Delta Invariants of Singular del Pezzo Surfaces

Ivan Cheltsov\textsuperscript{1,2} · Jihun Park\textsuperscript{3,4} · Constantin Shramov\textsuperscript{1,5}

Received: 27 May 2019  © The Author(s) 2020

Abstract
We use the methods introduced by Cheltsov–Rubinstein–Zhang (Sel Math (N.S.) 25(2):25–34, 2019) to estimate $\delta$-invariants of the seven singular del Pezzo surfaces with quotient singularities studied by Cheltsov–Park–Shramov (J Geom Anal 20(4):787–816, 2010) that have $\alpha$-invariants less than $\frac{2}{3}$. As a result, we verify that each of these surfaces admits an orbifold Kähler–Einstein metric.

Keywords  Delta invariant · del Pezzo surfaces · K-stability · Kähler–Einstein metric · Quotient singularities

Mathematics Subject Classification  14J17 · 14J45 · 32Q20

All varieties are assumed to be complex, projective and normal unless otherwise stated.

Ivan Cheltsov
I.Cheltsov@ed.ac.uk

Jihun Park
wlog@postech.ac.kr

Constantin Shramov
costya.shramov@gmail.com

1 School of Mathematics, The University of Edinburgh, Edinburgh EH9 3JZ, UK
2 Laboratory of Algebraic Geometry, National Research University Higher School of Economics, 6 Usacheva street, Moscow, Russia 117312
3 Center for Geometry and Physics, Institute for Basic Science, 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk 37673, Korea
4 Department of Mathematics, POSTECH, 77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk 37673, Korea
5 Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina street, Moscow, Russia 119991

Published online: 30 January 2020
1 Introduction

Let $S_d$ be a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $d$, where $a_0 \leq a_1 \leq a_2 \leq a_4$. Then $S_d$ is given by a quasihomogeneous polynomial equation of degree $d$:

$$f(x, y, z, t) = 0 \subset \mathbb{P}(a_0, a_1, a_2, a_3) \cong \text{Proj} \left( \mathbb{C}[x, y, z, t] \right),$$

where $wt(x) = a_0$, $wt(y) = a_1$, $wt(z) = a_2$, and $wt(t) = a_3$. Here, being quasismooth simply means that the above equation defines a hypersurface that is singular only at the origin in $\mathbb{C}^4$, which implies that $S_d$ has at most cyclic quotient singularities. On the other hand, being well-formed implies that $K_{S_d} \sim \mathcal{O}_{\mathbb{P}(a_0, a_1, a_2, a_3)}(d - a_0 - a_1 - a_2 - a_3)$, see [15, Theorem 3.3.4], and [18, 6.14].

Put $I = a_0 + a_1 + a_2 + a_3 - d$ and suppose that $I$ is positive. Then $S_d$ is a del Pezzo surfaces with at most quotient singularities. If $S_d$ is smooth, then it always admits a Kähler–Einstein metric by Tian [33] (see also [7,25,30,35]). Singular del Pezzo surfaces with orbifold Kähler–Einstein metrics drew attention from Riemannian geometers because they may lift to Sasakian–Einstein 5-manifolds through $S^1$-bundle structures. Through this passage, Boyer, Galicki and Nakamaye yielded a significant amount of examples towards classification of simply-connected Sasakian–Einstein 5-manifolds (see [5,6]).

For $I = 1$, Johnson and Kollár presented an algorithm in [19] that produces the (infinite) list of all possibilities for the quintuple $(a_0, a_1, a_2, a_3, d)$. They also proved the following result:

**Theorem 1.1** ([19, Theorem 8]) Suppose that $S_d$ with $I = 1$ is singular and the quintuple $(a_0, a_1, a_2, a_3, d)$ is not one of the following four quintuples:

$$(1, 2, 3, 5, 10), \quad (1, 3, 5, 7, 15), \quad (1, 3, 5, 8, 16), \quad (2, 3, 5, 9, 18). \quad (1.2)$$

Then $S_d$ admits an orbifold Kähler–Einstein metric.

Its proof uses the criterion given by the $\alpha$-invariant (for the definition, see [9, Definition 1.2]) of the surface $S_d$, see [14,24,32]. It says that the surface $S_d$ admits an (orbifold) Kähler–Einstein metric if the inequality

$$\alpha(S_d) > \frac{2}{3} \quad (1.3)$$

holds, where $\alpha(S_d)$ is the $\alpha$-invariant of the surface $S_d$. Indeed, Johnson and Kollár verified (1.3) in the case when $I = 1$, the surface $S_d$ is singular, and the quintuple $(a_0, a_1, a_2, a_3, d)$ is not one of the four exceptions (1.2). Two of the four remaining cases (1.2) have been treated in [1] by Araujo, who proved the following result:

---

Springer
Theorem 1.4 ([1, Theorem 4.1]) In the following two cases:

\[ (a_0, a_1, a_2, a_3, d) = (1, 2, 3, 5, 10), \]
\[ (a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15) \]

and the equation of \( S_d \) contains \( yzt \),

the inequality \( \alpha(S_d) > \frac{2}{3} \) holds. In particular, \( S_d \) admits an orbifold Kähler–Einstein metric.

The remaining two cases of (1.2) have been dealt with in the paper [8]. In this paper, we succeeded in estimating their \( \alpha \)-invariants from below by large enough numbers for the criterion (1.3). To be precise, we proved the following result:

Theorem 1.5 ([8, Theorem 1.10]) Suppose that

\[ (a_0, a_1, a_2, a_3, d) = (1, 3, 5, 8, 16), \]
\[ (2, 3, 5, 9, 18) \]

or

\[ (a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15) \]

and the defining equation of the surface \( S_d \) does not contain \( yzt \). Note that in the latter case one has

\[ \alpha(S_d) = \frac{8}{15} < \frac{2}{3} \]

by [8, Theorem 1.10] so that the criterion by the \( \alpha \)-invariant could not be applied.

Meanwhile, since 2010 we have witnessed dramatic developments in the study of the Yau–Tian–Donaldson conjecture concerning the existence of Kähler–Einstein metrics on Fano manifolds and stability. The challenge to the conjecture has been heightened by Chen, Donaldson, Sun and Tian who have completed the proof for the case of Fano manifolds with anticanonical polarizations [13,34]. Following this celebrated achievement, useful technologies have been introduced to determine whether given Fano varieties are Kähler–Einstein or not, via the theorem of Chen–Donaldson–Sun and Tian.

Recently Fujita and Odaka introduced a new invariant of Fano varieties, which they called \( \delta \)-invariant (for the definition, see [16, Definition 1.2]), that serves as a strong criterion for uniform K-stability (see [16]).

Theorem 1.6 ([4,16]) Let \( X \) be a Fano variety with at most Kawamata log terminal singularities. Then \( X \) is uniformly K-stable if and only if \( \delta(X) > 1 \).

This powerful tool has been practiced for del Pezzo surfaces in [11,12,27]. Around the same time, Li et al. proved in [22,23] that the result of Chen, Donaldson, Sun and Tian also holds for some singular Fano varieties. In particular, it holds for del Pezzo surfaces with quotient singularities. Thus, if \( \delta(S_d) > 1 \), then the surface \( S_d \) admits an (orbifold) Kähler–Einstein metric. Note that \( 3\alpha(S_d) \geq \delta(S_d) \geq \frac{3}{2}\alpha(S_d) \) by [4, Theorem A].

Now we are strongly reinforced by these new technologies, so that we could complete the study of existence of an (orbifold) Kähler–Einstein metric on the surface \( S_d \) in the case \( I = 1 \) started by Johnson and Kollár in [19]. In this paper, we prove the following result:
Theorem 1.7 Let $S_d$ be a quasi-smooth hypersurface in $\mathbb{P}(1, 3, 5, 7)$ of degree 15 such that its defining equation does not contain $yzt$. Then $\delta(S_d) \geq \frac{6}{5}$. In particular, the surface $S_d$ admits an orbifold Kähler–Einstein metric.

Corollary 1.8 If $I = 1$, then $S_d$ admits an orbifold Kähler–Einstein metric.

For $I \geq 2$, the problem of existence of an orbifold Kähler–Einstein metric on the surface $S_d$ was first studied by Boyer et al. in [6]. In this case, there is no reasonable classification similar to that obtained by Johnson and Kollár in [19]; note however that Paemurru [26] presents an algorithm that produces the (infinite) list of all possibilities for the quintuple $(a_0, a_1, a_2, a_3, d)$ for every fixed $I \geq 2$. In [6,8,10], the existence of an orbifold Kähler–Einstein metric has been proved for (infinitely) many surfaces $S_d$ with $I \geq 2$. However, in the following six cases their method did not work:

1. $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and the equation of $S_d$ does not contain $yzt$;
2. $(a_0, a_1, a_2, a_3, d) = (7, 10, 15, 19, 45)$;
3. $(a_0, a_1, a_2, a_3, d) = (7, 18, 27, 37, 81)$;
4. $(a_0, a_1, a_2, a_3, d) = (7, 15, 19, 32, 64)$;
5. $(a_0, a_1, a_2, a_3, d) = (7, 19, 25, 41, 82)$;
6. $(a_0, a_1, a_2, a_3, d) = (7, 26, 39, 55, 117)$.

In this paper, we use $\delta$-invariants to show that $S_d$ is Kähler–Einstein in these six cases as well:

Theorem 1.9 Suppose that $(a_0, a_1, a_2, a_3, d)$ is one of the six quintuples listed above. Then $\delta(S_d) \geq \frac{65}{64}$. In particular, the surface $S_d$ admits an orbifold Kähler–Einstein metric.

According to the similarity of the proofs, we handle the seven types of del Pezzo surfaces in Theorems 1.7 and 1.9 into three cases as follows:

Case A. $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the equation of $S_d$ does not contain $yzt$;
$(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and the equation of $S_d$ does not contain $yzt$;

Case B. $(a_0, a_1, a_2, a_3, d) = (7, 15, 19, 32, 64)$;
$(a_0, a_1, a_2, a_3, d) = (7, 19, 25, 41, 82)$;

Case C. $(a_0, a_1, a_2, a_3, d) = (7, 10, 15, 19, 45)$;
$(a_0, a_1, a_2, a_3, d) = (7, 18, 27, 37, 81)$;
$(a_0, a_1, a_2, a_3, d) = (7, 26, 39, 55, 117)$.

We will handle each of these cases separately in Sects. 3, 4 and 5, respectively; see Corollaries 3.3, 4.3 and 5.6. In Sect. 2, we will present some results that will be used in the proofs of Theorems 1.7 and 1.9.

