Structural mathematical modelling: the concept of feedback in the dynamics of mechanical oscillation systems

S V Eliseev¹a, R S Bolshakov ²b, A I Orlenko³c and A N Trofimov⁴d

¹, ², ⁴Irkutsk State Transport University, 15 Chernyshevsky, Irkutsk, Russia
³Krasnoyarsk Institute of Railway Transport, the branch of ISTU, 89 Lado Ketskhoveli, Krasnoyarsk, Russia

ᵃeliseev_s@inbox.ru, ᵇbolshakov_rs@mail.ru, ᶜorlenko_ai@krsk.irgups.ru and ᵈtrofimov_an@irgups.ru

Abstract. The paper considers a concept of feedbacks in the dynamics of mechanical oscillation systems. The authors propose a generalized theory of dynamic damping of oscillations as a process of introducing additional constraints in a basic model that looks like a system with one or two degrees of freedom. This article describes a method for constructing mathematical models and the results of studying the features of the dynamic properties of systems in which dynamic oscillation damping modes can be implemented.

Keywords: feedback concept, dynamics of vibration protection systems, dynamic damping of oscillations, features of modes of dynamic oscillation damping.

1. Introduction

The issues of evaluation, control and management of the dynamic state of machines, equipment and apparatus operating under conditions of intense dynamic loading are given considerable attention, which has been reflected in a number of works that describe theoretical and practical aspects [1 – 3]. Particular attention is paid to the tasks of vibration protection while ensuring the vehicle operation safety.

As the computational schemes of technical objects, mechanical oscillating systems with several degrees of freedom are widely used, having in their composition various elements whose dynamic interactions determine the features of the formation of conditions for reducing loads on the nodes and components of machines operating under the influence of periodic disturbances. Mathematical modeling in the evaluation of dynamic states is represented by numerous works [4, 5]. One of the challenging directions in the evolution of mathematical modeling methods is the use of approaches oriented to applying the principles of dynamic analogies and ideas about the potential of comparing mechanical oscillatory systems with dynamically equivalent automatic control systems. In recent years, one of the directions in the development of methods and means for changing dynamic states has been dynamic damping of oscillations [6], which initiates the research directions related to the potentialities of using the analytical tools of the automatic control theory.

The proposed article develops a methodological basis for solving problems associated with the evolution of ideas and principles of dynamic oscillation damping, as a form of implementing the effects of introducing additional constraints, including feedbacks, into mechanical oscillatory systems.
2. Some key points

With all the variety of methods and means, the search for and development of new technical solutions, the most common methods are those based on the selection of tasks that interpret the idea of dynamic damping as the inclusion of additional mass by means of an elastic element. At the same time, the results associated with other forms of dynamic damping of oscillations, based on the use of pendulum devices, lever mechanisms, motion transformation devices, additional rotations of links creating centrifugal forces, are fairly widely known. There are also techniques for balancing and balancing rotating masses. In a generalized way, dynamic damping can also be considered as the introduction of additional constraints in the interpretations admitted by the automatic control theory (Fig. 1, a, b).

![Figure 1](image1.png)

**Figure 1.** A diagram that reflects the introduction of an additional constraints: a) at the level of constructional and technical forms; b) structural interpretation

Dynamic damping of oscillations requires rather complex adjustment to external influences that have non-stationary parameters. Problems arise also with allowance for the influence of the resistance forces that limit the efficiency of dynamic oscillation damping as a whole. The approach in which a dynamic analog of a linear mechanical oscillatory system becomes an automatic control system is associated with an expansion of the ideas of a set of typical elementary links of the vibration protection system. This, in particular, is related to the development of a technology for connecting typical elementary links to obtain the corresponding structures. Active development of the idea of dynamic damping of oscillations was obtained in the balancing and equilibration of rotating masses, which predetermined the emergence of automatic adjustment systems. Interest in the potential of constructing active control systems for oscillatory processes in mechanical systems promoted the development of mechatronics as an interdisciplinary scientific direction in the problems of machine dynamics. The capabilities of dynamic oscillation dampers, for example, the frequency of dynamic damping modes, are determined by the frequency equation of the numerator of the system transfer function [7].

