Newtonian Limits of the Relativistic Cosmological Perturbations

Jai-chan Hwang  
Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Taegu, Korea

Hyerim Noh  
Korea Astronomy Observatory, San 36-1, Whaam-dong, Yusung-gu, Daejon, Korea

Relativistic cosmological perturbation analyses can be made based on several different fundamental gauge conditions. In the pressureless limit the variables in certain gauge conditions show the correct Newtonian behaviors. We consider the general curvature and the cosmological constant in the background medium. The perturbed density in the comoving gauge, and the perturbed velocity and the perturbed potential in the zero-shear gauge show the same behavior as the Newtonian ones in a general scale. Far inside horizon, except for the uniform-density gauge, density perturbations in all the fundamental gauge conditions show the correct Newtonian behavior. In this paper we elaborate these Newtonian correspondences. We also present the relativistic results considering general pressures in the background and perturbation.

I. INTRODUCTION

The analysis of gravitational instability in the expanding universe model was first presented by Lifshitz (1946) in a general relativistic context. Historically, the much simpler, and in hindsight, more intuitive Newtonian study followed later (Bonner 1957). The pioneering study by Lifshitz is based on a gauge choice which is commonly called as the synchronous gauge. As the later studies have shown, the synchronous gauge is only one way of fixing the gauge freedom out of several available fundamental gauge conditions (Bardeen 1980; Kodama & Sasaki 1984; Hwang 1991b). As will be summarized in the following, out of the several gauge conditions only the synchronous gauge fails to fix the gauge mode completely, thus often needs more involved algebra. Though, as long as one is careful of the algebra this gauge choice does not cause any kind of intrinsic problem. The gauge condition which turns out to be especially suitable for handling the perturbed density is the comoving gauge (Nariai 1969; Sakai 1969). Since the comoving gauge condition completely fixes the gauge transformation property, the variables in this gauge can be equivalently considered as gauge invariant ones. As mentioned, there exist several such fundamental gauge conditions each of which completely fixes the gauge transformation properties. Thus, the variables in such gauge conditions are equivalently gauge invariant. Using the gauge freedom as an advantage for handling the problem was emphasized in Bardeen (1988). In order to use the gauge choice as an advantage a gauge ready method was proposed in Hwang (1991b) which will be adopted in the following.

The variables which characterize the self gravitating Newtonian fluid flow are the density, the velocity and the gravitational potential (the pressure is specified by an equation of state). Whereas, the relativistic flow may be characterized by various components of the metric (or curvature) and the energy momentum tensor. Since the relativistic gravity theory is a constrained system we have the freedom of imposing certain conditions on the metric or the energy momentum tensor as coordinate conditions. In the perturbation analyses the freedom arises because we need to introduce a fictitious background system in order to describe the physical perturbed system. The correspondence of a given spacetime point between the perturbed spacetime and the fictitious background one could be made with certain degrees of freedom. This freedom can be fixed by the suitable gauge conditions based on the spacetime coordinate transformation. Studies in the literature show that a certain variable in a certain gauge condition correctly reproduces the corresponding Newtonian behavior. Although the perturbed density in the comoving gauge shows the Newtonian behavior, the perturbed potential and the perturbed velocity in the same gauge do not behave like the Newtonian ones; e.g., in the comoving gauge the perturbed velocity vanishes by the coordinate (gauge) condition. It is known that both the perturbed potential and the perturbed velocity in the zero-shear gauge correctly behave like the corresponding Newtonian variables (Bardeen 1980).

In this paper we will elaborate establishing such correspondences between the relativistic and Newtonian perturbed variables. Our previous work on this subject is presented in Hwang (1994a; H1 hereafter) and Hwang & Hyun (1994). In the following, we will derive the relativistic equations which describe the perturbed density, potential and velocity variables in several available gauge conditions and will compare the equations with the corresponding equations satisfied by the Newtonian system. In our analyses we will include both the spatial curvature and the cosmological constant in the background medium. Based on such correspondences we will extend our result to the situations with
general pressures in the background and perturbations. We will present the relativistic equations satisfied by the gauge invariant combinations and will derive the general solutions valid in the large scale considering both the spatial curvature ($K$) and the cosmological constant ($\Lambda$).

In §II we present the closed form equations and general solutions for the Newtonian perturbed variables in a pressureless medium. The solutions are valid for general $K$ and $\Lambda$. In §III we summarize a complete set of equations describing the relativistic perturbations with general pressures (including both the adiabatic, entropic and anisotropic ones). The equations are presented in a gauge ready form and the method of handling the gauge issue is briefly described. In §IV we consider a pressureless limit. We derive the equations for the density, the potential and the velocity in several different fundamental gauge conditions. By comparing these relativistic equations in various gauges with the Newtonian ones in §II, in §V we identify the gauge conditions which reproduce the correct Newtonian behavior for certain variables. In §VI we present the equations for the gauge invariant variables which have correct Newtonian limits, now, considering the general pressures in the background and perturbations. We derive the general large scale solutions valid for general $K$ and $\Lambda$ in an ideal fluid medium (thus, valid for general equation of state of the form $p = p(\mu)$, but with negligible entropic and anisotropic pressures). We also present a quantity which is conserved in the large scale under general changes of the background equation of state for $K = 0$. §VII is a brief discussion. We set $c \equiv 1$.

