Necessary non-local conditions for a time-fractional diffusion-wave equation

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Abstract

In this paper the time-fractional diffusion-wave equation with Riemann-Liouville fractional derivative is studied. The integral operators with the Wright function in the kernel, associated with the studied equation are introduced and their properties are investigated. In terms of these operators the necessary non-local conditions binding traces of solution and its derivatives on the boundary of a rectangular domain are found. By using the limiting properties of the Wright function the necessary non-local conditions for wave equation are obtained. With the help of the mentioned integral operator’s properties a unique solvability of the problem with Samarskii integral condition for the diffusion-wave equation is proved. The solution is obtained in explicit form.

Keywords: non-local BVP, diffusion-wave equation, wave equation, fractional differential equations, necessary non-local conditions, Samarskii problem, integral condition, derivative of fractional order.

1 Introduction

In the domain \( \Omega = \{(x, y): 0 < x < l, 0 < y < T\} \) we consider the equation

\[
\frac{\partial^2 u}{\partial x^2}(x, y) - D^\alpha_{y} u(x, y) = 0,
\]

(1)

where \( 0 < \alpha < 2, D^\alpha_{y} \) is the Riemann-Liouville fractional integro-differentiation operator of order \( \nu \) \cite[p. 9]{11}, which determined as

\[
D^\nu_{y} g(y) = \frac{\text{sgn}(y - a)}{\Gamma(-\nu)} \int_{a}^{y} \frac{g(s)ds}{|y - s|^{\nu+1}}, \quad \nu < 0,
\]

for \( \nu \geq 0 \) the operator \( D^\nu_{y} \) can be determined by recursive relation

\[
D^\nu_{y} g(y) = \text{sgn}(y - a) \frac{d}{dy} D^{\nu-1}_{y} g(y), \quad \nu \geq 0,
\]

\( \Gamma(z) \) is the gamma-function.

It is well-known that the fractional partial differential equations are appeire in mathematical models describing various processes in a mediums with a fractal structure (see for example \cite[chapter 5]{11}). Equations of the form (1), due to their numerous applications in the modeling of processes of a different nature, have been actively investigated by many authors using different methods during the last decades. Such equations describe anomalous diffusion and subdiffusion processes, relaxation phenomena in complex viscoelastic materials, and so on.

For an extensive bibliography on this subject see for example references in papers \cite{16,17,8,9}.

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2 Integral operators associated with the diffusion-wave equation

In the paper [4] (see also [6]) the fundamental solution \( \Gamma(x - t, y - s) \) of the equation (1) was represented in terms of the function

\[
\Gamma(x, y) = \frac{y^{\beta - 1}}{2} \phi(-\beta, \beta; -|x-y|),
\]

where \( \beta = \alpha/2 \), \( \phi(\rho, \mu; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\rho k + \mu)} \) is the Wright function [20].

Let us introduce the following operators \( N_{\theta,x,y} \) and \( R_{\delta,x,y} \), which acting by the formulas

\[
N_{\theta,x,y} \nu(t) = \frac{1}{2} \int_{x_1}^{x_2} \nu(t) y^{\theta - 1} \phi(-\beta, \theta; -|x-t| y^{-\beta}) dt, \tag{3}
\]

\[
R_{\delta,x,y} \mu(y) = \frac{1}{2} \int_{0}^{y} \mu(s)(y-s)^{\delta - 1} \phi(-\beta, \delta; -|x-s| y^{-\beta}) ds. \tag{4}
\]

The following properties of the operators \( N_{\theta,x,y} \) and \( R_{\delta,x,y} \) are hold.

**Property 1.** The relation

\[
(2R_{\delta,x,y}^\delta) (2R_{\theta,x,y}^\theta) \mu(y) = 2R_{\theta,x,y}^{\delta+\theta,x_1+x_2} \mu(y) \tag{5}
\]

holds for all \( x_1 > 0, x_2 > 0 \).

**Property 2.** Let \( a \geq 0, \delta + \beta > 0, \theta + \beta > 0 \). Then the following relations hold

\[
R_{\delta,x,y}^{\delta+a+b,y} \tau(t) = \begin{cases} N_{\delta+a,0}^{\delta+a+b,y} \tau(t), & b \leq 0, \\ [N_{\delta+a+b,0}^{\delta+a,b} + N_{\delta+a,0}^{\delta+a+b,y}] \tau(t), & 0 < b < l, \\ N_{\delta+a+b,0}^{\delta+a,b,y} \tau(t), & b \geq l. \end{cases} \tag{6}
\]

**Property 3.** Let \( y^{1-\nu} \mu(y) \in C[0,T] \), \( \nu + \delta \geq 0 \), then

\[
\lim_{x \to 0} R_{\delta,x,y}^{\delta,x} \mu(y) = \frac{1}{2} D_{\delta,x,y}^{\delta} \mu(y). \tag{7}
\]

**Property 4.** Let \( \tau(t) \in C[0,T] \), then

\[
\lim_{y \to 0} D_{\delta,x,y}^{2,\delta-n} N_{\delta,0}^{\delta,k+1,x,y} \tau(x) = \begin{cases} \tau(x), & k = n, \\ 0, & k < n. \end{cases} \tag{8}
\]

