HOMOLOGY OF TWISTED QUIVER
BUNDLES WITH RELATIONS

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Abstract. We study the Ext modules in the category of left modules over a twisted algebra of a finite quiver over a ringed space \((X, \mathcal{O}_X)\), allowing for the presence of relations. We introduce a spectral sequence which relates the Ext modules in that category with the Ext modules in the category of \(\mathcal{O}_X\)-modules. Contrary to what happens in the absence of relations, this spectral sequence in general does not degenerate at the second page. We also consider local Ext sheaves. Under suitable hypotheses, the Ext modules are represented as hypercohomology groups.

Contents

1. Introduction 2
2. Twisted quiver algebras with relations 3
3. Representations of a twisted quiver algebra 7
4. A spectral sequence 12
5. Hypercohomology 20
References 22

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1. Introduction

In 1977 Barth used monads (3-term complexes of vector bundles that have nonzero cohomology only at their middle term) to construct moduli spaces of rank 2, degree 0 vector bundles on the projective plane \( \mathbb{P}^2 \). That construction was generalized by Barth and Hulek \cite{14, 3}, and in 1985 Drézet and Le Potier \cite{9} eventually constructed the moduli space of all Gieseker-semistable torsion-free sheaves \( \mathbb{P}^2 \). Their work used in a critical way the fact that the relevant monad is a Kronecker complex, and indeed the moduli space can be constructed as a moduli space of semistable Kronecker complexes modulo the action of a reductive group. Now semistable Kronecker complexes can be regarded as semistable representations of a quiver with relations, whose moduli spaces were constructed by King \cite{15}. Moreover, moduli spaces of semistable sheaves on \( \mathbb{P}^1 \times \mathbb{P}^1 \) were constructed using quiver moduli spaces in \cite{18}.

Another approach to the use of quiver moduli spaces to construct moduli spaces of sheaves uses the theory of Bridgeland stability conditions \cite{7}, see \cite{22, 2}.

Quivers have been of particular importance in the study of framed sheaves, whose moduli spaces can be regarded as higher rank generalizations of Hilbert schemes of points, and for some spaces are resolutions of singularities of instanton moduli spaces \cite{8, 21}. The description of these spaces by means of the so-called ADHM data indeed often allows one to identify them as (components) of a moduli space of quiver representations \cite{20, 16, 4}; the reader may refer to \cite{5} for a review of this aspect, and to \cite{10} for an introduction to quiver varieties.

Other moduli spaces that have been treated by means of quiver techniques are Higgs bundles, stable pairs and triples (see e.g. \cite{1}).

So, once the significance of the moduli spaces of quiver representations for moduli problems has been established, it becomes important to study their main properties, in the first instance their deformations. This requires the knowledge of the main homological properties of moduli of quiver representations. In \cite{11} Gothen and King studied the homological algebra of the category of representations of a twisted quiver without relations over the category of \( \mathcal{O}_X \)-modules on a ringed space \((X, \mathcal{O}_X)\). Their main result is a long exact sequence which relates the Ext groups in the category of representations of the quiver algebra with the Ext groups in the category of \( \mathcal{O}_X \)-modules, thus making the former potentially computable. In this paper we generalize Gothen and King’s work allowing for the presence of relations. We also consider local Ext sheaves in addition to global Ext modules.

Let \( Q \) be a quiver, and let \((X, \mathcal{O}_X)\) be a ringed space. Let \( \mathcal{B} \) be the sheaf of \( \mathcal{O}_X \)-algebras generated by the vertices of the quiver, and let \( \mathcal{M} \) be a sheaf of \( \mathcal{O}_X \)-modules, graded over the arrows of the quiver; it has a natural \( \mathcal{B} \)-module structure. The \( \mathcal{M} \)-twisted quiver algebra \( \mathcal{M} Q \) of \( Q \) is the tensor algebra of \( \mathcal{M} \) over \( \mathcal{B} \). The \( \mathcal{M} \)-twisted quiver algebra \( Q \) with relations will be a quotient \( \mathcal{A} = \mathcal{M} Q / \mathcal{K} \), where \( \mathcal{K} \) is a sheaf of two-sided ideals (the ideal of relations). Moreover we set \( \mathcal{N}_\alpha = \mathcal{M}_\alpha / (\mathcal{M}_\alpha \cap \mathcal{K}) \), and \( \mathcal{N} = \oplus_\alpha \mathcal{N}_\alpha \). We shall denote by \( \mathfrak{R}(\mathcal{A}) \) the
category \(\mathcal{M}\)-twisted representations of \(Q\) with relations. Finally, let \(\mathcal{N}Q\) be the tensor algebra of \(\mathcal{N}\) over \(B\) (more detailed definitions will be given in the next section).

Our first result is a technical one: under the hypothesis that the \(O_X\)-modules \(\mathcal{N}_\alpha\) are finitely presented, in Theorem 3.2 we prove that the category \(\mathcal{R}(\mathcal{A})\) is equivalent to the category \(\mathcal{N}Q\text{-mod}\): in particular, \(\mathcal{A}\text{-mod}\) is a full subcategory of \(\mathcal{R}(\mathcal{A})\). In the absence of relations we recover the result of [1, 11] that \(\mathcal{R}(\mathcal{M}Q)\) is equivalent to \(\mathcal{M}Q\text{-mod}\), with a mild generalization, due to our slightly weaker assumptions. It may be interesting to note that one has an equivalence of categories between a suitable subcategory \(\tilde{\mathcal{R}}(\mathcal{A})\) of \(\mathcal{R}(\mathcal{A})\) and \(\mathcal{A}\text{-mod}\) even in the case with relations but with a trivial twist.

In Section 4 we address the problem of relating the Ext modules in the category \(\mathcal{A}\text{-mod}\) with those in the category \(O_X\text{-mod}\). While in [11] this leads to a long exact sequence, here we only find a spectral sequence relating the two Exts, which in general does not degenerate at the second step. This happens because the spectral sequence is associated with a first quadrant double complex that in general is unbounded in both directions, which in turn depends from the fact that in our case the resolution (4.6) is infinite.

Theorem 5.1 generalizes the analogous result from [11], namely, the Ext modules – when their first argument is locally free as an \(O_X\)-module – can be realized as hypercohomology groups. Under the same hypothesis, Corollary 4.7 expresses this result in local form, i.e., for the local Ext sheaves.

Natural developments of these first results would be a base change formula for the local Exts, and a study of the deformation theory of these moduli spaces, which would allow one to characterize their tangent spaces and tangent sheaf. We shall address these issues in a future publication.

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2. Twisted quiver algebras with relations

Let \(Q = (I, E, h, t)\) be a quiver, where the set \(I\) labels the vertices of \(Q\), \(E\) labels the arrows, and the maps \(t, h: E \to I\) associate with any arrow its origin (tail) and destination (head). We will assume throughout this paper that \(Q\) is finite, in the sense that both sets \(I\) and \(E\) are finite. For each \(i \in I\), we denote by \(e_i\) the trivial path starting and ending at the vertex \(i\). The free Abelian group \(B\) generated by \(\{e_i\}_{i \in I}\) has a natural ring structure given by

\[
e_i e_j = \delta_{ij} e_i \quad \text{for all } i, j \in I.
\]
Note that $\sum_{i \in I} e_i = 1_B$, so that the elements $\{e_i\}_{i \in I}$ are a complete set of orthogonal idempotents. We remark that, because of the relations (2.1), one has the identification

$$T_B\left(\bigoplus_{\alpha \in E} \mathbb{Z}\alpha\right) = \mathbb{Z}Q,$$  \hspace{1cm} (2.2)

where $T_B(-)$ is the tensor algebra functor on the category of $B$-bimodules and $\mathbb{Z}Q$ is the path algebra of $Q$ over $\mathbb{Z}$.

Let $(X, O_X)$ be a ringed space. The free $O_X$-module

$$B = \bigoplus_{i \in I} O_X e_i$$

generated by $\{e_i\}_{i \in I}$ can be endowed with a structure of a sheaf of $B$-algebras by imposing conditions that are formally the same as those in eq. (2.1).

Consider now a collection $\{M_\alpha\}_{\alpha \in E}$ of $O_X$-modules labeled by the arrows of $Q$. Each $M_\alpha$ can be endowed with a structure of $B$-bimodule in the following way: for every open subset $U \subseteq X$ and for every section $x \in M_\alpha(U)$,

$$e_i x = \begin{cases} x & \text{if } h(\alpha) = i \\ 0 & \text{otherwise} \end{cases}, \quad xe_i = \begin{cases} x & \text{if } t(\alpha) = i \\ 0 & \text{otherwise} \end{cases}$$

**Definition 2.1.** Let $\mathcal{M} = \bigoplus_{\alpha \in E} M_\alpha$. The $\mathcal{M}$-twisted quiver algebra of $Q$ is the tensor algebra of $B$ over $\mathcal{M}$:

$$\mathcal{M}Q = T_B(\mathcal{M}).$$

In the particular case where $M_\alpha = O_X \alpha$ for all $\alpha \in E$, there is the following isomorphism of $B$-algebras, which generalises that in eq. (2.2),

$$T_B(\bigoplus_{\alpha \in E} O_X \alpha) \simeq O_X Q,$$

where $O_X Q = O_X \otimes_\mathbb{Z} ZQ$ is the path algebra of $Q$ over $O_X$.

Let $\mathcal{X} \hookrightarrow \mathcal{M}Q$ be a sheaf of two-sided ideals in the $\mathcal{M}$-twisted quiver algebra of $Q$.

**Definition 2.2.** The $\mathcal{M}$-twisted quiver algebra of $Q$ over $X$ with relations $\mathcal{X}$ is the quotient $\mathcal{A} = \mathcal{M}Q / \mathcal{X}$.

