METALLIC STRUCTURES ON DIFFERENTIABLE MANIFOLDS

MUSTAFA ÖZKAN AND FATMA YILMAZ

Abstract. In this paper, we study metallic structures, i.e. polynomial structures with the structure polynomial \( Q(J) = J^2 - aJ - bI \) on manifolds using the metallic ratio, which is a generalization of the Golden proportion. We investigate for integrability and parallelism conditions of metallic structures. Also, we give some properties of the metallic Riemannian metric and an example of the metallic structure.

1. Introduction

The Golden ratio is well known by everyone and this ratio possesses rich application areas such as architecture, painting, design, music, nature, optimization, and perceptual studies. However, in fact, it is probably fair to say that the golden ratio has inspired thinkers of all disciplines like no other number in the history of mathematics.

Crasmareanu and Hretcanu [2] researched the geometry of the golden structure on a manifold using a corresponding almost product structure. Özkan [15], Özkan, Çıtlak and Taylan [16] and Özkan and Yılmaz [17] studied the prolongations of golden structure. Also, many studies made on golden structures and golden Riemannian manifolds [7, 9, 22, 24].

In this paper, we use the metallic ratio, which is a generalization of the golden proportion. This generalization was introduced in 1999 by Vera W. de Spinadel [23] and named as the metallic means family or metallic proportions. Recently, many authors studied on metallic structures [8, 12].

We are inspired by [2] and follow its steps. By this way, we use metallic structure on a differentiable manifold. We are interested in finding the properties of the metallic structure.

The outline of this paper is as follows. In section 2, we give basic definitions about metallic means and metallic means family. In the next section, some relations between the metallic structure and the almost product structure, almost tangent structure, almost complex structure are given. Moreover, we show some properties of the metallic structure. In section 4, we give some examples of the metallic structure. After that, we study a few connections on the metallic structure. In section 6, we search for the integrability and parallelism of the metallic structure with respect to the Schouten and Vranceanu connections. Section 7 deals with the metallic Riemannian manifold and its properties. At the end of this section, we give an example of the metallic structure on manifold \( \mathbb{R}^2 \).

2000 Mathematics Subject Classification. 53C15, 57N16, 53C25.

Key words and phrases. Metallic ratio, metallic structure, polynomial structure, almost product structure, Riemannian manifold.
2. Preliminaries

In this section, we give a brief information about the metallic means family using the Generalized Secondary Fibonacci sequence.

Fibonacci sequence provides the following relation:
\[ F(k + 1) = F(k) + F(k - 1). \]
This relation can be generalized using
\[ G(k + 1) = aG(k) + bG(k - 1), \quad k \geq 1 \]
where \( a, b, G(0) = c, \) and \( G(1) = d \) are real numbers.
\[ c, d, ad + bc, a(ad + bc) + bd, ... \]
are the members of the Generalized Secondary Fibonacci Sequence.

Let \( a \) and \( b \) be positive integers. The positive root of the equation
\[ x^2 - ax - b = 0 \]
is called a member of the metallic means family \([23] \). This root is denoted by
\[ \rho_{a,b} = \frac{a + \sqrt{a^2 + 4b}}{2} \]
and it is known as metallic ratio.

\( \rho_{a,b} \) for different values of \( a \) and \( b \), we obtain the following ratios:

- If \( a = b = 1 \), \( \rho_{1,1} = \frac{1 + \sqrt{5}}{2} \), we get the Golden mean, which is the ratio of two consecutive classical Fibonacci numbers \([12] \).
- If \( a = 2 \) and \( b = 1 \), \( \rho_{2,1} = 1 + \sqrt{2} \), we get the Silver mean, which is the ratio of two consecutive Pell numbers \([6, 12, 18, 19] \).
- If \( a = 3 \) and \( b = 1 \), \( \rho_{3,1} = \frac{3 + \sqrt{13}}{2} \), we get the Bronze mean \([12, 27] \).
- If \( a = 4 \) and \( b = 1 \), \( \rho_{4,1} = 2 + \sqrt{5} \), we get the Subtle mean \([5, 12] \).
- If \( a = 1 \) and \( b = 2 \), \( \rho_{1,2} = 2 \), we get the Copper mean \([12] \).
- If \( a = 1 \) and \( b = 3 \), \( \rho_{1,3} = \frac{1 + \sqrt{13}}{2} \), we get the Nickel mean \([12] \).

Throughout this paper, all manifolds are assumed to be of class \( C^{\infty} \) unless otherwise stated.

3. Metallic Structures on Manifolds

Before making the description of the metallic structures on manifolds, we give some definitions in order to state the main result of this paper.

**Definition 1** \([12] \). Let \( M_n \) be an \( n \)-dimensional manifold. A polynomial structure on a manifold \( M_n \) is called a metallic structure if it is determined by an \( (1, 1) \)-tensor field \( J \), which satisfies the equation
\[ J^2 = aJ + bI \]
where \( a, b \) are positive integers and \( I \) is the identity operator on the Lie algebra \( \chi(M_n) \) of the vector fields on \( M_n \).

The following proposition gives the main properties of a metallic structure.
Proposition 1 ([12]). (i) The eigenvalues of \( J \) are the metallic ratio \( \rho_{a,b} \) and \( a - \rho_{a,b} \).

