THE MOTIVIC IGUSA ZETA FUNCTION OF A SPACE MONOMIAL CURVE WITH A PLANE SEMIGROUP

HUSSIN MOURTADA, WILLEM VEYS, AND LENA VOS

ABSTRACT. In this paper, we compute the motivic Igusa zeta function of a space monomial curve that appears as a special fibre of an equisingular family whose generic fibres are complex plane branches. To this end, we determine the irreducible components of the jet schemes of such a space monomial curve. This approach does not only yield a closed formula for the motivic zeta function, but also allows to determine its poles. We show that, while the family of the jet schemes of the fibres is not flat, the number of poles of the motivic zeta function associated to the space monomial curve is equal to the number of poles of the motivic zeta function associated to a generic curve in the family.

INTRODUCTION

The history of the motivic Igusa zeta function goes back to the seventies when Igusa studied the \( p \)-adic Igusa zeta function, which is related to the classical problem in number theory of computing the number of solutions of congruences. More precisely, the original Igusa zeta function counts, for a non-constant polynomial \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) and a prime \( p \), the \( \mathbb{Z}/(p^m+1)\mathbb{Z} \)-points of \( X = \{ f = 0 \} \), when \( m \) varies in \( \mathbb{N} \). It was introduced by Weil [Wei], and its basic properties, such as rationality, were first investigated by Igusa [Igu]. In analogy with the \( p \)-adic zeta function, Denef and Loeser [DL2] introduced the ‘more general’ motivic Igusa zeta function in which \( f \in \mathbb{C}[x_1, \ldots, x_n] \) is a complex polynomial, and the \( \mathbb{Z}/(p^m\mathbb{Z}) \)-points of \( X = \{ f = 0 \} \) are replaced by its \( \mathbb{C}[t]/(t^{m+1}) \)-points. It is more general in the sense that one can obtain the earlier zeta function from the motivic one.

The space of \( \mathbb{C}[t]/(t^{m+1}) \)-points of \( X = \{ f = 0 \} \), or equivalently, of morphisms \( \text{Spec}(\mathbb{C}[t]/(t^{m+1})) \rightarrow X \), has a natural scheme structure; it is denoted by \( X_m \) and called the \( m \)-th jet scheme of \( X \). Geometrically, if we consider \( X \) in the affine space \( \mathbb{C}^n \), the space \( X_m \) can be thought of as the moduli space parameterising germs of curves in \( \mathbb{C}^n \) which have a ‘contact’ with \( X \) larger than \( m \). Simple invariants of the \( X_m \) (e.g., their irreducible components and their dimensions) encode deep information on the singularities of \( X \) [Mou1, Mou3, Mus1]. In terms of these jet schemes, the motivic Igusa zeta function \( Z^{mot}_X(T) \) associated to \( X \) (or to \( f \)) can be written as

\[
Z^{mot}_X(T) = 1 - \frac{1}{T} J_X(T),
\]

where \( J_X(T) \) is the Poincaré series

\[
J_X(T) := \sum_{m \geq 0} [X_m]([\mathbb{L}^{-n}T]^{m+1}) \in \mathcal{M}_\mathbb{C}[[T]].
\]

Here, \( \mathcal{M}_\mathbb{C} \) is a localization of the Grothendieck ring of complex varieties, and \([X_m]\) and \( \mathbb{L} \) are the classes of \( X_m \) and of the affine line \( \mathbb{C} \) in this Grothendieck ring, respectively. Clearly, this expression also makes sense when \( X \) is any subscheme of \( \mathbb{C}^n \) given by some

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ideal \( I \) in \( \mathbb{C}[x_1, \ldots, x_n] \), instead of just a hypersurface. Furthermore, the motivic zeta function turns out to be a rational function in \( T \), and it is natural to study its poles.

The motivic Igusa zeta function for one polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) can also be expressed in terms of an embedded resolution of singularities of \( f \); the analogous expression for an ideal is in terms of a principalisation of the ideal. This formula in the hypersurface case can be found in [DL2], and its generalisation to ideals is mentioned in [VZ]. It is the most classical way to compute the motivic zeta function and allows to determine a complete list of candidates poles of this zeta function. However, it is in general very difficult to calculate a principalisation and to verify whether the candidate poles are actual poles; usually, ‘most’ of the candidates are in fact no actual poles. Therefore, in this paper, in order to determine the motivic zeta function and its poles, we will compute the above Poincaré series, based on the structure of the jet schemes.

We will apply this approach to the class of monomial space curves whose semigroup is the semigroup of a complex plane branch. Such a curve naturally appears as a special fibre of an equisingular family, whose generic fibre is isomorphic to an irreducible plane (germ of a) curve. More precisely, let \( \mathcal{C} = \{f = 0\} \subset (\mathbb{C}^2, 0) \) be a germ of a complex plane branch defined by an irreducible series \( f \) whose generic fibre is isomorphic to an irreducible plane branch defined by an irreducible series \( f = 0 \), and let

\[
\nu_C : R := \mathbb{C}[[x_0, x_1]](f) \longrightarrow \mathbb{N}
\]

be the associated valuation, where \( \nu_C(h) = (f, h)_0 \) is the local intersection multiplicity of the curve \( C \) and the curve \( \{ h = 0 \} \). The semigroup \( \Gamma(C) := \{\nu_C(h) \mid h \in R \setminus \{0\}\} \subset \mathbb{N} \) is finitely generated, and we can identify a unique minimal system of generators \((\beta_0, \ldots, \beta_g)\) of \( \Gamma(C) \). Let \( (Y, 0) \subset (\mathbb{C}^{g+1}, 0) \) be the image of the monomial map \( M : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^{g+1}, 0) \) given by \( M(t) = (t^{\beta_0}, t^{\beta_1}, \ldots, t^{\beta_g}) \). It is an irreducible curve with the ‘plane’ semigroup \( \Gamma(C) \) as its semigroup and it is the special fibre of a flat family \( \eta : (\chi, 0) \subset (\mathbb{C}^{g+1}, 0) \longrightarrow (\mathbb{C}, 0) \) whose generic fibre is isomorphic to \( C \). The explicit equations defining \( Y \) in \( \mathbb{C}^{g+1} \) are

\[
\begin{align*}
{x_1}^{n_1} - x_0^{n_0} &= 0 \\
{x_2}^{n_2} - x_0^{b_{02}}x_1^{b_{21}} &= 0 \\
&\vdots \\
x_{g}^{n_g} - x_0^{b_{0g}}x_1^{b_{g1}} \cdots x_{g-1}^{b_{g(g-1)}} &= 0,
\end{align*}
\]

where \( n_i > 1 \) and \( b_{ij} \geq 0 \) are integers that are defined in terms of \((\beta_0, \ldots, \beta_g)\). It is worth mentioning that the family \((\chi, 0)\) is equisingular in the sense that it admits a simultaneous embedded resolution [Tei1, LMR].

We will first study the jet schemes of \( Y \); we determine their irreducible components and we associate with them a natural graph which is similar to the one in [Mon1, Mon3, CM], and which we use to encode the computation of the motivic Igusa zeta function of \( Y \), see Theorem 3.7 and Figure 1, respectively. With this point of view, we are then able to compute a closed formula for the motivic zeta function in Theorem 4.7:

\[
Z^{mot}_Y(T) = \frac{1 - (L - 1)L^{-(g+1)} - L^{-(g+1)}T}{1 - L^{-g}T} + \frac{P_0(T)}{1 - L^{-\nu_1}T^{N_1}} + \sum_{i=1}^{g-1} \frac{P_i(T)}{(1 - L^{-\nu_i}T^{N_i})(1 - L^{-\nu_{i+1}}T^{N_{i+1}})} - \frac{(L - 1)L^{-(\nu_g + g + 1)}(1 - T)T^{N_g}}{(1 - L^{-g}T)(1 - L^{-\nu_g}T^{N_g})},
\]
where $P_i(T)$ for $i = 0, \ldots, g - 1$ are concrete polynomials with coefficients in the ring $\mathbb{Z}[L, L^{-1}]$, and $(N_i, \nu_i)$ for $i = 1, \ldots, g$ are couples of known positive integers with

$$
\frac{\nu_i}{N_i} = \frac{1}{n_i \beta_i} \left( \sum_{l=0}^{i} \beta_l - \sum_{l=1}^{i-1} n_l \beta_l \right) + (i - 1) + \sum_{l=i+1}^{g} \frac{1}{n_l}.
$$

Except for some ‘small’ concrete cases, we do not see how one can obtain such a formula using a principalisation. Furthermore, we obtain only $g + 1$ candidate poles:

$$
L^g, \quad L^{\frac{N_i}{n_i}}, \quad i = 1, \ldots, g.
$$

Using residues and the related topological Igusa zeta function, we prove that, in contrary to formulas that one could obtain using a principalisation, all these candidate poles are actual poles. In particular, we can determine the log canonical threshold of $Y \subset \mathbb{C}^{g+1}$ given by

$$
\frac{\nu_i}{N_1} = \sum_{l=0}^{g} \frac{1}{n_l}.
$$

Note that the number of poles of the motivic zeta function of $Y$ is the same as the number of poles of the motivic zeta function of the plane branch $C$. This is a fascinating result as the induced family on the level of jet schemes is not flat. More precisely, let $\phi_m : \chi_m \to \chi$ for every $m \in \mathbb{N}$ be the natural map defined by truncating the jets, and consider the induced family $\eta \circ \phi_m : (\chi_m, 0) \to (\mathbb{C}, 0)$. Then, although the family $\chi$ is equisingular (in particular, flat), we show in Theorem 3.8 that the family $\chi_m$ is not flat for $m$ large enough. We would like to point out that in the hypersurface case, an equisingular family of hypersurfaces does induce a flat family on the jet schemes [Ley].

The poles of the motivic Igusa zeta function associated to a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ are the subject of an intriguing open problem, the so-called monodromy conjecture, which relates number theoretical invariants and topological invariants of $f$. Roughly speaking, it predicts a relation between the poles of the motivic zeta function and the action of the monodromy of $f$, seen as a function $f : \mathbb{C}^n \to \mathbb{C}$, on the cohomology of its Milnor fibre at some point $x \in X \subset \mathbb{C}^n$. For an ideal $I$, one can state the generalised monodromy conjecture in which Verdier monodromy replaces the classical monodromy. To date, both the classical and the generalised conjecture have only been proven in full generality for polynomials and ideals in two variables, see [Loe1] and [VV], respectively. In higher dimension, there are various partial results in the hypersurface case (we refer to the introduction of [BV] for a list of references), while in the non-hypersurface case, the conjecture has only been proven for monomial ideals [HMY]. As the results in this paper make the first part of this conjecture for curves $Y$ of the above type very explicit, the logical next step is to study the monodromy part for such curves. This, together with proving the monodromy conjecture, is work in progress; this will solve the conjecture for an interesting class of binomial ideals in arbitrary dimension.

The paper is organised as follows. We assume $\mathbb{N}$ to be the set of non-negative integers. We begin in Section 1 with introducing the curves $Y \subset \mathbb{C}^{g+1}$ in which we are interested. Section 2 consists of a brief discussion of the jet schemes and motivic zeta function associated to an affine variety. In Section 3, we determine the irreducible components of the jet schemes $Y_m$ for $m \in \mathbb{N}$ and show that the induced family $\chi_m$ on the jet schemes is not flat for most $m$. Based on the structure of $Y_m$, we compute the motivic zeta function of $Y$, find its $g + 1$ candidate poles, and provide some examples in Section 4. Finally, in Section 5, we prove that all candidate poles are actual poles.
1. Space monomial curves with plane semigroups

In this section, we introduce the type of singularities that we will consider in this article. They arise as (equisingular) deformations of germs of irreducible plane curves. We begin with an irreducible series \( f \in \mathbb{C}[[x_0, x_1]] \) in two variables over the complex numbers satisfying \( f(0) = 0 \). We denote by \( C := \{ f = 0 \} \subset (\mathbb{C}^2, 0) \) the germ at the origin of the curve defined by \( f \). We can assume, modulo a change of variables, that the curve \( \{ x_0 = 0 \} \) is transversal to \( C \) and that the curve \( \{ x_1 = 0 \} \) has maximal contact (among smooth curves) with \( C \). To \( C \), one can relate a valuation

\[
\nu_C : \frac{\mathbb{C}[[x_0, x_1]]}{\{ f \}} \setminus \{ 0 \} \longrightarrow \mathbb{N}
\]
defined by

\[
\nu_C(h) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_0, x_1]]}{(f, h)}.
\]

We denote by \( \Gamma(C) \) the semigroup associated with \( \nu_C \):

\[
\Gamma(C) := \left\{ \nu_C(h) \mid h \in \frac{\mathbb{C}[[x_0, x_1]]}{\{ f \}} \setminus \{ 0 \} \right\} \subset \mathbb{N}.
\]

Then, \( \Gamma(C) \) is a finitely generated semigroup with which we can associate the following data [Zar]:

1. the unique system of minimal generators \( (\beta_0, \ldots, \beta_g) \) of \( \Gamma(C) \), with \( \beta_0 < \cdots < \beta_g \) and \( \gcd(\beta_0, \ldots, \beta_g) = 1 \) (\( \gcd \) being the greatest common divisor);
2. the integers \( e_i := \gcd(\beta_0, \ldots, \beta_i) \) for \( i = 0, \ldots, g \), where \( e_0 = \beta_0, e_g = 1 \) and \( e_0 > \cdots > e_g \); and
3. the integers \( n_i := \frac{e_{i-1}}{e_i} \geq 2 \) for \( i = 1, \ldots, g \).

It follows from [Aze] that for every \( i = 1, \ldots, g \), the integer \( n_i \beta_i \) belongs to the semigroup generated by \( \beta_0, \ldots, \beta_{i-1} \). Hence, for \( i = 1, \ldots, g \), there exists a unique system of integers \( \{ b_{ij} \}_{0 \leq j < i} \) such that

\[
b_{ij} < n_j \quad \text{for} \quad j \neq 0 \quad \text{and} \quad n_i \beta_i = b_{i0} \beta_0 + \cdots + b_{i(i-1)} \beta_{i-1};
\]

the uniqueness follows from the inequalities \( b_{ij} < n_j \). For notational reasons, we introduce \( n_0 := b_{i0} \). Additionally, one can show that \( n_i \beta_i < \beta_{i+1} \) for all \( i = 1, \ldots, g - 1 \). It is also worth noting that \( e_i = n_{i+1} \cdots n_g \) for \( i = 0, \ldots, g - 1 \), that \( n_0 > n_1 \geq 2 \), and that \( n_0 = \frac{\beta_1}{e_1} \) and \( n_1 = \frac{\beta_2}{e_1} \) are coprime. Furthermore, one can choose a system of approximate roots or a minimal generating sequence \( (x_0, \ldots, x_g) \) of \( \nu_C \), where \( x_i \in \mathbb{C}[[x_0, x_1]] \) such that \( \nu_C(x_i) = \beta_i \) for \( i = 0, \ldots, g \), see for example [AM], [Mou2], [Spi] or [Tei1]. For \( i = 0, 1 \), the condition \( \nu_C(x_i) = \beta_i \) is equivalent to the assumptions that we put above on the variables \( x_0 \) and \( x_1 \), respectively. These elements satisfy identities of the form

\[
v x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma = (\gamma_0, \ldots, \gamma_i)} c_{i, \gamma} v x_0^{\gamma_0} \cdots x_i^{\gamma_i}, \quad i = 0, \ldots, g,
\]

where \( v = 1, x_{g+1} = 0, c_i \in \mathbb{C} \setminus \{ 0 \}, c_{i, \gamma} \in \mathbb{C}, 0 \leq \gamma_j < n_j \) for \( 1 \leq j \leq i \), and \( \sum_{j=0}^i \gamma_j \beta_j > n_i \beta_i \).

