Research Article

Multiple Positive Solutions for Fractional Three-Point Boundary Value Problem with p-Laplacian Operator

Dong Li, Yang Liu, and Chunli Wang

1. Department of Mathematics, College of Science Jiamusi University, Jiamusi, Heilongjiang 154007, China
2. School of Mathematics and Statistics, Hefei Normal University, Hefei, Anhui 230061, China
3. Institute of Information Technology of GUET, Guilin, Guangxi 540004, China

Correspondence should be addressed to Chunli Wang; wangchunliwcl821222@sina.com

Received 23 May 2020; Accepted 20 June 2020; Published 9 July 2020

Academic Editor: Jia-Bao Liu

Copyright © 2020 Dong Li et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we investigate the existence of multiple positive solutions or at least one positive solution for fractional three-point boundary value problem with p-Laplacian operator. Our approach relies on the fixed point theorem on cones. The results obtained in this paper essentially improve and generalize some well-known results.

1. Introduction

Nowadays, fractional calculus has been adapted to numerous fields, such as engineering, mechanics, physics, chemistry, and biology. Many essays and monographs studying various issues in fractional calculus have been researched (see [1–6]). In particular, fractional differential equations have been found to be a powerful tool in modeling various phenomena in many areas of science and engineering such as physics, fluid mechanics, and heat conduction. More details about research achievement on fractional differential equations and their applications are shown in [7–10].

Recently, fractional differential equations have gained considerable attention (see [11–15] and the references therein). Fractional differential equations and differential equations with p-Laplacian operators have attracted much attention from many mathematicians. As a result, meaningful research results have been drawn [16–20]. Fractional-order boundary value problems involving classical, multipoint, high-order, and integral boundary conditions have extensively been studied by many researchers and a variety of results can be found in recent literature on the topic [21–25].

In [15], Chai studied the boundary value problems of fractional differential equations with p-Laplacian operator as follows:

\[
\begin{aligned}
D_0^\alpha \left( \phi_p \left( D_0^\beta u(t) \right) \right) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\
u(0) = 0, u(1) + \sigma D_0^\gamma u(1) &= 0, \quad D_0^\mu u(0) = 0,
\end{aligned}
\]

where \( \phi_p(s) = |s|^{p-2}s, \quad p > 1, \quad f \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+), \quad \alpha \in (1, 2], \quad \beta \in (0, 1], \quad \gamma \in (0, 1], \quad \alpha - \gamma \geq 1, \quad \sigma > 0, \) and \( D_0^\alpha, \quad D_0^\beta, \quad D_0^\gamma, \quad D_0^\mu, \) are the standard Riemann–Liouville derivatives. Some existence results of positive solutions are obtained by using the monotone iterative method.

In [16], by using Krasnosel’skii’s fixed point theorem, Tian et al. obtained the existence of positive solutions for a boundary value problem of fractional differential equations with p-Laplacian operator as follows:

\[
\begin{aligned}
D_0^\alpha \left( \phi_p \left( D_0^\beta u(t) \right) \right) &= f(t, u(t)), \quad 0 < t < 1, \\
u(0) = u(0) = u(1) = D_0^\mu u(0) = 0, \quad D_0^\mu u(1) = \lambda D_0^\mu u(\eta),
\end{aligned}
\]

where \( \phi_p(s) = |s|^{p-2}s, \quad p > 1, \quad f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+), \quad \alpha \in (2, 3], \quad \beta \in (1, 2], \quad \eta \in (0, 1), \quad \lambda \in (0, \infty), \) and \( D_0^\alpha, \quad D_0^\beta, \) are the Riemann–Liouville fractional derivatives.