Let us briefly explain how we estimate $\delta(S_d)$ in the proofs of Theorems 1.7 and 1.9. In our old paper [8], we developed a technique how to study possible singularities of log pairs $(S_d, D)$, where $D$ is an effective $\mathbb{Q}$-divisor on the surface $S_d$ such that $D \sim_{\mathbb{Q}} -K_{S_d}$. This resulted in explicit values of $\alpha(S_d)$ in all considered cases. To estimate $\delta(S_d)$, one has to study singularities of similar log pairs with an additional condition: the $\mathbb{Q}$-divisor $D$ has to be of $k$-basis type for $k \gg 1$ (for the definition,
see [16, Definition 1.1]). By [16, Lemma 2.2] (see also Theorem 2.9 below), this extra condition provides strong upper bounds on multiplicities of the $\mathbb{Q}$-divisor $D$ in various curves on $S_d$. We use these bounds (for some very particular curves in $S_d$) together with our original methods developed in [8], to obtain the required estimates for $\delta(S_d)$. This approach was first used in [11] to estimate $\delta$-invariants of the so-called asymptotically del Pezzo surfaces. Nevertheless, in our case we have an additional difficulty arising from the singularities of the surface $S_d$, while all surfaces considered in [11] are smooth.

It would be interesting to study the problem of existence of an orbifold Kähler–Einstein metric on $S_d$ in the remaining cases. In some of these cases, the del Pezzo surface $S_d$ is indeed not Kähler–Einstein. For instance, the surface $S_d$ does not admit an orbifold Kähler–Einstein metric in the case when $I > 3a_0$. This follows from the obstruction found by Gauntlett et al. [17]. On the other hand, we expect the following to be true:

**Conjecture 1.10** If $I = 2$ or $I = 3$, then $S_d$ admits an orbifold Kähler–Einstein metric.

We believe that this conjecture can be proved using a similar approach to the one we use in the proofs of Theorems 1.7 and 1.9.

### 2 Basic Tools

Let $S$ be a surface with at most cyclic quotient singularities, let $C$ be an irreducible reduced curve on $S$, let $P$ be a point of the curve $C$, and let $D$ be an effective $\mathbb{R}$-divisor on the surface $S$. In this section, we present a few of well-known (local and global) results that will be used in the proof of Theorem 1.9. We start with

**Lemma 2.1** ([20]) Suppose that $P$ is a smooth point of the surface $S$, and the singularities of the log pair $(S, D)$ are not log canonical at $P$. Then $\text{mult}_P(D) > 1$.

This immediately implies

**Corollary 2.2** If $P$ is a smooth point of the surface $S$, the log pair $(S, D)$ is not log canonical at $P$, and $C$ is not contained in the support of the divisor $D$, then $D \cdot C > 1$.

To state an analogue of this result in the case when $S$ is singular at $P$, recall that $S$ has a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ at the point $P$, where $a$ and $b$ are coprime positive integers that are also coprime to $n$. Thus, if $n = 1$, then $P$ is a smooth point of the surface $S$. For $n > 1$, Corollary 2.2 can be generalized as follows:

**Lemma 2.3** Suppose that the log pair $(S, D)$ is not log canonical at $P$, and $C$ is not contained in the support of the divisor $D$. Then $D \cdot C > \frac{1}{n}$.

**Proof** This follows from [8, Lemma 2.2] and [8, Lemma 2.3], cf. [2].

In general, the curve $C$ may be contained in the support of the divisor $D$. Thus, we write

$$D = aC + \Delta,$$
where $a$ is a non-negative real number, and $\Delta$ is an effective $\mathbb{R}$-divisor on $S$ whose support does not contain the curve $C$. Then we have the following useful result:

**Lemma 2.4** Suppose that $a \leq 1$, the surface $S$ is smooth at the point $P$, the curve $C$ is also smooth at $P$, and the log pair $(S, D)$ is not log canonical at $P$. Then

$$C \cdot \Delta \geq (C \cdot \Delta)_P > 1,$$

where $(C \cdot \Delta)_P$ is the local intersection number of $C$ and $\Delta$ at $P$.

**Proof** This is a special case of a much more general result, known as the inversion of adjunction (see [28,31]).

By our assumption, the surface $S$ has a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ at the point $P$. Thus, locally near $P$, the surface $S$ is a quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}_n$ that acts on $\mathbb{C}^2$ as

$$(x, y) \mapsto (\omega^ax, \omega^by),$$

where $\omega$ is a primitive $n$th root of unity. We can consider $x$ and $y$ as weighted coordinates around the point $P$.

**Remark 2.6** The pair $(S, C)$ has purely log terminal singularity at $P$ if and only if $C$ is given by $x = 0$ or $y = 0$ for an appropriate choice of weighted coordinates $x$ and $y$. This follows from [28, Theorem 2.1.2], see also [20, Sect. 9.6]. Geometrically, this means that $C$ is smooth at $P$, and its proper transform on the minimal resolution of singularities of the singular point $P$ intersects the tail curve in the chain of exceptional curves. If $(S, C)$ has purely log terminal singularities, then

$$\left(K_S + C\right) \cdot C = -2 + \sum_{O \in C} \frac{n_O - 1}{n_O},$$

where we assume that $S$ has a cyclic quotient singularity of index $n_O$ at the point $O$. 
Let $f : \tilde{S} \rightarrow S$ be the weighted blow up of the point $P$ with $\text{wt}(x) = a$ and $\text{wt}(y) = b$, and let $E$ be the exceptional curve of the morphism $f$. Then $\tilde{S}$ has at most cyclic quotient singularities, one has $E \cong \mathbb{P}^1$, and the log pair $(\tilde{S}, E)$ has purely log terminal singularities. Moreover, the curve $E$ has at most two singular points of the surface $\tilde{S}$. One of them is a singular point of type $\frac{1}{a}(n, -b)$, and another is a singular point of type $\frac{1}{b}(-a, n)$. Furthermore, we have

$$K_{\tilde{S}} \sim_{\mathbb{Q}} f^*(K_S) + \frac{a + b - n}{n}E.$$ 

If the curve $C$ is locally given by $x = 0$ near the point $P$, then

$$\tilde{C} \sim_{\mathbb{Q}} f^*(C) - \frac{a}{n}E,$$

where $\tilde{C}$ is the proper transform of the curve $C$ on the surface $\tilde{S}$. For more properties of weighted blow ups and their defining equations, see [28, Sect. 3] or [2].

Denote by $\tilde{D}$ the proper transform of the divisor $D$ via $f$. Then

$$\tilde{D} \sim_{\mathbb{R}} f^*(D) - mE$$

for some non-negative rational number $m$. If $C$ is not contained in the support of the divisor $D$, we can estimate $m$ using

$$0 \leq \tilde{D} \cdot \tilde{C} = (f^*(D) - mE) \cdot \tilde{C} = D \cdot C - mE \cdot \tilde{C},$$

where $D \cdot C$ and $E \cdot \tilde{C}$ can be computed in every case. Note that

$$K_{\tilde{S}} + \tilde{D} + \left(m - \frac{a + b - n}{n}\right)E \sim_{\mathbb{R}} f^*(K_S + D).$$

This implies

**Proposition 2.7** The log pair $(S, D)$ is log canonical at $P$ if and only if the log pair

$$\left(\tilde{S}, \tilde{D} + \left(m - \frac{a + b - n}{n}\right)E\right)$$

is log canonical along the curve $E$.

So far, we considered only local properties of the divisor $D$ on the surface $S$. These properties will be used later to prove Theorem 1.9. However, the nature of this theorem is global so that we will need one global result that is due to Fujita and Odaka. To state it, we remind the reader of what the volume $\text{vol}(D)$ of the $\mathbb{R}$-divisor $D$ is. If $D$ is a Cartier divisor, then its volume is simply the number

$$\text{vol}(D) = \lim_{k \to \infty} \sup_{k \in \mathbb{N}} \frac{h^0(\mathcal{O}_S(kD))}{k^2/2!},$$
where the lim sup can be replaced by limit (see [21, Example 11.4.7]). Likewise, if $D$ is a $\mathbb{Q}$-divisor, we can define its volume using the identity

$$\text{vol}(D) = \frac{\text{vol}(\lambda D)}{\lambda^2}$$

for an appropriate positive rational number $\lambda$. One can show that the volume $\text{vol}(D)$ only depends on the numerical equivalence class of the divisor $D$. Moreover, the volume function can be continuously extended to $\mathbb{R}$-divisors (see [21] for details).

If $D$ is not pseudoeffective, then $\text{vol}(D) = 0$. If $D$ is pseudoeffective, its volume can be computed using its Zariski decomposition [3,29]. Namely, if $D$ is pseudoeffective, then there exists a nef $\mathbb{R}$-divisor $N$ on the surface $S$ such that

$$D \sim_R N + \sum_{i=1}^r a_i C_i,$$

where each $C_i$ is an irreducible curve on $S$ with $N \cdot C_i = 0$, each $a_i$ is a non-negative real number, and the intersection form of the curves $C_1, \ldots, C_r$ is negative definite. Such a decomposition is unique, and it follows from [3, Corollary 3.2] that

$$\text{vol}(D) = \text{vol}(N) = N^2. \quad (2.8)$$

Recall that $D = aC + \Delta$, where $a$ is a non-negative real number, and $\Delta$ is an effective divisor whose support does not contain the curve $C$. Let

$$\tau = \sup \left\{ x \in \mathbb{R}_{>0} \mid D - xC \text{ is pseudoeffective} \right\}.$$

Then $a \leq \tau$. However, to prove Theorem 1.9, we have to find a better bound for $a$ under an additional assumption that $D$ is an ample $\mathbb{Q}$-divisor of $k$-basis type for $k \gg 1$ (for the definition, see [16, Definition 1.1] and the proof of Theorem 2.9 below). One such estimate is given by the following very special case of [16, Lemma 2.2].

**Theorem 2.9** Suppose that $D$ is a big $\mathbb{Q}$-divisor of $k$-basis type for $k \gg 1$. Then

$$a \leq \frac{1}{D^2} \int_0^\tau \text{vol}(D - xC) \, dx + \epsilon_k,$$

where $\epsilon_k$ is a small constant depending on $k$ such that $\epsilon_k \to 0$ as $k \to \infty$.

**Proof** Let us give a sketch of the proof that shows the nature of the required bound. First, recall from [16] that being $k$-basis type simply means that

$$D = \frac{1}{kd_k} \sum_{i=1}^{d_k} \{ s_i = 0 \},$$
where \( d_k = h^0(S, O_S(kD)) \) and \( s_1, \ldots, s_{dk} \) are linearly independent sections in \( H^0(S, O_S(kD)) \). Here, we assume that \( kD \) is a Cartier divisor and \( k \gg 0 \).

