On an entropy of $\mathbb{Z}^k_+$-actions

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Abstract: In this paper, a definition of entropy for $\mathbb{Z}^k_+$-actions due to S. Friedland [4] is studied. Unlike the traditional definition, it may take a nonzero value for actions whose generators have finite (even zero) entropy as single transformations. Some basic properties are investigated and its value for the $\mathbb{Z}^k_+$-actions on circles generated by expanding endomorphisms is given. Moreover, an upper bound of this entropy for the $\mathbb{Z}^k_+$-actions on tori generated by expanding endomorphisms is obtained via the preimage entropies, which are entropy-like invariants depending on the “inverse orbits” structure of the system.

1 Introduction

Based on the need in the study of lattice statistical mechanics, Ruelle [13] introduced the concept of entropy for $\mathbb{Z}^k(k \geq 2)$-actions. A necessary condition for this entropy to be positive is that the generators should have infinite entropy as single transformations. In [4], Friedland gave a new definition of entropy for $\mathbb{Z}^k$-actions (or, more generally, $\mathbb{Z}^k_+$-actions, here $\mathbb{Z}^k_+ = \{0, 1, 2, \cdots \}$) which is appropriate for that whose generators have finite entropy as single transformations.

We begin by recalling the definition of topological entropy for $\mathbb{Z}^k_+$-actions. Let $(X, d_X)$ be a compact metric space and $C^0(X, X)$ the set of continuous maps on $X$. Any $f \in C^0(X, X)$ naturally generates a $\mathbb{Z}^k_+$-action: $\mathbb{Z}^k_+ \to C^0(X, X), n \mapsto f^n$. Let $K$ be a compact subset of $X$. For any $\varepsilon > 0$, a subset $E \subset X$ is said to be an $(f, n, \varepsilon)$-spanning set of $K$, if for any $x \in K$, there exists $y \in E$ such that

$$\max_{0 \leq i \leq n-1} d_X(f^i(x), f^i(y)) \leq \varepsilon.$$

Let $r_{d_X}(f, n, \varepsilon, K)$ denote the smallest cardinality of any $(f, n, \varepsilon)$-spanning set of $K$. A subset $F \subset K$ is said to be an $(f, n, \varepsilon)$-separated set of $K$, if $x, y \in F, x \neq y$, implies

$$\max_{0 \leq i \leq n-1} d_X(f^i(x), f^i(y)) > \varepsilon.$$

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Let \( s_{d_X}(f, n, \varepsilon, K) \) denote the largest cardinality of any \((f, n, \varepsilon)\)-separated set of \(K\). Let

\[
    h(f, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{d_X}(f, n, \varepsilon, K).
\]

By a standard discussion, we can give \( h(f, K) \) using spanning set, i.e., we can replace \( s_{d_X}(f, n, \varepsilon, K) \) by \( r_{d_X}(f, n, \varepsilon, K) \) in the above equation. The topological entropy of \(f\) is defined by \( h(f) = h(f, X) \).

Now we recall the traditional definition and Friedland’s definition of entropy for \(\mathbb{Z}_+^k\)-actions. Let \((X, d_X)\) be a compact metric space and \(T : \mathbb{Z}_+^k \to \mathbb{C}(X, X)\) a continuous \(\mathbb{Z}_+^k\)-action on \(X\). Denote the cube \( \prod_{i=1}^k \{0, \ldots, n-1\} \subset \mathbb{Z}_+^k \) by \( Q_n \). A set \( E \subset X \) is \((T, n, \varepsilon)\)-spanning if for every \(x \in X\) there exists a \(y \in E\) with \(d_X(T^i(x), T^i(y)) \leq \varepsilon\) for all \(i \in Q_n\). Let \(r_{d_X}(T, n, \varepsilon, X)\) be the smallest cardinality of any \((T, n, \varepsilon)\)-spanning set. A set \( F \subset X \) is an \((T, n, \varepsilon)\)-separated set if for any \(x, y \in F\), \(x \neq y\) implies \(d_X(T^i(x), T^i(y)) > \varepsilon\) for some \(i \in Q_n\). Let \(s_{d_X}(T, n, \varepsilon, X)\) be the largest cardinality of any \((T, n, \varepsilon)\)-separated set. The traditional definition of \(\hat{h}(T)\) is given by

\[
    \hat{h}(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n^k} \log s_{d_X}(T, n, \varepsilon, X). \tag{1.1}
\]

By a standard discussion, we can give \(\hat{h}(T)\) using spanning set, i.e., we can replace \(s_{d_X}(T, n, \varepsilon, X)\) by \(r_{d_X}(T, n, \varepsilon, X)\) in (1.1). For the general theory of entropy of \(\mathbb{Z}^k\)-actions, see Schmidt’s comprehensive book [14], and for the more general theory of entropy for countable amenable group actions, see for example [11] and [6].

It is well known that a necessary condition for \(\hat{h}(T)\) to be positive is that the generators \(\{T_i := T(0, \ldots, 0, 1_{(i)}), 0, \ldots, 0\}\) should have infinite entropy as single transformations. In contrast to the traditional definition, Friedland [11] introduced another definition of the topological entropy as follows. Define the orbit space of \(T\) by

\[
    X_T = \{ \bar{x} = \{x_n\}_{n \in \mathbb{Z}_+} \in \prod_{n \in \mathbb{Z}_+} X : \text{for any } n \in \mathbb{Z}_+, T_{i_n}(x_n) = x_{n+1} \text{ for some } i_n \in \{1\}_{i=1}^k \}.
\]

This is a closed subset of the compact space \(\prod_{n \in \mathbb{Z}_+} X\) and so is again compact. A natural metric on \(X_T\) is defined by

\[
    d_{X_T}(\bar{x}, \bar{y}) = \sum_{n=0}^{\infty} \frac{d_X(x_n, y_n)}{2^n} \tag{1.2}
\]

for \(\bar{x} = \{x_n\}_{n \in \mathbb{Z}_+}, \bar{y} = \{y_n\}_{n \in \mathbb{Z}_+} \in X_T\). We can define a natural shift map \(\sigma_T : X_T \to X_T\) by \(\sigma_T(\{x_n\}_{n \in \mathbb{Z}_+}) = \{x_{n+1}\}_{n \in \mathbb{Z}_+}\). Thus we have associated in a natural way a \(\mathbb{Z}_+\)-action with the \(\mathbb{Z}_+^k\)-action.

