Stable and Efficient Structures for the Content Production and Consumption in Information Communities

Larry Yueli Zhang\(^1\) and Peter Marbach\(^2\)

\(^1\) University of Toronto
yizhang@cs.toronto.edu
\(^2\) University of Toronto
marbach@cs.toronto.edu

Abstract. Real-world information communities exhibit inherent structures that characterize a system that is stable and efficient for content production and consumption. In this paper, we study such structures through mathematical modelling and analysis. We formulate a generic model of a community in which each member decides how they allocate their time between content production and consumption with the objective of maximizing their individual reward. We define the community system as “stable and efficient” when a Nash equilibrium is reached while the social welfare of the community is maximized. We investigate the conditions for forming a stable and efficient community under two variations of the model representing different internal relational structures of the community. Our analysis results show that the structure with “a small core of celebrity producers” is the optimally stable and efficient for a community. These analysis results provide possible explanations to the sociological observations such as “the Law of the Few” and also provide insights into how to effectively build and maintain the structure of information communities.

Key words: modelling and analysis, communities, social networks

1 Introduction

Communities are an important structure that widely exists in real-world online and offline social networks. A common type of community is the information community in which the members of the community produce content and consume the content produced by other members, with the most popular example being Reddit \([11]\) where each “subreddit” is essentially an information community with a specific topic of interest. Real-world communities often exhibit inherent structures such as the high density of interactions within the community and the existence of a core set of active members who would contribute the majority of the content in the community (“the Law of the Few”) \([3, 13]\). There have been a large body of research work on community detection algorithms based on such structures. However, there is still a lack of the formal understanding of
why these structures would consistently and naturally emerge during the formation process of real-world communities. Understanding the formation process of these natural structures is important as it provides us a microscopic view of the working mechanisms of communities and would enable us to utilize communities more efficiently.

Our overall hypothesis is the following: real-world social network structures have been going through an evolutionary process, and as a result of that only the optimal structure (in terms of stability and efficiency) can survive, sustain therefore exist widely in real-world social networks. In other words, if we observe a widely existing structure in real-world social networks, then this structure must be optimal in the sense that it has stable user behaviours and it is efficient for the purpose of the network.

In the case of information communities, each member in the community is an agent who can choose to spend certain portions of their time in producing content items or in consuming content produced by other members. In order for the community to be stable, all members’ time allocation strategies should collectively form a Nash equilibrium, i.e., each member would get penalized by deviating from the equilibrium strategy. A member in the community can be rewarded by either production or consumption. For consumption, the rewarded is from the consumed content itself; for production, a member is rewarded when the content she produces is consumed by other members of the community (the reputation effect). A community structure is called “efficient” when it can provide its members the highest possible amount of reward. If we use a mathematical model to formulate the above behaviours and efficiency measures, we will then be able to formally analyze the condition under which the community structure is optimally stable and efficient, therefore obtain a mathematical description of the “surviving and sustaining” community structure. The validity of the model would be verified if the result of the analysis happens to agree with the widely-existing structures observed in real-world communities. Compared to the empirical observations, the formal analytical results would provide us more refined understanding of the microscopic working mechanisms of the real-world communities.

In this paper, we formulate a model that captures the production and consumption behaviours inside an information community. Our analysis results show that the structure with a small set of “celebrity producers” is the optimally stable and efficient structure. These analysis results provide possible explanations to the sociological observations such as “the Law of the Few” and also provide insights into how to effectively build and maintain the structure of information communities.

2 Related Works

Social network analysis has been one of the fastest growing research fields in the 21st century. We refer readers to Scott et al. [12] for a comprehensive coverage of the development of the subject, rather than listing the large collection
of references in this paper. Experimental works observed interesting properties of real-world complex networks such as the power-law degree distribution, the small-world phenomena and the community structure. These observations lead to modelling works that tried to explain why the observed properties would emerge, such models include the preferential attachment models, the copying model and the forest fire model. However, most of these works were studying macroscopic structural properties rather than looking into the internal microscopic structures of the network.

The community structure has been an interesting topic for researcher in the field of social network analysis. A large body of work has been devoted to modelling and detecting community structures in large scale social networks (e.g., [2, 8, 10, 6, 5]). The networks are often represented by graphs in which the vertices represent underlying social entities and the edges represent some sort of social tie or interaction between pairs of vertices. Our model differs in the sense that it also considers the user behaviours on top of the network connections.

In [1], the efficiency of a network in terms of information diffusion is studied, a mathematical analysis is perform to investigate the optimal network structure to achieve the best efficiency for information diffusion (high precision, high recall and low diameter), and the result shows that a Kronecker-graph [6] would satisfy such conditions. The approach taken in [1] is similar to the approach we take in this paper except that we are more focussed on the community related aspects. The work in [3] used a game-theoretic model to study the emergence of the “Law of the Few” but it is also in the context of information diffusion rather than about communities. The work in [7] is the closest to the interest of this paper. In [7], a game theoretic model is formulated to analyze the community structures in terms of content production and consumption. Each member’s strategy involves choosing a particular interest to produce or consume content on. The result shows that in the Nash equilibrium of the model the members’ choices form community structures. The difference of our work from [7] is that we focus on the internal structure of a single community rather than on the scale of multiple communities, and besides the Nash equilibrium, we also take the social welfare into consideration.

3 Model

We will first describe the general configuration of the model and the payoff/reward functions, then in Section 3.2 and Section 3.3 we introduce two variations of modelling the internal relations between the community members. Both models will be analyzed and the results will be compared in Section 4.

3.1 General Configuration

We have a single community with n members indexed by 1 ≤ i ≤ n. Each member is capable of both producing and consuming content items. The produced
content items could be chosen by all members or a subset of the members of the community for consumption. Each member has a limited total amount of time which could be allocated to either production or consumption, and each member make a decision about how much of their time to allocate to production and consumption. A member is rewarded if their products are consumed by members of the community (the production reward), or if the member consumes an item that is produced by a member of the community (the consumption reward). Each member’s objective is to maximize their total individual reward from both production and consumption.

