ELLIPtical weighted HOLics for WEAK LENsING SHEAR MEASUREMENT. III. THE EFFECT OF RANDOM COUNT NOISE ON IMAGE MOMENTS IN WEAK LENsING ANALYSIS

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ABSTRACT

This is the third paper on the improvement of systematic errors in weak lensing analysis using an elliptical weight function, referred to as E-HOLICs. In previous papers, we succeeded in avoiding errors that depend on the ellipticity of the background image. In this paper, we investigate the systematic error that depends on the signal-to-noise ratio of the background image. We find that the origin of this error is the random count noise that comes from the Poisson noise of sky counts. The random count noise makes additional moments and centroid shift error, and those first-order effects are canceled in averaging, but the second-order effects are not canceled. We derive the formulae that correct this systematic error due to the random count noise in measuring the moments and ellipticity of the background image. The correction formulae obtained are expressed as combinations of complex moments of the image, and thus can correct the systematic errors caused by each object. We test their validity using a simulated image and find that the systematic error becomes less than 1% in the measured ellipticity for objects with an IMCAT significance threshold of $\nu \sim 11.7$.

Key word: gravitational lensing: weak

1. INTRODUCTION

The importance of weak lensing analysis is now widely recognized because it provides us with direct and unbiased information about the mass distribution of lensing objects. Weak lensing analysis measures shapes (called ellipticity, which has two components interpreted as direction and magnitude) of many background images (galaxies) that are then averaged over an appropriate number of images to remove the intrinsic random ellipticity of images and to obtain the ellipticity from the gravitational tidal effect (shear) of the lensing object. The shear carries information about the mass structure of the lensing object. Thus, an accurate shape measurement of the background images is critically important to accurately measure the mass distribution. Thus far, weak lensing is very successful for cluster lensing (ellipticity due to shear is on the order of 5%) and provides us with rich information regarding mass structures of clusters and our understanding of structure formation in the universe. Recently, the cosmic shear, i.e., weak lensing due to large-scale structure, attracted much attention because of its ability to study the nature of dark energy, which is supposed to be the source of the accelerated expansion of the universe. In fact, several projects that will measure the cosmic shear have been proposed and some of them are almost ready to begin the observation phase (Hyper Suprime-Cam: http://www.naoj.org/Projects/HSC/HSCProject.html; Dark Energy Survey: http://www.darkenergysurvey.org/; Euclid: http://sci.esa.int/euclid, etc.). However, the signal of cosmic shear is very weak (on the order of 1%) compared with cluster lensing and thus needs special treatment. We must develop a very accurate shape measurement scheme that avoids various systematic errors. For example, the measured gravitational shear depends on the ellipticity and signal-to-noise ratio (S/N) of the background image. Usually such dependence becomes small when averaging many of the images, but it is critically important to realize that these dependencies are correlated with the redshift distribution of the image, which is also important in order to have an accurate measurement of the shear. Thus, we cannot simply average the images without having a method free from such systematic errors. The required accuracy for the measurement of ellipticity is less than 1% in order to have useful information regarding dark energy.

There have been many studies in this area and various measurement schemes have been proposed (Kaiser et al. 1995; Bernstein & Jarvis 2002; Refregier 2003; Kuijken 2006; Miller et al. 2007; Kitching et al. 2008; Melchior 2011). The accuracy of these methods has been tested using simulated data provided by STEP1 (Heymans et al. 2006), STEP2 (Massey et al. 2007), GREAT08 (Bridle et al. 2010), and GREAT10 (Kitching et al. 2012). Although much progress has been reported, none of the methods described to date achieve the required accuracy and are free from any sort of systematic errors.

We have also developed a new scheme based on the KSB method (Kaiser et al. 1995) using an elliptical weight function (which we have called E-HOLICs) to measure the background image as accurately as possible (Okura & Futamase 2011, Paper I). We show in Paper II that the E-HOLICs approach can improve the systematic error dependent on ellipticity. In this paper, we study the systematic error that depends on the S/N.

There are some previous studies regarding this systematic error (Hirata et al. 2004; Kacprzak et al. 2012; Refregier et al. 2012; Okura & Futamase 2012, Paper II; Melchior & Viola 2012). These results show that the systematic error comes from random count noise (hereafter RCN). In particular, Melchior and Viola introduced a probabilistic approach to the effect of ellipticity measurement against various noises and tested their method using simulated data. We take a different approach by explicitly constructing the correction formulae for systematic bias due to the RCN as functions of S/N and moments of individual objects applicable to KSB and E-HOLICs. For this purpose we calculate the second-order effects of the RCN in the measurement of moments and ellipticity for Gaussian-weighted images with both the KSB method and the E-HOLICs method (i.e., without PSF correction). The correction formulae are expressed using parameters of the S/N and the complex moments of the objects and are used to correct the systematic effects.
error due to RCN in each object. Using the simulated data from GREAT08, we find that the derived formulae correct the S/N-dependent bias within 1% for images with $v \gg 11.7$.

This paper is organized as follows. In Section 2, we explain the notation used and provide some definitions. In Section 3, we calculate the second-order effects of the RCN and obtain general formulae to correct these effects. We test the formula in the case of the KSB method with a Gaussian weight function. The correction formula for the E-HOLICs method is presented in Section 4. In Section 5, we test the E-HOLICs method using GREAT08 simulated data. In Section 6, we summarize our results. In the Appendices A–F, we show detailed calculations for deriving the correction formulae. They are rather complicated expressions in the general situation, but reduce to simple expressions when the objects are approximated as having a nearly Gaussian shape.

2. BASIS AND DEFINITIONS

In this section, we present the notation and definitions used for the E-HOLICs method. Some of them were defined in Paper II, but here we add the effect of the RCN and the centroid shift error (hereafter CSE).

2.1. Random Count Noise

First, we write the observed brightness distribution of an object as “$I^{\text{obs}}(\theta)$,” which is the sum of the object “$I^{\text{obj}}(\theta)$” and the RCN “$I^{\text{RCN}}(\theta)$,” so

$$I^{\text{obs}}(\theta) = I^{\text{obj}}(\theta) + I^{\text{RCN}}(\theta),$$

where, “$\theta$” is the position angle in the complex coordinate whose origin is at the centroid of the object “$I^{\text{obj}}(\theta)$”

$$\theta \equiv \theta_1^1 \equiv \theta_1 + i\theta_2,$$

and the products of the positions are written as

$$\theta_N^M = (\theta_1^1)^N \theta_2^M,$$

where $N$ means order and $M$ means spin-number.

We assume that the RCN is the Poisson noise of sky counts and all pixels have the same root-mean-square (rms) value of the RCN “$\sigma_{\text{RCN}}$.” We do not consider the Poisson noise of the object itself in this paper. If an object has a photon count of $N_{\text{obj}}$, then the Poisson noise is on the order of $\sqrt{N_{\text{obj}}}$, and thus the order of errors is reduced by $1/\sqrt{N_{\text{obj}}}$. Therefore, if the object is bright enough to be able to neglect sky noise, we can also neglect its own Poisson noise. On the other hand, if an object is faint, the Poisson noise from the object is much smaller than the Poisson noise from the sky $N_{\text{sky}}(N_{\text{obj}} \ll N_{\text{sky}})$, so we can also neglect it. However, this requires further consideration in the situation where $N_{\text{obj}} \sim N_{\text{sky}}$ which will be discussed in another paper.

2.2. Notation

In measuring the moments of the image using the E-HOLICs method, we use an elliptical Gaussian weight function for measuring the complex moments, and we define the weight function with an arbitrary ellipticity $\delta_W \equiv \delta_{W1} + i\delta_{W2}$ as

$$W(\theta, \delta_W) \equiv e^{-\frac{\theta_W^2 + (\theta_2^2 + \delta_{W1}^2\theta_2^2)}{\sigma_W^2}},$$

where $\sigma_W^2$ is a size parameter of the weight function. $\sigma_W$ can have an arbitrary scale and it is not restricted in the E-HOLICs method, but we think $\sigma_W$ should be chosen to obtain the maximum S/N of objects in a real analysis.

The complex moments and HOLICs of an arbitrary brightness distribution are defined as

$$Z_N^M(I, \delta_W) \equiv \int d^2\theta \theta_N^M I(\theta) W(\theta, \delta_W)$$

and

$$\mathcal{H}_M^N(I, Z_P^O, \delta_W) \equiv \frac{Z_N^M(I, \delta_W)}{Z_P^O(I, \delta_W)}.$$

For simplicity, we use the following notation for combinations of functions:

$$Z_R^O(I, \delta_W) = \left[ AZ_M^N + A(I, \delta_W)Z_P^O(I, \delta_W) \right]_{(I, \delta_W)}.$$

In this paper, we define the origin of the coordinate at the centroid of $I^{\text{obj}}(\theta)$; therefore, the following dipole moment is 0, so

$$Z_1^1(I^{\text{obj}}, \delta_W) \equiv 0.$$

However, RCN causes a CSE, so we cannot obtain Equation (8) in a real analysis. We write the CSE as $\Delta \theta = \Delta \theta_1^1$; then the complex moments with RCN and CSE measured in a real analysis are defined as

$$\tilde{Z}_M^N(I^{\text{obs}}, \delta_W) \equiv \int d^2\theta (\theta - \Delta \theta)_N^M (I^{\text{obj}}(\theta) + I^{\text{RCN}}(\theta)) W(\theta - \Delta \theta, \delta_W).$$
and HOLICs are measured as

$$\hat{H}_M(I_{\text{abs}}, Z_p^O, \delta_w) \equiv \frac{\hat{Z}_M(I_{\text{abs}}, \delta_w)}{Z_p^O(I_{\text{abs}}, \delta_w)}. \tag{10}$$

The details of the CSE are shown in Section 3.

We use $\delta_i$ as the ellipticity of the object, and $\delta_{\text{obj}}(\delta_{\text{abs}})$ as the measured ellipticity of the object’s image (the observed image with RCN). Thus, $\delta_{\text{obj}}(\delta_{\text{abs}})$ depends on the choice of the weight function and thus the measurement method.

2.3. WSN and Weight Size

Here, we define the weighted signal-to-noise ratio (WSN) with an elliptical weight function as a parameter for the S/N

$$\text{WSN} \equiv \frac{\int d^2\theta I^{\text{obj}}(\theta)W(\theta, \delta_w)}{\sigma_{\text{RCN}}\sqrt{\int d^2\theta W(\theta, \delta_w)}} \approx \frac{Z_0^0(I^{\text{obj}}, \delta_w)}{\sigma_{\text{RCN}}\sqrt{S_w}}. \tag{11}$$

where $S_w$ is the integral over the weight function, i.e., the weighted area

$$S_w = \frac{\sigma_w^2\pi}{\sqrt{1 - \delta_w^2}}. \tag{12}$$

WSN appears frequently in the following calculations, so using WSN instead of S/N gives the correction formulae a simpler form.

The E-HOLICs method does not restrict the scale $\sigma_w$ of the weight function, and theoretically we can calculate the moments using any scale. However, in practice, the scale should be determined to maximize the WSN. This condition is obtained using the following equation:

$$\frac{\partial}{\partial \sigma_w^2} \text{WSN} = \frac{1}{\sigma_R^2} \frac{1}{\sqrt{S_w}} \left( -\frac{2}{\sigma_w^2} \right) \left[ Z_0^2 - \text{Re} \left[ \delta_w Z_2^* Z_2^0 \right] - \frac{\sigma_w^2 Z_0^0}{2} \right]_{(I^{\text{obj}}, \delta_w)} = 0. \tag{14}$$

We use the scale of the weight function, which satisfies this equation in measuring the correlation of WSN and IMCAT’s $\nu$ and in the simulation test in Section 5.

