Vortex dynamics for 2D Euler flows with unbounded vorticity

Stefano Ceci and Christian Seis
Institut für Analysis und Numerik, Westfälische Wilhelms-Universität Münster, Germany.
E-mails: ceci@wwu.de, seis@wwu.de

Abstract: It is well-known that the dynamics of vortices in an ideal incompressible two-dimensional fluid is described by the Kirchhoff–Routh point vortex system. In this paper, we revisit the classical problem of how well solutions to the Euler equations approximate these vortex dynamics and extend previous rigorous results to the case where the vorticity field is unbounded. More precisely, we establish estimates for the 2-Wasserstein distance between the vorticity and the empirical measure associated with the point vortex dynamics. In particular, we derive an estimate on the order of weak convergence of the Euler solutions to the solutions of the point vortex system.

1 Introduction

The aim of this paper is to study the motion and interaction of vortices in an ideal incompressible two-dimensional fluid. More precisely, we study the evolution by the Euler equations of vortex patches with possibly unbounded vorticity under mild concentration assumptions on the initial configuration. In our main result, we quantify the convergence of solutions to the Euler equations towards a system of interacting point vortices.

The investigation of the dynamics of such idealized vortex systems goes back to the pioneering work of Helmholtz in the middle of the 19th century [16]. Helmholtz (implicitly) introduced the point vortex system and derived some of its most fundamental properties. In his lectures on mathematical physics [22], Kirchhoff later demonstrated that the system is of Hamiltonian form, which was later extended to the case of vortices in a bounded domain by Routh [27]. The system of equations governing the motion of point vortices in a bounded region is today accordingly referred to as the Kirchhoff–Routh system.

The first rigorous connection between the Euler equations and the Kirchhoff–Routh model was established by Marchioro and Pulvirenti [25]. The authors consider vortex patches that are initially confined in small regions in the plane and they...
show that the size of these regions can be suitably controlled during the evolution. The argument exploits the symmetry properties of the two-dimensional Biot–Savart kernel and relies on the regularity of the velocity field away from the vortex patches. Once the preservation of vorticity concentration is proved, the convergence towards the point vortex system follows immediately. The Marchioro–Pulvirenti method was subsequently gradually refined in [26, 8, 6].

In situations in which the Biot–Savart kernel lacks certain symmetry features, the method from [25] seems to fail. This is the case, for instance, for the axisymmetric Euler equations without swirl, which describe the motion and interaction of vortex rings, or for the lake equations, which serve as a shallow water model in the regime of small Froude numbers. In such systems, the dynamics of point vortices could be derived via energy expansion methods [4, 7, 12]; see also [30] for an analogous result for the two-dimensional Euler equations. These techniques, however, seem not to be suitable for capturing the interaction of vortices. Indeed, the results in [30, 11, 12] are restricted to single vortices while [7] describes the evolution of travelling wave type solutions. In particular, what remains an open problem today is the rigorous derivation of the leapfrogging dynamics of interacting vortex rings, which was already announced by Helmholtz [16] and later explicitly formulated by Hicks [17] under the metaphorical title *The mutual threading of vortex rings*.

Quite recently, Davila, Del Pino, Musso and Wei [11] proposed a third ansatz for constructing vortex solutions to the two-dimensional Euler equations that contain precise information on the vortex cores. Their approach relies on what is commonly referred to as the gluing method — a method that was previously successfully applied to a number of desingularization problems, including concentration phenomena along curves for the Schrödinger equation [14] and non-affine solutions to the stationary Allen–Cahn equation in large dimensions [13].

In the present paper, we reconsider the method of Marchioro and Pulvirenti and extend the results on vortex dynamics for the two-dimensional Euler equations to the case of unbounded vorticity fields. Such a result was, in a certain sense, already foreshadowed by a recent contribution of Caprini and Marchioro [8], in which the authors prove the connection between the Euler equations and the point vortex model in situations where the vorticity is bounded but of arbitrary large amplitude. To be more specific, we consider vorticity fields with a (suitably small) control in $L^p$ for some $p > 2$. Our main result, Theorem 1, states that any solution to the Euler equation whose vorticity initially concentrates around a finite collection of points, remains (in a scale-independent time interval) concentrated at the same scale and the center of concentration can be chosen as the vortex points that evolve by the Kirchhoff–Routh model. This extension is particularly remarkable, as our integrability setting falls outside the range of known well-posedness results in $L^\infty$ or BMO established by Yudovich [21] and Vishik [32]; see also [23, 3] for alternative and simplified proofs. Yet, it is crucial to note that the range of integrability exponents we are dealing with is precisely the one in which the fluid velocity is (Hölder) continuous and thus bounded as a consequence of Calderón–Zygmund and Sobolev inequalities.

As a measure of concentration, we consider the Wasserstein distance $W_2$ between the vorticity field and the empirical measure associated with the point vortices.
Because Wasserstein distances metrize weak convergence, cf. [31, Theorem 7.12], our result provides an estimate on the order of weak convergence of (possibly non-unique) solutions to the Euler equations towards the unique solution to the Kirchhoff–Routh model. We remark that the Wasserstein distance played a crucial role already in the original work of Marchioro and Pulvirenti [25], even though it has been considered rather as an auxiliary second moment function in there, cf. [7].

Weak concentration measures were recently introduced in the context of the three-dimensional Euler vortex filament dynamics in [19]. In this work, the flat norm is used to measure the distance between the three dimensional vorticity vector from a curve in $\mathbb{R}^3$. Note that in two dimensions, it is nothing but the Wasserstein (or Kantorovich–Rubinstein) distance $W_1$, or, equivalently, the norm associated to the negative Sobolev space $W^{-1,1}$. Weak concentration measures proved to be suitable tools already in the context of the dynamics of Ginzburg–Landau-type vortices, see for example [9, 20].

In a certain sense, our concentration estimate can be considered as a stability result for the two-dimensional Euler equations. Indeed, our main estimate provides a control of the distance between the vorticity field and the empirical measure associated with the point vortex dynamics — which can be formally considered as a singular solution to the Euler equations — in terms of the distance of the corresponding initial configurations. We remark that until today, general stability estimates for the Euler equations are not available. To the best of our knowledge, the estimates that are closest to stability estimates are those by Loeper [23], who reproves Yudovich’s uniqueness result by using optimal transportation techniques. (He mainly works with the Wasserstein distance $W_2$ as well.) Even for linear transport equations with general Sobolev vector fields, stability estimates were obtained only recently in [28, 29]; see also [10] for the corresponding results for Lagrangian flows.

In the following section, we will present the precise mathematical setting and state our main result, the proofs of which can be found in Section 3.

