Formulation of a new elastoviscoplastic model for time-dependent behavior of clay

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Abstract
This paper presents a generalized, elastoviscoplastic constitutive model, MIT-SR, that is capable of describing a wide range of time-dependent characteristics observed in clays from creep to strain-rate-dependent shear behavior. The key component of the proposed model is a novel evolution equation that attributes viscoplastic deformations to a state variable, $R_\alpha$, referred to as internal strain rate, which represents the perturbation of the clay particle assembly due to historical straining. This state variable is driven by external straining actions (under compression or shear), which can be intrinsically linked to the loading step in classical plasticity theory, and decays with time representing a fading memory process. The proposed framework can be used to extend existing time-independent elastoplastic models. In this case, MIT-SR is built upon a prior elastoplastic model (MIT-S1), which uses 3-D stress-space surfaces and hardening laws to represent anisotropic effective stress-strain-strength properties, and a paraelastic approach to describe nonlinear hysteretic behavior at small strains. The paper highlights the versatility of the proposed MIT-SR model in representing a wide range of time-dependent characteristics for normally consolidation behavior and undrained shear behavior. By varying a strain-rate sensitivity parameter, $\beta$, the model can capture a full spectrum from temporary material response to changing strain rate, to isotache-type behavior where the normal consolidation and critical state lines are functions of the applied steady strain rate. The paper also showcases the model prediction for undrained creep and undrained relaxation behavior, and its promising capability in describing rate-effects under cyclic direct simple shear.

KEYWORDS
clay behavior, constitutive relations, rate-dependency, viscoplasticty

1 | INTRODUCTION

Time-dependent behavior of clay has widely been observed under a variety of loading conditions. Examples include creep behavior upon one-dimensional loading (eg, Ref. ¹²) and a variety of strain rate effects in shear experiments (eg, Refs. ³–⁶). Many geotechnical applications are sensitive to the time-dependent behavior of clay. For instance, excessive long-term settlement of clay beneath levee or dam can increase the risk of overtopping during a flood event. Reliable prediction of
these settlements is challenging, as the compression is typically accompanied by lateral spreading, suggesting a coupling between volumetric and deviatoric behavior. It is therefore important to model the general time-dependent behavior of clay as accurately as possible.

A number of elastoviscoplastic (EVP) formulations have been developed for geomaterials in the past four decades. Most of these models are created on a similar basis that integrates an elastoplastic yield surface in stress space (eg, Modified Cam-Clay [MCC] with evolution equation for viscoplastic strain (eg, in the form of an overstress or isotache-type power-law equation). Some recent developments also add representations of anisotropic effects through rotational yield surface. However, there is little development in the fundamental assumption for the evolution law of the viscoplastic strain.

To date, many existing EVP soil models have employed a variety of evolution laws based on the isotache concept, which assumes that the viscoplastic strain rate is uniquely defined by the current effective stress and strain states. However, this empirical assumption leads to flawed predictions for various rate effects observed in clays. It has been recognized that isotache models with the stress history state and void ratio determined in laboratory tests significantly overestimate the creep rate of normally consolidated clays at field scale. When modeling rate effects under shearing, the isotache models often predict similar rate dependency in the shear stress conditions at large strain and peak strength conditions, but are not able to explain the observations of small rate dependency at critical state for reconstituted or uncemented clays. There are also some temporary strain rate effects observed for clays in CRS consolidation and triaxial shear tests where there is a step change in strain rates which deviate from the isotache concept. This diversity of strain rate effects suggests the need for a unified framework that can predict a more comprehensive range of rate-dependent characteristics for clay.

This paper presents a new 3-D EVP formulation, referred to as the MIT-SR soil model. The proposed model uses a novel evolution equation generalized from a 1-D formulation for time-dependent compression behavior of clays, which attributes viscoplastic strain rate to an internal state variable representing the perturbation of clay particle assembly due to historical compression or shear. This novel evolution equation allows the model to represent a diverse range of rate-dependent characteristics observed under compression and shear conditions. In addition, the development of MIT-SR model succeeds the long-standing research of fundamental soil behavior at MIT, and benefits through the development of previous elastoplastic models, MIT-E3, and MIT-S1. Many of their iconic features were extended through the development of the proposed EVP model, such as the capability to describe the evolved anisotropy in stress-strain behavior and the nonlinear variation in small-strain stiffness. The following sections elaborate the proposed formulation and illustrate its unique features in predicting a variety of rate-dependent behavior of clays under compression and shear.

### 2 FORMULATION OF MIT-SR

#### 2.1 Generalization from a novel 1-D model

Inspired by Taylor’s premise that associates the creep behavior of clay with the remolding effects on the soil skeleton caused by the preceding primary consolidation process, the authors proposed an EVP model for 1-D compression behavior of clays. In this model, the total strain rate is additively decomposed into elastic, \( \dot{\varepsilon}_{el} \), and viscoplastic components, \( \dot{\varepsilon}_{vp} \):

\[
\dot{\varepsilon} = \dot{\varepsilon}_{el} + \dot{\varepsilon}_{vp},
\]

\[
\dot{\varepsilon}_{vp} = R_a \cdot f \left( \frac{\sigma_v}{\sigma_p} \right).
\]

The key feature of this model is that the definition of the viscoplastic strain rate (Equation 2) includes a new state variable, \( R_a \), referred to as Internal Strain Rate, as well as the ratio of the current effective stress \( \sigma_v \) to the yield stress \( \sigma_p \). The internal strain rate, \( R_a \), represents the perturbation of clay particles assembly through the past history of compression. It can be equated with the thermodynamic concept of Granular Temperature describing the kinetic energy level (associated with the random motions of particle) for a granular assembly after being perturbed from an equilibrium state. The
proposed 1-D formulation can logically explain the creep or relaxation after loading or straining, and is consistent with Taylor’s qualitative premise.

There is an important difference between the proposed evolution equation (Equation 2) and most existing isochore/isochronous equations [e.g., Equations 9-14 and 18-24] that define the viscoplastic strain rate, \( \dot{\varepsilon}_{vp} \) as a unique function of the current states of \( \sigma' \) and the viscoplastic strain, \( \varepsilon_{vp} \) (via the yield stress \( \sigma'_p \)).

\[
\dot{\varepsilon}_{vp} = \dot{\varepsilon}_{ref} \cdot f \left[ \sigma'_1, \sigma'_p \left( \varepsilon_{vp} \right) \right],
\]

(3)

where \( \dot{\varepsilon}_{ref} \) is a material constant representing reference strain rate, often obtained as strain rate at the end of a 24 hour interval in an incrementally loaded (IL) oedometer test.

In contrast, the internal strain rate (Equation 2) evolves with time based on an Activation and Decay mechanism:

\[
\dot{R}_a = \left[ f \left( \dot{\varepsilon} \right) - R_a \right] \cdot m_t.
\]

(4)

Here, the activation function is governed by the total strain rate, \( f(\dot{\varepsilon}) \), to reflect the external perturbations to the soil skeleton (due to compression for the 1-D formulation\(^{25}\)). Simultaneously, there is an autodecay process controlled by the current value of the internal strain rate (\( -R_a \)), which allows rational descriptions of creep and relaxation process. The net change of \( R_a \) is scaled by a transient coefficient, \( m_t \).

