An attractive $\phi^4$ theory in light-front coordinates

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We study an attractive $\phi^4$ interaction using Tamm-Dancoff truncation with light-front coordinates in 3+1 dimensions. The truncated theory requires a coupling constant renormalization, we compute its $\beta$ function non-perturbatively, show that the model is asymptotically free, and find the corresponding Callan-Symanzik equations. The model supports bound states, we find the wave function for the ground state of the two-particle sector. We also give a bound for the $N$-particle ground state energy within a mean field approximation, including the corresponding result for the case of $2 + 1$ dimensions where the model does not require renormalization.

1. Introduction

It is well-known that light-front coordinates have some advantages for interacting field theory problems, especially for finding bound states. For example, in his seminal work [Hoo74] ’t Hooft derived an integral equation for the bound state spectrum of 1 + 1 dimensional QCD within a light-front quantization. Similarly, the solution of the WZW theory in two dimensions is understood in light-front coordinates in Witten’s work [Wit84] by means of its connection to a fermionic theory. The literature on light-front theories is very extensive, we will not be able to do justice to this vast field, for reviews of the basic ideas the reader may consult [Har96, Hei01, Per94]. Our aim is to use this approach
in the truncated version of a well-known theory and study possible bound states in the
spirit of Tamm-Dancoff.

It was first remarked by Weinberg in [Wei66] that a non-covariant approximation scheme
like Tamm-Dancoff could become meaningful in the infinite momentum frame, which was
later understood to be equivalent to quantizing a theory in light-front coordinates. The
main advantage of using light-front coordinates is that the light-front vacuum does not
change as a result of interactions [Hei01]. This follows from the fact that \( p^+ \) must be
positive, which implies that the interacting vacuum cannot contain nonzero momentum
particles. Otherwise the momentum operator \( P^+ \) acting on the vacuum would add up
to a nonzero value, which contradicts the requirement that the vacuum must be a zero
eigenstate of the momentum operator. In contrast, the vacuum of an interacting theory
in the equal time formalism, the so called instant form, is drastically different from that
of the free theory. The two can only be linked by an adiabatic switching on and off of the
interaction which is properly implemented only in perturbation theory calculations. While
containing many particles, one can still keep the canonical vacuum as a zero eigenstate of
momentum operator. We remark that the vacuum structure of a light-front theory is not
unitarily equivalent to that of the usual canonical quantization of an interacting theory,
therefore there is no conflict between these two pictures.

Based on this fundamental difference between the two quantization methods, it is
proposed in [PHW90] that the Tamm-Dancoff truncation could be a better approximation
in light-front coordinates. Implementation of this idea for a Yukawa coupling in 1 + 1
dimensions is presented in [HPS92]. One possibility is to truncate the Fock space on
which the Hamiltonian acts. A very restrictive form of truncation is to remove parts
of the interaction that do not conserve particle number. This is what we think of
as an approximation to the fully interacting theory. Further examination is needed to
check the consistency and validity of this approximation, for this work we only study the
consequences of this approach.

Following this idea, we propose an attractive version of relativistic contact interactions.
Just like we have a theory of scalar relativistic particles with repulsive contact interactions,
one expects that there should be an attractive version as well. The difference from the
repulsive case is that this theory should be constructed directly as a quantum theory and
not be built around the classical vacuum solution. To reiterate, one cannot start with a
classical Lagrangian in this case since the vacuum does not correspond to the zero field
configuration unlike the repulsive model, classically the theory is unstable. This is why
one needs a fine tuning of the coupling. As we add higher and higher momenta modes for
the interaction, the coupling should be tuned down to zero in a precise manner. Due to
this fine tuning, a residual interaction is left, sufficiently weak to keep the theory stable at
the quantum level. This is a well-known phenomenon in the non-relativistic version of our
model, bosons with contact interactions in two dimensions [Hop82; Ber94; Ber92; DFT94].
It has been revised and extended by Rajeev in [Raj99; DR04] using a novel approach.

\footnote{In this work, we use a non-orthogonal set of light-front coordinates where this is replaced by \( p_3 \), but
the main arguments remain exactly the same.}

\footnote{A careful examination shows that in \( \phi^4 \) there are two terms with three creation and one annihilation
operators and the Hermitian conjugate.}
The asymptotic freedom of the theory in two dimensions is studied in \[Raj99; Ber94\], and a lower bound for the ground state energy is also given in \[DR04\] for all particle numbers. In this work, we stay within an extreme Tamm-Dancoff approximation. We show that we define an asymptotically free theory based on an approach developed in \[Raj99\]. Moreover, we claim that one can find a non-perturbative solution to possible bound states, we construct this solution for the two particle sector. It may then be possible to build the fully interacting theory as a perturbation of the truncated theory, which now contains bound states, energies of which begin below the spectrum of the free theory. One can then envisage the complete bound spectrum of the full theory as dressed versions of these basic bound states, as happens in some condensed matter systems \[Mud14; Wen07\]. The above truncation will break the locality of the theory but we assume that this is a viable approximation to a Poincaré invariant interacting theory. It is an interesting challenge to extend the approach proposed here to a more exact description of the problem at hand.

In the rest of this section we set the basics of our problem. First, we give the free Lagrangian in 1.1, prescribe the quantization in 1.2 and in 1.3 we add the interaction term. Next, in 2.1 we construct the resolvent and associated to it the principal operator. We find that this operator diverges and renormalize it in 2.2. Our calculations depend on the flow of this operator’s eigenvalues, we prove in 2.3 that they do constitute a flow. We demonstrate certain scaling properties of this operator, compatible with renormalization-group ideas, in 2.4. We then find the wave function for the ground state of the two-particle sector in 3. In 4 we provide a lower bound for the ground state energy of the N-particle sector, include the case for 2 + 1 dimensions in 4.1, and conclude in 5. In the appendix we provide some details of the coordinates we use, list the associated Lorentz transformations and the Poincaré generators. We also briefly summarize the principal operator and the orthofermion there, fundamental to this work.

1.1. Lagrangian formulation

We begin with the free theory, since an attractive interaction must be introduced directly within the quantum theory. We use light-front coordinates in the form as it is used in \[Raj94; KR98; Raj00\] and for which we provide some details in appendix A. The action $S$ is

$$S = \int dx^3 dx^- d^2x_\perp \left( \phi (-\partial_3) \partial_- \phi - \frac{1}{2} \phi \left( -\nabla_\perp^2 - \partial_3^2 + m^2 \right) \phi \right).$$

(1)

We realize that the field $\phi$ is conjugate to itself, therefore there are no momentum variables in these coordinates, we are already in the phase space formalism. The quantization can be accomplished by replacing the Poisson brackets of the classical fields with the corresponding commutators of the field operators.

We note that when we talk about particles of mass $m$, the on-shell condition for physical momenta in this coordinate system is

$$p_- = \frac{m^2 + p_1^2 + p_2^2}{2p_3} \equiv \omega(p).$$

(2)
The positivity of energy requires
\[ p_3 > 0. \] (3)

Incidentally this is consistent with the field being real valued, we only need half the degrees of freedom, as we verify below. That is, the decomposition of the field into its Fourier modes in the kinematical variables \( p_\mu \) requires that \( p_3 > 0 \) and \( p_3 < 0 \) modes are linked by complex conjugation. In the quantized theory there are no creation operators with negative \( p_3 \) and similarly no positive ones for the annihilation operators.

1.2. Quantization of the scalar field

In this part, we briefly review the quantization of the free theory. In order to obtain the commutation relations we decompose the real scalar field into its Fourier modes. We then check causality and afterwards we describe the interaction and its truncation.