1. Let us consider the features of the selection and description of feedbacks (Fig. 2, a-f) in mechanical oscillatory systems.

In the development of the concept of feedback, an approach is proposed to consider the main stages of the formation of mathematical models on a system level.

![Figure 2](image2.png)

**Figure 2.** Schematic diagram of the association of notions in the concept of feedback
Let us note that structural representations (fig. 2, f) make force and kinematic external influences equivalent; this served as the basis for the development of ideas about generalized problems of vibration protection and vibration isolation. The block diagram in fig. 2, f can be called the basic circuit of mechanical oscillation systems, in the sense that more developed systems can be built by building up the base system through the introduction of additional constraints or connecting several blocks with each other. To describe the relationship between output and input quantities, one can use transfer functions. In fig. 2, g, respectively, $W_1(p)$ is the transfer function "object displacement by mass $m$ – base displacement $z$" and $W_2(p)$ is "displacement of object by mass $m$ – external force $F$". In the simplest mechanical oscillatory system (fig. 3, a), consisting of two elements with masses $m_1$ and $m_2$, the linkage is provided by an elastic coupling in the form of a spring with stiffness $k$. At point A (fig. 3b) an external force $F$ is applied; $y_1$ and $y_2$ are the coordinates of the displacement of masses. The system consists of two partial systems I and II, marked by the corresponding contours in fig. 3, b. The mathematical model of the system is shown in Fig. 3 and is a system of two differential equations of the second order. In fig. 3, d there are transfer functions $W_1(p)$ and $W_2(p)$, which make it possible to find the frequency of natural oscillations (expression $-(1)$) and the frequency of dynamic damping (expression $-(2)$). The fact that the partial systems I and II have cross-couplings represented in the structural diagram (fig. 3, b) by two channels of interaction with the gain $k$ is important for further consideration. The structural diagram (fig. 3, b) allows us, through formal transformations, to construct a structural model reflecting this type of dynamic action, as dynamic damping of oscillations under the action of an external perturbing monoharmonic force $F$. The dynamic damping mode for the computational model shown in fig. 3, a corresponds to the introduction of positive feedback in the structural scheme (fig. 3, b). In general, the general diagram (fig. 3, a – f) of dynamic interactions in the free-motion system of two masses $m_1$ and $m_2$ connected by an elastic coupling $k$ gives an idea of the formation of constraints and their functional purpose [1].

**Figure 3.** General diagram of dynamic interaction in a two-mass system

2. The structural approach developed in the article predetermines the use of the transfer function as the main dynamic characteristic of mechanical oscillation systems and vibration protection systems, in particular. Introduction of the notion of an extended set of elementary typical links of mechanical oscillation systems make it possible to change the structures of the basic models of the initial system. In this case, additional feedbacks are formed in the form of mechanical circuits of different complexity. In this interpretation, the additional feedback can be considered as a generalized spring (or quasi-spring) and have a transfer function in the form of a fractional-rational expression. In the works [1, 8] one can find some examples of the implementation of additional feedbacks for various systems.
When the additional masses are attached to construct a dynamic oscillation absorber for several frequencies of external kinematic influences in a coordinate system, associated with a fixed reference frame, it results in an increase in the frequencies of the natural oscillations of the partial system containing the protection object.

