Expressiveness of Extended Bounded Response \begin{math}\text{LTL}\end{math}

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Extended Bounded Response \begin{math}\text{LTL}\end{math} with \begin{math}\text{Past}\end{math} \begin{math}\text{(LTL}^{\text{EBR}}+\text{P})\end{math} is a safety fragment of Linear Temporal Logic with \begin{math}\text{Past}\end{math} \begin{math}(\text{LTL}+\text{P})\end{math} that has been recently introduced in the context of reactive synthesis. The strength of \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math} is a fully symbolic compilation of formulas into symbolic deterministic automata. Its syntax is organized in four levels. The first three levels feature (a particular combination of) future temporal modalities, the last one admits only past temporal operators. At the base of such a structuring there are algorithmic motivations: each level corresponds to a step of the algorithm for the automaton construction. The complex syntax of \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math} made it difficult to precisely characterize its expressive power, and to compare it with other \begin{math}\text{LTL}+\text{P}\end{math} safety fragments.

In this paper, we first prove that \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math} is expressively complete with respect to the safety fragment of \begin{math}\text{LTL}+\text{P}\end{math}, that is, any safety language definable in \begin{math}\text{LTL}+\text{P}\end{math} can be formalized in \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math}, and vice versa. From this, it follows that \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math} and Safety-\begin{math}\text{LTL}\end{math} are expressively equivalent. Then, we show that past modalities play an essential role in \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math}: we prove that the future fragment of \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math} is strictly less expressive than full \begin{math}\text{LTL}^{\text{EBR}}+\text{P}\end{math}.

1 Introduction

Linear Temporal Logic (\begin{math}\text{LTL}\end{math}) was introduced in the late seventies \cite{13} as a modal logic for reasoning over computer programs, modeling their computations as state sequences (\emph{i.e.}, linear orders) that represent the state a computer program is in at a given time. \begin{math}\text{LTL}\end{math} originally used temporal modalities for moving only in the future of a time point. Later, it turned out that adding modalities for moving in the past (we refer to this logic as \begin{math}\text{LTL}+\text{P}\end{math}) does not add expressive power to \begin{math}\text{LTL}\end{math} \cite{10}, but only succinctness \cite{11}. The definition of the operators in the syntax of \begin{math}\text{LTL}+\text{P}\end{math} was proved to be carefully designed. In fact, Kamp \cite{8} as well as Gabbay \emph{et al.} \cite{6} proved that the properties that one can formalize in \begin{math}\text{LTL}+\text{P}\end{math} are exactly those definable in the first-order fragment of the \begin{math}\text{monadic second-order theory of one successor}\end{math} (S1S, for short), which is in turn decidable \cite{1, 2}.

Among the different properties that one can define in \begin{math}\text{LTL}+\text{P}\end{math}, two notable classes are the set of \begin{math}\text{safety}\end{math} and \begin{math}\text{co-safety}\end{math} properties. Safety properties express the intuitive requirement that \begin{math}\text{something bad never happens}\end{math}, and thus each counterexample of a safety property is finite. Co-safety properties are duals of safety properties: each state sequence that satisfies the property has a finite witness. The safety and...
co-safety classes play a crucial role in verification and synthesis, since their main feature of having finite witnesses makes in general the problems much simpler \cite{9,13}.

Several safety fragments of LTL have been introduced over the years. One of the most natural examples is Safety-LTL \cite{3,16,18}. The Safety-LTL logic is defined as the set of all and only those formulas of LTL (with only future modalities) such that, when in negated normal form, do not contain existential temporal operators (like the until operator). In \cite{3}, Chang et al. proved that all the safety properties definable in LTL are expressible in Safety-LTL as well, and vice versa.

Extended Bounded Response LTL with Past (LTL\textsubscript{EBR}+P) is a recently introduced safety fragment of LTL+P with an efficient reactive synthesis problem. In addition to the fact that realizability from LTL\textsubscript{EBR}+P specifications is EXPTIME-complete (while LTL realizability is 2EXPTIME-complete), in practice realizability and synthesis from LTL\textsubscript{EBR}+P specifications turned out to be much more efficient than other approaches \cite{4}. The syntax of LTL\textsubscript{EBR}+P is articulated over layers: the first three layers comprise a combination of future temporal modalities, while the last layer includes only past temporal operators. Each of the layers was carefully designed in order to correspond to a step of the algorithm for constructing a symbolic automaton starting from an LTL\textsubscript{EBR}+P specification. This results into a great performance improvement in practice, but the syntax of LTL\textsubscript{EBR}+P makes it hard to find its exact expressive power, and, consequently, makes it hard also to compare it with other safety fragments of LTL+P, like, for instance, Safety-LTL.

In this paper we prove that LTL\textsubscript{EBR}+P is expressively complete with respect to the safety fragment of LTL+P. As a by-product, we obtain that LTL\textsubscript{EBR}+P and Safety-LTL are expressively equivalent. The core of the proof exploits a normal form theorem for each safety property definable in LTL+P \cite{3,17}, which establishes a correspondence between safety properties definable in LTL+P and properties of the form G\textsubscript{α}, where G is the globally operator of LTL and α is a pure past formula. Consequently, it is clear that the pure past layer of LTL\textsubscript{EBR}+P plays a crucial role for the expressive equivalence of LTL\textsubscript{EBR}+P.

We show that this layer is really necessary. In fact, we prove that LTL\textsubscript{EBR}, that is LTL\textsubscript{EBR}+P devoid of the pure past layer, is strictly less expressive than full LTL\textsubscript{EBR}+P. This is shown by proving that all the formulas of LTL\textsubscript{EBR} can constrain, for any time point \(i\) in an infinite state sequence, only a bounded prefix before (or interval around) \(i\). This implies that formulas that are able to constrain, for each time point \(i\), a prefix of unbounded (although finite) length before \(i\), like for instance \(G(p_1 \rightarrow Hp_2)\) (where \(H\) is the historically past operator of LTL+P), are not definable in LTL\textsubscript{EBR}.

The rest of the paper is organized as follows. In Section 2, we give the necessary background. The expressive power of LTL\textsubscript{EBR}+P is proved in Section 3. In Section 4, we prove that the future fragment of LTL\textsubscript{EBR}+P is strictly less expressive than LTL\textsubscript{EBR}+P. Finally, we summarize the results of the paper in Section 5.

## 2 Preliminaries

In this section, we give the definitions that are necessary throughout the paper.

### 2.1 Linear Temporal Logic

Linear Temporal Logic (LTL) is a modal logic interpreted over infinite, discrete linear orders \cite{5,13}. Syntactically, LTL can be seen as an extension of propositional logic with the addition of the next operator (\(X\phi\), i.e., at the next state \(\phi\) holds) and the until operator (\(\phi_1 \mathcal{U} \phi_2\), i.e., \(\phi_2\) will eventually hold and \(\phi_1\) will hold until then).
LTL with Past (LTL+P) extends LTL with the addition of temporal operators able to talk about what happened in the past with respect to the current time, and it is obtained from LTL by adding the following past temporal operators: (i) the yesterday operator (Yφ, i.e., there exists a previous state in which φ holds); (ii) the weak yesterday operator (Zφ, i.e., either a previous state does not exists or in the previous state φ holds); (iii) and the since operator (φ₁ S φ₂, i.e., there was a past state where φ₂ held, and φ₁ has held since then). We will now briefly recall the syntax and semantics of LTL+P, which encompasses that of LTL as well. Formally, given a set Σ of proposition letters, LTL+P formulas over Σ are generated by the following grammar:

$$\phi ::= p \mid \neg \phi \mid \phi₁ \lor \phi₂ \mid \phi₁ \land \phi₂ \mid X\phi₁ \mid \phi₁ U \phi₂ \mid \phi₁ R \phi₂ \mid F\phi₁ \mid G\phi₁ \mid Y\phi₁ \mid \phi₁ S \phi₂ \mid \phi₁ T \phi₂ \mid O\phi₁ \mid H\phi₁ \mid Z\phi₁$$

where $p \in \Sigma$ and $\phi₁$ and $\phi₂$ are LTL+P formulas. Most of the temporal operators of the language can be defined in terms of a small number of basic ones. We refer to [4] for the definition of these shortcuts. We say that an LTL+P formula is pure past if and only if all the temporal operators inside the formula are past operators. We call pure past LTL, written as LTLₚ, the fragment of LTL+P containing only pure past formulas.

