Stability of Oscillating Gaseous Masses in Massive Brans-Dicke Gravity

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Abstract

This paper explores the instability of gaseous masses for the radial oscillations in post-Newtonian correction of massive Brans-Dicke gravity. For this purpose, we derive linearized perturbed equation of motion through Lagrangian radial perturbation which leads to the condition of marginal stability. We discuss radius of instability of different polytropic structures in terms of the Schwarzschild radius. It is concluded that our results provide a wide range of difference with those in general relativity and Brans-Dicke gravity.

Keywords: Brans-Dicke Theory; Hydrodynamics; Instability; Newtonian and post-Newtonian regimes.

PACS: 04.25.Nx; 04.40.Dg; 04.50.Kd.

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1 Introduction

The study of evolution and formation of stellar structures has been issue of great interest in gravitational physics and cosmology. In this context, the phenomenon of dynamical stability of celestial objects has important implications in the analysis. It is believed that different stability ranges for stellar bodies lead to different phases of evolution or structure formation of the astronomical models. In general relativity (GR), Chandrasekhar [1, 2, 3] was the first who described a mechanism to explain dynamical instability of stellar structure in weak field approximation (at post-Newtonian (pN) limits). He used equation of state involving adiabatic index ($\gamma$) and concluded that the fluid remains unstable for $\gamma < \frac{4}{3}$. Herrera et al. [4] investigated dynamical evolution of self-gravitating fluids in different configurations (anisotropic fluid, adiabatic, non adiabatic as well as shearing viscous fluid). Sharif and his collaborators [5] also explored characteristics of different celestial fluid configurations in weak regimes through stability analysis.

The mystery of accelerating expansion of the universe has taken a remarkable attention in the last decade. In this context, the mechanism of modified theories of gravity has become a fascinated candidate. Modified theory of gravity means theory of gravity followed by modified Einstein-Hilbert action. The viability of these theories is an issue of great importance. For this reason, these theories are tested on different gravitational scales such as strong as well as weak field gravitational regime [6]. In this regard, the evolution and formation of celestial structure are considered to be the most suitable test-beds for modified theories. It is believed that modification of GR introduces some new astrophysical insights which can explain hidden parts of the universe. In this context, a large number of researchers have discussed modified astrophysical analysis [7]. Nutku [8] studied modified fluid hydrodynamics that affects the results of Chandraskhar. Recently, we have discussed modified dynamics of self-gravitating system in both weak and strong fields [9].

Brans-Dicke (BD) gravity (natural generalization of GR) [10] is one of the most explored examples of modified theory which is considered as a solution of many cosmic issues. This theory modifies the Einstein-Hilbert action according the Dirac hypothesis, i.e., it allows dynamical gravitational coupling (converts Newtonian gravitational constant into dynamical one) by means of dynamical massless scalar field ($G = \frac{1}{\phi}$). In this gravity, gravitational effects are described by coupling a massless scalar field $\phi$ with the curvature part
(Ricci scalar). One of the main features of this theory is that it contains a constant tuneable parameter $\omega_{BD}$ which is a coupling constant and can adjust required results. This theory provides suitable solutions of various cosmic problems but remains unable to probe "graceful exist" problem of old inflationary cosmology. The inflationary phenomenon described by BD gravity shows unacceptably large microwave background perturbations (by collisions between big bubbles) which can be controlled with the help of specific values of coupling parameter $\omega_{BD} \leq 25$ [11]. But these defined ranges of parameter are in conflict with observational limits [12].

In order to solve this problem, a massive scalar field is introduced in the framework of BD gravity [13] via a potential function $V(\phi)$. This new gravity is known as massive BD (MBD) gravity or self-interacting BD gravity. Moreover, BD gravity investigates all strong field issues (cosmological issues) for negative and small values of $\omega_{BD}$ [14] but satisfies all weak field tests (related to solar system) for large and positive values of $\omega_{BD}$ [15]. The MBD gravity provides a consistency with weak field gravitational test, i.e., explains cosmic acceleration for positive and large values of $\omega_{BD}$ [16]. There has been a large body of literature which describes dynamics of MBD gravity in many cosmic issues [17, 18]. Olmo [19] calculated pN limits of MBD equations but he converted only lowest-order (order of $c^{-2}$) limits of solutions in terms of potential functions to explore $f(R)$ gravity as a special case of scalar-tensor gravity. Recently, we have explored hydrodynamics of different celestial configurations in complete pN correction of MBD gravity that modify the results of GR and BD gravity [20].

In this paper, we investigate gaseous system in MBD gravity and compare the results with GR and BD gravity. For this purpose, we explore stability of gaseous masses for radial oscillations in weak field approximation of MBD gravity. The paper is organized as follows. The next section represents complete pN approximation of MBD theory in terms of potential as well as super-potential functions and the dynamical equations. Section 3 explores instability of gaseous systems for radial oscillations by means of Lagrangian perturbation and variational principle. In section 4, we evaluate instability conditions of different polytropes in MBD theory. Finally, section 5 summarizes the results.
2 Massive Brans-Dicke Gravity and Dynamical Equations

The action of MBD gravity with \((\kappa^2 = \frac{8\pi}{c^2})\) \([16]\) is given by

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\left[\phi R - \frac{\omega_{BD}}{\phi} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi)\right] + L_m, (1)
\]

where \(L_m\) represents matter distribution depending upon metric. By varying the above action with respect to \(g_{\alpha\beta}\) and \(\phi\), we obtain MBD equations as follows

\[
G_{\alpha\beta} = \frac{\kappa^2}{\phi} T_{\alpha\beta} + \left[\phi_{,\alpha\beta} - g_{\alpha\beta} \Box \phi\right] + \frac{\omega_{BD}}{\phi} \left[\phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\mu} \phi_{,\mu}\right] - \frac{V(\phi)}{2} g_{\alpha\beta},
\]

\[
\Box \phi = \frac{\kappa^2 T}{3 + 2\omega_{BD}} + \frac{1}{3 + 2\omega_{BD}} \left[\phi \frac{dV(\phi)}{d\phi} - 2V(\phi)\right],
\]

where \(T_{\alpha\beta}\) shows the energy-momentum tensor, \(T = g^{\alpha\beta} T_{\alpha\beta}\) and \(\Box\) represents the d’Alembertian operator. Equations \([2]\) and \([3]\) indicate MBD field equations as well as evolution equation for the scalar field, respectively. We assume matter distribution as a perfect fluid which can be compatible with pN regime

\[
T_{\alpha\beta} = [\rho c^2(1 + \frac{\pi}{c^2}) + p] u_{\alpha} u_{\beta} - pg_{\alpha\beta},
\]

where \(\rho, \rho_{\pi}, p, u_\alpha\) indicate density, thermodynamics density, pressure and four velocity, respectively.

2.1 Post-Newtonian Approximation

The weak-field limits of any relativistic theory explain the order of deviations of the local system from its isotropic and homogenous background. The parameterized pN approximations are widely used as weak field approximated solutions that are obtained by using the following Taylor expansion of the metric functions \([21]\)

\[
g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta},
\]

with

\[
h_{00} \approx h_{00}^{(2)} + h_{00}^{(4)}; \quad h_{0i} \approx h_{0i}^{(3)}; \quad h_{ij} \approx h_{ij}^{(2)}.
\]
Here $\eta_{\alpha\beta}$ shows the Minkowski metric (describing isotropic and homogenous background of $g_{\alpha\beta}$), $h_{\alpha\beta}$ indicates deviation of $g_{\alpha\beta}$ from background values ($\eta_{\alpha\beta}$), $i, j = 1, 2, 3$ and the superscripts $(2), (3)$ and $(4)$ describe approximation of order $(c^{-2}), (c^{-3})$ as well as $(c^{-4})$. In this approximation scheme, the field equations are solved formally and the metric functions are expressed as a sequences of pN functions of source variables (source of metric function like matter) coupled to coefficients (pN parameter). These coefficients are based upon the matching conditions between the local system and cosmological models or on other constants of the theory. The pN functions are basically metric potentials which are chosen under reasonable assumption of Poisson’s equations and gauge conditions to have unique solutions according to pN order of correction [21].