In [17], by using the monotone iterative method, Tian et al. obtained the existence of positive solutions for a boundary value problem of fractional differential equations with p-Laplacian operator as follows:
\[
\begin{align*}
\left\{ \begin{array}{l}
D_\phi^\beta (\phi(D_\phi^\alpha u(t))) = f(t, u(t)), \quad 0 < t < 1, \\
u(0) = D_\phi^\alpha u(0) = 0, D_\phi^\alpha u(1) = aD_\phi^\alpha u(\xi), \quad D_\phi^\alpha u(1) = bD_\phi^\alpha u(\eta),
\end{array} \right.
\end{align*}
\]

where \( \phi_p(s) = |s|^{p-2}s, \ p > 1, \ a \in (1, 2], \ 0 < \beta \leq \alpha - 1, \ \xi, \eta \in (0, 1), \ a, b \in [0, \infty) \) and \( 1 - a^\alpha \beta^{-1} > 0, \ 1 - b^\alpha \beta^{-1} > 0, \ f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+), \) and \( D_\phi^\alpha \) is the Riemann–Liouville fractional derivative.

In [18], by means of the \( p \)-Laplacian operator, Han et al. obtained the existence of positive solutions for the boundary value problem of fractional differential equation as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
D_\phi^\beta (\phi(D_\phi^\alpha u(t))) = f(t, u(t)), \quad 0 < t < 1, \\
u(0) = u(0) = 0, \quad D_\phi^\alpha u(1) = \delta D_\phi^\alpha u(\eta), \quad \phi(D_\phi^\alpha u(0)) = (\phi(D_\phi^\alpha u(1)))', \quad \ni 0,
\end{array} \right.
\end{align*}
\]

where \( \phi(s) = |s|^{p-2}s, \ p > 1, \ a \in (2, 3], \ \beta \in (1, 2], \ \mu \in (1, \alpha - 1), \ f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+), \ \delta \geq 0, \ 0 < \eta < 1, \) and \( \Delta = 1 - \delta \eta^{\mu-\mu_1} > 0. \)

The aim is to establish some existence and multiplicity results of positive solutions for BVP (5). This paper is organized as follows. In Section 2, some properties of Green’s function will be given, which are needed later. In Section 3, the existence of multiplicity results of positive solutions of BVP will be discussed (5).

2. Preliminary Knowledge and Lemmas

Lemma 1 (see [14]). Assume that \( D_\phi^\alpha u \in L^1(a, b) \) with a fractional derivative of order \( \alpha > 0. \) Then,

\[
\begin{align*}
\int_a^b D_\phi^\beta D_\phi^\alpha u(t) = u(t) + c_1 (t-a)^{\alpha-1} + c_2 (t-a)^{\alpha-2} + \cdots + c_n (t-a)^{\alpha-n},
\end{align*}
\]

for some \( c_i \in \mathbb{R}, i = 1, 2, \ldots, n, \) where \( n \) is the smallest integer greater than or equal to \( \alpha. \)

Lemma 2. If \( y \in C[0, 1], \) then the fractional boundary value problem is

\[
\begin{align*}
\left\{ \begin{array}{l}
D_\phi^\alpha u(t) + y(t) = 0, \quad 0 < t < 1, \\
u(0) = u(0) = 0, \quad D_\phi^\alpha u(1) = \delta D_\phi^\alpha u(\eta),
\end{array} \right.
\end{align*}
\]

The unique solution is \( u(t) = \int_0^1 G(t,s)y(s)ds, \) where \( G(t,s) = G_1(t,s) + (\delta/\Delta)(s^\alpha-1)G_2(\eta,s), \)

\[
\begin{align*}
G_1(t,s) &= \frac{1}{\Gamma(\alpha)} \left\{ \begin{array}{l}
t^{\alpha-1} - (s^\alpha-1), \quad 0 \leq s \leq t, \\
t^{\alpha-1} - (s^\alpha-1), \quad t \leq s,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
G_2(\eta,s) &= \frac{1}{\Gamma(\alpha)} \left\{ \begin{array}{l}
(1-s^\alpha+\eta^\alpha-1), \quad 0 \leq s \leq \eta, \\
(1-s^\alpha+\eta^\alpha-1), \quad \eta \leq s,
\end{array} \right.
\end{align*}
\]

Proof. The general solution to the problem (7) is

\[
\begin{align*}
D_\phi^\alpha (\phi(D_\phi^\alpha u(t))) = \lambda f(u(t)), \quad 0 < t < 1, \\
u(0) = u'(0) = u'(1) = 0, \quad \phi(D_\phi^\alpha u(0)) = (\phi(D_\phi^\alpha u(1)))', \quad \ni 0,
\end{align*}
\]

where \( \phi(s) = |s|^{p-2}s, \ p > 1, \ a \in (2, 3], \ \beta \in (1, 2], \ f : (0, +\infty) \rightarrow (0, +\infty) \) is continuous, and \( D_\phi^\alpha, D_\phi^\beta \) are the Riemann–Liouville fractional derivatives.

