PUNCTUAL EQUIVALENCE RELATIONS AND THEIR 
(PUNCTUAL) COMPLEXITY

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Abstract. The complexity of equivalence relations has received much attention in the recent literature. The main tool for such endeavor is the following reducibility: given equivalence relations $R$ and $S$ on natural numbers, $R$ is computably reducible to $S$ if there is a computable function $f : \omega \to \omega$ that induces an injective map from $R$-equivalence classes to $S$-equivalence classes. In order to compare the complexity of equivalence relations which are computable, researchers considered also feasible variants of computable reducibility, such as the polynomial-time reducibility. In this work, we explore $\mathbf{Peq}$, the degree structure generated by primitive recursive reducibility on punctual equivalence relations (i.e., primitive recursive equivalence relations with domain $\omega$). In contrast with all other known degree structures on equivalence relations, we show that $\mathbf{Peq}$ has much more structure: e.g., we show that it is a dense distributive lattice. On the other hand, we also offer evidence of the intricacy of $\mathbf{Peq}$, proving, e.g., that the structure is neither rigid nor homogeneous.

1. Introduction

The classification of equivalence relations according to their complexity is a major research thread in logic. The following two examples stand out from the existing literature.

- In descriptive set theory, one often deals with equivalence relations defined on Polish spaces (e.g., $2^\omega$ or $\omega^\omega$), which are classified in terms of Borel embeddings. The corresponding theory is now a consolidated field of modern descriptive set theory, which shows deep connections with topology, group theory, combinatorics, model theory, and ergodic theory (see, e.g., [17, 21, 13, 24]).

- On the other hand, in computability theory, it is common to concentrate on equivalence relations on the natural numbers and compare their algorithmic content in terms of computable reductions (see, e.g., [14, 15, 19, 1, 12]).

Computable reducibility (for which we use the symbol $\leq_c$, and call $c$-degrees the elements of the corresponding degree structure) has been adopted to calculate the complexity of natural equivalence relations on $\omega$, proving, e.g., that provable

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equivalence in Peano Arithmetic is $\Sigma^0_1$ complete [10], Turing equivalence on c.e.
sets is $\Sigma^0_1$ complete [24], and the isomorphism relations on several familiar classes
of computable structures (e.g., trees, torsion abelian groups, fields of characteristic
0 or $p$) are $\Sigma^1_1$ complete [16]. In parallel, there has been a growing interest in the
abstract study of the poset of degrees generated by computable reducibility on the
collection of equivalence relations of a certain complexity. Most notably, the poset
Ceers of the $c$-degrees of computably enumerable equivalence relations (commonly
known by the acronym ceers) has been thoroughly explored [6, 2]: e.g., it has been
recently shown that its first-order theory is as complicated as true arithmetic [5].
Less is known about larger structures of $c$-degrees; but recent studies considered
the $\Delta^0_2$ case [30, 9] and the global structure ER of all $c$-degrees [3].

Yet, despite its classificatory power, computable reducibility has an obvious
shortcoming: it is simply too coarse for measuring the relative complexity of
computable equivalence relations. Indeed, it is easy to note that any two com-
putable equivalence relations $R$ and $S$ with infinitely many classes are computably
bi-reducible. A natural way to overcome this limitation is by adopting feasible re-
ducibilities, as is done in [11, 20], where the authors prove that, relatively to these
reducibilities, the isomorphism relations of finite fields, finite abelian groups, and
finite linear orders have all the same complexity.

This paper focuses on a subcollection of computable equivalence relations, namely
primitive recursive equivalence relations with domain $\omega$, called punctual. To clas-
sify punctual equivalence relations, we adopt primitive recursive reducibility. More
precisely, a punctual equivalence relation $R$ is pr-reducible to a punctual equiva-
lence relation $S$ (notation: $R \preceq_{pr} S$) if there exists a primitive recursive function
$f$ such that

$$(\forall x, y \in \omega)[x R y \iff f(x) S f(y)].$$

So, the main object of study of this paper is Peq, i.e., the degree structure of
pr-degrees consisting of punctual equivalence relations.

In the sequel, we hope to convince the reader that Peq is a remarkable struc-
ture. For instance, in constrast with Ceers and all other established structures of
$c$-degrees, Peq is surprisingly well-behaved, being a dense distributive lattice (The-
orems 5.3 and 1.3). On the other hand, we also offer evidence of the intricacy of
Peq, proving, e.g., that the structure is neither rigid nor homogeneous (Theorems
1.10 and 1.11). Furthermore, working in a primitive recursive setting will affect
our proof strategies: as any primitive recursive check converges, our constructions
will be typically injury-free and requirements will be solved one by one. That said,
to build primitive recursive objects, we won’t need to worry about coding, and in
fact we will rely on a restricted form of Church-Turing thesis (see Remark 2.2),
which will give to the paper more of a computability-theoretic flavour rather than
a complexity-theoretic one.

A final piece of motivation comes from the rapid emergence of online structure
theory (see, e.g., [1, 13, 29, 28]), a subfield of computable structure theory which
deals with algorithmic situations in which unbounded search is not allowed, that
is formalized by focusing, e.g., on punctual presentations of structures, rather than
computable ones.

The rest of the paper is organized as follows. In Section 2, we review background
material. In Section 3, we observe a simple yet important fact: punctual equivalence
relations have a normal form. In the central sections of the paper, Sections 4-7,
we prove that \textbf{Peq} has several desirable features (which are unusual for degree structures of equivalence relations): we show, in particular, that \textbf{Peq} is a dense distributive lattice, in which each degree is join-reducible and each degree below the top is meet-reducible. The last two sections should convince the reader that, despite being surprisingly well-behaved, \textbf{Peq} remains intricate: e.g., we prove that it contains intervals which do not embed the diamond lattice; that it is neither rigid nor homogeneous; and that it contains nonisomorphic lowercones. Several questions remain open.

2. Background, terminology, and notations

We review some background material and terminology. For background on computability theory, especially primitive recursive functions, the reader is referred to [31].

2.1. Equivalence relations. As aforementioned, punctual equivalence relations are primitive recursive equivalence relations with domain \( \omega \). Unless stated otherwise, all our equivalence relations will be punctual. A transversal of an equivalence relation \( R \) is any set \( T \) such that \( xRy \) for any distinct \( x,y \in T \). The principal transversal \( T_R \) of a given equivalence relation \( R \) is the following transversal:

\[
T_R := \{ y \neq 0 : (\forall x < y)(xRy) \}.
\]

Clearly, if \( R \) is punctual, then \( T_R \) is a primitive recursive set. As is customary in theory of ceers, we denote by \( \text{Id} \) the identity on the natural numbers. We often write \( f : R \leq_{\text{pr}} S \) to mean that \( f \) is a primitive recursive reduction from \( R \) to \( S \).

2.2. Finite equivalence relations. We define an equivalence relation \( R \) to be finite if it has only finitely many equivalence classes; \( R \) is non-finite otherwise. As in the case of ceers [19] (but differently from the case of \( \Delta^0_2 \) equivalence relations [9]), \textbf{Peq} has an initial segment of order type \( \omega \) consisting of the finite punctual equivalence relations. More precisely, it is immediate to observe that

\[
\text{Id}_1 \leq_{\text{pr}} \text{Id}_2 \leq_{\text{pr}} \cdots \leq_{\text{pr}} \text{Id}_n \leq_{\text{pr}} \cdots ,
\]

where \( \text{Id}_n \) is equality mod \( n \). Clearly each \( \text{Id}_n \) and \( \text{Id} \) are punctual equivalence relations. In addition, any finite punctual \( R \) is \( \text{pr} \)-equivalent to \( \text{Id}_n \), for some \( n \geq 1 \), and \( \text{Id}_n \leq_{\text{pr}} R \), for every \( n \geq 1 \) and every non-finite punctual \( R \).

Remark 2.1. (Terminology) In the rest of the paper, we shall assume that all our equivalence relations are non-finite. Henceforth, by a punctual equivalence relation we will mean a primitive recursive relation with infinitely many equivalence classes. Likewise, by a punctual degree we will mean the \( \text{pr} \)-degree of a punctual equivalence relation: of course, the equivalence relations lying in a punctual degree are all punctual.

2.3. A listing of the primitive recursive functions. Throughout the paper we will refer to an effective listing \( \{p_e\}_{e \in \omega} \) of the primitive recursive functions, which can be found in many textbooks: see e.g. [22] for a detailed definition of such a listing. Let \( T(e, x, z) \) and \( U \) denote respectively Kleene’s (primitive recursive) predicate and a primitive recursive function such that for every \( e, \varphi_e(x) = U(\mu z T(e, x, z)) \) (where \( \varphi_e \) denotes the partial computable function with index \( e \))
in the standard listing of the partial computable functions), and let \( p \) be a recursive function such that \( p_\epsilon = \varphi_{p(\epsilon)} \): since \( p \) comes from the \( s\)-\( m\)-\( n\)-theorem we may assume that \( p \) is primitive recursive. Let \( V \) be the primitive recursive predicate

\[
V(e,x,y,s) \iff (\exists z < s)(T(p(e),x,z) \& U(z) = y):
\]

we will refer to \( V(e,x,y,s) \) by saying that “\( p_\epsilon \) has converged to \( y \) in \( < s \) steps”, and we will denote this by \( p_\epsilon(x)[s] \downarrow = y \). Similarly, it is primitive recursive to check whether “\( p_\epsilon(x) \) has converged in \( < s \) steps” (denoted by \( p_\epsilon(x)[s] \downarrow \)), i.e. \((\exists y < s)V(e,x,y,s)\).

2.4. **Binary strings.** We will use the standard notations and terminology about **finite binary strings**, which are the elements of the set \( 2^{\omega} \). Let \( \sigma, \tau \) be finite binary strings: we will denote by \( l^\sigma \) the length of \( \sigma \); the concatenations of \( \sigma \) and \( \tau \) will be denoted by \( \sigma \tau \); if \( i \in \{0,1\} \) is a number then \( \langle i \rangle \) denotes the string of length 1, consisting of the single bit \( i \); the symbol \( \lambda \) denotes the empty string.

We will freely identify a set \( X \subseteq \omega \) with its characteristic function, thus viewing \( X \) as a member of \( 2^{\omega} \). If \( k \in \omega \) and \( X \) is a set, then the symbol \( X \upharpoonright k \) denotes the initial segment of \( X \) (thus a member of \( 2^{< \omega} \)) of length \( k \). Given \( \sigma \in 2^{< \omega} \) and a set \( X \), let

\[
\sigma \ast X = \begin{cases} 
\sigma(i), & \text{if } i < l^\sigma \\
X(i - l^\sigma), & \text{otherwise}. 
\end{cases}
\]

In analogy with the common usage for sets where \( \overline{X} = \{ x : X(x) = 0 \} \), given a string \( \sigma \in 2^{\omega} \) we denote \( \overline{\sigma} = \{ i : i < l^\sigma \& \sigma(i) = 0 \} \), and we let \( \#(0^s) = \#(\overline{\sigma}) \), i.e. the number of 0’s in the range of \( \sigma \).

We will assume a “primitive recursive coding” of the binary strings, with primitive recursive length function, projections, etc.

**Remark 2.2.** Throughout this paper we will often build primitive recursive functions or primitive recursive sets. It is sometimes convenient to use the following analogue of the Church-Turing thesis for primitive recursive functions:

Let \( \varphi_\epsilon \) be the \( \epsilon \)-th partial (general) computable function. Then (in accordance with what stated in Section 2.3) every primitive recursive function is equal to some \( \varphi_\epsilon \) where the computation for \( \varphi_\epsilon \) runs in primitive recursively many steps. Thus, to define a primitive recursive function \( f \) (or a set), it is enough to specify a general algorithm for computing \( f(n) \) as long as the number of steps taken to decide \( f(n) \) is bounded by a primitive recursive function.

This helps to avoid overly formal and cumbersome definitions, since it is often easier to see that the time bound is primitive recursive.

3. **The normal form theorem and some structural properties**

In the literature about computable reducibility, it is common to use the following way of encoding sets of numbers by equivalence relations: given \( X \subseteq \omega \), one defines

\[
x \mathbin{R_X} y \iff x, y \in X \text{ or } x = y.
\]

Equivalence relations of the form \( R_X \) are called **1-dimensional** in [19], while \( c \)-degrees containing 1-dimensional equivalence relations are called **set-induced** in [8]. The interesting feature of set-induced degrees is that they offer algebraic and logical information about the overall structure of \( c \)-degrees: for example, in [5] it is proved that the first-order theory of ceers is undecidable, by showing that the interval
[deg_c(Id), deg_c(R_K)] of the c-degrees is isomorphic to the interval [0_1, 0'_1] of the 1-degrees, where 0_1 is the 1-degree of an infinite and co-infinite computable set and 0'_1 is the 1-degree of the halting set K.

Yet, set-induced degrees are far from exhausting the collection of all c-degrees. In fact, if R is an equivalence relation with two non-computable R-classes, then there is obviously no X such that R ≡_c R_X. When dealing with punctual equivalence relations, the situation changes: all pr-degrees are set-induced. Specifically, the next easy theorem shows that the punctual complexity of R is entirely encoded in its principal transversal T_R.

**Theorem 3.1 (Normal Form Theorem).** For any punctual R, we have that

\[ R \equiv_{pr} R_{T_R} \]

**Proof.** It is immediate to see that the function \( f(x) = (\mu y \leq x) [y R x] \) is primitive recursive and reduces R to \( R_{T_R} \). On the other hand, the following primitive recursive function g reduces \( R_{T_R} \) to R,

\[
g(x) = \begin{cases} 
0, & \text{if } x \in T_R, \\
x, & \text{otherwise}.
\end{cases}
\]

Hence, \( R \equiv_{pr} R_{T_R} \).

So, whenever is needed, one can assume that a given punctual equivalence relation is in normal form. Furthermore, the next lemma ensures that, if two punctual equivalence relations R and S are in normal form and \( R \leq_{pr} S \), then there is a reduction from R to S that “respects” the normal form.

**Lemma 3.2.** If \( R_X \leq_{pr} R_Y \), then there is a reduction g from \( R_X \) to \( R_Y \) such that \( g[X] \subseteq Y \).

**Proof.** Let f be a primitive recursive function reducing \( R_X \) to \( R_Y \) such that \( f[X] = \{a\} \) with \( a \neq Y \). Fix \( y \in Y \) and define

\[
g(x) = \begin{cases} 
y, & \text{if } x \in X, \\
a, & \text{if } x \notin X \text{ and } f(x) \in Y, \\
f(x), & \text{if } x \notin X \text{ and } f(x) \notin Y.
\end{cases}
\]

It is easy to see that g is primitive recursive, maps X to Y and reduces \( R_X \) to \( R_Y \).

Another assumption that one can make without loss of generality is that a pr-reduction between punctual equivalence relations is surjective on the equivalence classes of its target. This contrasts with the case of ceers where (see [6]) if R, S are such that \( R \leq_c S \) via a computable function f whose range hits all the equivalence classes of S, then the reduction can be inverted, i.e., \( S \leq_c R \) as well. This inversion lemma fails for punctual equivalence relations, in fact:

**Lemma 3.3.** If \( R \leq_{pr} S \) then there exists g such that \( g : R \leq_{pr} S \) and g hits all the S-classes.

**Proof.** Let \( R \leq_{pr} S \) via \( f \): by Theorem 3.1 we may assume that \( R = R_X \) and \( S = R_Y \) for some pair of primitive recursive sets and by Lemma 3.2 we may assume
that $f$ maps $X$ to $Y$. Define

$$g(x) = \begin{cases} f(x), & \text{if } x \in X, \\ \mu y \{ y \leq \max\{ f(i) : i \leq x \} \} & \text{if } x \notin X. \end{cases}$$

Then $g$ gives the reduction and is onto $Y$: to show primitive recursiveness of $g$ we use the fact that if $x$ is the $n$-th element in $X$, then the $n$-th element of $Y$ in order of magnitude is $\leq \max\{ f(i) : i \leq x \}$ by injectivity of $f$ on $X$. \hfill \Box

**Remark 3.4.** Note that the function $g : R_X \leq_{pr} R_Y$ constructed in the last lemma is nondecreasing on $X$, i.e., if $x < y$ and $x, y \in X$ then $g(x) < g(y)$.

We may comment on the results presented so far by saying that investigating punctual equivalence relations under $\leq_{pr}$ turns out to be the same as investigating primitive recursive sets under a primitive recursive reducibility that is required to be bijective on the complements of the sets. In the rest of the paper, we will take advantage of this correspondence by often constructing, instead of a full punctual equivalence relation $R$, only a primitive recursive set $Y$ corresponding to its main class in normal form (i.e. $R \equiv_{pr} R_Y$).

### 4. Incomparability of punctual degrees

In this and the following section, we tackle some of the most natural questions that one can formulate about a new degree structure.

#### 4.1. The greatest punctual degree

We begin by proving that there is a greatest punctual degree.

