Factorization of the current algebra and integrable top-like systems.

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Abstract

A hierarchy of integrable hamiltonian nonlinear ODEs is associated with any decomposition of the Lie algebra of Laurent series with coefficients being elements of a semi-simple Lie algebra into a sum of the subalgebra consisting of the Taylor series and some complementary subalgebra. In the case of the Lie algebra $so(3)$ our scheme covers all classical integrable cases in the Kirchhoff problem of the motion of a rigid body in an ideal fluid. Moreover, the construction allows us to generate integrable deformations for known integrable models.

Key words: Integrable nonlinear ODE, Lax pair, current algebra

1 Introduction

In the paper we consider integrable nonlinear ODEs with Lax representations defined by a vector space decomposition of the algebra $\mathcal{G}_\lambda$ of Laurent series with respect to the parameter $\lambda$ with coefficients being elements of a semi-simple Lie algebra $\mathcal{G}$ into a direct sum

$$\mathcal{G}_\lambda = \mathcal{G}_+ \oplus \mathcal{G}_-,$$

(1.1)

where $\mathcal{G}_-$ is the subalgebra of Taylor series and $\mathcal{G}_+$ is some complementary subalgebra. Following the terminology by I. Cherednik [1], we shall call such complementary subalgebras factorizing ones. For the standard situation, $\mathcal{G}_+$ coincides with the set $\mathcal{G}_+^{st}$ of all polynomials in $\lambda^{-1}$ with zero free term.
In the case when the factorizing subalgebra is isotropic with respect to the invariant nondegenerate form

\[ <X(\lambda), Y(\lambda) > = \text{res}(X(\lambda), Y(\lambda)), \quad X(\lambda), Y(\lambda) \in \mathcal{G}_\lambda, \]  

(1.2)

where \((\cdot, \cdot)\) is the Killing form on \(\mathcal{G}\), there exist deep interconnections between factorizing subalgebras and solutions of the classical Yang-Baxter equation [2]. Non-isotropic subalgebras probably have not been seriously investigated.

Equations of the Landau-Lifshitz type form the most known class of integrable PDEs related to the factorizing subalgebras [3]. The Lax representation for these equations has the form

\[ L_t = A_x + [A, L], \]  

(1.3)

where \(L, A\) are elements of a factorizing algebra \(\mathcal{G}_+\). A relationship between factorizing subalgebras and systems of the chiral model type has been established in [6].

In the case if a solution of the equation of the Landau-Lifshitz type does not depend on \(x\), relation (1.3) is reduced to

\[ L_t = [A, L]. \]  

(1.4)

While for equation (1.3) it is naturally to suppose that both \(L\) and \(A\) belong to the same Lie algebra, equation (1.4) is self-consistent also in the case when \(A\) belongs to a Lie algebra whereas \(L\) lies in some module over this algebra. One of our main observations is that besides equations (1.4), where \(L, A \in \mathcal{G}_+\), there is a second, maybe more interesting, class of integrable nonlinear ODEs, for which \(A \in \mathcal{G}_+, L \in \mathcal{G}_+^\perp\). Here \(\perp\) stands for the orthogonal complement with respect to form (1.2). These two classes coincide for isotropic factorizing subalgebras. It is easy to see that \(\mathcal{G}_+^\perp\) is a module over \(\mathcal{G}_+\). It is especially important for us that equations on \(\mathcal{G}_+^\perp\) possesses a natural hamiltonian structure defined by a linear Poisson bracket. In general, equations of the second class do not admit any two-dimensional generalization (1.3).

Apparently, the Lax equations with \(L \in \mathcal{G}_+^\perp\) slip expert’s minds. It is shown in Section 4 that precisely such Lax representations for \(\mathcal{G} = so(3)\) correspond to classical integrable homogeneous quadratic Hamiltonians in rigid body dynamics (namely, to spinning tops by Euler, Kirchhoff, Clebsch and Steklov-Lyapunov). All factorizing subalgebras for \(\mathcal{G} = so(3)\) have been classified in the paper [4]. It turns out that the classical tops are in one-to-one correspondence with the factorizing subalgebras. In addition, various deformations of known integrable models found recently in [7, 8, 10] turn out to be generated by non-isotropic factorizing subalgebras (see Section 5).

In Sections 2,3 a general algebraic theory for integrable models related to factorizing subalgebras is developed. In particular, a description of commuting flows and linear Poisson brackets on finite-dimensional orbits is given. It is interesting to note that if the factorizing subalgebra satisfies the following condition

\[ \lambda^{-m}\mathcal{G}_+ \subset \mathcal{G}_+ \]  

(1.5)

for some \(m \geq 1\), then the equations of motion are hamiltonian with respect to one more linear Poisson bracket. This fact explains the existence of linear transformations found by A. Bobenko [11], which interconnect known integrable cases on \(e(3)\) and \(so(4)\). The compatibility
of two linear Poisson brackets has been taking as a basis for the approach developed in [3].

If condition (1.5) is fulfilled with \( m = 1 \), the factorizing subalgebra \( G_+ \) is said to be homogeneous. In Section 6 the homogeneous subalgebras are considered. In particular, we show that the factorizing subalgebra associated with any finite group of reduction (see [12]) is equivalent to a homogeneous one.

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## 2 General scheme.

Let \( \mathcal{A} \) be an associative algebra equipped with nondegenerate semi-Frobenius bilinear form \( \langle \cdot, \cdot \rangle \). The latter means that

\[
\langle a, b \rangle = \langle b, a \rangle, \quad \langle a, b \ast c \rangle = \langle a \ast b, c \rangle.
\]

By \( \mathcal{A}^{(-)} \) denote the adjoint Lie algebra with the bracket \( [a, b] = a \ast b - b \ast a \). It is easy to see that \( \langle \cdot, \cdot \rangle \) induces the following invariant form on \( \mathcal{A}^{(-)} \). In other words,

\[
\langle [a, b], c \rangle = -\langle b, [a, c] \rangle. \tag{2.6}
\]

Let \( \mathcal{U} \) be a Lie subalgebra in \( \mathcal{A}^{(-)} \) such that the restriction \( \langle \cdot, \cdot \rangle_\mathcal{U} \) of the form \( \langle \cdot, \cdot \rangle \) to \( \mathcal{U} \) is nondegenerate and

\[
\mathcal{A}^{(-)} = \mathcal{U} \oplus \mathcal{U}^\perp. \tag{2.7}
\]

Suppose that

\[
\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-, \tag{2.8}
\]

where \( \mathcal{U}_- \) and \( \mathcal{U}_+ \) are Lie subalgebras. By \( \pi_- \) and \( \pi_+ \) denote the projectors onto \( \mathcal{U}_- \) and \( \mathcal{U}_+ \), respectively.

Let \( I_- \) and \( I_+ \) be ideals of finite codimension in \( \mathcal{U}_- \) and \( \mathcal{U}_+ \). Consider a vector subspace in \( \mathcal{U} \) of the form

\[
\mathcal{O} = I_-^\perp \cap I_+^\perp, \tag{2.9}
\]

where \( \perp \) stands for the orthogonal complement in \( \mathcal{U} \) with respect to the form \( \langle \cdot, \cdot \rangle_\mathcal{U} \). Since

\[
I_-^\perp \cap I_+^\perp = (I_- + I_+)^\perp,
\]

the vector space \( \mathcal{O} \) is finite-dimensional.