Based on the above research, this paper analyzed the following fractional three-point boundary value problem with the \( p \)-Laplacian operator:

\[
\begin{align*}
\left\{ \begin{array}{l}
D_\phi^\beta (\phi(D_\phi^\alpha u(t))) = f(t, u(t)), \quad 0 < t < 1, \\
u(0) = u'(0) = u'(1) = 0, \quad \phi(D_\phi^\alpha u(0)) = (\phi(D_\phi^\alpha u(1)))', \quad \ni 0,
\end{align*}
\]

From the boundary value condition of (7), \( C_2 = C_3 = 0, \)

\[
\begin{align*}
C_1 = \frac{1}{\Delta} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.
\end{align*}
\]

Therefore,

\[
\begin{align*}
u(t) = -\int_0^1 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \int_0^1 \frac{t^\alpha-s^\alpha}{\Gamma(\alpha)} y(s)ds,
\end{align*}
\]

\[
\begin{align*}
\text{Therefore,}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
G(t,s) = \frac{1}{\Gamma(\alpha)} \left\{ \begin{array}{l}
t^{\alpha-1} - (s^\alpha-1), \quad 0 \leq s \leq t, \\
t^{\alpha-1} - (s^\alpha-1), \quad t \leq s,
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
G_2(s) = \frac{1}{\Gamma(\alpha)} \left\{ \begin{array}{l}
(1-s^\alpha+\eta^\alpha-1), \quad 0 \leq s \leq \eta, \\
(1-s^\alpha+\eta^\alpha-1), \quad \eta \leq s,
\end{array} \right.
\end{align*}
\]

Proof. Problem (12) is equivalent to

\[
\begin{align*}
D_\phi^\alpha (\phi(D_\phi^\alpha u(t))) = \lambda f(u(t)), \quad 0 < t < 1, \\
u(0) = u'(0) = u'(1) = 0, \quad \phi(D_\phi^\alpha u(0)) = (\phi(D_\phi^\alpha u(1)))', \quad \ni 0,
\end{align*}
\]
\[
\phi(D^\alpha_0 u(t)) = \int_0^t \frac{(t - \tau)^{\beta_1 - 1}}{\Gamma(\beta_1)} w(\tau) d\tau + C_1 t^{\beta_1 - 1} + C_2 t^{\beta_2 - 2}.
\]

(14)

From the boundary value condition of (12), \(C_2 = 0\) and \(C_1 = -\int_0^1 ((1 - \tau)^{\beta_2 - 2}) / \Gamma(\beta) w(\tau) d\tau\); then,

\[
\phi(D^\alpha_0 u(t)) = \int_0^t \frac{(t - \tau)^{\beta_1 - 1}}{\Gamma(\beta_1)} w(\tau) d\tau - \int_0^1 \frac{(1 - \tau)^{\beta_2 - 2}}{\Gamma(\beta)} w(\tau) d\tau
\]

\[= -\int_0^1 H(t, \tau) w(\tau) d\tau.\]

(15)

Therefore,

\[D^\alpha_0 u(t) + \phi^{-1}\left(\int_0^1 H(t, \tau) w(\tau) d\tau\right) = 0.\]

(16)

Based on Lemma 2,

\[u(t) = \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) w(\tau) d\tau\right) ds.\]

(17)

**Lemma 5.** Let \(E = (E, ||\cdot||)\) be a Banach space and let \(K \subseteq E\) be a cone in \(E\). Assume \(\Omega_1\) and \(\Omega_2\) are open subsets of \(E\) with \(0 \in \Omega_1\) and \(\overline{\Omega_1} \subset \Omega_2\) and let \(T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow K\) be a continuous and completely continuous. In addition, suppose either

(1) \(||Tu|| \leq ||u||, u \in K \cap \partial \Omega_1\), and \(||Tu|| > ||u||, u \in K \cap \partial \Omega_2\),

(2) \(||Tu|| \geq ||u||, u \in K \cap \partial \Omega_1\), and \(||Tu|| < ||u||, u \in K \cap \partial \Omega_2\).

