Long Twins in Random Words

Andrzej Dudek, Jarosław Grytczuk and Andrzej Ruciński

Abstract. Twins in a finite word are formed by a pair of identical subwords placed at disjoint sets of positions. We investigate the maximum length of twins in a random word over a k-letter alphabet. The obtained lower bounds for small values of $k$ significantly improve the best estimates known in the deterministic case. Bukh and Zhou in 2016 showed that every ternary word of length $n$ contains twins of length at least $0.34n$. Our main result states that in a random ternary word of length $n$, with high probability, one can find twins of length at least $0.41n$. In the general case of alphabets of size $k \geq 3$ we obtain analogous lower bounds of the form $\frac{1.64}{k+1}n$ which are better than the known deterministic bounds for $k \leq 354$. In addition, we present similar results for multiple twins in random words.

1. Introduction

Looking for twin objects has had a long tradition in all branches of mathematics. For discrete structures a first question of this kind is attributed to Ulam (see, e.g., [7]) who proposed to measure similarity of two graphs in terms of their edge decompositions into pairwise isomorphic subgraphs. If the number of subgraphs in a decomposition is small, then there must be a large pair among them, which leads to the following general question: given a combinatorial structure (or a pair of structures), how large disjoint isomorphic substructures does it (do they) contain?

This problem has been studied in various forms: for graphs [1,16], words [3,6], and permutations [5,8–10]. Lots of interesting results and challenging open problems can be found in these papers and their references. Some of them are collected in a survey [2].
1.1. Twins in Words: History and Background

In this paper, we study the problem of twins in random words. To put our work in a broader context, we briefly recall what is known on twins in words in general.

The first who introduced and studied this problem were Axenovich, Per- son, and Puzynina [3]. We say that a word \( w = w_1w_2 \cdots w_n \in A^n \), \(|A| < \infty \), contains twins of length \( t \) if there exist disjoint subsets \( \{i_1 < i_2 < \cdots < i_t\} \) and \( \{j_1 < j_2 < \cdots < j_t\} \) of indices such that \( w_{i_1}w_{i_2} \cdots w_{i_t} = w_{j_1}w_{j_2} \cdots w_{j_t} \). Let \( t(w) \) denote the maximum length of twins in \( w \) and let \( t(n, k) \) denote the minimum of \( t(w) \) over all words \( w \) of length \( n \) over a \( k \)-element alphabet \( A \). They proved in [3] that \( t(n, 2) = n/2 - o(n) \). In other words, every finite binary word can be split into a pair of identical subwords, up to an asymptotically negligible remainder. To get this striking result they developed a regularity lemma for words—an interesting tool with a great potential for further applications (see, e.g., [12]).

At present, it is not known whether the same is true for alphabets of size 3. By considering the sub-word formed by two most frequent letters, the above mentioned estimate \( t(n, 2) = n/2 - o(n) \) from [3] implies that \( t(n, k) \geq n/k - o(n) \). For \( k \geq 3 \), this was slightly improved by Bukh and Zhou [6] to

\[
t(n, k) \geq 1.02 \cdot \frac{n}{k} - o(n) \tag{1.1}
\]

and to

\[
t(n, k) \geq \left( \frac{k}{81} \right)^{1/3} \cdot \frac{n}{k} - (k/3)^{1/3}, \tag{1.2}
\]

the latter being a (much) better estimate for larger \( k \). In [3,6] there are also some upper bounds on \( t(n, k) \) valid for \( k \geq 4 \). In particular, \( t(n, 4) \leq 0.4932n \).

Twins in random words were not much studied before, although the proofs of the upper bounds in [3,6] were obtained via the probabilistic method. The only work addressing this issue directly is a recent paper by He et al. [13], where it is proved that, with high probability, binary words of length \( n \) contain twins of size \( n/2 - \omega \sqrt{n} \), for any function \( \omega = \omega(n) \) tending to infinity with \( n \). Moreover, the authors of [13] formulate a striking conjecture that almost every binary word with even numbers of ones and zeros is a perfect pair of twins. If true, this would imply that, with high probability, a random binary word of an even length \( n \) contains twins of size at least \( n/2 - 1 \) (when the number of ones is odd, this is the best one can get).

1.2. Our Results

In this paper, we study twins in a random word \( W_k(n) \) obtained by drawing one with probability \( k^{-n} \) out of all \( k \)-ary words of length \( n \). Equivalently, one could toss a \( k \)-sided fair die, independently, \( n \) times. Either way, this is an equiprobable space.

Our main result improves the lower bound (1.1) for all but very few ternary (\( k = 3 \)) words of length \( n \).
Table 1. Comparing bound (1.2) of Bukh and Zhou [6] (for all words) with Theorem 1.2 (for almost all words)

| $k$    | $3$   | $4$   | $5$   | $10$  | $50$  | $100$ | $200$ | $400$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| $(k/8^3)^{1/3}$ | 0.333 | 0.367 | 0.395 | 0.498 | 0.851 | 1.073 | 1.352 | 1.703 |
| $1.64k/k+1$  | 1.230 | 1.312 | 1.367 | 1.491 | 1.608 | 1.624 | 1.632 | 1.636 |

**Theorem 1.1.** With probability $1 - e^{-\Omega(n/(\log n))}$,

\[ t(W_3(n)) \geq 0.411n. \]

The proof involves computer-assisted calculations (see Appendix A). It seems plausible that substantially stronger computational devices could bring further improvements of the bound. Moreover, by using classical methods (without computers) we provide a slightly worse bound, namely, $0.375n$ (cf. Example 3.8 in Sect. 3.4).

For larger alphabets we have the following result. We present it in a form which emphasizes the improvement over the deterministic bound (1.1). We say that an event $E_n$ holds asymptotically almost surely (a.a.s.) if $\Pr(E_n) \to 1$ as $n \to \infty$.

**Theorem 1.2.** For each $k \geq 3$ and large $n$, a.a.s.

\[ t(W_k(n)) \geq \frac{1.64k}{k+1} \cdot \frac{n}{k}, \]

This gives an improvement upon the (deterministic) estimates (1.1) and, for all $k \leq 354$, upon (1.2) (see Table 1).