**Property 5.** The relation

\[
\int_{0}^{l} R_{\delta,x,y}^{\theta,x} \mu(y) dx = \pm \left[R_{\delta,x,y}^{\theta,x} - R_{\delta,x,y}^{\theta,x} \right] \mu(y) \tag{9}
\]

holds.
Property 6. The following relations hold

\[
\int_0^l N_{0l}^{\delta, a \pm x, y} \tau(t) dt = \pm \text{sign} a \left[ N_{0l}^{\delta + \beta, a, y} - N_{0l}^{\delta + \beta, a \pm x, y} \right] \tau(t), \quad |a| \geq 2l,
\]

\[
\int_0^l N_{0l}^{\delta, -x, y} \tau(t) dt = \left[ N_{0l}^{\delta + \beta, 0, y} - N_{0l}^{\delta + \beta, -y} \right] \tau(t),
\]

\[
\int_0^l N_{0l}^{\delta, x, y} \tau(t) dt = - \left[ N_{0l}^{\delta + \beta, 0, y} + N_{0l}^{\delta + \beta, x, y} \right] \tau(t) + \frac{y^{\delta + \beta - 1}}{\Gamma(\delta + \beta)} \int_0^l \tau(x) dx.
\]

For the proof of properties 1 and 2 it is enough to change the integration order, use the definitions of the operators \( R_{0y}^{\delta, x} \) and \( N_{0l}^{\theta, x, y} \) and the following convolution formula for Wright functions [18]

\[
\int_0^y (\delta - 1) \phi(-\alpha, \delta; -x_1 \xi - \alpha)(y - \xi)^{\mu - 1} \phi(-\alpha, \mu; -x_2(y - \xi)^{-\alpha}) d\xi =
\]

\[
= y^{\delta + \mu - 1} \phi(-\alpha, \delta + \mu; -(x_1 + x_2)y^{-\alpha}), \quad \forall x_1, x_2 > 0. \quad (8)
\]

The property 3 is proved in [16, p. 35]. We rewrote it in the terms of the operator \( R_{0y}^{\delta, x} \).

The property 4 follows from the formula [16, p. 24]

\[
D_{0y}^\nu y^{\delta - 1} \phi(-\beta, \delta; -cy^{-\beta}) = y^{\delta - \nu - 1} \phi(-\beta, \delta - \nu; -cy^{-\beta}), \quad c > 0, \quad \delta + \beta > 0,
\]

the relation [6]

\[
\int_0^\infty \phi(-\beta, 1 - \beta; -s) ds = 1, \quad (10)
\]

and the estimates [16, p. 29]

\[
|y^{\delta - 1} \phi(-\beta, \delta; -xy^{-\beta})| \leq Cx^{-\theta} y^{\delta + \beta \theta - 1}, \quad x > 0, \quad y > 0,
\]

(11)

where \( \theta \geq 0 \) for \( \delta \notin \mathbb{N}_0 \); and \( \theta \geq -1 \) for \( \delta \in \mathbb{N}_0 \); \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( C \) is a positive constant.

For the proof of properties 5 and 6 it is enough to use the definitions of operators \( R_{0y}^{\delta, x} \) and \( N_{0l}^{\theta, x, y} \), and formula [20]

\[
\frac{d}{dz} \phi(-\beta, \delta; z) = \phi(-\beta, \delta - \beta; z).
\]

In view of [11] from properties 1 and 2 follow the relations

\[
D_{0y}^{-\theta} R_{0y}^{\delta, x} \mu(y) = R_{0y}^{\delta, x} D_{0y}^{-\theta} \mu(y) = R_{0y}^{\delta + \theta, x} \mu(y),
\]

(13)

\[
D_{0y}^{-\delta} N_{0l}^{\theta, x, y} \nu(x) = N_{0l}^{\delta + \theta, x, y} \nu(x).
\]

(14)
3 Necessary non-local conditions for the time-fractional diffusion-wave equation

Let \( n \in \{1, 2\} \) is a number such that \( n - 1 < \alpha \leq n \). A function \( u = u(x, y) \) of the class \( D_{0y}^{\alpha-k}u(x, y) \in C(\Omega) \), \( 1 \leq k \leq n \), \( u_{xx}(x, y) \), \( D_{0y}^{\alpha}u(x, y) \in C(\Omega) \), satisfying the equation (1) at all points \((x, y) \in \Omega\) is called a regular solution of equation (1) in the domain \( \Omega \) [16, p. 103].

The following assertion hold [7].

Theorem 1. Let \( u = u(x, y) \) is a regular in the domain \( \Omega \) solution of equation (1), satisfying the condition

\[
\lim_{y \to 0} D_{0y}^{\alpha-k} u(x, y) = \tau_k(x), \quad 1 \leq k \leq n, \quad 0 < x < l,
\]

such that \( u_x \in C([0, l] \times (0, T)) \) and \( u_x(0, y), u_x(l, y) \in L[0, T] \). Then the function \( u(x, y) \) fulfill the non-local conditions

\[
\begin{align*}
u(0, y) &= 2 \sum_{k=1}^{n} N_{0y}^{\beta-k+1,0,y} \tau_k(t) + 2 R_{0y}^{\beta,l} u_x(l, s) - D_{0y}^{\alpha} u_x(0, s) + 2 R_{0y}^{0,l} u(l, s), \\
u(l, y) &= 2 \sum_{k=1}^{n} N_{0y}^{\beta-k+1,l,y} \tau_k(t) - 2 R_{0y}^{\beta,l} u_x(0, s) + D_{0y}^{\alpha} u_x(l, s) + 2 R_{0y}^{0,l} u(0, s).
\end{align*}
\]