(ii) A metallic structure \( J \) is an isomorphism on the tangent space of the manifold \( T_x (M_n) \) for every \( x \in M_n \).

(iii) \( J \) is invertible and its inverse \( J^{-1} = \hat{J} \) satisfies
\[
b J^2 + a\hat{J} - I = 0.
\]

An almost tangent structure \( T \), an almost product structure \( F \), and an almost complex structure \( C \) appear in pairs. In other words, \( -T \), \( -F \) and \( -C \) are also an almost tangent structure, an almost product structure, and an almost complex structure, respectively [2].

Proceeding from this point, it can be said that metallic structures appear in pairs:

Proposition 2. If \( J \) is a metallic structure, then \( \tilde{J} = aI - J \) is also a metallic structure.

It is known from [10] that a polynomial structure on a manifold \( M_n \) induces a generalized almost product structure \( F \). Therefore, we make a connection between the metallic structure and the almost product structure on \( M_n \).

Proposition 3 ([12]). If \( F \) is an almost product structure on \( M_n \), then
\[
J = \frac{a}{2} I + \left( \frac{2\rho_{a,b} - a}{2} \right) F \tag{3.2}
\]
is a metallic structure on \( M_n \).

Contrarily, if \( J \) is a metallic structure on \( M_n \), then
\[
F = \left( \frac{2}{2\rho_{a,b} - a} J - \frac{a}{2\rho_{a,b} - a} I \right) \tag{3.3}
\]
is an almost product structure on \( M_n \).

By the use of the relationship between the almost product structure and the metallic structure in Proposition [3] we can give the following definitions:

Definition 2. Let \( T \) be an almost tangent structure on \( M_n \). Then
\[
J_t = \frac{a}{2} I + \left( \frac{2\rho_{a,b} - a}{2} \right) T
\]
is called a tangent metallic structure on \( (M_n, T) \).

This structure verifies the following equation:
\[
J_t^2 - aJ_t + \frac{a^2}{4} I = 0.
\]

Considering this equation in the real field \( \mathbb{R} \), i.e. \( x^2 - ax + \frac{a^2}{4} = 0 \), we have the tangent real metallic ratio \( \rho_t = \frac{a}{2} \).

Definition 3. Let \( C \) be an almost complex structure on \( M_n \). Then
\[
J_c = \frac{a}{2} I + \left( \frac{2\rho_{a,b} - a}{2} \right) C
\]
is called a complex metallic structure on \((M_n, C)\).

The polynomial equation satisfied by \(J_c\) is

\[J_c^2 - aJ_c + \frac{a^2 + 2b}{2} = 0.\]

For the case of \(M_n = \mathbb{R}\), we get the equation

\[x^2 - ax + \frac{a^2 + 2b}{2} = 0\]

with solutions,

\[x_1 = \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} i, \quad x_2 = \frac{a}{2} - \frac{\sqrt{a^2 + 4b}}{2} i.\]

The positive root of equation

\[\rho_c = \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} i\]

is called a complex metallic ratio.

4. Examples of The Metallic Structures

In this section, we give some examples of the metallic structure.

Example 1 (Clifford algebras). Let \(C'(n)\) be the real Clifford algebra of the positive definite form \(\sum_{i=1}^{n} (x^i)^2\) of \(\mathbb{R}^n\) \cite{14}. According to the Clifford product, the standard base of \(\mathbb{R}^n\) satisfies the following relations:

\[e_i^2 = 1, \quad e_i e_j + e_j e_i = 0, \quad i \neq j.\]

Therefore, using \(J_i = \frac{1}{2} (a + \sqrt{a^2 + 4be_i})\) and \(4.1\), we derive a new representation of the Clifford algebra:

\[\begin{align*}
J_i, & \quad \text{metallic structure} \\
J_i J_j + J_j J_i & = a (J_i + J_j) - \frac{a^2}{2}, \quad i \neq j.
\end{align*}\]

In \cite{14}, \(e_1\) and \(e_2\), orthonormal basis vectors of \(\mathbb{R}^2\), are determined by

\[1 = I_2, \quad e_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\]

and hence we get

\[\begin{align*}
(i) \quad J_1 & = \frac{1}{2} (a + \sqrt{a^2 + 4be_1}) = \begin{pmatrix} \frac{a + \sqrt{a^2 + 4b}}{2} & 0 \\ 0 & \frac{a - \sqrt{a^2 + 4b}}{2} \end{pmatrix} \\
(ii) \quad J_2 & = \frac{1}{2} (a + \sqrt{a^2 + 4be_2}) = \frac{1}{2} \begin{pmatrix} a & \sqrt{a^2 + 4b} \\ \sqrt{a^2 + 4b} & a \end{pmatrix}.
\end{align*}\]

Example 2 (2D Metallic matrices). For \(J \in \mathbb{R}^n\), if \(J\) provides the equation

\[J^2 = aJ + bI_n\]

then this matrix is called a metallic matrix where \(I_n\) is the identity matrix on \(\mathbb{R}^n\).
In $\mathbb{R}^2$, we obtain a two-parametric family of the metallic structures by solving (4.3).

