Large time behavior of strong solutions for stochastic Burgers equation with transport noise

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Abstract: We consider the large time behavior of strong solutions to the stochastic Burgers equation with transport noise. It is well known that both the rarefaction wave and viscous shock wave are time-asymptotically stable for deterministic Burgers equation since the pioneer work of A. Ilin and O. Oleinik [31] in 1964. However, the stability of these wave patterns under stochastic perturbation is not known until now. In this paper, we give a definite answer to the stability problem of the rarefaction and viscous shock waves for the 1-d stochastic Burgers equation with transport noise. That is, the rarefaction wave is still stable under white noise perturbation and the viscous shock is not stable yet. Moreover, a time-convergence rate toward the rarefaction wave is obtained. To get the desired decay rate, an important inequality (denoted by Area Inequality) is derived. This inequality plays essential role in the proof, and may have applications in the related problems for both the stochastic and deterministic PDEs.

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1 Introduction

The one dimensional Burgers equation reads

\[ u_t + uu_x = \nu u_{xx}, \]  
\[ (1.1) \]

where the viscosity coefficient \( \nu \) is a positive constant, and becomes the inviscid Burgers equation as \( \nu = 0 \), i.e.,

\[ u_t + uu_x = 0. \]  
\[ (1.2) \]
It is well known that the inviscid Burgers equation (1.2) has rich wave phenomena such as shock and rarefaction wave, cf. [42]. Consider the Riemann initial data

\[ u(x,0) = \begin{cases} 
  u_-, & x < 0, \\
  u_+, & x > 0, 
\end{cases} \]  

(1.3)

then the inviscid Burgers equation (1.2) admits shock or rarefaction wave depending on the sign of \( u_- - u_+ \). If \( u_- < u_+ \), the solution of (1.2) is rarefaction wave given by

\[ u^r(t,x) = u^r\left(\frac{x}{t}\right) = \begin{cases} 
  u_-, & x < u_-t, \\
  \frac{x}{t}, & u_-t < x < u_+t, \\
  u_+, & x > u_+t, 
\end{cases} \]  

(1.4)

and if \( u_- > u_+ \), the solution is shock wave given by

\[ u^s(t,x) = \begin{cases} 
  u_-, & x < x - st, \\
  u_+, & x > x - st, 
\end{cases} \]  

(1.5)

where \( s = \frac{u_+ + u_-}{2} \) is the propagation speed of shock due to Rankine-Hugoniot (RH) condition. Both the shock and rarefaction waves are nonlinearly stable for the equation (1.2), cf. [42]. Since the Burgers equation (1.1) is a viscous version of (1.2), it is commonly conjectured that the viscous fundamental wave patterns, i.e., viscous shock wave and rarefaction wave, are stable for the Burgers equation (1.1). The conjecture was first verified for one space dimension in [31], see also [24], and [28], [29], [32], [40], [44] for multi-dimensional scalar viscous conservation laws. The conjecture is also valid for general systems of viscous conservation laws such as the compressible Navier-Stokes equations, cf. [26], [35], [37]-[38], [45] and the Boltzmann equation, cf. [26], [33], [34] and the references therein, in which new techniques like weighted characteristic energy method, approximate Green’s function and Evans function approach were developed. Nevertheless, we would ask a natural question:

**(Q):** Would shock and rarefaction waves be still stable under the stochastic perturbation?

From the physical point of view, these basic wave patterns might be perturbed by stochastic noise in the real environment. As a starting point, we focus on the following stochastic Burgers equation with transport noise,

\[ du + uu_x dt = \mu u_x dt + \sigma u_x dB(t), \]  

(1.6)

where \( B(t) \) is one-dimensional standard Brownian motion on some probability space \((\Omega, \mathcal{F}, P)\). The stochastic term \( \sigma u_x dB(t) \) can be explained as follows. Let \( u(t,x) \) be the smooth solution of the deterministic Burgers equation (1.1). When the position \( x \) is perturbed by a Brownian motion \( \sigma B(t) \), then \( u(t,x + \sigma B(t)) \) satisfies the equation (1.6) due to Itô formula, where \( \mu = \)
\[ \nu + \frac{1}{2} \sigma^2, \text{see [12] for the details, see also [8], [10] and [43].} \]

The shock formation, local \((\nu = 0)\) and global existence \((\nu > 0)\) of smooth solutions to (1.6) was investigated in [1]. The regularity effect of noise recently attracts considerable attentions, see the interesting papers [13, 14, 15, 3, 17, 18, 19, 21] and the references therein.

In this paper, we focus on the large time behavior of strong solutions toward the viscous shock and rarefaction waves, and try to give a definite answer to the above question (Q) for the stochastic Burgers equation (1.6). Roughly speaking, we show that the rarefaction wave is still stable under transport noise perturbation and the viscous shock is not stable yet. Now we formulate the main results.

Since the rarefaction wave \(u'(\frac{\tau}{t})\) given in (1.4) is only Lipschitz continuous, we follow the method of [39] to introduce approximate rarefaction wave, which is a smooth solution of the following problem

\[
\begin{cases}
\bar{u}_t + \bar{u}_x = 0, \\
\bar{u}(0, x) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2} \int_0^x (1 + \xi^2)^{-\frac{1}{2}} d\xi \to u_\pm, \text{ as } x \to \pm\infty,
\end{cases}
\]

where \(k = \left( \int_0^{+\infty} (1 + \xi^2)^{-\frac{1}{2}} d\xi \right)^{-1}\). The main result is stated as follows,

**Theorem 1.1 (Rarefaction wave).** Let \(\sigma^2 < 2\mu\) and \(u_0(x)\) be the initial data of the stochastic Burgers equation (1.6). Set \(\phi(t, x) = u(t, x) - \bar{u}(t, x)\). If \(\phi(0, x) \in H^2(\mathbb{R})\), then there exists a unique strong solution of (1.6) satisfying

\[
\mathbb{E}\|u(t, \cdot) - u'(\cdot, t)\|_{L^p(\mathbb{R})} \leq C_p (2 + t)^{-\frac{p^2}{2p}} \ln(2 + t), \quad \forall p \in [2, +\infty),
\]

and

\[
\mathbb{E}\|u(t, \cdot) - u'(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_\epsilon (2 + t)^{-\frac{1}{2} + \epsilon}, \quad \forall \epsilon > 0,
\]

where \(u'(t, x)\) is the rarefaction wave given in (1.4). Moreover, it holds that for any \(\epsilon > 0\), there exists a \(\mathcal{F}_\infty\) measurable random variable \(C_\epsilon(\omega) \in L^2(\Omega)\) such that

\[
\|u(t, \cdot) - u'(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_\epsilon(\omega)(2 + t)^{-\frac{1}{2} + \epsilon}, \quad \text{a.s.}
\]

**Remark 1.** Theorem 1.1 answers the question (Q) in the case of rarefaction wave for the stochastic Burgers equation (1.6), i.e., the rarefaction wave is nonlinearly stable under transport noise.

**Remark 2.** The time-decay rate (1.8) in \(L^p\) norm is almost optimal! Indeed, even for the deterministic heat equation

\[
u_t = u_{xx}, \quad u(0, x) \in L^2(\mathbb{R}),
\]

the optimal decay rate of \(u(t, x)\) in \(L^p\) is \((2 + t)^{-\frac{p^2}{2p}}\). In fact, the term \(\ln(2 + t)\) in (1.8) is coming from the Brownian motion \(u_x dB(t)\).
Remark 3. The assumption $\sigma^2 < 2\mu$ is equivalent to $\nu > 0$ which is the viscosity of the deterministic Burgers equation (1.1). Hence the assumption $\sigma^2 < 2\mu$ is necessary.

The proof of Theorem 1.1 relies on a key inequality below, denoted by Area Inequality.

**Theorem 1.2 (Area Inequality).** Assume that a Lipschitz continuous function $f(t) \geq 0$ satisfies

$$f'(t) \leq C_0(1 + t)^{-\alpha}, \quad (1.12)$$

and

$$\int_0^tf(s)ds \leq C_1(1 + t)^\beta \ln^\gamma(1 + t), \quad \gamma \geq 0, \quad (1.13)$$

for some constants $C_0$ and $C_1$, where $0 \leq \beta < \alpha$. Then it holds that if $\alpha + \beta < 2$,

$$f(t) \leq 2 \sqrt{C_0C_1(1 + t)^{\beta - \alpha} \ln^\gamma(1 + t)} \quad \text{as} \quad t \gg 1. \quad (1.14)$$

Moreover, if $\beta = \gamma = 0$, i.e., $f(t) \in L^1[0, \infty)$ and $0 < \alpha \leq 2$, it holds that

$$f(t) = o(t^{-\frac{\alpha}{2}}), \quad \text{as} \quad t \gg 1, \quad (1.15)$$

and the index $\frac{\alpha}{2}$ is optimal.

Remark 4. The time-decay rate (1.14) is surprising even for the case $0 < \alpha < 1$, $\beta = \gamma = 0$, in which the condition (1.13) becomes

$$\int_0^{+\infty} f(t)dt \leq C_1 < +\infty. \quad (1.16)$$

To get the decay rate of $f(t)$, the usual way (see [25]) is to multiply (1.12) by $1 + t$, then we have

$$[(1 + t)f(t)]' \leq f(t) + C_0(1 + t)^{1-\alpha}. \quad (1.17)$$

Integrating (1.17) on $[0, T]$ implies that

$$(1 + t)f(T) \leq f(0) + \int_0^T f(t)dt + C_0 \int_0^T (1 + t)^{1-\alpha} dt \leq C + \frac{C_0}{\alpha - 2}(1 + t)^{2-\alpha}, \quad (1.18)$$

which gives

$$f(t) \leq C(1 + t)^{-\alpha}, \quad \text{as} \quad t \gg 1. \quad (1.19)$$

It is impossible from (1.19) to get the time-decay rate as $0 < \alpha < 1$ through the usual way (1.17) to (1.19). Also note that for $1 < \alpha < 2$, the decay rate $t^{-\frac{\alpha}{2}}$ in (1.15) is faster than $t^{1-\alpha}$ in (1.19).
Remark 5. Since the inequality (1.12) might be derived only for some $0 < \alpha < 1$ in the stability analysis, where $f(t)$ usually corresponds to the norm of some Sobolev spaces, we can expect that the Area Inequality might have applications in the time-decay rate of solutions for both the deterministic and stochastic PDEs, see recent work [27].

