A PRIORI ESTIMATES FOR THE OBSTACLE PROBLEM OF
HESSIAN TYPE EQUATIONS ON RIEMANNIAN MANIFOLDS

WEISONG DONG
Department of Mathematics, Harbin Institute of Technology
Harbin, 150001, China

TINGTING WANG1,2 AND GEJUN BAO1
1 Department of Mathematics, Harbin Institute of Technology
Harbin, 150001, China
2 Jiuquan Satellite Launch Center, Jiuquan, 735000, China

(Communicated by Changfeng Gui)

ABSTRACT. We are concerned with a priori estimates for the obstacle problem of a wide class of fully nonlinear equations on Riemannian manifolds. We use new techniques introduced by Bo Guan and derive new results for a priori second order estimates of its singular perturbation problem under fairly general conditions. By approximation, the existence of a $C^{1,1}$ viscosity solution is proved.

1. Introduction. This is one of a series of papers in which we study the obstacle problem for Hessian type equations on Riemannian manifolds. Let $(M^n, g)$ be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary $\partial M$, $\bar{M} := M \cup \partial M$, and $\nabla$ denote its Levi-Civita connection. In this paper we study the obstacle problem

$$\max \left\{ u - h, -(f(\lambda(\nabla^2 u + A[u])) - \psi[u]) \right\} = 0 \text{ in } M$$

with the Dirichlet boundary condition

$$u = \varphi \text{ on } \partial M,$$

where $h \in C^3(\bar{M})$ is called an obstacle, $\varphi \in C^4(\partial M)$, $h > \varphi$ on $\partial M$, $\psi[u] = \psi(x, u, \nabla u)$ is a positive function in $C^3$ and $A[u] = A(x, u, \nabla u)$ is a smooth $(0, 2)$ tensor which may depend on $u$ and $\nabla u$, $f$ is a symmetric function of $\lambda \in \mathbb{R}^n$, and for a $(0, 2)$ tensor $X$ on $M$, $\lambda(X)$ denotes the eigenvalues of $X$ with respect to the metric $g$.

In this paper, we prove the existence of a viscosity solution of (1) and (2) in $C^{1,1}(M)$ (see [5, 28] for the definition of viscosity solution).

Following [4], let $\Gamma$, a proper subset of $\mathbb{R}^n$, be an open and convex cone, with vertex at the origin, containing the positive cone: $\Gamma_+ := \{ \lambda \in \mathbb{R}^n : \forall i, \lambda_i > 0 \}$. $\Gamma$ is also assumed to be invariant under interchange of any two $\lambda_i$; i.e. it is symmetric in the $\lambda_i$. Let $\sigma_k$ be the $k$th-elementary symmetric function. Define $\Gamma_k := \{ \lambda \in \mathbb{R}^n : \forall i, \lambda_i > 0 \}$. $\Gamma$ and $\sigma_k$ are used throughout this paper.

2000 Mathematics Subject Classification. Primary: 35B45, 35J60; Secondary: 58J05, 35D40.
Key words and phrases. Obstacle problem, A priori estimates, Hessian, fully nonlinear elliptic equations, Riemannian manifolds.

* Corresponding author: Weisong Dong.
\[ \mathbb{R}^n : \sigma_k(\lambda) > 0 \}. \] Then \( \Gamma_n = \Gamma_+ \) and \( \Gamma \subset \Gamma_1 \). The function \( f \in C^2(\Gamma) \cap C^0(\overline{\Gamma}) \) is assumed to satisfy the fundamental structure conditions

\[ f_i = \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \ 1 \leq i \leq n, \tag{3} \]

\[ f \text{ is a concave function in } \Gamma, \tag{4} \]

\[ f > 0 \text{ in } \Gamma, \ f = 0 \text{ on } \partial \Gamma, \tag{5} \]

and

\[ \lim_{R \to \infty} f(R1) = \infty, \text{ where } 1 = (1, \ldots, 1) \in \mathbb{R}^n. \tag{6} \]

A function \( u \in C^2(M) \) is called admissible at \( x \in M \) if \( \lambda(\nabla^2 u + A[u])(x) \in \overline{\Gamma} \) and we call it admissible in \( M \) if it is admissible at each \( x \in M \). It is shown in [4] that (3) implies that (1) is elliptic for admissible solutions, and (4) ensures that \( F \) defined by \( F(r) = f(\lambda(r)) \) for \( r = \{r_{ij}\} \in S^{n \times n} \) with \( \lambda(r) \in \Gamma \) is concave, where \( S^{n \times n} \) is the set of \( n \times n \) symmetric matrices.

