Dynamical behavior in mimetic $F(R)$ gravity

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Received January 10, 2015
Revised March 10, 2015
Accepted March 23, 2015
Published April 20, 2015

Abstract. We investigate the cosmological behavior of mimetic $F(R)$ gravity. This scenario is the $F(R)$ extension of usual mimetic gravity classes, which are based on re-parametrizations of the metric using new, but not propagating, degrees of freedom, that can lead to a wider family of solutions. Performing a detailed dynamical analysis for exponential, power-law, and arbitrary $F(R)$ forms, we extracted the corresponding critical points. Interestingly enough, we found that although the new features of mimetic $F(R)$ gravity can affect the universe evolution at early and intermediate times, at late times they will not have any effect, and the universe will result at stable states that coincide with those of usual $F(R)$ gravity. However, this feature holds for the late-time background evolution only. On the contrary, the behavior of the perturbations is expected to be different since the new term contributes to the perturbations even if it does not contribute at the background level.

Keywords: modified gravity, dark energy theory

ArXiv ePrint: 1501.00488
1 Introduction

In order to explain the late-time universe acceleration one can follow two main directions. The first is to introduce the concept of dark energy in the framework of General Relativity (for reviews see [1, 2]), while the second is to modify the gravitational sector itself (for reviews see [3, 4]). The latter approach has an additional motivation, namely to improve the UltraViolet behavior that arises from the non-renormalizability of General Relativity and the difficulties of its quantization [5]. However, we mention that one can transform between the above directions, partially or completely, or construct various combined scenarios such as those with nonminimal couplings [6].

In order to construct gravitational modifications one usually adds higher-order corrections to the Einstein-Hilbert action. Amongst them the simplest model is that of $F(R)$ gravity, where one replaces the Ricci scalar $R$ in the action by an arbitrary function $F(R)$ [7], which proves to have interesting cosmological implication such as the successful description of inflation [8] (see [9, 10] for the analysis of the cosmological density perturbations), of late-time acceleration [11–32], or of both in a unified picture [33–35]. Furthermore, other higher-curvature models are those using the Gauss-Bonnet term $G$ [36, 37] or functions of it [37–39], Lovelock combinations [40, 41], Weyl combinations [42, 43], Galileon modifications [44–47], higher spatial-derivatives as in Hořava-Lifshitz gravity [48–64], suitable self-interacting gravitational terms as in nonlinear massive gravity [65–68] etc.
One interesting class of gravitational modification, is that of mimetic gravity [69–76]. In these constructions one parametrizes the metric using new, but not propagating, degrees of freedom, and thus he obtains modified field equations which may admit a wider family of solutions. Usually, one can obtain solutions with an extra term proportional to $a^{-3}$, and that is why many authors talk about “mimetic dark matter”, i.e. a matter-like term of gravitational origin. In these lines, in [77] the authors added an $F(R)$ modification in the framework of mimetic gravity, and showed that the resulting cosmology can accept new solutions comparing to usual $F(R)$ gravity or usual mimetic gravity. Thus, “mimetic $F(R)$ gravity” corresponds to a new class of gravitational modification that deserves further investigation.

In the present work we are interested in studying in detail the cosmological behavior is scenarios governed by mimetic $F(R)$ gravity. In order to bypass the complexity of the involved equations that do not allow for analytical solutions, we use the powerful method of dynamical analysis, which extracts information about the global behavior of the scenario [78, 79]. However, a significant difference comparing to usual mimetic gravity is that now in the Friedmann equations, apart from the term proportional to $a^{-3}$, we obtain the $F(R)$ contributions. Since both these contributions can only be observed through gravitational observations, it is impossible to separate them, and hence one must include them in a unified, dark-energy sector. The situation is similar to the case of “dark radiation”, i.e. a term of gravitational origin proportional to $a^{-4}$, that appears in many models, which is usually considered a part of the dark energy sector [80–84], even if in this case one can use Big Bang Nucleosynthesis in order to constrain it. Therefore, in the scenario at hand, where such a constrain is moreover absent, the incorporation of the new terms in an effective dark energy sector is the only self-consistent approach.

The plan of the work is the following: in section 2 we review the scenario of mimetic $F(R)$ gravity and we apply it in a cosmological framework. In section 3 we investigate the dynamics in the case of an exponential $F(R)$ form, while in section 4 we perform the analysis for a power-law ansatz. In section 5, for completeness, we provide the tools for a general analysis for arbitrary $F(R)$ forms. Finally, in section 6 we discuss the physical features of the obtained results, while section 7 is devoted to the Conclusions.

## 2 Mimetic $F(R)$ gravity and cosmology

In this section we provide a brief review of mimetic $F(R)$ gravity [77]. As we mentioned in the Introduction, the idea behind the general class of mimetic gravities [69–76] is that parametrizing the metric using new (but not propagating) degrees of freedom one can obtain modified field equations which may admit a wider family of solutions. For instance, after the action of a metric theory is given, a convenient parametrization of the metric $g_{\mu\nu}$ is [69]

$$
 g_{\mu\nu} = -\hat{g}^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \hat{g}_{\mu\nu}, \tag{2.1}
$$

and thus the action variation will be performed in terms of both $\hat{g}_{\mu\nu}$ and $\phi$ (for an equivalent formulation using Lagrange multipliers see [70, 71, 85]). We stress that relation (2.1) implies that

$$
 g \left( \hat{g}_{\mu\nu}, \phi \right)^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = -1, \tag{2.2}
$$

which shows that the scalar field will not be a propagating degree of freedom [69, 77, 85, 86]. Additionally, due to the above parametrization, the mimetic extension of the initial theory has become conformally invariant. In summary, variation with respect to $\hat{g}_{\mu\nu}$ will give rise to
the traceless part of the Einstein equations, while variation with respect to \( \phi \) gives the trace part of Einstein equations modified by an extra prefactor and thus allowing for a wider class of solutions. Such solutions may have an effective dark-matter-like component, and since in some sense the whole theory mimics a dark matter sector, the theory is named “mimetic” gravity. Lastly, note that \( \hat{g}_{\mu \nu} \) does not appear in the final equations of motion, since it can be eliminated in terms of the initial metric \( g_{\mu \nu} \) and \( \phi \).

Let us apply the above general instructions in the usual metric \( F(R) \) gravity following [77]. We start from the standard \( F(R) \)-action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{F(R)}{2\kappa^2} + \mathcal{L}_m \right],
\]

where \( \kappa^2 \) is the gravitational constant, \( R \) is the Ricci scalar calculated by the metric \( g_{\mu \nu} \), and \( \mathcal{L}_m \) stands for the matter Lagrangian. Parametrizing the metric according to (2.1) we obtain

\[
S = \int d^4x \sqrt{-\hat{g}} \left( \hat{g}_{\mu \nu}, \phi \right) \left[ \frac{F(R(\hat{g}_{\mu \nu}, \phi))}{2\kappa^2} + \mathcal{L}_m \right].
\]

Hence, variation with respect to \( \hat{g}_{\mu \nu} \) gives [77] :

\[
\frac{1}{2} g_{\mu \nu} F(R(\hat{g}_{\mu \nu}, \phi)) - R(\hat{g}_{\mu \nu}, \phi) \mu F_R(R(\hat{g}_{\mu \nu}, \phi))
+ \nabla \left( g(\hat{g}_{\mu \nu}, \phi) \right)_{\mu} \nabla \left( g(\hat{g}_{\mu \nu}, \phi) \right)_{\nu} F(R(\hat{g}_{\mu \nu}, \phi))
- g(\hat{g}_{\mu \nu}, \phi) \Box (\hat{g}_{\mu \nu}, \phi) F_R(R(\hat{g}_{\mu \nu}, \phi)) + \kappa^2 T_{\mu \nu}
+ \partial_\mu \phi \partial_\nu \phi \left[ 2 F(R(\hat{g}_{\mu \nu}, \phi)) - R(\hat{g}_{\mu \nu}, \phi) F_R(R(\hat{g}_{\mu \nu}, \phi)) \right]
- 3 \Box \left( g(\hat{g}_{\mu \nu}, \phi) \right) F_R(R(\hat{g}_{\mu \nu}, \phi)) + \kappa^2 T = 0,
\]

where \( F_R \) stands for \( \partial F(R)/\partial R \), \( \nabla_\mu \) and \( \Box \) are respectively the covariant derivative and box operators with respect to \( g_{\mu \nu} \) (first expressed in terms of \( g_{\mu \nu} \) and its derivatives and then expanded in terms of \( \hat{g}_{\mu \nu}, \phi \) through (2.1)), and \( T_{\mu \nu} \) is the matter energy-momentum tensor arising from \( \mathcal{L}_m \). Additionally, variation of (2.4) with respect to \( \phi \) leads to

\[
\nabla \left( g(\hat{g}_{\mu \nu}, \phi) \right)_{\mu} \left\{ \partial_\phi \left[ 2 F(R(\hat{g}_{\mu \nu}, \phi)) - R(\hat{g}_{\mu \nu}, \phi) F_R(R(\hat{g}_{\mu \nu}, \phi)) \right]
- 3 \Box \left( g(\hat{g}_{\mu \nu}, \phi) \right) F_R(R(\hat{g}_{\mu \nu}, \phi)) + \kappa^2 T \right\} = 0,
\]

with \( T = g(\hat{g}_{\mu \nu}, \phi)^{\mu \nu} T_{\mu \nu} \) the trace of the matter energy-momentum tensor \( T_{\mu \nu} \). Since the above equations do not contain \( \hat{g}_{\mu \nu} \) explicitly, but only through the combination (2.1), in the following we omit the \( \hat{g}_{\mu \nu} \) and \( \phi \) dependence of the various quantities.

We mention that every solution of standard \( F(R) \) gravity is a solution of the above mimetic \( F(R) \) gravity, however the opposite is obviously not true. Apart from the wider class of solutions, the advantage of the new theory is that it is conformally invariant.

Since we are interested in investigating the cosmological implications of mimetic \( F(R) \) gravity, in the following we consider the flat Friedmann- Robertson-Walker (FRW) metric

\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,
\]

\[
\text{(2.7)}
\]
where \( a(t) \) is the scale factor. Since \( \phi \) is homogeneous in this case, the constraint (2.2) leads to \( \phi = t \), which simplifies significantly the equations. Moreover, the Ricci scalar as usual becomes \( R = 6\dot{H} + 12H^2 \), where \( H \equiv \dot{a}/a \) is the Hubble parameter and dots denoting differentiation with respect to the cosmic time \( t \). Under the above considerations, the 00 and \( ii \) components of (2.5) lead to the same equation

\[
0 = 2F_{RRR}\ddot{R}^2 + 2F_{RR}\ddot{R} + 4HF_{RRR}\dot{R} - 2\left(\dot{H} + 3H^2\right)F_R + F'(R) + 2\kappa^2p_m, \tag{2.8}
\]

while (2.6) gives

\[
\frac{C_\phi}{a^3} = 2F(R) - 6\left(\dot{H} + 2H^2\right)F_R + 3F_{RRR}\dot{R}^2 + 3F_{RR}\ddot{R} + 9HF_{RRR}\dot{R} + \kappa^2(3p_m - \rho_m), \tag{2.9}
\]

where \( C_\phi \) is a constant of integration, and \( \rho_m \) and \( p_m \) are respectively the energy density and pressure of the perfect-fluid matter sector \( (T = -\rho_m + 3p_m) \). These are the Friedmann equations of the scenario at hand. They can be rewritten as

\[
\dot{H} = -H^2 - \frac{C_\phi}{3a^3F_R} - \frac{\kappa^2\rho_m}{3F_R} + \frac{F(R)}{6F_R} + \frac{H\dot{F}_{RR}}{F_R}, \tag{2.10a}
\]

\[
F_{RRR} = -\frac{C}{3a^3\dot{R}^2} - \frac{HF_{RR}}{R} - \frac{\dot{F}_{RR}}{R^2} + 2\frac{HF_R}{R^2} - \frac{F'(R)}{3R^2} - \frac{\kappa^2(3p_m + \rho_m)}{3R^2}, \tag{2.10b}
\]

where we have reduced the Raychaudhuri equation (2.8) to its simpler form (2.10a) by eliminating the third-order derivative \( F_{RRR} \) through (2.9). Finally, for the purpose of the following analysis, it proves convenient to re-express the first Friedmann equation as

\[
\left[H + \frac{\dot{F}_{RR}}{2F_R}\right]^2 + \frac{F'(R)}{6F_R} = \frac{\kappa^2\rho_m}{3F_R} + \frac{C_\phi}{3a^3F_R} + \frac{R}{6} + \frac{\dot{\dot{R}}^2F_{RR}^2}{4F_R^2}. \tag{2.11}
\]

In the case where \( C_\phi = 0 \) we re-obtain the equations of motion of standard \( F(R) \) gravity, however in the general case we obtain a correction-term proportional to \( a^{-3} \). Hence, once again we verify that all solutions of standard \( F(R) \) gravity are solutions of the above theory for \( C_\phi = 0 \).