In the case that $u_+ > u_-$, let $u^\dagger(t, x) := \bar{u}(\xi), \xi = x - st$ be the viscous shock wave of the deterministic Burgers equation (1.1) satisfying

$$
\begin{cases}
-s\bar{u}' + \bar{u}'' = v\bar{u}'', \\
\bar{u}(\xi) \to u_\pm, \text{ as } \xi \to \pm\infty,
\end{cases}
$$

(1.20)

where $\xi = \frac{d}{d\xi}, s = \frac{u_+ u_-}{2}$. Without loss of generality, let $s = 0$, i.e., $u_- = -u_+ > 0$. It is known in [31] that the equation (1.20) admits a unique solution $\bar{u}(\xi)$ up to a shift. Since the position $x$ is perturbed by $\sigma B(t)$, the perturbed viscous shock is $\bar{u}^B(t, x) := \bar{u}(x + \sigma B(t))$ and the two viscous shock waves coincide at the initial time, i.e., $\bar{u}(x) = \bar{u}^B(0, x)$. Let

$$d(t) = \mathbb{E}[\|\bar{u}(x) - \bar{u}^B(t, x)\|_{L^\infty(\mathbb{R})}, \ d(0) = 0.$$

We have the following instability theorem.

**Theorem 1.3 (Instability for shock wave).** $d(t)$ is an increasing function of $t$. Moreover, it holds that

$$\lim_{t \to +\infty} d(t) = u_- - u_+.$$

(1.21)

**Remark 6.** Theorem 1.3 indicates that the viscous shock wave is not stable under transport noise perturbation.

We outline the proof of Theorem 1.1. One of the main difficulties comes from the stochastic perturbation $u_x dB(t)$ in the whole space $\mathbb{R}$ such that mild solution approach and some compact methods might not be available anymore. This difficulty is overcome by combining energy method, iteration approach and new $L^p$ and martingale estimates. One of the advantages of energy method is that the stochastic integral term can be cancelled in the expectation, while it is not clear in the mild solution formula. Precisely speaking, we first apply a cut-off technique to control the nonlinear term $u^2$. Then we instead consider the cut-off equation (6.1) whose global existence of strong solution is shown by the contraction mapping principle and energy method. It is noted that the quadratic variation for the derivative is understood in Krylov’s theory [30]. Once the global existence of (6.1) is obtained, the global existence of strong solution to the original equation (1.1) is proved through a new martingale estimate and stopping time method.

Another difficulty is from the rarefaction wave. The quadratic variation generates a bad term $\|\bar{u}\|^2 \approx \frac{1}{2^t}$ which is not integrable over $[0, \infty)$. By the energy method, $\mathbb{E}[\|\phi(t, \cdot)\|^2_{L^2(\mathbb{R})}$ may increase with time $t$ (probably $\ln^\frac{1}{2}(2 + t)$), while $\|\phi(t, \cdot)\|^2_{L^2(\mathbb{R})}$ is uniformly bounded for the
deterministic Burgers equation (1.1)). To get the desired a priori estimates, the time-decay rate of $\mathbb{E}\|\phi_x(t, \cdot)\|_{L^2(\mathbb{R})}$ for the derivative is necessarily achieved. Fortunately, we observe that for any $2 < p < +\infty$, $\mathbb{E}\|\phi(t, \cdot)\|_{L^p(\mathbb{R})}$ decays with a rate by a new $L^p$ energy method and martingale estimates although the $L^2$ norm may increase. We further observe that $\mathbb{E}\|\phi_x(t, \cdot)\|_{L^p(\mathbb{R})}$ decays with a rate by a new $L^p$ energy method and martingale estimates although the $L^2$ norm may increase. We further observe that $\mathbb{E}\|\phi_x(t, \cdot)\|_{L^p(\mathbb{R})}$ provides a time-decay rate with some $0 < \alpha < 1$ in the energy inequality for $\phi_x$, so that the Area Inequality can be applied, where $f(t) = \mathbb{E}\|\phi_x(t, \cdot)\|^2_{L^2(\mathbb{R})}$. The time-decay rate of $\mathbb{E}\|\phi(t, \cdot)\|_{L^p(\mathbb{R})}$ in (1.9) is derived by the celebrated Gargliadio-Nirenberg inequality. Finally, the decay rate (1.10) a.s. is obtained by combining the martingale estimates and the decay (1.9) in expectations.

For the other works of the Burgers equation and conservation laws with stochastic force, see the interesting papers [2, 4, 5, 6, 7, 9, 11, 20, 21, 22, 23, 41] and the references therein.

The rest of the paper is organized as follows. Sections 2-6 are devoted to the proof of Theorem 1.1. Among them, some preliminaries on the approximate rarefaction waves are given in section 2, while the Area Inequality is proved in section 3. Section 4 is devoted to the a priori estimates through a delicate $L^p$, $p \geq 2$ energy method and then the time-decay rates (1.9) and (1.10) of strong solution toward the rarefaction wave are given in section 5. In Section 6, the global existence of strong solution is proved by combining the martingale estimates and the global existence of the cut-off equation (6.1) through the contraction mapping principle and stopping time method. Finally the instability of viscous shock wave is given in Section 7.

2 Preliminaries

In this section, some properties of the approximate rarefaction wave $\bar{u}(t, x)$ given in (1.7) are listed as follows.

Lemma 2.1 ([39], [32]).  

i) $u_- < \bar{u}(t, x) < u_+$, $\bar{u}_x(t, x) > 0$ for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$,

ii) For all $p \in [1, \infty]$, there exist constants $C_p$ and $C_{p, d}$ such that for large $t$,

$$
\begin{align*}
\|\bar{u}_x(t, \cdot)\|_{L^p(\mathbb{R})} & \leq C_p \min(d, d^{1/p} t^{-1+1/p}), \\
\|\bar{u}_{xx}(t, \cdot)\|_{L^p(\mathbb{R})} & \leq C_p \min(d, d^{-(p-1)/2p} t^{-1+(p-1)/p}), \\
\|\bar{u}_{xxx}(t, \cdot)\|_{L^p(\mathbb{R})} & \leq C_{p, d} (2 + t)^{-(1+2(p-1)/2p)},
\end{align*}
$$

where $d = u_+ - u_-$ is the strength of rarefaction wave.

iii) For all $p \in (1, \infty)$, there is a constant $C_{p, d}$ such that for large $t$,

$$
\|\bar{u}(t, \cdot) - \bar{u}'(t, \cdot)\|_{L^p(\mathbb{R})} \leq C_{p, d} t^{-(p-1)/2p},
$$

where $\bar{u}'(t, x)$ is the rarefaction wave given in (1.4).
Next, we give the Gagliardo-Nirenberg (GN) inequality which reads as, for any \(1 \leq p \leq +\infty\) and integer \(0 \leq j < m\),
\[
\|\nabla_j u\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla_m u\|_{L^\theta(\mathbb{R}^n)}^{\theta \gamma} \|u\|_{L^1(\mathbb{R}^n)}^{1-\theta \gamma},
\]
where \(\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right) \theta + \frac{1}{q} (1 - \theta)\), \(\frac{j}{m} \leq \theta \leq 1\). Based on the GN inequality, we also list an interesting interpolation inequality, which will be used later.

**Lemma 2.2** ([29]). It holds that,
\[
\|u\|_{L^p} \leq C(p,q,n) \|\nabla (|u|^{\frac{p}{2}})\|_{L^2}^{\frac{1}{2\gamma}} \|u\|_{L^1}^{\frac{1}{2\gamma}},
\]
for \(2 \leq p < \infty\) and \(1 \leq q \leq p\), where \(\gamma = (n/2)(1/q - 1/p)\), and \(C(p,q,n)\) is a positive constant.

### 3 Area inequality

This section is devoted to the Area Inequality, which plays a key role to get the convergence rate of strong solution toward the rarefaction wave for the stochastic Burgers equation (1.6).

**Proof of Theorem 1.2.** We first prove (1.14), i.e.,
\[
f(t) \leq 2 \sqrt{C_0 C_1} (1 + t)^{-\alpha} \ln^\gamma (1 + t), \ t >> 1,
\]
under the conditions
\[
f'(t) \leq C_0 (1 + t)^{-\alpha},
\]
and
\[
\int_0^t f(s)ds \leq C_1 (1 + t)^{\beta} \ln^\gamma (1 + t), \ \gamma \geq 0,
\]
where \(0 \leq \beta < \alpha\) and \(\alpha + \beta < 2\). The proof is provided by the way of contradiction.

If the inequality (3.2) does not hold, then there exists a sequence \(\{t_n\}_{n=1}^\infty\) with \(t_n \uparrow \infty\) such that \(f(t_n) > C_2 (1 + t_n)^{-\alpha} \ln^\gamma (1 + t_n)\), where \(C_2 := 2 \sqrt{C_0 C_1}\). Note that the inequality (3.2) in the interval \([0, t_n]\) is equivalent to
\[
\frac{df(\tau)}{d\tau} \geq -C_0 (1 + t_n - \tau)^{-\alpha}, \ f(\tau)|_{\tau=0} = f(t_n), \ \tau \geq 0,
\]
where \(\tau = t_n - \tau\). Then we construct a function \(g_n(\tau)\) satisfying
\[
\frac{dg_n(\tau)}{d\tau} = -C_0 (1 + t_n - \tau)^{-\alpha}, \ g_n(\tau)|_{\tau=0} = C_2 (1 + t_n)^{-\alpha} \ln^\gamma (1 + t_n), \ \tau \geq 0.
\]
It is obvious that \(f(\tau) \geq g_n(\tau)\) for any \(\tau \in [0, t_n]\) since \(f(\tau)|_{\tau=0} \geq g_n(\tau)|_{\tau=0}\), see Figure 1 below.
The ODE (3.5) is exactly a backward ordinary differential equation starting from $t_n$, i.e.,

\[
\begin{cases}
\frac{dg_n(\tau)}{d\tau} = C_0(1 + \tau)^{-\alpha}, & 0 \leq \tau \leq t_n, \\
g_n(t_n) := C_2(1 + t_n)^{\frac{\beta - \alpha}{1 - \alpha}} \ln^{\frac{\gamma}{1 - \alpha}}(1 + t_n).
\end{cases}
\] (3.6)

A direct computation gives the formula of $g_n(\tau)$ for $\alpha \neq 1$ on $0 < \tau \leq t_n$,

\[
g_n(\tau) = g_n(t_n) - C_0 \int_\tau^{t_n} (1 + s)^{-\alpha} ds = g_n(t_n) - \frac{C_0}{1 - \alpha} [(1 + t_n)^{1-\alpha} - (1 + \tau)^{1-\alpha}].
\] (3.7)

