TORSORS AND THE QUILEN-BARR-BECK COHOMOLOGY FOR RESTRICTED LIE ALGEBRAS

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ABSTRACT. Given an exact sequence of restricted Lie algebras using Duskin’s torsors theory, we establish an eight term exact sequence for Quillen-Barr-Beck cohomology of restricted Lie algebras. As an application we obtain for any extension of algebraic groups over an algebraic closed field of prime characteristic an eight term exact sequence for the corresponding restricted Lie algebras extension.

1. INTRODUCTION

Let $0 \rightarrow H \rightarrow G \rightarrow A \rightarrow 0$ be an extension of commutative algebraic groups over an algebraic closed field. If $B$ is another commutative algebraic group, then J.-P. Serre in [28] and M. Rosenlicht in [25] proved that there is an exact sequence

$$0 \rightarrow \text{Hom}(A, B) \rightarrow \text{Hom}(G, B) \rightarrow \text{Hom}(H, B) \rightarrow \text{Ext}(A, B) \rightarrow \text{Ext}(G, B) \rightarrow \text{Ext}(H, B)$$

If the ground field has prime characteristic then the extension of commutative algebraic groups induces a short exact sequence of abelian restricted Lie algebras. In this paper we associate to this exact sequence of abelian restricted algebras, an eight term exact sequence for the Quillen-Barr-Beck cohomology of restricted Lie algebras. Using Duskin’s torsors theory we construct in Theorem 4.4, for any short exact sequence $0 \rightarrow N \rightarrow g \rightarrow b \rightarrow 0$ of restricted Lie algebras, an eight term exact sequence for Quillen-Barr-Beck cohomology with coefficients any Beck $b$-module.

Here we use the cohomology defined in [10] following the general scheme of Quillen-Barr-Beck cohomology theory for universal algebras (see [2], [31], [30]). In contrast to the cases of Groups, Associative algebras, Lie algebras, where we obtain the classical cohomologies theories, we do not obtain Hochschild (co)-homology for restricted Lie algebras [17]. We obtain different cohomology which classifies more general abelian extensions of restricted Lie algebras and not strongly abelian extensions which are classified by Hochschild cohomology.

The Hochschild-Serre spectral sequence for (co)-homology in various algebraic categories gives rise to exact sequences of terms of low degree. For the category of restricted Lie algebras it is proved by Eckmann and Stammbach in [13] that there is a five-term exact sequence for Hochschild (co)-homology of restricted Lie algebras.

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In Quillen-Barr-Beck cohomology this is replaced by the eight-term exact sequence of Theorem 4.4.

We now give an outline of the methods used in our proof. Duskin (cf. [11]), develops the theory of \( n \)-torsors in order to give an interpretation of the \( n \)-th cotriple cohomology group. Later, Glenn in [16] defines torsor cohomology groups and gives a slightly different definition of \( n \)-th torsor which coincides with the one given by Duskin in the context of a monadic category over \( \text{Sets} \) whose objects have an underlying group structure (cf. Corollary 7.2.4 in [16]). Cegarra and Aznar in [7] proved in the framework of a Barr-exact category that there is an eight term exact sequence for torsor cohomology. We use this result in order to obtain exact sequences for Quillen-Barr-Beck cohomology for restricted Lie algebras.

In Lemma 2.1 of Section 2 we compute the 0-torsors group for the category of restricted Lie algebras. Thus, we obtain a 5-term exact sequence for Quillen-Barr-Beck cohomology of restricted Lie algebras.

In section 3 in order to study 2-torsors groups we are led to consider crossed modules and internal groupoids in the category of restricted Lie algebras. In Theorem 3.3 it is obtained that the notions of crossed modules and internal groupoids are equivalent.

In section 4 we study 2-fold extensions for the category of restricted Lie algebras. Two fold extensions in certain categories of interest are studied by M. Gerstenhaber, in [15] and is given cohomological interpretation. Besides, J.-L. Loday and C. Kassel in [21] consider 2-fold extensions of Lie algebras in connection to the study of relative cohomology of Lie algebras. In Lemma 4.2 we prove that the 2-torsors cohomology group is isomorphic to the group of equivalence classes of 2-fold extensions of restricted Lie algebras. Thus we obtain an interpretation of the second Quillen-Barr-Beck cohomology group in terms of 2-fold extensions. As a consequence using Cegarra-Aznar’s eight term exact sequence we establish in Theorem 4.4 an eight-term exact sequence for Quillen-Barr-Beck cohomology of restricted Lie algebras which extends the 5-term exact sequence of Section 2. In the last subsection 4.4 we apply Theorem 4.4 to extensions of algebraic groups in prime characteristic.

1.1. Restricted Lie algebras. In modular Lie theory in order to extent theorems which are valid in characteristic zero we are led to consider restricted Lie algebras introduced by N. Jacobson in [19]. Let \( k \) denote a field of characteristic \( p \neq 0 \) and \( \text{Lie} \) the category of Lie algebras over \( k \).

**Definition 1.1.** A restricted Lie algebra \((L, \langle \cdot \rangle_{[pL]}^{\cdot})\) over \( k \) is a Lie algebra \( L \in \text{Lie} \) together with a map \( \langle \cdot \rangle_{[pL]}^{\cdot} : L \to L \) called the \( p \)-map such that the following relations hold:

\[
\begin{align*}
(\alpha x)^{[pL]} &= \alpha^p x^{[pL]} \\
[x, y]^{[pL]} &= [x, y]_1^{(p)} + \ldots + y^{(p)} \\
(x + y)^{[pL]} &= x^{[pL]} + y^{[pL]} + \sum_{i=1}^{p-1} s_i(x, y)
\end{align*}
\]
where \(is_i(x, y)\) is the coefficient of \(x^{i-1}\) in \(ad_{x+y}^{i-1}(x)\), where \(ad_x : L \to L\) denotes the adjoint representation given by \(ad_x(y) := [y, x]\) and \(x, y \in L\), \(\alpha \in k\). A Lie algebra homomorphism \(f : L \to L'\) is called restricted if \(f(x^p) = f(x)^p\). We denote by \(RLie\) the category of restricted Lie algebras over \(k\).

**Remark 1.2.** Let \(L \in RLie\) be a restricted Lie algebra and \(x, y \in L\). If \(L_{x,y}\) is the Lie algebra generated by the elements \(x, y\) then \(s_i(x, y) \in L_{x,y}^p\) where \(L_{x,y}^1 := L_{x,y}\) and

\[
L_{x,y}^n := [L_{x,y}^{n-1}, L_{x,y}]
\]

for \(0 \leq i \leq p - 1\).

**Remark 1.3.** There is a notion of free restricted Lie algebra over a set. Therefore, the category \(RLie\) of restricted Lie algebras is a monadic category over Sets. It follows that \(RLie\) is a Barr exact category.

