ISOMORPHY CLASSES OF INVOLUTIONS OF SO\((n, k, \beta), \ n > 2\)

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Abstract. A first characterization of the isomorphism classes of \(k\)-involutions for any reductive algebraic group defined over a perfect field was given in [Helm2000] using 3 invariants. In [HWD04] a full classification of all involutions on \(SL(n, k)\) for \(k\) algebraically closed, the real numbers, the \(p\)-adic numbers or a finite field was provided. In this paper, we build on these results to develop a detailed characterization of the involutions of \(SO(n, k, \beta)\), where \(\beta\) is any non-degenerate symmetric bilinear form and \(k\) is any field not of characteristic 2. We use these results to classify the isomorphism classes of involutions of \(SO(n, k, \beta)\) for some bilinear forms and some fields \(k\).

1. Introduction

Let \(G\) be a connected reductive algebraic group defined over a field \(k\) of characteristic not 2, \(\vartheta\) an involution of \(G\) defined over \(k\), \(H\) a \(k\)-open subgroup of the fixed point group of \(\vartheta\) and \(G_k\) (resp. \(H_k\)) the set of \(k\)-rational points of \(G\) (resp. \(H\)). The variety \(G_k/H_k\) is called a symmetric \(k\)-variety. For \(k = \mathbb{R}\) these symmetric \(k\)-varieties are also called real reductive symmetric spaces. These varieties occur in many problems in representation theory, geometry and singularity theory. To study these symmetric \(k\)-varieties one needs first a classification of the related \(k\)-involutions. A characterization of the isomorphism classes of the \(k\)-involutions was given in [Helm2000] essentially using the following 3 invariants:

(i) classification of admissible \((\Gamma, \vartheta)\)-indices,
(ii) classification of the \(G_k\)-isomorphism classes of \(k\)-involutions of the \(k\)-anisotropic kernel of \(G\).
(iii) classification of the \(G_k\)-isomorphism classes of \(k\)-inner elements of \(G\).

For more details, see [Helm2000]. The admissible \((\Gamma, \vartheta)\)-indices determine most of the fine structure of the symmetric \(k\)-varieties and a classification of these was included in [Helm2000] as well. For \(k\) algebraically closed or \(k\) the real numbers the full classification can be found in [Hel88]. For other fields a full classification of the remaining two invariants is still lacking. In particular the case of symmetric \(k\)-varieties over the \(p\)-adic numbers is of interest. We note that the above characterization was only proven for \(k\) a perfect field.

In [HWD04] a full characterization of the isomorphism classes of \(k\)-involutions was given in the case that \(G = SL(n, k)\) which does not depend on any of the results in [Helm2000]. It was also shown how one may construct an outer-involution from a given non-degenerate symmetric or skew-symmetric bilinear form \(\beta\) of \(k^n\). Using this characterization the possible isomorphism classes for \(k\) algebraically closed, the real numbers, the \(p\)-adic numbers and finite fields were classified.

In this paper we build upon the results of [HWD04] to give a characterization of involutions of \(SO(n, k, \beta)\), the special orthogonal group with respect to a symmetric bilinear form \(\beta\) on \(k^n\). We first show that if an automorphism \(\vartheta = \text{Inn}_A\) where \(A \in GL(n, \mathbb{R})\) leaves \(SO(n, k, \beta)\) invariant, then we can assume \(A\) in \(SO(n, k[\sqrt{\alpha}], \beta)\) where \(k[\sqrt{\alpha}]\) is a quadratic extension of \(k\). Thus, to classify the involutions of \(SO(n, k, \beta)\) it suffices to determine which \(A\) in \(SO(n, k[\sqrt{\alpha}], \beta)\) induce involutions of \(SO(n, k, \beta)\) and to then determine the isomorphy classes of these involutions over \(O(n, k, \beta)\). Using these results, we give a full classification (mostly for the standard orthogonal group) of involutions of \(SO(n, k, \beta)\) when
$k$ is algebraically closed, the real numbers, or a finite field of order odd $p > 3$. We provide a partial classification when $p = 3$.

2. Preliminaries

Our basic reference for reductive groups will be the papers of Borel and Tits [BT65], [BT72] and also the books of Borel [Bor91], Humphreys [Hum75] and Springer [Spr81]. We shall follow their notations

Our main reference for results regarding involutions of $\text{SL}(n, k)$ will be [HWD04]. Let $k$ be a field of characteristic not 2, $k$ is algebraically closed, the real numbers, or a finite field of order odd $2$. ROBERT W. BENIM, CHRISTOPHER E. DOMETRIUS, ALOYSIUS G. HELMINCK, AND LING WU

Theorem 2.2.

Let $G$ be an algebraic group defined over a field $k$, and let $G_k$ be the set of $k$-rational points. Let $\text{Aut}(G_k)$ denote the set of all automorphisms of $G_k$. For $A \in \text{GL}(n, k)$ let $\text{Inn}_A$ denote the inner automorphism defined by $\text{Inn}_A(X) = A^{-1}XA$ for all $X \in \text{GL}(n, k)$. Let $\text{Inn}_k(G_k) = \{ \text{Inn}_A \mid A \in G_k \}$ denote the set of all inner automorphisms of $G_k$ and let $\text{Inn}(G_k)$ denote the set of automorphisms $\text{Inn}_A$ of $G_k$ with $A \in G$ such that $\text{Inn}_G(G_k) = G_k$. If $\text{Inn}_A$ is order 2, then $\text{Inn}_A$ is an inner involution of $G_k$. We say that $\vartheta$ and $\tau$ in $\text{Aut}(G_k)$ are $\text{Inn}(G_k)$-isomorphic if there is a $\varphi$ in $\text{Inn}(G_k)$ such that $\tau = \varphi^{-1} \vartheta \varphi$. Equivalently, we say that $\tau$ and $\vartheta$ are in the same isomorphy class.

Let $\text{Inn}_k(G_k)$ denote the set of automorphisms of $\text{G}$ and let $\text{Inn}(G_k)$ be the set of inner involution of $G_k$. We shall follow their notations.

$\text{Inn}_k(G_k)$ denotes the set of all $k$-automorphisms of $G_k$. We will sometimes use $I$ instead of $I_n$ when the dimension of the identity matrix is clear.

We recall some important definitions and theorems from [HWD04].

Definition 2.1. Let $G$ be an algebraic group defined over a field $k$, and let $G_k$ be the set of $k$-rational points. Let $\text{Aut}(G_k)$ denote the set of all automorphisms of $G_k$. For $A \in \text{GL}(n, k)$ let $\text{Inn}_A$ denote the inner automorphism defined by $\text{Inn}_A(X) = A^{-1}XA$ for all $X \in \text{GL}(n, k)$. Let $\text{Inn}_k(G_k) = \{ \text{Inn}_A \mid A \in G_k \}$ denote the set of all inner automorphisms of $G_k$ and let $\text{Inn}(G_k)$ denote the set of automorphisms $\text{Inn}_A$ of $G_k$ with $A \in G$ such that $\text{Inn}_G(G_k) = G_k$. If $\text{Inn}_A$ is order 2, then $\text{Inn}_A$ is an inner involution of $G_k$. We say that $\vartheta$ and $\tau$ in $\text{Aut}(G_k)$ are $\text{Inn}(G_k)$-isomorphic if there is a $\varphi$ in $\text{Inn}(G_k)$ such that $\tau = \varphi^{-1} \vartheta \varphi$. Equivalently, we say that $\tau$ and $\vartheta$ are in the same isomorphy class.

In [HWD04], the isomorphy classes of the inner-involutions of $\text{SL}(n, k)$ were classified, and they are as follows:

Theorem 2.2. Suppose the involution $\vartheta \in \text{Aut}(\text{SL}(n, k))$ is of inner type. Then up to isomorphism $\vartheta$ is one of the following:

(i) $\text{Inn}_Y | G$, where $Y = I_{n-i,i} \in \text{GL}(n, k)$ where $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \}$ where

\[
I_{n-i,i} = \begin{pmatrix} I_{n-i} & 0 \\ 0 & I_i \end{pmatrix}
\]

(ii) $\text{Inn}_Y | G$, where $Y = L_{x, x} \in \text{GL}(n, k)$ where $x \in k*/k^{*2}$, $x \not\equiv 1 \mod k^{*2}$ and

\[
L_{n, x} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ x & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & x & 0 \end{pmatrix}
\]

Note that (ii) can only occur when $n$ is even.

For the purposes of this paper, we will use matrices of the form $L_{x, x}$ (and there multiples) rather than $L_{x, x}$. Either of these serves as a member of the isomorphy class listed in the previous theorem. We will eventually see that all of the isomorphy classes of $\text{SO}(n, k, \beta)$ are just isomorphy classes of $\text{SL}(n, k)$ that have been divided into multiple isomorphy classes.
We now begin to define the notion of a special orthogonal group. Let \( M \) be the matrix of a non-degenerate bilinear form \( \beta \) over \( k^n \) with respect to a basis \( \{e_1, \ldots, e_n\} \) of \( V \). We will say that \( M \) is the matrix of \( \beta \) if the basis \( \{e_1, \ldots, e_n\} \) is the standard basis of \( k^n \).

The typical notation for the orthogonal group is \( O(n, k) \), which is the group

\[
O(n, k) = \{ A \in M(n, k) \mid (Ax)^T A y = x^T y \}.
\]

This group consists of the matrices which fix the standard dot product. This can be generalized to any non-degenerate symmetric bilinear \( \beta \), which will yield the group

\[
O(n, k, \beta) = \{ A \in M(n, k) \mid \beta(Ax, Ay) = \beta(x, y) \}.
\]

If \( M \) is the matrix of \( \beta \) with respect to the standard basis, then we can equivalently say

\[
O(n, k, \beta) = \{ A \in M(n, k) \mid A^T M A = M \}.
\]

It is clear from this definition that all matrices in \( O(n, k, \beta) \) have determinant 1 or -1. We also define

\[
SO(n, k, \beta) = O(n, k, \beta) \cap SL(n, k).
\]

We note a couple of important facts, the first of which will be used repeatedly throughout this paper.

(i) Symmetric matrices are congruent to diagonal matrices, where the entries are representatives of \( k^*/(k^*)^2 \).

(ii) If \( \beta_1 \) and \( \beta_2 \) correspond to \( M_1 \) and \( M_2 \), then \( SO(n, k, \beta_1) \) and \( SO(n, k, \beta_2) \) are isomorphic via

\[
\Phi : SO(n, k, \beta_1) \to SO(n, k, \beta_2) : X \to Q^{-1} X Q
\]

for some \( Q \in GL(n, k) \) if and only if \( Q^T M_1 Q = M_2 \) (\( M_1 \) and \( M_2 \) are congruent via \( Q \)).

So, we will assume that \( \beta \) is such that \( M \) is diagonal. Then, to classify the involutions of an orthogonal group where \( M \) is not diagonal, one can apply the classification that will follow by simply using the isomorphism given above.

Lastly, two vectors \( x, y \in k^n \) are said to be orthogonal with respect to the bilinear form \( \beta \) if \( \beta(x, y) = 0 \). We will eventually see that orthogonal vectors play an important role in the structure of involutions of \( SO(n, k, \beta) \).

3. Automorphisms of \( SO(n, k, \beta) \)

It follows from a proposition on page 191 of [Bor91] that the outer automorphism group

\[
Out(SO(n, k, \beta)) = Aut(SO(n, k, \beta))/\text{Inn}(SO(n, k, \beta))
\]

must be a subgroup of the diagram automorphisms of the associated Dynkin diagram. If \( n = 2m + 1 \) and \( m \geq 2 \), then this Dynkin diagram is \( B_m \) which has only the trivial diagram automorphism. Thus, there are no outer automorphisms of \( SO(n, k, \beta) \) when \( n \) is odd. If \( n = 2m \) and \( m \geq 4 \), then this Dynkin diagram is \( D_m \). The group of automorphisms of this Dynkin diagram is \( \mathbb{Z}_2 \) when \( m > 4 \), and is \( S_6 \) when \( m = 4 \), because of triality. But, it can be shown that the order 3 outer-automorphisms do not lift from \( D_4 \) to the orthogonal group. For the details, see Chapter 30 of [Bump2013]. So, when \( n \) is even, \( Out(SO(n, k, \beta)) = \mathbb{Z}_2 \). We will see that the outer automorphisms are of the form \( \text{Inn}_A \) where \( A \in O(n, k, \beta) \) and \( \det(A) = -1 \). When \( k \) is not algebraically closed, then all automorphisms of \( SO(n, k, \beta) \) will still be of the form \( \text{Inn}_A \) for some \( A \in O(n, k, \beta) \) since \( \text{Inn}_A \) must also be an automorphism of \( SO(n, k, \beta) \). Thus, the classifications and characterizations that follow in this paper consider all automorphisms and involutions of \( SO(n, k, \beta) \), assuming that \( n \) is sufficiently large.

We now examine which automorphisms will act as the identity on \( SO(n, k, \beta) \). This will prove to be useful when we classify matrix representatives for automorphisms.

**Lemma 3.1.** Assume \( n > 2 \). Let \( A \in GL(n, k) \). If \( \text{Inn}_A \) is the identity on \( SO(n, k, \beta) \), then \( A \) is a diagonal matrix.
Proof. Suppose $A$ is such that $\text{Inn}_A$ is the identity on $\SO(n,k,\beta)$. For $1 \leq r < s \leq n$, let $X_{rs}$ be the diagonal matrix with all 1’s, except in the in the $r$th and $s$th diagonal entries, where instead there are -1’s. This matrix always lies in $\SO(n,k,\beta)$. So, we must have $AX_{rs} = X_{rs}A$. On the left side, the matrix is the same as $A$, but with the $r$th and $s$th columns negated. On the right side, the matrix is the same as $A$, but with the $r$th and $s$th rows negated. So, all entries of $A$ on these rows and columns which aren’t in the $(r,r)$, $(r,s)$, $(s,r)$ or $(s,s)$ components must be equal to 0, since this is the only number which equals its negative. To see that the $(r,s)$ and $(s,r)$ components of $A$ must also equal 0, we can repeat this process for $X_{rs}$, where $t$ is distinct from both $r$ and $s$. (Note that this is where we use the fact that $n > 2$.) Thus, all off-diagonal elements of $A$ are 0, which means $A$ is diagonal. □

We want to be able to say more about the matrix $A$ when $\text{Inn}_A$ acts as the identity. It turns out that if we make the following assumption on the orthogonal group $\SO(n,k,\beta)$, then we can show that $A$ is a multiple of the identity.

**Definition 3.2.** Let $k$ be a field and suppose $\beta$ is a bilinear form on $k^n$ such that $M = (\beta(\cdot,\cdot))$ is diagonal, with diagonal entries $m_1, \ldots, m_n$, which are representatives of $k^*/(k^*)^2$. If for each $m_i, m_j \in k^*/(k^*)^2$, $x^2 + \frac{m_i}{m_j}y^2 = 1$ has a solution $(x,y)$ such that $y \neq 0$, then we call $\SO(n,k,\beta)$ a friendly orthogonal group.

With this new terminology in mind, we get the following result.

**Lemma 3.3.** Assume $n > 2$. Suppose $\SO(n,k,\beta)$ is a friendly orthogonal group. Let $A \in \GL(n,\overline{k})$. Then, $\text{Inn}_A$ is the identity on $\SO(n,k,\beta)$ if and only if $A = \alpha I$ for some $\alpha \in \overline{k}^*$.

Proof. We know from the previous lemma that $A$ is diagonal. Let $a_i$ represent the $i$th diagonal entry of $A$. Recall that we are assuming that $M$ is diagonal. Label $M$’s $i$th diagonal as $a_i$. Then, there exists $a,b \in k$ where $b \neq 0$ such that $a^2 + \frac{m_i}{m_j}b^2 = 1$. For $1 \leq i < j \leq n$, let

$$Y_{ij} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & & & \\
\vdots & & \ddots & & \\
1 & a & 0 & \cdots & b \\
0 & 1 & & & \\
\vdots & & \ddots & & \\
0 & 0 & 1 & & \\
-\frac{m_i}{m_j}b & 0 & \cdots & 0 & a \\
\vdots & & & \ddots & 0 \\
0 & \cdots & & \cdots & 0 & 1
\end{pmatrix},$$

where the noteworthy entries occur in the $i$th and $j$th rows and columns. It is a simple calculation to show that $Y_{ij}^TMY_{ij} = M$, and that $\det(Y_{ij}) = 1$. So, $Y_{ij} \in \SO(n,k,\beta)$. Then, we know that $AY_{ij} = Y_{ij}A$. By comparing both sides of this equality and inspecting the $(i,j)$ entry, we see that $ba_j = ba_i$. Since we are assuming that $b \neq 0$, then it follows that $a_i = a_j$. Since we can repeat this for all $i$ and $j$, then it is clear that $A$ is a multiple of the identity. □

This result is only useful if we can show that $\SO(n,k,\beta)$ is commonly a friendly orthogonal group. In the following theorem, we see that most $\SO(n,k,\beta)$ are friendly.

**Theorem 3.4.**

(i) If $\text{Char}(k) \neq 2,3$, then $\SO(n,k,\beta)$ is a friendly orthogonal group.
Definition 3.6. Fix a bilinear form \(X\). We can rewrite this as
\[
\text{If we transpose both sides, then we see that}
\]
\[
\text{By the previous lemma, it follows that (}
\]
Lemma 3.7. Assume \(A \in GL(n, k)\), then \(\text{Inn}_A(\text{SO}(n, k, \beta)) \subseteq \text{SO}(n, k, \beta)\) if and only if \(A\) is \(\alpha\)-orthogonal and \(A = p\bar{A}\) where \(p, \alpha \in \overline{k}\) and
\[
(i) \quad \bar{A} \in \text{SO}(n, \overline{k}, \beta) \text{ if } n \text{ odd}, \text{ or}
\]
\[
(ii) \quad \bar{A} \in \text{O}(n, \overline{k}, \beta) \text{ if } n \text{ is even}.
\]
Example 3.5. Suppose \(k = \mathbb{F}_3\) and that \(\beta\) is such that \(M = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}\). Then, \(\text{SO}(n, \mathbb{F}_3, \beta)\) is not a friendly orthogonal group, because there is no solution to \(x^2 - y^2 = 1\) where \(y \neq 0\).