Clearly, there is a short exact sequence of $\mathcal{M}Q$-bimodules

$$0 \longrightarrow \mathcal{X} \longrightarrow \mathcal{M}Q \longrightarrow \mathcal{A} \longrightarrow 0,$$  \hspace{1cm} (2.3)

where $\pi$ is the canonical projection. We set

$$\mathcal{N}_\alpha = \pi(\mathcal{M}_\alpha) = \mathcal{M}_\alpha / (\mathcal{M}_\alpha \cap \mathcal{X})$$  \hspace{0.5cm} \text{for each } \alpha \in E,

$$\mathcal{N} = \bigoplus_{\alpha \in E} \mathcal{N}_\alpha.$$  \hspace{1cm} (2.4)
Note that \( \mathcal{N} \) is not contained in \( \mathcal{A} \), but there is a surjective \( \mathcal{B} \)-bimodules morphism \( \mathcal{N} \rightarrow \pi(\mathcal{M}) \), which induces a surjective homomorphisms of \( \mathcal{B} \)-algebras \( \varpi_1 : \mathcal{N}Q = T_B\mathcal{N} \rightarrow T_B\pi(\mathcal{M}) \).

Moreover, since \( \mathcal{A} \) is generated by the \( \mathcal{B} \)-sub-bimodule \( \pi(\mathcal{M}) \), there exists a surjective homomorphism \( \varpi_2 : T_B\pi(\mathcal{M}) \rightarrow \mathcal{A} \). So the composite homomorphism \( \varpi = \varpi_2 \circ \varpi_1 : \mathcal{N}Q \rightarrow \mathcal{A} \) fits into the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}Q & \longrightarrow & \mathcal{N}Q \\
\downarrow & \nearrow \pi & \downarrow \mathcal{A} \\
\end{array}
\]

where the horizontal morphism is naturally induced by \( \pi \).

Let \( \mathcal{V} \) be a left \( \mathcal{A} \)-module and denote by \( \rho_{\mathcal{V}} : \mathcal{A} \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{V}) \) the morphism induced by the multiplication. If we set

\[
\mathcal{V}_i = \rho_{\mathcal{V}}(e_i)\mathcal{V}
\]

for each \( i \in I \), then \( \mathcal{V}_i \) is an \( \mathcal{O}_X \)-module and, since the \( e_i \) are mutually orthogonal,

\[
\mathcal{V} \simeq \bigoplus_{i \in I} \mathcal{V}_i . \tag{2.5}
\]

For each \( \alpha \in E \) the presheaf morphism

\[
\mathcal{N}_\alpha(U) \otimes_{\mathcal{O}_X(U)} \mathcal{V}_{t(\alpha)}(U) \longrightarrow \mathcal{V}_{h(\alpha)}(U)
\]

\[
n \otimes v \longmapsto \rho_{\mathcal{V}}(n)(v) \tag{2.6}
\]

induces a morphism \( \rho_{\mathcal{V},\alpha} : \mathcal{N}_\alpha \otimes_{\mathcal{O}_X} \mathcal{V}_{t(\alpha)} \rightarrow \mathcal{V}_{h(\alpha)} \) of \( \mathcal{O}_X \)-modules.

**Lemma 2.3.** Let \( \mathcal{V}, \mathcal{W} \) be left \( \mathcal{A} \)-modules, and suppose that

- (R1) for each \( \alpha \in E \) the sheaf \( \mathcal{N}_\alpha \) is a locally free \( \mathcal{O}_X \)-module;
- (R2) the natural surjective morphism \( \mathcal{N} \rightarrow \pi(\mathcal{M}) \) is a \( \mathcal{B} \)-bimodule isomorphism.

Let

\[
\gamma_{\mathcal{A}';\mathcal{V},\mathcal{W}} : \bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X}(\mathcal{V}_i, \mathcal{W}_i) \longrightarrow \bigoplus_{\alpha \in E} \text{Hom}_{\mathcal{O}_X}(\mathcal{N}_\alpha \otimes_{\mathcal{O}_X} \mathcal{V}_{t(\alpha)}, \mathcal{W}_{h(\alpha)})
\]

be the morphism locally defined by

\[
\gamma_{\mathcal{A}';\mathcal{V},\mathcal{W}}(\{f_i\}_{i \in I}) = \{f_{h(\alpha)} \circ \rho_{\mathcal{V},\alpha} - \rho_{\mathcal{W},\alpha} \circ (\text{id}_{\mathcal{N}_\alpha} \otimes f_{t(\alpha)})\}_{\alpha \in E} ; \tag{2.7}
\]

where \( \{f_i\}_{i \in I} \) is a section on \( U \). Then there is a natural identification

\[
\ker \gamma_{\mathcal{A}';\mathcal{V},\mathcal{W}} = \text{Hom}_{\mathcal{A}'}(\mathcal{V}, \mathcal{W}) .
\]

**Proof.** \( \text{Hom}_{\mathcal{A}'}(\mathcal{V}, \mathcal{W}) \) can be embedded into \( \bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X}(\mathcal{V}_i, \mathcal{W}_i) \) as an \( \mathcal{O}_X \)-module. In fact for every \( \mathcal{A} \)-linear morphism \( f : \mathcal{V} \rightarrow \mathcal{W} \) one has \( f(\mathcal{V}_i) \subseteq \mathcal{W}_i \).
Condition (R1) implies that each point of $X$ has an open neighbourhood $U$ such that
\[
\mathcal{N}_a(U) \otimes_{\mathcal{O}_X(U)} \mathcal{V}_{t(a)}(U) \simeq (\mathcal{N}_a \otimes_{\mathcal{O}_X} \mathcal{V}_{t(a)})(U).
\] (2.8)
Let $g = \bigoplus_{i \in I} g_i \in \bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}_i, \mathcal{W}_i)(U)$. Since $\mathcal{A}$ is generated by $\pi(\mathcal{M})$, by condition (R2) $g$ is $\mathcal{A}$-linear if and only if
\[
g(\rho_V(n)v) = \rho_V(n)g(v)
\] for each open subset $U' \subset U$, each $n \in \mathcal{N}(U')$ and each $v \in \mathcal{V}(U')$. From the definition of $\mathcal{N}$ and from eq. (2.5) it follows that
\[
n = \bigoplus_{\alpha \in E} n_\alpha, \quad v = \bigoplus_{i \in I} v_i,
\] (2.10)
with $n_\alpha \in \mathcal{N}_a(U')$ and $v_i \in \mathcal{V}_i(U')$. By substituting eq. (2.10) into eq. (2.9) one gets
\[
0 = fg(\rho_V(n)v) - \rho_V(n)g(v) = \bigoplus_{\alpha \in E} \left(gh_{t(a)}(\rho_V(n_\alpha)v_i) - \rho_V(n_\alpha)g_i(v_i)\right)
\]
\[
= \bigoplus_{\alpha \in E} \left(gh_{t(a)}(\rho_V(n_\alpha)v_i) - \rho_{V,a}(n_\alpha \otimes v_i(t(a)))\right)
\]
\[
= \bigoplus_{\alpha \in E} \left([g_{h(a)} \circ \rho_{V,a} - \rho_{V,a} \circ (\id_{\mathcal{N}_a} \otimes g_{t(a)})](n_\alpha \otimes v_i(t(a)))\right),
\] (2.11)
where the equality (2.11) relies on eq. (2.8). The thesis follows.

We now prove another property of the morphism $\gamma_{\mathcal{A},\mathcal{V},\mathcal{W}}$ (cf. (2.7)), which will be helpful in the proof of Lemma 4.3. It is immediate that, for any $\mathcal{A}$-bimodule $\mathcal{V}$ and any left $\mathcal{A}$-module $\mathcal{W}$, the sheaves
\[
\bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}_i, \mathcal{W}_i) \quad \text{and} \quad \bigoplus_{\alpha \in E} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{N}_a \otimes_{\mathcal{O}_X} \mathcal{V}_{t(a)}, \mathcal{W}_{h(a)})
\]
have an induced structure of left $\mathcal{A}$-modules: in fact, if $i \in I$ and if $U' \subseteq U \subseteq X$ are open subsets, it is enough to set
\[
(a f_i)(v_i) = f_i(v_i a).
\]
for every $f_i \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}_i, \mathcal{W}_i)(U)$, every $a \in \mathcal{A}(U)$, and every $v_i \in \mathcal{V}_i(U')$.

**Lemma 2.4.** Let $\mathcal{V}$ be an $\mathcal{A}$-bimodule and $\mathcal{W}$ a left $\mathcal{A}$-module. If condition (R1) of Lemma 2.3 is satisfied, then the morphism $\gamma_{\mathcal{A},\mathcal{V},\mathcal{W}}$ is $\mathcal{A}$-linear.

**Proof.** To make the notation more agile we set $\gamma = \gamma_{\mathcal{A},\mathcal{V},\mathcal{W}}$. Condition (R1) implies that each point of $X$ has an open neighbourhood $U$ where eq. (2.8) holds true. Let $f = \bigoplus_{i \in I} f_i \in \bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{V}_i, \mathcal{W}_i)(U)$, let $U'$ be any open subset of $U$, and let $a \in \mathcal{A}(U')$, $n_\alpha \in \mathcal{N}_a(U')$
and \( v_i \in \mathcal{V}_i(U') \), with \( \alpha \in E \) and \( i \in I \). Then

\[
[\gamma(af)]_\alpha(n_\alpha \otimes v_{t(\alpha)}) = [af_h(\alpha)](\rho_\nu(n_\alpha)v_{t(\alpha)}) - \rho_\nu(n_\alpha)[af_t(\alpha)](v_{t(\alpha)}) = \\
= f_h(\alpha)(\rho_\nu(n_\alpha)v_{t(\alpha)}a) - \rho_\nu(n_\alpha)f_t(\alpha)(v_{t(\alpha)}a) = \\
= [(f_h(\alpha) \circ \rho_\nu, a) - \rho_\nu \circ (\text{id}_{\mathcal{N}_\alpha} \otimes f_t(\alpha))](n_\alpha \otimes v_{t(\alpha)}a) = \\
= [\gamma(f)]_\alpha(n_\alpha \otimes v_{t(\alpha)}a) = [a\gamma(f)]_\alpha(n_\alpha \otimes v_{t(\alpha)}) ,
\]

where in the first and in the third step we have used the isomorphism (2.8).

