ON QUATERNION ALGEBRAS OVER THE COMPOSITE OF
QUADRATIC NUMBER FIELDS
AND OVER SOME DIHEDRAL FIELDS

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Abstract. Let \( p \) and \( q \) be two positive primes. In this paper we obtain a complete characterization of quaternion division algebras \( H_K(p, q) \) over the composite \( K \) of \( n \) quadratic number fields. Also, in Section 6, we obtain a characterization of quaternion division algebras \( H_K(p, q) \) over some dihedral fields \( K \).

1. Introduction

Let \( F \) be a field with \( \text{char}(F) \neq 2 \) and let \( a, b \in F \setminus \{0\} \). The generalized quaternion algebra \( H_F(a, b) \) over the field \( F \) is the algebra having basis \( \{1, i, j, k\} \) and multiplication table:

\[
\begin{array}{c|cccc}
  & 1 & i & j & k \\
\hline
1 & 1 & i & j & k \\
i & i & a & k & a \\
j & j & -k & b & -b \\
k & k & -a & b & -ab \\
\end{array}
\]

If \( x = x_1 + x_2i + x_3j + x_4k \in H_F(a, b) \), with \( x_i \in F \), the conjugate \( \overline{x} \) of \( x \) is defined as \( \overline{x} = x_1 - x_2i - x_3j - x_4k \), and the norm of \( x \) as \( n(x) = x\overline{x} = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2 \).

Quaternion algebras turn out to be central simple algebras over \( F \) (i.e. associative and noncommutative algebras without two sided ideals whose center is precisely \( F \)) of dimension 4 over \( F \). Recall that the dimension \( d \) of a central simple algebra \( A \) over a field \( F \) is always a perfect square, and its square root \( n \) is defined to be the degree of \( A \).

The theory of central simple algebras (in particular quaternion algebras and cyclic algebras) has strong connections with algebraic number theory, combinatorics, algebraic geometry, coding theory, computer science and signal theory.

If the equations \( ax = b, \ ya = b \) have unique solutions for all \( a, b \in A, \ a \neq 0 \), then the algebra \( A \) is called a division algebra. If \( A \) is a finite-dimensional algebra, then \( A \) is a division algebra if and only if \( A \) has no zero divisors (\( x \neq 0, \ y \neq 0 \Rightarrow xy \neq 0 \)). In the case of generalized quaternion algebras there is a simple criterion that guarantees them to be division algebras: \( H_F(a, b) \) is a division algebra if and only if there is a unique element of zero norm, namely \( x = 0 \).

Let \( L \) be an extension field of \( F \), and let \( A \) be a central simple algebra over \( F \). We recall that \( A \) is said to split over \( L \), and \( L \) is called a splitting field for \( A \), if \( A \otimes_F L \) is isomorphic with a matrix algebra over \( L \).

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Several results are known about the splitting behavior of quaternion algebras over specific fields [7, 11, 18]. Explicit conditions which guarantee that a quaternion algebra splits over the field of rationals numbers - or else is a division algebra - were studied in [2]. In [16] the second author studied the splitting behavior of some quaternion algebras over some specific quadratic and cyclotomic fields. Moreover, in [17] the second author found some sufficient conditions for a quaternion algebra to split over a quadratic field.

In this paper we obtain a complete characterization of quaternion division algebras $H_K(p, q)$ over quadratic and biquadratic number fields $K$, when $p$ and $q$ are two positive primes.

In this paper, unless otherwise stated, when we say "prime integer" we mean "positive prime integer".

The structure of this paper is the following. In Section 2 we state some results about quaternion algebras and quadratic fields which we will need later. In Section 3 we find some necessary and sufficient conditions for a quaternion algebra over a quadratic field to be a division algebra. In Section 4 we find some necessary and sufficient conditions for a quaternion algebra over a biquadratic field to be a division algebra. In Section 5 we extend the previous results to any composite of $n$ quadratic number fields. In the last section we compare our approach with the classical ones, from a computational point of view.

2. Preliminaries

In this section we recall some basic facts about quadratic and biquadratic fields, as well as some important results concerning quaternion algebras.

Let us recall first the decomposition behavior of an integral prime ideal in the ring of integers of quadratic number fields [8, Chapter 13].

**Theorem 2.1 (Decomposition of primes in quadratic fields).** Let $d \neq 0, 1$ be a square free integer. Let $O_K$ be the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $\Delta_K$ be the discriminant of $K$. Let $p$ be an odd prime integer. Then, we have:

(i) $p$ is ramified in $O_K$ if and only if $p|\Delta_K$. In this case $pO_K = (p, \sqrt{d})^2$;

(ii) $p$ splits totally in $O_K$ if and only if $(\frac{\Delta_K}{p}) = 1$. In this case $pO_K = P_1 \cdot P_2$, where $P_1$ and $P_2$ are distinct prime ideals in $O_K$;

(iii) $p$ is inert in $O_K$ if and only if $(\frac{\Delta_K}{p}) = -1$;

(iv) the prime 2 is ramified in $O_K$ if and only if $d \equiv 2 \pmod{4}$ or $d \equiv 3 \pmod{4}$. In the first case $2O_K = (2, \sqrt{d})^2$, while in the second case $2O_K = (2, 1 + \sqrt{d})^2$;

(v) the prime 2 splits totally in $O_K$ if and only if $d \equiv 1 \pmod{8}$. In this case $2O_K = P_1 \cdot P_2$, where $P_1, P_2$ are distinct prime ideals in $O_K$, with $P_1 = (2, 1 + \sqrt{d})^2$;

(vi) the prime 2 is inert in $O_K$ if and only if $d \equiv 5 \pmod{8}$.

Now, let $d_1$ and $d_2$ be two distinct squarefree integers not equal to one. It is well known that $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ is a Galois extension of $\mathbb{Q}$ with Galois group isomorphic to the Klein 4-group. There are three quadratic subfields of $K$, namely $\mathbb{Q}(\sqrt{d_1}), \mathbb{Q}(\sqrt{d_2})$ and $\mathbb{Q}(\sqrt{d_3})$, where $d_3 = \text{lcm}(d_1, d_2)/\text{gcd}(d_1, d_2)$. The next result concerning prime ideals which split completely in composita of extensions of the base field is quite general; we state it only in the case of our interest, i.e. the
biquadratic number fields \cite{14} p. 46, which are the composita of two quadratic number fields.

**Theorem 2.2** (Splitting of primes in biquadratic fields). Let \( d_1 \) and \( d_2 \) be two distinct squarefree integers not equal to one, and let \( d_3 = \text{lcm}(d_1, d_2)/\gcd(d_1, d_2) \). Let \( \mathcal{O}_K \) denote the ring of integers of the biquadratic field \( K \) and \( \mathcal{O}_{K_i} \) the ring of integers of the quadratic subfield \( K_i = \mathbb{Q}(\sqrt{d_i}), i \in \{1, 2, 3\} \). Let \( p \) be a prime integer. Then \( p \) splits completely in \( \mathcal{O}_K \) if and only if \( p \) splits completely in each \( \mathcal{O}_{K_i}, (i = \{1, 2, 3\}) \).

Next, let \( K \) be a number field and let \( \mathcal{O}_K \) be its ring of integers. If \( v \) is a place of \( K \), let us denote by \( K_v \) the completion of \( K \) at \( v \). We recall that a quaternion algebra \( H_K(a, b) \) is said to ramify at a place \( v \) of \( K \) - or \( v \) is said to ramify in \( H_K(a, b) \) - if the quaternion \( K_v \)-algebra \( H_v = K_v \otimes H_K(a, b) \) is a division algebra. This happens exactly when the Hilbert symbol \( (a, b)_v \) is equal to \(-1\), i.e. when the equation \( ax^2 + by^2 = 1 \) has no solutions in \( K_v \). We recall that the reduced discriminant \( D_{H_K(a, b)} \) of the quaternion algebra \( H_K(a, b) \) is defined as the product of those prime ideals of the ring of integers \( \mathcal{O}_K \) of \( K \) which ramify in \( H_K(a, b) \). The following splitting criterion for a quaternion algebras is well known \cite{21} Corollary 1.10):

**Proposition 2.3.** Let \( K \) be a number field. Then, the quaternion algebra \( H_K(a, b) \) is split if and only if its discriminant \( D_{H_K(a, b)} \) is equal to the ring of integers \( \mathcal{O}_K \) of \( K \).

