Abstract

We work on the relation between the local thermodynamic instability and the dynamical instability of large black holes in four-dimensional anti-de Sitter space proposed by Gubser and Mitra. We find that all perturbations suppressing the metric fluctuations at linear order become dynamically unstable when black holes lose the local thermodynamic stability. We discuss how dynamical instabilities can be explained by the Second Law of Thermodynamics.

1 Introduction

Black holes are very interesting objects from their causal structures in general relativity to their quantum mechanical properties. To figure out their physical relevance, we need to answer if the complete gravitational collapse of a body results in a black hole rather than a naked singularity. The conjecture [1] that nature censors naked singularity was proposed in this respect. One of motivation of this is from the fact that black holes in 4-dimensional asymptotic Minkowski space are stable: linear perturbations around black hole solutions do not give any evolution.

However, it was found in [2, 3] that black strings and p-branes are unstable. The basic idea of the Gregory-Laflamme instability is that whatever has the biggest entropy is favored. Since a black string has a different topology of horizon as that of a black hole and entropy is proportional to the area of horizon, array of black holes has bigger entropy than a uncompactified black string when they have the same mass [4]. The instability of black strings was shown [2, 3] by doing perturbation theory. It is a very interesting question to see what would happen during the transition between them. It has been argued that violation of cosmic censorship does occur during this process.
Recently it has been suggested that a black string settles down to a new static black string solution which is not translationally invariant along the string [4].

Entropy argument used above was revisited by Gubser and Mitra to propose that a black brane becomes dynamically unstable when it is locally thermodynamically unstable [5, 6]. Local thermodynamic stability is defined as having an entropy which is concave down as a function of the mass and the conserved charges [8]. This conjecture was made from the perspective of AdS/CFT correspondence [4, 10, 11], which identifies two low energy excitations, both of which are decoupled from supergravity in flat space, in two low energy descriptions of superstring theory [12]. Some unstable fluctuation modes may be excited when there is a thermodynamic instability in the field theory and according to AdS/CFT the same thing would happen in AdS [7]. A semi-classical proof of above conjecture using the Euclidean path integral approach to quantum gravity was given in [13].

The motivation of Gubser-Mitra (GM) conjecture is that Lorentzian time evolution should proceed so as to increase the entropy. In this paper, we find that their argument holds for both of the cases in which the metric fluctuations are suppressed and in which only the metric fluctuations are turned on. In the former case, any perturbation for all equal charges of $AdS_4$-RN solutions becomes dynamically unstable when the system loses the thermodynamic stability and all evolutions increase entropy. In the latter case, there is no dynamical instability even though the system is thermodynamically unstable. The stability can be explained by the fact that entropy would be decreasing if the perturbation is unstable. We discuss these in section 2 and section 3. We try to explain dynamical instabilities of black holes from the Second Law of Thermodynamics in section 4.

2 Gubser-Mitra analysis and its generalization

2.1 $AdS_4$-RN black hole

An electrically charged black hole in the asymptotically $AdS_4$ was found in [14]. Starting from $\mathcal{N} = 8$ supergravity in 4-dimensions, they gauged the rigid SO(8) symmetry of 28 gauge boson [15] and the potential induced from this gauging makes $AdS_4$ a vacuum solution of this theory. The $AdS_4$ black hole solution is made by focusing on $U(1)^4$ Cartan subgroup of SO(8), which is believed to be a consistent truncation. Only three scalar fields of 70 scalar fields in the original theory are kept by working in symmetric gauge. The Lagrangian of this truncated theory is,

$$\mathcal{L} = \frac{\sqrt{g}}{2k^2} \left[ R - \sum_{i=1}^{3} \left( \frac{1}{2} (\partial \phi_i)^2 - \frac{2}{L^2} \cosh \phi_i \right) - 2 \sum_{A=1}^{4} e^{\alpha_A^i \phi_i} (F_{\mu \nu}^{(A)})^2 \right]$$