Formulas from LTL+P are interpreted over state sequences. A state sequence $\sigma = \langle \sigma₀, \sigma₁, \ldots \rangle$ is an infinite, linearly ordered sequence of states, where each state $\sigmaᵢ$ is a set of proposition letters, that is $\sigmaᵢ \in 2^{Σ}$ for $i \in \mathbb{N}$. We will interchangeably use also the term ω-word over the alphabet $2^{Σ}$ for referring to a state sequence. A set of ω-words is called ω-language. Given two indices $i, j \in \mathbb{Z}$, with $i \leq j$, we denote as $\sigma[i, j]$ the interval of $\sigma$ from index $i$ to index $j$, that is $\langle \sigmaᵢ, \ldots, \sigmaⱼ \rangle$ if $i \geq 0$, or $\langle \sigma₀, \ldots, \sigmaᵢ \rangle$ otherwise. With $\sigma[i, ∞]$ we denote the (infinite) suffix of $\sigma$ starting from $i$.

Given a state sequence $\sigma$, a position $i \geq 0$, and an LTL+P formula $\phi$, we inductively define the satisfaction of $\phi$ by $\sigma$ at position $i$, written as $\sigma, i \models \phi$, as follows:

1. $\sigma, i \models p$ \text{ iff } $p \in \sigmaᵢ$;
2. $\sigma, i \models \neg \phi$ \text{ iff } $\sigma, i \not\models \phi$;
3. $\sigma, i \models \phi₁ \lor \phi₂$ \text{ iff } $\sigma, i \models \phi₁$ or $\sigma, i \models \phi₂$;
4. $\sigma, i \models X\phi$ \text{ iff } $\sigma, i+1 \models \phi$;
5. $\sigma, i \models Y\phi$ \text{ iff } $i > 0$ and $\sigma, i-1 \models \phi$;
6. $\sigma, i \models \phi₁ U \phi₂$ \text{ iff there exists } $j \geq i$ such that $\sigma, j \models \phi₂$, and $\sigma, k \models \phi₁$ for all $k$, with $i \leq k < j$;
7. $\sigma, i \models \phi₁ S \phi₂$ \text{ iff there exists } $j \leq i$ such that $\sigma, j \models \phi₂$, and $\sigma, k \models \phi₁$ for all $k$, with $j \leq k \leq i$;

We say that $\sigma$ satisfies $\phi$, written as $\sigma \models \phi$, if it satisfies the formula at the first state, i.e., if $\sigma, 0 \models \phi$: in this case, we call $\sigma$ a model of $\phi$. We say that two formulas $\phi$ and $ψ$ are equivalent ($\phi \equiv ψ$) if and only if they are satisfied by the same set of state sequences.

If $\phi$ is a full LTL+P formula, then we define the language of $\phi$, written $L(\phi)$, as $L(\phi) = \{ \sigma \in (2^{Σ})^ο \mid \sigma \models \phi \}$. If, instead, $\phi$ contains only past operators, we change the definition of language as follows: for all $\phi \in LTLₚ$, we define the language over finite words of $\phi$ as $L^{<ω}(\phi) := \{ \sigma \in (2^{Σ})^ω \mid \sigma = \langle σ₀, \ldots, σₙ \rangle \land σₙ \models \phi \}$. 


Notation  From now on, given a linear temporal logic $\mathbb{L}$, with some abuse of notation, we will denote with $\mathbb{L}$ also the set of formulas that syntactically belong to $\mathbb{L}$. Conversely, we denote with $[\mathbb{L}]$ the set of all and only those languages $\mathcal{L}$ of infinite words for which there exists a formula $\phi \in \mathbb{L}$ (i.e., $\phi$ syntactically belongs to $\mathbb{L}$) such that $\mathcal{L} = \mathcal{L}(\phi)$. For the $\text{LTL}_P$ logic, we write $[\text{LTL}_P]^{<\omega}$ for denoting the set of languages $\mathcal{L}$ over finite words such that $\mathcal{L} = \mathcal{L}^{<\omega}(\phi)$ for some $\phi \in \text{LTL}_P$.

It is known that past modalities do not add expressive power to $\text{LTL}$ $[6,10,11]$, therefore writing $[\text{LTL}]$ is the same as writing $[\text{LTL}+P]$.

2.2 $\omega$-regular expressions and (co-)Safety classes

We denote as $\text{REG}$ the set of regular languages of finite words $[7]$. An $\omega$-regular language is a set of $\omega$-words recognized by an $\omega$-regular expression, that is, an expression of the form $\bigcup_{i=1}^n U_i \cdot (V_i)^\omega$, where $n \in \mathbb{N}$ and $U_i, V_i \in \text{REG}$ for $i = 1, \ldots, n$. With $\omega$-$\text{REG}$, we denote the set of the $\omega$-regular languages. One of the seminal results in automata theory is the correspondence between $\omega$-regular languages and Büchi automata $[1,2]$. An important class of $\omega$-regular languages comprises those languages that express the fact that something “bad” (like for instance a deadlock, or a simultaneous access into a critical section by two different processes) never happens. For this reason, they are called safety languages (or safety properties).

Definition 1 (Safety language $[9]$). Let $\mathcal{L} \subseteq \Sigma^\omega$ be an $\omega$-regular language. We say that $\mathcal{L}$ is a safety language if and only if for all the words $\sigma \in \Sigma^\omega$ it holds that, if $\sigma \notin \mathcal{L}$, then $\exists i \in \mathbb{N} \cdot \forall \sigma' \in \Sigma^\omega \cdot \sigma[i \cdot] \cdot \sigma' \notin \mathcal{L}$. The class of safety $\omega$-regular languages is denoted as SAFETY.

Given some temporal logic $\mathbb{L}$, we say that $\mathbb{L}$ is a safety fragment of $\text{LTL}$ iff $\phi \in \mathbb{L}$ implies that $\phi \in \text{LTL}$, and $\mathcal{L}(\phi)$ is a safety language (Def. 1), for all formulas $\phi$. The class of the $\omega$-regular co-safety languages, that we call coSAFETY, is defined as the dual of SAFETY, that is the set of languages $\mathcal{L}$ such that $\mathcal{L} \in \text{coSAFETY}$ iff $\overline{\mathcal{L}} \in \text{SAFETY}$, where $\overline{\mathcal{L}}$ is the complement language of $\mathcal{L}$.

The $\text{SAFETY-LTL}$ logic $[3,16,18]$ is defined as the set of $\text{LTL}$ formulas such that, when in negated normal form, do not contain existential temporal operators (i.e., U and F). $\text{SAFETY-LTL}$ is a safety fragment of $\text{LTL}$ $[16]$.

We give an alternative and equivalent definition of the SAFETY class of [Def. 1] that will be useful in the following sections: $\text{SAFETY} := \{ \mathcal{L} \subseteq \Sigma^\omega \mid \overline{\mathcal{L}} = K \cdot \Sigma^\omega \land K \in \text{REG} \}$.

We define the class $\text{SAFETY}^{\text{SF}}$ (coSAFETY$^{\text{SF}}$) as the set obtained from SAFETY (resp. coSAFETY) by restricting $K$ to be a star-free expression, that is, a regular expression devoid of the Kleene star $[12]$. In particular, $\text{coSAFETY}^{\text{SF}} := \{ \mathcal{L} \subseteq \Sigma^\omega \mid \mathcal{L} = K \cdot \Sigma^\omega \land K \in \text{SF} \}$, where $\text{SF} \subseteq \text{REG}$ is the set of star-free regular expressions. With $\omega$-$\text{SF}$ we denote the set of star-free $\omega$-regular expressions. We now state some equivalence results that will be helpful later. Star-free expressions (SF) and pure-past $\text{LTL}$ ($\text{LTL}_{P}$) have the same expressive power. The same holds for the $\omega$-$\text{SF}$ class and $\text{LTL}$.