If \( \mathcal{O}_K \) is a principal ideal domain, then we may identify the ideals of \( \mathcal{O}_K \) with their generators, up to units. Thus, in a quaternion algebra \( H \) over \( \mathbb{Q} \), the element \( D_H \) turns out to be an integer, and \( H \) is split if and only if \( D_H = 1 \).

The next proposition gives us a geometric interpretation of splitting \cite{7} Proposition 1.3.2):

**Proposition 2.4.** Let \( F \) be a field. Then, the quaternion algebra \( H_K(a, b) \) is split if and only if the conic \( C(\alpha, \beta) : ax^2 + by^2 = z^2 \) has a rational point in \( K \), i.e. there are \( x_0, y_0, z_0 \in K \) such that \( ax_0^2 + by_0^2 = z_0^2 \).

The next proposition relates the norm group of extensions of the base field to the splitting behavior of a quaternion algebra \cite{7} Proposition 1.1.7):

**Proposition 2.5.** Let \( F \) be a field. Then, the quaternion algebra \( H_F(a, b) \) is split if and only if \( \alpha \) is the norm of an element of \( F(\sqrt{b}) \).

For quaternion algebras it is true the following \cite{7} Proposition 1.1.7):

**Proposition 2.6.** Let \( K \) be a field with char \( K \neq 2 \) and let \( \alpha, \beta \in K \setminus \{0\} \). Then the quaternion algebra \( H_K(\alpha, \beta) \) is either split or a division algebra.

In particular, this tells us that a quaternion algebra \( H_K(a, b) \) is a division algebra if and only if there is a prime \( p \) such that \( p|D_{H_K(a, b)} \). We end up this section with two statements following from the classical Albert-Brauer-Hasse-Noether theorem. Proofs of specific formulations of this theorem can be found in \cite{13} [4].

**Theorem 2.7.** Let \( H_F \) be a quaternion algebra over a number field \( F \) and let \( K \) be a quadratic extension of \( F \). Then there is an embedding of \( K \) into \( H_F \) if and only if no prime of \( F \) which ramifies in \( H_F \) splits in \( K \).
Proposition 2.8. Let $F$ be a number field and let $K$ be a quadratic extension of $F$. Let $H_F$ be a quaternion algebra over $F$. Then $K$ splits $H_F$ if and only if there exists an $F$-embedding $K \rightarrow H_F$.

3. Division quaternion algebras over quadratic number fields

In [10] the second author obtained the following result about quaternion algebras over the field $\mathbb{Q}(i)$:

Proposition 3.1. Let $p \equiv 1 \pmod{4}$ be a prime integer and let $m$ be an integer which is not a quadratic residue modulo $p$. Then the quaternion algebra $H_{\mathbb{Q}(i)}(m, p)$ is a division algebra.

In [17] the second author obtained some sufficient conditions for a quaternion algebra $H_{\mathbb{Q}(i)}(p, q)$ to split, where $p$ and $q$ are two distinct primes:

Proposition 3.2. Let $d \neq 0, 1$ be a squarefree integer such that $d \not\equiv 1 \pmod{8}$, and let $p$ and $q$ be two primes, with $q \geq 3$ and $p \neq q$. Let $\mathcal{O}_K$ be the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{d})$, and let $\Delta_K$ be the discriminant of $K$.

(i) if $p \geq 3$ and both $(\frac{p}{d})$ and $(\frac{p}{q})$ are not equal to 1, then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ splits;

(ii) if $p = 2$ and $(\frac{p}{q}) \neq 1$, then the quaternion algebra $H_{\mathbb{Q}(\sqrt{d})}(2, q)$ splits.

From the aforementioned results we deduce easily a necessary and sufficient condition for a quaternion algebra $H_{\mathbb{Q}(i)}(p, q)$ to be a division algebra:

Proposition 3.3. Let $p$ and $q$ be two distinct odd primes, such that $(\frac{q}{p}) \neq 1$. Then the quaternion algebra $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra if and only if $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$.

Proof. To prove the necessity, note that if $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra, then Proposition 3.2 and Proposition 2.6 tell us that $(\frac{p}{q}) = 1$ or $(\frac{q}{p}) = 1$. This is equivalent to $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$.

To prove the sufficiency, we must distinguish amongst two cases:

- $p \equiv 1 \pmod{4}$:
  Since $(\frac{q}{p}) \neq 1$, Proposition 3.1 tells us that $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra.

- $q \equiv 1 \pmod{4}$:
  Since $(\frac{q}{p}) = -1$, the quadratic reciprocity law implies $(\frac{q}{p}) = -1$. Proposition 3.1 then tells us that $H_{\mathbb{Q}(i)}(p, q)$ is a division algebra.

We ask ourselves whether we can obtain a necessary and sufficient explicit condition for $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ to be a division algebra when $d$ is arbitrary. From Proposition 3.2 we obtain a necessary explicit condition for $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ to be a division algebra, namely: if $H_{\mathbb{Q}(\sqrt{d})}(p, q)$ is a division algebra, then $(\frac{p}{d}) = 1$ or $(\frac{q}{d}) = 1$. However this condition is not sufficient: for example, if we let $K = \mathbb{Q}(\sqrt{3})$, $p = 7$, $q = 47$, then $(\frac{p}{d}) \neq 1$ and $(\frac{q}{d}) = 1$, the quaternion algebra $H_{\mathbb{Q}(7, 47)}$ is a division algebra, but the quaternion algebra $H_{\mathbb{Q}(\sqrt{3})}(7, 47)$ is not a division algebra.

It is known [18] Section 14.1) that if a prime integer $p$ divides $D_{H(a, b)}$ then it must divide $2ab$, hence we may restrict our attention to these primes. In other words,
in order to obtain a sufficient condition for a quaternion algebra \( H_{Q(\sqrt{\alpha})}(p, q) \) to be a division algebra, it is important to study the ramification of the primes \( p, q \) in the algebra \( H_{Q}(p, q) \). The following lemma [2, Lemma 1.20] gives us a hint:

**Lemma 3.4.** Let \( p \) and \( q \) be two primes, and let \( H_{Q}(p, q) \) be a quaternion algebra of discriminant \( D_H \).

(i) if \( p \equiv q \equiv 3 \pmod{4} \) and \( \left( \frac{p}{q} \right) \neq 1 \), then \( D_H = 2p \);
(ii) if \( q = 2 \) and \( p \equiv 3 \pmod{8} \), then \( D_H = pq = 2p \);
(iii) if \( p \) or \( q \equiv 1 \pmod{4} \), with \( p \neq q \) and \( \left( \frac{p}{q} \right) = -1 \), then \( D_H = pq \).

In addition, the following lemma [2, Lemma 1.20] tells us precisely when a quaternion algebra \( H_{Q}(p, q) \) splits.

**Lemma 3.5.** Let \( p \) and \( q \) be two prime integers. Then \( H_{Q}(p, q) \) is a matrix algebra if and only if one of the following conditions is satisfied:

(i) \( p = q = 2 \);
(ii) \( p = q \equiv 1 \pmod{4} \);
(iii) \( q = 2 \) and \( p \equiv \pm 1 \pmod{8} \);
(iv) \( p \neq q, p \neq 2, q \neq 2, \left( \frac{p}{q} \right) = 1 \) and either \( p \) or \( q \) is congruent to 1 \pmod{4}.

The next theorem [2, Theorem 1.22] describes the discriminant of \( H_{Q}(p, q) \), where \( p \) and \( q \) are primes.

**Theorem 3.6.** Let \( H = \left( \frac{a, b}{Q} \right) \) be a quaternion algebra. Then

(i) \( D_H = 1 \) then \( H \) splits;
(ii) \( D_H = 2p, p \) prime and \( p \equiv 3 \pmod{4} \) then \( H \cong \left( \frac{p-1}{Q} \right) \);
(iii) \( D_H = pq, p, q \) primes, \( q \equiv 1 \pmod{4} \) and \( \left( \frac{p}{q} \right) = -1 \) then \( H \cong \left( \frac{p-1}{Q} \right) \).

If \( a \) and \( b \) are prime numbers the algebra \( H \) satisfies one and only one of the above statements.