Let \( M \) be a positive rational number such that \( M \geq \tau \). We may assume that \( kM \) is an integer. Then there is a filtration of vector spaces

\[
0 = H^0\left(S, O_S(kD - (kM + 1)C)\right) \subseteq H^0\left(S, O_S(kD - kMC)\right) \\
\subseteq H^0\left(S, O_S(kD - (kM - 1)C)\right) \subseteq \ldots \subseteq H^0\left(S, O_S(kD - 3C)\right) \\
\subseteq H^0\left(S, O_S(kD - 2C)\right) \subseteq H^0\left(S, O_S(kD - C)\right) \subseteq H^0\left(S, O_S(kD)\right).
\]

Let \( r_i = h^0\left(S, O_S(kD - iC)\right) \). Then

\[
0 = r_{kM+1} \leq r_{kM} \leq r_{kM-1} \leq \ldots \leq r_3 \leq r_2 \leq r_1 \leq r_0 = d_k.
\]

Since the sections \( s_1, \ldots, s_{dk} \) are linearly independent, we see that at most \( r_1 \) of them are contained in

\[
H^0\left(S, O_S(kD - C)\right).
\]

Among them at most \( r_2 \) are contained in \( H^0\left(S, O_S(kD - 2C)\right) \). Among them at most \( r_3 \) are contained in \( H^0\left(S, O_S(kD - 3C)\right) \), etc. Finally, at most \( r_{kM} \) sections among \( s_1, \ldots, s_{dk} \) are contained in

\[
H^0\left(S, O_S(kD - kMC)\right),
\]

and there are no sections in \( H^0\left(O_S(kD - (kM + 1)C)\right) = 0 \). Then

- at most \( r_1 \) sections among \( s_1, \ldots, s_{dk} \) vanish at \( C \);
- at most \( r_2 \) sections among \( s_1, \ldots, s_{dk} \) vanish at \( C \) with order \( \geq 2 \);
- at most \( r_3 \) sections among \( s_1, \ldots, s_{dk} \) vanish at \( C \) with order \( \geq 3 \);
- \ldots
- at most \( r_{kM-1} \) sections among \( s_1, \ldots, s_{dk} \) vanish at \( C \) with order \( \geq kM - 1 \);
- at most \( r_{kM} \) sections among \( s_1, \ldots, s_{dk} \) vanish at \( C \) with order \( kM \);
- no sections among \( s_1, \ldots, s_{dk} \) vanish at \( C \) with order \( \geq kM + 1 \).

This immediately implies that the order of vanishing of the product \( s_1 \cdot s_2 \cdot s_3 \ldots \cdot s_{dn} \) at the curve \( C \) is at most

\[
kMr_{kM} + (kM - 1)(r_{kM-1} - r_{kM}) + (kM - 2)(r_{kM-2} - r_{kM-1}) \\
\ldots + 4(r_4 - r_5) + 3(r_3 - r_4) + 2(r_2 - r_3) + (r_1 - r_2) = \sum_{i=1}^{kM} r_i.
\]
Then we have
\[ a \leq \frac{r_1 + r_2 + \ldots + r_{kM-1} + r_{kM}}{kr_0}. \]
As \( k \to \infty \), the right-hand side in this inequality converges to
\[ \frac{1}{D^2} \int_0^\tau \text{vol}(D - xC)dx, \]
which gives the upper bound on \( a \). For a detailed proof, we refer the reader to [16]. □

**Corollary 2.10** Suppose that \( D \) is a big \( \mathbb{Q} \)-divisor of \( k \)-basis type for \( k \gg 1 \), and
\[ C \sim_{\mathbb{Q}} \mu D \]
for some positive rational number \( \mu \). Then
\[ a \leq \frac{1}{3\mu} + \epsilon_k, \]
where \( \epsilon_k \) is a small constant depending on \( k \) such that \( \epsilon_k \to 0 \) as \( k \to \infty \).

**Proof** Using Theorem 2.9, we get
\[ a \leq \frac{1}{D^2} \int_0^\infty \text{vol}(D - \lambda C)d\lambda + \epsilon_k, \]
where \( \epsilon_k \) is a small constant depending on \( k \) such that \( \epsilon_k \to 0 \) as \( k \to \infty \). But
\[ \int_0^\infty \text{vol}(D - \lambda C)d\lambda = \int_0^\infty \text{vol}((1 - \lambda\mu)D)d\lambda = D^2 \int_0^{1/\mu} (1 - \lambda\mu)^2d\lambda = \frac{D^2}{3\mu}. \]
This implies the assertion. □

### 3 Case A

In this section, we consider two types of quasismooth hypersurfaces as follows:
- \( S_{15} \): a quasismooth hypersurface in \( \mathbb{P}(1, 3, 5, 7) \) of degree 15;
- \( S_{12} \): a quasismooth hypersurface in \( \mathbb{P}(2, 3, 4, 5) \) of degree 12.

By suitable coordinate changes, \( S_{15} \) may be assumed to be given by
\[
\begin{align*}
z^3 + y^5 + x^2 + b_1 y z t + b_2 x y^3 z + b_3 x^2 y z^2 + b_4 x^2 y^2 t + b_5 x^3 z t + b_6 x^3 y^4 + b_7 x^4 y^2 z + b_8 x^5 z^2 + b_9 x^5 y t + b_{10} x^6 y^3 + b_{11} x^7 y z + b_{12} x^8 t + b_{13} x^9 y^2 + b_{14} x^{10} z + b_{15} x^{12} y + b_{16} x^{15} &= 0
\end{align*}
\]

\( \copyright \) Springer
Delta Invariants of Singular del Pezzo Surfaces

and $S_{12}$ by

$$z(z - x^2)(z - \epsilon x^2) + y^4 + xt^2 + b_1 yzt + b_2 x y^2 z + b_3 x^2 y t + b_4 x^3 y^2 = 0,$$

where $\epsilon$ ($\epsilon \neq 0$ and $\epsilon \neq 1$), $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}, b_{15}$
and $b_{16}$ are constants. Note that the surface $S_{15}$ has the only singular point at $O_t = [0 : 0 : 0 : 1]$. Meanwhile, $S_{12}$ has exactly four singular points at $O_x = [1 : 0 : 0 : 0], O_t = [0 : 0 : 0 : 1], Q_1 = [1 : 0 : 1 : 0]$ and $Q_2 = [1 : 0 : \epsilon : 0].$

In the sequel, we use $S$ for the surfaces $S_{15}$ and $S_{12}$ if properties or conditions are satisfied by both the surfaces.

Denote by $C_x$ the curve in $S$ cut out by the equation $x = 0$. Then the curve $C_x$ is reduced and irreducible in both the cases. It is easy to check

$$\text{lct}(S_{15}, C_x) = \begin{cases} 1 & \text{if } a_1 \neq 0, \\ \frac{8}{15} & \text{if } a_1 = 0, \end{cases}$$

$$\text{lct}(S_{12}, C_x) = \begin{cases} 1 & \text{if } a_1 \neq 0, \\ \frac{7}{12} & \text{if } a_1 = 0, \end{cases}$$

where $\text{lct}(S, C_x)$ is the log canonical threshold of $C_x$ on $S$. Moreover, one has $\alpha(S) = \text{lct}(S, C_x)$ by [8, Theorem 1.10]. Thus, since $\delta(S) \geq \frac{3}{2}\alpha(S)$, we obtain

**Corollary 3.1** If $b_1 \neq 0$, then $\delta(S) \geq \frac{3}{2}$.

From now on, we suppose that $b_1 = 0$.

**Proposition 3.2** Let $D$ be an effective $\mathbb{Q}$-divisor on $S$ such that

$$D \sim_{\mathbb{Q}} -K_S.$$

Write $D = aC_x + \Delta$, where $a$ is a non-negative number, and $\Delta$ is an effective $\mathbb{Q}$-divisor on the surface $S$ whose support does not contain the curve $C_x$. Suppose also that $a \leq \frac{8}{27}$. Then the log pair $(S, \frac{6}{5}D)$ is log canonical.

**Corollary 3.3** One has $\delta(S) \geq \frac{6}{5}$.

**Proof** Let $D$ be a $\mathbb{Q}$-divisor of $k$-basis type divisor on $S$ with $k \gg 0$. Write $D = aC_x + \Delta$, where $a$ is a non-negative number, and $\Delta$ is an effective $\mathbb{Q}$-divisor on the surface $S$ whose support does not contain the curve $C_x$. By Corollary 2.10, we have $a \leq \frac{8}{27}$ for $k \gg 0$. Thus, the log pair $(S, \frac{6}{5}D)$ is log canonical for $k \gg 0$ by Proposition 3.2. This implies that $\delta(S) \geq \frac{6}{5}$ by Corollary 3.1.

To prove Proposition 3.2, we fix an effective $\mathbb{Q}$-divisor $D$ on the surface $S$ such that

$$D \sim_{\mathbb{Q}} -K_S.$$
Write \( D = aC_x + \Delta \), where \( a \) is a non-negative number, and \( \Delta \) is an effective \( \mathbb{Q} \)-divisor on the surface \( S \) whose support does not contain the curve \( C_x \). Suppose also that \( a \leq \frac{8}{21} \). Let us show that the log pair \( (S, \frac{6}{5} D) \) is log canonical.

**Lemma 3.4** The log pair \( (S, \frac{6}{5} D) \) is log canonical outside \( C_x \).

**Proof** The required assertion follows from [8, Lemma 2.7]. For convenience of the reader, let us give the detailed proof here. Let \( P \) be a point in \( S \setminus C_x \). Since \( P \not\in C_x \), there are complex numbers \( c_1 \) and \( c_2 \) such that \( P \) satisfies the following system of equations:

\[
\begin{align*}
  z + c_1 x^5 &= 0 & \text{for } S_{15}, \\
  y + c_2 x^3 &= 0 & \text{for } S_{12}, \\
  y^2 + c_1 x^3 &= 0 \\
  z + c_2 x^2 &= 0 & \text{for } S_{12}.
\end{align*}
\]

Let \( \mathcal{P} \) be the pencil of curves that is given by

\[
\begin{align*}
  v(z + c_1 x^5) + \mu(yx^2 + c_2 x^5) &= 0 & \text{on } S_{15}, \\
  v(y^2 + c_1 x^3) + \mu(zx + c_2 x^3) &= 0 & \text{on } S_{12}
\end{align*}
\]

for \([v : \mu] \in \mathbb{P}^1\). Then the base locus of the pencil \( \mathcal{P} \) consists of finitely many points. Moreover, by construction, the point \( P \) is one of them. Let \( C \) be a general curve in \( \mathcal{P} \). Then

\[
C \cdot D \leq \frac{5}{6}
\]

so that \( (S, \frac{6}{5} D) \) is log canonical at \( P \) by Corollary 2.2 if \( P \) is a smooth point of the surface \( S \). This verifies the statement for \( S_{15} \).

For \( S_{12} \), we suppose that \( (S_{12}, \frac{6}{5} D) \) is not log canonical at \( P \). Then \( P \) must be one of the points \( O_x, Q_1, Q_2 \). Observe that the point \( P \) belongs to the curve \( C_y \) cut by \( y = 0 \). Moreover, the curve \( C_y \) is irreducible and the log pair \( (S_{12}, \frac{6}{5} \cdot 2 C_y) \) is log canonical. Thus, it follows from [9, Remark 2.22] that there exists an effective \( \mathbb{Q} \)-divisor \( D' \) on the surface \( S_{12} \) such that

\[
D' \sim_{\mathbb{Q}} -K_{S_{12}},
\]

the log pair \( (S_{12}, \frac{6}{5} D') \) is not log canonical at the point \( P \), and the support of the divisor \( D' \) does not contain the curve \( C_y \). However,

\[
D' \cdot C_y = \frac{6}{10},
\]

which is impossible by Lemma 2.3 since \( (S_{12}, \frac{6}{5} D') \) is not log canonical at the point \( P \). This completes the proof for \( S_{12} \). \( \square \)
Lemma 3.5 The log pair \((S, \frac{6}{5}D)\) is log canonical at a point in \(C_x \setminus \{O_t\}\).

