CHARACTERISTIC NUMBERS OF ELLIPTIC SPACE CURVES

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ABSTRACT. We solve the problem of characteristic numbers of elliptic curves in any dimensional projective space. The answers are given in the form of effective recursions. A C++ program implementing most of the recursions is available upon request.

1. INTRODUCTION

Computing the characteristic numbers of curves in projective spaces is a classical problem in algebraic geometry: how many curves in \( \mathbb{P}^r \) of given degree and genus that pass through a general set of linear subspaces, and are tangent to a general set of hyperplanes? These types of problems have provided a wealth of inspiration for classical algebraic geometers, such as Schubert, Zenthen, Mailard, and have been one of the main drives of the development of intersection theory. They have achieved remarkable success despite the technology of their time (in mathematical and literal sense). For example, Schubert has correctly computed the number of twisted cubics in \( \mathbb{P}^3 \) that are tangent to 12 quadric surfaces \((5, 819, 539, 783, 680)\), and this represents the summit of enumerative geometry at the time (see [KSX]).

However, not until recently are characteristic numbers computed in certain generality, as all of the results obtained by classical algebraic geometers are bounded by degree. Moduli space of stable maps provide a powerful tool to attack characteristic numbers problems of curves with low genus. The case of rational space curve of any degree was solved in [P1]. The author actually gave algorithms to compute intersections of divisors on \( \overline{\mathcal{M}}_{0,n}(r, d) \) and derived characteristic numbers as a corollary. However, if one is only interested in the characteristic numbers, then a small tweak would give a much faster and easier to implement algorithm. The case of elliptic plane curves, any degree, was solve in [V2]. The difficulty to extend that result to \( \mathbb{P}^r \) was due to two factors. Firstly, there was yet a result calculating incidence-only (no tangency) numbers for elliptic space curves in any dimensional projective space. Secondly, because the tangency divisor on \( \overline{\mathcal{M}}_{1,n}(r, d) \) is related to the divisor of elliptic curves with fixed \( j \)-invariant, hence one also needs to compute the characteristic numbers of elliptic curves with fixed \( j \)-invariant. The latter was solved in a separate paper by the author ([N1]). The algorithms for characteristic numbers of genus two plane curves were given in [GKP], although no actual numbers were shown.

In this paper, we solve the full characteristic numbers problem for elliptic curves in projective spaces by using the moduli space of stable maps. Note that, even in the incidence-only case our result is already new. For example, no incidence-only characteristic numbers for elliptic curves in \( \mathbb{P}^4 \) and \( \mathbb{P}^5 \) have been computed. The number of elliptic curves in \( \mathbb{P}^3 \) were computed in [G] and the proof was scheduled to appear in another paper. However, the
The author of this paper was unable to locate it. The incidence-only numbers of elliptic curves in $\mathbb{P}^3$ were also computed in [V1]. In that paper, only the numbers up to degree 4 elliptic curves were computed, but the method could compute numbers of any degree. The incidence-only numbers of degree 4 elliptic curves in $\mathbb{P}^3$ were first computed in [AV] using a classical argument.

The approach of this paper is as follows. We first compute incidence-only numbers for elliptic space curves using Getzler relation. This is analogous to the enumeration of rational space curves using WDVV equation, but with a small twist. First we will not be using the entire moduli space of stable maps $\overline{M}_{1,n}(r, d)$ as intersecting enumerative classes with the virtual fundamental class will have unwanted contributions from other components (this space is not irreducible). In fact, it was claimed in [G] that the actual count of elliptic curves is a linear combination of genus one and genus zero Gromov-Witten invariants. We will only use the main component, that is the closure of the locus of maps with smooth source curves, denoted by $\overline{M}_{1,r}(r, d)^\ast$. For our purpose, knowledge of the maps on the loci where the main component intersect other components is required, and this is described in Section 4.

In the case of rational curve, the WDVV equation is a rational equivalence of divisors, hence we can pull back via the forgetful morphism and obtain a relation on the stable map space. This is no longer true for the genus one stable map space. The reason is that Getzler relation is a rational equivalence of codimension 2 strata on $\overline{M}_{1,4}$, and that the forgetful morphism $\overline{M}_{1,4}(r, d)^\ast \to \overline{M}_{1,4}$ have fibre dimensions that could jump: for example, the preimage of the stratum $\delta_{2,2}$, has two components, one of which is of codimension 2, and the other is a Weil divisor. We can get around this as follows. First we use enough enumerative constraints to cut down the space $\overline{M}_{1,n}(r, d)^\ast$ into a 2-dimensional family. Then we pushforward via the forgetful morphism, and then intersect with Getzler’s relation. As a result we obtain a relation of the enumerative invariants of elliptic curves with that of rational curves, rational cuspidal curves, and elliptic curves with fixed $j$-invariants. The first is well-known, and the latter two were computed in [N2] and [N1] respectively.

To go from incidence-only to full characteristic numbers, we study the relation between the tangency divisor and the incident and boundary divisors. We will not obtain a rational equivalence, due to the present of enumeratively irrelevant divisors, those that are intersections of $\overline{M}_{1,n}(r, d)^\ast$ with other components. However, we obtain a numerical equivalence whenever we intersect with only curves in $\overline{M}_{1,n}(r, d)^\ast$ that has empty intersection with the irrelevant divisors. This is true for 1-dimensional families of elliptic curves cut down by enumerative constraints, so this is enough for our purpose.

The structure of this paper is as follows. In Section 2, we introduce basic notions such as various stacks of stable maps and the enumerative constraints. In Section 3 we review Getzler’s relation and give an example of using the relation to count elliptic curves. In Section 4, we give the recursion computing incidence-only numbers of elliptic space curves based on Getzler’s relation. In Section 5, we give the recursions computing full characteristic numbers of elliptic space curves. We end with some tables with numerical examples in Section 6.
The author would like to thank Ravi Vakil for many helpful suggestions and some of the arguments used in this paper, and for suggesting this problem.

2. Definitions and Notations

2.1. The moduli space of rational and elliptic curves in $\mathbb{P}^r$. We denote $\overline{M}_{0,n}(r,d)$ the Kontsevich compactification of the moduli space of genus zero curves with $n$ marked points of degree $d$ in $\mathbb{P}^r$. Let $\overline{M}_{1,n}(r,d)^*$ be the main component of the moduli space of stable maps of genus one, that is, the closure of the locus of maps with smooth domains. We will also use the notation $\overline{M}_{0,S}(r,d)$ and $\overline{M}_{1,S}(r,d)$ where the markings are indexed by a set $S$.

2.2. The constraints and the ordering of constraints. We will be concerned with the number of curves passing through a constraint, and each constraint is denoted by a $(r+1)$-tuple $\Delta$ as follows:

(i) $\Delta(0)$ is the number of hyperplanes that the curves need to be tangent to.

(ii) For $0 < i \leq r$, $\Delta(i)$ is the number of subspaces of codimension $i$ that the curves need to pass through.

In [N1] and [N2], the constraints may have $(r+2)$ coordinates because we want to impose conditions on the node or cusp, but we will not need that here.

Note that because in general a curve of degree $d$ will always intersect a hyperplane at $d$ points, introducing an incident condition with a hyperplane has the same effect as that of multiplying the enumerative number by $d$. For example, if we ask how many genus zero curves of degree 4 in $\mathbb{P}^3$ that pass through the constraint $\Delta = (1,2,3,4,0)$, that means we ask how many genus zero curves of degree 4 pass through three lines, four points, are tangent to one hyperplane, and then multiply that answer by $4^2$. We will also refer to $\Delta$ as a set of linear spaces, hence we can say, pick a space $a$ in $\Delta$.