We recall that a small ramified \( Q \)-algebra is a rational quaternion algebra having the discriminant equal to the product of two distinct prime numbers. If we take into account the previous results and Lemma 3.4, we obtain the following necessary and sufficient explicit condition for a small ramified \( Q \)-algebra \( H_{Q}(p, q) \) to be a division algebra over a quadratic field \( Q(\sqrt{\Delta}) \):

**Proposition 3.7.** Let \( p \) and \( q \) be two distinct odd primes, with \( p \) or \( q \equiv 1 \pmod{4} \) and \( \left( \frac{q}{p} \right) = -1 \). Let \( K = Q(\sqrt{\Delta}) \) and let \( \Delta_K \) be the discriminant of \( K \). Then the quaternion algebra \( H_{Q(\sqrt{\Delta})}(p, q) \) is a division algebra if and only if \( \left( \frac{\Delta_K}{p} \right) = 1 \) or \( \left( \frac{\Delta_K}{q} \right) = 1 \).

**Proof.** If \( H_{Q(\sqrt{\Delta})}(p, q) \) is a division algebra then Proposition 5.2 gives \( \left( \frac{\Delta_K}{p} \right) = 1 \) or \( \left( \frac{\Delta_K}{q} \right) = 1 \).

Conversely, assume that either \( \left( \frac{\Delta_K}{p} \right) = 1 \) or \( \left( \frac{\Delta_K}{q} \right) = 1 \). By hypothesis \( p \) or \( q \equiv 1 \pmod{4} \) and \( \left( \frac{q}{p} \right) = -1 \). According to Lemma 3.4 (iii), \( D_H = pq \). This means that the primes which ramify in the quaternion algebra \( H_{Q}(p, q) \) are precisely \( p \) and \( q \). Since either \( \left( \frac{\Delta_K}{p} \right) = 1 \) or \( \left( \frac{\Delta_K}{q} \right) = 1 \), by Theorem 2.1 it follows that either \( p \) or \( q \) splits in the ring of integers of the quadratic field \( K \). Finally, Theorem 2.7 and Proposition 2.8 imply that the quaternion algebra \( H_{Q(\sqrt{\Delta})}(p, q) \) does not split, hence, according to Proposition 2.6 \( H_{Q(\sqrt{\Delta})}(p, q) \) is a division algebra. \( \square \)
When \( q = 2 \) and \( p \) is a prime such that \( p \equiv 3 \pmod{8} \), then, according to Lemma 3.4, the discriminant \( D_{H_{Q(p,q)}} \) is equal to \( 2p \), so \( H_{Q(p,q)} \) is a division algebra. The next proposition shows what happens when we extend the field of scalars from \( Q \) to \( Q(\sqrt{d}) \):

**Proposition 3.8.** Let \( p \) be an odd prime, with \( p \equiv 3 \pmod{8} \). Let \( K = Q(\sqrt{d}) \) and let \( \Delta_K \) be the discriminant of \( K \). Then \( H_{Q(\sqrt{d})}(p,2) \) is a division algebra if and only if \( (\frac{\Delta_K}{p}) = 1 \) or \( d \equiv 1 \pmod{8} \).

**Proof.** If \( H_{Q(\sqrt{d})}(p,2) \) is a division algebra then, from Proposition 3.2 Proposition 2.6, Theorem 2.7 and Proposition 2.8, we conclude that \( (\frac{\Delta_K}{p}) = 1 \) or \( d \equiv 1 \pmod{8} \).

Conversely, since \( p \equiv 3 \pmod{8} \) then, according to Lemma 3.4(ii) we must have \( D_H = 2p \). It follows that the primes which ramify in \( H_{Q(p,2)} \) are precisely \( p \) and 2. Since either \( (\frac{\Delta_K}{p}) = 1 \) or \( d \equiv 1 \pmod{8} \) then, after applying Theorem 2.7, we obtain that either \( p \) or 2 splits in the ring of integers of \( K \). From Theorem 2.7, Proposition 2.8 and Proposition 2.6 we conclude that \( H_{Q(\sqrt{d})}(p,2) \) is a division algebra.

We study next the case where \( p \) and \( q \) are primes, both congruent to 3 modulo 4. If \( (\frac{q}{p}) \neq 1 \), then, according to Lemma 3.4(i), the discriminant \( D_{H_{Q(p,q)}} \) is equal to \( 2p \), so \( H_{Q(p,q)} \) is a division algebra. The next proposition tells us when the quaternion algebra \( H_{Q(\sqrt{d})}(p,q) \) is still a division algebra.

**Proposition 3.9.** Let \( p \) and \( q \) be two odd prime integers, with \( p \equiv q \equiv 3 \pmod{4} \) and \( (\frac{q}{p}) \neq 1 \). Let \( K = Q(\sqrt{d}) \) and let \( \Delta_K \) be the discriminant of \( K \). Then the quaternion algebra \( H_{Q(\sqrt{d})}(p,q) \) is a division algebra if and only if \( (\frac{\Delta_K}{p}) = 1 \) or \( d \equiv 1 \pmod{8} \).

**Proof.** If \( H_{Q(\sqrt{d})}(p,q) \) is a division algebra then from Proposition 3.2(i) it follows that either \( (\frac{\Delta_K}{p}) = 1 \) or \( (\frac{\Delta_K}{q}) = 1 \). But, according to Lemma 3.4(i) we must have \( D_{H_{Q(p,q)}} = 2p \). So the integral primes which ramify in \( H_{Q(p,q)} \) and could split in \( K \) are precisely \( p \) and 2. Finally, after applying Proposition 2.6 Theorem 2.7 Proposition 2.8 and Theorem 2.1 we obtain that either \( (\frac{\Delta_K}{p}) = 1 \) or \( d \equiv 1 \pmod{8} \).

The proof of the converse is similar to the proof of sufficiency of Proposition 3.8.

Taking into account these results and Proposition 2.6 we are able to understand when a quaternion algebra \( H_{Q(\sqrt{d})}(p,q) \) splits. It is clear that in of the cases covered by Lemma 3.5 a quaternion algebra \( H_{Q(\sqrt{d})}(p,q) \) splits. Moreover, using Proposition 3.7 Proposition 3.8 Proposition 3.9 and Proposition 2.6 we obtain the following necessary and sufficient explicit condition for a small ramified \( Q \)-algebra \( H_{Q(p,q)} \) to be a split algebra over a quadratic field \( Q(\sqrt{d}) \):

**Corollary 3.10.** Let \( p \) and \( q \) be two distinct odd primes, with \( p \) or \( q \equiv 1 \pmod{4} \) and \( (\frac{q}{p}) = -1 \). Let \( \Delta_K \) be the discriminant of \( K = Q(\sqrt{d}) \). Then the quaternion algebra \( H_{Q(\sqrt{d})}(p,q) \) splits if and only if \( (\frac{\Delta_K}{p}) \neq 1 \) and \( (\frac{\Delta_K}{q}) \neq 1 \).
Corollary 3.11. Let \( p \) be an odd prime, with \( p \equiv 3 \pmod{8} \). Let \( \Delta_K \) be the discriminant of \( K = \mathbb{Q}(\sqrt{d}) \). Then \( H_{\mathbb{Q}(\sqrt{d})}(p, 2) \) splits if and only if \( \left( \frac{\Delta_K}{p} \right) \neq 1 \) and \( d \neq 1 \pmod{8} \).

Corollary 3.12. Let \( p \) and \( q \) be two odd prime integers, with \( p \equiv q \equiv 3 \pmod{4} \) and \( \left( \frac{q}{p} \right) \neq 1 \). Let \( \Delta_K \) be the discriminant of \( K = \mathbb{Q}(\sqrt{d}) \). Then the quaternion algebra \( H_{\mathbb{Q}(\sqrt{d})}(p, q) \) splits if and only if \( \left( \frac{\Delta_K}{p} \right) \neq 1 \) and \( d \neq 1 \pmod{8} \).

The only case left out is \( q = 2, p \equiv 5 \pmod{8} \). We consider first the quaternion algebra \( H_{\mathbb{Q}}(p, q) \), and we get the following result:

Lemma 3.13. Let \( p \equiv 5 \pmod{8} \) be a prime integer. Then the discriminant of the quaternion algebra \( H_{\mathbb{Q}}(p, 2) \) is equal to \( 2p \), and hence \( H_{\mathbb{Q}}(p, 2) \) is a division algebra.

