T-duality in Massive Integrable Field Theories: 
The Homogeneous and Complex sine-Gordon Models

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Abstract

The T-duality symmetries of a family of two-dimensional massive integrable field theories defined in terms of asymmetric gauged Wess-Zumino-Novikov-Witten actions modified by a potential are investigated. These theories are examples of massive non-linear sigma models and, in general, T-duality relates two different dual sigma models perturbed by the same potential. When the unperturbed theory is self-dual, the duality transformation relates two perturbations of the same sigma model involving different potentials. Examples of this type are provided by the Homogeneous sine-Gordon theories, associated with cosets of the form \( G/U(1)^r \) where \( G \) is a compact simple Lie group of rank \( r \). They exhibit a duality transformation for each element of the Weyl group of \( G \) that relates two different phases of the model. On-shell, T-duality provides a map between the solutions to the equations of motion of the dual models that changes Noether soliton charges into topological ones. This map is carefully studied in the complex sine-Gordon model, where it motivates the construction of Bogomol’nyi-like bounds for the energy that provide a novel characterisation of the already known one-solitons solutions where their classical stability becomes explicit.
1 Introduction.

It seems difficult to exaggerate the importance of duality in the investigation of the non-perturbative aspects of quantum field theory, statistical mechanics, and string theory. In brief, duality symmetries are discrete transformations that relate either two apparently different theories, or the same one at different values of its coupling constants. In some cases, this makes possible to investigate the strong coupling regime of a given theory from the knowledge of the weak coupling behaviour of its dual – and sometimes the original and the dual models coincide. Concerning two-dimensional non-linear sigma models, the most characteristic type of duality transformation is target-space duality, or ‘T-duality’ for short, which is the generalisation of the well known equivalence of the theories of a single bosonic field compactified on circles of radii $R$ and $1/R$. Two apparently different sigma models are said to be T-dual to each other if there is a canonical transformation between the phase spaces that preserves the respective Hamiltonians. Due to the important role of target-space duality in the study of (super) string vacua, there is a lot known about T-duality for massless sigma models, and we address the interested reader to [1] where comprehensive reviews and references to the original literature can be found. In contrast, T-duality of massive sigma models has received much less attention.

The purpose of this paper is to discuss T-duality in a particular family of two-dimensional massive integrable field theories defined by the gauged Wess-Zumino-Novikov-Witten (WZW) actions associated with cosets of the form $G/(H \times U(1))$ modified by a potential. It includes the Homogeneous sine-Gordon (HSG) theories [2, 3, 4], which correspond to cosets of the form $G/U(1)^r$ where $G$ is a compact simple Lie group of rank $r$. They are generalisations of the complex sine-Gordon (CSG) model [5, 6], which is recovered for $G = SU(2)$ [7, 8]. Other theories included in this family are the Symmetric-space sine-Gordon theories constructed in [2, 9], and the models studied by Gomes et al. in [10] associated with cosets of the form $SL(2) \times U(1)^n/U(1)$.

All these two-dimensional theories admit a Lagrangian description in terms of a massive non-linear sigma model like

$$\mathcal{L} = \frac{1}{2} \mathcal{G}_{ij}(X) \left( \frac{\partial X^i}{\partial t} \frac{\partial X^j}{\partial t} - \frac{\partial X^i}{\partial x} \frac{\partial X^j}{\partial x} \right) + \mathcal{B}_{ij}(X) \frac{\partial X^i}{\partial t} \frac{\partial X^j}{\partial x} - \mathcal{U}(X), \quad (1.1)$$

where $i = 1 \ldots n$ with $n$ the dimension of the target space, $\mathcal{G}$ is a metric, $\mathcal{B}$ is an antisymmetric tensor, and $\mathcal{U}$ is a potential. They exhibit a global $U(1)$ symmetry, which gives rise to an abelian T-duality transformation that relates off-shell two different massive sigma models associated with the same coset. An important feature is that the potential always remains invariant under the $U(1)$ symmetry and, therefore, the resulting duality transformations are actually a reflection of the T-duality symmetries of the unperturbed gauged WZW models, which are well established in the literature [11, 12]. In particular, if the coset is of the form $G/U(1)$, T-duality relates two different Lagrangians involving gauge transformations of vector or axial form, which is just a consequence of the well known duality between the $U(1)$ vector-gauged WZW model and the axially-gauged one.

Our discussion will be mostly classical, but our results are expected to be helpful to elucidate the consequences of duality in the corresponding quantum theories. At the quantum level, the formulation in terms of gauged WZW actions leads to a natural non-perturbative definition of the theory as a perturbed coset conformal field theory (CFT).
specified by an action of the form [13]

\[ S = S_{\text{CFT}} + \mu_a \int d^2 x \Phi^a. \]  

(1.2)

Here \( S_{\text{CFT}} \) denotes an action for the unperturbed two-dimensional CFT corresponding to the gauged WZW model, \( \Phi^a \) are spinless operators found in the operator algebra of \( S_{\text{CFT}} \), and \( \mu_a \) are dimensionful real parameters (coupling constants). The Lagrangian (1.1) describes the theory in the semiclassical limit, so that the massless sigma model \( \mathcal{L}|_{U=0} \) provides a Lagrangian description for the CFT corresponding to \( S_{\text{CFT}} \), and the potential becomes identified with the perturbation. In general, duality transformations relate two different dual CFTs perturbed by the same potential. However, in some cases the non-linear sigma models corresponding to the dual CFTs coincide, up to a change of the field variables. Then, the CFT is said to be self-dual, and the duality transformation becomes equivalent to a change of the potential or, equivalently, of the coupling constants \( \{\mu_a\} \) in (1.2). This can be understood as a relationship between different phases of the same perturbed CFT akin to the Kramers-Wannier duality between the high and low temperature phases of the Ising model [14]. Recall that the scaling \( T \to T_c \) limit of the Ising model is described by an action of the form (1.2) where the unperturbed CFT is the critical Ising model, and the perturbation is defined by a single operator (the thermal or energy-density operator), such that \( \mu \sim T - T_c \). Then, the duality between the high \( (T > T_c) \) and low \( (T < T_c) \) temperature phases corresponds to the duality of the perturbed CFT theory under \( \mu \to -\mu \).

The paper is organised as follows. In section 2 we provide a brief description of the duality transformations exhibited by the family of integrable theories using Buscher’s formulation of abelian T-duality. In section 3, we introduce the particular class of integrable theories. In section 4, we explicitly construct the canonical transformation that relates a given theory to its dual. In section 5, the duality transformation is considered on-shell, which provides a map between the solutions to the equations of motion of the two dual massive sigma models that interchanges Noether and topological charges. Finally, in sections 6 and 7, we illustrate our results with a number of examples associated with cosets of the form \( G/U(1)^p \). In particular, we discuss T-duality in the HSG and the CSG models. In the HSG models, we also characterise their phase structure to show that T-duality actually provides a relation between different phases. In the CSG model, our discussion motivates the construction of Bogomol’nyi-like bounds for the energy saturated by the already known solitons which, to our knowledge, have not been reported before in the literature.

### 2 Potentials and T-duality.

A useful description of abelian T-duality in the massless case was introduced originally by Buscher [15, 1]. It can be summarised as follows. Consider the 1+1 dimensional bosonic non-linear sigma model defined by the Lagrangian (1.1) with \( U = 0 \). Assume that the sigma model has an abelian isometry and that we have chosen coordinates adapted to the isometry such that it is represented simply by a translation in the coordinate \( X^1 \), which requires that \( \mathcal{G} \) and \( \mathcal{B} \) are independent of \( X^1 \). Then, T-duality is a transformation that
relates the non-linear sigma model corresponding to \((G, B)\) to another one specified by 
\[ 
G_{11}^D = \frac{1}{G_{11}}, \quad G_{ij}^D = \frac{B_{ij}}{G_{11}}, \quad G_{ij} = G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}, \quad B_{1i} = \frac{G_{1i}}{G_{11}}, \\
B_{ij} = B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}, \quad i, j \neq 1; 
\]
moreover, \((G^D)^D = G\) and \((B^D)^D = B\). Both sigma models are related by a canonical transformation between the phase spaces that preserves the respective Hamiltonians \([16, 17]\). Consequently, even though they are generally defined by completely different Lagrangians, the sigma models specified by \((G, B)\) and \((G^D, B^D)\) describe the same physics.

In the massive case, conformal invariance is explicitly violated by the presence of the potential \(U\). Then, a general description of T-duality is not available, but we shall study a family of theories where the duality properties directly follow from those of massless sigma models. Suppose that \(U(X)\) is a function of the coordinates and not of their derivatives, and that it preserves the abelian isometry, so that \(U(X)\) is also independent of \(X^1\). Then, the potential does not change under the canonical transformation corresponding to (2.1) and T-duality relates the massive models specified by \((G, B, U)\) and \((G^D, B^D, U)\). This will be indicated as follows 
\[ 
(G, B, U) \xrightarrow{T\text{-duality}} (G^D, B^D, U), 
\]
which provides a relationship between two different dual sigma models perturbed by the same potential. Explicit examples of massive integrable sigma models related by duality transformations like (2.2) can be found in \([10, 18]\).

A interesting particular case occurs when the unperturbed non-linear sigma model is ‘self-dual’. By this we mean that there is a change of field variables \(X^i \rightarrow \tilde{X}^i\) (a point transformation in the context of canonical transformations) that provides a relationship of the form \((G^D, B^D, U) \rightarrow (G, B, \tilde{U})\). Then, eq. (2.2) becomes 
\[ 
(G, B, U) \xrightarrow{T\text{-duality}} (G^D, B^D, U) \xrightarrow{X^i \rightarrow \tilde{X}^i} (G, B, \tilde{U}), 
\]
which relates two perturbations of the same non-linear sigma model by different potentials. The relevance of (2.3) becomes clearer when considering the non-perturbative definition of the model at the quantum level by means of an action of the form (1.2). Then, the potential is identified with the perturbation, and eq. (2.3) can be understood as a duality symmetry between two different phases of the same model characterised by the domain where the coupling constants \(\mu_a\) take values. The HSG and CSG models, which will be discussed in sections 6 and 7, provide examples where T-duality is of the form (2.3).