Proof. We use the general representation of regular solutions of equation (1) in the rectangular domain [16, c.116]

\[
u(x, y) = \sum_{k=1}^{n} (-1)^{k-1} \int_{x_1}^{x_2} \tau_k(t) \frac{\partial^{k-1}}{\partial t^{k-1}} G(x, y; t, 0) dt + \\
+ \sum_{i=1}^{2} (-1)^{i} \int_{0}^{y} [G(x, y; x_1, s) u_1(x_1, s) - G_1(x, y; x_1, s) u(x_1, s)] ds.
\]

By setting in relation (18) \( G(x, y; t, s) = \Gamma(x - t, y - s) \), taking into account the equalities

\[
\frac{\partial}{\partial t} \Gamma(x, y) = \frac{\text{sgn} x}{2y} \phi(-\beta, 0; -|x|y^{-\beta}),
\]

\[
\frac{\partial}{\partial s} \Gamma(x, y) = -\frac{y^{\beta-2}}{2} \frac{\text{sgn} x}{2y} \phi(-\beta, \beta - 1; -|x|y^{-\beta}),
\]

which hold in view of relations (9) and (12), we obtain

\[
u(x, y) = \frac{1}{2} \sum_{k=1}^{n} \int_{x_1}^{x_2} \tau_k(t) y^{\beta-k} \phi(-\beta, \beta - k + 1; -|x - t|y^{-\beta}) dt + \\
+ \frac{1}{2} \int_{0}^{y} u_1(x_2, s) (y - s)^{\beta-1} \phi(-\beta, \beta; - (x_2 - x)(y - s)^{-\beta}) ds -
\]

4
The following lemma holds [17].

**Lemma 1.** Let the function $g(t)$ be absolutely integrable on any finite interval of the semiaxis $t > 0$, is continuous at the point $t = 1$ and grows with $t \to \infty$ no faster than $\exp(\sigma t^\beta), \sigma > 0, \delta < \frac{1}{\beta}$. Then

$$\lim_{\beta \to 1} \int_0^\infty g(t)\phi(-\beta, 0; -t)dt = g(1), \quad \lim_{\beta \to 1} \int_0^\infty g(t)\phi(-\beta, \beta; -t)dt = \int_0^1 g(t)dt.$$

4 Necessary non-local conditions

for the wave equation

The following lemma holds [17].

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$$\lim_{\beta \to 1} \int_0^\infty g(t)\phi(-\beta, 0; -t)dt = g(1), \quad \lim_{\beta \to 1} \int_0^\infty g(t)\phi(-\beta, \beta; -t)dt = \int_0^1 g(t)dt.$$

$$\begin{align*}
\frac{1}{2} \int_0^y u_t(x_1, s)(y-s)^{-\beta} \phi(-\beta, \beta; -(x-x_1)(y-s)^{-\beta})ds + \\
+ \frac{1}{2} \int_0^y \frac{u(x_2, s)}{y-s} \phi(-\beta, 0; -(x_2 - x)(y-s)^{-\beta})ds + \\
+ \frac{1}{2} \int_0^y \frac{u(x_1, s)}{y-s} \phi(-\beta, 0; -(x-x_1)(y-s)^{-\beta})ds.
\end{align*}$$

(19)

In the terms of the operators (9) and (11), the relation (19) can be rewritten in the form

$$u(x, y) = \sum_{k=1}^n \mathcal{R}_{x_1 x_2}^{\beta-k+1, y} \tau_k(t) + \mathcal{R}_{y}^{\beta, x_2-x} u_t(x_2, s) - \mathcal{R}_{y}^{\beta, y-x_1} u_t(x_1, s) +$$

$$+ \mathcal{R}_{y}^{\beta, x_2-x} u(x_2, s) + \mathcal{R}_{y}^{\beta, x_2-x} u(x_1, s).$$

(20)

By using the fact that in view of the property 3 of the operator $\mathcal{R}_{y}^{\beta, x}$

$$\lim_{x \to 0} \mathcal{R}_{y}^{\beta, x} \mu(y) = \frac{1}{2} D_{y}^{-\beta} \mu(y), \quad \lim_{x \to 0} \mathcal{R}_{y}^{0, x} \mu(y) = \frac{\mu(y)}{2},$$

from (20) we obtain

$$\frac{1}{2} u(x_1, y) = \sum_{k=1}^n \mathcal{R}_{x_1 x_2}^{\beta-k+1, y} \tau_k(t) + \mathcal{R}_{y}^{\beta, x_2-x} u_t(x_2, s) -$$

$$- \frac{1}{2} D_{y}^{-\beta} u_t(x_1, s) + \mathcal{R}_{y}^{0, x_2-x} u(x_2, s),$$

(21)

$$\frac{1}{2} u(x_2, y) = \sum_{k=1}^n \mathcal{R}_{x_1 x_2}^{\beta-k+1, y} \tau_k(t) - \mathcal{R}_{y}^{\beta, y-x_1} u_t(x_1, s) +$$

$$+ \frac{1}{2} D_{y}^{-\beta} u_t(x_2, s) + \mathcal{R}_{y}^{0, x_2-x} u(x_1, s).$$

(22)

By passing in the relations (21) and (22) to the limit as $(x_1, x_2) \to (0, l)$, we obtain (16) and (17). The proof of Theorem 1 is complete.

