Dynamical Borel–Cantelli lemma for recurrence theory

MUMTAZ HUSSAIN†, BING LI‡, DAVID SIMMONS§ and BAOWEI WANG¶

† La Trobe University, P.O. Box 199, Bendigo, VIC 3552, Australia
(e-mail: M.Hussain@latrobe.edu.au)
‡ School of Mathematics, South China University of Technology, Guangzhou 510640, China
(e-mail: scbingli@scut.edu.cn)
§ Department of Mathematics, University of York, Heslington, York YO10 5DD, UK
(e-mail: David.Simmons@york.ac.uk)
¶ School of Mathematics and Statistics, Huazhong University of Science and Technology, 430074 Wuhan, China
(e-mail: bwei_wang@hust.edu.cn)

(Received 1 November 2020 and accepted in revised form 15 February 2021)

Abstract. We study the dynamical Borel–Cantelli lemma for recurrence sets in a measure-preserving dynamical system $(X, \mu, T)$ with a compatible metric $d$. We prove that under some regularity conditions, the $\mu$-measure of the following set

$$R(\psi) = \{x \in X : d(T^n x, x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N} \}$$

obeys a zero–full law according to the convergence or divergence of a certain series, where $\psi : \mathbb{N} \to \mathbb{R}^+$. The applications of our main theorem include the Gauss map, $\beta$-transformation and homogeneous self-similar sets.

Key words: recurrence, dynamical Borel–Cantelli lemma, zero-full law, exponentially mixing

2020 Mathematics Subject Classification: 37A05 (Primary); 28D05 (Secondary)

1. Introduction

Poincaré’s recurrence theorem is one of the most fundamental results for dynamical systems and concerns the properties of the distribution of orbits. More precisely, let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system with a compatible metric $d$, that is, $(X, d)$ is a metric space, $\mathcal{B}$ is a Borel $\sigma$-algebra of $X$, and $\mu$ is a $T$-invariant probability measure.
If \((X, d)\) has a countable base then Poincaré’s recurrence theorem states that \(\mu\)-almost every \(x \in X\) is recurrent in the sense that

\[
\liminf_{n \to \infty} d(T^n x, x) = 0.
\]

However, this result gives no information about the speed at which a generic orbit \(\{T^n x\}_{n \geq 0}\) comes back to the starting point or the shrinking neighbourhood. A question of great importance is how to determine conditions under which the rate of recurrence can be quantified for general dynamical systems. In particular, the focus is on the size of the following set:

\[
R(\psi) := \{x \in X : d(T^n x, x) < \psi(n) \text{ for i.m. } n \in \mathbb{N}\}
\]

where \(\psi : \mathbb{N} \to \mathbb{R}^+\) is a positive function and i.m. denotes infinitely many.

The most significant and one of the first quantitative recurrence results is due to Boshernitzan [3].

**Theorem 1.1.** [3] Let \((X, \mathcal{B}, \mu, T)\) be a measure-preserving system with a compatible metric \(d\). Assume that the \(\alpha\)-dimensional Hausdorff measure of \(X\) is \(\sigma\)-finite for some \(\alpha > 0\). Then, for \(\mu\)-almost all \(x \in X\),

\[
\liminf_{n \to \infty} n^{1/\alpha} d(T^n x, x) < \infty. \tag{1.1}
\]

In the concluding remarks of the same paper, Boshernitzan stated another theorem which deals with the situation where no a priori size of \((X, d)\) is known.

**Theorem 1.2.** [3] Let \((X, \mathcal{B}, \mu, T)\) be a measure-preserving system with a compatible metric \(d\) such that \((X, d)\) is \(\sigma\)-compact. Then there exists a sequence \(\{a_n\}_{n \geq 1}\) with \(a_n \to \infty\) as \(n \to \infty\) depending on \((X, d)\), such that for almost all \(x \in X\),

\[
\liminf_{n \to \infty} a_n d(T^n x, x) = 0.
\]

Theorem 1.1 was improved by Barreiro and Saussol [2], who showed that the exponent \(\alpha\) could be replaced by the lower local dimension of a measure at \(x\). For piecewise \(C^2\) expanding maps with the ergodic measure equivalent to Lebesgue measure, Kirsebom, Kunde and Persson [6] improved the speed in (1.1) from \(n\) to \(n (\log n)^\theta\) with \(\theta < 1/2\).

As far as a general error function \(\psi\) is concerned, hardly anything is known. The only known results for \(\mu\)-measure of \(R(\psi)\) were recently proven by Chang, Wu and Wu [4], Baker and Farmer [1] and Kirsebom, Kunde and Persson [6]. Chang, Wu and Wu [4] considered homogeneous self-similar sets satisfying the strong separation condition. Baker and Farmer [1] generalized the results of Chang, Wu and Wu to finite conformal iterated function systems with the open set condition. Among other interesting results, Kirsebom, Kunde and Persson [6] presented the recurrence and shrinking target theory when \(T\) is an integer matrix action. However, all of these results are not applicable to some well-known dynamical systems, for example, \(\beta\)-transformation or the Gauss map. We remedy this shortfall in this paper by providing a criterion for the size of \(R(\psi)\) applicable to general dynamical systems satisfying certain conditions.
Throughout, we take $X$ to be a compact subset of $\mathbb{R}^d$. Let $\{X_i\}_{i \in \mathcal{I}}$ be a countable family of non-empty pairwise disjoint subsets of $X$ such that each $X_i$ is open in $X$. Suppose that $T : X \to X$ is Borel measurable, and for all $i \in \mathcal{I}$, $T|_{X_i}$ is a $C^1$ map. Furthermore, we assume that $T$ is expanding, meaning that $\|(D_x T)^{-1}\|^{-1} > 1$ for any $x \in X$. By the notation $D_x T$ we mean the derivative of $T$ at a point $x \in X$ and $\|\cdot\|$ denotes the norm of $\mathbb{R}^d$. Let $\mu$ be a $T$-invariant probability measure and 
\[
\mu\left(X \setminus \bigcup_{i \in \mathcal{I}} X_i\right) = 0.
\]
We will make use of the following conditions.