**Example 2.1** Let \( \mathcal{A} \) be arbitrary associative algebra, \( \mathcal{U} \) be the Lie algebra of Laurent series with respect to the parameter \( \lambda \) with coefficients from \( \mathcal{A} \), and \( r \) be a fixed constant element of \( \mathcal{A} \). It is easy to see that

\[
\mathcal{U}_+ = \{ \sum_{i=1}^{p} a_i \lambda^{-i}(1 + \lambda r) \mid a_i \in \mathcal{A}, \ p \in \mathbb{N} \} \tag{2.10}
\]
is a factorizing Lie subalgebra in $U$, which is complementary to the subalgebra $U_-$ of all Taylor series. One can verify that

$$U_+^\perp = \{ \sum_{i=1}^q (1 + \lambda r)^{-1}a_i \lambda^{-i} \mid a_i \in \mathcal{A}, \; q \in \mathbb{N} \}. \quad (2.11)$$

Define the orbit $O_n$ by the following pair of ideals

$$I_+ = U_+, \quad I_- = \lambda^n U_-.$$ 

It follows from (2.9) that

$$O_n = \lambda^{-n} U_- \cap U_+^\perp.$$ 

Thus we have

$$O_n = \{ \sum_{i=1}^n (1 + \lambda r)^{-1}a_i \lambda^{-i} \mid a_i \in \mathcal{A} \}.$$ 

Let $f(L, \lambda_1, \cdots \lambda_k)$ be a polynomial with constant coefficients such that $L \in O$ and $\lambda_i$ are arbitrary elements from the center of $\mathcal{A}$. By $\bar{f}$ denote the projection of $f$ onto $U$ corresponding to the decomposition (2.7). Expressions like $\bar{f}$ we shall call spectral invariants of $L$. Since $[f, L] = 0$, we have also $[\bar{f}, L] = 0$. If $\bar{g}$ is one more spectral invariant, then $[\bar{f}, L] = 0$ implies $[\bar{f}, g] = 0$. Now it follows from (2.7) that $[\bar{f}, \bar{g}] = 0$.

**Theorem 2.1** The set $O$ is a finite-dimensional orbit with respect to the flow defined by the Lax equation

$$L_t = [\pi_+ (\bar{f}), L], \quad (2.12)$$

where $\bar{f}$ is arbitrary spectral invariant of $L$.

**Proof.** In order to prove that $O$ is an orbit, we have to verify that $L \in O$ implies $[\pi_+ (\bar{f}), L] \in O$. By definition, $\pi_+ (\bar{f}) \in U_+$. It follows from the invariance of $\langle \cdot, \cdot \rangle$ that $I_+^\perp$ is a module over $U_+$. Therefore $[\pi_+ (\bar{f}), L] \in I_+^\perp$. From $[f, L] = 0$, we get $[\pi_+ (\bar{f}), L] = -[\pi_-(\bar{f}), L]$. As $I_-^\perp$ is a module over $U_-$, we have $[\pi_+ (\bar{f}), L] \in I_-^\perp$. This implies that $[\pi_+ (\bar{f}), L] \in O$.

As we will see below, the Lax equation possesses a rich store of first integrals and commuting flows.

**Proposition 2.1** Expression $\langle \bar{g}_1, \bar{g}_2 \rangle$ is a first integral of equation (2.12) for any two spectral invariants $h_1 = \bar{g}_1$ and $\bar{g}_2$.

**Proof.** Since $g_i$ is a polynomial in $L$, we have $(g_i)_t = [\pi_+(\bar{f}), g_i]$. It follows from the decomposition (2.7) that

$$(g_i)_t = [\pi_+(\bar{f}), g_i]. \quad (2.13)$$

The invariance (2.6) of the form $\langle \cdot, \cdot \rangle$ guarantees that the expression

$$h_t = \langle \bar{g}_1, [\pi_+(\bar{f}), \bar{g}_2] \rangle + \langle \bar{g}_2, [\pi_+(\bar{f}), \bar{g}_1] \rangle.$$ 

is equal to zero.
Theorem 2.2  For any spectral invariants \( \bar{g}_1 \) and \( \bar{g}_2 \), the flows

\[
L_t = [\pi_+ (\bar{g}_1), L] \tag{2.14}
\]

and

\[
L_r = [\pi_+ (\bar{g}_2), L] \tag{2.15}
\]
on the orbit \( \mathcal{O} \) commute each other.

Proof. Using (2.13), (2.14), (2.15), we get

\[
L_t L_r - L_r L_t = [\pi_+ [\pi_+ (\bar{g}_2), \bar{g}_1] - [\pi_+ (\bar{g}_2), \pi_+ (\bar{g}_1)], L]. \tag{2.16}
\]
The right hand side of this relation vanishes because, as it was mentioned above, \([\bar{g}_2, \bar{g}_1] = 0\).

It turns out that under the following natural (cf. (2.9)) additional condition

\[
\mathcal{O}^\perp = I_- + I_+,
\]
equation (2.12) is a hamiltonian one. Define \( \mathcal{H} \) by the formula

\[
\mathcal{H} = U_- / I_- \bigoplus U_+ / I_+,
\]
where the algebra \( \bar{U}_+ \) is obtained from \( U_+ \) by the replacement of the bracket \([a, b] \) by \([b, a] \). Stress that the symbol \( \bigoplus \) in this formula stands for the direct sum of Lie algebras in contrast with (2.8), where it means the vector space direct sum.

Define a nondegenerate bilinear form, pairing the algebra \( \mathcal{H} \) and the orbit \( \mathcal{O} \) as follows

\[
(\langle a_- + I_- + I_+, L \rangle) = \langle a_+ + a_-, L \rangle. \tag{2.17}
\]
Since \( \langle I_- \bigoplus I_+, \mathcal{O} \rangle = 0 \), the form (2.17) is well defined. The Kirillov bracket on \( \mathcal{O} \) is given by

\[
\{\varphi, \psi\}(L) = ([\text{grad}_L \varphi, \text{grad}_L \psi], L), \quad L \in \mathcal{O}, \quad \text{grad}_L \varphi, \text{grad}_L \psi \in \mathcal{H}. \tag{2.18}
\]
Suppose that the bases \( f^i \) and \( e_i \) in \( \mathcal{O} \) and \( \mathcal{H} \) are adjoint with respect to the form (2.17),

\[
[e_i, e_j] = c_{ij}^k e_k
\]
and \( L = q_i f^i \). Then, because of the formula \( \text{grad} q_i = e_i \), the Poisson brackets between the coordinate functions are given by

\[
\{q_i, q_j\} = c_{ij}^k q_k. \tag{2.19}
\]

Theorem 2.3 (cf. [13])

- i) The functionals of the form

\[
H_g = \langle g, 1 \rangle, \tag{2.20}
\]

where \( g(L, \lambda_1, \ldots, \lambda_k) \) is a polynomial in all arguments with constant coefficients and \( L \in \mathcal{O} \), are in involution with respect to the bracket (2.19);
• ii) The Lax equation

\[ L_t = [\pi_+(\bar{f}), L], \]  

(2.21)

where

\[ f = -\frac{dg}{dL}(L, \lambda_1, \cdots, \lambda_k), \]  

(2.22)

on the orbit \( O \) is hamiltonian with respect to the bracket (2.19) and the Hamiltonian function \( H_g \).

Proof. Let \( H_{g_1} \) and \( H_{g_2} \) be functionals of the form (2.20). Then

\[ \text{grad} H_{g_i} = -\left( \pi_-(\bar{f}_i) + I, \pi_+(\bar{f}_i) + J \right), \]

where \( f_i \) and \( g_i \) are related by the formula (2.22). According to (2.18), we have

\[ \{H_{g_1}, H_{g_2}\} = \left< -[\pi_+(f_1), \pi_+(f_2)] + [\pi_-(f_1), \pi_-(f_2)], L \right> = \]

\[ \frac{1}{2} < [(-\pi_+(\bar{f}_1) \pi_-(\bar{f}_1)), \bar{f}_2] - [(-\pi_+(\bar{f}_2) + \pi_-(\bar{f}_2)), \bar{f}_1], L > . \]

It follows from the invariance of the form \( \left< \cdot, \cdot \right> \) and from the relations \( \{\bar{f}_1, L\} = [\bar{f}_2, L] = 0 \), that the latter expression is equal to zero. The item 1 is proved. The Hamiltonian system with the Hamiltonian \( H_g \) has the form

\[ \frac{d\Psi(L)}{dt} = \{\Psi, H_g\}(L) = (\text{grad}_L(\Psi), L_t). \]