For the remainder of the paper, we will assume that all orthogonal groups are friendly.

Definition 3.6. Fix a bilinear form \(\beta\) with matrix \(M\). If \(A \in GL(n, k)\) is a matrix such \(M^{-1}ATMA = \alpha I_n\), then we call \(A\) \(\alpha\)-orthogonal.

Note that orthogonal matrices are 1-orthogonal.

We now have the following preliminary result that characterizes automorphisms of \(\text{SO}(n, k, \beta)\).

Lemma 3.7. Assume \(n > 2\). If \(A \in GL(n, k)\), then \(\text{Inn}_A(\text{SO}(n, k, \beta)) \subseteq \text{SO}(n, k, \beta)\) if and only if \(A\) is \(\alpha\)-orthogonal and \(A = p\bar{A}\) where \(p, \alpha \in \overline{k}\) and
\[
(i) \quad \bar{A} \in \text{SO}(n, \overline{k}, \beta) \text{ if } n \text{ odd}, \text{ or}
\]
\[
(ii) \quad \bar{A} \in \text{O}(n, \overline{k}, \beta) \text{ if } n \text{ is even}.
\]
Proof. Suppose \(A \in GL(n, k)\) and \(\text{Inn}_A(\text{SO}(n, k, \beta)) \subseteq \text{SO}(n, k, \beta)\). Choose \(X \in \text{SO}(n, k, \beta)\). Then, \(A^{-1}XA \in \text{SO}(n, \overline{k}, \beta)\). So,
\[
I = (A^{-1}XA)^{-1}A^{-1}XA \\
= M^{-1}(A^{-1}XA)^TMA^{-1}X \\
= M^{-1}AT(X^{-1})^TMA^{-1}X.
\]
This implies that
\[
A^{-1}X^{-1} = M^{-1}ATX^{-1}(A^{-1})^TMA^{-1},
\]
which means
\[
X^{-1} = AM^{-1}ATX^{-1}(A^{-1})^TMA^{-1}.
\]
We can rewrite this as
\[
M^{-1}X^TAM = AM^{-1}ATX^{-1}(A^{-1})^TMA^{-1}.
\]
If we transpose both sides, then we see that
\[
M^TXM^{-1} = (A^{-1})^TMA^{-1}XAM^{-1}AT.
\]
Solving for the \(X\) term on the left, we get that
\[
X = M^{-1}(A^{-1})^TMA^{-1}XAM^{-1}ATM \\
= (AM^{-1}ATM)^{-1}X(AM^{-1}ATM).
\]
By the previous lemma, it follows that \((AM^{-1}ATM) = \alpha I\) for some \(\alpha \in \overline{k}\). Thus, \(A\) is \(\alpha\)-orthogonal. Let \(p = \sqrt[3]{\alpha}\). Then let \(\bar{A} = \frac{1}{p}A\). It follows that \(M^{-1}ATM\bar{A} = \frac{1}{p^2}M^{-1}ATM\bar{A} = \frac{1}{\alpha}(\alpha I) = I\), which shows
that $\tilde{A}$ is orthogonal. That is, $\tilde{A} \in O(n, \overline{k}, \beta)$. If $n$ is odd and det($\tilde{A}$) = $-1$, then we can replace $\tilde{A}$ with $-\tilde{A}$, and instead have a matrix inside $SO(n, \overline{k}, \beta)$.

Since the converse is clear, we have proven the statement. 

In the following theorem which completes the characterization of automorphisms on $SO(n, k, \beta)$, we see that we do not need the algebraic closure of the field $k$, but either the field itself, or a quadratic extension.

**Theorem 3.8.** Assume $n > 2$.

(i) If $n$ is odd and $A$ is in $O(n, \overline{k}, \beta)$, then Inn$_A$ keeps $SO(n, k, \beta)$ invariant if and only if we can choose $A \in SO(n, k, \beta)$.

(ii) If $n$ is even and $A$ is in $O(n, \overline{k}, \beta)$, then Inn$_A$ keeps $SO(n, k, \beta)$ invariant if and only if there exists $p \in \overline{k}$ and $B \in GL(n, k)$ such that $B = pA$ and $B$ is $\alpha$-orthogonal for some $\alpha \in \overline{k}$.

Further, we can show $A \in O(n, k[\sqrt{\alpha}], \beta)$ where each entry of $A$ is a $k$-multiple of $\sqrt{\alpha}$.

**Proof.** Let $n > 2$ be arbitrary, and suppose $A$ is in $O(n, \overline{k}, \beta)$ such that Inn$_A$ keeps $SO(n, k, \beta)$ invariant. Let $a_{ij}$ be the $(i, j)$ entry of $A$. Let $X_{rs}$ be the diagonal matrix with all entries $-1$, except for the $(r, r)$ and $(s, s)$ entries, which are $1$. Since $M$ is diagonal, it is clear that $X_{rs} \in O(n, k, \beta)$. If $n$ is even, then $X \in SO(n, k, \beta)$. If $n$ is odd, then $-X \in SO(n, k, \beta)$. So, we know that Inn$_A(X_{rs})$ or $-\text{Inn}_A(X_{rs})$ must lie in $SO(n, k, \beta)$. It is also clear that Inn$_A(I)$ is in $SO(n, k, \beta)$. So, both Inn$_A(X_{rs})$ and Inn$_A(I)$ have entries in $k$. Let us examine the entries of Inn$_A(X_{rs})$:

$$\text{Inn}_A(X_{rs}) = A^{-1}X_{rs}A = (M^{-1}A^T)(MX_{rs}A)$$

$$= \left(\begin{array}{ccc}
\frac{a_{11}}{m_1} & \cdots & \frac{a_{s-1}}{m_{s-1}} \\
\vdots & \ddots & \vdots \\
\frac{a_{n-1}}{m_{n-1}} & \cdots & \frac{a_{n}}{m_{n}}
\end{array}\right)
\left(\begin{array}{ccc}
-m_1a_{11} & \cdots & -m_1a_{1n} \\
\vdots & \ddots & \vdots \\
-m_{s-1}a_{s-1,n} & \cdots & -m_{s-1}a_{s-1,n}
\end{array}\right)
\left(\begin{array}{ccc}
m_1 & \cdots & m_{s-1} \\
\vdots & \ddots & \vdots \\
m_{n-1} & \cdots & m_n
\end{array}\right)
\left(\begin{array}{ccc}
\frac{a_{11}}{m_1} & \cdots & \frac{a_{s-1}}{m_{s-1}} \\
\vdots & \ddots & \vdots \\
\frac{a_{n-1}}{m_{n-1}} & \cdots & \frac{a_{n}}{m_{n}}
\end{array}\right).$$

Since Inn$_A(X_{rs})$ and $\text{Inn}_A(I)$ have entries in $k$, then so does the matrix $\text{Inn}_A(I) + \text{Inn}_A(X_{rs})$. Using a similar calculation to the above, we can see that this matrix has entries $2a_{ir}a_{jr} \frac{m_r}{m_i} + 2a_{si}a_{sj} \frac{m_s}{m_i}$. It follows that $m_r a_{ir}a_{jr} + m_s a_{si}a_{sj} \in k$ for all $i, j, r, s$.

So, we have that

$$m_r a_{ir}a_{jr} - m_s a_{ti}a_{tj} = (m_r a_{ir}a_{jr} + m_s a_{si}a_{sj}) - (m_r a_{ir}a_{jr} + m_t a_{ti}a_{tj}) \in k,$$

which means that

$$m_r a_{ir}a_{jr} = \frac{1}{2} (m_r a_{ir}a_{jr} - m_s a_{ti}a_{tj}) + \frac{1}{2} (m_r a_{ir}a_{jr} + m_s a_{ti}a_{tj}) \in k.$$

Since $m_r \in k$, then $a_{ir}a_{jr} \in k$ for all $i, j, r$.

Now, we consider the bilinear form $\beta_1$ which has matrix $M^{-1}$. We know that $X \in SO(n, \overline{k}, \beta)$ if and only $X^T MX = M$. But, if that is the case for a given $X$, then it follows that

$$X^{-1} M^{-1} (X^{-1})^T = M^{-1}.$$
Thus, $(X^{-1})^T \in \text{SO}(n, \mathbb{F}, \beta)$. Since this is a group, then we in fact know that $X^T \in \text{SO}(n, \mathbb{F}, \beta)$. It is then easy to see that $X \in \text{SO}(n, k, \beta)$ if and only if $X^T \in \text{SO}(n, k, \beta)$.

We further claim that $\text{Inn}_{A^T}$ is an automorphism of $\text{SO}(n, k, \beta)$. Suppose $Y \in \text{SO}(n, k, \beta)$ and consider $\text{Inn}_{A^T}(Y) = (A^T)^{-1}YA^T$. This matrix lies in $\text{SO}(n, k, \beta)$ if and only if its inverse-transpose lies in $\text{SO}(n, k, \beta)$. It’s inverse transpose is $A(Y^{-1})^TA^{-1} = \text{Inn}_{A^{-1}}((Y^{-1})^T)$. Since $\text{Inn}_{A^{-1}}$ must be an automorphism of $\text{SO}(n, k, \beta)$ because $\text{Inn}_A$ is, and since $(Y^{-1})^T \in \text{SO}(n, k, \beta)$ because $Y \in \text{SO}(n, k, \beta)$, then we have proven our claim.

Since $\text{Inn}_{A^T}$ is an automorphism of $\text{SO}(n, k, \beta)$, then it follows from our earlier work that $a_{is}, a_{jt} \in k$ for all $i, j, r$.

We now recall from earlier the matrices $Y_{ij} \in \text{SO}(n, k, \beta)$. So, it must be the case that $\text{Inn}_A(Y_{ij}) \in \text{SO}(n, k, \beta)$. Let us examine the entries of $\text{Inn}_A(Y_{ij})$:

$$\text{Inn}_A(Y_{ij}) = A^{-1}Y_{ij}A$$

$$= (M^{-1}A^T)(MY_{ij}A)$$

$$= \left(\begin{array}{cccc}
\frac{a_{11}}{m_1} & \cdots & \frac{a_{i1}}{m_1} & \\
\frac{a_{12}}{m_1} & \cdots & \frac{a_{i2}}{m_1} & \\
\vdots & \ddots & \vdots & \\
\frac{a_{1n}}{m_1} & \cdots & \frac{a_{in}}{m_1} & \\
\end{array}\right) \left(\begin{array}{c}
m_1a_{11} \\
m_1a_{i1} \\
\vdots \\
m_1a_{in} \\
\end{array}\right) \left(\begin{array}{c}
m_1a_{1n} \\
m_1a_{in} \\
\vdots \\
m_1a_{nn} \\
\end{array}\right)$$

$$= \left(\begin{array}{c}
\frac{a_{is}}{m_s} \left(\frac{am_isa_{it} + bm_isa_{jt}}{m_s}\right) + \frac{a_{js}}{m_s} \left(\frac{-bm_isa_{it} + am_jsa_{jt}}{m_s}\right) + \sum_{l \neq i,j} a_{ls}a_{lt} \frac{m_l}{m_s} \\
\end{array}\right)\in k(s,t).$$

We know that $a, b, m_s, a_{is}a_{it}, a_{js}a_{jt} \in k$. Since each of the matrix entries of $\text{Inn}_A(Y_{ij})$ must lie in $k$, then it follows that

$$\frac{a_{is}}{m_s} \left(\frac{am_isa_{it} + bm_isa_{jt}}{m_s}\right) + \frac{a_{js}}{m_s} \left(\frac{-bm_isa_{it} + am_jsa_{jt}}{m_s}\right) + \sum_{l \neq i,j} a_{ls}a_{lt} \frac{m_l}{m_s} \in k(s,t).$$

If we further apply the facts we just stated about what lies in $k$, then we see that

$$a_{is}a_{jt} - a_{js}a_{it} \in k,$$

for all $i, j, s, t$.

We now want to show that $a_{is}a_{jt} \in k$ for all $i, j, s, t$. Without loss of generality, we will assume that both $a_{is}$ and $a_{jt}$ are nonzero. We may assume this since each row and column must have at least one nonzero entry. If both $a_{is}, a_{jt} \in k$, then $a_{is}a_{jt} \in k$ is obvious, so we assume $a_{is} \notin k$. Recall that $a_{is}^2 \in k$.

It follows that

$$a_{is}^2a_{jt} - (a_{is}a_{it})a_{js} = a_{is}(a_{is}a_{jt} - a_{js}a_{it}) \in k[a_{is}].$$

Since we also know that $a_{is}^2a_{is}a_{it} \in k$ and $a_{jt}, a_{it} \in k[a_{jt}]$, then it follows that

$$a_{is}(a_{is}a_{jt} - a_{js}a_{it}) \in k[a_{jt}]$$

as well. Recall that $a_{jt}^2 \in k$. So, there exists $c, d, e, f \in k$ such that

$$c + da_{is} = a_{is}(a_{is}a_{jt} - a_{js}a_{it}) = e + fa_{jt}.$$
Since we are assuming that $a_{is} \notin k$, then it follows that both $d$ and $f$ are nonzero, and that $a_{jt} \notin k$. We see that
\[ 0 = (c - e) + (da_{is} - fa_{jt}). \]

It follows that $da_{is} - fa_{jt} \in k$. So, $da_{is} = fa_{jt} + g$ for some $g \in k$. Therefore, we can write
\[ a_{is} = ua_{jt} + v \text{ where } u = \frac{f}{d}, v = \frac{g}{d} \in k. \]
We see that $k$ contains
\[ a_{is}^2 = (ua_{jt} + v)^2 = u^2a_{jt}^2 + 2uva_{jt} + v^2. \]

Since $a_{jt} \notin k$, then it follows that $uv = 0$. Since $a_{is} = ua_{jt} + v \notin k$, then $u \neq 0$. So, $v = 0$ and $a_{is} = ua_{jt}$. Thus, $a_{is}a_{jt} = \frac{1}{a}a_{jt} \in k$, as desired. From this it is clear that $k[a_{is}] = k[a_{jt}]$ for all $i, j, s, t$ (assuming that $a_{is}$ and $a_{jt}$ are both nonzero). So, let $\alpha = a_{is}^2$ where $a_{is}$ is a fixed nonzero entry of $A$. Then, we have shown that all the entries of $A$ are in $k[\sqrt{\alpha}]$. This means that $A \in O(n, k[\sqrt{\alpha}], \beta)$, and all of the entries of $A$ are $k$-multiples of $\sqrt{\alpha}$, as desired.

If $n$ is odd, then we can replace $A$ with $-A$ to get a matrix in $SO(n, k[\sqrt{\alpha}], \beta)$. So, assume that $A \in SO(n, k[\sqrt{\alpha}], \beta)$. We now show that we do not need a quadratic extension of $k$ when $n$ is odd. Proceed by contradiction and assume $A \in SO(n, k[\sqrt{\alpha}], \beta)$ inducing the automorphism. Then, from our work above, we know that $\sqrt{\alpha}A \in GL(n, k)$. Then,
\[ \det(\sqrt{\alpha}A) = [\sqrt{\alpha}]^n \det(A) = \alpha^n \notin k, \]
which is a contradiction. So, if $n$ is odd, we may assume that $A \in SO(n, k, \beta)$. \hfill \Box

4. Involutions of $SO(n, k, \beta)$

We now begin to focus on involutions and their classification. We will distinguish different types of involutions. First, we note that for some involutions, $\varphi = \text{Inn}_A$, but not in all cases. Sometimes we must settle for $A \in SO(n, k[\sqrt{\alpha}], \beta) \setminus O(n, k, \beta)$.

This is not the only way in which we can distinguish between different types of involutions. If $\text{Inn}_A$ is an involution, then $\text{Inn}_{A^2} = (\text{Inn}_A)^2$ is the identity map. We know from earlier that this means that $A^2 = \gamma I$ for some $\gamma \in k$. But, we know for certain that $A$ is orthogonal. So, $A^2$ is also orthogonal. That means that $(A^2)^TM(A^2) = M$, which implies $(\gamma I)^TM(\gamma I) = M$, which means $\gamma^2 = 1$. So, $\gamma = \pm 1$. Thus, we can also distinguish between different types of involutions by seeing if $A^2 = I$ or $A^2 = -I$. This gives the four types of involutions, which are outlined in Table 1. It follows from our characterization of automorphisms that when $n$ is odd, that Type 2 and Type 4 involutions do not occur. But, we also see that if $n$ is odd and $A$ is orthogonal, then $A^2$ must have determinant 1. So, we see in addition that Type 3 involutions can also only occur when $n$ is even.

4.1. Type 1 Involutions. We now find a structured form for the matrices of all types of involutions. We begin with Type 1 involutions. When $n$ is odd, these are the only involutions.

**Lemma 4.1.** Suppose $\vartheta$ is a Type 1 involution of $SO(n, k, \beta)$. Then, there exists $A \in O(n, k, \beta)$ such that $A = X \left( \begin{smallmatrix} I_s & 0 \\ 0 & -I_t \end{smallmatrix} \right) X^{-1}$ where $s + t = n$ and
\[ X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \in GL(n, k), \]
where the $x_i$ are orthogonal eigenvectors of $A$, meaning $X^T MX$ is diagonal, and $s \leq t$. 

\begin{table}[h]
\centering
\caption{The various possible types of involutions of $SO(n, k, \beta)$}
\begin{tabular}{|c|c|}
\hline
\text{Type} & \text{Set} \\
\hline
1 & $A \in O(n, k, \beta)$ \\
2 & $A \in O(n, k[\sqrt{\alpha}], \beta) \setminus O(n, k, \beta)$ \\
3 & $A^2 = I$ \\
4 & $A^2 = -I$ \\
\hline
\end{tabular}
\end{table}
Lemma 4.2. Symmetric matrices $M_1$ and $M_2$ with entries in $\mathbb{Q}_p$ are congruent if and only if

$$\det(M_1) = \gamma^2 \det(M_2) \quad \text{and} \quad c_p(M_1) = c_p(M_2)$$

where $c_p(M)$ denotes the Hasse symbol of matrix $M$.

