Introduction. Quantum spin liquids (QSLs) [1, 2] are the embodiment of topological orders and offer the ideal platform for the systematic investigations of the fractional anyonic excitations and statistics therein [3, 4]. While the experimental progress of QSL and topological orders are difficult and often hampered by the complexity of materials and limitation of probing techniques, such as how to remove the impurity scattering of kagome antiferromagnets Herbertsmithite and Zn-doped Barlowite [5–11], theoretical progress are fast and promising both on topological orders and quantum phase transitions between them. However, most theoretical studies on topological phase transitions either stay at the algebraic level of anyon condensation [12–14], or are based on perturbed exactly-solvable yet unrealistic models such as the string-net models [15–21]. Realistic models and its fully quantum body solution of topological orders are rare, for obvious reasons: the lack of imagination in model construction and the lack of unbiased numerical methodology to handle these correlated systems.

Here, we hit the two birds with one stone. By designing a lattice model of coupled Kagome QSLs that involve only two-spin interactions and solving it with large scale quantum Monte Carlo simulations, amended with an analysis on anyon condensation transitions, we found that our model offer a three-stage anyon condensation process, of a vestigial type [22], from a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ topological order QSL to a $\mathbb{Z}_2$ topological order QSL and eventually to a trivial symmetric phase. These results provide concrete examples of phase transitions between topological orders in quantum magnets. The designed quantum spin liquid model and its numerical solution offer a playground for further investigations on vestigial anyon condensation. 

Model and Method. We design a lattice model that hosts a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ Kagome QSL phase and solve it with large scale QMC simulations. As shown in Fig. 1 (a), the model lives on a bilayer Kagome lattice with a 6-site unit cell and Hamiltonian reads

$$H = -J_x \sum_{\langle i,j \rangle} (S_i^x S_j^x + \text{H.c.}) + J_z \sum_{r \in \mathcal{O}} \left( \sum_{i \in r} S_i^z \right)^2 + J \sum_{\langle i,j \rangle} S_i \cdot S_j$$

(1)

where $J_x$ is the ferromagnetic transverse nearest neighbor interaction in the Kagome plane, $J_z$ is the antiferromagnetic longitudinal interactions between any two spins in the hexagon of the Kagome plane, and $J$ is the interlayer antiferromagnetic Heisenberg interaction. Throughout the paper, we set $J_z = 1$ as the energy unit. The first two terms constitute two layered Balents-Fisher-Girvin (BFG) model [23], which are coupled together by the last term. Since the BFG model in each layer only has $U(1)$ symmetry, even if the interlayer interaction is SU(2) symmetric, the global spin rotational symmetry of the system is still $U(1)$.

To investigate the ground state phase diagram of Eq. (1), we employ large-scale stochastic series expansion (SSE) QMC simulations [24, 25] with a plaquette update and generalized balance condition [26–29]. As for the spectral functions, we employ the stochastic analytic continuation (SAC) method [30–39] to obtain the real frequency spin excitation spectra from the QMC imaginary time correlation functions. Details of the implementation of the numerics are presented in the Supplemental Materials(SM) [40].