The proposed formulation differs from the preexisting 1-D EVP formulations, as it accommodates historical straining effects and provides a more versatile description of rate-dependent compression behavior. This 1-D model was found\(^{25}\) to capture a full range of observations from isochore behavior to temporary strain rate effects under constant rate of strain (CRS) consolidation conditions. It provides a unifying framework that can characterize the clay behavior throughout long-term consolidation according to either hypothesis A (consolidation of clay layers with different thickness producing similar level of strain at the end of primary [EOP] consolidation) or hypothesis B (the strain at EOP increases with the thickness of clay and the secondary compression behavior is unique for clay layers of different thickness),\(^{25}\) or any behavior in between.

Furthermore, the proposed 1-D formulation can be readily generalized to govern the evolution of viscoplastic strain in both compression (volumetric) and shear (deviatoric) directions, and hence, paves the way for a new 3-D EVP model. The key of this generalization is to associate the internal strain rate, \( R_a \), with an overall perturbation on the soil skeleton due to general loading including both compression and shear contributions. In fact, there has been a phenomenological representation in the standard plasticity framework to account for the overall perturbation on a continuum soil body under general loading. A typical plasticity model defines a yield surface in the 3-D stress space, and prescribes a consistency condition for stresses to conform to the yield surface under loading conditions. This standard procedure results in a plastic multiplier, which can be rationally interpreted as an representation of the overall perturbation on the soil skeleton under instantaneous loads. As a result, the generalized internal strain rate, \( R_a \), that governs the magnitude of the viscoplastic strain rate, can be intrinsically linked to the plastic multiplier calculated in a 3-D elastoplastic soil model. This interpretation bridges the gap between two types of constitutive frameworks and leads to a tangible approach to create EVP soil models from existing elastoplastic formulations.

A new EVP soil model, MIT-SR, has been created on the basis of 3-D elastoplastic models, MIT-E3 and MIT-S1. It consists of (a) 3-D stress surfaces including a loading surface, a reference surface, and a critical state surface; (b) kinematic and isotropic hardening laws to allow the description of evolving anisotropy; (c) viscoplastic evolution equation generalized from the above-referenced 1-D formulation; and (D) hysteretic formulation to allow nonlinear variation in small-strain stiffness in both volumetric and deviatoric components. The following sections detail the formulation of each component.

### 3.2 3-D Stress surface

The MIT-SR model incorporates: (a) a conical surface to represent the critical state condition, (b) a loading surface to govern volumetric and deviatoric states, and (c) a reference surface (homothetic projection of the loading surface) to represent the influence of consolidation history and realize isotropic and kinematical hardening effects. Figure 1A shows the configuration for the proposed 3-D surfaces in principal effective stress space (\( \sigma'_1, \sigma'_2, \sigma'_3 \); where all principal stress are compressive), while Figure 1B exposes their relative position by cutting the surfaces with the plane where \( \sigma'_2 = \sigma'_3 \).
The critical state surface is an isotropic failure criterion, \( h_f \), based on the generalization of Matsuoka and Nakai criterion:

\[
h_f(\eta) = k^2 - \eta : \eta = 0, \tag{5a}
\]

\[
k^2 = k_a^2 + \left(3 - \frac{k_a^2}{2}\right) J_{3\eta}, \quad k_a^2 = \frac{8\sin^2 \phi'_c}{3 + \sin^2 \phi'_c}, \tag{5b}
\]

where \( \eta = S/\sigma' \) is the tensor of the deviatoric stress ratios, \( S \) is the deviatoric stress tensor, and \( \sigma' \) is the mean effective stress; \( k \) is defined in terms of the third invariant of the deviatoric stress ratio, \( J_{3\eta} (= \det(\eta)) \) and the friction angle, \( \phi'_c \), measured at large strain (corresponding to critical state) in triaxial compression tests.

The loading and reference surfaces are oriented along a direction, \( \mathbf{b} \), which represents the inherent anisotropy associated with the current stress state, and is able to evolve through kinematic hardening. The sizes of the reference and loading surfaces are represented in terms of the maximum mean effective stresses, \( \alpha' \) and \( \alpha'_1 \), respectively. Changes in \( \alpha' \) reflect strain hardening or softening, while the ratio of \( \alpha'/\alpha'_1 \) is related to the overconsolidation of the clay.

The loading surface regulates the current effective stress states:

\[
f = (\sigma')^2 \left\{ (\eta - \mathbf{b}) : (\eta - \mathbf{b}) - \zeta^2 \left( \frac{\alpha'_1}{\alpha'} - 1 \right) \left( \frac{\sigma'}{\alpha'_1} \right)^m \right\}, \tag{6}
\]

\[
\zeta^2 = c^2 + \mathbf{b} : \mathbf{b} - \eta : \mathbf{b}, \tag{7}
\]

\[
c^2 = \frac{24\sin^2 \phi'_m}{(3 - \sin \phi'_m)^2}. \tag{8}
\]

Two parameters \( m \) and \( \zeta^2 \) control the surface geometry. Figure 2A illustrates how the parameter \( m \) affects the shape of the loading surface under isotropic and \( K_0 \)-normally consolidated conditions. The results are presented in a triaxial stress space in terms of the mean effective stress, \( \sigma' \) [ = \((\sigma'_v + 2\sigma'_h)/3\) and shear stress, \( q \) [ = \((\sigma'_v - \sigma'_h)/2\)], where \( \sigma'_v \) and \( \sigma'_h \)
are vertical and horizontal effective stresses, respectively. When \( m = 0 \), the surface has an ellipsoidal geometry (similar to MIT-E3), while \( m = 1 \) corresponds to a lemniscate (similar to MIT-S1). This function allows the MIT-SR model to combine the advantages of both prior formulations and to provide more flexibility to capture complex clay behavior. The parameter \( \zeta^2 \) describes the frictional characteristics of clays and is defined in terms of a friction angle \( \phi_m', \eta, \) and \( b \), while increases in \( \phi_m' \) enlarges the aperture of the loading surface, as shown in Figure 2B.

The reference surface is a homothetic projection of the loading surface, which has a similar formulation to Equation (6), but with \( \sigma^r_{\alpha} \) and \( \alpha^r_1 \) substituting for \( \sigma' \) and \( \alpha'_1 \), respectively. The \( \sigma^r_{\alpha} \) represents image stresses, which are obtained by radially mapping the current stress state \( \sigma' \) with constant \( \eta = S/\sigma' \) onto the reference surface. The \( \alpha' \) controls the size of the reference surface, and the ratio of size of reference surface to loading surface is \( \alpha'/\alpha'_1 \) (Figure 1B), can be equated with the overconsolidation ratio. The reference surface is primarily used to (a) calculate a plastic multiplier at image stress point, which drives the internal strain rate, and (b) to represent isotropic and kinematic hardening effects.

### 2.3 Plastic loading and hardening on reference surface

The proposed EVP model retains the calculation of plastic response from the prior MIT-S1 model, and carries it out on the reference surface. This entails calculating a plastic multiplier, \( \Lambda \), under plastic loading condition, \( CL > 0 \) (see Equation 9), and then evaluating isotropic and kinematic hardening effects of the reference surface. In contrast to elastoplastic models, \( \Lambda \) does not directly lead to an instantaneous plastic deformation in the EVP formulation. Instead, it is used to represent the intensity of external loading on perturbing clay particle assembly, and drives a new internal state variable that controls the magnitude of viscoplastic deformation, as elaborated in the following section.