Due to $0 \leq \beta < \alpha$ and $\alpha + \beta < 2$, $\frac{\beta - \alpha}{1 - \alpha} < 1 - \alpha$ is always true. Taking $\tau = \frac{t_n}{2}$ gives that

\[
g_n\left(\frac{t_n}{2}\right) = g_n(t_n) - \frac{C_0}{1 - \alpha} [(1 + t_n)^{1-\alpha} - (1 + \frac{t_n}{2})^{1-\alpha}] < 0, \text{ as } t_n \gg 1,
\] (3.8)

Since $g_n(\tau)$ is monotonically increasing, there exists a unique $s_n \in \left(\frac{t_n}{2}, t_n\right)$ such that $g_n(s_n) = 0$. Taking $\tau = s_n$ in (3.7), it follows from the mean value theorem that

\[
s_n = \left[(1 + t_n)^{1-\alpha} - \frac{1-\alpha}{C_0}g_n(t_n)\right]^{\frac{1}{1-\alpha}} - 1
\]

\[
= (1 + t_n)[1 - \frac{1-\alpha}{C_0}g_n(t_n)(1 + t_n)^{\alpha-1}]^{\frac{1}{1-\alpha}} - 1
\]

\[
= (1 + t_n)[1 - \frac{1}{C_0}g_n(t_n)(1 + t_n)^{\alpha-1}(1 - \xi_n)^{\frac{\alpha}{1-\alpha}}] - 1
\]

\[
\leq t_n - \frac{2}{3C_0}g_n(t_n)(1 + t_n)^{\alpha},
\] (3.9)

where $\xi_n \in (\min\{0, 1 - (\frac{2}{3})^{\frac{\alpha}{1-\alpha}}\}, \max\{0, 1 - (\frac{2}{3})^{\frac{\alpha}{1-\alpha}}\})$ is a constant, and we have used the fact that

\[
g_n(t_n)(1 + t_n)^{\alpha-1} = C_2(1 + t_n)^{\frac{\alpha\beta}{1 - \alpha}} \ln^{\frac{\gamma}{1 - \alpha}}(1 + t_n) = o(1), \text{ as } t_n \to \infty.
\] (3.10)

Then we have

\[
t_n - s_n \geq \frac{2}{3C_0}g_n(t_n)(1 + t_n)^{\alpha}.
\] (3.11)

Note that the curve $g_n(\tau)$ is concave due to the fact that the derivative of $(1 + \tau)^{-\alpha}$ in (3.6) is negative. Thus the region $S_{ACDE}$ surrounded by the segments $AC$, $CD$, $DE$ and the curve
and a sequence \( \{\epsilon\} \) and we expect a better result (1.15). If (1.15) does not hold, then there exist a small constant which yields the inequality (3.11). As in (3.12), the same argument implies (3.1). This is a contradiction and thus the inequality (3.1) holds for \( \alpha \neq 1 \).

For the case \( \alpha = 1 \), the same argument implies that there exists a unique \( s_n \in (\frac{n}{2}, t_n) \) such that \( g_n(s_n) = 0 \). Then the formula of \( g_n(\tau) \) gives

\[
g_n(t_n) = g_n(s_n) + C_0 \int_{s_n}^{t_n} (1 + s)^{-1} ds = C_0 \ln \frac{1 + t_n}{1 + s_n}.
\]

(3.13)

Note that \( g_n(t_n) = C_2(1 + t_n)^{\frac{\beta - 1}{\alpha}} \ln (1 + t_n) = o(1) \), we get

\[
s_n = (1 + t_n)e^{-\frac{\alpha g_n(t_n)}{C_0}} - 1 \approx (1 + t_n)(1 - \frac{g_n(t_n)}{C_0}) - 1 = t_n - \frac{1}{C_0} g_n(t_n)(1 + t_n),
\]

(3.14)

which yields the inequality (3.11). As in (3.12), the same argument implies (3.1).

Next we consider the case that \( \beta = \gamma = 0 \) and \( 0 < \alpha \leq 2 \), i.e.,

\[
\int_0^{\infty} f(t) dt < +\infty,
\]

(3.15)

and we expect a better result (1.15). If (1.15) does not hold, then there exist a small constant \( \epsilon \) and a sequence \( \{t_n\}_{n=1}^{\infty} \) with \( t_n \uparrow \infty \) such that \( f(t_n) > \epsilon(1 + t_n)^{-\frac{3}{2}} \). In the same way as in (3.6) and (3.8), we can construct a function \( g_n(\tau) \) with \( g_n(t_n) := \epsilon(1 + t_n)^{-\frac{3}{2}} \) and \( g_n(s_n) = 0 \). Note that \( g_n(t_n)(1 + t_n)^{\gamma - 1} = \epsilon(1 + t_n)^{-\frac{3}{2} - 1} \) is small as \( 0 < \alpha \leq 2 \) so that the inequality (3.11) still holds for \( t_n - s_n \). By the same argument as in (3.12), we have as \( t_n \gg 1 \),

\[
\int_{s_n}^{t_n} f(\tau) d\tau \geq \frac{1}{2} g_n(t_n)(t_n - s_n) \geq \frac{2}{3C_0} (g_n(t_n))^2 (1 + t_n)^\alpha = \frac{2\epsilon^2}{3C_0}.
\]

(3.16)

Note that the left hand side of (3.16) tends to zero as \( t_n \to +\infty \) due to (3.15), while the right one is a fixed constant. This is a contradiction and hence (1.15) holds, i.e., \( f(t) = o(t^{-\frac{3}{2}}) \), \( t \gg 1 \).

To prove Theorem 1.2 it remains to show that the index in the decay rate (1.15) is optimal. For this, we only need to prove that for any \( \epsilon > 0 \), there exist a sequence \( \{t_n\}_{n=1}^{\infty} \) with \( t_n \uparrow \infty \) and a function \( g(t) \) satisfying all conditions of Theorem 1.2 for \( \beta = \gamma = 0 \) such that \( g(t_n) = 1 + t_n)^{-\frac{3}{2} - \epsilon} \). The function \( g(t) \) is constructed as follows.

Let \( s_n = e^n \). As in (3.6), we consider the following ordinary differential equation on \([s_n, t_n]\),

\[
g_n'(\tau) = C_0(1 + \tau)^{-\alpha}, \quad g_n(t_n) = (1 + t_n)^{-\frac{3}{2} - \epsilon}, \quad g_n(s_n) = 0,
\]

(3.17)
which gives
\[ t_n - s_n \approx \frac{1}{C_0} g_n(t_n)(1 + t_n)\alpha = \frac{1}{C_0} (1 + t_n)^{\frac{\alpha}{2} - \epsilon}, \quad n \gg 1. \] (3.18)

Let \( g_n(t) \) monotonically decrease to zero on \([t_n, z_n]\) where \( z_n \) is close to \( t_n \) so that \( z_n < s_{n+1} \) and

\[ \sum_{n=1}^{\infty} \int_{t_n}^{z_n} g_n(t)dt < \infty, \] (3.19)

see Figure 2 below.

\[ \text{Figure 2} \]

\[ \begin{align*}
0 & \quad s_n \quad t_n \quad z_n \\
\square & \quad g_n(t_n) \\
D & \quad \Delta_n \\
C & \quad g_n(\tau) \\
A & \\
B & \\
\end{align*} \]

Define
\[ g(t) = \begin{cases} 
  g_n(t), & t \in [s_n, z_n], \\
  0, & \text{otherwise}. 
\end{cases} \] (3.20)

It is obvious to see that \( g'(t) = C_0(1 + t)^{-\alpha} \) on \([s_n, t_n]\) and \( g'(t) \leq 0 \) as \( t \notin [s_n, t_n] \) so that (1.12) is satisfied.

On the other hand, the integral \( \int_{s_n}^{t_n} g(t)dt \) is less than the area of the rectangle \( \square ABCD \) with the width \( t_n - s_n \) and the height \( g_n(t_n) = (1 + t_n)^{\frac{\alpha}{2} - \epsilon} \), i.e.,

\[ \int_{s_n}^{t_n} g(t)dt \leq (t_n - s_n)g_n(t_n) \leq \frac{2}{C_0} (1 + t_n)^{-2\epsilon} \leq \frac{2}{C_0} s_n^{-2\epsilon} \leq Ce^{-2\epsilon}, \] (3.21)

which implies, together with (3.19),

\[ \int_{0}^{\infty} g(t)dt = \sum_{n=1}^{\infty} \int_{s_n}^{t_n} g(t)dt + \sum_{n=1}^{\infty} \int_{t_n}^{z_n} g(t)dt \leq C \sum_{n=1}^{\infty} e^{-2\epsilon n} + C < \infty. \] (3.22)

In particular, \( g(t_n) = (1 + t_n)^{\frac{\alpha}{2} - \epsilon}. \) Thus the index in (1.15) is optimal. Therefore Theorem 1.2 is completed. \( \square \)
4 The a priori estimates

First we introduce a useful lemma concerning the regularity of strong solution to (1.6).

**Lemma 4.1** (Theorem 4.10 and 5.1 in [30]). Let \( f \in L^2([0, T] \times \Omega, H^1(\mathbb{R})) \) and \( g \in L^2([0, T] \times \Omega, H^2(\mathbb{R})) \) be adapted process, then for any \( \phi_0 \in H^2(\mathbb{R}) \) and any \( T > 0 \), the stochastic equation

\[
\begin{align*}
\frac{d\phi}{dt} &= \mu \phi_{xx} dt + \sigma \phi_x dB(t) + f dt + g dB(t), \\
\phi(0) &= \phi_0,
\end{align*}
\] (4.1)

with \( \sigma^2 < 2\mu \), admits a unique strong solution in

\[ W_T := \left\{ \phi \in C((0, T); H^3(\mathbb{R})) \cap L^2((0, T); H^2(\mathbb{R})) \right\}. \]

Let \( \phi(x, t) = u(t, x) - \bar{u}(t, x) \), then the stochastic Burgers equation (1.6) can be reduced into the following perturbation equation

\[
\begin{align*}
\frac{d\phi}{dt} + (\phi \bar{u})_x dt + \frac{1}{2}(\phi^2)_x dt &= \mu \phi_{xx} dt + \mu \bar{u}_x dt + \sigma (\phi_x + \bar{u}_x) dB(t), \quad \text{in } \mathbb{R} \times [0, \infty), \\
\phi|_{t=0}(x) &= \phi_0(x), \quad \text{in } \mathbb{R}.
\end{align*}
\] (4.2)

The global existence of strong solution to (4.2) will be proved in Section 6 through a cut-off equation (6.1). The rest of this section is devoted to the a priori estimates of solutions in the following solution space

\[ X_T := \left\{ \phi \in C((0, T); H^1(\mathbb{R})), \phi_x \in L^2((0, T); H^1(\mathbb{R})) \right\} \supset W_T. \]