(1)

where $\alpha_A^i = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$. The metric signature is $(-+++)$ and $G_4 = \frac{1}{4}$. The electrically charged solutions are
\[
\begin{align*}
    ds^2 &= -\frac{F}{\sqrt{H}} dt^2 + \frac{\sqrt{H}}{F} dz^2 + \sqrt{H} z^2 d\Omega^2 \\
    e^{2\phi_A} &= \frac{h_1 h_2}{h_3 h_4} e^{2\phi_2} = \frac{h_1 h_3}{h_2 h_4} e^{2\phi_3} = \frac{h_1 h_4}{h_2 h_3} \\
    F_{0z}^{(A)} &= \pm \frac{1}{\sqrt{8h_A^2}} \frac{Q_A}{z^2} \\
    H &= \prod_{A=1}^{4} h_A \\
    F &= 1 - \frac{\mu}{z} + \frac{z^2}{L^2} H \\
    h_A &= 1 + \frac{q_A}{z} \\
    Q_A &= \mu \cosh \beta_A \sinh \beta_A \\
    q_A &= \mu \sinh^2 \beta_A 
\end{align*}
\]

where the quantities \(Q_A\) are the physical conserved charges. The mass is \[16\]

\[
    M = \frac{\mu}{2} + \frac{1}{4} \sum_{A=1}^{4} q_A, \tag{3}
\]

and the entropy is

\[
    S = \pi z_H^2 \sqrt{H(z_H)} \tag{4}
\]

where \(z_H\) is the largest root of \(F(z_H) = 0\). It is possible to express \(M\) directly in terms of the entropy and the physical charges in the large black hole limit, \(M \gg L\), as \[7\]

\[
    M = \frac{1}{2\pi^\frac{3}{2} L^2 \sqrt{S}} \left[ \prod_{A=1}^{4} (S^2 + \pi^2 L^2 Q_A^2) \right]^{\frac{1}{4}}. \tag{5}
\]

We are going to study in the case where all four charges are equal, \(q_A = q\). In this case the solution can be written in term of a new radial variable, \(r = z + q\), and it becomes

\[
\begin{align*}
    ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \\
    F_{0r} &= \frac{Q}{\sqrt{8r^2}} \\
    f &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2}.
\end{align*} \tag{6}
\]

It is known \[17\] that the consistent \(S^7\) truncation of 11-dimensional supergravity is equivalent to \(\mathcal{N} = 8\) 4-dimensional gauged supergravity. Also the equivalence of large R-charged black holes in D=4, D=5 and D=7 with spinning near-extreme M2, D3 and M5 branes are respectively demonstrated in \[16\]. It is important to check that our black holes can be embedded to higher dimensional black objects because GM conjecture requests the non-compact translational symmetry. Any instability found in the large black hole limit, \(M \gg L\) in our case implies the instability of M2-brane.
2.2 Thermodynamic instability and adiabatic evolution

Local thermodynamic stability is defined as having an entropy which is concave down as a function of the extensive variables. This means that the Hessian matrix,

\[
H_{M,Q}^S = \begin{pmatrix}
\frac{\partial^2 S}{\partial M^2} & \frac{\partial^2 S}{\partial MQ} \\
\frac{\partial^2 S}{\partial MQ} & \frac{\partial^2 S}{\partial Q^2}
\end{pmatrix}
\]

has no positive eigenvalues. It is straightforward to express \(H_{M,Q}^S\) in terms of derivatives of \(M(S,Q_A)\). From this definition, if we introduce the dimensionless variable

\[
\chi = \frac{Q}{M^{3/2}}
\]

for all equal charges, \(Q_A = Q\), the thermodynamic instability is present when \(\chi > 1\) \(^1\). We can see that the most positive eigenvector\(^1\) of Hessian increases entropy most when entropy is at its extremum.

\[
S(M + \delta M, Q_A + \delta Q_A) = S(M, Q_A) + \frac{1}{2}(\delta M, \delta Q_A)H_{M,Q}^S\left(\frac{\delta M}{\delta Q_B}\right)
\]

Even though entropy is not at its extremum, we can forget about the first derivative parts by energy and charge conservation in microcanonical ensemble. With this it was found that in the positive eigenvector direction of Hessian for all equal charges, the dynamical instability coincides with the thermodynamic instability with a small discrepancy due to numerical errors. It will be interesting to see what would happen in other directions.

