High Power $n$ of $m_b$ in Beauty Widths and $n = 5 \to \infty$ Limit

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Abstract

The leading term in the semileptonic width of heavy flavor hadrons depends on the fifth power of the heavy quark mass. We present an analysis where this power can be self-consistently treated as a free parameter $n$ and the width can be studied in the limit $n \to \infty$. The resulting expansion elucidates why the small velocity (SV) treatment is relevant for the inclusive semileptonic $b \to c$ transition. The extended SV limit (ESV limit) is introduced. The leading terms in the perturbative $\alpha_s$ expansion enhanced by powers of $n$ are automatically resummed by using the low-scale Euclidean mass. The large-$n$ treatment explains why the scales of order $m_b/n$ are appropriate. On the other hand, the scale cannot be too small since the factorially divergent perturbative corrections associated with running of $\alpha_s$ show up. Both requirements are met if we use the short-distance mass normalized at a scale around $m_b/n \sim 1$ GeV. A convenient definition of such low-scale OPE-compatible masses is briefly discussed.
1 Introduction

A true measure of our understanding of heavy flavor decays is provided by our ability to extract accurate values for fundamental quantities, like the CKM parameters, from the data. The inclusive semileptonic widths of $B$ mesons depend directly on $|V_{cb}|$ and $|V_{ub}|$, with the added bonus that they are amenable to the OPE treatment, and the non-perturbative corrections to $\Gamma_{\text{sl}}(B)$ are of order $1/m_b^2$, i.e. small. Moreover, they can be expressed, in a model independent way, in terms of fundamental parameters of the heavy quark theory – $\mu_*^2$ and $\mu_G^2$ – which are known with relatively small uncertainties.

Therefore, the main problem in the program of a precise determination of, say, $|V_{cb}|$ from $\Gamma_{\text{sl}}(B)$ is the theoretical understanding of the perturbative QCD corrections and the heavy quark masses. The theoretical expression for $\Gamma_{\text{sl}}(B)$ depends on a high power of the quark masses $m_b$ (and $m_c$): $\Gamma_{\text{sl}} \propto m_b^n$ with $n = 5$. The quark masses are not observables, and the uncertainties in their values are magnified by the power $n$. For this reason it is often thought that the perturbative corrections in the inclusive widths may go beyond theoretical control.

The quark masses in the field theory are not constant but rather depend on the scale, and in this respect, are similar to other couplings like $\alpha_s$ which define the theory. Of course, $m_Q$ and $\alpha_s$ enter the expansion differently. Moreover, in contrast to $\alpha_s$ in QCD the heavy quark mass $m_Q$ has a finite infrared limit to any order in the perturbation theory, $m_Q^{\text{pole}}$, which is routinely used in the calculations.

If $\Gamma_{\text{sl}}(B)$ is expressed in terms of the pole mass of the $b$ quark treated as a given number, the expression for $\Gamma_{\text{sl}}(B)$ contains a factorially divergent series in powers of $\alpha_s$

$$\Gamma_{\text{sl}} \sim \sum_k k! \left( \frac{\beta_0 \alpha_s}{2 \pi} \right)^k$$

due to the $1/m_Q$ infrared (IR) renormalon. This series gives rise to an unavoidable uncertainty which is linear rather than quadratic in $1/m_b$. The related perturbative corrections are significant already in low orders. They are directly associated with running of $\alpha_s$ and their growth reflects the fact that the strong coupling evolves into the nonperturbative domain at a low enough scale. A strategy allowing one to circumvent this potentially large uncertainty was indicated in Refs. [8, 9] – instead of the pole quark mass one should pass to a Euclidean mass, according to Wilson’s OPE. This mass is peeled off from the IR part; the IR domain is also absent from the width up to effects $1/m_Q^2$, and the (IR-related) factorial divergence disappears. Following this observation, it became routine to express $\Gamma_{\text{sl}}(B)$ in terms of the Euclidean quark masses in the $\overline{\text{MS}}$ scheme, say, $\overline{m}_b(m_b)$.

Killing IR renormalons on this route, however, one does not necessary have a fast convergent perturbative series: in general, it contains corrections of the type $(n\alpha_s)^k$ which are not related to running of $\alpha_s$ – they are present even for the vanishing $\beta$ function. We will see that such large corrections inevitably appear if one
works with the $\overline{\text{MS}}$ masses like $\overline{m}_b(m_b)$ and $\overline{m}_c(m_c)$. Due to the high value of the power $n$ they constitutes quite an obvious menace to the precision of the theoretical predictions. The standard alternative procedure of treating inclusive heavy quark decays, which gradually evolved in the quest for higher accuracy, is thus plagued by its own problems.

The aim of the present work is to turn vices into virtues, by treating $n$ as a free parameter and developing a $1/n$ expansion. In this respect our approach is conceptually similar to the $1/N_c$ expansion or the expansion in the dimensions of the space-time, etc. which are quite common in various applications of field theory (for a discussion of the virtues of expansion in ‘artificial’ parameters, see [11]). The main advantage is the emergence of a qualitative picture which guides the theoretical estimates in the absence of much more sophisticated explicit higher-order calculations.

Certain unnaturalness of the $\overline{\text{MS}}$ masses normalized, say, at $\mu = m_b$, for inclusive widths is rather obvious before a dedicated analysis, in particular in $b \to c \ell \nu$ inclusive decays. The maximal energy fed into the final hadronic system and available for exciting the final hadronic states, which determines the “hardness” of the process, is limited by $m_b - m_c \simeq 3.5$ GeV. Moreover, in a typical decay event, leptons carry away a significant energy $E_{\ell \nu} > \sqrt{q^2}$, since the lepton phase space emphasizes the larger $q^2$. This is illustrated in Fig. 1 where the distribution over invariant mass of the lepton pair $\sqrt{q^2}$ and energy release $E_r = m_b - m_c - \sqrt{q^2}$ is shown. The situation is less obvious a priori for $b \to u$, but again the typical energy of the final state hadrons is manifestly smaller than $m_b$. We use the parameter $n$ to quantify these intuitive observations.

Since the power $n = 5$ is of a purely kinematic origin, technically it is quite easy to make $n$ a free parameter. To this end it is sufficient, for instance, to modify the lepton current by introducing fictitious additional leptons emitted by the $W$ boson, along with the standard $\ell \nu$ pair. At the very end the number of these fictitious leptons is to be put to zero.
The emergence of the large perturbative terms containing powers of $n$ in a general calculation is rather obvious and can be illustrated in the following way. Consider, for example, the $b \rightarrow u$ decay rate, and use the pole mass $m_{b\text{pole}}$. The series will have a factorial divergence in high orders due to the $1/m$ IR renormalon, but we are not concerned about this fact now since we reside in purely perturbative domain and we do not study high orders $k \sim 1/\alpha_s$ of the perturbation theory. We can consider, for example, the case of vanishing $\beta$ function when nothing prevents one from defining the pole mass with the arbitrary precision.

Let us assume then that using the pole mass, $m_{b\text{pole}}$, the perturbative expansion of the width

$$\Gamma \simeq d_n m_{b\text{pole}}^n A^{\text{pt}}(\alpha_s) = d_n m_{b\text{pole}}^n \left[ 1 + a_1 \frac{\alpha_s}{\pi} + a_2 \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots \right]$$

(1)

has all coefficients $a_k$ completely $n$-independent (we will show below that it is the case in the leading-$n$ approximation). Then, being expressed in terms of a different mass $\tilde{m}_b$,

$$\tilde{m}_b = m_b \left( 1 - c \frac{\alpha_s}{\pi} \right)$$

(2)

one has

$$\Gamma \simeq d_n \tilde{m}_b^n \tilde{A}^{\text{pt}}(\alpha_s)$$

(3)

with

$$\tilde{A}^{\text{pt}}(\alpha_s) = \frac{A^{\text{pt}}(\alpha_s)}{(1 - c \frac{\alpha_s}{\pi})^n} = 1 + (nc + a_1) \frac{\alpha_s}{\pi} + \left( \frac{n(n+1)}{2} c^2 + nca_1 + a_2 \right) \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots .$$

(4)

Just using a different mass generates $n$-enhanced terms $\sim (n\alpha_s)^k$. In the case of the $\overline{\text{MS}}$ mass one has, basically, $c = 4/3$. In order to have a good control over the perturbative corrections in the actual width, one needs to resum these terms (at least, partially). This can be readily done.

These $n$-enhanced perturbative corrections are only one, purely perturbative, aspect of a general large-$n$ picture. The total width is determined by integrating $n$-independent hadronic structure functions with the $n$-dependent kinematic factors, the latter being saturated at large $\sqrt{q^2}$ close to the energy release $m_b - m_c$. As a result, the energy scale defining the effective width of integration, is essentially smaller than the energy release; a new, lower momentum scale automatically emerges in the problem at large $n$.

The question of the characteristic scale in inclusive decays was discussed in the literature. The discussion was focused, however, almost exclusively on the problem of choosing the normalization point for the running coupling $\alpha_s$. We emphasize that for the inclusive decays it is a secondary question; the principal one is the normalization point for quark masses which run as well. Although their relative
variation is smaller, this effect is enhanced, in particular, by the fifth power occurring in the width.

With \( n = 5 \) treated as a free parameter, we arrive at the following results concerning semileptonic \( B \) decays driven by \( b \to l\nu q, \ q = c \) or \( u \):

1. The leading \( n \)-enhanced perturbative corrections to the total width are readily resummed by using the low-scale quark masses. No significant uncertainty in the perturbative corrections is left in the widths.

2. The surprising proximity of the \( b \to c\ell \nu \) transition to the small velocity (SV) limit \([12]\) which seems to hold in spite of the fact that \( m_c^2/m_b^2 \ll 1 \), becomes understood. The typical kinematics of the transition are governed by the parameter \((m_b - m_c)/nm_c\) rather than by \((m_b - m_c)/m_c\). This “extended SV” parameter shows why the SV expansion is relevant for actual \( b \to c \) decays.

