The Kauffman bracket and the Jones polynomial in quantum gravity

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Abstract

An analysis of the action of the Hamiltonian constraint of quantum gravity on the Kauffman bracket and Jones knot polynomials is proposed. It is explicitly shown that the Kauffman bracket is a formal solution of the Hamiltonian constraint with cosmological constant ($\Lambda$) to third order in $\Lambda$. The calculation is performed in the extended loop representation of quantum gravity. The analysis makes use of the analytical expressions of the knot invariants in terms of the two and three point propagators of the Chern-Simons theory. Some particularities of the extended loop calculus are considered and the implications of the results to the case of the conventional loop representation are discussed.

1 Introduction

In a companion article [1] the action of the vacuum Hamiltonian constraint of quantum gravity on the third coefficient of the Jones knot polynomial $J_3(\gamma)$ was studied. In that opportunity a formal (analytical) expression for $H_0 J_3(\gamma)$ was indirectly obtained form the fact that the Kauffman bracket is annihilated by $H_\Lambda$ to third order in the cosmological constant. The result shows that $J_3(\gamma)$ can in principle be annihilated by $H_0$ in the loop representation if the topology of the loop $\gamma$ is appropriately restricted at the intersecting points.

To complete the analysis developed in [1] one has to demonstrate that the Kauffman bracket is effectually annihilated by the Hamiltonian constraint with cosmological constant to $O(\Lambda^3)$. The explicit confirmation of this fact is one of the purposes of this article. We make use of the extended loop representation of quantum gravity, where the calculation can be fully developed taking into consideration the analytical expressions of the knot invariants. In spite that
the regularization and renormalization problems associated with the formal calculus can be solved using the point splitting method, we shall limitate the analysis to the formal level for the sake of simplicity.

Another purpose of this article is to present some features of the extended loop calculus. There already exists in the literature detailed references about the extended loop representation and its relationship with the conventional loop representation of quantum gravity. We do not give here a self contained introduction on this subject (see [2] for a general reference). Nevertheless, the use of extended loops involves a particular methodology that is emphasized in this paper. In particular, it is shown that a systematic of calculation exists for the constraints in this representation.

The article is organized as follows: in Sect. 2 the relationship between the Kauffman bracket and the Jones polynomial and their role in quantum gravity are briefly discussed. Also some mathematical background necessary for the following sections are introduced. In Set. 3 the constraints are analyzed from a general point of view in the extended loop framework. In Sect. 4 we evaluate the Hamiltonian on the third coefficient of the Jones polynomial. We compute separately the contributions that appear in the analytical expression of the $J_3$ diffeomorphism invariant. In Sect. 5 the result of $H_0 J_3$ in terms of extended loops is analyzed. In Sect. 5.1 we see that the Hamiltonian does not annihilate the third coefficient of the Jones polynomial for general extended loops. In Sect. 5.2 a geometrical interpretation in terms of ordinary loops is developed. The action of the Hamiltonian with cosmological constant on the Kauffman bracket is considered in Sect. 6. We confirm that this knot invariant is annihilated by $H_{\Lambda}$ to $O(\Lambda^3)$. The conclusions are in Sect. 7 and three appendixes with useful results are added.

2 Preliminars

The introduction of the loop representation [3, 5] has allowed to make substantial advances in the knowledge of the state of space of quantum gravity. For one hand, the diffeomorphism invariance of the theory is automatically implemented by the requirement of the knot invariance of the loop wavefunctions. By this way, knots and the quantum states of gravity appears to be related. This relationship is highlighted by the fact that nonperturbative solutions of the Wheeler-DeWitt equation can be found in terms of knot invariants [3, 4]. The common feeling about these facts is that knot theory and general relativity seems to have a profound relationship when one intends to describe the quantum properties of space-time [6].

One of the ways that this relationship is manifested is the following: for one hand there exists a general link between Chern-Simons and knot theories [7]; on the other the exponential of the Chern-Simons form constructed with the Ashtekar connections is annihilated by the constraints of gravity with cosmological constant in the connection representation [5]. As a result, some knot
polynomials could be associated with the solutions of the Hamiltonian constraint when loops are introduced as the underlying geometrical structure of quantum gravity. The argumentation is essentially formal (the loop transform is a formal relationship between the connection and the loop representations for quantum gravity) but its heuristic use was proved to be useful in the search of nondegenerate solutions of the Hamiltonian constraint in the conventional loop representation \[9\].

More precisely, the loop transform of the exponential of the Chern-Simons form is related to the Kauffman bracket knot polynomial \[7, 10, 11\]. The Kauffman bracket can be expressed as a power series in the cosmological constant. Each coefficient of the series is a knot invariant that is, at least formally, promoted as a solution of the Hamiltonian constraint with cosmological constant in the loop representation.

There exists a close relationship between the Kauffman bracket and the Jones polynomials. If \(K_\Lambda\) and \(J_\Lambda\) denote these knot polynomials, one finds that

\[
K_\Lambda (\gamma) = e^{-\Lambda \varphi_G (\gamma)} J_\Lambda (\gamma)
\]

where \(\varphi_G (\gamma)\) is the Gauss self-linking number of the loop \(\gamma\). The phase factor \(e^{-\Lambda \varphi_G (\gamma)}\) contains all the framing dependence of the Kauffman bracket invariant and it is by itself annihilated by the \(H_\Lambda [12, 13]\). Using this fact one can write the following expression for the action of the Hamiltonian on the Kauffman bracket \[1\]

\[
H_\Lambda K_\Lambda (\gamma) = \left(H_0 + \Lambda \det q\right) K_\Lambda (\gamma) = \\
\sum_{m=2}^{\infty} \Lambda^m \{H_0 J_m + \sum_{n=1}^{m-1} (-1)^n \left[H_0 (\varphi_G J_{m-n}) - n \det q (\varphi_G J_{m-n})\right]\} = 0
\]

where \(\det q\) is the determinant of the three metric and \(J_m\) gives the coefficients of the expansion of the Jones polynomial in terms of the cosmological constant. The cancellation of this expression is expected from the formal arguments given before and it has to be checked explicitly in the loop representation. Notice that for \(m = 2\) the above result reduces to (recall that \(J_1 \equiv 0\))

\[
H_0 J_2 (\gamma) \equiv 0
\]

By this way the coefficients of the Jones knot polynomial appear to be related to the solutions of the vacuum Hamiltonian constraint. This fact and the possibility that a similar result could hold for higher orders in the cosmological constant was first advanced by Brügmann, Gambini and Pullin \[9\]. These authors have confirmed the annulment of equation \(2\) up to the second order in \(\Lambda\). The next equation reads as follows

\[
H_\Lambda K_\Lambda^{(3)} = \Lambda^3 \{H_0 J_3 + \det q J_2 - H_0 (\varphi_G J_2)\} \equiv 0
\]
As it was mentioned, an analysis of this equation was just developed in \[1\]. In that opportunity a result for $H_0 \cdot J_3 (\gamma)$ was derived from the annulment of \(4\). Now we are interested to confirm the fact that $H_\lambda \cdot K_2 (\gamma) = 0$. This means that we have to compute explicitly $H_0 \cdot J_3 (\gamma)$. This calculation confronts with hard computational and regularization problems in the conventional loop representation. The extended loop representation offers a new way to solve this problem both at the formal and the renormalized level.

The relationship between the extended loop and the loop representations can be formulated in general. The group of loops is a discrete subgroup of the extended loop group \[14\], so any result obtained in the extended loop framework can be reduced to ordinary loops. In particular it is demonstrated that the extended version of the constraints of quantum gravity reduce to the corresponding of the conventional loop representation once the specialization is performed \[15\]. We introduce now some convenient notation related to extended loops. The elements of the extended loop group are given by infinite strings of multivector density fields of the form

$$X = (X, X^{\mu_1}, \ldots, X^{\mu_1 \ldots \mu_n}, \ldots) \tag{5}$$

where $X$ is a real number and a greek index $\mu_i : (a_i, x_i)$ represents a paired vector index $a_i$ and space point $x_i$. The number of paired indices defines the rank of the multivector field. The constraints have the following expressions in terms of extended loops

$$C_{ax} \psi(R) = \psi(\mathcal{F}_{ab}(x) \times R^{(bx)}) \tag{6}$$

$$\frac{1}{2} H_0(x) \psi(R) = \psi(\mathcal{F}_{ab}(x) \times R^{(ax, bx)}) \tag{7}$$

where $\times$ indicates the extended group product, $\mathcal{F}_{ab}(x)$ are some elements of the algebra of the group and

$$[R^{(bx)}]^{\mu_1 \ldots \mu_n} \equiv R^{(bx)\mu_1 \ldots \mu_n} := R^{(bx \mu_1 \ldots \mu_n)c} \tag{8}$$

$$[R^{(ax, bx)}]^{\mu_1 \ldots \mu_n} \equiv R^{(ax, bx)\mu_1 \ldots \mu_n} := \sum_{k=0}^{n} R^{(ax \mu_1 \ldots \mu_k bx \mu_{k+1} \ldots \mu_n)c} \tag{9}$$