Proposition 1 (Thomas $[17]$, Lichtenstein et al. $[10]$). $[\text{LTL}_{P}]^{<\omega} = \text{SF}$ and $[\text{LTL}] = \omega$-$\text{SF}$.

Finally, we will use the following normal-form theorem, stated in $[3]$, that proves that any LTL-definable safety (resp. co-safety) language can be expressed by a formula of the form $G\alpha$ (resp. $F\alpha$), and vice versa. An independent proof of this theorem can be derived also from the results by Thomas in $[17]$.

Theorem 1 (Chang et al. $[3]$). $[\text{LTL}] \cap \text{SAFETY} = [G\alpha]$ and $[\text{LTL}] \cap \text{coSAFETY} = [F\alpha]$.

Fig. 1 summarizes the expressive power of the various fragments and logics, included $\text{LTL}_{EBR}+P$ and $\text{LTL}_{EBR}$ (that are the subject of this paper).
Figure 1: Comparison of expressiveness between the various formalisms. For ease of exposition, we highlighted the rectangle corresponding to LTL with thick borders.

2.3 Extended Bounded Response LTL

Extended Bounded Response LTL with Past (LTL_{EBR}+P, for short) is a fragment of LTL+P, recently introduced in the context of reactive synthesis [4]. Here below, we recall its syntax.

**Definition 2** (The logic LTL_{EBR}+P [4]). Let $a, b \in \mathbb{N}$. An LTL_{EBR}+P formula $\chi$ is inductively defined as follows:

\[
\begin{align*}
\eta & := p \mid \neg \eta \mid \eta_1 \lor \eta_2 \mid Y \eta \mid \eta_1 S \eta_2 & \text{Pure Past Layer} \\
\psi & := \eta \mid \neg \psi \mid \psi_1 \lor \psi_2 \mid X \psi \mid \psi_1 U^{[a,b]} \psi_2 & \text{Bounded Future Layer} \\
\phi & := \psi \mid \phi_1 \land \phi_2 \mid X \phi \mid G \phi \mid \psi R \phi & \text{Future Layer} \\
\chi & := \phi \mid \chi_1 \lor \chi_2 \mid \chi_1 \land \chi_2 & \text{Boolean Layer}
\end{align*}
\]
We define the bounded until operator \( \psi_1 \cup_{[a,b]} \psi_2 \) as a shortcut for the LTL formula
\[
\bigvee_{i=a}^{b}(X_1 \ldots X_i(\psi_2) \land \bigwedge_{j=0}^{i-1} X_1 \ldots X_j(\psi_1)).
\]
This means that LTL\(_{EBR+P}\) features really only universal temporal modalities (i.e., \( X, G, \) and \( R \)), and thus it is a syntactical fragment of LTL+P and also a safety fragment (see Theorem 3.1 in [16]). We define LTL\(_{EBR}\) as the fragment of LTL\(_{EBR+P}\) devoid of the full past layer. The syntax of LTL\(_{EBR+P}\) is articulated over layers, that impose some syntactical restrictions on the formulas that can be generated from the grammar. For example, LTL\(_{EBR+P}\) forces the leftmost argument of any \emph{release} operator to contain no universal temporal modalities (i.e., \( R \) and \( G \)). Originally, the layered structure was guided by the steps of the algorithm for the construction of symbolic automata starting from LTL\(_{EBR+P}\)-formulas. We refer the reader to [4] for more details.

All formulas in LTL\(_{EBR+P}\) can be transformed into a \emph{canonical form} (defined here below) by maintaining the equivalence.

**Definition 3** (Canonical Form of LTL\(_{EBR+P}\) [4]). The canonical form of LTL\(_{EBR+P}\) is the set of all and only the formulas of the following type:

\[
\begin{align*}
X^i \alpha_i \otimes \cdots \otimes X^j \alpha_j \otimes \\
X^{i+1} G \alpha_{i+1} \otimes \cdots \otimes X^k \alpha_k \otimes \\
X^{i+k+1}(\alpha_{i+k+1}, R \beta_{i+k+1}) \otimes \cdots \otimes X^{i+h}(\alpha_{i+h}, R \beta_{i+h})
\end{align*}
\]

where each \( \alpha_i, \beta_i \in \text{LTL}_P \), \( \otimes \in \{\land, \lor\} \), and \( i, j, k, h \in \mathbb{N} \).

### 3 Expressive power of LTL\(_{EBR+P}\)

In this section, we study the expressiveness of the LTL\(_{EBR+P}\) logic. In particular, we compare the set of languages definable in LTL\(_{EBR+P}\) with the set of safety languages expressible in LTL, and prove that the two sets are equal, that is \( [\text{LTL\(_{EBR+P}\)}] = [\text{LTL}] \cap \text{SAFETY} \). Consequently, LTL\(_{EBR+P}\) and Safety-LTL are expressively equivalent (i.e., \( [\text{LTL\(_{EBR+P}\)}] = [\text{Safety-LTL}] \)).

First we recall the normal-form theorem stated in [Th. 1] establishing that \( [\text{LTL}] \cap \text{SAFETY} = [G \alpha] \). Proving that \( [\text{LTL\(_{EBR+P}\)}] = [\text{LTL}] \cap \text{SAFETY} \) is straightforward. In [16], Sistla proved that any fragment of LTL+P with only universal (future) temporal operators (i.e., \( X, R, \) and \( G \)) defines only safety properties, and thus is a safety fragment of LTL+P. Since LTL\(_{EBR+P}\)-formulas contain only universal (future) temporal operators, it follows that LTL\(_{EBR+P}\) is a safety fragment of LTL+P (this corresponds to the left-to-right direction). For the right-to-left direction it suffices to show that the normal form \( G \alpha \) is syntactically definable in LTL\(_{EBR+P}\) (i.e., \( G \alpha \in \text{LTL\(_{EBR+P}\)} \) and thus also \( \mathcal{L}(G \alpha) \in [\text{LTL\(_{EBR+P}\)}] \), for any \( \alpha \in \text{LTL}_P \).}

**Theorem 2.** \([\text{LTL\(_{EBR+P}\)}] = [\text{LTL}] \cap \text{SAFETY} \).

**Proof.** We first prove that \( [\text{LTL\(_{EBR+P}\)}] \subseteq [\text{LTL}] \cap \text{SAFETY} \). Let \( \phi \in [\text{LTL\(_{EBR+P}\)}] \). By Def. 2, \( \phi \in \text{LTL}+P \), and thus, since \( [\text{LTL}] = [\text{LTL}+P] \), it holds that \( \mathcal{L}(\phi) \in [\text{LTL}] \). Moreover, since \( \text{LTL\(_{EBR+P}\)} \) contains only universal temporal operators, by Theorem 3.1 in [16], it is a safety fragment of LTL, and we have that \( \mathcal{L}(\phi) \in \text{SAFETY} \). Therefore, \( \mathcal{L}(\phi) \in [\text{LTL}] \cap \text{SAFETY} \).