**Condition I.** (Ahlfors regularity) The measure $\mu$ is Ahlfors regular of dimension $\delta > 0$, that is, there exist positive constants $\eta_1, \eta_2$ such that for any ball $B(x, r) \subset X$ with $x \in X$,
\[
\eta_1 r^\delta \leq \mu(B(x, r)) \leq \eta_2 r^\delta.
\]

**Condition II.** (Exponential mixing) There exist constants $C > 0$ and $0 < \gamma < 1$ such that for any ball $E \subset X$ and measurable set $F \subset X$,
\[
|\mu(E \cap T^{-n} F) - \mu(E)\mu(F)| \leq C \gamma^n \mu(F) \quad \text{for all } n \geq 1.
\]

**Condition III.** (Bounded distortion) There exists $K > 0$ such that
\[
K^{-1} \leq \frac{d(T^m x, T^m y)}{d(x, y)\|D_x(T^n)\|} \leq K
\]
for any $n \in \mathbb{N}$ and $x, y$ in the same cylinder $J_n \in \mathcal{F}_n$. Here, $\mathcal{F}_n$ denotes the collection of cylinders of order $n$, that is,
\[
\mathcal{F}_n := \{X_{i_0} \cap T^{-1} X_{i_1} \cap \cdots \cap T^{-(n-1)} X_{i_{n-1}} : i_0, i_1, \ldots, i_{n-1} \in \mathcal{I}\}.
\]

**Condition IV.** Denote $K_{J_n} := \inf_{x \in J_n} \|D_x(T^n)\|$. Assume that $\inf_{J_n \in \mathcal{F}_n} K_{J_n} > 1$ for some $n \in \mathbb{N}$ and there exists a universal constant $K > 0$ such that
\[
\sum_{J_n \in \mathcal{F}_n} (K_{J_n})^{-\delta} \leq K \quad \text{for all } n \in \mathbb{N}.
\]

**Condition V.** (Conformality) There exists a constant $C > 0$ such that for any $J_n \in \mathcal{F}_n$ and ball $B(x_0, r) \subset J_n$, 
\[
B(T^n(x_0), C^{-1} K_{J_n} r) \subset T^n B(x_0, r) \subset B(T^n(x_0), CK_{J_n} r).
\]

Our main result is the following.

**THEOREM 1.3.** Let $\mu$ be a probability measure and $\psi$ a positive function on $\mathbb{N}$. Suppose that $\mu$ satisfies conditions I–V. Then
\[
\mu(R(\psi)) = \begin{cases} 
0 & \text{if } \sum_{n=1}^{\infty} \psi^\delta(n) < \infty, \\
1 & \text{if } \sum_{n=1}^{\infty} \psi^\delta(n) = \infty.
\end{cases}
\]

An immediate consequence of this theorem is the following strengthening of Boshernitzan’s results.
COROLLARY 1.4. Under the setting above, for $\mu$-almost every $x \in X$,
\[
\liminf_{n \to \infty} \psi(n)^{-1} d(T^n x, x) = 0 \text{ or } \infty,
\]
if $\sum_{n=1}^{\infty} \psi(n) < \infty$ or $\sum_{n=1}^{\infty} \psi(n) = \infty$, respectively.

We give some remarks on a comparison of our result with those of Boshernitzan, Chang, Wu and Wu and Baker and Farmer. Recall the recurrence set
\[
R(\psi) = \{ x \in X : d(T^n x, x) < \psi(n) \text{ for i.m. } n \in \mathbb{N} \}.
\]

Remark 1.5. (Boshernitzan [3])
- Theorems 1.1 and 1.2 provide a convergence speed $\psi$ such that $d(T^n x, x) \to 0$, but we do not know whether the convergence speed is optimal.
- The general convergence speed in Theorem 1.2 depends on the underlying dynamical system $(X, T)$.
- Boshernitzan also noted, in the concluding remarks, that there is no ‘universal’ convergence speed suitable for all dynamical systems. This indicates, more or less, that if a universal function is required, the system must satisfy some additional conditions.

Remark 1.6. (Chang, Wu and Wu [4], Baker and Farmer [1]) The results of Chang, Wu and Wu [4] and Baker and Farmer [1] are applicable to finite conformal iterated function systems with the open set condition and the map $T : X \to X$ induced by the left shift. Generally speaking, the set
\[
A_n := \{ x \in X : d(T^n x, x) < \psi(n) \}
\]
concerns the distribution of the periodic points
\[
P_n := \{ x \in X : T^n x = x \}.
\]

For a finite conformal iterated function system with the open set condition, this corresponds to a finite full shift symbolic space, making it convenient to study the corresponding set.
- In this full shift setting, the set $P_n$ can be precisely expressed and the points in $P_n$ are sufficiently well distributed in $X$.
- The finiteness of the iterated function system ensures that a ball is equivalent to a cylinder set. Therefore, everything can be translated to a finite full shift symbolic space.
- The natural measure supported on $X$ is a Gibbs measure, so it has a nice Bernoulli property which leads to quasi-independence of the sets in question.