II. NEWTONIAN COSMOLOGICAL PERTURBATIONS

The background evolution is governed by

$$H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \rho \propto a^{-3}, \tag{1}$$

where we allowed the general curvature (total energy) and the cosmological constant; $a(t)$ is a cosmic scale factor, $H(t) \equiv \dot{a}/a$, and $\rho(t)$ is the mass density. Perturbed parts of the mass conservation, the momentum conservation and the Poisson’s equations are (see eqs. [43,46] of H1):

$$\delta \dot{\rho} + 3H \delta \rho = -\frac{k}{a} \delta \rho_v, \quad \delta \dot{\rho}_v + H \delta \rho_v = \frac{k}{a} \delta \Phi, \quad -\frac{k^2}{a^2} \delta \Phi = 4\pi G \delta \rho, \tag{2}$$

where $\delta \rho(k,t)$, $\delta \rho_v(k,t)$ and $\delta \Phi(k,t)$ are the Fourier mode of the perturbed mass density, velocity and gravitational potential, respectively. Equation (3) can be arranged into the closed form equations for $\delta$ (\(\equiv \delta \rho/\rho\)), $\delta \rho_v$ and $\delta \Phi$ as:

$$\ddot{\delta} + 2H \dot{\delta} - 4\pi G \rho \delta = \frac{1}{a^2 H} \left[ a^2 H^2 \left( \frac{\delta}{H} \right) \right] = 0, \tag{3}$$

$$\ddot{\delta} \rho_v + 3H \dot{\delta} \rho_v + \left( \dot{H} + 2H^2 - 4\pi G \rho \right) \delta \rho_v = 0, \tag{4}$$

$$\ddot{\delta} \Phi + 4H \dot{\delta} \Phi + \left( \dot{H} + 3H^2 - 4\pi G \rho \right) \delta \Phi = \frac{1}{a^2 H} \left[ a^2 H^2 \left( \frac{\delta \Phi}{H} \right) \right] = 0. \tag{5}$$

We note that equations (3-5) are valid for general $K$ and $\Lambda$. The general solutions for $\delta$, $\delta \rho_v$ and $\delta \Phi$ immediately follow as (see also Table 1 of H1):

$$\delta(k,t) = (k^2 - 3K) \left[ H C(k) \int_0^t \frac{dt}{a^2} + \frac{H}{4\pi G \rho a^3} d(k) \right], \tag{6}$$

$$\delta \rho_v(k,t) = -\frac{k^2 - 3K}{k^2} \left[ \frac{k}{aH} C(k) \left( 1 + a^2 \dot{H} \right) \int_0^t \frac{dt}{a^2} + \frac{k\dot{H}}{4\pi G \rho a^2} d(k) \right], \tag{7}$$

$$\delta \Phi(k,t) = -\frac{k^2 - 3K}{k^2} \left[ 4\pi G \rho a^2 H C(k) \int_0^t \frac{dt}{a^2} + \frac{H}{a} d(k) \right]. \tag{8}$$

The $C$ and $d$ terms indicate the growing and decaying modes, respectively; the coefficients are matched in accordance with the solutions with general pressure in equations (3-5).
III. RELATIVISTIC COSMOLOGICAL PERTURBATIONS

For the background we have (eq.[6] of Hwang 1993):

\[ H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \dot{\mu} = -3H(\mu + p), \quad \dot{H} = -4\pi G(\mu + p) + \frac{K}{a^2}. \]  

(9)

The third equation follows from the first two equations. For \( p = 0 \) and replacing \( \mu \) with \( \rho \) equation (9) reduces to equation (1). The relativistic cosmological perturbation equations in a gauge ready form were presented in Hwang (1991b). We introduce familiar variables for the perturbed density and the perturbed velocity variables as

\[ \delta \equiv \varepsilon \mu, \quad v \equiv -\frac{k}{\mu a} \Psi \mu + p. \]  

(10)

The gauge ready form of relativistic perturbation equations can be written as (see eqs.[8-14] of Hwang 1993):

\[ \dot{\varphi} = H\alpha - \frac{1}{3}\chi + \frac{1}{3}\frac{k^2}{a^2} \chi, \]  

(11)

\[ -\frac{k^2 - 3K}{a^2} \varphi + H\kappa = -4\pi G\mu \delta, \]  

(12)

\[ \kappa - \frac{k^2 - 3K}{a^2} \chi = 12\pi G(\mu + p) \frac{a}{k} v, \]  

(13)

\[ \dot{\chi} + H\chi - \alpha - \varphi = 8\pi G\sigma, \]  

(14)

\[ \dot{\kappa} + 2H\kappa = \left( \frac{k^2}{a^2} - 3\dot{H} \right) \alpha + 4\pi G \left( 1 + 3c_s^2 \right) \mu \delta + 12\pi G e, \]  

(15)

\[ \dot{\delta} + 3H \left( c_s^2 - w \right) \delta + 3H \frac{e}{\mu} (1 + w) \left( \kappa - 3H\alpha - \frac{k}{a} v \right), \]  

(16)

\[ \dot{\nu} + \left( 1 - 3c_s^2 \right) Hv = \frac{k}{\mu} \alpha + \frac{k}{a(1 + w)} \left( c_s^2 \delta + \frac{e}{\mu} - \frac{2k^2 - 3K}{3a^2} \mu \sigma \right). \]  

(17)

According to their origins, equations (11)-(17) can be called as the definition of \( \kappa \), ADM energy constraint, momentum constraint, ADM propagation, Raychaudhuri, energy conservation, and momentum conservation equations, respectively. The perturbed metric variables \( \varphi(k, t), \kappa(k, t), \chi(k, t) \) and \( \alpha(k, t) \) are the perturbed part of the three space curvature, expansion, shear and lapse function, respectively. The perturbed fluid variables \( \delta(k, t), v(k, t), e(k, t) \) and \( \sigma(k, t) \) are the relative density perturbation, (frame independent) velocity variable, entropic and anisotropic pressures, respectively. The isotropic pressure is decomposed as

\[ \pi(k, t) \equiv c_s^2(t)\varepsilon(k, t) + e(k, t), \quad \sigma \equiv \frac{\dot{\rho}}{\mu}, \quad w(t) \equiv \frac{p}{\mu}. \]  