We now prove that \( [\text{LTL}] \cap \text{SAFETY} \subseteq [\text{LTL\(_{EBR+P}\)}] \). Let \( \phi \) be a formula such that \( \mathcal{L}(\phi) \in [\text{LTL}] \cap \text{SAFETY} \). By [Th. 1], \( \mathcal{L}(\phi) \in [G \alpha] \). Now, \( G \alpha \) (for any \( \alpha \in \text{LTL}_P \)) is a formula that syntactically belongs to LTL\(_{EBR+P}\), that is \( G \alpha \in \text{LTL\(_{EBR+P}\)} \), and thus \( [G \alpha] \subseteq [\text{LTL\(_{EBR+P}\)}] \). It follows that \( \mathcal{L}(\phi) \in [\text{LTL\(_{EBR+P}\)}] \). \( \square \)
3.1 Comparison between $\text{LTL}_{\text{EBR}+P}$, $G\alpha$ and Safety-LTL

**Comparison with $G\alpha$**  
Previously, we proved that the set of languages definable in $\text{LTL}_{\text{EBR}+P}$ is exactly the set of safety languages definable in $\text{LTL}+P$. In turn, [Th. 1] shows that these sets correspond to languages definable by a formula of type $G\alpha$, where $\alpha \in \text{LTL}$. Despite being equivalent fragments, we think that $\text{LTL}_{\text{EBR}+P}$ offers a more natural language for safety properties than the $G\alpha$ fragment. Consider for example the following property, expressed in natural language: either there exists two time points $t' \leq t$ such that (i) $p_1$ holds in $t$, (ii) $p_2$ holds in $t'$, and (iii) $p_2$ holds from time point 0 to $t$. The property can be easily formalized in $\text{LTL}_{\text{EBR}+P}$ by the formula $p_1 \cdot (p_2 \cdot O(p_1)) \cdot H(p_3)$. The equivalent formula in the $G\alpha$ fragment is $G(H(p_3) \lor O(p_2 \land O(p_1)) \land H(p_3))$, which is arguably more intricate.

**Comparison with Safety-LTL**  
Safety-LTL is the fragment of $\text{LTL}$ (thus with only future temporal modalities) containing all and only the $\text{LTL}$-formulas that, when in negated normal form, do not contain any until or eventually operator. In [15], Sistla proved that this fragment expresses only safety properties, that is $[\text{Safety-LTL}] \subseteq [\text{LTL}] \cap \text{SAFETY}$. The converse direction, that is $[\text{LTL}] \cap \text{SAFETY} \subseteq [\text{Safety-LTL}]$, is reported in [3]. It immediately follows that $\text{LTL}_{\text{EBR}+P}$ and Safety-LTL are expressively equivalent, namely $[\text{LTL}_{\text{EBR}+P}] = [\text{Safety-LTL}]$.

Differently from $\text{LTL}_{\text{EBR}+P}$, Safety-LTL does not impose any syntactic restriction on the nesting of the logical operators; as a matter of fact, $G(p_1 \lor Gp_2)$ belongs to the syntax of Safety-LTL but not to the syntax of $\text{LTL}_{\text{EBR}+P}$, even though $G(p_1 \lor Gp_2) \equiv G(\neg p_2 \rightarrow Hp_1) \in \text{LTL}_{\text{EBR}+P}$. The restrictions on the syntax of $\text{LTL}_{\text{EBR}+P}$ are due to algorithmic aspects: each layer of the syntax of $\text{LTL}_{\text{EBR}+P}$ (recall Def. 2) corresponds to a step of the algorithm for the symbolic automata construction starting from $\text{LTL}_{\text{EBR}+P}$-formulas. As a matter of fact, in practice, $\text{LTL}_{\text{EBR}+P}$ has shown to avoid an exponential blowup in time with respect to known algorithms for automata construction for safety specifications [4]. Last but not least, the realizability problem of $\text{LTL}_{\text{EBR}+P}$ is $\text{EXPTIME}$-complete [1], as opposed to the realizability of $\text{LTL}+P$, which is $2\text{EXPTIME}$-complete [14][15]. Consider now $\text{LTL}_{\text{EBR}}$, that is the fragment of $\text{LTL}_{\text{EBR}+P}$ devoid of past operators. Since each formula of $\text{LTL}_{\text{EBR}}$ syntactically belongs to Safety-LTL, it immediately follows that $[\text{LTL}_{\text{EBR}}] \subseteq [\text{Safety-LTL}]$. In the next section, we will prove that the converse direction does not hold, that is $\text{LTL}_{\text{EBR}}$ is strictly less expressive than $\text{LTL}_{\text{EBR}+P}$, and thus less expressive than Safety-LTL as well.

4 $\text{LTL}_{\text{EBR}}$ is strictly less expressive than full $\text{LTL}_{\text{EBR}+P}$

In the previous sections, we have seen that:

$$[\text{LTL}_{\text{EBR}+P}] = [G\alpha] = [\text{LTL}] \cap \text{SAFETY} = [\text{Safety-LTL}]$$

In particular, thanks to the use of the pure past layer (recall Def. 2), $\text{LTL}_{\text{EBR}+P}$ can easily capture the whole class of $[G\alpha]$, and thus the whole class of $[\text{LTL}] \cap \text{SAFETY}$. However, one may wonder whether the pure past layer is really necessary, or whether the class $[G\alpha]$ can be expressed in $\text{LTL}_{\text{EBR}+P}$ without the use of past operators.

$\text{LTL}_{\text{EBR}}$ is defined as the fragment of $\text{LTL}_{\text{EBR}+P}$ devoid of the pure past layer (recall Section 2.3). In this section, we investigate the problem of establishing whether $\text{LTL}_{\text{EBR}}$ has the same expressive power of $\text{LTL}_{\text{EBR}+P}$, or equivalently, whether $\text{LTL}_{\text{EBR}}$ can express every language in $[\text{Safety-LTL}]$. We will
prove that this is not the case, that is
\[ [\text{LTL}_{\text{EBR}}] \subseteq [\text{LTL}_{\text{EBR}} + \text{P}] \]  

(1)

This result proves that past modalities, although being not important for the expressiveness of full LTL (since \([\text{LTL}] = [\text{LTL} + \text{P}]\) \cite{6,10,11}), can play a crucial role for the expressive power of fragments of LTL, like, for instance, \(\text{LTL}_{\text{EBR}}\).

4.1 The general idea

We will prove Eq. (1) by showing that \( [\text{LTL}_{\text{EBR}}] \subseteq [\text{Safety-LTL}] \). The result in Eq. (1) follows from the fact that \([\text{Safety-LTL}] = [\text{LTL}_{\text{EBR}} + \text{P}] \). We will prove that the language of the Safety-LTL-formula \( \phi_C := \text{G}(p_1 \lor \text{G}(p_2)) \) cannot be expressed by any \(\text{LTL}_{\text{EBR}}\)-formula. The formula \( \phi_C \) belongs syntactically to Safety-LTL, and thus \( \mathcal{L}(\phi_C) \in [\text{Safety-LTL}] \). We also note that \( \phi_C \) can be expressed in \(\text{LTL}_{\text{EBR}} + \text{P} \). In fact, it holds that:
\[ \text{G}(p_1 \lor \text{G}(p_2)) \equiv \text{G}(\neg p_2 \rightarrow \text{H}(p_1)) \]  

(2)

Since \( \text{G}(\neg p_2 \rightarrow \text{H}(p_1)) \in [\text{LTL}_{\text{EBR}} + \text{P}] \), it holds that \( \mathcal{L}(\phi_C) \in [\text{LTL}_{\text{EBR}} + \text{P}] \). It is worth noticing the following points: (i) \( \text{G}(\neg p_2 \rightarrow \text{H}(p_1)) \) is of the form \( \text{G}\alpha \), where \( \alpha \in \text{LTL}_P \) (\( \alpha \) is a pure past formula); (ii) the formula \( \phi_C \) is equivalent to \( \text{G}(p_2) \lor ((X\text{G}p_2) \text{R} p_1) \), but the latter formula does not syntactically belong to \(\text{LTL}_{\text{EBR}}\), due to the restriction that forces the leftmost argument of any release operator to contain no universal temporal operators (i.e., R and G). In fact, in the following, we will prove that \( \mathcal{L}(\phi_C) \not\subseteq [\text{LTL}_{\text{EBR}}]\).

The proof of the undefinability of \( \phi_C \) is based on the fact that each formula of \(\text{LTL}_{\text{EBR}}\) cannot constrain an arbitrarily long prefix of a state sequence, but only a finite prefix whose maximum length depends on the maximum number of next operators.

