Upper and lower bounds for the function $S(t)$ on the short intervals$^1$

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Abstract. We prove under RH the existence of a very large positive and negative values of the argument of the Riemann zeta function on a very short intervals.

In this paper, we study an upper and lower bounds for the function $$S(t) = \pi^{-1} \arg \zeta\left(\frac{1}{2} + it\right)$$ on the short intervals. We refer to [1] for the definition and for the basic properties of $S(t)$ and to [2] for the history of the question. Here we mention only the recent result of R.N. Boyarinov [3]:

Theorem (R.N. Boyarinov). Let $T > T_0 > 0$ and let $$\sqrt{\log \log T} < H \leq (\log T)(\log \log T)^{-\frac{3}{2}}.$$ If the Riemann hypothesis is true then the inequalities

$$\sup_{T-H \leq t \leq T+2H} \{\pm S(t)\} \geq \frac{1}{900} \frac{\sqrt{\log H}}{\log \log H}$$

hold.

In what follows, we prove the similar assertion for the case when $H$ is essentially smaller than $\sqrt{\log \log T}$. Namely, we prove

Theorem. Let $m \geq 2$ be any fixed integer, $T > T_0(m) > m$ and let

$$\frac{2(2m \log \log T)^{2m}}{\log \log \log \log T} \leq H \leq \sqrt{\log \log T}.$$ If the Riemann hypothesis is true then the inequalities

$$\sup_{T-H \leq t \leq T+2H} \{\pm S(t)\} \geq \frac{1}{50\pi} \frac{\sqrt{\log H}}{(8m \log \log H)^m}$$

hold.

Notations. We use the following notations:
- $m \geq 2$ is any fixed integer;
- $\Phi(u) = \exp\left(-\frac{u^{2m}}{2m}\right)$;
- $\Lambda(n)$ denotes von Mangoldt’s function: $\Lambda(n) = \log p$ if $n = p^m$ and $p$ is prime, and $\Lambda(n) = 0$ otherwise;

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– \hat{f} denotes the Fourier transform of \( f \), that is 
\[
\hat{f}(\lambda) = \int_{-\infty}^{+\infty} f(u) e^{-i\lambda u} du;
\]
– \( \theta, \theta_1, \theta_2, \ldots \) denote complex numbers whose absolute values do not exceed one, different in different relations.

We need some auxiliary assertions.

**Lemma 1.** The function \( \hat{\Phi}(\lambda) \) decreases monotonically on the segment \( 0 \leq \lambda \leq 1 \) from the value 
\[
\hat{\Phi}(0) = \frac{2\Gamma(1/(2m))}{(2m)^{1-1/(2m)}}
\]
\[
\Phi\left(\frac{\pi}{4}\right),
\]
Moreover, there exists the constant \( \lambda_0 = \lambda_0(m) \) such that the inequality 
\[
|\hat{\Phi}(\lambda)| < \frac{5}{\sqrt{m}} |\lambda|^{-\beta} \exp\left( -\frac{|\lambda|^\alpha}{\alpha} \sin(\pi \kappa) \right)
\]
holds for any real \( \lambda, |\lambda| > \lambda_0 \), with 
\[
\alpha = \frac{2m}{2m-1}, \quad \beta = \frac{m-1}{2m-1}, \quad \kappa = \frac{1}{2(2m-1)}.
\]