2 Mathematical setting and result

We consider the Euler equations in a bounded smooth domain $\Omega$ in $\mathbb{R}^2$. It is well-known that the evolution can be stated in terms of the scalar vorticity field $\omega = \omega(t,x) \in \mathbb{R}$, which is simply transported by the flow of the fluid velocity $u = u(t,x) \in \mathbb{R}^2$ and thus mathematically described by the transport equation

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{in } \Omega. \tag{1}$$

By assuming no-penetration boundary conditions, that is, $u \cdot \nu = 0$ on $\partial \Omega$, where $\nu$ is the outer normal, we ensure that there is no flow across the boundary of the domain. We moreover suppose that the fluid is incompressible, which translates into the mathematical condition $\nabla \cdot u = 0$ in $\Omega$. While the vorticity can be computed from the velocity field by taking the curl, that is, $\omega = \partial_1 u_2 - \partial_2 u_1$, the velocity field
can be reconstructed from the vorticity with the help of the Biot–Savart law
\[ u(t, x) = K * \omega(t, x) + \nabla^\perp \eta(t, x) = \int_{\Omega} K(x - y) \omega(t, y) \, dy + \nabla^\perp \eta(t, x), \]
where \( K \) is the rotated gradient of the Newtonian potential \( G(z) = -\frac{1}{2\pi} \log \frac{1}{|z|} \), that is, \( K = -\nabla^\perp G \) or
\[ K(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2} \quad \text{for} \quad z^\perp = (-z_2, z_1), \]
and \( \eta \) is the harmonic extension of \( G*\omega \) (this is the stream function in \( \mathbb{R}^2 \)) restricted to the boundary,
\[ \begin{cases} -\Delta \eta(t, \cdot) = 0 & \text{in } \Omega \\ \eta(t, \cdot) = G*\omega(t, \cdot) & \text{on } \partial \Omega. \end{cases} \]
Notice that \( \psi = G*\omega - \eta \) defines the stream function associated with \( u \), that is, \( u = -\nabla^\perp \psi \) and \( \psi \) solves the Poisson equation
\[ -\Delta \psi = \omega \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial \Omega. \]
We finally equip the Euler vorticity equation with an initial condition,
\[ \omega(0) = \bar{\omega} \quad \text{in } \Omega. \]

Throughout this article, we assume that the vorticity field belongs to the Lebesgue space \( L^\infty((0, T); L^p(\Omega)) \) for some \( p > 2 \). Thanks to Calderón–Zygmund theory, the associated velocity field is thus Sobolev regular in the spatial variable, \( u \in L^\infty((0, T); W^{1,p}(\Omega)) \), and then bounded, \( u \in L^\infty((0, T) \times \Omega) \), by Sobolev embedding. It follows that the product \( u\omega \) is integrable in \( \Omega \) and thus, the transport equation \( (1) \) can be interpreted in the sense of distributions. It is known that such a solution exists for every \( p \geq 1 \), see e.g. [24] and references therein for the case \( p > 4/3 \), however still nothing is known about uniqueness if \( p < \infty \). Moreover, every solution is renormalized in the sense of DiPerna and Lions [15], that is
\[ \partial_t \beta(\omega) + u \cdot \nabla \beta(\omega) = 0 \quad \text{in } \Omega \]
for every bounded \( \beta \in C^1(\mathbb{R}) \) that vanishes near 0 and has suitable decay properties at infinity. Furthermore, from the theory in [15] (and [1]) it follows that the vorticity is transported by the (regular) Lagrangian flow \( \phi \) of the velocity field \( u \), that is
\[ \omega(t, \phi_t(x)) = \bar{\omega}(x), \]
where \( \phi = \phi_t(x) \in \mathbb{R}^2 \) solves the ordinary differential equation
\[ \partial_t \phi_t(x) = u(t, \phi_t(x)), \quad \phi_0(x) = x, \quad \phi_1(x), \quad \text{see [1]} \]
for a precise definition in the case of rough velocity fields.

We shall now make our choice of initial data more specific in order to be able to capture the vortex dynamics that we described in the introduction. We suppose
that the vorticity can be decomposed into \( N \) separated patches, that is, we suppose that
\[
\bar{\omega} = \sum_{i=1}^{N} \bar{\omega}_i,
\]
and the patches are disjointly supported and not touching the boundary,
\[
\min_{i \neq j} \text{dist} (\text{spt} \bar{\omega}_i, \text{spt} \bar{\omega}_j) \geq \delta, \quad \min \text{dist} (\text{spt} \bar{\omega}_i, \partial \Omega) \geq \delta,
\] (3)
for some \( \delta > 0 \). We also assume that every patch has fixed sign, that is for every \( i \) it must hold either \( \bar{\omega}_i \geq 0 \) or \( \bar{\omega}_i \leq 0 \). At any later time, we may write
\[
\omega(t) = \sum_{i=1}^{N} \omega_i(t),
\]
where \( \omega_i \) is the unique (cf. [15]) solution to the linear transport equation \( \partial_t \omega_i + u \cdot \nabla \omega_i = 0 \) with the initial datum \( \omega_i(0) = \bar{\omega}_i \), and thus, \( \omega_i \) transported by the flow \( \phi \).

In particular, because \( u \) is bounded, there exists a maximal time \( T \in (0, \infty] \) such that the supports of the vortex patches remain separated and in distance to the boundary in the sense that
\[
\min_{i \neq j} \text{dist} (\omega_i(t), \text{spt} \omega_j(t)) \geq \frac{\delta}{2}, \quad \min \text{dist} (\omega_i(t), \partial \Omega) \geq \frac{\delta}{2}
\] (4)
for all \( t \in [0, T) \).

We denote the intensity of the \( i \)th vortex patch by \( a_i \). It is preserved by the evolution and given by
\[
a_i = \int_{\Omega} \bar{\omega}_i \, dx = \int_{\Omega} \omega_i(t) \, dx.
\] (5)
We suppose that each vortex patch is initially concentrated around a certain point \( \bar{Y}_i \) in \( \Omega \) in the sense that
\[
W_2 \left( \frac{\bar{\omega}_i}{a_i}, \delta \bar{Y}_i \right) \leq \varepsilon,
\] (6)
where \( W_2 \) is the 2-Wasserstein distance and the concentration scale \( \varepsilon \) is much smaller than the separation scale \( \delta \), that is,
\[
\varepsilon \ll \delta.
\]
Notice that the Wasserstein distance is well-defined because, by (5), \( \bar{\omega}_i/a_i \) and \( \delta \bar{Y}_i \) are both probability measures. For a comprehensive introduction into Wasserstein distances (and the theory of optimal transportation in general), we refer to Villani’s monograph [31] and Chapter 7 therein. Notice that if one of the marginals is an atomic measure as in (5), the Wasserstein distance reduces to a simple second moment function
\[
W_2 \left( \frac{\bar{\omega}_i}{a_i}, \delta \bar{Y}_i \right) = \left( \frac{1}{a_i} \int_{\Omega} |x - \bar{Y}_i|^2 \bar{\omega}_i \, dx \right)^{\frac{1}{2}}.
\] (7)
Considering this expression as a function of $\bar{Y}_i$, it is easily seen that the Wasserstein distance is minimized by locating $\bar{Y}_i$ at the center of vorticity. Indeed, setting 

$$X_i = \frac{1}{a_i} \int_{\Omega} x \bar{\omega}_i(x) \, dx,$$

it holds that 

$$W_2 \left( \frac{\bar{\omega}_i}{a_i}, \delta_{\bar{X}_i} \right) \leq \inf_{\bar{Y}_i} W_2 \left( \frac{\bar{\omega}_i}{a_i}, \delta_{\bar{Y}_i} \right). \quad (8)$$