Now we show conditions equivalent to isomorphy.
Theorem 4.3. Suppose \( \vartheta \) and \( \varphi \) are two Type 1 \textit{Involutions} of \( \text{SO}(n,k,\beta) \) where \( \vartheta = \text{Inn}_A \) and \( \varphi = \text{Inn}_B \). Then, \( A = X \begin{pmatrix} -I_{m_A} & 0 \\ 0 & I_{n-m_A} \end{pmatrix} X^{-1} \) and \( B = Y \begin{pmatrix} -I_{m_B} & 0 \\ 0 & I_{n-m_B} \end{pmatrix} Y^{-1} \) where \( m_A, m_B \leq \frac{n}{2} \), and
\[ X = (x_1 \; x_2 \; \cdots \; x_n), \quad Y = (y_1 \; y_2 \; \cdots \; y_n) \in \text{GL}(n,k) \]
have columns that are orthogonal eigenvectors of \( A \) and \( B \) respectively. We also have the diagonal matrices
\[ X^T MX = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \]
and
\[ Y^T MY = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}. \]

The following are equivalent:
\( (i) \) \( \vartheta \) is isomorphic to \( \varphi \) over \( \text{SO}(n,k,\beta) \).
\( (ii) \) \( A \) is conjugate to \( B \) or \( -B \) over \( \text{SO}(n,k,\beta) \).
\( (iii) \) \( X_1 \) is congruent to \( Y_1 \) over \( \text{GL}(m,k) \) and \( X_2 \) is congruent to \( Y_2 \) over \( \text{GL}(n-m,k) \), or \( X_1 \) is congruent to \( Y_2 \) over \( \text{GL}(\frac{n}{2},k) \) and \( X_2 \) is congruent to \( Y_1 \) over \( \text{GL}(\frac{n}{2},k) \).
\( (iv) \) \( \vartheta \) is isomorphic to \( \varphi \) over \( \text{O}(n,k,\beta) \).
\( (v) \) \( A \) is conjugate to \( B \) or \( -B \) over \( \text{O}(n,k,\beta) \).
\( (vi) \) if \( k = \mathbb{Q}_p \), there exists some \( \gamma \in \mathbb{Q}_p \), such that
\[ \det(X_1) = \gamma^2 \det(Y_1), \quad \det(X_2) = \gamma^2 \det(Y_2), \quad c_p(X_1) = c_p(Y_1), \quad \text{and} \quad c_p(X_2) = c_p(Y_2) \]
or
\[ \det(X_1) = \gamma^2 \det(Y_2), \quad \det(X_2) = \gamma^2 \det(Y_1), \quad c_p(X_1) = c_p(Y_2), \quad \text{and} \quad c_p(X_2) = c_p(Y_1). \]

Proof. First we note that in the event that \( k = \mathbb{Q}_p \), then \( (vi) \) will be equivalent to \( (iii) \) by Lemma 4.2.

Next, we prove \( (i) \) is equivalent to \( (ii) \). First suppose \( A \) is conjugate to \( B \) or \( -B \) over \( \text{SO}(n,k,\beta) \). So, we can choose \( Q \in \text{SO}(n,k,\beta) \) such that \( \pm B = Q^{-1}AQ \), and let \( \chi = \text{Inn}_{Q^{-1}} \). Then, for all \( U \in \text{SO}(n,k,\beta) \), we have
\[ \chi^{-1} \vartheta \chi(U) = Q^{-1}A^{-1}QUQ^{-1}AQ \]
\[ = (Q^{-1}AQ)^{-1}U(Q^{-1}AQ) \]
\[ = (\pm B)^{-1}U(\pm B) \]
\[ = B^{-1}UB \]
\[ = \varphi(U). \]

So, \( \chi^{-1} \vartheta \chi = \varphi \). That is, \( \vartheta \) is congruent to \( \varphi \) over \( \text{SO}(n,k,\beta) \).

Since the center of \( \text{SO}(n,k,\beta) \) is \( \{ I, -I \} \), then the converse follows similarly, so \( (i) \) and \( (ii) \) are equivalent. Likewise, we can show that \( (iv) \) and \( (v) \) are equivalent.

Next we show that \( (ii) \) implies \( (iii) \). First suppose that \( Q^{-1}AQ = B \) for some \( Q \in \text{SO}(n,k,\beta) \). \( Q^{-1}AQ = B \) implies
\[ Q^{-1}X \begin{pmatrix} -I_{m_A} & 0 \\ 0 & I_{n-m_A} \end{pmatrix} X^{-1}Q = Y \begin{pmatrix} -I_{m_B} & 0 \\ 0 & I_{n-m_B} \end{pmatrix} Y^{-1}. \]
Since the matrices on both sides of the equality above must have the same eigenvalues with the same multiplicities, then we see that \( m_A = m_B \), so we will just write \( m \), and note that
\[ -I_{m,n-m} = \begin{pmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}. \]
Rearranging the previous equation, we have
\[ I_{m,n-m}X^{-1}QY = X^{-1}QYI_{m,n-m}. \]
which tells us that $X^{-1}QY = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$, where $R_1 \in \text{GL}(m,k)$ and $R_2 \in \text{GL}(n-m,k)$. Rearranging, we have that $QY = X \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$. Since $Q \in O(n,k,\beta)$, then we know that $Q^T MQ = M$.

So,
\[
Y^T MY = Y^T Q^T MQY
= \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}^T (X^T MX) \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}.
\]

From here we see that $Y_1 = R_1^T X_1 R_1$ and $Y_2 = R_2^T X_2 R_2$.

Now suppose that $Q^{-1}AQ = -B$ for some $Q \in \text{SO}(n,k,\beta)$. This implies
\[
Q^{-1}X \begin{pmatrix} -I_{m_A} & 0 \\ 0 & I_{n-m_A} \end{pmatrix} X^{-1}Q = Y \begin{pmatrix} I_{m_B} & 0 \\ 0 & -I_{n-m_B} \end{pmatrix} Y^{-1}.
\]

Since the matrices on both sides of the equality above must have the same eigenvalues with the same multiplicities, then we see that $m_A = n - m_B$. Since $m_A, m_B \leq \frac{n}{2}$, then it follows that $m_A = m_B = \frac{n}{2}$.

Rearranging the previous equation, we have
\[
I_{\frac{n}{2}, \frac{n}{2}} X^{-1}QY = X^{-1}QY I_{\frac{n}{2}, \frac{n}{2}}.
\]

which tells us that $X^{-1}QY = \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}$, where $R_1, R_2 \in \text{GL}(\frac{n}{2},k)$. Rearranging, we have that $QY = X \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}$. Since $Q \in O(n,k,\beta)$, then we know that $Q^T MQ = M$.

So,
\[
Y^T MY = Y^T Q^T MQY
= \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}^T (X^T MX) \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}.
\]

From here we see that $Y_2 = R_1^T X_1 R_1$ and $Y_1 = R_2^T X_2 R_2$.

This shows that (ii) implies (iii).

Now we show that (iii) implies (ii). Assume that (iii) is the case. Specifically, assume that $R_1 \in \text{GL}(m,k)$ and $R_2 \in \text{GL}(n-m,k)$ such that $Y_1 = R_1^T X_1 R_1$ and $Y_2 = R_2^T X_2 R_2$. Let $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$. So, we have $Y^T MY = R^T (X^T MX) R$. Let $Q = XRY^{-1}$. We will now show that $Q^{-1}AQ = B$ and that $Q \in \text{SO}(n,k,\beta)$.

\[
Q^{-1}AQ = (XRY^{-1})^{-1}A(XRY^{-1})
= Y R^{-1} X^{-1} A X Y^{-1}
= Y R^{-1} (-I_{m,n-m}) Y^{-1}
= Y (-I_{m,n-m}) Y^{-1} = B.
\]

Next, we must show that $Q \in \text{SO}(n,k,\beta)$. We first show that $Q^T MQ = M$. Recall that $Y^T MY = R^T (X^T MX) R$. So,
\[
Q^T MQ = (XRY^{-1})^T M (XRY^{-1})
= (Y^{-1})^T (R^T X^T MX R) Y^{-1}
= (Y^{-1})^T (Y^T MY) Y^{-1}
= M.
\]

In the event that det$(Q) = -1$, then we can replace the first column of $X$ with its negative. This will have no effect on $R$ or $Y$, so the new $Q = XRY^{-1}$ have determinant 1, and it will still be the case that $Q^{-1}AQ = B$ and $Q^T MQ = M$. So, $Q \in \text{SO}(n,k,\beta)$.

If instead we assume that $R_1 \in \text{GL}(m,k)$ and $R_2 \in \text{GL}(n-m,k)$ such that $Y_2 = R_1^T X_1 R_1$ and $Y_1 = R_2^T X_2 R_2$, then if we let $R = \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}$, then we can let $Q = XRY^{-1}$ and get that $Q^{-1}AQ = -B$ and $Q \in \text{SO}(n,k,\beta)$. This shows that (iii) implies (ii).
We now show that (iv) and (v) are equivalent to the previous three conditions. First, we note that it is clear that (i) implies (iv). So, we need only show that (iv) or (v) implies one of the other three conditions. But, (v) implies (iii) from an argument very similar to the argument where we showed that (iii) implies (ii). Thus, all the conditions are equivalent. □

We note that the equivalence of conditions (i) and (ii) in the previous theorem show that a Type 1 involution cannot be in the same isomorphy class as an involution of a different type. This will be the same for all types. That is, isomorphic involutions must be of the same type. We also note that this Theorem shows that isomorphy over \( SO(n, k, \beta) \) and \( O(n, k, \beta) \) are the same for Type 1 involutions. We will show in an explicit example that this does not occur in the Type 2 case. For the remaining three types of involutions, we will only find conditions for isomorphy over \( O(n, k, \beta) \). Again, recall that these three Types of involutions occur when \( n \) is even.

### 4.2. Type 2 Involutions

We have a similar characterization of the matrices and isomorphy classes in the Type 2 case. We first prove a result about that characterizes the eigenvectors in the Type 2 case.

**Lemma 4.4.** Suppose \( A \in O(n, k[\sqrt{\alpha}], \beta) \setminus O(n, k, \beta) \) induces a Type 2 involution of vectors \( SO(n, k, \beta) \) where \( \sqrt{\alpha} \notin k \). Also suppose \( x, y \in k^n \) such that \( x + \sqrt{\alpha}y \in E(A, -1) \). Then, \( x - \sqrt{\alpha}y \in E(A, 1) \). Likewise, if \( u, v \in k^n \) such that \( u + \sqrt{\alpha}v \in E(A, 1) \). Then, \( u - \sqrt{\alpha}v \in E(A, -1) \). Further, \( \dim(E(A, 1)) = \dim(E(A, -1)) \).

**Proof.** First, we observe that “\( \sqrt{\alpha} \)-conjugation,” similar to the familiar complex conjugation (i-conjugation), preserves multiplication. That is,

\[
(a + \sqrt{\alpha}b)(c + \sqrt{\alpha}d) = (ac + abd) + \sqrt{\alpha}(ad + bc)
\]

and

\[
(a - \sqrt{\alpha}b)(c - \sqrt{\alpha}d) = (ac + abd) - \sqrt{\alpha}(ad + bc).
\]

So, “\( \sqrt{\alpha} \)-conjugation” will preserve multiplication on the matrix level as well. Because of this and since

\[A(x + \sqrt{\alpha}y) = -x - \sqrt{\alpha}y,\]

then it follows that

\[(-A)(x - \sqrt{\alpha}) = -x + \sqrt{\alpha}y.\]

We can multiply both sides to see that

\[A(x - \sqrt{\alpha}) = x - \sqrt{\alpha}y.\]

That is, \( x - \sqrt{\alpha}y \in E(A, 1) \). This proves the first statement. An analogous argument proves the second.

To see that \( \dim(E(A, 1)) = \dim(E(A, -1)) \) is the case, note that the first statement tells us that \( \dim(E(A, 1)) \leq \dim(E(A, -1)) \), and that the second statement tells us that \( \dim(E(A, 1)) \geq \dim(E(A, -1)) \), since “\( \sqrt{\alpha} \)-conjugation” is an invertible operator on \( k[\sqrt{\alpha}] \).

We are now able to characterize the Type 2 involutions.

**Lemma 4.5.** Suppose \( \vartheta \) is a Type 2 involution of \( SO(n, k, \beta) \). Let \( A \) be the orthogonal matrix in \( O(n, k[\sqrt{\alpha}], \beta) \) such that \( \vartheta = \text{Inn}_A \). Then,

\[
A = -\frac{\sqrt{\alpha}}{\alpha}X \begin{pmatrix} 0 & I_\frac{n}{2} \\ \alpha I_\frac{n}{2} & 0 \end{pmatrix} X^{-1}
\]

where

\[X = \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_\frac{n}{2} \\ y_1 & y_2 & \cdots & y_\frac{n}{2} \end{array} \right) \in \text{GL}(n, k),\]

where for each \( i \), we have orthogonal vectors \( x_i + \sqrt{\alpha}y_i \in E(A, -1) \) and orthogonal vectors \( x_i - \sqrt{\alpha}y_i \in E(A, 1) \). Further,

\[X^T MX = \begin{pmatrix} X_1 & X_2 \\ X_2 & \frac{1}{\alpha}X_1 \end{pmatrix}\]

where \( X_1 \) and \( X_2 \) are diagonal matrices.
Proof. We wish to construct bases for \( E(A, 1) \) and \( E(A, -1) \) such that all the vectors lie in \( k[\sqrt{\alpha}]^n \). From the previous lemma, we know that \( \dim(E(A, 1)) = \dim(E(A, -1)) = \frac{n}{2} \). (Note that this means that \( n \) must be even for a Type 2 involution to occur.) Since \( \text{Im}_A \) is a Type 1 involution of \( \text{SO}(n, k[\sqrt{\alpha}], \beta) \), then we can apply Lemma 4.1 to find an orthogonal basis \( \{x_1 + \sqrt{\alpha}y_1, ..., x_{\frac{n}{2}} + \sqrt{\alpha}y_{\frac{n}{2}}\} \) of \( E(A, -1) \), where \( x_1, ..., x_{\frac{n}{2}}, y_1, ..., y_{\frac{n}{2}} \in k^n \). By the previous lemma, we know that \( \{x_1 - \sqrt{\alpha}y_1, ..., x_{\frac{n}{2}} - \sqrt{\alpha}y_{\frac{n}{2}}\} \) must be a basis for \( E(A, 1) \). Let \( X = (x_1, x_2, ..., x_{\frac{n}{2}}, y_1, y_2, ..., y_{\frac{n}{2}}) \in \text{GL}(n, k) \).

We now make a couple of observations. Suppose \( u = x + \sqrt{\alpha}y \) is a -1-eigenvector of \( A \) such that \( x, y \in k^n \). Then, we know \( v = x - \sqrt{\alpha}y \) is a 1-eigenvector of \( A \). Observe that
\[
Ax = \frac{1}{2} A(u + v) = \frac{1}{2} (-u + v) = -\sqrt{\alpha}y.
\]
It follows from this that
\[
Ay = -\frac{\sqrt{\alpha}}{\alpha} x.
\]
Since \( Ax = -\sqrt{\alpha}y \) and \( Ay = -\frac{\sqrt{\alpha}}{\alpha} x \), then it follows that
\[
X^{-1}AX = \begin{pmatrix} 0 & -\frac{\sqrt{\alpha}}{\alpha} I_{\frac{n}{2}} \\ -\frac{\sqrt{\alpha}}{\alpha} I_{\frac{n}{2}} & 0 \end{pmatrix}.
\]
Rearranging this, we see that
\[
A = -\frac{\sqrt{\alpha}}{\alpha} X \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ 0 & 0 \end{pmatrix} X^{-1}.
\]

Now, we need only prove the last statement to prove the Lemma. Since \( \{x_1 + \sqrt{\alpha}y_1, ..., x_{\frac{n}{2}} + \sqrt{\alpha}y_{\frac{n}{2}}\} \) is an orthogonal set of vectors, then we know when \( i \neq j \) that
\[
0 = \beta(x_i + \sqrt{\alpha}y_i, x_j + \sqrt{\alpha}y_j)
= (\beta(x_i, x_j) + \alpha \beta(y_i, y_j)) + \sqrt{\alpha}(\beta(x_i, y_j) + \beta(x_j, y_i))
\]
This tells us that
\[
\beta(x_i, x_j) = -\alpha \beta(y_i, y_j)
\]
and
\[
\beta(x_i, y_j) = -\beta(x_j, y_i).
\]
Since vectors from \( E(A, 1) \) and \( E(A, -1) \) are orthogonal, then we also know that
\[
0 = \beta(x_i + \sqrt{\alpha}y_i, x_j - \sqrt{\alpha}y_j)
= (\beta(x_i, x_j) - \alpha \beta(y_i, y_j)) + \sqrt{\alpha}(-\beta(x_i, y_j) + \beta(x_j, y_i))
\]
regardless of if \( i \) and \( j \) are distinct or equal.
This tells us that
\[
\beta(x_i, x_j) = \alpha \beta(y_i, y_j)
\]
and
\[
\beta(x_i, y_j) = \beta(x_j, y_i).
\]
So, when \( i \neq j \), then we know that
\[
\beta(x_i, y_j) = 0,
\]
\[
\beta(x_i, x_j) = 0,
\]
and
\[
\beta(y_i, y_j) = 0.
\]
When \( i = j \), we note that
\[
\beta(x_i, x_i) = \alpha \beta(y_i, y_i).
\]
Then, we have

\[ X^T MX = \begin{pmatrix} X_1 & X_2 \\ X_2 & \frac{1}{\alpha} X_1 \end{pmatrix} \]

where \( X_1 \) and \( X_2 \) have been shown to be diagonal. \( \square \)

We now show an example of a Type 2 involution, and apply the previous lemma to it.