Proof. We give here a simple proof which is independent of the theorems stated above. We know that if a prime divides the discriminant of \( H_{\mathbb{Q}}(a, b) \) then it must divide \( 2ab \). Since \( p \equiv 5 \pmod{8} \), from the properties of the Hilbert symbol and of the Legendre symbol we obtain:

\[
(2, p)_p = \left( \frac{2}{p} \right) = (-1)^{\frac{p-1}{2}} = -1
\]

and

\[
(2, p)_2 = (-1)^{\frac{p-1}{2} + \frac{p-1}{2}} = -1
\]

Hence the primes which ramify in \( H_{\mathbb{Q}}(p, 2) \) are exactly 2 and \( p \). Therefore, the reduced discriminant of \( H_{\mathbb{Q}}(p, 2) \) must be equal to \( 2p \). \( \square \)

We turn now our attention to the quaternion algebra \( H_{\mathbb{Q}(\sqrt{d})}(p, q) \), where \( q = 2 \) and \( p \equiv 5 \pmod{8} \).

Proposition 3.14. Let \( p \) be an odd prime, with \( p \equiv 5 \pmod{8} \). Let \( K = \mathbb{Q}(\sqrt{d}) \) and let \( \Delta_K \) be the discriminant of \( K \). Then \( H_{\mathbb{Q}(\sqrt{d})}(p, 2) \) is a division algebra if and only if \( \left( \frac{\Delta_K}{p} \right) = 1 \) or \( d \equiv 1 \pmod{8} \).

Proof. The proof is similar to the proof of Proposition 3.8 after replacing Lemma 3.4 with Lemma 3.13. \( \square \)

From Proposition 3.14 and Proposition 2.6 we obtain:

Corollary 3.15. Let \( p \) be an odd prime, with \( p \equiv 5 \pmod{8} \). Let \( \Delta_K \) be the discriminant of \( K = \mathbb{Q}(\sqrt{d}) \). Then \( H_{\mathbb{Q}(\sqrt{d})}(p, 2) \) splits if and only if \( \left( \frac{\Delta_K}{p} \right) \neq 1 \) and \( d \neq 1 \pmod{8} \).

Theorem 3.16 (Classification over quadratic fields). Let \( d \) be a squarefree integer not equal to one, and let \( K = \mathbb{Q}(\sqrt{d}) \), with discriminant \( \Delta_K \). Let \( p \) and \( q \) be two positive primes. Then the quaternion algebra \( H_K(p, q) \) is a division algebra if and only if one of the following conditions holds:

(i) \( p \) and \( q \) are odd and distinct, and
   \( \left( \frac{p}{q} \right) = -1 \), and
   \( p \equiv 1 \pmod{4} \) or \( q \equiv 1 \pmod{4} \), and
   \( \left( \frac{\Delta_K}{p} \right) = 1 \) or \( \left( \frac{\Delta_K}{q} \right) = 1 \);

(ii) \( q = 2 \), and
   \( p \equiv 5 \pmod{8} \) or \( p \equiv 5 \pmod{8} \), and
   either \( \left( \frac{\Delta_K}{p} \right) = 1 \) or \( d \equiv 1 \pmod{8} \);
(iii) $p$ and $q$ are odd, with $p \equiv q \equiv 3 \pmod{4}$, and

- $(\frac{q}{p}) \neq 1$, and
- either $(\frac{\Delta_K}{p}) = 1$ or $d \equiv 1 \pmod{8}$;

or

- $(\frac{p}{q}) \neq 1$, and
- either $(\frac{\Delta_K}{q}) = 1$ or $d \equiv 1 \pmod{8}$;

Proof. The theorem follows easily from the four propositions above, after taking into account Lemma 3.14 and Lemma 8.11

\[ \square \]

4. Division quaternion algebras over biquadratic number fields

Let us start with a proposition that will be useful in this section:

Proposition 4.1. Let $a$ and $b$ be distinct nonzero integers and let $p$ be an odd prime.

(i) If $a$ and $b$ are quadratic residues modulo $p$, then $\text{lcm}(a,b)/\text{gcd}(a,b)$ is also a quadratic residue modulo $p$;

(ii) If $a$ and $\text{lcm}(a,b)/\text{gcd}(a,b)$ are quadratic residues modulo $p$, then $b$ is a quadratic residue modulo $p$.

Proof. Since $a$ and $b$ are both quadratic residues modulo $p$ it follows that $(\frac{ab}{p}) = 1$.

Let $c = \frac{\text{lcm}(a,b)}{\text{gcd}(a,b)} = \frac{ab}{\text{gcd}(a,b)}$. We have now $(\frac{\text{gcd}(a,b)^2}{p}) = 1$ and $(\frac{ab}{p}) = (\frac{\text{gcd}(a,b)^2}{p}) (\frac{a}{p}) = 1$. Therefore $(\frac{a}{p}) = 1$, so $\frac{\text{lcm}(a,b)}{\text{gcd}(a,b)}$ is a quadratic residue modulo $p$.

The proof of the second case is similar to the proof of the first case. \[ \square \]

Let $K/L$ be an extension of number fields. If a quaternion algebra $H_L(p, q)$ splits, then the quaternion algebra $H_K(p, q)$ splits as well. If the quaternion algebra $H_L(p, q)$ is a division algebra then the quaternion algebra $H_K(p, q)$ could still be a division algebra or else split.

We consider now a division quaternion algebra $H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})}(p, q)$ over a base field $L = \mathbb{Q}(\sqrt{d_1})$, and try to find some conditions which guarantee that it is still a division algebra over the biquadratic field $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$.

Proposition 4.2. Let $d_1$ and $d_2$ be distinct squarefree integers not equal to one. Let $p$ and $q$ be distinct odd prime integers such that $(\frac{p}{q}) = -1$, and $p$ or $q$ is congruent to one modulo 4. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ and let $K_i = \mathbb{Q}(\sqrt{d_i})$ ($i = 1, 2$), with discriminant $\Delta_K_i$. Then the quaternion algebra $H_K(p, q)$ is a division algebra if and only if $(\frac{\Delta_{K_1}}{p}) = (\frac{\Delta_{K_2}}{p}) = 1$ or $(\frac{\Delta_{K_1}}{q}) = (\frac{\Delta_{K_2}}{q}) = 1$.

Proof. Let us assume that $H_K(p, q)$ is a division algebra. Hence the quaternion algebras $H_{\mathbb{Q}(p, q)}$, $H_{K_1}(p, q)$, $H_{K_2}(p, q)$ must all be division algebras.

Let $K_3 = \mathbb{Q}(\sqrt{d_3})$ be the third quadratic subfield of $K$, where $d_3 = \frac{\text{lcm}(d_1, d_2)}{\text{gcd}(d_1, d_2)}$, with discriminant $\Delta_{K_3}$. Since $H_K(p, q)$ is a division algebra, it follows that $H_{K_1}(p, q)$ must be a division algebra as well. Proposition 3.7 tells us that one of the following conditions must be verified:

(i) $(\frac{\Delta_{K_1}}{p}) = (\frac{\Delta_{K_2}}{p}) = (\frac{\Delta_{K_3}}{p}) = 1$;

(ii) $(\frac{\Delta_{K_1}}{q}) = (\frac{\Delta_{K_2}}{q}) = (\frac{\Delta_{K_3}}{q}) = 1$;
(iii) There are \( i,j \in \{1,2,3\}, i \neq j \) such that \( \left( \frac{\Delta_{K_i}}{p} \right) = \left( \frac{\Delta_{K_j}}{p} \right) = 1 \) and there is \( \ell \in \{1,2,3\}, \ell \neq i, \ell \neq j \) such that \( \left( \frac{\Delta_{K_\ell}}{q} \right) = 1 \). Then \( d_i \) and \( d_j \) are quadratic residues modulo \( p \). From Proposition 4.1 it follows that \( d_i, d_2, d_3 \) are quadratic residues modulo \( p \), so \( \left( \frac{\Delta_{K_i}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \); 

(iv) There are \( i,j \in \{1,2,3\}, i \neq j \) such that \( \left( \frac{\Delta_{K_i}}{q} \right) = \left( \frac{\Delta_{K_j}}{q} \right) = 1 \) and there is \( \ell \in \{1,2,3\}, \ell \neq i, \ell \neq j \) such that \( \left( \frac{\Delta_{K_\ell}}{p} \right) = 1 \). From Proposition 4.1 it follows that \( \left( \frac{\Delta_{K_i}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \).

Conversely, let us assume that \( \left( \frac{\Delta_{K_i}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \). By Proposition 4.1 it follows that \( \left( \frac{\Delta_{K_1}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \). By Theorem 2.1 ii) it follows that \( p \) splits in each \( \mathcal{O}_{K_i} \). According to Theorem 2.2 \( p \) must split in \( \mathcal{O}_K \) as well. Since \( \left( \frac{\Delta_{K_1}}{p} \right) = 1 \), Proposition 3.7 tells us that \( H_{K_1}(p,q) \) must be a division algebra, and that \( p \) must ramify in \( H_{K_1}(p,q) \). By Theorem 2.7 Proposition 2.8 and Proposition 2.6 the quaternion algebra \( H_K(p,q) \) must be a division algebra.