**Proof.** Differentiating the equation 
\[
\hat{\Phi}(\lambda) = 2 \int_{-\infty}^{+\infty} \Phi(u) \cos(\lambda u) du
\]
with respect to \( \lambda \), we obtain \( \hat{\Phi}'(\lambda) = -2j(\lambda) \) where 
\[
j(\lambda) = \int_{-\infty}^{+\infty} u \Phi(u) \sin(\lambda u) du.
\]
Suppose that \( \lambda > 0 \). Using the inequalities \( \sin(\lambda u) \geq \frac{2}{\pi} \lambda u \) for \( 0 \leq u \leq \frac{\pi}{2\lambda} \) and \( \sin(\lambda u) \geq 0 \) for \( \frac{\pi}{2\lambda} \leq u \leq \frac{\pi}{\lambda} \) and splitting the integral \( j(\lambda) \) into the sum 
\[
\left( \int_{\frac{\pi}{2\lambda}}^{\frac{\pi}{\lambda}} + \int_{\frac{\pi}{\lambda}}^{+\infty} \right) u \Phi(u) \sin(\lambda u) du = j_1 + j_2 + j_3,
\]
we get 
\[
j_1 \geq \frac{2\lambda}{\pi} \int_{0}^{\frac{\pi}{2\lambda}} u^2 \Phi(u) du = \frac{2\lambda}{\pi} \Phi\left(\frac{\pi}{2\lambda}\right) \int_{0}^{\frac{\pi}{2\lambda}} u^2 du = \frac{1}{3} \left(\frac{\pi}{2\lambda}\right)^2 \Phi\left(\frac{\pi}{2\lambda}\right), \quad j_2 > 0;
\]
\[ |j_3| \leq \int_\frac{\pi}{4}^{+\infty} u\Phi(u)du = \left(2m\right)^{\frac{1}{2}}-1 \int_1^{+\infty} w^{\frac{1}{2m}-1}e^{-w}dw < \]
\[ < \left(\frac{\lambda}{\pi}\right)^{2(m-1)} \int_0^{+\infty} e^{-w}dw = \left(\frac{\lambda}{\pi}\right)^{2(m-1)}\Phi\left(\frac{\pi}{\lambda}\right). \]

Hence,

\[ j(\lambda) > j_1 - |j_3| > \frac{1}{3}\left(\frac{\pi}{2\lambda}\right)^2\Phi\left(\frac{\pi}{2\lambda}\right)\left\{1 - 12\left(\frac{\lambda}{\pi}\right)^{2m}\exp\left(-\frac{1}{2m}(1 - 2^{-2m})\left(\frac{\pi}{\lambda}\right)^{2m}\right)\right\}. \]

If \(0 < \lambda \leq 1\) then the expression in the figure brackets is bounded from below by the value

\[ 1 - 12\frac{\pi^4}{\pi^4} \exp\left(-\frac{15}{4}\left(\frac{\pi}{2}\right)^4\right) > 1 - 2 \cdot 10^{-10} > 0 \]

uniformly for \(m \geq 2\). Therefore, \(\Phi'(\lambda) < 0\) for \(0 < \lambda \leq 1\). Thus we prove the first assertion of the lemma.

Next, splitting the expression for \(\widehat{\Phi}(1)\) into the sum

\[ 2 \int_{-\infty}^{+\infty} \Phi(u) \cos u\;du = 2\left(\int_0^{\frac{\pi}{4}} + \int_\frac{\pi}{4}^{\frac{\pi}{2}} + \int_\frac{\pi}{2}^{+\infty}\right) \Phi(u) \cos u\;du = 2(j_1 + j_2 + j_3) \]

and using the same arguments as above, we obtain:

\[ j_1 > \Phi\left(\frac{\pi}{4}\right)\int_0^{\frac{\pi}{4}} \cos u\;du = \frac{1}{\sqrt{2}}\Phi\left(\frac{\pi}{4}\right), \quad j_2 > 0, \]

\[ |j_3| \leq (2m)^{\frac{1}{2m}-1} \int_1^{+\infty} w^{\frac{1}{2m}-1}e^{-w}dw \leq \left(\frac{2}{\pi}\right)^{2m} \Phi\left(\frac{\pi}{2}\right), \]

and hence

\[ \widehat{\Phi}(1) > j_1 - |j_3| > 2\left\{\frac{1}{\sqrt{2}}\Phi\left(\frac{\pi}{4}\right) - \left(\frac{2}{\pi}\right)^4 \Phi\left(\frac{\pi}{2}\right)\right\} = \]

\[ = \sqrt{2}\Phi\left(\frac{\pi}{4}\right)\left\{1 - \sqrt{2}\left(\frac{2}{\pi}\right)^4 \Phi\left(\frac{\pi}{2}\right)\right\}. \]
One can note that
\[
\frac{\Phi\left(\frac{\pi}{2}\right)}{\Phi\left(\frac{\pi}{4}\right)} = \exp\left\{ -\frac{1}{2m} \left(1 - 2^{-2m}\right) \left(\frac{\pi}{2}\right)^{2m} \right\} \leq \exp\left( -\frac{15}{64} \left(\frac{\pi}{2}\right)^4 \right)
\]
for any \( m \geq 2 \). Thus we arrive at the desired bound for \( \hat{\Phi}(1) \).