The extra term proportional to \( a^{-3} \) is present in all mimetic gravity versions \([69–77]\) and since it mimics an effective matter sector it gave the name “mimetic” to this class of theories. However, since this term is an effective term of gravitational origin, and thus it will not appear in the future experimental (direct) verification of dark matter, and since in the Friedmann equations in appears alongside the \( F(R) \) terms, in the present work, and in contrast with the usual mimetic considerations, we prefer to incorporate it inside the effective dark energy sector. The situation is similar to the case of “dark radiation”, i.e. a term of gravitational origin proportional to \( a^{-3} \) that appears in many models. Dark radiation is considered a part of the effective dark energy sector and not a part of the radiation sector (the physics of which is more or less known), although one can use Big Bang Nucleosynthesis data in order to constrain it independently of the rest dark energy sector \([80]\). Hence, in the present model we will consider the term \( C_\phi/a^3 \) as part of the effective dark energy sector, alongside with the \( F(R) \) terms, although one might use gravitational lenses data in order to constraint it independently of the \( F(R) \) terms.
In these lines, we can rewrite the Friedmann equations (2.8), (2.9) in the usual form

\[ H^2 = \frac{\kappa^2}{3} (\rho_m + \rho_{DE}) \]  \\
\[ \dot{H} = -\frac{\kappa^2}{2} (\rho_m + p_m + \rho_{DE} + p_{DE}), \]  \\

(2.12a) \\
(2.12b)

defining the energy density and pressure of the effective dark energy sector as

\[ \rho_{DE} \equiv \frac{1}{\kappa^2} \left[ \frac{RF_R - F}{2} - 3H \dot{R}F_{RR} + 3H^2 (1 - FR) + \frac{C_{\phi}}{a^3} \right] \]  \\
\[ p_{DE} \equiv \frac{1}{\kappa^2} \left[ \dot{R}^2 F_{RRR} + 2H \dot{R}F_{RR} + \dot{R}F_{RR} + \frac{F - RF_R}{2} - \left( 2\dot{H} + 3H^2 \right) (1 - FR) \right]. \]  \\

(2.13a) \\
(2.13b)

Hence, as we can see, the effect of the mimetic \( F(R) \) gravity comparing to the standard \( F(R) \) one, is an extra term in the effective dark energy density, while its pressure remains unaffected. Additionally, we can define the dark energy equation-of-state parameter as usual as

\[ w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}}, \]  \\

(2.14)

while the corresponding quantity of the matter sector is \( w_m = \rho_m/p_m \), which satisfies \(-1 \leq w_m \leq 1\). We mention here that we wrote the above Friedmann equations and we defined the dark energy quantities using the initial gravitational constant \( \kappa^2 \) and not the effective one \( \kappa^2/F_R \), in order to ensure the separate conservation of the dark energy and the matter sectors [7], namely

\[ \dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0, \]  \\
\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0. \]  \\

(2.15a) \\
(2.15b)

Lastly, as usual, the \( F(R) \) form is forced to satisfy the following general conditions [87]: the existence of a stable Newtonian limit requires

\[ |F(R) - R| \ll R, \ |FR - 1| \ll 1, \ RF_{RR} \ll 1, \]  \\

(2.16)

in order for the non-GR corrections to a space-time metric to remain small (the last condition implies that the Compton wavelength is much less than the radius of curvature of the background space-time) [87]. The ghost avoidance and classical and quantum stability requires [88, 89] (see also [90])

\[ F_R > 0, F_{RR} > 0. \]  \\

(2.17)

Note that if \( F_{RR} \) becomes zero for a finite \( R = R_c \), then a weak (sudden) curvature singularity is generally formed [87]. In the absence of matter, the asymptotic future stability of the de Sitter solutions requires

\[ \frac{F_R|_{R=R_1}}{F_{RR}|_{R=R_1}} > R_1, \]  \\

where \( R_1 \) satisfies \( RF_R - 2F(R) = 0 \) [91].

Similarly to the standard \( F(R) \) case, one can in principle impose the desired \( a(t) \) behavior and suitably reconstruct the \( F(R) \) form that generates it [77]. However, in this work we are interested in the inverse procedure, that is first consider a specific \( F(R) \) form and
then investigate the induced universe evolution. In order to achieve this independently of the specific initial conditions, in the following section we apply the powerful method of dynamical analysis \cite{92–99}. In particular, we first transform the cosmological equations into their autonomous form and we extract the corresponding critical points. Then, we linearize the perturbations around these critical points, and we express them in terms of the perturbation matrix. Hence, the eigenvalues of this perturbation matrix for each critical point, determine its type and stability.

3 Mimetic $F(R)$ gravity with exponential form

In this section we examine the behavior of mimetic $F(R)$ gravity, under an exponential $F(R)$ ansatz of the form

$$F(R) = \Lambda \left[ \exp (pR) - 1 \right], \quad (3.1)$$

which is one of the most well-studied in standard $F(R)$ gravity \cite{7}. For convenience we focus on the physically interesting $\Lambda > 0$ and $p > 0$ cases, although the investigation of the general case is straightforward. Additionally, we parametrize this $F(R)$ form as

$$F(R) = f(R) - \Lambda, \quad (3.2)$$

with

$$f(R) = \Lambda \exp (pR), \quad (3.3)$$

The Friedmann equations can now be expressed as:

$$H^2 = \frac{C_\phi}{3a^3f_R} + \frac{\kappa^2\rho_m}{3f_R} - \frac{f}{6f_R} - \frac{H\dot{f}_{RR}}{f_R} \frac{R}{6} + \frac{\Lambda}{6f_R}, \quad (3.2a)$$

$$\dot{H} = -H^2 - \frac{C_\phi}{3a^3f_R} - \frac{\kappa^2\rho_m}{3f_R} + \frac{f}{6f_R} + \frac{H\dot{f}_{RR}}{f_R} - \frac{\Lambda}{6f_R}, \quad (3.2b)$$

while equation (2.10b) becomes

$$\ddot{R} = -\frac{C_\phi e^{-pR}}{3pa^3} + \frac{2H^2}{p} - H\dot{R} - \frac{e^{-pR}(e^{pR} - 1)}{3p^2} - p\dot{R}^2 - \frac{\kappa^2(3w_m + 1)\rho_m e^{-pR}}{3p}. \quad (3.3)$$

In order to transform these equations into their autonomous form, we need to introduce suitably defined auxiliary variables \cite{92–99}. Thus, we define the normalized variables

$$P = \frac{C_\phi}{3a^3D^2f_R}, \quad Q = \frac{H}{D}, \quad x = \frac{\dot{R}f_{RR}}{2Df_R}, \quad y = \frac{f}{6D^2f_R}, \quad z = \frac{\kappa^2\rho_m}{3D^2f_R}, \quad (3.4)$$

where

$$D = \sqrt{\left(\frac{H + \frac{\dot{R}f_{RR}}{2f_R}}{2Df_R}\right)^2 + \frac{f}{6f_R}} = \sqrt{\left(\frac{H + \frac{1}{2p}\dot{R}}{2Df_R}\right)^2 + \frac{1}{6p}}. \quad (3.5)$$

Moreover, we define two more auxiliary variables, which in the present example are related, namely

$$r \equiv -\frac{Rf_R}{f} = -pR, \quad m \equiv \frac{Rf_{RR}}{f_R} = pR = -r. \quad (3.6)$$

The role of these two variables will become clear in section 5.

\footnote{Note that at late times, i.e. for small curvatures ($R \ll 1$), we have $F(R) \sim R + \frac{\kappa^2R^2}{3} + \mathcal{O}(R^3)$.}
Additionally, the constraint equation becomes

\[ \Omega_A - e^r \left[ 1 - (Q + x)^2 \right] = 0. \]  

(3.8)

Additionally, the constraint equation becomes

\[ P + x^2 - ry + \Omega_A + z = 1. \]  

(3.9)

The appearance of the above two constraints and the constraint \((Q + x)^2 + y = 1\) allows us to eliminate three auxiliary variables, for instance \(y\) and \(z\) and \(r\). Hence, the dynamical equations for the remaining variables write as

\[
P' = P \left\{ x - (Q + x) \left\{ 3r [(Q + x)^2 - 1] + 2Qx - x^2 + 3\Omega_A \right\} \right.
\]

\[
- 3Pw_m(Q + x) \left\{ P + r \left[ (Q + x)^2 - 1 \right] + x^2 + \Omega_A - 1 \right\}, \tag{3.10a}
\]

\[
Q' = - \frac{1}{2} \left\{ 3Q^3r + Q^2(9r + 2)x + Q^2 \left[ r (9x^2 - 5) + x^2 + 3\Omega_A + 1 \right] \right.
\]

\[
- Qx \left[ r (7 - 3x^2) + x^2 - 3\Omega_A + 3 \right] - 2r (x^2 - 1) \right\}
\]

\[
- \frac{3}{2} Qw_m(Q + x) \left\{ P + r \left[ (Q + x)^2 - 1 \right] + x^2 + \Omega_A - 1 \right\}, \tag{3.10b}
\]

\[
x' = - \frac{1}{2} \left\{ 3Q^3rx - Q^2 \left[ -(9r + 2)x^2 + r + 4 \right] + Qx \left[ r (9x^2 - 5) + x^2 + 3\Omega_A - 5 \right] \right.
\]

\[
+ (x^2 - 1) \left[ r (3x^2 - 1) - x^2 + 3\Omega_A - 3 \right] \right\}
\]

\[
- \frac{3}{2} w_m \left[ x(Q + x) - 1 \right] \left\{ P + r \left[ (Q + x)^2 - 1 \right] + x^2 + \Omega_A - 1 \right\}, \tag{3.10c}
\]

\[
\Omega_A' = - \Omega_A \left\{ 3Q^3r + Q^2(9r + 2)x + Q \left[ r (9x^2 - 3) + x^2 + 3\Omega_A - 3 \right] \right.
\]

\[
+ x \left[ 3r (x^2 - 1) - x^2 + 3\Omega_A - 1 \right] \right\}
\]

\[
- 3\Omega_A w_m(Q + x) \left\{ P + r \left[ (Q + x)^2 - 1 \right] + x^2 + \Omega_A - 1 \right\}. \tag{3.10d}
\]

where \(r\) is expressed in terms of the other variables as

\[ r = \ln \left[ \frac{\Omega_A}{1 - (Q + x)^2} \right]. \]  

(3.11)