The above equations with \( v = 1 \) allow us to embed \( C \) as a complete intersection in \( (\mathbb{C}^{g+1}, 0) \) with coordinates \( x_0, \ldots, x_g \). Making \( v \) vary in \( (\mathbb{C}, 0) \) defines a family \( (\chi, 0) \subset (\mathbb{C}^{g+1} \times \mathbb{C}, 0) \) of germs of curves, which is equisingular for instance in the sense that all branches in the family have the same semigroup \( \Gamma(C) \). We denote by \( \eta : (\chi, 0) \longrightarrow (\mathbb{C}, 0) \) the restriction to \( (\chi, 0) \) of the projection on the second factor \( (\mathbb{C}^{g+1} \times \mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0) \).
The special fibre $Y := \eta^{-1}(0)$ is the curve which is of interest to us and is defined by the equations

$$
\begin{align*}
  f_1 := x_1^{n_1} - c_1 x_0^{n_0} &= 0 \\
  f_2 := x_2^{n_2} - c_2 x_0^{n_0} x_1^{b_1} &= 0 \\
  \vdots \\
  f_g := x_g^{n_g} - c_g x_0^{b_0} x_1^{b_1} \cdots x_g^{b_{g-1}} &= 0.
\end{align*}
$$

(1)

After a change of variables in the coordinates $x_0, \ldots, x_g$, we can assume that every $c_i$ for $i = 1, \ldots, g$ is equal to 1; the coefficients $c_i$ are important to see that any irreducible plane curve is a (equisingular) deformation of a curve of type $Y$. For simplicity, throughout this paper, we will consider $c_i = 1$ for $i = 1, \ldots, g$.

**Remark 1.1.** It is worth mentioning that the above embedding of $C$ in $(\mathbb{C}^{g+1}, 0)$ as a complete intersection is Newton non-degenerate in the sense of [Tev1, AGS]. Such an embedding always exists for a singularity in characteristic 0 [Tev2], and is conjectured to exist in positive characteristic [Tei2].

The curve $Y$ is called the *monomial curve associated with $C$* because it is the image in $(\mathbb{C}^{g+1}, 0)$ of the monomial map $M : (\mathbb{C}, 0) \to (\mathbb{C}^{g+1}, 0)$ given by

$$M(t) = (t^{\beta_0}, t^{\beta_1}, \ldots, t^{\beta_g}).$$

In particular, $Y$ is an irreducible curve with its semigroup equal to the ‘plane’ semigroup $\Gamma(C)$, see [Tei1] for these and other properties of $Y$. Finally, note that, even though $Y$ has been defined as a deformation of a germ, we can consider the global curve in $\mathbb{C}^{g+1}$ defined by the above polynomial (actually binomial) equations of $Y$. This is still an irreducible curve and from now on, we will denote by $Y \subset \mathbb{C}^{g+1}$ this global curve.

## 2. Jet schemes and motivic Igusa zeta function

This section provides a short introduction to the jet schemes and motivic zeta function associated to an affine variety. By a *(complex)* variety, we mean a reduced, separated scheme of finite type over $\mathbb{C}$, which is not necessarily irreducible. Throughout this section, let $X = V(I) \subset \mathbb{C}^{g+1}$ be an affine variety defined by an ideal $I = (f_1, \ldots, f_r) \subset \mathbb{C}[x_0, \ldots, x_g]$.

For every $m \in \mathbb{N}$, the *m-th jet scheme* of $X$ is the $\mathbb{C}$-scheme $X_m$ whose $\mathbb{C}$-points are

$$X_m(\mathbb{C}) = \{\text{Spec}(\mathbb{C}[[t]]/(t^{m+1})) \to X\}.$$ 

It immediately follows that $X_0 = X$. For general $m$, one can derive the defining equations of $X_m$ in its natural ambient space $\mathbb{C}^{(g+1)(m+1)}$ as follows. Let $x_i^{(j)}$ for $i = 0, \ldots, g$ and $j = 0, \ldots, m$ be the coordinates in $\mathbb{C}^{(g+1)(m+1)}$. We will denote by $\overline{x}^{(j)}$ the $(g+1)$-tuple $(x_0^{(j)}, \ldots, x_g^{(j)})$ and by $\overline{x}(t)$ the element

$$\overline{x}(t) := \overline{x}(0) + \overline{x}(1)t + \cdots + \overline{x}(m)t^m$$

in $\mathbb{C}[\overline{x}^{(j)}; j = 0, \ldots, m][t]/(t^{m+1})$. For $k = 1, \ldots, r$ and $l = 0, \ldots, m$, let $F_k^{(l)} \in \mathbb{C}[\overline{x}^{(j)}; j = 0, \ldots, l]$ be the elements which satisfy the identity

$$f_k(\overline{x}(t)) = F_k^{(0)} + F_k^{(1)}t + \cdots + F_k^{(m)}t^m \mod (t^{m+1}).$$

(2)
Then, we have
\[ X_m = \text{Spec} \frac{\mathbb{C}[x^{(j)}; j = 0, \ldots, m]}{(F_k^{(l)}; k = 1, \ldots, r; l = 0, \ldots, m)} \].

For \( m, p \in \mathbb{N} \) with \( m \geq p \), there is a natural map \( \pi_{m,p} : X_m \to X_p \) induced by the truncation map \( \mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^{p+1}) \). We will put \( \pi_{m,0} \) for \( \pi_m \). Note that for \( m, p, q \in \mathbb{N} \) with \( m \geq p \geq q \), we have \( \pi_{p,q} \circ \pi_{m,p} = \pi_{m,q} \).

In order to define the motivic zeta function associated to \( X \), we first give a brief introduction to the Grothendieck ring of complex varieties and fix some notation. Let \( \text{Var}_C \) be the category of complex varieties. The Grothendieck group \( K_0(\text{Var}_C) \) is the abelian group generated by the symbols \([V]\) for \( V \in \text{Var}_C \) with the following two relations: 

1. \([V] = [W]\) for isomorphic \( V \) and \( W \), and
2. \([V] = [W] + [V \setminus W]\) for \( W \) closed in \( V \).

By using the multiplication \([V] \cdot [W] := [V \times W]\), the Grothendieck group becomes a commutative ring with 1 := \([\text{Spec } \mathbb{C}]\) as unit element, and we still denote the Grothendieck ring by \( K_0(\text{Var}_C) \). We write \( L := [\mathbb{C}] \) for the class of the affine line and \( M_C := K_0(\text{Var}_C)[L^{-1}] \) for the ring obtained by inverting \( L \).

**Remark 2.1.** For a constructible subset \( W \) of a variety \( V \) (i.e. \( W \) is a finite union of locally closed subvarieties of \( V \)), we can define its class in \( K_0(\text{Var}_C) \) as follows. First, we can always write \( W \) as a finite disjoint union \( W_1 \sqcup \cdots \sqcup W_r \) of locally closed subvarieties of \( V \). Then, one can show that \([W] := \sum_{i=1}^r [W_i]\) is well-defined as element in \( K_0(\text{Var}_C) \). In particular, this definition implies for a locally trivial fibration \( p : V \to B \) with fibre \( F \) that \([V] = [B] \cdot [F]\).

Note that for every \( m \in \mathbb{N} \), a point \( \gamma \in \text{Spec } \mathbb{C}[x^{(j)}; j = 0, \ldots, m] \) corresponds to a jet \( \gamma(t) = (\gamma_0(t), \ldots, \gamma_g(t)) \in (E[t]/(t^{m+1}))^{g+1} \) for some field extension \( E \) of \( \mathbb{C} \). We will often also denote this jet by \( \gamma := \gamma(t) \). Hence, we can define
\[ \text{ord}_t f_k(\gamma) := \text{ord}_t(f_k(\gamma(t))) \]
for \( k = 1, \ldots, g \), and
\[ X_m := \{ \gamma \in \text{Spec } \mathbb{C}[x^{(j)}; j = 0, \ldots, m] \mid \min_{k=1,\ldots,r} \text{ord}_t(f_k(\gamma)) = m \} \].

For each \( m \), the set \( X_m \) is a locally closed subvariety of \( \text{Spec } \mathbb{C}[x^{(j)}; j = 0, \ldots, m] \), and it thus defines a class \([X_m]\) in the Grothendieck ring.

With this notation, the *(global) motivic Igusa zeta function* associated to the variety \( X \) (or to the ideal \( I \)) is the formal power series
\[ Z^\text{mot}_X(T) := L^{-g+1} \sum_{m \geq 0} [X_m](L^{-g+1}T)^m \in M_C[[T]]. \]

There is also a local version where \( X_m \) is replaced by \( X_{m,0} \) consisting of those \( \gamma \in X_m \) with \( x_i^{(0)} = 0 \) for \( i = 0, \ldots, g \), or equivalently, with associated jet \( \gamma(t) \) having the origin 0 as center (i.e. \( \gamma(0) = 0 \)). For \( X \) defined by one polynomial (i.e. \( I = (f) \)), the motivic zeta function was introduced and shown to be rational by Denef and Loeser in [DL2]. The definition for general ideals can be found in [VZ].

To find the motivic zeta function \( Z^\text{mot}_Y(T) \) associated to the monomial curve \( Y \subset \mathbb{C}^{g+1} \), we will not compute the above series directly. Instead, we will compute the Poincaré series
\[ J_Y(T) := \sum_{m \geq 0} [Y_m](L^{-g+1}T)^{m+1}, \]
where $[Y_m] \subset \mathbb{C}^{(g+1)(m+1)}$ is the $m$-th jet scheme of $Y$. This is well-defined because of the fact that $[Y_m] = [(Y_m)_{\text{red}}]$ in $K_0(\text{Var}_{\mathbb{C}})$. Using the relations $[Y_0] = \mathbb{L}^{g+1} - [Y_0]$ and $[Y_m] = \mathbb{L}^{g+1}[Y_{m-1}] - [Y_m]$ for $m \geq 1$, it is not hard to see that the Poincaré series is related to $Z^{\text{mot}}_Y(T)$ by

$$Z^{\text{mot}}_Y(T) = 1 - \frac{1 - T}{T} J_Y(T).$$

**Remark 2.2.** One often considers the more natural Poincaré series

$$\sum_{m \geq 0} [Y_m] T^m.$$

We choose to work with the above series $J_Y(T)$ because the factor $\mathbb{L}^{-(g+1)(m+1)}$ implies, in some sense, that we need to look for the codimension of $Y$.

3. Jet Schemes of Space Monomial Curves Whose Semigroup is Plane

In this section, we will study the jet schemes $Y_m$ of the monomial curve $Y \subset \mathbb{C}^{g+1}$ defined in Section 1. We keep the notations from the previous sections and we will be concerned with the case $g > 1$; if $g = 1$, then $Y$ is a plane curve with one pinceaux pair and the structure of $Y_m$ has been studied in [Mou1, Corollary 4.4]. The information is mainly concentrated in the fibres $\pi^{-1}_{m}(0)$ of $\pi_m : Y_m \to Y$ for $m \geq 1$; indeed, the restriction of $\pi_m$ to $\pi^{-1}_{m}(Y \setminus \{0\})$ is a trivial fibration whose fibres are isomorphic to $\mathbb{C}^n$. In the next proposition, we see that the fibres $\pi^{-1}_{m}(0)$ for $m \leq n_0 n_1$ are irreducible and rather easy to understand. In fact, we will be concerned with their reduced structure $\pi^{-1}_{m}(0)_{\text{red}}$. For a rational number $\frac{a}{b}$, we denote by $[\frac{a}{b}]$ its integer part.

**Proposition 3.1.** (1) For $m \in \mathbb{N}$ satisfying $0 < m < n_0 n_1$, we have

$$\pi^{-1}_{m}(0)_{\text{red}} = \text{Spec} \frac{\mathbb{C}[x(j); j = 0, \ldots, m]}{(x_i(0), \ldots, x_i([\frac{m}{i}]); i = 0, \ldots, g)}.$$

(2) The fibre $\pi^{-1}_{n_0 n_1}(0)_{\text{red}}$ is given by

$$\text{Spec} \frac{\mathbb{C}[x(j); j = 0, \ldots, n_0 n_1]}{(x_0(0), \ldots, x_0([n_1-1]), x_1(0), \ldots, x_1([n_1]), \ldots, x_i(0), \ldots, x_i([\frac{n_0 n_1}{i}]); i = 2, \ldots, g)}.$$

The following lemma is used in the proof of Proposition 3.1.

**Lemma 3.2.** For $i = 2, \ldots, g$, we have $b_{ij} > n_0$.

**Proof.** Fix $i \in \{2, \ldots, g\}$. We prove the inequality in the lemma by contradiction; assume that $b_{ij} \leq n_0$. On one hand, we have

$$b_{i0} \beta_0 + b_{i1} \beta_1 + \cdots + b_{i(i-1)} \beta_{i-1} \leq n_0 \beta_0 + n_1 \beta_1 + \cdots + n_{i-1} \beta_{i-1} - (\beta_1 + \cdots + \beta_{i-1}),$$

where we used that $b_{ij} \leq n_j - 1$ for $j = 1, \ldots, i - 1$. On the other hand, by repeatedly using that $\beta_i > n_{i-1} \beta_{i-1}$ and that $n_{i} \geq 2$, we find that

$$n_{i-1} \beta_i > n_i (n_{i-1} \beta_{i-1})$$

$$\geq n_{i-1} \beta_{i-1} + n_{i-1} \beta_{i-1}$$

$$> n_{i-1} \beta_{i-1} + n_{i-1} (n_{i-2} \beta_{i-2})$$

$$\geq n_{i-1} \beta_{i-1} + n_{i-2} \beta_{i-2} + n_{i-2} \beta_{i-2}$$

$$\vdots$$

$$\geq n_{i-1} \beta_{i-1} + n_{i-2} \beta_{i-2} + \cdots + n_2 \beta_2 + n_1 \beta_1 + n_1 \beta_1$$

$$= n_{i-1} \beta_{i-1} + n_{i-2} \beta_{i-2} + \cdots + n_2 \beta_2 + n_1 \beta_1 + n_1 \beta_1 + n_0 \beta_0,$$
where the last equality follows from \( n_0\beta_0 = b_{10}\beta_0 = n_1\beta_1 \). The two families of inequalities give

\[
b_{i0}\beta_0 + b_{i1}\beta_1 + \cdots + b_{i(i-1)}\beta_{i-1} \leq n_{i0}\beta_0 + n_{i1}\beta_1 + \cdots + n_{i-1}\beta_{i-1} - (\beta_1 + \cdots + \beta_{i-1})
\]

\[
< n_{i0}\beta_0 + n_{i1}\beta_1 + \cdots + n_{i-1}\beta_{i-1}
\]

\[
< n_i\beta_i.
\]

This contradicts the equality \( b_{i0}\beta_0 + b_{i1}\beta_1 + \cdots + b_{i(i-1)}\beta_{i-1} = n_i\beta_i \) from Section 1. \( \square \)

**Proof of proposition 3.1.** We begin with proving the first part; assume that \( 0 < m < n_0n_1 \). Recall that \( f_1, \ldots, f_g \) are the defining equations of \( Y \) given in (1), and that a closed point \( \gamma \in \text{Spec } \mathbb{C}[x^{(j)}; j = 0, \ldots, m] \) corresponds to a jet that we also call \( \gamma = \gamma(t) = (\gamma_0(t), \ldots, \gamma_g(t)) \) with

\[
\gamma_i(t) = \sum_{l=0}^{m} x_i^{(l)} t^l,
\]

where \( x_i^{(l)} \) are the coordinates of \( \gamma \). With this notation, \( \gamma \in \pi^{-1}_m(0)_{\text{red}} \) if and only if the coordinates \( x_i^{(0)} \) for \( i = 0, \ldots, g \) are zero (i.e., the center of \( \gamma \) is the origin) and \( \text{ord}_i(f_k(\gamma)) \geq m+1 \) for \( k = 1, \ldots, g \). From Proposition 4.1 in [Mou1] applied to the curve given by the equation \( x_i^{n_i} - x_0^{n_0} = 0 \), we find that

\[
x_i^{(l)} = 0, \quad i = 0, 1 ; \quad l = 0, \ldots, \left[ \frac{m}{n_i} \right].
\]

Taking into consideration that \( m < n_0n_1 \), one can see that these conditions are the only conditions coming from the equation \( f_1 = x_1^{n_1} - x_0^{n_0} \). In particular, assuming that the center of \( \gamma \) is the origin, the condition \( \text{ord}_i(f_1(\gamma)) \geq m+1 \) implies that \( \text{ord}_i(\gamma(0)) \geq \left[ \frac{m}{n_i} \right] + 1 \). We will prove that \( \text{ord}_i(\gamma(0)) \geq \left[ \frac{m}{n_i} \right] + 1 \) in turn implies for every \( i = 2, \ldots, g \) that \( \text{ord}_i(f_i(\gamma)) \geq m+1 \) is equivalent to \( \text{ord}_i(\gamma_i) \geq \left[ \frac{m}{n_i} \right] + 1 \}. \) Recall that \( f_i = x_i^{n_i} - x_0^{b_{0i}} \cdot \cdots \cdot x_{i-1}^{b_{i-1}} \). Since, by Lemma 3.2, we have \( b_0 > n_0 \) and since \( \text{ord}_i(\gamma(0)) \geq \left[ \frac{m}{n_i} \right] + 1 \), we obtain

\[
\text{ord}_i(x_0^{b_{0i}} \cdots x_i^{b_{i-1}}(\gamma)) = \text{ord}_i(\gamma_0^{n_0} \cdots \gamma_i^{n_i}) \geq \text{ord}_i(\gamma_0^{n_0}) \]

\[
\geq \text{ord}_i(\gamma_0^{n_0+1})
\]

\[
\geq (n_0 + 1) \left( \left[ \frac{m}{n_i} \right] + 1 \right)
\]

\[
\geq m + 1.
\]

Hence, \( \text{ord}_i(f_i(\gamma)) \geq m+1 \) if and only if \( \text{ord}_i(x_i^{n_i}(\gamma)) = \text{ord}_i(\gamma_i^{n_i}) \geq m+1 \). This latter is equivalent to \( \text{ord}_i(\gamma_i) \geq \left[ \frac{m}{n_i} \right] + 1 \}, \) and this ends the proof of the first part.