We suppose that the intensities $a_i$ and the separation scale $\delta$ are independent of the concentration scale $\varepsilon$. Then $(6)$ means that, up to rescaling with the scale independent constant $a_i$, the vortex patch $\bar{\omega}_i$ approximates the Dirac measure $\delta_{\bar{Y}_i}$ if $\varepsilon \ll 1$. A prototype vorticity field is thus a Dirac sequence or a constant vortex patch of the form $\varepsilon^{-2} \chi_{B_{\varepsilon}(\bar{Y}_i)}$. In this paper, we consider unbounded perturbations of such sequences in the sense that $\bar{\omega} = \bar{\omega}^p + \bar{\omega}^\infty$ and

$$\|\bar{\omega}^p\|_{L^p} \lesssim \frac{1}{\varepsilon^{2(1 - \frac{2}{p})}}, \quad \|\bar{\omega}^\infty\|_{L^\infty} \lesssim \frac{1}{\varepsilon^2} \quad (9)$$

for some $p > 2$. Here and in the following, $A \lesssim B$ means that $A$ is bounded by $B$ up to a multiplicative constant independent of $\varepsilon$ and $t$. Notice that these perturbations are small in the sense that $\|\varepsilon^{-2} \chi_{B_{\varepsilon}(\bar{Y})}\|_{L^p} \sim \varepsilon^{2(\frac{1}{2} - 1)} \gg \varepsilon^{2(\frac{2}{p} - 1)} \gtrsim \|\bar{\omega}^p\|_{L^p}$. We also remark that, as a consequence of the DiPerna–Lions theory, if $\omega^p_i, \omega^\infty_i$ are defined as the solutions to the linear transport equation with velocity $u$ and initial datum $\bar{\omega}^p_i$ and $\bar{\omega}^\infty_i$, respectively, so that $\omega_i = \omega^p_i + \omega^\infty_i$ by the uniqueness for the linear equation, we have thanks to the renormalization property that

$$\|\omega^p_i(t)\|_{L^p} = \|\bar{\omega}^p_i\|_{L^p} \leq \|\bar{\omega}^p\|_{L^p}$$

and

$$\|\omega^\infty_i(t)\|_{L^\infty} = \|\bar{\omega}^\infty_i\|_{L^\infty} \leq \|\bar{\omega}^\infty\|_{L^\infty}.$$

Thus the bound imposed in $(9)$ on the initial datum carries over to the solution $\omega_i(t)$.

Our main goal in this paper is to establish a rigorous link between the Euler equations and the Kirchhoff–Routh point vortex system

$$\frac{dY_i}{dt}(t) = \sum_{j \neq i} a_j K(Y_i(t) - Y_j(t)) + \nabla_{\perp} \theta(t, Y_i(t)), \quad Y_i(0) = \bar{Y}_i, \quad (10)$$

where $\theta$ represents the interaction with the boundary and is defined as the solution to the Laplace equation

$$\begin{cases} -\Delta \theta(t, \cdot) = 0 & \text{in } \Omega \\ \theta(t, \cdot) = \sum_j a_j G(\cdot - Y_j(t)) & \text{on } \partial \Omega. \end{cases}$$

Upon choosing $T$ smaller if necessary, we will furthermore assume that

$$\min_{i \neq j} |Y_i(t) - Y_j(t)| \geq \frac{\delta}{2}, \quad \min \text{dist} \left( Y_i(t), \partial \Omega \right) \geq \frac{\delta}{2} \quad (11)$$

$$6$$
for all $t \in [0, T)$. Notice that for certain initial configurations, a collapse of vortices is possible. Choosing $T$ with the above condition, however, avoids this scenario. We refer to the nice survey paper [2] for a discussion.

Under the scaling estimate (9), we show that if the vorticity initially concentrates around the points $\bar{Y}_1, \ldots, \bar{Y}_N$ in the sense of (6), then at any later time $t \in [0, T)$, the vorticity concentrates around the solution $Y_1(t), \ldots, Y_N(t)$ of the Kirchhoff–Routh system (10).

**Theorem 1.** Let $T$ be given such that (4) and (11) hold. Let $\bar{Y}_1, \ldots, \bar{Y}_N \in \Omega$ be such that (6) holds for any $i \in \{1, \ldots, N\}$ and suppose that $Y_1, \ldots, Y_N$ solve the Kirchhoff–Routh system. Then $T$ is independent of $\varepsilon$, i.e, $T \gtrsim 1$, and there exists a constant $C < \infty$ independently of $\varepsilon$ such that, for any $i \in \{1, \ldots, N\}$,

$$W_2\left(\frac{\omega_i(t)}{a_i}, \delta_{Y_i(t)}\right) \lesssim e^{Ct} \varepsilon \quad \text{for all } t \in [0, T).$$

The result in Theorem 1 translates into an estimate between the full vorticity field $\omega = \sum \omega_i$ and the empirical measure associated with the point vortex system. Indeed, interpreting the Kantorovich–Rubinstein distance $W_1$ as the dual norm $W^{-1,1}$, that is,

$$W_1(f, g) = \sup \left\{ \int_\Omega (f - g)\zeta \, dx : \|\nabla \zeta\|_{L^\infty} \leq 1 \right\},$$

the distance function can be readily extended to a distance between two not necessarily nonnegative functions of equal mean by setting

$$W_1(f, g) = W_1((f - g)_+, (f - g)_-),$$

where the subscript plus and minus signs indicate the positive and negative parts of a function. We then have the following estimate.

**Corollary 1.** Under the assumptions of Theorem 1, it holds that

$$W_1\left(\omega(t), \sum_{i=1}^N a_i \delta_{Y_i(t)}\right) \lesssim e^{Ct} \varepsilon \quad \text{for all } t \in [0, T).$$

In particular, because Kantorovich–Rubinstein distances metrize weak convergence, cf. [31, Theorem 7.12], the result can be interpreted as an estimate on the order of weak convergence of the vorticity field $\omega(t)$ towards the empirical measure $\sum_i a_i \delta_{Y_i(t)}$: For any $t \in [0, T)$,

$$\omega(t) \rightharpoonup \sum_{i=1}^N a_i \delta_{Y_i(t)} \quad \text{weakly with order at most } \varepsilon.$$

Here weak convergence has to be understood in the sense of weak convergence of measures. We recall that our result holds true for any solution to the Euler equation.
Thus, in the event that it turns out that the two-dimensional Euler equations are not uniquely solvable, we regain uniqueness in the singular limit $\varepsilon \to 0$.