To avoid imposing any geometric restrictions on \( \partial M \) in deriving Hessian estimates for admissible solutions, we suppose, in addition to (3)-(6), that there exists an admissible subsolution \( \underline{u} \in C^2(M) \) satisfying

\[ \begin{cases} f(\lambda(\nabla^2 \underline{u} + A[\underline{u}])) \geq \psi[\underline{u}] \text{ in } M, \\ \underline{u} = \varphi \text{ on } \partial M, \end{cases} \tag{7} \]

and \( \underline{u} \leq h \) in \( M \). We remark here that the existence of \( \underline{u} \) in some cases can be found in [17] (See Theorem 1.3 of [17]). Besides, we suppose that

\[ -\psi(x, z, p) \text{ and } A^{\xi}(x, z, p) \text{ are concave in } p, \tag{8} \]

and

\[ -\psi_z, A_{z}^{\xi} \geq 0, \ \forall \xi \in T_x M. \tag{9} \]

For the interior gradient estimates, we need some growth conditions as usual and assume that

\[ \begin{cases} p \cdot \nabla_x A^{\xi}(x, z, p) + |p|^2 A_{z}^{\xi}(x, z, p) \leq \bar{\omega}_1(x, z)|\xi|^2(1 + |p|^{\gamma_1}), \\ p \cdot \nabla_x \psi(x, z, p) + |p|^2 \psi_z(x, z, p) \geq -\bar{\omega}_2(x, z)(1 + |p|^{\gamma_2}), \end{cases} \tag{10} \]

for some constants \( 0 < \gamma_1, \gamma_2 < 4 \) and some continuous functions \( \bar{\omega}_1, \bar{\omega}_2 \geq 0 \). In addition to (10), assume that

\[ f_j(\lambda) \geq \nu_0 \left(1 + \sum f_i(\lambda)\right) \text{ for any } \lambda \in \Gamma \text{ with } \lambda_j < 0, \tag{11} \]

where \( \nu_0 \) is a uniform positive constant. Note that (11) is commonly used in deriving gradient estimates, see e.g. [10, 29] and references therein. We also need the following growth conditions:

\[ p \cdot D_p \psi(x, z, p), -p \cdot D_p A^{\xi}(x, z, p)/|\xi|^2 \leq \bar{\omega}(x, z)(1 + |p|^{\gamma}) \tag{12} \]

and

\[ |A^{\eta}(x, z, p)| \leq \bar{\omega}(x, z)|\xi||\eta|(1 + |p|^{\gamma}), \ \forall \xi, \eta \in T_x M, \xi \perp \eta, \tag{13} \]

for some constant \( \gamma \in (0, 2) \) and some continuous function \( \bar{\omega} \geq 0 \).

At last, in deriving the \( C^0 \) estimates and the boundary \( C^1 \) estimates, we will use

\[ |A^{\xi}(x, z, p)| \leq \bar{\omega}(x, z)|\xi|^2(1 + |p|^2) \tag{14} \]

for any \( \xi \in T_x M \) when \( |p| \) is sufficiently large, where \( \bar{\omega} \geq 0 \) is a continuous function. We remark that when \( A(x, z, p) \equiv A(x, p) \), we do not use this condition. See Theorem 4.1 in [2].
With all the above assumptions we were able to prove the following result.

**Theorem 1.1.** Suppose that (3)–(13) hold. Then there exists a viscosity solution $u \in C^{1,1}(M)$ to the obstacle problem (1) and (2) under any of the following additional conditions: (i) $A(x, z, p) \equiv A(x, p)$; (ii) (14) and $\text{tr}A(x, z, 0) \leq 0$ when $z$ is sufficiently large. Furthermore, $u$ belongs to $C^{3,\alpha}$ on $\{x \in M : u(x) < h(x)\}$, for any $\alpha \in (0, 1)$.

Our motivation to study equation (1) comes partly from its geometric applications. In [8] Gerhardt considered hypersurfaces having prescribed mean curvature $H$ that are bounded from below by an obstacle. The case $H = 0$ (minimal surfaces) had been studied by for example Kinderlehrer [19, 20] and Giusti [15]. Xiong and Bao [30] studied the problem of finding the greatest hypersurface below a given obstacle, whose Gauss-Kronecker curvature (accordingly, $f = \sigma_n^{1/n}$) is bounded from below by a positive function, and established $C^{1,1}$ regularity in nonconvex domains in $\mathbb{R}^n$. Lee [21] considered the obstacle problem for Monge-Ampère Equation in the case when $A \equiv 0$, $\psi \equiv 1$, $\varphi \equiv 0$, and proved the $C^{1,1}$ regularity of the viscosity solution and $C^{1,\alpha}$ regularity of free boundary in a strictly convex domain in $\mathbb{R}^n$.

The interest in (1) also arises from its connection to the optimal transportation problem, see e.g. Savin [26, 27], Caffarelli and McCann [3]. Moreover, Liu and Zhou [23] treated an obstacle problem for Monge-Ampère type functionals whose Euler-Lagrange equations including the affine maximal surface equation and Abreu’s equation. Oberman [24, 25] showed that the convex envelope is a viscosity solution of a partial differential equation in the form of a nonlinear obstacle problem.

For $\sigma > 0$ let $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$ and $\partial \Gamma^\sigma$ denote its boundary which is a smooth and convex hypersurface in $\mathbb{R}^n$ by assumptions (3) and (4). For $\lambda \in \Gamma$ let $T_\lambda = T_\lambda \partial \Gamma^f(\lambda)$ denote the tangent plane at $\lambda$ to the level surface $\partial \Gamma^f(\lambda)$. Assumption

$$\partial \Gamma^\sigma \cap T_\lambda \partial \Gamma^f(\lambda) \text{ is nonempty and compact, } \forall \sigma > 0, \lambda \in \Gamma^\sigma, \quad (15)$$

which is introduced in [11] was used by Bao, Dong, and Jiao in [2] to consider the problem (1) and (2). This assumption excludes the linear function $f = \sigma_1$. (See [11].) Also, the obstacle problem for Hessian equations on Riemannian manifolds was studied by Jiao and Wang [17], where they considered the case when $A \equiv \kappa_{\text{ug}}$ under conditions on $f$ which however exclude the case that $f = (\sigma_k / \sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n$.