The time slot: In our model, we investigate everything that happens within a unit time. The assumption is that the long term behaviour of a member is the repetition of their behaviour within a unit time.

Rates of content production and consumption: We define $N_p \geq 0$ to be the number of content items that a member can produce if they were to spend 100% of their unit time on production; and we let $N_c \geq 0$ be the number of items that a member can consume within a unit time slot if they were to spend 100% of their time on consumption. We assume that all members share the same values of $N_p$ and $N_c$, and we assume the following inequality:

$$0 \leq \frac{N_c}{nN_p} \leq 1$$

This assumption is reasonable because $nN_p$ is the largest possible number of content items that can be produced, an $N_c$ value that is larger than $nN_p$ would be unrealistic.

A member’s time allocation strategy: Let $\alpha_i \ (0 \leq \alpha \leq 1)$ be the portion of the unit time that member $i$ allocates to production (therefore $1 - \alpha$ is allocated to consumption). Each member chooses their own $\alpha_i$, we will investigate if a set of choices of $\alpha_i$ would lead to a Nash equilibrium. Within the unit time, a member can consume at most $(1 - \alpha_i) \cdot N_c$ items. If the number of available items is less than or equal to this number, then each member would consume all the available items without any choice; if the total number of available items is greater than this number, then the member would choose a subset (of size $(1 - \alpha_i) \cdot N_c$) of the available items to consume, uniformly at random.

The production reward models the “reputation effect” in social networks, i.e., having content products consumed by other people is rewarding for the producer of the content. The reward for each item that a member produces is proportional to the number of members who consume the item, with a constant factor $r_p$, i.e., if an item is consumed by $m$ members, then the reward for this item is $r_p \cdot m$. The total production reward for a member is the sum of the rewards of all items that the member produces. The constant factor $r_p$ is the same for all members. The consumption reward of a given item is a constant $r_c$. The total consumption reward of a member is $r_c$ multiplied by the number of items consumed by the member. The total individual reward of a member in the community is the sum of their production reward and consumption reward. The sum of the total individual rewards of all members in the community is the social
welfare. While each member tries to maximize their own individual reward, the overall efficiency of the community is measured by its social welfare.

The following two subsections will define two variations of the internal relational structure of the community.

### 3.2 The Celebrity-Follower Community Structure

Under the celebrity-follower relational structure, a subset of the members of the community are “celebrities” that are followed by everyone in the community, i.e., the content items produced by a celebrity member can be seen by all members of the community. A non-celebrity member has zero followers, i.e., an item produced by a non-celebrity member cannot be seen or consumed by any member.

Let $\eta$ be the portion of celebrity members, i.e., the number of celebrity members is $\eta n$. When $\eta = 1$, all members are connected via a complete graph. When $\eta$ is small, we have a small core of celebrities that would be responsible for producing all content items in the community. If visualized as a directed graph, the structure would have $\eta \cdot n^2$ edges in total. Note that we are assuming a member can be a follower of themselves so the graph can have self-pointing edges. This would lead to cleaner analysis results.

Note that we are not making any assumptions about how large the value of $\eta$ is, and it is interesting to see whether the efficiency of the community system can be different with $\eta$’s value being in different ranges. In real-world communities, we often observe patterns that are similar to the celebrity-follower structure, i.e., a small subset of “elite contributors” would produce most of the content items that are consumed by all members of the community, and a community typically has a significant portion of “lurkers”. We will be able to provide a theoretical explanation to this real-world phenomenon.

### 3.3 The Uniform Community Structure

In contrast to the celebrity-follower structure where the members play unequal roles in the community, the uniform relational structure has all members with the equal role, i.e., every member has the same number of followers and follows the same number of other members. In terms of a graph, it is a regular graph where every vertex has the same in-degrees and out-degrees.

To make this structure comparable with the celebrity-follower structure, we let it have the same number of edges as the celebrity-follower graph. The celebrity-follower graph discussed in the previous section has $\eta \cdot n^2$ edges, therefore, in the uniform graph, we let each vertex have in-degree $\eta \cdot n$ as well as out-degree $\eta \cdot n$.

### 3.4 Summary of the model

Overall, our model is a game-theoretic model where each agent (member of the community) chooses a strategy ($\alpha_i$) with the objective of optimizing their
individual reward. The efficiency of the whole community is measured by the social welfare (total reward of all members). The stability of the community is indicated by whether the strategies of all members collectively form a Nash equilibrium.

4 Analysis

Our hypothesis is that, in order to exist and sustain in the real world, a social structure must be stable and efficient. For an information community, this means that the members’ strategies form a Nash equilibrium while the social welfare of the community is maximized. Therefore, our analysis will take the following approach: we first derive the set of members’ strategies that would maximize the social welfare of the community, then we investigate the condition for this set of strategies to form a Nash equilibrium.

In Section 4.1 and Section 4.2 we perform the analyses for the celebrity-follower and uniform structures, respectively, then we will compare and discuss the analysis results.

4.1 Analysis of the Celebrity-Follower Structure

The following theorem summarizes the analysis results for communities with the celebrity-follower structure.

Theorem 1. For a community with the celebrity-follower structure where there are \( \eta \cdot n \) celebrity members, the maximum social welfare and the Nash equilibrium are described in the following cases.

Case 1: if \( \eta < \min\left(\frac{N_c}{nN_p}, 1 - \frac{N_c}{nN_p}, 1 - \frac{N_c r_c}{nN_p r_p}\right) \), then the maximum social welfare is reached when a member \( i \) of the community takes the following strategy:

\[
\alpha_i = \begin{cases} 
1 & \text{if member } i \text{ is a celebrity} \\
0 & \text{otherwise} 
\end{cases} 
\]  

(2)

The maximum social welfare \( G_{\text{max}} \) is the following:

\[
G_{\text{max}} = \eta(1 - \eta)n^2N_p(r_p + r_c) 
\]  

(3)

This set of strategies always form a Nash equilibrium under this case.

