Global existence for reaction–diffusion systems with dissipation of mass and quadratic growth

PHILIPPE SOULET

Abstract. We consider the Neumann and Cauchy problems for positivity preserving reaction–diffusion systems of \( m \) equations enjoying the mass and entropy dissipation properties. We show global classical existence in any space dimension, under the assumption that the nonlinearities have at most quadratic growth. This extends previously known results which, in dimensions \( n \geq 3 \), required mass conservation and were restricted to the Cauchy problem. Our proof is also simpler.

1. Introduction and main result

We consider the reaction–diffusion system

\[
\begin{aligned}
\frac{\partial}{\partial t} u_i - d_i \Delta u_i &= f_i(u), \quad x \in \Omega, \quad 0 < t < T \quad (1 \leq i \leq m), \\
\frac{\partial}{\partial n} u_i &= 0, \quad x \in \partial \Omega, \quad 0 < t < T, \\
u_i(x, 0) &= u_{i,0}(x), \quad x \in \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^n \) is either a smoothly bounded domain, or \( \Omega = \mathbb{R}^n \), with \( n \geq 1 \). Here \( m \geq 2, d_i > 0, u = (u_1, \ldots, u_m) \), and the functions \( f_i : \mathbb{R}^m \to \mathbb{R} \) are of class \( C^1 \) for \( i = 1, \ldots, m \). As for the initial data, we assume

\[
u_0 \in (L^\infty(\Omega))^m, \quad \text{with } u_0 \geq 0 \text{ in } \overline{\Omega} \text{ and } u_{0,i} \not\equiv 0 \text{ for } i = 1, \ldots, m
\]  

(in this paper vector inequalities such as \( u \geq 0 \) or \( u > 0 \) will be understood component-wise). Then, problem (1) admits a unique, maximal solution, classical for \( t > 0 \). Its existence time will be denoted by \( T_{\text{max}} \).

We assume the following structure conditions:

(Preservation of positivity) \( u \geq 0, \quad u_i = 0 \implies f_i(u) \geq 0 \), \( i = 1, \ldots, m \) \( (3) \)

(Dissipation of mass) \( \sum_{i=1}^{m} f_i(u) \leq 0, \quad u > 0 \) \( (4) \)

Assumption (3) implies that \( u > 0 \) in \( \overline{\Omega} \times (0, T_{\text{max}}) \) by the maximum principle. Assumption (4) guarantees that the total mass \( \sum_{i} \int_{\Omega} u_i(x, t) \, dx \) is nonincreasing in
time (in case \( \Omega \) is bounded), and such systems are frequently encountered in models of chemical reactions. In the equidiffusive case \( d_i \equiv d \), Assumptions (3), (4) ensure global existence and boundedness of the solution, as an immediate consequence of the maximum principle applied to \( \sum_i u_i \).

It is well known that global existence is no longer trivial when the \( d_i \) are not equal (a case which is indeed relevant in chemical reactions) and there is an abundant mathematical literature on this question (see, e.g., [10], [19, Section 33] and [16] for surveys, and see also further references in Remark 1 below). Various sufficient conditions on the nonlinearities \( f_i \) for global classical existence have been found, as well as examples of finite time blowup for certain systems. The case of systems with at most quadratic growth

\[
|f_i(u)| \leq M(1 + |u|^2), \quad u \geq 0, \quad 1 \leq i \leq m, \tag{5}
\]

has received particular interest, especially in view of the special case

\[
m = 4, \quad f_i(u) = (-1)^i(u_1u_3 - u_2u_4), \tag{6}
\]

which corresponds to the reversible binary reaction

\[
A_1 + A_3 \rightleftharpoons A_2 + A_4.
\]

It was proved by Kanel [13] that if \( \Omega = \mathbb{R}^n \), (2), (5) are satisfied and Assumption (3) is replaced with the stronger mass conservation assumption

\[
\sum_{i=1}^{m} f_i(u) = 0, \quad u \geq 0, \tag{7}
\]

then global existence is true in any space dimension. This in particular covers the case (6). See also [2–4,11,12] for related results and alternative approaches. In particular, the case (6) in a bounded domain in dimension 2 is covered in [11].