Here, in this paper the assumption is taken off due to the new Lemma 2.1 which together with (25) is essential for a priori second order estimates. Jiao in [16] considered the problem with a more general obstacle, but with $A \equiv \chi$ where $\chi$ is a smooth tensor on $M$ and $\psi = \psi(x)$. Compared with these, we study the obstacle problem of the general case (1) and (2), and derive a priori estimates without condition (15). Moreover, our problem (1) covers the case that $f = (\sigma_k / \sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n$ and $f = \sigma_1$.

2. **Beginning of proof.** To prove the existence of viscosity solutions to (1) and (2), we use a penalization technique and consider the following singular perturbation problem

$$\begin{cases}
  f(\lambda(\nabla^2 u + A[u])) = \psi[u] + \beta_\epsilon(u - h) & \text{in } M, \\
  u = \varphi & \text{on } \partial M, 
\end{cases} \quad (16)$$
where the penalty function $\beta_\varepsilon$ is defined by

$$
\beta_\varepsilon(z) = \begin{cases} 
0, & z \leq 0, \\
z^3/\varepsilon, & z > 0,
\end{cases}
$$

for $\varepsilon \in (0, 1)$. Obviously $\beta_\varepsilon \in C^2(\mathbb{R})$ satisfies

$$
\beta_\varepsilon, \beta'_\varepsilon, \beta''_\varepsilon \geq 0;
\beta_\varepsilon(z) \to \infty \text{ as } \varepsilon \to 0^+ \text{ whenever } z > 0;
\beta_\varepsilon(z) = 0 \text{ whenever } z \leq 0.
$$

Observe that $u$ is also a subsolution to (16). Let

$$
U = \{ u_\varepsilon \mid u_\varepsilon \in C^4(\bar{M}) \text{ is an admissible solution of (16) with } u_\varepsilon \geq u \text{ on } \bar{M} \}.
$$

We aim to derive the uniform bound

$$
|u_\varepsilon|_{C^2(\bar{M})} \leq C
$$

for $u_\varepsilon \in U$, where $C$ is independent of $\varepsilon$.

Once (19) is obtained, the Evans-Krylov theorem [6, 18] and the Schauder theory [7] ensure the smooth regularity of admissible solutions of (16), while the existence is guaranteed by the continuity method [7] and the degree theory [22]. We then conclude that there exists a function $C^{1,1}(\bar{M})$ satisfying (1) and (2), see [2, 30].

For simplicity, we may drop the subscript $\varepsilon$ in the following when there is no possible confusion.

We use ideas and notations from [12]. (See also [13, 14].) Write

$$
\mu(x) = \lambda(\nabla^2 u(x) + A[u](x))
$$

and note that $\{ \mu(x) : x \in \bar{M} \}$ is a compact subset of $\Gamma$. For all $\lambda \in \Gamma$, let $\nu_\lambda = Df(\lambda)/|Df(\lambda)|$ denote the unit normal vector to the level hypersurface of $f$ through $\lambda$. There exists a uniform constant $\zeta_0 \in (0, \frac{1}{2n})$ such that

$$
\nu_\mu(x) - 2\zeta_0 1 \in \Gamma_n, \forall x \in \bar{M}
$$

where $1 = (1, \ldots, 1) \in \mathbb{R}^n$.

We need the following lemma which is crucial in deriving $a \ priori$ $C^2$ estimates.

**Lemma 2.1** ([12, 14]). Let $K$ be a compact subset of $\Gamma$ and $\zeta > 0$. There is a constant $\theta > 0$ such that for any $\mu \in K$ and $\lambda \in \Gamma$, when $|\nu_\mu - \nu_\lambda| \geq \zeta$, 

$$
\sum f_i(\lambda)(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) + \theta(1 + \sum f_i(\lambda)).
$$

We use the notation

$$
A^\xi = A(x, \cdot, \cdot)(\xi, \eta), \quad \xi, \eta \in T^*_x M,
$$

and

$$
U := \nabla^2 u + A[u], \quad F(U) = f(\lambda(U)).
$$

Under a local frame $e_1, \ldots, e_n$,

$$
U_{ij} := U(e_i, e_j) = \nabla_{ij} u + A^{ij}[u]
$$

and

$$
F^{ij} = \frac{\partial F}{\partial U_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial U_{ij} \partial U_{kl}}(U).
$$

Let $\mathcal{L}$ be the linear operator locally defined by

$$
\mathcal{L}v := F^{ij} \nabla_{ij} v + (F^{ij} A^{ij}_{pk} - \psi_{pk}) \nabla_k v, \quad v \in C^2(M),
$$
here and thereafter we use Einstein summation convention. In the process of deriving a priori second order estimates, see Section 3 below, we apply Lemma 2.1 with \( \zeta = \zeta_0 \) in (21) (we will explain this in Remark 1), and an immediate result shows that:

**Lemma 2.2.** Assume that (8) and (9) hold. Then if \( |\nu_\mu - \nu_\lambda| \geq \zeta_0 \), we have

\[
\mathcal{L}(u - u) \geq \theta(1 + \sum F_{ii}) - \beta_\varepsilon(u - h),
\]

where \( u \) is an admissible solution.

