CONVEXITY IN MULTIVALUED HARMONIC FUNCTIONS

Abstract

We investigate variants of a three circles type Theorem in the context of \(Q\)-valued functions. We prove some convexity inequalities related to the \(L^2\)-growth function in the \(Q\)-valued settings. Optimality of these inequalities and comparison to the case of real valued harmonic functions is also discussed.

1 Introduction

1.1 Background

The study of multi valued harmonic functions was originated in the pioneering work of Almgren \[2\] on Plateau’s problem, which asks for a surface of minimal area among all surfaces stretched across a given closed contour. Almgren’s theory was further extended and simplified in \[5\]. The profound geometric applications of Almgren’s theory to minimal surfaces are not addressed here. Instead, we shall connect the theory of \(Q\)-valued functions to a classical results from complex analysis, which has some modern reflections. Let us begin by providing some background to the material that motivated this work. Let \(0 \neq u\) be harmonic on the unit ball \(B_1(0)\). Then one can associate to \(u\) real valued functions \(H_u, D_u, H_u, I_u : (0, 1) \to \mathbb{R}\) by letting

\[
H_u(r) := \int_{\partial B_r(0)} u^2(x) d\sigma,
\]

\[
D_u(r) := \int_{B_r(0)} |
abla u|^2 dx,
\]

\[
\overline{H}_u(r) := \frac{1}{|\partial B_r(0)|} H_u(r),
\]

and
\[ I_u(r) := \frac{rD_u(r)}{\mathcal{H}_u(r)}. \]

\( \mathcal{H}_u(r) \) is called the \( L^2 \)-growth function of \( u \) and \( I_u(r) \) is called the frequency function of \( u \). The motivation behind the definition of the function \( \mathcal{H}_u \) comes from a classical result in complex analysis known as the three circles Theorem:

Given a holomorphic function \( f \) on the unit ball \( B_1 \), let \( M(r) := \max_{B_r(0)} |f| \).

The three circles Theorem, proved by Hadamard, states that the function \( \tilde{M} : (-\infty, 0) \to \mathbb{R} \) defined by \( \tilde{M}(t) := M(e^t) \) is log convex, that is \( \log \tilde{M}(t) \) is convex.

It is therefore natural to seek for a three circles type Theorem for real harmonic functions \( u : B_1(0) \subset \mathbb{R}^n \to \mathbb{R} \). It was first observed by Agmon \([1]\) that such a theorem holds if the function \( M \) is replaced by an appropriate \( L^2 \)-version on the sphere. Namely, Agmon proves that the function \( t \mapsto \tilde{H}(e^t) \) is log convex.

In 2015, Lippner and Mangoubi observed the following stronger result:

**Theorem 1.1.** \([6], \text{Theorem 1.4}\). Let \( u : B_1(0) \to \mathbb{R} \) be harmonic. Then \( \mathcal{H}_u \) is absolutely monotonic, that is \( \mathcal{H}^{(k)}(r) \geq 0 \) for all \( k \in \mathbb{N} \). In particular, \( H_u^{(k)} \geq 0 \) for all \( k \in \mathbb{N} \).

Since Theorem (1.1) in \([6]\) is carried out in a discrete setting, for the sake of completeness a proof of the continuous version (as stated above) is presented in the appendix. The second statement in Theorem (1.1) is an immediate consequence of the first one. It is an exercise to verify that absolute monotonicity of \( \mathcal{H} \) implies log convexity of \( t \mapsto \mathcal{H}(e^t) \) (see \([7], \text{II, Problem 123}\) ). Roughly speaking, we are interested in the question whether a Lippner-Mangoubi type theorem can be obtained in the more general setting of multi valued harmonic functions. Let us emphasize this could have fascinating applications in the regularity theory of these objects, since absolutely monotonic functions are real analytic (due to a celebrated theorem of Bernstein. See \([3]\)). In some sense, the nonlinear nature of the problem is the main obstacle in obtaining elliptic regularity type results for multi valued harmonic functions. We hope that approaching the problem via Bernstein’s theorem may be useful in overcoming some of the difficulties that are created by the lack of linearity.

### 1.2 Main Results

Given some \( P \in \mathbb{R}^n \) we denote by \( \|[P]\| \) the Dirac mass in \( P \in \mathbb{R}^n \) and define

\[ \mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} \|[P_i]\| |P_i| \in \mathbb{R}^n, 1 \leq i \leq Q \right\}. \]

The set \( \mathcal{A}_Q(\mathbb{R}^n) \) is endowed with a metric \( G \), not specified for the moment, such that the space \( (\mathcal{A}_Q(\mathbb{R}^n), G) \) is a complete metric space. We then consider functions \( f : \Omega \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n) \), where \( \Omega \) is some domain in \( \mathbb{R}^m \). We call such functions \( Q \)-valued functions. One key fact is the existence of a notion of a harmonic \( Q \)-valued function.
We adapt the terminology of [5] and call such functions Dir-minimizing. As their name suggests, Dir-minimizing functions are defined as functions minimizing a certain class of integrals, by analogy with the classical Dirichlet principle. For each $f : B_1(0) \subset \mathbb{R}^m \to A_Q(\mathbb{R}^n)$ Dir-minimizing we associate a real valued function $\overline{H}_f : (0, 1) \to \mathbb{R}$ by letting

$$\overline{H}_f(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} |f|^2 d\sigma.$$ 

The function $\overline{H}_f$ is a generalization of the function introduced in the beginning. Our first aim would be to generalize Agmon’s Theorem to the multi valued case. We will prove

**Proposition 1.2.** Let $f : B_1(0) \subset \mathbb{R}^m \to A_Q(\mathbb{R}^n)$ be Dir minimizing such that $H(r) > 0$. Define $a : (-\infty, 0) \to \mathbb{R}$ by $a(t) = \log H(e^t)$. Then

(i) $a'(t) \geq 0$ for all $t \in (-\infty, 0)$.