For $\alpha = 1$ the conditions (16) and (17) coincide with necessary non-local conditions for the heat equation [10] p. 275. 
Next, we transform the integrals
\[ \alpha \to 2. \]
For this purpose we rewrite conditions (16) and (17) in the following form
\[
\begin{align*}
I(0, y) &= \int_0^{l/y} \tau_1(\xi) y^{\beta-1} \psi(-\beta, \beta; -\xi y^{-\beta}) d\xi + \int_0^{l/y} \tau_2(\xi) y^{\beta-2} \phi(-\beta, \beta; -1; -\xi y^{-\beta}) d\xi + \\
&+ \int_0^{y} \frac{u_x(l, \eta)}{(y - \eta)^{1-\beta}} \phi(-\beta, \beta; -l(y - \eta)^{-\beta}) d\eta - \frac{1}{\Gamma(\beta)} \int_0^{y} \frac{u_x(0, \eta)}{(y - \eta)^{1-\beta}} d\eta + \\
&+ \int_0^{y} \frac{u(l, \eta)}{y - \eta} \phi(-\beta, 0; -l(y - \eta)^{-\beta}) d\eta = \sum_{i=1}^5 I_i(y). \quad (23)
\end{align*}
\]
\[
\begin{align*}
u(l, y) &= \int_0^{l/y} \tau_1(\xi) y^{\beta-1} \phi(-\beta, \beta; -(l - \xi)y^{-\beta}) d\xi + \\
&+ \int_0^{l/y} \tau_2(\xi) y^{\beta-2} \phi(-\beta, \beta; -1; -(l - \xi)y^{-\beta}) d\xi - \\
&- \int_0^{y} \frac{u_x(0, \eta)}{(y - \eta)^{1-\beta}} \phi(-\beta, \beta; -l(y - \eta)^{-\beta}) d\eta + \frac{1}{\Gamma(\beta)} \int_0^{y} \frac{u_x(l, \eta)}{(y - \eta)^{1-\beta}} d\eta + \\
&+ \int_0^{y} \frac{u(0, \eta)}{y - \eta} \phi(-\beta, 0; -l(y - \eta)^{-\beta}) d\eta = \sum_{i=1}^5 J_i(y). \quad (24)
\end{align*}
\]
Next, we transform the integrals \( I_i(y) \)
\[
\begin{align*}
I_1(y) &= \int_0^{l/y} \tau_1(\xi) y^{\beta-1} \phi(-\beta, \beta; -\xi y^{-\beta}) d\xi = \int_0^{l/y} \tau_1(y^\beta \eta) y^{2\beta-1} \phi(-\beta, \beta; -\eta) d\eta = \\
&= \int_0^{l/y} \tau_1(y^\beta \eta) y^{2\beta-1} \phi(-\beta, \beta; -\eta) H(l - y^\beta \eta) d\eta,
\end{align*}
\]
\[
\begin{align*}
I_2(y) &= \int_0^{l/y} \tau_2(\xi) y^{\beta-2} \phi(-\beta, \beta; -\xi y^{-\beta}) d\xi = \frac{d}{dy} \int_0^{l/y} \tau_2(\xi) y^{\beta-1} \phi(-\beta, \beta; -\xi y^{-\beta}) d\xi, \quad (25)
\end{align*}
\]
\[
\begin{align*}
I_3(y) &= \int_0^{y} \frac{u_x(l, \eta)}{(y - \eta)^{1-\beta}} \phi(-\beta, \beta; -l(y - \eta)^{-\beta}) d\eta = \\
&= \int_0^{\frac{1}{y^\beta}} u_x \left( l, y - \frac{l/\xi}{y^\beta} \right) \frac{l}{\beta \xi^2} \phi(-\beta, \beta; -\xi) d\xi = \\
\end{align*}
\]
Using Lemma 1, we obtain the following relations

\[
I_1(y) = \int_0^1 u_x(0, \eta) \frac{y}{y - \eta} \phi(-\beta, 0; -\xi) d\eta = \int_0^1 u_x(l, \eta) \frac{y}{y - \eta} \phi(-\beta, 0; -\xi) d\eta
\]

\[
= \int_0^\infty u_x(l, \eta) \frac{y}{y - \eta} \phi(-\beta, 0; -\xi) d\eta.
\]

Using Lemma 1, we obtain the following relations

\[
\lim_{\beta \to 1} I_1(y) = \int_0^1 \tau_1(y \eta) \frac{y}{y - \eta} d\eta = \int_0^1 \tau_1(\xi) \frac{y}{y - \eta} d\eta = 0.
\]

\[
\lim_{\beta \to 1} I_2(y) = \frac{d}{dy} \int_0^1 \tau_2(\xi) \frac{y}{y - \eta} d\eta = \frac{d}{dy} \int_0^1 \tau_2(\xi) \frac{y}{y - \eta} d\eta = \frac{d}{dy} \int_0^1 \tau_2(\xi) \frac{y}{y - \eta} d\eta.
\]

\[
\lim_{\beta \to 1} I_3(y) = \int_0^1 u_x(l, y - \frac{l}{\xi}) \frac{1}{\xi^2} \phi(-\beta, 0; -\xi) d\eta.
\]

\[
\lim_{\beta \to 1} I_5(y) = u_0(y) \frac{y}{y - \eta} \phi(-\beta, 0; -\xi) d\eta = \int_0^1 u_x(l, y - \frac{l}{\xi}) \frac{1}{\xi^2} \phi(-\beta, 0; -\xi) d\eta.
\]

\[
\lim_{\beta \to 1} I_5(y) = u_0(y) \frac{y}{y - \eta} \phi(-\beta, 0; -\xi) d\eta.
\]

where \(H(x)\) is the Heaviside function.

