On $\alpha$-largeness and the Paris–Harrington principle in $\text{RCA}_0$ and $\text{RCA}_0^*$

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Abstract

We examine, within $\text{RCA}_0$, the treatment by Ketonen and Solovay on the use of $\alpha$-largeness for giving an upper bound for the Paris–Harrington principle. We also show how to modify the arguments to work within $\text{RCA}_0^*$. To the author’s knowledge, this is the first time that it is confirmed that the treatment can be done within EFA without some transfinite induction added.

Keywords: reverse mathematics, $\alpha$-largeness, Paris–Harrington principle, Ramsey theory, elementary function arithmetic.

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1 Introduction

In [2] Ketonen and Solovay prove the following theorem:

**Theorem 1 (Ketonen–Solovay)** If $X \geq 3$ is $\omega_{d+1}(c+5)$-large, then every colouring $C: [X]^{d+2} \to c$ has homogeneous $H \subseteq X$ of size $> \min H$.

This note consists of a modified version of the original presentation which should be better suited for reading from the reverse mathematics viewpoint. This may be of interest in light of the theorem’s use, by Patey and Yokoyama, in the conservativity result for $\text{RT}_2^2$ in [3]. As stated there, once it is understood, the original proof, for $d = 0$ and standard $c$ (Lemma 13 in this note), is not hard to be seen to be formalisable in $\text{RCA}_0$, since one can restrict the uses of transfinite induction to transfinite induction on $\omega^{c+4}$. 

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However, for readers unfamiliar with the subject matter, it is somewhat tedious to check this due to the distribution of the proof throughout the paper.

Thanks to arithmetic conservativity (Corollary IX.1.11 from [6]), this is also a confirmation that the Ketonen–Solovay theorem is provable within \( I\Sigma_1 \), as asked for in Problem 3.37 in [1]. This shows that Ketonen and Solovay’s copious use of transfinite induction on ordinals is not necessary for the theorem in question.

Finally, we also confirm that one can weaken the base theory to \( RCA^*_0 \). Thanks to \( \Pi^0_2 \)-conservativity (Corollary 4.9 in [7]), this also implies that the Ketonen–Solovay theorem is provable in elementary function arithmetic, \( EFA \).

The presentation within \( RCA_0 \) is suitable for advanced master level students and those who are unfamiliar with the Ketonen–Solovay paper [2]. We assume only basic knowledge on reverse mathematics in \( RCA_0 \) as in II.1-II.3 from [6]. At some places we favour an intuitive description and we leave many of the details as exercises for the reader. The changes compared to Ketonen–Solovay are concentrated in Section 3, with the rest of the proof, in Section 4, being only slightly modified from the originals.

In the last section we will describe how to modify our arguments to work within \( RCA^*_0 \).

2 Ordinals below \( \varepsilon_0 \) in \( RCA_0 \)

We will define the ordinals below \( \varepsilon_0 \) within \( RCA_0 \) as in Definition 2.3 in [5].

**Definition 2** We define the set \( \mathcal{E} \) of notations of ordinals \(< \varepsilon_0 \) and order \(< \) on \( \mathcal{E} \) as follows:

1. If \( \alpha_0 \geq \cdots \geq \alpha_n \in \mathcal{E} \), then \( \omega^{\alpha_0} + \cdots + \omega^{\alpha_n} \in \mathcal{E} \).
2. \( \omega^{\alpha_0} + \cdots + \omega^{\alpha_n} < \omega^{\beta_0} + \cdots + \omega^{\beta_m} \) if and only if:
   a. \( n < m \) and \( \alpha_i = \beta_i \) for all \( i \leq n \), or:
   b. there is \( i \leq \min\{n, m\} \) with \( \alpha_j = \beta_j \) for all \( j < i \) and \( \alpha_i < \beta_i \).

We use 0 to denote the empty sum, \( 0 < \alpha \) for all \( \alpha \neq 0 \), \( 1 = \omega^0 \), \( n = 1 + \cdots + 1 \), \( \omega = \omega^1 \), \( \omega_0(\alpha) = \alpha \), \( \omega_{d+1}(\alpha) = \omega^{\omega_d(\alpha)} \) and \( \omega_d = \omega_d(1) \).

