Analysis of anisotropy of time-dependent and nonlinear properties of unidirectional CFRP

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Abstract Analysis of unidirectional carbon fiber reinforced plastic AS4/3501-6 regularities under strain-rate compression was given. Constitutive equations based on nonlinear hereditary mechanics relationships under shear in plane of the layer were elaborated. In order to obtain the constitutive equations, which are allowing describing the stress-strain curves of unidirectional carbon-fiber reinforced specimens under compression at different angles to the direction of reinforcement in the linear region, the algebra of resolvent operators and the relations of the elasticity theory of an anisotropic body were used. The possibility of correct using of the elaborated approach was shown.

1. Introduction
Carbon fiber reinforced plastics (CFRP) are widely used in different structures. It should be noted that most of the papers devoted to behavior under quasi-static loading [1]. In recent years, an increase number of works related to the study of the time-dependent properties of polymer-based composite materials have been observing. It is assumed that the main problem of the use of composite materials in structures is associated with the anisotropy of the mechanical properties causing the accumulation of damage during operation and the assessment of the residual resource. This is due to the presence of a polymeric binder leading to a noticeable time dependence of the properties, manifestation of the creep effects, relaxation, and strain-rate effect. In particular, in some of publications the effect of strain-rate loading on the stress-strain curves of CFRP was investigated [2-3]. Obviously, time-dependent properties most pronounced when loading in a direction, which is not coincide with the direction of reinforcement.

2. Constitutive equations
A constitutive equation of unidirectional layer takes a form depending on the direction of loading application. Clearly, in the reinforcement direction the time-dependent properties can be neglected, i.e.

\[ \varepsilon_1 = \frac{1}{E_1} \sigma_1. \]

In the direction of transverse to reinforcement time-dependent properties revealed and hereditary constitutive equation can be taken in linear hereditary elastic form

\[ \varepsilon_2 = \frac{1}{E_2} \left[ 1 + F_z \right] \sigma_z, \]

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where \( F'_2 \) is a hereditary operator, which is defined by the following relationship

\[
F'_2 \sigma = \int_0^1 F_2(t-\tau)\sigma(\tau)\,d\tau,
\]

\( F_2 \) is a kernel of the operator.

The most complex form of the constitutive equation is for shear in plane of the layer because of the time-dependent and nonlinear properties. Rabotnov nonlinear constitutive equation shows the following form [4]

\[
\varphi(\gamma_{12}) = \left[1 + F'_1\right]\tau_{12},
\]

where \( \varphi(\gamma_{12}) \) is a function of instantaneous stress-strain curve which is along with time-dependent properties allows one to describe nonlinear properties.

3. Model description

The anisotropy of the elasticity of a unidirectional material describing by a known relationship can be described by equation of the following form

\[
\frac{1}{E_{\varphi}} = \frac{c^4}{E_1} + \left( \frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) c^2 s^2 + \frac{s^4}{E_2},
\]

(4)

Where \( E_1, E_2, G_{12} \) are the elastic moduli of the layer, \( \nu_{12} \) is Poisson ratio, \( c = \cos \theta \) and \( s = \sin \theta \) are the trigonometric functions of the angle between the direction of the loading and the reinforcement.

Using the Volterra correspondence principle [4], which consists of replacing the elastic modulus with the corresponding operator expression, one can describe the deformation under a loading variable in time. In particular, substituting in the expression for the elastic modulus (4), we can obtain an operator expression for the elastic modulus at an angle \( \theta \) to the direction of reinforcement

\[
\frac{1}{E_{\varphi}} = \frac{c^4}{E_1} + \left( \frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) c^2 s^2 + \frac{s^4}{E_2}.
\]

(5)

The expressions for the elastic modulus, expressed by the hereditary operators, will generally have the following form [4]:

\[
\frac{1}{E^*} = \frac{1}{E_0}(1+kF^*)
\]

Substituting the Volterra representation in the expression for the elastic modulus (5), we obtain the operator expression for the elastic modulus

\[
\frac{1}{E^*} = \frac{c^4}{E_1} + \left( \frac{1+F'_1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) c^2 s^2 + \frac{s^4}{E_2}(1+F'_2)
\]