3 The integrable theories.

Eqs. (2.2) and (2.3) summarise the form of the abelian T-duality transformations exhibited by a large family of two-dimensional integrable theories. They are particular cases of an even larger class of integrable models associated with generic cosets constructed in the literature by several authors with different purposes. Namely, description of deformed coset models \([8, 19]\), Hamiltonian reduction of two-loop WZW models \([10, 18, 20]\), Lagrangian formulation of reduced symmetric space sigma models \([21]\), or, like in \([2, 22]\),
simply as the Lagrangian models whose classical equations of motion are the non-abelian affine Toda (NAAT) equations of Leznov and Saveliev [23], which can be recognised as their distinctive common feature. Following [21], these theories can be defined in a systematic way in terms of a triplet of Lie algebras \((\mathfrak{h}, \mathfrak{g}, \mathfrak{j})\), with respective associated Lie groups \(H \subseteq G \subseteq F\), as follows – in the following, we will always assume that \(F\) is simple, although the construction is not restricted to this case. If \(G \neq F\), we shall assume that \(F/G\) is a symmetric space, which means that there is a Lie algebra decomposition \(\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{t}\) that satisfies
\[
[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}, \quad [\mathfrak{g}, \mathfrak{t}] \subseteq \mathfrak{t}, \quad [\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{g}.
\] (3.1)

Then, we choose two arbitrary (adjoint-diagonalisable) constant elements \(T_+\) and \(T_-\) in \(\mathfrak{t}\). Correspondingly, if \(G = F\) we choose \(T_\pm\) in \(\mathfrak{g} = \mathfrak{f}\). Finally, \(\mathfrak{h}\) is defined as the simultaneous centraliser of \(T_+\) and \(T_-\) in \(\mathfrak{g}\), namely \(\mathfrak{h} = \{u \in \mathfrak{g} \mid [u, T_+] = [u, T_-] = 0\}\), and the model is defined by the action [2]
\[
S^{(\tau)}[h, A_\pm] = k \left( S^{(\tau)}_{\mathfrak{g}WZW}[h, A_\pm] - \int d^2x \ U(h) \right)
\] (3.2)
\[
S^{(\tau)}_{\mathfrak{g}WZW}[h, A_\pm] = S_{\mathfrak{g}WZW}[h] + \frac{1}{\pi} \int d^2x \left( -\langle A_+ \mid \partial_- hh^{-1} \rangle - \langle A_- \mid \partial_+ hh^{-1} \rangle \right)
\]
\[
+ \langle \tau(A_-) \mid h^{-1} \partial_+ h \rangle + \langle h^{-1} A_+ h \mid \tau(A_-) \rangle - \langle A_+ \mid A_- \rangle \right),
\] (3.3)
\[
U(h) = \lambda \langle T_+ \mid h^{-1} T_- h \rangle.
\] (3.4)

Here, \(kS_{\mathfrak{g}WZW}[h]\) is the usual WZW action at level \(k\) for the bosonic field \(h\) taking values in (some faithful representation of) \(G \subseteq F\); \(A_\pm\) are non-dynamical gauge fields taking values in \(\mathfrak{h}\), \(\lambda\) is a dimensionful parameter, \(\langle \mid \rangle\) is the invariant bilinear form of \(\mathfrak{f}\) normalised such that \(S_{\mathfrak{g}WZW}[h]\) is defined modulo \(2\pi \mathbb{Z}\) [24], and we are using the notation \(\partial_\pm = \partial/\partial x_\pm\) with \(x_\pm = t \pm x\). In this action, \((\lambda, T_+, T_-)\) play the role of coupling constants.

The action \(S^{(\tau)}_{\mathfrak{g}WZW}\) is invariant under the group of gauge transformations
\[
h \to \alpha \ h \ \tilde{\tau}(\alpha^{-1}), \quad A_\pm \to \alpha \ A_\pm \alpha^{-1} - \partial_\pm \alpha \ \alpha^{-1},
\] (3.5)
where \(\alpha = \alpha(t, x)\) takes values in \(H\), and \(\tilde{\tau}\) is the lift of a suitable automorphism \(\tau\) of \(\mathfrak{h}\) that leaves the restriction of the bilinear form \(\langle \mid \rangle\) to \(\mathfrak{h}\) invariant so that the gauge group is ‘anomaly free’ [2, 25]. The lift \(\tilde{\tau}\) is defined by \(\tilde{\tau}(e^u) = e^{\tau(u)}\), for any \(u \in \mathfrak{h}\). In particular, taking \(\tau = +I\) or \(-I\) leads to gauge transformations of vector or axial type, respectively. Then, \(kS^{(\tau)}_{\mathfrak{g}WZW}\) is the action of a \(G/H\) coset conformal field theory (CFT) at level \(k\) [25]. The particular coset model depends on \(\tau\). The usual one is recovered with \(\tau = +I\), which is the only case considered in [21], while for \(\tau \neq I\) the models are examples of the asymmetric coset models constructed in [26]. Taking the relationship between the constant elements \(T_\pm\) and \(H\) into account, it is easy to check that \(U(\alpha h \beta) = U(h)\) for any \(\alpha, \beta \) in \(H\), which shows that the potential is uniquely defined on the coset \(G/H\) independently of the choice of \(\tau\).

The connection of these models with the NAAT equations is obtained as follows. The variation of the action (3.2) with respect to the field \(h\) yields the equations of motion that
\footnote{This case can equivalently be described in terms of the symmetric space \(G \times G/G\) of type II in Cartan’s classification.}
can be expressed in a zero-curvature form [2]

\[
\left[ \partial_+ + h^{-1} \partial_+ h + h^{-1} A_+ h - \pi \lambda \xi T_+ , \partial_- + \tau(A_-) + \xi^{-1} h^{-1} T_- h \right] = 0 ,
\]

(3.6)

where \( \xi \) is a constant spectral parameter. Correspondingly, the variations with respect to \( A_\pm \) lead to the constraints

\[
P_h \left( h^{-1} \partial_+ h + h^{-1} A_+ h \right) - \tau(A_+) = 0 ,
\]

\[
P_h \left( -\partial_- h^{-1} + h \tau(A_-) h^{-1} \right) - A_- = 0 ,
\]

(3.7)

where \( P_h \) is the projector onto the subalgebra \( h \). Next, projecting (3.6) onto \( h \) and using (3.7), it can be checked that the gauge field is flat on-shell: \( [\partial_+ + A_+ , \partial_- + A_-] = 0 \).

Using (3.7) and the constraints (3.7) simplify to

\[
\partial_- \left( h^{-1} \partial_+ h \right) = -\pi \lambda \left[ T_+ , h^{-1} T_- h \right] ,
\]

\[
P_h \left( h^{-1} \partial_+ h \right) = P_h \left( \partial_- h h^{-1} \right) = 0 ,
\]

(3.8)

which is a system of non-abelian affine Toda equations [23, 2, 20].

Notice that (3.3) is quadratic in the non-dynamical gauge fields. Therefore, they can be integrated out by solving (3.7) for \( A_\pm \) to obtain a sigma model description of the coset CFT and, hence, of the massive theory in terms of a Lagrangian like (1.1). However, in order to leave the choice of the gauge fixing prescription completely free, we will keep the gauge fields and work directly with (3.2).

Since we are interested in abelian T-duality, we will restrict ourselves to cases where \( H \) contains, at least, a \( U(1) \) factor; i.e., \( H \) will be of the form \( \tilde{H} \times U(1) \). If we call \( T^0 \) the generator of the \( U(1) \) factor, we can choose a basis of generators \( \{ t^a , T^0 \} \) for the Lie algebra \( h = \tilde{h} \oplus u(1) \) such that

\[
\langle T^0 | T^0 \rangle = -1 \quad \text{and} \quad \langle t^a | T^0 \rangle = 0 \ \forall \ a .
\]

(3.9)

In this basis, the gauge fields split in components as follows

\[
A_\pm = A^a_\pm t^a + a_\pm T^0 \equiv \hat{A}_\pm + a_\pm T^0 .
\]

(3.10)

We will also assume that the \( U(1) \) factor is compact and \( T^{0 \dagger} = -T^0 \). Then, the WZW field can be parametrised as

\[
h = e^{\beta T^0} h_0 e^{\gamma T^0} ,
\]

(3.11)

with \( \beta \) and \( \gamma \) real bosonic fields, so that the \( \tau \)-dependent \( U(1) \) gauge transformations (3.5) generated by the elements of \( H \) of the form \( \alpha = e^{\rho T^0} \) correspond to

\[
\beta \rightarrow \beta + \rho , \quad \gamma \rightarrow \gamma - \rho , \quad a_\pm \rightarrow a_\pm - \partial_\pm \rho ,
\]

(3.12)

while \( h_0 \) remains invariant. This way, and using the Polyakov-Wiegmann formula

\[
kS_{\text{WZW}}[gh] = kS_{\text{WZW}}[g] + kS_{\text{WZW}}[h] - \frac{k}{\pi} \int d^2x \langle g^{-1} \partial_+ g | \partial_- h h^{-1} \rangle ,
\]

(3.13)
the action (3.3) can be written in the form

\[ S^{(T)}[e^{\beta T^0} h_0 e^{\gamma (T^0)}, A_{\pm}] = \]
\[ = \frac{k}{2\pi} \int d^2 x \left( -\partial_+ \phi \partial_- \phi + 2\tilde{a}_+ \partial_- \phi - 2\tilde{a}_- \partial_+ \phi \right) + \Delta S^{(T)}[h_0, A_{\pm}], \tag{3.14} \]

where

\[ \Delta S^{(T)}[h_0, A_{\pm}] = k \left( \int d^2 x U(h_0) \right) \]
\[ + \frac{k}{\pi} \left( (1 + R^\tau(h_0)) \tilde{a}_+ \tilde{a}_- - \tilde{a}_+ J_{\tau}^+(h_0) + \tilde{a}_- J_{\tau}^-(h_0) \right) \]
\[ + \left( \tau T_0 h_0^{-1} \right) \tilde{a}_- + \tau T_0 h_0^{-1} \tilde{a}_+ , \tag{3.15} \]

and we have introduced the gauge invariant fields (see (3.12))

\[ \phi = \beta + \gamma, \quad \tilde{a}_+ = a_+ + \partial_+ \gamma, \quad \tilde{a}_- = a_- - \partial_- \gamma, \tag{3.16} \]

together with

\[ R^\tau(h_0) = \langle \tau(T_0) | h_0^{-1} T_0 h_0 \rangle \]
\[ J_{\tau}(h_0) = \langle \tau T_0 h_0^{-1} \rangle, \quad J_{\tau}^+(h_0) = \langle \tau(T_0) | h_0^{-1} \partial_+ h_0 \rangle. \tag{3.17} \]