We measured the S/N and WSN for background objects detected from A1689 using real data taken by the Subaru Suprime-Cam, and used only objects with $\nu \geq 7$ as detected by IMCAT (http://www.ifa.hawaii.edu/~kaiser/imcat; the definition of $\nu$ can be found in Kaiser et al. 1995). The plots of $\nu$ and WSN are shown in Figure 1 and we estimate the following relationship in this paper:

$$\text{WSN} \leq 0.85\nu. \tag{15}$$

We believe that the noise in these relationships comes from differences of definition when measuring the object parameters (for example, the definition of weight function size) between the IMCAT measurement and the E-HOLICs method.

2.4. Averaging

In weak lensing analysis, the term “averaging” is the averaging of a parameter of many different objects to reduce the intrinsic shear, but in this paper “averaging” means averaging of a parameter of the same object with different RCNs. This requires observing the same object many times. Therefore, the difference between the value measured without RCN and the averaged value with RCN is the systematic error.

The averaged values of the complex moments of the RCN vanish. The averaged complex moments also vanish.

$$\overline{Z}_M(I^{\text{RCN}}, \delta_w) = 0. \tag{16}$$

On the other hand, the squares of the moments are not 0, because the RCN has self-correlation.

The standard deviation of RCN (where we assume that RCN has no correlation in each pixel) at an arbitrary position $\theta_a$ is obtained as

$$1 \overline{N} \sum_i I^{\text{RCN}}(\theta_a) I^{\text{RCN}}(\theta_a') = I^{\text{RCN}}(\theta_a) I^{\text{RCN}}(\theta_a') = \sigma_{\text{RCN}}^2 \delta D(\theta_a - \theta_a'). \tag{17}$$

where “$i$” means the $i$th set of RCN and $\delta D(\theta)$ is the Dirac delta function. Here, we define the square of the complex moments of $I^{\text{RCN}}(\theta)$. The square of the moments are often used in this paper, because the square of the RCN is important in determining systematic error. The square of the complex moments is defined as

$$G_{M+P}^{N+0}(\delta_w) \equiv \frac{Z_M^N(I^{\text{RCN}}, \delta_w) Z_P^O(I^{\text{RCN}}, \delta_w)}{\sigma_{\text{RCN}}^2}. \tag{18}$$
Figure 1. Plots of $\nu$ vs. WSN. We used background objects obtained using real data from A1689 and rejected objects with a $\nu$ lower than 7. Some objects have a correlation of about WSN = $\nu$ and almost all objects have WSN $\leq \nu$.

and the averaged value of $G_M^N$ is calculated as the integral of the square of the weight function:

$$
G_M^N(\delta_W) = \int d^2 \theta \theta (W(\theta, \delta_W))^2.
$$

(19)

Since we use an elliptical Gaussian for the weight function, $W(\theta, \delta_W)$, $G_M^N(\delta_W)$ can be calculated analytically. The detailed calculation and explicit expressions for $G_M^N(\delta_W)$ are given in Appendix B where the averaged value of the square of the complex moments is obtained by the product of $\sigma^2_{W}$ and $G_M^N(\delta_W)$.

2.5. Simulation Test

In this paper, we show formulae that predict systematic error due to the RCN and test the validity of these formulae using a simulation test. In the simulation, we use the Gaussian weight function and an elliptical Gaussian image (hereafter EGI or $I_{EG}$), because these functions can be integrated analytically. EGI is defined as

$$
I_{EG}(\theta, \delta_I) \equiv Ae^{-\theta^2/\sigma^2_W - Re[\delta_I^\dagger \theta]}.
$$

(20)

where we use $\sigma^2_W = 200(\text{pixel}^2)$ and $\delta_I = (0.5, 0)$. The reason for using such a large image is to avoid negative effects of small images (e.g., pixelization). We then use a different set of 10,000 RCN images for each object. Therefore, we have 10,000 different images that are the sum of the same object’s image and a different RCN image for each WSN. We estimate the systematic errors due to RCN in measuring the complex moments and ellipticity from these 10,000 images and compare them with the correction formulae obtained below. We will refer to this simulation test as the “EGI simulation test.”

3. CENTROID SHIFT ERROR, COMPLEX MOMENTS, AND ELLIPTICITY WITH RANDOM COUNT NOISE

In this section, we present the general formulae for the CSE and the complex moments with RCN. Detailed calculations are shown in Appendices C and D. We then show the validity of the formulae in the EGI simulation using the KSB method.

3.1. Centroid Shift Error with Random Count Noise

Here, we present the formula that corrects the CSE due to the RCN. We define the centroid of $I^{\text{obj}}$ as the origin of the complex coordinate, so $\theta = 0$ is the true centroid; however, the observed centroid of $I^{\text{obs}}(\theta)$ is different from the origin and this difference $\Delta \theta$ is the CSE. We measure the centroid as the position at which the dipole moment of $I^{\text{obs}}(\theta)$ vanishes:

$$
\hat{Z}_1(I^{\text{obs}}, \delta_W) = \int d^2 \theta (\theta - \Delta \theta)^I I^{\text{obj}}(\theta) W(\theta - \Delta \theta, \delta_W) + \hat{Z}_1(I^{\text{RCN}}, \delta_W) = 0.
$$

(21)
except for low WSN images. In a real analysis we cannot measure the complex moments of EGIs, which is shown in Appendix F.1 in detail. By comparing them, we see that these formulae give a very good approximation, because the difference is third order and higher.

Figure 2. Averaged square of the CSE normalized by \( \sigma_W^2 \) (so \( W^2(\Delta \theta, 0)\Delta \theta^2_M/\sigma_W^2 \)) of the EGI and KSBGW (\( \delta_W = 0 \)). The horizontal axis is the WSN and the vertical axis is \( W^2(\Delta \theta, 0)\Delta \theta^2_M/\sigma_W^2 \). A square (circle) indicates \( W^2(\Delta \theta, 0)\Delta \theta^2_M/\sigma_W^2 \) measured from simulated data and a solid (dashed) line means an analytical prediction, i.e., Equation (F4) (Equation (F5)).

By expanding Equation (21) into the power series of \( \Delta \theta \) and neglecting higher order terms, we obtain the CSE as a function of complex moments of the object and the RCN as follows:

\[
W(\Delta \theta, \delta_W)\Delta \theta(I^{\text{obj}}, \delta_W) \approx 2 \left[ \tilde{\xi}_0^0 I^{\text{RCN}}(\delta_W) - \tilde{\xi}_1^0 I^{\text{RCN}}(\delta_W) \right] \tag{22}
\]

where \( \tilde{\xi}_N^M \) is some combination of complex moments. The definition is shown in Appendix A. The averaged value of Equation (22) is 0, but this value squared does not vanish by averaging and is calculated as follows:

\[
\frac{W^2(\Delta \theta, \delta_W)\Delta \theta^2(I^{\text{obj}}, \delta_W)}{\text{WSN}} \approx \frac{1}{\text{WSN}} \frac{\sigma_W^2}{1 - \delta_W^2} \left[ (\tilde{\xi}_0^0)^2 + (\tilde{\xi}_2^0)^2 - 2 \text{Re}[\tilde{\xi}_0^0 \tilde{\xi}_2^0 \delta_W] \right] \tag{23}
\]

\[
\frac{W^2(\Delta \theta, \delta_W)\Delta \theta^2(I^{\text{obj}}, \delta_W)}{\text{WSN}} \approx \frac{1}{\text{WSN}} \frac{\sigma_W^2}{1 - \delta_W^2} \left( \tilde{\xi}_0^0 \right)^2 \delta_W - 2 \tilde{\xi}_0^0 \tilde{\xi}_2^0 + (\tilde{\xi}_2^0)^2 \delta_W^* \tag{24}
\]

The squares of the CSE do not vanish by averaging, so these are important for measuring the systematic error due to the RCN. We then define the ellipticity for the distribution of the CSE as

\[
\delta_C(I^{\text{obj}}, \delta_W) \equiv \frac{W^2(\Delta \theta, \delta_W)\Delta \theta^2(I^{\text{obj}}, \delta_W)}{W^2(\Delta \theta, \delta_W)\Delta \theta^2(I^{\text{obj}}, \delta_W)}. \tag{25}
\]

We can obtain the expected value of the scale and ellipticity of the distribution of the CSE from Equations (23) and (25), i.e., the WSN and complex moments of objects. Figure 2 shows the EGI simulation test for Equations (23) and (24) and an analytic approximation for EGIs, which is shown in Appendix F.1 in detail. By comparing them, we see that these formulae give a very good approximation, except for low WSN images. In a real analysis we cannot measure the complex moments of \( I^{\text{obj}} \), and thus we approximate them by the complex moments of \( I^{\text{obj}} \):

\[
\frac{W^2(\Delta \theta, \delta_W)\Delta \theta^2(I^{\text{obj}}, \delta_W)}{\text{WSN}} \approx \frac{W^2(\Delta \theta, \delta_W)\Delta \theta^2(I^{\text{obj}}, \delta_W)}{W^2(\Delta \theta, \delta_W)\Delta \theta^2(I^{\text{obj}}, \delta_W)}. \tag{26}
\]

We can obtain \( \Delta \theta \) from \( W(\Delta \theta)\Delta \theta \) (e.g., by using iteration), but in the following section we approximate it as \( W(\Delta \theta)\Delta \theta \approx \Delta \theta \) because the difference is third order and higher.
3.2. Complex Moments with Random Count Noise

In this section, we consider systematic error due to the RCN in measuring the complex moments. Detailed calculations are shown in Appendix D.

The relationship between the complex moments of the object’s image and the observed image are obtained by expanding Equation (9). Here, we define the effects due to the RCN in measuring the complex moments up to the second order of the RCN as

\[
\hat{Z}_M^{(I_{\text{obs}}, \delta_W)} \approx Z_M^{(I_{\text{obj}}, \delta_W)} + \Delta Z_M^{(1)}(I_{\text{obj}}, \delta_W) + \Delta Z_M^{(2)}(I_{\text{obj}}, \delta_W),
\]

where the subscript \( X \) in the \( \Delta Z_M^{(N)} \) term indicates the \( X \)th order in \( I_{\text{RCN}}(\theta) \). The first and second-order effects for the complex moments from \( I_{\text{RCN}}(\theta) \) are calculated as

\[
\Delta Z_M^{(1)}(I_{\text{obj}}, \delta_W) = Z_M^{(I_{\text{RCN}}, \delta_W)}
\]

\[
\Delta Z_M^{(2)}(I_{\text{obj}}, \delta_W) \approx \frac{1}{8} \left[ Z_0^N \left( F_N^M - \Delta \theta_2^2 + F_{M\pm}^N \Delta \theta_0^2 + F_{M+}^N \Delta \theta_2^{2+} \right) \right]_{(I_{\text{obj}}, \delta_W)}
\]

\[
- \frac{1}{WSN^2 W_S} \left[ Z_0^0 \left( I_{\text{obs}} G_N^M - \epsilon_2^0 G_{2,M-2}^N - \epsilon_0^0 G_{2,M-2}^N \right) \right]_{(I_{\text{obj}}, \delta_W)}
\]

\[
\Delta Z_M^{(2)}(I_{\text{obj}}, \delta_W) \approx \frac{1}{8} \left[ Z_0^N \left( F_N^M - \Delta \theta_2^2 + F_{M\pm}^N \Delta \theta_0^2 + F_{M+}^N \Delta \theta_2^{2+} \right) \right]_{(I_{\text{obj}}, \delta_W)}
\]

\[
- \frac{1}{WSN^2 W_S} \left[ Z_0^0 \left( I_{\text{obs}} G_N^M - \epsilon_2^0 G_{2,M-2}^N - \epsilon_0^0 G_{2,M-2}^N \right) \right]_{(I_{\text{obj}}, \delta_W)}
\]

where we neglect the odd order moments of \( I_{\text{obj}} \) (\( N \) is even, so \( N+1 \) is neglected), \( F_N^M \) indicates some combination of the complex moments, and \( GX \) is also some combination of \( G_N^M \). The definitions of these terms are shown in Appendix A. Therefore, the systematic error in measuring the complex moments is given by

\[
\Delta Z_M^{(N)}(I_{\text{obj}}, \delta_W) \approx \Delta \hat{Z}_M^{(N)}(I_{\text{obs}}, \delta_W)
\]

because the differences in the approximation are a higher order in the RCN. Finally, the correction formula for the complex moments is defined as

\[
Z_M^{(N)}(I_{\text{obj}}, \delta_W) \approx Z^{(\text{COR})}_M^{(N)}(I_{\text{obs}}, \delta_W) \equiv \left[ \hat{Z}_M^{(N)} - \Delta \hat{Z}_M^{(N)}(I_{\text{obs}}, \delta_W) \right]
\]

where \( Z^{(\text{COR})}_M^{(N)} \) only has a random error from the RCN, so the averaged value of \( Z^{(\text{COR})}_M^{(N)} \) has no error from the RCN.

