Speeding up the Euler scheme for killed diffusions

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Outline of talk

1. Background on linear diffusions
2. Recurrent transformations
3. Explicit Euler-Maruyama schemes for diffusions
4. A new Euler-Maruyama scheme for killed diffusions
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Background on linear diffusions

Recurrent transformations

Explicit Euler-Maruyama schemes for diffusions

A new Euler-Maruyama scheme for killed diffusions

Numerical experiments
Setup

- $X$ is a regular diffusion on $(\ell, r) \subset \mathbb{R}$ adapted to right continuous $(\mathcal{F}_t)_{t \geq 0}$. $\ell$ and $r$ are allowed to be infinite.
- If any of the boundaries are reached in finite time, the process is killed and sent to the cemetery state, $\Delta$, i.e no reflecting boundaries.
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$$\zeta := \inf\{t \geq 0 : X_t = \Delta\}.$$

- We shall write $E^x[Y]$ to denote expectation of $Y$ with respect to $P^x$. Recall that Markov property means $E^x[f(X_{t+s})|\mathcal{F}_t] = E^{X_t}[f(X_s)]$ while the strong Markov property amounts to $E^x[f(X_{T+s})|\mathcal{F}_T] = E^{X_T}[f(X_s)]$ for all stopping times.
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  while the strong Markov property amounts to
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  for all stopping times.
- Diffusion assumption entails $X$ is continuous and strong Markov, while the regularity amounts to $P^x(T_y < \infty) > 0$ whenever $x$ and $y$ belongs to the open interval $(\ell, r)$, where
  \[ T_y := \inf \{ t > 0 : X_t = y \} \text{ for } y \in (\ell, r). \]
Examples

1. Brownian motion living on an interval \((a, b)\) and killed as soon as it reaches the boundary.

2. \(\delta\)-dimensional Bessel process on \((0, \infty)\).

3. Solution of a stochastic differential equation:

\[
X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \quad t < \zeta, \quad (1)
\]

where \(\zeta := \inf\{t \geq 0 : X_t \in \{l, r\}\}\).
- \( X \) is completely determined by its \textit{scale function}, \( s \), and speed measure, \( m \): \( s \) is any increasing function such that \( s(X) \) is a local martingale, which leads to

\[
P_x(X(T) < T_a < T_b) = s(b) - s(x) - s(a), \quad l < a < b < r
\]

Af = df \, ds, \quad f \in D(A),

\[
Pt_f(x) := E_x[f(X_t)] = \int_{l}^{r} f(y) p(t, x, y) m(dy)
\]

for any non-negative \( f \) that vanishes on the boundary, where \( p \) corresponds to the transition density with respect to \( m \). If \( p \) is symmetric, i.e., \( p(t, x, y) = p(t, y, x) \), \( X \) is recurrent iff \( -s(l) = s(r) = \infty \). Otherwise, it is called transient.
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P^x(T_a < T_b) = \frac{s(b) - s(x)}{s(b) - s(a)}, \quad l < a < b < r
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Af = \frac{d}{dm} \frac{df}{ds}, \quad f \in \mathcal{D}(A),
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X is completely determined by its *scale function*, s, and speed measure, m: s is any increasing function such that s(X) is a local martingale, which leads to

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More on transience

- A consequence (in fact another formulation) of transience is that for any continuous $f$ compactly supported in $(l, r)$

$$E^x \int_0^\xi f(X_t)dt < \infty.$$  

- Consequently, there exists a potential kernel, $u$, such that

$$E^x \int_0^\xi f(X_t)dt = \int_l^r u(x, y)f(y)m(dy).$$

Moreover, $u$ is continuous and for $x \leq y$,

$$u(x, y) = u(y, x) = \frac{(s(x) - s(l))(s(r) - s(y))}{s(r) - s(l)} \leq u(y, y).$$

In particular,

$$P^x(T_y < \infty) = \frac{u(x, y)}{u(y, y)}.$$
Recurrent transformations

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Barrier option simulation
Recurrent transformations (Çetin (2018))

**Definition 1**

Let $X$ be a regular diffusion satisfying (1) and $h : (l, r) \mapsto (0, \infty)$ be a continuous function. $(h, M)$ is said to be a recurrent transform (of $X$) if the following are satisfied:

1. $M$ is an adapted process of finite variation.
2. $h(X)M$ is a nonnegative local martingale.
3. There exists a unique weak solution to
   \[
   X_t = x + \int_0^t \sigma(X_s) dB_s + \int_0^t \left\{ b(X_s) + \sigma^2(X_s) \frac{h'(X_s)}{h(X_s)} \right\} ds.
   \] (2)
4. The (scale) function $s_r$ is finite for all $x \in (l, r)$ with $-s_r(l+) = s_r(r-) = \infty$, where
   \[
   s_r(x) := \int_c^x \frac{s'(y)}{h^2(y)} dy, \quad x \in (l, r),
   \] (3)
Suppose $X$ is transient, let $y \in (l, r)$ be fixed and consider

$$h(x) := u(x, y), \ x \in (l, r), \ \text{and} \ M_t = \exp \left( \frac{s'(y) L^y_t}{2u(y, y)} \right).$$
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Then, $(h, M)$ is a recurrent transform of $X$. In particular, there exists a non-explosive unique weak solution to

