Quantum Diffusion and Tunneling with Parametric Banded Random Matrix Hamiltonians

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Abstract

The microscopic origin of dissipation of a driven quantum many body system is addressed in the framework of a parametric banded random matrix approach. We find noticeable violations of the fluctuation–dissipation theorem and we observe also that the energy diffusion has a markedly non–Gaussian character. Within the Feynman–Vernon path integral formalism and in the Markovian limit, we further consider the time evolution of a slow subsystem coupled to such a “bath” of intrinsic degrees of freedom. We show that dissipation leads to qualitative modifications of the time evolution of the density matrix of the slow subsystem. In either the spatial, momentum or energy representation the density distribution acquires very long tails and tunneling is greatly enhanced.

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1 Introduction

It is by now an accepted fact that spectral fluctuations of many quantum systems are identical to those resulting from random matrix theory. Random Hamiltonians have initially been introduced by Wigner to describe nuclear properties at excitation energies of the order of the nucleon binding energy \[1\]. They have found applications in many areas of physics. Just to cite several directions, they have been used to describe properties of simple quantum chaotic systems \[2\], spectra of complex molecules \[3\], condensed matter physics \[4\], nuclear properties and spectral fluctuations obtained in QCD calculations \[5\]. There is enough experimental evidence from all these physical systems to justify the assignment of the qualifier universal to their spectral properties.

In this article we address the problem of the evolution of a quantum mechanical system for which the spectral fluctuations are assumed to be universal, the average density of states has the proper thermodynamic behaviour and which, in addition, is capable of changing its “shape”. By “shape” we understand either some parameters describing the actual geometrical shape of the system or some other similar global characteristics, or externally applied fields. The shape can either be controlled by the experimenter, or can have its own quantum dynamics. Some classic examples of the first type of system are those studied in thermodynamics, while examples of the second type are atomic nuclei and complex molecules. We are interested in what happens when the shape changes at a finite rate. In other words, we are interested in the microscopic nature and character of dissipation in finite quantum many–body systems.

The paper is organized as follows. In Section II we describe the structure of the Hamiltonian and the evolution equations for driven systems and what we often refer to as a simple quantum system in interaction with a complex environment. The theoretical framework in this last case was developed many years ago by Feynman and Vernon \[6\] and used extensively, particularly in condensed matter physics, in Caldeira–Leggett type of treatments of dissipative tunneling \[7\]. There also exists in the literature another type of approach, somewhat more phenomenological in nature, namely the quantum Langevin equation \[8\], which is claimed to lead to a dissipative dynamics similar to that of the more popular Caldeira–Leggett formalism. Our main contribution consists in the introduction of a new type of environment or “bath”, based on a parametric banded random matrix approach \[9, 10, 11\]. In our opinion, such an approach, besides being extremely flexible, has a better microscopic foundation and is better suited to describe finite many–body quantum systems.

In Section III we study the temporal evolution of driven complex quantum systems which we describe using a parametric banded random matrix approach. The shape is an externally modulated parameter. Should this shape change infinitely slowly, the evolution of the system would be reversible. However, at any finite rate of change, it is for all practical purposes irreversible. A simple example should suffice. Assume that we have a pump, inside of which the motion is chaotic. In the pump there is one single particle,
bouncing elastically off the wall and off the piston. At some point in time someone starts moving the piston, eventually bringing it back to its initial position. We assume that during the entire time the piston is in motion, the person is not permitted/allowed or able to get any information about the position or momentum of the particle. Further, the person is not allowed to act in any direct way on the particle. For almost all closed trajectories of the piston, covered with a finite velocity, the particle inside will not return to its initial state. One can then classify the “state transformation” of the particle in the pump as irreversible. This in some sense paraphrases the almost century old argument between Boltzmann on the one side and Zermelo and Loschmidt on the other [12]. In that argument the role of the particle inside the pump was played by the infinite number of atoms in a gas. Part of the Boltzmann’s argument was that because their number is infinite, one cannot in any conceivable way actually reverse their velocities, and thus irreversibility arises. We have only changed “cannot” with “is not permitted” and arrived at the same result, but now for perhaps one of the most simple systems possible. This is essentially what one often encounters in real situations. Even with the intrinsic system having a finite number of degrees of freedom, there is no obvious way by which one can control their state or acquire information about their microscopic state without disturbing them, and in particular reverse their momenta. One can only control some “external” parameters.

In Section IV we analyse a few cases of a simple quantum system, coupled to such a complex, but finite, “environment” within the adiabatic approximation. In this case the energy transfer is allowed in both directions, from one to the other subsystem and back as well. However, if one of the subsystems has a large number of active degrees of freedom the energy flow will most of the time occur in one direction. This is a generic situation, encountered in numerous quantum finite many–body systems.

A short summary of our results and an outlook for future investigations are presented in the last Section V.

2 Evolution equations

The Hamiltonian governing the dynamics of a quantum system coupled to a complex environment is assumed to be of the form

\[ H(X, x) = H_0(X) + H_1(X, x). \]  

(1)

We refer to \( X \) as shape variables. In the ensuing formulas we shall not display explicitly the dependence of the Hamiltonian on the intrinsic variables \( x \) of the environment, but rather discuss its matrix elements in a fixed intrinsic basis. The part of the total Hamiltonian (1) which depends on the intrinsic coordinates \( H_1(X) \) is defined as a parametric banded random matrix, whose matrix elements depend on the “slow” coordinate \( X \)

\[ [H_1(X)]_{ij} = [h_0]_{ij} + [h_1(X)]_{ij}. \]  

(2)
\( h_0 \) is taken to be diagonal and defines the average density of states, with \( \langle k | h_0 | l \rangle = [h_0]_{kl} = \varepsilon_k \delta_{kl} \). We refer to these eigenstates as “typical states” of the intrinsic system with an energy \( \varepsilon \). In Refs. [9, 10, 11] we have discussed at length the reasons why one chooses this specific form of the Hamiltonian. For an intrinsic subsystem with a large number of degrees of freedom the average density of states, 

\[
\rho(\varepsilon) = \text{Tr} \delta(H_1(X) - \varepsilon),
\]

for each given shape \( X \) increases sharply with energy. The overline denotes here a procedure for extracting the smooth part of \( \rho(\varepsilon) \) as a function of energy and it amounts essentially to an ensemble average, to be introduced below. For a many Fermion system, \( \rho(\varepsilon) \) has a roughly exponential behaviour. Recall that \( \ln \rho(\varepsilon) \) is approximately proportional to the thermodynamic entropy of the intrinsic system, which is an extensive quantity. The fact that the average density of states for the intrinsic subsystem has such a behaviour is a key element of the entire approach. This is equivalent to stating that the intrinsic subsystem has a large heat capacity and thus can play the role of a “reservoir”, although not necessarily ideal. In principle \( \rho(\varepsilon) \) can be \( X \)-dependent as well, but we shall ignore this aspect here. Without an \( X \)-dependence of the average density of states, mechanical work cannot be performed on or by the model environment we study here, and only heat exchange is allowed.