**Definition 1.1.** We define the topological entropy \(h(T)\) of the \(\mathbb{Z}_+^k\)-action \(T\) to be the topological entropy of the map \(\sigma_T : X_T \to X_T\), i.e.,

\[
    h(T) = h(\sigma_T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{d_{X_T}}(\sigma_T, n, \varepsilon, X_T),
\]

where \(s_{d_{X_T}}(\sigma_T, n, \varepsilon, X_T)\) is the largest cardinality of any \((\sigma_T, n, \varepsilon)\)-separated set in \(X_T\). (Similarly, we can replace \(s_{d_{X_T}}(\sigma_T, n, \varepsilon, X_T)\) by \(r_{d_{X_T}}(\sigma_T, n, \varepsilon, X_T)\), the smallest cardinality of any \((\sigma_T, n, \varepsilon)\)-spanning set in \(X_T\).)

The main purpose of this paper is to investigate some fundamental properties of the entropy \(h(T)\) of \(\mathbb{Z}_+^k\)-actions and evaluate its values for some standard examples.
In section 2, some basic properties of $h(T)$ are given. It is well known that for any map $f : X \to X$, the power rule for its entropy holds, i.e., $h(f^m) = mh(f)$ for any positive integer $m$. However, we can only get $h(T^m) \leq mh(T)$ for any $\mathbb{Z}_+^k$-action $T$ (Proposition 2.2). We also show in Proposition 2.2 that the entropy of any subgroup action (especially, each generator) of $T$ is less than or equal to that of $T$. When each generator $T_i$ is Lipschitzian with Lipschitz constant $L(T_i)$, then we can get an upper bound of $h(T)$ by \( \log \sum_{i=1}^k L(T_i)D(x) \), where $L(T_i) = \max\{1, L(T_i)\}$ and $D(x)$ is the ball dimension of $X$ (Proposition 2.4). The entropy of a skew product transformation which is an extension of $(X_T, \sigma_T)$ is also considered (Proposition 2.5).

In Section 3, we use Gellar and Pollicott’s method [5] to show (in Theorem 3.1) that for the $\mathbb{Z}_+^k$-action $T$ on the unit circle $X = \mathbb{S}^1$ generated by pairwise different endomorphisms $T_i(x) = L_i x \mod 1$, where $L_i, 1 \leq i \leq k$ are all positive integers greater than 1,

\[
h(T) = \log \sum_{i=1}^k L_i. \tag{1.3}
\]

In Section 4, we use other entropy-like invariants, the so called preimage entropies which rely on the preimage structure of the system, to show (in Theorem 4.1) that for the $\mathbb{Z}_+^k$-action $T$ on the torus $\mathbb{T}^n$ generated by pairwise different matrices $\{A_i\}_{i=1}^k$ whose eigenvalues $\{\lambda_i^{(1)}, \cdots, \lambda_i^{(n)}\}_{i=1}^k$ are of modulus greater than 1,

\[
h(T) \leq \log \left(\sum_{i=1}^k \prod_{j=1}^n |\lambda_i^{(j)}|\right). \tag{1.4}
\]

## 2 Some basic properties of $h(T)$

Throughout this section we always assume that $T : \mathbb{Z}_+^k \to C^0(X, X)$ is a continuous $\mathbb{Z}_+^k$-action with the generators $\{T_i : 1 \leq i \leq k\}$.

It is well known that topological entropy of a map (i.e., a $\mathbb{Z}_+$-action) is invariant under conjugacy. Now we can show that for any $\mathbb{Z}_+^k$-action a similar property holds true. We call another $\mathbb{Z}_+^k$-action $T'$ is topologically conjugate to $T$, if their generators $\{T_i : 1 \leq i \leq k\}$ and $\{T'_i : 1 \leq i \leq k\}$ are pairwise conjugate under the same homeomorphism $h$, i.e. we have the following commutative diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{T_i} & X \\
\downarrow h & & \downarrow h \\
X & \xrightarrow{T'_i} & X
\end{array}
\]

for each $i$. Since we can express $T'_i$ by $hT_ih^{-1}$, by the definition of the entropy we can get the following property immediately.

**Proposition 2.1.** Let $T$ and $T'$ be two conjugate $\mathbb{Z}_+^k$-actions with generators $\{T_i : 1 \leq i \leq k\}$ and $\{T'_i : 1 \leq i \leq k\}$ respectively, then

\[h(T) = h(T').\]

The following proposition concerns the relation between the entropy of the power $T^m$ and that of $T$, and the relation between the entropy of the subgroup action $T^{(l)}$ and that of $T$.

**Proposition 2.2.** Let $T : \mathbb{Z}_+^k \to C^0(X, X)$ be a continuous $\mathbb{Z}_+^k$-action with the generators $\{T_i : 1 \leq i \leq k\}$. We have the following properties of the entropy $h(T)$.
(1) For $m > 1$, we have $h(T^m) \leq mh(T)$, where $T^m$ is the $\mathbb{Z}_+^k$-action with the generators $\{T_{i}^m : 1 \leq i \leq k\}$.

(2) For any $1 \leq l < k$ and any $\mathbb{Z}_+^l$-action $T^{(l)}$ generated by some subcollection $\{T_{i_1}, \cdots, T_{i_l}\} \subset \{T_1, \cdots, T_k\}$, we have

$$h(T^{(l)}) \leq h(T).$$

In particular, $h(T_i) \leq h(T)$ for any $1 \leq i \leq k$.

Proof. (1) Let $\tilde{X} = \{\{x_n\}_{n \in \mathbb{Z}_+} \in \prod_{n \in \mathbb{Z}_+} X :$ for any $j \in \mathbb{Z}_+$, there exists some $1 \leq i \leq k$, such that for all $0 \leq s \leq m, x_{jm+s} = T_i^s(x_{jm})\}$. It is obvious that $\tilde{X} \subset X_T$. Define $\pi : \tilde{X} \to X_T$ by $\pi(\{x_n\}_{n \in \mathbb{Z}_+}) = \{x_{nm}\}_{n \in \mathbb{Z}_+}$. It is easy to obtain that $\pi$ is continuous and

$$\sigma_T^m \circ \pi(\{x_n\}_{n \in \mathbb{Z}_+}) = \pi \circ \sigma_T^m(\{x_n\}_{n \in \mathbb{Z}_+})$$

for any $\{x_n\}_{n \in \mathbb{Z}_+} \in \tilde{X}$. Therefore,

$$h(\sigma_T^m) \leq h(\sigma_T^m|_{\tilde{X}}) \leq h(\sigma_T^m) = mh(\sigma_T),$$

in which the last equality is from the well known power rule for topological entropy (see [15] for example).