Remark 1.7. (Chernov and Kleinbock [5]) We take this opportunity to compare the recurrence set above with the shrinking target set. Define the shrinking target set
\[
S(\psi) = \{ x \in X : d(T^n x, x_0) < \psi(n), \text{ i.m. } n \in \mathbb{N} \} = \limsup_{n \to \infty} T^{-n} B(x_0, \psi(n))
\]
for some fixed $x_0 \in X$. A dynamical Borel–Cantelli lemma for this setting was presented by Chernov and Kleinbock [5]. For this set:

\[
\text{Dynamical Borel–Cantelli lemma for recurrence theory} \quad 1997
\]
• since $\mu$ is $T$-invariant, the measure of events $B_n := T^{-n}B(x_0, \psi(n))$ can be calculated easily;
• the mixing property, Condition II, together with the invariance property of $\mu$, can be applied directly to verify the quasi-independence of the events $\{B_n\}_{n \geq 1}$.

Remark 1.8. (Our method) In our setting, the set $P_n$ cannot be constructed easily. The events in our setting cannot be expressed as the $T$-inverse image of some sets, so the invariance of $\mu$ and the mixing property cannot be used directly. The way to overcome these difficulties is to look at the set $A_n$ locally, as locally $A_n$ behaves like $T^{-n}B_n$ for some $B_n$. Then the invariance and the mixing property of $\mu$ can be applied. It should be noted that this will lead to a superposition of the error terms, which makes the problem more involved.

2. Proof of Theorem 1.3
We split the proof of the theorem into several subsections for convenience. Let

$$A_n := \{x \in X : d(T^n x, x) < \psi(n)\};$$

then, $R(\psi) = \limsup_{n \to \infty} A_n$.

For notational simplicity, we use $a \lesssim b$ or $a = O(b)$ to say $a \leq Cb$ for some unspecified constant $C > 0$, and $a \asymp b$ if $a \lesssim b$ and $b \lesssim a$.

2.1. The measure of $A_n$. In this subsection, the main aim is to prove the following proposition.

**Proposition 2.1.** Assume that Conditions I and II hold. Then

$$\sum_{n=1}^{\infty} \psi^\delta(n) = \infty \iff \sum_{n=1}^{\infty} \mu(A_n) = \infty.$$ 

As previously stated, the set $A_n$ cannot be expressed in the form $T^{-n}B_n$ for some $B_n$. However, if considered locally, it can be expressed in this form.

**Lemma 2.2.** Let $B = B(x_0, r)$ be a ball centred at $x_0 \in X$ and radius $r > 0$. Then for any $n \in \mathbb{N}$ with $\psi(n) > r$ and any subset $E$ of $B$,

$$E \cap T^{-n}(B(x_0, \psi(n) - r)) \subset E \cap A_n \subset E \cap T^{-n}(B(x_0, \psi(n) + r)).$$

**Proof.** Fix a point $x \in E \cap A_n$. Then $d(x, x_0) < r$ and $d(T^n x, x) < \psi(n)$. By using the triangle inequality,

$$d(T^n x, x_0) \leq d(T^n x, x) + d(x, x_0) < \psi(n) + r.$$ 

That is, $x \in T^{-n}(B(x_0, \psi(n) + r))$. Therefore,

$$E \cap A_n \subset E \cap T^{-n}(B(x_0, \psi(n) + r)).$$

The left-hand-side inclusion follows similarly.
Remark 2.3. The lemma above gives us a way to write the set $A_{n}$ as the inverse of a ball with a fixed centre by restricting it to a smaller ball. If we choose the ball $B = B(x_{0}, \epsilon \psi(n))$ with $0 < \epsilon < 1$, then the above lemma yields

$$B \cap T^{-n}(B(x_{0}, (1 - \epsilon)\psi(n))) \subset B \cap A_{n} \subset B \cap T^{-n}(B(x_{0}, (1 + \epsilon)\psi(n))).$$

For any ball $B$, with Lemma 2.2 at our disposal, we are in a position to estimate the measure of $B \cap A_{n}$.

**Lemma 2.4.** Let $0 < \epsilon \leq 1/2$ and $B = B(x_{0}, \epsilon \psi(n))$ with fixed $x_{0} \in X$. Assume that Conditions I and II hold. Then

$$\mu(B \cap A_{n}) \geq C_{1} \mu(B)\psi^{\delta}(n) - C_{2}\gamma^{n}\psi^{\delta}(n),$$

$$\mu(B \cap A_{n}) \leq C_{3} \mu(B)\psi^{\delta}(n) + C_{3}\gamma^{n}\psi^{\delta}(n),$$

where $C_{1} = \eta_{1}(1 - \epsilon)^{\delta}$, $C_{2} = \eta_{2}C(1 - \epsilon)^{\delta}$ and $C_{3} = \max\{\eta_{2}(1 + \epsilon)^{\delta}, \eta_{2}C(1 + \epsilon)^{\delta}\}$ are constants, and $\eta_{1}, \eta_{2}$ and $C$ are constants arising from Conditions I and II.

**Proof.** We prove inequality (2.1) only, as the proof of inequality (2.2) follows similarly.

Using the left inclusion in Lemma 2.2 and then the mixing property of $\mu$ (Condition II), we have

$$\mu(B \cap A_{n}) \geq \mu(B \cap T^{-n}B(x_{0}, (1 - \epsilon)\psi(n))) \geq \mu(B) \cdot \mu(B(x_{0}, (1 - \epsilon)\psi(n))) - C\gamma^{n} \mu(B(x_{0}, (1 - \epsilon)\psi(n))).$$

Now, using the Ahlfors regularity of $\mu$ (Condition I), we conclude that

$$\mu(B \cap A_{n}) \geq \eta_{1}(1 - \epsilon)^{\delta}\mu(B)\psi^{\delta}(n) - \eta_{2}C(1 - \epsilon)^{\delta}\gamma^{n}\psi^{\delta}(n).$$

The next lemma estimates the $\mu$-measure for the set $A_{n}$.