As usual, if \( \alpha_n = 0 \), then \( \alpha \) is called a successor, otherwise, when not equal to 0, it is called a limit. One can define primitive recursive functions for ordinal-addition,
natural (Hessenberg) sum, and ordinal multiplication on $E$. Recall that, for $\alpha$ and $\beta$ as in Definition 2, the natural sum is:

$$\alpha \oplus \beta = \omega^{\gamma_0} + \cdots + \omega^{\gamma_m + n + 1},$$

where the $\gamma_i$’s are all the $\alpha_i$’s and $\beta_i$’s in descending order. The natural sum has the important property that none of the terms are lost, which can happen with ordinal addition. For example:

$$\omega + \omega^2 = \omega^2 \neq \omega^2 + \omega = \omega \oplus \omega^2.$$

Every ordinal in $E$ has a Cantor Normal Form:

$$\alpha = \text{CNF} \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_n} \cdot a_n,$$

where the $a_i$’s are positive integers and $\alpha_0 > \cdots > \alpha_n$.

**Definition 3 (Maximal coefficient)** MC(0) = 0 and, given $\alpha = \text{CNF} \omega^{\alpha_0} \cdot a_0 + \cdots + \omega^{\alpha_n} \cdot a_n > 0$:

$$\text{MC}(\alpha) = \max\{a_i, \text{MC}(\alpha_i)\}.$$

**Definition 4 (Fundamental sequence)** For $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n} \in E$ and $x \in \mathbb{N}$, take $0[x] = 0$, $(\alpha + 1)[x] = \alpha$, and:

1. If $\alpha_n = \beta + 1$, then $\alpha[x] = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n - 1} + \omega^\beta \cdot x$,
2. If $\alpha_n$ is a limit, then $\alpha[x] = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}[x]$.

**Definition 5** A finite set $X = \{x_0 < \cdots < x_{|X| - 1}\}$ is called $\alpha$-large if:

$$\alpha[x_0] \cdots [x_{|X| - 1}] = 0.$$

Any $X$ is 0-large.

Any $\omega$-large set $X$ has size $> \min X$. Unless otherwise specified, we will assume $\alpha$-large sets to be strictly above 2.

At first glance one may think that we require transfinite induction to demonstrate properties of the fundamental sequences and $\alpha$-large sets. In the remainder of this section we avoid this usage to treat some properties for later use.

**Lemma 6** If $\omega_d > \alpha > \beta$ and $x > \text{MC}(\beta)$, then $\alpha[x] \geq \beta$, where the inequality is strict if $\alpha$ is a limit.

**Proof:** Induction on $d$. 3
Lemma 7 For any $\alpha$ and any $x_0 < \cdots < x_R$, MC($\alpha$) < $y_0 < \cdots < y_R$ with $0 < x_i < y_i$ for all $i \leq R$: if $\alpha[x_0] \ldots [x_R] > 0$, then $\alpha'[y_0] \ldots [y_R] > 0$.

Proof: Use induction on $R$ with the aid of Lemma 6 to show $\alpha'[y_0] \ldots [y_R] \geq \alpha[x_0] \ldots [x_R]$.

Lemma 8 For any $\alpha > \beta > 0$ and any MC($\beta$) < $x_0 < \cdots < x_R$, we have that $\alpha[x_0] \ldots [x_R] > \beta[x_0] \ldots [x_R]$.

Proof: Use induction on $R$ with the aid of Lemma 6.

Define the thrice iterated exponential:

$$E(x) = 2^{2^x}.$$

One can check that:

1. The smallest $\omega$-large interval which contains $x$ as its minimal element is $[x, 2x]$.  
2. For $\omega^2$ this is bigger than $[x, 2^x \cdot x]$.  
3. If $x \geq 3$, then $\omega^3[x] \ldots [E(x) + x + 8] > 0$.

The following lemma shows that $\omega^3$-large sets $X$ are larger than $E(\text{min } X)$. This is a rather weak lower bound, since Ketonen and Solovay showed in their original proof that one can use the tower function instead of $E$.

Lemma 9 For any $3 \leq x_0 < x_1 < \ldots$ we have $\omega^3[x_0] \ldots [x_{E(x_0)+8}] > 0$.

