On Projection Robust Optimal Transport: Sample Complexity and Model Misspecification

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Abstract

Optimal transport (OT) distances are increasingly used as loss functions for statistical inference, notably in the learning of generative models or supervised learning. Yet, the behavior of minimum Wasserstein estimators is poorly understood, notably in high-dimensional regimes or under model misspecification. In this work we adopt the viewpoint of projection robust (PR) OT, which seeks to maximize the OT cost between two measures by choosing a $k$-dimensional subspace onto which they can be projected. Our first contribution is to establish several fundamental statistical properties of PR Wasserstein distances, complementing and improving previous literature that has been restricted to one-dimensional and well-specified cases. Next, we propose the integral PR Wasserstein (IPRW) distance as an alternative to the PRW distance, by averaging rather than optimizing on subspaces. Our complexity bounds can help explain why both PRW and IPRW distances outperform Wasserstein distances empirically in high-dimensional inference tasks. Finally, we consider parametric inference using the PRW distance. We provide an asymptotic guarantee of two types of minimum PRW estimators and formulate a central limit theorem for max-sliced Wasserstein estimator under model misspecification. To enable our analysis on PRW with projection dimension larger than one, we devise a novel combination of variational analysis and statistical theory.

1 Introduction

Recent years have witnessed an ever-increasing role for ideas from optimal transport (OT) [Villani, 2008] in machine learning. Combining OT distances with the general principles of minimal distance estimation (MDE) [Wolfowitz, 1957, Basu et al., 2011] yields a powerful basis for various statistical inference problems, such as density estimation Bassetti et al. [2006], training of generative model [Arjovsky et al., 2017, Gulrajani et al., 2017, Montavon et al., 2016, Adler and Lunz, 2018, Cao et al., 2019], auto-encoders [Tolstikhin et al., 2018], clustering [Cuturi and Doucet, 2014, Bonneel et al., 2016, Ho et al., 2017, Ye et al., 2017], multitask regression [Janiati et al., 2020], trajectory inference [Hashimoto et al., 2016, Schiebinger et al., 2017, Yang et al., 2020, Tong et al., 2020] or nonparametric testing [Ramdas et al., 2017]; see Peyré and Cuturi [2019] and Panaretos and Zemel [2019] for reviews on these topics.

For OT ideas to continue to bear fruit in machine learning, it will be necessary to tackle two characteristic challenges: (1) high dimensionality and (2) model misspecification. Initial progress has been made on the latter problem by Bernton et al. [2019], who showed that in the misspecified case the minimum Wasserstein estimator (MWE) outputs the Wasserstein projection of the data-generating
distribution onto the fitted model class. These authors also obtained results on robustness and the asymptotic distribution of the projection, while these results only apply to the one-dimensional setting. High-dimensional settings are challenging; indeed, it is known that the sample complexity of estimating the Wasserstein distance can grow exponentially in dimension [Dudley, 1969, Fournier and Guillin, 2015, Singh and Póczos, 2018, Weed and Bach, 2019, Lei, 2020].

We focus on a promising approach to treating high-dimensional problems: Compute the OT distance between low-dimensional projections of high-dimensional input measures. The simplest and most representative example of this approach is the sliced Wasserstein distance [Rabin et al., 2011, Bonnotte, 2013, Bonneel et al., 2015, Deshpande et al., 2019, Kolouri et al., 2019a, Nadjahi et al., 2020, Manole et al., 2019], which is defined as the average OT distance obtained between random 1-dimensional projections, and which is shown practical in real applications [Deshpande et al., 2018, 2019, Kolouri et al., 2016, 2019b, Carriere et al., 2017, Wu et al., 2019, Liutkus et al., 2019]. In an important extension, Paty and Cuturi [2019] and Niles-Weed and Rigollet [2019] proposed very recently to seek the $k$-dimensional subspace ($k > 1$) that would maximize the OT distance between two measures after projection. The quantity is named as projection robust Wasserstein (PRW) distance\(^1\), which is conceptually simple and does solve the curse of dimensionality in the so-called spiked model as proved in [Niles-Weed and Rigollet, 2019, Theorem 1] by recovering the $n^{-1/k}$ rate under the Talagrand transport inequality. This result suggests that PRW can be significantly more useful than the OT distance for inference tasks when the dimension is large. From a computational point of view, PRW becomes the max-sliced Wasserstein distance when the projection dimension is $k = 1$ and has an efficient implementation [Deshpande et al., 2019]. For general $k \geq 1$, Lin et al. [2020] proposed to compute PRW using Riemannian optimization toolbox and provided theoretical guarantee and encouraging empirical results. However, it is desirable to understand its statistical behavior which mostly determines the practical performance of PRW.

**Contributions.** In this paper, we study the statistical properties of PRW and another so-called integrated PRW (IPRW), which replaces the maximum in the original PRW with an average of OT distance over $k$-dimensional projections. Our contributions can be summarized as follows.

1. We prove that the empirical measure $\hat{\mu}_n$ converges to true measure $\mu_*$ under both PRW and IPRW with different rates. These rates are new to our knowledge. For example, when the order $p = 3/2$ and the projected dimension $k \geq 3$, the rate is $n^{-1/k}$ for IPRW. For PRW, the rate is $(n^{-1/k} + n^{-1/6} \sqrt{dk \log(n)} + n^{-2/3} d \log(n))$ when $\mu_*$ satisfies a projection Bernstein tail condition and $(n^{-1/k} + n^{-1/2} \sqrt{dk \log(n)} + n^{-2/3} d \log(n))$ when $\mu_*$ satisfies a projection Poincaré inequality.

2. We derive the concentration results when $\mu_*$ satisfies a Bernstein tail condition or a projection Poincaré inequality. In terms of tail conditions, our Bernstein condition and Poincare inequality handle subexponential tail while Talagrand inequality in Niles-Weed and Rigollet [2019] addresses subgaussian tail. Our assumptions are thus weaker than those used in Bernton et al. [2019], not requiring the nonsingularity of the Jacobian or the separability of the parameters. Our\(^1\)This quantity is also named as Wasserstein Projection Pursuit (WPP) [Niles-Weed and Rigollet, 2019]. For simplicity, we refer from now on to PRW/WPP as PRW.

3. We establish asymptotic guarantees for the minimal PRW and expected minimal PRW estimators under model misspecification. For minimal PRW estimator with the order $p = 1$ and the projected dimension $k = 1$, we derive an asymptotic distribution for arbitrary dimension $d$ with the $n^{-1/2}$ rate in the Hausdorff metric. Our assumptions are weaker than those used in Bernton et al. [2019], not requiring the nonsingularity of the Jacobian or the separability of the parameters.
techniques for CLT in misspecified settings did not appear in Nadjahi et al. [2019] and complete the analysis in Bernton et al. [2019].

4. We conduct experiments on synthetic data and neural networks to validate our theory. We also present a simple optimization algorithm that can efficiently compute the PRW distance in practice even when \( k \geq 2 \); see Appendix F or Appendix B of the concurrent work by Lin et al. [2020].

## 2 Preliminaries on Projected Optimal Transport

In this section, we provide some technical background materials on projection optimal transport. Throughout the paper, we denote \( \| \cdot \| \) as the Euclidean norm (in the corresponding vector space) and \( \Rightarrow \) as the convergence in the weak sense.

**Wasserstein and sliced Wasserstein.** Let \( p \geq 1 \) and define \( \mathcal{P}(\mathbb{R}^d) \) and \( \mathcal{P}_p(\mathbb{R}^d) \) as the set of all Borel measures on \( \mathbb{R}^d \) and the subset that satisfies \( M_p(\mu) := \int_{\mathbb{R}^d} \|x\|^p d\mu(x) < +\infty \). For two probability measures \( \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d) \), their Wasserstein distance of order \( p \) is defined as follows:

\[
W_p(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p d\pi(x, y) \right)^{1/p},
\]

where the infimum is taken over \( \Pi(\mu, \nu) \subseteq \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \)—the set of probability measures with marginals \( \mu \) and \( \nu \). In the 1D case, Rachev and Rüschendorf [1998, Theorem 3.1.2.(a)] have shown that \( W_p(\mu, \nu) = (\int_0^1 |F^{-1}_\mu(t) - F^{-1}_\nu(t)|^p dt)^{1/p} \), where \( F^{-1}_\mu \) and \( F^{-1}_\nu \) are the quantile functions of \( \mu \) and \( \nu \). This 1D formula motivates the *sliced Wasserstein (SW) and max-sliced Wasserstein (max-SW) distances* [Bonnotte, 2013, Bonneel et al., 2015, Deshpande et al., 2019]. In particular, the idea is to use as a proxy of \( W_p(\mu, \nu) \) the average or maximum of a set of 1D Wasserstein distances constructed by projecting \( d \)-dimensional measures to a random collection of 1D spaces. Computationally appealing, both SW and max-SW distances are widely used in practice, especially in generative modeling [Kolouri et al., 2019b, Deshpande et al., 2019, Liutkus et al., 2019]. Practitioners observe that the SW distance only outputs a good Monte-Carlo approximation with a large number of projections, while the max-SW distance achieves similar results with fewer projections [Kolouri et al., 2019a, Nguyen et al., 2020].

Encouraged by the success of SW and max-SW, Paty and Cuturi [2019] asked whether we can gain more by using a subspace of dimension \( k \geq 2 \), define the projection robust Wasserstein (PRW) distance, and prove that this quantity is well posed if the order is \( p \geq 1 \). More specifically, let \( S_{d,k} = \{ E \in \mathbb{R}^{d \times k} : E^\top E = I_k \} \) be the set of \( d \times k \) orthogonal matrices and \( E^* \) be the linear transformation associated with \( E \) for any \( x \in \mathbb{R}^d \) by \( E^*(x) = E^\top x \). For any measurable function \( f \) and \( \mu \in \mathcal{P}(\mathbb{R}^d) \), we denote \( f_#\mu \) as the push-forward of \( \mu \) by \( f \), so that \( f_#\mu(A) = \mu(f^{-1}(A)) \) where \( f^{-1}(A) = \{ x \in \mathbb{R}^d : f(x) \in A \} \) for any Borel set \( A \). For any given subspace dimension \( K \), the PRW distance of order \( p \) between \( \mu \) and \( \nu \) is defined by

\[
PW_{p,k}(\mu, \nu) := \sup_{E \in S_{d,k}} W_p(E_#^*\mu, E_#^*\nu).
\]

The PRW distance has better discriminative power than the SW or max-SW distances since it can extract more geometric information from high-dimensional projections than that from 1-dimensional projections; see Paty and Cuturi [2019] for more details.
As an alternative, we define the IPRW distance, which replaces the supremum in Eq. (2.2) with an average. The IPRW distance of order $p$ between $\mu$ and $\nu$ is

$$
\bar{PW}_{p,k}(\mu, \nu) := \left( \int_{\mathbb{S}_{d,k}} W^p_{p,k}(E^*_{\#}\mu, E^*_{\#}\nu) d\sigma(E) \right)^{1/p},
$$

(2.3)

where $\sigma$ is the uniform distribution on $\mathbb{S}_{d,k}$. Note that IPRW is well defined for comparing two measures and match our intuition. For example, given three Gaussian distributions $\mu_i = \mathcal{N}(u_i, I_d)$ for $i = 1, 2, 3$, we have $\bar{PW}_{p,2}(\mu_i, \mu_j) = c\|u_i - u_j\|$ where $c > 0$ only depend on $p$ and the dimension $d$.

The IPRW and PRW distances generalize the SW and max-SW distances to the high-dimensional projection setting. Both PRW and and IPRW are distances and satisfy the triangle inequality: the proof for PRW is in Paty and Cuturi [2019, Proposition 1], while that for IPRW is the same as that for SW in Bonnotte [2013]. Compared to the PRW distance, the IPRW distance performs better statistically but remains unfavorable in computational sense. Indeed, a large amount of projections from $\mathbb{S}_{d,k}$ are necessary to approximate the IPRW distance. However, if the intrinsic dimension of data distribution is small, the required number of random projections is small; see Nadjah et al. [2019].

Let $X_{1:n} = (X_1, \ldots, X_n)$ be independent and identically distributed samples according to the true measure $\mu_* \in \mathcal{P}(\mathbb{R}^d)$. The empirical measure of $X_{1:n}$ is defined by $\hat{\mu}_n := (1/n) \sum_{i=1}^n \delta_{X_i}$. It is known that $\hat{\mu}_n \Rightarrow \mu_*$ almost surely, and $W_p(\hat{\mu}_n, \mu_*) \rightarrow 0$ almost surely since Wasserstein distances metrizes weak convergence [Villani, 2008, Theorem 6.9] (note that $q \geq p \geq 1$). However, $\mathbb{E}[W_p(\hat{\mu}_n, \mu_*)] \simeq n^{-1/d}$ whenever $\mu$ is absolutely continuous with respect to Lebesgue measure and $d > 2p$ [Dudley, 1969, Fournier and Guillin, 2015, Weed and Bach, 2019] ($\simeq$ means “equal to” with a constant independent of $n$). The convergence is slow when the dimension is high — an instance of the well-known curse-of-dimensionality phenomenon.

Due to the low-dimensional structure of the IPRW and PRW distances, the rate of IPRW and PRW distances is expected to be of $n^{-1/k}$ in the large-$n$ limit. Similar rates have been derived for $\mathbb{E}[|\bar{PW}_{k,p}(\hat{\mu}_n, \hat{\nu}_n) - W_p(\mu, \nu)|]$ as a function of $n$ under a spiked transport model for both $\mu$ and $\nu$; see Niles-Weed and Rigollet [2019, Theorem 8]. Their bound depends on problem dimension $d$ and requires $\mu$ and $\nu$ to satisfy the Talagrand transport inequality [Talagrand, 1996]. For the special case when $k = 1$, the rate for the IPRW distance was studied in [Nadjahi et al., 2020] and the minimax confidence intervals were established in Manole et al. [2019]. To our knowledge, there has been no other paper on the statistical properties of IPRW and PRW distances for $k \geq 2$.

**Parametric modeling and inference.** A statistical model is a family of distributions, $\mathcal{M} = \{\mu_\theta \in \mathcal{P}(\mathbb{R}^d) \mid \theta \in \Theta\}$, where $\Theta$ is the parameter space. A minimal set of the conditions of a proper family of distribution are: (i) $(\Theta, \| \cdot \|_{1})$ is a Polish space, (ii) $\Theta$ is $\sigma$-compact, i.e., it is the union of countably many compact subspaces, and (iii) parameters are identifiable, i.e., $\mu_\theta = \mu_{\theta'}$ implies $\theta = \theta'$ for all $\theta, \theta' \in \Theta$. Since the space $\mathcal{P}(\mathbb{R}^d)$ endowed with the distance $W_p$, is a Polish space, we estimate model coefficients using minimum distance estimation (MDE) [Wolfowitz, 1957, Basu et al., 2011], where the distance we consider here is PRW. The main reason why we do not choose IPRW in this setting is computational. The minimum project robust Wasserstein (MPRW) estimator is defined as follows:

$$
\hat{\theta}_n := \arg\min_{\theta \in \Theta} \bar{PW}_{p,k}(\hat{\mu}_n, \mu_\theta).
$$

(2.4)

Note that the probability density function of $\mu_\theta$ can be difficult to evaluate in practice, especially when $\mu_\theta$ is a generative model. Nevertheless, in various settings, even if the density is not available, one
can generate samples \( Z_{1:m} \) from \( \mu_\theta \) and use them to approximate \( \mu_\theta \). With this approximation, a natural alternative is the \textit{minimum expected projection robust Wasserstein} (MEPRW) estimator, which is defined as follows [Bernton et al., 2019, Nadjahi et al., 2019]:

\[
\hat{\theta}_{n,m} := \arg\min_{\theta \in \Theta} \mathbb{E}[\overline{PW}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) \mid X_{1:n}],
\]

(2.5)

where \( n \) is the number of samples from the data distribution \( \mu_* \), \( m \) is the number of samples from the parametric distribution \( \mu_\theta \), and \( \hat{\mu}_{\theta,m} \) is an empirical version of \( \mu_\theta \) based on samples \( Z_{1:m} \).

Existing works have established asymptotic guarantees for minimal Wasserstein and sliced Wasserstein estimators [Bernton et al., 2019, Nadjahi et al., 2019]. Despite the similar proof paths, our results for the MPRW and MEPRW estimators are new and derived under weaker assumptions and more general settings than previous work; see Sections 3.3 and 3.4.

3 Main Results on Projection Robust Optimal Transport Estimation

Throughout this section, we assume \( p \geq 1 \) and \( k \in [d] \triangleq \{1, 2, \ldots, d\} \) unless stated otherwise. Focusing on the IPRW and PRW distances, we prove that they are lower semi-continuous and metrize weak convergence. Through a new sample complexity analysis, we derive the convergence rate of empirical measures under both distances as well as an improved rate for the PRW distance when \( \mu_* \) satisfies either a Bernstein tail condition or the Poincaré inequality. For the generative models with the PRW distance, we study the misspecified setting where the limit \( \theta_* \) is not necessarily the limit of the maximum likelihood estimator. We establish the asymptotic properties of the MPRW and MEPRW estimators and formulate a central limit theorem when \( p = 1 \) and \( k = 1 \).

3.1 Topological properties

We begin with the results on the relationship between the IPRW, PRW and Wasserstein distances. The following lemma demonstrates their equivalence in a topological sense.

\textbf{Lemma 3.1} The IPRW, PRW and Wasserstein distances are equivalent. That is, for any sequence of probability measures \( \{\mu_i\}_{i \in \mathbb{N}} \) and probability measure \( \mu \) in \( \mathcal{P}_p(\mathbb{R}^d) \), we have \( \mathbb{PW}_{p,k}(\mu_i, \mu) \to 0 \) if and only if \( \overline{PW}_{p,k}(\mu_i, \mu) \to 0 \) if and only if \( W_p(\mu_i, \mu) \to 0 \).

Lemma 3.1 is a generalization of Bayraktar and Guo [2019, Theorem 1] where the projection dimension is \( k = 1 \). By Lemma 3.1 and Villani [2008, Theorem 6.9], we obtain the following result regarding the topology induced by the IPRW and PRW distances of order \( p \).

\textbf{Theorem 3.2} The IPRW and PRW distances both metrize weak convergence. In other words, for any sequence of probability measures \( \{\mu_i\}_{i \in \mathbb{N}} \) and probability measure \( \mu \) in \( \mathcal{P}_p(\mathbb{R}^d) \), we have \( \mathbb{PW}_{p,k}(\mu_i, \mu) \to 0 \) if and only if \( \overline{PW}_{p,k}(\mu_i, \mu) \to 0 \) if and only if \( W_p(\mu_i, \mu) \to 0 \).

Theorem 3.2 generalizes Villani [2008, Theorem 6.9] since the PRW distance is the Wasserstein distance when the projection dimension \( k = d \). When \( k = 1 \), Theorem 3.2 recovers the results presented by Bayraktar and Guo [2019] which implies that the SW and max-SW distances metrize weak convergence. It is worthy noting that this implication is stronger than Nadjahi et al. [2019, Theorem 1], which only provides a one-sided argument.
Theorem 3.3 The IPRW and PRW distances are both lower semi-continuous in the usual weak topology. In other words, if the sequences of probability measures \( \{\mu_i\}_{i \in \mathbb{N}}, \{\nu_i\}_{i \in \mathbb{N}} \subseteq \mathcal{P}(\mathbb{R}^d) \) satisfy \( \mu_i \Rightarrow \mu \) and \( \nu_i \Rightarrow \nu \) for probability measures \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \), then we have \( \mathcal{PW}_{p,k}(\mu_i, \nu_i) \leq \lim_{i \to +\infty} \mathcal{PW}_{p,k}(\mu_i, \nu_i) \) and \( \mathcal{PW}_{p,k}(\mu, \nu) \leq \lim_{i \to +\infty} \mathcal{PW}_{p,k}(\mu_i, \nu_i) \).

The above theorem generalizes Nadjahi et al. [2019, Lemma S6] and is pivotal to our asymptotic analysis for the MPRW and MEPRW estimators.

3.2 Convergence and concentration of empirical measures

We provide the rate of the empirical measures under the IPRW and PRW distances of order \( p \) with the projection dimension \( k \). We present our main result on convergence rates in the following theorem.

Theorem 3.4 Let \( \mu_* \in \mathcal{P}_q(\mathbb{R}^d) \) and \( M_q(\mu_*) < +\infty \) for some \( q \geq 1 \). Then we have\(^2\)

\[
\mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n, \mu_*)] \lesssim_{p,q} n^{-(\frac{1}{2(p+q)} \wedge \frac{1}{q} - \frac{1}{q})} (\log(n))^\frac{\zeta_{p,q,k}}{p},
\]

where \( \lesssim_{p,q} \) refers to “less than” with a constant depending only on \( p, q \) and

\[
\zeta_{p,q,k} = \begin{cases} 
2 & \text{if } k = q = 2p, \\
1 & \text{if } (k \neq 2p \text{ and } q = \frac{kp}{k-p}) \text{ or } (q > k = 2p), \\
0 & \text{otherwise}.
\end{cases}
\]

Remark 3.1 Theorem 3.4 shows that our bound does not depend on \( d \), while all bounds for the Wasserstein distance grow exponentially in \( d \) when \( d \geq 2p \) [Lei, 2020, Theorem 3.1]. This improvement shows that the PRW distance does not suffer from the curse of dimensionality while retaining flexibility via the choice of \( k \). We are also aware of concurrent work [Nath and Jawanpuria, 2020] in which the sample complexity has no dependence on dimensionality.

Definition 3.1 We say \( \mu \in \mathcal{P}(\mathbb{R}^d) \) satisfies a projection Bernstein tail condition if there exist \( \sigma, V > 0 \) for all \( E \in \mathbb{S}_{d,k} \) and \( X \sim E_*^\mu \) such that \( \mathbb{E}[||X||^r] \leq (1/2)\sigma^2 r! V^{r-2} \) for all \( r \geq 2 \).

Theorem 3.5 Suppose \( \mu_* \in \mathcal{P}_q(\mathbb{R}^d) \) satisfies a projection Bernstein tail condition and assume the same setting as in Theorem 3.4. For all \( n \geq 1 \), the following inequality holds true:

\[
\mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n, \mu_*)] \lesssim_{p,q} n^{-(\frac{1}{2(p+q)} \wedge \frac{1}{q} - \frac{1}{q})} (\log(n))^\frac{\zeta_{p,q,k}}{p} + n^{\frac{1}{p} - \frac{1}{q}} \sqrt{d k \log(n)} + n^{-\frac{1}{p}} dk \log(n).
\]

Definition 3.2 We say \( \mu \in \mathcal{P}(\mathbb{R}^d) \) satisfies a projection Poincaré inequality if there exists \( M > 0 \) for all \( E \in \mathbb{S}_{d,k} \) and \( X \sim E_*^\mu \) such that \( \text{Var}(f(X)) \leq M \mathbb{E}[||\nabla f(X)||^2] \) for any \( f : \mathbb{R}^d \to \mathbb{R} \) satisfying that \( \mathbb{E}[f(X)^2] < +\infty \) and \( \mathbb{E}[||\nabla f(X)||^2] < +\infty \).