Case 2: if \( \frac{1}{2} < \frac{N_c}{nN_p} \leq 1 \) and \( 1 - \frac{N_c}{nN_p} \leq \eta \leq \frac{N_c}{nN_p} \), then the maximum social welfare is reached when a member \( i \) of the community follows the following strategy.

\[
\alpha_i = \begin{cases} 
\frac{N_c}{N_c + r_p N_p} & \text{if member } i \text{ is a celebrity} \\
0 & \text{otherwise} 
\end{cases} 
\]  

(4)

The maximum social welfare \( G_{\text{max}} \) is the following:
Stable and Efficient Structures

\[ G_{\text{max}} = \frac{\eta^2 N_c N_p (r_p + r_c)}{N_c + \eta n N_p} \]  

(5)

However, this set of strategies \textbf{never} forms a Nash equilibrium under this case.

Case 3: if \( 0 \leq \frac{N_c}{N_p} \leq \frac{1}{2} \) and \( \frac{N_c}{N_p} \leq \eta \leq 1 - \frac{N_c}{n N_p} \), then the maximum social welfare is reached when a member \( i \) of the community follows the following strategy.

\[ \alpha_i = \begin{cases} \frac{N_c}{\eta n N_p} & \text{if member } i \text{ is a celebrity} \\ 0 & \text{otherwise} \end{cases} \]  

(6)

The maximum social welfare under this strategy is

\[ G_{\text{max}} = N_c \left( n - \frac{N_c}{N_p} \right) (r_p + r_c) \]  

(7)

This set of strategies \textbf{never} forms a Nash equilibrium under this case.

Case 4: if \( \eta > \max(\frac{N_c}{n N_p}, 1 - \frac{N_c}{n N_p}, 1 - \frac{N_c r_c}{n N_p r_p}) \), the social welfare is maximized when a member \( i \) of the community follows the following strategy.

\[ \alpha_i = \begin{cases} \frac{N_c}{N_c + \eta n N_p} & \text{if member } i \text{ is a celebrity} \\ 0 & \text{otherwise} \end{cases} \]  

(8)

The maximum social welfare under this strategy is

\[ G_{\text{max}} = \frac{\eta^2 N_c N_p (r_p + r_c)}{N_c + \eta n N_p} \]  

(9)

This set of strategies \textbf{never} forms a Nash equilibrium under this case.

The detailed proof of Theorem 1 can be found in the appendix. This theorem shows that Case 1 is the only case where the members’ strategies reach a Nash equilibrium while the social welfare is maximized. In other words, in order for the community to be optimally stable and efficient, the portion of celebrity members must be small enough, i.e., \( \eta < \min(\frac{N_c}{n N_p}, 1 - \frac{N_c}{n N_p}, 1 - \frac{N_c r_c}{n N_p r_p}) \).

4.2 Analysis of the Uniform Structure

The following theorem summarizes the analysis results for communities with the celebrity-follower structure.

**Theorem 2.** For a community with the uniform structure where each member has \( \eta n \) followers and follows \( \eta n \) members, the maximum social welfare is reached when the following set of strategies is applied.

\[ \alpha_i = \frac{N_c}{N_c + \eta n N_p} \quad \forall 1 \leq i \leq n \]  

(10)

The maximum social welfare \( G_{\text{max}} \) is the following:
The above set of strategies forms a Nash equilibrium if and only if the following condition is true.

\[ \eta \leq \left( \frac{N_c r_c}{n N_p r_p} + \frac{1}{n} \right) \]  \hspace{1cm} (12)

The detailed proof of Theorem 2 can be found in the appendix. This result shows that, assuming the uniform community structure, there exist a simple set of strategies that is stable while the social welfare is maximized. What we are interested in is how the optimal efficiency of the uniform structure compares with that of a community with the celebrity-follower structure. The following theorem provides us a formal result.

Theorem 3. Let \( G_{\text{max - celebrity}} \) be the maximum social welfare with a Nash equilibrium for the celebrity-follower community structure (Eq (3)) and \( G_{\text{max - uniform}} \) be the maximum social welfare with a Nash equilibrium for the uniform community structure (Eq (11)). The following is always true:

\[ G_{\text{max - celebrity}} \geq G_{\text{max - uniform}} \]  \hspace{1cm} (13)

The detailed proof of Theorem 3 can be found in the appendix. This theorem provides a simple and clear result: given being in its optimally stable and efficient state, a community with the celebrity-follower structure always has a better optimal social welfare than a community with the uniform structure.

5 Discussions

The combination of the analysis results in Section 4.1 and 4.2 provide us two different angles of explaining the common “law-of-the-few” structural patterns that widely exist in real-life information communities. A given community structure, in order to exist and sustain, must be both stable and efficient, meaning that the community can stably stay at the state with the maximum social welfare. Theorem 1 tells us that the community can only be stable and efficient if there is a small enough “core” of celebrity members who will actively contribute all the content to be consumed by all members of the community, while the majority of the community members would simply consume the content produced by the core members. A community structure that does not satisfy this condition would not be stable therefore would not commonly exist in reality.

Moreover, among the different possible structure that are both stable and efficient, some structures are more efficient than others. Theorem 3 shows that the small-core celebrity-follower structure is not only stable and efficient, but also it is more efficient than other stable structures such as the uniform structure.

With the above two factors taken into account, the celebrity-follower structure with a small set of celebrities becomes the winner, therefore becomes the commonly existing structure in real-world information communities.
In the equilibrium state, the strategies of the celebrity and non-celebrity members are clearly differentiated: the celebrity members should dedicate all of their time to production whereas the non-celebrity members should spend all of their time on consumption. These specialized producing and consuming behaviours also coincide with real-world observations: in a web service such as Reddit, the visitors of a typical subreddit would often separate into two different roles, i.e., the “active contributors” who frequently post content in the subreddit and the “lurkers” who would always just consume content silently.