Alternatively, we can express the estimate in terms of the centers of vorticity $X(t) = \frac{1}{a_i} \int_\Omega x\omega(t)\,dx$
in the following way:

**Theorem 2.** Under the assumptions of Theorem 1, for any $i \in \{1, \ldots, N\}$, it holds that

$$|X_i(t) - Y_i(t)| \lesssim e^{Ct} \varepsilon \quad \text{and} \quad \left| \frac{d}{dt} X_i(t) - \frac{d}{dt} Y_i(t) \right| \lesssim e^{Ct} \varepsilon$$

for all $t \in [0, T)$.

Hence, both position and velocity of the centers of vorticity deviate from those of the point vortex system (10) only by a constant of order $\varepsilon$ as $\varepsilon \ll 1$.

We also remark that the Kirchhoff–Routh system (10) features leapfrogging in simple geometric situations, for instance, if $\Omega$ is a ball. Indeed, if we place two vortices of equal sign (for simplicity) into this ball and both vortices are located sufficiently close to each other, the vortices start spinning around each other while traveling along the domain boundary. Conditions for leapfrogging in the case of the half-plane were already computed by Hicks [17]. (In fact, Hicks studies the problem of four vortices in $\mathbb{R}^2$, whose location is symmetric with respect to one axis. Therefore, Hicks computations also apply to the half-plane problem, if the two vortices outside the half-plane are treated as mirror vortices.) To the best of our knowledge, the present paper is the first to rigorously derive leapfrogging dynamics for the Euler equation. Yet, the available techniques seem not to be sophisticated enough in order to study the much more interesting leapfrogging problem for vortex rings, that we mentioned in the introduction.

The remainder of the article is devoted to the proofs.

3 Proofs

We first show how Theorem 1 implies Corollary 1.

**Proof of Corollary 1.** We apply metric properties of the Kantorovich–Rubinstein distance $W_1$ to the effect that

$$W_1\left(\omega(t), \sum_i a_i \delta_{Y_i(t)}\right) \leq \sum_i W_1(\omega_i(t), a_i \delta_{Y_i}) = \sum_i |a_i| W_1\left(\frac{\omega_i(t)}{a_i}, \delta_{Y_i(t)}\right)$$

$$= \sum_i |a_i| \int |x - Y_i(t)| \frac{\omega_i(t, x)}{a_i} \,dx.$$

It remains to use Jensen’s inequality to observe that $W_1 \leq W_2$ and the statement of the corollary follows from Theorem 1. \hfill \blacksquare
We will now turn to the proofs of Theorems 1 and 2 simultaneously. Our overall strategy is strongly inspired by that of Marchioro and Pulvirenti [25]. However, in order to be able to derive the statement for any solution (recall that uniqueness of solutions is not known in the regularity setting under consideration), we will not follow the regularization procedure performed in the original paper, but we approach the problem in a more direct way. Yet, most of the individual lemmas that we derive in the following have their analogues in [25].

Notice that in view of (8), we may always assume that \(Y_i\) holds with \(Y_i\) replaced by \(X_i\). We will first establish concentration estimates around the centers of vorticity.

We start by introducing some notation. We know that the velocity in the Euler equation takes the form \(u = K * \omega + \nabla \perp \eta\) and, since we are working with solutions of the type \(\omega = \sum_{i=1}^{N} \omega_i\), we have

\[
u = \sum_{i=1}^{N} K * \omega_i + \nabla \perp \eta.
\]

We call \(u_i(t, x) := K * \omega_i(t, x)\) the velocity generated by the \(i\)-th patch, and \(u_b(t, x) := \nabla \perp \eta(t, x)\) the velocity generated by the interaction with the boundary. We furthermore write \(\Omega_i(t) := \text{spt} \omega_i(t)\) and \(\bar{\Omega}_i := \text{spt} \bar{\omega}_i\).

Our first concern is a control of the velocity field generated by the \(j\)-th patch in \(\Omega_i(t)\).

**Lemma 1.** Let \(i \in \{1, \ldots, N\}\) be given. Then for any \(j \neq i\) it holds that

\[
\|u_j(t)\|_{C^{0,1}(\Omega_i(t))} \lesssim 1 \text{ for any } t \in [0, T).
\]

**Proof.** Let us prove first the \(C^0\) bound. By the definition of \(T\) in (4), we have that \(|x - y| \geq \delta/2\) for any \(x \in \Omega_i(t)\) and \(y \in \Omega_j(t)\), and thus, by the definition of \(u_j\) and \(K\),

\[
|u_j(t, x)| \leq \frac{1}{\pi \delta} \int_{\Omega} |\omega_j(t, y)| dy = \frac{|a_j|}{\pi \delta} \sim 1.
\]

Now to the Lipschitz bound. Since \(K\) is Lipschitz on \(B_{\delta/2}(0)^c\) with Lipschitz constant of order \(\delta^{-2}\), we have for any \(x, z \in \Omega_i(t)\),

\[
|u_j(t, x) - u_j(t, z)| \leq \int_{\Omega} |K(x - y) - K(z - y)| |\omega_j(t, y)| dy \lesssim \frac{|a_j|}{\delta^2} |x - z| \sim |x - z|.
\]

This concludes the proof. 

We now show that the velocity field that is due to the boundary interaction is bounded uniformly in the support of the vortex patch.

**Lemma 2.** Let \(i \in \{1, \ldots, N\}\) be given. Then

\[
\|u_b(t)\|_{C^{0,1}(\Omega_i(t))} \lesssim 1 \text{ for any } t \in [0, T).
\]
Proof. Because the support of $\omega$ has a distance at least $\delta/2$ to the boundary, it is clear that $G * \omega$ is smooth on $\partial \Omega$. Moreover, since $G$ is Lipschitz on $B_{\delta/2}(0)$ with constant of order $1/\delta$, it holds

$$\|G * \omega\|_{L^\infty(\partial \Omega)} \lesssim \frac{1}{\delta} \sum_i |a_i| \sim 1.$$ 

By the maximum principle for harmonic functions, we deduce that $\|\eta\|_{L^\infty(\Omega)} \lesssim 1$, and, by standard estimates for harmonic functions, this bound carries over to the gradient on any compact subset of $\Omega$. In particular, by the definition of $T$ in (4),

$$\|\nabla \eta\|_{L^\infty(\Omega_i(t))} \lesssim 1,$$

which is what we had to prove. \hfill \blacksquare

These two first lemmas give us bounds on those velocity contributions that are generated by far vortex patches and the interaction with the boundary (the so called far field), but we still know nothing about the velocity generated by the patch itself (the so called near field). What we have for the moment, however, is already sufficient to prove concentration in terms of the 2-Wasserstein distance around the centers of vorticity $X_i(t)$. Before stating the next lemma, we compute their velocity. Notice first that there is no self-induced motion,

$$\int_\Omega u_i(t, x) \omega_i(t, x) \, dx = \int\int_{\Omega \times \Omega} K(x - y) \omega_i(x) \omega_i(y) \, dx \, dy = 0,$$

because the Biot–Savart kernel on $\mathbb{R}^2$ $K$ is odd, that is, $K(z) = -K(-z)$. Hence, a direct computation reveals that

$$\frac{d}{dt} X_i(t) = \frac{1}{a_i} \int u(t, x) \omega_i(t, x) \, dx = \frac{1}{a_i} \int F_i(t, x) \omega_i(t, x) \, dx,$$ \hfill (12)

where

$$F_i(t, x) := \sum_{j \neq i} u_j(t, x) + u_b(t, x) = u(t, x) - u_i(t, x)$$

is the velocity far field associated to the $i$-th vorticity patch.