The norm \( \| \cdot \|_T \) is defined as

\[
\|\phi\|_T := \left( \mathbb{E} \sup_{0 \leq s \leq T} \|\phi(s)\|^2_{H^1(\mathbb{R})} + \mathbb{E} \int_0^T \|\phi_x(s)\|^2_{H^1(\mathbb{R})} ds \right)^{\frac{1}{2}}.
\]

We have

**Lemma 4.2.** Assume \( \sigma^2 < 2\mu \) and \( \phi(t, x) \in X_T \) is the strong solution of (4.2), it holds that

\[
\begin{align*}
\|\phi(t)\|^2 + \int_0^t \|\phi_x\|^2 ds + \int_0^t \int_\mathbb{R} \phi^2 \bar{u}_x dx dt &\leq C_1 \left( \|\phi_0\|^2 + \ln(2 + t) \right) + C_2 \int_0^t \int_\mathbb{R} \phi \bar{u}_x dB(t),
\end{align*}
\] (4.3)

**Proof.** Multiply (4.2) by \( \phi \), it holds in the sense of Itô integral that

\[
\phi d\phi + \phi^2 \bar{u}_x dt + \phi \phi_x \bar{u} dt + \phi^2 \phi_x dt = \mu \phi \phi_x dt + \mu \phi \bar{u}_x dt + \sigma \phi (\phi_x + \bar{u}_x) dB(t),
\] (4.4)
i.e.,
\[
\int_0^t \phi(s)d\phi(s) + \int_0^t \phi^2 \ddu_s ds + \int_0^t \phi_s \ddu_s ds + \int_0^t \phi^2 \phi_s ds
\]
= \int_0^t \mu \phi_s x_s ds + \int_0^t \mu \ddu_s x_s ds + \sigma \int_0^t \phi(x_s + \ddu_s) dB(s).
\]

Using Itô formula and stochastic Fubini theorem yield that
\[
\frac{1}{2}d\|\phi\|^2 - \frac{1}{2}d\langle\|\phi\|^2\rangle_t + \frac{1}{2} \int_{\mathbb{R}} \phi^2 \ddu_s dx dt
\]
= -\mu\|\phi_s\|^2 dt + \mu \int_{\mathbb{R}} \phi \ddu_s x dt + \sigma \int_{\mathbb{R}} \phi \ddu_s dB(t).
\]

By Lemma 2.1 ii), we obtain
\[
\left| \int_{\mathbb{R}} \phi \ddu_s x dx \right| = \left| \int_{\mathbb{R}} \phi \ddu_s x dx \right| \leq \epsilon\|\phi_s\|^2 + C_\epsilon\|\ddu_s\|^2 \leq \epsilon\|\phi_s\|^2 + C_\epsilon(2 + t)^{-1}.
\]

In addition, from Itô formula, the quadratic variation reads
\[
d\langle\|\phi\|^2\rangle_t = \sigma^2(1 + \epsilon)\|\phi_s\|^2 dt + C_\epsilon(2 + t)^{-1} dt, \quad \forall 0 < \epsilon < 1,
\]
which gives that
\[
\|\phi(t)\|^2 + \int_0^t \int_{\mathbb{R}} \phi^2 \ddu_s x dx dt + (2\mu - \sigma^2 - 4\epsilon) \int_0^t \|\phi_s\|^2 ds
\]
\leq \|\phi_0\|^2 + C_\epsilon \ln(2 + t) + 2\sigma \int_0^t \int_{\mathbb{R}} \phi \ddu_s x dB(t).
\]

Choosing \(\epsilon\) small enough implies that
\[
\|\phi(t)\|^2 + \int_0^t \|\phi_s\|^2 ds + \int_0^t \int_{\mathbb{R}} \phi^2 \ddu_s x ds
\]
\leq C_1 (\|\phi_0\|^2 + \ln(2 + t)) + C_2 \int_0^t \int_{\mathbb{R}} \phi \ddu_s x dB(s).
\]

\[\Box\]

**Lemma 4.3.** Assume \(\sigma^2 < 2\mu\) and \(\phi(t, x) \in X_T\) is the strong solution of (4.2), it holds that
\[
d\|\phi_s\|^2 + \|\phi_{sx}\|^2 dt + \int_{\mathbb{R}} \phi^2 \ddu_s x dx dt
\]
\leq C_1 \left( (2 + t)^{-2} dt + (2 + t)^{-2}\|\phi\|^2 dt + \|\phi\|_{L^\infty(\mathbb{R})} dt \right) - C_2 \int_{\mathbb{R}} \phi_{sx} \ddu_s x dB(t).
\]
Proof. Multiply \([4.2]\) by \(\phi_{xx}\), it holds that,
\[
-\phi_{xx}d\phi - \phi_{xx}(\phi_x \bar{u} + \phi_t \bar{u}_x) dt - \phi_x \phi_{xx} dt = -\mu \phi^2_{xx} dt - \mu \phi_{xx} \bar{u}_{xx} dt - \phi_{xx} \sigma(\phi_x + \bar{u}_x) dB(t).
\]
Note that from Lemma [2.1] ii, we have
\[
\int_{\mathbb{R}} \phi_{xx}(\bar{u} + \phi \bar{u}_x) dx = \frac{1}{2} \int_{\mathbb{R}} \phi^2_x \bar{u}_x dx - \int_{\mathbb{R}} \phi_{xx} \phi \bar{u}_x dx
\geq \frac{1}{2} \int_{\mathbb{R}} \phi^2_x \bar{u}_x dx - \epsilon \|\phi_{xx}\|^2 - C_\epsilon(2 + t)^{-2}\|\phi\|^2,
\]
\[
\int_{\mathbb{R}} \phi_{xx}^2 dx \leq \epsilon \|\phi_{xx}\|^2 + C_\epsilon(2 + t)^{-2},
\]
\[
\int_{\mathbb{R}} \phi \phi_{xx} \phi_{xx} dx \leq \frac{1}{2} \|\phi_{xx}\|^3 \leq C\|\phi_{xx}\|^2 \|\phi\|^\frac{3}{2} \leq \epsilon \|\phi_{xx}\|^2 + C_\epsilon \|\phi\|^6_{L^6(\mathbb{R})}.
\]
On the other hand, from Lemma [4.1] in which \(f := (-(\phi \bar{u})_x - \frac{1}{2}(\phi^2)_x) \in L^2([0, T] \times \Omega, \mathbb{H}^1(\mathbb{R}))\) and \(g := \sigma \bar{u}_x \in L^2([0, T] \times \Omega, \mathbb{H}^2(\mathbb{R}))\), we conclude that \(\|\phi_x\|^2\) is semi-martingale. Thus we get
\[
d\langle \|\phi_x\|^2 \rangle_t = \sigma^2 \|\phi_{xx} + \bar{u}_{xx}\|^2 dt \leq \sigma^2 (1 + \epsilon) \|\phi_{xx}\|^2 dt + \sigma^2 C_\epsilon(2 + t)^{-2} dt,
\]
which indicates that by choosing \(\epsilon\) small,
\[
d\|\phi_x\|^2 + \|\phi_{xx}\|^2 dt + \int_{\mathbb{R}} \phi^2_x \bar{u}_x dx dt
\leq C_1 \left( (2 + t)^{-2} dt + (2 + t)^{-2} \|\phi\|^2 dt + \|\phi\|^6_{L^6(\mathbb{R})} dt \right) - C_2 \int_{\mathbb{R}} \phi_{xx} \bar{u}_x dx dB(t).
\]

Next we give an interesting \(L^p(\mathbb{R})\) estimate which is crucial to close the a priori estimates.

**Lemma 4.4.** Assume \(\sigma^2 < 2\mu\) and \(\phi(t, x) \in X_T\) is the strong solution of \([4.2]\), it holds that for any \(p > 2\),
\[
d\|\phi\|^p_{L^p(\mathbb{R})} + \int_{\mathbb{R}} |\phi|^p_{xx} \bar{u}_x + |\phi|^p \phi_{xx}^2 dx dt \leq C_1 (2 + t)^{-\frac{p}{2}} dt + C_2 \int_{\mathbb{R}} |\phi|^p \phi_{xx} dx dB(t).
\]

Proof. Multiply \([4.2]\) by \(|\phi|^p \phi_{xx}\), we have
\[
|\phi|^p \phi_{xx} d\phi + |\phi|^p \phi_{xx} \sigma(\phi_x + \bar{u}_x) dB(t),
\]
and
\[
\int_{\mathbb{R}} |\phi|^p \phi_{xx} dx = \int_{\mathbb{R}} |\phi|^p \bar{u}_x dx + \int_{\mathbb{R}} |\phi|^p \phi_{xx} \bar{u}_x dx = \frac{p-1}{p} \int_{\mathbb{R}} |\phi|^p \bar{u}_x dx,
\]
\[
\int_{\mathbb{R}} |\phi|^p \phi_{xx} dx = \int_{\mathbb{R}} |\phi|^p [\phi \phi_{xx} - \phi^2_{xx}] dx = -(p-1) \int_{\mathbb{R}} |\phi|^p \phi_{xx}^2 dx,
\]
\[
\int_{\mathbb{R}} |\phi|^p \phi_{xx} dx \leq (p-1) \int_{\mathbb{R}} |\phi|^p \phi_{xx}^2 dx \leq \epsilon \int_{\mathbb{R}} |\phi|^p \phi_{xx}^2 dx + \epsilon \int_{\mathbb{R}} |\phi|^p \bar{u}_x dx + C_\epsilon \int_{\mathbb{R}} \bar{u}_x^{p+1} dx
\leq \epsilon \int_{\mathbb{R}} |\phi|^p \phi_{xx}^2 dx + \epsilon \int_{\mathbb{R}} |\phi|^p \bar{u}_x dx + C_\epsilon (2 + t)^{-\frac{p}{2}}.
\]
On the other hand, set $F_p(u) = \int_{\mathbb{R}} |u|^p(x)dx$, then $F_p \in C^2([L^1(\mathbb{R}), \mathbb{R})]$ and its first and second derivatives are

$$DF_p(u) = p|u|^{p-2}u \in L^2(\mathbb{R}),$$
$$D^2F_p(u) = (p^2 - p)|u|^{p-2} \in \mathcal{L}(L^2(\mathbb{R})).$$