We usually call these combinations the one-point-R and the two-point-R respectively, being an $R$ the following combination of $X$’s

$$R^{\mu_1 \ldots \mu_n} := \frac{1}{2} [X^{\mu_1 \ldots \mu_n} + X^{\mu_1 \ldots \mu_n}] \tag{10}$$

The overline operation is defined as follows

$$X^{\mu_1 \ldots \mu_n} := (-1)^n X^{\mu_n \ldots \mu_1} \tag{11}$$

The subscript $c$ indicates cyclic permutation and in \(3\) the sequences $\mu_1 \ldots \mu_0$ and $\mu_{n+1} \ldots \mu_n$ for $k = 0$ and $k = n$ respectively are assumed to be the null set.
of indices. Notice that the diffeomorphism and the Hamiltonian constraints have very similar expressions in the extended loop representation. The only difference is the object that one puts into the group product with $F_{ab}(x)$. The one- and the two-point-R have basically different symmetry and regularity properties. The extended loop wavefunctions are linear in the multivector fields and they are written in general as follows

$$\psi(X) \equiv \psi(R) = \sum_{n=0}^{\infty} D_{\mu_1...\mu_n} R_{\mu_1...\mu_n}$$

where the propagators $D_{\mu_1...\mu_n}$ satisfy a set of identities of the Mandelstam type and a generalized convention of sum are assumed for the repeated paired (greek) indices. Usually we refer the $\mu_i$ indices of the propagators as covariant whereas the $\mu_i$ indices of the multivector fields will be called contravariant. This terminology is suggested from the behavior of these objects under general coordinate transformations. Using (12) we get from (7) the following general expression for the action of the Hamiltonian on the extended loop wavefunctions

$$\frac{1}{2}H_0(x)\psi(R) = \sum_{n=0}^{\infty} D_{\mu_1...\mu_n} [F_{ab}^{\mu_1}(x) R^{(ax, bx)_{\mu_2...\mu_n}} + F_{ab}^{\mu_1\mu_2}(x) R^{(ax, bx)_{\mu_3...\mu_n}}]$$

where we have explicitly used the fact that $F_{ab}(x)$ has only two nonvanishing components $F_{ab}^{\mu_1}(x)$ and $F_{ab}^{\mu_1\mu_2}(x)$, given by

$$F_{ab}^{\mu_1}(x) = \delta_{ab}^{\alpha_1\alpha_2} \partial_{\alpha_1} \delta(x_1 - x)$$
$$F_{ab}^{\mu_1\mu_2}(x) = \delta_{ab}^{\alpha_1\alpha_2} \delta(x_1 - x) \delta(x_2 - x)$$

The propagators are cyclic: $D_{\mu_1...\mu_n} \equiv D_{(\mu_1...\mu_n)c}$. This means that the indices of the propagator that are contracted with the $F$’s really lie in any position of the $D$’s. It is a remarkable fact that for all the (known) wavefunctions of quantum gravity, the propagators $D_{\mu_1...\mu_n}$ are expressed completely in terms of the two and three point propagators of the Chern-Simons theory $g_{\mu_1\mu_2}$ and $h_{\mu_1\mu_2\mu_3}$. We write the Chern-Simons propagators in the following way

$$g_{\mu_1\mu_2} := \epsilon_{\alpha_1\alpha_2\alpha_3} g_{\mu_1\alpha_1} g_{\mu_2\alpha_2} g_{\mu_3\alpha_3}$$
$$h_{\mu_1\mu_2\mu_3} := \epsilon_{\alpha_1\alpha_2\alpha_3} \delta(z_1 - z_2) \delta(z_1 - z_3)$$

with

$$\phi_{x_2}^{kx_1} := -\frac{1}{4\pi} \frac{(x_1 - x_2)^k}{|x_1 - x_2|^3} = \frac{\partial^k}{\nabla^2} \delta(x_1 - x_2)$$

and

$$\epsilon^{\alpha_1\alpha_2\alpha_3} := \epsilon^{\alpha_1\alpha_2\alpha_3} \delta(z_1 - z_2) \delta(z_1 - z_3)$$

In the above equations we have used a combined notation for the indices $\mu_i = (a_i, x_i)$ and $\alpha_k = (c_k, z_k)$. Also notice that in the greek indices of $F_{ab}^{\mu_1}(x)$
and $F_{ab}^{\mu_1 \mu_2}(x)$ are contracted with $g$’s and $h$’s (the building blocks of the $D$’s) in different arrangement of the covariant indices. In the next section we develop the result of these contractions for a generic case.

### 3 The $F_{ab}(x)$’s on the Chern-Simons propagators

The components of $F_{ab}(x)$ are distributional objects of rank one and two and their action on the propagators $g$ and $h$ can be worked explicitely at the formal level. According to (15) the component of rank two includes only (discrete and continuous) delta functions, so its action on the propagators is straightforward

\[
F_{ab}^{\mu_1 \mu_2}(x) g_{\mu_3 \mu_4} g_{\mu_2 \mu_4} = g_{\mu_3} [ax g \delta x]_{\mu_4}
\]

\[
F_{ab}^{\mu_1 \mu_2}(x) h_{\mu_1 \mu_2 \mu_3} = 2 h_{ax} bx_{\mu_3}
\]

Notice that in the case that the two contravariant indices of $F_{ab}^{\mu_1 \mu_2}(x)$ are joined to the same $g$, the (divergent) contribution vanishes by symmetry. In order to take into account the derivative that appears in (14) it is convenient to write $F_{ab}^{\mu_1}(x)$ in terms of the inverse of the two point propagator, given by

\[
g_{\mu_1 \mu_2} = \delta_{\alpha_1 \alpha_2} k \partial_k \delta(x_1 - x_2)
\]

From (14) we have

\[
F_{ab}^{\mu_1}(x) = -\epsilon_{abc} \epsilon^{c \alpha_1 d} \partial_d \delta(x - x_1) = -\epsilon_{abc} g^{cx \mu_1}
\]

The action of $F_{ab}^{\mu_1}(x)$ over the two point propagator is simply given by

\[
F_{ab}^{\mu_1}(x) g_{\mu_1 \mu_2} = -\epsilon_{abc} g^{cx \mu_1} g_{\mu_1 \mu_2} = -\epsilon_{abc} \delta^{cx \mu_2}_{T}
\]

where $\delta^{cx \mu_2}_{T}$ is the projector on the space of transverse (divergence free) multivector density fields [14], given by

\[
\delta^{cx \mu_2}_{T} = \delta_{\alpha_2} \delta(x - x_2) - \phi^{cx}_{x_2, \alpha_2}
\]

Notice now in which way $F_{ab}^{\mu_1}(x)$ operates on a generic term of a wavefunction when it is contracted with a two point propagator:

\[
F_{ab}^{\mu_1}(x) g_{\mu_1 \alpha_k} D^{........\alpha_k} = -\epsilon_{abc} \delta^{cx \alpha_k}_{T} D^{........\alpha_k} = -\epsilon_{abc} D^{........}[R^{cx \alpha_k} - \phi^{cx}_{z_k, c_k} R^{........\alpha_k}] = -\epsilon_{abc} D^{........}[R^{cx \alpha_k} + \phi^{cx}_{z_k} \partial_{\alpha_k} R^{........\alpha_k}] 
\]

where only the indices of interest were written and in the last step the second term was integrated by parts with respect to the $z_k$ variable ($\partial_{\alpha_k} \equiv \partial_{c_k z_k} \equiv$...).
\[ \partial \phi \big|_{z^k} \). The integration by parts induced by the longitudinal projector \( \phi^{cx}_{k, \alpha_k} \) produces the divergence of the multivectors with respect to the \( \alpha_k \)-entry. The divergence taking with respect to any index of an extended loop of rank \( n \) generates a contribution of rank \( n - 1 \) in the following way

\[
\partial_{\alpha_k} R^{... \alpha_k ...} = [\delta(z_k - z_{k-1}) - \delta(z_k - z_{k+1})] R^{... \alpha_{k-1} \alpha_{k+1} ...} \tag{27}
\]

where \( \alpha_{k-1} \) and \( \alpha_{k+1} \) are the neighbors of \( \alpha_k \) in the sequence of indices of \( R \) (for \( k = 1 \) and \( k = n \) we have \( z_0 = z_{n+1} \equiv o \), with \( o \) a reference spatial point). The above property is the “differential constraint” of the elements of the extended loop group \([14]\). Applying (27) to (26) we get

\[
\mathcal{F}_{ab}^{\mu_1}(x) g_{\mu_1 \alpha_k} D^{... \alpha_k ...} = -\epsilon_{abc} D^{...} [R^{... cx ...} + (\phi^{cx}_{z_k-1} - \phi^{cx}_{z_k+1}) R^{... \alpha_{k-1} \alpha_{k+1} ...}] \tag{28}
\]