We now turn to the key concentration lemma, that was already found in [25].

**Lemma 3** ([25]). Let $i \in \{1, \ldots, N\}$ be given. There exists a constant $C < \infty$ dependent only on $\delta$ such that

$$W_2 \left( \frac{\omega_i(t)}{a_i}, \delta X_i(t) \right) \leq e^{C t} \epsilon \quad \text{for any } t \in [0, T).$$

**Proof.** We recall from (11) that the 2-Wasserstein distance reduces to a simple second moments function if one of the marginals is an atomic measure, that is

$$W_2(t) := W_2 \left( \frac{\omega_i(t)}{a_i}, \delta X_i(t) \right) = \left( \frac{1}{a_i} \int |x - X_i(t)|^2 \, \omega_i(t, x) \, dx \right)^{1/2}.$$
Computing its time derivative, we get (forgetting about the $t$’s)
\[
\frac{d}{dt} W^2 = 2 \int (x - X_i) \cdot \left( u(x) - \frac{dX_i}{dt} \right) \omega(x) \, dx
\]
\[
= 2 \int (x - X_i) \cdot u_i(x) \omega_i(x) \, dx + 2 \int (x - X_i) \cdot \left( F_i(x) - \frac{dX_i}{dt} \right) \omega_i(x) \, dx.
\]
The first integral vanishes because $K$ is odd, while into the second we plug the expression for the derivative of $X_i$:
\[
\frac{d}{dt} W^2 = 2 \int (x - X_i) \cdot \left( \omega_i(x) \right) dt.
\]
Since we are considering times $t \leq T$, from Lemmas 1 and 2 we know that $F_i$ is Lipschitz on $\Omega_i(t)$. Therefore, assuming that $\omega_i \geq 0$ for notational simplicity,
\[
\left| \frac{d}{dt} W^2 \right| \lesssim \int |x - X_i| |x - y| \omega_i(x) \omega_i(y) \, dy \, dx \\
\lesssim \int |x - X_i|^2 \omega_i(x) \, dx + \int \int_{\Omega \times \Omega} |x - X_i| |y - X_i| \omega_i(x) \omega_i(y) \, dy \, dx \\
\lesssim W^2,
\]
where we also used the triangle in the second and Jensen’s inequality in the third estimate. Using a Gronwall argument and keeping in mind that by hypothesis (6) and (8) the initial Wasserstein distance is bounded by $\varepsilon$, we obtain our thesis. 

Lemma 3 gives us the concentration result in terms of $\varepsilon \ll 1$, if we can ensure that the time $T = T_\varepsilon$ stays bounded away from 0, i.e., $T \gtrsim 1$, as $\varepsilon$ becomes small. To do so, we need to bound also the near field $u_i$, so that we can estimate the velocity with which the patch $\Omega_i$ moves. We begin with a result that allows us to estimate the near velocity field of a point on the boundary of the patch in terms of its distance from the center of vorticity.

**Lemma 4.** Let $i \in \{1, \ldots, N\}$ and $x \in \partial \Omega_i(t)$ be given. There exists a constant $C < \infty$ dependent only on $\delta$ such that
\[
|u_i(t,x)| \lesssim 1 + \frac{1}{|x - X_i(t)|} + \frac{e^{Ct}}{|x - X_i(t)|^2}
\]
for any $t \in [0,T)$.

**Proof.** Since the time $t$ is fixed, in this proof we will frequently omit it. Setting
\[
R := |x - X_i|,
\]
we have, assuming again that $\omega_i$ is nonnegative for notational convenience,
\[
|u_i(x)| \leq \frac{1}{2\pi} \int \frac{1}{|x - y|} \omega_i(y) \, dy \\
\lesssim \int_{B_{R/2}(X_i)} \frac{1}{|x - y|} \omega_i(y) \, dy + \int_{B_{R/2}(X_i)^c} \frac{1}{|x - y|} \omega_i(y) \, dy =: I_1 + I_2.
\]
The term \( I_1 \) is easily bounded by \( 2|a_i|/R \), because on its domain of integration it holds that \( |x-y| \geq R/2 \). To bound the other term, we make use of the interpolation-type estimate

\[
\int_A \frac{1}{|x-y|} \omega_i(y) \, dy \lesssim \ell^{1 - \frac{2}{p}} \|\omega^p_i\|_{L^p(A)} + \ell \|\omega^\infty_i\|_{L^\infty(A)} + \frac{1}{\ell} \|\omega_i\|_{L^1(A)},
\]  

(14)

that holds true for any measurable subset \( A \) of \( \Omega \) and any \( \ell > 0 \). Notice that this is a refinement of the estimate

\[
\int_A \frac{1}{|x-y|} \omega_i(y) \, dy \lesssim \|\omega_i\|_{L^p(A)} \|\omega_i\|_{L^1(A)},
\]  

(15)

that holds for \( p > 2 \) and that, to the best of our knowledge, has been first derived in [18]. Indeed, choosing \( \omega^\infty_i = 0 \) in (14) and optimizing in \( \ell \) yields (15).

We postpone the simple proof of (14) and proceed with the estimate of \( I_2 \). Keeping in mind Lemma 3, we have

\[
\|\omega_i\|_{L^1(B_{R/2}(X_i)^c)} \lesssim \frac{1}{R^2} W_2 \left( \frac{\omega_i}{a_i}, \delta_{X_i} \right)^2 \lesssim \frac{\varepsilon^2}{R^2} e^{Ct},
\]

and thus, by assumption (9),

\[
I_2 \lesssim \frac{\ell (1 - \frac{2}{p})}{\varepsilon^{2(1 - \frac{2}{p})}} + \frac{\ell}{\varepsilon^2} + \frac{\varepsilon^2}{\ell R^2} e^{Ct}.
\]

Setting \( \ell = \varepsilon^2 \), the statement in (13) follows.