**Lemma 6.** Let \(K\) be a cone in a real Banach space \(E\), \(K_r = \{x \in K : ||x|| < r\}\), \(\psi\) be nonnegative continuous concave functional on \(K\) such that \(\psi(x) \leq ||x||, \forall x \in K_r\), and

\[K(\psi, d, e) = \{x \in K : d \leq \psi(x), ||x|| \leq e\}.\]

(22)

Suppose \(T : K_r \longrightarrow K_r\) is completely continuous and there exist constants \(0 < c < d < e \leq \rho\) such that

(i) \(\{x \in K(\psi, d, e) : \psi(x) > d\} \neq \emptyset\) and \(\psi(Tx) > d\) for \(x \in K(\psi, d, e)\),

(ii) \(||Tx|| < c\) for \(x < c\),

(iii) \(\psi(Tx) > d\) for \(x \in K(\psi, d, e)\) with \(||Tx|| > e\).

Then, \(T\) has at least three fixed points \(x_1, x_2, \text{ and } x_3\) with \(||x_1|| < c, d < \psi(x_2),\) and \(c < ||x_3|| < \psi(x_3) < d\).

**3. Main Results**

When \(E = [0, 1]\), any \(u \in E, ||u|| = \max_{t \in [0, 1]} |u(t)|\), then \(E\) is a real Banach space. \(K \subseteq E\) is a cone, which can be defined as \(K = \{u \in E; \min_{t \in [0, 1]} u(t) \geq 0, \min_{t \in [0, 1]} u(t) \geq (1/4)^{-1} ||u||\}\).

Defining the operator \(T : E \longrightarrow E\), for any \(u \in E,

\[Tu(t) = \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau\right) ds,
\]

(23)

and for convenience, the following notation is introduced:
Lemma 4, we have completely continuous. 

If there are two positive numbers \(0 < r_1 < r_2\) such that the following conditions hold:

\((B_1)\) \(f (t, u) \geq \phi (r_1 N_{a^{\alpha}}^{a^2-2})\), for \((t, u) \in [0, 1] \times [0, r_1]\)

\((B_2)\) \(f (t, u) \leq \phi (r_2 M)\), for \((t, u) \in [0, 1] \times [0, r_2]\)

then the fractional three-point boundary value problem (5) has at least one positive solution \(u\) and \(r_1 \leq \|u\| \leq r_2\).

**Proof.** From the continuity of \(G, H, f\), it can be concluded that \(T: K \rightarrow K\) is continuous. For \((t, s) \in I \times (0, 1), u \in K\), by Lemma 4, we have

\[
\min_{n \in I} T u(t) = \min_{n \in I} \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) f (\tau, u(\tau)) d\tau\right) ds \\
\geq \left(\frac{1}{4}\right)^{a-1} \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) f (\tau, u(\tau)) d\tau\right) ds \\
\geq \left(\frac{1}{4}\right)^{a-1} \|T u\|.
\]

(25)

It means that \(T(K) \subseteq K\). Therefore, the Arzela–Ascoli theorem can prove that the operator \(T: K \rightarrow K\) is completely continuous.

Let \(\Omega_0 = \{u \in K: \|u\| \leq \lambda_2\}\), for \(u \in \partial \Omega_0\). From Lemma 4 and \((B_1)\), we can conclude that

\[
T u(t) = \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) f (\tau, u(\tau)) d\tau\right) ds \\
\geq r_1 N_{a^{\alpha}}^{a^2-2} \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) d\tau\right) ds \\
\geq r_1 N_{a^{\alpha}}^{a^2-2} \int_0^1 \left(\frac{1}{4}\right)^{a-1} G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) d\tau\right) ds \\
= r_1 N \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) d\tau\right) ds \\
r_1.
\]

(26)

Then, when \(u \in \partial \Omega_0\), \(\|T u\| \geq \|u\|\).