The analysis results also provide insights into how to effectively build and maintain information communities. The most important takeaway from our analysis results is that there should be mechanisms that encourage the formation of a small-core celebrity-follower structure inside the community. For example, many online social network applications use features such as “thumbs-up” or “upvote” to promote and reward high quality content that are liked by many community members. Besides providing effective content filtering (ranking by votes), this voting mechanism also encourages the optimally stable and efficient community structure: since the production reward is only earned when a post is upvoted, the members who would produce low-quality content would not be rewarded and would essentially become the non-celebrity members in the celebrity-follower structure. The members who produce high-quality content would be rewarded by the upvotes and become the celebrities in the community. The size of the core of celebrities will tend to be small if the display of the content in the community is ranked by popularity: most members will only consume a small portion of the top-ranked content items therefore only a small set of high quality producers would actually be rewarded and become the real core of the community. This analysis would lead to an interesting and counter-intuitive hypothesis: if the content display of the community is such that different members would see a diverse range of different items, then this would cause the formation of a larger-sized celebrity core or a uniform-like structure in the community which would make the community structure less stable. It would be interesting to empirically verify if this hypothesis is true in practice.

Another interesting insight is that, in the optimal community structure, the number of celebrity members in the core, i.e., \( \eta \cdot n \), must satisfy that \( \eta \cdot n < N_c/N_p \). This means that the size of the core of celebrities does not increase as the size of the community \( n \) increases. This could be a possible reason of why we have communities in the first place: having a large number of people communicating in a single giant community is inefficient in terms of the total amount of production because it only allows a small number of core members to contribute in content production. Larger total production rate can be achieved by dividing people into different smaller communities each of which has its own core members, since the total number of people who will contribute in content production would be multiplied by the number of communities.
6 Conclusions

This paper attempts to obtain a formal understanding of the natural structural patterns of real-world information communities. We formulate a mathematical model that describes the generic content production and consumption behaviours in a community. The analysis result shows that the small-core celebrity-follower structure is the optimal structure that would lead to the optimally efficient and stable community. These analytical results agree with the sociological observations on real-world information communities. Besides providing a refined microscopic view of the working mechanisms of information communities, the analysis results also provide useful insights into how to better build and maintain the structure of information communities. Designing efficient mechanisms that encourage the formation of stable and efficient communities would be an interesting topic for future works.

References

1. Bosagh Zadeh, Reza, et al. "On the precision of social and information networks." Proceedings of the first ACM conference on Online social networks. ACM, 2013.
2. Fortunato, Santo, and Darko Hric. Community detection in networks: A user guide. Physics Reports 659: 1-44 (2016)
3. Galeotti, Andrea, and Sanjeev Goyal. The law of the few. The American Economic Review 100.4: 1468-1492 (2010)
4. Gladwell, Malcolm. The tipping point: How little things can make a big difference. Little, Brown (2006)
5. Kumar, Ravi, Jasmine Novak, and Andrew Tomkins. Structure and evolution of online social networks. Link mining: models, algorithms, and applications. Springer New York, 337-357 (2010)
6. Leskovec, Jure, et al. Kronecker graphs: An approach to modeling networks. Journal of Machine Learning Research 11.Feb (2010): 985-1042.
7. Marbach, Peter. The structure of communities in information networks. Information Theory and Applications Workshop (ITA) (2016)
8. Massoulié, Laurent. Community detection thresholds and the weak Ramanujan property. Proceedings of the forty-sixth annual ACM symposium on Theory of computing. ACM (2014)
9. Mislove, Alan, et al. Measurement and analysis of online social networks. Proceedings of the 7th ACM SIGCOMM conference on Internet measurement. ACM (2007)
10. Newman, M. E. J. Community detection in networks: Modularity optimization and maximum likelihood are equivalent. arXiv preprint [arXiv:1606.02319] (2016)
11. Reddit: [http://www.reddit.com] (2017)
12. Scott, John. Social network analysis. Sage (2017)
13. Wenger, Etienne. Communities of practice: A brief introduction (2011)
14. Zhang, Jun, Mark S. Ackerman, and Lada Adamic. Expertise networks in online communities: structure and algorithms. Proceedings of the 16th international conference on World Wide Web. ACM (2007)
Appendices

A Proof of Theorem 1

In the following two subsections we will prove the two cases separately.

A.1 Proof of Case 1 and 2

We will first analyze the strategies for maximizing the social welfare and then investigate the Nash equilibriums of the such strategies.

Maximum Social Welfare for Case 1 and Case 2

In this case, $\eta < \frac{N_c}{nN_p} \Rightarrow \eta nN_p < N_c$, i.e., the celebrity members are not able satisfy everyone’s consumption demand even if they are producing at full capacity. First, we present the following lemma.

Lemma 1. The optimal maximum social welfare is reached only if all non-celebrity members have $\alpha_i = 0$.

Proof. This result is easy to see since a product of a non-celebrity member never gets consumed by others, a non-celebrity member’s optimal strategy is always to spend 100% of their time on consumption.

Lemma 2. When the social welfare is maximized, all celebrity members must choose the same $\alpha_i$ value.

Proof. The proof is very similar to part of the proof of Theorem 2 in Appendix B, therefore is omitted for succinctness. The idea is to compare an arbitrary configuration of $\alpha_i$ values with the configuration where everyone choose the same value.

Now let’s investigate the $\alpha_i$ value that would maximize the social welfare for Case 1 and Case 2. There are two cases that are possible when calculating the social welfare of the community: whether the celebrity members are under-supplied or over-supplied. For the celebrity members to be over-supplied, we need the following to hold:

\[(1 - \alpha_i)N_c \leq \alpha_i \eta N_p \]  
\[\Leftrightarrow \alpha_i \geq \frac{N_c}{N_c + \eta N_p} \]  

Otherwise, the celebrity members are under-supplied.

