HIROTA-KIMURA TYPE DISCRETIZATION OF THE CLASSICAL NONHOLONOMIC SUSLOV PROBLEM

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Abstract. We constructed Hirota-Kimura type discretization of the classical nonholonomic Suslov problem of motion of rigid body fixed at a point. We found a first integral proving integrability. Also, we have shown that discrete trajectories asymptotically tend to a line of discrete analogies of so-called steady-state rotations. The last property completely corresponds to well-known property of the continuous Suslov case. The explicite formulae for solutions are given. In $n$-dimensional case we give discrete equations.

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1. INTRODUCTION

There are several methods for constructing discrete counterparts of integrable dynamical systems in classical mechanics. One is well-known Veselov-Moser discretization (see [12]). By this method, based on discrete variational principle, many of discrete integrable systems are found (see [13]). In cases with discrete Lagrangian, corresponding discrete map is Poisson with respect to certain Poisson structure. Usually the discrete map is multi-valued. Recently, Hirota and Kimura constructed explicit integrable discretizations of the Euler and the Lagrange cases of motion of a heavy rigid body fixed at a point using Hirota’s bilinear method [5, 9]. They found first integrals of motion and they solved equations in terms of elliptic functions. Suris and Petrera in [13] found bi-Hamiltonian structure for discrete Euler top.

Our goal is to apply Hirota-Kimura method to the classical nonholonomic Suslov problem. Suslov in [15] considered motion of a rigid body fixed at a point with projection of angular velocity to an axis fixed in the body equal
to zero. Solutions in terms of trigonometric and exponential functions are given as well in [15]. There are various generalizations of classical Suslov problem: integrable potential perturbations ([10, 2]) and higher-dimensional generalizations ([3, 8, 7, 16]). In [4] Fedorov and Zenkov presented certain discretization of Suslov problem based on the Veselov-Moser discretization and its extension to nonholonomic cases suggested in [1, 11]. Obtained map is multi-valued. Discrete trajectories asymptotically tend to discrete analogies of so-called steady-state rotations, the property characteristic for continuous case considered by Suslov. Moreover, Fedorov and Zenkov [4] gave the equations in $n$-dimensional case and proved similar asymptotic behavior as in three-dimensional case.

In the present paper we use the Hirota-Kimura method and get explicit discretization of the reduced Suslov problem on the linear subspace of algebra $so(3)$. The corresponding map is not multi-valued. We gave one first integral which appears to be enough for integration. Using a linear change of variables, we have found explicit solutions in terms of exponential functions. (Notice, that in the case of discretization given in [4], the explicit solutions are not known.) Presented discrete version of Suslov case has similar asymptotic behavior as the continuous one.

The paper is organized as follows. In Section 2 basic facts about the classical Suslov problem are given. In Section 3 we present discrete equations of Hirota-Kimura type and we construct first integral of motion. Integration procedure for discrete equations is performed in Section 4. In Section 5 we give the discrete equations for $n$-dimensional Suslov case.

## 2. A brief account of the classical Suslov problem

The classical nonholonomic Suslov problem is defined in [15]. Configuration space is Lie group $SO(3)$. In a bases fixed in the body, the equations of the motion are:

\[
\dot{\mathbf{M}} = \mathbf{M} \times \mathbf{\Omega} + \lambda \mathbf{a} \\
\langle \mathbf{a}, \mathbf{\Omega} \rangle = 0.
\]

Here $\mathbf{\Omega}$ is the angular velocity, and $\mathbf{M} = I \mathbf{\Omega}$ is the angular momentum, $I$ is the inertia operator, $\mathbf{a}$ is a unit vector fixed in the body and $\lambda$ is the Lagrange multiplier. In a bases chosen such that $\mathbf{a} = (0, 0, 1)$ and

\[
I = \begin{bmatrix}
I_1 & 0 & I_{13} \\
0 & I_2 & I_{23} \\
I_{13} & I_{23} & I_3
\end{bmatrix},
\]
the equations (1) become:

\[ I_1 \dot{\Omega}_1 = -I_{13} \Omega_1 \Omega_2 - I_{23} \Omega_2^2 \]
\[ I_2 \dot{\Omega}_2 = I_{13} \Omega_1^2 + I_{23} \Omega_1 \Omega_2 \]
(2)

\[ 0 = -I_{13} \dot{\Omega}_1 - I_{23} \dot{\Omega}_2 + (I_1 - I_2) \Omega_1 \Omega_2 + \lambda \]

\[ \Omega_3 = 0. \]