4 Off-shell T-duality.

The action (3.2) or, equivalently, (3.14) is invariant under the global $U(1)$ transformation

\[ h \rightarrow e^{\rho T^0} h e^{\gamma (T^0)}, \tag{4.1} \]

which corresponds to $\phi(t, x) \rightarrow \phi(t, x) + 2\rho$ and, therefore, $\phi$ is an adapted coordinate for this symmetry transformation. As explained in [16], in order to find the dual action associated with the abelian isometry (4.1) by means of a canonical transformation we can use the Routh function with respect to $\phi$, which means that the Legendre transformation is only performed with respect to this coordinate. Let us write $S^{(T)}[h, A_{\pm}] = \int d^2 x L^{(T)}[h, A_{\pm}]$. Taking (3.14) into account, the relevant canonical momentum is

\[ \Pi_\phi = \frac{\partial L^{(T)}}{\partial \phi} = \frac{k}{4\pi} \left( -\partial_+ \phi + 2(\tilde{a}_+ - \tilde{a}_-) \right), \tag{4.2} \]

and the Routh function is given by

\[ R^{(T)}(\phi, \Pi_\phi) = \partial_\phi \Pi_\phi - L^{(T)} \]
\[ = -\frac{2\pi}{k} \Pi_\phi^2 + 2(\tilde{a}_+ - \tilde{a}_-) \Pi_\phi - \frac{k}{8\pi} (\partial_\phi)^2 + \frac{k}{2\pi}(\tilde{a}_+ + \tilde{a}_-) \partial_\phi \]
\[ - \frac{k}{2\pi} (\tilde{a}_+ - \tilde{a}_-)^2 - \Delta L^{(T)}, \tag{4.3} \]

\[ 2 \text{It is well known that the Polyakov-Wiegmann formula only holds if the WZW model is defined on a simply connected compact manifold; otherwise, it has to be corrected by adding topological terms [27]. In our case, we have used this freedom to remove a term of the form } -\frac{k}{2\pi} \int d^2 x \left( \partial_+ (\beta \partial_- \gamma) - \partial_- (\beta \partial_+ \gamma) \right) \]
\[ \text{in order to ensure explicit gauge invariance.} \]
where $\Delta \mathcal{L}^{(\tau)}$ is the piece of $\mathcal{L}^{(\tau)}$ corresponding to (3.15). Now, following [16], we consider the canonical transformation generated by

$$F = -\frac{k}{8\pi} \int_{-\infty}^{+\infty} dx \left( \partial_x \phi \phi^D - \phi \partial_x \phi^D \right); \quad (4.4)$$

namely,$^3$

$$\Pi_\phi = \frac{\delta F}{\delta \phi} = \frac{k}{4\pi} \partial_x \phi^D, \quad \Pi_{\phi^D} = -\frac{\delta F}{\delta \phi^D} = \frac{k}{4\pi} \partial_x \phi. \quad (4.5)$$

This transformation preserves the Hamiltonian and, hence, the dual Routh function is obtained just by making these substitutions in $\mathcal{R}^{(\tau)}$, $\Pi \phi D \left( \phi, \Pi \phi \right) = \mathcal{R}^{(\tau)} D \left( \phi^D \right)$,

$$\mathcal{R}^{(\tau)} \left( \phi, \Pi \phi \right) = \mathcal{R}^{(\tau)} D \left( \phi^D, \Pi_{\phi^D} \right)$$

$$= -\frac{2\pi}{k} \Pi_{\phi^D}^2 + 2(\tilde{a}_+ + \tilde{a}_-) \Pi_{\phi^D} - \frac{k}{8\pi} (\partial_x \phi^D)^2 + \frac{k}{2\pi} (\tilde{a}_+ - \tilde{a}_-) \partial_x \phi^D$$

$$- \frac{k}{2\pi} (\tilde{a}_+ - \tilde{a}_-)^2 - \Delta \mathcal{L}^{(\tau)}. \quad (4.6)$$

Performing the inverse Legendre transform, it is not difficult to check that (4.6) corresponds to the dual action

$$S^{(\tau)} D \left[ e^{\beta T^0} h_0 e^{\gamma \tau(T^0)} A_+, A_- \right] = S^{\tau \cdot \sigma T_0} \left[ e^{\beta T^0} h_0 e^{-\gamma \tau(T^0)} A_+, A_- \right], \quad (4.7)$$

where $A_- = \sigma T_0 (A_-) + 2 \partial_- \gamma T^0$, and $\sigma T_0$ is the following involutive automorphism of $\mathfrak{h}$:

$$\sigma T_0 (u) = u - 2 \frac{\langle u \mid T_0 \rangle}{\langle T_0 \mid T_0 \rangle} T_0, \quad \forall u \in \mathfrak{h}; \quad (4.8)$$

i.e., $\sigma T_0$ is a reflection on the subspace of $\mathfrak{h}$ orthogonal to $T_0$, and $\sigma^2 T_0 = I$. Therefore, up to the trivial field transformations

$$h = e^{\beta T^0} h_0 e^{\gamma \tau(T^0)} \longrightarrow h' = e^{\beta T^0} h_0 e^{-\gamma \tau(T^0)}, \quad A_- \rightarrow A_-', \quad (4.9)$$

the only effect of the T-duality transformation associated with the $U(1)$ transformation (4.1) is to change the action corresponding to the automorphism $\tau$ into the action corresponding to $\tau \cdot \sigma T_0$,

$$S^{(\tau)} \xrightarrow{\text{T-duality}} S^{\tau \cdot \sigma T_0} \quad (4.10)$$

Notice that the potential (3.4) remains invariant and, hence, the duality transformation summarised by (4.7) and (4.10) is precisely of the form (2.2).

5 On-shell T-duality.

On-shell, eq. (4.5) provides a map between the solutions to the equations of motion of the two dual massive sigma models, which can be characterised as a pseudoduality transformation using the terminology of [28]. In the string theory context, it is well known

$^3$This canonical transformation corresponds to a duality transformation of the form (2.1) together with a trivial change of the normalisation of the dual field $\phi^D \rightarrow -(k/4\pi) \phi^D$. 

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that T-duality changes trivial boundary conditions (momentum modes) into non-trivial ones (winding modes). In our case, even though we consider theories defined on Minkowski space where winding numbers are not defined, the on-shell T-duality transformations will admit a similar interpretation when restricted to soliton solutions, changing Noether (electric) soliton charges into topological (magnetic) ones.

The (gauge invariant) solutions to the equations of motion of (3.14) will be specified by \((h_0, \phi)\). This way, the map provided by (4.5) corresponds to \((h_0, \phi) \rightarrow (h_0, \phi^D)\), where \(\phi^D\) is the solution to the equations of the canonical transformation understood as partial differential equations for \(\phi^D\):

\[
\frac{\partial_x \phi^D}{(1 - R^\tau)} = \frac{(1 - R^\tau)\partial_t \phi - 2(\mathcal{J}_+^\tau + \mathcal{J}_-^\tau)}{1 + R^\tau}, \quad \frac{\partial_t \phi^D}{(1 - R^\tau)} = \frac{(1 - R^\tau)\partial_x \phi - 2(\mathcal{J}_+^\tau - \mathcal{J}_-^\tau)}{1 + R^\tau},
\]

where

\[
\mathcal{J}_+^\tau = \mathcal{J}_+^\tau(h_0) + \langle \hat{A}_+ | h_0 \tau(T^0) h_0^{-1} \rangle, \quad \mathcal{J}_-^\tau = \mathcal{J}_-^\tau(h_0) - \langle \tau(\hat{A}_-) | h_0 T^0 h_0^{-1} \rangle,
\]

and \(R^\tau = R^\tau(h_0), J_+^\tau(h_0)\) and \(J_-(h_0)\) have been previously defined in (3.17). The \(U(1)\) global symmetry (4.1) leads to the conserved Noether current \(J^{\mu N} = 2 \partial \mathcal{L}^{(\tau)} / \partial (\partial_{\mu} \phi)\), and the eqs. (5.1) can be written as

\[
\frac{\partial_x \phi^D}{k} = \frac{2\pi}{k} J_0^N, \quad \frac{\partial_t \phi^D}{k} = \frac{2\pi}{k} J_1^N, \quad (5.3)
\]

whose integrability is equivalent to the conservation of \(J^{\mu N}\), which holds on-shell.

Consider a soliton solution to the equations of motion of \(S^{(\tau)}\) that carries a definite value of the \(U(1)\) Noether charge \(Q^N = \int_{-\infty}^{+\infty} dx J_0^N\). Integrating the equations (5.3), we obtain

\[
\phi^D(t, +\infty) - \phi^D(t, -\infty) = \frac{2\pi}{k} \int_{-\infty}^{+\infty} dx J_0^N = \frac{2\pi}{k} Q^N
\]

\[
Q^{(D) N} = \int_{-\infty}^{+\infty} dx J_0^{(D) N} = \frac{k}{2\pi} (\phi(t, +\infty) - \phi(t, -\infty)).
\]

It is important to notice that, due to the continuous \(U(1)\) symmetry of the potential, the values of the boundary conditions \(\phi(t, \pm\infty)\) are not quantised. This means that, in general, their value will change continuously with \(t\) and, thus, they do not provide proper conserved topological charges which can be used to classify the solutions of the equations of motion. In contrast, for each pair of dual soliton solutions, and since the Noether charges \(Q^N\) and \(Q^{(D) N}\) are conserved, the eqs. (5.4) show that \(\omega = \phi(t, +\infty) - \phi(t, -\infty)\) is indeed time independent. In other words, \(\omega\) provides a conserved quantity, but its conservation is not of topological nature; it follows from the equations of motion in the dual phase. With this caveat, \(\omega\) can be understood as a topological charge carried by the soliton solutions associated with the current

\[
J_\mu^T = -\varepsilon_{\mu\nu} \partial^\nu \phi, \quad \varepsilon_{01} = +1,
\]

so that \(Q^T = \int_{-\infty}^{+\infty} dx J_0^T = \phi(t, +\infty) - \phi(t, -\infty) = \omega\). This way, soliton solutions can be specified by \([\omega, Q^N]\), where \(\omega\) and \(Q^N\) are constant, and (5.4) shows that the solutions
to the equations of motion of $S^{(\tau)}$ characterised by $[\omega, (k/2\pi)\omega^D]$ are T-dual to those of $S^{(\tau\cdot\sigma_0)}$ labelled by $[\omega^D, (k/2\pi)\omega]$.

Recall now that $\phi$ is a compact field, of angular nature, which is identified with $\phi + 2\pi \Delta$ for some constant real number $\Delta$ that depends on the normalisation of $\phi$. Consequently, the topological charges $\omega$ and $\omega^D$ are defined modulo $2\pi\Delta$ and $2\pi\Delta^D$, respectively, and T-duality implies that the value of the Noether charge $Q^N$ is uniquely defined only modulo $k\Delta^D$.

In contrast to the semiclassical description which has been considered so far, under quantisation the Noether charges carried by the solitons become quantised in terms of some unit charges $q$ and $q^D$; i.e., $Q^N \in q\mathbb{Z}$ and $Q^{(D)N} \in q^D\mathbb{Z}$. This, together with the identification $Q^N \sim Q^N + k\Delta^D$, can be seen as an indication that the global $U(1)$ symmetry will be broken to a discrete symmetry associated with a finite subgroup of $U(1)$ characterised by $k$—e.g., $\mathbb{Z}_k$—, which is actually expected to occur in the quantum theory. Moreover, through the duality transformation, the boundary conditions become quantised too: $\omega \in (2\pi q^D/k)\mathbb{Z}$ and $\omega^D \in (2\pi q/k)\mathbb{Z}$, which together with $\omega \sim \omega + 2\pi\Delta$ indicates that the solitonic quantum field configurations will be parafermionic.

6 Examples.

We now describe T-duality in some specific models associated with cosets of the form $G/U(1)^p$, with $p \geq 1$.

6.1 $G/U(1)$ models.