**Proposition 4.1.** $\text{Peq}$ has a greatest element. In fact, for all punctual $R$, $R \leq_{pr} \text{Id}$.

**Proof.** Given $R$, let $f$ be the primitive recursive function from the proof of the Normal Form Theorem (i.e., the function reducing $R$ to $R_{\text{Id}}$). Note that $f$ is also a reduction from $R$ to Id, as it maps equivalence classes to singletons. \hfill \Box

The punctual degree of Id contains many natural examples of equivalence relations. For example, in the literature about polynomial-time reducibility (see, e.g., [1]) researchers considered the isomorphism relations of familiar classes of finite structures, such as graphs, groups, trees, linear orders, Boolean algebras, and so forth. It is not difficult to see that all these relations turn out to be $pr$-equivalent to Id. Consider for instance GI, the isomorphism relation between finite graphs. On one hand, the problem of deciding whether two finite graphs are isomorphic is primitive recursive (in fact, it belongs to NP). On the other hand, a $pr$-reduction from Id to GI is readily obtained by assigning, to each $n$, the empty graph on the domain $\{ i : i < n \}$. Hence, GI is $pr$-equivalent to Id.

Anyway, the fact that the punctual degree of Id is large does not come as a surprise. In fact, to construct a computable function which is not primitive recursive requires non-trivial work (recall Ackermann’s famous construction [4]). And we show next that a punctual equivalence relation $R$ lies strictly below Id only if, when presented in normal form, the set of its singletons cannot be enumerated in a primitive recursive way without repetitions.

To be more precise, let us introduce first the following analogue of immunity for primitive recursive sets.

**Theorem 4.2.**
Definition 4.2. A set $X \subseteq \omega$ is primitive recursively enumerable (abbreviated by p.r.e.) if $X$ is the range of an injective primitive recursive function. $X$ is primitive recursively immune (abbreviated by p.r.-immune) if $X$ is infinite and it has no infinite p.r.e. subset.

Remark 4.3. Note that we define a set as p.r.e. only if it has a primitive recursive enumeration which is injective. Without injectivity one would obtain all c.e. sets: it follows easily from Kleene’s Normal Form Theorem that any c.e. set can be enumerated by a primitive recursive function. In contrast, Theorem 4.5 shows that the p.r.e. sets (as just defined) form a proper subclass of the c.e. sets, and in fact even of the primitive recursive sets.

Proposition 4.4. If $X$ is any set of numbers then $\text{Id} \in \text{pr} \mathcal{R} X$ if and only if $X$ is p.r.-immune.

Proof. $(\Rightarrow)$: If $X$ contains a p.r.e. set $A$, then $\text{Id} \in \text{pr} \mathcal{R} X$ via any primitive recursive function that injectively lists $A$.

$(\Leftarrow)$: If $\text{Id} \in \text{pr} \mathcal{R} X$ via some primitive recursive function $f$, then, with the exception of at most one element, the range of $f$ is contained in $X$. Therefore, $\text{range}(f) \cap X$ is an infinite p.r.e. set showing that $X$ is not p.r.-immune.

Theorem 4.5. There exists a primitive recursive set $Y$ whose complement is p.r.-immune.

Proof. We construct $Y$ in stages by approximating its characteristic function, i.e., $Y = \bigcup_{s \in \omega} \sigma_s$, where $l^\sigma_s = s$. The construction of $Y$ is similar to that of a simple set and is rather straightforward. We describe it in some detail to let the reader familiarize themselves with the sort of machinery that we employ in more intricate constructions.

We aim at satisfying the following requirements:

$P_e$: if $p_e$ is injective, then $\text{range}(p_e) \cap Y \neq \emptyset$,

$M$: $Y$ is co-infinite,

where $\{p_e\}_{e \in \omega}$ is a computable list of all primitive recursive functions, as in Section 2.3.

The strategy for a $P_e$-requirement works as follows. During the so-called “$P_e$-cycle” we enlarge the set $Y$ (by setting $\sigma_{s+1} = \sigma_s \backslash \langle 1 \rangle$ at the current stage $s$) until we see that one of the following conditions holds: either the function $p_e$ is not injective, or we have $p_e(z) \in Y$ for some $z$. Let $s_0 + 1$ be the first stage at which we witness such a situation. We set $\sigma_{s_0+1} = \sigma_{s_0} \backslash \langle 0 \rangle$, close the $P_e$-cycle and move to satisfying the $P_{e+1}$-strategy, by opening the $P_{e+1}$-cycle.

The construction. At the beginning of any nonzero stage of the construction we assume that there exists exactly one open $P$-cycle. Section 2.3 will guarantee that the various checks involving $p_e$-computations and their convergence are primitive recursive.

Stage 0. Define $\sigma_0 = \lambda$; open the $P_0$-cycle. Thus at the beginning of the next stage, there will be exactly one open $P$-cycle, namely the $P_0$-cycle.
Stage $s + 1$. Assume that the $P_e$-cycle is the currently open cycle. We distinguish three cases:

1. There are $l, m \leq s$ such that $p_e(l)[s] \downarrow = p_e(m)[s] \downarrow$: if so, close the $P_e$-cycle, define $\sigma_{s+1} = \sigma_s \uparrow(0)$, and open the $P_{e+1}$-cycle.
2. There is $m \leq s$ such that $p_e(m)[s] \downarrow = z$ and $\sigma_s(z) \downarrow = 1$: do the same as in (1).
3. Otherwise: keep the $P_e$-cycle open and define $\sigma_{s+1} = \sigma_s \uparrow(1)$.

Again it is immediate to see that the next stage will inherit from this stage exactly one open $P$-cycle.

The verification. The verification relies on the following lemmas.

Lemma 4.6. Every $P_e$-cycle is eventually opened and is later closed forever.

Proof. Notice that the $P_e$-cycle is opened at stage $s$ if and only if $e = 0$ and $s = 0$, or $e > 0$ and we close at $s$ the $P_{e-1}$-cycle.

So assume by induction that the lemma is true of every $i < e$: thus there exists a unique $s_0$ at which we open the $P_e$-cycle ($s_0 = 0$ if $e = 0$, or otherwise $s_0$ is the stage at which we close the $P_{e-1}$-cycle). If the $P_e$-cycle does not satisfy the claim then the construction implies the following: the function $p_e$ is injective and $\sigma_{s+1} = \sigma_s \uparrow(1)$ for all $s \geq s_0 + 1$. Therefore since $p_e$ is injective and $Y$ is cofinite, there would be infinitely many elements $m$ with $p_e(m) \in Y$. Thus, at some stage $s_1 + 1 \geq s_0 + 1$, the $P_e$-cycle would be closed by the item (2) of the construction, and never opened again. We obtain a contradiction, showing that the $P_e$-cycle satisfies the claim.

Lemma 4.7. $Y$ is primitive recursive and co-infinite.

Proof. It is enough to observe that for all $s$, the value of $\sigma_s$ is decided at stage $s$, and therefore the function $s \mapsto \sigma_s$ is primitive recursive. Moreover, it is immediate to check that $l^{\sigma_s} = s$, for every $s$. Hence, $Y = \bigcup_{s \in \omega} \sigma_s$ is primitive recursive, as $Y(s) = \sigma_{s+1}(s)$.

Since every $P_e$-cycle eventually closes, there are infinitely many stages $s$ with $\sigma_{s+1} = \sigma_s \uparrow(1)$. This implies that the set $\overline{Y}$ is infinite.

Lemma 4.8. All $P$-requirements are satisfied.

Proof. Suppose that $p_e$ is an injective function. Consider the stage $s_0$ at which the $P_e$-cycle closes. Clearly, the closure of the $P_e$-cycle is triggered by item (2). Thus, there is a number $m \leq s_0$ with $\sigma_{s_0}(p_e(m)) \downarrow = 1$. Hence, we have $p_e(m) \in Y$ and $\text{range}(p_e) \cap Y \neq \emptyset$.

Theorem 4.5 is proved.

Combining the last three propositions we immediately obtain that there exists a punctual degree strictly below the identity.

Corollary 4.9. There exists a punctual $R$ such that $R <_{pr} \text{Id}$.
4.2. **Counterexamples to reducibilities.** In the rest of the paper, we will often need to build a punctual $S$ such that, for a given $R$, we have $R \not\leq_{pr} S$. To do so, we construct a primitive recursive set $Y$ such that for every $e$, the requirement

$$P_e : p_e \text{ does not reduce } R \text{ to } R_Y,$$

is satisfied, and thus $S = R_Y$ is our desired equivalence relation. Typically, we construct an increasing sequence $\{\sigma_s : s \in \omega\}$ of strings in $2^{<\omega}$ so that $Y = \bigcup_s \sigma_s$.

At a stage $s + 1$ of the construction we say that $p_e$ shows a counterexample to $R \leq_{pr} R_Y$ if there exist $l, m \leq s$ so that $p_e(l)[s] \downarrow$, $p_e(m)[s] \downarrow$, and $p_e(l)[s], p_e(m)[s] < l^{\sigma_s}$, and

$$l \ R \ m \iff (\sigma_s(p_e(l)) \neq \sigma_s(p_e(m))) \text{ or } (p_e(l) \neq p_e(m) \& \sigma_s(p_e(l)) = \sigma_s(p_e(m)) = 0).$$

A counterexample will guarantee, for the final $Y$, that

$$l \ R \ m \iff p_e(l) \not\leq_{pr} p_e(m),$$

as desired. For a given $\sigma_s$, primitive recursiveness of $R$ and Section 2.3 will guarantee that checking if a counterexample is being shown is primitive recursive.

Similarly, if for every $e$ we want to satisfy the requirement

$$Q_e : p_e \text{ is not a reduction from } R_Y \text{ to } R,$$

at a stage $s + 1$ of the construction we say that $p_e$ shows a counterexample to $R_Y \leq_{pr} R$ if there exist $l, m < l^{\sigma_s}$ so that $l \neq m$, $p_e(l)[s] \downarrow$, $p_e(m)[s] \downarrow$, and

$$\sigma_s(l) = \sigma_s(m) = 1 \iff p_e(l) \not\leq_{pr} p_e(m) :$$

a counterexample guarantees for the final $Y$ that $R_Y \not\leq_{pr} R$.

Finally, sometimes we build two primitive recursive sets $X, Y$ in the form $X = \bigcup_s \sigma^X_s$ and $Y = \bigcup_s \sigma^Y_s$. At a stage $s + 1$ of the construction we say that $p_e$ shows a counterexample to $X \leq_{pr} Y$ if there exist $l, m < l^{\sigma^X_s}$ so that $l \neq m$, $p_e(l)[s] \downarrow$, $p_e(m)[s] \downarrow$, and $p_e(l)[s], p_e(m)[s] < l^{\sigma^Y_s}$, and

$$\sigma^X_s(l) = \sigma^X_s(m) = 1 \iff (\sigma^Y_s(p_e(l)) \neq \sigma^Y_s(p_e(m))) \text{ or } (p_e(l) \neq p_e(m) \& \sigma^X_s(p_e(l)) = \sigma^X_s(p_e(m)) = 0).$$

4.3. **Incomparability.** Among non-finite punctual equivalence relations, there is no least punctual degree. In fact, Id is the only punctual equivalence relation which is non-finite and pr-comparable with all other punctual equivalence relations.

The next result will be a consequence also of Theorem 6.1 and Theorem 6.7 (to be discussed later). However, it may be useful to give a direct proof in order to introduce the incomparability strategy, which will be exploited in other constructions.

**Theorem 4.10.** For any punctual $R \leq_{pr} \text{Id}$, there is punctual $S$ such that $S|_{pr} R$.

**Proof.** Given $R \leq_{pr} \text{Id}$, we build by stages a primitive recursive set $Y$ such that $S = R_Y$ satisfies the claim. The requirements to be satisfied are

$$P_e : p_e \text{ is not a reduction from } R \text{ to } R_Y,$$

$$Q_e : p_e \text{ is not a reduction from } R_Y \text{ to } R.$$
The strategy. To satisfy $P_e$, one continues putting more and more fresh elements into $Y$, thus not increasing the number of $R_Y$-classes. By doing so, we will witness eventually that $p_e$ maps two $R$-classes to a single $R_Y$-class, since the number of $R$-classes will outgrow the number of $R_Y$-classes (recall that $R$ is non-finite, see Remark 2.1).

To satisfy a given $Q_e$-requirement, we follow a dual strategy: for any fresh element, we declare the corresponding singleton as an $R_Y$-class. This ensures that we will find a pair of witnesses that show that $p_e$ is not a reduction of $R_Y$ to $R$, since otherwise we would have a reduction of $\text{Id}$ to $R$ as well.

The construction. We construct set $Y$ in stages: in fact at a stage $s$ we define its initial segment $\sigma_s$ of length $s$, and eventually we take $Y = \bigcup_{s \in \omega} \sigma_s$.

Stage 0. Define $\sigma_0 = \lambda$; open the $P_0$-cycle.

Stage $s + 1$. Assume that $R$ is the currently open cycle.

1. $R = P_e$, for some $e$: If so, we distinguish two cases.
   
   (a) If $p_e$ shows a counterexample to $R \leq_{pr} R_Y$ (see Section 4.2) then close the $P_e$-cycle. Define $\sigma_{s+1} = \sigma_s \langle 0 \rangle$. Open the $Q_e$-cycle.
   
   (b) Otherwise, keep the $P_e$-cycle open and define $\sigma_{s+1} = \sigma_s \langle 1 \rangle$.

2. If $R = Q_e$, for some $e$, then distinguish two more cases.
   
   (a) If $p_e$ shows a counterexample to $R_Y \leq_{pr} R$ (see Section 4.2) then close the $Q_e$-cycle. Define $\sigma_{s+1} = \sigma_s \langle 0 \rangle$. Open the $P_{e+1}$-cycle.
   
   (b) Otherwise, keep the $Q_e$-cycle open and define $\sigma_{s+1} = \sigma_s \langle 1 \rangle$.

This concludes the construction.

The verification. The verification relies on the following lemmas.

Lemma 4.11. $Y$ is primitive recursive.

Proof. The function $s \mapsto \sigma_s$ is primitive recursive; moreover $l^{\sigma_e}. Y(s) = \sigma_{s+1}(s)$. \hfill \qed

Lemma 4.12. All $P$-requirements are satisfied.

Proof. The proof follows the lines of the analogous claim in the proof of Theorem 4.2. First of all, it easily follows by induction that the $P_e$-cycle is opened at stage 0 if $e = 0$, and at the stage at which the $Q_{e-1}$-cycle is closed if $e > 0$; and the $Q_e$-cycle is opened at the stage at which the $P_e$-cycle is closed. Assume that for every $i < e$ the $P_i$-cycle and the $Q_i$-cycle have been closed, and the corresponding requirements are satisfied. Then at the stage $s_0$ (with $s_0 = 0$ if $e = 0$, or $s_0$ is the stage when we close the $Q_{e-1}$-cycle) we open the $P_e$-cycle. Failure to close the $P_e$-cycle would entail that $Y$ is cofinite, thus $R_Y$ would have only finitely many classes, but $p_e$ never showing a counterexample would give that $p_e : R \leq_{pr} R_Y$, a contradiction as $R$ is not finite. Thus, at some stage, $p_e$ shows a counterexample to $R \leq_{pr} R_Y$, whence $P_e$ is satisfied. \hfill \qed

Lemma 4.13. All $Q$-requirements are satisfied.

Proof. Assume that for every $i < e$ the $Q_i$-cycle has been closed, and for every $i \leq e$ the $P_i$-cycle has been closed, and the corresponding requirements are satisfied. If $s_0$ is the stage at which we open the $Q_e$-cycle (that is when we close the $P_e$-cycle) and for every $s \geq s_0 + 1$ we never close the cycle then this would entail that $Y$ is
finite (whence \( R_Y \equiv_{\text{pr}} \text{Id} \)), but as \( p_e \) never shows a counterexample this would give that \( p_e : R_Y \leq_{\text{pr}} R \), whence \( \text{Id} \leq_{\text{pr}} R \), a contradiction. Thus, at some stage, \( p_e \) shows a counterexample to \( R_Y \leq_{\text{pr}} R \), whence \( Q_e \) is satisfied.

This concludes the proof of Theorem 4.10.

From the last theorem, it follows that there is an infinite antichain of punctual degrees.

**Corollary 4.14.** There are punctual equivalence relations \( \{S_i\}_{i \in \omega} \) such that \( S_i \mid_{\text{pr}} S_j \) for all \( i \neq j \).

**Proof.** The proof relies on the following observation

**Observation.** Suppose that \( \{R_i : i \in \omega\} \) is a family of punctual equivalence relations none of which is \( \text{pr} \)-equivalent to \( \text{Id} \), and for which there exists a primitive recursive predicate \( U(i, x, y) \) such that \( x R_i y \) if and only if \( U(i, x, y) \). Then a straightforward modification of the proof of Theorem 4.10 will show that there exists a punctual \( S \) such that \( R_i \mid_{\text{pr}} S \), for every \( i \). The construction in this case aims to build a primitive recursive set \( Y \) satisfying the following requirements \( R_{<e,i} \) indexed by the values of the primitive recursive Cantor pairing function:

\[
P_{<e,i} : p_e \text{ does not reduce } R_i \text{ to } R_Y, \\
Q_{<e,i} : p_e \text{ does not reduce } R_Y \text{ to } R_i.
\]

The construction goes by opening and closing \( R \)-cycles as in the proof of the previous theorem. Checking if a counterexample is being shown is primitive recursive, since this can be done by using the primitive recursive predicate \( U \), together with Section 2.3.