Measurements and Results. To obtain the groundstate phase diagram as shown in Fig. 1 (c), we first plot the kinetic energy density $E_k / N = \langle - \sum_{\langle i,j \rangle} (S_i^x S_j^x + \text{H.c.}) \rangle / N$, which is the expectation value of the transverse part of the Hamiltonian in Eq. (1), with the lattice size $N = 6 \times L^2$ and the linear size $L = 18$ and inverse temperature $\beta = 2L/J_x$. The results are shown in Fig. 2. We fix different initial values of $J_x$ and scan
FIG. 1. (a) Bilayer Kagome lattice model with the 6-site unit cell and lattice vectors \( \mathbf{r}_{1,2} \). The nearest-neighbor ferromagnetic in-plane transversal interaction \( J_\perp \) (black bonds), the hexagonal antiferromagnetic in-plane longitudinal interaction \( J_z \) (shaded hexagon) and the interlayer antiferromagnetic Heisenberg interaction \( J \) (blue bonds) are present. (b) Brillouin zone (BZ) of the bilayer kagome lattice, with the reciprocal space vectors \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \), the high-symmetry points \( \Gamma, M \) and \( K \). (c) The QMC phase diagram of the model spanned by the axes of \( J_\perp/J_z \) and \( J/J_z \). The two different quantum spin liquids (QSL-I and QSL-II), ferromagnetic (FM) and singlet phase are shown. The dashed lines are continuous phase transitions and the solid line is first order phase transition. The symbols stand for the places where the QMC parameter scans are performed. (d) Schematic process of the vestigial anyon condensation from QSL-I to QSL-II and eventually to the trivial singlet phase. \( 1, e, m \) and \( \psi \) are the four anyons of \( \mathbb{Z}_2 \) topological order and the red crosses stand for the confinement of the corresponding anyons at the phase transition.

FIG. 2. The density of the kinetic energy \( E_k/N \) as function of \( J \) for \( J_\perp = 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, 0.12 \), respectively, with system size \( L = 18 \) and the inverse of temperature \( \beta J_\perp = 2L \).

the value of \( J \) to monitor how the \( E_k/N \) behave. At \( J_\perp = 0.06 \) and 0.07, where the single layer Kagome model is inside a \( \mathbb{Z}_2 \) QSL phase as shown in our previous work [27], the kinetic energy density demonstrates a turning point around \( J = 0.01 \) and 0.022 in a continuous manner. As shown schematically in Fig. 1 (d), this is the anyon condensation transition between the \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) topological order QSL to the \( \mathbb{Z}_2 \) topological order QSL. Further increase \( J_\perp = 0.08, 0.09 \), an discontinuous jump appears in the \( J \) scans and this is the first order phase transition from FM phase to the QSL-II in the phase diagram of Fig. 1 (c). The data of spin stiffness across these transitions are shown in the SM [40]. It is interesting to observe that, for the curves of \( J_\perp = 0.06 \), 0.07, 0.08 and 0.09, there exist another turning point at larger values of \( J \), which signifies the second anyon condensation transition from QSL-II to a trivial product state of interlayer singlets. This condensation transition is again schematically shown in Fig. 1 (d). At larger value of \( J_\perp = 0.10, 0.11 \) and 0.12, there exist only one transition between the FM phase and the singlet phase.

Next, we focus on the two QSLs, and reveal the main discovery of this work – the vestigial anyon condensation. We first recall that the single-layer BFG model [23] realizes the \( \mathbb{Z}_2 \) topological order [27, 36, 41, 42], with four types of anyons: the trivial anyon 1, the bosonic spinon \( e \), the vison \( m \) and the fermionic spinon \( \psi \), which is obtained by fusing \( e \) and \( m \). The operators \( S^x \) and \( S^z \) in Hamiltonian Eq. (1) create/annihilate a pair of \( e \) and \( m \) anyons, respectively. Hence, their spectra reflect the two-particle continuum of the corresponding anyons, and the two operators can be used to construct Wilson-loop operators which we use below. Furthermore, \( e \) and \( m \) anyons both carry nontrivial symmetry fractionalization: \( e \) anyons carry half-integer spins and \( m \) anyons carry fractionalized crystalline momentum [43]. Consequently, condensing either type of anyons leads to spontaneous breaking of the corresponding symmetries: condensing \( e \) (\( m \)) anyons gives a continuous
transition to an FM phase (a valence-bond-solid phase) [36], respectively.