i) For $r, t \in \mathbb{R}$ and $s \in \mathbb{R} - \{0\}$,

\[
J_{r,s} = \left( \begin{array}{cc}
 r & \frac{1}{s} (r^2 - ar - b) \\
 a - r & s \\
 a - t & \frac{1}{t} (t^2 - at - b) \\
s & 1 \\
 \end{array} \right)
\]

or

\[
J_{s,t} = \left( \begin{array}{cc}
 a & 0 \\
 b & a \\
 -b & b \\
 0 & 1 \\
 \end{array} \right).
\]

ii) For $r = \rho_{a,b}$, $s \in \mathbb{R}$,

\[
J_{\rho_{a,b},s} = \left( \begin{array}{cc}
 \rho_{a,b} & 0 \\
 s & a - \rho_{a,b} \\
 0 & a - \rho_{a,b} \\
 \end{array} \right) \quad \text{or} \quad J_{a-\rho_{a,b},s} = \left( \begin{array}{cc}
 a - \rho_{a,b} & 0 \\
 s & \rho_{a,b} \\
 0 & \rho_{a,b} \\
 \end{array} \right).
\]

iii) For $r = \rho_{a,b}$, $s = 0$,

\[
J_{\rho_{a,b},0} = \left( \begin{array}{cc}
 \rho_{a,b} & 0 \\
 0 & a - \rho_{a,b} \\
 \end{array} \right) \quad \text{or} \quad J_{a-\rho_{a,b},0} = \left( \begin{array}{cc}
 a - \rho_{a,b} & 0 \\
 0 & \rho_{a,b} \\
 \end{array} \right).
\]

Then, from (4.2) and (4.4), we get

\[
J_1 = \lim_{s \to 0} J_{\rho_{a,b},s} \quad \text{and} \quad J_2 = J_{\sqrt{a^2 + 4}, 0}.
\]

Example 3 (Metallic reflection). In an Euclidean space $E$, the reflection with respect to a hyperplane $H$ with the normal $v \in E \setminus \{0\}$ has the formula

\[
r_v(x) = x - \frac{2 \langle x, v \rangle}{\langle v, v \rangle} v, \quad x \in E.
\]

For $r_0^2 = I_E$, $I_E$ is the identity on $E$. We can define a metallic reflection with respect to $v$ as

\[
J_v = \frac{1}{2} \left( aI_E + \sqrt{a^2 + 4b} v \right)
\]

where $v$ is an eigenvector of $J_v$ with the corresponding eigenvalue $a - \rho_{a,b}$. In [20, p.314], under the condition that $X$ is an orthogonal group of $E$, we can give the following:

\[
J_v(x) = \rho_{a,b} x - \frac{\sqrt{a^2 + 4b} \langle x, v \rangle}{\langle v, v \rangle} v.
\]

Example 4 (Triple structure in terms of metallic structures). Let $F$ and $T$ denote two tensor fields of type $(1, 1)$ on $M_n$. Using the triple $(F, T, K = T \circ F)$, we obtain the following structures:

1) $F^2 = T^2 = I$ and $T \circ F - F \circ T = 0$; then $J^2 = I$,
2) $F^2 = T^2 = I$ and $T \circ F + F \circ T = 0$; then $J^2 = -I$,
3) $F^2 = T^2 = -I$ and $T \circ F - F \circ T = 0$; then $J^2 = I$,
4) $F^2 = T^2 = -I$ and $T \circ F + F \circ T = 0$; then $J^2 = -I$.

These structures are called almost hyperproduct (ahp), almost biproduct complex (abpc), almost product bicomplex (apbc), and almost hypercomplex (ahc), respectively.
Taking into account (3.2), we get
\[ J_F = \frac{a}{2} I + \left( \frac{2\rho_{a,b} - a}{2} \right) F, \quad J_T = \frac{a}{2} I + \left( \frac{2\rho_{a,b} - a}{2} \right) T, \quad J_K = \frac{a}{2} I + \left( \frac{2\rho_{a,b} - a}{2} \right) K. \]

Then, we find a relation between \( J_F, J_T, J_K \) as,
\[ \sqrt{a^2 + 4b} J_K = 2J_T J_F - a J_T - a J_F + \rho_{a,b} I - b I. \]

Hence, the triple \((J_F, J_T, J_K)\) is,

1') An (ahp)-structure if and only if \( J_F, J_T \) are metallic structures and \( J_F J_T = J_T J_F \), then \( J_K \) is a metallic structure.

2') An (abpc)-structure if and only if \( J_F, J_T \) are metallic structures and \( J_T J_F + J_F J_T = a (J_T + J_F) - \frac{1}{2} a^2 I \), then \( J_K \) is a complex metallic structure.

3') An (apbc)-structure if and only if \( J_F, J_T \) are complex metallic structures and \( J_T J_F = J_T J_F \), then \( J_K \) is a metallic structure.

4') An (ahc)-structure if and only if \( J_F, J_T \) are complex metallic structures and \( J_T J_F + J_F J_T = a (J_T + J_F) - \frac{1}{2} a^2 I \) then, \( J_K \) is a complex metallic structure.