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t \left\{ b(X_s) + \sigma^2(X_s) \frac{u_x(X_s, y)}{u(X_s, y)} \right\} ds.$$ \hspace{1cm} (4)

If $R^{h,x}$ denotes the law of the solution and $T$ is an $R^{h,x}$-a.s. finite stopping time, then for any $F \in \mathcal{F}_T$

$$P^x(\zeta > T, F) = u(x, y)E^{h,x} \left[ 1_F \frac{1}{u(X_T, y)} \exp \left( - \frac{s'(y)}{2u(y, y)} L^y_T \right) \right].$$ \hspace{1cm} (5)
Suppose \( X \) is transient, let \( y \in (l, r) \) be fixed and consider

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h(x) := u(x, y), \quad x \in (l, r), \text{ and } M_t = \exp\left(\frac{s'(y)L^{y}_t}{2u(y, y)}\right).
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\]
Suppose $X$ is a Brownian motion that is killed when hitting 0 or 1. Then,

$$u(x, y) = x(1 - y), \ 0 < x \leq y < 1.$$  

Thus, if we apply the recurrent transform from Example 1 with $y = 1/2$, we obtain the following SDE:

$$dX_t = dB_t + \left\{ \frac{1}{X_t} 1_{[X_t \in (0, \frac{1}{2})]} - \frac{1}{1 - X_t} 1_{[X_t \in (\frac{1}{2}, 1)]} \right\} dt.$$  

Recall that the recurrent transformation implies that the solution to the above SDE never hits 0 or 1, which is also clear from the SDE representation.
Let $\mu$ be a Borel probability measure on $(l, r)$ such that
$$\int_{(l, r)} |s(y)| \mu(dy) < \infty.$$ Suppose $X$ is transient and define
$$h(x) := \int_{(l, r)} u(x, y) \mu(dy).$$

$(h, M)$ is a recurrent transform of $X$, where
$$M_t := \exp \left( \int_0^t \frac{1}{h(X_s)} dA_s \right) \quad \text{and} \quad A_t := \int_{(l, r)} \frac{s'(x)L^x_t}{2} \mu(dx).$$

If $R^{h,x}$ denotes the law of the solution of (2) and $T$ is a stopping time such that $R^{h,x}(T < \infty) = 1$, then for any $F \in \mathcal{F}_T$
$$P^x(\zeta > T, F) = h(x) E^{h,x} \left[ 1_F \frac{1}{h(X_T)} \exp \left( - \int_0^t \frac{1}{h(X_s)} dA_s \right) \right],$$
where $E^{h,x}$ is the expectation operator with respect to the probability measure $R^{h,x}$. 
Explicit Euler-Maruyama schemes for diffusions
Suppose the solution of (1) has infinite lifetime and we are interested in $E^x[g(X_T)]$ for some bounded function $g$. 

Then, an approximation of $E^x[g(X_T)]$ is found by averaging $g(X^N_{t_n})$ over a sufficiently large number of simulations.
Suppose the solution of (1) has infinite lifetime and we are interested in $E^x[g(X_T)]$ for some bounded function $g$.

We in general don’t know the transition density explicitly, so we must resort to some approximation algorithms.
The case of no-killing

- Suppose the solution of (1) has infinite lifetime and we are interested in $E^x[g(X_T)]$ for some bounded function $g$.
- We in general don’t know the transition density explicitly, so we must resort to some approximation algorithms.
- The most popular and straightforward algorithm is the *explicit* Euler-Maruyama scheme:

$$X_{t_n}^N = X_{t_{n-1}}^N + b(X_{t_{n-1}}^N) \frac{T}{N} + \sigma(X_{t_{n-1}}^N)(B_{t_n} - B_{t_{n-1}}), \quad t_n = \frac{nT}{N}, \ n \in \{0, N\}, \ X_0^N = x. \quad (6)$$

Then, an approximation of $E^x[g(X_T)]$ is found by averaging $g(X_{t_n}^N)$ over a sufficiently large number of simulations.
A relevant question in above algorithm is ‘how fine do we need to discretize in order to get a ‘negligible’ error for practical purposes?’