(6)

Expression (6) can be divided into two parts. The first one, which allows describing the instantaneous component of strain that does not depend on time and the second one, which allows describing the time-dependent properties

\[
\frac{1}{E^*} = \frac{c^4}{E_1} + \left( \frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} \right) c^2 s^2 + \frac{s^4}{E_2} + \frac{1}{G_{12}} c^2 s^2 F'_1 + \frac{1}{E_2} s^4 F'_2.
\]

(7)

Further, the problem is to identify the type and values of the hereditary operator parameters. The problem can be significantly simplified if we choose the resolvent type of the operators [4]. Linear combination of exponential functions pertains to such type of the operators presented by Prony series, as well as Rabotnov fractional exponential function [4].

In this paper, we take the condition of the hereditary operators in (2) and (3) that are similar and are Abel’s operators where power function is used as a kernel. It means that \( F'_1 \) and \( F'_2 \) have the following
kernel of operator: \( I_\alpha = \frac{1}{\Gamma(1+\alpha)} t^\alpha, \) \(-1<\alpha<0\), where \( \alpha \) is singularity parameter, \( \Gamma(\ ) \) is gamma-function. Substituting \( F_{12}^* = k_{12}I_\alpha^* \) and \( F_{12} = k_{12}I_\alpha \) in (7) we get the operator expression of the modulus 
\[
\frac{1}{E_0} = \frac{1}{E_0} + \lambda_\alpha I_\alpha^*,
\]
where \( \lambda_\alpha = \frac{k_{12}E_1^0}{E_2^0} \) and \( \frac{1}{E_0} = \frac{c^4}{E_1} + \left( \frac{1}{E_1} - \frac{2\nu_{12}}{E_1} \right) c^4 s^2 + \frac{s^4}{E_2^0}.
\)

Taking into account the assumption made earlier, we can write the following form of the constitutive equation 
\[
e_\alpha = \frac{1}{E_0} \left( 1 + k_{12}I_\alpha^* \right) \sigma_\alpha\),
\]
where \( k_{12} = \lambda_\alpha E_0^0 \). It should be noted that from (9) we can obtain the operator expression for modulus at angle \( \theta \) to the reinforcement direction.

Further, the values of the hereditary operators’ parameters must be identified. This has been accomplished taking the experimental data obtained in [3]. Necessary to inverse constitutive equation (9), which can be done with the help of the algebra relationship for the resolvent operator [4] 
\[
\left(1 + k_{12}I_\alpha^*\right)^{-1} = 1 - k_{12}\Omega_\alpha(-k_{12})^n,
\]
where \( \Omega_\alpha(-k_{12})^n = \frac{t^{1+\alpha}}{1+\alpha(1+n)} \sum_{n=0}^{\infty} \left( \frac{(-k_{12})^{1+\alpha}}{1+\alpha(1+n)} \right)^n \) is Rabotnov fractional-exponential function with the kernel expression: \( \Omega_\alpha(-k_{12},t) = t^n \sum_{n=0}^{\infty} \left( \frac{(-k_{12})^{1+\alpha}}{1+\alpha(1+n)} \right)^n \). Relationship (10) means that Rabotnov fractional-exponential function is the resolvent of Abel’s operator.

Then, constitutive equation (9) can be inverted in the following form 
\[
\sigma_\alpha = E_0^0 \left( 1 - k_{12}\Omega_\alpha(-k_{12}) \right) e_\alpha.
\]

Interrelated constitutive equations (9) and (11) describe anisotropy of mechanical properties in linear elastic area. The experimental data digitized in [3] show noticeable effect of sensitivity to strain-rate and manifest physical nonlinearity at all values of the angles to the loading direction. To a less extent, they appear at angles equaling to 15°, 30° and to a more extent at angles equaling to 30°, 45° and 60°.