Finally, the last assertion of the lemma follows from the formula
\[
\int_{-\infty}^{+\infty} \exp\left( -\frac{u^2}{2m} + i\lambda u \right) du =
\]
\[
= 4\sqrt{\pi\kappa} \exp\left( -\frac{\lambda}{\alpha} \sin(\pi\kappa) \right) \left\{ \cos\left( \frac{\lambda}{\alpha} \cos(\pi\kappa) \right) + O(\lambda^{-\alpha}) \right\},
\]
where \( \lambda \to +\infty \) and \( \alpha, \beta, \kappa \) are defined as above (see [4, §7.1]).

**Lemma 2.** The following inequalities hold true:
\[
|S(t)| \leq \begin{cases} 1, & \text{if } |t| \leq 280, \\ 1.05 \log |t|, & \text{if } |t| > 280. \end{cases}
\]

**Proof.** The first inequality follows from the data of Table 1 from [5] and the second one follows from the classical estimate of R.J. Backlund [6]:
\[
|S(t)| < 0.1361 \log |t| + 0.4422 \log \log |t| + 4.3451 \leq (\log |t|) \left( 0.1361 + 0.4422 \frac{\log \log 280}{\log 280} + \frac{4.3451}{\log 280} \right) < 1.05 \log |t|.
\]
The lemma is proved.

Let \( \tau > 1 \) and \( f(u) = \Phi(\tau u) \).

**Lemma 3.** If the Riemann hypothesis is true then the formula
\[
\int_{-\infty}^{+\infty} f(u)S(t + u) du = -\frac{1}{\pi} \sum_{n=2}^{+\infty} \Lambda(n) \frac{\sin(t \log n)}{\log n} \hat{f}(\log n) - 2 \int_{0}^{1/2} f(-t - iu) du
\]
holds.

**Proof.** The proof of this assertion repeats word-by-word (with the minor changes) the proof of Theorem 3 from [1]. The difference is that we need to use the explicit formula for \( f(z) \) instead of the inequality \( |f(z)| \leq c(|z| + 1)^{-(1+\alpha)} \).

**Lemma 4.** Let \( y > y_0 > 0, \mu, \nu \geq 0, k \geq 1, k = \mu + \nu, \) and let \( p_1, \ldots, p_\mu, q_1, \ldots, q_\nu \) range over the primes in the interval \((1, y]\) that satisfy the condition \( p_1 \ldots p_\mu \neq q_1 \ldots q_\nu \).
If \( a(p) \) is a sequence of complex numbers in which \( |a(p)| \leq \delta \) for any prime \( p \leq y \), then the integral
\[
I = \int_T^{T+H} \sum_{p_1, \ldots, p_\mu, q_1, \ldots, q_\nu} \frac{a(p_1) \ldots a(p_\mu)}{\sqrt{p_1 \ldots q_\mu}} \frac{\overline{a(q_1) \ldots \overline{a(q_\nu)}}}{\sqrt{q_1 \ldots q_\nu}} \left( \frac{p_1 \ldots p_\mu}{q_1 \ldots q_\nu} \right)^{it} dt
\]
satisfies the estimate \( |I| \leq (\delta^2y^3)^k \).

**Proof.** It is lemma 2 from [7, §2.1].

**Lemma 5.** Let \( k \geq 1 \) be an integer, let \( M > 0 \) and let a real function \( W(t) \) satisfies the inequalities
\[
\int_T^{T+H} W^{2k}(t) dt \geq HM^{2k}, \quad \int_T^{T+H} W^{2k+1}(t) dt \leq \frac{1}{2} HM^{2k+1}.
\]
Then
\[
\sup_{T \leq t \leq T+H} \{ \pm W(t) \} \geq \frac{M}{2}.
\]