Note that $N$ is the number of discretizations and the ‘weak error’ is given by

$$e(T) = E^x[g(X_{t_N}^N)] - E^x[g(X_T)].$$
Convergence rate for the explicit scheme

- A relevant question in above algorithm is ‘how fine do we need to discretize in order to get a ‘negligible’ error for practical purposes?’
- Note that $N$ is the number of discretizations and the ‘weak error’ is given by

$$e(T) = E^x[g(X_{t_N}^{N})] - E^x[g(X_T)].$$

- Under some regularity conditions on the coefficients of the SDE and $g$, there exists a bounded function $u$ with bounded derivatives such that $E^x[g(X_T)|\mathcal{F}_t] = u(t, X_t)$. In particular, $u(T, x) = g(x)$ and

$$u_t + bu_x + \frac{1}{2}\sigma^2 u_{xx} = 0.$$
Moreover,

\[ e(T) = E^x[u(T, X_T^N)] - E^x[u(T, X_T)] = E^x[u(T, X_T^N)] - u(0, x) \]

\[ = \sum_{n=0}^{N-1} E^x[u(t_{n+1}, X_{t_{n+1}}^N) - u(t_n, X_{t_n}^N)] \]

With the help of Ito's formula, the regularity conditions imply

\[ \left| E^x[u(t_{n+1}, X_{t_{n+1}}^N) - u(t_n, X_{t_n}^N)] \right| \leq \frac{K}{N^2}, \]

where \( K \) is a constant independent of \( N \).
Moreover,

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where \( K \) is a constant independent of \( N \).

Therefore, \( |e(T)| \sim O\left(\frac{1}{N}\right) \).
The case of killed diffusions

The above breaks down if the lifetime is not infinite and we are interested in $E^x[g(X_T)1_{[T<\zeta]}]$, e.g. the price of a barrier option.
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- The reason is that (6) produces a process that can exit $(\ell, r)$, and the most straightforward explicit scheme would be $E^x[g(X^N_T)1_{T<\zeta_N}]$, where $\zeta_N$ is the first time that the discretized process exits $(\ell, r)$. 
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- Again one can find a function \( v \) vanishing at the accessible boundaries such that such that \( e(T) = E^x[u(T, X_T) - u(T, X^N_T)] \). However, \( u_x \) does not vanish at the boundaries, and thus, the application of a generalized Ito’s formula yields local time terms.

- The local time terms result in a lower weak convergence rate, \( O(N^{-1/2}) \) (see Gobet (1999)).
A new Euler-Maruyama scheme for killed diffusions
The case of killing

- Now suppose the solution of (1) has a finite lifetime and we are interested in $E^x[g(X_T)1_{T<\zeta}]$ for some bounded function $g$.

- We can assume without loss of generality that $X$ is on natural scale by considering $s(X)$ if necessary. This amounts to assuming that $X$ is a local martingale, i.e. $b \equiv 0$. Note that there is one-to-one correspondence between $X$ and $s(X)$ since $s$ is strictly increasing.
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Since at least one of the boundaries is accessible, by considering $-X$ if necessary, we may assume $\ell$ is an accessible boundary. Moreover, by a further translation, we may assume $\ell = 0$. 

Given the aforementioned problems with killed diffusions, can recurrent transformations help us to improve the convergence rate?
The case of killing

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- We can assume without loss of generality that \( X \) is on natural scale by considering \( s(X) \) if necessary. This amounts to assuming that \( X \) is a local martingale, i.e. \( b \equiv 0 \). Note that there is one-to-one correspondence between \( X \) and \( s(X) \) since \( s \) is strictly increasing.

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- Given the aforementioned problems with killed diffusions, can recurrent transformations help us to improve the convergence rate?
Recurrent transformation via bounded potentials

Let $h$ be a potential such that $h(x) = \int u(x, y) f(y) m(dy)$, where $f \geq 0$ is continuous and $\int f(y) m(dy)$ as well as $\int f(y) y m(dy)$ are finite. Moreover, $\frac{1}{2} \sigma^2 h'' = -f$.

$h$ is bounded, concave, and $(h, \exp(\int_0^T f(X_s) ds))$ is a recurrent transformation. The resulting law $R^{h, x}$ is the law of the following process:

$$dX_t = \sigma(X_t) dW_t + \sigma^2(X_t) \frac{h'(X_t)}{h(X_t)} dt.$$  

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Let $h$ be a potential such that $h(x) = \int u(x, y)f(y)m(dy)$, where $f \geq 0$ is continuous and $\int f(y)m(dy)$ as well as $\int f(y)ym(dy)$ are finite. Moreover, $\frac{1}{2}\sigma^2h'' = -f$.