We now prove the second part of the proposition. Let \( \gamma \in \text{Spec } \mathbb{C}[x^{(j)}; j = 0, \ldots, n_0n_1] \) be again associated with a \((g + 1)\)-tuple \( \gamma = \gamma(t) \). It follows from the first part that \( \gamma \in \pi^{-1}_{n_0n_1}(0)_{\text{red}} \) implies that

\[
x_i^{(l)} = 0, \quad i = 0, \ldots, g ; \quad l = 0, \ldots, \left[ \frac{n_0n_1 - 1}{n_i} \right].
\]
Noticing that $f_1$ is weighted homogeneous of degree $n_0 n_1$ if we give $x_0$ the weight $n_1$ and $x_1$ the weight $n_0$, we can write
\[
f_1(\gamma) \equiv f_1\left(t^{n_0}(x_0^{(n_1)} + x_0^{(n_1+1)} t + \cdots + x_0^{(n_0 n_1)} t^{n_0 n_1 - n_1}),
\right.
\]
\[
\left. t^{n_0}(x_1^{(n_0)} + x_1^{(n_0+1)} t + \cdots + x_1^{(n_0 n_1)} t^{n_0 n_1 - n_0}) \right).
\]
\[
\equiv t^{n_0 n_1} f_1\left(x_0^{(n_1)} + x_0^{(n_1+1)} t + \cdots + x_0^{(n_0 n_1)} t^{n_0 n_1 - n_1},
\right.
\]
\[
\left. x_1^{(n_0)} + x_1^{(n_0+1)} t + \cdots + x_1^{(n_0 n_1)} t^{n_0 n_1 - n_0} \right).
\]
\[
\equiv t^{n_0 n_1} f_1(x_0^{(n_1)}, x_1^{(n_0)}).
\]
\[
\equiv t^{n_0 n_1} (x_0^{(n_1)} - x_0^{(n_0)}) \mod (t^{n_0 n_1 + 1}).
\]

Thus, modulo that $\gamma$ is centred at the origin and that $\text{ord}_t(f_1(\gamma)) \geq n_0 n_1$, the condition $\text{ord}_t(f_1(\gamma)) \geq n_0 n_1 + 1$ is equivalent to $x_0^{(n_1)} - x_0^{(n_0)} = 0$. For $i = 2, \ldots, g$, again modulo that $\gamma$ is centred at the origin and that $\text{ord}_t(f_i(\gamma)) \geq n_0 n_1$, one can now see as in the first part of the proof that $\text{ord}_t(f_i(\gamma)) \geq n_0 n_1 + 1$ if and only if $x_i^{(n_1)} - x_i^{(n_0)} = 0$. This proves the second part of the proposition. □

The same reasoning as in the proof of Proposition 3.1 gives us the following corollary.

**Corollary 3.3.** Let $l \in \mathbb{N}$. For $m \in \mathbb{N}$ such that $l n_0 n_1 < m < (l + 1) n_0 n_1$, we have
\[
\pi_{m,n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(n_1)} = 0\}) \subseteq \text{Spec} \frac{\mathbb{C}[x_{j}^{(j)}; j = 0, \ldots, m]}{(x_{i}^{(0)}, \ldots, x_{i}^{(\lfloor \frac{m}{n_1} \rfloor)}; i = 0, \ldots, g)}.
\]

The ideal defining the embedding of $\pi_{m,n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(n_1)} = 0\}) \subseteq \text{Spec} \mathbb{C}[x_{j}^{(j)}; j = 0, \ldots, m, (l + 1) n_0 n_1]$ is generated by
\[
x_0^{(0)}, \ldots, x_0^{((l + 1) n_1 - 1)}, x_0^{(0)}, \ldots, x_1^{((l + 1) n_1 - 1)}, x_1^{((l + 1) n_0) n_1} - x_0^{((l + 1) n_0 n_1)},
\]
\[
x_i^{(0)}, \ldots, x_i^{((l + 1) n_1 - 1)}; i = 2, \ldots, g.
\]

Before pursuing in describing the jet schemes of $Y$, let us give a description of the picture as we understand it until now. Recall that we want to study the geometry of $\pi_{m}^{-1}(0)_{\text{red}}$ for $m \geq 1$. From Proposition 3.1, we know that $\pi_{m}^{-1}(0)_{\text{red}}$ for $0 < m < n_0 n_1$ is irreducible and isomorphic to an affine space. We also know that $\pi_{m}^{-1}(0)_{\text{red}}$ is the product of an affine space and a hypersurface defined by the equation $x_0^{(n_1)} - x_0^{(n_0)} = 0$. We can stratify $\pi_{m}^{-1}(0)_{\text{red}} \subseteq \mathbb{C}[g]_{(n_1)}(n_0 n_1)$ as follows:
\[
\pi_{m,n_0 n_1}^{-1}(0)_{\text{red}} = (\pi_{m,n_0 n_1}^{-1}(0)_{\text{red}} \cap \{x_0^{(n_1)} = 0\}) \cup (\pi_{m,n_0 n_1}^{-1}(0)_{\text{red}} \cap \{x_0^{(n_1)} \neq 0\}).
\]

For $n_0 n_1 < m \leq 2 n_0 n_1$, Corollary 3.3 shows that $\pi_{m,n_0 n_1}^{-1}(\pi_{m,n_0 n_1}^{-1}(0)_{\text{red}} \cap \{x_0^{(n_1)} = 0\}) = \pi_{m,n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(n_1)} = 0\})$ is irreducible and has a rather simple geometry. Noticing, therefore, that $\pi_m = \pi_{m,n_0 n_1} \circ \pi_{m,n_0 n_1}$, we need to study the inverse image under $\pi_{m,n_0 n_1}$ of $\pi_{m,n_0 n_1}^{-1}(0)_{\text{red}} \cap \{x_0^{(n_1)} \neq 0\}$ to obtain a stratification of $\pi_{m}(0)_{\text{red}}$ in two strata of which we understand the geometry. For $2 n_0 n_1 < m$, Corollary 3.3 does not immediately give us an easy description of $\pi_{m,n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(n_1)} = 0\})$. However, because $\pi_{m,n_0 n_1} = \pi_{m,n_0 n_1} \circ \pi_{m,n_0 n_1}$, understanding this inverse image boils down to understanding the inverse image by $\pi_{m,n_0 n_1}$ of the known $\pi_{m,n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(n_1)} = 0\})$, which we again stratify in two strata corresponding to $x_0^{(2 n_0)} = 0$ and $x_0^{(2 n_0)} \neq 0$. For general $m \geq 1$, the stratification of $\pi_{m,n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(n_1)} = 0\})_{\text{red}}$ for
$k \geq 0$ as
\[
\begin{align*}
(\pi_{(k+1)n_0n_1}, &\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(km_1)} = 0\})_{red} \cap \{x_0^{(k+1)n_1)} = 0\}) \\
\sqcup &\{(\pi_{(k+1)n_0n_1}, k_{n_0n_1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(km_1)} = 0\})_{red} \cap \{x_0^{(k+1)n_1)} \neq 0\}) \}
\end{align*}
\]
yields the following stratification of $\pi^{-1}_m(0)_{red}$: let $l \in \mathbb{N}$ such that $ln_0n_1 < m \leq (l+1)n_0n_1$, then
\[
\pi^{-1}_m(0)_{red} = \left( \bigcup_{k=1}^l D_{m,k} \right) \sqcup B_m,
\]
where
\[
\begin{align*}
D_{m,k} &:= \pi^{-1}_{m,kn_0n_1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(kn_1-1)} = 0\} \cap \{x_0^{(kn_1)} \neq 0\})_{red}, \\
B_m &:= \pi^{-1}_{m,ln_0n_1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(ln_1)} = 0\})_{red}.
\end{align*}
\]
This stratification will allow us to determine the irreducible components of $Y_m$ and will be crucial for our computation of the motivic zeta function associated to $Y$. It is important to notice, as we will see later, that some of the above strata may be empty. Furthermore, $B_m$ is a closed subvariety of $\pi^{-1}_m(0)_{red}$ of which Corollary 3.3 provides the geometric structure. In particular, we know its codimension in $\mathbb{C}^{(g+1)(m+1)}$.

**Corollary 3.4.** Let $l \in \mathbb{N}$. For $m \in \mathbb{N}$ such that $ln_0n_1 < m \leq (l+1)n_0n_1$, the codimension of $B_m$ in $\mathbb{C}^{(g+1)(m+1)}$ is equal to
\[
g + 1 + \sum_{i=0}^g \left[ \frac{m}{n_i} \right].
\]

For $m = (l+1)n_0n_1$, the codimension of $B_m$ in $\mathbb{C}^{(g+1)(m+1)}$ is equal to
\[
g + (l+1)(n_0 + n_1) + \sum_{i=2}^g \left[ \frac{(l+1)n_0n_1}{n_i} \right].
\]

Still, we need to understand the geometry of the strata $D_{m,k}$ for $k = 1, \ldots, l$, which are locally closed subvarieties of $\pi^{-1}_m(0)_{red}$. We begin by introducing some useful notations. Firstly, for $k \geq 1$ and $m \geq kn_0n_1$, let $C_{m,k} := \overline{D_{m,k}}$ be the Zariski closure of $D_{m,k}$ in $\pi^{-1}_m(0)_{red}$. Secondly, for $k \geq 1$, let $j(k) \in \mathbb{N}$ be defined by
\[
j(k) := \begin{cases} 2 & \text{if } n_2 \nmid k \\ \max_{i \in \mathbb{N}} \{n_2 \cdots n_{i-1} | k\} & \text{otherwise.} \end{cases}
\]
Note that $2 \leq j(k) \leq g + 1$. For $1 \leq i < j(k)$ and for $m \in \mathbb{N}$ satisfying
\[
\frac{kn_i\beta_i}{\epsilon_1} \leq m < \frac{kn_{i+1}\beta_{i+1}}{\epsilon_1},
\]
where, by convention, $\beta_{g+1} := +\infty$, we define
\[
c_{i,k}(m) := k(n_0 + n_1) + \sum_{l=2}^i \frac{k\beta_l}{\epsilon_1} + \sum_{l=1}^i \left( m - \frac{kn_l\beta_l}{\epsilon_1} + 1 \right) + \sum_{l=i+1}^g \left( \left[ \frac{m}{n_l} \right] + 1 \right).
\]
Proposition 3.5. Let \( k \geq 1 \) and \( 1 \leq i < j(k) \). For \( m \in \mathbb{N} \) with \( \frac{k_n \beta}{e_1} < m < \frac{k_{n+1} \beta + 1}{e_1} \), the stratum \( D_{m,k} \) is isomorphic to
\[
(Y^i \setminus \{0\}) \times \mathbb{C}^{(g+1)(m+1) - c_{i,k}(m) - 1} \simeq (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{(g+1)(m+1) - c_{i,k}(m) - 1}.
\]
In particular, \( C_{m,k} \) is irreducible and its codimension in \( \mathbb{C}^{(g+1)(m+1)} \) is equal to \( c_{i,k}(m) \).

For \( m \geq \frac{kn_j \beta e_1 e_1}{e_1} \), we have \( D_{m,k} = \emptyset \).

Before giving a proof of this proposition, we show the next lemma.

Lemma 3.6. Let \( i, j \in \mathbb{N} \) be such that \( i + 1 \leq j \leq g \). We have
\[ b_{j_0} \beta_0 + \cdots + b_{j_l} \beta_i \geq n_{i+1} \beta_{i+1}, \]
and the inequality is strict if \( i + 1 < j \).

Proof. For \( i + 1 = j \), the inequality is an equality and there is nothing to prove. Assume that \( i + 1 < j \). On one hand, we have
\[ b_{j_0} \beta_0 + \cdots + b_{j_l} \beta_i = n_j \beta_j - b_{j_{i+1}} \beta_{i+1} - \cdots - b_{j_{j-1}} \beta_{j-1} \geq n_j \beta_j - n_{i+1} \beta_{i+1} - \cdots - n_{j-1} \beta_{j-1}. \]

The inequality follows from the fact that \( b_{j_l} < n_l \) for \( l = 1, \ldots, j-1 \). On the other hand, as in the proof of Lemma 3.2, we have
\[ n_j \beta_j > n_{j-1} \beta_{j-1} + n_{j-2} \beta_{j-2} + \cdots + n_{i+1} \beta_{i+1} + n_{i+1} \beta_{i+1} + n_{i+1} \beta_{i+1}. \]

The two series of inequalities give the strict inequality in the lemma. \(\square\)

Proof of Proposition 3.5. In this proof, we still denote by \( f_1, \ldots, f_g \) the defining equations of \( Y \) and by \( F_h^{(l)} \) for \( h = 1, \ldots, g \) and \( l \in \mathbb{N} \) the polynomials defined from \( f_h \) by the identity (2) in Section 2. We will prove that for \( m \in \mathbb{N} \) with \( \frac{kn_h \beta}{e_1} < m < \frac{k_{n+1} \beta + 1}{e_1} \) for some \( 1 \leq i < j(k) \), the ideal defining the embedding of \( D_{m,k} \) in \( \text{Spec} \mathbb{C}^{[x(j); j = 0, \ldots, m]} \) is generated by
\[
F_h^{(l)} = x_h^{(\frac{k_h \beta}{e_1})} - x_h^{(\frac{k_h \beta}{e_1})} \beta_{h_{n-1}} x_h^{(\frac{k_h \beta - 1}{e_1})} \beta_{h_{n-1}} \beta_{h_{n-1}} \]
and for \( l = 1, \ldots, m - \frac{kn_h \beta}{e_1} \),
\[
F_h^{(l)} = \alpha_l x_h^{(\frac{k_h \beta}{e_1})} x_h^{(\frac{k_h \beta + 1}{e_1})} - H_l(x_h^{(\frac{k_h \beta}{e_1})}, \ldots, x_h^{(\frac{k_h \beta + 1}{e_1})}, x_h^{(\frac{k_h \beta + l}{e_1})}, x_h^{(\frac{k_h \beta + l - 1}{e_1})})
\]
for some \( \alpha_l \in \mathbb{C} \setminus \{0\} \) and \( H_l \) a polynomial. From this result, the claim in the proposition will easily follow.