![Diagram](image)

**FIG. 3.** Two types of Wilson loop $W_{A(B)}$ as function of the side length of the loop $L_{\text{loop}}$ (proportional to the perimeter of the encircled region $M$) at different phase: QSL-I ($J_\kappa = 0.06, J = 0.004$), QSL-II ($J_\kappa = 0.06, J = 0.20$), Singlet ($J_\kappa = 0.06, J = 0.40$), FM ($J_\kappa = 0.09, J = 0.06$), respectively, with system size $L = 18$ and the inverse of temperature $\beta J_\kappa = 2L$.

In our bilayer model, the QSL-I phase is smoothly connected to the $J = 0$ limit where the two layers decouple. Hence, this topological order is a stacking of two $\mathbb{Z}_2$ topological orders. We abuse the notation of Deligne product $\boxtimes$ to represent such stacking and denote this topological order by $\mathbb{Z}_2 \boxtimes \mathbb{Z}_2$. Anon excitations in this topological order has the form of $a \boxtimes b$, where $a$ and $b$ are anyons of the two layers, respectively.

The second phase, the QSL-II, is a single $\mathbb{Z}_2$ topological order, which can be obtained from the QSL-I phase through condensing the anyon $m \boxtimes m$, as shown in Fig. 1 (d). This condensation has two consequences: First, it means that the visons on the two layers, $m \boxtimes 1$ and $1 \boxtimes m$, are identified as the same type of anyons, and become the vison ($m$ anyon) in the $\mathbb{Z}_2$ topological order. Second, the condensation of $m \boxtimes m$ confines the spinons on each layer, which are denoted by $e \boxtimes 1$ and $1 \boxtimes e$, because they have nontrivial mutual braiding statistics with $m \boxtimes m$. On the other hand, the bound state $e \boxtimes e$ is still deconfined after the condensation, and becomes the $e$ anyon in the $\mathbb{Z}_2$ topological order. We notice that unlike the spinons in the QSL-I phase, the $e$ anyon in the QSL-II phase does not carry a fractional spin because it is a bound state of spinons on each layer. As a result, a further condensation of $e$ anyons brings the QSL-II phase to a trivial paramagnetic phase without a topological order or spontaneous symmetry breaking, which is the singlet phase in the phase diagram.

![Graph](image)

**FIG. 4.** $S^z(q, \omega)$ spectra along the high symmetry path with $J_\kappa = 0.06$. (a) $J = 0.004$ inside the QSL-I, (b) $J = 0.2$ inside the QSL-II and (c) $J = 0.4$ inside the Singlet phases, with $\beta = 600$ and the system size $L = 18$. $S^z(q, \omega)$ spectra along the high symmetry path with the same parameter sets (d) inside the QSL-I phase, (e) inside the QSL-II phase and (f) inside the Singlet phase.

To reveal this theoretical understanding of the vestigial anyon condensation process, we designed the measurement of Wilson loops $W_A$ as shown in Fig. 3 (a). $W_A = \langle \prod_{i \in M} S_i^z \rangle$ measures the probability of moving a $m \boxtimes 1$ anyon along the loop $M$ in one of the Kagome plane. The results are shown in Fig. 3 (c), in a semi-log plot versus $L_{\text{loop}}$ (proportional to the perimeter), the perimeter-law decay of $W_A$ is present both in QSL-I and QSL-II, indicating the deconfinement of such anyons, and in the Singlet and FM phases, the $W_A$ decays in a area-law indicating the confinement of the anyons [44].

To distinguish the QSL-I and QSL-II, we further designed another Wilson loop $W_B$ shown in Fig. 3 (b). It is a vison loop that is half-way in the upper layer and half-way in the lower layer. As illustrated in Fig. 3(b), $W_B$ only exhibits perimeter-law decay in the QSL-II phase, indicating that the visons on the two layers belong to the same type. In other words, the vison in one layer is able to hop to another layer. In the QSL-I phase, on the other hand, $W_B$ decays with an area law, indicating that visons on the two layers belong to different types and they cannot hop from one layer to another. Therefore, $W_B$ does not form a closed Wilson loop in this phase. The area law decays are also present in the Singlet and FM phases.