In the area of linear strain it is necessary to determine the following parameters: \( E_1, \nu_{12}, E_2^0, k_1, G_{12}, k_{12}, \alpha \). The modulus of elasticity and Poisson ratio were taken from the data [1] \( E_1 = 126 \text{ GPa}, \nu_{12} = 0.28 \). The singularity parameter \( \alpha \) of the kernel was taken from previous analysis as known: \( \alpha = -0.9 \). With the help of the experimental data the remaining parameters \( E_2^0, k_1 \) and \( G_{12}, k_{12} \) can be determined. The first couple of the parameters were determined on the compression of the specimens at the angle 90°.

4. Compressive stress-strain curves

For further practical use, we give the following formula
\[
\int_0^t (t-\xi)^\alpha \xi^\beta d\xi = \Gamma(1+\alpha,1+\beta) = \frac{\Gamma(1+\eta)}{\Gamma(2+\eta+\beta)},
\]
where \( B(\ ) \) is beta function. Then, using (12) the expression for fractional exponential function acting on a power function is specified by
\[(1 - \Theta'(k_{12})) \cdot t^b = t^b - t^{1+\alpha \cdot \alpha} \sum_{n=0}^{\infty} \frac{(-k_{12} t^{1+\alpha})^n}{\Gamma[1 + b + (1 + \alpha)(1 + n)]}. \tag{13}\]

The constitutive equation describing strain-rate loading can be derived from relationship (13) when \(b = 1\) and may be written in the following form

\[
\sigma_2 = E_2^0 \left[1 - k_2 \left(\frac{\varepsilon_2}{\dot{\varepsilon}}\right)^{1+\alpha} \sum_{n=0}^{\infty} \frac{\left(-k_2 \left(\frac{\varepsilon_2}{\dot{\varepsilon}}\right)^{1+\alpha}\right)^n}{\Gamma[2 + (1 + \alpha)(1 + n)]}\right] \varepsilon_2. \tag{14}\]

From (14) we define \(E_2^0 = 15.5\) GPa, \(k_2 = 0.2018\) s\(^{(1+\alpha)}\). Comparison experimental data and calculation presented in Figure 1.

**Figure 1.** Stress-strain curves of unidirectional CFRP AS4/3501-6 under transverse strain-rate compression. (Upper line is instantaneous strain curve: \(\sigma_2 = E_2^0 \varepsilon_2\)).

It should be noted that a strain value of about 1.3% is a relatively small deviation from the linear law. At higher values of the strain, it is necessary to establish nonlinear constitutive equation.

The parameters characterizing the deformation under in-plane shear of the layer were determined from the testing of specimens compressed at angle 45° with taking into account the previously determined numerical values. When calculating the characteristics of in-plane shear of the layer, it was assumed that the modulus of elasticity in the direction of reinforcement weakly depends on time and can be taken equal to a constant value.

The values \(G_{12}^0\) and \(k_{12}\) were determined by using the CFRP deformation curves at angle of 45° and strain-rate of 380, 0.8 and 10\(^{-4}\) s\(^{-1}\) with the equation similar to (14)

\[
\sigma_{45} = E_{45}^0 \left[1 - k_{45} \left(\frac{\varepsilon_{45}}{\dot{\varepsilon}}\right)^{1+\alpha} \sum_{n=0}^{\infty} \frac{\left(-k_{45} \left(\frac{\varepsilon_{45}}{\dot{\varepsilon}}\right)^{1+\alpha}\right)^n}{\Gamma[2 + (1 + \alpha)(1 + n)]}\right] \varepsilon_{45}. \tag{15}\]
The parameters values can be derived as $E_{45}^0 = 25,15$ GPa, $k_{45} = 0,2564$ s$^{-1+\alpha}$. Comparisons of the calculated data and the experimental ones in area of linear strain are in satisfactory agreement.

The numerical values of parameters for in-plane shear in the plane of the layer were obtained by using the relationships for $E_{45}^0$ and $k_{45}$ proved to be $G_{12}^0 = 10,1$ GPa, $k_{12} = 0,301$ s$^{-1+\alpha}$. It should be emphasized that the above-mentioned values of the parameters were determined from the experimental data in the linear part of the stress-strain diagrams.

