{-1}\mathbb{Z}}_\theta} = \|\langle g, g \rangle^{a^{-1}\mathbb{Z}}\|_{\text{op}}^{1/2}, \quad g \in \ell^1(a^{-1}\mathbb{Z}),$$

$$\|g\|_{A^\mathbb{Z}_d} = \|\langle g, g \rangle^{\mathbb{Z}_d}\|_{\text{op}}^{1/2}, \quad g \in \mathbb{C}^d,$$
If $g$ is a function in $S_0(\mathbb{R})$ such that $\langle g, g \rangle^\mathbb{R}$ is a projection in $A^\mathbb{R}_\theta$, then the following holds.

(i) The module norm of $g$ satisfies

$$\|g\|_{A^\mathbb{R}_\theta} \leq C := \theta^{-1} \sum_{m,n \in \mathbb{Z}} |\langle g, e^{2\pi i m(\cdot)}g(\cdot - n\theta^{-1}) \rangle|.$$  

(ii) The sequence $\tilde{g} := \{\sqrt{a^{-1}} g(t)\}_{t \in a^{-1}\mathbb{Z}}$ belongs to $\ell^1(a^{-1}\mathbb{Z})$ and is such that $\langle \tilde{g}, \tilde{g} \rangle^{a^{-1}\mathbb{Z}}$ is a projection in $A^{a^{-1}\mathbb{Z}}_\theta$. Moreover, the module norm of $\tilde{g}$ satisfies $\|\tilde{g}\|_{A^{a^{-1}\mathbb{Z}}_\theta} \leq C$.

(iii) The finite sequence $\tilde{g}(t) := \sqrt{a^{-1}} \sum_{k \in \mathbb{Z}} g(a^{-1}(t - kd))$, $t \in \{0, 1, \ldots, d - 1\}$ belongs to $\mathbb{C}^d$ and is such that $\langle \tilde{g}, \tilde{g} \rangle^{\mathbb{Z}_d}$ is a projection in $A_{\theta}^{\mathbb{Z}_d}$. Moreover, the module norm of $\tilde{g}$ satisfies $\|\tilde{g}\|_{A_{\theta}^{\mathbb{Z}_d}} \leq C$. 

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Sampling and periodization

For $\gamma \in \mathbb{R}\backslash\{0\}$ we let $\mathcal{R}_{\gamma \mathbb{Z}}$ be the sampling/restriction operator that takes a function in $\mathcal{S}_0(\mathbb{R})$ to a sequence in $\ell^1(\gamma \mathbb{Z})$ defined by

$$(\mathcal{R}_{\gamma \mathbb{Z}} f)(k) = f(k), \text{ for } k \in \gamma \mathbb{Z}.$$ 

Furthermore, for $d \in \mathbb{N}$ we let $\mathcal{P}_{\gamma d \mathbb{Z}}$ be the periodization operator

$\mathcal{P}_{\gamma d \mathbb{Z}} : \ell^1(\gamma \mathbb{Z}) \to \mathbb{C}^d$, $(\mathcal{P}_{\gamma d \mathbb{Z}} f)(t) = \sum_{k \in \gamma d \mathbb{Z}} f(\gamma t - k), \text{ } t \in \{0, 1, \ldots, d - 1\}.$
If $g$ be a function in $S_0(\mathbb{R})$ such that $\langle g, g \rangle$ is a projection in $A_\theta^R$, then the following holds.

(i) The module norm of $g$ satisfies

$$\|g\|_\Lambda \leq C := \theta^{-1} \sum_{\lambda^\circ \in \theta^{-1}\mathbb{Z} \times \mathbb{Z}} |\langle g, \pi(\lambda^\circ)g \rangle|.$$

(ii) The sequence in $\ell^1(a^{-1}\mathbb{Z})$ defined by $\tilde{g} = (a^{-1/2} \mathcal{R}_{a^{-1}\mathbb{Z}}g)$ is such that $\langle \tilde{g}, \tilde{g} \rangle$ is a projection in $A_{\theta}^{a^{-1}\mathbb{Z}}$. Moreover, the module norm of $\tilde{g}$ satisfies $\|\tilde{g}\| \tilde{\Lambda} \leq C$.