Theorem 3.6 Suppose \( \mu_* \in \mathcal{P}_q(\mathbb{R}^d) \) satisfies a projection Poincaré inequality and assume the same setting as in Theorem 3.4. For all \( n \geq 1 \), the following inequality holds true:

\[
\mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n, \mu_*)] \lesssim_{p,q} n^{-(\frac{1}{2(p+q)} \wedge \frac{1}{q} - \frac{1}{q})} (\log(n))^\frac{\zeta_{p,q,k}}{p} + n^{-\frac{1}{p}} \sqrt{d k \log(n)} + n^{-\frac{1}{p}} dk \log(n).
\]

\(^2a \vee b = \max\{a, b\} \) and \( a \wedge b = \min\{a, b\} \) here.
We present concentration results when $\mu_*$ satisfies stronger conditions than Definition 3.1 and 3.2.

**Definition 3.3** A measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfies a Bernstein tail condition if there exists $\sigma, V > 0$ such that $\mathbb{E}_{X \sim \mu}[\sup_{E \in \mathcal{S}_{d,k}} \|E^\top X\|_r^r] \leq (1/2)\sigma^2 r! V^{r-2}$ for all $i = 1, 2, \ldots, n$ and all $r \geq 2$.

**Theorem 3.7** If $\mu_* \in \mathcal{P}(\mathbb{R}^d)$ satisfies a Bernstein tail condition then the following statement holds true for both $W = \mathcal{PW}_{p,k}$ and $W = \mathcal{PW}_{p,k}$:

$$\mathbb{P}(\|W(\hat{\mu}_n, \mu_*) - \mathbb{E}[W(\hat{\mu}_n, \mu_*)]\| \geq t) \leq 2 \exp\left(-\frac{t^2}{8\sigma^2 n^{1-2/p} + 4tVn^{-1/p}}\right).$$

**Definition 3.4** $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfies a Poincaré inequality if there exists $M > 0$ for $X \sim \mu$ such that $\mathbb{V}[f(X)] \leq M\mathbb{E}[\|\nabla f(X)\|^2]$ for any $f$ satisfying $\mathbb{E}[f(X)^2] < +\infty$ and $\mathbb{E}[\|\nabla f(X)\|^2] < +\infty$.

**Theorem 3.8** If $\mu_* \in \mathcal{P}(\mathbb{R}^d)$ satisfies Poincaré inequality then the following statement holds true for both $W = \mathcal{PW}_{p,k}$ and $W = \mathcal{PW}_{p,k}$:

$$\mathbb{P}(\|W(\hat{\mu}_n, \mu_*) - \mathbb{E}[W(\hat{\mu}_n, \mu_*)]\| \geq t) \leq 2 \exp(-K^{-1} \min\{n^{\frac{1}{2}}t, n^{\frac{1}{2p}}t^2\}),$$

where $K > 0$ only depends on $M$ (cf. Definition 3.4).

**Discussions.** We demonstrate that the Bernstein-type tail conditions in Definition 3.1 and 3.3 are not strong enough to give an effective bound for all $p \geq 1$. The similar results for the Wasserstein distance have been recently derived by Lei [2020] and recognized as the standard limitation for the Bernstein-type tail conditions. This is also the motivation which drives us to consider a Poincaré inequality.

For Theorem 3.5 and 3.6, the first term matches that in Theorem 3.4 while the extra two terms come from bounding the gap $\mathbb{E}[\sup_{E \in \mathcal{S}_{d,k}} (W_p(E^{n\mu}_* \hat{\mu}_n, E^{n\mu}_* \mu_*) - \mathbb{E}[W_p(E^{n\mu}_* \hat{\mu}_n, E^{n\mu}_* \mu_*)])]$. Compared with Niles-Weed and Rigollet [2019, Theorem 8], where $\mu_*$ satisfies the Talagrand transport inequality, our conditions are weaker but our rate matches their $n^{-1/k} + n^{-1/2} \sqrt{dk \log(n)}$ rate in the large-$n$ limit when $p = 1$. For Theorem 3.7 and 3.8, the latter bound is better than the former bound when $p > 1$. Moreover, the tail condition in Definition 3.3 is stronger than that in Definition 3.1 yet weaker than the standard Bernstein tail condition where $X \sim \mu$ inside the expectation without a sup; see Wainwright [2019]. The Poincaré inequality is weaker than the log-Sobolev inequality and is satisfied by various exponential measures and the measures induced by Markov processes [Ledoux, 1999]. Intuitively, these two conditions handle subexponential tail while Talagrand inequality in Niles-Weed and Rigollet [2019] addresses subgaussian tail; see Ledoux [1999] and Talagrand [1996] for the details.

### 3.3 Properties of MPRW and MEPRW estimators

We derive the asymptotic properties of the MPRW and MEPRW estimators under model misspecification, which is common in practice. Our setting is more general than that considered in [Nadjahi et al., 2019] and our results support the applications in real-world scenario better. Specifically, while Nadjahi et al. [2019] focused on the well-specified setting, the statistical models can be misspecified in many real-world applications. We also use the Wasserstein distance in Assumptions 3.1 and 3.4 since these assumptions have been shown valid for many real-world application problems [Bernton et al., 2019].
Assumption 3.1 There exists a probability measure $\mu_* \in \mathcal{P}(\mathbb{R}^d)$ such that the data-generating process satisfies that $\lim_{n \to +\infty} W_p(\hat{\mu}_n, \mu_*) = 0$ almost surely.

Assumption 3.2 The map $\theta \mapsto \mu_\theta$ is continuous: $\|\theta_n - \theta\|_\Theta \to 0$ implies $\mu_{\theta_n} \Rightarrow \mu_\theta$.

Assumption 3.3 There exists a constant $\tau > 0$ such that the set $\Theta_\tau(\tau) \subseteq \Theta$ is bounded where $\Theta_\tau(\tau) = \{\theta \in \Theta : W_p(\mu_*, \mu_\theta) \leq \inf_{\theta \in \Theta} W_p(\mu_*, \mu_\theta) + \tau\}$.

Theorem 3.9 Under Assumption 3.1-3.3, there exists a sample space $\Omega$ with $\mathbb{P}(\Omega) = 1$ such that, for all $\omega \in \Omega$,

$$\lim_{n \to +\infty} \inf_{\theta \in \Theta} W_p(\hat{\mu}_n(\omega), \mu_\theta) = \inf_{\theta \in \Theta} W_p(\mu_*, \mu_\theta),$$

and

$$\limsup_{n \to +\infty} \arg\min_{\theta \in \Theta} W_p(\hat{\mu}_n(\omega), \mu_\theta) \subseteq \arg\min_{\theta \in \Theta} W_p(\mu_*, \mu_\theta).$$

In addition, $\arg\min_{\theta \in \Theta} W_p(\hat{\mu}_n(\omega), \mu_\theta) \neq \emptyset$ for all $n \geq n(\omega)$ with some $n(\omega) > 0$.

Assumption 3.4 If $\|\theta_n - \theta\|_\Theta \to 0$, then $\mathbb{E}[W_p(\hat{\mu}_{\theta_n,n}, \mu_{\theta_n}) | X_{1:n}] \to 0$.

In the next result, we present an analogous version of Theorem 3.9 for the MEPRW estimator as $\min\{n, m\} \to +\infty$. For the simplicity, we set $m \coloneqq m(n)$ such that $m(n) \to +\infty$ as $n \to +\infty$.

Theorem 3.10 Under Assumption 3.1-3.4, there exists a sample space $\Omega$ with $\mathbb{P}(\Omega) = 1$ such that, for all $\omega \in \Omega$,

$$\lim_{n \to +\infty} \inf_{\theta \in \Theta} \mathbb{E}[W_p(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}] = \inf_{\theta \in \Theta} W_p(\mu_*, \mu_\theta),$$

and

$$\limsup_{n \to +\infty} \arg\min_{\theta \in \Theta} \mathbb{E}[W_p(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}] \subseteq \arg\min_{\theta \in \Theta} W_p(\mu_*, \mu_\theta).$$

In addition, $\arg\min_{\theta \in \Theta} \mathbb{E}[W_p(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}] \neq \emptyset$ for $n \geq n(\omega)$ with some $n(\omega) > 0$.

Assumption 3.5 There exists a constant $\tau > 0$ such that the set $\Theta_\tau(\tau) \subseteq \Theta$ is bounded where $\Theta_\tau(\tau) = \{\theta \in \Theta : W_p(\hat{\mu}_n, \mu_\theta) \leq \inf_{\theta \in \Theta} W_p(\hat{\mu}_n, \mu_\theta) + \tau\}$.

Theorem 3.11 Under Assumption 3.2, 3.4 and 3.5, the following statement holds true,

$$\lim_{m \to +\infty} \inf_{\theta \in \Theta} \mathbb{E}[W_p(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}] = \inf_{\theta \in \Theta} W_p(\mu_*, \mu_\theta),$$

and

$$\limsup_{m \to +\infty} \arg\min_{\theta \in \Theta} \mathbb{E}[W_p(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}] \subseteq \arg\min_{\theta \in \Theta} W_p(\mu_*, \mu_\theta).$$

In addition, $\arg\min_{\theta \in \Theta} \mathbb{E}[W_p(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}] \neq \emptyset$ for $m \geq m_n$ with some $m_n > 0$.

To this end, the MPRW and MEPRW estimators both asymptotically converge to $\theta_* \in \Theta$, which is a minimizer of $\theta \mapsto W_p(\mu_*, \mu_\theta)$, assuming its existence. Moreover, $\theta_*$ is not the limit of maximum likelihood estimator and satisfies $\mu_{\theta_*} = \mu_*$ in a well-specified setting. Our consistency results support the success of generative modelling using the max-SW distance.
3.4 Rate of convergence and asymptotic distribution

We investigate the asymptotic distribution of the MPRW estimator under model misspecification and establish the rate of convergence when $k = p = 1$. For any $u \in S^{d-1}$ and $t \in \mathbb{R}$, we define

$$F_{\theta}(u,t) = \int_{\mathbb{R}^d} 1_{(-\infty,t]}(\langle u, x \rangle) \, d\mu_{\theta}(x),$$

$$\hat{F}_n(u,t) = (1/n)\{i \in [n] : \langle u, X_i \rangle \leq t\}.$$  

The functions $F_{\theta}(u,\cdot)$ and $\hat{F}_n(u,\cdot)$ are the cumulative distribution functions of $u_{\theta}^* \mu_{\theta}$ and $u_{\theta}^* \hat{\mu}_n$ where $u \in S^{d-1}$ is a unit vector. Let $L(S^{d-1} \times \mathbb{R})$ be the class of functions on $S^{d-1} \times \mathbb{R}$ such that $f(\cdot,t)$ is continuous and $f(u,\cdot)$ is absolutely integrable, with the norm $\|f\|_L = \sup_{u \in S^{d-1}} \int_{\mathbb{R}} |f(u,t)| \, dt$.

**Assumption 3.6** There exists a measurable function $D_* : S^{d-1} \times \mathbb{R} \to \mathbb{R}^{d_0}$ such that $\|F_{\theta}(u,t) - F_{\theta_*}(u,t) - (\theta - \theta_*) D_*(u,t)\|_L = o(\|\theta - \theta_*\|_\Theta)$.

**Assumption 3.7** There exists a random element $G_* : S^{d-1} \times \mathbb{R} \to \mathbb{R}^{d_0} \delta$ such that the stochastic process $\sqrt{n}(\hat{F}_n - F_*)$ converges weakly in $L(S^{d-1} \times \mathbb{R})$ to $G_*^3$.

**Assumption 3.8** There exists a neighborhood $\mathcal{N}$ of $\theta_* \in \Theta$ and a positive constant $c_*$ such that $\overline{\mathcal{W}}_{1,1}(\mu_{\theta}, \mu_*) \geq \overline{\mathcal{W}}_{1,1}(\mu_{\theta_*}, \mu_*) + c_* \|\theta - \theta_*\|_\Theta$ for all $\theta \in \mathcal{N}$.

**Remark 3.2** Assumption 3.6 is strictly weaker than a norm-differentiation condition where $D_*$ has to be nonsingular. Assumption 3.7 permits model misspecification where there is no $\theta_* \in \Theta$ such that $F_{\theta_*} = F_*$ and thus is more general than Nadjaoui et al. [2019, A8]. Assumption 3.8 accounts for local strong identifiability for the model $\mu_{\theta}$ around $\theta_*$ and is necessary for the fast rate of $n^{-1/2}$ under model misspecification. (Bernton et al. [2019] assumes the analogous condition for the Wasserstein distance. However, their analysis depends on a much stronger version with $\mathcal{N} = \Theta$. ) Thanks to Assumption 3.8, we do not require the condition that the parameters are weakly separable in the PRW sense.

**Remark 3.3** In well-specified setting where there exists $\theta_* \in \Theta$ such that $F_* = F_{\theta_*}$, it is straightforward to derive the norm-differentiation condition from Assumption 3.6 and 3.8. This is not true, however, under model misspecification. Moreover, there are minor technical issues in the proof of Bernton et al. [2019, Theorem B.8]; see Appendix E.4. Fixing them would be straightforward but require additional assumptions. Fortunately, we can overcome this gap using some new techniques. Thus, with some refinement, our results can be interpreted as an improvement of Bernton et al. [2019] with fewer assumptions.

To study the asymptotic distributions in the misspecified setting, we employ definitions from Pollard [1980, Section 7]. (Note, however, that our proof technique is different from Pollard [1980], which depends on the nonsingularity of $D_*$ and requires $\mu_* = \mu_{\theta_*}$ for some $\theta_*$ in the interior of $\Theta$.)

**Definition 3.5 (Hausdorff metric)** Let $S$ be the class of convex and compact sets in $L(S^{d-1} \times \mathbb{R})$ equipped with $\|\cdot\|_L$. The Hausdorff metric on $S$ is defined by $d_H(S_1,S_2) = \inf\{\delta > 0 : S_1 \subseteq S_\delta^1, S_2 \subseteq S_\delta^2\}$, where $S_\delta^\delta = \bigcup_{x \in S} \{z \in L(S^{d-1} \times \mathbb{R}) : \|z - x\|_L \leq \delta\}$.

\footnote{As pointed by Nadjaoui et al. [2019], one can prove that Assumption 3.7 holds in general by extending [Dede, 2009, Proposition 3.5] and [del Barrio et al., 1999, Theorem 2.1(a)] with some mild conditions on the tails of $u_{\theta}^* \mu_*$. Using the same argument, this extension can also be done for $\|\cdot\|_L$ in our paper.}
Definition 3.6 (Approximate MPRW estimators)  The set of approximate MPRW estimators is defined by
\[ M_n = \{ \theta \in \Theta : \mathcal{PW}_{1,1}(\hat{\mu}_n, \mu_{\theta}) \leq \inf_{\theta' \in \Theta} \mathcal{PW}_{1,1}(\hat{\mu}_n, \mu_{\theta'}) + \eta_n / \sqrt{n} \} \]
where \( \eta_n > 0 \) such that \( \mathbb{P}(\eta_n \to 0) = 1 \) and \( M_n \) is nonempty.

Theorem 3.12 Suppose Assumption 3.1-3.3 and 3.6-3.8 hold for some \( \theta^* \) in the interior of \( \Theta \) and let \( G_n = \sqrt{n}(\hat{F}_n - F_{\theta^*}) \) and \( G^*_n = G^* + \sqrt{n}(F_{\theta^*} - F_{\theta^*}) \). We also define \( K(x, \beta) = \{ \theta \in \mathcal{N}_1 : \| x - \sqrt{n}(\theta - \theta^*, D_{\theta^*}) \|_L \leq \inf_{\theta' \in \mathcal{N}_1} \| x - \sqrt{n}(\theta' - \theta^*, D_{\theta^*}) \|_L + \beta \} \) where
\[ \mathcal{N}_1 = \left\{ \theta \in \mathcal{N} : \frac{\| F_{\theta} - F_{\theta^*} - (\theta - \theta^*, D_{\theta^*}) \|_L}{\| \theta - \theta^* \|_{\Theta}} \leq c_\epsilon / 2 \right\} . \]

Then there exists a sequence satisfying \( \lim_{n \to +\infty} \beta_n = 0 \) such that \(^4\mathbb{P}_* (M_n \subseteq K(G^*_n, \beta_n)) \to 1 \) as \( n \to +\infty \). For any \( \epsilon > 0 \), we have \( \mathbb{P}(d_H(K(G^*_n, 0), K(G_n, \beta_n)) < \epsilon) \to 1 \) as \( n \to +\infty \).

Theorem 3.12 provides the theoretical guarantee for statistical inference with the max-SW distance under model misspecification. Indeed, since \( K(G^*_n, 0) = \arg\min_{\theta \in \mathcal{N}_1} \| G^* + \sqrt{n}(F_{\theta} - F_{\theta^*} - (\theta - \theta^*, D_{\theta^*}) \|_L \), the results indicate that the distributional limit of the approximate MPRW estimator set is close to the limit of the sets \( \arg\min_{\theta \in \mathcal{N}_1} \| G^* + \sqrt{n}(F_{\theta} - F_{\theta^*} - (\theta - \theta^*, D_{\theta^*}) \|_L \) in the Hausdorff metric. Note that \( d > 1 \) is allowed but we need \( k = 1 \). This is necessary for our techniques since the current analysis heavily depends on the explicit form of PRW using cumulative distribution functions. Deriving CLT when \( k > 1 \) is important but out of the scope of this paper.

Remark 3.4 In the well-specified setting, Assumption 3.8 can be replaced by Assumption A.1-A.2. Under certain conditions, we derive the CLT (cf. Theorem A.3) which is analogous to Nadjahi et al. [2019, Theorem 6] for the minimum sliced Wasserstein estimators. We refer to Theorem A.3 in Appendix A for a simplified version in well-specified setting.

\(^4\mathbb{P}_* \) denotes the (inner) probability; see Pollard [1980] for details.
Discussions. We make some additional remarks on the relationship between our work and the existing works by Bernton et al. [2019] and Nadjahi et al. [2019]. Since PRW is a type of Wasserstein, the consistency proof roadmap is essentially similar to that in Bernton et al. [2019] and Nadjahi et al. [2019]. However, we remark that (i) the sample complexity bounds of PRW are new; (ii) the techniques for CLT in misspecified settings did not appear in Nadjahi et al. [2019] and complete the analysis in Bernton et al. [2019]. Remark 3.2 states that our Assumption 3.8 is weaker than that is used in Bernton et al. [2019]. In particular, \( N \) is the neighborhood defined in Assumption 3.8 and accounts for a local strong identifiability. In contrast, Bernton et al. [2019] requires a global strong identifiability (\( N = \Theta \)). Remark 3.3 states that our setting is more general than the well-specified setting which is discussed by Nadjahi et al. [2019] from a technical point of view.

4 Experiments

We empirically validate our theoretical findings through several experiments on synthetic and real data. Given the space limit, we present the experimental setup in Appendix G and explain an optimization algorithm for computing the PRW distance and estimators in Appendix F. We defer the additional results on other dataset to Appendix H.

Let \( \mu = \nu = \mathcal{U}([-v,v]^d) \) be an uniform distribution over a hypercube and we study the convergence and computation of \( \mathcal{PW}_{2,k}(\hat{\mu}_n, \hat{\nu}_n) \) and \( \mathcal{PW}_{2,k}(\hat{\mu}_n, \hat{\nu}_n) \) for \( n \in \{20, 100, 250, 500, 1000\} \). Figure 1 presents average distances and computational times for \( (d, v) \in \{(10, 1), (30, 5), (50, 5)\} \), where the shaded areas show the max-min values over 100 runs. First, the IPRW distance is smaller than the PRW distance for small \( n \) when \( d \) and \( v \) are large. This confirms that the IPRW distance is independent of \( d \) (cf. Theorem 3.4). Second, the PRW distance nearly matches the IPRW distance when \( n \) is large. This confirms Theorem 3.6 since the uniform distribution with its bounded domain satisfies the Poincaré inequality. Finally, the computation of the PRW distance is faster than that of the IPRW distance.
Consider the parametric inference using Gaussian models $\mathcal{M} = \{N(\mathbf{m}, \sigma^2 I) : \mathbf{m} \in \mathbb{R}^2, \sigma^2 > 0\}$ and a collection of i.i.d. observations generated from a mixture of 8 Gaussian distributions in $\mathbb{R}^2$. This simple setting is useful since the closed-form expression of Gaussian density makes the computation of the MPRW estimator of order 1 tractable. Following the setup in Nadjahi et al. [2019, Section 4], we illustrate the consistency of the MPRW and MEPRW estimators of order 1 and the convergence of MEPRW estimator of order 1 to MPRW estimator of order 1. Results are shown in Figure 2; they are consistent with Theorem 3.9, 3.10 and 3.11, where $\mathbf{m} = \hat{\mathbf{m}}_{10^5}$. Despite the model misspecification, our estimators still converge as the number of observations increases and the MEPRW estimator converges to the MPRW estimator as we generate more samples. We also verify our central limit theorem by estimating the density of $\hat{\sigma}_n^2$ with a kernel density estimator over 100 runs. Figure 3 shows the distribution centered and rescaled by $\sqrt{n}$ for each $n$, where $\sigma^2 = \hat{\sigma}_{10^5}^2$, and it confirms the convergence rate we derived in Theorem 3.12; see Appendix H for the case with 12 or 25 distributions.

We conduct experiments on image generation using the PRW generator of order 2, as an alternative to the SW generator [Deshpande et al., 2018]. Here we focus on the case of $k = 1$, where the PRW generator is exactly max-SW generator. We train the neural networks (NNs) with $(n, m) \in \{(100, 20), (1000, 40), (5000, 60), (10000, 100)\}$ where $n$ is the number of training samples and $m$ is the number of generated samples. We compare their testing losses to that of a NN trained using $n = 10^5$ (i.e. whole training dataset) and $m = 200$. All testing losses are evaluated using the trained models on the testing dataset ($n = 10^4$) with $m = 250$ generated samples. Figure 4 presents the mean testing loss on ImageNet200 over 10 runs, where the shaded areas show the max-min values over the runs.

**Discussions.** First, PRW has better discriminative power than max-SW or SW since it considers high-order summaries and extract more geometric information from two high-dimensional distributions, in order to distinguish them better; see Paty and Cuturi [2019] for the details. Moreover, we have presented in Figure 1 (top row) and Figure 5 (top row) that the PRW/IPRW value increase as $k$ increases. Thus, PRW/IPRW based on larger $k$-dimensional projections have better discriminative power.

Second, the IPRW computation generally requires many random projections and is thus more time-consuming than PRW for a desired accuracy.

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5The approach we apply here is the same as used by Nadjahi et al. [2019].
when $k = 1$; see Kolouri et al. [2019a, Page 4]. Fortunately, it may require much fewer for certain application problems when the intrinsic dimension of data distribution is small, and is easily amenable to parallel computation. Thus, IPRW can serve as a practical alternative to PRW. Moreover, the reported PRW and IPRW values in Figure 1 and Figure 5 (appendix) are computed by using 30 iterations for PRW and 100 projections for IPRW. Therefore, the statistical/simulation error contributes to the flip of order between IPRW and PRW when their true values are close.