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In particular,

$$\mathbb{E}^x\left[ \frac{g(X_T)1_{[T<\zeta]}}{h(x)} \right] = \mathbb{E}^{h,x}\left[ \frac{g(X_T)}{h(X_T)} \exp \left( -\int_0^T \frac{\sigma^2(X_s)h''(X_s)}{2h(X_s)} ds \right) \right].$$
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In particular,

$$E^x[g(X_T)1_{[T<\zeta]}] = E^{h,x}\left[\frac{g(X_T)}{h(X_T)} \exp \left(-\int_0^T \frac{\sigma^2(X_s)h''(X_s)}{2h(X_s)} ds\right)\right]$$

Now consider the following explicit scheme

$$X_{t_n}^N = X_{t_{n-1}}^N + \sigma^2(X_{t_{n-1}}^N)\frac{h'(X_{t_{n-1}}^N)}{h(X_{t_{n-1}}^N)}\frac{T}{N} + \sigma(X_{t_{n-1}}^N)(B_{t_n} - B_{t_{n-1}}). \quad (8)$$
If one wants to study the explicit scheme using a PDE method as before, the other object of interest is

$$\nu(T-t, x) = E^{h,x} \left[ \frac{g(X_t)}{h(X_t)} \exp \left( - \int_0^t \frac{\sigma^2(X_s) h''(X_s)}{2h(X_s)} ds \right) \right], \ x \in (0, r).$$

Although the numerical experiments converge, there are two immediate difficulties in proving the weak convergence rate for (8):

1. $\frac{h'}{h}$ is neither bounded nor Lipschitz.
Explicit scheme for the recurrent transformation

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  1. \( \frac{h'}{h} \) is neither bounded nor Lipschitz.
  2. \( X^N \) can exit \((0, r)\) with positive probability.

- The first issue is somewhat controllable as we shall see later by choosing \( h \) accordingly.
If one wants to study the explicit scheme using a PDE method as before, the other object of interest is

\[ v(T-t, x) = E^{h,x} \left[ \frac{g(X_t)}{h(X_t)} \exp \left( -\int_0^t \frac{\sigma^2(X_s)h''(X_s)}{2h(X_s)} \, ds \right) \right], \quad x \in (0, r). \] (9)

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1. \( \frac{h'}{h} \) is neither bounded nor Lipschitz.
2. \( X^N \) can exit \((0, r)\) with positive probability.

The first issue is somewhat controllable as we shall see later by choosing \( h \) accordingly.

However, the second difficulty does not go away and one needs to impose ad hoc boundary specifications.
As before, let $t_n = \frac{n}{N} T$ for $n = 0, \ldots, N$. Set $\hat{X}_0 = X_0$ and proceed inductively by setting

$$
\hat{X}_t = \hat{X}_{t_n} + \sigma(\hat{X}_{t_n})(W_t - W_{t_n}) + (t - t_n)\sigma^2(\hat{X}_{t_n}) \frac{h'(\hat{X}_t)}{h(\hat{X}_t)} \quad (10)
$$

for $t \in (t_n, t_{n+1}]$ and $n = 0, \ldots N - 1$. Typically implicit schemes require a small $\Delta t$. However, due to the concavity of $h$ the mapping $H: x \in (0, r) \to x - z h'(x) h(x)$ is invertible and has full range for any $z > 0$. Indeed, $H' \geq 1$. We shall call this well-defined scheme continuous backward Euler-Maruyama (BEM) scheme.
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Typically implicit schemes requires a small $\Delta t$. 

As before, let $t_n = \frac{n}{N} T$ for $n = 0, \ldots, N$. Set $\hat{X}_0 = X_0$ and proceed inductively by setting

$$\hat{X}_t = \hat{X}_{t_n} + \sigma(\hat{X}_{t_n})(W_t - W_{t_n}) + (t - t_n)\sigma^2(\hat{X}_{t_n}) \frac{h'(\hat{X}_t)}{h(\hat{X}_t)}$$  \hspace{1cm} (10)

for $t \in (t_n, t_{n+1}]$ and $n = 0, \ldots N - 1$.

Typically implicit schemes requires a small $\Delta t$.

However, due to the concavity of $h$ the mapping $H : x \in (0, r) \mapsto x - z \frac{h'(x)}{h(x)}$ is invertible and has full range for any $z > 0$. Indeed, $H' \geq 1.$
A drift-implicit scheme

As before, let $t_n = \frac{n}{N} T$ for $n = 0, \ldots, N$. Set $\hat{X}_0 = X_0$ and proceed inductively by setting

$$
\hat{X}_t = \hat{X}_{t_n} + \sigma(\hat{X}_{t_n})(W_t - W_{t_n}) + (t - t_n)\sigma^2(\hat{X}_{t_n})\frac{h'(\hat{X}_t)}{h(\hat{X}_t)} \tag{10}
$$

for $t \in (t_n, t_{n+1}]$ and $n = 0, \ldots N - 1$.

Typically implicit schemes requires a small $\Delta t$.

However, due to the concavity of $h$ the mapping $H : x \in (0, r) \mapsto x - z \frac{h'(x)}{h(x)}$ is invertible and has full range for any $z > 0$. Indeed, $H' \geq 1$.