To finish the proof of the corollary, define by induction the following infinite antichain \( \{S_n\}_{n \in \omega} \) of punctual equivalence relations: Pick \( S_0 \prec_{\text{pr}} \text{Id} \); having found \( S_0, \ldots, S_n \) apply the above observation to the family \( \{R_i\}_{i \in \omega} \), where \( R_i = S_i \) if \( i < n \), and \( R_i = S_n \) if \( i \geq n \).

\[\Box\]

5. **The punctual degrees form a distributive lattice**

In this section, we study the structure of the punctual degrees under joins and meets. By the Normal Form Theorem we will confine ourselves to equivalence relations of the form \( R_X \), where \( X \) is a primitive recursive set.

The first result of this section provides us with a useful characterization of the reducibility \( \leq_{\text{pr}} \). Informally speaking, the characterization connects the punctual degree of a relation \( R_X \) with the growth rate of the function

\[
\#(0^X)[s] := \#(0^{X^1(s+1)}),
\]

i.e. the function which counts the number of singleton \( R_X \)-classes.

We emphasize that for a primitive recursive \( X \), the corresponding function \( \#(0^X)[s] \) is also primitive recursive.

**Proposition 5.1.** Let \( X \) and \( Y \) be cofinite primitive recursive sets. Then we have \( R_X \leq_{\text{pr}} R_Y \) if and only if there exists a primitive recursive function \( h(x) \) such that for all \( s \),

\[
\#(0^X)[s] \leq \#(0^Y)[h(s)].
\]
Claim 5.4. Recall that the functions \(\#\) is easy to show that the set \(Z\) is the supremum and infimum. To prove the theorem, it is sufficient to show that the relations \(X \preceq X_Y\) prove a primitive recursive reduction \(h(s)\) as follows:

\[
h(s) = \begin{cases} 
  f(k^*), & \text{if } k^* \text{ is the greatest recursive number such that } k^* \leq s \text{ and } k^* \in X, \\
  0, & \text{if } (\forall k \leq s)(k \in X).
\end{cases}
\]

Indeed, suppose that \(#(0^X)[s] = N > 0\). Consider the set

\[
X \cap \{0, 1, \ldots, s\} = \{k_1 < k_2 < \cdots < k_N\}.
\]

Then we have \(f(k_1) < f(k_2) < \cdots < f(k_N) = h(s)\), and hence, \(#(0^Y)[h(s)] \geq N = #(0^X)[s]\).

To show the converse, let \(h\) be a primitive recursive function such that \(#(0^X)[s] \leq #(0^Y)[h(s)]\) for all \(s\). Without loss of generality, one may assume that \(0 \in Y\). The desired primitive recursive reduction \(g: X \preceq X_Y\) is defined by recursion on \(x \in \omega\) as follows:

\[
g(0) = \begin{cases} 
  0, & \text{if } 0 \in X, \\
  \text{the least element from } Y, & \text{if } 0 \in X; \\
\end{cases}
\]

\[
g(x + 1) = \begin{cases} 
  0, & \text{if } x + 1 \in X, \\
  \mu y[y \leq h(x + 1) \& y \in Y \setminus \{g(z) : z \leq x\}], & \text{if } x + 1 \in X.
\end{cases}
\]

Suppose that \(x \in X \setminus \{0\}\) and the set \(X \cap \{0, 1, \ldots, x\}\) equals \(\{k_1 < k_2 < \cdots < k_N = x\}\). Then the set \(Y \cap \{1, 2, \ldots, h(x)\}\) contains at least \(N\) elements, and this set has at most \(N - 1\) elements from \(\text{range}(g \upharpoonright x)\). Therefore, the function \(g\) is well-defined. In addition, it is easy to observe that \(g\) provides a reduction \(X \preceq X_Y\).

Proposition 4.4 can now be restated as:

**Corollary 5.2.** \(\text{Id} \preceq X_Y\) if and only if there is a primitive recursive function \(h(x)\) such that \(s \leq #(0^Y)[h(s)]\) for all \(s \in \omega\).

5.1. **Joins and meets.** Now we are ready to prove that the partial order \(\text{Peq}\) has joins and meets, which make the structure a lattice. By slightly abusing notations, we will talk about suprema and infima of punctual equivalence relations (referring of course to the poset \(\text{Peq}\) of the \(pr\)-degrees).

**Theorem 5.3.** The structure \(\text{Peq}\) is a lattice.

**Proof.** Suppose that \(X\) and \(Y\) are cofinite primitive recursive sets such that \(X \preceq Y\). Without loss of generality, we assume that \(0 \in X \cap Y\). In order to prove the theorem, it is sufficient to show that the relations \(X \) and \(Y\) have supremum and infimum.

We define a set \(Z_0 \subseteq \omega\) as follows: \(0 \in Z_0\), and

\[
s + 1 \notin Z_0 \iff \max(\#(0^X)[s + 1], \#(0^Y)[s + 1]) > \max(\#(0^X)[s], \#(0^Y)[s]).
\]

Recall that the functions \(#(0^X)[s]\) and \(#(0^Y)[s]\) are primitive recursive. Hence, it is easy to show that the set \(Z_0\) is primitive recursive and cofinite.

**Claim 5.4.** \(Z_0\) is the supremum of \(X\) and \(Y\).
These observations (together with an easy induction argument) imply that

$$
\#(0^U)[s + 1] = \begin{cases}
\#(0^U)[s] + 1, & \text{if } s + 1 \notin U, \\
\#(0^U)[s], & \text{if } s + 1 \in U.
\end{cases}
$$

These observations (together with an easy induction argument) imply that

$$
\#(0^Z_0)[s] = \max(\#(0^X)[s], \#(0^Y)[s]).
$$

Thus, one can apply Proposition 5.1 for the function \( h(x) = x \), and deduce that

\( R_{Z_0} \) is an upper bound for both \( R_X \) and \( R_Y \).

By Proposition 5.1, we choose primitive recursive functions \( h_X \) and \( h_Y \) such that \( \#(0^X)[s] \leq \#(0^Y)[h_X(s)] \) and \( \#(0^Y)[s] \leq \#(0^Y)[h_Y(s)] \). Then by (1), we obtain

$$
\#(0^{Z_0})[s] \leq \max(\#(0^X)[h_X(s)], \#(0^Y)[h_Y(s)]) = \#(0^V)[\max(h_X(s), h_Y(s))].
$$

Hence, we deduce that \( R_{Z_0} \preceq_{pr} R_V \), and \( R_{Z_0} \) is join of \( R_X \) and \( R_Y \).

Claim 5.5. \( R_{Z_0} \) is the infimum of \( R_X \) and \( R_Y \).

Proof. As in the previous claim, one can easily show that

$$
\#(0^{Z_1})[s] = \min(\#(0^X)[s], \#(0^Y)[s]).
$$

By Proposition 5.1, \( R_{Z_1} \) is a lower bound for \( R_X \) and \( R_Y \).

Let \( R_Y \) be a lower bound of \( R_X \) and \( R_Y \). We fix primitive recursive functions \( q_X \) and \( q_Y \) such that \( \#(0^V)[s] \leq \#(0^X)[q_X(s)] \) and \( \#(0^V)[s] \leq \#(0^Y)[q_Y(s)] \). Then by (1),

$$
\#(0^V)[s] \leq \min(\#(0^X)[q_X(s)], \#(0^Y)[q_Y(s)]) \leq \min(\#(0^X)[\max(q_X(s), q_Y(s))], \#(0^Y)[\max(q_X(s), q_Y(s))]) = \#(0^{Z_1})[\max(q_X(s), q_Y(s))].
$$

Therefore, \( R_{Z_1} \) is meet of \( R_X \) and \( R_Y \).

Theorem 5.3 is proved.

Definition 5.6. Given primitive recursive sets \( X \) and \( Y \), let us denote \( R_X \lor R_Y \) the relation \( R_{Z_0} \), constructed in the proof of Theorem 5.3, giving the supremum of \( R_X \) and \( R_Y \). In addition, let us denote \( Z_0 = X \lor Y \). Likewise, let us denote \( R_X \land R_Y \) the relation \( R_{Z_1} \), constructed in the proof above, giving the infimum of \( R_X \) and \( R_Y \). We also denote \( Z_1 = X \land Y \).

Corollary 5.7. There are no minimal pairs of punctual degrees.

Proof. Immediate.

Note that in the proof of Theorem 5.3, we gave an explicit algorithm for building suprema and infima. This allows us to easily obtain the following:

Theorem 5.8. The lattice \( \text{Peq} \) is distributive.
Proof. Let $X$, $Y$, and $Z$ be co-infinite primitive recursive sets. As discussed above, by $R_X \vee R_Y$ we denote the supremum of $R_X$ and $R_Y$, and $R_X \wedge R_Y$ is the infimum of $R_X$ and $R_Y$. We sketch the proof for the following distributivity law:

$$R_X \vee (R_Y \wedge R_Z) \equiv_{pr} (R_X \vee R_Y) \wedge (R_X \vee R_Z).$$

Suppose that $Q \equiv_{pr} R_X \vee (R_Y \wedge R_Z)$ and $S \equiv_{pr} (R_X \vee R_Y) \wedge (R_X \vee R_Z)$. We may assume that $Q = R_U$ and $S = R_V$, where $U$ and $V$ are primitive recursive sets such that $0 \in U \cap V$, and for every $s \in \omega$,

$$
\begin{align*}
\#(0^U)[s] &= \max(\#(0^X)[s], \min(\#(0^Y)[s], \#(0^Z)[s])), \\
\#(0^V)[s] &= \min(\max(\#(0^X)[s], \#(0^Y)[s]), \max(\#(0^X)[s], \#(0^Z)[s])).
\end{align*}
$$

Since the structure $(\omega, \leq)$ is a linear order, for any numbers $x, y, z \in \omega$, we have

$$\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z)).$$

Hence, it is clear that $\#(0^U)[s] = \#(0^V)[s]$ for every $s$, and $R_U = R_V$. This concludes the proof of Theorem 5.8. \hfill \Box

6. Density

We prove now that the distributive lattice of punctual degrees is dense. This contrasts with the case of Ceers and ER, where each degree has a minimal cover (see [1, 3] for details). However, density is a phenomenon that often shows up when focusing on the subrecursive world. Mehlorn [27] proved that the degree structures induced by many subrecursive reducibilities on sets (including the primitive recursive one) are dense. Similarly, Ladner [26] proved that, if $P \neq NP$, then the poset of NP sets under polynomial-time reducibility is dense.

Density emerges also in the study of the online content of structures. More precisely, for a structure $\mathcal{A}$, $\text{FPR}(\mathcal{A})$ denotes the degree structure generated by primitive recursive isomorphisms on the collection of all punctual copies of $\mathcal{A}$. Bazhenov, Kalimullin, Melnikov, and Ng [11] recently proved the following: if a punctual infinite $\mathcal{A}$ is finitely generated, then the poset $\text{FPR}(\mathcal{A})$ is either one-element or dense.

**Theorem 6.1 (Density).** If $R_X \leq_{pr} R_Z$ are punctual equivalence relations then there exists a primitive recursive set $Y$ such that $R_X \leq_{pr} R_Y \leq_{pr} R_Z$.

**Proof.** We will satisfy the following requirements, for every $e \in \omega$:

- $P_e : p_e$ does not reduce $R_Y$ to $R_X$,
- $Q_e : p_e$ does not reduce $R_Z$ to $R_Y$,
- $M : R_X \leq_{pr} R_Y$,
- $N : R_Y \leq_{pr} R_Z$,

where $\{p_e\}_{e \in \omega}$ is a computable listing of all primitive recursive functions, see Remark 2.3.

Assume that $f : R_X \leq_{pr} R_Z$. Assume also, without losing generality, that $0 \in X \cap Z$, and that $Z$ is both infinite and co-infinite.
The environment. At stage \( s + 1 \) we inherit from stage \( s \) a finite binary string \( \sigma^Y_s \) of length \( s + 1 \); moreover we will let \( \sigma^X_s = X|s + 1 \) and \( \sigma^Z_s = Z|s + 1 \). For \( U \in \{ X,Y,Z \} \) we will denote \( \# (0^U_s) [s] = \# (0^U) [s] \) (see Section 2.4 for the notation \( \# (0^\tau) \), where \( \tau \) is any finite binary string).

The strategies. Let us sketch the strategy to achieve \( R_Y \preceq_{pr} R_X \). When we attack for the first time the requirement at stage \( s_0 \) we are given the strings \( \sigma^X_{s_0}, \sigma^Y_{s_0}, \sigma^Z_{s_0} \), for which we have guaranteed that \( \# (0^Y) [s_0] = \# (0^Z) [s_0] \).

We open the so called \( P_e \)-cycle: until \( p_e \) does not show a counterexample to \( R_Y \preceq_{pr} R_X \), we keep copying larger and larger pieces of \( Z \) in \( Y \), so that starting from the input \( s_0 + 1 \) the set \( Y \) looks like \( Z \) from the input \( s_0 + 1 \). If this process goes on forever, then we would eventually get \( R_Z \preceq_{pr} R_Y \): the initial segment \( \sigma^Z_X \) of \( Z \) which is not copied by the copying procedure can be mapped by the reduction to \( \sigma^Z_{s_0} \) as the two strings have the same number of 0’s; if \( i < s_0 + 1 \) is such that \( \sigma^Z_{s_0}(i) = 1 \) (i.e. \( Z(i) = 1 \)) then the reduction maps \( i \) to \( 0 \in Y \). Thus, eventually we get that \( p_e \) does show a counterexample to \( R_Y \preceq_{pr} R_X \), otherwise \( R_Y \preceq_{pr} R_X \), but then \( R_Z \preceq_{pr} R_Y \preceq_{pr} R_X \).

When a counterexample shows up, we close the \( P_e \)-cycle and we move to next requirement, opening the \( Q_e \)-cycle. (In fact before opening the \( Q_e \)-cycle, we have to go through a transition phase to reach a stage \( t \) at which \( \# (0^Y) [t] = \# (0^X) [t] \).) This also shows that \( P_e \) is eventually satisfied.

The strategy to achieve \( R_Z \preceq_{pr} R_Y \) is similar, opening and closing the so called \( Q_e \)-cycle: until \( p_e \) does not show a counterexample to \( R_Z \preceq_{pr} R_Y \) we keep copying larger and larger pieces of \( X \) in \( Y \), so that starting from the input \( s_0 + 1 \) (where \( s_0 \) is when the cycle was opened) the set \( Y \) looks like \( X \) from \( s_0 + 1 \). In order to implement this procedure in a correct way, we require the following: when we start the \( Q_e \)-cycle at \( s_0 \), we have \( \# (0^Y) [s_0] = \# (0^X) [s_0] \).

Again, the \( Q_e \)-cycle cannot go on forever, otherwise we would get \( R_Z \preceq_{pr} R_Y \) (since \( p_e \) never shows a counterexample), but on the other hand the copying procedure would give \( R_Y \preceq_{pr} R_X \), yielding a contradiction. After a counterexample shows up, there will be a transition phase, at the end of which we will reach a stage \( t \) at which \( \# (0^X) [t] = \# (0^Z) [t] \).

It remains to explain how we achieve that \( R_X \preceq_{pr} R_Y \preceq_{pr} R_Z \). For this, we guarantee that at each step \( s \) we have

\[
\# (0^X) [s] \leq \# (0^Y) [s] \leq \# (0^Z) [s],
\]

so that we can search in a bounded way for the images in \( Y \) of the 0’s in \( \sigma^X_s \), and for the images in \( Z \) of the 0’s in \( \sigma^Y_s \). This, together with the facts that \( s \) is in the domains of both \( \sigma^X_s \) and \( \sigma^Y_s \), and the mappings \( s \mapsto \sigma^U_s \) are primitive recursive, will give the desired reductions.

Remark 6.2. As \( R_X \preceq_{pr} R_Z \), we may assume that if \( \sigma \in X, \tau \in Z \) have the same length then \( \# (0^\sigma) \leq \# (0^\tau) \): for this, one can replace \( Z \) with the join \( X \lor Z \) if needed.

The construction. The construction is in stages.
Case 1) Suppose that we are within a previously opened $P_e$-cycle which has not been declared closed yet. We assume by induction that when we opened (say at $s_0$) the cycle, we had $\# (0^X)[s_0] \leq \# (0^Y)[s_0] = \# (0^Z)[s_0]$.

**Copying phase.** (Copy $R_Z$ in $R_Y$.) If we have not yet moved to the $P_e \rightarrow Q_e$-transition phase, then let

$$\sigma^Y_{s+1} = \sigma^Y_s \langle Z(s + 1) \rangle.$$ 

Notice that by the assumption in Remark 6.2 after this we still have

$$\# (0^X)[s + 1] \leq \# (0^Y)[s + 1] = \# (0^Z)[s + 1].$$

After this, if $p_e$ has shown a counterexample to $R_Y \leq_{pr} R_X$ (as defined in Section 4.2) then enter the $P_e \rightarrow Q_e$-transition phase:

**Transition phase.** Carry out the following.

1. If $\# (0^X)[s + 1] = \# (0^Y)[s + 1]$ then exit from the transition phase. We close the $P_e$-cycle and open the $Q_e$-cycle.
2. If $\# (0^X)[s + 1] < \# (0^Y)[s + 1]$ then let

$$\sigma^Y_{s+1} = \sigma^Y_s \langle 1 \rangle :$$

(when $X(s+1) = 0$ this has the effect of making $\# (0^X)[s+1] = \# (0^X)[s]$ + 1, whereas $\# (0^Y)[s + 1] = \# (0^Y)[s]$) and go to (1), remaining in this transition phase.