Given total strain rates (containing a volumetric component \( \dot{\varepsilon} \) and a deviatoric tensor \( \dot{E} \)), the plastic loading condition, \( CL \), is evaluated at an image stress state on the reference surface:

\[
CL = (K'Q'\dot{\varepsilon} + 2G'Q'^r : \dot{E}), \tag{9}
\]

where \( K' \) and \( G' \) are the bulk modulus and shear modulus at the image point, respectively; \( Q' = \partial f'/\partial \sigma'^r \) and \( Q'^r = \partial f'/\partial S'^r \) are the volumetric and deviatoric components of the gradient of the reference surface, respectively.

When plastic loading occurs, \( CL > 0 \), a plastic multiplier, \( \Lambda \), is calculated from the consistency condition, \( \dot{f}' = 0 \), as follows:

\[
\Lambda = \frac{\langle CL \rangle}{H + K'Q'P + 2G'Q'^r : P'}, \tag{10}
\]

\[
H = -\frac{\partial f'}{\partial \alpha'} \frac{\alpha'}{\Lambda} - \frac{\partial f'}{\partial b} \frac{b}{\Lambda}, \tag{11}
\]
where \( \langle \rangle \) are Macaulay brackets (i.e., \( < x > = x \) for \( x \geq 0 \); \( < x > = 0 \) for \( x < 0 \)); \( H \) is the plastic modulus containing contributions from isotropic and kinematic hardening; \( P \) and \( P' \) represent the volumetric and deviatoric components of flow directions, respectively, and are defined subsequently in Equation (19).

The model uses the same isotropic hardening laws as the prior MIT-S1 model, where the size of the reference surface, \( \alpha' \), evolves according to the following relations:

\[
\frac{\dot{\alpha}'}{\alpha'} = \frac{\Lambda P}{(\rho_c - \rho_r) \cdot n} - \frac{2 \mathbf{b} : \mathbf{b}}{a^2 + \mathbf{b} : \mathbf{b}},
\]

\[
a^2 = \frac{24 \sin^2 \phi'_{cs}}{(3 - \sin \phi'_{cs})^2}.
\]

The first term in Equation (12) corresponds to the density hardening process, where \( n \) is the porosity; \( \rho_c \), corresponds to the slope of the Limiting Compression Curve (LCC) for normal consolidation (in a loge-log\( \sigma' \) space after \(34\)); \( \rho_r \) governs the small strain unloading/reloading behavior under isotropic or \( K_0 \) conditions, with its specific form given in Equation (27).

The second term in Equation (12) controls the relative locations of normal consolidation/LCC curves for specimens consolidated at different stress ratios (\( \eta \)).\(^{34}\)

On the other hand, if unloading occurs with \( CL < 0 \) and \( \dot{\varepsilon} < 0 \), the reference surface contracts with:

\[
\frac{\dot{\alpha}'}{\alpha'} = \frac{\dot{\varepsilon}}{\rho_c \cdot n}.
\]

The model also adopts nearly the same kinematic hardening law from MIT-S1, where the orientation of the reference and loading surfaces, \( \mathbf{b} \), rotate toward the principal stress axes, where the material constant, \( \psi \), controls the evolving rate of anisotropy:

\[
\dot{\mathbf{b}} = \psi \frac{\Lambda}{\alpha' - n} \langle r_x \rangle \langle Q' \cdot P \rangle (\eta - \mathbf{b}),
\]

\[
r_x = \frac{(k^2 + \mathbf{b} : \mathbf{b} - 2 \eta : \mathbf{b})}{k^2_{\alpha'}}.
\]

### 2.4 Viscoplastic evolution equation

The MIT-SR model decomposes both the volumetric and deviatoric components of the total strain rate into elastic and viscoplastic contributions (denoted by superscripts \( \text{el} \) and \( \text{vp} \), respectively):

\[
\begin{pmatrix}
\dot{\varepsilon} \\
\dot{\mathbf{E}}
\end{pmatrix} = \begin{pmatrix}
\dot{\varepsilon}^{\text{el}} \\
\dot{\mathbf{E}}^{\text{el}} + \dot{\mathbf{E}}^{\text{vp}}
\end{pmatrix}.
\]

The viscoplastic strain rates are determined by projecting a viscoplastic multiplier \( \Lambda^{\text{VP}} \) onto the volumetric and deviatoric flow directions \( P \) and \( P' \), respectively:

\[
\begin{pmatrix}
\dot{\varepsilon}^{\text{vp}} \\
\dot{\mathbf{E}}^{\text{vp}}
\end{pmatrix} = \Lambda^{\text{VP}} \begin{pmatrix}
P \\
P'
\end{pmatrix},
\]

\[
\begin{pmatrix}
P \\
P'
\end{pmatrix} = \begin{bmatrix}
(k^2 - \eta : \eta) \frac{\sigma'_{rr}}{\alpha'} \\
\chi P \eta + \frac{\eta^2}{\alpha'} \parallel \eta \parallel Q'_{rr}
\end{bmatrix},
\]
\[ \chi = \left( \frac{\rho_c}{\rho_c - \rho_r} \right) \left[ \frac{1}{3} \left( \frac{1 + 2K_{INC}}{K_{INC}} \right) - \frac{K}{2G} \left( \frac{\rho_r}{\rho_c} \right) \right]. \]  

(20)

The viscoplastic multiplier, \( \Lambda^{\text{vp}} \), controls the magnitude of the viscoplastic strain rate and is a generalized form of the 1-D viscoplastic evolution equation (Equation 2). It is defined as the product of the internal strain rate, \( R_a \), and a function of the stress ratio, \( f(\alpha'/\alpha'_1) \):

\[ \Lambda^{\text{vp}} = R_a \cdot f \left( \frac{\alpha'}{\alpha'_1} \right). \]  

(21)

The generalized internal strain rate, \( R_a \), now represents the overall perturbation of clay particle assembly due to historical straining in both compression and shear. Under general loading conditions \( (CL > 0) \), \( R_a \) evolves with an evolution equation similar to Equation (4) representing an activation-decay mechanism:

\[ \dot{R}_a = [f_A - R_a] \cdot m_t. \]  

(22)

Equation (22) suggests that \( R_a \) increases with an activation function, \( f_A \), which represents the intensity of external straining effects on perturbing clay particle assembly. Simultaneously, there is a temporal decay in \( R_a \) driven by the current value of \( (\dot{R}_a) \), as the clay particles adjust toward equilibrium after perturbation, where the external strain rate effects phase out. This self-decay process rationally explains the observation of creep and relaxation behavior after interrupting compression or shear action on clays. Equation (22) also implies a steady state at \( R_a = f_A \) when the activation and decay terms balance out. The steady state of \( R_a \) corresponds to the laboratory observation of normally consolidation at constant rate of strain (CRS) conditions or the response at large shear strains under constant shear strain rate.

To represent an overall perturbation effects, the activation function, \( f_A \), is defined as a function of the plastic multiplier, \( \Lambda \), which is obtained from the plastic calculation and accounts for the contributions of all strain rate components \( (\dot{\varepsilon}, \dot{E}) \):

\[ f_A = \Lambda \left( \frac{\Lambda \cdot P_0}{\dot{\varepsilon}_{\text{ref}}} \right)^{-\beta}. \]  

(23)

Equation (23) describes a power-law-type rate dependency for the steady state of \( R_a \) from the Equation (22). The parameter \( \beta (\geq 0) \) is a material constant that controls rate sensitivity; \( P_0 \) is the volumetric component of flow direction evaluated at the tip of the reference surface (ie, at \( \eta = b \) and \( \sigma'' = \sigma' \)); and \( \dot{\varepsilon}_{\text{ref}} \) is a reference strain rate.