If \( \left( \frac{\Delta_{K_1}}{q} \right) = \left( \frac{\Delta_{K_2}}{q} \right) = \left( \frac{\Delta_{K_3}}{q} \right) = 1 \), the same argument shows that \( H_K(p,q) \) must be a division algebra. \( \square \)

**Proposition 4.3.** Let \( d_1 \) and \( d_2 \) be distinct squarefree integers not equal to one. Let \( p \) be an odd prime integer, such that \( p \equiv 3 \pmod{8} \). Let \( K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \), and let \( K_i = \mathbb{Q}(\sqrt{d_i}) \) \( (i = 1, 2, 3) \), with discriminant \( \Delta_{K_i} \). Then the quaternion algebra \( H_K(p,2) \) is a division algebra if and only if \( \left( \frac{\Delta_{K_1}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \) or \( d_1, d_2 \equiv 1 \pmod{8} \).

**Proof.** Let us assume that \( H_K(p,2) \) is a division algebra. Then the quaternion algebras \( H_Q(p,2) \), \( H_{K_1}(p,2) \), \( H_{K_2}(p,2) \) are division algebras as well.

Let \( K_3 = \mathbb{Q}(\sqrt{d_3}) \) be the third quadratic subfield of \( K \), where \( d_3 = \frac{\gcd(d_1,d_2)}{\gcd(d_1,d_2)} \), with discriminant \( \Delta_{K_3} \). Since \( H_K(p,q) \) is a division algebra, it follows that \( H_{K_3}(p,q) \) must be a division algebra as well.

Let \( D_H \) be the discriminant of the quaternion algebra \( H_Q(p,q) \). According to the hypothesis and to Lemma 4.4 ii), we have \( D_H = 2p \). Proposition 3.8 tells us that one of the following conditions must be verified:

(i) \( \left( \frac{\Delta_{K_1}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \);

(ii) \( d_1, d_2, d_3 \equiv 1 \pmod{8} \);

(iii) There are \( i,j \in \{1,2,3\}, i \neq j \) such that \( \left( \frac{\Delta_{K_i}}{p} \right) = \left( \frac{\Delta_{K_j}}{p} \right) = 1 \) and there is \( \ell \in \{1,2,3\}, \ell \neq i, \ell \neq j \) such that \( d_i \equiv 1 \pmod{8} \). Then, \( d_i \) and \( d_j \) are quadratic residues modulo \( p \). From Proposition 4.1 it follows that \( d_i, d_2, d_3 \) are quadratic residues modulo \( p \), so \( \left( \frac{\Delta_{K_1}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \);

(iv) There are \( i,j \in \{1,2,3\}, i \neq j \) such that \( d_i, d_j \equiv 1 \pmod{8} \) and there is \( \ell \in \{1,2,3\}, \ell \neq i, \ell \neq j \) such that \( \left( \frac{\Delta_{K_\ell}}{p} \right) = 1 \). Since \( d_3 = \frac{\gcd(d_1,d_2)}{\gcd(d_1,d_2)} \), it follows that \( d_1, d_2, d_3 \equiv 1 \pmod{8} \).

Let us now prove the converse. If \( \left( \frac{\Delta_{K_1}}{p} \right) = \left( \frac{\Delta_{K_2}}{p} \right) = \left( \frac{\Delta_{K_3}}{p} \right) = 1 \) the argument is the same used in the proof of sufficiency of Proposition 4.2.

If \( d_1, d_2 \equiv 1 \pmod{8} \), it follows easily that \( d_3 \equiv 1 \pmod{8} \). According to Proposition 3.8 the algebras \( H_{K_1}(p,2) \), \( H_{K_2}(p,2) \) and \( H_{K_3}(p,2) \) are all division algebras and \( 2 \) ramifies there. From Theorem 2.1 it follows that 2 splits completely in \( K_1 \),
Let $d_1$ and $d_2$ be distinct squarefree integers not equal to one. Let $p$ and $q$ be distinct odd prime integers, with $p \equiv q \equiv 3 \pmod{4}$ and $(\frac{q}{p}) = -1$. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, and let $K_i = \mathbb{Q}(\sqrt{d_i})$ ($i = 1, 2$), with discriminant $\Delta_{K_i}$. Then the quaternion algebra $H_K(p, q)$ is a division algebra if and only if $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1$ or $d_1, d_2 \equiv 1 \pmod{8}$.

Proof. The proof is similar to the proof of Proposition 4.3. □

Proposition 4.5. Let $d_1$ and $d_2$ be distinct squarefree integers not equal to one. Let $p$ be an odd prime integer, such that $p \equiv 5 \pmod{8}$. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, and let $K_i = \mathbb{Q}(\sqrt{d_i})$ ($i = 1, 2$), with discriminant $\Delta_{K_i}$. Then the quaternion algebra $H_K(p, 2)$ is a division algebra if and only if $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{2}) = 1$ or $d_1, d_2 \equiv 1 \pmod{8}$.

Proof. The proof is similar to the proof of Proposition 4.3. □

After gluing the last three proposition together, we obtain the main theorem of our paper:

Theorem 4.6 (Classification over biquadratic fields). Let $d_1$ and $d_2$ be distinct squarefree integers not equal to one. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, and let $K_i = \mathbb{Q}(\sqrt{d_i})$ ($i = 1, 2$) with discriminant $\Delta_{K_i}$. Let $p$ and $q$ be two positive primes. Then the quaternion algebra $H_K(p, q)$ is a division algebra if and only if one of the following conditions holds:

(i) $p$ and $q$ are odd and distinct, and
$(\frac{q}{p}) = -1$, and $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$, and
$(\frac{\Delta_{K_1}}{p}) = (\frac{\Delta_{K_2}}{p}) = 1$ or $(\frac{\Delta_{K_1}}{q}) = (\frac{\Delta_{K_2}}{q}) = 1$;

(ii) $q = 2$, and $p \equiv 3 \pmod{8}$ or $p \equiv 5 \pmod{8}$, and either $(\frac{\Delta_{K_1}}{p}) = (\frac{\Delta_{K_2}}{p}) = 1$ or $d_1, d_2 \equiv 1 \pmod{8}$;

(iii) $p$ and $q$ are odd, with $p \equiv q \equiv 3 \pmod{4}$, and
• $(\frac{q}{p}) \neq 1$, and
• either $(\frac{\Delta_{K_1}}{q}) = (\frac{\Delta_{K_2}}{q}) = 1$ or $d_1, d_2 \equiv 1 \pmod{8}$;

or

• $(\frac{q}{p}) \neq 1$, and
• either $(\frac{\Delta_{K_1}}{q}) = (\frac{\Delta_{K_2}}{q}) = 1$ or $d_1, d_2 \equiv 1 \pmod{8}$;

Proof. The theorem follows easily from the four propositions above, after taking into account Lemma 3.4 and Lemma 3.13. □

Let’s point out that in the first case of the classification theorem we have a manifest symmetry, i.e. by Legendre’s statement of the quadratic reciprocity law $p$ is a quadratic residue modulo $q$ if and only if $q$ is a quadratic residue modulo $p$, while in the third case the setting is asymmetrical, i.e., again by Legendre’s statement of the quadratic reciprocity law, $p$ is a quadratic residue modulo $q$ if and only if $q$ is
not a quadratic residue modulo p.

5. Extensions

In this section we show how to apply the technique of proof shown in the previous section can be applied to classify quaternion algebras over the composite of \(n\) quadratic fields. Let \(d_1, d_2, \ldots, d_n\) be distinct squarefree integers not equal to one. It is known that \(K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})\) is a Galois extension of \(\mathbb{Q}\) with Galois group isomorphic to the group \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \mathbb{Z}_2\).

Let's start with the smallest case, i.e. with \(n = 3\). For this purpose we take a quaternion algebra \(H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})}(p, q)\) over a base field \(L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})\), and we try to find some conditions which guarantee that it is still a division algebra over the field \(K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})\).

Let us recall the following well known theorem ([19], p.360):

**Theorem 5.1.** Suppose that \(p\) is a prime of \(\mathbb{Q}\) which splits completely in each of two fields \(F_1\) and \(F_2\). Then \(p\) splits completely in the composite field \(F_1F_2\).