The proof is by induction on \( i \). We begin with the case \( i = 1 \); let \( kn_1 \leq m < \frac{kn_2 \beta}{e_1} \). As in Proposition 3.1, a closed point \( \gamma \in \text{Spec} \mathbb{C}^{[x^{(j)}; j = 0, \ldots, m]} \) corresponds to a jet that we also call \( \gamma = \gamma(t) = (\gamma_0(t), \ldots, \gamma_m(t)) \) with
\[
\gamma_i(t) = \sum_{l=0}^{m} x_i^{(l)} t^l.
\]
where $x_i$ are the coordinates of $\gamma$. The condition that $\gamma \in D_{m,k}$ is equivalent to the conditions $x_0 = \cdots = x_{k(n+1)-1} = 0$, $x_{k(n+1)} \neq 0$, and $ord_l(f_s(\gamma)) \geq m + 1$ for $s = 1, \ldots, g$.

From a little argument using Corollary 3.3, one can see that this implies the equalities $x_0 = \cdots = x_{k(n+1)-1} = 0$ and $x_1^{(k(n+1))} - x_0^{(k(n))} = 0$. Since $x_0^{(k(n+1))} \neq 0$, the last equation tells us that $x_1^{(k(n+1))} \neq 0$.

Let us first examine the condition $ord_l(f_1(\gamma)) \geq m + 1$. We have

\[
f_1(\gamma) = \sum_{l=k(n+1)}^{m} x_l \cdot f_l = \sum_{l=k(n+1)}^{m} x_l \cdot (m-k(n+1)) = \sum_{l=k(n+1)}^{m} x_l \cdot \beta_l \cdot \ln \equiv t^{k(m+1)} \mod (m+1),
\]

where $F_l := \sum_{l=k(n+1)}^{m} x_l \cdot \beta_l \cdot \ln$. The second equality follows from the weighted-homogeneity of $f_1$ with weights $n_1$ and $n_0$ for $x_0$ and $x_1$, respectively. Hence, the condition that $ord_l(f_1(\gamma)) \geq m + 1$ is equivalent to the annihilation of $F_l$ for $l = 0, \ldots, m - k(n+1)$ with the condition that $x_0^{(k(n+1))} \neq 0$. These are the defining equations of the jet schemes of the regular part of the curve defined by $x_0^{(k(n+1))} - x_0^{(k(n))} = 0$.

Clearly, $F_l = x_1^{(k(n+1))} - x_0^{(k(n))}$, and for $l = 1, \ldots, m - k(n+1)$, they are of the form

\[
F_l = \alpha_l x_1^{(k(n+1))} - H_l(x_0^{(k(n+1))}, x_1^{(k(n))}, \ldots, x_l^{(k(n+1))}, x_l^{(k(n-1))}),
\]

where $\alpha_l = \ln \{0\}$ and $H_l$ is a polynomial. Because $x_1^{(k(n+1))} \neq 0$, we can divide each $F_l$ for $l = 1, \ldots, m - k(n+1)$ by $x_1^{(k(n+1))}$ to see that $F_l$ is linear in $x_1^{(k(n+1))}$, and that the expression of $x_1^{(k(n+1))}$ does not depend on the variables $x_1^{(k(n+1))}$ for $h > k(n+1)$. In other words, the system of equations given by $F_l$ for $l = 0, \ldots, m - k(n+1)$ is triangular in the variables $x_1^{(k(n+1))}, \ldots, x_1^{(k(n+1-m-k(n+1))})$.

Let us now examine the conditions $ord_l(f_s(\gamma)) \geq m + 1$ for $s = 2, \ldots, g$. We already know that $ord_l(\gamma_0) = k(n+1) = \frac{k_0}{e_1}$ and $ord_l(\gamma_1) = k(n+1) = \frac{k_2}{e_1}$. Therefore,

\[
ord_l(x_0^{(b_0)} \cdots x_{s-1}^{(b_{s-1})}(\gamma)) \geq ord_l(\gamma_0^{(b_0)} \gamma_1^{(b_{s-1})}) = b_0 \frac{k_0}{e_1} + b_{s-1} \frac{k_2}{e_1} \geq m + 1,
\]

where the last two inequalities follow from Lemma 3.6 and our assumption on $m$, respectively. Since $f_s = x_s^{(r_n)} - x_s^{(r_m)} \cdots x_s^{(r_{s-1})}$, it follows that $ord_l(f_s(\gamma)) \geq m + 1$ is equivalent to $ord_l(x_s^{(r_m)}(\gamma)) = ord_l(\gamma_0^{(r_n)}) \geq m + 1$, which is in turn equivalent to $x_s^{(0)} = \cdots = x_s^{(\frac{m}{c_1})} = 0$.

To recapitulate, the embedding of $D_{m,k}$ in Spec $\mathbb{C}[x^{(j)}; j = 0, \ldots, m]_{x_0^{(k(n+1))}}$ is defined by the ideal

\[
\left(x_0^{(r_m)} \cdots x_r^{(k(n+1)) - 1}, F_1^{(l)}, x_0^{(r_m)} \cdots x_s^{(k(n+1))}; r = 0, 1; l = 0, \ldots, m - k(n+1); s = 2, \ldots, g\right),
\]

which is exactly the same as we claimed. The codimension of $D_{m,k}$ is equal to $c_{1,k}(m)$; indeed, the above ideal is a complete intersection because the system of equations given by $F_1^{(l)}$ for $l = 0, \ldots, m - k(n+1)$ is triangular in the variables $x_1^{(k(n+1))}, \ldots, x_1^{(k(n+1-m-k(n+1))}}$.

As $F_0 = x_1^{(k(n+1))} - x_0^{(k(n+1))}$, we clearly also have

\[
D_{m,k} \simeq (Y^1 \setminus \{0\}) \times \mathbb{C}^{(g+1)(m+1)-c_{1,k}(m)-1}.
\]
The case where \( i < j(k) \). We need to prove that the description for the ideal of \( D_{m,k} \) is also true if \( \frac{kn_i \beta}{e_1} < m < \frac{kn_{i+1} \beta + 1}{e_1} \). Let \( \gamma \in D_{m,k} \) be identified with its corresponding jet \( \gamma(t) \). Because

\[
\pi_{m, kn_0 n_1} = \pi_{m, \frac{kn_i \beta}{e_1} - 1} \circ \pi_{\frac{kn_i \beta}{e_1} - 1, kn_0 n_1}^{-1},
\]

we have

\[
D_{m,k} = \pi^{-1}_{m, \frac{kn_i \beta}{e_1} - 1} \left( D_{m, \frac{kn_i \beta}{e_1} - 1, kn_0 n_1} \right).
\]

From the induction hypothesis, it follows that the coordinates of \( \gamma \) satisfy, among others, the equations

\[
x_r^{(0)} = \cdots = x_r^{(\frac{kn_i \beta}{e_1} - 1)} = x_r^{(\frac{kn_i \beta}{e_1})} \cdots x_r^{(h-1)} = 0;
\]

\[
r = 0, \ldots, i - 1; \; h = 1, \ldots, i - 1.
\]

Since \( x_0^{(\frac{kn_i \beta}{e_1})} \neq 0 \), the equation for \( h = 1 \) gives us that \( x_1^{(\frac{kn_i \beta}{e_1})} \neq 0 \). Then, the equation for \( h = 2 \) gives that \( x_2^{(\frac{kn_i \beta}{e_1})} \neq 0 \). We can repeat this to conclude that

\[
x_r^{(\frac{kn_i \beta}{e_1})} \neq 0; \; r = 0, \ldots, i - 1.
\]

Together with the other equations, this implies that \( ord_t(\gamma_r) = \frac{kn_i \beta}{e_1} \) for \( r = 0, \ldots, i - 1 \). The induction hypothesis also tells us that \( ord_t(\gamma_i) \geq \frac{kn_i \beta}{e_1} \). We now investigate, modulo the defining ideal of \( D_{\frac{kn_i \beta}{e_1} - 1, k} \), the condition \( ord_t(f_i(\gamma)) \geq m + 1 \).

Note that the equation \( f_i \) is weighted homogeneous of degree \( \frac{kn_i \beta}{e_1} \) if we give \( x_r \) the weight \( \frac{kn_i \beta}{e_1} \) for \( r = 0, \ldots, i \). Therefore,

\[
f_i(\gamma) \equiv f_i \left( \sum_{l=1}^{m} x_i^{(\frac{kn_i \beta}{e_1})} l^t, \sum_{l=1}^{m} x_i^{(\frac{kn_i \beta}{e_1})} l^t, \ldots, \sum_{l=1}^{m} x_i^{(\frac{kn_i \beta}{e_1} + l)} l^t \right) \equiv t^{\frac{kn_i \beta}{e_1} - 1} \sum_{l=0}^{m - \frac{kn_i \beta}{e_1}} \mathcal{F}^{(l)}_i l^t \pmod{(t^{m+1})},
\]

where \( \mathcal{F}^{(l)}_i := F_i^{(l)} \left( x_0^{(\frac{kn_i \beta}{e_1})}, \ldots, x_i^{(\frac{kn_i \beta}{e_1})}, \ldots, x_0^{(\frac{kn_i \beta}{e_1} + l)}, \ldots, x_i^{(\frac{kn_i \beta}{e_1} + l)} \right) \). More precisely,

\[
\mathcal{F}^{(0)}_i = x_i^{(\frac{kn_i \beta}{e_1})} \cdots x_i^{(\frac{kn_i \beta}{e_1} + b_0)} \cdots x_i^{(\frac{kn_i \beta}{e_1} + b_{i-1})},
\]

and, for \( l = 1, \ldots, m - \frac{kn_i \beta}{e_1} \),

\[
\mathcal{F}^{(l)}_i = \alpha_i x_i^{(\frac{kn_i \beta}{e_1})} x_i^{(\frac{kn_i \beta}{e_1} + l)} - H_l \left( x_0^{(\frac{kn_i \beta}{e_1})}, \ldots, x_i^{(\frac{kn_i \beta}{e_1})}, \ldots, x_0^{(\frac{kn_i \beta}{e_1} + l)}, \ldots, x_i^{(\frac{kn_i \beta}{e_1} + l)}, x_i^{(\frac{kn_i \beta}{e_1} + l)} \right)
\]

for some \( \alpha_i \in \mathbb{C} \setminus \{0\} \) and a polynomial \( H_l \). The condition \( ord_t(f_i(\gamma)) \geq m + 1 \) is thus given by the annihilation of \( \mathcal{F}^{(l)}_i \) for \( l = 0, \ldots, m - \frac{kn_i \beta}{e_1} \). Because \( x_r^{(\frac{kn_i \beta}{e_1})} \neq 0 \) for \( r = 0, \ldots, i - 1 \), the equation \( \mathcal{F}^{(0)}_i = 0 \) gives that \( x_i^{(\frac{kn_i \beta}{e_1})} \neq 0 \). Dividing \( x_i^{(\frac{kn_i \beta}{e_1})} \) in \( \mathcal{F}^{(l)}_i \) for \( l \geq 1 \), we again see that the system of equations is triangular in the variables \( x_i^{(\frac{kn_i \beta}{e_1})}, \ldots, x_i^{(\frac{kn_i \beta}{e_1} + m - \frac{kn_i \beta}{e_1})} \), and both the description of the ideal and the statement of the proposition follow.
The case where $i = j(k)$. In this case, it is enough to prove that $D_{\frac{kn_i\bar{\beta}}{e_1},k} = \emptyset$. Suppose there exists an element $\gamma \in D_{\frac{kn_i\bar{\beta}}{e_1},k}$ and identify $\gamma$ once more with its jet $\gamma(t)$. On one hand, like in the previous case, the induction hypothesis implies that 

$$x_r^{\frac{k\bar{\beta}}{e_1}} \neq 0; \ r = 0, \ldots, i - 1.$$ 

Therefore,

$$\text{ord}_t(x_0^{b_0} \cdots x_{i-1}^{b_{(i-1)}}(\gamma)) = \frac{kn_i\bar{\beta}}{e_1}.$$ 

On the other hand, from the assumption $i = j(k)$ and the fact that $e_i = n_{i+1} \cdots n_g$, we have $\frac{kn_i\bar{\beta}}{e_1} = \frac{k'\bar{\beta}}{e_1}$ for some $k' \geq 1$ which is not a multiple of $n_i$. Since $n_i = \frac{e_i-1}{e_1}$ and $e_i = \gcd(e_i-1,\bar{\beta}_i)$, we also know that $n_i$ and $\frac{\beta_i}{e_1}$ are coprime. It follows that $n_i$ does not divide $\frac{kn_i\beta}{e_1} = k'\bar{\beta}_i$. As $f_i = x_i^{n_i} - x_0^{b_0} \cdots x_{i-1}^{b_{(i-1)}}$ and $\text{ord}_t(\gamma_i) \geq \frac{k\bar{\beta}_i}{e_1}$, by the induction hypothesis, we can conclude that $\text{ord}_t(f_i(\gamma)) = \frac{kn_i\beta_i}{e_1}$. This contradicts that $\gamma \in D_{\frac{kn_i\bar{\beta}}{e_1},k}$.

In other words, $D_{\frac{kn_i\bar{\beta}}{e_1},k} = \emptyset$. 

We are now able to give the decomposition of $\pi_{m}^{-1}(0)_{\text{red}}$ into irreducible components.

**Theorem 3.7.** Consider $m \geq 1$. Let $l \in \mathbb{N}$ be such that $ln_0n_1 < m < (l+1)n_0n_1$, and let

$$D_{m,k} = \pi_{m,\text{mon}}^{-1}\left(\{x_0^{(0)} = x_1^{(1)} = \cdots = x_0^{(kn_1)} = 0\} \cap \{x_0^{(kn_1)} \neq 0\}\right)_{\text{red}}, \ k = 1, \ldots, l,$$

$$B_m = \pi_{m,\text{mon}}^{-1}\left(\{x_0^{(0)} = x_1^{(1)} = \cdots = x_0^{(ln_1)} = 0\}\right)_{\text{red}}.$$ 

The irreducible components of $\pi_{m}^{-1}(0)_{\text{red}}$ are $C_{m,k} = \overline{D_{m,k}}$ for $k = 1, \ldots, l$ such that $m < \frac{kn_i\beta_i}{e_1}$ and $B_m$. Furthermore, $B_m$ is a component of maximal dimension.

**Proof.** If $l = 0$, then $\pi_{m}^{-1}(0)_{\text{red}} = B_m$ is irreducible, and there is nothing to prove. For $l \geq 1$, the stratification (3) and Proposition 3.5 tell us that

$$\pi_{m}^{-1}(0)_{\text{red}} = \left(\bigcup_{k=1}^{l} C_{m,k}\right) \cup B_m$$

is a decomposition in irreducible, closed subvarieties, and that the extra condition on $k$ comes from the fact that $C_{m,k} = \emptyset$ for $m \geq \frac{kn_i\beta_i}{e_1}$. We still need to prove that there are no inclusions between the closed sets in this union. We already have the following two non-inclusions (assuming that $C_{m,k}, C_{m,k'} \neq \emptyset$):

1. $C_{m,k} \not\subset C_{m,k'}$ for $k < k' \leq l$, because $C_{m,k'} \subset \{x_0^{(kn_1)} = 0\}$ but $C_{m,k} \not\subset \{x_0^{(kn_1)} = 0\}$ by the definition of these closed sets; and
2. $C_{m,k} \not\subset B_m$ for all $k \leq l$ because $B_m \subset \{x_0^{(ln_1)} = 0\}$ but $C_{m,k} \not\subset \{x_0^{(ln_1)} = 0\}$, again by the definition of the sets.