\( \square \)

Lemmas 2.3 and 2.4 hold true with only slight modifications also in the case of left \( \mathcal{M}Q \)- and \( \mathcal{N}Q \)-modules. Let \( \mathcal{Y}Q \) be either \( \mathcal{M}Q \) or \( \mathcal{N}Q \). A left \( \mathcal{Y}Q \)-module \( \mathcal{V} \) admits a decomposition

\[
\mathcal{V} \cong \bigoplus_{i \in I} \mathcal{V}_i ,
\]

(2.12)

analogous to the decomposition (2.5) of a left \( \mathcal{A} \)-module \( \mathcal{V} \). By proceeding in a similar way as in eq. (2.6), one can define morphisms

\[
\{ \rho_{\nu, \alpha} : \mathcal{L}_\alpha \otimes \mathcal{O}_X \mathcal{V}_t(\alpha) \rightarrow \mathcal{V}_{h(\alpha)} \}_{\alpha \in E}
\]

(2.13)

where \( \mathcal{L}_\alpha = \mathcal{M}_\alpha \) when \( \mathcal{Y}Q = \mathcal{M}Q \), while \( \mathcal{L}_\alpha = \mathcal{N}_\alpha \) when \( \mathcal{Y}Q = \mathcal{N}Q \). For every left \( \mathcal{Y}Q \)-module \( \mathcal{W} \) we can then introduce a morphism

\[
\gamma_{\mathcal{Y}Q, \mathcal{Y}'; \mathcal{W}} : \bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X}(\mathcal{V}_i, \mathcal{W}_i) \rightarrow \bigoplus_{\alpha \in E} \text{Hom}_{\mathcal{O}_X}(\mathcal{L}_\alpha \otimes \mathcal{O}_X \mathcal{V}_t(\alpha), \mathcal{V}_{h(\alpha)}),
\]

formally defined, at the local level, as in eq. (2.7). One has the following result.

**Lemma 2.5.** Suppose, with the same notation as above, that

(R1') for each \( \alpha \in E \) the sheaf \( \mathcal{L}_\alpha \) is a locally free \( \mathcal{O}_X \)-module.

Then \( \ker \gamma_{\mathcal{Y}Q, \mathcal{Y}'; \mathcal{W}} = \text{Hom}_{\mathcal{Y}Q}(\mathcal{Y}', \mathcal{W}) \). Moreover, if \( \mathcal{Y} \) is a \( \mathcal{Y}Q \)-bimodule and \( \mathcal{W} \) is a left \( \mathcal{Y}Q \)-module, then the morphism \( \gamma_{\mathcal{Y}Q, \mathcal{Y}; \mathcal{W}} \) is \( \mathcal{Y}Q \)-linear.

Note that condition (R2) of Lemma 2.3 is no more needed.

3. Representations of a twisted quiver algebra

Let \( \mathcal{A} \) be an \( \mathcal{M} \)-twisted quiver algebra with relations as above. We introduce a category \( \mathfrak{R}(\mathcal{A}) \), whose objects we call “\( \mathcal{M} \)-twisted representations of \( Q \) with relations”, and contains the category of left \( \mathcal{A} \)-modules as a full subcategory.

**Definition 3.1.** An object of \( \mathfrak{R}(\mathcal{A}) \) is a pair \( (\{\mathcal{V}_i\}_{i \in I}, \{\phi_\alpha\}_{\alpha \in E}) \), where each \( \mathcal{V}_i \) is an \( \mathcal{O}_X \)-module and each

\[
\phi_\alpha : \mathcal{N}_\alpha \otimes \mathcal{O}_X \mathcal{V}_t(\alpha) \rightarrow \mathcal{V}_{h(\alpha)}
\]

is a morphism of \( \mathcal{O}_X \)-modules.
A morphism \( f: \{\{V_i\}, \{\phi_i\}\} \rightarrow \{\{W_i\}, \{\psi_i\}\} \) is a collection of morphisms

\[ \{f_i: V_i \rightarrow W_i\}_{i \in I} \]

of \( \mathcal{O}_X \)-modules such that the diagram

\[
\begin{array}{ccc}
\mathcal{N}_\alpha \otimes_{\mathcal{O}_X} V_{t(\alpha)} & \xrightarrow{\phi_\alpha} & V_{h(\alpha)} \\
\text{id} \otimes f_{t(\alpha)} & \downarrow & \downarrow f_{h(\alpha)} \\
\mathcal{N}_\alpha \otimes_{\mathcal{O}_X} W_{t(\alpha)} & \xrightarrow{\psi_\alpha} & W_{h(\alpha)}
\end{array}
\] (3.1)

commutes for each \( \alpha \in E \).

The following theorem is main result of this section. It generalizes Proposition 2.3 in [11] and clarifies the relation between the category \( \mathcal{R}(\mathcal{A}) \) and the category \( \mathcal{A} \text{-mod} \) of left \( \mathcal{A} \)-modules.

**Theorem 3.2.** Suppose that the sheaves \( \mathcal{N}_\alpha \) satisfy condition (R1) of Lemma 2.3. Then the category \( \mathcal{R}(\mathcal{A}) \) is equivalent to the category \( \mathcal{N}Q \text{-mod} \). In particular \( \mathcal{A} \text{-mod} \) is a full subcategory of \( \mathcal{R}(\mathcal{A}) \).

**Proof.** A functor \( F: \mathcal{N}Q \text{-mod} \rightarrow \mathcal{R}(\mathcal{A}) \) is defined as follows: for each left \( \mathcal{N}Q \)-module \( V \)

\[ F(V) = (\{V_i\}_{i \in I}, \{\rho_{V,i,\alpha}\}_{\alpha \in E}) \]

(see eq. (2.12) and (2.13)) and for each morphism \( f: V \rightarrow W \) of left \( \mathcal{N}Q \)-modules

\[ F(f) = \{f|_{V_i}\}_{i \in I}. \]

Note that by Lemma 2.5 \( F(f) \) is indeed a morphism in the category \( \mathcal{R}(\mathcal{A}) \).

Since \( \mathcal{N}Q \) is freely generated by \( \mathcal{N} \) over \( \mathcal{B} \), any \( \mathcal{O}_X \)-algebra homomorphism \( \chi: \mathcal{N}Q \rightarrow \mathcal{R} \) is uniquely determined by its restrictions \( \chi_0: \mathcal{B} \rightarrow \mathcal{R} \) and \( \chi_1: \mathcal{N} \rightarrow \mathcal{R} \). So, any object \( (V, \phi) \) of \( \mathcal{R}(\mathcal{A}) \) can be associated with the homomorphism

\[ \rho_\phi: \mathcal{N}Q \rightarrow \text{End}_{\mathcal{O}_X}(\bigoplus_{i \in I} V_i) \] (3.2)

of \( \mathcal{O}_X \)-algebras such that:

- \( \rho_{\phi,0}: \mathcal{B} \rightarrow \text{End}_{\mathcal{O}_X}(\bigoplus_{i \in I} V_i) \) is given by
  \[ \rho_{\phi,0}(e_j)(\bigoplus_{i \in I} V_i) = V_j \]
  for each \( j \in I \);
- if \( U \) is a small enough open set in \( X \), then, for each section \( n_\alpha \in \mathcal{N}_\alpha(U) \) and for each open subset \( U' \subseteq U \), \( \rho_{\phi, 1}(n_\alpha): V_{t(\alpha)}(U') \rightarrow V_{h(\alpha)}(U') \) is given by
  \[ \rho_{\phi, 1}(n_\alpha)(v) = \phi_\alpha((n_\alpha|_{U'}) \otimes_{\mathcal{O}_X(U')} v) \]
(we have make implicit use of the isomorphism
\[ \mathcal{N}_\alpha(U) \otimes_{\mathcal{O}_X(U)} \mathcal{V}_{t(\alpha)}(U) \cong (\mathcal{N}_\alpha \otimes_{\mathcal{O}_X} \mathcal{V}_{t(\alpha)})(U), \]
which holds true on some open neighbourhood \( U \) of every point of \( X \) by virtue of condition (R1)).

It is immediate to check the compatibility conditions
\[ \rho_{\phi,0}(e_i) \circ \rho_{\phi,1}(n) = \rho_{\phi,1}(e_in) \quad \text{and} \quad \rho_{\phi,1}(n) \circ \rho_{\phi,0}(e_i) = \rho_{\phi,1}(ne_i) \]
for each section \( n \in \mathcal{N}(U) \) and for each \( i \in I \).

Thus, we can map each object \((\mathcal{V},\phi)\) of \( \mathcal{R}(\mathcal{A}) \) to the left \( \mathcal{N}Q \)-module
\[ G((\mathcal{V},\phi)) = (\bigoplus_{i \in I} \mathcal{V}_i, \rho_\phi), \tag{3.3} \]
(we mean that the left \( \mathcal{N}Q \)-module structure of \( \bigoplus_{i \in I} \mathcal{V}_i \) is induced by the homomorphism \( \rho_\phi \) defined in (3.2)). In this way, we obtain a functor
\[ G: \mathcal{R}(\mathcal{A}) \rightarrow \mathcal{N}Q\text{-mod}, \tag{3.4} \]
with
\[ G(\{f_i\}) = \bigoplus_{i \in I} f_i \tag{3.5} \]
for each morphism \( \{f_i\}: (\mathcal{V},\phi) \rightarrow (\mathcal{W},\psi) \). The commutativity of the diagram (3.1), together with Lemma 2.5, imply that \( G(\{f_i\}) \) is \( \mathcal{N}Q \)-linear. To conclude the proof, one checks that the functor \( F \) is a quasi-inverse to \( G \) and conversely. □

**Remark 3.3.** When there are no relations, i.e., when \( \mathcal{A} \simeq \mathcal{M}Q \), Theorem 3.2 is a slight generalization of [1, Prop. 5.1], since Theorem 3.2 requires the sheaves \( \mathcal{M}_\alpha \) to be locally free \( \mathcal{O}_X \)-modules, but not necessarily of finite rank. △

**Remark 3.4.** When \( X \) is a scheme, Lemmas 2.3 and 2.4 hold true even if the condition (R1) is replaced by the weaker condition
\[(R1') \text{ for each } \alpha \in E \text{ the sheaf } \mathcal{N}_\alpha \text{ is a quasi-coherent } \mathcal{O}_X\text{-module} \]
provided that the sheaves \( \mathcal{V} \) and \( \mathcal{W} \) are quasi-coherent \( \mathcal{O}_X \)-modules as well. The key point is that Eq. (2.8) holds for all open affine subsets \( U \subseteq X \). Also the statement of Lemma 2.5 can be modified in an analogous way.