Proof: This follows from item (3) directly above and Lemma 7.

3 Theorem 4.11 replacement:

Take:

$$\Phi(\alpha) = \omega^3 \cdot \alpha + \omega^3 + l + 2.$$
Lemma 10 (Theorem 4.11-Replacement) Suppose that $2 < X = \{x_0, \ldots, x_{|X|-1}\}$ is $\Phi(\gamma_0)$-large and $\gamma_0 > \gamma_1 > \cdots > \gamma_j$ is such that $MC(\gamma_i) \leq E(x_i + l)$. Then $j \leq |X| - 1$.

Before we start with the proof we give the idea: Take $\alpha_0 = \Phi(\gamma_0)$ and $\alpha_{i+1} = \alpha_i[x_i]$. By $\Phi(\gamma_0)$-largeness we know that $\alpha_{|X|} = 0$. In the original proof of Theorem 4.11 in [2] it is shown that the $\Phi(\gamma_i)$’s are a subsequence of the $\alpha_i$’s. We will show that the $\alpha_i$’s contain a subsequence whose $i$th elements are larger than the corresponding $\Phi(\gamma_i)$’s, thus demonstrating the conclusion of the lemma. The core of this lemma, namely pointing out the subsequence which has this property, is contained in the Claim.

Proof: Notice that:

1. $MC(\omega^3 \cdot \alpha + x) \leq \max\{MC(\alpha) + 3, x\},$
2. $MC(\alpha[x]) \leq \max(MC(\alpha), x),$
3. $E(x + 1) > E(x) + 4,$
4. Take $\beta_0 = \omega^3 \cdot \beta + \omega^3$ and $\beta_{k+1} = \beta_k[x_i+k]$, then, by Lemma [2] $\beta E(x_i)+8 > \omega^3 \cdot \beta$.

Take:

$$a_i = \begin{cases} 0 & \text{if } i = 0, \\ E(x_{i+l+1}) & \text{otherwise.} \end{cases}$$

Claim: $\alpha_{a_i} > \Phi(\gamma_i)$ for all $0 < i \leq j$.

Proof of the claim: Induction on $i$. We show both the case $i = 1$ and the induction step simultaneously.

We have the following, if $i = 0$ by notice (4), otherwise by induction hypothesis and all four notices:

$$\alpha_{a_i+E(x_{a_i+l+1})+l+8} > \omega^3 \gamma_i,$$

Therefore, thanks to $\gamma_{i+1} < \gamma_i$:

$$\alpha_{a_i+E(x_{a_i+l+1})+l+8} > \omega^3 \gamma_i \geq \omega^3 \cdot (\gamma_{i+1} + 1) = \omega^3 \cdot \gamma_{i+1} + \omega^3.$$

Hence:

$$\alpha_{a_{i+1}} \geq \alpha_{a_i+E(x_{a_i+l+1})} > \Phi(\gamma_{i+1}),$$

thus ending the proof of the claim, hence the lemma.

\[\square\]
Remark As a side note, the claim in the lemma also implies that the $\Phi(\gamma_i)$’s are a subsequence of the $\alpha_i$’s by using the following fact which can be shown using induction on $i$:

If $\alpha_{j-i-1} > \beta \geq \alpha_j$ and $x_{j-i-1} > MC(\beta)$, then $\beta = \alpha_l$ for some $j - i \leq l \leq j$.

4 How to prove the Ketonen–Solovay theorem

We proceed with, essentially, the proofs from Section 5 and 6 of [2]. The proofs have been streamlined into our setting, defining trees as sets of sequences, as is usual in reverse mathematics. The use of tree arguments for proving Ramsey-type theorems is attributed to Erdős and Rado. The outline is as follows:

1. Show that, if $X$ is $(\omega \cdot c)$-large, then for every colouring $C: X \to c$ there exists $\omega$-large $C$-homogeneous $H \subseteq X$. This is Lemma 11.

2. Given $C: [X]^{d+2} \to c$, construct Erdős–Rado trees $T_i$ from the first $i$ elements of $X$. Derive, from these trees, a decreasing sequence of ordinals of length $|X|$. Use Lemma 11 to determine that, if $X$ is “large enough” compared to $\alpha$, then $T_{|X|}$ contains an $\alpha$-large branch $Y$ such that the value of $C(x)$ on $[Y]^{d+2}$ does not depend on $\max x$. The case $d = 0$ is handled in Lemma 13, the case $d > 0$ is treated in Lemma 14.