The proof uses a tool called the Boosting Lemma (Lemma 3.4). It is stated in terms of a special model of random words which allows for iterative enhancement of the twin length, while adding a new letter to the alphabet. This new model assumes that the numbers of occurrences of letters are fixed in such a way that the model is asymptotically equivalent to $W_k(n)$.

### 1.3. Multiple Twins

We also consider a more general notion of multiple twins. By $r$-twins in a word $w$ we mean $r$ disjoint identical subwords of $w$. Let $t^{(r)}(w)$ be the maximum length of $r$-twins in $w$ and let $t^{(r)}(n,k)$ be the minimum of $t^{(r)}(w)$ over all words $w$ of length $n$ from a $k$-letter alphabet. By the results of [3,6] we know, respectively, that $t^{(r)}(n,k) \sim n/r$ when $r \geq k$, and

\[ t^{(r)}(n,k) \geq C_r \cdot k^{1/(2^{r-1})} \cdot \frac{n}{k} - O(1), \quad (1.3) \]

for $k > r \geq 3$, where $C_r = \left(\frac{1}{2^{r-1}}\right)^{1+1/(2^{r-1})}$ and the $O(1)$ term depends on $r$ and $k$ only. Estimate (1.3) was only mentioned in the concluding remarks in [6] and the explicit form of $C_r$ was not given. However, based on the proof of inequality (1.2) therein, it is quite obvious how it should be computed.
Table 2. Comparing bound (1.3) of Bukh and Zhou [6] (for all words) with Theorem 1.3 (for almost all words)

| \((r, k)\)   | (3, 4) | (3, 10) | (3, 100) | (3, 1000) | (3, 10^10) | (4, 10^10) | (4, 10^40) |
|--------------|--------|---------|----------|-----------|------------|------------|------------|
| \(C_r \cdot k^{1/(r-1)}\) | 0.196  | 0.214   | 1.036    | 0.340     | 1.703      | 0.261      | 1.878      |
| \(\Pi_{r,k}\) | 1.016  | 1.036   | 1.041    | 1.041     | 1.003      | 1.003      |            |

For a random word \(W_k(n)\) we get the following estimates which, again, for small \(r\) and \(k\) yield better lower bounds than (1.3) (see Table 2). For \(r, k \geq 2\), let \(\Pi_{r,k} = \prod_{j=r+1}^{k} \frac{j^r}{j-1}\).

**Theorem 1.3.** For every \(k > r \geq 3\), a.a.s.

\[
t^{(r)}(W_k(n)) \geq \Pi_{r,k} \cdot \frac{n}{k} - o(n).
\]

1.4. Organization

In the next section we prove Theorem 1.1. Section 3 begins with a short proof of the known bound \(t^{(r)}(n, k) \sim n/r, r \geq k\), for random words. We do so for self-containment, as the proof in [3] (for all words) is quite involved. The next subsection contains a standard proof of an asymptotic equivalence between \(W_k(n)\) and another model of random words. Then comes the crucial Boosting Lemma, while the last subsection of Sect. 3 brings applications of the Boosting Lemma, among them short proofs of Theorems 1.2 and 1.3, the former utilizing also Theorem 1.1. The last section contains some remarks and open problems, while the Appendix presents a Maple code used for derivation of the data collected in Table 3 as well as a proof of a technical estimate needed in Sect. 3.3.

2. Computer Assisted Bound

In this section, we prove Theorem 1.1.

**Proof of Theorem 1.1.** Fix a positive integer \(s\) and split the set of positions of a ternary random word \(W_3(n)\) into \(m := n/s\) segments of length \(s\) (we assume \(s|n\)). For \(1 \leq j \leq m\) and \(1 \leq t \leq \lfloor s/2 \rfloor\), let \(X_t^j\) be the indicator random variable such that \(X_t^j = 1\) if the \(j\)-th segment of a random word \(W_3(n)\) contains twins of length \(t\) but not \(t + 1\); otherwise \(X_t^j = 0\). Furthermore, define \(X_t := \sum_{j=1}^{m} X_t^j\).

First we calculate the expected value of \(X_t\). Let \(\lambda_t\) count the number of ternary words of length \(s\) with twins of length \(t\) but not \(t + 1\). Clearly, \(\sum_{t=1}^{\lfloor s/2 \rfloor} \lambda_t = 3^s\). In general, finding (or even tightly approximating) \(\lambda_t\) does not seem to be an easy problem. However, for small \(s\) one can use a computer program to determine \(\lambda_s\). In Table 3 we present all values of \(\lambda_t\) for all \(6 \leq s \leq 14\) and \(1 \leq t \leq \lfloor s/2 \rfloor\) (see Appendix A for more details). Since \(X_t\) has the binomial distribution \(\text{Bin}(m, \lambda_t/3^s)\), its expectation is
Table 3. The exact values of $\lambda_t$ for all $6 \leq s \leq 14$ and $1 \leq t \leq \lfloor s/2 \rfloor$

| $s$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-----|---|---|---|---|----|----|----|----|----|
| $\lambda_1$ | 42 | 6 | | | | | | | |
| $\lambda_2$ | 594 | 1086 | 822 | 288 | 42 | | | | |
| $\lambda_3$ | 93 | 1095 | 5118 | 11100 | 10806 | 5292 | 1350 | 162 | |
| $\lambda_4$ | 621 | 8385 | 43776 | 106032 | 123750 | 75810 | 24894 | | |
| $\lambda_5$ | | 4425 | 65823 | 373638 | 992244 | 1312530 | | | |
| $\lambda_6$ | Zero | | 32703 | 526107 | 3196644 | | | | |
| $\lambda_7$ | | | | | | | | | 248901 |

Moreover, a standard application of the Chernoff inequality (see, e.g., ineq. (2.9) in [15]) together with the union bound yields that for all $1 \leq j \leq m$ and $1 \leq t \leq \lfloor s/2 \rfloor$, $X_t$ is highly concentrated around its mean, i.e.,

$$X_t = \frac{\lambda_t}{s^{3s}}(1 + o(1))n$$

with probability $1 - e^{-\Omega(n/(\log n))}$.