The goal of the present paper is to extend the global existence result for system (1) with at most quadratic growth, under the weaker mass dissipation condition (3), as well as to cover the case of bounded domains with Neumann conditions. As a counterpart, we will need an additional structure assumption, the so-called dissipation of entropy property:

\[
\sum_{i=1}^{m} f_i(u) \log u_i \leq 0, \quad u > 0. \tag{8}
\]

Such a condition is satisfied by many systems corresponding to reversible reactions, and this is for instance true in the case of (6). As for the at most quadratic growth condition, we will assume it under the following slightly stronger form

\[
|\nabla f_i(u)| \leq M(1 + |u|), \quad u \geq 0, \quad 1 \leq i \leq m, \tag{9}
\]

for some constant \( M > 0 \). Our main result is the following.
THEOREM 1. Let $m \geq 2$, $n \geq 1$, let $\Omega \subset \mathbb{R}^n$ be either a smoothly bounded domain, or $\Omega = \mathbb{R}^n$, and let $u_0$ satisfy (2). Assume that the nonlinearities $f_i \in C^1(\mathbb{R}^m, \mathbb{R})$ satisfy properties (3), (4), (8), (9). Then, problem (1) has a global classical solution, i.e. $T_{\text{max}} = \infty$.

Our proof is based on suitable modifications of Kanel’s methods in [12,13]. Namely, we combine the argument from [12], based on interpolation inequalities and suitable auxiliary problems, with the entropy structure provided by (8). This enables one to reach nonlinearities with quadratic growth under a mere mass dissipation structure, without requiring mass conservation. Actually, the mass conservation structure (7) is crucially used in [3,13], via delicate Hölder estimates for suitable parabolic equations with bounded coefficients (based on De Giorgi type iteration). We can here take advantage of the entropy structure to avoid such difficulties, resulting in a much simpler argument. We are also able to extend this approach to the Neumann problem.

REMARK 1. (a) Theorem 1 remains valid, with simple proof changes, if one considers Dirichlet instead of Neumann boundary conditions in (1).

(b) Assumption (9) in Theorem 1 can be replaced by the slightly weaker condition:

$$|f_i(u)| \leq M(1 + |u|^2) \quad \text{and} \quad \frac{\partial f_i}{\partial u_j}(u) \geq -M(1 + |u|), \quad u \geq 0, \ 1 \leq i, j \leq m.$$  \hfill (9')

Also, Assumption (8) can be replaced with

$$\sum_{i=1}^m f_i(u)(1 + \log u_i) \leq C \sum_{i=1}^m u_i \log(1 + u_i), \quad \text{for all } u \text{ with } u_i \geq 1, \ 1 \leq i \leq m.$$  

(c) In the case $\Omega$ is bounded, conditions (3) and (4) guarantee that $u$ remains bounded in $L^1$. But it is an open problem whether $u$ is globally bounded in $L^\infty$ under the assumptions of Theorem 1 with $n \geq 3$, even for the case of system (6). For positive results when $n \leq 2$, see [17] and the references therein.

(d) The existence of a global weak solution was shown in [6,16] under the mere Assumptions (3)–(5) (without conservation of mass or dissipation of entropy).

(e) A related, active topic is the study, by means of entropy methods, of the stabilization as $t \to \infty$ of (classical or weak) global solutions of systems with dissipation of mass. For this, we refer to, e.g., [1,5,7–9,14,17,18].

2. Proof of Theorem 1

We shall use the following interpolation lemma. It was proved in [12] in the case $\Omega = \mathbb{R}^n$ and we here extend it to the case of bounded domains with Neumann boundary conditions. The proof will be given in Sect. 3.

For $T > 0$, we denote $Q_T = \Omega \times (0, T)$. For $k = 1, 2$, we set $E_k = \{ \psi \in BC^k(\Omega) : \partial_\nu \psi = 0 \text{ on } \partial \Omega \}$ (where the boundary conditions are omitted if $\Omega = \mathbb{R}^n$, and where
$\psi$ may be real- or vector-valued). We also denote $\|U\|_{k,T} = \sup_{t \in (0,T)} \|U(t)\|_{C^k(\Omega)}$ for $k \geq 0$ integer and, if $u$ is vector-valued, $\|u\|_{k,T} = \max_{1 \leq i \leq m} \|u_i\|_{k,T}$.

