Safety of Flow Decompositions in DAGs

Shahbaz Khan
Department of Computer Science, University of Helsinki, Finland

Alexandru I. Tomescu
Department of Computer Science, University of Helsinki, Finland

Abstract

Network flows are some of the most studied combinatorial optimization problems with innumerable applications. Any flow on a directed acyclic graph (DAG) \( G \) having \( n \) vertices and \( m \) edges can be decomposed into a set of \( O(m) \) paths, with applications ranging from network routing to the assembly of biological sequences. In some of these applications, the flow decomposition corresponds to some particular data that need to be reconstructed from the flow. Thus, such applications require finding paths (or subpaths) appearing in all possible flow decompositions, referred to as safe paths.

Recently, Ma, Zheng, and Kingsford [WABI 2020] addressed a related problem in a probabilistic framework. In a follow-up work, they gave a quadratic-time algorithm based on a global criterion, for a generalized version (AND-Quant) of the corresponding verification problem, i.e., reporting if a given flow path is safe. Our contributions are as follows:

1. A simple characterization for the safety of a given path based on a local criterion, which can be directly adapted to give an optimal linear time verification algorithm.

2. A simple enumeration algorithm that reports all maximal safe paths on a flow network in \( O(mn) \) time. The algorithm reports all safe paths using a compact representation of the solution (called \( \mathcal{P}_v \)), which is \( \Omega(mn) \) in the worst case, but merely \( O(m + n) \) in the best case.

3. An improved enumeration algorithm where all safe paths ending at every vertex are represented as funnels using \( O(n^2 + |\mathcal{P}_v|) \) space. These can be computed and used to report all maximal safe paths, using time linear in the total space required by funnels, with an extra logarithmic factor.

Overall we present a simple characterization for the problem leading to an optimal verification algorithm and a simple enumeration algorithm. The enumeration algorithm is improved using the funnel structures for safe paths, which may be of independent interest.

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1 Introduction

Network flows are a central topic in computer science, with countless practical applications. Assuming that the flow network has a unique source \( s \) and a unique sink \( t \), every flow can be decomposed into a collection of weighted \( s-t \) paths and cycles [12]. For DAGs such a decomposition contains only paths. Such path (and cycle) view of a flow indicates how information optimally passes from \( s \) to \( t \). For example, this decomposition is a key step in network routing problems (e.g. [16, 9, 15, 23]), transportation problems (e.g. [24, 25]), or in the more recent and prominent application of reconstructing biological sequences (RNA transcripts, see e.g. [26, 31, 14, 6, 30, 36], or viral quasi-species genomes, see e.g. [3, 2]).
Finding the decomposition with the minimum number of paths (and cycles) is NP-hard, even if the flow network is a directed acyclic graph (DAG) \cite{33}. On the theoretical side, this hardness result lead to research on approximation algorithms \cite{15, 29, 27, 23, 1, 5}, and FPT algorithms \cite{17}. On the practical side, many approaches usually employ a standard greedy width heuristic \cite{33}, of repeatedly removing an \textit{s-t} path carrying the most amount of flow. Another pseudo-polynomial-time heuristic was recently proposed by \cite{28}, in connection with biological data, that tries to iteratively simplify the graph such that the flow decomposition problem can be solved locally at some vertices.

In the routing and transportation applications, an optimal flow decomposition indicates how to send some information from \textit{s} to \textit{t}, and thus any optimal decomposition is satisfactory. However, this is not the case in the prominent application of reconstructing biological sequences, since each flow path represents a reconstructed sequence: a different optimal set of flow paths encodes different biological sequences, which may differ from the real ones. For a concrete example, consider the following application. In complex organisms, a gene may produce more RNA molecules (\textit{RNA transcripts}, i.e., strings over an alphabet of four characters), each having a different abundance. Currently, given a sample, one can read the RNA transcripts and find their abundances using high-throughput sequencing \cite{35}. This technology produces short overlapping substrings of the RNA transcripts. The main approach for recovering the RNA transcripts from such data is to build an edge-weighted DAG from these fragments and to transform the weights into flow values by various optimization criteria, and then to decompose the resulting flow into an “optimal” set of weighted paths (i.e., the RNA transcripts and their abundances in the sample) \cite{21}. Clearly, if there are multiple optimal flow decomposition solutions, then the reconstructed RNA transcripts may not match the original ones, and thus be incorrect.

\subsection{Problem Definition and Related Work}

Recently, Ma et al. \cite{19} were the first to address the issue of multiple solutions to the flow decomposition problem, under a probabilistic framework. Later, they \cite{20} solve a problem (\textit{AND-Quant}), which, in particular, leads to a quadratic-time algorithm for the following problem: given a flow in a DAG, and edges \textit{e}_1, \textit{e}_2, \ldots, \textit{e}_k, decide if in every flow decomposition there is always a decomposed flow path passing through all of \textit{e}_1, \textit{e}_2, \ldots, \textit{e}_k. Thus, by taking the edges \textit{e}_1, \textit{e}_2, \ldots, \textit{e}_k to be the edges of a path \textit{P}, the AND-Quant problem can decide if a path \textit{P} (i.e., a given biological sequence) appears in all flow decompositions. This indicates that \textit{P} is likely part of some original RNA transcript.