(iii) The vector in $\mathbb{C}^d$ given by $\tilde{\tilde{g}} = (a^{-1/2} \mathcal{P}_{a^{-1}d\mathbb{Z}} \mathcal{R}_{a^{-1}\mathbb{Z}}g)$ is such that $\langle \tilde{\tilde{g}}, \tilde{\tilde{g}} \rangle$ is a projection in $A_{\theta}^{\mathbb{Z}^d}$. Moreover, the module norm of $\tilde{\tilde{g}}$ satisfies $\|\tilde{\tilde{g}}\| \tilde{\tilde{\Lambda}} \leq C$. 
Setting

Let $\Lambda$ be the lattice in $\mathbb{R}^2$ given by

$$\Lambda = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \cdot \begin{bmatrix} \mathbb{Z} \\ \mathbb{Z} \end{bmatrix}, \quad \alpha, \beta > 0.$$ 

and let $g$ and $h$ be functions in $S_0(\mathbb{R})$ such that $\langle g, h \rangle^{\mathbb{R}}$ is an idempotent element of $\mathcal{A}^\mathbb{R}$, i.e. $\{\pi(\lambda)g\}_{\lambda \in \Lambda}$ and $\{\pi(\lambda)h\}_{\lambda \in \Lambda}$ are dual Gabor frames for $L^2(\mathbb{R})$.

If $\alpha$ and $\beta$ are such that $\alpha \beta = a/M = b/N$ for some $a, b, M, N \in \mathbb{N}$, we put $d = Mb(=aN)$, and let

$$\tilde{\Lambda} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} \{0, 1, \ldots, N-1\} \\ \{0, 1, \ldots, M-1\} \end{bmatrix} \subset \mathbb{Z}_d \times \mathbb{Z}_d,$$
Let \( \tilde{g} \) and \( \tilde{h} \) in \( \mathbb{C}^d \) given by

\[
\tilde{g}(t) = \sqrt{\alpha a^{-1}} \sum_{k \in \mathbb{Z}} g(\alpha a^{-1}(t - kd)),
\]

\[
\tilde{h}(t) = \sqrt{\alpha a^{-1}} \sum_{k \in \mathbb{Z}} h(\alpha a^{-1}(t - kd)),
\]

for \( t \in \{0, 1, \ldots, d - 1\} \).

**Theorem**

Then \( \langle \tilde{g}, \tilde{h} \rangle^{\Lambda} \) is an idempotent element of \( \mathcal{A}^{\mathbb{Z}_d} \), i.e., the two Gabor systems \( \{ \pi(\lambda)\tilde{g} \}_{\lambda \in \tilde{\Lambda}} \) and \( \{ \pi(\lambda)\tilde{h} \}_{\lambda \in \tilde{\Lambda}} \) are dual frames for \( \mathbb{C}^d \).
From the discrete to the continuous

Want to construct a function in $S_0(\mathbb{R})$ from a sequence in $\ell^1$ is by use of linear interpolation.

**Quasi-interpolation**

For a given $\gamma > 0$ let $\wedge_\gamma$ be the triangular-function

$$\wedge_\gamma(x) = (1 - |\gamma^{-1}x|) \cdot 1_{[-\gamma,\gamma]}.$$\[10pt]

Furthermore, for any $\gamma > 0$ we define the operator

$$Q_{\gamma\mathbb{Z}}^{\mathbb{R}} : \ell^1(\gamma\mathbb{Z}) \to S_0(\mathbb{R}), \quad (Q_{\gamma\mathbb{Z}}^{\mathbb{R}} c)(t) = \sum_{k \in \gamma\mathbb{Z}} c(k) \cdot \wedge_\gamma(t - k).$$