This equation exhibits the effect of \( \mathcal{F}_{ab}^{\mu_1}(x) \) on the wavefunction when the \( \mu_1 \) index lies on a two point propagator. We see that for any term of rank \( n \) of the expansion \([12]\), a term of rank \( n - 1 \) is always generated by means of the differential constraint (27). These type of contributions are characterized by the appearance of a difference of \( \phi \) functions. Also notice that in the case that \( \mathcal{F}_{ab}^{\mu_1}(x) \) acts on a three point propagator \( h \) instead of a two point \( g \), the only difference with the previous result is that the \( \alpha_k \) index is now linked with others \( g \)’s through \( \epsilon^{\alpha_k ...} \) according to (17). In effect, we have in this case

\[
\mathcal{F}_{ab}^{\mu_1}(x) h_{\mu_1 \pi_1 \pi_j} = -\epsilon_{abc} \delta^{cx}_{T \alpha_k} \epsilon^{\alpha_k \alpha_1 \alpha_m} g_{\pi_1 \alpha_1} g_{\pi_j \alpha_m} \tag{29}
\]

In the appendix I it is shown that

\[
\delta^{cx}_{T \alpha_k} \epsilon^{\alpha_k \alpha_1 \alpha_m} = \epsilon^{cx \alpha_1 \alpha_1 \alpha_m} + (\phi^{cx}_{z_1} - \phi^{cx}_{z_m}) g^{\alpha_1 \alpha_m} \tag{30}
\]

Using this fact equation (29) can be put in the form

\[
\mathcal{F}_{ab}^{\mu_1}(x) h_{\mu_1 \pi_1 \pi_j} = -g_{\pi_1 [ax \ yb]} \pi_j + \epsilon_{abc} \phi^{cx}_{z} \delta^{dx}_{\pi_1, g_{\pi_j} \pi_j} \tag{31}
\]

where in the last term an integration in \( z \) is assumed. When the \( \pi \equiv (e, y) \) indices are contracted with contravariant indices of multivector fields, the differential constraint (27) will be induced by the longitudinal projectors \( \phi^{dz}_{y_i \epsilon_i} \) and \( \phi^{dz}_{y_j \epsilon_j} \). The following result is obtained for this case

\[
\mathcal{F}_{ab}^{\mu_1}(x) h_{\mu_1 \pi_1 \pi_j} D^{... \pi_1 \pi_j ...} = \\
\{ -g_{\pi_1 [ax \ yb]} \pi_j + \epsilon_{abc} (\phi^{cx}_{y_i} - \phi^{cx}_{y_j}) g_{\pi_1 \pi_j} \} D^{... \pi_1 \pi_j ...} + \epsilon_{abc} \phi^{cx}_{z} (\phi^{dz}_{y_{i-1}} - \phi^{dz}_{y_{j+1}}) g_{\pi_1, dz} D^{... \pi_1 \pi_j ...} + \epsilon_{abc} \phi^{cx}_{z} (\phi^{dz}_{y_{i+1}} - \phi^{dz}_{y_{j-1}}) g_{\pi_j, dz} D^{... \pi_1 \pi_j ...} \tag{32}
\]

We see that a larger number of \( \phi \)’s appear when the \( \mu_1 \) index of the propagator belongs to an \( h \). This is a general result: suppose that the index \( \pi_j \) of the \( h \)
Simons theory) is the same the properties of gravity. It is a remarkable fact that the systematic (that is based essentially on contravariant multivector indices) exists for the calculation in the extended loop representation of quantum gravity. The analytical expression of the Hamiltonian on $J_3$ [17] is connected with others $g$’s through an $e^{\pi \mu \nu \rho \sigma}$ (instead to be contracted with a contravariant multivector index as in (32)); then a greater number of integrations by parts involving longer chains of $\phi$’s would appear in the final result. Notice that this is the case for the term of lower rank of $J_3(R)$ (see equation (34) below). In the appendix II we shall prove that

$$F^{\mu \nu}(x) h_{\mu_1 \mu_2 \alpha} g^{\alpha \beta} h_{\mu_3 \mu_4 \beta} =$$

$$-g_{\mu_2 \nu \alpha \beta} h_{\mu_3 \mu_4} + \epsilon_{abc} \phi^{cd}_{x_{\alpha \beta}} h_{\mu_3 \mu_4} - \epsilon_{abc} \phi^{cd}_{x_a} h_{\mu_3 \mu_4} + \epsilon_{abc} \phi^{cd}_{x_a} (\phi_{x_3} - \phi_{x_4}) g_{\mu_2 \mu_3} g_{\mu_4}$$

$$+ \epsilon_{abc} \phi^{cd}_{x_a} \phi_{y} (\phi_{x_4 \alpha} g_{\mu_3 \alpha} - \phi_{x_3 \alpha} g_{\mu_4 \alpha}) g_{\mu_2 \mu_3}$$

In this case we will have three integrations by parts once the free covariant indices of the above expression are contracted with contravariant greek indices of multivector fields. The above discussion puts in evidence that a systematization exists for the calculation in the extended loop representation of quantum gravity. It is a remarkable fact that the systematic (that is based essentially on the properties of $F_{\alpha \beta}(x)$ and the two and three point propagators of the Chern-Simons theory) is the same for both the Hamiltonian and diffeomorphism constraints. This last property is a consequence of the fact that the combinations $R^{(a_\beta \nu \alpha \beta) \mu_1 \ldots \mu_n}$ and $R^{(a_\beta \nu \alpha \beta) \mu_1 \ldots \mu_n}$ satisfies the differential constraint with respect to the $\mu$ indices in the two cases. Another aspects of the methodology of the extended loop calculus will be pointed out in the following sections.

4 The Hamiltonian on $J_3$

The analytical expression of the $J_3$ diffeomorphism invariant is [17]

$$J_3(R) = -6 \{ (2 g_{\mu_1 \mu_4} g_{\mu_2 \mu_5} g_{\mu_3 \mu_6} + \frac{1}{2} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} g_{\mu_5 \mu_6}) R^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6}$$

$$+ g_{(\mu_1 \mu_2 \mu_3) \mu_4 \mu_5 \mu_6} R^{(\mu_1 \mu_2 \mu_3) \mu_4 \mu_5 \mu_6}$$

$$+ (h_{\mu_1 \mu_2} g^{\alpha \beta} h_{\mu_3 \mu_4} - h_{\mu_1 \mu_3} g^{\alpha \beta} h_{\mu_2 \mu_4}) R^{(\mu_1 \mu_2 \mu_3) \mu_4 \mu_5 \mu_6} \}$$

(34)

In order to evaluate the Hamiltonian on $J_3$ it is convenient to consider the contributions of the different ranks for separate. In the next subsections we shall develop the partial results for the different ranks.