(18)

The perturbed variables used in equations (11)-(17) are designed so that any one of the following conditions can be used to fix the freedom based on the temporal gauge transformation: \( v \equiv 0 \) (the comoving gauge), \( \chi \equiv 0 \) (the zero-shear gauge), \( \kappa \equiv 0 \) (the uniform-expansion gauge), \( \alpha \equiv 0 \) (the synchronous gauge), \( \varphi \equiv 0 \) (the uniform-curvature gauge), and \( \delta \equiv 0 \) (the uniform-density gauge). Each of these six gauge conditions, except for the synchronous gauge, fixes the temporal gauge transformation property completely. Thus, each variable in these five gauge conditions is equivalent to a corresponding gauge invariant combination. Due to the spatial homogeneity of the background, the effect from the spatial gauge transformation has been been trivially handled; \( \chi \) is a spatial gauge invariant combination, and the other metric and fluid variables are naturally spatially gauge invariant (Bardeen 1988). The variables \( e \) and \( \sigma \) are gauge invariant.

We proposed a convenient way of writing the gauge invariant variables (Hwang 1991b). For example, we let

\[ \delta_v \equiv \delta + 3(1 + w) \frac{aH}{k} v \equiv 3(1 + w) \frac{aH}{k} v_s, \]

\[ \varphi_\chi \equiv \varphi - H\chi \equiv -H\varphi_v, \quad v_\chi \equiv v - \frac{k}{a} \chi \equiv -\frac{k}{a} \chi_v. \]  

(19)

The variables \( \delta_v, v_\chi \) and \( \varphi_\chi \) are gauge invariant combination; \( \delta_v \) becomes \( \delta \) in the comoving gauge which takes \( v = 0 \) as the gauge condition, etc. In this manner we can systematically construct the corresponding gauge invariant
combination for any variable based on a gauge condition which fixes the temporal gauge transformation property completely. For the gauge transformation properties, see §2.2 of Hwang (1991b). A variable evaluated in different gauges can be considered as different variables, and they show different behaviors in general. In this sense, except for the synchronous gauge, the variables in the rest of the five gauges can be considered as the gauge invariant variables. (Thus, the variables with a subindex \( \alpha \) are not gauge invariant, because those are equivalent to variables in the synchronous gauge.) Although \( \delta_v \) is a gauge invariant variable which becomes \( \delta \) in the comoving gauge, \( \delta_v \) itself can be evaluated in any gauge with the same value. The complete solutions for these six different gauge conditions are presented in an ideal fluid case (Hwang 1993) and in a pressureless medium (H1).

In the following we consider a pressureless fluid with \( p = 0 \) and \( \pi = 0 = \sigma \). Equations (11)-(17) become:

\[
\begin{align*}
\dot{\varphi} &= H\alpha - \frac{1}{3}\kappa + \frac{1}{3}\frac{k^2}{a^2}\chi, \\
-\frac{k^2 - 3K}{a^2}\varphi + H\kappa &= -4\pi G\mu \delta, \\
\kappa - \frac{k^2 - 3K}{a^2}\chi &= 12\pi G\mu \frac{a}{k} v, \\
\dot{\chi} + H\chi - \alpha - \varphi &= 0, \\
\dot{\kappa} + 2H\kappa &= \left(\frac{k^2}{a^2} - 3H\right)\alpha + 4\pi G\mu \delta, \\
\dot{\delta} &= \kappa - 3H\alpha - \frac{k}{a} v, \\
\dot{v} + Hv &= \frac{k}{a}\alpha.
\end{align*}
\]

In H1 and Hwang & Hyun (1994) we made arguments on the correspondences between the Newtonian and the relativistic analyses. In order to reinforce the Newtonian correspondences of certain (gauge invariant) variables in certain gauges, in the following we will present the closed form differential equations for \( \delta, v \) and \( \varphi \) in the six different gauge conditions.

### IV. EQUATIONS IN SIX FUNDAMENTAL GAUGE CONDITIONS

In this section we consider a pressureless medium. Thus, equations (20)-(26) are the basic set of perturbation equations in a gauge ready form.

#### A. Comoving Gauge

As the gauge condition we set \( v = 0 \). Equivalently, we can set \( v = 0 \) and let every other variable as the gauge invariant combinations with subindices \( v \). From equation (26) we have \( \alpha_v = 0 \). Thus the comoving gauge is a case of the synchronous gauge; this is true only in a pressureless situation. From equations (20)-(26) we can derive:

\[
\begin{align*}
\ddot{\delta}_v + 2H\dot{\delta}_v - 4\pi G\mu \delta_v &= 0, \\
\ddot{\varphi}_v + 3H\dot{\varphi}_v - \frac{K}{a^2}\varphi_v &= 0.
\end{align*}
\]

Thus, equation (24) has the identical form as equation (4). We can show that the variables \( a\kappa \) and \( \chi/a \) satisfy the same equation for \( \delta_v \) in equation (4); see equations (15) and (11).