Therefore, we can estimate and correct the systematic error from the combination of complex moments by averaging Equation (29) for each object. Figures 3–5 show the systematic errors of various complex moments in the EGI simulation tests. In these figures we plot the systematic error ratio (SER), defined as follows:

\[
\text{SER} \equiv \frac{\hat{Z}_M^{(N)}(I_{\text{obs}}, 0) - Z_M^{(N)}(I_{\text{EG}}, 0)}{Z_M^{(N)}(I_{\text{EG}}, 0)}.
\]

We also plot the second-order systematic error ratio (2ndSER) for EGI defined as

\[
\text{2ndSER} \equiv \frac{\Delta \hat{Z}_M^{(N)}(I_{\text{obs}}, 0)}{Z_M^{(N)}(I_{\text{EG}}, 0)}.
\]

where the explicit expressions for 2ndSER are shown in Appendix F1. Figures 3, 4, and 5 show the monopole, spin-0 quadrupole, and spin-2 quadrupole, respectively. As seen in these figures, Equation (29) is a good estimation for the systematic error except for low WSN objects.

3.3. Ellipticity with Random Count Noise

We now show the systematic error from the RCN in measuring ellipticity, as used in weak lensing analysis. In the previous section, we showed the effects for complex moments from RCN, but ellipticity is defined using quadrupole moments with normalization. Therefore, we need the effects for normalization. Detailed calculations are shown in Appendix D.
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Figure 3. SER due to the RCN for monopole moments measured in EGI and KSBGW. The WSN is plotted on the horizontal axis and the SER is plotted on the vertical axis. These plots are the simulated results of SER of $\Delta Z^0_0$ and the line is the 2ndSER, shown in Equation (F7).

Figure 4. SER due to RCN for $Z^2_0$ in EGI and KSBGW. The WSN is plotted on the horizontal axis and the SER is plotted on the vertical axis. These plots are the simulated results of SER of $\Delta Z^0_0$ and the line is the 2ndSER shown in Equation (F8).

The observed ellipticity and ellipticity of the object are defined as

$$\delta_{\text{obs}} = \mathcal{H}^2_Z(I_{\text{obs}}, Z^2_0, \delta_W) = \left[ \frac{Z^2_0}{Z^2_0}_{(I_{\text{obs}}, \delta_W)} \right]$$  (35)

$$\delta_{\text{obj}} = \mathcal{H}^2_Z(I_{\text{obj}}, Z^2_0, \delta_W) = \left[ \frac{Z^2_0}{Z^2_0}_{(I_{\text{obj}}, \delta_W)} \right]$$  (36)

where $\delta_{\text{obs}}$ is the ellipticity we measure and $\delta_{\text{obj}}$ is ellipticity we want.
Therefore, the difference of these ellipticities is due to RCN. We consider the effect up to the second order in the RCN as follows:

\[ \delta_{\text{obs}} \approx \delta_{\text{obj}} + \Delta \delta_{(1)}(I_{\text{obj}}, \delta_W) + \Delta \delta_{(2)}(I_{\text{obj}}, \delta_W), \]  

(37)

where the subscript number indicates the order of the RCN effect. Note that we cannot neglect the first-order effects in this calculation because there are combinations of first-order effects for complex moments. For example, the RCN effects for the observed ellipticity is calculated by

\[ \Delta \delta_{(1)}(I_{\text{obj}}, \delta_W) = \left[ \frac{\Delta Z^2_{2(1)} - \delta_{\text{obj}} \Delta Z^2_{2(0,1)}}{Z_0^2} \right]_{(I_{\text{obj}}, \delta_W)} \]  

(38)

\[ \Delta \delta_{(2)}(I_{\text{obj}}, \delta_W) = \left[ \frac{\Delta Z^2_{2(2)} - \delta_{\text{obj}} \Delta Z^2_{2(0,2)}}{Z_0^2} + \frac{1}{2WSN^2} \left( \frac{Z_0^2}{Z_0^2} \right)^2 \frac{\sigma_W^2}{S_W} \frac{(2\sqrt{2} - 1)}{\delta_W} \right]_{(I_{\text{obj}}, \delta_W)}, \]  

(39)

so the observed ellipticity has the following systematic error:

\[ \overline{\Delta \delta}_{(2)}(I_{\text{obj}}, \delta_W) = \left[ \frac{\Delta Z^2_{2(2)} - \delta_{\text{obj}} \Delta Z^2_{2(0,2)}}{Z_0^2} - \frac{\delta_W}{8WSN^2} \left( \frac{Z_0^2}{Z_0^2} \right)^2 \frac{\sigma_W^4}{S_W} \frac{1}{\delta_W} \right]_{(I_{\text{obj}}, \delta_W)}. \]  

(40)

In a real analysis, we approximate Equation (40) using the complex moments of \( I_{\text{obs}} \), which can be measured as:

\[ \overline{\Delta \delta}_{(2)}(I_{\text{obj}}, \delta_W) \approx \Delta \delta_{(2)}(I_{\text{obs}}, \delta_W). \]  

(41)

The difference is expected to be higher order. Thus, we arrive at the following correction formula for ellipticity:

\[ \delta_{\text{obj}} \approx \delta_{\text{obj}}^{\text{(COR)}} = \delta_{\text{obs}} - \overline{\Delta \delta}_{(2)}(I_{\text{obs}}, \delta_W). \]  

(42)

Figure 6 shows the result of the EGI simulation test for the observed ellipticity in the case of the KSB method. There we plot SER, defined as \( (\delta_{\text{obs}} - \delta_{\text{obj}})/\delta_{\text{obj}} \) and calculated by Equation (40), and the 2ndSER is defined as \( \Delta \delta_{(2)}(I_{\text{obj}}, \delta_W) \) for EGI as calculated in Appendix F.1. We can see that Equation (40) results in good agreement except for low WSN images, and the ellipticity of objects having WSN = 5 is measured with about a 4% underestimation.
4. THE E-HOLICs METHOD WITH RANDOM COUNT NOISE

In this section, we present the systematic error of the complex moments and ellipticity with the RCN using the E-HOLICs method. The E-HOLICs method uses the ellipticity of objects as the ellipticity of the weight function (in a real analysis, we need to find this ellipticity by iterating the calculation, for example). In a realistic situation, we cannot measure the true ellipticity due to the RCN. Therefore, we must consider the RCN effect on the ellipticity of the weight function, which will be discussed in the next section. In this section, we will show calculations using the true ellipticity for the weight function. Although this is not realistic, it is still useful for later discussions.

4.1. The E-HOLICs Method with Random Count Noise and True Ellipticity for the Weight Function

In this section, we use the true ellipticity ($\delta_I$) for the weight function. Thus, ellipticities are defined as

$$\delta_{\text{obs}} = \tilde{\chi}_2^2(\mathcal{I}_{\text{obs}}, Z_0^2, \delta_I)$$

$$\delta_{\text{obj}} = \chi_2^2(\mathcal{I}_{\text{obj}}, Z_0^2, \delta_I) = \delta_I,$$

where the true ellipticity is equal to complex distortion in the E-HOLICs method.

The CSE and the averaged values of CSE with an arbitrary ellipticity for the weight function are given by Equations (22)–(24), which are rather complicated. In the E-HOLICs method, the definitions of $\mathcal{E}_{\text{M}}^N$ and $\delta_I$ lead to $\mathcal{E}_{\text{C}}^N(\mathcal{I}_{\text{obs}}, \delta_I) = 0$, and thus the correction formulae for the CSE become as follows. If we use the weight scale, which satisfies Equation (14), we obtain the following expressions for CS (see Appendix E.1):

$$\Delta \theta_2^2(\mathcal{I}_{\text{obj}}, \delta_W) = \frac{1}{\text{WSN}^2} \frac{\sigma_W^2}{1 - \delta_I^2}$$

$$\Delta \theta_2^2(\mathcal{I}_{\text{obj}}, \delta_W) = \frac{1}{\text{WSN}^2} \frac{\sigma_W^2}{1 - \delta_I^2}$$

Figure 7 shows the validity of the above expressions using the EGI simulation where we plot $\mathcal{W}^2(\Delta \theta, \delta_I) \Delta \theta_2^2(\mathcal{I}_{\text{obj}}, \delta_W)$ and $\mathcal{W}^2(\Delta \theta, \delta_I) \Delta \theta_2^2(\mathcal{I}_{\text{obj}}, \delta_W)$ applied to an EGI (see Appendix F.2).

From Equations (28) and (29), the effects of the RCN in measuring the complex moments with an elliptical weight having $\delta_I$ are obtained as $\Delta Z_{M(1)}^N(\mathcal{I}_{\text{obj}}, \delta_I)$ and $\Delta Z_{M(2)}^N(\mathcal{I}_{\text{obj}}, \delta_I)$. We tested the SER in measuring the complex moments using the EGI simulation test, where the SER and 2ndSER are the same as Equations (33) and (34) except for the use of the elliptic weight function. Figures 8 and 9 show the results of the EGI tests and an analytic approximation using an EGI is given in Appendix F.2. As seen in the figure, the analytic approximation gives a good approximation for SER except for low WSN objects.
Figure 7. Averaged square of the CSE normalized by $\sigma_W^2$ (so $\bar{W}^2(\Delta \theta, \delta_I)/\sigma_W^2$) of EGI and the E-HOLICs method $\delta_W = \delta_I$. WSN is plotted on the horizontal axis and the vertical axis shows $\bar{W}^2(\Delta \theta, \delta_I)/\sigma_W^2$. The squares (circle) indicate $\bar{W}^2(\Delta \theta, \delta_I)/\sigma_W^2$ (Re$[\bar{W}^2(\Delta \theta, \delta_I)/\sigma_W^2]$), measured from the simulated data and the solid (dashed) line is an analytical prediction, i.e., Equation (F26) (Equation (F27)).

Figure 8. SER due to the RCN for monopole moments measured using EGI and the E-HOLICs method and the true ellipticity for the weight function. WSN is plotted on the horizontal axis, and the systematic error is plotted on the vertical axis. The dots with error bars are the simulated results with an SER of $\Delta Z_0^0$ and the curve is the 2ndSER given by Equation (F29).