4.1 $H_0 (g \ldots g \ldots R \ldots)$

Let us start by considering the action of the rank one component of $F_{\alpha \beta}(x)$ on the terms with three two point propagators. We have

$$F_{\alpha \beta}(x) g \ldots g \ldots R^{(a_\beta \nu \alpha \beta) \mu_1 \ldots \mu_n} = F_{\alpha \beta}(x) (2 g_{\mu_1 \mu_4} g_{\mu_2 \mu_5} g_{\mu_3 \mu_6} + g_{(\mu_1 \mu_2)} g_{\mu_3 \mu_4} g_{\mu_5 \mu_6}$$

$$+ g_{\mu_1 \mu_5} g_{\mu_2 \mu_4} g_{\mu_3 \mu_6} + g_{\mu_1 \mu_4} g_{\mu_2 \mu_5} g_{\mu_3 \mu_5}) R^{(a_\beta \nu \alpha \beta) \mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6}$$

(35)
This expression can be simplified using the following property of the two-point-R:

\[ R^{(ax, bx)\mu_1...\mu_n} = R^{(bx, ax)\mu_1...\mu_n} \]  \hspace{1cm} (36)

This property follows directly from the fact that the \( R \)'s are even with respect to the overline operation. Notice now that

\[ F_{ab}^{\mu_1}(x)g_{\mu_1\mu_5}g_{\mu_2\mu_4}g_{\mu_3\mu_6}R^{(ax, bx)\mu_2\mu_3\mu_4\mu_5\mu_6} \]

\[ = -F_{ab}^{\mu_1}(x)g_{\mu_1\mu_5}g_{\mu_2\mu_4}g_{\mu_3\mu_6}R^{(bx, ax)\mu_6\mu_5\mu_3\mu_2} \]

\[ = \mp F_{ab}^{\mu_1}(x)g_{\mu_1\mu_5}g_{\mu_2\mu_4}g_{\mu_3\mu_6}R^{(ax, bx)\mu_2\mu_3\mu_4\mu_5\mu_6} \]  \hspace{1cm} (37)

This kind of symmetry operation will be used repeatedly along the calculation. From (24) and (25) we get

\[ F_{ab}(x)g_{..g..}R^{(ax, bx)\cdots} = -\epsilon_{abc}\{\delta_T^{cx}_{\mu_4}(2g_{\mu_2\mu_5}g_{\mu_3\mu_6} + g_{\mu_2\mu_4}g_{\mu_3\mu_5}) \]

\[ + 2\delta_T^{cx}_{\mu_3}(g_{\mu_2\mu_5}g_{\mu_4\mu_6})R^{(ax, bx)\mu_2\mu_3\mu_4\mu_5\mu_6} \]

\[ = -2\epsilon_{abc}g_{\mu_1\mu_3}g_{\mu_2\mu_4}(R^{(ax, bx)\mu_1}cx_{\mu_2\mu_3\mu_4} + R^{(ax, bx)\mu_1\mu_2}cx_{\mu_3\mu_4}) \]

\[ -\epsilon_{abc}g_{\mu_1\mu_4}g_{\mu_2\mu_3}R^{(ax, bx)\mu_1\mu_2\mu_3\mu_4} \]

\[ -\epsilon_{abc}(2(\phi^{cx}_{x_2} - \phi^{cx}_{x_1})g_{\mu_1\mu_3}g_{\mu_2\mu_4} + (\phi^{cx}_{x_2} - \phi^{cx}_{x_1})g_{\mu_1\mu_4}g_{\mu_2\mu_3})R^{(ax, bx)\mu_1\mu_2\mu_3\mu_4} \]  \hspace{1cm} (38)

For the component of rank two one obtains

\[ F_{ab}(x)g_{..g..}R^{(ax, bx)\cdots} = \{2g_{\mu_1\mu_3}g_{\mu_2[ax\bx]}\mu_4 + g_{\mu_1\mu_4}g_{\mu_2[ax\bx]\mu_3}\}R^{(ax, bx)\mu_1\mu_2\mu_3\mu_4} \]  \hspace{1cm} (39)

The partial result for the rank six of \( J_3 \) is

\[ \frac{1}{12}H_0 (g..g..R\cdots) = \epsilon_{abc}g_{\mu_1\mu_4}g_{\mu_2\mu_3}R^{(ax, bx)\mu_1\mu_2cx_{\mu_3\mu_4}} \]

\[ + 2\epsilon_{abc}g_{\mu_1\mu_5}g_{\mu_2\mu_4}[R^{(ax, bx)\mu_1cx_{\mu_2\mu_3\mu_4}} + R^{(ax, bx)\mu_1\mu_2cx_{\mu_3\mu_4}}] \]

\[ + \{2g_{\mu_1\mu_3}g_{\mu_2[ax\bx]}\mu_4 - g_{\mu_1\mu_4}g_{\mu_2[ax\bx]\mu_3} + \epsilon_{abc}(2(\phi^{cx}_{x_2} - \phi^{cx}_{x_1})g_{\mu_1\mu_3}g_{\mu_2\mu_4} \]

\[ + (\phi^{cx}_{x_2} - \phi^{cx}_{x_1})g_{\mu_1\mu_4}g_{\mu_2\mu_3})R^{(ax, bx)\mu_1\mu_2\mu_3\mu_4} \]  \hspace{1cm} (40)

We get three types of terms: one of rank seven and two of rank six. The terms of rank seven have the multivector fields with three spatial indices evaluated at the point \( x \). The contribution of rank six has two different sources: for one hand the terms generated by the application of the differential constraint \((25)\) on the multivector fields of rank seven and on the other the terms generated by the action of \( F_{ab}^{\mu_1\mu_2}(x) \) on the propagators.

**4.2 \( H_0 (g..h..R\cdots) \)**

We have in this case

\[ F_{ab}(x)g..h..R^{(ax, bx)\cdots} \]
The partial result for the rank five of $J_3$ is
\[
\mathcal{F}_{ab}^{\mu_1}(x) (2g_{\mu_1\mu_3}h_{\mu_2\mu_4\mu_5} + 2g_{\mu_2\mu_4}h_{\mu_1\mu_3\mu_5} + g_{\mu_2\mu_5}h_{\mu_1\mu_3\mu_4}) R^{(ax, bx)}_{\mu_2\mu_3\mu_4\mu_5}
\]
\[
= -2\epsilon_{abc}\delta_T^{cx} h_{\mu_2\mu_4\mu_5} - 2g_{\mu_2\mu_4}g_{\mu_3[ax \ g_{bx}]_\mu_5} + 2\epsilon_{abc}g_{\mu_2\mu_4}\phi_z^{cz} \delta_T^{d_\mu_5} R^{(ax, bx)}_{\mu_3 g_{\mu_5} dz}
\]
\[
=g_{\mu_2\mu_5}g_{\mu_3[ax \ g_{bx}]_\mu_4} + \epsilon_{abc}g_{\mu_2\mu_5}\phi_z^{cz} \delta_T^{d_\mu_5} R^{(ax, bx)}_{\mu_3 g_{\mu_5} dz}
\]
where in the last step we have used (24) and (31).

Introducing now (25) and performing the integration by parts indicated by the longitudinal projectors one obtains after a few direct manipulations
\[
\mathcal{F}_{ab}^c(g... \ R^{(ax, bx)}... = -2\epsilon_{abc}h_{\mu_1\mu_3} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
\[
+ \{ -2g_{\mu_1\mu_3}g_{\mu_2[ax \ g_{bx}]_\mu_4} - g_{\mu_1\mu_4}g_{\mu_2[ax \ g_{bx}]_\mu_3} + 2\epsilon_{abc}(\phi^{cx}_z - \phi^{cx}_x)g_{\mu_1\mu_4}g_{\mu_2\mu_3}
\]
\[
+ 2\epsilon_{abc}(\phi^{cx}_z - \phi^{cx}_x)g_{\mu_1\mu_2\mu_3} + \epsilon_{abc}\phi_z^{cx}g_{\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
\[
+ \epsilon_{abc}\phi_z^{cx}g_{\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
(42)

The component of rank two of $\mathcal{F}_{ab}(x)$ gives the following contribution
\[
\mathcal{F}_{ab}^c (g... \ R^{(ax, bx)}... = 2(g_{\mu_1[ax \ h_{bx}]_\mu_2} + g_{\mu_1\mu_3}h_{ax bx \ mu_2}) R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
(43)

The partial result for the rank five of $J_3$ is
\[
\frac{1}{12} H_0 (g... \ R^{...} = 2\epsilon_{abc}h_{\mu_1\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
\[
- \{ -2g_{\mu_1\mu_3}g_{\mu_2[ax \ g_{bx}]_\mu_4} - g_{\mu_1\mu_4}g_{\mu_2[ax \ g_{bx}]_\mu_3} + 2\epsilon_{abc}(\phi^{cx}_z - \phi^{cx}_x)g_{\mu_1\mu_4}g_{\mu_2\mu_3}
\]
\[
+ \epsilon_{abc}(\phi^{cx}_z - \phi^{cx}_x)g_{\mu_1\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
\[
- 2(g_{\mu_1[ax \ h_{bx}]_\mu_2} + g_{\mu_1\mu_3}h_{ax bx \ mu_2}) R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
\[
+ \epsilon_{abc}\phi_z^{cx}(\phi^{dz}_x - \phi^{d_\mu_5}g_{\mu_1\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3}
\]
(44)

where we have used (33). Three different sort of terms appear in this case: two of rank six and one of rank five. One of the contributions of rank six have the multivector fields with three spatial indices evaluated at the point $x$ (like the rank seven of (33)). The other cancels exactly the contribution of rank six that comes from $\frac{1}{12} H_0 (g... \ R^{...})$. Moreover, we will see in the next subsection that the rank five of (44) is canceled by terms that appears when the Hamiltonian acts on the term of rank four of $J_3$.