We build upon the AND-Quant problem, by addressing the flow decomposition problem under the \textit{safety} framework \cite{32}. For a problem admitting multiple solutions, a partial solution is said to be \textit{safety} if it appears in all solutions to a problem. For example, a path \textit{P} is safe for the flow decomposition problem, if for any flow decomposition into paths \{\textit{P}_1, \ldots, \textit{P}_k\}, it holds that \textit{P} is a subpath of some \textit{P}_i. In this paper, we will consider any flow decomposition as a valid solution, not only the ones of minimum cardinality. Dropping the minimality criterion is motivated by both theory and practice. On the one hand, since finding one minimum-cardinality flow decomposition is NP-hard, we believe that finding all safe paths for them is also intractable. On the other hand, given the various issues with sequencing data, practical methods usually incorporate different variations of the minimum-cardinality criterion, see e.g. \cite{6, 3, 2}. Thus, safe paths for all flow decompositions are likely correct to many practical variations of the flow decomposition problem.

The safety framework was introduced by Tomescu and Medvedev \cite{32} for the genome assembly problem from bioinformatics, where e.g. a solution may be a circular walk covering
every edge of a graph at least once. Safety has precursors in combinatorial optimization, under the name of persistency. For example, persistent edges present in all maximum bipartite matching were studied by Costa [10]. Incidentally, persistency has also been studied for the maximum flow problem, by finding persistent edges always having a non-zero flow value in any maximum flow solution ([8] acknowledges [18] for first addressing the problem); which is easily verified if the maximum flow decreases after removing the corresponding edge.

1.2 Our Results

Our contributions can be succinctly described as follows

1. **A simple local characterization resulting in an optimal verification algorithm:**
   We give a characterization for a safe path \( P \) based on excess flow, a local property of \( P \).

   ▶ **Theorem 1.** A path \( P \) is \( w \)-safe iff its excess flow \( f_P \geq w > 0 \).

   The previous work [20] on AND-Quant describes a global characterization using the maximum flow of the entire graph transformed according to \( P \), requiring \( O(mn) \) time. The excess flow is a local property of \( P \) which is thus computable in time linear in the length of \( P \). This also directly gives a simple verification algorithm which is optimal.

   ▶ **Theorem 2.** Given a flow graph (DAG) having \( n \) vertices and \( m \) edges, it can be preprocessed in \( O(m) \) time to verify the safety of any path \( P \) in \( O(|P|) = O(n) \) time.

2. **Simple Enumeration Algorithm:** The characterization results in a simple algorithm for reporting all maximal safe paths using a flow decomposition of the graph.

   ▶ **Theorem 3.** Given a flow graph (DAG) having \( n \) vertices and \( m \) edges, all its maximal safe paths can be reported in \( O(|P_f|) = O(mn) \) time, where \( P_f \) is some flow decomposition.

   This approach starts with a candidate solution and simply uses the characterization on its subpaths in an efficient manner (a similar approach was previously used by [10, 1]). The solution of the algorithm is reported using a compact representation (referred as \( P_c \)), whose size can be \( \Omega(mn) \) is the worst case but merely \( O(m + n) \) in the best case.

3. **Improved enumeration algorithm:** The enumeration algorithm can be improved by collectively considering all the safe paths ending at a vertex, which form a unique structure called a funnel [22], that can be used to report all maximal safe paths.

   ▶ **Theorem 4.** Given a flow graph (DAG) with \( n \) vertices and \( m \) edges, the funnel structures for the left maximal safe paths ending at its vertices

   (a) requires total \( O(n^2 + |P_c|) \) space for all vertices,

   (b) can be computed in \( O(m \log n + n^2 + |P_c|) \) time,

   (c) can be used to report all maximal safe paths in \( O(n^2 + |P_c| \log n) \) time,

   where \( P_c \) is the concise representation of the solution.

   This approach exploits the structure of the safe paths (unlike the simple algorithm) to identify how maximal safe solutions overlap, and thus results in efficient algorithms. This often leads to output sensitive enumeration algorithms, whose running time is parametrized by the size of the output (a similar approach was previously used by [7]).
2 Preliminaries and Notations

We consider a DAG $G = (V, E)$ with $n$ vertices and $m$ edges, where each edge $e$ has a flow $f(e)$ passing through it (also called its weight). Any two edges are called siblings if they share either their source vertex, or their target vertex. For each vertex $u$, $f_{in}(u)$ and $f_{out}(u)$ denotes the total flow on its incoming edges and total flow on its outgoing edges, respectively. A vertex $v$ in the graph is called a source if $f_{in}(v) = 0$ and a sink if $f_{out}(v) = 0$. Every other vertex $v$ satisfies the conservation of flow $f_{in}(v) = f_{out}(v)$, making the graph a flow graph.

For a path $P$ in the graph, $|P|$ represents the number of its edges. For a set of paths $P = \{P_1, \ldots, P_k\}$ we denote its total size (number of edges) by $|P| = |P_1| + \cdots + |P_k|$. Similarly, for any subgraph $F$ of the graph, we denote its total size (number of edges) by $|F|$.

For any flow graph (DAG), its flow decomposition is a set of weighted paths $P_f$ such that the flow on each edge in the flow graph equals the sum of the weights of the paths containing it. A flow decomposition of a graph can be computed in $O(|P_f|) = O(mn)$ time using the classical Ford Fulkerson algorithm \[12\].

A path $P$ is called $w$-safe if, in every possible flow decomposition, $P$ is a subpath of some paths in $P_f$ whose total weight is at least $w$. If $P$ is $w$-safe with $w > 0$, we call $P$ a safe flow path, or simply safe path. Intuitively, for any edge $e$ with non-zero flow, we consider where did the flow on $e$ come from? We would like to report the maximal path ending with $e$ along which at least $w > 0$ weight always “flows” to $e$ (see Figure 1). A safe path is left maximal (or right maximal) if extending it to the left (or right) with any edge makes it unsafe. A safe path is maximal if it is both left and right maximal.