**Example 4.6.** Assume that \( \beta \) is the standard dot product. Then, \( \text{Inn}_A \) can be a Type 2 involution of \( \text{SO}(4, \mathbb{Q}) \) if \( A \) is symmetric and orthogonal, since this will imply that \( A^2 = I \), and if the entries of \( A \) are all \( k \)-multiples of some \( \sqrt{\alpha} \) such that \( \sqrt{\alpha} \notin k \) but \( \alpha \in k \). Observe that the matrix

\[
A = \frac{\sqrt{3}}{3} \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}
\]

is both symmetric and orthogonal. Since each entry is the \( \mathbb{Q} \)-multiple of \( \sqrt{3} \), then it is clear that \( \text{Inn}_A \) is a Type 2 involution of \( \text{SO}(4, \mathbb{Q}) \). It can be shown that \( E(A, -1) \) has dimension 2. An orthogonal basis for this subspace is formed by the vectors

\[
v_1 = \begin{pmatrix} 1 \\ \frac{\sqrt{3}}{2} \\ 0 \\ 1 \end{pmatrix} + \sqrt{3} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}
\]

and

\[
v_2 = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{3}}{2} \\ 0 \\ 1 \end{pmatrix} + \sqrt{3} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}.
\]

It can be shown that

\[
v_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} - \sqrt{3} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}
\]

and

\[
v_4 = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} - \sqrt{3} \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ 0 \end{pmatrix}
\]

are orthogonal 1-eigenvectors of \( A \), where these are the \( \sqrt{3} \)-conjugates of \( v_1 \) and \( v_2 \), respectively.

Following the notation of the previous lemma, we have

\[
X = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

where \( X^T X = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \) and \( A = -\frac{\sqrt{3}}{3} X \begin{pmatrix} 0 \\ 3 \sqrt{3} \\ 0 \\ 0 \end{pmatrix} X^{-1} \).

We now find conditions in the Type 2 case that are equivalent to isomorphy.
Theorem 4.7. Suppose $\vartheta$ and $\varphi$ are two Type 2 Involutions of $\text{SO}(n, k, \beta)$ where $\vartheta = \text{Inn}_A$ and $\varphi = \text{Inn}_B$. Then,

$$A = -\frac{\sqrt{n}}{\alpha} X \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} X^{-1} \in \text{O}(n, k[\sqrt{\alpha}], \beta)$$

where

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_{\frac{n}{2}} & y_1 & y_2 & \cdots & y_{\frac{n}{2}} \end{pmatrix} \in \text{GL}(n, k)$$

and the $x_i + \sqrt{\alpha} y_i$ are the orthogonal basis of $E(A, -1)$, and

$$X^T M X = \begin{pmatrix} X_1 & X_2 \\ \frac{1}{\alpha} X_1 \end{pmatrix}$$

where $X_1$ and $X_2$ are diagonal matrices, and

$$B = -\frac{\sqrt{\gamma}}{\gamma} Y \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \gamma I_{\frac{n}{2}} & 0 \end{pmatrix} Y^{-1} \in \text{O}(n, k[\sqrt{\gamma}], \beta)$$

where

$$Y = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_{\frac{n}{2}} & \tilde{y}_1 & \tilde{y}_2 & \cdots & \tilde{y}_{\frac{n}{2}} \end{pmatrix} \in \text{GL}(n, k)$$

and the $\tilde{x}_i + \sqrt{\gamma} \tilde{y}_i$ is the orthogonal eigenvectors of $E(B, -1)$, and

$$Y^T M Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & \frac{1}{\alpha} Y_1 \end{pmatrix}$$

where $Y_1$ and $Y_2$ are diagonal matrices, and the following are equivalent:

(i) $\vartheta$ is isomorphic to $\varphi$ over $\text{O}(n, k, \beta)$.

(ii) $A$ is conjugate to $B$ or $-B$ over $\text{O}(n, k, \beta)$.

(iii) $\alpha = \gamma$ and $Y^T M Y = R^T X^T M X R$ where $R = \frac{R_1 R_2}{\alpha R_2 R_1}$ is in $\text{GL}(n, k)$, or $\alpha = \gamma$ and $Y^T M Y = R^T X^T M X R$ where $R = \frac{R_1 R_2}{-\alpha R_2 R_1}$ is in $\text{GL}(n, k)$.

(iv) We can choose $X$ and $Y$ such that $\alpha = \gamma$, and for $R = \frac{R_1 R_2}{\alpha R_2 R_1}$ is in $\text{GL}(n, k)$ we have

$$Y_1 = R_1^T X_1 R_1 + \alpha R_2^T X_2 R_1 + \alpha R_2^T X_1 R_2 + \alpha R_2^T X_1 R_2$$

and

$$Y_2 = R_2^T X_1 R_1 + R_1^T X_2 R_1 + \alpha R_2^T X_2 R_2 + R_1^T X_1 R_2,$$

or for $R = \frac{R_1 R_2}{-\alpha R_2 R_1}$ we have

$$Y_1 = R_1^T X_1 R_1 - \alpha R_2^T X_2 R_1 - \alpha R_1^T X_2 R_2 + \alpha R_2^T X_1 R_2$$

and

$$Y_2 = R_2^T X_1 R_1 - R_1^T X_2 R_1 - \alpha R_2^T X_2 R_2 + R_1^T X_1 R_2.$$

Proof. Proving the equivalence of (i) and (ii) is identical to the proof in the previous theorem. So, we begin by showing that (ii) implies (iii). First suppose there exists $Q \in \text{O}(n, k, \beta)$ such that $Q^{-1} A Q = B$. So, we have

$$Q^{-1} \frac{\sqrt{n}}{\alpha} X \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} X^{-1} Q = \frac{\sqrt{\gamma}}{\gamma} Y \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \gamma I_{\frac{n}{2}} & 0 \end{pmatrix} Y^{-1}.$$  

Also, we know that since $A \in \text{SO}(n, k[\sqrt{\alpha}], \beta)$ and $B \in \text{SO}(n, k[\sqrt{\gamma}], \beta)$ are congruent over $\text{O}(n, k, \beta)$, then we must be able to make a choice of $\gamma$ such that $\alpha = \gamma$. Thus,

$$Q^{-1} X \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} X^{-1} Q = Y \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} Y^{-1}.$$  

Rearranging, we see that

$$\begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} X^{-1} Q Y = \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} Y^{-1}.$$
Let $R = X^{-1}QY$, and note that $R \in \text{GL}(n, k)$. Since $(0 \ a \ I_{\frac{n}{2}} \ 0 \ b)R = R (0 \ -aR_2 - R_1 \ 0 \ b)$, then $R = (R_1 \ R_2 \ R_3)$. Observe that $XR = QY$. Also, observe that since $Q \in \text{SO}(n, k, \beta)$, then we know that $Q^TMQ = M$. It follows from these observations that

$$R^TX^TXMR = (XR)^TMXR$$

$$= (QY)^TMQY$$

$$= Y^T(Q^TMQ)Y$$

$$= Y^TMX.$$

If instead we assume that there exists $Q \in \text{O}(n, k, \beta)$ such that $Q^{-1}AQ = -B$, then we can similarly show that $\alpha = \gamma$ and $Y^TMY = R^TX^TXMR$ where $R = (-\alphaR_2 \ R_1) \in \text{GL}(n, k)$ for $R_1, R_2 \in M(\frac{n}{2}, k)$. This proves that $(ii)$ implies $(iii)$.

We now show that $(iii)$ implies $(ii)$. First assume $\alpha = \gamma$ and $X^TXM$ is congruent to $Y^TMX$ over $\text{GL}(n, k)$ where $Y^TMY = R^TX^TXMR$ for $R = (R_1 \ R_2 \ R_3)$, where $R_1, R_2 \in \text{GL}(\frac{n}{2}, k)$. Let $Q = XRY^{-1}$. Then, we observe that

$$Q^{-1}AQ = (XRY^{-1})^{-1}A(XRY^{-1})$$

$$= YR^{-1}(X^{-1}AX)RY^{-1}$$

$$= -\frac{\sqrt{\alpha}}{\alpha}YR^{-1} \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} RY^{-1}$$

$$= -\frac{\sqrt{\alpha}}{\alpha}Y \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} R^{-1}RY^{-1}$$

$$= -\frac{\sqrt{\alpha}}{\alpha}Y \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ \alpha I_{\frac{n}{2}} & 0 \end{pmatrix} Y^{-1} = B.$$

To show that $(ii)$ is indeed the case, we need only show that $Q \in \text{O}(n, k, \beta)$. By construction, we know that $Q \in \text{GL}(n, k)$. So, it is suffice to show $Q^TMQ = M$. But,

$$Q^TMQ = (XRY^{-1})^T(MX)Y^{-1}$$

$$= (Y^{-1})^T(R^TX^TXMR)Y^{-1}$$

$$= (Y^{-1})^T(Y^TMY)Y^{-1} = M.$$ 

If we instead assume that $\alpha = \gamma$ and $X^TXM$ is congruent to $Y^TMX$ over $\text{GL}(n, k)$ where $Y^TMY = R^TX^TXMR$ for $R = (-\alphaR_2 \ R_1)$, where $R_1, R_2 \in \text{GL}(\frac{n}{2}, k)$, then if we let $Q = XRY^{-1}$, we can similarly show that $Q^{-1}AQ = -B$ and $Q \in \text{O}(n, k, \beta)$. This shows that $(iii)$ implies $(ii)$.

Lastly, matrix multiplication shows that $(iii)$ and $(iv)$ are equivalent.

The reader will notice that in the Type 1 case, our conditions gave us congruency of involutions over $\text{SO}(n, k, \beta)$, but the Type 2 case gave us congruency of involutions over $\text{O}(n, k, \beta)$. In the following example, we give an example that shows that the above Theorem cannot be strengthened by replacing $\text{O}(n, k, \beta)$ with $\text{SO}(n, k, \beta)$.

**Example 4.8.** Consider the group $\text{SO}(4, F_3)$. That is, consider the case where $k$ is the group of three elements, and the bilinear form is the standard dot product. A Type 2 involution is induced by the matrix

$$A = i \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \in \text{SO}(4, F_3[i]).$$
By analyzing suitable eigenvectors for this matrix, we see that

\[ X = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \]

where

\[ A = iX^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} X. \]

We also see that

\[ X^T X = \begin{pmatrix} X_1 & X_2 \\ X_2 & 2X_1 \end{pmatrix} \]

where \( X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( X_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \).

Now, we also consider the Type 2 involution of \( SO(4, F_3) \) that is induced by the matrix

\[ B = i \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} \in SO(4, F_3[i]). \]

By analyzing suitable eigenvectors for this matrix, we see that for

\[ Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \]

we have

\[ B = iY^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} Y. \]

We also see that

\[ Y^T Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & 2Y_1 \end{pmatrix} \]

where \( Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

We have two ways of showing that these Type 2 involutions are congruent over \( O(4, F_3) \). First, we consider the matrix

\[ Q = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix} \in O(4, F_3) \setminus SO(4, F_3). \]

Then, \( B = Q^{-1} A Q \). This is condition (ii) of the previous theorem.

Secondly, if we let \( R = \left( \begin{pmatrix} R_1 \\ 2R_2 \\ R_3 \end{pmatrix} \right) \in GL(4, F_3) \) where \( R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( R_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), then we get that \( R^T Y^T Y R = X^T X \). This is condition (iii) of the previous theorem.

We note that during the author’s computations, \( R \) and \( Y^T Y \) were discovered by using a Maple loop that used condition (iv) from the previous Theorem, and simply just tried every value of \( R_1 \) and \( R_2 \). The calculation considered thousands of possibilities. This is a brute force method that is not ideal, and certainly only possible when \( k \) is a finite field. \( Y \), and then \( B \) were then calculated, and \( Q \) was computed using the formula \( Q = X R Y^{-1} \), as in the proof of the theorem.

We now show that there does not exist \( W \in SO(4, F_3) \) such that \( B = W^{-1} A W \). We proceed by contradiction and suppose that these does exist such an \( W \). It then follows that \( A \) and \( QW^{-1} \in O(4, F_3) \)
are commuting matrices. It is a simple matter to show that matrices that commute with $A$ must be of the form
\[
\begin{pmatrix}
a & b & c & d \\
b & a+b & d & c+d \\
e & f & g & h \\
f & e+f & h & g+h
\end{pmatrix}.
\]

One such matrix is
\[
\begin{pmatrix}
1 & 1 & 2 & 2 \\
1 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 \\
2 & 1 & 2 & 1
\end{pmatrix} \in SO(4, \mathbb{F}_3).
\]

But, all other such orthogonal matrices differ from this matrix only in that an even number of rows and/or columns have been multiplied by $2 = -1$ or an even number of rows and/or columns have been swapped. All of these actions create matrices that will also have determinant 1. Thus, all the matrices in $O(4, \mathbb{F}_3)$ which commute with $A$ are also members of $SO(4, \mathbb{F}_3)$, which contradicts $QW^{-1} \in O(4, \mathbb{F}_3)$ commuting with $A$. So, no such $W \in SO(4, \mathbb{F}_3)$ can exist, which means we cannot strengthen the above theorem by replacing $O(n, k, \beta)$ with $SO(n, k, \beta)$ in conditions (i) and (ii).

**Corollary 4.9.** Suppose $\text{Inn}_A$ and $\text{Inn}_B$ are both Type 2 involutions of $SO(n, k, \beta)$. Then, $\text{Inn}_A$ and $\text{Inn}_B$ are isomorphic over $O(n, k, \beta)$ if and only if they are isomorphic over $SO(n, k, \beta)$.

**Proof.** When viewed as Type 2 involutions of $SO(n, k, \beta)$, we can write
\[
A = -\frac{\sqrt{\alpha}}{\alpha} U \begin{pmatrix} 0 & I_2 \\ \alpha I_2 & 0 \end{pmatrix} U^{-1} \text{ where } U^T MU = \begin{pmatrix} U_1 & U_2 \\ \frac{1}{\alpha} U_1 \end{pmatrix}
\]
and
\[
B = -\frac{\sqrt{\alpha}}{\alpha} V \begin{pmatrix} 0 & I_2 \\ \alpha I_2 & 0 \end{pmatrix} V^{-1} \text{ where } V^T MV = \begin{pmatrix} V_1 & V_2 \\ \frac{1}{\alpha} V_1 \end{pmatrix}
\]
and $U_1, U_2, V_1$ and $V_2$ are diagonal matrices.

When $\text{Inn}_A$ and $\text{Inn}_B$ are viewed as involutions of $SO(n, k[\sqrt{\alpha}], \beta)$, then they are Type 1 involutions. Further, we can choose $X$ and $Y \in GL(n, k[\sqrt{\alpha}])$ such that $A = X \begin{pmatrix} -I \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & I \frac{1}{\sqrt{\alpha}} \end{pmatrix} X^{-1}$, $B = Y \begin{pmatrix} -I \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & I \frac{1}{\sqrt{\alpha}} \end{pmatrix} Y^{-1}$, and
\[
X_1 = \frac{1}{2}(U_1 + \sqrt{\alpha} U_2),
\]
\[
X_2 = \frac{1}{2}(U_1 - \sqrt{\alpha} U_2),
\]
\[
Y_1 = \frac{1}{2}(V_1 + \sqrt{\alpha} V_2),
\]
and
\[
Y_2 = \frac{1}{2}(V_1 - \sqrt{\alpha} V_2).
\]

This follows from the way in which $U$ and $V$ are constructed from the eigenvalues of $A$ and $B$. We need to simply have $X$ and $Y$ consist of the appropriate eigenvectors, and mandate that the last $\frac{n}{2}$ columns of $X$ and $Y$ are the $\sqrt{\alpha}$-conjugates of the first $\frac{n}{2}$ columns. (The only exception to this is that we may need to negate the first column of $X$, so that we can preserve isomorphy of $\text{Inn}_A$ and $\text{Inn}_B$ over $SO(n, k[\sqrt{\alpha}], \beta)$, if we are assuming that. But, this will not change the value of $X_1$.)

Now, suppose $\text{Inn}_A$ and $\text{Inn}_B$ are isomorphic over $SO(n, k[\sqrt{\alpha}], \beta)$ as Type 1 involutions. Then, from Theorem 4.3 we know that $Y_1$ is congruent to either $X_1$ or $X_2$. 

In the first case, we see that
\[
\frac{1}{2}(V_1 + \sqrt{\alpha}V_2) = Y_1
\]
\[
= (R_1 + \sqrt{\alpha}R_2)^T X_1(R_1 + \sqrt{\alpha}R_2)
\]
\[
= (R_1 + \sqrt{\alpha}R_2)^T \frac{1}{2}(U_1 + \sqrt{\alpha}U_2)(R_1 + \sqrt{\alpha}R_2),
\]
where \( R_1 \) and \( R_2 \) are over \( k \).

In the second case, we see that
\[
\frac{1}{2}(V_1 + \sqrt{\alpha}V_2) = Y_1
\]
\[
= (R_1 + \sqrt{\alpha}R_2)^T X_2(R_1 + \sqrt{\alpha}R_2)
\]
\[
= (R_1 + \sqrt{\alpha}R_2)^T \frac{1}{2}(U_1 - \sqrt{\alpha}U_2)(R_1 + \sqrt{\alpha}R_2),
\]
where \( R_1 \) and \( R_2 \) are over \( k \).

It follows from this that
\[
V_1 = R_1^T U_1 R_1 + \alpha R_2^T U_2 R_1 + \alpha R_1^T U_2 R_2 + \alpha R_2^T U_1 R_2
\]
and
\[
V_2 = R_2^T U_1 R_1 + R_1^T U_2 R_1 + \alpha R_2^T U_2 R_2 + R_1^T U_1 R_2,
\]
or
\[
V_1 = R_1^T U_1 R_1 - \alpha R_2^T U_2 R_1 - \alpha R_1^T U_2 R_2 + \alpha R_2^T U_1 R_2,
\]
and
\[
V_2 = R_2^T U_1 R_1 - R_1^T U_2 R_1 - \alpha R_2^T U_2 R_2 + R_1^T U_1 R_2.
\]

The previous theorem tells us that this means that \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic over \( O(n, k, \beta) \). Since the converse is clear, then we have shown what was needed.

\[\square\]

4.3. Type 3 Involutions. We now examine the Type 3 case. Recall that \( \varphi \) is a Type 3 involution if \( \varphi = \text{Inn}_A \), where \( A \in O(n, k, \beta) \) and \( A^2 = -I \). Such matrices have eigenvalues \( \pm i \), and are diagonalizable because the minimal polynomial has no repeated roots. We begin by proving a couple of results about the eigenvectors of such matrices.