In Figure 2 it is shown the dependence of the parameters characterizing anisotropy of instantaneous modulus and time-dependent properties. The dependence of the instantaneous modulus is similar to the corresponding to anisotropic theory of elasticity.

![Figure 2. Dependencies of the constitutive equation anisotropy parameters.](image)

The obtained constitutive equations allow one describing the mechanical behavior of CFRP under arbitrary loading conditions. For example, under creep, relaxation, etc.

However, the experimental data given in [3] were not enough to obtain all the necessary parameters of nonlinear constitutive equation. The value of threshold strain of the linear area can be determined as $\gamma_* = 0,75$ %. The nonlinear constitutive equation for in-plane shear of the layer with Abel’s hereditary operator can be written as

$$\varphi(\gamma_{12}) = (1 + k_{12}I^\alpha_{\alpha})\tau_{12}.$$  \hspace{1cm} (16)

By using the algebra of resolvent operator equation (16), can be inversed

$$\tau_{12} = (1 - k_{12}\Omega^\alpha_{\alpha} (-k_{12}))\varphi(\gamma_{12}).$$ \hspace{1cm} (17)

The instantaneous stress-strain curve should be a piecewise smooth function being composed of a linear section and a non-linear section, which is the best way to describe nonlinear strain. With the help of the power approximation in the nonlinear strain area, the instantaneous deformation curve equation is given as following

$$\varphi(\gamma_{12}) = G_{12}^0\gamma_{12}\left[ H(\gamma_{12}) - H(\gamma_{12} - \gamma_*) \right] + \left[ \tau_* + a(\gamma_{12} - \gamma_*)^b \right]H(\gamma_{12} - \gamma_*),$$. \hspace{1cm} (18)
where \( \tau_\gamma = G_{ij}^0 \gamma_\gamma \), parameters \( a, b \) define nonlinear strain under shear, \( H(\ ) \) is Heaviside unit function of the corresponding argument.

The shear stress-strain curve is determined from the instant strain curve by the time dependence of the strain

\[
\tau_{12} = \varphi(\gamma_{12}) - k_{12} \Omega_{\gamma}^\alpha (-k_{12}) \varphi(\gamma_{12})
\]

(19)

Using expression (13) and denoting: \( \psi^*_{\alpha}(t) = (1-k_{12} \Omega^\alpha (-k_{12})) \cdot \dot{t}^\alpha \), we can write the constitutive equation, which allows one to describe the deformation at a strain-rate loading: \( \gamma_{12} = \dot{\gamma} \cdot t \). Substituting the last expression into (18) and then into (17) we get

\[
\tau_{12} = G_{ij}^0 \psi^*_{\alpha}(t) \dot{\gamma} \left[ H(\dot{\gamma} t) - H(\dot{\gamma}(t-t_\gamma)) \right] + \left[ \tau_* + \psi^*_{\alpha}(t-t_\gamma) \dot{\gamma}^\alpha \right] H(\dot{\gamma}(t-t_\gamma)),
\]

(20)

where \( \tau_* = G_{ij}^0 \psi^*(t_\gamma) \cdot \dot{\gamma} \cdot \dot{t}_\gamma = \gamma_* \). The following parameters of the instantaneous stress-strain curve were obtained: \( a = 30 \) GPa, \( b = 0.8 \). A comparison of experimental data and the calculated strain curve is shown in Figure 3.

![Figure 3](image_url)

**Figure 3.** Stress-strain curves under shear strain-rate loading. The upper solid line is an instantaneous stress-strain curve.

5. Conclusions

The model based on constitutive equations of hereditary mechanics of solids has been derived. The method allows one to predict anisotropy of mechanical properties of unidirectional CFRP under time-variable loading has been suggested. To derive constitutive equations, the relations of anisotropic theory of elasticity, Volterra correspondence principle and relationships of resolvent operators’ algebra have been used. The offered approach has been applied to experimental readings of unidirectional CFRP AS4/3501-6. The developed approach can serve as a basis for the description of the mechanical behavior regularities of CFRP.

References

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