The simplest examples where T-duality is described by (4.10) are provided by the models corresponding to cosets of the form $G/U(1)$. In particular, they include the Complex sine-Gordon theory, which will be discussed in detail in section 7, and the models studied by Gomes et al. in [10]. They are associated with $SU(2)/U(1)$ and with cosets of the form $SL(2) \times U(1)^n/U(1)$, respectively. In this case, there are only two possibilities for the automorphism $\tau$. It can be either $\tau = +I$ or $\tau = -I$, which lead to $U(1)$ gauge transformations of vector or axial form. Correspondingly, the automorphism defined in (4.8) is $\sigma_0 = -I$ and, hence, T-duality is simply a reflection of the well known fact that the $U(1)$ vector-gauged WZW model is dual to the axially-gauged one [11].

6.2 $G/U(1)^p$ models with $p > 1$: T-duality in the HSG models.

Naively, when $H$ contains more than one $U(1)$ factor, each $T^0 \in \mathfrak{h}$ gives rise to a T-duality transformation of the form (4.10), which will be denoted $D_{T^0}$. These transformations can be multiplied. The product of two of them can be defined simply as the result of performing one after the other, so that $D_{T^0}D_{V^0}$ is specified by the diagram

$$
S^{(\tau)} \xrightarrow{D_{V^0}} S^{(\tau\cdot\sigma_0)} \xrightarrow{D_{T^0}} S^{(\tau\cdot\sigma_0\cdot\sigma_0)} \xrightarrow{D_{T^0}D_{V^0}} (6.1)
$$
which relates the models corresponding to $\tau$ and $\tau \cdot \sigma_{V_0} \cdot \sigma_{T_0}$ without changing the potential. The resulting set of duality transformations forms a non-abelian group where $D_{T_0} D_{T_0} = I$ and $D_{T_0} D_{V_0} \neq D_{V_0} D_{T_0}$ unless $\langle T^0 | V^0 \rangle = 0$. However, already at the classical level, not all the resulting transformations are consistent, which will now be illustrated in the particular case of the Homogeneous sine-Gordon theories.

The HSG theories were constructed in [2] at the classical level, in [3] as multiparameter quantum integrable deformations of conformal field theories, and in [4] as factorised $S$-matrix theories. Some of their non-perturbative properties have been recently investigated in [29, 30, 31, 32]. They are associated with cosets of the form $G/H$, where $G$ is a compact simple Lie group of rank $r$, and $H \simeq U(1)^r$ is a maximal torus of $G$. In the construction of section 3, they correspond to triplets $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $G$ and $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ associated with the maximal torus $H$. Moreover, $T_\pm$ are chosen such that the centraliser of each of them in $g$ is $\mathfrak{h}$. In these theories, it is easy to understand why not any $T_0 \in \mathfrak{h}$ leads to a consistent duality transformation. The reason is that the set of possible automorphisms $\tau$ in (3.2) forms a discrete group, and only the choices of $T_0$ such that both $\tau$ and $\tau \cdot \sigma_{T_0}$ are in this group lead to consistent transformations.

An admissible automorphism $\tau$ has to satisfy two conditions. The first one is that it leaves the restriction of the bilinear form $\langle \langle \cdot | \cdot \rangle \rangle$ to $\mathfrak{h}$ invariant, namely $\langle \langle \tau(u) | \tau(v) \rangle \rangle = \langle u | v \rangle$ for all $u, v \in \mathfrak{h}$, to ensure that the group of gauge transformations (3.5) is anomaly free [25]. Since $\mathfrak{h}$ is a Cartan subalgebra and $G$ is simple, the restriction of $\langle \langle \cdot | \cdot \rangle \rangle$ to $\mathfrak{h}$ is (proportional to) the Euclidean metric on $\mathbb{R}^r$. Therefore, this condition constrains $\tau$ to be an orthogonal $O(r)$ transformation acting on $\mathfrak{h}$ [2, 3]. The second condition arises by noticing that the gauge transformations (3.5) are not defined in terms of $\tau$, but in terms of $\tilde{\tau}$, which is the lift of $\tau$ into $H$ defined as follows. Let $\{h^1 \ldots h^r\}$, with $(h^i)^t = h^i$ and $\langle h^i | h^j \rangle = \delta_{ij}$, be a basis of $\mathfrak{h}$, and write a generic element of $H$ as $\exp(2\pi i \vec{\phi} \cdot \vec{\bar{h}})$, where $\vec{\phi}$ is an $r$-dimensional real vector. Then,

$$\tilde{\tau} \left( \exp(2\pi i \vec{\phi} \cdot \vec{\bar{h}}) \right) = \exp(2\pi i \tau(\vec{\phi}) \cdot \vec{\bar{h}}),$$

where we have used the same notation for the automorphism $\tau$ acting on $\vec{\phi} \cdot \vec{\bar{h}} \in \mathfrak{h}$ and for the corresponding linear transformation acting on $\vec{\phi} \in \mathbb{R}^r$; i.e., $\tau(\vec{\phi} \cdot \vec{\bar{h}}) \equiv \tau(\vec{\phi}) \cdot \vec{\bar{h}}$. This way, the second condition to be satisfied by $\tau$ is simply that $\tilde{\tau}$ is well defined on $H$.\footnote{I thank Patrick Dorey for pointing out this condition, which determines that $\tau$ has to be an element of a discrete group, and was missed in the original papers about the HSG models.}

In order to solve it, we have to make the torus structure of $H \subset G$ explicit. Notice that $\exp(2\pi i \vec{\phi} \cdot \vec{\bar{h}})$ furnishes a map from $\mathbb{R}^r$, where $\vec{\phi}$ takes values, onto $H$. Therefore, $H$ can be identified with $\mathbb{R}^r$ factored out by the kernel of this map. This is the set of vectors $\vec{\phi} \in \mathbb{R}^r$ mapped onto the unit element of $G$; i.e., the vectors that satisfy

$$\exp \left( 2\pi i \vec{\phi} \cdot \vec{\bar{h}} \right) = 1.$$  (6.3)

This identity has to hold in any representation, and it is convenient to write it in terms of the weights of $G$. Recall that a weight $\vec{w} = (w^1 \ldots w^r) \in \mathbb{R}^r$ is the eigenvalue of $\{h^1 \ldots h^r\}$ corresponding to a common eigenvector in some representation of $G$. The set of these weights is the ‘weight lattice’ of $G$, which will be denoted $\Lambda_w(G)$. Then, (6.3) is equivalent to

$$\vec{\phi} \cdot \vec{w} \in \mathbb{Z}, \quad \forall \vec{w} \in \Lambda_w(G).$$  (6.4)
The vectors that satisfy (6.4) span another lattice \( \Lambda^*_w(G) \) known as the ‘dual lattice’ to \( \Lambda_w(G) \). Consequently, there is a solution to (6.3) for each \( \vec{\phi} \in \Lambda^*_w(G) \). This provides the identification\(^5\)

\[
H \simeq \mathbb{R}^r/\Lambda^*_w(G) .
\]

Consequently, the requirement that the lift of \( \tau \) specified by (6.2) is well defined on \( H \) constrains \( \tau \) to be an element of the group of automorphisms of \( \Lambda^*_w(G) \), denoted \( \text{Aut} \Lambda^*_w(G) \), which is a discrete subgroup of \( O(r) \).

The conclusion is that the action (3.2) specifies a different HSG model for each choice of \( \tau \in \text{Aut} \Lambda^*_w(G) \) acting on \( \mathfrak{h} \) according to \( \tau(\vec{\phi} \cdot \vec{h}) = \tau(\vec{\phi}) \cdot \vec{h} \), and that the models corresponding to \( \tau \) and \( \tau \cdot \sigma_{T_0} \) are related by a T-duality transformation of the form (4.10) for each \( T_0 \in \mathfrak{h} \) such that \( \sigma_{T_0} \) is also in \( \Lambda^*_w(G) \).

An important set of transformations that leave \( \Lambda^*_w(G) \) invariant is provided by the Weyl group of \( G \), denoted \( \mathcal{W}(G) \). It is generated by the reflections in the hyperplanes orthogonal to the roots of \( G \), known as Weyl reflections, which are the linear transformations

\[
w_{\vec{\alpha}}(\vec{\phi}) = \vec{\phi} - 2 \frac{\vec{\alpha} \cdot \vec{\phi}}{\vec{\alpha} \cdot \vec{\alpha}} \vec{\alpha}
\]

defined for each root \( \vec{\alpha} \) of \( G \). Acting on the Cartan subalgebra, \( w_{\vec{\alpha}} \) corresponds to the automorphism \( \sigma_{T_0} \) defined in (4.8) for \( T_0 = \vec{\alpha} \cdot \vec{h} \),

\[
w_{\vec{\alpha}}(\vec{\phi}) \cdot \vec{h} = \sigma_{\vec{\alpha} \cdot \vec{h}}(\vec{\phi} \cdot \vec{h}) .
\]

Therefore, since \( w_{\vec{\alpha}} \in \text{Aut} \Lambda^*_w(G) \), there is a T-duality transformation of the form (4.10) associated with \( T_0 = \vec{\alpha} \cdot \vec{h} \) for each root \( \vec{\alpha} \), which will be denoted \( D_{T_0} = D_{\vec{\alpha} \cdot \vec{h}} \) using the notation introduced just before (6.1). These transformations can be multiplied according to (6.1), so that \( D_{\vec{\beta} \cdot \vec{h}} D_{\vec{\alpha} \cdot \vec{h}} \) relates the HSG models corresponding to the automorphisms \( \tau \) and \( \tau \cdot (w_{\vec{\alpha}} \cdot w_{\vec{\beta}}) \). This allows one to associate a T-duality transformation to each Weyl transformation. Recall that a generic element of \( \mathcal{W}(G) \) is obtained as the product of a finite number of Weyl reflections, say \( \omega = w_{\vec{\alpha}^{(1)}} \cdot w_{\vec{\alpha}^{(2)}} \cdot \ldots \cdot w_{\vec{\alpha}^{(n)}} \), where \( \vec{\alpha}^{(1)}, \ldots, \vec{\alpha}^{(n)} \) are roots of \( G \). Then, the duality transformation associated with \( \omega \) is defined by

\[
D_{\omega} = D_{\vec{\alpha}^{(n)} \cdot \vec{h}} \ldots D_{\vec{\alpha}^{(2)} \cdot \vec{h}} D_{\vec{\alpha}^{(1)} \cdot \vec{h}} ,
\]

and it relates the HSG models corresponding to \( \tau \) and \( \tau \cdot \omega \).