We consider the following ordering on the set of constraints, in order to prove that our algorithm will terminate later on. Let $r(\Delta) = -\sum_{i \geq 1} i^2 \Delta[i]$, and this will be our rank function. We compare two constraints $\Delta, \Delta'$ using the following criteria, whose priority are in the following order. We only proceed to using the next criterion if using the current one give us a tie.

- If $\Delta(0) > \Delta'(0)$ then $\Delta < \Delta'$.
- If $\Delta(0) = \Delta'(0)$ and $\Delta$ has fewer non-hyperplane elements than $\Delta'$ does, then $\Delta < \Delta'$.
- If $r(\Delta) < r(\Delta')$ then $\Delta < \Delta'$.

Informally speaking, characteristic numbers where the constraints are more spread out at two ends are computed first in the recursion. We write $\Delta = \Delta_1 \Delta_2$ if $\Delta = \Delta_1 + \Delta_2$ as a partition of the set of linear spaces in $\Delta$. 

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2.3. The stacks $\mathcal{R}, \mathcal{N}, \mathcal{C}, \mathcal{N}\mathcal{R}, \mathcal{R}\mathcal{R}, \mathcal{E}, \mathcal{E}\mathcal{R}, \mathcal{E}\mathcal{R}\mathcal{R}_2$. We list the following definitions of stacks of stable maps that will occur in our recursions.

1) Let $\mathcal{R}(r,d)$ be the usual moduli space of genus zero stable maps, $\overline{M}_{0,0}(r,d)$.

2) Let $\mathcal{N}(r,d)$ be the closure in $\overline{M}_{0,\{A,B\}}(r,d)$ of the locus of maps of smooth rational curves $\gamma$ such that $\gamma(A) = \gamma(B)$. Informally, $\mathcal{N}(r,d)$ parametrizes degree $d$ rational nodal curves in $\mathbb{P}^r$.

3) Let $\mathcal{C\mathcal{U}}(r,d)$ be the closure in $\overline{M}_{0,\{C\}}(r,d)$ of the locus of maps of smooth rational curves $\gamma$ such that the differential $\gamma'(C)$ is zero. Informally, $\mathcal{C}(r,d)$ parametrizes degree $d$ rational cuspidal curves in $\mathbb{P}^r$.

4) Let $\mathcal{E}(r,d)$ be the main component of the moduli space of genus one stable maps $\overline{M}_{1,0}(r,d)$. That is $\mathcal{E}(r,d) = \overline{M}_{1,0}(r,d)^*$.

5) For $d_1, d_2 > 0$, let $\mathcal{R}\mathcal{R}(r,d_1,d_2)$ be $\overline{M}_{0,\{C\}}(r,d_1) \times \overline{M}_{0,\{C\}}(r,d_2)$ where the fibre product is taken over evaluation maps $e_{C}$ to $\mathbb{P}^r$.

6) Similarly we can define $\mathcal{N}\mathcal{R}(r,d_1,d_2)$, $\mathcal{E}\mathcal{R}(r,d_1,d_2)$ (see figure 1).

7) For $d_1, d_2 > 0$, let $\mathcal{R}\mathcal{R}_2(r,d_1,d_2)$ be the closure in $\overline{M}_{0,\{A,C\}}(r,d_1) \times \mathbb{P}^r \overline{M}_{0,\{B,C\}}(r,d_2)$ (the projections are evaluation maps $e_{C}$) of the locus of maps $\gamma$ such that $\gamma(A) = \gamma(B)$. We call maps in this family rational two-nodal reducible curves.

8) For $d_1, d_2, d_3 > 0$, let $\mathcal{E}\mathcal{R}\mathcal{R}(r,d_1,d_2,d_3)$ be $\overline{M}_{0,\{C\}}(r,d_1) \times e_{C} \overline{M}_{1,\{C,D\}}(r,d_2) \times e_{D} \overline{M}_{0,D}(r,d_3)$. Similarly, we can define $\mathcal{E}\mathcal{R}\mathcal{R}(d_1,d_2,d_3)$ for $d_1, d_2, d_3 > 0$.

9) We define $\mathcal{J}(r,d)$ to be the closure in $\mathcal{E}(r,d)$ of the locus of maps whose domains are smooth and have a fixed but generic $j$-invariant. The enumerative geometry of this stack is studied in [N1].

Fig 1. Pictorial description of a general curve in the stacks $\mathcal{R}, \mathcal{C\mathcal{U}}, \mathcal{N}, \mathcal{R}\mathcal{R}, \mathcal{N}\mathcal{R}, \mathcal{R}\mathcal{R}_2$. 
2.4. **Stacks of stable maps with constraints.** Let $\mathcal{F}$ be a stack of stable maps of curves into $\mathbb{P}^r$. For a constraint $\Delta$, we define $(\mathcal{F}, \Delta)$ be the closure in $\mathcal{F}$ of the locus of maps that satisfy the constraint $\Delta$. For maps of reducible source curves, tangency condition include the case where the image of the node lies on the tangency hyperplane, as the intersection multiplicity is 2 in this case. For a stack $\mathcal{F}$ that is supported on a finite number of points then we denote $\#\mathcal{F}$ to be the stack-theoretic length of $\mathcal{F}$.

If $\mathcal{F}$ is a closed substack of the stacks $\mathcal{N}\mathcal{R}, \mathcal{R}\mathcal{R}, \mathcal{E}\mathcal{R}$ then we denote $(\mathcal{F}, \Gamma_1, \Gamma_2, k)$ to be the closure in $\mathcal{F}$ of the locus of maps $\gamma$ such that the restriction of $\gamma$ on the $i$–th component satisfies constraint $\Gamma_i$, and that the image of the node lies on a codimension $k$ subspace. We use the notation $(\mathcal{F}, \Delta, k)$ if we don’t want to distinguish the conditions on each component. If $k$ is 0 we omit it from the notation.

If $\mathcal{F}$ is a closed substack of the stack $\mathcal{R}\mathcal{R}_2$ then we denote $(\mathcal{F}, \Gamma_1, \Gamma_2)$ to be the closure in $\mathcal{F}$ of the locus of maps $\gamma$ such that the restriction of $\gamma$ on the $i$–th component satisfies constraint $\Gamma_i$. We use the notation $(\mathcal{F}, \Delta)$.

If $\mathcal{F}$ is a closed substack of the stacks $\mathcal{E}\mathcal{R}\mathcal{R}, \mathcal{E}\mathcal{E}\mathcal{R}$ then we denote $(\mathcal{F}, \Gamma_1, \Gamma_2, \Gamma_3)$ to be the closure in $\mathcal{F}$ of the locus of maps $\gamma$ such that the restriction of $\gamma$ on the $i$–th component satisfies constraint $\Gamma_i$. We use the notation $(\mathcal{F}, \Delta)$ if we don’t want to distinguish the conditions on each component.

The enumerative geometry of all the stacks defined above are known, except for the stacks that involve $\mathcal{E}$. But the enumerative geometry of the stacks $\mathcal{E}\mathcal{R}, \mathcal{E}\mathcal{E}\mathcal{R}, \mathcal{E}\mathcal{R}\mathcal{R}$ can be easily deduced from that of $\mathcal{E}$ (see [N1], Section 3). Note the small difference between the notation here and in [N1], [N2], as we have remove some conditions on the nodes as they are not necessary.

### 3. Getzler’s relation on $\overline{M}_{1,4}$

We review some of the basic intersection theory on $\overline{M}_{1,4}$ and especially Getzler’s relation. For a more complete treatment see [G]. Consider the moduli space $\overline{M}_{1,4}$ with the $\mathbb{S}_4$ action
by permuting the marked points. In [G], the following $S_4$-invariant codimension 2 strata are defined:

Fig 3. The $S_4$-invariant codimension 2 strata of $\overline{M}_{1,4}$

Getzler computed the intersection matrix of these cycles and found a non-trivial null vector

$$12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_{\beta} \cong 0$$

These cycles are understood stack-theoretically. Pandharipande gave a beautiful argument in [P3] showing that the above relation is actually a rational equivalence, by using the auxiliary space of admissible covers.