Gubser and Mitra analyzed the linear perturbation in which fluctuations of the metric are suppressed. The most unstable eigenvector is \((\delta M, \delta Q_A) = (0, 1, 1, -1, -1)\) for all equal charges. The condition that the metric decouples at linear order is that \(Q_A \cdot \delta Q_A = 0\). In this case \(\delta T_{ab}\) vanishes at linear order and we can also see from (2) the metric does not change at linear order. It is not difficult to make the linear perturbation equations in this decoupling case beyond the eigenvector direction. Our original motivation was to see two things: first, even though the system loses the thermodynamic stability, it would not be dynamically unstable if we perturb the system in the way of decreasing its entropy. Second, because the eigenvector direction increases entropy most, it would be the fastest way of increasing entropy.

The general perturbation in which the metric decouples for all equal charges is that

\[
\delta Q_A = (1, a, b, -a - b - 1)\delta Q
\]

where \(a, b\) can be any real numbers. From (2), we can make an ansatz about a relevant perturbation

\[
\delta \phi_i = (1 + a, 1 + b, -a - b)\frac{\delta \phi}{2}
\]

\[
\delta F^{(A)} = (1, a, b, -a - b - 1)\delta F.
\]

This ansatz relating three scalar fields to one scalar field and four U(1) fields to the other U(1) field should be checked if it is consistent with equations of motion and

\(^1\)The positive eigenvector here means an eigenvector with a positive eigenvalue.
it turns out to be consistent. The linear perturbation for each $\phi_i$ is the same up to overall factor
\[
\left[ \nabla_\mu \nabla^\mu + \frac{2}{L^2} - 8F^2_{\mu\nu} \right] \delta \phi - 16F^{\mu\nu} \delta F_{\mu\nu} = 0
\]  
(11)
and the linear perturbation for each $F^{(A)}_{\mu\nu}$ is also the same up to overall factor
\[
d\delta F = 0 \quad d* \delta F + d\delta \phi \wedge *F = 0.
\]  
(12)
Here $F$ is the background field strength in (6): it is the same for four $F^{(A)}$. It is remarkable that all directions have the same perturbation equation. The case of $a=1, b=-1$ is the unstable eigenvector and we can see that (11) and (12) are exactly what Gubser and Mitra found \cite{6}. From their analysis, we can conclude that all perturbations suppressing metric fluctuation at linear order have dynamical instabilities when the system is thermodynamically unstable. We can see that the eigenvector direction is the fastest way of increasing entropy not because it evolves fastest but because it increases entropy most. It was suspected that some of perturbations (9) would decrease entropy by the continuity of (8) in the case that $\chi$ is slightly greater than 1, but the Second Law of Thermodynamics is not violated in another remarkable way\cite{23}.

The second derivative parts of (8) with perturbation (9) is
\[
\frac{1}{2}(\delta M, \delta Q_A)H^S_{M,Q_A}\left(\frac{\delta M}{\delta Q_B}\right) = A(M, Q_A)(\chi - 1){(1 + a^2 + b^2 + (1 + a + b)^2)} \]  
(13)
Here $A(M, Q_A)$ is a positive when $\chi \sim 1$. We can see that all perturbations (9) increase entropy when $\chi > 1$. The factorization of the eigenvalue part $\chi - 1$ can explain why we have the same linear perturbation equations (11) and (12) for all perturbations (9).