![Figure 1](image)

Prefix of the blue path up to $e$ contributes at least 2 units of flow to $e$, as the rest may enter the path by the edges (red) with flow 4 and 2. Similarly, the suffix of the blue path from $e$ maintains at least 1 unit of flow from $e$, as the rest may exit the path from the edges (red) with flow 5 and 2. Both these safe paths are maximal as they cannot be extended left or right.

3 Characterization and Properties of Safe Paths

The safety of a path can be characterized by its excess flow (see Figure 2), defined as follows.

![Figure 2](image)

Figure 2 The excess flow of a path is the incoming or outgoing flow (blue) that passes through the path despite the flow (red) leaking at its internal vertices.

Definition 5 (Excess flow). The excess flow $f_P$ of a path $P = \{u_1, u_2, ..., u_k\}$ is

$$f_P = f(u_1, u_2) - \sum_{u_i \in \{u_2, ..., u_{k-1}\}} f(u_i, v) = f(u_{k-1}, u_k) - \sum_{u_i \in \{u_2, ..., u_{k-1}\}} f(v, u_i)$$

where the former and later equations are called diverging and converging criterion, respectively.
The converging and diverging criteria are equivalent by the conservation of flow on internal vertices. The idea behind the excess flow $f_P$ is the amount of the incoming flow (or outgoing flow) of a path $P$ necessitates $f_P$ flow to pass through $P$ despite the flow leaking on internal vertices (Figure 2). Since the path decomposition before entering a vertex does not affect the path decomposition on leaving a vertex, the absence of positive excess flow renders the path unsafe. The following results give the simple characterization and some additional properties of safe paths.

**Theorem 1.** A path $P$ is $w$-safe iff its excess flow $f_P \geq w > 0$.

**Proof.** The excess flow $f_P$ of a path $P$ trivially makes it $w \leq f_P$-safe as it is necessitated by conservation of flow. If $f_P < w$, then $P$ is surely not $w$-safe, because a path decomposition can have the paths entering $P$ which exit $P$ using out-edge of an intermediate note without passing completely through $P$. In the worst-case, such exiting paths would leave only $f_P$ flow (by definition) passing completely through $P$, which is $< w$. Additionally, for a path to be safe, it must hold that $w > 0$.

**Lemma 6.** For any path in a flow graph (DAG), we have the following properties:

(a) Any subpath of a $w$-safe path is also at least $w$-safe.

(b) The converging and diverging criteria for a path $P = \{u_1, \ldots, u_k\}$ are equivalent to

$$f_P = \sum_{i=1}^{k-1} f(u_i, u_{i+1}) - \sum_{i=2}^{k-1} f_{\text{out}}(u_i) = \sum_{i=1}^{k-1} f(u_i, u_{i+1}) - \sum_{i=2}^{k-1} f_{\text{in}}(u_i).$$

(c) Adding an edge $(u, v)$ to the start, or the end, of a path reduces its excess flow by $f_{\text{in}}(v) - f(u, v)$, or $f_{\text{out}}(u) - f(u, v)$, respectively.

(d) A safe path starting (or ending) with an edge $e$ implies that the path replacing $e$ with its maximum weight sibling is also safe.

(e) Two safe paths cannot merge at an intermediate vertex (or vertices) and then diverge.

**Proof.** (a) Since all flows are positive, any suffix subpath $P'$ of $P$ has $f_{P'} \geq f_P$ by truncating the corresponding negative terms in the diverging criterion of the excess flow. Similarly, any prefix subpath $P'$ of $P$ has $f_{P'} \geq f_P$ by truncating the corresponding negative terms in the converging criterion of the excess flow.

(b) Expanding the values of $f_{\text{in}}(u_i)$ and $f_{\text{out}}(u_i)$ results in the original criteria.

(c) Using the converging criterion in (b) adding an edge at the start of a path modifies its excess flow by $f(u, v) - f_{\text{in}}(v)$. Similarly, using the diverging criterion in (b) adding an edge at the end of a path modifies its excess flow by $f(u, v) - f_{\text{out}}(u)$.

(d) Let the safe path $P$ start (or end) with $e$, and consider an alternate path that replaces $e$ with its maximum weight sibling $e'$. The excess flow of the alternate path can be computed by the diverging (or converging) criterion to be $f_P - f(e) + f(e') \geq f_P > 0$, making it safe.

(e) Let two safe paths $P$ and $P'$ merge at a vertex $v_1$, entering $v_1$ respectively by $e_1$ and $e'_1$, and then diverge at a vertex $v_2$, leaving $v_2$ respectively by $e_2$ and $e'_2$ (see Figure 3). Using Lemma (c), the subpaths $\{e_1, \ldots, e_2\}$ and $\{e'_1, \ldots, e'_2\}$ of safe paths $P$ and $P'$ respectively, are also safe. By diverging criterion of the safe path $P$ we have $f(e_1) > f(e'_2)$. On the other hand, by converging criterion of the safe path $P'$ we have $f(e'_2) > f(e_1)$, which is a contradiction.
Figure 3 The diverging and converging criterion applied to path $P$ and $P'$ respectively.

4 Simple Verification and Enumeration Algorithms

The characterization of a safe path in a flow graph can be directly adapted to simple algorithms for verification and enumeration of safe paths.

4.1 Verification Algorithm

The characterization (Theorem 1) can be directly adapted to verify the safety of a path optimally. We preprocess the graph to compute the incoming flow $f_{in}(u)$ and outgoing flows $f_{out}(u)$ for each vertex $u$ in total $O(m)$ time. Using Lemma 6(b) the time taken to verify the safety of any path $P$ is $O(|P|) = O(n)$, resulting in following theorem.

▶ Theorem 2. Given a flow graph (DAG) having $n$ vertices and $m$ edges, it can be preprocessed in $O(m)$ time to verify the safety of any path $P$ in $O(|P|) = O(n)$ time.