(2) For the $\mathbb{Z}_+^l$-action $T^{(l)}$ generated by some $\{T_{i_1}, \cdots, T_{i_l}\}$, denote

$$X_T^{(l)} = \{\{x_n\}_{n \in \mathbb{Z}_+} \in \prod_{n \in \mathbb{Z}_+} X : x_{n+1} = T_{i_j}(x_n) \text{ for some } 1 \leq j \leq l\}.$$

It is obvious that

$$h(T^{(l)}) = h(\sigma_T^{(l)}) = h(\sigma_T|_{X_T^{(l)}}) \leq h(\sigma_T) = h(T).$$

Remark 2.3. For (1) of Proposition 2.2, either of the equality and strictly inequality in $h(T^m) \leq mh(T)$ can possibly hold. For example, for the $\mathbb{Z}_+^k$-action $T$ in Theorem 3.1, $h(T^m) = mh(T)$; for the $\mathbb{Z}_+^k$-action $T$ on the unit circle $\mathbb{S}^1$ whose generators $\{T_i : 1 \leq i \leq k\}$ are pairwise different rotations, by Theorem 4 of [3] we have $h(T^m) = h(T) = \log k < mh(T)$.

From (2) of Proposition 2.2, for any $\mathbb{Z}_+^k$-action $T$ with generators $\{T_i : 1 \leq i \leq k\}$, the entropy of each generator is less than or equal to that of $T$, i.e., $h(T_i) \leq h(T), 1 \leq i \leq k$. In general, $h(T_i)$ is strict less than $h(T)$, even for the actions with some trivial generators. For example, for the above $\mathbb{Z}_+^k$-action $T$ on the unit circle $\mathbb{S}^1$ whose generators are pairwise different rotations, it is obvious that $h(T_i) = 0$ for each $i$, but $h(T) = \log k$.

However, for any $\mathbb{Z}_+^k$-action $T$ with positive traditional entropy, i.e., $\hat{h}(T) > 0$, such as the full $k$-dimensional $m$-shift transformation $T$ on the space

$$X = \{0, \cdots, m-1\}_{\mathbb{Z}_+^k} = \{\{x_{(i_1, \cdots, i_k)}\}_{(i_1, \cdots, i_k) \in \mathbb{Z}_+^k} \in \prod_{(i_1, \cdots, i_k) \in \mathbb{Z}_+^k} \{0, 1, \cdots, m-1\} \}
$$

generated by

$$T_j : \{x_{(i_1, \cdots, i_k)}\} \mapsto \{x_{(i_1, \cdots, i_j+1, \cdots, i_k)}\}, 1 \leq j \leq k,$$

we have $h(T) = h(T_i) = \infty$ for any $1 \leq i \leq k$. 

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Let $(X, d)$ be a compact metric space and $b(\varepsilon)$ the minimum cardinality of covering of $X$ by $\varepsilon$-balls. Then

$$D(X) = \limsup_{\varepsilon \to 0} \frac{\log b(\varepsilon)}{\log \varepsilon} \in \mathbb{R} \cup \{\infty\}$$

is called the ball dimension of $X$. It is well known that if a map $f : X \to X$ is Lipschitzian with Lipschitz constant $L(f)$, then

$$h(f) \leq D(X) \log(\max\{1, L(f)\}),$$

see Theorem 3.2.9 of [8] for example. In the following, we give the corresponding inequalities for $\mathbb{Z}^k$-actions.

**Proposition 2.4.** Let $T : \mathbb{Z}^k_+ \to C^0(X, X)$ be a continuous $\mathbb{Z}^k_+$-action with the generators $\{T_i : 1 \leq i \leq k\}$. If the ball dimension of $X$ is finite, i.e., $D(X) < \infty$, and each $T_i : X \to X$ is Lipschitzian with Lipschitz constant $L(T_i)$, then

$$h(T) \leq \log \sum_{i=1}^k L_+(T_i)^{D(X)},$$

where $L_+(T_i) = \max\{1, L(T_i)\}$.

In particular, if $X$ is an $m$-dimensional compact Riemannian manifold and each $T_i, 1 \leq i \leq k$, is differentiable, then

$$h(T) \leq \log \sum_{i=1}^k \max\{1, \sup_{x \in X} \|d_x T_i \|^m\}.$$

**Proof.** It is well known that the topological entropy is unchanged by taking uniformly equivalent metrics (Theorem 7.4 of [15]). Here we say two metrics $d_\mathcal{X}$ and $d_\mathcal{X}'$ on $X$ are uniformly equivalent if

$$id : (X, d_\mathcal{X}) \to (X, d_\mathcal{X}') \quad \text{and} \quad id : (X, d_\mathcal{X}') \to (X, d_\mathcal{X})$$

are both uniformly continuous.

Take $\rho > \max_{1 \leq i \leq k} L_+(T_i)$. Clearly, $\rho > 1$. Now we define two metrics $d_{\mathcal{X}_T}$ and $d_{\mathcal{X}_T}'$ on $X_T$ by

$$d_{\mathcal{X}_T}(\bar{x}, \bar{y}) = \sum_{n=0}^{\infty} \frac{d_\mathcal{X}(x_n, y_n)}{\rho^n} \quad \text{and} \quad d_{\mathcal{X}_T}'(\bar{x}, \bar{y}) = \sup_{n \geq 0} \frac{d_\mathcal{X}(x_n, y_n)}{\rho^n}$$

for any $\bar{x} = \{x_n\}_{n \in \mathbb{Z}^+}, \bar{y} = \{y_n\}_{n \in \mathbb{Z}^+}$. Since $\rho > 1$, the metrics $d_{\mathcal{X}_T}$ and $d_{\mathcal{X}_T}'$ are both uniformly equivalent to $d_{X_T}$ which is defined in (1.2).

In the following we will estimate the entropy $h(T)$ with respect to the metric $d_{\mathcal{X}_T}'$. For any $\varepsilon > 0$, consider a maximal $(\sigma_T, m, \varepsilon)$-separated set $E$ of $X_T$ with cardinality $s_{d_{\mathcal{X}_T}'}(\sigma_T, m, \varepsilon, X_T)$. Obviously, for any $\bar{x} = \{x_n\}_{n \in \mathbb{Z}^+}, \bar{y} = \{y_n\}_{n \in \mathbb{Z}^+} \in E$ with $\bar{x} \neq \bar{y}$, we have

$$\max_{0 \leq s \leq m-1} \sup_{n \geq s} \frac{d_\mathcal{X}(x_n, y_n)}{\rho^{n-s}} > \varepsilon.$$