Notice that at each stage $t$ within a $P_e$-cycle we have by the assumption in Remark 6.2

$$\# (0^X)[t] \leq \# (0^Y)[t] \leq \# (0^Z)[t],$$

and when we close the $P_e$-cycle, we have

$$\# (0^X)[t] = \# (0^Y)[t] \leq \# (0^Z)[t].$$

Case 2) Suppose that we are within a previously opened $Q_e$-cycle which has not been declared closed yet. We assume by induction that when we opened (say at $s_0$) the cycle we had $\# (0^X)[s_0] = \# (0^Y)[s_0] \leq \# (0^Z)[s_0]$.

**Copying phase.** (Copy $R_X$ in $R_Y$.) Let

$$\sigma^Y_{s+1} = \sigma^Y_s \langle X(s + 1) \rangle.$$ 

Notice that by the assumption in Remark 6.2 after this we still have $\# (0^X)[s+1] = \# (0^Y)[s+1] \leq \# (0^Z)[s+1]$ if we had $\# (0^X)[s] = \# (0^Y)[s] \leq \# (0^Z)[s]$. After this, if $p_e$ has shown a counterexample to $R_Z \leq_{pr} R_Y$ (as defined in Section 4.2) then enter the $Q_e \rightarrow P_{e+1}$-transition phase:
Transition phase. Carry out the following.

1. If \( \#(0^X)[s+1] = \#(0^Z)[s+1] \) then exit from the transition phase. We close the \( Q_e \)-cycle and open the \( P_{e+1} \)-cycle.
2. If \( \#(0^Y)[s+1] < \#(0^Z)[s+1] \) then let
   \[
   \sigma^Y_{s+1} = \sigma^Y_s \upharpoonright (0^r),
   \]
   (when \( Z(s+1) = 1 \) this has the effect of making \( \#(0^Y)[s+1] = \#(0^Y)[s] + 1 \) whereas \( \#(0^Z)[s+1] = \#(0^Z)[s] \)) and go to (1), remaining in this transition phase.

Notice that at each stage \( t \) within a \( Q_e \)-cycle we have by the assumption in Remark 6.3
\[
\#(0^X)[t] \leq \#(0^Y)[t] \leq \#(0^Z)[t],
\]
and when we close the \( Q_e \)-cycle we have
\[
\#(0^X)[t] \leq \#(0^Y)[t] = \#(0^Z)[t].
\]

The verification. The verification relies on the following lemmas.

Lemma 6.3. For each \( e \), the requirements \( P_e \) and \( Q_e \) are satisfied.

Proof. As in the proof of Theorem 4.10, it easily follows by induction that the \( P_e \)-cycle is opened at stage 0 if \( e = 0 \), and at the stage at which the \( Q_{e-1} \)-cycle is closed if \( e > 0 \). The \( Q_e \)-cycle is opened at the stage at which the \( P_e \)-cycle is closed.

Assume that for every \( i < e \) the \( P_i \)-cycle and the \( Q_i \)-cycle have been closed, and the corresponding requirements are satisfied. Then at the stage \( s_0 \) (with \( s_0 = 0 \) if \( e = 0 \), or \( s_0 \) is the stage when we close the \( Q_{e-1} \)-cycle if \( e > 0 \)) we open the \( P_e \)-cycle. If \( p_e \) never shows a counterexample to \( R_Y \leq_{pr} R_X \), then we claim that \( R_Z \leq_{pr} R_X \), a contradiction.

To show this claim, notice that in this case (i.e. should \( p_e \) never show a counterexample to \( R_Y \leq_{pr} R_X \), implying that \( R_Y \leq_{pr} R_X \), we would have \( Y = \sigma^Y_{s_0} \ast Z \). Then \( R_Z \leq_{pr} R_Y \) by a primitive recursive function \( q \) which matches up the zeros in \( \sigma^Z_{s_0} \) with those of \( \sigma^Y_{s_0} \) (using that both strings have the same number of zeros, since \( \#(0^X)[s_0] = \#(0^Z)[s_0] \), \( q(i) = 0 \) if \( Z(i) = 1 \) and \( i \leq s_0 \), and \( q(i) = i \) for \( i > s_0 + 1 \). It would follow that \( R_Z \leq_{pr} R_X \), a contradiction.

Thus, at some stage \( p_e \) shows a counterexample to \( R_Y \leq_{pr} R_X \), whence \( P_e \) is satisfied. Moreover, since \( X \) is infinite the transition phase of the cycle will end, since eventually \( X \) will produce enough 0’s to match up with those which are present in \( \sigma^Y \) at the beginning of the \( P_e \rightarrow Q_e \)-transition phase of the \( P_e \)-cycle. Therefore, the cycle will be closed.

Similarly, assume that for every \( i < e \) the \( Q_i \)-cycle has been closed, and for every \( i \leq e \) the \( P_i \)-cycle has been closed, and the corresponding requirements are satisfied. If \( s_0 \) is the stage at which we open the \( Q_e \)-cycle (that is when we close the \( P_e \)-cycle) and for every \( s \geq s_0 \) we never close the cycle, then \( R_Z \leq_{pr} R_Y \), and thus an argument similar to the one given above would entail that \( R_Z \) would be \( pr \)-reducible to \( R_X \), as the construction would ensure in this case that \( Y = \sigma^Y_{s_0} \ast X \) and \( \#(0^X)[s_0] = \#(0^Y)[s_0] \), giving that \( R_Y \leq_{pr} R_X \). Finally, the \( Q_e \rightarrow P_{e+1} \)-transition phase ends, since \( Z \) is infinite.

Hence, all \( P \)- and \( Q \)- requirements are satisfied. \( \square \)

Claim 6.4. \( Y \) is primitive recursive.
Proof. The function \( s \mapsto \sigma^Y_s \) is primitive recursive and \( Y(s) = \sigma^Y_{s+1}(s) \). \( \square \)

**Lemma 6.5.** \( R_X \leq_{pr} R_Y \leq_{pr} R_Z \).

Proof. We need to define two primitive recursive functions \( g, h \) which provide reductions \( g: R_X \leq_{pr} R_Y \) and \( h: R_Y \leq_{pr} R_Z \). Using that the functions \( q^Y, q^Z \) where \( q^Y(s) = \sigma^Y_s \) and \( q^Z(s) = \sigma^Z_s \) are primitive recursive, and at each stage \( t \) we have that \( \#(0^X)[t] \leq \#(0^Y)[t] \) define

\[
g(s) = \begin{cases} 0, & \text{if } X(s) = 1, \\ \min\{i < l^Y_s : \sigma^Y_s(i) = 0 \& (\forall j < s)[i \neq g(j)]\}, & \text{if } X(s) = 0. \end{cases}
\]

Similarly, using that at each stage \( t \) we have \( \#(0^Y)[t] \leq \#(0^Z)[t] \) we can define

\[
h(s) = \begin{cases} 0, & \text{if } Y(s) = 1, \\ \min\{i < l^Z_s : \sigma^Z_s(i) = 0 \& (\forall j < s)[i \neq h(j)]\}, & \text{if } Y(s) = 0. \end{cases}
\]

It is not hard to see that \( g \) and \( h \) provide the desired \( pr \)-reductions. \( \square \)

The last lemma ensures that the global requirements \( M \) and \( N \) are both satisfied. In combination with Lemma 6.3, this means that \( R_Y \) lies strictly in between \( R_X \) and \( R_Z \), as desired. Theorem 6.1 is proved. \( \square \)

Upwards and downwards density are immediate consequences of Theorem 4.10, the existence of infima (Theorem 5.3), and Theorem 6.1:

**Corollary 6.6.** If \( R \leq_{pr} Id \), then there are \( S_0, S_1 \) such that

\[ S_0 \leq_{pr} R \leq_{pr} S_1 \leq_{pr} Id. \]

Proof. Upwards density (i.e. existence of \( S_1 \)) is a particular case of Theorem 6.1. For downward density (i.e. existence of \( S_0 \)), recall that if \( R \) is a punctual equivalence relation, then by Theorem 4.10, there exists \( S \) such that \( R \mid_{pr} S \): thus \( S_0 := R \wedge S \) is a punctual equivalence relation such that \( S_0 \leq_{pr} R \). \( \square \)

We now combine the density strategy of the previous theorem with the incomparability strategy exploited in Theorem 4.10,

**Theorem 6.7** (Density plus incomparability). If \( R_X \leq_{pr} R_T \leq_{pr} R_Z \) are punctual equivalence relations, then there exists a primitive recursive set \( Y \) such that \( R_X \leq_{pr} R_Y \leq_{pr} R_Z \) and \( R_T \mid_{pr} R_Y \).

Proof. Suppose that \( R_X \leq_{pr} R_T \leq_{pr} R_Z \) are punctual equivalence relations. To build \( Y \), a trivial modification of Theorem 6.1 suffices.

In the previous proof, we close the \( P_s \)-cycle in Case 1 of Step \( s+1 \) when we see that \( p_e \) has shown a counterexample to \( R_Y \leq_{pr} R_X \). For the purpose of the present proof, we now ask to close the \( P_s \)-cycle in Case 1 of Step \( s+1 \) when we have seen that \( p_e \) has shown a counterexample to \( R_Y \leq_{pr} R_T \), and we have matched up through the transition phase \( \#(0^X) = \#(0^Y) \): should \( p_e \) never show a counterexample to \( R_Y \leq_{pr} R_T \), then (as in the proof of Theorem 6.1) our copying phase of Case 1 would end up with making \( R_Z \leq_{pr} R_Y \), giving \( R_Z \leq_{pr} R_T \), a contradiction.

Similarly, here we ask to close the \( Q_s \)-cycle in Case 2 of Step \( s+1 \) when we see that \( p_e \) shows a counterexample to \( R_T \leq_{pr} R_Y \), and we have matched up through the transition phase \( \#(0^Y) = \#(0^Z) \). Should \( p_e \) never show a counterexample to \( R_T \leq_{pr} R_Y \), then (as in the proof of Theorem 6.1) our copying phase of Case 2 would end up with making \( R_Y \leq_{pr} R_X \), giving \( R_T \leq_{pr} R_X \), a contradiction. \( \square \)
Remark 6.8. Notice that the two previous theorems provide another proof of Theorem 4.10. Indeed, given a punctual \( R \prec_{pr} \text{Id} \), it is enough to pick by Corollary 5.4 \( S_0, S_1 \) such that \( S_0 \prec_{pr} R \prec_{pr} S_1 \), so that by density plus incomparability there exists \( S \parallel R \), with the stronger specification that \( S \) lies between \( S_0 \) and \( S_1 \).

Notice also that by a straightforward extension of the argument in Theorem 5.7 (in the same vein as in the argument for Corollary \[ \text{4.14} \]), one can show that if \( R \prec_{pr} S \) then one can build an infinite antichain whose members all lie between \( R \) and \( S \).

7. Join- and meet-reducibility

In a poset \( \langle P, \leq \rangle \) an element \( a \in P \) is \emph{join-reducible} if in \( P \) there are \( b, c < a \) such that \( a \) is the join of \( b, c \), and \( a \) is \emph{meet-reducible} if there are \( b, c > a \) such that \( a \) is the meet of \( b, c \).

Before showing that in \( \text{Peq} \) every element is join-reducible, and every \( R \prec_{pr} \text{Id} \) is meet-reducible, let us introduce some notations and simple observations which will be useful in the rest of this section.

In analogy with the principal function \( p_X \) of the complement \( \overline{X} \) (where \( X \subseteq \omega \)), given a string \( \sigma \in 2^{<\omega} \), let also \( p_{pr} \) denote the order preserving finite bijection \( p_{pr} : \{ n : n < \#(0^n) \} \rightarrow \overline{\sigma} \) (the notation \( \overline{\sigma} \) has been introduced in Section 2.4). Again in analogy with what we have done for sets (see Definition 5.6), we give the following definition.

Definition 7.1. Given \( \sigma, \tau \in 2^{<\omega} \) such that \( l^\sigma = l^\tau = h \) and \( \#(0^\sigma) = \#(0^\tau) = m \) let \( \sigma \lor \tau \) be the string with \( l^{\sigma \lor \tau} = h \) and such that \((\sigma \lor \tau)(i) = 0 \) if and only if \( i = \min(p_{pr}(n), p_{pr}(n)) \), for some \( n < m \). Dually, define \( \sigma \land \tau \) to be the string with \( l^{\sigma \land \tau} = h \) and such that \((\sigma \land \tau)(i) = 0 \) if and only if \( i = \max(p_{pr}(n), p_{pr}(n)) \), for some \( n < m \).

Notice that for \( \sigma, \tau \) as in the definition, we have \( \#(0^\sigma \lor \tau) = \#(0^\sigma \land \tau) = m \).

Lemma 7.2. Let \( (\sigma_0, \sigma_1) \) be a pair of strings such that \( l^{\sigma_0} = l^{\sigma_1} = h \) and \( \#(0^{\sigma_0}) = \#(0^{\sigma_1}) \); let \( (\tau_0, \tau_1) \) be another pair of strings such that \( l^{\tau_0} = l^{\tau_1} = h' \) and \( \#(0^{\tau_0}) = \#(0^{\tau_1}) \); finally, let \( Y_0, Y_1 \) be a pair of sets such that \( \sigma_0 \subseteq Y_0 \) and \( \sigma_1 \subseteq Y_1 \). Then, for an operation \( \circ \in \{ \lor, \land \} \),

1. for every \( i < h' \) we have \( (\sigma_0 \land \tau_0 \circ \sigma_1 \land \tau_1)(h + i) = (\tau_0 \circ \tau_1)(i) \);
2. \( \sigma_0 \circ \sigma_1 \subseteq Y_0 \circ Y_1 \), and for every \( i < h \), we have \( (Y_0 \circ Y_1)(i) = (\sigma_0 \circ \sigma_1)(i) \).

Proof. The proof is immediate. Let \( \#(0^{\sigma_0}) = \#(0^{\sigma_1}) = m \), and \( \#(0^{\tau_0}) = \#(0^{\tau_1}) = m' \). Item (1) follows from the fact that \( \sigma_0 \land \tau_0 \circ \sigma_1 \land \tau_1 \) has \( m + m' \) zeros: the first \( m \) ones of them (in order of magnitude) come from comparing the pairs \((p_{pr}(n), p_{pr}(n))\) with \( n < m \); and the last \( m' \) ones of them come from comparing the pairs \((p_{pr}(n), p_{pr}(n))\) with \( n < m' \), which amounts to comparing the pairs \((p_{pr}(n), p_{pr}(n))\) with \( n < m' \).

Finally (2) follows easily from (1). \( \square \)

Theorem 7.3. Each punctual \( R \not\parallel \) is join-reducible.
Proof. Let $R_Z$ be a punctual equivalence relation in normal form. As $\text{Id} \equiv_{pr} R_Z$, with $E$ denoting the set of even numbers, we can always assume that $Z$ is infinite and cofinite.

We will build $R_{Y_0}, R_{Y_1} \prec_{pr} R_Z$ such that $R_{Y_0} \lor R_{Y_1} = R_Z$. To do so, we will satisfy the following requirements:

$P_e$: $p_e$ does not reduce $R_Z$ to $R_{Y_1}$,

$Q_e$: $p_e$ does not reduce $R_Z$ to $R_{Y_0}$,

$N$: $R_Z = R_{Y_0} \lor R_{Y_1}$.

Notice that the $N$-requirement is actually requiring that $R_Z = R_{Y_0} \lor R_{Y_1}$, not just $R_Z \equiv_{pr} R_{Y_0} \lor R_{Y_1}$.

We will build $Y_0, Y_1$ in stages by approximating their characteristic functions, i.e., $Y_i = \bigcup_{x \in \omega} \sigma^Y_{x^i}$ for $i \in \{0, 1\}$. In the construction at each stage $s$ we will use also the string $\sigma^R_s$ which, we recall, is the initial segment of $Z$ with length $s$.

The construction. We adopt the same terminology and notations as those employed in Theorem \[\text{[1]}\]. As in the proof of that theorem, at each stage, the construction can be either in a copying phase or in a transition phase.

Stage 0. $\sigma^0_{Y_0} = \sigma^0_{Y_1} = \lambda$. Open the $P_0$-cycle, which will be implemented starting from next stage.

Stage $s + 1$. We distinguish two cases.

Case 1. Suppose that we are within a previously opened cycle $P_e$, which has not been declared closed yet. We assume by induction that we have $\# \langle Y_i \rangle[s + 1] \leq \# \langle Y_0 \rangle[s + 1] \leq \# \langle Y_2 \rangle[s + 1]$.