We construct a quantized scalar field \( \phi(x) \) via
\[
\phi(+)(x) = \int \frac{[dp]a(p)e^{-ipx}}{(2\pi)^{3/2}\sqrt{2p_3}}, \\
\phi(-) \equiv (\phi(+))^\dagger, \\
\phi(x) \equiv \phi(+)(x) + \phi(-)(x). (4)
\]

Note that the notation \( [dp] \) is for \( dp_3 d^2 p_\perp \) with integration over \( p_3 > 0 \) also implied.\(^3\)

We emphasize that the condition \( \phi^\dagger(x) = \phi(x) \) implies that the creation and annihilation operators should only be defined for \( p_3 > 0 \). They have the commutator
\[
[a(p), a^\dagger(q)] = \delta(p - q). (5)
\]

We check the causal structure of this theory. Note the following commutator:
\[
[\phi^+(x), \phi^-(y)] = \frac{1}{(2\pi)^3} \int \frac{[dp]}{2p_3} e^{-ip(x-y)} = \Delta(x-y). (6)
\]

This is Lorentz invariant. We evaluate it at equal light-front time, \( x^- = y^- \). With \( b \equiv x - y \) we have \( (b)^2 = -(b^\perp)^2 \). This gives
\[
\Delta(b^- = 0, b) = \frac{1}{(2\pi)^3} \int_0^\infty \frac{dp_3}{2p_3} e^{-ip_3b^3} \int d^2 p_\perp e^{-ip_\perp \cdot b^\perp} \\
= \delta(b^\perp) \frac{1}{2\pi} \int_0^\infty \frac{dp_3}{2p_3} e^{-ip_3b^3}. (7)
\]

\(^3\)There is some subtlety about the zero mode, in some cases it should be treated separately. For the time being we set this issue aside, since this state can be approached by arbitrarily small momentum values, moreover its energy goes to infinity. As a result, we may assume a symmetric cut around \( p_3 = 0 \), the integrals are defined by the principal value prescription and \( p_3 = 0 \) is avoided.
The field commutator is
\[
[\phi(x), \phi(y)] = \left[\phi^+(x), \phi^-(y)\right] - \left[\phi^+(y), \phi^-(x)\right].
\] (8)

We get
\[
[\phi(x^-, x), \phi(x^-, y)] = \delta(x^\perp - y^\perp) \frac{1}{2\pi} \int_0^\infty \frac{dp_3}{2p_3} (2\pi)^3 (x^3 - y^3)
\] (9)

This vanishes for \(x^\perp \neq y^\perp\). Therefore the field commutes with itself at space-like intervals, it is causal.

A more detailed analysis of light-front coordinates is given in [Hei01, Har96]. Further details regarding our coordinates are given in appendix A.

1.3. Hamiltonian

We have the free Hamiltonian
\[
H_0 = \int [dp] \omega(p) a_1 a(p).
\] (10)

To this we add an attractive \(\phi^4\) interaction,
\[
H_I = -\lambda \int d^3 x d^2 x^\perp \phi^4(x).
\] (11)

As proposed in [PHW90], the Tamm-Dancoff approximation can be more reliable in the light-front formalism to calculate bound state energies and wave functions. That is what we follow here. To this end we normal order and then truncate this interaction, keeping only the \(\phi^- \phi^- \phi^+ \phi^+\) terms as an extreme limit of truncation. These are the only terms in the interaction that conserve particle number. There is also a mass term coming from the normal ordering which we absorb into the definition of mass. The resulting interaction Hamiltonian is
\[
H_1(x^-) = -\lambda (2\pi)^3 \int d^3 x d^2 x^\perp \phi^(-) \phi^(-) \phi^+ \phi^+(x).
\] (12)

At constant light-front time this is
\[
H_I = -\lambda \int \left( \prod_{i=1}^4 \frac{[dp_i]}{\sqrt{2p_3}} \right) a_1 a_2 a_3 a_4 \delta(p_1 + p_2 - p_3 - p_4).
\] (13)

Here and for what follows \(a_1\) is a shorthand for \(a(p_1)\), \(\omega_1\) for \(\omega(p_1)\) etc. We take as the approximate quantum Hamiltonian \(H = H_0 + H_I\).
2. Analysis of the Hamiltonian

Since our Hamiltonian conserves the particle number, an analysis of the corresponding resolvent is of use. A novel approach to such contact interaction problems is suggested in [Raj99], which we follow here. We summarize parts of this approach in appendix B, more details can be found in the original work. The essential idea is to extend the bosonic Fock space into \( F(\mathcal{H}) \oplus F(\mathcal{H}) \otimes L^2(\mathbb{R}^3) \). Here \( L^2(\mathbb{R}^3) \) refers to the Hilbert space of orthofermions. The algebra of orthofermions requires that either there is no orthofermion or that there is only one of them at any given moment. Here, this extension allows us to convert a multiplicative renormalization to an additive one, which facilitates the removal of this divergence to obtain a finite theory. By means of a redefinition of a bare parameter, in this case the coupling constant, we find a finite operator to work with.

2.1. Principal operator

In order to separate the coupling constant from its multiplicative form in the interaction and remove the divergence additively, we define the following operator:

\[
\tilde{H} = H_0 \Pi_0 + \left( \int \frac{[dp_1 dp_2]}{\sqrt{4p_{13}p_{23}}} a_1^\dagger a_2^\dagger \chi(p_1 + p_2) + \text{H. C.} \right) + \frac{\Pi_1}{\lambda}.
\]

Here \( \Pi_0 \) refers to the projection onto the no-orthofermion subspace and similarly \( \Pi_1 \) refers to the one for the one-orthofermion subspace. The reason behind defining \( \tilde{H} \) this way becomes clear in later steps. We decompose it with respect to its action on the orthofermion subspaces:

\[
\tilde{H} - E \Pi_0 = \begin{pmatrix}
(H_0 - E) \Pi_0 & \int \frac{[dp_1 dp_2]}{\sqrt{4p_{13}p_{23}}} a_1^\dagger a_2^\dagger \chi(p_1 + p_2) \Pi_1 \\
\int \frac{[dp_3 dp_4]}{\sqrt{4p_{33}p_{43}}} a_3 a_4^\dagger \chi(p_3 + p_4) & \frac{\Pi_1}{\lambda}
\end{pmatrix} \equiv \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix}.
\]

This has the inverse

\[
(\tilde{H} - E \Pi_0)^{-1} \equiv \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix}.
\]

Using the first expression of (109) we find

\[
\alpha = \left( H_0 - E - \lambda \int \frac{[dp_1 dp_2]}{\sqrt{4p_{13}p_{23}}} a_1^\dagger a_2^\dagger \chi(p_1 + p_2) \right) \left( \int \frac{[dp_3 dp_4]}{\sqrt{4p_{33}p_{43}}} a_3 a_4^\dagger \chi(p_3 + p_4) \right)^{-1} \Pi_0
\]

\[
= (H_0 - E + H_1(0))^{-1} \Pi_0 = (H - E)^{-1} \Pi_0
\]

\[
\equiv R(E) \Pi_0.
\]

For the second line we use (105). This convenient recombination of individual terms to reproduce the original Hamiltonian is the reason behind the particular form of \( \tilde{H} \).
Here $R(E)$ is the formal resolvent of the Hamiltonian $H$. Again using (109) we have an alternate form of the same expression,

$$\alpha = \left( \frac{1}{H_0 - E} + \frac{1}{H_0 - E} b^\dagger \Phi(E)^{-1} b \frac{1}{H_0 - E} \right) \Pi_0.$$  \hspace{1cm} (18)

Here $\Phi(E)$, as defined in (110), is the principal operator

$$\Phi(E) = \frac{\Pi_1}{\lambda} - \int \left( \prod_{i=1}^{4} \frac{[dp_i]}{\sqrt{2p_{i3}}} \right) \chi^\dagger (p_3 + p_4) a_3 a_4 (H_0 - E)^{-1} a_1 a_2^\dagger \chi (p_1 + p_2).$$ \hspace{1cm} (19)

The possible zeros of this operator, as long as they are below the $N$ particle threshold of the free part, correspond to bound states, as all the other terms in the resolvent (18) are regular at $E$. Note that when $R(E)$ acts on a $N$-particle state, $\Phi(E)$ sees a $N-2$-particle state with one orthofermion, since $b$ annihilates two particles and creates an orthofermion.

We realize that this expression is not normal ordered. We normal order it, which gives

$$\Phi(E) = \frac{\Pi_1}{\lambda} - (2K(E) + 4U_1(E) + U_2(E)),$$ \hspace{1cm} (20)

where

$$K(E) = \int \frac{[dp_1 dp_2]}{4p_{13}p_{23}} \chi^\dagger (p_1 + p_2) (H_0 - E + \omega_1 + \omega_2)^{-1} \chi (p_1 + p_2),$$

$$U_1(E) = \int \frac{[dp_1 dp_2 dp_3]}{2p_{13} \sqrt{4p_{23}p_{33}}} \chi^\dagger (p_1 + p_3) a_2^\dagger (H_0 - E + \omega_1 + \omega_2 + \omega_3)^{-1} a_3 \chi (p_1 + p_2),$$

$$U_2(E) = \int \left( \prod_{i=1}^{4} \frac{[dp_i]}{\sqrt{2p_{i3}}} \right) \chi^\dagger (p_3 + p_4) a_1 a_3^\dagger (H_0 - E + \omega_1 + \omega_2 + \omega_3 + \omega_4)^{-1} a_3 a_4 \chi (p_1 + p_2).$$ \hspace{1cm} (21)

Note that for $R(E)$ acting on a two-particle state, that is $\Phi(E)$ acting on the vacuum of the bosonic system and one orthofermion state, the terms $U_1$ and $U_2$ are irrelevant. They contain particle annihilation terms so they evaluate to zero acting on such a state. Therefore for a description of the two-particle sector $K(E)$ alone is in effect.