3. The ‘large-\( n \)’ remarks are relevant even if \( \alpha_s \) does not run. In actual QCD, the better control of the perturbative effects meets a conflict of interests: the normalization point for masses cannot be taken either very low or too high, thus neither \( \overline{\text{MS}} \) nor pole masses are suitable. We discuss a proper way to define a short-distance heavy quark mass \( m_q(\mu) \) with \( \mu \ll m_Q \) which is a must for nonrelativistic expansion in QCD. A similar definition of the effective higher-dimension operators is also given.

4. The limits \( m_Q \to \infty \) and \( n \to \infty \) cannot freely be interchanged. Quite different physical situations arise in the two cases, as indicated later. The perturbative regime takes place if \((m_b - m_c)/n \gg \Lambda_{\text{QCD}}\), and the process is not truly short-distance if the opposite holds. Nevertheless, at \( m_c \gg \Lambda_{\text{QCD}} \) even in the latter limit the width is determined perturbatively up to effects \( \sim \Lambda_{\text{QCD}}^2/m_c^2 \).

We hasten to add that analyzing only terms leading in \( n \) may not be fully adequate for sufficiently accurate predictions in the real world where \( n = 5 \), which turns out to be not large enough. Still, it is advantageous to start from the large \( n \) limit and subsequently include (some of the) subleading terms.

## 2 The theoretical framework: large-\( n \) expansion in inclusive semileptonic decays

The new tool we bring to bear here is the following: the actual value \( n = 5 \) of the power \( n \) in the width arises largely through the integration over the phase space for the lepton pair. One can draw up simple physical scenarios where \( n \) appears as a free parameter and the limit \( n \to \infty \) can be analyzed. The simplest example is provided by \( l \) (massless) scalar “leptons” \( \phi \), emitted in the weak vertex of the
lepton-hadron interaction,

\[ \mathcal{L}_{\text{weak}} = \frac{G_l}{\sqrt{2}} V_{Qq} \phi^i \bar{\ell} \gamma_\alpha (1 - \gamma_5) \nu_l \overline{\nu} \gamma^\alpha (1 - \gamma_5) Q, \]  

(5)

where \(G_l\) generalizes the Fermi coupling constant to \(l \neq 0\). At \(l = 0\) the coupling \(G_0 = G_F\). Then by dimensional counting one concludes that \(\Gamma(Q) \sim |G_l|^2 \cdot m_Q^n\), where \(n = 2l + 5\). It should be kept in mind that the details of the physical scenario are not essential for our conclusions.

We will discuss \(b \to q\) transitions with an arbitrary mass ratio, \(m_q/m_b\), thus incorporating both \(b \to u\) and \(b \to c\) decays. Following Refs. [2, 13] we introduce a hadronic tensor \(h_{\mu\nu}(q_0, q^2)\), its absorptive part \(W_{\mu\nu}(q_0, q^2) = \frac{1}{i} \text{disc} h_{\mu\nu}(q_0, q^2)\) and its decomposition into five covariants with structure functions \(w_i(q_0, q^2), i = 1, \ldots, 5\).

Variables \(q_0\) and \(q^2\) are the energy and the effective mass of lepton pair. Only \(w_1\) and \(w_2\) contribute when lepton masses are neglected and the semileptonic decay width is then given by

\[ \Gamma_{\text{sl}} \equiv |V_{cb}|^2 \frac{G_F^2}{8\pi^3} \cdot \gamma, \]

(6)

where

\[ q^2_{\text{max}} = (M_B - M_D)^2, \quad q_0_{\text{max}} = \frac{M_B^2 + q^2 - M_D^2}{2M_B}. \]

Note that the upper limits of integration over \(q^2\) and \(q_0\) are determined by vanishing of the structure functions for the invariant mass of the hadronic system less than \(M_D^2\). The lower limit of integration over \(q_0\) and \(q^2\) is due to the constraints on the momentum of the lepton pair. It has nothing to do with the properties of the structure functions. In particular, the structure functions exist also in the scattering channel where the constraint on the lepton momentum would be different. In what follows we will see that the distinction between the leptonic and hadronic kinematical constraints is important.

By adding \(l\) extra scalar “leptons” emitted in the weak vertex we make \(n = 5 + 2l\) a free parameter. The quantity \(\gamma\) then becomes \(n\) dependent, \(\gamma \to \gamma(n)\),

\[ \gamma(n) = \frac{1}{2\pi} \int_0^{q^2_{\text{max}}} dq^2 \int_{q_0_{\text{max}}}^{q_0_{\text{max}}} dq_0 \sqrt{q_0^2 - q^2} \left\{ q^2 w_1(q_0, q^2) + \frac{1}{3} (q_0^2 - q^2) w_2(q_0, q^2) \right\}, \]

(7)

In the real world \(l = 0, n = 5\), but as far as the QCD part is concerned we are free to consider any value of \(l\). The extra factor \((q^2)^l\) appeared due to the phase space
of the scalar “leptons” 1

A different choice of variables is more convenient for our purposes. Instead of $q_0$ and $q^2$ we will use $\epsilon$ and $T$,

$$\epsilon = M_B - q_0 - \sqrt{M_B^2 + q_0^2 - q^2}, \quad T = \sqrt{M_B^2 + q_0^2 - q^2 - M_D}.$$

(8)
The variable $T$ is simply related to the spatial momentum $\vec{q}$. More exactly, $T$ has the meaning of the minimal kinetic energy of the hadronic system for the given value of $\vec{q}$. This minimum is achieved when the $D$ meson is produced. The variable $\epsilon$ has the meaning of the excitation energy, $\epsilon = (M_X - M_D) + (T_X - T)$ where $T_X = \sqrt{M_X^2 + q_0^2 - q^2 - M_X}$ is the kinetic energy of the excited state with mass $M_X$. The following notations are consistently used below:

$$\Delta = M_B - M_D, \quad \vec{q}^2 = q_0^2 - q^2, \quad |\vec{q}| = \sqrt{\vec{q}^2}.$$

In terms of the new variables one obtains

$$\gamma(n) = \frac{1}{\pi} \int_0^{T_{\text{max}}} dT (T+M_D) \sqrt{T^2 + 2M_DT} \int_0^{\epsilon_{\text{max}}} d\epsilon \left( \Delta^2 - 2M_BT - 2\epsilon + 2T\epsilon + \epsilon^2 \right)^l \left\{ \left( \Delta^2 - 2M_BT - 2\epsilon + 2T\epsilon + \epsilon^2 \right) w_1 + \frac{1}{3} (T^2 + 2M_DT) w_2 \right\}$$

(9)

where

$$T_{\text{max}} = \frac{\Delta^2}{2M_B}, \quad \epsilon_{\text{max}} = \Delta - T - \sqrt{T^2 + 2M_DT}.$$

It is implied that the structure functions $w_{1,2}$ depend on $\epsilon$ and $T$.

2.1 Cancelation of the infrared contribution

Before submerging in the large $n$ limit we will address a question which naturally comes to one’s mind immediately upon inspection of Eq. (6) or Eq. (7). Indeed, these expressions give the total decay probabilities in terms of an integral over the physical spectral densities over the physical phase space. Both factors depend on the meson masses, and know nothing about the quark mass. This is especially clear in the case of the phase space, which seems to carry the main dependence for large $n$. And yet, in the total probability the dependence on the meson masses must disappear, and the heavy quark mass must emerge as a relevant parameter. In other words, the large-distance contributions responsible for making the meson mass out of the quark one must cancel each other.

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1 Strictly speaking, the extra “leptons” produce a change in the leptonic tensor, in particular, making it non-transversal. This variation leads, in turn, to a different relative weight of the structure function $w_2$, and the emergence of the structure functions $w_4$ and $w_5$ in the total probability. These complications are irrelevant, and we omit them. We also omit an overall $l$ dependent numerical factor, which merely redefines $G_l$. 

6
It is instructive to trace how this cancelation occurs. We will study the issue for arbitrary value of \( l \), assuming for simplicity that the final quark mass vanishes (i.e. \( b \to u \) transition). This assumption is not crucial; one can consider arbitrary ratio \( m_c/m_b \), as a result of cancelation of the infrared contributions residing in \( M_B \) and in the integrals over the spectral densities, is the heavy flavor sum rules presented in great detail in Ref. [14], see Eqs. (130) – (135).

We start from expanding the integrand in Eq. (9) in \( \epsilon \), keeping the terms of the zeroth and first order in \( \epsilon \),

\[
\gamma(n) = \frac{1}{\pi} \int_0^{M_B/2} dT T^2 M_B^l(M_B - 2T)^l \times \\
\left[ M_B(M_B - 2T) \int_0^{M_B - 2T} d\epsilon w_1 - 2(l + 1)(M_B - T) \int_0^{M_B - 2T} d\epsilon \epsilon w_1 \right] + \\
\frac{1}{3\pi} \int_0^{M_B/2} dT T^4 M_B^{l-1}(M_B - 2T)^l \times \\
\left[ M_B(M_B - 2T) \int_0^{M_B - 2T} d\epsilon w_2 - 2l(M_B - T) \int_0^{M_B - 2T} d\epsilon \epsilon w_2 \right].
\]  

(10)

The integrals over \( \epsilon \) are given by the sum rules mentioned above,

\[
\frac{1}{2\pi} \int d\epsilon w_1 = 1, \quad \frac{1}{2\pi} \int d\epsilon \epsilon w_1 = M_B - m_b = \overline{\Lambda},
\]  

(11)

and

\[
\frac{1}{2\pi} \int d\epsilon w_2 = \frac{2m_b}{T}, \quad \frac{1}{2\pi} \int d\epsilon \epsilon w_2 = \frac{2m_b}{T}\overline{\Lambda}.
\]  

(12)

The expressions (11) and (12) imply that the integration over \( \epsilon \) is saturated at small \( \epsilon \), of order \( \overline{\Lambda} \). After substituting the sum rules in Eq. (10) and integrating over \( T \) we arrive at

\[
\gamma(n) = \frac{M_B^{2l+5}}{2(l + 4)(l + 3)(l + 2)} \left[ \left( 1 - (2l + 5)\frac{\overline{\Lambda}}{M_B} \right) + \right] + \\
\frac{m_b M_B^{2l+4}}{2(l + 4)(l + 3)(l + 2)(l + 1)} \left[ \left( 1 - (2l + 4)\frac{\overline{\Lambda}}{M_B} \right) \right].
\]  

(13)

The first term here comes from \( w_1 \) and the second from \( w_2 \). This expression explicitly demonstrates that the meson mass is substituted by the quark one at the level of \( 1/m \) terms under consideration, \( M_B(1 - \overline{\Lambda}/M_B) = m_b \).