From the other hand, \( \text{grad}_L(\Psi) = (\pi_-(a) + I_-, \pi_+(a) + I_+) \) for some \( a \in \mathcal{U} \). Therefore,

\[ (\text{grad}_L(\Psi), [\pi_+(\bar{f}), L]) = \frac{1}{2} \left( (\pi_-(a) + I_-) \left[ \pi_+(\bar{f}) - \pi_-(\bar{f}), L \right] \right) = \]

\[ \frac{1}{2} < [\pi_-(a) + \pi_+(a), \pi_+(\bar{f}) - \pi_-(\bar{f})] + [\pi_-(a) + \pi_+(a), \bar{f}], L > = \]

\[ < [\pi_+(a), \pi_+(\bar{f})] - [\pi_-(a), \pi_-(\bar{f})], L >= \]

\[ \left( [\text{grad}_L(\Psi), \text{grad}_L(H_g)], L \right) = \{\Psi, H_g\}(L). \]

In the next to last equality we used the formula

\[ \text{grad} H_g = -\left( \pi_-(\bar{f}) + I_-, \pi_+(\bar{f}) + I_+ \right), \]

where \( g \) and \( f \) are related by (2.22). The nondegeneracy of the form \( \left< \cdot, \cdot \right> \) implies that the equation \( L_t = [\pi_+(\bar{f}), L] \) is hamiltonian with respect to the bracket (2.18). This completes the proof of Theorem.

Suppose the ideals \( I_+ \) and \( I_- \) satisfy the following additional conditions:

\[ \mu I_+ \subset \mathcal{G}_+, \quad \mu I_- \subset \mathcal{G}_-, \quad \mu^{-1}O \subset \mathcal{U} \]  

(2.23)

for some invertible element \( \mu \) from the center of the associative algebra \( \mathcal{A} \). Then:
Remark 2.1 The orbit $\mathcal{O}_\mu$ constructed by means of the ideals $\mu I_+$ and $\mu I_-$, is related to $\mathcal{O}$ by the formula

$$\mathcal{O}_\mu = \mu^{-1} \mathcal{O}.$$  

Remark 2.2 The equation $\tilde{L}_t = \left[ \pi_+ (\bar{F}(L, \lambda_1, \cdots \lambda_k)), \bar{L} \right]$ on the orbit $\mathcal{O}_\mu$ coincides with equation (2.21) on the orbit $\mathcal{O}$, where $\bar{L} = \mu^{-1} L$, $f(L, \lambda_1, \cdots \lambda_k) = F(\bar{L}, \lambda_1, \cdots \lambda_k)$.

Remark 2.3 Under condition (2.23), Lax equation (2.21) has two hamiltonian structures described in Theorem 3. The first structure corresponds to the pair of ideals $I_+, I_-$ whereas the second is related to the pair $\mu I_+, \mu I_-$. The question whether these Poisson brackets are compatible remains still open.

3 The case of the current algebra.

In this paper we consider the current algebra over a semi-simple Lie algebra $G$ as a basic example of the Lie algebra $U$. We assume that $G$ is embedded into the matrix algebra $Mat_{k \times k}$ by the adjoint representation. In this case the associative algebra $A$ is the set of all Laurent series with respect to the parameter $\lambda$ with coefficients from $Mat_{k \times k}$. The bilinear form on $A$ is defined by the formula

$$<X(\lambda), Y(\lambda) > = \text{res} \left( \text{trace}(XY) \right).$$  

We take

$$G_\lambda = \left\{ \sum_{i=-n}^{\infty} g_i \lambda^i \mid g_i \in G, \ n \in \mathbb{Z} \right\}$$  

and

$$G_- = \left\{ \sum_{i=0}^{\infty} g_i \lambda^i \mid g_i \in G \right\}$$  

for $U$ and $U_-$. Suppose that as a vector space, $G_\lambda$ is a direct sum of $G_-$ and some complementary factorizing subalgebra $G^+$. The standard factorizing subalgebra is given by

$$G^+ = \left\{ \sum_{i=1}^{n} g_i \lambda^{-i} \mid g_i \in G, \ n \in \mathbb{N} \right\}.$$  

It is easy to see that both $G_-$ and $G^+$ are isotropic with respect to the form

$$<X, Y> = \text{res} \left( \text{trace}(ad_X \cdot ad_Y) \right), \quad X, Y \in G_\lambda.$$  

Define the orbit $\mathcal{O}_n$ by the pair of ideals

$$I_+ = G_+, \quad I_- = \lambda^n G_-.$$  

It follows from (2.9) that

$$\mathcal{O}_n = \lambda^{-n} G_- \cap G^+.$$  

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For the standard factorizing subalgebra $G^st_+$ we have

$$\mathcal{O}_n = \{ \sum_{i=1}^{n} g_i \lambda^{-i} \}.$$  

The spectral invariants are supposed to be polynomials of the form $f(L, \lambda_1, \lambda_2)$, where $\lambda_1 = \lambda, \lambda_2 = \lambda^{-1}$.

The corresponding linear Poisson bracket (2.19) is related to the finite-dimensional Lie algebra

$$\mathcal{H} = \mathcal{G}_- / \lambda^n \mathcal{G}_-.$$  \hspace{1cm} (3.8)

Since $\mathcal{G}_-$ coincides with the algebra of Taylor series, the algebra $\mathcal{H}$ is isomorphic to the polynomial algebra $\mathcal{G}[\varepsilon]$, where $\varepsilon^n = 0$. In particular, if $\mathcal{G} = so(3)$ and $n = 2$, then $\mathcal{H}$ is isomorphic to the Lie algebra $\mathfrak{e}(3)$ of motions of Euclidean space $\mathbb{R}^3$.

Remark 2.3 states that if the subalgebra $G^+_\lambda$ satisfies condition (1.5) for some $0 \leq m \leq n$, then the corresponding linear Poisson bracket $\{ \cdot, \cdot \}_m$ can be constructed such that the spectral invariants of the operator $L$ are in involution with respect to this bracket. The Poisson bracket $\{ \cdot, \cdot \}_m$ corresponds to the finite-dimensional Lie algebra

$$\mathcal{H}_m = \mathcal{G}_- / I \oplus \mathcal{G}_+/J,$$  \hspace{1cm} (3.9)

where $I = \lambda^{n-m} \mathcal{G}_-$, $J = \lambda^{-m} \mathcal{G}_+$. The algebra $\mathcal{G}_+$ is obtained from $\mathcal{G}_+$ by the replacement of $[a, b]$ with $[b, a]$. The symbol $\oplus$ in (3.9) means the direct sum of Lie algebras. If $\mathcal{G} = so(3)$, $n = 2$, and $m = 1$, then in generic case $\mathcal{H}_1$ is isomorphic to the Lie algebra $so(4)$.

When $m = 0$, condition (1.5) is always fulfilled and we find ourselves in the situation described by (3.8). A subalgebra is called homogeneous if condition (1.5) with $m = 1$ holds. Section 6 is devoted to homogeneous subalgebras.

### 4 The current algebra over $so(3)$ and classical spinning tops.

In this section we consider the case $G^\lambda = so(3)$. Let us fix the basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the Lie algebra $so(3)$. It is easy to check that

$$[e_2, e_1] = e_3, \quad [e_3, e_2] = e_1, \quad [e_1, e_3] = e_2.$$  

Define the invariant bilinear form on $so(3)$ as

$$(x_1 e_1 + x_2 e_2 + x_3 e_3, y_1 e_1 + y_2 e_2 + y_3 e_3) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$  

The scalar product in the current algebra $G^\lambda$ is defined by

$$\langle X, Y \rangle = \text{res}(X, Y), \quad X, Y \in G^\lambda.$$
Evidently, any factorizing subalgebra $G_+$ contains unique elements of the form

$$E_1 = \frac{1}{\lambda} e_1 + O(\lambda), \quad E_2 = \frac{1}{\lambda} e_2 + O(\lambda), \quad E_3 = \frac{1}{\lambda} e_3 + O(\lambda) \quad (4.1)$$

that generate $G_+$ as a Lie algebra. It is also clear that if we take arbitrary series (4.1) and generate a Lie subalgebra $G_+$, then the sum of this subalgebra and the algebra of all Taylor series coincides with the whole current algebra. The only problem is that, in general, this sum is not direct.