As a consequence, if \(p\) splits completely in a field \(F\), then \(p\) also splits completely in the minimal normal extension of \(\mathbb{Q}\) containing \(F\). We obtain the following result:

**Proposition 5.2.** Let \(d_1, d_2\) and \(d_3\) be distinct squarefree integers not equal to one. Let \(p\) and \(q\) be distinct odd prime integers such that \((\frac{d_i}{p}) = -1\), \(p\) does not divide \(d_i\) and \(q\) does not divide \(d_i\), \((i = 1, 2, 3)\) and \(p\) or \(q\) is congruent to one modulo 4. Let \(K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})\) and let \(K_i = \mathbb{Q}(\sqrt{d_i})\) \((i = 1, 2, 3)\), with discriminant \(\Delta_{K_i}\). Then the quaternion algebra \(H_{\mathbb{Q}}(p, q)\) is a division algebra if and only if \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1\) or \((\frac{\Delta_{K_j}}{p}) = (\frac{\Delta_{K_j}}{q}) = 1}\).

**Proof.** If \(H_{\mathbb{Q}}(p, q)\) is a division algebra, then, according to Proposition 4.2,

- \(H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})}(p, q)\) is a division algebra, and this is equivalent to \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_j}}{q}) = 1\);
- \(H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_3})}(p, q)\) is a division algebra, and this is equivalent to \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_j}}{q}) = 1\);
- \(H_{\mathbb{Q}(\sqrt{d_2}, \sqrt{d_3})}(p, q)\) is a division algebra, and this is equivalent to \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_j}}{q}) = 1\).

Therefore, one of the following conditions must be satisfied:

(i) \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_j}}{p}) = 1\);
(ii) \((\frac{\Delta_{K_i}}{q}) = (\frac{\Delta_{K_j}}{q}) = 1\);
(iii) There are \(i, j \in \{1, 2, 3\}\), \(i \neq j\) such that \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1\) and there is \(l \in \{1, 2, 3\}\), \(l \neq i, l \neq j\) such that \((\frac{\Delta_{K_l}}{q}) = (\frac{\Delta_{K_l}}{q}) = 1\).
(iv) There are \(i, j, k \in \{1, 2, 3\}\), \(i \neq j \neq k \neq i\) such that \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{p}) = 1\) and \((\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{p}) = 1\).
Conversely, let us suppose that \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = (\Delta_{K_3}^p) = 1 \). Since \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = 1 \), from Theorem 2.2 it follows that \( p \) splits in \( \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \). Since \( (\Delta_{K_2}^p) = 1 \), from Theorem 2.1 it follows that \( p \) splits in \( \mathbb{Q}(\sqrt{d_3}) \). According to Theorem 5.1, \( p \) must split in \( K \). According to Proposition 5.2, \( H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})}(p, q) \) is a division algebra and \( p \) ramifies in \( H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})}(p, q) \). By Theorem 2.7, Proposition 2.8 and Proposition 2.9, the quaternion algebra \( H_K(p, q) \) must be a division algebra.

If \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = (\Delta_{K_3}^p) = 1 \), the same argument shows that \( H_K(p, q) \) is a division algebra.

We can generalize now Proposition 4.2 and Proposition 5.2.

Proposition 5.3. Let \( n \) be a positive integer, \( n \geq 2 \) and let \( d_1, d_2, \ldots, d_n \) be distinct squarefree integers not equal to one. Let \( p \) and \( q \) be distinct odd prime integers such that \( (\frac{q}{p}) = -1 \), \( p \) does not divide \( d_i \) and \( q \) does not divide \( d_i \), \( i = 1, \ldots, n \), and \( p \) or \( q \) is congruent to one modulo 4. Let \( K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}) \) and let \( K_i = \mathbb{Q}(\sqrt{d_i}) \) \( i = 1, \ldots, n \), with discriminant \( \Delta_{K_i} \). Then the quaternion algebra \( H_K(p, q) \) is a division algebra if and only if \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = \ldots = (\Delta_{K_n}^p) = 1 \) or \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = \ldots = (\Delta_{K_n}^p) = 1 \).

Proof. The proof is by mathematical induction over \( n \), for \( n > 2 \). The inductive step is based on the same argument that we used to go from Proposition 4.2 to Proposition 5.2.

Proposition 5.4. Let \( d_1, d_2 \) and \( d_3 \) be distinct squarefree integers not equal to one. Let \( p \) be an odd prime integer such that \( p \) does not divide \( d_i \), \( i = 1, 2, 3 \), \( p \equiv 3 \) (mod 8). Let \( K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}) \) and let \( K_i = \mathbb{Q}(\sqrt{d_i}) \) \( i = 1, 2, 3 \), with discriminant \( \Delta_{K_i} \). Then the quaternion algebra \( H_K(p, 2) \) is a division algebra if and only if \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = (\Delta_{K_3}^p) = 1 \) or \( d_1, d_2, d_3 \equiv 1 \) (mod 8).

Proof. ”\( \Rightarrow \)” If \( H_K(p, 2) \) is a division algebra, it results that \( H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})}(p, q) \) is a division algebra and this is equivalent with \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = 1 \) or \( d_1, d_2, d_3 \equiv 1 \) (mod 8); \( H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_3})}(p, q) \) is a division algebra and this is equivalent with \( (\Delta_{K_1}^p) = (\Delta_{K_3}^p) = 1 \) or \( d_1, d_3 \equiv 1 \) (mod 8); \( H_{\mathbb{Q}(\sqrt{d_2}, \sqrt{d_3})}(p, q) \) is a division algebra and this is equivalent with \( (\Delta_{K_2}^p) = (\Delta_{K_3}^p) = 1 \) or \( d_2, d_3 \equiv 1 \) (mod 8) (according to Proposition 4.3). Considering these we can have one of the following cases:

(i) \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = (\Delta_{K_3}^p) = 1 \);

(ii) \( d_1, d_2, d_3 \equiv 1 \) (mod 8);

(iii) There are \( i, j \in \{1, 2, 3\}, i \neq j \) such that \( (\Delta_{K_i}^p) = (\Delta_{K_j}^p) = 1 \) and there is \( l \in \{1, 2, 3\}, l \neq i, l \neq j \) such that \( d_i, d_j \equiv 1 \) (mod 8) and \( d_l \equiv 1 \) (mod 8).

It results that \( d_1, d_2, d_3 \equiv 1 \) (mod 8).

(iv) There are \( i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i \) such that \( (\Delta_{K_i}^p) = (\Delta_{K_j}^p) = (\Delta_{K_k}^p) = 1 \) and \( (\Delta_{K_l}^p) = (\Delta_{K_m}^p) = 1 \) and \( d_i, d_j, d_l \equiv 1 \) (mod 8). It results that \( (\Delta_{K_i}^p) = (\Delta_{K_j}^p) = (\Delta_{K_k}^p) = 1 \).

”\( \Leftarrow \)” If \( (\Delta_{K_1}^p) = (\Delta_{K_2}^p) = (\Delta_{K_3}^p) = 1 \), the argument is the same used in the proof of sufficiency of Proposition 4.2.
If $d_1, d_2, d_3 \equiv 1 \pmod{8}$, applying to Proposition 5.3 it results that the quaternion algebra $H_{Q(\sqrt{d_1}, \sqrt{d_3})}(p, q)$ is a division algebra and 2 ramifies this algebra. According to Theorem 2.1 it follows that 2 splits completely in $K_1$, $K_2$, and $K_3$. From Theorem 2.2 it results that 2 splits completely in $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_3})$. Using this and the fact that 2 splits completely in $K_3$, applying Proposition 5.1 we conclude that 2 splits completely in $K$. According to Theorem 2.1 Proposition 2.5, and Proposition 2.6, it follows now that $H_K(p, 2)$ is a division algebra.

Proposition 5.5. Let $d_1, d_2$, and $d_3$ be distinct squarefree integers not equal to one. Let $p, q$ be two odd prime integers such that $p$ does not divide $d_i$, $i = 1, 2, 3$, $p \equiv q \equiv 3 \pmod{4}$, $(\frac{q}{p}) \neq 1$. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$ and let $K_i = \mathbb{Q}(\sqrt{d_i})$ ($i = 1, 2, 3$), with discriminant $\Delta_K$. Then the quaternion algebra $H_K(p, q)$ is a division algebra if and only if $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1$ or $d_1, d_2, d_3 \equiv 1 \pmod{8}$.

Proof. The proof is similar to the proof of Proposition 5.4.