### 2.2. Renormalized principal operator

We note that there is a divergence in $K(E)$, therefore the operator $\Phi(E)$ is not well defined as it is. We now demonstrate this divergence and then renormalize it by introducing a similar divergence to the inverse coupling constant. A remarkable aspect of this extended Fock space construction is the possibility to renormalize the Hamiltonian non-perturbatively.\footnote{As long as we stay within the truncated theory.}
We make the following coordinate changes for the integral in $K(E)$:

$$P = p_{13} + p_{23}, \quad Q = \frac{p_{13} - p_{23}}{2},$$

$$\xi = p_{1\perp} + p_{2\perp}, \quad \eta = \frac{p_{23}p_{1\perp} - p_{13}p_{2\perp}}{p_{13} + p_{23}}.$$  \hfill (22)

This gives

$$K(E) = \frac{1}{2} \int dP dQ d^2 \xi d^2 \eta \chi^1(P, \xi) \left( [2(H_0 - E) + P + \frac{\xi^2}{P}]\left(\frac{P^2}{4} - Q^2\right) + m^2 P + \eta^2 P \right)^{-1} \chi(P, \xi).$$

We focus on the following part of this integral:

$$\int dQ d^2 \eta \left( [2(H_0 - E) + P + \frac{\xi^2}{P}]\left(\frac{P^2}{4} - Q^2\right) + m^2 P + \eta^2 P \right)^{-1} = \frac{1}{2} \int_{-1}^{1} d\tau \int d^2 \eta \left( [2(H_0 - E)P + P^2 + \xi^2]\frac{1 - \tau^2}{4} + m^2 + \eta^2 \right)^{-1}. \hfill (24)$$

Here we used the coordinate transformation $Q = \frac{P\tau}{2}$. We note that the $\eta$ integral diverges logarithmically. It is possible to renormalize this divergence via a coupling constant redefinition. As expected the truncated model does not require a mass renormalization.

### 2.2.1. Renormalization

A possible choice for $\lambda$ is

$$\frac{1}{\lambda} = \frac{1}{\lambda_R(M)} + \int d^2 \eta (M^2 + \eta^2)^{-1}. \hfill (25)$$

This gives a finite combination

$$\frac{\Pi_1}{\lambda} - 2K(E) = \frac{\Pi_1}{\lambda_R(M)} - 2K_R(E; M), \hfill (26)$$

where

$$K_R(E; M) = -\frac{\pi}{4} \int dP d^2 \xi \chi^1(P, \xi) \int_{-1}^{1} d\tau \ln \left( \frac{[2(H_0 - E)P + P^2 + \xi^2](1 - \tau^2) + 4m^2}{4M^2} \right) \chi(P, \xi). \hfill (27)$$

Here $M$ is an arbitrary scale for the system, and the renormalized coupling constant $\lambda_R(M)$ runs with it. It has the $\beta$ function

$$\beta(\lambda_R(M)) = -2\pi\lambda_R^2. \hfill (28)$$

This is derived using the fact that physical results should be independent of the choice of this scale $M$. The $\beta$-function is absolutely negative, therefore the model is asymptotically free. We remark that this is not the only possible renormalization scheme, we can also
use a physical parameter like the binding energy in place of the arbitrary scale $M$. This is utilized for the bound state wave function calculation in section 3.

The renormalized principal operator becomes

$$
\Phi_R(E) \equiv \frac{\Pi_1}{\lambda_R(M)} - (2K_R(E; M) + 4U_1(E) + U_2(E)).
$$

(29)

We assert that the full expression involving $\Phi_R(E)$ now defines a resolvent through (18), to be called $R_R(E)$. However the corresponding renormalized Hamiltonian $H_R$ cannot be written down explicitly.

As it stands we have a finite resolvent, however this doesn’t guarantee that the theory is finite. To secure this, we need to show that the ground state energies of all particle sectors are finite. We give a bound for the ground state energy of the N-particle sector within a mean field theory approximation in section 4. Our calculations depend on the flow of eigenvalues of the principal operator, we now concentrate on this property.

### 2.3. Eigenvalue flow

The resolvent (18) shows that the bound states below the spectrum of $H_0$ can only come from the zeros of $\Phi(E)$. In order to show that we can indeed find these zeros and locate the ground state, we prove that

$$
\frac{\partial w(E)}{\partial E} = \left\langle \frac{\partial \Phi_R(E)}{\partial E} \right\rangle < 0,
$$

(30)

where $w(E)$ is an eigenvalue of $\Phi_R(E)$. That is, we prove that a given eigenvalue of $\Phi_R(E)$ decreases with increasing $E$. Recall that the zero eigenvalues of $\Phi_R(E)$ correspond to possible bound states, this means that if we find any eigenvalue of $\Phi_R(E)$ below zero, we can increase $E$ to make it zero and find the corresponding state. We assume that the flow of eigenvalues are differentiable and that no crossings occur. This also implies that the ground state energy of the system corresponds to the zero of the minimum eigenvalue of $\Phi_R(E)$, as it reaches zero with the smallest $E$. This observation is essential to obtain a mean field estimate of the ground state energy for large number of particles.

We remark that $P_3$ is always positive, therefore the minimum of the spectrum of $H$ is always the invariant mass for the corresponding state, higher values within the same composition will give the translational energy increase of the system as a whole, as we see below explicitly for the two particle sector.

To proceed with the proof we first show that the derivative of $\Phi(E)$ is negative definite. Then we show that the derivatives of $\Phi(E)$ and $\Phi_R(E)$ are exactly the same. Note that this only requires the derivatives of $K(E)$ and $K_R(E)$ to be equal.

We rewrite $\Phi(E)$ in terms of

$$
I(E; s) \equiv \int \left( \prod_{i=1}^{2} \frac{[dp_i]}{\sqrt{2p_{i,3}}} \right) e^{-\frac{3}{2}(H_0-E)} a_1^\dagger a_2^\dagger \chi(p_1 + p_2).
$$

(31)
Using this in (19) we get

\[
\Phi(E) = \frac{\Pi_1}{\lambda} - \int \frac{[dp_i]}{\sqrt{2p_i}} \chi^+(p_3 + p_4)a_3 a_4 \left( \int_0^\infty ds e^{-s(H_0 - E)} \right) a_\downarrow a_\uparrow \chi(p_1 + p_2)
\]

\[
= \frac{\Pi_1}{\lambda} - \int_0^\infty ds I^1I(E; s).
\]

If we take the derivative formally, we get

\[
\frac{\partial \Phi(E)}{\partial E} = - \int_0^\infty ds s I^1I(E; s).
\]

This is a negative definite operator. We now compare the derivatives of \(K(E)\) and \(K_R(E; M)\).

The derivate of \(K_R(E; M)\), using (27), is

\[
-\frac{\pi}{4} \int dP d^2\xi \chi^+(P, \xi) \int_{-1}^1 d\tau (-2P)(1-\tau^2) \left( [2(H_0 - E)P + P^2 + \xi^2](1 - \tau^2) + 4m^2 \right)^{-1} \chi(P, \xi),
\]

and the derivative of \(K(E)\), using (24), is

\[
\frac{1}{4} \int dP d^2\xi \chi^+(P, \xi) \int_{-1}^1 d\tau \int d^2\eta (2P) \frac{1 - \tau^2}{4} \left( [2(H_0 - E)P + P^2 + \xi^2][1 - \tau^2] + m^2 + \eta^2 \right)^{-1} \chi(P, \xi)
\]

\[
= -\frac{\pi}{4} \int dP d^2\xi \chi^+(P, \xi) \int_{-1}^1 d\tau (-2P)(1-\tau^2) \left( [2(H_0 - E)P + P^2 + \xi^2](1 - \tau^2) + 4m^2 \right)^{-1} \chi(P, \xi).
\]

These two expressions match. The other derivatives, those of \(U_1\) and \(U_2\), agree trivially. Therefore our claim on the eigenvalue flow of \(\Phi_R(E)\) holds.