Consider again the formula \( \phi_C := \text{G}(p_1 \lor \text{G}(p_2)) \). The language \( \mathcal{L}(\phi_C) \) is expressed by the \(\omega\)-regular expression \( \{\{p_1\}\}^0 + \{\{p_1\}\}^* \cdot \{\{p_2\}\}^\omega \). Written in natural language, each model of \( \phi_C \) cannot contain a position in which \( \neg p_2 \) holds preceded by a position in which \( \neg p_1 \) holds.

**Remark 1.** Let \( \sigma \subseteq (2^\Sigma)^\omega \) be a state sequence. It holds that:
\[ \sigma \models \phi_C \Rightarrow \neg \exists i, j (i \leq i \land \sigma_j \models \neg p_1 \land \sigma_i \models \neg p_2) \]

We define \( i^k\sigma_j \) as the state sequence such that at the time points \( i \) and \( k \) it holds \( p_1 \land \neg p_2 \), at time point \( j \) it holds \( \neg p_1 \land p_2 \), and for all the other time points \( p_1 \land p_2 \) holds. The membership of \( i^k\sigma_j \) to \( \mathcal{L}(\phi_C) \) depends on the value of the three indices \( i, j \) and \( k \), as follows.

**Remark 2.** If \( i < j \) and \( k < j \), then \( i^k\sigma_j \models \phi_C \). Conversely, if \( i \geq j \) or \( k \geq j \), then \( i^k\sigma_j \not\models \phi_C \).

As we will see, given a generic formula \( \psi \in \text{LTL}_{\text{EBR}} \), one can always find some values for the indices \( i, j \) and \( k \) such that (a) \( i \) is chosen sufficiently greater than \( j \); (b) \( k \) is chosen sufficiently greater than \( j \); (c) \( \psi \) is not able to distinguish the state sequence \( i^i\sigma_j \) from \( i^k\sigma_j \). Since, by Remark 2 \( i^i\sigma_j \in \mathcal{L}(\phi_C) \) but \( i^k\sigma_j \not\in \mathcal{L}(\phi_C) \), this proves the undefinability of \( \phi_C \) in \(\text{LTL}_{\text{EBR}}\). The rationale is that the \(\text{LTL}_{\text{EBR}}\) logic combines bounded future formulas (i.e., formulas obtained by a Boolean combination of propositional atoms and X operators) and universal temporal operators (i.e., G and R). This implies the fact that, for a generic model \( \sigma \) of an \(\text{LTL}_{\text{EBR}}\)-formula \( \psi \), at each time point \( i \geq 0 \) of \( \sigma \) (this corresponds to the universal temporal operators) only a finite and bounded suffix after \( i \) (this corresponds to the \(\text{LTL}_{\text{EBR}}\)-formulas) can be constrained by \( \psi \) (this can be thought of as a sort of bounded memory property of this logic). Equivalently, this means that each \(\text{LTL}_{\text{EBR}}\)-formula is not able to constrain any finite but arbitrarily long (unbounded) prefix of a state sequence, contrary, for instance, to the case of the formula \( \text{G}(\neg p_2 \rightarrow \text{H}(p_1)) \) (that is equivalent to \( \phi_C \), see Eq. (2)).
4.2 The Canonical Form

The limitation of LTL\text{EBR}-formulas mentioned before is more evident in the canonical form for the LTL\text{EBR} logic, that we will define in this part. We first give some preliminaries definitions. We define Bounded Past LTL\text{P} (LTL\text{BP}, for short) as the set of all and only the LTL\text{EBR}+\text{P} formulas that are a Boolean combination of propositional atoms and yesterday operators (Y). We use the shortcut $\psi_1 \mathcal{S}\{a,b\} \psi_2$ for denoting the formula $\bigvee_{i=a}^b (Y_1 \ldots Y_i(\psi_2) \land \bigwedge_{j=0}^{i-1} Y_j(\psi_1))$. Given a formula $\alpha \in \text{LTL}\text{BP}$, we define its temporal depth, denoted as $D(\alpha)$, as follows:

- $D(p) = 0$, for all $p \in \Sigma$
- $D(\neg \alpha_1) = D(\alpha_1)$
- $D(\alpha_1 \land \alpha_2) = \max\{D(\alpha_1), D(\alpha_2)\}$
- $D(Y\alpha_1) = 1 + D(\alpha_1)$
- $D(\alpha_1 \mathcal{S}\{a,b\} \alpha_2) = b + \max\{D(\alpha_1), D(\alpha_2)\}$

For each $\alpha \in \text{LTL}\text{BP}$, the language $\mathcal{L}^{\leq \omega}(\alpha)$ consists only of words of length at most $D(\alpha) + 1$. Recall from Section 2 that, given a infinite state sequence $\sigma = (\alpha_0, \alpha_1, \ldots)$ and some $n \geq 0$, $\sigma|_{[n-d,n]}$ is the interval of $\sigma$ of length at most $d$ ending at index $n$. The crucial property of LTL\text{BP}-formulas, that can be shown with a simple induction, is that their truth over a state sequence $\sigma$ can be checked by considering only a finite and bounded interval of $\sigma$, whose length depends on the temporal depth of the formula.

Remark 3. For any $\alpha \in \text{LTL}\text{BP}$, with temporal depth $d = D(\alpha)$, and for any $n \geq 0$, it holds that $\sigma, n \models \alpha$ if and only if $\sigma|_{[n-d,n]} \models \alpha$.

We give now the canonical form for LTL\text{EBR}, and we refer to it as Canonical-LTL\text{EBR}. The canonical form of LTL\text{EBR} forces any universal unbounded operator, like globally or release, to contain only LTL\text{BP}-formulas. Formally, we define Canonical-LTL\text{EBR} as the canonical form described in Def. 3 but such that each $\alpha_i, \beta_i$ is a bounded past LTL formula. By applying the same transformation from LTL\text{EBR}+\text{P} to its canonical form given in [4], one obtain the following lemma.

Lemma 1. $[\text{LTL}\text{EBR}] = [\text{Canonical-LTL}\text{EBR}]$.

Proof. Obviously $[\text{Canonical-LTL}\text{EBR}] \subseteq [\text{LTL}\text{EBR}]$, since each formula $\psi$ that belongs to Canonical-LTL\text{EBR} can be turned into an equivalent one $\psi^\prime \in \text{LTL}\text{EBR}$ by expanding each bounded past operators into conjunctions/disjunctions of yesterday operators.

For proving $[\text{LTL}\text{EBR}] \subseteq [\text{Canonical-LTL}\text{EBR}]$, it is sufficient to apply the transformations described in [4] for the translation of LTL\text{EBR}+\text{P} into canonical form. In particular, since by definition $\psi$ has no past temporal operators, the only past operators in $\psi^\prime$ are the ones introduced by the pastification step described in [4], which are all bounded, that is either Y or S\{a,b\}.

The canonical form of LTL\text{EBR} makes it easier to prove Eq. (1). Take for example the formula $XXG(p \lor Y p \lor YY p)$, that belongs to Canonical-LTL\text{EBR}. It is clear that, at each time point, this formula can constrain only the interval consisting of the current state and its two previous states (in fact its temporal depth is 3).

4.3 The main proof

In this part, we show the undefinability of the formula $\varphi_C$ in the Canonical-LTL\text{EBR} logic. The undefinability in LTL\text{EBR} follows from Lemma 1.
prove the property for formulas of type Cimatti, Geatti, Gigante, Montanari and Tonetta that, for any interval of

\[ d \]

Lemma 2.