**Lemma 2.** Let $T > 0$, $U_0 \in E_1$, $g \in BC(\overline{Q}_T; \mathbb{R})$ and let $U : \overline{Q}_T \to \mathbb{R}$ be a classical solution of

$$
\begin{cases}
U_t - d\Delta U = g, & x \in \Omega, \ 0 < t < T, \\
U^\nu = 0, & x \in \partial \Omega, \ 0 < t < T, \\
U(x, 0) = U_0(x), & x \in \Omega.
\end{cases}
$$

(i) Then we have

$$
\|U\|_{1,T} \leq C(\Omega, d, T) \left[ \|U_0\|_{C^1} + \|U\|_{0,T}^{1/2} \|g\|_{0,T}^{1/2} \right].
$$

(10)

Here and below, the constants $C(\Omega, d, T) > 0$ remain bounded for $T > 0$ bounded.

(ii) Assume in addition that $U_0 \in E_2$ and $g \in C([0, T]; E_1)$. Then we have

$$
\|U\|_{2,T} \leq C(\Omega, d, T) \left[ \|U_0\|_{C^2} + \|U\|_{1,T}^{1/2} \|g\|_{1,T}^{1/2} \right]
$$

and

$$
\|U\|_{2,T} \leq C(\Omega, d, T) \left[ \|U_0\|_{C^2} + \|g\|_{1,T}^{1/2} \left[ \|U_0\|_{C^1} + \|U\|_{0,T}^{1/2} \|g\|_{0,T}^{1/2} \right] \right]^{1/2}.
$$

(12)

**Proof of Theorem 1.** By a time shift we may assume without loss of generality that $u_0 \in BC^2(\Omega)$, with $\partial \nu u_0 = 0$ on $\partial \Omega$ if $\Omega$ is bounded. Also we shall write $\sum_i$ for $\sum_{i=1}^m$.

**Step 1. Passage to entropy variables.** We set $L_i = \partial_t - d_i \Delta$ and define the new unknowns

$$
v_i := (1 + u_i) \log(1 + u_i) > 0, \quad w_i := v_i e^{-Kt}.
$$

We claim that for suitable constant $K > 0$, the functions $w_i$ satisfy

$$
\sum_i L_i w_i \leq 0.
$$

(13)

To this end we compute

$$
\partial_t v_i = (1 + \log(1 + u_i)) \partial_t u_i, \quad \nabla v_i = (1 + \log(1 + u_i)) \nabla u_i
$$

and

$$
\Delta v_i = (1 + \log(1 + u_i)) \Delta u_i + (1 + u_i)^{-1} |\nabla u_i|^2,
$$

hence

$$
L_i v_i \leq (1 + \log(1 + u_i)) L_i u_i = (1 + \log(1 + u_i)) f_i(u).
$$
Set \( e = (1, \ldots, 1) \) and denote by \( | \cdot |_\infty \) the max norm on \( \mathbb{R}^m \). It follows from (4), (8), (9) (or (9')) and the mean value theorem that

\[
\sum_i L_i v_i \leq \sum_i \log(1 + u_i) f_i(u) = \sum_i \log(1 + u_i)(f_i(u) - f_i(e + u)) + \sum_i \log(1 + u_i) f_i(e + u) \\
\leq \sum_i \log(1 + u_i)(f_i(u) - f_i(e + u)) \leq mM|e|(1+|e|+|u|) \log(1+|u|_\infty) \\
\leq K(1+|u|_\infty) \log(1+|u|_\infty) \leq K \sum_i (1+u_i) \log(1 + u_i) = K \sum_i v_i,
\]

with \( K = m^{3/2}M|e|(1+|e|) \), and (13) follows.