Lemma 4.10. Suppose \( A \in O(n, k, \beta) \) induces a Type 3 involution of \( SO(n, k, \beta) \). Also suppose \( x, y \in k^n \) such that \( x + iy \in E(A, -i) \). Then, \( x - iy \in E(A, i) \). Likewise, if \( u, v \in k^n \) such that \( u + iv \in E(A, i) \), then \( u - iv \in E(A, -i) \). Further, \( \dim(E(A, i)) = \dim(E(A, -i)) \).

Proof. Suppose \( x, y \in k^n \) such that \( x + iy \in E(A, -i) \). Then,
\[
A(x + iy) = -i(x + iy)
\]
implies
\[
Ax + iAy = y - ix.
\]
If we take the complex conjugate, then we see that
\[
Ax - iAy = y + ix.
\]
This implies
\[
A(x - it) = i(x - it),
\]
which shows that \( x - iy \in E(A, i) \). A similar proof will show that if \( u, v \in k^n \) such that \( u + iv \in E(A, i) \), then \( u - iv \in E(A, -i) \).

Since \( x + iy \in E(A, -i) \) implies \( x - iy \in E(A, i) \) and vice versa, then we see that \( \dim(E(A, i)) = \dim(E(A, -i)) \). \[\square\]
Lemma 4.11. Suppose $\vartheta = \text{Inn}_A$ is a Type 3 involution of $\text{SO}(n,k,\beta)$ where $A \in \text{O}(n,k,\beta)$. Then, we can find $x_1, \ldots, x_\frac{n}{2}, y_1, \ldots, y_\frac{n}{2} \in k^n$ such that the $x_j + iy_j$ are a basis for $E(A, -i)$ and the $x_j - iy_j$ are a basis for $E(A, i)$.

Proof. Since $\text{Inn}_A$ is Type 3, then we are assuming that $A \in \text{O}(n,k,\beta)$ and $A^2 = -I$. Note that this also means that $n$ is even. It follows that all eigenvalues of $A$ are $\pm i$. Since there are no repeated roots in the minimal polynomial of $A$, then we see that $A$ is diagonalizable. We wish to construct bases for $E(A, i)$ and $E(A, -i)$ such that all the vectors lie in $k[i]^n$. Let $\{z_1, \ldots, z_n\}$ be a basis for $k^n$. For each $j$, let $u_j = z_j + iA z_j$. Note that

$$Au_j = A(z_j + iAz_j)$$

$$= (A + iA^2)z_j$$

$$= (A - iI)z_j$$

$$= -i(z_j + iAz_j)$$

$$= -iu_j.$$ 

So, $\{u_1, \ldots, u_n\}$ must span $E(A, -i)$. Thus, we can appropriately choose $\mathcal{B}$ of these vectors and form a basis for $E(A, -i)$. Note that each of these vectors lies in $k[i]^n$. Label these basis vectors as $v_1, \ldots, v_\frac{n}{2}$. We can write each of these vectors as $v_j = x_j + iy_j$. By the previous lemma, we know that $x_j - iy_j \in E(A, i)$.

Since these vectors will be linearly independent, then they form a basis for $E(A, i)$. We are now able to prove results that characterize the matrices that induce Type 3 involutions, and then use these characterizations to find conditions on these involutions that are equivalent to isomorphy. We will have to prove our result by looking at separate cases, depending on whether or not $i = \sqrt{-1}$ lies in $k$. We begin by assuming that $i \in k$.

Lemma 4.12. Assume $i \in k$ and suppose $\vartheta = \text{Inn}_A$ is a Type 3 involution of $\text{SO}(n,k,\beta)$, where $A \in \text{O}(n,k,\beta)$. Then, $A = X \left( \begin{array}{cc} -iI & 0 \\ 0 & iI \end{array} \right) X^{-1}$ for some $X \in \text{GL}(n,k)$, where $X^T M X = \left( \begin{array}{cc} 0 & X_1 \\ X_1 & 0 \end{array} \right)$, where $X_1$ is a diagonal matrix.

Proof. We know from Lemma 4.11 that we have bases for $E(A, -i)$ and $E(A, I)$ that lie in $k^n$. We will show that we can in fact choose bases $a_1, \ldots, a_\frac{n}{2}$ for $E(A, -i) \cap k^n$ and $b_1, \ldots, b_\frac{n}{2}$ for $E(A, i) \cap k^n$ such that $\beta(a_j, a_i) = 0 = \beta(b_j, b_i)$ and $\beta(a_j, b_i)$ is nonzero if and only if $j = i$. We will build these bases recursively.

First, we know that we can choose some nonzero $a_1 \in E(A, -i) \cap k^n$. Then, since $\beta$ is non degenerate, we can choose a vector $t$ such that $\beta(a_1, t) \neq 0$. Note that $E(A, -i) \oplus E(A, i) = k^n$, so we can choose $t_{-i} \in E(A, -i) \cap k^n$ and $t_i \in E(A, i) \cap k^n$ such that $t = t_{-i} + t_i$. Since $\beta(a_1, t_{-i}) = 0$, then it follows that $\beta(a_1, t_i) \in k$ is nonzero. Let $b_1 = t_i$.

Let $E_1 = \text{Span}_k(a_1, b_1)$ and let $F_1$ be the orthogonal complement of $E_1$ in $k^n$. Since the system of linear equations

$$\beta(a_1, x) = 0$$

$$\beta(b_1, x) = 0$$

has $n - 2$ free variables, then we see that $F_1$ has dimension $n - 2$.

We now wish to find $a_2 \in F_1 \cap E(A, -i)$. Similar to the construction in the previous lemma, we can choose $x \in F_1$, and let $a_2 = x + iAx$. It follows that $a_2 \in F_1 \cap E(A, -i)$. Now we want $b_2 \in F_2 \cap E(A, i)$ such that $\beta(a_2, b_2) = 1$. Since $\beta|_{E_1}$ is non degenerate, then there exists some $y \in F_2$ such that $\beta(a_2, y) \neq 0$. Similar to the construction of $b_1$, we see that this implies the existence a vector $b_2$ that fits our criteria.

Now, we let $E_2 = \text{Span}_k(a_1, a_2, b_1, b_2)$ and let $F_2$ be the orthogonal complement of $E_2$ in $k^n$. We continue this same argument $\frac{n}{2}$ times, until we have the bases that we wanted to find. Let

$$X = (a_1, \ldots, a_\frac{n}{2}, b_1, \ldots, b_\frac{n}{2}).$$

Then, the result follows. \qed
Theorem 4.13. Assume that $i \in k$. Then, if $\text{Inn}_A$ and $\text{Inn}_B$ are both Type 3 involutions of $\text{SO}(n,k,\beta)$, then $\text{Inn}_A$ and $\text{Inn}_B$ are isomorphic over $\text{O}(n,k,\beta)$.

Proof. Suppose we have two such involutions of $\text{SO}(n,k,\beta)$. Let them be represented by matrices $A, B \in \text{O}(n,k,\beta)$. By the previous Lemma, we can choose diagonal $X, Y \in \text{GL}(n,k)$ such that

$$X^{-1}AX = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} = Y^{-1}BY,$$
$$X^TMX = \begin{pmatrix} 0 & X_1 \\ X_1 & 0 \end{pmatrix},$$

and

$$Y^TMY = \begin{pmatrix} 0 & Y_1 \\ Y_1 & 0 \end{pmatrix}.$$

Since $X_1$ and $Y_1$ are both invertible diagonal matrices, then we can choose $R_1$ and $R_2 \in \text{GL}(\frac{n_i}{\beta},k)$ such that $Y_1 = R_1^TX_1R_2$. Let $R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$ and $Q = XRY^{-1}$. It follows from this that $R^TX^TMXR = Y^TMY$. We will show that $Q \in \text{O}(n,k,\beta)$ and $Q^{-1}AQ = B$. This will then prove that $\text{Inn}_A$ and $\text{Inn}_B$ lie in the same isomorphy class.

We first show that $Q \in \text{O}(n,k,\beta)$. Note that

$$Q^TMQ = (XRY^{-1})^TM(XRY^{-1}) = (Y^{-1})^T R^T(X^TMX)RY^{-1} = (Y^{-1})^T(Y^TMY)Y^{-1} = M,$$

which proves this claim.

Lastly, we show that $Q^{-1}AQ = B$. We first note that $R$ and $\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}$ commute. Then, we see that

$$Q^{-1}AQ = (XRY^{-1})^{-1}A(XRY^{-1}) = YR^{-1}(X^{-1}AX)RY^{-1}$$
$$= YR^{-1}\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}RY^{-1}$$
$$= YR^{-1}R\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}Y^{-1}$$
$$= Y\begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}Y^{-1}$$
$$= B.$$

We have shown what was needed. \qed

We now begin examining the case where $i \notin k$.

Lemma 4.14. Assume $i \notin k$ and suppose $\vartheta = \text{Inn}_A$ is a Type 3 involution of $\text{SO}(n,k,\beta)$. Then, $A = U \begin{pmatrix} 0 & -i \vartheta \\ i \vartheta & 0 \end{pmatrix}U^{-1}$ for

$$U = \begin{pmatrix} a_1 & a_2 & \cdots & a_\vartheta \\ a_1 & a_2 & \cdots & a_\vartheta \\ b_1 & b_2 & \cdots & b_\vartheta \end{pmatrix} \in \text{GL}(n,k),$$

where the $a_j + ib_j$ are a basis for $E(A,-i)$, the $a_j - ib_j$ are a basis for $E(A,i)$, and $U^TMU = \begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix}$ is a diagonal matrix.

Proof. We know from Lemma 4.11 that we have bases for $E(A,-i)$ and $E(A,i)$ that lie in $k[i]^n$. We will show that we can in fact choose bases $a_1 + ib_1, \ldots, a_\vartheta + ib_\vartheta$ for $E(A,-i) \cap k[i]^n$ and $a_1 - ib_1, \ldots, a_\vartheta - ib_\vartheta$ for $E(A,i) \cap k[i]^n$ such that $\beta(a_j + ib_j, a_l - ib_l)$ is nonzero if and only if $j = l$. From this, we will be able to show that $\beta(a_j, a_l) = 0 = \beta(b_j, b_l)$ when $j \neq l$ and $\beta(a_j, b_l) = 0$ for all $j$ and $l$. We will build these bases recursively.
Recall that given any vector \( x \in k^n \), we know that \( x + iAx \in E(A, -i) \). We want to choose \( x \in k^n \) such that \( \beta(x, x) \neq 0 \). (The reasons for this will become apparent.) \( M \) is an invertible matrix, so there are at least \( n \) instances of \( e_j^T M e_j \neq 0 \). If there is an instance where \( j = l \), let \( x = e_j \). If not, then instead we have \( e_j^T M e_j = 0 = e_l^T M e_j \), and we let \( x = e_j + e_l \). Then,

\[
\beta(x, x) = \beta(e_j + e_l, e_j + e_l) = 2\beta(e_j, e_l) \\
\neq 0.
\]

So, we have \( x \in k^n \) such that \( \beta(x, x) \neq 0 \), and we have \( x + iAx \in E(A, -i) \). Let \( a_1 = x \) and \( b_1 = Ax \). So, \( a_1 + ib_1 \in E(A, -i) \) and \( a_1 - ib_1 \in E(A, i) \). From this, it follows that

\[
\beta(a_1 + ib_1, a_1 - ib_1) = (\beta(a_1, a_1) + \beta(b_1, b_1)) + i(-\beta(a_1, b_1) + \beta(a_1, b_1)) \\
= 2\beta(a_1, a_1) = 2\beta(x, x) \\
\neq 0.
\]

Let \( E_1 = \text{Span}_{k[i]}(a_1 + ib_1, a_1 - ib_1) = \text{Span}_{k[i]}(a_1, b_1) \), and let \( F_1 \) be the orthogonal complement of \( E_1 \) over \( k[i] \). \( F_1 \) has dimension \( n - 2 \), and \( \beta_{|F_1} \) is nondegenerate. So, we can find a nonzero vector \( x \in F_1 \cap k^n \) such that \( \beta_{|F_1}(x, x) = 0 \). So, as in the last case, let \( \alpha_2 = x \) and \( \beta_2 = Ax \). As before, we have \( \beta(a_1 + ib_1, a_1 - ib_1) \neq 0. \)

Let \( E_2 = \text{Span}_{k[i]}(a_2, b_2, b_1, b_2) \), and let \( F_2 \) be the orthogonal complement of \( E_2 \) over \( k[i] \). In this manner, we can create the bases that we noted in the opening paragraph of this proof.

Note that we always have

\[
0 = \beta(a_j + ib_j, a_l + ib_l) = (\beta(a_j, a_l) - \beta(b_j, b_l)) + i(\beta(a_j, b_l) + \beta(b_j, a_l)),
\]

and when \( j \neq l \) we have

\[
0 = \beta(a_j + ib_j, a_l - ib_l) = (\beta(a_j, a_l) + \beta(b_j, b_l)) + i(-\beta(a_j, b_l) + \beta(b_j, a_l)).
\]

This tells us that when \( j \neq l \) that

\[
\beta(a_j, b_l) = \beta(a_l, a_j) = \beta(b_j, b_l) = 0.
\]

When \( j = l \), we see that \( \beta(b_j, b_j) = \beta(a_j, a_j) \) and that \( \beta(a_j, b_j) = -\beta(b_j, a_j) \). The last of these shows that \( \beta(a_j, b_l) = 0 \), regardless of the values of \( j \) and \( l \).

Let

\[
U = (a_1, ..., a_\frac{n}{2}, b_1, ..., b_\frac{n}{2}).
\]

Then, it follows that \( U^T MU = \begin{pmatrix} U_1 & 0 \\ 0 & -I_{\frac{n}{2}} \end{pmatrix} \) where \( U_1 \) is a diagonal \( \frac{n}{2} \times \frac{n}{2} \) matrix.

Lastly, since \( b_j = Aa_j \), then it follows that \( Ab_j = -a_j \). So, we have that

\[
A = U \begin{pmatrix} 0 & -I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0 \end{pmatrix} U^{-1}.
\]

We now look at an example that highlights some of these results that we have just proven in the Type 3 case.

**Example 4.15.** Assume that \( \beta \) is the standard dot product. Then, \( \text{Im}_{|A} \) can be a Type 3 involution of \( \text{SO}(4, \mathbb{R}) \) only if we can choose \( A \) such that it is skew-symmetric and orthogonal since this will imply that \( A^2 = -I \), and if the entries of \( A \) lie in \( k \). Observe that the matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]
is both skew-symmetric and orthogonal with entries in $k$, so it induces a Type 3 involution. It can be shown that $E(A, -i)$ has dimension 2. A basis for this subspace is formed by the vectors

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be shown that

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

are $i$-eigenvectors of $A$, where these are the conjugates of $v_1$ and $v_2$, respectively.

Following the notation of the previous lemma, we have

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where $U^T U = I$ and $A = U \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} U^{-1}$. We note that $U_1 = I$.

We now find conditions on Type 3 involutions that are equivalent to isomorphy, in the case that $i \notin k$.

**Theorem 4.16.** Assume $i \notin k$. Then, if $\text{Inn}_A$ and $\text{Inn}_B$ are both Type 3 involutions of $\text{SO}(n, k, \beta)$, then $\text{Inn}_A$ and $\text{Inn}_B$ are isomorphic over $\text{O}(n, k, \beta)$.

**Proof.** By the previous Lemma, we can choose a matrix $U \in \text{GL}(n, k)$ such that

$$A = U \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} U^{-1}$$

for $U = \begin{pmatrix} a_1 & a_2 & \cdots & a_{\frac{n}{2}} & b_1 & b_2 & \cdots & b_{\frac{n}{2}} \end{pmatrix} \in \text{GL}(n, k)$, where the $a_j + ib_j$ are a basis for $E(A, -i)$, the $a_j - ib_j$ are a basis for $E(A, i)$, and $U^T M U = \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix}$ is a diagonal matrix.

Let

$$X = (a_1 + ib_1, ..., a_{\frac{n}{2}} + ib_{\frac{n}{2}}, a_1 - ib_1, ..., a_{\frac{n}{2}} - ib_{\frac{n}{2}}),$$

and consider $\text{Inn}_A$ and $\text{Inn}_B$ as involutions of $\text{SO}(n, k[i], \beta)$. By construction, we see that $X$ is a matrix that satisfies the conditions of Lemma 4.12 for the group $\text{SO}(n, k[i], \beta)$. We note that $X_1 = 2U_1$. We also know by the previous Theorem that $\text{Inn}_A$ and $\text{Inn}_B$ are isomorphic (when viewed as involutions of $\text{SO}(n, k[i], \beta)$) over $\text{O}(n, k[i], \beta)$. So, we can choose $Q_i \in \text{O}(n, k[i], \beta)$ such that $Q_i^{-1} A Q_i = B$. Let $Y = Q_i^{-1} X$. We now show a couple of facts about $Y$.