To finish the proof of Theorem 1.1 we construct twins in $W_3(n)$ as follows. Let, for each $1 \leq j \leq m$, $A_j$ and $B_j$ be a pair of the longest twins in the $j$-th segment. Observe that concatenating $A_j$ and $B_j$ over all $m$ segments yields, a.a.s., twins $A_1A_2\ldots A_m$ and $B_1B_2\ldots B_m$ of length

$$\sum_{t=1}^{\lfloor s/2 \rfloor} tX_t = \sum_{t=1}^{\lfloor s/2 \rfloor} \frac{t\lambda_t}{s^{3s}}(1 + o(1))n = g_s(1 + o(1))n,$$

where

$$g_s := \sum_{t=1}^{\lfloor s/2 \rfloor} \frac{t\lambda_t}{s^{3s}}.$$

From the last column of Table 3 we compute that

$$g_{14} = \frac{4 \cdot 24894 + 5 \cdot 1312530 + 6 \cdot 3196644 + 7 \cdot 248901}{14 \cdot 3^{14}} = \frac{27584397}{66961566} > 0.4119.$$

This completes the proof. \[\square\]
3. General Bounds

3.1. More Twins than Letters

It was shown in [3] that for all $r \geq k$

$$t^{(r)}(n, k) \geq \frac{n}{r} - O \left( \left( \frac{\log \log n}{\log n} \right)^{1/4} \right).$$

For self-containment we insignificantly improve this lower bound for almost all words. The new bound holds with probability so close to 1 that a passage to the alternative model employed in Sect. 3.4 is still possible.

To this end, let us recall the main observation in [3]: if there is a partition of a $k$-word $w$ of length $n$ into $m$ segments $w_1w_2 \cdots w_m$ of sizes $N := n/m$ (we assume $m|n$) and in each segment each letter occurs at least $\mu$ times, then one can construct $r$-twins in $w$ via a natural interlacing. Namely, Twin 1 consists of $\mu$ elements $a_1$ of $w_1$, $\mu$ elements $a_2$ of $w_2$, ..., $\mu$ elements $a_k$ of $w_k$, followed by $\mu$ elements $a_1$ of $w_{r+1}$, $\mu$ elements $a_2$ of $w_{r+2}$, and so on, for as long as there are still at least $r - 1$ segments ahead. Twin 2 follows the same pattern except that it begins with $\mu$ elements $a_1$ of $w_2$, that is, it is shifted by one segment with respect to Twin 1. Then Twin 3 begins with $\mu$ elements $a_1$ of $w_3$, etc.

This way all $r$ twins are disjoint and they use together $k\mu$ elements from each segment except the first $k - 1$ ones where the consumption gradually grows from $\mu$ to $(k-1)\mu$, and, in the worst case, when $m = r - 1 \pmod r$, the last $r - 1$ ones, where the consumption declines from $(k-1)\mu$ to $\mu$, leaving the last $r - k$ segments completely untouched. The exact worst case count yields that together twins $T_1, \ldots, T_r$ cover at least

$$[m - (k - 1) - (r - 1)] k\mu + 2 \binom{k}{2} \mu = (m - r + 1)k\mu \quad (3.1)$$

elements of $w$.

The R-H-S of (3.1) is, given $m \to \infty$ as $n \to \infty$, very close to $mk\mu$, which, in turn, will be close to $n$, provided $\mu \sim N/k$. To get these relations, the authors of [3] developed a regularity lemma for words. Here we are in a more comfortable situation as we are dealing with random words. Therefore, a simple application of Chernoff’s bound yields the following result.

**Lemma 3.1.** For $r \geq k \geq 2$, with probability at least $1 - O(n^{-k+1/3})$,

$$t^{(r)}(W_k(n)) = n/r - O \left( n^{2/3} \sqrt{\log n} \right).$$

**Proof.** Split the set of positions of $W_k(n)$ into $m := n^{1/3}$ consecutive segments of length $N := n^{2/3}$. For $j = 1, \ldots, m$ and $i = 1, \ldots, k$, let $X := X_i^j$ be the number of elements $a_i$ in the $j$-th segment. Then $EX = N/k$ and, by Chernoff’s bound from [15], ineq. (2.6), we have

$$\mathbb{P}(X \leq (1 - \varepsilon)EX) \leq n^{-k},$$
where \( \varepsilon = kn^{-1/3} \sqrt{2 \log n} \). As we only have \( km = O(n^{1/3}) \) random variables \( X_i^j \), with probability at least \( 1 - O(n^{-k+1/3}) \), they all satisfy the opposite bound, that is, for all \( j = 1, \ldots, m \) and \( i = 1, \ldots, k \),

\[
X_i^j \geq \mu := (1 - \varepsilon)n^{2/3} / k = n^{2/3} / k - O(n^{1/3} \sqrt{\log n}).
\]

Upon applying (3.1) with this \( \mu \), we conclude that the number of elements uncovered by the \( r \)-twins is, indeed, at most \( n - (m - r + 1)k\mu = O(n^{2/3} \sqrt{\log n}) \). \( \square \)

Notice that for \( r = k = 2 \), the estimate for \( t(W_2(n)) \) in the lemma is slightly weaker than the one from [13] mentioned in the introduction. However, this fact does not affect our results, as all we really need in the proofs (see Example 3.8 and the proof of Theorem 1.3) is an estimate \( t^{(r)}(W_k(n)) = n/r - o(1) \). Besides, it is not clear how to generalize the result from [13] to other values of \( k \) and \( r \).

### 3.2. Equivalence of Models of Random Words

Guided by analogy to random graphs, we consider two basic models of random words: the binomial and the fixed-letter-count model. Given positive integers \( k \) and \( n \), an alphabet \( A = \{ a_1, \ldots, a_k \} \), and constants \( 0 \leq p_1, \ldots, p_k \leq 1 \), where \( p_1 + \cdots + p_k = 1 \), the binomial random word \( W(n; p_1, \ldots, p_k) \) is a sequence of independent random variables \( (X_1, \ldots, X_n) \), where for each \( j = 1, \ldots, n \) and \( i = 1, \ldots, k \) we have \( P(X_j = a_i) = p_i \). Here we are exclusively interested in the special, equiprobable instance \( W_k(n) := W(n; 1/k, \ldots, 1/k) \).

As a technical tool rather than an object of genuine interest we also define another model of a random word. Given integers \( 0 \leq M_1, \ldots, M_k \leq n \), where \( M_1 + \cdots + M_k = n \), the fixed-letter-count random word \( W(n; M_1, \ldots, M_k) \) is obtained by taking uniformly at random a permutation (with repetitions) of \( n \) elements, among which, \( i = 1, \ldots, k \), there are \( M_i \) elements \( a_i \). Thus, in the latter model, we restrict ourselves to words with prescribed numbers of each letter and every such word has the same probability \( \binom{n}{M_1, \ldots, M_k}^{-1} \) to be chosen.