Note that it is easy to get relations on $\overline{M}_{1,n}$ induced by WDVV. Let $\delta_0$ be the boundary stratum of irreducible nodal curves on $\overline{M}_{1,n}$. We have a map $\eta : \overline{M}_{0,n+2} \to \overline{M}_{1,n}$ by identifying the last two marked points. Pushing forward the WDVV relation on $\overline{M}_{0,n+2}$ yields a relation on $\overline{M}_{1,n}$. Unfortunately, these type of relations does not help us counting elliptic curves cause the general curves in the strata involved in the relations do not have a smooth elliptic component.

Now as an example, we use Getzler’s relation to compute a trivial example.

**Example 3.1.** There is 1 elliptic cubic in $\mathbb{P}^3$ that passes through 3 points and 6 lines.

**Proof.** The 3 point conditions determine the plane of the elliptic cubic. The other 6 conditions translate to 6 other point conditions for a smooth plane cubic. There is 1 elliptic plane cubic through 9 points is a fact followed easily from linear algebra. We give another proof using Getzler’s relation. Call the number of elliptic cubics satisfying the given constraint $X$. First we consider a 2-dimensional of $F$ maps $\gamma$ on $\overline{M}_{1,A,B,C,D}(3,3)^*$ satisfying the following conditions:
- $\gamma(A), \gamma(B), \gamma(C), \gamma(D)$ each belongs to a general plane.
- The image of $\gamma$ passes through 3 point and 4 lines. By elementary dimension counting it is easy to see that $\mathcal{F}$ is 2 dimension. Let $p : \overline{M}_{1,4}(r, d)^* \to \overline{M}_{1,4}$ be the forgetful morphism. Now we intersection $\mathcal{G} = p_\ast(\mathcal{F})$ with the Getzler’s relation.

- $\mathcal{G} \cap \delta_{2,2}$: There are no map of elliptic curves with degree 1. Maps of degree 2 are double covers, but with incidence-only constraints, these will not contribute. Thus the degree on the elliptic components are 3 and the two rational twigs are contracted. (We will see in Section 4 and 5 that the maps that contracted the elliptic component do not contribute either). For each possible way of distributing the marked points on the two rational twigs we see that the count is $X$, thus we have $3X$ in total. Multiply this by the coefficient of $\delta_{2,2}$ we get $36X$.

- $\mathcal{G} \cap \delta_{2,3}$: Argue similarly, the two rational twigs must contract. Then the node on the elliptic curve must satisfy incidence conditions of 3 of the points of $A, B, C, D$ (whichever on the contracted components). Thus we get the numbers of elliptic curves passing through 4 points and 4 lines, multiplied by 3 due to a marked point on the elliptic component satisfying a hyperplane condition. But this is 0 cause an elliptic cubic is planar and cannot pass through 4 general points.

- Similarly $\mathcal{G} \cap \delta_{2,2}$ and $\mathcal{G} \cap \delta_{2,4}$ are 0.

- $\mathcal{G} \cap \delta_{0,3}$: If the rational twig is contracted, then we get the numbers of nodal plane cubics passing through 4 points and 3 lines, which is 0. If the genus 1 component is contracted, then by a result in section 4, the resulting image curve has a cusp (corresponding to the node on the domain). The cusp must satisfy an hyperplane incidence condition. By dimension counting, the contribution in this case is also 0.

- $\mathcal{G} \cap \delta_{0,4}$: Similarly, if the rational twig get contracted, then there is no contribution. If the genus 1 component gets contracted, then we the number of rational cuspidal cubic curves passing through 3 points and 4 lines. Translating this to the plane curve counting problem we get the number of rational cuspidal cubic passing through 7 points, which is 24. There are 4 marked points each satisfying a hyperplane condition, and each map has automorphism group of order 2. Thus the total contribution is $3^4 \cdot 24 / 2 = 972$.

- $\mathcal{G} \cap \delta_5$: In general, counting maps with this type of domains ($\mathcal{R}\mathcal{R}_2$) is not trivial, and an algorithm for this is given in [N1]. In this particular example, however, it is easy. If any of the twig gets contracted, we have rational nodal curve with conditions on the node, but by simple dimension counting we see that there is no contribution. Thus the pair has to be line-conic intersecting at 2 points. The 3 points conditions determine the plane for this pair, so we can translate this into a plane curve counting problem. How many pair of line-conics passing through 7 points. The answer is 1, multiplied by the number of ways to distribute the conditions, which is $\binom{7}{2} = 21$. We also have incidence on the marked points, especially on the conics (each contributes a factor of 2). For each marked points distribution, the contribution is $2^2 = 4$. There are six total ways to distribute the marked points. Thus the total contribution of this stratum is $21 \cdot 4 \cdot 6 = 504$. Multiplying with the coefficient of
the stratum, we get $504 \cdot (-2) = -1008$.

Thus we end up with the equation

$$36X + 972 - 1008 = 0$$

hence $X = 1$.

4. Incidence-only characteristic numbers of elliptic space curves via Getzler’s relation

First we need a lemma to tell us which maps can occur as boundary of the main component $\overline{M}_{1,n}(r,d)^*$. 

The Euler characterestic is an important invariant in a family of curves. It is 1 minus the arithmetic genus. The Euler characteristic can be defined for any scheme $X$ as the Euler characteristic of its structure sheaf $\chi(X)$. The following lemma is useful to compute Euler characteristic of reducible curves.

**Lemma 4.1.** Euler characteristic satisfies the inclusion-exclusion property. That is, if $S_1, S_2$ are two subschemes of an ambient scheme, then $\chi(S_1 \cup S_2) + \chi(S_1 \cap S_2) = \chi(S_1) + \chi(S_2)$

**Proof.** We have the following exact sequence of sheaf on $S_1 \cup S_2$:

$$0 \to \mathcal{O}_{S_1 \cup S_2} \to \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \to \mathcal{O}_{S_1 \cap S_2} \to 0$$

where all the sheaves on smaller schemes are pushforwarded to $S_1 \cup S_2$. The identity follows from the fact that Euler characteristic is additive along an exact sequence. □

**Proposition 4.2.** Let $\gamma : (C,S) \to \mathbb{P}^r$ be a map in $\overline{M}_{1,n}(r,d)^*$ ($d \neq 1$) from a nodal, $n$-pointed curve $C$ of arithmetic genus one that contracts a connected union of irreducible components having total arithmetic genus one. Let $R_j$'s ($1 \leq j \leq m$) be the non-contracting rational tails and let $n_j$ be the intersection of $R_j$ with the contracted subcurve. Then the images of the tangent vectors to $R_j$'s at $n_j$'s are not independent.

**Proof.** First note that contracted subcurve has arithmetic genus one, hence each $R_j$ intersects it at exactly one point. Let $\mathcal{C} \to B$ be a one dimensionsinal family with general fibers smooth of genus 1 and the central fiber being $C$. If $\gamma$ contracts the central fiber then it factors through a contraction $c : \mathcal{C} \to \mathcal{C}'$. The central fiber of $\mathcal{C}'$, denoted by $C'$ consists $m$ rational twigs connected at one node. However, the tangent vectors to these twigs at the node are not independent for the following reason. The Euler characteristic of the general fiber is 0 (equals 1− the arithmetic genus), and this is a constant of the fibers in the family. If the tangent vectors were independent, then the node (with the intersection scheme structure ) is reduced, and by a repeat application of Lemma 4.1, we see that the Euler characteristic would be 1. It follows that the images of the tangent vectors in $\mathbb{P}^r$ are also not independent. □
Using the lemma, one can see that the following loci are intersections of the main component with other components. For $1 \leq m \leq r$, $\overline{M}_{1,n}^m(r,d)$ is the closure of the locus of maps $\gamma$ having the following property:
- The domain of $\gamma$ has a smooth elliptic component joined with $m$ rational tails $R_j$ at the node $n_j$.
- $\gamma$ has degree 0 on the elliptic component, and the images of the tangent vectors to $R_j$ at $n_j$ are linearly dependent.

