WEAK TIME DISCRETIZATION FOR SLOW-FAST STOCHASTIC REACTION-DIFFUSION EQUATIONS

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Abstract. This paper derives a weak convergence theorem of the time discretization of the slow component for a two-time-scale stochastic evolutionary equations on interval [0,1]. Here the drift coefficient of the slow component is cubic with linear coupling between slow and fast components.

1. Introduction. Multiscale problems arising from material science, chemistry, climate dynamics, fluids dynamics and so on, have received a great deal of interest in recent years [2, 11, 12, 14, 17, 23]. With the development of the computer science, the computation for complex systems with multiple time scales is possible. Especially numerical analysis for stochastic differential equations attracted more and more attentions and different methods are developed [7, 10, 13]. Multiscale methods is effective and important for slow-fast systems, recent work [13] presented a time discretization scheme of averaging equation for stochastic slow-fast ordinary differential equations. Under bounded assumptions on coefficients, weak and strong convergence between the Euler approximating solution of the averaging equation and the solution of slow component are constructed. However such result for stochastic slow-fast partial differential equations are still rare. For this we intend to construct a time discretization approximation solution for the following slow-fast stochastic partial differential equations on interval $D = [0,1]$

$$\begin{align*}
\partial_t u^\epsilon(t,x) &= \partial_{xx} u^\epsilon(t,x) + f(u^\epsilon(t,x), v^\epsilon(t,x)) \\
\partial_t v^\epsilon(t,x) &= \frac{1}{\epsilon}[\partial_{xx} v^\epsilon(t,x) + g(u^\epsilon(t,x), v^\epsilon(t,x))] + \frac{1}{\sqrt{\epsilon}} \partial_t W(t) \\
u^\epsilon(0,x) &= u_0(x), \quad v^\epsilon(0,x) = v_0(x), \quad x \in D, \\
u^\epsilon(t,0) &= u^\epsilon(t,1) = 0, \quad v^\epsilon(t,0) = v^\epsilon(t,1) = 0, \quad t > 0.
\end{align*}$$

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Here nonlinearity \( f(u,v) = u - u^3 + uv \), \( \{W(t) : t \geq 0\} \) is some \( L^2(D) \)-valued Wiener process and \( g(\cdot, \cdot) \) is a Lipschitz nonlinearity which are detailed in next section.

For small \( \epsilon > 0 \), a direct computation for the slow component is difficult and expensive due to the singularity \( \epsilon^{-1} \). So simple and effective models are desirable. Averaging principle is a useful method to reduce the slow-fast system to a simple model without fast component, which avoids the singularity \( \epsilon^{-1} \), provide the fast component satisfies Krylov–Bogoliubov–Mitropolsky (KBM) condition [20]. In fact under the KBM condition, \( u^\epsilon \) converges to \( \bar{u} \) in some sense with [25]

\[
\begin{align*}
\partial_t \bar{u}(t,x) &= \partial_{xx} \bar{u}(t,x) + \bar{f}(\bar{u}(t,x)), \\
\bar{u}(0,x) &= u_0(x), \quad \bar{u}(t,0) = \bar{u}(t,1) = 0,
\end{align*}
\]

Here

\[
\bar{f}(u) = \int_{L^2(D)} f(u,v) \mu^u(dv)
\]

and \( \mu^u \) is the unique stationary measure for fast component with frozen slow component \( u \).

Averaging principle plays a vital role for qualitative analysis of dynamical systems with two time scales and has a long history and numerous results. Krylov and Bogoliubov [19] presented a rigorous mathematical justification to averaging method for deterministic dynamical system which is now called Krylov–Bogoliubov method. There are several versions of averaging principle based on Krylov and Bogoliubov’s work for ordinary differential equation [21, 24]. The averaging result for stochastic differential equation of Itô-type in finite dimensions was firstly established by Khasminskii [18]. Cerrai and Freidlin [6] and Cerrai [5] proved the averaging principle for a general class of stochastic reaction-diffusion equation, which extended the classical Khasminskii-type averaging principle for finite systems to infinite dimensional systems. Further extensions to stochastic parabolic equations with non-Gaussian stable noise are also active research area [1]. For more related, interesting results on averaging principle we refer to [16, 25, 26, 27].

Although the averaged model is a simpler system, the averaged term \( \bar{f} \) is difficult to be calculated explicitly. So an acceptable approximation solution which is useful in computation is desirable. Here we aim to construct a time discretization solution \( u_n \) (section 3) by an Euler scheme and Mont–Carlo method to approximate the averaged term \( \bar{f} \). In fact we present the error estimates in a weak sense, that is for any \( \phi \in C_0^\infty(H) \) and \( T > 0 \), to estimate the deviation (Theorem 3.1)

\[
|\mathbb{E}\phi(u^\epsilon(T)) - \mathbb{E}\phi(u_n)|.
\]

To do this, we split the estimate into three parts

\[
|\mathbb{E}\phi(u^\epsilon(T)) - \mathbb{E}\phi(u_n)| \leq |\mathbb{E}\phi(u^\epsilon(T)) - \phi(\bar{u}(T))| \\
+ |\phi(\bar{u}(T)) - \phi(\bar{u}_n)| \\
+ |\phi(\bar{u}_n) - \mathbb{E}\phi(u_n)|,
\]

where \( \bar{u}_n \) is the Euler scheme approximation for \( \bar{u}(t) \) which is the solution to the averaging equation (5). For the first part of the estimate (Theorem 5.1) we follow the classical asymptotic expansion method which has been applied to weak order of an averaging principle for some slow-fast stochastic reaction-diffusion equation [3, 15, e.g.] under Lipschitz continuity assumption of the drift coefficient. We adapt the
method to system (1)–(4). More precisely, we decompose $\mathbb{E}\varphi(u^e(t))$ with respect to the scale parameter $\epsilon$ in form

$$
\mathbb{E}\varphi(u^e(t)) = X_0 + \epsilon X_1 + Y^\epsilon,
$$

where the functions $X_0, X_1, Y^\epsilon$ are determined recursively and comply some linear evolution equations. On the one hand, we identify leading term $X_0$ with $\mathbb{E}\varphi(\bar{u}(t))$ by an uniqueness argument. For this purpose, we present the Kolmogorov operators with parameter to construct an evolution equation that describes $X_0$ and $\mathbb{E}\varphi(\bar{u}(t))$. On the other hand, with the above argument, Gronwall inequality and a Poisson equation associated with the generator of fast component, an explicit expression of $X_1$ is derived. As a consequence, we obtain the boundedness estimate of $X_1$ by some a priori estimates. The next key step is to estimate the remainder term $Y^\epsilon$ described by a linear equation consisting of $L_2 X_1$ and $\frac{\partial X_1}{\partial t}$, where $L_2$ is the Kolmogorov operator for the slow equation with frozen fast component. After obtaining the boundedness of the terms $L_2 X_1$ and $\frac{\partial X_1}{\partial t}$, the remainder $Y^\epsilon$ in the expansion can be estimated by standard evolution equation method. The second part is the time discretization approximation of deterministic equations (see (27)). It has the same result as the ordinary differential equation [13, e.g. Theorem 2.4]. For the last part (Theorem 4.5), we adapt the arguments [13] of slow-fast stochastic ordinary differential equations (SODEs) to the slow-fast stochastic partial differential equation (SPDEs).

The rest of the paper is organized as follows. Section 2 is devoted to the general notations and assumptions. The ergodicity of fast process and the main results are given in section 3. Some a priori estimates to the solutions of slow and fast components are presented in section 4. We give an asymptotic expansion scheme and the proof of the main result in section 5.

2. Assumptions and notations. Let $H = L^2(D)$ with the usual scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Denote by $\mathcal{L}(H)$ the Banach space of linear and bounded operators from $H$ to itself, equipped with usual operator norm. $\mathcal{L}_1(H)$ denotes the Banach space of trace-class operators, endowed with the norm

$$
\|A\|_1 = \text{Tr}[\sqrt{AA^*}],
$$

and $\mathcal{L}_2(H)$ denotes the Hilbert space of Hilbert-schmidt operators on $H$, endowed with the scalar product

$$
\langle A, B \rangle_2 = \text{Tr}[AB^*]
$$

and the corresponding norm $\|A\|_2 = \sqrt{\text{Tr}[AA^*]}$.

We use test functions $\phi$ in the space $C^2_c(H)$ of functions from $H$ to $\mathbb{R}$ that are twice continuously differentiable, bounded, with first and second order bounded derivatives.

In the sequel, we identify the first derivative $D\phi(x) \in \mathcal{L}(H)$ with the gradient in the Hilbert space $H$, and the second derivative $D^2\phi(x)$ with a linear operator on $H$

$$
\langle D\phi(x), h \rangle = D\phi(x) \cdot h, \quad \text{for each } h \in H,
$$

$$
\langle D^2\phi(x) \cdot h, k \rangle = D^2\phi(x) \cdot (h, k), \quad \text{for each } h, k \in H.
$$

Let $A = \Delta$ on $D$ with zero Dirichlet boundary condition, then there are $\{e_k(x) \}_{k \geq 1}$ a complete orthonormal system of in $H$ such that, for $k = 1, 2, \ldots$

$$
Ae_k(x) = -\lambda_k e_k(x), \quad e_k(0) = e_k(1) = 0,
$$

Ae_k(x) = -\lambda_k e_k(x), \quad e_k(0) = e_k(1) = 0,
with \(0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots\). Denote by \(\{e^{At}\}_{t \geq 0}\) the strongly continuous semigroup generated by \(A\) on \(H\), then
\[
e^{At}h = \sum_{k \in \mathbb{N}} e^{-\lambda_k t} \langle e_k, h \rangle e_k.
\]

For \(\alpha \in [0, 1]\) we defined the operator \((-A)\alpha\) by
\[
(-A)\alpha x = \sum_{k \in \mathbb{N}} \lambda_k^{2\alpha} x_k e_k
\]
with domain
\[
D((-A)\alpha) = \left\{ x = \sum_{k \in \mathbb{N}} x_k e_k \in H : \|x\|_{(-A)\alpha} := \sum_{k \in \mathbb{N}} \lambda_k^{2\alpha} x_k^2 < \infty \right\}.
\]

Recalling that \(U := H_0^1(D)\) is continuously and densely embedded in \(L^2(D)\), consider the Gelfand triple: \(U \subset L^2 \subset U^*\), where \(U^*\) is the dual space of \(U\). According to the Poincaré inequality, for every \(x \in U\),
\[
u^* \langle Ax, x \rangle_U = -\|\nabla x\|^2 \leq -\lambda_1 \|x\|^2. \tag{7}
\]

Concerning the drift coefficients \(g\), we impose the following assumptions
\((H_1)\). Nonlinear function \(g(\cdot, \cdot)\) is Lipschitz continuous with respect to two variables with Lipschitz constant \(L_g\) satisfying \(-\lambda_1 + L_g < 0\) and \(g \in C^2_b(H)\). Moreover, we assume that \(l_g = \sup_{v \in H} \|g_n(u, v)\|\) satisfying \(\lambda_1 - L_g - l_g > 0\).