**Proof of Lemma 2.2.** For any \( x \in M \), choose smooth orthonormal local frames \( e_1, \ldots, e_n \) about \( x \) such that \( \{U_{ij}(x)\} \) is diagonal, so is \( \{F^{ij}(U)(x)\} \). If \( |\nu_\mu - \nu_\lambda| \geq \zeta_0 \), then by Lemma (2.1), we have

\[
F^{ii}(U)(U_{ii} - U_{ii}) \geq \psi[|u| - \psi[|u| - \beta_\varepsilon(u - h) + \theta(1 + \sum F_{ii})].
\]

It follows from (8) and (9) that

\[
A_{ik} v_k (u - u) \geq A_{ik} |u| - A_{ik} |u| \quad \text{and} \quad -\psi_{ik} v_k (u - u) \geq -\psi[|u| + \psi[u].
\]

Thus (23) is obtained. \( \square \)

**Remark 1.** If instead \( |\nu_\mu - \nu_\lambda| < \zeta_0 \), we have by (21) that \( \nu_\lambda - \zeta_0 1 \in \Gamma_n \), and therefore

\[
F^{ii} \geq \frac{\zeta_0}{\sqrt{n}} \sum F^{kk}, \quad \forall 1 \leq i \leq n.
\]

We also have in this case that, by the concavity of \( F \),

\[
F^{ii}(U)(U_{ii} - U_{ii}) \geq F(U) - F(U) \geq \psi[u] - \psi[u] - \beta_\varepsilon(u - h).
\]

Then combining with (24) we obtain

\[
\mathcal{L}(u - u) \geq -\beta_\varepsilon(u - h). \quad (26)
\]

**Remark 2.** Note that (23) and (26) are the highlight of the paper.

3. **Estimates for second order derivatives.** In this section, we prove a priori estimates of second order derivatives for an admissible solution \( u \in \mathfrak{U} \). We see that \( \text{tr}(A[u]) \leq C \) on \( M \), where \( C \) is independent of \( \varepsilon \), but \( C \) depends on \( |u|_{C^1(M)} \). Let \( G \) be the solution to

\[
\begin{cases}
\Delta G + C = 0 \quad &\text{in } M, \\
G = \varphi \quad &\text{on } \partial M.
\end{cases}
\]

Then we have \( u \leq G \) in \( M \) by the maximum principle since \( \Delta u + C \geq \Delta u + \text{tr}(A[u]) > 0 \) in \( M \). Since \( h > \varphi \) on \( \partial M \), we have \( h > G \geq u \) in a neighborhood of \( \partial M \) in which \( \beta_\varepsilon(u - h) \equiv 0 \). Thus, in such a neighborhood of \( \partial M \), the Dirichlet problem (16) reduces to

\[
\begin{cases}
f(A(\nabla^2 u + A[u])) = \psi[|u|] \quad &\text{in a neighborhood of } \partial M, \\
u = \varphi \quad &\text{on } \partial M,
\end{cases}
\]

and hence by the arguments of Section 3 in [14], we obtain the boundary estimates for second order derivatives

\[
|\nabla^2 u| \leq C \quad \text{on } \partial M \quad (28)
\]

under assumptions (3)-(5) and (7)-(9), where the constant \( C \) in (28) is independent of \( \varepsilon \), but depends on \( |u|_{C^1(M)} \).
Therefore, it remains to estimate the interior second order derivatives $|\nabla^2 u|_{C^0(M)}$ for the global estimates of second derivatives $|\nabla^2 u|_{C^0(M)}$. For that we have

**Theorem 3.1.** Assume that $f$ satisfies (3)-(9). Let $u \in \mathcal{U}$. Then

$$|\nabla^2 u|_{C^0(M)} \leq C$$

where $C$ depends on $|u|_{C^1(M)}$, $|u|_{C^2(M)}$ and other known data.

The following lemma will be needed which is key in both the second derivatives estimates and the gradient estimates.

**Lemma 3.2** ([2, 30]). There exists a positive constant $c_0$, which is independent of $\varepsilon$, but depends on $|u|_{C^0(M)}$, such that

$$0 \leq \beta_\varepsilon(u-h) \leq c_0 \text{ in } M.$$  

(30)

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** We are going to prove the following inequality by computation follows essentially [2] that

$$b\mathcal{L}(u - u) \leq -c_1 F^{ii} U_{ii}^2 + CB^2 \sum_{i \in J} F^{ii}$$

$$+ C(1 + \sum_{i \in J} F^{ii}) + CB^2 \beta + c_1 \beta_\varepsilon(u-h).$$

(31)

Set

$$W(x) = \max_{i \in T_x, \psi_i \xi = 1} (A^{\xi \xi}(x, u, \nabla u) + \nabla \xi \xi u)\phi, \quad x \in \bar{M},$$

where $\phi$ is a function to be determined. Assume that $W$ is achieved at an interior point $x_0 \in M$ in the unit direction $\xi_0 \in T_x M$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about $x_0$ such that $\xi_0 = e_1$, $\nabla_i e_j(x_0) = 0$ and that $U_{ij}(x_0)$ is diagonal. Assume $U_{11}(x_0) > 0$ and $U_{11}(x_0) \geq \cdots \geq U_{nn}(x_0)$.