For \( K = 0 \) equation (28) leads to two solutions which are \( \varphi_v \propto \text{constant} \) and \( \int_0^t a^{-3} dt \). From equation (26) we have \( \alpha_v = 0 \). Thus, for \( K = 0 \), from equations (21) and (22) we have \( \dot{\varphi}_v = 0 \). This implies that, for \( K = 0 \), the second solution, \( \varphi_v \propto \int_0^t a^{-3} dt \), should have the vanishing coefficient. The general solution of equation (28) will be presented later in equation (60).
B. Zero-shear Gauge

We let $\chi \equiv 0$, and substitute the other variables into the gauge invariant combinations with subindices $\chi$. We can derive:

$$
\ddot{\delta}_\chi + \frac{2k^2/a^2 - 36\pi G\mu}{k^2/a^2 - 12\pi G\mu [1 + 3H^2a^2/(k^2 - 3K)]} H \left( \dot{\delta}_\chi - 12\pi G\mu \frac{a^2}{k^2 - 3K} H \delta_\chi \right)
- 4\pi G\mu \left[ 1 - 3(H^2 - 2\dot{H}) \frac{a^2}{k^2 - 3K} \right] \delta_\chi = 0,
$$

(29)

$$
\ddot{v}_\chi + 3H \dot{v}_\chi + \left( \dot{H} + 2H^2 - 4\pi G\mu \right) v_\chi = 0,
$$

(30)

$$
\ddot{\varphi}_\chi + 4H \dot{\varphi}_\chi + \left( \dot{H} + 3H^2 - 4\pi G\mu \right) \varphi_\chi = 0.
$$

(31)

Thus, equations (30) and (31) have identical forms as equations (3) and (3), respectively. Only in the small scale limit the behavior of $\delta$ is the same as the Newtonian one.

C. Uniform-expansion Gauge

We let $\kappa \equiv 0$, and substitute the remaining variables into the gauge invariant combinations with subindices $\kappa$. We can derive:

$$
\ddot{\delta}_\kappa + \left( 2H - \frac{12\pi G\mu H}{k^2/a^2 - 3H} \right) \dot{\delta}_\kappa - \left[ 4\pi G\mu \frac{k^2/a^2 + 6H^2}{k^2/a^2 - 3H} + \left( \frac{12\pi G\mu H}{k^2/a^2 - 3H} \right) \right] \delta_\kappa = 0,
$$

(32)

$$
\ddot{v}_\kappa + \left[ 2H - \frac{12\pi G\mu H}{k^2/a^2 - 3H} + \frac{4\pi G\mu}{k^2/a^2 - 3H} \left( \frac{k^2/a^2 + 3H^2}{4\pi G\mu} \right) \right] \dot{v}_\kappa
+ \left[ \dot{H} + H^2 - 4\pi G\mu \frac{k^2/a^2 + 3H^2}{k^2/a^2 - 3H} + H \frac{4\pi G\mu}{k^2/a^2 - 3H} \left( \frac{k^2/a^2 + 3H^2}{4\pi G\mu} \right) \right] v_\kappa = 0,
$$

(33)

$$
\ddot{\varphi}_\kappa + \left( 4H - \frac{12\pi G\mu H}{k^2/a^2 - 3H} \right) \dot{\varphi}_\kappa
+ \left[ \dot{H} + 3H^2 - 4\pi G\mu \frac{k^2/a^2 + 9H^2}{k^2/a^2 - 3H} - \left( \frac{12\pi G\mu H}{k^2/a^2 - 3H} \right) \right] \varphi_\kappa = 0.
$$

(34)

In the small scale limit we can show that equations (32)-(34) reduce to equations (1)–(2). Thus, in the small scale limit, all three variables $\delta_\kappa$, $v_\kappa$ and $\varphi_\kappa$ correctly reproduce the Newtonian behavior. However, outside or near horizon scale, the behaviors of all these variables strongly deviate from the Newtonian ones.

In §84 of Peebles (1980) we find that in order to get the usual Newtonian equations a coordinate transformation is made so that we have $\tilde{h} \equiv 0$ in the new coordinate. We can show that $\tilde{h} = 2\kappa$ in our notation. Thus the new coordinate is in fact the uniform-expansion gauge.

D. Synchronous Gauge

We let $\alpha \equiv 0$. This gauge condition does not fix the temporal gauge transformation property completely. Thus, although we can still indicate the variables in this gauge condition using subindices $\alpha$ without ambiguity, these variables are not gauge invariant (see §3.2.1 of Hwang 1991b). Equation (26) leads to

\footnote{This was incorrectly pointed out below eq.[49] of H1; in H1 it was mentioned that in §84 of Peebles (1980) the gauge transformations were made into the comoving gauge for $\delta$ and into the zero-shear gauge for $v$ and $\varphi$, respectively.}
\[ v_\alpha = c_g \frac{\kappa}{a}, \]  
(35)

which is a pure gauge mode. Thus, fixing \( c_g = 0 \) exactly corresponds to the comoving gauge. We can show that the following two equations are not affected by the remaining gauge mode.

\[ \ddot{\delta}_\alpha + 2H \dot{\delta}_\alpha - 4\pi G \mu \delta_\alpha = 0, \]  
(36)

\[ \ddot{\varphi}_\alpha + 3H \dot{\varphi}_\alpha - \frac{K}{a^2} \varphi_\alpha = 0. \]  
(37)

Equation (36) is identical to equation (3). This is because the behavior of the gauge mode happens to coincide with the behavior of one of the physical mode for \( \delta_\alpha \) and \( \varphi_\alpha \). However, for the variables \( \kappa_\alpha \) and \( \chi_\alpha \) the gauge mode contribution appears explicitly. We can derive:

\[ (a\kappa_\alpha)'' + 3H (a\kappa_\alpha) + \left( \dot{H} + 2H^2 - 4\pi G \mu \right) (a\kappa_\alpha) = -4\pi G \mu k v_\alpha = -4\pi G \mu \frac{k^2}{a} c_g, \]  
(38)

\[ (\chi_\alpha/a)'' + 3H (\chi_\alpha/a) + \left( \dot{H} + 2H^2 - 4\pi G \mu \right) (\chi_\alpha/a) = -4\pi G \mu \frac{1}{k} v_\alpha. \]  
(39)

In the comoving gauge the right hand sides of both equations vanish. Thus, in a pressureless medium, variables in the synchronous gauge behave the same as the ones in the comoving gauge, except for the additional gauge modes which appear in the synchronous gauge. In a pressureless medium, we can simultaneously impose both the comoving gauge and the synchronous gauge conditions. However, this is possible only in a pressureless medium. (see §VI).