**Proof** Let \(P\) be a point in \(C_x \setminus \{O_t\}\). Observe that \(P\) is a smooth point of the surface \(S\), and \(C_x\) is smooth at the point \(P\). Note also that \(\frac{6}{5}a < 1\). Thus, we can apply Lemma 2.4 to \((S, \frac{6}{5}D)\) and the curve \(C_x\) at the point \(P\). Indeed, since

\[
\left( C_x \cdot \Delta \right)_P \leq C_x \cdot \Delta = \frac{1-a}{7} \leq \frac{5}{6} \text{ on } S_{15},
\]

\[
\left( C_x \cdot \Delta \right)_P \leq C_x \cdot \Delta = \frac{1-2a}{5} \leq \frac{5}{6} \text{ on } S_{12},
\]

the log pair \((S, \frac{6}{5}D)\) must be log canonical at \(P\). \(\square\)

Note that \(S_{15}\) (resp. \(S_{12}\)) has singularity of type \(\frac{1}{3}(3, 5)\) (resp. \(\frac{1}{4}(3, 4)\)) at the point \(O_t\). In the chart defined by \(t = 1\), the surface \(S_{15}\) is given by

\[
z^3 + y^5 + x + b_2xy^3z + b_3x^2yz^2 + b_4x^2y^2 + b_5x^3z + b_6x^3y^4 + b_7x^5y^2z + b_8x^5z^2 + b_9x^5y + b_{10}x^6y^3 + b_{11}x^7yz + b_{12}x^8 + b_{13}x^9y^2 + b_{14}x^{10}z + b_{15}x^{12}y + b_{16}x^{15} = 0,
\]

and \(S_{12}\) by

\[
z(z - x^2)(z - \epsilon x^2) + y^4 + x + a_1yz + a_2xy^2z + a_3x^2y + a_4x^3y^2 = 0.
\]

Thus, in a neighborhood of the point \(O_t\), we may regard \(y\) and \(z\) as local weighted coordinates with \(\text{wt}(y) = 3\) and \(\text{wt}(z) = 5\) for \(S_{15}\) and with \(\text{wt}(y) = 3\) and \(\text{wt}(z) = 4\) for \(S_{12}\).

Let \(f: \tilde{S} \to S\) be the weighted blow up at the singular point \(O_t\) with weights \(\text{wt}(y) = 3\), \(\text{wt}(z) = 5\) for \(S_{15}\) and with weights \(\text{wt}(y) = 3\), \(\text{wt}(z) = 4\) for \(S_{12}\). Denote by \(E\) the exceptional curve of the blow up \(f\). Then

\[
K_{\tilde{S}_{15}} \sim_{\mathbb{Q}} f^*(K_{S_{15}}) + \frac{1}{7}E,
\]

\[
K_{\tilde{S}_{12}} \sim_{\mathbb{Q}} f^*(K_{S_{12}}) + \frac{2}{5}E.
\]

The surface \(S\) has two singular points in \(E\). One is a point of type \(\frac{1}{3}(1, 1)\), and the other is a singular point of type \(\frac{1}{4}(1, 1)\) on \(\tilde{S}_{15}\) (resp. \(\frac{1}{4}(1, 1)\) on \(\tilde{S}_{12}\)). Denote the former by \(O_3\) and the latter by \(O\). Observe that

\[
E^2 = -\frac{7}{15} \text{ on } \tilde{S}_{15},
\]

\[
E^2 = -\frac{5}{12} \text{ on } \tilde{S}_{12},
\]

and \(E \cong \mathbb{P}^1\).
Let $\tilde{C}_x$ be the proper transform of the curve $C_x$ on the surface $\tilde{S}$. Then

$$\tilde{C}_x \sim_\mathbb{Q} f^*(C_x) - cE \quad \text{for } S_{15},$$

where $c = \frac{15}{7}$ for $S_{15}$ and $c = \frac{12}{5}$ for $S_{12}$, and the intersection $E \cap \tilde{C}_x$ consists of a single point, which is different from $O_3$ and $O$. Note that the curves $E$ and $\tilde{C}_x$ intersect transversally at the point $E \cap \tilde{C}_x$.

Denote by $\tilde{\Delta}$ the proper transform of the $\mathbb{Q}$-divisor $\Delta$ on the surface $\tilde{S}$. Then

$$\tilde{\Delta} \sim_\mathbb{Q} f^*(\Delta) - mE$$

for some non-negative rational number $m$. To estimate it, observe that

$$0 \leq \tilde{C}_x \cdot \tilde{\Delta} = \left( f^*(C_x) - cE \right) \cdot \left( f^*(\Delta) - mE \right) C_x \cdot \Delta - m$$

so that $m \leq \frac{1-a}{7}$ for $S_{15}$ and $m \leq \frac{1-2a}{5}$ for $S_{12}$. Now we are ready to prove

**Lemma 3.6** The log pair $(S, \frac{6}{5} D)$ is log canonical at $O_i$.

**Proof** Suppose that the log pair $(S, \frac{6}{5} D)$ is not log canonical at $O_i$. Let us seek for a contradiction. Let $\lambda = \frac{6}{5}$. Then

$$K_{\tilde{S}} + \lambda a \tilde{C}_x + \lambda \tilde{\Delta} + \mu E \sim_\mathbb{Q} f^*(K_S + \lambda D),$$

where

$$\mu = \frac{15\lambda a}{7} + \lambda m - \frac{1}{7} \quad \text{for } S_{15},$$

$$\mu = \frac{12\lambda a}{5} + \lambda m - \frac{2}{5} \quad \text{for } S_{12}.$$

Thus, the log pair

$$\left( \tilde{S}, \lambda a \tilde{C}_x + \lambda \tilde{\Delta} + \mu E \right) \quad (3.7)$$

is not log canonical at some point $Q \in E$. Note that $\mu \leq 1$ because $m \leq \frac{1-a}{7}$ (or $m \leq \frac{1-2a}{5}$) and $a \leq \frac{8}{21}$.

We first apply Lemmas 2.4 or 2.5–(3.7) and the curve $E$ at the point $Q$. Indeed,

$$E \cdot \tilde{\Delta} = E \cdot (f^*(\Delta) - mE) = -mE^2 = \begin{cases} \frac{7m}{15} & \leq \frac{1-a}{15} \leq \frac{1}{6} \quad \text{on } \tilde{S}_{15}, \\ \frac{5m}{12} & \leq \frac{1-2a}{12} \leq \frac{5}{24} \quad \text{on } \tilde{S}_{12}. \end{cases}$$

This shows that $Q$ must be the intersection point of $E$ and $\tilde{C}_x$. 

\[\Box\] Springer
Applying Lemma 2.4 again, we see that
\[
\frac{5}{6} = \frac{1}{\lambda} < \left( a\tilde{C}_x + \tilde{\Delta} \right) \cdot E = a + \tilde{\Delta} \cdot E = \begin{cases} 
\frac{a + 7m}{15} & \text{on } \tilde{S}_{15}, \\
\frac{a + 5m}{12} & \text{on } \tilde{S}_{12}.
\end{cases}
\]

However, these inequalities contradict our assumption \( a \leq \frac{8}{21} \). Therefore, the log pair \((S, \frac{6}{5}D)\) is log canonical at \( O_t \). \qed

Proposition 3.2 is completely verified.

4 Case B

The way to evaluate \( \delta \)-invariants for Case B is almost same as that of Case A. In spite of this, we write the proof for the readers’ convenience.

In this section, we consider the following two types of quasismooth hypersurfaces:

- \( S_{64} \): a quasismooth hypersurface in \( \mathbb{P}(7, 15, 19, 32) \) of degree 64;
- \( S_{82} \): a quasismooth hypersurface in \( \mathbb{P}(7, 19, 25, 41) \) of degree 82.

As in the previous section, we use \( S \) for the surfaces \( S_{64} \) and \( S_{82} \) if properties or conditions are satisfied by both the surfaces.

We may assume that the surface \( S_{64} \) is given by the equation
\[
t^2 + y^3z + xz^3 + x^7y = 0
\]
in \( \mathbb{P}(7, 15, 19, 32) \) and \( S_{82} \) by the equation
\[
t^2 + y^3z + xz^3 + x^9y = 0
\]
in \( \mathbb{P}(7, 19, 25, 41) \). The surface \( S \) is singular at the points \( O_x = [1 : 0 : 0 : 0] \), \( O_y = [0 : 1 : 0 : 0] \) and \( O_z = [0 : 0 : 1 : 0] \), and is smooth away from them. Moreover, the surface \( S_{64} \) (resp. \( S_{82} \)) has quotient singularity of types \( \frac{1}{7}(5, 4) \), \( \frac{1}{15}(7, 2) \), \( \frac{1}{19}(2, 3) \) (resp. \( \frac{1}{7}(2, 3) \), \( \frac{1}{19}(7, 3) \), \( \frac{1}{19}(2, 3) \)) at the points \( O_x \), \( O_y \), \( O_z \), respectively.

Let \( C_x \) be the curve in \( S \) cut out by \( x = 0 \) and \( C_y \) by \( y = 0 \). Then both the curves \( C_x \) and \( C_y \) are irreducible. We have
\[
\frac{35}{54} = \text{lct}(S_{64}, \frac{9}{7}C_x) < \text{lct}(S_{64}, \frac{9}{15}C_y) = \frac{25}{18},
\]
\[
\frac{7}{12} = \text{lct}(S_{82}, \frac{10}{7}C_x) < \text{lct}(S_{82}, \frac{10}{19}C_y) = \frac{19}{12},
\]
which imply \( \alpha(S_{64}) \leq \frac{35}{54} \) and \( \alpha(S_{82}) \leq \frac{7}{12} \). In fact, we have \( \alpha(S_{64}) = \frac{35}{54} \) and \( \alpha(S_{82}) = \frac{7}{12} \) by [8, Theorem 1.10].
Proposition 4.1 Let \( D \) be an effective \( \mathbb{Q} \)-divisor on \( S \) such that

\[
D \sim_{\mathbb{Q}} -K_S.
\]

Write \( D = aC_x + \Delta \), where \( a \) is a non-negative number, and \( \Delta \) is an effective \( \mathbb{Q} \)-divisor on the surface \( S \) whose support does not contain the curve \( C_x \). Suppose also that \( a \leq \frac{1}{2} \). Then the log pair \((S, \frac{19}{18} D)\) is log canonical.

Proof Suppose also that \( a \leq \frac{1}{2} \).