The first two equations are closed in \( \Omega_1 \) and \( \Omega_2 \). After solving them one finds the Lagrange multiplier \( \lambda \) as a function of time from the third equation. Hence, for complete integrability by quadratures, only one first integral of motion is necessary. This is the energy integral as follows from (2) easily. Suslov in [15] gave solutions of the system in terms of trigonometric and exponential functions. He observed a remarkable fact (as the referee observed to be known before to Walker and Routh): the motion of the body asymptotically tends to a line of rotations with constant angular velocities which satisfy \( I_{13} \Omega_1 + I_{23} \Omega_2 = 0 \).

3. HIROTA-KIMURA TYPE DISCRETIZATION OF THE SUSLOV PROBLEM

In the spirit of Hirota-Kimura, a discrete counterpart of the first two equations and nonholonomic constraint of (1) is:

\[ I_1 (\tilde{\Omega}_1 - \Omega_1) + I_{13} (\tilde{\Omega}_3 - \Omega_3) = \epsilon \frac{I_2}{2} (\tilde{\Omega}_2 \Omega_3 + \Omega_2 \tilde{\Omega}_3) \]
\[ + I_{23} \Omega_3 \tilde{\Omega}_3 - \frac{I_3}{2} (\tilde{\Omega}_2 \Omega_3 + \Omega_2 \tilde{\Omega}_3) - \frac{I_{13}}{2} (\tilde{\Omega}_1 \Omega_2 + \Omega_1 \tilde{\Omega}_2) - I_{23} \Omega_2 \tilde{\Omega}_2 \]
(3)

\[ I_2 (\tilde{\Omega}_2 - \Omega_2) + I_{23} (\tilde{\Omega}_3 - \Omega_3) = \epsilon \frac{I_3}{2} (\tilde{\Omega}_1 \Omega_3 + \Omega_1 \tilde{\Omega}_3) \]
\[ - I_{13} \Omega_3 \tilde{\Omega}_3 - \frac{I_2}{2} (\tilde{\Omega}_1 \Omega_3 + \Omega_1 \tilde{\Omega}_3) + \frac{I_{23}}{2} (\tilde{\Omega}_1 \Omega_2 + \Omega_1 \tilde{\Omega}_2) + I_{13} \Omega_1 \tilde{\Omega}_1 \]

\[ \tilde{\Omega}_3 = -\Omega_3. \]

Here \( \Omega_i = \Omega_i(t) \), \( \tilde{\Omega}_i = \Omega_i(t + \epsilon) \) and \( \epsilon \) is the time step. The limit when \( \epsilon \) goes to 0 should reconstruct equations (3) of the continuous Suslov problem [2]. Thus one concludes that \( \Omega_3 \) should be equal to zero, and equations (3) become:

\[ I_1 (\tilde{\Omega}_1 - \Omega_1) = \epsilon \left[ -\frac{I_{13}}{2} (\tilde{\Omega}_1 \Omega_2 + \Omega_1 \tilde{\Omega}_2) - I_{23} \Omega_2 \tilde{\Omega}_2 \right] \]
(4)

\[ I_2 (\tilde{\Omega}_2 - \Omega_2) = \epsilon \left[ \frac{I_{23}}{2} (\tilde{\Omega}_1 \Omega_2 + \Omega_1 \tilde{\Omega}_2) + I_{13} \Omega_1 \tilde{\Omega}_1 \right] \]

\[ \Omega_3 = 0. \]