Given three indices \( i, j, k \in \mathbb{N} \) such that \( i \neq j \) and \( k \neq j \), we formally define the state sequence \( i^k \sigma_j = \{ i^k \sigma_0^j, i^k \sigma_1^j, \ldots \} \) as follows:

\[
i^k \sigma_h^j = \begin{cases} 
\{ p_1 \} & \text{if } h \in \{ i, k \} \\
\{ p_2 \} & \text{if } h = j \\
\{ p_1, p_2 \} & \text{otherwise}
\end{cases}
\]

The core of the main theorem is based on the fact that any formula of type \( G \alpha \) or \( \alpha R \beta \), where \( \alpha \) and \( \beta \) are bounded past LTL formulas, is not able to distinguish the state sequence \( i^i \sigma_j \) with \( i < j \) (which is a model of \( \varphi_G \)) from \( i^k \sigma_j \) with \( k > j \) (which is not a model of \( \varphi_G \)), for sufficiently large values of \( i, j \) and \( k \). The choice for the values of the three indices is based on the values of the temporal depth of \( \alpha \) and \( \beta \). Since the globally operator is a special case of the release operator, that is \( G \alpha \equiv \perp R \alpha \), it suffices to prove the property for formulas of type \( \alpha R \beta \). We first prove the two fundamental properties that show that, for any interval of \( i^i \sigma_j \) of length at most \( d \) (for any \( d \in \mathbb{N} \)), we can find the exact same interval in \( i^k \sigma_j \), and vice versa. Fig. 2 shows the idea of this correspondence.

**Lemma 2.** Let \( d \in \mathbb{N} \). For all \( i \geq d \), for all \( j \geq i + d \), and for all \( k \geq j + d \), it holds that:

\[
\text{Property 1: } \forall n' \geq 0. \exists n \geq 0. i^i \sigma_j^{[n-d,n]} = i^k \sigma_j^{[n-d,n']}
\]

\[
\text{Property 2: } \forall n \geq 0. \exists n' \geq 0. i^i \sigma_j^{[n-d,n]} = i^k \sigma_j^{[n-d,n']}
\]

**Proof.** Take any value for \( i, j, \) and \( k \) such that: (i) \( i \geq d \), (ii) \( j \geq i + d \), (iii) \( k \geq j + d \). Given any interval of length \( d \) of the state sequence \( i^i \sigma_j \), we show how to find an exact same one in \( i^k \sigma_j \), and vice versa.

The constraints above on the three indices ensure that both the state sequences \( i^i \sigma_j \) and \( i^k \sigma_j \) contain only three types of intervals of length at most \( d \). Consider \( i^k \sigma_j \) (the case for \( i^i \sigma_j \) is specular). The three types are the following:

- **Type 1:** \( \{(p_1, p_2)\}^n \) for some \( 0 \leq n \leq d \);
- **Type 2:** \( \{(p_1, p_2)\}^n \cdot \{(p_1)\} \cdot \{p_1, p_2\}^{d-n-1} \), for some \( 0 \leq n < d \);
- **Type 3:** \( \{(p_1, p_2)\}^n \cdot \{(p_2)\} \cdot \{p_1, p_2\}^{d-n-1} \), for some \( 0 \leq n < d \);

The situation is depicted in Fig. 2. Given any interval of any of the three types above, we show below how to find the very same interval in \( i^i \sigma_j \) (Fig. 2) tries to show visually this correspondence:

- each interval of \( i^k \sigma_j \) of type \( \{(p_1, p_2)\}^n \) is equal to \( i^i \sigma_j^{i,n} \);
- each interval of \( i^k \sigma_j \) of type \( \{(p_1, p_2)\}^n \cdot \{(p_1)\} \cdot \{p_1, p_2\}^{d-n-1} \) is equal to \( i^i \sigma_j^{i,n,i+d-n-1} \).

---

**Figure 2**
• each interval of \(i^k \sigma^j\) of type \((\{p_1, p_2\})^d \cdot (\{p_2\}) : (\{p_1, p_2\})^{d-n-1}\) is equal to \(i^j \sigma^j_{[j-n,j+d-n-1]}\).

This proves \textit{Property 1.}

Similarly, the correspondence between intervals of \(i^i \sigma^j\) and intervals of \(i^k \sigma^j\) is the following:

• each interval of \(i^i \sigma^j\) of type \((\{p_1, p_2\})^n\) is equal to \(i^k \sigma^j_{[0,n]}\);

• each interval of \(i^i \sigma^j\) of type \((\{p_1, p_2\})^n : (\{p_1\}) : (\{p_1, p_2\})^{d-n-1}\) is equal to \(i^k \sigma^j_{[i-n,i+d-n-1]}\);

• each interval of \(i^i \sigma^j\) of type \((\{p_1, p_2\})^n : (\{p_2\}) : (\{p_1, p_2\})^{d-n-1}\) is equal to \(i^k \sigma^j_{[j-n,j+d-n-1]}\).

This proves \textit{Property 2.} \(\square\)

We can now prove that the state sequences \(i^i \sigma^j\) and \(i^k \sigma^j\) are indistinguishable for each formula of type \(\alpha \mathcal{R} \beta\) (and, consequently, of type \(G \alpha\), with \(\alpha, \beta \in \text{LTL}_{\text{BP}}\).

\textbf{Lemma 3.} Let \(\alpha, \beta \in \text{LTL}_{\text{BP}}\), and let \(d = \max\{D(\alpha), D(\beta)\}\) be the maximum between the temporal depths of \(\alpha\) and \(\beta\). It holds that \(i^j \sigma^j \models \alpha \mathcal{R} \beta \iff i^k \sigma^j \models \alpha \mathcal{R} \beta\), for all \(i \geq d\), for all \(j \geq i + d\), and for all \(k \geq j + d\).

\textbf{Proof.} Take any value for \(i\), \(j\), and \(k\) such that: (i) \(i \geq d\), (ii) \(j \geq i + d\), (iii) \(k \geq j + d\).

We first prove the left-to-right direction. Suppose that \(i^j \sigma^j \models \alpha \mathcal{R} \beta\). We divide in cases:

1. Suppose that \(i^j \sigma^j, n \models \beta\) for all \(n \geq 0\). Since \(\beta \in \text{LTL}_{\text{BP}}\) and \(D(\beta) \leq d\), it holds that \(i^j \sigma^j_{[n-d,n]} \models \beta\), for all \(n \geq 0\). Suppose by contradiction that there exists some \(n' \geq 0\) such that \(i^k \sigma^j_{[n'-d,n']} \models \neg \beta\).

By \textit{Property 1 of Lemma 2} this means that there exists some \(n'' \geq 0\) such that \(i^i \sigma^j_{[n''-d,n'']} \models \neg \beta\). But this is a contradiction. Thus, it holds that \(i^k \sigma^j_{[n'-d,n']} \models \beta\) for all \(n' \geq 0\), that is, for all \(n' \geq 0\), and thus \(i^k \sigma^j \models \alpha \mathcal{R} \beta\).

2. Suppose that \(\exists n \geq 0. (i^j \sigma^j, n \models \alpha \land \forall 0 \leq m \leq n. i^i \sigma^j, m \models \beta)\). We divide again in cases:

(a) Suppose that \(n < k\). Then \(i^j \sigma^j_{[0,n]} = i^k \sigma^j_{[0,n]}\). Clearly, it holds that \(i^k \sigma^j, n \models \alpha\) and \(i^k \sigma^j, m \models \beta\) for all \(0 \leq m \leq n\). Therefore \(i^k \sigma^j \models \alpha \mathcal{R} \beta\).

(b) Suppose that \(n \geq k\). In particular, it holds that \(i^j \sigma^j_{[n-d,n]} \models \alpha \land \beta\). We use a \textit{contraction argument} for proving that in this case there exists a smaller index at which the \textit{release} satisfies its existentail part (\(i.e.,\) the formula \(\alpha\)). Consider the time point \(i - 1\). It holds that \(i^j \sigma^j_{[i-1-d,i-1]} = i^j \sigma^j_{[n-d,n]}\) and thus, since \(i^j \sigma^j_{[n-d,n]} \models \alpha \land \beta\) and \(\alpha, \beta \in \text{LTL}_{\text{BP}}\), we have that \(i^j \sigma^j_{[i-1-d,i-1]} \models \alpha \land \beta\). Moreover, \(i^j \sigma^j_{[0,i-1]}\) is a prefix of \(i^j \sigma^j_{[0,n]}\), and thus, given that \(i^j \sigma^j_{[p-d,p]} \models \beta\) for all \(0 \leq p \leq n\), it holds that \(i^j \sigma^j_{[p-d,p]} \models \beta\) for all \(0 \leq p \leq i - 1\). From this, it follows that \(i^j \sigma^j, i-1 \models \alpha\) and \(i^j \sigma^j, m \models \beta\) for all \(0 \leq m \leq i - 1\). Since \(i - 1 < k\), by \textbf{Item 2a}, it holds that \(i^k \sigma^j \models \alpha \mathcal{R} \beta\).