For $m = 1$, the image curve has a cusp, for $m = 2$, the image curve is a two-component rational curves meeting at a tacnode. In general, for $m \leq r$ the image curve has a $m$-fold elliptical singularity. For $m > r$, $\overline{M}_{1,n}^m(r,d)$ lies entirely on $\overline{M}_{1,n}(r,d)^*$.

We are now ready to derive a recursion counting elliptic space curves. First we need some notations. Let $\Delta$ be a constraint such that $\Delta(0) = 0$, and assume $i \geq 2$ is an index such that $\Delta(i) \geq 2$. Let $p', q'$ be two subspaces in $\Delta$ of codimension $i$. Let $h, k$ be two general hyperplanes in $\mathbb{P}^r$ and let $p, q$ be subspaces of codimension $i - 1$. The following constraints are derived from $\Delta$:
- $\tilde{\Delta}$ by removing $p'$ and $q'$.
- $\Delta_0$ by removing $p', q'$ but adding a codimension $i - 1$ and a codimension $i + 1$ subspaces.
- $\Delta_1$ by removing $p', q'$ but adding a codimension $2$ and a codimension $2(i - 1)$ subspaces.
- $\Delta_2$ by removing $p', q'$ but adding a hyperplane and a codimension $2i - 1$ subspace.
- $\Delta_c$ by removing $p', q'$ but adding $p, q, h, k$. We denote $1_{ST}$ be the indicator function where $ST$ is a logical statement. Let $S_4$ acts on the set $\{p, q, h, k\}$. It is easy to see that $\Delta_i, 0 \leq i \leq 2$ is of lower rank than that of $\Delta$.

**Theorem 4.3.** We have the following recursive formula, providing the left-hand side is finite (it is understood that if a constraint contains the empty subspace then the corresponding enumerative term is automatically 0).
\[
\#(\mathcal{E}(d, r), \Delta) = \frac{1}{12(2 + 1_{i=2})} \left[ -12 \#(\mathcal{E}(r, d), \Delta_1) \right. \\
- \frac{12}{4} \sum_{d_1 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1, r), \tilde{\Delta}), (\beta(h) \cap \beta(k)), (\beta(p), \beta(q))) \\
- \frac{12}{8} \sum_{d_1 + d_2 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_1, d_e, d_2), \tilde{\Delta}), (\beta(h)), (\beta(k))), \emptyset, (\beta(p), \beta(q))) \\
+ \frac{4}{2} \sum_{d_1 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1, r), \tilde{\Delta}), (\beta(h)), (\beta(k)), (\beta(p) \cap \beta(q))) \\
+ \frac{4}{2} \sum_{d_1 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1, r), \tilde{\Delta}), (\beta(h)), (\beta(p), \beta(q)), \beta(k)) \\
+ 24 \#(\mathcal{E}(d, r), \Delta_0) + 24 \#(\mathcal{E}(d, r), \Delta_2) \\
+ \frac{4}{2} \sum_{d_1 + d_2 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1, d_2), \tilde{\Delta}), (\beta(h)), (\beta(k)), (\beta(p), \beta(q))) \\
+ \frac{2}{4} \sum_{d_1 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1), \tilde{\Delta}), \emptyset, (\beta(h), \beta(k), \beta(p) \cap \beta(q))) \\
+ \frac{2}{4} \sum_{d_1 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1), \tilde{\Delta}), (\beta(h), (\beta(p), \beta(q)))) \\
- \frac{6}{6} \sum_{d_1 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1), \tilde{\Delta}), (\beta(h), (\beta(k), (\beta(p), (\beta(q)))) \\
- \frac{6}{6} \sum_{d_1 + d_2 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1, d_2), \tilde{\Delta}), (\beta(h), (\beta(k), (\beta(p), (\beta(q)))) \\
- \frac{6}{6} \sum_{d_1 + d_2 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1, d_2), \tilde{\Delta}), (\beta(h), (\beta(k), (\beta(p), (\beta(q)))) \\
- \frac{6}{6} \sum_{d_1 + d_2 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_e, d_1, d_2), \tilde{\Delta}), (\beta(h), (\beta(k), (\beta(p), (\beta(q)))) \\
+ \frac{2}{8} \sum_{d_1 + d_2 + d_e = d} \sum_{d \in S_4} \#((\mathcal{ER}(d_1, d_2), \tilde{\Delta}), (\beta(h), (\beta(k), (\beta(p), (\beta(q))))]
\]

**Proof.** We consider the moduli space \(\overline{M}_{1,4}(r, d)^*\), where there is a one-to-one correspondence \(\mu: \text{set of marked points} \to \{h, k, p, q\}\). Let \(\mathcal{F} \subset \overline{M}_{1,4}(r, d)^*\) be the subfamily cut out by the constraint \(\tilde{\Delta}\) and the condition that \(\gamma(P_i) \in \mu(P_i)\) for each mark point \(P_i\). \(\mathcal{F}\) is either empty or 2-dimensional. If \(\mathcal{F}\) is empty then all the summands in the equation are 0 so there is nothing to prove. Assume \(\mathcal{F}\) is 2-dimensional. Let \(\mathcal{G}\) be the pushforward of \(\mathcal{F}\) under the forgetful morphism \(\overline{M}_{1,4}(r, d) \to \overline{M}_{1,4}\). We intersect \(\mathcal{G}\) with Getzler’s relation.

- \(\mathcal{G} \cap \delta_{2,2}\): If the elliptic component is contracted, then by Proposition 4.2, there are two possibilities for the image curves. It can be a two-component rational curves intersecting at a
tacnode. By simple dimension counting (it lies in a family of one fewer dimension than that of a rational cuspidal curve) we see that the image curves (of those in $\mathcal{F}$ that contract the elliptic component) move in a family of 3 dimension fewer than $\mathcal{F}$. But $\mathcal{F}$ is 2-dimensional so, this loci must be empty. The other possibility is again two-component rational curve, but one has a cusp at the intersection. Dimension counting again shows that this locus on $\mathcal{F}$ is empty. (Note that if $\mathcal{F}$ has at least 3 dimensional , then these loci are generally not empty and are in fact a divisor on $\mathcal{F}$. However, a contributor to the dimensions of these loci comes from varying the two nodes on the elliptic components, which is irrelevant for enumerative reason). If both rational components are contracted, then we either get $(2 + 1_{i=2}) \#(\mathcal{E}(r, d), \Delta)$ depending on if $p$ is 1-dimensional. If $p$ is not 1-dimensional, then we also get $\#(\mathcal{E}(r, d), \Delta_1)$. Next, if 1 of the rational tails is contracted, then we get

$$\frac{1}{4} \sum_{d_1 + d_2 = d} \sum_{\beta \in \mathcal{S}_4} \#((\mathcal{E}(d_2, d_1), r), (\beta(h) \cap \beta(k)), (\beta(p), \beta(q)))$$

The factor $1/4$ comes from the fact that permuting marked points on the same rational tail gives us the same stratum. If non of the rational tails are contracted, then the contribution is

$$\frac{1}{8} \sum_{d_1 + d_2 + d_3 = d} \sum_{\beta \in \mathcal{S}_4} \#((\mathcal{E}(d_1, d_2, r), \Delta), (\beta(h), \beta(k)), 0, (\beta(p), \beta(q)))$$

The factor $1/8$ comes from the fact that permuting marked points on each tail and permuting the two tails give us the same stratum (given that we consider all possible degree distributions, contrary to the situation in the previous remark, where one of the tail is contracted while the other is not). 12 is the coefficient of $\delta_{2,2}$ in Getzler’s relation.