3. Using induction on $d$, derive Theorem 1 from the above.

Lemma 11 If $X$ is $(\omega \cdot c)$-large then for every colouring $C: X \to c$ there exists $\omega$-large $C$-homogeneous $H \subseteq X$.

Proof: Since $X$ is $(\omega \cdot c)$-large it is the disjoint union of $\omega$-large sets: $X = X_0 \cup \cdots \cup X_{c-1}$. Assume, without loss of generality, that the min $C^{-1}(i)$’s are increasing. Assume, for a contradiction, that no $C^{-1}(i)$ is $\omega$-large. By induction on $i < c - 1$ we have:

$$\min C^{-1}(i) \leq \min X_i \& |\bigcup_{j \leq i} C^{-1}(j)| < |\bigcup_{j \leq i} X_j|,$$

the latter being implied by the first, as the $X_i$’s are $\omega$-large whilst the $C^{-1}(i)$’s are not. This implies $|\bigcup_{j \leq c-1} C^{-1}(j)| < |X|$, a contradiction.

\[\square\]
**Definition 12** For $0 < i \leq d + 1$, $C : [X]^{d+1} \to c$, we say $Y \subseteq X$ is $\min_i$-$C$-homogeneous if the value of $C$, on $[Y]^{d+1}$, depends only on the first $i$ elements of its input:

$$C(x_0, \ldots, x_i, y_i, \ldots y_d) = C(x_0, \ldots, x_i, z_i, \ldots, z_d)$$

for all $x_0 < \cdots < x_i-1 < y_i < \cdots < y_d, x_i-1 < z_i < \cdots < z_d$ from $Y$.

**Lemma 13** If $X$ is $(\omega^{c+3} + \omega^3 + c + 4)$-large, then every colouring $C : [X]^2 \to c$ has homogeneous $H \subseteq X$ of size $> \min H$.

**Proof:** Given $X = \{x_0 < \cdots < x_{|X|-1}\}$ and $C : [X]^2 \to c$, by the previous lemma it is sufficient so show that $X$ has an $(\omega \cdot c)$-large $\min_1$-$C$-homogeneous subset. Assume, for a contradiction, that is not the case.

Define $T_0 \subset \cdots \subset T_{|X|}$ as follows: $T_0 = \{\emptyset\}$ and

$$T_{i+1} = T_i \cup \{\sigma \dashv x_i\},$$

where $\sigma \in T_i$ is the leftmost of maximum length such that $\{\sigma_0, \ldots, \sigma_{\lh(\sigma)-1}, x_i\}$ is $\min_1$-$C$-homogeneous. By construction, if $\sigma \dashv y, \sigma \dashv z \in T_i$ then:

$$C(\sigma_{\lh(\sigma)-1}, y) \neq C(\sigma_{\lh(\sigma)-1}, z).$$

So the number of branches of $\sigma$ has upper bound $c$.

Let $(\omega \cdot c)[\sigma_0] \ldots [\sigma_{\lh(\sigma)-1}] = \omega \cdot d_\sigma + r_\sigma > 0$

Define: $n_{\sigma,i} = (c + 1)^r_\sigma (c - \# \text{branches of } \sigma \text{ in } T_i)$.

Take $\gamma_0 = \omega^c$ and, for $i > 0$:

$$\gamma_i = \bigoplus_{\substack{j < c \\theta \neq \sigma \in T_i \\ , \ j d_\sigma}}^\omega_j \cdot n_{\sigma,i}.$$ 

One can check that: $MC(\gamma_i) \leq E(x_i + c)$.

Notice that, by the absence of an $(\omega \cdot c)$-large subset of $X$: $\gamma_{i+1} > \gamma_i$ and $\gamma_{|X|} > 0$.

This is a contradiction due to Lemma 10.