Similarly, for the terms on the right-hand side of (24), we obtain

\[
\lim_{\beta \to 1} J_1(y) = \int_{l-y}^l \tau_1(\xi) H(\xi) d\xi,
\]

\[
\lim_{\beta \to 1} J_2(y) = \frac{d}{dy} \int_{l-y}^l \tau_2(\xi) H(\xi) d\xi = \frac{d}{dy} \int_{l-y}^l \tau_2(\xi) H(\xi) d\xi.
\]

\[
\lim_{\beta \to 1} J_3(y) = \int_{l+y}^y u_x(0, \eta) d\eta,
\]

\[
\lim_{\beta \to 1} J_5(y) = u_0(y - \eta) \frac{y}{y - \eta} \phi(-\beta, 0; -\xi) d\eta.
\]
Using relations (25) – (32), from (23) and (24) we obtain necessary non-local condition for wave equation

\[ u(0, y) = \int_0^l \tau_1(t) H(y - t) dt + \tau_2(y) H(l - y) + \int_0^{y-l} u_x(l, s) ds - \int_0^y u_x(0, s) ds + u(l, y - l) H(y - l), \]

\[ u(l, y) = \int_{l-y}^l \tau_1(t) H(t) dt + \tau_2(l - y) H(l - y) - H(y - l) \int_0^{y-l} u_x(0, s) ds + \int_0^y u_x(l, s) ds + u(0, y - l) H(y - l). \]

For \( y \leq l \) the conditions take the form:

\[ u(0, y) = \int_0^y \tau_1(t) dt + \tau_2(y) - \int_0^y u_x(0, s) ds, \quad (33) \]

\[ u(l, y) = \int_{l-y}^l \tau_1(t) dt + \tau_2(l - y) + \int_0^y u_x(l, s) ds. \quad (34) \]

For \( y \geq l \) we have

\[ u(0, y) = \int_0^l \tau_1(t) dt + \int_0^{y-l} u_x(l, s) ds - \int_0^y u_x(0, s) ds + u(l, y - l), \quad (35) \]

\[ u(l, y) = \int_0^l \tau_1(t) dt - \int_0^{y-l} u_x(0, s) ds + \int_0^y u_x(l, s) ds + u(0, y - l). \quad (36) \]

In case \( l = T \) the conditions (33) and (34) was obtained in paper [14].

5 Samarskii problem

5.1 Samarskii problem for the time-fractional diffusion-wave equation

Using Theorem 1, we investigate the Samarskii problem for the equation (1) in the following formulation.

**Problem 1.** In the domain \( \Omega \) find a solution \( u(x, y) \) of equation (1), satisfying the condition (15) and boundary conditions

\[ a_1 u(0, y) + a_2 u(l, y) = \varphi(y), \quad 0 < y \leq T, \quad (37) \]
\[
\int_0^t u(x, y)dx = \mu(y), \quad 0 < y \leq T;
\]  

(38)

where \( \tau_1(x) \), \( \varphi(y) \), \( \mu(y) \) are given functions, \( a_1, a_2 \) are given numbers, and \( a_1 \neq a_2 \).

The condition (38) is called Samarskii condition [10, p. 140].

In paper [11] Duhamel-type representation of the solution of problem 1 with fractional derivative in Caputo sense was obtained in case \( a_1 = 1, a_2 = 0, \varphi \equiv 0, \mu \equiv 0 \), and the initial conditions given in the form

\[
\frac{\partial^k}{\partial y^k} u(x, y) \big|_{y=0} = f_k(x), \quad k = 1, n,
\]

where \( f_k(x) \) are the functions choosed in a special way. In paper [13] in case \( 0 < \alpha < 1, a_1 = 1, a_2 = 0, \mu(y) \equiv \text{const} y^{\alpha-1} \), and when the condition (15) given in local form, problem 1 was studied by using the separation of variables method.

Note that the solving of Problem 1 by the reduction to the first boundary value problem with the help of conditions (16) and (17) was announced in paper [7].

By \( C^{1, q}[0, l] \) we denote the space of continuously differentiable functions on \( [0, l] \) whose derivatives satisfy the Holder condition with exponent \( q \). Following assertion hold.

**Theorem 2.** Let \( \tau_1(x) \in C[0, l] \); \( \tau_2(x) \in C^{1, q}[0, l], \) \( q > \frac{1-\beta}{\beta} \), for \( n = 2 \);

\[
y^{n-\alpha}\varphi(y) \in C[0, T], \quad D_0^\mu \mu(y) \in C[0, T]
\]

and the matching conditions

\[
\lim_{y \to 0} D_0^{\alpha-n} \varphi(y) = a_1 \tau_n(0) + a_2 \tau_n(l),
\]

\[
\lim_{y \to 0} D_0^{\alpha-k} \mu(y) = \int_0^l \tau_k(x)dx, \quad k = 1, n.
\]

(39)