In the basis of the eigenstates of \( h_0 \), we define \( h_1(X) \) as a parameter dependent \( N \times N \) real Gaussian random matrix, which is completely specified by its first two moments

\[
[h_1(X)]_{kl} = 0, \quad [h_1(X)]_{ij}[h_1(Y)]_{kl} = [\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]G_{ij}(X - Y).
\]  

The overline stands for the statistical average over the ensemble of random Gaussian matrices. Even though we shall limit all the formulas in this work to the Gaussian orthogonal ensemble (GOE), with minor changes, the formalism is equally applicable to other Gaussian ensembles. In Eq. (4), \( G_{ij}(X - Y) \) is a “bell shaped” correlation function with a characteristic width \( X_0 \). Physically this is simply the formal expression of the fact that intrinsic states corresponding to vastly different values of the shape \( X \) have statistically independent compositions. The dependence on \( i, j \) allows for the description of banded matrices, where an effective number of states \( N_0 \leq N \) are coupled by \( h_1(X) \). It is convenient to use an explicit parameterization of \( G_{ij} \), which explicitly incorporates the average density of states and the bandwidth of the statistical fluctuations [13]:

\[
G_{ij}(X) = \frac{\Gamma^\dagger}{2\pi\sqrt{\rho(\varepsilon_i)\rho(\varepsilon_j)}} \exp \left[ -\frac{(\varepsilon_i - \varepsilon_j)^2}{2\kappa_0^2} \right] G\left( \frac{X}{X_0} \right).
\]  

Here \( G(x) = G(-x) = G^*(x) \leq 1 \), \( G(0) = 1 \) and \( \Gamma^\dagger \) is the spreading width for the intrinsic subsystem. \( \kappa_0 \) (linked with the effective band width \( N_0 \approx \kappa_0 \rho(\varepsilon) \)) and \( X_0 \) are also characteristics of the intrinsic system. This parameterization is consistent with the
Figure 1: A portion of the instantaneous eigenvalue spectrum $E_n(X)$ as a function of the shape $X$. The Hamiltonian is defined through Eqs. (2), (4) and (5), with unit average density of states and $\Gamma^\downarrow = 2\pi$, $\kappa_0 = \infty$, $G(x) = \exp(-x^2/2)$ and $X_0 = 1$.

Experimental evidence that in many-body systems the spreading width $\Gamma^\downarrow$ changes relatively slowly with the excitation energy\cite{13}. Moreover, distributions of matrix elements extracted from various theoretical many-body models show strong deviations from a pure Gaussian distribution\cite{14} and are in qualitative agreement with Eq. (5). A typical Born–Oppenheimer spectrum (with an average unit level density) as a function of the shape $X$ is shown in Fig. 1 for a particular realization of the random Hamiltonian $H_1(X)$. For each fixed value of the shape variable $X$ the spectrum is characterized by fluctuations very similar to GOE.

2.1 Driven subsystem

A particular situation of definite physical interest is that when the time dependence of the “slow” variables is known and/or controlled externally as for example in the case of an applied external electric or/and magnetic field(s). This is a typical situation in thermodynamics, when the controlled parameters are changed adiabatically, insuring the reversibility of the transformation. At any finite rate of change the transformation looses its reversible character and dissipation sets in. In an experiment, if one has some control over the parameters, as a rule, one has little control on the intrinsic subsystem. Thus, if we move the parameter $X(t)$ at finite velocity along a closed circuit in parameter space,
returning to the initial point, the final intrinsic states will in general be distinct from the initial one. These changes of the intrinsic subsystem, which not only depend on the rate of change of the controlled parameters, but also on the specific paths in the parameter space, are referred to as dissipative effects. State changes of the intrinsic subsystem, which depend on the path in the parameter space, but do not depend on either the rate at which this path is covered or on the direction in which the path is travelled, can be incorporated into reversible type of transformations, by slightly generalizing the definition of adiabatic transformations \[15\]. Such effects can be linked with the appearance of effective abelian or nonabelian gauge fields \[16\], besides the familiar (in the Born–Oppenheimer approximation) effective potential forces.

For the sake of simplicity we assume that \( X(t) = V_0 t \) and that the average level density has an exponential form: \( \rho(\varepsilon) = \rho_0 \exp(\beta \varepsilon) \). Thus \( \beta = 1/T = d \ln \rho(\varepsilon)/d\varepsilon \) can be interpreted as the inverse thermodynamic temperature of the intrinsic subsystem. (Note that the spectrum extends to infinity in both directions from \( \varepsilon = 0 \).)

The time evolution of the fast subsystem is found by solving the time-dependent Schrödinger equation in the form \[9, 10, 11\]:

\[
\phi(t) = T \exp \left[ -\frac{i}{\hbar} \int_0^t ds H_1(X(s)) \right] \phi(0) = \mathcal{U}(X(t))\phi(0). \tag{6}
\]

where \( T \) is the time ordering operator, and \( \mathcal{U}(X(t)) \) the propagator. (We assume that the initial state \( \phi(0) \) is uncorrelated with the Hamiltonian \( H_1(X(t)) \) at later times; correlated initial conditions have been discussed elsewhere \[9\].) One can show that in the leading order in an expansion in \( 1/N_0 \) the average propagator \( \mathcal{U}(X(t)) = \mathcal{U}(X(t)) \) is diagonal in the representation we have chosen. Its diagonal matrix elements have the following form

\[
U_k(X(t)) = \langle k|\mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_0^t ds H_1(X(s)) \right]|k \rangle = \exp \left( -\frac{\imath \varepsilon_k t}{\hbar} \right) \sigma(X(t)) \tag{7}
\]

( note that \( \sigma(X(t)) \) is state independent) and \( \sigma(X(t)) \) satisfies the following integral equation \[11\]:

\[
\sigma(X(t)) = 1 - \frac{\Gamma^i}{\hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 \sigma(X(s_1 - s_2)) \sigma(X(s_2)) P(s_1 - s_2) G \left( \frac{(s_1 - s_2)V_0}{X_0} \right). \tag{8}
\]

Here \( P(s) \) is given by

\[
P(s) = P^*(-s) = \frac{\kappa_0}{\sqrt{2\pi\hbar}} \exp \left[ -\frac{\kappa_0^2}{2\hbar^2} \left( s + \frac{\hbar \beta}{2} \right)^2 \right], \tag{9}
\]

when the correlator \( [h_1(X)]_{ij}[h_1(Y)]_{kl} \) is defined as in Eq. \(5\). In order to be able to compute averages of observables, we need to introduce the set of generalized occupation number probabilities

\[
\mathcal{N}_k(X(t_1), X(t_2)) = \langle \phi(t_1)|k\rangle \langle k|\phi(t_2) \rangle = \sum_l \langle l|\mathcal{U}^\dagger(X(t_1))|k\rangle \langle k|\mathcal{U}(X(t_2))|l \rangle n_l(0). \tag{10}
\]
Thus \( n_k(t) \equiv \mathcal{N}_k(X(t), X(t)) \) is the occupation probability of the state \(|k\rangle\). The generalized occupation number probabilities can be determined by solving an evolution equation for the characteristic functional

\[
\mathcal{N}(X(t_1), X(t_2), \tau) = \langle \phi(t_1) | \exp \left[ \frac{i \hat{h}_0(\tau - t_1 + t_2)}{\hbar} \right] | \phi(t_2) \rangle 
\]

\[
= \int d\varepsilon_k p(\varepsilon_k) \mathcal{N}_k(X(t_1), X(t_2)) \exp \left[ \frac{i \varepsilon_k(\tau - t_1 + t_2)}{\hbar} \right].
\]