We first consider a point \( P \) that lies neither on \( C_x \) nor on \( C_y \). Observe that \( P \) is a smooth point of the surface \( S \). Since \( P \not\in C_x \), there are complex numbers \( c_1 \) and \( c_2 \) such that \( P \) satisfies the following system of equations:

\[
\begin{align*}
{y}^7 + c_1 x^{15} &= 0 \\
{y}^2 z + c_2 x^7 &= 0 & \text{for } S_{64} \\
{y}^4 + c_1 x^5 t &= 0 \\
{y}^3 + c_2 x z^2 &= 0 & \text{for } S_{82}.
\end{align*}
\]

Moreover, since \( P \not\in C_y \), we have \( c_1 \neq 0 \). Let \( \mathcal{P} \) be the pencil given by

\[
\begin{align*}
\nu \left( y^7 + c_1 x^{15} \right) + \mu x^8 \left( y^2 z + c_2 x^7 \right) &= 0 & \text{on } S_{64}; \\
\nu \left( y^4 + c_1 x^5 t \right) + \mu y^3 + c_2 x z^2 &= 0 & \text{on } S_{82}
\end{align*}
\]

for \([\nu : \mu] \in \mathbb{P}^1\). The base locus of the pencil \( \mathcal{P} \) consists of finitely many points. Furthermore, by construction, the point \( P \) is one of them. Let \( C \) be a general curve in \( \mathcal{P} \). Then

\[
\text{mult}_P(D) \leq C \cdot D \leq \frac{18}{19}.
\]

It immediately follows from Corollary 2.2 that the log pair \((S, \frac{19}{18} D)\) is log canonical outside \( C_x \) and \( C_y \).

We next consider a point \( P \) on \( C_x \) different from \( O_z \). Since \( a \leq \frac{1}{2} \), we apply Lemmas 2.4 and 2.5 to the log pair \((S, \frac{18}{19} a C_x + \frac{18}{19} \Delta)\). Indeed, since

\[
\begin{align*}
\left( C_x \cdot \Delta \right)_P &\leq C_x \cdot \Delta = \frac{18 - 14a}{285} \leq \frac{6}{95} & \text{on } S_{64}, \\
\left( C_x \cdot \Delta \right)_P &\leq C_x \cdot \Delta = \frac{20 - 14a}{475} \leq \frac{18}{19 \cdot 19} & \text{on } S_{82},
\end{align*}
\]

the log pair \((S, \frac{19}{18} D)\) must be log canonical at \( P \).

We now let \( P \) be a point on \( C_y \) different from \( O_z \). Suppose that the log pair \((S, \frac{19}{18} D)\) is not log canonical at the point \( P \). Recall that \((S_{64}, \frac{19}{18} \cdot \frac{9}{15} C_y)\) and \((S_{82}, \frac{19}{18} \cdot \frac{10}{19} C_y)\) are...
log canonical, and the curve $C_y$ is irreducible. Thus, it follows from [9, Remark 2.22] that there exists an effective $\mathbb{Q}$-divisor $D'$ on the surface $S$ such that

$$D' \sim_{\mathbb{Q}} -K_S,$$

the log pair $(S, \frac{19}{18} D')$ is not log canonical at the point $P$ and the support of the divisor $D'$ does not contain the curve $C_y$. Observe

$$C_y \cdot D' = \begin{cases} \frac{18}{19 \cdot 7} & \text{on } S_{64} \\ \frac{4}{35} & \text{on } S_{82} \end{cases} \leq \frac{18}{19 \cdot 7}.$$

This implies that the log pair $(S, \frac{19}{18} D')$ is log canonical at the point $P$. This contradicts our assumption. Thus, we see that $(S, \frac{19}{18} D)$ is log canonical away from $O_z$. Hence, to complete the proof of Proposition 4.1, we have to show that $(S, \frac{19}{18} D)$ is log canonical at the point $O_z$.

Recall that $S_{64}$ (resp. $S_{82}$) has singularity of type $\frac{1}{19}(2, 3)$ (resp. $\frac{1}{25}(2, 3)$) at the point $O_z$. In the chart $z = 1$, the surface $S_{64}$ is given by

$$t^2 + y^3 + x + x^7 y = 0$$

and $S_{82}$ by

$$t^2 + y^3 + x + x^9 y = 0.$$

In a neighborhoods of the point $O_z$, we can consider $y$ and $t$ as local weighted coordinates such that $\text{wt}(y) = 2$ and $\text{wt}(t) = 3$.

Let $f: \tilde{S} \to S$ be the weighted blow up at the singular point $O_z$ with weights $\text{wt}(y) = 2$ and $\text{wt}(t) = 3$. Denote by $E$ the exceptional curve of the blow up $f$. Then

$$K_{\tilde{S}_{64}} \sim_{\mathbb{Q}} f^*(K_{S_{64}}) - \frac{14}{19}E$$

$$K_{\tilde{S}_{82}} \sim_{\mathbb{Q}} f^*(K_{S_{82}}) - \frac{20}{25}E.$$

The surface $S$ has two singular points in $E$. One is a point of type $\frac{1}{2}(1, 1)$ and the other is of type $\frac{1}{4}(1, 1)$. Denote the former by $O_2$ and the latter by $O_3$. Observe

$$E^2 = -\frac{19}{6} \text{ on } \tilde{S}_{64}$$

$$E^2 = -\frac{25}{6} \text{ on } \tilde{S}_{82}$$

and $E \cong \mathbb{P}^1$. 

\[ Springer \]
Let $\tilde{C}_x$ be the proper transform of the curve $C_x$ on the surface $\tilde{S}$. Then

$$\tilde{C}_x \sim_Q f^*(C_x) - cE,$$

where $c = \frac{6}{19}$ for $S_{64}$ and $c = \frac{6}{25}$ for $S_{82}$, and the intersection $E \cap \tilde{C}_x$ consists of a single point different from $O_2$ and $O_3$. Note that the curves $E$ and $\tilde{C}_x$ intersect transversally.

Denote by $\tilde{\Delta}$ be the proper transform of the $\mathbb{Q}$-divisor $\Delta$ on the surface $\tilde{S}$. Then

$$\tilde{\Delta} \sim_Q f^*(\Delta) - mE$$

for some non-negative rational number $m$. To estimate it, observe

$$0 \leq \tilde{C}_x \cdot \tilde{\Delta} = \left( f^*(C_x) - cE \right) \cdot \left( f^*(\Delta) - mE \right) = C_x \cdot \Delta - m = C_x \cdot (D - aC_x) - m.$$

This implies $m \leq \frac{18 - 14a}{285}$ for $S_{64}$ and $m \leq \frac{20 - 14a}{1925}$ for $S_{82}$.

We finally suppose that the log pair $(S, \frac{19}{18}D)$ is not log canonical at $O_z$. Let $\lambda = \frac{19}{18}$. Then

$$K_S + \lambda a\tilde{C}_x + \lambda \tilde{\Delta} + \mu E \sim_Q f^*(K_S + \lambda D),$$

where

$$\mu = \frac{6\lambda a}{19} + \lambda m + \frac{14}{19} \quad \text{for } S_{64},$$

$$\mu = \frac{6\lambda a}{25} + \lambda m + \frac{20}{25} \quad \text{for } S_{82}.$$

Thus, the log pair

$$\left( \tilde{S}, \lambda a\tilde{C}_x + \lambda \tilde{\Delta} + \mu E \right)$$

is not log canonical at some point $Q \in E$.

Using $m \leq \frac{18 - 14a}{15.19}$ for $S_{64}$, $m \leq \frac{20 - 14a}{1925}$ for $S_{82}$ and $a \leq \frac{1}{2}$, we get

$$\frac{6\lambda a}{19} + \lambda m + \frac{14}{19} \leq \frac{4\lambda a}{15} + \frac{6\lambda}{95} + \frac{14}{19} \leq \frac{56\lambda}{285} + \frac{14}{19} = \frac{2422}{2565} < 1,$$

$$\frac{6\lambda a}{25} + \lambda m + \frac{20}{25} \leq \frac{4\lambda a}{19} + \frac{4\lambda}{95} + \frac{4}{5} \leq \frac{14\lambda}{95} + \frac{4}{5} = \frac{817}{855} < 1.$$

Since

$$E \cdot \tilde{\Delta} = E \cdot (f^*(\Delta) - mE) = -mE^2 = \begin{cases} \frac{19m}{6} \leq \frac{9 - 7a}{45} \leq \frac{6}{19} & \text{on } \tilde{S}_{64}, \\ \frac{25m}{6} \leq \frac{20 - 14a}{6 \cdot 19} \leq \frac{6}{19} & \text{on } \tilde{S}_{82}. \end{cases}$$
Lemmas 2.4 and 2.5 imply that \( Q \) must be the intersection point of \( E \) and \( \tilde{C}_x \). It then follows from Lemma 2.4 that

\[
\frac{18}{19} = \frac{1}{\lambda} < \left( a \tilde{C}_x + \tilde{\Delta} \right) \cdot E = a + \tilde{\Delta} \cdot E = \begin{cases} 
\frac{a + \frac{19m}{6}}{45} \leq a + \frac{9 - 7a}{45} & \text{on } \tilde{S}_{64}, \\
\frac{a + \frac{25m}{6}}{6 \cdot 19} \leq a + \frac{20 - 14a}{6 \cdot 19} & \text{on } \tilde{S}_{82}.
\end{cases}
\]

This contradicts our assumption \( a \leq \frac{1}{2} \). The obtained contradiction completes the proof. \( \square \)

**Corollary 4.3** One has \( \delta(S) \geq \frac{19}{18} \).

**Proof** See the proof of Corollary 3.3. \( \square \)

### 5 Case C

In this section, we consider the following three types of quasismooth hypersurfaces:

- \( S_{45} \): a quasismooth hypersurface in \( \mathbb{P}(7, 10, 15, 19) \) of degree 45;
- \( S_{81} \): a quasismooth hypersurface in \( \mathbb{P}(7, 18, 27, 37) \) of degree 81;
- \( S_{117} \): a quasismooth hypersurface in \( \mathbb{P}(7, 26, 39, 55) \) of degree 117.

As in the previous sections, we use \( S \) for all the surfaces \( S_{45}, S_{81}, \) and \( S_{117} \) if properties or conditions are satisfied by all the surfaces.

By appropriate coordinate changes, we may assume that the surface \( S_{45} \) is defined by the equation

\[
z^3 - y^3z + xt^2 + x^5y = 0
\]

in \( \mathbb{P}(7, 10, 15, 19) \), the surface \( S_{81} \) by

\[
z^3 - y^3z + xt^2 + x^9y = 0
\]

in \( \mathbb{P}(7, 18, 27, 37) \), and the surface \( S_{117} \) by

\[
z^3 - y^3z + xt^2 + x^{13}y = 0
\]

in \( \mathbb{P}(7, 26, 39, 55) \).