Finally, our experimental results show that the max-sliced Wasserstein estimator works well in practice and converges to some point as the number of samples grow. This supports our consistency results since the max-sliced Wasserstein distance is PRW with $k = 1$. Note that there are many existing works on the empirical comparison between max-SW and SW using generative modeling and we refer the interested readers to Kolouri et al. [2019a] and the reference therein.

5 Conclusion

We study in this paper the statistical aspect of the projection robust Wasserstein (PRW) distance. Our work provides an enhanced understanding of two PRW distances and the associated minimal distance estimators under model misspecification, complementing the existing literature [Niles-Weed and Rigollet, 2019, Bernton et al., 2019, Nadjahi et al., 2019, 2020]. Experiments on synthetic and real datasets highlight some aspects of our theoretical results. Future work includes theory for entropic PRW and the applications of PRW with $k \geq 2$ to deep generative models.

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K. D. Yang, K. Damodaran, S. Venkatachalapathy, A. C. Soylemezoglu, G. V. Shivashankar, and C. Uhler. Predicting cell lineages using autoencoders and optimal transport. *PLoS computational biology*, 16(4):e1007828, 2020. (Cited on page 1.)

J. Ye, P. Wu, J. Z. Wang, and J. Li. Fast discrete distribution clustering using Wasserstein barycenter with sparse support. *IEEE Transactions on Signal Processing*, 65(9):2317–2332, 2017. (Cited on page 1.)
A Further Results on the MPRW and MEPRW Estimators

In this section, we discuss the measurability of the MPRW and MEPRW estimators. For a generic function \( f \) on the domain \( X \), we define \( \delta \)-\( \arg\min_{x \in X} \{ x \in X : f(x) \leq \inf_{x \in X} f + \delta \} \). Our results are summarized in the following two theorems.

**Theorem A.1** Under Assumption 3.1, for any \( n \geq 1 \) and \( \delta > 0 \), there exists a Borel measurable function \( \tilde{\theta}_n : \Omega \to \Theta \) such that
\[
\tilde{\theta}_n(\omega) \in \left\{ \begin{array}{ll}
\arg\min_{\theta \in \Theta} \mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \mu_\theta) & \text{if this set is nonempty,} \\
\delta \cdot \arg\min_{\theta \in \Theta} \mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \mu_\theta) & \text{otherwise.}
\end{array} \right.
\]

**Theorem A.2** Under Assumption 3.1, for any \( n \geq 1, m \geq 1 \) and \( \delta > 0 \), there exists a Borel measurable function \( \tilde{\theta}_{n,m} : \Omega \to \Theta \) such that
\[
\tilde{\theta}_{n,m}(\omega) \in \left\{ \begin{array}{ll}
\arg\min_{\theta \in \Theta} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}) | X_{1:n}] & \text{if this set is nonempty,} \\
\delta \cdot \arg\min_{\theta \in \Theta} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}) | X_{1:n}] & \text{otherwise.}
\end{array} \right.
\]

We also present the asymptotic distribution of the goodness-of-fit statistics as well as the MPRW estimator in the well-specified setting and establish the rate of convergence. For this we require the well separability of the model in Assumption A.1 and the non-singularity of \( D_* \) in Assumption A.2 to take place of the local strong identifiability in Assumption 3.8.

**Assumption A.1** For any \( \epsilon > 0 \), there exists \( \delta > 0 \) so that \( \inf_{\theta \in \Theta : \| \theta - \theta_* \|_\Theta \geq \epsilon} \mathcal{P}W_{1,1}(\mu_\theta, \mu_\theta) \geq \delta \).

**Assumption A.2** There exists a non-singular \( D_* \) such that Assumption 3.6 holds true.

**Theorem A.3** Suppose that \( \mu_* = \mu_{\theta_*} \) for some \( \theta_* \) in the interior of \( \Theta \). Under Assumption 3.1-3.3, 3.6-3.7 and A.1-A.2, the goodness-of-fit statistics satisfies
\[
\sqrt{n} \inf_{\theta \in \Theta} \mathcal{P}W_{1,1}(\tilde{\mu}_n, \mu_\theta) \Rightarrow \inf_{\theta \in \Theta} \max_{u \in S^{d-1}} \int_{\mathbb{R}} |G_*(u, t) - \langle \theta, D_*(u, t) \rangle| dt, \quad \text{as } n \to +\infty.
\]

Suppose also that the random map \( \theta \to \max_{u \in S^{d-1}} \int_{\mathbb{R}} |G_*(u, t) - \langle \theta, D_*(u, t) \rangle| dt \) has a unique infimum almost surely. Then the MPRW estimator of order 1 satisfies
\[
\sqrt{n}(\tilde{\theta}_n - \theta_*) \Rightarrow \arg\min_{\theta \in \Theta} \max_{u \in S^{d-1}} \int_{\mathbb{R}} |G_*(u, t) - \langle \theta, D_*(u, t) \rangle| dt, \quad \text{as } n \to +\infty.
\]

Both the weak convergence results are valid for the metric induced by the norm \( \| \cdot \|_L \).

B Postponed Proofs in Subsection 3.1

This section lays out the detailed proofs for Lemma 3.1, Theorem 3.2 and 3.3.
B.1 Preliminary technical results

For completeness, we collect several preliminary technical results\textsuperscript{6} which will be used in the proofs.

**Theorem B.1 (Prokhorov’s theorem)** Let $\mathcal{P}(\mathbb{R}^d)$ denote the collection of all probability measures defined on $\mathbb{R}^d$ with the Borel $\sigma$-algebra and $\{\mu_i\}_{i \in \mathbb{N}}$ is a tight sequence in $\mathcal{P}(\mathbb{R}^d)$. Then every subsequence of $\{\mu_i\}_{i \in \mathbb{N}}$ has a subsequence that converges weakly in $\mathcal{P}(\mathbb{R}^d)$. Moreover, if every weakly convergent subsequence has the same limit, the whole sequence converges weakly to this limit.

**Theorem B.2 (Theorem 4.1 in Villani [2008])** Let $(\mathcal{X}, \mu)$ and $(\mathcal{Y}, \nu)$ be two Polish probability spaces; let $a : \mathcal{X} \to \mathbb{R} \cup \{-\infty\}$ and $b : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ be upper semi-continuous such that $a$ and $b$ are absolutely integrable with respect to the measures $\mu$ and $\nu$ respectively. Let $c : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous, such that $c(x, y) \geq a(x) + b(y)$ for all $x, y$. Then there exists an optimal coupling $\pi \in \Pi(\mu, \nu)$ which minimizes the total cost $\mathbb{E}[c(X, Y)]$.

**Lemma B.3 (Lemma 4.4 in Villani [2008])** Let $\mathcal{X}$ and $\mathcal{Y}$ be two Polish spaces. Let $P \subseteq \mathcal{P}(\mathcal{X})$ and $Q \subseteq \mathcal{P}(\mathcal{Y})$ be tight subsets of $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$ respectively. Then the set of all transportation plans whose marginals lie in $P$ and $Q$ respectively, is itself tight in $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$.

**Theorem B.4 (Theorem 6.9 in Villani [2008])** Let $(\mathcal{X}, d)$ be a Polish space and $p \in [1, +\infty)$. The Wasserstein distance $\mathcal{W}_p$ metrizes the weak convergence in $\mathcal{P}_p(\mathcal{X})$. That is, if $\{\mu_i\}_{i \in \mathbb{N}}$ is a sequence of measures in $\mathcal{P}_p(\mathcal{X})$ and $\mu \in \mathcal{P}_p(\mathcal{X})$, then $\mu_i \Rightarrow \mu$ if and only if $\mathcal{W}_p(\mu_i, \mu) \to 0$.

**Definition B.1 (Lower semi-continuity)** We say that $f : \mathcal{X} \to \mathbb{R}$ is lower semi-continuous if for any $x_0 \in \mathcal{X}$ and any $y < f(x_0)$, there exists a neighborhood $U$ of $x_0$ such that $f(x) > y$ for all $x \in U$. In the case of a metric space, this is equivalent to $\liminf_{x \to x_0} f(x) \geq f(x_0)$ for any $x_0 \in \mathcal{X}$.

B.2 Proof of Lemma 3.1

We first show that, for any $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ and $\nu \in \mathcal{P}_p(\mathbb{R}^d)$, the following inequality holds true,

$$\mathcal{PW}_{p,k}(\mu, \nu) \leq \mathcal{PW}_{p,k}(\mu, \nu) \leq \mathcal{W}_p(\mu, \nu). \quad (B.1)$$

Indeed, by the definition of $\mathcal{PW}_{p,k}$ and $\mathcal{PW}_{p,k}$, the first inequality is trivial. For the second inequality, we derive from the definition of $\mathcal{PW}_{p,k}$ that

$$\mathcal{PW}_{p,k}^\mu(\mu, \nu) = \sup_{E \in S_{d,k}} \mathcal{W}_p^\mu(E^\#_\mu, E^*_\#\nu) = \sup_{E \in S_{d,k}} \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|E^\top(x - y)\|^p \, d\pi(x, y).$$

Since $E \in S_{d,k}$, we have $\|E^\top(x - y)\| \leq \|x - y\|$. Thus, we have $\mathcal{PW}_{p,k}^\mu(\mu, \nu) \leq \mathcal{W}_p^\mu(\mu, \nu)$. Putting these pieces together yields Eq. (B.1). For any sequence $\{\mu_i\}_{i \in \mathbb{N}} \subseteq \mathcal{P}_p(\mathbb{R}^d)$ and $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, we conclude from Eq. (B.1) that $\mathcal{W}_p(\mu_i, \mu) \to 0$ implies $\mathcal{PW}_{p,k}(\mu_i, \mu) \to 0$ and $\mathcal{PW}_{p,k}(\mu_i, \mu) \to 0$.

The remaining step is to show that if $\mathcal{PW}_{p,k}(\mu_i, \mu) \to 0$ implies $\mu_i \Rightarrow \mu$. Let $Z_i \sim \mu_i$, we have $E^\top Z_i \sim E^*_\# \mu_i$. By the definition of the IPRW distance (cf. Definition 2.3) and using the fact that $\mathcal{PW}_{p,k}(\mu_i, \mu) \to 0$, we have $\left(\|E^\top Z_i\|^p\right)_{i \in \mathbb{N}}$ is

\textsuperscript{6}For the Prokhorov’s theorem, we only present the results on the Euclidean space. For more results on general separable metric space, we refer the interested readers to Billingsley [2013].
uniformly integrable for all $E \in S_{d,k}$. Since $S_{d,k}$ is compact, there exists a finite set $\{E_1, E_2, \ldots, E_I\} \subseteq S_{d,k}$ so that $\|x\| \leq \sum_{j=1}^I \|E_j^T x\|$ for all $x \in \mathbb{R}^d$. Therefore, we have

$$ \|Z_i\|^p \leq \left( \sum_{j=1}^I \|E_j^T Z_i\|^p \right)^{\frac{1}{p}} \leq \mathbb{P} \left( \max_{1 \leq j \leq I} \|E_j^T Z_i\| \right)^p \leq \mathbb{P} \left( \sum_{j=1}^I \|E_j^T Z_i\|^p \right)^{\frac{1}{p}}. $$

Therefore, we deduce that $(\|Z_i\|^p)_{i \in \mathbb{N}}$ is uniformly integrable which implies the tightness of $\{\mu_i\}_{i \in \mathbb{N}}$. Using the Prokhorov’s theorem (cf. Theorem B.1), we obtain that every subsequence of $\{\mu_i\}_{i \in \mathbb{N}}$ has a weakly convergent subsequence.

The next step is to show that all the weakly convergent subsequences converge to the same probability measure $\mu$. We fix an arbitrary subsequence and for simplicity abbreviate the subscripts and still denote it by $\{\mu_i\}_{i \in \mathbb{N}}$. Let $\tilde{\mu}_i$ be the limit of any given weakly convergent subsequence $\langle \mu_{i_j} \rangle_{j \in \mathbb{N}}$, we need to prove that $\tilde{\mu}_i = \mu$. In particular, we define the characteristic function for any probability measure $\nu$ as follows,

$$ \Phi_{\nu}(z) := \int_{\mathbb{R}^d} e^{i(z,x)} \, d\nu(x) \quad \text{for all } z \in \mathbb{R}^d. $$

Since $\mu_{ij} \Rightarrow \tilde{\mu}_i$, we have $\Phi_{\mu_{ij}}(z) \to \Phi_{\tilde{\mu}_i}(z)$ for all $z \in \mathbb{R}^d$. Thus, we need to show that $\Phi_{\mu_{ij}}(z) \to \Phi_{\mu}(z)$ for all $z \in \mathbb{R}^d$. This is trivial when $z = 0_d$ since $\Phi_{\mu_{ij}}(0_d) = \Phi_{\mu}(0_d) = 1$ for all $j \in \mathbb{N}$. Otherwise, let $r := \|z\|$ and $v := z/\|z\|$, we have

$$ \lim_{j \to +\infty} \Phi_{\mu_{ij}}(z) = \lim_{j \to +\infty} \int_{\mathbb{R}^d} e^{i(z,x)} \, d\mu_{ij}(x) = \lim_{j \to +\infty} \int_{\mathbb{R}^d} e^{ir \langle v, x \rangle} \, d\mu_{ij}(x). $$

Since $\|v\| = 1$, we define $\tilde{E} \in S_{d,k}$ whose first column is $v$. Let $\tilde{r}$ be a $k$-dimensional vector whose first coordinate is $r$ and other coordinates are zero. Then we have $r \langle v, x \rangle = \langle \tilde{r}, \tilde{E}^T x \rangle$. Putting these pieces together yields that

$$ \lim_{j \to +\infty} \Phi_{\mu_{ij}}(z) = \lim_{j \to +\infty} \int_{\mathbb{R}^k} e^{i\langle \tilde{r}, y \rangle} \, d\tilde{E}^\ast_{\#} \mu_{ij}(y). $$

For such fixed $\tilde{E}$, we claim that $\mathcal{W}_p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu) \to 0$ holds true. More specifically, $\mathcal{P} \mathcal{W}_{p,k}(\mu_{ij}, \mu) \to 0$ implies that $\int \mathcal{W}_p^p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu) \, d\sigma(E) \to 0$. Since $\mathcal{W}_p^p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu)$ is non-negative, it is easy to derive that $\mathcal{W}_p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu) \to 0$ for almost every $E$. Nonetheless, by the continuity of $\mathcal{W}_p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu)$ with respect to $E$, we can obtain that $\mathcal{W}_p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu) \to 0$ for all fixed $E$. Indeed, by the proof by contradiction, we assume that $\mathcal{W}_p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu) \to 0$ for some fixed $E$. Then, there exists a neighborhood $S$ of $E$ (it is fixed) such that $\int_S \mathcal{W}_p^p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu) \, d\sigma(E) \to 0$. This contradicts $\int_S \mathcal{W}_p^p(\tilde{E}^\ast_{\#} \mu_{ij}, \tilde{E}^\ast_{\#} \mu) \, d\sigma(E) \to 0$ since the inside term is non-negative. Thus, we achieve the desired claim.

Using Theorem B.4, we have $\tilde{E}^\ast_{\#} \mu_{ij} \Rightarrow \tilde{E}^\ast_{\#} \mu$. Since $\langle r, x \rangle = \langle \tilde{r}, \tilde{E}^T x \rangle$, we have

$$ \lim_{j \to +\infty} \int_{\mathbb{R}^k} e^{i\langle \tilde{r}, y \rangle} \, d\tilde{E}^\ast_{\#} \mu_{ij}(y) = \int_{\mathbb{R}^k} e^{i\langle \tilde{r}, y \rangle} \, d\tilde{E}^\ast_{\#} \mu(x) = \int_{\mathbb{R}^d} e^{ir \langle v, x \rangle} \, d\mu(x) = \int_{\mathbb{R}^d} e^{i(z,x)} \, d\mu(x). $$

Putting these pieces together yields that $\Phi_{\mu_{ij}}(z) \to \Phi_{\mu}(z)$ for all $z \in \mathbb{R}^d/\{0_d\}$ and $\tilde{\mu}_i = \mu$ for all $i \in \mathbb{N}$. Using the Prokhorov’s theorem again yields that the whole sequence $\{\mu_i\}_{i \in \mathbb{N}}$ has the limit $\mu$ in weak sense. Therefore, $\mathcal{P} \mathcal{W}_{p,k}(\mu_i, \mu) \to 0$ implies $\mu_i \Rightarrow \mu$. Since the Wasserstein distances metrize the weak convergence (cf. Theorem B.4), we conclude that $\mathcal{P} \mathcal{W}_{p,k}(\mu_i, \mu) \to 0$ implies $\mathcal{W}_p(\mu_i, \mu) \to 0$. This completes the proof.
B.3 Proof of Theorem 3.2

By Lemma 3.1, we have \( \overline{PW}_{p,k}(\mu_i, \mu) \to 0 \) if and only if \( \overline{PW}_{p,k}(\mu_i, \mu) \to 0 \) if and only if \( W_p(\mu_i, \mu) \to 0 \). By Theorem B.4, we have \( \mu_i \Rightarrow \mu \) if and only if \( W_p(\mu_i, \mu) \to 0 \). Putting these pieces together yields the desired result.

B.4 Proof of Theorem 3.3

Fixing \( E \in S_{d,k} \), the mapping \( x \mapsto E^\top x \) is continuous from \( \mathbb{R}^d \) to \( \mathbb{R}^k \). Since \( \mu_i \Rightarrow \mu \) and \( \nu_i \Rightarrow \nu \), the continuous mapping theorem implies that \( E^*_{\#} \mu_i \Rightarrow E^*_{\#} \mu \) and \( E^*_{\#} \nu_i \Rightarrow E^*_{\#} \nu \). The next step is the key ingredient in the proof and we hope to show that

\[
\liminf_{i \to +\infty} W_p^p(E^*_{\#} \mu_i, E^*_{\#} \nu_i) = \liminf_{i \to +\infty} W_p^p(E^*_{\#} \mu_i, E^*_{\#} \nu_i)
\]  

for all \( E \in S_{d,k} \). From Theorem B.2, there exists a coupling \( \pi_i \in \Pi(E^*_{\#} \mu_i, E^*_{\#} \nu_i) \) such that \( W_p^p(E^*_{\#} \mu_i, E^*_{\#} \nu_i) = \int_{\mathbb{R}^k \times \mathbb{R}^k} ||x-y||^p \, d\pi_i(x, y) \). By the definition of \( \liminf \), there exists a subsequence of \( \{\pi_i\}_{i \in \mathbb{N}} \) such that \( \int_{\mathbb{R}^k \times \mathbb{R}^k} ||x-y||^p \, d\pi_i(x, y) \) converges to \( \liminf_{i \to +\infty} W_p^p(E^*_{\#} \mu_i, E^*_{\#} \nu_i) \). For the simplicity, we still denote it by \( \{\pi_{i_j}\}_{j \in \mathbb{N}} \). By Lemma 3.3 and Prokhorov’s theorem (cf. Theorem B.1), \( \{\pi_{i_j}\}_{j \in \mathbb{N}} \) is sequentially compact in weak sense. Thus, there exists a subsequence \( \{\pi_{i_j}\}_{j \in \mathbb{N}} \) such that \( \pi_{i_j} \Rightarrow \tilde{\pi} \in \mathcal{P}(\mathbb{R}^k \times \mathbb{R}^k) \). Putting these pieces together yields that

\[
\liminf_{i \to +\infty} W_p^p(E^*_{\#} \mu_i, E^*_{\#} \nu_i) = \int_{\mathbb{R}^k \times \mathbb{R}^k} ||x-y||^p \, d\tilde{\pi}(x, y).
\]

By the definition of the Wasserstein distance, it suffices to show that \( \tilde{\pi} \in \Pi(E^*_{\#} \mu, E^*_{\#} \nu) \). Indeed, let \( f : \mathbb{R}^k \to \mathbb{R} \) be a continuous and bounded function, we have

\[
\int_{\mathbb{R}^k \times \mathbb{R}^k} f(x) \, d\pi(x, y) = \lim_{j \to +\infty} \int_{\mathbb{R}^k \times \mathbb{R}^k} f(x) \, d\pi_{i_j}(x, y).
\]

Since \( \pi_{i_j} \in \Pi(E^*_{\#} \mu_{i_j}, E^*_{\#} \nu_{i_j}) \) and \( E^*_{\#} \mu_{i_j} \Rightarrow E^*_{\#} \mu \), we have

\[
\lim_{j \to +\infty} \int_{\mathbb{R}^k \times \mathbb{R}^k} f(x) \, d\pi_{i_j}(x, y) = \lim_{j \to +\infty} \int_{\mathbb{R}^k} f(x) \, dE^*_{\#} \mu_{i_j}(x) = \int_{\mathbb{R}^k} f(x) \, dE^*_{\#} \mu(x).
\]

Since \( E^*_{\#} \mu_{i_j} \Rightarrow E^*_{\#} \mu \), the same argument implies that \( \int_{\mathbb{R}^k \times \mathbb{R}^k} f(y) \, d\tilde{\pi}(x, y) = \int_{\mathbb{R}^k} f(y) \, dE^*_{\#} \nu(y) \). Putting these pieces together yields Eq. (B.2).

For the IPRW distance, we derive from Eq. (B.2) and the Fatou’s lemma that

\[
\mathcal{PW}_{p,k}^p(\mu, \nu) = \int_{S_{d,k}} W_p^p(E^*_{\#} \mu, E^*_{\#} \nu) \, d\sigma(E) \leq \liminf_{i \to +\infty} \int_{S_{d,k}} W_p^p(E^*_{\#} \mu_i, E^*_{\#} \nu_i) \, d\sigma(E) = \liminf_{i \to +\infty} \mathcal{PW}_{p,k}^p(\mu_i, \nu_i).
\]

Since \( \mathcal{PW}_{p,k}^p(\mu, \nu) \) and \( \mathcal{PW}_{p,k}^p(\mu_i, \nu_i) \) are both nonnegative, we take the \( p \)-th root of both sides of the above inequality and have \( \mathcal{PW}_{p,k}^p(\mu, \nu) \leq \liminf_{i \to +\infty} \mathcal{PW}_{p,k}^p(\mu_i, \nu_i) \).

For the PRW distance, we derive from Eq. (B.2) and the fact that the supremum of a sequence of lower semi-continuous mappings is lower semi-continuous that

\[
\overline{PW}_{p,k}^p(\mu, \nu) = \sup_{E \in S_{d,k}} W_p^p(E^*_{\#} \mu, E^*_{\#} \nu) \leq \liminf_{i \to +\infty} \overline{PW}_{p,k}^p(\mu_i, \nu_i).
\]

where the first equality holds true since the Wasserstein distance is nonnegative. Since \( \overline{PW}_{p,k}^p(\mu, \nu) \) and \( \overline{PW}_{p,k}^p(\mu_i, \nu_i) \) are both nonnegative, we have \( \overline{PW}_{p,k}^p(\mu, \nu) \leq \liminf_{i \to +\infty} \overline{PW}_{p,k}^p(\mu_i, \nu_i) \).
C Postponed Proofs in Subsection 3.2

In this section, we provide the detailed proofs for Theorem 3.4-3.8.