We now prove the right-to-left direction. Suppose that \(i^k \sigma^j \models \alpha \mathcal{R} \beta\). We divide in cases:

1. Suppose that \(i^k \sigma^j, n \models \beta\). This case is specular to \textbf{Item 1}.

2. Suppose that \(\exists n \geq 0. (i^k \sigma^j, n \models \alpha \land \forall 0 \leq m \leq n. i^k \sigma^j, m \models \beta)\). Since \(\alpha, \beta \in \text{LTL}_{\text{BP}}\) and \(D(\alpha), D(\beta) \leq d\), it holds that \(\exists n \geq 0. (i^k \sigma^j_{[n-d,n]} \models \alpha \land \forall 0 \leq m \leq n. i^k \sigma^j_{[m-d,m]} \models \beta)\). We divide again in cases:

(a) If \(n < k\), then \(i^k \sigma^j_{[0,n]} = i^j \sigma^j_{[0,n]}\) and thus \(i^j \sigma^j, n \models \alpha\) and \(i^j \sigma^j, m \models \beta\) for all \(0 \leq m \leq n\), that is \(i^j \sigma^j \models \alpha \mathcal{R} \beta\).
(b) If \( k \leq n \leq k + d \), then \( i_k \sigma^j_{[n-d,n]} = i_k \sigma^j_{[n-k-i-d,n-k-i]} \) (we used again a contraction argument). Since by hypothesis \( i_k \sigma^j_{[n-d,n]} \models \alpha \), it holds also that \( i_k \sigma^j_{[n-k-i-d,n-k-i]} \models \alpha \). Moreover, \( i_k \sigma^j_{[0,n-k]} \) is a prefix of \( i_k \sigma^j_{[0,n]} \), and thus, since by hypothesis \( i_k \sigma^j_{[p-d,p]} \models \beta \) for all \( 0 \leq p \leq n \), it also holds that \( i_k \sigma^j_{[p-d,p]} \models \beta \) for all \( 0 \leq p \leq n - k - i \). Therefore \( i_k \sigma^j_{[n-k-i-d,n-k-i]} \models \alpha \) and \( i_k \sigma^j_{[m-d,m]} \models \beta \) for all \( 0 \leq m \leq n - k - i \). Since \( l + n - i < k \), by [Item 2a] it holds that \( i_i \sigma^j \models \alpha R \beta \).

(c) Otherwise \( n > k + d \). We have that \( i_k \sigma^j_{[n-d,m]} = i_k \sigma^j_{[i,1-i-d]} \) (also in this case we used a contraction argument). Since by hypothesis \( i_k \sigma^j_{[n-d,m]} \models \alpha \), it also hold that \( i_k \sigma^j_{[i-1,i-1-d]} \models \alpha \). Moreover \( i_k \sigma^j_{[0,i-1]} \) is a prefix of \( i_k \sigma^j_{[0,n]} \) and thus, since by hypothesis \( i_k \sigma^j_{[p-d,p]} \models \beta \) for all \( 0 \leq p \leq n \), it also holds that \( i_k \sigma^j_{[p-d,p]} \models \beta \) for all \( 0 \leq p \leq i - 1 \). Therefore \( i_k \sigma^j, i-1 \models \alpha \) and \( i_k \sigma^j, m \models \beta \) for all \( 0 \leq m \leq i - 1 \). Since \( i - 1 < k \), by [Item 2a] it holds that \( i_i \sigma^j \models \alpha R \beta \).

By using [Lemma 3] as the proof for the base case, we prove by induction on the structure of the formula that any formula in Canonical-LTL_{EBR} is not able to distinguish the state sequences \( i_i \sigma^j \) and \( i_k \sigma^j \) for sufficiently large values of \( i, j, k \). In the following, given a formula \( \psi \in \text{Canonical-LTL}_{EBR} \), we will denote with \( m_\psi \) the maximum number of nested next operators in \( \psi \), and with \( d_\psi \) the maximum temporal depth between all its LTL_{BP}-subformulas.

**Lemma 4.** Let \( \psi \in \text{Canonical-LTL}_{EBR} \). It holds that \( i_i \sigma^j \models \psi \) iff \( i_k \sigma^j \models \psi \), for all \( i \geq m_\psi + d_\psi \), for all \( j \geq i + d_\psi \), and for all \( k \geq j + d_\psi \).

**Proof.** Take any value for \( i, j, \) and \( k \) such that: (i) \( i \geq m_\psi + d_\psi \), (ii) \( j \geq i + d_\psi \), (iii) \( k \geq j + d_\psi \). We proceed by induction on the structure of the formula \( \psi \).

For the base case, we consider three cases: (i) formulas in LTL_{BP}, that is such that all its temporal operators refer to the past and are bounded; (ii) formulas of type \( G \alpha \), where \( \alpha \in \text{LTL}_{BP} \); (iii) formulas of type \( \alpha R \beta \), where \( \alpha, \beta \in \text{LTL}_{BP} \).

We consider the case of a formula \( \alpha \in \text{LTL}_{BP} \), and suppose that \( i_i \sigma^j \models \alpha \). By definition of \( i_i \sigma^j \) and \( i_k \sigma^j \), it always holds that \( i_i \sigma^j = i_k \sigma^j \). Since \( \alpha \in \text{LTL}_{BP} \) refers only to the current state or to the past, it follows that \( i_i \sigma^j \models \alpha \) if and only if \( i_k \sigma^j \models \alpha \).

Consider now the case for \( \alpha R \beta \), where \( \alpha, \beta \in \text{LTL}_{BP} \). Since \( m_{\alpha R \beta} = 0 \) (i.e., the are no next operators in this formula), we can apply [Lemma 3] having that \( i_i \sigma^j \models \alpha R \beta \) if and only if \( i_k \sigma^j \models \alpha R \beta \). Since \( G \alpha = \bot R \alpha \), this proves also the case for the globally operator.

For the inductive step, since by hypothesis \( \psi \) belongs to the canonical form of LTL_{EBR}, it suffices to consider only the case for the next operator, conjunctions and disjunctions.

Consider first the case for the next operator, and suppose that \( i_i \sigma^j \models X \psi' \). For any indices \( i, j \) such that \( i \geq m_{X \psi'} + d_{X \psi'} \), \( j \geq i + d_{X \psi'} \), and \( k \geq j + d_{X \psi'} \), we want to prove that \( i_k \sigma^j \models X \psi' \). By definition of the next operator, it holds that \( i_i \sigma^j, 1 \models \psi' \). Now, let \( \tau \) be the state sequence obtained from \( i_i \sigma^j \) by discarding its initial state, that is \( \tau := i_i \sigma^j_{[1,\omega]} \). Obviously, \( \tau \models \psi' \). We observe that \( \tau \) is equal to the state sequence \( i^{-1} j^{-1} \sigma^{j^{-1}} \). Since the maximum number \( m_{\psi'} \) of nested next operators in \( \psi' \) is \( m_{X \psi'} - 1 \) (while \( \alpha_{\psi'} \) remains the same), we can apply the inductive hypothesis on \( \psi' \), having that \( i^{-1} j^{-1} \sigma^{j^{-1}} \models \psi' \). By definition of \( \tau \), it follows that \( i_k \sigma^j \models X \psi' \).