**Step 2. Linear auxiliary problem.** Pick any finite \( T < T_{\text{max}} \). Following [12,13] (in slightly modified form), we fix \( d = 1 + \max_i d_i \), set \( L = \partial_t - d\Delta \) and, for each \( 1 \leq i \leq m \), we introduce the (classical) solution \( z_i \geq 0 \) of the auxiliary problem

\[
\begin{aligned}
Lz_i &= w_i, \quad x \in \Omega, \; 0 < t < T, \\
\partial_\nu z_i &= 0, \quad x \in \partial\Omega, \; 0 < t < T, \\
z_i(x, 0) &= 0, \quad x \in \Omega. 
\end{aligned}
\]

(14)

We claim that there exists a constant \( C_1 > 0 \) independent of \( T \) such that

\[
w_i \leq C_1 - \sum_i (d - d_i) \Delta z_i \quad \text{in } Q_T \quad \text{for } 1 \leq i \leq m
\]

(15)

and

\[
z_i \leq dC_1T \quad \text{in } Q_T \quad \text{for } 1 \leq i \leq m.
\]

(16)

To show (15) and (16), we set

\[
\phi = \sum_i L_i z_i.
\]

Using (13), we first notice that \( \phi \) satisfies

\[
L\phi = \sum_i L_i(Lz_i) = \sum_i L_iw_i \leq 0.
\]

(17)

Next, in the case \( \Omega \) bounded, observing that \( z_i \) is sufficiently smooth up to the boundary and that \( \partial_\nu \partial_t z_i = \partial_t \partial_\nu z_i = 0 \) on \( \partial\Omega \), we get

\[
\partial_\nu \Delta z_i = -d^{-1} \partial_\nu Lz_i = -d^{-1} \partial_\nu w_i = 0 \quad \text{on } \partial\Omega.
\]
Therefore, \( \partial_v(L_i z_i) = -d_i \partial_v \Delta z_i = 0 \), hence \( \partial_v \phi = 0 \), on \( \partial \Omega \). In both cases \( \Omega \) bounded and \( \Omega = \mathbb{R}^n \), it thus follows from (17) and the maximum principle that
\[
\phi \leq C_1 \quad \text{in} \quad Q_T,
\]
with \( C_1 \) independent of \( T \). Then eliminating \( \partial_t z_i \) between \( L \) and \( L_i \), by writing
\[
\sum_i (d - d_i) \Delta z_i = \sum_i (L_i z_i - L z_i) = \phi - \sum_i w_i,
\]
we see that (18) and \( w_i \geq 0 \) guarantee (15).

On the other hand, we may eliminate \( \Delta z_i \) by writing
\[
\sum_i (d - d_i) \partial_t z_i = \sum_i (d L_i z_i - d_i L z_i) = d \phi - \sum_i d_i w_i \leq d C_1.
\]
Integrating in time and using \( z_{i,0} = 0 \) and \( d = 1 + \max_i d_i \), we get
\[
\sum_i z_i \leq \sum_i (d - d_i) z_i \leq \sum_i (d - d_i) z_{i,0} + d C_1 T \leq d C_1 T,
\]
hence (16), owing to \( z_i \geq 0 \).

**Step 3. Interpolation and feedback argument.** We shall now use inequalities (15), (16), along with a feedback argument, to bound \( w_i \) (hence \( u_i \)). To this end, we shall suitably estimate the diffusion terms \( \Delta z_i \) by means of the interpolation Lemma 2. In this Step 3, \( C(T) \) will denote a generic positive constant (possibly depending on the solution \( u \)), which remains bounded for \( T > 0 \) bounded.

