Dissipative Viscous Cylindrical Collapse in $f(R)$ Gravity With Full Causal Approach

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Abstract

The plan of this study is to inspect the effects of dynamics of dissipative gravitational collapse in cylindrical symmetric non-static spacetime by using Misner-Sharp concept in context of metric $f(R)$ theory of gravity. For more generic isotropic fluid distribution of cylindrical object by dissipative nature of dark source of the fluid due to energy matter tensor, the Misner-Sharp approach technique has been used to illustrate the heat flux with free radiating out flow, bulk and shear viscosity. Furthermore, dynamical equation have been associated with full causal heat transportation equations in framework of Müller-Israel-Stewart formalism. The present study explain the effects of thermodynamics viscid/heat radiating coupling factors on gravitational collapse in Müller-Israel-Stewart notion and matches with the consequences of prior astrophysicists by excluding such coefficients and viscosity variables.

Keywords: Gravitational viscous collapse, Cylindrical structure, Dynamical equations, Transport equations, Modified $f(R)$ gravity.

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1 Introduction

A provoking still inquisitive problem for cosmological, gravitating physics and along with astronomical in general theory of relativity (GR) is to identify the concluding providence of a incessant dissipative gravitating implosion. The firm arrangement of a gigantic object endures longer the inmost pull of gravity is nullified by the external stress of the fissile fuel at the centre of the object. Later, once the object has wakened its fissile fuel there is no extended any thermodynamical scorching and there will be interminable gravitating implosion. The configuration of the objects are made of explosive mass which incessantly gravitational and is fascinated to its core due to gravitating interface of its particles. Later astrophysicist [1]-[3] galactic perceptions like the investigation of type Ia supernova demonstrate that the extension of the cosmos is rushing and its alleged this might be expected to a continuous positive cosmological invariant. These perceptions have raised in thoughtfulness regarding the investigation of astral bodies with cosmological invariant. Oppenheimer and Snyder [4] are pioneers of this field of gravitating source long decades ago. They made an activity stride in the field of gravitating collapse and examined a nitty gritty work regarding the issue of gravitational collapse. At this order following a quarter century, a brief sensible examination was accomplished by Misner and Sharp [5] with isotropic matter in the inside of a imploding object and as well as outside the object in vacuum. Vaidya [6] explain the gravitational collapse of radiating source.

Various paradigms have been opened to explicate the enigmatic behavior of dark energy (DE). Using the modified concepts of gravity for instance \( f(R) \), \( f(R, T) \) and \( f(G) \) etc. can be well elucidated for the expanding problem of universe. Capozziello [7] studied the metric \( f(R) \) theory of gravity is the usual form of general relativity (GR) got by presenting an uniformed function addiction on R (Ricci invariant). Sharif and Yousaf [8] explored the constancy of an imperfect fluid expansion region with non-static parameter of spherical symmetric model in \( f(R) \) framework. Mak and Harko [9] has investigated the notional indications from many astrophysical procedures that establish the significance of natural dissimilarities in pressure. Herrera and Santos [10] determined the impacts of imperfect fluid in astral spherical objects for the behavior of leisurely gyration. Webber [11] examined the effects of anisotropy pressure in dense models with magnetic field. Chakraborty et al. [12] analyzed the gears of tangential pressure along with radial pres-
sure in astrophysical structures by quasi-spherical paradigms in the form of implosion. Garattini [13] explored the configuration of exposed peculiarity through mass limitation in metric $f(R)$ theory of gravity. Sharif and his traitors [13]-[21] investigated the various fascinating consequence by the association of inhomogeneous energy density and Weyl tensor in dense objects during the collapse in GR as well as in modified theories of gravity. Cognola et al. [22] have introduced the results of the spherical imploding object for black hole (BH) in $f(R)$ context by the parameter of positive Ricci invariant $R$. Capozziello et al. [23] examined the modified Lane-Emden equation by $N$ parameter in context of $f(R)$ and study the hydrostatic periods of astral models. Copeland et al. [24] have analyzed the several concepts to confer the expanding behavior of universe. Amendola et al. [25] discussed the feasible paradigms of metric $f(R)$ gravity in context of both Einstein and Jordan frames.

Chandrasekhar [26] introduced the impacts of perfect fluid collapsing source and deliberate the constancy of dynamics. Herrera et al. [27] discussed the effects of dissipative variables on the gravitating imploding source and investigate that heat dissipation makes the scalar structure constant. Abbas et al. [28] presented the results of compact star models of anisotropic fluid distribution with cylindrical symmetric static spacetime geometry. Mak and harko [29] studied the kind of exact results of the field equations by taking spherical structure. They also describe that energy density along with tangential and radial pressure are limited and increase at the core of imperfect fluid object. Rahaman et al. [30] prolonged the Krori-Barua solution with spherical surface of static spacetime for the examination of charge imperfect fluid. Herrera and his collaborators [31]-[41] analyzed the stability and dynamics of many gravitating source using various form of fluid. In few investigations they have used the non-casual as well as casual approaches to discuss the dynamics of gravitating source. Nolan [42] discussed the gravitational collapse of counter gyrating dust cloud with cylindrical paradigms to find the naked peculiarity. Hayward [43] produced the results of cylindrical geometry for black holes, gravitational waves and cosmic strings. Chan [44] has investigated the role of shear viscosity act in the imploding course.

In this paper, we analyze the dynamics of gravitational collapse with cylindrical symmetry in the context of $f(R)$ gravity. The main persistence of this study is to examine the viscous dissipative collapse in cylindrical spacetime with full causal approach.

The proposal of this work as follows: In section 2, we describe the cylin-
drical symmetric spacetime and isotropic collapsing matter distribution in $f(R)$ theory of gravity. In section 3 Einstein modified field equations and the dynamical equations have been acquired by gravitational cylindrical source. Section 4 is devoted to the transportation equations attained in frame of Müller-Israel-Stewart formalism [45]-[47], then transportation equations are associated by dynamical equations. Finally, the conclusions of the study have been discussed in the last section.