**Example 5 (Quaternion algebras).** Let \( \tilde{H} \) be a quaternion algebra. The base of \( \tilde{H} \) is \( \{1, e_1, e_2, e_3\} \), which verifies
\[ e_1^2 = e_2^2 = e_3^2 = -1 \]
\[ e_1 e_2 = -e_2 e_1 = e_3, \quad e_3 e_1 = -e_1 e_3 = e_2, \quad e_2 e_3 = -e_3 e_2 = e_1. \]

is a quaternion, which can be divided into two parts, such as \( S_q \) is a scalar part and \( V_q \) is a vector part. In this way, \( q = S_q + V_q \) where \( S_q = a_0 \) and \( V_q = a_1 e_1 + a_2 e_2 + a_3 e_3 \).

The norm of a quaternion is determined by \( N_q = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2} \). Also, \( q_0 = \frac{q}{N_q} \) is a unit quaternion \((q \neq 0)\). A unit form can be written by \( q_0 = \cos \alpha + \tilde{S}_0 \sin \alpha \) where \( \tilde{S}_0 \) is a unit vector verifying the equality \( \tilde{S}_0^2 = -1 \).

Hence, by the inspire of [25] we can define the metallic biquaternion structure as
\[ J_q = \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} \tilde{S}_0 \] where \( \tilde{S}_0^2 = -1 \) and \( i^2 = -1 \).

Similarly, we can define the metallic split quaternion structure as
\[ J_q = \frac{a}{2} + \tilde{S}_0 \frac{\sqrt{a^2 + 4b}}{2} \] where \( \tilde{S}_0^2 = 1 \).

5. **Connection As Metallic Structure**

5.1. **Connections in principal fibre bundles.** Let \( P(\pi, M_n, G) \) be a principal fibre bundle where \( \pi : P \to M_n \) is a fibre projection and \( G \) is a Lie group. \( V = \ker \pi_\ast \) is the vertical distribution on \( P \), \( H \) is the complementary distribution of \( V \), i.e. \( TP = V \oplus H \), and \( H \) is \( G \)-invariant. The corresponding projectors of \( V \) and \( H \) are \( v \) and \( h \), respectively. So, the tensor field of type \((1,1)\)
\[ F = v - h \]
is an almost product structure on \( P \). From [21], \( F \) defines a connection if and only if the followings are satisfied:
1) $F(X) = X \iff X \in V$

2) $dR_e \circ F_u = F_{ue} \circ dR_e$ for all $e \in G$ and $u \in P$.

By use of the relation between the almost product structure and the metallic structure, we get the following assertion:

**Proposition 4.** The metallic structure $J$ on $P$ indicates a connection if and only if the following properties are satisfied:

1') $X \in \chi(P)$, where $\chi(P)$ is the Lie algebra of vector fields on $P$, is an eigenvector of $J$ with respect to the eigenvalue $\rho_{a,b}$ if and only if $X$ is a vertical vector field.

2') $dR_e \circ J_u = J_{ue} \circ dR_e$ for all $e \in G$ and $u \in P$.

Let $\omega \in \Omega^1(P, g)$ be the connection 1-form of $H$ and suppose that $\Omega \in \Omega^2(P, g)$ is the curvature form of $\omega$ where $g$ is the Lie algebra of $G$. Then, we get [2],

\[
\Omega(X, Y) = -\frac{1}{4} \omega(N_F(X, Y))
\]

where $N_F$ is the Nijenhuis tensor of $F$.

**Proposition 5.** Using (5.2), we obtain

\[
N_F = \frac{4}{(2\rho_{a,b} - a)^2} N_J
\]

and then we get

\[
\Omega(X, Y) = -\frac{1}{(2\rho_{a,b} - a)^2} \omega(N_J(X, Y)).
\]

The connection is flat if its curvature form $\Omega \equiv 0$.

**Proposition 6.** The connection is flat if and only if the associated metallic structure is integrable, i.e. $N_J \equiv 0$.

This connection yields a lift $I_\omega : \chi(M_n) \to \chi(P)$ verifying

\[
[I_\omega X^*, I_\omega Y^*] - I_\omega [X^*, Y^*] = N_F(I_\omega X^*, I_\omega Y^*)
\]

for every $X^*, Y^* \in \chi(M_n)$ [2].

Hence, using (5.1) and (5.3):

**Proposition 7.** The lift defined by $I_\omega$ is a morphism if and only if the associated metallic structure is integrable.

5.2. **Connection in tangent bundles.** Let $\pi : TM_n \to M_n$ be the tangent bundle of $M_n$. $V(M_n) = \ker \pi_*(\pi_* : TTM_n \to TM_n)$ is called vertical distribution of $M_n$. $(x^i)_{1 \leq i \leq n}$ is a local coordinate system on $M_n$ and $(x, y) = (x^i, y^i)_{1 \leq i \leq n}$ is a local coordinate system on $TM_n$. For an atlas on $TM_n$ with these local coordinates, the tangent structure of $TM_n$ is $T = \frac{\partial}{\partial y^i} \otimes dx^i$, i.e.

\[
T \left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad T \left(\frac{\partial}{\partial y^i}\right) = 0.
\]

$v : \chi(M_n) \to \chi(M_n)$ is a $(1, 1)$ tensor field verifying

\[
\begin{cases}
T \circ v = 0 \\
v \circ T = T
\end{cases}
\]

which is called the vertical projector.
Definition 4 \([2]\). A complementary distribution \(N\) to the vertical distribution \(V(M_n)\)
\[(5.4) \chi(M_n) = N \oplus V(M_n)\]
is called normalization, horizontal distribution or non-linear connection.