(ii) $a''(t) \geq 0$ for a.e. $t \in (-\infty, 0)$.

Furthermore, $a$ is convex.

Since (ii) holds merely up to a null set, the convexity of $a$ does not follow directly. This requires an additional consideration. The following theorem, is the main result of this work

**Theorem 1.3.** Let $f : B_1(0) \subset \mathbb{R}^2 \to A_Q(\mathbb{R}^n)$ be a Dir minimizing function. Suppose $f|_{\partial B_r} \in W^{1, 2}(\partial B_r(0), A_Q(\mathbb{R}^n))$ for a.e. $0 < r < 1$. For each $N > 0$ define $\overline{h}_{N,f} : (0, 1) \to \mathbb{R}$ by $\overline{h}_{N,f}(r) := \overline{H}_f(r^N)$. Then

(i) $\overline{h}_{N,f}''(r) \geq 0$ for all $r \in (0, 1)$.

(ii) $\overline{h}_{N,f}'''(r) \geq 0$ for a.e. $r \in (0, 1)$.

Furthermore, $\overline{h}_{N,f}$ is convex.

The proof of Theorem 1.3 will be followed by a higher dimensional version, that is, when the domain is the $m$–dimensional unit ball for arbitrary $m > 2$. In the higher dimensional version, the constant $\frac{Q}{2}$ will be replaced by some constant depending on $m$ which does not have a simple closed formula. It should be remarked that unlike in the scenario of Theorem 1.1, the fact that $\overline{H}_f$ (and hence $a$ and $\overline{h}_{N,f}$) is a.e. twice differentiable (and moreover $C^1$) is nontrivial.

A naive version of Theorem 1.1 for $Q$-valued functions is not valid, as witnessed by the example $f(z) = \sum_{w^2 = z} [[w]]$, for which the associated $\overline{H}$ function has a negative second derivative for all $0 < r < 1$. In addition, the following proposition demonstrates that we do not have an obvious third derivative version of Theorem 1.3:

**Proposition 1.4.** (i) Define $f : B_1(0) \subset \mathbb{R}^2 \to A_2(\mathbb{R}^2)$ by $f(z) = \sum_{w^2 = 2z - 1} [[w]]$. Then $f$ is Dir minimizing, $f|_{\partial B_r} \in W^{1, 2}(\partial B_r, A_2(\mathbb{R}^n))$ for all $r \neq \frac{1}{2}$ and $\overline{h}'''_{N,f}(r) = \overline{H}_f''(r) < 0$ for all $\frac{1}{2} < r < 1$. 

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Both Proposition (1.4) and the above mentioned example will be proved and explained in section 4. In Proposition (1.4), our main contribution is performing the computation which shows that \( H'''_{f} < 0 \) for all \( \frac{1}{2} < r < 1 \) and showing that the boundary condition \( f|_{\partial B_{r}} \in W^{1,2}(\partial B_{r}, \mathcal{A}_{Q}(\mathbb{R}^{n})) \) is indeed satisfied for all \( r \neq \frac{1}{2} \). Proving that \( f \) is Dir minimizing is a difficult task, and relies on some rather heavy machinery from geometric measure theory. It should be emphasized that the domain of \( f \) in both counterexamples is the planar unit disk. Thus, we did not rule out the possibility that in higher dimensions the \( L^{2} \)-growth function of \( f \) is more well behaved.

Organization of the paper. In section 2 we fix some notation and briefly review the frequency function and its relatives. A detailed exposition may be found in [5]. Section 3 is devoted mainly to the proof of Proposition (1.2), Theorem (1.3) and other related convexity inequalities. In Section 4 we perform the calculations required to establish the counterexample given in Proposition (1.4). In the same context we will prove that the boundary condition \( f|_{\partial B_{r}} \in W^{1,2}(\partial B_{r}(0), \mathcal{A}_{Q}(\mathbb{R}^{n})) \) in Theorem (1.3) is in fact verified for a certain class of Dir-minimizing functions on the unit disk.

2 Preliminaries

2.1 Notations

\( d\sigma = \) The surface measure.

\( \Delta u = \) The Laplacian of \( u \).

\( \cdot = \) The standard scalar product on \( \mathbb{R}^{m} \).

\( \nu = \) The unit normal to the sphere.

\( C_{m} = \) Surface area of the \( m \)-dimensional unit sphere.

\( B_{R}(x) = \) The ball of radius \( R \) centered at \( x \).

We assume that the reader is familiar with the basic theory of \( Q \)-valued functions. Following [5], we recall some basic notions and terminology. Given some \( P \in \mathbb{R}^{n} \) we denote by \( [[P]] \) the Dirac mass in \( P \in \mathbb{R}^{n} \) and define

\[
\mathcal{A}_{Q}(\mathbb{R}^{n}) := \{ \sum_{i=1}^{Q} [[P_{i}]] | P_{i} \in \mathbb{R}^{n}, 1 \leq i \leq Q \}.
\]

We endow \( \mathcal{A}_{Q}(\mathbb{R}^{n}) \) with a metric \( \mathcal{G} \), defined as follows: For each \( T_{1}, T_{2} \in \mathcal{A}_{Q}(\mathbb{R}^{n}) \) with \( T_{1} = \sum_{i=1}^{Q} [[P_{i}]], T_{2} = \sum_{i=1}^{Q} [[S_{i}]] \) we define

\[
\mathcal{G}(T_{1}, T_{2}) := \min_{\sigma \in \mathcal{P}_{Q}} \sqrt{\sum_{i=1}^{Q} |P_{i} - S_{\sigma(i)}|^{2}},
\]

where \( \mathcal{P}_{Q} \) is the permutation group on \( \{1, ..., Q\} \). The space \( (\mathcal{A}_{Q}(\mathbb{R}^{n}), \mathcal{G}) \) is a complete metric space. A \( Q \)-valued function is a function \( f : \Omega \subset \mathbb{R}^{m} \rightarrow \mathcal{A}_{Q}(\mathbb{R}^{n}) \).
\(A_Q(\mathbb{R}^n)\). Of course, this formalism was designed to capture the notion of a function attaining multiple values at each point. A regularity theory can be developed for \(Q\)-valued functions. In particular, the notion of a Sobolev space \(W^{1,p}(\Omega, A_Q(\mathbb{R}^n))\) and the notion of an approximate differential denoted by \(Df\).