Thus, the cancelation of the infrared contribution is evident. It is worth emphasizing that the infrared contributions we speak of need not necessarily be of nonperturbative nature. The cancelation of infrared \( 1/m_b \) effects takes place for
perturbative contributions as well as long as they correspond to sufficiently small momenta, $\lesssim m_b/n$.

Let us parenthetically note that Eq. (13) is valid for any $l$; in particular, we can put $l$ equal to zero. One may wonder how one can get in this case a non-vanishing correction associated with $\int d\epsilon \epsilon w^2$, while the weight factor for $w^2$ in the original integrand (9) at $l=0$ seemingly has no dependence on $\epsilon$ at all. This case is actually singular: the domain of $T \rightarrow M_B/2$ contributes to the integral at the level $\Lambda_{\text{QCD}}/M_B$. For such $T$ the interval of integration over $\epsilon$ shrinks to zero, but nevertheless the integral of $w^2$ stays finite (unity) according to Eq. (12); this signals a singularity. It is directly seen from Eq. (10) where at $l=0$ one has logarithmic divergence at $T \rightarrow M_B/2$. The most simple correct way to deal with $w^2$ at $l=0$ is inserting a step-function $\theta(M_B-2T-\epsilon)$ in the integrand instead of the upper limits of integration. This step-function does provide an $\epsilon$ dependence, the term linear in $\epsilon$ is $\epsilon \delta(M_B-2T)$. At non-vanishing values of $l$ the $\delta$-function above produces no effect. As usual, analytic continuation from the nonsingular case $l > 0$ to $l = 0$ leads to the same result.

In principle, it is instructive to trace the $l$ dependence of other non-perturbative terms, e.g. those proportional to $\mu^2$ and $\mu^2_G$ which are known for $b \rightarrow u$ at arbitrary $l$ from explicit calculations of Refs. [4, 5, 6] and have a very simple form. We will not dwell on this issue here.

Our prime concern in this work is the interplay of the infrared effects in the perturbative corrections. The main new element appearing in the analysis of the perturbative corrections is the fact that the integrals over $\epsilon$ are not saturated in the domain $\epsilon \sim \Lambda$. That is why we need to introduce the normalization point $\mu$ which will divide the entire range of the $\epsilon$ integration in two domains, $\epsilon > \mu$ and $\epsilon < \mu$.

The domain $\epsilon < \mu$ provides us with a clear-cut physical definition of $\Lambda$, which becomes $\mu$ dependent,

$$
\Lambda(\mu) + \frac{\mu^2_G}{2|\vec{q}|} + \frac{\mu^2 - \mu^2_G}{3|\vec{q}|} \left(1 - \frac{|\vec{q}|}{m_b}\right) + O\left(\frac{\Lambda_{\text{QCD}}^3}{|\vec{q}|^2}\right) = \frac{1}{f_0^\mu} \int d\epsilon \, w_1^{b \rightarrow u}(\epsilon, |\vec{q}|) + \int d\epsilon \, w_1^{b \rightarrow u}(\epsilon, |\vec{q}|) ,
$$

where the variable $\epsilon$ is defined by Eq. (8) with $M_D$ set equal to zero, the second variable $T = |\vec{q}|$ in the $b \rightarrow u$ transition. Equation (14) is a consequence of the sum rules derived in Ref. [14]. There, the integrals over the spectral densities are given in the form of a “condensate” expansion. The currents inducing given spectral densities (and the spectral densities themselves) acquire certain normalization factors due to short-distance renormalization of the weak vertex, which are irrelevant for our present purposes. To get rid of these normalization factors we consider the ratio of the sum rules. The power corrections to the denominator in the right-hand side of Eq. (14) shows up only in $O\left(\frac{\Lambda_{\text{QCD}}^3}{|\vec{q}|^2}\right)$ terms.

The quark mass, $m_b(\mu) = M_B - \Lambda(\mu)$, becomes in this way a well-defined and experimentally measurable quantity. A very similar definition of $\Lambda(\mu)$ does exist also in the $b \rightarrow c$ transition, through the integral over $w_1^{b \rightarrow c}$, see Ref. [14]. This definition generalizes the Voloshin sum rule [13] to the arbitrary values of $\vec{q}$, and
includes corrections $O(\Lambda_{\text{QCD}}^2/m)$. See Section 4 for the further discussion of $\overline{\Lambda}(\mu)$ definition.

### 2.2 The large $n$ limit

Now when we understand how the infrared parts cancel, we are ready to address the practical issue of what particular normalization point $\mu$ is convenient to use in the analysis of the total widths. Certainly, the theoretical predictions can be given for any value of $\mu \gg \Lambda_{\text{QCD}}$. Our goal is to choose $\mu$ in such a way that the domain of momenta above $\mu$ does not give the enhanced perturbative corrections. Then all perturbative contributions enhanced by large $n$ will be automatically included in the definition of $m_b(\mu)$.

To this end we invoke the large $n$ limit, as was explained in the Introduction. For $l \gg 1$ one has

\[
(\Delta^2 - 2M_B T - 2\Delta \epsilon + 2T \epsilon + \epsilon^2)^l \simeq \Delta^{2l} \exp \left[-2l \left(\frac{T M_B}{\Delta^2} + \frac{\epsilon}{\Delta} - \frac{T \epsilon}{\Delta^2} - \frac{\epsilon^2}{2\Delta^2}\right)\right],
\]

and the dominant domains in the integration variables are given by

\[
T \lesssim \frac{1}{l} \frac{\Delta^2}{M_B}, \quad \epsilon \lesssim \frac{1}{l} \Delta.
\]

All expressions then simplify, and the two integrations decouple

\[
\gamma(n) = \frac{1}{8\pi} \Delta^{8+2l} \times
\]

\[
\left\{ \int_0^\infty d\tau \frac{\Delta}{12M_B^2} \right\} \left( \int_0^\infty d\epsilon e^{-2l t \frac{2}{2M_B}} w_1 \left( \epsilon, \frac{\tau \Delta^2}{2M_B} \right) + \int_0^\infty d\tau \frac{\Delta}{12M_B^2} \right\}
\]

where

\[
\tau = \frac{2M_B T}{\Delta^2}, \quad \tau_0 = \frac{2M_B M_B}{\Delta^2}.
\]

The integrals over the excitation energy $\epsilon$

\[
J_i(T; \sigma) \equiv \frac{1}{2\pi} \int_0^\infty d\epsilon e^{-\frac{\epsilon}{2}} w_i(\epsilon, T)
\]

are a combination of the sum rule considered in Refs. [13, 14]. They are related to the Borel transform of the forward scattering amplitude of the weak current off the $B$ meson. The quantity $\sigma$ is the Borel parameter, and expanding $J_i(T; \sigma)$ in $1/\sigma$ gives us the first, second, and so on sum rules of Ref. [14]. The Borelized version has an advantage, however, of providing an upper cut off in $\epsilon$ in a natural way. This
is important in the analysis of the perturbative corrections; the second argument of \( J \) defines the normalization point.

One readily reads off from Eq. (15) that only a fraction \( \sim 1/n \) of the total energy release is fed into the final hadronic system. The three-momentum carried by the final state hadrons is likewise small, namely of order \( \sqrt{m_c \Delta}/n \) for \( b \to c \) and \( \Delta/n \) for \( b \to u \), respectively; the latter case is obtained from the former by setting \( M_D = 0 \). This implies that by evaluating the various quantities at a scale \( \sim \Delta/n \) rather than \( \Delta \) one includes the potentially large higher order corrections. These conclusions can be made more transparent by considering two limiting cases.

(a) If \( m_c \) stays fixed, the limit \( n \to \infty \) leads to the SV regime, as is expressed by \( v \sim (m_b - m_c)/nm_c < 1 \). Neglecting \( \tau \) compared to \( \tau_0 \), one gets

\[
\gamma(n) = \Delta^{2l+5} \left( \frac{M_D}{M_B} \right)^{3/2} \int_0^\infty d\tau \tau^{l+1} e^{-l(l+1)\tau} J_1 \left( \frac{\tau \Delta^2}{2M_B}; \frac{\Delta}{2(l+1)} \right) + \frac{1}{3} \Delta^{2l+5} \left( \frac{M_D}{M_B} \right)^{5/2} \int_0^\infty d\tau \tau^{3/2} e^{-l\tau} J_2 \left( \frac{\tau \Delta^2}{2M_B}; \frac{\Delta}{2l} \right). \tag{17}
\]

The behavior of \( \Gamma_\text{sv} \) at \( n \to \infty \) then is determined by the integrals \( J_{1,2} \) near zero argument. The second term is clearly subleading because the weight function contains an extra power of \( \tau \). It is not difficult to see that the axial current contribution is dominant, and one obtains for the reduced width

\[
\gamma(n)|_{n \to \infty} \simeq \Delta^n \sqrt{2\pi} \left( \frac{M_D}{M_B} \right)^{3/2} \frac{1}{n^{3/2}} \xi_A \left( \frac{\Delta}{n} \right) \tag{18}
\]

where \( \xi_A(\mu) = 1 + \mathcal{O}(\alpha_s(\mu)) \) is the coefficient function of the unit operator in the first sum rule for the axial current at zero recoil (for further details see Ref. [14]). To the degree of accuracy pursued here (power corrections are switched off so far) one has \( \xi_A \simeq \eta_A^2 \) where the factor \( \eta_A \) incorporates the perturbative corrections to the axial current at zero recoil. Eq. (18) provides a decent approximation to the true width even for \( n = 5 \). The vector current contribution is subleading; to take it into account at \( n = 5 \) we need a refined expansion to be described in Appendix.