In the above equations, the primes denote derivatives with respect to the new time variable \(\eta\) defined as \(d\eta = Ddt\). Thus, the system (3.10) determines a flow on the region of the phase space defined as

\[ \Psi_1 = \left\{ (P, Q, x, \Omega_A) \in \mathbb{R}^5 : |Q + x| \leq 1, 0 \leq P + x^2 - r \left[ 1 - (Q + x)^2 \right] + \Omega_A \leq 1, \right. \]

\[ r = \ln \left[ \frac{\Omega_A}{1 - (Q + x)^2} \right], P \geq 0 \}. \]  

(3.12)

\[ ^2 \text{Note that from (3.10a) it follows that the sign of } P \text{ is invariant, and recall that we have assumed that } f_R > 0 \text{ which implies that } P \geq 0. \]
Lastly, in terms of the auxiliary variables (3.4) and $r$, explicitly given by (3.11), the matter and dark-energy density parameters from (2.12a), (2.13a), the deceleration parameter, the dark-energy equation-of-state parameter (2.14), and the total equation-of-state parameter, are written as

$$\Omega_m = \frac{\kappa^2 \rho_m}{3H^2} = \frac{e^{-r} \left\{ 1 - P + (e^r - r) \left[ (Q + x)^2 - 1 \right] - x^2 \right\}}{Q^2},$$

$$\Omega_{DE} = \frac{\kappa^2 \rho_{DE}}{3H^2} = -\frac{e^{-r} \Delta_1}{Q^2},$$

$$q \equiv -1 - \frac{\dot{H}}{H^2} = \frac{r \left[ 1 - (Q + x)^2 \right]}{Q^2} + 1,$$

$$w_{DE} = \frac{e^{r} \left\{ Q^2(2r + 3w_m - 1) + Q(4rx + 6wx_m) + (x^2 - 1)(2r + 3w_m) \right\}}{3\Delta_1} - \frac{w_m \left\{ P + r \left[ (Q + x)^2 - 1 \right] + x^2 - 1 \right\}}{\Delta_1},$$

$$w_{tot} = -1 - \frac{2\dot{H}}{3H^2} = \frac{2q - 1}{3} = \frac{2r \left[ (Q + x)^2 - 1 \right] + 1}{3\Delta_1},$$

where $\Delta_1 = e^r \left( 2Qx + x^2 - 1 \right) - \left\{ P + r \left[ (Q + x)^2 - 1 \right] + x^2 - 1 \right\}$.

The scenario of mimetic $F(R)$ gravity with the exponential form (3.1), i.e. the system (3.10) that lies on the reduced phase space (3.12), admits three isolated physical critical points (note that the appearance of the constraint (3.11) reduces their number significantly), which are displayed in table 1 along with their existence and stability conditions. The details of the analysis and the calculation of the various eigenvalues of the $5 \times 5$ perturbation matrix are presented in appendix A. Furthermore, for each critical point we calculate the values of various observables, such as the density parameters, the deceleration parameter and the dark-energy and total equation-of-state parameters, given by (3.13)-(3.17), and we summarize the results in table 2.

| Name | $P$ | $Q$ | $r$ | $x$ | $\Omega_\Lambda$ | Existence | Stability |
|------|-----|-----|-----|-----|-----------------|-----------|-----------|
| $\Sigma_1$ | 0   | 0   | 0   | 0   | 1               | always    | nonhyperbolic (see numerics) |
| $\Sigma_2$ | 0   | $Q_{c1}$ | $\frac{2Q_{c1}^2}{Q_{c1}^2 - 1}$ | 0   | $1 - 2Q_{c1}^2$ | always    | saddle    |
| $\Sigma_3$ | 0   | $Q_{c2}$ | $\frac{2Q_{c2}^2}{Q_{c2}^2 - 1}$ | 0   | $1 - 2Q_{c2}^2$ | always    | saddle    |

Table 1. The real critical points of the system (3.10) of mimetic $F(R)$ gravity with the exponential form (3.1) and their existence and stability conditions. The parameters $Q_{c1}$ and $Q_{c2}$ correspond to the two roots of the transcendental equation $2Q_c^2 - e^{\frac{Q_c^2}{Q_c^2 - 1}} (Q_c^2 - 1) - 1 = 0$, which numerically are found to be $Q_{c1} = -Q_{c2} \approx 0.666$, which belong to the interval $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and thus $\Omega_\Lambda > 0$.

4 Mimetic $F(R)$ gravity with power-law form

In this section we study the behavior of mimetic $F(R)$ gravity under a power-law $F(R)$ ansatz of the form

$$F(R) = R + \alpha R^n - \Lambda,$$

(4.1)
Table 2. The real critical points of the system (3.10) of mimetic $F(R)$ gravity with the exponential form (3.1), and the corresponding values of the matter and dark energy density parameters, of the deceleration parameter, and of the dark-energy and total equation-of-state parameters, calculated through (3.13)-(3.17).

| Name | $\Omega_m$ | $\Omega_{DE}$ | $q$ | $w_{DE}$ | $w_{tot}$ |
|------|------------|--------------|-----|----------|----------|
| $\Sigma_1$ | arbitrary | arbitrary | arbitrary | arbitrary | arbitrary |
| $\Sigma_2$ | 0 | 1 | -1 | -1 | -1 |
| $\Sigma_3$ | 0 | 1 | -1 | -1 | -1 |

which is also one of the most well-studied in standard $F(R)$ gravity [7]. We focus on the physically interesting $\Lambda > 0$ case, although the analysis of the general case is straightforward.

Furthermore, we parametrize this $F(R)$ form as $F(R) = f(R) - \Lambda$ with $f(R) = R + \alpha R^n$.

The Friedmann equations can now be expressed as:

$$H^2 = \frac{C_\phi}{3a^3 f_R} + \frac{\kappa^2 \rho_m}{3f_R} - \frac{f}{6f_R} - \frac{\dot{H}f_{RR}}{f_R} + \frac{R}{6} + \frac{\Lambda}{6f_R}, \quad (4.2a)$$

$$\dot{H} = -H^2 - \frac{C_\phi}{3a^3 f_R} - \frac{\kappa^2 \rho_m}{3f_R} + \frac{f}{6f_R} + \frac{\dot{H}f_{RR}}{f_R} - \frac{\Lambda}{6f_R}, \quad (4.2b)$$

while equation (2.10b) becomes

$$\ddot{R} = -\frac{C_\phi R^{2-n}}{3\alpha(n-1)n a^3} + H^2 \left[ \frac{2R^{2-n}}{\alpha(n-1)n} + \frac{2R}{n-1} \right] - H\dot{R} - \frac{(n-2)\dot{R}^2}{R} - \frac{\kappa^2(3w_m + 1)\rho_m R^{2-n}}{3\alpha(n-1)n} - \frac{\Lambda R^{2-n}}{3\alpha(n-1)n} - \frac{R^3}{3(n-1)n}. \quad (4.3)$$

In order to transform these equations into their autonomous form we introduce the normalized variables

$$P = \frac{C_\phi}{3a^3 D^2 f_R}, \quad Q = \frac{H}{D}, \quad x = \frac{\dot{f}_{RR}}{2f_R}, \quad y = \frac{f}{6D^2 f_R}, \quad z = \frac{\kappa^2 \rho_m}{3D^2 f_R}, \quad (4.4)$$

with

$$D = \sqrt{\left( H + \frac{\dot{f}_{RR}}{2f_R} \right)^2 + \frac{f}{6f_R}}. \quad (4.5)$$

Moreover, we define two additional auxiliary variables, which in the present example are related, namely

$$r \equiv -\frac{R f_R}{f} = -\frac{R (\alpha n R^{n-1} + 1)}{\alpha R^n + R},$$

$$m \equiv \frac{R f_{RR}}{f_R} = \frac{n(1 + r)}{r}. \quad (4.6)$$

Finally, similarly to the previous section, we define

$$\Omega_\Lambda = \frac{\Lambda}{6D^2 f_R}. \quad (4.7)$$
Hence, from the definitions (4.4) and the first Friedmann equation (4.2a) we deduce that the above auxiliary variables satisfy the constraints

\[ P + x^2 - ry + z + \Omega\Lambda = 1, \]  

(4.8)

and

\[ (Q + x)^2 + y = 1. \]  

(4.9)

Using the above two constraint equations in order to eliminate two auxiliary variables, namely \( y \) and \( z \), we finally result to the following autonomous dynamical system:

\[
P' = - \frac{P^3}{n(r + 1)} \left[ 3nr^2 + 2r^2 + 4nr + n \right] + \frac{PQx^2}{n(r + 1)} \left[ 4 + n(r + 1)(9r + 5) \right] + \frac{P}{n(r + 1)} \left[ 2r^2 + 3n(r + 1)(r - \Omega\Lambda + 1) \right] - \frac{PQx}{n(r + 1)} \left[ 2r^2 + n(r + 1)(9r + 4) \right] + 3PQ \left[ (r - \Omega\Lambda) - Q^2r \right] - \frac{1}{w_m} \left\{ 3PQx \left[(r + 1)x^2Q + (3r + 1)x^2Q^2 + x [3rQ^2 + PQ - Q(r - \Omega\Lambda + 1)] \right] + 3Qx^2 \left[(9r + 3) - 3PQ(r - \Omega\Lambda + 1 - Q^2r - Q) \right] \right\},
\]

(4.10a)

\[
Q' = - \frac{Qx^2}{2n(r + 1)} \left[ (3n + 2)r^2 + 4nr + n \right] + \frac{Q^2x^2}{2n(r + 1)} \left[ 4r^2 + n(r + 1)(9r + 5) \right] + \frac{Q}{2n(r + 1)} \left[ 2r^2 + n(r + 1)(7r - 3\Omega\Lambda + 5) \right] - \frac{Qx}{n(r + 1)} \left[ 2r^2 + n(r + 1)(9r + 4) \right] - 3Qx \left[ (9r + 3) - 3PQ(r - \Omega\Lambda + 1 - Q^2r - Q) \right]
\]

(4.10b)

\[
x' = - \frac{nx^2}{2n(r + 1)} \left[ (3n + 2)r^2 + 4nr + n \right] + Qx^2 \left[ 4r^2 + n(r + 1)(9r + 5) \right] + \frac{x^2}{2n(r + 1)} \left[ 2r^2 + n(r + 1)(4r - 3\Omega\Lambda + 4) \right] - Q^2 \left[ 2r^2 + n(r + 1)(9r + 4) \right] \left\{ (r + 1)x^4 - Q(3r + 1)x^3 + x \left[-3rQ^2 - P + 2(r + 1) - \Omega\Lambda \right] \right\} + \frac{Qx}{2n(r + 1)} \left[ (9r + 3) - 3Q^2r \right] + \frac{Q^2}{2(r + 4)} - \frac{1}{2} \left[ (r - 3\Omega\Lambda + 3) \right] + \frac{3Qx^2}{2} \left[ (9r + 3) - 3Q^2r \right] + \frac{Q^2}{2} \left[ (9r + 3) - 3Q^2r \right] - 3Qx \left[ 3Q^2 + P - 3r + \Omega\Lambda - 1 \right] + P + Q^2r - r + \Omega\Lambda - 1 \right\},
\]

(4.10c)

\[
\Omega\Lambda' = - \frac{\Omega\Lambda x^2 \left[ 4r^2 + n(r + 1)(9r + 5) \right] + \left[ (3n + 2)r^2 + 4nr + n \right] x\} \frac{\Omega\Lambda x}{\left[ 2r^2 + 3n(r + 1)(r - \Omega\Lambda + 1) \right] - Q^2 \left[ 2r^2 + n(r + 1)(9r + 4) \right]} + \frac{\Omega\Lambda x}{n(r + 1)} \left[ 2r^2 + 3n(r + 1)(r - \Omega\Lambda + 1) \right] - Q^2 \left[ 2r^2 + n(r + 1)(9r + 4) \right] \left\{ x (3rQ^2 + P\Omega\Lambda - r + \Omega\Lambda - 1) \right\} + (r + 1)x^3 - Q \left[ rQ^2 + (3r + 1)x^2 + P - r + \Omega\Lambda - 1 \right] \right\},
\]