By (15), (14), (12) in Lemma 2 and (16), we have
\[
\| w_i \|_{0,T} \leq C \left[ 1 + \| z_i \|_{2,T} \right] \leq C(T) \left[ 1 + \| w_i \|_{1,T}^{1/2} \| z_i \|_{0,T}^{1/4} \| w_i \|_{0,T}^{1/4} \right] \\
\leq C(T) \left[ 1 + \| w_i \|_{1,T}^{1/2} \| w_i \|_{0,T}^{1/4} \right],
\]
hence
\[
\| w_i \|_{0,T} \leq C(T) \left[ 1 + \| w_i \|_{1,T}^{2/3} \right]. \tag{19}
\]
On the other hand, since \( |f_i(u)| \leq C(1 + |u|^2) \) due to (9), we deduce from (1) and (10) in Lemma 2 that
\[
\| u_i \|_{1,T} \leq C(T) \left[ \| u_{i,0} \|_{C^1} + \| u_i \|_{0,T}^{1/2} \| f_i(u) \|_{0,T}^{1/2} \right] \leq C(T) (1 + \| u \|_{0,T}^{3/2}.
\]
Since \( \nabla w_i = e^{-Kt}(1 + \log(1 + u_i)) \nabla u_i \), it follows that
\[
\| w_i \|_{1,T} \leq (1 + \log(1 + \| u_{i,0} \|_{0,T})) \| u_i \|_{1,T} \leq C(T) (1 + \| u \|_{0,T}^{3/2} \log(2 + \| u \|_{0,T}).
\]
Combining this with (19) and taking maximum over \( i \in \{1, \ldots, m\} \), we obtain
\[
(1 + \| u \|_{0,T} \log(1 + \| u \|_{0,T}) \leq e^{Kt} \| w \|_{0,T} \leq C(T) (1 + \| u \|_{0,T}^{2/3}) \log(2 + \| u \|_{0,T})^{2/3},
\]
hence \( \| u \|_{0,T} \leq C(T) \). We conclude that \( T_{\text{max}} = \infty \), since \( T_{\text{max}} < \infty \) would imply the blowup of \( \| u \|_{0,T} \) as \( T \to T_{\text{max}} \), whereas \( C(T) \) remains bounded for \( T \) bounded. \( \square \)
3. Proof of Lemma 2

In this proof, $C$ denotes a generic positive constant depending only on $\Omega, d, T$, and remaining bounded for $T$ bounded.

(i) For $k > 0$ to be chosen later, we note that $U$ solves $U_t - d \Delta U + kU = g + kU$, hence $(\partial_t - d \Delta)(e^{kt}U) = e^{kt}(g + kU)$. By the variation of constants formula, we deduce that

$$U(t) = e^{-kt}e^{t \Delta}U_0 + \int_0^t e^{(t-s) \Delta}e^{-k(t-s)}(g + kU)(s) \, ds,$$

where $(e^{t \Delta})$ denotes the Neumann or the Cauchy heat semigroup. We have the estimates

$$\|e^{t \Delta} \psi\|_{C^1} \leq C\|\psi\|_{C^1}, \quad 0 < t < T, \ \psi \in E_1,$$

and

$$\|e^{t \Delta} \psi\|_{C^1} \leq Ct^{-1/2}\|\psi\|_{L^\infty}, \quad 0 < t < T, \ \psi \in BC(\overline{\Omega}),$$

(see e.g. [15] and the references therein). We deduce that

$$\|U(t)\|_{C^1} \leq Ce^{-kt}\|U_0\|_{C^1} + C \int_0^t (t-s)^{-1/2}e^{-k(t-s)}\|(g + kU)(s)\|_{C^1} \, ds.$$  

Using

$$\int_0^\infty (t-s)^{-1/2}e^{-k(t-s)} \, ds = k^{-1/2} \int_0^\infty \tau^{-1/2}e^{-\tau} \, d\tau = Ck^{-1/2},$$

we obtain

$$\|U\|_{1,T} \leq C\|U_0\|_{C^1} + C\|k^{-1/2}g\|_{0,T} + k^{1/2}\|U\|_{0,T}.$$ 

Inequality (10) then follows by choosing $k = \|g\|_{0,T}\|U\|_{0,T}^{-1}$.

(ii) We have the estimates

$$\|e^{t \Delta} \psi\|_{C^2} \leq C\|\psi\|_{C^2}, \quad 0 < t < T, \ \psi \in E_2,$$

and

$$\|e^{t \Delta} \psi\|_{C^2} \leq Ct^{-1/2}\|\psi\|_{C^1}, \quad 0 < t < T, \ \psi \in E_1$$

(see e.g. [15] and the references therein). It follows from (20) that

$$\|U(t)\|_{C^2} \leq Ce^{-kt}\|U_0\|_{C^2} + C \int_0^t (t-s)^{-1/2}e^{-k(t-s)}\|(g + kU)(s)\|_{C^1} \, ds.$$ 

By the argument in part (i), we deduce (11). Property (12) then follows by combining (10) and (11). \qed
REFERENCES

[1] M. Bisi, L. Desvillettes and G. Spiga, Exponential convergence to equilibrium via Lyapunov functionals for reaction–diffusion equations arising from non reversible chemical kinetics, *ESAIM Math. Model. Numer. Anal.* 43 (2009), 151–172.