The above discussion exhibits another property of the constraints in the extended loop representation. A chain mechanism of cancellations appears when the constraints act on a wavefunction. This mechanism links intimately the successive ranks of the wavefunction. One can summarize this mechanism in the following way for the case of the Hamiltonian: the Hamiltonian acting on the rank $n$ of the wavefunction generates a contribution of rank $n + 1$ and other of rank $n$. In the (remnant) contribution of rank $n + 1$ the multivector fields have always three spatial points evaluated at $x$ (they are of the general form $R^{(ax, bx)... \ cx...})$. The contribution of rank $n$ is canceled by terms that are generated when the operator acts on the rank $n – 1$ of the wavefunction. We will see in the next subsection that the last term of $J_3$ closes the chain in a consistent way. It is important to remark that exactly the same cancellations take place in the case of the diffeomorphism operator.
4.3 \( H_0(h_\alpha g^{**}h_\alpha R^{***}) \)

The action of the Hamiltonian on the term of rank four of \( J_3 \) is given by the following expression

\[
-\frac{1}{12} H_0(h_\alpha g^{**}h_\alpha R^{***}) = 2 F_{ab\mu_1}(x) h_{\mu_1 \mu_2 \alpha} g^{\alpha \beta} h_{\mu_3 \mu_4 \beta} R^{(a \mu_1 b \mu_2 \mu_3 \mu_4)} + (2 h_{ax \alpha} h_{\mu_1 \mu_2 \beta} g^{\alpha \beta} - h_{\mu_2 \alpha} [a x h_{bx}]_{\mu_1 \beta} g^{\alpha \beta}) R^{(a \mu_1 \mu_2 \mu_3 \mu_4)}
\]

where we have used (36) and (21). Using (33) and performing the integrations by parts indicated by the longitudinal projectors we get

\[
\frac{1}{12} H_0(h_\alpha g^{**}h_\alpha R^{***}) = 2 \{ g_{\mu_1 [a x h_{bx}] \mu_2 \mu_3} - \epsilon_{abc} \phi^{cx}_{x_1} h_{\mu_1 \mu_2 \mu_3} \\
+ \epsilon_{abc} \phi^{cx}_{x_1} (\phi^{dx}_{x_3} - \phi^{dx}_{x_2}) g_{\mu_1 \mu_2 \mu_3} \} R^{(a \mu_1 b \mu_2 \mu_3 \mu_4)} - (2 h_{ax \alpha} h_{\mu_1 \mu_2 \beta} g^{\alpha \beta} - h_{\mu_2 \alpha} [a x h_{bx}]_{\mu_1 \beta} g^{\alpha \beta} + 2 \epsilon_{abc} \phi^{cx}_{x_1} (\phi^{dx}_{x_3} - \phi^{dx}_{x_2}) h_{dx \mu_1 \mu_2} \\
+ 2 \epsilon_{abc} \phi^{cx}_{x_1} (\phi^{dy}_{x_3} + \phi^{dy}_{x_2}) g_{\mu_1 \mu_2 \mu_3 \mu_4} \} R^{(a \mu_1 b \mu_2 \mu_3 \mu_4)}
\]

We observe that in this expression does not appear contributions of the type \( R^{(a \mu_1 b \mu_2 \mu_3 \mu_4 ... c x ...)} \). This is due to the fact that there are no “free” two point propagators present in \( h_\alpha g^{**}h_\alpha \). Also notice that the rank five contribution cancels the term of rank five of (44) as it was advanced in the preceding subsection.

What happens with the rank four of (46)? We do not have at our disposal a term of rank three in \( J_3 \) to continue the chain of cancellations. This contribution has necessarily to vanish. This condition follows from the relationship existing between the Hamiltonian and the diffeomorphism in the extended loop representation. All the results obtained heretofore for the Hamiltonian are valid for the diffeomorphism with the replacement of the two-point-\( R \) by the one-point-\( R \). This means that the annulment of the rank four of (46) is required from the diffeomorphism invariance of \( J_3 \). One can see this requirement as a condition of consistency for the analytical expression (34) to represent a diffeomorphism invariant. The demonstration of this fact will be given in the appendix III.

To conclude this subsection we resume: there is no remnant contribution corresponding to the lower rank of \( J_3 \) and this term is responsible of the closure (in a consistent way) of the chain of cancellations associated with the action of the constraints in the extended loop representation. This punctualization is also valid for the conventional loop representation.

5 Collecting the partial results

The results of the preceding section show that it is possible to write the following expression for the action of the Hamiltonian on the third coefficient of the Jones
polynomial
\[
\frac{1}{12} H_0(x) J_3(R) = 2\epsilon_{abc} h_{\mu_1\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} \\
+ 2\epsilon_{abc} g_{\mu_1\mu_3} g_{\mu_2\mu_4} [R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_4} + R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_4}] \\
+ \epsilon_{abc} g_{\mu_1\mu_4} g_{\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_4} 
\]
(47)

This expression admits several ulterior manipulations in the extended representation: one that exploits the symmetry properties of the propagators and the \( R \)'s under permutation of the indices and other that points out towards a "geometrical" interpretation of the result. As both are important and pursue different goals we will analyze them for separate.

5.1 Trying to annihilate \( H_0 J_3 \)

At this point it is convenient to remember the initial motivation of the calculation. From the general arguments of Sect. 2 it would be expected the cancellation of (47). But how can this expression be zero?

The only general way to answer this question is by using symmetry considerations. Notice that the symmetries must work in the case of the one-point-\( R \) in order to make \( J_3 \) a solution of the diffeomorphism constraint. The symmetry operations performed with covariant indices include the properties of \( \epsilon_{abc} \) and of the propagators \( g \) and \( h \). The contravariant indices can be moved using the overline operation (like in (36)) and the cyclicity.

The properties of the two-point-\( R \) under the overline of the indices were totally exploited in the calculation. We have now to decompose these combinations according to (9). For example, for the contribution of rank six of (47) one has

\[
R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} = R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} - R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} c \\
+ R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} c - R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} c + R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} c 
\]
(48)

This expression is reduced to the following using the cyclicity and the symmetry properties of \( h \) and \( \epsilon_{abc} \):

\[
\epsilon_{abc} h_{\mu_1\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} = \epsilon_{abc} h_{\mu_1\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} c 
\]
(49)

Repeating the procedure for the others terms of (47) one gets

\[
\frac{1}{12} H_0(x) J_3(R) = 2\epsilon_{abc} h_{\mu_1\mu_2\mu_3} R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_3} c \\
- 6\epsilon_{abc} g_{\mu_1\mu_2} g_{\mu_3\mu_4} [R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_4} c + R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_4} c] \\
- R^{(ax, bx)}_{\mu_1 cx \mu_2\mu_4} c 
\]
(50)

\[2\text{Remember that the results of the extended representation can be always specialized to the case of ordinary loops. This means that a geometrical interpretation can in principle be possible for the analytical result (47).} \]
No further reduction is possible. This is the final result.

We see that the Hamiltonian does not annihilate the third coefficient of the Jones polynomial for general multivector fields. As it was emphasized before, the only difference between the Hamiltonian and the diffeomorphism operators in the extended loop representation is to change the two-point-R by the one-point-R. To end this subsection we shall verify that the diffeomorphism operator annihilates $J_3$. From (47) one can write

$$
\frac{1}{6} C_{ax} J_3(R) = 2\epsilon_{abc} h_{\mu_1 \mu_2 \mu_3} R^{(bx \mu_1 cx \mu_2 \mu_3)_c} + \epsilon_{abc} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} [R^{(bx \mu_1 cx \mu_2 \mu_3 \mu_4)_c} + R^{(bx \mu_1 \mu_2 cx \mu_3 \mu_4)_c}] + \epsilon_{abc} g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} R^{(bx \mu_1 \mu_2 cx \mu_3 \mu_4)_c} \quad (51)
$$

This expression can be rewritten in the following way

$$
\frac{1}{6} C_{ax} J_3(R) = \epsilon_{abc} h_{\mu_1 \mu_2 \mu_3} [R^{(bx \mu_1 cx \mu_2 \mu_3)_c} + R^{(cx \mu_1 bx \mu_2 \mu_3)_c}] + \epsilon_{abc} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} [R^{(bx \mu_1 cx \mu_2 \mu_3 \mu_4)_c} + R^{(cx \mu_1 bx \mu_2 \mu_3 \mu_4)_c}] + \epsilon_{abc} g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} [R^{(bx \mu_1 \mu_2 cx \mu_3 \mu_4)_c} + R^{(cx \mu_1 \mu_2 bx \mu_3 \mu_4)_c}] \equiv 0 \quad (52)
$$

that vanishes identically in a formal sense.