With the establishment of the vestigial anyon condensation, we then discuss the phase transitions. As mentioned above, the continuous transition between QSL-I and QSL-II is driven by the condensation of $m \boxtimes m$. This transition reduces the topological order from $\mathbb{Z}_2 \boxtimes \mathbb{Z}_2$ to $\mathbb{Z}_2$ but does not break any global symmetry because $m \boxtimes m$ carries no symmetry fractionalization. Therefore, it belongs to the $(2 + 1)$d Ising* universality class, meaning that the critical exponent $\nu$ is the same as the 3D Ising transition, but there is no local order parameter. Similarly, the transition from QSL-II to the singlet phase is driven by the condensation of $e$ anyon, and also belongs to the Ising*
class. The continuous transition from the QSL-I phase to the FM phase is driven by the condensation of spinons (e 1 and 1 e), which eliminates the topological order completely and breaks the U(1) global symmetry, this is denoted as the 3D XY** transition and have been revealed in our previous studies [27, 28, 45]. The transition from the QSL-II phase to the FM phase, however, is first order, consistent with the fact that there is no anyon in the QSL-II phase that carries fractional U(1) charge. Last, the continuous transition between the singlet and the FM phase is of the Landau type and belongs to the 3D XY universality class [46, 47].

Finally, we illustrate the experimental signature of the vestigial anyon condensation via the dynamical spin spectra that can be probed from neutron scattering. As shown in Fig. 4, we compute the dynamical spin-spin correlation functions $S_{ap}(q, \tau) = \langle S^z_{q,a}(\tau)S^z_{q,p}\rangle$ and $S_{ap}(q, \tau) = \langle S^z_{q,a}(\tau)S^z_{q,p}\rangle$ where $q$ moves along the high-symmetry path in the BZ (Fig. 1 (b)) and $\alpha, \beta$ stand for the site index of the 6-site unit cell. From the SAC process and by taking the trace of the site indices, we obtain the spectra $S^z(q, \omega)$ and $S^z(q, \omega)$ with the former probing the spinon-pair and the later probing the vison-pair. The spectra in Fig. 4 (a) and (d) inside the QSL-I phase are consistent with those in previous works [36, 42] and the spinon and vison continua are clearly visible with the former acquiring larger gap and wider spread in the frequency and the later acquiring much smaller gap and signature of translational symmetry fractionalization (a finite momentum minima at $M$ point). Going into the QSL-II phase in Fig. 4 (b) and (e), due to the confinement of the spinon, i.e., condensation of $m$ $m$, the $S^z(q, \omega)$ loses its continuum and becomes a sharp trion band with a big gap, above which there are multi-trion bands. On the other hand, since the visons on the two layers are now identical, the $S^z(q, \omega)$ can still detect their continua as shown in Fig. 4 (e), with even smaller gap and spread in energy than those in Fig. 4 (d). Further increase $J$ to the singlet phase, both spectra are now sharp and present the typical $S = 1$ trion dispersion in an anisotropic singlet-product paramagnet. The entire Fig. 4 therefore demonstrates the dynamical signature of the vestigial anyon condensation.

**Discussion.** In this work, we construct a concrete coupled Kagome QSL model for anyon condensation, and solve it with unbiased QMC numerics. Our vestigial anyon condensation process from QSL-I to QSL-II and eventually to the trivial singlet phase, is in full consistency with the topological field theory analysis, and our dynamics spectra provide the experimental relevant signature for its detection. This work paves the way of investigating anyon condensation in more realistic quantum many-body models and eventually real materials.

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In this supplementary material, we give a brief introduction to the SSE-QMC method and present more details of our numerical calculations.