We consider now the case for conjunctions, and suppose that \( i_i \sigma^j \models \psi_1 \land \psi_2 \), for generic indices \( k, i \) and \( j \) such that \( i \geq m_{\psi_1 \land \psi_2} + d_{\psi_1 \land \psi_2} \), \( j \geq i + d_{\psi_1 \land \psi_2} \), and \( k \geq j + d_{\psi_1 \land \psi_2} \). It holds that \( i_i \sigma^j \models \psi_1 \) and
i,j,σj ∈ ψ2. Moreover, mψ1 ≤ mψ1∧ψ2 and mψ2 ≤ mψ1∧ψ2. Similarly, dψ1 ≤ dψ1∧ψ2 and dψ2 ≤ dψ1∧ψ2. This means that we can apply the inductive hypothesis both on ψ1 and ψ2 on the current indices k, i and j. By inductive hypothesis, we have that i,j,σj ψ1 and i,k,σj ψ2. It follows that i,k,σj ψ1 ∧ ψ2. The case for ψ1 ∨ ψ2 is specular. □

Thanks to Lemma 4 it is simple to prove the undefinability of $G(p_1 ∨ G(p_2))$ in LTL_EBR, that proves that LTL_EBR is strictly less expressive than Safety-LTL.

Theorem 3. $[[\text{LTL}_{\text{EBR}}]] ⊆ [[\text{Safety-LTL}]].$

Proof. Consider the formula $φ_G := G(p_1 ∨ G(p_2))$. We prove that there does not exist a formula $ψ ∈ LTL_{EBR}$ such that $L(ψ) = L(φ_G)$. We proceed by contradiction. Suppose that there exists a formula $ψ ∈ LTL_{EBR}$ such that $L(ψ) = L(φ_G)$. By Lemma 1 there exists a formula $ψ' ∈ \text{Canonical-LTL}_{EBR}$ such that $L(ψ) = L(ψ')$. Let $m_ψ$ be the maximum number of nested next operators in $ψ'$, and let $d_ψ$ be the maximum temporal depth between all the $LTL_{EBR}$-subformulas in $ψ'$. Let $k, j$ and $j$ be three indices such that: (i) $i ≥ m_ψ + d_ψ$; (ii) $i ≥ j + d_ψ$; (iii) $k ≥ j + d_ψ$. Consider the two state sequences $i,j,σ_j ∈ LTL_{EBR}$ and $i,k,σ_j$. By Lemma 4, $i,j,σ_j ∈ L(ψ')$ if and only if $i,k,σ_j ∈ L(ψ')$, that is $i,j,σ_j ∈ L(φ_G)$ if and only if $i,k,σ_j ∈ L(φ_G)$. Since it holds that $i,j,σ_j ∈ L(φ_G)$ but $i,k,σ_j ∉ L(φ_G)$, this is clearly a contradiction. □

Corollary 1. $[[\text{LTL}_{\text{EBR}}]] ⊆ [[\text{LTL}_{\text{EBR}}+\text{P}]].$

5 Conclusions

We considered the logic LTL_EBR+P, a recently introduced safety fragment of LTL with an efficient realizability problem. The syntax of LTL_EBR+P made it difficult to exactly characterize its expressible power. We studied the expressible power of LTL_EBR+P and of its pure future fragment, LTL_EBR, and compare it with other safety fragments of LTL. It turned out that LTL_EBR+P is expressively complete with respect to the safety fragment of LTL, and, consequently, it is expressively equivalent to Safety-LTL. We found out that past modalities are crucial for the expressible power of LTL_EBR+P. In fact, LTL_EBR is strictly less expressive than full LTL_EBR+P. This was somehow surprising, since it proves that, despite not being fundamental for the expressiveness of full LTL, past modalities are crucial for fragments of LTL, like, for instance, LTL_EBR+P.

References

[1] J Richard Büchi (1960): Weak Second-Order Arithmetic and Finite Automata. Mathematical Logic Quarterly 6(1-6), pp. 66–92, doi:10.1002/malq.19600060105

[2] J. Richard Büchi (1960): On a Decision Method in Restricted Second Order Arithmetic. pp. 425–435. doi:10.1007/978-1-4613-8928-6_23 Available at https://doi.org/10.1007%2F978-1-4613-8928-6_23

[3] Edward Y. Chang, Zohar Manna & Amir Pnueli (1992): Characterization of Temporal Property Classes. In Werner Kuich, editor: Proceedings of the 19th International Colloquium on Automata, Languages and Programming, Lecture Notes in Computer Science 623, Springer, pp. 474–486, doi:10.1007/3-540-55719-9_97

[4] Alessandro Cimatti, Luca Geatti, Nicola Gigante, Angelo Montanari & Stefano Tonetta (2020): Reactive Synthesis from Extended Bounded Response LTL Specifications. In: 2020 Formal Methods in Computer Aided Design, FMCAD 2020, Haifa, Israel, September 21-24, 2020, IEEE, pp. 83–92, doi:10.34727/2020/isbn.978-3-85448-042-6_15
[5] Stéphane Demri, Valentin Goranko & Martin Lange (2016): Temporal logics in computer science: finite-state systems. 58, Cambridge University Press, doi:10.1017/CBO9781139236119.

[6] Dov M. Gabbay, Amir Pnueli, Saharon Shelah & Jonathan Stavi (1980): On the Temporal Analysis of Fairness. In Paul W. Abrahams, Richard J. Lipton & Stephen R. Bourne, editors: Conference Record of the Seventh Annual ACM Symposium on Principles of Programming Languages, Las Vegas, Nevada, USA, January 1980, ACM Press, pp. 163–173, doi:10.1145/567446.567462.

[7] John E Hopcroft, Rajeev Motwani & Jeffrey D Ullman (2001): Introduction to automata theory, languages, and computation. Acm Sigact News 32(1), pp. 60–65, doi:10.1145/568438.568455.

[8] Johan Anthony Wilem Kamp (1968): Tense logic and the theory of linear order.

[9] Orna Kupferman & Moshe Y Vardi (2001): Model checking of safety properties. Formal Methods in System Design 19(3), pp. 291–314, doi:10.1023/A:1011254632723.

[10] Orna Lichtenstein, Amir Pnueli & Lenore Zuck (1985): The glory of the past. In: Workshop on Logic of Programs, Springer, pp. 196–218, doi:10.1007/3-540-15648-8_16.

[11] Nicolas Markey (2003): Temporal logic with past is exponentially more succinct.

[12] Robert McNaughton & Seymour A Papert (1971): Counter-Free Automata (MIT research monograph no. 65). The MIT Press.

[13] Amir Pnueli (1977): The temporal logic of programs. In: 18th Annual Symposium on Foundations of Computer Science (sfcs 1977), IEEE, pp. 46–57, doi:10.1109/SFCS.1977.32.

[14] Amir Pnueli & Roni Rosner (1989): On the synthesis of an asynchronous reactive module. In: International Colloquium on Automata, Languages, and Programming (ICALP), Springer, pp. 652–671, doi:10.1016/0022-0000(86)90026-7.

[15] Roni Rosner (1992): Modular synthesis of reactive systems. Ph.D. thesis, PhD thesis, Weizmann Institute of Science.

[16] A Prasad Sistla (1994): Safety, liveness and fairness in temporal logic. Formal Aspects of Computing 6(5), pp. 495–511, doi:10.1007/BF01211865.

[17] Wolfgang Thomas (1988): Safety-and liveness-properties in propositional temporal logic: characterizations and decidability. Banach Center Publications 1(21), pp. 403–417, doi:10.4064/BC1-21-1-403-417.

[18] Shufang Zhu, Lucas M. Tabajara, Jianwen Li, Geguang Pu & Moshe Y. Vardi (2017): A Symbolic Approach to Safety LTL Synthesis. In Ofir Strichman & Rachel Tzoref-Brill, editors: Proceedings of the 13th International Haifa Verification Conference, Lecture Notes in Computer Science 10629, Springer, pp. 147–162, doi:10.1007/978-3-319-70389-3_10.