The asymptotic equivalence between the two models goes smoothly one way (from fixed-letter-count to binomial), but to proceed the other way the models lack monotonicity, so we are forced to recourse to an analog of Pittel’s inequality (see, e.g., ineq. (1.6) in [15]) which bring, however, some limitations.

Let \( Q \) be a property (subset) of words of length \( n \) over alphabet \( A \). Although more general statements can be easily proved, we restrict ourselves to the special case of the model \( W_k(n) \) and also to limiting probabilities equal to 1 only. The proofs follow those for random graphs (cf. Section 1.4 in [15], in particular, proofs of Prop. 1.12 and of Pittel’s inequality (1.6) therein) and rely on the law of total probability and the fact that the space of \( W_k(n) \) conditioned on \( M_i \)'s being the numbers of occurrences of the elements \( a_i \in A \), \( i = 1, \ldots, k \), coincides with that of \( W(n; M_1, \ldots, M_k) \).

**Proposition 3.2.** If for all \( M_1, \ldots, M_k \) such that \( M_1 + \cdots + M_k = n \) and \( M_i = n/k + O(\sqrt{n}) \), \( i = 1, \ldots, k \), \( P(W(n; M_1, \ldots, M_k) \in Q) \to 1 \) as \( n \to \infty \), then \( P(W_k(n) \in Q) \to 1 \) as \( n \to \infty \).
Proof. Let $C$ be a large constant and define (for each $n$)
\[ M_C = \{(M_1, \ldots, M_k) : M_1 + \cdots + M_k = n \quad \text{and} \quad |M_i - n/k| \leq C\sqrt{n}\}. \]

Let $(M^*_1, \ldots, M^*_k)$ minimize $\mathbb{P}(W(n; M_1, \ldots, M_k) \in Q)$ over $M_C$. Finally, let $X_i$ be the number of occurrences of letter $a_i$ in $W_k(n)$. Note that each $X_i$ has binomial distribution with expectation $n/k$ and variance less than $n/k$. Then, by the law of total probability
\[
\mathbb{P}(W_k(n) \in Q) \geq \mathbb{P}(W(n; M^*_1, \ldots, M^*_k) \in Q)\mathbb{P}((X_1, \ldots, X_k) \in M_C).
\]

By assumption, $\mathbb{P}(W(n; M^*_1, \ldots, M^*_k) \in Q) \to 1$ as $n \to \infty$. By Chebyshev’s inequality applied together with the union bound,
\[
\mathbb{P}((X_1, \ldots, X_k) \notin M_C) \leq k \frac{n/k}{C^2 n} = C^{-2}.
\]

Thus, $\liminf_{n \to \infty} \mathbb{P}(W_k(n) \in Q) \geq 1 - C^{-2}$. As this is true for every $C$, letting $C \to \infty$ yields $\lim_{n \to \infty} \mathbb{P}(W_k(n) \in Q) = 1$. \hfill \Box

Proposition 3.3. If $\mathbb{P}(W_k(n) \in Q) = 1 - o(n^{-k/2})$ as $n \to \infty$, then for all $M_i = n/k + \omega_i$, where $|\omega_i| \leq \sqrt{(n \log n)/(3k^2)}$ and $\sum_i \omega_i = 0$, $\mathbb{P}(W(n; M_1, \ldots, M_k) \in Q) \to 1$ as $n \to \infty$.

Proof. Fix $M_1, \ldots, M_k$ as in the statement of the proposition. By the law of total probability, we obtain
\[
\mathbb{P}(W_k(n) \notin Q) = \sum_{M'_1, \ldots, M'_k} \mathbb{P}(W(n; M'_1, \ldots, M'_k) \notin Q) \left(\binom{n}{M'_1, \ldots, M'_k}\right) \frac{1}{k^n}
\]

from which we get
\[
\mathbb{P}(W(n; M_1, \ldots, M_k) \notin Q) \leq \frac{k^n}{(M_1, \ldots, M_k)} \mathbb{P}(W_k(n) \notin Q).
\]

It remains to estimate the ratio $\frac{k^n}{(M_1, \ldots, M_k)}$. Using Stirling’s formula several times, we get
\[
\frac{k^n}{(M_1, \ldots, M_k)} = O(n^{(k-1)/2}) \prod_{i=1}^k (1 + k\omega_i/n)^{M_i} = O(n^{(k-1)/2}) \exp \left\{ \sum_{i=1}^k k\omega_i M_i/n \right\}.
\]

Now since $M_i = n/k + \omega_i$ and $\sum_i \omega_i = 0$, we get
\[
\sum_{i=1}^k k\omega_i M_i/n = \sum_{i=1}^k k\omega_i \left(\frac{1}{k} + \frac{\omega_i}{n}\right) = \sum_{i=1}^k k\omega_i^2/n \leq \frac{\log n}{3}
\]

and so $k^n/(M_1, \ldots, M_k) = O(n^{k/2-1/6})$, which yields that $\mathbb{P}(W_k(n) \notin Q) = o(1)$. \hfill \Box
3.3. Boosting Lemma

Fix $2 \leq r \leq k$ and observe that if for some $\lambda = \lambda(n) > 0$ and all $n \geq n_0$, we have $t^{(r)}(n, k) \geq \lambda n$, then also, by dropping the least frequent letter, $t^{(r)}(N, k + 1) \geq \frac{\lambda k}{k+1} N$, provided $N \geq \frac{k+1}{k} n_0$. In this section, we show that for random words this trivial bound can be improved if one considers the fixed-letter-count model.

To get a similar result for the binomial model $W_k(n)$ one has to switch first to the fixed-letter-count model $W(n; M_1, \ldots, M_k)$ and then back to $W_{k+1}(N)$, the switches facilitated, respectively, by Propositions 3.3 and 3.2. The reason for switching is that the fixed-letter-count model can be broken into two phases allowing the enlargement of the twins.