We shall call this well-defined scheme continuous backward Euler-Maruyama (BEM) scheme.
Suppose that $h \in C^2_b((0, r), (0, \infty))$, $h^{(3)}$ exists and satisfies $|h^{(3)}| \leq K(1 + h^{-p})$ for some constant $K$ and $p \in [0, 1)$. Define $H(t_n, z; t, x) = x - \sigma^2(z)(t - t_n)\frac{h'}{h}(x)$. Then for $t \in (t_n, t_{n+1}]$

\[
d\hat{X}_t = \frac{\sigma(\hat{X}_{t_n})}{H_x(t_n, \hat{X}_{t_n}; t, \hat{X}_t)}dW_t + \frac{\sigma^2(\hat{X}_{t_n})}{H_x^2(t_n, \hat{X}_{t_n}; t, \hat{X}_t)} \left\{ \frac{h'}{h}(\hat{X}_t) + \mu(t_n, \hat{X}_{t_n}; t, \hat{X}_t) \right\} dt. \tag{11}
\]
The first key lemma

Suppose that \( h \in C^2_b((0, r), (0, \infty)) \), \( h^{(3)} \) exists and satisfies \(|h^{(3)}| \leq K(1 + h^{-p})\) for some constant \( K \) and \( p \in [0, 1) \). Define \( H(t_n, z; t, x) = x - \sigma^2(z)(t - t_n)\frac{h'}{h}(x) \). Then for \( t \in (t_n, t_{n+1}] \)

\[
d\hat{X}_t = \frac{\sigma(\hat{X}_{tn})}{H_x(t_n, \hat{X}_{tn}; t, \hat{X}_t)} dW_t
+ \frac{\sigma^2(\hat{X}_{tn})}{H_x^2(t_n, \hat{X}_{tn}; t, \hat{X}_t)} \left\{ \frac{h'}{h}(\hat{X}_t) + \mu(t_n, \hat{X}_{tn}; t, \hat{X}_t) \right\} dt. \tag{11}
\]

Consider the sets \( O_1 := \{x : h'(x) > 0\} \) and \( O_2 := \{x : h'(x) < 0\} \). Then

\[
\inf_{x \in O_1} \mu(t_n, z; t, x) \geq c_1 \quad \text{and} \quad \sup_{x \in O_2} \mu(t_n, z; t, x) \leq c_2
\]

for some constants \( c_1 \leq 0 \leq c_2 \) that do not depend on \( t_n, t \) or \( z \).
Consider the expected associated error

\[ E^{h,X_0} \left[ v(T, \hat{X}_T) \pi_N \right] - v(0, X_0), \]

where

\[ \pi_k(s) := \exp \left( \sum_{n=0}^{k-1} s \sigma^2(\hat{X}_{tn}) \frac{h''(\hat{X}_{tn})}{2h(\hat{X}_{tn})} \right), \quad k = 1, \ldots, N, \]

with the convention that \( \pi_k = \pi_k(TN^{-1}) \). Then

\[ E^{h,X_0}[e(N)] = \sum_{n=0}^{N-1} E^{h,X_0} \left[ v(t_{n+1}, \hat{X}_{t_{n+1}}) \pi_{n+1} - v(t_n, \hat{X}_{t_n}) \pi_n \right] \]

\[ = \sum_{n=0}^{N-1} E^{h,X_0} \pi_n \left( v(t_{n+1}, \hat{X}_{t_{n+1}}) \exp \left( T \frac{\sigma^2(\hat{X}_{tn}) h''(\hat{X}_{tn})}{2Nh(\hat{X}_{tn})} \right) - v(t_n, \hat{X}_{t_n}) \right) \]
\[ \nu(t_{n+1}, \hat{X}_{t_{n+1}}) \exp \left( T \frac{\sigma^2(\hat{X}_{t_n}) h''(\hat{X}_{t_n})}{2Nh(\hat{X}_{t_n})} \right) - \nu(t_n, \hat{X}_{t_n}) = M + I_1 + I_2 + I_3, \]

where \( M \) is a (local) martingale increment,

\[
I_1 = \int_{t_n}^{t_{n+1}} \frac{\pi_{n+1}(t - t_n)}{\pi_n(t - t_n)} \frac{\sigma^2(\hat{X}_{t_n}) \nu_x(t, \hat{X}_t) \mu(t_n, t; \hat{X}_{t_n}; t, \hat{X}_t)}{H_x^2(t_n, \hat{X}_{t_n}; t, \hat{X}_t)} dt
\]

and \( I_1 \) and \( I_2 \) are similarly complicated integrals containing

\[
\frac{1}{h(\hat{X}_t)} \text{ and } \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\hat{X}_{t_n}) h^{-p}(\hat{X}_t)}{H_x^2(t_n, \hat{X}_{t_n}; t, \hat{X}_t)} dt
\]

for some \( p \in (0, 3) \).
Computing inverse moments

- Typical approach in the literature towards computing uniform bounds on moments is via Ito’s formula and controlling the (local) martingale terms using BDG inequality.