(40)

are satisfied. Then there exists unique regular in the domain \( \Omega \) solution of problem 1. This solution has the form

\[
u(x, y) = \sum_{k=1}^n \sum_{m=-\infty}^{\infty} \left[ \mathcal{R}_{0y}^{\alpha-k+1, 2m+l-x, y} - \mathcal{R}_{0y}^{\alpha-k+1, 2m-l-x, y} \right] \tau_k(x) - 2 \sum_{m=1}^{\infty} \left[ \mathcal{R}_{0y}^{0, 2m-l-x} - \mathcal{R}_{0y}^{0, 2m-l-x} \right] \varphi_1(y) + 2 \sum_{m=1}^{\infty} \left[ \mathcal{R}_{0y}^{0, 2m-l-x} - \mathcal{R}_{0y}^{0, 2m-l-x} \right] \varphi_0(y),
\]

(41)

where

\[
\varphi_0(y) = \frac{a_2}{a_2 - a_1} \psi(y) + \frac{1}{a_2 - a_1} \varphi(y), \quad \varphi_1(y) = \frac{a_1}{a_1 - a_2} \psi(y) + \frac{1}{a_1 - a_2} \varphi(y),
\]

\[
\psi(y) = 2 \sum_{k=1}^n \sum_{m=-\infty}^{\infty} \mathcal{R}_{0y}^{\alpha-k+1, ml-y} \tau_k(\xi) + 4 \sum_{m=1}^{\infty} \mathcal{R}_{0y}^{\alpha, ml} D_0^\mu \mu(y) + D_0^{-\beta} D_0^\mu \mu(y).
\]

(42)

**Proof.** By integrating both sides of equality (11) by \( x \) in view of (38), we obtain the condition

\[
u_x(l, y) - u_x(0, y) = D_0^\alpha \mu(y), \quad 0 \leq y \leq T.
\]

(43)
From (16) and (17) by taking into account the conditions (37) and (43) we obtain

$$\psi(y) - 2R_{0y}^{0,l} \psi(\eta) = \Phi(y),$$

(44)

with respect to the function $\psi(y) = u(0, y) + u(l, y)$, where

$$\Phi(y) = 2\sum_{k=1}^{n} \left[ N_{0l}^{\alpha,0,y} + N_{0l}^{\beta,1,y} \right] \tau_k(\xi) + 2 \left[ R_{0y}^{0,0} + R_{0y}^{\alpha,l} \right] D_{0y}^{\alpha} \mu(y),$$

$$\delta_k = \beta - k + 1.$$

Let us show that $y^{\alpha - \alpha} \Phi(y) \in C[0, T]$. Since $\tau_k(x) \in C[0, l]$, then in view of the formula (12) we get

$$N_{0l}^{\beta-k+1,x,y} \tau_k(t) = \frac{1}{2} \int_{0}^{l} \tau_k(t) y^{\beta-k} \phi(-\beta, \beta - k + 1; -x - t |y^{-\beta}) dt \leq$$

$$\leq M \int_{0}^{l} y^{\beta-k} \phi(-\beta, \beta - k + 1; -x - t |y^{-\beta}) dt =$$

$$= \frac{M}{2} y^{2\beta-k} \phi(-\beta, 2\beta - k + 1; -x - t |y^{-\beta}) \bigg|_{t=0}^{t=1},$$

where $M = \max_{x \in [0, l]} \tau_k(x)$. From last relation and the estimates

$$|y^{\alpha-k} \phi(-\beta, \alpha - k + 1; -x y^{-\beta})| \leq C x^{-\theta} y^{\alpha-k+\beta\theta},$$

(45)

which holds in view of (11), follows that $y^{k-\alpha} N_{0l}^{\beta-k+1,x,y} \tau_k(\xi) \in C[0, T], k = 1, n$. From $D_{0y}^{\alpha} \mu(y) \in C[0, T]$ follow that $D_{0y}^{\alpha} D_{0y}^{\alpha} \mu(y), R_{0y}^{0,0} D_{0y}^{\alpha} \mu(y) \in C[0, T]$.

The equation (44) is an integral Volterra equation of the second kind. Its unique solution can be written in the form

$$\psi(y) = 2 \sum_{m=0}^{\infty} R_{0y}^{0,m} \Phi(\eta).$$

(46)

Indeed, by virtue of (5) we get

$$\psi(y) - 2R_{0y}^{0,l} \psi(\eta) = +2 \sum_{m=0}^{\infty} R_{0y}^{0,m} \Phi(\eta) =$$

$$= \Phi(y) + 2 \sum_{m=1}^{\infty} R_{0y}^{0,m} \Phi(\eta) - 2 \sum_{m=0}^{\infty} R_{0y}^{0,(m+1)} \Phi(\eta) =$$

$$= \Phi(y) + 2 \sum_{m=1}^{\infty} R_{0y}^{0,m} \Phi(\eta) - 2 \sum_{m=1}^{\infty} R_{0y}^{0,m} \Phi(\eta) = \Phi(y).$$

We rewrite equality (46) in the form

$$\psi(y) = \Phi(y) + \int_{0}^{y} W(y - \eta) \Phi(\eta) d\eta,$$
Hence using the property 3 we obtain (42).

From (42) by using (13) and (14) we obtain the form

\[ y = \text{solution of the system} \]

On condition

From estimate (11) follows

Now, when the function \( y \) is found, we can find \( u(0, y) \) and \( u(l, y) \), as a solution of the system

On condition \( a_1 \neq a_2 \), the unique solution of this system is

It is obvious that \( y^{n-\alpha} \varphi_0(y), y^{n-\alpha} \varphi_1(y) \in C[0, T] \).