The equal time functional \( \mathcal{N}(X(t), X(t), \tau) \) is thus the Fourier transform of the occupation number probabilities \( n_k(t) \). \( \mathcal{N}(t, t, \tau) \) is an extremely useful quantity, since it provides the following cumulant expansion \[13\]:

\[
\mathcal{N}(X(t), X(t), \tau) = \sum_k n_k(t) \exp \left( \frac{i \varepsilon_k \tau}{\hbar} \right) = \exp \left[ \sum_n \langle \phi(t) | \hat{h}_0^n | \phi(t) \rangle \frac{(i \tau)^n}{\hbar^n n!} \right],
\]

where \( \langle \phi(t) | \hat{h}_0^n | \phi(t) \rangle \) are the cumulants. If we assume that the initial occupation number probabilities are \( n_0(0) = 1 \) and \( n_l(0) = n_{-l}(0) = 0 \) for \( l \neq 0 \) (remember that the spectrum of \( H_1(X) \) is infinite in both directions), then we find that \( \mathcal{N}(X(t_1), X(t_2), \tau) \) satisfies the evolution equation \[11\]

\[
\mathcal{N}(X(t_1), X(t_2), \tau) = \sigma^*(X(t_1)) \sigma(X(t_2)) + \frac{\Gamma^4}{4} \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \mathcal{N}(X(s_1), X(s_2), \tau)
\]

\[
\times P(s_1 - s_2 - \tau) G \left( \frac{(s_1 - s_2) \bar{V}_0}{X_0} \right) \sigma^*(X(t_1 - s_1)) \sigma(X(t_2 - s_2)).
\]

### 2.2 The path integral approach

When the shape variables \( X \) become dynamical variables, there is energy exchange between the two subsystems, and the dynamics becomes more complicated. A formalism to tackle the case when both subsystems have to be treated quantum mechanically has been put forward by Feynman and Vernon \[3\]. One can write the following double path integral representation for the density matrix of the entire system

\[
\mathcal{R}(X, x, Y, y, t) = \int dX_0 dY_0 \psi(X_0) \psi^*(Y_0) \int_{X(0)=X_0}^{X(t)=X} D X(t) \int_{Y(0)=Y_0}^{Y(t)=Y} D Y(t)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \left[ S_0(X(t)) - S_0(Y(t)) \right] \right\}
\]

\[
\times \langle x | T \exp \left[ -\frac{i}{\hbar} \int_0^t dt' H_1(X(t')) \right] | \phi \rangle \langle \phi | T_a \exp \left[ \frac{i}{\hbar} \int_0^t dt'' H_1(Y(t'')) \right] | y \rangle,
\]

where \( T \) and \( T_a \) represent the time ordering and time anti-ordering operators respectively and \( S_0(X(t)) \) is the classical action corresponding to the Hamiltonian \( H_0(X) \). The particular form for the initial state wave function we have used here, namely

\[
\Psi(X, x) = \psi(X) \phi(x)
\]

(15)
is not unique and other choices are equally possible, for example, a density matrix. By introducing the influence functional

$$\mathcal{L}(X(t), Y(t), t) = \langle \phi | \left\{ T_a \exp \left[ \frac{i}{\hbar} \int_0^t dt'' H_1(Y(t'')) \right] \right\} \left\{ T_{\text{exp}} \left[ -\frac{i}{\hbar} \int_0^t dt' H_1(X(t')) \right] \right\} | \phi \rangle$$

(16)

one readily obtains the following double path integral representation for the density matrix of the “slow” subsystem

$$\rho(X, Y, t) = \int dX_0 dY_0 \psi(X_0) \psi^*(Y_0) \int_{X(0)=X_0}^{X(t)=X} D X(t) \int_{Y(0)=Y_0}^{Y(t)=Y} D Y(t) \times \exp \left\{ \frac{i}{\hbar} \left[ S_0(X(t)) - S_0(Y(t)) \right] \right\} \mathcal{L}(X(t), Y(t), t).$$

(17)

The formulation of the problem through a path integral representation serves only as a very convenient vehicle to obtain an evolution equation for the density matrix $\rho(X, Y, t)$. (Our usage of the same Greek letter $\rho$ for two different quantities, average level density for the intrinsic subsystem and density matrix for the “slow” subsystem, should not lead to confusion.)

Using the formalism developed in Ref. [11] one can derive relatively simple analytical expressions for the influence functional. For the case of an adiabatic evolution of the slow subsystem, it was shown in Ref. [11] that the influence functional has the simple form

$$\mathcal{L}(X(t), Y(t), t) = \mathcal{N}(X(t), Y(t), 0) = \exp \left\{ \frac{\Gamma^\downarrow}{\hbar} \int_0^t \left[ G(X(t'), Y(t')) - 1 \right] dt' \right\}. \quad (18)$$

By combining the double path integral representation for the density matrix $\rho(X, Y, t)$ with the above expression for the influence functional in the adiabatic approximation, one easily derives that the density matrix satisfies the following Schrödinger–like equation (for similar examples see Refs. [7])

$$i\hbar \partial_t \rho(X, Y, t) = \{ H_0(X) - H_0(Y) + i\Gamma^\downarrow [G(X, Y) - 1] \} \rho(X, Y, t)$$

(19)

with the initial condition

$$\rho(X, Y, 0) = \psi(X) \psi^*(Y).$$

(20)

This equation for the density matrix describes a quantum mechanical Markovian process and it satisfies the conditions of the Lindblad’s theorem [17]. Therefore the solutions of this evolution equation with meaningful physical initial conditions can be given a probabilistic interpretation at all subsequent times. In particular this means that

$$\rho(X, X, t) \geq 0, \quad \int dX \rho(X, X, t) = 1$$

(21)

(22)

for any $t \geq 0$. 


3 Temporal evolution of a driven system

In this section we shall present analytical results for two limits, adiabatic and diabatic, and numerical results for the intermediate situation. In order to quantitatively determine whether the evolution of the driven subsystem \( X(t) = V_0 t \) is in either of these limits it is useful to introduce two time scales: i) the characteristic time scale for the slow motion \( \tau_{\text{slow}} = X_0 / V_0 \) and ii) the characteristic time scale for the fast degrees of freedom, \( \tau_{\text{fast}} = \hbar / \kappa_0 \). The adiabatic limit corresponds to \( \tau_{\text{slow}} \gg \tau_{\text{fast}} \). (We shall also assume that in the adiabatic limit the condition \( \kappa_0 \beta \ll 1 \) is also fulfilled.) The diabatic limit is obtained when \( \tau_{\text{slow}} \ll \tau_{\text{fast}} \).