(4.10d)
In the above equations the primes denote derivatives with respect to the new time variable \( \eta \) defined as \( d\eta = Ddt \). Hence, the system (4.10) defines a flow on the region of the phase space

\[
\Psi_2 := \{(P, Q, r, x, \Omega_\Lambda) : 0 \leq P - r \left[1 - (Q + x)^2\right] + x^2 + \Omega_\Lambda \leq 1, |Q + x| \leq 1, P \geq 0\}.
\]

(4.11)

Finally, the matter and dark-energy density parameters from (2.12a), (2.13a), the deceleration parameter, the dark-energy equation-of-state parameter (2.14), and the total equation-of-state parameter read as

\[
\Omega_m \equiv \frac{\kappa^2 \rho_m}{3H^2} = \frac{(n - 1)r \left\{P + r \left[(Q + x)^2 - 1\right] + x^2 + \Omega_\Lambda - 1\right\}}{Q^2(n + r)},
\]

(4.12)

\[
\Omega_{DE} \equiv \frac{\kappa^2 \rho_{DE}}{3H^2} = -\frac{\Delta_2}{Q^2(n + r)^3},
\]

(4.13)

\[
q \equiv -1 - \frac{\dot{H}}{H^2} = \frac{r \left[1 - (Q + x)^2\right] + 1}{Q^2},
\]

(4.14)

\[
w_{DE} \equiv \frac{(n + r)\left[Q^2(2r - 1) + 4Qrx + 2r(2x^2 - 1)\right]}{3\Delta_2} + \frac{w_m(n - 1)r \left\{P + r \left[(Q + x)^2 - 1\right] + x^2 + \Omega_\Lambda - 1\right\}}{\Delta_2},
\]

(4.15)

\[
w_{tot} \equiv -1 - \frac{2\ddot{H}}{3H^2} = \frac{2q - 1}{3} = \frac{2r \left[(Q + x)^2 - 1\right]}{3Q^2} + \frac{1}{3},
\]

(4.16)

where \( \Delta_2 = (n - 1)r \left\{P + 2Qrx + (r + 1)\left(x^2 - 1\right) + \Omega_\Lambda\right\} + Q^2(r + 1)[n(r - 1) - r] \).

The scenario of mimetic \( F(R) \) gravity with the power-law form (4.1), i.e. the system (4.10), admits \( 14 \times 2 + 1 = 29 \) isolated physical critical points and three curves of critical points (one of them, namely \( T_{14} \), exist only for a specific value of the parameter \( w_m \)), which are displayed in table 3 along with their existence and stability conditions. The details of the analysis and the calculation of the various eigenvalues of the \( 5 \times 5 \) perturbation matrix are presented in appendix B. Furthermore, for each critical point we calculate the values of various observables, such as the density parameters, the deceleration parameter and the dark-energy and total equation-of-state parameters, given by (4.12)-(4.16), and we summarize the results in table 4. Observe that for some specific points having either \( Q = 0 \) or \( r = -n \) the expressions (4.12)-(4.16) are not well defined (NWD), since the involved limits depend on the limit order.

5 Dynamical analysis for general \( F(R) \) forms

As we saw in the previous sections, in order to perform the stability analysis one needs to choose a specific \( F(R) \) ansatz. However, this is restricting since for different \( F(R) \) forms one must repeat the whole analysis from the start. Hence, in the present section, for completeness, we extend the usual procedure in order to be able to perform the analysis for arbitrary \( F(R) \) forms. Following the generalized method of [100], the idea is to suitably parametrize an arbitrary \( F(R) \) function and perform the dynamical analysis in general. Therefore, after

---

\(^3\)Note that from (4.10a) it follows that the sign of \( P \) is invariant, and recall that we have assumed that \( f_R > 0 \) which implies that \( P \geq 0 \).

---
always stable (unstable) for $\epsilon > 0$, and $n < 1$ or $n > \frac{3}{2}$

saddle unstable (stable) otherwise

always saddle

always unstable (stable) for $w_m < \frac{1}{2}$ and $n < 1$ or $n > \frac{3}{2}$
saddle otherwise

always saddle

always saddle

stable (unstable) for $w_m > -1, n > 2$
saddle otherwise

stable (unstable) for $\frac{1}{2} < n < 2, -1 \leq w_m \leq \frac{1 - 2n + 3n^2}{4-2n}$

non-hyperbolic with 4D stable (unstable) manifold for $0 < n < 2, -1 < w_m$

numerical determination (see appendix B)

saddle

non-hyperbolic

stable for $n < 0, -1 < w_m \leq 1.5 \sqrt{\frac{1}{w_m}} < Q_{17} < \frac{1}{4}$

or $0 < n \leq \frac{1}{2}, -1 < w_m \leq 1, Q_{17} > \frac{1}{2}$

or $n > \frac{1}{2}, -1 < w_m \leq 1, \frac{1}{4} < Q_{17} < \sqrt{\frac{1}{w_m}}$

stable (unstable) for $w_m > -1, 0 < n < 1$
saddle otherwise

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Name & $P$ & $Q$ & $r$ & $x$ & $\Omega_1$ & Existence & Stability \\
\hline
$T_1$ & 0 & 0 & 0 & $\epsilon$ & 0 & always & unstable (stable) \\
$T_2$ & 0 & 0 & $-n$ & $\epsilon$ & 0 & always & saddle \\
$T_3$ & 0 & 2$\epsilon$ & 0 & $-\epsilon$ & 0 & always & saddle \\
$T_4$ & 0 & 2$\epsilon$ & $-n$ & $-\epsilon$ & 0 & always & unstable (stable) for $w_m < \frac{1}{2}$ and $n < 1$ or $n > \frac{3}{2}$

saddle otherwise \\
$T_5$ & $\frac{7}{2}$ & $\frac{7}{2}$ & 0 & $\frac{1}{2}$ & 0 & always & saddle \\
$T_6$ & $\frac{7}{2}$ & $\frac{7}{2}$ & $-n$ & $-\frac{1}{2}$ & 0 & always & saddle \\
$T_7$ & $\frac{7}{2}$ & $\frac{7}{2}$ & $-n$ & $-\frac{1}{2}$ & 0 & always & saddle \\
$T_8$ & 0 & (2$n$ - 1)$\Gamma_2$ & $-n$ & (n - 2)$\Gamma_2$ & 0 & $\frac{1}{2} \leq n \leq 1$ or $n \geq \frac{3}{2}$ & stable (unstable) for $w_m > -1, n > 2$

saddle otherwise \\
$T_9$ & 0 & $-w_3$ & 0 & $w_3$ & 0 & $-1 \leq w_m \leq \frac{1}{2}$ & saddle \\
$T_{10}$ & 0 & $-w_3$ & $-n$ & $-w_3$ & 0 & $-1 \leq w_m \leq \frac{1}{2}$ & saddle \\
$T_{11}$ & 0 & $\frac{2n}{n-1}$ & $-n$ & $-\frac{2(n-1)(w_m+1)}{n^2}$ & 0 & $\frac{1}{2} < n < 2, -1 \leq w_m \leq \frac{1}{4} \frac{n^2}{w_m}$ or $0 < n \leq \frac{1}{2}, -1 \leq w_m \leq \frac{1}{4} (4-n)$ or $n = 2, w_m = -1$ & stable (unstable) for $\frac{1}{2} < n < 2, -1 \leq w_m \leq \frac{1}{4} \frac{n^2}{w_m}$

or $0 < n \leq \frac{1}{2}, -1 \leq w_m \leq \frac{1}{4} (4-n)$

or $n = 2, w_m = -1$

saddle otherwise \\
$T_{12}$ & 0 & $\frac{2\Gamma_2}{\sqrt{n-1}}$ & $-2$ & 0 & 0 & always & non-hyperbolic with 4D stable (unstable) manifold for $0 < n < 2, -1 < w_m$

$T_{13}$ & 0 & $\sqrt{\frac{1}{w_m}}$ & $-n$ & $\frac{1}{w_m}$ & 0 & $n > 2$ & numerical determination (see appendix B) \\
$T_{14}$ & 0 & 1 & $r_{14}$ & 0 & 0 & $w_m = \frac{1}{2}$ & saddle \\
$T_{15}$ & 0 & 0 & 0 & 0 & 1 & $-1 < w_m \leq 1$ & non-hyperbolic \\
$T_{16}$ & 0 & $\frac{2\Gamma_2}{\sqrt{n-1}}$ & 0 & $Q_{16}$ & $1 - Q_{16}^2$ & $-\frac{1}{2} \leq Q_{16} \leq \frac{1}{2}$ & saddle \\
$T_{17}$ & 0 & $Q_{17}$ & $\frac{2w_0}{Q_{17}^{2n-1}}$ & 0 & $1 - 2Q_{17}^2$ & $Q_{17}^2 \leq 1$ & stable for $n < 0, -1 < w_m \leq 1.5 \sqrt{\frac{1}{w_m}} < Q_{17} < \frac{1}{4}$

or $0 < n \leq \frac{1}{2}, -1 < w_m \leq 1, Q_{17} > \frac{1}{4}$

or $n \geq \frac{1}{2}, -1 < w_m \leq 1, \frac{1}{4} < Q_{17} < \sqrt{\frac{1}{w_m}}$

saddle otherwise \\
$T_{18}$ & 0 & $\frac{1}{2}$ & $-n$ & $\frac{1}{2}$ & 0 & always & stable (unstable) for $w_m > -1, 0 < n < 1$

saddle otherwise \\
\hline
\end{tabular}
\caption{The real critical points and curves of critical points of the system \eqref{eq:4.10} of mimetic $F(R)$ gravity with the power-law form \eqref{eq:4.1}. We use the notation $\epsilon = \pm 1$, where $\epsilon = +1$ corresponds to expanding universe and $\epsilon = -1$ to contracting one, with the stability conditions outside parentheses corresponding to $\epsilon = +1$ while those inside parentheses to $\epsilon = -1$. We have defined $\Gamma_1 = \frac{2n(13 - 8n) - 6}{n(n+2)^2 + 3}$, $\Gamma_2 = \sqrt{n/n[9(n-19)+13] - 4}$ and $\Gamma_3 = n^2 + 9(n - 1)^2 w_m^2 + 6 [(n - 4)n + 2] w_m + 2n + 3$. Additionally, $r_{c14}, Q_{c16}$ and $Q_{c17}$ are the parameters of the corresponding curves.}
Table 4. The real critical points and curves of critical points of the system (4.10) of mimetic $F(R)$ gravity with the power-law form (4.1), and the corresponding values of the matter and dark energy density parameters, of the deceleration parameter, and of the dark-energy and total equation-of-state parameters, calculated through (4.12)-(4.16). We use the notation $\epsilon = \pm 1$, where $\epsilon = +1$ corresponds to expanding universe and $\epsilon = -1$ to contracting one, with the stability conditions outside parentheses corresponding to $\epsilon = +1$ while those inside parentheses to $\epsilon = -1$. NWD stands for “Not well-defined”.

This general analysis one can just substitute the specific $F(R)$ form in the obtained results, without the need to repeat the whole dynamical elaboration from the beginning.