We note that the leading-\( n \) expansion of the width yields \( (M_B - M_D)^n \) which differs from \( (m_b - m_c)^n \) by terms \( (n \alpha_s)^k \) that we target, but coincides to this accuracy with \( (m_b(\mu) - m_c(\mu))^n \) if \( \mu \lesssim m_b/n \),

\[
m_b(\mu) - m_c(\mu) \simeq (M_B - M_D) + (m_b - m_c) \frac{\mu^2(\mu) - \mu^2_c(\mu)}{2m_b m_c} + \ldots
\]

\[
\simeq (M_B - M_D) \left( 1 - \frac{4\alpha_s}{3\pi} \frac{\mu^2}{2m_b m_c} + \ldots \right). \tag{19}
\]

\( ^2 \)The quantity \( \eta_A^2 \equiv \lim_{\mu \to 0} \xi_A^{\text{pert}}(\mu) \) is ill-defined once power corrections are addressed; at the same time \( \xi_A(\mu) \) is even then a well defined quantity provided that \( \mu \gg \Lambda_{\text{QCD}} \). Yet, to any finite order in perturbation theory, one can work with \( \eta_A \).
In the last line we used the perturbative expression for $\mu^2(\mu) - \mu^2_{\xi}(\mu)$.

(b) Another case of interest is $m_q \ll m_b/n$, when the final-state quark is ultrarelativistic. In this case the axial and vector currents contribute equally. The reduced width now takes the form
\[
\gamma(n) = \frac{1}{4} M_B^n \int_0^\infty \tau \tau^2 e^{-(l+1)\tau} J_1 \left( \frac{\tau M_B}{2} ; \frac{M_B}{2(l+1)} \right) .
\]
(20)
The quantity $J_1(\tau M_B/2)$ stays finite (unity at the tree level) at $\tau \to 0$. The second structure function again formally yields only subleading in $1/n$ contributions. Thus
\[
\gamma_n(b \to u)|_{n \to \infty} \simeq M_B^n \frac{4}{n^3} \xi_u(M_B/n) ,
\]
where
\[
\xi_u(n) = J_1 \left( \frac{M_B}{n} ; \frac{M_B}{n} \right) ; \quad \xi_{u\text{tree}} = 1 .
\]
(22)
This width decreases faster with $n$ than for massive quarks due to the higher power of $|\vec{q}| \sim m_b/n$. This underlies the fact that the numerical factor in $\Gamma_{sl}$ in front of $\Delta^5$ is much smaller for $b \to u$ than in the SV limit, namely, 1/192 vs. 1/15. For the same reason the simple expansion described above provides a poor approximation for $b \to u$ when evaluated at $l = 0$, i.e. $n = 5$.

(c) One can consider the third large-$n$ regime when $m_c/m_b \sim 1/n$. Although it appears to be most close to the actual situation, we do not dwell on it here: the analysis goes in the very same way, but expressions are more cumbersome since one has to use the full relativistic expression for the kinetic energy of the $D$ meson. No specific new elements appear in this case.

(d) Let us discuss now the normalization point for quark masses at large $n$. Eqs. (18), (22) show the kinematically-generated dependence on the masses. The non-trivial hadronic dynamics are encoded in the factors $\xi_A$ and $\xi_u$. Our focus is the heavy quark masses. In the SV case (a) the dependence on them is trivial as long as the low-scale masses are employed. In the case of the $b \to u$ transitions we showed in Sect. 2.1 that the factor $\xi_u(M_B/n)$ actually converted $M_B^n$ into $m_b(\mu)^n$, and, thus, effectively established the normalization point for the latter, $\mu \lesssim m_b/n$. Using the effective running mass $m_b(\mu \sim m_b/n)$ does not incorporate, however, the domain of the gluon momenta above $\sim m_b/n$. This contribution has to be explicitly included in the perturbative corrections. It is not enhanced by powers of $n$. Physically, it is nothing but a statement that for $\Gamma(b \to u)$ the proper normalization scale for masses is $\mu \sim m_b/n$.

We hasten to emphasize that the statement above is not a question of normalization of $\alpha_s$ used in the perturbative calculations. One can use $\alpha_s$ at any scale to evaluate $m_b(\mu)$ as long as this computation has enough accuracy. The fact that
using inappropriate $\alpha_s$ can lead to an apparent instability in $m_b(\mu)$, is foreign to the evaluation of the width.

Let us summarize the main features which are inferred for inclusive widths by analyzing the straightforward large-$n$ expansion introduced via Eq. (13). The integral over the lepton phase space carries the main dependence on $n$. The QCD corrections – the real theoretical challenge – are only indirectly sensitive to $n$: for the kinematics determines the energy and momentum scales at which the hadronic part has to be evaluated.

The characteristic momentum scale $\mu$ for the inclusive decay width is smaller than the naive guess $\mu \sim m_b$. In the large $n$ limit it scales like $m_b/n$. This momentum defines the relevant domain of integration of the structure function in $q_0$, i.e. the scale at which the forward transition amplitude appears in the total decay rate. In particular, by evaluating the quark masses that determine the phase space at this scale $\sim m_b/n$ one eliminates the strongest dependence of the radiative corrections on $n$.

The expansion in $n$ derived from Eq. (13) allows a transparent discussion of the underlying physics. Unfortunately it yields, as already stated, a decent numerical approximation only for very large values of $n$; i.e., for $n = 5$ the non-leading terms are still significant. Not all those terms are dominated by kinematical effects, and their treatment poses non-trivial problems. A more refined expansion can be developed that effectively includes the kinematics-related subleading contributions and leads to a good approximation already for $n = 5$. This treatment, however, is rather cumbersome and less transparent. It will be briefly described in the Appendix. The purpose is only to demonstrate that the essential features of our expansion leading to the qualitative picture we rely on – that the characteristic momentum scale is essentially lower than $m_b$ – hold already for the actual case $n = 5$. The reader who is ready to accept this assertion, can skip the technicalities of the refined $1/n$ expansion in the Appendix.

3 Applications

We now briefly consider the consequences of the $1/n$ expansion for a few problems of interest.

3.1 The extended SV limit

In many respects one observes that the inclusive $b \to c$ decays seem to lie relatively close to the SV limit. Most generally, the various characteristics of the decay depend on the ratio $m_c^2/m_b^2$; although it is rather small, the actual characteristics often do not differ much from the case where it approaches 1, when the average velocity of the final-state $c$-quark was small. The proximity of the inclusive $b \to c$ decays to the SV limit has two aspects. First, it is obvious that the nonperturbative effects
work to suppress the effective velocity of the final state hadrons; the most obvious changes are kinematical replacement $m_b \rightarrow M_B$ and $m_c \rightarrow M_{D,D^*,D^{**}}$ when passing from quarks to actual hadrons. The impact of increasing the mass is much more effective for the final-state charm than for the initial-state beauty. This decreases the velocities of the final state hadrons significantly at $\Delta = m_b - m_c \simeq 3.5 \text{ GeV}$. Yet this simple effect would not be numerically large enough were it not considerably enhanced by the properties of the lepton phase space, which can be easily seen comparing it, for example, with the one for semileptonic decays at fixed $q^2 = 0$.

On the other hand, without any nonperturbative effects, theoretical expressions at the purely parton level are known to favor the SV kinematics even for actual quark masses. The simplest illustration is provided by the tree-level phase space,

$$z_0(m_b, m_c) = m_b^5 \left( 1 - 8 \frac{m_c^2}{m_b^2} - 12 \frac{m_c^4}{m_b^4} \log \frac{m_c^2}{m_b^2} + 8 \frac{m_c^6}{m_b^6} - \frac{m_c^8}{m_b^8} \right). \quad (23)$$

It is most instructive to analyze the sensitivity of this expression to $m_b$ and $\Delta = m_b - m_c$ rather than $m_b$ and $m_c$, as expressed through

$$\kappa_\Delta \equiv \frac{\Delta}{z_0} \frac{\partial z_0(m_b, \Delta)}{\partial \Delta}, \quad \kappa_b \equiv \frac{m_b}{z_0} \frac{\partial z_0(m_b, \Delta)}{\partial m_b}, \quad (24)$$

where

$$\kappa_\Delta + \kappa_b = 5.$$  

In the light quark limit $-m_c^2/m_b^2 \rightarrow 0$ one has $z_0|_{\text{light}} = m_b^5 \left( 1 - \mathcal{O} \left( \frac{m_c^2}{m_b^2} \right) \right)$ and, thus

$$\kappa_\Delta|_{\text{light}} = 0, \quad \kappa_b|_{\text{light}} = 5. \quad (25)$$

In the SV limit, on the other hand, one finds

$$z_0|_{\text{SV}} \simeq \frac{64}{5} (m_b - m_c)^5. \quad (26)$$

Therefore, the SV limit is characterized by

$$\kappa_\Delta|_{\text{SV}} = 5, \quad \kappa_b|_{\text{SV}} = 0 \quad (27)$$

For actual quark mass values $-m_c/m_b \simeq 0.28$ one finds

$$\kappa_\Delta \simeq 3, \quad \kappa_b \simeq 2 \quad (28)$$

i.e., even for $m_c^2/m_b^2 \simeq 0.08 \ll 1$ one is still closer to the SV than the light-quark limit. The “half-way” point $-\kappa_\Delta = \kappa_b = 2.5$ lies at $m_c^2/m_b^2 \simeq 0.05$! This is a consequence of the large parameter $n = 5$. A similar pattern persists also for the low-order perturbative corrections \cite{18, 19}, and on the level of the $1/m^2$ power corrections \cite{4, 6, 7}.

The relevance of the SV approximation in the actual inclusive decays thus finds a rational explanation in the $1/n$ expansion.
3.2 Resummation of large perturbative corrections

In computing the perturbative corrections for large \( n \) one can potentially encounter large \( n \)-dependent corrections of the type \((n\alpha_s/\pi)^k\), which makes them sizable for the actual case \( n = 5 \). The related practical concern of certain numerical ambiguity of the one-loop calculations of the widths was raised in Ref. [20]. Fortunately, the leading subseries of these terms can readily be summed up. The summation essentially amounts to using the quark masses normalized at a low scale, \( \mu \sim (m_b, \Delta)/n \). As long as one does not go beyond a relatively low order in \( \alpha_s \) one can ignore renormalon divergences and simply use the “\( k \)-th order pole mass”. Quite obviously, the terms \( \sim (n\alpha_s/\pi)^k \) appear only if one uses deep Euclidean masses. If the result is expressed in terms of low-scale Euclidean masses (i.e. those normalized at \( \mu \sim \Delta/n \) or \( \mu = \text{several units} \times \Lambda_{\text{QCD}} \)) the terms \( \sim (n\alpha_s/\pi)^k \) enter with coefficients proportional to \( \mu/\Delta \) and, therefore, contribute only on the subleading level.