3.1 Adiabatic limit

The generalized occupation number probabilities \( \mathcal{N}(X(t), X(t), \tau) \) can be computed by integrating Eqs. (8) and (13). The technical trick which is used is to replace in these equations the quantity \( P(s) \) introduced in Eq. (9) with an appropriately chosen Dirac \( \delta \)-function \([11]\). In the adiabatic limit \( P(t) \) is much narrower than \( G(V_0 t / X_0) \). In the leading order in the parameter \( \kappa_0 \beta \ll 1 \) one obtains for \( t \geq |\tau| \)

\[
\mathcal{N}(X(t), X(t), \tau) = \exp \left\{ -\frac{\Gamma}{\hbar} \left[ 1 - G \left( \frac{\tau V_0}{X_0} \right) \right] \left( t - |\tau| \right) - \frac{\Gamma |\tau|}{\hbar} \right\}.
\]  

(23)

In the strict adiabatic limit \( V_0 \rightarrow 0, G(\tau V_0 / X_0) \rightarrow 1 \), and the first term in the exponential vanishes. The occupation numbers \( n_k(t) \) reach rather quickly the asymptotic distribution

\[
n_k = \frac{1}{\pi} \frac{\Gamma_i}{\varepsilon_k^2 + \left( \frac{\Gamma_i}{2} \right)^2},
\]  

(24)

This Lorentzian shape is identical with the constant random matrix theory result\([3, 11]\). During a time \( t \approx \tau_{\text{fast}} \), the slow variables hardly change and the dynamics of the fast system is almost identical to the dynamics governed by a constant random Hamiltonian. Our initial state in the middle of the spectrum, chosen as \( n_0(0) = 1 \), is thus spread over an energy interval \( \approx \Gamma_i \) and the distribution has a Lorentzian shape. If the Hamiltonian is time independent, after this time there would be essentially no further evolution of the average occupation number probabilities. The subsequent dynamical evolution of the fast subsystem occurs only because the Hamiltonian \( H_1(X(t)) \) is time dependent, and only the subsequent time evolution of the system leads to dissipation and entropy production in the long time limit.

One can now explicitly evaluate the cumulants \( \langle \langle \phi(t) | h_0^\dagger | \phi(t) \rangle \rangle \) \([19]\). All odd moments of \( h_0 \) vanish identically (since \( G(x) = G(-x) \) and thus there are only even powers of \( \tau \) in the expansion in Eq. (12)). The reason for this is our assumption that \( \kappa_0 \beta \rightarrow 0 \), which we lift below. In the limit \( t \rightarrow \infty \), all even cumulants of \( h_0 \) increase linearly in time. If
\[ G(x) = \exp(-x^2/2) \] (we shall use this form hereafter for illustrative purposes) then in the limit \( t \to \infty \)

\[
\langle \langle \phi(t)|h_0^{2n}|\phi(t) \rangle \rangle = \frac{\Gamma t}{\hbar} \left( \frac{\hbar V_0}{X_0} \right)^{2n} \frac{(2n)!}{2^{n}n!},
\]

\[
D(V_0) = \frac{\hbar \Gamma}{2X_0^2} V_0^2 = \frac{\hbar \Gamma}{2\tau_{\text{slow}}^2},
\]

resulting in a non–Gaussian distribution. A Gaussian process would have only nonvanishing linear and quadratic cumulants. The energy diffusion constant is extracted from the time dependence of the second cumulant when \( t \to \infty \) according to

\[
\Delta_2^2(t) = \langle \phi(t)|[H_1(X(t)) - E(t)]^2|\phi(t) \rangle \approx \langle \phi(t)|h_0^2|\phi(t) \rangle \approx \text{const} + 2D(V_0)t,
\]

where \( E(t) = \langle \phi(t)|H_1(X(t))|\phi(t) \rangle \). Note that the energy variance is time dependent only for a time dependent Hamiltonian. As the result of the symmetric initial distribution and since \( \beta = 0 \) (remember that \( \kappa_0 \beta \ll 1 \), we obtain no friction (i.e. \( E(t) = \text{const} \) and only nonvanishing even cumulants. To get friction we have to consider the next order corrections to the adiabatic limit \( \kappa_0 \beta \ll 1 \). Since for \( \beta > 0 \) the average level density is increasing with energy, there will be on the average more transitions upward in energy than downward and thus the driven subsystem is heated up. One can show that the odd cumulants are then given by the following expressions\[11\]

\[
\langle \langle \phi(t)|h_0^{2n-1}|\phi(t) \rangle \rangle = \frac{\beta}{2} \langle \langle \phi(t)|h_0^{2n}|\phi(t) \rangle \rangle. \tag{27}
\]

The case \( n = 1 \) corresponds to the Einstein fluctuation–dissipation theorem. In familiar diffusive processes, all cumulants of order higher than the second are vanishing. The existence of large higher order cumulants results in energy tails of the energy distribution significantly longer than in traditional phenomenological transport approaches, like Fokker–Planck or Langevin equations. One can show that in the tails the energy distribution has the following behaviour

\[
P(\varepsilon) \propto \exp(-\alpha|\varepsilon| \ln^{1/2}|\varepsilon|), \tag{28}
\]

where \( \alpha \) is some (time dependent) constant. A somewhat similar functional form has been determined for the distribution of conductance fluctuations in mesoscopic systems \[21\]. The presence of these longer than expected tails is a clear indication that the excitation mechanism cannot be reduced to a simple random walk in energy space.

### 3.2 Diabatic limit

Another simple analytical solution can be obtained in the diabatic limit, when \( \tau_{\text{slow}} = X_0/V_0 \ll \tau_{\text{fast}} = \hbar/\kappa_0 \). In this case \( G(V_0s/X_0) \) becomes much narrower than \( P(s) \) and
can be replaced with an appropriately chosen Dirac $\delta$–function. The evolution equations (8) and (13) can be solved again and one obtains:

$$N(t, t, \tau) = \exp \left\{ -\frac{\Gamma X_0 \kappa_0}{\hbar^2 V_0} \left[ \exp \left( \frac{\kappa_0^2 \beta^2}{8} \right) - \exp \left( \frac{\kappa_0^2}{2 \left( \frac{\beta}{2} + i\tau \bar{h}\sqrt{2} \right)^2} \right) \right] t \right\}.$$  

(Note that in this case what we denote as the slow degrees of freedom are actually faster than the intrinsic ones.) This is similar to the functional form found in the adiabatic limit. Again, all the cumulants of $h_0$ increase linearly in time

$$\langle \langle \phi(t) | h_0^n | \phi(t) \rangle \rangle = \left[ \frac{\Gamma X_0 \kappa_0}{\hbar^2 V_0} \exp \left( \frac{\beta^2 \kappa_0^2}{8} \right) \left( \frac{i \kappa_0}{\sqrt{2}} \right)^n \right] H_n \left( -\frac{i \kappa_0 \beta}{2 \sqrt{2}} \right) t,$$

where $H_n(x)$ are Hermite polynomials, resulting in a non–Gaussian diffusion of the occupation numbers. From Eq. (29) and the second cumulant, we find in this limit a completely different velocity dependence:

$$D(V_0) = \left[ \frac{\Gamma X_0 \kappa_0^3 (\beta^2 \kappa_0^2 + 4)}{8 \hbar^2} \exp \left( \frac{\beta^2 \kappa_0^2}{8} \right) \right] \frac{1}{V_0}.$$  

The fluctuation–dissipation theorem, which provides the relation between the first and the second cumulants is in this case:

$$\beta D = \gamma \left( 1 + \frac{\beta^2 \kappa_0^2}{4} \right),$$

where $\gamma t = \langle \langle \phi(t) | h_0 | \phi(t) \rangle \rangle$. For a Gaussian diffusive process one would have obtained $\beta D = \gamma$. Thus, in both adiabatic and diabatic limits we have obtained significant deviations from an excitation mechanism corresponding to a simple random walk in the energy space. One might expect that the intermediate regime will be very similar in this respect, as we shall substantiate in the next subsection. We thus conclude that the energy transfer to a “complex” quantum system is definitely not a Markovian process. The presence of the above mentioned long tails shows that the system retains for some time the memory of the direction it was proceeding and transitions into the same direction in energy are somewhat favored over transitions into the other direction.