Having in mind that a vertical projector \(v\) is \(C^\infty(M_n)\)–linear with \(imv = V(M_n)\), we get:

Proposition 8 \([2]\). With the help of \(\ker v = N(v)\), a vertical vector \(v\) yields a non-linear connection. Viceversa, if \(N\) is a non-linear connection, then \(h_N\) and \(v_N\) are horizontal and vertical projectors, respectively, with respect to the decomposition \[(5.4).\]

Then we give the following proposition:

Proposition 9 \([2]\). If \(N\) is a non-linear connection then \(v_N\) is a vertical projector with \(N(v_N) = N\).

Definition 5 \([2]\). If the following relations hold, a \((1, 1)\) tensor field \(\Gamma\) is called non-linear connection of an almost product type:
\[
\begin{cases}
\Gamma \circ T = -T \\
T \circ \Gamma = T
\end{cases}
\]

Let \(\Gamma\) be a nonlinear connection of an almost product type, the following assertions are true:

i) \(v \Gamma = \frac{1}{2} (I_{\chi(M_n)} - \Gamma)\) is a vertical vector,

ii) While \(N(v \Gamma)\) is the \((+1)\)–eigenspace of \(\Gamma\), \(V(M_n)\) is the \((-1)\)–eigenspace of \(\Gamma\).

As a result, every vertical projector \(v\) yields a non-linear connection of an almost product type as \(\Gamma = I_{\chi(M_n)} - 2v\). Using the last relation, we have \(\Gamma^2 = I_{\chi(M_n)}\) (\(\Gamma\) tensor field is an almost product structure on \(M_n\)).

Hence, we associate this result with the metallic structure.

Proposition 10. A non-linear connection \(N\) on \(M_n\), given by the vertical vector \(v\), can also be defined by a metallic structure \(J (= J_\Gamma)\)
\[J = \rho_{a,b} I_{\chi(M_n)} - \sqrt{a^2 + 4bv}\]
with \(N\) the \(\rho_{a,b}\)–eigenspace and \(V(M_n)\) the \((a - \rho_{a,b})\)–eigenspace.

6. Integrability and Parallelism of Metallic Structures

In this section, we give integrability and parallelism conditions of the metallic structure. Recall the Nijenhuis tensor of \(J\),
\[N_J(X, Y) = J^2 [X, Y] + [JX, JY] - J [JX, Y] - J [X, JY]\]
for every \(X, Y \in \chi(M_n)\).

Let \(L\) and \(M\) be two complementary distributions on \(M_n\) corresponding to \(\rho_{a,b}\) and \(a - \rho_{a,b}\), respectively. The corresponding projection operators of \(L\) and \(M\) are denoted by \(l\) and \(m\), which results in
\[
\begin{cases}
l^2 = l, & m^2 = m \\
lm = ml = 0, & l + m = I
\end{cases}
\]
According to the above conditions and based on a straightforward computation from (3.2), we obtain the following equations:

\begin{align}
(6.2) \quad l &= \frac{1}{2\rho_{a,b} - a} J - \frac{a - \rho_{a,b}}{2\rho_{a,b} - a} I \\
\quad m &= -\frac{1}{2\rho_{a,b} - a} J + \frac{\rho_{a,b}}{2\rho_{a,b} - a} I.
\end{align}

**Theorem 1.** i) If \(N_J = 0\), \(J\) is integrable.

ii) If \(m [lX, lY] = 0\), the distribution \(L\) is integrable. Similarly, if \(l [mX, mY] = 0\), the distribution \(M\) is integrable for all vector fields \(X, Y\) on \(\chi (M_n)\).

**Remark 1.** From (3.1) and (6.2),

\begin{align}
(6.3) \quad Jl &= lJ = \frac{\rho_{a,b}}{2\rho_{a,b} - a} J + \frac{b}{2\rho_{a,b} - a} I = \rho_{a,b} l \\
\quad Jm &= mJ = -\frac{a - \rho_{a,b}}{2\rho_{a,b} - a} J - \frac{b}{2\rho_{a,b} - a} I = (a - \rho_{a,b}) m.
\end{align}

Using (6.3), we can easily see that

\begin{align}
l [mX, mY] &= \frac{1}{(2\rho_{a,b} - a)^2} lN_J (mX, mY), \\
m [lX, lY] &= \frac{1}{(2\rho_{a,b} - a)^2} mN_J (lX, lY).
\end{align}

**Proposition 11.** Using (5.2), a metallic structure \(J\) is integrable if and only if the associated almost product structure \(F\) is integrable.

**Proposition 12.** Let \(X, Y \in \chi (M_n)\). \(L\) is integrable if and only if \(mN_J (lX, lY) = 0\) and \(M\) is integrable if and only if \(lN_J (mX, mY) = 0\). If \(J\) is integrable, then the distributions \(L\) and \(M\) are integrable.