Proposition 5.6. Let $d_1, d_2$, and $d_3$ be distinct squarefree integers not equal to one. Let $p$ be an odd prime integer such that $p$ does not divide $d_i$, $i = 1, 2, 3$, $p \equiv 5 \pmod{8}$, $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$ and let $K_i = \mathbb{Q}(\sqrt{d_i})$ ($i = 1, 2, 3$), with discriminant $\Delta_K$. Then the quaternion algebra $H_K(p, 2)$ is a division algebra if and only if $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1$ or $d_1, d_2, d_3 \equiv 1 \pmod{8}$.

Proof. The proof is similar to the proof of Proposition 5.4.

Taking into account the results obtained in Proposition 5.2, Proposition 5.4, Proposition 5.5, and Proposition 5.6, we obtain the following classification theorem.

Theorem 5.7 (Classification over the composite of three quadratic fields). Let $d_1, d_2, d_3$ be distinct squarefree integers not equal to one. Let $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$, and let $K_i = \mathbb{Q}(\sqrt{d_i})$ ($i = 1, 2, 3$), with discriminant $\Delta_K$. Let $p$ and $q$ be two positive primes. Then the quaternion algebra $H_K(p, q)$ is a division algebra if and only if one of the following conditions holds:

(i) $p$ and $q$ are odd and distinct, and $(\frac{q}{p}) = -1$, and $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$, and $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1$ or $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1$;

(ii) $q = 2$, and $p \equiv 3 \pmod{8}$ or $p \equiv 5 \pmod{8}$, and either $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1$ or $d_1, d_2, d_3 \equiv 1 \pmod{8}$;

(iii) $p$ and $q$ are odd, with $p \equiv q \equiv 3 \pmod{4}$, and

$•$ $(\frac{q}{p}) \neq 1$, and $•$ either $(\frac{\Delta_{K_i}}{p}) = (\frac{\Delta_{K_i}}{q}) = 1$ or $d_1, d_2, d_3 \equiv 1 \pmod{8}$;

or

$•$ $(\frac{q}{p}) \neq 1$, and $•$ either $(\frac{\Delta_{K_i}}{q}) = (\frac{\Delta_{K_i}}{p}) = 1$ or $d_1, d_2, d_3 \equiv 1 \pmod{8}$;

Proof. The theorem follows easily from the propositions above, after taking into account Lemma 3.13 and Lemma 5.13.
By mathematical induction over \( n \), for \( n \geq 2 \), we can generalize Proposition 5.4. Proposition 5.5 and Proposition 5.6 and obtain the following classification for division quaternion algebras over a composite of \( n \) quadratic fields.

**Theorem 5.8** (Classification over a composite of \( n \) quadratic fields). Let \( d_1, d_2, \ldots, d_n \) be distinct squarefree integers not equal to one, with \( n \geq 2 \). Let \( p \) and \( q \) be distinct odd prime integers such that \( p \) does not divide \( d_i \) and \( q \) does not divide \( d_i \) (\( i = 1, \ldots, n \)). Let \( K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}) \) and \( K_i = \mathbb{Q}(\sqrt{d_i}) \) \( (i = 1, \ldots, n) \), with discriminant \( \Delta_{K_i} \). Then the quaternion algebra \( H_K(p, q) \) is a division algebra if and only if one of the following conditions holds:

(i) \( p \) and \( q \) are odd and distinct, and
\[
\left( \frac{\Delta_{K_i}}{p} \right) = -1, \text{ and } \left( \frac{\Delta_{K_i}}{q} \right) = \left( \frac{\Delta_{K_i}}{q} \right) = \ldots = \left( \frac{\Delta_{K_i}}{q} \right) = 1;
\]

(ii) \( q = 2 \), and
\[
p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4}, \text{ and } \left( \frac{\Delta_{K_i}}{p} \right) = \ldots = \left( \frac{\Delta_{K_i}}{p} \right) = 1 \text{ or } \left( \frac{\Delta_{K_i}}{q} \right) = \ldots = \left( \frac{\Delta_{K_i}}{q} \right) = 1;
\]

(iii) \( p \) and \( q \) are odd, with \( p \equiv q \equiv 3 \pmod{4} \), and
\[
\text{ either } \left( \frac{\Delta_{K_i}}{p} \right) = \ldots = \left( \frac{\Delta_{K_i}}{p} \right) = 1 \text{ or } d_1, d_2, \ldots, d_n \equiv 1 \pmod{8};
\]
\[
\text{ or } \left( \frac{\Delta_{K_i}}{q} \right) = \ldots = \left( \frac{\Delta_{K_i}}{q} \right) = 1 \text{ or } d_1, d_2, \ldots, d_n \equiv 1 \pmod{8}.
\]

6. Division quaternion algebras over the Hilbert class field of a quadratic field

We ask ourselves what happens when consider a quaternion algebra over a field \( K \), which is a Galois extension of \( \mathbb{Q} \), with Galois group nonabelian, with the order \( 2l \), where \( l \) is an odd prime integer. For to study this we use the following remark, which can be found in [11], p.77.

**Remark 6.1.** Let \( K/F \) be a finite extension of fields, with the degree \([K : F] = odd\) and let \( a, b \in F \) (where \( F \) is the multiplicative group of the field \( F \)). Then, the quaternion algebra \( H_K(a, b) \) splits if and only if \( H_F(a, b) \) splits.

First, we study the case when the Galois group \( \text{Gal}(K/\mathbb{Q}) \) is isomorphic to the permutations group \( S_3 \) (i.e. the dihedral group \( D_3 \)) and we obtain the following results:

**Proposition 6.2.** Let \( \epsilon \) be a primitive root of order 3 of the unity and let \( F = \mathbb{Q}(\epsilon) \) be the 3 th cyclotomic field. Let \( a \in K \setminus \{0\} \) be a cubicfree integer not equal to one, let the Kummer field \( K = F(\sqrt[3]{a}) \) and let \( p \) and \( q \) be distinct odd prime integers, \( \left( \frac{\Delta_{K_i}}{p} \right) = -1 \), and \( p \) or \( q \) is congruent to one modulo 4. Then the quaternion algebra \( H_K(p, q) \) is a division algebra if and only if \( \left( \frac{\Delta_{K_i}}{p} \right) = 1 \) or \( \left( \frac{\Delta_{K_i}}{q} \right) = 1 \).

**Proof.** \( F = \mathbb{Q}(\epsilon) = \mathbb{Q}(i\sqrt{3}) \) and the degree \([K : F] = 3\). \( K/\mathbb{Q} \) is a Galois extension and the Galois group \( \text{Gal}(K/\mathbb{Q}) \) is isomorphic to the group \( S_3 \). According to Remark 6.1 and Proposition 2.6, \( H_K(p, q) \) is a division algebra if and
only if \( H_F(p, q) \) is a division algebra. Applying to Proposition 6.7 this happens if and only if \( \left( \frac{-3}{p} \right) = 1 \) or \( \left( \frac{-3}{q} \right) = 1 \).}

**Proposition 6.3.** Let \( \epsilon \) be a primitive root of order 3 of the unity and let \( F = \mathbb{Q}(\epsilon) \) be the 3 th cyclotomic field. Let \( \alpha \in K \setminus \{0\} \) be a cubicfree integer not equal to one, let the Kummer field \( K = F(\sqrt[3]{\alpha}) \) and let \( p \) and \( q \) be odd prime integers, \( p \equiv 3 \pmod{8} \). Then the quaternion algebra \( H_K(p, 2) \) is a division algebra if and only if \( \left( \frac{-3}{p} \right) = 1 \).

**Proof.** The proof is similar with the proof of Proposition 6.2, but instead of Proposition 3.7 we use Proposition 3.8. \( \square \)

**Proposition 6.4.** Let \( \epsilon \) be a primitive root of order 3 of the unity and let \( F = \mathbb{Q}(\epsilon) \) be the 3 th cyclotomic field. Let \( \alpha \in K \setminus \{0\} \) be a cubicfree integer not equal to one, let the Kummer field \( K = F(\sqrt[3]{\alpha}) \) and let \( p \) and \( q \) be odd prime integers, \( (\frac{q}{p}) \neq 1 \), and \( p \equiv q \equiv 3 \pmod{4} \). Then the quaternion algebra \( H_K(p, q) \) is a division algebra if and only if \( \left( \frac{-3}{p} \right) = 1 \).