It remains to provide the argument for (14), in which we drop the index \( i \) for convenience. We decompose

\[
\int_A \frac{1}{|x-y|} \omega(y) \, dy = \int_{A \cap B_t(x)} \frac{1}{|x-y|} \omega^p(y) \, dy + \int_{A \cap B_t(x)} \frac{1}{|x-y|} \omega^\infty(y) \, dy + \int_{A \setminus B_t(x)} \frac{1}{|x-y|} \omega(y) \, dy.
\]

The first two integrals can be estimated with the help of the Hölder inequality (\( p' \) being the conjugate exponent to \( p \)) and a change of variables,

\[
\int_{A \cap B_t(x)} \frac{1}{|x-y|} \omega^p(y) \, dy \leq \left( \int_{B_t(0)} \frac{1}{|y|^{p'}} \, dy \right)^{\frac{1}{p'}} \|\omega^p\|_{L^p(A)} \lesssim \ell^{1 - \frac{2}{p'}} \|\omega^p\|_{L^p(A)}
\]

because \( p > 2 \), and similarly for the second integral. For the third integral, we simply observe that

\[
\int_{A \setminus B_t(x)} \frac{1}{|x-y|} \omega(y) \, dy \leq \frac{1}{\ell} \|\omega\|_{L^1(A)}.
\]

This concludes the proof. ■
Remark 1. We remark here that a small modification of the proof would yield the same control of the velocity field if \( \omega_i \) belonged to \( L^\infty((0,T); \text{BMO}) \) with

\[
\| \omega_i(t) \|_{\text{BMO}} \lesssim \varepsilon^{-2} \quad \text{for all } t \in [0,T].
\] (16)

We omit this case here, because it is in general not known if the BMO-norm is preserved in time, and therefore assumption (16) may not be attainable under general assumptions on the initial datum. We refer to [5] for (optimal) estimates on the BMO norm for the two-dimensional Euler equations.

We still need one more ingredient in order to bound \( T \) from below uniformly in \( \varepsilon \). With Lemma 4 we have estimated the near velocity field on boundary points in terms of their distance from the center of vorticity. Should this distance vanish in \( \varepsilon \), then this result would be useless. We need therefore to estimate also the velocity of the center of vorticity. In this way, we will be able to bound the velocity of all points on the boundary of the patch (and therefore of the patch itself): If the points stay away from the center, then we use Lemma 4; otherwise we use the bound on the center of vorticity.

Lemma 5. Let \( i \in \{1, \ldots, N\} \) be given. Then

\[
|X_i(t) - \bar{X}_i| \lesssim t \quad \text{for any } t \in [0,T).
\]

Proof. Keeping in mind the expression for the velocity of the vorticity centers (12), we simply compute

\[
|X_i(t) - \bar{X}_i| = \left| \int_0^t \frac{d}{ds} X_i(s) \, ds \right| = \frac{1}{|a_i|} \left| \int_0^t \int_{\Omega} F_i(s, x) \omega_i(s, x) \, dx \, ds \right| \lesssim t,
\]

because we know by Lemmas 1 and 2 that \( F_i \) is bounded on \( \Omega_i(t) \) uniformly in \( \varepsilon \) for times \( t \in [0,T) \) and \( \| \omega_i \|_{L^1(\Omega)} = |a_i| \).

We are now ready to bound \( T \) from below uniformly in \( \varepsilon \).

Lemma 6. It holds that \( T \gtrsim 1 \).

The proof of Lemma 6 is quite elementary.

Proof. We may without loss of generality suppose that \( T \lesssim 1 \), because otherwise there is nothing to prove. Moreover, we may assume that \( T \) is solely defined through (4), because any time implicitly determined by (11) is independent of \( \varepsilon \) by definition. In particular, one of the inequalities in (4) has to be an equality, and thus, in view of the assumption on the initial data in (3), there are fluid parcels carrying nonzero vorticity that are transported over a distance at least \( \delta/4 \) in the time interval \( [0,T] \).

We denote the minimal time in which a fluid parcel moves over that distance by \( T' \). More precisely, for every initial vortex patch \( \bar{\Omega}_i \), we consider the \( \delta/4 \)-neighborhood

\[
\bar{\Omega}_i^{\delta} := \left\{ y \in \Omega : \text{dist}(y, \bar{\Omega}_i) \leq \delta/4 \right\},
\]

where \( \text{dist}(y, \bar{\Omega}_i) \) is the Euclidean distance between \( y \) and \( \bar{\Omega}_i \). We have

\[
\bar{\Omega}_i^{\delta} \subset \{ y \in \Omega : \text{dist}(y, \bar{\Omega}_i) \leq \delta/2 \},
\]

and thus

\[
|\bar{\Omega}_i^{\delta}| \lesssim \varepsilon.
\]

It follows that

\[
\frac{1}{\varepsilon} \int_0^T \| \omega_i \|_{L^1(\Omega)} \, dt \lesssim 1.
\]

This implies

\[
\frac{1}{\varepsilon} \int_0^T \| \omega_i \|_{L^1(\Omega)} \, dt \lesssim 1,
\]

and hence

\[
\frac{1}{\varepsilon} \int_0^T \| \omega_i \|_{L^1(\Omega)} \, dt \lesssim 1.
\]

Therefore, \( T \gtrsim 1 \), as desired.
and define $T' = T'_\varepsilon$ as the first instant, when some patch $\Omega_i(t)$ touches $\partial \Omega^\delta_i$. Hence,

$$T' := \min \left\{ t \geq 0 : \phi_t(\Omega_i) \cap \partial \Omega^\delta_i \neq \emptyset \text{ for some } i \in \{1, \ldots, N\} \right\} ,$$

where $\phi$ is the flow associated to $u$, cf. (2). Observe that $T' \leq T$, and thus $T' \lesssim 1$ by assumption. We will prove the lower bound $T' \gtrsim 1$, which is stronger than the statement of the lemma.

Let $\varepsilon$ be arbitrarily fixed, and let $\Omega_i$ be the first patch that touches $\partial \Omega^\delta_i$. Hence, since the flow is (Lipschitz) continuous in the time variable thanks to the boundedness of the velocity field, there exists an $x \in \partial \Omega_i$ such that

$$|x - \phi_{T'}(x)| \geq \frac{\delta}{4}. \quad (17)$$

Let us divide the proof into two cases.

**Case 1.** There exists $t \in [0, T']$ such that $|X_i(t) - X_i| > \frac{\delta}{16}$. This means that $X_i(t)$ has covered a distance of at least $\delta/16$. In this case we use Lemma 5, obtaining

$$\frac{\delta}{16} \lesssim t \leq T'$$

uniformly $\varepsilon$, which is what we have to prove.

**Case 2.** For every $t \in [0, T']$ it holds that $|X_i(t) - X_i| \leq \frac{\delta}{16}$. In this case, $X_i(t)$ has covered a smaller distance than $\phi_t(x)$. Notice that because $\frac{\delta}{4} \leq \text{dist}(\Omega_i, \partial \Omega^\delta_i) \leq |\phi_{T'}(x) - \bar{X}_i|$, we then have that

$$|\phi_{T'}(x) - X_i(T')| \geq |\phi_{T'}(x) - \bar{X}_i| - |X_i(T') - \bar{X}_i| \geq \frac{\delta}{4} - \frac{\delta}{16} = \frac{3}{16} \delta. \quad (18)$$

We split our argument into two further subcases.