In order to discuss stability of gaseous system in MBD gravity and check the compatibility of our results with the analysis of GR [12], we approximate the system in pN limits. For this purpose, we use complete pN approximations (upto order of $(c^{-4})$) of MBD gravity. The parameterized pN limits of MBD solutions has been evaluated by using the following expansion of metric and dynamical scalar field [19, 20]

\[
\begin{align*}
    g_{\alpha\beta} &\approx \eta_{\alpha\beta} + h_{\alpha\beta}, \\
    \phi &\approx \phi_0(t_0) + \varphi^{(2)}(t, x) + \varphi^{(4)}(t, x), \\
    V(\phi) &\approx V_0 + \varphi \frac{dV_0}{d\phi_0} + \varphi^2 \frac{d^2V_0}{d^2\phi_0} + \ldots.
\end{align*}
\]

Here $t_0$ represents time of isotropic and homogenous background of local system. The term $\phi_0 = \phi(t_0)$ shows unperturbed or initial value of scalar field in isotropic and homogenous background of local system which vary very slowly with respect to $t_0$. This implies that the cosmological considerations would allow a slow evolution of $\phi_0$ on cosmological timescales. Since these timescales are much larger than the solar system timescales, so its evolution may be ignored for physical setup in weak-field and it is considered as constant. The term $V_0 = V(\phi_0)$ shows the potential function of scalar field at $t_0$ and $\varphi(t, x)$ is the local deviation of scalar field from $\phi_0$.

The parameterized pN approximations of MBD solutions are given by [20]

\[
g_{ij} \approx (-1 - \frac{2\gamma_{BD}U}{c^2} - \frac{\Lambda_{BD}r^2}{3c^2})\delta_{ij},
\]

\[\text{(5)}\]
\[ g_{00} \approx 1 - \frac{2U}{f(r)c^2} + \frac{1}{2c^4} \left( -2U + \frac{\Lambda_{BD} r^2}{3} \right)^2 - \left( \frac{-2U - A(r)}{3 + 2\omega_{BD} + A(r)} \right)^2 \] 
\[ g_{0i} \approx \frac{1}{c^3} \left( \frac{4U_i}{f(r)} - \frac{1}{2} \frac{\partial^2 \chi}{\partial t \partial x_i} \right). \] (6)

Here \( U = G_{\text{eff}} \frac{M_\odot}{r} (M_\odot = \int d^3 x^* \rho_{\text{sun}}(t, x^*) \) is the Newtonian mass of the sun) is the effective gravitational potential determined by Poisson’s equation

\[ \nabla^2 U = -4\Pi \rho G_{\text{eff}}, \] (8)

where \( G_{\text{eff}} \) indicates the effective gravitational constant (dynamical Newtonian gravitational constant) for massive scalar field defined by \[18, 19\]

\[ G_{\text{eff}} = \frac{\kappa^2}{8\Pi \phi_0} f(r) = \frac{\kappa^2}{8\Pi 0} \left( 1 + \frac{A(r)}{3 + 2\omega_{BD}} \right), \] (9)

where

\[ A(r) = \begin{cases} 
    e^{-m_0 r} & m_0^2 > 0 \\
    \cos(m_0 r) & m_0^2 < 0,
\end{cases} \quad m_0 = \left( \frac{\phi_0 \frac{d^2 V_0}{d\phi_0} - \frac{dV_0}{d\phi_0}}{3 + 2\omega_{BD}} \right)^{1/2}. \]

Here the term \( m_0 \) is the mass of the massive scalar field and “\( r \)” represents scale of experiments and observations. It is actually a distance between two points in the local system and can be used to express radius of configuration (spherical or cylindrical) under consideration. The term \( \gamma_{BD} \) represents the parameterized pN parameter given by

\[ \gamma_{BD} = \frac{3 + 2\omega_{BD} - A(r)}{3 + 2\omega_{BD} + A(r)}. \] (10)

The oscillatory solutions \( A(r) = \cos(m_0 r), \ m_0^2 < 0 \) are unacceptable \[23\]. In this case, the inverse-square law modifies as

\[ \frac{M_\odot}{r^2} \rightarrow \left( 1 + \cos(m_0 r) + (m_0 r) \sin(m_0 r) \right) \frac{M_\odot}{r^2}, \] (11)

and for very light fields (showing long-range interactions), the arguments of cosine and sine are very small in solar system scales \( (m_0 r << 1) \) which
provide \( \cos(m_0 r) \approx 1 \) and \( \sin(m_0 r) \approx 0 \). These approximations lead to usual Newtonian limits up to an irrelevant redefinition of Newtonian Constant. This also yields \( \gamma_{BD} \approx 1/2 \) for \( \omega_{BD} = 0 \) which is observationally unacceptable since \( \gamma_{obs} \approx 1 \). If the scalar interaction is short-range or mid-range, the Newtonian limits would dramatically be modified. In fact, the leading order term is then oscillating, \( \sin(m_0 r) \frac{M}{r} \), and is clearly incompatible with observations. That is why, we consider only the damped solutions \( A(r) = e^{-m_0 r} \), \( m_0^2 > 0 \).

The Yukawa-type correction in the Newtonian potential has not been observed over distances that range from meters to planetary scales. In addition, since the post-Newtonian parameter \( \gamma_{BD} \) is observationally very close to unity, the mass function present in Eqs. (10) and (11) satisfy the constraint \( m_0 >> \frac{1}{r} \) (\( r \) shows the scale of the observations or experiments testing the scalar field). For solar system scale observations, the relevant scale is the Astronomical Unit (\( r \approx AU \approx 10^{-8} \text{km} \)) corresponding to a mass scale \( m_{AU} \approx 10^{-27} \text{GeV} \). Although this scale is small for particle physics considerations, but it is still much larger than the Hubble mass scale \( m_{H0} \approx 10^{-42} \text{GeV} \) required for nontrivial cosmological evolution of \( \phi \). Current solar system constraints [21, 23] of the parameter \( \omega_{BD} \) have been obtained under one of the following assumptions [19, 18]

- When the background value of \( m_0 \) is very small (\( m_0 << \frac{1}{r} \)) (negligible mass of the field) and \( m_0 << m_{AU} \), MBD system reduces to simple BD gravity (massive scalar field becomes massless scalar field) having
  \[
  G_{eff} = \frac{\kappa^2}{8\Pi \phi_0} \left( 4 + 2\omega_{BD} \right), \quad \gamma_{BD} = \frac{1 + \omega_{BD}}{2 + \omega_{BD}}.
  \]
  That is why the BD theory (massless scalar field) is consistent with solar system constraints of the Cassini mission for \( \omega_{BD} > 40000 \).

- For \( m = 0 \) and \( m_0 \approx m_{AU} \approx 10^{-27} \text{GeV} \), the observational constraints on \( \omega_{BD} \) are same as discussed for the case \( m_0 << m_{AU} \).

- For massive scalar field (\( m_0 >> m_{AU} \) and \( m_0 >> \frac{1}{r} \)), the dynamics of the spatial part of \( \phi \) is frozen on the solar system scale through potential function of scalar field and all values of \( \omega_{BD} \) are observationally acceptable [27]. It can be noticed that further limit (\( \omega_{BD} \to \infty \)) reduces the value of \( G_{eff} \) to simple Newtonian gravitational constant \( G \) and \( \gamma_{BD} = 1 \) which is consistent with GR.
For $m_0 \gtrsim 200 m_{AU}$, all values of $\omega_{BD} > -\frac{3}{2}$ are observationally allowed.