Now, we give the parallelism conditions of the metallic structures.

Let \(\nabla\) be a linear connection on \(M_n\). We associate the pair \((J, \nabla)\) with other linear connections, which are Schouten and Vrânceanu connections [1, 3].

i) The Schouten connection,

\begin{align}
(6.4) \quad \tilde{\nabla}_X Y &= l (\nabla_X lY) + m (\nabla_X mY).
\end{align}

ii) The Vrânceanu connection,

\begin{align}
(6.5) \quad \hat{\nabla}_X Y &= l (\nabla_{lX} lY) + m (\nabla_{mX} mY) + l [mX, lY] + m [lX, mY].
\end{align}

**Proposition 13.** The projectors \(l, m\) are parallels in terms of Schouten and Vrânceanu connections for every linear connection \(\nabla\) on \(M_n\). Moreover, \(J\) is parallel in terms of Schouten and Vrânceanu connections.

**Proof.** For every vector fields \(X, Y\) on \(\chi (M_n)\) and with the help of (6.1),

\begin{align}
(\tilde{\nabla}_X l) Y &= \tilde{\nabla}_X lY - l (\tilde{\nabla}_X Y) = l (\nabla_X lY) - l (\nabla_X lY) = 0,
\end{align}
\[
(\hat{\nabla}_XY) = l(\nabla_XY) + l[mX, lY] - l(\nabla_{lX}lY) - l[mX, lY] = 0.
\]
Therefore, \( l \) is parallel relevant to \( \nabla \) and \( \hat{\nabla} \).

Similarly, the projector \( m \) is parallel with respect to the Schouten and Vrâncanu connections. From (6.3), \( J \) is parallel in terms of Schouten and Vrâncanu connections.

**□**

**Definition 6 ([4]).** If \( \nabla_XY \in L \) where \( X \in \chi(M_n) \) and \( Y \in L \), the distribution \( L \) is called parallel with respect to linear connection \( \nabla \).

**Definition 7 ([4]).** If \( (\Delta J)(X, Y) \in L \) where \[ (6.6) \quad (\Delta J)(X, Y) = J\nabla_XY - J\nabla_YX - \nabla_J(XY) \] for \( X \in L, Y \in \chi(M_n) \), the distribution \( L \) is called \( \nabla - \text{half parallel} \).

**Definition 8 ([4]).** If \( (\Delta J)(X, Y) \in M \) where \( X \in L, Y \in \chi(M_n) \), the distribution \( L \) is called \( \nabla - \text{anti half parallel} \).

**Proposition 14.** The distributions \( L \) and \( M \) are parallel in terms of Schouten and Vrâncanu connections for the linear connection \( \nabla \) on \( M_n \).

**Proof.** Let \( X \in \chi(M_n) \) and \( Y \in L \). \( mY = 0 \) and \( lY = Y \), using (6.4) and (6.5) we obtain
\[
\hat{\nabla}_XY = l(\nabla_XY) \in L,
\]
\[
\hat{\nabla}_XY = l(\nabla_{lX}lY) + l[mX, lY] \in L.
\]
Thus, \( L \) is parallel with respect to \( \nabla \) and \( \hat{\nabla} \).

In the same manner, we can see that \( M \) satisfies the similar relations. \( \square \)

**Proposition 15.** The distributions \( L \) and \( M \) are parallel with respect to \( \nabla \) linear connection if and only if \( \nabla \) and \( \hat{\nabla} \) are equal.

**Proof.** If \( L, M \) are \( \nabla - \text{parallel} \) then \( \nabla_X(lY) \in L \) and \( \nabla_X(mY) \in M \) where \( X, Y \in \chi(M_n) \). For that reason
\[
\nabla_X(lY) = l(\nabla_XY) \quad \text{and} \quad \nabla_X(mY) = m(\nabla_XY).
\]
Since, \( l + m = I \) and from (6.4),
\[
\nabla_XY = l(\nabla_XY) + m(\nabla_XY) = \hat{\nabla}_XY.
\]
Therefore \( \nabla = \hat{\nabla} \).

The converse can be shown easily. \( \square \)

**Proposition 16.** If \( \{lX, mY\} \in L \) where \( X \in L, Y \in \chi(M_n) \), the distribution \( L \) is half parallel with respect to the Vrâncanu connection.

**Proof.** In the view of the equation (6.6) for \( \hat{\nabla} \), we get
\[
m(\Delta J)(X, Y) = mJ\hat{\nabla}_X Y - mJ\hat{\nabla}_YX - m\hat{\nabla}_JX Y + m\hat{\nabla}_Y(JX)
\]
where \( X \in L, Y \in \chi(M_n) \).

As a result, by (6.3) and (6.4), we have
\[
m(\Delta J)(X, Y) = (a - 2\rho_{a,b}) m[X, mY]
\]
which proves the proposition. \( \square \)
Similarly, we have the following proposition for distribution $M$:

**Proposition 17.** If $[mX, lY] \in M$ where $X \in M$, $Y \in \chi (M_n)$, the distribution $M$ is half parallel with respect to the Vranceanu connection.