The above assertion is most easily inferred by applying Eq. (15) to \( w_i \) computed through order \( k \) (with \( k < n \)); they are self-manifest in Eqs. (18) and (22). Upon using the thresholds in the perturbative \( w_i \) to define \( m_b \) and \( m_c \), the moments of \( w_i \) are smooth and independent of \( n \) in the scaling limit \( m, \Delta \sim n \), which ensures the absence of the leading power of \( n \). The perturbative thresholds, on the other hand, are given just by the pole masses as they appear in the perturbation theory to the considered order.

It is worth clarifying why these terms emerge if one uses different masses. The reason is that the moments of the structure functions (or more general weighted averages in the ‘refined’ expansion) intrinsically contain the reference point for the energy \( q_0 \) and \( q^2_{\text{max}} \). At \( q^2 > q^2_{\text{max}} \) the integral over \( q_0 \) merely vanishes whereas for \( q^2 < q^2_{\text{max}} \) it is unity plus corrections. A similar qualification applies to evaluation of the integral over the energy \( q_0 \) itself. These properties which single out the proper ‘on-shell’ masses, were tacitly assumed in carrying out the expansion in \( 1/n \).

On the formal side, if one uses a mass other than the perturbative pole mass, the perturbative \( w_i \) contain singular terms of the type

\[
\delta(\epsilon - \delta m) = \delta(\epsilon) + \sum_{k=1}^{\infty} (-\delta m)^k \frac{\delta^{(k)}(\epsilon)}{k!}.
\]  

where \( \delta m \sim \alpha_s \cdot \Delta \) is the residual shift in the energy release. (A similar shift in the argument of the step-functions emerges for the perturbative continuum contributions.) Each derivative of the \( \delta \)-function generates, for example, a power of \( n \) in the integral over \( \epsilon \) in the representation (13). For \( b \to u \) widths considered in detail in Sect. 2, using \( \overline{m}_b(m_b) \) would yield \( -\delta m \sim \frac{4\alpha_s}{3\pi} m_b \) leading to the series \( \left(\frac{4\alpha_s}{3\pi}\right)^k \overline{m}_b^k(\mu) \) with \( \mu \ll m_b \) which sums into \( e^{n\alpha_s/3\pi} \) and converts \( \overline{m}_b(m_b)^n \) into \( m_b^n(\mu) \) with \( \mu \ll m_b \).

A more detailed analysis can be based on the refined \( 1/n \) expansion described in the Appendix, Eqs. (16)–(18); it will be presented elsewhere. For our purposes here it is enough to note that using masses normalized at a scale \( \sim \Delta/n \) does not
generate terms $\sim (n\alpha_s/\pi)^k$.  

A qualifying comment is in order. Including higher order terms in the expansions in $1/n$ and in $\alpha_s$, one cannot find a universal scale most appropriate for all perturbative corrections simultaneously, including the normalization of the strong coupling. Even upon resumming all $n$-dependent effects one would end up with the ordinary corrections to the structure functions, e.g. those describing the short-distance renormalization of the vector and axial currents. The relevant scale for such effects is obviously $\sim m_b$; we are not concerned about them here since they are not enhanced by the factor $n$ and, thus, not large.

3.3 Power corrections and duality in inclusive widths

Using $n$ as an expansion parameter one can estimate the importance of the higher order nonperturbative corrections by summing up the leading terms in $n$. The simple estimates show, however, that these effects do not exceed a percent level, and thus are not of practical interest.\footnote{It is important that here we consider large $n$ for a given order in the perturbative expansion.} It is still instructive to mention a qualitative feature drawn from this analysis.

The characteristic momentum scale decreases for large $n$; at $\Delta/n \sim \Lambda_{\text{QCD}}$ one is not in the short-distance regime anymore. Correspondingly, the overall theoretical accuracy of the calculations naturally seems to deteriorate with increasing $n$ (and/or increasing $m_c$ up to $m_b$). However, using the sum rules derived in \cite{14} it is not difficult to show that even in the limit $n \to \infty$ the width is defined by short-distance dynamics unambiguously up to $1/m_Q^2$ terms. Moreover, in this limit a straightforward resummation of the leading nonperturbative effects yields the width in terms of the zero-recoil formfactor $|F_D|^2$ and the physical masses $M_B$ and $M_{D^*}$, without additional uncertainties. Thus, in the worst limit for inclusive calculations, the theoretical computation of the width merely reduces to the calculation of the exclusive $B \to D^*$ transition, the best place for application of the heavy quark symmetry \cite{12, 21}.

The large-$n$ arguments may seem to suggest that the width can depend on the meson masses rather than on the quark ones, in contradiction to the OPE. In fact, since in the $b \to c$ transitions one encounters the SV limit when $n \to \infty$, the leading term is given by $(M_B - M_D)^n$ which differs from $(m_b - m_c)^n$ only by $O\left(\Lambda_{\text{QCD}}^2/m_Q^2\right)$, in accordance with the OPE. The problem can emerge in the next-to-leading order in $1/n$. These corrections can be readily written up in terms of the transition formfactors at small velocity transfer, and take the form of the sums of the transition probabilities which one encounters in the small velocity sum rules \cite{14}. The latter ensure that the terms $\sim \Lambda_{\text{QCD}}/m_Q$ are absent. By the same token, using the third

\footnote{The $b \to c \tau \nu$ decays represent a special case: due to the sizable $\tau$ mass the energy release is smaller than in $b \to c \mu \nu$ and $b \to c e \nu$ and a SV scenario is realized in a rather manifest way. The $n$-enhanced nonperturbative corrections can then reach the ten percent level, but are readily resummed using the exact meson masses in the phase space factors.}
sum rule, one observes that the leading-$n$ term, $n\mu_\pi^2/m_Q^2$, is absent from the width although it is obviously present in $(M_B - M_D)^n$ (we assume that the quark masses are fixed as the input parameters). Similar properties hold to all orders in $1/n$, however they are not very obvious in this expansion.

If the infrared part of the quark masses cancels in the widths, why is it not the short-distance masses like $\overline{MS}$ that emerge? The answer is clear: the excited final states in the width enter with the weight $\sim \exp(-n\epsilon/\Delta)$, and the Coulomb part of the mass cancels in the width only up to momenta $\sim \Delta/n$.

The above consideration only interprets the known OPE results. The general question at which scale duality is violated is more instructive. We immediately see that, generically, the characteristic momenta are given by the scale $m_Q/n$ rather than by $m_Q$. For example, considering the calculation of the relevant forward transition amplitude for $b \rightarrow u$ in coordinate space \cite{Cleymans:1975ia}, we have

$$T \sim 1/x^{n+4}. \quad (30)$$

Its Fourier transform $\sim p^n \log p^2$ ($+$ terms analytic in $p^2$) is saturated at $|x| \sim r_0 \sim p/n$, corresponding to a time separation $t_0 \sim n/m_Q$. In the case of the massive final state quarks one has $t_0 \sim n/\Delta$.

In the semileptonic $b \rightarrow c$ decays one may naively conclude that the scale is too low. However, here the heavy quark symmetry ensures vanishing of all leading corrections even at $\Delta \rightarrow 0$, i.e. in the SV limit; this applies as well to the ESV limit. The nontrivial corrections appear only at the $1/\Delta^2$ or $n/(m\Delta)$ level and they are still small.

The situation is, in principle, different in the nonleptonic decays, where the color flow can be twisted (instead of the color flow from $b$ to $c$, it can flow from $b$ to $d$). In the nonleptonic $b \rightarrow c \bar{u}d$ decay, with $\Delta \simeq 3.5$ GeV, one is only marginally in the asymptotic freedom domain and \textit{a priori} significant nonperturbative corrections as well as noticeable violations of duality could be expected. In this respect it is more surprising that so far no prominent effects have been identified on the theoretical side \cite{Beneke:1992hw}. We note that a particular model for the violation of duality in the inclusive decays suggested in Ref. \cite{Jaffe:1988pm} explicitly obeys the large-$n$ scaling above.

The importance of the duality-violating terms is governed by the ratio $\Delta/n$,

$$\frac{\Gamma_{\text{dual.viol}}}{\Gamma_{\text{tree}}} \sim e^{-\frac{1}{\rho}(\frac{1}{\Delta} \ln \frac{\Delta}{\rho})}, \quad (31)$$

with $\rho$ being a hadronic scale parameter. The corrections blow up when $\Delta/n \sim \mu_{\text{had}}$. We hasten to remind, however, that this consideration is only qualitative, since we do not distinguish between, say $n = 5$ and $n - 2 = 3$. The arguments based on the refined expansion in the ESV case suggest that the momentum scale is governed by an effectively larger energy than literally given by $\Delta/n$ with $n = 5$.

\footnote{A priori the effect of the chromomagnetic interaction in order $1/m_Q^2$ could have been significant, but it gets suppressed due to the specific chiral and color structure of weak decay vertices \cite{Beneke:1992hw}, properties that are external to QCD.}

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4 Low-scale heavy quark mass; renormalized effective operators

Addressing nonperturbative power corrections in $b$ decays via Wilson’s OPE one must be able to define an effective nonrelativistic theory of a heavy quark normalized at a scale $\mu \ll m_Q$; the heavy quark masses in this theory (which enter, for example, equations of motion of the $Q$ fields) are low-scale (Euclidean) masses $m_Q(\mu)$. Moreover, even purely perturbative calculations benefit from using them, as was shown above: it allows one to solve two problems simultaneously – resum the leading $n$-enhanced terms and avoid large but irrelevant IR-renormalon–related corrections. For the perturbative calculations *per se*, therefore, the question arises of what is the proper and convenient definition of a low-scale Euclidean mass? Of course, one can always define it through an off-shell quark propagator. Generically this will lead to a mass parameter which is not gauge invariant; this feature, though not being a stumbling block, is often viewed as an inconvenience.