Let $K(\varepsilon) = \lceil \log_\rho \frac{\text{diam}(X, d_\mathcal{X})}{\varepsilon} \rceil$. In order to estimate the cardinality of $E$, we will write it into the union of subsets which consists of the points in $E$ with the first $m + K(\varepsilon)$ elements lie in the same orbit space of some sequence of $(T_{iz}, T_{iz}, \cdots, T_{im+K(\varepsilon)-1})$. From the definition of $d_{\mathcal{X}_T}'$ and the choice of $K(\varepsilon)$, we can estimate the cardinality of each of these subsets easily.
For any \( (i_1, \ldots, i_{m+K(\varepsilon)}) \in \prod_{n=1}^{m+K(\varepsilon)} \{1, \ldots, k\} \), denote
\[
\tilde{X}_{(i_1, \ldots, i_{m+K(\varepsilon)})} = \{ \tilde{x} = \{x_n\}_{n \in \mathbb{Z}_+} : x_n = T_n(x_{n-1}) \text{ for } 1 \leq n \leq m + K(\varepsilon) - 1 \}
\]
and
\[
\tilde{E}_{(i_1, \ldots, i_{m+K(\varepsilon)})} = E \cap \tilde{X}_{(i_1, \ldots, i_{m+K(\varepsilon)})}.
\]
Clearly,
\[
X_T = \bigcup_{(i_1, \ldots, i_{m+K(\varepsilon)}) \in \prod_{n=1}^{m+K(\varepsilon)} \{1, \ldots, k\}} \tilde{X}_{(i_1, \ldots, i_{m+K(\varepsilon)})}
\]
and
\[
E = \bigcup_{(i_1, \ldots, i_{m+K(\varepsilon)}) \in \prod_{n=1}^{m+K(\varepsilon)} \{1, \ldots, k\}} \tilde{E}_{(i_1, \ldots, i_{m+K(\varepsilon)})}.
\]
(Note that, each of them may be not a disjoint union.) Moreover, by the choice of \( \rho \) and \( K(\varepsilon) \), we can see that for any \( \overline{x} = \{x_n\}_{n \in \mathbb{Z}_+} \), \( \overline{y} = \{y_n\}_{n \in \mathbb{Z}_+} \in \tilde{E}_{(i_1, \ldots, i_{m+K(\varepsilon)})} \) if \( x_n = y_n \) for any \( 0 \leq n \leq m + K(\varepsilon) - 1 \) then \( \overline{x} = \overline{y} \). Therefore, if we denote the projection from \( \prod_{n \in \mathbb{Z}_+} X \) to its factor \( \prod_{n=0}^{m+K(\varepsilon)-1} X \) by \( \text{Proj}_{m+K(\varepsilon)} \) and let
\[
E_{(i_1, \ldots, i_{m+K(\varepsilon)})} = \text{Proj}_{m+K(\varepsilon)}(\tilde{E}_{(i_1, \ldots, i_{m+K(\varepsilon)})}),
\]
then
\[
\text{card}(E_{(i_1, \ldots, i_{m+K(\varepsilon)})}) = \text{card}(\tilde{E}_{(i_1, \ldots, i_{m+K(\varepsilon)})}).
\]
Define a metric \( d_{(i_1, \ldots, i_{m+K(\varepsilon)})} \) on \( X_{(i_1, \ldots, i_{m+K(\varepsilon)})} := \text{Proj}_{m+K(\varepsilon)}(\tilde{X}_{(i_1, \ldots, i_{m+K(\varepsilon)})}) \) by
\[
d_{(i_1, \ldots, i_{m+K(\varepsilon)})}(\tilde{z}, \tilde{z}') = \max_{0 \leq s \leq m + K(\varepsilon) - 1} \frac{d_X(z_s, z'_s)}{\rho^s}
\]
for any \( \tilde{z} = \{z_s\}_{s=0}^{m+K(\varepsilon)-1} \) and \( \tilde{z}' = \{z'_s\}_{s=0}^{m+K(\varepsilon)-1} \in X_{(i_1, \ldots, i_{m+K(\varepsilon)})} \). From the choice of \( \rho \), we can see that \( (X_{(i_1, \ldots, i_{m+K(\varepsilon)})}, d_{(i_1, \ldots, i_{m+K(\varepsilon)})}) \) is isometric to \( (X, d_X) \). That is,
\[
d_{(i_1, \ldots, i_{m+K(\varepsilon)})}(\tilde{z}, \tilde{z}') = d_X(z_0, z'_0).
\]
Then the ball dimension of \( (X_{(i_1, \ldots, i_{m+K(\varepsilon)})}, d_{(i_1, \ldots, i_{m+K(\varepsilon)})}) \) is equal to \( D(X) \).
Let \( \varepsilon_{(i_1, \ldots, i_{m+K(\varepsilon)})} = \frac{3}{m + K(\varepsilon) - 1} \). Then for any \( \tilde{z}, \tilde{z}' \in E_{(i_1, \ldots, i_{m+K(\varepsilon)})} \) with \( \tilde{z} \neq \tilde{z}' \), we have
\[
Bd_{(i_1, \ldots, i_{m+K(\varepsilon)})}(\tilde{z}, \varepsilon_{(i_1, \ldots, i_{m+K(\varepsilon)})}) \cap Bd_{(i_1, \ldots, i_{m+K(\varepsilon)})}(\tilde{z}', \varepsilon_{(i_1, \ldots, i_{m+K(\varepsilon)})}) = \emptyset.
\]
Therefore,
\[
\text{card}(E_{(i_1, \ldots, i_{m+K(\varepsilon)})}) \leq \frac{\alpha}{\varepsilon_{(i_1, \ldots, i_{m+K(\varepsilon)})} D(X)}.
\]
where $\alpha$ is a constant independent of $(i_1, \cdots, i_{m+K(\varepsilon)})$. Hence

$$s_{d_X}^{\sigma_T}(\sigma_T, m, \varepsilon, X'_T) = \text{card}(E)$$

$$\leq \sum_{(i_1, \cdots, i_{m+K(\varepsilon)})} \text{card}(E_{(i_1, \cdots, i_{m+K(\varepsilon)})})$$

$$\leq \sum_{(i_1, \cdots, i_{m+K(\varepsilon)})} \frac{\alpha \cdot 3^D(X)}{\varepsilon^D(X)} \cdot \prod_{n=1}^{m+K(\varepsilon)-1} \text{card}(\text{E}(T_i)_{i=1}^n)$$

Thus,

$$h(T) = \lim_{\varepsilon \to 0} \lim_{m \to \infty} \frac{1}{m} \log s_{d_X}^{\sigma_T}(\sigma_T, m, \varepsilon, X'_T) \leq \log \sum_{i=1}^k L(T_i)^D(X).$$

In the following, we will consider a skew product transformation such that $(\sigma_T, X'_T)$ is its factor, and we will use it to evaluate the entropy of $\mathbb{Z}_+^k$-action on circles in the next section.