C.1 Preliminary technical results

To facilitate reading, we collect several preliminary technical results which will be used in the postponed proofs in subsection 3.2.

**Theorem C.1 (Tonelli’s theorem)** if \((X, A, \mu)\) and \((Y, B, \nu)\) are \(\sigma\)-finite measure spaces, while \(f : X \times Y \to [0, +\infty]\) is non-negative measurable function, then

\[
\int_X \left( \int_Y f(x, y) \, dy \right) \, dx = \int_Y \left( \int_X f(x, y) \, dx \right) \, dy = \int_{X \times Y} f(x, y) \, d(x, y). 
\]

The following proposition provides the state-of-the-art general bound for the Wasserstein distance between the true measure and its empirical version in \(\mathbb{R}^d\). Note that we do not assume any additional structures of the true measure. Similar results can be found in many classical works, e.g., Fournier and Guillin [2015, Theorem 1], Weed and Bach [2019, Theorem 1] and Lei [2020, Theorem 3.1]. Since \(p \geq 1\), we present the following results which directly follows the proof of Lei [2020, Theorem 3.1].

**Proposition C.2** Let \(\mu^* \in \mathcal{P}_q(\mathbb{R}^d)\) and \(M_q := M_q(\mu^*) < +\infty\). Then we have

\[
\mathbb{E}[W_p(\widehat{\mu}_n, \mu^*)] \leq \left( \mathbb{E}[W_p^p(\widehat{\mu}_n, \mu^*)] \right)^{1/p} \lesssim_{p,q} n^{-\frac{1}{2\left(p\wedge \min\{\frac{1}{p}, \frac{1}{q}\}\right)}} (\log(n))^{\frac{\zeta_{p,q,d}}{p}}, \quad \text{for all } n \geq 1. \tag{C.1}
\]

where \(\lesssim_{p,q}\) refers to “less than” with a constant depending only on \((p, q)\) and

\[
\zeta_{p,q,d} = \begin{cases} 
2 & \text{if } d = q = 2p, \\
1 & \text{if } “d \neq 2p \text{ and } q = \frac{dp}{d-p},” \text{ or } “q > d = 2p” , \\
0 & \text{otherwise.}
\end{cases}
\]

The following proposition provides a bound for the covering number of \(S_{d,k}\) in the operator norm of a matrix, denoted by \(\|\cdot\|_{\text{op}}\). This is a straightforward consequence of the classical results on the covering number of the unit sphere in \(\mathbb{R}^d\) in Euclidean norm. For the proof details, we refer the interested readers to Niles-Weed and Rigollet [2019, Lemma 4]. For the background materials on the covering number, we refer the interested readers to Wainwright [2019, Chapter 5]. For the ease of presentation, we provide a formal definition of covering number of \(S_{d,k}\) in \(\|\cdot\|_{\text{op}}\) as follows.

For any \(\epsilon \in (0, 1)\), the \(\epsilon\)-covering number of \(S_{d,k}\) in \(\|\cdot\|_{\text{op}}\) is defined by

\[
N(S_{d,k}, \epsilon, \|\cdot\|_{\text{op}}) = \inf \left\{ N \in \mathbb{N} : \exists x_1, x_2, \ldots, x_N \in S_{d,k}, \text{ s.t. } S_{d,k} \subseteq \bigcup_{i=1}^N B(x_i, \epsilon) \right\},
\]

where \(B(x, r) = \{y \in S_{d,k} : \|y - x\|_{\text{op}} \leq r\}\) is the ball of radius \(r > 0\) centered at \(x \in S_{d,k}\) in the operator norm of a matrix.

**Proposition C.3** There exists a universal constant \(c > 0\) such that for all \(\epsilon \in (0, 1)\), the \(\epsilon\)-covering number of \(S_{d,k}\) in \(\|\cdot\|_{\text{op}}\) satisfies that \(N(S_{d,k}, \epsilon, \|\cdot\|_{\text{op}}) \leq (c\sqrt{kd})^d\).
The following theorem [Lei, 2020] summarizes the concentration results assuming the Bernstein tail condition under product measure. Indeed, let \( \{X_i\}_{i \in [n]} \) be independent samples from probability measure \( \mu_i \) on spaces \( \mathcal{X}_i \) and \( X'_i \) be independent copies of \( X_i \) for all \( i \in [n] \). Denote \( X = (X_1, \ldots, X_n) \) and \( X'_i = (X_1, \ldots, X'_i, \ldots, X_n) \) which is identical to \( X \) except for \( X'_i \). Let \( f : \prod_{i=1}^n \mathcal{X}_i \to \mathbb{R} \) be a function such that \( \mathbb{E}[|f(X)|] < +\infty \), and define \( D_i = f(X) - f(X'_i) \).

**Theorem C.4** Suppose that there exists some \( \sigma_i, M > 0 \) so that \( \mathbb{E}[|D_i||X_{-i}|] \leq (1/2)\sigma_i^2 k!M^{k-2} \) for all \( k \geq 2 \). Then the following statement holds,

\[
P(f(X) - \mathbb{E}(f(X)) > t) \leq \exp\left(-\frac{t^2}{2(\sum_{i=1}^n \sigma_i^2) + 2tM}\right).
\]

The following theorem summarizes the concentration results assuming the Poincaré inequality under product measure. We denote by \( \|\nabla_i f\| \) the length of the gradient with respect to the \( i \)th coordinate.

**Theorem C.5** *(Corollary 4.6 in Ledoux [1999])* Denote by \( \mu^n \) the product of \( \mu \) on \( \otimes_{i=1}^n \mathbb{R}^d \) and \( \mu \in \mathcal{P}(\mathbb{R}^d) \) satisfies the Poincaré inequality (cf. Definition 3.4). For every function \( f \) on \( \otimes_{i=1}^n \mathbb{R}^d \) satisfying \( \mathbb{E}(|f(X)|) < +\infty \), and \( \sum_{i=1}^n \|\nabla_i f(X)\|^2 \leq \alpha^2 \) and \( \max_{1 \leq i \leq n} \|\nabla_i f(X)\| \leq \beta \) almost surely. Then the following statement holds true for \( X \sim \mu^n \) that,

\[
P(f(X) - \mathbb{E}(f(X)) > t) \leq \exp\left(-\frac{1}{K} \min\left\{ \frac{t}{\beta}, \frac{t^2}{\alpha^2} \right\}\right),
\]

where \( K > 0 \) only depends on the constant \( M \) in the Poincaré inequality.

### C.2 Proof of Theorem 3.4

Note that \( \mu_* \in \mathcal{P}_q(\mathbb{R}^d) \) and \( M_q := M_q(\mu_*) < +\infty \). Fixing \( E \in S_{d,k} \), we have \( E_{\# \mu_*}^* \in \mathcal{P}_q(\mathbb{R}^k) \) and \( M_q(E_{\# \mu_*}^*) \leq M_q < +\infty \). Then Proposition C.2 implies that

\[
(\mathbb{E}[\mathcal{W}_p(E_{\# \mu_*}^*)])^{1/p} \leq_{p,q} n^{-\frac{1}{(2p)^\alpha}} \left( \frac{\log(n)}{p} \right)^{-\frac{1}{p}} \frac{\zeta_{p,q,k}}{\rho} \quad \text{for all } n \geq 1.
\]

Since \( \mathcal{W}_p(E_{\# \mu_*}^*) \geq 0 \) for any \( E \in S_{d,k} \) and \( \mu_* \in \mathcal{P}_q(\mathbb{R}^d) \), Theorem C.1 implies that

\[
\mathbb{E}[\mathcal{W}_p^*(E_{\# \mu_*}^*)] = \int_{S_{d,k}} \mathcal{W}_p^*(E_{\# \mu_*}^*) \, d\sigma(E) = \int_{S_{d,k}} \mathbb{E}[\mathcal{W}_p^*(E_{\# \mu_*}^*)] \, d\sigma(E).
\]

Note that \( \zeta_{p,q,k} = \zeta'_{p,q,k} \) where \( \zeta'_{p,q,k} \) is defined in Theorem 3.4. Moreover, \( p \geq 1 \). By the Jensen’s inequality, we have

\[
\mathbb{E}[\mathcal{W}_p(E_{\# \mu_*}^*)] \leq (\mathbb{E}[\mathcal{W}_p^*(E_{\# \mu_*}^*)])^{1/p}.
\]  

Putting these pieces together yields the desired result.
C.3 Proof of Theorem 3.5

By the definition of $\overline{PW}_{p,k}(\hat{\mu}_n, \mu_*)$, we have

$$\mathbb{E}[\overline{PW}_{p,k}(\hat{\mu}_n, \mu_*)] \leq \sup_{E \in \mathbb{S}_{d,k}} \mathbb{E}[W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)] + \mathbb{E} \left[ \sup_{E \in \mathbb{S}_{d,k}} (W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*) - \mathbb{E}[W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)]) \right].$$

(C.3)

Using the same arguments for proving Theorem 3.4, we have

$$\sup_{E \in \mathbb{S}_{d,k}} \mathbb{E}[W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)] \lesssim_{p,q} n^{-\left(\frac{1}{2p}\vee (\frac{1}{p} - \frac{1}{q})\right)} \log(n)^{\frac{C_{p,q,k}}{p}}$$

for all $n \geq 1$. (C.4)

The remaining step is to bound the gap $\mathbb{E}[\sup_{E \in \mathbb{S}_{d,k}} (W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*) - \mathbb{E}[W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)])]$. We first claim that $W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*) - \mathbb{E}[W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)]$ is sub-exponential with parameters $(2\sigma n^{1/2-1/p}, 2Vn^{-1/p})$ for all $E \in \mathbb{S}_{d,k}$ if the true measure $\mu_*$ satisfies the projection Bernstein-type tail condition (cf. Definition 3.1). Indeed, let $f(X) = W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)$, we have

$$D_i = f(X) - f(X'_i) \leq W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu'_n) \leq n^{-1/p} (\|E^*_\#(X_i) - E^*_\#(X'_i)\|).$$

By the triangle inequality and using the projection Bernstein-type tail condition, we have

$$\mathbb{E}[|D_i|^k | X_{-i}] \leq 2k n^{-k/p} \mathbb{E}[\mathbb{E}_{X \sim E^*_\#}(|X|^k)] \leq 2k-1 n^{-k/p} \sigma^2 k! V^{k-2} = \frac{(2n^{-1/p} \sigma)^2 k! (2n^{-1/p} V)^{k-2}}{2}.$$   

This implies that the condition in Theorem C.4 holds true with $\sigma_i = 2n^{-1/p} \sigma$ and $M = 2n^{-1/p} V$. Equipped with Theorem C.4 yields that

$$\mathbb{P} \left( W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*) - \mathbb{E}[W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)] \geq t \right) \leq \exp \left( -\frac{t^2}{8\sigma^2 n^{1-2/p} + 4t V n^{-1/p}} \right).$$

For the simplicity, let $Z_E = W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*) - \mathbb{E}[W_p(E^*_\# \hat{\mu}_n, E^*_\# \mu_*)]$. Then we have $\mathbb{E}[Z_E] = 0$ and

$$\mathbb{P}(Z_E \geq t) \leq \exp(-t^2/(8\sigma^2 n^{1-2/p} + 4t V n^{-1/p})).$$

This together with the definition of $Z_E$ and Wainwright [2019, Theorem 2.2] yields the desired claim.

We then interpret $\{Z_E\}_{E \in \mathbb{S}_{d,k}}$ as an empirical process indexed by $E \in \mathbb{S}_{d,k}$ and claim that there exists a random variable $L$ satisfying $\mathbb{E}[L] \leq 4M_q(\mu_*)$ so that $|Z_U - Z_V| \leq L\|U - V\|_{op}$ for all $U, V \in \mathbb{S}_{d,k}$. More specifically, it follows from the definition that

$$Z_U - Z_V = (W_p(U^*_\# \hat{\mu}_n, U^*_\# \mu) - W_p(V^*_\# \hat{\mu}_n, V^*_\# \mu)) - \mathbb{E} \left[ W_p(U^*_\# \hat{\mu}_n, U^*_\# \mu) - W_p(V^*_\# \hat{\mu}_n, V^*_\# \mu) \right].$$

Since the Wasserstein distance is nonnegative and satisfies the triangle inequality, we have

$$W_p(U^*_\# \hat{\mu}_n, U^*_\# \mu) - W_p(V^*_\# \hat{\mu}_n, V^*_\# \mu) = W_p(U^*_\# \hat{\mu}_n, U^*_\# \mu) - W_p(U^*_\# \hat{\mu}_n, V^*_\# \mu) + W_p(U^*_\# \hat{\mu}_n, V^*_\# \mu) - W_p(V^*_\# \hat{\mu}_n, V^*_\# \mu) \leq W_p(U^*_\# \mu, V^*_\# \mu) + W_p(U^*_\# \mu, V^*_\# \hat{\mu}_n).$$

Putting these pieces together yields that

$$Z_U - Z_V \leq W_p(U^*_\# \hat{\mu}_n, V^*_\# \hat{\mu}_n) + W_p(U^*_\# \mu_*, V^*_\# \mu_*) + \mathbb{E} \left[ W_p(U^*_\# \hat{\mu}_n, V^*_\# \hat{\mu}_n) + W_p(U^*_\# \mu_*, V^*_\# \mu_*) \right].$$

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Since the Wasserstein distance is symmetrical, we have
\[ Z_V - Z_U \leq W_p(U_{\#}^\ast \hat{\mu}_n, V_{\#}^\ast \hat{\mu}_n) + W_p(U_{\#}^\ast \mu_\ast, V_{\#}^\ast \mu_\ast) + \mathbb{E} \left[ W_p(U_{\#}^\ast \hat{\mu}_n, V_{\#}^\ast \hat{\mu}_n) + W_p(U_{\#}^\ast \mu_\ast, V_{\#}^\ast \mu_\ast) \right]. \]

Therefore, we conclude that
\[ |Z_U - Z_V| \leq W_p(U_{\#}^\ast \hat{\mu}_n, V_{\#}^\ast \hat{\mu}_n) + W_p(U_{\#}^\ast \mu_\ast, V_{\#}^\ast \mu_\ast) + \mathbb{E} \left[ W_p(U_{\#}^\ast \hat{\mu}_n, V_{\#}^\ast \hat{\mu}_n) + W_p(U_{\#}^\ast \mu_\ast, V_{\#}^\ast \mu_\ast) \right]. \]

Let \( X \sim \mu \), we have
\[
|Z_U - Z_V| \leq 2 \left( \mathbb{E}(\|(U - V)X\|^p) \right)^{1/p} + \left( \frac{1}{n} \sum_{i=1}^{n} \|(U - V)X_i\|^p \right)^{1/p} + \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \|(U - V)X_i\|^p \right)^{1/p} \right]
\]
\[
\leq \|U - V\|_{\text{op}} \left( 2 \mathbb{E}(\|X\|^p) \right)^{1/p} + \left( \frac{1}{n} \sum_{i=1}^{n} \|X_i\|^p \right)^{1/p} + \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} \|X_i\|^p \right)^{1/p} \right]
\]
\[
:= L\|U - V\|_{\text{op}}.
\]

Note that \( X_{1:n} = (X_1, \ldots, X_n) \) are independent and identically distributed samples according to \( \mu_\ast \).
By the Jensen’s inequality and using the fact that \( q > p \geq 1 \), we have
\[
\mathbb{E}[L] \leq 4 \mathbb{E}(\|X\|^p) \leq 4 \mathbb{E}(\|X\|^q)^{1/q} = 4M_q(\mu_\ast).
\]

Thus, by a standard \( \epsilon \)-net argument, we obtain that
\[
\mathbb{E} \left[ \sup_{E \in \mathbb{S}_{d,k}} Z_E \right] \leq \inf_{\epsilon > 0} \left\{ \epsilon \mathbb{E}[L] + 4\sigma n^{1/2-1/p} \sqrt{\log(N(S_{d,k}, \epsilon, \| \cdot \|_{\text{op}}))} + 2Vn^{-1/p} \log(N(S_{d,k}, \epsilon, \| \cdot \|_{\text{op}})) \right\}
\]

Proposition C.3 shows that there exists a universal constant \( c > 0 \) such that
\[
\log(N(S_{d,k}, \epsilon, \| \cdot \|_{\text{op}})) \leq dk \log \left( \frac{c \sqrt{k}}{\epsilon} \right).
\]

Putting these pieces together and choosing \( \epsilon = \sqrt{k} n^{-1/p} \) (it is chosen to achieve the tight bound) yields that
\[
\mathbb{E} \left[ \sup_{E \in \mathbb{S}_{d,k}} Z_E \right] \lesssim_{p,q} \inf_{\epsilon > 0} \left\{ \epsilon + n^{1/2-1/p} \sqrt{dk \log \left( \frac{\sqrt{k}}{\epsilon} \right)} + n^{-1/p} dk \log \left( \frac{\sqrt{k}}{\epsilon} \right) \right\}
\]
\[
\lesssim_{p,q} n^{1/2-1/p} \sqrt{dk \log(n)} + n^{-1/p} dk \log(n).
\]

Therefore, we conclude that
\[
\mathbb{E} \left[ \sup_{E \in \mathbb{S}_{d,k}} \left( W_p(E_{\#}^\ast \hat{\mu}_n, E_{\#}^\ast \mu_\ast) - \mathbb{E}[W_p(E_{\#}^\ast \hat{\mu}_n, E_{\#}^\ast \mu_\ast)] \right) \right] \lesssim_{p,q} n^{1/2-1/p} \sqrt{dk \log(n)} + n^{-1/p} dk \log(n).
\]

This together with Eq. (C.3) and Eq. (C.4) yields the desired inequality.
C.4 Proof of Theorem 3.6

Using the same arguments as in Theorem 3.5, we obtain Eq. (C.3) and Eq. (C.4). So it suffices to bound the gap $\mathbb{E}[\sup_{E \in \mathbb{S}_{d,k}} (W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*) - \mathbb{E}[W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*)])$ under different conditions.

We first claim that $W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*) - \mathbb{E}[W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*)]$ is sub-exponential with parameters $(\sqrt{K^2/2n - 1/(2p)}, (K/2)n^{-1/p})$ for all $E \in \mathbb{S}_{d,k}$ if the true measure $\mu_*$ satisfies the projection Poincaré inequality (cf. Definition 3.2). Indeed, we consider $X = (X_1, \ldots, X_n)$ and $X' = (X'_1, \ldots, X'_n)$ where $X_i, X'_i$ are independent samples from $E_{#}^{*} \mu_*$. Let $f(X) = W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*)$, we have $\mathbb{E}(|f(X)|) < +\infty$. By the triangle inequality, we have

$$|f(X) - f(X')| \leq \frac{1}{p} \left( \sum_{i=1}^n ||X_i - X'_i||^p \right)^{1/p} \leq n^{-\frac{1}{p}} \|X - X'\|.$$

This implies that the following statement holds almost surely,

$$\sum_{i=1}^n ||\nabla_i f(X)||^2 \leq n^{-\frac{2}{p}} \quad \text{and} \quad \max_{1 \leq i \leq n} ||\nabla_i f(X)|| \leq n^{-\frac{1}{p}}, \quad \text{almost surely}.$$

In addition, the probability measure $E_{#}^{*} \mu_* \in \mathcal{P}(\mathbb{R}^k)$ is assumed to satisfy the Poincaré inequality. Equipped with Theorem C.5 yields that

$$\mathbb{P}(W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*) - \mathbb{E}[W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*)] \geq t) \leq \exp\left( -\frac{1}{K} \min \left\{ \frac{t}{n^{-1/p}}, \frac{t^2}{n^{-2/(2p)}} \right\} \right),$$

For the simplicity, let $Z_E = W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*) - \mathbb{E}[W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*)]$. Then we have $\mathbb{E}[Z_E] = 0$ and $\mathbb{P}(Z_E \geq t) \leq \exp(-K^{-1} \min\{n^{1/p} t, n^{2/(2p)} t^2\})$. This together with the definition of $Z_E$ and Wainwright [2019, Theorem 2.2] yields the desired claim.

Using the same argument in Theorem 3.5, we can interpret $\{Z_E\}_{E \in \mathbb{S}_{d,k}}$ as an empirical process indexed by $E \in \mathbb{S}_{d,k}$ and show that there exists a random variable $L$ satisfying $\mathbb{E}[L] \leq 4M_q(\mu_*)$ so that $|Z_U - Z_V| \leq L ||U - V||_{\text{op}}$ for all $U, V \in \mathbb{S}_{d,k}$. By a standard $\epsilon$-net argument, we obtain that

$$\mathbb{E}\left[\sup_{E \in \mathbb{S}_{d,k}} Z_E\right] \leq \inf_{\epsilon > 0} \left\{ \epsilon \mathbb{E}[L] + \sqrt{2K} n^{-1/(2p)} \sqrt{\log(N(\mathbb{S}_{d,k}, \epsilon, || \cdot ||_{\text{op}}))} + (K/2)n^{-1/p} \log(N(\mathbb{S}_{d,k}, \epsilon, || \cdot ||_{\text{op}})) \right\}.$$

Combining Proposition C.3 and choosing $\epsilon = \sqrt{n} n^{-1/p}$ (it is chosen to achieve the tight bound) yields that

$$\mathbb{E}\left[\sup_{E \in \mathbb{S}_{d,k}} Z_E\right] \leq_{p,q} \inf_{\epsilon > 0} \left\{ \epsilon + \sqrt{2K} n^{-1/(2p)} \sqrt{\log\left(\frac{\sqrt{K}}{\epsilon}\right)} + n^{-1/p} \log\left(\frac{\sqrt{K}}{\epsilon}\right) \right\} \leq_{p,q} \sqrt{2K} n^{-1/(2p)} \sqrt{\log(n)} + n^{-1/p} \log(n).$$

Therefore, we conclude that

$$\mathbb{E}\left[\sup_{E \in \mathbb{S}_{d,k}} (W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*) - \mathbb{E}[W_p(E_{#}^{*} \hat{\mu}_n, E_{#}^{*} \mu_*)]) \right] \leq_{p,q} n^{-1/(2p)} \sqrt{\log(n)} + n^{-1/p} \log(n).$$

This together with Eq. (C.3) and Eq. (C.4) yields the desired inequality.
C.5 Proof of Theorem 3.7

Since the arguments in this proof hold true for both IPRW and PRW distances, we denote $W = W_{p,k}$ or $W = W_{p,k}$ for short. Let $f(X) = W(\hat{\mu}_n, \mu_\star)$, we have

$$D_i = f(X) - f(X'_i) \leq W(\hat{\mu}_n, \hat{\mu}'_n) \leq n^{-1/p} \left( \sup_{E \in \mathcal{S}_{d,k}} \| E'_\#(X_i) - E'_\#(X'_i) \| \right).$$

By the triangle inequality, we have

$$\mathbb{E} \left[ |D_i|^k \mid X_{-i} \right] \leq 2^k n^{-k/p} \left( \mathbb{E} \left[ \sup_{E \in \mathcal{S}_{d,k}, X \sim E_\#} |X|^k \right] \right).$$

Since the true measure $\mu_\star$ satisfies the Bernstein-type tail condition (cf. Definition 3.3), we have

$$\mathbb{E} \left[ |D_i|^k \mid X_{-i} \right] \leq 2^{k-1} n^{-k/p} \sigma^2 k! V^{k-2} = \frac{(2n^{-1/p} \sigma)^2 k!(2n^{-1/p} V)^{k-2}}{2}$$

This implies that the condition in Theorem C.4 holds true with $\sigma = 2n^{-1/p} \sigma$ and $M = 2n^{-1/p} V$. Equipped with Theorem C.4 yields the desired inequality.