**Proof.** It is a slight modification of lemma 4 from the paper of K.-M. Tsang [8].

**Proof of the theorem.** Let \( \tau = 2 \log \log H \) and suppose that \( T \leq t \leq T + H \). Applying lemma 3 to the function \( f(u) = \Phi(\tau u) \) we obtain
\[
\tau \int_{-\infty}^{+\infty} \Phi(\tau u) S(t + u) du = - \frac{1}{\pi} V(t) + R_1(t),
\]
where
\[
V(t) = \sum_{n=2}^{+\infty} \frac{\Lambda(n)}{\sqrt{n}} \frac{\sin(t \log n)}{\log n} \tilde{\Phi} \left( \frac{\log n}{\tau} \right), \quad R_1(t) = 2 \int_0^{1/2} \Phi(\tau(t + i u)) du.
\]
Since
\[
\text{Re}(t + i u)^{2m} = t^{2m} \left( 1 + \sum_{\nu=1}^{m} (-1)^\nu \left( \frac{2m}{2\nu} \right) \left( \frac{u}{t} \right)^{2m-\nu} \right)
\]
and since the absolute value of the last sum is bounded from above by the values
\[
\sum_{\nu=1}^{2m} \left( \frac{2m}{\nu} \right) (2t)^{-2\nu} = \left( 1 + \frac{1}{2t} \right)^{2m} - 1 \leq \frac{1}{2t} \left( 1 + \frac{1}{2t} \right)^{2m-1} < \frac{e}{2t},
\]
we find that
\[
|\Phi(\tau(t + i u))| = \exp \left\{ - \frac{\tau^{2m}}{2m} \text{Re}(t + i u)^{2m} \right\} < \exp \left\{ - \frac{(\tau t)^{2m}}{8m} \right\},
\]
\[
|R_1(t)| \leq 2 \cdot \frac{1}{2} \exp \left\{ - \frac{(\tau t)^{2m}}{8m} \right\}.
\]
Further, let \( X = \exp ((4m\tau)^{2m}) \) and let \( R_2(t) \) be the contribution to the sum \( V(t) \) from the terms with \( n > X \). Since \( \frac{\log n}{\tau} > \lambda_0 \) for any \( n > X \) (\( \lambda_0 \) is defined in lemma 1) then lemma 1 implies that

\[
|\hat{\Phi}(\frac{\log n}{\tau})| \leq \frac{5}{\sqrt{m}} \left( \frac{\tau}{\log n} \right)^\beta \exp \left\{ -\frac{\sin (\pi \kappa)}{\alpha} \left( \frac{\log n}{\tau} \right)^\alpha \right\},
\]

where the values \( \alpha, \beta \) and \( \kappa \) are defined above. By the inequality

\[
\sin (\pi \kappa) \geq \frac{2\pi}{\pi \kappa} = \frac{1}{2m}
\]

we conclude from (1) that

\[
|R_2(t)| < \frac{5}{\sqrt{m}} \left( \frac{\tau}{\log X} \right)^\beta \sum_{n>X} \frac{1}{\sqrt{n}} \exp \left\{ -\frac{1}{2m} \left( \frac{\log n}{\tau} \right)^\alpha \right\}.
\]

In order to estimate the last sum, we take \( r_0 = \left\lceil \frac{\log X}{\tau} \right\rceil = [4m\tau^{2m-1}] \) and split the domain of summation into the segments of the type \( e^{r\tau} < n \leq e^{(r+1)\tau} \), \( r = r_0, r_0 + 1, \ldots \). The sum over such segment is bounded by the values

\[
\exp \left\{ -\frac{r^\alpha}{2m} \right\} \sum_{e^{r\tau} < n \leq e^{(r+1)\tau}} \frac{1}{\sqrt{n}} < 2 \exp \left\{ \frac{(r+1)\tau}{2} - \frac{r^\alpha}{2m} \right\} <
\]

\[
< \exp \left\{ r\tau - \frac{r^\alpha}{2m} \right\} < \exp \left\{ -\frac{r^\alpha}{4m} \right\}.
\]