Since these equations are linear in \( \tilde{\Omega}_i \), the map defined by (4) is explicit and unique-valued:

\[
\begin{bmatrix}
\tilde{\Omega}_1 \\
\tilde{\Omega}_2
\end{bmatrix} = 
\begin{bmatrix}
1 + \frac{\epsilon I_{13}}{I_2} \Omega_2 & \frac{\epsilon I_{23}}{I_1} \Omega_1 + \frac{\epsilon I_{23}}{I_2} \Omega_2 \\
-\frac{\epsilon I_{13}}{I_2} \Omega_1 - \frac{\epsilon I_{23}}{I_2} \Omega_2 & 1 - \frac{\epsilon I_{23}}{I_2} \Omega_1
\end{bmatrix}^{-1}
\begin{bmatrix}
\Omega_1 \\
\Omega_2
\end{bmatrix},
\]
giving
\[
\Omega_1 = \frac{1}{\Delta} (\Omega_1 - \frac{\epsilon I_{23}}{2I_1} \Omega_1^2 - \frac{\epsilon I_{13}}{2I_1} \Omega_1 \Omega_2 - \frac{\epsilon I_{23}}{I_1} \Omega_2^2),
\]
\[
\Omega_2 = \frac{1}{\Delta} (\Omega_2 + \frac{\epsilon I_{13}}{2I_2} \Omega_2^2 + \frac{\epsilon I_{23}}{2I_2} \Omega_1 \Omega_2 + \frac{\epsilon I_{13}}{I_2} \Omega_1^2),
\]
where
\[
\Delta = \left(1 + \frac{\epsilon I_{13}}{2I_1} \Omega_2\right) \left(1 - \frac{\epsilon I_{23}}{2I_2} \Omega_1\right) + \frac{\epsilon^2}{I_1 I_2} \left(\frac{I_{13} \Omega_1}{2} + I_{23} \Omega_2\right) \left(I_{13} \Omega_1 + \frac{I_{23} \Omega_2}{2}\right).
\]
As we have already mentioned, in continuous case equations (2) have the energy integral. But for considered discretization the energy is not an integral anymore, but there exists a first integral as it can be seen from the following statement.

**Lemma 1.** The function
\[
F = \frac{I_1 \Omega_1^2 + I_2 \Omega_2^2}{4I_1 I_2 + \epsilon^2 (I_{13} \Omega_1 + I_{23} \Omega_2)^2}
\]
is a first integral of equations (5)

**Proof.** Proof follows by direct calculations. □

In the limit when \(\epsilon\) goes to zero, integral (6) tends to the energy integral divided by constant.

### 4. Integration

In order to integrate discrete Suslov equations, we introduce new coordinates:
\[
x = I_{13} \Omega_1 + I_{23} \Omega_2, \quad y = I_{23} I_1 \Omega_1 - I_{13} I_2 \Omega_2.
\]
The Jacobian of the change of coordinates (7) is:
\[-(I_{13}^2 I_2 + I_{23}^2 I_1).\]
Thus, it is equal to zero only in the case \(I_{13} = I_{23} = 0\) when \(\bar{\Omega}_1 = \Omega_1\) and \(\bar{\Omega}_2 = \Omega_2\), giving equilibrium position.

In the new coordinates (7) equations (4) are:
\[
\bar{x} - x = \frac{\epsilon}{2I_1 I_2} (\bar{x} y + x \bar{y})
\]
\[
\bar{y} - y = -\epsilon x \bar{x}.
\]
The first integral (6) becomes
\[
F = \frac{I_1 I_2 x^2 + y^2}{4I_1 I_2 + \epsilon^2 x^2}.
\]
The curve \( F(x, y) = h \) can be parameterized by introducing:

\[
x = 2 \sqrt{\frac{I_1 I_2 h}{I_1 I_2 - h \epsilon \cos^2 \phi}} \cos \phi,
\]

\[
y = 2 I_1 I_2 \sqrt{\frac{h}{I_1 I_2 - h \epsilon \cos^2 \phi}} \sin \phi.
\]

Putting (10) into the second equation of (8) and denoting

\[
u = \sqrt{\frac{I_1 I_2}{I_1 I_2 - h \epsilon^2}} \tan \phi
\]

we get:

\[
u \sqrt{\tilde{u}^2 + 1} - \tilde{u} \sqrt{u^2 + 1} = \frac{2 \epsilon \sqrt{I_1 I_2 h}}{I_1 I_2 - h \epsilon^2}.
\]

Let us suppose the form of a solution of (11):

\[
u(n) = \text{sh}(k_1(t_0 + n \epsilon) + k_2),
\]

where \( k_1 \) and \( k_2 \) are constants. By plugging the form into (11), one gets:

\[
\text{sh}(-k_1 \epsilon) = \frac{2 \epsilon \sqrt{I_1 I_2 h}}{I_1 I_2 - h \epsilon^2}.
\]

So, we have

**Proposition 1.** Let constant \( k_1 \) satisfy (12) and \( k_2 \) be arbitrary constant. Then the function \( u(n) = \text{sh}(k_1(t_0 + n \epsilon) + k_2) \) gives solutions of equation (11).