2.4. Scaling properties

We now demonstrate the scaling properties of the principal operator. Given a positive real number \(\gamma\), one can construct a unitary operator acting on the Fock space. Here \(U(\gamma)\) is a unitary operator that scales the momentum of the particle and orthofermion operators:

\[
U(\gamma)a(p)U^\dagger(\gamma) = \gamma^2 a(\gamma p), \quad U(\gamma)\chi(p)U^\dagger(\gamma) = \gamma^2 \chi(\gamma p).
\]

This in turn can be used to establish the following scaling property of the principal operator \(\Phi_R(E)\):

\[
U^\dagger(\gamma)\Phi_R(\gamma E; M, \lambda_R(M), m)U(\gamma) = \Phi_R(E; \gamma^{-1}M, \lambda_R(\gamma M), \gamma^{-1}m)
\]

\[
= \Phi_R(E; M, \lambda_R(\gamma M), \gamma^{-1}m).
\]

These follow from such scalings of equations (5) and (105). We derive (37) now. (38) will follow afterwards. To get (37), scale \(E\) and all momenta by \(\gamma\) in \(\Phi_R(E)\) and insert
$U(\gamma)U^\dagger(\gamma)$ as appropriate. We demonstrate this for $K_R(E; M)$, it can be shown for $U_1(E)$ and $U_2(E)$ precisely in the same manner:

\[
U^\dagger(\gamma)K_R(\gamma E; M, m)U(\gamma)
= -\frac{\pi}{4} U^\dagger(\gamma) \int dP \gamma^2 d^2 \xi \chi^\dagger(\gamma P, \gamma \xi) \times \\
\left\{ \int_{-1}^{1} d\tau \ln \left( \frac{2(\gamma \gamma^{-1} H_0 - \gamma E) \gamma P + \gamma^2 P^2 + 2\gamma \xi^2 (1 - \tau^2) + \gamma^2 4(\gamma^{-1} m)^2}{4 \gamma^2 (\gamma^{-1} M)^2} \right) \chi(\gamma P, \gamma \xi) \right\} U(\gamma)
= -\frac{\pi}{4} \int dP d^2 \xi \chi^\dagger(P, \xi) \int_{-1}^{1} d\tau \ln \left( \frac{2(H_0 - E) P + P^2 + \xi^2 (1 - \tau^2) + 4(\gamma^{-1} m)^2}{4 (\gamma^{-1} M)^2} \right) \chi(P, \xi)
= K_R(E; \gamma^{-1} M, \gamma^{-1} m).
\] (39)

Note that here we have used

\[
U^\dagger(\gamma)H_0 U(\gamma) = \gamma H_0.
\] (40)

Below, we obtain (38) in a more conventional way by means of the renormalization group equation.

### 2.4.1. Callan-Symanzik equation

It is instructive to look at the exact scaling properties of the principal operator $\Phi_R(E)$ from a more conventional perspective. A well-known approach to scaling in field theories, which is particularly suitable for renormalized correlation functions, is given by the Callan-Symanzik equation [PS95]. We obtain an analogous expression in our case directly for the principal operator.

Observe that the operator

\[
\gamma \frac{\partial}{\partial \gamma} + M \frac{\partial}{\partial M} + m \frac{\partial}{\partial m},
\] (41)

which can be thought of as a scale-invariant derivative, leads to

\[
\left( \gamma \frac{\partial}{\partial \gamma} + M \frac{\partial}{\partial M} + m \frac{\partial}{\partial m} \right) \Phi_R(E; \gamma^{-1} M, \lambda_R(M), \gamma^{-1} m) = 0.
\] (42)

Therefore we get

\[
\left( \gamma \frac{\partial}{\partial \gamma} - \beta \frac{\partial}{\partial \lambda_R} + m \frac{\partial}{\partial m} \right) \Phi_R(E; \gamma^{-1} M, \lambda_R(M), \gamma^{-1} m) = 0,
\] (43)

with $\beta$ as found before, using the definition of the $\beta$-function. If we consider this as an equation to be obeyed by the principal operator, we can look for a solution. As a simple ansatz, we propose that

\[
\Phi_R(E; \gamma^{-1} M, \lambda_R(M), \gamma^{-1} m) = f(\gamma)\Phi_R(E; M, \lambda_R(M), \gamma^{-1} m).
\] (44)
Acting on this with the operator just defined we get

\[ \left( \gamma \frac{\partial}{\partial \gamma} - \beta \frac{\partial}{\partial \lambda R} + m \frac{\partial}{\partial m} \right) f(\gamma) \Phi_R(E; M, \lambda_R(\gamma M), \gamma^{-1} m) = 0. \]  

(45)

This gives the condition

\[ \frac{\partial f(\gamma)}{\partial \gamma} = 0, \]  

(46)

which has the solution \( f(\gamma) = 1 \). We note that these results agree perfectly with the non-relativistic version of this theory \cite{Ber92, Ber94, AF95}. This renormalization group point of view works along similar lines on a manifold as well, as shown in \cite{ET13}.

2.4.2. Exact scaling result

The former derivation required an ansatz for the solution. However we can verify the same result directly as well since we renormalize the principal operator and solve the \( \beta \) function (28) non-perturbatively. It has the solution

\[ \lambda_R(\gamma M) = \frac{\lambda_R(M)}{1 + 2\pi \ln \gamma \lambda_R(M)}. \]  

(47)

We use this to verify (38) directly. We need to show that

\[ \frac{\Pi_1}{\lambda_R(M)} - 2K_R(E; \gamma^{-1} M) = \frac{\Pi_1}{\lambda_R(\gamma M)} - 2K_R(E; M). \]  

(48)

The remaining parts, \( U_1(E) \) and \( U_2(E) \) match trivially as they do not run with \( M \). We get

\[ \frac{\Pi_1}{\lambda_R(\gamma M)} - 2K_R(E; M) = \frac{\Pi_1}{\lambda_R(M)} - 2K_R(E; \gamma^{-1} M) + 2\pi \ln \gamma \Pi_1 - 2K_R(E; M) + 2K_R(E; \gamma^{-1} M), \]  

(49)

where we add and subtract the same term. With

\[ -2K_R(E; M) + 2K_R(E; \gamma^{-1} M) = -\frac{\pi}{2} \int dP d^2 \xi \chi^*(P, \xi) \int_{-1}^{1} d\tau \ln \left( \frac{4M^2}{4(\gamma^{-1} M)^2} \right)^{1/2} \chi(P, \xi) \]

\[ = -2\pi \ln \gamma \Pi_1, \]

(50)

we get the claimed equality.

3. Wave function of the two-particle bound state

In this section we follow an alternative but equivalent renormalization scheme that works better for energy estimates. Once again looking at (24), we see that choosing \( \lambda \) in terms of the binding energy of two particles \( \mu \) via

\[ \frac{1}{\lambda} = \frac{1}{2} \int_{-1}^{1} d\tau \int d^2 \eta \left( -\mu^2 \frac{1 - \tau^2}{4} + m^2 + \eta^2 \right)^{-1} \]  

(51)
leads to the following finite combination in the resolvent:

$$\Pi_1 - 2K(E) = \frac{\pi}{2} \int dP \, d^2\xi \, \chi^\dagger(P, \xi) \int_{-1}^1 d\tau \ln \left( \frac{[2(H_0 - E)P + P^2 + \xi^2](1 - \tau^2) + 4m^2}{-\mu^2(1 - \tau^2) + 4m^2} \right) \chi(P, \xi).$$

(52)

Using this, we find the wave function of the two-particle bound state from the discontinuity of the resolvent above and below its continuum of eigenvalues. It obeys the following for small $\epsilon$ near an eigenvalue $E$:

$$R_R(E + i\epsilon) - R_R(E - i\epsilon) = \frac{1}{H_R - (E + i\epsilon)} - \frac{1}{H_R - (E - i\epsilon)} = 2\pi i \delta(H_R - E)$$

(53)

$$= 2\pi i |\psi(E)\rangle\langle\psi(E)|.$$  

Since the spectrum of $H_0$ begins at $2m$ for two-particle states and we seek $0 < E < 2m$, we can replace $\frac{1}{H_0 - E + i\epsilon}$ with $\frac{1}{H_0 - E}$ for the following, we are never near an eigenvalue of $H_0$. We have, with (18),

$$R_R(E + i\epsilon) - R_R(E - i\epsilon) = \frac{1}{H_0 - E} b^\dagger \left( \frac{1}{\Phi_R(E + i\epsilon)} - \frac{1}{\Phi_R(E - i\epsilon)} \right) b \frac{1}{H_0 - E}.$$  