**Lemma 2.5.** Let $0 < \epsilon \leq 1/2$ and $n \in \mathbb{N}$. Assume that Conditions I and II hold. Then

$$C_{4}\psi^{\delta}(n) - C_{5}\gamma^{n}\epsilon^{-\delta} \leq \mu(A_{n}) \leq C_{6}\psi^{\delta}(n) + C_{6}\gamma^{n}\epsilon^{-\delta},$$

where $C_{4} = \eta_{1}^{-1}5^{-\delta}C_{1}$, $C_{5} = 5^{-\delta}\epsilon^{-\delta}C_{2}$ and $C_{6} = \max\{\eta_{1}^{-1}\eta_{2}C_{3}5^{\delta}, \eta_{1}^{-1}\epsilon^{-\delta}C_{3}\}$ are constants.

**Proof.** Consider the collection of balls

$$\{B(x, \epsilon \psi(n)) : x \in X\},$$

which naturally covers $X$. By Vitali’s covering theorem (commonly known as the 5r covering lemma), we can find countably many disjoint balls $\{B(x_{j}, \epsilon \psi(n))\}_{j \in \mathcal{J}}$ such that

$$\bigcup_{j \in \mathcal{J}}B(x_{j}, \epsilon \psi(n)) \subset X \subset \bigcup_{j \in \mathcal{J}}B(x_{j}, 5\epsilon \psi(n)).$$

(2.3)

By the left inclusion of (2.3) and the disjointness of $\{B(x_{j}, \epsilon \psi(n))\}_{j \in \mathcal{J}}$, we have

$$\sum_{j \in \mathcal{J}}\eta_{1}(\epsilon \psi(n))^{\delta} \leq \sum_{j \in \mathcal{J}}\mu(B(x_{j}, \epsilon \psi(n)))$$
\[
\begin{align*}
\mu \left( \bigcup_{j \in J} B(x_j, \varepsilon \psi(n)) \right) \\
\leq \mu(X) = 1.
\end{align*}
\]

So the cardinality \( N \) of \( J \) is bounded from above by \( \eta_1^{-1} (\varepsilon \psi(n))^{-\delta} \). Similarly, by the right inclusion of (2.3), we have

\[
1 = \mu(X) = \mu \left( \bigcup_{j \in J} B(x_j, 5\varepsilon \psi(n)) \right) \\
\leq \sum_{j \in J} \mu(B(x_j, 5\varepsilon \psi(n))) \\
\leq \sum_{j \in J} \eta_2 5^\delta (\varepsilon \psi(n))^{\delta}.
\]

Thus \( N \) is bounded from below by \( \eta_2^{-1} 5^{-\delta} (\varepsilon \psi(n))^{-\delta} \).

It is clear that

\[
A_n \subset \bigcup_{j \in J} (B(x_j, 5\varepsilon \psi(n)) \cap A_n).\quad (2.4)
\]

Thus, by Lemma 2.4,

\[
\mu(A_n) \leq \sum_{j \in J} \mu(B(x_j, 5\varepsilon \psi(n)) \cap A_n) \\
\leq N \cdot [C_3 \mu(B(x_j, 5\varepsilon \psi(n))) \psi^\delta(n) + C_3 \gamma^n \psi^\delta(n)] \\
\leq \eta_1^{-1} \eta_2 C_3 5^\delta \psi^\delta(n) + \eta_1^{-1} \varepsilon^{-\delta} C_3 \gamma^n.
\]

The other inequality concerning \( \mu \) can be proved by replacing (2.4) with

\[
A_n \supset \bigcup_{j=1}^N (B(x_j, \varepsilon \psi(n)) \cap A_n).
\]

**Proof of Proposition 2.1.** Take \( \varepsilon = 1/2 \). Then, in view of Lemma 2.5, we have that

\[
\sum_{n=1}^\infty \mu(A_n) \asymp \sum_{n=1}^\infty \psi^\delta(n) + \sum_{n=1}^\infty \gamma^n.
\]

Since \( 0 < \gamma < 1 \), the second term on the right converges, and the proof of the proposition is complete.

### 2.2. Estimating the measure of \( A_m \cap A_n \) with \( m < n \)

Recall that \( \mathcal{F}_m \) denotes the collection of cylinders of order \( m \),

\[
\mathcal{F}_m := \{ X_{i_0} \cap T^{-1} X_{i_1} \cap \cdots \cap T^{-(m-1)} X_{i_m-1} : i_0, i_1, \ldots, i_{m-1} \in \mathcal{I} \}.
\]

**Lemma 2.6.** Let \( J_m \) be a cylinder in \( \mathcal{F}_m \). For any open set \( U \subset J_m \), we have

\[
\mu(T^m U) \asymp K_{J_m} 5^\delta \mu(U).
\]
\textbf{Proof.} For a ball $B(x_0, r) \subset J_m$, Condition V implies that
\[ B(T^m x_0, C^{-1} K J_m r) \subset T^m B(x_0, r) \subset B(T^m x_0, C K J_m r). \]
Then, the Ahlfors regularity of $\mu$ implies that
\[ \mu(T^m B(x_0, r)) \asymp K_{J_m}^\delta \mu(B(x_0, r)). \]
Together with the fact that every open set can be written as the disjoint union of at most countably many balls, the desired result follows. \hfill \Box