It follows from the invariance of the scalar product $< \cdot, \cdot >$ that $G_+^\perp$ is a module over $G_+$. Obviously, $G_+^\perp$ contains unique elements of the form

$$R_1 = \frac{1}{\lambda} e_1 + O(\lambda), \quad R_2 = \frac{1}{\lambda} e_2 + O(\lambda), \quad R_3 = \frac{1}{\lambda} e_3 + O(\lambda) \quad (4.2)$$

that generate $G_+^\perp$ as a module over $G_+$.

**Example 1.** It can be checked that the generators

$$E_1 = \frac{1}{\lambda} e_1(1 + q\lambda)^{1/2}(1 + r\lambda)^{1/2},$$

$$E_2 = \frac{1}{\lambda} e_2(1 + p\lambda)^{1/2}(1 + r\lambda)^{1/2},$$

$$E_3 = \frac{1}{\lambda} e_3(1 + p\lambda)^{1/2}(1 + q\lambda)^{1/2}$$

defines a factorizing subalgebra (see [6]). The orthogonal complement is generated by

$$R_1 = \frac{1}{\lambda} e_1(1 + q\lambda)^{-1/2}(1 + r\lambda)^{-1/2},$$

$$R_2 = \frac{1}{\lambda} e_2(1 + p\lambda)^{-1/2}(1 + r\lambda)^{-1/2},$$

$$R_3 = \frac{1}{\lambda} e_3(1 + p\lambda)^{-1/2}(1 + q\lambda)^{-1/2}.$$

The simplest orbit is given by (3.7) with $n = 1$. It consists of the operators of the form

$$L = M_1 R_1 + M_2 R_2 + M_3 R_3.$$ 

The function $H_\lambda = (L, L)$ is a polynomial in $\mu = \lambda^{-1}$, whose coefficients are functions

$$H_1 = \text{res}(\lambda H_\lambda) = M_1^2 + M_2^2 + M_3^2$$

and

$$H_2 = \text{res}(H_\lambda) = pM_1^2 + qM_2^2 + rM_3^2.$$

Since in this case the algebra (3.8) is isomorphic to $so(3)$, these polynomials commute each other with respect to the Poisson bracket

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad (4.3)$$

where $\varepsilon_{ijk}$ is the totally skew-symmetric tensor.
Taking for $A$ the operator

$$A = \pi_+(L) = M_1E_1 + M_2E_2 + M_3E_3,$$

related to the Hamiltonian $H_2$ by (2.22), we get a Lax pair for the Euler top

$$M_t + M \times VM = 0,$$

where

$$V = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix}.$$ (4.4)

The orbit (3.7) with $n = 2$ consists of the operators of the form

$$L = \gamma_1S_1 + \gamma_2S_2 + \gamma_3S_3 + M_1R_1 + M_2R_2 + M_3R_3.$$ (4.5)

If we choose the adjoint bases in $\mathcal{O}$ and $\mathcal{H}$ (see section 2), then

$$S_1 = [R_3, E_2] + \frac{1}{2}(q-r)R_1, \quad S_2 = [R_1, E_3] + \frac{1}{2}(r-p)R_2, \quad S_3 = [R_2, E_1] + \frac{1}{2}(p-q)R_3.$$ 

The function $H_\lambda = (L, L)$ is (up to a multiplier) a polynomial in $\mu = \lambda^{-1}$. As algebra (3.8) is isomorphic to $\mathfrak{e}(3)$, the coefficients of this polynomial commute each other with respect to the Poisson bracket

$$\{M_i, M_j\} = \varepsilon_{ijk}M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0.$$ (4.6)

Two non-trivial coefficients are as follows

$$H_1 = M_1^2 + M_2^2 + M_3^2 + pM_1\gamma_1 + qM_2\gamma_2 + rM_3\gamma_3 + \frac{1}{4}(q-r)^2\gamma_1^2 + \frac{1}{4}(p-r)^2\gamma_2^2 + \frac{1}{4}(p-q)^2\gamma_3^2$$

and

$$H_2 = pM_1^2 + qM_2^2 + rM_3^2 + p(q+r)M_1\gamma_1 + q(p+r)M_2\gamma_2 + r(p+q)M_3\gamma_3$$

$$+ \frac{1}{4}p(q+r)^2\gamma_1^2 + \frac{1}{4}q(p+r)^2\gamma_2^2 + \frac{1}{4}r(p+q)^2\gamma_3^2.$$ 

This commuting pair of quadratic polynomials defines the integrable Steklov-Lyapunov case in the Kirchhoff problem of the motion of a rigid body in an ideal fluid. The equations of motion defined by the Hamiltonian $H_1$ correspond to $A$-operator of the form

$$A = \pi_+(\lambda L) = M_1E_1 + M_2E_2 + M_3E_3.$$ 

The flow defined by the Hamiltonian $H_2$ corresponds to $A = \pi_+(L)$.

**Example 2.** One can verify that the generators

$$E_1 = \frac{1}{\lambda}e_1\sqrt{1-p\lambda^2}, \quad E_2 = \frac{1}{\lambda}e_2\sqrt{1-q\lambda^2}, \quad E_3 = \frac{1}{\lambda}e_3\sqrt{1-r\lambda^2}$$

give rise to a factorizing subalgebra. This subalgebra is isotropic with respect to the form $<\cdot, \cdot>$ and therefore $R_1 = E_i$. The orbit (3.7) with $n = 2$ consists of operators of the form (4.5), where

$$S_1 = [E_2, E_3] \quad S_2 = [E_3, E_1], \quad S_3 = [E_1, E_2].$$
The function $H_\lambda = (L, L)$ leads to two non-trivial quadratic polynomial
\[ H_1 = M_1^2 + M_2^2 + M_3^2 - (q + r)\gamma_1^2 - (p + r)\gamma_2^2 - (p + q)\gamma_3^2 \]
and
\[ H_2 = pM_1^2 + qM_2^2 + rM_3^2 - qr\gamma_1^2 - pr\gamma_2^2 - pq\gamma_3^2. \]
This pair of quadratic polynomials commuting with respect to bracket (4.6) defines the Clebsch integrable case in the Kirchhoff problem of the motion of a rigid body in an ideal fluid. Equations of motions for the Hamiltonian $H_1$ corresponds to operator $A = \pi_+(\lambda L)$. The flow for $H_2$ is given by $A = \pi_+(L)$.

**Example 3.** Let
\[ R = \begin{pmatrix} 0 & r_3 & r_1 \\ -r_3 & 0 & r_2 \\ -r_1 & -r_2 & 0 \end{pmatrix} \quad (4.7) \]
be arbitrary constant element of $so(3)$. Then the generators
\[
E_1 = \frac{e_1}{\lambda} + \nu [R, e_1] + \frac{1}{2}[R, [R, e_1]], \\
E_2 = \frac{e_2}{\lambda} + \nu [R, e_2] + \frac{1}{2}[R, [R, e_2]], \\
E_3 = \frac{e_3}{\lambda} + \nu [R, e_3] + \frac{1}{2}[R, [R, e_3]]
\]
define a factorizing subalgebra. The generators of the orthogonal complement are given by
\[
R_1 = \frac{e_1}{\lambda} + \nu [R, e_1] - \frac{1}{2}[R, [R, e_1]] - (R, R)e_1, \\
R_2 = \frac{e_2}{\lambda} + \nu [R, e_2] - \frac{1}{2}[R, [R, e_2]] - (R, R)e_2, \\
R_3 = \frac{e_3}{\lambda} + \nu [R, e_3] - \frac{1}{2}[R, [R, e_3]] - (R, R)e_3.
\]
The orbit with $n = 2$ is described by formula (1.5), where
\[
S_1 = [E_2, R_3]+\frac{1}{2}(r_2^2-r_3^2)R_1, \quad S_2 = [E_3, R_1]+\frac{1}{2}(r_3^2-r_1^2)R_2, \quad S_3 = [E_1, R_2]+\frac{1}{2}(r_1^2-r_2^2)R_3.
\]
Coefficients of $H_\lambda = (L, L)$ yield two non-trivial quadratic polynomials $H_1$ and $H_2$, which commute with respect to the Poisson bracket (4.6). It is not difficult to find a linear combination of $H_1$ and the Casimir functions for bracket (4.6) that equals the square of the following linear integral
\[ I = r_1M_1 + r_2M_2 + r_3M_3 + \frac{1}{2}r_1(r_2^2-r_3^2)\gamma_1 + \frac{1}{2}r_2(r_3^2-r_1^2)\gamma_2 + \frac{1}{2}r_3(r_1^2-r_2^2)\gamma_3. \]
Thus the factorizing subalgebra from Example 3 gives rise to a Lax representation for the Kirchhoff integrable case in the problem of the motion of a rigid body in an ideal fluid.