Copying phase. If we have not yet moved to the $P_e \rightarrow Q_e$-transition phase, then we copy $R_Z$ into $R_{Y_0}$: let $\sigma^Y_{s+1} = \sigma^Y_{s} \langle Z(s) \rangle$ and $\sigma^Y_{s+1} = \sigma^Y_{s} \langle 1 \rangle$. After this, if $p_e$ shows a counterexample to $R_Z \equiv_{pr} R_{Y_1}$ then go to the $P_e \rightarrow Q_e$-transition phase which will be implemented starting from the next stage.

Transition phase. Suppose that we are within the $P_e \rightarrow Q_e$-transition phase. Let $Y_1(s + 1) = 0$ and $Y_0(s + 1) = Z(s)$. After this, if $\# \langle Y_0 \rangle[s + 1] = \# \langle Y_1 \rangle[s + 1]$ then close the $P_e$-cycle and open the $Q_e$-cycle which will be implemented starting from next stage; otherwise, stay in this transition phase.

Case 2. Suppose that we are within a previously opened $Q_e$-cycle, which has not been declared closed yet. We assume by induction that we have $\# \langle Y_0 \rangle[s + 1] \leq \# \langle Y_1 \rangle[s + 1] \leq \# \langle Y_2 \rangle[s + 1]$.

Copying phase. If we have not yet moved to the $Q_e \rightarrow P_{e+1}$-transition phase, then we copy $R_Z$ into $R_{Y_1}$: let $\sigma^Y_{s+1} = \sigma^Y_{s} \langle 1 \rangle$ and $\sigma^Y_{s+1} = \sigma^Y_{s} \langle Z(s) \rangle$. After this, if $p_e$ shows a counterexample to $R_Z \equiv_{pr} R_{Y_0}$ then go to the $Q_e \rightarrow P_{e+1}$-transition phase which will be implemented starting from the next stage; otherwise, stay in this transition phase.
Transition phase. Suppose that we are within the $Q_e \to P_{e+1}$-transition phase. Let $Y_0(s+1) = 0$ and $Y_1(s+1) = Z(s)$. After this, if $\#(0^{Y_0})[s+1] = \#(0^{Y_1})[s+1]$ then close the $Q_e$-cycle and open the $P_{e+1}$-cycle which will be implemented starting from next stage; otherwise, stay in this transition phase.

Notice that for every $s$, the constructed initial segment $\sigma^Y_{s+1}$ of $Y_1$ has length $s$. (We note that the distinction between “copying phase” and “transition phase” can be misleading, as in the transition phase of Case 1 we still keep copying into $Y$.)

So, suppose that at stage $s$, mapping $s \mapsto \sigma^Y_{s+1}$ is primitive recursive.

The rest of the verification is based on the following lemmas.

Lemma 7.4. The $P$- and $Q$- requirements are satisfied. Moreover, if $s$ is a stage at which we close a cycle, then $\#(0^{Y_0})[s] = \#(0^{Y_1})[s]$.

Proof. As in the proof of Theorem 4.10 and Theorem 5.1, if $s$ is any stage, then at $s$ we are either in an open $P_e$- or $Q_e$-cycle for exactly one $e$.

Eventually any $P$- or $Q$- cycle will be closed. This is easily seen by induction. Suppose that at stage $s_0$ we open the $P_e$-cycle (the $P_0$-cycle is opened at stage 0). Then eventually $p_e$ shows a counterexample to $R_Z \preceq_{pr} R_{Y_1}$ (thus $P_e$ is satisfied), otherwise our copying procedure would put all fresh elements into $Y_1$, giving that $R_{Y_1}$ is finite, contradicting that $R_Z$ is punctual and thus non-finite. After $p_e$ has shown a counterexample, we start the $P_e \to Q_e$-transition phase. During the transition we keep all fresh elements out of $Y_1$. This makes the number of 0’s of $Y_1$ growing as fast as possible, while we copy $Z$ in $Y_0$. Eventually, we will witness a stage $s$ at which $\#(0^{Y_0})[s] = \#(0^{Y_1})[s]$; otherwise, $Z$ would be finite contradicting the fact that $Z$ is infinite. This shows that the $P_e$-cycle is eventually closed, and we open the $P_e$-cycle.

By a similar argument we can prove that each $Q_e$-cycle is eventually opened, then closed, and the corresponding requirement is satisfied.

Lemma 7.5. The $N$-requirement is satisfied.

Proof. We want to show that $Z = Y_0 \lor Y_1$. Let $s_0 < s_1 < \ldots$ be the sequence of stages at which we open a cycle, with $s_0 = 0$. Our argument is by induction on the index $n$ of $s_n$. Assume by induction that, for every $i < s_n$ we have $(Y_0 \lor Y_1)(i) = Z(i)$: this is true if $n = 0$. Assume that at $s_n$ we open a $P_e$-cycle, the other case being similar: this cycle is closed at the stage $s_{n+1}$.

By construction, $\sigma^{Y_0}_{s_{n+1}} = \sigma^{Y_0}_{s_n} \tau_0$ and $\sigma^{Y_1}_{s_{n+1}} = \sigma^{Y_1}_{s_n} \tau_1$, where

$$\tau_0 = \langle Z(s_0), \ldots, Z(s_{n+1} - 1) \rangle \text{ and } \tau_1 = 1^h 0^k,$$

for some $h, k$ with $h + k = s_{n+1} - s_n$ (here, for $i \in \{0, 1\}$, $i^m$ denotes the string of length $m$ giving value $i$ on all its inputs). As in $\tau_1$ the bit 0 appears only in the final segment $0^k$, for every $i < s_{n+1} - s_n$ we have that

$$(\tau_0 \lor \tau_1)(i) = Z(s_n + i).$$
It then follows from Lemma 7.2 that for every \( j < s_{n+1} - s_n \) we have
\[(Y_0 \lor Y_1)(s_n + j) = (\tau_0 \lor \tau_1)(j) = Z(s_n + j),\]
giving that \((Y_0 \lor Y_1)(i) = Z(i)\) for all \( i < s_{n+1} \).
\[\square\]

This concludes the verification. \[\square\]

By a symmetric argument, one can also show the following.

**Theorem 7.6.** Any \( R_Z \preceq_{pr} \text{Id} \) is meet-reducible.

*Proof.* The requirements are

\[
P_e: p_e \text{ does not reduce } R_{Y_1} \text{ to } R_Z, \\
Q_e: p_e \text{ does not reduce } R_{Y_0} \text{ to } R_Z, \\
M: R_Z = R_{Y_0} \land R_{Y_1}. 
\]

The proof and the construction are similar to the previous theorem, with the modifications that whenever in the previous theorem in a copying or transition phase we added the bit \( i \) to \( Y_0 \) or \( Y_1 \), we now add the bit \( 1 - i \). Notice for instance that for every \( e, p_e \) eventually shows a counterexample to \( R_{Y_1} \preceq_{pr} R_Z \) as otherwise now \( Y_1 \) would be eventually finite, thus \( R_{Y_1} \equiv_{pr} \text{Id} \), and thus \( \text{Id} \preceq_{pr} R_Z \), a contradiction.

So \( P_e \) is satisfied. A similar argument shows that each \( Q_e \) is satisfied.

In order to show that \( R_Z = R_X \land R_Y \), notice that this time (assuming that at \( s_n \) we open a \( P_e \)-cycle, the other case being similar) \( \sigma_{s_{n+1}}^{Y_0} = \sigma_{s_n}^{Y_0} \land \tau_0 \) and \( \sigma_{s_{n+1}}^{Y_1} = \sigma_{s_n}^{Y_1} \land \tau_1 \), where \( \tau_1 = 0^h \land 1^k \), for some \( h, k \) with \( h + k = s_{n+1} - s_n \), and \( \tau_0 = \langle Z(s_n), \ldots, Z(s_{n+1} - 1) \rangle \). It then follows from Lemma 7.2 (as in \( \tau_1 \) the 0's show up before the 1's) that for every \( j < s_{n+1} - s_n \) we have
\[(Y_0 \land Y_1)(s_n + j) = (\tau_0 \land \tau_1)(j) = Z(s_n + j).\]

Thus by induction on the index \( n \) of \( s_n \) we can show that \((Y_0 \land Y_1)(i) = Z(i)\) for all \( i < s_n \).
\[\square\]

8. **Embedding of the diamond lattice**

So far, we highlighted that \( \text{Peq} \) is a remarkably well-behaved degree structure, being a dense distributive lattice. Moreover, we proved that degrees below the top are not distinguishable with respect to join- or meet-reducibility. In these two remaining sections, we will turn the perspective upside down, focusing on some fairly unexpected ill-behaviour of \( \text{Peq} \). In particular, in this section we show that some intervals of punctual equivalence relations \( R \preceq_{pr} S \) embed the diamond lattice, and some other don’t. In fact, we will offer a complete characterization of the intervals which embed the diamond lattice, by relying on a combinatorial property of primitive recursive sets \( X \) and \( Y \), named \( \diamond \)-property, which intuitively says that there are infinitely many initial segments of the natural numbers up to which \( X \) and \( Y \) have an equivalent number of zeros.

Given a set \( U \), throughout the section we agree, as in the proof of Theorem 6.1, that \( \sigma_s^U \) denotes the initial segment of \( U \) having length \( s + 1 \) and \( \#(0^U)[s] \) denotes the cardinality of \( \sigma_s^U \). To avoid trivial cases, we also assume that here we consider only primitive sets that are infinite and cofinite.
Definition 8.1. We say that a pair \((X,Y)\) of primitive recursive sets satisfies the \(\diamond\)-property if there is a pair \((X^*,Y^*)\) of primitive recursive sets such that \(R_{X^*} \equiv_{pr} R_X\) and \(R_{Y^*} \equiv_{pr} R_Y\) and
\[
(\forall \omega)(\exists s)[\#(0^{X^*})[s] = \#(0^{Y^*})[s]].
\]

We say that a stage \(s\) is an equilibrium point for a pair \((X,Y)\) of primitive recursive sets if
\[
\#(0^X)[s] = \#(0^Y)[s].
\]

Theorem 8.2. Suppose that a pair \((X,Z)\) does not satisfy the \(\diamond\)-property, and \(R_X \prec_{pr} R_Z\). Then, there are no \(Y_0\) and \(Y_1\) such that
\[
R_X \prec_{pr} R_{Y_i}, \prec_{pr} R_Z, R_X \equiv_{pr} R_{Y_0} \wedge R_{Y_1}, \text{ and } R_Z \equiv_{pr} R_{Y_0} \vee R_{Y_1}.
\]

Proof. We show the contrapositive statement. Suppose that \(R_X, R_{Y_0}, R_{Y_1}\), and \(R_Z\) form a diamond. Without loss of generality, one may assume that \(0 \in Y_0\).

Lemma 8.3. The pair \((Y_0,Y_1)\) has infinitely many equilibrium points.

Proof. Suppose that \(s^\ast\) is the last equilibrium point for \((Y_0,Y_1)\). Then, without loss of generality, we may assume that for any \(s > s^\ast\),
\[
\#(0^Y)[s] > \#(0^Y)[s].
\]

Let \(n^\ast := \#(0^Y)[s^* + 1]\). For a number \(m \geq n^\ast\), as \(p_{\omega^*_{Y_0}}(m) > s^* + 1\), we have
\[
m + 1 = \#(0^Y)[p_{\omega^*_{Y_0}}(m)] > \#(0^Y)[p_{\omega^*_{Y_0}}(m)],
\]
and thus \(p_{\omega^*_{Y_0}}(m) < p_{\omega^*_{Y_1}}(m)\). By the definition of \(Y_0 \vee Y_1\), we deduce that for every \(m \geq n^\ast\), we have \(p_{\omega^*_{Y_0} \vee Y_1}(m) = p_{\omega^*_{Y_0}}(m)\). Therefore, the function
\[
f(x) := \begin{cases} 0, & \text{if } x \in Y_0 \vee Y_1, \\ p_{\omega^*_{Y_0}}(l), & \text{if } x = p_{\omega^*_{Y_0} \vee Y_1}(l) \text{ for some } l < n^\ast, \\ x, & \text{otherwise,} \end{cases}
\]
provides a \(pr\)-reduction from \(R_{Y_0} \vee R_{Y_1}\) into \(R_{Y_0}\), which gives a contradiction. \(\square\)

Let \(s_0 < s_1 < s_2 < \ldots\) be the sequence of all equilibrium points for \((Y_0,Y_1)\). We choose an infinite subsequence of equilibrium points
\[
s^\ast_0 < s^*_1 < s^*_2 < \ldots
\]
such that for every \(i \in \omega\), \(#(0^Y)[s^*_i] < #(0^Y)[s^*_i] [s^*_i + 1]\).

Let \(t_i := \#(0^Y)[s^*_i]\). By Lemma 8.2, it is clear that
\[
\#(0^Y \vee Y_1)[s^*_i] = \#(0^{Y_0 \wedge Y_1})[s^*_i] = t_i,
\]
and hence, the pair \((X,Z)\) satisfies the \(\diamond\)-property. Theorem 8.2 is proved. \(\square\)

On the other hand, the next result shows that the \(\diamond\)-property is sufficient for embedding the diamond:

Theorem 8.4. Let \(R_X \prec_{pr} R_Z\) such that \((X,Z)\) satisfies the \(\diamond\)-property. There are punctual \(R_{Y_0}, R_{Y_1}\) such that the infimum (resp. supremum) of \(R_{Y_0}\) and \(R_{Y_1}\) is \(pr\)-equivalent to \(R_X\) (resp. \(R_Z\)).

Proof. Without loss of generality, assume that \(X\) and \(Z\) are both infinite, co-infinite, and \(0 \in X \cap Z\). Observe also that we may assume that the following hold
(1) the pair $(X, Z)$ has infinitely many equilibrium points;
(2) for all $s$, $\#(0^Z)[s] \geq \#(0^X)[s]$.

To see this that this can be assumed, first choose $X, Z$ with infinitely many equilibrium points; they must exist since $(X, Z)$ has the \( \star \)-property. Second, replace $Z$ with $X \lor Z$ if needed; note that if $t$ is an equilibrium point for $(X, Z)$, then by Lemma 7.2 it should be an equilibrium point for $(X, X \lor Z)$ as well.

We will build sets $Y_0, Y_1$ in stages, satisfying the following requirements:

\[ Q_e: p_e \text{ does not reduce } R_Z \text{ to } R_{Y_0}, \]
\[ P_e: p_e \text{ does not reduce } R_Z \text{ to } R_{Y_1}, \]
\[ M: R_X = R_{Y_0} \land R_{Y_1}, \]
\[ N: R_Z = R_{Y_0} \lor R_{Y_1}. \]

It is easy to see that the above requirements are sufficient; in particular, we do not need to prove that $R_{Y_i} \preceq_{pr} R_X$. Notice also that we require $R_X$ and $R_Z$ to be in fact equal, and not just $pr$-equivalent, to $R_{Y_0} \land R_{Y_1}$ and $R_{Y_0} \lor R_{Y_1}$, respectively: therefore we get for free that $R_{Y_0}$ and $R_{Y_1}$ lie in the interval determined by $R_X$ and $R_Z$. As always, we will build $Y_0, Y_1$ in stages by approximating their characteristic functions, i.e., $Y_i = \bigcup_{s \in \omega} \sigma^Y_s$ for $i \in \{0, 1\}$. Each string $\sigma^Y_s, \sigma^X_s, \sigma^Z_s$ we define or deal with at a stage $s$ has length $s + 1$.

The strategies. In order to achieve that $p_e$ does not reduce $R_Z$ to $R_{Y_0}$, we open the $Q_e$-cycle and employ a copying procedure, copying $R_X$ into $R_{Y_0}$, until we see that $p_e$ shows a counterexample to $R_Z \preceq_{pr} R_{Y_0}$. Meanwhile we employ a corresponding copying procedure, copying $R_Z$ into $R_{Y_1}$.

When seeing that $p_e$ shows a counterexample at stage, say, $s + 1$, by our assumption on always being $\#(0^X)[t] \leq \#(0^Z)[t]$ it may happen that $\#(0^X)[s + 1] < \#(0^Z)[s + 1]$. If so, before closing the $Q_e$-cycle we open the so-called $Q_e \rightarrow P_e$-transition phase, which consists (still copying $R_X$ into $R_{Y_0}$ and $R_Z$ into $R_{Y_1}$) in prolonging bit by bit $\sigma^Y_{s+1}$ (which the construction has guaranteed to have the same number of 0’s as $\sigma^X_{s+1}$) with the bits of $X$, and in prolonging bit by bit $\sigma^Y_{s+1}$ (which the construction has guaranteed to have the same number of 0’s as $\sigma^Z_{s+1}$) with the bits of $Z$, until we reach the next equilibrium point of $(X, Z)$: at this point we close the $Q_e$-cycle and we open the $P_e$-cycle.

The described procedure has the goal of making it possible to apply Lemma 7.2 and conclude that the bits added to $\sigma^Y_{s+1}$ and $\sigma^Y_{s+1}$ since when we opened the $Q_e$-cycle satisfy

\[ (\sigma^Y_{s+1} \lor \sigma^Y_{s+1})(i) = (X \lor Z)(i) \text{ and } (\sigma^Y_{s+1} \land \sigma^Y_{s+1})(i) = (X \land Z)(i), \]

so as to eventually get $Y_0 \lor Y_1 = X \lor Z$ and $Y_0 \land Y_1 = X \land Z$.