The surface \( S \) is singular at the points

\[
O_x = [1 : 0 : 0 : 0], \quad O_y = [0 : 1 : 0 : 0], \quad O_t = [0 : 0 : 0 : 1], \quad Q = [0 : 1 : 1 : 0],
\]

and is smooth away from them. Moreover, the surface \( S_{45} \) (resp. \( S_{81} \) and \( S_{117} \)) has quotient singularity of types \( \frac{1}{7}(1, 5), \frac{1}{10}(7, 9), \frac{1}{19}(2, 3), \frac{1}{5}(1, 2) \) (resp. \( \frac{1}{7}(3, 1), \frac{1}{18}(7, 1), \frac{1}{27}(2, 3), \frac{1}{5}(7, 1) \) and \( \frac{1}{7}(2, 3), \frac{1}{26}(7, 3), \frac{1}{53}(2, 3), \frac{1}{13}(7, 3) \)) at the points \( O_x, O_y, O_t, Q \), respectively.
Let $C_x$ be the curve in $S$ that is cut out by $x = 0$. Then

$$C_x = L_{xz} + R_x,$$

where $L_{xz}$ is the curve given by $x = z = 0$ and $R_x$ by $x = z^2 - y^3 = 0$ in the ambient weighted projective space. These two curves $L_{xz}$ and $R_x$ meet each other at the point $O_I$. Also, we have

$$L_{xz}^2 = -\frac{23}{10 \cdot 19}, \quad R_x^2 = -\frac{8}{5 \cdot 19}, \quad L_{xz} \cdot R_x = \frac{3}{19} \quad \text{on } S_{45},$$

$$L_{xz}^2 = -\frac{47}{18 \cdot 37}, \quad R_x^2 = -\frac{20}{9 \cdot 37}, \quad L_{xz} \cdot R_x = \frac{3}{37} \quad \text{on } S_{81}, \quad (5.1)$$

$$L_{xz}^2 = -\frac{71}{26 \cdot 55}, \quad R_x^2 = -\frac{32}{13 \cdot 55}, \quad L_{xz} \cdot R_x = \frac{3}{55} \quad \text{on } S_{117}.$$

Note also that the curve $R_x$ is singular at the point $O_I$.

Let $C_y$ be the curve in $S$ cut out by $y = 0$. Then $C_y$ is irreducible and

$$\frac{35}{54} = \lct \left( S_{45}, \frac{6}{7} C_x \right) < \lct \left( S_{45}, \frac{6}{10} C_y \right) = \frac{25}{18},$$

$$\frac{35}{72} = \lct \left( S_{81}, \frac{8}{7} C_x \right) < \lct \left( S_{81}, \frac{8}{18} C_y \right) = \frac{15}{8},$$

$$\frac{7}{18} = \lct \left( S_{117}, \frac{10}{7} C_x \right) < \lct \left( S_{117}, \frac{10}{26} C_y \right) = \frac{13}{6}.$$

In fact, in these three cases $\alpha(S)$ is given by the numbers $\frac{35}{54}$, $\frac{35}{72}$, and $\frac{7}{18}$ on the left-hand sides by [8, Theorem 1.10].

To estimate $\delta(S)$, we fix an effective $\mathbb{Q}$-divisor $D$ on the surface $S$ such that

$$D \sim_{\mathbb{Q}} -K_S$$

and write $D = aL_{xz} + bR_x + \Delta$, where $a$ and $b$ are non-negative numbers, and $\Delta$ is an effective $\mathbb{Q}$-divisor on the surface $S$ whose support does not contain the curves $L_{xz}$ and $R_x$.

**Lemma 5.2** If the $\mathbb{Q}$-divisor $D$ is of $k$-basis type with $k \gg 0$, then

$$a \leq \begin{cases} \frac{2}{5} & \text{on } S_{45} \\ \frac{1}{2} & \text{on } S_{81} \\ \frac{1}{11} & \text{on } S_{117} \end{cases}, \quad b \leq \begin{cases} \frac{1}{3} & \text{on } S_{45} \\ \frac{1}{5} & \text{on } S_{81} \\ \frac{12}{25} & \text{on } S_{117} \end{cases}.$$
Proof Suppose that $D$ is of $k$-basis type with $k \gg 0$. Theorem 2.9 implies that

$$a \leq \frac{1}{(-K_S)^2} \int_0^{\infty} \text{vol}(-K_S - \lambda L_{xz})d\lambda + \epsilon_k,$$

where $\epsilon_k$ is a small constant depending on $k$ such that $\epsilon_k \to 0$ as $k \to \infty$. Since

$$-K_S - \lambda L_{xz} \sim Q \left\{ \begin{array}{ll}
\left(\frac{6}{7} - \lambda\right) L_{xz} + \frac{6}{7} R_x & \text{on } S_{45}, \\
\left(\frac{8}{7} - \lambda\right) L_{xz} + \frac{8}{7} R_x & \text{on } S_{81}, \\
\left(\frac{10}{7} - \lambda\right) L_{xz} + \frac{10}{7} R_x & \text{on } S_{117}
\end{array} \right.$$

and $R_x^2 < 0$, we have $\text{vol}(-K_S - \lambda L_{xz}) = 0$ for $\lambda \geq \frac{6}{7}$ on $S_{45}$, $\lambda \geq \frac{8}{7}$ on $S_{81}$ and $\lambda \geq \frac{10}{7}$ on $S_{117}$. Similarly, using (5.1), we see that

$$(-K_S - \lambda L_{xz}) \cdot R_x = \left\{ \begin{array}{ll}
\left(\left(\frac{6}{7} - \lambda\right) L_{xz} + \frac{6}{7} R_x\right) \cdot R_x = \frac{6 - 15\lambda}{19 \cdot 5} & \text{on } S_{45}, \\
\left(\left(\frac{8}{7} - \lambda\right) L_{xz} + \frac{8}{7} R_x\right) \cdot R_x = \frac{8 - 27\lambda}{37 \cdot 9} & \text{on } S_{81}, \\
\left(\left(\frac{10}{7} - \lambda\right) L_{xz} + \frac{10}{7} R_x\right) \cdot R_x = \frac{10 - 39\lambda}{13 \cdot 55} & \text{on } S_{117}.
\end{array} \right.$$

This shows that the divisor $-K_S - \lambda L_{xz}$ is nef for $\lambda \leq \frac{2}{5}$ on $S_{45}$, $\lambda \leq \frac{8}{27}$ on $S_{81}$ and $\lambda \leq \frac{10}{39}$ on $S_{117}$. Thus, we have

$$\text{vol}(-K_S - \lambda L_{xz}) = (-K_S - \lambda L_{xz})^2 = \left\{ \begin{array}{ll}
\frac{54}{665} - \frac{6\lambda}{95} - \frac{23\lambda^2}{190} & \text{for } \lambda \leq \frac{2}{5} \text{ on } S_{45}, \\
\frac{32}{777} - \frac{8\lambda}{333} - \frac{47\lambda^2}{666} & \text{for } \lambda \leq \frac{8}{27} \text{ on } S_{81}, \\
\frac{200}{7007} - \frac{12\lambda}{1001} - \frac{36\lambda^2}{715} & \text{for } \lambda \leq \frac{10}{39} \text{ on } S_{117}.
\end{array} \right.$$

To compute $\text{vol}(-K_S - \lambda L_{xz})$ for $\frac{2}{5} < \lambda < \frac{6}{7}$ on $S_{45}$, $\frac{8}{27} < \lambda < \frac{8}{7}$ on $S_{81}$ and $\frac{10}{39} < \lambda < \frac{10}{7}$ on $S_{117}$, we let

$$N = \left\{ \begin{array}{ll}
\left(\frac{6}{7} - \lambda\right) L_{xz} + \left(\frac{6}{7} - \frac{15\lambda - 6}{8}\right) R_x & \text{for } S_{45}, \\
\left(\frac{8}{7} - \lambda\right) L_{xz} + \left(\frac{8}{7} - \frac{27\lambda - 8}{20}\right) R_x & \text{for } S_{81}, \\
\left(\frac{10}{7} - \lambda\right) L_{xz} + \left(\frac{10}{7} - \frac{39\lambda - 10}{32}\right) R_x & \text{for } S_{117}.
\end{array} \right.$$
Then, using (5.1) again, we see that $N \cdot R_x = 0$ and $N \cdot L_{xz} \geq 0$. Thus, we conclude that the divisor $N$ is nef on the respective interval for $\lambda$. This shows that

$$
-K_S - \lambda L_{xz} \sim_Q \begin{cases} 
N + \frac{15\lambda - 6}{8} R_x & \text{on } S_{45}, \\
N + \frac{27\lambda - 8}{20} R_x & \text{on } S_{81}, \\
N + \frac{39\lambda - 10}{32} R_x & \text{on } S_{117}
\end{cases}
$$

is the Zariski decomposition of the divisor $-K_S - \lambda L_{xz}$. Hence, we have

$$
\text{vol}(-K_S - \lambda L_{xz}) = N^2 = \begin{cases} 
\frac{1}{280}(6 - 7\lambda)^2 & \text{on } S_{45}, \\
\frac{1}{1260}(8 - 7\lambda)^2 & \text{on } S_{81}, \\
\frac{369}{1121120}(10 - 7\lambda)^2 & \text{on } S_{117}
\end{cases}
$$

by (2.8). Thus, integrating, we get

$$
a \leq \frac{1}{(-K_S)^2} \int_0^\infty \text{vol}(-K_S - \lambda L_{xz}) d\lambda + \epsilon_k = \begin{cases} 
\frac{118}{315} + \epsilon_k & \text{for } S_{45}, \\
\frac{760}{1701} + \epsilon_k & \text{for } S_{81}, \\
\frac{8780}{17199} + \epsilon_k & \text{for } S_{117}
\end{cases}
$$

This gives us the asserted bounds for $a$.