(54)

We expand $\frac{1}{\Phi_R}$ in terms of the eigenspace of $\Phi_R$:

$$\frac{1}{\Phi_R(E + i\epsilon)} - \frac{1}{\Phi_R(E - i\epsilon)} = \int dw \, \rho(w) |w\rangle\langle w| \left( \frac{1}{w(E + i\epsilon)} - \frac{1}{w(E - i\epsilon)} \right).$$  

(55)

Here $\rho(w)$ is the density of states for the eigenspace of $\Phi_R$. We can expand a given eigenvalue $w$ near its zero $E^*(w)$, $w(E^*) = 0$:

$$w(E + i\epsilon) = \frac{\partial w}{\partial E} \bigg|_{E^*} (E + i\epsilon - E^*), \quad w(E - i\epsilon) = \frac{\partial w}{\partial E} \bigg|_{E^*} (E - i\epsilon - E^*).$$  

(56)

This gives

$$\frac{1}{w(E + i\epsilon)} - \frac{1}{w(E - i\epsilon)} = \frac{2\pi i \delta(E - E^*)}{-\frac{\partial w}{\partial E} \bigg|_{E^*}} = 2\pi i \delta(w(E)).$$  

(57)

Here the last equality follows from expanding a Dirac-delta function with the zeros of its argument, and for a given $w$ there is only one such zero. Therefore

$$\frac{1}{\Phi_R(E + i\epsilon)} - \frac{1}{\Phi_R(E - i\epsilon)} = \int dw \, \rho(w) |w\rangle\langle w| 2\pi i \delta(w(E))$$

$$= 2\pi i \rho(E) |w(E)\rangle\langle w(E)|.$$  

(58)

5Generalized eigenvalues, more properly continuous spectrum, but we continue to use physics terminology.
Using (15) for \(b\) and \(b^\dagger\) and moving \(\frac{1}{i\hbar - E}\) terms inside, we have for a two-particle state

\[
R_R(E + i\epsilon) - R_R(E - i\epsilon) = \int \frac{[dp_1 \, dp_2]}{\sqrt{4p_{13}p_{23}}} \frac{a_1^\dagger a_2^\dagger}{\omega_1 + \omega_2 - E} \chi(p_1 + p_2) \left( \frac{1}{\Phi_R(E + i\epsilon)} - \frac{1}{\Phi_R(E - i\epsilon)} \right) \]

\[
\times \int \frac{[dp_3 \, dp_4]}{\sqrt{4p_{34}p_{43}}} \frac{\chi^\dagger(p_3 + p_4)}{\omega_3 + \omega_4 - E} \chi(p_1 + p_2) - 2\pi i \rho(w(E)) |w(E)\rangle \langle w(E)|
\]

\[
\times \int \frac{[dp_3 \, dp_4]}{\sqrt{4p_{34}p_{43}}} \frac{\chi^\dagger(p_3 + p_4)}{\omega_3 + \omega_4 - E} \chi(p_1 + p_2) \sqrt{\rho(w(E))} |w(E)\rangle \langle w(E)|
\]

\[
= 2\pi i \left( \int \frac{[dp_1 \, dp_2]}{\sqrt{4p_{13}p_{23}}} \frac{a_1^\dagger a_2^\dagger}{\omega_1 + \omega_2 - E} \chi(p_1 + p_2) \sqrt{\rho(w(E))} |w(E)\rangle \langle w(E)| \right) (H.C.) .
\]

Using this with (53) we get the wave function

\[
|\psi(E)\rangle = \int \frac{[dp_1 \, dp_2]}{\sqrt{4p_{13}p_{23}}} \frac{a_1^\dagger a_2^\dagger}{\omega_1 + \omega_2 - E} \chi(p_1 + p_2) \sqrt{\rho(w(E))} |w(E)\rangle .
\]

For the ground state \(E = \mu\) we have

\[
|w(\mu)\rangle = \int dP \, d^2\xi \, \chi^\dagger(P, \xi) \delta(P - \mu) \delta(\xi) |0\rangle
\]

\[
= \chi^\dagger(\mu, 0) |0\rangle .
\]

This is an eigenstate of \(\Phi_R(E)\). This can be verified by acting on it with (29). Note that due to translational invariance we expect a continuum of states. Using this we have

\[
|\psi(\mu)\rangle = \int \frac{[dp_1 \, dp_2]}{\sqrt{4p_{13}p_{23}}} \frac{a_1^\dagger a_2^\dagger}{\omega_1 + \omega_2 - \mu} \delta(p_{13} + p_{23} - \mu) \delta(p_1 + p_2 - \mu) \sqrt{\rho(w(\mu))} |0\rangle .
\]

Therefore the momentum-space wave function is

\[
\Psi(p_1, p_2) = \sqrt{2\rho(w(\mu))} \frac{\delta(p_{13} + p_{23} - \mu) \delta(p_1 + p_2 - \mu)}{\omega_1 + \omega_2 - \mu} .
\]

We take its Fourier transform to find the position-space wave function

\[
\Psi(x_1, x_2) = \sqrt{\frac{2\rho(w(\mu))}{(2\pi)^3}} \int [dp_1 \, dp_2] e^{-i(p_{13}x_1^3 + p_{23}x_2^3 + p_1 \cdot x_1^\dagger + p_2 \cdot x_2^\dagger)} \delta(p_{13} + p_{23} - \mu) \delta(p_1 + p_2 - \mu) .
\]

\[\overset{\text{Strictly speaking not eigenvectors, but they are called so in the generalized sense.}}{\text{(61)}}\]
We apply the same coordinate transformations that we have used before \(^{(22)}\), which gives

\[
\frac{\Psi(x_1, x_2)}{\sqrt{2\rho(w(\mu))}} = \int dP \ dQ \ d^2 \xi \ d^2 \eta \ e^{-i(P \frac{r_1^2 + r_2^2}{2} + Q(x_1^2 - x_2^2) + \xi \cdot \frac{x_1 + x_2}{2} + \eta \cdot \frac{x_1 - x_2}{2})} \frac{\delta(P - \mu)\delta(\xi)}{\omega_1 + \omega_2 - \mu}
\]

\[
= \int_{-1}^{1} \frac{d\tau}{\mu} \int_{0}^{\infty} \eta \ d\eta \int_{0}^{2\pi} d\phi \ e^{-i(\frac{\mu r_1^2 + r_2^2}{2} + \frac{\mu^2}{T}(x_1^2 - x_2^2) + \eta \cdot |x_1 - x_2| \cos \phi)} \frac{m^2 + \eta^2 + \frac{\mu^2}{T}(1 - \tau^2)}{2\omega_1(1 - \tau^2)} - \mu
\]

\[
= e^{-i(\frac{\mu r_1^2 + r_2^2}{2})} \frac{\pi}{2} \mu^2 \int_{-1}^{1} d\tau \ e^{-i\frac{\mu^2}{T}(x_1^2 - x_2^2)(1 - \tau^2)} K_0 \left( \frac{1}{2} |x_1 - x_2| \sqrt{4m^2 - (1 - \tau^2)\mu^2} \right) .
\] (65)

A two-particle system is separable in the center-of-mass and relative coordinates. Using

\[
x_{CM} = \frac{x_1 + x_2}{2}, \quad x_r = x_1 - x_2,
\] (66)

we get

\[
\Psi(x_{CM}, x_r) = N e^{-i\mu x_{CM}^2} \int_{-1}^{1} d\tau \ e^{-i\mu x_r^2(1 - \tau^2)} K_0 \left( \frac{1}{2} x_r \sqrt{4m^2 - (1 - \tau^2)\mu^2} \right) .
\] (67)

Here \(N\) is a normalization constant. We recall that the bound state wave function of two non-relativistic particles interacting via a delta function potential is exactly \(K_0(\sqrt{2m|E_b|}|x_1 - x_2|)\), up to a normalization constant \([\text{Raj99, Ber94, Hop82}]\). Here \(E_b\) refers to the binding energy. Note that for a relativistic system the absolute value of the binding energy is \(\sqrt{4m^2 - \mu^2}\), in our case we have a convolution over all such possible differences\(^7\). The convolution takes \(x_r^2\) into account in a subtle way, in the transverse direction the system behaves very much like a two dimensional delta potential, whereas in the light-front direction we have an oscillatory superposition of this two dimensional wave function with a weighted energy difference.