Let \(L\) and \(L'\) be restricted Lie algebras. The direct product of \(L\) and \(L'\) is as a Lie algebra their direct product in the category of Lie algebras

\[
L \times L' = \{(x, y); x \in L, y \in L'\}
\]

equipped with the \(p\)-map given by \((x, y)^p := (x^p, y^p)\) for all \((x, y) \in L \times L'\). If \(f : L \to R\) and \(f' : L' \to R\) are restricted Lie homomorphisms, then the pullback of \(L\) and \(L'\) over \(R\) is given by the following commutative diagram in \(RLie\)

\[
\begin{array}{ccc}
L \times_R L' & \xrightarrow{pr'} & L' \\
\downarrow{pr} & & \downarrow{f'} \\
L & \xrightarrow{f} & R
\end{array}
\]

where

\[
L \times_R L' := \{(x, y) \in L \times L'; f(x) = f'(y)\}
\]

Let \(L \in RLie\) be a restricted Lie algebra and \(U(L)\) its enveloping algebra. We denote by \(u(L) := U(L)/ \langle x^p - x^p \rangle\), \(x \in L\) the restricted enveloping algebra of \(L\). A \(L\)-module \(A\) is called **restricted** if \(x^p a = (x \cdots (x a) \cdots)\). The category of restricted Lie \(L\)-modules is equivalent to the category of \(u(L)\)-modules.

**Example 1.4.** Let \(\mathfrak{A}\) be any associative algebra over a field \(k\) with characteristic \(p \neq 0\). We denote by \(\mathfrak{A}_{Lie}\) the induced Lie algebra with the bracket given by \([x, y] := xy - yx\), for all \(x, y \in \mathfrak{A}\). Then \((\mathfrak{A}, (\cdot)^p)\) is a restricted Lie algebra where \((\cdot)^p\) is the Frobenious map given by \(x \mapsto x^p\). Thus, there is a functor \((\cdot)^p : \mathfrak{A}_{Lie} \to RLie\) from the category of associative algebras to the category of restricted Lie algebras.

**Example 1.5.** Let \(\mathfrak{A}\) be an associative algebra over \(k\). Then \(gl_n(\mathfrak{A})\) the Lie algebra of \(n \times n\) matrices with coefficients in \(\mathfrak{A}\) is a restricted Lie algebra. Moreover, one proves that \(sl_n(\mathfrak{A})\) is a restricted Lie subalgebra of \(gl_n(\mathfrak{A})\).

**Example 1.6.** Let \(V\) be a \(k\)-vector space then a map \(f : V \to V\) such that \(f(x + y) = f(x) + f(y)\) and \(f(\alpha x) = \alpha^p f(x)\) for all \(x, y \in V\) and \(\alpha \in k\) is called \(p\)-semi-linear. Any pair \((V, f)\) where \(V\) is \(k\)-vector space and \(f : V \to V\) a \(p\)-semi-linear map is an abelian restricted Lie algebra.
Example 1.7. Let \( \mathfrak{B} \) be an \( k \)-algebra not necessarily associative then the set of \( k \)-derivations \( \text{Der}(\mathfrak{B}) \) is endowed with the structure of restricted Lie algebra. In particular, if \( D \in \text{Der}(\mathfrak{B}) \) by the Leibniz formula we one has

\[
D^p(xy) = \sum_{i=0}^{i=p} \binom{p}{i} D^i(x)D^{p-i}(y)
\]

for all \( x, y \in \mathfrak{B} \). Since the \( \text{char} \ k = p \) we have \( \binom{p}{i} = 0 \) for \( 1 \leq i \leq p - 1 \), therefore \( D^p \in \text{Der}(\mathfrak{B}) \).

Let \( (L, [p]) \in \text{RLie} \) be a restricted Lie algebra then a derivation \( D \in \text{Der}(L) \) is called the restricted derivation if \( D(x[p]) = ad^{p-1}(x)(D(x)) \) for all \( x \in L \).

Example 1.8. Let \( G \) be an algebraic group over \( k \). The associated Lie algebra \( \text{Lie}(G) \) of \( G \) is endowed with the structure of restricted Lie algebra (cf. \([4]\)).

Let \( (L, (-)[p]), (N, (-)[p']) \) be two restricted Lie algebras, and \( \eta : L \rightarrow \text{Der}(N) \) a restricted homomorphism such that \( \eta(x) \) is a restricted derivation for every \( x \in L \). We recall that the Lie product of the semi-direct product of \( L \) and \( N \) is given by

\[
[(l, n), (l', n')] = ([l, l'], \eta(l)n' - \eta(l')n + [n, n'])
\]

Then it follows from Jacobson’s Theorem 11 in \([19]\) that the semi-direct product of \( L \) and \( N \) is endowed with the \( p \)-map extending the \( p \)-maps on \( L \) and \( N \) (cf. Theorem 2.5 in \([14]\)). We call this restricted Lie algebra the semi-direct product of \( L \) and \( N \) in the category of restricted Lie algebras and we denote it by \( L \rtimes N \).

Let us recall some definitions and results which we use in the next section. We refer the reader to \([10]\) for details.

In his doctoral dissertation, Beck incorporates the various notions of module over enveloping algebras to the general notion of a Beck module.

1.2. Beck-modules for \( \text{RLie} \). Let \( L \in \text{RLie} \) be a restricted Lie algebra, \( A \) a restricted \( L \)-module and \( f : A \rightarrow A^L \) a \( p \)-semi-linear map from \( A \) to the invariants \( A^L = \{ a \in M : xa = 0 \text{ for all } x \in L \} \). We denote by \( L \rtimes_f A \) the semi-direct product in \( \text{RLie} \) of \( L \) and \( A \). In particular, \( L \rtimes_f A \) is the semi-direct product in \( \text{Lie} \) together with the \( p \)-map \( (l, a)^[p] = ([l^p], \cdots, l^p a + f(a)) \) where \( x \in L \) and \( a \in A \). For \( L \in \text{RLie} \), a restricted Lie algebra the next theorem gives a characterization of the category of abelian group objects of the slice category \( \text{ab}(\text{RLie}/L) \) i.e the category Beck \( L \)-modules in \( \text{RLie} \).

Theorem 1.9. The category of abelian group objects \( \text{ab}(\text{RLie}/L) \) is equivalent to the category \( A \) whose objects are pairs \( (A, f) \) where \( A \) is a restricted \( L \)-module and \( f : A \rightarrow A^L \) is a \( p \)-semi-linear map from \( A \) into its submodule of invariants \( A^L \) and whose morphisms \( (A_1, f_1) \rightarrow (A_2, f_2) \) are \( L \) homomorphisms \( \alpha : A_1 \rightarrow A_2 \) such that \( f_2 \alpha = \alpha f_1 \).