4.2 Enumeration of All Maximal Safe Paths

The maximal safe paths can be reported in $O(mn)$ time by computing a candidate decomposition of the flow into paths, and verifying the safety of its subpaths using the characterization and a scan with the commonly used two-pointer approach.

The candidate flow decomposition can be computed in $O(mn)$ time using the classical flow decomposition algorithm [12] resulting in $O(m)$ paths $P_f$ each of $O(n)$ length. Now, we compute the maximal safe paths along each path $P \in P_f$ by a two-pointer scan as follows. We start with the subpath containing the first two edges of the path $P$. We compute its excess flow $f$, and if $f > 0$ we append the next edge to the path on the right and incrementally compute its excess flow by Lemma 6(c). Otherwise, if $f \leq 0$ we remove the first edge of the path from the left and incrementally compute the excess flow similarly by Lemma 6(c) (removing an edge $(u, v)$ would conversely modify the flow by $f_{in}(v) - f(u, v)$). We stop when the end of $P$ is reached with a positive excess flow.

The excess flow can be updated in $O(1)$ time when adding an edge to the subpath on the right or removing an edge from the left. If the excess flow of a subpath $P'$ is positive and on appending it with the next edge it ceases to be positive, we report $P'$ as a maximal safe path by reporting only its two indices on the path $P$. Thus, given a path of length $O(n)$, all its maximal safe paths can be reported in $O(n)$ time, and hence require total $O(mn)$ time for the $O(m)$ paths in the flow decomposition $P_f$, resulting in the following theorem.

▶ Theorem 3. Given a flow graph (DAG) having $n$ vertices and $m$ edges, all its maximal safe paths can be reported in $O(|P_f|) = O(mn)$ time, where $P_f$ is some flow decomposition.

Concise representation

The solution can be reported using a concise representation (referred as $P_c$) having a set of paths as follows. We add to $P_c$ every subpath of each path $P \in P_f$ that contains maximal safe paths, along with the indices of the solution on the path. Thus, for one or more overlapping
maximal safe subpaths from $P$ we add a single path in $\mathcal{P}_c$ which is the union of all such maximal safe paths, making the paths added to $\mathcal{P}_c$ of minimal length. Finally, we also remove the duplicates among the maximal safe subpaths reported from different paths in $\mathcal{P}_f$ using a Trie \[11\], making the set of paths in $\mathcal{P}_c$ minimal. Thus, we define $\mathcal{P}_c$ as follows.

**Definition 7** (Concise representation $\mathcal{P}_c$). A minimal set of paths having a minimal length such that every safe path of the flow network is a subpath of some path in the set.

**Remark 8.** In the worst case, the algorithm is optimal for some DAGs having $|\mathcal{P}_c| = |\mathcal{P}_f| = \Omega(mn)$, but in general $|\mathcal{P}_c|$ can be as small as $O(m+n)$ (see Figure [1]). Thus, improving this bound requires us to not use a flow decomposition (and hence a candidate solution).

![Figure 4](image.png) The example demonstrates the worst case (left) and the best case (right) graphs where the simple enumeration algorithm is optimal, and inefficient, respectively. We have two paths $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. The set $C = \{c_1, \ldots, c_k\}$ has edges from $a_1$ and the set $D = \{d_1, \ldots, d_k\}$ has edges to $b_1$. Choosing $k = n/4$ and any subset of connections between $C \times D$ we get a graph with any $n$ and $m$. Let there be flow $k$ on the black edges and unit flow on the red edges. (a) In the worst case graph (left) the flow on remaining edges is according to flow conservation assuming $a_i$ as the source and $b_k$ as the sink. Each edge in $C \times D$ necessarily has a separate path in $\mathcal{P}_f$ from $a_1$ to $b_k$, with $k$ maximal safe paths between $\{a_i, b_i\}$ for all $1 \leq i \leq k$ because every path between $a_i$ to $b_1$ has excess flow $i$. This ensures that $|\mathcal{P}_c| = |\mathcal{P}_f| = \Omega(mn)$.

(b) In the best case graph (right) the two edges from $a_{k-1}$ to $a_k$ and from $b_1$ to $b_2$ carry equal flow, and the remaining edges have flow according to conservation of flow. Each edge in $C \times D$ has a safe path of $O(1)$ size from $a_k$ to $b_1$. Here we have still have $|\mathcal{P}_f| = \Omega(mn)$ but $|\mathcal{P}_c| = O(m+n)$.

## 5 Efficient Computation of All Maximal Safe Paths

We compute the maximal safe paths more efficiently by using dynamic programming along the topological ordering in DAGs. Lemma 4[1] gives a unique structure to the left maximal safe paths ending at a vertex $v$ because there is no possibility of two safe paths diverging after converging. Such a structure was previously studied for an entire DAG satisfying the no diverging after converging property, and referred to as a funnel [22].

We shall first describe the funnel structure $\mathcal{F}_u$ of the left maximal safe paths ending at a vertex $u$. Thereafter, we will describe how this $\mathcal{F}_u$ can be efficiently used to incrementally build $\mathcal{F}_v$ for each out-neighbour $v$ of $u$. Finally, we will describe how to report all the maximal safe paths ending at a vertex $v$ using $\mathcal{F}_v$. The key idea behind the efficiency of our algorithm is bounding the overall size of the funnel structures on all the vertices and associating the work done with the total size of all the funnel structures. Our algorithm computes all maximal safe paths of a DAG in $O(n^2 + (m + |\mathcal{P}_c|) \log n)$ time, where $\mathcal{P}_c$ is the concise representation of the solution reported by the simple algorithm.