**Case 2.1.** For every $t \in [0, T']$ it holds that $|\phi_t(x) - X_i(T')| > \frac{5}{32} \delta$. In this case we have

$$|\phi_t(x) - X_i(t)| \geq |\phi_t(x) - X_i(T')| - |X_i(t) - X_i(T')| \geq \frac{5}{32} \delta - \frac{\delta}{8} = \frac{1}{32} \delta \quad (19)$$

for every $t \in [0, T']$, because $|X_i(t) - X_i(T')| \leq \delta/8$ from the hypothesis of case 2. Inequalities (17) and (19) yield the bound on $T'$. Indeed, thanks to Lemma 4 and (19), for every $t \in [0, T']$

$$|u_i(t, \phi_t(x))| \lesssim 1 + \frac{1}{|\phi_t(x) - X_i(t)|} + \frac{e^{CT'}}{|\phi_t(x) - X_i(t)|^2} \lesssim 1$$

(where we also used that $T' \lesssim 1$) and this, coupled with Lemmas 1 and 2 gives

$$|u(t, \phi_t(x))| \lesssim 1 \quad \text{for all } t \in [0, T'].$$

But then, using also (17) yields

$$\frac{\delta}{4} \leq |\phi_{T'}(x) - x| = \left| \int_0^{T'} u(t, \phi_t(x)) \, dt \right| \lesssim T',$$
uniformly in $\varepsilon$, as desired.

**Case 2.2.** There exists $t \in [0, T')$ such that $|\phi_t(x) - X_i(t')| \leq \frac{5}{32} \delta$.

Consider the maximal time for which this happens, i.e.,

$$T_1 := \max \left\{ t \in [0, T') : |\phi_t(x) - X_i(t')| \leq \frac{5}{32} \delta \right\}.$$  

From (18) it follows that $T_1 < T'$ and

$$|\phi_t(x) - X_i(t')| \geq \frac{5}{32} \delta \quad \text{for all } t \in [T_1, T').$$

From this we infer that

$$|\phi_t(x) - X_i| \geq |\phi_t(x) - X_i(t')| - |X_i(t) - X_i(t')| \geq \frac{5}{32} \delta - \frac{\delta}{8} = \frac{\delta}{32} \quad \text{(20)}$$

for any $t \in [T_1, T']$, where we have used $|X_i(t) - X_i(t')| \leq \delta/8$ by the general hypothesis of Case 2. Moreover, using (18) again,

$$|\phi_{T'}(x) - \phi_{T_1}(x)| \geq |\phi_{T'}(x) - X_i(t')| - |\phi_{T_1}(x) - X_i(t')| \geq \frac{3}{16} \delta - \frac{5}{32} \delta = \frac{\delta}{32} \quad \text{(21)}$$

From now on, we can conclude exactly like in Case 2.1 using inequalities (20) and (21) instead respectively of (19) and (17), and working on the interval $[T_1, T']$ instead of $[0, T')$, to obtain

$$\frac{\delta}{32} \lesssim (T' - T_1) \leq T'$$

for this case. 

Until now, we have established that the vorticity field remains concentrated around the center of vorticity during the evolution for time intervals independent of $\varepsilon$. In order to prove Theorems 1 and 2, it remains to show that the centers of vorticity are $\varepsilon$-close to the point vortices.

In a first step, we have to establish a bound on the difference of the boundary contributions.

**Lemma 7.** For any $t \in [0, T)$ and any $i \in \{1, \ldots, N\}$ it holds that

$$|\nabla \eta(t, Y_i(t)) - \nabla \theta(t, Y_i(t))| \lesssim e^{C_1 \varepsilon} + \sum_j |X_j(t) - Y_j(t)|.$$

**Proof.** Let us drop the $t$'s in this proof. Consider $x \in \partial \Omega$, then

$$|\eta(x) - \theta(x)| = \left| \int_\Omega G(x - y) \omega(y) \, dy - \sum_j a_j G(x - Y_j) \right|$$

$$\leq \sum_j \int_\Omega |G(x - y) - G(x - Y_j)| |\omega_j(y)| \, dy.$$
Since we are considering times smaller than \( T \), we know that \(|x - y| \geq \delta/2 \) and \(|x - Y_i| \geq \delta/2 \) and, in particular,

\[
|G(x - y) - G(x - Y_j)| \leq \|\nabla G\|_{L^\infty(B_{\delta/2}(0))} |y - Y_j| \lesssim |y - Y_j|.
\]

Using the triangular inequality we have then

\[
|\eta(x) - \theta(x)| \lesssim \sum_j \int |y - X_j| |\omega_j(y)| \, dy + \sum_j |a_j| |X_j - Y_j|.
\]

Bounding the integral in the first term of the right-hand side by the Jensen inequality and Lemma 3 we see that

\[
\int |y - X_j| |\omega_j(y)| \, dy \leq |a_j| W_2 \left( \frac{\omega_j}{a_j}, \delta_{X_j} \right) \leq |a_j| \epsilon \epsilon.
\]

Hence

\[
\|\eta(t) - \theta(t)\|_{L^\infty(\theta\Omega)} \lesssim \epsilon \epsilon + \sum_j |X_j(t) - Y_j(t)|
\]

for any \( t \in [0, T) \). On the other hand, because \( \eta - \theta \) is harmonic, this bound carries over to all of \( \Omega \) by the maximum principle. Standard gradient estimates for harmonic functions in the interior of \( \Omega \) then yield the desired estimate.

We are now in the position to prove Theorems 1 and 2.

Proof of Theorems 1 and 2 We start with an estimate on the rate of change of the distance of \( X_i(t) \) and \( Y_i(t) \). Using the velocity formula of the vorticity centers and the definition of the point vortex system \( \Omega \), we find that

\[
\frac{d}{dt} |X_i(t) - Y_i(t)| \leq \frac{d}{dt} X_i(t) - \frac{d}{dt} Y_i(t) \leq \sum_{j \neq i} \frac{1}{a_i} \int_{\Omega \times \Omega} K(x - y) \omega_i(x) \omega_j(y) \, dx \, dy - a_j K(Y_i(t) - Y_j(t))
\]

\[
\leq \frac{1}{a_i} \sum_{j \neq i} \int_{\Omega \times \Omega} |K(x - y) - K(Y_i(t) - Y_j(t))| |\omega_i(x)| |\omega_j(y)| \, dx \, dy
\]

\[
+ \frac{1}{a_i} \int_{\Omega} |\nabla \eta(x) - \nabla \eta(Y_i(t))| |\omega_i(x)| \, dx + |\nabla \eta(Y_i(t)) - \nabla \theta(Y_i(t))|,
\]

where we have dropped most of the \( t \)'s for notational convenience. Using the Lipschitz property of the Biot–Savart kernel away from the origin, the Lipschitz estimate of Lemma 3 on \( \eta \), the requirements (11) and (11) on \( T \) and Lemma 7 we find that

\[
\frac{d}{dt} |X_i(t) - Y_i(t)| \leq \frac{d}{dt} X_i(t) - \frac{d}{dt} Y_i(t) \leq \sum_{j \neq i} \int_{\Omega} |x - Y_i(t)| |\omega_i(x)| \, dx + e^{C_\epsilon} \epsilon + \sum_j |X_j(t) - Y_j(t)|,
\]
and thus, using triangle and Jensen’s inequality and the concentration estimate from Lemma 3,

\[
\left| \frac{d}{dt} |X_i(t) - Y_i(t)| \right| \leq \left| \frac{d}{dt} X_i(t) - \frac{d}{dt} Y_i(t) \right| \lesssim e^{Ct} \epsilon + \sum_j |X_j(t) - Y_j(t)|.
\]

