On separation of variables for reflection algebras

J M Maillet and G Niccoli

Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France
E-mail: maillet@ens-lyon.fr and giuliano.niccoli@ens-lyon.fr

Received 16 May 2019
Accepted for publication 28 June 2019
Published 27 September 2019

Abstract. We implement our new separation of variables (SoV) approach for open quantum integrable models associated to higher rank representations of the reflection algebras. We construct the (SoV) basis for the fundamental representations of the $Y(gl_n)$ reflection algebra associated to general integrable boundary conditions. Moreover, we give the conditions on the boundary $K$-matrices allowing for the transfer matrix to be diagonalizable with simple spectrum. These SoV basis are then used to completely characterize the transfer matrix spectrum for the rank one and two reflection algebras. The rank one case is developed for both the rational and trigonometric fundamental representations of the 6-vertex reflection algebra. On the one hand, we extend the complete spectrum characterization to representations previously lying outside the SoV approach, e.g. those for which the standard algebraic Bethe Ansatz applies. On the other hand, we show that our new SoV construction can be reduced to the generalized Sklyanin’s one whenever it is applicable. The rank two case is developed explicitly for the fundamental representations of the $Y(gl_3)$ reflection algebra associated to general integrable boundary conditions. For both rank one and two our SoV approach leads to a complete characterization of the transfer matrix spectrum in terms of a set of polynomial solutions to the corresponding quantum spectral curve equation. Those are finite difference functional equations of order equal to the rank plus one, i.e. here two and three respectively for the $Y(gl_2)$ and $Y(gl_3)$ reflection algebras.

Keywords: algebraic structures of integrable models, integrable spin chains and vertex models, quantum integrability (Bethe Ansatz), symmetries of integrable models

1 Author to whom any correspondence should be addressed.
1. Introduction

In this article we continue the development of our new method [1] for constructing separation of variables (SoV) bases of quantum integrable lattice models. While in [1–4] we have considered various representations of the Yang–Baxter algebra for integrable models with quasi-periodic boundary conditions, we explore in the present article representations of the reflection algebras describing open quantum integrable models with general boundary conditions preserving integrability. We use the framework of the quantum inverse scattering method (QISM) [5–13] extended to the reflection algebras.
situation [14, 15] to describe these open quantum integrable lattice models. Their exact solution has become a subject of intense studies [14–96] in quantum integrability. They have attracted a large research interest both because of their potential relevance in the description of the non-equilibrium and transport properties of quantum integrable systems [97–107] and the fact that their solution by standard methods has represented a longstanding challenge for the most general integrable boundary conditions. Important instances are the integrable XXZ (and XYZ) quantum spin 1/2 chains with the most general integrable boundary magnetic fields. Their analytic or algebraic Bethe Ansatz descriptions have been first confined to the case of z-oriented boundary magnetic fields [15, 21] and then to the case in which the left and right boundary magnetic fields satisfy some special relation [39, 48–50, 53, 63]. The problem of the most general integrable boundaries has been overcome in the framework of the Sklyanin’s quantum version of the separation of variables (SoV) [108–111]. More in detail in [89–91] a generalized version of the Sklyanin’s SoV approach has been introduced by the combined used of SoV and of the (Vertex-IRF) Baxter’s gauge transformations [112, 113]. This generalized version of the Sklyanin’s SoV approach has led to the complete characterization of the transfer matrix spectrum with unconstrained integrable boundaries. While in [92] it has been first proven the equivalence of this SoV characterization with a second order difference type functional equation of Baxter’s form containing an inhomogeneous term under the most general integrable boundary conditions2.

The Sklyanin’s separation of variables method, or some minor generalizations of it, has by now found a large set of applications producing cutting edge results on the construction of the full eigenvalue and eigenvector spectrum of the transfer matrices associated to a large class of integrable quantum models [1–4, 85–96, 114–138], while also providing some fundamental steps towards the computation of their form factors and correlation functions [94, 96, 126–134]. Nevertheless, one has to mention that some applicability issues have been encountered in particular in relation with the quantum integrable models associated to higher rank representations of the Yang–Baxter and reflection algebras. Moreover, these issues appeared to exist already for the rank one quantum integrable models when some special integrable boundary conditions are considered. This is in particular the case for boundary conditions allowing for the application of the standard algebraic Bethe Ansatz approach, e.g. the XXZ quantum spin 1/2 chain with closed and open boundary conditions associated to diagonal twist and diagonal boundary matrices, respectively. This may have generated the wrong perception that SoV and ABA methods have disjoint applicability ranges.

Our new SoV method [1] allows to overcome these problems while it is proven to be reducible to the original Sklyanin’s SoV approach when this last one is applicable. We have already proven its efficiency to completely characterize the spectrum for fundamental representations of both the rational and trigonometric Yang–Baxter algebra for any positive integer rank [1–3] and under general closed integrable boundary conditions (see also [138–140] for alternative methods using Sklyanin B operator). Moreover, in [4] we have described its application to non-fundamental representations of the Yang–Baxter algebras. Already for the rank one case our analysis [1] has proven the

2 For these unconstrained integrable boundaries, see also [80–82] for a subsequent analysis by a generalized version of the algebraic Bethe Ansatz approach and [75] for another Ansatz description of the transfer matrix eigenvalue spectrum, where in fact inhomogeneous Baxter’s type functional equations first appeared.

https://doi.org/10.1088/1742-5468/ab357a
applicability of the SoV method beyond the limits of the original one, being applicable as well for diagonal and non-diagonal boundary twists, overcoming for these representations the apparent dualism between standard ABA and SoV approaches.

In the present article, we show that our SoV approach can be applied to higher rank fundamental representations of the reflection algebra and under the most general integrable boundary conditions. Once again we get new results already for the rank one case leading to the extension of the SoV approach to representations previously unattainable by the generalized version of the Sklyanin’s method.

More in detail the article is organized as follows. In section 2, we develop the analysis for general fundamental representations of $Y(gl_2)$ reflection algebra. First, we analyze the applicability of SoV in a generalized Sklyanin’s approach. Then we present our new SoV approach. We compare them and we show their consistence in the overlapping region of applicability, while proving that our SoV construction does not suffer the limitations of the original one. Then, we derive the transfer matrix spectrum in our SoV scheme by the quantum spectral curve equation and we define general criteria to establish the diagonalizability and simplicity of the transfer matrix. Section 3 is devoted to the generalization of our SoV analysis and results to the fundamental representations of $U_q(gl_2)$ reflection algebra. By carefully taking the rational limit of the trigonometric 6-vertex $R$ and $K$ matrices, we are able to obtain the trigonometric results from their rational counterparts. In section 4, we develop our SoV analysis of the fundamental representations of $Y(gl_3)$ reflection algebra. We construct our SoV basis and we obtain complete characterization of the transfer matrix spectrum both by solutions to a discrete system of equations and to the quantum spectral curve equation. Finally, section 5 is devoted to the SoV basis construction for the fundamental representations of $Y(gln)$ reflection algebra and to the identification of general criteria for the diagonalizability and simplicity of the transfer matrix. In section 6 we present some conclusions. Finally, in the appendix, the scalar products of separate vectors and co-vectors are determined for the rank one case in our new SoV approach.

2. SoV for fundamental representations of $Y(gl_2)$ reflection algebra

The transfer matrix associated to the fundamental representations of $gl_2$ reflection algebra reads:

$$T(\lambda) = \text{tr}_0\{K_+(\lambda) M(\lambda) K_-(\lambda) \hat{M}(\lambda)\} = \text{tr}_0\{K_+(\lambda) U_-(\lambda)\}. \quad (2.1)$$

It defines a one-parameter family of commuting operators [15] on the quantum space $\mathcal{H} = \otimes_{i=1}^N V_i$, with $V_i \simeq \mathbb{C}^2$, of the $N$ sites bidimensional fundamental representations of the reflection algebra [14]. The transfer matrix is introduced in terms of the following definitions. First we define the boundary matrices

$$K_+(\lambda) = K_-(\lambda + \eta; \zeta_+, \kappa_+, \tau_+), \quad (2.2)$$

and

https://doi.org/10.1088/1742-5468/ab357a
where $K_{-}(\lambda)$ is the most general scalar solution \cite{30, 32–34} of the rational 6-vertex reflection equation:

\begin{equation}
\begin{aligned}
R_{ab}(\lambda - \mu) K_{-a}(\lambda) R_{a,b}(\lambda + \mu - \eta) K_{-b}(\mu) R_{ab}(\lambda + \mu - \eta) K_{-a}(\lambda) R_{ab}(\lambda - \mu) \in \text{End}(V_a \otimes V_b),
\end{aligned}
\end{equation}

w.r.t. the rational 6-vertex $R$-matrix:

\begin{equation}
R_{ab}(\lambda) = \begin{pmatrix}
\lambda + \eta & 0 & 0 & 0 \\
0 & \lambda & \eta & 0 \\
0 & \eta & \lambda & 0 \\
0 & 0 & 0 & \lambda + \eta
\end{pmatrix} \in \text{End}(V_a \otimes V_b).
\end{equation}

Then using it we can define the boundary monodromy matrix:

\begin{equation}
\mathcal{U}_{-0}(\lambda) = M_0(\lambda) K_{-0}(\lambda) \hat{M}_0(\lambda) = \begin{pmatrix} A_{-}(\lambda) & B_{-}(\lambda) \\ C_{-}(\lambda) & D_{-}(\lambda) \end{pmatrix} \in \text{End}(V_0 \otimes \mathcal{H}),
\end{equation}

an operator solution to the same reflection equation \cite{15}

\begin{equation}
R_{ab}(\lambda - \mu) \mathcal{U}_{-a}(\lambda) R_{ab}(\lambda + \mu - \eta) \mathcal{U}_{-b}(\mu) = \mathcal{U}_{-b}(\mu) R_{ab}(\lambda + \mu - \eta) \mathcal{U}_{-a}(\lambda) R_{ab}(\lambda - \mu) \in \text{End}(V_a \otimes V_b \otimes \mathcal{H}),
\end{equation}

where we have defined:

\begin{equation}
\hat{M}_0(\lambda) = (-1)^N \sigma_0^\eta M_0^\eta(-\lambda) \sigma_0^\eta,
\end{equation}

in terms of the bulk monodromy matrix:

\begin{equation}
M_0(\lambda) = R_{0N}(\lambda - \xi^{(0)}_N) \ldots R_{01}(\lambda - \xi^{(0)}_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},
\end{equation}

satisfying the rational 6-vertex Yang–Baxter algebra:

\begin{equation}
R_{ab}(\lambda - \mu) M_a(\lambda) M_b(\mu) = M_b(\mu) M_a(\lambda) R_{ab}(\lambda - \mu) \in \text{End}(V_a \otimes V_b \otimes \mathcal{H}),
\end{equation}

with

\begin{equation}
\xi^{(h)}_n = \xi_n + \eta/2 - h\eta, \quad 1 \leq n \leq N, \quad h \in \{0, 1\}.
\end{equation}

The commutativity of the transfer matrix family $T^{(K_{+,-})}(\lambda)$ has been first proven by Sklyanin \cite{15} as a consequence of the reflection equation satisfied by $\mathcal{U}_{-}(\lambda)$ and $K_{+}(\lambda)$.

It is worth noticing that the most general boundary matrices are in fact of the following general form:

\begin{equation}
K_{\pm}(\lambda) = I + \frac{\lambda \pm \eta/2}{\xi_{\pm}} \mathcal{M}^{(\pm)},
\end{equation}

where

\begin{equation}
\mathcal{M}^{(\pm)^2} = r^{(\pm)} I, \quad \text{with} \quad r^{(\pm)} = 1 - \delta_{n_{2,\pm}^1}^{1/4},
\end{equation}

https://doi.org/10.1088/1742-5468/ab357a
\[ \tilde{\zeta}_\pm = \zeta_\pm \delta_{\kappa_\pm^2} - 1/4 + \zeta_\pm \mu^{\pm} / \sqrt{1 + 4\kappa_\pm^2}. \]  

Moreover, in the case \( r^{(\pm)} = 1 \) and \( K_\pm(\lambda) \) not proportional to the identity, there exist \( S^{(\pm)} \) invertible \( 2 \times 2 \) matrices such that \( \mathcal{M}^{(\pm)} = S^{(\pm)} \sigma^{\pm} \left( S^{(\pm)} \right)^{-1} \). Then, one can easily verify that \( K_\pm(\lambda) \) are one-parameter families of commuting matrices:

\[ [K_+(\lambda), K_-(\mu)] = \frac{(\lambda + \eta/2)(\mu - \eta/2)}{\zeta_+ \zeta_-} \left[ \mathcal{M}^+, \mathcal{M}^- \right] = 0 \]  

if and only if:

\[ \kappa_- e^{\tau_-} = \kappa_+ e^{\tau_+} \equiv \kappa e^{\tau}, \]  

so that \( K_\pm(\lambda) \) are simultaneously diagonalizable if and only if the above conditions are satisfied and \( \kappa^2 \neq -1/4 \). Finally, as already described in [134], let us observe that if the matrices \( K_\pm(\lambda) \) are non-commuting, i.e. if there exists \( a \in \{-1, 1\} \) such that:

\[ \kappa_- e^{\alpha_-} \neq \kappa_+ e^{\alpha_+}, \]  

then there exists a couple \( (\epsilon_+, \epsilon_-) \in \{-1, 1\}^2 \) such that the following matrix is invertible:

\[ W^{(K_+, -)} \equiv \frac{1}{\kappa_- \kappa_+ \epsilon_- \epsilon_+} \left( \begin{array}{cc} 1 & -\epsilon_- \sqrt{1 + 4\kappa_-^2} \\ -\epsilon_+ \sqrt{1 + 4\kappa_+^2} & 1 \end{array} \right), \]  

and we can define the following similarity transformation

\[ \tilde{K}_\mp(\lambda) = W^{(K_+, -)} \left( K_\mp(\lambda) \right) W^{(K_+, -)}^{-1} = \left( \begin{array}{cc} \tilde{a}_\mp(\lambda) & \tilde{b}_\mp(\lambda) \\ \tilde{c}_\mp(\lambda) & \tilde{d}_\mp(\lambda) \end{array} \right), \]  

where it holds:

\[ \tilde{K}_+(\lambda) = I + \frac{\lambda + \eta/2}{\zeta_+} (\sigma^z + \tilde{c}_+ \sigma^z), \quad \tilde{K}_-(\lambda) = I + \frac{\lambda - \eta/2}{\zeta_-} (\sigma^z + \tilde{b}_- \sigma^z), \]  

with

\[ \tilde{\zeta}_\pm = \epsilon_\pm \frac{\zeta_\pm}{\sqrt{1 + 4\kappa_\pm^2}}, \]  

\[ \tilde{c}_+ = \epsilon_+ \frac{2\kappa_+ e^{-\tau_+}}{\sqrt{1 + 4\kappa_+^2}} \left[ 1 + \frac{(1 - \epsilon_+ \sqrt{1 + 4\kappa_+^2})(1 + \epsilon_- \sqrt{1 + 4\kappa_-^2})}{2\kappa_+ \kappa_- e^{-\tau_-}} \right], \]  

\[ \tilde{b}_- = \epsilon_- \frac{2\kappa_- e^{-\tau_-}}{\sqrt{1 + 4\kappa_-^2}} \left[ 1 + \frac{(1 - \epsilon_+ \sqrt{1 + 4\kappa_+^2})(1 + \epsilon_- \sqrt{1 + 4\kappa_-^2})}{2\kappa_+ \kappa_- e^{-\tau_+}} \right], \]

and\(^3 \tilde{b}_- \neq 0.\)

\(^3\) Note that the assumption that the boundary matrices are non-commuting implies that they are not simultaneously diagonalizable. At least one of the conditions \( \epsilon_\pm \neq 0 \) or \( \tilde{b}_- \neq 0 \) must be satisfied and with a proper choice of \( (\epsilon_+, \epsilon_-) \in \{-1, 1\}^2 \) we can obtain that the second inequality holds.

https://doi.org/10.1088/1742-5468/ab357a
2.1. Applicability of SoV in generalized Sklyanin’s approach

The Sklyanin’s approach to define SoV can be generalized to the reflection algebra case. We can summarize the applicability of this approach\(^4\) as developed in \([89, 91]\) and \([94]\) by the following:

**Proposition 2.1.** Let us assume that for any \(a\) and \(b\) in \(\{1, \ldots, N\}\), with \(a \neq b\), the following condition on inhomogeneity parameters \(\xi\)’s
\[
\xi_a \neq \xi_b + \epsilon \eta \quad \forall \epsilon \in \{-1, 0, 1\},
\]
holds and that the boundary matrix \(K_-(\lambda)\) and \(K_+(\lambda)\) are non-commuting ones, namely that (2.17) is satisfied. Then, defining:
\[
\hat{U}_-(\lambda) = W_0^{(K_+, -)} U_-(\lambda) (W_0^{(K_+, -)})^{-1} = \begin{pmatrix} \hat{A}_-(\lambda) & \hat{B}_-(\lambda) \\ \hat{C}_-(\lambda) & \hat{D}_-(\lambda) \end{pmatrix},
\]
with \(W_{K_+, -} = \otimes_{a=1}^{N} W_{a}^{(K_+, -)}\), the generalized Sklyanin’s left and right SoV basis for the transfer matrix \(T(\lambda)\) are the left and right eigenbasis of \(\hat{B}_-(\lambda)\):
\[
\langle h_- | \equiv \langle 0 |W_{K_+, -} \prod_{n=1}^{N} \left( \frac{\hat{A}_-(\xi_n/2 - \xi_n)}{\hat{A}_-(\eta/2 - \xi_n)} \right)^{1-h_n} | h_- \rangle \equiv \prod_{n=1}^{N} \left( \frac{\hat{D}_-(\xi_n + \eta/2)}{K_n \hat{A}_-(\eta/2 - \xi_n)} \right)^{h_n} W_{K_+, -}^{-1} \langle 0 \rangle,
\]
with eigenvalues:
\[
b_-, h_-(\lambda) = (-1)^N \tilde{b}_- \frac{\lambda - \eta/2}{\xi_-} \prod_{n=1}^{N} (\lambda - \xi_n^{(h_n)})(-\lambda - \xi_n^{(h_n)}), \quad \tilde{\zeta}_- = \epsilon_- \zeta_- / \sqrt{1 + 4\kappa_-^2}.
\]

Here \(\langle 0 |\) is the co-vector with all spin up and \(| 0 \rangle\) is the vector with all spin down and
\[
A_-(\lambda) = (-1)^N \frac{\tilde{\zeta}_- + \lambda - \eta/2}{\xi_-} a(\lambda) d(-\lambda), \quad K_n = (\xi_n + \eta)/(\xi_n - \eta),
\]
and
\[
a(\lambda) \equiv \prod_{n=1}^{N} (\lambda - \xi_n + \eta/2), \quad d(\lambda) \equiv \prod_{n=1}^{N} (\lambda - \xi_n - \eta/2).
\]

As presented here, the Sklyanin’s SoV basis can be defined only in the case of non-commuting boundary matrices. Instead, as we will prove in the next section, our new SoV approach works in the completely general case. So we can use it also in the case of commuting boundary matrices for which algebraic Bethe Ansatz \([15]\) also works in the special case of simultaneously diagonalizable boundary matrices.

\(^4\) See also \([86, 88]\) for an earlier purely functional version of SoV (i.e. without the construction of the SoV basis) for these representations.
2.2. Our SoV approach

Let us define:

\[ A_{\tilde{\zeta}^+, \tilde{\zeta}^-}(\lambda) \equiv (-1)^N \frac{\lambda^2 + \eta}{2\lambda} \left( \frac{\lambda - \eta/2 + \tilde{\zeta}^+ (\lambda - \eta/2 + \tilde{\zeta}^-)}{\tilde{\zeta}^+ \tilde{\zeta}^-} \right) a(\lambda) d(-\lambda), \]  

(2.29)

then the following theorem holds:

**Theorem 2.1.**

(i) Let \( K_-(\lambda) \) and \( K_+(\lambda) \) be non-commutative boundary matrices (2.17) and \( T(\lambda) \) be the associated one-parameter family of transfer matrix, then

\[ \langle h_1, ..., h_N | \equiv \langle S | \prod_{n=1}^N \left( \frac{T(\xi_n - \eta/2)}{A_{\tilde{\zeta}^+, \tilde{\zeta}^-}(\eta/2 - \xi_n)} \right)^{1-h_n}, \]  

(2.30)

for any \( \{ h_1, ..., h_N \} \subset \{0, 1\}^N \), is a co-vector basis of \( \mathcal{H} \) for almost any choice of the co-vector \( \langle S | \) and of the inhomogeneity parameters satisfying the condition (2.24).

(ii) Let \( K_-(\lambda) \) and \( K_+(\lambda) \) be commutative boundary matrices (2.16), moreover not both proportional to the identity. Then, for any fixed choice of the boundary parameters \( \{ \zeta^+, \kappa, \tau \} \) (or \( \{ \zeta^-, \kappa, \tau \} \)), the set (2.30) is a co-vector basis of \( \mathcal{H} \) for almost any choice of the co-vector \( \langle S | \) of the inhomogeneity parameters satisfying the condition (2.24) and of \( \zeta^- \) (or \( \zeta^+ \)).

**Proof.**

Let us prove (i). Let us consider the following choice on the inhomogeneity parameters:

\[ \xi_a = a\xi \text{ } \forall a \in \{1, ..., N\}, \]  

(2.31)

where \( \xi \) is some complex parameter, then, by exactly the same steps followed in the proof of the general proposition 2.4 of [1], we can prove that \( T(\xi_l - \eta/2) \) are polynomials of degree \( 2N + 1 \) in \( \xi \) for all \( l \in \{1, ..., N\} \) with maximal degree coefficient given by:

\[ \frac{(-1)^N \eta l(N - l)!(N + l)!}{\tilde{\zeta}^+ \tilde{\zeta}^-} M_i(\xi)^{-} M_i(\xi)^{+}. \]  

(2.32)

Let us choose \( \langle S | \) of tensor product form

\[ \langle S | = \otimes_{l=1}^N \langle S, l | \text{ where } \langle S, l | = (s_+, s_-)_l, \]  

(2.33)

then the set of co-vectors (2.30) is a basis as soon as

\[ \langle S, l | \left( \frac{M_i(-) M_i(+) \tilde{\zeta}^- + A_{\tilde{\zeta}^+, \tilde{\zeta}^-}(\eta/2 - \xi_l)}{\tilde{\zeta}^- \tilde{\zeta}^+ A_{\tilde{\zeta}^+, \tilde{\zeta}^-}(\eta/2 - \xi_l)} \right)^{1-h} \text{ with } h \in \{0, 1\}, \]  

(2.34)

https://doi.org/10.1088/1742-5468/ab357a
is a basis on the local space $V_l \simeq \mathbb{C}^2$ for all $l \in \{1, \ldots, N\}$. This is indeed the case as it holds:

$$\det_2 \left| \begin{pmatrix} M_{-}^{(i)}(\tau) M_{+}^{(i)} \end{pmatrix}
\begin{pmatrix} \zeta \zeta + A_{\zeta+} \zeta+ (\eta/2 - \xi) \end{pmatrix}^{i-1} \left| e_j(l) \right\rangle \right|_{i,j \in \{1,2\}} = 2^{s_2} (\kappa e^{-\tau} + \kappa e^{-\tau}) + 2^{s_2} (\kappa e^{-\tau} + \kappa e^{-\tau}) + 2 \kappa s_- e^{-\tau} e^{-\tau} + e^{-\tau} e^{-\tau}$$

(2.35)

where $|e_j(l)\rangle$ is the element $j \in \{1,2\}$ of the natural basis in $V_l$, which can be always chosen different from zero by an appropriate choice of $s_+$ and $s_-$ under the condition (2.17).