Proof. See Theorem 1.7 in \([10]\). \( \square \)

Let \( R_f \) be the polynomial ring consisting of the set of polynomials \( \sum_{i=0}^{i=m} a_i f^i \) where \( a_i \in k \), \( f \) an indeterminate and \( f a = a^p f \). We denote by \( w(L) \) the ring which as \( k \)-vector space is \( w(L) := R_f \otimes_k u(L) \) and such that \( R_f \rightarrow w(L) \) and \( u_L \rightarrow w(L) \) are algebra homomorphisms and

\[
(P \otimes 1)(1 \otimes l) := P \otimes l \text{ and } (l \otimes 1)(P \otimes 1) := 0
\]
for all $P \in R_f$ and $l \in L$. The following theorem gives another characterization of the category of abelian group objects $ab(RLie/L)$.

**Theorem 1.10.** The category of abelian group objects $ab(RLie/L)$ is equivalent to the category of $w(L)$-modules.

**Proof.** See Theorem 1.8 in [10].

**Remark 1.11.** By Theorems 1.9, 1.10 we note that if $L \in RLie$, then each $w(L)$-module $A$ is associated to a couple $(\bar{A}, f)$ where $\bar{A}$ is a restricted $L$-module and $f$ a $p$-semi-linear map such that $l.f(\bar{a}) = 0$ for all $l \in L$ and $\bar{a} \in \bar{A}$.

1.3. **Beck-derivations for RLie.** Let $g \in RLie/L$ be a restricted Lie algebra and $A$ an $w(L)$-module. The group of Beck derivations is defined as follows

$$\text{Der}_p(g, A) := \{ d \in \text{Der}(g, \bar{A}) : d(x^p) = x \cdots x \text{d}x + f(dx), \ x \in g \}$$

It is proved in [10] the isomorphism

$$(1.4) \quad \text{Hom}_{RLie/L}(g, L \times f \bar{A}) \simeq \text{Der}_p(g, A)$$

given by

$$f \mapsto pr_A f$$

where $pr_A$ denotes the canonical projection.

Cartan and Eilenberg in their book [6] describe a general context for the definition of (co)-homology groups for various algebraic structures. In each case is used an appropriate notion of enveloping algebra and the definition is given in terms of $Ext$ and $Tor$ functors. In this context Hochschild in [17] and B. Pareigis in [22] defined (co)-homology groups for the category of restricted Lie algebras. In the paper of Barr and Beck [2] is given the relation between cotriple cohomology groups and the cohomology theories described in the Cartan-Eilenberg context. It is proved that for the categories of Groups, Associative algebras, Lie algebras the two theories coincide (considering a shift in dimension). Also, cohomology as a derived functor of derivations are studied for several algebraic categories by Barr and Rinehart in [3]. Besides, D. Quillen in [31], [30] develops an axiomatic homotopy theory and cohomology groups are defined in the context of model categories. Through this development are defined (co)-homology groups for universal algebras (see section 2 in [30]). In [10] is defined Quillen-Barr-Beck cohomology for the category of restricted Lie algebras.

1.4. **Quillen-Barr-Beck cohomology for RLie.** Let $L \in RLie$ be a restricted Lie algebra and $A$ a $w(L)$-module. Let $F : Sets \to RLie$ be the free functor, left adjoint to the forgetful functor $U : RLie \to Sets$. The adjunction $(F, U)$ give rise to a cotriple $G$ in $RLie/L$. In [10] are defined cotriple cohomology groups

$$H^n_p(L, A) := H^n(\text{Der}_p(G^*(L), A))$$

Bellow we recall the definition of a $n$-torsor, for details and terminology we refer the reader to [11], [12] and [16].
1.5. Torsors, interpretation of cotriple cohomology. Duskin in [11] gave an interpretation of the cotriple cohomology in terms of $n$-dimensional torsors generalizing to any dimension Beck's interpretation of dimension 1. Let $E$ be a monadic category over Sets and $G = FU$ the associated cotriple. The $n$-truncating functor 

$$tr^n : \text{Simpl}(E) \to tr^n\text{Simpl}(E)$$

called Verdier's coskeleton functor. The coskeleton functor is constructed by iterating simplicial kernels. We denote by $\text{Cosk}^n$ the composition functor $\text{Cosk}^n := \text{cosk}^n \circ tr^n : \text{Simpl}(E) \to \text{Simpl}(E)$

Let $A \in \text{ab}(E)$ be an abelian group object, then the simplicial object $K(A,n)$ is defined as the $(n+1)$-coskeleton of the following $(n+1)$-truncated simplicial object

$$\begin{array}{c}
A^{n+1} \\
\frac{k_n}{pr_0} \\
A \\
\end{array} \rightarrow \begin{array}{c}
1 \\
\cdot \cdot \cdot \\
1
\end{array}$$

where $1$ denotes the terminal object and $k_n = (-1)^n \sum_{i=0}^{i=n} (-1)^i pr_i$ for all $n \geq 1$.

**Definition 1.12.** Let $X \in E$, then a $K(A,n)$-torsor in $E$ over $X$ relative to $U$ is defined as an augmented over $X$ simplicial object $(X, d_i, s_i)$ together with a simplicial morphism $\chi : X \rightarrow K(A,n)$ such that

1. $X \rightarrow X$ is a $U$-split augmented simplicial object
2. $\chi : X \rightarrow K(A,n)$ is a simplicial morphism such that the following squares are pullbacks, for each $m \geq n$ and all $0 \leq i \leq m$

$$X_m \xrightarrow{\chi_m} K(A,n)_m$$

$$(d_0, \ldots, d_i, \ldots, d_m)$$

$$\Lambda^i(m-1)(X) \xrightarrow{} \Lambda^i(m-1)(K(A,n))$$

where $\Lambda^i(m-1)$ denotes the $i$-horn simplicial object of the $(m-1)$-truncated simplicial object
3. the canonical map $X \rightarrow \text{Cosk}^{n-1}_{\text{aug}}(X)$ is an isomorphism, where $\text{Cosk}^n_{\text{aug}}$ denotes augmented $n$-coskeleton functor.

If $(X, s, \chi)$ and $(X', s', \chi')$ are two $K(A,n)$-torsors over $X$ then a morphism of $K(A,n)$-torsors is an $X$-map $f : X \rightarrow X'$ of augmented simplicial objects such that $\chi = \chi'f$. The set of the connected components of $n$-torsors is denoted by $\text{Tors}^n(X,A)$. Let $H^n_G(X,A)$ be the cotriple cohomology groups, then Duskin in [11] proved the following theorem.

**Theorem 1.13.** If $X \in E$ and $A$ an abelian group object then there is a bijection between the set $\text{Tors}^n(X,A)$ of equivalence classes of $n$-torsors and the $n$-th cotriple cohomology $H^n_G(X, A)$ for $n \geq 1$.