3. Dynamic damping of oscillations as introduction of feedbacks

1. A vibration protection system using dynamic vibration damping can consist of several elements. In Fig. 4 shows several calculation schemes, which can be divided into 4 groups: dynamic absorbers have linkages between masses $m_1$ and $m_2$, but the element of mass has no linkage ($k_{12} = 0$) with the object of protection (fig. 4, a); there are two dynamic absorbers, and $m_1$ and $m_2$ are linked with the object of protection, but they are not linked with each other ($k_{12} = 0$) (fig. 4, b); additional masses $m_1$ and $m_2$ are linked with each object of protection, but also interlinked ($k_{12} \neq 0$), (fig. 4, c); the dynamic absorber is not two additional masses $m_1$ and $m_2$, but a solid body having mass $M$ and moment of inertia $I$ with elastic supports $k_1 \neq 0, k_2 \neq 0$ (fig. 4, d).

Figure 4. Computational models of vibration protection systems with two dynamic vibration dampers

As an example, Fig. 5 shows the structural diagram of the dynamic vibration damper according to the scheme corresponding to fig. 4, a. Dynamic properties of the system (fig. 5) can be estimated from the analysis of the structural scheme: at the frequency $\omega_1^2 = \frac{k_{12}}{m_2}$ the feedback is "reset", and the object of protection performs oscillations that are independent of the dynamic oscillation dampers $m_1$ and $m_2$. In turn, the denominator of the transfer function of the feedback loop is a frequency equation of the form

$$m_1 m_2 p^4 + [m_2 (k_1 + k_{12}) + m_1 k_{12}] + k_{12} = 0.$$  \hspace{1cm} (1)

The roots of equation (1) determine the frequencies of dynamic damping

$$\omega_{0,1,2}^2 = \frac{m_2 (k_1 + k_{12}) + m_1 k_{12}}{2 m_2} \pm \sqrt{\frac{[m_2 (k_1 + k_{12}) + m_1 k_{12}]^2 - 4 m_1 m_2 k_{12}}{4 (m_2)^2}}.$$  \hspace{1cm} (2)

Figure 5. Structural diagram of the system with a dynamic oscillation damper, consisting of two masses ($m_1$ and $m_2$)

Similarly, we can consider the calculation schemes shown in fig. 4, b and c.
It can be noted that, in comparison with the diagram in fig. 4, a the element \( k_{12} \) does not make it possible to obtain simplifications in such a form as in fig. 4, b due to the presence of nonplanar constraints, and this requires the use of special techniques. Taking into consideration the connectivity in the motions of the dynamic dampers \( m_1 \) and \( m_2 \) alters the parameters of the dynamic damping mode and others, but, in general, the dynamic properties of the system remain the same if one has in mind the number of resonances and the number of dynamic damping modes.

The computational scheme of this absorber with associated motions is shown in fig. 4, d. The motion of the dynamic damper is considered in the coordinate system \( y_1 \) and \( y_2 \), and also in the coordinate system \( y_0 \) and \( \phi \), referring to the center of gravity of the solid body. Let us consider the features of the choice of coordinate systems. Fig. 6 shows the computational model of a dynamic vibration damper for a solid body (a model problem of transport dynamics) with a lever dynamic vibration damper.

![Computational model of a system with a simplified DD and taking into account the displacement of the center of masses at point \( A_1 \)](image)

**Figure 6.** The computational model of a system with a simplified DD and taking into account the displacement of the center of masses at point \( A_1 \)

The solution of the problem was made taking into account the change in the position of the center of masses of the system during the installation of the damper. Fig. 7 shows the amplitude-frequency characteristics (AFC); a family of curves that reflect the effect of a change in the displacement of the DD installation point on the system properties. The system is characterized by the existence of two modes of dynamic damping in the pre- and interresonance frequencies of the region (fig. 7, a). In Fig. 7, b, the relative position of the frequency response is shown in more detail.

![Amplitude-frequency characteristics](image)

**Figure 7.** The family of the amplitude-frequency characteristics of the system with respect to the \( y \) coordinate, for different values of \( l_0 \) (a); the relative position of the amplitude-frequency characteristic in the pre-resonance region (b)

The effect of the change of \( l_0 \) on the character of the relative position of the AFC depends, in a significant way, on the ratio of the parameters and the selection of the coordinate system of the protection object.