### 5.2 A geometrical interpretation for $H_0 J_3$

The connection between the extended loop and the loop representations involves some technology that was introduced in [15] and developed in [1] for the particular case of the third coefficient of the Jones polynomial. In what follows we will incorporate the main consequences of this procedure of reduction into the context of the present discussion.

Perhaps the main virtue of the procedure of reduction developed in [1] is to show that new combinations of multivector fields of the general form $R^{(ax, bx), \ldots, cx, \ldots}$ appear and that they have a direct and simple geometrical interpretation when extended loops are particularized to ordinary loops. The combinations are

$$
R^{(ax, bx)\mu_1 \ldots \mu_n} := \sum_{k=0}^{n} R^{(ax, bx)\mu_1 \ldots \mu_k cx \mu_{k+1} \ldots \mu_n} \quad (53)
$$

and

$$
R^{(ax, bx)\mu_1 \ldots \mu_n} := \sum_{k=0}^{n} R^{(ax, bx)\mu_1 \ldots \mu_k cx \mu_{k+1} \ldots \mu_n} \quad (54)
$$

3The property (36) used to simplify the results for the Hamiltonian corresponds to the following for the diffeomorphism: $R^{(bx \mu_1 \ldots \mu_n)_c} = (-1)^{n+1} R^{(bx \mu_n \ldots \mu_1)_c}$.

4The same procedure in (50) generates antisymmetric expressions with respect to the interchange of $bx$ with $cx$ and $ax$ with $cx$. 

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The way that ordinary loops appear is simple: one has only to establish that the multivector fields are now the “multitangent fields” \( X^{\mu_1 \cdots \mu_n}(\gamma) \) associated with the loop \( \gamma \). Besides the properties of the multivector fields that belong to the general extended loop group, the multitangents have another particularities like the algebraic constraint and the possibility to express the loop as a composition of open paths. These last properties are in the root of the procedure of reduction mentioned above.

In order to gain simplicity we limit the discussion to a particular case. Let \( \gamma \) be a loop with a triple intersection at the point \( x \) and denote \( \gamma = \gamma_1 \gamma_2 \gamma_3 \), being \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) the three “petals” of the trefoil. It is assumed that the origin \( o \) of the loop lies in \( \gamma_1 \). A multitangent field with three spatial points fixed at \( x \) decomposes in the following way

\[
X^{\mu_1 \cdots \mu_n}(\gamma) = X^{\mu_1 \cdots \mu_j}(\gamma_1) T^{ax}_1 X^{\mu_{j+1} \cdots \mu_n}(\gamma_2) T^{bx}_2 X^{\mu_{k+1} \cdots \mu_n}(\gamma_3) T^{cx}_3 X^{\mu_{k+1} \cdots \mu_n}(\gamma_1) \quad (55)
\]

where \( T^{ax}_m \) is the tangent at \( x \) when one crosses the time \( m \) to this point and \( \gamma_1, \gamma_2, \gamma_3 \) indicates the portion of the loop from \( y \) to \( z \). Using this result it is possible to show that

\[
\epsilon_{abc} R^{(ax, bx) \mu_1 \cdots \mu_n}(\gamma) = -2 \epsilon_{abc} T_{1}^{ax} T_{2}^{bx} T_{3}^{cx} \left[ R^{\mu_1 \cdots \mu_n}(\gamma_1 \gamma_2 \gamma_3) + R^{\mu_1 \cdots \mu_n}(\gamma_1 \gamma_2 \gamma_3) + R^{\mu_1 \cdots \mu_n}(\gamma_1 \gamma_2 \gamma_3) \right] \quad (56)
\]

and

\[
\epsilon_{abc} R^{(ax, bx) \mu_2 \cdots \mu_n}(\gamma) = -2 \epsilon_{abc} T_{1}^{ax} T_{2}^{bx} T_{3}^{cx} \left[ R^{\mu_1 \cdots \mu_n}(\gamma_2 \gamma_1 \gamma_3) + R^{\mu_1 \cdots \mu_n}(\gamma_3 \gamma_2 \gamma_1) + R^{\mu_1 \cdots \mu_n}(\gamma_1 \gamma_2 \gamma_3) \right] \quad (57)
\]

if the set of indices \( \mu_1 \ldots \mu_n \) is cyclic (that is, if \( \mu_1 \ldots \mu_n = (\mu_1 \ldots \mu_n)_c \)). The relevant fact here is that the multitangents of rank \( n \) have been reconstructed and that they appear instead of a product of multitangent fields like in [53]. The reconstruction is generated by the sum in \( k \) that defines the quantities \( R^{(ax, bx) \mu_2 \cdots \mu_n} \) and \( R^{(ax, bx) \mu_3 \cdots \mu_n} \). Suppose now that the greek indices of the above expressions are contracted with appropriate covariant indices \( D_{\mu_1 \cdots \mu_n} \) and let us perform the sum in \( n \) of the resulting expression. We get for the first case:

\[
\epsilon_{abc} \sum_{n=0}^{\infty} D_{\mu_1 \cdots \mu_n} R^{(ax, bx) \mu_1 \cdots \mu_n}(\gamma) = -2 \epsilon_{abc} T_{1}^{ax} T_{2}^{bx} T_{3}^{cx} \left[ \psi(\gamma_1 \gamma_2 \gamma_3) + \psi(\gamma_1 \gamma_2 \gamma_3) + \psi(\gamma_1 \gamma_2 \gamma_3) \right] \quad (58)
\]

with

\[
\psi(\gamma) \equiv \psi[R(\gamma)] = \sum_{n=0}^{\infty} D_{\mu_1 \cdots \mu_n} R^{\mu_1 \cdots \mu_n}(\gamma) \quad (59)
\]
and \( \gamma \) is the rerouted loop. If \( \psi(\gamma) \) is a knot invariant, then the left hand side of (88) acquires a natural geometrical meaning. This is precisely the case of the result (17). Notice that

\[
h_{\mu_1 \mu_2 \mu_3} \left[ R^{(ax, bx) \mu_1 \mu_2 \mu_3} - R^{(ax, bx) \mu_1 \mu_2 \mu_3} \right] =
2h_{\mu_1 \mu_2 \mu_3} \left[ R^{(ax, bx) \mu_1 \mu_2 \mu_3} - R^{(bx, ax) \mu_1 \mu_2 \mu_3} \right]
\] (60)

and

\[
g_{\mu_1 \mu_3 \mu_4} g_{\mu_2 \mu_4} \left[ R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} - R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} \right] =
g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} \left[ 2R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} - 2R^{(bx, ax) \mu_1 \mu_2 \mu_3 \mu_4} \right]
+ R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} - R^{(bx, ax) \mu_1 \mu_2 \mu_3 \mu_4}
\] (61)

Using these results it is immediate to show that

\[
\frac{1}{6} H_0 J_3(R) = \epsilon_{abc} h_{\mu_1 \mu_2 \mu_3} \left[ R^{(ax, bx) \mu_1 \mu_2 \mu_3} - R^{(ax, bx) \mu_1 \mu_2 \mu_3} \right]
+ \epsilon_{abc} g_{\mu_1 \mu_3 \mu_4} g_{\mu_2 \mu_4} \left[ R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} - R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} \right]
+ 3\epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} \left[ R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} + R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} \right]
\] (62)

The analytical expression of the second coefficient of the Jones polynomial is

\[
J_2(R) = -3 \left\{ h_{\mu_1 \mu_2 \mu_3} R^{\mu_1 \mu_2 \mu_3} + g_{\mu_1 \mu_3 \mu_4} g_{\mu_2 \mu_4} R^{\mu_1 \mu_2 \mu_3 \mu_4} \right\}
\] (63)

so,

\[
H_0 J_3(R) = 2\epsilon_{abc} \left\{ J_2 \left[ R^{(ax, bx)} \mu_1 \mu_2 \mu_3 \mu_4 \right] \right\}
+ 18\epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} \left[ R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} + R^{(ax, bx) \mu_1 \mu_2 \mu_3 \mu_4} \right]
\] (64)