**Proof.** Theorem (8.9) in [11]. □
Later Glenn in \[16\] defined the notion of \(n\)-dimensional hypergroupoids and gave a slightly different definition of \(n\)-torsor. If \(E\) is a monadic category over \(Sets\) whose objects have an underlying group structure, then by Corollary 7.2.4 in \[16\] the two notions of \(n\)-torsors coincide. Let \(p : B \to R\) be a regular epimorphism in \(E\) and \(B_p\) the simplicial object (with augmentation \(p\)), obtained by iterating the simplicial kernel construction. Then, A.M Cegarra and E. R. Aznar in \[7\] define (Definition 1.1) the abelian groups

\[
Tor^0(p, A) := \text{Hom}_{\text{simp}(E)}(B^p, K(A, 1))
\]

and \(Tors^1(p, A)\) of connected components of 2-torsors over \(R\) with fixed augmentation \(p\) (Definition 2.1). Moreover, it is proved in \[7\] Theorem 7.3 that there is an eight term exact sequence

\[
\begin{align*}
\text{Hom}_E(R, A) & \xrightarrow{p^*} \text{Hom}_E(B, A) \to Tor^0(p, A) \to Tors^1(R, A) \to Tors^1(B, A) \\
& \quad \to Tors^1(p, A) \to Tors^2(R, A) \to Tors^2(B, A).
\end{align*}
\]

\section{A 5-term exact sequence for Quillen-Barr-Beck cohomology}

The Hochschild-Serre spectral sequence for (co)-homology of Lie algebras gives rise to exact sequences of terms of low degree. In particular, if

\[
0 \to N \to g \to b \to 0
\]

is an exact sequence of Lie algebras and if \(A\) is a left (resp. right) \(b\)-module, then we have the following exact sequences

\[
0 \to \text{Der}(b, A) \to \text{Der}(g, A) \to \text{Hom}_U(g)(N_{ab}, A) \to H^2_b(b, A) \to H^2(g, A)
\]

and

\[
H^2(g, A) \to H^2_b(b, A) \to N_{ab} \otimes_U(b) A \to H_1(g, A) \to H_1(b, A) \to 0
\]

where \(N_{ab} := N/[N,N]\).

For the case of Hochschild (co)-homology of restricted Lie algebras, there are analogue sequences. Precisely, if

\[
0 \to N \to g \to b \to 0
\]

is an exact sequence of restricted Lie algebras and if \(M\) is a right restricted \(b\)-module then B. Eckmann and U. Stammback proved in \[13\] the existence of the following 5-term exact sequence

\[
H^{Hoch}_2(g, A) \to H^{Hoch}_2(b, A) \to N_{ab} \otimes_U(b) A \to H^{Hoch}_1(g, A) \to H^{Hoch}_1(b, A) \to 0
\]

where \(N_{ab} := N/[N,N]\) and \([N,N]\)' denotes the ideal generated by the elements \([x,y], z \in N\) where \(x, y, z \in N\).

If \(N\) is a restricted Lie algebra we denote by \(N_{ab} := N/\langle [N,N]\rangle_p\) the quotient restricted Lie algebra, where \(\langle [N,N]\rangle_p\) is the \(p\)-ideal generated by the elements \([x,y]\) where \(x, y \in N\). If \(p : g \to L\) is a restricted Lie epimorphism with kernel \(N\) then the \(p\)-map on \(N\) induce an \(R_p\)-action on \(N_{ab} := N/\langle [N,N]\rangle_p\) given by \(f.(n + \langle [N,N]\rangle_p) := n^{[p]} + \langle [N,N]\rangle_p\). Besides, \(N_{ab}\) is a \(L\)-module via the
action \( x.(n^+ + [N, N] >_p) := [s(x), n^+ + [N, N] >_p \] where \( s : L \to g \) denotes a section of \( p \) and \( x \in L, n \in N \). Moreover,

\[
x.(n^+[p] + <N, N >_p) = [s(x), n^+[p] + <N, N >_p
\]

\[
= [s(x), n^p + ... + n] + <N, N >_p
\]

\[
= 0
\]

It follows that, \( N_{ab} \) is equipped with the structure of a \( w(L) \)-module.

**Lemma 2.1.** If \( p : g \to L \) is a restricted Lie epimorphism with kernel \( N \) and \( A \) a \( w(L) \)-module, then we have an isomorphism

\[
Tor^0(p, A) \simeq Hom_{w(L)}(N_{ab}, A)
\]

**Proof.** Let \( g_* \) be the simplicial restricted Lie algebra which is obtained by iterating the simplicial kernel construction i.e

\[
g_* : \ldots \times L g \times L g \xrightarrow{d_2} g \times L g \xrightarrow{d_1} g \xrightarrow{d_0} L
\]

If we denote by \( g \rtimes N \) the semi-direct product of \( g \) and \( N \) in RLie, we get an isomorphism of restricted Lie algebras

\[
g \rtimes N \simeq g \times L g
\]

given by

\[(x, n) \mapsto (x, x + n)\]

Therefore we get the following simplicial object

\[
g_* : \ldots \times N \rtimes N \xrightarrow{d_2} g \times N \xrightarrow{d_1} g \xrightarrow{d_0} L
\]

where

\[
d_0(x, n, n') := (x, n)
\]

\[
d_1(x, n, n') := (x, n')
\]

\[
d_2(x, n, n') := (x + n, n' - n)
\]

The abelian group \( Tor^0(p, A) \) is defined by

\[
Tor^0(p, A) := Hom_{simpl(RLie/L)}(g_*, K(L \times f \bar{\bar{A}}, 1))
\]

and by Lemma 2.1 in [7] we obtain

\[
Tor(p, A) = ker(Hom_{RLie/L}(g \rtimes N, L \times f \bar{\bar{A}}) \xrightarrow{d_0 - d_1 + d_2} Hom_{RLie/L}(g \rtimes N \rtimes N, L \times f \bar{A}))
\]

Let \( \phi \in Tor(p, A) \) then by isomorphism (1.4) any \( \phi \in Tor(p, A) \) is associated to a Beck derivation \( d_\phi \in Der_p(g \rtimes N, A) \) such that \( d_\phi d_0 - d_\phi d_1 + d_\phi d_2 = 0 \), i.e

\[
d_\phi(x, n) - d_\phi(x, n') + d_\phi(x + n, n' - n) = 0
\]
for all \( x \in g \) and \( n, n' \in N \). Since \( N = \ker p \) we get

\[
d_{\phi}(0, [n, n']) = d_{\phi}([\{0, n\}, \{0, n'\}])
\]  
\[
= (0, n)d_{\phi}(0, n') - (0, n')d_{\phi}(0, n)
\]
\[
= 0
\]

Besides,

\[
d_{\phi}(0, [n, n'][p]) = d_{\phi}(\{0, [n, n'][p]\})
\]  
\[
= (0, [n, n'][p]) \cdots (0, [n, n'][p])d_{\phi}(0, [n, n']) + f(d_{\phi}(0, [n, n']))
\]
\[
= 0
\]