- $\mathcal{G} \cap \delta_{2,3}$, $\mathcal{G} \cap \delta_{2,4}$, $\mathcal{G} \cap \delta_{3,4}$ can be found similarly, and these summands explain all the terms until the first $\mathcal{N}(r, d)$ term.

- $\mathcal{G} \cap \delta_{0,3}$: If the nodal elliptic component is contracted, then by Proposition 4.2, we must have a cuspidal curve joined with a smooth rational curve by a node. But this locus on $\mathcal{F}$ must be empty since a smooth rational cuspidal curve is moving in a 2 dimensions fewer family than that of a smooth elliptic curve (a general member of $\mathcal{F}$). The rational tail can not be contracted also : if it were contracted , then moving the marked points on the rational tail (one-dimensional family) change the map but does not change the image curves, meaning than the image curves can not satisfy all the enumerative constraint. Thus non of the components can be contracted, and we have the contribution from this stratum is

$$\frac{1}{6} \sum_{d_1 + d_2 + d_3 = d} \sum_{\beta \in \mathcal{S}_4} \frac{1}{2} \#((\mathcal{N}(d_1, d_2), \Delta), (\beta(h)), (\beta(k), \beta(p)), (\beta(q)))$$

The factor $1/6$ comes from the fact that we can permute the marked points on the rational component, and the factor $1/2$ comes from the fact that we can permute the branches over the node of the image for maps in $\mathcal{N}(r, d)$.

- $\mathcal{G} \cap \delta_{0,4}$: Argue similarly, the rational tail can not be contracted. If the nodal elliptic component is not contracted, then the contribution is

$$\sum_{d_1 + d_2 = d} \frac{1}{2} \#((\mathcal{N}(d_1, d_2), \Delta), 0, (h, k, p, q))$$
If the nodal elliptic component is contracted, then the contribution is
\[ \frac{1}{2} \#(S, \Delta_c) \]

The factor 1/2 comes from the fact that any map that contracts the nodal elliptic component has automorphism group of order 2.

- $\mathcal{G} \cap \delta_b$: Argue as above, we can show that none of the rational bridges can be contracted. The contribution is
\[ \frac{1}{8} \sum d_1 + d_2 = d \sum_{\beta \in S_4} \frac{1}{2} \#((\mathcal{R}\mathcal{R}_2(d_1, d_2, r), \tilde{\Delta}), (\beta(h), \beta(k)), (\beta(p), \beta(q))) \]

We can permute the marked points on each component and permute the two component themselves, hence the factor 1/8. The factors 1/2 comes from the fact that we can permute two branches over one of the node of the image for maps in $\mathcal{R}\mathcal{R}_2(r, d_1, d_2)$.

Except the terms involving $\mathcal{E}(r, d)$, all other terms can be computed either from algorithms in [N1] and [N2] or recursively. For example $\mathcal{E}\mathcal{R}$ terms $\mathcal{E}\mathcal{R}\mathcal{R}$ and $\mathcal{R}\mathcal{E}\mathcal{R}$ can be computed using the “splitting the diagonal” method described in Section 3 of [N1]. The term $\mathcal{E}(r, d_i)$ for $i = 0, 1, 2$ can be assumed known by induction since $\Delta_i$ is of lower rank than that of $\Delta$.

Note that we can choose $p', q'$ being any two subspaces in $\Delta$ having same codimensions: they could be two points, two lines, two planes etc. For each choice we have a different recursion. Hence a good check of the formulas is that different choices giving same numbers. We have confirmed this is true for all elliptic space curves in $\mathbb{P}^3, \mathbb{P}^4, \mathbb{P}^5$ of degree at most 5, and so far no contradiction has been found.

5. Characteristic numbers of elliptic space curves

In this section we give a recursive formula counting elliptic space curves with tangency conditions. First we need the following lemma.

**Lemma 5.1.** Let $\Delta$ be a constraint, and $\mathcal{F}$ a family of maps in $\overline{M}_{1,0}(r, d)^*$ such that $\mathcal{F} = (\overline{M}_{1,0}(r, d)^*, \Delta)$ is one-dimensional. Then the maps in $\mathcal{F}$ do not contract an elliptic component. Moreover, the number of maps in $\mathcal{F}$ whose images are tangent to a general hyperplane is given by
\[ I + \frac{dJ}{12} + \sum_{i=0}^{r-1} iR_i \]

where $I$ is the number of maps in $\mathcal{F}$ satisfying top incident condition, $J$ is the number of maps in $\mathcal{F}$ whose domain is a smooth elliptic curve having a fixed $j$-invariant ($j$ is not 0 or 1728), and $R_i$ is the number of rational tails of domains of maps in $\mathcal{F}$ that are mapped with degree $i$.

**Proof.** The loci of maps that contract an elliptic component are at most divisors on $\overline{M}_{1,0}(r, d)$. Since we can change such maps by moving the nodes on the elliptic component
without changing the images, the images must move in a family of at most 2 dimensions fewer than those of maps of smooth elliptic curves. Since \( \mathcal{F} \) is one-dimensional, and enumerative constraints impose conditions on the image curves only, this shows those loci must be empty. Let \( \mathcal{T}, \mathcal{H} \) be the divisor on \( \mathcal{F} \) corresponding to tangency condition and top incidence condition respectively. Assume the \( \mathcal{F} \) is represented by the total family \( \pi : \mathcal{C} \to \mathcal{F} \) and a map \( \mu : \mathcal{C} \to \mathbb{P}^r \). Let \( H \) be a general hyperplane in \( \mathbb{P}^r \). Then \( \mathcal{D} = \mu^{-1}(H) \) is a smooth curve in \( \mathcal{C} \) and \( \mathcal{T} \) is given by the branch divisor of the covering \( \pi : \mathcal{D} \to \mathcal{F} \). Thus

\[ \mathcal{T} \cdot \mathcal{F} = \pi_*(K_\mathcal{D} - \pi^*K_\mathcal{F}) = \pi_*(K_\mathcal{D} - (K_\mathcal{C} + \omega_\pi)_\mathcal{D}) = \pi_*(\mathcal{D}(\mathcal{D} + \omega_\pi)) \]

where the last equality follows from adjunction. It is easy to see that \( I = \mathcal{H} \cdot \mathcal{F} = \pi_*(\mathcal{D}^2) \). Let \( R \) be the divisor of \( \mathcal{C} \) corresponding to rational tails, and let \( \delta_0 \) be the divisor of \( \mathcal{C} \) corresponding to nodal elliptics. Then we have

\[ \omega_\pi = \frac{\delta_0}{12} + R \]

This follows from Theorem 12.1 in [BPV], corrected by the rational tails. Since \( \delta_0 \) is equivalent to the locus of fibers having fixed \( j \)-invariant, we have

\[ \mathcal{T} \cdot \mathcal{F} = \pi_*(\mathcal{D}^2) + \pi_*(\mathcal{D}\omega_\pi) = I + \frac{dJ}{12} + \sum_{i=1}^{r-1} iR_i \]

Using the lemma, we can easily deduce the following result.

**Theorem 5.2.** Let \( \Delta \) be a constraint with \( \Delta(0) > 0 \). Let \( \Delta' \) be the constraint derived from \( \Delta \) by removing a tangency condition and adding a top incidence condition. Let \( \Delta'' \) be the constraint derived from \( \Delta \) by removing a tangency condition. Then we have the following equality, provided the left-hand side is finite.

\[ \#(\mathcal{E}(r,d), \Delta) = \#(\mathcal{E}(d,r), \Delta') + \frac{d}{12} \#(\mathcal{J}(r,d), \Delta') + \sum_{d_1+d_e=d} d_1 \#(\mathcal{ER}(d_e,d_1), \Delta''). \]

This gives a recursive formula for all characteristic numbers of elliptic space curves with at least one tangency condition. The characteristic numbers \( \#(\mathcal{J}(r,d), \Delta') \) were computed in [NT].