Let $\Sigma_k = \prod_{n \in \mathbb{Z}_+} \{1, \cdots, k\}$ be the standard symbolic space with the product topology. A natural metric $d_{\Sigma_k}$ on $\Sigma_k$ is defined by

$$d_{\Sigma_k}((i_n)_{n \in \mathbb{Z}_+}, (j_n)_{n \in \mathbb{Z}_+}) = \sum_{n=0}^{\infty} \frac{d(i_n, j_n)}{2^n}$$

for $(i_n)_{n \in \mathbb{Z}_+}, (j_n)_{n \in \mathbb{Z}_+} \in \Sigma_k$, where $d(i_n, j_n) = 0$ when $i_n = j_n$, and $d(i_n, j_n) = 1$ when $i_n \neq j_n$. Let $Y = \Sigma_k \times X$ be endowed with the product topology and define a map $\tilde{\sigma} : Y \to Y$ by

$$\tilde{\sigma}(\{i_n\}_{n \in \mathbb{Z}_+}, x) = (\{i_{n+1}\}_{n \in \mathbb{Z}_+}, T_0x).$$

This is a skew product over the shift transformation

$$\sigma_k : \Sigma_k \to \Sigma_k, \quad \{i_n\}_{n \in \mathbb{Z}_+} \mapsto \{i_{n+1}\}_{n \in \mathbb{Z}_+}.$$ 

The basis transformation $\sigma_k$ is a natural factor of this skew product, hence by Bowen’s entropy inequality in [1], we have that

$$h(\tilde{\sigma}) \leq h(\sigma_k) + \sup_{\{i_n\}_{n \in \mathbb{Z}_+} \in \Sigma_k} h(\tilde{\sigma}, Y_{\{i_n\}_{n \in \mathbb{Z}_+}}),$$

where $Y_{\{i_n\}_{n \in \mathbb{Z}_+}} = \{\{i_n\}_{n \in \mathbb{Z}_+}, x \in Y : x \in X\}$. Clearly, $h(\sigma_k) = \log k$. Moreover, if the ball dimension of $X$ is finite, i.e., $D(X) < \infty$, and each $T_i : X \to X$ is Lipschitzian with Lipschitz constant $L(T_i)$, then from the proof of Proposition 1.4 we have that

$$h(\tilde{\sigma}, Y_{\{i_n\}_{n \in \mathbb{Z}_+}}) \leq \limsup_{n \to \infty} \frac{1}{n} \log \prod_{j=1}^n L(T_i)^D(X) \leq D(X) \log L,$$

in which $L = \max_{1 \leq i \leq k} L(T_i)$, and hence

$$h(\tilde{\sigma}) \leq \log k + D(X) \log L.$$

In the following we can see that $\sigma_T$ is another factor of $\tilde{\sigma}$.  

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Proposition 2.5. Let $T : \mathbb{Z}_+^k \longrightarrow C^0(\mathbb{X}, \mathbb{X})$ be a continuous $\mathbb{Z}_+^k$-action with the generators $\{T_i : 1 \leq i \leq k\}$ and $\tilde{\sigma} : Y \rightarrow Y$ be as above. Then $\tilde{\sigma}$ is an extension of $\sigma_T$, and hence

$$h(T) = h(\sigma_T) \leq h(\tilde{\sigma}).$$

Proof. Define a map $\pi : Y \rightarrow X_T$ by

$$\tilde{\pi}(\{i_n\} \in \mathbb{Z}_+, x) = \{x_n\} \in \mathbb{Z}_+,$$

where $x_0 = x$ and $x_n = T_{i_{n-1}} \circ \cdots \circ T_{i_1} \circ T_{i_0}(x)$ for $n \geq 1$. We claim that it is a semi-conjugacy between $\tilde{\sigma}$ and $\sigma_T$. In fact, we firstly have that $\pi \circ \tilde{\sigma} = \sigma_T \circ \pi$ from the definitions of $\pi, \tilde{\sigma}$ and $\sigma_T$. Secondly, $\pi$ is surjective since for any point $\{x_n\} \in X_T$ we can construct $\{i_n\} \in \mathbb{Z}_+$ by setting $x = x_0$ and choosing $i_n$ for $n \geq 0$ inductively such that $i_n = j$ if $x_{n+1} = T_j(x_n)$ for some $1 \leq j \leq k$ (Please note that the choice of $i_n$ may not be unique). Finally, $\pi$ is continuous since for any sequence of points $\{\mathbb{Y}^{(i)}\} = \{\{i_n\} \in \mathbb{Z}_+, x^{(i)}\} \in \mathbb{Z}_+$ tends to $\mathbb{Y} = \{\{i_n\} \in \mathbb{Z}_+, x\}$ as $i \rightarrow \infty$, we have $\{i_n\} \in \mathbb{Z}_+$ and $x^{(i)} \rightarrow x$ as $i \rightarrow \infty$, and hence from the definition of $\pi$, the uniform continuity of $T_i, 1 \leq i \leq k$, and the topologies of $Y$ and $X_T$, $\pi(\mathbb{Y}^{(i)}) \rightarrow \pi(\mathbb{Y})$ as $i \rightarrow \infty$. Therefore, $\sigma_T$ is a factor of $\tilde{\sigma}$, and hence $h(T) = h(\sigma_T) \leq h(\tilde{\sigma})$. \hfill $\square$

3 $\mathbb{Z}_+^k$-actions on circles generated by expanding endomorphisms

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle with the “geodesic” metric $d_{S^1}$, i.e., for any $x, y \in S^1$, $d_{S^1}(x, y)$ is the length of the shorter path joining them.

As one of the simplest system $f : S^1 \longrightarrow S^1, x \mapsto px(\text{mod 1})$, its entropy $h(f) = \log p$. However, for $\mathbb{Z}_+^k$-action on circles which is generated by this kind of endomorphisms, its entropy is not easy to be calculated. In [4], Friedland conjectured that for a $\mathbb{Z}_+^k$-action on the circle $S^1$ whose generators are $T_1 = px(\text{mod 1})$ and $T_2 = qx(\text{mod 1})$, where $p$ and $q$ are two co-prime integers, its entropy $h(T) = \log(p + q)$. Soon afterwards Geller and Pollicott [5] answered this conjecture affirmatively under a weaker condition “$p, q$ are all integers greater than 1”. In this section we generalize the main result in [5] to $\mathbb{Z}_+^k$-action on circles.