At the point $x_0$ where the function $\log U_{11} + \phi$ (defined near $x_0$) attains its maximum, we have

$$\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i \phi = 0, \quad i = 1, \ldots, n$$

(32)

and

$$\frac{\nabla_i U_{11}}{U_{11}} = \left(\frac{\nabla_i U_{11}}{U_{11}}\right)^2 + \nabla_i \phi \leq 0.$$  

(33)

Differentiating equation (16) twice and using (32), we obtain at $x_0$,

$$F^{ii} \nabla_{11} U_{ii} + F^{ij,kl} \nabla_i U_{ij} \nabla_{11} U_{kl}$$

$$\geq \psi_p \nabla_k U_{11} + \psi_p q_{kl} \nabla_{11} u + \beta_\varepsilon'(u-h)\nabla_1 (u-h)^2$$

$$\geq \beta_\varepsilon'(u-h)\nabla_1 (u-h) - C U_{11}$$

$$\geq -U_{11} \psi_p \nabla_k \phi - C U_{11} + \psi_p q_{kl} U_{11}^2 + (U_{11} - C) \beta_\varepsilon'(u-h)$$

(34)

provided $U_{11}$ is sufficiently large. Recall the formula for interchanging order of covariant derivatives

$$\nabla_{ijkl} v = R_{ijkl}^m \nabla_{mk} v + \nabla_i R_{ijkl}^m \nabla_m v$$

$$+ R_{ijk}^l \nabla_{lm} v + R_{ijl}^m \nabla_{jm} v + R_{jik}^m \nabla_{jm} v + R_{jkl}^m \nabla_{km} v + \nabla_k R_{jil}^m \nabla_m v.$$
Moreover, we use the formula calculated in [2] (see Equation (3.3) of [2])

$$F^{ii} \nabla_{ii} U_{11} = F^{ii} \nabla_{ii} (\nabla_{11} u + A^{11}(x, u, \nabla u))$$

$$\geq F^{ii} \nabla_{11} U_{ii} + F^{ii}(\nabla_{ii} A^{11} - \nabla_{11} A^{ii}) - C U_{11} \sum F^{ii}. \quad (35)$$

Differentiating equation (16) once, we obtain

$$F^{ii} (\nabla_{kii} u + \nabla_k A^{ii}) = \nabla_k \psi + \nabla_k \beta(u - h) \quad (36)$$

Moreover, we use the formula calculated in [2] (see Equation (3.3) of [2])

$$\nabla_{ikj} v - \nabla_{ijk} v = R^{i}_{kj} \nabla_{l} v, \quad (37)$$

to derive that

$$F^{ii} (\nabla_{ii} A^{11} - \nabla_{11} A^{ii})$$

$$\geq F^{ii} (A^{11}_{pk} \nabla_{ikj} u - A^{ii}_{pk} \nabla_{11k} u) + F^{ii} (A^{11}_{pij} U_{ii}^2 - A^{ii}_{pij} U_{11})$$

$$- C U_{11} \sum F^{ii} - C \sum F^{ii}$$

$$\geq U_{11} F^{ii} A^{ii}_{pk} \nabla_k \phi - C \sum F^{ii} U_{ii}^2 - U_{11} \sum F^{ii} A^{ii}_{pij}$$

$$- C U_{11} (1 + \sum F^{ii}) - C \beta'(u - h). \quad (38)$$

Thus, by substituting (35) into (33) and using (34) and (38), we obtain

$$\mathcal{L} \phi \leq E - \psi_{pi} U_{11} + \frac{C}{U_{11}} \sum F^{ii} U_{ii}^2 + U_{11} \sum F^{ii} A^{ii}_{pij}$$

$$+ \left( \frac{C}{U_{11}} - 1 \right) \beta'(u - h) + C \sum F^{ii} + C \quad (39)$$

where

$$E = \frac{1}{U_{11}^2} F^{ii} (\nabla_i U_{11})^2 + \frac{1}{U_{11}} F^{jj,kl} \nabla_i U_{ij} \nabla_j U_{kl}. $$

Let

$$\phi = \frac{\delta|\nabla u|^2}{2} + b(u - u)$$

where $b, \delta$ are undetermined constants satisfying $0 < \delta < 1 \leq b$. Direct computation yields

$$\nabla_i \phi = \delta \nabla_k u \nabla_{ik} u + b \nabla_i (u - u)$$

and

$$\nabla_{ii} \phi = \delta (\nabla_{ik} u)^2 + \delta \nabla_k u \nabla_{ik} u + b \nabla_{ii} (u - u)$$

$$\geq \frac{\delta}{2} U_{ii}^2 - C \delta + \delta \nabla_k u \nabla_{ik} u + b \nabla_{ii} (u - u).$$

From (36), we have

$$F^{ii} \nabla_{k} u \nabla_{k} U_{ii} = \nabla_{k} u \psi_{k} + \psi_{k} \nabla_k u |u|^2 + \psi_{pi} \nabla_{k} u \nabla_{kl} u + \beta'(u - h) (|\nabla u|^2 - \nabla u \cdot \nabla h).$$

We then have by (37) that

$$F^{ii} \nabla_{k} u \nabla_{ik} u \geq (\psi_{pi} - F^{ii} A^{ii}_{pi}) \nabla_{k} u \nabla_{kl} u - C (1 + \sum F^{ii}) - C \beta'(u - h).$$