Equation (22) uses a transient coefficient, \( m_t \), to control the overall changing rate of \( R_a \), which is assumed to be proportional to the magnitude of the viscoplastic strain rate, \( \| \dot{\varepsilon}^{\text{vp}} \| \):\n
\[ \dot{m}_t = \left( \frac{\rho_c}{\rho_{us}} - 1 \right) \left[ \frac{\| \dot{\varepsilon}^{\text{vp}} \|}{\rho_r \cdot n} \right] + O(\| \dot{\varepsilon} \|) \],  

(24)

where \( \rho_{us}/\rho_c \) is a material constant that controls the transient change in \( R_a \), and plays a primary role in predicting creep and relaxation, as well as temporary strain rate effects in compression and shear. The term of \( O(\dot{\varepsilon}) = 0.1 \| \dot{\varepsilon} \| \) (default assumption) is a contingency term to ensure \( m_t > 0 \) when \( \| \dot{\varepsilon}^{\text{vp}} \| \approx 0 \).

When simulating CRS consolidation tests, the proposed model can capture a diverse range of rate-dependent characteristics of compression behavior. Figure 3A shows typical predictions of the model with the rate-sensitivity parameter \( \beta = \rho_{us}/\rho_c > 0 \), where the model simulates a set of parallel LCC lines at different strain rates, and parallel shifts between the LCC lines when there is a step change in strain rate. When \( \beta = 0 \) and \( \rho_{us}/\rho_c > 0 \) (Figure 3B), the model is characterized by a unique rate-independent LCC locus for CRS tests, while step changes in strain rate are associated with transient jumps in effective stress. The former result conforms to the isotachic framework of soil behavior,\(^9\)-\(^{14},^{18}-^{24}\) while the latter is consistent with the Temporary Effect of Strain rate and Acceleration (TESRA) behavior observed in Refs. \(^{30}\) and \(^{31}\). The above two cases conform broadly to limiting conditions of 1-D primary consolidation (often referred to as Hypotheses A and B), as discussed in Ref. \(^{25}\), the general MIT-SR model is also able to capture intermediate time-dependent characteristics for \( 0 < \beta < \rho_{us}/\rho_c \).
The function of ratio, \( f(\alpha'/\alpha'_1) \), in Equation (21) allows a nonlinear variation in the rate dependency with stress history and is derived from empirical observations of the variation in creep rate after unloading of 1-D surcharge on clays:\(^{40,41}\)

\[
f \left( \frac{\alpha'}{\alpha'_1} \right) = \left( \frac{\alpha'}{\alpha'_1} \right)^{-1/2} \exp \left\{ \frac{1}{2\beta^2} \left[ 1 - \left( \frac{\alpha'}{\alpha'_1} \right)^{\beta_2} \right] \right\},
\]

(25)

where the material constant, \( \beta_2 (\geq 0) \), controls the nonlinearity.

Figure 4A illustrates typical predictions of LCC/normal consolidation states at different strain rates for a specified value of \( \beta_2 = 6.8 \). In contrast to Figure 3A where there is a uniform offset in \( \log(\sigma'/p_{atm}) \) with each log cycle of strain rate, results for \( \beta_2 > 0 \) show larger rate effects at higher strain rates (manifested by the variable spacing between LCC lines). Figure 4B illustrates how the parameter \( \beta_2 \) affects the predicted effective stresses versus strain rates, \( \sigma'/\sigma'_1 = \dot{\varepsilon} \) relations, where the stresses are obtained at void ratio, \( e = 1.0 \), and normalized by the stress, \( \sigma'_1 \), at \( \dot{\varepsilon} = 0.05% / h \). This relation reduces to a power
Influence of parameters $\beta_2$ and $\beta_3$ on the predicted creep behavior after unloading in 1-D oedometer test: (A) temporal evolution of volumetric strain and (B) volumetric strain–viscoplastic strain rate relations.

The evolution relation for the internal strain rate $\dot{R}_a$ in Equation (22) is valid for loading scenarios. For unloading situations (i.e., $CL < 0$ and $\Lambda = 0$), it is assumed that $\dot{R}_a$ decreases with the size of loading surface, $\alpha'_1$, in an incremental form of a power-law relation:

$$\frac{\dot{R}_a}{R_a} = (\beta_3 - 1) \left( \frac{\alpha'_1}{\alpha'_1} \right)$$

with a material parameter, $\beta_3$, that characterizes the decreasing rate of $R_a$. The above evolution law governs the postunloading deformation process of clay, and is especially relevant in the predictions of reduced creep associated with the application of temporary surcharge loads in soft ground construction.\(^{40}\)

Figures 5A and 5B illustrate the predicted creep behavior for sample of clay after being unloaded in an oedometer test, and the influence of parameters $\beta_2$ and $\beta_3$ on the predictions. This test closely mimics the standard surcharging procedure in the field. The specimen is first normally consolidated to an EOP consolidation state and then is unloaded to OCR $\approx 1.5$. Figure 5A shows the predicted temporal evolution of volumetric strain at different values of $\beta_2$ and $\beta_3$. The resulted curves all exhibit continuous creep development after temporary swelling. Increasing $\beta_3$ effectively levels off the creep curves immediately after swelling, while $\beta_2$ controls the creep rates over longer time periods. Figure 5B plots the relations of volumetric strain versus logarithmic viscoplastic strain rate, $\epsilon - \log \dot{\epsilon}^{vp}$, for the same set of predictions. It can be seen that increasing $\beta_3$ effectively reduces $\dot{\epsilon}^{vp}$ after swelling and causes a shift in the $\epsilon - \log \dot{\epsilon}^{vp}$ relation, while $\beta_2$ controls the slope of the $\epsilon - \log \dot{\epsilon}^{vp}$ relation for creep after swelling. Figures 5A and 5B show that the prediction using $\beta_2 \approx 6$ and $\beta_3 = 19$ matches closely the postunloading creep behavior measured for a specimen of Salt Lake City clay (SLC; $w_p = 22.3\%$ and $w_L = 43.6\%$).\(^{44}\) A systematic interpretation on the postsurcharge creep behavior of SLC clay has been presented in Refs.\(^{40}\) and\(^{41}\).
2.5 Nonlinear small strain and hysteretic behavior

The MIT-SR model adapts the paraelastic approach\textsuperscript{45} to describe the nonlinear stress-strain behavior at small strain levels, which is similar to the elastoplastic MIT-E\textsuperscript{3},\textsuperscript{32,33} and MIT-S\textsuperscript{134} formulations. The model introduces an intrinsic property, \( \rho_{r0} \), to represent maximum elastic stiffness that is typically observed immediately after a load reversal:

\[
\rho_{r0} = \frac{1}{C_b \left( 1 + \frac{K}{2G} \eta \right)^{1/6}} \left( \frac{\sigma'}{p_{atm}} \right)^{2/3},
\]  

(27)

where \( C_b \) is a dimensionless constant; \( p_{atm} \) is atmosphere pressure (\( \approx 101 \) kPa); and the ratio \( K/2G \) can be related to the Poisson’s ratio, \( \nu' \), via \( K/2G = (1+\nu')/[3(1-2\nu')] \).