Hence,

\[
|R_2(t)| < \frac{5}{\sqrt{m}} \left( \frac{\tau}{\log X} \right)^\beta \int_{(4m\tau)^{2m-1}}^{+\infty} \exp \left\{ -\frac{u^\alpha}{2m} \right\} du =
\]

\[
= \frac{5}{\sqrt{m}} \left( \frac{\tau}{(4m\tau)^{2m}} \right)^\beta \left( \frac{2m}{\pi} \right)^\frac{\beta}{\alpha} \int_{(4m\tau)^{2m}}^{+\infty} w^{\frac{1}{\alpha}-1} e^{-w} dw <
\]

\[
< \frac{5}{\sqrt{m}} \left( \frac{\tau}{(4m\tau)^{2m}} \right)^\beta \frac{1}{2}\exp \left\{ -\frac{(4m\tau)^{2m}}{2m} \right\} =
\]

\[
= \frac{5(4m)^{2(2m-1)}}{(4m\tau)^m} \exp \left\{ -\frac{(4m\tau)^{2m}}{2m} \right\} \leq \frac{5\sqrt{2}}{(4m\tau)^m} \exp \left\{ -\frac{(4m\tau)^{2m}}{2m} \right\}.
\]

Finally, let \( R_3(t) \) be the sum over \( n = p^\nu \leq X \) with \( \nu \geq 2 \). Then the obvious estimate

\[
|\hat{\Phi}(\frac{\log n}{\tau})| = \left| \int_{-\infty}^{+\infty} \Phi(u)n^{-iu} du \right| \leq \hat{\Phi}(0)
\]
yields

\[
|R_3(t)| \leq \hat{\Phi}(0) \sum_{\nu=2}^{+\infty} \sum_{p \leq X^{1/\nu}} \frac{p^{-\nu/2}}{\nu} \leq \hat{\Phi}(0) \left( \frac{1}{2} \sum_{p \leq \sqrt{X}} \frac{1}{p} + \frac{1}{3} \sum_{\nu=3}^{+\infty} \frac{p^{-\nu/2}}{\nu} \right) < \\
\hat{\Phi}(0) \left( \frac{1}{2} \log \log X + c \right) = \hat{\Phi}(0)(m \log (4m\tau) + c),
\]

where \( c > 0 \) is a sufficiently large absolute constant.

Thus we get

\[
\tau \int_{-\infty}^{+\infty} \Phi(\tau u)S(t+u) \, du = -\frac{1}{\pi} W(t) + \theta_1 Q_1,
\]

where

\[
W(t) = \sum_{p \leq x} \frac{a(p)}{\sqrt{p}} \sin(t \log p), \quad a(p) = \Phi\left(\frac{\log p}{\tau}\right),
\]

\[
Q_1 = \exp \left\{ -\frac{(\tau t)^{2m}}{8m} \right\} + \frac{5\sqrt{2}}{(4m\tau)^m} \exp \left\{ -\frac{(4m\tau)^{2m}}{2m} \right\} + \hat{\Phi}(0)(m \log (4m\tau) + c) < \\
\hat{\Phi}(0)(m \log (4m\tau) + 2c).
\]

Now let us consider the integrals

\[
I_1 = \int_{H}^{+\infty} \Phi(\tau u)S(t+u) \, du, \quad I_2 = \int_{-\infty}^{-H} \Phi(\tau u)S(t+u) \, du.
\]

Splitting \( I_1 \) into the sum

\[
I_1 = \left( \int_{H}^{T} + \int_{T}^{+\infty} \right) \Phi(\tau u)S(t+u) \, du = I_1^{(1)} + I_1^{(2)}
\]

and applying lemma 2 we obtain:

\[
|I_1^{(1)}| \leq 1.05 \int_{H}^{T} \Phi(\tau u) \log (t+u) \, du < \frac{1.1 \log T}{\tau} \int_{H\tau}^{+\infty} \Phi(v) \, dv \leq \frac{1.1}{\tau} \Phi(H\tau) \log T \frac{T}{(H\tau)^{2m-1}},
\]

\[
|I_1^{(2)}| \leq 1.05 \int_{T}^{+\infty} \Phi(\tau u) \log (t+u) \, du < \frac{1.1}{\tau} \int_{T\tau}^{+\infty} \Phi(v) \log v \, dv \leq \frac{1.1}{\tau} \frac{2m\Phi(T\tau) \log (T\tau)}{(T\tau)^{2m-1}},
\]

and hence

\[
|I_1| < \frac{1.2}{\tau} \frac{\Phi(H\tau) \log T}{(H\tau)^{2m-1}}.
\]
Next, we split the integral $I_2$ into the sum
\[ \int_{H}^{+\infty} \phi(\tau u) S(t-u) \, du = \left( \int_{H}^{t-10^2} + \int_{t-10^2}^{t+10^2} + \int_{t+10^2}^{+\infty} \right) \phi(\tau u) S(t-u) \, du = I_2^{(1)} + I_2^{(2)} + I_2^{(3)}. \]