Let us now prove (ii). Let us observe that $T(\lambda)$ is a polynomial of degree 1 in $1/\zeta_-$ with constant term which coincides with the transfer matrix $T^{(K_+, l)}(\lambda)$ associated to the same $K_+(\lambda)$ and $K_-(\lambda) = I$. Here, we show that the set of co-vectors (2.30) generated by $T^{(K_+, l)}(\lambda)$ is a basis for almost any choice of the co-vector $\langle S \rangle$ and of the inhomogeneity parameters satisfying (2.24). This implies the statement of the theorem once we recall that the co-vectors (2.30) generated by $T(\lambda)$ are polynomials of maximal degree $N$ in $1/\zeta_-$. Let us impose on the inhomogeneity parameters the same condition (2.31) then $T^{(K_+, l)}(\xi_l - \eta/2)$ are polynomials of degree $2N$ in $\xi$ for all $l \in \{1, \ldots, N\}$ with maximal degree coefficient given by:

$$\frac{(-1)^N \eta l (N - l)!(N + l)!}{\zeta_+} \begin{pmatrix} 1 & 2keh \end{pmatrix}^{1-h} \begin{pmatrix} 2keh \end{pmatrix}^{-1}$$

(2.36)

Then for a chosen $\langle S \rangle$ of tensor product form (2.33) the set of co-vectors (2.30) generated by $T^{(K_+, l)}(\lambda)$ is a basis as soon as

$$\langle S, l \rangle \begin{pmatrix} 1 & 2keh \end{pmatrix}^{-1-h} \text{ with } h \in \{0,1\},$$

(2.37)

is a basis on the local space $V_l \simeq \mathbb{C}^2$ for all $l \in \{1, \ldots, N\}$. This is indeed the case as it holds:

$$\det_2 \left| \begin{pmatrix} \langle S, l \rangle \begin{pmatrix} 1 & 2keh \end{pmatrix}^{i-1} \left| e_j(l) \right\rangle \right|_{i,j \in \{1,2\}} = 2^{s_2} (\kappa e^{-\tau} + \kappa e^{-\tau}) + 2^{s_2} (\kappa e^{-\tau} + \kappa e^{-\tau}) + 2 \kappa s_- e^{-\tau} e^{-\tau} + e^{-\tau} e^{-\tau}$$

(2.38)

which can be always chosen different from zero for any fixed $\kappa$ and $\tau$ for an appropriate choice of $s_+$ and $s_-$. 

\[\square\]
2.3. Comparison of the two SoV constructions

Here we want to show that under some special choice of the co-vector \( \langle S \rangle \), our SoV left basis reduces to the SoV basis associated to the generalized Sklyanin’s approach when this last one is applicable, i.e. when the two boundary matrices are non-commuting.

Let us introduce the following gauged transformed monodromy matrix:

\[
\left( \begin{array}{cc}
\tilde{A}_-(\lambda) & \tilde{B}_-(\lambda) \\
\tilde{C}_-(\lambda) & \tilde{D}_-(\lambda)
\end{array} \right) = M(\lambda) \tilde{K}_-(\lambda) \tilde{M}(\lambda) = W_0^{(K_{+,-})} \mathcal{W}_{K_{+,-}} \mathcal{U}_{-}(\lambda) W_{K_{+,-}}^{-1} (W_0^{(K_{+,-})})^{-1},
\]

(2.39)

then the associated transfer matrix:

\[
\tilde{T}(\lambda) = \text{tr}_0 \left\{ \tilde{K}_+(\lambda) M(\lambda) \tilde{K}_-(\lambda) \tilde{M}(\lambda) \right\} = \tilde{c}_+(\lambda) \tilde{B}_-(\lambda) + \frac{(2\lambda + \eta) (\lambda - \frac{\eta}{2} + \zeta_+)}{2\lambda \zeta_+} \tilde{A}_-(\lambda) + \frac{(2\lambda - \eta) (-\lambda - \frac{\eta}{2} + \zeta_+)}{2\lambda \zeta_+} \tilde{A}_-(\lambda),
\]

(2.40)

is related to the original transfer matrix by:

\[
T(\lambda) = \mathcal{W}_{K_{+,-}}^{-1} \tilde{T}(\lambda) \mathcal{W}_{K_{+,-}},
\]

(2.41)

and the generalized Sklyanin’s SoV basis can be rewritten as:

\[
\langle \mathbf{h}_- | \equiv \langle 0 | \prod_{n=1}^{N} \left( \frac{\tilde{A}_-(\eta/2 - \xi_n)}{A_-(\eta/2 - \xi_n)} \right)^{1-h_a} \mathcal{W}_{K_{+,-}}.
\]

(2.42)

Here we want to show that the co-vector \( \langle \mathbf{h}_- | \) and the co-vector \( \langle h_1, ..., h_N | \) of our SoV basis (2.30) are proportional for any \( \{h_1, ..., h_N \} \in \{0, 1\}^\otimes N \) when we set:

\[
\langle S | \equiv \langle 0 | \mathcal{W}_{K_{+,-}}.
\]

(2.43)

The proof is done by induction on \( l = N - \sum_{a=1}^{N} h_a \), just using the identity:

\[
\langle 0 | \tilde{A}_-(\xi_a - \eta/2) = 0 \quad \forall a \in \{1, ..., N\}
\]

(2.44)

and the following reflection algebra commutation relations:

\[
\tilde{A}_-(\mu) \tilde{A}_-(\lambda) = \tilde{A}_-(\lambda) \tilde{A}_-(\mu) + \frac{\eta}{\lambda + \mu - \eta} (\tilde{B}_-(\lambda) \tilde{C}_-(\mu) - \tilde{B}_-(\mu) \tilde{C}_-(\lambda)).
\]

(2.45)

First, the statement is obviously true for \( l = 0 \). Let us assume that our statement holds for any state:

\[
\langle \mathbf{h}_- | = N_{\pi} \langle h_1, ..., h_N | \quad \text{with } l = N - \sum_{a=1}^{N} h_a \leq N - 1,
\]

(2.46)

for some given \( l \). Then, let us show it for any state with \( l + 1 \). To this aim we fix a state in the above set and we denote with \( \pi \) a permutation on the set \( \{1, ..., N\} \) such that:

\[
h_{\pi(a)} = 0 \quad \text{for } a \leq l \quad \text{and } h_{\pi(a)} = 1 \quad \text{for } l < a.
\]

(2.47)
Now, let us take $c \in \{\pi(l + 1), ..., \pi(N)\}$ and let us compute:

$$
\langle h_- | T(\xi^{(1)}_c) \rangle = \langle 0 | \prod_{n=1}^{l} \frac{\bar{A}_-(\eta/2 - \xi_{\pi(n)})}{A_-(\eta/2 - \xi_{\pi(n)})} \bar{T}(\xi^{(1)}_c) \mathcal{W}_{K_{+,-}} = \langle 0 | \prod_{n=1}^{l} \frac{\bar{A}_-(\eta/2 - \xi_{\pi(n)})}{A_-(\eta/2 - \xi_{\pi(n)})} \left( \frac{(2\xi^{(1)}_c + \eta)(\xi^{(1)}_c - \frac{\eta}{2} + \bar{\zeta}_+)}{2\xi^{(1)}_c \bar{\zeta}_+} + \frac{(2\xi^{(1)}_c - \eta)(-\xi^{(1)}_c - \frac{\eta}{2} + \bar{\zeta}_-)}{2\xi^{(1)}_c \bar{\zeta}_-} \right) \mathcal{W}_{K_{+,-}},
$$

(2.48)

(2.49)

so that we have just to prove:

$$
\langle 0 | \prod_{n=1}^{l} \frac{\bar{A}_-(\eta/2 - \xi_{\pi(n)})}{A_-(\eta/2 - \xi_{\pi(n)})} \bar{A}_-(\xi^{(1)}_c) = 0.
$$

(2.50)

From the commutation relation (2.45), the above co-vector can be rewritten as:

$$
\langle 0 | \prod_{n=1}^{l-1} \frac{\bar{A}_-(\eta/2 - \xi_{\pi(n)})}{A_-(\eta/2 - \xi_{\pi(n)})} A^{-1}_-(\eta/2 - \xi_{\pi(l)}) \bar{\mathcal{A}}_-(\xi^{(1)}_c) \bar{A}_-(\xi^{(1)}_c) \rangle + \eta(\bar{B}_-(\xi^{(1)}_c) \bar{C}_-(\xi^{(1)}_c) - \bar{B}_-(\xi^{(1)}_c) \bar{C}_-(\xi^{(1)}_c))/(\xi_c + \xi_{\pi(l)} - \eta),
$$

(2.51)

which reduces to:

$$
\langle 0 | \prod_{n=1}^{l-1} \frac{\bar{A}_-(\eta/2 - \xi_{\pi(n)})}{A_-(\eta/2 - \xi_{\pi(n)})} A^{-1}_-(\eta/2 - \xi_{\pi(l)}) \bar{A}_-(\xi^{(1)}_c) \bar{A}_-(\xi^{(1)}_c) \rangle = \langle 0 | \prod_{n=1}^{l-1} \frac{\bar{A}_-(\eta/2 - \xi_{\pi(n)})}{A_-(\eta/2 - \xi_{\pi(n)})} A^{-1}_-(\eta/2 - \xi_{\pi(l)}) \bar{A}_-(\xi^{(1)}_c) \bar{A}_-(\xi^{(1)}_c) \rangle.
$$

(2.52)

once we observe that the co-vector on the left of $\bar{B}_-(\xi^{(1)}_c)$ and $\bar{B}_-(\xi^{(1)}_c)$ are left eigenvectors of $\bar{B}_-(\lambda)$ with eigenvalue zeros at $\lambda = \pm \xi^{(1)}_{\pi(l)}$, $\pm \xi^{(1)}_c$. That is we can commute in the co-vector $\bar{A}_-(\xi^{(1)}_c)$ and $\bar{A}_-(\xi^{(1)}_c)$ and by the same argument $\bar{A}_-(\xi^{(1)}_c)$ and $\bar{A}_-(\xi^{(1)}_c)$ for any $r \leq l - 1$ up to bring $\bar{A}_-(\xi^{(1)}_c)$ completely to the left acting on $\langle 0 |$ which proves (2.50) as a consequence of (2.44).

### 2.4. Transfer matrix spectrum in our SoV approach

In our SoV basis the separate relations are given directly by the particularization of the fusion relations at the spectrum of the separate variables. In the case at hand these fusion relations just reduces to the following identities:

$$
T(\xi^{(0)}_n)T(\xi^{(1)}_n) = A_{\xi^{(0)}_n,\xi^{(1)}_n}A_{-\xi^{(0)}_n,\xi^{(1)}_n}, \forall a \in \{1, ..., N\},
$$

(2.53)

which are proven by direct computations using the reduction of the rational 6-vertex $R$-matrix to the permutation operator and to a 1-dimensional projector at $\lambda = 0$ and $-\eta$, respectively. To these relations now one has to add the knowledge of the analytic properties of the transfer matrix that we can easily derive. In fact, $T(\lambda)$ is a polynomial of degree 2 in all the $\xi_a$ and of degree $N + 1$ in $\lambda^2$ with the following leading central coefficient:

https://doi.org/10.1088/1742-5468/ab357a
\[
\lim_{\lambda \to +\infty} \lambda^{-2(N+1)} T(\lambda) = t_{N+1} I, \quad \text{with } t_{N+1} = \frac{2(1 + 4\kappa_+\kappa_- \cosh(\tau_+ - \tau_-))}{\zeta_+\zeta_-} = \frac{2 + \bar{\zeta}_+}{\zeta_+\zeta_-},
\]
whose values in \( \pm \eta/2 \) are central:
\[
T(\pm \eta/2) = 2(-1)^{N}\det_q M(0) \equiv t(\eta/2),
\]
with \( \det_q M(\lambda) = a(\lambda + \eta/2) d(\lambda - \eta/2) \). Let us define the following set of functions:
\[
r_{a,b}(\lambda) = \frac{\lambda^2 - (\eta/2)^2}{(\xi_a^{(h)})^2 - (\eta/2)^2} \prod_{b \neq a, b=1}^{N} \frac{\lambda^2 - (\xi_b^{(h)})^2}{(\xi_a^{(h)})^2 - (\xi_b^{(h)})^2},
\]
\[
s_\eta(\lambda) = \prod_{b=1}^{N} \frac{\lambda^2 - (\xi_b^{(h)})^2}{(\eta/2)^2 - (\xi_b^{(h)})^2},
\]
\[
u_\eta(\lambda) = (\lambda^2 - (\eta/2)^2) \prod_{b=1}^{N} (\lambda^2 - (\xi_b^{(h)})^2),
\]
then the following theorem holds:

**Theorem 2.2.** Under the same conditions of theorem 2.1, ensuring the existence of the left SoV basis, the spectrum of \( T(\lambda) \) is characterized by:
\[
\Sigma_T = \left\{ t(\lambda) : t(\lambda) = t_{N+1}u_{h=0}(\lambda) + t(\eta/2)s_{h=0}(\lambda) + \sum_{a=1}^{N} r_{a,h=0}(\lambda)x_a, \quad \forall \{x_1, ..., x_N\} \in S_T \right\},
\]
\( S_T \) is the set of solutions to the following system of \( N \) quadratic equations:
\[
x_n[t_{N+1}u_{h=0}(\xi_n^{(1)}) + t(\eta/2)s_{h=0}(\xi_n^{(1)}) + \sum_{a=1}^{N} r_{a,h=0}(\xi_a^{(0)})x_a] = A_{\xi_+\xi_-}(\xi_n^{(0)})A_{\xi_+\xi_-}(\xi_n^{(1)}), \quad \forall n \in \{1, ..., N\},
\]
in \( N \) unknown \( \{x_1, ..., x_N\} \). Moreover, \( T(\lambda) \) has simple spectrum and for any \( t(\lambda) \in \Sigma_T \) the associated unique (up-to normalization) eigenvector \( |t\rangle \) has the following separated wave-function in the left SoV basis:
\[
\langle h_1, ..., h_N|t\rangle = \prod_{n=1}^{N} \left( \frac{t(\xi_n - \eta/2)}{A_{\xi_+\xi_-}(\eta/2 - \xi_n)} \right)^{1-h_n}.
\]

**Proof.** The system of \( N \) quadratic equations (2.60) in \( N \) unknown \( \{x_1, ..., x_N\} \) is nothing else but the rewriting of the transfer matrix fusion equations for the eigenvalues. Any transfer matrix eigenvalue is then a solution of this system and the associated right eigenvector \( |t\rangle \) admits the characterization (2.61) in our left SoV basis. Let us now prove the reverse statement. This is done by proving that any polynomial \( t(\lambda) \) satisfying this system is an eigenvalue. For this, we prove that the vector \( |t\rangle \) characterized by (2.61) is a transfer matrix eigenvector, namely:

https://doi.org/10.1088/1742-5468/ab357a
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\[ \langle h_1, \ldots, h_N | T(\lambda) | t \rangle = t(\lambda) \langle h_1, \ldots, h_N | t \rangle, \quad \forall \{h_1, \ldots, h_N\} \in \{0, 1\}^\otimes N. \quad (2.62) \]

Let us write the following interpolation formula for the transfer matrix:

\[ T(\lambda) = t_{N+1} u_h(\lambda) + t(\eta/2) s_h(\lambda) + \sum_{a=1}^{N} r_{a,h}(\lambda) T(\zeta_h^{(h_a)}), \quad (2.63) \]

and use it to act on the generic element of the left SoV basis. Then, we have:

\[ \langle h_1, \ldots, h_a, \ldots, h_N | T(\zeta_h^{(h_a)}) | t \rangle = \begin{cases} A_{\zeta, \zeta_\bar{e}}(\eta/2 - \xi_a) \langle h_1, \ldots, h_1', 0, \ldots, h_N | t \rangle & \text{if } h_a = 1 \\ \frac{A_{\zeta, \zeta_\bar{e}}(\xi_0) A_{\zeta, \zeta_\bar{e}}^{(-\xi_0)}(\xi_a)}{A_{\zeta, \zeta_\bar{e}}^{(-\xi_0)}(\xi_0)} \langle h_1, \ldots, h_1', 1, \ldots, h_N | t \rangle & \text{if } h_a = 0 \end{cases} \quad (2.64) \]

which by the definition of the state |t\rangle can be rewritten as:

\[ \langle h_1, \ldots, h_a, \ldots, h_N | T(\zeta_h^{(h_a)}) | t \rangle = \begin{cases} t(\xi_0) \prod_{n \neq a,n=1}^{N} \left( \frac{t(\xi_n - \eta/2)}{A_{\zeta, \zeta_\bar{e}}(\eta/2 - \xi_a)} \right)^{1-h_n} & \text{if } h_a = 1 \\ \frac{A_{\zeta, \zeta_\bar{e}}(\xi_0) A_{\zeta, \zeta_\bar{e}}^{(-\xi_0)}(\xi_a)}{A_{\zeta, \zeta_\bar{e}}^{(-\xi_0)}(\xi_0)} \prod_{n \neq a,n=1}^{N} \left( \frac{t(\xi_n - \eta/2)}{A_{\zeta, \zeta_\bar{e}}(\eta/2 - \xi_a)} \right)^{1-h_n} & \text{if } h_a = 0 \end{cases} \quad (2.65) \]

and finally, by the fusion equation satisfied by the \( t(\lambda) \), it reads:

\[ \langle h_1, \ldots, h_a, \ldots, h_N | T(\zeta_h^{(h_a)}) | t \rangle = \begin{cases} t(\xi_0) \prod_{n \neq a,n=1}^{N} \left( \frac{t(\xi_n - \eta/2)}{A_{\zeta, \zeta_\bar{e}}(\eta/2 - \xi_a)} \right)^{1-h_n} & \text{if } h_a = 1 \\ t(\xi_1) \prod_{n=1}^{N} \left( \frac{t(\xi_n - \eta/2)}{A_{\zeta, \zeta_\bar{e}}(\eta/2 - \xi_a)} \right)^{1-h_n} & \text{if } h_a = 0 \end{cases} \quad (2.66) \]

and so:

\[ \langle h_1, \ldots, h_a, \ldots, h_N | T(\zeta_h^{(h_a)}) | t \rangle = t(\xi_h^{(h_a)}) \langle h_1, \ldots, h_a, \ldots, h_N | t \rangle, \quad (2.67) \]

from which we have:

\[ \langle h_1, \ldots, h_N | T(\lambda) | t \rangle = \left( t_{N+1} u_h(\lambda) + t(\eta/2) s_h(\lambda) + \sum_{a=1}^{N} r_{a,h}(\lambda) t(\xi_h^{(h_a)}) \right) \langle h_1, \ldots, h_N | t \rangle, \quad (2.68) \]

proving our statement. \( \square \)

The previous characterization of the spectrum allows to introduce a functional equation characterization of it, the so-called quantum spectral curve equation. This is in the current case a second order Baxter’s type difference equation. In particular, this result coincides with the theorem 3.2 of [94], the only difference being on the applicability of the result that is now extended to the case of commuting boundary matrices.

**Theorem 2.3.** Under the same conditions of theorem 2.1, ensuring the existence of the left SoV basis, an entire function \( t(\lambda) \in \Sigma_T \) if and only if there exists a unique polynomial

https://doi.org/10.1088/1742-5468/ab357a
\[ Q_t(\lambda) = \prod_{b=1}^{p_{K,+,-}} (\lambda^2 - \lambda_b^2), \quad \lambda_1, \ldots, \lambda_{p_{K,+,-}} \in \mathbb{C} \setminus \{ \pm \zeta_1^{(0)}, \ldots, \pm \zeta_N^{(0)} \}, \quad (2.69) \]

such that
\[ t(\lambda) \ Q_t(\lambda) = A_{\xi+,\xi-}(\lambda) \ Q_t(\lambda - \eta) + A_{\xi+,\xi-}(-\lambda) \ Q_t(\lambda + \eta) + F(\lambda), \quad (2.70) \]

with
\[ F(\lambda) = \frac{\delta_0 \bar{c}_+}{\zeta_0 - \zeta_0} (\lambda^2 - (\eta/2)^2) \prod_{b=1}^{N} \prod_{h=0}^{1} \left( \lambda^2 - (\xi_b^{(h)})^2 \right). \quad (2.71) \]

Similarly, an entire function \( t(\lambda) \in \Sigma_T \) if and only if there exists a unique polynomial
\[ P_t(\lambda) = \prod_{b=1}^{q_{K,+,-}} (\lambda^2 - \mu_b^2), \quad \mu_1, \ldots, \mu_{q_{K,+,-}} \in \mathbb{C} \setminus \{ \pm \zeta_1^{(0)}, \ldots, \pm \zeta_N^{(0)} \}, \quad (2.72) \]

such that
\[ t(\lambda) \ P_t(\lambda) = A_{-\xi+,\xi-}(\lambda) \ P_t(\lambda - \eta) + A_{-\xi+,\xi-}(-\lambda) \ P_t(\lambda + \eta) + F(\lambda). \quad (2.73) \]

Here, it holds
\[ p_{K,+,-} = (1 - \delta_{0,\bar{c}_+})N + \delta_{0,\bar{c}_+} \ p \quad (2.74) \]
\[ q_{K,+,-} = (1 - \delta_{0,\bar{c}_+})N + + \delta_{0,\bar{c}_+} \ q \quad (2.75) \]

with \( p \) and \( q \) non negative integers such that
\[ p + q = N, \quad (2.76) \]

and the following Wronskian equation is satisfied in the case \( \delta_0 \bar{c}_+ = 0 \):
\[ 2(-1)^N(\bar{c}_+ + \bar{c}_- + (p - q)\eta)(\lambda - \eta/2) \ a(\lambda) \ d(\lambda) = (\lambda - \eta/2 + \bar{c}_+)(\lambda - \eta/2 + \bar{c}_-) \ Q_t(\lambda - \eta) \ P_t(\lambda) \]
\[ - (\lambda - \eta/2 - \bar{c}_+)(\lambda - \eta/2 - \bar{c}_-) \ Q_t(\lambda) \ P_t(\lambda - \eta). \quad (2.77) \]

**Proof.** Under the conditions ensuring the existence of the left SoV basis, the equivalence of the first discrete SoV characterization with these functional equations is proven by the standard arguments as introduced in [92, 124], see e.g. the proof of theorem 2.3 of [94]. \( \square \)

### 2.5. Diagonalizability and simplicity of the transfer matrix

Our SoV approach implies that the transfer matrix spectrum is simple as soon as our SoV basis can be constructed. Here, we show that the transfer matrix is indeed diagonalizable with simple spectrum for generic values of boundary parameters. One can

https://doi.org/10.1088/1742-5468/ab357a
adapt the general proposition 2.5 of [1] to the present case and in fact this result is just a special case of the general theorem 5.1 of section 5, derived for the fundamental representations of the $\mathfrak{g}_n$ reflection algebra. However, in this section we present a slightly different proof based on the explicit form of the transfer matrix scalar product formula, as re-derived in the appendix A within the current SoV framework.