The degradation of elastic stiffness is represented by a transient equation below, where \( \rho_r \) is equal to \( \rho_{r0} \) multiplied by a scaling factor. The factor contains volumetric and deviatoric components, which are defined in terms of two nondimensional scalar stress ratios, \( \xi \) and \( \xi_s \), and scaled by two material constants, \( D \) and \( w_s \), respectively.

\[
\rho_r = \rho_{r0} \cdot (1 + \frac{D}{\rho_{r0}} \cdot \xi + w_s \cdot \xi_s).
\]  

(28)

Equation (29a) defines \( \xi \) in terms of the difference in natural logarithmic scale between mean effective stress, \( \sigma' \), and a hysteretic stress state, \( \sigma^h \), and Equation (29b) defines \( \xi_s \) in terms of the difference between deviatoric stress ratio, \( \eta \), and a hysteretic stress ratio, \( \eta^h \). In contrast to the conventional paraelastic approach, which relates stiffness degradation to stress reversal points (a key limitation of prior elastoplastic formulations), the proposed formulation assumes that stiffness degradation occurs whenever the normalized total strain rate, \( \dot{\varepsilon}/\| \dot{\varepsilon} \| \) (or \( \dot{E}/\| \dot{E} \| \)), has the same sign as \( \ln(\sigma'/\sigma^h) \) or \( (\eta - \eta^h) \):

\[
\xi = \left< r \frac{\dot{\varepsilon}}{\| \dot{\varepsilon} \|} \cdot \ln \left( \frac{\sigma'}{\sigma^h} \right) \cdot \left( 1 + \frac{b : b}{d_a^2} \right) \right>,
\]  

(29a)

\[
\xi_s = \left< \frac{\dot{E}}{\| \dot{E} \|} : (\eta - \eta^h) \right>,
\]  

(29b)

\[
d_a^2 = \frac{2(1 - K_{O_{NC}})^2}{(1 + K_{O_{NC}} + K_{O_{NC}}^2)},
\]  

(29c)

where the constant, \( r \), is a scaling factor for \( \xi \); and the hysteretic state variables, \( \sigma^h \) and \( \eta^h \), mainly evolve with the elastic strain components:

\[
\frac{\sigma^h}{\sigma^h} = \frac{\xi}{\rho_r \cdot n} \dot{\varepsilon}^{el},
\]  

(30a)

\[
\eta^h = \frac{2G}{K} \frac{\xi_s}{\rho_r \cdot n} \dot{E}^{el} + \frac{|\dot{\varepsilon}|}{\rho_{r0} \cdot n} (\eta - \eta^h).
\]  

(30b)

The proposed hysteretic formulation has the following implications:

I During \( K_{O_{NC}} \) or isotropic consolidation, \( \eta^h \rightarrow \eta \), and \( \xi_s \) becomes negligible, as suggested by Equations (29b) and (30b). Hence, the response of \( \xi \) controls the hysteretic behavior in consolidation.

Figure 6 illustrates the prediction of a \( K_{O_{NC}} \)-consolidation test with a cycle of unloading and reloading and the evolution of \( \sigma' \) and \( \sigma^h \) with void ratio \( e \) (curves U-L-F and U’-L’-F’). Initially, it is assumed \( \sigma^h = \sigma' \) at an initial overconsolidated stress state (Point O). The state variable \( \sigma^h \) reaches a steady state (O-U’) parallel to the normal consolidation response.
FIGURE 6  Prediction of hysteretic behavior for unloading-reloading in CRS tests

(O-U). During unloading, nonlinear swelling occurs for \( \sigma' \leq \sigma^h \) near Point U', and nonlinear reloading response initiates near Point L' when \( \sigma' \geq \sigma^h \). As reloading continues, both \( \sigma' \) and \( \sigma^h \) join their respective steady-state paths again.

II In a consolidated undrained shear test, \( \xi_s \approx 0 \) prior to shearing (from Equation 29b), while during undrained shearing, \( \dot{\varepsilon} \approx 0 \) and \( \xi \approx 0 \) (Equation 29a). Thus, the model can readily capture stiff, small strain shear behavior upon initiation of shearing (characterized by maximum “elastic” shear modulus, \( G_{\text{max}} \)). Stiffness then degrades as \( \xi_s \) increases with strain, where the parameter, \( w_s \), plays a primary role.

Figure 7 plots the predicted degradation curves for undrained secant shear modulus, \( G \), normalized by maximum shear modulus, \( G_{\text{max}} \), over a small strain range (\( \varepsilon_a = 10^{-3}\% \) to \( 10^{-1}\% \)). The MIT-SR formulation allows \( G/G_{\text{max}} \) to degrade from a very small-strain level (\( \varepsilon_a = 10^{-3}\% \)), while higher values of \( w_s \) cause more rapid degradation. These trends generally match the typical range of data for reseated Boston Blue Clay. 46

2.6 | Incremental stresses

The MIT-SR model assumes isotropic elastic behavior and evaluates the changes in the effective stresses as

\[
\left( \begin{array}{c} \dot{\sigma}' \\ \dot{\mathbf{S}} \end{array} \right) = \left[ \begin{array}{c} K \left( \dot{\varepsilon}^{\text{el}} + \dot{\varepsilon}^{\text{dil}} \right) \\ 2G \dot{\mathbf{E}}^{\text{el}} \end{array} \right] = \left[ \begin{array}{c} K \left( \dot{\varepsilon} - \dot{\varepsilon}^{\text{up}} + \dot{\varepsilon}^{\text{dil}} \right) \\ 2G \left( \dot{\mathbf{E}} - \dot{\mathbf{E}}^{\text{up}} \right) \end{array} \right],
\]

where the elastic bulk and shear moduli (\( K \) and \( G \)) are defined as:

\[
K = \frac{\sigma'}{n \cdot \rho_r},
\]

\[
G = K \frac{3(1-2v')}{2(1+v')}.
\]
Equation (31) contains an additional strain rate term, $\dot{\varepsilon}^{\text{dil}} (> 0)$, to represent the dilative behavior observed in various undrained shearing modes of OC clays. It is defined in terms of the elastic deviatoric strain rate, $\dot{E}^{\text{el}}$, deviatoric stress ratio, $\eta$, and the stress ratio, $\sigma'/\alpha'$, where $D_L$ is a material constant controlling the magnitude of the dilation. This dilation strain rate term can result in negative shear-induced pore pressure during undrained shear at OC, leading to increase in the peak and critical shear strengths, but has negligible influence on the shear behavior for NC states:

$$\dot{\varepsilon}^{\text{dil}} = D_L \cdot \dot{E}^{\text{el}} : \eta \left[ 1 - \left( \frac{\sigma'}{\alpha'} \right) \right].$$ (33)

In summary, the proposed MIT-SR model incorporates 17 material constants and 5 internal state variables, as listed in Tables 1 and 2, respectively. Among the material parameters, 12 of them are adapted from the ones used in MIT-S1 model and can be determined in a similar manner to the original formulation. Tables 1 and 2 briefly indicate how to determine these parameters from standard laboratory tests, while a companion paper details the calibration procedure for the MIT-SR, using the experimental data of resedimented Boston Blue Clay.