Let \(\Omega_1 = \{u \in K: \|u\| \leq \lambda_2\}\), for \(u \in \partial \Omega_1\). Then, we also can conclude from Lemma 4 and \((B_2)\) that

\[
T u(t) = \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) f (\tau, u(\tau)) d\tau\right) ds \\
\leq r_2 M \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) d\tau\right) ds \\
= r_2.
\]

(27)

Therefore, for \(u \in \partial \Omega_2\), \(\|T u\| \leq \|u\|\). In summary, by Lemma 5, the fractional three-point boundary value problem (5) has at least one positive solution \(u\) and \(r_1 \leq \|u\| \leq r_2\).

**Theorem 2.** If there exist positive real numbers \(0 < c < d < (1/4)^{a-1}\) such that the following conditions hold:

\((B_3)\) \(f (t, u) < \phi (r_1 M), \) for \((t, u) \in I \times [0, 1] \times [0, r_1]\)

\((B_4)\) \(f (t, u) > \phi (d N / (1/4)^{a^2-2}), \) for \((t, u) \in I \times [d, d/(1/4)^{a^2-1}]\)

\((B_5)\) \(f (t, u) < \phi (r_2 M), \) for \((t, u) \in I \times [a, d] \times [0, r_2]\)

then the fractional three-point boundary value problem (5) has at least three positive solutions \(u_1, u_2, \) and \(u_3\) with

\[
\max_{0 \leq t \leq 1} |u_1(t)| < c,
\]

\[
d < \min_{t \in I} |u_2(t)| < r,
\]

\[
c < \max_{0 \leq t \leq 1} |u_3(t)| < r,
\]

\[
\min_{t \in I} |u_3(t)| < d.
\]

(28)

**Proof.** Firstly, if \(u \in \overline{K}_r\), then we may assert that \(T: \overline{K}_r \rightarrow \overline{K}_r\) is a completely continuous operator. To see this, suppose \(u \in \overline{K}_r\); then, \(\|u\| \leq r\). It follows from Lemma 4 and \((B_3)\) that

\[
\|T u\| = \max_{0 \leq t \leq 1} |T u(t)| \\
\leq \max_{0 \leq t \leq 1} \left(\int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) f (\tau, u(\tau)) d\tau\right) ds\right) \\
= r_1 N \int_0^1 G(t, s) \phi^{-1}\left(\int_0^1 H(s, \tau) d\tau\right) ds \\
= r_2.
\]

(29)

Therefore, \(T: \overline{K}_r \rightarrow \overline{K}_r\). This together with Lemma 5 implies that \(T: \overline{K}_r \rightarrow \overline{K}_r\) is a completely continuous operator. In the same way, if \(u \in \overline{K}_r\), then assumption \((B_4)\) yields \(\|T u\| < c\). Hence, condition (ii) of Lemma 6 is satisfied.

To check condition (i) of Lemma 6, we let \(u(t) = d (1/4)^{a^2-1} t\) for \(t \in [0, 1]\). It is easy to verify that \(u(t) = d (1/4)^{a^2-1} \in K(\psi, d, d/(1/4)^{a^2-1})\) and

\[
\psi (u) = (d (1/4)^{a^2-1}) > d, \text{ and so}
\]

\[
\left\{u \in K \left(\psi, d, \frac{d}{(1/4)^{a^2-1}}\right) | \psi (u) > d\right\} \neq \emptyset.
\]

(30)

Thus, for all \(u \in K(\psi, d, d/(1/4)^{a^2-1})\), we have that

\[
d \leq u(t) \leq d (1/4)^{a^2-1} \text{ for } t \in I \text{ and } Tu \in K.
\]

From Lemma 4 and \((B_4)\), one has
Lemma 6, we have solutions from Lemma 6, it follows that there exist three positive conditions (iii) of Lemma 6 holds.