Meanwhile, we have

$$
\text{vol}(-K_S - \lambda R_x) = (-K_S - \lambda R_x)^2 = \begin{cases} 
\frac{54}{665} - \frac{12\lambda}{95} - \frac{8\lambda^2}{95} & \text{for } 0 \leq \lambda \leq \frac{1}{5} \text{ on } S_{45}, \\
\frac{1}{9 \cdot 37 \cdot 49} - \frac{32 \cdot 21}{9 \cdot 37} - \frac{16\lambda}{9 \cdot 37} - \frac{20\lambda^2}{9 \cdot 37} & \text{for } 0 \leq \lambda \leq \frac{4}{27} \text{ on } S_{81}, \\
\frac{30}{1001} - \frac{4\lambda}{143} - \frac{32\lambda^2}{715} & \text{for } 0 \leq \lambda \leq \frac{5}{39} \text{ on } S_{117}
\end{cases}
$$

since the divisor $-K_S - \lambda R_x$ is nef for the values $\lambda$ in the respective interval. The Zariski decomposition of the divisor $-K_S - \lambda R_x$ is given by
\[
\begin{align*}
&\left\{ \begin{array}{l}
\left( \frac{6}{7} - \frac{30\lambda - 6}{23} \right) L_{xz} + \left( \frac{6}{7} - \lambda \right) R_x + \frac{30\lambda - 6}{23} L_{xz} \\
\text{nef R-divisor}
\end{array} \right. \\
&\left\{ \begin{array}{l}
\left( \frac{8}{7} - \frac{54\lambda - 8}{47} \right) L_{xz} + \left( \frac{8}{7} - \lambda \right) R_x + \frac{54\lambda - 8}{47} L_{xz} \\
\text{nef R-divisor}
\end{array} \right. \\
&\left\{ \begin{array}{l}
\left( \frac{10}{7} - \frac{78\lambda - 10}{71} \right) L_{xz} + \left( \frac{10}{7} - \lambda \right) R_x + \frac{78\lambda - 10}{71} L_{xz} \\
\text{nef R-divisor}
\end{array} \right.
\end{align*}
\]

so that we could obtain

\[
\text{vol}( -K_S - \lambda R_x ) = \left\{ \begin{array}{c}
\left( \frac{6}{7} - \frac{30\lambda - 6}{23} \right) L_{xz} + \left( \frac{6}{7} - \lambda \right) R_x \bigg|^2 = \frac{2}{5} \cdot 23 \cdot (6 - 7\lambda)^2, \\
\left( \frac{8}{7} - \frac{54\lambda - 8}{47} \right) L_{xz} + \left( \frac{8}{7} - \lambda \right) R_x \bigg|^2 = \frac{2}{7} \cdot 47 \cdot (8 - 7\lambda)^2, \\
\left( \frac{10}{7} - \frac{78\lambda - 10}{71} \right) L_{xz} + \left( \frac{10}{7} - \lambda \right) R_x \bigg|^2 = \frac{2}{7} \cdot 13 \cdot 71 \cdot (10 - 7\lambda)^2.
\end{array} \right.
\]

Finally, \( \text{vol}( -K_S - \lambda R_x ) = 0 \) for \( \lambda > \frac{6}{7} \) on \( S_{45} \), for \( \lambda > \frac{8}{7} \) on \( S_{81} \), and for \( \lambda > \frac{10}{7} \) on \( S_{117} \) since \( -K_S - \lambda R_x \) is not pseudoeffective for these values \( \lambda \). Thus, by Theorem 2.9, we have

\[
b \leq \frac{1}{(-K_S)^2} \int_{\lambda_0}^{\infty} \text{vol}( -K_S - \lambda R_x ) d\lambda + \varepsilon_k = \left\{ \begin{array}{c}
\frac{97}{315} + \varepsilon_k \text{ for } S_{45}, \\
\frac{107090068}{52821363} + \varepsilon_k \text{ for } S_{81}, \\
\frac{1205}{2457} + \varepsilon_k \text{ for } S_{117}.
\end{array} \right.
\]

This yields the required bounds for \( b \). \( \square \)

Now we prove the main assertion in this section.

**Proposition 5.3** If \( a \) and \( b \) satisfy the bounds in Lemma 5.2 then the log pair \((S, \frac{65}{64} D)\) is log canonical.

**Proof** We suppose that \( a \) and \( b \) satisfying the bounds in Lemma 5.2.

We first claim that the log pair \((S, \frac{65}{64} D)\) is log canonical outside of \( C_x \) and \( C_y \). This immediately follows from the same argument as in the beginning of the proof of Proposition 4.1 with the pencil \( \mathcal{P} \) given by

\[
\begin{align*}
\nu(x^{10} + c_1 y^7) + \mu y^4(z^2 + c_2 y^3) &= 0 \text{ on } S_{45}, \\
\nu(x^{18} + c_1 y^7) + \mu y^4(z^2 + c_2 y^3) &= 0 \text{ on } S_{81}, \\
\nu(x^{26} + c_1 y^7) + \mu y^4(z^2 + c_2 y^3) &= 0 \text{ on } S_{117},
\end{align*}
\]
where \( c_1 \) and \( c_2 \) are appropriate constants, for \([v : \mu] \in \mathbb{P}^1\). For a general member \( C \) in \( \mathcal{P} \) we obtain

\[
C \cdot D \leq \frac{64}{65},
\]

which verifies the claim. Notice that the surface \( S \) is smooth outside \( C_x \) and \( C_y \).

We now consider a point \( P \) on \( C_y \) different from \( O_t \). Suppose that the log pair \((S, \frac{65}{64} D)\) is not log canonical at the point \( P \). Recall that \((S, \frac{65}{64} C_y)\) is log canonical, where \( e \) is the positive rational number such that \(-K_S \sim_{\mathbb{Q}} e C_y\), and that the curve \( C_y \) is irreducible. Thus, it follows from [9, Remark 2.22] that there exists an effective \( \mathbb{Q} \)-divisor \( D' \) on the surface \( S \) such that

\[
D' \sim_{\mathbb{Q}} -K_S,
\]

the log pair \((S, \frac{65}{64} D')\) is not log canonical at the point \( P \), and the support of the divisor \( D' \) does not contain the curve \( C_y \). Observe that

\[
C_y \cdot D' \leq \frac{64}{7 \cdot 65}.
\]

This implies that the log pair \((S, \frac{65}{64} D')\) is log canonical at the point \( P \). This contradiction shows that the log pair \((S, \frac{65}{64} D)\) is log canonical outside \( C_x \).

Let \( P \) be a point on \( C_x \) other than \( O_t \). We have two cases for the location of \( P \), i.e., when \( P \) lies on \( L_{xz} \) and when it lies on \( R_x \). Note that we always have \( \frac{65a}{64} < 1 \) and \( \frac{65b}{64} < 1 \).

We first consider the case where \( P \) belongs to \( L_{xz} \). Then the log pair \((S, L_{xz} + \frac{65b}{64} R_x + \frac{65}{64} \Delta)\) is log canonical at \( P \). Indeed,

\[
(bR_x + \Delta) \cdot L_{xz} = (D - aL_{xz}) \cdot L_{xz} = \begin{cases} 6 + \frac{23a}{190} & \leq \frac{64}{65 \cdot 10} \text{ for } S_{45}, \\ 8 + \frac{47a}{37 \cdot 18} & \leq \frac{64}{65 \cdot 18} \text{ for } S_{81}, \\ 10 + \frac{71a}{55 \cdot 26} & \leq \frac{64}{65 \cdot 26} \text{ for } S_{117}. \end{cases}
\]

Lemmas 2.4 or 2.5 implies that \((S, \frac{65}{64} D)\) is log canonical at the point \( P \). If the point \( P \) must lie on \( R_x \), then we consider

\[
(aL_{xz} + \Delta) \cdot R_x = (D - bR_x) \cdot R_x = \begin{cases} 3 + \frac{8b}{95} & \leq \frac{64}{65 \cdot 5} \text{ for } S_{45}, \\ 8 + \frac{20b}{9 \cdot 37} & \leq \frac{64}{65 \cdot 9} \text{ for } S_{81}, \\ 10 + \frac{32b}{13 \cdot 55} & \leq \frac{64}{65 \cdot 13} \text{ for } S_{117}. \end{cases}
\]
Lemmas 2.4 or 2.5 then show that \((S, \frac{65}{64}D)\) is log canonical at the point \(P\). Now it is enough to shows that \((S, \frac{65}{64}D)\) is log canonical at \(O_t\).

Recall that \(S_{45}\) (resp. \(S_{81}\) and \(S_{117}\)) has singularity of type \(\frac{1}{19}(2, 3)\) (resp. \(\frac{1}{37}(2, 3)\) and \(\frac{1}{55}(2, 3)\)) at the point \(O_t\). In the chart given by \(t = 1\), the surface \(S_{45}\) is given by

\[
z^3 - y^3z + x + x^5y = 0,
\]

the surface \(S_{81}\) by

\[
z^3 - y^3z + x + x^9y = 0,
\]

and the surface \(S_{117}\) by

\[
z^3 - y^3z + x + x^{13}y = 0.
\]

In a neighborhood of the point \(O_t\), we can consider \(y\) and \(z\) as local weighted coordinates such that \(\text{wt}(y) = 2\) and \(\text{wt}(z) = 3\).

Let \(f : \tilde{S} \to S\) be the weighted blow up at the singular point \(O_t\) such that \(\text{wt}(y) = 2\) and \(\text{wt}(z) = 3\). Denote by \(E\) the exceptional curve of the blow up \(f\). Then

\[
K_{\tilde{S}_{45}} \sim_{\mathbb{Q}} f^*(K_{S_{45}}) - \frac{14}{19}E,
\]

\[
K_{\tilde{S}_{81}} \sim_{\mathbb{Q}} f^*(K_{S_{81}}) - \frac{32}{37}E,
\]

\[
K_{\tilde{S}_{117}} \sim_{\mathbb{Q}} f^*(K_{S_{117}}) - \frac{10}{11}E.
\]

The surface \(S\) has two singular points in \(E\). One is of type \(\frac{1}{2}(1, 1)\) and the other is of type \(\frac{1}{3}(1, 1)\). Denote the former one by \(O_2\) and the latter one by \(O_3\). Observe

\[
E^2 = \begin{cases} 
- \frac{19}{6} & \text{on } \tilde{S}_{45}, \\
- \frac{37}{6} & \text{on } \tilde{S}_{81}, \\
- \frac{55}{6} & \text{on } \tilde{S}_{117},
\end{cases}
\]

and \(E \cong \mathbb{P}^1\).

Let \(\tilde{L}_{xz}\) and \(\tilde{R}_x\) be the proper transforms of the curve \(L_{xz}\) and \(R_x\) to the surface \(\tilde{S}\), respectively. Then

\[
\tilde{L}_{xz} \sim_{\mathbb{Q}} f^*(L_{xz}) - \frac{3}{c}E, \quad \tilde{R}_x \sim_{\mathbb{Q}} f^*(R_x) - \frac{6}{c}E,
\]
where \( c \) is the index of singularity \( O_t \). The intersection \( E \cap \tilde{L}_{xz} \) consists of the point \( O_2 \) and the intersection \( E \cap \tilde{R}_x \) consists of a single smooth point. Note that \( \tilde{L}_{xz} \cdot E = \frac{1}{2} \) and the curves \( E \) and \( \tilde{R}_x \) intersect transversally.