In order to achieve that $p_e$ does not reduce $R_Z$ to $R_{Y_1}$, we use (in an obvious way) a similar strategy, this time copying $R_X$ into $R_{Y_1}$ and $R_Z$ into $R_{Y_0}$; we go into the $P_e \rightarrow Q_{e+1}$-transition phase when $p_e$ shows a counterexample. Finally, after reaching the next equilibrium point, we close the $P_e$-cycle and open the $Q_{e+1}$-cycle.

The construction. Unless otherwise specified, we adopt the same terminology and notation employed in Theorem 6.1.

Stage 0. $\sigma^Y_0 = \sigma^Y_0 = (1)$. Open the $Q_0$-cycle.
Stage $s + 1$. There are two cases.

Case 1. Suppose that we are within a previously opened $Q_e$-cycle, which has not been declared closed. We assume by induction that we have

$$\#(0^X)[s + 1] = \#(0^Y)[s + 1] \leq \#(0^Y_1)[s + 1] = \#(0^Z)[s + 1],$$

and the first stage $s'$ of this particular $Q_e$-cycle has the following property

$$\#(0^X)[s'] = \#(0^Y)[s'] = \#(0^Y_1)[s'] = \#(0^Z)[s'].$$

Copying phase. If we have not yet moved to the $Q_e \rightarrow P_e$-transition phase, then we copy $R_X$ into $R_Y$ and copy $R_Z$ into $R_Y$: let $\sigma_{s+1}^{Y_0} = \sigma_{s+1}^{Y_0} \triangleq (X(s+1))$ and $\sigma_{s+1}^{Y_1} = \sigma_{s+1}^{Y_1} \triangleq (Z(s+1))$. After this, if $p_e$ shows a counterexample to $R_Z \leq_{pr} R_Y$, then go to the $Q_e \rightarrow P_e$-transition phase, which will be implemented starting from the next stage.

Transition phase. Suppose that we are within the transition phase of a previously opened $Q_e$-cycle, which has not been declared closed yet. Let $\sigma_{s+1}^{Y_0} = \sigma_{s+1}^{Y_0} \triangleq (X(s+1))$ and $\sigma_{s+1}^{Y_1} = \sigma_{s+1}^{Y_1} \triangleq (Z(s+1))$. After this, if $\#(0^Y_0)[s + 1] = \#(0^Y_1)[s + 1]$, then close the $Q_e$-cycle, and open the $P_e$-cycle, which will be processed starting from next stage.

Case 2. Suppose that we are within a previously opened $P_e$-cycle, which has not been declared closed yet. We assume by induction that

$$\#(0^X)[s + 1] = \#(0^Y)[s + 1] \leq \#(0^Y_1)[s + 1] = \#(0^Z)[s + 1],$$

and when we had opened this cycle, we had $\#(0^Y_1)[s'] = \#(0^Y)[s']$.

Copying phase. If we have not yet moved to the $P_e \rightarrow Q_{e+1}$-transition phase, then let $\sigma_{s+1}^{Y_0} = \sigma_{s+1}^{Y_0} \triangleq (Z(s+1))$ and $\sigma_{s+1}^{Y_1} = \sigma_{s+1}^{Y_1} \triangleq (X(s+1))$. After this, if $p_e$ has shown a counterexample for $p_e: R_Z \leq_{pr} R_Y$, then move to the $P_e \rightarrow Q_{e+1}$-transition phase.

Transition phase. Suppose that we are within the transition phase of a previously opened $P_e$-cycle, which has not been declared closed yet. Let $\sigma_{s+1}^{Y_0} = \sigma_{s+1}^{Y_0} \triangleq (Z(s+1))$ and $\sigma_{s+1}^{Y_1} = \sigma_{s+1}^{Y_1} \triangleq (X(s+1))$. After this, if $\#(0^Y_0)[s + 1] = \#(0^Y_1)[s + 1]$ then close the $P_e$-cycle, and open the $Q_{e+1}$-cycle, which will be processed starting from next stage.

The verification. The sets $Y_0, Y_1$ are primitive recursive as $Y_i(s) = \sigma_{s}^{Y_i}(s)$ (recall that $l^{Y_i} = s + 1$) and the mapping $s \mapsto \sigma_{s}^{Y_i}$ is primitive recursive. The rest of the verification is based on the following lemmas.

**Lemma 8.5.** The $P$- and $Q$- requirements are satisfied.

**Proof.** Similarly to the proofs of Theorem 5.3 and Theorem 6.3, it is easily seen by induction that every $P$- or $Q$-cycle is opened and later closed, and exactly one cycle is open at each stage. Consider for instance a $Q_e$-cycle, and assume that it was opened at stage $s_0$, and we started processing the cycle from $s_0 + 1$. Assume also that

$$\#(0^X)[s_0] = \#(0^Y_0)[s_0].$$
Should \( p_e \) never show a counterexample to \( R_Z \leq_{pr} R_{Y_0} \) then it would be \( R_Z \leq_{pr} R_X \). Indeed, in this case we would eventually get \( Y_0 = \sigma_{Y_0}^e \ast X \) (see Section 2.4 for the notation): thus, \( p_e : R_Z \leq_{pr} R_{Y_0} \) would imply \( R_Z \leq_{pr} R_X \). Therefore, eventually we do get a counterexample, and requirement \( Q_e \) is satisfied.

After \( p_e \) shows a counterexample, we start the transition phase: when we open it (say, at \( s_1 \)) we have
\[
\# (0^{Y_0}) [s_1] \leq \# (0^{Y_1}) [s_1].
\]
By our assumption that \((X, Z)\) has the \( \Diamond \)-property it follows (by prolonging \( \sigma_{Y_0} \) as \( X \), and \( \sigma_{Y_1} \) as \( Z \)) that eventually \( \# (0^{Y_0}) \) catches up with \( \# (0^{Y_1}) \), thus we reach a stage \( s + 1 \) when \( \# (0^{Y_0}) [s + 1] = \# (0^{Y_1}) [s + 1] \). At this stage, we close the \( Q_e \)-cycle and we open the \( P_e \)-cycle.

A similar claim holds for \( P_e \)-cycles. Note that in the second part of the cycle, the transition phase waits until the inequality \( \# (0^{Y_1}) [t] \leq \# (0^{Y_0}) [t] \) reaches a stage \( s + 1 \) such that \( \# (0^{Y_1}) [s + 1] = \# (0^{Y_0}) [s + 1] \).

We also conclude that all \( P \) - and \( Q \) -requirements are satisfied. 

\[ \square \]

Lemma 8.6. The \( M \)-requirement and the \( N \)-requirement are satisfied.

Proof. Let \( 0 = s_0 < s_1 < \ldots \) be an infinite sequence of stages \( s \) at which we have
\[
\# (0^{X}) [s] = \# (0^{Y_0}) [s] = \# (0^{Y_1}) [s] = \# (0^Z) [s].
\]
For instance, this happens when we open cycles.

Let \( i \in \omega \) and let \( n \) be such that \( i < s_{n+1} - s_n \). Suppose that at \( s_n \) we open a \( Q \)-cycle — then \( \sigma_{Y_0}^{s_{n+1}} = \sigma_{Y_0}^{s_n} \uparrow \tau_0 \) and \( \sigma_{Y_1}^{s_{n+1}} = \sigma_{Y_1}^{s_n} \uparrow \tau_1 \), where \( \tau_0 (i) = X (s_n + 1 + i) \) and \( \tau_1 (i) = Z (s_n + 1 + i) \). Since the pairs \( (\sigma_{s_n}^{Y_0}, \sigma_{s_n}^{Y_1}), (\tau_0, \tau_1), \) and \( (Y_0, Y_1) \) satisfy the assumptions of Lemma 7.2, it follows by induction on the index \( n \) of \( s_n \) that
\[
(Y_0 \land Y_1)(i) = (X \land Z)(i) \\
(Y_0 \lor Y_1)(i) = (X \lor Z)(i).
\]

Hence, we deduce that \( R_{Y_0} \land R_{Y_1} = R_X \) and \( R_{Y_0} \lor R_{Y_1} = R_Z \). Lemma 8.1 is proved.

This concludes the verification. Theorem 8.1 is proved. 

Combining Theorem 8.2 and Theorem 8.3, we obtain the following.

Corollary 8.7. An interval \([R, S]\) of \( \text{Peq} \) embeds the diamond lattice preserving 0 and 1 if and only if \((R, S)\) has the \( \Diamond \)-property.

9. ON THE INTRICACY OF \( \text{Peq} \)

In this final section, we deepen the analysis of \( \text{Peq} \), unveiling further structural complexity. Most notably, we will focus on the automorphisms of \( \text{Peq} \), proving that such a degree structure is neither rigid nor homogeneous. We will also show that \( \text{Peq} \) contains nonisomorphic lowercones. These results will require both to further explore the consequences of the \( \Diamond \)-property defined above and to introduce another property, named slowness, concerning the rate at which a primitive recursive set shows its zeros. We conclude the section by collecting a number of interesting open questions, which may motivate future work. We are particularly interested in whether the theory of \( \text{Peq} \) is decidable or not.
Remark 9.1. (Redefining the symbol \(\#(0^X)[s]\).) In this section, for technical reasons, we will take \(\#(0^X)[s]\) to be the number of \(i \leq k\) such that \(X(i) = 0\), where \(k\) is the largest such that \(X(j) \downarrow\) in at most \(s\) many steps for all \(j \leq k\). So \(\#(0^X)[s]\) is the number of zeroes that we can see in the characteristic function of \(X\) after evaluating it for \(s\) many steps.

The next lemma is an analogue of Proposition 5.1 — it expresses \(R_X \leq_{pr} R_Y\) in terms of the growth rates of \(\#(0^X)[s]\) and \(\#(0^Y)[t]\):

**Lemma 9.2.** Given any \(X, Y, R_X \leq_{pr} R_Y\) if and only if there exists a primitive recursive function \(p\) such that for every \(s, \#(0^X)[s] \leq \#(0^Y)[p(s)]\).

**Proof.** Suppose that \(R_X \leq_{pr} R_Y\) via \(f\). For each \(s\) we let \(p(s)\) be the least stage \(t > s\) such that \(Y(n)[t] \downarrow\) for all \(n \leq f(s)\). Then \(\#(0^X)[s] \leq \#(0^Y)[p(s)]\).

Now conversely fix \(p\). For each \(m\) we find the first stage \(t\) for which we have \(X \uparrow (m+1)[t] \downarrow\). Let \(h(m) = p(t)\). Then \(h(p_{\Gamma}(n)) \geq p_{\Gamma}(n)\) for all \(n\) and by Proposition 5.1, \(R_X \leq_{pr} R_Y\). \(\Box\)

It is easy to see that there are \(R <_{pr} S\) such that the pair \((R, S)\) satisfies the \(\clubsuit\)-property: By Theorem 1.1 take a pair of incomparable \(Y_0 \uparrow_{pr} Y_1\), then \(R = Y_0 \wedge Y_1 <_{pr} S = Y_0 \vee Y_1\) has the \(\clubsuit\)-property. In fact, by Theorems 7.3 and 7.6, given any \(R <_{pr} \text{Id}\) there is some \(S >_{pr} R\), and given any \(S\) there is some \(R <_{pr} S\) such that \((R, S)\) has the \(\clubsuit\)-property. So every punctual degree is the top and (if it is not \(\text{Id}\)) the bottom of an interval with the \(\clubsuit\)-property.

However, since the \(\clubsuit\)-property is a property of a \(pr\)-degree and not of a set, it is not totally obvious why there should be an interval that does not satisfy the \(\clubsuit\)-property. We prove a lemma which expresses the \(\clubsuit\)-property as a property about sets. Note that the \(\clubsuit\)-property does not apriori require the two sets to be comparable, and the characterization below holds in general.

**Lemma 9.3.** A pair \((X, Y)\) satisfies the \(\clubsuit\)-property if and only if there exist primitive recursive functions \(p\) and \(q\) such that \(\#(0^X)[s] = \#(0^Y)[p(s)] = \#(0^Y)[t] = \#(0^X)[q(t)]\) for infinitely many \(s, t\).

**Proof.** Suppose that \((X, Y)\) satisfies the \(\clubsuit\)-property. Fix \((X^*, Y^*)\) witnessing that the pair \((X, Y)\) has the \(\clubsuit\)-property, so that

\[
(\forall s)(\exists t \geq s) \left[\#(0^{X^*})[t] = \#(0^{Y^*})[t]\right],
\]

and functions \(f_{X,X^*}, f_{X^*,X}, f_{Y,Y^*}\) and \(f_{Y^*,Y}\) satisfying

\[
\#(0^X)[s] \leq \#(0^{Y^*})[f_{X,X^*}(s)], \quad \#(0^{X^*})[s] \leq \#(0^X)[f_{X^*,X}(s)],
\]
\[
\#(0^Y)[s] \leq \#(0^{Y^*})[f_{Y,Y^*}(s)], \quad \#(0^{Y^*})[s] \leq \#(0^Y)[f_{Y^*,Y}(s)]
\]

for every \(s\), respectively (applying Lemma 9.2). Then obviously we should take \(p\) and \(q\) to be the composition of the given functions in the correct order. More specifically, we let \(p(s) = \) the least stage \(t \leq f_{Y^*,Y}(f_{X,X^*}(s + 1))\) such that \(\#(0^X)[s] = \#(0^Y)[t]\), and if \(t\) cannot be found then let \(p(s) = s\). Similarly, let \(q(t) = \) the least stage \(u \leq f_{X^*,X}(f_{Y,Y^*}(t + 1))\) such that \(\#(0^Y)[t] = \#(0^X)[u]\), and if \(u\) cannot be found then let \(q(t) = t\).

We first check that \(p\) works. Let \(w\) and \(j\) be such that

\[
\#(0^{X^*})[w] = \#(0^{Y^*})[w] = j.
\]
Let \( s \) be the greatest stage such that \( \# \left( 0^X \right) [s] = j \). Then as \( \# \left( 0^{X^*} \right) [w] = j \) and \( \# \left( 0^X \right) [s + 1] = j + 1 \), we certainly have \( w < f_{X,X^*}(s + 1) \). Then
\[
f_{Y^*,Y}(f_{X,X^*}(s + 1)) \geq f_{Y^*,Y}(w)
\]
and also
\[
\# \left( 0^Y \right) [f_{Y^*,Y}(w)] \geq \# \left( 0^{Y^*} \right) [w] = j.
\]
Therefore, the bound \( f_{Y^*,Y}(f_{X,X^*}(s + 1)) \) is large enough, and we have
\[
\# \left( 0^X \right) [s] = \# \left( 0^Y \right) [p(s)] = j.
\]
A similar argument holds for \( q \).

Now conversely, assume that \( p \) and \( q \) exist. It is easy to see that we can make \( p \) and \( q \) nondecreasing. We wish to show that \((X,Y)\) satisfies the \( \lozenge \)-property. An obvious candidate for \( X^* \) is a set satisfying \( \# \left( 0^{X^*} \right) [p(s)] = \# \left( 0^X \right) [s] \) for every \( s \), and then we can take \( Y^* = Y \), so that \( \# \left( 0^{X^*} \right) [p(s)] = \# \left( 0^X \right) [p(s)] \) holds for infinitely many \( s \). Unfortunately, in order to do this, we will need to compute \( p^{-1} \) which in general is not primitive recursive. So we will have to use both \( p \) and \( q \) to define \( X^* \) and \( Y^* \).

We call \((s,t)\) a **good pair** if
\[
\# \left( 0^X \right) [s] = \# \left( 0^Y \right) [p(s)] = \# \left( 0^Y \right) [t] = \# \left( 0^X \right) [q(t)];
\]
by the hypothesis there are infinitely many good pairs.

First, suppose that there are infinitely many good pairs \((s,t)\) such that \( p(s) \geq s \). Define \( c(w) \) to be the largest value of \( u \leq w \) such that \( \# \left( 0^X \right) [u] \leq \# \left( 0^Y \right) [w] \) or \( p(u) < w \). Notice that the function \( c \) is primitive recursive and non-decreasing. Therefore, so is the function \( d(w) = \min \{d(w - 1) + 1, \# \left( 0^X \right) [c(w)]\} \). Furthermore, \( d \) has the property that for any \( w \), \( d(w + 1) \leq d(w) + 1 \), and that for any \( w \) there is some \( t \) satisfying \( w \leq t \leq 2w \) such that \( d(t) = \# \left( 0^X \right) [c(w)] \). (Recall our convention that for any set \( Z \) and any stage \( t \), \( \# \left( 0^Z \right) [t + 1] = \# \left( 0^Z \right) [t] + 1 \). Therefore we can define the primitive recursive set \( X^* \) satisfying \( \# \left( 0^{X^*} \right) [w] = d(w) \) for all \( w \). Take \( Y^* = Y \).