2 Isotropic gravitational collapsing matter distribution fenced with cylindrical astral structure

In general relativity (GR) Einstein-Hilbert (EH) action can be expressed as,

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R. \quad (1)$$

The prolonged formulation of Einstein-Hilbert (EH) in $f(R)$ context is,

$$S_{modif} = \frac{1}{2\kappa} \int d^4x \sqrt{-gf(R)}, \quad (2)$$

where $f(R)$ is arbitrary function of Ricci scalar $R$, $T_{\alpha\beta}$ is isotropic matter tensor and $\kappa$ is coupling constant. The devising of equation (2) in form of $f(R)$ theory have the following Einstein field equations,

$$F(R)R_{\alpha\beta} - \frac{1}{2} f(R)g_{\alpha\beta} - \nabla_\alpha \nabla_\beta F(R) + g_{\alpha\beta} \nabla^\alpha \nabla_\alpha F(R) = \kappa T_{\alpha\beta}, \quad (3)$$

Here $F(R) = \frac{df(R)}{dR}$, $\nabla^\alpha \nabla_\alpha$ is the D’Alembert operator and $\nabla_\alpha$ denotes the covariant derivative. Above equations may be expressed in Einstein tensor as given below,

$$G_{\alpha\beta} = \frac{\kappa}{F} \left( T^{m}_{\alpha\beta} + T^{D}_{\alpha\beta} \right), \quad (4)$$

where

$$T^{D}_{\alpha\beta} = \frac{1}{\kappa} \left[ \frac{f(R) - RF(R)}{2} g_{\alpha\beta} + \nabla_\alpha \nabla_\beta F(R) - g_{\alpha\beta} \nabla^\alpha \nabla_\alpha F(R) \right], \quad (5)$$
represent the stress action matter energy tensor. The cylindrically symmetric non-static spacetime structure which contains the dissipative fluid defines interior metric given by,

\[ ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 d\phi^2 + D^2 dz^2, \] (6)

where \( A = A(t, r), B = B(t, r), C = C(t, r), D = D(t, r) \). We labeled the coordinates \( x^0 = t, x^1 = r, x^2 = \phi \) and \( x^3 = z \). The exterior metric is given by,

\[ ds^2 = -\left( -\frac{2M(\nu)}{R} \right) d\nu^2 - 2d\nu d\bar{R} + \bar{R}^2 (d\phi^2 + \gamma^2 dZ^2), \] (7)

where \( M(\nu) \) is mass and \( \gamma^2 = -\frac{\Lambda}{3} \), \( \Lambda \) is cosmological constant. The tensor \( T_{\alpha\beta}^- \) of the dissipative collapsing matter distribution is as,

\[ T_{\alpha\beta}^- = (\mu + p + \Pi)V_\alpha V_\beta + (p + \Pi)g_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha + \epsilon l_\alpha l_\beta + \pi_{\alpha\beta}, \] (8)

Here \( \mu \) represent energy density, \( p \) is the isotropic pressure, \( \Pi \) the bulk viscosity, \( q_\alpha \) the heat flux, \( \pi_{\alpha\beta} \) is the shear viscosity, \( \epsilon \) the radiation density, \( V_\alpha \) the four velocity of the fluid. Furthermore, \( l_\alpha \) denotes radial null four vector. The above extents gratify the results as below.

\[ V_\alpha V_\alpha = -1, \quad V_\alpha q_\alpha = 0, \quad l_\alpha V_\alpha = -1, \quad l_\alpha l_\alpha = 0, \] (9)

\[ \pi_{\mu\nu} V^\nu = 0, \quad \pi_{[\mu\nu]} = 0, \quad \pi_\alpha^\alpha = 0. \] (10)

In general non-reversible thermodynamics in [48, 49],

\[ \pi_{\alpha\beta} = -2\eta \sigma_{\alpha\beta}, \quad \Pi = -\zeta \Theta, \] (11)

wherever \( \eta \) is the factor of shear viscosity and \( \zeta \) denotes the factor of bulk viscosity, \( \sigma_{\alpha\beta} \) and \( \Theta \) are the shear tensor and heat flow expansion respectively.

The shear tensor \( \sigma_{\alpha\beta} \) is defined by,

\[ \sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha} V_{\beta)} - \frac{1}{3} \Theta h_{\alpha\beta}, \] (12)

where the acceleration \( a_\alpha \) and the expansion \( \Theta \) are recognized by,

\[ a_\alpha = V_{\alpha;\beta} V^\beta, \quad \Theta = V_\alpha^\alpha. \] (13)
and $h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta$ is the projector onto the hypersurface orthogonal to the four velocity. Let us define the following quantities for the given metric,

$$V^\alpha = A^{-1} \delta_0^\alpha, \quad q^\alpha = qB^{-1} \delta_1^\alpha, \quad l^\alpha = A^{-1} \delta_0^\alpha + B^{-1} \delta_1^\alpha,$$

where $q$ depending on $t$ and $r$. Also it trails from (9) so that,

$$\pi_{0\alpha} = 0, \pi_{11} = -2\pi_2^2 = -2\pi_3^3. \quad (15)$$

In a more explicit formation we can write,

$$\pi_{\alpha\beta} = \Omega \left( \chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta} \right), \quad (16)$$

Here $\chi^\alpha$ is a unit four vector along the radial direction, sustaining

$$\chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0, \quad \chi^\alpha = B^{-1} \delta_1^\alpha, \quad (17)$$

and $\Omega = \frac{3}{2}\pi_1^1$. With Eq. (14), we obtain the following non null components of shear tensor.

$$\sigma_{11} = \frac{B^2}{3A} [\Sigma_1 - \Sigma_3], \sigma_{22} = \frac{C^2}{3A} [\Sigma_2 - \Sigma_1], \sigma_{33} = \frac{D^2}{3A} [\Sigma_3 - \Sigma_2]. \quad (18)$$

Now

$$\sigma_{\alpha\beta} \sigma^{\alpha\beta} = \sigma^2 = \frac{1}{3A^2} [\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2] \quad (19)$$

Here

$$\Sigma_1 = \frac{\dot{B}}{B} - \frac{\dot{C}}{C}, \Sigma_2 = \frac{\dot{C}}{C} - \frac{\dot{D}}{D}, \Sigma_3 = \frac{\dot{D}}{D} - \frac{\dot{B}}{B}, \quad (20)$$

and here dot denotes derivative with respect to $t$.