Theorem 3.1. Let $T : \mathbb{Z}_+^k \longrightarrow C^0(S^1, S^1)$ be a continuous $\mathbb{Z}_+^k$-action on the circle $X = S^1$ with the generators $T_i(1 \leq i \leq k)$ defined by $T_i(x) = L_{i}(x)(\text{mod 1})$ where $L_i, 1 \leq i \leq k$, are all integers greater than 1 and are pairwise different. Then the formula (1.3) holds, i.e.,

$$h(T) = \log \sum_{i=1}^k L_i.$$

Proof. As we have done in Proposition 2.5, denote $Y = \Sigma_k \times X$ endowed with the product topology and define a map $\tilde{\sigma} : Y \rightarrow Y$ by

$$\tilde{\sigma}(\{i_n\} \in \mathbb{Z}_+, x) = (\{i_{n+1}\} \in \mathbb{Z}_+, T_{i_0}x).$$

This is a skew product over the shift transformation

$$\sigma_k : \Sigma_k \rightarrow \Sigma_k, \quad \{i_n\} \in \mathbb{Z}_+ \rightarrow \{i_{n+1}\} \in \mathbb{Z}_+.$$
Consider a cover of $\Sigma_k \times S^1$ consisting of the closed sets

$$\left\{ [i]_0 \times \left[ \frac{j}{M}, \frac{j+1}{M} \right] : 1 \leq i \leq k, \ 0 \leq j \leq M-1 \right\},$$

where $[i]_0 = \{ i_n \}_{n \in \mathbb{Z}} \in \Sigma_k : i_0 = i$ and $M = \prod_{i=1}^k L_i$. It is clear that this cover consists of $kM$ elements. We label them by $\{ B_l \}_{l=1}^{kM}$, where

$$B_l = [i]_0 \times \left[ \frac{j}{M}, \frac{j+1}{M} \right]$$

for $l = (i-1)M+j+1, 1 \leq i \leq k, 0 \leq j \leq M-1$. Now we can define a $kM \times kM$ transition matrix $A$ by

$$A(s, t) = \begin{cases} 1 & \text{if } \hat{\sigma}(\text{int}B_s) \supset \text{int}B_t \\ 0 & \text{if } \hat{\sigma}(\text{int}B_s) \cap \text{int}B_t = \emptyset. \end{cases}$$

This will take the form

$$A = \begin{pmatrix} Q_1 \\ \vdots \\ Q_k \end{pmatrix},$$

where $Q_s(1 \leq s \leq k)$ is an $M \times kM$ matrix with the form

$$Q_s = \begin{pmatrix} P_s \cdots P_s \\ \vdots \\ P_s \cdots P_s \end{pmatrix},$$

in which $P_s$ is a $\prod_{j \neq s} L_j \times M$ matrix given by

$$P_s = \begin{pmatrix} 1 \cdots 1 & 0 \cdots 0 & \cdots & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 & \cdots & 0 \cdots 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 \cdots 0 & 0 \cdots 0 & \cdots & 1 \cdots 1 \end{pmatrix}_{\times L_s \times L_s \times L_s}.$$

By calculating, we can get that $A$ is irreducible. Denote

$$\Lambda = \{ \{z_n\}_{n \in \mathbb{Z}} \in \Sigma_{kM} : A(z_n, z_{n+1}) = 1 \text{ for } n \geq 0 \}$$

and let $\hat{\sigma} : \Lambda \to \Lambda$ denote the associated subshift of finite type. We observe that the column sums in $A$ are all equal to $\sum_{i=1}^k L_i$. Therefore, by Perron-Frobenius Theorem and Theorem 7.13 of [15], we obtain that

$$h(\hat{\sigma}) = \log \sum_{i=1}^k L_i. \quad (3.1)$$

Consider the map $\hat{\pi} : \Lambda \to X_T$ defined by $\hat{\pi}(\{z_n\}) = \{x_n\}$ with $x_0 = \bigcap_{m=0}^{\infty} I_m$ where

$$I_m = \bigcap_{n=0}^{m-1} (T_{i_n} \circ \cdots \circ T_{i_1})^{-1} \left[ \frac{l_n}{M}, \frac{l_n+1}{M} \right]$$
and \(x_{n+1} = T_{i_n}(x_n)\), in which \(T_{i_n} = T_i\) if the element in the cover according to \(z_n\) is \((|t|_0, \left[ \frac{l_n}{M}, \frac{l_n + 1}{M} \right])\).

Since \(I_0 \supset I_1 \supset \cdots \supset I_m \supset \cdots\) is a nested sequence of closed sets we have by compactness that \(\bigcap_{m=0}^{\infty} I_m \neq \emptyset\). Moreover, since each \(L_i\) is greater than 1, each \(T_i\) is expanding and hence \(\lim_{m \to \infty} \text{diam } I_m = 0\). In particular, this intersection consists of a single point, say \(x_0\). Therefore the map \(\tilde{\pi}\) is well defined. Similar to what we have done to \(\pi\) in the proof of Proposition 2.4 we can show that \(\tilde{\pi}\) is a semi-conjugacy, i.e., \(\sigma_T \circ \tilde{\pi} \equiv \hat{\pi} \circ \dot{\sigma}\). Hence

\[
h(\sigma_T) \leq h(\hat{\sigma}). \tag{3.2}
\]

It only remains to show that \(h(\sigma_T) \geq \log \sum_{i=1}^{k} L_i\). Observe that although \(\tilde{\pi}\) is surjective, it can fail to be injective. We claim that the set on which injectivity fails is “small”. Assume that \(\tilde{\pi}(\{z_n\}) = \tilde{\pi}(\{z'_n\}) = \{x_n\}\) but \(\{z_n\} \neq \{z'_n\}\). In particular, assume that \(z_i = z'_i\) for \(0 \leq i \leq n - 1\), but \(z_n \neq z'_n\). This can only happen if \(x_n \in \left\{ \frac{i}{M} : 0 \leq i \leq M \right\}\), since \(L_i, 1 \leq i \leq k\) are pairwise different. In particular, we see that

\[
\Omega = \left\{ \{z_n\} \in \Lambda : \text{card}(\tilde{\pi}^{-1}(\tilde{\pi}(\{z_n\}))) \geq 2 \right\}
\]

is a countable set.

Since \(A\) is irreducible then \(\hat{\sigma} : \Lambda \to \Lambda\) is a transitive subshift of finite type. Hence from [12], there is a unique measure of maximal entropy, i.e., \(\nu\) is the unique \(\hat{\sigma}\)-invariant probability measure with entropy \(h(\hat{\sigma}) = \log \sum_{i=1}^{k} L_i\). Moreover, \(\nu\) is the Markov measure, it is clear that \(\nu(\Omega) = 0\) and so \(\tilde{\pi} : (\Lambda, \nu) \to (X_T, \pi^*\nu)\) is an isomorphism. By the variational principle we see that

\[
h(\sigma_T) = \sup \{h_\mu(\sigma_T) : \mu \text{ is any } \sigma_T \text{ invariant probability measure}\}
\]

\[
\geq h_{\pi^*\nu}(\sigma_T) = h(\hat{\sigma}). \tag{3.3}
\]

Combining (3.1), (3.2) and (3.3) we obtain the desired equality (1.3). \(\square\)

### 4 \(\mathbb{Z}_+^k\)-actions on tori generated by expanding endomorphisms

Let \(A : \mathbb{R}^n \to \mathbb{R}^n\) be a non-singular integer matrix. There is a natural induced endomorphism of the \(n\)-dimensional torus \(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n\), for simplicity, we also denote it by \(A\). It is well known that for any endomorphism \(A\) on the torus \(\mathbb{T}^n\), we have

\[
h(A) = \sum_{|\lambda^{(i)}| > 1} \log |\lambda^{(j)}|, \tag{4.1}
\]

where \(\lambda^{(1)}, \ldots, \lambda^{(n)}\) are the eigenvalues of \(A\), counted with their multiplicities.