**Proposition 18.** The distributions $L$ and $M$ are anti half parallel with respect to Vranceanu connection.

**Proof.** Taking account of the equation (6.6) for $\hat{\nabla}$, we get

$$l(\Delta J)(X, Y) = lJ \hat{\nabla}_X Y - lJ \hat{\nabla}_Y X - l\hat{\nabla}_{JX} Y + l\hat{\nabla}_Y (JX)$$

where $X \in L$, $Y \in \chi (M_n)$.

Using (6.3) and (6.5), we get

$$l(\Delta J)(X, Y) = (2\rho_{a,b} - a) l[mX, lY].$$

Because $mX = 0$, we get $l(\Delta J)(X, Y) = 0$. Thus, $(\Delta J)(X, Y) \in M$.

In the same way, we can show that $M$ is anti-half parallel with respect to the Vranceanu connection. \qed

7. Metallic Riemannian Manifolds

In this section, we study a metallic Riemannian manifold with respect to the Riemannian metric and we give an example of the metallic structure on manifold $\mathbb{R}^2$.

**Definition 9** ([11], [21], [26]). Let $F$ be an almost product structure on $M_n$ and $g$ be a Riemannian metric (or a semi-Riemannian metric) given by

$$g(F(X), F(Y)) = g(X, Y), \quad \forall X, Y \in \chi (M_n)$$

or equivalently, $F$ be a $g$-symmetric endomorphism, such as

$$g(F(X), F(Y)) = g(X, Y).$$

So, the pair $(g, F)$ is called a Riemannian almost product structure (or a semi-Riemannian almost product structure).

By the help of (5.3) and (5.5), the following proposition can be given:

**Proposition 19.** The almost product structure $F$ is a $g$-symmetric endomorphism if and only if the metallic structure $J$ is also a $g$-symmetric endomorphism.

**Definition 10** ([12]). Let $g$ be a Riemannian metric (or a semi-Riemannian metric) on $M_n$, such as

$$g(J(X), Y) = g(X, J(Y)), \quad \forall X, Y \in \chi (M_n).$$

Then, $(g, J)$ is called a metallic Riemannian structure (or a metallic semi-Riemannian structure). Also, $(M_n, g, J)$ is named a metallic Riemannian manifold (or a metallic semi-Riemannian manifold).

**Corollary 1.** On a metallic Riemannian manifold,

a) The projectors $l$, $m$ are $g$-symmetric, i.e.

$$g(l(X), Y) = g(X, l(Y))$$

$$g(m(X), Y) = g(X, m(Y)).$$

b) The distributions $L$, $M$ are $g$-orthogonal, i.e.

$$g(l(X), m(Y)) = 0.$$
The metallic structure is \( N_J \)-symmetric, i.e.
\[
N_J(J(X), Y) = N_J(X, J(Y)).
\]

A Riemannian almost product structure is a locally product structure if \( F \) is parallel with respect to the Levi-Civita connection \( \nabla \) of \( g \), i.e. \( \nabla F = 0 \) and if \( \nabla \) is a symmetric linear connection, then the Nijenhuis tensor of \( F \) satisfies
\[
N_F(X, Y) = \left( \nabla_X F X \right) Y - \left( \nabla_Y F Y \right) X - \left( \nabla_Y F X \right) Y + F \left( \nabla_Y F X \right).
\]

If \((M_n, g, J)\) is a locally product metallic Riemannian manifold, then the metallic structure \( J \) is integrable.

**Theorem 2.** The set of linear connections \( \nabla \) for which \( \nabla J = 0 \) is
\[
\nabla_X Y = \frac{1}{a^2 + 4b} \left[ \left( a^2 + 2b \right) \nabla_X Y + 2J \left( \nabla_X Y \right) - aJ \left( \nabla_X Y \right) - a \left( \nabla_X JY \right) \right] + O_F Q(X, Y)
\]
where \( \tilde{\nabla} \) is an arbitrary fixed linear connection and \( Q \) is a \((1, 2)\)-tensor field for which \( O_F Q \) is an associated Obata operator
\[
O_F Q(X, Y) = \frac{1}{2} \left[ Q(X, Y) + F Q(X, FY) \right]
\]
for the corresponding almost product structure \((3.2)\).

We conclude with the following example for the metallic structure.

**Example 6.**
\[
\begin{align*}
l &= \frac{x^2}{x^2 + y^2} \frac{\partial}{\partial x} \otimes dx + \frac{xy}{x^2 + y^2} \frac{\partial}{\partial x} \otimes dy + \frac{xy}{x^2 + y^2} \frac{\partial}{\partial y} \otimes dx + \frac{y^2}{x^2 + y^2} \frac{\partial}{\partial y} \otimes dy \\
m &= \frac{y^2}{x^2 + y^2} \frac{\partial}{\partial x} \otimes dx - \frac{xy}{x^2 + y^2} \frac{\partial}{\partial x} \otimes dy - \frac{xy}{x^2 + y^2} \frac{\partial}{\partial y} \otimes dx + \frac{x^2}{x^2 + y^2} \frac{\partial}{\partial y} \otimes dy
\end{align*}
\]
are projection operators in \( \mathbb{R}^2 \) which satisfy the conditions \((6.1)\).