**Theorem 2.4.** Let $\langle t | t \rangle$ and $\langle t' | t' \rangle$ be the unique eigenco-vector and eigenvector associated to any fixed eigenvalue $t(\lambda)$ of $T(\lambda)$, then it holds:

$$\langle t | t \rangle \neq 0,$$

and $T(\lambda)$ is diagonalizable with simple spectrum almost for any value of $\eta$ and of the inhomogeneities satisfying (2.24) in the following two general cases:

(i) $K_-(\lambda)$ and $K_+(\lambda)$ are non-commutative boundary matrices (2.16) while $M^{(-)}_I M^{(+)}_I$ is diagonalizable with simple spectrum$^5$.

(ii) $K_-(\lambda)$ and $K_+(\lambda)$ are simultaneously diagonalizable, i.e. it holds (2.16) and

$$\kappa^2 \neq -1/4,$$

for any fixed choice of the boundary parameters $\{\zeta_-, \kappa, \tau\}$ and for almost any value of $\zeta_-$, with $\epsilon \in \{-1, 1\}$.

**Proof.** Let us denote with:

$$K = \begin{cases} M^{(-)} \in \text{the case (i)} \\ \left( \begin{array}{cc} 1 & 2\kappa e^\tau \\ 2\kappa e^{-\tau} & -1 \end{array} \right) \in \text{the case (ii)} \end{cases}$$

and let us denote with $k_0$ and $k_1$ the associated eigenvalues. Then in the case (i) the matrix $K$ is diagonalizable and with simple spectrum by assumption while in the case (ii) the requirement (2.79) implies that $K$ is diagonalizable with simple spectrum as it holds:

$$k_1 = -k_0 = \sqrt{1 + 4\kappa^2} \neq 0.$$

We can now proceed to compute the scalar product:

$$\langle t | t \rangle = \sum_{h_1, \ldots, h_N = 0}^1 \prod_{a=1}^N \left( \frac{\xi_a - \eta}{\xi_a + \eta} \right)^{h_a} \frac{t(\xi_a + \eta/2)^{1-h_a} V(\xi_a^{(h_a)})}{V(\xi_a, \ldots, \xi_N)}$$

$$= \sum_{h_1, \ldots, h_N = 0}^1 \prod_{a=1}^N \left( \frac{\xi_a - \eta}{\xi_a + \eta} \right)^{h_a} \frac{t(\xi_a - \eta/2)^{1-h_a} V(\xi_a^{(h_a)})}{V(\xi_a, \ldots, \xi_N)}.$$

$^5$ Note that this is the case for any fixed value of the boundary parameters $\kappa_-, \tau_-$ and $\kappa_-$, up two values of $\kappa_-$, for $\epsilon \in \{-1, 1\}$.

https://doi.org/10.1088/1742-5468/ab357a
then the leading coefficient of $\langle t \mid t \rangle$ in $\xi$ is given by the following limit:

$$
\lim_{\xi \to \infty} \langle t \mid t \rangle = \lim_{\xi \to \infty} \sum_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} \left( \frac{A_{\xi, \xi} (\xi_a + \eta/2)}{t(\xi_a - \eta/2)} \right)^{h_a} \frac{t(\xi_a - \eta/2)}{A_{\xi, \xi} (\eta/2 - \xi_a)}^{1-h_a},
$$

(2.84)

once we impose on the inhomogeneity parameters the condition (2.31). Let us now distinguish between the two cases.

In the case (i), let us first observe that it holds:

$$
\det M^{(-)}M^{(+)} = \det M^{(-)} \det M^{(+)} = (-1)(-1) = 1
$$

(2.85)

and so

$$
k_0k_1 = 1,
$$

(2.86)

taking that into account we have:

$$
\lim_{\xi \to \infty} \langle t \mid t \rangle = \sum_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} \left( \frac{\alpha}{k_{a\theta}} \right)^{h_a} (-a k_{\theta a})^{1-h_a} = N! \prod_{a=1}^{N} (k_{a\theta}^{-1} - k_{\theta a})
$$

(2.87)

$$
= N!(k_0 - k_1) \prod_{a=1}^{N} (1 - 2\delta_{\theta a, 0}) \neq 0,
$$

(2.88)

where we have defined $\hat{\theta}_a = \{0, 1\}\backslash \theta_a$ for any $a \in \{1, \ldots, N\}$ and the $\{\theta_1, \ldots, \theta_N\} \in \{0, 1\}^N$ are uniquely fixed by:

$$
\frac{(-1)^{N-\eta l}(N-l)!(N+l)!}{\zeta_+ \zeta_-} k_{\theta a} = \lim_{\xi \to \infty} \xi^{-(2N+1)} t(\xi_a - \eta/2).
$$

(2.89)

Here, we have used that $T(\xi_a - \eta/2)$ are polynomials of degree $2N+1$ in $\xi$ for all $a \in \{1, \ldots, N\}$ with maximal degree coefficient given by (2.32).

In the case ii), the scalar product $\langle t \mid t \rangle$ is computed for the eigenstates associated to the transfer matrix $T(K^{(+)}f(\lambda))$. That is we have to take first the limit $\zeta_- \to \infty$, then the leading coefficient of $\langle t \mid t \rangle$ in $\xi$ is given by the following limit:

$$
\lim_{\xi \to \infty} \langle t \mid t \rangle = \lim_{\xi \to \infty} \sum_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} \left( \frac{A_{\xi, \xi} (\xi_a + \eta/2)}{t(\xi_a - \eta/2)} \right)^{h_a} \frac{t(\xi_a - \eta/2)}{A_{\xi, \xi} (\eta/2 - \xi_a)}^{1-h_a},
$$

(2.91)

where

$$
A_{\xi, \xi}(\lambda) \equiv \lim_{\zeta_- \to \infty} A_{\zeta_+ \zeta_-}(\eta/2 \pm \xi_n) = (-1)^{N} \frac{2\lambda + \eta}{2\lambda} \left( \frac{\lambda - \eta}{\zeta_+} + \bar{\zeta}_+ \right) a(\lambda) d(-\lambda).
$$

(2.92)
The following identities hold:

\[
\frac{(-1)^{N-l} \eta l (N-l)!}{\zeta_+} \kappa_{\theta_a} = \lim_{\xi \to \infty} \xi^{-2N} t(\xi - \eta/2),
\]

(2.93)

as \(T^{(K_+,J)}(\xi - \eta/2)\) are polynomials of degree \(2N\) in \(\xi\) for all \(l \in \{1, \ldots, N\}\) with maximal degree coefficient given by (2.37).

So that taking now the limit \(\xi \to \infty\), we get:

\[
\lim_{\xi \to \infty} \langle t|t \rangle \equiv \sum_{h_1, \ldots, h_N = 0}^{1} \prod_{a=1}^{N} \left( \frac{\sqrt{1+4\kappa^2}}{k_{\theta_a}} \right)^{h_a} \left( \frac{\kappa_{\theta_a}}{k_1} \right)^{1-h_a}
\]

(2.94)

\[
= \sum_{h_1, \ldots, h_N = 0}^{1} \prod_{a=1}^{N} \left( \frac{k_1}{k_{\theta_a}} \right)^{h_a} \left( \frac{\kappa_{\theta_a}}{k_1} \right)^{1-h_a}
\]

(2.95)

\[
= 2^N \prod_{a=1}^{N} (1 - 2\delta_{\theta_a,0}) \neq 0.
\]

(2.96)

This proves that

\[
\langle t|t \rangle \neq 0
\]

(2.97)

for almost any values of the inhomogeneities, of \(\eta\) and for any choice of the transfer matrix eigenvalue \(t(\lambda)\). Finally, given an eigenvalue \(t(\lambda)\) it is associated with a non trivial Jordan block if and only if the eigenco-vector and eigenvector associated to \(t(\lambda)\) are orthogonal. Since we have shown that this is not the case, it implies that the transfer matrix is diagonalizable and has simple spectrum as already proven.

\[\square\]

3. SoV for fundamental representations of \(\mathbf{U}_q(\hat{\mathfrak{gl}}_2)\) reflection algebra

In this section we consider the representation of the reflection algebra associated to the trigonometric 6-vertex \(R\)-matrix:

\[
R_{ab}(\lambda) = \begin{pmatrix}
\sinh(\lambda + \eta) & 0 & 0 & 0 \\
0 & \sinh \lambda & \sinh \eta & 0 \\
0 & \sinh \eta & \sinh \lambda & 0 \\
0 & 0 & 0 & \sinh(\lambda + \eta)
\end{pmatrix} \in \text{End}(V_a \otimes V_b).
\]

(3.1)

As in the rational case also in the trigonometric case the transfer matrix, defined by

\[
T(\lambda) = \text{tr}_a \{ K_{+,a}(\lambda) M_a(\lambda) K_{-,a}(\lambda) \hat{M}_a(\lambda) \} = \text{tr}_a \{ K_{+,a}(\lambda) \mathcal{U}_{-,a}(\lambda) \},
\]

(3.2)

generates a one-parameter family of commuting operators on the quantum space \(\mathcal{H} = \otimes_{i=1}^{N} V_i\), with \(V_i \simeq \mathbb{C}^2\). Here, we have defined...
\[ K_+ (\lambda) = K_- (\lambda + \eta; \zeta_+, \kappa_+, \tau_+) , \]  

(3.3)

and

\[ K_{-\alpha} (\lambda; \zeta, \kappa, \tau) = \frac{1}{\sinh \zeta} \begin{pmatrix} \sinh(\lambda - \eta/2 + \zeta) & \kappa e^{\tau} \sinh(2\lambda - \eta) \\ \kappa e^{-\tau} \sinh(2\lambda - \eta) & \sinh(\lambda + \eta/2) \end{pmatrix} \in \text{End}(V_a \simeq \mathbb{C}^2), \]  

(3.4)

which is the most general scalar solution to the trigonometric 6-vertex reflection equation [30, 32–34]. The same definitions as in the rational case hold for the boundary monodromy matrix

\[ U_{-\alpha} (\lambda) = M_0 (\lambda) K_- (\lambda) \hat{M}_0 (\lambda) = \begin{pmatrix} A_-(\lambda) & B_-(\lambda) \\ C_-(\lambda) & D_-(\lambda) \end{pmatrix} \in \text{End}(V_a \otimes \mathcal{H}), \]  

(3.5)

an operator solution to the same reflection equation, for the bulk monodromy matrix:

\[ M_0 (\lambda) = R_{0 N} (\lambda - \xi_n^{(0)}) \cdots R_{01} (\lambda - \xi_1^{(0)}) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \in \text{End}(V_a \otimes \mathcal{H}), \]  

(3.6)

an operator solution of the trigonometric 6-vertex Yang–Baxter equation, and for

\[ \hat{M}_0 (\lambda) = (-1)^N \sigma_{\alpha}^y M_0^y (\lambda) \sigma_{\alpha}^y. \]  

(3.7)

For the trigonometric 6-vertex reflection algebra the fusion of transfer matrices leads to the following quantum determinant relations:

\[ T(\xi_n^{(0)}) T(\xi_1^{(1)}) = A_{\alpha, \beta} (\xi_n^{(0)}) A_{\alpha, \beta} (-\xi_1^{(1)}) , \forall \alpha \in \{1, ..., N\}, \]  

(3.8)

which are proven by direct computations using the reduction of the trigonometric 6-vertex \( R \)-matrix to the permutation operator and to a 1-dimensional projector for \( \lambda = 0 \) and \( -\eta \), respectively. Here, we have defined:

\[ A_{\alpha, \beta} (\lambda) = (-1)^N \frac{\sinh(2\lambda + \eta)}{\sinh 2\lambda} g_+(\lambda) g_-(\lambda) a(\lambda) d(-\lambda), \]  

(3.9)

\[ d(\lambda) = a(\lambda - \eta), \quad a(\lambda) = \prod_{n=1}^N \sinh(\lambda - \xi_n + \eta/2), \]  

(3.10)

and

\[ g_{\pm} (\lambda) = \begin{cases} \sinh(\lambda + \alpha_{\pm} - \eta/2) \cosh(\lambda \mp \beta_{\pm} - \eta/2) / (\sinh \alpha_{\pm} \cosh \beta_{\pm}) & \text{if } \kappa_{\pm} \neq 0 \\ \sinh(\lambda + \zeta_{\pm} - \eta/2) / \sinh \zeta_{\pm} & \text{if } \kappa_{\pm} = 0, \end{cases} \]  

(3.11)

where \( \alpha_{\pm} \) and \( \beta_{\pm} \) are defined in terms of the boundary parameters by:

\[ \sinh \alpha_{\pm} \cosh \beta_{\pm} = \frac{\sinh \zeta_{\pm}}{2\kappa_{\pm}}, \quad \cosh \alpha_{\pm} \sinh \beta_{\pm} = \frac{\cosh \zeta_{\pm}}{2\kappa_{\pm}}. \]  

(3.12)

Moreover, the transfer matrix is an even function of the spectral parameter \( \lambda \) and it is central in the following values:

\[ \lim_{\lambda \to \pm \infty} e^{\mp 2\lambda(N+2)} T(\lambda) = 2^{- (2N+1)} \frac{\kappa_{\pm} \cosh (\tau_{\pm} - \tau_-)}{\sinh \zeta_+ \sinh \zeta_-}, \]  

(3.13)
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\[ T(\pm \eta/2) = (-1)^N 2 \cosh \eta \, a(\eta/2) d(-\eta/2), \]

\[ T(\pm (\eta/2 - i \pi/2)) = -2 \cosh \eta \, \coth \zeta_- \, \coth \zeta_+ \, a(i \pi/2 + \eta/2) d(i \pi/2 - \eta/2). \]

Let us define the functions:

\[ g_{a, \h}(\lambda) = \frac{\cosh^2 2\lambda - \cosh^2 \eta}{\cosh^2 2\xi_a^{(h_a)} - \cosh^2 \eta} \prod_{b \neq a}^{N} \cosh 2\lambda - \cosh 2\xi_b^{(h_b)} \quad \text{for } a \in \{1, \ldots, N\}, \]

and

\[ f_{\h}(\lambda) = \frac{\cosh 2\lambda + \cosh \eta}{2 \cosh \eta} \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\xi_b^{(h_b)}}{\cosh \eta - \cosh 2\xi_b^{(h_b)} A_{\alpha_{\pm, \beta_{\pm}}}(\eta/2)} \]

\[ - (-1)^{N} \frac{\cosh 2\lambda - \cosh \eta}{2 \cosh \eta} \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\xi_b^{(h_b)}}{\cosh \eta + \cosh 2\xi_b^{(h_b)} A_{\alpha_{\pm, \beta_{\pm}}}(\eta/2 + i\pi/2)} \]

\[ + 2^{(1-N)} \frac{\kappa_+ \kappa_- \cosh(\tau_+ - \tau_-)}{\sinh \zeta_+ \sinh \zeta_-} \left( \cosh^2 2\lambda - \cosh^2 \eta \right) \prod_{b=1}^{N} \left( \cosh 2\lambda - \cosh 2\xi_b^{(h_b)} \right), \]

Then for any \( \h = \{h_1, \ldots, h_N\} \in \{0, 1\}^N \) the following interpolation formula for the transfer matrix holds:

\[ T(\lambda) = f_{\h}(\lambda) + \sum_{a=1}^{N} g_{a, \h}(\lambda) T(\xi_a^{(h_a)}). \]

### 3.1. The rational limit of trigonometric 6-vertex \( R \)-matrix and \( K \)-matrix

Let us remark that both the trigonometric 6-vertex \( R \)-matrix and the \( K \)-matrix general scalar solution of the trigonometric 6-vertex reflection equation admit a well defined limit to their rational counterparts. To shorten the notations, we denote all objects related to the rational case with an index \( XXX \) and while an index \( XXZ \) will refer to the same objects in the trigonometric case. In particular, defining:

\[ \lambda = \epsilon \hat{\lambda}, \eta = \epsilon \hat{\eta}, \xi_n = \epsilon \hat{\xi}_n, \zeta_{\pm} = \epsilon \hat{\zeta}_{\pm}, \]

\[ \kappa_{\pm} = \hat{\kappa}_{\pm} + O(\epsilon), \tau_{\pm} = \hat{\tau}_{\pm} + \epsilon \hat{\tau}_{\pm} + O(\epsilon^2), \]

then it holds:

\[ \lim_{\epsilon \to 0} \frac{R^{(XXZ)}(\lambda|\eta)}{\sinh \epsilon} = R^{(XXX)}(\hat{\lambda}|\hat{\eta}), \]

\[ \lim_{\epsilon \to 0} \frac{K^{(XXZ)}_{\pm}(\lambda|\eta, \zeta_{\pm}, \kappa_{\pm}, \tau_{\pm})}{\sinh \epsilon} = K^{(XXX)}_{\pm}(\hat{\lambda}|\hat{\eta}, \hat{\zeta}_{\pm}, \hat{\kappa}_{\pm}, \hat{\tau}_{\pm}), \]

so that we have:

https://doi.org/10.1088/1742-5468/ab357a
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\[
\lim_{\varepsilon \to 0} \frac{U_{XXZ}(\lambda | \eta, \{ \xi \}, \zeta, \kappa, \tau)}{\sinh^{2N+1} \varepsilon} = U_{XXX}(\lambda | \eta, \{ \xi \}, \zeta, \kappa, \tau),
\]

(3.23)

\[
\lim_{\varepsilon \to 0} \frac{T_{XXX}(\lambda | \eta, \{ \xi \}, \zeta, \kappa, \tau)}{\sinh^{2N+2} \varepsilon} = T_{XXX}(\lambda | \eta, \{ \xi \}, \zeta, \kappa, \tau).
\]

(3.24)

Moreover, we get the following prescriptions on the parameters \(\alpha_\pm\) in the rational limit:

\[
\alpha_\pm(\varepsilon) = \varepsilon \tilde{\alpha}_\pm + \tilde{\alpha}_\pm
\]

(3.25)

which induces the following functional form for

\[
\beta_\pm(\varepsilon) = \tilde{\beta}_\pm + \varepsilon \tilde{\beta}_\pm
\]

(3.26)

with:

\[
\cosh \tilde{\beta}_\pm = \lim_{\varepsilon \to 0} \frac{\sinh \zeta}{2 \kappa \sinh \alpha} = \frac{\sqrt{1 + 4\kappa^2}}{2 \kappa},
\]

(3.27)

\[
\sinh \tilde{\beta}_\pm = \lim_{\varepsilon \to 0} \frac{\cosh \zeta}{2 \kappa \cosh \alpha} = \frac{1}{2 \kappa},
\]

(3.28)

and which lead to the following rational limit:

\[
\lim_{\varepsilon \to 0} \frac{A_{a_\pm, b_\pm}(\lambda)}{\sinh^{2N+2} \varepsilon} = A_{\xi, \tilde{\xi}}(\lambda),
\]

(3.29)

where the \(A_{\xi, \tilde{\xi}}(\lambda)\) are the coefficients (2.29) defined for the rational 6-vertex reflection algebra. This is in agreement with the preservation of the transfer matrix fusion equations under the rational limit:

\[
0 = \lim_{\varepsilon \to 0} \frac{T_{XXX}(\xi_0(0))T_{XXX}(\xi_1(1)) - A_{a_\pm, b_\pm}(\xi_0(0))A_{a_\pm, b_\pm}(-\xi_1(1))}{\sinh^{2N+4} \varepsilon},
\]

(3.30)

\[
= T_{XXX}(\xi(0))T_{XXX}(\xi(1)) - A_{\xi, \tilde{\xi}}(\xi(0))A_{\xi, \tilde{\xi}}(-\xi(1)).
\]

(3.31)

3.2. Applicability of our new approach and comparison with Sklyanin’s SoV

The general proposition 2.6 of our first paper [1] applies to these representations and it allows us to define the left SoV basis.

**Theorem 3.1.** Let \(T(\lambda)\) be the one-parameter family of transfer matrix associated to a generic couple \(K_-(\lambda)\) and \(K_+(\lambda)\) of boundary matrices then

\[
\langle h_1, ..., h_N \rangle \equiv \langle S \rangle \prod_{n=1}^{N} \left( \frac{T(\xi_n - \eta/2)}{A_{a_\pm, b_\pm}(\eta/2 - \xi_n)} \right)^{1-h_n}
\]

(3.32)

for any \(\{ h_1, ..., h_N \} \in \{0, 1\}^{\otimes N}\), is a co-vector basis of \(\mathcal{H}\) for almost any choice of the co-vector \(\langle S \rangle\), of the inhomogeneity parameters satisfying the condition (2.24) modulo \(\pi\), of the parameter \(\eta\) and of the boundary parameters.

https://doi.org/10.1088/1742-5468/ab357a

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Proof. The proof can be given as a consequence of the fact that the rational limit of the
trigonometric transfer matrix coincides with the rational transfer matrix and then
we can apply the theorem proven in the rational case.