### 3 ILLUSTRATION OF MODEL PREDICTIVE CAPABILITY

This section showcases key features of the proposed model in predicting a variety of rate-dependent characteristics of clay under shear conditions including: (1) strain rate effects in undrained triaxial shear tests, (2) undrained creep behavior, (3) undrained relaxation behavior, and (4) strain rate effects in undrained cyclic direct simple shear (DSS) test. The MIT-SR model is implemented in an elemental-level simulator to predict the considered shear behavior. The model integration uses the Modified Euler scheme, which consists of two consecutive explicit Forward Euler substeps and uses the difference between the results of the two substeps to guide the integration accuracy and adjust the time-step size. Appendix I presents the flowcharts for this integration algorithm. More details about the implementation of the MIT-SR model are also available in Ref. These simulations generally follow the sequence of typical experiment phases to reproduce the stress history state and the internal strain rate prior to the shearing phase. Table 3 lists the parameter sets used for these demonstrations, respectively.
### TABLE 1  Material parameters for general MIT-SR model

| Symbols | Physical meaning | Suggested calibration method |
|---------|------------------|-----------------------------|
| $\rho_c$ | Compressibility for LCC behavior in loge-logo$'\sigma'$ plot | Measure from loge-logo$'\sigma'$ compression curve |
| $C_b$ | Small-strain elastic compressibility | Derive from high-quality small-strain measurement of G |
| $D$ | Nonlinear volumetric and deviatoric hysteretic behavior | Measure from 1-D swelling loge-logo$'\sigma'$ curve |
| $w_i$ | Alternative measure of Poisson’s ratio at small strain | Measure from shear stiffness degradation curve |
| $2G/K$ | | Measure from 1-D swelling stress path |
| $K_{\text{ONC}}$ | Lateral earth pressure ratio in LCC region | Measure for NC clays using SHANSEP consolidation |
| $\phi_c'$ | (Large-strain) critical state friction angle | Measure in triaxial compression tests |
| $m$ | Geometry of loading/reference surfaces | Fit ESPs for triaxial compression and extension tests on NC clays |
| $\Psi$ | Rate of evolution of anisotropy due to stress history | Fit ESP for triaxial extension test on NC clays |
| $D_L$ | Dilation behavior | Fit ESP for triaxial compression test at OCR $> 2$ |

#### Parameters for time-dependent behavior

| $\rho_a$ | Compressibility in secondary compression | Fit 1-D secondary compression curve or inferred from reported $C_a/C_c$ ratio |
| $\beta$ | Rate sensitivity of steady state of $R_a$ | Measure from the rate sensitivity of CRS tests |
| $\dot{\varepsilon}_{\text{ref}}$ | Reference strain rate | Select typically the strain rate used in CRS test or measured at the end of 24 hours interval IL oedometer test |
| $\beta_2$ | Nonlinear variation of rate dependency with stress history | Measure reduction of the postunloading creep property with OCR in 1-D swelling |
| $\beta_3$ | Reduction of creep rate during unloading | Measure decrease in creep rate with OCR in 1-D swelling |

### TABLE 2  Internal state parameters for MIT-SR

| Symbol | Physical meaning | Estimation of initial values for $K_0$ condition |
|--------|------------------|-------------------------------------------------|
| $\alpha'$ | Size of bounding surface | $\alpha'_0 = \sigma'_p \left( 1 + 2K_{\text{ONC}} \right) / 3$ \( \text{where } \sigma'_p = \text{preconsolidation pressure} \) |
| $b$ | Orientation of anisotropic bounding surface in stress space | $b_0$ governed by $K_{\text{ONC}}$ and $b_0 = \eta_0$ for NC clays |
| $R_a$ | Activated rate [1/time] due to historic straining | Strain rate from the preceding consolidation on a clay layer with known drainage length |
| $\sigma^h$ | Volumetric hysteretic state parameter | For NC: Equations (28)-(30) $\rightarrow$ steady state; For OC: $\sigma^h = \sigma_0$ |
| $\eta^h$ | Deviatoric hysteretic state parameter | $\eta^h = \eta_0$ for NC and OC clays |

### TABLE 3  MIT-SR parameters used in the demonstration simulations

| Demo # | $\rho_c$ | $C_b$ | $D$ | $w_i$ | $2G/K$ | $K_{\text{ONC}}$ | $\phi_c'$ | $\phi_m$ | $m$ | $\psi$ | $D_L$ | $\beta$ | $\rho_a/\rho_c$ | $\dot{\varepsilon}_{\text{ref}}$ (1/s) | $\beta_2$ | $\beta_3$ |
|---------|---------|-------|-----|-------|--------|-------------|----------|---------|-----|-------|-------|--------|-------------|------------------|--------|--------|
| 1, 2)   | 0.4     | 500   | 0.03| 1     | 32     | 1.2        | 0.55     | 33°     | 26° | 0.5   | 30    | 0      | 0.03        | 0.03              | 1E-7   | 1      | 19    |
| 3)      | 0.2     | 300   | 0.03| 1     | 40     | 1.2        | 0.5      | 34°     | 25° | 0.2   | 30    | 1      | 0.04        | 0.04              | 1E-7   | 1      | 19    |
| 4)      | 0.5     | 500   | 0.03| 1     | 30     | 1.2        | 0.5      | 30°     | 25° | 0.8   | 0     | 1      | 0.02        | 0.02              | 1E-7   | 1      | 15    |

Note: #1, #2, undisturbed/natural Haney Clay; #3, remolded Osaka-Nankoclay.
3.1 Rate effects in undrained triaxial shear

The MIT-SR is used to predict the strain rate effects in isotropically consolidated undrained compression (CIUC) test. This set of simulations emulates CIUC experiments on undisturbed Haney Clay samples ($w_p = 26\%$ and $w_L = 44\%)$. Each test consists of isotropically consolidating the sample to 5.25ksc, holding stresses and allowing the sample to creep for 12 hours, and then running undrained triaxial shear at five strain rates ($\dot{\varepsilon}_a = 1.1, 0.15, 1.4 \times 10^{-2}, 2.8 \times 10^{-3},$ and $9.4 \times 10^{-4}/\text{min}$).

Similar to the prediction for CRS tests (Figure 3), a parametric study was conducted on the rate-sensitivity parameter $\beta$ to evaluate its influence on the predicted rate effects of undrained shear behavior. Figures 8A to 8C plot a series of shear stress versus axial strain curves for different set of values for $\beta$ and $\rho_a/\rho_c$. The key observations are as follows:

(i) For $\beta = \rho_a/\rho_c > 0$ ($= 0.03$), Figure 8A shows that the predicted undrained shear strength, $s_u$, increases with strain rates, and the rate dependency extends over the postpeak softening responses, which suggests rate-dependent critical state conditions. It is also noticed that the predicted undrained strengths, $s_u$, and the postpeak softening responses closely match the experimental data.

(ii) For $\beta = 0$ and $\rho_a/\rho_c > 0$ ($= 0.03$), the proposed model still predicts rate-dependent undrained shear strength, $s_u$, that increases with strain rate. However, the rate dependency for the postpeak responses becomes less pronounced and all five curves converge toward the critical state condition with much smaller rate dependency (Figure 8B).

(iii) For $\beta = \rho_a/\rho_c = 0$, the MIT-SR model computes nearly rate-independent shear responses throughout the pre- and postpeak regions across all five strain rates (Figure 8C).