As a consequence we have the following statement:

**Theorem 1.** Let constant \( k_1 \) satisfy (12) and \( k_2 \) be arbitrary constant. The functions:

\[
\Omega_1(n) = \frac{2 \sqrt{h}}{(I_{13}^2 I_1 + I_{13}^2 I_2) \text{ch}(k_1(t_0 + n \epsilon) + k_2)} \left( \frac{I_{13} I_2}{\sqrt{I_1 I_2 - h \epsilon^2}} + I_{23} \text{sh}(k_1(t_0 + n \epsilon) + k_2) \right)
\]

\[
\Omega_2(n) = \frac{2 \sqrt{h}}{(I_{13}^2 I_1 + I_{13}^2 I_2) \text{ch}(k_1(t_0 + n \epsilon) + k_2)} \left( \frac{I_{23} I_1}{\sqrt{I_1 I_2 - h \epsilon^2}} - I_{13} \text{sh}(k_1(t_0 + n \epsilon) + k_2) \right)
\]

give the solutions of equations (5).

**Proof.** From Proposition II and (10) one gets:

\[
x(n) = 2 \sqrt{\frac{h I_1 I_2}{I_1 I_2 - h \epsilon^2 \text{ch}(k_1(t_0 + n \epsilon) + k_2)}} \frac{1}{\text{ch}(k_1(t_0 + n \epsilon) + k_2)},
\]

\[
y(n) = 2 \sqrt{\frac{h I_1 I_2}{I_1 I_2 - h \epsilon^2 \text{ch}(k_1(t_0 + n \epsilon) + k_2)}} \frac{\text{sh}(k_1(t_0 + n \epsilon) + k_2)}{\text{ch}(k_1(t_0 + n \epsilon) + k_2)}.
\]

Proof follows from (7).

**Proposition 2.** The discrete trajectories of motion of the body asymptotically tend to a line of discrete analogies of so-called steady-state rotations that satisfy:

\[
I_{13} \Omega_1 + I_{23} \Omega_2 = 0.
\]
Proof. In the limit $n \to \pm \infty$, we have that $x$ goes to zero giving that $I_{13} \Omega_1 + I_{23} \Omega_2$ goes to zero.

The last statement is illustrated by the following pictures. In all four cases, the line $I_{13} \Omega_1 + I_{23} \Omega_2 = 0$ is clearly indicated. Following parameters are chosen for picture 1: $\epsilon = 0.2, I_1 = 4, I_2 = 1, I_{13} = -0.5, I_{23} = -0.3$, for picture 2: $\epsilon = 0.2, I_1 = 4, I_2 = 3, I_{13} = -0.4, I_{23} = -0.2$, for picture 3: $\epsilon = 0.02, I_1 = 4, I_2 = 2, I_{13} = 0, I_{23} = -0.2$ and for picture 4: $\epsilon = 1, I_1 = 3, I_2 = 3, I_{13} = -0.2, I_{23} = -0.2$. 
Remark 1. Equation (11) splits on two equations:

\[ \tilde{u} = -c\sqrt{u^2 + 1} + u\sqrt{c^2 + 1}, \quad \tilde{u}' = -c\sqrt{u^2 + 1} - u\sqrt{c^2 + 1} \]

where \( c = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2 - h} \). The solutions given in Proposition 1 correspond to the first equation. One easily concludes that for the second equation the limit when \( \epsilon \) goes to zero is not defined well.