In section 3, we use Geller and Pollicott’s method to get a formula of Friedland’s entropy for expanding \(\mathbb{Z}_+^k\)-actions on circles. However, it seems that it is not easy to use a similar strategy to deal with the high dimensional cases. In this section, we use other entropy-like invariants, the so-called preimage entropies, to estimate the Friedland’s entropy for the \(\mathbb{Z}_+^k\)-actions on tori generated by expanding endomorphisms. In the following, we first state some basic notions and facts for preimage entropies. For more information about them, please refer to [7], [9] and [10].

Let \(f\) be a continuous surjective map on a compact metric space \((X, d_X)\). There are many types of preimage entropies and we only present two of them here. The first one is the pointwise preimage
entropy $h_n(f)$ which is defined by

$$h_n(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} s_{d_X}(f, n, \varepsilon, f^{-n}(x))$$

$$= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} r_{d_X}(f, n, \varepsilon, f^{-n}(x)).$$

The second one is the preimage branch entropy which is defined as follows. For any $x \in X$ and $n \in \mathbb{Z}_+$, the $n$-th order preimage tree of $x$ under $f$ is defined by

$$T_n(x, f) = \{[z_n, z_{n-1}, \ldots, z_1, z_0 = x] : f(z_j) = z_{j-1} \text{ for all } 1 \le j \le n\}.$$  

Each ordered set $\xi = [z_n, z_{n-1}, \ldots, z_1, z_0 = x] \in T_n(x, f)$ is called a branch of $T_n(x, f)$. For any two branches

$$\xi = [z_n, z_{n-1}, \ldots, z_1, z_0 = x] \in T_n(x, f) \text{ and } \eta = [z'_n, z'_{n-1}, \ldots, z'_1, z'_0 = y] \in T_n(y, f),$$

the branch distance between them is defined as

$$d^b_X(\xi, \eta) = \max_{0 \le j \le n} d_x(z_j, z'_j).$$

Let $T_n(X, f) = \bigcup_{x \in X} T_n(x, f)$. We can define a branch-Hausdorff metric $d^b_X$ on $T_n(X, f)$ by

$$d^b_X(T_n(x, f), T_n(y, f)) = \max\{\max_{\xi \in T_n(x, f)} \min_{\eta \in T_n(y, f)} d^b_X(\xi, \eta), \max_{\eta \in T_n(y, f)} \min_{\xi \in T_n(x, f)} d^b_X(\eta, \xi)\} \quad (4.2)$$

for $T_n(x, f)$ and $T_n(y, f)$ in $T_n(X, f)$. Intuitively, $d^b_X(T_n(x, f), T_n(y, f)) < \varepsilon$ if and only if each branch of either tree is $d^b_X$ within $\varepsilon$ of at least one branch of the other tree. Let $s_{d^b_X}(f, n, \varepsilon, T_n(X, f))$ denote the maximum cardinality of any $d^b_X$-$\varepsilon$-separated collection of trees in $T_n(X, f)$, and $r_{d^b_X}(f, n, \varepsilon, T_n(X, f))$ denote the minimum cardinality of any $d^b_X$-$\varepsilon$-spanning collection of trees in $T_n(X, f)$. Then the preimage branch entropy $h_i(f)$ is defined by

$$h_i(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log s_{d^b_X}(f, n, \varepsilon, T_n(X, f))$$

$$= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log r_{d^b_X}(f, n, \varepsilon, T_n(X, f)).$$

Similar to that for the entropy $h(f)$, it is easy to see that $h_n(f)$ and $h_i(f)$ are unchanged by taking an equivalent metric on $X$. From Theorem 3.1 in [7], we have the following inequalities relating these entropies

$$h_n(f) \le h(f) \le h_n(f) + h_i(f). \quad (4.3)$$

For some recent progress in the study of preimage entropies in different forms and in different settings, we refer to [2], [17], [16], [18] and [19].

**Theorem 4.1.** Let $T$ be a $\mathbb{Z}^k_+$-action on the torus $\mathbb{T}^n$ with the generators $\{T_i = A_i\}_{i=1}^k$ which are pairwise different. If $\{A_i\}_{i=1}^k$ are all non-singular and all eigenvalues of $A_i$ are of modulus greater than 1, then the inequality (1.2) holds, i.e.,

$$h(T) \le \log\left(\sum_{i=1}^k \prod_{j=1}^n |\lambda_i^{(j)}|\right),$$

where $\lambda_i^{(1)}, \ldots, \lambda_i^{(n)}$ are the eigenvalues of $A_i$, counted with their multiplicities.
Proof. Firstly, we show that
\[ h_i(\sigma_T) = 0. \] (4.4)
Since \( \{A_i\}_{i=1}^k \) are all non-singular, for any \( x \in \mathbb{T}^n \) and \( 1 \leq i \leq k \),
\[ \text{card}(T_i^{-1}(x)) = | \det A_i | = \prod_{j=1}^n |\lambda_i^{(j)}|. \]
For simplicity of notation, we denote \( \prod_{j=1}^n |\lambda_i^{(j)}| \) by \( N_i \) and for \( x \in \mathbb{T}^n \) denote
\[ T_i^{-1}(x) = \{x^{(1)}, x^{(2)}, \ldots, x^{(N_i)}\}. \]
Since for each \( 1 \leq i \leq k \) all eigenvalues of \( A_i \), are of modulus greater than 1, each \( T_i \) is expanding. Hence, we can take \( 0 < \varepsilon_0 < \frac{1}{2} \) and \( \lambda > 1 \), such that for any \( 0 < \varepsilon \leq \varepsilon_0, x \in \mathbb{T}^n \) and \( 1 \leq i \leq k \),
\[ T_i^{-1}(B_{d_X}(x, \varepsilon)) = \bigcup_{j=1}^{N_i} U(x^{(j)}), \]
where \( U(x^{(j)}) \) is an open neighborhood of \( x^{(j)} \) for \( 1 \leq j \leq N_i \), and \( \{U(x^{(j)})\}_{j=1}^{N_i} \) are pairwise disjoint, moreover, the restriction \( T_i|_{U(x^{(j)})} : U(x^{(j)}) \to B_{d_X}(x, \varepsilon) \) is a homeomorphism and for any \( z, z' \in U(x^{(j)}) \),
\[ d_X(T_i(z), T_i(z')) \geq \lambda \cdot d_X(z, z'). \]
So for any \( y \in B_{d_X}(x, \varepsilon) \), the corresponding 1-th order preimage tree \( T_i(y, T_i) \) lies in the \( d_X^{\varepsilon} \)-neighborhood of the 1-th order preimage tree \( T_i(x, T_i) \) and actually
\[ d_X(T_i(x, T_i), T_i(y, T_i)) = d_X(x, y) \]
since \( \lambda > 1 \). If for any finite sequence of endomorphisms \( \{T_n, \{T_i, \cdots, T_k\}_{i=1}^l \} \) and \( x \in \mathbb{T}^n \), let
\[ T_l(x, \{T_n\}_{i=1}^l) = \{[z_l, z_l-1, \cdots, z_1, z_0 = x] : T_{n_j}(z_j) = z_{j-1} \text{ for all } 1 \leq j \leq l\} \]
be the \( l \)-th order preimage tree of \( x \) with respect to \( \{T_n\}_{i=1}^l \). Then for any \( 0 < \varepsilon \leq \varepsilon_0 \) and \( y \in B_{d_X}(x, \varepsilon) \), we can inductively conclude that the \( l \)-th order preimage tree \( T_l(y, \{T_n\}_{i=1}^l) \) lies in the \( d_X^{\varepsilon} \)-neighborhood \( T_l(x, \{T_n\}_{i=1}^l) \), and actually
\[ d_X^{\varepsilon}(T_l(x, \{T_n\}_{i=1}^l), T_l(y, \{T_n\}_{i=1}^l)) = d_X(x, y), \]
where \( d_X^{\varepsilon} \) is analogues to that in (4.2).