The RCN effect on ellipticity is obtained from Equation (39), i.e., $\bar{\Delta} \delta_3(\bar{I}^{(0)}, \delta_I)$. We test this formula in measuring the ellipticity using the EGI simulation test, where the SER and 2ndSER have the same expressions as Figure 6 except for the use of the elliptic weight function. Figure 10 shows the result of the EGI test. The ellipticity is defined by quadrupole moments and the averaged effect for quadrupole moments is 0 (Equations (F30) and (F31)), however, the ellipticity has a non-zero average. This results from the combination of first-order effects in the complex moments due to $I^{\text{RCN}}(\theta)$. In this situation, we observe that the ellipticity is underestimated by 1% on average for WSN = 5. We can see that these expressions give formulae with a reasonably good fit except for low WSN objects.
4.2. The E-HOLICs Method with Random Count Noise and Observed Ellipticity for the Weight Function

In the previous section, we derived the correction formulae for the systematic error using the E-HOLICs method in an ideal situation with the true ellipticity. When applying the E-HOLICs method to a realistic situation, we cannot use the true ellipticity of the object because of the RCN effect. Thus, in reality, we define the ellipticities as follows:

$$\delta_{\text{obj}} = \mathcal{H}_{2}^{2}(\bar{I}_{\text{obj}}, Z_{0}^{2}, \delta_{I}) = \delta_{I}$$  \hspace{1cm} (47)
\[ \delta_{\text{obs}} = \tilde{H}_2^I(I^{\text{obs}}, Z_{0}^2, \delta_O) \]  

(48)

where we define the observed ellipticity \( \delta_O \) as follows:

\[ \tilde{H}_2^I(I^{\text{obs}}, Z_{0}^2, \delta_O) = \delta_O. \]  

(49)

Since \( \tilde{H}_2^I(I^{\text{obj}}, Z_{0}^2, \delta_f) \) is equal to the complex distortion, we would like to have a correction formula for the RCN in \( \tilde{H}_2^I(I^{\text{obs}}, Z_{0}^2, \delta_O) \) to derive \( \tilde{H}_2^I(I^{\text{obj}}, Z_{0}^2, \delta_f) \) (i.e., a method to derive \( \delta_f \) from \( \delta_0 \)). We express the error due to the RCN in measuring the complex moments and ellipticity as follows:

\[ \hat{Z}_M^N(I^{\text{obs}}, \delta_O) \approx Z_M^N(I^{\text{obj}}, \delta_f) + \Delta \hat{Z}_{(1)}(I^{\text{obj}}, \delta_f) + \Delta \hat{Z}_{(2)}(I^{\text{obj}}, \delta_f) \]  

(50)

\[ \delta_O \approx \delta_f + \Delta \delta_{(1)}(I^{\text{obj}}, \delta_f) + \Delta \delta_{(2)}(I^{\text{obj}}, \delta_f). \]  

(51)

4.2.1. The Correction Formula for the Centroid Shift Error

Note that the CSE \( \Delta \theta \) is already first order in \( I^{\text{RCN}}(\theta) \) and we use the square of the CSE (i.e., \( \Delta \theta_0^2 \)) to calculate the complex moments. Therefore, we can neglect the differences between \( \delta_f \) and \( \delta_O \) in the calculation of the CSE because it is a higher order. Thus, we obtain

\[ \Delta \theta(I^{\text{obj}}, \delta_O) \approx \Delta \theta(I^{\text{obj}}, \delta_f) \]  

(52)

and the averaged values are also obtained as

\[ \Delta \theta(I^{\text{obj}}, \delta_O) \approx \Delta \theta(I^{\text{obj}}, \delta_f) = 0 \]  

(53)

\[ \Delta \theta_0^2(I^{\text{obj}}, \delta_O) \approx \Delta \theta_0^2(I^{\text{obj}}, \delta_f) \]  

(54)

\[ \Delta \theta_0^2(I^{\text{obj}}, \delta_O) \approx \Delta \theta_0^2(I^{\text{obj}}, \delta_f). \]  

(55)

Figure 11 shows the results of the EGI simulation test for the CSE where an analytic approximation is given in Appendix F.3.

4.2.2. The Correction Formulae for Complex Moments and Ellipticity

In this section, we derive the corrections to the systematic error in measuring the complex moments and ellipticity. First, the elliptical weight function with the observed ellipticity \( W(\theta, \delta_O) \) can be expanded up to the second order in the RCN as follows:

\[ W(\theta, \delta_O) = W(\theta, \delta_f + \Delta \delta_{(1)}(I^{\text{obj}}, \delta_f) + \Delta \delta_{(2)}(I^{\text{obj}}, \delta_f)) \approx W(\theta, \delta_f) \left[ 1 + \frac{\text{Re}[\Delta \delta_{(1)} + \Delta \delta_{(2)}]}{2\sigma_W^2} \frac{\delta_f^2}{\delta_f^2} \right]^{(I^{\text{obj}}, \delta_f)}. \]  

(56)

By expanding \( \hat{Z}_M^N(I^{\text{obs}}, \delta_O) \) with \( \Delta \theta \) and \( \Delta \tilde{\delta} \), we obtain the error due to RCN in measuring the complex moments as

\[ \Delta \hat{Z}_{(1)}^N(I^{\text{obj}}, \delta_f) = \left[ \Delta Z_{(M+1)}^N + \frac{1}{2\sigma_W^2} \left( \Delta \delta_{(1)}^* Z_{M+2}^{N+2} + \Delta \delta_{(2)}^* Z_{M-2}^{N+2} \right) \right]^{(I^{\text{obj}}, \delta_f)}. \]  

(57)

\[ \Delta \hat{Z}_{(2)}^N(I^{\text{obj}}, \delta_f) = \left[ \Delta Z_{(M+2)}^N + \frac{1}{2\sigma_W^2} \left( \Delta \delta_{(1)}^* Z_{M+2}^{N+2} + \Delta \delta_{(2)}^* Z_{M-2}^{N+2} \right) + \frac{1}{8\sigma_W^4} \left( 2|\Delta \delta_{(1)}|^2 Z_{M+4}^{N+4} + 2|\Delta \delta_{(2)}|^2 Z_{M-4}^{N+4} \right) \right]^{(I^{\text{obj}}, \delta_f)}. \]  

(58)

Then, from the definition of \( \delta_O \), \( \Delta \tilde{\delta} \) can be expressed by \( \Delta \hat{Z}_{M,N}^2(I^{\text{obj}}, \delta_f) \) as follows:

\[ \Delta \tilde{\delta}_{(1)}(I^{\text{obj}}, \delta_f) = \left[ \frac{\Delta \hat{Z}_{2(1)}^2(I^{\text{obj}}, \delta_f)}{Z_0^2} \right]^{(I^{\text{obj}}, \delta_f)}. \]  

(59)
Detailed calculations to derive Equations (57)–(60) are given in Appendix E.2. Therefore, we obtain the correction formula for measuring ellipticity with the E-HOLICs method as

$$\delta_{\text{obj}} \approx \delta_{\text{obj}}^{(\text{COR})} = \delta_{\text{obs}} - \Delta \delta_{(2)}(I^{\text{obj}}, \delta_I) \approx \delta_{\text{obs}} - \Delta \delta_{(2)}(I^{\text{obs}}, \delta_O).$$

We can also obtain correction formulae for the complex moments. By adopting Equations (59) and (60) for Equation (58), the systematic error in measuring the complex ellipticity can be obtained as follows:

$$\Delta \hat{Z}_M^{(N)}(I^{\text{obj}}, \delta_I) \approx \left[ \frac{\Delta Z_{M(2)}^N}{Z_0^Z} + \frac{1}{2\sigma_W^2} (\Delta \hat{Z}_{(2)}^{N+2} Z_{N+2}^N + \Delta \hat{Z}_{(2)}^{N-2} Z_{N-2}^N) \\
+ \frac{1}{2\sigma_W^2} \left( \frac{1}{Z_0^Z} \frac{Z_0^Z}{Z_0^Z} (2(2-\delta_I^2)) Z_M^{N+4} - Z_{M+4}^{N+4} - Z_{M-4}^{N+4} \right) \\
+ \frac{2}{\sigma_W^2} \left( 2G_M^{N+4} - \delta_I^2 G_{M+2}^{N+4} - 2G_{M-2}^{N+4} \right) \right].$$

and we arrive at the following correction formula for measuring the complex ellipticity using the E-HOLICs method:

$$Z_M^{(\text{COR})}^{(N)}(I^{\text{obj}}, \delta_I) \equiv \hat{Z}_M^{(N)}(I^{\text{obj}}, \delta_O) - \Delta \hat{Z}_M^{(N)}(I^{\text{obj}}, \delta_I) \approx \hat{Z}_M^{(N)}(I^{\text{obj}}, \delta_O) - \Delta \hat{Z}_M^{(N)}(I^{\text{obs}}, \delta_O).$$

We test the above formulae using the EGI simulation test in Figure 12 (Figures 13 and 14) where we plot the SER and 2ndSER. They take the same expression as Figure 3 for the complex moments and Figure 6 for the ellipticity except for the use of the elliptic weight function. The analytic approximations for the EGI used in the 2ndSER in these figures are given in Appendix F.3. These figures show that the systematic error in the ellipticity and in the complex moments are well approximated by Equation (E17) except for low WSN objects.

4.3. Summary

In this section, we show the steps for correcting systematic error due to the RCN in real analysis. The systematic error in ellipticity is notated by Equation (61), and the components of the systematic error have complex forms as shown in Appendix E.2 in detail. However, the components are defined by combinations of complex moments; we can calculate the systematic error from complex moments of $I^{\text{obs}}$ in the following steps:

1. Measuring the centroid, size, and ellipticity of $I^{\text{obs}}$.
2. Measuring the complex moments of $I^{\text{obs}}$.
3. Measuring the WSN from the variance count, which the image has, and complex moments of $I^{\text{obs}}$ (Equation (11)), and calculating the combinations of complex moments (Appendix A).
4. Obtaining Equation (E16) by calculating Equations (E8) and (E15) by combining the WSN and the combinations of complex moments.

5. TESTS USING GREAT08 SIMULATION IMAGES

In this section, we will explain a test of our correction formulae derived in the previous section using the “LowNoise_Known set0001.fits” data in the GREAT08 simulation data set. We applied the correction formulae to 10,000 different images with WSN = 10, created by adding 10,000 sets of the RCN (the same rms but different patterns) to a low noise image for each of the 10,000 low noise images in the GREAT08 data set. We compared the averaged ellipticities with the correction formula (61) image by image and we arrive at the following correction formula for measuring the complex ellipticity using the E-HOLICs method:

$$\Delta \hat{Z}_M^{(N)}(I^{\text{obj}}, \delta_I) = \left[ \frac{\Delta Z_{M(2)}^N}{Z_0^Z} - \delta_I \Delta \hat{Z}_{(2)}^{(N)}(I^{\text{obj}}, \delta_I) - \frac{(\Delta \hat{Z}_{(2)}^{(N)})^2}{(Z_0^Z)^2} \right].$$

Table 1 shows the SER and 1σ error for the complex moments; the second column is the SER and the third column is the SER and 1σ error corrected by Equation (63), respectively, where objects have WSN = 10.
Table 1
Systematic Error Ratio (SER) Using Images with an Ellipticity of 0.0 \sim 0.8 in the GREAT08 Simulated Data

| Parameter | SER | Corrected SER |
|-----------|-----|---------------|
| Z^0_0     | 0.0260 ± 2.95e-08 | −2.90e-04 ± 6.67e-09 |
| Z^2_0     | 0.0729 ± 3.45e-07 | −5.62e-03 ± 5.60e-08 |
| Z^2_2     | 0.1424 ± 4.17e-06 | −0.0129 ± 1.04e-06 |

**Notes.** The first column shows the parameters evaluated. The second column shows the SER and 1\(\sigma\) error without correction. The third column shows the SER and 1\(\sigma\) error corrected by the correction formula in Equation (63).

6. CONCLUSIONS AND FUTURE WORK

Based on our previous studies, this paper evaluates the systematic error caused by the S/N of the observed image in weak lensing analysis using a method referred to as E-HOLICs. Our previous results (Paper I), demonstrated that the shear is underestimated when low S/N background images are used and overestimated when only the high S/N background images are used in weak lensing analysis. An improvement in the former type of error is important because if we have such an improvement, we can use many faint background sources, which improves the overall statistical accuracy of weak lensing analysis.

As previous studies showed, we also identified the origin of the systematic error as the photon RCN in the sky. Although its first-order effect vanishes by averaging, the second-order effects are not canceled in measuring the moments and centroid of the images. We investigated this effect carefully and obtain the formulae using the KSB method and the E-HOLICs method to correct the effect in measuring moments and ellipticity. Although general expressions for these formulae are complicated, they reduce to relatively simple forms for images with an elliptical Gaussian form (EGI). We tested the validity of the correction formula in Equations (61) and (63) for EGIs using simulated data. Furthermore, we applied the general formulae to the GREAT08 data set and confirmed that the systematic error reduces to less than 1\% when measuring ellipticity for images with WSN = 10, which roughly corresponds to a \(v = 11.7\) object.