E. Uniform-curvature Gauge

We let \( \varphi = 0 \), and substitute the other variables into the gauge invariant combinations with subindices \( \varphi \). We have

\[ \ddot{\delta}_\varphi + 2H \dot{\delta}_\varphi - 4\pi G \mu \delta_\varphi = 0, \]  
(40)

\[ \ddot{\varphi}_\varphi + \left( 5H + 2\frac{\dot{H}}{H} \right) \dot{\varphi}_\varphi + \left( 3\dot{H} + 4H^2 \right) \varphi_\varphi = 0. \]  
(41)

In the small scale limit equation (40) reduces to equation (3). In the uniform-curvature gauge, the perturbed potential is set equal to zero by the gauge condition. The uniform-curvature gauge condition has distinguished properties in handling the scalar field or the dilaton field which appears in some very general classes of the generalized gravity theories (Hwang 1994b; Hwang & Noh 1996).

F. Uniform-density Gauge

We let \( \delta = 0 \), and substitute the other variables into the gauge invariant combinations with subindices \( \delta \). We have

\[ \ddot{\delta}_\delta + 2 \left( 2H + \frac{\dot{H}}{H} \right) \dot{\delta}_\delta + 3 \left( H^2 + \dot{H} \right) v_\delta = 0, \]  
(42)

\[ \ddot{\varphi}_\delta + 2H \left( \frac{k^2 - 3K}{a^2} + 12\pi G \mu \right) \dot{\varphi}_\delta - \frac{4\pi G \mu k^2/a^2}{(k^2 - 3K)/a^2 + 12\pi G \mu} \varphi_\delta = 0. \]  
(43)

These equations differ from equations (4) and (5). Of course, we have no equation for \( \delta \) which is set equal to zero by our choice of the gauge condition.
V. NEWTONIAN CORRESPONDENCES

We found that equations for $\delta$ in the comoving gauge ($\delta_v$), and for $v$ and $\varphi$ in the zero-shear gauge ($v_\chi$ and $\varphi_\chi$) show the same forms as the corresponding Newtonian equations. Using the gauge invariant combinations in equation (19), equations (21), (22), (23), (25) and (26) can be combined to give:

$$\dot{\delta}_v = -\frac{k^2 - 3K}{ak}v_\chi - Hv_\chi = -\frac{k}{a}\varphi_\chi, \quad \frac{k^2 - 3K}{a^2}v_\chi = 4\pi G\mu \delta_v. \quad (44)$$

Comparing equation (44) with equation (2) we can identify the following correspondences:

$$\delta_v \leftrightarrow \delta|_{\text{Newtonian}}, \quad \frac{k^2 - 3K}{k^2}v_\chi \leftrightarrow \delta v|_{\text{Newtonian}}, \quad \frac{k^2 - 3K}{k^2}\varphi_\chi \leftrightarrow \delta \Phi|_{\text{Newtonian}}. \quad (45)$$

For the potential and density variables we can also identify variables which behave similarly

$$\delta_v = 3H\frac{a}{k}\delta, \quad v_\chi = \frac{k}{a}\chi v = -\frac{ak}{k^2 - 3K}\kappa v, \quad \varphi_\chi = -\alpha v = -H\chi \varphi, \quad (46)$$

where we used equations (19), (22), (23) and (24).

From a given solution we can derive all the rest of the solutions even in other gauge conditions. This can be done either by using the gauge invariant combination of variables or directly through gauge transformations. General solutions in a pressureless medium are presented in H1. From the complete solutions presented in Tables 1 and 2 of H1 we can identify variables in certain gauges which correspond to the Newtonian ones. These are summarized in Table 1.

[[TABLE 1.]]

From Table 1 we notice that in the small scale limit, except for the uniform-density gauge where $\delta \equiv 0$, $\delta$ in all gauge conditions behaves in the same way. Thus, as in equation (45), it may be natural to identify $\delta$ in the comoving gauge with the corresponding Newtonian one.

Notice that, although we have horizons in the relativistic analysis the equations for $\delta_v$, $v_\chi$ and $\varphi_\chi$ keep the same form as the corresponding Newtonian equations in the general scale. Considering this as an additional point we regard $\delta_v$, $v_\chi$ and $\varphi_\chi$ as most closely corresponding ones to the Newtonian variables.

VI. RELATIVISTIC COSMOLOGICAL HYDRODYNAMICS

In the previous sections we have shown that the gauge invariant combinations $\delta_v$, $v_\chi$ and $\varphi_\chi$ behave most similarly to the Newtonian $\delta \equiv \delta \varphi/\varphi$, $\delta v$ and $\delta \Phi$. The equations remain the same in a general scale which includes the superhorizon scales in the relativistic situation. In this section, we will present the equations for $\delta_v$, $v_\chi$ and $\varphi_\chi$, including the effects of the general pressure terms. Equations (11)–(17) are the basic set of perturbation equations in a gauge ready form.

From equations (13), (16) and (17) we have

$$\dot{\delta}_v - 3Hw\delta_v = -(1 + w) \frac{k^2 - 3K}{k^2}v_\chi - 2H\frac{k^2 - 3K}{a^2}\sigma \mu. \quad (47)$$

From equations (14) and (17) we have

$$\dot{v}_\chi + Hv_\chi = -\frac{k}{a}\varphi_\chi + \frac{k}{a(1 + w)} \left[ c_s^2\delta_v + \frac{e}{\mu} - 8\pi G(1 + w)\sigma - \frac{2}{3}\frac{k^2 - 3K}{a^2}\sigma \mu \right]. \quad (48)$$

From equations (12) and (13) we have

$$\frac{k^2 - 3K}{a^2}\varphi_\chi = 4\pi G\mu \delta_v. \quad (49)$$

From equations (11), (13) and (14) we have

$$\dot{\varphi}_\chi + Hv_\chi = -4\pi G(\mu + p)\frac{a}{k}v_\chi - 8\pi G H\sigma. \quad (50)$$
Considering the correspondences in equation (43) we can immediately see that equations (47)-(50) have the correct Newtonian limit expressed in equations (3).