Observe that $Q_{\gamma\mathbb{Z}}^{\mathbb{R}}$ interpolates linearly in between the points $(k, c(k))_{k \in \gamma\mathbb{Z}}$. 
Quasi-interpolation–ctd

The procedure to turn a vector in $\mathbb{C}^d$ into a sequence in $\ell^1(\gamma\mathbb{Z})$ is similar:

$$Q_{\gamma d} : \mathbb{C}^d \to \ell^1(\gamma\mathbb{Z}), \quad (Q_{\gamma d} c)(k) = \begin{cases} 0 & \text{if } k \notin \gamma\{-\lfloor \frac{d-1}{2} \rfloor, \ldots, \lfloor \frac{d}{2} \rfloor\}, \\ c(\gamma^{-1} k \mod d) & \text{if } k \in \gamma\{-\lfloor \frac{d-1}{2} \rfloor, \ldots, \lfloor \frac{d}{2} \rfloor\}. \end{cases}$$

Here $c \in \mathbb{C}^d$ is treated as a function that can be evaluated on the set $\{0, 1, \ldots, d - 1\}$.

Feichtinger-Kaiblinger

(i) For any $f \in \mathcal{S}_0(\mathbb{R})$

$$\lim_{\gamma \to 0} \left\| f - Q_{\gamma\mathbb{Z}}^R R_{\gamma\mathbb{Z}} f \right\|_{\mathcal{S}_0(\mathbb{R})} = 0.$$ 

(ii) For any $f \in \mathcal{S}_0(\mathbb{R})$

$$\lim_{n \to \infty} \left\| f - Q_{n^{-1/2}\mathbb{Z}}^R Q_{n^{-1/2}\mathbb{Z}}^n P_{n^{1/2}\mathbb{Z}} R_{n^{-1/2}\mathbb{Z}} f \right\|_{\mathcal{S}_0(\mathbb{R})} = 0.$$
Observation

We know already that the samples of generator of a projection in $A^R_\theta$ also generate a projection in $A^{a^{-1}Z}_\theta$. If these samples are dense enough, and linear interpolated to a function on $\mathbb{R}$ by the operator $Q^{R}_{\gamma Z}$, then the in this way constructed collections of functions generate a projection in $A^R_\theta$, again.

Theorem

For all $a \in \mathbb{N}$ let $\tilde{g}$ in $\ell^1(a^{-1}\mathbb{Z})^n$ be given by

$$\tilde{g} = (R_{a^{-1}\mathbb{Z}}g).$$

For all $a$ that are sufficiently large, $k := (Q^{R}_{a^{-1}\mathbb{Z}}\tilde{g})$ in $S_0(\mathbb{R})$ is such that $b_k := \langle k, k \rangle \in B^{R}_{\theta^{-1}}$ is invertible on $L^2(\mathbb{R})$ and on $S_0(\mathbb{R})$.

In that case, $k \cdot b_k^{-1} \in S_0(\mathbb{R})$ is such that $\langle k, k \cdot b_k^{-1} \rangle$ is a projection in $A^R_\theta$. 
Theorem–ctd

Furthermore, we also have that \( k = Q^{\mathbb{R}}_{a^{-1} \mathbb{Z}} \tilde{g} \) converges towards \( g \) in the module norm as \( a \to \infty \), that is

\[
\lim_{a \to \infty} \| g - k \|_\Lambda = 0.
\]

Theorem

For each \( d \in \mathbb{N} \) let \( \tilde{g} \) be the vector in \( \mathbb{C}^d \) given by

\[
\tilde{g} = \mathcal{P}_{d^{1/2} \mathbb{Z}} \mathcal{R}_{d^{-1/2} \mathbb{Z}} g.
\]