Using Eqs.(13) and (14), we get

$$a_1 = \frac{\dot{A}}{A}, \quad \Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} \right), \quad (21)$$

where prime represents derivative with respect to $r.$
3 Dynamical Equations in \( f(R) \) gravity

The modified field Eqs. (4) for cylindrical symmetric spacetime in \( f(R) \) metric formalism gives the following system of equations.

\[
\begin{align*}
\left( \frac{A}{B} \right)^2 & \left[ -\frac{C''}{C} - \frac{D''}{D} + \frac{B'}{B} \left( \frac{C'}{C} + \frac{D'}{D} \right) - \frac{C'D'}{CD} \right] \\
+ \left( \frac{B'C'}{BC} + \frac{B'D'}{BD} + \frac{C'D'}{CD} \right) & = \kappa \left[ (\mu + \epsilon) A^2 + \frac{A^2}{\kappa} \left\{ - \left( \frac{f(R) - RF(R)}{2} \right) \right\} \right] \\
- \frac{\ddot{F}}{A^2} & \left( \frac{B}{B} + \frac{C}{C} + \frac{D}{D} \right) = \kappa \left[ - (q + \epsilon) AB + \frac{1}{\kappa} \left( \dot{F}' - \frac{A'F}{A} - \frac{B'F'}{B} \right) \right], \\
\end{align*}
\]
(22)

\[
\begin{align*}
- \left( \frac{\dot{C}'}{C} + \frac{\dot{D}'}{D} - \frac{\dot{B'C'}}{BC} - \frac{\dot{B'D'}}{BD} - \frac{\dot{C'A'}}{AC} - \frac{\dot{D'A'}}{AD} \right) & = \kappa \left[ - (q + \epsilon) AB + \frac{1}{\kappa} \left( \dot{F}' - \frac{A'F}{A} - \frac{B'F'}{B} \right) \right], \\
\end{align*}
\]
(23)

\[
\begin{align*}
- \left( \frac{\dot{B}'D'}{BD} \right) & = \kappa \left[ (p + \Pi - \frac{\Omega}{3}) \right] B^2 + \frac{B^2}{\kappa} \left\{ \left( \frac{f(R) - RF(R)}{2} \right) \right\} + \frac{\dot{F}}{A^2} \\
- \frac{\ddot{F}}{A^2} & \left( \frac{C}{C} + \frac{D}{D} - \frac{\dot{A}}{A} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{C'}{C} + \frac{D'}{D} \right) \right], \\
\end{align*}
\]
(24)

\[
\begin{align*}
- \left( \frac{D}{A} \right)^2 & \left[ \dot{B} + \frac{C}{C} - \frac{A}{A} \left( \dot{B} + \frac{D}{D} \right) + \frac{B'D}{BD} \right] + \left( \frac{C}{B} \right)^2 \left[ \frac{A''}{A} + \frac{D''}{D} - \frac{A'}{A} \left( \frac{B'}{B} - \frac{D'}{D} \right) \right] \\
- \frac{B'D'}{BD} & = \kappa \left[ (p + \Pi - \frac{\Omega}{3}) C^2 - \frac{C^2}{\kappa} \left\{ - \left( \frac{f(R) - RF(R)}{2} \right) \right\} \right] \\
- \frac{\ddot{F}}{A^2} & + \frac{F''}{B^2} \left( \frac{A}{A} + \frac{B'}{B} - \frac{C}{C} - \frac{D'}{D} \right) + \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{B'}{B} + \frac{C'}{C} + \frac{D'}{D} \right) \right], \\
\end{align*}
\]
(25)

\[
\begin{align*}
- \left( \frac{D}{A} \right)^2 & \left[ \dot{B} + \frac{C}{C} - \frac{A}{A} \left( \dot{B} + \frac{D}{D} \right) + \frac{B'C}{BC} \right]
\end{align*}
\]
\[
\left( \frac{D}{B} \right)^2 \left[ \frac{A''}{A} + \frac{C''}{C} - \frac{A'}{A} \left( \frac{B'}{B} - \frac{C'}{C} \right) - \frac{B'C''}{BC} \right] = \frac{\kappa}{F} \left[ \left( p + \Pi - \frac{\Omega}{3} \right) D^2 
\right. \\
- \frac{D^2}{\kappa} \left\{ - \left( \frac{f(R) - RF(R)}{2} \right) - \frac{\dot{F}}{A^2} + \frac{F''}{B^2} + \frac{\ddot{F}}{A^2} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} - \frac{\dot{D}}{D} \right) 
\right.
\left. + \frac{\dot{F'}}{B^2} \left( \frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} + \frac{D'}{D} \right) \right\}.
\]

(26)

The energy of gravitating source per specific length defined by cylindrical symmetric space-time is given by [43], [50]-[52].

\[
E = \frac{(1 - l^{-2} \nabla^a r \nabla_a r)}{8}.
\]

(27)

For a cylindrical symmetric paradigms by killing vectors, the circumference radius \( \rho \) and specific length \( l \) and moreover, arial radius \( r \) are described in [43], [50]-[52].

\[
\rho^2 = \xi_{(1)} a \xi_{(1)}, \quad l^2 = \xi_{(2)} a \xi_{(2)}, \quad \text{so that} \quad r = \rho l.
\]

In the whole interior region C-energy has the following form [19]

\[
m(r, t) = El = \frac{l}{8} + \frac{1}{8D} \left[ \frac{1}{A^2} \left( C\dot{D} + \dot{C}D \right)^2 - \frac{1}{B^2} \left( CD' + C'D \right)^2 \right].
\]

(28)

The proper time and radial derivatives are given by, through Misner and Sharp technique [5].

\[
D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad D_C = \frac{1}{C'} \frac{\partial}{\partial r},
\]

(29)

Where \( C \) is the areal radius of a spherical surface inside the limit. The velocity of the collapsing matter is described by the proper time derivative of \( C \) and \( D \) [20], i.e.,

\[
U = D_TC = \frac{\dot{C}}{A}, \quad V = D_TD = \frac{\dot{D}}{A},
\]