4. Some applications of the theory of dynamic dampers

1. The development of a generalized methodology for mathematical modeling of dynamic damping of oscillations and some applications of the theory require certain clarifications.
The proposed suspension (more precisely, its model) consists (fig. 8) of the object of protection of mass $M$ with allowance for moments of inertia $I$.

The center of gravity of a solid body is located at $p$. In the suspension system, two levers with masses $m_1$ and $m_2$ are involved; their moments of inertia with respect to $p$ are denoted respectively by $I_1$ and $I_2$.

As an example, a bogie truck with two traction motors for an electric locomotive is given to such a computational model.

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![Figure 8](image_url)  
**Figure 8.** Calculation scheme of the bogie truck with inertial levers

The system of differential equations of motion in the coordinate system $y$, $\phi$ has the form:

$$
\ddot{y}(M + M_1a_1^2 + M_2a_2^2 + I_c^1 + I_c^2) + \phi(k_1 + k_2 + k_3(c_1 - c_2)^2) + 
+ \phi[k_1l_1 - k_2l_2 - k_2(c_1 - c_2)(c_1l_1 - c_1l_2)] = z_1(I_c^2 - M_1a_1^2) + 
+ \phi(k_1l_1 - k_2l_2 - k_2(c_1 - c_2)(c_1l_1 - c_1l_2)] + z_2(k_1c_1^2(c_1 - c_2) + k_2c_2^2(c_1 - c_2) = 0.
$$

In the coordinates $y_1$ and $y_2$, the system of equations (2), (3) is transformed to another form, which allows us to search for the ratio of the system parameters, which simultaneously provides damping along two coordinates. The choice of the dynamic damping mode is determined by the conditions of the vibration protection task and depends on the type of the transfer function of the adjustment link $W_0(p) = k_3$. As an example, fig. 9, a, b shows the choice of properties of the amplitude-frequency characteristics (the Mathcad application package was used here).

![Figure 9](image_url)  
**Figure 9.** Amplitude-frequency characteristics of the system with respect to the $y$ coordinate:

a) two modes of dynamic damping, b) dynamic damping mode before the first resonance

A sequential system of possible variants for constructing systems with a dynamic lever-type damper is shown in fig. 10. Amplitude-frequency characteristics, reflecting the features of dynamic properties, are shown in fig. 11.

![Figure 10](image_url)  
**Figure 10.** Sequential system of possible variants for constructing systems with a dynamic lever-type damper

![Figure 11](image_url)  
**Figure 11.** Amplitude-frequency characteristics, reflecting the features of dynamic properties
Figure 10. Computational model for vibration protection system with articulations: a – corresponds to the articulation $k_4 \to \infty$; b – corresponds to the articulation $k_{30} \to \infty$; c – simultaneously corresponds to articulations $k_4 \to \infty$ and $k_{30} \to \infty$

Figure 11. Types of amplitude-frequency characteristics of the system for various ratios of the parameters (the values of the parameters are shown in the figure field)

2. As one of the applications, the possibilities of a dynamic vibration damper obtained on the basis of the application of the generalized method for constructing mathematical models of systems can be considered.

The diagrams, shown in fig. 12, are related to each other by the possibilities of transformations based on the method of eliminating variables.

The general form of the amplitude-frequency characteristic of a system with a dynamic arm-type damper according to fig. 12, a is shown in fig. 13.
Figure 12. Computational models of vibration protection systems with a lever dynamic vibration damper: a) the elasticity of the lever and hinges are not taken into account; b) the elasticities of all the hinges are taken into account; c) allowance is made for the elasticity of the mounting hinge with the base and the elasticity of the lever.

The transfer function of the vibration protection system in fig. 12a has the form:

\[ W(p) = \frac{\varphi}{\varphi_0} = \frac{mi(i+1)p^2 + k}{(M + mi^2)p^2 + k}. \]  

(5)

Figure 13 shows the amplitude-frequency characteristic (AFC) constructed on the basis of (5), which shows three characteristic cases.