The inverse moments are especially painful (cf. some works by Alfonsi and Neuenkirch & Szpruch). One difficulty with the first approach in the present case is that the local martingale term in the decomposition of, e.g., \( h^{-1}(X) \), is a strict local martingale. The works of Alfonsi and Neuenkirch & Szpruch study in particular the inverse moments of

\[
dY_t = dB_t + f(Y_t) \, dt
\]

for a large class of conservative diffusions in a given interval but their conditions on \( f \) cannot be satisfied when \( f = h' \) with \((h, M)\) being a recurrent transformation, as it implies the Radon-Nikodym density \( dR/dP \) is an \( \mathcal{R} \)-martingale.
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for a large class of conservative diffusions in a given interval but their conditions on $f$ cannot be satisfied when $f = \frac{h'}{h}$ with $(h, M)$ being a recurrent transformation, as it implies the Radon-Nikodym density $\frac{dR}{dP}$ is an $R$-martingale.
A comparison result

Consider the case $r = \infty$, and define $A$ by $A_0 = 0$ and

$$dA_t = \frac{\sigma^2(\hat{X}_{tn})}{H_x^2(t_n, \hat{X}_{tn}; t, \hat{X}_t)} dt, \quad t \in (t_n, t_{n+1}].$$

Also assume that $\sigma$ is bounded. Thus, $A_t \leq t\|\sigma^2\|_{\infty}$. 
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\]

Also assume that \( \sigma \) is bounded. Thus, \( A_t \leq t\|\sigma^2\|_{\infty} \).

Set \( \hat{Y}_t = \hat{X}_{A_t^{-1}} \) and recall (11). DD-S Theorem yields

\[
d\hat{Y}_t = d\beta_t + \left( \frac{h'}{h}(\hat{Y}_t) + \mu_t \right) dt,
\]

for some \( \mu_t \) with \( \mu_t \geq c_1 \), where \( \beta \) is \((\mathcal{F}_{t^{-1}A})\)-Brownian motion.

By comparison, for any non-increasing \( \phi \),

\[
E^{h,X_0}(\phi(\hat{X}_t)) \leq E^{h,X_0}(\phi(\hat{Y}_{A_t})),
\]

where

\[
Y_t = X_0 + \beta_t + \int_0^t \left( \frac{h'}{h}(Y_s) + c_1 \right) ds.
\]
Inverse moments

- Since \( h \) is increasing when \( r = \infty \), the above in particular allows us to bound \( E^{h,X_0}(\frac{1}{h}(\hat{X}_t)) \), uniformly in \( N \), via \( Y \).

- A difficulty, however, is that we need the moment of \( \frac{1}{h(Y)} \) at a rather arbitrary stopping time.
Inverse moments

Since $h$ is increasing when $r = \infty$, the above in particular allows us to bound $E^{X_0} \left( \frac{1}{h}(\hat{X}_t) \right)$, uniformly in $N$, via $Y$.

A difficulty, however, is that we need the moment of $\frac{1}{h(Y)}$ at a rather arbitrary stopping time.

The potential theory developed for Schrödinger semigroups comes to our rescue.

Let's allow again $r$ to be finite and consider

$$dY_t = dW_t + \left\{ \frac{h'(Y_t)}{h(Y_t)} + c \right\} dt, \quad t < \zeta(Y), \quad (13)$$

where $c \leq 0$ if $r = \infty$ and is unconstrained otherwise. $\zeta(Y)$ above denotes the first time that $Y$ exits $(\ell, r)$. 
The second key lemma

Let $Y$ be the process defined by (13) with $Y_0 = X_0$. Then the following statements are valid:

1. $R^{h,X_0}(\zeta(Y) = \infty) = 1$. 
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2. For any stopping time $S$ that is bounded $R^{h,X_0}$-a.s. there exists a constant $K$ that does not depend on $X_0$ such that

$$
E^{h,X_0} \left[ \frac{1}{h(Y_S)} \right] < \frac{K}{h(X_0)}.
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The second key lemma

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$$E^{h,X_0} \left[ \frac{1}{h(Y_S)} \right] < \frac{K}{h(X_0)}.$$

3. For any $t > 0$ and $p \in [0, 1)$

$$E^{h,X_0} \left[ \int_0^t \frac{1}{h^2 + p(Y_s)} \, ds \right] < \infty.$$
Some moment estimates for the BEM scheme