(30)

which is always negative. Using this result, Eq. (28), indicates that

\[
\dot{E} \equiv \frac{C'}{B} \left[ \left( \frac{VC + UD}{D} \right)^2 - \frac{8}{D} \left( m(r, t) - \frac{l}{8} \right) \right] \frac{1}{2} - \frac{CD'}{BD}.
\]

(31)
The rate of change of mass with respect to proper time in equation (28), with the use of Eqs. (22), (23), (24)-(26) is given by

\[
D_T m(r, t) = \frac{CD}{F} \left[ -4\pi \left\{ \left( \mu + 2\epsilon - p - \Pi + \frac{4\Omega}{3} \right) U + \tilde{E}(q + \epsilon) \right\} \right.
\]

\[
- U \left\{ \tilde{E}^2 \left( \frac{F' C'}{C} + \frac{F' D'}{D} - \frac{3\tilde{E}F'D_C B}{2} \right) + \frac{\tilde{E}^2}{2} \left( \frac{D_C A D_T F}{C} + \frac{3F''}{C'^2} \right) \right.
\]

\[
+ \frac{\tilde{E}^2 F}{D} \left( D_{CC} D + \frac{(D_C D)^2}{4D} + \frac{D_C D}{4C} \right) + \frac{\tilde{E}^2 F D_{CC} C}{C} - \frac{\tilde{E}^3 F D_C B}{C'} \left( \frac{1}{C} + \frac{D_C D}{D} \right)
\]

\[
- \frac{F}{B} \left( \frac{V D_T B}{D} + D_{TT} B \right) - \frac{F}{4} \left( \frac{D_{TT} C}{C} + \frac{D_{TT} D}{D} \right) + \frac{R F(R)}{2} - \frac{f(R)}{2} \}
\]

\[
- U^3 \left\{ \frac{\tilde{F}}{C} \left( \frac{D_T A}{2C^2} - \frac{1}{C} \right) - \frac{\tilde{F}^3}{C} + \frac{F D_T A}{4CC'} \right\}
\]

\[
+ U^2 \left\{ \frac{\tilde{E}^2 F}{C} \left( \frac{3F D_T B}{2EC'} - \frac{F'D_C A}{2C'} \right) + \frac{\tilde{F}^3}{C'D} + \frac{F D_T B}{B} \left( \frac{1}{C} - \frac{D_T A}{C} \right) \right.
\]

\[
- \frac{\tilde{E}^2 F}{C} \left( D_{CC} A - \frac{\tilde{E} D_C A D_C B}{C'} + \frac{D_C A}{2C} \right) + \frac{V F}{2D} \left( \frac{D_T A}{C} - \frac{D_T A}{C} \right) \}
\]

\[
- \frac{V^2 F}{2D} \left( \frac{C D_{TT} D}{D} - \frac{U}{2D} \right) - \frac{V^3 F}{4D^2} \left( \frac{C(D_C D)^2}{2D^2} + \frac{D_T A}{D} \right)
\]

\[
+ \frac{V F}{4D} \left\{ \frac{C D_{TT} D}{D} + D_{TT} C + \tilde{E}^2 \left( \frac{C(D_C D)^2}{2D^2} - \frac{1}{2C} \right) \right. \}
\]

\[
- \frac{\tilde{E}^2}{2C'} \left( \tilde{E} D_T B \left( C_D F + \frac{F}{2C} - \frac{(D_C D)^2 F}{2D^2} \right) - \frac{F}{D} \left( \frac{D_T C}{2C} + \frac{D_T D'}{2D} \right) \right)
\]

\[
+ \frac{F D_C D}{2D} \left( \frac{C D_T D'}{D} + D_T C' \right) - D_T F' \right] \right\} \right.
\]

(32)

The overhead formation chiefs us to the distinction rate of mass energy in the cylinder having radius C and the R.H.S of this equation pronounces the raising in energy in the inside boundary of radius C. The value (\(\mu + 2\epsilon - p - \Pi + \frac{4\Omega}{3}\)) is the effective radial pressure whereas \(\mu, \epsilon \) and \(\Pi\) are the energy density, the radiation pressure and the bulk viscosity. The usual thermodynamic result is \(\pi_{\alpha\beta}\) in the relaxation period and the second value of the right-hand side illustrates the matter energy of the fluid, which is releasing the cylindrical exterior. Moreover, all extra terms reflects the behavior of the dark source of the dissipative matter under the formation of modified theory of gravity.
and with its differentiation, lessening the pressure in the core of star due to continuous collapsing phase of the dark energy. Similarly, we can calculate

\[ D_C m(r, t) = \frac{CD}{2F} \left[ \kappa \left\{ \mu + 2\epsilon + p + \Pi + \frac{2\Omega}{3} + \frac{U}{E} (q + \epsilon) \right\} \right. \]

\[ + \frac{U^3 F D_C A}{2CC} + U^2 \left\{ \frac{D_T F D_C A}{C} + \frac{F}{C} \left( \frac{D_C D}{4D} - \frac{D_T A}{C} \right) \right\} - U \left\{ \tilde{E} D_C F \left( \frac{\tilde{E} D_C A}{C} - \frac{D_T B}{C'} \right) \right. \]

\[ + \frac{D_T F D_T A}{C} + F \left( \frac{1}{C'} \left( \frac{D_T C'}{2C} + \frac{D_T D'}{2D} - \frac{\tilde{E} D_T B D_C D}{D} \right) + \frac{\tilde{E}^2}{C} \left( \frac{D_C A}{C} + \frac{D_C A D_C D}{D} \right) \right) \}

\[ - \frac{V^2 F}{2D^2} \left\{ C \left( \frac{U D_C A}{C} + \frac{D_C D}{2D} \right) - 1 \right\} + \frac{V F}{2C'D} \left( \frac{C D_T D'}{D} + D_T C' \right) + \frac{V F}{D} \left( \frac{1}{2C} - \frac{D_T A}{C} \right) \]