\section{3 Stability from the metric perturbation analysis}

\subsection{3.1 Metric perturbation equation}

In the previous section, we observed that there is a dynamical instability in any case that the metric perturbation is suppressed. It would be very interesting to see what would happen in the case that the metric is also involved. This is very difficult to do and we have not succeeded in doing this in the case that all fields are involved.

In this section, we analyze a simple case: three scalar fields and four U(1) fields are suppressed. This perturbation is in the direction of $\delta M \neq 0, \delta Q_A = 0$ for all equal charges. From (5) and (8) we can see that entropy is decreasing in this perturbation.

\footnote{We thank a referee of JHEP for pointing out this and we would like to apologize to the authors of \cite{22} for incorrectly disclaiming their argument in the previous version of this paper.}

\footnote{The violation of the Second Law of Thermodynamics was suspected from the fact that AdS is not globally hyperbolic. We thought the area law for black holes might not hold. However the area law needs only a partial Cauchy surface \cite{20}, so it holds in AdS. $\phi_i$ do not satisfy the dominant energy condition, but they satisfy the strong energy condition defined in \cite{18} or the timelike convergence condition defined in \cite{20}. Therefore, the area of horizon can not be decreasing with classical evolutions by the area law.}

\section{5
Varying (1) yields the equations of motion

\[
\nabla_a \nabla^a \phi_i + \frac{2}{L^2} \sinh \phi_i - 2 \sum_{A=1}^{4} \alpha_A^i e^{\alpha_A^i \phi_i} (F_{ab}^{(A)})^2 = 0 \tag{14.a}
\]

\[
\partial_a (\sqrt{g} e^{\alpha_A^i \phi_i} (F^{(A)ab})) = 0 \tag{14.b}
\]

\[
R_{ab} = -3 \sum_{i=1}^{3} \left\{ \frac{1}{L^2} \cosh \phi_i g_{ab} + \frac{1}{2} \partial_a \phi_i \partial_b \phi_i \right\} + 4 \sum_{A=1}^{4} \left\{ F_{ac}^{(A)} F_{b}^{(A)c} - \frac{1}{4} g_{ab} (F^{(A)})^2 \right\} . \tag{14.c}
\]

We expect a relevant perturbation is

\[
\delta \phi_i = 0, \quad \delta F_{ab}^{(A)} = 0, \quad \delta g_{ab} = \gamma_{ab} \tag{15}
\]

We need to check if above ansatz is consistent with equations of motion and it is so if

\[
\gamma = 0 \quad \gamma^t + \gamma^r = 0 \tag{16}
\]

where \(\gamma = \gamma^a = g^{ab} \gamma_{ba}\). The linear perturbation equations from (14.a) and (14.b) are automatically satisfied for all equal charges. Now we need to check if (15) is consistent with (14.c). The linear perturbation from (14.c) is

\[
-\frac{1}{2} \nabla_a \nabla^a \gamma - \frac{1}{2} \nabla^b \nabla^c \gamma_{ac} + \nabla^b \nabla^c \gamma_{(a)b} = \Lambda \gamma_{ac} + 16 \left( -F_{a}^b F_{c}^d \gamma_{bd} - \frac{1}{4} \gamma_{ac} F^2 + \frac{1}{2} g_{ac} F_{bd} F_{d}^{e} \gamma_{be} \right) \tag{17}
\]

where we use the totally symmetric notation for \(\cdot\), \(\Lambda = -\frac{3}{L^2}\) and \(F^2 = F_{ab} F_{ab}\). Four U(1) fields become the same. It is a well-known fact from electromagnetism that if there is a source term, a simple gauge choice is not easy. However, if we take the trace of (17), it becomes

\[
-\frac{1}{2} \nabla_a \nabla^a \gamma = \Lambda \gamma - 16 \gamma F^2. \tag{18}
\]