Usually, one works with the mass defined in the \( \overline{\text{MS}} \) scheme, which is computationally convenient. However, due to the well-known unphysical features of this scheme it is totally meaningless at the low normalization point $\mu = \text{several units} \times \Lambda_{\text{QCD}}$. For example, if we formally write the \( \overline{\text{MS}} \) mass at this point

$$m_Q^{\overline{\text{MS}}} (\mu) \sim m_Q^{\overline{\text{MS}}} (m_Q) \left( 1 + \frac{2\alpha_s}{\pi} \ln \frac{m_Q}{\mu} \right)$$

we get large logarithms $\ln m_b/\mu$ which do not reflect any physics and are not there under any reasonable definition of the heavy quark mass.

Thus, one must find another definition. We suggest the most physical one motivated by the spirit of the Wilson’s OPE, i.e. the mass which would be seen as a ‘bare’ quark mass in the completely defined effective low-energy theory.

The basic idea is to follow the way suggested in Ref. [14], and is based on the consideration of the heavy flavor transitions in the SV kinematics, when only the leading term in the recoil velocity, $\sim \vec{v}^2$, is kept. Note that in Subsection 2.1 we have used a similar purpose a different kinematical limit when the final particle is ultrarelativistic.

Let us consider the heavy-quark transition $Q \rightarrow Q'$ induced by some current $J_{QQ'} = \bar{Q}' \Gamma Q$ in the limit when $m_{Q,Q'} \rightarrow \infty$ and $\vec{v} = \vec{q}/m_{Q'} \ll 1$ is fixed. The Lorentz structure $\Gamma$ of the current can be arbitrary with the only condition that $J_{QQ'}$ has a nonvanishing tree-level nonrelativistic limit at $\vec{v} = 0$. The simplest choice is also to consider $Q' = Q$ (which, for brevity of notation, is assumed in what follows). Then one has [14]

$$\Lambda(\mu) \equiv \lim_{m_Q \rightarrow \infty} \left[ M_{HQ} - m_Q(\mu) \right] = \lim_{\vec{v} \rightarrow 0} \lim_{m_Q \rightarrow \infty} \frac{2}{\vec{v}^2} \int_0^\mu \frac{d\epsilon}{\epsilon} w(\epsilon, m_Q \vec{v}) ,$$

where $w(\epsilon, |\vec{q}|)$ is the structure function of the hadron (the probability to excite the
state with the mass \( M_{H_Q} + \epsilon \) for a given spatial momentum transfer \( \vec{q} \); \( M_{H_Q} \) is the mass of the initial hadron.

The OPE and the factorization ensures that

a) For the given initial hadron the above \( \overline{\Lambda}(\mu) \) does not depend on the chosen weak current.

b) The value \( m_Q(\mu) = M_{H_Q} - \overline{\Lambda}(\mu) \) does not depend on the choice of the hadron \( H_Q \). This holds if \( \mu \) is taken above the onset of duality.

To define perturbatively the running heavy quark mass one merely considers Eq. (33) in the perturbation theory, to a given order:

\[
m_Q(\mu) = \left[ M_{H_Q} \right]_{\text{pert}} - \left[ \overline{\Lambda}(\mu) \right]_{\text{pert}} .
\]

In the perturbation theory \( H_Q \) is the quasifree heavy quark state and \( M_{H_Q} \) is the pole mass \( m_Q^{\text{pole}} \) to this order; \( w(\epsilon) \) at \( \epsilon > 0 \) is a perturbative probability to produce \( Q \) and a number of gluons in the final state which belongs to the perturbative continuum. It is a continuous function (plus \( \delta(\epsilon) \) at \( \epsilon = 0 \)). Both numerator and denominator in the definition of \( \overline{\Lambda} \) are finite. All quantities are observables and, therefore, are manifestly gauge-invariant. For example, the first order

\[
\left[ \overline{\Lambda}(\mu) \right]_{\text{pert}} = \frac{16 \alpha_s}{9 \pi} \mu , \quad m_Q(\mu) = m_Q^{\text{pole}_{1\text{-loop}}} - \frac{16 \alpha_s}{9 \pi} \mu ,
\]

\[
m_Q(\mu) = \overline{m}_Q^{\text{MS}}(m_Q) \left( 1 + \frac{4 \alpha_s}{3 \pi} - \frac{16 \alpha_s}{9 \pi} \frac{\mu}{m_Q} \right) .
\]

The IR renormalon-related problems encountered in calculation of \( m_Q^{\text{pole}} \) cancel in Eq. (34) against the same contribution entering \( \overline{\Lambda}(\mu) \) in Eq. (33).

In a similar way one can define matrix elements of renormalized at the point \( \mu \) heavy quark operators \([14, 23, 24]\). The zeroth moment of the structure function is proportional to \( \langle H_Q | Q \bar{Q} | H_Q \rangle / (2M_{H_Q}) \) and to the short-distance renormalization factor. In the limit \( m_Q \to \infty \) this matrix element equals unity up to terms \( \mu^2 / m_Q^2 \). Then it is convenient to define matrix elements of higher dimension operators in terms of the ratios (for which all normalization factors go away):

\[
\mu_\pi^2(\mu) = \frac{\langle H_Q | \bar{Q}(i\vec{D})^2 Q | H_Q \rangle_{\mu}}{\langle H_Q | Q \bar{Q} | H_Q \rangle_{\mu}} = \lim_{\vec{v} \to 0} \lim_{m_Q \to \infty} \frac{3}{v^2} \int_0^\mu d\epsilon \frac{e^2 w(\epsilon, m_Q v)}{\int_0^\mu d\epsilon w(\epsilon, m_Q v)} ,
\]

\[
\rho_D^3(\mu) = -\frac{1}{2} \frac{\langle H_Q | \bar{Q}(\vec{D} \vec{E}) Q | H_Q \rangle_{\mu}}{\langle H_Q | Q \bar{Q} | H_Q \rangle_{\mu}} = \lim_{\vec{v} \to 0} \lim_{m_Q \to \infty} \frac{3}{v^2} \int_0^\mu d\epsilon \frac{e^3 w(\epsilon, m_Q v)}{\int_0^\mu d\epsilon w(\epsilon, m_Q v)} ,
\]

and so on. All these objects are well-defined, gauge-invariant and do not depend on the weak current probe.

Returning to the heavy quark mass, the literal definition given above based on Eq. (34) is most suitable for \( \mu \ll m_Q \) when the terms \( \sim (\alpha_s / \pi) (\mu^2 / m_Q) \) are
inessential. One can improve it including higher-order terms in \( \mu/m_Q \), if necessary, in the following general way.

Consider the \( 1/m_Q \) expansion of the heavy hadron mass

\[
\left( M_{H_Q} \right)_{\text{spin-av}} = m_Q(\mu) + \overline{\lambda}(\mu) + \frac{\mu_2(\mu)}{2m_Q(\mu)} + \frac{\rho_3(\mu)}{4m_Q^2(\mu)} + \ldots ; \tag{38}
\]

We took the average hadronic mass over the heavy quark spin multiplets here. Besides \( \mu_2 \) and \( \rho_3 \) defined above the new quantity \( \rho_3 \) enters the \( 1/m_Q^2 \) term. It is a non-local correlator of two operators \( \bar{Q}(\vec{\sigma}\vec{D})^2Q \) defined in Eq. (24) of [14]. We apply the relation (38) to the perturbative states and get

\[
m_Q(\mu) = m^\text{pole}_Q - \left[ \overline{\lambda}(\mu) \right]_\text{pert} - \frac{[\mu_2(\mu)]_\text{pert}}{2m_Q(\mu)} - \frac{[\rho_3(\mu)]_\text{pert}}{4m_Q^2(\mu)} - \ldots . \tag{39}
\]

For example, through order \( 1/m_Q \) one has

\[
\left[ \mu_2(\mu) \right]_\text{pert} \approx \frac{4\alpha_s}{3\pi} \mu^2 . \tag{40}
\]

To renormalize the non-local correlators entering the definition of \( \rho_3 \), one represents them as a QM sum over the intermediate states and cuts it off at \( E_n \geq M_{H_Q} + \mu \). For example, \( [\rho_3(\mu)]_\text{pert} \approx (8\alpha_s/9\pi)\mu^3 \). One subtlety is worth mentioning here: including higher orders in \( 1/m_Q \) and going beyond one-loop perturbative calculations one must be careful calculating the coefficient functions in the expansion Eq. (38).

The general definition above is very transparent physically and most close to the Wilson’s procedure for a theory in Minkowski space. It corresponds to considering the QM nonrelativistic system of slow heavy flavor hadrons and literally integrating out the modes with \( \omega = E - m_Q > \mu \). Considering the SV limit and keeping only terms linear in \( \vec{\nu}^2 \) is important: in this case no problem arises with choosing the reference energy in \( \epsilon \) or in imposing the cutoff (energy vs. invariant mass, etc.) – they are all equivalent in this approximation; the differences appear starting terms \( \sim \vec{\nu}^4 \). The conceptual details will be further illuminated in subsequent publications.

5 Determination of \( |V_{cb}| \) (|\( V_{ub} \)|) from the inclusive widths

We now address the central phenomenological problem – how accurately one can extract \( |V_{cb}| \) and \( |V_{ub}| \) from inclusive semileptonic widths. \( \mathcal{O}(\alpha_s) \) terms are known, and higher order ones are expected to be small for \( b \) decays. However radiative corrections to the mass get magnified due to the high power with which the latter enters the width. This concern has been raised in [20]. The effect of large \( n \) indeed shows up as the series of terms \( \sim (n\alpha_s/\pi)^k \). Therefore one needs to sum up such
terms. The resummation of the running $\alpha_s$ effects in the form $\frac{\alpha_s}{\pi} \left( \frac{\beta_0 \alpha_s}{\pi} \right)^k$ has been carried out in [10] (the term with $k = 1$ in this series was earlier calculated in [25]); with $\beta_0/2 = 4.5$, accounting for the large-$n$ series seems to be at least as important. We emphasize that the BLM-type [26] improvements leave out these large-$n$ terms.