### 3.3 Arbitrary driving velocity

For the intermediate velocity regime we have to resort to a numerical solution of the evolution equations Eqs. (8) and (13). In Fig. 2 we show $n_k(t)$ as function of the level number $k$. The tails have an almost pure exponential behavior, in agreement with the theoretical expectations, see Eq. (28). In Fig. 3 we show the behavior of the diffusion constant, $D(V_0)$, from the adiabatic to the diabatic limit for some values of the parameter $\beta$. In all cases, $D(V_0)$ evolves from quadratic (in the adiabatic limit) to an inverse velocity
Figure 2: The time dependence of the occupation probabilities $n_k(t)$ (note the vertical logarithmic scale) plotted as functions of the level number $k$, for $V_0 = 4$, $\beta = 0$ and $\kappa_0 \approx 20$. The values of the remaining parameters are specified in the text. The narrower distribution corresponds to $t = 1$ with widening curves for $t = 2$, 3 and 3.5 respectively. The small notch in the uppermost curve is a remnant of the initial conditions $n_0(0) = 1$ and $n_{k\neq0}(0) = 0$. 
Figure 3: Velocity dependence of the diffusion constant $D(V_0)$ from the adiabatic to the diabatic limits. We use $X_0 = 1$, $\beta/\rho_0 = 0.1$, $\rho_0 \Gamma^* = 2\pi$ ($\rho_0$ thus defines the energy units), $\kappa_0 \rho_0 = 5$ (the lowest curve) and $\kappa_0 \rho_0 = 15$ (the highest curve).
Figure 4: Deviation from the fluctuation–dissipation theorem as a function of $\beta = 1/T$. For low velocities, the theorem is largely satisfied. However, for large velocities it is violated even at moderate temperatures. Here we use $X_0 = 1$, $\rho_0 \Gamma^\downarrow = 2\pi$ and $\kappa_0 \rho_0 = 5$.

dependence (in the diabatic limit). At high velocities, the system becomes increasingly transparent, as reflected in the decrease of the diffusion constant. A similar behaviour is observed for the first cumulant, i.e. the average heating rate, which as a function of the velocity $V_0$ has a similar profile with $D(V_0)$. This is reminiscent of the motional narrowing phenomenon in NMR. At low enough velocities the fast system has sufficient time to “accommodate” to the new environment, while the shape $X$ changes. In the opposite limit of high velocities, the shape $X$ evolves so rapidly that the system can barely react to the changes. Consequently, the energy diffusion is maximal only for some intermediate velocity regime, when the “slow” motion is in “resonance” with the “fast” dynamics, namely when $\tau_{\text{slow}} = X_0/V_0$ is comparable to $\tau_{\text{fast}} = h/\kappa_0$. In Fig. 4, we plot the ratio of $\beta D/\gamma$, where $D$ and $\gamma$ are computed from the first and second order cumulants as a function of velocity $V_0$. When the ratio is unity, the Einstein limit of the fluctuation–dissipation theorem is recovered. Noticeable differences occur at large velocities. Higher order cumulants have also been extracted from our numerical results and in all cases their magnitudes and temporal behavior is similar to the analytical results discussed in the previous two subsections. Higher order cumulants increase essentially linearly with time and their magnitudes increase with the order of the cumulant.
4 Dynamical evolution of a simple quantum system in a complex environment

In the adiabatic approximation (discussed in the previous section), it is possible to analytically determine the influence functional. Since the characteristic time scale for the intrinsic subsystem $\tau_{fast} = \hbar/\kappa_0$ is much shorter than any expected time scale of the “slow” subsystem in Eq. (16) one obtains significant contributions only from “left” $Y(t'')$ and “right” $X(t')$ paths corresponding to $t'' = t'$, and the influence functional acquires the particularly simple form given in Eq. (18). For a derivation of this form we refer the reader to Ref. [11]. As a result the evolution equation for the density matrix of the “slow” subsystem is local in time and memory effects are absent. Among the conditions we have listed for the applicability of our adiabatic results, we had $\beta = 0$, which corresponds to an infinite temperature of the intrinsic subsystem (since $\beta = 1/T$). As we have discussed in Section II, in the present formulation only heat transfer is allowed between the two subsystems. Since the temperature of the intrinsic system is infinite in this limit, one naturally expects energy transfer only from the intrinsic system towards the slow system and no mechanical work. Even though there are no memory effects in the time evolution of the “slow” subsystem, memory effects are still present in the evolution of the intrinsic subsystem (see Section III).

It is worth noting that the functional form of the influence functional derived by us is entirely different from the popular Caldeira–Leggett form [7], which is a quadratic expression in $X(t')$ and $Y(t')$. If we were to use only the first term in a Taylor expansion of $G(X(t'), Y(t')) - 1$ we would obtain an expression similar to Caldeira–Leggett form for the influence functional.

It is well known that most of the trajectories in a path integral are very jagged and one might question the applicability of an adiabatic approximation. One can try to make a long argument in support of the adiabatic approximation using “circumstantial” evidence, e.g. the success of the Born–Oppenheimer approximation in studying various physical systems. We only note that relatively smooth paths, obtained by taking into account only low frequencies, seem to be adequate, as is clearly shown in the partial Fourier smoothing methods for computing path integrals [18].

In this section we shall discuss several cases: that of a linear potential, a quadratic potential and a double well potential where dissipative tunneling can occur. Surprisingly enough, the first two cases can be solved explicitly. To our knowledge, these represent new cases of time dependent quantum mechanical problems, which can be given an entirely analytical treatment [23]. Only the case of a double well potential is treated numerically.
4.1 Linear potential

The case when the slow variables evolve in a linear potential

\[ H_0(X) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial X^2} - FX \]  

(33)

is particularly instructive. We shall assume furthermore that the “effective potential” is “translation invariant”, namely \( G(X, Y) = G(X - Y) \). In terms of the variables \( s = X - Y \) and \( r = (X + Y)/2 \) the evolution equation for \( \rho(X, Y, t) \) becomes

\[(i\hbar \partial_t + \frac{\hbar^2}{m} \partial_r \partial_s) \rho(r, s, t) = \{-Fs + i\Gamma^\downarrow[G(s) - 1]\} \rho(r, s, t).\]  

(34)

We seek a solution in the form

\[ \rho(r, s, t) = \int\int \frac{dk}{2\pi\hbar} \exp\left(\frac{ikr}{\hbar}\right) d(k, s, t). \]  

(35)

The function \( d(s, t, k) \) satisfies the equation

\[ \left( \partial_t + \frac{k}{m} \partial_s \right) d(k, s, t) = \left\{ \frac{iFs}{\hbar} + \frac{\Gamma^\downarrow m}{\hbar} \right\} d(k, s, t). \]  

(36)

For either \( s = 0 \) or \( k = 0 \), \( d(k, s, t) \) is the characteristic function [19] for the spatial or momentum distribution of the slow subsystem, respectively.