Let \( \tilde{d}(w) \) be the largest value \( \leq w \) such that \( \# \left( 0^X \right) [\tilde{d}(w)] = d(w) \), which is also primitive recursive. Therefore, by Lemma 1.3 we obviously have \( R_{X^*} \leq_{pr} R_X \). Now we observe that for each \( s, t \leq c(w) \) where \( w = \max \{s, p(s) + 1\} \). Therefore
\[
\# \left( 0^{X^*} \right) [2w] = d(2w) \geq \# \left( 0^X \right) [c(w)] \geq \# \left( 0^X \right) [s],
\]
showing that \( R_{X^*} \geq_{pr} R_X \). Now it follows by a straightforward induction on \( w \) that the following claim is true:
\[
d(w) \geq \max \{\# \left( 0^X \right) [u] | \# \left( 0^X \right) [u] \leq \# \left( 0^Y \right) [w] \}
\]
for some \( u \leq w \) (using the fact that
\[
\# \left( 0^X \right) [c(w)] \geq \max \{\# \left( 0^X \right) [u] | \# \left( 0^X \right) [u] \leq \# \left( 0^Y \right) [w] \}
\]
for some \( u \leq w \)). Now take \((s,t)\) to be a good pair with \( p(s) \geq s \). By the claim above, we have \( \# \left( 0^{X^*} \right) [p(s)] \geq \# \left( 0^X \right) [s] \). If they were not equal then \( d(p(s)) \)
would have to be larger than \( \#(0^X)[s] \) which means that
\[
\#(0^X)[s] < d(p(s)) \leq \#(0^X)[c(p(s))].
\]
But then as \( c(p(s)) \geq s \) and \( p \) is nondecreasing, we have \( p(c(p(s))) \geq p(s) \), which means, by the definition of \( c(p(s)) \), that
\[
\#(0^X)[c(p(s))] \leq \#(0^Y)[p(s)] = \#(0^X)[s],
\]
a contradiction. Thus we conclude that
\[
\#(0^X)[p(s)] = \#(0^X)[s] = \#(0^Y)[p(s)] = \#(0^Y)[p(t)].
\]
If there are infinitely many good pairs \((s, t)\) such that \( q(t) \geq t \) then we repeat the above, now taking \( X^* = X \) and \( Y^* \) defined analogously, using \( q \) in place of \( p \). So we assume that there are infinitely many good pairs \((s, t)\) such that \( p(s) < s \) and \( q(t) < t \). We claim that we can take \( X = X^* \) and \( Y = Y^* \). Fix a good pair \((s, t)\) such that \( p(s) < s \), \( q(t) < t \) and
\[
\#(0^X)[s] = \#(0^Y)[p(s)] = \#(0^Y)[t] = \#(0^X)[q(t)] = j.
\]
Suppose that
\[
\min\{w \mid \#(0^X)[w] = j\} \leq \min\{w \mid \#(0^Y)[w] = j\}.
\]
Now this means that
\[
p(s) \geq \min\{w \mid \#(0^X)[w] = j\}
\]
and since \( p(s) < s \) we have that
\[
\#(0^X)[p(s)] = j = \#(0^Y)[p(s)].
\]
On the other hand if
\[
\min\{w \mid \#(0^X)[w] = j\} \geq \min\{w \mid \#(0^Y)[w] = j\}
\]
then
\[
\#(0^Y)[q(t)] = j = \#(0^X)[q(t)].
\]
\[\square\]

**Corollary 9.4.** Let \( R_X \leq_{pr} R_Y \). Then \((X, Y)\) satisfies the \( \bullet \)-property if and only if there exists a primitive recursive function \( q \) such that \( \#(0^Y)[t] = \#(0^X)[q(t)] \) for infinitely many \( t \). Furthermore, we can take \( q \) to be nondecreasing, and we may also replace “\#(0^Y)[t] = \#(0^X)[q(t)]” with “\#(0^Y)[t] \leq \#(0^X)[q(t)]”.

**Proof.** If \( R_X \leq_{pr} R_Y \) then we fix by Lemma 1.2 a function \( g \) such that \( \#(0^X)[s] \leq \#(0^Y)[g(s)] \) for every \( s \). But \( p(s) \leq g(s) \) for every \( s \), where \( p(s) \) is the least such that \( \#(0^X)[s] = \#(0^Y)[p(s)] \). \[\square\]

**Corollary 9.5.** If \( R_X \leq_{pr} R_Y \) and \((X, Y)\) has the \( \bullet \)-property then every subinterval of \((X, Y)\) also has the \( \bullet \)-property.

**Proof.** Suppose \( R_X \leq_{pr} R_{X'} \leq_{pr} R_{Y'} \leq_{pr} R_Y \). Restricting the diamond \( R_C, R_D \) to the subinterval \((X', Y')\) does not automatically do it, since for instance, \( R_C \lor R_{X'} \) could be above \( R_{Y'} \).

We may assume that \( \#(0^X)[t] = \#(0^Y)[t] \) for infinitely many \( t \). We fix functions \( f \) and \( g \) such that for every \( t \), \( f(t) \) and \( g(t) \) are the least such that \( \#(0^Y)[t] = \#(0^Y)[g(t)] \) and \( \#(0^X)[t] = \#(0^X)[f(t)] \). Now given any \( t \) let \( q(t) = f(u) \)
where \( u \) is the largest such that \( u < g(t+1) \) and \( \# (0^X)[u] = \# (0^Y)[u] \). If \( u \) cannot be found, let \( q(t) = t \). By Corollary 9.4 it remains to check that \( \# (0^Y)[w] = \# (0^X)[q(w)] \) for infinitely many \( w \). Suppose that \( \# (0^X)[t] = \# (0^Y)[t] = j \) for some \( t,j \). Let \( w \) be the largest such that \( \# (0^Y)[w] = j \). Then

\[
\# (0^Y)[g(w+1)] = \# (0^Y)[w+1] = j + 1
\]

and therefore \( t < g(w+1) \). By the minimality of \( g(w+1) \), we have

\[
\# (0^X)[u] = \# (0^Y)[u] = j
\]

for the chosen \( u \), and therefore

\[
\# (0^X)[q(w)] = \# (0^X)[f(u)] = \# (0^X)[u] = j = \# (0^Y)[w]. \quad \square
\]

Lemma 9.3 characterizes the \( \clubsuit \)-property in terms of the relative growth rates of \( \# (0^X) \) and \( \# (0^Y) \). For our next purpose it shall be convenient to express the \( \clubsuit \)-property in terms of the relative growth rates of \( p_X \) and \( p_Y \). The term “\( n_+1 \)” in the next lemma is important; by Remark 9.14 we cannot replace \( p_Y(n+1) \) with \( p_X(n) \).

**Lemma 9.6.** Let \( R_X \preceq_{pr} R_Y \). Then \( (X,Y) \) satisfies the \( \clubsuit \)-property if and only if there exists a primitive recursive function \( r \) such that \( p_X(n) \leq r(p_Y(n+1)) \) for infinitely many \( n \).

**Proof.** Suppose that \( (X,Y) \) has the \( \clubsuit \)-property. Fix \( q \) as in Corollary 9.4. Let \( r(m) = q(u) \), where \( u \) is the least stage such that \( Y(i)[u] \downarrow \) for all \( i \leq m \). Now let \( t \) and \( n \) be such that \( \# (0^Y)[t] = \# (0^X)[q(t)] = n \). Notice that \( p_X(n) < q(t) \). Let \( m = p_Y(n+1) \) and \( u \) be such that \( r(m) = q(u) \). Then since \( \# (0^Y)[t] = n \), we have \( t < u \), which means that \( r(m) = q(u) \geq q(t) > p_Y(n) \).

Now suppose that \( r \) exists; obviously we may assume that \( r \) is nondecreasing. Define \( q(t) \) to be the largest stage \( u \leq v \) such that \( \# (0^X)[u] = \# (0^Y)[t] \), where \( v \) is the least stage such that \( X(j)[v] \downarrow \) for all \( j \leq r(t+1) \); if this does not exist, let \( q(t) = t \). Now let \( n \) be such that \( p_X(n) \leq r(p_Y(n+1)) \), and let \( t \) be the largest such that \( \# (0^Y)[t] = n \). We check that \( \# (0^Y)[t] = \# (0^X)[q(t)] \). We have \( \# (0^Y)[t+1] = n + 1 \) and thus \( p_Y(n+1) < t + 1 \) which means that

\[
r(t+1) \geq r(p_Y(n+1)) \geq p_X(n).
\]

This means that \( q(t) \) will be equal to some largest \( u \) such that

\[
\# (0^X)[u] = \# (0^Y)[t] = n.
\]

Hence \( \# (0^X)[q(t)] = \# (0^Y)[t] \). \( \square \)

Returning to our question as to whether every interval has the \( \clubsuit \)-property, we can now make use of Lemma 9.6 to construct an interval that does not have the \( \clubsuit \)-property. In fact, we can show that every punctual degree is the top of an interval that does not satisfy the \( \clubsuit \)-property.

**Proposition 9.7.** Given any \( R_Y \) there is some \( R_X \preceq_{pr} R_Y \) such that \( (X,Y) \) does not have the \( \clubsuit \)-property.
Definition 9.8. Given any primitive recursive set $X$, we define $X^{[-1]}$ to be the set defined by:

$$X^{[-1]}(n) = \begin{cases} 1, & \text{if } n \text{ is the least such that } X(n) = 0, \\
X(n), & \text{otherwise.} \end{cases}$$

In particular, $\# \left(0^{X^{[-1]}}\right)[t] = \max\{0, \#(0^X)[t] - 1\}$ for every stage $t$.

An immediate consequence of the definition is:

Lemma 9.9. $R_{X^{[-1]}} \leq_{pr} R_X$ if and only if $R_X \leq_{pr} \text{Id}$.

Proof. Since $\# \left(0^{X^{[-1]}}\right)[t] \leq \# \left(0^X\right)[t]$ for every $t$, we have $R_{X^{[-1]}} \leq_{pr} R_X$ (by Lemma 9.3), so we have to show that $R_{X^{[-1]}} \geq_{pr} R_X$ if and only if $R_X \geq_{pr} \text{Id}$. Suppose that $g$ reduces Id to $R_X$. Let $n$ be the least element not in $X$. If $n \notin \text{rng}(g)$ then $g$ is already a reduction from Id to $R_{X^{[-1]}}$, and if $g(m) = n$ then the function $h(k) = g(k + m + 1)$ will reduce Id to $R_{X^{[-1]}}$.

Conversely suppose that $f$ reduces $R_X$ to $R_{X^{[-1]}}$. Define the function $F$ by the following. Let $F(0) = \max f([0, n])$ where $n$ is the second element not in $X$. Let $F(k + 1) = \max f([0, F(k)])$. Then for each $k$, there are at least $k + 1$ many distinct elements not in $X$ which are smaller than $F(k)$. The function $F$ can easily be used to define a reduction of Id to $R_X$. \qed

Our first question about rigidity is easily answered by Lemma 9.9:

Theorem 9.10. ($\text{Peq} \leq$) is not rigid.

Proof. The map $\operatorname{deg}(R_X) \mapsto \operatorname{deg}(R_{X^{[-1]}})$ is a non-trivial automorphism. In fact, it fixes $\operatorname{deg}(\text{Id})$ and moves every other degree to a strictly smaller degree. \qed

Corollary 9.11. The only definable degree is the greatest degree, $\operatorname{deg}(\text{Id})$. No finite set of degrees is definable except for $\{\operatorname{deg}(\text{Id})\}$.
We turn now our attention to the question of how homogeneous the structure \((\text{Peq}, \leq)\) is. (Un)fortunately, the structure \((\text{Peq}, \leq)\) is neither rigid nor homogeneous, which indicates that the structure is not as trivial as might seem at first glance. This justifies further investigations into the degree structure of \(\text{Peq}\).

**Definition 9.12.** We call a coinfinite primitive recursive set \(X\) slow if for every primitive recursive function \(r\), \(p_X(n + 1) > r(p_X(n))\) holds for almost every \(n\).

A slow set generates its zeros slower than any primitive recursive function can predict (infinitely often). Slow sets obviously exist. An immediate consequence of Lemma 9.6 is:

**Corollary 9.13.** If \(X\) is slow then \((X, \text{Id})\) does not satisfy the \(\text{♦}\)-property.

Thus, if \(X\) is slow and \((Y, \text{Id})\) satisfies the \(\text{♦}\)-property \((Y\) exists, by Theorem 2.3), then no automorphism of \((\text{Peq}, \leq)\) can map \(\text{deg}(Y)\) to \(\text{deg}(X)\). This fact also means that uppercones of \(\text{Peq}\) are not always isomorphic to each other.

We also note that the converse to Corollary 9.13 is false: Given any \(X\) we can easily find some \(Y\) such that \(R_{X[-1]} \leq_{pr} R_Y \leq_{pr} R_X\) and \(Y\) is not slow; to do this we can arrange for \(p_{X(n + 1)} = p_{X(n) + 1}\) for infinitely many \(n\). Then for each such \(X\), \((Y, \text{Id})\) cannot satisfy the \(\text{♦}\)-property, by Corollary 9.13.

When we turn to lowercones however, the situation is less obvious. Since every punctual degree is the top of an interval with the \(\text{♦}\)-property as well as the top of (another) interval without the \(\text{♦}\)-property, it is not clear how we can immediately distinguish two lowercones from each other using the \(\text{♦}\)-property, similarly to how we separated uppercones. In fact, the lowercone \(\{\text{deg}(R_Y) \mid R_Y \leq_{pr} R_X\}\) is isomorphic to the lower cone \(\{\text{deg}(R_Y) \mid R_Y \leq_{pr} R_{X[-1]}\}\).

Hence it is entirely conceivable that every lowercone \(\{\text{deg}(R_Y) \mid R_Y \leq_{pr} R_X\}\) is isomorphic to \((\text{Peq}, \leq)\). From the point of view of each degree of \(R_Y\) where \(R_Y \leq_{pr} R_X\), the set \(X\) has no delay, since the zeros of \(X\) are always generated no slower than the zeros of \(Y\). Hence we might expect to always be able to extend any partial embedding of \(\text{Peq}\) into \(\{\text{deg}(R_Y) \mid R_Y \leq_{pr} R_X\}\). We will show that this is not the case. The key to our analysis lies (again!) in the operator \(X \mapsto X^{-1}\).

**Remark 9.14.** By Lemma 2.4, \((X^{-1}, X)\) will have the \(\text{♦}\)-property for any \(X\). Consequently, we cannot replace “\(p_X(n + 1)\)” in Lemma 9.6 with “\(p_X(n)\)” ; for instance, if \(Y\) is slow and \(X = Y^{-1}\).

Even though an interval of the form \((X^{-1}, X)\) will always satisfy the \(\text{♦}\)-property, the same is not true of an interval of the form \((X^{-2}, X)\), where \(X^{-2} = (X^{-1})^{-1}\).

**Lemma 9.15.** \((X^{-2}, X)\) satisfies the \(\text{♦}\)-property if and only if \(X\) is not slow.

**Proof.** We apply Lemma 9.6 and noting that \(p_{X^{-2}}(n) = p_{X}(n + 2)\).

**Lemma 9.16.** Given any \(R_Y \leq_{pr} R_X\), either \((Y, X)\) satisfies the \(\text{♦}\)-property or \(R_Y \leq_{pr} R_{X[-1]}\).

**Proof.** If \(R_Y \not\leq_{pr} R_{X[-1]}\) then \(\#(0^Y)[s] > \#(0^{X[-1]})[s]\) for infinitely many \(s\), which means that \(\#(0^Y)[s] \geq \#(0^X)[s]\) for infinitely many \(s\). Apply Corollary 9.4.

\(^1\)Strictly speaking, we should write \((X, 2\omega)\) instead of \((X, \text{Id})\) here.
Lemma 9.16 tells us that $R_{X_{1]}^{-1}}$ bounds all $R_Y$ below $R_X$ such that $(Y, X)$ does not have the ♦-property. This will allow us to define the map $\operatorname{deg}(R_X) \mapsto \operatorname{deg}(R_{X_{1]}^{-1})$. Towards this, we prove another lemma:

**Lemma 9.17.** Let $R_X$ and $R_Y$ be punctual equivalence relations with $R_{X_{1]}^{-1}} \not\preceq_{pr} R_Y$. Then there is some $Z$ such that $R_Z \preceq_{pr} R_{X_{1]}^{-1}}, (Z, X)$ does not have the ♦-property and $R_Z \not\preceq_{pr} R_Y$.

**Proof.** Fix a computable listing $\{p_e\}_{e\omega}$ of all primitive recursive functions as in Section 2.3. By making the function values larger, we may assume that for every $p_e$ is strictly increasing, and that $p_{e+1}(n) > p_e(n)$. This listing $(e, n) \mapsto p_e(n)$ is total computable but of course not primitive recursive. We will also assume that $p_e(x) = \psi(x)$ for some total computable function which halts in fewer than $\hat{p}_e(x)$ many steps, where $\hat{p}_e$ is some primitive recursive function. All indices can be found effectively.

We now define the set $Z$ in stages. Since $Z$ must be primitive recursive, at every stage $s$ we must decide $Z \uparrow s + 1$. In the below construction, at each stage $s + 1$, we will declare $\# \left(0^2\right)[s + 1] = \# \left(0^2\right)[s]$ or $\# \left(0^2\right)[s] + 1$; in the former case we mean that we set $Z(s + 1) = 1$ and in the latter case we set $Z(s + 1) = 0$.