Then the Itô formula implies that

$$dF_p(\phi) = \langle DF_p(\phi), d\phi \rangle + \frac{1}{2} \left( \|D^2F_p(\phi)\|^2 \sigma(\phi_x + \bar{u}_x) \right) dt$$
$$= p\langle |\phi|^{p-2}\phi, d\phi \rangle + \frac{1}{2} (p^2 - p)\sigma^2 \left( \|\phi^{\frac{p}{2}-1}(\phi_x + \bar{u}_x)\|^2 \right) dt$$
$$\leq p\langle |\phi|^{p-2}\phi, d\phi \rangle + \left( \frac{1}{2} (p^2 - p)\sigma^2 + \epsilon \right) \left( \|\phi^{\frac{p}{2}-1}\|_{L^2}^2 + C \|\phi^{\frac{p}{2}-1}u_x\|_{L^2}^2 \right) dt,$$

and

$$\left( \|\phi^{\frac{p}{2}-1}u_x\|_{L^2}^2 \right) \leq \epsilon \int_{\mathbb{R}} |\phi|^p \bar{u}_x dx + C \epsilon \int_{\mathbb{R}} \bar{u}_x^{\frac{p}{2}+1} dx \leq \epsilon \int_{\mathbb{R}} |\phi|^p \bar{u}_x dx + C \epsilon (2 + t)^{-\frac{p}{2}}.$$

Collecting all estimates above, we get that

$$d\|\phi\|_{L^p(\mathbb{R})}^p + (p - 1 - \epsilon) \int_{\mathbb{R}} |\phi|^p \bar{u}_x dx dt + \frac{1}{2} (p^2 - p)(2\mu - \sigma^2 - \epsilon) \int_{\mathbb{R}} |\phi|^{p-2}\phi_x^2 dx dt$$
$$\leq C \epsilon (2 + t)^{-\frac{p}{2}} dt + p\epsilon \int_{\mathbb{R}} |\phi|^{p-2}\phi \bar{u}_x dx dB(t).$$

Note that $\sigma^2 < 2\mu$, choosing $\epsilon$ small gives (4.17). Thus the proof of Lemma 4.4 is completed.

\[\square\]

## 5 Decay estimates

This section is devoted to the decay rate of strong solution toward the rarefaction wave given in (1.4).

### 5.1 Decay rate in expectation

**Theorem 5.1.** Let $\phi \in X_T$ be the unique strong solution of (4.2), it holds that for any $0 \leq t \leq T$,

$$\mathbb{E}\|\phi(t)\|_2^2 + \mathbb{E} \int_0^t \|\phi_x\|_2^2 dt + \mathbb{E} \int_0^t \phi^2 \bar{u}_x dx dt \leq C \ln(2 + t),$$

and

$$\mathbb{E}\|\phi\|_{L^p(\mathbb{R})}^p \leq C (2 + t)^{-\frac{p^2}{4}} \ln^p(2 + t).$$

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We can check that $E$ estimate (5.1) for $E$ directly follows from Lemma 4.2 by taking expectation. For $L^p$ estimate (5.2), multiplying (4.17) by $(2 + t)\frac{p}{2}$ gives that

$$
\begin{align*}
\frac{d}{dt}(2 + t)^\frac{p}{2}\|\phi(t)\|_{L^p(R)}^p + (2 + t)^\frac{p}{2} &\int_R |\phi|^p \bar{u}_x + |\phi|^{p-2} \phi_x^2 dx dt \\
\leq C_1 dt + C_1(2 + t)^{-\frac{p}{2}} ||\phi||_{L^p(R)}^p dt + C_2(2 + t)^\frac{p}{2} \int_R |\phi|^{p-2} \phi_x dxdB(t).
\end{align*}
$$

(5.3)

Thanks the inequality (2.4), we have

$$
(2 + t)^\frac{p}{2} ||\phi||_{L^p(R)}^p \leq C_p(2 + t)^{\frac{p}{2}} \|\phi\|_{L^2(R)}^p \|\phi\|_{L^2(R)}^{\frac{4p}{2p-4}} \leq \frac{1}{2}(2 + t)^{\frac{p}{2}} \int_R |\phi|^{p-2} \phi_x^2 dx + C_p(2 + t)^{\frac{p}{2}} |\phi|_{L^p(R)}^p.
$$

(5.4)

Integrating (5.3) over $[0, t]$ gives that

$$
(2 + t)^{\frac{p}{2}} ||\phi(t)||_{L^p(R)}^p \leq C_1 t + C_1 \int_0^t (2 + s)^{-\frac{p}{2}} \|\phi(t)\|_{L^p(R)}^p ds + C_2 \int_0^t (2 + s)^{\frac{p}{2}} \int_R |\phi|^{p-2} \phi_x dxdB(s).
$$

(5.5)

Taking expectation, we have

$$
\mathbb{E}||\phi(t)||_{L^p(R)}^p \leq C(2 + t)^{\frac{p}{2}} + C(2 + t)^{\frac{p}{2}} \int_0^t (2 + s)^{\frac{p}{2}} \mathbb{E}||\phi(t)||_{L^p(R)}^p ds.
$$

(5.6)

We estimate $\mathbb{E}||\phi(t)||_{L^p(R)}^p$ through $\mathbb{E}||\phi(t)||_{L^p(R)}^4$ and the BDG inequality. In fact, (4.3) gives that

$$
||\phi(t)||_{L^p(R)}^4 \leq \left[ C_1 \left[ (1 + ||\phi_0||^2) + \ln(2 + t) \right] + C_2 \int_0^t \int_R \phi \bar{u}_x dxdB(s) \right]^2,
$$

(5.7)

and

$$
\mathbb{E}||\phi||_{L^p(R)}^4 \leq C_1 \ln^2(2 + t) + C_2 \mathbb{E} \int_0^t \left( \int_R \phi \bar{u}_x dxdB \right)^2 ds \\
= C_1 \ln^2(2 + t) + C_2 \mathbb{E} \int_0^t \left( \int_{u_x} \phi d\tilde{u} \right)^2 ds \\
\leq C_1 \ln^2(2 + t) + C_2 (u_x - u_{-}) \mathbb{E} \int_0^t \int_R \phi^2 \bar{u}_x dxdB \\
\leq C \ln^2(2 + t).
$$

(5.8)

We can check that $\mathbb{E}||\phi||_{L^p(R)}^p \leq C \ln^\frac{p}{2}(2 + t)$, $\forall p \in [2, 4]$ by the Hölder inequality. Next we shall use the BDG inequality and the decay property $\bar{u}_x \leq \frac{C}{2\pi t}$ to estimate $\mathbb{E}||\phi||_{L^p(R)}^p$ for any $p > 4$.

We claim that

$$
\mathbb{E}||\phi||_{L^p(R)}^p \leq C_p \ln^p(2 + t), \quad \forall p \geq 4.
$$

(5.9)
The claim will be proved by the following induction principle: find a sequence \( p_n \to \infty \) so that (5.9) is valid for all \( p_n \). Since (5.9) holds for \( p_0 = 4 \) due to (5.8), we can suppose that (5.9) is true for \( p_k, k = 0, 1, \cdots, n \) until \( p_n \) starting with \( p_0 = 4 \). Now we try to find \( p_{n+1} > p_n \). Note that \( ||\phi||^2 \) is semi-martingale, the BDG inequality and (4.3) yield that

\[
\mathbb{E}[||\phi||^{p_{n+1}}] \leq \mathbb{E}(\sup_{0 \leq s \leq t} ||\phi||^2)^{\frac{p_{n+1}}{2}} \leq C \left( \ln^{\frac{p_{n+1}}{2}} (2 + t) + \mathbb{E} \left[ \int_0^t \left( \int_R |\phi|^2 d\mu_s \right)^2 ds \right] \right)^{\frac{p_{n+1}}{2}}
\]  

(5.10)

and for some \( \frac{1}{2} < \beta_n < 1 \) determined later,

\[
\mathbb{E} \left[ \int_0^t \left( \int_R |\phi|^2 d\mu_s \right)^2 dx \right] \leq \mathbb{E} \left[ \int_0^t \left( \int_R |\phi|^{2(1-\beta_n)} d\mu_s \right) \right]^{\frac{p_{n+1}}{2}} 
\]

\[
\leq \mathbb{E} \left\{ \int_0^t \left( \int_R |\phi|^{2(1-\beta_n)} d\mu_s \right) ||\bar{u}_s||_{L_\infty} ||\phi||^{2\beta_n} ds \right\}^{\frac{p_{n+1}}{2}} 
\]

\[
\leq C \mathbb{E} \int_0^t \int_R |\phi|^2 \bar{u}_s dxds + C \mathbb{E} \int_0^t (2 + s)^{-1} ||\phi||^2 ds \right\}^{\frac{p_{n+1}}{2}} 
\]

\[
\leq C \ln(2 + t) + \ln^{\frac{p_{n+1}}{2}-1} (2 + t) \mathbb{E} \sup_{0 \leq s \leq t} ||\phi||^{\frac{2p_{n+1}}{2-(2-1-\beta_n)p_{n+1}}} (s).
\]  

Choosing \( \beta_n = 1 - \frac{2}{3p_n} \) and

\[
p_{n+1} := \frac{3p_n}{2 - \frac{1}{p_n}} \geq \frac{3}{2} p_n
\]  

(5.12)

so that \( \frac{2\beta_n p_{n+1}}{4-(1-\beta_n)p_{n+1}} = p_n \), we obtain from (5.9)-(5.11) that

\[
\mathbb{E}[||\phi||^{p_{n+1}}] \leq C \ln^{\frac{p_{n+1}}{2}} (2 + t) + \ln^{\frac{p_n}{2}} (2 + t) \mathbb{E}[||\phi||^{p_n}] \leq C \ln^{\frac{1}{2}} p_n (2 + t) \leq C \ln^{p_{n+1}} (2 + t). 
\]  

(5.13)

Thus we find the sequence \( \{p_n\}, n = 1, \cdots \) and verify the claim (5.9) for all \( p_n \). The claim for \( p \in (p_n, p_{n+1}) \) can be justified by the Hölder inequality. Hence (5.9) is indeed true for all \( p > 2 \).