It remains to show that there are no inclusions in the other directions. This will follow from the following inequalities in codimensions, considered in $\mathbb{C}^{(g+1)(m+1)}$ (again assuming that $C_{m,k}, C_{m,k'} \neq \emptyset$):

1. $\text{codim}(C_{m,k}) \geq \text{codim}(C_{m,k'})$ for $k < k' \leq l$, and
2. $\text{codim}(C_{m,k}) \geq \text{codim}(B_m)$ for all $k \leq l$.

Indeed, the above non-inclusions tell us in particular that all these closed sets are not equal. Because they are also irreducible, the inequalities in codimensions imply that there are no inclusions in the other directions.
We begin with remarking that the inequalities $\bar{\beta}_2 > n_i\bar{\beta}_1$ and $n_2 \geq 2$ imply that

$$\frac{\ln n_2\bar{\beta}_2}{e_1} > (h + 1)n_0n_1$$

for all $h \geq 1$. In particular, we have $ln_0n_1 < m \leq (l + 1)n_0n_1 < \frac{\ln n_2\bar{\beta}_2}{e_1}$ such that $C_{m,l} \neq \emptyset$ with $\text{codim}(C_{m,l}) = c_{i,l}(m)$. One can now verify with the formulas of the codimension that $\text{codim}(C_{m,l}) \geq \text{codim}(B_m)$, but this can also been seen from the following short argument. First, it is easy to check that $C_{mn_0n_1+1,l}$ and $B_{ln_0n_1+1}$ have the same codimension (note that $\text{codim}(C_{mn_0n_1+1,l}) \neq \emptyset$ of codimension $c_{i,l}(ln_0n_1 + 1)$). Second, for $n \in \mathbb{N}$ satisfying $ln_0n_1 + 2 \leq n \leq (l + 1)n_0n_1 < \frac{\ln n_2\bar{\beta}_2}{e_1}$, it follows from Corollary 3.3 that the equation $F_1^{(n)}$ contributes to the codimension of $B_n$ if and only if $n_0$ or $n_1$ divides $n$ (i.e. there is an extra variable $x_0(\frac{\beta}{e_1})$ or $x_1(\frac{\beta}{e_1})$ equal to 0), while from the proof of Proposition 3.5, $F_1^{(n)}$ always contributes to the codimension of $C_{n,l}$. Additionally, the equations $F_j^{(n)}$ for $j = 2, \ldots, g$ contribute to the codimension of $B_n$ if and only if they contribute to the codimension of $C_{n,l}$. Therefore, we have the inequality $\text{codim}(C_{n,l}) \geq \text{codim}(B_n)$ for $ln_0n_1 < n \leq (l + 1)n_0n_1 < \frac{\ln n_2\bar{\beta}_2}{e_1}$, and in particular for $n = m$.

If $l = 1$, we are done; we assume from now on that $l > 1$. For $k = 1, \ldots, l$, we define

$$c_k(m) := k(n_0 + n_1) + \sum_{i=2}^{m} \frac{k\bar{\beta}_1}{e_1} + \sum_{i=1}^{m} \left(m - \frac{kn_i\bar{\beta}_1}{e_1} + 1\right) + \sum_{i=1}^{g} \left(\left\lceil \frac{m}{n_i} \right\rceil + 1\right),$$

where $i \in \{1, \ldots, g\}$ such that $\frac{kn_i\bar{\beta}_1}{e_1} \leq m < \frac{kn_{i+1}\bar{\beta}_1}{e_1}$. We can always find such a unique integer $i$ because $\bar{\beta}_{i+1} = +\infty$ by convention. Furthermore, we know by Proposition 3.5 that $\text{codim}(C_{m,k}) = c_{i,k}(m) = c_k(m)$ if $C_{m,k} \neq \emptyset$, or equivalently, if $i < j(k)$. Consider now a fixed $k \in \{2, \ldots, l\}$. We will show that

$$c_{k-1}(m) \geq c_k(m).$$

This leads to the series of inequalities

$$c_1(m) \geq c_2(m) \geq \cdots \geq c_l(m) = \text{codim}(C_{m,l}) \geq \text{codim}(B_m),$$

from which the above-mentioned inequalities in the codimension follow, and hence, the non-inclusions that we wanted to prove. In particular, our proof implies that $B_m$ is a component of maximal dimension, being of smallest codimension.

We first compare $c_{k-1}(kn_0n_1 + 1)$ and $c_k(kn_0n_1 + 1)$. Applying the above reasoning to $(k - 1)n_0n_1 < kn_0n_1 \leq ((k - 1) + 1)n_0n_1$ gives that $c_{k-1}(kn_0n_1) = \text{codim}(C_{kn_0n_1,k-1}) \geq \text{codim}(B_{kn_0n_1})$. From inequality (5) for $h = k - 1$, it follows that $i = 1$ in $c_{k-1}(kn_0n_1)$, and that $i = 1$ in $c_{k-1}(kn_0n_1 + 1)$ if $kn_0n_1 + 1 < \frac{(k-1)n_2\bar{\beta}_2}{e_1}$ or $i = 2$ in $c_{k-1}(kn_0n_1 + 1)$ if $kn_0n_1 + 1 = (\frac{(k-1)n_2\bar{\beta}_2}{e_1})$. Using this, one can check that

$$c_{k-1}(kn_0n_1 + 1) - c_{k-1}(kn_0n_1) \geq \text{codim}(B_{kn_0n_1+1}) - \text{codim}(B_{kn_0n_1}).$$

Because $c_k(kn_0n_1 + 1) = \text{codim}(C_{kn_0n_1+1,k}) = \text{codim}(B_{kn_0n_1+1})$, we obtain that

$$c_{k-1}(kn_0n_1 + 1) \geq c_k(kn_0n_1 + 1).$$

Now, note that

(1) the value of $c_k(n)$ increases when $n$ varies in the interval $\left[\frac{(k-1)n_2\bar{\beta}_2}{e_1}, \frac{(k-1)n_{i+1}\bar{\beta}_1}{e_1}\right] \cap \mathbb{N}$ and it grows faster when $n$ varies in $\left[\frac{kn_i\bar{\beta}_1}{e_1}, \frac{kn_{i+1}\bar{\beta}_1}{e_1}\right] \cap \mathbb{N}$ for greater $i$;
Remark 3.1. is the same as in Corollary 3.4. For these reasons, we can apply Mustata’s formula \([Mus1]\) to obtain the log canonical threshold \(D\) of \(k\).

We will obtain its value in Corollary 5.2 from the well-known fact that is a component of maximal dimension, or minimal codimension, of \(\mathbb{C}\). This (flat) family is also equisingular in the sense that the latter family is flat outside the special fibre. We will show below that the whole family \(\eta\) is not flat for \(g \geq 2\) (in most cases). This can be compared with the result in [Ley] which states that a family of hypersurfaces admitting a simultaneous embedded resolution of singularities does induce a flat family on the level of jet schemes for every \(m\): in our case, while the generic fibres are hypersurfaces, the embedding dimension of \(Y\) is equal to \(g\).

Theorem 3.8. For \(g \geq 3\) and \(m \geq 1\), the family \(\eta \circ \phi_m : (\chi_m, 0) \rightarrow (\mathbb{C}, 0)\) is not flat. For \(g = 2\) and \(m\) big enough, the family \(\eta \circ \phi_m : (\chi_m, 0) \rightarrow (\mathbb{C}, 0)\) is not flat.
Proof. For \( g \geq 2 \) and \( m \) satisfying the conditions in the statement, we will prove that the dimension of the special fibre of \( \eta \circ \phi_m : (\chi_m, 0) \rightarrow (\mathbb{C}, 0) \) is strictly larger than the dimension of its generic fibre, or equivalently, that the codimension of the generic fibre is strictly larger than the codimension of the special fibre, both considered in \( \mathbb{C}^{(g+1)(m+1)} \). The codimension of the special fibre in \( \mathbb{C}^{(g+1)(m+1)} \) is equal to the codimension of \( B_m \), which we have found in Corollary 3.4. The codimension of the generic fibre in \( \mathbb{C}^{2(m+1)} \) is given in Corollary 4.10 from [Mou1]; it is equal to

\[
2 + \left[ \frac{m}{\beta_0} \right] \quad \text{or} \quad 1 + \frac{m}{\beta_1},
\]

depending on some conditions on \( m \). Since the jet schemes are independent of the embedding, we can compute the codimension of the generic fibre in \( \mathbb{C}^{(g+1)(m+1)} \) by using the fact that the dimension is the same in both embedding spaces. More precisely, we have

\[
(g + 1)(m + 1) - \text{(codimension in } \mathbb{C}^{(g+1)(m+1)}) = 2(m + 1) - \text{(codimension in } \mathbb{C}^{2(m+1)}).
\]

We distinguish three cases.

The case \( g \geq 4 \). It is enough to prove for every \( m \geq 1 \) that

\[
(g - 1)(m + 1) + 1 + \left[ \frac{m}{\beta_0} \right] \geq g + 1 + \frac{m}{n_0} + \left[ \frac{m}{n_1} \right] + \cdots + \left[ \frac{m}{n_g} \right],
\]

which is equivalent to

\[
(g - 1)m > 1 + \left[ \frac{m}{n_0} \right] + \left[ \frac{m}{n_1} \right] + \cdots + \left[ \frac{m}{n_g} \right] - \left[ \frac{m}{\beta_0} \right] - \left[ \frac{m}{\beta_1} \right].
\]

Since \( n_i \geq 2 \), this is clearly true if \( m = 1 \). For \( m \geq 2 \), we can use that \( n_i \geq 2 \) and \( n_0 > n_1 \) to find that

\[
(g + 1)\frac{m}{2} > \left[ \frac{m}{n_0} \right] + \left[ \frac{m}{n_1} \right] + \cdots + \left[ \frac{m}{n_g} \right] - \left[ \frac{m}{\beta_0} \right] - \left[ \frac{m}{\beta_1} \right].
\]

Therefore, it is sufficient to show that \( (g - 1)m \geq 1 + (g + 1)\frac{m}{2} \), which is true for \( g \geq 4 \) and \( m \geq 2 \).

The case \( g = 3 \). For \( m \geq \beta_0 \), we can again prove the inequality (7); because \( \left[ \frac{m}{\beta_0} \right] \geq 1 \), this follows from decreasing the upper bound in (8) to \( 2m - 1 \). For \( m < \beta_0 \), the inequality (7) is not true in general. However, the codimension in \( \mathbb{C}^{2(m+1)} \) of the generic fibre in this case is given by

\[
2 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right],
\]

so that it is enough to show that

\[
2m > \left[ \frac{m}{n_0} \right] + \left[ \frac{m}{n_1} \right] + \left[ \frac{m}{n_2} \right] + \left[ \frac{m}{n_3} \right] - \left[ \frac{m}{\beta_0} \right] - \left[ \frac{m}{\beta_1} \right],
\]

which is true by (8).

The case \( g = 2 \). In this case, our claim does not hold in general; it is easy to find examples in which the (co)dimension of the generic fibre and the special fibre are equal for some small \( m \geq 1 \). However, we will prove that for \( m \) big enough, the claim does always hold. We again consider the inequality (7). By using that \( \left[ \frac{m}{n} \right] \leq \frac{m}{n} < \left[ \frac{m}{n} \right] + 1 \) for any positive integer \( n \), it is enough to investigate

\[
m \geq 3 + \frac{m}{n_0} + \frac{m}{n_1} + \frac{m}{n_2} - \frac{m}{\beta_0} - \frac{m}{\beta_1}.
\]
Because $\tilde{\beta}_0 = n_1 n_2$ and $\tilde{\beta}_1 = n_0 n_2$ if $g = 2$, we can rewrite this as

$$n_0 n_1 n_2 m > 3n_0 n_1 + (n_1 n_2 + n_0 n_2 + n_0 - n_1)m,$$

which is equivalent to $m \geq \frac{3n_0 n_1 n_2}{(n_2 - 1)(n_0 n_1 - n_0 - n_1)}$ (note that $n_0 n_1 - n_0 - n_1 > 0$ as $n_0 > n_1 \geq 2$ are coprime). Hence, for $m$ satisfying this lower bound, the codimension of the generic fibre is certainly bigger than the codimension of the special fibre. □

4. Motivic zeta function of space monomial curves with plane semigroups

Using the results from the previous section, we are now able to compute the series

$$J_Y(T) = \sum_{m \geq 0} [Y_m](\mathbb{L}^{-(g+1)} T)^{m+1}$$

and to deduce the motivic Igusa zeta function associated to the monomial curve $Y \subset \mathbb{C}^{g+1}$.

Assume first that $g \geq 2$. We start with recalling the stratification (3) of $\pi_m^{-1}(0)_{\text{red}}$ for $m \in \mathbb{N}$ and $l \in \mathbb{N}$ such that $ln_0 n_1 < m \leq (l+1)n_0 n_1$ given by

$$\pi_m^{-1}(0)_{\text{red}} = \left( \bigcup_{k=1}^l D_{m,k} \right) \sqcup B_m,$$

where

$$D_{m,k} = \pi_{m,k\cdot n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(kn_1-1)} = 0\} \cap \{x_0^{(kn_1)} \neq 0\})_{\text{red}},$$

$$B_m = \pi_{m\cdot n_0 n_1}^{-1}(\{x_0^{(0)} = x_0^{(1)} = \cdots = x_0^{(ln_1)} = 0\})_{\text{red}}.$$

If $ln_0 n_1 < m < (l+1)n_0 n_1$, Corollary 3.3 implies that $B_m \simeq \mathbb{C}^{(g+1)(m+1)} - c(m)$ with

$$c(m) := g + 1 + \sum_{i=0}^g \left\lfloor \frac{m}{n_i} \right\rfloor.$$

For $m = (l+1)n_0 n_1$, the same corollary gives us the defining equations of $B_m$, which is not isomorphic to $\mathbb{C}^{(g+1)(m+1)} - c(m)$, but to the product of an affine space and the hypersurface defined by $x_1^{((l+1)n_0)n_1} - x_0^{((l+1)n_1)n_0} = 0$. However, the singular part of $B_{(l+1)n_0 n_1}$,

$$B_{(l+1)n_0 n_1} \cap \{x_0^{((l+1)n_1)} = 0\},$$

is isomorphic to $\mathbb{C}^{(g+1)((l+1)n_0 n_1) - c((l+1)n_0 n_1)}$. Furthermore, it is easy to see that the regular part, $B_{(l+1)n_0 n_1} \cap \{x_0^{((l+1)n_1)} \neq 0\}$, is equal to $D_{(l+1)n_0 n_1,l+1}$. Hence, if we define

$$B_m := \begin{cases} \{0\} & \text{if } m = 0 \\ B_m \cap \{x_0^{(ln_1)} = 0\} & \text{if } m = ln_0 n_1 \text{ for some } l > 0 \\ B_m & \text{if } ln_0 n_1 < m < (l+1)n_0 n_1 \text{ for some } l \geq 0, \end{cases}$$

then each $B_m \simeq \mathbb{C}^{(g+1)(m+1)} - c(m)$, and we find for all $m, l \in \mathbb{N}$ with $ln_0 n_1 \leq m < (l+1)n_0 n_1$ that

$$\pi_m^{-1}(0)_{\text{red}} = \left( \bigcup_{k=1}^l D_{m,k} \right) \sqcup B_m.$$

This is a stratification in locally closed subvarieties for which $D_{m,k} = \emptyset$ if $m > \frac{kn_0 n_1 \tilde{\beta}_k}{c_1}$.