However, the present analysis has not taken into account the PSF correction, which is necessary for observations from the ground. The PSF correction uses complicated combinations of higher moments and will be very complicated using the E-HOLICs approach. The results described here are very encouraging and represent a worthwhile challenge. Finally, we should point out that the present work will be applicable to space-based observations because the PSF from instruments is expected to be small for such observations. It will be very interesting to confirm these expectations using data from sources such as COSMOS.

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APPENDIX A
DEFINITIONS

In this section, we show the definitions of complex moments and the combination of complex moments used in this paper. The relationships between complex moments and the moments defined in the KSB method are obtained as follows:

\[ Z^0_0(I, 0) = Q \]  
\[ Z^1_1(I, 0) = Q_1 + iQ_2 \]  
\[ Z^2_0(I, 0) = Q_{11} + Q_{22} \]  
\[ Z^2_2(I, 0) = Q_{11} - Q_{22} + i2Q_{12} \]  
\[ Z^3_1(I, 0) = (Q_{111} + Q_{122}) + i(Q_{112} + Q_{222}) \]  
\[ Z^3_3(I, 0) = (Q_{111} - 3Q_{122}) + i(3Q_{112} - Q_{222}) \]  
\[ Z^4_0(I, 0) = Q_{1111} + 2Q_{1122} + Q_{2222} \]  
\[ Z^4_2(I, 0) = Q_{1111} - Q_{2222} + i2(Q_{1112} + Q_{1222}) \]
\[ Z^4_4(I, 0) = Q_{1111} - 6Q_{1122} + Q_{2222} + i4(Q_{1112} - Q_{1222}) \]  

(A9)

where

\[ Q_{ij...kl} = \int d^2\theta I(\theta)\theta_i\theta_j...\theta_k\theta_lW(\theta, 0) \] 

(A10)

\( \mathcal{E}^N_M \) is defined as

\[ \mathcal{E}^N_M(I^{\text{obj}}, \delta) \equiv \left( (N - M + 2)\mathcal{H}^N_M - \frac{2}{\sigma^2_w} \left( \mathcal{H}^{N+2}_M \delta^*_w \mathcal{H}^{N-2}_M \right) \right) \] 

(A11)

\[ \mathcal{E}^N_M(I^{\text{obj}}, \delta) \equiv \left( (N + M + 2)\mathcal{H}^N_M - \frac{2}{\sigma^2_w} \left( \mathcal{H}^{N+2}_M \delta^*_w \mathcal{H}^{N-2}_M \right) \right) \] 

(A12)

These are non-dimensional and have spin-M, with \( \tilde{\mathcal{E}}^N_M \) defined as

\[ \tilde{\mathcal{E}}^0_{0-}(I^{\text{obj}}, \delta) \equiv \left[ \frac{\mathcal{E}^0_{0-}}{\mathcal{E}^0_{0-} - \mathcal{E}^0_{2-}} \right] \] 

(A13)

\[ \tilde{\mathcal{E}}^2_{2-}(I^{\text{obj}}, \delta) \equiv \left[ \frac{\mathcal{E}^2_{2-}}{\mathcal{E}^2_{0-} - \mathcal{E}^2_{2-}} \right] \] 

(A14)

\[ \tilde{\mathcal{E}}^0_{0-}(I^{\text{obj}}, \delta) \equiv \left[ \frac{\mathcal{E}^0_{0-}}{\mathcal{E}^0_{0-} - \mathcal{E}^0_{2-}} \right] \] 

(A15)

\[ \tilde{\mathcal{E}}^2_{2-}(I^{\text{obj}}, \delta) \equiv \left[ \frac{\mathcal{E}^2_{2-}}{\mathcal{E}^2_{0-} - \mathcal{E}^2_{2-}} \right] \] 

(A16)

\( \mathcal{F}^N_M \) is defined as

\[ \mathcal{F}^N_{M-}(I^{\text{obj}}, \delta) \equiv \left[ \frac{Z^N_{0-}}{Z^N_0} \left( (N - M)\mathcal{E}^{N-2}_{M-2} - \frac{2}{\sigma^2_w} \left( \mathcal{E}^{N-2}_{M-2} - \delta^*_w \mathcal{E}^N_{M+} \right) \right) \right] \] 

(A17)

\[ \mathcal{F}^N_{M+}(I^{\text{obj}}, \delta) \equiv \left[ \frac{Z^N_{0-}}{Z^N_0} \left( (N - M)\mathcal{E}^{N-2}_{M+2} - \frac{2}{\sigma^2_w} \left( \mathcal{E}^{N-2}_{M+2} - \delta^*_w \mathcal{E}^N_{M-} \right) \right) \right] \] 

(A18)
where $2 \leq N$. $G^N_M$ is defined as

$$G^N_M(\delta_W) \equiv (N + M - 2 + X) G_M^N - \frac{2}{\sigma_W^2} (G^N_{M+2} - \delta_W G^N_{M+2})$$

(A20)

where we can calculate their averaged value analytically as shown in Appendix B.

APPENDIX B

$G^N_M(\delta_W)$

In this section, we show that integration of the squared Gaussian is calculated analytically by

$$\overline{G^N_0(\delta_W)} = \frac{1}{2} \frac{\pi \sigma_W^2}{\sqrt{1 - \delta_W^2}} = S_W$$

(B1)

$$\overline{G^2_0(\delta_W)} = \frac{1}{2} \frac{\sigma_W^2}{1 - \delta_W^2} S_W = \frac{1}{2} \overline{G^N_0(\delta_W)}$$

(B2)

$$\overline{G^2_2(\delta_W)} = \frac{1}{2} \overline{\delta_W \sigma_W^2} S_W = \frac{1}{2} \overline{G^N_0(\delta_W)}$$

(B3)

$$\overline{G^4_0(\delta_W)} = \frac{1}{4} \left(2 + 3\delta_W^2\right) \sigma_W^4 S_W = \frac{1}{4} \overline{G^N_0(\delta_W)}$$

(B4)

$$\overline{G^2_2(\delta_W)} = \frac{1}{4} \left(3\delta_W \sigma_W^4 S_W = \frac{1}{4} \overline{G^N_0(\delta_W)}$$

(B5)

$$\overline{G^6_0(\delta_W)} = \frac{3}{8} \left(2 + 3\delta_W^2\right) \sigma_W^6 S_W$$

(B7)

$$\overline{G^2_2(\delta_W)} = \frac{3}{8} \left(4 + \delta_W^2\right) \delta_W \sigma_W^6 S_W$$

(B8)

$$\overline{G^8_0(\delta_W)} = \frac{3}{8} \overline{5\delta_W^2 \sigma_W^6 S_W}$$

(B9)
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\[
\frac{G_0^8(\delta_w)}{8} = \frac{3}{2} \frac{5 \delta_v^3 \sigma_w^6}{1 - \delta_w^2} S_w \tag{B10}
\]

\[
\frac{G_0^8(\delta_w)}{16} = \frac{3}{16} \frac{8 + 24 \delta_v^2 + 3 \delta_v^4}{(1 - \delta_w^2)^4} \sigma_w^8 S_w \tag{B11}
\]

\[
\frac{G_0^8(\delta_w)}{16} = \frac{3}{16} \frac{8 + 24 \delta_v^2 + 3 \delta_v^4}{(1 - \delta_w^2)^4} \sigma_w^8 S_w \tag{B12}
\]

\[
\frac{G_0^8(\delta_w)}{16} = \frac{3}{16} \frac{8 + 24 \delta_v^2 + 3 \delta_v^4}{(1 - \delta_w^2)^4} \sigma_w^8 S_w \tag{B13}
\]

\[
\frac{G_0^8(\delta_w)}{16} = \frac{3}{16} \frac{8 + 24 \delta_v^2 + 3 \delta_v^4}{(1 - \delta_w^2)^4} \sigma_w^8 S_w \tag{B14}
\]

\[
\frac{G_0^8(\delta_w)}{16} = \frac{3}{16} \frac{8 + 24 \delta_v^2 + 3 \delta_v^4}{(1 - \delta_w^2)^4} \sigma_w^8 S_w \tag{B15}
\]

and

\[
G_{\theta \delta_w} = \frac{N \pm M - 6 + 2X}{2} G_M(\delta_w). \tag{B16}
\]

APPENDIX C

DETAILED CALCULATIONS OF CENTROID SHIFT ERROR

In this section, we present detailed calculations of the CSE \( \Delta \theta \) with the elliptical Gaussian weight having an arbitrary ellipticity \( \delta_w \).

We miss setting the centroid of the objects on the condition that the measured dipole moment is 0, and the dipole moment is measured as

\[
\hat{Z}_i^1(I_{\text{obs}}, \delta_w) = \hat{Z}_i^1(I_{\text{obj}}, \delta_w) + \hat{Z}_i^1(I_{\text{RCN}}, \delta_w)
\]

\[
= \int d^2 \theta (\theta - \Delta \theta_i^1 I_{\text{obj}}(\theta - \Delta \theta, \delta_w) + \hat{Z}_i^1(I_{\text{RCN}}, \delta_w) = 0, \tag{C1}
\]

and by expanding this with \( \Delta \theta \) and neglecting higher order terms, we obtain

\[
\hat{Z}_i^1(I_{\text{obs}}, \delta_w) \approx \hat{Z}_i^1(I_{\text{RCN}}, \delta_w)
\]

\[
+ W(\Delta \theta, \delta_w) \left[ -\Delta \theta_i^1 \left( Z_0^0 - \frac{Z_0^2}{\sigma_w^2} \right) - \Delta \theta_i^1 \left( -\frac{Z_0^2}{\sigma_w^2} \right) \right]_{i(I_{\text{obj}}, \delta_w)}
\]

\[
\equiv - \frac{W(\Delta \theta, \delta_w)}{2} \left[ Z_0^0 (\theta_i^1 \Delta \theta + \theta_i^1 \Delta \theta^*) I_{\text{obj}}(\delta_w) + \hat{Z}_i^1(I_{\text{RCN}}, \delta_w) = 0, \tag{C2}
\]

where the expansion of the elliptical Gaussian weight function is

\[
W(\theta - \Delta \theta, \delta_w) \approx W(\Delta \theta, \delta_w) \left( 1 + \frac{\Delta \theta_i^1 - \Delta \theta_i^1 \Delta \theta^* + (\Delta \theta_i^1 - \Delta \theta_i^1 \Delta \theta^*)^*}{\sigma_w^2} \right) W(\theta, \delta_w). \tag{C3}
\]

Here, we neglect odd-numbered orders of the complex moments of \( I_{\text{obj}}(\delta_w) \) (i.e., \( Z_0^1(I_{\text{obj}}, \delta_w) = Z_0^1(I_{\text{obj}}, \delta_w) = 0 \)). Finally, we obtain \( \Delta \theta \) as

\[
W(\Delta \theta, \delta_w) \Delta \theta(I_{\text{obj}}, \delta_w) = 2 \left[ \frac{\hat{Z}_0^0 \hat{Z}_1^1(I_{\text{RCN}}, \delta_w) - \hat{Z}_0^1 \hat{Z}_1^1(I_{\text{RCN}}, \delta_w)}{Z_0^1} \right]_{i(I_{\text{obj}}, \delta_w)}, \tag{C4}
\]
and $\Delta T_M^2$ are obtained as

$$W^2(\Delta \theta, \delta_W) \Delta T_M^2(I^{\text{obj}}, \delta_W) \approx \frac{1}{\text{W} \text{S}^2} \frac{4}{8} \frac{\text{h}^2}{\text{S}_W} \left[ (\vert \tilde{\xi}_0 \vert^2 + \vert \tilde{\xi}_2 \vert^2) G_0^2 - 2\text{Re}[\tilde{\xi}_0 \tilde{\xi}_2^* G_0^2] \right]_{(I^{\text{obj}}, \delta_W)}$$  \hfill (C5)

$$W^2(\Delta \theta, \delta_W) \Delta T_M^2(I^{\text{obj}}, \delta_W) \approx \frac{1}{\text{W} \text{S}^2} \frac{4}{8} \frac{\text{h}^2}{\text{S}_W} \left[ (\tilde{\xi}_0^2)^2 G_0^2 + 2\text{Re}[\tilde{\xi}_0 \tilde{\xi}_2^* G_0^2] \right]_{(I^{\text{obj}}, \delta_W)}. $$  \hfill (C6)