The above observations suggest that the value of $\beta$ closely corresponds to the rate sensitivity of the critical state response. This is because that as the specimen reaches critical state condition, the internal strain rate $R_a$ approaches a steady-state condition, for which the rate effect is governed by $\beta$. On the other hand, Figure 8B shows that the computed undrained strengths still show noticeable rate dependency with $\beta = 0$ and $\rho_a/\rho_c > 0$. The peak-undrained shear strengths $s_u$ for these CIUC tests are mobilized at a strain range from $\varepsilon_{ap} = 2\%$ to $3\%$, where the $R_a$ is likely in a transient state. The rate dependency for the shear response over this range of strain can be possibly attributed to temporary strain rate effects, for which the ratio of $\rho_a/\rho_c$ has an influential role.

The above predictions for the rate effects in CIUC tests echo the results of the CRS simulations in Figure 3. For both CRS compression and triaxial shear, the proposed MIT-SR model with $\beta = \rho_a/\rho_c$ (Figure 8A) reproduces the typical predictions of existing Isotache-type models [eg, Equations 9, 10, 12-14, and 20-23], whereas the MIT-SR predictions with $\beta = 0$ and $\rho_a/\rho_c > 0$ (Figure 8B) correspond to the TESRA. Lastly, the proposed MIT-SR model can reduce to a rate-independent shear behavior with both $\rho_a/\rho_c \approx 0$ and $\beta \approx 0$, as shown in Figure 8C.
3.2 Undrained creep

The proposed model is also used to simulate the undrained creep behavior for Haney Clay during CIUC test. The simulation uses the same set of input parameters as in the above CIUC benchmark (with $\beta = \rho_d / \rho_c = 0.03$) and follows the reported experiment phases. During the test, the sample is isotropically consolidated and held for 12 hours before shearing. Each of the 11 specimens is then sheared (in undrained compression) to a different level of shear stress ($q / \sigma'_v$), and allowed to creep (at constant shear stress levels) under undrained condition for up to 40 days or until a creep rupture occurred.

Figure 9A shows the axial strain versus logarithmic elapsed time curves for the predicted undrained creep phases at all the shear stress levels. During undrained creep, the axial strain increases monotonically with time, and large creep strains occur at a higher shear stress level (at a given time). Figure 9B shows the corresponding axial strain rate versus elapsed time curves in a double-logarithm plot. At the beginning of these creep processes, the axial strain rate decreases with time, which corresponds to the well-known “primary creep or transient creep.” As creep continues, the acceleration in creep rate occurs for the series of creep at higher stress levels ($q / \sigma'_v$ from 0.636 to 0.518), which leads to the development of uncontrolled axial strain, often referred to as “tertiary creep or acceleration creep.” And the result also indicates that the creep at higher shear stress level leads to a sooner creep rupture. The MIT-SR predictions well capture these characteristics for the primary creep and the tertiary creep, and also show good agreement with the experiment data in both Figures 9A and 9B.

3.3 Undrained relaxation

This benchmark uses MIT-SR model to simulate the undrained relaxation tests measured for remolded Nanko-Osaka Clay ($w_P = 27.4\%$ and $w_L = 63.5\%$). In these simulations, each specimen is isotropically consolidated to 2ksc and sheared separately under triaxial compression (CIUC) and extension (CIUE) conditions. During shearing, the samples are allowed to relax at undrained (constant strain conditions) at three initial levels of axial strain over periods up to 10 hours.

Figures 10A and 10B compare the computed and measured effective stress paths and shear stress vs elapsed time curves for all of the relaxation phases, respectively. The MIT-SR predicted effective stress paths converge toward the mean stress axis and tend to overestimate the measured changes in pore pressures (Figure 10A). The model is generally in very good agreement with the measured evolution of shear stresses in these experiments (Figure 10B).

3.4 Strain rate effects on cyclic shear test

The MIT-SR integrates a small strain hysteretic formulation together with an EVP framework. This combination intrinsically opens a potential for the proposed model to represent the rate-dependent characteristics of the hysteretic stress-strain response under cyclic shearing. The present section offers an attempt of using the MIT-SR model to illustratively simulate
the strain rate effects in a cyclic DSS test. This hypothetical experiment first consolidates the specimen under $K_0$ condition at an axial strain rate of 0.1%/h to a specified vertical stress level, $\sigma'_{vc}$, allows the specimen to creep for 24 hours, and then conducts cyclic shear under undrained DSS conditions at three different shear strain rates (0.5%, 5%, and 50%/h). The cyclic shear phase contains the shear stress level between $|\tau/\sigma'_{vc}| < 0.15$, and continues for 10 cycles or until failure.

Figures 11A to 11C plot the simulated shear stress, $\tau/\sigma'_{vc}$, versus shear strain, $\gamma$, curves for cyclic shear at three different strain rates, respectively. The proposed model clearly captures the typical hysteretic features observed during cyclic shearing, including the stiffer response upon each reversal in stress and succeeding degradation in stiffness as shear continues. In addition, the slower rates of shearing lead to larger accumulated shear strain due to the increase in viscoplastic shear strain within each cycle, and the slowest cyclic shear with 0.5%/h results in failure (uncontrolled development of shear strain) after five cycles.

Figures 11D to 11F plot the corresponding shear stress, $\tau/\sigma'_{vc}$, versus vertical effective stress, $\sigma'_v/\sigma'_{vc}$, curves. The predicted stress paths show the accumulation of pore pressure with each load cycle. The spacing between the paths in each consecutive load cycles shows changes in the development of excess pore pressure (converging toward shakedown behavior). It is also apparent that the predicted pore pressure accumulation depends on the strain rate. The slower shear process results in more excess pore pressure at a given cycle, and the slowest shear results in failure as stress path reaches the critical state line. The above prediction agrees qualitatively with the experimental observations\(^{53}\) and shows promising capabilities of the model for representing rate effects associated with cyclic shearing of clays.

4 SUMMARY

This paper presents a new EVP soil model, MIT-SR. The key features and implications for the proposed model are summarized as follows:
1. The proposed model uses a novel evolution equation for viscoplastic strain, which is generalized from the 1-D formulation presented by the authors. This new relation attributes the viscoplastic strain rate to an internal strain rate, \( R_a \), representing the perturbation of clay particle assembly due to historical straining. The evolution of the generalized \( R_a \) can be intrinsically linked to a plastic loading process that is calculated on the reference stress surface in accordance with the classical plasticity theory. This interpretation leads to a tangible means to develop new EVP soil models based on existing plastic formulations.

2. The MIT-SR model is built upon an advanced elastoplastic formulation, MIT-SI, and successfully retains its key components including (a) 3-D stress surfaces including a loading surface, a reference surface, and a critical state surface; (b) kinematic and isotropic hardening laws to allow the description of evolving anisotropy; and (c) hysteretic formulation to allow nonlinear variation in small-strain stiffness in both volumetric and deviatoric components.

3. The proposed model is versatile and can represent a wide range of rate-dependent characteristics for volumetric compression and shear behavior. The model predictions with its rate-sensitivity parameter \( \beta = \rho_a / \rho_c > 0 \) are qualitatively similar to the existing isotache models with rate-dependent normal consolidation lines and critical state shear strength. For \( \beta = 0 \), the MIT-SR predicts transient strain rate effects that generally affect the undrained shear strength, but rate-independent steady-state properties (including unique critical state and normal consolidation properties).

4. The MIT-SR model also incorporates the parameters \( \beta_2 \) and \( \beta_3 \) to allow in-depth representations for the nonlinearity of rate dependency and variation in creep rate after surcharge load.

