The Aharonov-Bohm Problem Revisited

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Abstract

The properties of a nonrelativistic charged particle in two-dimensions in the presence of an arbitrary number of nonquantized magnetic fluxes are investigated in free space as well as in a uniform magnetic field. The fluxes are represented mathematically as branch points in one of the complex@ coordinates. To construct many-flux solutions without a magnetic field, however, the fluxes must be treated as dynamical objects dual to the charges. A medium made up of fluxes acts like an anti-magnetic field and tends to expel the charges.

1 Introduction

In a classic paper, Aharonov and Bohm [1] have pointed out that a locally trivial vector potential of a gauge field can lead to observable effects in a nontrivial topology through the phase of the wave function of a charged particle. This has spawned many important theoretical concepts in later developments in gauge theories, like instanton, monopole, and confinement by monopole condensation. On the experimental side, confirmation of the original Aharonov Bohm (A-B) effect due to a nonquantized magnetic flux tube has been somewhat controversial because of the difficulty of setting up ideal conditions, but there have been several experiments, notably those by Tonomura et al [2] showing the shifts in diffraction fringes due to magnetic fluxes, as expected by the A-B effect. (See [3] for a general review of the problem.) The present work was motivated by the theoretical question: What would be the properties of a medium filled with many nonquantized magnetic
fluxes (vortices)? Assume, for simplicity, that the fluxes are all parallel so that problem can be reduced to that of an electrically charged particle in a 2-plane pierced by pointlike magnetic fluxes. Then, for example, what would be the behavior of a charged particle in a lattice of such fluxes? What would be its energy spectrum? Would the fluxes cause more drastic effects than a phase shift in the wave function? Although the semiclassical arguments suffice to discuss and analyze most situations realistically, it requires a more rigorous treatment of the Schrödinger equation to answer these questions. This turns out to be a highly nontrivial problem. To our knowledge, the solution of the Schrödinger equation in the presence of two or more fluxes has not been explicitly constructed. (A qualitative discussion of special cases was made by Peshkin et al. [3]. Some other theoretical problems related to the present one have been discussed by Aharonov and Casher [7], Dubrovin and Novikov [8], Jackiw [11], and Lewis [14].)

We develop here a method of solving the many-flux problem by adapting the work of Sommerfeld [5] [6]. He solved the problem of diffraction of light by a semi-infinite wall in two dimensions by regarding the wall as a branch cut in a two-sheeted Riemann surface. The incoming plane wave in the physical plane dives into the unphysical second sheet as it hits the cut, while the reflected wave emerges from the second sheet through the cut. The solutions were constructed as a contour integral of a kernel in a complex angular variable over the two sheets.

The A-B problem can be posed in a similar fashion. The pure gauge vector potential of a flux is singular at the flux site. In the complex variables $z = x + iy$ it behaves like $(\alpha/2)(i/z, -i/\bar{z})$, where $2\pi\alpha$ is the flux strength. After removing the potential by a gauge transformation, we get a wave function that satisfies a free Schrödinger equation, but it is singular in the sense that the function acquires a phase $\eta = \exp(2i\pi\alpha)$ after encircling the flux once, so the wave function must vanish at the flux if $\alpha$ is noninteger. Regarded as a function in the complex 2-plane, the problem then reduces to that of solving the free Schrödinger equation on a Riemann surface where the flux is a branch point, and the solution must satisfy the boundary condition that it gets a phase change $\eta$ when going around it once, and vanishes at the branch point.

The paper is organized as follows. We first establish the basic formalism of complex integral representation, apply it to rederive the known results for the case of a single flux with or without the presence of a uniform magnetic field. [1] [14]. Then the problem of many fluxes is addressed by two methods.
which yield different types of solutions. One of them is to use multiple integral representations, and is applicable only if a magnetic field is present. The other is to treat the fluxes as dynamical objects having properties dual to the charges. An important general property that emerges is that the nontrivial phase information around all fluxes must be represented as branch points in either the $z$ or the $\bar{z}$ coordinate, but not a mixture of them. It means that, since the wave function must vanish at the flux sites, the (fractional) angular momentum around each flux, as opposed to the intrinsic flux strength, has the same sign, either positive or negative. Thus for example, if the two flux strengths are $\alpha_1 > 0$ and $\alpha_2 < 0$, the solution is constructed as having positive angular momenta $\alpha_1 + l_1 > 0$ and $l_2 + \alpha_2 > 0, l_i = \text{integer}$, in the $z$ variable (and similarly negative angular momenta in $\bar{z}$). Otherwise the solution would have to be represented by a function in $z$ in a region around one flux, and by a function in $\bar{z}$ in another, but it is not possible to match the values of the wave functions and their derivatives at the interface.

The fact that angular momenta around all the fluxes must have the same sign is not intuitively clear, but may be inferred from the zero energy solutions, where the nontrivial information about all the fluxes must be encoded in terms of either analytic or antianalytic functions. It is also understandable from the following consideration. Suppose $\alpha_1 = -\alpha_2$, and let them approach each other and collapse to nothing. The wave function originally vanished at both sites and was finite elsewhere. So it will vanish at the point of collapse, hence it will be in a $p$ or higher angular momentum state.