### 5.1 Funnel Structure of Safe Paths Ending at a Vertex

We describe $\mathcal{F}_v$ using the notion of source or sink paths of a vertex $v$ in a DAG. A source path is any path to the vertex $v$ from a source of the DAG, whereas the sink path is any
path from the vertex \( v \) to a sink of the DAG. The funnel structure (see Figure 5) and its properties are described as follows, which also proves Theorem 4(a).

**Figure 5** The funnel structures of safe paths ending at \( v \) having the diverging trees (blue) and a converging tree (green). The characteristic edge (black) of a path is the first edge incident on the converging tree.

▶ **Theorem 9 (Funnel).** The left maximal safe paths entering a vertex \( v \) form a funnel, where:

(a) Every vertex has either a unique source path or a unique sink path, or both.
(b) The funnel has a single converging tree (having vertices with unique sink path) succeeding possibly multiple diverging trees (having remaining vertices with unique source path).
(c) The characteristic edge of each flow path is its first edge incident on the converging tree.

The size of the funnel structure \( F_v \) at \( v \) is \( O(n_v + m_v) \), where \( n_v \) is the number of vertices having a safe path ending at \( v \), and \( m_v \) is the number of paths in \( P_c \) containing \( v \). For all \( v \in V \), the total size is \( O(n^2 + |P_c|) \).

**Proof.** We first prove the properties of the funnel (also derivable from previous work [22]) and then the space complexity.

(a) Any vertex having both multiple source paths and multiple sink paths has two different safe paths that merge at (or before) the vertex and then diverge at (or after) it, which contradicts Lemma 6(e).
(b) All the vertices having a unique sink path (terminating at \( v \) as the funnel contains only the safe paths ending at \( v \)), form a single converging tree rooted at \( v \) (see Figure 5). Similarly, the remaining vertices having unique sink paths (Theorem 4(a)) form a set of disjoint diverging trees since there can be multiple sources. Finally, since a vertex with a unique sink path cannot reach a vertex with multiple sink paths, the vertices of the converging tree always succeed those of diverging trees in any path.
(c) The first incident edge on the converging tree for a path in \( F_v \) would be from a vertex with a unique source path to a vertex with a unique sink path. So the edge can be a part of exactly one path, which it characterizes (see Figure 5). Such an edge always exists for a path because it always starts from a source (hence with a unique source path) and ends at \( v \) (hence with a unique sink path).

The number of vertices (and hence edges of the converging and diverging trees) in the funnel is \( O(n_v) \) (by definition). Further, the non-tree edges (if any) are the characteristic edges for some safe path containing \( v \), and each such safe path is present in a unique path in \( P_c \) (as each path contains \( v \) exactly once). Thus, each characteristic edge is uniquely associated with a path in \( P_c \) containing \( v \), resulting in total \( O(m_v) \) non-tree edges in the funnel. Overall, since \( n_v = O(n) \) and \( v \) contributes to the length of at least \( m_v \) paths in \( P_c \), that total size of all funnels is \( O(n^2 + |P_c|) \).
5.2 Incremental Construction of All Funnels

The funnel of a vertex is incrementally built on processing its incoming neighbours. We process the vertices of the DAG in topological order, ensuring that before we process a vertex its funnel is completely built while processing its incoming neighbours.

We now describe how we build the funnels of the out neighbours of a vertex \( u \) using \( \mathcal{F}_u \) (see Algorithm 1). Since every edge is trivially safe, every outgoing edge \((u, v)\) of \( u \) is added to corresponding \( \mathcal{F}_v \). For updating any \( \mathcal{F}_v \) with the safe paths through \( u \), we traverse \( \mathcal{F}_u \) in the reverse topological order and verify the safety of each such path using the converging criterion. An edge from \( \mathcal{F}_u \) is thus added to \( \mathcal{F}_v \) if some safe path contains the edge and \((u, v)\). We distinguish the unique maximum weighted outgoing edge \((u, v^*)\) of \( u \) (if any), and use different approaches to incrementally build \( \mathcal{F}_{v^*} \) and \( \mathcal{F}_v \) \((v \neq v^*)\). This is because being unique we spend \( O(|\mathcal{F}_u|) \) time for \( \mathcal{F}_{v^*} \), and only \( O(|\Delta \mathcal{F}_v|) \) time for other \( \mathcal{F}_v \), where \( \Delta \mathcal{F}_v \) are the new edges added to \( \mathcal{F}_v \) while processing \( u \). Thus, the time required to build all \( \mathcal{F}_v \) is bounded by the total size of all \( \mathcal{F}_v \), i.e., \( O(n^2 + |\mathcal{P}_e|) \).

**Algorithm 1** Incrementally building funnels

```
BUILD-FUNNELS():
  forall u \in V \text{ in topological order do}
  BUILD-FROM(u)

BUILD-FROM(u):
  forall (u, v) \in E do Add (u, v) to \( \mathcal{F}_v \)
  /* Computing \( \mathcal{F}_v \) for \( v \neq v^* \) */
  forall (u, v) \in E do Add v to \( \mathcal{N}_s, \mathcal{N}_p \)
  Sort v \in \mathcal{N}_p \text{ by weights of } (u, v)
  y \leftarrow u, P \leftarrow \{ y \}
  while \( \mathcal{N}_p \neq \emptyset \) do
    (x, y) \leftarrow \text{Max weight edge entering } y
    P \leftarrow \{ x \} \cup P
    f[x] \leftarrow f[x] - f(u, v^*)
    forall v \in \mathcal{N}_s do
      f' \leftarrow f[x] + f(u, v)
      if \( f' > 0 \) and \((x, y) \notin \mathcal{F}_v\) then
        Add (x, y) to \( \mathcal{F}_v \) // suffix path
      else Remove v from \( \mathcal{N}_s \)
    while \( f[x] + f(u, \text{head}(\mathcal{N}_p)) < 0 \) do
      v \leftarrow \text{pop}(\mathcal{N}_p), y' \leftarrow z \leftarrow y
      while y' \notin \mathcal{F}_v do
        y' \leftarrow \text{Next of } y' \text{ in } P
      Add \( P[z, y'] \) in \( \mathcal{F}_v \) // prefix path
      y \leftarrow x
```