\textbf{Lemma 2.7.} Let $J_m$ be a cylinder in $\mathcal{F}_m$. Then
\[ \text{rad}(J_m) \lesssim K_{J_m}^{-1} \text{ and } \mu(J_m) \lesssim K_{J_m}^{-\delta}. \]
\textbf{Proof.} The proof follows immediately from the expanding rate of $T^m|_{J_m}$ and then the Ahlfors regularity of $\mu$. \hfill \Box

\textbf{Lemma 2.8.} Let $J_m$ be a cylinder in $\mathcal{F}_m$. Then there is a ball of radius $r = K_{J_m}^{-1} \psi(m)$, say $B(z, r)$, such that
\[ J_m \cap A_m \subset B(z, r) \cap J_m := J_m^*. \]
\textbf{Proof.} Choose $z \in J_m \cap A_m$. For any $x \in J_m \cap A_m$, on the one hand we have
\[ d(T^m x, T^m z) \asymp \|D_z(T^m)\| \cdot d(x, z); \]
and on the other hand,
\[ d(T^m x, T^m z) \leq d(T^m x, x) + d(x, z) + d(z, T^m z) < 2 \psi(m) + d(x, z). \]
Since $T$ is expanding,
\[ \|D_x(T^m)\| \geq \|(D_x(T^m))^{-1}\|^{-1} \gtrsim 1, \]
thus
\[ d(x, z) \lesssim \|D_z(T^m)\|^{-1} \psi(m). \] \hfill \Box

\textbf{Proposition 2.9.} Let $m < n$. Then
\[ \mu(A_m \cap A_n) \lesssim \psi^\delta(m) \psi^\delta(n) + \gamma^{n-m} \psi^\delta(n) + O(\gamma^n) \psi^\delta(m). \]
\textbf{Proof.} Write
\[ A_m = \bigsqcup_{J_m \in \mathcal{F}_m} J_m \cap A_m = \bigsqcup_{J_m \in \mathcal{F}_m} J_m^*. \]
Now we estimate $\mu(J_m^* \cap A_n)$ for any fixed $J_m \in \mathcal{F}_m$. Take $r = K_{J_m}^{-1} \psi(m)$ and the ball $B(z, r)$ as in Lemma 2.8. There are two cases.

\textit{Case (i)}: $r \leq \psi(n)$. Applying Lemma 2.2 to $J_m^*$, we have
\[ J_m^* \cap A_n \subset J_m^* \cap T^{-n}(B(z, 2 \psi(n))). \] \hfill (2.5)

Applying Lemma 2.6 to the right-hand side of the inequality (2.5), we have
\[ \mu(J_m^* \cap A_n) \lesssim K_{J_m}^{-\delta} \cdot \mu(T^m(J_m^*) \cap T^{-(n-m)}(B(z, 2 \psi(n)))). \] \hfill (2.6)
By the conformality of $T$ (Condition V), we claim that

$$T^m J_m^* = T^m (B(z, r) \cap J_m) \subset B(T^m z, (C + K) \psi(m)).$$

In fact, write the open set $B(z, r) \cap J_m$ as a disjoint union of balls in $J_m$, saying

$$B(z, r) \cap J_m = \bigcup_{i \geq 1} B(z_i, r_i). \tag{2.7}$$

Trivially, $r_i \leq r$ for all $i$. Then, applying Condition V to the balls on the right-hand side of (2.7), one has

$$T^m (B(z, r) \cap J_m) \subset \bigcup_{i \geq 1} B(T^m z_i, C \psi(m)).$$

On the other hand, since both $z$ and $z_i$ are in $J_m$ and $T^m |_{J_m}$ is $C^1$, it follows that

$$|T^m z - T^m z_i| = |Dz^m| \cdot |z - z_i| \leq K \cdot J_m \cdot |z - z_i| \leq K \cdot \psi(m).$$

This shows the claim. Thus, recalling (2.6) and writing $\tilde{C} = C + K$, one has

$$\mu(J_m^* \cap A_n) \leq K_{J_m}^{-\delta} \mu(B(T^m z, \tilde{C} \psi(m)) \cap T^{-n-m}(B(z, 2\psi(n))).$$

Finally by the mixing property of $\mu$ (Condition II), it follows that

$$\mu(J_m^* \cap A_n) \leq K_{J_m}^{-\delta} \mu(B(T^m z, \tilde{C} \psi(m)) \cdot \psi(m)) + C \gamma^{n-m} \mu(B(z, 2\psi(n))).$$

So

$$I_1 := \sum_{J_m \in F_m, 1 \leq i \leq p_{m,n}} \mu(J_m^* \cap A_n) \lesssim \psi^\delta(m) \cdot \psi^\delta(n) + \gamma^{n-m} \psi^\delta(n),$$

where we have used the boundedness (Condition IV) of $\sum_{J_m \in F_m} K_{J_m}^{-\delta}$.

Case (ii): $r > \psi(n)$. We replace the ball $B(z, r)$ by a collection of balls of radius $\psi(n)$. To achieve this, choose a maximum of $\psi(n)$-separated points in $B(z, r)$, denoted by $\{z_i\}_{1 \leq i \leq p_{m,n}}$. Then it is clear that

$$B(z, r) \subset \bigcup_{i=1}^{p_{m,n}} B(z_i, \psi(n)) \quad \text{and} \quad \bigcup_{i=1}^{p_{m,n}} B(z_i, \psi(n)) \subset B(z, 2r).$$