Indeed, let $M_1, \ldots, M_{k+1}$ be given such that $\sum_{i=1}^{k+1} M_i = N$. We generate the random word $W(N; M_1, \ldots, M_{k+1})$ by first permuting all $n := M_1 + \cdots + M_k$ letters from $A \setminus \{a_{k+1}\}$ (Phase 1). This can be done in precisely $M_1^{\binom{n}{M_1}}$ ways. Then we throw in the $M_{k+1}$ letters $a_{k+1}$ which can go anywhere between the previously distributed letters (Phase 2). This can be done, by the formula for the number of ways to allocate $M_{k+1}$ balls into $n + 1$ bins, in $\binom{N}{M_{k+1}}$ ways.

Note that the product of these two numbers is, indeed, $(M_1, \ldots, M_{k+1})$ and that the outcome of Phase 1 is precisely the random word $W(n; M_1, \ldots, M_k)$.

Lemma 3.4. (Boosting Lemma) For $2 \leq r \leq k$ and $n$ sufficiently large, let a partition $n = M_1 + \cdots + M_k$ into nonnegative integers be given. If $M_{k+1} = n/k + O(\sqrt{n})$ and, for some $\lambda = \lambda(n) > 0$, a.a.s. $t^{(r)}(W(n; M_1, \ldots, M_k)) \geq \lambda n$, then a.a.s.

$$t^{(r)}(W(N; M_1, \ldots, M_k, M_{k+1})) \geq \left(1 + \frac{1}{(k+1)^r - 1}\right) \frac{\lambda k}{k+1} N \left(1 - o(1)\right),$$

where $N = n + M_{k+1}$.

Note that the factor of $\frac{\lambda k}{k+1} N$ comes for free already after Phase 1, so that the actual improvement sits in the parentheses. Also, notice that

$$\frac{n}{N} = \frac{k}{k+1} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right).$$

In the proof of Lemma 3.4 we will need a technical estimate on a ratio of binomial coefficients, the proof of which is deferred to Appendix B.

Fact 3.5. Let $M = \Theta(N)$ and $\ell^2 = o(N)$. Then,

$$\frac{\binom{N-\ell}{M-\ell}}{\binom{M}{N}} = \left(\frac{M}{N}\right)^\ell \left(1 + O\left(\frac{\ell^2}{N}\right)\right).$$

Proof of Lemma 3.4. We generate $W(N; M_1, \ldots, M_{k+1})$ in two phases as described prior to the statement of the lemma. Let $W'$ be the outcome of Phase 1, that is, an instance of $W(n; M_1, \ldots, M_k)$. Further, let $Q'$ be the event that $t^{(r)}(W(n; M_1, \ldots, M_k)) \geq \lambda n$ and let $Q$ be the ultimate event that

$$t^{(r)}(W(N; M_1, \ldots, M_k, M_{k+1})) \geq \left(1 + \frac{1}{(k+1)^r - 1}\right) \frac{\lambda k}{k+1} N \left(1 - o(1)\right).$$
By the law of total probability,
\[
\mathbb{P}(Q) = \sum_{W'} \mathbb{P}(Q|W') \mathbb{P}(W') = \sum_{W' \in Q'} \mathbb{P}(Q|W') \mathbb{P}(W') + o(1). \tag{3.2}
\]

We now focus on \( \mathbb{P}(Q|W') \) with \( W' \in Q' \), that is, fixing an instance \( W' \) of Phase 1 with \( t^{(r)}(W) \geq \lambda n \), we are going to thoroughly investigate the outcome of Phase 2. Fix \( W' = w_1 \cdots w_n \) and \( r \)-twins \( T_1, \ldots, T_r \) therein of length \( |T_1| = \cdots = |T_r| = \lambda n \) (we may assume that \( \lambda n \) is an integer). We treat the \( n + 1 \) spaces before, between, and after the letters of \( W' \) as bins and the \( M_{k+1} \) letters \( a_{k+1} \) as balls. Precisely, bin \( b_0 \) is in front of \( w_1 \), for \( i = 1, \ldots, n - 1 \), bin \( b_i \) lies between \( w_i \) and \( w_{i+1} \), and bin \( b_n \) is to the right of \( w_n \).

We group the bins lying immediately to the right of the elements of the \( r \)-twins \( T_1, \ldots, T_r \) into \( r \)-tuples of bins denoted by \( R_1, \ldots, R_{\lambda n} \). Formally, if \( T_j = w^{(j)}_{i_1} \cdots w^{(j)}_{i_{\lambda n}}, j = 1, \ldots, r \), then the \( r \)-tuple of bins \( R_\ell \) consists of the bins \( b^{(1)}_{\ell}, \ldots, b^{(r)}_{\ell} \), for \( \ell = 1, \ldots, \lambda n \).

**Example 3.6.** Let \( k = r = 3 \), \( n = 27 \), and
\[
W' = w_1 \cdots w_{27} = b a a c a c b b a b c a c a c a b b c b a c c b a b.
\]
There are 3-twins of length 5 here, each forming the word \( aabcc \), namely, \( T_1 = w_2w_3w_8w_{11}w_{19}, T_2 = w_5w_9w_{10}w_{13}w_{23}, \) and \( T_3 = w_{12}w_{16}w_{20}w_{22}w_{24} \). For instance, consider the first triple of bins, \( R_1 = \{b_2, b_5, b_{12}\} \). If the letter \( d \) is inserted into each of these three bins, a longer word
\[
b a a d a a c a a d c b b a b c a a d c a a b b a c b c c b a b,
\]
is obtained, and the length of the twins, each forming now the word \( adabcc \), increases by one.

Clearly, if each bin from an \( r \)-tuple receives \( s \) balls (read: \( s \) letters \( a_{k+1} \)), then the length of the twins can be extended by \( s \). Since we would like to utilize several \( r \)-tuples of bins simultaneously, we categorize them with respect to the minimum number of balls.

For \( 1 \leq s \leq \log n \), let \( X_s \) be the number of \( r \)-tuples of bins with at least \( s \) balls in each bin and exactly \( s \) balls in some bin (i.e. there is a bin with exactly \( s \) balls). We call each such \( r \)-tuple of bins an \( s \)-provider. For each \( \ell = 1, \ldots, \lambda n \), let \( I_\ell \) be the indicator random variable such that \( I_\ell = 1 \) if \( R_\ell \) is an \( s \)-provider and 0 otherwise. Hence, \( X_s = \sum_{\ell=1}^{\lambda n} I_\ell \).