Using the method of characteristics for wave equations [20], Eq. (36) can be solved through quadratures and the density matrix is determined to be

\[ \rho(r, s, t) = \int \int \frac{dr'dk}{2\pi\hbar} \rho_0\left( r', s - \frac{kt}{m} \right) \exp\left[ \frac{ik(r - r')}{\hbar} \right] \times \exp\left\{ \frac{iFst}{\hbar} - \frac{iFt^2k}{2hm} + \frac{\Gamma^\downarrow m}{\hbar k} \int_s^{s'} ds' [G(s') - 1] \right\}. \]  

(37)

where \( \rho_0(r, s) = \rho(r, s, 0) \) is the initial density matrix. From Eqs. (33) and (37), the characteristic function for the momentum distribution can be identified as \( D(s, t) = d(0, s, t) \), where:

\[ D(s, t) = \int dr \rho(r, s, t) = \int dr \rho_0(r, s) \exp\left\{ \frac{iFst}{\hbar} + \frac{\Gamma^\downarrow t}{\hbar} [G(s) - 1] \right\}. \]  

(38)

One extremely economical and intuitive way to characterize the momentum distribution of the collective subsystem is through its cumulants

\[ \langle \langle p^n \rangle \rangle_t = \left. \left( \frac{hd}{ids} \right)^n \ln D(s, t) \right|_{s=0} \]  

(39)
For the case of a Gaussian correlation function $G(X) = \exp[-X^2/2X_0^2]$, we find

\begin{align}
\langle \langle p \rangle \rangle_t &= \langle \langle p \rangle \rangle_{t=0} + Ft, \\
\langle \langle p^2 \rangle \rangle_t &= \langle \langle p^2 \rangle \rangle_{t=0} + \frac{\Gamma \hbar t}{X_0^2}, \quad (40) \\
\langle \langle p^{2n} \rangle \rangle_t &= \langle \langle p^{2n} \rangle \rangle_{t=0} + (2n-1)!! \frac{\Gamma \hbar t}{\hbar} \left( \frac{\hbar}{X_0} \right)^{2n}, \\
\langle \langle p^{2n+1} \rangle \rangle_t &= \langle \langle p^{2n+1} \rangle \rangle_{t=0}. \quad (42)
\end{align}

The meaning of the "correlation length" $X_0$ is that intrinsic shapes separated by $|X - Y| > X_0$ are statistically uncorrelated. Notice that only the first cumulant is affected by the presence of a linear potential in the expected manner, namely a uniform acceleration of the slow subsystem. The "bath" of intrinsic degrees of freedom affects only higher order even cumulants of the momentum distribution while the odd cumulants of order higher than one remain unchanged.

The cumulants of the spatial distribution can be obtained from the characteristic function $d(k,0,t)$. From Eq. (37) it immediately follows that

\begin{equation}
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-ik}{\hbar} \right)^n \langle \langle r^n \rangle \rangle = \ln d_0(k,0,t) + \left( \frac{-ik}{\hbar} \right) \frac{Ft^2}{2m} + \frac{\Gamma \hbar m}{\hbar k} \int_{-\infty}^{0} ds' [G(s') - 1]. \quad (44)
\end{equation}

The term $\ln d_0(k,0,t)$ gives the contributions to the cumulant expansion arising from free expansion of the initial wave packet, in the absence of both the linear potential and the coupling to the internal degrees of freedom. The linear potential leads to the expected (classical) behaviour of the center of the wave packet (see the second term on the rhs of Eq. (44)). The contribution to the even cumulants arising from dissipation alone is

\begin{equation}
\langle \langle r^{2n} \rangle \rangle_{\text{diss}} = \frac{(2n-1)!!}{2n+1} \frac{\Gamma \hbar t}{\hbar} \left( \frac{\hbar t}{mX_0} \right)^{2n}. \quad (45)
\end{equation}

Of particular interest is the second cumulant

\begin{equation}
\langle \langle r^2 \rangle \rangle_{\text{diss}} = \frac{\Gamma \hbar t}{3X_0^2 m}, \quad (46)
\end{equation}

which shows that dissipation leads to a super diffusive expansion of the wave packet. This behaviour is to be contrasted with the free expansion or ballistic propagation, in which case $\langle \langle r^2 \rangle \rangle \propto t^2$ and with normal diffusion, for which $\langle \langle r^2 \rangle \rangle \propto t$. It is instructive to estimate also the "size" of this state in phase space. In the limit $t \to \infty$ the dissipative contribution dominates, and one has

\begin{equation}
\Delta r \Delta p \approx \left[ \langle \langle r^2 \rangle \rangle_{\text{diss}} \langle \langle p^2 \rangle \rangle_{\text{diss}} \right]^{1/2} = \frac{2\pi W_0 \hbar t^2}{\sqrt{3mX_0^2}}. \quad (47)
\end{equation}
Thus the “blob” spreads in phase space at a much faster rate than in the case of a simple quantum mechanical system.

Similarly, one can show that with logarithmic accuracy for large values of the variable 
\(
\theta(r, t) = |r - Ft^2/2m|/t
\)
(assuming vanishing initial average linear momentum) the spatial distribution behaves as
\[
\rho(r, 0, t) \propto \exp \left[ -\nu \theta(r, t) \ln^{1/2} \theta(r, t) \right],
\]
where \(\nu\) is some constant. Thus the effect of dissipation is undeniably not only significant, but also leads to qualitatively new features.

### 4.2 Quadratic potential

Another case that is susceptible of an analytical treatment is that of a quadratic potential for the collective subsystem

\[
H_0(X) = -\frac{\hbar^2}{2m} \partial_X^2 + \frac{m\omega^2 X^2}{2}.
\]

Using the representation for \(\rho(r, s, t)\) of Eq. (35), the equation for the transformed density now becomes

\[
\left( \partial_t + \frac{k}{m} \partial_s - m\omega^2 s \partial_k \right) d(k, s, t) = \frac{\Gamma \downarrow}{\hbar} [G(s) - 1] d(k, s, t).
\]