By the Ahlfors regularity of $\mu$, a volume argument implies that

$$p_{m,n} \asymp \left(\frac{r}{\psi(n)}\right)^\delta \asymp \left(\frac{K_{J_m}^{-1} \psi(m)}{\psi(n)}\right)^\delta.$$
Finally, summing over all \(1 \leq i \leq p_{m,n}\), we have
\[
\mu(J_m^* \cap A_n) \leq \sum_{i=1}^{p_{m,n}} \mu(B(z_i, \psi(n)) \cap A_n) \lesssim [\psi^\delta(n) + O(\gamma^n)]\psi^\delta(m)K^{-\delta}_{J_m}.
\]
Therefore,
\[
I_2 := \sum_{J_m \in F_m, r > \psi(n)} \mu(J_m^* \cap A_n) \lesssim [\psi^\delta(n) + O(\gamma^n)]\psi^\delta(m) \cdot \sum_{J_m \in F_m} K^{-\delta}_{J_m}
\]
\[
\lesssim \psi^\delta(m)\psi^\delta(n) + O(\gamma^n)\psi^\delta(m).
\]
Hence,
\[
\mu(A_m \cap A_n) = \sum_{J_m \in F_m} \mu(J_m^* \cap A_n) = I_1 + I_2
\]
\[
\lesssim \psi^\delta(m)\psi^\delta(n) + \gamma^{n-m}\psi^\delta(n) + O(\gamma^n)\psi^\delta(m). \tag*{\square}
\]

2.3. Completing the proof of Theorem 1.3. There are two parts of the proof: the convergence part and the divergence part. The convergence part, however, is a straightforward application of the first Borel–Cantelli lemma and Proposition 2.1 by noting that
\[
\sum_{n=1}^{\infty} \psi^\delta(n) < \infty \implies \sum_{n=1}^{\infty} \mu(A_n) < \infty.
\]

The main ingredient in proving the divergence part is the use of the well-known Paley–Zigmund inequality, which enables us to conclude the positiveness of \(\mu(\lim \sup A_n)\). Then, with a technical argument, we conclude the full measure property.

2.3.1. Positive measure. Let \(N \in \mathbb{N}\) and \(Z_N(x) = \sum_{n=1}^{N} \chi_{A_n}(x)\), where \(\chi\) is the characteristic function. We first estimate the lower bound for the first moment and then the upper bound for the second moment of the random variable \(Z_N\).

- The first moment: by Lemma 2.5 and choosing \(\varepsilon = 1/2\), for \(N\) sufficiently large, one has
  \[
  \mathbb{E}(Z_N) = \sum_{n=1}^{N} \mu(A_n) \geq \sum_{n=1}^{N} (C_4 \psi^\delta(n) - C_5 \gamma^n \varepsilon^{-\delta})
  \]
  \[
  \geq C_4 \sum_{n=1}^{N} \psi(n)^\delta - C_5' \geq C_4 \sum_{n=1}^{N} \psi(n)^\delta,
  \]
  where for the second inequality we used the divergence of \(\sum_{n \geq 1} \psi(n)^\delta\).

- The second moment:
  \[
  \mathbb{E}(Z_N^2) = \mathbb{E} \left( \sum_{n=1}^{N} \chi_{A_n} + 2 \sum_{1 \leq m < n \leq N} \chi_{A_m} \chi_{A_n} \right)
  \]
  \[
  = \sum_{n=1}^{N} \mu(A_n) + 2 \sum_{1 \leq m < n \leq N} \mu(A_m \cap A_n).
  \]
Summing over \( m, n \) \((1 \leq m < n \leq N)\) in Proposition 2.9 gives
\[
\sum_{1 \leq m < n \leq N} \mu(A_m \cap A_n) \lesssim \left( \sum_{1 \leq n \leq N} \psi^\delta(n) \right)^2 + \sum_{1 \leq n \leq N} \psi^\delta(n).
\]

Therefore,
\[
\mathbb{E}(Z_N^2) = \sum_{n=1}^N \mu(A_n) + \sum_{1 \leq m < n \leq N} \mu(A_m \cap A_n) \leq C \left( \sum_{1 \leq n \leq N} \psi^\delta(n) \right)^2 + (1 + C \sum_{1 \leq n \leq N} \psi^\delta(n)).
\]

By the Paley–Zygmund inequality, for any \( \lambda > 0 \), we obtain
\[
\mu(Z_N > \lambda \mathbb{E}(Z_N)) \geq (1 - \lambda)^2 \frac{\mathbb{E}(Z_N)^2}{\mathbb{E}(Z_N)} \geq (1 - \lambda)^2 \frac{\left( \sum_{1 \leq n \leq N} (C/2) \psi^\delta(n) \right)^2}{C \left( \sum_{1 \leq n \leq N} \psi^\delta(n) \right)^2 + (1 + C \sum_{1 \leq n \leq N} \psi^\delta(n))}.
\]

Letting \( N \to \infty \) we get
\[
\mu(\limsup A_n) \geq \mu(\limsup(Z_N > \lambda \mathbb{E}(Z_N))) \geq \limsup \mu(Z_N > \lambda \mathbb{E}(Z_N)) > 0.
\]

2.3.2. Full measure. Consider a subset of \( X \):
\[
R'(\psi) = \{ x \in X : \lim inf_{n \to \infty} \psi(n)^{-1} |T^nx - x| < \infty \}.
\]

We check that the set \( R'(\psi) \) is invariant in the sense that
\[
\mu(R'(\psi) \setminus T^{-1}R'(\psi)) = 0. \tag{2.8}
\]

More precisely, take a point \( x \in R'(\psi) \cap (\bigcup_{i \geq 1} X_i) \). Let \( i \geq 1, c(x) > 0 \) and \( \{n_k\}_{k \geq 1} \subset \mathbb{N} \) be such that
\[
x \in X_i \quad \text{and} \quad |T^{n_k}x - x| < c(x) \cdot \psi(n_k) \quad \text{for all} \ k \geq 1.
\]