The event \( \{I_\ell = 1\} \) is the set difference of two events: that each bin in \( R_\ell \) has at least \( s \) balls and that each bin in \( R_\ell \) has at least \( s + 1 \) balls. Thus, setting \( M := M_{k+1} \),
\[
\mathbb{P}(I_1 = 1) = \frac{\binom{N-rs}{M-rs}}{\binom{N}{M}} - \frac{\binom{N-rs-r}{M-rs-r}}{\binom{N}{M}}
\]
and Fact 3.5 yields
\[
\mathbb{P}(I_1 = 1) = \left(\frac{M}{N}\right)^{rs} \left(1 - \left(\frac{M}{N}\right)^r\right) \left(1 + O\left(\frac{s^2}{N}\right)\right)
\]
Thus, setting \( \kappa = (k + 1)^{-r} \), we have
\[
\mathbb{E}X_s = \lambda n (1 - \kappa) \kappa^s (1 + O (1/\sqrt{n})).
\]

Our goal is to show a.a.s. simultaneous concentration of each \( X_s, s \leq \log n \), near its expectation. We are going to use Chebyshev's inequality (followed by the union bound over all \( s \))
\[
\mathbb{P}(|X_s - \mathbb{E}X_s| \geq \gamma \mathbb{E}X_s) \leq \frac{\text{Var} X_s}{(\gamma \mathbb{E}X_s)^2}
\]
with \( \gamma = 1/\log n \). To facilitate the future use of the union bound, we need to show that \( \text{Var} X_s = o((\mathbb{E}X_s)^2/\log^3 n) \), which will imply that \( \frac{\text{Var} X_s}{(\gamma \mathbb{E}X_s)^2} = o(1/\log n) \). To this end we write
\[
\text{Var} X_s = \mathbb{E}(X_s(X_s - 1)) + \mathbb{E}X_s - (\mathbb{E}X_s)^2.
\]
Note that \( \{I_{\ell_1} = I_{\ell_2} = 1\} = A \setminus (B_1 \cup B_2) \), where \( A \) is the event that all 2\( r \) bins in \( R_{\ell_1} \) and \( R_{\ell_2} \) each contains at least \( s \) balls, while \( B_i, i = 1, 2 \), is the event that each bin in \( R_{\ell_i} \) contains at least \( s \) balls and each bin in \( R_{\ell_3 - i} \) contains at least \( s + 1 \) balls. Thus,
\[
\mathbb{P}(I_{\ell_1} = I_{\ell_2} = 1) = \mathbb{P}(A) - \mathbb{P}(B_1) - \mathbb{P}(B_2) + \mathbb{P}(B_1 \cap B_2)
\]
which by Fact 3.5 is equal to
\[
(\kappa^{2s} - 2\kappa^{2s+1} + \kappa^{2s+2}) \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) = ((1 - \kappa) \kappa^s)^2 \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right).
\]
Thus,
\[
\text{Var} X_s = \lambda n (\lambda n - 1) ((1 - \kappa) \kappa^s)^2 \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right)
\]
\[
+ \lambda n (1 - \kappa) \kappa^s \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) - \left( \lambda n (1 - \kappa) \kappa^s \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \right)^2
\]
\[
= (\lambda n (1 - \kappa) \kappa^s)^2 + O(n^{3/2})
\]
\[
+ O(n) - (\lambda n (1 - \kappa) \kappa^s)^2 - O(n^{3/2}) = O(n^{3/2}) = o((\mathbb{E}X_s)^2/\log^3 n),
\]
with a big margin.

Hence,
\[
\sum_{s=1}^{[\log n]} \mathbb{P}(|X_s - \mathbb{E}X_s| \geq \gamma \mathbb{E}X_s) = o(1),
\]
and, in particular, a.a.s., for all \( 1 \leq s \leq \log n \),
\[
X_s \geq \mathbb{E}X_s \left( 1 - \frac{1}{\log n} \right) = \lambda n (1 - \kappa) \kappa^s \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) \left( 1 - \frac{1}{\log n} \right)
\]
\[
= \lambda n (1 - \kappa) \kappa^s \left( 1 - O \left( \frac{1}{\log n} \right) \right).
\]
As each $X_s$ contributes $s$ towards an enlargement of the twins $T_1, \ldots, T_r$, we need to calculate $\sum_{s=1}^{\lfloor \log n \rfloor} sX_s$. Since for every positive integer $p$

$$\sum_{s=1}^{\infty} sk^s = \sum_{s=p+1}^{\infty} sk^s = \frac{\kappa}{(1-\kappa)^2} - \frac{(1 + (1 - k)p)\kappa^{p+1}}{(1 - \kappa)^2},$$

we get

$$\sum_{s=1}^{\lfloor \log n \rfloor} sX_s \geq \lambda n (1 - \kappa) \left( 1 - O \left( \frac{1}{\log n} \right) \right) \sum_{s=1}^{\lfloor \log n \rfloor} sk^s = \frac{\lambda \kappa}{1 - \kappa} n(1 - o(1)) = \frac{\lambda}{(1 + k)^r} n(1 - o(1)).$$

Hence, still conditioning on $W'$, a.a.s., there are in $W(N; M_1, \ldots, M_k, M_{k+1})$ twins of length at least

$$\lambda n + \frac{\lambda}{(1 + k)^r - 1} n(1 - o(1)) = \left( 1 + \frac{1}{(1 + k)^r - 1} \right) \lambda n(1 - o(1)).$$

As $n = \frac{k}{k+1} N(1 + o(1))$, this means that $P(Q|W') = 1 - o(1)$ uniformly over all $W' \in Q'$, and, by (3.2), the lemma follows.

**Remark 3.7.** By considering only one random variable $X$ counting $r$-tuples of bins with no bin empty, the proof becomes a bit simpler, but the result is slightly weaker: a.a.s.

$$t^{(r)}(W(N; M_1, \ldots, M_k, M_{k+1})) \geq \left( 1 + \frac{1}{(k+1)^r} \right) \frac{\lambda k}{k+1} N.$$ We get rid of $o(1)$, but lose $-1$ in the denominator, so overall the bound is lower, though for large $k$ the difference is insignificant.