[2] J.A. Cañizo, L. Desvillettes and K. Fellner, Improved duality estimates and applications to reaction–diffusion equations, *Comm. Partial Differential Equations* 39 (2014), 1185–1204.

[3] M.C. Caputo, T. Goudon, A. Vasseur, Solutions of the 4-species quadratic reaction–diffusion system are bounded and $C^\infty$-smooth, in any space dimension, Preprint arXiv:1709.05694 (2017).

[4] M.C. Caputo and A. Vasseur, Global regularity of solutions to systems of reaction–diffusion with sub-quadratic growth in any dimension, *Comm. Partial Differential Equations* 34 (2009), 1228–1250.

[5] L. Desvillettes and K. Fellner, Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations, *J. Math. Anal. Appl.* 319 (2006), 157–176.

[6] L. Desvillettes, K. Fellner, M. Pierre and J. Vovelle, Global existence for quadratic systems of reaction–diffusion, *Adv. Nonlinear Stud.* 7 (2007), 491–511.

[7] K. Fellner and E.-H. Laamri, Exponential decay towards equilibrium and global classical solutions for nonlinear reaction–diffusion systems, *J. Evol. Equ.* 16 (2016), 681–704.

[8] K. Fellner, W. Prager and B.Q. Tang, Exponential decay towards equilibrium and global classical solutions for nonlinear reaction–diffusion systems, *Kinet. Relat. Models* 10 (2017), 1055–1087.

[9] K. Fellner and B.Q. Tang, Explicit exponential convergence to equilibrium for nonlinear reaction–diffusion systems with detailed balance condition, *Nonlinear Anal.* 159 (2017), 145–180.

[10] M. Fila and H. Ninomiya, Reaction versus diffusion: blow-up induced and inhibited by diffusivity, *Russian Math. Surveys* 60 (2005), 1217–1235.

[11] T. Goudon and A. Vasseur, Regularity analysis for systems of reaction-diffusion equations, *Ann. Sci. Éc. Norm. Supér.* (4) 43 (2010), 117–142.

[12] J.I. Kanel, The Cauchy problem for a system of semilinear parabolic equations with balance conditions, *Differentsial’nye Uravneniya* 20 (1984), 1753–1760 (English translation: *Differential Equations* 20 (1984), 1260–1266).

[13] J.I. Kanel, Solvability in the large of a system of reaction–diffusion equations with the balance condition, *Differentsial’nye Uravneniya* 26 (1990), 448–458 (English translation: *Differential Equations* 26 (1990), 331–339).

[14] A. Mielke, J. Haskovec and P.A. Markowich, On uniform decay of the entropy for reaction–diffusion systems, *J. Dynam. Differential Equations* 27 (2015), 897–928.

[15] X. Mora, Semilinear parabolic equations define semiflows on $C^k$ spaces, *Trans. Amer. Math. Soc.* 278 (1983), 21–55.

[16] M. Pierre, Global existence in reaction–diffusion systems with control of mass: a survey, *Milan J. Math.* 78 (2010), 417–455.

[17] M. Pierre, T. Suzuki and Y. Yamada, Dissipative reaction diffusion systems with quadratic growth, *Indiana Univ. Math. J.* (2018), to appear (Preprint hal: 01671797).

[18] M. Pierre, T. Suzuki and R. Zou, Asymptotic behavior of solutions to chemical reaction–diffusion systems, *J. Math. Anal. Appl.* 450 (2017), 152–168.

[19] P. Quittner, Ph. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhäuser Advanced Texts, 2007, 584 p.+xi.

Philippe Souplet
Université Paris 13, Sorbonne Paris Cité, CNRS UMR 7539,
Laboratoire Analyse Géométrie et Applications
93430 Villetaneuse
France
E-mail: souplet@math.univ-paris13.fr