\[
L = Sp \left\{ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right\} \quad \text{and} \quad M = Sp \left\{ y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right\}
\]
are complementary distributions corresponding to the projection operators \( l \) and \( m \), respectively. The distributions \( L, M \) are orthogonal with respect to the Euclidean metric of \( \mathbb{R}^2 \). Furthermore, these distributions are connected to the metallic structure
\[
\begin{align*}
J \left( \frac{\partial}{\partial x} \right) &= \frac{\rho_{a,b} x^2 + (a - \rho_{a,b}) y^2}{x^2 + y^2} \frac{\partial}{\partial x} + \frac{(2 \rho_{a,b} - a) xy}{x^2 + y^2} \frac{\partial}{\partial y} \\
J \left( \frac{\partial}{\partial y} \right) &= \frac{(2 \rho_{a,b} - a) xy}{x^2 + y^2} \frac{\partial}{\partial x} + \frac{(a - \rho_{a,b}) x^2 + \rho_{a,b} y^2}{x^2 + y^2} \frac{\partial}{\partial y}
\end{align*}
\]
which is integrable because \( N_J \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = 0 \).
References

[1] Bejancu A., Farran H.R., Foliations and geometric structures, Mathematics and Its Applications, vol.580, Springer, 2006.
[2] Crasmareanu M., Hretcanu C. E., Golden differential geometry, Chaos, Solitons & Fractals, 38(2008), 1229–1238.
[3] Cruceanu V., On almost biproduct complex manifolds, An. St. Univ. I. Cuza, Iași, Mat, 52(1)(2006), 5–24.
[4] Das L.S., Nikic J., Nivas R., Parallelism of distributions and geodesics on $F(a_1, a_2, ..., a_n)$—structure Lagrangian manifolds, Diff. Geom. Dyn. Syst., 8(2006), 82–89.
[5] El Naschie M. S., COBE satellite measurement, hyperspheres, superstrings and the dimension of space time, Chaos, Solitons & Fractals, 9(8)(1998), 1445–1471.
[6] Falcon S., Plaza A., On 3-dimensional k-Fibonacci spirals, Chaos, Solitons & Fractals, 38(4)(2008), 993–1003.
[7] Gezer, A., Cengiz, N., Salimov, A., On integrability of golden Riemannian structures, Turk J. Math., 37(4)(2013), 693–703.
[8] Gezer, A., Karaman, C., On metallic Riemannian structures, Turk J. Math., 39(6)(2015), 954–962.
[9] Gezer, A., Karaman, C., Golden-Hessian Structures, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 86(1)(2016), 41–46.
[10] Goldberg S.I., Petridis N. C., Differentiable solutions of algebraic equations on manifolds, Kodai Math Sem. Rep., 25(1973) 111-128.
[11] Gray A., Pseudo-Riemannian almost product and submersions, J. Math. Mech.16(1967) 715-737.
[12] Htercanu, C.E., Crasmareanu, M., Metallic structures on Riemannian manifolds, Rev. Un. Mat. Argentina, 54(2)(2013), 15–27.
[13] Ianus S., Some almost product structures on manifolds with linear connection, Kodai Math. Sem. Rep., 23(1971) 305-310.
[14] Louestoe P., Clifford Algebras and Spinors, Cambridge University Press, United Kingdom, 2001.
[15] Ozkan, M., Prolongations of golden structure to tangent bundle, Differ. Geom. Dyn. Syst., 16(2014), 227-238.
[16] Ozkan M., Citlak A. A., Taylan E., Prolongations of golden structure to tangent bundle of order 2, GU J Sci., 28(2)(2015), 253-258.
[17] Ozkan M., Yilmaz F., Prolongations of golden structures to tangent bundles of order r, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 65(1)(2016), 35-47.
[18] Ozkan M., Peltek B., A new structure on manifolds: Silver structure, Int. Electron. J. Geom., 9(2)(2016), 59–69.
[19] Ozkan, M., Taylan, E., Citlak, A.A., On lifts of silver structure, Journal of Science and Arts, 2(39)(2017), 223–234.
[20] Procesi C., Lie groups: An approach through invariants and representations, Universitext, Springer, 2007.
[21] Pripoae G. T., Classification of semi-Riemannian almost product structure, BSG Proc. 11, Geometry Balkan Press, Bucharest, 243–251, 2004.
[22] Savas M., Ozkan, M., Iscan M., On 4–dimensional golden-Walker structures, Journal of Science and Arts, 2(35) (2016), 89-100.
[23] Spinadel V. W., de, The family of metallic means, Vis. Math. 1, 3(1999). [http://vismath1.tripod.com/spinadel/]
[24] Şahin, B., Akyol, M. A., Golden maps between Golden Riemannian manifolds and constancy of certain maps, Math. Commun. 19(2014), 1-10.
[25] Yardumcu, E. H., Yaylı, Y., Golden quaternionic structures, Int. Electron. J. Pure Appl. Math. 7 (2014), 109-125.
[26] Yano K., Kon M., Structures on manifolds, Series in Pure Mathematics, Vol 3, World Scientific, Singapore, 1984.
[27] Yilmaz, F., Ozkan, M., Bronze structure, 14. Mathematics Symposium, Niğde, Turkey, 109–111, 2015.