This solution explains how variational quantities effect the matter distribution between the neighboring exteriors in the object of radius C. The first two quantities on R.H.S and their picture relates with the overhead result expect for the factor \( p + \Pi + \frac{2\Omega}{3} \). The presence of this issue is due to the complex field equations. The outstanding values signify input of dark energy (DE) because of the curvature matter. Applying integration of Eq. (33) with C, we establish

\[ m(r, t) = \frac{1}{2} \int_0^C \frac{CD}{2F} \left[ \kappa \left\{ \mu + 2\epsilon + p + \Pi + \frac{2\Omega}{3} + \frac{U}{E} (q + \epsilon) \right\} \right. \]

\[ + \frac{U^3 F D_C A}{2CC} + U^2 \left\{ \frac{D_T F D_C A}{C} + \frac{F}{C} \left( \frac{D_C D}{4D} - \frac{D_T A}{C} \right) \right\} - U \left\{ \tilde{E} D_C F \left( \frac{\tilde{E} D_C A}{C} - \frac{D_T B}{C'} \right) \right. \]

\[ + \frac{D_T F D_T A}{C} + F \left( \frac{1}{C'} \left( \frac{D_T C'}{2C} + \frac{D_T D'}{2D} - \frac{\tilde{E} D_T B D_C D}{D} \right) + \frac{\tilde{E}^2}{C} \left( \frac{D_C A}{C} + \frac{D_C A D_C D}{D} \right) \right) \}

\[ - \frac{V^2 F}{2D^2} \left\{ C \left( \frac{U D_C A}{C} + \frac{D_C D}{2D} \right) - 1 \right\} + \frac{V F}{2C'D} \left( \frac{C D_T D'}{D} + D_T C' \right) + \frac{V F}{D} \left( \frac{1}{2C} - \frac{D_T A}{C} \right) \]
\[-E^2 F \left\{ \frac{D_C D}{2D^2} \left( CD_{CC} D + D_C D - \frac{C(D_C D)^2}{2D} - \frac{\hat{E}CD_CBD_C D}{C'} \right) \right\} \]
\[-\frac{1}{2C} \left\{ D_{CC} C - \frac{3D_C D}{2D} - \frac{\hat{E}CD_B}{C'} \right\} + \frac{1}{2D} \left( D_C DD_{CC} C - D_{CC} D \right) \}
\[-\frac{\hat{E}}{C''} \left( D_T F D_T B + \hat{E}^2 D_C F D_C B + \frac{F D_T B D_T D}{D} \right) + D_{TT} F + \frac{F'' \hat{E}^2}{C''} \]
\[-\frac{U D_T F'}{C''} + \frac{F D_{TT} C}{C} + \frac{F D_{TT} D}{D} \right\} dC. \tag{34} \]

The above equation gives the total mass in terms of C-energy inside the cylinder with the contribution of \( f(R) \) dark source term. To perceive the dynamical action by assuming Misner and Sharp [5, 53] concept, the contracted Bianchi identities are given as,

\[ \left( T_{\alpha \beta}^{(m)} + T_{\alpha \beta}^{(D)} \right) ;_\beta V_{\alpha} = 0, \quad \left( T_{\alpha \beta}^{(m)} + T_{\alpha \beta}^{(D)} \right) ;_\beta \chi_{\alpha} = 0, \tag{35} \]

which produce

\[ \frac{1}{A} \left[ (\mu + \dot{\epsilon}) + (\mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3}) \frac{\dot{B}}{B} + \left( \mu + p + \Pi + \epsilon - \frac{\Omega}{3} \right) \left( \frac{\dot{C}}{C} + \frac{\dot{D}}{D} \right) \right] \]
\[ + \frac{1}{B} \left[ (q + \dot{\epsilon})' + (q + \epsilon) \left( 2A' + \frac{(CD)'}{CD} \right) \right] - D_1 = 0. \tag{36} \]

\[ \frac{1}{A} \left[ (q + \dot{\epsilon}) + (q + \epsilon) \left( 2\frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} \right) \right] \]
\[ + \frac{1}{B} \left[ \left( p + \Pi + \epsilon + \frac{2\Omega}{3} \right)' + \left( \mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3} \right) \frac{A'}{A} + (\epsilon + \Omega) \frac{(CD)'}{CD} \right] + D_2 = 0, \tag{37} \]

Here \( D_1 \) and \( D_2 \) are the expressions of dark energy having in Appendix of equations (57) and (58) respectively.

Using Eqs. (24), (29)-(31), it follows that

\[ D_T (\mu + \epsilon) + \frac{1}{3} (3\mu + 3p + 3\Pi + 4\epsilon) \Theta + \frac{1}{3} (\epsilon + \Omega) \left( \frac{2\dot{B}}{AB} - \frac{U}{C} - \frac{V}{D} \right) \]
\[ + \hat{E} D_C (q + \epsilon) + (q + \epsilon) \left( 2\frac{a_1}{B} + \frac{\hat{E}}{C} + \frac{D'}{BD} \right) - D_1 = 0. \tag{38} \]
The acceleration $D_T U$ of the dissipative viscous collapsing source is achieved by using Eqs. (24), (29)-(31), are

$$D_T U = \frac{m(r, t)}{CD} + \frac{l}{8CD} - \frac{4\pi C}{F} \left( p + \Pi + \epsilon + \frac{2\Omega}{3} \right) + \frac{\tilde{E} C}{2B} a_1 + \left( \mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3} \right) \left( \Theta + \frac{D_T B}{B} \right) + D_2 = 0. \quad (39)$$

After, inserting $a_1$ from (40) into (39), we get

$$\left( \mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3} \right) D_T U = -\left( \mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3} \right) \left[ \frac{m(r, t)}{CD} - \frac{l}{8CD} - \frac{4\pi C}{F} \left( p + \Pi + \epsilon + \frac{2\Omega}{3} \right) + \frac{\tilde{E} C}{2B} a_1 + \left( \mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3} \right) \left( \Theta + \frac{D_T B}{B} \right) + D_2 \right].$$

Now, to evaluate the term $\left( \mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3} \right)$ seems on the L.H.S and as well as comes on R.H.S, this is the effective inertial mass, and rendering to the equivalence principle it is too recognized as passive gravitational mass. On
the R.H.S, the first term of square bracket quantity explicates the impacts of collapsing variables on the active gravitational mass of the cylindrical collapsing object in dark source with f(R) metric theory, this datum has been pointed out early by Herrera et al. [38] in the context of GR. Now in the second square bracket there are the gradient of the whole active pressure which is prejudiced with collapsing variables, radiating density. The final bracket comprises unalike additions because of the collapsing matter nature of the structure. The second value in this bracket is positive inferring that outgoing of $q > 0$ and $\epsilon > 0$ decreases the total energy of diminishing matter, which reduces the degree of collapse. The last term of this bracket describes the dark energy source of the dissipative gravitating collapse.