5. In addition, the proposed model can offer comprehensive predictions of the primary and tertiary creep characteristics during undrained creep, and the temporal evolution in shear stress during undrained relaxation. The model is also promising for capturing the strain rate effects on the hysteretic response and the accumulation of excess pore pressure under cyclic shear.

**Figure 11** MIT-SR predictions for rate-effects on cyclic undrained direct simple shear test (A-C) shear stress-strain and (D-F) effective stress paths.
6. Lastly, the MIT-SR model is developed with multiple modules and can be simplified by deactivating specific parameters, e.g., due to the lack of data for calibration or for comparative study with other formulations. For instance, setting (a) $D$, $r$, and $w = 0$, to neglect nonlinear hysteretic feature; (b) $D_L = 0$ to remove dilation; and (c) $\beta = \rho_a/\rho_c$, $\beta_2 = 0$, and $\beta_3 \approx (\rho_c - \rho_r)/\rho_a$ to allow typical isotache-type predictions, the MIT-SR model is essentially reduced to an anisotropic isotache formulation similar to Refs. 21 and 22. Further assuming $\beta = \rho_a/\rho_c \approx 0$ can lead to a rate-independent elastoplastic prediction.

A companion paper will elaborate the calibration procedure and present a more detailed validation for the proposed model, 47 using the experimental data of Resedimented Boston Blue clay. 48

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DATA AVAILABILITY STATEMENT

The current research uses existing datasets, which are openly available in the literature as cited in the references section. The authors plan to make the code for the MIT-SR soil model available on GitHub.

Nomenclature

\begin{align*}
&\alpha(\phi'_{cs}) \quad \text{constant defining LCC spacing at different stress ratios} \\
&\dot{E}, \dot{E}^e, \dot{E}^{vp} \quad \text{deviatoric total, elastic and viscoplastic strain rate tensor} \\
&\dot{\varepsilon}, \dot{\varepsilon}^e, \dot{\varepsilon}^{vp} \quad \text{volumetric total, elastic, and viscoplastic strain rate} \\
&\dot{\varepsilon}_{dl} \quad \text{dilation strain rate} \\
&\dot{\varepsilon}_a \quad \text{axial strain rate} \\
&\dot{\varepsilon}_h, \dot{E}_h \quad \text{volumetric and deviatoric component of hysteretic strain rate} \\
&k, k_a(\phi'_{cs}) \quad \text{constants defining shape of Matsuoka-Nakai critical surface} \\
&Q, Q^e \quad \text{volumetric and deviatoric components for gradient of reference surface} \\
&\xi, \xi_s \quad \text{volumetric and deviatoric hysteretic state} \\
&\zeta(\phi'_{m}) \quad \text{constant defining shape of loading/reference surface} \\
&\alpha, \alpha_1 \quad \text{size of reference surface and loading surface} \\
&b \quad \text{orientation of reference/loading surface} \\
&c \quad \text{scalar defining the shape of loading surface} \\
&C_b \quad \text{dimensionless parameter controlling compressibility at maximum stiffness} \\
&D \quad \text{parameters control nonlinear swelling behavior} \\
&d_a \quad \text{constant that controls the spacing between } K_0 \text{ and Hydrostatic LCC} \\
&D_L \quad \text{dimensionless parameter for dilation behavior} \\
&e \quad \text{void ratio} \\
&f_A \quad \text{Activation function in evolution law of } R_a \\
&G, G^e \quad \text{shear modulus at current stress and image stress} \\
&G_{\max} \quad \text{maximum shear modulus} \\
&H \quad \text{plastic modulus} \\
&J_{30}(=\det|\eta|) \quad \text{third invariant of the deviatoric stress ratio} \\
&K, K^e \quad \text{bulk modulus at current stress and image stress} \\
&K_{\text{ONC}} \quad \text{lateral earth pressure coefficient at normally consolidated state} \\
&m_t \quad \text{transient coefficient} \\
&n \quad \text{porosity} \\
&P, P^e \quad \text{volumetric and deviatoric components of plastic flow direction} \\
&p_{\text{atm}} \quad \text{atmosphere pressure} \\
&q \quad \text{shear stress } (=\sigma'_v - \sigma'_h) \\
\end{align*}
$r$ parameter controlling nonlinearity of 1-D swelling behavior

$R_a$ internal strain rate

$r_x$ constant controlling kinematic hardening law

$S$ deviatoric stress tensor

$s_u$ undrained shear strength

$\mu$ shear stress in DSS test

$\phi^c_{cs}$ critical state friction angle

$\phi^m_{in}$ friction angle defining the shape of loading/reference surface

$\chi(K_{ONC})$ parameter ensuring $K_0$ straining

$\Psi$ kinematic hardening parameter

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APPENDIX I
Integration Algorithms for MIT-SR

Chart I.1 Modified Euler integration flowchart

START MODIFIED EULER INTEGRATION

GIVEN \( \dot{x}, \dot{\varepsilon} \) for time step, \( \Delta t_{m+1} \)
\( \sigma', \eta, \) and \( SV \) from time step, \( \Delta t_m \)

1st FORWARD EULER STEP

GIVEN \( \dot{x}, \dot{\varepsilon} \) for time step, \( \Delta t_{m+1} \)
\( \sigma_1', \eta_1, \) and \( SV_1 \), from 1st Euler step

2nd FORWARD EULER STEP

EVALUATE Error between two steps, \( R \)

UPDATE time step size for next step
\( \Delta t_{n+2} = q_t \cdot \Delta t_{n+1} \)
\( q_t = \frac{TOL}{R} \)

CHECK IF Error \( R < TOL \)

NO

UPDATE Stress \( \sigma', \eta, \) and \( SV \)

YES

OUTPUT \( \sigma_1', \eta_1, \) and \( SV_1 \), for 1st Euler step

OUTPUT \( \sigma_2', \eta_2, \) and \( SV_2 \), for 2nd Euler step

END MODIFIED EULER INTEGRATION
Chart I.2 Substep forward Euler integration flowchart

START FORWARD EULER STEP

GIVEN
\( \dot{\varepsilon}, \ddot{E} \) for time step, \( \Delta t_{n+1} \)
\( \sigma^*, \eta, \) and \( SV \) from step, \( \Delta t_n \)

FIND image stress \( \sigma^r \) and radial mapping ratio \( \sigma^r/\sigma = \alpha/\alpha' \)

CALCULATE hysteretic states \( \xi, \xi_g \)
based on \( \sigma^h, \eta^h \)

EVALUATE \( \rho, K^*, G^* \) at image point

EVALUATE
Surface gradient \( Q^f, Q^* \)
Plastic flow direction \( P, P' \)
Hardening modulus, \( H \)
Plastic loading condition, \( CL \)
Plastic multiplier \( \Lambda \) at image point

CALCULATE viscoplastic strain rate \( \dot{\varepsilon}^{vp}, \ddot{E}^{vp} \)

EVALUATE dilation strain rate \( \dot{\varepsilon}^{dil} \)

CALCULATE stresses increment \( \sigma^r, \dot{S} \)

CHECK
IF \( CL > 0 \)

NO
Unloading

UPDATE internal strain rate
\( \dot{R}_a = \sigma^r_1 \)

YES
Plastic loading

CALCULATE internal strain rate
\( \dot{R}_a = \Lambda \)

CALCULATE Hardening \( SV \)
\( \dot{\alpha}, \dot{b} = \Lambda \)

UPDATE Hysteretic state \( \sigma^h, \eta^h \)

END EULER STEP