### 3.4. Applications

In this subsection, we present applications of the Boosting Lemma (Lemma 3.4), most importantly, in the proofs of Theorems 1.2 and 1.3. Each time the scenario is the same: we pick an existing lower bound on the length of twins in $W_k(n)$ (e.g., the bound in Theorem 1.1 above or any deterministic result from [3] or [6]), translate it via Proposition 3.3 to the fixed-letter-count model and then apply (iteratively) the Boosting Lemma to get a bound, still in the fixed-letter-count model, for the targeted value of $k > k'$. At the end we go back to the binomial model $W_k(n)$ via Proposition 3.2. For this we need to have our estimate valid for all $M_i, i = 1, \ldots, k'$.

Let us now analyze what it really means. Fix $k$ and let $M_i = n/k + w_i, i = 1, \ldots, k$, where $|w_i| = O(\sqrt{n})$ and $\sum_{i=1}^k w_i = 0$. How these assumptions
alter when we drop $M_k$. Let $n_{k-1} = M_1 + \cdots + M_{k-1}$. Then, for each $i = 1, \ldots, k-1$,
\[
n_{k-1} = \frac{k-1}{k} n + \frac{k-1}{k} \sum_{i=1}^{k-1} w_i = (k-1)M_i - (k-1)w_i - w_k,
\]
thus
\[
M_i = \frac{n_{k-1}}{k-1} + w_i + \frac{w_k}{k-1} = \frac{n_{k-1}}{k-1} + w_i^{(k-1)},
\]
where $w_i^{(k-1)} = w_i + \frac{w_k}{k-1}$. Note that $\sum_{i=1}^{k-1} w_i^{(k-1)} = 0$ as it should. We may iterate this relation all the way down to $k'$ (where we want to begin the process of applying Lemma 3.4), obtaining for $j = k-1, \ldots, k'$ and $i = 1, \ldots, k'$, with $n_j = M_1 + \cdots + M_j$,
\[
M_i = \frac{n_j}{k} + w_i^{(j)},
\]
where
\[
w_i^{(j)} = w_i + \frac{1}{j} \sum_{q=j+1}^{k} w_q.
\]
Observe that $w_i^{(k')} = O(\sqrt{n})$, so we stay within the range required in Proposition 3.3. Below we illustrate how to apply the Boosting Lemma.

**Example 3.8.** Let $r = 2$ and $k = 3$. We know, either by our Lemma 3.1 or by the result in [3], that a.a.s. a random binary word $W_2(n)$ contains twins of length $(\frac{1}{2} - o(1))n$. We would like to deduce from this, via Lemma 3.4, a lower bound on the length of twins in the random ternary word $W_3(n)$ (so, we take $k' = 2$ here).

Fix $M_i = n/3 + w_i$, $i = 1, 2, 3$, where $|w_i| = O(\sqrt{n})$ and $w_1 + w_2 + w_3 = 0$. Suppressing $M_3$, we get, with $n' = n - M_3$, an instance of $W(n'; M_1, M_2)$ which satisfies the assumptions of Proposition 3.3, that is, for $i = 1, 2$, we have $M_i = n'/2 + w_i^{(2)}$, where $w_i^{(2)} = w_i + \frac{w_3}{2} = O(\sqrt{n})$. Thus, we may conclude that a.a.s. $t^{(2)}(W(n'; M_1, M_2)) \geq \lambda n'$ with $\lambda = \frac{1}{2} - o(1)$. In turn, by Lemma 3.4, a.a.s.
\[
t^{(2)}(W(n; M_1, M_2, M_3)) \geq \left(1 + \frac{1}{8}\right) \frac{2\lambda}{3} n(1 - o(1)) \geq 0.375n(1 - o(1)).
\]
Since this is true for all choices of $M_i$ as above, by Proposition 3.2, we finally get that a.a.s. $t^{(2)}(W_3(n)) \geq 0.375n(1 - o(1))$. This is much less than the bound in Theorem 1.1, so the result has some value only for computer-skeptical readers. On the other hand, it is still better than the bound $t^{(2)}(W_3(n)) \geq 0.34n(1 - o(1))$ in (1.1), though the latter holds for all ternary words.

Continuing with this example, let us iterate applications of Lemma 3.4 till, say, $k = 10$. Skipping details, we see that the obtained bound is a.a.s.
\[
t^{(2)}(W_{10}(n)) \geq \frac{9}{8} \cdot \frac{16}{15} \cdot \frac{25}{24} \cdot \frac{36}{35} \cdot \frac{49}{48} \cdot \frac{64}{63} \cdot \frac{81}{80} \cdot \frac{100}{99} \cdot \frac{n}{10}(1 + o(1))
\]
Let us conclude the paper with some open questions and suggestions for future studies. The first problem naturally concerns twins in ternary words.

**Problem 4.1.** Does \( t(n, 3) = n/2 - o(n) \)?

Even if the answer is negative it can still be true that *almost all* ternary words contain twins of such length, that is, a.a.s. \( t(W_3(n)) = n/2 - o(n) \). It is known, as demonstrated in [6], that a.a.s. \( t(W_4(n)) \) is \( 0.4932n \) which makes a similar statement for random *quaternary* words false. In general, it is not clear how close \( t(W_k(n)) \) and \( t(n, k) \) are from each other.
Problem 4.2. Does $t(W_k(n)) = t(n, k) + o(n)$ hold a.a.s. for all $k \geq 2$?

By the result of Axenovich, Person, and Puzynina [3] we know that this is the case for $k = 2$.

One may also consider more restricted notions of twins in words reflecting their placement in the word. For instance, it may happen that twins occupy together a connected segment forming a \textit{shuffle square}, as in the example below:

\[
\begin{array}{cccccccccc}
  a & b & c & a & c & b & a & b & c & a.
\end{array}
\]

As proved recently by Bulteau, Jugé, and Vialette [4], there exist arbitrarily long words over a 6-letter alphabet containing no shuffle squares. It is not known, however, if the size of the alphabet in this result is optimal and, more generally, what is the maximum length of a shuffle square guaranteed to be present in \textit{every} (long) $k$-letter word, $k = 2, \ldots, 5$.

Here, we formulate this question for random words.

Problem 4.3. What is the expected maximum length of a shuffle square in a random word $W_k(n)$?