More in detail, let us denote by \( \{ \hat{\eta}, \{ \xi_n \}, \zeta_\pm, \tilde{\kappa}_\pm, \tilde{\tau}_\pm \} \in \mathbb{C}^{7+N} \) a choice of parameters
such that the set of SoV co-vectors

\[
\langle h_1, \ldots, h_N; \hat{\eta}, \{ \xi_n \}, \zeta_\pm, \tilde{\kappa}_\pm, \tilde{\tau}_\pm | (XXX) \rangle \equiv \langle S | \prod_{n=1}^{N} \left( \frac{T_{XXX}(\xi_n - \hat{\eta}/2|\hat{\eta}, \{ \xi_n \}, \zeta_\pm, \tilde{\kappa}_\pm, \tilde{\tau}_\pm) \right)^{1-h_n} \rangle
\]

(3.33)
of the rational \( gl_2 \) reflection algebra (2.30) is a co-vector basis of \( \mathcal{H} \). Then, under the
prescriptions:

\[
\eta(\varepsilon) = \varepsilon \hat{\eta}, \xi_n(\varepsilon) = \varepsilon \hat{\xi}_n, \zeta_\pm(\varepsilon) = \varepsilon \hat{\zeta}_\pm, \kappa_\pm(\varepsilon) = \hat{\kappa}_\pm + O(\varepsilon), \quad \tau_\pm(\varepsilon) = \tilde{\tau}_\pm + \varepsilon \tilde{\tau}_\pm + O(\varepsilon^2),
\]

(3.34)
the SoV co-vectors of the trigonometric \( U_q(gl_2) \) reflection algebra (3.32) admit the
following power expansions in \( \varepsilon \)

\[
\langle h_1, \ldots, h_N; \eta(\varepsilon), \{ \xi_n(\varepsilon) \}, \zeta_\pm(\varepsilon), \kappa_\pm(\varepsilon), \tau_\pm(\varepsilon) | (XXX) \rangle = \langle h_1, \ldots, h_N; \hat{\eta}, \{ \hat{\xi}_n \}, \zeta_\pm, \tilde{\kappa}_\pm, \tilde{\tau}_\pm | (XXX) \rangle
+ \varepsilon^2(O_n + O(\varepsilon^4))
\]

(3.35)
where \( \langle h_1, \ldots, h_N |_1 \rangle \) is some finite co-vector, as it holds:

\[
\frac{T_{XXX}(\xi_n(\varepsilon) - \eta(\varepsilon)/2|\eta(\varepsilon), \{ \xi_n(\varepsilon) \}, \zeta_\pm(\varepsilon), \kappa_\pm(\varepsilon), \tau_\pm(\varepsilon))}{A_{\alpha_\pm, \beta_\pm}(\eta(\varepsilon)/2 - \xi_n(\varepsilon)|\eta(\varepsilon), \{ \xi_n(\varepsilon) \})}
= \frac{T_{XXX}(\xi_n - \hat{\eta}/2|\hat{\eta}, \{ \hat{\xi}_n \}, \zeta_\pm, \tilde{\kappa}_\pm, \tilde{\tau}_\pm) + \varepsilon^2O_n + O(\varepsilon^4)}{A_{\hat{\alpha_\pm}, \hat{\kappa}_\pm}(\hat{\eta}/2 - \xi_n|\hat{\eta}, \{ \xi_n \})}
\]

(3.36)
for some finite operator \( O_n \). Clearly, the above power expansions in \( \varepsilon \) and the fact
that the co-vectors (3.33) form by assumption a basis imply that there exists a
positive \( \bar{\varepsilon} \) such that the set of co-vectors (3.35) is also a basis for any \( \varepsilon \) such that
0 < \( \varepsilon \) < \( \bar{\varepsilon} \). The statement that (3.32) is basis for almost any choice of the parameters
\( \{ \eta, \{ \xi_n \}, \zeta_\pm, \kappa_\pm, \tau_\pm \} \in \mathbb{C}^{7+N} \) is then mainly a consequence of the fact that these co-vectors
are rational function of polynomials in the variables:

\[
E = e^{2\eta}, \quad \{ X_n = e^{2\xi_n} \}, \quad Z_\pm = e^{2\zeta_\pm}, \quad \kappa_\pm, \quad T_\pm = e^{2\tilde{\kappa}_\pm}.
\]

(3.37)
More precisely, let us define the \( n^N \times n^N \) matrices:

\[
\mathcal{M}^{(XXX)}_{i,j}(\langle S |, \eta, \{ \xi_n \}, \zeta_\pm, \kappa_\pm, \tau_\pm) \equiv \frac{\langle h_1(\hat{i}), \ldots, h_N(\hat{i}); \eta, \{ \xi_n \}, \zeta_\pm, \kappa_\pm, \tau_\pm | (XXX) e_j \rangle}{\prod_{n=1}^{N} A_{\alpha_\pm, \beta_\pm}(\eta/2 - \xi_n|\eta, \{ \xi_n \})},
\]

(3.38)
and

\[
\mathcal{M}^{(XXX)}_{i,j}(\langle S |, \hat{\eta}, \{ \hat{\xi}_n \}, \hat{\zeta}_\pm, \hat{\kappa}_\pm, \hat{\tau}_\pm) \equiv \frac{\langle h_1(\hat{i}), \ldots, h_N(\hat{i}); \hat{\eta}, \{ \hat{\xi}_n \}, \hat{\zeta}_\pm, \hat{\kappa}_\pm, \hat{\tau}_\pm | (XXX) e_j \rangle}{\prod_{n=1}^{N} A_{\hat{\alpha_\pm}, \hat{\kappa}_\pm}(\hat{\eta}/2 - \xi_n|\hat{\eta}, \{ \xi_n \})},
\]

(3.39)
for any $i,j \in \{1, ..., n^N\}$, where we have defined uniquely the $N$-tuple $(h_1(i), ..., h_N(i)) \in \{1, ..., n\}^{2N}$ by:

$$1 + \sum_{a=1}^{N} h_a(i) n^{a-1} = i \in \{1, ..., n^N\}, \tag{3.40}$$

and $|e_j| \in \mathcal{H}$ is the element $j \in \{1, ..., n^N\}$ of the elementary basis in $\mathcal{H}$. Then the condition that the set (3.32) form a basis of co-vector in $\mathcal{H}$ is equivalent to the condition:

$$\text{det}_n^n ||M^{(XXZ)}_{i,j}(\langle S \rangle, \eta, \{\xi_n\}, \zeta_\pm, \kappa_\pm, \tau_\pm)|| \neq 0. \tag{3.41}$$

Note that the above determinant is polynomial in the variables $E, \{X_n\}, Z_\pm, \kappa_\pm$ and Laurent polynomial in $T_\pm = e^{2\tau_\pm}$. So to prove that (3.41) indeed holds for almost any values of the parameters it is enough to prove that it holds in just one point. Now by using the power expansion in $\varepsilon$ of the trigonometric transfer matrices we have:

$$\text{det}_n^n ||M^{(XXZ)}_{i,j}(\langle S \rangle, \eta(\varepsilon), \{\xi_n(\varepsilon)\}, \zeta_\pm(\varepsilon), \kappa_\pm(\varepsilon), \tau_\pm(\varepsilon))|| = \varepsilon^{2N(2N+2)}(\text{det}_n^n ||M^{(XXX)}_{i,j}(\langle S \rangle, \hat{\eta}, \{\hat{\xi}_n\}, \hat{\zeta}_\pm, \hat{\kappa}_\pm, \hat{\tau}_\pm)|| + O(\varepsilon^2)), \tag{3.42}$$

which is nonzero for any $\varepsilon$ such that $0 < \varepsilon \leq \varepsilon$. This complete the proof of the theorem.

It is important to recall that Sklyanin’s SoV approach [108] or its generalized version by Baxter’s like gauge transformations [89, 91] works only in the case in which at least one of the two boundary matrices is non-diagonal and furthermore the boundary parameters satisfy the requirements

$$\tau_+ - \tau_- + (N + 1 - 2r) \eta \neq \varepsilon_-(\alpha_- + \beta_-) - \varepsilon_+(\alpha_+ - \beta_+) + \frac{i(\varepsilon_+ + \varepsilon_-)\pi}{2}, \tag{3.43}$$

for any $(r, \varepsilon_+, \varepsilon_-) \in \{1, ..., N\} \times \{-1, 1\}^2$. In our SoV approach we can define the above basis even in the case of both diagonal boundary matrices or in non-diagonal cases which are forbidden in the generalized Sklyanin’s SoV approach.

Let us impose that there exists $(r, \varepsilon_+, \varepsilon_-) \in \{1, ..., N\} \times \{-1, 1\}^2$:

$$\tau_+(\varepsilon) - \tau_-(\varepsilon) + (N + 1 - 2r) \eta(\varepsilon) = \frac{i(\varepsilon_+ + \varepsilon_-)\pi}{2} + \sum_{l=1,2} \epsilon_l(\hat{\beta}_l(\varepsilon) - l\alpha_l(\varepsilon)) \quad \forall \varepsilon \in \mathbb{C}, \tag{3.44}$$

where $\eta(\varepsilon)$, $\tau_+(\varepsilon)$, $\alpha_\pm(\varepsilon)$ and $\beta_\pm(\varepsilon)$ satisfy the prescription on the rational limit, i.e. (3.19), (3.20) and (3.25)–(3.28), so the above equation is equivalent to:

$$\hat{\tau}_+ - \hat{\tau}_- = \frac{i(\varepsilon_+ + \varepsilon_-)\pi}{2} + \epsilon_+ \hat{\beta}_+ + \epsilon_- \hat{\beta}_-, \tag{3.45}$$

$$\tilde{\tau}_+ - \tilde{\tau}_- = (N + 1 - 2r) \tilde{\eta} + \sum_{l=1,2} \epsilon_l(\tilde{\beta}_l - l\tilde{\kappa}_\pm). \tag{3.46}$$

Then taking the rational limit, we obtain $T_{(XXX)}(\lambda|\tilde{\eta}, \{\hat{\xi}\}, \hat{\kappa}_\pm, \hat{\tau}_\pm)$ where (3.45) is just imposing one condition on the parameters $\hat{\kappa}_\pm$ and $\hat{\tau}_\pm$ which has no effect on the
definition of the SoV basis in our approach for the rational case. So by the polynomiality argument above developed, it follows that also in the trigonometric case our approach is defining a basis for almost any value of $\varepsilon$ and of the boundary parameters satisfying (3.45) and (3.46). This finally implies that our set of co-vector stays a co-vector basis for almost any choice of the boundary parameters satisfying the constrain (3.44).

Here we want to show that under some special choice of the co-vector $\langle S \rangle$, our SoV left basis reduces to the SoV basis associated to Sklyanin’s approach when this last one is applicable. For simplicity we show this statement only in the case:

$$K_+(\lambda) = \begin{pmatrix} a_+(\lambda) & b_+(\lambda) \\ 0 & d_+(\lambda) \end{pmatrix}$$

(3.47)

where $b_+(\lambda)$ may be also zero. In this case Sklyanin’s approach directly applies and the associate co-vector basis is the eigenbasis of $B_-(\lambda)$, which reads (up to normalization):

$$\langle h_- | \equiv \langle 0 | \prod_{n=1}^{N} (A_- (\eta/2 - \xi_n))^{1-h_n}.$$  

(3.48)

The proof is done by induction just using the identity:

$$\langle 0 | A_- (\xi_a - \eta/2) = 0 \ \forall a \in \{1, \ldots, N\}$$  

(3.49)

and the following reflection algebra commutation relations:

$$A_- (\mu) A_- (\lambda) = A_- (\lambda) A_- (\mu) + \frac{\sinh \eta}{\sinh (\lambda + \mu - \eta)} (B_- (\lambda) C_- (\mu) - B_- (\mu) C_- (\lambda)).$$  

(3.50)

The steps in the proof for this trigonometric case are mainly the same as those presented in the rational case so we do not repeat them here.

3.3. Transfer matrix spectrum in our SoV approach

Let us show here how in our SoV schema it is characterized the transfer matrix spectrum.

**Theorem 3.2.** Under the same general conditions of theorem 3.1, ensuring the existence of the left SoV basis, the spectrum of $T(\lambda)$ is characterized by:

$$\Sigma_T = \left\{ t(\lambda) : t(\lambda) = f_{h=0}(\lambda) + \sum_{a=1}^{N} g_{a,h=0}(\lambda) x_a, \ \forall \{x_1, \ldots, x_N\} \in S_T \right\},$$

(3.51)

$S_T$ is the set of solutions to the following system of $N$ quadratic equations:

$$x_n f_{h=0}(\xi_n^{(1)}) + \sum_{a=1}^{N} g_{a,h=0}(\xi_n^{(1)}) x_a = A_{a_\pm, \beta_\pm} (\xi_n^{(0)}) A_{a_\pm, \beta_\pm} (-\xi_n^{(1)}), \ \forall n \in \{1, \ldots, N\},$$

(3.52)

in $N$ unknown $\{x_1, \ldots, x_N\}$. Moreover, $T(\lambda)$ has $w$-simple spectrum and for any $t(\lambda) \in \Sigma_T$ the associated unique (up-to normalization) eigenvector $|t\rangle$ has the following factorized wave-function in the left SoV basis:
\[
\langle h_1, \ldots, h_N | T(\lambda) | t \rangle = t(\lambda) \langle h_1, \ldots, h_N | t \rangle, \quad \forall \{h_1, \ldots, h_N\} \in \{0, 1\}^\otimes N.
\] (3.54)

We compute first the following matrix elements:

\[
\langle h_1, \ldots, h_a, \ldots, h_N | T(\xi_a^{(h_a)}) | t \rangle = \left\{ \begin{array}{ll}
\left. \frac{A_{n+1, n} (\xi/2 - \xi_n) (h_1, \ldots, h_a = 0, \ldots, h_N | t)}{A_{n+1, n} (\xi/2 - \xi_n)} \right|^{1-h_n} & \text{if } h_a = 1 \\
\left. \frac{A_{n+1, n} (\xi/2 - \xi_n) (h_1, \ldots, h_a = 1, \ldots, h_N | t)}{A_{n+1, n} (\xi/2 - \xi_n)} \right|^{1-h_n} & \text{if } h_a = 0
\end{array}\right.
\] (3.55)

which by the definition of the state \( | t \rangle \) can be rewritten as:

\[
\langle h_1, \ldots, h_a, \ldots, h_N | T(\xi_a^{(h_a)}) | t \rangle = \left\{ \begin{array}{ll}
t(\xi_a^{(1)}) \Pi_{n \neq a, n=1}^{N} \left( \frac{t(\xi_n - \xi/2)}{A_{n+1, n} (\xi/2 - \xi_n)} \right)^{1-h_n} & \text{if } h_a = 1 \\
0 & \text{if } h_a = 0
\end{array}\right.
\] (3.56)

and finally, by the fusion equation satisfied by \( t(\lambda) \), it reads:

\[
\langle h_1, \ldots, h_a, \ldots, h_N | T(\xi_a^{(h_a)}) | t \rangle = \left\{ \begin{array}{ll}
t(\xi_a^{(1)}) \Pi_{n \neq a, n=1}^{N} \left( \frac{t(\xi_n - \xi/2)}{A_{n+1, n} (\xi/2 - \xi_n)} \right)^{1-h_n} & \text{if } h_a = 1 \\
t(\xi_a^{(0)}) \Pi_{n=1}^{N} \left( \frac{t(\xi_n - \xi/2)}{A_{n+1, n} (\xi/2 - \xi_n)} \right)^{1-h_n} & \text{if } h_a = 0
\end{array}\right.
\] (3.57)

and so:

\[
\langle h_1, \ldots, h_a, \ldots, h_N | T(\xi_a^{(h_a)}) | t \rangle = t(\xi_a^{(h_a)}) \langle h_1, \ldots, h_a, \ldots, h_N | t \rangle.
\] (3.58)

From these identities and by using the interpolation formula:

\[
T(\lambda) = f_h(\lambda) + \sum_{a=1}^{N} g_a h(\lambda) T(\xi_a^{(h_a)}),
\] (3.59)

we get

\[
\langle h_1, \ldots, h_N | T(\lambda) | t \rangle = \left( f_h(\lambda) + \sum_{a=1}^{N} g_a h(\lambda) t(\xi_a^{(h_a)}) \right) \langle h_1, \ldots, h_N | t \rangle,
\] (3.60)

proving our statement.

\[\square\]

https://doi.org/10.1088/1742-5468/ab357a
The previous characterization of the spectrum allows to introduce an equivalent description in terms of a functional equation, the so-called quantum spectral curve equation, which in the case at hand is a second order Baxter’s type difference equation. In particular, this result coincides with the theorem 3.1 of [92], the only difference being that the applicability of the result extends now to the case of both diagonal boundary matrices and non-diagonal boundary matrices even satisfying the condition (3.43).

**Theorem 3.3.** Under the same conditions of theorem 3.1, ensuring the existence of the left SoV basis, \( t(\lambda) \in \Sigma_T \) if and only if there exists and is unique the polynomial

\[
Q_t(\lambda) = \prod_{a=1}^{p_{K_{+-}}} (\cosh 2\lambda - \cosh 2\lambda_a), \quad \lambda_1, \ldots, \lambda_{p_{K_{+-}}} \in \mathbb{C} \setminus \{ \pm \xi_1^{(0)}, \ldots, \pm \xi_N^{(0)} \},
\]

such that

\[
t(\lambda) Q_t(\lambda) = A_{\alpha_+,\beta_+}(\lambda) Q_t(\lambda - \eta) + A_{\alpha_-,\beta_-}(\lambda) Q_t(\lambda + \eta) + F(\lambda),
\]

with

\[
F(\lambda) = F_0 (\cosh^2 2\lambda - \cosh^2 \eta) \prod_{b=1}^N \prod_{i=0}^{1} (\cosh 2\lambda - \cosh 2\xi_b^{(i)}),
\]

where

\[
F_0 = \frac{\kappa_+ \kappa_- (\cosh(\tau_+ - \tau_-) - \cosh(\alpha_+ + \alpha_- - \beta_+ + \beta_- - (N + 1)\eta))}{2^{N-1} \sinh \zeta_+ \sinh \zeta_-},
\]

\[
p_{K_{+-}} = (1 - \delta_{0,F_0})N + \delta_{0,F_0} p, \text{ with } p \leq N.
\]

**Proof.** Under the conditions ensuring the existence of the left SoV basis, the equivalence of the first discrete SoV characterization with this functional equation is proven by standard arguments as introduced in [92, 124], see for example the proof of the theorem 3.1 of [92].

We have already proven that the existence of our SoV basis implies that the transfer matrix spectrum is simple. Now following the general proposition 2.6 of [1] we can also show that in general the transfer matrix is diagonalizable with simple spectrum.

**Theorem 3.4.** For almost any couple \( K_{-}(\lambda) \) and \( K_{+}(\lambda) \) of boundary matrices, any choice of the co-vector \( \langle S \rangle \) of the inhomogeneity parameters satisfying the condition (2.24) and of the parameter \( \eta \), we have that for any eigenvalue \( t(\lambda) \) of \( T(\lambda) \), it holds:

\[
\langle t | t \rangle \neq 0,
\]

where \( |t \rangle \) and \( \langle t | \) are the unique eigenvector and eigenco-vector associated to \( t(\lambda) \), and \( T(\lambda) \) is diagonalizable with simple spectrum.
Proof. The proof follows taking the rational limit and using the result proven in this case and then by using the fact that eigenvalues and eigenstates are algebraic functions in the parameter of the representations to deduce that the statement is true for almost any values of the parameters in the trigonometric case.

4. SoV for fundamental representations of $\mathcal{Y}(gl_3)$ reflection algebra

Here, we develop the SoV approach, from the construction of the SoV basis up to the functional equation characterization of the transfer matrix spectrum, for the most general fundamental representations of the $\mathcal{Y}(gl_3)$ reflection algebra. This spectral problem has been already studied in the Analytic and Nested algebraic Bethe Ansatz framework in [29, 31, 54–56, 69, 70], under some special type of boundary conditions. More recently, it has been analyzed in [79] under general boundary conditions by a modified version of Analytic Bethe Ansatz producing an Ansatz for the transfer matrix eigenvalues.

4.1. Fundamental representations of $\mathcal{Y}(gl_3)$ reflection algebra

We consider here the reflection algebra associated to the rational $gl_3$ R-matrix:

$$R_{a,b}(\lambda) = \lambda I_{a,b} + \eta P_{a,b} = \begin{pmatrix} a_1(\lambda) & b_1 \\ c_1 & a_2(\lambda) & b_2 \\ c_2 & a_3(\lambda) \\ b_3 \end{pmatrix} \in \text{End}(V_a \otimes V_b), \quad (4.1)$$

where $V_a \cong V_b \cong \mathbb{C}^3$ and we have defined:

$$a_j(\lambda) = \begin{pmatrix} \lambda + \eta \delta_{j,1} & 0 & 0 \\ 0 & \lambda + \eta \delta_{j,2} & 0 \\ 0 & 0 & \lambda + \eta \delta_{j,3} \end{pmatrix}, \quad \forall j \in \{1, 2, 3\};$$

$$b_1 = \begin{pmatrix} 0 & 0 & 0 \\ \eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix};$$

$$c_1 = \begin{pmatrix} 0 & \eta & 0 \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 0 & \eta \\ 0 & \eta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \eta \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.2)$$

which satisfies the Yang–Baxter equation:

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu) \in \text{End}(V_1 \otimes V_2 \otimes V_3). \quad (4.3)$$

Let us introduce the following boundary matrices [28, 30, 38, 47]:

$$K_\pm(\lambda) = I \mp \frac{\lambda - 3\delta_{\pm1,1}\eta/2}{\zeta_\pm} \mathcal{M}(\pm), \quad (4.4)$$
where
\[(\mathcal{M}^{(\pm)})^2 = r^{(\pm)} I, \ r^{(\pm)} = 1, 0, \] (4.5)
and in the case \(r^{(\pm)} = 1\)
\[
\mathcal{M}^{(\pm)} = W^{(\pm)} \begin{pmatrix} \epsilon_1^{(\pm)} & 0 & 0 \\ 0 & \epsilon_2^{(\pm)} & 0 \\ 0 & 0 & \epsilon_3^{(\pm)} \end{pmatrix} (W^{(\pm)})^{-1},
\] (4.6)
for any fixed invertible \(W^{(\pm)} \in \text{End}(V)\), where:
\[\epsilon_j^{(\pm)} = 1 \text{ for } j \in \{1, \ldots, p_{\pm}\}, \quad \epsilon_j^{(\pm)} = -1 \text{ for } j \in \{p_{\pm} + 1, \ldots, 3\},
\] (4.7)
for some \(p_{\pm} \in \{0, 1, 2, 3\}\). These \(K_{\pm}\)-matrices satisfy the following reflection equations:
\[R_{ab}(\lambda - \mu) \ K_{-a}(\lambda) \ R_{ba}(\lambda + \mu) \ K_{-b}(\mu) = K_{-b}(\mu) \ R_{ab}(\lambda + \mu) \ K_{-a}(\lambda) \ R_{ba}(\lambda - \mu),\] (4.8)
and
\[R_{ab}(\mu - \lambda) \ K_{+a}(\lambda) \ R_{ba}(-\lambda - \mu - 3\eta) \ K_{+b}(\mu) = K_{+b}(\mu) \ R_{ab}(-\lambda - \mu - 3\eta) \ K_{+a}(\lambda) \ R_{ba}(\mu - \lambda).\] (4.9)

We can define the following bulk monodromy matrix:
\[M_a(\lambda) \equiv R_{a,N}(\lambda - \xi_N) \cdots R_{a,1}(\lambda - \xi_1) \in \text{End}(V_a \otimes \mathcal{H}),\] (4.10)
satisfying the Yang–Baxter algebra associated to \(R\), where \(\mathcal{H} = \bigotimes_{n=1}^{N} V_n\) is the Hilbert space of a lattice model with \(N\) sites, having in each lattice site a local Hilbert space given by a fundamental representation. We can then define the boundary monodromy matrix:
\[\mathcal{U}_{-a}(\lambda) = M_a(\lambda) \ K_{-a}(\lambda) \ \hat{M}_a(\lambda) \in \text{End}(V_a \otimes \mathcal{H}),\] (4.11)
satisfying the above reflection equation, where we have defined:
\[\hat{M}_a(\lambda) \equiv R_{a,1}(\lambda + \xi_1) \cdots R_{a,N}(\lambda + \xi_N) \in \text{End}(V_a \otimes \mathcal{H}).\] (4.12)

Then, the transfer matrix,
\[T(\lambda) = \text{tr}_{V_a} \{K_{+a}(\lambda) \ M_a(\lambda) \ K_{-a}(\lambda) \ \hat{M}_a(\lambda)\} = \text{tr}_{V_a} \{K_{+a}(\lambda) \mathcal{U}_{-a}(\lambda)\} \in \text{End}(\mathcal{H}),\] (4.13)
defines a one-parameter family of commuting operators on \(\mathcal{H}\) [15].