When the condition \( M = mi \) is satisfied, the vibration protection system is locked, and for \( M \neq mi \) one of the variants of the amplitude-frequency characteristic is implemented.

To examine the features of dynamic damping in systems with lever linkages, an experiment was carried out on a model of a vibration protection system with a motion transformation device.

A general view of the experimental installation is shown in fig. 14.

Figure 13. Amplitude-frequency characteristics of the system for different mass ratios:
- curve a corresponds to the condition \( M > mi \);
- curve b corresponds to \( M = mi \);
- curve c corresponds to \( M < mi \).

To carry out the experiment, we used a C-004 vibration stand (frequency range 0.1-20 Hz), a multichannel synchronous set of the vibration measuring equipment “Atlant-8” (serial number 070), sensors for vibration measurement “Vikont” VK-310A.

Fig. 15 shows the characteristic records of the motion of the object in the pre-resonance, resonance and resonance regions. The amplitude-frequency characteristics for different values of the transfer ratio are shown in fig. 16a, b.

The results of the experiment were processed on the basis of a known technique using the root-mean-square values of the measured quantities.
The best coincidence of the results was observed in the low-frequency region (2-7 Hz).

"Locking" of the system, characteristic for high frequencies, gives a coincidence of results within 12%. With an increase in frequencies $> 20$ Hz, the experiment is distorted by the influence of nonlinear factors related to the distortion of the shape of the external disturbing signal.

In general, it can be assumed that the experiment gives a satisfactory congruence with theoretical calculations in terms of determining the feasibility of implementing dynamic effects in different frequency ranges.

5. Conclusion

Based on the conducted studies, a method is proposed for constructing mathematical models for vibration protection systems using dynamic absorbers of oscillations with several degrees of freedom, which consists in the fact that dynamic absorbers are interpreted in the form of additional feedbacks. The results of studying the properties of dynamic vibration dampers of various design options allow us to expand the concept of the dynamic damping mode and to determine the significance of the choice of systems of generalized coordinates. The authors propose and develop a technique for transforming the structural schemes of mechanical oscillatory systems on the basis of their simplification through the articulation of links, which made it possible to form a justification for the appearance and use of lever linkages in mechanical systems. A technique is proposed for estimating the properties of mechanical oscillatory systems in the modes of dynamic damping of oscillations simultaneously along several coordinates of the motion of the protection object.

Structural mathematical modeling is based on the notion that a structural diagram of a dynamically equivalent automatic control system can be compared to a mechanical oscillatory system as a computational model of a technical object under the conditions of the influence of periodic external forces. In the framework of this approach, an object, whose dynamic state is evaluated, can be considered as an integrating link of the second order, with respect to which the corresponding additional constraints are formed. In the physical sense, the additional constraint can be interpreted as an elastic element entering the system. In more complex cases, the feedback is implemented in the form of a certain structural formation from several typical elements. Such structural formation can be considered as a quasi-spring with corresponding dynamic stiffness. In the general case, the dynamic damping mode is implemented in systems with more than one degree of freedom. In this case, the feedback with respect to the object of protection can be represented by a fractional-rational expression. The dynamic feedback stiffness, equal to infinity, corresponds to the mode of dynamic damping of oscillations, which occurs at the corresponding frequency.

When using the effects of introducing additional constraints in the form of motion transformation devices (which can be implemented by some mechanisms, for example, non-locking screw mechanisms), it is also possible to create dynamic damping modes if the system has formally one degree of freedom. However, for the dynamic damping oscillation modes to emerge, a kinematic perturbation is necessary.
As for more complex mechanical oscillation systems, the emergence of dynamic damping modes is ultimately determined by the feedback structure relative to the protection object. This is related to the representation of the feedback transfer function as an expression of a fractional-rational form.

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