Since \( X_i \) is open, then for all \( k \) large, \( T^{n_k}x \in X_i \). So, for each \( k \geq 1 \) large,
\[
|T^{n_k}(Tx) - Tx| = |T(T^{n_k}x) - T(x)| \leq K \|D_x(T)\| \cdot |T^{n_k}x - x| < \tilde{c}(x) \cdot \psi(n_k).
\]

This means that
\[
R'(\psi) \cap (\bigcup_{i \geq 1} X_i) \subset T^{-1}R'(\psi),
\]

which proves (2.8) since
\[
\mu(X \setminus \bigcup_{i \geq 1} X_i) = 0.
\]
It is clear that $R(\psi) \subset R'(\psi)$. The exponential mixing property (Condition II) implies that $T$ is ergodic. Thus, together with the invariance of $R'(\psi)$, we have shown that

$$\sum_{n \geq 1} \psi(n)^{\delta} = \infty \implies \mu(R(\psi)) > 0 \implies \mu(R'(\psi)) > 0 \implies \mu(R'(\psi)) = 1. \quad (2.9)$$

Next, we show that $\mu(R(\psi)) = 1$. Take a sequence of positive numbers $\{\ell(n) : n \geq 1\}$ such that

$$\sum_{n=1}^{\infty} \left( \frac{\psi(n)}{\ell(n)} \right)^{\delta} = \infty, \quad \lim_{n \to \infty} \ell(n) = \infty.$$  

Applying (2.9) to $\tilde{\psi}(n) = \psi(n)/\ell(n)$, we have that for $\mu$-almost all $x \in X$,

$$\liminf_{n \to \infty} \frac{\ell(n)}{\psi(n)} d(T^n x, x) < \infty.$$  

By Egorov’s theorem, for any $\epsilon > 0$, there exists $M > 0$ such that the set

$$R_M = \left\{ x \in X : \frac{\ell(n)}{\psi(n)} d(T^n x, x) < M \text{ for i.m. } n \in \mathbb{N} \right\}$$  

is of measure at least $1 - \epsilon$. It is clear that

$$R_M \subset R(\psi) \quad \text{since } \ell(n) > M \text{ for large } n \in \mathbb{N}.$$  

Since $\epsilon$ is arbitrary, we conclude that

$$\mu(R(\psi)) = 1.$$  

3. Applications

In this section we present some applications of Theorem 1.3. There may be more applications, but we have restricted ourselves to some well-known examples. In particular, Theorems 3.1 and 3.3 given below are new and have never appeared in the literature before. These two theorems give the dichotomy laws for the Lebesgue measures of the recurrence sets for $\beta$-transformation and the Gauss map, respectively. By contrast, the dynamical Borel–Cantelli lemma for the shrinking target problems was studied over 50 years ago by Philipp [8], who considered the dynamics of $N$-adic transformation, $\beta$-transformation and the Gauss map.

3.1. $\beta$-transformation. For a real number $\beta > 1$, define the transformation $T_\beta : [0, 1] \to [0, 1]$ by

$$T_\beta : x \mapsto \beta x \mod 1.$$  

This map generates the $\beta$-transformation dynamical system $([0, 1], T_\beta)$. It is well known that $\beta$-expansion is a typical example of an expanding non-finite Markov system whose properties are reflected by the orbit of some critical point. General $\beta$-expansions have been widely studied in the literature, beginning with the pioneering works of Rényi [9], Parry [7], Schmeling [10], Tan and Wang [11] and others.

For this application we first check that the $\beta$-dynamical system satisfies all the conditions stated in our framework.
(1) Partition:

\[ X_i = \left( \frac{i-1}{\beta}, \frac{i}{\beta} \right) \text{ and } 1 \leq i \leq \lfloor \beta \rfloor, \quad X_{\lfloor \beta \rfloor + 1} = \left( \frac{\lfloor \beta \rfloor}{\beta}, 1 \right). \]

(2) Ahlfors regularity of the measure. Let \( \mu \) be the Parry measure which is equivalent to the Lebesgue measure \( L \) with density \( h(x) = \left( \int_0^1 \sum_{n:T^n1 < x} 1/\beta^n \, dx \right)^{-1} \sum_{n:T^n1 < x} 1/\beta^n \).

(3) The strong mixing property is due to Philipp [8].

(4) Bounded distortion. Restricted to a cylinder \( J_n \) of order \( n \), \( T^n_\beta \) is a linear map with slope \( \beta^n \).

(5) \[ \sum_{J_n \in \mathcal{F}_n} (K_{J_n})^{-\delta} = \sum_{J_n \in \mathcal{F}_n} \beta^{-n} = \beta^{-n} \cdot \#\mathcal{F}_n \leq \frac{\beta}{\beta - 1}, \]

where the inequality follows from the fact that \( \beta^n \leq \#\mathcal{F}_n \leq ((\beta^n+1)/(\beta - 1)); \) see [9].

Hence, all the conditions in the main theorem are fulfilled for \( \beta \)-transformation. Thus, as an application of our theorem, we are able to prove the full Lebesgue measure of the recurrence set

\[ R(T_\beta, \psi) := \{x \in [0, 1] : |T^n_\beta x - x| < \psi(n) \text{ for i.m. } n \in \mathbb{N}\}, \]

in the \( \beta \)-dynamical system. 2ex

**THEOREM 3.1.** Let \( \mu \) be the Parry measure. Then

\[ \mu(R(T_\beta, \psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(n) = \infty. \end{cases} \]

**Remark 3.2.** It should be noted that the \( \beta \)-transformation for a general \( \beta > 1 \) is neither a self-similar set nor a finite conformal iterated function system with the open set condition. So the results of Baker and Farmer [1] and Chang, Wu and Wu [4] are not applicable to \( \beta \)-dynamical systems. Nor are their results applicable to the dynamical systems generated by the Gauss map, as stated below.