For all \( d \) that are sufficiently large, the function \( k \) in \( S_0(\mathbb{R}) \) given by

\[
k := Q^{\mathbb{R}}_{d^{-1/2} \mathbb{Z}} Q_{d}^{d-1/2 \mathbb{Z}} \tilde{g}
\]

is such that \( b_k := \langle k, k \rangle_\bullet \in \mathcal{B}^{\mathbb{R}}_{\theta-1} \) is invertible on \( L^2(\mathbb{R}) \) and on \( S_0(\mathbb{R}) \). In that case, \( k \cdot b_k^{-1} \in S_0(\mathbb{R}) \) is such that \( \bullet \langle k \cdot b_k^{-1}, k \cdot b_k^{-1} \rangle \) is a projection in \( \mathcal{A}_\theta^{\mathbb{R}} \).
Theorem—ctd

In that case, \( k \cdot b_k^{-1} \in S_0(\mathbb{R}) \) is such that \( \langle k \cdot b_k^{-1}, (k \cdot b_k^{-1}) \rangle \) is a projection in \( A^\mathbb{R}_\theta \).

Furthermore, we also have that \( k \) converges towards \( g \) in the module norm as \( d \to \infty \), that is

\[
\lim_{d \to \infty} \| g - k \|_\Lambda = 0.
\]

\[
\| g - k \|_\Lambda^2 = \| g - k \|_{\Lambda^\circ}^2 = \| \langle g - k, g - k \rangle \|_{\text{op}}
\]
\[
\leq \theta^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \left| \langle g - k, \pi(\lambda^\circ)^*(g - k) \rangle_{L^2(\mathbb{R})} \right|
\]
\[
= \theta^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \left| V_{g-k}(g - k)(\lambda^\circ) \right|
\]
\[
\leq c \| g - k \|_{S_0}^2
\]

for some \( c > 0 \) that does not depend on \( a \) or \( g \).
Irrational case

Feichtinger-Kaiblinger

For a given $\theta$ we put $\Lambda = \mathbb{Z} \times \theta\mathbb{Z}$ and $\Lambda^\circ = \theta^{-1}\mathbb{Z} \times \mathbb{Z}$. If $g$ is function in $S_0(\mathbb{R})$ such that $\langle g, g \rangle$, is a projection in $A^R_\theta$ for some irrational $\theta$, then, for every rational $\tilde{\theta}$ that is sufficiently close to $\theta$, the element $b_g := \langle g, g \rangle \in B^R_\tilde{\theta}$ is invertible as an operator on $L^2(\mathbb{R})$ and on $S_0(\mathbb{R})$. Moreover, the element $h := g \cdot b_g^{-1}$ in $S_0(\mathbb{R})$ is such that $\langle g, h \rangle$ is a projection in $A^R_{\tilde{\theta}}$. 
Irrational case

Theorem

[Jakobsen-L.] Let $\theta$ be irrational and let $g$ in $S_0(\mathbb{R})$ be a generator of the Heisenberg module $E$ over $C^*(\mathbb{Z} \times \theta \mathbb{Z}, c)$, recall that $\langle g, g \rangle$ is a projection in $A_{\mathbb{R}}^R$. If $(\theta_i)_{i \in \mathbb{N}}$ is a sequence of rational numbers such that 
$$\lim_{i \to \infty} |\theta - \theta_i| = 0,$$
and $(a_i), (b_i), (N_i)$ and $(M_i)$ are sequences of natural numbers such that $\theta_i = a_i/M_i = b_i/N_i$ for all $i \in \mathbb{N}$ and such that the sequence $(d_i) = (a_i \cdot N_i)$ is increasing and unbounded, then there exists a sequence of vectors $(\tilde{g}_i)_i$ such that $\langle \tilde{g}_i, \tilde{g}_i \rangle$ is a projection in $A_{\mathbb{Z}_{d_i}}^R$. There exist functions $k^{(i)}$ in $S_0(\mathbb{R})$ such that $\langle k^{(i)}, k^{(i)} \rangle$ is a projection in $A_{\mathbb{R}}^R$. Furthermore, we also have that $(k^{(i)})$ converges towards $g$ in the module norm as $i \to \infty$, that is

$$\lim_{i \to \infty} \| g - k^{(i)} \|_\Lambda = 0.$$