C.6 Proof of Theorem 3.8

Since the arguments in this proof hold true for both IPRW and PRW distances, we denote $W = W_{p,k}$ or $W = W_{p,k}$ for short. We consider $X = (X_1, X_2, \ldots, X_n)$ and $X' = (X'_1, X'_2, \ldots, X'_n)$ where $X_i, X'_i$ are independent samples from $\mu_\star$. Let $f(X) = W(\hat{\mu}_n, \mu_\star)$, we have $\mathbb{E}(|f(X)|) < +\infty$. By the triangle inequality, we have

$$|f(X) - f(X')| \leq n^{-1/p} \left( \sum_{i=1}^n \| X_i - X'_i \|^p \right)^{1/p} \leq n^{-2/p} \| X - X' \|.$$ 

This implies that the following statement holds almost surely,

$$\sum_{i=1}^n \| \nabla_i f(X) \|^2 \leq n^{-2} \text{ and } \max_{1 \leq i \leq n} \| \nabla_i f(X) \| \leq n^{-1/p}.$$

In addition, the true measure $\mu_\star$ satisfies the Poincaré inequality (cf. Definition 3.4). Equipped with Theorem C.5 yields the desired inequality.

D Postponed Proofs in Subsection 3.3

In this section, we provide the detailed proofs for Theorem 3.9-3.11 and Theorem A.1-A.2. Our results are derived analogously to the proof in Bernton et al. [2019] for the estimators based on Wasserstein distance and the proof in Nadjahi et al. [2019] for the estimators based on sliced-Wasserstein distance.
D.1 Preliminary technical results

To facilitate the reading, we collect several preliminary technical results which will be used in the postponed proofs in subsection 3.3.

Theorem D.1 (Theorem 2.43 in Aliprantis and Border [2006]) A real-valued lower semi-continuous function on a compact space attains a minimum value, and the nonempty set of minimizers is compact. Similarly, an upper semicontinuous function on a compact set attains a maximum value, and the nonempty set of maximizers is compact.

Definition D.1 (epiconvergence) Let \( \mathcal{X} \) be a metric space and \( \{f_i\}_{i \in \mathbb{N}} \) be a sequence of real-valued function from \( \mathcal{X} \) to \( \mathbb{R} \). We say that the sequence \( \{f_i\}_{i \in \mathbb{N}} \) epiconverges to a function \( f : \mathcal{X} \to \mathbb{R} \) if for each \( x \in \mathcal{X} \), the following statement holds true,

\[
\liminf_{i \to +\infty} f_i(x_i) \geq f(x) \text{ for every sequence } \{x_i\}_{i \in \mathbb{N}} \text{ such that } x_i \to x,
\]

\[
\limsup_{i \to +\infty} f_i(x_i) \leq f(x) \text{ for some sequence } \{x_i\}_{i \in \mathbb{N}} \text{ such that } x_i \to x.
\]

Proposition D.2 (Proposition 7.29 in Rockafellar and Wets [2009]) Let \( \mathcal{X} \) be a metric space and \( \{f_i\}_{i \in \mathbb{N}} \) be a sequence of real-valued function from \( \mathcal{X} \) to \( \mathbb{R} \) with a lower semi-continuous function \( f : \mathcal{X} \to \mathbb{R} \). Then the sequence \( \{f_i\}_{i \in \mathbb{N}} \) epiconverges to \( f \) if and only if

\[
\liminf_{i \to +\infty} (\inf_{x \in K} f_i(x)) \geq \inf_{x \in K} f(x) \text{ for every compact set } K \subseteq \mathcal{X},
\]

\[
\limsup_{i \to +\infty} (\sup_{x \in O} f_i(x)) \leq \sup_{x \in O} f(x) \text{ for every open set } O \subseteq \mathcal{X}.
\]

Recall that \( \delta\text{-argmin}_{x \in \mathcal{X}} f = \{ x \in \mathcal{X} : f(x) \leq \inf_{x \in \mathcal{X}} f + \delta \} \) for a generic function \( f : \mathcal{X} \to \mathbb{R} \). The following theorem gives asymptotic properties for the infimum and \( \delta\text{-argmin} \) of epiconvergent functions and thus a standard approach to prove the existence and consistency of the estimators.

Theorem D.3 (Theorem 7.31 in Rockafellar and Wets [2009]) Let \( \mathcal{X} \) be a metric space and \( \{f_i\}_{i \in \mathbb{N}} \) be a sequence of function which epiconverges to a lower semi-continuous function \( f \) with \( \inf_{x \in \mathcal{X}} f \in (-\infty, +\infty) \). Then we have the following statements,

1. \( \inf_{x \in \mathcal{X}} f_i \to \inf_{x \in \mathcal{X}} f \) if and only if for every \( \delta > 0 \) there exists a compact set \( B \subseteq \mathcal{X} \) and \( N \in \mathbb{N} \) such that \( \inf_{x \in B} f_i \leq \inf_{x \in \mathcal{X}} f_i + \delta \) for all \( i \geq N \).

2. \( \limsup_{i \to +\infty} (\delta\text{-argmin}_{x \in \mathcal{X}} f_i) \subseteq \delta\text{-argmin}_{x \in \mathcal{X}} f \) for any \( \delta \geq 0 \) and \( \limsup_{i \to +\infty} (\delta\text{-argmin}_{x \in \mathcal{X}} f_i) \subseteq \argmin_{x \in \mathcal{X}} f \) whenever \( \delta_i \downarrow 0 \).

3. Assume that \( \inf_{x \in \mathcal{X}} f_i \to \inf_{x \in \mathcal{X}} f \), there exists a sequence \( \delta_i \downarrow 0 \) such that \( \delta_i\text{-argmin}_{x \in \mathcal{X}} f_i \to \argmin_{x \in \mathcal{X}} f \). Conversely, if \( \argmin_{x \in \mathcal{X}} f \neq \emptyset \) and if such a sequence exists, then \( \inf_{x \in \mathcal{X}} f_i \to \inf_{x \in \mathcal{X}} f \).

The following theorem summarizes the well-known Skorokhod’s representation theorem.

Theorem D.4 (Skorokhod’s representation theorem) Let \( \{\mu_n\}_{n \in \mathbb{N}} \) be a sequence of probability measures on a metric space \( \mathcal{X} \) such that \( \mu_n \) converges weakly to some probability measure \( \mu_\infty \) on \( \mathcal{X} \) as \( n \to \infty \). Suppose also that the support of \( \mu_\infty \) is separable. Then there exist random variables \( X_n \) defined on a common probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that the law of \( X_n \) is \( \mu_n \) for all \( n \) (including \( n = \infty \)) and such that \( X_n \) converges to \( X_\infty \) almost surely.
The following theorem presents the classical results which lead to a standard approach for proving the measurability of the estimators. Note that the projection \( \text{proj}(D) = \{ x \in X : \exists y \in Y, \text{s.t.} (x, y) \in D \} \) for each \( D \subseteq X \times Y \) and the section \( D_x = \{ y \in Y : (x, y) \in D \} \) for each \( x \in \text{proj}(D) \).

**Theorem D.5 (Corollary 1 in Brown and Purves [1973])** Let \( X, Y \) be complete separable metric spaces and \( f \) be a real-valued Borel measurable function defined on a Borel subset \( D \) of \( X \times Y \). Suppose that for each \( x \in \text{proj}(D) \), the section \( D_x \) is \( \sigma \)-compact and \( f(x, \cdot) \) is lower semi-continuous with respect to the relative topology on \( D_x \). Then

1. The sets \( G = \text{proj}(D) \) and \( I = \{ x \in G : \exists y \in D_x \text{ s.t.} y = \arg\min_{z \in Y} f(x, z) \} \) are Borel.

2. For each \( \epsilon > 0 \), there exists a Borel measure function \( \varphi_{\epsilon} \) satisfying, for \( x \in G \) that,

\[
\begin{align*}
f(x, \varphi_{\epsilon}(x)) = \left\{
\begin{array}{ll}
\inf_{y \in G} f(x, y), & x \in I, \\
\leq \epsilon + \inf_{y \in G} f(x, y), & x \notin I \text{ and } \inf_{y \in G} f(x, y) \neq -\infty, \\
\leq -\epsilon^{-1}, & x \notin I \text{ and } \inf_{y \in G} f(x, y) = -\infty.
\end{array}
\right.
\end{align*}
\]

To show that the MEPRW estimator is measurable, we establish the lower semi-continuity of the expectation of empirical PRW distance in the following lemma.

**Lemma D.6** The expected empirical PRW distance is lower semi-continuous in the usual weak topology. If the sequences \( \{ \mu_i \}_{i \in \mathbb{N}}, \{ \nu_i \}_{i \in \mathbb{N}} \subseteq \mathcal{P}^d(d) \) satisfying that \( \mu_i \Rightarrow \mu \in \mathcal{P}(d) \) and \( \nu_i \Rightarrow \nu \in \mathcal{P}(d) \), we have \( E[\overline{\mathcal{W}}_{p,k}(\mu, \hat{\nu}_m)] \leq \liminf_{i \to +\infty} E[\overline{\mathcal{W}}_{p,k}(\mu_i, \nu_i)] \), where \( \hat{\nu}_m = (1/m) \sum_{j=1}^m \delta_{z_j} \) for i.i.d. samples \( Z_{1:m} \) according to \( \nu \) and \( \{ \hat{\nu}_i \}_{i \in \mathbb{N}} \) are defined similarly.

### D.2 Proof of Theorem 3.9

We first prove that \( \arg\min_{\theta \in \Theta} \overline{\mathcal{W}}_{p,k}(\mu_\theta, \mu_\theta) \neq \emptyset \). Indeed, by Assumption 3.2 and Theorem 3.3, the mapping \( \theta \mapsto \overline{\mathcal{W}}_{p,k}(\mu_\theta, \mu_\theta) \) is lower semi-continuous. By Assumption 3.3, the set \( \Theta_\tau(\tau) \) for some \( \tau > 0 \). By the definition of \( \inf \), there exists \( \theta' \in \Theta \) such that \( \overline{\mathcal{W}}_{p,k}(\mu_\theta, \mu_{\theta'}) = \inf_{\theta \in \Theta} \overline{\mathcal{W}}_{p,k}(\mu_\theta, \mu_{\theta'}) + \tau/2 \). This implies that \( \theta' \in \Theta_\tau(\tau) \) and \( \Theta_\tau(\tau) \) is nonempty. By the lower semi-continuity of the mapping \( \theta \mapsto \overline{\mathcal{W}}_{p,k}(\mu_\theta, \mu_\theta) \), the set \( \Theta_\tau(\tau) \) is closed. Putting these pieces together yields that \( \Theta_\tau(\tau) \) is compact. Therefore, we conclude the desired result from Theorem D.1.

Then we show that there exists a set \( E \subseteq \Theta \) with \( P(E) = 1 \) such that, for all \( w \in E \), the sequence of mappings \( \theta \mapsto \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta) \) epiconverges to the mapping \( \theta \mapsto \overline{\mathcal{W}}_{p,k}(\mu_\theta, \mu_\theta) \) as \( n \to +\infty \). Indeed, we only need to prove that the conditions in Proposition D.2 hold true.

Fix \( K \subseteq \Theta \) as a compact set. By the lower semi-continuity of the mapping \( \theta \mapsto \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta) \) (cf. Assumption 3.2 and Theorem 3.3), Theorem D.1 implies that

\[
\inf_{\theta \in K} \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta) = \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_{\theta_n})
\]

for some sequence \( \theta_n = \theta_n(\omega) \in K \). Thus, we have

\[
\lim_{n \to +\infty} \inf_{\theta \in K} \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta) = \lim_{n \to +\infty} \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_{\theta_n}).
\]

By the definition of \( \lim \inf \), there exists a subsequence of \( \{ \theta_n \}_{n \in \mathbb{N}} \) such that \( \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_{\theta_n}) \) converges to \( \lim_{n \to +\infty} \overline{\mathcal{W}}_{p,k}(\hat{\mu}_n(\omega), \mu_{\theta_n}) \) along this subsequence. By the compactness of \( K \), this subsequence
must have a convergent subsequence. We denote this subsequence as \( \{ \theta_{n_j} \}_{j \in \mathbb{N}} \) and its limit as \( \hat{\theta} \in K \). Then

\[
\liminf_{n \to +\infty} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n_j}}(\omega), \mu_{\theta_{n_j}}) = \lim_{j \to +\infty} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n_j}}(\omega), \mu_{\theta_{n_j}}).
\]

Since \( \omega \in E \) where \( \mathbb{P}(E) = 1 \), Assumption 3.1 and 3.2 imply \( \mu_{\theta_{n_j}}(\omega) \Rightarrow \mu_{\ast} \) and \( \mu_{\theta_{n_j}} \Rightarrow \mu_{\hat{\theta}} \). These pieces together with the lower semi-continuity of the PRW distance (cf. Theorem 3.3) yields that \( \lim_{j \to +\infty} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n_j}}(\omega), \mu_{\theta_{n_j}}) \geq \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\hat{\theta}}) \). Putting these pieces together yields that

\[
\liminf_{n \to +\infty} \inf_{\theta \in K} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu(\omega), \mu_{\theta}) \geq \inf_{\theta \in K} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}).
\]

Fix \( O \subseteq \Theta \) as an arbitrary open set. By the definition of inf, there exists a sequence \( \theta_{n} = \theta'_{n}(\omega) \in O \) such that \( \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta}, \mu_{\theta_{n}}) \rightarrow \inf_{\theta \in O} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}) \). In addition, \( \inf_{\theta \in O} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta_{n}}(\omega)). \) Thus, we have

\[
\limsup_{n \to +\infty} \inf_{\theta \in O} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \limsup_{n \to +\infty} \inf_{\theta \in O} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) = \limsup_{n \to +\infty} \inf_{\theta \in O} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \inf_{\theta \in O} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}).
\]

Proposition D.2 guarantees that there exists a set \( E \subseteq \Omega \) with \( \mathbb{P}(E) = 1 \) such that, for all \( \omega \in E \), the sequence of mappings \( \theta \mapsto \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \) epiconverges to the mapping \( \theta \mapsto \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}) \) as \( n \to +\infty \). Then the second statement of Theorem D.3 implies that

\[
\limsup_{n \to +\infty} \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}). \tag{D.1}
\]

The next step is to show that, for every \( \delta > 0 \), there exists a compact set \( B \subseteq \Theta \) and \( N \in \mathbb{N} \) such that \( \inf_{\theta \in B \cap \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) + \delta \). In what follows, we prove a stronger statement which states that the above inequality holds true with \( \delta = 0 \). Indeed, by the same reasoning for the open set case in the proof of epiconvergence, we have

\[
\limsup_{n \to +\infty} \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}).
\]

By Assumption 3.3 and using previous argument, \( \Theta_{\ast}(\tau) \) is nonempty and compact for some \( \tau > 0 \). The above inequality implies that there exists \( n_1(\omega) > 0 \) such that, for all \( n \geq n_1(\omega) \), the set \( \{ \theta \in \Theta : \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) + \tau \} \) is nonempty. For any \( \theta \) in this set and let \( n \geq n_1(\omega) \), we have

\[
\frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta_{n}}(\omega)) \leq \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta_{n}}(\omega)) + \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}) + \tau / 2.
\]

By Assumption 3.1, there exists \( n_2(\omega) > 0 \) such that, for all \( n \geq n_2(\omega) \), we have

\[
\frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta_{n}}(\omega)) \leq \mathcal{W}_{p}(\mu_{\ast}, \mu_{\theta_{n}}(\omega)) \leq \frac{\tau}{2}.
\]

Putting these pieces together yields that, for all \( n \geq \max\{n_1(\omega), n_2(\omega)\} \), we have \( \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta_{n}}(\omega)) \leq \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}) + \tau \). This implies that, for all \( n \geq \max\{n_1(\omega), n_2(\omega)\} \) that,

\[
\left\{ \theta \in \Theta : \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\theta_{n}}(\omega), \mu_{\theta}) \leq \inf_{\theta \in \Theta} \frac{\partial \mathcal{W}_{p,k}}{\partial \mathcal{W}_{p,k}}(\mu_{\ast}, \mu_{\theta}) + \frac{\tau}{2} \right\} \subseteq \Theta_{\ast}(\tau).
\]
Therefore, we have $\inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta) = \inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta)$. This together with the compactness of $\Theta_* \in \Theta$ yields the desired result.

The first statement of Theorem D.3 implies that

$$\inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta) \to \inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\mu_*, \mu_\theta), \quad \text{as } n \to +\infty. \quad (D.2)$$

By Assumption 3.2 and Theorem 3.3, the mapping $\theta \mapsto \mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta)$ is lower semi-continuous. Theorem D.1 implies $\arg\min_{\theta \in \Theta} \mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \mu_\theta)$ are nonempty for all $n \geq \max\{n_1(\omega), n_2(\omega)\}$. Together with Eq. (D.1) and (D.2) yields the desired results.

Finally, we remark that these results hold true for $\delta_n$-argmin$_{\theta \in \Theta} \mathcal{PW}_{p,k}(\hat{\mu}_n, \mu_\theta)$ with $\delta_n \to 0$. For Eq. (D.1) and (D.2), the analogous results can be derived by using the second and third statements of Theorem D.3. To show that $\delta_n$-argmin$_{\theta \in \Theta} \mathcal{PW}_{p,k}(\hat{\mu}_n, \mu_\theta)$ is nonempty, we notice it contains the nonempty set $\arg\min_{\theta \in \Theta} \mathcal{PW}_{p,k}(\hat{\mu}_n, \mu_\theta)$.

**D.3 Proof of Theorem 3.10**

Following the same approach used for analyzing Theorem 3.9, it is straightforward to derive that $\arg\min_{\theta \in \Theta} \mathcal{PW}_{p,k}(\mu_*, \mu_\theta) \neq \emptyset$. Then we show that there exists a set $E \subseteq \Omega$ with $\mathbb{P}(E) = 1$ such that, for all $\omega \in E$, the sequences $\theta \mapsto \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n]$ epiconterges $\theta \mapsto \mathcal{PW}_{p,k}(\mu_*, \mu_\theta)$ as $n \to +\infty$. Indeed, it suffices to verify the conditions in Proposition D.2.

Fix $K \subseteq \Theta$ as an arbitrary compact set. By Assumption 3.2 and Lemma D.6, the mapping $\theta \mapsto \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n]$ is lower semi-continuous. Then Theorem D.1 implies that

$$\inf_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n] = \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n]$$

for some sequence $\theta_n = \theta_n(\omega) \in K$. Thus, we have

$$\lim_{n \to +\infty} \inf_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n] = \lim_{n \to +\infty} \inf_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n].$$

Following the same approach used in the proof of Theorem 3.9, there exists a subsequence of $\{\theta_n\}_{n \in \mathbb{N}}$, denoted by $\{\theta_{n_j}\}_{j \in \mathbb{N}}$ with the limit $\theta \in K$, such that

$$\lim_{n \to +\infty} \inf_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n] = \lim_{j \to +\infty} \inf_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_{n_j}(\omega), \hat{\mu}_{\theta,m(n_j)}) | X_1:n_j]$$

$$\geq \lim_{j \to +\infty} \inf_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_{n_j}(\omega), \mu_\theta) | X_1:n_j] - \lim_{j \to +\infty} \sup_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\mu_\theta, \hat{\mu}_{\theta,m(n_j)}) | X_1:n_j].$$

Since $\omega \in E$ where $\mathbb{P}(E) = 1$, Assumption 3.1 and 3.2 imply $\hat{\mu}_{n_j}(\omega) \Rightarrow \mu_*$ and $\mu_{\theta_{n_j}} \Rightarrow \mu_\theta$. These pieces together with the lower semi-continuity of the PRW distance (cf. Theorem 3.3) yields that $\lim_{j \to +\infty} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_{n_j}(\omega), \mu_{\theta_{n_j}})] \geq \mathbb{E}[\mathcal{PW}_{p,k}(\mu_*, \mu_\theta)]$. By Assumption 3.4 and using $\theta_{n_j} \to \theta$, we have $\limsup_{j \to +\infty} \mathbb{E}[\mathcal{PW}_{p,k}(\mu_{\theta_{n_j}}, \hat{\mu}_{\theta,m(n_j)}) | X_1:n_j] \to 0$. Putting these pieces together yields that

$$\lim_{n \to +\infty} \inf_{\theta \in K} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n] \geq \inf_{\theta \in K} \mathcal{PW}_{p,k}(\mu_*, \mu_\theta).$$

Fix $O \subseteq \Theta$ as an arbitrary open set. By the definition of $\inf$, there exists a sequence $\theta'_n = \theta'_n(\omega) \in O$ such that $\mathcal{PW}_{p,k}(\mu_*, \mu_{\theta'_n}) \to \inf_{\theta \in O} \mathcal{PW}_{p,k}(\mu_*, \mu_\theta)$. In addition, we have

$$\inf_{\theta \in O} \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_1:n] \leq \mathbb{E}[\mathcal{PW}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta'_n,m(n)}) | X_1:n].$$
Thus, we have
\[
\limsup_{n \to +\infty} \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \leq \limsup_{n \to +\infty} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta',m}(n)) \mid X_{1:n}]
\]
\[
\leq \limsup_{n \to +\infty} \mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \mu_\ast) + \limsup_{n \to +\infty} \mathcal{P}W_{p,k}(\mu_\ast, \mu_{\theta'}) + \limsup_{n \to +\infty} \mathbb{E}[\mathcal{P}W_{p,k}(\mu_{\theta'}, \tilde{\mu}_{\theta',m}(n)) \mid X_{1:n}].
\]

Since \( \omega \in E \) where \( \mathbb{P}(E) = 1 \), Assumption 3.1 implies \( \limsup_{n \to +\infty} \mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \mu_\ast) = 0 \). By the definition of \( \theta'_n \), we have \( \limsup_{n \to +\infty} \mathcal{P}W_{p,k}(\mu_\ast, \mu_{\theta'_n}) = \inf_{\theta \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_{\theta}) \). Using Assumption 3.4 and \( \lim_{j \to +\infty} \theta_{m_j} = \theta \), we have \( \limsup_{n \to +\infty} \mathbb{E}[\mathcal{P}W_{p,k}(\mu_{\theta'}, \tilde{\mu}_{\theta',m}(n)) \mid X_{1:n}] = 0 \). Putting these pieces together yields that \( \limsup_{n \to +\infty} \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \leq \inf_{\theta \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta) \).