We used the condition, \(\gamma^t = -\gamma^r\) for this. Because (18) is a homogeneous equation for \(\gamma\), we can choose a transverse traceless gauge whereby

\[
\nabla^a \gamma_{ab} = 0 \quad \gamma = 0. \tag{19}
\]

See [18] for a detail about this gauge choice. It should be noted that this gauge choice is only possible with our ansatz that scalar fields and U(1) fields are not fluctuating. Finally we get the perturbation equation for the metric from (17) following [18].

\[
(\nabla^b \nabla_b + 2 \Lambda - 8 F^2) \gamma_{ac} - (R^d_{c} \gamma_{ad} + R^d_{a} \gamma_{cd}) - (2 R^b_{ac} d + 32 F^a_{b} F^d_{c}) \gamma_{bd} = 0 \tag{20}
\]
It is the even wave in the canonical form [19] which is relevant to our case:

\[
\gamma_{ab} = \begin{pmatrix}
\tilde{\gamma}_{tt} & \tilde{\gamma}_{tr} & 0 & 0 \\
\tilde{\gamma}_{rt} & \tilde{\gamma}_{rr} & 0 & 0 \\
0 & 0 & r^2 k & 0 \\
0 & 0 & 0 & r^2 k \sin^2 \theta
\end{pmatrix} e^{i wt} P_l(\cos \theta).
\tag{21}
\]

It can be easily checked that in this form, \(\gamma_{tt}\) and \(-\gamma_{rr}\) have the same equation in (20) and this proves that our ansatz (15)-(16) is completely consistent with equations of motion. This equality between \(\gamma_{tt}\) and \(-\gamma_{rr}\) is expected from the \(\delta M\) perturbation of the metric in (6). The equations for \(\gamma_{tt}\) and \(\gamma_{tr}\) are coupled:

\[
\begin{align*}
-\frac{\partial^2}{\partial t^2} + f \partial_r^2 + \left(2\frac{\tilde{f}}{r} - \frac{f'}{r} \right) \partial_r - \frac{2 f'}{r} + \frac{\partial_\theta \sin \theta \partial_\theta}{r^2 \sin \theta} \gamma_{tt} + 2 f' \partial_t \gamma_{tr} &= 0 \\
-\frac{\partial^2}{\partial t^2} + f \partial_r^2 + \left(2\frac{\tilde{f}}{r} + \frac{f'}{r} \right) \partial_r - \frac{2 f'}{f} - \frac{(f')^2}{f} - f'' + \frac{\partial_\theta \sin \theta \partial_\theta}{r^2 \sin \theta} \gamma_{tr} + \frac{2 f' \partial_t}{f^2} \gamma_{tt} &= 0
\end{align*}
\tag{22}
\]

Here \(f\) is defined in (6) and \(f' = \partial_r f\). Using the form (21) we can make the forth order ordinary differential equation for \(\tilde{\gamma}_{tr}\), which is what we are going to study by numerics.

### 3.2 Numerical analysis

To carry out a numerical study of (22), we can cast the equation in terms of a dimensionless radial variable \(u\), a dimensionless charge parameter \(\chi\), a dimensionless mass parameter \(\sigma\), and a dimensionless frequency \(\tilde{w}\) introduced in [4]:

\[
u = \frac{r}{M^{\frac{1}{2}} L^{\frac{3}{2}}} \quad \chi = \frac{Q}{M^{\frac{1}{2}} L^{\frac{3}{2}}} \quad \sigma = \left( \frac{L}{M} \right)^{\frac{2}{3}} \quad \tilde{w} = \frac{w L^2}{M^{\frac{4}{3}}}
\tag{23}
\]