\((H_2)\). Wiener process \(\{W(t) : t \geq 0\}\) is \(L^2(D)\)-valued with covariance operator \(Q\) satisfying \(\text{Tr}(\langle -A \rangle Q) < \infty\). Now the system \((1)-(4)\) is in the following abstract form
\[
du^\varepsilon(t) = \left[ A u^\varepsilon(t) + u^\varepsilon(t) - (u^\varepsilon(t))^3 + u^\varepsilon(t)v^\varepsilon(t) \right] dt
\]
\[
dv^\varepsilon(t) = \frac{1}{\varepsilon} \left[ A v^\varepsilon(t) + g(u^\varepsilon(t), v^\varepsilon(t)) \right] dt + \frac{1}{\sqrt{\varepsilon}} dW(t) \tag{8}
\]
\[
u^\varepsilon(0) = u_0, \quad v^\varepsilon(0) = v_0.
\]

Next let’s consider the Galerkin approximation system for equation \((8)\). Therefore, subsequently we actually work with the Galerkin approximations. Let \(H_n = \text{span}\{e_1, \ldots, e_n\}\), \(P_n : H \to H_n\) is orthogonal projection operator, and \(P_nv = \sum_{i=1}^n \langle v, e_i \rangle e_i, v \in H\). Then the Glkerkin approximation system is
\[
du_n^{\varepsilon}(t) = \left[ A_n u_n^{\varepsilon}(t) + u_n^{\varepsilon}(t) - (u_n^{\varepsilon}(t))^3 + u_n^{\varepsilon}(t)v_n^{\varepsilon}(t) \right] dt \tag{9}
\]
\[
dv_n^{\varepsilon}(t) = \frac{1}{\varepsilon} \left[ A_n v_n^{\varepsilon}(t) + g_n(u_n^{\varepsilon}(t), v_n^{\varepsilon}(t)) \right] dt + \frac{1}{\sqrt{\varepsilon}} dW_n(t) \tag{10}
\]
\[
u_n^{\varepsilon}(0) = u_n(0), \quad v_n^{\varepsilon}(0) = v_n(0) \tag{11}
\]
where \(A_n = P_n A\) is bounded linear operator on \(H_n\), \(g_n = P_ng, u_n(0) = P_n u_0, v_n(0) = P_n v_0, W_n\) is a wiener process with the covariance \(Q_n = P_n Q P_n\). It is well known that Galerkin approximations \(u_n^{\varepsilon}(t)\) and \(v_n^{\varepsilon}(t)\) both mean square converge to \(u(t)\) and \(v(t)\) respectively. Let us replace \(u^\varepsilon(0), v^\varepsilon(0), u^\varepsilon, v^\varepsilon, A, g\) and \(H\) with \(u_n^{\varepsilon}(0), v_n^{\varepsilon}(0), u_n^{\varepsilon}, v_n^{\varepsilon}, A_n, g_n\) and \(H_n\). For notation simplicity, we omit its dependency on \(n\). Finally, the following estimates are all still remain true for Galerkin approximations and the constants \(C\) don’t depend on dimension \(n\) and differ from line to line. We also write the dependence of constant on parameters explicitly if it is necessary.
3. **Averaging dynamics and numerical scheme.** For fixed \( u \in H \) consider the problem associate to fast motion with frozen slow component

\[
dw(t) = Au(t)dt + g(u, v(t))dt + dW(t),
\]

\[
v^0(0) = v_0.
\]

Notice that the drift \( g : H \times H \to H \) is Lipschitz continuous. For fixed slow component \( u \in H \) and any initial data \( v_0 \in H \), equation (12) has a unique mild solution denoted by \( v^u(t, v_0) \). Now, we consider the transition semigroup \( P^u_t \) associated with process \( v^u(t, v_0) \), by setting for any \( \psi \in \mathcal{B}_b(H) \) the space of bounded measurable functions on \( H \),

\[
P^u_t \psi(v_0) = \mathbb{E}(\psi(v^u(t, v_0))).
\]

A similar approach \([6]\) yields

\[
\mathbb{E}\|v^u(t, v_0)\|^2 \leq C(e^{-(\lambda_1 - L_o)t}\|v_0\|^2 + \|u\|^2 + 1), \quad t > 0,
\]

where \( C > 0 \) is a constant. This implies the existence of an invariant measure \( \mu^u \) for the Markov semigroup \( P^u_t \) associated with equation (12) on \( H \) such that

\[
\int_H P^u_t \psi d\mu^u = \int_H \psi d\mu^u, \quad t \geq 0.
\]

Moreover,

\[
\int_H \|\psi\|^2 \mu^u(dy) \leq C(1 + \|u\|^2).
\]

On the other hand, by \((H_1)\), for any \( t \geq 0 \)

\[
\mathbb{E}\|v^u(t, v_0) - v^u(t, v'_0)\|^2 \leq C\|v_0 - v'_0\|^2 e^{-\beta t},
\]

where \( \beta = (\lambda_1 - L_o) > 0 \) which implies \( \mu^u \) is the unique invariant measure of \( P^u_t \).

Then we can define a \( H \)-valued mapping \( \bar{f} \)

\[
\bar{f}(u) = \int_H f(u, v)\mu^u(dv), \quad u \in H,
\]

and the averaging equation

\[
d\bar{u}(t) = [A\bar{u}(t) + \bar{f}(\bar{u}(t))]dt,
\]

where \( \bar{f}(\bar{u}(t)) = \bar{u}(t) - (\bar{u}(t))^3 + \bar{u}(t)\mathbb{E}e^{\nu u} \) and \( \nu^u \) denotes the stationary solution to the fast equation for fixed slow component \( u \), whose distribution is \( \mu^u \).

According to the invariance of \( \mu^u \), we have

\[
\|\mathbb{E}f(u, v^u(t, v_0)) - \bar{f}(u)\|^2 = \left\| \int_H \left( \mathbb{E}f(u, v^u(t, v_0)) - \mathbb{E}f(u, v^u(t, z)) \right)\mu^u(dz) \right\|^2
\]

\[
\leq \|u\|^2 \int_H \mathbb{E}\|v^u(t, v_0) - v^u(t, z)\|^2 \mu^u(dz)
\]

\[
\leq \|u\|^2 \int_H C\|v_0 - z\|^2 e^{-\beta t} \mu^u(dz)
\]

\[
\leq Ce^{-\beta t}(1 + \|u\|^4_{H^1} + \|v_0\|^4),
\]

which implies that

\[
\bar{f}(u) = \lim_{t \to +\infty} \mathbb{E}f(u, v^u(t, v_0)), \quad u \in H.
\]
Introduce the implicit Euler scheme of the equation (15)

\[ u_{n+1} = S_{\Delta t} u_n + \Delta t S_{\Delta t} \tilde{f}(u_n), \]  

(18)

where \( \Delta t \) is macrotime step, \( S_{\Delta t} = (I - A \Delta t)^{-1} \). Here and after, \( n \) is the number of macrotime steps. At each macroscopic time step \( n \), (18) has a numerical solution \( u_n \), in order to move to step \( n+1 \), we need estimate \( \tilde{f}(u_n) \), so we use the Monte–Carlo approximation \( f(u_n) \) to \( \tilde{f}(u_n) \)

\[ \tilde{f}(u_n) = \frac{1}{MN} \sum_{j=1}^{M} \sum_{m=nT}^{nT+N-1} f(u_n, v_{n,m,j}), \]  

(19)

where \( M \) is Monte–Carlo step, that is the number of replicas, \( N \) is the number of steps in the time averaging, and \( nT \) is the number of steps we skip to eliminate transients. \( v_{n,m,j} \) satisfies the following Euler scheme

\[ v_{n,m+1,j} = S_{\Delta t}^{*} v_{n,m,j} + \Delta t \frac{1}{\epsilon} S_{\Delta t}^{*} g(u_n, v_{n,m,j}) + \sqrt{\Delta t} \frac{1}{\epsilon} S_{\Delta t}^{*} \sqrt{Q} \chi_{m+1,j}, \]

(20)

\[ v_{n,0,j} = v_{n-1,nT+N-1}, \quad v_{0,0,j} = 0, \]

where \( S_{\Delta t}^{*} = (I - A \Delta t)^{-1}, \chi_{m+1,j} = \frac{1}{\sqrt{\Delta t}} [W^j((m+1)\Delta t) - W^j(m\Delta t)], W^j(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k(x), \{ \beta_k \}_{k,j} \) is independent identically distributed standard Brownian motion. Then we derive approximation numerical scheme of the equation (18)

\[ u_{n+1} = S_{\Delta t} u_n + \Delta t S_{\Delta t} \tilde{f}(u_n). \]

(21)

Now we give the main result of the paper.

**Theorem 3.1.** Assume that \( u_0 \in H^2, v_0 \in H_0^1 \). Under the assumptions (H1) and (H2), for any \( T > 0, \phi \in C^2_0 \), there exists a constant \( C > 0 \) such that

\[
\| E\phi(u')(T) - E\phi(u_n) \| \\
\leq \sqrt{\epsilon} C_{T,\phi} (1 + \| u_0 \|_{H_0^1}^{32} + \| v_0 \|_{H_0^1}^{32} + \| Au_0 \|^4) + C_{T,\phi} \frac{1 + \lambda_1 \Delta t}{\lambda_1} \sqrt{1 + \frac{1}{N}} \\
\left[ \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + \Delta t^k + e^{-\frac{1}{2} nT \Delta t} \left( e^{-\frac{1}{2} (n-1)N_m \Delta t} + R + \Delta t \right) \right]
\]

where

\[ R = \frac{\Delta t}{1 - e^{-\frac{1}{2} \beta N_m \Delta t}}. \]

4. Some a priori estimates. This section gives the moment estimate of slow-fast systems (9)–(10). Let \( u^c \) and \( v^c \) be the solution to the slow system and the fast system respectively. For notation simplicity, we omit the space variable \( x \). Now we present two lemma which is useful in the following part.

**Lemma 4.1.** [8] Let \( u(t) \) and \( k(t) \) be positive continuous functions on interval \([c, d]\) and \( a, b \) be nonnegative constants. Further, let \( g(z) \) be a positive nondecreasing function for \( z \geq 0 \). If

\[ u(t) \leq a + b \int_c^t k(s) g(u(s)) ds, \quad t \in [c, d], \]

then

\[ u(t) \leq G^{-1}(G(a) + b \int_c^t k(s) ds), \quad c \leq t \leq d, \]
where
\[ G(\lambda) = \int_{\xi}^{\lambda} \frac{ds}{g(s)} \quad (\xi > 0, \lambda > 0) \]
and \( d_1 \) is defined such that
\[ G(a) + b \int_{c}^{t} k(s)ds \]
belongs to the domain of \( G^{-1} \) for \( t \in [c, d_1] \).

**Lemma 4.2.** [6] There exists \( \gamma < 1 \), such that
\[ \sum_{k=1}^{\infty} e^{-\lambda k t} \leq C(t \wedge 1)^{-\gamma} e^{-\lambda t}, \quad t \geq 0. \]
In particular,
\[ \|e^{tA}\|_{L^2(\Omega)} \leq C(t \wedge 1)^{-\gamma} e^{-\lambda t}, \quad t \geq 0. \]

**Lemma 4.3.** For any \( u_0, v_0 \in H \), and \( m \geq 1, T > 0, \epsilon > 0 \),
\[ \sup_{\epsilon} \sup_{t \in [0, T]} \mathbb{E} \|u^\epsilon(t)\|^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}), \]
\[ \sup_{\epsilon} \sup_{t \in [0, T]} \mathbb{E} \|u^\epsilon(t)\|^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}), \]
where \( C_{m,T} > 0 \) is a constant that only depends on \( m \) and \( T \).