The factorizing subalgebras from Examples 2,3 are invariant with respect to multiplying by $\lambda^{-2}$, whereas the subalgebra from Example 1 admits multiplication by $\lambda^{-1}$. According to
remark [23] the corresponding flows are hamiltonian with respect to one more linear bracket defined by the Lie algebra \( \mathcal{H} = \mathcal{G}_+/\lambda^{-2}\mathcal{G}_+ \), which is isomorphic to \( \text{so}(4) \) for generic values of parameters. This fact explains the existence of linear transformations found by A. Bobenko [11], which interconnect known integrable cases on \( \mathfrak{e}(3) \) and \( \text{so}(4) \).

As it was shown in [3], any factorizing subalgebra for \( \text{so}(3) \) yields an integrable PDE of the Landau-Lifshitz type for one vector unknown \( s = (s_1, s_2, s_3) \), where \(|s| = 1\). In this case the \( L \)-operator is of the form

\[
L = s_1E_1 + s_2E_2 + s_3E_3, \tag{4.8}
\]

and the \( A \)-operator has the following structure:

\[
A = s_1E_2 \times E_3 + s_2E_3 \times E_1 + s_3E_1 \times E_2 + q_1E_1 + q_2E_2 + q_3E_3, \tag{4.9}
\]

where \( q_i \) are some (individual for each factorizing subalgebra) differential polynomials in the components of the vector \( s \). Their explicit form can be easily determined from the Lax equation (1.3).

It turns out that for Example 2 relation (1.3) is equivalent (up to change of signs for independent variables) to the Landau-Lifshitz equation

\[
s_t = s \times s_{xx} + s \times Vs,
\]

where the matrix \( V \) is given by (4.4). For the first time, this Lax representation for the Landau-Lifshitz equation was found in [14].

The subalgebra from Example 1 yields the equation

\[
s_t = s \times \left( s_{xx} + \frac{1}{2}(V[s, s] + [s, s]V) + \frac{1}{2}[s, Vss + s_sV] + [s, Vs + sV] \right). \tag{4.10}
\]

At last, the subalgebra from Example 3 corresponds to equation

\[
s_t = s \times \left( s_{xx} + \frac{1}{2}(Z[s, s] + [s, s]Z) + \frac{1}{2}[s, Zss + s_sZ] + [s, Zs + sZ] + cZs \right),
\]

where

\[
Z = \begin{pmatrix}
r_1^2 & r_1r_2 & r_1r_3 \\
r_1r_2 & r_2^2 & r_2r_3 \\
r_1r_3 & r_2r_3 & r_3^2
\end{pmatrix}, \quad c = \nu^2 + \frac{r_1^2 + r_2^2 + r_3^2}{4}.
\]

In other words, one can say that \( Z \) is arbitrary matrix of rank 1 and \( c \) is arbitrary constant. The above list of three equations coincides with the list of the paper [15], where all equations of the Landau-Lifshitz type having higher conservation laws have been found.

## 5 Other examples.

### 5.1 Generalization of Example 1 to the \( \text{so}(n) \)-case.

The factorizing subalgebra from Example 1 can be defined by the formula

\[
\mathcal{U}_+ = (1 + \lambda V)^{1/2} \mathcal{U}^{s*}(1 + \lambda V)^{1/2}, \tag{5.1}
\]
where $\mathcal{U}^{st}$ denotes the set of polynomials in $\lambda^{-1}$ with coefficients from $so(3)$, and the matrix $V$ is given by (4.1). It can easily be checked that formula (5.1), where $V$ is arbitrary diagonal matrix, defines a factorizing sublagebra for $G = so(n)$ as well. The orthogonal complement to $\mathcal{U}_+$ is given by

$$\mathcal{U}_+^\perp = (1 + \lambda V)^{-1/2} \mathcal{U}^{st}(1 + \lambda V)^{-1/2}.$$ 

The simplest orbit (3.7) corresponding to $n = 1$ gives rise to the following hamiltonian equation

$$M_t = [V, M^2], \quad M \in so(n)$$

on $so(n)$ with the Lax pair

$$L = (1 + \lambda V)^{-1/2} \frac{M}{\lambda} (1 + \lambda V)^{-1/2}, \quad A = (1 + \lambda V)^{1/2} \frac{M}{\lambda} (1 + \lambda V)^{1/2}.$$ 

The equation

$$M_t = [V, M^2] + [M, \Gamma], \quad \Gamma_t = VM\Gamma - \Gamma MV, \quad M, \Gamma \in so(n)$$

corresponding to the orbit with $n = 2$, possesses the Lax pair

$$L = (1 + \lambda V)^{-1/2} \left( \frac{M}{\lambda^2} + \frac{\Gamma}{\lambda} \right) (1 + \lambda V)^{-1/2}, \quad A = (1 + \lambda V)^{1/2} \frac{M}{\lambda} (1 + \lambda V)^{1/2}.$$ 

One can regard this equation as a $so(n)$-generalization of the Steklov-Lyapunov top. In this case the dynamical variables are formed in a pair of matrices from $so(n)$ and the linear Poisson bracket is defined by the Lie algebra $so(n) \oplus \epsilon so(n)$, where $\epsilon^2 = 0$.

For the generalized Landau-Lifshitz equation

$$S_t = P_x + PV S - SV P, \quad S_x = [S, P],$$

related to subalgebra (5.1), the Lax representation is given by

$$L = (1 + \lambda V)^{1/2} \frac{S}{\lambda} (1 + \lambda V)^{1/2}, \quad A = (1 + \lambda V)^{1/2} \left( \frac{S}{\lambda^2} + \frac{P}{\lambda} \right) (1 + \lambda V)^{1/2}.$$ 

For the Lie algebra $so(3)$ on the orbit $(S, S) = 1$, we have $P = [S, S_x] + qS$, where the unknown function $q$ has to be found from the condition $(S, S_t) = 0$. As the result, we get (up to $t \to -t$) equation (4.10).

### 5.2 An example on the Kac-Moody algebra related to $so(n, m)$. 

Let

$$\mathcal{G} = \{ A \in \text{Mat}_{(n+m)\times(n+m)} | A^T = -SAS \},$$

where

$$S = \begin{pmatrix} E_n & 0 \\ 0 & -E_m \end{pmatrix}.$$ 

It is clear that the Lie algebra $\mathcal{G}$ is isomorphic to $so(n+m)$. Consider the subalgebra $\mathcal{U}$ of the current algebra over $\mathcal{G}$ consisting of such Laurent series that the coefficients of even (respectively, odd) powers of $\lambda$ belong to $V_1$ (respectively, $V_{-1}$). Here by $V_{\pm 1}$ denote eigenspaces.
of the inner second order automorphism $G \to SGS^{-1}$, corresponding to eigenvalues $\pm 1$. Actually, this means that the coefficients of even powers of $\lambda$ have the following block structure

\[
\begin{pmatrix}
v_1 & 0 \\
0 & v_2
\end{pmatrix},
\]

where $v_1 \in so(n)$, $v_2 \in so(m)$, and the coefficients of odd powers are of the form

\[
\begin{pmatrix}
0 & w \\
w^t & 0
\end{pmatrix},
\]

where $w \in \text{Mat}_{n,m}$.