Therefore,

$$\mathcal{L} \phi \geq b \mathcal{L} (u - u) + \frac{\delta}{2} F^{ii} U_{ii}^2 - C \delta \beta'(u - h) - C \sum F^{ii} - C. \quad (40)$$
Now we estimate $E$ in (39) following [11] (see also [30]) by using an inequality shown by Andrews [1] and Gerhardt [9]. For fixed $0 < s \leq 1/3$, let
$$ J = \{ i : U_{ii} \leq -sU_{11} \}, \quad K = \{ i : U_{ii} > -sU_{11} \}. $$

Similar to [11], we have
$$ -F^{ij,kl} \nabla_i U_{ij} \nabla_k U_{kl} \geq \frac{2(1 - s)}{(1 + s)U_{11}} \sum_{i \in K} (F^{ii} - F^{11})(|\nabla_i U_{11}|^2 - CU_{11}^2/s). $$

Then,
$$ E \leq \frac{1}{U_{11}} \sum_{i \in J} F^{ii}(\nabla_i U_{11})^2 + C \sum_{i \in K} F^{ii} + \frac{CF^{11}}{U_{11}} \sum_{i \in K} (\nabla_i U_{11})^2 $$
$$ \leq \sum_{i \in J} F^{ii}(\nabla_i \phi)^2 + C \sum_{i \in K} F^{ii} + CF^{11} \sum_j (\nabla_j \phi)^2 $$
$$ \leq Cb^2 \sum_{i \in J} F^{ii} + C\delta^2 \sum_{i \in J} F^{ii}U_{ii}^2 + C \sum_{i \in K} F^{ii} + C(\delta^2 U_{11}^2 + b^2)F^{11}. $$

Combining (39), (40) and (41), we obtain
$$ b\mathcal{L}(u) - u \leq \left( C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}} \right) F^{ii}U_{ii}^2 + Cb^2 \sum_{i \in J} F^{ii} + C(1 + \sum_{i \in J} F^{ii}) $$
$$ + Cb^2 F^{11} + \left( C\delta - 1 + \frac{C}{U_{11}} \right) \beta^2(u - h). $$

Taking $\delta < 1$ small enough such that
$$ c_1 := -\frac{1}{2} \max \{ C\delta^2 - \frac{\delta}{2}, C\delta - 1 \} > 0 $$

Then we may assume
$$ \max \{ C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}, \frac{C}{U_{11}} + C\delta - 1 \} \leq -c_1, $$

otherwise, we have $U_{11} \leq C/c_1$ and we are done. Therefore, we proved (31).

From now on we use the new method introduced by Guan [12].

We let $\tilde{\mu} = \mu(x_0)$ and recall that $\mu$ is defined in equation (20) in Section 2 and $\tilde{\lambda} = \lambda(U(x_0))$. If $|\nu_{\tilde{\mu}} - \nu_{\tilde{\lambda}}| \geq \zeta_0$, we apply (23) to (31) and obtain that
$$ (b\theta - C)(1 + \sum_{i \in J} F^{ii}) \leq -c_1 F^{ii}U_{ii}^2 + Cb^2 F^{11} $$
$$ + Cb^2 \sum_{i \in J} F^{ii} + b\beta^2(u - h) - c_1 \beta^2(u - h). $$

Fix $b > 1$ sufficiently large such that $b\theta - C > 0$. It follows from Lemma (3.2) that
$$ b\beta^2(u - h) - c_1 \beta^2(u - h) \leq \frac{(u - h)^2}{\varepsilon} (b(\varepsilon))^{1/3} - 3c_1 \leq 0 $$
if $\varepsilon \leq (3c_1/b^{1/3})^3$. Then (42) yields
$$ c_1 F^{ii}U_{ii}^2 - Cb^2 \sum_{i \in J} F^{ii} - Cb^2 F^{11} \leq 0 $$
when $\varepsilon$ is small. Note that $|U_{ii}| \geq sU_{11}$ for $i \in J$. It follows that
$$ (c_1 s^2 U_{11}^2 - Cb^2) \sum_{i \in J} F^{ii} + (c_1 U_{11}^2 - Cb^2) F^{11} \leq 0 $$
This implies a bound $U_{11}(x_0) \leq Cb^2/(c_1 s^2)$.

Next suppose $|µ̄ - ν_λ| < ζ_0$. We then obtain by applying (26) to (31) that

$$c_1 F^{ii} U_{11}^2 \leq Cb^2(1 + \sum F^{ii}) + bβ_e(u - h) - c_1 β_e'(u - h).$$

Again we can choose $ε$ small enough such that $bβ_e(u - h) - c_1 β_e'(u - h) \leq 0$. Thus we have by (25),

$$\frac{c_1 ζ_0 |\hat{λ}|^2}{\sqrt{n}} \sum F^{ii} \leq c_1 F^{ii} U_{11}^2 \leq Cb^2(1 + \sum F^{ii})$$