Then we combine two equations in (22)

\[
\left[ \left\{ \frac{\tilde{w}^2}{f} + \tilde{f} \partial_u^2 + \left(2\frac{\tilde{f}}{u} - \frac{f'}{u} \right) \partial_u - \frac{2 \tilde{f}}{u} - \frac{\sigma l(l+1)}{u^2} \right\} \frac{\tilde{f}^2}{2 f'} \right] \tilde{\gamma}_{tr} = 0
\tag{24}
\]

\[
\tilde{f} = \sigma - \frac{2}{u} + \frac{\chi^2}{u^2} + u^2.
\]

Now we need to specify the boundary condition for \(\tilde{\gamma}_{rt}\). We want to place an initial data surface touching the horizon at one end and ending on the boundary of \(AdS\) at the other end. Because \(AdS\) does not have a Cauchy surface [20], the future domain of dependence of this initial data lies inside of Cauchy horizon of \(AdS\). To define ‘small’ for the perturbation at the horizon, we can use non-singular coordinates, Kruskal coordinates [21]. Dropping the \(S^2\) piece in (6) and introducing a tortoise coordinate \(r_*\) and Kruskal coordinates \((T, R)\) according to

\[
\frac{dr_*}{dr} = \frac{1}{f} \quad e^{\frac{1}{2} f'(r_H)(\pm t + r_*)} = \pm T + R.
\tag{25}
\]
The near-horizon metric is regular

\[ ds^2 \approx \frac{4}{e^{f(r_H)r_H} f'(r_H)} (-dT^2 + dR^2). \]  

(26)

We can express the Kruskal components \( \gamma'_{ab} \) in terms of the original components \( \gamma_{ab} \)

\[ \gamma'_{tt} = \frac{4}{f'(r_H)(-T^2 + R^2)^2} \left[ R^2 \gamma_{tt} - 2 f RT \gamma_{tr} + f^2 T^2 \gamma_{rr} \right] \]

\[ \gamma'_{tr} = \frac{4}{f'(r_H)(-T^2 + R^2)^2} \left[ -TR \gamma_{tt} + 2 f (T^2 + R^2) \gamma_{tr} - f^2 TR \gamma_{rr} \right] \]  

(27)

\[ \gamma'_{rr} = \frac{4}{f'(r_H)(-T^2 + R^2)^2} \left[ T^2 \gamma_{tt} - 2 f RT \gamma_{tr} + f^2 R^2 \gamma_{rr} \right] \]

all of which should be finite as \( r \to r_H \) on our initial data surface. To avoid the issue of mode superposition and a better physical sense in which black holes would form in a collapse situation, we would require a surface ending on a future horizon \[ [22] \]. When we approach the future horizon from outside of a black hole region, \( R = T + O(r - r_H) \).

This implies that normalizable wavefunctions \( \gamma_{tr} \) must be \( O(r - r_H) \) as we approach the event horizon. We also want it falling off like \( 1/r^2 \) near the boundary of \( AdS_4 \). This boundary condition is taken from the asymptotic behavior \( \gamma_{tt} \approx r^3 \gamma_{tr} \). Using Maple, we solved (24) numerically. We did not find any unstable mode. At \( \sigma = 0 \), thermodynamic stability is lost at \( \chi = 1 \). The smallest \( \tilde{w}^2 \) in this case is \( \tilde{w}^2 = 2 \). There is no normalizable wavefunction with negative \( \tilde{w}^2 \). Negative mode is found at \( \chi = 3.7 \), which lies in naked singularity regions and therefore is not relevant. We can conclude that in the perturbation, \( \delta M \neq 0, \delta Q_A = 0 \) there is no dynamical instability, which makes sense because this perturbation would decrease entropy if it were unstable.

4 Discussions and Perspectives

We have been working on Gubser-Mitra conjecture, which relates thermodynamics to dynamics in black holes. When the metric fluctuations are suppressed at linear order, all perturbations are dynamically unstable if black holes are thermodynamically unstable and all of evolutions increase entropy. However, when only the metric fluctuations are turned on, they are dynamically stable. Entropy is not decreasing in this case. This result strengthens both the motivation of Gubser-Mitra conjecture, which claims that Lorentzian time evolution should go so as to increase entropy, and the validity of their conjecture. The Second Law of Thermodynamics is not a rigorous law of nature. It is correct in a certain macroscopic time scale. Therefore it is remarkable if dynamics obeys this law, even though we are interested in classical instabilities.