As a consequence, when \( X \) is a scheme, also Theorem 3.2 has an alternative, slightly more general version. We call
- \( \mathcal{R}_{qc}(\mathcal{A}) \) the full subcategory of \( \mathcal{R}(\mathcal{A}) \) whose objects \( \{\mathcal{V}_i, \{\phi_\alpha\}\} \) are such that the sheaves \( \mathcal{V}_i \) are quasi-coherent \( \mathcal{O}_X \)-modules;
- \( \mathcal{Qcoh}_{NQ}(X) \) (resp. \( \mathcal{Qcoh}_{N}(X) \)) the full subcategory of \( \mathcal{N}Q\text{-mod} \) (resp. of \( \mathcal{A}\text{-mod} \)) whose objects are quasi-coherent \( \mathcal{O}_X \)-modules.
If the sheaves \( N_\alpha \) satisfy condition (R1”), then the categories \( \mathcal{R}_{\mathfrak{q}_0}(\mathcal{A}) \) and \( \mathcal{Qcoh}_{\mathcal{M}'}(X) \) are equivalent. In particular, \( \mathcal{Qcoh}_{\mathcal{M}'}(X) \) is a full subcategory of \( \mathcal{R}_{\mathfrak{q}_0}(\mathcal{A}) \). △

3.1. The untwisted case. We now study in more detail the case when \( M_\alpha = \mathcal{O}_{X\alpha} \) for each \( \alpha \in E \). So we have that \( \mathcal{M} = \bigoplus_{\alpha \in E} \mathcal{O}_{X\alpha} \) and \( \mathcal{M} Q \simeq \mathcal{O}_X Q \). We require the following condition to be satisfied:

\[ (K2) \ M_\alpha \cap \mathcal{K} = 0 \]

Condition (K2) implies \( \mathcal{M}_\alpha \cap \mathcal{K} = 0 \) for all \( \alpha \in E \), so that \( N_\alpha = M_\alpha = \mathcal{O}_{X\alpha} \).

Thus condition (R1) of Lemma 2.3 (which coincides, in the present case, with condition (R1’) of Lemma 2.5) is satisfied. Moreover, condition (K2) implies that \( \pi(M) = M \) (see eq. (2.3)), and this, along with eq. (3.6), implies that also condition (R2) of Lemma 2.3 is satisfied.

Eq. (3.6) entails that, for every left \( \mathcal{A} \)-module \( V \), there are canonical isomorphisms

\[ N_\alpha \otimes_{\mathcal{O}_X} V_{t(\alpha)} \simeq V_{t(\alpha)} \]

for all \( \alpha \in E \). This fact simplifies matters. In particular, for each \( \alpha \in E \), the morphism \( \rho_{V,\alpha} \) defined in (2.6) simply becomes multiplication by \( \pi(\alpha) \), i.e.

\[ \rho_{V,\alpha} = \rho_V(\pi(\alpha)) \in \text{End}_{\mathcal{O}_X}(V) \].

Moreover, by Theorem 3.2, there is an equivalence \( G : \mathcal{R}(\mathcal{A}) \rightarrow \mathcal{O}_X Q \text{-mod} \). The subcategory of \( \mathcal{R}(\mathcal{A}) \) equivalent to the subcategory \( \mathcal{A} \text{-mod} \) of \( \mathcal{O}_X Q \text{-mod} \) can be described in an explicit way, as we now show. Let \((U, \phi)\) be an object in \( \mathcal{R}(\mathcal{A}) \). For any path \( p \) in the quiver \( Q \), we set

\[ \phi_p = \begin{cases} \text{id}_U & \text{if } p = e_i \\ \phi_{\alpha_1} \circ \phi_{\alpha_2} \circ \cdots \circ \phi_{\alpha_n} & \text{if } p = \alpha_1 \alpha_2 \cdots \alpha_n. \end{cases} \]

So, if the path \( p \) starts at the vertex \( i \) and ends at the vertex \( j \), one has

\[ \phi_p \in \text{Hom}_{\mathcal{O}_X(U_i, U_j)}. \]

Let \( U \) be an open subset of \( X \). In view of the natural inclusion

\[ \mathcal{K}(U) \subseteq (\mathcal{O}_X Q)(U) \simeq (\mathcal{O}_X(U)) Q, \]

each section \( \eta \in \mathcal{K}(U) \) can be thought of as an element of the path algebra of \( Q \) with coefficients in the ring \( \mathcal{O}_X(U) \). In other words, each section \( \eta \in \mathcal{K}(U) \) can be written as a finite sum

\[ \eta = \sum_{p \text{ path in } Q} r_p p, \]
with \( r_p \in \mathcal{O}_X(U) \). With this remark in mind, we set
\[
\eta(\phi) = \sum_{p \in \mathcal{P}} r_p(\phi_p|_U) \in \bigoplus_{i,j \in I} \text{Hom}_{\mathcal{O}_U}(\mathcal{U}_i|_U, \mathcal{U}_j|_U) = \text{End}_{\mathcal{O}_U}(\bigoplus_{i \in I} \mathcal{U}_i)|_U)
\] (3.8)

**Definition 3.5.** The category \( \tilde{\mathcal{R}}(\mathcal{A}) \) is the full subcategory of \( \mathcal{R}(\mathcal{A}) \) whose objects \((\mathcal{U}, \phi)\) satisfy the condition
\[
\eta(\phi) = 0 \quad \text{for all sections } \eta \text{ of } \mathcal{H} \] (3.9)

The main result of this Subsection is the following theorem.

**Theorem 3.6.** Assume that \( \mathcal{M}_\alpha = \mathcal{O}_X \alpha \) for all \( \alpha \in E \) and that condition (K2) is satisfied. Then there is an equivalence \( G_0 : \tilde{\mathcal{R}}(\mathcal{A}) \to \mathcal{A}-\text{mod} \) that fits into the commutative diagram of functors
\[
\begin{array}{ccc}
\tilde{\mathcal{R}}(\mathcal{A}) & \xrightarrow{G_0} & \mathcal{A}-\text{mod} \\
\downarrow J & & \downarrow R \\
\mathcal{R}(\mathcal{A}) & \xrightarrow{G} & \mathcal{O}_X Q-\text{mod}
\end{array}
\] (3.10)

where the functor \( G \) is the equivalence introduced in eq. (3.4), \( J \) is the canonical embedding of the full subcategory \( \tilde{\mathcal{R}}(\mathcal{A}) \) into \( \mathcal{R}(\mathcal{A}) \), and the functor \( R \) is the embedding induced by restriction of scalars via the homomorphism \( \pi \) in the exact sequence (2.3).

**Proof.** Eq. (3.6) implies \( \mathcal{M}Q = \mathcal{M}Q = \mathcal{O}_X Q \), so that exact sequence (2.3) becomes
\[
0 \to \mathcal{H} \to \mathcal{O}_X Q \xrightarrow{\pi} \mathcal{A} \to 0
\]

The surjectivity of \( \pi \) implies that the functor \( R : \mathcal{A}-\text{mod} \to \mathcal{O}_X Q-\text{mod} \) is a full embedding. For each object \((\mathcal{U}, \phi)\) in \( \tilde{\mathcal{R}}(\mathcal{A}) \) let
\[
\rho_\phi : \mathcal{O}_X Q \to \text{End}_{\mathcal{O}_X}(\bigoplus_{i \in I} \mathcal{V}_i)
\]
be the homomorphism introduced in eq. (3.2). It is easy to check that, for each section \( \eta \) of \( \mathcal{O}_X Q \), one has
\[
\rho_\phi(\eta) = \eta(\phi)
\]
(see eq. (3.8)). Since \( \phi \) fulfills condition (3.9), one has \( \rho_\phi|_\mathcal{H} = 0 \) and there is an induced homomorphism \( \tilde{\rho}_\phi : \mathcal{A} \to \text{End}_{\mathcal{O}_X}(\bigoplus_{i \in I} \mathcal{U}_i) \), which fits into the following commutative diagram
\[
\begin{array}{ccc}
\mathcal{O}_X Q & \xrightarrow{\pi} & \mathcal{A} \\
\downarrow \rho_\phi & & \downarrow \tilde{\rho}_\phi \\
\text{End}_{\mathcal{O}_X}(\bigoplus_{i \in I} \mathcal{U}_i) & \xrightarrow{\rho_\phi} & \mathcal{O}_X Q
\end{array}
\] (3.11)
By analogy with the definition of the functor $G$ (cf. eq. (3.3) and (3.5)), let $G_0 : \tilde{\mathcal{R}}(\mathcal{A}) \rightarrow \mathcal{A}\text{-mod}$ be the functor given by

$$G_0((\mathcal{U}, \phi)) = (\bigoplus_{i \in I} \mathcal{U}_i, \tilde{\rho}_i) \quad \text{for each object } (\mathcal{U}, \phi)$$

$$G_0(\{f_i\}) = \bigoplus_{i \in I} f_i \quad \text{for each morphism } \{f_i\} : (\mathcal{U}, \phi) \rightarrow (\mathcal{W}, \psi).$$

The functor $G_0$ is well defined because the morphism $\bigoplus_{i \in I} f_i$ is $\mathcal{A}$-linear. To prove this claim one can argue as follows: first, the commutativity of (3.11) implies that the $\mathcal{O}_X Q$-module structure induced on $\bigoplus_{i \in I} \mathcal{U}_i$ (resp. on $\bigoplus_{i \in I} \mathcal{W}_i$) by restriction of scalars is precisely the same as that provided by $\rho_\phi$ (resp. $\rho_\psi$); second, condition (K2) implies conditions (R1) and (R2), and Theorem 3.2 shows that $\bigoplus_{i \in I} f_i$ is $\mathcal{O}_X Q$-linear; third, $R$ is a full embedding, so that $\bigoplus_{i \in I} f_i$ is $\mathcal{A}$-linear.

To prove that $G_0$ is an equivalence we show that it has a quasi-inverse. For each $\mathcal{A}$-module $V$ let

$$F_0(V) = (\{V_i\}_{i \in I}, \{\rho_V(\pi(\alpha))\}_{\alpha \in E})$$

(see eq. (2.5), (2.6) and (3.7)). Now, $F_0(V)$ is an object of $\mathcal{R}(\mathcal{A})$; but for each section $\eta$ of $\mathcal{O}_X Q$ one has

$$\eta(\{\rho_V(\pi(\alpha))\}) = \rho_V(\pi(\eta)),$$

(same notation as in eq. (3.8)), so (3.9) is satisfied and $F_0(V)$ is actually an object of $\tilde{\mathcal{R}}(\mathcal{A})$.