The inclusion \( y^{n-\alpha} u(x, y) \in C(T) \) is valid if the conditions

Let us show it. From the relations (17) and (18) follows that

From (12) by using (13) and (14) we obtain

where

\[
W(y) = \sum_{m=1}^{\infty} \frac{(-1)^m}{y} \phi(-\beta, 0; -mly^{-\beta}).
\]
We integrate the equality (55) on the interval \([0, l]\) with respect to \(x\). Using properties 5 and 6, we get

\[
\int_0^l u(x, y)dx = 2 \sum_{k=1}^n \sum_{m=1}^{\infty} (-1)^{m+1} \left[ \Lambda_{0t}^{\delta_k+\beta,(m+1)l,y} + \Lambda_{0t}^{\delta_k+\beta,-ml,y} \right] \tau_k(x) + \\
+ \frac{1}{\Gamma(\delta + \beta)} \int_0^l \tau_k(x)dx + 2 \mathcal{R}_{0y}^{\beta,0} \psi(y) + 4 \sum_{m=1}^{\infty} (-1)^m \mathcal{R}_{0y}^{\beta,m} \psi(y),
\]

where \(\psi(y) = \varphi_0(y) + \varphi_1(y)\) is a solution of the Volterra equation (44).

From (56) we obtain

\[
2 \mathcal{R}_{0y}^{\beta,0} \psi(y) = D_{0y}^{\beta} \psi(y) = 2 \sum_{m=0}^{\infty} \mathcal{R}_{0y}^{\beta,m} \Phi(y).
\]
We transform the last two addends in right side of \( \text{(50)} \) with the help of \( \text{(5)} \)
\[
2\mathcal{R}_{0y}^\beta \psi(y) + 4 \sum_{m=1}^{\infty} (-1)^m \mathcal{R}_{0y}^{\beta,m} D_{0y}^{-\beta} \psi(y) =
\]
\[
= 2 \sum_{m=0}^{\infty} \mathcal{R}_{0y}^{\beta,m} \Phi(y) + 8 \sum_{m=1}^{\infty} (-1)^m \mathcal{R}_{0y}^{\beta,m} \mathcal{R}_{0y}^{\beta,s} \Phi(y) =
\]
\[
= 2 \sum_{m=0}^{\infty} \mathcal{R}_{0y}^{\beta,m} \Phi(y) + 8 \sum_{m=1}^{\infty} \sum_{s=1}^{m} (-1)^s \mathcal{R}_{0y}^{\beta,s} \mathcal{R}_{0y}^{\beta, (m-s)} \Phi(y) =
\]
\[
= 2 \sum_{m=0}^{\infty} \mathcal{R}_{0y}^{\beta,m} \Phi(y) + 4 \sum_{m=1}^{\infty} \mathcal{R}_{0y}^{\beta,m} \Phi(y) \sum_{s=1}^{m} (-1)^s =
\]
\[
= 2 \mathcal{R}_{0y}^{\beta,0} \Phi(y) + \sum_{m=1}^{\infty} (-1)^m \mathcal{R}_{0y}^{\beta,m} \mathcal{R}_{0y}^{\beta,m} \Phi(y) = 2 \sum_{m=1}^{\infty} (-1)^m \mathcal{R}_{0y}^{\beta,m} \Phi(y). \quad \text{(57)}
\]
By denoting \( \mu_1(y) = D_{0y}^{-\beta} D_{0y}^{\alpha} \mu(y) \) and using the properties of operator \( \mathcal{R}_{0y}^{\beta,0} \), we obtain
\[
2 \sum_{m=0}^{\infty} \mathcal{R}_{0y}^{\beta,m} \mu_1(y) + 2 \sum_{m=1}^{\infty} (-1)^m \mathcal{R}_{0y}^{\beta,m} \mathcal{R}_{0y}^{\beta,m} \mu_1(y) +
\]
\[
+ 2 \sum_{m=0}^{\infty} (-1)^m \mathcal{R}_{0y}^{\beta,m} \sum_{k=1}^{n} \left[ \mathcal{N}_{0l}^{\delta_k,0,m} + \mathcal{N}_{0l}^{\delta_k,1,m} \right] \tau_k(\xi) = D_{0y}^{-\beta} \mu_1(y) +
\]
\[
+ 2 \sum_{k=1}^{n} \sum_{m=0}^{\infty} (-1)^m \left[ \mathcal{N}_{0l}^{\delta_k,\beta, -m,1} + \mathcal{N}_{0l}^{\delta_k,\beta, (m+1),1} \right] \tau_k(\xi). \quad \text{(58)}
\]
By virtue of the fractional analogue of Newton-Leibniz formula \([11, p. 11]\), we have
\[
D_{0y}^{-\beta} \mu_1(y) = D_{0y}^{-\alpha} D_{0y}^{\alpha} \mu(y) = \mu(y) - \sum_{k=1}^{n} \frac{y^{\delta_k + \beta - 1}}{\Gamma(\delta_k + \beta)} \lim_{y \to 0} D_{0y}^{\alpha-k} \mu(y).
\]
By taking into account last relation, from \([55], [57], \) and \([58] \) we obtain
\[
\int_{0}^{l} u(x, y) dx = \mu(y) + \sum_{k=1}^{n} \frac{y^{\delta_k + \beta - 1}}{\Gamma(\delta_k + \beta)} \left[ \int_{0}^{l} \tau_k(x) dx - \lim_{y \to 0} D_{0y}^{\alpha-k} \mu(y) \right].
\]
Thus, under matching conditions \([40] \) function \( u(x, y) \) satisfies Samarskii integral condition. The proof of Theorem 2 is complete.
5.2 Samarskii problem for the wave equation

Consider Problem 1 in case when $\alpha = 2$, $a_1 = 1$, $a_2 = 0$ and $T < l$. In general case this problem can be solved in similar way.