Substituting (5.9) into (5.6) gives (5.2). Therefore the proof is complete. \( \square \)

**Lemma 5.1.** Assume \( \sigma^2 < 2\mu \). Let \( \phi \in X_T \) be the solution of (4.2), then it holds that for any \( \epsilon > 0 \) and any \( 0 \leq t \leq T \),

\[
\mathbb{E}[||\phi_{\epsilon}(t)||^2] \leq C_{\epsilon}(2 + t)^{-\frac{1}{2} + \epsilon}.
\]  

(5.14)

**Proof.** Take expectation on (4.11), we have

\[
d\mathbb{E}[||\phi||^2] + \mathbb{E}[||\phi_{\epsilon+1}||^2] dt \leq C(2 + t)^{-2}(1 + \mathbb{E}[||\phi||^2]) dt + C\mathbb{E}[||\phi||_{L_\infty}^6] dt.
\]  

(5.15)
Choosing $p = 6$ in (5.2) gives that
\[
\frac{d}{dt} \mathbb{E} \| \phi_x(t) \|^2 \leq C(2 + t)^{-1} \ln^6(2 + t) \leq C(2 + t)^{-1+2\epsilon}.
\] (5.16)

Note that
\[
\int_0^t \mathbb{E} \| \phi_x \|^2 dt \leq C \ln(2 + t),
\] (5.17)
the area inequality (Theorem 1.3, $f(t) = \mathbb{E} \| \phi_x \|^2$) implies that
\[
\mathbb{E} \| \phi_x(t) \|^2 \leq C(2 + t)^{-\frac{1}{2}+\epsilon}.
\]

\[\square\]

**Theorem 5.2.** Assume $\sigma^2 < 2\mu$. Let $\phi \in X_T$ be the solution of (4.2), then for any $\epsilon > 0$,
\[
\mathbb{E} \| \phi \|_{L^\infty(\mathbb{R})} \leq C(2 + t)^{-\frac{1}{2}+\epsilon}.
\] (5.18)

**Proof.** Thanks the G-N inequality, we have that
\[
\| \phi \|_{L^\infty(\mathbb{R})} \leq C_p \| \phi \|_{L^p(\mathbb{R})}^{\frac{p}{p-2}} \| \phi_x \|_{L^\frac{p}{p-2}}^{\frac{2}{p-2}},
\]
which gives that
\[
\mathbb{E} \| \phi \|_{L^\infty(\mathbb{R})} \leq C_p \mathbb{E} \left( \| \phi \|_{L^p(\mathbb{R})}^{\frac{p}{p-2}} \| \phi_x \|_{L^\frac{p}{p-2}}^{\frac{2}{p-2}} \right) \leq C_p \left( \mathbb{E} \| \phi \|_{L^p(\mathbb{R})}^{p} \right)^{\frac{1}{p}} \left( \mathbb{E} \| \phi_x \|^2 \right)^{\frac{1}{2}},
\]
(5.19)
by choosing $p$ sufficiently large. Thus the proof is completed. \[\square\]

### 5.2 Decay rate a.s.

**Lemma 5.2.** Let $\phi \in X_T$ be the strong solution of (4.2), it holds that for any $p > 2$, there exists a $\mathcal{F}_\infty$ measurable random variable $C_p(\omega) \in L^2(\Omega)$ such that
\[
\| \phi \|_p^p \leq C_p(\omega)(2 + t)^{-\alpha}, \text{ a.s. } \forall \alpha < \frac{p-2}{4}.
\] (5.20)

**Proof.** For any $\epsilon > 0$, multiply (4.6) by $(2 + t)^{-\epsilon}$, one has that
\[
\frac{1}{2} \frac{d}{dt} (2 + t)^{-\epsilon} \| \phi \|^2 + \frac{\epsilon}{2} (2 + t)^{-\epsilon-1} \| \phi \|^2 dt + (2 + t)^{-\epsilon} \left[ \| \phi_x \|^2 dt + \frac{1}{2} \int_\mathbb{R} \phi^2 \bar{u}_x dx dt \right]
= (2 + t)^{-\epsilon} \int_\mathbb{R} \phi \bar{u}_x dx dt + \frac{1}{2} (2 + t)^{-\epsilon} \sigma^2 \| \phi_x + \bar{u}_x \|^2 dt + (2 + t)^{-\epsilon} \int_\mathbb{R} \phi \bar{u}_x dxdB(t)
\]
(5.21)
which gives that

\[ \mathbb{E} \int_0^t (2 + s)^{-\epsilon} \int_\mathbb{R} \phi^2 \tilde{u}_s dx ds \leq C_\epsilon. \quad (5.22) \]

Define \( M^\epsilon(t) = \int_0^t (2 + s)^{-\epsilon} \int_\mathbb{R} \phi \tilde{u}_s dx dB(s) \), we have

\[
EM^\epsilon(t)^2 = \mathbb{E} \int_0^t (2 + s)^{-2\epsilon} \left( \int_\mathbb{R} \phi(x) \tilde{u}(x) dx \right)^2 ds \\
= \mathbb{E} \int_0^t (2 + s)^{-2\epsilon} \left( \int_{\tilde{u}}^\infty \phi(\tilde{u}) d\tilde{u} \right)^2 ds \\
\leq (u_+ - u_-) \mathbb{E} \int_0^t (2 + s)^{-2\epsilon} \int_{\tilde{u}_-}^{\tilde{u}_+} \phi^2(\tilde{u}) d\tilde{u} ds \\
= (u_+ - u_-) \mathbb{E} \int_0^t (2 + s)^{-2\epsilon} \int_\mathbb{R} \phi^2 \tilde{u}_s dx ds \leq C_\epsilon,
\]

which implies from Doob’s \( L^p \) inequality that there exists a \( \mathcal{F}_\infty \) measurable random variable \( C_\epsilon(\omega) \in L^2(\Omega) \) such that

\[ |M^\epsilon(t)| \leq C_\epsilon(\omega), \ a.s. \]

Again using (5.21) yields that

\[ \|\phi\|^2 \leq C_\epsilon(2 + t)^\epsilon(1 + M^\epsilon(t)) \leq C_\epsilon(\omega)(2 + t)^\epsilon, \ a.s. \quad (5.24) \]

On the other hand, for any \( 0 < \alpha < \frac{p-2}{4} \), define

\[ N_{\alpha,p}(t) = \int_0^t (2 + s)^\alpha \int_\mathbb{R} |\phi|^{p-2} \phi \tilde{u}_s dx dB(s). \quad (5.25) \]

A direct computation gives that

\[
\mathbb{E}[N_{\alpha,p}^2(t)] = \mathbb{E} \int_0^t (2 + s)^{2\alpha} \left( \int_\mathbb{R} |\phi|^{p-2} \phi \tilde{u}_s dx \right)^2 dsr \\
= \int_0^t (2 + s)^{2\alpha} \mathbb{E} \left( \int_{\tilde{u}_-}^{\tilde{u}_+} |\phi(\tilde{u})|^{p-2} \phi d\tilde{u} \right)^2 ds \\
\leq C \int_0^t (2 + s)^{2\alpha} \mathbb{E} \int_\mathbb{R} |\phi|^{2p-2} \tilde{u}_s dx ds \\
\leq C \int_0^t (2 + t)^{2\alpha-1} (2 + t)^{\frac{2p-2}{2p-2}} \ln^{2(p-1)}(2 + t) dt \leq C,
\]

where we have used the fact that \( \mathbb{E}[|\phi(t)|^{2p-2}] \leq C(2 + t)^{\frac{2p-2}{2p-2}} \ln^{2(p-1)}(2 + t) \) due to (5.2) and \( |\tilde{u}_s| \leq C(2 + t)^{-1} \). Then there exists a \( \mathcal{F}_\infty \) measurable random variable \( C_\rho(\omega) \in L^2(\Omega) \) such that

\[ |N_{\alpha,p}(t)| \leq C_\rho(\omega), \ a.s. \]
Multiplying (4.17) by \((2 + t)^\alpha\) and using (5.4) and (5.24) imply that
\[
d(2 + t)^\alpha\|\phi\|_p^p + (2 + t)^\alpha \int_\mathbb{R} |\phi|^p \bar{u}_s dxdt + (2 + t)^\alpha \int_\mathbb{R} |\phi|^{p-2} \phi_s^2 dxdt
\leq C_1(2 + t)^{\alpha - \frac{p+2}{2}} dt + C_2(2 + t)^{\alpha - \frac{p+2}{2}} \int_\mathbb{R} |\phi|^{p-2} \phi s dxdB(t)\] (5.27)
\[
\leq C_1(2 + t)^{\alpha - \frac{p+2}{2}} dt + C_2(2 + t)^{\alpha - \frac{p+2}{2} + \frac{p}{2}} C_\epsilon(\omega) + C_3(2 + t)^{\alpha} \int_\mathbb{R} |\phi|^{p-2} \phi s dxdB(t), \ a.s.
\]
Integrating over \([0, t]\) and choosing \(\epsilon\) sufficiently small so that \(\alpha - \frac{p+2}{4} + \frac{p}{2} < -1\), one has that
\[
(2 + t)^\alpha\|\phi\|_p^p \leq C(1 + C(\omega)) + CN_{\alpha, p}(t) \leq C_\epsilon(\omega), \ a.s.
\] (5.28)
which gives (5.20). Thus the proof is completed. \(\square\)

**Lemma 5.3.** Let \(\phi \in X_T\) be the strong solution of (4.2), it holds that for any \(\epsilon > 0\), there exists a \(\mathcal{F}_\infty\) measurable random variable \(C_\epsilon(\omega) \in L^2(\Omega)\) such that
\[
\|\phi_\epsilon\|^2 \leq C_\epsilon(\omega)(2 + t)^\epsilon, \ a.s.
\] (5.29)

**Proof.** Multiplying (4.16) by \((2 + t)^{-\frac{1}{2}}\), we have
\[
d(2 + t)^{-\frac{1}{2}}\|\phi_x\|^2 + (2 + t)^{-\frac{1}{2}}\|\phi_{xx}\|^2 dt
\leq C_1(2 + t)^{-\frac{1}{2}} \left((2 + t)^{-2} + (2 + t)^{-2}\|\phi\|^2 + \|\phi\|_{L^\infty(\mathbb{R})}^2\right) dt\] (5.30)
\[
- C_2(2 + t)^{-\frac{1}{2}} \int_\mathbb{R} \phi_{xx} \bar{u}_s dxdB(t),
\]
which implies that
\[
\mathbb{E} \int_0^t (2 + s)^{-\frac{1}{2}}\|\phi_{xx}\|^2 dxdt \leq C.
\]
Define \(N(t) = - \int_0^t \int_\mathbb{R} \phi_{xx} \bar{u}_s dxdB(s)\), then in the same way as (5.23), one has that
\[
\mathbb{E} N(t)^2 = \mathbb{E} \int_0^t \left(\int_\mathbb{R} \phi_{xx}(x) \bar{u}_s(x) dx\right)^2 ds \leq (u_+ - u_-) \int_\mathbb{R} \|\bar{u}_x\|_{L^\infty(\mathbb{R})} \mathbb{E} \int_\mathbb{R} \phi_{xx}^2(x) dxdx s
\leq C \mathbb{E} \int_0^t (2 + s)^{-\frac{1}{2}} \int_\mathbb{R} \phi_{xx}^2 dxdx s \leq C.
\] (5.31)
Thus there exists a \(\mathcal{F}_\infty\) measurable random variable \(C(\omega) \in L^2(\Omega)\) such that
\[
|N(t)| \leq C(\omega), \ a.s.
\]
Integrating (4.16) on \([0, t]\), and choosing \(p = 6\) and \(\alpha = -1 + \epsilon\) (\(\epsilon > 0\) is any constant) in (5.28), we get (5.29). The proof is completed. \(\square\)
Theorem 5.3. Let \( \phi \in X_T \) be the solution of (4.2), then for any \( \epsilon > 0 \), there exists a \( \mathcal{F}_\infty \) measurable random variable \( C_\epsilon(\omega) \in L^2(\Omega) \) such that

\[
\|\phi\|_{L^\infty(\mathbb{R})} \leq C_\epsilon(\omega)(2 + t)^{-\frac{1}{4} + \epsilon}, \text{ a.s.}
\] (5.32)