We see that the analytical expression of the \( J_2 \) diffeomorphism invariant appears naturally in \( H_0 J_3 \). The remaining contributions generate Gauss-linking numbers when ordinary loops are introduced. In effect, according the results of reference \( [1] \) one finds the following result for the case of the trefoil

\[
\frac{1}{12} H_0(x) J_3(\gamma) = -\epsilon_{abc} T_1^{ax} T_2^{bx} T_3^{cx} \left\{ J_2(\gamma_1 \gamma_2 \gamma_3) - J_2(\gamma_2 \gamma_1 \gamma_3) \right\}
+ 2[\varphi_G(\gamma_1, \gamma_2) - \varphi_G(\gamma_1, \gamma_3)]^2 + 2[\varphi_G(\gamma_2, \gamma_3) - \varphi_G(\gamma_2, \gamma_1)]^2
+ 2[\varphi_G(\gamma_3, \gamma_1) - \varphi_G(\gamma_3, \gamma_2)]^2
\] (65)

where

\[
\varphi_G(\gamma_i, \gamma_j) := g_{\mu_1 \mu_2} X^{\mu_1}(\gamma_i) X^{\mu_2}(\gamma_j)
\] (66)

gives the linking number of the loops \( \gamma_i \) and \( \gamma_j \). The geometrical content of (88) is quite nontrivial. This result valid formally for ordinary loops follows from the properties of the quantities (53) and (54) at the level of extended loops.
The form of (65) suggests that the cancellation of the pairs of $J_2$ and $\varphi_G$ invariants could take place for certain particular topologies of the loop. The possibility that $J_3(\gamma)$ could represent a nonperturbative quantum state of gravity acquires then a new significance. This possibility remembers now the basic property of the “smoothened loops” [18], that is the annihilation of the loop wavefunction by the Hamiltonian when the domain of definition is restricted to loops without intersections. A first approach has not revealed any immediate solution of this kind. This question is currently under progress.

6 The Kauffman bracket

We consider now the Kauffman bracket. As it was shown in Sect. 2, the relevance of the Kauffman bracket in quantum gravity is based on the fact that the exponential of the Chern-Simons form is a solution of the constraints in the connection representation. The properties of the Wilson loops (which are essential for both the Kauffman bracket and the loop transform) are in the root of the fact that

$$H_\Lambda (\text{Kauffman bracket}) \equiv 0$$ \hspace{1cm} (67)

This observation points out to the following: the fact that the Kauffman bracket is annihilated by $H_\Lambda$ is in concordance with the coherence and simplicity of the (conventional) loop representation. Any failure of the condition (67) will immediately rebound on the relationship between the connection and the loop representations through the loop transform.

In what follows we verify (67) to third order in the cosmological constant. For this we need the results of the preceding sections as well as those of [1]. In reference [1] it was demonstrated by direct calculation that

$$H_0 (\varphi_G J_2) - \det q(x) J_2 = 2\epsilon_{abc} \left\{ J_2 [R^{(ax, bx)}]_{cx} - J_2 [R^{(ax, bx)}]_{cx} \right\} + 9\epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3 \mu_4}$$ \hspace{1cm} (68)

where the underline of the two indices in the second term of the r.h.s. represents the following combination of multivector fields:

$$R^{\mu_1 \mu_2 \mu_3 \ldots \mu_n} := \sum_{k=2}^{n-1} \sum_{i=k}^{n} R^{\mu_1 \mu_2 \ldots \mu_k \mu_1 \mu_k+1 \ldots \mu_i \mu_2 \mu_i+1 \ldots \mu_n}$$ \hspace{1cm} (69)

It is possible to prove without difficulty that

$$\epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} R^{(ax, bx)}_{\mu_1 \mu_2 \mu_3 \mu_4} = 2\epsilon_{abc} g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} \times$$

$$[R^{(ax, bx)}_{\mu_1} \epsilon (\mu_2 \mu_3 \mu_4) + R^{(ax, bx)}_{\mu_1 \mu_3 \mu_4} \epsilon \mu_2 + R^{(ax, bx)}_{\mu_1 \mu_3 \mu_4} \epsilon \mu_4 + R^{(ax, bx)}_{\mu_1 \mu_3 \mu_4} \epsilon \mu_2 \mu_4]$$ \hspace{1cm} (70)

and that
\[\epsilon_{abc}g_{\mu_1 \mu_2}g_{\mu_3 \mu_4}K^{(ax,bx)}_{(\mu_1 \mu_2 \mu_3 \mu_4)c} \equiv 0 \quad (71)\]

by symmetry considerations. Introducing (70) and (71) into (68) and using (4) and (64) we obtain

\[H_{\Lambda} K_{\Lambda}^{(3)} = 0 \quad (72)\]

This result reinforces the role of the (conventional) loop transform for quantum gravity.

7 Conclusions

The result (72) follows after a long chain of calculations in the extended loop representation. As we have seen, the calculations are highly nontrivial and require a methodology proper of extended loops. Moreover, the intermediate result (17) admits an interesting geometrical interpretation for ordinary loops that by itself opens a new question about the solutions of the vacuum Hamiltonian constraint. The question is: Is it possible that a topological restriction of the domain of definition of \(J_3(\gamma)\) makes \(H_0 J_3(\gamma) = 0?\) An intriguing fact that points in the same direction is the following: in general \(J_3\) does not satisfy all the Mandelstam identities that are required for the quantum states of gravity. However, it is possible to demonstrate that the Mandelstam identities will be recovered by \(J_3\) precisely for those loops that would make \(H_0 J_3(\gamma) = 0\).

In the analysis, a special care was taken to point out the particularities of the extended loop calculus. The existence of a systematic for the operation of the constraints as well as the intimate relationship existing between the diffeomorphism and the Hamiltonian constraints in the extended loop framework were emphasized in several opportunities. With respect to this point one can add the following: the power of calculation of the extended loop representation allows to raise several important questions about knot theory and quantum gravity. These questions can be summarized as follows: 1) Is it possible to construct in a systematic way analytical expressions of diffeomorphism invariants using the two and three point propagators of the Chern-Simons theory? 2) Are other propagators (besides \(g\) and \(h\)) relevant for quantum gravity? 3) Is it possible to check the Mandelstam identities of the diffeomorphism invariants in a systematic way? 4) There exist analytical expressions constructed in terms of \(g\)'s and \(h\)'s different to that of the exponential of the Gauss self-linking and the Kauffman bracket that are invariant under diffeomorphisms and that satisfy

\[5\text{The restriction of the domain of definition of the loop wavefunctions introduces a new problem: the use of characteristic functions in the loop space. This problem is not completely understood at present and it is shared by the smoothened loops of reference [18].}\]

\[6\text{In reference [1] it was shown that the topological condition required for } H_0 J_3(\gamma) = 0 \text{ (in particular, that } J_2(\gamma_1 \gamma_2 \gamma_3) = J_2(\gamma_2 \gamma_1 \gamma_3) \text{ for the case of the unknot trefoil) makes } J_3(\gamma) \text{ to satisfy all the Mandelstam identities.}\]
the Mandelstam identities? This last point is specially relevant for quantum gravity. These topics are currently under study.

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**Appendix I**

In this appendix we shall see that

\[ \delta_T^{\alpha_k \alpha_l \alpha_m} = \epsilon^{\alpha_k \alpha_l \alpha_m} + (\phi_{z_l} - \phi_{z_m}) g^{\alpha_l \alpha_m} \]  

(73)

From (25) we have

\[ \partial_{\alpha_k} \epsilon^{\alpha_k \alpha_l \alpha_m} = \epsilon^{\alpha_k \alpha_l \alpha_m} - \phi_{z_k, \alpha_k} \epsilon^{\alpha_k \alpha_l \alpha_m} \]

= \epsilon^{\alpha_k \alpha_l \alpha_m} + \phi_{z_k} \partial_{\alpha_k} \epsilon^{\alpha_k \alpha_l \alpha_m} \]  

(74)

The divergence of \( \epsilon^{\alpha_k \alpha_l \alpha_m} \) with respect to the \( \alpha_k \)-entry can be written in the following way

\[ \partial_{\alpha_k} \epsilon^{\alpha_k \alpha_l \alpha_m} = \epsilon^{\alpha_k \alpha_l \alpha_m} \delta(z_k - z_l) \delta(z_k - z_m) \]

\[ = \delta(z_k - z_l) \epsilon^{\alpha_l \alpha_m} \partial_{\alpha_k} \delta(z_k - z_m) - \delta(z_k - z_m) \epsilon^{\alpha_l \alpha_m} \partial_{\alpha_k} \delta(z_k - z_l) \]

where in the last step we have used (22). Introducing (75) into (74) we get directly (73).