These new stratifications can be visualised with a tree as in Figure 1. On the vertical axis, we collect $B_m$ for all $m \in \mathbb{N}$, but to shorten the notation, we only write $m$ instead of $B_m$ in the tree. This axis will be referred to as the main axis or main branch. For each $k \geq 1$, we construct a side branch at $kn_0 n_1$ consisting of $D_{m,k}$ for all $m$ such that
\( kn_0n_1 \leq m < \frac{kn_j(k)\beta_j(k)}{e_1} \), where \( \beta_{g+1} = +\infty \). We again use a shorter notation and call this the side branch associated to \( k \). If \( j(k) < g + 1 \), this side branch stops at \( D_{m,k} \) for \( m = \frac{kn_j(k)\beta_j(k)}{e_1} - 1 \); if \( j(k) = g + 1 \) (i.e. \( n_2 \cdots n_g \) divides \( k \)), this side branch never stops, and the part starting from \( \frac{kn_j(k)\beta_j(k)}{e_1} \) is called the infinite branch associated to \( k \). In such a general picture, it is hard to give the side branches the correct length, and the tree should be interpreted as if the decomposition of \( \pi^{-1}_m(0)_{\text{red}} \) for some \( m \in \mathbb{N} \) can be reconstructed by drawing a horizontal line starting from the main axis and taking all intersections with the side branches.

\[ \text{Figure 1. Visualisation of the stratification of } Y_m \text{ for all } m \geq 0. \]
Proposition 4.2. The contribution of the main branch to $J_Y(T)$ for all $m \in \mathbb{N}$ and $l \in \mathbb{N}$ satisfying $ln_0n_1 \leq m < (l+1)n_0n_1$ that

$$[Y_m] = [\pi_m^{-1}(Y \setminus \{0\})] + [\pi_m^{-1}(0)_{\text{red}}] = [\pi_m^{-1}(Y \setminus \{0\})] + [B_m] + \sum_{k=1}^{l} [D_{m,k}].$$

Hence, to sum $[Y_m]$ over all $m \geq 0$, we can first consider the side branch of 0, the main branch, and the side branches for $k \geq 1$ separately, and then collect these totals to find the whole series $J_Y(T)$.

Let us first take a look at the side branch of 0. By using, for example, the monomial map $M$ from Section 1, we can see that $Y \setminus \{0\} \simeq \mathbb{C} \setminus \{0\}$. Because the restriction of $\pi_m$ to $\pi_m^{-1}(Y \setminus \{0\})$ is a trivial fibration with fibres isomorphic to $\mathbb{C}^m$, this implies that $[\pi_m^{-1}(Y \setminus \{0\})] = (\mathbb{L} - 1)\mathbb{L}^m$ by Remark 2.1, and a simple calculation leads to the following expression.

**Proposition 4.1.** We have

$$\sum_{m \geq 0} [\pi_m^{-1}(Y \setminus \{0\})](\mathbb{L}^{-(g+1)}T)^{m+1} = \frac{(\mathbb{L} - 1)\mathbb{L}^{-(g+1)}T}{1 - 1T}.$$

The computations for the main axis are also easy. Let $L_i$ for $i = 0, \ldots, g$ be the least common multiple of $n_i, n_{i+1}, \ldots, n_g$. Put $N_1 := L_0$ and $\nu_1 := \sum_{l=0}^{g} \frac{N_1}{n_l}$.

**Proposition 4.2.** The contribution of the main branch to $J_Y(T)$ is

$$\sum_{m \geq 0} [B_m](\mathbb{L}^{-(g+1)}T)^{m+1} = \frac{\mathbb{L}^{-(g+1)}T}{1 - \mathbb{L}^{-\nu_1}T^{N_1}} \sum_{r=0}^{L_0-1} \mathbb{L}^{-\frac{m}{N_1}} T^r.$$

**Proof.** Because $B_m \simeq \mathbb{C}^{(g+1)(m+1) - c(m)}$, we need to compute

$$\sum_{m \geq 0} \mathbb{L}^{-c(m)} T^{m+1} = \mathbb{L}^{-(g+1)} T \sum_{m \geq 0} \mathbb{L}^{-\tilde{c}(m)} T^m,$$

where $c(m) = g + 1 + \sum_{i=0}^{g} \frac{m}{n_i}$ and $\tilde{c}(m) = \sum_{i=0}^{g} \left[ \frac{m}{n_i} \right]$. To this end, note that

$$\nu_1 = \tilde{c}(m + L_0) - \tilde{c}(m)$$

for all $m \in \mathbb{N}$. Hence, we can rewrite

$$\sum_{m \geq 0} \mathbb{L}^{-\tilde{c}(m)} T^m = \sum_{r=0}^{L_0-1} \sum_{m \geq 0} \mathbb{L}^{-\tilde{c}(mL_0+r)} T^{mL_0+r}$$

$$= \sum_{r=0}^{L_0-1} \sum_{m \geq 0} \mathbb{L}^{-\tilde{c}(mL_0+r)} T^{mL_0+r}$$

$$= \frac{1}{1 - \mathbb{L}^{-\nu_1} T^{N_1}} \sum_{r=0}^{L_0-1} \mathbb{L}^{-\tilde{c}(r)} T^r,$$

which gives the desired expression. \qed

**Remark 4.1.** In the proof of Proposition 4.2, we found that $\nu_1 = \tilde{c}(m + L_0) - \tilde{c}(m)$ for all $m \in \mathbb{N}$ by looking for a positive integer $N$ such that $\tilde{c}(m)$ is linear on congruence classes modulo $N$. That is,

$$\tilde{c}(m + N) = \tilde{c}(m) + \tilde{c}(N)$$
for all \( m \in \mathbb{N} \). In order to make \( \tilde{c}(m) = \sum_{i=0}^{g} \left\lfloor \frac{m}{m_i} \right\rfloor \) linear on congruence classes modulo \( N \) for any choice of \( n_0, \ldots, n_g \), we need to impose that \( n_i \) divides \( N \) for all \( i = 0, \ldots, g \). Clearly, \( N = L_0 \), the least common multiple of \( n_0, \ldots, n_g \), is the smallest integer satisfying this condition. In fact, the ‘period’ \( N \) can be any common multiple of \( n_0, \ldots, n_g \). This does not make any difference for the poles of the motivic zeta function because the ratio \( \frac{\tilde{c}(N)}{N} \) stays the same. However, it is more natural to take the smallest period as this leads to the smallest remaining sum \( \sum_{r=0}^{N-1} L_r \sum_{\ell=0}^{2g} \frac{1}{n_i^r} T^r \).

The rest of this section will be mainly devoted to the contribution of the side branches associated to \( k \geq 1 \), which is by Proposition 3.5 given by

\[
\sum_{k \geq 1} \sum_{i=1}^{j(k)-1} \sum_{m=\frac{kn_i \beta_i}{1}}^{\frac{kn_i + 1 \beta_i + 1}{e_i} - 1} [D_{m,k}](L^{-(p+1)}T)^{m+1} = \sum_{k \geq 1} \sum_{i=1}^{j(k)-1} \sum_{m=\frac{kn_i \beta_i}{1}}^{\frac{kn_i + 1 \beta_i + 1}{e_i} - 1} (L - 1)(L^{-(c_{i,k}(m)+1)}T)^{m+1},
\]

where \( c_{i,k}(m) \) is defined in (4). Let us consider for a moment the part where \( i = 1 \) for all \( k \geq 1 \):

\[
\sum_{k \geq 1} \sum_{m=\frac{kn_1 \beta_1}{1}}^{\frac{kn_1 + 1 \beta_1 + 1}{e_1} - 1} (L - 1)(L^{-(c_{1,k}(m)+1)}T)^{m+1}.
\]

Each interval \([kn_1 \beta_1, \frac{kn_2 \beta_2}{e_1}] \cap \mathbb{N} \) can be partitioned in \( k \) intervals

\[
I_{1,k}^{(p)} := \left[ \frac{kn_1 \beta_1}{e_1} + (p - 1)\left( \frac{n_2 \beta_2}{e_1} - n_0 \right), \frac{kn_1 \beta_1}{e_1} + p\left( \frac{n_2 \beta_2}{e_1} - n_0 \right) \right] \cap \mathbb{N},
\]

\( p = 1, \ldots, k \), of the same length \( l_k := \frac{n_2 \beta_2}{e_1} - n_0 \) as in Figure 2. Using these intervals, the part for \( i = 1 \) can be rewritten as

\[
\sum_{p \geq 1} \sum_{k \geq 0} \sum_{I_{1,k}^{(p)} \subset I_{i,k}} (L - 1)(L^{-(c_{1,p+k}(m)+1)}T)^{m+1},
\]

where, from now on, a sum over all \( m \in I \) for some interval \( I \subset \mathbb{N} \) is written in a shorter way as \( \sum_{I} \). We will first concentrate on computing the sum over all \( \kappa \) and \( m \) for a fixed \( p \). In other words, we will first sum for each interval \( I_{1,k}^{(p)} \) over all sufficient \( k \), meaning that \( I_{1,k}^{(p)} \) appears in the partition of \([kn_0 \beta_1, \frac{kn_2 \beta_2}{e_1}] \cap \mathbb{N}. \) We will refer to this as vertical summation inspired by Figure 2. Afterwards, we will sum these totals over all \( p \geq 1 \), called horizontal summation.

For \( i = 2, \ldots, g - 1 \), we need to consider all \( k \geq 1 \) with \( i < j(k) \), or in other words, all multiples of \( n_2, \ldots, n_i \). For each such \( k = kn_i \beta_i e_i \), we can partition the interval \([\frac{kn_i \beta_i}{e_i}, \frac{kn_i + 1 \beta_i + 1}{e_i}] \cap \mathbb{N} \) in \( k' \) intervals

\[
I_{1,k}^{(p)} := \left[ \frac{kn_i \beta_i}{e_i} + (p - 1)\left( \frac{n_2 \beta_2}{e_i} - n_i \beta_i \right), \frac{kn_i \beta_i}{e_i} + p\left( \frac{n_2 \beta_2}{e_i} - n_i \beta_i \right) \right] \cap \mathbb{N},
\]

\( p = 1, \ldots, k' \), of length \( l_k := n_2 \cdots n_i \left( \frac{n_2 \beta_2}{e_i} - n_i \beta_i \right) = \frac{n_2 \beta_2}{e_i} + n_i \beta_i e_i \). With this notation, the part for \( i \) is equal to

\[
\sum_{p \geq 1} \sum_{k \geq 0} \sum_{I_{1,k}^{(p)}} (L - 1)(L^{-(c_{1,p+k}(n_2 \cdots n_i(m)+1)}T)^{m+1}.
\]

We will again first sum vertically (for fixed \( p \)) and then horizontally (over \( p \geq 1 \)).
For $i = g$, we need to consider the infinite branches associated to $k = k' n_2 \cdots n_g$ and $j(k) = g + 1$. With

$$I_{g,k}^{(1)} := \left[ \frac{kn_g \beta_g}{e_1}, +\infty \right) \cap \mathbb{N},$$

the infinite branches lead to

$$\sum_{\kappa \geq 0} \sum_{j(1), (1+) n_2 \cdots n_g} (L - 1)^{\nu_{j(1), (1+) n_2 \cdots n_g}} L^{-1} \nu_{j(1), (1+) n_2 \cdots n_g}^{(m+1)} T^{m+1}.$$

This consists of only one vertical sum.

To find the vertical summation, we start with a lemma. Recall that $L_i$ for $i = 0, \ldots, g$ is the least common multiple of $n_i, n_{i+1}, \ldots, n_g$; that $N_1 = L_0$; and that $\nu_1 = \sum_{i=0}^g \frac{N_1}{n_i}$. For $i = 2, \ldots, g$, we introduce the positive integers

$$N_i := L_i n_2 \cdots n_{i-1} \frac{n_i \beta_i}{e_1} = L_i \frac{\beta_i}{e_i},$$

and

$$\nu_i := N_i \left( \frac{1}{n_i \beta_i} \left( \sum_{l=0}^{i-1} \beta_l - \sum_{l=1}^{i-1} n_l \beta_l \right) + (i - 1) + \sum_{l=i+1}^g \frac{1}{n_l} \right).$$
It is worth noting that
\[ \frac{\nu_i}{N_i} = \frac{1}{n_i \beta_i} (\bar{\beta}_0 + \beta_1) + \sum_{l=2}^{g} \frac{1}{n_l}, \]
or in other words, that
\[ \frac{\nu_i}{N_i} = \frac{1}{n_i \beta_i} \left( \sum_{l=0}^{i} \bar{\beta}_l - \sum_{l=1}^{i-1} n_l \beta_l \right) + (i - 1) + \sum_{l=i+1}^{g} \frac{1}{n_l} \]
holds for all \( i = 1, \ldots, g \). Using the definition (4) of \( c_{i,k}(m) \), the next lemma is a straightforward calculation.

**Lemma 4.3.** (1) For all \( k \geq 1 \) and \( 0 \leq m < kl_1 = k\left(\frac{n_2 \beta_2}{e_1} - n_0 n_1\right) \), we have
\[ \nu_1 = c_{1,k} + \frac{L_0}{n_0 n_1} \left( (k + \frac{L_0}{n_0 n_1}) n_0 n_1 + m \right) - c_{1,k}(k n_0 n_1 + m). \]

(2) For \( i = 2, \ldots, g \) and \( k \geq 1 \), we have for every \( m \in \mathbb{N} \) lying in the interval \([0, kl_i] = [0, kn_2 \cdots n_i \left(\frac{n_{i+1} \beta_{i+1}}{e_1} - \frac{n_i \beta_i}{e_1}\right)]\) that
\[ \nu_i = c_{i,(k+\frac{L_i}{n_i})n_2 \cdots n_i} \left( (k + \frac{L_i}{n_i}) n_2 \cdots n_i \frac{n_i \beta_i}{e_1} + m \right) - c_{i,n_2 \cdots n_i} \left( kn_2 \cdots n_i \frac{n_i \beta_i}{e_1} + m \right). \]

Note that the integer \( K_1 := \frac{L_0}{n_0 n_1} \) is the smallest integer (for general \( n_0, \ldots, n_g \)) making the sum \( \sum_{l=2}^{g} \left[ \frac{m}{n_l} \right] \) in \( c_{1,k}(m) \) linear on congruence classes modulo \( K_1 n_0 n_1 \). The integers \( K_i := \frac{L_i}{n_i} \) for \( i = 2, \ldots, g \) are chosen in a similar way. This idea was used in the proof of Proposition 4.2, and we will continue following the approach of this proof in showing the results of the next proposition. The first two results say that for each \( i = 1, \ldots, g - 1 \) and \( p \geq 1 \), we only need to know the behaviour on the first \( K_i \) side branches associated to sufficient \( k \) (i.e. the partition of the side branch of \( k \) contains the interval \( J_{i,k}^{(p)} \)) in order to know the whole vertical summation. This motivates our choice for the smallest integers \( K_i \), and it is again easy to check that this does not influence the ratios \( \frac{\nu_i}{N_i} \).