The method of characteristics [20] can again be used to determine its solution

\[
\rho(r, s, t) = \int \frac{dk}{2\pi\hbar} \exp \left[ \frac{ikr}{\hbar} \right]
\times d_0 \left( s \cos \omega t - \frac{k}{m\omega} \sin \omega t, m\omega s \sin \omega t + k \cos \omega t \right)
\times \exp \left\{ \frac{\Gamma \downarrow}{\hbar} \int_0^t dt' [G(s \cos \omega(t - t') - \frac{k}{m\omega} \sin \omega(t - t')) - 1] \right\},
\]

where \(d_0(s, k) = d(s, k, 0)\). In a similar manner to the one described in the previous subsection, one can determine various cumulants. For both spatial and momentum distributions only even cumulants are affected by dissipation [22]

\[
\langle \langle p^{2n} \rangle \rangle_{\text{diss}} = (2n - 1)!! \frac{\Gamma \downarrow}{\hbar \omega} \left( \frac{\hbar}{X_0} \right)^{2n} \int_0^\omega d\tau \cos^{2n} \tau
\]
\[
\approx \frac{[(2n - 1)!!]^2}{2^n n!} \frac{\Gamma \downarrow \tau}{\hbar} \left( \frac{\hbar}{X_0} \right)^{2n},
\]

\[
\langle \langle r^{2n} \rangle \rangle_{\text{diss}} = (2n - 1)!! \frac{\Gamma \downarrow}{\hbar \omega} \left( \frac{\hbar}{m\omega X_0} \right)^{2n} \int_0^\omega d\tau \sin^{2n} \tau
\]
\[
\approx \frac{[(2n - 1)!!]^2}{2^n n!} \frac{\Gamma \downarrow \tau}{\hbar} \left( \frac{\hbar}{m\omega X_0} \right)^{2n}.
\]
There is a noticeable difference with the case of a linear potential, in that all cumulants increase now only linearly with time. It looks as if the quadratic potential has a “focusing” effect on the spatial distribution. The fact that the momentum and spatial distributions are so similar should come as no surprise in the case of a harmonic oscillator, which possesses an obvious symmetry between the momenta and coordinates. In the limit \( t \to \infty \) we obtain that the “uncertainty relation” has the following expression

\[
\Delta r \Delta p \approx \left[ \langle \langle p^2 \rangle \rangle_{\text{diss}} \langle \langle r^2 \rangle \rangle_{\text{diss}} \right]^{1/2} = \frac{\pi \hbar W_0 t}{m \omega X_0^2},
\]

which has to be contrasted with the quadratic time behaviour in the linear or no potential cases, see Rel. (48).

It is a simple matter to analytically continue these expression to the case of an inverted parabolic potential or barrier. This rather innocuous procedure, leads however to an entirely different time dependence of the cumulants, all of them increasing exponentially with time in this case (as \( \cos \tau \) and \( \sin \tau \) become \( \cosh \tau \) and \( \sinh \tau \) respectively).

### 4.3 Tunneling in a symmetric double well potential

The double well potential we analyse has the form

\[
V(X) = a \left( X^2 - \frac{b}{2a} \right)^2.
\]

(57)

For a strong enough barrier the spectrum of a relatively large number of low lying eigenstates is made up of doublets in the absence of the coupling to the intrinsic subsystem. The corresponding eigenfunctions are approximately symmetric and antisymmetric combinations of wave functions localized in the two wells. The first nine eigenvalues are shown in Fig. 5 for the case \( a = 1/2 \) and \( b = 5 \) together with the potential. We have chosen as an initial state a linear combination of the eigenfunctions of the ground and first excited states \( (n = 0, 1) \)

\[
\begin{align*}
H_0(X) &= -\frac{\hbar^2}{2m} \partial_X^2 + V(X), \\
H_0(X)\psi_n(X) &= E_n \psi_n(X) \\
\rho(X,Y,0) &= \frac{1}{2} \langle \psi_0(X) - \psi_1(X) | \psi_0(Y) - \psi_1(Y) \rangle.
\end{align*}
\]

(58) \hspace{1cm} (59) \hspace{1cm} (60)

In this case the particle is initially localized in one well with a probability almost equal to one, and only an exponentially small amount is present in the other well. In the absence of coupling to the intrinsic subsystem this state will tunnel almost entirely to the other well in an exponentially long time \( \tau_{\text{tunnel}} = \pi \hbar / (E_1 - E_0) \), since the splitting between the two states is exponentially small. For the particular choice of parameters we have chosen \( (\hbar = 1, m = 1/2, a = 1/2 \) and \( b = 5 ) \) \( \tau_{\text{tunnel}} \approx 2,300 \).
Figure 5: The first nine eigenvalues for the Schrödinger equation with the potential (57) with $\hbar = 1$, $m = 1/2$, $a = 1/2$ and $b = 5$. (The splitting of the first two levels is approximately $10^{-3}$.)
The coupling to the intrinsic subsystem is characterized by two parameters $\Gamma \downarrow$ and $X_0$, which we have varied independently. We retain a Gaussian correlator $G(x) = \exp(-x^2/2)$. As we show in Figs. 6 and 7, the effect of dissipation on the tunneling process is profound. Not only the tunneling rate is changed by orders of magnitude, but the shape of the wave packet is qualitatively different from the one in the absence of coupling. In the case of an usual quantum mechanical tunneling, the wave packet would have simply gradually changed from a state localized in one well to a state in the other well. Since the potential well is symmetric, the shape of the wave packet after a time $\tau_{\text{tunnel}} = \pi \hbar/\left(E_1 - E_0\right)$ would have been simply the mirror image with respect to the origin of the initial one, see Fig. 6.

The role of the spreading width $\Gamma \downarrow$ is rather simple to understand. Since the coupling between the two subsystems is defined by it, see Eq. (5), it is natural to expect that with increasing $\Gamma \downarrow$ the rate of tunneling increases as well. We observe an approximate power law relationship between the probability to find the particle in the other well at a given time

$$P_+(t) = \int_0^\infty dX \rho(X,X,t)$$

and the spreading width $\Gamma \downarrow$ (see Fig. 7), namely $P_+(t) \propto \Gamma \downarrow^\alpha$, where $\alpha \approx 3/2$.

The role played by the correlation length $X_0$ is however somewhat more subtle. The rate of tunneling is increasing dramatically when $X_0$ is decreasing. At the formal level this can be easily understood from the evolution equation (19) for the density matrix. In the $XY$–plane when $|X - Y| < X_0$, the term responsible for dissipation is negligible. The smaller the correlation length $X_0$ the bigger is the area where dissipation is directly effective. In the region $|X - Y| > X_0$ the term $i\Gamma \downarrow[G(X,Y) - 1] \approx -i\Gamma \downarrow$ dominates the time evolution of the density matrix. Thus, in the long time limit the density matrix tends to become almost diagonal. This tendency could have been inferred as well from the time dependence of the cumulants for the momentum distribution, which we have obtained in the previous two subsections. Since these cumulants increase with time, that means that the derivatives of the density matrix $\rho(X,Y,t)$ with respect to $s = X - Y$ become larger and larger for $X = Y$.

Another and perhaps a better way to understand the role played by the correlation length $X_0$ on tunneling is by returning to the initial picture, see Fig. 1. With decreasing $X_0$ the intrinsic subsystem undergoes more and more transitions on its way from one well to another, since the density of (avoided) level crossings per unit length increases. We remind the reader, however, that trying to understand the role of the energy exchange between the two subsystems in terms of isolated jumps at level crossings is not quite correct, as it was discussed at length in Ref. [9]. This also follows from the results presented in Section II. As we stressed there, had the picture of a random walk in the energy space been an appropriate one, the emergence of rather long tails in the energy distribution would have been rather difficult to explain.