Combining equations (47)-(50) we can derive the closed form expressions for the $\delta_v$, $v_\chi$ and $\varphi_\chi$ which are the relativistic counterpart of equations (3). We have:

$$
\ddot{\delta}_v + (2 + 3c_s^2 - 6w) H \dot{\delta}_v + \left[ c_s^2 k^2 a^2 - 4\pi G \mu \left( 1 - 6c_s^2 + 6w - 3w^2 \right) + 12 \left( w - c_s^2 \right) \frac{K}{a^2} + (3c_s^2 - 5w) \Lambda \right] \delta_v \\
= \frac{1 + w}{a^2 H} \left[ \frac{H^2}{a(\mu + p)} \left( \frac{a^3 \mu}{H} \right) \right] \dot{\delta}_v + c_s^2 k^2 \delta_v \\
= - \frac{k^2 - 3K}{\mu} \left\{ e + 2H \dot{\sigma} + 2 \left[ - \frac{1}{3} \frac{k^2}{a^2} + 2H \right] \left( 1 + c_s^2 \right) \right\}.
$$

$$
\ddot{\varphi}_\chi + (4 + 3c_s^2) \frac{H}{a} \dot{\varphi}_\chi + \left[ c_s^2 k^2 \frac{a^2}{a^2} + 8\pi G \mu \left( c_s^2 - w \right) - 2 \left( 1 + 3c_s^2 \right) \frac{K}{a^2} + (1 + c_s^2) \Lambda \right] \varphi_\chi \\
= \frac{\mu + p}{H} \left[ \frac{H^2}{a(\mu + p)} \left( \frac{a^3 \mu}{H} \right) \right] \dot{\varphi}_\chi + c_s^2 k^2 \varphi_\chi \\
= - 4\pi G \left\{ e + 2H \dot{\sigma} + 2 \left[ - \frac{1}{3} \frac{k^2}{a^2} + 2H \right] \right\}.
$$

It may be an interesting exercise to show that the above equations are indeed valid for general $K$ and $\Lambda$, and for the general equation of state $p = p(\mu)$; use $w = -3H(1 + w)(c_s^2 - w)$.

If we ignore the entropic and anisotropic pressures ($e = 0 = \sigma$) on scales larger than Jeans scale equation (52) immediately leads to a general integral form solution as (see §3.2.3 of Hwang 1991b, and §V of Hwang & Vishniac 1990)

$$
\varphi_\chi(k, t) = 4\pi G C(k) \frac{H}{a} \int_0^t \frac{a(\mu + p)}{H^2} dt + \frac{H}{a} d(k) \\
= C(k) \left[ 1 - \frac{H}{a} \int_0^t a \left( 1 - \frac{K}{a^2} \right) dt \right] + \frac{H}{a} d(k).
$$

Solutions for $\delta_v$ and $v_\chi$ follow from equations (39) and (50), respectively, as

$$
\delta_v(k, t) = \frac{k^2 - 3K}{4\pi G \mu a^2} \varphi_\chi(k, t),
$$

$$
v_\chi(k, t) = - \frac{k}{4\pi G(\mu + p)a^2} \left\{ C(k) \left[ \frac{K}{a} - H \int_0^t a \left( 1 - \frac{K}{a^2} \right) dt \right] + H d(k) \right\}.
$$

We stress that these large scale asymptotic solutions are valid for general $K$ and $\Lambda$, and for general $p = p(\mu)$. For $K = 0 = \Lambda$ and $w = \text{constant}$ we have $a \propto t^{2/[3(1+w)]}$ and equations (39)-(42) become:

$$
\varphi_\chi(k, t) = \frac{3(1 + w)}{5 + 3w} C(k) + \frac{2}{3(1 + w)} \frac{1}{a} d(k) \propto \text{constant}, \quad t^{- \frac{3 + 3w}{3(1 + w)}},
$$

$$
\delta_v(k, t) \propto t^{- \frac{3(1 + 3w)}{5 + 3w}}, \quad t^{- \frac{3 + 3w}{3(1 + w)}},
$$

$$
v_\chi(k, t) \propto t^{- \frac{3 + 3w}{3(1 + w)}}, \quad t^{- \frac{3 + 3w}{3(1 + w)}}.
$$

We note that $C(k)$ and $d(k)$ are integration constants which are independent of the temporal evolution of the background equation of state, i.e., constants for general $p = p(\mu)$.

The equations in this section and equations (1)-(3) include general pressures which may account for the nonequilibrium or dissipative effects in hydrodynamic flows in the cosmological context (with general $K$ and $\Lambda$). The equations are expressed in general forms so that they can include the case of the scalar field and classes of generalized gravity theories. Application of the gauge ready formalism to the minimally coupled scalar field was made in Hwang (1994b), and to the generalized gravity theories in Hwang & Noh (1996).

\[\text{We correct a typographical error in equation (129) of Hwang & Vishniac: } c_0/a \text{ should be replaced by } c_0.\]
A. A Conserved Quantity

It is known that the curvature fluctuation in the comoving gauge, $\varphi_v$, is conserved in the large scale limit independently of the changes in the background equation of state (Hwang & Hyun 1994). From equation (19) we have

$$\varphi_v = \varphi_{\chi} - \frac{aH}{k} v_{\chi}. \quad (57)$$

Thus, from equations (53) and (55) we can derive

$$\varphi_v(k, t) = C(k) \left[ 1 + \frac{K}{a^2} \frac{1}{4\pi G(\mu + p)} \left[ 1 - \frac{H}{a} \int_0^t a \left( 1 - \frac{K}{a^2} \right) dt \right] \right] + \frac{K}{a^2} \frac{H/a}{4\pi G(\mu + p)} d(k). \quad (58)$$