First, we note that since $Y$ was obtained from $X$ via row operations, then for $1 \leq j \leq \frac{n}{2}$, the $j$th and $\frac{n}{2} + j$th columns are $i$-conjugates of one another.
Also, note that
\[ Y^{-1}BY = (Q^{-1}_i X)^{-1} B(Q^{-1}_i X) \]
\[ = X^{-1} Q_i B Q_i^{-1} X \]
\[ = X^{-1} AX \]
\[ = \begin{pmatrix} -i I_n & 0 \\ 0 & i I_n \end{pmatrix}. \]

Lastly, we see that
\[ Y^T M Y = (Q^{-1}_i X)^T M (Q^{-1}_i X) \]
\[ = X^T ((Q^{-1}_i)^T M Q_i) X \]
\[ = X^T M X \]
\[ = \begin{pmatrix} 0 & X_1 \\ X_1 & 0 \end{pmatrix} \]
\[ = \begin{pmatrix} 0 & 2U_1 \\ 2U_1 & 0 \end{pmatrix}. \]

We can write
\[ Y = (c_1 + id_1, ..., c_{\frac{n}{2}} + id_{\frac{n}{2}}, c_1 - id_1, ..., c_{\frac{n}{2}} - id_{\frac{n}{2}}) \]
where \( c_j, d_j \in k^n \). So, let
\[ V = (c_1, ..., c_{\frac{n}{2}}, d_1, ..., d_{\frac{n}{2}}) \in \text{GL}(n, k). \]

It follows from what we have shown that \( B = V \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} V^{-1} \) where
\[ V^T M V = \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} = U^T M U. \]

Now, let \( Q = UV^{-1} \). We will show that \( Q^{-1} A Q = B \) and \( Q \in O(n, k, \beta) \). This will prove that \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic over \( O(n, k, \beta) \).

We first show that \( Q \in O(n, k, \beta) \).
\[ Q^T M Q = (UV^{-1})^T M U V^{-1} \]
\[ = (V^{-1})^T (U^T M U) V^{-1} \]
\[ = (V^{-1})^T (V^T M V) V^{-1} \]
\[ = M. \]

Lastly, we show that \( Q^{-1} A Q = B \).
\[ Q^{-1} A Q = (UV^{-1})^{-1} A (UV^{-1}) \]
\[ = V U^{-1} A U V^{-1} \]
\[ = V \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} V^{-1} \]
\[ = B. \]

We have shown what was needed. \( \square \)

Combining the results from this section, we get the following corollary.

**Corollary 4.17.** If \( \text{Inn}_A \) and \( \text{Inn}_B \) are both Type 3 involutions of \( \text{SO}(n, k, \beta) \), then \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic over \( O(n, k, \beta) \). That is, \( \text{SO}(n, k, \beta) \) has at most one isomorphy class of Type 3 involutions.
4.4. Type 4 Involutions. We now move on to a similar classification in the Type 4 case. First, we characterize the eigenvectors of the matrices that induce these involutions. Recall that we can choose $A \in O(n, k[\sqrt{\alpha}], \beta)$ such that each entry of $A$ is a $k$-multiple of $\sqrt{\alpha}$, and that we know $A^2 = -I$. We begin by proving a couple of lemmas about the eigenspaces of these matrices.

Lemma 4.18. Suppose $A \in O(n, k[\sqrt{\alpha}], \beta)$ induces a Type 4 involution of $SO(n, k, \beta)$. Also suppose $x, y \in k^n$ such that $x + \sqrt{-\alpha}y \in E(A, -i)$. Then, $x - \sqrt{-\alpha}y \in E(A, i)$. Likewise, if $u, v \in k^n$ such that $u + \sqrt{-\alpha}v \in E(A, i)$, then $u - \sqrt{-\alpha}v \in E(A, -i)$. Further, $\dim(E(A, i)) = \dim(E(A, -i))$.

Proof. Suppose $x, y \in k^n$ such that $x + \sqrt{-\alpha}y \in E(A, -i)$. Then, $A(x + \sqrt{-\alpha}y) = -i(x + \sqrt{-\alpha}y)$ which implies $Ax + \sqrt{-\alpha}Ay = \sqrt{-\alpha}y - ix$. Then, complex conjugation tells us that $Ax - \sqrt{-\alpha}Ay = \sqrt{-\alpha}y + ix$, which tells us that $A(x - \sqrt{-\alpha}y) = i(x - \sqrt{-\alpha}y)$. A similar argument shows that if $u, v \in k^n$ such that $u + \sqrt{-\alpha}v \in E(A, i)$, then $u - \sqrt{-\alpha}v \in E(A, -i)$.

Since $x + \sqrt{-\alpha}y \in E(A, -i)$ implies $x - \sqrt{-\alpha}y \in E(A, i)$ and vice versa, then we see that $\dim(E(A, i)) = \dim(E(A, -i))$. $\square$

Lemma 4.19. Suppose $\vartheta = \text{Inn}_A$ is a Type 4 involution of $SO(n, k, \beta)$ where $A \in O(n, k[\sqrt{\alpha}], \beta)$. Then, we can find $x_1, \ldots, x_\frac{n}{2}, y_1, \ldots, y_\frac{n}{2} \in k^n$ such that the $x + \sqrt{-\alpha}y$ are a basis for $E(A, -i)$ and the $x - \sqrt{-\alpha}y$ are a basis for $E(A, i)$.

Proof. Since $\text{Inn}_A$ is Type 4, then we are assuming that $A \in O(n, k[\sqrt{\alpha}], \beta)$ and $A^2 = -I$. Note that this also means that $n$ is even. It follows that all eigenvalues of $A$ are $\pm i$. Since there are no repeated roots in the minimal polynomial of $A$, then we see that $A$ is diagonalizable. We wish to construct bases for $E(A, i)$ and $E(A, -i)$ such that all the vectors lie in $k[i]^n$. Let $\{z_1, \ldots, z_n\}$ be a basis for $k^n$. For each $j$, let $u_j = (\sqrt{\alpha}A - \sqrt{-\alpha}I)z_j$. Note that $Au_j = A(\sqrt{\alpha}A - \sqrt{-\alpha}I)z_j = (\sqrt{\alpha}A^2 - \sqrt{-\alpha}A)z_j = -i(\sqrt{\alpha}A - \sqrt{-\alpha}I)z_j = -iu_j$.

So, $\{u_1, \ldots, u_n\}$ must span $E(A, -i)$. Thus, we can appropriately choose $\frac{n}{2}$ of these vectors and form a basis for $E(A, -i)$. Note that each of these vectors lies in $k[i]^n$. Label these basis vectors as $v_1, \ldots, v_{\frac{n}{2}}$. We can write each of these vectors as $v_j = x_j + \sqrt{-\alpha}y_j$. By the previous lemma, we know that $x_j - \sqrt{-\alpha}y_j \in E(A, i)$, and it follows that these will be linearly independent. Since there are $\frac{n}{2}$ of them, then they form a basis for $E(A, i)$. $\square$

We are now able to prove results that characterize the matrices that induce Type 4 involutions, and then use these characterizations to find conditions on these involutions that are equivalent to isomorphism. We will have separate cases, depending on whether or not $\sqrt{-\alpha}$ lies in $k$. We begin by assuming that $\sqrt{-\alpha} \notin k$. Since we are also assuming that $\sqrt{\alpha} \notin k$, then it follows from these two assumptions that $\alpha$ and $-1$ lie in the same square class of $k$. Thus, we can assume in this case that $\alpha = -1$, which means $\sqrt{-\alpha} = 1$.

Lemma 4.20. Assume $\sqrt{-\alpha} \notin k$ and suppose $\vartheta = \text{Inn}_A$ is a Type 4 involution of $SO(n, k, \beta)$. Then, $A = X \left( \begin{array}{c} -i \\ 0 \\ i \\ \sqrt{2} \end{array} \right) X^{-1}$ for some $X \in \text{GL}(n, k)$, where $X^T MX = \left( \begin{array}{c} 0 & X_1 \\ X_1 & 0 \end{array} \right)$, and $X_1$ is diagonal.
Proof. We know from Lemma 4.19 that we have bases for $E(A, -i)$ and $E(A, i)$ that lie in $k^n$. We will show that we can in fact choose bases $a_1, \ldots, a_n$ for $E(A, -i) \cap k^n$ and $b_1, \ldots, b_n$ for $E(A, i) \cap k^n$ such that $\beta(a_j, a_i) = 0 = \beta(b_j, b_i)$ and $\beta(a_j, b_i)$ is nonzero if and only if $j = i$. We will build these bases recursively.

First, we know that we can choose some nonzero $a_1 \in E(A, -i) \cap k^n$. Then, since $\beta$ is non degenerate, we can choose a vector $t$ such that $\beta(a_1, t) \neq 0$. We note that $E(A, -i) \oplus E(A, i) = k^n$, so we can choose $t_{-i} \in E(A, -i) \cap k^n$ and $t_i \in E(A, i) \cap k^n$ such that $t = t_{-i} + t_i$. Since $\beta(a_1, t_{-i}) = 0$, then it follows that $\beta(a_1, t_i) \in k$ is nonzero. Let $b_1 = t_i$.

Let $E_1 = \text{Span}_k(a_1, b_1)$ and let $F_1$ be the orthogonal complement of $E_1$ in $k^n$. Since the system of linear equations

\[
\beta(a_1, x) = 0 \\
\beta(b_1, x) = 0
\]

has $n - 2$ free variables, then we see that $F_1$ has dimension $n - 2$.

We now wish to find $a_2 \in F_1 \cap E(A, -i)$. Similar to the construction in the previous lemma, we can choose $x \in F_1$, and let $a_2 = (\sqrt{-\alpha}A - \sqrt{-\alpha}I)x$. It follows that $a_2 \in F_1 \cap E(A, -i)$. Now we want $b_2 \in F_2 \cap E(A, i)$ such that $\beta(a_2, b_2)$ is nonzero. Since $\beta|_{F_1}$ is non degenerate, then there exists some $y \in F_2$ such that $\beta(a_2, y) \neq 0$. Similar to the construction of $b_1$, we see that this implies the existence of a vector $b_2$ that fits our criteria.

Now, we let $E_2 = \text{Span}_k(a_1, a_2, b_1, b_2)$ and let $F_2$ be the orthogonal complement of $E_2$ in $k^n$. We continue this same argument $\frac{n}{2}$ times, until we have the bases that we wanted to find. Let

\[
X = (a_1, \ldots, a_\frac{n}{2}, b_1, \ldots, b_\frac{n}{2}).
\]

Then, the result follows. \qed

Here is an example of a Type 4 involution when $\sqrt{-\alpha} \in k$.

**Example 4.21.** Assume that $\beta$ is the standard dot product and that $k = \mathbb{F}_3$, the field of three elements. So, the square roots of 2 are $\pm i$. Observe that the matrix

\[
A = i \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 \\
2 & 2 & 0 & 0 \\
2 & 1 & 0 & 0
\end{pmatrix}
\]

is both skew-symmetric and orthogonal. Since each entry is $\mathbb{F}_3$-multiple of $i$, then it follows from our work on automorphisms that $\text{Inn}_A$ is an involution of $SO(4, \mathbb{F}_3)$ of Type 4. A basis for $E(A, -i)$ is formed by the vectors

\[
v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}
\]

and

\[
v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.
\]

It can be shown that

\[
v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.
\]
and

\[ v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} \]

are \( i \)-eigenvectors of \( A \).

Following the notation of the previous lemma, we have

\[ X = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \]

where \( X^T X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \) and \( A = -iX \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} X^{-1} \). We also note that \( X_1 = I \).

Now we characterize the isomorphy classes of Type 4 involutions in the case where \( \sqrt{-\alpha} \in k \).

**Theorem 4.22.** Assume that \( \sqrt{-\alpha} \in k \). Then, if \( \text{Inn}_A \) and \( \text{Inn}_B \) are both Type 4 involutions of \( \text{SO}(n, k, \beta) \) where the entries of \( A \) and \( B \) are \( k \)-multiples of \( \sqrt{\alpha} \), then \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic over \( \text{O}(n, k, \beta) \).

**Proof.** Suppose we have two such involutions of \( \text{SO}(n, k, \beta) \). Let them be represented by matrices \( A, B \in \text{O}(n, k, \beta) \). By the previous Lemma, we can choose \( X, Y \in \text{GL}(n, k) \) such that

\[ X^{-1} A X = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix} = Y^{-1} B Y, \]

\[ X^T M X = \begin{pmatrix} 0 & X_1 \\ X_1 & 0 \end{pmatrix}, \]

and

\[ Y^T M Y = \begin{pmatrix} 0 & Y_1 \\ Y_1 & 0 \end{pmatrix}, \]

where \( X_1 \) and \( Y_1 \) are diagonal.

Since \( X_1 \) and \( Y_1 \) are both invertible diagonal matrices, then we can choose \( R_1 \) and \( R_2 \in \text{GL}(\frac{n}{2}, k) \) such that \( Y_1 = R_1^T X_1 R_2 \). Let \( R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \) and \( Q = X R Y^{-1} \). It follows from this that \( R^T X^T M X R = Y^T M Y \). We will show that \( Q \in \text{O}(n, k, \beta) \) and \( Q^{-1} A Q = B \). This will then prove that \( \text{Inn}_A \) and \( \text{Inn}_B \) lie in the same isomorphy class.

First we show that \( Q \in \text{O}(n, k, \beta) \). Note that

\[ Q^T M Q = (X R Y^{-1})^T M (X R Y^{-1}) = (Y^{-1})^T R^T (X^T M X) R Y^{-1} = (Y^{-1})^T (Y^T M Y) Y^{-1} = M, \]

which proves this claim.
Lastly, we show that $Q^{-1}AQ = B$. We first note that $R$ and $(-t I \ 0 \ I)$ commute. Then, we see that
\[ Q^{-1}AQ = (XRY^{-1})^{-1}A(XRY^{-1}) \]
\[ = YR^{-1}(X^{-1}AX)RY^{-1} \]
\[ = YR^{-1} \left( \begin{array}{cc} -t I & 0 \\ 0 & I \end{array} \right) \]
\[ = YR^{-1}R \left( \begin{array}{cc} -t I & 0 \\ 0 & I \end{array} \right) Y^{-1} \]
\[ = Y \left( \begin{array}{cc} -t I & 0 \\ 0 & I \end{array} \right) Y^{-1} \]
\[ = B. \]

We have shown what was needed. \qed

We now examine the case where $\sqrt{-\alpha} \notin k$.

**Lemma 4.23.** Assume $\sqrt{-\alpha} \notin k$ and suppose $\vartheta = \text{Inn}_A$ is a Type 4 involution of $\text{SO}(n,k,\beta)$. Then, $A = -\frac{\sqrt{-\alpha}}{\alpha} U \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) U^{-1}$ for
\[ U = \left( \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{array} \right) \in \text{GL}(n,k), \]
where the $a_j + \sqrt{-\alpha}b_j$ are a basis for $E(A, -i)$, the $a_j - \sqrt{-\alpha}b_j$ are a basis for $E(A, i)$, and $U^T MU = \left( \begin{array}{cc} v_1 & 0 \\ 0 & \frac{1}{v_1} \end{array} \right)$ is diagonal.

**Proof.** We know from Lemma 4.19 that we have bases for $E(A, -i)$ and $E(A, i)$ that lie in $k[\sqrt{-\alpha}]^n$. We will show that we can in fact choose bases $a_1 + \sqrt{-\alpha}b_1, \ldots, a_n + \sqrt{-\alpha}b_n$ for $E(A, -i) \cap k[i]^n$ and $a_1 - \sqrt{-\alpha}b_1, \ldots, a_n - \sqrt{-\alpha}b_n$ for $E(A, i) \cap k[\sqrt{-\alpha}]^n$ such that $\beta(a_j + \sqrt{-\alpha}b_j, a_l - \sqrt{-\alpha}b_l)$ is nonzero if and only if $j = l$. From this, we will be able to show that $\beta(a_j, a_l) = 0 = \beta(b_j, b_l)$ when $j \neq l$ and $\beta(a_j, b_l) = 0$ for all $j$ and $l$. We will build these bases recursively.

Given any vector $x \in k^n$, we know that $x + iAx \in E(A, -i)$. We want to choose $x \in k^n$ such that $\beta(x, x) \neq 0$. (The reasons for this will become apparent.) $M$ is an invertible matrix, so there are at least $n$ instances of $e_j^T Me_\ell \neq 0$. If there is an instance where $j = l$, let $x = e_j$. If instead we have $e_j^T Me_\ell = 0 = e_l^T Me_j$, then let $x = e_j + e_l$. We note that this works because
\[ \beta(x, x) = \beta(e_j + e_l, e_j + e_l) = 2\beta(e_j, e_l) \neq 0. \]

So, we have $x \in k^n$ such that $\beta(x, x) \neq 0$, and we have $x + iAx \in E(A, -i)$. Let $a_1 = x$ and $b_1 = \frac{1}{\sqrt{-\alpha}}Ax$. So, $a_1 + \sqrt{-\alpha}b_1 \in E(A, -i)$ and $a_1 - \sqrt{-\alpha}b_1 \in E(A, i)$. From this, it follows that
\[ \beta(a_1 + \sqrt{-\alpha}b_1, a_1 - \sqrt{-\alpha}b_1) = (\beta(a_1, a_1) + \sqrt{-\alpha}(-\beta(a_1, b_1) + \beta(a_1, b_1)) \]
\[ = \beta(x, x) + \alpha \beta \left( \frac{1}{\sqrt{-\alpha}}Ax, \frac{1}{\sqrt{-\alpha}}Ax \right) \]
\[ = 2\beta(x, x) \]
\[ \neq 0. \]

Let $E_1 = \text{Span}_{k[\sqrt{-\alpha}]}(a_1 + \sqrt{-\alpha}b_1, a_1 - \sqrt{-\alpha}b_1) = \text{Span}_{k[\sqrt{-\alpha}]}(a_1, b_1)$, and let $F_1$ be the orthogonal complement of $E_1$ over $k[\sqrt{-\alpha}]$. $F_1$ has dimension $n - 2$, and $\beta|_{F_1}$ is nondegenerate. So, we can find a nonzero vector $x \in F_1 \cap k^n$ such that $\beta|_{F_1}(x, x) = 0$. So, as in the last case, let $a_2 = x$ and $b_2 = \frac{1}{\sqrt{-\alpha}}Ax$. As before, we have $\beta(a_2 + \sqrt{-\alpha}b_2, a_2 - \sqrt{-\alpha}b_2) \neq 0$.

Let $E_2 = \text{Span}_{k[\sqrt{-\alpha}]}(a_1, a_2, b_1, b_2)$, and let $F_2$ be the orthogonal complement of $E_2$ over $k[\sqrt{-\alpha}]$. In this manner, we can create the bases that we noted in the opening paragraph of this proof.