As suggested in \[ [7] \], entropic arguments can give a good information not only on the existence of dynamical instabilities but also on the direction they point. We can have a very heuristic argument why there is a dynamical instability when the perturbation \( (\delta M, \delta Q_A) \) increases entropy. Entropy can be understood as a functional of the fields describing black hole. Let \( X_i(t) \) denote metric, scalar and U(1) fields. If we write down the equation (8) in terms of \( X_i(t) \), it becomes
\[ S(X_i(t_0 + dt)) = S(X_i(t_0)) + \frac{1}{2} \left( \frac{\delta^2 S}{\delta X_i \delta X_j} \right)_{t=t_0} \frac{dX_i}{dt} \frac{dX_j}{dt} dt^2. \]  

(28)

Here \( X_i(t_0) \) is the solution describing our black hole and the derivative of entropy with respect to \( X_i \) is a functional derivative. \( \delta X_i(t) \) can be understood as a solution of the linear perturbation equation in this case. If we assume that \( \left( \frac{\delta^2 S}{\delta X_i \delta X_j} \right) \) gives a positive definite inner product between fields, we can see that entropy increases if and only if the time dependence of \( X_i(t) \) is \( e^{iwt} \) with \( w^2 \) negative. This argument does not hold for a Schwarzschild black hole and this might be due to our assumption on the positivity of \( \left( \frac{\delta^2 S}{\delta X_i \delta X_j} \right) \). It will be a very interesting problem to understand this from the quantum theory of gravitation and it might explain the reason why we need a non-compact translational symmetry of a black object in Gubser-Mitra conjecture.

This is also very interesting in mathematical point of view. Suppose we have a hypersurface defined by \( S = S(M, Q_A) \). Each perturbation \( (\delta M, \delta Q_A) \) corresponds to a tangent vector originated from \( p = (M_0, Q_{0A}) \) up to a normalization factor in a tangent space \( T_p S \) of the hypersurface at \( p \). This tangent space can be understood as the real projective space \( \mathbb{RP}^4 \). The second derivative part of equation (8) is a homogeneous polynomial of \( (\delta M, \delta Q_A) \) of degree 2 and the zero locus of this polynomial gives us an algebraic subvariety of \( \mathbb{RP}^4 \). This variety separates \( \mathbb{RP}^4 \) into two parts: \( \delta S > 0 \) and \( \delta S < 0 \). We can also separate \( \mathbb{RP}^4 \) in another way: A stable region in which a perturbation \( (\delta M, \delta Q_A) \) gives no dynamical evolution and an unstable region in which a perturbation \( (\delta M, \delta Q_A) \) gives a dynamical evolution. If our argument in the previous paragraph is correct, this means that both of separations are the same and it will give a very interesting relation between algebraic equations and differential equations.

It was argued that an unstable black string settles down to a new static black string solution which is not translationally invariant along the string and can be viewed as a local entropy maximum but not a global one. If the final stage of the evolution is a local entropy maximum, we can not say that evolutions from different perturbations end up with the same final solution. In our case, we have 3 eigenvectors of Hessian with the same positive eigenvalues for all equal charge: \( (a = 1, b = -1) \), \( (a = -1, b = 1) \) and \( (a = -1, b = -1) \) in (9). Considering the sign of these vectors, there are six most unstable perturbation vectors in the tangent space. It would be very interesting to see that what will be the final solutions for these perturbations. Finally it is an open question to see if evolutions from perturbations around the eigenvectors would result in the same final solutions in which the evolutions from the nearby eigenvectors result. We leave these questions for future work.

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