Building \( \mathcal{F}_{v^*} \)

We traverse \( \mathcal{F}_u \) in reverse topological order, computing the maximum excess flow \( f[x] \) of a path from \( x \) to \( v^* \) through \( u \). This is computed by considering \( f[y] \) of each out-neighbour \( y \) of \( x \) in \( \mathcal{F}_u \), where the excess flow of \( x-v^* \) path is computed in \( O(1) \) time using the \( y-v^* \) path by adding the edge \((x, y)\) to its start (Lemma 3). Every such edge \((x, y)\) resulting in a safe \( x-v^* \) path is added to \( \mathcal{F}_{v^*} \), if not already present (may have been added while processing some other \( \mathcal{F}_u \)). Note that if \( x \) is in the converging tree, there is a unique sink.
10 Safety of Flow Decompositions in DAGs

path from \( x \) (Theorem 11), hence \( x \) considers a single \( y \) where \( f[y] \) represents a single safe \( y-v^* \) path. However, in the diverging trees \( y \) may have multiple sink paths (Theorem 11), but we process only \( f[y] \) representing the \( y-v^* \) safe path with the maximum excess flow. This is because such a path would always be extendable to add \( (x, y) \) if any other \( y-v^* \) path is extendable (Lemma 11). Since we cannot process \( y \) for each path separately in \( O(|F_u|) \) time, we process \( y \) only once using \( f[y] \) for its \( y-v^* \) safe path with the maximum excess flow. Thus, we process each edge of \( F_u \) once to build \( F_v \), taking total \( O(|F_u|) \) time.

Building \( F_v \) for \( v \neq v^* \)

We limit the paths in \( F_u \) that are processed to build \( F_v \) using the following property.

\[ \text{Lemma 10. For a vertex } u \text{ and an edge } (u, v) \text{ where } v \neq v^*, F_u \text{ adds a single path to } F_v. \]

\[ \text{Proof. If } F_u \text{ adds multiple paths to } F_v, \text{ by Lemma 11 such paths are also extendable to some maximum weighted outgoing edge } (u, v') \text{ of } u, \text{ where } v' \neq v. \text{ This contradicts Lemma 11 as two safe paths containing } u \text{ converge in } F_u \text{ and then diverge to } v \text{ and } v'. \]

Thus, we process exactly one path in \( F_u \) to be added to \( F_v \) in the reverse topological order. Using Lemma 11 this path, say \( P \), contains \((u, v)\) preceded by the maximum weight incoming edge \((y, u)\) of \( u \), and the maximum weight incoming edge \((x, y)\) of \( y \), and so on. For a vertex \( v \), a suffix of this path, \( P_v \), is safe as long as its excess flow is positive. This can be verified efficiently using Lemma 12 (similar to \( F_v \)). Moreover, maximum excess flow of a safe \( x-v \) path can be computed using \( f[x] \) (computed for \( F_v \)) and adding \( f(u, v^*) \) and adding \( f(u, v) \) (Lemma 12). However, \( P \) cannot be simply added to \( F_v \) using \( O(|P_v|) \) time, because as described earlier we aim to use only \( O(|\Delta F_v|) \) time.

Consider Figure 6 if \( P_v \) has a subpath already present in \( F_v \), we cannot traverse the entire \( P_v \) independently for \( v \). So, we traverse (and add to \( F_v \)) only the suffix and the prefix (if any) of \( P_v \), not in \( F_v \) independently, and the rest only once for all out-neighbours of \( u \). The suffix of \( P_v \) is simply added in the reverse order until it reaches a vertex \( y \) already present in \( F_v \). Using Lemma 11 \( y \) is in the diverging tree of \( F_v \) as the new suffix containing \((u, v)\) always exists. Hence \( y \) has a unique source path which either contains the rest of \( P_v \), or begins from some vertex \( y' \in P_v \). For the later case, we compute the start (say \( z \)) of \( P_v \) for all out-neighbours of \( u \) together, and then add the corresponding prefix of \( P_v \), say \( P_v[z, y'] \), for each \( P_v \) independently. Note that the out-neighbours of \( u \) may have different \( y \), \( y' \), and \( z \).

To compute \( y \), \( y' \), and \( z \) for all out-neighbours of \( u \), we add the out-neighbours to lists \( N_s \) and \( N_p \) to compute the corresponding suffix and prefix, respectively. We traverse \( P \) backwards, adding the suffix edges to \( F_v \) for all \( v \in N_s \). When for some \( v \), the corresponding \( y \in F_v \) is reached we simply remove \( v \) from \( N_s \) and continue for the other vertices adding each suffix path to the corresponding \( F_v \) (until \( N_s = \emptyset \)). For adding prefix paths we need to compute \( z \) for each \( P_v \), beyond which \( P_v \) is not safe. Thus, we traverse backwards in \( P \) together for all \( v \) until some \( P_v \) is unsafe. This \( v \) would always be the one with lowest weight \((u, v)\) among \((u, x)\) for \( x \in N_p \) by Lemma 11 as the remaining path is the same for all. Hence we initially sort \( v \in N_p \) in the increasing order of the weights of \((u, v)\), so that while traversing \( P \) backwards we check the safety of \( P_v \) only for the first element in \( N_p \) (\text{head}(N_p)).
When \( P_v \) for head(\( N_p \)) becomes unsafe we repeatedly remove the head(\( N_p \)) from \( N_p \) until \( P_v \) is safe for the rest of \( v \in N_p \). For a removed vertex \( v \), we traverse its prefix path (if any) forwards from \( z \) until we reach \( y' \in F_v \) and add it to \( F_v \). Again, we continue traversing back on \( P \) until all prefixes are added (until \( N_p = \emptyset \)). The process thus requires \( O(|\Delta F_v|) \) for adding only new edges (prefixes and suffixes) to \( F_v \), \( O(|F_v|) \) for traversing the path \( P \), and \( O(deg(u) \log n) \) for considering all out-neighbours independently and sorting them.