Applying the first inequality of lemma 2 to $I_2^{(2)}$ and the second one to the estimation of $I_2^{(1)}$ and $I_2^{(3)}$ we obtain
\[ |I_2| < \frac{1.2 \phi(H\tau) \log T}{\tau (H\tau)^{2m-1}}, \quad |I_1| + |I_2| < \frac{2.4 \phi(H\tau) \log T}{\tau (H\tau)^{2m-1}}. \]

Finally, we have
\[ j(t) = \tau \int_{-H}^{H} \phi(\tau u) S(t+u) \, du = -\frac{1}{\pi} W(t) + \theta Q_2, \]
where
\[ Q_2 = \hat{\phi}(0) \left( m \log (4m\tau) + 2c \right) + \frac{2.4 \phi(H\tau) \log T}{\tau (H\tau)^{2m-1}}. \]

Since $H\tau > (2m \log \log T)^{1/2m}$ for $H$ and $m$ under considering, we find that
\[ \Phi(H\tau) < (\log T)^{-1}, \quad Q_2 < 2\hat{\Phi}(0)m \log (4m\tau). \]

Now let us take $k = \left\lfloor \frac{\log H}{5 \log X} \right\rfloor$ and define the integrals $I(k)$ and $J(k)$ by the following relations:
\[ I(k) = \int_{T}^{T+H} W^{2k}(t) \, dt, \quad J(k) = \int_{T}^{T+H} W^{2k+1}(t) \, dt. \]

Writing $W(t)$ as
\[ \frac{1}{2i} \left( U(t) - \overline{U}(t) \right), \quad U(t) = \sum_{p \leq X} \frac{a(p)}{\sqrt{p}} p^it, \]
we find that
\[ I(k) = (2i)^{-2k} \sum_{\nu=0}^{2k} (-1)^{\nu} \binom{2k}{\nu} j_\nu, \quad j_\nu = \int_{T}^{T+H} U^\nu(t) \overline{U}^{2k-\nu}(t) \, dt. \]

The application of lemma 4 with $\delta = \hat{\Phi}(0)$ to the case $\nu \neq k$ yields:
\[ |j_\nu| < \left( \hat{\Phi}(0) X^{3/2} \right)^{2k} < X^{4k} \leq H^{\delta}. \]
The same estimate is valid for the contribution to \( j_k \) of the terms under the condition \( p_1 \ldots p_k \neq q_1 \ldots q_k \). Hence,

\[
I(k) = 2^{-2k} \binom{2k}{k} H \mathcal{S}_k + \theta H^\frac{4}{5},
\]

where

\[
\mathcal{S}_k = \sum_{\substack{p_1 \ldots p_k = q_1 \ldots q_k \leq X \atop p_1, \ldots, p_k \neq q_1, \ldots, q_k}} \frac{a^2(p_1) \ldots a^2(p_k)}{p_1 \ldots p_k}.
\]

In order to estimate the sum \( \mathcal{S}_k \) from the below, we truncate the sum by replacing the upper bound for \( p_1, \ldots, q_k \) by the value \( Y = e^\tau = (\log H)^2 < X \). Then it follows from lemma 1 that the inequalities

\[
a(p) = \Phi\left(\frac{\log p}{\tau}\right) \geq \Phi(1)
\]

hold for any \( p \leq Y \). Further, if we retain the terms in the truncated sum that correspond to the tuples \( (p_1, \ldots, p_k) \) involving no repetitions and using the fact that the number of solutions \( (q_1, \ldots, q_k) \) of the equation \( p_1 \ldots p_k = q_1 \ldots q_k \) is equal to \( k! \), we obtain that

\[
\mathcal{S}_k = (\Phi(1))^{2k} \sum_{p_1 \ldots p_k = q_1 \ldots q_k \leq Y} (p_1 \ldots p_k)^{-1} \geq k! \Phi(1)^{2k} \sum_{p_1, \ldots, p_k \leq Y \atop p_1, \ldots, p_k \text{ are distinct}} (p_1 \ldots p_k)^{-1}.
\]