Summing over \(i\) and using a Gronwall argument yields

\[
\sum_i |X_i(t) - Y_i(t)| \lesssim e^{Ct} \epsilon + e^t \sum_i |\bar{X}_i - \bar{Y}_i|.
\]

Notice that

\[
|\bar{X}_i - \bar{Y}_i| = \frac{1}{a_i} \int |\bar{X}_i - \bar{Y}_i| \bar{\omega}_i(x) \, dx,
\]

then by the triangle and Jensen’s inequalities, \(\square\) and \(\square\) it follows that \(|\bar{X}_i - \bar{Y}_i| \lesssim e^{Ct} \epsilon\). A combination of the previous bounds yields the full statement of Theorem 2.

To derive Theorem 1 we have to combine Theorem 2 and Lemma 3. \(\square\)

Acknowledgement

The second author acknowledges inspiring discussions with Bob Jerrard on the topic of this paper. This work is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044 – 390685587, Mathematics Münster: Dynamics–Geometry–Structure.

References

[1] Ambrosio, L. Transport equation and Cauchy problem for \(BV\) vector fields. Invent. Math. 158, 2 (2004), 227–260.

[2] Aref, H. Point vortex dynamics: a classical mathematics playground. J. Math. Phys. 48, 6 (2007), 065401, 23.

[3] Azzam, J., and Bedrossian, J. Bounded mean oscillation and the uniqueness of active scalar equations. Trans. Amer. Math. Soc. 367, 5 (2015), 3095–3118.

[4] Benedetto, D., Caglioti, E., and Marchioro, C. On the motion of a vortex ring with a sharply concentrated vorticity. Math. Methods Appl. Sci. 23, 2 (2000), 147–168.

[5] Bernicot, F., and Keraani, S. Sharp constants for composition with a bi-Lipschitz measure-preserving map. Math. Res. Lett. 21, 5 (2014), 937–952.

[6] Buttà, P., and Marchioro, C. Long time evolution of concentrated Euler flows with planar symmetry. SIAM J. Math. Anal. 50, 1 (2018), 735–760.
[7] Buttà, P., and Marchioro, C. Time evolution of concentrated vortex rings. arXiv:1904.04785v1 (2019).

[8] Caprini, L., and Marchioro, C. Concentrated Euler flows and point vortex model. Rend. Mat. Appl. (7) 36, 1-2 (2015), 11–25.

[9] Colliander, J. E., and Jerrard, R. L. Vortex dynamics for the Ginzburg-Landau-Schrödinger equation. Internat. Math. Res. Notices, 7 (1998), 333–358.

[10] Crippa, G., and De Lellis, C. Estimates and regularity results for the DiPerna-Lions flow. J. Reine Angew. Math. 616 (2008), 15–46.

[11] Davila, J., del Pino, M., Musso, M., and Wei, J. Gluing methods for vortex dynamics in euler flows. arXiv:1803.00066v2 (2018).

[12] Dekeyser, J., and Van Schaftingen, J. Vortex motion for the lake equations. arXiv:1901.01717v1 (2019).

[13] del Pino, M., Kowalczyk, M., and Wei, J. On De Giorgi’s conjecture in dimension $N \geq 9$. Ann. of Math. (2) 174, 3 (2011), 1485–1569.

[14] del Pino, M., Kowalczyk, M., and Wei, J.-C. Concentration on curves for nonlinear Schrödinger equations. Comm. Pure Appl. Math. 60, 1 (2007), 113–146.

[15] DiPerna, R. J., and Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. 98, 3 (1989), 511–547.

[16] Helmholtz, H. Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. J. Mathematik 55 (1858), 25–55.

[17] Hicks, W. M. On the mutual threading of vortex rings. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 102, 715 (1922), 111–131.

[18] Iftimie, D. Évolution de tourbillon à support compact. In Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1999). Univ. Nantes, Nantes, 1999, pp. Exp. No. IV, 8.

[19] Jerrard, R. L., and Seis, C. On the vortex filament conjecture for Euler flows. Arch. Ration. Mech. Anal. 224, 1 (2017), 135–172.

[20] Jerrard, R. L., and Spirn, D. Hydrodynamic limit of the Gross-Pitaevskii equation. Comm. Partial Differential Equations 40, 2 (2015), 135–190.

[21] Judovič, V. I. Non-stationary flows of an ideal incompressible fluid. Ž. Vyčisl. Mat i Mat. Fiz. 3 (1963), 1032–1066.

[22] Kirchhoff, G. R. Vorlesungen über mathematische Physik. Teubner, Leipzig, 1876.
[23] **Loeper, G.** A fully nonlinear version of the incompressible Euler equations: the semigeostrophic system. *SIAM J. Math. Anal.* 38, 3 (2006), 795–823.

[24] **Majda, A.** Vorticity and the mathematical theory of incompressible fluid flow. *Comm. Pure Appl. Math.* 39, S, suppl. (1986), S187–S220. Frontiers of the mathematical sciences: 1985 (New York, 1985).

[25] **Marchioro, C., and Pulvirenti, M.** Euler evolution for singular initial data and vortex theory. *Comm. Math. Phys.* 91, 4 (1983), 563–572.

[26] **Marchioro, C., and Pulvirenti, M.** Vortices and localization in Euler flows. *Comm. Math. Phys.* 154, 1 (1993), 49–61.

[27] **Routh, E. J.** Some applications of conjugate functions. *Proc. Lond. Math. Soc.* 12 (1880/1881), 73–89.

[28] **Seis, C.** A quantitative theory for the continuity equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34, 7 (2017), 1837–1850.

[29] **Seis, C.** Optimal stability estimates for continuity equations. *Proc. Roy. Soc. Edinburgh Sect. A* 148, 6 (2018), 1279–1296.

[30] **Turkington, B.** On the evolution of a concentrated vortex in an ideal fluid. *Arch. Rational Mech. Anal.* 97, 1 (1987), 75–87.

[31] **Villani, C.** *Topics in optimal transportation*, vol. 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[32] **Vishik, M.** Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type. *Ann. Sci. École Norm. Sup. (4)* 32, 6 (1999), 769–812.