**Proof.** The proof is similar with the proof of Proposition 6.2, but instead of Proposition 3.7 we use Proposition 3.9. \( \square \)

Now, we pay attention to the case when the Galois group \( \text{Gal}(K/\mathbb{Q}) \) is isomorphic to a dihedral group \( D_l \), with \( l \) prime, \( l \geq 5 \), if this case exists. Here appears the inverse Galois problem. From class field theory, we know that this case exists, that is a dihedral group \( D_l \), with \( l \) prime can be realized as a Galois group over \( \mathbb{Q} \) (9). In [9] (p. 352-353) appears the following theorem:

**Theorem 6.5.** For any prime \( l \) and any quadratic field \( F = \mathbb{Q}\left(\sqrt{d}\right) \) there exist infinitely many dihedral fields of degree 2l containing \( F \) (where a dihedral field of degree 2l is a normal extension of degree 2l over \( \mathbb{Q} \) with dihedral Galois group \( D_l \)).

Let \( l \) be an odd prime integer and let \( F = \mathbb{Q}\left(\sqrt{d}\right) \) be an imaginary quadratic field with class number \( h_F = l \). Let \( H_F \) be the Hilbert class field of \( F \). If the quaternion algebra \( H_F(p, q) \) is a division algebra, we are interested when \( H_{H_F}(p, q) \) when is still a division algebra. We obtain the following results:

**Proposition 6.6.** Let \( d \) be a squarefree integer not equal to one and let \( F = \mathbb{Q}\left(\sqrt{d}\right) \) be an imaginary quadratic field, with class number \( h_F = l \), let \( H_F \) be the Hilbert class field of \( F \) and let \( \Delta_F \) be the discriminant of \( F \). Let \( p \) and \( q \) be distinct odd prime integers, \( (\frac{q}{p}) = -1 \), and \( p \) or \( q \) is congruent to one modulo 4. Then the quaternion algebra \( H_{H_F}(p, q) \) is a division algebra if and only if \( \left( \frac{\Delta_F}{p} \right) = 1 \) or \( \left( \frac{\Delta_F}{q} \right) = 1 \).

**Proof.** The degree \( [H_F : F] = l = \text{odd} \). It is known that the Hilbert class field \( H_F \) over \( F \) has degree \( l \), \( H_F/\mathbb{Q} \) is a Galois extension of fields and the Galois group \( \text{Gal}(H_F/\mathbb{Q}) \) is isomorphic to the dihedral field \( D_l \) (see [1], p. 348).

According to Remark 6.4 and Proposition 2.6 \( H_{H_F}(p, q) \) is a division algebra if and only if \( H_F(p, q) \) is a division algebra. Applying to Proposition 6.7 this happens if and only if \( \left( \frac{\Delta_F}{p} \right) = 1 \) or \( \left( \frac{\Delta_F}{q} \right) = 1 \). \( \square \)
Proposition 6.7. Let \( d \) be a squarefree integer not equal to one and let \( F = \mathbb{Q} \left( \sqrt{d} \right) \) be an imaginary quadratic field, with class number \( h_F = 1 \), let \( H_F \) be the Hilbert class field of \( F \) and let \( \Delta_F \) be the discriminant of \( F \). Let \( p \) be an odd prime integer, \( p \equiv 3 \pmod{8} \). Then the quaternion algebra \( H_{HF}(p,q) \) is a division algebra if and only if \( \left( \frac{\Delta_F}{p} \right) = 1 \) or \( d \equiv 1 \pmod{8} \).

Proof. The proof is similar with the proof of Proposition 6.6, when instead of Proposition 3.7 we use Proposition 3.8. □

Proposition 6.8. Let \( d \) be a squarefree integer not equal to one and let \( F = \mathbb{Q} \left( \sqrt{d} \right) \) be an imaginary quadratic field, with class number \( h_F = 1 \), let \( H_F \) be the Hilbert class field of \( F \) and let \( \Delta_F \) be the discriminant of \( F \). Let \( p \) and \( q \) be distinct odd prime integers, \( (\frac{q}{p}) \neq 1 \), and \( p \equiv q \equiv 3 \pmod{4} \). Then the quaternion algebra \( H_{HF}(p,q) \) is a division algebra if and only if \( \left( \frac{\Delta_F}{p} \right) = 1 \) or \( d \equiv 1 \pmod{8} \).

Proof. The proof is similar with the proof of Proposition 6.6, when instead of Proposition 3.7 we use Proposition 3.9. □

Using Theorem 6.5, we remark that Proposition 6.6, Proposition 6.7, Proposition 6.8 remain valid when instead of the Hilbert class field of an imaginary quadratic field \( F \) with class number an odd prime \( l \) we consider a dihedral field of degree \( 2l \) containing \( F \).

7. Final remarks

The task of deciding whether a quaternion algebra over a number field is a division algebra is computationally a feasible one, thanks to the facilities included in computational algebra packages like Magma [10], Pari, Sage, etc; however, different approaches lead to very different execution times.

A naive way to check if a quaternion algebra is a division algebra is to show that a norm equation has no solution, thanks to Proposition 2.5 and Proposition 2.6. The problem of determining whether a norm equation over an extension of number fields has a solution has been extensively investigated in the past, both over arbitrary and over specific extensions of number fields [1, 5, 6, 15]. Algorithms included in Magma, Pari and Sage allow one to find out whether a norm equation has or not at least a solution, and, sometime, to find a sought solution. However in the general case this is not an easy task.

Let us consider first the apparently efficient approach based on Proposition 2.6. For this purpose we wrote two small functions in SAGE, release 8.1, to test the efficiency of this method, and we used them to test an increasing number of cases (100 - 1000 - 10000 - 100000) of quaternion algebras.

In order to construct the cases to check, we considered \( r \) unordered couples \( \{p, q\} \) of positive primes \( p \) and \( q \), for increasing values of \( p \) and \( q \) starting from 2, and

- in the case \( K = \mathbb{Q} \left( \sqrt{d} \right) \), we took approximately an equal number \( r \) of squarefree integers \( \pm d \), for increasing values of \( d \) starting from one, omitting the trivial case which would give \( K = \mathbb{Q} \);
- in the case \( K = \mathbb{Q} \left( \sqrt{d_1}, \sqrt{d_2} \right) \), we took approximately an equal number \( r \) of unordered couples \( \{ \pm d_1, \pm d_2 \} \) of squarefree integers, for increasing...
values of $d_1$ and $d_2$ starting from 1, omitting the trivial cases which would give $K = \mathbb{Q}(\sqrt{t})$ or $K = \mathbb{Q}$.

We ran our tests on a 2.6 Ghz - i7 quad core - mac mini (late 2012), equipped with 8 GB of RAM. The running time is shown in the second column of Tables 1 and 2. The value "n.a" means that the computation was taking too long and hence we forced the termination of the program.

| # of algebras | Norm approach | Discriminant approach | Our approach |
|---------------|---------------|-----------------------|--------------|
| 100           | 1136          | 727                   | 3            |
| 1000          | 16117         | 7941                  | 35           |
| 10000         | n.a.          | 82966                 | 328          |
| 100000        | n.a.          | 855620                | 3151         |

Table 1. Running time in ms. required to test $H_{\mathbb{Q}(\sqrt{d})}(p, q)$.

A different approach to check if a quaternion algebra is a division algebra, based on Proposition 2.3, is to show that the discriminant ideal of the algebra is not equal to the full ring of integers of the base field. Again we wrote two small functions in SAGE to test the efficiency of this approach. The running time is shown in the third column of Tables 1 and 2.

| # of algebras | Norm approach | Discriminant approach | Our approach |
|---------------|---------------|-----------------------|--------------|
| 100           | 2501          | 1044                  | 4            |
| 1000          | 43329         | 9428                  | 42           |
| 10000         | n.a.          | 97049                 | 384          |
| 100000        | n.a.          | 1000279               | 3818         |

Table 2. Running time in ms. required to test $H_{\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})}(p, q)$.

The approach described in this paper does not require one to solve norm equations over relative extensions of number fields, neither to compute the discriminant of quaternion algebras defined over an arbitrary number field. In fact, all we need is to compute a few Legendre symbols as well as the discriminants of quadratic extensions of $\mathbb{Q}$ involved, which is a very easy task: indeed, for a nonzero square free integer $d$, the discriminant of the quadratic field $\mathbb{Q}(\sqrt{d})$ is $d$ if $d$ is congruent to 1 modulo 4, otherwise $4d$. We wrote two small functions in SAGE to test the efficiency of our approach. The running time is shown in the fourth column of Tables 1 and 2.

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