Because the dipole moments of the RCN vanish upon averaging and the averaged value of $G_M^N$ can be obtained analytically, the averaged CSE can be obtained as

$$W(\Delta \theta, \delta_W) \Delta \theta(I^{\text{obj}}, \delta_W) = 0$$  \hfill (C7)

$$W^2(\Delta \theta, \delta_W) \Delta T_M^2(I^{\text{obj}}, \delta_W) = \frac{1}{\text{W} \text{S}^2} \frac{\text{h}^2}{\text{S}_W} \left[ (\tilde{\xi}_0^2)^2 \delta_W^2 - 2\text{Re}[\tilde{\xi}_0 \tilde{\xi}_2^* \delta_W^2] \right]_{(I^{\text{obj}}, \delta_W)}. $$  \hfill (C8)

$$W^2(\Delta \theta, \delta_W) \Delta T_M^2(I^{\text{obj}}, \delta_W) = \frac{1}{\text{W} \text{S}^2} \frac{\text{h}^2}{\text{S}_W} \left[ (\tilde{\xi}_0^2)^2 \delta_W^2 - 2\text{Re}[\tilde{\xi}_0 \tilde{\xi}_2^* \delta_W^2] \right]_{(I^{\text{obj}}, \delta_W)}. $$  \hfill (C9)

Therefore, we can calculate the averaged CSE for each object from the WSN and the complex moments of the objects.

APPENDIX D

DETAILED CALCULATIONS OF COMPLEX MOMENTS AND ELLIPTICITY WITH AN ARBITRARY ELLIPTICITY FOR THE WEIGHT FUNCTION

In this section, we show the detailed derivation of observed complex moments and ellipticity with the RCN. The observed complex moments with the RCN are measured as Equation (9), and by expanding it by the CSE, we obtain Equations (28) and (29) as follows:

$$Z_M^N(I^{\text{obj}}, \delta_W) = \int d^2 \theta (\theta - \Delta \theta)^N M(I^{\text{obj}}(\theta)) W(\theta - \Delta \theta, \delta_W)$$

$$= \int d^2 \theta (\theta - \Delta \theta)^N M(I^{\text{obj}}(\theta) + I^{\text{RCN}}(\theta)) W(\theta, \delta_W) \left( \frac{1}{\sigma_W^2} \left( \begin{array}{c} \Delta \theta_0^2 - \text{Re}[\delta_W \Delta \theta_0^2] \\ \end{array} \right) \right)$$

$$\times \left( \begin{array}{c} (\Delta \theta^* - \delta_W \Delta \theta)^2 \delta_W^2 + (\Delta \theta_0^2 - \delta_W \Delta \theta_0^2) \delta_W^2 + (\Delta \theta - \delta_W \Delta \theta_0^2) \delta_W^2 \end{array} \right)$$

$$= Z_M^N(I^{\text{obj}}, \delta_W) + Z_M^N(I^{\text{RCN}}, \delta_W)$$

$$+ \left( \begin{array}{c} (N + M)(N + M - 2) \frac{Z_{M-2} \Delta \theta_0^2}{8} + \frac{(N + M)(N - M)}{8} Z_M^{N-2} \Delta \theta_0^2 + \frac{(N - M)(N - M - 2)}{8} Z_{M+2}^{N-2} \Delta \theta_0^2 \\ \end{array} \right)$$

$$- \frac{1}{2 \sigma_W^2} \left( \begin{array}{c} ((N + M) Z_M^{N-2} - \delta_W \Delta \theta_0^2) \Delta \theta_0^2 + (N - M) (Z_M^{N} - \delta_W \Delta \theta_0^2) \Delta \theta_0^2 \\ \end{array} \right)$$

$$+ \left( \begin{array}{c} (N + M) Z_M^{N} - \delta_W \Delta \theta_0^2 + (N - M) (Z_M^{N} - \delta_W \Delta \theta_0^2) \Delta \theta_0^2 \\ \end{array} \right)$$

$$+ \frac{1}{2 \sigma_W^2} \left( \begin{array}{c} \left( Z_M^{N-2} - \delta_W \Delta \theta_0^2 + \delta_W \Delta \theta_0^2 \right) \Delta \theta_0^2 + \left( Z_M^{N} - \delta_W \Delta \theta_0^2 + \delta_W \Delta \theta_0^2 \right) \Delta \theta_0^2 \\ \end{array} \right)$$

$$+ \left( \begin{array}{c} (2(1 + \delta_W^2) Z_M^{N-2} - \delta_W \Delta \theta_0^2 + 2 \delta_W \Delta \theta_0^2) \Delta \theta_0^2 \\ \end{array} \right)$$

$$= Z_M^N(I^{\text{obj}}, \delta_W) + Z_M^N(I^{\text{RCN}}, \delta_W)$$

$$+ \left[ \frac{Z_0^0}{8} \frac{Z_0^0}{M} \right]_{(I^{\text{obj}}, \delta_W)}$$

$$- \frac{1}{\text{W} \text{S}^2} \frac{\text{h}^2}{\text{S}_W} \left[ (\tilde{\xi}_0 \tilde{\xi}_2^* G_{M+2}^N - \tilde{\xi}_2 \tilde{\xi}_2^* G_{M-2}^N + \tilde{\xi}_0^* \tilde{\xi}_2^* G_{M+2}^N - \tilde{\xi}_2 \tilde{\xi}_2^* G_{M-2}^N) \right]_{(I^{\text{obj}}, \delta_W)}$$

$$\equiv Z_M^N(I^{\text{obj}}, \delta_W) + \Delta Z_M^N(I^{\text{obj}}, \delta_W) + \Delta Z_M^N(I^{\text{obj}}, \delta_W).$$  \hfill (D1)
Therefore, the averaged value of the systematic error can be obtained from

\[
\Delta Z_{M(2)}^N(I_{\text{obj}}, \delta_W) = \left[ \frac{Z_0^N}{8} \left( \mathcal{F}_M^N \Delta \theta_0^2 + \mathcal{F}_M^N \Delta \theta_0^2 + \mathcal{F}_M^N \Delta \theta_0^2 \right) \right]_{(I_{\text{obj}}, \delta_W)} - \frac{1}{W S W^2} \frac{Z_0^N}{S_W} \left[ Z_0^N (\mathcal{F}_0^N G_{2M}^N - \mathcal{F}_0^N G_{2M}^N) - \mathcal{F}_0^N (G_{2M}^N - G_{2M}^N) - \mathcal{F}_0^N G_{2M}^N \right]_{(I_{\text{obj}}, \delta_W)}.
\] (D2)

The ellipticity with the RCN is measured as in Equation (35) and calculated as

\[
\delta_{\text{obs}} \equiv \hat{H}^2_\delta(I_{\text{obj}}, \delta_W, Z_0) \approx \left[ \frac{Z_0^2 + \Delta Z_{2(1)}^2 + \Delta Z_{2(2)}^2}{Z_0^2 + \Delta Z_{0(1)}^2 + \Delta Z_{0(2)}^2} \right]_{(I_{\text{obj}}, \delta_W)} \approx \delta_{\text{obj}} + \frac{\Delta Z_{2(1)}^2 + \Delta Z_{2(2)}^2 - \delta_{\text{obj}} \Delta Z_{0(1)}^2 - \delta_{\text{obj}} \Delta Z_{0(2)}^2}{2 S_W \frac{Z_0^2}{Z_0^2}} + \frac{\sigma_W^4}{2 S_W} \frac{Z_0^2}{Z_0^2} \left[ \frac{Z_0^2}{Z_0^2} \right]_{(I_{\text{obj}}, \delta_W)}
\]
and the systematic error is obtained from

\[
\Delta \hat{H}_{\delta(2)}(I_{\text{obj}}, \delta_W) = \left[ \frac{Z_0^2 - \delta_{\text{obj}} \Delta Z_{0(2)}^2}{Z_0^2} \right]_{(I_{\text{obj}}, \delta_W)}
\] (D3)

APPENDIX E

DETAILED CALCULATIONS USING THE E-HOLICs METHOD WITH OBSERVED ELLIPTICITY FOR THE WEIGHT FUNCTION

E.1. Using the True Ellipticity for the Weight Function

In this section, we show a simple form for the formulae calculated. These can be obtained if we use \( \delta \), a weight scale that satisfies Equation (14) for the weight function parameters.

From the definitions of ellipticity Equation (47) and \( \mathcal{E}_M^N \) (Equations (A11) and (13)), we obtain

\[
\mathcal{E}_0^N(I_{\text{obj}}, \delta) = 1
\] (E1)

\[
\mathcal{E}_2^N(I_{\text{obj}}, \delta) = 0
\] (E2)

from an arbitrary brightness distribution \( I_{\text{obj}} \). Therefore, some of the formulae calculated above have a simpler form. The formulae for CSE (Equations (C5), (C6), (C8), and (C9)) are

\[
\Delta \theta_0^2(I_{\text{obj}}, \delta_W) = \frac{1}{W S W^2} \frac{4}{S_W} \frac{G_0^2}{S_W}
\] (E3)

\[
\Delta \theta_2^2(I_{\text{obj}}, \delta_W) = \frac{1}{W S W^2} \frac{4}{S_W} \frac{G_2^2}{S_W}
\] (E4)

\[
\Delta \theta_0^2(I_{\text{obj}}, \delta_W) = \frac{1}{W S W^2} \frac{\sigma_W^2}{Z_0^2} \frac{Z_0^2}{Z_0^2} \frac{1}{1 - \delta^2_1}
\] (E5)

\[
\Delta \theta_2^2(I_{\text{obj}}, \delta_W) = \frac{1}{W S W^2} \frac{\sigma_W^2}{Z_0^2} \frac{Z_0^2}{Z_0^2} \frac{1}{1 - \delta^2_1}
\] (E6)

and formulae of complex moments are obtained from

\[
\hat{Z}_M^N(I_{\text{obj}}, \delta) = Z_M^N(I_{\text{obj}}, \delta) + Z_M^N(I_{\text{RCN}}, \delta)
\]

and the averaged value of the systematic error can be obtained from

\[
\Delta \hat{Z}_{M(2)}^N(I_{\text{obj}}, \delta_W) = Z_0^N \left[ \frac{\sigma_W^2}{8} \frac{2}{1 - \delta^2_1} \left( \frac{\mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1}{\mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1} \right) - \frac{Z_0^N}{S_W} (N - 2) \frac{\sigma_W^2}{Z_0^N} \right]_{(I_{\text{obj}}, \delta_W)}
\] (E7)