**Proposition 4.4.** (1) For \( i = 1 \) and all \( p \geq 1 \), the vertical summation gives
\[ \frac{L-1}{1-L^{-\nu_1} T^{N_1}} \sum_{r=0}^{\frac{L_0}{n_0 n_1} - 1} \sum_{J_{1,p+r}} L^{-(c_{1,p+r}(m)+1)} T^{m+1}. \]

(2) For every \( i = 2, \ldots, g - 1 \) and \( p \geq 1 \), the vertical summation gives
\[ \frac{L-1}{1-L^{-\nu_i} T^{N_i}} \sum_{r=0}^{\frac{L_i}{n_i} - 1} \sum_{J_{i,p+r}} L^{-(c_{i,(p+r)n_2 \cdots n_i}(m)+1)} T^{m+1}. \]

(3) For the infinite branches, we find
\[ \frac{(L-1) L^{-(\nu_g+g+1)} T^{N_g+1}}{(1-L^{-g} T)(1-L^{-\nu_g} T^{N_g})}. \]
Proof. As in Proposition 4.2, we can rewrite the sum
\[
\sum_{\kappa \geq 0} \sum_{i \geq 0} (L - 1) L^{-(c_{1,p}+\kappa)(m+1)} T^{m+1}
\]
\[
= \sum_{\kappa \geq 0} \sum_{m=(p-1)l_1}^{pl_1-1} (L - 1) L^{-(c_{1,p}+\kappa)((p+\kappa)n_0n_1+m+1)} T^{(p+\kappa)n_0n_1+m+1}
\]
with the period \( K_1 = \frac{L_0}{n_0n_1} \) found in Lemma 4.3 as
\[
\frac{L - 1}{1 - L^{-L_0} T^{N_1}} \sum_{r=0}^{l_1-1} \sum_{m=(p-1)l_1}^{pl_1-1} L^{-(c_{1,p}+r)((p+r)n_0n_1+m+1)} T^{(p+r)n_0n_1+m+1}.
\]
This is equal to the result in (1). The second part of the proposition follows from the same arguments applied to
\[
\sum_{\kappa \geq 0} \sum_{i \geq 0} (L - 1) L^{-(c_{1,p}+\kappa)n_2 \cdots n_i (m+1)} T^{m+1}.
\]
To prove (3), we can again use Lemma 4.3 (note that \( L_g = n_g \)) for
\[
\sum_{\kappa \geq 0} \sum_{g \geq 1} (L - 1) L^{-(c_{g,1}+\kappa)n_2 \cdots n_g (m+1)} T^{m+1}
\]
\[
= \sum_{\kappa \geq 0} \sum_{m \geq 0} (L - 1) L^{-(c_{g,1}+\kappa)n_2 \cdots n_g ((1+\kappa)n_2 \cdots n_g n_g \beta_g m+1)} T^{(1+\kappa)n_2 \cdots n_g \beta_g m+1}
\]
together with
\[
\sum_{m \geq 0} L^{-(c_{g,n_2 \cdots n_g} (n_2 \cdots n_g n_g \beta_g m+1)+1)} T^{n_2 \cdots n_g \beta_g m+1} = \sum_{m \geq 0} L^{-(\nu_g g+gm+1)} T^{n_g \beta_g m+1}
\]
\[
= \frac{L^{-(\nu_g g+1)} T^{n_g \beta_g +1}}{1 - L^{-\nu_g g T}}.
\]
\[\square\]

It remains to sum the first two expressions of Proposition 4.4 horizontally over all \( p \geq 1 \).

We again begin with a lemma that follows from simple computations.

Lemma 4.5. (1) For all \( k \geq 1 \), \( r \geq 0 \) and \( 0 \leq m < (r+1)l_1 = (r+1)(\frac{n_g \beta_g}{e_1} - n_0n_1) \), we have
\[
\nu_2 = c_{1,k+L_2+r}((k+L_2+r)n_0n_1 + (k+L_2-1)l_1 + m) - c_{1,k+r}((k+r)n_0n_1 + (k-1)l_1 + m).
\]

(2) For \( i = 2, \ldots, g-1 \) and \( k \geq 1 \), we have for all \( r \geq 0 \) and \( m \in \mathbb{N} \) in the interval
\[\[0,(r+1)l_1[ = [0,(r+1)n_2 \cdots n_i(\frac{n_{i+1} \beta_{i+1}}{e_1} - \frac{n_i \beta_i}{e_1})] \]

that
\[
\nu_{i+1} = c_{i,k+L_{i+1}+r}n_2 \cdots n_i \left( (k+L_{i+1}+r)n_2 \cdots n_i \frac{n_i \beta_i}{e_1} + (k + L_{i+1} - 1)l_i + m \right)
\]
\[
- c_{i,k+r}n_2 \cdots n_i \left( (k+r)n_2 \cdots n_i \frac{n_i \beta_i}{e_1} + (k-1)l_i + m \right).
\]

Using similar arguments as in the proof of Proposition 4.4 with this lemma, the horizontal summation leads to the next result. With the visualisation of Figure 2, we can rephrase the first line as follows. In order to calculate the whole contribution of the first intervals \([kn_0n_1, \frac{kn_g \beta_g}{e_1}] \cap \mathbb{N} \) for \( k \geq 1 \), we only need to consider the block of intervals \( I_{1,k}^{(p)} \)
for \( p = 1, \ldots, L_2 \) and the first \( \frac{L_0}{n_0 \bar{e}_1} \) sufficient \( k \) (i.e. \( k = p, \ldots, p + \frac{L_0}{n_0 \bar{e}_1} \)). The second line can be interpreted in a similar way, and the last line is the part of the infinite branches.

**Proposition 4.6.** The contribution to \( J_Y(T) \) of the side branches associated to \( k \geq 1 \) is

\[
\frac{L_0 - 1} {(1 - L^{-\nu_i T N_1})(1 - L^{-\nu_2 T N_2})} \sum_{r=0}^{L_0 - 1} \sum_{r'=1}^{L_2} \sum_{\nu \in \mathbb{Z}} L_-^{(c_{i, r'+r}(m))} T^{m+1} \\
+ \sum_{i=2}^{g-1} \frac{L_0 - 1} {(1 - L^{-\nu_i T N_i})(1 - L^{-\nu_{i+1} T N_{i+1}})} \sum_{r=0}^{L_0 - 1} \sum_{r'=1}^{L_2} \sum_{\nu \in \mathbb{Z}} L_-^{(c_{i, (r'+r)n_2 - n_i}(m))} T^{m+1} \\
+ \frac{(L_0 - 1) L_-^{(\nu_0 + 1)} T^{N_0 + 1}} {(1 - L^{-\nu T})(1 - L^{-\nu T N_0})}.
\]

Combining all propositions of this section with the relation

\[ Z_Y^{\text{mot}}(T) = 1 - \frac{1 - T}{T} J_Y(T), \]

we are now ready to give an explicit expression for the motivic zeta function of \( Y \).

**Theorem 4.7.** Consider a monomial curve \( Y \subset \mathbb{C}^{g+1} \) defined by the equations (1). Denote by \( L_i \) for \( i = 0, \ldots, g \) the least common multiple of \( n_i, n_{i+1}, \ldots, n_g \). Let \( N_1 := L_0 \), let \( \nu_1 := \sum_{\ell=0}^{g} \frac{N_1}{n_\ell} \), and let \( N_i \) and \( \nu_i \) for \( i = 2, \ldots, g \) be the positive integers defined as

\[
N_i = \frac{L_i \bar{\beta}_i}{e_i} \quad \text{and} \quad \nu_i = N_i \left( \frac{1}{n_i \bar{\beta}_i} \left( \sum_{\ell=0}^{i-1} \bar{\beta}_\ell - \sum_{\ell=1}^{i} n_i \bar{\beta}_\ell \right) + (i - 1) + \sum_{l=i+1}^{g} \frac{1}{n_l} \right).
\]

The motivic Igusa zeta function associated to \( Y \subset \mathbb{C}^{g+1} \) is given by

\[
Z_Y^{\text{mot}}(T) = 1 - (1 - T) \left( \frac{(L_0 - 1) L_-^{(g+1)}} {1 - L^{-\nu T}} + \frac{L_-^{-(g+1)}} {1 - L^{-\nu T N_1}} \sum_{r=0}^{L_0 - 1} \sum_{\nu \in \mathbb{Z}} L_-^{g} T^\nu \right) \\
+ \frac{(L_0 - 1) L_-^{(\nu_0 + 1)} T^{N_0 + 1}} {(1 - L^{-\nu T})(1 - L^{-\nu T N_0})}.
\]

Here, \( Z_i(T) \) for \( i = 1, \ldots, g \) are polynomials with coefficients in \( \mathbb{Z}[L, L_-^{-1}] \). More precisely,

\[
Z_1(T) := \sum_{r=0}^{L_0 - 1} \sum_{r'=1}^{L_2} \sum_{\nu \in \mathbb{Z}} L_-^{(c_{1, r'+r}(m))} T^{m},
\]

\[
Z_i(T) := \sum_{r=0}^{L_0 - 1} L_i \sum_{r'=1}^{L_{i+1}} \sum_{\nu \in \mathbb{Z}} L_-^{(c_{i, (r'+r)n_2 - n_i}(m))} T^{m}, \quad i = 2, \ldots, g - 1,
\]

where, for \( i = 1, \ldots, g - 1 \) and \( k, m, p \in \mathbb{N} \),

\[
I_{i,k}^{(p)} = \left[ \begin{array}{c} k n_i \bar{\beta}_i \\ e_i \end{array} \right] + (p - 1) \left( \frac{n_{i+1} \bar{\beta}_{i+1}} {e_i} - \frac{n_i \bar{\beta}_i}{e_i} \right), \quad k n_i \bar{\beta}_i + p \left( \frac{n_{i+1} \bar{\beta}_{i+1}} {e_i} - \frac{n_i \bar{\beta}_i}{e_i} \right) \right] \cap \mathbb{N},
\]

and

\[
c_{i,k}(m) = k(n_0 + n_1) + \sum_{l=2}^{i} \frac{k \bar{\beta}_l}{e_l} + \sum_{l=1}^{i} \left( m - \frac{k n_l \bar{\beta}_l}{e_l} + 1 \right) + \sum_{l=i+1}^{g} \left( \left[ \frac{m}{n_l} \right] + 1 \right).
\]
We see that the motivic zeta function is a rational function in \( T \), and it is natural to take a look at its poles. Since \( \mathcal{M}_C \) is not an integral domain, see for instance [Cau], one has to be careful with defining a pole. However, for example with the definition of Rodrigues and Veys in [RV], one can see that a complete list of possible poles for \( Z_{Y_1}^{\text{mot}}(T) \) is given by

\[
\mathbb{L}^g, \quad \mathbb{L}^{\nu_i}, \quad i = 1, \ldots, g,
\]

which could intuitively be expected from the above expression. In the next section, we will prove that all these candidates are actual poles, as we already notice in the following examples.

**Example 4.1.** (1) Consider the irreducible plane curve given by \((x_1^2 - x_0^3)^2 - x_0^5 x_1 = 0\). Its semigroup has \((4, 6, 13)\) as unique set of minimal generators, and the corresponding space curve \( Y_1 \subseteq \mathbb{C}^3 \) in three variables \((g = 2)\) is defined by

\[
\begin{align*}
  x_1^2 - x_0^3 &= 0, \\
  x_2^2 - x_0^5 x_1 &= 0.
\end{align*}
\]

Using Theorem 4.7, one can compute that

\[
Z_{Y_1}^{\text{mot}}(T) = \frac{(\mathbb{L} - 1) P_1(T)}{\mathbb{L}^{47}(1 - \mathbb{L}^{-2} T)(1 - \mathbb{L}^{-8} T^6)(1 - \mathbb{L}^{-37} T^{26})}
\]

where \( P_1(T) \) is the polynomial

\[
\begin{align*}
  (\mathbb{L} + 1) T^{31} - \mathbb{L}^3 T^{30} + \mathbb{L}^3 T^{29} - (\mathbb{L}^6 + \mathbb{L}^5) T^{28} + (\mathbb{L}^6 + \mathbb{L}^5) T^{27} - 2 \mathbb{L}^8 T^{26} + (-\mathbb{L}^9 + \mathbb{L}^8) T^{25} \\
  + (\mathbb{L}^12 - \mathbb{L}^{11}) T^{24} + (-\mathbb{L}^{12} + \mathbb{L}^{11}) T^{23} + (\mathbb{L}^{15} - \mathbb{L}^{14}) T^{22} + (-\mathbb{L}^{15} + \mathbb{L}^{14}) T^{21} + \mathbb{L}^{18} T^{20} \\
  - \mathbb{L}^{18} T^{19} - \mathbb{L}^{25} T^{14} + \mathbb{L}^{25} T^{13} + (\mathbb{L}^{29} - \mathbb{L}^{28}) T^{12} + (-\mathbb{L}^{29} + \mathbb{L}^{28}) T^{11} + (\mathbb{L}^{32} - \mathbb{L}^{31}) T^{10} \\
  + (-\mathbb{L}^{32} + \mathbb{L}^{31}) T^9 + (\mathbb{L}^{35} - \mathbb{L}^{34}) T^8 + (-\mathbb{L}^{35} + \mathbb{L}^{34}) T^7 + (\mathbb{L}^{38} + \mathbb{L}^{37}) T^5 + \mathbb{L}^{40} T^4 \\
  - \mathbb{L}^{40} T^3 + (\mathbb{L}^{43} + \mathbb{L}^{42}) T^2 - (\mathbb{L}^{43} + \mathbb{L}^{42}) T + \mathbb{L}^{46} + \mathbb{L}^{45}.
\end{align*}
\]

This expression has three poles, \( \mathbb{L}^2, \mathbb{L}^5 \) and \( \mathbb{L}^{25} \), corresponding to \( g = 2, \ \frac{\nu_2}{N_1} = \frac{8}{6} \) and \( \frac{\nu_2}{N_2} = \frac{37}{26} \), respectively. These are precisely the set of candidate poles from Theorem 4.7.

(2) The polynomial \[((x_1^2 - x_0^3)^2 - x_0^5 x_1)^2 - x_0^{10}(x_1^2 - x_0^3)\] defines an irreducible plane curve whose semigroup is minimally generated by \((8, 12, 26, 53)\), and it induces the curve \( Y_2 \subseteq \mathbb{C}^4 \) given by

\[
\begin{align*}
  x_1^2 - x_0^3 &= 0, \\
  x_2^2 - x_0^5 x_1 &= 0, \\
  x_3^2 - x_0^{10} x_2 &= 0.
\end{align*}
\]

For this curve, Theorem 4.7 gives

\[
Z_{Y_2}^{\text{mot}}(T) = \frac{(\mathbb{L} - 1) P_2(T)}{\mathbb{L}^{299}(1 - \mathbb{L}^{-3} T)(1 - \mathbb{L}^{-11} T^6)(1 - \mathbb{L}^{-50} T^{26})(1 - \mathbb{L}^{-235} T^{106})},
\]

for a concrete polynomial \( P_2(T) \) of degree 137 with coefficients in \( \mathbb{Z}[\mathbb{L}] \), which occupies more than half a page. Again, all four candidate poles, \( \mathbb{L}^g = \mathbb{L}^3, \quad \mathbb{L}^{\frac{11}{11}} = \mathbb{L}^{\frac{11}{11}}, \quad \mathbb{L}^{\frac{53}{53}} = \mathbb{L}^{\frac{53}{53}}, \) and \( \mathbb{L}^{\frac{299}{299}} = \mathbb{L}^{\frac{299}{299}} \) turn out to be actual poles.

Both these results were also obtained using other methods in [Pot]; there, the local \( p \)-adic zeta function of \( Y_i \) was calculated in terms of a principalisation of its defining ideal. From the data of the same principalisation, one can deduce an expression for the global \( p \)-adic zeta function of \( Y_i \), to which the above expression for the global motivic zeta function specialises.
The approach in this section also provides a way to compute the local motivic zeta function associated to $Y$. As in Section 2, one can check that the relations $[X_{0,0}] = \emptyset$ and $[X_{m,0}] = \mathbb{L}^{T+1}[\pi_m^{-1}(0)_{\text{red}}] - [\pi_m^{-1}(0)_{\text{red}}]$ imply that

$$\mathbb{L}^{-(g+1)} \sum_{m \geq 0} [X_{m,0}] (\mathbb{L}^{-(g+1)}T)^{m} = \mathbb{L}^{-(g+1)} - \frac{1-T}{T} \sum_{m \geq 0} [\pi_m^{-1}(0)_{\text{red}}] (\mathbb{L}^{-(g+1)}T)^{m+1}.$$ 

Therefore, the local version is equal to the above expression with the first 1 replaced by $\mathbb{L}^{-(g+1)}$ and without the term

$$\frac{(1-T)(\mathbb{L} - 1)\mathbb{L}^{-(g+1)}}{1 - \mathbb{L}^{-gT}},$$

which comes from the side branch of 0 consisting of $[Y \setminus \pi_m^{-1}(0)_{\text{red}}]$.