This question sounds particularly intriguing in the light of a recent conjecture by He, Huang, Nam, and Thaper [13] (mentioned already in Sect. 1.1), that almost every binary word (with even numbers of ones and zeros) is a shuffle square. For more on counting shuffle squares and their variants see, e.g., [14]. Similar problems can also be considered for \textit{shuffle cubes}, or more generally, for arbitrary \textit{shuffle r-powers}, in analogy to general $r$-twins.

An even more restricted version of twins is obtained when each twin in a shuffle square occupies itself a connected segment, like in the example below:

\[
\begin{array}{cccccccccc}
  a & b & c & a & b & c & a & b & c & a.
\end{array}
\]

This basic structure is well-known and widely studied in combinatorics on words under the name of a \textit{square} or a \textit{repetition} (see [17,18]). By the famous result of Thue [19] we know that there exist ternary words of any length with no squares altogether. In the binary case, it is known that there exist arbitrarily long words avoiding squares of length greater than 2, as proved by Fraenkel and Simpson [11]. However, not much is known about squares in random words.

Problem 4.4. What is the expected maximum length of a square in a random word $W_k(n)$?

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Appendix A: Maple Code

Here we present a simple program written in Maple that counts the number of ternary words of length $s$ that contain twins of length at least $t$. This easily allows one to calculate the number of ternary words of length $s$ with twins of length $t$ but not $t+1$ (defined as $\lambda_t$ in the proof of Theorem 1.1). Indeed, $\lambda_t$ is equal to the difference between the number of words having twins of length at least $t$ minus the one with twins of length at least $t+1$.

For the sake of clarity, we did not attempt to optimize our program.

```maple
with(combinat): # Load combinatorial functions

# This procedure returns the number of ternary words
# of length s that contain twins of length at least t

count_twins := proc(s, t)
local P_list, P, sizeP, K, sizeK, L, sizeL, L1, L2, counter, p, k, l, i:

# Generate all ternary words of length s
P_list := [seq(1, i = 1 .. s), seq(2, i = 1 .. s), seq(3, i = 1 .. s)]:
P := permute(P_list, s):
sizeP := nops(P):

# Generate all 2t−subsets of {1,...,s}
K := choose([seq(i, i = 1 .. seq_length)], 2*t):
sizeK := nops(K):

# Partition {1,...,2t} into two sets, each of size t
L := setpartition([seq(i, i = 1 .. 2*t)], t):
sizeL := nops(L):

counter := 0: # Number of words with twins of length at least t

for p to sizeP do
    for k to sizeK do
        for l to sizeL do
            L1 := L[l][1]: # Indices for the first possible twin
            L2 := L[l][2]: # Indices for the second possible twin
```
if evalb(P[p][K[k][L1]] = P[p][K[k][L2]]) # Twins?
    then
        # Twins found
        counter := counter + 1:
        # Move to the next word in P
        l := sizeL: # Break the loop over l
        k := sizeK: # Break the loop over k
    end if:
end do: # Over l
end do: # Over k
end do: # Over p
end proc:

Listing 1. The main procedure.

In Listing 2 we show how the procedure from Listing 1 can be used yielding results summarized in Table 3.

seq_length := 6: # Length of the ternary words
min_twin_length := 1: # Min length of possible twins
max_twin_length := floor(seq_length/2): # Max length of possible twins
for i from max_twin_length by -1 to min_twin_length do
    c[i] := count_twins(seq_length, i): # Number of twins of length >= i
    if i = max_twin_length
        then
            d[i] := c[i]: # Since there are no twins of length max_twin_length+1, c[max_twin_length] stores the number of words with the longest twins of length max_twin_length
        else
            d[i] := c[i] - c[i + 1]:
        end if:
    printf("Number of words with the longest twins of length %d = %d
", i, d[i]):
end do:

Listing 2. Calling the main procedure.

Appendix B: Proof of Fact 3.5

Proof of Fact 3.5. We are going to use the following form of Stirling’s formula:

\[ N! = \sqrt{2\pi N} \left( \frac{N}{e} \right)^N \left( 1 + O \left( \frac{1}{N} \right) \right). \]

Hence,

\[ \frac{N!}{(N - \ell)!} = \sqrt{\frac{N}{N - \ell}} \cdot \frac{N^{N - \ell}}{(N - \ell)^{N - \ell} \cdot e^{-\ell}} \left( 1 + O \left( \frac{1}{N} \right) \right). \]
Since
\[ \sqrt{\frac{N}{N - \ell}} = \left(1 + \frac{\ell}{N - \ell}\right)^{1/2} = 1 + O\left(\frac{\ell}{N}\right), \]
we get
\[ \frac{N!}{(N - \ell)!} = \frac{N^N}{(N - \ell)^N - \ell} \cdot e^{-\ell} \left(1 + O\left(\frac{\ell}{N}\right)\right). \]
Consequently, as
\[ \frac{(N-\ell)!}{(M-\ell)!} = \frac{M!}{N!} \cdot \frac{(N-\ell)!(\ell/N)}{(M-\ell)!} \cdot \left(1 + O\left(\frac{\ell}{N}\right)\right), \]
we obtain
\[ \frac{(N-\ell)!}{(M-\ell)!} = \frac{M^M}{N^N} \cdot \frac{(N-\ell)(\ell/N)}{(M-\ell)(\ell/M)} \cdot \left(1 + O\left(\frac{\ell}{N}\right)\right). \]
Further, since \(1 + x = e^{x + O(x^2)}\) whenever \(x \to 0\),
\[ \frac{M^M}{N^N} \cdot \frac{(N-\ell)(\ell/N)}{(M-\ell)(\ell/M)} \cdot e^{O(\ell^2/N)}, \]
where the last equality follows, because \(-\ell(N - \ell)/N + \ell(M - \ell)/M = \ell^2(M - N)/(MN)\) and \(M = \Theta(N)\). Finally, as by assumption \(\ell^2 = o(N)\), we have \(e^{O(\ell^2/N)} = 1 + O(\ell^2/N)\) and so
\[ \frac{(N-\ell)!}{(M-\ell)!} = \left(\frac{M}{N}\right)^{\ell} \left(1 + O\left(\frac{\ell^2}{N}\right)\right) \left(1 + O\left(\frac{\ell}{N}\right)\right) = \left(\frac{M}{N}\right)^{\ell} \left(1 + O\left(\frac{\ell^2}{N}\right)\right), \]
as required. \(\square\)

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