□

**Remark** Any $\omega^{c+4}$-large $X > 2$ is also $(\omega^{c+3} + \omega^3 + c + 4)$-large.

**Proof:** This follows from $\omega^{c+4}[x_0] \ldots [x_{c+4}] > (\omega^{c+3} + \omega^3 + c + 4)$ with Lemmas 7 and 8.
Lemma 14 Suppose $X$ is $(\omega^3 \cdot \omega^\alpha + \omega^3 + \max\{c, MC(\alpha)\} + 3)$-large, then for every colouring $C: [X]^{d+1} \to c$ there exists $\alpha$-large $\min_d C$-homogeneous subset of $X$.

Proof: Assume, for a contradiction, that the colouring $C: [X]^{d+1} \to c$ is such that it does not have $\alpha$-large $\min_d C$-homogeneous subset of $X = \{x_0 < \cdots < x_{|X|-1}\}$. Define $T_0 \subset \cdots \subset T_{|X|}$ as follows: $T_0 = \{\emptyset\}$ and

$$T_{i+1} = T_i \cup \{\sigma^{-1}x_i\},$$

where $\sigma \in T_i$ is the leftmost of maximum length such that $\{\sigma_0, \ldots, \sigma_{\text{lh}(\sigma)-1}, x_i\}$ is $\min_d C$-homogeneous. By construction, if $\sigma^- y, \sigma^- z \in T_i$ then there are $\sigma_{j_0} < \cdots < \sigma_{j_{d-1}}$ with

$$C(\sigma_{j_0}, \ldots, \sigma_{j_{d-1}}, y) \neq C(\sigma_{j_0}, \ldots, \sigma_{j_{d-1}}, z).$$

So the number of branches of $\sigma$ is bound by the number of colourings $[\sigma_0, \ldots, \sigma_{\text{lh}(\sigma)-1}]^d \to c$, which has upper bound $c^{\sigma_{\text{lh}(\sigma)-1}}$. Define: $m_{\sigma,i} = c^{\sigma_{\text{lh}(\sigma)-1}} - \#\text{branches of } \sigma \text{ in } T_i$. By the comment directly above $m_{\sigma,i} \geq 0$.

Take $\gamma_0 = \omega^\alpha$ and:

$$\gamma_i = \bigoplus_{\emptyset \neq \sigma \in T_i} \omega^{\alpha[\sigma_0] \cdots [\sigma_{\text{lh}(\sigma)-1}]} \cdot m_{\sigma,i}.$$

One can check that: $MC(\gamma_i) \leq E(x_i + \max\{c, MC(\alpha)\} + 1)$. We can see that, by the absence of $\min_d$-homogeneous $\alpha$-large subsets of $X$: $\gamma_i > \gamma_{i+1}$ and $\gamma_{|X|} > 0$.

This is a contradiction by Lemma 10.

Theorem 1 can now be shown using induction on $d$ to prove:

If $X \geq 3$ is $\omega_0(\omega^{c+4} + d)$-large, then every colouring $C: [X]^{d+2} \to c$ has homogeneous $H \subseteq X$ of size $> \min H$.

Use Lemma 13 for the base case and Lemma 14 in the induction step. Use Lemmas 7 and 8 to bridge the differences in largeness.

Corollary 15 Theorem 7 is provable in RCA$_0$. Remark Above proof is also fine in RCA$_0^*$ if we fix a standard $d$.
5 Starring the proofs

We work in $\text{RCA}_0^*$ as defined in X.4 from [6] and elaborated in Section 2 from [7]. Since every elementary function’s existence is proven within $\text{RCA}_0^*$ we recommend Sections 2 and 3 from Chapter 1 of [4] as background material. Defining our ordinals as in Section 2 poses no problem, however note that, in the proof in $\text{RCA}_0$, we used implicitly that $(\alpha, x) \mapsto \alpha[x]$ is primitive recursive.