Therefore, the averaged value of the systematic error can be obtained from

\[
\Delta \hat{Z}_{M(2)}^N(I_{\text{obj}}, \delta_W) = \frac{1}{W S W^2} \left[ \frac{Z_0^N}{8} \frac{\sigma_W^2}{1 - \delta^2_1} \left( \frac{\mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1}{\mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1 + \mathcal{F}_M^N \Delta \theta_1} \right) - \frac{Z_0^N}{S_W} (N - 2) \frac{\sigma_W^2}{Z_0^N} \right]_{(I_{\text{obj}}, \delta_W)}
\] (E8)
\[
\mathcal{F}^N_{E,SM}(I^{\text{obj}}, \delta_I) = \left[ Z_0^N \sigma^N W \frac{\sigma^N_{E,SM}}{1 - \delta_I^2} \left( \mathcal{F}^N_{SEM, \delta_I} + \mathcal{F}^N_{SM, \delta_I} \right) \right]_{(I^{\text{obj}}, \delta_I)} \\
= Z_0^{N-2} \sigma^2 W \left( \left( N + M \right) \left( e^{N-2} + \delta_I e^{M-2} \right) + \left( N - M \right) \left( e^{N-2} + \delta_I^* e^{M+2} \right) \right) \\
- 2Z_0^N \left( e^{N} + e^{N-2} \right) \right]_{(I^{\text{obj}}, \delta_I)}.
\]

\[
E.2. \text{ Using Observed Ellipticity for the Weight Function}
\]

In this section, we show detailed calculations using the E-HOLICs method using the observed ellipticity for the weight function. The relationship between complex moments using the true ellipticity for the weight function \( \tilde{Z}_M^N(I^{\text{obs}}, \delta) \) and with the observed ellipticity for the weight function \( \tilde{Z}_M^N(I^{\text{obs}}, \delta) \) is obtained using the relationship between the weight functions (Equation (56)). By using that relationship, \( \tilde{Z}_M^N(I^{\text{obs}}, \delta) \) can be expressed using \( \delta \) and \( \Delta Z^N_M \) as

\[
\tilde{Z}_M^N(I^{\text{obs}}, \delta) \approx \left[ Z_M^N + \Delta Z^N_{M(1)} + \Delta Z^N_{M(2)} + \frac{1}{2\sigma_W^2} \left( \Delta \delta_I^2 \Delta Z^N_{M+2(1)} + \Delta \delta_I^2 \Delta Z^N_{M-2(1)} \right) \right. \\
+ \left. \frac{1}{2\sigma_W^2} \left( \left( \Delta \delta_I^2 + \Delta \delta_I^2 \right) Z^N_{M+2(1)} + \left( \Delta \delta_I^2 + \Delta \delta_I^2 \right) Z^N_{M-2(1)} \right) \right. \\
+ \left. \frac{1}{8\sigma_W^4} \left( \Delta \delta_I^4 Z^N_{M+4(1)} + 2(\Delta \delta_I^2 \Delta \delta_I^2) Z^N_{M+2(1)} + \Delta \delta_I^4 Z^N_{M-4} \right) \right]_{(I^{\text{obj}}, \delta_I)} \\
= Z_M^N(I^{\text{obj}}, \delta_I) + \Delta \tilde{Z}_M^N(I^{\text{obs}}, \delta_I) + \Delta \tilde{Z}_M^N(I^{\text{obj}}, \delta_I).
\]

The differences between \( \delta_I \) and \( \delta_O \) can be calculated by expanding \( \tilde{H}_2^2(I^{\text{obs}}, \delta_O) \) with \( \Delta \theta \) and \( \Delta \delta \) as

\[
\delta_O = \tilde{H}_2^2(I^{\text{obs}}, Z_0^2, \delta_O) \approx \delta_I + \left[ \Delta Z^2_{2(1)} - \delta_I \Delta Z^2_{0(1)} + \Delta Z^2_{2(2)} - \delta_I \Delta Z^2_{0(2)} \right] \\
+ \frac{1}{2\sigma_W^2} \left( \Delta \delta_I^2 \left( Z^4_{2(1)} - \delta_I Z^4_{0(1)} + \Delta \delta_I^2 \left( Z^4_{2(2)} - \delta_I Z^4_{0(2)} \right) \right) \right. \\
+ \frac{1}{8\sigma_W^4} \left( \Delta \delta_I^4 \left( Z^6_{2(1)} - \delta_I Z^6_{0(1)} + \Delta \delta_I^4 \left( Z^6_{2(2)} - \delta_I Z^6_{0(2)} \right) \right) \right. \\
+ \left. \frac{1}{2\sigma_W^2} \left( \Delta \delta_I^2 \left( Z^2_{2(1)} - \delta_I Z^2_{0(1)} + \Delta \delta_I^2 \left( Z^2_{2(2)} - \delta_I Z^2_{0(2)} \right) \right) \right) \right]
\]

\[
\equiv \delta_I + \Delta \delta_I(I^{\text{obj}}, \delta_I) + \Delta \delta_I(I^{\text{obj}}, \delta_I), \tag{E11}
\]

Therefore, we obtain the first-order effect of the RCN on ellipticity from

\[
\frac{1}{4} \left[ \Delta \tilde{\delta}_I^2 \tilde{e}^2_{0(1)} - \Delta \tilde{\delta}_I^2 \tilde{e}^2_{-4} \right]_{(I^{\text{obj}}, \delta_I)} = \left[ \Delta Z^2_{2(1)} - \delta_I \Delta Z^2_{0(1)} \right]_{(I^{\text{obj}}, \delta_I)} \tag{E12}
\]

\[
\Delta \tilde{\delta}_I = 4 \left[ \tilde{e}^2_{0(1)} \left( \Delta Z^2_{2(1)} - \delta_I \Delta Z^2_{0(1)} \right) - \tilde{e}^2_{-4} \left( \Delta Z^2_{2(2)} - \delta_I \Delta Z^2_{0(2)} \right) \right]_{(I^{\text{obj}}, \delta_I)}. \tag{E13}
\]
and the second-order effect of the RCN on ellipticity from

\[
\frac{1}{4} \left[ \Delta \tilde{\delta}^2_{(2)} \epsilon_{0-}^2 + \Delta \tilde{\delta}^2_{(2)} \epsilon_{4-}^2 \right]_{(I^{\text{obj}}, \delta_I)} = \left[ \Delta Z_{2(2)}^2 - \delta_I \Delta Z_{0(2)}^2 \right] \frac{Z_0^2}{Z_0^2} - \frac{1}{8\sigma_W^2 S_W} \left( \frac{Z_0^2}{Z_0^2} \right)^2 \\
\times \left( \tilde{\epsilon}_{0-}^2 \left( G_{4-}^2 - \delta_I G_{2-}^2 \right) - \tilde{\epsilon}_{4-}^2 \left( G_{0-}^2 - \delta_I G_{2-}^2 \right) + \Delta \tilde{\delta}^2 \left( 2 - \delta_I^2 \right) \right) \\
- \frac{1}{16\sigma_W^2} \left( \frac{Z_0^4}{Z_0^2} \right)^2 \left[ \Delta \tilde{\delta}^2_{(1)} \epsilon_{2-}^2 + 2 \left| \Delta \tilde{\delta}^2_{(1)} \epsilon_{6-}^2 \right| \right]_{(I^{\text{obj}}, \delta_I)}
\]

\[
\equiv \Delta \tilde{\delta}^2_{(2)} (I^{\text{obj}}, \delta_I).
\]

(E14)

The averaged value of the second-order effect on ellipticity is calculated from

\[
\frac{1}{4} \left[ \Delta \tilde{\delta}^2_{(2)} \epsilon_{0-}^2 + \Delta \tilde{\delta}^2_{(2)} \epsilon_{4-}^2 \right]_{(I^{\text{obj}}, \delta_I)} = \frac{1}{8\sigma_W^2} \left( \frac{Z_0^2}{Z_0^2} \right)^2 \left( \tilde{\epsilon}_{0-}^2 \left( 4 - \delta_I^2 \right) \delta_I + \tilde{\epsilon}_{4-}^2 \left( 2 + \delta_I^2 \right) \delta_I^2 - 3 \tilde{\epsilon}_{0-}^2 \left( 2 - \delta_I^2 \right) \delta_I - 3 \tilde{\epsilon}_{4-}^2 \delta_I^2 \right) \\
+ \frac{1}{2\sigma_W^2} \left( \tilde{\epsilon}_{0-}^2 \left( \frac{1}{2} \tilde{\epsilon}_{0-}^2 \left( 2 + \delta_I^2 \right) \delta_I^2 + \tilde{\epsilon}_{4-}^2 \delta_I^2 \right) \\
- \tilde{\epsilon}_{0-}^2 \left( \tilde{\epsilon}_{4-}^2 \left( 2 - \delta_I^2 \right) \delta_I^2 + \tilde{\epsilon}_{0-}^2 \left( 2 - \delta_I^2 \right) \delta_I^2 \right) \right)_{(I^{\text{obj}}, \delta_I)}
\]

(E15)

and therefore

\[
\Delta \tilde{\delta}^2_{(2)} (I^{\text{obj}}, \delta_I) = 4 \left[ \tilde{\epsilon}_{0-}^2 \left( \tilde{\delta}_{0-}^2 - \tilde{\delta}_{4-}^2 \right) \right]_{(I^{\text{obj}}, \delta_I)} \approx 4 \left[ \tilde{\epsilon}_{0-}^2 \Delta \tilde{\delta}^2_{(2)} - \tilde{\epsilon}_{4-}^2 \Delta \tilde{\delta}^2_{(2)} \right]_{(I^{\text{obj}}, \delta_I)},
\]

(E16)

which is the systematic error in measuring ellipticity using the E-HOLICs method.

The systematic error of complex moments is obtained by adopting Equations (E13) and (E16) as

\[
\Delta Z_{M(2)}^N \approx \left[ \Delta Z_{M(2)}^N + \frac{1}{2\sigma_W^2} \left( \Delta \tilde{\delta}_{(2)}^2 - \tilde{\epsilon}_{0-}^2 - \Delta \tilde{\delta}_{(2)}^2 \right) \right]_{(I^{\text{obj}}, \delta_I)}
\]

(E17)
APPENDIX F
KSB METHOD AND E-HOLICs METHOD WITH ELLIPTICAL GAUSSIAN IMAGE

In this section, we show systematic errors using the KSB method and the E-HOLICs method with an elliptical Gaussian weight and an EGI, analytically.

F.1. The KSB Method with an Elliptical Gaussian Image

Using an EGI and the KSB method, \( \varepsilon_{M \pm}^N(I_{EG}, 0) \) are obtained analytically as

\[
\varepsilon_{M \pm}^N(I_{EG}, 0) = \left( N \pm M + 2 \right) - \frac{2 + M + N}{2} - \frac{1 + N + \delta_i^2}{4} \mathcal{H}_M(I_{EG}, Z_0^N, 0) \tag{F1}
\]

where \( N = 0, 2 \) or \( 4 \) and \( |M| \leq |N| \), but the following equation cannot be used in upper general forms

\[
\varepsilon_{0 \pm}^0(I_{EG}, 0) = - \frac{\delta_i}{2} \tag{F2}
\]

\[
\varepsilon_{0 \pm}^4(I_{EG}, 0) = 3 \left( 1 - \frac{\delta_i^2}{2} \right) \tag{F3}
\]

The CSE scale and ellipticity are obtained analytically from Equations (23) and (24) as

\[
W^2(\Delta \theta, 0) \Delta \theta_0^2(I_{EG}, 0) = \frac{\sigma_w^2}{\text{WSN}^2} \left( 1 + \frac{5}{4} \delta_i^2 \right) \tag{F4}
\]

\[
W^2(\Delta \theta, 0) \Delta \theta_2^2(I_{EG}, 0) = \frac{\sigma_w^2 \delta_i}{\text{WSN}^2} \left( 1 + \frac{3}{2} \delta_i^2 \right) \tag{F5}
\]

\[
\delta_c(I_{EG}, 0) = \delta_i \frac{1 + \frac{3}{2} \delta_i^2}{1 + \frac{5}{4} \delta_i^2} \approx \left( 1 + \frac{1}{4} \delta_i^2 \right) \delta_i \tag{F6}
\]

and the results of the test are shown in Figure 2. The systematic error in the complex moments is obtained from Equation (30) as

\[
\Delta Z_0(2)(I_{EG}, 0) \approx \frac{1}{\text{WSN}^2} \left( \frac{2 + \delta_i^2}{4} \right) Z_0^0(I_{EG}, 0) \tag{F7}
\]

\[
\Delta Z_0^2(2)(I_{EG}, 0) \approx - \frac{1}{\text{WSN}^2} \left( \frac{3}{2} + \frac{3}{4} \delta_i^2 \right) Z_0^2(I_{EG}, 0) \tag{F8}
\]

\[
\Delta Z_2(2)(I_{EG}, 0) \approx - \frac{1}{\text{WSN}^2} \left( \frac{3}{2} + \frac{3}{4} \delta_i^2 \right) Z_2^2(I_{EG}, 0) \tag{F9}
\]

\[
\Delta Z_0^4(2)(I_{EG}, 0) \approx - \frac{1}{\text{WSN}^2} \left( \frac{1}{2} + \frac{3}{4} \delta_i^2 \right) Z_0^4(I_{EG}, 0) \tag{F10}
\]

\[
\Delta Z_2^4(2)(I_{EG}, 0) \approx - \frac{1}{\text{WSN}^2} \left( \frac{7}{4} \delta_i^2 \right) Z_2^4(I_{EG}, 0) \tag{F11}
\]

\[
\Delta Z_4(2)(I_{EG}, 0) \approx - \frac{1}{\text{WSN}^2} \left( \frac{1}{2} + \frac{9}{4} \delta_i^2 \right) Z_4^4(I_{EG}, 0) \tag{F12}
\]
The results of the test are shown in Figures 3–5. Then, Equation (39) can be calculated analytically using:

\[
\Delta \delta(2, I_{\text{obj}}, 0) = \frac{\delta_{\text{obj}}}{\text{WSN}^2} \left( \frac{3}{4} + \frac{1}{8} \delta_i^2 \right).
\] (F13)

The results of the test are shown in Figure 6.