When the celebrity members are over-supplied, the social welfare is calculated as the following:
\[ G = \alpha_i \eta \eta N_p \left[ (1 - \eta)n + \frac{(1 - \alpha_i)N_c}{\alpha_i \eta N_p} \cdot \eta n \right] (r_p + r_c) \] (16)

\[ = \eta n (r_p + r_c) \left[ \alpha_i ((1 - \eta)nN_p - N_c) + N_c \right] \] (17)

When the celebrity members are under-supplied, the social welfare is calculated as the following:

\[ G = \alpha_i \eta n^2 N_p \] (18)

Combining Eq (16) and (18), the social welfare for Case 1 and Case 2 is the following.

\[ G = \begin{cases} 
\alpha_i \eta n^2 N_p & \text{if } 0 \leq \alpha_i < \frac{N_c}{N_c + \eta n N_p} \\
\eta n (r_p + r_c) \left[ \alpha_i ((1 - \eta)nN_p - N_c) + N_c \right] & \text{if } \frac{N_c}{N_c + \eta n N_p} \leq \alpha_i \leq 1 
\end{cases} \] (19)

When the celebrity members are under-supplied, \( G \) is always an increasing function of \( \alpha_i \). When the celebrity members are over-supplied, \( G \) could be either an increase or decreasing function of \( \alpha_i \) depending on whether the coefficient \( ((1 - \eta)nN_p - N_c) \) is positive or negative. Note that

\[ (1 - \eta)nN_p - N_c > 0 \iff \eta < 1 - \frac{N_c}{n N_p} \] (20)

Therefore, for Case 1 where \( \eta < 1 - \frac{N_c}{n N_p} \), the social welfare \( G \) is maximized when \( \alpha_i = 1 \), and the maximum social welfare is

\[ G_{\text{max}} = \eta (1 - \eta)n^2 N_p (r_p + r_c) \] (21)

Therefore, for Case 2 where \( \eta \geq 1 - \frac{N_c}{n N_p} \), the social welfare \( G \) is maximized when \( \alpha_i = \frac{N_c}{N_c + \eta n N_p} \), and the maximum social welfare is

\[ G_{\text{max}} = \frac{\eta \eta^2 N_c N_p (r_p + r_c)}{N_c + \eta n N_p} \] (22)

Nash Equilibrium for Case 1

Now consider the Nash equilibrium for Case 1. For a non-celebrity member, \( \alpha_i = 0 \) is clearly the best strategy since there is no additional reward if they spent any time in production.

For a celebrity member, consider the small deviation from the equilibrium strategy, i.e., rather than choosing \( \alpha_i = 1 \), it chooses \( \alpha_i = 1 - \delta \) for some \( \delta > 0 \). This member’s product reward is decreased by \( \delta N_p (1 - \eta)nr_p \). This member’s consumption reward is increased by up to \( \delta N_c r_c \). We want to show that the change of the member’s individual reward to be negative, i.e.,

\[ \Delta R \leq \delta N_c r_c - \delta N_p (1 - \eta)nr_p < 0 \] (23)

\[ \iff \eta < 1 - \frac{N_c r_c}{n N_p r_p} \] (24)

which is true according to the assumption of Case 1. Therefore, the Nash equilibrium of Case 1 is proven.
Nash Equilibrium for Case 2

Consider a celebrity member who would change their $\alpha_i$ to $\alpha_i + \delta$ and $\alpha_i - \delta$ for some $\delta > 0$. In order for a Nash equilibrium to be formed, it is necessary that both ways of changing result in a smaller individual reward.

If the strategy is changed to $\alpha_i + \delta$, then the gain of production reward is up to $\delta N_p n r_p$ and the loss of consumption reward is $\delta N_c r_c$. In order for the individual reward to decrease, we need:

$$\Delta R \leq \delta N_p n r_p - \delta N_c r_c < 0 \quad (25)$$

$$\iff \frac{N_c r_c}{n N_p r_p} > 1 \quad (26)$$

If the strategy is changed to $\alpha_i - \delta$, then the loss of production reward is $\delta N_p n r_p$ and the gain of consumption reward is up to $\delta N_c r_c$. In order for the individual reward to decrease, we need:

$$\Delta R \leq \delta N_c r_c - \delta N_p n r_p < 0 \quad (27)$$

$$\iff \frac{N_c r_c}{n N_p r_p} < 1 \quad (28)$$

A Nash equilibrium requires that both (26) and (28) to be true, which is impossible, therefore the Nash equilibrium cannot exist. The Nash equilibrium of Case 1 is proven, which completes all proofs for Case 1 and Case 2.

A.2 Proof of Case 3 and Case 4

Case 3 and Case 4 are the cases where the celebrity members are capable of supplying any member in community, i.e., $\eta n N_p \geq N_c$. First of all, Lemma 1 and Lemma 2 both apply to this part of the proof. And we have the following additional lemma.

Lemma 3. When the social welfare is maximized, a non-celebrity member must not be over-supplied, and a celebrity member must not be under-supplied.

Proof. Suppose the non-celebrity members were over-supplied, then the high-quality members must also be over-supplied. Reducing production time will increase the total number of consumptions for sure, therefore the global welfare would be increased.

Similarly, suppose the celebrity members were under-supplied, then the non-celebrity members must also be under-supplied. Increasing production time will increase the total number of consumptions for sure, therefore the global welfare would be increased.
Maximum Social Welfare for Case 3

We now write down the expression of the social welfare. Because of Lemma 2, we let $\alpha_h$ be the common $\alpha_i$ value of all celebrity members, then the total number of produced items is $\alpha_h \eta n N_p$. For each item, the expected number of consumptions from the non-celebrity members is $(1-\eta)n$ since each non-celebrity member is not over-supplied; the expected number of consumptions from the celebrity members is the following (because they are not under-supplied):

$$\frac{(1-\alpha_h)N_c}{\alpha_h N_p \eta m} \cdot \eta m = \frac{N_c}{N_p} \left( \frac{1}{\alpha_h} - 1 \right)$$

(29)

Therefore, total reward is the $(r_p + r_c)$ multiplied by the total number of consumptions, which is

$$R = (r_p + r_c) \cdot \alpha_h N_p \eta m \cdot \left[ (1-\eta)n + \frac{N_c}{N_p} \left( \frac{1}{\alpha_h} - 1 \right) \right]$$

(30)

$$= (r_p + r_c) \cdot [\alpha_h \left( N_p \eta (1-\eta)n^2 \right) - N_c \eta m] + N_c \eta m]$$

(31)

The expression in (31) could be an increasing or decreasing function of $\alpha_h$. It depends the sign of coefficient of $\alpha_h$. Therefore, we need to divide into two sub-cases, which are exactly Case 3 and Case 4.