Suppose \(\Omega \subset \mathbb{R}^m\) is a bounded domain with smooth boundary. By analogy with the Dirichlet principle we say that a function \(f : \Omega \rightarrow A_Q(\mathbb{R}^n)\) is Dir-minimizing if \(f \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n))\) and

\[
\int_{\Omega} |Df|^2 \leq \int_{\Omega} |Dg|^2,
\]

for all \(g \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n))\) whose trace on \(\partial \Omega\) agrees with that of \(f\). We shall always assume \(m \geq 2\).

### 2.2 Frequency Function

We recall the frequency function and its relatives in the context of \(Q\)-valued functions. We have the following Holder regularity type theorem for Dir-minimizing functions:

**Theorem 2.1.** ([5], Theorem 6.2) There are constants \(\alpha = \alpha(m, Q) \in (0, 1)\) and \(C = C(m, n, Q, \delta)\) with the following property. If \(f : B_1(0) \subset \mathbb{R}^m \rightarrow A_Q(\mathbb{R}^n)\) is Dir-minimizing then

\[
\sup_{x \neq y \in B_\delta(0)} \frac{G(f(x), f(y))}{|x - y|^\alpha} \leq C \text{Dir}(f)^{\frac{1}{2}}
\]

for all \(0 < \delta < 1\).

In light of Theorem (2.1) \(|f|^2\) is continuous on \(B_1(0)\). Fix \(0 < \delta < 1\). Then, according to Theorem (2.1) \(f\) is continuous on \(B_\delta(0)\). Since \(G\) is a metric, by the triangle inequality:

\[
||T| - |S|| = |G(T, Q[[0]]) - G(S, Q[[0]])| \leq G(T, S).
\]

So \(|f|\) is the composition of \(f\) with a Lipschitz function, which implies that \(|f|\) is continuous on \(B_\delta(0)\), and so the same is true for \(|f|^2\). This is true for all \(0 < \delta < 1\) from which we deduce continuity on \(B_1(0)\). Thus, both \(H_f\) and \(\overline{H_f}\) are well defined on all \(r \in (0, 1)\). If furthermore \(H_f(r) > 0\) for all \(r\) then

\[
I_f(r) := \frac{rD_f(r)}{H_f(r)}
\]

is well defined and is called the frequency function of \(f\). When there is no ambiguity, we shall omit the subscript \(f\).
3 Proof of Main Results

In this section we give a proof of Proposition (1.2) and Theorem (1.3). The following identities will play a crucial role in the study of convexity of the frequency function and its relatives.

**Proposition 3.1.** ([5], Proposition 5.2) Let \( f : B_R(x) \subset \mathbb{R}^m \rightarrow \mathcal{A}_Q(\mathbb{R}^n) \) be Dir-minimizing.

Then for a.e. \( 0 < r < R \) we have: (i) \( (m-2) \int_{B_r(x)} |Df|^2 dx = r \int_{\partial B_r(x)} |Df|^2 d\sigma - 2r \int_{\partial B_r(x)} \sum_{i=1}^Q |\partial^r f_i|^2 d\sigma \).

(ii) \( \int_{B_r(x)} |Df|^2 dx = \int_{\partial B_r(x)} \sum_{i=1}^Q (\partial^r f_i) \cdot f_i d\sigma. \)

Our starting point is the following theorem.

**Theorem 3.2.** ([5], Theorem 7.5) Let \( f : B_1(0) \subset \mathbb{R}^m \rightarrow \mathcal{A}_Q(\mathbb{R}^n) \) be Dir-minimizing. Then:

(a) \( H \in C^1(0,1) \) and the following identity holds for all \( r \in (0,1) : \)

\[ H'(r) = \frac{m-1}{r} H(r) + 2D(r). \quad (3.1) \]

(b) If \( H(r) > 0 \) then \( I(r) \) is absolutely continuous and non decreasing. In particular, \( I'(r) \geq 0 \) for a.e. \( r \).

Statement (b) is known as Almgren’s monotonicity formula. Since \( D \) is absolutely continuous, it is a.e. differentiable. Therefore, in view of equation (3.1), we see that \( H' \) is a.e. differentiable. Otherwise put, the second derivative of \( H \) exists a.e. The regularity properties of \( H \) clearly apply for \( \overline{H} \) as well.

With the aid of Almgren’s monotonicity formula we are able to extend Agmon’s convexity result [1] in the context of \( Q \)-valued functions. This method of proof differs from Agmon’s original approach for real valued harmonic functions, which involves ODEs on Banach spaces.