3.2. **Gauss map.** Let \( T_G \) be the Gauss map on \([0, 1)\). It was shown by Philipp [8] that the system \(([0, 1), T_G)\) is exponentially mixing with respect to the Gauss measure \( \mu \) given by \( d\mu = dx/(1 + x) \log 2 \). Since the Gauss measure \( \mu \) is equivalent to the Lebesgue measure \( (L) \), Condition I is satisfied with \( \delta = 1 \). For any irrational \( x \in [0, 1) \),

\[ q_n^2(x) \leq |(T_G^n(x))'| \leq 4q_n^2(x), \tag{3.1} \]

where \( q_n(x) \) is the denominator of the \( n \)th convergent of the continued fraction expansion of \( x \). It follows that given any cylinder \( I(a_1, a_2, \ldots, a_n) \) with \( a_1, \ldots, a_n \in \mathbb{N} \), for any
x, y ∈ I(a_1, a_2, \ldots, a_n),
\[ \frac{1}{4} \leq \frac{|(T^n_G(x))'|}{|(T^n_G(y))'|} \leq 4. \]

So Condition III also holds.

For any \( J_n = I(a_1, a_2, \ldots, a_n) \) ∈ \( \mathcal{F}_n \),
\[ q_n^2(a_1, \ldots, a_n) \leq K_{J_n} = \inf_{x \in J_n} |(T^n_G(x))'| \leq 4q_n^2(a_1, \ldots, a_n). \tag{3.2} \]

Note that
\[ \frac{1}{2q_n^2} \leq |I(a_1, a_2, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}. \]

We have
\[ \sum_{J_n \in \mathcal{F}_n} K_{J_n}^{-1} \leq \sum_{J_n \in \mathcal{F}_n} q_n^{-2} \leq \sum_{J_n \in \mathcal{F}_n} 2|I(a_1, a_2, \ldots, a_n)| \leq 2. \]

That is, Condition IV is satisfied. Since \( T^n_G|J_n \) is monotonic and \( C^1 \), combining (3.1) and (3.2) gives that Condition V holds with \( C = 4 \).

Define the recurrence set as
\[ R(T_G, \psi) = \{ x \in [0, 1) : |T^n_Gx - x| < \psi(n) \text{ for i.m. } n \in \mathbb{N} \}. \]

Thus we can apply Theorem 1.3 to this set.

**Theorem 3.3.** Let \( \psi \) be a positive function and let \( T_G \) be the Gauss map. Then
\[ \mathcal{L}(R(T_G, \psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(n) < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(n) = \infty. \end{cases} \]

3.3. Homogeneous self-similar sets. Our result is applicable to a range of self-similar sets, but here we demonstrate it for the classical middle-third Cantor set \( K \). Let \( T_3 \) be the 3-adic transformation on \( K \), \( \mu \) the Cantor measure restricted on \( K \), \( \delta = \log_3 2 \). Then all the conditions are fulfilled for Theorem 1.3. Let
\[ R(T_3, \psi) = \{ x \in K : |T^n_3x - x| < \psi(n) \text{ for i.m. } n \in \mathbb{N} \}. \]

We have the following theorem.

**Theorem 3.4.** Let \( \psi \) be a positive function. Then
\[ \mu(R(T_3, \psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \psi(n)^\delta < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \psi(n)^\delta = \infty. \end{cases} \]

**Acknowledgements.** We thank Professor Dmitry Kleinbock for helpful remarks and the anonymous referee for valuable comments. M.H. was supported by the Australian Research Council Discovery Project (200100994). B.L. was supported partially by NSFC 11671151 and Guangdong Natural Science Foundation 2018B0303110005. D.S. was supported by
the Royal Society Fellowship. B.W. was supported by NSFC 11722105 and 11831007. Part of this work was carried out when B.L. and D.S. visited La Trobe University. We thank La Trobe University and MATRIX research institute for travel support. B.L. is the corresponding author.

REFERENCES

[1] S. Baker and M. Farmer. Quantitative recurrence properties for self-conformal sets. *Proc. Amer. Math. Soc.* 149(3) (2021), 1127–1138.
[2] L. Barreira and B. Saussol. Hausdorff dimension of measures via Poincaré recurrence. *Comm. Math. Phys.* 219 (2001), 443–463.
[3] M. D. Boshernitzan. Quantitative recurrence results. *Invent. Math.* 113(3) (1993), 617–631.
[4] Y. Chang, M. Wu and W. Wu. Quantitative recurrence properties and homogeneous self-similar sets. *Proc. Amer. Math. Soc.* 147(4) (2019), 1453–1465.
[5] N. Chernov and D. Kleinbock. Dynamical Borel–Cantelli lemmas for Gibbs measures. *Israel J. Math.* 122 (2001), 1–27.
[6] M. Kirsebom, P. Kunde and T. Persson. On shrinking targets and self-returning points. *Preprint*, 2020, arXiv:2003.01361v2.
[7] W. Parry. On the $\beta$-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* 11 (1960), 401–416.
[8] W. Philipp. Some metrical theorems in number theory. *Pacific J. Math.* 20 (1967), 109–127.
[9] A. Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* 8 (1957), 477–493.
[10] J. Schmeling. Symbolic dynamics for $\beta$-shifts and self-normal numbers. *Ergod. Th. & Dynam. Sys.* 17(3) (1997), 675–694.
[11] B. Tan and B. Wang. Quantitative recurrence properties for beta-dynamical system. *Adv. Math.* 228(4) (2011), 2071–2097.