For $K = 0$ (but for general $\Lambda$) we have

$$\varphi_v(k, t) = C(k), \quad (59)$$

with the vanishing decaying mode; the disappearance of the decaying mode in equation (59) implies that the dominating decaying modes in equations (53) and (55) cancel out for $K = 0$. Thus, for $K = 0$, $\varphi_v$ is conserved for generally varying background equation of state, i.e., for general $p = p(\mu)$. This conservation property of the curvature variable in a certain gauge remains to be true for models which are based on a minimally coupled scalar field or even on classes of generalized gravity theories; in the generalized gravity the uniform-field gauge is more suitable for handling the conservation property, and the uniform-field gauge coincides with the comoving gauge in the minimally coupled scalar field (Hwang 1994b and Hwang & Noh 1996). For a pressureless medium equation (58) reduces to

$$\varphi_v(k, t) = C(k) \left[ 1 + KH \int_0^t \frac{dt}{a^2} \right] + KH \frac{1}{4\pi G\mu a^3} d(k). \quad (60)$$

VII. DISCUSSION

We would like to make comments on related works in the books by Weinberg (1972) and Peebles (1993). Equation (15.10.57) in Weinberg (1972) and equation (10.118) in Peebles (1993) are in error. The correction in the case of Weinberg’s was made in Hwang (1991a); in a medium with nonvanishing pressure the equation for the density fluctuation in the synchronous gauge becomes a third order because of the presence of a gauge mode in addition to the physical growing and decaying modes. The truncated second order equations in Weinberg and Peebles will pick up a gauge mode instead of the physical decaying mode in the synchronous gauge. The error in Peebles is based on imposing the synchronous gauge and the comoving gauge simultaneously. In a medium with nonvanishing pressure one cannot impose the two gauge conditions simultaneously even in the large scale limit. In the Appendix we elaborate our point.

In this paper we have tried to identify the variables in the relativistic perturbation analysis which reproduce the correct Newtonian behavior in the pressureless limit. In §IV we have shown that $\delta$, $\delta v$ and $\varphi$ in the uniform-expansion gauge reproduce the Newtonian behaviors in the small scale limit (i.e., on scales smaller than the visual horizon). However, the variables change their behaviors near and outside horizon scale. Meanwhile, $\delta$ in the comoving gauge and $v$ and $\varphi$ in the zero-shear gauge show the same behavior as the corresponding Newtonian variables in a general scale. In the small scale limit the density perturbation in most of the gauge conditions correctly reproduces the Newtonian behavior. In fact, these results were already presented in H1. Comparing with H1, in the present work we tried to reinforce the correspondence by showing explicitly the second order differential equations which are satisfied by the variables in several gauge conditions. Various general and asymptotic solutions for every variable in the pressureless medium are presented in the Tables of H1.

JH wishes to thank Prof. R. Brandenberger for interesting correspondences. This work was supported by the KOSEF, Grants No. 95-0702-04-3 and No. 961-0203-013-1, and through the SRC program of SNU-CTP.
APPENDIX: ANALYSIS IN THE SYNCHRONOUS GAUGE.

In the following we correct a minor confusion in the literature concerning perturbation analyses in the synchronous gauge. The argument is based on Hwang (1991a; H2 hereafter). We consider a situation with \( K = 0 = \Lambda, e = 0 = \sigma \) and \( w = \text{constant} \) (thus \( c_s^2 = w \)).

The equation for the density perturbation in the comoving gauge is given in equation (51). In our case we have

\[
\ddot{\delta}_v + (2 - 3w)H \dot{\delta}_v + \left[ c_s^2 k^2/a^2 - 4\pi G \mu (1 - w)(1 + 3w) \right] \delta_v = 0. \tag{A1}
\]

The solution in the large scale limit is presented in equation (56).

In the synchronous gauge, from equations (15)-(17) we have

\[
\ddot{\delta}_\alpha + 2H \dot{\delta}_\alpha + \left[ c_s^2 k^2/a^2 - 4\pi G \mu (1 + w)(1 + 3w) \right] \delta_\alpha + 3w(1 + w) \frac{k}{a} Hv_\alpha = 0, \tag{A2}
\]

\[
\dot{v}_\alpha + (1 - 3w)Hv_\alpha - \frac{k}{a(1 + w)} \delta_\alpha = 0. \tag{A3}
\]

From these two equations we can derive a third order differential equation for \( \delta_\alpha \) as

\[
\dddot{\delta}_\alpha + \frac{11 - 3w}{2}H \ddot{\delta}_\alpha + \left( \frac{5 - 24w - 9w^2}{2}H^2 + c_s^2 \frac{k^2}{a^2} \right) \dot{\delta}_\alpha + \left[ \frac{3}{4} (1 + w) (1 + 3w) (-1 + 9w) H^3 + \frac{3}{2} (1 + w) Hc_s^2 \frac{k^2}{a^2} \right] \delta_\alpha = 0. \tag{A4}
\]

In the large scale limit the solutions are

\[
\delta_\alpha \propto t^{\frac{2(1 + 3w)}{3(1 + w)}}, \; t^{\frac{2(1 + 3w)}{9(1 + w)}}, \; t^{-1}. \tag{A5}
\]

In the Appendix B of H2 we made an argument that the third solution with \( \delta_\alpha \propto t^{-1} \) is nothing but a gauge mode for a medium with \( w \neq 0 \). (For a pressureless case the physical decaying mode also behaves as \( t^{-1} \) and the second solution in eq.[A5] is invalid; see §4 of H2). (As mentioned before, the combination \( \delta_\alpha \) is not gauge invariant. We have \( \dot{\delta}_\alpha \equiv \delta + 3H(1 + w) \int^t \alpha \, dt \) and the lower bound of the integration gives rise to the gauge mode, thus behaving as proportional to \( t^{-1} \); see eq.[44] of Hwang 1991b). In Eq. (B5) of H2 we derived the relation between solutions in the two gauges explicitly; the growing modes are the same in both gauges whereas the decaying modes differ by a factor \( (k/aH)^2 \).