For each morphism $f : V \rightarrow W$ of left $\mathcal{A}$-modules let

$$F_0(f) = \{f|_{V_i}\}_{i \in I},$$

which is a morphism in $\tilde{\mathcal{R}}(\mathcal{A})$ by Lemma 2.3. We have therefore defined a functor $F_0 : \mathcal{A}\text{-mod} \rightarrow \tilde{\mathcal{R}}(\mathcal{A})$, and it is not hard to check that $G_0$ and $F_0$ are quasi-inverse to each other.

Finally, the commutativity of (3.10) follows at once from the commutativity of (3.11). \hfill \Box

4. A spectral sequence

In this and the next section the sheaves $\mathcal{M}_\alpha$ are assumed to be locally free $\mathcal{O}_X$-modules (for instance, when they are flat finitely generated $\mathcal{O}_X$-modules, and $(X, \mathcal{O}_X)$ is a locally ringed space, they are indeed locally free of finite rank [19, Thm. 7.10]).

Theorem 4.1. Let $V$ and $W$ be two left $\mathcal{A}$-modules, and suppose the following three conditions are fulfilled:

(K1) $\mathcal{A}^p$ is a locally free $\mathcal{O}_X$-module for all $p \geq 0$;

(K2) $\mathcal{K} \cap \mathcal{M} = 0$;

(K3) $\mathcal{K}$ is flat as a right $\mathcal{M} Q$-module.

Then:
a) there exists a convergent first quadrant spectral sequence

\[ E^p,q_\bullet (V, W) \Rightarrow \text{Ext}^{p+q}_A (V, W), \]

whose first page is

\[ E^p,q_1 (V, W) = \begin{cases} \bigoplus_{i \in I} \text{Ext}^q_{O_X} \left( (\mathcal{A}/2 \otimes_{\mathcal{A}} V)_i, W_i \right) & \text{if } p \text{ is even} \\ \bigoplus_{\alpha \in E} \text{Ext}^q_{O_X} \left( M_\alpha \otimes_{O_X} (\mathcal{A}/(p-1)/2 \otimes_{\mathcal{A}} V)_{t(\alpha)}, W_{h(\alpha)} \right) & \text{if } p \text{ is odd} \end{cases} \] (4.1)

b) there exists a convergent first quadrant spectral sequence

\[ E^p,q_\bullet (V, W) \Rightarrow \text{Ext}^{p+q}_A (V, W), \]

whose first page is

\[ E^p,q_1 (V, W) = \begin{cases} \bigoplus_{i \in I} \text{Ext}^q_{O_X} \left( (\mathcal{A}/2 \otimes_{\mathcal{A}} V)_i, W_i \right) & \text{if } p \text{ is even} \\ \bigoplus_{\alpha \in E} \text{Ext}^q_{O_X} \left( M_\alpha \otimes_{O_X} (\mathcal{A}/(p-1)/2 \otimes_{\mathcal{A}} V)_{t(\alpha)}, W_{h(\alpha)} \right) & \text{if } p \text{ is odd} \end{cases} \]

To prove Theorem 4.1 we need to establish some preliminary results.

First, we fix a convention to distinguish the two different structures of left \(A\)-module on the sheaf \(\mathcal{H} = \mathcal{H}(\mathcal{F}, \mathcal{G})\) when \(\mathcal{F}\) is an \(\mathcal{A}\)-bimodule and \(\mathcal{G}\) is a left \(\mathcal{A}\)-module. Let \(U' \subseteq U \subseteq X\) be open sets, and let \(f \in \mathcal{H}(U), a \in \mathcal{A}(U)\) and \(s \in \mathcal{F}(U')\). We denote by \(af\) the left multiplication (our usual one) given by

\[(af)(s) = f(sa)\]

and by \(as\) the \(-\)-left multiplication given by

\[(as)(s) = a(f(s)) = f(as).\] (4.2)

We adopt an analogous convention for \(M_Q\)-modules.

Next, we notice that for each \(p \geq 0\) there is a short exact sequence of \(M_Q\)-bimodules:

\[ 0 \rightarrow \mathcal{H}/p+1 \overset{\mathcal{t}_p}{\rightarrow} \mathcal{H}/p \overset{\pi_p}{\rightarrow} \mathcal{A}/p \rightarrow 0, \] (4.3)

where \(\mathcal{t}_p\) is the canonical injection and \(\pi_p\) is the canonical projection (of course, \(\pi_0 = \pi\)). Since both \(\mathcal{H}/p\) and \(\mathcal{A}/p\) can be decomposed as in eq. (2.5), we set

\[ t_{p,i} = t_p|_{\mathcal{H}/p}^i, \quad \pi_{p,i} = \pi_p|_{\mathcal{A}/p}^i. \]

Each left \(\mathcal{A}\)-module \(\mathcal{V}\) has a natural left \(M_Q\)-module structure induced by the restriction of scalars. So we can apply the functor \(\mathcal{H}(\mathcal{A}, \mathcal{V})\) to eq. (4.3), getting an injection

\[ 0 \rightarrow \mathcal{H}(\mathcal{A}, \mathcal{V}) \overset{\pi_p}{\rightarrow} \mathcal{H}(\mathcal{H}/p, \mathcal{V}) \] (4.4)

of left \(M_Q\) and \(-\)-left \(\mathcal{A}\)-modules (cf. eq. (4.2)).
Lemma 4.2. For any left $\mathcal{A}$-module $\mathcal{V}$, the injection (4.4) is an isomorphism of left $\mathcal{M}Q$- and $s$-left $\mathcal{A}$-modules.

Proof. We show that for each point $x \in X$ the morphism $\pi_{ps,x}$ is surjective. Let $U$ be an open neighbourhood of $x$ and $f \in \text{Hom}_{\mathcal{M}Q}(\mathcal{X}^p, \mathcal{V})(U)$. We observe that

$$f(\mathcal{X}^{p+1}|_U) = 0.$$  

In fact, for any open subset $U' \subset U$, every section of $\mathcal{X}^{p+1}(U')$ can be written as a finite sum of products of the form $s = s_1s_2 \cdots s_{p+1}$, with $s_i \in \mathcal{X}(U')$ for $i = 1, \ldots, p + 1$. But one has

$$f(s) = s_1f(s_2 \cdots s_{p+1}) = \pi(s_1)f(s_2 \cdots s_{p+1}) = 0.$$  

Thus $f$ induces a morphism $\tilde{f}: \mathcal{A}^P|_U \to \mathcal{V}|_U$ such that

$$\pi_{ps}(\tilde{f}) = f.$$  

This proves that $\pi_{ps}|_U$ is surjective, and so is $\pi_{ps,x}$. $\square$

Proposition 4.3. Suppose that conditions (K1) and (K2) are satisfied.

- For all left $\mathcal{M}Q$-modules $\mathcal{V}$ and for all $p \geq 0$ the morphism $\gamma_{\mathcal{M}Q,\mathcal{X}^p,\mathcal{V}}$ is surjective.
- Every left $\mathcal{A}$-module $\mathcal{V}$ fits into the following exact sequences of left $\mathcal{A}$-modules:

$$0 \to \text{Hom}_{\mathcal{A}}(\mathcal{A}^P, \mathcal{V}) \to \bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X}(\mathcal{A}^P, \mathcal{V}_i) \to \bigoplus_{\alpha \in E} \text{Hom}_{\mathcal{O}_X}(\mathcal{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{A}^P|_{t(\alpha)}, \mathcal{V}_{h(\alpha)}) \to \text{Hom}_{\mathcal{A}}(\mathcal{A}^{P+1}, \mathcal{V}) \to 0 \ (4.5)$$

(this makes sense by Lemma 2.4).

Proof. Throughout this proof we write $\otimes$ for $\otimes_{\mathcal{O}_X}$ and let $\mathcal{M} = \mathcal{M}Q$. Let $\mathcal{V}$ be a left $\mathcal{A}$-module; starting from the exact sequence (4.3) it is easy to build up the commutative diagram of left $\mathcal{M}$-modules shown in Figure 1, where we let

$$\theta_{p,11} = \bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X}(\pi_{p,i}, \mathcal{V}_i), \quad \theta_{p,12} = \bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X}(\nu_{p,i}, \mathcal{V}_i)$$

$$\theta_{p,21} = \bigoplus_{\alpha \in E} \text{Hom}_{\mathcal{O}_X}(\text{id}_\alpha \otimes \pi_{p,t(\alpha)}, \mathcal{V}_{h(\alpha)}), \quad \theta_{p,22} = \bigoplus_{\alpha \in E} \text{Hom}_{\mathcal{O}_X}(\text{id}_\alpha \otimes \nu_{p,t(\alpha)}, \mathcal{V}_{h(\alpha)}).$$

The surjectivity of $\theta_{p,12}$ and $\theta_{p,22}$ follows from condition (K1). The exactness of the rows is a consequence of Lemmas 2.3 and 2.5. In particular, to build up the first row we have used condition (K2), which implies $\mathcal{N}_\alpha = \mathcal{M}_\alpha$ for all $\alpha \in E$ (see eq. (2.4)), and condition (R2).