### 3.1. Normalizability of the wave function

We integrate the square of the wave function in the relative coordinates and show that it is finite and positive. For what follows we drop the center of mass part of the wave function.

---

\(^7\)Note that since \(\hbar c\) has dimensions of length times energy, we get a dimensionless combination inside \(K_0\). In the non-relativistic theory, to get a dimensionless variable from the relative distance, we need mass, energy and \(\hbar\).
function since that is what leads to a continuous spectrum and it is not expected to be normalizable:

\[
\int dx_r^3 d^2 x_r^\perp |\Psi(x_r)|^2 \propto \int dx_r^3 \int d^2 x_r^\perp \int_{-1}^{1} d\tau \int_{-1}^{1} d\tau' e^{-i2ma(\tau-\tau')x_3^r}(1-\tau^2)(1-\tau'^2)
\]
\[
\times \left\{ K_0(x_r^\perp m\sqrt{1-(1-\tau^2)a^2}) K_0(x_r^\perp m\sqrt{1-(1-\tau'^2)a^2}) \right\}
\]
\[
\propto \int_{-1}^{1} d\tau \int_{-1}^{1} d\tau' \delta(\tau-\tau')(1-\tau^2)(1-\tau'^2)
\]
\[
\times \left\{ \int_{0}^{\infty} \eta d\eta K_0(\eta \sqrt{1-(1-\tau'^2)a^2}) K_0(\eta \sqrt{1-(1-\tau'^2)a^2}) \right\}
\]
\[
\propto \int_{-1}^{1} d\tau \int_{-1}^{1} d\tau' \delta(\tau-\tau')(1-\tau^2)(1-\tau'^2) \ln \left( \frac{1-(1-\tau^2)a^2}{1-(1-\tau'^2)a^2} \right)
\]
\[
\propto \int_{-1}^{1} dt \frac{(1-t^2)^2}{1-(1-t^2)a^2}.
\]

This is clearly a positive, finite quantity, therefore the wave function is normalizable.

4. Large number of particles

In order to get an idea on bound states for large number of bosons, we propose that the mean field theory is a good approximation. We search for the smallest eigenvalue of \( \Phi_R(E) \) using the following variational ansatz:

\[
|\Omega_0\rangle = \int \frac{dq}{\sqrt{2q^3} \sqrt{(N-2)!}} \left( \prod_{i=1}^{N-2} \frac{dp_i}{\sqrt{2p_i}} \right) u(p_1) a_1(p_1) \psi(q) \chi_1(q) |0\rangle.
\]

Here we have the unknown wave functions \( u \) for the bosons and \( \psi \) for the orthofermion. This ansatz is for the sought after ground state wave function, therefore we take the expectation value of the principal operator \( \Phi_R(E) \) and get

\[
\langle \Omega_0 | \Phi_R(E) | \Omega_0 \rangle = \Omega(E).
\]

We minimize this eigenvalue by choosing \( u \) and \( \psi \) to find the smallest eigenvalue \( \Omega_*(E) \) of \( \Phi_R(E) \) and solve \( \Omega_*(E) = 0 \) for \( E \) to get a variational estimate on the bound state energy. This approach works thanks to the flow of eigenvalues of \( \Phi_R(E) \) that we discuss in section 2.3.

In principle, working out the variations with respect to the unknown functions \( u \) and \( \psi \) leads to variational equations to be solved. At present, we use a simpler approach and proceed by choosing \( u(p) \), leaving \( \psi(q) \) unknown, with both normalized:

\[
u(p) = \frac{4\alpha^2}{m^2\sqrt{\pi}p_3^2 e^{-a^2p_3^2/m^2} e^{-a^2p_1^2/m^2}},
\]
\[
\int \frac{|dp|}{2p_3}|u(p)|^2 = \int \frac{|dq|}{2q_3}|\psi(q)|^2 = 1.
\]

Using \([52], (20)\) and \([21]\), we take the expectation value of the principal operator under the given variational ansatz \(|\Omega_0\rangle\). Looking at \([21]\), we note that the operator \(U_2\) contains two pairs of ladder operators, which under this expectation value would come with \(N^2\), whereas \(U_1\) contains just one pair, which would come with \(N\). Therefore, as we consider the large \(N\) limit, we drop terms coming from \(U_1\). We also replace expectation values of functions of \(H_0\) as follows: Given a function \(f(H_0)\), we use the approximation

\[
\langle f(H_0) \rangle \approx f(N \langle h_0 \rangle),
\]

where

\[
\langle h_0 \rangle \equiv \int |dp|\omega(p)|u(p)|^2 \frac{2}{2p_3}.
\]

We note that this can be improved for example by means of a cumulant expansion, since both functions have integral representations using the exponential of \(H_0\). Using these we get in the large-\(N\) limit

\[
\Omega(E) \approx \frac{\pi}{2} \int dP d^2\xi \frac{\langle P, \xi \rangle^2}{2P} \int_{-1}^{1} d\tau \ln \left( \frac{2(N \langle h_0 \rangle - E)P(1 - \tau^2) + [(P^2 + \xi^2)(1 - \tau^2) + 4m^2]}{-\mu^2(1 - \tau^2) + 4m^2} \right)
\]

\[
- \frac{N^2}{16} \left( \frac{4\alpha^2}{m^2\sqrt{\pi}} \right)^4 \int \left[ \prod_{i=1}^{4} \langle dp_i | e^{-\alpha^2(p_{3i}^2 + p_{4i}^2)/m^2} \rangle \right] \frac{\psi^*(P_3 + P_4)\psi(P_1 + P_2)}{\sqrt{2}(p_{3i} + p_{4i})\sqrt{2}(p_{1i} + p_{2i})}.
\]

We drop the terms in square brackets, which are all positive, leading to

\[
\Omega(E) \approx \frac{\pi}{2} \int dP d^2\xi \frac{\langle P, \xi \rangle^2}{2P} \int_{-1}^{1} d\tau \ln \left( \frac{2(N \langle h_0 \rangle - E)P(1 - \tau^2)}{-\mu^2(1 - \tau^2) + 4m^2} \right)
\]

\[
- \frac{N^2}{16(N \langle h_0 \rangle - E)} \int \left[ \prod_{i=1}^{4} \langle dp_i | e^{-\alpha^2(p_{3i}^2 + p_{4i}^2)/m^2} \rangle \right] \frac{\psi^*(P_3 + P_4)\psi(P_1 + P_2)}{\sqrt{2}(p_{3i} + p_{4i})\sqrt{2}(p_{1i} + p_{2i})}.
\]

For the evaluation of the integral in the second term above we use the following coordinates:

\[
P = P_1 + P_2,
\]

\[
Q = \frac{1}{2}(P_1 - P_2).
\]

We define

\[
I = \int d^3P \, d^3Q \frac{\psi(P)}{\sqrt{2P_3}} e^{-\alpha^2(P^2/2 + 2Q^2)/m^2},
\]

where \(Q_3 \in (-\frac{1}{2}P_3, \frac{1}{2}P_3)\) is to be taken. With this, we have for the above integral

\[
\int \left[ \prod_{i=1}^{4} \langle dp_i | e^{-\alpha^2(p_{3i}^2 + p_{4i}^2)/m^2} \rangle \right] \frac{\psi^*(P_3 + P_4)\psi(P_1 + P_2)}{\sqrt{2}(p_{3i} + p_{4i})\sqrt{2}(p_{1i} + p_{2i})} = I^*I.
\]
In order to bound this integral, observe that
\[
|I| \leq \int d^3 P \, d^3 Q \, \left| \frac{\psi(P)}{\sqrt{2 P_3}} \right| e^{-\alpha^2 (P^2/2 + 2Q^2)/m^2}
\]
\[
\leq \left( \int d^3 P \, \left| \frac{\psi(P)}{\sqrt{2 P_3}} \right|^2 \right)^{1/2} \left( \int d^3 P e^{-\alpha^2 P^2/m^2} \left( \int d^3 Q e^{-\alpha^2 2Q^2/m^2} \right)^2 \right)^{1/2}.
\]
(80)

Therefore,
\[
I \leq m^9 (2\pi)^3 \frac{\sqrt{2\pi}}{64\alpha^9} \int_0^\infty dx \, \text{erf}(x) e^{-2x^2}
\]
\[
< m^9 \frac{\pi^7}{72\alpha^9}.
\]
(81)

Here \text{erf}(x) is the error function,
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \, e^{-t^2}.
\]
(82)