Secondly, we verify that (iii) of Lemma 6 is satisfied. By \( f_h \) this shows that condition (i) of Lemma 6 holds.

\[
\begin{align*}
\psi(Tu(t)) &= \min_{t \in I} |Tu(t)| \\
&= \min_{t \in I} \left( \int_0^1 G(t,s) \phi^{-1} \left( \int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \\
&\geq \frac{1}{4} \left( \int_0^1 G(1,s) \phi^{-1} \left( \int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \\
&\geq \frac{1}{4} \frac{dN}{(1/4)^{\alpha-2}} \left( \int_0^1 G(1,s) \phi^{-1} \left( \int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \\
&= d.
\end{align*}
\]

This shows that condition (i) of Lemma 6 holds. Secondly, we verify that (iii) of Lemma 6 is satisfied. By Lemma 6, we have

\[
\begin{align*}
\min_{t \in I} |Tu(t)| &= \min_{t \in I} \left( \int_0^1 G(t,s) \phi^{-1} \left( \int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \\
&\geq \frac{1}{4} \left( \int_0^1 G(1,s) \phi^{-1} \left( \int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \\
&\geq \frac{1}{4} \frac{dN}{(1/4)^{\alpha-2}} \left( \int_0^1 G(1,s) \phi^{-1} \left( \int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau \right) ds \right)
\end{align*}
\]

for \( u \in K(\psi, d, r) \) with \( \|Tu\| > (d/(1/4)^{\alpha-1}) \), which shows that condition (iii) of Lemma 6 holds.

To sum up, all the conditions of Lemma 6 are satisfied; from Lemma 6, it follows that there exist three positive solutions \( u_1, u_2, \) and \( u_3 \) with

\[
\begin{align*}
\max_{0 \leq t \leq 1} |u_1(t)| &< c, \\
d &< \min_{t \in I} |u_2(t)| < r, \\
c &< \max_{0 \leq t \leq 1} |u_3(t)| < r, \\
\min_{t \in I} |u_3(t)| &< d.
\end{align*}
\]

\( \square \)

4. Conclusion

The existence of solutions to three-point boundary value problems of fractional differential equations with the \( p \)-Laplacian operators is discussed by using the fixed point exponential theorem and fixed point theorem of cone compression and cone expansion. By extending the existence of solutions to boundary value problems, we have obtained the sufficient condition that the boundary value problem has multiple positive solutions or at least one positive solution.

\[ \psi(Tu(t)) = \min_{t \in I} |Tu(t)| \]

\[ = \min_{t \in I} \left( \int_0^1 G(t,s) \phi^{-1} \left( \int_0^1 H(s,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \]

\[ \geq \frac{1}{4} \left( \int_0^1 G(1,s) \phi^{-1} \left( \int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \]

\[ \geq \frac{dN}{(1/4)^{\alpha-2}} \left( \int_0^1 G(1,s) \phi^{-1} \left( \int_0^1 H(1,\tau)f(\tau,u(\tau))d\tau \right) ds \right) \]

\[ = d. \]

Data Availability

The data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by 2019 Project of Foundational Research Ability Enhancement for Young and Middle-Aged University Faculties of Guangxi (2019KY1046), Nature and Science Foundation of Anhui (2008085QA08), Scientific Research Projects of Institute of Information Technology of GUET (B201911), and Science and Technology Research Project of Heilongjiang Provincial Department of Education (12543079).