4 Transport Equations

The purpose of this section is to confer a full causal technique for the viscous dissipative gravitating collapse of astral paradigms accompanied by heat transference. This infers that all collapsing variables should please the transportation equations attained from causal thermodynamics. Therefore, we utilize the transportation equations for heat, bulk and shear viscosity from Müller-Israel-Stewart formalism [45]-[47] for dissipative source. Herrera et al. [38] discussed transportation equations for heat, bulk and shear viscosity. The entropy flux is given by

$$S^\mu = S n V^\mu + \frac{q^\mu}{T} - (\beta_0 \Pi^2 + \beta_1 q_\nu q^\nu + \beta_2 \pi_\nu \pi^\nu) \frac{V^\mu}{2T} + \frac{\alpha_0 q^\mu}{T} + \frac{\alpha_1 \pi^{\mu\nu} q_\nu}{T},$$

(42)

where $\beta_1$ and $\beta_2$ are thermodynamic factors for unalike additions to entropy density, $\alpha_0$ and $\alpha_1$ are thermodynamics heat coupling factors, $T$ is temperature.

Furthermore, from the Gibbs equation and Bianchi identities, it follows that

$$T S_{\alpha}^\alpha = -\Pi \left[ V_{;\alpha}^\alpha - \alpha_0 q_\alpha^\alpha + \beta_0 \Pi_{;\alpha} V^\alpha + \frac{T}{2} \left( \frac{\beta_0}{T} V^\alpha \right)_{;\alpha} \Pi \right]$$

$$-q^\alpha [h^\mu_\alpha (ln T)_{;\mu} (1 + \alpha_0 \Pi) + V_{\alpha;\mu} V^\mu - \alpha_0 \Pi_{;\alpha}$$

$$-\alpha_1 \pi^{\mu;\alpha} + \alpha_1 \pi^{\mu}_\alpha h^\alpha_\mu (ln T)_{;\beta} + \beta_1 q_{\alpha;\mu} V^\mu + \frac{T}{2} \left( \frac{\beta_1}{T} V^\mu \right)_{;\mu} q_\alpha]$$
Finally, by the standard procedure, the constitutive transport equations follow from the requirement $S^\alpha_v \geq 0$

\[
\tau_0 \Pi_{;\alpha} V^\alpha + \Pi = -\zeta \theta + \alpha_0 \zeta q^\alpha_{;\alpha} - \frac{1}{2} \zeta T \left( \frac{\tau_0}{\zeta T} V^\alpha \right)_{;\alpha} \Pi. \tag{44}
\]

\[
\tau_1 \beta_{;\alpha} = -\kappa \left[ h_{\alpha}^\beta T_{;\beta}(1 + \alpha_0 \Pi) + \alpha_1 \pi_{\alpha}^\beta h_{\mu}^\beta T_{;\beta} + T(a_\alpha - \alpha_0 \Pi_{;\alpha} - \alpha_1 \pi_{\alpha;\mu}^\beta) \right] - \frac{1}{2} \kappa T^2 \left( \frac{\tau_1}{\kappa T^2} V^\beta \right)_{;\beta} q_{\alpha} \tag{45}
\]

\[
\tau_2 h_{\alpha}^\mu h_{\beta}^\kappa \pi_{\mu;\rho} V^\rho + \pi_{\alpha} = -2\eta \sigma_{\alpha} + 2\eta_1 q_{;\alpha} - \eta T \left( \frac{\tau_2}{2\eta T} V^\nu \right)_{;\nu} \pi_{\alpha \beta}, \tag{46}
\]

with

\[
q_{;\alpha} = h_{\beta;\mu}^\nu \left( \frac{1}{2} (q_{\mu;\nu} + q_{\nu;\mu}) - \frac{1}{3} q_{\sigma;\mu} h_{\sigma}^\kappa h_{\sigma;\mu} \right), \tag{47}
\]

and where the relaxational times are given by

\[
\tau_0 = \zeta b_0, \quad \tau_1 = \kappa T \beta_1, \quad \tau_2 = 2\eta \beta_2, \tag{48}
\]

Here, $\zeta$ and $\eta$ are the factors of bulk and shear viscosity. Using the interior metric of cylindrical structure into eqs. (44)-(46) have following set of equations,

\[
\tau_0 \Pi = - \left( \frac{\zeta}{2} + \frac{\tau_0}{2} \right) A \Theta + \frac{A}{B} \alpha_0 \zeta \left[ q' + q \left( \frac{A'}{A} + \frac{C'}{C} + \frac{D'}{D} \right) \right]
\]

\[-\Pi \left[ \frac{\zeta T^2}{2} \left( \frac{\tau_0}{\zeta T^2} \right)_{;\alpha} + A \right], \tag{49}
\]

\[
\tau_1 \dot{q} = -\frac{A}{B} \kappa \left\{ T' \left( 1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega \right) + T \left[ \frac{A'}{A} - \alpha_0 \Pi \right] - \frac{2}{3} \alpha_1 \left( \Omega' + \frac{A'}{A} \Omega + \frac{3}{2} \left( \frac{C'}{C} + \frac{D'}{D} \right) \Omega \right) \right\}
\]

\[-q \left[ \frac{\kappa T^2}{2} \left( \frac{\tau_1}{\kappa T^2} \right)_{;\alpha} + \frac{\tau_1}{2} A \theta + A \right], \tag{50}
\]
\[ \tau_2 \dot{\Omega} = -\eta \left( \frac{2B}{B} \frac{\dot{C}}{C} - \frac{\dot{D}}{D} \right) + 2\eta \alpha_1 A B q' - \Omega \left[ \eta \left( \frac{\tau_2}{2\eta T} \right) + \frac{\tau_2}{2} A \Theta + A \right], \]