As we are now working within the weaker system, we need this function to be elementary, which is non-obvious for nonstandard $d$. For example, if we encode the ordinals using prime numbers, as in [7], then we may need a nonstandard amount of iterations of the exponential function to determine the code of $\alpha[x]$, which is not available in $\text{RCA}_0^*$. In the following definition, starting with the pairing map $j(x, y) = \frac{1}{2}(x+y+1)(x+y) + y$ with projections $j_1$ and $j_2$, we use:

$$
\pi_i^n(x) = \begin{cases}
   j_1j_2^{(n-i)}(x) & \text{if } 0 < i < n, \\
   j_2^{(n)}(x) & \text{if } i = 1.
\end{cases}
$$

**Definition 16 (Codes of ordinals)** Using bounded recursion, we define the codes of ordinals from $E$ and relation $\prec$ on the codes as follows:

1. $a$ is a code whenever $j_1(a) = n > 0$ and $\pi_i^n(a) \succeq \cdots \succeq \pi_i^n(a)$ are codes.
2. $a \prec b$, with $n = j_1(a)$ and $m = j_1(b)$, if and only if:
   - (a) $n < m$ and $\pi_i^n(a) = \pi_i^m(b)$ for all $0 < i \leq m$, or
   - (b) there is $0 < i \leq \min\{n, m\}$ with $\pi_i^n(a) = \pi_i^m(b)$ for all $0 < j < i$ and $\pi_i^m(a) \preceq \pi_i^m(b)$

$0$ is the code for $0$, $w_0 = j(1, 0)$, $w_{i+1} = j(1, w_i)$ and $0 \prec a$ for all codes of ordinals $a$. We sometimes use $a_i = \pi_i^n(a)$.

One can define ordinal addition, multiplication, natural (Hessenberg) sum, the Cantor Normal Form (coded version) and the maximum coefficient on the codes of the ordinals using bounded recursion.

Using bounded recursion one can define a function $\text{code}: E \mapsto \mathbb{N}$ such that it preserves the order and operations on the ordinals. If $\omega_d$ exists, $\text{code}(\omega_d) = w_d$.

Our next step is to define the fundamental sequences on the codes of ordinals.
**Definition 17** On the codes of ordinals, following the definition of fundamental sequences on ordinals, define:

\[
 a[x] = \begin{cases} 
 0 & \text{if } a = 0, \\
 j(n - 1, j_2^{(2)}(a)) & \text{if } n = j_1(a) > 0 \text{ and } a_n = 0, \\
 j(n + x - 1, f(x, b, a)) & \text{if } n = j_1(a) > 0, m = j_1(a_n) > 0, (a_n)_m = 0 \\
 & \text{and } b = j(m - 1, j_2^{(2)}(a_n)), \\
 j(n, j(a_n[x], j_2^{(2)}(a))) & \text{otherwise},
\end{cases}
\]

where \( f(0, b, a) = j_2^{(2)}(a) \) and \( f(i + 1, b, a) = j(b, f(i, b, a)) \).

**Lemma 18** The functions \((a, x) \mapsto a[x]\) and \((a, \{x_0 < \cdots < x_n\}) \mapsto a[x_0] \ldots [x_n]\) are elementary.

**Proof:** One can check:

\[ f(i, b, a) \leq 2^i(a + b + 1)^{2i} \]

hence, for \(0 < a < w_d\) and \(x > 0\):

\[ a[x] \leq 3^d(2^{x+1}a^{2x})^{2d}. \]

So:

\[ a[x_0] \ldots [x_n] \leq (6a)^d(2^{x_n+2})^{(n+1)(d+1)}. \]

Therefore, these functions are elementary by bounded recursion.

\[ \square \]

Taking care to use \( \Delta^0_0 \)-induction every time induction was used, one can proceed with the proofs as described, using codes of ordinals instead of \( \mathcal{E} \) where necessary.

As an example of dealing with the induction steps, examine Lemma 6 modified to the ordinal codes:

**Lemma 19** If \( w_d \succ a \succ b \) and \( x > \text{MC}(b) \), then \( a[x] \succeq b \), where the inequality is strict if \( a \) is a limit.

**Proof:** Given \( x, a \) and \( b \), use \( \Delta^0_0 \)-induction on \( d \) to prove the following statement:

If \( w_d \succ a' \succ b', a' \leq a, b' \leq b, \) and \( x > \text{MC}(b) \), then \( a'[x] \succeq b' \), where the inequality is strict if \( a' \) is a limit.

To show that the statement is \( \Delta^0_0 \), notice that the characteristic functions of \( a \prec b \) and \( a[x] \succeq b \) are elementary.

\[ \square \]

**Corollary 20** Theorem 7 is provable in \( \text{RCA}^*_0 \).  

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