It is interesting to remark that given a couple of integers \(p_{\pm} \in \{0, 1, 2, 3\}\), then the following identity holds:
\[T(\lambda|p_{\pm}, \zeta_{\pm}) = \text{tr}_{V_a} \{K_{+a}(\lambda|p_{+}, \zeta_{+}) \ M_a(\lambda) \ K_{-a}(\lambda|p_{-}, \zeta_{-}) \ \hat{M}_a(\lambda)\} = CT(\lambda|p'_{\pm} = 3 - p_{\pm}, \zeta'_{\pm} = -\zeta_{\pm})C,\] (4.14)
where we have used that
\[K_{\pm,a}(\lambda|p_{\pm}, \zeta_{\pm}) = C_{a} K_{\pm,a}(\lambda|p'_{\pm} = 3 - p_{\pm}, \zeta'_{\pm} = -\zeta_{\pm})C_{a},\] (4.16)
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In the following we consider only couples of \( p_\pm \in \{0, 1, 2, 3\} \) which are not complementary in this sense as those complementary follows by the above identity. Let us remark moreover that in the case \( r^{(+)} = 1 \), it holds:

\[
\mathcal{M}^{(+)} \mathcal{M}^{(-)} = W^{(+,-)} \begin{pmatrix} (-1)^{(3-p_+)(3-p_-)} & 0 & 0 \\ 0 & e^{-\alpha} & 0 \\ 0 & 0 & e^{\alpha} \end{pmatrix} (W^{(+,-)})^{-1},
\]

(4.18)

for some invertible \( 3 \times 3 \) matrix \( W^{(+,-)} \) and \( \alpha \in \mathbb{C} \), being

\[
\det \mathcal{M}^{(+)} \mathcal{M}^{(-)} = (-1)^{(3-p_+)(3-p_-)},
\]

(4.19)

\[
\text{tr} \left( \mathcal{M}^{(+)} \mathcal{M}^{(-)} \right)^k = \text{tr} \left( \mathcal{M}^{(-)} \mathcal{M}^{(+)} \right)^k = \text{tr} \left( \mathcal{M}^{(+)} \mathcal{M}^{(-)} \right)^{-k} \quad \forall k \in \mathbb{Z}.
\]

(4.20)

Let us here follow the standard fusion procedure of \( R \)-matrices [141–144] and boundary \( K \)-matrices [28]. We define, the following antisymmetric projectors:

\[
P_{1,\ldots,m} = \sum_{\pi \in S_m} (-1)^{\sigma_\pi} \frac{P_{\pi}}{m!} \in \text{End}(V_1 \otimes \cdots \otimes V_m),
\]

(4.21)

where \( S_m \) is the set of the permutations \( \pi \) of \( \{1,\ldots,m\} \), \( \sigma_\pi \) is the signature of \( \pi \), and we have defined

\[
P_{\pi}(v_1 \otimes \cdots \otimes v_m) = v_{\pi(1)} \otimes \cdots \otimes v_{\pi(m)} \in V_1 \otimes \cdots \otimes V_m,
\]

(4.22)

with \( P_{1}^{-} = I \). Now, by using them we can introduce the fused transfer matrices. In particular, the second fused transfer family reads:

\[
T_{2}(\lambda) = \text{tr}_{V_{(ab)}} \left\{ K_{(ab)}^{+}(\lambda) M_{(ab)}(\lambda) K_{(ab)}^{-}(\lambda) \hat{M}_{(ab)}(\lambda) \right\},
\]

(4.23)

where \( V_{(ab)} = P_{ab}^{-} V_a \otimes V_b \), and we have defined the fused boundary matrices:

\[
K_{(ab)}^{+}(\lambda) = P_{ab}^{-} K_{+,b}(\lambda - \eta) R_{ab}(2\lambda - 2\eta) K_{+,a}(\lambda) P_{ab}^{-},
\]

(4.24)

\[
K_{(ab)}^{-}(\lambda) = P_{ab}^{-} K_{-,a}(\lambda) R_{ba}(2\lambda - \eta) K_{-,b}(\lambda - \eta) P_{ab}^{-},
\]

(4.25)

and the fused bulk monodromy matrices:

\[
M_{(ab)}(\lambda) = P_{ab}^{-} M_{a}(\lambda) M_{b}(\lambda - \eta) P_{ab}^{-},
\]

(4.26)

\[
\hat{M}_{(ab)}(\lambda) = P_{ab}^{-} \hat{M}_{a}(\lambda) \hat{M}_{b}(\lambda - \eta) P_{ab}^{-}.
\]

(4.27)

Then, we can define the further fused boundary matrices:

\[
K^{\ast}_{(abc)}(\lambda) = P_{ace}^{-} K^{\ast}_{(bc)}(\lambda - \eta) R_{ac}(2\lambda - \eta) R_{ab}(2\lambda - 2\eta) K_{+,a}(\lambda) P_{abc}^{-},
\]

(4.28)

\[\text{Here, clearly it holds } P_{1,\ldots,m} = 0 \text{ for } m \geq 4 \text{ for the current case } V_i \cong \mathbb{C}^3 \text{ for any } i \in \{1,\ldots,N\}.\]

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\[ K_{(abc)}^-(\lambda) = P_{abc}^- K_{(bc)}^-(\lambda) R_{ba}(2\lambda - \eta) R_{ca}(2\lambda - 2\eta) K_{-a}(\lambda - \eta) P_{abc}^-, \]  

(4.29)

and the further fused bulk monodromy matrices:

\[ M_{(abc)}(\lambda) = P_{abc}^- M_a(\lambda) M_b(\lambda - \eta) M_c(\lambda - 2\eta) P_{abc}^-, \]  

(4.30)

\[ \hat{M}_{(abc)}(\lambda) = P_{abc}^- \hat{M}_a(\lambda) \hat{M}_b(\lambda - \eta) \hat{M}_c(\lambda - 2\eta) P_{abc}^-, \]  

(4.31)

and from them the third family of quantum spectral invariants:

\[ T_3(\lambda) = \text{tr}_{V_{(abc)}} \left\{ K_{(abc)}^+(\lambda) M_{(abc)}(\lambda) K_{(abc)}^-(\lambda) \hat{M}_{(abc)}(\lambda) \right\}, \]  

(4.32)

where \( V_{(abc)} = P_{abc}^- V_a \otimes V_b \otimes V_c \), which is also called the quantum determinant.

### 4.2. Properties of the transfer matrices

In this subsection we regroup some properties satisfied by the transfer matrices associated to the fundamental representations of the rank two reflection algebras which play an important role in our SoV construction.

**Property 4.1.** The transfer matrices \( T(\lambda) \) and \( T_2(\lambda) \) defines two one parameter families of mutually commuting operators:

\[ [T(\lambda), T(\mu)] = [T(\lambda), T_2(\mu)] = [T_2(\lambda), T_2(\mu)] = 0. \]  

(4.33)

Moreover, the quantum determinant \( T_3(\lambda) \) is a central element of the reflection algebra, i.e.

\[ [T_3(\lambda), U_{-a}(\mu)] = 0. \]  

(4.34)

The transfer matrix properties described in the following can be directly derived by using the known \( R \)-matrix properties like its reduction to the permutation operator and to the antisymmetric and symmetric projectors for special values of its arguments \((0, -\eta) \) and \( \eta \), respectively. These explicit computations have been presented recently in [79].

**Property 4.2.** The quantum spectral invariants have the following polynomial form:

(i) \( T(\lambda) \) is a degree \( 2N + 2 \) polynomial in \( \lambda \) with the central asymptotics:

\[ \lim_{\lambda \to \infty} \lambda^{-(2+2N)} T(\lambda) = -\text{tr}_a \mathcal{M}_a^{(+)} \mathcal{M}_a^{(-)}, \]  

(4.35)

and the following two central values:

\[ T(0) = \zeta_- d(\eta) \text{tr}_a K_a^{(+)}(0), \quad T(-3\eta/2) = \zeta_+ d(3\eta/2) \text{tr}_a K_a^{(-)}(-3\eta/2). \]  

(4.36)

(ii) \( T_2^{(K)}(\lambda) \) is a degree \( 4N + 6 \) polynomial in \( \lambda \) with the \( 2N + 2 \) central zeros:

\[ T_2(\lambda) = (\lambda - \eta)(\lambda + 3\eta/2)d(\lambda - \eta) T_2(\lambda), \]  

(4.37)
and the central asymptotic behaviour:
\[
\lim_{\lambda \to \infty} \lambda^{-(6+4N)} T_2(\lambda) = \text{tr}_{ab} P_{ab}^\dagger M_a^+ M_a^- M_b^+ M_b^- P_{ab},
\]

(4.38)

furthermore it has the following two central values:
\[
T_2(\eta/2) = \eta(\eta^2/4 - \zeta^2) d(\eta/2) d(3\eta/2) \text{tr}_{(ab)} K_{(ab)}^+(\eta/2),
\]

(4.39)
\[
T_2(-\eta) = \eta(\eta^2/4 - \zeta^2) d(\eta) d(2\eta) \text{tr}_{(ab)} K_{(ab)}^-(\eta). 
\]

(4.40)

It has also two known values in terms of the transfer matrix:
\[
T_2(0) = r(-\eta) T(0) T(-\eta),
\]

(4.41)
\[
T_2(-\eta/2) = r(-2\eta) T(-\eta/2) T(-3\eta/2).
\]

(4.42)

(iii) The quantum determinant explicitly reads:
\[
T_3(\lambda) = (-1)^{p_+ p_-} \big(2\lambda - 2\eta\big) \big(2\lambda + \eta\big) \big(2\lambda - 3\eta\big) \big(2\lambda + 2\eta\big) \big(2\lambda - 4\eta\big) \big(2\lambda + 3\eta\big)
\times d(\lambda + \eta) d(\lambda - \eta) \prod_{h=0}^{2-p_+} (\eta/2 - \zeta_+ - h\eta - \lambda) \prod_{h=0}^{p_+-1} (\eta/2 + \zeta_+ - h\eta - \lambda)
\times d(\lambda - 2\eta) \prod_{h=0}^{2-p_-} (\lambda - \zeta_- - h\eta) \prod_{h=0}^{p_-1} (\lambda + \zeta_- - h\eta).
\]

(4.43)

Moreover, the following fusion identities holds:
\[
r(\pm 2\xi_a - \eta) r(\pm 2\xi_a - 2\eta) T(\pm \xi_a) T_2(\pm \xi_a - \eta) = T_3(\pm \xi_a),
\]

(4.45)
\[
r(\pm 2\xi_a - \eta) T(\pm \xi_a) T(\pm \xi_a - \eta) = T_2(\pm \xi_a),
\]

(4.46)

where we have defined
\[
d(\lambda) = \prod_{a=1}^{N} (\lambda - \xi_a)(\lambda + \xi_a),
\]

(4.47)

\[
a(\lambda) = d(\lambda + \eta),
\]

\[
r(\lambda) = -\lambda(\lambda + 3\eta).
\]

Moreover, the transfer matrix satisfies the following important set of inversion relations:

**Lemma 4.1.** The following identities holds:
\[
T(\xi_l) = R_{l, l-1}(\xi_l - \xi_{l-1}) \cdots R_{l, 1}(\xi_l - \xi_1) K_{-l}(\xi_l) R_{1, l}(\xi_l + \xi_l) \cdots R_{l, l-1}(\xi_l + \xi_{l-1})
\]

\[
\cdot R_{l+1, l}(\xi_l + \xi_{l+1}) \cdots R_{l, N}(\xi_l + \xi_N) \text{tr}_{a} \left[ K_{a, l}(\xi_l) R_{a, l}(0) R_{a, l}(2\xi_l) \right]
\]

\[
\cdot R_{l, N}(\xi_l - \xi_N) \cdots R_{l, l+1}(\xi_l - \xi_{l+1}),
\]

(4.48)

https://doi.org/10.1088/1742-5468/ab357a
and
\[
T(-\xi_l) = R_{l,l+1}(\xi_{l+1} - \xi_l) \cdots R_{a,N}(\xi_N - \xi_l) \text{tr}_{V_a} \left[ K_{+,a}(-\xi_l) R_{a,l}(0) R_{a,l}(-2\xi_l) \right] \\
\cdot R_{l,N}(-\xi_l + \xi_N) \cdots R_{l,l+1}(-\xi_l + \xi_{l+1}) R_{l,l-1}(-\xi_l + \xi_{l-1}) \cdots R_{l,1}(-\xi_l + \xi_1) \quad (4.49)
\]
for any \( l \in \{1, \ldots, N\} \). Moreover, the following \( N \) centrality conditions hold:
\[
T(\xi_a) T(-\xi_a) = r_a 
\]
where
\[
r_a = \frac{(\xi_a - 3\eta/2)(\xi_a + 3\eta/2)}{(\xi_a - \eta/2)(\xi_a + \eta/2)} a(\xi_a) a(-\xi_a) (\xi_+ + \eta/2)^2 - \xi_a^2 (\xi_a^2 - \xi_a^2). \quad (4.51)
\]

**Proof.** Let us prove the above identities for the transfer matrix evaluated at the inhomogeneities values. Let us introduce the following short notations:
\[
K_{+,a}^{(l)} = K_{+,a}(\xi_l), \quad K_{-,a}^{(l)} = K_{-,a}(\xi_l), \quad K_{-,l}^{(l)} = K_{-,l}(-\xi_l), \quad (4.52)
\]
\[
R_{a,h}^{(-)} = R_{a,h}(\xi_l - \xi_h), \quad R_{l,h}^{(-)} = R_{l,h}(\xi_l - \xi_h), \quad (4.53)
\]
\[
R_{a,h}^{(+)} = R_{a,h}(\xi_l + \xi_h), \quad R_{l,h}^{(+)} = R_{l,h}(\xi_l + \xi_h), \quad (4.54)
\]
and
\[
\hat{K}_{+,a}^{(l)} = K_{+,a}(-\xi_l), \quad \hat{K}_{-,a}^{(l)} = K_{-,a}(-\xi_l), \quad \hat{K}_{-,l}^{(l)} = K_{-,l}(-\xi_l), \quad (4.55)
\]
\[
\hat{R}_{a,h}^{(-)} = R_{a,h}(\xi_h - \xi_l), \quad \hat{R}_{l,h}^{(-)} = R_{l,h}(\xi_h - \xi_l), \quad (4.56)
\]
\[
\hat{R}_{a,h}^{(+)} = R_{a,h}(-\xi_l + \xi_h), \quad \hat{R}_{l,h}^{(+)} = R_{l,h}(-\xi_l + \xi_h). \quad (4.57)
\]
By definition, then \( T(\xi_l) \) reads:
\[
\eta \text{tr}_{V_a} \left[ K_{+,a}^{(l)} R_{a,N}^{(-)} \cdots R_{a,l+1}^{(-)} P_{a,l}(0) R_{a,l-1}^{(-)} \cdots R_{a,1}^{(-)} K_{-,a}^{(l)} R_{a,1}^{(+)} \cdots R_{a,N}^{(+)} \right] \\
= \eta R_{l,l-1}^{(-)} \cdots R_{l,l+1}^{(-)} P_{l,l}^{(+)} \cdots P_{l,l-1}^{(+)} \text{tr}_{V_a} \left[ K_{+,a}^{(l)} R_{a,N}^{(-)} \cdots R_{a,l+1}^{(-)} P_{a,l}(0) R_{a,l}^{(-)} \cdots R_{a,N}^{(-)} \right] \\
= \eta R_{l,l-1}^{(-)} \cdots R_{l,l+1}^{(-)} P_{l,l}^{(+)} \cdots P_{l,l-1}^{(+)} \text{tr}_{V_a} \left[ K_{+,a}^{(l)} P_{a,l}(0) R_{a,N}^{(-)} \cdots R_{a,l+1}^{(-)} R_{a,l}^{(+)} \cdots R_{a,N}^{(+)} \right] \quad (4.58)
\]
In the following, we use the Yang–Baxter equation:
\[
R_{l,l+1}^{(-)} R_{a,l}^{(+)} R_{a,l+1}^{(-)} = R_{a,l+1}^{(+)} R_{a,l}^{(+) R_{l,l+1}^{(-)}}, \quad (4.59)
\]
and:
\[
R_{l,N}^{(-)} \cdots R_{l,l+2}^{(-)} R_{a,l+1}^{(+)} = R_{a,l+1}^{(+)} R_{l,N}^{(-)} \cdots R_{l,l+2}^{(-)}, \quad (4.60)
\]
https://doi.org/10.1088/1742-5468/ab357a
to rewrite (4.58) as it follows:
\[
\eta R_{ij}^{(-)} \cdots R_{i_1}^{(-)} \eta_{R}^{(-)} R_{i_1}^{(+)} \cdots R_{i_{j+1}}^{(-)} R_{i_{j+2}}^{(+)} \cdots R_{a,N} R_{i_{j+1}}^{(-)} = R_{a,N} R_{i_{j+1}}^{(+)} \cdots R_{a,1}^{(-)} R_{i_{j+1}}^{(-)}
\]
(4.61)

now making the same steps for \( R_{ij}^{(-)} R_{i_1}^{(+)} R_{i_2}^{(+)} \) for all the \( j \) from \( l + 2 \) up to \( N \), we end up with our formula (4.48). Similarly, we have that \( T(-\xi) \) reads:
\[
\eta \text{tr}_{V_a_1} \left[ \hat{K}_{+,a}^{(l)} \hat{R}_{a,N}^{(+)1} \cdots \hat{R}_{a,1}^{(+)N} \right] = \eta \text{tr}_{V_a_1} \left[ \hat{K}_{+,a}^{(l)} \hat{R}_{a,N}^{(+)1} \cdots \hat{R}_{a,1}^{(+)N} \right] \hat{R}_{a,1}^{(-)} \cdots \hat{R}_{a,N}^{(-)} \hat{R}_{a,1}^{(-)} \cdots \hat{R}_{a,N}^{(-)}
\]
(4.63)

where in the last line we have used:
\[
\hat{R}_{a,1}^{(+)1} \hat{P}_{a,(0)} = \hat{P}_{a,(0)} \hat{R}_{a,1}^{(+)}
\]
(4.64)

In the following, we use the Yang–Baxter equation:
\[
\hat{R}_{i_{j+1}}^{(+)} \hat{R}_{i_{j+2}}^{(+)} \hat{R}_{a,N}^{(-)} = \hat{R}_{a,N}^{(-)} \hat{R}_{a,1}^{(+)} \hat{R}_{i_{j+1}}^{(+)}
\]
(4.65)

and the commutativities:
\[
\hat{R}_{i_{j+1}}^{(+)} \cdots \hat{R}_{a,N}^{(+)} \hat{R}_{a,1}^{(-)} = \hat{R}_{a,1}^{(-)} \cdots \hat{R}_{a,N}^{(+)}
\]
(4.66)

\[
\hat{R}_{i_{j+1}}^{(+)} \cdots \hat{R}_{a,N}^{(+)} \hat{R}_{a,1}^{(-)} = \hat{R}_{a,1}^{(-)} \cdots \hat{R}_{a,N}^{(+)}
\]
(4.67)

to rewrite (4.63) as it follows:
\[
\eta \text{tr}_{V_a_1} \left[ \hat{K}_{+,a}^{(l)} \hat{R}_{a,N}^{(+)1} \cdots \hat{R}_{a,1}^{(+)N} \right] \hat{R}_{a,1}^{(-)} \cdots \hat{R}_{a,N}^{(-)} \hat{R}_{a,1}^{(-)} \cdots \hat{R}_{a,N}^{(-)}
\]
(4.68)

now making the same steps for \( R_{ij}^{(+)} R_{i_1}^{(+)} R_{i_2}^{(+)} \) for all the \( j \) from \( l + 2 \) up to \( N \), we end up with our formula (4.49).

Let us now prove the inversion relations. By direct computation one can prove that the following identities hold:
\[
K_{-,a}(\xi) \text{tr}_{V_a} \left[ K_{+,a}(\xi) R_{a,N}(0) R_{a,1}(2\xi) \right] \text{tr}_{V_a} \left[ K_{+,a}(-\xi) R_{a,N}(0) R_{a,1}(-2\xi) \right] K_{-,a}(-\xi) = \left( \xi^2 - (3\eta/2)^2 \right) \eta(\eta - 2\xi)((\xi^2 + \eta/2) - \xi^2)/((\xi^2 - \xi^2)/(\xi^2 - (\eta/2)^2)
\]
(4.69)

and by using the unitarity property of the \( R \)-matrix:

https://doi.org/10.1088/1742-5468/ab357a
we get:
\[ R^{(-)}_{i,j} \tilde{R}^{(-)}_{i,j} = -(\xi_l - \xi_j + \eta)(\xi_l - \xi_j - \eta), \quad (4.71) \]
\[ R^{(+)}_{i,j} \tilde{R}^{(+)}_{i,j} = -(\xi_l + \xi_j + \eta)(\xi_l + \xi_j - \eta). \quad (4.72) \]

Finally, by taking the product of the rhs of formulae (4.48) and (4.49) and by using the above identities we derive (4.50).