Now we work on the first term of (76). Observe that
\[
\pi^2 \int dP d^2 \xi \left| \frac{\psi(P, \xi)}{2P} \right|^2 \int_{-1}^1 d\tau \ln \left( \frac{2(N \langle h_0 \rangle - E) P(1 - \tau^2)}{-\mu^2 (1 - \tau^2) + 4m^2} \right) = \pi \ln \left( \frac{N \langle h_0 \rangle - E}{\sqrt{4m^2 - \mu^2}} \right)
\]
\[
+ \frac{\pi}{2} \int dP d^2 \xi \left| \frac{\psi(P, \xi)}{2P} \right|^2 \int_{-1}^1 d\tau \ln \left( \frac{2P(1 - \tau^2) \sqrt{4m^2 - \mu^2}}{-\mu^2 (1 - \tau^2) + 4m^2} \right).
\]
(83)

Here the latter term is positive and finite, we drop it in what follows. Combining these results, we get a lower bound for the expectation value of the principal operator,
\[
\Omega(E) > \pi \ln \left( \frac{N \langle h_0 \rangle - E}{\sqrt{4m^2 - \mu^2}} \right) - \frac{2m\pi^7}{9\alpha} \frac{N^2}{N \langle h_0 \rangle - E}
\]
(84)

Note that the expectation value \langle h_0 \rangle for the given trial function \( u(p) \) can be calculated:
\[
\langle h_0 \rangle = m \sqrt{\frac{2\pi}{8}} \left( \frac{3}{\alpha} + 4\alpha \right).
\]
(85)

We can minimize the lower bound for \( \Omega \) by properly adjusting the parameter \( \alpha \). That leads to a complicated equation. We instead consider a simpler lower bound and estimate from below. Assuming \( E < 0 \) for simplicity, we get
\[
\Omega(E) > \pi \ln \left( \frac{Nm \sqrt{\frac{2\pi}{2}} + |E|}{\sqrt{4m^2 - \mu^2}} \right) - \frac{2m\pi^7}{9} \frac{N^2}{Nm \sqrt{\frac{2\pi}{8}}}
\]
(86)

We now solve for the zero of this lower bound. The true ground state will be bounded from below by this value, thanks to the flow of eigenvalues that we have shown in section 2.3. As a result we find
\[
E_{gr} > -\sqrt{4m^2 - \mu^2} e^{\frac{8\sqrt{2\pi}}{2\pi} N}
\]
(87)
4.1. Remarks on the $2 + 1$ dimensional model

We can follow the same steps of analysis for $2 + 1$ dimensions. Here, the term that diverges logarithmically in $3 + 1$ dimensions, (23), is finite. The coupling constant has dimensions of energy. It is well-known that in $3 + 1$ dimensions there is no coupling constant renormalization, apart from normal ordering there is a single two loop diagram for mass renormalization. In the Tamm-Dancoff truncation that we use, mass renormalization will not show up, therefore our model is expected to be a finite theory.

As a result one would find for the principal operator everything as before now without the need for renormalization. Using the same eigenstate (61) for the principal operator, we can search for the bound state energy within the two particle sector. As a result, we find the equation for the binding energy $\mu$,

$$\lambda = \frac{1}{2\pi} \frac{\mu}{\sinh^{-1} \left( \frac{\mu}{\sqrt{4m^2 - \mu^2}} \right)}.$$  

(88)

In principle, we can replace the coupling constant with this expression, by specifying the two particle binding energy.

In order to understand the large number of particles limit, we use the same variational ansatz with the same wave functions, now normalized for $2 + 1$ dimensions. Note that in the large $N$ limit, in addition to the $U_1$ term that we ignored, we can now ignore the $K$ term as well. This term comes with $\frac{1}{\sqrt{N}}$ and can be ignored compared to the $U_2$ term which comes with $N^2$. Unlike in the case of $3 + 1$ dimensions there is a separate positive term, $\frac{1}{\lambda}$, that we should keep. Following the same ideas and estimates, this leads to a lower bound for the lowest eigenvalue,

$$\Omega(E) > \frac{1}{\lambda} - \frac{2\pi^2}{9} \frac{N^2}{Nm\sqrt{\pi} + |E|}.$$  

(89)

Again we solve for the zero of this lower bound of the eigenvalue to get an estimate for the ground state energy, as a result we find

$$E_{gr} > -\frac{2\pi^2}{9} \lambda N^2.$$  

(90)

Due to our limited form of the trial wave function, this cannot be considered as a definitive proof, but it gives us some indication that the $2 + 1$ dimensional theory behaves much better than its higher dimensional counterpart. A more careful analysis should give a better estimate of this lower bound to the ground state energy. We remark that in the exact result for the $1 + 1$ dimensional non-relativistic version \cite{McG64}, the ground state energy goes with $-\frac{\lambda^2}{18} N^3$ to leading order, so the present theory seems to behave even better in $2 + 1$ dimensions.

5. Conclusion

Equation (87) gives us some indication that for large number of particles, the ground state energy of our truncated theory may be bounded from below, which makes the theory
well-defined as it stands. This result is in accordance with the previously found bound in
the $2 + 1$ dimensional non-relativistic system \cite{Raj99}. Interestingly, the non-relativistic
system in three dimensions requires further subtle fine tuning to obtain a well-defined
ground state energy.

Nevertheless, the bound that we present is far off from what we would like to find. A
ture relativistic theory with pair creation processes should not have bound state energies
well below zero. That would mean that one can create more particles and by binding
them reduce the total energy of the system, and the vacuum would then become unstable.
Our crude estimates cannot answer this question at the moment, moreover, the truncated
part of the Hamiltonian will lead to new divergent terms which should be cured by mass
renormalization. As a result, this question cannot be answered in a satisfactory manner at
this stage and requires a deeper scrutiny. The $2 + 1$ dimensional version of our truncation
behaves much better, it may be possible to think of it as a consistent theory by itself
describing some intermediate energy phenomena of light bosons. Alternatively, there is a
possibility that the terms that we drop due to truncation can be added to this version as
perturbations to gain a better approximation of the full version.

In any case, the present truncation can give us some insight about the spectrum of
few-body systems with sufficiently weak attractive interactions. We establish a well-
defined resolvent after a coupling constant renormalization. Moreover, we show that the
truncated theory is asymptotically free and the principal operator satisfies an operator
analog of Callan-Symanzik equation. How much of this can be retained for the full theory
remains as a future challenge.

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A. Coordinates

Given a four-vector $b^\mu = (b^0, b^1, b^2, b^3)$ in Cartesian coordinates, we make a coordinate
change to the light-front coordinates $b^\mu = (b^0 - b^3, b^1, b^2, b^3)$. This gives the Lorentz
invariant inner product of a four-vector $x$ to be

$$x_\mu x^\mu = 2x^- x^3 + (x^-)^2 - (x^1)^2 - (x^2)^2 = 2x^- x^3 - x^2_1 - x^2_2 - x^2_3.$$  \hfill (91)
Using $x_\mu y^\mu = g_{\mu\nu} x'^\nu y^\mu$ we find that

$$
x_- = x^- + x^3 = x^0, \quad x_3 = x^- - x^3.
$$

(92)

All vector components in this work are written in these coordinates. For simplicity we denote the components of an arbitrary four-vector $b$ as

$$
b_\mu = (b_0, b_1, b_2, b_3), \quad b^\mu = (b^0, b^1, b^2, b^3),
$$

$$
\equiv (b_-, b_\perp, b_3), \quad \equiv (b^-, b^\perp, b^3).
$$

(93)

Here we call $b_\perp \equiv (b_1, b_2)$ and $b^\perp \equiv (b^1, b^2)$ as the transverse components.

### A.1. Lorentz transformations

For consistency, we present a few key points related to our choice of coordinates. Note that three independent Lorentz generators can be chosen, depending on whether the velocity is in the $x^3$ direction or in the transverse directions.

In the first case, we have the Lorentz transformations, with the rapidity parameter $\tanh(\theta) = v$,

$$
x^- \mapsto e^{\theta} x^-, \quad x^3 \mapsto e^{\theta} x^3 + \sinh(\theta) x^-, \quad \text{with } x^+ \mapsto x^+.
$$

(94)

The dual variables transform as

$$
p_- \mapsto e^{\theta} p_- - \sinh(\theta) p_3, \quad p_3 \mapsto e^{-\theta} p_3, \quad \text{with } p_\perp \mapsto p_\perp.
$$

(95)

Note that this is important not only kinematically but also for the vacuum structure of the theory to remain intact, since $p_3$ only gets scaled under this transformation the condition in (3) is respected for all observers.