In general, the semiclassical duality transformations should not always be expected to correspond to exact duality symmetries of the quantum theories. However, since the potential remains invariant in the transformations of the form (4.10), it is natural to expect a correspondence between exact duality symmetries of the unperturbed \( G/U(1)^r \) coset CFT and the duality symmetries of the quantum HSG theories. The exact duality symmetries of WZW and coset models have been identified by Kiritsis in [12] by means

\(^5\)In the rather different context of gauge theories, eq. (6.3) can be recognised as the quantisation condition satisfied by the magnetic weights of monopoles, which has been solved long ago by Goddard, Nuyts and Olive in [33], where details about \( \Lambda^*_w(G) \) can be found. \( \Lambda^*_w(G) \) is the weight lattice of the ‘dual group’ to \( G \). In particular, if \( G \) is simply connected, in addition to semi-simple, compact and connected, \( \Lambda^*_w(G) \) is the co-root lattice of \( G \), denoted \( \Lambda^*_G(G) \) and defined as the integer span of the simple co-roots \( \vec{\alpha}^\vee_i = (2/\alpha_i^2)\vec{\alpha}_i \), where \( \vec{\alpha}_1, \ldots, \vec{\alpha}_r \) form a set of simple roots of \( G \). The group of automorphisms of \( \Lambda^*_G(G) \) is the semidirect product of the Weyl group of \( G \) and the group of automorphisms of the Dynkin diagram of \( G \) [34]. In more general cases where \( G \) is not simply connected, \( \Lambda^*_G(G) \) is always contained in \( \Lambda^*_w(G) \).
of the study of their partition function on the torus. Remarkably, for compact \( G/U(1)^r \)

cosets, the exact duality transformations are in one-to-one relation with the elements of
the Weyl group of \( G \), and they correspond to the transformations \( D_w \) constructed in the
previous paragraph. This leads to conjecture that, for each \( w \in \mathcal{W}(G) \), the semiclassical
T-duality transformation \( D_w \) provides an exact duality symmetry of the \( G/U(1)^r \) quantum
HSG models.

So far, the duality symmetries constructed in this section have been presented as
examples of transformations of the form (2.2), which relate two different dual sigma
models perturbed by the same potential. Remarkably, the transformations \( D_w \) associated
with the Weyl transformations can alternatively be written in the form (2.3), as duality
relations between two different perturbations of the same non-linear sigma model. In
order to make this explicit, let us indicate the dependence of (3.2) on \( T_+ \) and \( T_- \), the two
elements of the Cartan subalgebra \( h \) that specify the potential,

\[
S^{(\tau)}[h, A_\pm] \equiv S^{(\tau)}_{(T_+, T_-)}[h, A_\pm].
\]  

(6.9)

A Weyl transformation \( w \in \mathcal{W}(G) \) can always be lifted to an inner automorphism of \( G \),
which ensures the existence of a (non unique) constant group element \( \gamma_w \in G \) such that
\( w(u) = \gamma_w^{-1} u \gamma_w \) for all \( u \in h \). This leads to the identity

\[
S^{(\tau-w)}_{(T_+, T_-)}[h, A_\pm] = S^{(\tau)}_{(T_+, w(T_-))}[\gamma_w^{-1} h, w(A_\pm)].
\]  

(6.10)

Therefore, up to a change of the field variables, the duality transformation \( D_w \) becomes

\[
\begin{align*}
S^{(\tau)}_{(T_+, T_-)} & \xrightarrow{D_w} S^{(\tau-w)}_{(T_+, T_-)} \xrightarrow{h \mapsto \gamma_w h} \xrightarrow{A_\pm \mapsto w^{-1}(A_\pm)} S^{(\tau)}_{(T_+, w(T_-))},
\end{align*}
\]  

(6.11)

which is a duality relation of the form (2.3) where the potential transforms according to

\[
U \equiv \lambda \langle T_+ \mid h^{-1} T_- h \rangle \quad \mapsto \quad \tilde{U} \equiv \lambda \langle T_+ \mid h^{-1} w(T_-) h \rangle.
\]  

(6.12)

For the Complex sine-Gordon theory, which is recovered for \( G = SU(2) \), the only
non-trivial Weyl transformation is \( w = -I \). Then, (6.12) becomes simply \( U \rightarrow \tilde{U} = -U \).
This relates the two phases of the model, which are characterised by the sign of its unique
coupling constant. This case will be analysed in more detail in the next section.

6.2.1 T-duality and the phases of the HSG models.

For general HSG models, the transformations (6.12) also relate the different phases
of the model, which are in one-to-one relation with the elements of the Weyl group of \( G \).
This can be proved as follows.

First, we have to characterise the phases of the HSG models. This can be done by
studying the form of the manifold of vacuum field configurations, which correspond to the
minima of the the potential (3.4). A constant field configuration \( h_0 \) is a minimum of (3.4)
if it satisfies two conditions [2]. The first one is

\[
[T_+, h_0^{-1} T_- h_0] = 0,
\]  

(6.13)

which ensures that the potential is stationary at \( h_0 \). In the HSG models, the centralisers
of \( T_+ \) and \( T_- \) coincide with the Cartan subalgebra \( h \). Then, (6.13) implies that the inner
automorphism of \( g \) generated by \( h_0 \) leaves \( \mathfrak{h} \) fixed and, thus, it corresponds to a Weyl transformation of \( G \). Therefore, the solutions to (6.13) are of the form \( h_0 = \gamma_\sigma \), where \( \gamma_\sigma^{-1} u \gamma_\sigma = \sigma(u) \) for all \( u \in \mathfrak{h} \), and \( \sigma \in \mathcal{W}(G) \).

The second condition is needed to ensure that \( h_0 = \gamma_\sigma \) actually corresponds to a minimum of the potential, which is equivalent to require that all the small fluctuations around \( h_0 \) have real non-vanishing masses. Let \( T_\pm = \pm i\vec{\lambda}_\pm \cdot \vec{h} \), and write \( \gamma_\sigma^{-1} T_- \gamma_\sigma \equiv -i\sigma(\vec{\lambda}_-) \cdot \vec{h} \) using the conventions introduced just after (6.2). Then, the mass spectrum of the small fluctuations around \( h_0 = \gamma_\sigma \) is given by [2]

\[
m^2_{\vec{\alpha}} = 4\pi \lambda (\vec{\alpha} \cdot \vec{\lambda}_+) (\vec{\alpha} \cdot \sigma(\vec{\lambda}_-)) \tag{6.14}
\]

for each root \( \vec{\alpha} \) of \( g \). Thus, the condition that \( h_0 = \gamma_\sigma \) corresponds to a minimum is that all these numbers are strictly positive. If \( \lambda > 0 \), this requires that \( \vec{\lambda}_\pm \) and \( \sigma(\vec{\lambda}_-) \) are inside the same Weyl chamber of \( \mathfrak{h} \). Recall that the set of hyperplanes orthogonal to all the roots of \( G \) partitions the Euclidean space \( \mathbb{R}^{r} \) into disjoint connected components called Weyl chambers. The Weyl group of \( G \) permutes the Weyl chambers, so that each two chambers are related by a Weyl transformation. Once a system of simple roots \( \Delta = \lbrace \vec{\alpha}_1 \ldots \vec{\alpha}_r \rbrace \) is chosen, there is one Weyl chamber, denoted \( C(\Lambda) \), such that any \( \vec{\phi} \in C(\Lambda) \) satisfies \( \vec{\alpha}_i \cdot \vec{\phi} > 0 \) for all \( i = 1 \ldots r \). \( C(\Lambda) \) is called the ‘fundamental Weyl chamber’ [34]. We will choose the system of simple roots such that \( \vec{\lambda}_+ \in C(\Delta) \).

We will now argue that there is a different phase for each \( \sigma \in \mathcal{W}(G) \) that is specified by the domain where \( \vec{\lambda}_- \) takes values. In particular, the \( \sigma \)-phase corresponds to

\[
\vec{\lambda}_- \in \sigma^{-1}(C(\Delta)) , \tag{6.15}
\]

so that that \( \vec{\lambda}_- \) takes values in disjoint components of \( \mathbb{R}^{r} \) for different phases. This will be supported by showing that the form of the manifold of vacuum field configurations depends on \( \sigma \). By construction, the vacuum configurations in the \( \sigma \)-phase are of the form \( h_0 = \gamma_\sigma \), where \( \gamma_\sigma \) is specified by the condition \( \gamma_\sigma^{-1} u \gamma_\sigma = \sigma(u) \) for all \( u \in \mathfrak{h} \). Its general solution can be written as

\[
\gamma_\sigma = \tilde{\gamma}_\sigma e^v , \tag{6.16}
\]

where \( \tilde{\gamma}_\sigma \) is a fixed particular solution and \( v \) is any element of \( \mathfrak{h} \). This shows that the space of field configurations of this type is isomorphic to \( U(1)^r \). However, not all these configurations are physical. Some of them become identified under the action of the \( \tau \)-dependent group of gauge transformations (3.5). In particular,

\[
\tilde{\gamma}_\sigma \to e^v \tilde{\gamma}_\sigma e^{-\tau(v)} = \tilde{\gamma}_\sigma (\tilde{\gamma}_\sigma^{-1} e^v \tilde{\gamma}_\sigma) e^{-\tau(v)} = \tilde{\gamma}_\sigma e^{(\sigma-\tau)(v)} , \tag{6.17}
\]

which is not trivial for each \( v \in \mathfrak{h} \) such that \( (\sigma - \tau)(v) \neq 0 \). The rank of the linear transformation \( \sigma - \tau \) is the number of linear independent ‘\( v \)’ which are not in its kernel. Therefore, the set of field configurations of the form (6.16) that become identified with \( \tilde{\gamma}_\sigma \) under the group of gauge transformations is isomorphic to \( U(1)^{\text{rank}(\sigma-\tau)} \). This proves that the manifold of physical vacuum configurations is

\[
\lbrace h_0 \rbrace \simeq U(1)^{r-\text{rank}(\sigma-\tau)} \tag{6.18}
\]

which does depend on \( \sigma \), and justifies the proposed identification of the phases of the HSG models.
The comparison of (6.12) and (6.15) confirms that, for each \( w \in \mathcal{W}(G) \), the duality transformation \( D_w \) specified by (6.11) indeed provides a relationship between two different phases of the model; namely, it relates the \( \sigma \)-phase with the \( \sigma \cdot w^{-1} \)-phase.

Notice that (6.18) provides a condition for the existence of a phase where the vacuum of the model is not degenerate. It requires that \( \text{rank}(\sigma - \tau) = r \) for some \( \sigma \in \mathcal{W}(G) \). This clarifies the meaning of the condition deduced in [2] to ensure that the potential has no flat directions and, hence, that the theory has a mass gap. As an example where this condition is met, consider the HSG theories where the group of gauge transformations is of axial type, which corresponds to \( \tau = -I \). Then, the condition is trivially satisfied for \( \sigma = I \). In contrast, if \( -I \in \mathcal{W}(G) \), then \( \text{rank}(\sigma - \tau) = 0 \) for \( \sigma = -I \), and the vacuum of the theory is maximally degenerate in the corresponding phase.

7 T-duality in the CSG theory.

As a prototypical example, we now discuss in detail one of the simplest integrable theories that exhibits a duality symmetry of the form (2.3): the Complex sine-Gordon (CSG) model. It has two different phases corresponding to the two signs of its unique coupling constant, and they turn out to be related by T-duality. This duality symmetry was already pointed out by Bakas [7], and an explicit transformation rule was constructed by Park and Shin [35]. Nevertheless, the latter is only valid on-shell. In contrast, we will show that that the two phases are related off-shell by a canonical transformation between the phase spaces. The proper understanding of the T-duality symmetry helps to clarify the nature of the already known CSG soliton solutions. In one of the phases, due to the trivial vacuum structure, the soliton solutions are topologically trivial – they are of the form \([0, Q^N]\). Then, the duality map provides a topological interpretation for them in the other phase. This leads to the discovery of Bogomol’nyi-like bounds for the energy saturated by the usual one-soliton solutions in both phases which, to our knowledge, have been overlooked in the literature.