Applying the arguments from [2] (the estimate of the sum \( \Sigma \)) to the estimation of the last sum, we find that

\[
\mathcal{S}_k \geq k! \Phi(1)^{2k} \left( \sum_{2k \log k < p \leq Y} \frac{1}{p} \right)^k \geq k! \Phi(1)^{2k} \left( \sum_{\sqrt{Y} < p \leq Y} \frac{1}{p} \right)^k \geq k! \left( \frac{4}{5} \hat{\Phi}(1) \right)^{2k},
\]

\[
I(k) > \frac{(2k)!}{k!} \frac{H}{2^{2k}} \left( \frac{4}{5} \hat{\Phi}(1) \right)^{2k} - H^\frac{4}{5} > \frac{e}{2} \left( \frac{4k}{e} \right)^k \left( \frac{\Phi(1)}{5} \right)^{2k} - H^\frac{4}{5} > HM^{2k},
\]

where

\[
M = \frac{2}{5} \hat{\Phi}(1) \sqrt{\frac{k}{e}} > 2.
\]

Finally, lemma 4 yields:

\[
|J(k)| < \left( \hat{\Phi}(0) X^\frac{3}{2} \right)^{2k+1} < X^{4k} \leq H^\frac{4}{5} < \frac{1}{2} HM^{2k+1}.
\]

Now it follows from lemma 5 that there exist the values \( t_0 \) and \( t_1 \) such that \( T \leq t_0, t_1 \leq T + H \) such that \( W(t_0) < -0.5M \) and \( W(t_1) > 0.5M \). Thus we have

\[
j(t_0) > -\frac{1}{\pi} W(t_0) - Q_2 > \frac{M}{2\pi} - 2\hat{\Phi}(0)m \log (4m\tau),
\]

\[
j(t_1) < -\frac{1}{\pi} W(t_1) + Q_2 < -\frac{M}{2\pi} + 2\hat{\Phi}(0)m \log (4m\tau).
\]
Setting $M_j = \sup_{|u| \leq H} (-1)^j S(t_j + u)$ for $j = 0, 1$, we obviously have
\[
j(t_0) < M_0 \tau \int_{-H}^{H} \Phi(\tau u) \, du < M_0 \hat{\Phi}(0), \quad j(t_1) > M_1 \hat{\Phi}(0)
\]
and therefore
\[
(-1)^j M_j > \mu, \quad \mu = \hat{\Phi}(0) \frac{M}{2\pi} - 2m \log (4m\tau).
\]
Finally, applying lemma 1 together with the inequality
\[
\Gamma \left( \frac{1}{2m} \right) = 2m \Gamma \left( 1 + \frac{1}{2m} \right) \leq 2m,
\]
we obtain
\[
\mu \geq \frac{1}{5\pi} \frac{\hat{\Phi}(1)}{\hat{\Phi}(0)} \sqrt{\frac{k}{e}} - 2m \log (4m\tau) > \\
> \frac{1}{8\pi} \frac{\Phi \left( \frac{\pi}{4} \right)}{\Gamma \left( \frac{1}{2m} \right)} (2m)^{\frac{1}{2m}} \sqrt{\frac{1}{e}} \left( \frac{\log H}{5(4m\tau)^{2m}} - 1 \right) - 2m \log (4m\tau) > \\
> \frac{1}{32\pi} \frac{\Phi \left( \frac{\pi}{4} \right)}{2m} (2m)^{\frac{1}{2m}} \frac{\sqrt{\log H}}{(4m\tau)^{1}} \geq \\
\geq \frac{1}{32\pi \sqrt{2}} \exp \left\{ - \frac{1}{4} \left( \frac{\pi}{4} \right)^{4} \right\} \frac{\sqrt{\log H}}{(8m \log \log H)^{m}} > \frac{1}{50\pi} \frac{\sqrt{\log H}}{(8m \log \log H)^{m}}.
\]
The theorem is proved.

**Remark.** The assertion of the theorem can be generalized to the case when $m$ grows with $T$.

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