Case 3: The coefficient is non-negative, i.e.,

$$[N_p \eta (1-\eta)n^2 - N_c \eta m] \geq 0$$

(32)

$$\Leftrightarrow \eta \leq \frac{N_c}{N_p}$$

(33)

In this case, to maximize the global welfare, $\alpha_h$ should be as large as possible, i.e., $\alpha_h$ should be the largest possible value that keeps the non-celebrity members non-oversupplied. It is the value that generates exactly $N_c$ items, which is

$$\alpha_h = \frac{N_c}{\eta m N_p}$$

(34)

Plug the above value into Eq. (31), the maximum social welfare is

$$G_{max} = (r_p + r_c)N_c \left( n - \frac{N_c}{N_p} \right)$$

(35)

Nash Equilibrium for Case 3

Now investigate the Nash equilibrium for Case 3 by consider the following two types of changes (let $\delta > 0$ denote the change amount): a celebrity member increasing from $\alpha_i$ to $\alpha_h + \delta$; and a celebrity member decreasing from $\alpha_h$ to $\alpha_h - \delta$. 
If the strategy is changed to \( \alpha_h + \delta \), then the gain of production reward is \( \delta N_p((1 - \eta) n + \eta n(1 - \alpha_h)) r_p \) and the loss of consumption reward is \( \delta N_c r_c \). In order for the individual reward to decrease, we need:

\[
\Delta R = \delta N_p((1 - \eta) n + \eta n(1 - \alpha_h)) r_p - \delta N_c r_c < 0 \quad (36)
\]

\[
\Rightarrow \frac{N_c r_c}{(n N_p - N_c) r_p} > 1 \quad (37)
\]

If the strategy is changed to \( \alpha_h + \delta \), then the loss of production reward is \( \delta N_p((1 - \eta) n + \eta n(1 - \alpha_h)) r_p \) and the gain of consumption reward is \( \delta N_c r_c \). In order for the individual reward to decrease, we need:

\[
\Delta R = \delta N_c r_c \delta N_p((1 - \eta) n + \eta n(1 - \alpha_h)) r_p < 0 \quad (38)
\]

\[
\Rightarrow \frac{N_c r_c}{(n N_p - N_c) r_p} < 1 \quad (39)
\]

A Nash equilibrium requires that both (37) and (39) to be true, which is impossible, therefore the Nash equilibrium cannot exist. The Nash equilibrium of Case 3 is proven.

**Maximum Social Welfare for Case 4**

Case 4: The coefficient in (31) is negative, i.e.,

\[
[N_p \eta (1 - \eta) n^2 - N_c \eta n] < 0 \quad (40)
\]

\[
\Rightarrow \eta > 1 - \frac{N_c}{n N_p} \quad (41)
\]

In this case, to maximize the social welfare, \( \alpha_h \) should be as small as possible, i.e., \( \alpha_h \) should be the smallest possible value that keeps the celebrity members non-under-supplied. The number of produced items should be exactly \((1 - \alpha_h) N_c \) items, i.e.,

\[
\alpha_h N_p \eta n = (1 - \alpha_h) N_c \quad (42)
\]

\[
\Rightarrow \alpha_h = \frac{N_c}{N_c + \eta n N_p} \quad (43)
\]

Plug the above value into Eq. (31), the maximum global welfare is

\[
G_{\text{max}} = \frac{\eta n^2 N_c N_p (r_p + r_c)}{N_c + \eta n N_p} \quad (44)
\]

Note that, Case 4’s optimal state for social welfare is in fact exactly the same as that of Case 2. Therefore, the Nash equilibrium of Case 4 follows the same configuration as that of Case 2, which means this set of strategy cannot form a Nash equilibrium. Hence, we have completed the proof for Theorem 1.
B Proof of Theorem 2

The claim is that the maximum social welfare is reached when (1) all members have the same \( \alpha_i \) value and (2) the number of produced items is just enough every member’s consumption. We consider three cases when calculating the social welfare of the community: everyone is over-supplied, everyone is under-supplied and some people are over-supplied while others are under-supplied. Let \( s = \sum_{i=1}^{n} \alpha_i \) denote the sum of everyone’s \( \alpha_i \) value.

Case 1: Everyone is over-supplied with content. The consumption reward a member \( i \) is simply \( (1 - \alpha_i)N_cr_c \), therefore the global total consumption reward is

\[
\sum_{i=1}^{n} (1 - \alpha_i)N_cr_c = N_cr_c(n - \sum_{i=1}^{n} \alpha_i) = N_cr_c(n - s) \tag{45}
\]

The global total production reward is equal to the total number of consumption multiplied by \( r_p \), which is

\[
r_p \cdot \sum_{i=1}^{n} (1 - \alpha_i)N_c = N_cr_p \left( n - \sum_{i=1}^{n} \alpha_i \right) = N_cr_p(n - s) \tag{46}
\]

Therefore the total global welfare for Case 1 is the sum of (45) and (46) which is

\[
G(\alpha) = N_cr_c(n - s) + N_cr_p(n - s) = N_c(r_p + r_c) \left( n - \sum_{i=1}^{n} \alpha_i \right) \tag{47}
\]

We can see that, in the case of everyone being over-supplied, the social welfare would decrease if anybody increases their \( \alpha_i \); the optimal value for this case is reached when everyone reduces their \( \alpha_i \) as much as possible as long as everyone is still over-supplied.