(43)

where $|\hat{λ}| = \sum λ_j^2 = \sum U_{ii}$. By the concavity of $f$, we have

$$|\hat{λ}| \sum f_i \geq \sum f_i \hat{λ}_i + f(\hat{λ}|1) - f(\hat{λ})$$

$$\geq f((\hat{λ}|1) - |ψ|u(x_0) - c_0 - \frac{1}{4|\hat{λ}|} \sum f_i \hat{λ}_i^2 - |\hat{λ}| \sum f_i$$

where $c_0$ comes from Lemma 3.2. Therefore,

$$|\hat{λ}|^2 \sum f_i \geq \frac{|\hat{λ}|}{2} (f((\hat{λ}|1) - |ψ|u(x_0) - c_0) - \frac{1}{8} \sum f_i \hat{λ}_i^2$$

$$\geq |\hat{λ}| - \frac{1}{8} \sum f_i \hat{λ}_i^2$$

(44)

when $|\hat{λ}|$ is large enough satisfying $f((\hat{λ}|1) \geq 2 + c_0 + \max_{x \in M} |ψ|u$ by (6). Combining (43) and (44) we have

$$|\hat{λ}|^2 \sum F^{ii} + |\hat{λ}| \leq Cb^2(1 + \sum F^{ii}),$$

which gives $|\hat{λ}| \leq Cb^2$. \hfill \qed

4. Gradient estimates and existence.

**Theorem 4.1.** Assume that (3)-(5) and (8) hold. Let $u \in Ω$. Suppose that (10)-(13) hold. Then for $ε$ sufficiently small, we have

$$\max_{M} |∇u| \leq C(1 + \max_{∂M} |∇u|),$$

(45)

where $C$ depends on $|u|_{C^0(M)}, |u|_{C^2(M)}$ and other known data.

**Proof of Theorem 4.1.** We outline the proof here for completeness, and the reader can refer to [2] for more details and another group of assumptions that guarantees (45).

Suppose $|∇u|φ^{-1/2}$ achieves a maximum at an interior point $x_0 \in M$, where $φ$ is a positive function to be determined. As in Section 3 we choose smooth orthonormal local frames $e_1, \ldots, e_n$ about $x_0$ such that $∇_{e_i} e_j = 0$ at $x_0$ and $\{U_{ij}(x_0)\}$ is diagonal. Set $w = |∇u|$. Then at $x_0$, we have

$$\nabla_i \frac{w}{w} - \frac{\nabla_i \phi}{2φ} = 0,$$

(46)

and

$$\nabla_i \frac{w}{w} + \frac{|\nabla_i \phi|^2}{4φ^2} - \frac{\nabla_i \phi}{2φ} \leq 0$$

(47)
for $i = 1, \ldots, n$. We see that for each fixed $1 \leq i \leq n$, $w \nabla_i w = \nabla_i w \nabla_i u$, and by (37) and (46) that

\[
w \nabla_i w = (\nabla_{ii} u + R_{ii}^U \nabla_k u) \nabla_i u + \left( \delta_{kl} - \frac{\nabla_k u \nabla_l u}{w^2} \right) \nabla_{ik} u \nabla_i u
\]

\[
\geq \nabla_i u \nabla_i U_{ii} - \frac{w^2}{2\phi} (A_{ii}^{ii} \nabla_k \phi + 2\phi A_{ii}^{ii}) - \nabla_i u A_{ij}^{ii} - Cw^2,
\]

in which the inequality follows from that the last term in RHS is non-negative.

Differentiating the equation (16), by (46), we have

\[
F^{ii} \nabla_i u \nabla_i U_{ii} = \nabla_i u \psi_i + \psi_u \nabla u|^2
\]

\[
+ \frac{w^2}{2\phi} \psi_{pk} \nabla_k \phi + \beta_i'(u-h)(|\nabla u|^2 - \nabla u \cdot \nabla h).
\]

Take $\phi = -u + \sup_M u + 1$. By (8), we have

\[
A^{ii} = A^{ii}(x, u, \nabla u) \leq A^{ii}(x, u, 0) + A_{ii}^{ii}(x, u, 0) \nabla_k u.
\]

We note that (4) and (5) imply that

\[
\sum f_i \lambda_i \geq 0.
\]

It follows that

\[
-F^{ii} \nabla_i \phi = F^{ii} U_{ii} - F^{ii} A^{ii} \geq -C(1 + |\nabla u|) \sum F^{ii}.
\]

Thus, by plugging (48), (49) and (51) into (47), and applying (10) and (12), we obtain

\[
0 \geq C_0 F^{ii} |\nabla_i u|^2 - C(|\nabla u|^{\gamma+2} + |\nabla u|^{\gamma} + 1)
\]

\[
- C(1 + |\nabla u| + |\nabla u|^{\gamma} + |\nabla u|^{\gamma-2}) \sum F^{ii},
\]

where $C_0 = \min_{\bar{M}} 1/4\phi^2 > 0$ depends on $|u|_{C^0(\bar{M})}$. We may assume $\nabla_i u(x_0) \geq |\nabla u(x_0)|/n > 0$. From (46), (50) and (13), we see that

\[
U_{11} \leq -\frac{1}{2\phi} |\nabla u|^2 + C(1 + |\nabla u| + |\nabla u|^{\gamma}) < 0
\]

if $|\nabla u|$ is sufficiently large, which yields by (11) that

\[
F^{11} \geq \nu_0 \left( 1 + \sum F^{ii} \right).
\]

We then see from (52) that

\[
0 \geq C_0 |\nabla u(x_0)|^2 - C(|\nabla u|^{\gamma+2} + |\nabla u|^{\gamma} + 1)
\]

\[
- C(1 + |\nabla u| + |\nabla u|^{\gamma} + |\nabla u|^{\gamma-2}) \sum F^{ii}.
\]

Thus $|\nabla u(x_0)| \leq C$ and the proof of (45) is completed.