**Proof.** Let \( F(u) = \|u\|^{2m} \), direct calculation yields
\[ F_u(u_0) \cdot h = 2m\|u_0\|^{2m-2}(u_0, h), \]
\[ F_{uu}(u_0) \cdot (h, k) = 2m\|u_0\|^{2m-2}(h, k) + 4m(m-1)(u_0, h)(u_0, k), \]
then by (7), Young’s inequality and Gronwall’s inequality, we have
\[ \|u^\epsilon(t)\|^{2m} \leq e^{-m\lambda_1 t}\|u_0\|^{2m} + C_m \int_0^t e^{-m\lambda_1(t-s)}(1 + \|v^\epsilon(s)\|^{2m})ds. \] (22)

For any \( \epsilon > 0 \), let
\[ W^{\epsilon,A}(t) = \frac{1}{\sqrt{\epsilon}} \int_0^t e^{\frac{A(t-s)}{\epsilon}} dW(s), \]
and make a change of variable \( \frac{t}{\epsilon} = \rho \), by Lemma 4.2
\[ \sup_{\epsilon > 0} \sup_{t \geq 0} \mathbb{E} \|W^{\epsilon,A}(t)\|^{2m} < \infty. \] (23)

Now define \( \rho^\epsilon(t) = v^\epsilon(t) - W^{\epsilon,A}(t) \), by (7), \((H_1)\) and Young’s inequality,
\[ \frac{d\|\rho^\epsilon(t)\|^{2m}}{dt} \leq -m(\lambda_1 - L_g)\|\rho^\epsilon(t)\|^{2m} \]
\[ + \frac{C_m}{\epsilon}(1 + \|u^\epsilon(t)\|^{2m} + \|W^{\epsilon,A}(t)\|^{2m}), \]
which yields
\[ \|\rho^\epsilon(t)\|^{2m} \leq e^{-\frac{m(\lambda_1 - L_g)}{\epsilon} t}\|v_0\|^{2m} + \frac{C_m}{\epsilon} \int_0^t e^{\frac{m(\lambda_1 - L_g)(t-s)}{\epsilon}}(1 + \|u^\epsilon(s)\|^{2m} + \|W^{\epsilon,A}(s)\|^{2m})ds. \]
Further, by (22), a simple calculation yields
\[
\|v^\epsilon(t)\|^{2m} \leq 2^{2m-1}\|W_{\epsilon,A}(t)\|^{2m} + C_m(1 + \|u_0\|^{2m} + \|v_0\|^{2m}) \\
+ \frac{C_m}{\epsilon} \int_0^t e^{-\frac{m(\lambda_1 - \lambda_2)(t-s)}{2}} \int_0^s \|v^\epsilon(r)\|^{2m} dr ds \\
+ \frac{C_m}{\epsilon} \int_0^t e^{-\frac{m(\lambda_1 - \lambda_2)(t-s)}{2}} \|W_{\epsilon,A}(s)\|^{2m} ds,
\]
together with (23), exchanging integrals and setting \(s = \frac{t-s}{\epsilon}\), we have
\[
\mathbb{E}\|v^\epsilon(t)\|^{2m} \leq C_m(1 + \|u_0\|^{2m} + \|v_0\|^{2m}) + C_m \int_0^t \mathbb{E}\|v^\epsilon(r)\|^{2m} dr.
\]
Then Gronwall’s inequality yields
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|v^\epsilon(t)\|^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}). \tag{24}
\]
By (22) again,
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|v^\epsilon(t)\|^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}),
\]
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|u^\epsilon(t)\|^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}).
\]

**Lemma 4.4.** For any \(u_0 \in H_0^1, v_0 \in H\) and \(T > 0, \epsilon > 0\),
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|v^\epsilon(t)\|_{H_0^m}^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}),
\]
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|u^\epsilon(t)\|_{H_0^m}^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}),
\]
where \(C_{m,T} > 0\) is a constant that only depends on \(m\) and \(T\).

**Proof.** By Itô’s formula, (7), (H₁) and Young’s inequality, we have
\[
\frac{d}{dt}\|v^\epsilon(t)\|_{H_0^m}^{2m} \leq -\frac{\lambda_1 m}{\epsilon}\|\partial_x v^\epsilon(t)\|^{2m} \\
+ \frac{C_m}{\epsilon}\|\partial_x v^\epsilon(t)\|^{2m-2}(1 + \|u^\epsilon(t)\|^{2} + \|v^\epsilon(t)\|^{2}) \\
+ \frac{2m\|\partial_x v^\epsilon(t)\|^{2m-2}}{\sqrt{\epsilon}}(\langle (-A)v^\epsilon(t), dW(t) \rangle) \\
+ \frac{m(2m-1)\|\partial_x v^\epsilon(t)\|^{2m-2}}{\epsilon} \text{Tr}(-A)Q,
\]
then Young’s inequality, Gronwall’s inequality andLemma 4.3 yield
\[
\mathbb{E}\|v^\epsilon(t)\|_{H_0^m}^{2m} \leq C_{m,T}(1 + \|u_0\|^{2m} + \|v_0\|^{2m}).
\]
Next we give an estimate of \(\|u^\epsilon(t)\|_{H_0^m}^{2m}\) by that of \(\|v^\epsilon(t)\|_{H_0^m}^{2m}\).
By (7), Young’s inequality and Gronwall’s inequality, we have
\[ \|u^r(t)\|_{H_0^1}^{2m} \leq e^{C_{m,t}}\|u_0\|_{H_0^1}^{2m} + C_m \int_0^t e^{C_{m,(t-s)}}\|u^r(s)\|_{H_0^1}^{4m} ds \]
\[ + C_{m,T} + C_m \int_0^t e^{C_{m,(t-s)}}\|u^r(s)\|_{H_0^1}^{4m} ds. \]

Let
\[ \beta_t(\omega) = e^{C_{m,t}}\|u_0\|_{H_0^1}^{2m} + C_{m,T} + C_m \int_0^t e^{C_{m,(t-s)}}\|u^r(s)\|_{H_0^1}^{4m} ds. \]

By (24), we have
\[ E\beta_T(\omega) \leq e^{C_{m,T}}\|u_0\|_{H_0^1}^{2m} + C_{m,T} + C_m (1 + \|u_0\|^{4m} + \|v_0\|^{4m}), \]
so
\[ \|u^r(t)\|_{H_0^1}^{2m} \leq \beta_T(\omega) + C_m \int_0^t e^{C_{m,(t-s)}}\|u^r(s)\|_{H_0^1}^{4m} ds. \]

In the Lemma 4.1, let
\[ u(t) = \|u^r(t)\|_{H_0^1}^{2m}, a = \beta_T(\omega), b = C_m, k(s) = e^{C_{m,(t-s)}}, g(s) = s^2, \]
then
\[ u(t) \leq a + b \int_0^t k(s)u^2(s) ds, \quad t \in [0, T]. \]

Now Lemma 4.1 yields
\[ u(t) \leq G^{-1}(G(a) + b \int_0^t e^{C_{m,(t-s)}} ds), \quad 0 \leq t \leq t_1 \leq T, \]
where
\[ G(\lambda) = \int_\xi^\lambda \frac{ds}{g(s)} = \frac{1}{\xi} - \frac{1}{\lambda} \quad (\xi > 0, \lambda > 0), \]
\[ G^{-1}(\lambda) = \frac{\lambda}{1 - \frac{\lambda}{\xi}}, \quad \lambda \in \text{Ran}G = (-\infty, \frac{1}{\xi}), \]
and \( t_1 \) is defined as follows
\[ t_1 = \frac{1}{C_m} \log \left[ \frac{1}{\beta_T(\omega)} + 1 \right], \]
such that
\[ G(a) + b \int_0^t k(s) ds \in (-\infty, \frac{1}{\xi}). \]

By inequality
\[ \frac{1}{\xi} - \frac{1}{\beta_T(\omega)} + e^{C_{m,t}} - 1 < \frac{1}{\xi}, \]
we obtain
\[ t \leq \frac{1}{C_m} \log \left[ \frac{1}{\beta_T(\omega)} + 1 \right]. \]

Then
\[ u(t) \leq \frac{\xi}{1 - \xi \left[ \frac{1}{\beta_T(\omega)} + e^{C_{m,t}} + 1 \right]} \leq 2\beta_T(\omega), \quad t \in [0, t_1]. \]

For \( t_1 \leq t \leq t_2, \)
\[ u(t) \leq G^{-1}(G(a) + b \int_{t_1}^t e^{C_{m,(t-s)}} ds), \quad t_1 \leq t \leq t_2 \leq T, \]
$t_2$ is defined as follows
\[ t_2 = \frac{2}{C_m} \log \left[ \frac{1}{2\beta_T(\omega)} + 1 \right], \]
such that
\[ G(a) + b \int_{t_1}^t e^{C_m(t-s)} ds \in \text{Ran}G. \]
By the following inequality
\[ \frac{1}{\xi} - \frac{1}{\beta_T(\omega)} + e^{C_m(t-t_1)} - 1 < \frac{1}{\xi}, \]
we obtain
\[ t < \frac{1}{C_m} \log \left[ \left( \frac{1}{\beta_T(\omega)} + 1 \right) \left( \frac{1}{2\beta_T(\omega)} + 1 \right) \right], \]
then
\[ u(t) \leq \frac{\xi}{1 - \xi \left( \frac{1}{\xi} - \frac{1}{\beta_T(\omega)} + e^{C_m(t-t_1)} - 1 \right) \leq 2\beta_T(\omega), \quad t \in [t_1, t_2]. \]
For $t_2 \leq t \leq t_3$,
\[ u(t) \leq G^{-1}(G(a) + b \int_{t_2}^t e^{C_m(t-s)} ds), \quad t_2 \leq t \leq t_3 \leq T, \]
$t_3$ is defined as follows
\[ t_3 = \frac{3}{C_m} \log \left[ \frac{1}{2\beta_T(\omega)} + 1 \right], \]
such that
\[ G(a) + b \int_{t_2}^t e^{C_m(t-s)} ds \in \text{Ran}G, \]
thanks to the following inequality,
\[ \frac{1}{\xi} - \frac{1}{\beta_T(\omega)} + e^{C_m(t-t_2)} - 1 < \frac{1}{\xi}, \]
we obtain
\[ t < \frac{1}{C_m} \log \left[ \frac{1}{\beta_T(\omega)} + 1 \right] + t_2 = \frac{1}{C_m} \log \left[ \left( \frac{1}{\beta_T(\omega)} + 1 \right) \left( \frac{1}{2\beta_T(\omega)} + 1 \right) \right], \]
then
\[ u(t) \leq \frac{\xi}{1 - \xi \left( \frac{1}{\xi} - \frac{1}{\beta_T(\omega)} + e^{C_m(t-t_2)} - 1 \right) \leq 2\beta_T(\omega), \quad t \in [t_2, t_3]. \]
Repeating the above steps $n$ times, there is $n(\omega)$ such that for any $n \geq n(\omega)$
\[ T < \frac{n}{C_m} \log \left[ \frac{1}{2\beta_T(\omega)} + 1 \right], \]
and
\[ u(t) \leq 2\beta_T(\omega), \quad t \in [0, T]. \]
Therefore
\[ E \sup_{0 \leq t \leq T} \| u^t(t) \|_{H_0^m}^{2m} \leq 2E\beta_T(\omega) \leq C_m (1 + \| u_0 \|_{H_0^m}^{4m} + \| v_0 \|_{H_0^m}^{4m}). \]
Now we state the discretization approximation of the solution of the averaging equation (see section 3).