We choose $\text{res}(\lambda^{-1}\text{tr}(XY))$ for the non-degenerate invariant form on $U$. Note that in this case the form $\text{res}(\text{tr}(XY))$ is generate.

Let $U_-$ be the set of all Taylor series from $U$,

\[U_+ = (1 + \lambda r)^{1/2} U^{st} (1 + \lambda r)^{1/2},\]

where $U^{st}$ is the set of polynomials in $\lambda^{-1}$ from $U$ and $r$ is arbitrary constant matrix of the form

\[
r = \begin{pmatrix}
0 & r_1 \\
-r_1^t & 0
\end{pmatrix}.
\]

Such choice of factorizing subalgebra is a natural generalization of Example 2.1 to the case, when the structure of coefficients of series from $U$ are defined by an additional automorphism of second order.

Suppose $I_+ = \lambda^2 U_-$ and

\[I_+ = \{(1 + \lambda r)^{1/2} \sum_{i=2}^{k} q_i \lambda^{-i} (1 + \lambda r)^{1/2}\}.
\]

Then the orbit (2.9) consists of the elements of the form

\[L = (1 + \lambda r)^{-1/2}(\lambda^{-1} w + v + \lambda u)(1 + \lambda r)^{-1/2}.
\] (5.2)

The corresponding Lax equation

\[L_t = [\pi_+(L), L]\] (5.3)

is equivalent to the system of equations

\[w_t = [w, wr + rw - v], \quad v_t = [u, w] + vwr - rvw, \quad u_t = uwr - rvu.
\] (5.4)

It is easy to see that this system admits the reduction

\[
u = \begin{pmatrix}
0 & r_1 \\
r_1^t & 0
\end{pmatrix},
\]

which is equivalent to the model found in [10]. In the case $r = 0$ the system and it’s Lax representation have been considered in [16].
Let us consider the case $G = \text{so}(3, 2)$ in more detail. Without loss of generality the matrices $u$ and $r$ may be chosen as follows

\[
  u = \begin{pmatrix}
    0 & 0 & 0 & a_1 & 0 \\
    0 & 0 & 0 & 0 & a_2 \\
    0 & 0 & 0 & 0 & 0 \\
    a_1 & 0 & 0 & 0 & 0 \\
    0 & a_2 & 0 & 0 & 0 
  \end{pmatrix}, \quad r = \begin{pmatrix}
    0 & 0 & 0 & k_1 & 0 \\
    0 & 0 & 0 & 0 & k_2 \\
    0 & 0 & 0 & 0 & 0 \\
    -k_1 & 0 & 0 & 0 & 0 \\
    0 & -k_2 & 0 & 0 & 0 
  \end{pmatrix},
\]

(5.5)

where $a_1 k_2 = a_2 k_1$. Consider a special case $k_2 = k_1 = k$, $a_2 = a_1 = a$, which is remarkable since the corresponding Hamiltonian admits an additional Hamiltonian reduction (see [13]). In this case the $L$-operator is given by (5.2), where

\[
  w = \begin{pmatrix}
    0 & 0 & 0 & \gamma_1 & \delta_1 \\
    0 & 0 & 0 & \gamma_2 & \delta_2 \\
    0 & 0 & 0 & \gamma_3 & \delta_3 \\
    \gamma_1 & \gamma_2 & \gamma_3 & 0 & 0 \\
    \delta_1 & \delta_2 & \delta_3 & 0 & 0 
  \end{pmatrix},
\]

(5.6)

and, as a simple computation shows,

\[
  v = \begin{pmatrix}
    0 & -M_3 & M_2 & 0 & 0 \\
    M_3 & 0 & -M_1 & 0 & 0 \\
    -M_2 & M_1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & M_4 + k(\gamma_2 - \delta_1) \\
    0 & 0 & 0 & -M_4 - k(\gamma_2 - \delta_1) & 0 
  \end{pmatrix}.
\]

Among the spectral invariants, there is the integral $(M_4 - M_3)^2$. Reduction mentioned above is that we fix the integral value: $M_4 - M_3 = z$. Note that in order to the dynamics defined by Lax equation (1.4) to be consistent with the reduction, an appropriate skew-symmetric matrix has to be added to the operator $A = \pi_+(L)$ (cf. [13]).

One can check that under the reduction $\text{tr}L^2$ yields the Hamiltonian (see [10])

\[
  H = M_1^2 + M_2^2 + 2M_3^2 + 2k(M_3\gamma_2 - M_2\gamma_3) + 2k(M_1\delta_3 - M_3\delta_1) - 2a\gamma_1 - 2a\delta_2 + 2zM_3,
\]

while $\text{tr}L^4$ gives us two fourth degree integrals providing the Liouville integrability of this Hamiltonian. The further reduction $\delta_1 = \delta_2 = \delta_3 = 0$ leads to the integrable case for the Kirchhoff problem found in [7]. The Poisson bracket for this reduction corresponds to the Lie algebra $e(3)$ (see formula (4.6)).

One of possible integrable generalizations for the reduced Hamiltonian to the family of brackets

\[
  \{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = \kappa \varepsilon_{ijk} M_k.
\]

(5.7)

has been proposed in [9]. It turns out that

\[
  L = R^{-1/2}(\lambda^{-1}w + v + \lambda u)R^{-1/2}
\]

(5.8)
with

\[
R = \begin{pmatrix}
1 & 0 & 0 & k_1\lambda & 0 \\
0 & 1 & 0 & 0 & k_2\lambda \\
0 & 0 & 1 & 0 & 0 \\
-k_1\lambda & 0 & 0 & 1 - \kappa \lambda^{-2} & 0 \\
0 & -k_2\lambda & 0 & 0 & 1
\end{pmatrix},
\]

can be taken for a Lax operator for this model. The parameter \(\kappa\) from the Poisson bracket (5.7) is related to \(k_1, k_2\) by the formula

\[
\kappa = \frac{k_1^2 - k_2^2}{k_2^4}.
\]

The matrix \(u\) is defined by (5.5), where \(a_1 k_2 = a_2 k_1\). The matrix \(w\) is given by (5.6) with \(\delta_1 = \delta_2 = \delta_3 = 0\). At last, the matrix \(v\) is of the form

\[
v = \begin{pmatrix}
0 & -M_3 & M_2 & 0 & 0 \\
M_3 & 0 & -M_1 & 0 & 0 \\
-M_2 & M_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \tau M_3 + k_2\gamma_2 + z & 0 \\
0 & 0 & 0 & -\tau M_3 - k_2\gamma_2 - z & 0
\end{pmatrix},
\]

where \(\tau = \frac{k_1}{k_2}\). The spectral invariants of the operator \(L\) commute each other with respect to bracket (5.7). The Hamiltonian

\[
H = \tau^2 M_1^2 + M_2^2 + 2\tau^2 M_3^2 + 2k_1(M_3\gamma_2 - M_2\gamma_3) - 2a_1\gamma_1 + 2\tau z M_3
\]

is one of them. If \(\tau = 1\), the Poisson bracket corresponds to the algebra \(\mathfrak{e}(3)\) and the \(L\)-operator coincides with the one described above. In general case, the algebraic nature of the matrix \(R\) remains to be mysterious (cf. [17]).

Explicit formulas for other integrable deformations of models from the paper [16] can be found in [10, 18]. However the Lax representations presented in these papers are not as algebraic and elegant as (5.2). In spite of that they looks similar to (5.2), the matrix \(r\) is variable.