As our numerical results suggest and the structure of the evolution equation (19) also...
Figure 6: The density profiles for $t = 0, 1, 2, 3, 4$ and $5$ for the initial condition defined in Eq. (60), in the double well potential of Fig. 5 for two different values of $\Gamma^\downarrow$. The dashed line shows the density profile in the absence of dissipation after a time $t = \tau_{\text{tunnel}} = \pi \hbar/(E_1 - E_0) \approx 2,300$. 

---

**Figure 6**

The density profiles for $t = 0, 1, 2, 3, 4$ and $5$ for the initial condition defined in Eq. (60), in the double well potential of Fig. 5 for two different values of $\Gamma^\downarrow$. The dashed line shows the density profile in the absence of dissipation after a time $t = \tau_{\text{tunnel}} = \pi \hbar/(E_1 - E_0) \approx 2,300$. 

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Figure 7: The probability to find the particle in the other well has the approximate power law behaviour $P_+(t) \propto \Gamma^{3/2}$. Different curves correspond to times $t = 1, 2, 3, 4$ and 5 respectively in ascending order.
seems to confirm, for \( t \to \infty \) the density matrix has the limit \( \rho(X,Y,t) \to \delta(X - Y) \), which is likely the only stationary solution of Eq. (19). At some point in time however, the adiabatic approximation we have used to derive the evolution equation (19) will cease to be valid, namely when the characteristic momenta will become of the order of \( mX_0\kappa_0/\hbar \). In Section III we established that for velocities of the order of \( X_0\kappa_0/\hbar \) one has the crossover from the adiabatic to the diabatic regime, when the “motional narrowing” sets in. For higher velocities dissipation becomes weaker, though not at all negligible. At this point there is however one essential new element which enters the evolution of the “slow” subsystem, the memory effects. We have commented on this previously in the text. In the absence of the memory effects in the dynamics of the “slow” subsystem “thermalization” between the two subsystems apparently cannot be achieved.

5 Conclusions

We are still quite a distance from solving the problem we have set out to do in the introduction. Nevertheless we have hopefully succeeded in clarifying several aspects. Here is the best place to draw the line and evaluate what we have achieved so far and what remains to be done.

We have developed a formalism, which describes a relatively simple quantum mechanical system coupled to a “bath” of intrinsic excitations. We solved the dynamical evolution equations for these systems, without making any uncontrollable approximations or assumptions. We have analysed two types of problems, each of which being interesting in its own right and also relevant for understanding various types of experiments. The first class of systems are the so called driven systems, when a certain number of externally controlled parameters are changed. Systems ranging from complex molecules to quantum dots in variable magnetic or electric external fields can be studied in this way. One question which we hope we have shed light on, is the velocity dependence of the diffusion constant. During the years there have been quite a range of answers to this question, some of them rather intriguing [24].

There are situations, when such parameters become dynamical variables, as for example in nuclear fission, for which one has to use the second type of methods described above. Our approach is based on an almost entirely microscopic description of the intrinsic system, using parametric random banded matrices. We have established that the dynamics show some new features. Perhaps the most prominent one is the appearance of extremely long tails in either energy, momentum or spatial distributions for the subsystems in interaction and the manifestly non–Gaussian character of the dissipative dynamics. These features raise significant doubts concerning the applicability of various phenomenological transport approaches, such as Fokker–Planck and Langevin equations [25], to finite many–body quantum systems.

Refinements of details of the present scheme are desirable, for example, the partic-
ular parametrization suggested in Ref. [13] for the correlator (5) and a more realistic parametrization for the energy dependence of the average density of states $\rho(\varepsilon)$. The introduction of a correlator that is not translationally invariant, which is well inside the scope of our formalism, allows one to tackle the problem of the scattering by a localized complex system.

We do not have yet a solution for the two subsystems in interaction outside the adiabatic limit and consequently, we do not have an understanding of the dynamical evolution for this very important for experiments case. The main difficulty here seems to reside in finding a practical scheme to solve the evolution equations, which are already known. An extension of the present formalism to open quantum systems (to include particle emission) is desirable as well. Apparently, from the formal point of view, the only new element would be the replacement of the Hermitian Hamiltonian $H_1(X)$ by a non–Hermitian one and that would not lead to unsurmountable difficulties. A non–Hermitian $H_1(X)$ would lead however also to a formalism in which the probability is not conserved anymore, but alternative approaches can be also envisioned.

One limitation of our approach is the neglect of the shape dependence of the average density of states for the intrinsic subsystem $\rho(\varepsilon, X)$. This limitation excludes the description of processes in which mechanical work is exchanged. There is no apparent technical or methodological difficulty in taking this additional feature into account and its implementation should be straightforward. Dissipative effects arising from the shape dependence of the average density of states alone, i.e. due to the time dependence in $\rho(\varepsilon, X(t))$, are expected to give rise to the so called one–body type of friction [26] or the wall-formula. A simple example is an ideal gas in a container, which changes its shape.

We have not explicitly specified the number of degrees of freedom of the “slow” subsystem. The reader might be left with the impression that our approach is limited to one dimensional “slow” motion only. Actually, that is not the case as it would be clear from a closer analysis. There is however an element of the dynamics, which would appear only when the dimensionality is two or higher, namely the emergence of abelian and/or nonabelian gauge fields [15, 16]. By their definition, gauge fields couple to momenta and thus are expected to appear in the next to the leading order of an adiabatic approximation. Naively one would incorrectly conclude that gauge fields are therefore non–adiabatic in nature. Gauge fields would lead undoubtly to new and interesting dynamical effects. The generalization of the present approach to include them seems to raise no technical or methodological issues.

There always is the nagging question about a microscopic justification of the entire parametric random matrix approach and especially about trying to get an idea about where different key parameters come from, e.g. $X_0, \kappa_0, \Gamma$. Obviously, that question lies outside the scope of this work. We think, however, that one can get a pretty good idea about the appropriate values for such parameters from both many–body theoretical models and experiment as well. For example, one can easily consider an ensemble of
many interacting Fermions in various continuously changing containers and thus extract $X_0$. We do not see a principal obstacle here. Concerning the applicability of the very idea of a random matrix, in order to model a many–body system, that seems to have been answered convincingly [1, 2, 3, 4, 5].

A more ambitious goal is to extend the formalism as to attain a more complete characterization of the two subsystems in interaction. So far the formalism conserves only the probability. One might want to have a formalism in which the total energy or other relevant observables, such as linear and/or angular momentum, is conserved. For example, one can devise evolution equations for a slightly generalized density matrix, such as $\rho(X, Y, \varepsilon, t)$, which is obtained by summing over intrinsic states with a given energy $\varepsilon$ only. This aspect has been barely touched upon by us [27] and appears to be a most daunting task. One can write down with relative ease an impressive number of evolution equations, satisfying such requirements. However the analysis and the solution of these equations seems at the present moment quite difficult, to say the least. Another direction, which seems to be the most difficult technically at the present time, is to extend the formalism so as to be able to compute fluctuation characteristics of the dynamics.

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