Proposition D.2 guarantees that there exists a set \( E \subseteq \Omega \) with \( \mathbb{P}(E) = 1 \) such that, for all \( \omega \in E \), the sequence of mappings \( \theta \mapsto \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \) epiconverges to the mapping \( \theta \mapsto \mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta) \) as \( n \to +\infty \). Then the second statement of Theorem D.3 implies that
\[
\limsup_{n \to +\infty} \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \leq \inf_{\theta \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta).
\]  

The next step is to show that, for every \( \delta > 0 \), there exists a compact set \( B \subseteq \Theta \) and \( N \in \mathbb{N} \) such that \( \inf_{\theta \in B} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \leq \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] + \delta \). In what follows, we prove a stronger statement which states that the above inequality holds true with \( \delta = 0 \). Indeed, by the same reasoning for the open set case in the proof of epiconvergence, we have
\[
\limsup_{n \to +\infty} \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \leq \inf_{\theta \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta).
\]

By Assumption 3.3 and using previous argument, \( \Theta_*(\tau) \) is nonempty and compact for some \( \tau > 0 \). The above inequality implies that there exists \( n_1(\omega) > 0 \) such that, for all \( n \geq n_1(\omega) \), the set \( \{ \theta \in \Theta : \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \leq \inf_{\theta' \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_{\theta'}) + \tau/3 \} \) is nonempty. For any \( \theta \) in this set and let \( n \geq n_1(\omega) \), we have
\[
\mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta) \leq \mathcal{P}W_{p,k}(\mu_\ast, \tilde{\mu}_n(\omega)) + \mathbb{E}[\mathcal{P}W_{p,k}(\mu_{\theta}, \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] + \inf_{\theta \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta) + \frac{\tau}{3}.
\]

By Assumption 3.1, there exists \( n_2(\omega) > 0 \) such that, for all \( n \geq n_2(\omega) \), we have
\[
\mathcal{P}W_{p,k}(\mu_\ast, \tilde{\mu}_n(\omega)) \leq W_p(\mu_\ast, \tilde{\mu}_n(\omega)) \leq \frac{\tau}{3}.
\]

By Assumption 3.4, there exists \( n_3(\omega) > 0 \) such that, for all \( n \geq n_3(\omega) \), we have
\[
\mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_{\theta,m}(n), \mu_\theta) \mid X_{1:n}] \leq \mathbb{E}[W_p(\mu_{\theta,m}(n), \mu_\theta) \mid X_{1:n}] \leq \frac{\tau}{3}.
\]

Putting these pieces together yields that, for all \( n \geq \max\{n_1(\omega), n_2(\omega), n_3(\omega)\} \) that,
\[
\mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta) \leq \inf_{\theta \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_\theta) + \tau.
\]

This implies that, for all \( n \geq \max\{n_1(\omega), n_2(\omega), n_3(\omega)\} \) that,
\[
\left\{ \theta \in \Theta : \mathbb{E}[\mathcal{P}W_{p,k}(\tilde{\mu}_n(\omega), \tilde{\mu}_{\theta,m}(n)) \mid X_{1:n}] \leq \inf_{\theta' \in \Theta} \mathcal{P}W_{p,k}(\mu_\ast, \mu_{\theta'}) + \frac{\tau}{3} \right\} \subseteq \Theta_*(\tau).
\]
Therefore, we have \(\inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}] = \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}]\). This together with the compactness of \(\Theta_0(\tau)\) yields the desired result.

The first statement of Theorem 3.11 implies that

\[
\inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}] \rightarrow \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\mu_*, \mu_\theta), \quad \text{as } n \rightarrow +\infty. \tag{D.4}
\]

By Assumption 3.2 and Lemma D.6, the mapping \(\theta \mapsto \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}]\) is lower semi-continuous. Theorem D.1 implies \(\arg\inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}]\) are nonempty for all \(n \geq \max\{n_1(\omega), n_2(\omega), n_3(\omega)\}\). Together with Eq. (D.3) and (D.4) yields the desired results.

Finally, we remark that these results hold true for \(\delta_n \rightarrow 0\). For Eq. (D.3) and (D.4), the analogous results can be derived by using the second and third statements of Theorem D.3. To show that \(\delta_n \rightarrow \arg\inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}]\) is nonempty, we notice it contains the nonempty set \(\arg\inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n(\omega), \hat{\mu}_{\theta,m(n)}) | X_{1:n}]\).

### D.4 Proof of Theorem 3.11

We first prove that \(\arg\inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_\theta)] \neq \emptyset\). Indeed, by Assumption 3.2 and Theorem 3.3, the mapping \(\theta \mapsto \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_\theta)]\) is lower semi-continuous. By Assumption 3.5, the set \(\Theta_0(\tau)\) is compact. By the definition of inf, there exists \(\theta_n' \in \Theta\) such that \(\inf_{\Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_\theta)] = \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_\theta)] + \tau_n/2\). This implies that \(\theta_n' \in \Theta_0(\tau)\) and \(\Theta_0(\tau)\) is empty. By the lower semi-continuity of the mapping \(\theta \mapsto \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_\theta)]\), the set \(\Theta_0(\tau)\) is closed. Putting these pieces together yields that \(\Theta_0(\tau)\) is compact. Therefore, we conclude the desired result from Theorem D.1.

Then we show that the sequences \(\theta \mapsto \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}]\) epiconverges to \(\theta \mapsto \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_\theta)]\) as \(m \rightarrow +\infty\). Indeed, it suffices to verify the conditions in Proposition D.2.

Fix \(K \subseteq \Theta\) as an arbitrary compact set. By Assumption 3.2 and Lemma D.6, the mapping \(\theta \mapsto \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}]\) is lower semi-continuous. Then Theorem D.1 implies that

\[
\inf_{\theta \in K} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}] = \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}]
\]

for some sequence \(\theta_m \in K\). Thus, we have

\[
\lim \inf_{m \rightarrow +\infty} \inf_{\theta \in K} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}] = \lim \inf_{m \rightarrow +\infty} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) | X_{1:n}].
\]

Following up the same approach used in the proof of Theorem 3.9, there exists a subsequence of \(\{\theta_m\}_{m \in \mathbb{N}}\), denoted by \(\{\theta_n\}_{j \in \mathbb{N}}\) with the limit \(\bar{\theta} \in K\), such that

\[
\lim_{m \rightarrow +\infty} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_{\theta_m,m}) | X_{1:n}] = \lim_{j \rightarrow +\infty} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_{\theta_{m,j},m_j}) | X_{1:n}] 
\geq \lim_{j \rightarrow +\infty} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\mu_{\theta_m,m_j}, \mu_{\theta_{m_j},m_j}) | X_{1:n}] - \lim \sup_{j \rightarrow +\infty} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\mu_{\theta_m,m}, \mu_{\theta_{m},m_{m_j}}) | X_{1:n}].
\]

Assumption 3.1 and 3.2 imply \(\mu_{\theta_{m_j}} \Rightarrow \mu_*\) and \(\mu_{\theta_{m}} \Rightarrow \mu_{\bar{\theta}}\). Together with the lower semi-continuity of the PRW distance yields that \(\lim \inf_{j \rightarrow +\infty} \mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_{\theta_{m,j},m_j}) \geq \mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_{\bar{\theta}})\). By Assumption 3.4 and using \(\theta_{m_j} \rightarrow \bar{\theta}\), we have \(\lim \sup_{j \rightarrow +\infty} \mathcal{P}_n \mathcal{W}_{p,k}(\mu_{\theta_{m_j},m_j}, \mu_{\theta_{m},m_{m_j}}) | X_{1:n} = 0\). Thus, we conclude that

\[
\lim \inf_{m \rightarrow +\infty} \mathbb{E}[\mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m,m}) | X_{1:n}] \geq \inf_{\bar{\theta} \in K} \mathcal{P}_n \mathcal{W}_{p,k}(\hat{\mu}_n, \mu_{\bar{\theta}}).
\]
Fix $O \subseteq \Theta$ as an arbitrary open set. By the definition of inf, there exists a sequence $\theta'_m \in O$ such that $\mathcal{PW}_{p,k}(\mu_n, \mu_{\theta'_m}) \rightarrow \inf_{\theta \in O} \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta})$. In addition, we have

$$\inf_{\theta \in O} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] \leq \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}].$$

Thus, we have

$$\limsup_{m \rightarrow +\infty} \inf_{\theta \in O} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] \leq \limsup_{m \rightarrow +\infty} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}]
\leq \limsup_{m \rightarrow +\infty} \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) + \limsup_{n \rightarrow +\infty} \mathbb{E}[\mathcal{PW}_{p,k}(\mu_{\theta}, \mu_{\theta}, m) | X_{1:n}].$$

By the definition of $\theta'_m$, we have $\limsup_{m \rightarrow +\infty} \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta'_m}) = \inf_{\theta \in O} \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta})$. Using Assumption 3.4 and $\lim_{j \rightarrow +\infty} \theta_{m_j} = \theta$, we have $\limsup_{m \rightarrow +\infty} \mathbb{E}[\mathcal{PW}_{p,k}(\mu_{\theta'_m}, \mu_{\theta'_m}) | X_{1:n}] = 0$. Putting these pieces together yields that $\limsup_{m \rightarrow +\infty} \inf_{\theta \in O} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] = \inf_{\theta \in O} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta})].$

Proposition D.2 guarantees that the sequence of mappings $\theta \mapsto \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}]$ epiconverges to the mapping $\theta \mapsto \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta})$ as $m \rightarrow +\infty$. Then the second statement of Theorem D.3 implies that

$$\limsup_{m \rightarrow +\infty} \arg\min_{\theta \in \Theta} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] \subseteq \arg\min_{\theta \in \Theta} \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}). \quad (D.5)$$

The next step is to show that, for every $\delta > 0$, there exists a compact set $B \subseteq \Theta$ and $N \in \mathbb{N}$ such that $\inf_{\theta \in B} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] \leq \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}) | X_{1:n}] + \delta$. In what follows, we prove a stronger statement which states that the above inequality holds true with $\delta = 0$. Indeed, by the same reasoning for the open set case in the proof of epiconvergence, we have

$$\limsup_{m \rightarrow +\infty} \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] \leq \inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}).$$

By Assumption 3.5 and using previous argument, $\Theta_n(\tau)$ is nonempty and compact for some $\tau > 0$. The above inequality implies that there exists $m_1 > 0$ such that, for all $m \geq m_1$, the set $\{\theta \in \Theta : \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] \leq \inf_{\theta \in \Theta} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}) | X_{1:n}] + \tau/2\}$ is nonempty. For any $\theta$ in this set and let $m \geq m_1$, we have

$$\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}) \leq \mathbb{E}[\mathcal{PW}_{p,k}(\mu_{\theta}, m, \mu_{\theta}) | X_{1:n}] + \inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}) + \tau/2.$$

By Assumption 3.4, there exists $m_2 > 0$ such that, for all $m \geq m_2$, we have

$$\mathbb{E}[\mathcal{PW}_{p,k}(\mu_{\theta}, m, \mu_{\theta}) | X_{1:n}] \leq \mathbb{E}[\mathcal{W}_p(\mu_{\theta}, m, \mu_{\theta}) | X_{1:n}] \leq \frac{\tau}{2}.$$ 

Putting these pieces together yields that $\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}) \leq \inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\mu_{\theta}, \mu_{\theta}) + \tau$ for all $m \geq \max\{m_1, m_2\}$. This implies that, for all $m \geq \max\{m_1, m_2\}$ that,

$$\left\{ \theta \in \Theta : \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] \leq \inf_{\theta \in \Theta} \mathcal{PW}_{p,k}(\mu_{\theta}, \mu_{\theta}) + \frac{\tau}{2} \right\} \subseteq \Theta_n(\tau).$$

Therefore, we have $\inf_{\theta \in \Theta} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}] = \inf_{\theta \in \Theta_n(\tau)} \mathbb{E}[\mathcal{PW}_{p,k}(\bar{\mu}_n, \mu_{\theta}, m) | X_{1:n}]$. This together with the compactness of $\Theta_n(\tau)$ yields the desired result.
The first statement of Theorem D.3 implies that
\[
\inf_{\theta \in \Theta} \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) \mid X_{1:n}] \to \inf_{\theta \in \Theta} \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n, \mu_\theta), \text{ as } m \to +\infty. \tag{D.6}
\]

By Assumption 3.2 and Lemma D.6, the mapping \( \theta \mapsto \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) \mid X_{1:n}] \) is lower semi-continuous. Theorem D.1 implies \( \arg\min_{\theta \in \Theta} \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) \mid X_{1:n}] \) are nonempty for all \( m \geq \max\{m_1, m_2\} \). Together with Eq. (D.5) and Eq. (D.6) yields the desired results.

Finally, we remark that these results hold true for \( \delta_n\)-argmin\(\theta \in \Theta \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) \mid X_{1:n}] \) with \( \delta_n \to 0 \). For Eq. (D.5) and (D.6), the analogous results can be derived by using the second and third statements of Theorem D.3. To show that \( \delta_n\)-argmin\(\theta \in \Theta \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) \mid X_{1:n}] \) is nonempty, we notice it contains the nonempty set argmin\(\theta \in \Theta \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n, \hat{\mu}_{\theta,m}) \mid X_{1:n}] \).

D.5 Proof of Lemma D.6

Since \( \nu_1 \Rightarrow \nu \in \mathscr{P}(\mathbb{R}^d) \) and \( \mathbb{R}^d \) is separable, the Skorokhod’s representation theorem (cf. Theorem D.4) implies that there exists \( m \) sequences of random variables \( \{\{Z_i^k\}_{i \in \mathbb{N}}, k \in [m]\} \) and \( m \) random variables \( \{Z_i^k, k \in [m]\} \) such that the distribution of \( Z_i^k \) is \( \nu_i \), the distribution of \( Z^k \) is \( \nu \) and \( \{Z_i^k\}_{i \in \mathbb{N}} \) converges to \( Z^k \) almost surely for all \( k \in [m] \).

Suppose that \( \tilde{\nu}_{i,m} = (\frac{1}{m})(\sum_{k=1}^{m} \delta_{Z_i^k}) \) and \( \tilde{\nu}_m = (\frac{1}{m})(\sum_{k=1}^{m} Z_i^k) \), we proceed to the key part of the proof and show that \( \{\tilde{\nu}_{i,m}\}_{i \in \mathbb{N}} \) weakly converges to \( \tilde{\nu}_m \). Indeed, it suffices to consider the deterministic case where \( \tilde{\nu}_{i,m} = (\frac{1}{m})(\sum_{k=1}^{m} \delta_{z_i^k}) \) and \( \tilde{\nu}_m = (\frac{1}{m})(\sum_{k=1}^{m} z^k) \) where \( \{z_i^k\}_{i \in \mathbb{N}}, k \in [m] \) and \( \{z^k, k \in [m]\} \) are all deterministic such that \( \lim_{i \to +\infty} \max_{k \in [m]} \|z_i^k - z^k\| = 0 \). Since the Wasserstein distance metrizes the weak convergence (cf. Theorem B.4), we only need to show that \( \lim_{i \to +\infty} \mathcal{W}_2(\tilde{\nu}_{i,m}, \tilde{\nu}_m) = 0 \). By the definition of the Wasserstein distance, \( \{\tilde{\nu}_{i,m}\}_{i \in \mathbb{N}} \) and \( \tilde{\nu}_m \), we have \( \mathcal{W}_2^2(\tilde{\nu}_{i,m}, \tilde{\nu}_m) \leq \max_{k \in [m]} \|z_i^k - z^k\|^2 \). Putting these pieces together yields that \( \{\tilde{\nu}_{i,m}\}_{i \in \mathbb{N}} \) weakly converges to \( \tilde{\nu}_m \) almost surely.

Finally, we conclude from the lower semi-continuity of the PRW distance (cf. Theorem 3.3) and the Fatou’s lemma that
\[
\mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\mu, \tilde{\nu}_m)] \leq \mathbb{E}\left[ \liminf_{i \to +\infty} \bar{\mathcal{PW}}_{p,k}(\mu_i, \tilde{\nu}_{i,m}) \right] = \liminf_{i \to +\infty} \mathbb{E}[\bar{\mathcal{PW}}_{p,k}(\mu_i, \tilde{\nu}_{i,m})].
\]
This completes the proof.

D.6 Proof of Theorem A.1

Using Assumption 3.2 and Theorem 3.3, the mapping \( (\mu, \theta) \mapsto \bar{\mathcal{PW}}_{p,k}(\mu, \mu_\theta) \) is lower semi-continuous in \( \mathcal{P}(\mathbb{R}^d) \times \Theta \). It remains to verify that the conditions in Theorem D.5 are satisfied.

We notice that the empirical measure \( \hat{\mu}_n(\omega) \) depends on \( \omega \in \Omega \) only through \( X_{1:n} \in \otimes_{i=1}^n \mathbb{R}^d \). Thus, we can write \( \hat{\mu}_n(\omega) = \hat{\mu}_n(x) \) as a function in \( \otimes_{i=1}^n \mathbb{R}^d \). Let \( D = \left( \otimes_{i=1}^n \mathbb{R}^d \right) \times \Theta \), it is a Borel subset of \( \otimes_{i=1}^n \mathbb{R}^d \). Since \( \mathbb{R}^d \) is a Polish space, \( \mathbb{R}^d \times \ldots \times \mathbb{R}^d \) endowed with the product topology is a Polish space. \( D_x \) is \( \sigma \)-compact for any \( x \in \text{int}(D) \) since \( D_x \subseteq \Theta \) and \( \Theta \) is \( \sigma \)-compact.

Define \( f(x, \theta) = \bar{\mathcal{PW}}_{p,k}(\hat{\mu}_n(x), \mu_\theta) \), we claim that \( f \) is measurable on \( D \) and \( f(x, \cdot) \) is lower semi-continuous on \( D_x \). Indeed, we have shown that the mapping \( (\mu, \theta) \mapsto \bar{\mathcal{PW}}_{p,k}(\mu, \mu_\theta) \) is lower semi-continuous and thus measurable in \( \mathcal{P}(\mathbb{R}^d) \times \Theta \). The mapping \( x \mapsto \hat{\mu}_n(x) \) is measurable in \( \otimes_{i=1}^n \mathbb{R}^d \). Since the composition of measurable functions is measure, \( f \) is measurable on \( D \). Moreover, for any \( x \in \otimes_{i=1}^n \mathbb{R}^d \), \( f(x, \cdot) \) is lower semi-continuous on \( D_x \) since the mapping \( (\mu, \theta) \mapsto \bar{\mathcal{PW}}_{p,k}(\mu, \mu_\theta) \) is lower semi-continuous on \( D \). Putting these pieces together yields the desired results.
D.7 Proof of Theorem A.2

Using Assumption 3.2 and Lemma D.6, the mapping \((\nu, \theta) \mapsto E[\overline{W}_{p,k}(\nu, \hat{\mu}, m) \mid X_{1:n}]\) is lower semi-continuous in \(P(\mathbb{R}^d) \times \Theta\). Then the proof can be done similarly to the proof of Theorem A.1 using this result and Theorem D.5.

E Postponed Proofs in Subsection 3.4

In this section, we provide the detailed proofs for Theorem 3.12 and Theorem A.3. Our derivation is the refinement of the analysis in Bernton et al. [2019] for minimal Wasserstein estimators.

E.1 Preliminary technical results

To facilitate reading, we collect several preliminary technical results which will be used in the postponed proofs in subsection 3.4.

Let \((\mathcal{X}, \| \cdot \|_X)\) be a normed linear space and \(\theta \mapsto f_\theta\) be a map from a subset \(\Theta\) of \(\mathbb{R}^d\) into \(\mathcal{X}\). The statistical information comes from a sequence \(\{f_n\}_{n \in \mathbb{N}}\) of random elements of \(\mathcal{X}\), each of which is assumed to be measurable with respect to the \(\sigma\)-algebra generated by the balls in \(\mathcal{X}\). In some sense \(f_n\) should converge to \(f_\theta^*\) where \(\theta^*\) is some fixed (but unknown) point in the interior of \(\Theta\). To avoid the abuse of notation, we use \(K_1(x, \beta)\) here.

**Theorem E.1 (Theorem 4.2 in Pollard [1980])** Suppose the following assumptions hold:

1. \(\inf_{\theta \notin N} \|f_\theta - f_\theta^*\|_X > 0\) for every neighborhood \(N\) of \(\theta^*\).
2. \(\theta \mapsto f_\theta\) is norm differentiable with non-singular derivative \(D_{\theta^*}\) at \(\theta^*\).
3. There exists a random element \(G^* \in \mathcal{X}\) for which \(G_n := \sqrt{n}(f_n - f^*_{\theta_*}) \Rightarrow G^*\) in the sense for the metric induced by the norm \(\| \cdot \|_X\).

Then the limiting distribution of the goodness-of-fit statistic is given by

\[
\sqrt{n} \inf_{\theta \in \Theta} \|f_n - f_\theta\|_X \Rightarrow \inf_{\theta \in \Theta} \|G^* - \langle \theta, D_{\theta^*}\rangle\|_X.
\]

Let \(K_1(x, \beta) = \{\theta : \|x - \langle \theta, D_{\theta^*}\rangle\|_X \leq \inf_{\theta' \in \Theta} \|x - \langle \theta', D_{\theta^*}\rangle\|_X + \beta\}\) and \(M_n\) is defined by

\[
M_n = \left\{\theta \in \Theta : \|f_n - f_\theta\|_X \leq \inf_{\theta' \in \Theta} \|f_n - f_{\theta'}\|_X + \eta_n/\sqrt{n}\right\},
\]

where \(\eta_n > 0\) is any sequence such that \(P(\eta_n \to 0) = 1\) and \(M_n\) is nonempty.

**Theorem E.2 (Theorem 7.2 in Pollard [1980])** Under the conditions of Theorem E.1, there exists a sequence of real number \(\beta_n \downarrow 0\) satisfying

\[
P_*(M_n \subseteq \theta^* + n^{-1/2}K_1(G_n, \beta_n)) \to 1, \quad \text{as} \ n \to +\infty.
\]

Moreover, for any \(\epsilon > 0\), we have \(P(d_H(K_1(G_n^*, 0), K_1(G_n, \beta_n)) < \epsilon) \to 1\) as \(n \to +\infty\).
E.2 Proof of Theorem 3.12

First, we show that \( M_n \subseteq N_1 \) with (inner) probability approaching 1 as \( n \to +\infty \). Indeed, with inner probability approaching 1, we have

\[
\arg\min_{\theta \in \Theta} \overline{PW}_{1,1}(\hat{\mu}_n, \mu_{\theta}) \subseteq \arg\min_{\theta \in \Theta} \overline{PW}_{1,1}(\mu_{\ast}, \mu_{\theta}).
\]

By the definition of \( \overline{PW}_{1,1} \), we conclude that any minimizer of \( \| \hat{F}_n - F_{\theta} \|_L \) will be included in the set of minimizers of \( \| F_{\ast} - F_{\theta} \|_L \) with inner probability approaching 1. By Assumption 3.8, the minimizer of \( \| F_{\ast} - F_{\theta} \|_L \) is unique and \( N_1 \) is the neighborhood of this minimizer. Putting these pieces together yields that the set \( \inf_{\theta \in \Theta} \overline{PW}_{1,1}(\hat{\mu}_n, \mu_{\theta}) \) is contained in the set \( N_1 \) with (inner) probability approaching 1 as \( n \to +\infty \). By the definition of \( M_n \), we achieve the desired result.