For $g = 1$, we can repeat most steps of these computations: we can change the stratification (6) of $\pi_1^{-1}(0)_{\text{red}}$ in exactly the same way such that $B_m \simeq \mathbb{C}^{(g+1)(m+1)-c(m)}$ for every $m \in \mathbb{N}$; we can split the calculations in a side branch of 0 with $\pi_m^{-1}(Y \setminus \{0\})$ for all $m \geq 0$, a main branch containing $B_m$ for all $m \geq 0$, and side branches at $k\eta_0\eta_1$ for all $k \geq 1$; and we get the same results for the side branch of 0 and the main branch as in Proposition 4.1 and 4.2, respectively. The only difference is that each $k \geq 1$ has an infinite branch consisting of $D_{m,k}$ with codimension $c_{1,k}(m)$ for all $m \geq k\eta_0\eta_1$. However, this can be treated similarly as the infinite branches for $g \geq 2$, and we obtain the same expression as in Proposition 4.4, part (3). In other words, the motivic zeta function of the plane curve $Y = V(x_1^{n_1} - x_0^{n_0}) \subset \mathbb{C}^2$ is given by the expression in Theorem 4.7 with $g = 1$.

5. POLES OF THE MOTIVIC ZETA FUNCTION OF A SPACE MONOMIAL CURVE

The explicit expression for the motivic Igusa zeta function associated to $Y \subset \mathbb{C}^{g+1}$ in Theorem 4.7 provides the following $g + 1$ candidate poles for all $g \geq 1$:

$$\mathbb{L}^g, \quad \mathbb{L}^{k/N_i}, \quad i = 1, \ldots, g.$$ 

We will now show that all these possible poles are actual poles.

Instead of proving this for the motivic zeta function directly, we will work with the topological Igusa zeta function associated to $Y$. This zeta function was first introduced by Denef and Loeser [DL1] for one polynomial $f$ in terms of an embedded resolution of $f$. Such a resolution can also be used to express the motivic zeta function of $f$ and to show that this function specialises to the topological one, see for example [DL2]. In particular, a pole of the topological zeta function induces a pole of the motivic zeta function. For an ideal, one can obtain a similar formula in terms of a principalisation of the ideal, where the topological version is again a specialisation of the motivic one. The generalisation to ideals by using a principalisation is mentioned in [VZ].

Roughly speaking, we obtain the topological zeta function $Z_{Y}^{\text{top}}(s)$ for $s \in \mathbb{C}$ by substituting $T = \mathbb{L}^{-s}$ in $Z_{Y}^{\text{mot}}(T)$ and taking the limit $\mathbb{L} \to 1$. Formally, one should first specialise to the Hodge zeta function, using Hodge polynomials, and then to the topological zeta function. In this way, we get the following expression for the topological zeta
function associated to the monomial curve \( Y \):
\[
Z_Y^{\text{top}}(s) = \frac{\nu_1}{\nu_1 + sN_1} - \frac{L_0L_2 (n_{\beta_2}/e_1 - n_0n_1)}{n_0n_1 (\nu_1 + sN_1) (\nu_2 + sN_2)} - \sum_{i=2}^{g-1} \frac{L_iL_{i+1}}{n_i} \left( \frac{n_{\beta_{i+1}}}{e_i} - \frac{n_i\bar{\beta}_i}{e_i} \right) \frac{s}{(\nu_i + sN_i)(\nu_{i+1} + sN_{i+1})} - \frac{s}{(g+s)(\nu_g + sN_g)}.
\]

The candidate poles of this rational function are clearly \(-g\) and \(-\frac{\nu_i}{N_i}\) for \(i = 1, \ldots, g\), which correspond to the possible poles for the motivic zeta function. Therefore, we will prove the stronger result that each of these candidates is an actual pole of \(Z_Y^{\text{top}}(s)\), implying the same for the finer \(Z_Y^{\text{mot}}(T)\).

**Example 5.1.** We consider the two curves \( Y_1 \) and \( Y_2 \) from Example 4.1. Applying the above specialisation yields
\[
Z_{Y_1}^{\text{top}}(s) = \frac{4(47s^2 + 169s + 148)}{(2 + s)(8 + 6s)(37 + 26s)},
\]
and
\[
Z_{Y_2}^{\text{top}}(s) = \frac{2(14176s^3 + 103282s^2 + 246789s + 193875)}{(3 + s)(11 + 6s)(50 + 26s)(235 + 106s)},
\]
from which we clearly see that all candidate poles are actual poles. Again, both these topological zeta functions were also found using a principalisation in [Pot].

We start with remarking the following inequalities between the candidate poles.

**Lemma 5.1.** The candidate poles can be ordered as
\[-g < -\frac{\nu_g}{N_g} < -\frac{\nu_{g-1}}{N_{g-1}} < \cdots < -\frac{\nu_1}{N_1}.
\]
In particular, this implies that every candidate pole is possibly an actual pole of order 1.

**Proof.** Recall from Section 4 that
\[
\frac{\nu_i}{N_i} = \frac{1}{n_i\bar{\beta}_i} \left( \sum_{l=0}^{i} \bar{\beta}_l - \sum_{l=1}^{i-1} n_l\bar{\beta}_l \right) + (i - 1) + \sum_{l=i+1}^{g} \frac{1}{n_l}
\]
for all \(i = 1, \ldots, g\). Let \(g \geq 2\) and take \(i \in \{1, \ldots, g-1\}\) fixed. The difference between \(\frac{\nu_{i+1}}{N_{i+1}}\) and \(\frac{\nu_i}{N_i}\) can be rewritten as
\[
\frac{\nu_{i+1}}{N_{i+1}} - \frac{\nu_i}{N_i} = \left( \frac{1}{n_i\beta_i} - \frac{1}{n_{i+1}\beta_{i+1}} \right) \left( -\bar{\beta}_0 + \sum_{l=1}^{i} (n_l - 1)\bar{\beta}_l \right).
\]
Because \(n_0 = \frac{\bar{\beta}}{e_1}\) and \(n_1 = \frac{\bar{\beta}}{e_1}\) are coprime, we know that
\[-\bar{\beta}_0 + (n_1 - 1)\bar{\beta}_1 = n_1\bar{\beta}_1 \left( 1 - \frac{1}{n_0} - \frac{1}{n_1} \right) > 0.
\]
Together with \(n_i\bar{\beta}_i < \bar{\beta}_{i+1}\) and \(n_i > 1\) for all \(l = 1, \ldots, i\), we indeed see that \(\frac{\nu_{i+1}}{N_{i+1}} - \frac{\nu_i}{N_i} > 0\).

Similarly, both the inequality \(-g < -\frac{\nu_g}{N_g}\) and the case for \(g = 1\) follow from the positive difference
\[
g - \frac{\nu_g}{N_g} = \frac{1}{n_g\beta_g} \left( -\bar{\beta}_0 + \sum_{l=1}^{g} (n_l - 1)\bar{\beta}_l \right).
\]

This relation between the candidate poles immediately yields the log canonical threshold of the pair \((\mathbb{C}^{g+1}, Y)\). The log canonical threshold is an important invariant in birational geometry, and we refer to [Kol] or [Mus2] for more about it. See also Remark 3.1.
Corollary 5.2. The log canonical threshold of the pair \((\mathbb{C}^{g+1}, Y)\) is equal to \(\frac{g}{N_1} = \sum_{i=0}^{g} \frac{1}{n_i} \).

We can now state and prove the main result of this section.

**Theorem 5.3.** A complete list of the poles of the topological zeta function associated to the monomial curve \(Y \subset \mathbb{C}^{g+1}\) is given by

\[-g, \quad -\frac{\nu_i}{N_i} = - \left( \frac{1}{n_i} \left( \sum_{i=0}^{g-1} \frac{\beta_i}{\nu_i} - \sum_{i=1}^{g-1} n_i \frac{\bar{\beta}_i}{\nu_i} \right) + (i-1) + \sum_{l=i+1}^{g} \frac{1}{n_l} \right), \quad i = 1, \ldots, g,
\]

and all these poles have order 1. Consequently, the motivic zeta function associated to \(Y\) has poles

\[\mathbb{L}^g, \quad \mathbb{L}^{\frac{N}{N_1}}, \quad i = 1, \ldots, g,
\]

which are all poles of order 1.

**Proof.** We will show that the residue of each candidate pole is non-zero. For every \(g \geq 1\), this is trivial for the residue of the smallest pole, \(-g\), given by

\[R = \frac{g}{\nu_g - g N_g}.
\]

To investigate the remaining residues, we denote the residue corresponding to \(-\frac{\nu_i}{N_i}\) by \(R_i\) for \(i = 1, \ldots, g\), and we list them, up to a factor of \(\frac{\nu_i}{N_i}\), in the next table.

| \(g\) | residues |
|------|----------|
| \(g = 1\) | \(R_1 = N_1 + \frac{1}{g-\frac{\nu_1}{N_1}}\) |
| \(g = 2\) | \(R_1 = N_1 + \frac{L_0}{n_0 n_1} \left( \frac{\nu_2}{\nu_1} - \frac{1}{N_1 N_2} \right) + \frac{1}{g-\frac{\nu_1}{N_1}}\) |
| \(R_2 = \frac{L_0}{n_0 n_1} \left( \frac{\nu_2}{\nu_1} - \frac{1}{N_1 N_2} \right) + \frac{1}{g-\frac{\nu_1}{N_1}}\) |
| \(g \geq 3\) | \(R_1 = N_1 + \frac{L_0}{n_0 n_1} \left( \frac{\nu_2}{\nu_1} - \frac{1}{N_1 N_2} \right) + \frac{L_2}{n_2} \left( \frac{\nu_3}{\nu_2} - \frac{1}{N_2 N_3} \right)\) |
| \(R_{i, i=3, \ldots, g-1} = \frac{L_{i-1}}{n_{i-1}} \left( \frac{\nu_{i-1}}{\nu_i} - \frac{1}{N_{i-1} N_i} \right)\) |
| \(R_{i, i=3, \ldots, g-1} = \frac{L_{i-1}}{n_{i-1}} \left( \frac{\nu_{i-1}}{\nu_i} - \frac{1}{N_{i-1} N_i} \right)\) |
| \(R_g = \frac{L_{g-1}}{n_{g-1}} \left( \frac{\nu_{g-1}}{\nu_g} - \frac{1}{N_{g-1} N_g} \right)\) |

From the relation between the candidate poles in Lemma 5.1, it immediately follows for all \(g\) that \(R_1 > 0\), being the sum of two positive numbers. We claim that all the other residues are strictly negative, which is less trivial as they consist of a positive and a negative part.

We first take a look at \(R_i\) for \(i = 3, \ldots, g-1\) and \(g \geq 4\). The inequality \(R_i < 0\) is equivalent to

\[\frac{L_{i-1} N_{i+1}}{n_{i-1}} \left( \frac{n_i \bar{\beta}_i}{\nu_i} - n_i \frac{\bar{\beta}_i}{\nu_i} \right) > \frac{L_{i+1} N_{i-1}}{n_i} \left( \frac{n_{i+1} \bar{\beta}_{i+1}}{\nu_i} - n_{i+1} \frac{\bar{\beta}_{i+1}}{\nu_i} \right).
\]
With the definition \( N_i = \frac{L_i \beta_i}{e_i} \) for \( l = i - 1, i + 1 \), this is in turn equivalent to
\[
\frac{\beta_{i+1}}{e_{i+1} n_{i-1}} (n_i \beta_i - n_{i-1} \beta_{i-1}) \left( \frac{\nu_{i+1}}{N_{i+1}} - \frac{\nu_i}{N_i} \right) > \frac{\beta_{i-1}}{e_i n_i} (n_{i+1} \beta_{i+1} - n_i \beta_i) \left( \frac{\nu_i}{N_i} - \frac{\nu_{i-1}}{N_{i-1}} \right),
\]
which can be rewritten as
\[
\frac{\beta_{i+1}}{e_{i+1} n_{i-1}} (n_i \beta_i - n_{i-1} \beta_{i-1}) \left( \frac{1}{n_i \beta_i} - \frac{1}{n_{i+1} \beta_{i+1}} \right) \left( -\bar{\beta}_0 + \sum_{l=1}^{i} (n_l - 1) \beta_l \right)
> \frac{\beta_{i-1}}{e_i n_i} (n_{i+1} \beta_{i+1} - n_i \beta_i) \left( \frac{1}{n_{i-1} \beta_{i-1}} - \frac{1}{n_i \beta_i} \right) \left( -\bar{\beta}_0 + \sum_{l=1}^{i-1} (n_l - 1) \beta_l \right),
\]
using formula (9) from the proof of Lemma 5.1. Finally, multiplying both sides by \( e_{i+1} n_{i-1} n_i n_{i+1} \beta_i = e_i n_i n_{i+1} \beta_i \) gives the condition
\[
(n_i \beta_i - n_{i-1} \beta_{i-1}) (n_{i+1} \beta_{i+1} - n_i \beta_i) \left( -\bar{\beta}_0 + \sum_{l=1}^{i} (n_l - 1) \beta_l \right)
> \frac{1}{n_i} (n_{i+1} \beta_{i+1} - n_i \beta_i) (n_i \beta_i - n_{i-1} \beta_{i-1}) \left( -\bar{\beta}_0 + \sum_{l=1}^{i-1} (n_l - 1) \beta_l \right),
\]
which is easily seen to hold. Analogously, the condition \( R_2 < 0 \) for \( g \geq 3 \) is equivalent to
\[
\frac{\beta_3}{e_1 e_2 n_0 n_1} (n_2 \beta_2 - n_1 \beta_1) \left( \frac{1}{n_2 \beta_2} - \frac{1}{n_3 \beta_3} \right) \left( -\bar{\beta}_0 + \sum_{l=1}^{2} (n_l - 1) \beta_l \right)
> \frac{1}{e_2 n_2} (n_3 \beta_3 - n_2 \beta_2) \left( \frac{1}{n_1 \beta_1} - \frac{1}{n_2 \beta_2} \right) \left( -\bar{\beta}_0 + \sum_{l=1}^{1} (n_l - 1) \beta_l \right),
\]
for which multiplication by \( e_1 e_3 n_0 n_1 n_2 n_3 \beta_2 = e_2 n_1 n_2 \beta_1 \beta_2 \) leads to the true condition
\[
(n_2 \beta_2 - n_1 \beta_1) (n_3 \beta_3 - n_2 \beta_2) \left( -\bar{\beta}_0 + \sum_{l=1}^{2} (n_l - 1) \beta_l \right)
> \frac{1}{n_2} (n_3 \beta_3 - n_2 \beta_2) (n_2 \beta_2 - n_1 \beta_1) \left( -\bar{\beta}_0 + \sum_{l=1}^{1} (n_l - 1) \beta_l \right).
\]
The last residue \( R_g \) for \( g = 2 \) and \( g \geq 3 \) can also be treated in a similar way. \(\square\)

**Remark 5.1.** Applying the same specialisation to the local version of the motivic zeta function, one obtains a local version for the topological zeta function. Because the limit for \( L \to 1 \) of
\[
\frac{(1 - T)(L - 1)L^{-(g+1)}}{1 - L^{-gT}}|_{T=L^{-\nu}}
\]
is equal to 0, the global and local topological zeta function of \( Y \) are identical. Hence, the results in this section are true for both the global and the local motivic zeta function.

Theorem 5.3 implies in particular that the motivic zeta function of the special fibre \( Y \) of the family \( \chi \) has the same number of poles as the motivic zeta function of a generic fibre in the family, whose poles are equal to the poles associated to the plane branch with an integer shift of \(-g - 1\). This is intriguing as the induced family on the jet schemes is in most cases not flat by Theorem 3.8, and the motivic zeta function is calculated in terms of the codimensions of the (irreducible components of) the jet schemes. However, one can check, using for example the expressions in [NV], that the poles associated to \( Y \) and to a generic fibre are not equal, except for the two smallest poles \(-g\) and \(-\frac{1}{\nu_g}\).
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(H. Mourtada) Université Paris Diderot, Institut de Mathématiques de Jussieu-Paris Rive Gauche, Batiment Sophie Germain, case 7012, 75205 Paris Cedex 13, France.

*E-mail address*: hussein.mourtada@imj-prg.fr

(W. Veys and L. Vos) KU Leuven, Departement Wiskunde, Celestijnenlaan 200B, bus 2400, 3001 Leuven, Belgium.

*E-mail address*: wim.veys@kuleuven.be, lena.vos@kuleuven.be