Note that we always have
\[ 0 = \beta(a_j + \sqrt{-\alpha}b_j, a_l + \sqrt{-\alpha}b_l) = (\beta(a_j, a_l) - \alpha \beta(b_j, b_l)) + \sqrt{-\alpha}(-\beta(a_j, b_l) + \beta(b_j, a_l)), \]

where $a_j + \sqrt{-\alpha}b_j \in E(A, -i)$ and $a_l + \sqrt{-\alpha}b_l \in E(A, i)$. 

\[ \text{Proof.} \]
and when \( j \neq l \) we have
\[
0 = \beta(a_j + \sqrt{-\alpha} b_j, a_l - \sqrt{-\alpha} b_l) = (\beta(a_j, a_l) + \alpha \beta(b_j, b_l)) + \sqrt{-\alpha}(-\beta(a_j, b_l) + \beta(b_j, a_l)).
\]

This tells us that when \( j \neq l \) that
\[\beta(a_j, b_l) = \beta(a_j, a_l) = \beta(b_j, b_l) = 0.\]

When \( j = l \), we see that \( \beta(b_j, b_j) = \frac{1}{\alpha} \beta(a_j, a_j) \) and that \( \beta(a_j, b_j) = -\beta(b_j, a_j) \). The last of these shows that \( \beta(a_j, b_l) = 0 \), regardless of the values of \( j \) and \( l \).

Let
\[U = (a_1, ..., a_{\frac{n}{2}}, b_1, ..., b_{\frac{n}{2}}).\]

Then, it follows that \( U^T M U = \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} \) where \( U_1 \) is a diagonal \( \frac{n}{2} \times \frac{n}{2} \) matrix.

Lastly, since \( b_j = \frac{1}{\sqrt{\alpha}} A a_j \), then it follows that \( A b_j = -\frac{1}{\sqrt{\alpha}} A a_j \). So, we have that \( A = -\frac{\sqrt{-\alpha}}{\alpha} U \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -\alpha I_{\frac{n}{2}} & 0 \end{pmatrix} U^{-1} \)
\[\Box\]

Here is an example of a Type 4 involution in the case that \( \sqrt{-\alpha} \notin k \).

**Example 4.24.** Assume that \( \beta \) is the standard dot product. Then, \( \text{Inn}_A \) can be a Type 4 involution of \( \text{SO}(4, \mathbb{Q}) \) only if \( A \notin \text{O}(4, \mathbb{Q}) \) is skew-symmetric and orthogonal (if we scale \( A \) appropriately), since this will imply that \( A^2 = -I \). Observe that the matrix
\[
A = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}
\]
is both skew-symmetric and orthogonal. Since each entry is \( k \)-multiple of \( \sqrt{2} \), then it follows from our work on automorphisms that \( \text{Inn}_A \) is an involution of \( \text{SO}(4, \mathbb{Q}) \) of Type 4. It can be shown that \( E(A, -i) \) has dimension 2. A basis for this subspace is formed by the vectors
\[
v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \sqrt{-2} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}
\]

and
\[
v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \sqrt{-2} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}.
\]

It can be shown that
\[
v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \sqrt{-2} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}
\]

and
\[
v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \sqrt{-2} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}
\]

are \( i \)-eigenvectors of \( A \), where these are the conjugates of \( v_1 \) and \( v_2 \), respectively.
Following the notation of the previous lemma, we have

\[ U = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \]

where \( U^T U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \) and \( A = -\frac{\sqrt{\alpha}}{\alpha} U \begin{pmatrix} 0 & I_\frac{n}{2} \\ -\alpha I_\frac{n}{2} & 0 \end{pmatrix} U^{-1} \). We also note that \( U_1 = I \).

We now find conditions on Type 4 involutions that are equivalent to isomorphism in the case where \( \sqrt{-\alpha} \notin k \).

**Theorem 4.25.** Assume \( \sqrt{-\alpha} \notin k \). Then, if \( \text{Inn}_A \) and \( \text{Inn}_B \) are both Type 4 involutions of \( \text{SO}(n, k, \beta) \) where the entries of \( A \) and \( B \) are \( k \)-multiples of \( \sqrt{\alpha} \), then \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic over \( \text{O}(n, k, \beta) \).

**Proof.** By Lemma 4.23 we can choose a matrix \( U \in \text{GL}(n, k) \) such that

\[ A = -\frac{\sqrt{\alpha}}{\alpha} U \begin{pmatrix} 0 & I_\frac{n}{2} \\ -\alpha I_\frac{n}{2} & 0 \end{pmatrix} U^{-1} \]

for

\[ U = \begin{pmatrix} a_1 & a_2 & \cdots & a_{\frac{n}{2}} & b_1 & b_2 & \cdots & b_{\frac{n}{2}} \end{pmatrix}, \]

where the \( a_j + \sqrt{-\alpha}b_j \) are a basis for \( E(A, -i) \), the \( a_j - \sqrt{-\alpha}b_j \) are a basis for \( E(A, i) \), and \( U^T M U = \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots \\ 0 & \cdots & \cdots & u_1 \end{pmatrix} \) is diagonal.

Consider \( \text{Inn}_A \) and \( \text{Inn}_B \) as involutions of \( \text{SO}(n, k[\sqrt{-\alpha}], \beta) \). If \( k[\sqrt{-\alpha}] = k[\sqrt{\alpha}] \), then these are Type 3 involutions of \( \text{SO}(n, k[\sqrt{-\alpha}], \beta) \), since \( A \) and \( B \) would have entries in the field, and \( i \in k[\sqrt{-\alpha}] \). Otherwise, if \( k[\sqrt{-\alpha}] \neq k[\sqrt{\alpha}] \), then these are Type 4 involutions where \( \sqrt{-\alpha} \notin k[\sqrt{-\alpha}] \).

Let

\[ X = (a_1 + \sqrt{-\alpha}b_1, \ldots, a_{\frac{n}{2}} + \sqrt{-\alpha}b_{\frac{n}{2}}, a_1 - \sqrt{-\alpha}b_1, \ldots, a_{\frac{n}{2}} - \sqrt{-\alpha}b_{\frac{n}{2}}). \]

By construction, we see that \( X \) is a matrix that satisfies the conditions of Lemma 4.11 or Lemma 4.20 for the group \( \text{SO}(n, k[\sqrt{-\alpha}], \beta) \). We note that \( X_1 = 2U_1 \). We also know by Corollary 4.17 or Theorem 4.22 that \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic (when viewed as involutions of \( \text{SO}(n, k[\sqrt{-\alpha}], \beta) \)) over \( \text{O}(n, k[\sqrt{-\alpha}], \beta) \). So, we can choose \( Q_\alpha \in \text{O}(n, k[\sqrt{-\alpha}], \beta) \) such that \( Q_\alpha^{-1} AQ_\alpha = B \). Let \( Y = Q_\alpha^{-1}X \). Since \( Y \) is constructed by doing row operations on \( X \), then we can write

\[ Y = (c_1 + \sqrt{-\alpha}d_1, \ldots, c_{\frac{n}{2}} + \sqrt{-\alpha}d_{\frac{n}{2}}, c_1 - \sqrt{-\alpha}d_1, \ldots, c_{\frac{n}{2}} - \sqrt{-\alpha}d_{\frac{n}{2}}), \]

where \( c_j, d_j \in k^n \). We now show a couple of facts about \( Y \).

First, we note that since \( Y \) was obtained from \( X \) via row operations, then for \( 1 \leq j \leq \frac{n}{2} \), the \( j \)th and \( \frac{n}{2} + j \)th columns are \( i \)-conjugates of one another.

Next, we observe that

\[ Y^{-1}BY = (Q_\alpha^{-1}X)^{-1}B(Q_\alpha^{-1}X) = X^{-1}Q_\alpha BQ_\alpha^{-1}X = X^{-1}AX = \begin{pmatrix} -iI_{\frac{n}{2}} & 0 \\ 0 & iI_{\frac{n}{2}} \end{pmatrix}. \]
Lastly, we see that

\[ Y^TMY = (Q_\alpha^{-1}X)^T M(Q_\alpha^{-1}X) \]
\[ = X^T((Q_\alpha^{-1})^T MQ_\alpha)X \]
\[ = X^T MX \]
\[ = \begin{pmatrix} 0 & X_1 \\ X_1 & 0 \end{pmatrix} \]
\[ = \begin{pmatrix} 0 & 2U_1 \\ 2U_1 & 0 \end{pmatrix}. \]

Let

\[ V = (c_1, ..., c_{\frac{n}{2}}, d_1, ..., d_{\frac{n}{2}}) \in \text{GL}(n, k). \]

It follows from what we have shown that

\[ B = -\frac{\sqrt{\alpha}}{\alpha} V \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -\alpha I_{\frac{n}{2}} & 0 \end{pmatrix} V^{-1} \]
\[ \text{where } V^T MV = \begin{pmatrix} U_1 & 0 \\ 0 & \frac{1}{\alpha} U_1 \end{pmatrix} = U^T MU. \]

Now, let \( Q = UV^{-1}. \) We will show that \( Q^{-1}AQ = B \) and \( Q \in O(n, k, \beta). \) This will prove that \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic over \( O(n, k, \beta). \)

We first show that \( Q \in O(n, k, \beta). \)

\[ Q^T MQ = (UV^{-1})^T MUV^{-1} \]
\[ = (V^{-1})^T (U^T MU)V^{-1} \]
\[ = (V^{-1})^T (V^T MV)V^{-1} \]
\[ = M. \]

Lastly, we show that \( Q^{-1}AQ = B. \)

\[ Q^{-1}AQ = (UV^{-1})^{-1} A(UV^{-1}) \]
\[ = VU^{-1} A U V^{-1} \]
\[ = -\frac{\sqrt{\alpha}}{\alpha} V \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -\alpha I_{\frac{n}{2}} & 0 \end{pmatrix} V^{-1} \]
\[ = B. \]

We have shown what was needed. \( \square \)

Combining the results from this section, we get the following corollary.

**Corollary 4.26.** If \( \text{Inn}_A \) and \( \text{Inn}_B \) are both Type 4 involutions of \( \text{SO}(n, k, \beta), \) then \( \text{Inn}_A \) and \( \text{Inn}_B \) are isomorphic over \( O(n, k, \beta) \) if and only if \( A \) and \( B \) have entries lying in the same field extension of \( k. \)

That is, \( \text{SO}(n, k, \beta) \) has at most \( |k^*/(k^*)^2| - 1 \) isomorphy classes of Type 4 involutions.

5. Maximal Number of Isomorphy Classes

From the work we have done, it follows that the maximum number of isomorphy classes of involutions of \( \text{SO}(n, k, \beta) \) over \( O(n, k, \beta) \) is a function of the number of square classes of \( k, \) and the number of congruency classes of invertible diagonal matrices over \( k. \) We first define the following formulas.

**Definition 5.1.** Let \( \tau_1(k) = |k^*/(k^*)^2| - 1 \) and \( \tau_2(m, k) \) be the number of congruency classes of invertible symmetric matrices of \( \text{GL}(m, k) \) over \( \text{GL}(m, k). \)

Let \( C_1(n, k, \beta), C_2(n, k, \beta), C_3(n, k, \beta) \) and \( C_4(n, k, \beta) \) be the number of isomorphy classes of \( \text{SO}(n, k, \beta) \) involutions over \( O(n, k, \beta) \) of types 1, 2, 3, and 4, respectively.

From our previous work, we have the following:
Corollary 5.2.  

(i) If $n$ is odd, then

$$C_1(n, k, \beta) \leq \left( \sum_{m=1}^{n-1} \tau_2(n-m,k)\tau_2(m,k) \right).$$

If $n$ is even, then

$$C_1(n, k, \beta) \leq \left( \sum_{m=1}^{n-1} \tau_2(n-m,k)\tau_2(m,k) \right) + \tau_2\left(\frac{n}{2}, k\right).$$

(ii) If $n$ is even, then

$$C_2(n, k, \beta) \leq \tau_1(k) \left( \left( \tau_2\left(\frac{n}{2}, k\right) \right) + \tau_2\left(\frac{n}{2}, k\right) \right).$$

(iii) If $n$ is even, then

$$C_3(n, k, \beta) \leq 1.$$

(iv) If $n$ is even, then

$$C_4(n, k, \beta) \leq \tau_1(k).$$

(v) If $n$ is odd, then $C_2(n, k, \beta) = C_3(n, k, \beta) = C_4(n, k, \beta) = 0.$

We now list values of $\tau_1$ and $\tau_2$ for a few classes of fields.

**Table 2.** Some values of $\tau_1(k)$

| $k$  | $R$ | $\mathbb{F}_q$, $2 \not| q$ | $Q_2$, $p \not= 2$ | $Q_2$ |
|-----|-----|----------------|----------------|------|
| $\tau_1(k)$ | 0 | 1 | 3 | 7 |

**Table 3.** Some values of $\tau_2(m, k)$

| $k$  | $R$ | $\mathbb{F}_q$, $2 \not| q$ |
|-----|-----|----------------|
| $\tau_1(k)$ | 1 | 1 |

For the $Q_p$, $\tau_2$ is a bit more difficult. Here we have

$$\tau_2(m, Q_p) = \begin{cases} 1 + \cdots \left( \frac{3}{m} \right), & m \leq 3 \\ \frac{2^m}{m} & m \geq 3 \end{cases}$$

when $p \not= 2$ and

$$\tau_2(m, Q_2) = \begin{cases} 1 + \cdots \left( \frac{7}{m} \right), & m \leq 7 \\ \frac{2^m}{m} & m \geq 7 \end{cases}.$$

Based on these values of $\tau_1$ and $\tau_2$, it is a straightforward matter to compute the maximal value of $C_j(n, k, \beta)$ for the fields mentioned above. We do so explicitly for the fields $\overline{k}$, $R$, and $\mathbb{F}_q$ where $2 \not| q$.

**Corollary 5.3.** Suppose $k = \overline{k}$

(i) If $n$ is odd, then $C_1(n, \overline{k}, \beta) \leq \frac{n-1}{2}$. If $n$ is even, then $C_1(n, \overline{k}, \beta) \leq \frac{n}{2}$.

(ii) $C_2(n, \overline{k}, \beta) = 0$.

(iii) If $n$ is odd, then $C_3(n, \overline{k}, \beta) = 0$. If $n$ is even, then $C_3(n, \overline{k}, \beta) \leq 1$.

(iv) $C_4(n, \overline{k}, \beta) = 0$.

Now suppose $k = \mathbb{R}$
(i) If $n$ is odd, then
\[ C_1(n, \mathbb{R}, \beta) \leq \sum_{m=1}^{\frac{n-1}{2}} (m+1)(n-m+1). \]

If $n$ is even, then
\[ C_1(n, \mathbb{R}, \beta) \leq \left( \sum_{m=1}^{\frac{n-1}{2}} (m+1)(n-m+1) \right) + \left( \frac{n}{2} + 1 \right) + \frac{n}{2} + 1. \]

(ii) If $n$ is odd, then $C_2(n, \mathbb{R}, \beta) = 0$. If $n$ is even, then
\[ C_2(n, \mathbb{R}, \beta) \leq \left( \frac{n}{2} + 1 \right) + \frac{n}{2} + 1. \]

(iii) If $n$ is odd, then $C_3(n, \mathbb{R}, \beta) = 0$. If $n$ is even, then $C_3(n, \mathbb{R}, \beta) \leq 1$.

(iv) If $n$ is odd, then $C_4(n, \mathbb{R}, \beta) = 0$. If $n$ is even, then $C_4(n, \mathbb{R}, \beta) \leq 1$.

Lastly, suppose $k = \mathbb{F}_q$ such that $2 \not| q$.

(i) If $n$ is odd, then $C_1(n, \mathbb{F}_q, \beta) \leq 2n - 2$. If $n$ is even, then $C_1(n, \mathbb{F}_q, \beta) \leq 2n - 1$.

(ii) If $n$ is odd, then $C_2(n, \mathbb{F}_q, \beta) = 0$. If $n$ is even, then $C_2(n, \mathbb{F}_q, \beta) \leq 3$.

(iii) If $n$ is odd, then $C_3(n, \mathbb{F}_q, \beta) = 0$. If $n$ is even, then $C_3(n, \mathbb{F}_q, \beta) \leq 1$.

(iv) If $n$ is odd, then $C_4(n, \mathbb{F}_q, \beta) = 0$. If $n$ is even, then $C_4(n, \mathbb{F}_q, \beta) \leq 1$.

### 6. Explicit Examples

#### 6.1. Algebraically Closed Fields

We now find the exact number of isomorphy classes for some $\text{SO}(n, k, \beta)$. We begin by looking at the case where $k = \overline{k}$. Note that all symmetric non-degenerate bilinear forms are congruent to the dot product over an algebraically closed field.

**Corollary 6.1.** Assume $k = \overline{k}$. If $\vartheta$ is an involution of $\text{SO}(n, k)$, then $\vartheta$ is isomorphic to $\text{Inn}_A$ where $A = \begin{pmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$ and $0 \leq m < \frac{n}{2}$, or $A = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$.

**Proof.** Since $k$ is algebraically closed, we know that all involutions of $\text{SO}(n, k)$ are of Type 1 or 3. We first deal with the Type 1 case. We can write $A = X \begin{pmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{pmatrix} X^{-1}$, where we know $X^T X$ is diagonal. We know that $X^T X$ must be congruent to $I$. Since we are looking for representatives of our isomorphy class, we assume $X^T X = I$, and we can choose $X = I$, which means $A = \begin{pmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$ is a representative of our isomorphy class.

We see that Type 3 involutions will exist since $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ will induce a Type 3 involution. Thus, these is one isomorphy class of Type 3 involutions. \( \square \)

We note that in this case, that the maximal number of isomorphy classes do in fact exist. That is, in Corollary [7.3], for the case where $k = \overline{k}$, we have equality in every statement.

#### 6.2. The Standard Real Orthogonal Group

We now examine the case where $\beta$ is the standard dot product, and $k = \mathbb{R}$.