The term $\frac{\Lambda_{BD}}{3c^2} = \frac{V_0}{6\phi_0 c^2}$ indicates the cosmological term (where $\Lambda_{BD}$ is a cosmological constant) which is based on the potential of the scalar field. In order to be consistent with observational data (ranging from the solar system to clusters of stellar structures), the contribution due to scalar density should be very small and the following constraint must be satisfied

$$\frac{V_0 L^2}{\phi_0} << 1.$$ 

Here $L$ shows the length scale equal to or greater than the solar system. The term $(\Phi + \psi)$ represents super-potential $\tilde{\Phi}$ given by the following Poisson’s equations \[12\]

$$\nabla^2 \tilde{\Phi} = -4\Pi G_{\text{eff}} \rho \sigma, \quad \tilde{\Phi} = \Phi + 2\psi,$$

$$\nabla^2 \psi = -\frac{1}{2\phi_0} \left[ V_0 (1 + h_{[ij]}^{(2)} - \frac{\varphi^{(2)}}{\phi_0}) + \varphi^{(2)} \frac{dV_0}{d\phi_0} \right],$$

here

$$\sigma = \frac{1}{f(r)} \left[ \pi + 2v^2 + h_{[ij]}^{(2)} - \frac{\varphi^{(2)}}{\phi_0} + \frac{3p}{\rho} \right], \quad \nabla^2 \Phi = -4\Pi G \tilde{\sigma}, \quad \tilde{\sigma} = f(r)\sigma.$$ 

Similarly $\chi$ and $U_i$ are potential functions satisfying the following Poisson’s equations

$$\nabla^2 \chi = h_{00}^{(2)} = \frac{1}{c^2} (-2U + \frac{\Lambda_{BD} r^2}{3}),$$

$$\nabla^2 \left( \frac{U_i}{f(r)} \right) = -4\Pi G_{\text{eff}} \frac{\rho v_i}{f(r)},$$

where $\nabla^2 U_i = -4\Pi G \rho v_i \ [11]$. The effect of $\phi_0$ is taken approximately constant and the effects of $\dot{\phi}_0$ as well as $\ddot{\phi}_0$ are neglected. The solutions satisfy the following gauge condition

$$h_{\mu,\alpha}^{\alpha} - \frac{1}{2} h_{\alpha,\mu}^{\alpha} - \frac{1}{c^2 \phi_0} \frac{\partial \varphi}{\partial x_{\mu}} = 0.$$ 

All the assumptions and solutions are consistent with BD gravity in the limits ($m_0 << \frac{1}{r}$), $\frac{V_0}{\phi_0} \rightarrow 0 \ [8]$ and the system reduces to GR with $\omega_{BD} \rightarrow \infty \ [11]$. 

8
2.2 Hydrodynamics

According to pN approximation of MBD theory, the equation of continuity and equation of motion (generalized Euler equation of Newtonian hydrodynamics) are obtained using

\[ T_{ij} = 0. \]  

(17)

From Eqs. (6)-(17), the equation of continuity is given by [1, 8, 20]

\[ \frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\tilde{\rho} v_i) = 0, \]

where

\[ \tilde{\rho} = \rho \left( 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 - \frac{\Lambda_{BD} r^2}{3} + \frac{9 + 6 \omega_{BD} - e^{-n_{or}}}{3 + 2 \omega_{BD} + e^{-n_{or}} U} \right) \right). \]  

(18)

This shows that the mass function indicated by density \( \tilde{\rho} \) remains conserved. The spatial components of Eq. (17) provide the equation of motion is given by [20]

\[
\begin{align*}
\frac{\partial \eta v_i}{\partial t} + \frac{\partial \eta v_i v_j}{\partial x_j} &+ \frac{\partial}{\partial x_i} \left[ \left( 1 + 2 \gamma_{BD} U + \frac{\Lambda_{BD} r^2}{3} \right) p \right] + \frac{2 \rho}{c^2} \frac{d}{dt} [(2 \gamma_{BD} U + \frac{\Lambda_{BD} r^2}{3}) v_i] - 4 \rho \frac{d}{c^2} \frac{dt}{f(r)} \left[ U_i \right] - \frac{\rho}{c^2} \left[ f(r) \sigma \frac{\partial}{\partial x_i} \left( \frac{U}{f(r)} \right) + \frac{\partial \Phi}{\partial x_i} \right] \\
&- 4 \rho \frac{d}{c^2} \frac{dt}{f(r)} \left[ U_j \right] - \frac{\rho}{c^2} \frac{d}{dt} \left[ \frac{U}{f(r)} \right] + \frac{\rho}{2c^2} \frac{d}{dt} (U_i - U_{\alpha;\alpha}) \\
&- \frac{\rho}{2c^2} W_i + \frac{\rho}{2c^2} Z_i(BD) = 0,
\end{align*}
\]

(19)

where \( \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \) represents material derivative and

\[ \frac{\partial^3 \chi}{\partial t \partial \mathbf{x}_i} = \frac{d}{dt} (U_i - U_{\alpha;\alpha}), \]

\[ \eta = \rho \left( 1 + \frac{1}{c^2} (v^2 + 2U - \frac{2 \Lambda_{BD} r^2}{3} + \pi + \frac{p}{\rho}) \right), \]

the potential functions \( U_{\alpha;\alpha}, W_i \) and \( Z_i(BD) \) are given in Appendix A.
3 Dynamical Stability of Gaseous Masses

To discuss stability of gaseous masses in the presence of massive scalar field, we use Chandrasekhar technique [2] which has been also used to explain stability of gaseous system in BD gravity [8]. For this, we assume that initially the spherically symmetric distribution of matter field is in complete hydrostatic equilibrium. Using Eq.(19), the hydrostatic condition is given by

\[
\left(1 + 2\gamma_{BD} \frac{U}{c^2} + \frac{\Lambda_{BD} r^2}{3c^2}\right) \frac{\partial p}{\partial x_i} = \frac{\rho}{c^2} \left[f(r)\sigma \frac{\partial}{\partial x_i} \left(\frac{U}{f(r)}\right) + \frac{\partial \Phi}{\partial x_i}\right] - \frac{\rho}{c^2} \times \frac{\partial}{\partial x_i} \left(\frac{U}{f(r)}\right) - p \frac{\partial}{\partial x_i} \left(1 + 2\gamma_{BD} \frac{U}{c^2} + \frac{\Lambda_{BD} r^2}{3c^2}\right) - \frac{\rho}{2c^2} Z^h_{i(BD)},
\]

where \(Z^h_{i(BD)}\) represents hydrostatic case of \(Z_i(BD)\) and its value is mentioned in Appendix A. In hydrostatic equilibrium, the values of \(\sigma\) and density term \(\tilde{\rho}\) are free from velocity term \(v^2\).

3.1 Lagrangian Perturbation and Oscillations

In order to discuss stability of oscillating MBD fluid, we consider that the fluid is flowing according to Lagrangian description. In Lagrangian description of fluid flow, the spatial reference system is comoving with the fluid. The position of the particle (depending upon spatial coordinates) is not an independent variable and the material derivative reduces to simple partial derivative of time at specific constant position [25]. We assume that the system is initially in hydrostatic configuration. Then after certain time, the equilibrium configuration of the system is slightly perturbed such that the spherically symmetric distribution remains unchanged. The perturbed state is obtained by the following Lagrangian displacement [2,8]

\[\tilde{\xi} e^{i\alpha t},\]

where \(\tilde{\xi}\) is a displacement vector defined by \(\tilde{\xi} = \tilde{\mathbf{x}} - \mathbf{x},\) (\(\tilde{\mathbf{x}}, \mathbf{x}\) representing position vector of Lagrangian particles from their initial position at time \(t\)). The term \(\alpha\) shows the characteristic frequency of oscillations. In order to determine frequency of the oscillations, we evaluate linearized Lagrangian form of Eq.(19) (which governs small oscillations about the equilibrium) by
using lagrangian perturbation and Eq. [20] as follows
\[ \alpha^2 \left\{ \eta \xi_i + \frac{2}{c^2} \left( 2\gamma_{BD} \frac{U}{c^2} + \frac{\Lambda_{BD} r^2}{3c^2} \right) \xi_i - \frac{2}{f(r)} U_i \right\} \]
\[ + \frac{\rho}{c^2} (U_i - U_{\alpha\alpha}) \right\} = -\frac{\partial}{\partial x_i} \left[ (1 + 2\gamma_{BD} \frac{U}{c^2} + \frac{\Lambda_{BD} r^2}{3c^2}) \Delta p + 2\rho_{\gamma_{BD}} \Delta U \right] \]
\[ + \frac{\rho}{c^2} \left[ f(r)\sigma \frac{\partial}{\partial x_i} \left( \Delta U \right) - f(r)\Delta \sigma \frac{\partial}{\partial x_i} \left( \frac{U}{f(r)} \right) \right] \]
\[ - \frac{\partial}{\partial x_i} \Delta \tilde{\Phi} - \frac{\Delta \rho}{\rho} \frac{\partial}{\partial x_i} \left[ (1 + 2\gamma_{BD} \frac{U}{c^2} + \frac{\Lambda_{BD} r^2}{3c^2}) \rho \right] \]
\[ + \frac{\rho}{c^2} \frac{\partial}{\partial x_i} \left( \frac{\Delta U}{f(r)} \right) + \frac{\rho}{c^2} \xi_i \Delta Z_{i(BD)}. \]  
(21)