7.1 Basics of the CSG model.

The CSG model is defined by the relativistic two-dimensional Lagrangian

\[
\mathcal{L}_{\text{CSG}} = \frac{1}{4\pi \beta^2} \left( \frac{\partial_\mu \psi \partial^\mu \psi^*}{1 - \psi \psi^*} - \lambda \psi \psi^* \right),
\]

where \( \psi = \psi(t, x) \) is a complex scalar field, and \( \lambda \) and \( \beta \) are real coupling constants; \( \lambda \) is dimensionful, while \( \beta \) is dimensionless and plays no role in the classical theory. The Lagrangian is invariant under the global \( U(1) \) transformations \( \psi(t, x) \rightarrow e^{i\phi} \psi(t, x) \). A more convenient form of the CSG Lagrangian is obtained if we substitute \( \psi = \sin \eta e^{i\phi} \), with \( \eta \) and \( \phi \) real fields. Then, (7.1) becomes

\[
\mathcal{L}_{\text{CSG}} = \frac{1}{4\pi \beta^2} \left( \partial_\mu \eta \partial^\mu \eta + \tan^2 \eta \partial_\mu \phi \partial^\mu \phi - \lambda \sin^2 \eta \right) \equiv \mathcal{L}_{\text{CSG}}(\phi, \eta; \lambda),
\]

which allows one to make explicit the relationship with the usual sine-Gordon model: it is recovered just by taking the field \( \phi \) to be constant. Notice that both (7.1) and (7.2) are Lagrangians of the form (1.1).
The CSG model was originally introduced by Lund and Regge to describe relativistic vortices in a superfluid [5] and, independently, by Pohlmeyer as a reduction of the $O(4)$ non-linear sigma model [6]. The geometric interpretation of the model has been discussed by Bakas [7]. It depends crucially on the sign of $\lambda$, which manifests that the model has two different phases: one for $\lambda > 0$ and another one for $\lambda < 0$. More recently, the classical aspects of the CSG model in the presence of a boundary have been addressed by Bowcock and Tzamtzis [36].

At the quantum level, the CSG model has been investigated using both ordinary perturbation theory [37] and semiclassical techniques [38, 39]. However, the way to properly define the theory non-perturbatively follows from the work of Bakas, who showed at the classical level that the CSG model can be formulated in terms of a gauged WZW action [7] (see also [8]) so that the model is defined by an action of the form (3.2) associated with the coset $SU(2)/U(1)$. In this case, it is convenient to make the bosonic field $h$ take values in the fundamental representation of $SU(2)$, and to choose the bilinear form as $\langle u | v \rangle = \text{Tr}(uv)$. Then, if we call $\sigma_1, \sigma_2, \sigma_3$ the usual Pauli matrices, the generator of the $U(1)$ factor can be chosen to be $T_0 = i\sqrt{2} \sigma_3$, and the potential becomes

$$U(h) = +\frac{\lambda}{16\pi} \text{Tr} \left( h\sigma_3h^{-1}\sigma_3 \right). \quad (7.3)$$

Concerning the automorphism $\tau$, as explained in section 6.2 it has to belong to the group of automorphisms of the dual to the weight lattice of $SU(2)$, which coincides with the root lattice of $SU(2)$. This group is isomorphic to the cyclic group $\mathbb{Z}_2$ and, thus, there are only two choices. It can be either $\tau = +I$ or $\tau = -I$, which lead to $U(1)$ gauge transformations of vector or axial form, respectively. For this reason, it is convenient to introduce the notation

$$S^{\{+I\}}_{\text{CSG}} \equiv S^V_{\text{CSG}}, \quad S^{\{-I\}}_{\text{CSG}} \equiv S^A_{\text{CSG}}. \quad (7.4)$$

The connection between $S^{(\tau)}_{\text{CSG}}$ and the CSG Lagrangian $\mathcal{L}_{\text{CSG}}$ is recovered as follows. Consider the following parametrisation of the $SU(2)$ field $h$,

$$h = \begin{pmatrix} u & -iv^* \\ -iv & u^* \end{pmatrix}, \quad \text{with} \quad |u|^2 + |v|^2 = 1. \quad (7.5)$$

Then, integrating out the non-dynamical fields $A_\pm$ by means of their equations of motion, the two actions provided by (3.2) become

$$S^V_{\text{CSG}}[h, A_\pm; \lambda] = \frac{k}{4\pi} \int d^2x \left( \frac{\partial_\mu u \partial^\mu u^*}{1 - uu^*} - \lambda (uu^* - \frac{1}{2}) \right)$$

and

$$S^A_{\text{CSG}}[h, A_\pm; \lambda] = \frac{k}{4\pi} \int d^2x \left( \frac{\partial_\mu v \partial^\mu v^*}{1 - vv^*} + \lambda (vv^* - \frac{1}{2}) \right). \quad (7.6)$$

Up to a constant shift of the energy density, both of them correspond to the Lagrangian of the CSG model given by (7.1) if we identify the coupling constant $1/\beta^2$ with $k$, the level of the $SU(2)_k/U(1)$ coset CFT, and leave the sign of $\lambda$ free. It is worth noticing that, in this approach, the singularity of (7.1) at $|\psi| = 1$ comes from the elimination of $A_\pm$, but it does not correspond to a real singularity of the action (3.2).

At the quantum level, the action (3.2) for the coset $SU(2)/U(1)$ provides a Lagrangian description of the the theory of $\mathbb{Z}_k$ parafermions – the $SU(2)_k/U(1)$ coset conformal
field theory – perturbed by their first thermal operator, which is known to be quantum integrable [40]. To be specific, (3.2) is an action of the form (1.2) where, in this case, $S_{\text{CFT}}$ denotes an action for the $SU(2)/U(1)$ coset conformal field theory, with central charge $2(k - 1)/(k + 2)$, and the perturbation is defined by the spinless primary field corresponding to the first thermal operator, whose conformal dimension is $\Delta = 2/(k + 2)$.

It is important to stress that (7.6) gives a complete description of the perturbed gWZW model (3.2) only in the large $k$ limit, which corresponds to both the semiclassical and weak coupling limits. Therefore, the CSG model provides an explicit Lagrangian formulation of the theory of $Z_k$ parafermions perturbed by the first thermal operator when $k$ is large, provided that the CSG coupling constant $\beta^2$ is identified with $1/k$, which in this context is also required to make sense of the WZW action in (3.3). An independent motivation for the quantisation of the CSG coupling constant is provided by the semiclassical analysis of the CSG scattering amplitudes, where it appears as a condition to ensure that the theory admits a factorisable $S$-matrix [39].

In turn, the perturbed $Z_k$ parafermionic theory can be used as a non-perturbative definition of the CSG theory beyond the large $k$ limit. This integrable perturbed CFT develops a finite correlation length and is described by the minimal factorised $S$-matrix associated with the Lie algebra $a_{k-1}$. This is independent of the sign of its coupling constant [40], which confirms that the T-duality identification between the two phases of the model persists in the quantum theory, where it is a consequence of the order-disorder duality symmetry of the unperturbed $Z_k$ parafermionic theory [41, 7] (see also [42]).

### 7.2 Off-shell T-duality in the CSG model.

In this case the non-dynamical gauge fields have only one component, $A_\pm = a_\pm T^0$, and the automorphism defined in (4.8) is $\sigma_{T_0} = -I$. Therefore, the duality transformation (4.7) associated with the $U(1)$ global symmetry corresponding to $\psi(t, x) \rightarrow e^{i\alpha} \psi(t, x)$ reads

$$
(S_{\text{CSG}}^V)^D \left[ e^{\beta T^0} h_0 e^{\gamma T^0}, A_+, A_-; \lambda \right] = S_{\text{CSG}}^A \left[ e^{\beta T^0} h_0 e^{-\gamma T^0}, A_+, A_-; \lambda \right],
$$

where $A'_- = -A_- + 2\partial_\gamma T^0$, which, in terms of the Lagrangian (7.2), is equivalent to

$$
L_{\text{CSG}}(\phi, \eta; \lambda) \rightarrow L_{\text{CSG}}^D(\phi, \eta; -\lambda) = \frac{k}{4\pi} \left( \partial_\mu \eta \partial^\mu \eta + \cot^2 \eta \partial_\mu \phi \partial^\mu \phi - \lambda \sin^2 \eta \right),
$$

and we have already substituted the coupling constant $\beta^2$ by $1/k$. Notice that the potential $U = (k/4\pi) \lambda \sin^2 \eta$ does not change, which confirms that the duality transformation is of the form (2.2).

The dual Lagrangian (7.8) is related to the original one (7.2) by means of the trivial field transformation $\tilde{\eta} = \frac{\pi}{2} - \eta$ as follows

$$
L_{\text{CSG}}^D(\phi, \eta; \lambda) = L_{\text{CSG}}(\phi, \tilde{\eta}; -\lambda) - \frac{k}{4\pi} \lambda.
$$

Therefore, up to the constant shift of the energy density, the effect of the T-duality transformation is just to change the sign of the coupling constant $\lambda$ and, hence, to provide an

---

*The field $\phi$ introduced in (7.2) and used along section 7 is related to the parametrisation (3.11) by means of $\phi = (\beta + \gamma)/\sqrt{2}$. The canonical transformation that generates (7.7) reads $\Pi_\phi = (k/2\pi)\partial_x \phi^D$ and $\Pi_{\phi^D} = (k/2\pi)\partial_x \phi$, to be compared with (4.5).*
off-shell relationship between the two phases of the model. The resulting transformation is of the form (2.3) with \( \hat{U} = -U \).

In the gauged WZW formulation, the transformation (7.9) corresponds to (6.10) that, in this case, reads

\[
S_{CSG}^V [h, A_+, A_-; \lambda] = S_{CSG}^A [i\sigma_1 h, -A_+, -A_-; -\lambda].
\]  

(7.10)

Actually, it was already pointed out by Bakas [7] that this is a manifestation of the symmetry under the order-disorder (Kramers-Wannier) duality transformation of the theory of \( \mathbb{Z}_k \) parafermions, in the perturbed conformal field theory language. Indeed, according to (7.6), \( S_{CSG}^V \) provides a description of the theory in terms of \( u \), which is the gauge invariant component of the WZW field \( h \) in (7.5) with respect to vector gauge transformations, while the elementary field in \( S_{CSG}^A \) is \( v \), the gauge invariant component with respect to axial gauge transformations. In the context of the theory of parafermions [41, 7], \( u \) and \( v \) represent the spin, \( \hat{\sigma}_1 = \phi_{1,1}^{(1)} \), and dual-spin, \( \hat{\mu}_1 = \phi_{1,-1}^{(1)} \), fields, respectively. They are the diagonal and off-diagonal components of the WZW field in the fundamental representation of \( SU(2) \), whose conformal dimension is \( \Delta = \overline{\Delta} = (k - 1)/2k(k + 2) \). Similarly, the composite field \( \text{Tr} (h a_3 h^{-1} \sigma_3) = 2 (u u^* - v v^*) \) represents the first thermal operator \( \phi_0^{(2)} \), whose conformal dimension is \( 2/(k + 2) \). This way, the duality transformation (7.10) corresponds simply to \( u = \hat{\sigma}_1 \leftrightarrow \hat{\mu}_1 = v \).