6. Numerical examples

In this section we give numerical examples of characteristic numbers of elliptic curves of low degree in \( \mathbb{P}^2, \mathbb{P}^3, \mathbb{P}^4, \mathbb{P}^5 \). For elliptic curves in \( \mathbb{P}^2 \) and \( \mathbb{P}^3 \) of low degrees, we give all characteristic numbers. For curves in higher dimensional projective spaces, we give a random sample of these numbers as there are too many possible characteristic numbers (in addition to long running time). Note that degree 2 elliptic curves are understood as degree 2 covers of \( \mathbb{P}^1 \).
| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| (0,6)     | 0         | (1,5)     | 0         | (2,4)     | 0         |
| (3,3)     | 0         | (4,2)     | 2         | (5,1)     | 10        |
| (6,0)     | 45/2      |           |           |           |           |

Table 1. Degree 2 elliptic plane curves.

| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|-----------|-----------|-----------|-----------|
| (0,9)     | 1         | (1,8)     | 2         |
| (3,6)     | 64        | (4,5)     | 256       |
| (6,3)     | 3424      | (7,2)     | 9766      |
| (9,0)     | 33616     |           |           |

Table 2. Degree 3 elliptic plane curves.

| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|-----------|-----------|-----------|-----------|
| (0,12)    | 225       | (1,11)    | 1010      |
| (3,9)     | 18432     | (4,8)     | 73920     |
| (6,6)     | 994320    | (7,5)     | 3230956   |
| (9,0)     | 129996216 |           |           |

Table 3. Degree 4 elliptic plane curves.

| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|-----------|-----------|-----------|-----------|
| (0,15)    | 87192     | (1,14)    | 411376    |
| (3,12)    | 8197344   | (4,11)    | 34294992  |
| (6,9)     | 512271756 | (7,8)     | 1802742368|
| (9,6)     | 1766868832| (10,5)    | 48034104112|
| (12,3)    | 24898451648| (13,2)    | 463227482784|
| (15,0)    | 1048687299072|           |           |

Table 4. Degree 5 elliptic plane curves.
Table 5. Degree 6 elliptic plane curves.

| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|----------|------------|----------|------------|----------|------------|
| (0, 0, 6) | 0          | (0, 2, 5) | 0          | (0, 4, 4) | 0          |
| (0, 6, 3) | 1          | (0, 8, 2) | 14         | (0, 10, 1)| 150        |
| (0, 12, 0)| 1500       | (1, 1, 5) | 0          | (1, 3, 4) | 0          |
| (1, 5, 3) | 4          | (1, 7, 2) | 50         | (1, 9, 1) | 498        |
| (1, 11, 0)| 4740       | (2, 0, 5) | 0          | (2, 2, 4) | 0          |
| (2, 4, 3) | 16         | (2, 6, 2) | 176        | (2, 8, 1) | 1620       |
| (2, 10, 0)| 14640      | (3, 1, 4) | 0          | (3, 3, 3) | 64         |
| (3, 5, 2) | 608        | (3, 7, 1) | 5136       | (3, 9, 0) | 43944      |
| (4, 0, 4) | 0          | (4, 2, 3) | 256        | (4, 4, 2) | 2048       |
| (4, 6, 1) | 15744      | (4, 8, 0) | 127104     | (5, 1, 3) | 976        |
| (5, 3, 2) | 6464       | (5, 5, 1) | 45040      | (5, 7, 0) | 342720     |
| (6, 0, 3) | 3424       | (6, 2, 2) | 18560      | (6, 4, 1) | 116768     |
| (6, 6, 0) | 836480     | (7, 1, 2) | 47936      | (7, 3, 1) | 269440     |
| (7, 5, 0) | 1809040    | (8, 0, 2) | 114248     | (8, 2, 1) | 553176     |
| (8, 4, 0) | 3439024    | (9, 1, 1) | 1024404    | (9, 3, 0) | 5768584    |
| (10, 0, 1)| 1774680    | (10, 2, 0)| 8656240    | (11, 1, 0)| 11875120   |
| (12, 0, 0)| 15480640   |          |            |          |            |

Table 6. Degree 3 elliptic curves in $\mathbb{P}^3$.  

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| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|---|---|---|---|---|---|
| (0, 0, 8) | 1 | (0, 2, 7) | 4 | (0, 4, 6) | 32 |
| (0, 6, 5) | 310 | (0, 8, 4) | 3220 | (0, 10, 3) | 34674 |
| (0, 12, 2) | 385656 | (0, 14, 1) | 4436268 | (0, 16, 0) | 52832040 |
| (1, 1, 7) | 12 | (1, 3, 6) | 96 | (1, 5, 5) | 920 |
| (1, 7, 4) | 9408 | (1, 9, 3) | 99270 | (1, 11, 2) | 1081968 |
| (1, 13, 1) | 12224484 | (1, 15, 0) | 143419320 | (2, 0, 7) | 36 |
| (2, 2, 6) | 288 | (2, 4, 5) | 2720 | (2, 6, 4) | 27312 |
| (2, 8, 3) | 281004 | (2, 10, 2) | 2988144 | (2, 12, 1) | 33049512 |
| (2, 14, 0) | 381061200 | (3, 1, 6) | 864 | (3, 3, 5) | 8000 |
| (3, 5, 4) | 78656 | (3, 7, 3) | 783840 | (3, 9, 2) | 8087616 |
| (3, 11, 1) | 87221808 | (3, 13, 0) | 985671936 | (4, 0, 6) | 2592 |
| (4, 2, 5) | 23360 | (4, 4, 4) | 224256 | (4, 6, 3) | 2145024 |
| (4, 8, 2) | 21331136 | (4, 10, 1) | 223311840 | (4, 12, 0) | 246611936 |
| (5, 1, 5) | 67440 | (5, 3, 4) | 630720 | (5, 5, 3) | 5721424 |
| (5, 7, 2) | 54410016 | (5, 9, 1) | 550239168 | (5, 11, 0) | 591968800 |
| (6, 0, 5) | 191760 | (6, 2, 4) | 1743488 | (6, 4, 3) | 14766080 |
| (6, 6, 2) | 133095808 | (6, 8, 1) | 1293435904 | (6, 10, 0) | 1351435840 |
| (7, 1, 4) | 4724272 | (7, 3, 3) | 36626544 | (7, 5, 2) | 309751664 |
| (7, 7, 1) | 2876272592 | (7, 9, 0) | 29088348480 | (8, 0, 4) | 12532016 |
| (8, 2, 3) | 86940920 | (8, 4, 2) | 681603936 | (8, 6, 1) | 6007997008 |
| (8, 8, 0) | 58587710176 | (9, 1, 3) | 197671204 | (9, 3, 2) | 1413728496 |
| (9, 5, 1) | 11731399560 | (9, 7, 0) | 109792714096 | (10, 0, 3) | 435015624 |
| (10, 2, 2) | 2767553376 | (10, 4, 1) | 21376596768 | (10, 6, 0) | 190821802560 |
| (11, 1, 2) | 5150502848 | (11, 3, 1) | 36418237824 | (11, 5, 0) | 307505812160 |
| (12, 0, 2) | 9269345984 | (12, 2, 1) | 58355286272 | (12, 4, 0) | 460737967360 |
| (13, 1, 1) | 88904673408 | (13, 3, 0) | 645526598016 | (14, 0, 1) | 131356680480 |
| (14, 2, 0) | 853096310656 | (15, 1, 0) | 1076343432320 | (16, 0, 0) | 1321684733280 |

Table 7. Degree 4 elliptic curves in $\mathbb{P}^3$.  