To overcome the strong frustration of the system, we take the plaquette decomposition with 5 sites and 10 legs in a vortex of the Hamiltonian of Eq. (1), which are shown as in Fig. S1. Here, in each bilayer-hexagon, we consider both $C_6$ and layer-inversion symmetries of the lattice and make sure the 5-site plaquette unit fully cover all sites of the lattice and all the interactions of the Hamiltonian. Then Eq. (1) is decomposed into diagonal operators:

$$H_{\text{diag}} = C - \frac{J_2}{z_2} \left( S_i^z S_j^- + S_j^z S_k^- + S_k^z S_l^- + S_l^z S_m^- \right) - \frac{J_2}{z_2} \left( S_j^z S_i^- + S_i^z S_k^- + S_k^z S_l^- + S_l^z S_m^- \right) - \frac{J_2}{z_3} (S_i^z S_j^z),$$

where $z_1 = 3$, $z_2 = 2$ and $z_3 = 4$ are prefactors to avoid over-counting of bonds, and off-diagonal operators:

$$H_{\text{off-diag}} = \frac{J_k}{z_1} \left( S_i^z S_k^- + S_k^z S_i^- + S_i^z S_j^- + S_j^z S_i^- + h.c. \right) + \frac{J_k}{z_3} \left( S_i^z S_j^- + h.c. \right).$$

After the decomposition, we implement the MC method with the general balance condition without detail balance [48] to obtain the solution of probability equations which strongly reduces the unpreferred bounce update [27, 28].

To benchmark the SSE-QMC code, we also implemented a worm-type continuous time QMC [49, 50] for the same model. As show in Fig. S2, we use these two method to calculate the kinetic energy density $E_k/N$ as function of $J$ at $J_k = 0.06$ with system size $L = 12$ and the inverse temperature $\beta J_k = 2L$, and obtained the identical results. The more important reason of implementing the worm-QMC is that in the worm representation, the transverse dynamical spin correlation function $S_\alpha^z \eta^z(q, \tau)$ is easier to measure as the head (or tail) of the worm is born with the “exact” imaginary time.

**FIG. S1.** Plaquette decomposition with 5 sites and 10 legs in a vortex.
FIG. S2. The kinetic energy density $E_k/N$ as function of $J$ at $J_\pm = 0.06$ with system size $L = 12$ and the inverse of temperature $\beta J_\pm = 2L$ calculated with both SSE-QMC and worm QMC, the results are identical.

whereas the longitudinal dynamical spin correlation $S_{zz}^{\sigma\rho}(q,\tau)$ is easier to be measured in SSE-QMC since it is the diagonal measurement. This is how the data of dynamical spin correlation functions in Fig. 4 of the main text are obtained.

As mentioned in the main text, in order to construct the phase diagram of the model as shown in Fig. 1 (c), we also compute other observables such as the spin stiffness $\rho_s = (W_1^2 + W_2^2)/(4\beta J_\pm)$ through winding number fluctuations $W_1^2, W_2^2$ [51], where $r_1, r_2$ are the two lattice directions. The results of the spin stiffness $\rho_s$ as function of $J$ for different $J_\pm$ with system size $L = 18$ are shown in Fig. S3. At $J_\pm = 0.06$, $\rho_s = 0$ for all the value of $J$, meaning that system is always inside the QSLs and the Singlet phase, without transverse long-range order. Only when $J_\pm = 0.08$ and 0.09, the spin stiffness is finite at small $J$ which means the system is inside the Ferromagnetic (FM) phase. However, when $J$ increases to the transition point, $\rho_s$ decreases sharply to zero, which reveals the first order transition from FM phase to QSL-II. We also consider the finite size effect of this transition, as shown in Fig. S4, and find that when $L \geq 12$, it is then large enough to eliminate finite size effects. Going back to Fig. S3, when $J_\pm \geq 0.10$, the $J$-scans again show continuous drop, this is the 3d O(3) transition from FM phase to the Singlet phase.

FIG. S4. The spin stiffness $\rho_s$ as function of $J$ for $J_\pm = 0.08$ with system size $L = 6, 12, 18$ and the inverse of temperature $\beta J_\pm = 2L$. 