Proof. From the G-N inequality, we immediately obtain from (5.20) and (5.29) that

\[
\|\phi\|_{L^\infty(\mathbb{R})} \leq C_p \|\phi\|_{L_p} \|\phi_x\|_{L_\infty} \leq C_\epsilon(\omega)(2 + t)^{-\frac{1}{4} + \epsilon}, \text{ a.s.}
\]

by choosing \( p \) sufficiently large. The Hölder inequality implies that \( C_\epsilon(\omega) \in L^2(\Omega) \). \( \square \)

6 Global existence

It remains to prove the global existence of strong solution of the perturbed equation (4.2). Due to the effect of noise, the \( \|\phi\|_{L^\infty(\mathbb{R})} \) norm may not be uniformly bounded. We adopt a cut-off technique to prove the global existence of (4.2). That is, we consider the following cut-off equation

\[
\begin{aligned}
  d\phi + (\phi\bar{u})_x dt + \frac{1}{2}((\Pi_m \phi)^2)_x dt &= \mu \phi_{xx} dt + \mu \bar{u}_{xx} dt + \sigma(\phi_x + \bar{u}_x) dB(t), \quad \text{in } \mathbb{R} \times [0, \infty), \\
  \phi|_{t=0}(x) &= \phi_0(x), \quad \text{on } \mathbb{R},
\end{aligned}
\] (6.1)

where

\[
\Pi_m : H^1(\mathbb{R}) \rightarrow H^1(\mathbb{R}), \quad f \mapsto \min \left\{ m, \frac{\|f\|_{H^1(\mathbb{R})}}{\|f\|_{H^1(\mathbb{R})}} \right\} f.
\]

Before studying the equation (6.1), we first derive a useful lemma to treat the cut-off term.

Lemma 6.1. Let \( H \) be some Hilbert space. Define a cut-off mapping \( \Pi_m \) as follows,

\[
\Pi_m x = \frac{x}{\|x\|_H} \min \left\{ m, \|x\|_H \right\}, \quad \forall x \in H.
\]

Then for any \( m > 0 \) and \( x, y \in H \), it holds that

\[
\|\Pi_m x - \Pi_m y\|_H \leq \|x - y\|_H.
\] (6.2)

Proof. For brevity, we denote \( \|\cdot\|_H \) as \( |\cdot| \). Without loss of generality, we assume that \( |y| \leq |x| \) and \( |x| \neq 0 \). The proof is divided into three cases.

Case 1: \(|y| \leq m \leq |x|\). Define \( \rho(t) = |(1 + t)\Pi_m x - y|^2, t \geq 0 \). A direct computation yields that

\[
\rho'(t) = \frac{d}{dt} \langle (1 + t)\Pi_m x - y, (1 + t)\Pi_m x - y \rangle = \frac{d}{dt} [(1 + t)^2 m^2 + |y|^2 - 2(1 + t) \langle \Pi_m x, y \rangle]
\]

\[
= 2 \left[ (1 + t)m^2 - \langle \Pi_m x, y \rangle \right] \geq 2tm^2 \geq 0,
\]

\[
2tm^2 \geq 0.
\]

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which indicates that $\rho(t)$ is an increasing function, where we have used the fact that $|\langle \Pi_n x, y \rangle| \leq m^2$. Note that $\Pi_n y = y$ and $|\frac{1}{m} \Pi_n x| = x$, we have

$$|x - y|^2 = \rho\left(\frac{|x|}{m} - 1\right) \geq \rho(0) = |\Pi_n x - \Pi_n y|^2.$$  

Case 2: $m < |y| \leq |x|$. Let $\bar{x} = \frac{m}{|y|} x$ and $\bar{y} = \frac{m}{|y|} y$, then $m = |\bar{y}| \leq |\bar{x}|$. Since $\Pi_n x = \Pi_m \bar{x}$ and $\Pi_n y = \Pi_m \bar{y}$, we have from case 1 that,

$$|\Pi_n x - \Pi_n y| = |\Pi_m \bar{x} - \Pi_m \bar{y}| \leq |\bar{x} - \bar{y}| = \frac{m}{|y|} |x - y| \leq |x - y|.$$  

Case 3: $|y| \leq |x| \leq m$. In this case $\Pi_n x = x$, $\Pi_n y = y$. It is obvious that $|\Pi_n x - \Pi_n y| = |x - y|$.

Therefore the proof is complete. □

The local existence of (6.1) is given as follows.

**Theorem 6.1 (Local existence).** Assume $\sigma^2 < 2\mu$. Then for any $\phi_0 \in H^2(\mathbb{R})$, there exists a time $T(m) > 0$ such that the cut-off equation (6.1) has a unique solution in $X_T(m)$.

**Proof.** We use the iteration method to prove the local existence. Given $\phi^0 \in X_T$, where $T$ will be chosen later, let $\phi^{n+1}$ be the unique strong solution of

$$\begin{cases}
    d\phi^{n+1} - \mu \phi^{n+1}_{xx} dt = \sigma \phi^{n+1}_{x} dB(t) - (\phi^n \bar{u})_{x} dt - \frac{1}{2}[\langle \Pi_n \phi^n \rangle^2 - \langle \Pi_n \phi^n-1 \rangle^2]_x dt + \mu \bar{u}_{xx} dt + \sigma \bar{u}_{x} dB(t), \\
    \phi^{n+1}(0) = \phi_0, \quad \phi_0 \in H^2(\mathbb{R}), \quad s \in [0, T].
\end{cases} \tag{6.3}$$

Since $\phi^0 \in X_T$, it is straightforward to check that $f := -(\phi^n \bar{u})_{x} - \frac{1}{2}[\langle \Pi_n \phi^n \rangle^2 - \langle \Pi_n \phi^n-1 \rangle^2]_x + \mu \bar{u}_{xx} \in L^2([0, T] \times \Omega, H^1(\mathbb{R}))$ and $g := \sigma \bar{u}_{x} \in L^2([0, T] \times \Omega, H^2(\mathbb{R}))$. Thanks Lemma 4.1, $\phi^{n+1}$ belongs to $W_T \subset X_T$.

Thus we define a mapping

$$\mathcal{T}_T : X_T \longrightarrow X_T. \tag{6.4}$$

It remains to show that $\mathcal{T}_T$ is a contracting mapping for a suitably small $T$. Let $\Phi^n(t) = \phi^{n+1}(t) - \phi^n(t) (n \geq 1)$, then we have the following stochastic heat equation for $\Phi^n$,

$$\begin{cases}
    d\Phi^n - \mu \Phi^n_{xx} dt = -(\Phi^{n-1} \bar{u})_{x} dt - \frac{1}{2} \left[ \langle \Pi_n \phi^n \rangle^2 - \langle \Pi_n \phi^{n-1} \rangle^2 \right]_x dt + \sigma \Phi^n_{x} dB(t), \\
    \Phi^n(0) = 0, \quad \Phi^{n-1}, \Phi^n, \Phi^{n-1} \in X_T.
\end{cases} \tag{6.5}$$

Multiplying (6.5) by $\Phi^n$ and integrating the result on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} \Phi^n d\Phi^n dx + \mu \int_{\mathbb{R}} \left(\Phi^n\right)^2 dx dt = -\int_{\mathbb{R}} \Phi^n (\Phi^{n-1} \bar{u})_x dx dt - \int_{\mathbb{R}} \Phi^n \left[ \langle \Pi_n \phi^n \rangle^2 - \langle \Pi_n \phi^{n-1} \rangle^2 \right]_x dx dt, \tag{6.6}$$

which implies from Itô formula that

$$\frac{1}{2} dt \|\Phi^n\|^2 + \mu \|\Phi^n\|^2 dt = \frac{1}{2} \sigma^2 \langle \|\Phi^n\|^2 \rangle_t - \int_{\mathbb{R}} \Phi^n (\Phi^{n-1} \bar{u})_x dx dt - \int_{\mathbb{R}} \Phi^n \left[ \langle \Pi_n \phi^n \rangle^2 - \langle \Pi_n \phi^{n-1} \rangle^2 \right]_x dx dt. \tag{6.7}$$
We estimate the right hand side of (6.7) term by term. The Cauchy inequality gives that
\[
\left| \int_{\mathbb{R}} \Phi^n (\Phi^{n-1} \bar{u})_x \, dx \right| = \left| \int_{\mathbb{R}} \Phi^n_x \Phi^{n-1} \bar{u} \, dx \right| \leq \epsilon \| \Phi^n_x \|^2 + C_{\epsilon} \| \Phi^{n-1} \|^2.
\]