**Theorem 4.5.** Under the assumptions (H₁) and (H₂), for any \( \phi \in C^\infty_0(H) \) and \( T > 0 \), there exists a constant \( C > 0 \) independent of \( (\epsilon, \Delta t, n_T, M, N) \), such that

\[
\left| \mathbb{E}(\phi(u_n)) - \phi(\bar{u}_n) \right| \leq C_T \phi \frac{1 + \lambda_1 \Delta t}{\lambda_1} \sqrt{\frac{1}{N} \left[ \frac{(\Delta t)^{1/2}}{\epsilon} + e^{-\frac{1}{2} \beta(n-1)N_m \Delta t} (e^{-\frac{1}{2} \beta(n-1)N_m \Delta t} + R + \Delta t) \right]}. 
\]

**Proof.** First we define an auxiliary function \( u(k, x) \) for \( k \leq n \)

\[
u(n, x) = \phi(x), \quad u(k, x) = u(k + 1, S_{\Delta t}x + \Delta t \bar{f}(x)),
\]

then \( u(0, x) = \phi(\bar{u}_n) \). Now, for \( k > 0 \), introduce a new probability space \( (\Omega_k, \mathcal{F}_k, \mathbb{P}_k) \), where

\[
\Omega_k = \{ \omega \in \Omega : \| u_n(\omega) \|_{H^1_0} \leq K \},
\]

\[
\mathcal{F}_k = \{ S \cap \Omega_k : S \in \mathcal{F} \},
\]

\[
\mathbb{P}_k = \frac{\mathbb{P}(S \cap \Omega_k)}{\mathbb{P}(\Omega_k)}, \quad \text{for any } S \in \mathcal{F}.
\]

\( \mathbb{E}_k \) denotes the expectation with respect to \( \mathbb{P}_k \). Let \( \mathbb{P}(\Omega_k) > 1 - \epsilon_k \), where \( \epsilon_k \to 0 \), \( k \to \infty \). Now, we limit \( \omega \in \Omega_k \). Then the solution of (18) is uniformly bounded in a compact set. By the smoothness of \( \bar{f} \) and (71), one can check that

\[
\sup_{k, x} \{ |\partial_x u(k, x)| + |\partial_x^2 u(k, x)| \} \leq C
\]

is uniformly bounded for \( \Delta t > 0 \). Hence we have

\[
|\mathbb{E}(u(k + 1, u_{k+1}) - u(k, u_k))| = |\mathbb{E}(u(k + 1, S_{\Delta t}u_k + \Delta t \bar{f}(u_k)) - u(k + 1, S_{\Delta t}u_k + \Delta t \bar{f}(u_k)))|
\leq C \Delta t \| u \|_{H^1_0} \| \mathbb{E}[\partial_x u(k + 1, S_{\Delta t}u_k + \Delta t \bar{f}(u_k)) \cdot (\mathbb{E}_u \bar{f}(u_k) - \bar{f}(u_k))]
\leq \frac{C \Delta t}{1 + \lambda_1 \Delta t} \mathbb{E}[\| u \|_{H^1_0} \| \mathbb{E}_u \bar{f}(u_k) - \bar{f}(u_k) \| ^2],
\]

and

\[
|\mathbb{E}\phi(u_n) - \phi(\bar{u}_n)| = |\mathbb{E}u(n, u_n) - u(0, x)|
\leq \sum_{0 \leq k \leq n-1} \mathbb{E}[u(k + 1, u_{k+1}) - u(k, u_k)]
\leq \sum_{0 \leq k \leq n-1} \left[ \frac{C \Delta t}{1 + \lambda_1 \Delta t} \mathbb{E}[\| u \|_{H^1_0} \| \mathbb{E}_u \bar{f}(u_k) - \bar{f}(u_k) \| ^2] + \frac{C \Delta t}{1 + \lambda_1 \Delta t} \mathbb{E}[\| \bar{f}(u_k) - \bar{f}(u_k) \| ^2] \right].
\]
Now we estimate \( \mathbb{E}\|\mathbb{E}_{u_n} f(u_n) - \bar{f}(u_n)\| \) and \( \mathbb{E}\|\bar{f}(u_n) - \bar{f}(u_n)\| \). Note that

\[
\mathbb{E}\|\mathbb{E}_{u_n} f(u_n) - \bar{f}(u_n)\| = \frac{1}{MN} \mathbb{E}\|u_n \sum_{j=1}^{M} \sum_{m=n_T}^{n_T+N-1} (\mathbb{E}_{u_n} v_{n,m,j} - \int_{H} y \mu_{u_n}^\epsilon(dy))\| \leq \frac{K}{MN} \sum_{j=1}^{M} \sum_{m=n_T}^{n_T+N-1} (\mathbb{E}\|\mathbb{E}_{u_n} v_{n,m,j} - \int_{H} y \mu_{u_n}^\epsilon(dy)\|^2)^{\frac{1}{2}}.
\]

If \( \Delta t \) is small enough, for each \( n \) and \( j \), \( v_{n,m,j} \) is exponential mixing with unique invariant measure \( \mu_{u_n}^{\Delta t, \epsilon} \), and there exists a random variable \( \zeta_{u_n, \Delta t, \epsilon} \) with distribution \( \mu_{u_n}^{\Delta t, \epsilon} \) that is independent of the driving Wiener processes. Let \( \zeta_{n,m} \) be the solution of the discretized equation of the fast equation (20) with initial value \( \zeta_{n,m} \), then the distribution of \( \zeta_{n,m} \) is \( \mu_{u_n}^{\Delta t, \epsilon} \). Since

\[
\mathbb{E}\|\mathbb{E}_{u_n} v_{n,m,j} - \int_{H} y \mu_{u_n}^\epsilon(dy)\|^2 \leq \mathbb{E}\|\int_{H} y \mu_{u_n}^{\Delta t, \epsilon}(dy) - \int_{H} y \mu_{u_n}^\epsilon(dy)\|^2 \leq C\left(\frac{\Delta t}{\epsilon}\right)^{-2\kappa}, \quad 0 < \kappa < \frac{1}{2},
\]

we have [4, Corollary 1.2],

\[
\mathbb{E}\|\mathbb{E}_{u_n} \zeta_{n,nN_{m}+m} - \int_{H} y \mu_{u_n}^\epsilon(dy)\|^2 \leq \mathbb{E}\|\int_{H} y \mu_{u_n}^{\Delta t, \epsilon}(dy) - \int_{H} y \mu_{u_n}^\epsilon(dy)\|^2 \leq C\left(\frac{\Delta t}{\epsilon}\right)^{-2\kappa}, \quad 0 < \kappa < \frac{1}{2},
\]

and

\[
\left(\mathbb{E}\|\mathbb{E}_{u_n} v_{n,m,j} - \int_{H} y \mu_{u_n}^\epsilon(dy)\|^2\right)^{\frac{1}{2}} \leq C\left[\mathbb{E}\|v_{n,m,j} - \zeta_{n,nN_{m}+m}\|^2 + \left(\frac{\Delta t}{\epsilon}\right)^{-\frac{\kappa}{2}}\right]. \tag{25}
\]

Further we have [9, Lemma 5.27], [4, Remark 4.8]

\[
\left(\mathbb{E}\|v_{n,m,j} - \zeta_{n,nN_{m}+m}\|^2\right)^{\frac{1}{2}} \leq e^{-\frac{1}{2} \beta m \Delta t} \left(\mathbb{E}\|v_{n-1,N_{m,j} - \zeta_{n-1,nN_{m}}\|^2\right)^{\frac{1}{2}} \\
\leq e^{-\frac{1}{2} \beta m \Delta t} \left(\mathbb{E}\|v_{n-1,N_{m,j} - \zeta_{n-1,nN_{m}}\|^2\right)^{\frac{1}{2}} + e^{\frac{1}{2} \beta m \Delta t} \left(\mathbb{E}\|\zeta_{n,nN_{m} - \zeta_{n-1,nN_{m}}\|^2\right)^{\frac{1}{2}} \\
\leq e^{-\frac{1}{2} \beta m \Delta t} \left[\mathbb{E}\|v_{n-1,N_{m,j} - \zeta_{n-1,nN_{m}}\|^2\right)^{\frac{1}{2}} + C \Delta t]\right]
\]

where the last inequality follows

\[
\mathbb{E}\|\zeta_{n,nN_{m} - \zeta_{n-1,nN_{m}}\|^2} \leq C\mathbb{E}\|u_n - u_{n-1}\|^2 \leq C \Delta t^2, \quad \beta = \lambda_1 - L_g.
\]
Repeating the above argument at each macrotime step from \( n - 1 \) to \( n = 0 \),

\[
\left( \mathbb{E} \left\| v_{n-1,N,m,j} - \zeta_{n-1,n,N,m} \right\|^2 \right)^{\frac{1}{2}} \\
\leq e^{-\frac{1}{2} \beta N_n \Delta t} \left[ \left( \mathbb{E} \left\| v_{n-2,N,m,j} - \zeta_{n-2,(n-1)N_m} \right\|^2 \right)^{\frac{1}{2}} + C \Delta t \right] \\
\leq e^{-\frac{1}{2} \beta N_n \Delta t} \left[ e^{-\frac{1}{2} \beta N_n \Delta t} \left( \mathbb{E} \left\| v_{n-3,N,m,j} - \zeta_{n-3,(n-2)N_m} \right\|^2 \right)^{\frac{1}{2}} + C \Delta t \right] \\
= e^{-\frac{1}{2} \beta (2N_n) \Delta t} \left( \mathbb{E} \left\| v_{n-3,N,m,j} - \zeta_{n-3,(n-2)N_m} \right\|^2 \right)^{\frac{1}{2}} \\
+ e^{-\frac{1}{2} \beta (2N_n) \Delta t} \cdot C \Delta t + e^{-\frac{1}{2} \beta N_n \Delta t} \cdot C \Delta t \\
\leq \ldots \leq e^{-\frac{1}{2} \beta (n-1)N_n \Delta t} \left( \mathbb{E} \left\| v_0 - \zeta_{n_0,N,m,j} \right\|^2 \right)^{\frac{1}{2}} \\
+ \frac{1 - e^{-\frac{1}{2} \beta (n-2)N_n \Delta t}}{1 - e^{-\frac{1}{2} \beta N_n \Delta t}} e^{-\frac{1}{2} \beta N_n \Delta t} \cdot C \Delta t \\
\leq C \left( e^{-\frac{1}{2} \beta (n-1)N_n \Delta t} + R \right)
\]

where \( R = \frac{\Delta t}{1 - e^{-\frac{1}{2} \beta N_n \Delta t}} \). Inserting the above estimates into (25), we get

\[
\left( \mathbb{E} \left\| u_{n,m,j} - \int_H y \mu_{u_n}^\epsilon(dy) \right\|^2 \right)^{\frac{1}{2}} \\
\leq C \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + C e^{-\frac{1}{2} \beta m \Delta t} \left( e^{-\frac{1}{2} \beta (n-1)N_m \Delta t} + R + \Delta t \right).
\]

The inequality also holds for \( n = 0 \) with an appropriate choice of \( C \). Now take sum over \( m \in [n_T,n_T+N-1] \) and \( j \in [1,M] \),

\[
\mathbb{E} \left\| u_{n,m,j} \right\| \\
\leq K C \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + \frac{1}{MN} \sum_{j=1}^{M} \sum_{m=n_T}^{n_T+N-1} e^{-\frac{1}{2} \beta m \Delta t} \left( e^{-\frac{1}{2} \beta (n-1)N_m \Delta t} + R + \Delta t \right) \frac{1 - e^{-\frac{1}{2} \beta N \Delta t}}{N(1 - e^{-\frac{1}{2} \beta \Delta t})}.
\]

Assuming \( \frac{\Delta t}{\epsilon} \in (0,1) \), then

\[
\frac{1 - e^{-\frac{1}{2} \beta N \Delta t}}{N(1 - e^{-\frac{1}{2} \beta \Delta t})} \leq C \frac{1 - e^{-\frac{1}{2} \beta N \Delta t}}{N \frac{\Delta t}{\epsilon} + 1} \leq \frac{C'}{N \frac{\Delta t}{\epsilon} + 1}.
\]

So

\[
\mathbb{E} \left\| u_{n,m,j} \right\| \leq C \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + e^{-\frac{1}{2} \beta n_T \Delta t} \left( e^{-\frac{1}{2} \beta (n-1)N_m \Delta t} + R + \Delta t \right) \frac{1 - e^{-\frac{1}{2} \beta N \Delta t}}{N \frac{\Delta t}{\epsilon} + 1}.
\]