### 6 Homogeneous subalgebras.

To efficiently describe the finite-dimensional orbits corresponding a factorizing subalgebra \(\mathcal{G}_+\), it is necessary to find the vector space \(\mathcal{G}_+^+\) in an explicit form. It easily can be done in the case, when \(\mathcal{G}_+\) is a homogeneous subalgebra (see [6]).

A factorizing subalgebra \(\mathcal{G}_+\) is called \textbf{homogeneous} if the following condition holds

\[
\frac{1}{\lambda} \mathcal{G}_+ \subset \mathcal{G}_+.
\]

Examples of homogeneous subalgebras can be found in [5, 6].

Let \(A(\lambda)\) be a formal series of the form

\[
A = E + R \lambda + S \lambda^2 + \cdots
\]

where \(R, S, \ldots\) are linear constant operators from \(\mathcal{G}\) to \(\mathcal{G}\), \(E\) is the identity operator.
Theorem 6.1 (see [6])

- i) Any homogeneous subalgebra $G_+$ can be represented in the form

$$G_+ = \{ \sum_{i=1}^{k} \lambda^{-i} A(g_i), \ g_i \in G, \ k \in \mathbb{N} \}$$

(6.3)

where $A(\lambda)$ is a formal series of the form (6.2).

- ii) Vector space (6.3) is a subalgebra iff for any $X, Y \in G$ 

$$[A(X), A(Y)] = A \left( [X, Y] + \lambda [X, Y]_1 \right),$$

(6.4)

where $[\cdot, \cdot]_1$ is a Lie bracket compatible with the bracket $[\cdot, \cdot]$.

- iii) For any homogeneous subalgebra the bracket $[\cdot, \cdot]_1$ is given by

$$[X, Y]_1 = [R(X), Y] + [X, R(Y)] - R([X, Y]),$$

(6.5)

where $R$ is the corresponding coefficient in (6.2).

Recall that the compatibility of two Lie brackets means that arbitrary linear combination of these brackets is also a Lie bracket.

For all homogeneous subalgebras $G_+$ the orthogonal complement $G_+^\perp$ can be found by means of the same formula.

Theorem 6.2 Let $G_+$ be a homogeneous subalgebra. Then

$$G_+^\perp = (A^{-1})^T (G_+^{st}),$$

(6.6)

where the series $A$ generates $G_+$ by formula (6.3), the standard factorizing subalgebra $G_+^{st}$ is defined by (3.4), and $T$ stands for transposition with respect to the scalar product (3.1).

Proof. It follows from (6.3) that $G_+ = A(G_+^{st})$. Hence

$$< A(G_+^{st}), (A^{-1})^T (G_+^{st}) > = < G_+^{st}, G_+^{st} > = 0,$n

i.e. $G_+^\perp \supset (A^{-1})^T (G_+^{st})$. From other hand, if $g \in G_+^\perp$, then $< A(G_+^{st}), g > = 0$. This implies $A^T (g) \in (G_+^{st})^\perp = G_+^{st}$ or $g \in (A^{-1})^T (G_+^{st})$.

6.1 The orbits for the case of homogeneous subalgebras.

The following statement is convenient for finding of explicit form of Lax equations corresponding to homogeneous subalgebras.

Proposition 6.1 Suppose $A$ satisfies (6.4); then the following identity holds

$$[A(X), (A^{-1})^T (Y)] = (A^{-1})^T \left( [X, Y] + \lambda X * Y \right).$$

(6.7)

Here for any $X, Y \in G$

$$X * Y = [R(X), Y] - [X, R^t(Y)] + R^t([X, Y]),$$

(6.8)

where $R$ is the coefficient from (6.2), $t$ stands for transposition with respect to the invariant form on $G$. 

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To prove identity (6.7) it is suffice to multiply innerly both sides by \( A(Z) \) and transform the left hand side using the invariance of the scalar product \(<\cdot,\cdot>\) and identity (6.4).

Notice that
\[ X * Y = -(\text{ad}_1 X)^t(Y), \]
where \( \text{ad}_1 X(Y) = [X, Y]_1 \) and the bracket \([\cdot,\cdot]_1\) is given by (6.3). In other words, the operator \(-(\text{ad}_1 X)^t\) defines the coadjoint action of the Lie algebra equipped with the bracket \([\cdot,\cdot]_1\) on the Lie algebra with the bracket \([\cdot,\cdot]\).

Any element of the orbit
\[ \mathcal{O} = \lambda^{-1}G_- \cap \lambda^2 G^+_1, \]
corresponding to the homogeneous subalgebra (6.3) has the form
\[ L = (A^{-1})^T (\lambda u + v + \lambda^{-1} w). \]

It is easy to see that \( \pi_+(L) = A(\lambda^{-1}w). \)

Using identity (6.7), one can obtain that the component-wise form of the Lax equation (5.3) is
\[ w_t = [w, v] + w * w, \quad v_t = [w, u] + w * v, \quad u_t = w * u, \quad (6.9) \]

**Continuation of Example 2.1.** In the situation of Example 2.1, \( A \) is the operator of right multiplication by \( 1 + \lambda r \), \( (A^{-1})^T \) is the operator of left multiplication by \( (1 + \lambda r)^{-1} \) and an element of the orbit is given by
\[ L = (1 + \lambda r)^{-1} (\lambda u + v + \lambda^{-1} w). \quad (6.10) \]

Formula (6.8) reads as follows \( X * Y = rXY - YXr \). The Lax equation (5.3) written in components is equivalent to
\[ w_t = [w, wr + rw - v], \quad v_t = [u, w] + vwr - rwv, \quad u_t = uwr - rwu. \quad (6.11) \]

Note that the Lax equation is equivalent to the following Lax triad
\[ P_t = P \lambda^{-1} w (1 + \lambda r) - (1 + \lambda r) \lambda^{-1} w P, \quad (6.12) \]
where \( P = \lambda u + v + \lambda^{-1} w \).

Suppose that the algebra \( A \) is equipped with an involution \( * \). Then system (6.11) admits two following reductions:

- \( r \) is symmetric and \( u, v, w \) are skew-symmetric with respect to \(*\);
- \( r \) and \( v \) are skew-symmetric, \( u \) and \( w \) are symmetric.

As an example of such a reduction, one can consider the case \( u, v, w \in so(3), r = \text{diag}(r_1, r_2, r_3) \).

Another class of reductions for system (6.11) is determined by the condition \( u = \text{const} \).

It follows from (6.11) that in this case the identity \( uwr - rwu = 0 \) should be valid for all \( w \). The trivial reduction \( u = \text{const} \) can be reduced to \( u = 0 \) by a shift of \( v \). However there exist also non-trivial martix reductions.

\(^1\text{i.e. } (ab)^* = b^*a^* \text{ and } (a^*)^* = a\)
Suppose the elements $r$, $w$ and $v$ are of the following block structure:

$$r = \begin{pmatrix} 0 & r_1 \\ r_2 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & w_1 \\ w_2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}.$$  

Then choosing

$$u = \text{const} \begin{pmatrix} 0 & r_1 \\ -r_2 & 0 \end{pmatrix},$$

we get self-consistent reduction of (6.11). A further reduction

$$v_1^t = -v_1, \quad v_2^t = -v_2, \quad w_2 = w_1^t, \quad r_2 = -r_1^t$$

is available. The latter reduction and corresponding hamiltonian structures are described in subsection 5.2.

### 6.2 Groups of reduction and homogeneous subalgebras.

One can construct examples of homogeneous subalgebras starting with so called groups of reduction [12]. Let $T$ be a finite subgroup in the group of linear-fractional transformations of a parameter $\lambda$. Without loss of generality we assume that neither element of $T$ except the unity preserves the point $\lambda = \infty$.

Let us consider the vector subspace $\bar{G}$ in the current algebra $G_\lambda$ consisting of elements of the form

$$a = \sum_{n=0}^m \sum_{t \in T} q_{n,t} t(\lambda^{-n}) | q_{n,t} \in \mathcal{G}, \ m \in \mathbb{N}. \quad (6.13)$$

We define $\mathcal{G}_+$ as the set of all elements in $\bar{G}$ that are stable with respect to the group action of $T$. It is clear that the set of all Laurent expansions of elements from $\mathcal{G}_+$ at the point $\lambda = 0$ forms a subalgebra in $G_\lambda$. We denote this subalgebra by $\mathcal{G}_+$ as well.