**Proof of Proposition 1.2.** In light of the above discussion it is clear that \( a \) is \( C^1(-\infty,0) \) and that \( a'' \) exists almost everywhere. For (i), note that

\[ C_m \overline{H}^r (0) = \left( \frac{1}{r^{m-1}} H(r) \right)^r = \frac{1}{r^{m-1}} H'(r) - (m-1) r^{-m} H(r) \]

\[ = \frac{1}{r^{m-1}} \left( \frac{m-1}{r} H(r) + 2D(r) \right) - (m-1) \frac{1}{r^m} H(r) = \frac{2D(r)}{r^{m-1}} \geq 0, \]

Where the second equality is due to equation (3.1). So

\[ \overline{H} (r) = \frac{2D(r)}{C_m r^{m-1}} \geq 0. \]

(3.2)
Therefore, for all \( t \in (-\infty, 0) \)

\[
a'(t) = \log(H(e^t))' = \frac{H'(e^t)e^t}{H(e^t)} \geq 0.
\]  

(3.3)

For (ii), start by noting that

\[
a(t) = \log(H(e^t)) = \log\left(H(e^t) - \log(C_m e^{(m-1)t})\right) = \log(H(e^t)) - \log(C_m) - (m-1)t.
\]

Therefore, to prove \( a''(t) \geq 0 \) it suffices to prove \( (\log(H(e^t)))'' \geq 0 \). By Theorem (3.2), we have that \( I'(r) \geq 0 \) for a.e. \( 0 < r < 1 \). To spare some space, all equalities and inequalities from now on should be interpreted up to a null set.

By virtue of equation (3.1)

\[
0 \leq I'(r) = \left(\frac{rD(r)}{H(r)}\right)' = \left(\frac{D(r)}{r^{m-2}H(r)}\right)'
\]

\[
= \frac{1}{2} \left( \frac{r(H'(r) - (m-1)H(r))}{H(r)} \right)'
\]

\[
= \frac{1}{2} \left( \frac{rH'(r) - (m-1)H(r)}{H(r)} \right)'
\]

\[
= \frac{1}{2} \left( \frac{rH'(r)}{H(r)} \right)' = \frac{1}{2} \left( \frac{H'(r) + rH''(r)H(r) - r(H'(r))^2}{H^2(r)} \right).
\]

Thus, we get

\[
(H'(r) + rH''(r))H(r) - r(H'(r))^2 \geq 0.
\]  

(3.4)

On the other hand by a straightforward calculation:

\[
e^{-t}(\log(H(e^t)))'' = (H'(e^t) + e^tH''(e^t))H(e^t) - e^t(H'(e^t))^2.
\]  

(3.5)

Combining inequality (3.4) with equation (3.5) we arrive at \( e^{-t}(\log(H(e^t)))'' \geq 0 \) which is the same as \( (\log(H(e^t)))'' \geq 0 \). We are left to explain why \( a \) is convex. It is classical that a continuously differentiable function is convex iff its derivative is non decreasing, and so our task reduces to showing that \( a' \) is non decreasing. By equations (3.3) and (3.2) we get

\[
a'(t) = \frac{H'(e^t)e^t}{H(e^t)} = \frac{2D(e^t)}{C_m e^{(m-2)t}H(e^t)}.
\]

Since \( D(e^t) \) is a composition of an absolutely continuous function with a non decreasing smooth function, it is absolutely continuous. In addition, \( \frac{1}{C_m e^{(m-2)t}H(e^t)} \) is differentiable. So \( a'(t) \) is absolutely continuous function on any closed sub
interval of \((-\infty, 0]\), as a product of such function. Therefore, the fundamental theorem of calculus is applicable: if \(t_1, t_2 \in (-\infty, 0), t_1 < t_2\) then

\[
a'(t_2) - a'(t_1) = \int_{t_1}^{t_2} a''(t) dt \geq 0.
\]

\[\square\]

Remark 3.3. We draw the reader’s attention to a somewhat delicate point, which will be also relevant in what will come next. The implication “non negative derivative a.e.\(\Rightarrow\)non decreasing” is not true in general. In Proposition 1.2 we employed the fact that the first derivative of \(a\) is absolutely continuous in order to deduce that it is convex. The absolute continuity of the derivatives is a consequence of equation (3.1). Hence, we implicitly relied here on the Dir-minimization property. In the case of 1 valued harmonic functions, this technicality is not created because all functions involved are smooth. To the best of our knowledge, improved regularity for the frequency function and its relatives in the multi valued settings is still an open problem.

The inequality \(\overline{H}''(r) \geq 0\) a.e.” is not true in general, as witnessed by the counterexample in section 4. Nevertheless, we are still able to obtain a convexity result by reducing the power of the normalization of \(H\). More precisely we observe the weaker

Proposition 3.4. Let \(f : B_1(0) \subset \mathbb{R}^m \to A_\mathcal{Q}(\mathbb{R}^n)\) be Dir minimizing. Then

\((i)\) \((r\overline{H}(r))' \geq 0\) for all \(r \in (0, 1)\). \((ii)\) \((r\overline{H}(r))'' \geq 0\) for a.e. \(r \in (0, 1)\). Furthermore \(r \mapsto r\overline{H}(r)\) is convex.

Proof. As for (i) we can in fact derive a stronger result, namely \(\overline{H}(r) \geq 0\) for all \(r \in (0, 1)\). We compute:

\[
\overline{H}'(r) = \frac{1}{C_m}(-(m-1)r^{-m}H(r) + r^{-(m-1)}H'(r))
\]

\[
= \frac{r^{-(m-1)}}{C_m}(H'(r) - \frac{m-1}{r}H(r)) = \frac{2D(r)r^{-(m-1)}}{C_m} \geq 0,
\]

where the last equality is by equation (3.1). For (ii), start by noting that equation (3.6) implies the following equality for a.e. \(r\)

\[
\overline{H}''(r) = \frac{2}{C_m r^{m-1}}(D'(r) - \frac{(m-1)D(r)}{r}).
\]

Gathering our calculations one readily checks that \((r\overline{H}(r))'' \geq 0 \iff rD'(r) - (m-3)D(r) \geq 0\). Indeed

\[
rD'(r) - (m-3)D(r) \geq rD'(r) - (m-2)D(r)
\]
\[= r \int_{\partial B_r(0)} |Df|^2 \geq (m - 2) \int_{B_r(0)} |Df|^2 = 2r \int_{\partial B_r(0)} \sum_{i=1}^Q |\partial_\nu f_i|^2 \geq 0,\]

where the last equality is thanks to (i), (3.1). The convexity of \( r \mapsto rH(r) \) follows by a similar argument to the one demonstrated in Proposition (1.2).

\[\square\]

It is clear that Proposition (3.4) implies in particular that the same conclusion holds true for \( H \) (and this can also be derived directly from iterating equation (3.1)).