Therefore is defined a map \( \tilde{\phi} : N_{ab} \to A \) given by

\[
\tilde{\phi}(n+ < [N, N] > p) := d_{\phi}(0, n)
\]

for all \( n \in N \). Moreover for \( x \in L \) and \( n \in N \) we have

\[
\tilde{\phi}((x, (n+ < [N, N] > p))) = \tilde{\phi}([s(x), n]+ < [N, N] > p)
\]  
\[
= d_{\phi}(0, [s(x), n])
\]
\[
= d_{\phi}([s(x), 0], (0, n))
\]
\[
= (s(x), 0)d_{\phi}(0, n) - (0, n)d_{\phi}(s(x), 0)
\]
\[
= (s(x), 0)d_{\phi}(0, n)
\]
\[
= x.\tilde{\phi}(n+ < [N, N] > p)
\]

\[
\tilde{\phi}((n[p]+ < [N, N] > p)) = d_{\phi}(0, n[p])
\]  
\[
= d_{\phi}((0, n)[p])
\]
\[
= (0, n) \cdots (0, n)d_{\phi}(0, n) + f(d_{\phi}(0, n))
\]
\[
= f(d_{\phi}(0, n))
\]

It follows that \( \tilde{\phi} \) is a morphism of \( w(L) \)-modules. Conversely, let \( \tilde{\phi} : Hom_{w(L)}(N_{ab}, A) \) be a \( w(L) \)-homomorphism. We define \( \phi : g \times N \to L \times_f A \) given by

\[
\phi(x, n) := (p(x), \tilde{\phi}(n+ < [N, N] > p))
\]

Then the associated derivation \( d_{\phi}(x, n) := \tilde{\phi}(n+ < [N, N] > p) \) satisfies the relation (2.1) and it follows that \( \phi \in Tor^0(p, M) \). Therefore the map \( \phi \mapsto \tilde{\phi} \) is a well defined homomorphism, inverse to the homomorphism \( \phi \mapsto \tilde{\phi} \). \( \square \)

**Theorem 2.2.** Let \( 0 \to N \to g \to b \to 0 \) be an exact sequence of restricted Lie algebras and \( A \) a \( w(b) \)-module. Then the following sequence is exact

\[
0 \to \text{Der}_p(b, A) \to \text{Der}_p(g, A) \to \text{Hom}_{w(b)}(N_{ab}, A) \to H^1_G(b, A) \to H^1_G(g, A)
\]

*Proof.* Since RLie is a monadic category over Sets whose objects have an underlying group structure, it follows from the above lemma and the exact sequence (1.5). \( \square \)
3. Internal groupoids and crossed modules

Crossed modules in groups were introduced by Whitehead [32] in the study of relative homotopy groups. Brown and Spencer in [5] noted that internal categories within the category of groups are equivalent to crossed modules. In the more general context of categories of groups with operations, crossed modules and internal categories are studied by Porter in [24]. Moreover, internal categories in a Mal’tsev variety are studied by Janelidze in [20]. Below we consider the case of the category of restricted Lie algebras.

Let us recall the definition of internal category in a category $C$ with pullbacks. An internal category in $C$ is a diagram in $C$

$$
\begin{array}{ccc}
C \times C_0 & \xrightarrow{\theta} & C \\
\downarrow & & \downarrow \\
C & \xrightarrow{t} & C_0
\end{array}
$$

such that $te = se = id_{C_0}$ with a morphism $\theta : C \times C_0 \to C$ where $C \times C_0$ denotes the pullback

$$
\begin{array}{ccc}
C \times C_0 & \xrightarrow{p_2} & C \\
\downarrow & & \downarrow \\
C & \xrightarrow{t} & C_0
\end{array}
$$

satisfying: $t\theta = tp_2$ and $s\theta = sp_1$, the associative low relation

$$
\begin{array}{ccc}
C \times C_0 \times C_0 & \xrightarrow{\theta \times id} & C \times C_0 \\
\downarrow & & \downarrow \\
C \times C_0 & \xrightarrow{\theta} & C_0
\end{array}
$$

the left and right unit lows for composition of morphisms

$$
\begin{array}{ccc}
C & \xrightarrow{id \times \theta} & C \times C_0 \\
\downarrow & & \downarrow \\
C \times C_0 & \xrightarrow{\theta} & C_0
\end{array}
$$

The morphisms $t, s, e$ are called the target, source and unit morphisms respectively. An internal category is called internal groupoid if for any $c \in C$ there is a $c' \in C$ such that $\theta(c, c') = es(c)$ and $\theta(c', c) = et(c)$.

J.-L. Loday and C. Kassel define in [21] the notion of crossed module for the category of Lie algebras. In the same way are defined crossed modules for the category of restricted Lie algebras (cf. [9]).

Definition 3.1. Let $\mu : M \to N$ be a homomorphism of restricted Lie algebras. The triple $(M, N, \mu)$ is called a crossed module if there is a restricted homomorphism $\eta : N \to Der(M)$ such that $\eta(n)$ is a restricted derivation for all $n \in N$ and the
following relations hold:

\[(3.1) \quad \mu(\eta(n)(m)) = [n, \mu(m)], \quad n \in N, \ m \in M\]

\[(3.2) \quad \eta(\mu(m))(m') = [m, m'], \quad m, m' \in M\]

In order to simplify the notion we denote \(\eta(n)(m) := n.m\) for all \(m \in M\) and \(n \in N\).

**Example 3.2.** Let \(L\) be a restricted Lie algebra and \(I\) an ideal of \(L\). If \(i : I \hookrightarrow L\) denotes the inclusion homomorphism then the \((I, L, i)\) is a crossed module in \(RLie\).

**Theorem 3.3.** The notion of a crossed module is equivalent to the notion internal groupoid in the category of restricted Lie algebras.

**Proof.** Let \((M, N, \mu)\) be a crossed module in \(RLie\) then we associate to it the following diagram in \(RLie\):

\[
\begin{align*}
(M \rtimes N) \times (M \rtimes N) & \xrightarrow{\theta} (M \rtimes N) \\
N & \xrightarrow{e} (M \rtimes N) \\
N & \xrightarrow{s} (M \rtimes N) \\
N & \xrightarrow{t} (M \rtimes N)
\end{align*}
\]

where \(s, t : M \rtimes N \to N\) are restricted Lie homomorphisms given by

\[
s(m, n) := n
\]

\[
t(m, n) := n + \mu(m)
\]

and \(\sigma : N \to M \rtimes N\) by

\[
e(n) := (0, n)
\]

for all \(m \in M\) and \(n \in N\). The multiplication

\[
\theta : (M \rtimes N) \times (M \rtimes N) \to M \rtimes N
\]

is given by

\[
\theta((m, n), (m', n + \mu(m))) = (m + m', n)
\]