### A.1.1. Transverse Lorentz transformations

Similarly for the transverse directions, we find

$$
x^- \mapsto \cosh(\theta) x^- + \sinh(\theta) x^3, \quad x^3 \mapsto \cosh(\theta) x^3 + \sinh(\theta) x_- + \sinh(\theta) x^3,
$$

(96)

with $x^3 \mapsto x^3$. The dual variables in this case transform as

$$
p_- \mapsto \cosh(\theta) p_- - \sinh(\theta) p_\perp,
p_\perp \mapsto \cosh(\theta) p_\perp - \sinh(\theta) p_-,
p_3 \mapsto p_3 - \sinh(\theta) p_\perp + (\cosh(\theta) - 1) p_-.
$$

(97)

The invariance of the sign of $p_3$ is important to keep the definition of the vacuum intact. In the Hamiltonian formalism that we work with, we are on-shell, that is, $p_-$ is not an
independent variable, it is constrained by \[ (2). \] When we use this constraint in the transformation of \( p_3 \) and reorganize terms, we find that

\[
p_3 \mapsto \frac{1}{p_3} \left( \cosh \left( \frac{\theta}{2} \right) p_3 - \sinh \left( \frac{\theta}{2} \right) p_{\perp} \right)^2 + \sinh^2 \left( \frac{\theta}{2} \right) m^2 ,
\]

which is manifestly positive.

Thus the sign of \( p_3 \) determines the sign of the transformed \( p_3 \), which means the vacuum condition remains intact \textit{on-shell}, which is the only requirement in the Hamiltonian formalism.

For comparison, the conventional light-front coordinates with \( p_+ \) and \( p_- \) have a very similar on-shell condition \cite{Hei01, Har96}. If we apply the same transverse Lorentz transformations to the light-front momentum \( p_- \) in that setting, we get

\[
p_- \mapsto \frac{1}{2p_-} \left( \cosh \left( \frac{\theta}{2} \right) p_- - \sinh \left( \frac{\theta}{2} \right) p_{\perp} \right)^2 + \sinh^2 \left( \frac{\theta}{2} \right) m^2 ,
\]

which also preserves positivity.

\section*{A.2. Fock space realizations of the generators}

For completeness we list the Fock space realizations of the Poincaré generators of our coordinates, all given at light-front time \( x^- = 0 \):

The Hamiltonian is

\[
H = P_- = \int \left[ dp \right] \frac{m^2 + p_1^2 + p_3^2}{2p_3} a^\dagger a .
\]

The momenta are

\[
P_3 = \int \left[ dp \right] p_3 a^\dagger a ,
\]

\[
P_1 = \int \left[ dp \right] p_1 a^\dagger a ,
\]

\[
P_2 = \int \left[ dp \right] p_2 a^\dagger a .
\]

The angular momenta are

\[
J_3 = i \int \left[ dp \right] \left( p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1} \right) a^\dagger a ,
\]

\[
J_1 = i \int \left[ dp \right] \left( \frac{m^2 + p_1^2 + p_3^2}{2p_3} \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_3} - p_3 \frac{\partial}{\partial p_2} \right) a^\dagger a ,
\]

\[
J_2 = i \int \left[ dp \right] \left( - \frac{m^2 + p_1^2 + p_3^2}{2p_3} \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_3} + p_3 \frac{\partial}{\partial p_1} \right) a^\dagger a .
\]

\[ ^8 \text{We do not need to demand the positivity of } p_3 \text{ at this stage. It takes a careful calculation to check that } (2) \text{ remains invariant under the listed Lorentz transformations, as it should.} \]
The boosts are

\[ K_3 = i \int |dp| \left( p_3 \frac{\partial}{\partial p_3} a^\dagger \right) a, \]

\[ K_1 = i \int |dp| \left( \frac{m^2 + p_1^2 + p_2^2}{2p_3} \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_3} \right) a^\dagger a, \]

\[ K_2 = i \int |dp| \left( \frac{m^2 + p_1^2 + p_2^2}{2p_3} \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_3} \right) a^\dagger a. \]

Following Harindranath [Har96], we define linear combinations that have closed subalgebras:

\[ F_1 = -K_1 + J_2 = -2i \int |dp| \left( \frac{m^2 + p_1^2 + p_2^2}{2p_3} \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_3} - \frac{1}{2} p_3 \frac{\partial}{\partial p_1} \right) a^\dagger a, \]

\[ F_2 = -K_2 - J_1 = -2i \int |dp| \left( \frac{m^2 + p_1^2 + p_2^2}{2p_3} \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_3} - \frac{1}{2} p_3 \frac{\partial}{\partial p_2} \right) a^\dagger a, \]

\[ E_1 = -K_1 - J_2 = -i \int |dp| \left( p_3 \frac{\partial}{\partial p_1} \right) a^\dagger a, \]

\[ E_2 = -K_2 + J_1 = -i \int |dp| \left( p_3 \frac{\partial}{\partial p_2} \right) a^\dagger a. \]

We list the commutators of these generators at table 1. Those that are trivially zero are not noted for ease of reading.

| \( P_3 \) | \( P_1 \) | \( P_2 \) | \( K_1 \) | \( E_1 \) | \( E_2 \) | \( J_3 \) | \( F_1 \) | \( F_2 \) | \( H \) |
|---|---|---|---|---|---|---|---|---|---|
| \( P_3 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( P_1 \) | \( iP_3 \) | 0 | 0 | 0 | 0 | 0 | 0 | \( -2iP_3 \) | \( -2iP_3 \) |
| \( P_2 \) | 0 | \( -iP_3 \) | 0 | \( -iP_3 \) | 0 | \( -iP_3 \) | 0 | \( -iP_3 \) | 0 |
| \( K_1 \) | \( -iP_3 \) | 0 | 0 | \( -iE_1 \) | \( -iE_2 \) | 0 | \( iF_1 \) | \( iF_2 \) | \( -iP_3 + iH \) |
| \( E_1 \) | 0 | \( iP_3 \) | 0 | \( iE_1 \) | 0 | \( -iE_2 \) | \( -2iK_3 \) | \( -2iJ_3 \) | \( iF_1 \) |
| \( E_2 \) | 0 | 0 | \( iP_3 \) | \( iE_1 \) | 0 | \( -iE_2 \) | \( 2iK_3 \) | \( -2iK_3 \) | \( iF_2 \) |
| \( J_3 \) | 0 | \( iP_3 \) | \( -iP_3 \) | 0 | \( iE_2 \) | \( iF_2 \) | \( -iF_1 \) | 0 | \( 0 \) |
| \( F_1 \) | \( 2iP_3 \) | \( 2iH - iP_3 \) | 0 | \( -iF_1 \) | \( 2iK_3 \) | \( -2iF_2 \) | \( -iF_2 \) | 0 | \( -iP_3 \) |
| \( F_2 \) | \( 2iP_3 \) | \( 2iH - iP_3 \) | 0 | \( -iF_1 \) | \( 2iK_3 \) | \( 2iK_3 \) | \( iF_2 \) | 0 | \( iF_2 \) |
| \( H \) | \( iP_3 - iH \) | \( -iP_3 \) | \( -iP_3 \) | 0 | \( -iF_1 \) | \( -iF_2 \) | 0 | \( -iP_3 \) | \( -iP_3 \) |

Table 1: Commutators of the Poincaré generators.

### B. Orthofermion and principal operator

We very briefly summarize parts of Rajeev’s approach in [Raj99], introducing the orthofermion \( \chi \) and its algebra. It has the extreme number statistics:

\[ \chi(p)\chi^\dagger(q) = \delta(p - q)\Pi_0, \quad \chi(p)\chi(q) = 0. \]

\[ (105) \]
The following are projections onto the no-orthofermion and one-orthofermion subspaces:

\[ \Pi_0 = \int [dp] \chi(p) \chi(p), \quad \Pi_1 = \int [dp] \chi(p) \chi(p). \]  

(106)

A Hermitian operator \( O \) that acts on this space can be decomposed as follows with respect to the orthofermion number:

\[ O = \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix}. \]  

(107)

This has the inverse

\[ O^{-1} = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix}, \]  

(108)

where

\[ \alpha = (a - b^\dagger d^{-1} b)^{-1} = a^{-1} + a^{-1} b^\dagger (d - ba^{-1} b^\dagger)^{-1} ba^{-1}. \]  

(109)

We define the principal operator \( \Phi \) as

\[ \Phi = d - ba^{-1} b^\dagger. \]  

(110)

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