We present now a proof of Theorem (1.3), including a higher dimensional analog. The main ingredients of the proof are the variational formulas provided by Proposition (3.1) and the following estimates

**Proposition 3.5.** ([5], Proposition 6.3) Let \( f : B_1(0) \subset \mathbb{R}^m \to A_Q(\mathbb{R}^n) \) be Dir-minimizing and suppose that \( g_r := f|_{\partial B_r(0)} \in W^{1,2}(\partial B_r(0), A_Q(\mathbb{R}^n)) \) for a.e \( 0 < r < 1 \). Then for a.e \( r \):

(i) If \( m = 2 \), \( \text{Dir}(f, B_r(0)) \leq Qr \text{Dir}(g_r, \partial B_r(0)) \).

(ii) If \( m > 2 \), \( \text{Dir}(f, B_r(0)) \leq c(m)r \text{Dir}(g_r, \partial B_r(0)) \), where \( c(m) < \frac{1}{m-2} \).

Before going into the proof of Theorem (1.3), let us heuristically explain why it is reasonable to expect the validity of such a result through the following example.

**Example 3.6.** We recall that a function \( f : B_1(0) \to A_Q(\mathbb{R}^n) \) is \( \alpha \)-homogeneous (\( \alpha > 0 \)) if \( \forall y \in B_1, y \neq 0 : f(y) = |y|^\alpha f(\frac{y}{|y|}) \). Denote by \( \eta : A_Q(\mathbb{R}^n) \to \mathbb{R}^n \) the center of mass map defined by

\[\eta(\sum_{i=1}^Q [P_i]) := \frac{\sum_{i=1}^Q P_i}{Q}.\]

We can derive an explicit formula for the \( L^2 \)-growth function of a continuous \( \alpha \)-homogeneous map:

\[H(r) = \int_{\partial B_r} |f|^2 d\sigma = r^{m-1} \int_{\partial B_1} |f|^2(ry)d\sigma = r^{m-1} \int_{\partial B_1} \sum_{i=1}^Q |f_i|^2(ry)d\sigma = \kappa r^{2\alpha + m-1},\]

for some constant \( \kappa \geq 0 \). Hence \( \mathcal{T} \) takes the form \( \mathcal{T}(r) = \kappa r^{2\alpha} \). Assume now that \( m = 2, f : B_1(0) \to A_Q(\mathbb{R}^n) \) is a Dir-minimizing, nontrivial, \( \alpha \)-homogeneous map with \( \eta \circ f = 0 \) (the simplest example of such a map is
the 2-valued function \( f : B_1(0) \to A_2(\mathbb{R}^2) \) defined by \( f(z) = \sum_{w^2 = z} [w] \). See Theorem (4.1). It is proved in [5], Proposition 8.2 that in this case necessarily \( \alpha = \frac{p}{q} \in \mathbb{Q} \) for some \( q \leq Q \). So \( \overline{H}(r) = \kappa r^{2p} \), which implies \( \overline{H}(r^2) = \kappa r^p \), and the latter function is obviously absolutely monotonic. We are thus lead to speculate that more generally, composing \( \overline{H} \) with some suitable \( \frac{1}{2}|Q| \)-power produces a function which is more well behaved. Theorem (1.3) partially confirms this speculation.

**Proof of Theorem 1.3.** That \( h''_N(r) \geq 0 \) follows immediately from equation (3.2). Henceforth all equalities and inequalities should be interpreted up to a null set. A direct calculation gives:

\[
\overline{h}''_N(r) = N(N - 1)r^{N-2}\overline{H}'(r^N) + \overline{H}''(r^N)N^2r^{2N-2}.
\]

Writing \( \xi = r^N \), we see that

\[
\overline{h}''_N(\xi) \geq 0 \iff \left( \frac{N - 1}{N} \right)\overline{H}'(\xi) + \overline{H}''(\xi)\xi \geq 0.
\]

Taking \( N = \frac{Q}{2} \) we get

\[
\overline{h}''_N(\xi) \geq 0 \iff (Q - 2)\overline{H}'(\xi) + Q\overline{H}''(\xi)\xi \geq 0.
\]

Owing to equation (3.2) we can express \( \overline{H}', \overline{H}'' \) explicitly

\[
\overline{H}'(\xi) = \frac{2}{C\xi} \int_{B_\xi(0)} |Df|^2
\]

and

\[
\overline{H}''(\xi) = \frac{2}{C\xi} \left( \int_{\partial B_\xi(0)} |Df|^2 - \frac{1}{\xi} \int_{B_\xi(0)} |Df|^2 \right).
\]

We now have

\[
C\xi((Q - 2)\overline{H}'(\xi) + Q\overline{H}''(\xi)\xi)
\]

\[
= 2(Q - 2) \int_{B_\xi(0)} |Df|^2 + 2Q\xi \left( \int_{\partial B_\xi(0)} |Df|^2 - \frac{1}{\xi} \int_{B_\xi(0)} |Df|^2 \right)
\]

\[
= 2Q\xi \left( \int_{\partial B_\xi(0)} |Df|^2 - 4 \int_{B_\xi(0)} |Df|^2 \right).
\]

Proposition (3.5), (i) combined with Proposition (3.1) yield the following estimate
\[ \int_{B_\xi(0)} |Df|^2 \, dx \leq \xi Q \text{Dir}(g_\xi, \partial B_\xi(0)). \]

Therefore
\[ 2 \int_{B_\xi(0)} |Df|^2 \leq 2\xi Q \text{Dir}(g_\xi, \partial B_\xi(0)) \]
\[ = 2\xi Q \left( \int_{\partial B_\xi(0)} |Df|^2 - \sum_{i=1}^{Q} |\partial_i f_i|^2 \, ds \right) = Q \xi \int_{\partial B_\xi(0)} |Df|^2, \]

where the last equality is due to (3.1), (i).