Here \( v_i \) is converted into \( i\alpha e^{iat} \xi_i \) and \( \frac{d}{dt} \) becomes \( \frac{\partial}{\partial t} \). The terms \( \Delta \rho, \Delta p, \Delta \sigma, \Delta U, \Delta Z_{i(BD)} \) and \( \Delta \tilde{\Phi} \) denote the Lagrangian changes in the respective quantities. The value of \( v_i \) is replaced by \( \xi_i \) in the definition of \( U_i \) as well as in \( U_{\alpha\alpha} \) and \( \Delta Z_{i(BD)} \) is expressed in appendix A.

Now we express the Lagrangian changes of various dynamical quantities in terms of \( \tilde{\xi} \). Under Lagrangian perturbation, the equation of continuity becomes
\[ \Delta \tilde{\rho} = -\tilde{\rho} \nabla \tilde{\xi}. \]  
(22)

Equations (18) and (22) imply
\[ \Delta \tilde{\rho} = \Delta \rho \left( 1 + \frac{1}{c^2} \left( \frac{\Lambda_{BD} r^2}{3} + \frac{9 + 6\omega_{BD} e^{-mor}}{3 + 2\omega_{BD} + e^{-mor}} U \right) \right) + \frac{\rho}{c^2} \]
\[ \times \frac{9 + 6\omega_{BD} e^{-mor}}{3 + 2\omega_{BD} + e^{-mor}} \Delta U = -\rho \left( 1 + \frac{1}{c^2} \left( -\frac{\Lambda_{BD} r^2}{3} \right) \right) \nabla \tilde{\xi}. \]

From the above equation, the explicit expressions of Lagrangian change in density can be evaluated in terms of \( \xi \) as \[ 2, 8 \]
\[ \Delta \rho = -\rho \left( \nabla \xi + \frac{9 + 6\omega_{BD} e^{-mor}}{3 + 2\omega_{BD} + e^{-mor}} \Delta U \right), \]  
(23)

where only linear terms of \( U \) and \( \Delta U \) are considered. The definition of adiabatic index (\( \gamma \)) and the relation
\[ d\pi = \frac{p}{\rho^2} d\rho, \]  
(24)
yield
\[ \Delta p = \gamma \frac{p}{\rho} \Delta \rho, \quad \rho \Delta \pi = \frac{p}{\rho} \Delta \rho. \] (25)

Equations (23) and (25) give
\[ \Delta p = -\gamma p \left( \nabla \tilde{\xi} + \frac{9 + 6\omega_{BD} - e^{-m_0r}}{3 + 2\omega_{BD} + e^{-m_0r}} \Delta U \right), \] (26)
\[ \rho \Delta \pi = -p \left( \nabla \tilde{\xi} + \frac{9 + 6\omega_{BD} - e^{-m_0r}}{3 + 2\omega_{BD} + e^{-m_0r}} \Delta U \right). \] (27)

From Eqs. (14) and (23)-(27), it follows that
\[ \Delta \sigma = \frac{1}{f(r)} \left( \frac{e^{-m_0r}}{3 + 2\omega_{BD} + e^{-m_0r}} \Delta U \right. \\
- \left. \frac{p}{\rho} (3\gamma - 2) \left( \nabla \tilde{\xi} + \frac{1}{c^2} \frac{9 + 6\omega_{BD} - e^{-m_0r}}{3 + 2\omega_{BD} + e^{-m_0r}} \Delta U \right) \right), \] (28)
where only linear terms of \( \xi_i \) are considered. In order to obtain explicit expressions of \( \Delta U \) and \( \Delta \tilde{\Phi} \) in terms of \( \tilde{\xi} \), we use relation between Eulerian and Lagrangian changes given by [2, 8]
\[ \Delta U = \delta U + \tilde{\xi} \nabla U, \quad \Delta \tilde{\Phi} = \delta \tilde{\Phi} + \tilde{\xi} \nabla \tilde{\Phi}. \] (29)

Here \( \delta U \) and \( \delta \tilde{\Phi} \) represent Eulerian changes in the respective quantities that can be calculated from Eqs. (8) and (12) as follows
\[ \nabla^2 \delta U = -4\Pi G_{\text{eff}} \delta \rho, \quad \nabla^2 \delta \tilde{\Phi} = -4\Pi G_{\text{eff}} \delta (\rho \sigma). \]

Integration of the above equation gives [2, 8]
\[ \delta U = \int_v G(\text{eff}) \rho(\tilde{x}) \sigma(\tilde{x}) \xi_i(\tilde{x}) \frac{\partial}{\partial x_i} \frac{1}{|x - \tilde{x}|} d\tilde{x} \]
\[ - \frac{6}{c^2} \int_v \frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0r}} G(\text{eff}) \rho(\tilde{x}) \Delta U(\tilde{x}) \frac{1}{|x - \tilde{x}|} d\tilde{x}, \] (30)
\[ \delta \tilde{\Phi} = \int_v G(\text{eff}) \rho(\tilde{x}) \sigma(\tilde{x}) \xi_i(\tilde{x}) \frac{\partial}{\partial x_i} \frac{1}{|x - \tilde{x}|} d\tilde{x} \]
\[ - \int_v G(\text{eff}) \rho(\tilde{x}) \Delta \tilde{\Phi}(\tilde{x}) \frac{1}{|x - \tilde{x}|} d\tilde{x}. \] (31)
Equations (29)-(31) provide

\[ \Delta U = \bar{\xi} \nabla U + \int_v G_{(\text{eff})} \rho(\bar{x}) \xi_i(\bar{x}) \frac{1}{|x - \bar{x}|} d\bar{x} - \frac{6}{c^2} \int_v \frac{3 + 2\omega_{BD}}{3 + 2\omega_{BD} + e^{-m_0 r}} G_{(\text{eff})} \rho(\bar{x}) \Delta U(\bar{x}) \frac{1}{|x - \bar{x}|} d\bar{x}, \]  

(32)

\[ \Delta \tilde{\Phi} = \bar{\xi} \nabla \tilde{\Phi} + \int_v G_{(\text{eff})} \rho(\bar{x}) \sigma(\bar{x}) \xi_i(\bar{x}) \frac{1}{|x - \bar{x}|} d\bar{x} - \int_v G_{(\text{eff})} \rho(\bar{x}) \Delta \tilde{\Phi}(\bar{x}) \frac{1}{|x - \bar{x}|} d\bar{x}. \]  

(33)

With the help of Eqs. (23)-(33), Eq. (21) can be expressed explicitly in terms of \( \bar{\xi} \).