These properties imply that the second transfer matrix \( T_2(\lambda) \) can be written in terms of the transfer matrix \( T(\lambda) \). Let us introduce the functions

\[
g_{a,e}(\lambda) = \frac{\lambda(\lambda + 3\eta/2)}{\xi_a(\xi_a + 3\eta/2)} \prod_{b \neq a,b=1}^N \frac{(\lambda + e\xi_a)(\lambda^2 - \xi_b^2)}{2e\xi_a(\xi_a^2 - \xi_b^2)}, \quad (4.73)\]

\[
f_{a,e}(\lambda) = \frac{(\lambda^2 - \eta^2)(\lambda^2 - (\eta/2)^2)d(\lambda - \eta)}{(\xi_a^2 - \eta^2)(\xi_a^2 - (\eta/2)^2)d(\epsilon\xi_a - \eta)} g_{a,e}(\lambda), \quad (4.74)\]

and

\[
T^{(\infty)}(\lambda) = -\lambda(\lambda + 3\eta/2)d(\lambda) \text{tr}_a \mathcal{M}_a^{(+)} \mathcal{M}_a^{(-)} \quad (4.75)\]

\[
T_2^{(\infty)}(\lambda) = 4\lambda(\lambda^2 - (\eta/2)^2)(\lambda^2 - \eta^2)(\lambda + 3\eta/2)d(\lambda) \times d(\lambda - \eta)\text{tr}_{ab} P_{ab}^{-1} \mathcal{M}_a^{(+)} \mathcal{M}_b^{(+)} \mathcal{M}_b^{(-)} P_{ab}^{-1}, \quad (4.76)\]

then the following corollary holds:

**Corollary 4.1.** The transfer matrix \( T_2(\lambda) \) is completely characterized in terms of the fundamental transfer matrix \( T(\lambda) \) by the fusion equations, and the following interpolation formulae hold:

\[
T_2(\lambda) = T_2^{(\infty)}(\lambda) + \sum_{\epsilon = \pm 1} \sum_{a=1}^N f_{a,e}(\lambda) T(\epsilon\xi_a - \eta) T(\epsilon\xi_a) + V(\lambda|T) \quad (4.77)\]

where

\[
V(\lambda|T) = \frac{8(\lambda^2 - \eta^2)(\lambda^2 - (\eta/2)^2)(\lambda + 3\eta/2)d(\lambda - \eta)d(\lambda)}{3\eta^2d(\eta)d(0)} T_2(0)
- \sum_{\epsilon = \pm 1} \frac{16\lambda(\lambda^2 - \eta^2)(\lambda + e\eta/2)(\lambda + 3\eta/2)d(\lambda - \eta)d(\lambda)}{3e(3 + e)\eta^2d((\epsilon - 2)\eta/2)d(\eta/2)} T_2(\epsilon\eta/2)
+ \frac{4\lambda(\lambda - \eta)(\lambda^2 - (\eta/2)^2)(\lambda + 3\eta/2)d(\lambda - \eta)d(\lambda)}{3\eta^2d(2\eta)d(\eta)} T_2(-\eta), \quad (4.78)\]
and

\[ T(\lambda) = T(\infty)(\lambda) + \sum_{c=\pm 1} \sum_{a=1}^{N} g_{a,c}(\lambda) r_{a}^{(1-c)/2} (T(\xi_a))^c + \frac{(\lambda + 3\eta/2)d(\lambda)}{(3\eta/2)d(0)} T(0) \]

\[- \frac{\lambda d(\lambda)}{(3\eta/2)d(0)} T(-3\eta/2). \]  

(4.79)

**Proof.** The known central zeroes and asymptotics imply the above interpolation formulae once we use the fusion equations to write \( T_{2}(\pm \xi_a - \eta) \) and the inversion relations to write \( T(-\xi_a) \) in terms of \( T(\xi_a) \). \( \square \)

### 4.3. Our SoV co-vector basis

**Theorem 4.1.** Let us assume that both \( K_{+a}(\lambda) \) and \( K_{-a}(\lambda) \) are non-proportional to the identity\(^7\) and that one of the following requirements holds:

(i) \( K_{+a}(\lambda) \) and \( K_{-a}(\lambda) \) are non-commuting, (ii) \( K_{+a}(\lambda) \) and \( K_{-a}(\lambda) \) are commuting matrices and either \( r^{(+)} r^{(-)} = 0 \) or \( r^{(+)} r^{(-)} = 1 \), where in this last case, moreover it holds:

\[ M^{(-)} = \pm W \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} W^{-1}, \text{ with } W \in \text{End}(\mathbb{C}^{3}), \]  

(4.80)

and

\[ M^{(+)} = \pm W \begin{pmatrix} 1 & 0^a & 0 \\ 0 & (\pm 1)^a & 0 \\ 0 & 0 & (\pm 1)^{a+1} \end{pmatrix} W^{-1} \text{ with } a \in \{0, 1\}, \]  

(4.81)

then for almost any choice of \( \langle S \rangle \) and of the inhomogeneities under the condition (2.24), the following set of co-vectors:

\[ \langle h_1, \ldots, h_N | \equiv \langle S | \prod_{n=1}^{N} (T(\xi_n))^{h_n} \text{ for any } \{h_1, \ldots, h_N\} \in \{0, 1, 2\}^{\otimes N}, \]  

(4.82)

forms basis of \( \mathcal{H}^* \). In particular, we can take the state \( \langle S \rangle \) of the following tensor product form:

---

\(^7\) Note this means that \( p_{\pm} \in \{1, 2\} \) and that we have just two independent cases here \( (p_+, p_- = 2) \) (with the equivalent complementary one \( (p_+ = 2, p_- = 1) \)) and \( (p_+ = 1, p_- = 1) \) (with the equivalent complementary one \( (p_+ = 2, p_- = 2) \)).
\[ \langle S \rangle = \bigotimes_{a=1}^{N} (x, y, z) a \Gamma_{W}^{-1}, \quad \Gamma_{W} = \bigotimes_{a=1}^{N} W_{K,a}, \]  
\( \text{simply asking } x y z \neq 0. \)

**Proof.** This is a special case of the theorem 5.1 presented in the next section for the case \( n = 3 \). Following the proof there, we obtain that a sufficient condition to get the theorem is to prove that there exist \( \alpha_{\pm} \in \mathbb{C} \) such that the following matrices
\[ K_{a}^{(+,-)}(\alpha_{\pm}, \mathcal{M}^{(\pm)}) = (\alpha_{-} + a \mathcal{M}_{a}^{(-)})(\alpha_{+} + a \mathcal{M}_{a}^{(+)}) \quad \forall a \in \{1, ..., N\} \]  
have simple spectrum. Then, it is simple to observe that the set of conditions considered above just imply this property. \( \square \)

### 4.4. Transfer matrix spectrum in our SoV approach

The following characterization of the transfer matrix spectrum holds:

**Theorem 4.2.** Under the same assumptions ensuring that the set of SoV co-vectors form a basis, the spectrum of \( T(\lambda) \) is characterized by:

\[ \Sigma_{T(\kappa)} = \{ t_{1}(\lambda) : t_{1}(\lambda \{ x_{i} \}) = \begin{cases} \frac{\partial T^{(\infty)}(\lambda)}{\partial \lambda} & \text{for } \lambda \neq 0 \\ + \sum_{a=1}^{N} g_{a,\epsilon}(\lambda) r_{a}^{(1-\epsilon)/2} x_{a}^{\epsilon} + V(\lambda|t_{1}(\lambda \{ x_{i} \})) = T_{3}(\mu \xi_{a}), \end{cases} \]  
\( \forall \{x_{1}, ..., x_{N}\} \in S_{T} \}. \]  
Here, \( S_{T} \) is the set of solutions to the following system of \( N \) equations in \( N \) unknowns \( \{x_{1}, ..., x_{N}\} \):

\[ r_{a}^{(1-\mu)/2} x_{a}^{\epsilon} T_{2}^{(\infty)}(\mu \xi_{a} - \eta) + \sum_{\epsilon = \pm 1} \sum_{a=1}^{N} f_{a,\epsilon}(\mu \xi_{a}) t_{1}(\epsilon \xi_{a} - \eta) r_{a}^{(1-\epsilon)/2} x_{a}^{\epsilon} + V(\lambda|t_{1}(\lambda \{ x_{i} \})) = T_{3}(\mu \xi_{a}), \]  
\( \lambda \neq 0, \mu = \pm 1, n \in \{1, ..., N\}, \) where

\[ V(\lambda|t_{1}(\lambda \{ x_{i} \})) = \frac{8(\lambda^{2} - \eta^{2})(\lambda^{2} - (\eta/2)^{2})(\lambda + 3\eta/2)d(\lambda - \eta)d(\lambda)}{3\eta^{6}d(\eta)d(0)} t_{2}(0) \]

\[ - \sum_{\epsilon = \pm 1} \frac{16\lambda(\lambda^{2} - \eta^{2})(\lambda + \epsilon\eta/2)(\lambda + 3\eta/2)d(\lambda - \eta)d(\lambda)}{3\epsilon(3 + \epsilon)\eta^{2}d((\epsilon - 2)\eta/2)d(\eta/2)} t_{2}(\epsilon\eta/2) \]

\[ + \frac{4\lambda(\lambda - \eta)(\lambda^{2} - (\eta/2)^{2})(\lambda + 3\eta/2)d(\lambda - \eta)d(\lambda)}{3\eta^{6}d(2\eta)d(\eta)} t_{2}(-\eta), \]

and we have defined:

\[ t_{2}(0) = r(-\eta)t_{1}(0)t_{1}(-\eta), \]

\[ t_{2}(-\eta/2) = r(-2\eta)t_{1}(-\eta/2)t_{1}(-3\eta/2), \]
Moreover, $T(\lambda)$ has simple spectrum and for any $t_1(\lambda) \in \Sigma_{T(\lambda)}$ the associated unique (up-to normalization) eigenvector $|t\rangle$ has the following wave-function in the left SoV basis:

$$
\langle h_1, \ldots, h_N | t \rangle = \prod_{n=1}^{N} h_n^{t}(\xi_n).
$$

Proof. The proof is done according to the same lines of the proof of the theorem 5.1 for the fundamental representations of the $Y(gl_3)$ Yang–Baxter algebra in [1]. In fact, we have explicitly illustrated this for the rank one case, where the proof of the theorem 2.2 for the fundamental representation of the Yang–Baxter algebra follows the same lines of that for the Yang–Baxter algebra.

From the above discrete characterization of the transfer matrix spectrum in our SoV basis we can prove the following quantum spectral curve functional reformulation. Here, we consider explicitly only the case\(^8\) ($p_-=2, p_+=1$). Note that denoting by $\alpha$ the boundary parameter introduced in (4.18), the matrices $M^{(-)}$ and $M^{(+) in noncommuting for $\alpha \neq 0 \text{ mod } \pi$. While for $\alpha = 0$ we keep the transfer matrix simplicity asking that it holds:

$$
M^{(-)} = W \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} W^{-1}, \quad M^{(+)} = W \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} W^{-1}.
$$

In the above setting the following quantum spectral curve functional equation characterization of the spectrum holds

**Theorem 4.3.** The entire function $t_1(\lambda)$ satisfying the conditions (4.36) and (4.50) is a $T(\lambda)$ eigenvalue if and only if there exists a unique polynomial:

$$
\varphi_t(\lambda) = \prod_{a=1}^{M} (\lambda - \lambda_a)(\lambda + \lambda_a + \eta) \quad \text{with} \quad M \leq N,
$$

and $\lambda_a \neq \xi_n \forall (a,n) \in \{1, \ldots, M\} \times \{1, \ldots, N\}$ such that $t_1(\lambda)$,

$$
t_2(\lambda) = T_2^{(\infty)}(\lambda) + \sum_{\epsilon = \pm 1} \sum_{n=1}^{N} f_{n,\epsilon}(\lambda)t_1(\epsilon \xi_n - \eta) + V(\lambda|t_1(\lambda)),
$$

and $\varphi_t(\lambda)$ are solutions of the following quantum spectral curve functional equation:

$$
\alpha(\lambda)\varphi_t(\lambda - 3\eta) - \beta(\lambda)t_1(\lambda - 2\eta)\varphi_t(\lambda - 2\eta) - \gamma(\lambda)t_2(\lambda - \eta)\varphi_t(\lambda - \eta) + T_3(\lambda)\varphi_t(\lambda) = f(\lambda),
$$

\(^8\)Here, we have decided to implement the functional equation construction in this case as it is not covered in the existing literature. Indeed, the eigenvalue Ansatz construction presented in [79] has been developed only for the case ($p_-=1, p_+=1$). Note that anyhow we can also derive the quantum spectral curve in this last case and it has the same form of the ($p_-=2, p_+=1$) case with just modified coefficients and inhomogeneous term.
where:

\[ f(\lambda) = (1 - \cos \alpha) v_3(\lambda) a(\lambda) d(\lambda) d(\lambda - \eta) d(\lambda - 2\eta) \]  
\[ \alpha(\lambda) = v_2(\lambda) \gamma_0(\lambda - \eta) \gamma_0(\lambda - 2\eta), \]  
\[ \beta(\lambda) = v_1(\lambda) \gamma_0(\lambda - \eta), \]  
\[ \gamma(\lambda) = v_0(\lambda) \gamma_0(\lambda), \]  

and

\[ \gamma_0(\lambda) = (\eta/2 + \zeta_+ - \lambda)(\zeta_+ + \lambda)a(\lambda), \]  
\[ v_0(\lambda) = 2^4(\lambda^2 - \eta^2)(\lambda + 3\eta/2)(\lambda - \eta/2), \]  
\[ v_1(\lambda) = 2^6(\lambda^2 - \eta^2)(\lambda^2 - (3\eta/2)^2)(\lambda + \eta/2), \]  
\[ v_2(\lambda) = 2^8(\lambda^2 - \eta^2)(\lambda^2 - (\eta/2)^2)(\lambda + 3\eta/2), \]  
\[ v_3(\lambda) = 2^8(\lambda + \zeta_- - \eta)(\lambda - \eta)(\lambda - 2\eta)(\lambda^2 - (\eta/2)^2) \]  
\[ \times (\lambda^2 - \eta^2)(\lambda^2 - (3\eta/2)^2) \gamma_0(\lambda). \]  

Moreover, up to a normalization the common transfer matrix eigenvector \(|t\rangle\) admits the following separate wave-function representation:

\[ \langle h_1, \ldots, h_N|t\rangle = \prod_{a=1}^{N} \gamma^{h_a}(\xi_a) \varphi^{h_a}_a(\xi_a - \eta) \varphi^{2-h_a}_a(\xi_a). \]  

**Proof.** Let us start assuming that the entire function \(t_1(\lambda)\) satisfies with the polynomial \(t_2(\lambda)\) and \(\varphi(\lambda)\) the functional equation then it is a degree \(2N + 2\) polynomial in \(\lambda\) with leading coefficient \(t_{1,2N+2}\) satisfying the equation:

\[ -1 - t_{1,2N+2} + \text{tr}_{ab} P_{ab}^{\alpha} M_{\alpha}^{(a)} M_{\alpha}^{(-a)} M_{\beta}^{(a)} M_{\beta}^{(-a)} P_{ab} - 1 = 4(\cos \alpha - 1). \]  

By using the identity (4.18), we can compute now the following traces:

\[ \text{tr}_{ab} M_{\alpha}^{(+a)} M_{\alpha}^{(-a)} = 2\cos \alpha - 1, \]  
\[ \text{tr}_{ab} P_{ab}^{\alpha} M_{\alpha}^{(-a)} M_{\alpha}^{(a)} M_{\beta}^{(-a)} M_{\beta}^{(+a)} = 2\cos \alpha - 1, \]  

from which it follows:

\[ t_{1,2N+2} = 1 - 2\cos \alpha = -\text{tr}_{ab} M_{\alpha}^{(+a)} M_{\alpha}^{(-a)}, \]  

as it is required for transfer matrix eigenvalues. Let us observe now that it holds:

\[ \alpha(\pm \xi_a) = \beta(\pm \xi_a) = f(\pm \xi_a) = 0, \gamma(\pm \xi_a) \neq 0, T_\beta(\pm \xi_a) \neq 0, \]
so that the functional equation implies:
\[
\frac{\gamma(\pm \xi_a) \varphi_t(\pm \xi_a - \eta)}{\varphi_t(\pm \xi_a)} = \frac{T_3(\pm \xi_a)}{t_2(\pm \xi_a - \eta)}.
\]  
(4.111)

Moreover, we have
\[
\alpha(\pm \xi_a + \eta) = T_3(\pm \xi_a - \eta) = f(\pm \xi_a + \eta) = 0, \quad \beta(\pm \xi_a + \eta) \neq 0, \quad \gamma(\pm \xi_a + \eta) \neq 0,
\]  
(4.112)

so that the functional equation implies:
\[
\frac{\beta(\pm \xi_a + \eta) \varphi_t(\pm \xi_a - \eta)}{\gamma(\pm \xi_a + \eta) \varphi_t(\pm \xi_a)} = \frac{t_2(\pm \xi_a)}{t_1(\pm \xi_a - \eta)}.
\]  
(4.113)

Finally, we have:
\[
t_2(\pm \xi_a + \eta) = T_3(\pm \xi_a + 2\eta) = f(\pm \xi_a + 2\eta) = 0, \quad \beta(\pm \xi_a + 2\eta) \neq 0, \quad \alpha(\pm \xi_a + 2\eta) \neq 0,
\]  
(4.114)

so that the functional equation implies:
\[
\frac{\alpha(\pm \xi_a + 2\eta) \varphi_t(\pm \xi_a - \eta)}{\beta(\pm \xi_a + 2\eta) \varphi_t(\pm \xi_a)} = t_1(\pm \xi_a).
\]  
(4.115)

These identities imply the following ones:
\[
r(\pm 2\xi_a - \eta)r(\pm 2\xi_a - 2\eta)t_1(\pm \xi_a)t_2(\pm \xi_a - \eta) = T_3(\pm \xi_a), \quad \forall a \in \{1, \ldots, N\},
\]  
(4.116)

\[
r(\pm 2\xi_a - \eta)t_1(\pm \xi_a)t_1(\pm \xi_a - \eta) = t_2(\pm \xi_a), \quad \forall a \in \{1, \ldots, N\},
\]  
(4.117)

so that, by the SoV characterization obtained in our previous theorem, we have that \(t_1(\lambda)\) and \(t_2(\lambda)\) are eigenvalues of the transfer matrices \(T(\lambda)\) and \(T_2(\lambda)\), associated to the same eigenvector \(|t\rangle\).

Let us now prove the reverse statement, i.e. we assume that \(t_1(\lambda)\) is eigenvalue of the transfer matrix \(T(\lambda)\) and we want to show that there exists a polynomial \(\varphi_t(\lambda)\) which satisfies with \(t_1(\lambda)\) and \(t_2(\lambda)\) the functional equation. Here, we characterize \(\varphi_t(\lambda)\) by imposing that it satisfies the following set of conditions:
\[
\gamma(\pm \xi_a) \frac{\varphi_t(\pm \xi_a - \eta)}{\varphi_t(\pm \xi_a)} = t_1(\pm \xi_a).
\]  
(4.118)

The fact that this characterizes uniquely a polynomial of the form (4.93) can be shown just following the general proof given in [2]. Let us show that this characterization of \(\varphi_t(\lambda)\) implies that the functional equation is indeed satisfied. The functional equation is a degree \(8N + 12\) polynomial in \(\lambda\) so to show it we have just to prove that it is satisfied in \(8N + 12\) distinct points as the leading coefficient is zero, as we have shown above. We use the following \(8N\) points \(\pm \xi_a + k\alpha \eta\) for any \(a \in \{1, \ldots, N\}\) and \(k\alpha \in \{-1, 0, 1, 2\}\). Indeed, for \(\lambda = \pm \xi_a - \eta\) it holds:
\[
\alpha(\pm \xi_a - \eta) = \beta(\pm \xi_a - \eta) = \gamma(\pm \xi_a - \eta) = T_3(\pm \xi_a - \eta) = f(\pm \xi_a - \eta) = 0,
\]  
(4.119)
from which the functional equation is satisfied for any \( a \in \{1, \ldots, N\} \) and in the remaining \( 6N \) points the functional equation reduces to the \( 6N \) equations (4.111)–(4.115) which are equivalent to the discrete characterization (4.118), thanks to the fusion equations satisfied by the transfer matrix eigenvalues. Finally, by using the explicit form of the quantum determinant, we can show that the spectral curve equation factorizes the following polynomial of degree 6:

\[
(\lambda + \zeta - \eta)(\lambda^2 - \eta^2)(\lambda + 3\eta/2)\gamma_0(\lambda),
\]

as indeed it holds:

\[
T_2(-\zeta) = 0, \text{ being } K^{-1}_{(ab)}(-\zeta) = 0.
\]

Moreover, we can prove that this simplified quantum spectral equation (i.e. the one obtained after removing the above six common zeros) is satisfied in the following six points:

\[
\lambda = 0, \pm \eta/2, \eta, 3\eta/2, 2\eta,
\]

just using the known central zeros of the second transfer matrix (4.37) and the transfer matrix properties (4.36) and (4.39)–(4.42). This completes our proof of the equivalent rewriting of the spectrum in terms of the quantum spectral curve.

Finally, renormalizing the eigenvector \(|t\rangle\) multiplying it by the non-zero product of the \( \varphi^2(\xi_a) \) over all the \( a \in \{1, \ldots, N\} \) we get:

\[
\prod_{a=1}^{N} \varphi^2(\xi_a) \prod_{a=1}^{N} t^{h_a} (\xi_a) \overset{(4.118)}{=} \prod_{a=1}^{N} \gamma^{h_a}(\xi_a) \varphi^2(\xi_a - \eta) \varphi^{2-h_a}(\xi_a),
\]

which proves our statement on the SoV characterization of the transfer matrix eigenvectors presented in this theorem.

\[\blacksquare\]

5. SoV basis for fundamental representations of \( \mathcal{Y}(gl_n) \) reflection algebra

Here, we show that the transfer matrices of the fundamental representations of \( \mathcal{Y}(gl_n) \) reflection algebra can also be used for the general higher rank \( n \geq 3 \) case as the independent generators of the SoV basis.