Since topological sort takes linear time (edges), the total time required to build all the funnels is \( O(m \log n + \sum_v |F_v|) = O(m \log n + n^2 + |P_c|) \) (Theorem 9), proving Theorem 11.

\[ \text{Theorem 11. Given a flow graph (DAG) with} \ n \ \text{vertices and} \ m \ \text{edges, the funnel structures for the left maximal safe paths ending at its vertices can be computed in} \ O(m \log n + n^2 + |P_c|), \ \text{where} \ P_c \ \text{is the concise representation of the solution.} \]

### 5.3 Reporting Maximal Safe Paths from a Funnel

The funnel \( F_u \) contains all the left maximal safe paths ending at the vertex \( u \). To report all maximal safe paths ending at \( u \), we need to address the following two issues:

1. Some of these safe paths may not start from a source of \( F_u \), rather some vertex \( v \) in a diverging tree where residual source path was added by some overlapping safe path, as there can be multiple sink paths for \( v \) (Theorem 9(b)). Thus, to represent a maximal safe path ending at \( u \) we report its characteristic edge (Theorem 9) along with its start vertex in \( F_u \). For completeness, we also report its excess flow. Thus, all maximal safe paths ending at \( u \) are reported in \( \text{Sol}_u \) containing triplets (characteristic edge, start vertex, excess flow) for each such path.

2. Some of these paths may not be right maximal if the path from its start vertex is extendable to the right of \( u \) while remaining safe. Such paths must be removed from the solution, hence we create another funnel \( F^*_u \) (initialized with \( F_u \)) and remove such paths from it without affecting the other maximal safe paths.

Thus, the funnel \( F^*_u \) and \( \text{Sol}_u \) are reported as the solution.

**Approach**

The start vertices of these maximal safe paths can be easily computed by visiting the vertices of \( F_u \) in the reverse topological order, computing their excess flow incrementally using that of its suffix path (Lemma 9(c)). When such a path becomes unsafe, we know that it’s suffix path is left maximal. Also, if a safe path reaches a source of \( F_u \), it is left maximal. To verify if any such path is also right maximal, we compute its excess flow after adding the edge \((u, v^*)\) (Lemma 9(d)) to its right using Lemma 9(c). If the new path is not safe, the suffix path is reported as maximal safe using its characteristic edge, start vertex, and excess flow. Otherwise, the path is not right maximal (and hence not maximal), so we delete its characteristic edge with left and right extensions (not shared with any maximal safe paths) from \( F^*_u \) (see Figure 7).

Note that verifying if a path is right maximal safe takes \( O(1) \) time. If it is not right maximal safe, \( F^*_u \) is updated by deleting some edges, taking overall \( O(|F_u|) \) time. However, computing the start vertex for each maximal safe path independently is not efficient because a vertex \( v \) in the diverging tree (see Figure 7) will be processed multiple times as it can have multiple sink paths (Theorem 11(b)). Total time for processing such vertices in \( F_u \) can be \( O(m u_n u) \) (recall Theorem 9). We instead process \( v \) once for all such safe paths containing \( v \) as follows. While extending the safe paths starting from \( v \) to the left, if the path \( P \) with
Figure 7 Multiple safe paths (red) starting at a vertex $v$ in the diverging tree. Extensions (purple, dashed) of characteristic edges $e_1$ and $e_2$ to be deleted if their safe paths are not right maximal.

the minimum excess flow is safe, the remaining paths are also safe. Otherwise, $P$ is either removed from $F_v^*$ if it is not right maximal, or is reported as maximal. In either case, we evaluate the next minimum weight path, and so on. Thus, processing a vertex either proceeds backward from it or removes a safe path to be evaluated, taking total $O(n_u + m_u)$ time.

Heap Structure $\mathcal{H}_v$

For processing the diverging trees having multiple safe paths starting at each vertex, we use any mergeable-heap\cite{13,34} structure, which is a min-heap where all operations including merging two heaps take amortized $O(\log n)$ time. Additionally, we use the standard lazy update propagation technique, where some value can be added to every element of the heap in $O(1)$ time, which is propagated to individual elements when they are accessed. Now, every element of a heap characterizing a safe path contains:

- $edg$: Denotes the characteristic edge of the safe path.
- $val$: The excess flow of the path without applying the lazy update.
- $upd$: The lazy update which is to be reduced from the $val$.

For each vertex $v \in F_u$, we maintain such a mergeable-heap structure $\mathcal{H}_v$ which stores an element corresponding to each safe path from $v$ to $u$. For a vertex $v$ in the converging tree, having a single sink path (Theorem 9(b)) and hence $|\mathcal{H}_v| = 1$, we do not need lazy update, so we keep $upd = -1$ for its element. Also, for such vertices in the converging tree inserting an element takes $O(1)$ time, being inserted into an empty heap. Similarly, for a vertex $x$ with a single out-neighbour $y$ in a diverging tree merging $\mathcal{H}_y$ into an empty $\mathcal{H}_x$ takes $O(1)$ time.