In order to parametrize the arbitrary $F(R)$ functions, we introduce the auxiliary variables [16, 100]

$$r = -\frac{RF_R}{F},$$
$$m = \frac{RF_{RR}}{F_R}.$$  \hspace{1cm} (5.1)
Furthermore, we introduce the normalization factor

$$D = \sqrt{\left( H + \frac{RF_{RR}}{2F_R} \right)^2 + \frac{F}{6F_R}},$$

and the normalized variables

$$P = \frac{C_\phi}{3a^3D^2F_R}, \quad Q = \frac{H}{D}, \quad x = \frac{RF_{RR}}{2DF_R}, \quad y = \frac{F}{6D^2F_R}, \quad z = \frac{\kappa^2\rho_m}{3D^2F_R}, \quad u = F_R.$$  

(5.3)

Since the consistency conditions require $F(R) > 0$ and $F_R > 0$, it follows that $P > 0, y > 0, z \geq 0$. Using the above auxiliary variables, the Friedmann equation (2.11) leads to the constraint

$$P + x^2 - ry + z = 1,$$

while the definition of $D$ gives rise to the additional constraint

$$(Q + x)^2 + y = 1.$$  

(5.5)

Therefore, we can use the above two constraints in order to eliminate two variables, which for convenience are chosen to be $y$ and $z$, through

$$y = 1 - (Q + x)^2,$$
$$z = 1 - P - x^2 + r[1 - (Q + x)^2].$$  

(5.6)

Defining a new time variable $\eta$ through $d\eta = Ddt$, we can finally extract the autonomous form of the cosmological equations as

$$P' = \frac{M(r)}{r + 1} \left[ -4Qx^2 + 2(1 - Q^2)x - 2x^3 \right] - \frac{P\left(9Q^2x^2 + 11Q^2r + 4Q^2 - 3r^2 - 4r - 3\right)}{r + 1}$$
$$- 3w_mP \left[ x\left(P + (3Q^2r - r - 1)\right) + PQ + Q(2Q^2r - r - 1) + Q(3r + 1)x^2 + (r + 1)x^3 \right]$$
$$- \frac{PQ\left(9r^2 + 10r + 5\right)x^2}{r + 1} + 3P(1 - Q^2)Qr - \frac{P\left(3r^2 + 2r + 1\right)x^3}{r + 1},$$

(5.7a)

$$Q' = \frac{M(r)}{r + 1} \left[ x(1 - x^2) - Q(x - 2x) \right] - \frac{3Q^4r}{2} - \frac{Q^3[r(9r + 11) + 4|x]}{2(r + 1)}$$
$$- \frac{3}{2} w_m \left[ Q^2\left(P + r(3x^2 - 1) + x^2 - 1\right) + Qx\left(P + (r + 1)(x^2 - 1)\right) + Q^4 + 3Q^3rx \right]$$
$$- \frac{Q^2}{2} \left\{ \frac{r(9r + 10) + 5x^2}{r + 1} - 5r + 1 \right\} + \frac{Qx\left(7r^2 - |r(3r + 2) + 1|x^2 + 10r + 5\right)}{2(r + 1)}$$
$$+ r(x^2 - 1),$$

(5.7b)
from (5.7e) we deduce the two limiting situations at an equilibrium point, namely [100]:

\[
\begin{align*}
\dot{x}' &= \frac{M(r)}{r+1} \left[ -2Qx^3 + (1 - Q^2)x^2 - x^4 \right] - \frac{x^2 (9Q^2r^2 + 11Q^2r + 4Q^2 - 4r^2 - 6r - 4)}{2(r+1)} \\
- \frac{3}{2} w_m \left[ x^2 (2P + 3Q^2r - 2r - 2) + xQ (2P + Q^2r - 3r - 1) + Q(3r + 1)x^3 \\
+ (r + 1)x^4 - P - Q^2r + r + 1 \right] - \frac{1}{2} Qx (3Q^2r - 5r - 5) + \frac{1}{2} (Q^2r + 4Q^2 - r - 3) \\
- \frac{Q (9r^2 + 10r + 5) x^3}{2(r+1)} - \frac{(3r^2 + 2r + 1) x^4}{2(r+1)},
\end{align*}
\]

(5.7c)

\[
\begin{align*}
\dot{r}' &= 2M(r)x, \\
\dot{u}' &= 2xu.
\end{align*}
\]

(5.7d, 5.7e)

and we have the additional equation

\[
\begin{align*}
D' &= \frac{M(r)Dx}{r+1} \left[ 2Qx + (Q - 1)(Q + 1) + x^2 \right] \\
+ \frac{3}{2} D w_m \left[ x (2P + 3Q^2r - 2r - 1) + PQ + Q (2P - r - 1) + Q(3r + 1)x^2 + (r + 1)x^3 \right] \\
+ D \left[ \frac{x (9Q^2r^2 + 11Q^2r + 4Q^2 - 4r^2 - 6r - 5)}{2(r+1)} + \frac{3}{2} Q (Q^2r - r - 1) \right. \\
\left. + \frac{Q (9r^2 + 10r + 5) x^2}{2(r+1)} + \frac{(3r^2 + 2r + 1) x^3}{2(r+1)} \right],
\end{align*}
\]

(5.8)

where primes denoting derivatives with respect to \( \eta \), and with

\[
M(r) = \frac{r(1 + r + m)}{m},
\]

(5.9)

assuming that \( m \) can be expressed as a function of \( r \), namely \( m = m(r) \). Since the equation (5.8) is decoupled form the rest, we are allowed to investigate the restricted dynamical system defined in the phase space

\[
\Psi = \{(P, Q, x, r, u) : |Q + x| \leq 1, 0 \leq P - r [1 - (Q + x)^2] + x^2 \leq 1, P \geq 0 \}.
\]

(5.10)

Additionally, note that since the evolution equation for \( u \) is decoupled too, it follows that the Jacobian matrix of the extended dynamical system for \((P, Q, r, x, u)\) has an extra eigenvalue \( \lambda_u = \frac{\partial \Psi}{\partial u} |_{x=x_e, u=u_e} \), where \((x_e, u_e)\) are the values of \((x, u)\) at the equilibrium point. Hence, from (5.7e) we deduce the two limiting situations at an equilibrium point, namely [100]:

- For \( x_e = 0 \) it follows that \( \lambda_u = 0 \). Thus, at the equilibrium point \( f_R \) acquires a constant value, and the stability issue cannot be resorted by linear analysis.

- For \( x_e \neq 0 \) it is required that \( u_e = 0 \), which implies that \( f_R = 0 \) at the equilibrium point. Additionally, \( \lambda_u = 2x_e \) and thus perturbations along the \( u \)-axis are conditionally stable in the extended phase space for \( x_e < 0 \).

Lastly, the matter and dark-energy density parameters from (2.12a), (2.13a), the deceleration parameter, the dark-energy equation-of-state parameter (2.14), and the total equation-
of-state parameter, can be expressed as

$$\Omega_m = \frac{r^2 \rho_m}{3H^2} = -\frac{\{P + r [(Q + x)^2 - 1] + x^2 - 1\} u}{Q^2},$$

(5.11a)

$$\Omega_{DE} = \frac{r^2 \rho_{DE}}{3H^2} = \frac{\{P + r [(Q + x)^2 - 1] + x^2 - 1\} u + Q^2}{Q^2},$$

(5.11b)

$$q \equiv -1 - \frac{\dot{H}}{H^2} = \frac{r [1 - (Q + x)^2]}{Q^2} + 1,$$

(5.11c)

$$w_{DE} = \frac{w_m \{P + r [(Q + x)^2 - 1] + x^2 - 1\} u}{\{P + r [(Q + x)^2 - 1] + x^2 - 1\} u + Q^2},$$

(5.11d)

$$w_{tot} \equiv -1 - \frac{2\dot{H}}{3H^2} = \frac{2q - 1}{3} = \frac{r (2 - 2(Q + x)^2)}{3Q^2} + \frac{1}{3},$$

(5.11e)

Since equation (5.7e) is decoupled from the rest, we will study the stability of the reduced dynamical system (5.7a)-(5.7d). The scenario of mimetic $F(R)$ gravity with arbitrary $F(R)$ forms, i.e. the system of equation (5.7a)-(5.7d), admits eighteen classes of critical points (nine corresponding to expanding universe and nine corresponding to contracting one), where each class contains as many critical points as the roots of the equation $M(r) = 0$, with the exception of the curves $P_{8}^{\pm}$ which exist for the special value $w_m = \frac{1}{3}$, and $P_{9}^{\pm}$ for which $r = -2$ and $M(-2)$ is not necessarily zero. These are presented in table 5 along with their existence and stability conditions. The details of the analysis and the calculation of the various eigenvalues of the $4 \times 4$ perturbation matrix are presented in appendix C. Additionally, for each class of critical points, using (5.11a)-(5.11e) we can calculate the values of various observables, such as the density parameters, the deceleration parameter and the total equation-of-state parameter, and we present them in table 6.

The above results hold for arbitrary $F(R)$ forms. Hence, given a specific $F(R)$ ansatz, one first calculates its corresponding $M(r)$ using (5.1) and (5.9), then he finds $r^*$ by solving $M(r = r^*) = 0$, and finally one just substitutes $r^*$ in tables 5 and 6.

6 Physical implications

In the previous sections we performed a detailed dynamical analysis for the scenario of mimetic $F(R)$ gravity for exponential and power-law ansatzes, and moreover we presented the method for the general analysis for arbitrary $F(R)$ forms. In this section we discuss on the physical features of the stable solutions, that is of the solutions that can attract the universe at late times, independently of the initial conditions.

6.1 Mimetic $F(R)$ gravity with exponential form

The scenario of mimetic $F(R)$ gravity with the exponential $F(R)$ form of (3.1), exhibits two saddle critical points and one nonhyperbolic one, namely $\Sigma_1$. In the latter case the present linear analysis is not adequate to determine its stability, and thus one needs to apply the center manifold method [101]. However, we mention that all the above points exist also in usual $F(R)$ gravity [102], and this is explained since the extra parameter of mimetic gravity, namely $C_\phi$, in this case is zero. Therefore, we deduce that mimetic $F(R)$
| Name | $P$ | $Q$ | $r$ | $x$ | Existence | Stability |
|------|-----|-----|-----|-----|-----------|-----------|
| $P_{0}^s$ | 0 | 0 | $r^s$ | $\epsilon$ | always | unstable (stable) for $M'(r^s) > 0$, $r^s < -1$ or $M'(r^s) > 0$, $r^s > -\frac{1}{2}$ |
| $P_{1}^s$ | 0 | $2\epsilon$ | $r^s$ | $-\epsilon$ | always | unstable (stable) for $-1 \leq w_m < \frac{1}{4}$, $M'(r^s) < 0$, $r^s < -\frac{1}{2}$ or $-1 \leq w_m < \frac{1}{4}$, $M'(r^s) < 0$, $r^s > -1$ |
| $P_{2}^s$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $r^s$ | $\frac{1}{2}$ | always | stable (unstable) for $0 < w_m \leq 1$, $M'(r^s) < 0$, $-1 < r^s < -\frac{4}{3}$ saddle otherwise |
| $P_{3}^s$ | $P_0$ | $\frac{2\epsilon}{\sqrt{2}+2}\sqrt{\frac{R}{\epsilon}+\frac{\beta}{\alpha^2}}$ | $r^s$ | $\frac{f_m+1}{\sqrt{2}+2}\sqrt{\frac{R}{\epsilon}+\frac{\beta}{\alpha^2}}$ | $r^s \leq -\frac{1}{2}$ | numerical determination |
| $P_{4}^s$ | $Q_1$ | $r^s$ | $-\frac{1+r^s}{2}\sqrt{\frac{R}{\epsilon}+\frac{\beta}{\alpha^2}}$ | $-1 \leq r^s \leq -\frac{1}{2}$ | unstable (stable) for $-1 < r^s < -\frac{1}{2}$, $M'(r^s) > 0$ or $r^s \leq -\frac{1}{2}$ saddle otherwise |
| $P_{5}^s$ | $0$ | $-\frac{2\epsilon}{\beta}$ | $r^s$ | $\frac{2\beta^2}{\beta^2+1}\sqrt{\frac{R}{\epsilon}+\frac{\beta}{\alpha^2}}$ | $r^s = -2$, $w_m = -1$ or stable (unstable) for $M'(r^s) > 0$, $-1.64 < r^s \leq -1.328$, $-1 < w_m < w_m$ $-2 < r^s \leq -\frac{1}{2}$, $-1 \leq w_m \leq -\frac{2\beta^2-2\beta \beta \gamma}{2\beta^2-\beta \beta \gamma}$ or $M'(r^s) > 0$, $-1.328 < r^s < -1$, $-1 < w_m < 0$ or $-\frac{1}{2} < r^s < 0$, $-1 \leq w_m \leq \frac{1}{2}$, $-4$ $\frac{1}{2}$ $\frac{1}{2}$ saddle otherwise |
| $P_{6}^s$ | $0$ | $\epsilon$ | $r_{cs}$ | $0$ | $w_m = \frac{1}{2}$ | saddle |
| $P_{7}^s$ | $\frac{2}{3}$ | $-2$ | $0$ | always | stable (unstable) for $w_m > -1$, $M(-2) > 0$ |