Now, we continue to estimate \( \mathbb{E} \left\| \tilde{f}(u_n) - \tilde{f}(u_n) \right\|^2 \). Note that

\[
\mathbb{E} \left\| \tilde{f}(u_n) - \tilde{f}(u_n) \right\|^2 = \frac{1}{M^2 N^2} \sum_{j,m,k,l} \mathbb{E} \left[ \left( u_n \left( v_{n,m,j} - \int_H y \mu_{u_n}^\epsilon(dy) \right), u_n \left( v_{n,k,l} - \int_H y \mu_{u_n}^\epsilon(dy) \right) \right) \right].
\]
by independence between $v_{n,m,j}$ and $v_{n,m,k}$ for $j \neq k$ for given $\{u_{n'}\}_{n' \leq n}$ and $\{v_{n'}\}_{n' < n}$, and combining with (26), we have for $j \neq k$,

$$\left\| \sum_{j \neq k, l} \sum_{m,l} E \left\langle u_n \left( v_{n,m,j} \right), u_n \left( v_{n,l,k} \right) \right\rangle \right\| \leq K^2 \left\| \sum_{j \neq k, m,l} \left( E \left\| E_n \left( v_{n,m,j} \right) \right\|^2 \right) \right\|^\frac{1}{2}$$

$$\leq C \sum_{j \neq k, m,l} \left[ \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + e^{-\frac{1}{2} \beta m \frac{\Delta t}{\epsilon}} \left( e^{-\frac{1}{2} \beta (n-1) N_m \frac{\Delta t}{\epsilon}} + R + \Delta t \right) \right] \left[ \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + e^{-\frac{1}{2} \beta (n-1) N_m \frac{\Delta t}{\epsilon}} \left( e^{-\frac{1}{2} \beta (n-1) N_m \frac{\Delta t}{\epsilon}} + R + \Delta t \right) \right]$$

$$\leq CMN^2 \left[ \left( \frac{\Delta t}{\epsilon} \right)^{1 - 2\kappa} + e^{-\beta m \Delta t} \left( e^{-\beta (n-1) N_m \Delta t} + R^2 + \Delta t^2 \right) \right],$$

where $E_n$ denote the expectation conditioned on $\{u_{n'}\}_{n' \leq n}$ and $\{v_{n'}\}_{n' < n}$. For $j = k$ and $m \leq l$ we have

$$\left\| E \left\langle u_n \left( v_{n,m,j} \right), u_n \left( v_{n,l,j} \right) \right\rangle \right\| \leq \left\| E \left\langle u_n \left( v_{n,m,j} \right), u_n \left( v_{n,l,j} \right) \right\rangle \right\| \leq K^2 \left\| \left( E \left\| E_n \left( v_{n,m,j} \right) \right\|^2 \right) \right\|^\frac{1}{2}$$

Here $E_{n,m,j}$ denotes the conditional expectation with respect to $v_{n,m,j}$. Similar results hold for $m > l$. Notice

$$\left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + e^{-\frac{1}{2} \beta m \frac{\Delta t}{\epsilon}} \left( e^{-\frac{1}{2} \beta (n-1) N_m \frac{\Delta t}{\epsilon}} + R + \Delta t \right),$$

and

$$\left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + e^{-\frac{1}{2} \beta (l-m) \frac{\Delta t}{\epsilon}} \left( e^{-\frac{1}{2} \beta (n-1) N_m \frac{\Delta t}{\epsilon}} + R + \Delta t \right),$$

summing over $j \in [1, M]$ and $m, l \in [n_T, n_T + N - 1]$,

$$\left\| \sum_{j} \sum_{m,l} E \left\langle u_n \left( v_{n,m,j} \right), u_n \left( v_{n,l,j} \right) \right\rangle \right\| \leq C \sum_{j} \sum_{m,l} \left[ \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + e^{-\frac{1}{2} \beta m \frac{\Delta t}{\epsilon}} \left( e^{-\frac{1}{2} \beta (n-1) N_m \frac{\Delta t}{\epsilon}} + R + \Delta t \right) \right] \left[ \left( \frac{\Delta t}{\epsilon} \right)^{\frac{1}{2} - \kappa} + e^{-\frac{1}{2} \beta \beta (l-m) \frac{\Delta t}{\epsilon}} \left( e^{-\frac{1}{2} \beta (n-1) N_m \frac{\Delta t}{\epsilon}} + R + \Delta t \right) \right]$$

$$\leq CMN^2 \left[ \left( \frac{\Delta t}{\epsilon} \right)^{1 - 2\kappa} + e^{-\beta m \Delta t} \left( e^{-\beta (n-1) N_m \Delta t} + R^2 + \Delta t^2 \right) \right].$$
then
\[ \mathbb{E}\|\tilde{f}(u_n) - \tilde{f}(u_n)\|^2 \leq C\left(1 + \frac{1}{N}\right)\left[\left(\frac{\Delta t}{\epsilon}\right)^{1+2\kappa} + e^{-\beta n T \Delta t}\left(e^{-\beta(n-1)N_n \Delta t} + R^2 + \Delta t^2\right)\right] \]
and
\[ |\phi(\bar{u}_n) - \mathbb{E}\phi(u_n)| \leq C_{T,\phi} \frac{1 + \lambda_1 \Delta t}{\lambda_1} \left[1 + \frac{1}{N}\left[\left(\frac{\Delta t}{\epsilon}\right) \frac{1}{\beta n T} + e^{-\frac{1}{2}\beta n T \Delta t}\left(e^{-\frac{1}{2}\beta(n-1)N_n \Delta t} + R + \Delta t\right)\right]\right]. \]

For the discretization approximation of the deterministic averaging equation, a standard ODE estimate (13) states that assuming the (21) is stable and of \( k \)-th order accuracy for (18) and \( \Delta t \) is small enough, then
\[ |\phi(\bar{u}(T)) - \phi(\bar{u}(0))| \leq \sup_{\frac{t}{\Delta t}} \left|\phi(\bar{u}(\Delta t)) - \phi(\bar{u}_n)\right| \leq C\Delta^k. \]

5. **Asymptotic expansions.** This section give some estimates on \( |\mathbb{E}\varphi(u^\epsilon(t, u_0)) - \varphi(\bar{u}(t, u_0))| \). Here we write the dependence of the solution of the fast and slow system on the initial value.

**Theorem 5.1.** Assume that \( u_0 \in H^2, v_0 \in H_0^1 \) and (H1), (H2) hold, for any \( T > 0 \) and \( \phi \in C_0^5(H) \), there exists a constant \( C > 0 \) such that
\[ |\mathbb{E}\phi(u^\epsilon(T, u_0, v_0)) - \phi(\bar{u}(T, u_0))| \leq \sqrt{T}C_{T,\varphi}(1 + \|u_0\|^3_{H_0^1} + \|v_0\|^3_{H_0^1} + \|Au_0\|^4). \]

For any \( u_0, v_0 \in H, t \geq 0 \), let
\[ X^\epsilon(t, u_0, v_0) = \mathbb{E}\phi(u^\epsilon(t, u_0, v_0)), \quad \bar{X}(t, u_0) = \phi(\bar{u}(t, u_0)). \]

Now we give an asymptotic expansions of \( X^\epsilon \) with respect to \( \epsilon \)
\[ X^\epsilon = X_0 + \epsilon X_1 + Y^\epsilon, \]
where \( Y^\epsilon \) is the remainder of the expansion.

For any \( \psi(u_0, v_0) : H \times H \to \mathbb{R} \) and \( \psi \in C^2 \), we define the differential operator
\[ L_1\psi(u_0, v_0) = \langle Au_0 + g(u_0, v_0), D_{v_0}\psi(u_0, v_0) \rangle + \frac{1}{2} \text{Tr}[D_{v_0}^2\psi(u_0, v_0)Q], \]
\[ L_2\psi(u_0, v_0) = \langle Au_0 + f(u_0, v_0), D_{u_0}\psi(u_0, v_0) \rangle. \]
For any \( \psi : H \to \mathbb{R} \) and \( \psi \in C^1 \), let
\[ \tilde{L}\psi(u_0) = \langle Au_0 + \tilde{f}(u_0), D_{u_0}\psi(u_0) \rangle \]
and
\[ L^\epsilon = \frac{1}{\epsilon}L_1 + L_2. \]

Noticing that that \( X^\epsilon \) satisfies the Kolmogorov equation
\[ \frac{\partial X^\epsilon(t, u_0, v_0)}{\partial t} = L^\epsilon X^\epsilon(t, u_0, v_0), X^\epsilon(0, u_0, v_0) = \phi(u_0), \]
and \( \bar{X} \) is independent of \( v_0 \) with \( \bar{X} \) satisfying the following Kolmogorov equation
\[ \frac{\partial \bar{X}(t, u_0)}{\partial t} = \bar{L}\bar{X}(t, u_0), \bar{X}(0, u_0) = \phi(u_0). \]
Then
\[
\frac{\partial X_0}{\partial t} + \epsilon \frac{\partial X_1}{\partial t} + \frac{\partial Y^\epsilon}{\partial t} = \frac{1}{\epsilon} L_1 X_0 + L_2 X_0 + L_1 X_1 + \epsilon L_2 X_1 + \frac{1}{\epsilon} L_1 Y^\epsilon + L_2 Y^\epsilon,
\]
and compare the coefficients on both sides with respect to \(\epsilon\)
\[
L_1 X_0 = 0, \quad \frac{\partial X_0}{\partial t} = L_1 X_1 + L_2 X_0, \quad \frac{\partial Y^\epsilon}{\partial t} = L^\epsilon Y^\epsilon + \epsilon (L_2 X_1 - \frac{\partial X_1}{\partial t}). \tag{32}
\]
The equation \(L_1 X_0 = 0\) yields that \(X_0\) is independent of \(v_0\), then \(X_0(t, u_0, v_0) = X_0(t, u_0)\), and \(X_0(0, u_0) = \phi(u_0)\). For any fixed slow variable \(u \in H\), let \(\mu^u\) be the invariant measure of a Markov process whose generator is \(L_1\), then
\[
\int_H L_1 X_1(t, u_0, v_0) \mu^{u_0}(dv_0) = 0,
\]
and
\[
\frac{\partial X_0(t, u_0)}{\partial t} = \int_H \frac{\partial X_0(t, u_0)}{\partial t} \mu^{u_0}(dv_0) = \int_H L_2 X_0(t, u_0) \mu^{u_0}(dv_0)
\]
\[
= \left\langle A u_0 + \int_H f(u_0, v_0) \mu^{u_0}(dv_0), D_{u_0} X_0(t, u_0) \right\rangle = L X_0(t, u_0).
\]
So \(X_0\) and \(\bar{X}\) satisfy the same equation and initial condition, by the uniqueness of the solution, we obtain \(X_0 = \bar{X}\). Then we have the following lemma.