Equation (F13) is written using the KSB method as

\[
\Delta \delta(2, I_{\text{obj}}, 0) = -\frac{\chi}{\text{WSN}^2} \left( \frac{3}{4} + \frac{1}{8} \delta_i^2 \right).
\] (F14)

\[
\chi \equiv \frac{Q_{11} - Q_{22} + 2iQ_{12}}{Q_{11} + Q_{22}}.
\] (F15)

\[
\text{WSN} = \frac{Q}{\sigma_{\text{RCN}} \sqrt{S_w}}.
\] (F16)

\[
S_w = \sigma_w^2 \pi.
\] (F17)

F.2. The E-HOLICs Method Using the True Ellipticity for the Weight Function and an Elliptical Gaussian Image

Using an EGI and the E-HOLICs method with the true ellipticity, \(E_N^M \pm \) are obtained analytically from

\[
E_{N \pm}^M(I_{\text{EG}}, \delta_i) = \frac{N \pm M + 2}{2} \mathcal{H}_{M-2}(I_{\text{EG}}, Z_0^N, \delta_i)
\] (F18)

and

\[
\tilde{E}_{0 \pm}^0(I_{\text{EG}}, \delta_i) = \left[ \frac{E_{0 \pm}^0}{|E_{0 \pm}^0|^2 - |E_{-1 \pm}^0|^2}_{(I_{\text{EG}}, \delta_i)} \right] = 1
\] (F19)

\[
\tilde{E}_{2 \pm}^0(I_{\text{EG}}, \delta_i) = \left[ \frac{E_{2 \pm}^0}{|E_{2 \pm}^0|^2 - |E_{-2 \pm}^0|^2}_{(I_{\text{EG}}, \delta_i)} \right] = 0
\] (F20)

\[
\tilde{E}_{0 \pm}^2(I_{\text{EG}}, \delta_i) = \left[ \frac{E_{0 \pm}^2}{|E_{0 \pm}^2|^2 - |E_{-1 \pm}^2|^2}_{(I_{\text{EG}}, \delta_i)} \right] = 1
\] (F21)

\[
\tilde{E}_{4 \pm}^0(I_{\text{EG}}, \delta_i) = \left[ \frac{E_{4 \pm}^0}{|E_{4 \pm}^0|^2 - |E_{-2 \pm}^0|^2}_{(I_{\text{EG}}, \delta_i)} \right] = 0.
\] (F22)

Then \(\mathcal{F}_M^N\) is calculated from

\[
\mathcal{F}_{M-}^N(I_{\text{EG}}, \delta_i) = \frac{(N + M)(N + M - 2)}{4} \mathcal{H}_{M-2}^N + \frac{2\delta_i^2}{\sigma_w^2} \mathcal{H}_{M-2}^N
\] (F23)

\[
\mathcal{F}_{M+}^N(I_{\text{EG}}, \delta_i) = \frac{(N - M)(N - M - 2)}{4} \mathcal{H}_{M+2}^N + \frac{2\delta_i^2}{\sigma_w^2} \mathcal{H}_{M}^N
\] (F24)

\[
\mathcal{F}_{M \pm}^N(I_{\text{EG}}, \delta_i) = 2 \left[ \frac{(N + M)(N - M)}{4} \mathcal{H}_{M}^{N-2} - \frac{2}{\sigma_w^2} \mathcal{H}_{M}^N \right]
\] (F25)
where \( 2 \leq N \). The average of the CSEs are calculated analytically from Equations (23) and (24) as

\[
\Delta \theta_i^2(I^{EG}, \delta_i) = \frac{\delta_i}{\text{WSN}^2} \frac{\sigma_i^2}{1 - \delta_i^2} 
\]

(F26)

\[
\Delta \theta_2^2(I^{EG}, \delta_i) = \frac{\delta_i}{\text{WSN}^2} \frac{\sigma_i^2}{1 - \delta_i^2} 
\]

(F27)

\[
\delta_C(I^{EG}, \delta_i) = \delta_i(I^{EG}, \delta_i),
\]

(F28)

and the results of the test are shown in Figure 7. The systematic error in measuring complex moments is obtained using Equation (30) as

\[
\Delta Z_0^0(I^{EG}, \delta_i) \approx \frac{1}{2\text{WSN}^2} Z_0^0(I^{EG}, \delta_i)
\]

(F29)

\[
\Delta Z_2^2(I^{EG}, \delta_i) \approx 0
\]

(F30)

\[
\Delta Z_4^4(I^{EG}, \delta_i) \approx 0
\]

(F31)

\[
\Delta Z_2^2(I^{EG}, \delta_i) \approx -\frac{1}{2\text{WSN}^2} Z_2^2(I^{EG}, \delta_i)
\]

(F32)

\[
\Delta Z_4^4(I^{EG}, \delta_i) \approx -\frac{1}{2\text{WSN}^2} Z_4^4(I^{EG}, \delta_i)
\]

(F33)

\[
\delta_C(I^{EG}, \delta_i) = \delta_i(I^{EG}, \delta_i),
\]

(F34)

and the results of the test are shown in Figures 8 and 9. The measured ellipticity is calculated from Equation (39) as

\[
\Delta \delta_{(1)}(I^{obj}, \delta_i) = -\frac{\delta_i}{\text{WSN}^2} \frac{1 - \delta_i^2}{2}
\]

(F35)

and the results of the test are shown in Figure 10.

**F.3. The E-HOLICs Method with the Observed Ellipticity for the Weight Function and an Elliptical Gaussian Image**

The averaged CSEs are approximated by \( \Delta \theta_i^2(I^{EG}, \delta_O) \) as

\[
\Delta \theta_i^2(I^{EG}, \delta_O) \approx \frac{1}{\text{WSN}^2} \frac{\sigma_i^2}{1 - \delta_i^2} 
\]

(F36)

\[
\Delta \theta_2^2(I^{EG}, \delta_O) \approx \frac{1}{\text{WSN}^2} \frac{\sigma_i^2}{1 - \delta_i^2} \delta_i 
\]

(F37)

\[
\delta_C(I^{EG}, \delta_O) = \delta_i(I^{EG}, \delta_O),
\]

(F38)

and the results of the test are shown in Figure 11. The systematic error in measuring the ellipticity Equation (60) is calculated analytically from

\[
\Delta \delta_{(1)}(I^{EG}, \delta_i) = -2\delta_i \frac{(1 - \delta_i^2)}{\text{WSN}^2}
\]

(F39)
Figure 11. Averaged square of the CSE normalized by $\sigma_W^2$ (so $W^2(\Delta \theta, \delta O)\Delta \theta^2/\sigma_W^2$) of EGI and the E-HOLICs method using the observed ellipticity for the weight function ($\delta_W = \delta_O$). The squares (circles) represent $W^2(\Delta \theta, \delta O)\Delta \theta^2/\sigma_W^2$ measured from simulated data, and the solid (dashed) line is the analytic approximation for EGI, i.e., Equation (F36) (Equation (F37)).

Figure 12. SER due to the RCN for the observed ellipticity using EGI and E-HOLICs. WSN is plotted on the horizontal axis and systematic error $\Delta \delta_{(2)}/\delta_I$ is plotted on the vertical axis. The dots with error bars are simulated results and the curve is the analytic approximation for EGI given by Equation (F41).
Figure 13. SER due to the RCN for monopole moments measured with EGI and the E-HOLICs method. WSN is plotted on the horizontal axis and systematic error is plotted on the vertical axis. The dots with error bars are the simulated result of SER of $\Delta Z_0^0$ and the curve is the analytic approximation for EGI given by Equation (F42).

Figure 14. SER due to the RCN for quadrupole moments measured with EGI and the E-HOLICs method. WSN is plotted on the horizontal axis and systematic error is plotted on the vertical axis. The cross (square) dots with error bar are the simulated results of SER $\Delta Z_{02}^0$($\Delta Z_{22}^0$) and the solid (long dash) curve is the analytic approximation given by Equation (F43) (Equation (F44)).
Figure 15. SER and corrected SER in each bin of measured ellipticity of the observed images made from GREAT08 images. The error bars for the SER are shown, but these are too small to see. The ellipticity of the image is plotted on the horizontal axis, and SER is plotted on the vertical axis. The solid lines are the SER of the ellipticity, and we can see that there is about a 2% underestimate. The dashed lines are the corrected SER by Equation (61), and it can be seen that the corrected SER is smaller than 0.01.

\[
|\Delta \tilde{\delta}_{(1)}(I^{EG}, \delta_I)|^2 = (2 - \delta_I^2) \left(1 - \frac{\delta_I^2}{WSN^2}\right).
\]  
\[
\Delta \tilde{\delta}_{(2)}(I^{EG}, \delta_I) = -2 \left(1 - \frac{\delta_I^2}{WSN^2}\right) \delta_I.
\]  

The results of the test are shown in Figure 12. Then systematic error in measuring the complex ellipticity is calculated from Equation (62) as

\[
\Delta \tilde{Z}_{0(2)}^{0}(I^{EG}, \delta_I) \approx \frac{4 - \delta_I^2}{2WSN^2} Z_0^0(I^{EG}, \delta_I)
\]  
\[
\Delta \tilde{Z}_{0(2)}^{2}(I^{EG}, \delta_I) \approx \frac{9 + 3\delta_I^2}{2WSN^2} Z_0^2(I^{EG}, \delta_I)
\]  
\[
\Delta \tilde{Z}_{2(2)}^{2}(I^{EG}, \delta_I) \approx \frac{13 - \delta_I^2}{2WSN^2} Z_2^2(I^{EG}, \delta_I)
\]  
\[
\Delta \tilde{Z}_{0(2)}^{4}(I^{EG}, \delta_I) \approx \frac{17 + 18\delta_I^2}{2WSN^2} Z_0^4(I^{EG}, \delta_I)
\]  
\[
\Delta \tilde{Z}_{2(2)}^{4}(I^{EG}, \delta_I) \approx \frac{23 + 6\delta_I^2}{2WSN^2} Z_2^4(I^{EG}, \delta_I)
\]  
\[
\Delta \tilde{Z}_{4(2)}^{4}(I^{EG}, \delta_I) \approx \frac{31 + 3\delta_I^2}{2WSN^2} Z_4^4(I^{EG}, \delta_I).
\]  

The results of the test are shown in Figures 13 and 14.
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