Suppose $h$ satisfies the conditions of the first key lemma and $\sigma$ is bounded. Let $T > 0$, $p \in [0, 1)$ and $t(s) = s - t_n$. Then

$$\sup_{N, t \leq T} E^{h,X_0} \left( \frac{1}{h}(\hat{X}_t) + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\hat{X}_{t_n}) h^{-2-p}(\hat{X}_t)}{H^2_X(t_n, \hat{X}_{t_n}; t, \hat{X}_t)} dt + |\hat{X}_t|^m \right) < \infty$$
Some moment estimates for the BEM scheme

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\sup_{N, t \leq T} E^{h, X_0} \left( \frac{1}{h}(\hat{X}_t) + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\hat{X}_{t_n}) h^{-2-p}(\hat{X}_t)}{H^2_X(t_n, \hat{X}_{t_n}; t, \hat{X}_t)} dt + |\hat{X}_t|^m \right) < \infty
\]

2. Let $p \in [0, 1)$ and $m \geq 0$ be an integer. For each $n$

\[
E^{h, X_0} \left( \int_{t_n}^{t_{n+1}} \left| \frac{h^{1-p}(\hat{X}_t)(1 + \hat{X}_t^m)\mu(t_n, \hat{X}_{t_n}; t, \hat{X}_t)}{H^2_X(t_n, \hat{X}_{t_n}; t, \hat{X}_t)} \right| dt \bigg| \mathcal{F}_n \right)
\leq \frac{KT}{N} E^{h, X_0} \left( \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\hat{X}_{t_n})(h^{-2-p}(\hat{X}_t) + \hat{X}_t^m)}{H^2_X(t_n, \hat{X}_{t_n}; t, \hat{X}_t)} dt \bigg| \mathcal{F}_n \right).
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Some moment estimates for the BEM scheme

Suppose $h$ satisfies the conditions of the first key lemma and $\sigma$ is bounded. Let $T > 0$, $p \in [0, 1)$ and $t(s) = s - t_n$. Then

1. $\sup_{N, t \leq T} E^{h, X_0} \left( \frac{1}{h} (\hat{X}_t) + \sum_{n=0}^{N-1} \left| \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\hat{X}_{tn}) h^{-2-p}(\hat{X}_t)}{H_x^2(t_n, \hat{X}_{tn}; t, \hat{X}_t)} dt + |\hat{X}_t|^m \right) \right) < \infty$

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E^{h, X_0} \left( \int_{t_n}^{t_{n+1}} \left| \frac{h^{1-p}(\hat{X}_t)(1 + \hat{X}_t^m)}{H_x^2(t_n, \hat{X}_{tn}; t, \hat{X}_t)} \right| dt \right| \mathcal{F}_n \right) \leq \frac{KT}{N} E^{h, X_0} \left( \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\hat{X}_{tn})(h^{-2-p}(\hat{X}_t) + \hat{X}_t^m)}{H_x^2(t_n, \hat{X}_{tn}; t, \hat{X}_t)} dt \right| \mathcal{F}_n \right).

3. If $p \leq \frac{1}{2}$ and $\frac{h''}{h^{1-p}}$ is bounded, denoting $\sigma(\hat{X}_{tn})$ by $\sigma_n$,

\[
E^{h, X_0} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\{ 1 - e^{t(s)} \sigma_n^2 \frac{h''(\hat{X}_{tn})}{2n} \right\} \frac{\sigma_n^2(h^{-p}(\hat{X}_s) + \hat{X}_s^m)}{H_x^2(t_n, \hat{X}_{tn}; s, \hat{X}_s)} ds \leq \frac{KT}{N}.
\]
Suppose $\sigma \in C^4_b((0, r), \mathbb{R})$, $g \in C^6_b((0, r), \mathbb{R})$ with $g^{(k)}(0) = 0$ (and $g^{(k)}(r) = 0$ if $r < \infty$) for $k \in \{0, 1, 2, 3, 4\}$,

$$\frac{|h^{(k)}|}{h} < \frac{K_h}{h^{k-2+p}}, \quad k \in \{2, 3, 4\},$$

for some $K_h$ and $p \in (0, 1)$, and recall $v$ from (9).
Relevant PDE estimates

Suppose \( \sigma \in C^4_b((0, r), \mathbb{R}) \), \( g \in C^6_b((0, r), \mathbb{R}) \) with \( g^{(k)}(0) = 0 \) (and \( g^{(k)}(r) = 0 \) if \( r < \infty \)) for \( k \in \{0, 1, 2, 3, 4\} \),

\[
\frac{|h^{(k)}|}{h} < \frac{K_h}{h^{k-2+p}}, \quad k \in \{2, 3, 4\},
\]

for some \( K_h \) and \( p \in (0, 1) \), and recall \( v \) from (9). Then,

\[
v_t + \frac{\sigma^2}{2} v_{xx} + \sigma^2 \frac{h'}{h} v_x = -\sigma^2 v \frac{h''}{2h}. \quad (14)
\]

Moreover, \( v \) and \( v_t \) are uniformly bounded and there exists a constant \( K \) such that

\[
\sup_{t \leq T} \left| \frac{\partial^k}{\partial x^k} v_t(t, x) \right| + \sup_{t \leq T} \left| \frac{\partial^k}{\partial x^k} v(t, x) \right| \leq Kh^{2-p-k}(x), \quad k \in \{1, 2\}. \quad (15)
\]
Hypotheses for the weak convergence estimates

Assumption 1

The functions $\sigma$, $h$ and $g$ satisfy the following regularity conditions.