### 3.2 The Variational Principle

The stability criteria of oscillating body depends upon the behavior of frequency. For \( \alpha^2 = 0 \), the system becomes marginally stable, i.e., the model will expand and contract with homologous property. Therefore, in order to discuss the behavior of frequency, we use variational principle with the help of Eq. (21). For this purpose, we assume that on the boundary \( (r = R) \), \( \Delta p = 0 \) and at the origin \( r = 0 \), each quantity is nonsingular [2, 3]. In this way, Eq. (21) along with boundary conditions represent a self-adjoint characteristic value problem for \( \alpha^2 \). Thus a variational base is obtained by converting Eq. (21) into \( \bar{\xi} \) and then integrating over the configuration of fluid by contracting with \( \xi_i \) [2]. The resulting equation becomes

\[
Q\alpha^2 = \int_v (\nabla \bar{\xi}) p \left[ \left( 1 + 2\gamma_{BD} \frac{U}{c^2} + \frac{\Lambda_{BD} r^2}{3c^2} \right) \frac{9 + 6\omega_{BD} - e^{-m_0 r} \Delta U}{3 + 2\omega_{BD} + e^{-m_0 r}} \right. \\
+ \left. \gamma \nabla \bar{\xi} + 2\rho \gamma_{BD} \Delta U \right] dx + \frac{\rho}{c^2} \int_v f(r) \sigma \xi_i \frac{\partial}{\partial x_i} \left( \frac{\Delta U}{f(r)} \right) - f(r) \Delta \sigma \\
\times \xi_i \frac{\partial}{\partial x_i} \left( \frac{U}{f(r)} \right) + \xi_i \frac{\partial \tilde{\Phi}}{\partial x_i} \right] dx + \frac{\rho}{c^2} \int_v \left[ (\nabla \bar{\xi}) + \frac{9 + 6\omega_{BD} - e^{-m_0 r} \Delta U}{3 + 2\omega_{BD} + e^{-m_0 r}} \Delta \sigma \right. \\
\times f(r) \xi_i \frac{\partial}{\partial x_i} \left( 1 + 2\gamma_{BD} \frac{U}{3} + \frac{\Lambda_{BD} r^2}{3} \right) p \right) dx + \frac{\rho}{c^2} \int_v \xi_i \frac{\partial}{\partial x_i} \\
\times \left( \frac{\Delta U}{f(r)} \right) dx - \frac{2\rho}{c^2} \int_v \xi_i \frac{\partial}{\partial x_i} \left( \frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} \Delta U \right) dx.
\]
\[ Q \alpha^2 = \alpha^2 \left\{ \int_v \eta |\bar{\xi}|^2 d\mathbf{x} + \frac{\Lambda_{BD} r^2}{3} |\bar{\xi}|^2 d\mathbf{x} + \int_v \int_v G(\text{eff}) \rho(x) \rho(\bar{x}) \right. \]
\[ \times \left[ \frac{|\bar{\xi}(x) - \bar{\xi}(\bar{x})|}{|x - \bar{x}|} d\mathbf{x} d\bar{\mathbf{x}} \right] \left[ 2\gamma_{BD} - \frac{4(3 + 2\omega_{BD})}{3 + 2\omega_{BD} + e^{-m_0 r}} \right] \]
\[ - \int_v \int_v \frac{G(\text{eff})}{2} \rho(x) \rho(\bar{x}) \frac{|\bar{\xi}(x), (x - \bar{x})|}{|x - \bar{x}|} d\mathbf{x} d\bar{\mathbf{x}} \right\} , \]

where \( Q \) represents positive-definite quantity.

### 3.3 The Onset of Instability for the Radial Oscillations in the Post-Newtonian Approximation

Here, we discuss the criteria for the onset of dynamical instability in pN limits of MBD gravity. For this purpose, we consider radial oscillations having density as well as pressure distribution in the equilibrium conditions. According to definitions of vector spherical harmonics in radial oscillations, the Lagrangian displacement turns out to be

\[ \xi_r = r \bar{\eta}, \quad \xi_\perp = 0, \quad \xi_\theta = 0, \]

where \( \bar{\eta} \) is an unknown function. The radial components of \( \Delta U \), \( \Delta \bar{\Phi} \) and \( \Delta \sigma \) can be obtained from Eqs.(28), (29) and (36) as follows

\[ \Delta \sigma = \frac{1}{f(r)} \left[ \frac{e^{-m_0 r}}{3 + 2\omega_{BD} + e^{-m_0 r}} \Delta U \right]. \]
\[ -\frac{p}{\rho}(3\gamma - 2) \left( \frac{d}{dr}(r^3\tilde{\eta}) \right) \] + O(c^{-2}), \quad (37)

\[ \Delta U = \delta U + r\tilde{\eta}\frac{dU}{dr}, \quad \Delta \tilde{\Phi} = \delta \tilde{\Phi} + r\tilde{\eta}\frac{d\tilde{\Phi}}{dr}. \quad (38)\]

Here the values of $\delta U$ and $\delta \tilde{\Phi}$ for radial oscillations are given by [2]

\[ \delta U = 4\Pi G_{(e,f)} \left[ \int_r^R \rho(s)s\tilde{\eta}ds - 3 \left( \frac{1}{r} \int_0^r \rho(s)\Delta U(s)s^2ds \right) \right] + \int_r^R \rho(s)\Delta U(s)sds, \quad (39)\]

\[ \delta \tilde{\Phi} = 4\Pi G_{(e,f)} \left[ -\int_r^R \rho(s)\sigma ds + \left( \frac{1}{\gamma} \int_0^r \rho(s)\Delta \sigma(s)s^2ds \right) \right] + \int_r^R \rho(s)\Delta \sigma(s)sds. \quad (40)\]

Using Eqs. (20), (36)-(38) and boundary conditions, Eq. (41) simplifies to

\[ Q\alpha^2 = \int_0^R p \left[ 1 + 2\gamma_{BD} \frac{U}{c^2} + \Lambda_{BD}r^2 \right] \left[ \gamma r^4 \left( \frac{d\tilde{\eta}}{dr} \right)^2 + (3\gamma - 4) \frac{d}{dr}(r^3\tilde{\eta}^2) \right] dr \]

\[ + \frac{1}{c^2} \left\{ \int_0^R \rho(|\Delta U|)^2r^2dr + 2 \int_0^R \left( \frac{9 + 6\omega_{BD} - e^{-mar}}{3 + 2\omega_{BD} + e^{-mar}} - 2\gamma_{BD} \right) \right. \]

\[ \times p\Delta U \frac{d}{dr}(r^3\tilde{\eta})dr \]. \quad (41)\]

The condition for marginal stability will be derived from the above equation by setting $\alpha^2 = 0$. In particular, for $\alpha^2 = 0$, $\gamma = constant = 4/3$ and Newtonian limits (order less than $c^{-2}$) of equilibrium condition, Eq. (41) implies that $\tilde{\eta} = constant$ (as a solution of the respective equation). This implies that in Newtonian approximation, the marginal stability is obtained for $\gamma - 4/3 = 0$ and $\tilde{\eta} = constant$. Accordingly, in pN approximation, this leads to

\[ \gamma - 4/3 = O(c^{-2}) \quad \tilde{\eta} = constant + O(c^{-2}). \quad (42)\]

Consequently, from Eq. (41) the condition of marginal stability in pN limits is given by

\[ (3\gamma - 4) \int_0^R p \left( 1 + 2\gamma_{BD} \frac{U}{c^2} + \Lambda_{BD}r^2 \right) \frac{d}{dr}(r^3\tilde{\eta}^2)dr \]
\[-\frac{1}{c^2} \left( \int_0^R \rho (\Delta U)^2 r^2 dr + \frac{2}{3} \int_0^R \frac{15 + 10 \omega_{BD} - e^{-m_0 r}}{3 + 2 \omega_{BD} + e^{-m_0 r}} p(\Delta U) r^2 dr \right) \times p(\Delta U) r^2 dr \],

(43)

where terms up to \(O(c^{-4})\) are neglected and \(\Delta U\) is approximated as

\[ \Delta U = -4\Pi G_{(eff)} \int_r^R \rho s ds + r dU dr. \]

This equation up to \(O(c^{-2})\) is given by

\[ 9(\gamma - 4/3) \int_0^R pr^2 dr = -\frac{1}{c^2} \left( \int_0^R \rho (\Delta U)^2 r^2 dr \right) + \frac{2}{3} \int_0^R \frac{15 + 10 \omega_{BD} - e^{-m_0 r}}{3 + 2 \omega_{BD} + e^{-m_0 r}} p(\Delta U) r^2 dr \]

(44)