Algorithm

Our algorithm first adds a single edge path $(u, v)$ to the heap of each in-neighbour $v$ of $u$ (see Algorithm 2). Now, moving from $y$ to $x$ through an edge $(x, y)$ we consider two cases.

(a) If $y$ is in the converging tree, then adding $(x, y)$ is always safe (recall that a different start vertex is only possible for the paths starting from a diverging tree). So for the current path $edg$ is updated to $(x, y)$ (being incident on $y$). The change in the excess flow on adding $(x, y)$ (Lemma 6(c)) is updated either in $val$ (if $x$ is in converging tree) or in $upd$ (if $x$ is in diverging tree). This updated element is inserted to $\mathcal{H}_x$.

(b) If $y$ is in a diverging tree, first we verify the safety of adding $(x, y)$ to the minimum excess flow path in $\mathcal{H}_y$ (head($\mathcal{H}_y$)). If it is not safe, it is extracted from the heap and dealt with (explained later). We continue to check for unsafe paths in $\mathcal{H}_y$ and extract them until head($\mathcal{H}_y$) is safe. When all paths in $\mathcal{H}_y$ are safe, the change in excess flow by adding $(x, y)$ (Lemma 6(c)) is updated in head($\mathcal{H}_y$).$upd$, and $\mathcal{H}_y$ is merged into $\mathcal{H}_x$. 
Algorithm 2 Reporting maximal safe paths in $F_u$

forall $(x, u) \in F_u$ do Add $[(x, u), f(x, u), -1]$ to empty $H_x$

upd* ← $f_{out}(u) - f(u, v^*)$ /* update to check right maximal */

forall $y \neq u \in F_u$ in reverse topological order do

forall $(x, y) \in F_u$ do

upd ← $f_{in}(y) - f(x, y)$

if $H_y \neq \emptyset$ then // extendable safe paths in $H_y$

if head($H_y$).upd = -1 then // $y$ in converging tree

if $x$ has unit out-degree in $F_u$ then // $x$ in converging tree

Add $[(x, y), head(H_y).val, -1]$ to empty $H_x$ // $x$ in diverging tree

else Add $[(x, y), head(H_y).val, upd]$ to $H_x$ // $y$ in diverging tree

else // $y$ in diverging tree

while head($H_y$).val - head($H_y$).upd - upd $\leq$ 0 do // path unsafe

top ← Extract min from $H_y$

if $top.val - top.upd - upd^* > 0$ then // path not maximal

Remove top.edg with extensions from $F_u^*$

else Add [top.edg, y, top.val - top.upd] to $Sol_u$ // path maximal

head($H_y$).upd ← head($H_y$).upd + upd

if $x$ has unit out-degree in $F_u$ then Merge $H_y$ into empty $H_x$

else Merge $H_y$ into $H_x$

forall source $v \in F_u$ do

while $H_v \neq \emptyset$ do

top ← Extract min from $H_y$

if $top.val - top.upd - upd^* > 0$ then // path not maximal

Remove top.edg with extensions from $F_u^*$

else Add [top.edg, v, top.val - top.upd] to $Sol_u$ // path maximal

Report $F_u^*$ and $Sol_u$

The unsafe paths extracted from the heaps and the safe paths in the heaps of the sources of $F_u$, are processed so that they are either removed from $F_u^*$ (if not right maximal) or added to the solution after applying the lazy update.

Analysis

Now, each of $m_u$ safe paths in $F_u$ uses one insert and one extract min operation in a non-empty heap, while moving from converging tree to diverging tree, and while adding to (or discarding from) the solution respectively, using $O(m_u)$ operations. Also, every merge operation into non-empty heaps combines at least two different safe paths in a heap, using total $O(m_u)$ operations. The remaining operations (including those on empty heaps) take $O(1)$ time while traversing $F_u$ taking $O(|F_u|)$ time. Thus, processing each funnel takes $O(m_u \log n + |F_u|)$ time, requiring total $O(n^2 + |P_c| \log n)$ time, which also proves Theorem 12.

Theorem 12. Given a flow graph (DAG) with $n$ vertices and $m$ edges and the funnel structures for the left maximal safe paths ending at its vertices, the maximal safe paths can be reported in $O(n^2 + |P_c| \log n)$ time, where $P_c$ is the concise representation of the solution.
6 Conclusion

We study the safety of flow paths in a given flow graph (DAG), which has applications in various domains, including the more prominent assembly of biological sequences. The previous work characterized such paths (and their generalizations) using a global criterion. We presented a simpler characterization based on a more efficiently computable local criterion, that can be directly adapted into an optimal verification algorithm. Also, it results in a simple enumeration algorithm, which is optimal for some worst-case examples. However, it is inefficient if the size of the concise representation of the solution is small. We improve this algorithm by exploiting a unique structure of a set of safe paths, called a funnel, resulting in a running time parameterized by the size of this concise representation of the solution.

Our improved enumeration algorithm improved the running time of the simple algorithm only when the size of flow decomposition $O(|P_c|)$ is larger than $O(n^2)$. Also, it uses extra space if $|P_c| = o(n^2)$. In the future, it would be interesting to see if it is possible to report the solution using optimal space, because both the paths in $P_c$ and different $F_v$ may have a lot of redundancy. Ideally, we would like to see an algorithm taking linear time in addition to the optimal output size (similar to what was achieved by [7] for a different problem). Another interesting extension to this problem having practical significance is finding safe paths for those flow decompositions whose paths have a certain minimum weight threshold.

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