**Table 5.** The real critical points and curves of critical points of the system (5.7a)-(5.7d) of mimetic $F(R)$ gravity, for arbitrary $F(R)$ ansatzes. We use the notation $\epsilon = \pm 1$, where $\epsilon = +1$ corresponds to expanding universe and $\epsilon = -1$ to contracting one, with the stability conditions outside the parentheses corresponding to $\epsilon = +1$ while those inside the parentheses correspond to $\epsilon = -1$. The symbol $r^s$ denotes the roots of the equation $M(r) = 0$, i.e. $r^s = M^{-1}(0)$. Furthermore, we define $P_{bc} = -\frac{2\beta^2}{(r^s-2)r^s+3}\epsilon$, $Q_5 = \sqrt{\frac{3}{\epsilon}}(2r^s+1)$, $r_1 = \sqrt{9(r^s+1)^2w_m^2+6[r^s(r^s+4)+2]w_m+(r^s)^2-2r^s+3}$ and $w_m = \frac{-32(r^s)^4+110(r^s)^2-113r^s-27}{4(4(r^s)^2+24(r^s)^2+29r^s+9)}-\frac{4\gamma^2}{\beta^2+1}\sqrt{-\frac{48(r^s)^2+136(r^s)^2+115(r^s)^2+25(r^s)^2}{4(r^s)^2+24(r^s)^2+29r^s+9}}$. Additionally, $r_{cs}$ is the parameter of curve $P_{8}^s$. 

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Table 6. The real critical points and curves of critical points of the system (5.7a)-(5.7d) of mimetic $F(R)$ gravity for arbitrary $F(R)$ ansatze, and the corresponding values of the rescaled matter density parameter $\Omega_m/f_R$, of the deceleration parameter $q$, and of the total equation-of-state parameter $w_{\text{tot}}$, calculated through (5.11a)-(5.11e). We use the notation $\epsilon = \pm 1$, where $\epsilon = +1$ corresponds to expanding universe and $\epsilon = -1$ to contracting one. The symbol $r^*$ denotes the roots of the equation $M(r) = 0$, i.e. $r^* = M^{-1}(0)$. Furthermore, NWD stands for “Not well-defined”.

| Name | $\Omega_m/f_R$ | $q$ | $w_{\text{tot}}$ |
|------|----------------|-----|-----------------|
| $P_1^\epsilon$ | NWD | NWD | NWD |
| $P_2^\epsilon$ | 0 | 1 | $\frac{1}{3}$ |
| $P_3^\epsilon$ | 0 | 1 | $\frac{1}{3}$ |
| $P_4^\epsilon$ | 0 | $-\frac{3}{2r^*} - 1$ | $-\frac{1}{r^*} - 1$ |
| $P_5^\epsilon$ | 0 | $\frac{3}{2r^* + 1} - \frac{1}{r^* + 1} - 1$ | $\frac{2}{2r^* + 1} - \frac{2}{3(r^* + 1)} - 1$ |
| $P_6^\epsilon$ | $2 - 3w_m$ | 1 | $\frac{1}{3}$ |
| $P_7^\epsilon$ | $-r^* |r^*(6w_m + 8) + 9w_m + 13| + 3(w_m + 1) + 2(r^*)^2$ | $-\frac{3(w_m + 1)}{2r^*} - 1$ | $-r^* + w_m + 1$ |
| $P_8^\epsilon$ | 1 | 1 | $\frac{1}{3}$ |
| $P_9^\epsilon$ | 0 | $-1$ | $-1$ |

gravity with exponential ansatz, presents the same asymptotic behavior with standard $F(R)$ gravity, and thus it does not lead to novel asymptotically late-time features. Additionally, note that apart from the finite critical points of table 1, there could be stable points at “infinity”, which requires to apply the Poincaré central projection method [103]. However, since this investigation lies beyond the scope of the present work, and moreover since these points exist also in usual $F(R)$ gravity and thus are not new, we do not analyze them in more details. Finally, note that the two saddle points $\Sigma_2$ and $\Sigma_3$, which correspond to dark-energy-dominated ($\Omega_\Lambda = 1$), accelerating ($q = -1$) solutions, where dark energy behaves as cosmological constant ($w_{DE} = -1$), and hence they are de Sitter solutions, can be very good candidates for describing the inflationary phase of the cosmic evolution.

In order to present the above behavior more transparently, we numerically evolve the autonomous system at hand, and in figures 1-3 we depict the phase-space behavior. In this example, the critical point $\Sigma_1$ is the stable late-time state of the universe.

### 6.2 Mimetic $F(R)$ gravity with power-law form

The scenario of mimetic $F(R)$ gravity with the power-law $F(R)$ form of (4.1), focusing in the more physically interesting case of expanding universe, exhibits two stable critical points, namely $T_8^+, T_{18}^+$, as well as a stable curve of critical points, namely $T_{17}^+$. We mention here that there are four more critical points that might be stable in a small region of the parameter space (namely $T_{11}^+, T_{12}^+, T_{13}^+$, and $T_{15}^+$), however their exact behavior requires numerical examination.
Figure 1. The phase space of the system (3.10) of mimetic $F(R)$ gravity with the exponential form (3.1), for the choice $w_m = 0$. The point $\Sigma_1$ attracts an open set of orbits. The figure shows the existence of closed orbits too. The behavior is qualitatively the same for different choices of $w_m$.

Figure 2. Projection of the orbits of figure 1 on the $x-\Omega_\Lambda$ plane.

Point $T_8^+$ is stable for $n > 2$, however the corresponding $\Omega_m$ it not well defined. Point $T_{18}^+$ is a stable physical critical point, and thus it can be the late-time state of the universe. It corresponds to a dark-energy-dominated universe, which however is non-accelerating and the dark energy behaves as radiation, which are not favored by observations. This point exist also in standard $F(R)$ gravity [16, 104, 105] as it corresponds to $C_\phi = 0$.

The critical points of the curve $T_1^+$ correspond to dark-energy-dominated, accelerating solutions, where dark energy behaves as cosmological constant $w_{DE} = -1$, and hence they are de Sitter solutions. They exist also in standard $F(R)$ gravity [16, 104, 105] as they correspond to $C_\phi = 0$. 


Figure 3. Projection of the orbits of figure 1 on the $Q$-$\Omega$ plane.

Figure 4. Projection of the phase space of the system (4.10) of mimetic $F(R)$ gravity with the power-law $F(R)$ form of (4.1), on the invariant set $r = -n$, for the choice $w_m = 0, n = 2$. Point $T_{12}^+$ is the late-time attractor for the universe. Notice also the presence of heteroclinic orbits connecting the contracting de Sitter solution $T_{12}^-$ with the expanding one $T_{12}^+$, i.e. corresponding to bouncing orbits.

Finally, in order to study the stability of the points $T_{11}^+, T_{12}^+, T_{13}^+$ and $T_{15}$ that require numerical investigation, we numerically evolve the autonomous system for various parameter choices and we depict the resulting phase-space behavior. In figure 4 we can see that point $T_{12}^+$ is an attractor. Notice also the presence of heteroclinic orbits connecting the contracting de Sitter solution $T_{12}^-$ with the expanding one $T_{12}^+$, i.e. corresponding to bouncing orbits [32, 106–109]. In figure 5 we observe that point $T_{13}^+$ is stable and thus it can attract the universe at late times, with the presence of bouncing solutions also visible. Lastly, in figure 6 we show the stable behavior of point $T_{15}^+$.

As we observe, we do find many critical points, some of which exist also in the case of usual $F(R)$ gravity, and some of which are novel and characterized by a $C_\phi$-value different
Figure 5. Projection of the phase space of the system (4.10) of mimetic $F(R)$ gravity with the power-law $F(R)$ form of (4.1), on the invariant set $r = -n$, for the choice $w_m = 0, n = 1.2$. Point $T_{13}^+$ is the late-time attractor for the universe. Notice also the presence of heteroclinic orbits connecting the contracting de Sitter solution $T_{13}^-$ with the expanding one $T_{13}^+$, i.e. corresponding to bouncing orbits.

Figure 6. Projection of the phase space of the system (4.10) of mimetic $F(R)$ gravity with the power-law $F(R)$ form of (4.1), on the invariant set $r = 0$, for the choice $w_m = 0, n = 2$. Point $T_{15}^+$ is the attractor of an open set of orbits.
from zero. However, concerning the stable critical points, i.e. the points that can attract the universe at late times, we observe that they all have $C_\phi = 0$, that is they exist in usual $F(R)$ gravity too (the points that have $C_\phi \neq 0$, namely $T^+_5$, $T^+_6$, and $T_7$, are always not stable). This implies that, although the new features of mimetic $F(R)$ gravity can affect the universe evolution at early and intermediate times, that is affect the specific universe evolution, at late times they will not have any effect, and the universe will result at states that coincide with those of usual $F(R)$. Correspondingly, the involved observables in these late-time solutions, do not depend on $C_\phi$ either. Thus, although mimetic $F(R)$ gravity could drive inflation in a different way than usual $F(R)$ gravity, concerning the dark-energy era it cannot lead to a different behavior. From the dynamical system point of view this is expected, since the new term behaves as $\sim 1/a^3$, which is known to usually lead to saddle behavior [78, 79]. Hence, although this term can affect the phase-space evolution, it cannot affect the stable late-time attractors.

6.3 Mimetic $F(R)$ gravity with arbitrary $F(R)$ form

The scenario of mimetic $F(R)$ gravity with arbitrary $F(R)$ forms admits eighteen classes of critical points (nine corresponding to expanding universe and nine corresponding to contracting one), where almost each class contains as many critical points as the roots of the equation $M(r) = 0$. Amongst them, and focusing on expanding solutions, $P^+_3$, $P^+_6$, $P^+_7$, and $P^+_9$ can be stable, and thus they can attract the universe at late times. $P^+_3$ and $P^+_6$ correspond to non-accelerated universe, and thus they are not favored by observations. $P^+_9$ is the de Sitter solution, corresponding to dark-energy dominated, accelerating universe, where the dark energy behaves as a cosmological constant. Additionally, point $P^+_7$ is the most interesting solution, since it corresponds to dark-energy dominated, accelerating universe, with dark-energy equation-of-state parameter different than $-1$, which additionally can have $0 < \Omega_m < 1$ and thus it can alleviate the coincidence problem since dark energy and dark matter density parameters are of the same order. Finally, concerning the curve of points $P^+_8$ that exist for $w_m = 1/3$, physically corresponding to radiation, the fact that they are saddle and completely dominated by radiation energy density, may correspond to the radiation-dominated phase in which the universe transiently goes through its evolution, before departing towards the subsequent phases.