From the definition of mass function \(M\) and gravitational potential energy \(W\) in equilibrium configuration [2], the above equation turns out to be

\[ (\gamma - 4/3) = \frac{1}{3c^2 W} \left( \int_0^R (\Delta U)^2 dM \right) + \frac{2}{3} \int_0^R \frac{15 + 10 \omega_{BD} - e^{-m_0 r}}{3 + 2 \omega_{BD} + e^{-m_0 r}} p(\Delta U) dM \]

where \(dM = 4\Pi \rho r^2 dr\) and \(W = -12\Pi \int_0^R pr^2 dr\). This equation describes criteria for the onset of dynamical instability of gaseous masses in the pN limits of MBD gravity that involve no information about the equilibrium condition beyond the Newtonian framework. Notice that this derivation is for a special case \(\gamma = constant\) and the defined criteria depend upon \(\omega_{BD}\) as well as mass function \(m_0\). In the limits \((m_0 << \frac{1}{r})\) and \(\frac{v_0}{c_0} \rightarrow 0\), the system reduces to simple BD, whereas within the limits \((m_0 << \frac{1}{r})\), \(\frac{v_0}{c_0} \rightarrow 0\) and \(\omega_{BD} \rightarrow \infty\), the above equation becomes consistent with GR. The instability criteria obtained in Newtonian limits are the same as described by theories of GR and BD. However, in pN limits, the resulting criteria are changed due to the last term of the above equation.

4 Dynamical Instability of Polytropes

Polytropes being self-gravitating spheres represent an approximation of more relativistic stellar models [26]. In order to obtain criteria for the onset of
dynamical instability of polytropes in massive gravity, we convert all the quantities \((r, \rho, p)\) and \(\Delta U\) into standard Emden variables \((\varepsilon, \theta)\) defined by

\[
r = \beta \varepsilon, \quad \rho = \rho_c \theta^n, \quad p = p_c \theta^{n+1},
\]

where \(\beta, \rho_c, p_c\) represent a scale length, central density, central pressure and \(\theta^n = \theta^n(\varepsilon)\) is a Lane Emden function with \(n\) as a polytropic index. Under these conditions, Eq.(44) becomes \([2, 8]\)

\[
(\gamma - 4/3) = -\frac{2G_{(\text{eff})} M}{Rc^2} \frac{(5 - n)}{18(n + 1)\varepsilon_1^4|\theta_1'|^3} \left\{ (n + 1) \int_0^{\varepsilon_1} \theta^n(\Delta U(\varepsilon))^2 \varepsilon^2 d\varepsilon \\
+ \frac{2}{3} \int_0^{\varepsilon_1} \frac{15 + 10\omega_{BD} - e^{-m_0\beta \varepsilon}}{3 + 2\omega_{BD} + e^{-m_0\beta \varepsilon}} \theta^{n+1} \Delta U(\varepsilon) \varepsilon^2 d\varepsilon \right\}. \tag{45}
\]

Here \(\varepsilon_1\) shows the first zero of \(\theta^n\), \(\theta_1'\) represents the value of first derivative of \(\theta^n\) at \(\varepsilon_1\). The above equation describes conditions for marginal stability of polytropes in MBD gravity, which depend upon values of \(\omega_{BD}\) and \(m_0\). In order to analyze some results of physical interest, we apply approximation scheme (use series solutions of exponential and Lane-Emden function) on the last term of the above equation and use

\[
\Delta U(\varepsilon) = -\int_\varepsilon^{\varepsilon_1} \theta^n \varepsilon d\varepsilon + \varepsilon \frac{d\theta}{d\varepsilon} = -(\theta + \varepsilon_1|\theta_1'|).
\]

Equation \([45]\) provides the resultant conditions for marginal stability (or criteria of onset of instability) as follows

\[
R = \frac{K}{\gamma - 4/3} R_s, \tag{46}
\]

where \(R_s = \frac{2G_{(\text{eff})} M}{c^2}\) is the Schwarzschild radius and \(K\) is the constant term given by

\[
K = \frac{5 - n}{18(n + 1)\varepsilon_1^4|\theta_1'|^3} \left\{ 2(11 - n) \int_0^{\varepsilon_1} \theta \left( \frac{d\theta}{d\varepsilon} \right)^2 \varepsilon^2 d\varepsilon + 1 \right\} - 26(1 + \omega_{BD}) \\
\times \int_0^{\varepsilon_1} \theta^{n+2} \varepsilon^2 d\varepsilon - 26(1 + \omega_{BD}) \varepsilon_1^2 \theta_1' \left[ \int_0^{\varepsilon_1} \theta^{n+1} \varepsilon^2 d\varepsilon - (39 + 26\omega_{BD}) m_0 \beta \right] \\
\times \left( \int_0^{\varepsilon_1} \theta^{n+1} \varepsilon^3 d\varepsilon + \int_0^{\varepsilon_1} \theta^{n+2} \varepsilon^3 d\varepsilon \right). \tag{47}
\]
Equation (46) shows radius of the system where it becomes unstable or equivalently the system becomes unstable if the mass of the system is contracted to radius $R$. If the radius of gaseous mass is greater than $R$, it remains stable in MBD gravity. Since the obtained radius of instability is a factor of the Schwarzschild radius, so the ratio $R/R_s$ should be greater than or equal to zero for real and physical results.

It can be noticed that the instability analysis depends upon five parameters. Equation (46) describing radius of instability depends upon the values of $K$ and adiabatic index $\gamma$. The value of $K$ (given in Eq.(47)) in turn depends upon the polytropic index $n$, Lane-Emden function $\beta$, tuneable parameter $\omega_{BD}$ and mass function $m_0$. In order to avoid complexity, we use fixed values of some parameters. Literature shows that the instability criteria in GR and BD gravity usually depend upon adiabatic index $\gamma$ and $\omega_{BD}$, hence we cannot fix their values. Moreover, in the case of MBD gravity, the behavior of mass function on the instability criteria is also of great importance. It has been shown that for $m_0 \gtrsim 200m_{AU} = 200 \times 10^{-27}$, all values of $\omega_{BD} > -\frac{3}{2}$ represent observationally allowed regions [18].

Different polytropic indices lead to different stellar structures out of which configurations defined for $n = 0, 1, 1.5, 2, 3$ and $n < 5$ are considered to be realistic stars [26]. Thus, firstly we evaluate values of radius of instability by calculating $R/R_s$ for different polytropic indices ($n = 0, 1, 1.5, 2, 3, 5$) as well as $n > 5$ with fixed value of mass function $m_0 = 200m_{AU} = 200 \times 10^{-27}$ and $\beta = 1$. In this way, the resulting instability criteria will depend upon $\omega_{BD}$ as well as adiabatic index $\gamma$ and it can be easily comparable with GR [2] and BD [8] theories. Secondly, we choose a fixed value of $\omega_{BD}$ (from the extracted instability criteria) along with $\beta = 1$ and find behavior of increasing mass $m_0 \gtrsim 200m_{AU} = 200 \times 10^{-27}$ on instability analysis for polytropes.

4.1 Polytropes for $n = 0$

Polytropic structures for $n = 0$ represent incompressible configurations in which density remains constant throughout the surface and pressure varnishes at the surface of stellar structure. The ranges of instability for this type of star are shown in Figure 1. It can be observed that for $-1.5 \leq \omega_{BD} \leq 0.5$, $\gamma > 4/3$, the obtained radius is approximately 10 times more than $R_s$. The values $0.5 < \omega_{BD}$, $0.5 < \gamma \leq 4/3$ provide valid radii ranges while $\gamma \geq 4/3$ implies un-physical results. In GR and BD gravity, for $n = 0$, $\gamma > 4/3$, the obtained radii are $R \approx 678.57R_s$ and $R \approx 660.70R_s$, respectively. Thus for
Figure 1: The ratio of radius of instability and Schwarzschild radius “$R/R_s = K/(\gamma - 4/3)$” is plotted against ($-1 \leq \gamma \leq 2$), ($-1.5 \leq \omega_{BD} \leq 1$) for $n = 0$, $\beta = 1$ and $m_0 = 200m_{AU} = 200 \times 10^{-27}$.

$\gamma > 4/3$, the system collapses earlier in MBD gravity than that in GR and BD gravity.