Finally, by applying Theorem 4.1 in [2] which gives uniform bounds for $|u|_{C^0(\bar{M})}$ and $|\nabla u|_{C^0(\bar{M})}$, provided (i) $A(x, z, p) \equiv A(x, p)$ and $A^{\xi}(x, p)$ is concave in $p$ for each $\xi \in T_x M$ or (ii) $\text{tr} A(x, z, 0) \leq 0$ when $z$ is sufficiently large and (14) holds. We thus have derived (19).

We finally obtain a $C^{1,1}(\bar{M})$ viscosity solution satisfying (1) and (2), see [2, 30], by approximation. We omit the proof here as it is standard and well known.
REFERENCES

[1] B. Andrews, Contraction of convex hypersurfaces in Euclidean space, Calc. Var. Partial Differential Equations, 2 (1994), 151–171.

[2] G.-J. Bao, W.-S. Dong and H.-M. Jiao, Regularity for an obstacle problem of Hessian equations on Riemannian manifolds, J. Differential Equations, 258 (2015), 696–716.

[3] L. A. Caffarelli and R. McCann, Free boundaries in optimal transport and Monge-Ampère obstacle problems, Ann. of Math., 171 (2010), 673–730.

[4] L. A. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations III. Functions of the eigenvalues of the Hessian, Acta Math., 155 (1985), 261–301.

[5] M. Crandall, H. Ishii and P. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc., 27 (1992), 1–67.

[6] L. C. Evans, Classical solutions of fully nonlinear, convex, second order elliptic equations, Comm. Pure Appl. Math., 35 (1982), 333–363.

[7] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations Of Second Order, 2nd edition, Springer-Verlag, New York, 1983.

[8] C. Gerhardt, Hypersurfaces of prescribed mean curvature over obstacles, Math. Z., 133 (1973), 169–185.

[9] C. Gerhardt, Closed Weingarten hypersurfaces in Riemannian manifolds, J. Differential Geom., 43 (1996), 612–641.

[10] B. Guan, The Dirichlet problem for Hessian equations on Riemannian manifolds, Calc. Var. Partial Differential Equations, 8 (1999), 45–69.

[11] B. Guan, Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, Duke Math. J., 163 (2014), 1491–1524.

[12] B. Guan, The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds, preprint, arXiv:1403.2133.

[13] B. Guan, S.-J. Shi and Z.-N. Sui, On estimates for fully nonlinear parabolic equations on Riemannian manifolds, Anal. PDE, 8 (2015), 1145–1164.

[14] B. Guan and H.-M. Jiao, The Dirichlet problem for Hessian type elliptic equations on Riemannian manifolds, Discrete Conti. Dyn. Syst., 36 (2016), 701–714.

[15] E. Giusti, Superfici minime cartesiane con ostacoli discontiuiti, Arch. Ration. Mech. Anal., 35 (1969), 47–82.

[16] H.-M. Jiao, C^{1,1} regularity for an obstacle problem of Hessian equations on Riemannian manifolds, Proc. Amer. Math. Soc., 144 (2016), 3441–3453.

[17] H.-M. Jiao and Y. Wang, The obstacle problem for Hessian equations on Riemannian manifolds, Nonlinear Anal., 95 (2014), 543–552.

[18] N. V. Krylov, Boundedly nonhomogeneous elliptic and parabolic equations in a domain, Izvestia Math. Ser., 47 (1983), 75–108.

[19] D. S. Kinderlehrer, Variational inequalities with lower dimensional obstacles, Israel J. Math., 10 (1971), 339–348.

[20] D. S. Kinderlehrer, How a minimal surface leaves an obstacle, Acta Math., 130 (1973), 221–242.

[21] K. Lee, The obstacle problem for Monge-Ampère equation, Comm. Partial Differential Equations, 26 (2001), 33–42.

[22] Y.-Y. Li, Degree theory for second order nonlinear elliptic operators and its applications, Comm. Partial Differential Equations, 14 (1989), 1541–1578.

[23] J.-K. Liu and B. Zhou, An obstacle problem for a class of Monge-Ampère type functionals, J. Differential Equations, 254 (2013), 1306–1325.

[24] A. Oberman, The convex envelope is the solution of a nonlinear obstacle problem, Proc. Amer. Math. Soc., 135 (2007), 1689–1694.

[25] A. Oberman and L. Silvestre, The Dirichlet problem for the convex envelope, Trans. Amer. Math. Soc., 363 (2011), 5871–5886.

[26] O. Savin, A free boundary problem with optimal transportation, Comm. Pure Appl. Math., 57 (2004), 126–140.

[27] O. Savin, The obstacle problem for Monge Ampère equation, Calc. Var. Partial Differential Equations, 22 (2005), 303–320.

[28] N. S. Trudinger, The Dirichlet problem for the prescribed curvature equations, Arch. Ration. Mech. Anal., 111 (1990), 153–179.
[29] J. Urbas, Hessian equations on compact Riemannian manifolds, in Nonlinear Problems in Mathematical Physics and Related Topics, II, Kluwer/Plenum, (2002), 367–377.

[30] J.-G. Xiong and J.-G. Bao, The obstacle problem for Monge-Ampère type equations in non-convex domains, Commun. Pure Appl. Anal., 10 (2011), 59–68.

Received September 2015; revised March 2016.

E-mail address: dweeson@gmail.com
E-mail address: ttwanghit@gmail.com
E-mail address: baogj@hit.edu.cn