**Appendix II**

Here we consider in detail the calculation of \( F_{ab}(x) h_{\ldots g^{**} h_{\ldots R^{-}} \ldots} \). We have

\[ F_{ab}^{\mu_1}(x) g_{\mu_1 \mu_2} g^{\alpha \beta} h_{\mu_3 \mu_4} \]

= \[ -\epsilon_{abc} \delta_T^{\alpha \beta} g^{\alpha \beta} h_{\beta \mu_3 \mu_4} \]

(76)
\[ \delta_T \frac{dz}{\alpha} \delta_\beta h_{\mu_1 \mu_2} = \delta_T \frac{dz}{\alpha} \delta_\beta \pi_1 \pi_2 \pi_3 g_{\pi_2 \pi_3 \pi_4} \]

\[ = \{ \epsilon \frac{dz}{\pi_1} + (\phi \frac{dz}{\pi_2} - \phi \frac{dz}{\pi_3}) \} g_{\pi_2 \pi_3 \pi_4} \]

\[ = \epsilon \frac{dz}{\pi_2} g_{\pi_2 \pi_3 \pi_4} + \phi \frac{dz}{\pi_2} \left( \delta_T \epsilon \frac{dz}{\mu_4} g_{\epsilon \mu_3} - \delta_T \epsilon \frac{dz}{\mu_3} g_{\epsilon \mu_4} \right) \]  (77)

The chain stops at this point. Introducing (77) into (76) we get after a few direct manipulations the result (83).

**Appendix III**

In this appendix we shall demonstrate that the quantity

\[ I_{ax \cdot bx \cdot \mu_1 \mu_2} := 2h_{ax \cdot bx \cdot \alpha} g^{\alpha \beta} h_{\mu_1 \mu_2 \beta} - h_{\mu_2 \alpha} \{ ax h_{bx} \} \mu_1 \beta g^{\alpha \beta} \]

\[ + 2\epsilon_{abc} \phi^{cz} (\phi^{\frac{dz}{z}} - \phi^{\frac{dz}{\pi_1}}) h_{dz \mu_1 \mu_2} + 2\epsilon_{abc} \phi^{cz} \phi^{\frac{dz}{\mu_1}} \phi^{\frac{dz}{\mu_2}}  \]

\[ + \epsilon_{abc} \phi^{cz} (g_{\mu_1 dz \mu_2 ey} - g_{\mu_1 ey g_{\mu_2 dz}}) \]  (78)

vanishes identically (in a formal sense). In the above expression, \( x \) is the point where the Hamiltonian is applied, \( x_i \) is the spatial part of \( \mu_1 \) and \( z \) and \( y \) are continuous variables integrated in \( \mathcal{R}^3 \). We start by considering the following decompositon of the three point propagator:

\[ h_{ax \mu_\alpha} = \epsilon \frac{dz}{e_2} \epsilon \frac{dz}{e_3} \epsilon \frac{dz}{e_4} \phi^{\pi_1} \phi^{\pi_2} y_{\mu_1 \mu_2 \mu_3} = \phi^{\pi_1} (g_{\mu_1 ey}) \]  (79)

When \( \mu \equiv bx \) the above expression is reduced to

\[ h_{ax \cdot bx \cdot \alpha} = \epsilon_{abc} \phi^{cy} \phi^{ey} y_{\alpha ey} \]  (80)

Using this fact the first term of \( I_{ax \cdot bx \cdot \mu_1 \mu_2} \) can be written in the following way

\[ h_{ax \cdot bx \cdot \alpha} g^{\alpha \beta} h_{\mu_1 \mu_2 \beta} = -\epsilon_{abc} \phi^{cz} \phi^{\frac{dz}{\mu_1 \mu_2} h_{dz} \mu_1 \mu_2} \]  (81)

The next step is to develop the second contribution of (78) in terms of \( g \)'s. We have

\[ h_{\mu_2 \alpha} [ax h_{bx}] \mu_1 \beta g^{\alpha \beta} \]

\[ = h_{\alpha ax} \{ \mu_1 h_{\mu_2} \} bx \beta g^{\alpha \beta} \]

\[ = \epsilon \frac{dz}{e_2} \epsilon \frac{dz}{e_3} g_{e_2 e_3} \epsilon \frac{dz}{\mu_1 \mu_2 e_1 y g_{ax d_2 g_{bx e_2 g_{ax d_2 e_1 y g_{d_1 e_3 y}}}} \]  (82)

The \( b \) index can be moved using the following identity

\[ \epsilon \frac{dz}{e_2} \epsilon \frac{dz}{e_3} g_{bx e_2 g_{ax}} = \delta_{\epsilon_3} \phi^{e_1 y} - \delta_{\epsilon_1} \phi^{e_3 y} \]  (83)

We get from (82)
Applying the same procedure to the third term of (78) we obtain

\[ h_{\mu_2 \alpha} [ax h_{bx}]_{\mu_1 \beta} g^{\alpha \beta} = \epsilon^{d_1 d_2 d_3} g_{ax} d_{d_2} \{ g_{d_3 z} [\mu_1, g_{\mu_2}]_{\epsilon y} g_{dx z} by - g_{dx z} [\mu_1, g_{\mu_2}]_{by} g_{dx z} ey \} \phi_x^{ey} \]

\[ = - h_{ey ax} [\mu_1, g_{\mu_2}]_{by} \phi^{ey}_x + \epsilon^{d_1 d_2 d_3} g_{dx z} by g_{ax} d_{d_2} g_{dx z} [\mu_1, g_{\mu_2}]_{ey} \phi_x^{ey} \]  
\[ = - h_{ey ax} [\mu_1, g_{\mu_2}]_{by} \phi^{ey}_x + g_{ax} b_z g_{dz} [\mu_1, g_{\mu_2}]_{ey} \phi^{dz}_y \phi_x^{ey} \]  
\[ = - g_{ax} d_z g_{bx} [\mu_1, g_{\mu_2}]_{ey} \phi^{dz}_y \phi_x^{ey} \]  
(84)

Developing now \( \epsilon^{d_1 d_2 d_3} g_{dx z} by \) according to (83) we obtain from (84) a term with the \( a \) and \( b \) indices attached to the same \( g \):

\[ h_{\mu_2 \alpha} [ax h_{bx}]_{\mu_1 \beta} g^{\alpha \beta} = - h_{ey ax} [\mu_1, g_{\mu_2}]_{by} \phi^{ey}_x + g_{ax} b_z g_{dz} [\mu_1, g_{\mu_2}]_{ey} \phi^{dz}_y \phi_x^{ey} \]  
\[ = - g_{ax} d_z g_{bz} [\mu_1, g_{\mu_2}]_{ey} \phi^{dz}_y \phi_x^{ey} \]  
(85)

Notice that using (79) we can write

\[ h_{ey ax} [\mu_1, g_{\mu_2}]_{by} \phi^{ey}_x = g_{ax} [\mu_1, g_{\mu_2}]_{by} g_{dz} \phi^{dz}_x \phi^{ey}_x - g_{dx} [\mu_1, g_{\mu_2}]_{by} g_{dz} a_y \phi^{dz}_y \phi_x^{ey} \]  
(86)

and that

\[ g_{dx} e_y \phi^{dz}_x \phi_x^{ey} \equiv 0 \]  
(87)

Introducing then (86) into (85) one gets

\[ h_{\mu_2 \alpha} [ax h_{bx}]_{\mu_1 \beta} g^{\alpha \beta} = \epsilon_{abc} \phi^c x g_{dx} [\mu_1, g_{\mu_2}]_{ey} \phi^{dz}_y \phi_x^{ey} \]  
(88)

Applying the same procedure to the third term of (78) we obtain

\[ \epsilon_{abc} \phi^c x h_{dx z} g_{dx} \mu_1^{\mu_2} = \epsilon_{abc} \phi^c x \phi^{dz}_x \phi^{e_1 e_2 e_3}_x g_{e_1 y} g_{e_2 y} \mu_1 g_{e_3 y} \mu_2 \]

\[ = \epsilon_{abc} \phi^c x \phi^{dz}_x \phi^{e_1 e_2 e_3}_x (g_{dz} e_y g_{a_1 y} \mu_2 - g_{dz} a_1 y g_{e_3 y} \mu_2) \]

\[ = \epsilon_{abc} \phi^c x \phi^{dz}_y \phi^{e_1 e_2 e_3}_x g_{dx} \mu_1 g_{e_3 y} \mu_2 \]  
(89)

where in the last step we used the following identity

\[ \phi^{dz}_x g_{dz} a_1 y \equiv - \phi^{dz}_y g_{dz} a_1 x_1 \]  
(90)

Introducing now (81), (88) and (89) into (78) we get

\[ I_{ax bx \mu_1 \mu_2} = 0 \]  
(91)

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