### 4.2 Polytropes for $n = 1$

Polytropic configurations for $n = 1$ show fully convective types of stars such as neutron stars and very cool late-type stars. The stability ranges of such type of star are given in Figure 2. It is obvious from the graph that for $\omega_{BD} > -1.5$, $\gamma > 4/3$, the resulting radius of instability is $R \approx 400000R_s$ which is much greater than the Schwarzschild radius. In this case, GR has $R \approx 8.4807 \times 10^7R_s$ and BD has $R \approx 809.46R_s$. This implies that in MBD gravity, the system becomes unstable before the time mentioned by GR and much after the time described by BD gravity.

### 4.3 Polytropes for $n = 1.5$

Structures for $n = 1.5$ represent good models of stars having fully convective interior. Figure 3 represents instability ranges of such stars under MBD
Figure 2: The ratio of radius of instability and Schwarzschild radius \( R/R_s = K/(\gamma - 4/3) \) is plotted against \((-1 \leq \gamma \leq 2), \ (-1.5 \leq \omega_{BD} \leq 1) \) for \( n = 1, \ \beta = 1 \) and \( m_0 = 200m_{AU} = 200 \times 10^{-27} \).

Figure 3: The ratio of radius of instability and Schwarzschild radius \( R/R_s = K/(\gamma - 4/3) \) is plotted against \((-1 \leq \gamma \leq 2), \ (-1.5 \leq \omega_{BD} \leq 1) \) for \( n = 1.5, \ \beta = 1 \) and \( m_0 = 200m_{AU} = 200 \times 10^{-27} \).
Figure 4: The ratio of radius of instability and Schwarzschild radius \( R/R_s = K/(\gamma - 4/3) \) is plotted against \((-1 \leq \gamma \leq 2), (-1.5 \leq \omega_{BD} \leq 1)\) for \( n = 2, \beta = 1 \) and \( m_0 = 200m_{AU} = 200 \times 10^{-27} \).

gravity. For \( \omega_{BD} > -1.5, \gamma > 4/3 \), the resulting radius is \( R \approx 1 \times 10^6 R_s \) while in GR \( R \approx 9.67594 \times 10^8 \). Thus, for \( n = 1.5 \), the masses become unstable in MBD much before the limit predicted by GR.

4.4 Polytropes for \( n = 2 \)

The radii of instability for structures \( n = 2 \) are shown in Figure 4. It can be noticed that \( \omega_{BD} > -1.5, \gamma > 4/3 \) express real results and \( R \approx 5000 R_s \). The obtained radii in GR as well as BD are \( R \approx 1126.94 R_s \) and \( R \approx 1053.85 R_s \), respectively. In this case, the radius of instability in MBD gravity is much greater than that evaluated in GR and BD theory and hence the system is more stable in MBD gravity.

4.5 Polytropes for \( n = 3 \)

Polytropic index \( n = 3 \) represents main sequences of stars that have degenerated cores such as white dwarfs. The stability ranges of this type of structure in MBD gravity are given in Figure 5 which show that for
Figure 5: The ratio of radius of instability and Schwarzschild radius \( R/R_s = K/(\gamma - 4/3) \) is plotted against \((-1 \leq \gamma \leq 2), (-1.5 \leq \omega_{BD} \leq 1)\) for \( n = 3, \beta = 1 \) and \( m_0 = 200m_{AU} = 200 \times 10^{-27} \).

\( \omega_{BD} > -1.5, \gamma > 4/3, \) the obtained radius is \( R \approx 100000R_s. \) In GR and BD gravity, we have \( R \approx 1686.7R_s \) and \( R \approx 1544.1R_s, \) respectively. Thus the system is more stable in MBD gravity than in GR and BD theory.

4.6 Polytropes for \( n = 5 \) and \( n > 5 \)

As we have already discussed, various polytropic indices lead to different stellar configurations out of which structure discussed for \( n = 0 \) to \( n < 5 \) are proved to be realistic stars [26]. However, some expected behavior of \( n > 5 \) on instability criteria can be obtained from Eqs.(46) and (47). It can be noticed from Eqs.(46) and (47) that for \( n = 5 \) we have \( K = 0 \) which in turn give unphysical result \( R = 0. \) In the case \( n > 5, \) if the values of \( K \) become negative, the radius of instability \( (R) \) remains physically acceptable if \( \gamma < \frac{4}{3} \), otherwise \( \gamma > \frac{4}{3} \) is the instability criteria.
Figure 6: The ratio of radius of instability and Schwarzschild radius \( \frac{R}{R_s} = K/\left(\gamma - 4/3\right) \) is plotted against \((-1 \leq \gamma \leq 2)\), \((200 \times 10^{-27} \leq m_0 \leq 1000 \times 10^{-27})\) for \(n = 0\), \(\beta = 1\) and \(\omega_{BD} = 1\).

Figure 7: The ratio of radius of instability and Schwarzschild radius \( \frac{R}{R_s} = K/\left(\gamma - 4/3\right) \) is plotted against \((-1 \leq \gamma \leq 2)\), \((200 \times 10^{-27} \leq m_0 \leq 1000 \times 10^{-27})\) for \(n = 1\), \(\beta = 1\) and \(\omega_{BD} = 1\).
Figure 8: The ratio of radius of instability and Schwarzschild radius \( \frac{R}{R_s} = \frac{K}{(\gamma - 4/3)} \) is plotted against \((-1 \leq \gamma \leq 2)\), \((200 \times 10^{-27} \leq m_0 \leq 1000 \times 10^{-27})\) for \(n = 1.5\), \(\beta = 1\) and \(\omega_{BD} = 1\).

Figure 9: The ratio of radius of instability and Schwarzschild radius \( \frac{R}{R_s} = \frac{K}{(\gamma - 4/3)} \) is plotted against \((-1 \leq \gamma \leq 2)\), \((200 \times 10^{-27} \leq m_0 \leq 1000 \times 10^{-27})\) for \(n = 2\), \(\beta = 1\) and \(\omega_{BD} = 1\).
Figure 10: The ratio of radius of instability and Schwarzschild radius 
"$R/R_s = K/(\gamma - 4/3)$" is plotted against $(-1 \leq \gamma \leq 2)$, $(200 \times 10^{-27} \leq m_0 \leq 1000 \times 10^{-27})$ for $n = 3$, $\beta = 1$ and $\omega_{BD} = 1$.

4.7 Effects of Massive Scalar Field on Stability Criteria

In MBD gravity, we cannot ignore the behavior of scalar field mass upon the instability criteria. Figures 6-11 show plotting of $R/R_s = K/(\gamma - 4/3)$ versus increasing mass function $m_0 \geq 200m_{AU} = 200 \times 10^{-27}$ and $(-1 \leq \gamma \leq 2)$ for $n = 0$, 1, 1.5, 2, 3, with $\omega_{BD} = \beta = 1$. The value of $\omega_{BD}$ is chosen from the results of Figure 1-5. It can easily be noticed from these figures that the variation of mass function does not disturb the behavior of instability criteria. The constraints on $\gamma$ remains the same as discussed previously for $n = 0$, 1, 1.5, 2, 3, in Figure 1-5. However, it can be observed that the radii of instability defined for polytropes in MBD gravity are several orders of magnitude different from GR and BD theories. This is due to the coupling of self-interacting massive scalar field with the curvature term. It is believed that theories of gravity that deviate widely from GR can lead to the development of suitable modified theory of gravity. It has been shown that phenomenon in the presence of massive scalar field (in massive scalar-tensor theories) can differ drastically from the pure general relativistic one [27].

The above analysis indicates that all the cases except $n = 0$, $n = 5$
Figure 11: The values of $R/R_s = K/(\gamma - 4/3)$ is plotted against $(-1 \leq \gamma \leq 2)$ for $n=0, 1, 2, 3$ in MBD, BD and GR frameworks. Here, we have fixed $\omega_{BD} = 6$, $m_0 = 200m_{AU} = 200 \times 10^{-27}$ and $\beta = 1$. The red map shows MBD ranges, BD theory results are mapped in black while purple colour represents GR limits.

and $n > 5$ have stable region for $\gamma > 4/3$ which is consistent with GR and BD gravity. The comparison of instability ranges ($R/R_s = K/(\gamma - 4/3)$) in MBD, GR [2] and BD [8] are more clearly described in Figure 11. Here, MBD results are plotted in red, BD ranges are shown in black while purple mapping describes GR limits.