7 Conclusions

In the present work we investigated the cosmological behavior of mimetic $F(R)$ gravity. This scenario is the $F(R)$ extension of usual mimetic gravity classes, which are based on re-parametrizations of the metric using new, but not propagating, degrees of freedom, that can lead to a wider family of solutions. Indeed, in the cosmological equations one obtains a novel term of the form $C_\phi/a^3$, and when the new parameter $C_\phi$ goes to zero he re-obtains the solutions of usual $F(R)$ gravity. In order to bypass the complexity of the involved equations we performed a detailed dynamical analysis, for the cases of exponential and power-law $F(R)$ ansatzes, and we provided the tools to perform the analysis in the general case of arbitrary $F(R)$ forms. Hence, we first extracted the critical points of the system, and then, for each of these solutions, we calculated various observables, such as the dark-energy and matter density parameters, the dark-energy and total equation-of-state parameter, and the deceleration parameter.
In our analysis we found many critical points, some of which exist also in the case of usual $F(R)$ gravity, and some of which are novel and characterized by a $C_\phi$-value different from zero. However, concerning the stable critical points, i.e. the points that can attract the universe at late times, interestingly enough we found that they all have $C_\phi = 0$, that is they exist in usual $F(R)$ gravity too. This implies that, although the new features of mimetic $F(R)$ gravity can affect the universe evolution at early and intermediate times, that is affect the specific universe evolution, at late times they will not have any effect, and the universe will result at states that coincide with those of usual $F(R)$ gravity. Correspondingly, the involved observables in these late-time solutions do not depend on $C_\phi$ either. Thus, although mimetic $F(R)$ gravity could drive inflation in a different way than usual $F(R)$ gravity, concerning the dark-energy era it cannot lead to a different behavior. From the dynamical system point of view this was expected, since the new term behaves as $\sim 1/a^3$, which is known to usually lead to saddle behavior [78, 79]. Hence, although this term can affect the phase-space evolution, it cannot affect the stable late-time attractors.

However, we should mention that the dynamical analysis provides information for the background behavior only. Hence, although mimetic $F(R)$ gravity at late times leads to background solutions that exist in usual $F(R)$ gravity too, the behavior of the perturbations is expected to be different, since the new term contributes to the perturbations even if it does not contribute to the background level. Thus, it would be both necessary and interesting to study the effect of mimetic $F(R)$ gravity on perturbation-related observables, such as the growth-index. Since this investigation lies beyond the scope of the present work, it is left for a future project.

Acknowledgments

The authors would like to thank Anupam Mazumdar, Shinichi Nojiri, Sergei D. Odintsov and Alex Vikman for useful comments. GL was supported by COMISIÓN NACIONAL DE CIENCIAS Y TECNOLOGÍA through Proyecto FONDECYT DE POSTDOCTORADO 2014 grant 3140244 and by DI-PUCV grant 123.730/2013. Thanks are due to all the members of Grupo inter-universitario de Astrofísica, Gravitación y Cosmología, for their support in a warm working environment. ENS wishes to thank Maternité Port Royal in Paris, for the hospitality during the initial phases of this project, during the birth of his daughter.

A Stability of the critical points of mimetic $F(R)$ gravity with exponential form

The scenario of mimetic $F(R)$ gravity with the exponential form (3.1), i.e. the system (3.10), admits three isolated physical critical points which are presented in table 1. In this appendix we calculate the eigenvalues of the perturbation $5 \times 5$ perturbation matrix for each critical point. For $\Sigma_1$ the associated eigenvalues are $\{i\sqrt{2}, -i\sqrt{2}, 0, 0\}$. Hence, it is nonhyperbolic with two imaginary eigenvalues, and therefore one needs to apply the center manifold analysis [101], however such a study lies beyond the scope of the present work and thus we resorted to numerical examination (see figures 1, 2 and 3). For $\Sigma_{2,3}$ the eigenvalues must be obtained numerically, but at least one of them, with value 1.99778, is always positive. Thus, these two points cannot be attractors, and indeed numerical examination shows that these two de Sitter solutions are saddle points.
B  Stability of the critical points of mimetic $F(R)$ gravity with power-law form

The scenario of mimetic $F(R)$ gravity with the power-law form (4.1), that is the system (4.10), admits $14 \times 2 + 1 = 29$ isolated physical critical points (14 corresponding to expanding universe and their 14 counterparts that correspond to contracting universe, plus one more point without its “symmetric” counterpart) and three curves of critical points, which are presented in table 3 along with their existence conditions. In this appendix we calculate the eigenvalues of the $5 \times 5$ perturbation matrix for each critical point and curve of critical points. We use the notation $\epsilon = \pm 1$.

For the critical points $T_1^\epsilon$ the associated eigenvalues are $\{2\epsilon, 2\epsilon, 2\epsilon, 2\epsilon\}$. Thus, for $\epsilon = +1$ it is unstable, while for $\epsilon = -1$ it is stable.

For the critical points $T_2^\epsilon$ the eigenvalues read $\left\{(4n-2)\epsilon, 2\epsilon, 2\epsilon, -2\epsilon\right\}$, and thus they are saddle points.

For $T_3^\epsilon$ the eigenvalues are $\{10\epsilon, 10\epsilon, 4\epsilon, -2\epsilon, (4 - 6w_m)\epsilon\}$, and thus they are saddle points.

For $T_4^\epsilon$ the eigenvalues write as $\{2\epsilon, 4\epsilon, 10\epsilon, \frac{(8n-10)\epsilon}{n-1}, (4 - 6w_m)\epsilon\}$. Thus, for $\epsilon = +1$ (respectively $\epsilon = -1$) it is a unstable (respectively stable) for $w_m < \frac{2}{3}$ and $n < 1$ or $n > \frac{5}{4}$, otherwise it is a saddle point.

For $T_5^\epsilon$ the eigenvalues read $\{2\epsilon, 2\epsilon, -\frac{2}{3}\epsilon, \frac{2}{3}\epsilon, -2w_m\epsilon\}$, and therefore they are saddle points.

For $T_6^\epsilon$ the eigenvalues are $\left\{-\frac{2}{3}\epsilon, -\frac{4}{3}\epsilon, \frac{2}{3}(4\epsilon - 3), -2w_m\epsilon, 2\epsilon\right\}$, and thus they are saddle points.

For $T_7^\epsilon$ the eigenvalues are extracted as

$$
\left\{\frac{6(n-1)\epsilon}{\sqrt{n(n+2)+3}}, \frac{6n\epsilon}{\sqrt{n(n+2)+3}}, \frac{\lambda_1\epsilon}{2(n-1)[n(n+2)+3]^T}, \frac{\lambda_2\epsilon}{2(n-1)[n(n+2)+3]^4}, \frac{\lambda_3\epsilon}{2(n-1)[n(n+2)+3]^7}\right\},
$$

where $\lambda_i$ are the three roots of the polynomial

$$
P(\lambda) = \lambda^3 \sqrt{n(n+2)+3 + 6\lambda^2(n(n+2)+3)^3}(2nw_m+1) - 8\lambda(n(n+2)+3)^{15/2}[n^2(32n-9w_m-76) + n(9w_m+51) - 9] - 96w_m[n(n-1)^2(4n-3)[n(8n-13)+3][n(n+2)+3]^{11}.
$$

Thus, in the general case the signs of the eigenvalues cannot be determined analytically and one needs to examine them numerically. For instance, for $n = 2$ and $w_m = 0$ the eigenvalues becomes $\left\{\frac{12\epsilon}{\sqrt{11}}, \frac{-\sqrt{11}+\sqrt{451}}{22} \epsilon, \frac{3\sqrt{451} - 1\epsilon}{2\sqrt{11}}, \frac{6\epsilon}{\sqrt{11}}, 0\right\}$, and in this case the points are saddle. Furthermore, since $T_7^\epsilon$ exists for $n \in (0.75, 1.35)$ it follows that at least one eigenvalue for $T_7^\epsilon$ is positive, thus, it cannot be stable.

For $T_8^\epsilon$ the eigenvalues write as

$$
\left\{-\frac{2(n-1)(n-2)}{n^*} \epsilon, -\frac{\epsilon}{n}[n[-2n(3w_m+4) + 9w_m + 13] - 3(w_m + 1)], \frac{[2n(7 - 4n) - 5]\epsilon}{n^*}, \frac{2(n-2)n\epsilon}{n^*} [(13 - 8n)n - 3] \epsilon\right\},
$$

where $n^*$ is the minimal positive root of the equation

$$
\frac{6(n-1)\epsilon}{\sqrt{n(n+2)+3}}, \frac{6n\epsilon}{\sqrt{n(n+2)+3}}, \frac{\lambda_1\epsilon}{2(n-1)[n(n+2)+3]^T}, \frac{\lambda_2\epsilon}{2(n-1)[n(n+2)+3]^4}, \frac{\lambda_3\epsilon}{2(n-1)[n(n+2)+3]^7} = 0.
$$
where \( n^* = \sqrt{n - 1} \sqrt{n[n(9n - 19) + 1]} - 4 \). Hence, restricting ourselves to the physical case \(-1 \leq w_m \leq 1\), we deduce that for \( \epsilon = +1 \) (respectively \( \epsilon = -1 \)) the point is stable (respectively unstable) for \( w_m > -1 \) and \( n > 2 \), otherwise it is saddle.

For \( T_9^\pm \) the eigenvalues are found to be
\[
\left\{ \frac{2(3w_m - 1)\epsilon}{3(w_m - 1)}, -\frac{2(w_m + 1)\epsilon}{w_m - 1}, -\frac{2(w_m + 1)\epsilon}{w_m - 1}, -\frac{2(3w_m - 2)\epsilon}{3(w_m - 1)}, -\frac{2w_m\epsilon}{w_m - 1} \right\},
\]
and thus they are saddle points.

For \( T_{10}^\pm \) the eigenvalues are
\[
\left\{ \frac{2(3w_m - 1)\epsilon}{3(w_m - 1)}, -\frac{2(4w_m - 3w_m - 3)\epsilon}{3(n - 1)(w_m - 1)}, -\frac{2(3w_m - 2)\epsilon}{3(w_m - 1)}, -\frac{2w_m(1 + \epsilon)}{w_m - 1} \right\}.
\]

Therefore, for \( \epsilon = +1 \) (respectively \( \epsilon = -1 \)) they are stable (respectively unstable) for \( w_m < -1 \) and \( \frac{1}{4}(3w_m + 3) < n < 1 \), otherwise they are saddle points.

For \( T_{11}^\pm \) the eigenvalues write as
\[
\left\{ \frac{6(n - 1)(w_m + 1)\epsilon}{\sqrt{\Delta_1}}, \frac{6n(w_m + 1)\epsilon}{\sqrt{\Delta_1}}, \epsilon \left\{ \sqrt{n - 1} \sqrt{\Delta_2} + 3n[(2n - 3)w_m - 1] + 3w_m + 3 \right\}, \frac{6nw_m\epsilon}{\sqrt{\Delta_1}} \right\},
\]
where \( \Delta_1 = n^2 + 9(n - 1)^2w_m^2 + 6[(n - 4)n + 2]w_m + 2n + 3 \) and \( \Delta_2 = 4n^3(3w_m + 8)^2 - 4n^2[3w_m(18w_m + 55) + 152] + 3n(8n + 1)(87w_m + 139) - 81(w_m + 1)^2 \). Thus, \( T_{11}^+ \) (respectively \( T_{11}^- \)) is stable (respectively unstable) for \( n = 2, w_m = -1 \) or \( \frac{2}{3} < n < 2, -1 \leq w_m \leq \frac{-8n^2 + 13n - 3}{6n^2 - 9n + 3} \).