A straightforward calculation shows that \(\theta\) is a Lie algebra homomorphism. Moreover we note that

\[
\theta(((m, 0), (0, \mu(m))))^{[p]} = \theta(((m, 0)^{[p]}, (0, \mu(m)))^{[p]})
\]

\[
= \theta((m^{[p]}, 0), (0, \mu(m))^{[p]})
\]

\[
= \theta((m^{[p]}, 0), (0, \mu(m^{[p]})))
\]

\[
= (m^{[p]}), 0)
\]

\[
= (\theta((m, 0), (0, \mu(m))))^{[p]}
\]

and in the same way we see that

\[
\theta(((0, n), (0, n)))^{[p]} = (\theta((0, n), (0, n)))^{[p]}
\]

and

\[
\theta(((0, 0), (m', 0)))^{[p]} = (\theta((0, 0), (m', 0)))^{[p]}
\]

Since \(\theta\) is a Lie algebra homomorphism we deduce that is actually a restricted Lie homomorphism. Then one can easily check that the conditions of associativity and unit and right lows are satisfied. Besides, if we define

\[
(m, n)' := (-m, n + \mu(m))
\]
for all \( m \in M \) and \( n \in N \) then we see that the above diagram in RLie defines an internal groupoid in RLie.

Let

\[
\begin{array}{ccc}
N' \times_N N' & \xrightarrow{\theta} & N' \\
\downarrow{s} & & \downarrow{t} \\
N & & N
\end{array}
\]

be an internal groupoid in RLie with multiplication \( \theta \). Then we associate to it a crossed module the \((M, N, \mu)\) where \( M := \text{Kers} \) and \( \mu := t \mid_M \) and with action \( \eta : N \to \text{Der}(M) \) given by

\[
\eta(n)(m) = [e(n), m]
\]

for all \( n \in N \) and \( m \in M \). In effect,

\[
\eta(n^{[p]})(m) = [e(n^{[p]}), m] = [(e(n))^{[p]}, m] = (\eta(n))^p(m)
\]

and

\[
\eta(n)(m^{[p]}) = [e(n), m^{[p]}] = ad^{p-1}(m)(\eta(n)(m))
\]

Since \( te = se = id \mid_N \) we get

\[
\theta(m + e(n), (m' + e(n + \mu(m))) = \theta(m + e(n), et(m + e(n))) + \theta(es(m'), m')
\]

Besides by unities properties of the groupoid we have that

\[
\theta((m + e(n), et(m + e(n))) + \theta(es(m'), m') = m + e(n) + m'
\]

Thus we obtain

\[
\theta((m + e(n)), (m' + e(n + \mu(m))) = m + m' + e(n)
\]

since \( \theta \) is a Lie algebra homomorphism we have

\[
\theta([[(m, m' + e\mu(m)), (0, m'')]]) = [\theta(m, m' + e\mu(m)), \theta(0, m'')]
\]

and

\[
[e\mu(m), m''] + [m', m''] = [m, m''] + [m', m'']
\]

We deduce that

\[
\eta(\mu(m)) = [e(\mu(m)), m''] = [m, m'']
\]

Finally we have,

\[
\mu(\eta(n)(m)) = b([e(n), m]) = [te(n), t(m)] = [n, \mu(m)]
\]
4. The second cohomology group and 2-fold extensions

Gerstenhaber in [15] studies 2-fold extensions in certain categories of interest including the category of Lie algebras. Also, the case of Lie algebras is studied by Shimada-Uehara-Brenneman-Iwai in [29]. Besides, J.-L. Loday and C. Kassel in [21] consider two fold extensions of Lie algebras associated to a crossed module. In this section we study 2-fold extensions in the category of restricted Lie algebras.

Let \((M,N,\mu)\) be a crossed module in RLie and \((A,\begin{array}{|c|c|}
\begin{array}{c}
pA
\end{array}
\end{array}) := \ker \mu\). By relation 3.2 of Definition 3.1 we get that \(\mu(a).m = [a,m] = 0\) for all \(a \in A\) and \(m \in M\). Thus \(A \subset C(M)\) where \(C(M)\) denotes the center of \(M\). Moreover by relation 3.1 of Definition 3.1 we have that \(\mu(n.a) = [n,\mu(a)] = 0\) thus \(n.a \in A\) for all \(n \in N\) and \(a \in A\).

The restriction on \(A\) of the action of \(N\) on \(M\) endows \(A\) with the structure of restricted \(N\)-module. Moreover we have
\[
n.a[pA] = ad_{p-1}(a)(n.a) = 0
\]
thus the couple \((A,pA)\) is a Beck \(N\)-module. Besides \(\mu(n.m) = [n,\mu(m)]\) for all \(n \in N\) and \(m \in M\), thus we obtain that \(\text{Img}(\mu)\) is a restricted ideal of \(N\). Since \((n + \mu(m)).a = n.a + [m,a] = n.a + 0\) one sees that \(A\) is also a restricted \(R\)-module where \(R := \text{coker} \mu\). Therefore \((A,pA)\) becomes a Beck \(R\)-module.

4.1. 2-fold extensions. Let \((A,pA)\) be a Beck \(R\)-module. We denote by \(E^2(R,A)\) the category whose objects are exact sequences in RLie
\[
0 \rightarrow A \rightarrow M \xrightarrow{\mu} N \rightarrow R \rightarrow 0
\]
such that \((M,N,\mu)\) is a crossed module and the induced Beck \(R\)-module structure is the given one. The morphisms are:
\[
\begin{array}{cccccccc}
0 & \rightarrow & A & \rightarrow & M & \rightarrow & N & \rightarrow & R & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{f} & & \downarrow{g} & & \downarrow{id} & & \\
0 & \rightarrow & A & \rightarrow & M' & \rightarrow & N' & \rightarrow & R & \rightarrow & 0
\end{array}
\]
where the morphisms \(f\) and \(g\) respect the actions. Two objects \(E_1, E_2 \in E^2(R,A)\) are called equivalent if there is a morphism in \(E^2(R,A)\) from one to the other. We will denote by \(E^2(R,A)\) the set of equivalence classes in \(E^2(R,A)\).

For \(p : E \rightarrow R\) a fixed epimorphism in RLie we consider the set \(E^2(p,A)\) the set of 2-term extensions
\[
0 \rightarrow A \rightarrow M \xrightarrow{\mu} N \xrightarrow{p} R \rightarrow 0
\]
and morphisms
\[
\begin{array}{cccccccc}
0 & \rightarrow & A & \rightarrow & M & \rightarrow & N & \xrightarrow{p} & R & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{f} & & \downarrow{id} & & \downarrow{id} & & \\
0 & \rightarrow & A & \rightarrow & M' & \rightarrow & N & \xrightarrow{p} & R & \rightarrow & 0
\end{array}
\]
Two objects $E_1, E_2 \in E^2(p,A)$ are called equivalent if there is a morphism from one to the other. We denote by $E^1(p,A)$ the set of equivalence classes.