References

[1] K. B. Oldham and J. Spanier, The Fractional Calculus (Theory and Applications of Differentiation and Integration to Arbitrary Order), Academic Press, San Diego, CA, USA, 1974.
[2] S. G. Kilbas and A. A. Marichev, Fractional Integrals and Derivatives (Theory and Applications), Gordon & Breach, New York, NY, USA, 1993.
[3] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, NY, USA, 1993.
[4] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA, USA, 1999.
[5] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, New Jersey, NJ, USA, 2000.
[6] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, Netherlands, 2006.
[7] V. Lakshmikantham, S. Leela, and J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, UK, 2009.
[8] R. Herrmann, Fractional Calculus (An Introduction for Physicists), World Scientific, Singapore, 2014.
[9] T. M. Atanacković, S. Pilipović, B. Stanković, and D. Zorica, Fractional Calculus with Applications in Mechanics, Wiley, New York, NY, USA, 2014.
[10] Y. Liu, D. P. Xie, D. D. Yang, and C. Z. Bai, “Two generalized Lyapunov-type inequalities for a fractional \( p \)-Laplacian equation with fractional boundary conditions,” Journal of Inequalities and Applications, vol. 98, no. 1, pp. 1–16, 2017.
[11] L. S. Zhang, X. Wu, and Y. Caccetta, “Extremal solutions for \( p \)-Laplacian differential systems via iterative computation,” Applied Mathematics Letters, vol. 26, no. 12, pp. 1151–1158, 2013.
[12] S. N. Rao, M. Singh, and M. Z. Meetei, “Multiplicity of positive solutions for Hadamard fractional differential equations with \( p \)-Laplacian operator,” Boundary Value Problems, vol. 43, no. 1, pp. 1–25, 2020.
[13] Y. Wang, “Multiple positive solutions for mixed fractional differential system with \( p \)-Laplacian operators,” Boundary Value Problems, vol. 144, no. 1, pp. 1–17, 2019.
[14] Y. He and B. Bi, “Existence and iteration of positive solution for fractional integral boundary value problems with \( p \)-
Laplacian operator,” *Advances in Difference Equations*, vol. 415, pp. 1–15, 2019.

[15] G. Chai, “Positive solutions for boundary value problem of fractional differential equation with $p$-Laplacian operator,” *Boundary Value Problems*, vol. 18, no. 1, pp. 1–15, 2012.

[16] Y. Tian, Y. Wei, and S. Sun, “Multiplicity for fractional differential equations with $p$-Laplacian,” *Boundary Value Problems*, vol. 127, no. 1, pp. 1–18, 2018.

[17] Y. Tian, Z. Bai, and S. Sun, “Positive solutions for a boundary value problem of fractional differential equation with $p$-Laplacian operator,” *Advances in Difference Equations*, vol. 349, pp. 1–19, 2019.

[18] Z. Han, H. Lu, and C. Zhang, “Positive solutions for eigenvalue problems of fractional differential equation with generalized $p$-Laplacian,” *Applied Mathematics and Computation*, vol. 257, no. 1, pp. 526–536, 2015.

[19] C. Z. Bai, “Existence and uniqueness of solutions for fractional boundary value problems with $p$-Laplacian operator,” *Advances in Difference Equations*, vol. 4, pp. 1–12, 2018.

[20] Y. H Li, “Multiple positive solutions for nonlinear mixed fractional differential equation with $p$-Laplacian operator,” *Advances in Difference Equations*, vol. 112, no. 1, pp. 1–12, 2019.

[21] J. B. Liu, J. Zhao, and Z. Cai, “On the generalized adjacency, Laplacian and signless Laplacian spectra of the weighted edge corona networks,” *Physica A*, vol. 540, Article ID 123073, 2020.

[22] J. B. Liu, J. Zhao, and Z. X. Zhu, “On the number of spanning trees and normalized Laplacian of linear octagonal-quadrilateral networks,” *International Journal of Quantum Chemistry*, vol. 119, Article ID 25971, 2019.

[23] B. B. Zhou and L. L. Zhang, “Existence of positive solutions of boundary value problems for high-order nonlinear conformable differential equations with $p$-Laplacian operator,” *Advances in Difference Equations*, vol. 351, no. 1, pp. 1–14, 2019.

[24] K. S. Jong, “Existence and uniqueness of positive solutions of a kind of multi-point boundary value problems for nonlinear fractional differential equations with $p$-Laplacian operator,” *Mediterranean Journal of Mathematics*, vol. 15, no. 3, 2018.

[25] H. X. Wang and W. M. Hu, “Existence of solutions for a class of fractional differential equations with $p$-Laplacian operators and integral boundary conditions,” *Mathematics in Practice and Theory*, vol. 46, no. 16, pp. 228–236, 2016.