Now, we would like to point out that even in the large scale limit one cannot ignore the last term in equation (A2). If we ignore the last term in equation (A2) we recover equation (15.10.57) in Weinberg (1972) and equation (10.118) in Peebles (1993) which is

\[
\ddot{\delta}_\alpha + 2H \dot{\delta}_\alpha + \left[ c_s^2 k^2/a^2 - 4\pi G \mu (1 + w)(1 + 3w) \right] \delta_\alpha = 0. \tag{A6}
\]

In the large scale we have solutions

\[
\delta_\alpha \propto t^{\frac{2(1 + 3w)}{3(1 + w)}}, \; t^{\frac{2(1 + 3w)}{9(1 + w)}}, \; t^{-1}. \tag{A7}
\]

By ignoring the last term in equation (A2), in the large scale limit we happen to recover the fictitious gauge mode under the price of losing the physical decaying mode. Thus, in the large scale limit one cannot ignore the last term in equation (A2) in a medium with general pressure; this is apparent in equation (A4). Also, one cannot impose both the synchronous gauge condition and the comoving gauge condition simultaneously. If we simultaneously impose such two conditions, thus setting \( v \equiv 0 \equiv \alpha \), from equation (17) we have

\[
0 = \frac{k}{a(1 + w)} \left( c_s^2 \delta + \frac{e}{\mu} - \frac{2 k^2 - 3 K \sigma}{3 \frac{k^2}{a^2} - \mu} \right). \tag{A8}
\]

Thus, for \( e = 0 = \sigma \) and medium with nonvanishing pressure we have \( \delta = 0 \) which is a meaningless system; this argument remains valid even in the large scale limit.
REFERENCES

Bardeen, J. M. 1980, Phys. Rev. D, 22, 1882
———. 1988, in Particle Physics and Cosmology, ed. A. Zee (London: Gordon & Breach), 1p
Bonner, W. B. 1957, MNRAS, 107, 104
Hwang, J. 1991a, Gen. Rel. Grav., 23, 235 (H2)
———. 1991b, ApJ, 375, 443
———. 1993, ApJ, 415, 486
———. 1994a, ApJ, 427, 533 (H1)
———. 1994b, ApJ, 427, 542
Hwang, J., & Hyun, J. J. 1994, ApJ, 420, 512
Hwang, J., & Noh, H. 1996, Phy. Rev. D, 54, 1460
Hwang, J., & Vishniac, E. T. 1990, ApJ, 353, 1
Kodama, H., & Sasaki, M. 1984, Prog. Theor. Phys. Suppl., 78, 1
Lifshitz, E. M. 1946, J. Phys. (USSR), 10, 116
Nariai, H. 1969, Prog. Theor. Phys., 41, 686
Peebles, P. J. E. 1980, The Large-Scale Structure of the Universe (Princeton: Princeton Univ. Press)
———. 1993, Principles of Physical Cosmology (Princeton: Princeton Univ. Press)
Sakai, K. 1969, Prog. Theor. Phys., 41, 1461
Weinberg, S. 1972, Gravitation and Cosmology (Wiley: New York)

Table 1. Newtonian correspondences: For the synchronous gauge we ignore the gauge mode. Thus the synchronous gauge is equivalent to the comoving gauge. Dots (…) indicate that the behavior differs from the Newtonian one. The small scale implies the scale smaller than the visual horizon. Explicit forms of exact and asymptotic solutions for every variable are presented in H1.

| Gauge                  | Variable | General Scale | Small Scale |
|------------------------|----------|---------------|-------------|
| Comoving gauge         | \( \delta_v \) | Newtonian     | Newtonian   |
| Zero-shear gauge       | \( \delta_\chi \) | \ldots        | Newtonian   |
| Uniform-expansion gauge | \( \delta_\kappa \) | \ldots        | Newtonian   |
| Synchronous gauge      | \( \delta_\alpha \) | Newtonian     | Newtonian   |
| Uniform-curvature gauge | \( \delta_\varphi \) | \ldots        | Newtonian   |
| Uniform-density gauge  | \( \delta \equiv 0 \) | 0             | 0           |
| Comoving gauge         | \( v \equiv 0 \) | 0             | 0           |
| Zero-shear gauge       | \( v_\chi \) | Newtonian     | Newtonian   |
| Uniform-expansion gauge | \( v_\kappa \) | \ldots        | Newtonian   |
| Synchronous gauge      | \( v_\alpha \) | 0             | 0           |
| Uniform-curvature gauge | \( v_\varphi \) | \ldots        | \ldots      |
| Uniform-density gauge  | \( v_\delta \) | \ldots        | \ldots      |
| Comoving gauge         | \( \varphi_v \) | \ldots        | \ldots      |
| Zero-shear gauge       | \( \varphi_\chi \) | Newtonian     | Newtonian   |
| Uniform-expansion gauge | \( \varphi_\kappa \) | \ldots        | Newtonian   |
| Synchronous gauge      | \( \varphi_\alpha \) | \ldots        | \ldots      |
| Uniform-curvature gauge | \( \varphi_\equiv 0 \) | 0             | 0           |
| Uniform-density gauge  | \( \varphi_\delta \) | \ldots        | \ldots      |