**Lemma 5.2.** Assume \((H_1)\) and \((H_2)\) hold, for any \(u_0, v_0 \in H\) and \(T > 0\),
\[
X_0(T, u_0, v_0) = \bar{X}(T, u_0).
\]

Next we construct \(X_1\). Due to (31), (32) and \(X_0 = \bar{X}\), we have \(\bar{L} \bar{X} = L_1 \bar{X}_1 + L_2 \bar{X}\), and \(X_1\) satisfies the following elliptic equation
\[
L_1 X_1 = \bar{L} \bar{X} - L_2 \bar{X} = \left\langle \bar{f}(u_0) - f(u_0, v_0), D_{u_0} \bar{X}(t, u_0) \right\rangle + \rho(u_0, v_0)\] \tag{33}
Since \(f(u_0, v_0) = u_0 - u_0^3 + u_0 v_0\), \(\rho\) is \(C^2\) with respect to \(u_0\) and its derivatives are uniformly bounded. Moreover, for any \(t \geq 0\), \(u_0 \in H\), we have
\[
\int_H \rho(t, u_0, v_0) \mu^{u_0}(dv_0) = 0.
\]
For any \(v_0 \in H\), \(s > 0\), let
\[
P^{u_0}_s \rho(t, u_0, v_0) = E \rho(t, u_0, v_0^{u_0}(v_0)),
\]
then
\[
\frac{\partial P^{u_0}_s \rho(t, u_0, v_0)}{\partial s} = \left\langle A v_0 + g(u_0, v_0), D_{v_0} (P^{u_0}_s \rho(t, u_0, v_0)) \right\rangle + \frac{1}{2} \text{Tr} \left[ D_{v_0 v_0} (P^{u_0}_s \rho(t, u_0, v_0)) Q \right].
\]
Notice the fact (Appendix)
\[
\lim_{s \to \infty} E \rho(t, u_0, v_0^{u_0}(v_0)) = \int_H \rho(t, u_0, v_0) \mu^{u_0}(dv_0) = 0, \tag{34}
\]
Further by (7), (71), (73) and Lemma 4.4, we have

\[
\left\langle A v_0 + g(u_0, v_0), D v_0 \int_0^{+\infty} P_s^{u_0} \rho(t, u_0, v_0) ds \right\rangle \\
+ \frac{1}{2} \text{Tr} \left[ D^2 v_0 v_0 \int_0^{+\infty} P_s^{u_0} \rho(t, u_0, v_0) Q ds \right] \\
= \int_0^{+\infty} \frac{\partial P_s^{u_0} \rho(t, u_0, v_0)}{\partial s} ds - \lim_{s \to \infty} E \rho(t, u_0, v_s^{u_0}(v_0)) - \rho(t, u_0, v_0) \\
= \int_{\mathcal{H}} \rho(t, u_0, v_0) \mu^{u_0}(dv_0) - \rho(t, u_0, v_0) = -\rho(t, u_0, v_0).
\]

By the definition of \(L_1\), (33) and (35), we have

\[
L_1 \left( \int_0^{+\infty} P_s^{u_0} \rho(t, u_0, v_0) ds \right) = -\rho(t, u_0, v_0),
\]

and

\[
X_1 = \int_0^{+\infty} E \rho(t, u_0, v_s^{u_0}(v_0)) ds.
\]

For \(X_1\), we have the following estimates.

**Lemma 5.3.** For any \(u_0, v_0 \in H_0^1\), \(T > 0\) and \(\phi \in C_0^2(\mathcal{H})\), we have

\[
|X_1(t, u_0, v_0)| \leq C_{T, \phi} \left( 1 + \|u_0\|^2_{H_0^1} + \|v_0\|^2_{H_0^1} \right) e^{C_T \|u_0\|_{H_0^1}}.
\]

**Proof.** By (33) and (36), we have

\[
X_1(t, u_0, v_0) = \int_0^{+\infty} E \left( \hat{f}(u_0) - f(u_0, v_s^{u_0}(v_0)), D u_0 \bar{X}(t, u_0) \right) ds.
\]

Then the invariance of \(\mu^{u_0}\), (77) and Lemma 6.1 yield

\[
|X_1(t, u_0, v_0)| \leq C_{T, \phi} \left( 1 + \|u_0\|^2_{H_0^1} + \|v_0\|^2_{H_0^1} \right) e^{C_T \|u_0\|_{H_0^1}}.
\]

Now we determine \(Y^\epsilon\).

**Lemma 5.4.** For any \(u_0 \in H^2\), \(v_0 \in H_0^1\), \(T > 0\) and \(\phi \in C_0^2(\mathcal{H})\),

\[
\left\| \frac{\partial D u_0 \bar{X}(t, u_0)}{\partial t} \right\| \leq C_{T, \phi} \left( 1 + \|u_0\|^3_{H_0^1} + \|v_0\|^2_{H_0^1} + \|A u_0\| \right) e^{C_T \|u_0\|_{H_0^1}}.
\]

**Proof.** By (67) and (69), we have

\[
\frac{\partial D u_0 \bar{X}(t, u_0)}{\partial t} = \langle \eta_t^{h, u_0}, \phi \circ (\bar{u}(t, u_0)) \cdot [A \bar{u}(t, u_0) + \hat{f}(\bar{u}(t, u_0))] \\
+ \langle \phi''(\bar{u}(t, u_0)) \cdot (\partial_x \bar{u}(t, u_0), \partial_x \bar{u}(t, u_0)) + \phi''(\bar{u}(t, u_0)) \cdot A \bar{u}(t, u_0), \eta_t^{h, u_0} \\
+ \langle \phi'(\bar{u}(t, u_0)), \hat{f}_g(\bar{u}(t, u_0)) \rangle \cdot \eta_t^{h, u_0} \rangle.
\]

Further by (7), (71), (73) and Lemma 4.4, we have

\[
\|A \bar{u}(t, u_0)\|^2 \leq \|A u_0\|^2 + C_T \left( 1 + \|u_0\|^3_{H_0^1} + \|v_0\|^2_{H_0^1} \right) + C_T \int_0^t \|A \tilde{u}(s, u_0)\|^4 ds.
\]

Similarly, by Lemma 22, we obtain

\[
\|A \bar{u}(t, u_0)\| \leq C_T (1 + \|u_0\|^3_{H_0^1} + \|v_0\|^2_{H_0^1} + \|A u_0\|).
\]

(37)
By (71) and (73), we get
\[ \| \tilde{f}(\bar{u}(t, u_0)) \| \leq C_T \left( 1 + \| u_0 \|^2_{H^1_0} + \| u_0 \|^3_{H^1_0} \right) \]
and further by (73), (74) and (75),
\[ | \bar{f}(\bar{u}(t, u_0)) \cdot \eta(t, u) | \leq C_T \left( 1 + \| u_0 \|_{H^1_0} \right) e^{C_T \| u_0 \|_{H^1_0} \cdot \| h \|}. \]
Then combining with (37), (38),
\[ \left\| \frac{\partial D_{u_0} \bar{X}(t, u_0)}{\partial t} \right\| \leq C_{T, \phi} \left( 1 + \| u_0 \|^3_{H^1_0} + \| v_0 \|^2_{H^1_0} + \| A u_0 \| \right) e^{C_T \| u_0 \|_{H^1_0}}. \]

**Lemma 5.5.** For any \( u_0 \in H^2 \), \( v_0 \in H^1_0 \), and \( T > 0 \),
\[ \left\| \frac{\partial X(t, u_0, v_0)}{\partial t} \right\| \leq C_{T, \phi} \left( 1 + \| u_0 \|^6_{H^1_0} + \| v_0 \|^4_{H^1_0} + \| A u_0 \|^2 \right) e^{C_T \| u_0 \|_{H^1_0}}. \]

**Proof.** By \((\partial_t - L^\varepsilon)X^\varepsilon = 0\), (31) and (32), we obtain
\[ \begin{align*}
(\partial_t - L^\varepsilon)X^\varepsilon &= (\partial_t - L^\varepsilon)X_0 - \epsilon (\partial_t - L^\varepsilon)X_1 \\
&= - (\partial_t - \frac{1}{\varepsilon} L_1 - L_2)X_0 - \epsilon (\partial_t - \frac{1}{\varepsilon} L_1 - L_2)X_1 \\
&= \epsilon (L_2 X_1 - P(t) X_1),
\end{align*} \]
then (36), (33), Lemma 5.4 and (14) yield,
\[ \begin{align*}
\left\| \frac{\partial X(t, u_0, v_0)}{\partial t} \right\| &\leq \int_0^\infty \left\| f(u_0) - Ef(u_0, v^u_s(v_0)) \right\| \left\| \frac{\partial D_{u_0} \bar{X}(t, u_0)}{\partial t} \right\| ds \\
&\leq C_{T, \phi} \left( 1 + \| u_0 \|^6_{H^1_0} + \| v_0 \|^4_{H^1_0} + \| A u_0 \|^2 \right) e^{C_T \| u_0 \|_{H^1_0}}. \quad (39)
\end{align*} \]

Now we estimate \( \| L_2 X_1(t, u_0, v_0) \| \) where
\[ L_2 X_1(t, u_0, v_0) = \langle A u_0 + f(u_0, v_0), D_{u_0} X_1(t, u_0, v_0) \rangle. \quad (40) \]

For this, we firstly estimate \( \| D_{u_0} X_1(t, u_0, v_0) \| \).

**Lemma 5.6.** Assume \((H_1)\) hold, for any \( u_0, v_0 \in H^1_0 \), \( T > 0 \) and \( \phi \in C^2_b(H) \),
\[ \| D_{u_0} X_1(t, u_0, v_0) \| \leq C_{T, \phi} \left( 1 + \| u_0 \|^4_{H^1_0} + \| v_0 \|^4_{H^1_0} \right). \]

**Proof.** According to (33) and (36), for any \( k \in H \),
\[ \begin{align*}
D_{u_0} X_1(t, u_0, v_0) \cdot k &= \int_0^\infty \langle D_{u_0} (f(u_0) - Ef(u_0, v^u_s(v_0))) \cdot k, D_{u_0} \bar{X}(t, u_0) \rangle ds \\
&\quad + \int_0^\infty \langle f(u_0) - Ef(u_0, v^u_s(v_0)), D^2_{u_0 u_0} \bar{X}(t, u_0) \cdot k \rangle ds \\
&= I_1(t, u_0, v_0, k) + I_2(t, u_0, v_0, k).
\end{align*} \]

Now we estimate these two terms respectively.
\[ |I_1(t, u_0, v_0, k)| \]
\[ \leq C \| k \| \| D_{u_0} \bar{X}(t, u_0) \| \int_0^\infty \left( \| E v^u_s - E v^u_s(v_0) \| + \| u_0 \|_{H^1_0} \cdot E \| \xi^u_s - \xi^u_s(v_0) \| \right) ds \]
Now we estimate

So Lemma 6.1, (42) and (14) yield

By chain rules and (7), we have

\[ \eta_t \]

where \( \eta_t := \frac{1}{\epsilon} \frac{d}{dt} (\bar{u}(t,u_0)) \). By Young’s inequality, (7), \( (14) \) and Gronwall’s lemma, we have

\[ \eta_t \leq C_T (1 + \|u_0\| + \|v_0\|) e^{-(\lambda_t - L_\alpha - L_\beta) t}. \]

(42)

So Lemma 6.1, (42) and (14) yield

\[ |I_1(t,u_0,v_0,k)| \leq C_T e^{C_T \|u_0\|_{H_0^1}} (1 + \|u_0\|^2_{H_0^1} + \|v_0\|^2_2) \|k\|. \]

(43)

Now we estimate \( |I_2(t,u_0,v_0,k)| \).

By the invariance of \( \mu_{u_0} \) and (77), we have

\[
|I_2(t,u_0,v_0,k)| = \left| \int_0^\infty \left( \int_H \mu_{u_0} f(u_0,z) \mu_{u_0}(dz) - \mathbb{E} f(u_0,v_0)\right) D_{u_0u_0}^2 \bar{X}(t,u_0) \cdot k ds \right|
\]

\[
\leq C (1 + \|u_0\|^2_{H_0^1} + \|v_0\|^2_{H_0^1}) \|D_{u_0u_0}^2 \bar{X}(t,u_0)\| \|k\|. \]

(44)

We need estimate \( \|D_{u_0u_0}^2 \bar{X}(t,u_0)\| \).