Remark 2. One can use the change of coordinates \( x = I_{13}\Omega_1 + I_{23}\Omega_2, y = I_{23}I_1\Omega_1 - I_{13}I_2\Omega_2 \) also in the continuous case. Then equations (2) become:

\[ \dot{x} = \frac{xy}{I_1 I_2}, \quad \dot{y} = -x^2. \]

which are simpler then original ones. One can easily see that Hirota-Kimura type discretizations of last equations are equations (7), as linear change of variables commutes with Hirota-Kimura type discretization.
5. Higher-dimensional case

As it has already been mentioned, the higher-dimensional generalization of Suslov case was suggested by Kozlov and Fedorov (see [3, 4]). The configuration space is Lie group $SO(n)$ and the nonholonomic constraints are:

$$\Omega_{ij} = 0, \quad 1 \leq i, j \leq n - 1.$$  

The equations of motion are:

$$\dot{M} = [M, \Omega] + \Lambda$$  

where $M = I\Omega + \Omega I$, and

$$I = \begin{bmatrix} I_{11} & 0 & \ldots & I_{1n} \\ 0 & I_{22} & \ldots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{1n} & I_{2n} & \ldots & I_{nn} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & \lambda_{12} & \ldots & \lambda_{1,n-1} & 0 \\ -\lambda_{12} & 0 & \ldots & \lambda_{2,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\lambda_{1,n-1} & -\lambda_{2,n-1} & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

From (13) closed systems of equations in $\Omega_{in}$, $1 \leq i \leq n - 1$ follows:

$$(I_{ii} + I_{nn})\dot{\bar{\Omega}}_{in} = -I_{in}(\Omega_{1n} + \ldots + \Omega_{2(n-1),n}) + (I_{in}\Omega_{1n} + \ldots + I_{n-1,n}\Omega_{n-1,n})\Omega_{in} + \epsilon(I_{in}\Omega_{1n} + \ldots + I_{n-1,n}\Omega_{n-1,n})\bar{\Omega}_{in}$.

Similarly as in three-dimensional case, we give the Hirota-Kimura discretization in $n$ dimensions by the following system of equations:

$$(I_{ii} + I_{nn})(\bar{\Omega}_{in} - \Omega_{in}) = -\epsilon(I_{in}(\bar{\Omega}_{1n}\Omega_{1n} + \ldots + \bar{\Omega}_{n-1,n}\Omega_{n-1,n}) + \epsilon(I_{in}\bar{\Omega}_{1n} + \ldots + I_{n-1,n}\bar{\Omega}_{n-1,n})\frac{\Omega_{in}}{2} + \epsilon(I_{in}\bar{\Omega}_{1n} + \ldots + I_{n-1,n}\bar{\Omega}_{n-1,n})\frac{\bar{\Omega}_{in}}{2}.$$

As in three-dimensional case, map defined by (15) is explicit and unique-valued:

$$\begin{bmatrix} \bar{\Omega}_{1n} \\ \vdots \\ \bar{\Omega}_{n-1,n} \end{bmatrix} = A^{-1} \begin{bmatrix} \Omega_{1n} \\ \vdots \\ \Omega_{n-1,n} \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 - \frac{\epsilon(I_{2n}\Omega_{2n} + \ldots + I_{n-1,n}\Omega_{n-1,n})}{2(I_{11} + I_{nn})} & \ldots & \frac{\epsilon(I_{2n}\Omega_{n-1,n} - I_{n-1,n}\Omega_{1n})}{2(I_{11} + I_{nn})} \\ \frac{\epsilon(2I_{2n}\Omega_{n-1,n} - I_{n-1,n}\Omega_{2n})}{2(I_{11} + I_{nn})} & \ldots & \frac{\epsilon(2I_{2n}\Omega_{n-1,n} - I_{n-1,n}\Omega_{2n})}{2(I_{11} + I_{nn})} \\ \vdots & \ddots & \vdots \\ \frac{\epsilon(2I_{1n}\Omega_{n-1,n} - I_{n-1,n}\Omega_{1n})}{2(I_{n-1,n} + I_{nn})} & \ldots & 1 - \frac{\epsilon(I_{1n}\Omega_{1n} + \ldots + I_{n-2,n}\Omega_{n-2,n})}{2(I_{n-1,n} + I_{nn})} \end{bmatrix}.$$
Let us mention that equations (15) have a particular solution \( \Omega_{1,n} = \ldots = \Omega_{n-1,n} = 0 \) which corresponds to motion with constant angular velocity.

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