In this paper we discussed a prescription automatically resumming the $(n\alpha_s/\pi)^k$ terms. This eliminates the source of the perturbative uncertainty often quoted in the literature as jeopardizing the calculations of the widths.

Let us demonstrate this assertion in a simplified setting.

Let us consider two limiting cases, $m_q \ll m_b$ and $m_q \simeq m_b$. Dropping from the widths inessential overall factors $G_F^2 |V_{qb}|^2 / (192\pi^3)$ ( $G_F^2 |V_{qb}|^2 / (15\pi^3)$ ) we get

$$\Gamma \simeq \Delta^n \left( 1 + a_1 \frac{\alpha_s}{\pi} + a_2 \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots \right),$$  \hspace{1cm} (41)

with $a_1 = -\frac{2}{3} \left( \pi^2 - \frac{25}{4} \right)$ (or $a_1 = -1$, respectively) at $n = 5$. As was shown before, these coefficients do not contain factors that scale like $n$, nor $a_2$ contains $n^2$, etc., and we can neglect them altogether. However, when one uses, say, the MS masses, one has

$$\overline{m}_Q \simeq m_Q \left( 1 - \frac{4}{3} \frac{\alpha_s}{\pi} \right),$$  \hspace{1cm} (42)

and to the same order in $\alpha_s$ one arrives, instead, at a different numerical estimate

$$\Gamma = \Delta^n \left( 1 - \frac{4}{3} \frac{\alpha_s}{\pi} \right)^n \left( 1 + n \frac{4}{3} \frac{\alpha_s}{\pi} \right) \rightarrow$$

$$\Delta^n \left( 1 - \frac{(n + 1)n}{2} \left( \frac{4}{3} \frac{\alpha_s}{\pi} \right)^2 + \frac{(n + 1)n(n - 1)}{3} \left( \frac{4}{3} \frac{\alpha_s}{\pi} \right)^3 - \ldots \right);$$  \hspace{1cm} (43)

$\overline{\Delta}$ is written in terms of the $\overline{\text{MS}}$ scheme masses. The expressions in Eq.(41) and Eq.(42) are equivalent to order $\alpha_s$, yet start to differ at order $\alpha_s^2$ due to $n$-enhanced terms. The size of the difference has been taken as a numerical estimate of the impact of the higher order (non-BLM) corrections which then seemed to be large indeed: the coefficient in front of $(\alpha_s/\pi)^2$ is more than 25; only such huge enhancement could lead to an uncertainty in the width $\sim 15\%$ from the higher order corrections.

Since we are able to resum these terms, the width Eq.(41) has small second (and higher order) corrections. Alternatively, using the $\overline{\text{MS}}$ masses one is bound to have large higher-order perturbative coefficients, and their resummation returns one to using the low-scale masses.

We, thus, conclude that the uncertainty associated with the higher-order perturbative corrections discussed in Ref. [20] is actually absent and is an artifact of the

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These uncertainties do not apply to our previous analyses [16, 13] where we have employed quark masses evaluated at a low scale keeping in mind the natural momentum scale inherent for the actual inclusive widths, following the arguments under discussion. They are intrinsic, however, to some alternative estimates which can be found in the literature.
approach relying on the $\overline{\text{MS}}$ masses without a resummation of the large-$n$ effects. A similar numerical uncertainty, in a somewhat softer form, appeared as a difference between the “OS” and the “$\overline{\text{MS}}$” all-order BLM calculations of Ref. [10]; it is likewise absent if one uses the proper low-scale masses.

In our asymptotic treatment of the large-$n$ limit we cannot specify the exact value of the normalization scale $\mu$ to be used for masses: it can equally be 0.7 GeV or 1.5 GeV; the exact scale would be a meaningless notion. We also saw that the position of the saddle point varies depending on the structure function considered. All these nuances are rather unimportant in practice: the impact of changing $\mu$ has been studied in Ref. [19], and it was found that the values of $|V_{qb}|$ extracted from the widths vary by less than $\pm 1\%$ for any reasonable value of $\mu$.

The large factor of 5 also enhances the sensitivity of $|V_{cb}|$ to the expectation value of the kinetic energy operator $\mu^2_\pi$ in the currently used approach where $m_b - m_c$ is related to $M_B, M_{B^*}, M_D, M_{D^*}$ and $\mu^2_\pi$. This is certainly a disadvantage; however, the emerging dependence of $|V_{cb}|$ on $\mu^2_\pi$ is practically identical (but differs in sign) to the one in the determination of $|V_{cb}|$ from the zero recoil rate $B \to D^{*+}\ell\nu$ (for details see the recent review [24]). We do not see, therefore, a way to eliminate the uncertainty due to $\mu^2_\pi$ other than to extract its numerical value from the data. Having at hand its proper field-theoretic definition, one does not expect significant theoretical uncertainties in it. In the exclusive case, unfortunately, even this would not remove the sizable element of model-dependence.

The $1/n$ expansion can equally well be applied when the final-state quark mass is practically zero as in $b \to u$. The large corrections of order $\alpha_s^2$ that appear when $m_b$ is taken at the high scale $m_b$ turn out to be quite small when $m_b$ is evaluated at $\Delta/n \sim 1$ GeV with $\Delta$ denoting the energy release. Using the analysis of Ref. [19] one can calculate the inclusive total semileptonic width $\Gamma(B \to l\nu X_u)$ quite reliably in terms of $|V(ub)|$ and $m_b(1\text{GeV})$ as inferred from $\Upsilon$ spectroscopy. Numerically one has [24]

$$
|V_{ub}| \approx 0.00415 \left( \frac{\text{BR}(B \to X_u l\nu)}{0.0016} \right)^{1/2} \left( \frac{1.55 \text{ ps}}{\tau_B} \right)^{1/2}. \tag{44}
$$

The theoretical uncertainty here is smaller than the experimental error bars which can be expected in the near future.

6 Conclusions

There are three dimensional parameters relevant for semileptonic decays of beauty hadrons: $m_b$, the energy release $\Delta_{bq} = m_b - m_q$ and $\Lambda_{\text{QCD}}$. Since $m_b, \Delta_{bq} \gg \Lambda_{\text{QCD}},$

---

7 A more complete BLM calculation has been carried out in [10]; the numerical results were quoted, however, only for the OS and $\overline{\text{MS}}$ schemes; the latter is expected to suffer from large $n$-related corrections, whereas the OS scheme is affected by the IR renormalons when higher-order BLM corrections are included; the BLM resummation does not cure it completely, again due to the $n^2$-enhanced terms in the spurious $1/m^2$ corrections.
the heavy quark expansion in terms of powers of $\Lambda_{\text{QCD}}/\Delta_{bq}$ yields a meaningful treatment, with higher-order terms quickly fading in importance. It is natural then that uncertainties in the perturbative treatment limit the numerical reliability of the theoretical predictions. It was even suggested that the latter uncertainties are essentially beyond control.

Our analysis shows that such fears are exaggerated. First, we have reminded the reader that the dependence of the total width on the fifth power of $m_b$ largely reflects the kinematics of the lepton pair phase space. We then used an expansion in $n$ to resum higher order contributions of kinematical origin by identifying unequivocally the relevant dynamical scales at which the quark masses have to be evaluated. We found that the relevant scale is not set by the energy release, but is lower, parametrically of order $\Delta_{bq}/n$.

We demonstrated that the expansion in $1/n$ works reasonably well in simple examples. This is not the main virtue, in our opinion. More important is the fact that this expansion allows one to have a qualitative picture of different types of corrections based on the scaling behavior in $n$. In particular, it shows the emergence of new, essentially lower scales relevant in the semileptonic decays.

Even without a dedicated analysis it is obvious that the typical scale of the energy release in the semileptonic $b$ decays lies below $m_b$. Without a free parameter at hand it is not too convincing to defend say, the scale $m_b/2$ as opposed to $2m_b$. Needless to say that this scale variation has a noticeable impact on the final numbers. Using $n$ as an expansion parameter the ambiguity is resolved – the appropriate normalization scale for masses is even below $m_b/2$ – and already this, quite weak, statement is extremely helpful [19] in numerical estimates.

Based on the large-$n$ expansion, we arrived at a few concrete conclusions.

- One can resum the dominant $n^k$ terms in the perturbative expansion of the inclusive widths, by merely using the Euclidean low-scale quark masses (e.g. normalized at the scale $\sim \Delta/n$). Therefore, the previous calculations of the width [16, 19] are free of the large uncertainties noted in [20] and which were later claimed to be inherent to the inclusive widths.

- We introduced the so called “extended” SV limit. We show that $m_c$ need not be close to $m_b$ for the SV regime to emerge in the $b \to c$ inclusive decays. Large $n$ helps ensure this regime, which gives a rationale for the relevance of the SV consideration in the actual inclusive $b \to c$ decays, both at the quark-gluon and at the hadronic levels.

Recently, the complete second-order corrections to the zero-recoil formfactors have been computed [27]; the non-trivial non-BLM parts proved to be small. The arguments based on the large-$n$ expansion suggest then that the non-BLM perturbative corrections not computed so far for $\Gamma_{\text{al}}(b \to c)$ are not large either.

The approach suggested here is applicable to other problems as well. For example, we anticipate that the second-order QED corrections to the muon lifetime which have not been calculated so far, will not show large coefficients if expressed in
terms on the muon mass normalized at the scale \( \sim m_\mu/3 - m_\mu/2 \); moreover, using \( \alpha_{em} \) at a similar scale (in the \( V \) scheme) is also advantageous – though, clearly, it does not matter in practice in QED.

**Note added:** When this paper was prepared for publication the full two-loop calculation of the perturbative correction in \( b \to c \ell\nu \) at another extreme kinematic point \( q^2 = 0 \) was completed \[28\]. The result suggests that the non-BLM perturbative corrections to the width are indeed small if one relies on the low-scale quark masses, in accord with our expectations.