Now we want to show that the global welfare for this case is no greater than the maximum global welfare shown in in Equation (11) of Theorem 2 i.e., we want to show the following

\[
N_c(r_p + r_c) \left( n - \sum_{i=1}^{n} \alpha_i \right) \leq \frac{\eta m^2 N_p N_c (r_c + r_p)}{N_c + \eta m N_p} \tag{48}
\]

\[
\Leftrightarrow \left( n - \sum_{i=1}^{n} \alpha_i \right) \leq \frac{\eta m^2 N_p}{N_c + \eta m N_p} = n \cdot \left( 1 - \frac{N_c}{N_c + \eta m N_p} \right) \tag{49}
\]

\[
\Leftrightarrow \sum_{i=1}^{n} \alpha_i \geq n \cdot \frac{N_c}{N_c + \eta m N_p} \tag{50}
\]

Note that \( N_c/(N_c + \eta m N_p) \) is the optimal \( \alpha_i \) value which make the number of produced items exactly the same as the number needed by every member. The inequality in (50) must be true because if it were not true, it would cause
Case 2: Everyone is under-supplied with content. In this case, each member consumes all content items that are available for them, therefore the total consumption reward is

$$\eta n N_p r_c \sum_{i=1}^{n} \alpha_i = \eta n N_p r_c s$$ (51)

The total production reward is again the total number of consumptions multiplied by $r_p$. Each of the produced items is consumed $\eta n$ times, thus we have the total production reward as follows:

$$\eta n r_p \sum_{i=1}^{n} \alpha_i N_p = \eta N_p r_p n s$$ (52)

Summing up the total consumption reward and the total production rewards, the total global welfare for Case 2 is

$$G(\alpha) = \eta N_p (r_p + r_c) n s = \eta N_p (r_p + r_c) n \left( \sum_{i=1}^{n} \alpha_i \right)$$ (53)

We can see that, in the case of everyone being under-supplied, the social welfare would decrease if anybody decreases their $\alpha_i$; the optimal value for this case is reached when everyone increase their $\alpha_i$ as much as possible as long as everyone is still under-supplied.

Again, we want to show that the global welfare for this case is no greater than the maximum global welfare shown in in Equation (11) of Theorem 2, i.e., we want to show the following

$$\eta N_p (r_p + r_c) n \left( \sum_{i=1}^{n} \alpha_i \right) \leq \eta n^2 N_p N_c (r_c + r_p) N_c + \eta n N_p$$ (54)

$$\Leftrightarrow \sum_{i=1}^{n} \alpha_i \leq n \cdot \frac{N_c}{N_c + \eta n N_p}$$ (55)

Again, since $N_c/(N_c + \eta n N_p)$ is the $\alpha_i$ value that makes the number of produced items exactly the same as the number needed by every member. The inequality in (55) must be true because if it were not true, it would cause a member to be over-supplied, which violates the assumption for this case. Completing the proof for Case 2.

Case 3 is in between Case 1 and Case 2, i.e., we assume that a portion of the members in the community are over-supplied while others are under-supplied.

Let $\bar{\alpha} = \frac{\sum_{i=1}^{n} \alpha_i}{n}$ be the average value of all members $\alpha_i$ values. We will compare an arbitrary configuration in Case 3 with the one where everyone choose the same $\alpha_i = \bar{\alpha}$. Note that, if everyone choose the same $\alpha_i$, it is either Case
1 or Case 2, i.e., either everyone is over-supplied or everyone is under-supplied.
We want to show that any configuration in Case 3 has smaller social welfare than the configuration where everyone has the same $\alpha_i = \bar{\alpha}$. We call this two configurations the “original configuration” and the “average configuration”.

Because of the choice of the value of $\bar{\alpha}$, the sum of everyone’s $\alpha_i$ value stay the same, therefore the total number of produced items is the same between the two configurations. We will compare the social welfares of the two configurations in two cases.

Case 3.1: Everyone is over-supplied in the average configuration. Then everyone consumes exactly $N_c(1 - \bar{\alpha})$ items in the average configuration. Therefore, the consumption reward for each member is $N_c r_c(1 - \bar{\alpha})$. In the original configuration, some members have greater $\alpha_i$ values than $\bar{\alpha}$ and some others have smaller $\alpha_i$ values than $\bar{\alpha}$. Consider a pair of members of which member A has $\alpha_i = \bar{\alpha} + \delta$ and member B has $\alpha_i = \bar{\alpha} - \delta$. For member A, since she becomes even more over-supplied and spends less time on consumption, her consumption reward decreases by exactly $\delta N_c r_c$. On the other hand, member B’s consumption reward increases by at most $\delta N_c r_c$, depending on whether B becomes under-supplied after decreasing her $\alpha_i$ value from $\bar{\alpha}$. Therefore, the all-members total consumption reward of the original configuration is no greater than that of the average configuration.

In terms of the total production reward, recall that it is the total number of consumptions multiplied by $r_p$. Let $K$ denote the total number of items produced. In the average configuration, for each item, the expected number of consumptions contributed by each member is $(1 - \alpha_i N_c)/K$. In the original configuration, consider again member A with $\alpha_i = \bar{\alpha} + \delta$ and member B with $\alpha_i = \bar{\alpha} - \delta$. Member A’s contribution decreases by exactly $\delta N_c r_c/K$, while member B’s contribution increases by at most $\delta N_c r_c/K$. Therefore, the all-members total production reward of the original configuration is no greater that of the average configuration. Thus, we can conclude for Case 3.1 that the global welfare of the original configuration is no greater than that of the average configuration.

Case 3.2: Everyone is under-supplied in the average configuration. Then everyone consumes exactly $K = \eta N_p \sum_{i=1}^n \alpha_i$ items in the average configuration. Therefore, the total consumption reward is $nK r_c$. Now consider the original configuration with member A having $\alpha_i = \bar{\alpha} + \delta$ and member B having $\alpha_i = \bar{\alpha} - \delta$. For member B, she is still under-supplied and her consumption reward stays the same since the total number of items stays the same. For member A, her consumption reward stays the same if she is still under-supplied and it decreases if she becomes over-supplied and consumes less than $K$ items. Therefore, the all-members total consumption reward of the original configuration is no greater that of the average configuration.

In terms of the total production reward, the average configuration has a total production reward of $nK r_p$, whereas the original configuration can have the same total production rewards, or less if anyone becomes over-supplied by having a larger $\alpha_i$ and consumes less $K$ items. Therefore, the all-members total production reward of the original configuration is no greater that of the average
configuration. Thus, we can conclude for Case 3.2 that the social welfare of the original configuration is no greater than that of the average configuration.

Since the average configuration belongs to either Case 1 or Case 2, both of which are proven to have no greater global welfare than the maximum we claimed in (11). Hence, the global welfare in (11) is the maximum possible global welfare of the community.