(51)

Here, we have to analyze the effect of several dissipative variables on the cylindrical interior surface. For this persistence, we use Eq.(50) in Eq.(41) and get

\[
\begin{align*}
(\mu + p + \Pi + 2\epsilon + \frac{2}{3} \Omega)(1 - \Lambda) D_T U \\
= (1 - \Lambda) F_{grav} + F_{hyd} + \tilde{E} \left( \frac{E}{2} + \frac{D'C}{2BD} \right) \frac{\kappa}{\tau_1} \left( D_C T \left( 1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega \right) - T \left[ \alpha_0 D_C \Pi + \frac{2}{3} \alpha_1 \left( D_C \Omega + \frac{3(CD')'}{2CD' e}\Omega \right) \right] \right) \\
+ \left( \frac{E}{2} + \frac{D'C}{2BD} \right) \left[ \frac{\kappa T^2 q}{2\tau_1} D_T \left( \frac{\tau_1}{\kappa T^2} \right) - D_T e \right] \\
- \left( \frac{E}{2} + \frac{D'C}{2BD} \right) \left[ \left( \frac{q}{2} + 2\epsilon \right) \theta - \frac{q}{\tau_1} + (q + \epsilon) \frac{D_T B}{B} + D_2 \right],
\end{align*}

(52)

where \( F_{grav} \) and \( F_{hyd} \) are representing as

\[
\begin{align*}
F_{grav} &= - \left( \mu + p + \Pi + 2\epsilon + \frac{2\Omega}{3} \right) \left[ m(r, t) - \frac{1}{8} + 4\pi \left( p + \Pi + \epsilon + \frac{2\Omega}{3} \right) \frac{C^2 D}{F} \right] \\
&= \frac{C^2 D(f(R) - RF(R))}{4F} + \frac{C^2 D D_T T F}{2F} + \frac{UC^2 DD_T T F}{2F} \left( \frac{1}{C} - \frac{D_T A}{C^2} \right) + \frac{VC^2 D_T F}{2F} \\
&- \left( \tilde{E} - \frac{\tilde{E}}{2} \right) \frac{C^2 D D_T A}{2BD} - \frac{\tilde{E}}{2} \frac{C^2 D D_T A}{2BD} \left( \frac{1}{C} + \frac{UD_c A}{C} \right) - \frac{C^2 D D_T C}{2} + \frac{C^2 D T T D}{2} \\
&+ \frac{C D D_T A}{2A} \left( U - \frac{VC}{D} \right) + \frac{C}{4} \left( UV - \tilde{E} D' \right) - \frac{C}{8D} \left( V^2 C - \frac{C D^2}{B^2} \right) \\
&- \frac{D}{8} \left( U^2 - \tilde{E}^2 \right) - \frac{1}{C D},
\end{align*}

(53)

\[
F_{hyd} = - \left( \frac{\tilde{E}}{2} + \frac{D'C}{2BD} \right) \left[ \tilde{E} D_C \left( p + \Pi + \epsilon + \frac{2\Omega}{3} \right) + \frac{\tilde{E}}{C} \left( \epsilon + \Omega \right) \right] + \frac{D'}{BD} \left( \epsilon + \Omega \right),
\]
and $\Lambda$ is defined as

$$\Lambda = \frac{\kappa T}{\tau_1} \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega \right)^{-1} \left( 1 - \frac{2}{3}\alpha_1\Omega \right). \quad (54)$$

We use Eq. (49) into Eq. (52) and subsequently get

$$\left( \mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) (1 - \Lambda + \Delta) D_T U = (1 - \Lambda + \Delta) F_{grav} + F_{hyd}$$

\[+ \frac{\tilde{E}}{\tau_1} \left( \tilde{E} + \frac{D'}{2BD} \right) \left\{ D_C T \left( 1 + \alpha_0 \Pi + \frac{2}{3} \alpha_1 \Omega \right) - T \left[ \alpha_0 D_C \Pi + \frac{2}{3} \alpha_1 \left( D_C \Omega + \frac{3(CD)'\Omega}{2CD'} \right) \right] \right\} \]

$$- \tilde{E} \left( \frac{\tilde{E}}{2} + \frac{D'C}{2BD} \right) \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) \Delta \left( \frac{D_C q}{q} + \frac{(CD)'}{CDC'} \right)$$

$$+ \left( \tilde{E} + \frac{D'C}{2BD} \right) \left[ \frac{\kappa T^2 q}{2\tau_1} D_T \left( \frac{\tau_1}{\kappa T^2} \right) - D_T \epsilon \right] + \left( \frac{\tilde{E}}{2} + \frac{D'C}{2BD} \right) \left[ \frac{q}{\tau_1} - (q + \epsilon) \frac{D_T B}{B} - D_2 \right]$$

$$+ \left( \tilde{E} + \frac{D'C}{2BD} \right) \frac{\Delta}{\alpha_0 \zeta q} \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega \right) \left\{ \left[ 1 + \frac{\zeta T}{2} D_T \left( \frac{\tau_0}{\zeta T} \right) \right] \Pi + \tau_0 D_T \Pi \right\}, \quad (55)$$

where $\Delta$ is defined as

$$\Delta = \alpha_0 \zeta q \left( \mu + p + \Pi + 2\epsilon + \frac{2}{3}\Omega \right)^{-1} \left( \frac{q + 2\epsilon}{2\zeta + \tau_0 \Pi} \right). \quad (56)$$

From now, by captivating made about interpretation the causal transport equations and their association by the dynamical equation, we trace that the term $1 - \Lambda + \Delta$ distresses extensively the inner energy and effective inertial mass density. This consequence is covenant with the finding of Herrera et al. [38].

## 5 Conclusions

The maiden of the 20th century Weyl [54] and Levi-Civita [55] ongoing that work with such static cylindrical symmetric structures after Einstein’s notion of gravity. Later, the ample work of spherical surfaces, hypothetical astronomer were observed to reconnoiter the effects of astral dense objects that have spherical symmetry. The cosmological source including heat dissipation and viscosity are very noticeable for searching the progression of dens
objects. Therefore, it is significant to involve the cylindrical symmetry in gravitating implosion.