**Problem 2.** Find a solution of equation

$$u_{xx} - u_{yy} = 0,$$

(satisfying the conditions

$$u(x,0) = \tau(x), \quad u_y(x,0) = \nu(x), \quad 0 < x < l,$$

$$u(0,y) = \varphi_0(y), \quad \int_0^l u(x,y)dx = \mu(y), \quad 0 < y < T < l.$$  

where $\tau(x)$, $\nu(x)$, $\varphi_0(y)$, $\mu(y)$ are given functions.

Problem 2 for the wave equation was studied in the work [3] by the reduction to the problem with non-local Bitsadze-Samarskii condition. Note also the paper [5] in which the non-local initial boundary value problems with integral nonlocal boundary conditions are investigated for one-dimensional medium oscillation equations and solutions of the corresponding problems are constructed. More extensive overview of the subject of nonlocal boundary problems for wave equation can be found in works [3] and [2].

From (33) and (34) by virtue of the equality

$$\int_0^y \left[ u_x(l,s) - u_x(0,s) \right] ds = \mu'(y) - \mu'(0),$$

we express the value of $u(l,y)$ through the data of Problem 2:

$$u(l,y) = \int_0^y \nu(t)dt + \int_{l-y}^l \nu(t)dt + \tau(y) + \tau(l-y) + \mu'(y) - \mu'(0) - \varphi_0(y) \equiv \varphi_1(y).$$

Thus, Problem 2 is reduced to the local first boundary value problem for the equation (59), which solution has the form [19, c. 70]

$$u(x,y) = \frac{\tau(x+y) + \tau(x-y)}{2} + \frac{1}{2} \int_{x-y}^{x+y} \nu(t)dt + \varphi_0(y-x) - \varphi_1(y-x),$$

(60)

where $\varphi_0(y) = \varphi_0(y)H(y)$, $\varphi_1(y) = \varphi_1(y)H(y)$, $H(y)$ is a Heaviside function, and

$$\tau(-x) = -\tau(x), \quad \tau(2l-x) = -\tau(x), \quad \nu(-x) = -\nu(x), \quad \nu(2l-x) = -\nu(x).$$

(61)

Obviously, that the function (60) is a solution of the equation (59), and also that the first three conditions of Problem 2 are satisfied.

Let us show that the fourth also is satisfied. Integrating the relation (60) with respect to variable $x$ from 0 to $l$

$$\int_0^l u(x,y)dx = \frac{1}{2} \int_0^l \left[ \tau(x+y) + \tau(x-y) \right] dx + \frac{1}{2} \int_0^l \nu(t)dt + \int_0^l \varphi_0(y-x)dx,$$

(60)
\[
+ \int_0^l \varphi_l(y + x - l)dx = I_1 + I_2 + I_3 + I_4. \tag{62}
\]

We transform the integrals \( I_k \) \((k = 1, 4)\). By taking into account the equalities (61), we obtain

\[
2I_1 = \int_0^l \tau(t)dt - \int_0^{l+y} \tau(2l - t)dt - \int_{-y}^0 \tau(-t)dt + \int_{-y}^{l-y} \tau(t)dt = 2\int_{-y}^{l-y} \tau(t)dt, \tag{63}
\]

\[
2I_2 = \int_0^l (t + y)\nu(t)dt + \int_{-y}^{l-y} (t + y)\nu(t)dt + \int_0^y \nu(t)dt + \int_y^l \nu(t)dt + \int_{y}^{l-y} (l - t + y)\nu(t)dt + \int_{-y}^0 (l - t + y)\nu(t)dt = 2\int_{-y}^y \nu(t)dt, \tag{64}
\]

\[
I_3 = \int_0^y \varphi_0(y - x)dx = \int_0^y \varphi_0(t)dt, \tag{65}
\]

\[
I_4 = \int_{l-y}^l \varphi_l(y + x - l)dx = \int_{l-y}^y \varphi_l(t)dt = \int_0^y (y - t)\nu(t)dt + \int_y^l (y + t - l)\nu(t)dt + \int_0^l \tau(s)ds + \int_{l-y}^l \tau(s)ds - \int_0^y \varphi_0(s)ds + \mu(y) - \mu(0) - \mu'(0)y. \tag{66}
\]

By virtue of (63) – (66), from (62) we obtain

\[
\int_0^l u(x,y)dx = I_1 + I_2 + I_3 + I_4 = \left( \int_0^y + \int_{l-y}^l \int_{-y}^{l-y} \right) \tau(t)dt - \mu(0) + \int_0^y + \int_{l-y}^{l-y} \nu(t)dt - \mu'(0) + \mu(y).
\]

From the last we can see that the function (60) satisfy the condition

\[
\int_0^l u(x,y)dx = \mu(y),
\]

if the following matching conditions hold

\[
\int_0^l \tau(x)dx = \mu(0), \quad \int_0^l \nu(x)dx = \mu'(0).
\]

Thus the function (60) is the solution of Problem 2.
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