The surjectivity of $\gamma_{\mathcal{M},\mathcal{X}^p,\mathcal{V}}$ can be shown by arguing essentially as in [11, Prop. 3.1]. Then the surjectivity of $\gamma_{\mathcal{M},\mathcal{X}^p,\mathcal{V}}$ for all $p \geq 0$ can be proved inductively by using the diagram itself. This proves the first statement.
By applying the Snake Lemma to the previous diagram one deduces the existence of an exact sequence of left $\mathcal{M}$-modules of the following form:

$$0 \to \text{Hom}_\mathcal{A}(\mathcal{A}^p, \mathcal{V}) \xrightarrow{a_p} \text{Hom}_\mathcal{M}(\mathcal{X}^p, \mathcal{V}) \to \text{Hom}_\mathcal{M}(\mathcal{X}^{p+1}, \mathcal{V}) \xrightarrow{\partial_p} \text{coker} \gamma_{\mathcal{A};\mathcal{V}} \to 0.$$ 

It is easy to prove that $a_p = \pi_{p*}$ (see eq. (4.4)), so that Lemma 4.2 implies that $a_p$ is an isomorphism. It follows that $\partial_p$ is an isomorphism as well. So one gets the left $\mathcal{M}$-modules isomorphisms

$$\text{coker} \gamma_{\mathcal{A};\mathcal{V}} \simeq \text{Hom}_\mathcal{M}(\mathcal{X}^{p+1}, \mathcal{V}) \simeq \text{Hom}_\mathcal{M}(\mathcal{X}^p, \mathcal{V}) \simeq \text{Hom}_\mathcal{A}(\mathcal{A}^p, \mathcal{V}),$$

where the last step follows from $\mathcal{A}$ being a quotient of $\mathcal{M}$. The resulting isomorphism $\text{coker} \gamma_{\mathcal{A};\mathcal{V}} \to \text{Hom}_\mathcal{A}(\mathcal{A}^p, \mathcal{V})$ is $\mathcal{A}$-linear. \hfill $\Box$

Let $\mathcal{V}$ be a left $\mathcal{A}$-module; by splicing the exact sequences (4.5) for all $p \geq 0$, and using the natural isomorphism $\text{Hom}_\mathcal{A}(\mathcal{A}, \mathcal{V}) \simeq \mathcal{V}$, one obtains an exact sequence of left $\mathcal{A}$-modules

$$0 \longrightarrow \mathcal{V} \xrightarrow{d^0} \mathcal{C}^0(\mathcal{V}) \xrightarrow{d^1} \mathcal{C}^1(\mathcal{V}) \xrightarrow{d^2} \cdots \tag{4.6}$$

where

$$\mathcal{C}^p(\mathcal{V}) = \begin{cases} \bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X}(\mathcal{A}_{i}^{p/2}, \mathcal{V}_i) & \text{if } p \text{ is even} \\ \bigoplus_{\alpha \in E} \text{Hom}_{\mathcal{O}_X}(\mathcal{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{A}_{t(\alpha)}^{(p-1)/2}, \mathcal{V}_{h(\alpha)}) & \text{if } p \text{ is odd} \end{cases}$$

**Lemma 4.4.** Suppose that condition (K3) is satisfied, and let $\mathcal{V}$ be a left $\mathcal{A}$-module which is $\mathcal{V}$ injective as an $\mathcal{O}_X$-module. Then $\mathcal{C}^p(\mathcal{V})$ is an injective left $\mathcal{A}$-module for all $p \geq 0$.

**Proof.** We claim that $\mathcal{X}^p$ is a flat right $\mathcal{M}Q$-module for all $p \geq 0$. For $p = 0$ this is trivial, and for $p = 1$ this is condition (K3). By induction, suppose that $\mathcal{X}^p$ is flat as a right $\mathcal{M}Q$-module for $p = 1, \ldots, q$, for some $q \geq 1$. This implies that the product $\mathcal{X}^q \otimes_{\mathcal{M}Q} \mathcal{X}$ is a flat right $\mathcal{M}Q$-module. By [17, Prop. 4.12] one has the following isomorphism of $\mathcal{M}Q$-bimodules:

$$\mathcal{X}^q \otimes_{\mathcal{M}Q} \mathcal{X} \simeq \mathcal{X}^{q+1} \quad \text{for all } q \geq 0$$

This implies that $\mathcal{X}^{q+1}$ is a flat right $\mathcal{M}Q$-module as well. This proves the claim.

Due to the right $\mathcal{A}$-module isomorphism, $\mathcal{X}^p \otimes_{\mathcal{M}Q} \mathcal{A} \simeq \mathcal{A}^p$, and since flatness is stable under bases change, $\mathcal{A}^p$ is a flat right $\mathcal{A}$-module for all $p \geq 0$. By eq. (2.5) and [17, Prop. 4.2] the sheaves $\mathcal{A}_i^p$ and $\mathcal{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{A}_{t(\alpha)}^p$ are flat right $\mathcal{A}$-modules for all $i \in I, \alpha \in E$ and $p \geq 0$. Moreover eq. (2.5) and [12, Prop. 5.2.2] imply that the sheaves $\mathcal{V}_i$ are injective $\mathcal{O}_X$-modules for all $i \in I$.

Let $\mathcal{F}$ be a flat right $\mathcal{A}$-module; there is a natural isomorphism of functors

$$\text{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{V}) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{V}_i).$$
Since the functor on the right-hand side is exact, the thesis follows. □

Let \( \mathcal{W} \) be a left \( \mathcal{A} \)-module, so also an \( \mathcal{O}_X \)-module by restriction of scalars. We choose a resolution \((\mathcal{W}^*, \partial_\bullet)\) of \( \mathcal{W} \) by injective \( \mathcal{O}_X \)-modules:

\[
0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{W}^0 \xrightarrow{\partial_0} \mathcal{W}^1 \xrightarrow{\partial_1} \mathcal{W}^2 \xrightarrow{\partial_2} \cdots
\] (4.7)

By [24, Thm. 2.3.7] every \( \mathcal{W}^q \) inherits a structure of left \( \mathcal{A} \)-module such that the differentials \( \partial_q \) are \( \mathcal{A} \)-linear. In particular one can perform a decomposition as in eq. (2.5), and one has inclusions \( \partial_q(\mathcal{W}^q_i) \subseteq \mathcal{W}^{q+1}_i \) for all \( q \geq 0 \). We call \( \partial_{q,i} \) the restriction of \( \partial_q \) to \( \mathcal{W}^q_i \), for all \( i \in I \) and \( q \geq 0 \). By using the exact sequence (4.6) associated with each term of the resolution (4.7), one can define a double complex of left \( \mathcal{A} \)-modules \((C^{\bullet\bullet}(\mathcal{W}), d, \partial)\) by putting

\[
C^{p,q}(\mathcal{W}) = C^p(\mathcal{W}^q) \\
d_{p,q} = d_p^{\mathcal{W}^q} \\
\partial_{p,q} = \begin{cases} 
\bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}^{p/2}_{\mathcal{X}}, \mathcal{W}^q_i) & \text{if } p \text{ is even} \\
- \bigoplus_{\alpha \in E} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{A}^{(p-1)/2}_{\mathcal{X}}\mathcal{A}_{\mathcal{X}}, \mathcal{W}^q_{h(\alpha)}) & \text{if } p \text{ is odd}
\end{cases}
\]

for all \( p, q \geq 0 \). Let \((T^{\bullet}, D_\bullet)\) be the total complex associated to \((C^{\bullet\bullet}(\mathcal{W}), d, \partial)\):

\[
\begin{array}{c}
\cdots \longrightarrow T^2 \xrightarrow{D_2} T^1 \xrightarrow{D_1} T^0 \xrightarrow{D_0} \cdots
\end{array}
\]

(4.8)

**Lemma 4.5.** The complex \((T^{\bullet}, D_\bullet)\) is a resolution of \( \mathcal{W} \) by injective left \( \mathcal{A} \)-modules.

**Proof.** The double complex \((C^{\bullet\bullet}(\mathcal{W}), d, \partial)\) fits into the diagram

\[
0 \longrightarrow C^\bullet(\mathcal{W}) \longrightarrow C^{\bullet\bullet}(\mathcal{W})
\]

\((T^{\bullet}, D_\bullet)\) is exact in positive degree, and \( \ker D_0 \simeq \mathcal{W} \) (see e.g. [23, Lemma 8.5]).

Finally, Lemma 4.4 and [12, Prop. 5.2.2] imply that the sheaf \( T^p \) is an injective left \( \mathcal{A} \)-module for all \( p \geq 0 \). □

We have now all of the ingredients needed to prove Theorem 4.1.

**Proof of Theorem 4.1.** To prove the first statement we apply the functor \( \operatorname{Hom}_\mathcal{A}(\mathcal{V}, -) \) to the double complex \((C^{\bullet\bullet}(\mathcal{W}), d, \partial)\), getting another double complex, which we call \((E^{\bullet\bullet}_0(\mathcal{V}, \mathcal{W}), \hat{d}, \hat{\partial})\). So

\[
E^{p,q}_0 = E^{p,q}_0(\mathcal{V}, \mathcal{W}) = \operatorname{Hom}_\mathcal{A}(\mathcal{V}, C^{p,q}(\mathcal{W})) \\
\simeq \begin{cases} 
\bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}^{p/2}_{\mathcal{X}}\mathcal{A}_{\mathcal{X}}, \mathcal{W}^q_i) & \text{if } p \text{ is even} \\
- \bigoplus_{\alpha \in E} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{A}^{(p-1)/2}_{\mathcal{X}}\mathcal{A}_{\mathcal{X}}\mathcal{A}_{\mathcal{X}}, \mathcal{W}^q_{h(\alpha)}) & \text{if } p \text{ is odd}
\end{cases}
\]

\[
\hat{d}_{p,q} = \operatorname{Hom}_\mathcal{A}(\mathcal{V}, d_{p,q})
\]

(4.9) (4.10)
\[ \hat{\partial}_{p,q} = \text{Hom}_A(V, \partial_{p,q}) \simeq \begin{cases} \bigoplus_{i \in I} \text{Hom}_X ((\mathcal{A}/2 \otimes \mathcal{A})_i, \partial_{q,i}) & \text{if } p \text{ is even} \\ - \bigoplus_{\alpha \in E} \text{Hom}_X (\mathcal{M}_\alpha \otimes X ((\mathcal{A}/(p-1)/2 \otimes \mathcal{A})_{l(\alpha)}, \partial_{q,h(\alpha)}) & \text{if } p \text{ is odd.} \end{cases} \] (4.11)

We have used the isomorphism \( \mathcal{A}/^r \otimes \mathcal{A} \simeq (\mathcal{A}/^r \otimes \mathcal{A})_i \) for all \( r \geq 0 \) and \( i \in I \), and have applied \([6, \text{Prop. II.4.1.1.(a)}]\). Note that the \( O_X \)-modules \((\mathcal{A}/i \otimes \mathcal{A})\) may fail to be \( \mathcal{A} \)-modules, so that the sheaves \( E^0_{p,q}(V, W) \) are not \( \mathcal{A} \)-modules in general, and the natural isomorphism used in eq. (4.9) may be \( O_X \)-linear but not \( \mathcal{A} \)-linear.