4.2. Baer sum. The set $E^2(R,A)$ can be endowed with the structure of abelian group. The Baer sum of two restricted Lie algebra extensions is induced from their Baer sum viewed as Lie algebra extensions. In particular, let $E$ and $E'$ be 2-fold extensions of restricted Lie algebras

$$(E): \quad 0 \to A \to M \xrightarrow{d} N \to R \to 0$$

and

$$(E') : \quad 0 \to A \to M' \xrightarrow{d'} N' \to R \to 0$$

The Baer sum of $(E)$ and $(E')$ is defined as the extension

$$(E + E') : \quad 0 \to (A \times A)/K \to (M \times M')/K \to N \times_R N' \to R \to 0$$

where $K := \ker f$ and $f : A \times A \to A$ given by $f(a,a') := a + a'$ for all $a,a' \in A$. The restricted ideal $K$ is consisting of the elements $(a,-a)$.

Next we give an interpretation of 2-torsors over a restricted Lie algebra $R$ in terms of 2-fold extensions. We note that 2-torsors relative to $U$ in $RLie$ are 2-torsors relative to $U$ in $Lie$ viewed as simplicial objects in $Lie$.

**Lemma 4.1.** Let $R \in RLie$ be a restricted Lie algebra and $(A,p_A)$ a Beck $R$-module. Then we have

$$Tors^2(R,A) \simeq E^2(R,A)$$

and

$$Tors^1(p,A) \simeq E^1(p,A)$$

**Proof.** It suffices to construct maps from $Tors^2(R,A)$ to $E^2(R,A)$ inverse to each other. Let $(E,)$ be a 2-torsor over $R$ with simplicial morphism

$$\epsilon : E, \to K(A \times_{p_A} R, 2)$$

We consider the Moore complex $M(E,)$ of $(E,)$ given by $M(E,)_0 = R$ and $M(E,)_n = \cap^0_i \ker d_i$, for all $i \geq 1$ and with differential $d_n := d_0|_{M(E,)_n}$. By conditions (1) and (2) of the Definition 1.12 the associated Moore complex $M(E,)$ is given by the following exact sequence in $RLie$

$$M(E,): \quad 0 \to Ker d_1 \cap \ker d_0 \to ker d_1 \xrightarrow{d_0} E_0 \xrightarrow{d_0} R \to 0$$

It follows from condition (2) of the Definition 1.12 that

$$E_2 \simeq (A \times_{p_A} R) \times_R \Lambda^2(1)(E,).$$

where

$$\Lambda^2(1)(E,) = \{(x_0,x_1) \in E_1 \times E_1, d_0(x_0) = d_0(x_1)\}$$

We have the following commutative diagram
Therefore we obtain

\[ \text{Ker } d_1 \cap \text{ker } d_0 = (A, p_A) \]

Since 2-torsors in RLie are 2-torsors in Lie, it follows from [26] that the Lie action of \( R \) on \( A \) coincides with the Lie action induced from the above exact sequence. Moreover we have

\[ r^{[p]}a = [s(r)^{[p]}, a] = (r(r(\cdots (r.a)))) \]

where \( s \) denotes a section of the surjection \( p : E_0 \to R \). Therefore the induced Beck \( R \)-module structure by the two-term exact sequence, coincides with the initial structure and we obtain a two term exact sequence in RLie

\[ 0 \to A \to \text{Ker } d_1 \xrightarrow{d_0} E_0 \xrightarrow{p} R \to 0 \]

Conversely, let

\[ 0 \to A \to M \xrightarrow{\mu} E_0 \xrightarrow{p} R \to 0 \]

be a two-term extension in \( E^2(R, A) \). By Theorem 3.3 is associated an augmented over \( R \) groupoid \( \Gamma \):

\[ E_1 := (E_0 \times M) \xrightarrow{\mu} E_0 \xrightarrow{p} R \to 0 \]

where \( s(e_0, m) := e_0, t(e_0, m) := \mu(m) + e_0 \) and \( e(e_0) := (0, e_0) \) for all \( e_0 \in E_0 \) and \( m \in M \). Let \( E := \text{cosk}^1(\Gamma) \) be the 1-coskeleton of \( \Gamma \). If

\[ ((\mu(z_1) + e_0, x_1), (e_0, y_1), (e_0, z_1)) \in E_2 \]

where \( x_1, y_1, z_1 \in M \) and \( e_0 \in E_0 \) then \( x_1 + z_1 - y_1 \in A \). As in the case of Lie algebras is defined a Lie algebra homomorphism \( \delta_2 : E_2 \to K(A \times_{p_A} R, 2) \) given by

\[ \delta_2((\mu(z_1) + e_0, x_1), (e_0, y_1), (e_0, z_1)) = (x_1 + z_1 - y_1, p(e_0)) \]
We notice that \( z_1 + x_1 = y_1 + a \) for some \( a \in A \), thus the Lie algebra \( L_{(z_1 + x_1), y_1} \) generated by \((z_1 + x_1)\) and \( y_1 \) is zero. By Remark 1.2 we obtain
\[
(z_1 + x_1 - y_1)^{[p]} = (z_1 + x_1)^{[p]} - y_1^{[p]} + \sum_{i=1}^{i=p} s_i((z_1 + x_1), y_1) \\
= (z_1 + x_1)^{[p]} - y_1^{[p]} \\
= z_1^{[p]} + x_1^{[p]} + \sum_{i=0}^{i=1} s_i(z_1, x_1) - y_1^{[p]}
\]
By Relation (3.2) we have \( \mu(z_1), x_1 = [z_1, x_1] \) thus
\[
\delta_2 \left( ([\mu(z_1), x_1], (0, y_1), (0, z_1))^{[p]} \right) = \delta_2 \left( ([\mu(z_1), x_1]^{[p]}, (0, y_1)^{[p]}, (0, z_1)^{[p]}) \right) \\
= \delta_2 \left( ([\mu(z_1)]^{[p]}, 0) + (0, x_1^{[p]}) + \sum_{i=1}^{i=p} s_i((\mu(z_1), 0), (0, x_1)) \right), (0, y_1^{[p]}), (0, z_1^{[p]}) \right) \\
= \delta_2 \left( ([\mu(z_1)]^{[p]}, x_1^{[p]} + \sum_{i=1}^{i=p} s_i(z_1, x_1)), (0, y_1^{[p]}), (0, z_1^{[p]}) \right) \\
= (x_1^{[p]} + \sum_{i=1}^{i=p} s_i(z_1, x_1) + z_1^{[p]} - y_1^{[p]}, 0) \\
= \delta_2 \left( ([\mu(z_1), x_1], (0, y_1), (0, z_1))^{[p]} \right)
\]
Also, \( p \) is a restricted Lie homomorphism so
\[
\delta_2 \left( ([e_0, 0], (e_0, 0), (e_0, 0))^{[p]} \right) = \delta_2 \left( ([e_0, 0], (e_0, 0), (e_0, 0))^{[p]} \right)
\]
Since \( \delta_2 \) is a Lie algebra homomorphism we see that \( \delta_2 \) is actually a restricted Lie homomorphism. Besides, as for the case of Lie algebras (see [26]) \( \delta \) is a normalized cocycle. It follows from Section 4.1 in [11] that \( E \) is a \( K(A \times_{\mu A} R, 2) \) torsor over \( R \). Also, one can see that the Moore complex \( M(E) \) of \( E \) is the exact sequence
\[
0 \to A \to M \xrightarrow{p_0} E_0 \xrightarrow{p} R \to 0
\]
Consequently, we defined maps from
\[
\text{Tors}^2(R, A) \to E^2(R, A)
\]
and
\[
E^2(R, A) \to \text{Tors}^2(R, A)
\]
which are inverse to each other. Moreover we can see that equivalent two-term extensions correspond to the same connected component in the category of 2-torsors and conversely elements in the same component correspond to equivalent two-term extensions.