From the above discussion and the definition of \( d_X^{\varepsilon} \) on the collection of \( l \)-th order preimage trees under \( \sigma_T \), \( T_l(X_T, \sigma_T) \), we can see that for any \( \bar{x}, \bar{y} \in X_T \) with \( d(\bar{x}, \bar{y}) < \varepsilon \leq \varepsilon_0 \), we have
\[ d_X^{\varepsilon}(T_l(\bar{x}, \sigma_T), T_l(\bar{y}, \sigma_T)) = d_X(\bar{x}, \bar{y}). \]
So, if a finite set \( \{\bar{x}^{(i)}\} \) is \( d_{X_T}^{\varepsilon_0} \)-dense in the compact space \( X_T \), then for any \( l \in \mathbb{Z}_+ \), \( \{T_l(\bar{x}^{(i)}, \sigma_T)\} \) is \( d_X^{\varepsilon_0} \)-dense in \( T_l(X_T, \sigma_T) \). Therefore, \( s_{d_X^{\varepsilon_0}}(\sigma_T, l, \varepsilon_0, T_l(X_T, \sigma_T)) \) is independent of \( l \) and hence (4.3) holds.

By (4.1) and the inequalities in (4.3), we have that
\[ h(T) = h(\sigma_T) = h_m(\sigma_T). \]
Since for any \( x \in \mathbb{T}^n \) and any \( 1 \leq i \leq k \), \( \text{card}(T_i^{-1}(x)) = N_i \) and \( \text{card}(\bigcup_{i=1}^{k} T_i^{-1}(x)) \leq \sum_{i=1}^{k} N_i \), and hence for any \( l \in \mathbb{Z}_+ \) and \( \bar{x} = \{x_n\}_{n \in \mathbb{Z}^+} \in X_T \), \( \text{card}(\sigma_T^{-l}(\bar{x})) \leq \left( \sum_{i=1}^{k} N_i \right)^l \). Therefore,

\[
h_m(\sigma_T) \leq \log \sum_{i=1}^{k} N_i, \tag{4.5}
\]

and then the desired inequality \((1.4)\) holds.

By the way, we would like to say that our original intention is to show that the equality in \((1.4)\) holds for expanding \( \mathbb{Z}_+^k \)-actions on tori. So far we can conclude as follows that for almost every \( \bar{x} = \{x_n\}_{n \in \mathbb{Z}^+} \in X_T \) with \( x_0 \in \mathbb{T}^n \setminus F \), the cardinality of the set \( \bigcup_{i=1}^{k} T_i^{-1}(x) \) is exactly \( \sum_{i=1}^{k} N_i \). Therefore, for any \( l \in \mathbb{Z}_+ \) and \( \bar{x} = \{x_n\}_{n \in \mathbb{Z}^+} \in X_T \) with \( x_0 \in \mathbb{T}^n \setminus F \), the cardinality of the \( l \)-th preimage set \( \sigma_T^{-l}(\bar{x}) \) is exactly \( \left( \sum_{i=1}^{k} N_i \right)^l \). We believe that for any \( \bar{x} = \{x_n\}_{n \in \mathbb{Z}^+} \in X_T \) with \( x_0 \in \mathbb{T}^n \setminus F \) (even for any \( \bar{x} = \{x_n\}_{n \in \mathbb{Z}^+} \in X_T \)),

\[
\lim_{\varepsilon \to 0} \limsup_{l \to \infty} \frac{1}{l} \log s_{d_{X_T}}(\sigma_T, l, \varepsilon, \sigma_T^{-l}(\bar{x})) = \log \left( \sum_{i=1}^{k} N_i \right),
\]

and hence the equality in \((1.4)\) holds.

**Remark 4.2.** In [20], the upper and lower bounds of the entropy of the nonautonomous dynamical systems on tori which are generated by expanding endomorphisms (Theorem 2.8 of [20]) were given. More precisely, let \( A_1, \infty = \{A_i\}_{i=1}^{\infty} \) be a sequence of equi-continuous surjective endomorphisms of \( \mathbb{T}^n \). If for each \( i \in \mathbb{Z}_+ \), all eigenvalues of \( A_i \) are of modulus greater than 1, then

\[
\limsup_{l \to \infty} \frac{1}{l} \sum_{i=1}^{n} \log |\lambda_i^{(l)}| \leq h(A_1, \infty) \leq \limsup_{n \to \infty} \frac{n}{l} \sum_{i=1}^{l-1} \log \Lambda_i^{(1)}, \tag{4.6}
\]

where \( \lambda_i^{(1)}, \ldots, \lambda_i^{(n)} \) are the eigenvalues of \( A_i, i \in \mathbb{Z}_+ \), counted with their multiplicities, and \( \Lambda_i^{(1)} \) is the biggest eigenvalue of \( \sqrt{A_i A_i^T}, i \in \mathbb{Z}_+ \). From Theorem 3.1 and Theorem 5.1 in [17], we have that \( h(A_1, \infty) = h_m(A_1, \infty) \). Using the similar method in the proof of the above Theorem 4.4, we can improve \((4.6)\) to the following equality

\[
h(A_1, \infty) = h_m(A_1, \infty) = \limsup_{l \to \infty} \frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{l-1} \log |\lambda_i^{(j)}|.
\]
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