There are two first quadrant spectral sequences associated to \((E^\bullet, \partial^0)\), both abutting to \( H^n(\text{Hom}_A(V, \mathcal{T}^\bullet), \hat{\partial}) \), where \( \hat{\partial} = \text{Hom}_A(V, D) \).

Lemma 4.5 implies that \( H^n(\text{Hom}_A(V, \mathcal{T}^\bullet), \hat{\partial}) \simeq \text{Ext}^n_A(V, W) \) for all \( n \geq 0 \). Since \((\mathcal{W}^\bullet, \partial)\) is a resolution of \( \mathcal{W} \) by injective \( O_X \)-modules, by taking the cohomology of the double complex \( E^\bullet, \partial^0 \) with respect to the differential \( \hat{\partial} \) one gets eq. (4.1).

The proof of the second statement is analogous. \( \square \)

If \( \mathcal{A} = 0 \), so that \( \mathcal{A} = M_Q \), Theorem 4.1 reduces to \([11, \text{Thm. 4.1}]\), which we restate here in a slightly different form.

**Corollary 4.6.** Let \( \mathcal{V} \) and \( \mathcal{W} \) be two left \( M_Q \)-modules.

- There exist two collections of \( O_X \)-modules \( \{E^q_{-}(\mathcal{V}, \mathcal{W})\}_{q=0}^\infty \) and \( \{E^q_{+}(\mathcal{V}, \mathcal{W})\}_{q=0}^\infty \) that fit into exact sequences

\[ \begin{array}{cccccc} 0 & \to & E^q_{-}(\mathcal{V}, \mathcal{W}) & \to & \bigoplus_{i \in I} \text{Ext}^q_{O_X}(\mathcal{V}_i, \mathcal{W}_i) & \to \bigoplus_{\alpha \in E} \text{Ext}^q_{O_X}(\mathcal{M}_\alpha \otimes O_X \mathcal{V}_{l(\alpha)}, \mathcal{W}_{h(\alpha)}) & \to E^q_{+}(\mathcal{V}, \mathcal{W}) & \to 0 \end{array} \] (4.12)

for all \( q \geq 0 \) and such that

\[ \text{Ext}^q_{M_Q}(\mathcal{V}, \mathcal{W}) \simeq \begin{cases} E^q_{-}(\mathcal{V}, \mathcal{W}) & \text{if } q = 0 \\ E^q_{+}(\mathcal{V}, \mathcal{W}) \oplus E^q_{+}(\mathcal{V}, \mathcal{W}) & \text{if } q > 0 \end{cases} \] (4.13)

If \( q = 0 \), the middle morphism in the sequence (4.12) is \( \gamma_{M_Q}(\mathcal{V}, \mathcal{W}) \).
• There exist two collections of $\Gamma(X, O_X)$-modules $\{E^q_+(V, W)\}_{q=0}^\infty$ and $\{E^q_-(V, W)\}_{q=0}^\infty$ that fit into exact sequences

$$0 \rightarrow E^q_-(V, W) \rightarrow \bigoplus_{i \in I} \operatorname{Ext}^q_{O_X}(V_i, W_i) \rightarrow \bigoplus_{\alpha \in E} \operatorname{Ext}^q_{O_X}(M_{\alpha} \otimes_{O_X} V_{t(\alpha)}, W_{h(\alpha)}) \rightarrow E^q_+(V, W) \rightarrow 0,$$

for all $q \geq 0$ and such that

$$\operatorname{Ext}^q_{\#Q}(V, W) \simeq \begin{cases} E^0_+(V, W) & \text{if } q = 0, \\ E^q_+(V, W) \oplus E^{q-1}_-(V, W) & \text{if } q > 0 \end{cases}.$$

Proof. We prove only the first statement as the second is completely analogous.

If $\mathcal{A} = 0$, with the notation of eq. (4.1) one has

$$\mathcal{E}^p,q_1(V, W) = \begin{cases} \bigoplus_{i \in I} \mathcal{E}^p,O_X(V_i, W_i) & \text{if } p = 0, \\ \bigoplus_{\alpha \in E} \mathcal{E}^p,O_X(M_{\alpha} \otimes_{O_X} V_{t(\alpha)}, W_{h(\alpha)}) & \text{if } p = 1, \\ 0 & \text{if } p > 1 \end{cases}.$$

Thus the spectral sequence stabilizes at the second step, with

$$\mathcal{E}^p,q_\infty \simeq \begin{cases} \mathcal{E}^p,q_2 & \text{if } p = 0, 1, \\ 0 & \text{if } p > 1 \end{cases}.$$

Since the sheaves $\mathcal{E}^p,q_2$ fit into exact sequences of the form

$$0 \rightarrow \mathcal{E}^0,q_2 \rightarrow \mathcal{E}^1,q_1 \rightarrow \mathcal{E}^1,q_1 \rightarrow \mathcal{E}^1,q_2 \rightarrow 0,$$

eqs. (4.12) and (4.13) follow from Theorem 4.1.

One can check that, up to composition with the natural isomorphisms

$$\operatorname{Hom}_{\#Q}(V, \bigoplus_{i \in I} \operatorname{Hom}_{O_X}(M_i, W_i)) \simeq \bigoplus_{i \in I} \operatorname{Hom}_{O_X}(V_i, W_i)$$

and

$$\operatorname{Hom}_{\#Q}(V, \bigoplus_{\alpha \in E} \operatorname{Hom}_{O_X}(M_{\alpha} \otimes_{O_X} V_{t(\alpha)}, W_{h(\alpha)})) \simeq \bigoplus_{\alpha \in E} \operatorname{Hom}_{O_X}(M_{\alpha} \otimes_{O_X} V_{t(\alpha)}, W_{h(\alpha)}),$$

one has

$$\operatorname{Hom}_{\#Q}(V, \gamma_{\#Q,\#Q}W) = \gamma_{\#Q,\#Q}VW.$$

The statement in Theorem 4.1 simplifies substantially whenever the sheaf $V$ is locally free as an $O_X$-module. For any pair $(V, W)$ of left $\mathcal{A}$-modules we introduce the complex $C^\bullet(V, W) = \mathcal{E}^\bullet_1(V, W)$ (see eq. (4.1)) with the differentials $d^\bullet(V, W)$ induced by the spectral
sequence. Explicitly one has:
\[
C_p(\mathcal{V}, \mathcal{W}) = \begin{cases} 
\bigoplus_{i \in I} \text{Hom}_{\mathcal{O}_X} \left( (\mathcal{A}^p^2 \otimes _{\mathcal{A}} \mathcal{V})_i, \mathcal{W}_i \right) & \text{if } p \text{ is even} \\
\bigoplus_{\alpha \in E} \text{Hom}_{\mathcal{O}_X} \left( \mathcal{M}_\alpha \otimes _{\mathcal{O}_X} (\mathcal{A}^{(p-1)/2} \otimes _{\mathcal{A}} \mathcal{V})_{t(\alpha)}, \mathcal{W}_{h(\alpha)} \right) & \text{if } p \text{ is odd}
\end{cases}
\]

**Corollary 4.7.** Let \( \mathcal{V} \) and \( \mathcal{W} \) be two left \( \mathcal{A} \)-modules. Suppose that conditions (K1)–(K3) are satisfied, and that \( \mathcal{V} \) is locally free as an \( \mathcal{O}_X \)-module. For all \( p \geq 0 \) there are isomorphisms
\[
\mathcal{E}x^{p}(\mathcal{V}, \mathcal{W}) \simeq H^p(C^* (\mathcal{V}, \mathcal{W}), \partial^{\mathcal{V}, \mathcal{W}}).
\]

**Proof.** We show that the functors
\[
\text{Hom}_{\mathcal{O}_X} \left( (\mathcal{A}^p \otimes _{\mathcal{A}} \mathcal{V})_i, - \right) \quad \text{and} \quad \text{Hom}_{\mathcal{O}_X} \left( \mathcal{M}_\alpha \otimes _{\mathcal{O}_X} (\mathcal{A}^p \otimes _{\mathcal{A}} \mathcal{V})_{t(\alpha)}, - \right)
\]
are exact for all \( i \in I, \alpha \in E \) and \( p \geq 0 \). From the exact sequence (4.3), using condition (K1) and arguing by induction, one sees that the functor \( \text{Hom}_{\mathcal{O}_X}(\mathcal{A}^p, -) \) is exact for all \( p \geq 0 \). Hence Corollary 4.6 implies that for all left \( \mathcal{M}-\text{modules} \) \( \mathcal{W} \) one has
\[
\mathcal{E}x^{p}(\mathcal{V}, \mathcal{W}) \simeq \begin{cases} 
\text{coker} \gamma_{\mathcal{N}^p, \mathcal{V}, \mathcal{W}} & \text{if } i = 1 \\
0 & \text{if } i > 1
\end{cases}
\]
for all \( p \geq 0 \). Proposition 4.3 implies that the functor \( \text{Hom}_{\mathcal{O}_X}(\mathcal{A}^p, -) \) is exact for all \( p \geq 0 \).

Since \( \mathcal{A} \) is a quotient of \( \mathcal{M} \), using Lemma 4.2 one deduces a natural isomorphism of functors
\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{A}^p, -) \simeq \text{Hom}_{\mathcal{M}}(\mathcal{A}^p, \mathcal{M}(-)) \simeq \text{Hom}_{\mathcal{M}}(\mathcal{A}^p, \mathcal{M}(-)),
\]
where \( \mathcal{M}(-) : \mathcal{A} \text{-mod} \rightarrow \mathcal{M} \text{-mod} \) is the restriction of scalars. Since \( \mathcal{M}(-) \) is exact, the functor \( \text{Hom}_{\mathcal{O}_X}(\mathcal{A}^p, -) \) is exact as well. Let \( \mathcal{V} \) be a left \( \mathcal{A} \)-module; one has an isomorphism
\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{A}^p \otimes _{\mathcal{A}} \mathcal{V}, -) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{A}^p, \text{Hom}_{\mathcal{O}_X}(\mathcal{V}, -)).
\]
When \( \mathcal{V} \) is locally free as an \( \mathcal{O}_X \)-module, the functor \( \text{Hom}_{\mathcal{O}_X}(\mathcal{A}^p \otimes _{\mathcal{A}} \mathcal{V}, -) \) is exact. Since \( (\mathcal{A}^p \otimes _{\mathcal{A}} \mathcal{V})_i \) is a direct summand of \( \mathcal{A}^p \otimes _{\mathcal{A}} \mathcal{V} \) and the sheaves \( \mathcal{M}_\alpha \) are locally free \( \mathcal{O}_X \)-modules, one proves the claim by [13, Prop. I.3.4].