Besides, by definition of the group structure on the set of torsors (see [10], [12]), the group structure on 2-torsors \( R\text{Lie} \) is induced from the group structure on 2-torsors in \( \text{Lie} \). Moreover, the Baer sum of two restricted Lie algebra extensions is induced from their Baer sum viewed as Lie algebra extensions. It follows that above bijections are actually group isomorphisms. Therefore the theorem follows. \( \square \)
Theorem 4.2. Let $R \in \text{RLie}$ be a restricted Lie algebra and $(A, p_A)$ a Beck $R$-module. Then there is an isomorphism

$$H^2_G(R, A) \simeq \mathcal{E}^2(R, A)$$

Proof. The theorem follows from the above Lemma 4.1 and the Theorem 1.13. □

Remark 4.3. We note that in the Cartan-Eilenberg context crossed modules in various algebraic categories are associated with the third cohomology group. In the Quillen-Barr-Beck context crossed modules are associated to the second cohomology group since there is a shift by 1 in the notation. Besides, G. Hochschild in [18] gives an interpretation of the third Hochschild cohomology in terms of space of restricted kernel classes.

4.3. Eight term exact sequence. The five term exact sequence of Theorem 2.2 for Quillen-Barr-Beck cohomology for restricted Lie algebras can be extended to an eight term exact sequence by the following theorem.

Theorem 4.4. Let $0 \to N \to g \xrightarrow{p} b \to 0$ be an exact sequence of restricted Lie algebras and $M$ an $w(b)$-module. Then the following sequence is exact

$$0 \to \text{Der}_p(b, M) \to \text{Der}_p(g, M) \to \text{Hom}_{w(b)}(N_{ab}, M) \to H^1_G(b, M) \to H^1_G(g, M)$$

$$\to \mathcal{E}^1(p, M) \to H^2_G(b, M) \to H^2_G(g, M)$$

Proof. It follows from the Theorem 4.2 and the eight term exact sequence (1.5). □

4.4. Application to extensions of algebraic groups. Let $k$ be algebraic closed field of prime characteristic and $G$ an algebraic group over $k$. Since $\text{char} k = p$ we have that the associated Lie algebra $\text{Lie}(G)$ is actually a restricted Lie algebra. In fact in this way is defined a functor $\text{Lie} : \text{Gr} \to \text{RLie}$ from the category of algebraic groups to the category of restricted Lie algebras (see [4]).

Let $H, A$ be algebraic groups then M. Rosenlicht in [25] and J.-P. Serre in [28] define as an extension of $H$ by $A$ a short exact sequence of groups

$$0 \to A \xrightarrow{\alpha} G \xrightarrow{\gamma} H \to 0 \quad (E)$$

such that $\alpha, \gamma$ are separable rational homomorphisms. Thus $H$ can be identified with a normal algebraic subgroup of $G$ and $A$ with $G/A$.

Let

$$0 \to A \xrightarrow{\alpha'} G' \xrightarrow{\gamma'} H \to 0 \quad (E')$$

be an other extension of $H$ by $A$ then $(E), (E')$ are called equivalent if there is a rational homomorphism $\psi : G \to G'$ such that the following diagram is commutative

$$\begin{array}{ccc}
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
| & & id & & \psi & & id & & |
\end{array}$$

The class of equivalent extensions of $H$ by $A$ is denoted by $\text{Ext}(H, A)$. 
Proposition 4.5. Let $0 \to H \to G \xrightarrow{\gamma} A \to 0$ be an exact sequence of algebraic groups and $M$ an $w(Lie(A))$-module. Then the following sequence is exact

$$0 \to Der_p(Lie(A), M) \to Der_p(Lie(G), M) \to Hom_{w(Lie(A))}(Lie(H), M) \to$$

$$\to H^1_G(Lie(A), M) \to H^1_G(Lie(G), M) \to E^1(Lie(\gamma), M) \to$$

$$\to H^2_G(Lie(A), M) \to H^2_G(Lie(A), M)$$

Proof. If $0 \to H \to G \xrightarrow{\gamma} A \to 0$ is an exact sequence of algebraic groups then $0 \to Lie(H) \to Lie(G) \xrightarrow{Lie(\gamma)} Lie(A) \to 0$ is an exact sequence of Lie algebras (see [25]). Since $k$ is a field of characteristic of $p$ the induced Lie algebra exact sequence is actually a restricted Lie algebras sequence. Therefore the proof follows from the Theorem 4.4 above. \qed

If $0 \to H \to G \xrightarrow{\gamma} A \to 0$ is an exact sequence of commutative algebraic groups and $B$ a commutative algebraic group, then J.-P. Serre in [28] and M. Rosenlicht in [25] proved that there is an exact an exact sequence

$$0 \to Hom(A, B) \to Hom(G, B) \to Hom(H, B) \to$$

$$\to Ext(A, B) \to Ext(G, B) \to Ext(H, B)$$

Since $0 \to H \to G \xrightarrow{\gamma} A \to 0$ is an exact sequence of commutative algebraic groups it is induced an exact sequence of abelian restricted Lie algebras $0 \to Lie(H) \to Lie(G) \xrightarrow{Lie(\gamma)} Lie(A) \to 0$. Then $Lie(B)$ becomes a $w(Lie(A))$-module when $Lie(A)$ acts trivially and $R_f$ acts via $fb := f|_B$ for all $b \in Lie(B)$. Thus we get an exact sequence

$$0 \to Hom_{RLie}(Lie(A), Lie(B)) \to Hom_{RLie}(Lie(G), Lie(B)) \to Hom_{RLie}(Lie(H), Lie(B)) \to$$

$$\to H^1_G(Lie(A), Lie(B)) \to H^1_G(Lie(G), Lie(B)) \to E^1(Lie(\gamma), Lie(B)) \to$$

$$\to H^2_G(Lie(A), Lie(B)) \to H^2_G(Lie(A), Lie(B))$$

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