Let us consider the \( \mathcal{Y}(gl_n) \) \( R \)-matrix

\[
R_{ab}(\lambda_a - \lambda_b) = (\lambda_a - \lambda_b)I_{ab} + \eta P_{ab} \in \text{End}(V_a \otimes V_b),
\]

with \( V_a = \mathbb{C}^n, \ V_b = \mathbb{C}^n, \ n \in \mathbb{N}^* \), solution of the Yang–Baxter equation:

\[
R_{ab}(\lambda_a - \lambda_b)R_{ac}(\lambda_a - \lambda_c)R_{bc}(\lambda_b - \lambda_c) = R_{bc}(\lambda_b - \lambda_c)R_{ac}(\lambda_a - \lambda_c)R_{ab}(\lambda_a - \lambda_b) \in \text{End}(V_a \otimes V_b \otimes V_c),
\]

where \( P_{ab} \) is the permutation operator on the tensor product \( V_a \otimes V_b \) and \( \eta \) is an arbitrary complex number. Then, we can define the bulk monodromy matrix:

\[
M_a(\lambda) \equiv R_{aN}(\lambda_a - \xi_N) \cdots R_{a1}(\lambda_a - \xi_1) \in \text{End}(V_a \otimes H),
\]
satisfying the Yang–Baxter algebra:
\[
R_{ab}(\lambda_a - \lambda_b)M_a(\lambda_a)M_b(\lambda_b) = M_b(\lambda_b)M_a(\lambda_a)R_{ab}(\lambda_a - \lambda_b) \in \text{End}(V_a \otimes V_b \otimes \mathcal{H}),
\]
where \( \mathcal{H} \equiv \bigotimes_{l=1}^{N} V_l \). The boundary matrices:
\[
K_{\pm}(\lambda) = I \mp \frac{\lambda - n\delta_{\pm 1,1} \eta/2}{\zeta_{\pm}} \mathcal{M}(\pm),
\]
where
\[
(\mathcal{M}(\pm))^2 = r^{(\pm)} I, \quad r^{(\pm)} = 1, 0
\]
define the most general scalar solutions to the reflection and dual reflection equations:
\[
K_{-a}(\lambda_a)R_{ab}(\lambda_a - \lambda_b)K_{-b}(\lambda_b)R_{ab}(\lambda_a + \lambda_b) = K_{-b}(\lambda_b)R_{ab}(\lambda_a + \lambda_b)K_{-a}(\lambda_a)R_{ab}(\lambda_a - \lambda_b)
\]
and
\[
K_{+a}(\lambda_a)R_{ab}(\lambda_b - \lambda_a)K_{+b}(\lambda_b)R_{ab}(\lambda_b + \lambda_a - n\eta) = K_{+b}(\lambda_b)R_{ab}(\lambda_b + \lambda_a - n\eta)K_{+a}(\lambda_a)R_{ab}(\lambda_b - \lambda_a).
\]
By using them we can define the boundary transfer matrix:
\[
T(\lambda) \equiv tr_{V_a}[K_{+a}(\lambda)M_a(\lambda)K_{-a}(\lambda)\hat{M}_a(\lambda)],
\]
where
\[
\hat{M}_a(\lambda) \equiv R_{a1}(\lambda + \xi_1) \cdots R_{aN}(\lambda + \xi_N),
\]
is proven to be a one parameter family of commuting operators following Sklyanin’s paper [15].

5.1. Generating the SoV basis by transfer matrix action

In these fundamental representations for the \( Y(gl_n) \) reflection algebra the following theorem holds:

**Theorem 5.1.** The following set of co-vectors:
\[
\langle h_1, \ldots, h_N | \equiv \langle S | \prod_{a=1}^{N} (T(\xi_a))^{h_a} \text{ for any } \{h_1, \ldots, h_N \} \in \{0, \ldots, n - 1\}^{\otimes N},
\]
is a basis of \( \mathcal{H}^* \) for almost any choice of the co-vector \( \langle S \rangle \), of the value of \( \eta \in \mathbb{C} \), of the inhomogeneity parameters satisfying (2.24) and of the boundary parameters in \( K_{\pm,a}(\lambda) \).

In particular, for any choice of the boundary parameters such that there exist \( \alpha_\pm \in \mathbb{C} \) for which the following matrix
\[
K^{(+,-)}_{a}(\alpha_\pm, \mathcal{M}(\pm)) = (\alpha_+ I_a + a \mathcal{M}_a^{(-)})(\alpha_- I_a + a \mathcal{M}_a^{(+)}) \in \text{End}(V_a)
\]
has simple spectrum on \( V_a \) for any \( a \in \{1, \ldots, N\} \), then we can take
\[
\langle S \rangle = \bigotimes_{a=1}^{N} \langle S, a \rangle, \text{ with } \langle S, a \rangle \in V_a, \quad \forall a \in \{1, \ldots, N\},
\]

https://doi.org/10.1088/1742-5468/ab357a
such that

\[ \langle S, a \rangle (K_a^{(+, -)})^h \text{ with } h \in \{0, ..., n - 1\}, \]  

(5.14)

form a co-vector basis for \( V_a \) for any \( a \in \{1, ..., N\} \).

**Proof.** We can follow the method already presented in [1] for the proof of the general proposition 2.4. Here we use that the transfer matrix is a polynomial in \( \eta \), the inhomogeneities \( \{\xi_a\}_{a \in \{1, ..., N\}} \) and Laurent polynomial in the boundary parameters. So the determinant of the \( n^N \times n^N \) matrix, whose lines coincides with the component of the co-vectors (5.11) in the natural basis of \( \mathcal{H} \), is a polynomial in the component of the co-vector \( \langle S \rangle \in \mathcal{H}^* \), in \( \eta \), in the inhomogeneities \( \{\xi_a\}_{a \in \{1, ..., N\}} \) and a Laurent polynomial in the boundary parameters. Then it is enough to prove that it is nonzero for some special value of these parameters to prove that it is so for almost any value of these parameters.

Let us observe now that from (4.48), it follows that \( T(\xi_i)\xi_+\xi_- \) are polynomials of degree \( 2N + 1 \) in \( \xi \) for all \( l \in \{1, ..., N\} \) with maximal degree coefficient given by:

\[ d_{l,2N+1}K_l^{(+, -)}(\alpha_{\pm}, M(\pm)) \], with \( d_{l,2N+1} = \eta(-1)^{N-l}(N-l)!(N+l)! \),

(5.15)

once we impose:

\[ \xi_a = a\xi \ \forall a \in \{1, ..., N\} \text{ and } \xi_{\pm} = \alpha_{\pm}\xi. \]

(5.16)

So that the co-vectors \( \langle h_1, ..., h_N \rangle \) have the following expansion in \( \xi \):

\[ \langle h_1, ..., h_N \rangle \equiv \frac{\xi^{(2N+1)}\sum_{a=1}^{N} h_a \prod_{a=1}^{N} d_{a,2N+1}^{h_a} \langle S \rangle \prod_{a=1}^{N} (K_a^{(+, -)}(\alpha_{\pm}, M(\pm)))^{h_a} + O(\xi^{(2N+1)\sum_{a=1}^{N} h_a = 1})}{\prod_{a=1}^{N} \xi_a^{h_a} (\xi_a - \eta \xi/4)} \]  

(5.17)

Hence a sufficient condition to generate a basis is given by:

\[ \det_n | \left( \langle S \rangle \left( \prod_{a=1}^{N} (K_a^{(+, -)}(\alpha_{\pm}, M(\pm)))^{h_a(i)} \right)_{i,j \in \{1, ..., n\}} \right)_{i,j \in \{1, ..., n\}} | \neq 0, \]

(5.18)

where for any \( i \in \{1, ..., n^N\} \) the \( N \)-tuple \( (h_1(i), ..., h_N(i)) \in \{1, ..., n\}^{\otimes N} \) is uniquely defined by (3.40) and \( |e_j\rangle \) is the element \( j \in \{1, ..., n^N\} \) of the natural basis in \( \mathcal{H} \). If we take \( \langle S \rangle \) of the tensor product form (5.13) then the above determinant reduces to:

\[ \prod_{a=1}^{N} \det_n | \langle S, a \rangle (\langle \alpha_- + aM_a(\pm)\rangle^{i-1}(\langle \alpha_+ + aM_a(\pm)\rangle^{i-1}) |e_j(a)\rangle \rangle_{i,j \in \{1, ..., n\}} | \]

(5.19)

where \( |e_j(a)\rangle \) is the element \( j \in \{1, ..., n\} \) of the natural basis in \( V_a \). Let us now show that for general choice of the boundary parameters the matrices \( K_a^{(+, -)}(\alpha_{\pm}, M(\pm)) \) can be indeed taken with non-degenerate spectrum. Let us prove it for the special choice:

\[ 9 \text{ The existence of } \langle S, a \rangle \text{ is implied by the spectrum simplicity of } K_a^{(+, -)}. \]
for any $a$. The remaining boundary parameters are indeed contained in the choice of two matrices $M_{a}^{-}$ and $M_{a}^{+}$ consistently with the conditions $M_{a}^{(2)} = I_{a}$. In particular, we are free to take:

$$[M_{a}^{-}, M_{a}^{+}] \neq 0,$$  \hspace{1cm} (5.21)

so that we have to determine the conditions on the eigenvalues of the full matrix $M_{a}^{-} M_{a}^{+}$ and prove that it can have simple spectrum. Let us denote by $t_{j}$ an eigenvalue of the matrix $M_{a}^{-} M_{a}^{+}$ and $m_{j}$ the corresponding degeneracy, for $j \in \{1, ..., \bar{n}\}$ and $n = \sum_{j=1}^{\bar{n}} m_{j}$. Then, by

$$\det_{a} M_{a}^{-} M_{a}^{+} = \det_{a} M_{a}^{-} \det_{a} M_{a}^{+},$$  \hspace{1cm} (5.22)

it follows

$$\prod_{j=1}^{\bar{n}} t_{j}^{m_{j}} = (-1)^{s_{-}+s_{+}} \text{ with } (-1)^{s_{\pm}} = \det_{a} M_{a}^{(\pm)},$$  \hspace{1cm} (5.23)

while the identity

$$\sum_{j=1}^{\bar{n}} m_{j} t_{j}^{r} = \sum_{j=1}^{\bar{n}} m_{j} t_{j}^{-r} \text{ for positive integer } r,$$  \hspace{1cm} (5.24)

follows from the identities:

$$\text{tr}_{V_{a}} \left( (M_{a}^{-} M_{a}^{+})^{r} \right) = \text{tr}_{V_{a}} \left( M_{a}^{-} M_{a}^{+} \ldots M_{a}^{-} M_{a}^{+} \right) = \text{tr}_{V_{a}} \left( M_{a}^{+} M_{a}^{-} M_{a}^{+} \ldots M_{a}^{-} \right) = \text{tr}_{V_{a}} \left( (M_{a}^{+} M_{a}^{-})^{r} \right),$$  \hspace{1cm} (5.25)

and from the identity:

$$(M_{a}^{-} M_{a}^{+})^{-1} = M_{a}^{+} M_{a}^{-}.$$  \hspace{1cm} (5.26)

The conditions (5.23) and (5.24) imply that $t_{j} \neq 0$ for any $j \in \{1, ..., \bar{n}\}$ and

$$\forall j \in \{1, ..., \bar{n}\} : t_{j} \neq \pm 1 \rightarrow \exists! \ h(j) \in \{1, ..., \bar{n}\} : t_{j} = t_{h(j)}^{-1}, m_{j} = m_{h(j)}.$$  \hspace{1cm} (5.27)

It is easy now to show that these conditions are compatible with the simplicity of the spectrum of $M_{a}^{-} M_{a}^{+}$. Avoiding the case of non-trivial Jordan blocks for simplicity, for example we can ask directly $\bar{n} = n$, i.e. $m_{j} = 1$ for any $j \in \{1, ..., \bar{n}\}$. Then we can distinguish the cases, for $n$ odd we can choose the following solution to the above conditions:
\[ t_1 = (-1)^{s_- s_+}, \quad t_{1+j+(n-1)/2} = t_{1+j}^{-1}, \quad t_{1+j} \neq \pm 1, \quad t_{1+h} \neq t_{1+h} \forall h \notin \{1, \ldots, (n-1)/2\} \quad (5.28) \]

for \( n \) even and \( s_- + s_+ \) even, we can choose the following solution to the above conditions:

\[
t_{j+n/2} = t_j^{-1}, \quad t_j \neq \pm 1, \quad t_j \neq t_h, \quad \forall h \neq j \in \{1, \ldots, n/2\},
\quad (5.29)\]

while for \( n \) even and \( s_- + s_+ \) odd, we can choose:

\[
t_1 = 1, \quad t_2 = -1, \quad t_{2+j+(n-2)/2} = t_{2+j}^{-1}, \quad t_{2+j} \neq \pm 1, \quad t_{2+h} \neq t_{2+h} \forall h \neq j \in \{1, \ldots, n/2 - 1\}. \quad (5.30)\]

This prove the possibility to choose \( M_a^{(-)} M_a^{(+)} \) with simple spectrum which completes the proof.

**Remark.** Note that the results about the construction of the SoV basis and the simplicity and diagonalizability of the fundamental transfer matrix can be extended naturally to the fundamental representations of the \( U_q(\hat{gl}_n) \) reflection algebra just using the same arguments described in the general propositions 2.5 and 2.6 of [1]. In particular, the proof follows mainly the same lines described for the case \( n = 2 \) in theorem 3.1 of our current paper.

### 5.2. Diagonalizability and simplicity of the transfer matrix

We want to show that the transfer matrix associated to the fundamental representation of the \( Y(gl_n) \) reflection algebra is diagonalizable with simple spectrum under some further requirement on the boundary matrices.

**Theorem 5.2.** Let the boundary matrices \( M_a^{(-)} \) and \( M_a^{(+)} \) be non-commuting while the product matrix \( M_a^{(-)} M_a^{(+)} \) is diagonalizable and with simple spectrum, then, for almost any value of \( \eta \in \mathbb{C} \) and of the inhomogeneity parameters satisfying (2.24), it holds:

\[
\langle t | t \rangle \neq 0,
\quad (5.31)\]

where \( |t\rangle \) and \( \langle t | \) are the unique eigenvector and eigenco-vector associated to \( t(\lambda) \), a generic eigenvalue of \( T(\lambda) \), and \( T(\lambda) \) is diagonalizable with simple spectrum.

**Proof.** Let us impose here:

\[
\xi_a = a \xi \quad \forall a \in \{1, \ldots, N\},
\quad (5.32)\]

then it follows that \( T(\xi_l) \) are polynomials of degree \( 2N + 1 \) in \( \xi \) for all \( l \in \{1, \ldots, N\} \) with maximal degree coefficient given by:

\[
T_{l,2N+1} \equiv d_{l,2N+1}^{(+,-)} M_{l}^{(-)} M_{l}^{(+)} , \quad \text{with} \quad d_{l,2N+1}^{(+,-)} = t^2 d_{l,2N+1}/(\xi_+ \xi_-).
\quad (5.33)\]

The proof now proceed exactly as in the general proposition 2.5 of [1], for the rank \( n - 1 \) fundamental representations of the \( Y(gl_n) \) Yang–Baxter algebra. Indeed by assumption \( T_{l,2N+1} \) is diagonalizable and has simple spectrum and so we can just replace...
it to the asymptotic operator $T_{l,N-1}^{(K)}$ used in the proof of proposition 2.5 of [1].

We can also give a more general characterization of the boundary conditions leading to the diagonalizability and simplicity of the transfer matrix, as it follows:

**Theorem 5.3.** Let the boundary matrix product $K_{+\alpha}(\lambda)K_{-\alpha}(\lambda)$ be simple and diagonalizable, then for almost any choice of $\eta \in \mathbb{C}$ of the inhomogeneity parameters satisfying (2.24) and of the boundary parameters $\zeta_{\pm}$, it holds:

$$\langle t|t \rangle \neq 0,$$

(5.34)

where $|t\rangle$ and $\langle t|$ are the unique eigenvector and eigenvector- associated to $t(\lambda)$, a generic eigenvalue of $T(\lambda)$, and $T(\lambda)$ is diagonalizable with simple spectrum.

**Proof.** Let us start observing that if the boundary matrix product $K_{+\alpha}(\lambda)K_{-\alpha}(\lambda)$ is simple for a given value of the $\zeta_{\pm}$ then it stays simple for almost any value of these parameters being $K_{+\alpha}(\lambda)K_{-\alpha}(\lambda)$ a polynomial of degree one in $1/\zeta_{\pm}$. Moreover, for $K_{+\alpha}(\lambda)K_{-\alpha}(\lambda)$ simple and diagonalizable, we also have that $K_{+1}(\xi)K_{-1}(\xi)$ is simple and diagonalizable for almost all the values of $\xi$. Let us now remark that the following identity holds:

$$K_{1}^{(+,-)}(\alpha_{\pm},\mathcal{M}(\pm)) = \alpha_{-}\alpha_{+}K_{+,1}(\xi)K_{-,1}(\xi),$$

(5.35)

with

$$\alpha_{-} = \frac{\zeta_{-}}{\xi_{1}}, \quad \alpha_{+} = \frac{\zeta_{+}}{\xi_{1} - \eta/2},$$

(5.36)

so that $K_{1}^{(+,-)}(\alpha_{\pm},\mathcal{M}(\pm))$ is simple and diagonalizable for almost all the values of $\xi$, $\alpha_{\pm}$. This simplicity implies that the determinant

$$\det_{n}(\langle S,1|((K_{1}^{(+,-)}(\alpha_{\pm},\mathcal{M}(\pm)))^{i}|e_{j}(1)\rangle_{i,j \in \{1,...,n\}}),$$

(5.37)

is nonzero for almost all the values of $\xi$, $\alpha_{\pm}$ and $\langle S,1 \rangle \in V_{1}$. Note that this determinant is a nonzero polynomial in the $\alpha_{\pm}$, and so in the $\zeta_{\pm}$. Then we can always find values of $\alpha_{\pm}$ such that the following determinants

$$\det_{n}(\langle S,1|(K_{1}^{(+,-)}(\alpha_{\pm},\mathcal{M}(\pm)))^{i}|e_{j}(l)\rangle_{i,j \in \{1,...,n\}} = \det_{n}(\langle S,1|((K_{1}^{(+,-)}(\alpha_{\pm},\mathcal{M}(\pm)))^{i}|e_{j}(1)\rangle_{i,j \in \{1,...,n\}}),$$

(5.38)

are nonzero for almost any value of the $\alpha_{\pm}$, and so of the $\zeta_{\pm}$, being also polynomials.

Let us now impose:

$$\xi_{a} = a \xi \ \forall a \in \{1,...,N\},$$

(5.39)

the leading coefficient of $T(\xi)\zeta_{+}\zeta_{-}$ reads:

$$T_{l,2N+1} = d_{l,2N+1}K_{1}^{(+,-)}(\alpha_{\pm},\mathcal{M}(\pm)),$$

(5.40)

so that following the proof of the general proposition 2.5 of [1], our statements hold
being the operators $T_{l;2N+1}$ diagonalizable and with simple spectrum on $V_l$ for any $l \in \{1, ..., N\}$ for almost any value of the $\zeta_\pm$.

6. Conclusion

In this paper we have solved the longstanding open problem to define the quantum separation of variables for the class of integrable quantum models associated to the fundamental representations of the $Y(gl_n)$ reflection algebras. We have used the SoV basis to completely characterize the eigenvalue and eigenvector spectrum of the transfer matrix for the rank one and rank two cases and proven its equivalence to the so-called quantum spectral curve equation. The result on the construction of the SoV basis for any positive integer rank, indeed, allows us to extend the complete characterization of the transfer matrix spectrum as well as to introduce the quantum spectral curve characterization of it also to any higher rank $n$. In this article, we have seen explicitly how the results for the rational fundamental representation of the $Y(gl_2)$ reflection algebra can be used to prove the construction of the SoV basis for the general trigonometric case and how our new SoV basis allows for the characterization of the transfer matrix spectrum in these representations. This also includes their quantum spectral curve equation. The same results can also be similarly derived for the fundamental representations of the trigonometric $U_q(gl_n)$ reflection algebras for any integer $n$. Our current investigations are both on completing the spectral analysis of other important quantum integrable models in our new SoV approach and to implement the analysis of the dynamics for the models already solved in this SoV framework. Here, the first fundamental step is the derivation of the scalar product formulae for the separate states of the type that we have derived for the rank one case in the appendix below. Such results should give access to the computation of matrix elements of local operators on transfer matrix eigenstates, i.e. the first fundamental step toward the dynamics of quantum models in this higher rank cases.

Acknowledgments

JMM and GN are supported by CNRS and ENS de Lyon.

Appendix. Scalar products in $Y(gl_2)$ reflection algebra

In this appendix we derive the scalar product of separate states, which contain as particular cases the transfer matrix eigenvectors. Let us comment that in the main text of the article we have shown that for fundamental representations of $Y(gl_2)$ reflection algebra associated to general non-commuting boundary matrices $K_+(\lambda)$ and $K_-(\lambda)$ our SoV basis can be reduced to the generalized Sklyanin’s one under a proper choice of the generating co-vector in the SoV basis. This observation implies that for this set of representations the ‘measure’ of the left/right SoV vectors must coincide with the ‘Sklyanin’s measure’ and so the scalar product of separate states can be computed.
according to the known literature [89, 94, 96]. Here, we use this appendix to show how to compute these scalar products directly in the framework of our new SoV approach. This has the advantage to prove scalar product formulae also for the representation associated to commuting boundary matrices, showing that they keep the same form independently from the applicability of the Sklyanin’s original approach. Using the same type of computations presented in the following we can show that this same statement applies also for the fundamental representations of $U_q(\hat{gl}_2)$ reflection algebra. Hence, the results of [89, 94, 96] hold as well for diagonal boundary conditions and under conditions on the parameters which make the generalized version of the Sklyanin’s approach inapplicable.

A.1. Construction of the right SoV basis orthogonal to the left one

The following theorem allows to produce the orthogonal basis to the left SoV basis and show that it is also of SoV type just using the polynomial form of the transfer matrix and the fusion equation. Let us denote by $|S\rangle$ the nonzero vector orthogonal to all the SoV co-vectors with the exception of $\langle S|$, i.e.

$$
\langle h_1, ..., h_N| S \rangle = \frac{\prod_{n=1}^N \delta_{h_n,0}}{N_S \tilde{V}(\xi_1^{(0)}, ..., \xi_N^{(0)})} \quad \forall \{h_1, ..., h_N\} \in \{0, 1\}^{\otimes N}, \tag{A.1}
$$

for some nonzero normalization $N_S$ and with

$$
\tilde{V}(x_1, ..., x_N) = \det_{1 \leq i,j \leq N} [x_i^{2j-1}] = \prod_{1 \leq k < j \leq N} (x_k^2 - x_j^2). \tag{A.2}
$$

Moreover, being the set of SoV co-vectors a basis, then $|S\rangle$ is uniquely defined by the above normalization.

Similarly, we can introduce the nonzero vector $|\bar{S}\rangle$ orthogonal to all the SoV co-vectors with the exception of $\langle 1, ..., 1|$, i.e.

$$
\langle h_1, ..., h_N| \bar{S} \rangle = \frac{\prod_{n=1}^N \delta_{h_n,1}}{N_S \tilde{V}(\xi_1^{(1)}, ..., \xi_N^{(1)})} \quad \forall \{h_1, ..., h_N\} \in \{0, 1\}^{\otimes N}, \tag{A.3}
$$

which also defines completely $|\bar{S}\rangle$.