Combining the last inequality with equation (3.8) gives
\[ (Q - 2) \mathcal{H}'(\xi) + Q \mathcal{H}''(\xi) \xi \geq 0, \] as wanted. Finally, note that
\[ h_{\alpha_m}^2(r) = \frac{2D(r^{\frac{2}{m}})^{r^{\frac{2}{m}}(m-1)} - \frac{\xi}{2} - 1}{C_m} = \frac{2D(r^{\frac{2}{m}})^{r^{\frac{2}{m}(2-m)-1}}}{C_m}. \]

Since \( D(r^{\frac{2}{m}}) \) is a composition of absolutely continuous non-decreasing functions, it is absolutely continuous. In view of the previous equation we see that \( h_{\alpha_m}^2 \) is absolutely continuous. Thus, proceeding as in Proposition (1.2), it follows that \( h_{\alpha_m}^2 \) is convex.

\[ \square \]

We finish this section by stating an higher dimensional analog of (3.5)

**Theorem 3.7.** Let \( m > 2 \) and \( f : B_1(0) \subset \mathbb{R}^m \to A_Q(\mathbb{R}^n) \) be a Dir minimizing function. Suppose \( f|_{\partial B_r} \in W^{1,2}(\partial B_r, A_Q(\mathbb{R}^n)) \) for a.e. \( 0 < r < 1 \). Let \( c(m) = \frac{1}{m-2} - \epsilon_m, \ 0 < \epsilon_m < \frac{1}{m-2} \) be the constant obtained via proposition (3.17). Let \( \alpha_m = \frac{1-\epsilon_m(m-2)}{\epsilon_m(m-2)^2} \).

(i) \( h_{\alpha_m}^2, f(r) \geq 0 \) for all \( r \in (0, 1) \).

(ii) \( h_{\alpha_m}^2, f(r) \geq 0 \) for a.e. \( r \in (0, 1) \).

**Proof.** The proof is identical to that of Theorem (1.3), using estimate (ii) in Proposition (1.2) instead of (i). \[ \square \]

We can now conclude that nontrivial Dir minimizing, \( \alpha \)–homogeneous function have exponents \( \alpha \) far away from 0.

**Corollary 3.8.** There are constants \( \beta_m > 0 \) with the following property. Let \( f : B_1(0) \subset \mathbb{R}^m \to A_Q(\mathbb{R}^n) \) be a nontrivial Dir minimizing, \( \alpha \)–homogeneous function. Suppose \( f|_{\partial B_r} \in W^{1,2}(\partial B_r, A_Q(\mathbb{R}^n)) \) for a.e. \( 0 < r < 1 \). Then \( \alpha \geq \beta_m \).
Proof. Put $\beta_2 = \frac{1}{\alpha_2}$ and $\beta_m = \frac{1}{\alpha_m}, m > 2$. According to the computation performed in Example 3.6, $H(r) = \kappa r^{2\alpha}$ for some $\kappa > 0$. According to Theorem (1.3) and Theorem (3.7), we see that $0 \leq \kappa \beta_m (\beta_m - 1)r^{-2\alpha}$ for a.e. $r$, which in particular gives $\alpha \geq \beta_m$. 

Note that in the specific case that $m = 2$ and $\eta \circ f = 0$, Corollary (3.8) recovers a weaker form of Proposition 8.2, [5] which was already mentioned in Example 3.6.

4 Counterexamples

The following theorem allows us to produce nontrivial examples of Dir-minimizing functions:

**Theorem 4.1.** ([2], Theorem 2.20) Let $a \neq 0, b \in \mathbb{R}$. Define $u : B_1(0) \subset \mathbb{R}^2 \to A_\mathbb{Q}(\mathbb{R}^2)$ by $u(z) = \sum_{w=az+b} [w]$. Then $u$ is Dir-minimizing.

The original proof of Theorem (4.1) in [2] is highly nontrivial and relies heavily on the theory of mass minimizing currents. Spadaro ([8], theorem 0.1) found an alternative simpler proof to this result. We start by demonstrating that Theorem (1.1) cannot be naively extended to the $\mathbb{Q}$-valued setting.

**Example 4.2.** Define $f : B_1(0) \subset \mathbb{R}^2 \to A_3(\mathbb{R}^2)$ by $f(z) = \sum_{w^3=z} [w]$. That $f$ is Dir minimizing follows from Theorem (4.1). We compute

$$
\int_{B_\rho(0)} |f|^2 \, dx = \int_0^{2\pi} \int_0^\rho 3r^6 \, dr \, d\theta = 3 \int_0^{2\pi} \int_0^\rho r^6 \, dr \, d\theta = \frac{9\pi \rho^8}{4} = C \rho^8,
$$

where $C > 0$. Therefore, $H(\rho) = C\rho^8$ for some constant $C > 0$, and so $\overline{H}(\rho) = \frac{C\rho^8}{2\pi \rho} = C\rho^6$, hence $\overline{H}'(\rho) < 0$.

Our next aim is to show that the boundary regularity condition appearing in Proposition (1.4) is indeed verified for a certain Dir-minimizing functions on the planar unit disk. This will be the content of Lemma (4.4), which is of interest by its own right. Any $z \in \mathbb{C} \setminus \{0\}$ admits a representation of the form $z = Re^{i\omega}$ for some $R > 0, \omega \in [0, 2\pi)$. We shall use the convention $\sqrt{z} = \sqrt{Re^{\frac{i\omega}{2}}}$. 