**Proposition 6.2** Suppose $T$ acts on the Lie algebra $\mathcal{G}$ without stable non-zero elements. Then $\mathcal{G}_+$ is a factorizing subalgebra in the current algebra $G_\lambda$ complementary for the subalgebra of Taylor series $\mathcal{G}_-$. 

**Proof.** Consider the element

$$Z_q = \sum_{t \in T} t(q\lambda^{-n}).$$

It is clear that it belongs to $\mathcal{G}_+$ and has a Laurent expansion of the form $Z_q = q\lambda^{-n} + o(1)$. Thus $\mathcal{G}_+$ contains a series with arbitrary principle part and therefore $\mathcal{G}_+ = \mathcal{G}_+ \cap \mathcal{G}_-$. It remains to prove that $\mathcal{G}_+ \cap \mathcal{G}_- = \{0\}$. Suppose $a \in \mathcal{G}_+ \cap \mathcal{G}_-$ and $a$ contains a term of the form $qt(\lambda^{-n})$, $q \neq 0, n > 0$ in the decomposition (6.13). Since $a \in \mathcal{G}_+$, we have $a = t^{-1}(a)$. Therefore $a$ contains the term $t^{-1}(q)\lambda^{-n}$ and for this reason does not belong to the algebra of Taylor series $\mathcal{G}_-$. Suppose now that $a \in \mathcal{G}_+ \cap \mathcal{G}_-$ and $a$ does not depend on $\lambda$ (i.e. $a \in \mathcal{G}$). Then $a$ is an element of $\mathcal{G}$ stable with respect to $T$-action and by condition, $a = 0$.

Define $\bar{\lambda}$ by the formula

$$\bar{\lambda}^{-1} = \sum_{t \in T} t(\lambda^{-1}). \quad (6.14)$$
The function $\bar{\lambda}$ is invariant with respect to the action of the group $T$ and therefore

$$\bar{\lambda}^{-1} \mathcal{G}_+ \subset \mathcal{G}_+. \quad (6.15)$$

It can easily be shown that a change of parameter $\lambda$ of the form

$$\lambda \rightarrow \lambda + k_2 \lambda^2 + k_3 \lambda^3 + \cdots \quad (6.16)$$

takes each factorizing subalgebra to another factorizing subalgebra. Two subalgebras connected by such a transformation can be regarded as equivalent.

It is easy to see that $\bar{\lambda} = \lambda + o(\lambda)$. Thus formula (6.14) defines a transformation of the form (6.16). It follows from (6.15) that $\mathcal{G}_+$ is equivalent to a homogeneous subalgebra.

**Example 6.1** Consider the commutative group $T = \mathbb{Z}_2 \times \mathbb{Z}_2$, consisting of elements $1, t_1, t_2$ and $t_3 = t_1 t_2$. Define $T$-action on the Lie algebra $\mathfrak{so}(3)$ with the help of conjugations by diagonal orthogonal matrices

$$\text{diag}(1, 1, 1), \quad \text{diag}(-1, -1, 1), \quad \text{diag}(-1, 1, -1), \quad \text{diag}(1, -1, -1),$$

Consider the following embedding of $T$ into the group of linear-fractional transformation. Let

$$s_1(\lambda) = -\lambda, \quad s_2(\lambda) = \frac{1}{\lambda}, \quad s_3(\lambda) = -\frac{1}{\lambda}, \quad \tau(\lambda) = \frac{a\lambda + b}{c\lambda + d}.$$ 

Then elements of $T$ are identified with transformations $t_i = \tau^{-1} \circ s_i \circ \tau$. The condition $t_i(\infty) \neq \infty$ is equivalent to $ac \neq 0$ and $a^4 \neq c^4$. Generators (4.1) of subalgebra $\mathcal{G}_+$ are

$$E_1 = \frac{e_1}{\lambda} + \sum_{i=1}^{3} t_i \left( \frac{e_1}{\lambda} \right), \quad E_2 = \frac{e_2}{\lambda} + \sum_{i=1}^{3} t_i \left( \frac{e_2}{\lambda} \right), \quad E_3 = \frac{e_3}{\lambda} + \sum_{i=1}^{3} t_i \left( \frac{e_3}{\lambda} \right).$$

Transformation (6.16) defined by

$$\bar{\lambda}^{-1} = \frac{1}{\lambda} + \sum_{i=1}^{3} t_i \left( \frac{1}{\lambda} \right)$$

converts the subalgebra $\mathcal{G}_+$ into a homogeneous one.

**References**

[1] I.V. Cherednik, Functional realizations of basis representations of factorizing Lie groups and algebras, *Func. Anal. and Appl.* 19(3), 36–52, 1985.

[2] A.A. Belavin, V.G. Drinfeld, On solutions of the classical Yang-Baxter equation for simple Lie algebras, *Func. Anal. and Appl.* 16(3), 1-29, 1982.

[3] I. Z. Golubchik and V. V. Sokolov, Generalized Heizenberg equations on Z-graded Lie algebras, *Teoret. and Mat. Fiz.*, 120(2), 248-255, 1999.

[4] V. V. Sokolov, On decompositions of the current algebra over $\mathfrak{so}(3)$ into a sum of two subalgebras, *Dokl. RAS.*, () to be published.
[5] A.V. Bolsinov and A.V. Borisov, Lax representation and compatible Poisson brackets on Lie algebras, Math. Notes, 72(1), 11-34, 2002.

[6] I. Z. Golubchik and V. V. Sokolov, Compatible Lie brackets and integrable equations of the principle chiral model type, Func. Anal. and Appl., 36(3), 172–181, 2002.

[7] V. V. Sokolov, A new integrable case for the Kirchhoff equation, Theoret. and Math. Phys., 129(1), 1335–1340, 2001.

[8] V.V. Sokolov, Generalized Kowalewski Top: new integrable cases on e(3) and so(4), CRM Proceedings and Lecture Notes, 32, 307-313, 2002.

[9] A. V. Borisov, I. S. Mamaev and V. V. Sokolov A new integrable case on so(4), Dokl. RAS, 381(5), 614–615, 2001.

[10] V. V. Sokolov and A.V. Tsiganov, On Lax pairs for the generalized Kowalewski and Goryachev-Chaplygin tops, Theoret. and Math. Phys., 131(1), 543-549, 2002.

[11] A.I. Bobenko, Euler equations on so(4) and e(3). Isomorphysms of integrable cases, Func. Anal. and Appl. 20(1), 64-66, 1986.

[12] A.V. Mikhailov, The reduction problem and the inverse scattering method in ”Solitons”, Topics in Current Physics, R. Bullough and P. Caudrey eds., New York: Springer-Verlag, 17, 243–285, 1980.

[13] A.G. Reyman and M.A. Semenov-Tian-Shansky, Integrable system. Theoretically-group approach, Izheusk: RC Dynamics, 2003, 351 c.

[14] E.K. Sklyanian, On complete integrability of the Landau-Lifshitz equation. LOMI preprint, E-3, 1979.

[15] A.V. Mikhailov, A.B. Shabat. Integrable deformations of the Heisenberg model, Phys. Lett. A, 116, 191–194, 1986.

[16] A.I. Bobenko, A.G. Reyman and M.A. Semenov-Tian-Shansky, The Kowalewski top 99 years later: a Lax pair, generalizations and explicit solutions, Commun. Math. Phys., 122, 321, 1989.

[17] I. V. Komarov, V. V. Sokolov and A. V. Tsiganov, Poisson maps and integrable deformations of Kowalevski top, J. Phys. A, 36, 8035–8047, 2003

[18] A.V. Tsiganov, Integrable deformations of the tops related to algebra so(p,q). Theoret. and Math. Phys., , –, 2004