Noticing (67), for any \( h,k \in H \),

\[ D_{u_0u_0}^2 \bar{X}(t,u_0) \cdot (h,k) = \phi^{(\ast)}(\bar{u}(t,u_0)) \cdot (\eta^{h,k,u_0}_t, \eta^{k,u_0}_t) + \phi'(\bar{u}(t,u_0)) \cdot \zeta^{h,k,u_0}_t, \]

where \( \zeta^{h,k,u_0}_t := D_{u_0u_0}^2 \bar{u}(t,u_0) \cdot (h,k) \) satisfies

\[ d\zeta^{h,k,u_0}_t = [A\zeta^{h,k,u_0}_t + \bar{f}''_{uu}(\bar{u}(t,u_0)) \eta^{h,u_0}(\eta^{k,u_0} - \eta^{k,u_0}_t) + \bar{f}''_u(\bar{u}(t,u_0)) \cdot \zeta^{h,k,u_0}_t] dt, \]

\[ \zeta^{h,k,u_0}_0 = 0. \]

By chain rules and (7), we have

\[ \frac{1}{2} \frac{d}{dt} \|\zeta^{h,k,u_0}_t\|^2 \leq -\lambda_1 \|\zeta^{h,k,u_0}_t\|^2 + \|\zeta^{h,k,u_0}_t\| \|\bar{f}''(\bar{u}(t,u_0))\| \|\eta^{h,u_0}_t\| \|\eta^{k,u_0}_t\| + \|\zeta^{h,k,u_0}_t\|, \]

(45)

where

\[ \bar{f}'(\bar{u}(t,u_0)) = 1 - 3\bar{u}(t,u_0)^2 + \mathbb{E}v^*u + \bar{u}(t,u_0) \cdot \mathbb{E}D_{u}v^*, \]

and

\[ \bar{f}''_{uu}(\bar{u}(t,u_0)) = -6\bar{u}(t,u_0) + \mathbb{E}D_{u}v^*u + \bar{u}(t,u_0) \mathbb{E}D_{u}^2 v^*, \]

\[ \eta^{u}_{t} := D_{u}^2 v^*u = D_{u} \zeta^{h}_{t} \] satisfies the following equation

\[ d\eta^{u}_{t} = \frac{1}{\epsilon} \mathcal{A}\eta^{u}_{t} + \frac{1}{\epsilon} \bar{g}''_{uu}(\bar{u},v^*u) \cdot \zeta^{u}_{t} \zeta^{h}_{t} + \frac{1}{\epsilon} \bar{g}''_u(\bar{u},v^*u) \cdot \eta^{u}_{t} + \frac{1}{\epsilon} g''_{uu}(\bar{u},v^*u) dt, \eta^{u}_{0} = 0. \]

By Young’s inequality, (7), \( (H_1) \) and Gronwall’s lemma, we have

\[ \|\eta^{u}_{t}\|^2 \leq C, \quad \|\zeta^{h}_{t}\| \leq C. \]

(46)
Then with the proof of Lemma 6.1,
\[ \| f_{u,u}(\bar{u}(t, u_0)) \| \leq C(\| \bar{u}(t, u_0) \| + \| \mathbb{E} \xi_r \| + \| \mathbb{E} \eta_r \|) \]
\[ \leq C(1 + \| u_0 \|_{H^1_0}). \]  
(47)

Substitute (47) into (45) and combine with (75) and Gronwall’s inequality, we obtain
\[ \| c_{t,k} \| \leq (1 + \| u_0 \|_{H^1_0}) \frac{2}{3} e^{C_T \| u_0 \|_{H^1_0} \| k \|}. \]  
(48)

Combine with (75), we have
\[ \| D_{u,u}^2 \bar{X}(t, u_0) \| \leq C_{T, \phi} (1 + \| u_0 \|_{H^1_0}) e^{C_T \| u_0 \|_{H^1_0} \| k \|}. \]  
(49)

Substitute (49) into (44), we have
\[ | I_2(t, u_0, v_0, k) | \leq C_{T, \phi} (1 + \| u_0 \|_{H^1_0}^4 + \| v_0 \|_{H^1_0}^4) e^{C_T \| u_0 \|_{H^1_0} \| k \|}. \]  
(50)

By (43) and (50), we have
\[ \| D_{u} X_1(t, u_0, v_0) \| \leq C_{T, \phi} e^{C_T \| u_0 \|_{H^1_0}} (1 + \| u_0 \|_{H^1_0}^4 + \| v_0 \|_{H^1_0}^4). \]

Therefore, by (40), we have
\[ | L_2 X_1(t, u_0, v_0) | \leq C_{T, \phi} (1 + \| A u_0 \|_{H^1_0}^2 + \| u_0 \|_{H^1_0}^8 + \| v_0 \|_{H^1_0}^8) e^{C_T \| u_0 \|_{H^1_0}}. \]  
(51)

Now we estimate $Y^\epsilon$.

**Lemma 5.7.** For any $u_0 \in H^2, v_0 \in H^1_0$, $T > 0$ and $\epsilon \in (0, T)$, $\phi \in C^2_0(H),
\[ | Y^\epsilon(T, u_0, v_0) | \leq \sqrt{T} C_{T, \phi} (1 + \| u_0 \|_{H^1_0}^{32} + \| v_0 \|_{H^1_0}^{32} + \| A u_0 \|_{H^1_0}^4). \]

**Proof.** By a variation of constant formula,
\[
Y^\epsilon(T, u_0, v_0) = \mathbb{E} Y^\epsilon(\epsilon, u_{T^-}(u_0, v_0), v_{T^-}(u_0, v_0)) \\
+ \mathbb{E} \int_0^T (L_2 X_1(t, u_{T^-}(u_0, v_0), v_{T^-}(u_0, v_0)) dt,
\]
and noticing that
\[ X^\epsilon(0, u_0, v_0) = \bar{X}(0, u_0), \]
we have
\[
Y^\epsilon(\epsilon, u_0, v_0) = X^\epsilon(\epsilon, u_0, v_0) - \bar{X}(\epsilon, u_0) - \epsilon X_1(\epsilon, u_0, v_0) \\
= -\epsilon X_1(\epsilon, u_0, v_0) + [X^\epsilon(\epsilon, u_0, v_0) - X^\epsilon(0, u_0, v_0)] - [\bar{X}(\epsilon, u_0) - \bar{X}(0, u_0)].
\]
On the one hand, Lemma 5.3 implies that
\[ | \epsilon X_1(\epsilon, u_0, v_0) | \leq C_{T, \phi} (1 + \| u_0 \|_{H^1_0}^2 + \| v_0 \|_{H^1_0}^2) e^{C_T \| u_0 \|_{H^1_0}}. \]  
(52)

On the other hand, by (28)
\[
| X^\epsilon(\epsilon, u_0, v_0) - X^\epsilon(0, u_0, v_0) | \\
= | \mathbb{E} \int_0^\epsilon \phi'(u^\epsilon(s, u_0, v_0)) \cdot [A u^\epsilon(s, u_0, v_0) + f(u^\epsilon(s, u_0, v_0), v^\epsilon(s, u_0, v_0))] ds| \\
\leq C_\phi \int_0^\epsilon | \mathbb{E}[A u^\epsilon(s, u_0, v_0)] + \| u^\epsilon \| + \| u^\epsilon \|_{H^1_0}^2 + \| v^\epsilon \|_{H^1_0}^2 | ds. \]  
(53)
By the chain rules,
\[
\frac{1}{2} \frac{d\|u^\epsilon(t)\|_{H_0^2}^2}{dt} \leq -\frac{1}{2} \|A u^\epsilon(t)\|^2 + \|u^\epsilon(t)\|_{H_0^2}^2 + \frac{C}{2} \|u^\epsilon(t)\|_{H_0^2}^4 + \|v^\epsilon(t)\|_{H_0^4}^2.
\]
Taking the expectation and integrating both sides from 0 to $\epsilon$, we have
\[
\int_0^\epsilon \mathbb{E}\|A u^\epsilon(s)\|ds \leq \sqrt{\epsilon} C_T (1 + \|u_0\|_{H_0^4}^4 + \|v_0\|_4^4)
\]
and by Lemma 4.3 and Lemma 4.4,
\[
\int_0^\epsilon \mathbb{E}\|u^\epsilon(s)\|_{H_0^2}^2 ds \leq \int_0^\epsilon (\mathbb{E}\|u^\epsilon(s)\|_{H_0^2}^4)^{\frac{1}{2}} ds \leq \epsilon C_T (1 + \|u_0\|_{H_0^2}^6 + \|v_0\|_6^6),
\]
\[
\int_0^\epsilon \mathbb{E}\|u^\epsilon(s)\|_{H_0^4}^2 ds \leq \epsilon C_T (1 + \|u_0\|_{H_0^4}^4 + \|v_0\|_4^4),
\]
\[
\int_0^\epsilon \mathbb{E}\|u^\epsilon(s)\|_{H_0^4}^2 ds \leq \epsilon C_T (1 + \|u_0\|_2^2 + \|v_0\|^2).
\]
Now substituting (54)-(58) into (53) yields
\[
|X^\epsilon(\epsilon, u_0, v_0) - X^\epsilon(0, u_0, v_0)| \leq \sqrt{\epsilon} C_{T, \phi} \left( 1 + \|u_0\|_{H_0^4}^6 + \|v_0\|_6^6 \right)
\]
Further, (37) and (38) yield
\[
|X(\epsilon, u_0) - X(0, u_0)| \leq C_{\phi} \int_0^\epsilon \|\tilde{u}(s, u_0)\| + f(\tilde{u}(s, u_0)) \|ds
\]
\[
\leq \epsilon C_{T, \phi} (1 + \|u_0\|_{H_0^2}^3 + \|v_0\|_{H_0^2}^2 + \|Au_0\|).
\]
So by (52), (59) and (60),
\[
|Y^\epsilon(\epsilon, u_0, v_0)| \leq \|X^\epsilon(\epsilon, u_0, v_0)\| + \|X^\epsilon(\epsilon, u_0, v_0) - X^\epsilon(0, u_0, v_0)\| + \|X(\epsilon, u_0) - X(0, u_0)\|
\]
\[
\leq \epsilon C_{T, \phi} (1 + \|u_0\|_{H_0^2}^2 + \|v_0\|_{H_0^2}^2) e^{\epsilon \|u_0\|_{H_0^4}}
\]
\[
+ \epsilon C_{T, \phi} (1 + \|u_0\|_{H_0^4}^6 + \|v_0\|_{H_0^6}^6 + \|Au_0\|).
\]
Further still by the chain rules and Young’s inequality,
\[
\frac{1}{2} \frac{d\|Au^\epsilon(t)\|_{H_0^2}^2}{dt} \leq \|Au^\epsilon(t)\|_{H_0^2}^2 + C[\|u^\epsilon(t)\|_{H_0^4}^4 + \|u^\epsilon(t)\|_{H_0^4}^4 + \|v^\epsilon(t)\|_{H_0^4}^4].
\]
Taking expectation on both sides, by Gronwall’s inequality and Lemma 4.4,
\[
\mathbb{E}\|Au^\epsilon(t)\|^2 \leq C_T (1 + \|u_0\|_{H_0^2}^2 + \|v_0\|_{H_0^2}^2),
\]
and by Lemma 4.3, Lemma 4.4,
\[
\mathbb{E}\|Y^\epsilon(\epsilon, u^\epsilon_{T-\epsilon}(u_0, v_0), v^\epsilon_{T-\epsilon}(u_0, v_0))\|
\]
\[
\leq \sqrt{\epsilon} C_{T, \phi} (1 + \|u_0\|_{H_0^2}^2 + \|v_0\|_{H_0^2}^2 + \|Au_0\|).
\]
Then by (51) and (39), we have,
\[
[(L_2 X_1 - \partial_t X_1)(t, u^\epsilon_{T-\epsilon}(u_0, v_0), v^\epsilon_{T-\epsilon}(u_0, v_0))]
\]
\[
\leq C_T, \phi (1 + \|Au^\epsilon_{T-\epsilon}(u_0, v_0)\|_2^2 + \|u^\epsilon_{T-\epsilon}(u_0, v_0)\|_{H_0^2}^4)
\]
\[
+ \|v^\epsilon_{T-\epsilon}(u_0, v_0)\|_{H_0^2}^4) e^{\epsilon \|u^\epsilon_{T-\epsilon}(u_0, v_0)\|_{H_0^4}}.
\]
Furthermore, by (61), we have
\[ \frac{1}{2} \frac{d}{dt} \| Au'(t) \|_4^4 \leq C \| Au'(t) \|_4^4 + C(\| u'(t) \|_{H_0^1}^2 + \| u'(t) \|_{H_0^1}^{12} + \| u'(t) \|_{H_0^1}^{12}), \]
then Gronwall’s inequality and Lemma 4.4 yield,
\[ \mathbb{E} \| Au'(t) \|_4^4 \leq C_T (1 + \| Au_0 \|_4^4 + \| u_0 \|_{H_0^1}^{24} + \| v_0 \|_4^{24}). \quad (64) \]
By Lemma 4.3, Lemma 4.4 and (64)
\[ \epsilon \left| \mathbb{E} \int_\epsilon^T (h_X(t, u_{T-1}(u_0, v_0), v_{T-1}(u_0, v_0))dt \right| \leq \epsilon C_{T, \phi} (1 + \| Au_0 \|_4^4 + \| u_0 \|_{H_0^1}^{32} + \| v_0 \|^{32}), \quad (65) \]
then by (63) and (65), we have
\[ |Y'(T, u_0, v_0)| \leq \sqrt{\epsilon} C_{T, \phi} (1 + \| u_0 \|_{H_0^1}^{32} + \| v_0 \|^{32} + \| Au_0 \|_4^4). \]

Now we finish proof of the theorem 5.1.