### 7.3 On-shell T-Duality and CSG soliton solutions.

As explained in section 5, on shell, eqs. (4.5) and (5.1) provide a map between the solutions to the equations of motion of (7.1),

\[
\partial_\mu \partial^\mu \psi + \psi^* \frac{\partial_\mu \psi \partial^\mu \psi}{1 - \psi \psi^*} + \lambda \psi (1 - \psi \psi^*) = 0,
\]  

(7.11)

in the two phases of the model. Indeed, if we denote a generic solution of (7.11) by \( \psi^{(\lambda)} \), the duality transformation \( (\phi, \eta; \lambda) \rightarrow (\phi^D, \frac{\pi}{2} - \eta; -\lambda) \) corresponds to

\[
\psi^{(\lambda)} = \sin \eta \, e^{i\phi} \longrightarrow \psi^{(-\lambda)} = \cos \eta \, e^{i\phi^D},
\]  

(7.12)

where \( \phi^D \) is the solution to the equations of the canonical transformation understood as partial differential equations for \( \phi^D \),

\[
\partial_t \phi^D = + \tan^2 \eta \, \partial_t \phi, \quad \partial_t \phi^D = + \tan^2 \eta \, \partial_2 \phi.
\]  

(7.13)

This on-shell duality, or pseudoduality, transformation coincides with the transformation constructed by Park and Shin within the gauged WZW formulation of the CSG model using a particular choice of the gauge fixing prescription [35]. There, (7.12) reads simply \( h \rightarrow i\sigma_1 h \), where \( h \) is a solution to the equations of motion of (3.2) with the gauge fixed by the condition \( A_\pm = 0 \). Using the parametrisation (7.5), it is equivalent to interchange the roles of \( u \) and \( v \) and simultaneously change the sign of \( \lambda \).

\footnote{The equations (7.13) were originally written in [43, section 7] as transformations that change the boundary conditions satisfied by the soliton solutions, without making any reference to duality.}
The $U(1)$ global symmetry of the CSG Lagrangian (7.1) leads to the conserved Noether current
\[ J^N_\mu = \frac{k}{4\pi} \left( \psi \frac{\partial_\mu \psi^* - \psi^* \partial_\mu \psi}{1 - \psi \psi^*} \right) = \frac{k}{2\pi} \tan^2 \eta \partial_\mu \phi . \] (7.14)

As explained in section 5, there is also a topological current associated with the field $\phi$, which, in this case, is convenient to define as
\[ J^T_\mu = -\frac{1}{2} \varepsilon_{\mu\nu} \partial^\nu \phi , \quad \varepsilon_{01} = +1 . \] (7.15)

Then, the eqs. (7.13) can simply be written as
\[ J^{(D)T}_\mu = \frac{\pi}{k} J^N_\mu , \quad J^T_\mu = \frac{\pi}{k} J^{(D)N}_\mu . \] (7.16)

Let us briefly review the main features of the already known soliton solutions of the CSG model from the perspective of eqs. (7.16). First, consider the phase $\lambda > 0$, where the coupling constant $\lambda$ can be properly understood as a squared-mass parameter. Here, the potential of the CSG model has a unique minimum at $\psi = 0$, and the boundary conditions to be satisfied by the soliton solutions are
\[ \psi(t, x) \xrightarrow{x \to \pm \infty} 0 . \] (7.17)

Therefore, in this phase, they carry a non-trivial $U(1)$ Noether charge but no topological charge. One-soliton solutions of this kind were originally constructed by Getmanov [45]. In their rest frame, they are given by periodic time-dependent field configurations rotating in the internal $U(1)$ space of the form
\[ \psi(t, x) = \frac{\cos \alpha}{\cosh(\sqrt{\lambda} \cos \alpha x)} \exp(i\sqrt{\lambda} \sin \alpha t) , \] (7.18)

and there is a different one for each value of the real parameter $\alpha \in (-\pi/2, +\pi/2)$. The classical $U(1)$ Noether charge and mass carried by this solution are given by
\[ Q^N = \frac{k}{\pi} \left( \text{sign}[\alpha] \frac{\pi}{2} - \alpha \right) , \quad M = \frac{k}{\pi} \sqrt{\lambda} \cos \alpha = \frac{k}{\pi} \sqrt{\lambda} \sin (\pi Q^N/k) . \] (7.19)

Using the notation introduced in section 5, these solitons are labelled by $[\omega, Q^N] \equiv [0, Q^N]^{(+)1}$, where the superscript $(+)$ indicates that they are solutions to the equations of motion for $\lambda > 0$.

In contrast, since $|\psi|^2 \leq 1$, the minima of the potential in the phase $\lambda < 0$ correspond to $|\psi| = 1$, and there is an infinite number of them related to each other by the global $U(1)$ symmetry. In their rest frame, the soliton solutions interpolating between these minima are time-independent field configurations that satisfy boundary conditions of the form
\[ \psi(t, x) \xrightarrow{x \to \pm \infty} e^{i\phi_{\pm}} \] (7.20)

and, thus, carry the topological charge
\[ Q^T = \int_{-\infty}^{+\infty} dx \, J^T_0 = \frac{\phi_+ - \phi_-}{2} . \] (7.21)

*We will only consider relativistic solitons defined in 1+1 Minkowski space. Solutions of the CSG equation in Euclidean 0+2 space have been constructed in [44].
Solutions of this kind were constructed by Lund and Regge [5],

$$\psi(t,x) = i e^{i \phi} \frac{e^x}{\sqrt{1-x^2}} \left( \sin Q^T \tanh(\sqrt{-\lambda} |\sin Q^T| x) - i \cos Q^T \right).$$

(7.22)

Their mass is given by

$$M(Q^T) = \frac{k}{\pi} \sqrt{|\lambda| \sin Q^T}$$

(7.23)

and their $U(1)$ Noether charge vanishes. Using the notation introduced in section 5, these solitons are labelled by $[\omega, Q^N] \equiv [2Q^T, 0]^{(-)}$, where $^{(-)}$ indicates that they are solutions for $\lambda < 0$.

T-duality relates the soliton solutions in both phases. Namely, if $(k/\pi) Q^T$ equals the Noether charge carried by the solution (7.18), then (7.22) is mapped into (7.18) under the pseudoduality transformation (7.12),

$$[0, Q^N]^{(+)} \xrightarrow{T-duality} [(2\pi/k) Q^N, 0]^{(-)}.$$  (7.24)

As explained in section 5, the fact that $\phi$ is a compact field implies, through the duality transformation, that the value of the Noether charge $Q^N$ is defined modulo some period characterised by $k$. In our case, $\phi \sim \phi + 2\pi \mathbb{Z}$ translates into $Q^N \sim Q^N + k\mathbb{Z}$ which, in particular, resolves the apparent discontinuous dependence of $Q^N$ on the value of $\alpha$ in (7.19). Correspondingly, $Q^T \sim Q^T + \pi \mathbb{Z}$.

In the $\lambda > 0$ phase, solitons are obviously not topological in nature, and their Noether charge $Q^N$ can take any real value, which makes their stability unclear [35, 36]. In the other phase, $\lambda < 0$, the situation is similar because, as explained in section 5, the conservation of $Q^T$ does not rely on topological arguments and this charge can also take any real value. The classical stability of the CSG solitons will be clarified in the next section by showing that they saturate Bogomol’nyi-like bounds for the energy.

In the quantum theory, the situation is different. Under quantisation, the Noether charge carried by the solitons in the $\lambda > 0$ phase becomes quantised. The precise form of this quantisation was found in [38, 39] by applying the Bohr-Sommerfeld (BS) quantisation rule to the periodic soliton configurations provided by (7.18); i.e., $S + MT = 2\pi n$ where $n$ is a positive integer, $T$ is the period of the soliton solution, $M$ is its mass, and $S$ is its action. For the CSG model, $S + MT = 2\pi Q^N$, and the BS rule implies that $Q^N$ has to be integer. However, we have already shown that $Q^N$ is only defined modulo $k$, namely $Q^N \sim Q^N + k\mathbb{Z}$. This is consistent with the built-in ambiguity in the definition of the WZW action in (3.2) and, hence, in the combination $S + MT$, which is precisely of the form $2\pi k \mathbb{Z}$ with our normalisations. Therefore, the semiclassical quantisation of the solitons in the $\lambda > 0$ phase provides exactly $k-1$ non-topological stable solitons with $U(1)$ Noether (electric) charges

$$Q^N = n, \quad n = 1 \ldots k - 1 \mod k.$$  (7.25)

Correspondingly, via the T-duality transformation, the quantisation of the Noether charge carried by the solitons in the $\lambda > 0$ phase implies the quantisation of the topological (magnetic) charge carried by the dual solitons. Taking (7.24) into account, the resulting allowed values of the topological charge are

$$Q^T = \frac{\pi}{k} n, \quad n = 1 \ldots k - 1 \mod k.$$  (7.26)
In other words, and using (7.20), the CSG fields corresponding to the quantum soliton solutions in the \( \lambda < 0 \) phase satisfy the parafermionic boundary conditions

\[
\psi(t, +\infty) = \exp \left( i \frac{2\pi}{k} n \right) \psi(t, -\infty),
\]

which is in agreement with the well known breaking of the classical global \( U(1) \) symmetry into a discrete \( \mathbb{Z}_k \) symmetry after quantisation [46]. It is worth noticing that, in both phases, the resulting quantum spectrum (7.25) and (7.26), and the corresponding mass formulae (7.19) and (7.23), match the spectrum of stable particles of the minimal factorised \( S \)-matrix theory associated with the Lie algebra \( a_{k-1} \), which describes the integrable theory of \( \mathbb{Z}_k \) parafermions perturbed by the first thermal operator [40].