**Theorem A.1.** Under the same conditions ensuring that the set of SoV co-vectors is a basis, then the following set of vectors:

$$
|h_1, ..., h_N\rangle = \prod_{a=1}^N \left[ \frac{T(\xi_a + \eta/2)}{K_a A_{\xi_a} \tilde{\zeta}_- (\eta/2 - \xi_a)} \right]^{h_a} |S\rangle \quad \forall \{h_1, ..., h_N\} \in \{0, 1\}^{\otimes N} \tag{A.4}
$$

forms an orthogonal basis to the left SoV basis:

$$
\langle h_1, ..., h_N|k_1, ..., k_N\rangle = \frac{\prod_{n=1}^N \delta_{h_n,k_n}}{N_S \tilde{V}(\xi_1^{h_1}, ..., \xi_N^{h_N})}. \tag{A.5}
$$

Let $t(\lambda)$ be a transfer matrix eigenvalue, $t(\lambda) \in \Sigma_T$, then the uniquely defined eigenvector
$|t\rangle$ and co-vectors $\langle t|$ admit the following SoV representations:

$$
|t\rangle = \sum_{h_1,\ldots,h_N=0}^1 \prod_{a=1}^N \left[ \frac{t(\xi_a - \eta/2)}{\mathcal{A}_{\xi_a,\xi_a} (\eta/2 - \xi_a)} \right]^{1-h_a} \hat{V} (\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)}) | h_1, \ldots, h_N \rangle,
$$

(A.6)

$$
\langle t| = \sum_{h_1,\ldots,h_N=0}^1 \prod_{a=1}^N \left[ \frac{t(\xi_a + \eta/2)}{\mathcal{A}_{\xi_a,\xi_a} (\eta/2 - \xi_a)} \right]^{h_a} \hat{V} (\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)}) \langle h_1, \ldots, h_N |,
$$

(A.7)

where we have set their normalization by imposing:

$$
\langle S|t\rangle = \langle t|S\rangle = 1/N_S.
$$

(A.8)

**Proof.** Let us start proving the orthogonality condition:

$$
\langle h_1, \ldots, h_N | k_1, \ldots, k_N \rangle = 0 \text{ for } \forall \{k_1, \ldots, k_N\} \neq \{h_1, \ldots, h_N\} \in \{0, 1\}^\otimes N.
$$

(A.9)

The proof is done by induction. For any vector $|k_1, \ldots, k_N\rangle$ let us denote $l = \sum_{n=1}^N k_n$. The property is obviously true for $l = 0$. Assuming that it is true for any vector $|k_1, \ldots, k_N\rangle$ with $\sum_{n=1}^N k_n = l$ for some $l \leq N - 1$ let us prove it holds for vectors $|k'_1, \ldots, k'_N\rangle$ with $\sum_{n=1}^N k'_n = l + 1$. To this aim we fix a vector $|k_1, \ldots, k_N\rangle$ with $\sum_{n=1}^N k_n = l$ and we denote by $\pi$ a permutation on the set $\{1, \ldots, N\}$ such that:

$$
k_{\pi(a)} = 1 \text{ for } a \leq l \quad \text{and } k_{\pi(a)} = 0 \text{ for } l < a,
$$

(A.10)

and then we compute:

$$
\langle h_1, \ldots, h_N | T (\xi_{\pi(l+1)}^{(0)}) | k_1, \ldots, k_N \rangle = k_a \mathcal{A}_{\xi_a,\xi_a} (-\xi_n^{(1)}) \langle h_1, \ldots, h_N | k'_1, \ldots, k'_N \rangle,
$$

(A.11)

where we have defined:

$$
k'_{\pi(a)} = k_{\pi(a)} \quad \forall a \in \{1, \ldots, N\} \setminus \{l + 1\} \quad \text{and } k'_{\pi(l+1)} = 1,
$$

(A.12)

for any $\{h_1, \ldots, h_N\} \neq \{k'_1, \ldots, k'_N\} \in \{0, 1\}^\otimes N$. There are two cases, the first case is $h_{\pi(l+1)} = 0$, then it holds:

$$
\langle h_1, \ldots, h_N | T (\xi_{\pi(l+1)}^{(0)}) | k_1, \ldots, k_N \rangle = \mathcal{A}_{\xi_a,\xi_a} (\xi_n^{(0)}) \langle h'_1, \ldots, h'_N | k'_1, \ldots, k'_N \rangle,
$$

(A.13)

where we have defined:

$$
h'_{\pi(a)} = h_{\pi(a)} \quad \forall a \in \{1, \ldots, N\} \setminus \{l + 1\} \quad \text{and } h'_{\pi(l+1)} = 1.
$$

(A.14)

Then from $\{h_1, \ldots, h_N\} \neq \{k'_1, \ldots, k'_N\} \in \{0, 1\}^\otimes N$ it follows also that $\{h'_1, \ldots, h'_N\} \neq \{k_1, \ldots, k_N\} \in \{0, 1\}^\otimes N$ and so the induction hypothesis implies that the rhs of (A.13) is zero and so we get:

$$
\langle h_1, \ldots, h_N | k'_1, \ldots, k'_N \rangle = 0.
$$

(A.15)

The second case is $h_{\pi(l+1)} = 1$, then we can use the following interpolation formula:
\[ T(\xi^{(0)}_{\pi(t+1)}) = t_{N+1} u_h(\xi^{(0)}_{\pi(t+1)}) + t(\eta/2) s_h(\xi^{(0)}_{\pi(t+1)}) + \sum_{a=1}^{N} r_{a,h}(\xi^{(0)}_{\pi(t+1)}) T(\zeta^{(0)}_a), \]  
(A.16)

from which \( \langle h_1, ..., h_N | T(\xi^{(0)}_{\pi(t+1)}) | k_1, ..., k_N \rangle \) reduces to the following sum:

\[
\left( t_{N+1} u_h(\xi^{(0)}_{\pi(t+1)}) + t(\eta/2) s_h(\xi^{(0)}_{\pi(t+1)}) \right) \langle h_1, ..., h_N | k_1, ..., k_N \rangle 
+ \sum_{a=1}^{N} r_{a,h}(\xi^{(0)}_{\pi(t+1)}) (A_{\xi^{(0)}}, (-\xi^{(1)}))^{1-h_{\pi(a)}} (A_{\xi^{(0)}})^{h_{\pi(a)}} \langle h_1, ..., h_N | k_1, ..., k_N \rangle,
\]

where we have defined:

\[ h_{\pi(j)}^{(a)} = h_{\pi(j)} \forall j \in \{1, ..., N\} \setminus \{a\} \text{ and } h_{\pi(a)}^{(a)} = 1 - h_{\pi(a)}. \]  
(A.17)

Let us now note that from \( h_{\pi(t+1)} = 1 \) it follows that \( \{h_1, ..., h_N\} \neq \{k_1, ..., k_N\} \) as \( k_{\pi(t+1)} = 0 \) by definition and similarly \( \{h_1^{(a)}, ..., h_N^{(a)}\} \neq \{k_1, ..., k_N\} \) being by definition \( h_{\pi(t+1)}^{(a)} = h_{\pi(t+1)} = 1 \) for any \( a \in \{1, ..., N\}\). Finally from \( \{h_1, ..., h_N\} \neq \{k_1', ..., k_N'\} \) with \( h_{\pi(t+1)} = k_{\pi(t+1)}' = 1 \) clearly it follows that \( \{h_1, ..., h_N\} \neq \{k_1, ..., k_N\} \). So by using the induction argument we get that all the terms in the above sum are equal to zero. So that also in the case \( h_{\pi(t+1)} = 1 \), we get that \( (A.15) \) is satisfied, and so it is satisfied for any \( \{h_1, ..., h_N\} \neq \{k_1', ..., k_N'\} \) which proves the induction of the orthogonality to \( l + 1 \). Indeed, by changing the permutation \( \pi \) we can both take for \( \pi(1), ..., \pi(l) \) any subset of cardinality \( l \) in \( \{1, ..., N\} \) and with \( \pi(l+1) \) any element in its complement \( \{1, ..., N\} \setminus \{\pi(1), ..., \pi(l)\} \).

We can compute now the left/right normalization, and to do this we just need to compute the following type of ratio:

\[
\frac{\langle h_1^{(a)}, ..., h_N^{(a)} | h_1^{(a)}, ..., h_N^{(a)} \rangle}{\langle h_1^{(a)}, ..., h_N^{(a)} | h_1^{(a)}, ..., h_N^{(a)} \rangle} = A_{\xi^{(0)}}, (-\xi^{(1)})^{1-h_{\pi(a)}} (A_{\xi^{(0)}})^{h_{\pi(a)}} \langle h_1^{(a)}, ..., h_N^{(a)} | h_1^{(a)}, ..., h_N^{(a)} \rangle
\]

(A.19)

with \( \tilde{h}_j^{(a)} = h_j^{(a)} \) for any \( j \in \{1, ..., N\} \setminus \{a\} \) while \( \tilde{h}_a^{(a)} = 0 \) and \( h_a^{(a)} = 1 \). We can use now once again the interpolation formula \( (2.63) \) computed in \( \lambda = \xi^{(1)}_a \) which by the orthogonality condition produces only one non-zero term, the one associate to \( T(\zeta^{(0)}_{\pi_a}) \), i.e. it holds:

\[
\frac{\langle h_1^{(a)}, ..., h_N^{(a)} | h_1^{(a)}, ..., h_N^{(a)} \rangle}{\langle h_1^{(a)}, ..., h_N^{(a)} | h_1^{(a)}, ..., h_N^{(a)} \rangle} = \frac{1}{k_a r_a, h(\xi^{(1)}_a)} = \prod_{b \neq a, b=1}^{N} \frac{(\xi^{(0)}_a)^2 - (\xi^{(b)}_a)^2}{(\xi^{(1)}_a)^2 - (\xi^{(b)}_a)^2}.
\]

(A.20)

It is now standard \[126, 127\] to get the Vandermonde determinant for the normalization once we use the above result.

Let us note that the set of SoV co-vectors and vectors being both basis, it follows that for any transfer matrix eigenstates \( |\ell\rangle \) and \( |t\rangle \) there exist at least a \( \{v_1, ..., r_N\} \in \{0, 1\}^{\geq N} \) and a \( \{s_1, ..., s_N\} \in \{0, 1\}^{\geq N} \) such that:

https://doi.org/10.1088/1742-5468/ab357a
which together with the identities:
\[ \langle h_1, ..., h_N | t \rangle \propto \langle S | t \rangle, \quad \langle t | h_1, ..., h_N \rangle \propto \langle t | \bar{S} \rangle \quad \forall \{h_1, ..., h_N\} \in \{0, 1\}^\otimes N, \] (A.22)
imply that:
\[ \langle S | t \rangle \neq 0, \quad \langle t | \bar{S} \rangle \neq 0, \] (A.23)
so that we are free to fix the normalization of the eigenstates \(|t\rangle\) and \(\langle t|\) by (A.8).

Finally, the representations for these left and right transfer matrix eigenvectors follow from the use of the SoV decomposition of the identity:
\[ I = N_S \sum_{h_1, ..., h_N = 0} \hat{V}(\xi_1^{(h_1)}, ..., \xi_N^{(h_N)})|h_1, ..., h_N\rangle \langle h_1, ..., h_N|. \] (A.24)

**Corollary A.1.** Under the same conditions ensuring that the set of SoV co-vectors is a basis, then the vectors of the right SoV basis admit also the following representations:
\[ |h_1, ..., h_N\rangle = \prod_{a=1}^N \left( \frac{K_a T(\xi_a - \eta/2)}{A_{\tilde{\xi}_+}, \tilde{\xi}^{-}(\xi_a + \eta/2)} \right)^{1-h_a} |\bar{S}\rangle \quad \forall \{h_1, ..., h_N\} \in \{0, 1\}^\otimes N, \] (A.25)
as well as for any element of the spectrum of \(T(\lambda, \{\xi\})\) the unique associated eigenvector \(|t\rangle\) admit the following SoV representations:
\[ \langle t| = N_t \sum_{h_1, ..., h_N = 0} \prod_{a=1}^N \left( \frac{K_a T(\xi_a + \eta/2)K_a}{A_{\tilde{\xi}_+}, \tilde{\xi}^{-}(\xi_a - \eta/2)} \right)^{1-h_a} \hat{V}(\xi_1^{(h_1)}, ..., \xi_N^{(h_N)})|h_1, ..., h_N|, \] (A.26)
where we have defined:
\[ N_t = \langle t|\bar{S}\rangle = \frac{1}{N_S} \prod_{a=1}^N A_{\tilde{\xi}_+}, \tilde{\xi}^{-}(\xi_a + \eta/2)K_a t(\xi_a - \eta/2) \neq 0, \] (A.27)

once we fix the normalization by (A.8).

**Proof.** Taking into account the chosen normalizations clearly it holds:
\[ |\bar{S}\rangle = |h_1 = 1, ..., h_N = 1\rangle = \prod_{a=1}^N \frac{T(\xi_a + \eta/2)}{K_a A_{\tilde{\xi}_+}, \tilde{\xi}^{-}(\eta/2 - \xi_a)} |S\rangle, \] (A.28)
so that:
On separation of variables for reflection algebras

\[
\prod_{a=1}^{N} \frac{K_a T(\xi_a - \eta/2)}{A_{\xi_a,\xi_n}(\xi_n + \eta/2)} \prod_{a=1}^{N} \frac{T(\xi_a + \eta/2)}{K_a A_{\xi_a,\xi_n}(\eta/2 - \xi_a)} |S\rangle = \prod_{a=1}^{N} \frac{K_a T(\xi_a - \eta/2)}{A_{\xi_a,\xi_n}(\xi_n + \eta/2)} T(\xi_a + \eta/2) K_a A_{\xi_a,\xi_n}(\eta/2 - \xi_a) |S\rangle
\]

by the fusion identities (2.53). From this representation of the right SoV vectors and from the original one in (A.4), it follows that:

\[
\langle t|\bar{S}\rangle \equiv \langle t|1, \ldots, 1 \rangle = \prod_{a=1}^{N} \frac{t(\xi_a + \eta/2)}{K_a A_{\xi_a,\xi_n}(\eta/2 - \xi_a)} t|S\rangle,
\]

from which our result follows. □

A.2. Algebraic Bethe Ansatz form of separate states

Let us rewrite the left and right transfer matrix eigenstates in terms of the Q-functions. The following corollary holds:

**Corollary A.2.** Under the same conditions ensuring that the set of SoV co-vectors is a basis, then for any element of the spectrum of \(T(\lambda)\) the unique associated eigenvector \(|t\rangle\) admit the following SoV representations:

\[
|t\rangle = \sum_{h \in \{0,1\}^N} \prod_{n=1}^{N} Q_t(\xi_n^{(h_n)}) \hat{Q}(\xi_n^{(h_n)}) \hat{\xi}_n(\xi_n^{(h_n)}) |h_1, \ldots, h_N\rangle,
\]

\[
\langle t| = \sum_{h \in \{0,1\}^N} \prod_{n=1}^{N} \left[ \left( \frac{\xi_n - \eta}{\xi_n + \eta} A_{\xi_n,\xi_n}(\xi_n^{(0)}) \right)^{h_n} \right] Q_t(\xi_n^{(h_n)}) \hat{Q}(\xi_n^{(h_n)}) \hat{\xi}_n(\xi_n^{(h_n)}) \langle h_1, \ldots, h_N|.
\]

Let us remark that these representations for the left and right transfer matrix eigenstates formally coincide with that obtained in the schema of the generalized Sklyanin’s SoV approach [134] even when this last approach does not apply and without any requirement on the form of the co-vector \(\langle S|\). One can introduce the following class of left and right separate states:

\[
\langle \alpha| = \sum_{h_1, \ldots, h_N=0}^{1} \prod_{a=1}^{N} \alpha(\xi_a^{(h_a)}) \hat{Q}(\xi_a^{(h_a)}) \hat{\xi}_a(\xi_a^{(h_a)}) |h_1, \ldots, h_N|,
\]

\[https://doi.org/10.1088/1742-5468/ab357a\]
\[ \beta = \sum_{h_1, \ldots, h_N=0}^1 \prod_{a=1}^N \left( \frac{\xi_n - \eta A\xi \xi_n^{(0)}}{\xi_n + \eta A\xi \xi_n^{(-1)}} \right)^{h_n} \beta(h_a) \hat{V}(\xi^{(h_1)}_1, \ldots, \xi^{(h_N)}_N) |h_1, \ldots, h_N\rangle, \quad (A.34) \]

where \( \alpha(\lambda) \) and \( \beta(\lambda) \) are generic functions. It is then clear by the previous corollary that the left and right transfer matrix eigenstates are special elements in these classes.

Let us now introduce the one parameter family of commuting operators by:

\[ B(\lambda) = N_S \sum_{h_1, \ldots, h_N=0}^1 b_{h_1, \ldots, h_N}(\lambda) \hat{V}(\xi^{(h_1)}_1, \ldots, \xi^{(h_N)}_N) |h_1, \ldots, h_N\rangle |h_1, \ldots, h_N\rangle, \quad (A.35) \]

where we have defined:

\[ b_{h_1, \ldots, h_N}(\lambda) = \prod_{a=1}^N (\lambda^2 - (\xi^{(h_a)}_a)^2). \quad (A.36) \]

Clearly, if the two boundary matrices are non simultaneously diagonalizable and we take the special choice \( \langle S | = \langle 0 | W^{-1}_K \) then it holds:

\[ \hat{B}_-(\lambda) = (-1)^N \frac{\lambda - \eta/2}{\zeta_-} B(\lambda). \quad (A.37) \]

Let us assume that \( \alpha(\lambda) \) be the following polynomial:

\[ \alpha(\lambda) = \prod_{k=1}^R (\lambda^2 - \alpha_k^2), \quad (A.38) \]

then the left and right separate states \( \langle \alpha | \) and \( | \alpha \rangle \) associated admits the following algebraic Bethe Ansatz form:

\[ \langle \alpha | = (-1)^{RN} \prod_{k=1}^R \mathbb{B}(\alpha_k), \quad | \alpha \rangle = (-1)^{RN} \prod_{k=1}^R \mathbb{B}(\alpha_k) | 1 \rangle, \quad (A.39) \]

where we have defined \( \langle 1 | \) and \( | 1 \rangle \) to be the separate co-vector and vector associated to the identity polynomial:

\[ \langle 1 | = \sum_{h_1, \ldots, h_N=0}^1 \hat{V}(\xi^{(h_1)}_1, \ldots, \xi^{(h_N)}_N) |h_1, \ldots, h_N\rangle, \quad (A.40) \]

\[ | 1 \rangle = \sum_{h_1, \ldots, h_N=0}^1 \prod_{a=1}^N \left( \frac{\xi_n - \eta A\xi \xi_n^{(0)}}{\xi_n + \eta A\xi \xi_n^{(1)}} \right)^{h_n} \hat{V}(\xi^{(h_1)}_1, \ldots, \xi^{(h_N)}_N) |h_1, \ldots, h_N\rangle. \quad (A.41) \]

### A.3. Scalar product of separate states

Let us consider a couple of separate states \( \langle \alpha | \) and \( | \beta \rangle \), then it holds:
\[ \langle \alpha | \beta \rangle = \sum_{h_1, \ldots, h_N=0}^1 \prod_{a=1}^N \left( \frac{\xi_n - \eta}{\xi_n + \eta} A_{\xi_n, \xi_n} (\xi_n^{(0)}) \right) h_n \alpha(\xi_n^{(h_n)}) \beta(\xi_n^{(h_n)}) \frac{\hat{V}(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)})}{N_S} \]  
(A.42)

\[ = \sum_{h_1, \ldots, h_N=0}^1 \prod_{a=1}^N (g_n g_n^{(h_n)}) \alpha(\xi_n^{(h_n)}) \beta(\xi_n^{(h_n)}) \frac{\hat{V}(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)})}{N_S} , \]  
(A.43)

where:

\[ g_n \equiv g_{\xi_n, \xi_n} (\xi_n) = \frac{(\xi_n + \bar{\xi}_n)(\xi_n + \bar{\xi}_n)}{(\xi_n - \xi_n)(\xi_n - \xi_n)} , \]  
(A.44)

and

\[ f_n \equiv f(\xi_n, \{\xi\}) = -\prod_{a=1}^N \frac{(\xi_n - \xi_n + \eta)(\xi_n + \xi_n + \eta)}{(\xi_n - \xi_n - \eta)(\xi_n + \xi_n - \eta)} \]  

\[ = -\prod_{a=1}^N \frac{(\xi_n^{(0)})^2 - (\xi_n^{(1)})^2}{(\xi_n^{(1)})^2 - (\xi_n^{(0)})^2} \frac{[\xi_n^{(0)}]^2 - [\xi_n^{(1)}]^2}{[\xi_n^{(1)}]^2 - [\xi_n^{(0)}]^2} \]  
(A.45)

So that in our general SoV approach the scalar product of separate states admits the same representations which hold for the separate states in the generalized Sklyanin’s approach, as one can directly infer comparing (A.43) with the formula (4.12) of [94]. Moreover, our current result is not limited to the cases of non-commuting boundary matrices, where the generalized Sklyanin’s approach applies. In particular, setting the normalization as:

\[ N_S = \frac{\hat{V}(\xi_1, \ldots, \xi_N)}{\hat{V}(\xi_1^{(1)}, \ldots, \xi_N^{(1)})} \prod_{n=1}^N \frac{\xi_n}{\xi_n - \bar{\xi}_n} , \]  
(A.46)

we obtain

\[ \langle \alpha | \beta \rangle = \prod_{n=1}^N \frac{\xi_n - \bar{\xi}_n}{\xi_n} \sum_{h_1, \ldots, h_N=0}^1 \prod_{a=1}^N (-g_n)^{h_n} \alpha(\xi_n^{(h_n)}) \beta(\xi_n^{(h_n)}) \frac{\hat{V}(\xi_1^{(1-h_1)}, \ldots, \xi_N^{(1-h_N)})}{\hat{V}(\xi_1, \ldots, \xi_N)} , \]  
(A.47)

which coincides with the formula (4.13) of [94], up to the non required prefactor $1/\bar{b}_-$. This means that we can use in our more general SoV framework the manipulation of these scalar product formulae to obtain Izergin and Slavnov type scalar products [45, 145–147] and the generalized Gaudin type formula as done in [94]. The same statements apply as well to the trigonometric case in comparison with the results in [96].

**ORCID iDs**

J M Maillet  
[https://orcid.org/0000-0002-4396-8631](https://orcid.org/0000-0002-4396-8631)

[https://doi.org/10.1088/1742-5468/ab357a](https://doi.org/10.1088/1742-5468/ab357a)
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