Recall that \( D = aL_{xz} + bR_x + \Delta \). Denote by \( \tilde{\Delta} \) the proper transform of the \( \mathbb{Q} \)-divisor \( \Delta \) on the surface \( \tilde{S} \). Then

\[
\tilde{\Delta} \sim f^*(\Delta) - mE
\]

for some non-negative rational number \( m \). To estimate \( m \), consider the intersection

\[
0 \leq \tilde{R}_x \cdot \tilde{\Delta} = \tilde{R}_x \cdot \left( f^*(\Delta) - mE \right) = R_x \cdot \Delta - m.
\]

Applying (5.1), we are able to obtain

\[
m \leq \begin{cases}
\frac{6}{5} \cdot \frac{19}{19} - \frac{3a}{5} + \frac{8b}{5} & \leq \frac{6}{5} \cdot \frac{19}{19} & \leq \frac{26}{285} \\
\frac{8}{9} \cdot \frac{37}{37} - \frac{3a}{9} + \frac{20b}{9} & \leq \frac{8}{9} \cdot \frac{37}{37} & \leq \frac{4}{111} \\
\frac{2}{11} \cdot \frac{13}{13} - \frac{3a}{55} + \frac{32b}{13} & \leq \frac{2}{11} \cdot \frac{13}{13} & \leq \frac{634}{17875}
\end{cases}
\]

(5.4)

We now suppose that the log pair \((\tilde{S}, \frac{65}{64} D)\) is not log canonical at \( O_t \). Put \( \lambda = \frac{65}{64} \). Then

\[
K_{\tilde{S}} + \lambda a\tilde{L}_{xz} + \lambda b\tilde{R}_x + \lambda \tilde{\Delta} + \mu E \sim f^*(K_S + \lambda D),
\]

where

\[
\mu = \begin{cases}
\frac{3\lambda a}{19} + \frac{6\lambda b}{19} + \lambda m + \frac{14}{19} & \text{for } S_{45}, \\
\frac{3\lambda a}{37} + \frac{6\lambda b}{37} + \lambda m + \frac{32}{37} & \text{for } S_{81}, \\
\frac{3\lambda a}{55} + \frac{6\lambda b}{55} + \lambda m + \frac{10}{11} & \text{for } S_{117},
\end{cases}
\]

Thus, the log pair

\[
(\tilde{S}, \lambda a\tilde{L}_{xz} + \lambda b\tilde{R}_x + \lambda \tilde{\Delta} + \mu E)
\]

(5.5)

is not log canonical at some point \( O \in E \). Using (5.4) and bounds for \( b \), we can easily check
\[
\mu \leq \begin{cases}
\frac{3\lambda a}{19} + \frac{6\lambda b}{19} + \frac{6\lambda}{95} - \frac{3a\lambda}{19} + \frac{8\lambda b}{95} + \frac{14}{19} = \frac{2\lambda b}{5} + \frac{6\lambda}{95} + \frac{14}{19} \leq 1 & \text{for } S_{45}, \\
\frac{3\lambda a}{37} + \frac{6\lambda b}{37} + \frac{8\lambda}{9 \cdot 37} - \frac{3\lambda a}{37} + \frac{20\lambda b}{37} + \frac{32}{37} = 2 \cdot \frac{9\lambda b}{125} + \frac{8\lambda}{9 \cdot 37} + \frac{32}{37} \leq 1 & \text{for } S_{81}, \\
\frac{3\lambda a}{55} + \frac{6\lambda b}{55} + \frac{2\lambda}{11 \cdot 13} - \frac{3\lambda a}{55} + \frac{32\lambda b}{10} = \frac{2\lambda b}{13} + \frac{2\lambda}{11 \cdot 13} + \frac{10}{11} \leq 1 & \text{for } S_{117}.
\end{cases}
\]

If \( O = E \cap \tilde{R}_x \), then we apply Lemma 2.4 to (5.5) and \( E \). This yields

\[
\lambda b + \lambda \tilde{\Delta} \cdot E = \left( \lambda b \tilde{R}_x + \lambda \tilde{\Delta} \right) \cdot E > 1
\]

so that we could obtain absurd inequalities

\[
\frac{64}{65} = \frac{1}{\lambda} < b + \tilde{\Delta} \cdot E = b + \frac{cm}{6} \leq \begin{cases}
\frac{1}{3} + \frac{19}{60} = \frac{13}{20} & \text{for } S_{45}, \\
\frac{1}{5} + \frac{37}{6 \cdot 25} = \frac{67}{150} & \text{for } S_{81}, \\
\frac{12}{25} + \frac{1}{3} = \frac{61}{75} & \text{for } S_{117},
\end{cases}
\]

where \( c \) is the index of the singularity \( O_t \). The inequality

\[
\tilde{\Delta} \cdot E = \frac{cm}{6} \leq \begin{cases}
\frac{13}{45} & \text{for } S_{45}, \\
\frac{2}{9} & \text{for } S_{81}, \\
\frac{317}{975} & \text{for } S_{117},
\end{cases} \leq \frac{1}{3\lambda} = \frac{64}{3 \cdot 65}
\]

implies that \( O = O_2 \). However, using (5.4) and Lemma 2.5 (applied to (5.5) and \( E \)), we conclude that the log pair (5.5) is log canonical everywhere since

\[
\left( a \tilde{L}_{x^2} + \tilde{\Delta} \right) \cdot E = \frac{a}{2} + \tilde{\Delta} \cdot E = \frac{a}{2} + \frac{cm}{6} \leq \begin{cases}
\frac{1}{5} + \frac{4b}{15} \leq \frac{13}{45} & \text{for } S_{45}, \\
\frac{4}{27} + \frac{10b}{27} \leq \frac{2}{9} & \text{for } S_{81}, \\
\frac{5}{39} + \frac{16b}{39} \leq \frac{317}{975} & \text{for } S_{117},
\end{cases}
\]

\[
\leq \frac{1}{2\lambda} = \frac{32}{65}
\]

This completes the proof. \( \square \)
Corollary 5.6 The $\delta$-invariant of $S$ is at least $\frac{65}{64}$.

Proof This immediately follows from Proposition 5.3 and Lemma 5.2.

Acknowledgements We started to work on this paper during our stay at the Erwin Schrödinger International Institute for Mathematics and Physics in Vienna in August 2018. We are grateful to the institute for excellent working conditions. Ivan Cheltsov was supported by the Royal Society Grant No. IES R1 180205 and the Russian Academic Excellence Project 5-100. Jihun Park was supported by IBS-R003-D1, Institute for Basic Science in Korea. Constantin Shramov was supported by the Russian Academic Excellence Project 5-100 and Young Russian Mathematics award.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Araujo, C.: Kähler–Einstein metrics for some quasi-smooth log del Pezzo surfaces. Trans. AMS 354, 4303–4312 (2002)
2. Bartolo, E., Martín-Morales, J., Ortigas-Galindo, J.: Intersection theory on Abelian-quotient V-surfaces and Q-resolutions. J. Singul. 8, 11–30 (2014)
3. Bauer, Th, Küranya, A., Szemberg, T.: Zariski chambers, volumes, and stable base loci. J. Reine Angew. Math. 576, 209–233 (2004)
4. Blum, H., Jonsson, M.: Thresholds, valuations, and $K$-stability, preprint, arXiv:1706.04548 (2017)
5. Boyer, C., Galicki, K.: Sasakian Geometry. Oxford University Press, Oxford (2008)
6. Boyer, C., Galicki, K., Nakamaye, M.: On the geometry of Sasakian–Einstein 5-manifolds. Math. Annalen 325, 485–524 (2003)
7. Cheltsov, I.: Log canonical thresholds of del Pezzo surfaces. Geom. Funct. Anal. 11, 1118–1144 (2008)
8. Cheltsov, I., Park, J., Shramov, C.: Exceptional del Pezzo hypersurfaces. J. Geom. Anal. 20(4), 787–816 (2010)
9. Cheltsov, I., Shramov, C.: Log canonical thresholds of smooth Fano threefolds. Russ. Math. Surv. 63, 73–180 (2008)
10. Cheltsov, I., Shramov, C.: Del Pezzo zoo. Exp. Math. 22, 313–326 (2013)
11. Cheltsov, I., Rubinstein, Y., Zhang, K.: Basis log canonical thresholds, local intersection estimates, and asymptotically log del Pezzo surfaces. Sel. Math. (N.S.) 25(2), 25–34 (2019)
12. Cheltsov, I., Zhang, K.: Delta invariants of smooth cubic surfaces. Eur. J. Math. 5(3), 729–762 (2019)
13. Chen, X.-X., Donaldson, S., Sun, S.: Kähler–Einstein metrics on Fano manifolds. I, II, III. J. Amer. Math. Soc. 28(1), 183–197, 199–234, 235–278 (2015)
14. Demaillie, J.-P., Kollár, J.: Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds. Ann. Sci. École Norm. Sup. 34, 525–556 (2001)
15. Dolgachev, I.: Weighted Projective Varieties. Lecture Notes in Mathematics, vol. 956, pp. 34–71. Springer, Berlin (1982)
16. Fujita, K., Odaka, Y.: On the $K$-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. 70(4), 511–521 (2018)
17. Gauntlett, J., Martelli, D., Sparks, J., Yau, S.-T.: Obstructions to the existence of Sasaki–Einstein metrics. Commun. Math. Phys. 273, 803–827 (2007)
18. Iano-Fletcher, A.R.: Working with Weighted Complete Intersections, Explicit Birational Geometry of 3-Folds. Lecture Notes Series, vol. 281, pp. 101–173. Cambridge University Press, Cambridge (2000)
19. Johnson, J., Kollár, J.: Kähler–Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces. Ann. de l’Inst. Fourier 51, 69–79 (2001)
20. Kollár, J.: Singularities of pairs. Proc. Symp. Pure Math. 62, 221–287 (1997)
Delta Invariants of Singular del Pezzo Surfaces

21. Lazarsfeld, R.: Positivity in Algebraic Geometry, I, II. Springer, Berlin (2004)
22. Li, C., Tian, G., Wang, F.: On Yau–Tian–Donaldson conjecture for singular Fano varieties. Preprint, arXiv:1711.09530 (2017)
23. Li, C., Tian, G., Wang, F.: The uniform version of Yau–Tian–Donaldson conjecture for singular Fano varieties. Preprint, arXiv:1903.01215 (2019)
24. Nadel, A.: Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature. Ann. Math. 132, 549–596 (1990)
25. Odaka, Y., Spotti, C., Sun, S.: Compact moduli spaces of Del Pezzo surfaces and Kähler–Einstein metrics. J. Differ. Geom. 102, 127–172 (2016)
26. Paemurru, E.: Del Pezzo surfaces in weighted projective spaces. Proc. EMS. 61, 545–572 (2018)
27. Park, J., Won, J.: K-stability of smooth del Pezzo surfaces. Math. Ann. 372(3–4), 1239–1276 (2018)
28. Prokhorov, Y.: Lectures on Complements on Log Surfaces, MSJ Memoirs 10. Mathematical Society of Japan, Tokyo (2001)
29. Prokhorov, Y.: On the Zariski decomposition problem. Proc. Steklov Inst. Math. 240, 37–65 (2003)
30. Shi, Y.-L.: On the $\alpha$-invariants of cubic surfaces with Eckardt points. Adv. Math. 225, 1285–1307 (2010)
31. Shokurov, V.: 3-Fold log flips. Russ. Acad. Sci. Izv. Math. 40, 93–202 (1993)
32. Tian, G.: On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M)>0$. Inv. Math. 89, 225–246 (1987)
33. Tian, G.: On Calabi’s conjecture for complex surfaces with positive first Chern class. Invent. Math. 101, 101–172 (1990)
34. Tosatti, V.: K-stability and Kähler–Einstein metrics. Commun. Pure Appl. Math. 68, 1085–1156 (2015)
35. Tosatti, V.: Kähler–Einstein metrics on Fano surfaces. Expos. Math. 30, 11–31 (2012)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.