The exactness of the functors (4.14) implies that \( \mathcal{E}^{p,q}_1(\mathcal{V}, \mathcal{W}) = 0 \) if \( q > 0 \). One deduces easily that the spectral sequence in eq. (4.1) stabilizes at the second step, and that
\[
\mathcal{E}^{p,q}_\infty = \begin{cases} 
\mathcal{E}^{p,0}_2 & \text{if } q = 0 \\
0 & \text{if } q > 0
\end{cases}
\]
\[
\simeq \begin{cases} 
H^p(C^* (\mathcal{V}, \mathcal{W}), \partial^{\mathcal{V}, \mathcal{W}}) & \text{if } q = 0 \\
0 & \text{if } q > 0
\end{cases}
\]
The thesis follows from Theorem 4.1. \( \square \)

**An example.** Let \( k \) be a field, and \( J \subseteq kQ \) a two-sided ideal. The short exact sequence of \( kQ \)-bimodules
\[
0 \rightarrow J \rightarrow kQ \rightarrow \Lambda \rightarrow 0.
\]
(4.15)
defines a \( kQ \)-algebra structure on \( \Lambda \). Denote \( \Lambda^p = J^p/J^{p+1} \) for all \( p \geq 0 \). We assume that \( O_X \) is a sheaf of \( k \)-algebras; tensoring (4.15) by \( O_X \) over \( k \) we obtain

\[
0 \longrightarrow \mathcal{K} \longrightarrow O_X Q \longrightarrow \mathcal{A} \longrightarrow 0,
\]

where we have put \( \mathcal{K} = J \otimes_k O_X \) and \( \mathcal{A} = \Lambda \otimes_k O_X \).

**Proposition 4.8.** Assume that \( J \) satisfies the condition \( J \cap \langle E \rangle_k = 0 \), where \( \langle E \rangle_k \) is the linear span of the arrows over \( k \). Then the conditions (K1)–(K3) (required by Theorem 4.1) are fulfilled.

**Proof.** Condition (K1) is obvious, as \( \mathcal{A}^p = \Lambda^p \otimes_k O_X \). To prove condition (K2), note that \( \mathcal{M} = \langle E \rangle_k \otimes_k O_X \), so that

\[
\mathcal{K} \cap \mathcal{M} \simeq (J \cap \langle E \rangle_k) \otimes_k O_X.
\]

Concerning the condition (K3), as \( kQ \) is both a left and a right hereditary ring, \( J \) is projective (hence flat) both as a left and right \( kQ \)-module. As flatness is stable under bases change, \( \mathcal{K} \) is flat both as a left and right \( O_X Q \)-module. Condition (K3) follows.

□

When \( X = \text{Spec} \, k \), Corollary 4.7 immediately implies:

**Corollary 4.9.** Let \( V \) and \( W \) be two left \( \Lambda \)-modules. One has

\[
\text{Ext}^p_{\Lambda}(V, W) \simeq H^p(C^\bullet(V, W), d^{V, W})
\]

for all \( p \geq 0 \).

5. Hypercohomology

When \( V \) is a locally free \( O_X \)-module one is able to compute the functors \( \text{Ext}^p_{\mathcal{A}}(V, -) \) as the hypercohomology of a complex. This generalizes Theorem 5.1 in [11].

**Theorem 5.1.** Let \( V \) and \( W \) be two left \( \mathcal{A} \)-modules. Suppose that the conditions (K1)–(K3) in Theorem 4.1 are satisfied, and suppose that \( V \) is locally free as an \( O_X \)-module. Then one has

\[
\text{Ext}^p_{\mathcal{A}}(V, W) \simeq H^p(C^\bullet(V, W), d^{V, W})
\]

for all \( p \geq 0 \).

**Proof.** We pick up a resolution \( (W^\bullet, \partial) \) of \( W \) by injective left \( \mathcal{A} \)-modules:

\[
0 \longrightarrow W \longrightarrow W^0 \xrightarrow{\partial_0} W^1 \xrightarrow{\partial_1} W^2 \xrightarrow{\partial_2} \cdots
\]
Since $\mathcal{A}$ is flat as an $\mathcal{O}_X$-module, [17, Cor. 3.6A] implies that $(\mathcal{W}^\bullet, \partial)$ is also a resolution of $\mathcal{W}$ by injective $\mathcal{O}_X$-modules. Hence one can use $(\mathcal{W}^\bullet, \partial)$ to build up the spectral sequences in Theorem 4.1.

The exactness of the functors (4.14) implies that $\mathcal{E}_{1}^{p,q}(\mathcal{V}, \mathcal{W}) = 0$ if $q > 0$ (see eq. (4.1)), so that the double complex $\mathcal{E}_{0}^{\bullet,\bullet}(\mathcal{V}, \mathcal{W})$ introduced in eqs. (4.9), (4.10) and (4.11) is a resolution of $\mathcal{C}^{\bullet}(\mathcal{V}, \mathcal{W}) = \mathcal{E}_{1}^{\bullet,0}(\mathcal{V}, \mathcal{W})$:

$$0 \longrightarrow \mathcal{C}^{\bullet}(\mathcal{V}, \mathcal{W}) \longrightarrow \mathcal{E}_{0}^{\bullet,\bullet}(\mathcal{V}, \mathcal{W}). \quad (5.1)$$

Let $(\mathcal{E}_{0}^{\bullet}(\mathcal{V}, \mathcal{W}), \Delta)$ be the total complex associated to the double complex $(\mathcal{E}_{0}^{\bullet,\bullet}(\mathcal{V}, \mathcal{W}), \hat{d}, \hat{\partial})$. In particular

$$\mathcal{E}_{0}^{n}(\mathcal{V}, \mathcal{W}) = \bigoplus_{p+q=n} \mathcal{E}_{0}^{pq}(\mathcal{V}, \mathcal{W})$$

By [23, Lemma 8.5] the morphism (5.1) induces a quasi isomorphism

$$\mathcal{C}^{\bullet}(\mathcal{V}, \mathcal{W}) \longrightarrow \mathcal{E}_{0}^{\bullet}(\mathcal{V}, \mathcal{W})$$

The sheaves $\mathcal{A}^p$ are flat as right $\mathcal{A}$-modules, as shown in the proof of Lemma 4.4. The isomorphism

$$(\mathcal{A}^p \otimes_{\mathcal{A}} \mathcal{V}) \otimes_{\mathcal{O}_X} \mathcal{V} \simeq (\mathcal{A}^p \otimes_{\mathcal{A}} \mathcal{V}) \otimes_{\mathcal{O}_X} \mathcal{V}$$

implies that the sheaves $\mathcal{A}^p \otimes_{\mathcal{A}} \mathcal{V}$ are flat $\mathcal{O}_X$-modules for all $p \geq 0$. The sheaves $\mathcal{A}^p \otimes_{\mathcal{A}} \mathcal{V}$ can be decomposed according to eq. (2.5), and [17, Proposition 4.2] implies that the sheaves $(\mathcal{A}^p \otimes_{\mathcal{A}} \mathcal{V})_i$ are flat $\mathcal{O}_X$-modules for all $p \geq 0$ and $i \in I$. So [17, Lemma 3.5] implies that the sheaves $\mathcal{E}_{0}^{p,q}(\mathcal{V}, \mathcal{W})$ are injective $\mathcal{O}_X$-modules for all $p, q \geq 0$, and [17, Proposition 3.4] implies in turn that the sheaves $\mathcal{E}_{0}^{n}(\mathcal{V}, \mathcal{W})$ are injective $\mathcal{O}_X$-modules for all $n \geq 0$. Applying the functor $\Gamma(-)$ to the complex $\mathcal{E}_{0}^{\bullet}(\mathcal{V}, \mathcal{W})$ and taking cohomology one obtains

$$H^p(\Gamma(\mathcal{E}_{0}^{\bullet}(\mathcal{V}, \mathcal{W}))) \simeq R^p\Gamma(\mathcal{C}^{\bullet}(\mathcal{V}, \mathcal{W})) = H^p(\mathcal{C}^{\bullet}(\mathcal{V}, \mathcal{W})). \quad (5.2)$$

(cf. [23, 8.1.2]). Since

$$\mathcal{E}_{0}^{p,q}(\mathcal{V}, \mathcal{W}) = \text{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{C}^{p,q}(\mathcal{W}))$$

(see eq. (4.9)), it follows that

$$\mathcal{E}_{0}^{n}(\mathcal{V}, \mathcal{W}) = \text{Hom}_{\mathcal{A}}(\mathcal{V}, \bigoplus_{p+q=n} \mathcal{C}^{p,q}(\mathcal{W})) = \text{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{T}^n),$$

whence

$$\Gamma(\mathcal{E}_{0}^{n}(\mathcal{V}, \mathcal{W})) = \text{Hom}_{\mathcal{A}}(\mathcal{V}, \mathcal{T}^n), \quad (5.3)$$

where $(\mathcal{T}^{\bullet}, D_{\bullet})$ is the total complex associated to $(\mathcal{C}^{\bullet,\bullet}(\mathcal{W}), d, \partial)$ (see eq. (4.8)). Lemma 4.5 states that $(\mathcal{T}^{\bullet}, D_{\bullet})$ is a resolution of $\mathcal{W}$ by injective left $\mathcal{A}$-modules. Eq. (5.2) and (5.3)
imply that
\[ H^p(C^*(\mathcal{V}, \mathcal{W})) \cong H^p(\text{Hom}_{\mathcal{A}}(\mathcal{V}, T^*)) \cong \text{Ext}_{\mathcal{A}}(\mathcal{V}, \mathcal{W}). \]

\[ \square \]

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Figure 1