1. $h \in C^4((0, r), (0, \infty))$ such that
   \[
   \frac{|h^{(k)}|}{h} < \frac{K_h}{h^{p+k-2}}, \quad k \in \{2, 3, 4\},
   \]
   for some $K_h$ and $p \in [0, \frac{1}{2}]$.

2. $\sigma \in C^2_b((0, r), (0, \infty))$ is bounded away from 0.

3. $g$ is of polynomial growth with $g(0) = 0$ ($g(r) = 0$ if $r < \infty$).

4. $v \in C^{1,4}((0, r), \mathbb{R})$, satisfies (14) and for $k \in \{1, 2\}$
   \[
   \sup_{t \leq T} \left| \frac{\partial^k}{\partial x^k} v_t(t, x) \right| + \sup_{t \leq T} \left| \frac{\partial^k}{\partial x^k} v(t, x) \right| \leq K(1 + x^m)h^{2-p-k}(x),
   \]
   for some constant $K$ and integer $m \geq 0$. 
Under Assumption 1,

\[
\left| E^{h,X_0}[l_1 + l_2 + l_3|\mathcal{F}_n] \right| \leq K \frac{T}{N} E^{h,X_0} \left( \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\hat{X}_{t_n})(h^{-2-p}(\hat{X}_s) + \hat{X}_s^m)}{H_x^2(t_n, \hat{X}_{t_n}; s, \hat{X}_s)} ds \bigg| \mathcal{F}_n \right) \\
+ E^{h,X_0} \left( \int_{t_n}^{t_{n+1}} \left( 1 - \exp \left( t(s)\sigma_n^2 h''(\hat{X}_{t_n}) \right) \right) \frac{\sigma_n^2(h^{-p}(\hat{X}_s) + \hat{X}_s^m)}{H_x^2(t_n, \hat{X}_{t_n}; s, \hat{X}_s)} ds \bigg| \mathcal{F}_n \right) \\
+ K \frac{T}{N} E^{h,X_0} \left( \int_{t_n}^{t_{n+1}} \frac{\sigma(\hat{X}_{t_n})^2(h^{-2}(\hat{X}_s) + \hat{X}_s^m)}{H_x^2(t_n, \hat{X}_{t_n}; s, \hat{X}_s)} ds \bigg| \mathcal{F}_n \right) \leq K \frac{T}{N}
\]
Numerical experiments
We shall apply our scheme to a down-and-out option in the Black-Scholes model and a double barrier option in hyperbolic local volatility model, where the local volatility is given by

\[
\sigma(x) = \nu \left\{ \frac{(1 - \beta + \beta^2)}{\beta} x + \frac{(\beta - 1)}{\beta} \left( \sqrt{x^2 + \beta^2(1 - x)^2} - \beta \right) \right\}.
\]

To achieve \( \sigma \) away from zero on \((\ell, r)\), we shall consider log price in the Black-Scholes model.

\( h(x) = e^{-\ell} - e^{-x} \) in the one sided case whereas \( h(x) = (x - \ell)(r - x) \) in the double barrier case. Neither \( h \) satisfies the condition of Assumption 1.
Figure: Absolute discrepancy between the benchmark price for ATM down-and-out put and those calculated with different numerical schemes when $S_0 = 1$, $K = 1$, $T = 1$ year, $l = \log(b = 0.8)$, $r = +\infty$ and $\sigma = 20\%$. 
Figure: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ATM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$. 
Figure: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for OTM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$. 
Figures: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ITM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.15$. 

Figure 1. ITM double barrier call.
Figure: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ATM double barrier call when $S_0 = 1$, $K = 1$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.3$. 
Conclusion

- Introduced a novel drift-implicit scheme for killed diffusions that brings the weak convergence rate back to $O(1/N)$.
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- Moment estimates are calculated using potential theory.
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- Moment estimates are calculated using potential theory.
- The earlier drift-implicit works that rely on BDG type inequalities for moment estimates impose restrictions on $h'/h$, which in turn imply $\frac{1}{h(X)} \exp\left(\frac{1}{2} \int_0^T \frac{f(X_s)}{h(X_s)} \, ds\right)$ is a $R^{h,x}$-martingale. This is not possible.

- Numerical experiments are consistent with theoretical results despite $h$ not satisfying the stated conditions.
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