### 7.4 Bogomol’nyi-like bounds in the CSG model.

The results of section 5 suggests a topological interpretation of the one-soliton solutions in the \( \lambda < 0 \) phase, in the sense that they are characterised by the boundary values of the CSG field. This interpretation is supported by the fact that they actually saturate a Bogomol’nyi-like bound, which can be deduced as follows. The energy density corresponding to a time independent field configuration \( \psi = \psi(x) \) is

\[
\mathcal{H}_{CSG} = \frac{k}{4\pi} \left( \frac{\partial_x \psi \partial_x \psi^*}{1 - \psi \psi^*} - \lambda (1 - \psi \psi^*) \right) + \frac{k}{4\pi} \lambda,
\]

which can be written as

\[
\mathcal{H}_{CSG} = \frac{k}{4\pi} \left( \frac{\partial_x \psi - ie^{i\Omega} \sqrt{-\lambda} (1 - \psi \psi^*)^2}{1 - \psi \psi^*} + 2\sqrt{-\lambda} \partial_x \text{Im}(e^{-i\Omega} \psi) \right) + \frac{k}{4\pi} \lambda,
\]

where \( \Omega \) is an arbitrary constant real number. Taking the boundary conditions (7.20) into account, and recalling that \( |\psi|^2 \leq 1 \), eq. (7.29) leads to

\[
M^{(-)} = \int_{-\infty}^{+\infty} dx \left( \mathcal{H}_{CSG} - \frac{k}{4\pi} \lambda \right) \geq \frac{k}{2\pi} \sqrt{-\lambda} \left( \sin(\phi_+ - \Omega) - \sin(\phi_- - \Omega) \right),
\]

where the superscript \(^{(-)}\) indicates that \( \lambda < 0 \). The most stringent bound is achieved by choosing \( e^{i\Omega} = \text{sign} \left[ \sin Q^T \right] e^{i \phi^+_+ + \phi^-_- / 2} \), which leads to the Bogomol’nyi-like bound

\[
M^{(-)} \geq \frac{k}{\pi} \sqrt{|\lambda|} \left| \sin Q^T \right|.
\]

For each value of \( Q^T \), this bound is saturated by the solutions to the first order equation

\[
\partial_x \psi = i\sqrt{-\lambda} \text{ sign } \left[ \sin Q^T \right] e^{i \phi^+_+ + \phi^-_- / 2} (1 - \psi \psi^*),
\]

which yields the one-soliton solution (7.22). Its mass is given by (7.23), which is clearly fixed by the charge \( Q^T = (\phi_+ - \phi_-) / 2 \).

The duality relation (7.24) makes natural to ask whether there is an analogue of the bound (7.31) in the \( \lambda > 0 \) phase. We show below that the answer to this question is affirmative, and that the one-soliton solutions (7.18) saturate another bound characterised
by their Noether charge, which provides a novel interpretation for them as two-dimensional examples of Coleman’s Q-balls [47]. This bound can be deduced as follows.

We start with the energy density corresponding to a generic field configuration \( \psi = \psi(t, x) \)

\[
H_{\text{CSG}} = \frac{k}{4\pi} \left( \frac{\partial_t \psi \partial_t \psi^* + \partial_x \psi \partial_x \psi^*}{1 - \psi \psi^*} + \lambda \psi \psi^* \right). 
\]

Using \( x_\pm = t \pm x \) and \( \partial_\pm = \partial / \partial x_\pm \), it can be written as

\[
H_{\text{CSG}} = \frac{k}{8\pi} \left( \frac{\left| 2 \partial_+ \psi - e^{i\Gamma} \sqrt{\lambda} \psi \sqrt{1 - \psi \psi^*} \right|^2 + \left| 2 \partial_- \psi + e^{-i\Gamma} \sqrt{\lambda} \psi \sqrt{1 - \psi \psi^*} \right|^2}{1 - \psi \psi^*} \right.
\]

\[
+ 4\sqrt{\lambda} \Re \left( e^{i\Gamma} \frac{\psi \partial_+ \psi^* - \psi^* \partial_- \psi}{\sqrt{1 - \psi \psi^*}} \right), 
\]

which leads to

\[
E^{(+)} = \int_{-\infty}^{+\infty} dx \ H_{\text{CSG}} \geq \frac{k}{2\pi} \sqrt{\lambda} \int_{-\infty}^{+\infty} dx \ \Re \left( e^{i\Gamma} \frac{\psi \partial_+ \psi^* - \psi^* \partial_- \psi}{\sqrt{1 - \psi \psi^*}} \right), 
\]

for an arbitrary function \( \Gamma = \Gamma(t, x) \). Consider the choice where \( \Gamma \) is a solution to the equations

\[
\partial_\pm \Gamma = \pm \frac{i}{2} \left( \frac{\psi \partial_\pm \psi^* - \psi^* \partial_\pm \psi}{1 - \psi \psi^*} \right) = \pm \frac{2\pi}{k} J_\pm^N, 
\]

which are integrable because the Noether current (7.14) is conserved. This way, and taking the boundary conditions (7.17) into account, eq. (7.35) becomes

\[
E^{(+)} \geq -\frac{k}{2\pi} \sqrt{\lambda} \int_{-\infty}^{+\infty} dx \ \partial_x \Re \left( e^{i\Gamma} \sqrt{1 - \psi \psi^*} \right)
\]

\[
= -\frac{k}{2\pi} \sqrt{\lambda} \left( \cos \Gamma(t, +\infty) - \cos \Gamma(t, -\infty) \right). 
\]

Notice that \( \partial_t \Gamma = (2\pi/k)J_0^N \), which means

\[
\Gamma(t, +\infty) - \Gamma(t, -\infty) = \frac{2\pi}{k} \int_{-\infty}^{+\infty} dx \ J_0^N = \frac{2\pi}{k} Q^N; 
\]

otherwise the value of \( \Gamma(t, \pm\infty) \) is arbitrary. Therefore, \( E^{(+)} \) is actually bounded below by the maximal value of the right-hand-side of (7.37), which is attained for \( \Gamma(t, \pm\infty) = \pm \pi Q^N / k - \pi \text{sign} \[\sin(\pi Q^N / k)\] / 2 \). This finally leads to the Bogomol’nyi-like bound we were looking for

\[
E^{(+)} \geq \frac{k}{\pi} \sqrt{\lambda} \left| \sin \left( \frac{\pi Q^N}{k} \right) \right|. 
\]

For each value of the Noether charge \( Q^N \), the bound (7.39) is saturated by the solutions to the equations

\[
2 \partial_\pm \psi = \pm e^{\pm i\Gamma} \sqrt{\lambda} \psi \sqrt{1 - \psi \psi^*}, 
\]

where \( \Gamma \) is the solution to (7.36) with the boundary values specified just before (7.39). This yields the one-solitons (7.18) originally constructed by Getmanov, whose mass is given by (7.19), which is indeed fixed by the Noether charge.
The Bogomol’nyi-like bounds deduced in the previous paragraphs provide a novel alternative characterisation of the previously known one-solitons solutions (7.22) and (7.18) that makes their classical stability explicit. The reason is that the solutions whose energy saturates a bound like (7.31) or (7.39) are, in general, below threshold for decay into a multi-particle state of the same charge $Q^T$ or $Q^N$, respectively. In principle, they could still be at threshold for decay in other states that saturate the same bound. However, say for (7.31) and (7.23), this could only occur if

$$Q^T = Q_1^T + Q_2^T \quad \text{and} \quad M \left( Q_1^T + Q_2^T \right) = M \left( Q_1^T \right) + M \left( Q_2^T \right) \quad (7.41)$$

but, taking into account the identification $Q^T \sim Q^T + \pi Z$ deduced just after (7.24), it is straightforward to check that the only solution to these conditions is the trivial one where either $Q_1^T$ or $Q_2^T$ vanishes. Therefore, the solutions that saturate the bound in each phase, which in this case are the one-soliton solutions (7.18) and (7.22), are indeed classically stable.

Finally, let us point out that the first order partial differential equations (7.32) and (7.40) satisfied by the configurations that saturate the Bogomol’nyi-like bounds (7.31) and (7.39) coincide with the Bäcklund equations used in [35] to construct the one-soliton solutions.

8 Conclusions

We have studied T-duality in a family of massive integrable field theories associated with cosets of the form $G/(H \times U(1))$ whose classical equations of motion are particular examples of the non-abelian affine Toda equations of Leznov and Saveliev [23]. Among others, the family includes the complex sine-Gordon model [5, 6, 7], the Homogeneous and Symmetric Space sine-Gordon models constructed in [2, 3, 4, 9], and the models associated with cosets of the form $SL(2) \times U(1)^n/U(1)$ considered in [10]. For each coset, the different theories are defined in terms of an asymmetric gauged WZW action [25, 26] perturbed by a potential. Following [2], the different gauged left and right actions of $H \times U(1)$ on $G$ are labelled by an automorphism $\tau$ of $H \times U(1)$.

Our main result is that all these theories exhibit off-shell abelian T-duality symmetries. For each $U(1)$ generator in $H \times U(1)$, there is a duality transformation given by eqs. (4.7) and (4.10) that relates two different theories associated with the same coset. These theories admit an equivalent description in terms of massive non-linear sigma models. Then, the form of the duality transformations is summarised by eq. (2.2), which exhibits that the potential remains invariant, and that the transformation is actually a consequence of the duality symmetries of the unperturbed theory.

In some cases, the two dual unperturbed theories coincide, their Lagrangian actions are related by a change of field variables, and this change modifies the potential. Then, the duality transformation has a different interpretation. It relates two perturbations of the same self-dual conformal field theory by different potentials. The form of this kind of duality transformations is summarised by eq. (2.3).

Explicit examples of this type are provided by the HSG models, which are associated to cosets of the form $G/U(1)^r$, where $G$ is a compact simple Lie group. In this case, our results show that they exhibit a duality transformation of the form (2.3) for each Weyl transformation of $G$ given by (6.11), where the potential transforms according to (6.12).
These transformations relate different phases of the models, which we have also characterised by studying the manifold of vacuum field configurations. They are in one-to-one relation with the elements of the Weyl group of $G$. We also conjecture that these duality symmetries survive in the quantum theory, which would be interesting to check against the available non-perturbative results for the HSG models; in particular, those recently obtained by means of the thermodynamic Bethe ansatz in [31].

On-shell, the T-duality transformations provide a map between the solutions to the classical equations of motion of the dual models. Since they exhibit, at least, a global $U(1)$ symmetry leading to a conserved Noether current, their equations of motion have soliton solutions that carry a finite value of the corresponding $U(1)$ charge. Restricted to them, the duality map turns Noether soliton charges into topological ones, which is reminiscent of the string theory case, where T-duality changes trivial boundary conditions into non-trivial ones.

We have studied in detail this transformation in the particular case of the CSG model, which corresponds to the coset $SU(2)/U(1)$. It has two different phases corresponding to the two signs of its unique coupling constant, and solitons are known to be not of topological nature in one of them. Then, the duality transformation suggests a topological interpretation for them in the dual phase. Although the conservation of the relevant topological charge is not truly of topological nature, this interpretation leads to a previously unreported Bogomol’nyi-like bound for the energy that is saturated by these solitons. In the other, non-topological, phase, we also show that the already known one-soliton solutions saturate another Bogomol’nyi-like bound, which provides a novel interpretation for them as Q-balls [47], and makes their classical stability explicit.

The HSG models can be seen as generalisations of the CSG model. In fact, the one-solitons solutions of the $G/U(1)^r$ HSG model have been constructed in [48] by embedding the non-topological $SU(2)$ CSG solitons into the regular $SU(2)$ subgroups of $G$. Taking into account the description of the phases of the HSG models presented in section 6.2, the construction of [48] was performed in a phase where the vacuum configuration is not degenerate, similar to the non-topological phase of the CSG model. Therefore, the resulting solitons are not topological, and their classical stability is not clearly established. In spite of this, their properties determine the quantum HSG theories constructed in [4], and it would be extremely interesting to extend the results achieved in the CSG model to this case. In particular, to investigate the possible existence of Bogomol’nyi-like bounds saturated by the HSG solitons which could clarify their stability.

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