Then we make three key claims. First, we claim that \( M_n \subseteq \Theta_n \) with (inner) probability approaching 1 as \( n \to +\infty \), where \( \Theta_n \) is defined by

\[
\Theta_n = \left\{ \theta \in \Theta : \| \theta - \theta_{\ast} \|_{\Theta} \leq \frac{4\sqrt{n}\| \hat{F}_n - F_{\ast} \|_L + 2\eta_n}{c_{\ast}\sqrt{n}} \right\}.
\]

Indeed, for any \( \theta \in \mathcal{N}_1 \), we derive from the triangle inequality that

\[
\| \hat{F}_n - F_{\theta} \|_L \leq \| \hat{F}_n - F_{\theta_{\ast}} \|_L \geq \| F_{\ast} - F_{\theta} \|_L \geq \| F_{\ast} - F_{\theta_{\ast}} \|_L - 2\| \hat{F}_n - F_{\ast} \|_L.
\]

Using the definition of \( \overline{PW}_{1,1} \) together with Assumption 3.8, we have

\[
\| \hat{F}_n - F_{\theta} \|_L \leq \| \hat{F}_n - F_{\theta_{\ast}} \|_L \geq c_{\ast}\| \theta - \theta_{\ast} \|_{\Theta} - 2\| \hat{F}_n - F_{\ast} \|_L.
\]  
(E.1)

Since \( M_n \subseteq \mathcal{N}_1 \) with (inner) probability approaching one, Eq. (E.1) holds true for any \( \theta \in M_n \) with (inner) probability approaching one. Moreover, by the definition of \( M_n \), we have \( \theta \in M_n \) satisfies

\[
\| \hat{F}_n - F_{\theta} \|_L \leq \inf_{\theta \in \Theta} \overline{PW}_{1,1}(\hat{\mu}_n, \mu_{\theta}) + \frac{\eta_n}{\sqrt{n}} \leq \| \hat{F}_n - F_{\theta_{\ast}} \|_L + \frac{\eta_n}{\sqrt{n}}
\]  
(E.2)

Combining Eq. (E.1), Eq. (E.2) and the definition of \( \Theta_n \), we conclude that \( \theta \in \Theta_n \) if \( \theta \in M_n \) with (inner) probability approaching 1. This completes the proof the first claim.

Second, we claim that \( \arg\min_{\theta \in \mathcal{N}_1} \| G_n - \left\{ \sqrt{n}(\theta' - \theta_{\ast}), D_{\theta_{\ast}} \right\} \|_L \subseteq \mathcal{N}_1 \cap \Theta_n \) with (inner) probability approaching 1 as \( n \to +\infty \). Indeed, by the definition of \( G_n \), we have

\[
\| G_n - \left\{ \sqrt{n}(\theta' - \theta_{\ast}), D_{\theta_{\ast}} \right\} \|_L = \sqrt{n}\| \hat{F}_n - F_{\theta_{\ast}} - \left\{ \theta - \theta_{\ast}, D_{\theta_{\ast}} \right\} \|_L.
\]

For the simplicity of notation, we let \( R_{\theta} = F_{\theta} - F_{\theta_{\ast}} - \left\{ \theta - \theta_{\ast}, D_{\theta_{\ast}} \right\} \). By Assumption 3.6, we have \( \| R_{\theta} \|_L \leq \sigma(\| \theta - \theta_{\ast} \|_{\Theta}) \). By the definition of \( \mathcal{N}_1 \), we have \( \| R_{\theta} \|_L \leq (1/2)c_{\ast}\| \theta - \theta_{\ast} \|_{\Theta} \). Therefore, for any \( \theta \in \mathcal{N}_1 \), we have

\[
\| \hat{F}_n - F_{\theta_{\ast}} - \left\{ \theta - \theta_{\ast}, D_{\theta_{\ast}} \right\} \|_L \geq \| \hat{F}_n - F_{\theta} \|_L - \| R_{\theta} \|_L
\]  
Eq. (E.1)

\[
\geq \| \hat{F}_n - F_{\theta_{\ast}} \|_L + (1/2)c_{\ast}\| \theta - \theta_{\ast} \|_{\Theta} - 2\| \hat{F}_n - F_{\ast} \|_L.
\]

This implies that, for any \( \theta \in \mathcal{N}_1 \setminus \Theta_n \), we have

\[
\| \hat{F}_n - F_{\theta_{\ast}} - \left\{ \theta - \theta_{\ast}, D_{\theta_{\ast}} \right\} \|_L \geq \| \hat{F}_n - F_{\theta_{\ast}} \|_L \geq \inf_{\theta \in \mathcal{N}_1 \cap \Theta_n} \| \hat{F}_n - F_{\theta_{\ast}} - \left\{ \theta' - \theta_{\ast}, D_{\theta_{\ast}} \right\} \|_L.
\]

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This completes the proof of the second claim.

Thirdly, we claim that there is an uniform control over the difference between \(\theta \mapsto \sqrt{n} ||\hat{F}_n - F_\theta||_L\) and the convex map \(\theta \mapsto \|G_n - \sqrt{n}(\theta - \theta_\star, D_{\theta_\star})\|_L\) over the set \(\Omega_n\) with (inner) probability approaching 1 as \(n \to +\infty\). Indeed, we define

\[
\Gamma_n = \sup_{\theta \in \Omega_n} |\sqrt{n}||\hat{F}_n - F_\theta||_L - \|G_n - \sqrt{n}(\theta - \theta_\star, D_{\theta_\star})\|_L|.
\]

By the definition of \(G_n\), we have

\[
\Gamma_n = \sup_{\theta \in \Omega_n} |\sqrt{n}||\hat{F}_n - F_\theta - (\theta - \theta_\star, D_{\theta_\star}) - R_\theta||_L - \sqrt{n}||\hat{F}_n - F_\theta - (\theta - \theta_\star, D_{\theta_\star})||_L| = o \left( \sup_{\theta \in \Omega_n} \sqrt{n}||\theta - \theta_\star||_\Theta \right) = o(\sqrt{n}||\hat{F}_n - F_\star||_L).
\]

By Assumption 3.7, we have \(\Gamma_n \to 0\) as \(\|\theta - \theta_\star\|_\Theta \to 0\) with (inner) probability approaching 1 as \(n \to +\infty\). This completes the proof of the third claim.

By the definition of \(G_n\) and \(G_\star\), we have \(\|G_n - G_\star\|_L = \|\sqrt{n}(\hat{F}_n - F_\star) - G_\star\|_L\). By Assumption 3.7, there exists a sequence \(\tau_n^1 \to 0\) such that \(\mathbb{P}(\|G_n - G_\star\|_L > \tau_n^1) \to 0\). By the definition of \(\Gamma_n\) and \(\eta_n\), there exists two sequences \(\tau_n^2 \to 0\) and \(\tau_n^3 \to 0\) such that \(\mathbb{P}(\Gamma_n > \tau_n^2) \to 0\) and \(\mathbb{P}(\eta_n > \tau_n^3) \to 0\).

Let \(\beta_n = \max\{2\tau_n^2, 2\tau_n^2 + \tau_n^3\}\), we have \(\beta_n \to 0\) with (inner) probability approaching 1 as \(n \to +\infty\). It remains to show that \(M_n \subseteq K(G_n, \beta_n)\) with (inner) probability approaching 1 as \(n \to +\infty\). Indeed, we have

\[
\inf_{\theta' \in X_1} \|G_n - \langle \sqrt{n}(\theta' - \theta_\star), D_{\theta_\star}\rangle\|_L \geq \inf_{\theta' \in X_1} \sqrt{n}||\hat{F}_n - F_{\theta'}||_L - \tau_n^2.
\]

By the definition of \(M_n\), let \(\theta \in M_n\), the above inequality implies

\[
\inf_{\theta' \in X_1} \|G_n - \langle \sqrt{n}(\theta' - \theta_\star), D_{\theta_\star}\rangle\|_L \geq \sqrt{n}||\hat{F}_n - F_\theta||_L - \tau_n^2 - \tau_n^3.
\]

Since \(M_n \subseteq \Theta_n\) with (inner) probability approaching 1 as \(n \to +\infty\), we have

\[
\sqrt{n}||\hat{F}_n - F_\theta||_L \geq \|G_n - \langle \sqrt{n}(\theta - \theta_\star), D_{\theta_\star}\rangle\|_L - \tau_n^2.
\]

Putting these pieces together with \(\beta_n \geq 2\tau_n^2 + \tau_n^3\) yields that \(\theta \in K(G_n, \beta_n)\).

Finally, let \(\epsilon > 0\), we prove that \(\mathbb{P}(d_H(K(G_n, 0), K(G_n, \beta_n)) < \epsilon) \to 1\) as \(n \to +\infty\). Indeed, by the triangle inequality, \(\theta \in K(G_n, 0)\) implies \(\theta \in K(G_n, 2\|G_n - G_\star\|_L)\). Therefore, we conclude that \(K(G_n, 0) \subseteq K(G_n, \beta_n)\) with (inner) probability approaching one as \(n \to +\infty\). On the other hand, \(\theta \in K(G_n, \beta_n)\) implies \(\theta \in K(G_n, \beta_n + 2\|G_n - G_\star\|_L)\). By the definition of \(\beta_n\), \(G_n\) and \(G_\star\), we obtain that \(\beta_n + 2\|G_n - G_\star\|_L \to 0\) with (inner) probability approaching one as \(n \to +\infty\). By the definition of the Hausdorff metric, we conclude the desired result.

**E.3 Proof of Theorem A.3**

Different from Theorem 3.12, the proof of Theorem A.3 is relatively straightforward and based on Theorem E.1 and E.2. It is mostly because there exists \(\theta_\star\) in the interior of \(\Theta\) such that \(F_\star = F_{\theta_\star}\).

More specifically, we consider \(f_\theta = F_\theta\) and \(f_n = \hat{F}_n\) such that

\[
F_\theta(u, t) = \int_{\mathbb{R}^d} 1_{(-\infty, t]}(\langle u, x \rangle) \, d\mu_\theta(x), \quad \hat{F}_n(u, t) = (1/n)\{i \in [n] : \langle u, X_i \rangle \leq t\}.
\]

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Let \( \mathcal{X} = L(\mathbb{S}^{d-1} \times \mathbb{R}) \) and \( \| \cdot \|_X = \| \cdot \|_L \), we can check that \( (\mathcal{X}, \| \cdot \|_X) \) is a normed linear space. By the definition of \( \overline{PW}_{1,1} \), we have \( \overline{PW}_{1,1}(\hat{\mu}_n, \mu_\theta) = \| \hat{F}_n - F_\theta \|_X \). By Assumption 3.1, \( \hat{F}_n \) converges to \( F_* \). Moreover, in well-specified setting, \( F_* = F_{\theta_*} \) where \( \theta_* \) is some fixed (but unknown) point in the interior of \( \Theta \). Now we are ready to check the conditions of Theorem E.1.

First, Assumption A.1 and \( \overline{PW}_{1,1}(\hat{\mu}_n, \mu_\theta) = \| \hat{F}_n - F_\theta \|_X \) imply C1. Furthermore, by the definition of norm differentiable, Assumption 3.6 and Assumption A.2 imply C2. Finally, Assumption 3.7 and \( F_* = F_{\theta_*} \) imply C3. Therefore, we conclude from Theorem E.1 that

\[
\sqrt{n} \inf_{\theta \in \Theta} \overline{PW}_{1,1}(\hat{\mu}_n, \mu_\theta) = \sqrt{n} \inf_{\theta \in \Theta} \| \hat{F}_n - F_\theta \|_L \Rightarrow \inf_{t \in \Theta} \| G_* - \langle t, D_{\theta_*} \rangle \|_L.
\]

in the sense for the metric induced by the norm \( \| \cdot \|_L \). This together with the definition of the norm \( \| \cdot \|_L \) implies the desired result for the goodness-of-fit statistics.

On the other hand, Theorem E.2 can be applied with specific choice of \( \eta_n \). More specifically, we notice that the estimator \( \hat{\theta}_n \) is well defined by

\[
\hat{\theta}_n := \arg\min_{\theta \in \Theta} \overline{PW}_{1,1}(\hat{\mu}_n, \mu_\theta) = \arg\min_{\theta \in \Theta} \| \hat{F}_n - F_\theta \|_L.
\]

Let \( \eta_0 = 0 \), the set \( M_n = \{ \hat{\theta}_n \} \) is a singleton set. This implies that \( \sqrt{n}(\hat{\theta}_n - \theta_*) \Rightarrow K_1(G_*, 0) \) as \( n \to +\infty \) under its Hausdorff metric topology. Since the random map \( \theta \to \max_{u \in \mathbb{S}^{d-1}} \int_{\mathbb{R}} |G_*(u, t) - \langle \theta, D_{\theta_*}(u, t) \rangle| \, dt \) has a unique infimum almost surely, we have \( K_1(G_*, 0) \) is a singleton set defined by

\[
K_1(G_*, 0) = \arg\min_{\theta \in \Theta} \max_{u \in \mathbb{S}^{d-1}} \int_{\mathbb{R}} |G_*(u, t) - \langle \theta, D_{\theta_*}(u, t) \rangle| \, dt.
\]

In this case, the Hausdorff metric is simply induced by the norm \( \| \cdot \|_L \). Putting these pieces together yields the desired result for the MPRW estimator of order 1.

### E.4 Minor Technical Issues

We use the notations of Bernton et al. [2019, Theorem B.8] throughout this subsection. Indeed, in page 38-39 of the recent arxiv version of Bernton et al. [2019], the authors prove that \( m(H_n) = \inf_{u \in L_n} f(H_n, u) \), implicitly assuming that the minimizer of the map \( \theta \mapsto \sqrt{n}\|F_n - F_{\theta_*} - \langle \theta - \theta_*, D_{\theta_*} \rangle\|_L \) is contained in the set \( N_1 = \{ \theta \in \mathcal{N} : \| \theta - \theta_* \|_L \leq c_n / 2 \} \). However, this result is not obvious. Indeed, it seems difficult to derive such results from the existing fact that the minimizer of \( \theta \mapsto \sqrt{n}\|F_n - F_{\theta_*} - \langle \theta - \theta_*, D_{\theta_*} \rangle\|_L \) is contained in \( \mathcal{N} \). We only have the uniform control over the difference between \( \theta \mapsto \sqrt{n}\|F_n - F_{\theta_*} - \langle \theta - \theta_*, D_{\theta_*} \rangle\|_L \) and \( \theta \mapsto \sqrt{n}\|F_n - F_{\theta_*} - \langle \theta - \theta_*, D_{\theta_*} \rangle\|_L \) over the set \( S_n \) instead of the whole set. So there is few relationship between the minimizers of these two mappings. Moreover, the techniques from the proof of Pollard [1980, Theorem 7.2] can not be applicable to fix this issue here since the proof depends on the assumption that \( \mu_* = \mu_{\theta_*} \) which does not hold under model misspecification yet.

### F Computational Aspects

The computation of the PRW distance is in general computationally intractable when the projection dimension is \( k \geq 2 \) since this amounts to solving a nonconvex max-min optimization model. Despite several pessimistic results [Paty and Cuturi, 2019, Niles-Weed and Rigollet, 2019], we adopt the Riemannian optimization toolbox [Absil et al., 2009] to develop a Riemannian supergradient algorithm and
empirically show that our algorithm can approximate $\mathcal{PW}_{2,k}(\hat{\mu}_n, \hat{\nu}_n)$ when the projection dimension is $k \geq 2$. Part of results can be found in the appendix of concurrent work [Lin et al., 2020] and we provide the details for the sake of completeness.

**Approximation of $\mathcal{PW}_{2,k}$.** We consider the computation of $\mathcal{PW}_{2,k}$ between empirical measures. Indeed, let $\{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{R}^d$ and $\{y_1, y_2, \ldots, y_n\} \subseteq \mathbb{R}^d$ denote sets of $n$ atoms, and let $(r_1, r_2, \ldots, r_n) \in \Delta^n$ and $(c_1, c_2, \ldots, c_n) \in \Delta^n$ denote weight vectors, we define discrete measures $\hat{\mu}_n \equiv \sum_{i=1}^{n} r_i \delta_{x_i}$ and $\hat{\nu}_n \equiv \sum_{j=1}^{n} c_j \delta_{y_j}$. The computation of $\mathcal{PW}_{2,k}(\hat{\mu}_n, \hat{\nu}_n)$ is equivalent to solving a structured max-min optimization model where the maximization and minimization are performed over the Stiefel manifold $\text{St}(d,k) \equiv \{ U \in \mathbb{R}^{d \times k} \mid U^T U = I_k \}$ and the transportation polytope $\Pi(\mu, \nu) \equiv \{ \pi \in \mathbb{R}^{n \times n}_+ \mid r(\pi) = r, c(\pi) = c \}$ respectively. Formally, we have

$$\max_{U \in \text{St}(d,k)} \min_{\pi \in \Pi(\mu, \nu)} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{i,j} \|U^T x_i - U^T y_j\|^2 \quad \text{s.t.} \quad U^T U = I_k, \ r(\pi) = r, \ c(\pi) = c. \quad (F.1)$$

Eq. (F.1) is equivalent to the non-convex nonsmooth optimization model as follows,

$$\max_{U \in \text{St}(d,k)} \left\{ f(U) \equiv \min_{\pi \in \Pi(\mu, \nu)} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{i,j} \|U^T x_i - U^T y_j\|^2 \right\}. \quad (F.2)$$

Fixing $U \in \text{St}(d,k)$, Eq. (F.2) becomes a classical OT problem which can be either solved by the Sinkhorn iteration [Cuturi, 2013] or the variant of network simplex method in the POT package [Flamary and Courty, 2017]. The key challenge is the maximization over the Stiefel manifold $\text{St}(d,k) \equiv \{ U \in \mathbb{R}^{d \times k} \mid U^T U = I_k \}$. Eq. (F.2) is a special instance of the Stiefel manifold optimization problem. The dimension of $\text{St}(d,k)$ is equal to $dk - k(k+1)/2$ and the tangent space at the point $Z \in \text{St}(d,k)$ is defined by $T_Z \text{St} := \{ \xi \in \mathbb{R}^{d \times k} : \xi^T Z + Z^T \xi = 0 \}$. We endow $\text{St}(d,k)$ with Riemannian metric inherited from the Euclidean inner product $\langle X, Y \rangle$ for any $X, Y \in T_Z \text{St}$ and $Z \in \text{St}(d,k)$. Then the projection of $G \in \mathbb{R}^{d \times k}$ onto $T_Z \text{St}$ is given by Absil et al. [2009, Example 3.6.2]: $\text{Pr}_{T_Z \text{St}}(G) = G - Z(G^T Z + Z^T G)/2$.

We make use of the notion of a *retraction*, which is the first-order approximation of an exponential mapping on the manifold and which is amenable to computation [Absil et al., 2009, Definition 4.1.1]. For the Stiefel manifold, we have the following definition:

**Definition F.1** A retraction on $\text{St} \equiv \text{St}(d,k)$ is a smooth mapping $\text{Retr} : T\text{St} \to \text{St}$ from the tangent bundle $T\text{St}$ onto $\text{St}$ such that the restriction of $\text{Retr}$ onto $T_Z \text{St}$, denoted by $\text{Retr}_Z$, satisfies that (i) $\text{Retr}_Z(0) = Z$ for all $Z \in \text{St}$ where 0 denotes the zero element of $T\text{St}$, and (ii) for any $Z \in \text{St}$, it holds that $\lim_{\xi \in T_Z \text{St}, \xi \to 0} \|\text{Retr}_Z(\xi) - (Z + \xi)\|_F/\|\xi\|_F = 0$.

Our algorithm uses the retraction based on the QR decomposition as suggested by Liu et al. [2019]. More specifically, $\text{Retr}_Z^Q(\xi) = qr(Z + \xi)$ where $qr(A)$ is the Q factor of the QR factorization of $A$.

We start with a brief overview of the Riemannian supergradient ascent algorithm for nonsmooth Stiefel optimization, denoted by $\max_{U \in \text{St}(d,k)} F(U)$. A generic Riemannian supergradient ascent algorithm for solving this problem is given by

$$U_{t+1} \leftarrow \text{Retr}_{U_t}(\gamma_{t+1} \xi_{t+1}) \quad \text{for any } \xi_{t+1} \in \text{subdiff} F(U_t),$$

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To this end, we summarize the pseudocode of the RSGAN algorithm in Algorithm 1.

Algorithm 1 Riemannian SuperGradient Ascent with Network Simplex Iteration (RSGAN)

1: **Input:** measures \( \{(x_i, r_i)\}_i \in [n] \) and \( \{(y_j, c_j)\}_j \in [n] \), dimension \( k \) and tolerance \( \epsilon \).
2: **Initialize:** \( U_0 \in St(d, k) \) and \( \gamma_0 > 0 \).
3: for \( t = 0, 1, 2, \ldots, T - 1 \) do
   4: Compute \( \pi_{t+1} \leftarrow OT(\{(x_i, r_i)\}_i \in [n], \{(y_j, c_j)\}_j \in [n], U_t) \).
   5: Compute \( \xi_{t+1} \leftarrow P_{U_{t}S_{k}}(2V_{\pi_{t+1}}U_{t}) \).
   6: Compute \( \gamma_{t+1} \leftarrow \gamma_0/\sqrt{t+1} \).
   7: Compute \( U_{t+1} \leftarrow Retr_{U_{t}}(\gamma_{t+1} \xi_{t+1}) \).
8: end for

where subdiff \( F(U_t) \) is Riemannian subdifferential of \( F \) at \( U_t \) and Retr is any retraction on \( St(d, k) \). The step size is set as \( \gamma_{t+1} = \gamma_0/\sqrt{t+1} \) as suggested by [Li et al., 2019]. By the definition of Riemannian subdifferential, \( \xi_t \) can be obtained by taking \( \xi \in \partial F(U) \) and by setting \( \xi_t = P_{U_{t}S_{k}}(\xi) \). Thus, it is necessary for us to specify the subdifferential of \( f \) in Eq. (F.2). We define \( V_{\pi} = \sum_{i=1}^{\pi} \sum_{j=1}^{\pi} \pi_{i,j} (x_i - y_j)(x_i - y_j)^\top \in \mathbb{R}^{d \times d} \) which is symmetry and derive that

\[
\partial f(U) = \text{Conv}\{2V_{\pi}, U \mid \pi^* \in \text{argmin}_{\pi \in \Pi(\mu, \nu)} \langle UU^\top, V_{\pi} \rangle \}, \quad \text{for any } U \in \mathbb{R}^{d \times k}.
\]

It remains to solve an OT problem with a given \( U \) at each inner loop of the maximization and use the output \( \pi(U) \) to obtain a supergradient of \( f \). The network simplex method can exactly solve this LP. To this end, we summarize the pseudocode of the RSGAN algorithm in Algorithm 1.