For \( T_{12}^\pm \) the eigenvalues are extracted as
\[
\left\{ 0, -\frac{3\epsilon}{\sqrt{2}}, -\frac{2(3\sqrt{n} + \sqrt{25n - 32})\epsilon}{\sqrt{2}}, -\frac{2(\sqrt{n} \sqrt{25n - 32} - 3n)\epsilon}{\sqrt{2}}, -\frac{3(w_m + 1)\epsilon}{\sqrt{2}} \right\},
\]
thus \( T_{12}^+ \) (respectively \( T_{12}^- \)) has a 4D stable (respectively unstable) manifold for \( 0 < n < 2, w_m > -1 \). However, since there exist a zero eigenvalue the points are nonhyperbolic, thus in order to determine their stability we need to resort to numerical examination (see figure 4).

For \( T_{13}^\pm \) the eigenvalues are extracted as
\[
\left\{ 0, -3\epsilon \sqrt{n}/n + 2, \frac{\lambda_1\epsilon}{2(n - 1)(n + 2)^2}, \frac{\lambda_2\epsilon}{2(n - 1)(n + 2)^2}, \frac{\lambda_3\epsilon}{2(n - 1)(n + 2)^2} \right\},
\]
where \( \lambda_i \) are the three roots of the polynomial
\[
P(\lambda) = \lambda^3 + 6\lambda^2(n - 1)\sqrt{n + 2} (w_m + 2) + 4\lambda(n - 1)(n + 2)^3 [9(n - 1)w_m + 5n - 1] - 96(n - 2)(n - 1)^2n^{3/2}(n + 2)^{9/2}(w_m + 1).
\]

Hence, in the general case the signs of the eigenvalues cannot be determined analytically and one needs to perform a numerical investigation. For instance, for \( n = 2 \) and \( w_m = 0 \) the eigenvalues become \( \left\{-\frac{3\epsilon}{\sqrt{2}}, -\frac{3\epsilon}{\sqrt{2}}, -\frac{3\epsilon}{\sqrt{2}}, 0, 0\right\} \), and thus the points exhibit a 3D stable manifold.
A complete stability analysis requires to use the center manifold theorem [101], however since this lies beyond the scope of the present work, we resort instead to numerical elaboration (see figure 5).

For $T_{14}$ the eigenvalues write as $\{0, -1, 1, 4, 4\}$, and thus it is a saddle point.

For $T_{15}$ the eigenvalues write as $\{0, 0, 0, 0, 0\}$, and thus it is non-hyperbolic. In order to examine its stability one needs to apply the center manifold analysis [101], however since such a study lies beyond the scope of the present work. However, numerical elaboration allows to conclude that it is a local attractor (see figure 6).

For the curve of critical points $T_{16}$ the eigenvalues are

$$\begin{cases} 
0, \frac{1}{2}Q_{c17} \left[ -\frac{\sqrt{(75n - 32)Q_{c17}^2 - 25n}}{\sqrt{n}\sqrt{3Q_{c17}^2 - 1}} - 3 \right], \\
\frac{1}{2}Q_{c17} \left[ \frac{\sqrt{(75n - 32)Q_{c17}^2 - 25n}}{\sqrt{n}\sqrt{3Q_{c17}^2 - 1}} - 3 \right], -3Q_{c17}(w_m + 1), -3Q_{c17} \end{cases}$$

(B.1)

where $Q_{c17}$ is the parameter of the curve. Hence, all the points of this curve are saddle points.

For the curve of critical points $T_{17}$ the eigenvalues are

$$\begin{cases} 
0, - \frac{5Q_{c16}}{2}, 2Q_{c16}, - \frac{3}{2}Q_{c16}(w_m + 1), - \frac{3Q_{c16}}{2} \end{cases},$$

where $Q_{c16}$ is the parameter of the curve. Hence, all the points of this curve are saddle points.

Thus, for $\epsilon = -1$ (respectively $\epsilon = +1$) it is unstable (respectively stable) for $w_m > -1$ and $0 < n < 1$, otherwise it is a saddle point.

C Stability of the critical points of mimetic $F(R)$ gravity with arbitrary $F(R)$ forms

The scenario of mimetic $F(R)$ gravity with arbitrary potentials, i.e. the system of equation (5.7a)-(5.7d) admits eighteen classes of critical points (nine corresponding to expanding universe ($\epsilon = +1$) and nine corresponding to contracting universe ($\epsilon = -1$)), where each class contains as may critical points as the roots of the equation $M(r = r^*) = 0$, with the exception of the curves $P_5^*$ which exist for the special value $w_m = \frac{1}{3}$, and $P_3^*$ for which $r = -2$ and $M(-2)$ is not necessarily zero. These are presented in table 5 along with their existence conditions. In this appendix we calculate the eigenvalues of the $4 \times 4$ perturbation matrix for each critical point and curve of critical points.

For the critical points $P_1^*$ the associated eigenvalues are $\{2\epsilon, 2\epsilon, \frac{2\sqrt{(2r^*+1)\epsilon}}{r^* + 1}, 2\epsilon M'(r^*)\}$. Thus, for $\epsilon = +1$ (respectively $\epsilon = -1$) they are unstable (respectively stable) for $M'(r^*) > 0$, $r^* < -1$ or $M'(r^*) > 0$, $r^* > -\frac{1}{2}$, otherwise they are saddle points.
For the critical points $P^*_2$ the eigenvalues read \[
\left\{ 4\epsilon, \frac{2(4r^*+5r)}{r+1}, -2(3w_m-2)\epsilon, -2\epsilon M'(r^*) \right\}.
\]
Therefore, for $\epsilon = +1$ (respectively $\epsilon = -1$) they are unstable (respectively stable) for $-1 \leq w_m < -\frac{2}{3}, M'(r^*) < 0, r^* < -\frac{5}{4}$ or $-1 \leq w_m < -\frac{2}{3}, M'(r^*) < 0, r^* > -1$, otherwise they are saddle points.

For the critical points $P^*_3$ the eigenvalues write as \[
\left\{ \frac{-4\epsilon}{3}, \frac{2(4r^*+5r)}{3(r^*+3)}, -2w_m\epsilon, \frac{2}\epsilon M'(r^*) \right\}.
\]
Therefore, for $\epsilon = +1$ (respectively $\epsilon = -1$) they are stable (respectively unstable) for $0 < w_m \leq 1, M'(r^*) < 0, -1 < r^* < -\frac{3}{4}$, otherwise they are saddle points.

For the critical points $P^{-}_{3}$ the eigenvalues are the roots of the polynomial
\[
P(\lambda) = \Delta_3 \lambda^3 \left[ (r^*)^3 - (r^*)^2 + r^* + 3 \right] + 3\lambda^2 \left[ (r^*)^3 - 3(r^*)^2 + r^* + 3 \right] (2r^* w_m - 1)
-2\Delta_3 \lambda \left\{ r^* [r^* (32r^* + 9w_m + 76) + 9w_m + 51] + 9 \right\}
-384 (r^*)^3 w_m - 912 (r^*)^3 w_m - 612 (r^*)^2 w_m - 108r^* w_m (w_m + 1),
\]
where $\Delta_3 = \sqrt{(r^*)^2 - 2r^* + 3}$, and the fourth eigenvalue is $\frac{6(\epsilon M'(r^*))}{\sqrt{(r^*-2)r^*+3}}$. Hence, in the general case the signs of the eigenvalues cannot be determined analytically and one needs to perform a numerical investigation.

For $P^-_{3}(r^*)$ the eigenvalues are the roots of the polynomial
\[
P(\lambda) = \Delta_3 \lambda^3 \left[ (r^*)^3 - (r^*)^2 + r^* + 3 \right] - 3\lambda^2 \left[ (r^*)^3 - 3r^* + r^* + 3 \right] (2r^* w_m - 1)
-2\Delta_3 \lambda \left\{ r^* [r^* (32r^* + 9w_m + 76) + 9w_m + 51] + 9 \right\}
+384 (r^*)^3 w_m + 912 (r^*)^3 w_m + 612 (r^*)^2 w_m + 108r^* w_m (w_m + 1),
\]
and the fourth eigenvalue is $\frac{6(\epsilon M'(r^*))}{\sqrt{(r^*-2)r^*+3}}$. Thus, in the general case the signs of the eigenvalues cannot be determined analytically and one needs to perform a numerical investigation.

For $P^+_{3}(r^*)$ the eigenvalues are
\[
\left\{ \frac{(2r^* + 1)(4r^* + 5) \epsilon}{\sqrt{r^* + 1} \sqrt{(r^* (9r^* + 19) + 13)} + 4}, \frac{(2r^* + 1)(4r^* + 5) \epsilon}{\sqrt{r^* + 1} \sqrt{(r^* (9r^* + 19) + 13)} + 4}, \frac{(r^* (8r^* + 13) + 3) \epsilon}{\sqrt{r^* + 1} \sqrt{(r^* (9r^* + 19) + 13)} + 4} \right\}.
\]
Thus, $P^+_{3}$ (respectively $P^-_{3}$) is unstable (respectively stable) for $-1 < r^* < -\frac{3}{4}, M'(r^*) > 0$ or $r^* < -\frac{3}{4}$, otherwise they are saddle points.

For the critical points $P^+_6$ the eigenvalues write as
\[
\left\{ -\frac{2w_m\epsilon}{w_m - 1}, -\frac{2\epsilon (4r^* + 3w_m + 3)}{3 (r^* + 1)(w_m - 1)}, \frac{2(3w_m - 2)\epsilon}{3(w_m - 1)}, \frac{2(3w_m - 1)\epsilon M'(r^*)}{3(w_m - 1)} \right\}.
\]
Therefore, for $\epsilon = +1$ (respectively $\epsilon = -1$) they are stable (respectively unstable) for $-1 \leq w_m < 0, -1 < r^* < -\frac{3}{4}(w_m + 1), M'(r^*) < 0$, otherwise they are saddle points.

For the critical points $P^-_{6}$ the eigenvalues write as
\[
\left\{ \frac{6r^* w_m \epsilon \sqrt{\Delta_5} + 6r^* w_m \epsilon + 3(w_m + 1)\epsilon}{2\sqrt{\Delta_4}}, \frac{6r^* w_m \epsilon \sqrt{\Delta_5} + 6r^* w_m \epsilon + 3(w_m + 1)\epsilon}{2\sqrt{\Delta_4}}, \frac{6r^* w_m \epsilon \sqrt{\Delta_5} + 6r^* w_m \epsilon + 3(w_m + 1)\epsilon}{2\sqrt{\Delta_4}} \right\},
\]

\[\]
Thus, \( P_7^+ \) (respectively \( P_7^- \)) is stable (respectively unstable) for \( M'(r^*) > 0, -1.64 < r^* \lesssim -1.328, -1 < w_m < w_m^- \), where

\[
\begin{align*}
\Delta_4 &= 9 \, (r^* + 1)^2 \, w_m^2 + 6 \, [r^*(r^* + 4) + 2] \, w_m + (r^*)^2 - 2r^* + 3 \\
\Delta_5 &= (r^* + 1)^{-1} \left\{ 4r^* \left[ r^*(3w_m + 8)^2 + 3w_m(18w_m + 55) + 152 \right] + 3(w_m + 1)(87w_m + 139) \right\}. \quad (C.1)
\end{align*}
\]

It is a saddle otherwise.

For the critical points \( P_8^+ \) the eigenvalues write as \( \{4\epsilon, -\epsilon, 0\} \). Thus, they are saddle. For the critical points \( P_8^- \) the eigenvalues are

\[
\begin{align*}
\left\{ \frac{3(w_m + 1)\epsilon}{\sqrt{2}}, -\frac{3\epsilon}{\sqrt{2}}, \frac{\left(\sqrt{9 - 8M(-2)} + 3\right)\epsilon}{2\sqrt{2}}, \frac{\left(\sqrt{9 - 8M(-2)} - 3\right)\epsilon}{2\sqrt{2}} \right\}.
\end{align*}
\]

Thus, \( P_8^+ \) (respectively \( P_8^- \)) is stable (respectively unstable) for \( w_m > -1, M(-2) > 0 \).

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