**Lemma 4.3.** Fix $r \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and define $h : (0, 2\pi) \to \mathbb{C}$ by

$$
h(\theta) := \sqrt{2re^{i\theta} - 1}.
$$

Then $h \in W^{1,2}((0, 2\pi), \mathbb{C})$.  

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Combining these estimates with equation (4.1) we see that there is at most one \( \theta \) for which \( |2 re^{i\theta} - 1| > 0 \) for all \( \theta \in [0, 2\pi] \), hence the asserted.

We recall that \( f \in W^{1,2}(\Omega, \mathcal{A}_{Q}(\mathbb{R}^n)) \) if there are \( \{ \varphi_j \}_{j=1}^{m} \subset L^2(\Omega, \mathbb{R}_{\geq 0}) \) such that 1. \( x \mapsto G(f(x), T) \in W^{1,2}(\Omega) \) for all \( T \in \mathcal{A}_{Q}(\mathbb{R}^n) \) and 2. \( |\partial_j G(f(x), T)| \leq \varphi_j \) a.e. for all \( T \in \mathcal{A}_{Q}(\mathbb{R}^n) \) and \( 1 \leq j \leq m \).

**Lemma 4.4.** Fix \( r \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \) and define \( f : (0, 2\pi) \to A_2(\mathbb{R}^2) \) by

\[
  f(\theta) \coloneqq \sum_{z^2 = 2re^{i\theta} - 1} [z].
\]

Then \( f \in W^{1,2}((0, 2\pi), A_{2}(\mathbb{R}^2)) \).

**Proof.** Let \( T = \sum_{i=1}^{2} [T_i] \in A_{2}(\mathbb{R}^2) \). Set \( h(\theta) = \sqrt{2 re^{i\theta} - 1} \) and for \( P = (P_1, P_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \) denote by \( d_P : \mathbb{R}^2 \to \mathbb{R} \) the function

\[
  d_P(x) \coloneqq \sqrt{|x - P_1|^2 + |x - P_2|^2}.
\]

It is not difficult to see that for any fixed \( P \), \( d_P \) is Lipschitz. Put

\[
  \alpha_T(\theta) \coloneqq (d_{(-T_1, T_2)} \circ h)(\theta), \beta_T(\theta) = (d_{(T_1, -T_2)} \circ h)(\theta).
\]

Note that

\[
  G(f(\theta), T) = \frac{\alpha_T(\theta) + \beta_T(\theta) - |\alpha_T(\theta) - \beta_T(\theta)|}{2}. \quad (4.1)
\]

By Lemma 4.3, \( \alpha_T, \beta_T \) are a composition of a Lipschitz function on a \( W^{1,2}(0, 2\pi) \) function. Therefore \( \alpha_T, \beta_T \) are also \( W^{1,2}(0, 2\pi) \). In view of Equation (4.1) it is now apparent that \( \theta \mapsto G(f(\theta), T) \in W^{1,2}(0, 2\pi) \). In addition, there is at most one \( \theta_0 \in (0, 2\pi) \) for which \( \alpha_T(\theta_0) = 0 \). An elementary calculation shows that the following estimate is obeyed for \( \theta \in (0, 2\pi) - \{ \theta_0 \} \):

\[
  |(\partial_\theta \alpha_T(\theta))| \leq 2 |(\partial_\theta h_1(\theta))| + 2 |(\partial_\theta h_2(\theta))|,
\]

and \( \partial_\theta h_i \in L^2(0, 2\pi), i = 1, 2 \) by Lemma 4.3. The same is true for \( \beta_T \). Combining these estimates with equation (4.1) we see that

\[
  |\partial_\theta G(f(\theta), T)| \leq 8(|(\partial_\theta h_1(\theta))| + |(\partial_\theta h_2(\theta))|) \in L^2(0, 2\pi),
\]

for all but finitely many \( \theta \in (0, 2\pi) \). So strictly by definition,

\[
  f \in W^{1,2}((0, 2\pi), A_{2}(\mathbb{R}^2)).
\]
Proof of Proposition 1.4. That $f$ is Dir minimizing follows from Theorem (4.1). Furthermore, by Lemma (4.4) $f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_2(\mathbb{R}^2))$ for all $r \neq \frac{1}{2}$.

We compute:

$$\int_{B_r(0)} |f|^2 \, dx = 2 \int_0^r 2\pi \int_0^{2\pi} r \sqrt{(2r\cos \theta - 1)^2 + 4r^2 \sin^2(\theta)} \, d\theta \, dr.$$ 

Therefore

$$H(\rho) = 2 \int_0^{2\pi} \rho \sqrt{(2\rho \cos \theta - 1)^2 + 4\rho^2 \sin^2(\theta)} \, d\theta,$$

which implies

$$\mathcal{H}(\rho) = \frac{1}{\pi} \int_0^{2\pi} \sqrt{(2\rho \cos \theta - 1)^2 + 4\rho^2 \sin^2(\theta)} \, d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \sqrt{1 + 4\rho^2 - 4\rho \cos \theta} \, d\theta := \frac{1}{\pi} A(\rho).$$