At stage $s$ we will have a parameter $V(s)$ which stands for the index such that at stage $s$ we are attempting to make $\# \left(0^2\right)[s] \nleq \# \left(0^X\right)[p_{V(s)}(s)]$. At stage $s = 0$ we declare $0 \in Z$ (i.e. $\# \left(0^2\right)[0] = 0$) and set $V(0) = 0$. Suppose we have the value of $\# \left(0^2\right)[s]$ and $V(s) = e$. Compute $p_e(k)$ for one more step (where $k$ was the input that was last processed). If there is a new convergence $p_e(k)$ seen at this step such that $\# \left(0^X\right)[k] \nleq \# \left(0^2\right)[k + 1]$, we take

$$\# \left(0^2\right)[s + 1] = \min(\# \left(0^2\right)[s] + 1, \# \left(0^X\right)[s] - 1).$$

Check if $\# \left(0^2\right)[t] > \# \left(0^Y\right)[p_e(t)]$ for any $t \leq s$ for which we have already found the value of $p_e(t)$. If so we increase $V$ by one. In all other cases take $\# \left(0^2\right)[s + 1] = \# \left(0^2\right)[s]$, and go to the next stage with the same value of $V$.

The above gives a primitive recursive description of the set $Z$. Since $\# \left(0^2\right)[s] \nleq \# \left(0^X\right)[s]$ for every $s$, we have $R_Z \preceq_{pr} R_{X_{1]}^{-1}}$. Notice that the construction processes the inputs $k$ sequentially; namely, the construction begins with $k = 0$ and $V = 0$ and waits for $p_0(0)$ to converge (this takes $\hat{p}_0(0)$ many stages). It then moves on to $k = 1$ and waits for $p_0(1)$ to converge, and so on. If ever, the construction decides to increase the value of $V$ while waiting on, say, $p_0(5)$, then we will move on to wait for $p_1(5)$ to converge, then $p_1(6)$, and so on. Let $k^*(s)$ be the value of $k$ being processed by the construction at stage $s$. Since $\{p_e\}_{e\omega}$ are all total, $\lim_s p^*(s) = \infty$. Define the sequence $\{k_i\}_{i\omega}$ such that $\# \left(0^{X^{-i}}\right)[k_i] = i$ and $\# \left(0^{X^{-i}}\right)[k_i + 1] = i + 1$. Take $k_{-1} = -1$.

**Claim 9.18.** For every stage $s$ we have $\# \left(0^2\right)[s] = i$, where $k_{i-1} < k^*(s) \leq k_i$.

**Proof.** If $s = 0$ then $i = k^*(0) = 0$ and so $\# \left(0^2\right)[0] = 0$. Assume $\# \left(0^2\right)[s] = i$ where $k_{i-1} < k^*(s) \leq k_i$. Since the value of $\# \left(0^2\right)[s + 1]$ is decided at the end of stage $s$, we have to examine what the construction did at stage $s$. At stage $s$ we would increase the value of $\# \left(0^2\right)$ only if $p_{V(s)}(k^*(s))$ is found to converge at that stage and $k^*(s) = k_i$. In that case $k^*(s + 1) = k_i + 1$ and so $k_i < k^*(s + 1) \leq k_{i+1}$,
and so we have to check that \( \# (0^Z) [s + 1] = i + 1 \). But note that as \( k^*(s) < s \) we have
\[
\# (0^{X^{(i-1)}}) [s] \geq \# (0^{X^{(i-1)}}) [k^*(s + 1)] = i + 1 = \# (0^Z) [s] + 1
\]
and so
\[
\# (0^Z) [s + 1] = \min \{ \# (0^Z) [s] + 1, \# (0^X) [s] - 1 \} = i + 1. \qedhere
\]

Next, we verify that \( \lim_s V(s) = \infty \); suppose not. Let \( t_0 \) be the least such that \( V(t_0) = \varepsilon \) and \( V(s) = \varepsilon \) for almost all \( s \). Let \( t_0 < t_1 < t_2 < \cdots \) be the stages such that \( p_e(l_i) \) first converged at stage \( t_i \), where \( i > 0 \), \( k^*(t_i) = l_i < k^*(t_i + 1) \) and \( l_i = k_{i+1} \). Note that \( l_1 > t_0 \). By our convention above, we have that \( t_i = \hat{p}_e(l_i) \).

By Claim 9.18 we see that \( \# (0^Z) [t] = \# (0^Z) [t_{i+1}] \) for every \( t, i \) such that \( t_i < t \leq t_{i+1} \).

For every \( k \) and \( i > 0 \) such that \( l_i < k \leq l_{i+1} \), we have \( t_i + 1 = \hat{p}_e(l_i) + 1 \leq \hat{p}_e(l_{i+1}) = t_{i+1} \), and since \( \# (0^Z) [t_i + 1] = \# (0^Z) [t_{i+1}] \), we also have \( \# (0^Z) [\hat{p}_e(k)] = \# (0^Z) [t_{i+1}] \). Therefore, we have
\[
\# (0^{X^{(i-1)}}) [k] \leq \# (0^{X^{(i-1)}}) [t_{i+1}] \quad \text{(by Claim 9.18)}
= \# (0^Z) [t_{i+1}]
= \# (0^Z) [\hat{p}_e(k)] \quad \text{(as the construction never increased} V \text{ after} t_0 \text{)}
\leq \# (0^Y) [p_e(\hat{p}_e(k))].
\]

This shows that \( R_{X^{(i-1)}} \preceq_{pr} R_Y \), contrary to our assumption.

Now that we know \( \lim_s V(s) = \infty \), for every \( e \) there must be a stage \( s \) of the construction where we saw \( \# (0^Z) [t] > \# (0^Y) [p_e(t)] \) for some \( t \leq s \), which means that \( R_Z \preceq_{pr} R_Y \). Now consider some primitive recursive function \( p_e \). Let \( V(s) = \varepsilon \) for some \( s \). For each \( k > k^*(s) \) we let \( t > s \) be the stage where \( k^*(t) = k < k^*(t+1) \), and \( i \) be such that \( k_{i-1} < k \leq k_i \). By Claim 9.18 \( \# (0^Z) [t] = i \). We now have
\[
\# (0^Z) [p_e(k)] \leq \# (0^Z) [\hat{p}_e(k)] \\
\leq \# (0^Z) [\hat{p}_V(t)(k)] \quad \text{(at stage} t \text{ we saw} p_V(t)(k) \text{ converge)}
= \# (0^Z) [t]
= i
= \# (0^{X^{(i-1)}}) [k_i]
= \# (0^{X^{(i-1)}}) [k]
< \# (0^X) [k].
\]

Thus, \( (Z, X) \) does not have the \( \diamond \)-property. \( \square \)

Now we are ready to apply the analysis started above.

**Corollary 9.19.** The map \( \deg(R_X) \mapsto \deg(R_{X^{(i-1)}}) \) is definable in \( (\text{Peq}, \leq) \).

**Proof.** Apply Lemmas 9.16 and 9.17 to see the following. Given punctual \( R_X \) and \( R_Y \), we have \( R_Y \equiv_{pr} R_{X^{(i-1)}} \) if and only if the following hold:

1. \( R_Y \leq_{pr} R_X \),
(2) For every $R_Z \leq_{pr} R_X$ such that $(Z, X)$ does not have the ♦-property, $R_Z \leq_{pr} R_Y$, and

(3) If $R_Y$ has properties (1) and (2), then $R_Y \geq_{pr} R_Y$.

That is, we may define $R_X^{[1]}$ as the least degree below $R_X$ that is an upperbound for the set of all degrees $R_Z \leq_{pr} R_X$ such that $(Z, X)$ does not have the ♦-property. Since the ♦-property is definable, so is the set of all pairs $(\deg(R_X), \deg(R_X^{[1]}))$.

\[\square\]

**Corollary 9.20.** If $\psi$ is an automorphism of $(\text{Peq}, \leq)$, then for any $R_X$, we have that $\psi(\deg(R_X^{[1]})) = \deg(R_Y^{[1]})$, where $R_Y \equiv_{pr} \psi(R_X)$.

**Corollary 9.21.** If $\psi$ is an automorphism of $(\text{Peq}, \leq)$, then $R_X$ is slow if and only if $\psi(R_X)$ is slow.

**Proof.** By Lemma 9.14, $R_X$ is slow if and only if $(X^{[2]}, X)$ does not have the ♦-property, if and only if $(\psi(X^{[2]}), \psi(X))$ does not have the ♦-property. But then $\psi(X^{[2]}) = \psi(X)^{[2]}$ which is equivalent to the fact that $\psi(X)$ is slow. \[\square\]

We are now able to give a negative answer to our question above regarding whether every lowercone is isomorphic to Peq. In fact, no proper lowercone can be isomorphic to Peq. Even though each lowercone is principal, our intuition that we should be able to replicate the structure below $\text{Id}$ in any lowercone by “relativising” is entirely incorrect.

**Corollary 9.22.** The lowercone $\{\deg(R_Y) \mid R_Y \leq_{pr} R_X\}$ below $R_X$ can never be isomorphic to $(\text{Peq}, \leq)$ unless $R_X \equiv_{pr} \text{Id}$. Thus, no proper lowercone can be isomorphic to $(\text{Peq}, \leq)$.

**Proof.** If $R_X \leq_{pr} \text{Id}$ then by Lemma 9.9, $R_X^{[1]} \leq_{pr} R_X$. Thus, by Lemma 9.16, $\deg(R_X^{[1]})$ is a degree strictly below $\deg(R_X)$ that is an upperbound on the set of all degrees $R_Z \leq_{pr} R_X$ such that $(Z, X)$ does not have the ♦-property (and thus does not embed the diamond). By Lemma 9.17 (applied with $R_X = R_X^{[1]})$, no degree strictly below $\deg(\text{Id})$ can serve as the image of $\deg(R_X^{[1]})$. \[\square\]

Our analysis on lowercones exploits the unique property satisfied by $X^{[-1]}$, and thus can only be applied to separate an incomplete (or proper) lowercone from Peq. Using our analysis, we can still say a little more; we show that not every pair of proper lowercones are isomorphic:

**Corollary 9.23.** If $X$ is slow and $Y$ is not, then their lowercones cannot be isomorphic (as posets).

**Proof.** Because $X^{[-1]}$ (and similarly $Y^{[-1]}$) is the least upperbound of the set of all $R_Z \leq_{pr} R_X$ such that $(Z, X)$ does not embed the diamond, any isomorphism between the two lowercones must send $X^{[-1]}$ to $Y^{[-1]}$ and therefore must map $X^{[-2]}$ to $Y^{[-2]}$. However, as $X$ is slow and $Y$ is not, by Lemma 9.15, $(Y^{[-2]}, Y)$ satisfies the ♦-property whereas $(X^{[-2]}, X)$ does not. \[\square\]

Since it is not hard to construct a pair of incomparable degrees, one of which is slow and the other is not, we immediately have a pair of incomparable lowercones that are not isomorphic. This leaves the intriguing question as to whether any pair of incomparable lowercones are isomorphic. We leave this question open:

\[\text{2The notation is not formally correct, but has the obvious meaning here.}\]
Question 9.24. Are there incomparable degrees $R_X |_{pr} R_Y$ with isomorphic lower cones?

Question 9.25. Are there continuum many automorphisms of $(\mathsf{Peq}, \leq)$?

Given that (Corollary 9.11) no finite set of degrees except for $\{\deg(\text{Id})\}$ is definable (without parameters), it may be difficult to apply the analysis of \cite{6, 5} to encode finite graphs into $\mathsf{Peq}$, which would involve having to define finite sets of degrees, albeit with a parameter. So we also ask:

Question 9.26. Is the first order theory of $(\mathsf{Peq}, \leq)$ decidable?

References

\begin{enumerate}
\item W. Ackermann. Zum Hilbertschen Aufbau der reellen Zahlen. Math. Ann., 99:118–133, 1928.
\item U. Andrews, S. Badaev, and A. Sorbi. A survey on universal computably enumerable equivalence relations. In A. Day, M. Fellows, N. Greenberg, B. Khoussainov, A. Melnikov, and F. Rosamond, editors, Computability and Complexity, volume 10010 of Lect. Notes Comput. Sci., pages 418–451. Springer, Cham, 2017.
\item U. Andrews, D. Belin, and L. San Mauro. On the structure of computable reducibility on equivalence relations of natural numbers. \textit{arXiv preprint arXiv:2105.12534}, 2021.
\item U. Andrews, S. Lempp, J. S. Miller, K. M. Ng, L. San Mauro, and A. Sorbi. Universal computably enumerable equivalence relations. \textit{J. Symb. Logic}, 79(1):60–88, 2014.
\item U. Andrews, N. Schweber, and A. Sorbi. The theory of ceers computes true arithmetic. \textit{Ann. Pure Appl. Logic}, 171(8):102811, 2020.
\item U. Andrews and A. Sorbi. Joins and meets in the structure of ceers. \textit{Computability}, 8(3–4):193–241, 2019.
\item N. Bazhenov, R. Downey, I. Kalimullin, and A. Melnikov. Foundations of online structure theory. \textit{Bull. Symbolic Logic}, 25(2):141–181, 2019.
\item N. Bazhenov, I. Kalimullin, A. Melnikov, and K. M. Ng. Online presentations of finitely generated structures. \textit{Theoret. Comput. Sci.}, 844:195–216, 2020.
\item N. Bazhenov, M. Mustafa, L. San Mauro, A. Sorbi, and M. Yamaleev. Classifying equivalence relations in the Ershov hierarchy. \textit{Arch. Math. Logic}, 59(7–8):835–864, 2020.
\item C. Bernardi and A. Sorbi. Classifying positive equivalence relations. \textit{J. Symb. Logic}, 48(03):529–538, 1983.
\item S. Buss, Y. Chen, J. Flum, S-D. Friedman, and M. Müller. Strong isomorphism reductions in complexity theory. \textit{J. Symb. Logic}, 76(4):1381–1402, 2011.
\item S. Coskey, J. D. Hamkins, and R. Miller. The hierarchy of equivalence relations on the natural numbers under computable reducibility. \textit{Computability}, 1(1):15–38, 2012.
\item R. Downey, A. Melnikov, and K. M. Ng. Foundations of online structure theory II: The operator approach. arXiv:2007.07401.
\item Yu. L. Ershov. \textit{Theory of numberings}. Nauka, Moscow, 1977. In Russian.
\item Yu. L. Ershov. \textit{Theory of numberings}. In E. G. Griffor, editor, \textit{Handbook of Computability Theory}, volume 140 of \textit{Studies Logic Found. Math.}, pages 473–503. North-Holland, 1999.
\item E. B. Fokina, S-D. Friedman, V. Harizanov, J. F. Knight, C. McCoy, and A. Montalbán. Isomorphism relations on computable structures. \textit{J. Symb. Logic}, 77(1):122–132, 2012.
\item H. Friedman and L. Stanley. A Borel reducibility theory for classes of countable structures. \textit{J. Symb. Logic}, 54(03):894–914, 1989.
\item S. Gao. \textit{Invariant descriptive set theory}. CRC Press, 2008.
\item S. Gao and P. Gerdes. Computably enumerable equivalence relations. \textit{Stud. Log.}, 67(1):27–59, 2001.
\item S. Gao and C. Ziegler. On polynomial-time relation reducibility. \textit{Notre Dame J. Form. Log.}, 58(2):271–285, 2017.
\item L. A. Harrington, A. S. Kechris, and A. Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. \textit{J. Amer. Math. Soc.}, 3(4):903–928, 1990.
\item P. G. Hinman. \textit{Recursion-theoretic hierarchies}. Springer, Berlin, 1978.
\item G. Hjorth. Borel equivalence relations. In \textit{Handbook of set theory}, pages 297–332. Springer, 2010.
\end{enumerate}
[24] E. Ianovski, R. Miller, K. M. Ng, and A. Nies. Complexity of equivalence relations and preorders from computability theory. *J. Symb. Logic*, 79(3):859–881, 2014.
[25] I. Kalimullin, A. Melnikov, and K. M. Ng. Algebraic structures computable without delay. *Theoret. Comput. Sci.*, 674:73–98, 2017.
[26] E. Ladner, R. On the structure of polynomial time reducibility. *J. ACM*, 22(1):155–171, 1975.
[27] K. Mehlhorn. Polynomial and abstract subrecursive classes. *J. Comput. System Sci.*, 12(2):147–178, 1976.
[28] A. Melnikov and K. M. Ng. A structure of punctual dimension two. *Proc. Amer. Math. Soc.*, 148(7):3113–3128, 2020.
[29] A. G. Melnikov. Eliminating unbounded search in computable algebra. In J. Kari, F. Manea, and I. Petre, editors, *Unveiling Dynamics and Complexity*, volume 10307 of *Lecture Notes in Computer Science*, pages 77–87. Springer, Cham, 2017.
[30] K. M. Ng and H. Yu. On the degree structure of equivalence relations under computable reducibility. *Notre Dame J. Form. Log.*, 60(4):733–761, 2019.
[31] P. Odifreddi. *Classical recursion theory: The theory of functions and sets of natural numbers*. Elsevier, 1992.

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