Approximation of \( \mathcal{PW}_{2,k} \). We recall the definition of the IPRW distance of order 2 as follows,

\[
\mathcal{PW}_{2,k}(\mu, \nu) = \int_{S_{d,k}} W_{2}^{\#}(E_{\#}^\ast \mu, E_{\#}^\ast \nu) d\sigma(E),
\]

where \( \sigma \) is the uniform distribution on \( S_{d,k} \) and \( E^\ast \) is the linear transformation associated with \( E \) for any \( x \in \mathbb{R}^{d} \) by \( E^\ast(x) = E^\top x \). For any measurable function \( f \) and \( \mu \in \mathcal{P}(\mathbb{R}^{d}) \), we denote \( f_{\#} \mu \) as the push-forward of \( \mu \) by \( f \), so that \( f_{\#} \mu(A) = \mu(f^{-1}(A)) \) where \( f^{-1}(A) = \{ x \in \mathbb{R}^{d} : f(x) \in A \} \) for any Borel set \( A \). We approximate the integral by selecting a finite set of projections \( S \subseteq S_{d,k} \) and computing the empirical average:

\[
\mathcal{PW}_{2,k}^{\#}(\mu, \nu) \approx \frac{1}{\text{card}(S)} \sum_{E \in S} W_{2}^{\#}(E_{\#}^\ast \mu, E_{\#}^\ast \nu).
\]

The quality of this approximation depends on the sampling of \( S_{d,k} \). In this paper, we use random projections picked uniformly on \( S_{d,k} \), which is analogues to the approach proposed by Bonneel et al. [2015] for the case of \( k = 1 \); see Sampling schemes for the details.

Approximation of \( \mathcal{PW}_{p,1} \). We recall the definition of the PRW distance of order \( p \) with the projection dimension \( k = 1 \) as follows,

\[
\mathcal{PW}_{p,1}(\mu, \nu) := \sup_{u \in S_{d,1}} W_{p}^{\#}(u_{\#}^\ast \mu, u_{\#}^\ast \nu) = \sup_{u \in S_{d,1}} \int_{0}^{1} |F_{u_{\#}^\ast \mu}^{-1}(t) - F_{u_{\#}^\ast \nu}^{-1}(t)|^p dt.
\]
where $u \in \mathbb{S}_{d,1}$ is an unit $d$-dimensional vector, $u^*$ is the linear transformation associated with $u$ for any $x \in \mathbb{R}^d$ by $u^*(x) = u^\top x$, and $F_{\xi}^{-1}$ is the quantile function of $\xi$. This integral can be estimated using a Monte Carlo estimate and a linear interpolation of the quantile function. Following up Nadja [2019, Appendix 4], we consider two approximations of this quantity. The first one is given by,

$$\mathcal{PW}_{p,1}^p(\mu, \nu) = \sup_{u \in \mathbb{S}_{d,1}} \frac{1}{K} \sum_{k=1}^{K} |\hat{F}_{\mu,\nu}^{-1}(t_k) - \tilde{F}_{\mu,\nu}^{-1}(t_k)|^p,$$

where $\{t_k\}_{k=1}^{K}$ are uniform and independent samples from $[0, 1]$ and $\tilde{F}_{\xi}^{-1}$ is a linear interpolation of $F_{\xi}^{-1}$ which denotes either the exact quantile function of a discrete measure $\xi$, or an approximation by a Monte Carlo procedure. The second one is given by

$$\mathcal{PW}_{p,1}^p(\mu, \nu) = \sup_{u \in \mathbb{S}_{d,1}} \frac{1}{K} \sum_{k=1}^{K} |s_k - \tilde{F}_{\mu,\nu}^{-1}(\tilde{F}_{\mu,\nu}(s_k))|^p,$$

where $\{s_k\}_{k=1}^{K}$ are uniform and independent samples from $u^*\mu$ and $\tilde{F}_{\xi}$ (resp. $\tilde{F}_{\xi}^{-1}$) is a linear interpolation of $F_{\xi}$ (resp. $F_{\xi}^{-1}$) which denotes either the exact cumulative distribution function (resp. quantile function) of a discrete measure $\xi$, or an approximation by a Monte Carlo procedure.

**Sampling schemes.** We explain the methods that we use to generate the i.i.d. samples from the uniform distribution on the set of $d \times k$ orthogonal matrices, i.e., $\mathbb{S}_{d,k} = \{ E \in \mathbb{R}^{d \times k} : E^\top E = I_k \}$ and the i.i.d. samples from multivariate elliptically contoured stable distributions.

To sample from $\mathbb{S}_{d,k}$, we first construct the $(d \times k)$-dimensional matrix $Z$ by drawing each of its components from the standard normal distribution $\mathcal{N}(0, 1)$ and then perform the QR decomposition of it: $E = qr(Z)$. By the definition, $E \in \mathbb{S}_{d,k}$ is an uniform sample.

To sample from multivariate elliptically contoured stable distributions, we follow the approach presented in Nadja [2019, Appendix 4]. Indeed, we recall that if $Y \in \mathbb{R}^d$ is $\alpha$-stable and elliptically contoured, i.e., $Y \in \mathcal{E}_{\alpha}\mathcal{S}_c(\Sigma, \mathbf{m})$, then its joint characteristic function is defined as, for any $t \in \mathbb{R}^d$ that,

$$\mathbb{E} \left[ \exp(it^\top Y) \right] = \exp \left( -((t^\top \Sigma t)^{\alpha}/2) + it^\top \mathbf{m} \right),$$

where $\Sigma$ is a positive definite matrix (akin to a correlation matrix), $\mathbf{m} \in \mathbb{R}^d$ is a location vector (equal to the mean if it exists) and $\alpha \in (0, 2)$ controls the thickness of the tail. Elliptically contoured stable distributions are scale mixtures of multivariate elliptically contoured Gaussian distributions [Samoradnitsky, 2017, Proposition 2.5.2] with computationally intractable densities. Fortunately, it was shown by Nolan [2013] that sampling from multivariate elliptically contoured stable distributions is possible: let $A \sim \mathcal{S}_{\alpha/2}(\beta, \gamma, \delta)$ be a one-dimensional positive $(\alpha/2)$-stable random variable with $\beta = 1$, $\gamma = 2 \cos(\pi \alpha/4)^{2/\alpha}$ and $\delta = 0$, and $G \sim \mathcal{N}(0, \Sigma)$. By the definition, $Y = \sqrt{AG + \mathbf{m}}$ satisfies Eq. (F.5) and $Y \sim \mathcal{E}_{\alpha}\mathcal{S}_c(\Sigma, \mathbf{m})$.

**Optimization methods.** Computing the MPRW and MEPRW estimators are intractable in general. This is mainly because the PRW distance requires a maximization over infinitely many projections. Formally, we hope to solve the following minimax optimization model,

$$\min_{\theta \in \Theta} \mathcal{PW}_{p,1}^p(\mu_\theta, \mu_\star) = \min_{\theta \in \Theta} \max_{u \in \mathbb{S}_{d,1}} \int_0^1 |F_{u_\theta \mu_\theta}^{-1}(t) - F_{u_\star \mu_\star}^{-1}(t)|^p dt,$$
where \(\{\mu_\theta : \theta \in \Theta\}\) is the model and \(\mu_*\) is the data-generating process. Following up the approach presented in Nadjaï et al. [2019] together with the approximation of \(\overline{\text{PW}}_{p,1}\), we consider using the ADAM optimization method to minimize the (expected) PRW distance over the set of parameters while applying multiple projected supergradient ascent to find an approximate projection \(u\) which maximizes over \(S_{d,1}\) at each inner loop. The ADAM optimization method is associated with the default parameter setting as suggested by Kingma and Ba [2015]. At each inner loop, we run 5 projected supergradient ascent with the learning rate \(10^{-3}\).

**Gaussian models.** For the MPRW estimator, we consider the approximate \(\overline{\text{PW}}^2_{2,1}\) distance based on Eq. (F.4). Indeed, let \(\mu\) denote \(\mathcal{N}(\mathbf{m}, \sigma^2 \mathbf{I})\) and \(\hat{\nu}\) denote the empirical probability measures of \(n\) samples drawn from the data-generating process, we define the function \(f_1(\mathbf{m}, \sigma^2, u)\) as

\[
f_1(\mathbf{m}, \sigma^2, u) = \frac{1}{\text{card}(S)} \sum_{s \in S} |s - \hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(s))|^2 \mathcal{N}(s; u^\top \mathbf{m}, \sigma^2 \mathbf{I}),
\]

where \(S \subseteq \mathbb{R}\) and \(\mathcal{N}(s; u^\top \mathbf{m}, \sigma^2 \mathbf{I})\) refers to the density function of Gaussian of parameters \((u^\top \mathbf{m}, \sigma^2 \mathbf{I})\) evaluated at \(s \in S\). We compute the explicit gradient expression of \(f_1(\mathbf{m}, \sigma^2, u)\) with respect to the mean \(\mathbf{m}\), the variance \(\sigma^2\) and the projection vector \(u\) as follows,

\[
\nabla_\mathbf{m} f_1(\mathbf{m}, \sigma^2, u) = \frac{1}{\sigma^2 \text{card}(S)} \sum_{s \in S} \left( |s - \hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(s))|^2 \mathcal{N}(s; u^\top \mathbf{m}, \sigma^2 \mathbf{I})(s - u^\top \mathbf{m})u \right),
\]

\[
\nabla_{\sigma^2} f_1(\mathbf{m}, \sigma^2, u) = \frac{1}{2\sigma^4 \text{card}(S)} \sum_{s \in S} \left( |s - \hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(s))|^2 \mathcal{N}(s; u^\top \mathbf{m}, \sigma^2 \mathbf{I})((s - u^\top \mathbf{m})^2 - \sigma^2) \right),
\]

\[
\nabla_u f_1(\mathbf{m}, \sigma^2, u) = \frac{1}{\sigma^2 \text{card}(S)} \sum_{s \in S} \left( |s - \hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(s))|^2 \mathcal{N}(s; u^\top \mathbf{m}, \sigma^2 \mathbf{I})(s - u^\top \mathbf{m}) \mathbf{m} \right).
\]

For the MEPRW estimator, we consider the approximate \(\overline{\text{PW}}^2_{2,1}\) distance based on Eq. (F.3). Indeed, let \(\hat{\mu}\) and \(\hat{\nu}\) denote the empirical probability measures of \(m\) samples drawn from \(\mathcal{N}(\mathbf{m}, \sigma^2 \mathbf{I})\) and \(n\) samples drawn from the data-generating process, we define the function \(f_2(\mathbf{m}, \sigma^2, u)\) as

\[
f_2(\mathbf{m}, \sigma^2, u) = \frac{1}{K} \sum_{k=1}^{K} |\hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(t_k)) - \hat{F}_{\mu}^{-1}(\hat{F}_{\nu}^* \nu(t_k))|^2,
\]

where \(\{t_k\}_{k=1}^{K}\) are uniform and independent samples from \([0, 1]\). We compute the explicit gradient expression of \(f_2(\mathbf{m}, \sigma^2, u)\) with respect to the mean \(\mathbf{m}\), the variance \(\sigma^2\) and the projection vector \(u\) as follows,

\[
\nabla_\mathbf{m} f_2(\mathbf{m}, \sigma^2, u) = -\frac{2}{K} \sum_{k=1}^{K} |\hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(t_k)) - \hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \nu(t_k))|u,
\]

\[
\nabla_{\sigma^2} f_2(\mathbf{m}, \sigma^2, u) = -\frac{2}{K} \sum_{k=1}^{K} |\hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(t_k)) - \hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \nu(t_k))| \mathbf{m},
\]

\[
\nabla_u f_2(\mathbf{m}, \sigma^2, u) = -\frac{1}{\sigma^2 K} \sum_{k=1}^{K} \left( |\hat{F}_{\mu}^{-1}(\hat{F}_{\mu}^* \mu(t_k)) - \hat{F}_{\mu}^{-1}(\hat{F}_{\nu}^* \nu(t_k))| (u^\top \mathbf{m} - \hat{F}_{\mu}^{-1}(\hat{F}_{\nu}^* \nu(t_k))) \right).
\]
Figure 5: Mean values (Top) and mean computational time (Bottom) of the IPRW and PRW distances of order 2 between empirical measures \( \hat{\mu}_n \) and \( \hat{\nu}_n \) as the number of points \( n \) varies. Results are averaged over 100 runs.

*Elliptically contoured stable models.* When comparing the MEPRW estimator with the MPRW estimator using elliptically contoured stable models, we also approximate these estimators using the ADAM optimization method with the default parameter setting.

We consider the approximate \( \bar{PW}_{2,1}^2 \) distance based on Eq. (F.3). Indeed, let \( \tilde{\mu} \) and \( \tilde{\nu} \) denote the empirical probability measures of \( m \) samples drawn from \( \mathcal{E}\alpha\mathcal{S}_\gamma(I, m) \) and \( n \) samples drawn from the data-generating process, we define the function \( f_3(\mathbf{m}, \mathbf{u}) \) as

\[
    f_3(\mathbf{m}, \mathbf{u}) = \frac{1}{K} \sum_{k=1}^{K} | \tilde{F}_{u_y\tilde{\mu}}^{-1}(t_k) - \tilde{F}_{u_y\tilde{\nu}}^{-1}(t_k) |^2,
\]

where \( \{t_k\}_{k=1}^K \) are uniform and independent samples from \([0,1]\). We compute the explicit gradient expression of \( f(\mathbf{m}, \mathbf{u}) \) with respect to the location parameter \( \mathbf{m} \) and the projection vector \( \mathbf{u} \) as follows,

\[
    \nabla_\mathbf{m} f_3(\mathbf{m}, \mathbf{u}) = -\frac{2}{K} \sum_{k=1}^{K} | \tilde{F}_{u_y\tilde{\mu}}^{-1}(t_k) - \tilde{F}_{u_y\tilde{\nu}}^{-1}(t_k) | \mathbf{u},
\]

\[
    \nabla_\mathbf{u} f_3(\mathbf{m}, \mathbf{u}) = -\frac{2}{K} \sum_{k=1}^{K} | \tilde{F}_{u_y\tilde{\mu}}^{-1}(t_k) - \tilde{F}_{u_y\tilde{\nu}}^{-1}(t_k) | \mathbf{m}.
\]

*Generative modeling.* We use the ADAM optimizer provided Pytorch GPU.

**G Experimental Setup**

*Computing infrastructure.* For the experiments on the uniform distribution over hypercube, we implement in Python 3.7 with Numpy 1.18 on a workstation with an Intel Core i5-9400F (6 cores and 6
Convergence and concentration. We conduct the experiment on the uniform distribution over different hypercubes which are also used in the experiment [Paty and Cuturi, 2019]. In particular, we consider \( \mu = \nu = U([-v, v]^d) \) which is an uniform distribution over an hypercube and where \( d \) and \( v \) stand for the dimension and scale of the distribution respectively. \( \hat{\mu}_n \) and \( \hat{\nu}_n \) are empirical distributions corresponding to \( \mu \) and \( \nu \) with \( n \) samples. We evaluate the PRW and IPRW distance in terms of mean values and mean computational times over 100 runs for \((d, v) \in \{(10, 1), (10, 3), (30, 1), (30, 5), (50, 1), (50, 5)\} \). For the PRW distance, we run Algorithm 1 with emd solver in the POT package [Flamary and Courty, 2017] and terminate the algorithm either when the maximum number of iterations \( T = 30 \) is reached or when \( \|U_{t+1} - U_t\|_F \leq 10^{-6} \). For the IPRW distance, we draw 100 uniform and independent projections from \( S_{d,k} \) and compute each Wasserstein distances using emd solver in the POT package again.

Model misspecification. We conduct the experiments on three type of data: the mixture of 8, 12 and 25 Gaussian distributions with Gaussian models \( \mathcal{M}_1 = \{ \mathcal{N}(\mathbf{m}, \sigma^2 \mathbf{I}) : \mathbf{m} \in \mathbb{R}^2, \sigma^2 > 0 \} \) and elliptically contoured stable models \( \mathcal{M}_2 = \{ \mathcal{E}_\alpha S_c(\mathbf{I}, \mathbf{m}) : \mathbf{m} \in \mathbb{R}^2 \} \). For data-generating process, we fix \( k \) centers \( \{(a_i, b_i)\}_{1 \leq i \leq k} \). For each sample, we first randomly select \( \mathbf{m} \) from the centers at uniform and then draw the sample from \( \mathcal{N}(2\mathbf{m}, 0.01) \). For the mixture of 8 and 12 Gaussian distributions, the fixed set of centers are evenly distributed around a unit circle. For the mixture of 25 Gaussian distributions, the fixed set of centers are 25 grid points in \([-2, 2]^2\).

We use the ADAM optimization method with the default parameter setting to compute the MPRW and MEPRW estimators. At each inner loop, we run 5 projected supergradient ascent with the learning rate \( 10^{-3} \). For the Gaussian models, we estimate the densities of \( \hat{\sigma}^2_n \) with a kernel density estimator by computing 100 times MPRW estimator of order 1. The maximum number of ADAM iterations is set as 20000. To illustrate the consistency of MPRW and MEPRW estimators, we compute 100 times MPRW and MEPRW estimators of order 2, where the maximum number of ADAM iterations are set
as 20000 and 10000 respectively. We also verify the convergence of MEPRW to MPRW by computing 100 times these estimators on a fixed set of $n = 2000$ observations for different $m$ generated samples from the model. The maximum number of ADAM iterations for MPRW and MEPRW estimators are set as 20000 and 10000. For the elliptically contoured stable models, we verify the consistency property of MEPRW and the convergence of MEPRW to MPRW. For the former one, we compute 100 times MEPRW estimator of order 2 and set the maximum number of ADAM iterations as 10000. For the latter one, we compute 100 times MPRW and MEPRW estimators of order 2 on a fixed set of $n = 100$ observations for different $m$ generated samples from the model. The maximum number of ADAM iterations are set as 20000 and 10000. All of these settings are consistently used on the mixture of 8, 12 and 25 Gaussian distributions.

**Generative modeling.** The procedure of the max-SW generator is summarized as follows: we first sample a random variable $Z$ from a fixed distribution on the base space $\mathcal{Z}$, and then transforms $Z$ through a neural network parametrized by $\theta$. This provides a parametric function $T_\theta : \mathcal{Z} \to \mathbb{R}^d$ which allows us to generate images from a distribution $\mu_\theta$. Our goal is to optimize the neural network parameters $\theta$ by minimizing the max-SW distance [Deshpande et al., 2019] between $\mu_\theta$ and data-generating distribution. We use a neural network with the fully-connected configuration from Deshpande et al. [2018, Appendix D] and train our model with CIFAR10$^7$ and ImageNet200$^8$. The former one consists of 60000 and 10000 images of size $3 \times 32 \times 32$ for training and testing while the latter one consists of 100000 and 10000 images for training and testing. We use the minimal expected max-SW estimator of order 2 approximated with 50 projected gradient ascent steps and $10^{-4}$ learning rate. We train for 1000 iterations with the ADAM optimizer [Kingma and Ba, 2015] and $10^{-4}$ learning rate.

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7Available in https://www.cs.toronto.edu/~kriz/cifar.html
8Available in https://tiny-imagenet.herokuapp.com/
H Additional Experimental Results

Convergence and concentration. Figure 5 presents average distances and computational times for \((d, v) \in \{(10, 5), (30, 1), (50, 1)\}\), where the shaded areas show the max-min values over 100 runs. We also observe that the IPRW distance is smaller than the PRW distance for small \(n\), especially so when \(d\) and \(v\) are large. The two distances are close when \(n\) is large, supporting the theoretical results given by Theorem 3.4 and Theorem 3.6 in practice. The computation of the PRW distance is relatively faster than that of the IPRW distance in these computations.

Model misspecification: Gaussian models. Figure 6 shows the distributions centered and rescaled by \(\sqrt{n}\) for a range of moderately large \(n\), based on the two underlying models including the mixture of 12 Gaussian distributions and the mixture of 25 Gaussian distributions. The left figure supports the convergence rate and the limiting distribution of the estimator as derived in Theorem 3.12 on the mixture of 12 Gaussian distributions. The right figure suggests that the limiting distribution is not normal when the underlying model is given by the mixture of 25 Gaussian distributions. For the latter case, the result is not as anticipated by Theorem 3.12. This is possibly because we only conduct 5 projected supergradient ascent at each inner loop, which may not be enough to achieve a good approximate projection \(u \in S_{d,1}\).

Figure 7 and 8 demonstrate the large-sample consistency behavior of MPRW and MEPRW estimators on the mixture of 12 and 25 Gaussian distributions, which are expected since Assumption 3.1-3.3 are mild. The MEPRW estimator also converges to the MPRW estimator on the mixture of 12 Gaussian distributions, confirming Theorem 3.11. One exception in these experiments is the failure of convergence of MEPRW to MPRW on the mixture of 25 Gaussian distributions. Apparently, the results from Theorem 3.11 do not hold in this experiment setting. This is likely due to the violation of Assumption 3.5 that is necessary for Theorem 3.11 to hold.
Figure 9: Minimal expected PRW estimations using elliptically contoured stable models and $n$ samples from the mixture of 8 Gaussian distributions (top), 12 Gaussian distributions (middle) and 25 Gaussian distributions (bottom), and $m$ samples generated from the model. Results are averaged over 100 runs and shaded areas represent standard deviation.

Model misspecification: Elliptically contoured stable models. Figure 9 (a) illustrates the consistency of the MEPRW estimator $\hat{m}_{n,m}$, approximated with 5 projected supergradient ascent, the same way as for the Gaussian models. Figure 9 (b) confirms the convergence of $\hat{m}_{n,m}$ to the MPRW estimator $\hat{m}$, where we fix $n = 100$ observations and compute the mean squared error between these two estimators (using 5 projected supergradient ascent) for different values of $m$. Note that the MPRW estimator is approximated with the MEPRW obtained for a large enough value of $m$: $\hat{m} = \hat{m}_{n,10^4}$. To this end, our results on elliptically contoured stable models confirm Theorem 3.9, Theorem 3.10 and Theorem 3.11 in practice.
Generative modeling. Figure 10 presents the mean test loss on CIFAR10 over 10 runs, where the shaded areas show the max-min values over the runs. Here the minimal expected max-SW estimator of order 2 is approximated with 20 projected gradient ascent steps and $10^{-4}$ learning rate. We trained for 1000 iterations with the ADAM optimizer [Kingma and Ba, 2015] and $10^{-4}$ learning rate. We also train the NNs with $(n, m) \in \{(100, 20), (1000, 40), (5000, 60), (10000, 100)\}$ where $n$ is the number of training samples and $m$ is the number of generated samples and compute the testing losses using the trained models on the testing dataset ($n = 10000$) with $m = 250$ generated samples. We compare these testing losses to that of a NN trained using $n = 60000$ (i.e., the entire training dataset) and $m = 200$ and present them in Figure 10. Again, our results confirm Theorem 3.10 in practice.