5 Conclusion

According to observational and experimental surveys, stellar configurations are running far away from each other with an accelerating rate causing accelerating expansion in the universe. It is believed that this is due to the presence of dark energy (mysterious energy) in the universe. Thus we cannot ignore the role of dark energy in the evolution of stellar structure. Among
different dark energy candidates, BD gravity is considered as the first prototype and the most fascinated alternative theory of gravity. In order to be consistent with observational data, the BD gravity is generalized (refines to) to MBD gravity (dilaton gravity) in which the scalar field becomes massive and dilatons are self-interacting due to the presence of potential of scalar field.

In this paper, we have discussed stability of spherical gaseous masses for radial oscillations in the presence of dark energy by incorporating MBD gravity. For this purpose, we have calculated complete pN corrected hydrodynamics of MBD gravity in terms of potential and super-potential functions. It is found that the obtained solutions use some generalized potential functions that are not involved in GR and BD gravity. This implies that stellar configurations described by MBD gravity are more massive (have more potential) than those of GR and BD theory. In order to discuss radial oscillations of the system, we have perturbed the system by Lagrangian radial perturbation and obtained linearized perturbed dynamical equations. By applying variational principle on governing perturbed equation of motion, we have formulated the criteria of onset of dynamical instability for a special case \( \gamma = \text{constant} \). It is found that the results obtained for Newtonian approximation are consistent with those described by GR and BD gravity but are modified in pN correction.

In order to discuss realistic models in MBD theory, we have evaluated radius of instability for different polytropic structures. The resultant models depend upon the mass of scalar field and provide a drastic change in the results of GR and BD gravity. The system for \( n = 0 \) is less stable than the systems described by GR and BD. For \( n = 1 \), the system is less stable than GR but more stable than BD system. Polytropes for \( n = 1.5 \) are less stable in MBD than in GR. Structures for \( n = 2, 3 \) are more stable in MBD gravity than those described by GR and BD theory. For \( n = 0 \), the system can be stable for \( \gamma < 4/3 \) which is inconsistent with GR. The case \( n = 5 \) gives unphysical result while for \( n > 5 \), the stability range is either \( \gamma < 4/3 \) or \( \gamma < 4/3 \) depending upon the behavior of \( K \). We have also investigated the effects of scalar field mass \( (m_0 \gtrsim 200m_{AU} = 200 \times 10^{-27}) \) on the stability criteria. It is found that it does not affect the instability ranges defined on \( \gamma \). However, the massive scalar field changes the magnitude of radii of instability from BD and GR theories. It can be noticed from the above discussions that the dynamics of massive scalar field (MBD gravity) affects the hydrostatic timescales of stellar structures. This implies that presence of dark energy...
not only causes expansion in the universe but affects the evolution of stellar evolutions.

From the above analysis, it can be noticed that the MBD gravity is better option than BD gravity as it describes the most general description of stellar evolutions which can be reduced to simple BD (in the limits \( m_0 << \frac{1}{r} \) and \( \frac{1}{\omega} \rightarrow 0 \)) as well as GR case ((\( m_0 << \frac{1}{r} \), \( \frac{1}{\omega} \rightarrow 0 \) and \( \omega_{BD} \rightarrow \infty \)). The analysis in MBD theory deals with all types of situations such as massive scalar field, massless scalar field and zero scalar field.

**Appendix A**

The potential functions \( U_{\alpha, ij} \), \( W_i(x) \) and \( Z_{i(BD)} \) in Eq. (19) are defined by

\[
U_{\alpha, ij} = G_{\text{eff}} \int_v \rho(\bar{x}) \frac{(x_i - \bar{x}_i)(x_j - \bar{x}_j) d\bar{x}}{|x - \bar{x}|^3},
\]

\[
W_i(x) = v_\alpha \frac{\partial}{\partial x_\alpha} (U_i - U_{j;ij}) = -G_{\text{eff}} \int_v \rho(\bar{x}) \frac{(x_\alpha - \bar{x}_\alpha) d\bar{x}}{|x - \bar{x}|^3},
\]

\[
+ G_{\text{eff}} \int_v \rho(\bar{x}) [v_i(x) v_\alpha(\bar{x}) + v_i(\bar{x}) v_\alpha(x)] \frac{(x_\alpha - \bar{x}_\alpha) d\bar{x}}{|x - \bar{x}|^3},
\]

\[
+ 3G_{\text{eff}} \int_v \rho(\bar{x}) [v_\alpha(x) v_\beta(\bar{x})(x_\alpha - \bar{x}_\alpha)(x_\beta - \bar{x}_\beta)] \frac{(x_i - \bar{x}_i) d\bar{x}}{|x - \bar{x}|^5},
\]

\[
Z_{i(BD)} = -2 \frac{\partial}{\partial x_i} \left[ \left( 1 + \frac{e^{-\text{mor}}}{3 + 2\omega_{BD} + e^{-\text{mor}}} \right) U + \frac{\Lambda_{BD} r^2}{3} \right] + 2v_2 \left( \frac{\partial U}{\partial x_i} \right),
\]

\[
+ \frac{\partial}{\partial x_i} \left( \frac{e^{-\text{mor}}}{3 + 2\omega_{BD} + e^{-\text{mor}}} U \right) + \frac{\partial (\gamma_{BD} U)}{\partial x_i} + \frac{\Lambda_{BD} \partial r^2}{2} + \frac{\partial (\gamma_{BD} U)}{\partial x_i},
\]

\[
+ p \left( 2 \frac{\partial U}{\partial x_i} + 2 \frac{\partial}{\partial x_i} \left( \frac{e^{-\text{mor}}}{3 + 2\omega_{BD} + e^{-\text{mor}}} U \right) \right) + 2 \frac{\partial (\gamma_{BD} U)}{\partial x_i}.
\]

The values of \( Z_{ih(BD)} \) are given by

\[
Z_{ih(BD)}^h = -2 \frac{\partial}{\partial x_i} \left[ \left( 1 + \frac{e^{-\text{mor}}}{3 + 2\omega_{BD} + e^{-\text{mor}}} \right) U + \frac{\Lambda_{BD} r^2}{3} \right],
\]

\[
+ p \left( 2 \frac{\partial U}{\partial x_i} + 2 \frac{\partial}{\partial x_i} \left( \frac{e^{-\text{mor}}}{3 + 2\omega_{BD} + e^{-\text{mor}}} U \right) \right) + 2 \frac{\partial (\gamma_{BD} U)}{\partial x_i}.
\]
\[ + \left( \frac{\Lambda_{BD}}{2} \frac{\partial r^2}{\partial x_i} \right) + c^2 \left( U - \frac{\Lambda_{BD} r^2}{6} \right) \Lambda_{BD} \frac{\partial r^2}{\partial x_i}. \]

The values of \( \Delta Z_i(BD) \) are

\[
\Delta Z_i(BD) = -2 \frac{\partial}{\partial x_i} \left( \frac{1}{3 + 2 \omega_{BD} + e^{-\text{mar}} U} \Delta U \right) + \Delta p \left[ 2 \frac{\partial}{\partial x_i} U \right]
+ 2 \frac{\partial}{\partial x_i} \left( \frac{e^{-\text{mar}}}{3 + 2 \omega_{BD} + e^{-\text{mar}} U} \Delta U \right) + \frac{\Lambda_{BD}}{2} \frac{\partial r^2}{\partial x_i}
+ \left[ 2 \frac{\partial}{\partial x_i} \Delta U + 2 \frac{\partial}{\partial x_i} \left( \frac{e^{-\text{mar}}}{3 + 2 \omega_{BD} + e^{-\text{mar}} U} \Delta U \right) \right] + c^2 \frac{\Lambda_{BD}}{2} \frac{\partial r^2}{\partial x_i}.
\]

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