**Proof.** by (29), (52) and Lemma 5.7, we have
\[ |\mathbb{E} \phi(u'(T, u_0, v_0)) - \phi(\bar{u}(T, u_0))| = |X'(T, u_0, v_0) - \bar{X}(T, u_0)| \leq \epsilon |X_1(T, u_0, v_0)| + |Y'(T, u_0, v_0)| \leq \sqrt{\epsilon} C_{T, \phi} (1 + \| u_0 \|_{H_0^1}^{32} + \| v_0 \|^{32} + \| Au_0 \|_4^4). \]

**6. Appendix.** This section gives the proof of the (34). In fact, thanks to the invariance of \( \mu^{u_0} \) and Lemma (5.6),
\[ \left| \mathbb{E} \rho(t, u_0, v^{u_0}_s(v_0)) - \int_H \rho(t, u_0, z) \mu^{u_0}(dz) \right| \leq \int_H \mathbb{E} \| v^{u_0}_s(z) - v^{u_0}_s(v_0) \| \| D_{u_0} \bar{X}(t, u_0) \| \mu^{u_0}(dz). \quad (66) \]
To complete the proof, we continue estimate \( \| D_{u_0} \bar{X}(t, u_0) \| \) and \( \mathbb{E} \| v^{u_0}_s(z) - v^{u_0}_s(v_0) \| \). First we have the following estimate.

**Lemma 6.1.** For any \( u_0 \in H_0^1 \) and \( T > 0 \),
\[ \| D_{u_0} \bar{X}(t, u_0) \| \leq C_T e^{C_T \| u_0 \|_{H_0^1}}. \]

**Proof.** For any \( t \in [0, T], h \in H, \)
\[ D_{u_0} \bar{X}(t, u_0) \cdot h = \langle \varphi'(\bar{u}(t, u_0)), \eta_{t}^{h, u_0} \rangle \quad (67) \]
and
\[ |D_{u_0} \bar{X}(t, u_0) \cdot h| \leq \sup_{z \in H} |\varphi'(z)| \| \eta_{t}^{h, u_0} \|, \quad (68) \]
where \( \eta_{t}^{h, u_0} = D_{u_0} \bar{u}(t, u_0) \cdot h \) is the mild solution of the equation
\[ \frac{d}{dt} \eta_{t}^{h, u_0} = [A \eta_{t}^{h, u_0} + \bar{f}_u(\bar{u}(t, u_0)) \cdot \eta_{t}^{h, u_0}]dt, \quad \eta_{0}^{h, u_0} = h. \quad (69) \]
Multiplying both sides of this equation by \( \eta_{t}^{h, u_0} \) and by (7) yields
\[ \frac{1}{2} \frac{d}{dt} \| \eta_{t}^{h, u_0} \|^2 \leq -\lambda_1 \| \eta_{t}^{h, u_0} \|^2 + \langle \eta_{t}^{h, u_0}, \bar{f}_u(\bar{u}(t, u_0)) \cdot \eta_{t}^{h, u_0} \rangle. \quad (70) \]
In order to estimate \( ||\eta_{t,u_0}^n|| \), we firstly estimate \( ||\tilde{f}(u(t,u_0))|| \). Note that
\[
\tilde{f}(u(t,u_0)) = \tilde{u}(t,u_0) - \bar{u}(t,u_0) + \bar{u}(t,u_0) \cdot E v^u,
\]
where \( v^u \) represents the stationary solution to the fast equation for the fixed slow variable \( u \). By direct calculation, we have
\[
E \| v_{t_0}^{u_0}(v_0) \|^2 \leq C(\| u \|^2 + \| u_0 \|^2 + 1),
\]
then
\[
E \| v^u \|^2 \leq C(1 + \| u \|^2), \quad E \| v^u \| \leq C(1 + \| \bar{u} \|). \tag{71}
\]
Note that
\[
\tilde{f}_x(\bar{u}(t,u_0)) \leq 1 + E v^u + \bar{u}(t,u_0) \cdot E(v^u)_{\bar{u}},
\]
then
\[
\| \tilde{f}_x(\bar{u}(t,u_0)) \| \leq 1 + \| v^u \| + \| \bar{u}(t,u_0) \cdot E(v^u)_{\bar{u}} \|, \tag{72}
\]
where \( (v^u)_{\bar{u}} := \zeta_t^u \) satisfies the following equation
\[
ds_{\zeta} = \frac{1}{\epsilon} A \zeta + \frac{1}{\epsilon} g_u(\bar{u}, v^u) \cdot \zeta + \frac{1}{\epsilon} g_{u}'(\bar{u}, v^u) \zeta, \quad \zeta_{t_0} = 0.
\]
Multiply both sides of this equation by \( \zeta_t^u \), and by (7), (H1), Gronwall’s inequality, we have
\[
\| \zeta_t^u \| \leq \sqrt{\frac{l_g}{\lambda_1 - L_g - l_g}} \leq C.
\]
By the chain rules, (15), (7), (71) and Gronwall’s inequality, we have
\[
\| \bar{u}(t,u_0) \| \leq C_T(1 + \| u_0 \|).
\]
Now we estimate \( \| \bar{u}(t,u_0) \|_{H^1_0} \). By the chain rules, (7) and (71), we have
\[
\| \bar{u}(t,u_0) \|^2_{H^1_0} \leq \alpha_T(\omega) + C \int_0^t \| \bar{u}(s,u_0) \|^4_{H^3_0} ds,
\]
where \( \alpha_T(\omega) = \| u_0 \|^2_{H^1_0} + T \).

By Lemma 4.1, we have
\[
\| \bar{u}(t,u_0) \|^2_{H^1_0} \leq C_T(1 + \| u_0 \|^2_{H^1_0}).
\]
In fact, let \( u(t) = \| \bar{u}(t,u_0) \|^2_{H^1_0}, a = \alpha_T(\omega), b = C, k(t) = 1, t \in [0,T], \)
\( g(s) = s^2 \). Then
\[
u(t) \leq \alpha_T(\omega) + C \int_0^t u(s)^2 ds,
\]
and
\[
u(t) \leq G^{-1}(G(\alpha_T(\omega)) + C t), \quad 0 \leq t \leq t_1 \leq T,
\]
where
\[
G(\lambda) = \int_0^\lambda \frac{ds}{g(s)} = \frac{1}{\xi} - \frac{1}{\lambda}, \quad Ran(G) = (-\infty, \frac{1}{\xi}), \quad (\xi > 0, \lambda > 0),
\]
\[
G^{-1}(\lambda) = \frac{\xi}{1 - \lambda \xi}, \quad \lambda \in Ran(G),
\]
\( t_1 \) is defined as follows
\[
t_1 = \frac{1}{2} \frac{1}{C \alpha_T(\omega)}.
\]
such that
\[ \frac{1}{\xi} - \frac{1}{\alpha_T(\omega)} + Ct \in \text{Ran}(G), \quad t \in [0, t_1]. \]

By
\[ \frac{1}{\xi} - \frac{1}{\alpha_T(\omega)} + Ct < \frac{1}{\xi} \]
we obtain
\[ t < \frac{1}{C\alpha_T(\omega)}, \]
then
\[ u(t) \leq \frac{\xi}{1 - \xi[\frac{1}{\xi} - \frac{1}{\alpha_T(\omega)} + Ct]} \leq 2\alpha_T(\omega), \quad t \in [0, t_1], \]
for \( t_1 \leq t \leq T, \)
\[ u(t) \leq \alpha_T(\omega) + C \int_{t_1}^{t} u(s)^2 ds \]
\[ u(t) \leq G^{-1}(G(\alpha_T(\omega)) + Ct + Ct_1), \quad t_1 \leq t \leq t_2 \leq T, \]
where \( t_2 \) is defined as
\[ t_2 = \frac{3}{4} \frac{1}{C\alpha_T(\omega)} \]
such that
\[ G(\alpha_T(\omega)) + Ct - Ct_1 \in \text{Ran}(G), \quad t \in [t_1, t_2]. \]

By
\[ \frac{1}{\xi} - \frac{1}{\alpha_T(\omega)} + Ct - Ct_1 < \frac{1}{\xi} \]
we obtain
\[ t < \frac{1}{C\alpha_T(\omega)} + t_1 = \frac{3}{2} \frac{1}{C\alpha_T(\omega)}. \]
then
\[ u(t) \leq \frac{\xi}{1 - \xi[\frac{1}{\xi} - \frac{1}{C\alpha_T(\omega)} + Ct - Ct_1]} = \frac{1}{\frac{3}{2} \alpha_T(\omega)} - Ct \leq 2\alpha_T(\omega). \]
Repeating the above steps \( n \) times, we obtain
\[ T < \frac{2^n - 1}{2^n} \frac{1}{C\alpha_T(\omega)}, \]
and
\[ u(t) \leq 2\alpha_T(\omega), t \in [0, T], \]
so
\[ \|\tilde{u}(t, u_0)\|_{H_0^1}^2 \leq C_T(1 + \|u_0\|_{H_0^1}^2), \quad (73) \]
and by (72), (71) and (73),
\[ \|\tilde{f}(t, u_0)\| \leq 1 + C_T(1 + \|u_0\|) + C_T(1 + \|u_0\|_{H_0^1}) \]
\[ \leq C_T(1 + \|u_0\|_{H_0^1}). \quad (74) \]
Substitute (73) into (70), we have
\[ \|\eta_{t, u_0}\| \leq C_T \|\eta\|_1 e^{C_T \|u_0\|_{H_0^1}}. \quad (75) \]
So (69) yields
\[ \|D_{u_0} \tilde{X}(t, u_0)\| \leq C_{T, u} e^{C_T \|u_0\|_{H_0^1}}. \quad (76) \]
Secondly, by (13),
\[
\mathbb{E}\|u_0(v_s^u(z) - v_s^u(v_0))\| \leq C\|u_0\|_{H_0^1} \|z - v_0\| e^{-\frac{(\lambda_1 - L_g)s}{2}},
\tag{77}
\]
and by (66), Lemma 6.1 and (77)
\[
|\mathbb{E}\rho(t, u_0, v_s^u(v_0)) - \int_{H} \rho(t, u_0, z)\mu^{u_0}(dz)|
\leq C_{\rho, \phi}e^{-\frac{(\lambda_1 - L_g)s}{2}}(1 + \|u_0\|^2_{H_0^1} + \|v_0\|^2_{H_0^1}) e^{C_T\|u_0\|_{H_0^1}}.
\]

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