**Corollary 6.2.** If $\vartheta$ is an involution of $\text{SO}(n, \mathbb{R})$, then $\vartheta$ is isomorphic to $\text{Inn}_A$ where $A = \begin{pmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$ and $0 \leq m \leq \frac{n}{2}$, or $A = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$. There are no Type 2 or Type 4 involutions for this group.

**Proof.** We begin with involutions of Type 1 and proceed in a fashion similar to the previous corollary. So, we can write $A = X \begin{pmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{pmatrix} X^{-1}$, where we know $X^T X$ is diagonal. Based on the conditions of Lemma [6.3] we know that $X^T X$ congruent to a diagonal matrix with entries all 1’s and −1’s. Since we are looking for a representative of our congruence class, let us assume we have equality. But, we see that there can be no −1’s in the diagonal since $k = \mathbb{R}$ and $X^T X$ would have to have negative eigenvalues. So,
we assume $X^T X = I_n$, which means we can choose $X = I_n$. So, $A = \left( -I_m 0 \atop 0 I_{n-m} \right)$ is a representative of our isomorphy class.

We now assume we have an involution of Type 2, and we will make use of Lemma \ref{lem:2inv}. We can write $A = -\frac{\sqrt{2\alpha}}{\alpha} X \left( \begin{array}{cc} 0 & I_{\frac{n}{2}} \\ \alpha & 0 \end{array} \right) X^{-1}$ where $X^T X = \left( \begin{array}{cc} X_1 & 0 \\ 0 & \bar{X}_1 \end{array} \right)$ is diagonal. We recall that $\alpha \in \mathbb{R}^*$ but $\sqrt{\alpha} \notin \mathbb{R}^*$. So, $\alpha$ must be a negative number, and we can choose $\alpha = -1$. That is, $X^T X = \left( \begin{array}{cc} X_1 & 0 \\ 0 & -\bar{X}_1 \end{array} \right)$. But, this is a contradiction, because when $k = \mathbb{R}$, there does not exist any nonzero vectors $x$ such that $x^T x \leq 0$, so the whole diagonal of $X^T X$ must be positive, which is not possible. This shows that there are no Type 2 involutions in this case. In a similar way, we can show that there are also no Type 4 involutions in this case.

In the Type 3 case, we have $A = X \left( I_{\frac{n}{2}} 0 \atop 0 I_{\frac{n}{2}} \right) X^{-1}$ where $X^T X = \left( \begin{array}{cc} X_1 & 0 \\ 0 & X_1 \end{array} \right)$ is diagonal. Similar to the Type 1 case, we know that $X^T X$ must be congruent to $I$, so we may assume that $X^T X$ is $I$, and choose $X = I$ as our representative. Then, we get that $A = \left( I_{\frac{n}{2}} -I_{\frac{n}{2}} \right)$ is a representative of the only Type 3 isomorphy class. □

Unlike the algebraically closed case, we note that in this case, that the maximal number of isomorphy classes do not exist. That is, in Corollary \ref{cor:6.3} for the case where $k = \mathbb{R}$, we have an explicit example where we do not have equality. In fact, given that we have seen that the Type 1 and 3 cases must exist for this group, we actually have the minimal number of isomorphy classes possible.

### 6.3. Orthogonal Groups of $\mathbb{F}_q$

We begin by examining the Type 1 involutions where $k = \mathbb{F}_q$ where $q = p^h$ for all cases where $p \geq 3$. This is a complete classification of the involutions when $n$ is odd. We note that for these fields we have $|\langle k^* \rangle^2| = 2$. So, we will use 1 and $M_q$ as representatives of of of the distinct square classes. Based on properties of symmetric matrices over $k = \mathbb{F}_q$, we know that up to congruence, there are two possibilities for $M$: either $M = I_n$ or $M = \left( I_{n-1} 0 \atop 0 M_q \right)$.

**Corollary 6.3.** First, assume that $M = I_n$. Suppose $\vartheta$ is a Type 1 involution of $\text{SO}(n, \mathbb{F}_q)$. Then $\vartheta$ is isomorphy to $\text{Inn}_A$ where we can write $A = I_{n-m,m}$ for $0 \leq m \leq \frac{n}{2}$ or

$$A = \left( \begin{array}{ccc} -I_{m-1} & 0 & 0 \\ 0 & 1 - \frac{2\alpha^2}{M_q} & \frac{2ab}{M_q} \\ 0 & \frac{2ab}{M_q} & I_{n-m-1} \end{array} \right)$$

for $0 \leq m \leq \frac{n}{2}$, where $M_q$ is a nontrivial non-square in $\mathbb{F}_q$ where $a^2 + b^2 = M_q$ and $a, b \in \mathbb{F}_q$.

Now, assume that $M = \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & M_q \end{array} \right)$. Suppose $\vartheta$ is an involution of $\text{SO}(n, \mathbb{F}_q, \beta)$. Then $\vartheta$ is isomorphy to $\text{Inn}_A$ where we can write

$$A = I_{n-m,m} \quad \text{or} \quad A = \left( \begin{array}{cc} -I_{n-m-1} & I_m \\ I_m & -1 \end{array} \right)$$

for $0 \leq m \leq \frac{n}{2}$.

**Proof.** We will use the same methods that we used in the proofs of the previous corollaries. That is, we will use the equivalent conditions of Lemma \ref{lem:involu} to prove that the matrices listed above will be representatives of the isomorphy classes of the involutions of $\text{SO}(n, \mathbb{F}_q)$.

We can write $A = X \left( -I_t 0 \atop 0 I_r \right) X^{-1}$, where $s + t = n$ and we know

$$X^T M X = \left( \begin{array}{cc} X_1 & 0 \\ 0 & X_2 \end{array} \right)$$

must be diagonal, and $X_1$ is an $s \times s$ matrix, and $X_2$ is a $t \times t$ matrix. It is a result from \cite{HWD04} that any diagonal matrix over $\mathbb{F}_q$ must be congruent to either $I$ or $\left( I 0 \atop 0 M_q \right)$ where $M_q$ is some fixed
non-square in \( F_q \). So, we know from the equivalent conditions in Lemma 4.3 that \( X_1 \) and \( X_2 \) must each be congruent to \( I \) or \( \left( \begin{array}{ccc} I & 0 \\ 0 & M_q \end{array} \right) \) (sizing the matrices appropriately). Let us first assume that \( M = I \). Since \( \det(X^T X) = (\det(X))^2 \) is a square, we observe that \( X_1 \) and \( X_2 \) must be simultaneously congruent to either \( I \) or \( \left( \begin{array}{ccc} I & 0 \\ 0 & M_q \end{array} \right) \) (again, sizing appropriately).

Since we are searching for a representative of the congruence class, we can assume that \( X^T X \) is either \( I \) or \( \left( \begin{array}{ccc} I & 0 \\ 0 & M_q \end{array} \right) \). In the first case, we can let \( X = I \), which means \( A = \left( \begin{array}{cc} -I_s & 0 \\ 0 & I_t \end{array} \right) \).

In the latter case, we here we assume \( X^T X = \left( \begin{array}{ccc} I & 0 \\ 0 & M_q \end{array} \right) \), we can let

\[
X = \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & I \\ 0 & -b & 0 \\ a & 0 & b \\ -b & 0 & a \end{array} \right)
\]

where we choose \( a \) and \( b \) so that \( a^2 + b^2 = M_q \). It follows from this that

\[
A = \left( \begin{array}{ccc} -I_{m-1} & 0 & 0 & 0 \\ 0 & 1 - 2a^2/M_q & 0 & 2ab/M_q \\ 0 & 0 & I_{n-m-1} & 0 \\ 0 & 2ab/M_q & 0 & 1 - 2b^2/M_q \end{array} \right).
\]

If we now assume that \( M = \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & M_q \end{array} \right) \), then we have that \( X^T M X \) is congruent to \( \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & M_q \end{array} \right) \) or \( \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & M_q \end{array} \right) \). In the first case, since we are looking for a representative of our congruence class, we can assume \( X^T \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & 0 \\ 0 & M_q \end{array} \right) \) \( \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & M_q \end{array} \right) \). This means we can choose \( A = \left( \begin{array}{cc} -I_s & 0 \\ 0 & I_t \end{array} \right) \) as our representative.

If instead \( X^T M X \) is congruent to \( \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & M_q \end{array} \right) \), then we can choose \( X = \left( \begin{array}{cc} I_{n-1} & 0 \\ 0 & I_{t-1} \end{array} \right) \). This gives representative \( A = \left( \begin{array}{cc} -I_{s-1} & 0 \\ 0 & I_{t-1} \end{array} \right) \).

We note that in these cases, we have that if \( n \) is odd, then \( C_1(n, F_q, \beta) = 2n - 2 \), and if \( n \) is even, then \( C_1(n, F_q, \beta) = 2n - 1 \). That is, for \( k = F_q \), we always have the maximal number of Type 1 involutions. If \( n \) is even, does this occur for the other types of involutions? We now restrict our attention to the case where \( n \) is even and \( \beta \) is the standard dot product. We have seen that we can have at most one class of Type 3 involutions, and we note that in fact \( C_3(n, F_q) = 1 \) since the orthogonal matrix \( A = \left( \begin{array}{cc} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{array} \right) \) will always induce a Type 3 involution.

Lastly, we need only consider Type 2 and Type 4 involutions. We know that \( C_2(n, F_q) \leq 3 \) and \( C_4(n, F_q) \leq 1 \). We will specifically look at the cases where \( q = 3, 5, \) and 7. For these cases, we see that we have existence of both Type 2 and Type 4 involutions via the matrices in Table 4.1.

We note that these examples will all generalize to higher dimensions, so it is clear that for these fields, whenever \( n \) is even we have \( C_2(n, F_q), C_4(n, F_q) \geq 1 \). So, for these three specific fields, we know that \( C_4(n, F_q) = 1 \), and that the number of isomorphy classes of Type 4 involutions is maximized as well. But, for \( \text{SO}(4, F_p) \) where \( p = 3, 5, \) and 7, we have done computations in Maple which use the conditions of Theorem 4.7 that show that for these fields, \( C_2(4, F_p) = 1 \). So, the number of Type 2 isomorphy classes is not maximized in these cases, despite the other three types being maximized. While we have been unable to prove this up to this point, we believe that this is a pattern that would continue. That is, we have the following conjecture:
Type 2 and Type 4 examples for $\text{SO}(4, F_p)$

| $k$ | Type 2 | Type 4 |
|-----|--------|--------|
| $F_3$ | $i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ | $i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ |
| $F_5$ | $\sqrt{2} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ | $\sqrt{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |
| $F_7$ | $\sqrt{3} \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ | $\sqrt{3} \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ |

**Conjecture 6.4.** Suppose that $\text{SO}(n, k)$ is a finite orthogonal group and that $n$ is even. Then, $C_2(n, k) = C_4(n, k) = \tau_1(k)$.

6.4. p-adic numbers. We now turn our attention to the case where $k = \mathbb{Q}_p$. We will assume $M = I_n$. We show a classification of the isomorphy classes of the Type 1 involutions of $\text{SO}(n, \mathbb{Q}_p)$ where $p > 2$, using Lemma 4.3. Note that if $n$ is odd, then this is a complete classification of the isomorphy classes of all the involutions of $\text{SO}(n, \mathbb{Q}_p)$.

Recall that the classification of involutions breaks down to analyzing the diagonal matrix $X^T X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$, specifically by looking at the congruency classes of $X_1$ and $X_2$. The final condition of Lemma 4.3 gave us conditions on the square class of the determinant and the Hasse symbol to classify the isomorphy classes for $\text{SO}(n, \mathbb{Q}_p)$. Using these conditions, we have classified all of the possible isomorphy classes of Type 1 involutions based on what the values of $X_1$ and $X_2$ would be for a representative of the congruency class in Tables 4 and 5. We note that each isomorphy class is determined by the triple $(\det(X_1) = \det(X_2), c_p(X_1), c_p(X_2))$. To show that each of these possible congruency classes exists, one would need to find a matrix $X$ such that $X^T X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$. This would then determine $A$.

In the case where $-1 \not\in (\mathbb{Q}_p^*)^2$, this will always be the case. To see that this is true, note that $X^T X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$ will always be a symmetric matrix with a determinant that is in the same square class as 1. When $-1 \not\in (\mathbb{Q}_p^*)^2$, all such matrices are such that $c_p \left( \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \right) = 1$ is the case. So, $\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$ will be congruent to $I_n$, which gives us the existence of $X$ such that $X^T X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$.

In the case where $-1 \in (\mathbb{Q}_p^*)^2$, then it is possible that $c_p \left( \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \right) = -1$. For these cases, it is not clear (to the authors) that there exists $X$ such that $X^T X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$.

We now assume that $p = 2$, and we construct a classification of the Type 1 involutions. We again note that if $n$ is odd, then this is a complete classification. We see that $\pm 1, \pm 2, \pm 3$ and $\pm 6$ are representatives for all of the the distinct square classes of $(\mathbb{Q}_2^*)^2$. For this case, we have not constructed tables with complete classifications of the two sets of isomorphy classes. Instead, we have constructed a table, Table 7, where there is a diagonal matrix over $\mathbb{Q}_2$ for each possible pair of determinant square class and value of Hasse symbol. A potential isomorphy class is determined by choosing for $X_1$ and $X_2$ any pair of matrices on this table where the two given matrices have determinants in the same square class. So, given the different possible Hasse symbol values, there are at most 24 isomorphy classes of Type 1 involutions. As in some of the previous cases, it is not immediately clear that there does or does not exist a matrix $X$ in each of these cases such that $X^T X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}$.
Table 5. $X_1$ and $X_2$ values when $k = \mathbb{Q}_p$, $p > 2$, and $-1 \not\in (\mathbb{Q}_p^*)^2$

| $X_1$ | $X_2$ | $\det(X_1)$ and $\det(X_2)$ | $c_p(X_1)$ | $c_p(X_2)$ |
|-------|-------|-----------------------------|------------|------------|
| $I_n$ | $I_n$ | $I_{n-3}$ 0 0 0 0 0 0 | 1 | 1 | -1 |
|       |       | $I_{n-2}$ 0 0 0 0 0 0 | $p$ | 1 | 1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $p$ | 1 | -1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $N_p$ | 1 | 1 |
|       |       | $I_{n-2}$ 0 0 0 0 0 0 | $N_p$ | 1 | -1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $pN_p$ | 1 | 1 |
|       |       | $I_{n-2}$ 0 0 0 0 0 0 | $pN_p$ | 1 | -1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $pN_p$ | -1 | -1 |
|       |       | $I_{n-2}$ 0 0 0 0 0 0 | $pN_p$ | -1 | -1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | 1 | -1 | -1 |

Table 6. $X_1$ and $X_2$ values when $k = \mathbb{Q}_p$, $p > 2$ and $-1 \not\in (\mathbb{Q}_p^*)^2$

| $X_1$ | $X_2$ | $\det(X_1)$ and $\det(X_2)$ | $c_p(X_1)$ | $c_p(X_2)$ |
|-------|-------|-----------------------------|------------|------------|
| $I_n$ | $I_n$ | $I_{n-3}$ 0 0 0 0 0 0 | 1 | 1 | -1 |
|       |       | $I_{n-2}$ 0 0 0 0 0 0 | $p$ | -1 | -1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $p$ | -1 | 1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $N_p$ | 1 | 1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $pN_p$ | -1 | -1 |
|       |       | $I_{n-2}$ 0 0 0 0 0 0 | $pN_p$ | -1 | 1 |
|       |       | $I_{n-2}$ 0 0 0 0 0 0 | $pN_p$ | 1 | 1 |
|       |       | $I_{n-1}$ 0 0 0 0 0 0 | $p$ | 1 | 1 |
Table 7. $X_1$ and $X_2$ values when $k = \mathbb{Q}_2$

| $\det(Y)$ square class | $c_f(Y) = 1$ | $c_p(Y) = -1$ |
|-------------------------|--------------|----------------|
| 1                       | $I_n$        | $(I_{n-3} \ 0 \ 0 \ 0)$ |
|                         |              | $(0 \ -2 \ 0 \ 0)$ |
|                         |              | $(0 \ 0 \ 3 \ 0)$ |
|                         |              | $(0 \ 0 \ 0 \ -6)$ |
| $-1$                    | $(I_{n-2} \ 0 \ 0)$ | $(I_{n-1} \ 0)$ |
|                         | $(0 \ 2 \ 0)$ | $(0 \ -1)$ |
|                         | $(0 \ 0 \ -2)$ |                      |
| $2$                     | $(I_{n-2} \ 0 \ 0)$ | $(I_{n-1} \ 0)$ |
|                         | $(0 \ -1 \ 0)$ | $(0 \ 2)$ |
|                         | $(0 \ 0 \ -2)$ |                      |
| $-2$                    | $(I_{n-1} \ 0)$ | $(I_{n-3} \ 0 \ 0 \ 0)$ |
|                         | $(0 \ -2)$ | $(0 \ -1 \ 0 \ 0)$ |
|                         |              | $(0 \ 0 \ -3 \ 0)$ |
|                         |              | $(0 \ 0 \ 0 \ -6)$ |
| $3$                     | $(I_{n-1} \ 0)$ | $(I_{n-2} \ 0 \ 0)$ |
|                         | $(0 \ 3)$ | $(0 \ 2 \ 0)$ |
|                         |              | $(0 \ 0 \ 6)$ |
| $-3$                    | $(I_{n-2} \ 0 \ 0)$ | $(I_{n-1} \ 0)$ |
|                         | $(0 \ -1 \ 0)$ | $(0 \ -3)$ |
|                         | $(0 \ 0 \ -3)$ |                      |
| $6$                     | $(I_{n-1} \ 0)$ | $(I_{n-2} \ 0 \ 0)$ |
|                         | $(0 \ 6)$ | $(0 \ 2 \ 0)$ |
|                         |              | $(0 \ 0 \ 3)$ |
| $-6$                    | $(I_{n-2} \ 0 \ 0)$ | $(I_{n-1} \ 0)$ |
|                         | $(0 \ -1 \ 0)$ | $(0 \ -6)$ |
|                         | $(0 \ 0 \ 6)$ |                      |
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