In addition, we have
\[
\left| \int_{\mathbb{R}} \Phi^n \left[ (\Pi_m \phi^n)^2 - (\Pi_m \phi^{n-1})^2 \right] \, dx \right| = \left| \int_{\mathbb{R}} \Phi^n_x \left[ \Pi_m \phi^n + \Pi_m \phi^{n-1} \right] \left[ \Pi_m \phi^n - \Pi_m \phi^{n-1} \right] \, dx \right|
\leq \| \Pi_m \phi^n + \Pi_m \phi^{n-1} \|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \| \Pi_m \phi^n - \Pi_m \phi^{n-1} \| \, dx
\leq C_m \| \Phi^n_x \| \| \Pi_m \phi^n - \Pi_m \phi^{n-1} \|
\leq \epsilon \| \Phi^n_x \|^2 + C_{\epsilon} m^2 \| \Phi^{n-1} \|^2_{H^1(\mathbb{R})}
\]
where we have used the fact that
\[
\| \Pi_m \phi^n - \Pi_m \phi^{n-1} \|_{H^1(\mathbb{R})} \leq C \| \phi^n - \phi^{n-1} \|_{H^1(\mathbb{R})}
\]
due to Lemma 6.1. From Itô formula, the quadratic variation reads
\[
d(\|\Phi^n\|^2)_t = \sigma^2 \|\Phi^{n-1}_x\|^2 \, dt. \tag{6.8}
\]
Collecting all the estimates above, integrating on \([0, T]\) \times \Omega and choosing \(\epsilon\) small enough, we have that
\[
\mathbb{E} \sup_{0 \leq s \leq T} \| \Phi^n \|^2 + \mathbb{E} \int_0^T \| \Phi^n_x(s) \|^2 \, ds
\leq C m^2 \int_0^T \mathbb{E} \| \Phi^{n-1}(s) \|_{H^1(\mathbb{R})} \|^2 \, ds \leq C m^2 T \mathbb{E} \sup_{0 \leq s \leq T} \| \Phi^{n-1} \|^2_{H^1(\mathbb{R})}, \tag{6.9}
\]
On the other hand, multiplying (6.5) by \(-\Phi^n_{xx}\) and integrating the result on \(\mathbb{R}\) yield that
\[
\frac{1}{2} \sigma^2 d(\| \Phi^n_x \|^2)_t + \mu \| \Phi^n_x \|^2 dt
= \frac{1}{2} \sigma^2 d(\| \Phi^n_x \|^2)_t + \int_{\mathbb{R}} \Phi^n_{xx}(\Phi^{n-1} \bar{u})_x \, dxdt + \int_{\mathbb{R}} \Phi^n_x \left[ (\Pi_m \phi^n)^2 - (\Pi_m \phi^{n-1})^2 \right] \, dx dt, \tag{6.10}
\]
It is straightforward to check that
\[
\left| \int_{\mathbb{R}} \Phi^n_{xx}(\Phi^{n-1} \bar{u})_x \, dx \right| = \left| \int_{\mathbb{R}} \Phi^n_{xx}(\Phi^{n-1} \bar{u} + \Phi^n \bar{u}_x) \, dx \right| \leq \epsilon \| \Phi^n_x \|^2 + C_{\epsilon} \| \Phi^{n-1} \|^2_{H^1(\mathbb{R})}
\]
and
\[
d(\| \Phi^n_x \|^2)_t = \sigma^2 \| \Phi^{n-1}_x \|^2 \, dt. \tag{6.11}
\]
\[\]
In addition, we have

\[
\left| \int_{\mathbb{R}} \Phi^n_{xx} \left[ \left( \Pi_m \Phi^n \right)^2 - (\Pi_m \Phi^n)^2 \right] \, dx \right| \\
\leq \left| \int_{\mathbb{R}} \Phi^n_{xx} \left[ \Pi_m \Phi^n + \Pi_m \Phi^n - \Pi_m \Phi^n \right] \, dx \right| \\
+ \left| \int_{\mathbb{R}} \Phi^n_{xx} \left[ \Pi_m \Phi^n + \Pi_m \Phi^n - \Pi_m \Phi^n \right] \, dx \right|
\]

\[:: J_1 + J_2.\]

Direct computation and Lemma 6.1 imply that

\[J_1 \leq Cm \int_{\mathbb{R}} \Phi^n_{xx} \left[ \Pi_m \Phi^n \right] \left| \Pi_m \Phi^n - \Pi_m \Phi^n \right| \, dx \leq Cm \parallel \Phi^n_{xx} \parallel \parallel \Pi_m \Phi^n - \Pi_m \Phi^n \parallel_{H^1(\mathbb{R})},\]

and

\[J_2 \leq \parallel \Pi_m \Phi^n - \Pi_m \Phi^n \parallel_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \Phi^n_{xx} \left[ \Pi_m \Phi^n + \Pi_m \Phi^n - \Pi_m \Phi^n \right] \, dx \leq Cm \parallel \Phi^n_{xx} \parallel \parallel \Pi_m \Phi^n - \Pi_m \Phi^n \parallel_{H^1(\mathbb{R})} \]

\[\leq \epsilon \parallel \Phi^n_{xx} \parallel^2 + Cc m^2 \parallel \Phi^n-1 \parallel_{H^1(\mathbb{R})}^2,\]

Collecting all estimates above, we have

\[E \sup_{0 \leq s \leq T} \parallel \Phi^n_x(s) \parallel^2 + (2 \mu - 8 \epsilon) \int_0^T E \parallel \Phi^n_x(s) \parallel^2 \, ds \leq Cc m^2 T E \sup_{0 \leq s \leq T} \parallel \Phi^n-1(s) \parallel_{H^1(\mathbb{R})}^2 + \sigma^2 E \int_0^T \parallel \Phi^n-1 \parallel_{H^1(\mathbb{R})}^2 \, ds.\]

Combining (6.9) and (6.12), choosing \(\epsilon\) and \(T\) small so that \(Cc m^2 T\) is sufficiently small, and \(\frac{\sigma^2}{2 \mu - 8 \epsilon} < 1\), we have

\[\parallel \Phi^n \parallel_{L^\infty(T_{\mathbb{R}})}^2 = \mathbb{E} \left[ \sup_{0 \leq s \leq T} \parallel \Phi^n(s) \parallel_{H^1(\mathbb{R})}^2 + \int_0^T \parallel \Phi^n_x(s) \parallel_{H^1(\mathbb{R})}^2 \right] \leq \frac{\sigma^2}{2 \mu - 8 \epsilon} \parallel \Phi^n-1 \parallel_{L^\infty(T_{\mathbb{R}})}^2 \]

(6.13)

which implies that \(T_{\mathbb{R}}\) is contracting in \(X_T\), thus the equation (6.1) has a unique strong solution in \(X_T\).

\[\square\]

**Theorem 6.2.** Assume that \(\sigma^2 < 2 \mu\) and \(\phi_0 \in H^2(\mathbb{R})\), then for any \(T \geq 0\), there exists a unique solution of (6.1) in \(X_T\).
Proof. Note that in Theorem 6.1, the local time $T$ only depends on $m$, but is independent from the initial data. Thus we can extend the local solution to any time $\bar{T} > 0$ for the cut off equation (6.1). □

On the other hand, it is straightforward to check that all a priori estimates in sections 4 and 5 can be obtained for the cut-off equation (6.1) in the same way. Thus we conclude from (5.24) and (5.29) that

$$\sup_{t \geq 0} \frac{\|\phi(t)\|_{H^1(\mathbb{R})}}{(2 + t)^{\epsilon}} < +\infty \quad \text{a.s.} \quad \forall \epsilon > 0.$$  \hspace{1cm} (6.14)

Now we are ready to prove the global existence of strong solution to (4.2), that is,

**Theorem 6.3 (Global existence).** Assume that $\sigma^2 < 2\mu$ and $\phi_0 \in H^2(\mathbb{R})$. Then for any fixed $T > 0$, there exists a unique strong solution of (4.2) in $X_T$ a.s.

Proof. Let $\phi_m$ be the global solution obtained in Theorem 6.2, set

$$\tau_m = \inf \{ t \geq 0 : \|\phi_m(t)\|_{H^1(\mathbb{R})} \geq m \}.$$  \hspace{1cm} (6.15)

Notice that $\phi_m(t) = \phi_n(t)$ for $m \geq n$ and all $t \leq \tau_n$. Set $\phi(t) = \phi_n(t)$ be the solution of (4.2) as $t \leq \tau_n$. Then $\tau_\infty = \infty$ a.s. holds from (6.14). Thus we have

$$\mathbb{P}(\tau_\infty < \infty) = \bigcup_{N=1}^{\infty} \mathbb{P}\left(\tau_m < N, \forall m \in \mathbb{Z}^+\right) = 0.$$  \hspace{1cm} (6.15)

Therefore Theorem 6.3 is proved. □

**Proof of Theorem 1.1.** The global existence of the unique strong solution to (1.6) is proved in Theorem 6.3 and the decay rates (1.8), (1.9) and (1.10) are obtained in Theorems 5.1, 5.2 and 5.3 respectively. Therefore the proof of Theorem 1.1 is completed. □

### 7 Instability of viscous shock wave

**Proof of Theorem 1.3.** As shown in (1.20), $\tilde{u}(\xi), \xi = x - st$ is the viscous shock wave of the deterministic Burgers equation (1.1) satisfying

$$\begin{cases} -s\tilde{u}' + \tilde{u}'' = \nu\tilde{u}'', \\ \tilde{u}(\xi) \to u_\pm, \text{ as } \xi \to \pm\infty. \end{cases}$$ \hspace{1cm} (7.1)

For simplicity, we consider the case that $s = 0$ (i.e., $u_- = -u_+ > 0$). In this situation, $\tilde{u}(x)$ has the following explicit formula,

$$\tilde{u}(x) = -u_- + \frac{2u_-c}{e^{hx} + c},$$ \hspace{1cm} (7.2)
where \( h = \frac{u_− - u_+}{\nu} > 0 \), and \( c > 0 \) could be any constant concerning with the shift of shock wave. Moreover, \( \tilde{u}(x) \) is monotonically decreasing with respect to \( x \), see Figure 3 below in the \( x-u \) plane.

As explained before, the perturbed viscous shock is \( \tilde{u}^B(t, x) := \tilde{u}(x + \sigma B(t)) \) and the two waves coincide at the initial time, i.e., \( \tilde{u}(x) = \tilde{u}^B(0, x) \). A direct computation gives that

\[
d(t) =: \mathbb{E}[\|\tilde{u}(x) - \tilde{u}^B(t, x)\|_{L^\infty} = \int_R \|\tilde{u}(x) - \tilde{u}(x + \sigma y)\|_{L^\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_R \|\tilde{u}(x) - \tilde{u}(x + \sigma \sqrt{t}z)\|_{L^\infty} e^{-\frac{z^2}{2}} dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \|\tilde{u}(x) - \tilde{u}(x + \sigma \sqrt{t}z)\|_{L^\infty} e^{-\frac{z^2}{2}} dz
\]

\[
+ \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \|\tilde{u}(x) - \tilde{u}(x + \sigma \sqrt{t}z)\|_{L^\infty} e^{-\frac{z^2}{2}} dz.
\]

Note that \( \tilde{u}(x) \) is monotonically decreasing, it is straightforward to check that for any \( z < 0 \), \( \tilde{u}(x + \sigma \sqrt{t}z) \) moves forward as \( t \) increases, and \( \|\tilde{u}(x) - \tilde{u}(x + \sigma \sqrt{t}z)\|_{L^\infty} \) is monotonically increasing with respect to \( t \) and \( \lim_{t \to \infty} \|\tilde{u}(x) - \tilde{u}(x + \sigma \sqrt{t}z)\|_{L^\infty} = u_- - u_+ \), see Figure 3. The same argument works for \( z > 0 \). Thus we conclude from Lebesgue dominated convergence theorem that

\[
\lim_{t \to \infty} d(t) = \frac{1}{\sqrt{2\pi}} \int_R 2u_- e^{-\frac{z^2}{2}} dz = u_- - u_+.
\]

Therefore Theorem 1.3 is completed. \( \square \)
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