Nash Equilibrium for the Uniform Structure

When every member chooses $\alpha_i = N_c/(N_c + \eta n N_p)$, the consumption reward of a member is

$$R_c \left( \alpha_i = \frac{N_c}{N_c + \eta n N_p} \right) = \left( 1 - \alpha_i \right) N_c r_c = \frac{\eta n N_p N_c r_c}{N_c + \eta n N_p} \quad (56)$$

The production reward of a member is

$$R_p \left( \alpha_i = \frac{N_c}{N_c + \eta n N_p} \right) = \alpha_i N_p r_p = \frac{\eta n N_p N_c r_p}{N_c + \eta n N_p} \quad (57)$$

The total reward is the sum of the above two which is

$$R \left( \alpha_i = \frac{N_c}{N_c + \eta n N_p} \right) = \frac{\eta n N_p N_c (r_c + r_p)}{N_c + \eta n N_p} \quad (58)$$

Now we calculate the reward for member $i$ when she changes her strategy is changed to $\alpha_i + \delta$. We divide the analysis into two cases: $\delta > 0$ and $\delta < 0$.

Case 1: $\delta > 0$, i.e., member $i$ increases $\alpha_i$. This will make some members over-supplied. For member $i$, the consumption reward decreases because of the reduced amount of consumption time, i.e.,

$$\Delta R_c = -\delta N_c r_c \quad (59)$$

The total number of available items for the followers is now $\eta \alpha_i n N_p + \delta N_p$, the probability of an item being chosen is

$$\frac{\eta \alpha_i n N_p}{\eta \alpha_i n N_p + \delta N_p} = \frac{\eta \alpha_i n}{\eta \alpha_i n + \delta} \quad (60)$$

The production reward for member $i$ becomes

$$(\alpha_i + \delta) \eta n \cdot \frac{\eta \alpha_i n}{\eta \alpha_i n + \delta} \cdot N_p r_p \quad (61)$$

Therefore, the change in member $i$’s total reward is

$$\Delta R = (\alpha_i + \delta) \eta n \cdot \frac{\eta \alpha_i n}{\eta \alpha_i n + \delta} \cdot N_p r_p - \eta \alpha_i n N_p r_p - \delta N_c r_c \quad (62)$$

We want to check the condition for $\Delta R < 0$. In the following derivation, we let $K = N_p r_p / N_c r_c$. 
\[
\Delta R = (\alpha_i + \delta)\eta n \cdot \frac{\eta \alpha_i n}{\eta \alpha_i n + \delta} \cdot N_p r_p - \eta \alpha_i n N_p r_p - \delta N_c r_c < 0
\]  
(63)

\Rightarrow K \left( \frac{\alpha_i \eta n^2 (\alpha_i + \delta)}{\alpha_i \eta n + \delta} - \alpha_i \eta n \right) < \delta

(64)

\Rightarrow \frac{K \alpha_i \eta n (\eta n - 1)}{\alpha_i \eta n + \delta} < \delta

(65)

\Rightarrow \frac{K \alpha_i \eta n (\eta n - 1)}{\alpha_i \eta n + \delta} < 1 \quad \# \delta > 0

(66)

\Rightarrow \delta > \alpha_i \eta n (K \eta n - K - 1)

(67)

To make sure (67) is satisfied, we must have

\[
\alpha_i \eta n (K \eta n - K - 1) \leq 0
\]  
(68)

\Rightarrow (K \eta n - K - 1) \leq 0

(69)

\Rightarrow K \leq \frac{1}{\eta n - 1}

(70)

\Rightarrow \frac{N_p r_p}{N_c r_c} \leq \frac{1}{\eta n - 1}

(71)

\Rightarrow \eta \leq \left( \frac{N_c r_c}{n N_p r_p} + \frac{1}{n} \right)

(72)

Summary of Case 1: If (72) is true, then the total reward for member \(i\) decreases if she increases her \(\alpha_i\) to \(\alpha_i + \delta\) (\(\delta > 0\)).

Case 2: \(\delta < 0\), i.e., member \(i\) decreases her \(\alpha_i\), then everyone in the community become under-supplied. For member \(i\), the production reward is reduced simply because of the reduced value of \(\alpha_i\), i.e.,

\[
\Delta R_p = \delta \eta n N_p r_p
\]  
(73)

The consumption reward for member \(i\) is also reduced because of the reduced number of items available for consumption (even though member \(i\) tries to spend more time on consumption).

\[
\Delta R_c = \delta N_p r_c
\]  
(74)

Therefore the change in the total reward for member \(i\) is

\[
\Delta R = \delta \cdot N_p (\eta n r_p + r_c) < 0 \quad \# \delta < 0
\]  
(75)

Case 2 done.

Combining Case 1 and Case 2, if and only if (72) is true, the change of \(\alpha_i\) will result in a decrease of member \(i\)'s total reward, therefore the strategy of choosing the original \(\alpha_i\) results in a Nash equilibrium. Completing the proof for Theorem 2.
C Proof of Theorem 3

Proof. The $G_{\text{max}}$-celebrity in Eq (3) is greater than or equal to the $G_{\text{max}}$-uniform in Eq (11) if and only if the following is true.

\[
\eta(1 - \eta)n^2N_p(r_p + r_c) \geq \frac{\eta m^2N_pN_c(r_c + r_p)}{N_c + \eta mN_p}
\]

\[
\Leftrightarrow 1 - \eta \geq \frac{N_c}{N_c + \eta mN_p}
\]

\[
\Leftrightarrow (1 - \eta)(N_c + \eta mN_p) \geq N_c
\]

\[
\Leftrightarrow \eta(nN_p - N_c + n\eta N_p) \geq 0
\]

\[
\Leftrightarrow nN_p - N_c + n\eta N_p \geq 0
\]

\[
\Leftrightarrow \eta \geq \frac{N_c - nN_p}{nN_p}
\]

According to the modelling assumption in Eq (11), $N_c - nN_p \leq 0$, and since $\eta \geq 0$, the condition is (81) is always true. Completing the proof of Theorem 3.