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**Appendix:** The refined \( 1/n \) expansion

Let us start from Eq. (7) expressed in terms of the quark masses \( m_b \) and \( m_c \), with \( \Delta = m_b - m_c \), and introduce variables \( \eta \) and \( \omega \) defined by

\[
\eta = 1 - \frac{\sqrt{q^2}}{\Delta}, \quad \omega = \frac{1}{\Delta} \left[ \sqrt{m_b^2 - 2m_bq_0 + q^2} - m_c \right].
\]  
(45)

Thus, \( \omega \) measures the effective mass \( M_X \) in the final hadron state, \( \omega = (M_X - m_c)/\Delta \), (in particular, \( \omega = 0 \) represents the free quark decay) whereas \( \eta \) determines the difference between \( q^2 \) and \( \Delta^2 \). Then we have

\[
\gamma(n) = 2\Delta^n \sqrt{\frac{\Delta}{m_b}} \left\{ \int_0^1 \mathrm{d}(1-\eta)^{n-2} \times \right.
\]

\[
\left[ \eta \left( \eta + \frac{2m_c}{\Delta} \right) \left( 1 - \left( 1 - \frac{m_c}{2m_b} \right) \eta + \frac{\Delta}{4m_b} \eta^2 \right)^{1/2} v_1(\eta) + \right.
\]

\[
\frac{\Delta^2}{3m_b^2} \int_0^1 \mathrm{d}\eta \eta^{3/2}(1-\eta)^{n-4} \left( \eta + \frac{2m_c}{\Delta} \right)^{3/2} \left( 1 - \left( 1 - \frac{m_c}{2m_b} \right) \eta + \frac{\Delta}{4m_b} \eta^2 \right)^{3/2} v_2(\eta) \left\} \right.
\]  
(46)

where functions \( v_i(\eta) \) are defined through

\[
v_1(\eta) \cdot (2m_c\eta + \eta^2)^{1/2} \cdot \left[ 1 - \left( 1 - \frac{m_c}{2m_b} \right) \eta + \frac{\eta^2\Delta}{4m_b} \right]^{1/2} \equiv
\]
\[ \frac{\Delta}{m_b} \int_0^n d\omega (m_c + \omega \Delta)(\eta - \omega)^{1/2}(2m_c + \eta \Delta + \omega \Delta)^{1/2} \times \]
\[ \left[ 1 - \left( 1 - \frac{m_c}{2m_b} \right) \eta - \frac{m_c}{2m_b} \omega + \frac{\Delta(\eta^2 - \omega^2)}{4m_b} \right]^{1/2} \frac{w_1}{2\pi}, \]  

(47)

\[ v_2(\eta) \cdot (2m_c \eta + \eta^2 \Delta)^{3/2} \cdot \left[ 1 - \left( 1 - \frac{m_c}{2m_b} \right) \eta + \frac{\eta^2 \Delta}{4m_b} \right]^{3/2} \equiv \]
\[ \frac{\Delta}{m_b} \int_0^n d\omega (m_c + \omega \Delta) \times \]
\[ \left[ (\eta - \omega)(2m_c + \eta \Delta + \omega \Delta) \left( 1 - \left( 1 - \frac{m_c}{2m_b} \right) \eta - \frac{m_c}{2m_b} \omega + \frac{\Delta(\eta^2 - \omega^2)}{4m_b} \right) \right]^{3/2} \frac{w_2}{2\pi}. \]  

(48)

Equation (46) represents an identity with two gratifying features. First, the width is expressed through the quantities \( v_i \). Being weighted integrals of the structure functions \( w_i \) they are smoother analytically than \( w_i \) themselves. In a certain respect, \( v_i \) are generalizations of the moments \( I_i \), that are relevant for calculating inclusive width.

Second, the form of Eq. (46) is particularly well suited for deriving the large-\( n \) expansion. One typically encounters integrals of the form \( \int_0^1 d\eta \eta^a (1 - \eta)^k \). At \( k \to \infty \) the integral is saturated at \( \eta_0 \simeq 1 - a/(k + a) \simeq 1 - a/k \), i.e. \( 1 - \eta_0 \ll 1 \). (We loosely refer to this saturation as a saddle-point evaluation, although it is not really a saddle point calculation in its standard definition.) This was actually the approximation used in the simple large-\( n \) expansion. However, for \( k = 0, 2 \) and \( a \sim 1/2 \div 3/2 \) the ansatz \( \eta_0 \ll 1 \) is quite poor numerically. On the other hand \( \int_0^1 d\eta \eta^a (1 - \eta)^k \) still has a reliable ‘saddle point’ even for large \( a \) and \( k = 0 \), since the width of the distribution is governed by \( ak/(a + k)^3 \) and does not become large. The point only shifts somewhat upward as compared to the ‘naive’ approach. This is the idea lying behind the improved expansion. Its purpose is merely to determine the essential kinematics in the process at hand, and to use the hadronic averages expanded around it. Of course, the phase space integrals always can be taken literally for any \( n \), if necessary.

As it was with the simple \( n \) expansion, one finds that the explicit dependence on \( n \) is contained in the kinematical factors that can be treated separately from the QCD dynamics contained in the hadronic averages \( v_{1,2}(\eta) \). There is some residual dependence on \( n \) entering through the value of the scale \( \eta_0 \) at which \( v_{1,2}(\eta) \) are to be evaluated (i.e., the exact shape of the weight functions). Again, one has to treat the vector and axial current contributions separately for \( b \to c \) and \( b \to u \).

In Table 1 we compare the exact results with those obtained from the refined \( 1/n \) expansion for \( n = 5 \) as a function of \( m_c/m_b \) in the simplest setting of the tree-level decay where the cumbersome factors in Eq. (46) are replaced by their values at the ‘saddle’ point. We have used the following notation there. The quantities
In the SV regime one finds through order $\alpha_s = 5$ is a comparison with the known perturbative expression at the one-loop level. One defines branching ratios normalized to the exact ($n$ and 0, respectively) and integrates $w$ numerically results already for the physical value $m_c = 5$; the cases $m_c = 0$ and $m_c \geq 0.2 \cdot m_b$ are treated separately.

$\gamma_{1,2}^{(V,A)}$ denote the width factors for the vector and axial contributions obtained by integrating $w_1$ and $w_2$, respectively. We then have $\gamma_{sl} = \gamma_{1}^{(A)} + \gamma_{1}^{(V)} + \gamma_{2}^{(A)} + \gamma_{2}^{(V)}$. One defines branching ratios normalized to the exact ($n = 5$) tree level expression:

$$BR_{1,2}^{(A,V)}(m_c/m_b) = \frac{\gamma_{1,2}^{(A,V)}(m_c/m_b)}{\gamma_{sl,exact}(m_c/m_b)}$$

and calculates them using, on the one hand, the expansion in $n$ evaluated at $n = 5$, and on the other hand the exact $n = 5$ results. Since the axial and vector part of $w_2$ coincide in the tree approximation, only one of them is shown in Table 1. One sees that the $n$-expansion works with a typical accuracy of about 10% for the inclusive width, as expected. The weight of the nonleading terms decreases as $m_c \to m_b$, i.e. in the SV limit. It is remarkable that the expansion based on the SV kinematics works so well down to a rather small mass ratio of $m_q/m_b \simeq 0.2$! This illustrates the observation made above that it is the parameter $(m_b - m_c)/m_b$ that describes the proximity to the SV limit. For further analysis we note that at $m_c/m_b = 0.3$ the 'saddle points' for $n = 5$ occur at $\eta_s \equiv 1 - 1/\sqrt{(m_b - m_c)}$ equal to 0.16, 0.28 and 0.4 for $\gamma_{1}^{(A)}$, $\gamma_{1}^{(V)}$ and $\gamma_{2}$, respectively; for the $b \to u$ case $\eta_s$ is 0.33 and 0.55 for $\gamma_{1,2}$, respectively.

Another cross-check of the numerical reliability of the refined $1/n$ expansion at $n = 5$ is a comparison with the known perturbative expression at the one-loop level. In the SV regime one finds through order $\alpha_s/\pi$

$$\gamma_{sl,SV} \simeq \frac{8}{15} \Delta^5 \left(0.924 - 0.945 \cdot \frac{\alpha_s}{\pi}\right) = \frac{8}{15} \Delta^5 \cdot 0.924 \left(1 - 1.023 \cdot \frac{\alpha_s}{\pi}\right)$$

(50)

to be compared to the exact result to that order

$$\gamma_{sl,exact} \simeq \frac{8}{15} \Delta^5 \left(1 - \frac{\alpha_s}{\pi}\right),$$

(51)

i.e., a difference of two percent only.

From these comparisons we conclude that the refined $1/n$ expansion yields good numerical results already for the physical value $n = 5$.

| $m_c/m_b$ | 0  | 0.2 | 0.3 | 0.4 | 0.6 | 0.8 | 1  |
|-----------|----|-----|-----|-----|-----|-----|----|
| $BR_{1,n=5}^{(A)}$ | 0.223 | 0.524 | 0.462 | 0.462 | 0.476 | 0.488 | 0.494 |
| $BR_{1,exact}^{(A)}$ | 0.25 | 0.396 | 0.434 | 0.459 | 0.486 | 0.497 | 0.5 |
| $BR_{1,n=5}^{(V)}$ | 0.229 | 0.093 | 0.060 | 0.0374 | 1.26 $\cdot 10^{-2}$ | 2.48 $\cdot 10^{-3}$ | 0 |
| $BR_{1,exact}^{(V)}$ | 0.25 | 0.104 | 0.066 | 0.041 | 1.36 $\cdot 10^{-2}$ | 2.65 $\cdot 10^{-3}$ | 0 |
| $BR_{2,n=5}^{(A,V)}$ | 0.171 | 0.248 | 0.24 | 0.233 | 0.225 | 0.219 | 0.215 |
| $BR_{2,exact}^{(A,V)}$ | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| $\gamma_{sl,n=5}/\gamma_{sl,exact}$ | 0.80 | 1.11 | 1.00 | 0.97 | 0.94 | 0.93 | 0.924 |

Table 1: Comparing widths in the refined $n$ expansion with the exact results at $n = 5$; the cases $m_c = 0$ and $m_c \geq 0.2 \cdot m_b$ are treated separately.
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