We compute $A''(\rho)$ for all $\frac{1}{2} < \rho < 1$:

$$A'(\rho) = \int_0^{2\pi} \frac{8\rho - 4 \cos \theta}{2\sqrt{1 + 4\rho^2 - 4\rho \cos \theta}} \, d\theta = \int_0^{2\pi} \frac{4\rho - 2 \cos \theta}{\sqrt{1 + 4\rho^2 - 4\rho \cos \theta}} \, d\theta,$$

$$A''(\rho) = \int_0^{2\pi} \frac{4 \sqrt{1 + 4\rho^2 - 4\rho \cos \theta} - \frac{2(4\rho - 2 \cos \theta)^2}{2\sqrt{1 + 4\rho^2 - 4\rho \cos \theta}}}{1 + 4\rho^2 - 4\rho \cos \theta} \, d\theta$$

$$= \int_0^{2\pi} \frac{4(1 + 4\rho^2 - 4\rho \cos \theta) - (4\rho - 2 \cos \theta)^2}{(1 + 4\rho^2 - 4\rho \cos \theta)^\frac{3}{2}} \, d\theta$$

$$= \int_0^{2\pi} \frac{4 \sin^2 \theta}{(1 + 4\rho^2 - 4\rho \cos \theta)^\frac{3}{2}} \, d\theta.$$
\[ A''(\rho) = \int_0^{2\pi} -6\sin^2 \theta (1 + 4\rho^2 - 4\rho \cos \theta)^{-\frac{5}{2}} (8\rho - 4\cos \theta) d\theta \]

\[ = -24 \int_0^{2\pi} \sin^2 \theta (1 + 4\rho^2 - 4\rho \cos \theta)^{-\frac{5}{2}} (2\rho - \cos \theta) d\theta. \quad (4.2) \]

We note that as long as \( \frac{1}{2} < \rho < 1 \), the RHS of equation (4.2) is strictly negative, hence the claim.

\[ \square \]

5 Appendix

This section was explained to me by Dan Mangoubi. We present here a proof of Theorem (1.1).

**Proposition 5.1.** Let \( u \) be harmonic on \( B_1(0) \). Let \( \phi : [0,1] \to \mathbb{R} \) be smooth and non decreasing. Define \( d(r) = \int_{B_1(0)} u^2(rx) \phi'(||x||^2) dx \). Then \( d \) is absolutely monotonic.

**Proof.** We denote by \( u_i \) the derivative of \( u \) with respect to the \( i \)-th variable. Define \( \psi(\xi) = \phi(\frac{||\xi||^2}{r^2}) \).

\[ d'(r) = \int_{B_1(0)} (2 \sum_{i=1}^m u(rx) u_i(rx) x_i) \phi'(||x||^2) dx \]

\[ = \int_{B_r(0)} (2 \sum_{i=1}^m u(\xi) u_i(\xi) \frac{\xi_i}{r^{m+1}}) \phi'(\frac{||\xi||^2}{r^2}) d\xi \]

\[ = \frac{1}{r^{m-1}} \int_{B_r(0)} (2 \sum_{i=1}^m u(\xi) u_i(\xi) \frac{\xi_i}{r^2}) \phi'(\frac{||\xi||^2}{r^2}) d\xi = \frac{1}{2^{m-1}} \int_{B_r(0)} \nabla u^2 \cdot \nabla \psi d\xi \]

\[ = \frac{1}{2^{m-1}} (\int_{\partial B_r(0)} \psi \nabla \cdot u^2 d\sigma - \int_{B_r(0)} \psi \Delta u^2 d\xi) \]

\[ = \frac{1}{2^{m-1}} (\int_{\partial B_r(0)} \phi(1) \nabla \cdot u^2 d\sigma - \int_{B_r(0)} \psi \Delta u^2 d\xi) \]
\[ = \frac{1}{2r^{m-1}} \left( \int_{B_r(0)} \phi(1) \Delta u^2 d\xi - \int_{B_r(0)} \psi \Delta u^2 d\xi \right) = \frac{1}{2r^{m-1}} \int_{B_r(0)} (\phi(1) - \psi) \Delta u^2 d\xi. \]

The equality \( \ast \) is by Green’s identity and the equality \( \ast \ast \) is by the divergence theorem. Taking advantage of the identity \( \Delta u^2 = |\nabla u|^2 \) for \( u \) harmonic we obtain

\[ d'(r) = \frac{1}{2r^{m-1}} \int_{B_r(0)} (\phi(1) - \psi)|\nabla u|^2 d\xi = \frac{1}{2} \int_{B_r(0)} (\phi(1) - \phi(||x||^2))|\nabla u|^2(rx)dx. \]

Let \( \Phi \) be some anti derivative of \( \phi \) and define \( \varphi : [0, 1] \rightarrow \mathbb{R} \) by \( \varphi(\rho) := \phi(1)\rho - \Phi(\rho) \). Evidently \( \varphi \) is non decreasing and according to the computation we preformed

\[ d'(r) = \frac{1}{2} \int_{B_r(0)} |\nabla u|^2(rx)\varphi'(||x||^2)dx. \]

Since each \( u_i \) is harmonic we can iterate the the same argument in order to obtain \( d^{(k)}(r) \geq 0 \) for all \( k \).

As a corollary we obtain a proof of theorem (1.1)

**Corollary 5.2.** Let \( u \) be harmonic on \( B_1(0) \). Then \( \overline{H} \) is absolutely monotonic.

**Proof.**

\[ \overline{H}(r) = \frac{1}{C_m r^{m-1}} \int_{\partial B_r(0)} u^2(x) d\sigma \]

\[ = \frac{1}{C_m r^{m-1}} \frac{d}{dr} \left( \int_{B_r(0)} u^2(x)dx \right) = \frac{1}{C_m r^{m-1}} \frac{d}{dr} \left( \int_{B_r(0)} u^2(x)dx \right) \]

\[ = \frac{1}{C_m r^{m-1}} \frac{d}{dr} \left( r^m \int_{B_1(0)} u^2(rx)dx \right) \]

\[ = \frac{1}{C_m} (mr^{m-1}) \int_{B_1(0)} u^2(rx)dx + r^m \frac{d}{dr} \left( \int_{B_1(0)} u^2(rx)dx \right) \]

\[ = \frac{1}{C_m} (m) \int_{B_1(0)} u^2(rx)dx + r \frac{d}{dr} \left( \int_{B_1(0)} u^2(rx)dx \right). \]

Taking \( \phi(t) = t \) in Proposition (1.1), we see that last expression is a sum of absolutely monotonic functions, and hence absolutely monotonic.

\[ \square \]
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