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| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|-------|-----------|-------|-----------|-------|-----------|
| (0, 0.10) | 42 | (0, 0.4) | 354 | (0, 0.7) | 1094 |
| (0.6, 7) | 38049 | (0, 0.8) | 441654 | (0, 10.5) | 5378454 |
| (0, 12.4) | 68292324 | (0, 14.3) | 901654884 | (0, 16.2) | 12358163808 |
| (0, 18.1) | 17559963528 | (0, 20.0) | 2583319387968 | (1, 1.9) | 1094 |
| (1, 3.8) | 10476 | (1, 5.7) | 111774 | (1, 7.6) | 1271974 |
| (1.9, 5) | 15194034 | (1, 11.4) | 189441324 | (1, 13.3) | 2460171444 |
| (1, 15.2) | 33232962528 | (1, 17.1) | 466363099008 | (1.19.0) | 6789367904448 |
| (2, 0.9) | 3340 | (2, 2.8) | 31120 | (2, 4.7) | 324980 |
| (2, 6.6) | 3618784 | (2, 8.5) | 42266004 | (2, 10.4) | 516526368 |
| (2, 12.3) | 6582996888 | (2, 14.2) | 87478828368 | (2, 16.1) | 121057823788 |
| (2, 18.0) | 1741971253648 | (3, 1.8) | 91560 | (3, 3.7) | 935136 |
| (3, 5.6) | 10164264 | (3, 7.5) | 11588944 | (3, 9.4) | 1381799016 |
| (3, 11.3) | 17235919176 | (3, 13.2) | 224820084336 | (3, 15.1) | 306277747088 |
| (3, 17.0) | 43504838611968 | (4, 0.8) | 267008 | (4, 2.7) | 2663824 |
| (4, 4.6) | 28172256 | (4, 6.5) | 312141824 | (4, 8.4) | 3619891072 |
| (4, 10.3) | 44074594080 | (4, 12.2) | 562512674880 | (4, 14.1) | 7529361687168 |
| (4, 16.0) | 105419849015808 | (5, 1.7) | 7515344 | (5, 3.6) | 77029344 |
| (5, 5.5) | 825583024 | (5, 7.4) | 9266866944 | (5, 9.3) | 109570881504 |
| (5, 11.2) | 1365937000128 | (5, 13.1) | 17924770819968 | (5, 15.0) | 24698296518508 |
| (6, 0.7) | 21015744 | (6, 2.6) | 207770304 | (6, 4.5) | 21422543544 |
| (6, 6.4) | 23132708544 | (6, 8.3) | 264540003744 | (6, 10.2) | 3208327374528 |
| (6, 12.1) | 41175810566208 | (6, 14.0) | 557397334146048 | (7, 1.6) | 5532209344 |
| (7, 3.5) | 5451732864 | (7, 5.4) | 56197250608 | (7, 7.3) | 618092874928 |
| (7, 9.2) | 7264503650688 | (7, 11.1) | 90935869598208 | (7, 13.0) | 1207034339515008 |
| (8, 0.6) | 1456801224 | (8, 2.5) | 13615688984 | (8, 4.4) | 132655331216 |
| (8, 6.3) | 1393893072176 | (8, 8.2) | 15805391490496 | (8, 10.1) | 192394855764288 |
| (8, 12.0) | 2498937360190848 | (9, 1.5) | 33442906324 | (9, 3.4) | 304042170552 |
| (9, 5.3) | 302760730440 | (9, 7.2) | 32950822741600 | (9, 9.1) | 388703233243008 |
| (9, 11.0) | 4928894795256768 | (10, 0.5) | 81107025144 | (10, 2.4) | 677080161264 |
| (10, 4.3) | 6326216895824 | (10, 6.2) | 65828029012748 | (10, 8.1) | 747920741035008 |
| (10, 10.0) | 9234940126062048 | (11, 1.4) | 1469692262864 | (11, 3.3) | 12719562619344 |
| (11, 5.2) | 125046496125728 | (11, 7.1) | 1368071583917408 | (11, 9.0) | 16398336070362048 |
| (12, 0.4) | 31332954358848 | (12, 2.3) | 2465727975168 | (12, 4.2) | 227432721556608 |
| (12, 6.1) | 2377092153260928 | (12, 8.0) | 27559370814663168 | (13, 1.3) | 46380697328576 |
| (13, 3.2) | 395961856525056 | (13, 5.1) | 3925472901986688 | (13, 7.0) | 4382607405481728 |
| (14, 0.3) | 85453000115200 | (14, 2.2) | 662634098949120 | (14, 4.1) | 6173065500610048 |
| (14, 6.0) | 66008781304150528 | (15, 1.2) | 1073447856471168 | (15, 3.1) | 927739025688844 |
| (15, 5.0) | 94369223149298688 | (16, 0.2) | 1703768515184128 | (16, 2.1) | 13397726490436608 |
| (16, 4.0) | 128504356404712448 | (17, 1.1) | 18739152863106048 | (17, 3.0) | 167464728764784128 |
| (18, 0.1) | 2570101985252488 | (18, 2.0) | 21014427290822608 | (19, 1.0) | 255958547477177088 |
| (20, 0.0) | 306095919912649728 | | | | |
Table 8. Degree 5 elliptic curves in $\mathbb{P}^3$.

| $\Delta$   | $E_\Delta$ | $\Delta$   | $E_\Delta$ |
|------------|------------|------------|------------|
| (0, 0, 3, 3) | 0          | (0, 2, 2, 3) | 0          |
| (0, 3, 3, 2) | 0          | (0, 6, 0, 3) | 1          |
| (0, 7, 1, 2) | 14         | (0, 8, 2, 1) | 222        |
| (0, 9, 0, 2) | 114        | (1, 4, 2, 2) | 4          |
| (1, 4, 5, 0) | 190        | (1, 6, 4, 0) | 1488       |
| (2, 0, 2, 3) | 0          | (2, 0, 5, 1) | 0          |
| (2, 1, 3, 2) | 0          | (2, 3, 2, 2) | 16         |
| (2, 3, 5, 0) | 640        | (2, 6, 2, 1) | 2280       |
| (2, 7, 3, 0) | 31044      | (3, 4, 4, 0) | 13680      |
| (3, 8, 2, 0) | 536304     | (4, 0, 4, 1) | 512        |
| (4, 2, 3, 1) | 3328       | (4, 3, 1, 2) | 2048       |
| (4, 9, 1, 0) | 8041776    | (5, 2, 1, 2) | 6464       |
| (5, 2, 4, 0) | 101408     | (5, 4, 0, 2) | 31624      |
| (5, 4, 3, 0) | 582888     | (6, 0, 3, 1) | 26848      |
| (6, 2, 2, 1) | 144400     | (6, 3, 0, 2) | 83312      |
| (6, 7, 1, 0) | 39311360   | (7, 0, 4, 0) | 497216     |
| (7, 2, 0, 2) | 203968     | (7, 6, 1, 0) | 76501840   |
| (9, 0, 3, 0) | 8583182    | (9, 4, 1, 0) | 224882706  |
| (9, 6, 0, 0) | 1130248810 | (10, 0, 1, 1) | 10539980   |
| (11, 1, 0, 1) | 72275990  | (15, 0, 0, 0) | 5552993600 |

Table 9. Some characteristic numbers of elliptic space curves of degree 3 in $\mathbb{P}^4$.  