In this study, we have arranged the dynamical equations which play meaningful role in progressively stages of a viscid dissipative gravitational cylindrical region. In way to perceive the impacts of $f(R)$ terms on the dynamical progression of a gravitational stuff, we have alleged the expedient formation of the collapsing variables which pless the transport equations of heat radiation, bulk and shear viscosity ensuing from causal thermodynamical approach. Moreover, the collapsing factors due to viscosity and heat flux have been involved in the examination of dynamical equations. For extensive point of view, we are mainly concerned in relaxation phase whose direction might be lesser or equal to radiating time. Throughout the work of transportation equations for collapsing variables, we have chosen to take the hyperbolic concept of collapse due to this concept is much consistent and has rare problems than parabolic concept [56]-[58].

A full causal technique has been implemented in [38] to evaluate the impacts of collapsing variables on the spherically symmetric collapse: this work gives the evocative solutions which have noteworthy consequences in astrophysics. The application of these solutions to some astral objects infers that in a sidereal existence, the radiating transformation of the collapsing material might be huge to yield an noticeable lessening in the gravitating action of the structure. It is pretty related to allusion that thermodynamical viscid/heat radiating coupling factors have been kept as non-terminating due to this theory gives the momentous foundation for the designing of variational astral model. Current scenario [59] took a partial technique to confer the impacts of bulk dissipative gravitating collapse source in cylindrical surface of the star with deserting the thermodynamical heat radiating coupling factors in the transport equations.

Using full causal technique the importance of dynamical viscous gravitation collapse is to be seen that with dynamical equation (55), which explicates how the measure of passive gravitational is effected by the collapsing variables and thermodynamical heat radiating coupling factors. In astral objects, happening all collapsing variables have much impacts, for instance a galactic estimation of heat coupling factor $\kappa$ can present a rushing lessening in gravitational force, thus ensuing in the setback of implosion [38]. Herrera at al. [60] investigated the bouncing action envisaging in mathematical paradigms in the current arithmetic calculation. They have also, The dissipative collapse due to force of gravity and we ratify that thermodynamical
heat radiating coupling factors must be included as preceding in transport equations. The present work is utilized in future with other modified theories such as Gauss-Bonet and $f(R,T)$ framework. This work has been done for spherical symmetry with and without electromagnetic field [61].

6 Appendix

$$D_1 = \frac{A}{\kappa} \left\{ \frac{1}{A^2 B^2} \left( \frac{\dot{F}' - A' \dot{F}}{A} - \frac{\dot{B}F'}{B} \right) \right\},$$

$$+ \left\{ \frac{f(R) - RF(R)}{2A^2} - \frac{F''}{A^2 B^2} + \frac{\dot{F}}{A^4} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} \right) - \frac{F'}{A^2 B^2} \left( \frac{C'}{C} + \frac{D'}{D} - \frac{B'}{B} \right) \right\},$$

$$+ \frac{2\dot{A}}{A^3} \left\{ \frac{f(R) - RF(R)}{2} - \frac{F''}{B^2} + \frac{\dot{F}}{A^2} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} \right) - \frac{F'}{B^2} \left( \frac{C'}{C} + \frac{D'}{D} - \frac{B'}{B} \right) \right\},$$

$$+ \frac{\dot{B}}{A^2 B} \left\{ - \frac{\ddot{F}}{A^2} + \frac{\ddot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) + \frac{\dot{F}}{A^2 B} \left( \frac{A'}{A} + \frac{B'}{B} \right) \right\},$$

$$+ \frac{\dot{C}}{A^2 C} \left\{ - \frac{\ddot{F}}{A^2} + \frac{\ddot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) + \frac{\dot{F}}{A^2 D} \left\{ - \frac{\ddot{F}}{A} + \frac{\ddot{F}}{A} \left( \frac{A'}{A} + \frac{B'}{B} \right) \right\},$$

$$+ \frac{1}{A^2 B^2} \left( \dot{F}' - \frac{A' \dot{F}}{A} - \frac{\dot{B}F'}{B} \right) \left( \frac{3A'}{A} + \frac{B'}{B} + \frac{C'}{C} + \frac{D'}{D} \right).$$

(57)

$$D_2 = \frac{B}{\kappa} \left\{ - \left\{ \frac{1}{A^2 B^2} \left( \frac{\dot{F}' - A' \dot{F}}{A} - \frac{\dot{B}F'}{B} \right) \right\},$$

$$+ \left\{ \frac{f(R) - RF(R)}{2B^2} + \frac{\dot{F}}{A^2 B^2} + \frac{\dot{F}}{A^2} \left( \frac{\dot{C}}{C} + \frac{\dot{D}}{D} - \frac{\dot{A}}{A} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{C'}{C} + \frac{D'}{D} \right) \right\},$$

$$+ \frac{A'}{AB^2} \left\{ \frac{\dot{F}}{A^2} + \frac{F''}{A^2 B^2} - \frac{\dot{F}}{A^2} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{B'}{B} \right) \right\},$$

$$+ \frac{2B'}{B^3} \left\{ \frac{f(R) - RF(R)}{2} + \frac{\dot{F}}{A^2} + \frac{\dot{F}}{A^2} \left( \frac{\dot{C}}{C} + \frac{\dot{D}}{D} - \frac{\dot{A}}{A} \right) - \frac{F'}{B^2} \left( \frac{A'}{A} + \frac{C'}{C} + \frac{D'}{D} \right) \right\},$$

$$+ \frac{C'}{B^2 C} \left\{ \frac{F''}{B^2} - \frac{\dot{F}}{A^2 B} + \frac{F'}{B^2 B} \right\} + \frac{D'}{B^2 D} \left\{ \frac{F''}{B^2} - \frac{\dot{F}}{A^2 B} - \frac{F'}{B^2 B} \right\}.$$
\[-\frac{1}{A^2B^2} \left( \frac{\dot{A}}{A} + \frac{3\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{D}}{D} \right) \left( \dot{F}' - \frac{A'\dot{F}}{A} - \frac{BF'}{B} \right) \].

(58)

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