| $\Delta$ | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|---------|-----------|---------|-----------|
| (0, 0, 7, 2) | 29 | (0, 1, 2, 5) | 0 |
| (0, 2, 3, 4) | 4 | (0, 10, 2, 2) | 1004916 |
| (0, 13, 2, 1) | 144007483 | (1, 0, 5, 3) | 38 |
| (1, 3, 5, 2) | 4860 | (1, 4, 6, 1) | 81196 |
| (1, 5, 1, 4) | 920 | (1, 7, 0, 4) | 9408 |
| (1, 7, 6, 0) | 11043310 | (1, 8, 1, 3) | 156968 |
| (1, 8, 4, 1) | 5342984 | (1, 10, 3, 1) | 43094568 |
| (2, 7, 1, 3) | 420000 | (2, 7, 4, 1) | 13098172 |
| (2, 12, 3, 0) | 13916950104 | (2, 13, 1, 1) | 637711884 |
| (2, 14, 2, 0) | 113231319632 | (3, 0, 1, 5) | 0 |
| (3, 1, 2, 4) | 864 | (3, 1, 8, 0) | 958400 |
| (3, 3, 7, 0) | 7666224 | (3, 4, 2, 3) | 139312 |
| (4, 1, 3, 3) | 47136 | (4, 3, 5, 1) | 9606144 |
| (4, 4, 3, 2) | 5203072 | (5, 1, 1, 4) | 67440 |
| (5, 3, 3, 2) | 12588560 | (5, 6, 0, 3) | 49299816 |
| (6, 0, 4, 2) | 4092688 | (6, 6, 4, 0) | 3573253632 |
| (7, 8, 1, 1) | 257686679704 | (7, 13, 0, 0) | 214317637545920 |
| (8, 2, 2, 2) | 965893920 | (9, 2, 3, 1) | 19635766224 |
| (11, 7, 1, 0) | 194300922530150 | (12, 3, 1, 1) | 3865044261124 |
| (12, 5, 0, 1) | 23269075459780 | (13, 1, 0, 2) | 709454579680 |
| (13, 7, 0, 0) | 2593390019349960 | (15, 3, 1, 0) | 680573664677760 |
| (17, 1, 1, 0) | 1046525020177280 | (17, 3, 0, 0) | 6253213828581120 |

Table 10. Some characteristic numbers of elliptic space curves of degree 4 in $\mathbb{P}^4$. 

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### Table 11. Some characteristic numbers of elliptic space curves of degree 5 in $\mathbb{P}^4$. 

| $\Delta$   | $E_\Delta$ |
|------------|------------|
| (0, 4, 9, 1) | 17812920  |
| (1, 4, 7, 2) | 22141610  |
| (2, 1, 8, 2) | 5574120   |
| (2, 19, 2, 0) | 157704159607499400 |
| (3, 13, 0, 3) | 3852663999504 |
| (4, 12, 3, 1) | 32466268747600 |
| (5, 2, 9, 0) | 1521705558528 |
| (5, 11, 0, 3) | 17870726781040 |
| (6, 7, 6, 0) | 267364637315200 |
| (6, 13, 0, 2) | 5661536113375616 |
| (7, 10, 1, 2) | 1261503043660160 |
| (8, 8, 0, 3) | 152112968567744 |
| (10, 4, 1, 3) | 75941963996580 |
| (12, 2, 4, 1) | 4927564446146120 |
| (13, 0, 3, 2) | 686078803431960 |
| (13, 9, 0, 1) | 36003235532769890160 |
| (15, 7, 0, 1) | 76323334247130289600 |
| (18, 1, 0, 2) | 2695498962098743296 |

### Table 12. Some characteristic numbers of elliptic space curves of degree 3 in $\mathbb{P}^5$. 

| $\Delta$   | $E_\Delta$ |
|------------|------------|
| (0, 0, 6, 2, 0) | 0 |
| (0, 7, 2, 1, 1) | 294 |
| (1, 3, 2, 2, 1) | 8 |
| (1, 5, 3, 2, 0) | 2694 |
| (2, 0, 6, 0, 1) | 80 |
| (2, 3, 5, 0, 1) | 80 |
| (2, 4, 0, 4, 0) | 7008 |
| (3, 5, 1, 0, 2) | 6400 |
| (4, 0, 5, 0, 1) | 976 |
| (5, 0, 0, 3, 1) | 10000 |
| (5, 1, 1, 2, 1) | 235888 |
| (6, 0, 4, 0, 1) | 307313568 |
| (6, 7, 1, 1, 0) | 160210908480 |
| (8, 1, 0, 0, 0) | 617311049676 |
| (12, 6, 0, 0, 0) | 4866566832 |
| (14, 1, 0, 1, 0) | 135504 |
| (14, 2, 1, 0, 0) | 63034080 |
| (15, 0, 0, 0, 0) | 5158008 |
| (20, 2, 2, 2, 0) | 242570957664 |
| (23, 1, 1, 1, 1) | 31799786256 |
| (25, 0, 0, 0, 0) | 219076067112 |
| $\Delta$    | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|------------|-----------|----------|-----------|
| (0, 3, 7, 1, 1) | 150696   | (0, 8, 0, 0, 4) | 3220   |
| (0, 9, 0, 1, 3) | 57960    | (0, 10, 3, 0, 2) | 18866664 |
| (0, 17, 0, 1, 1) | 139530931995 | (0, 20, 0, 0, 1) | 11150743642205 |
| (1, 1, 2, 2, 3) | 12       | (1, 6, 3, 1, 2) | 585468   |
| (1, 10, 0, 3, 1) | 56249988 | (2, 0, 2, 6, 0) | 7544    |
| (2, 0, 5, 0, 3) | 1444     | (2, 1, 7, 1, 1) | 85680    |
| (2, 5, 4, 3, 0) | 75690880 | (2, 14, 4, 0, 0) | 36244480608680 |
| (3, 0, 3, 5, 0) | 203200   | (3, 4, 1, 5, 0) | 10274688 |
| (3, 4, 2, 3, 1) | 6044208  | (3, 11, 3, 0, 1) | 299599503152 |
| (4, 1, 6, 1, 1) | 30574272 | (4, 3, 7, 1, 0) | 540207552 |
| (4, 5, 2, 1, 2) | 56706560 | (5, 10, 3, 1, 0) | 22175099750880 |
| (6, 1, 7, 1, 0) | 19987349088 | (6, 9, 1, 1, 1) | 1477032997824 |
| (6, 14, 0, 0, 1) | 555756395786592 | (7, 0, 7, 1, 0) | 36471958824 |
| (8, 5, 2, 1, 1) | 803465851328 | (9, 1, 4, 2, 0) | 479197387872 |
| (9, 11, 0, 0, 1) | 260493865714080 | (11, 2, 2, 1, 1) | 395681280704 |
| (13, 2, 3, 1, 0) | 865530347368728 | (16, 0, 0, 0, 2) | 127384328451776 |

Table 13. Some characteristic numbers of elliptic space curves of degree 4 in $\mathbb{P}^5$. 

| $\Delta$    | $E_\Delta$ | $\Delta$ | $E_\Delta$ |
|------------|-----------|----------|-----------|
| (0, 4, 7, 0, 3) | 21370599 | (0, 5, 0, 3, 4) | 13368   |
| (0, 16, 1, 0, 3) | 6416662050785 | (1, 1, 1, 2, 5) | 120    |
| (2, 24, 2, 0, 0) | 92992972101933954474544 | (5, 6, 1, 3, 2) | 669439268976 |
| (6, 0, 0, 4, 3) | 70316304 | (8, 2, 4, 4, 0) | 24964253347264 |
| (8, 9, 0, 3, 1) | 59596479340997440 | (9, 11, 5, 0, 0) | 467080626814847812288 |
| (10, 4, 4, 0, 2) | 19278514559525472 | (11, 2, 0, 3, 2) | 27526502278424 |
| (11, 4, 0, 1, 3) | 1322240935084886 | (11, 15, 2, 0, 0) | 5731941475630570274830480 |
| (12, 3, 0, 5, 0) | 93646029368101232 | (12, 6, 0, 4, 0) | 798505641168359072 |
| (13, 7, 0, 2, 1) | 55781303615787140368 | (15, 1, 1, 0, 3) | 110267644926473616 |

Table 14 Some characteristic numbers of elliptic space curves of degree 5 in $\mathbb{P}^5$. 

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