One of the consequences of the above property is that a medium made up of fluxes tends to expel the charge as if it was placed in an anti-magnetic field. This behavior has a simple explanation. By construction the angular momentum around each flux is of the same sign, say $\alpha'$, irrespective of the sign of its intrinsic flux strength. The total angular momentum around a circle at a radius $R$ lattice units will be $\sim \alpha' \pi R^2$, hence the radial momentum $p(R)$ is $\pm (E - (\alpha' \pi R)^2)^{1/2}$. For large $R$, then, the wave function will go like

\footnote{To see this, consider at zero energy two fluxes of opposite sign located at $\pm ib$. Since any analytic function of $z$ or of $\bar{z}$ satisfies the Shroedinger equation, assume: $\psi = f_1 = (z - ib)^\alpha g(z)$ in the upper half plane ($y = \Im z > 0$), where $g(z)$ is regular, and $\psi = f_2 = (\bar{z} - ib)^\alpha g(\bar{z})$ in the lower half plane imposed, and analytically each can be extended to the whole plane. Obviously $f_1 = f_2$ on the real line, but their normal derivatives are opposite and $\neq 0$. (If the derivatives were zero, the functions would vanish identically.) The same conclusion is reached if the solution in each half plane is itself a sum of analytic and antianalytic parts.}
\[ \exp(\int pdR) \sim \exp(\pm \alpha' \pi R^2/2) \]. The minus sign is excluded by the boundary condition at the flux sites.

## 2 Basics

Let a flux of strength \( \alpha \) be located at the origin. For an ‘electron’ of charge \(-e\), the Aharonov-Bohm gauge potential \( A \) and the covariant derivative \( D \) in the standard circular gauge are given by

\[
\begin{align*}
(A_x, A_y) &= (\alpha/2)(y/r^2, -x/r^2) \\
\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n
\text{2.1)}
\]

For notational convenience, we are taking \( \hbar e = 1 \) so that integer \( \alpha \) corresponds to a quantized flux. In terms of the complex variables \( z = x + iy \) and \( \bar{z} = x - iy \), they become

\[
\begin{align*}
x &= (z + \bar{z})/2, \quad y = (z - \bar{z})/2i, \\
\partial/\partial z &= (1/2)(\partial/\partial x - i\partial/\partial y), \\
\partial/\partial \bar{z} &= (1/2)(\partial/\partial x + i\partial/\partial y), \\
(\partial^2/\partial x^2 + \partial^2/\partial y^2) &= 4(\partial/\partial z)(\partial/\partial \bar{z}), \\
A_z &= i\alpha/2z, \quad A_{\bar{z}} = -i\alpha/2\bar{z} \\
D_z &= \partial_z + \alpha/(2z), \quad D_{\bar{z}} = \partial_{\bar{z}} - \alpha/(2\bar{z})
\end{align*}
\]

The basic Schrödinger equation reads, in time dependent (energy Eigen-value) forms, after redefining \( mE/2 = k^2/4 \rightarrow E \)

\[
\begin{align*}
\{D_z, D_{\bar{z}}\}/2 + \partial_r \Psi &= 0 \\
((\{D_z, D_{\bar{z}}\}/2 + E)\Psi &= 0
\end{align*}
\]

(2.3)

with the condition that \( \psi \) is one-valued and finite. (We will not necessarily require finiteness of derivatives, though.) Here a Euclidean time is used for later convenience, \( z \) and \( \bar{z} \) are regarded as independent variables spanning a complex 2-dimensional plane. The physical space is its subspace where \( z \) and \( \bar{z} \) are complex conjugates of each other (real \( x\)-\( y \) plane). Hereafter we
will refer to this situation as "on shell". Since Eq.(2.3) has independent first
derivatives in $z$ and $\bar{z}$, the actual physical space also includes its tangential
neighborhoods. After solutions have been written down in $z$ and $\bar{z}$, however,
we can stay on shell by reverting to $x$ and $y$. The ordering of $D_z$ and $D_{\bar{z}}$
may be ignored since their potential noncommutativity at their singularities will
not arise. From Eq.(2.2) we see that Eq.(2.3) is invariant under the operation

$$z \leftrightarrow \bar{z}, \quad \alpha \to -\alpha \quad (2.4)$$

It corresponds to a reflection $y \to -y$. The interchange of $z$ and $\bar{z}$ is also
effected by complex conjugation of the wave function which corresponds to
time reversal.

We next eliminate from Eq.(2.3) the gauge potential by a singular gaug e
transformation $G$:

$$\Psi = G\psi, \quad G = (\bar{z}/z)^{\alpha/2}, \quad (2.5)$$

so that $\psi$ now satisfies a free Schroedinger equation.

$$\left(\partial_z\partial_{\bar{z}} + \partial_t\right)\psi = 0 \quad (2.6)$$

($A$ is pure gauge if its $z(\bar{z})$ component is a function of $z(\bar{z})$ only. See Appendix
1.) Since the original $\psi$ is one-valued, $\psi$ must be singular in such a way as to
cancel the singularity in the gauge function in Eq.(2.5). This means that $\psi$
must be defined as a free wave function on a Riemann surface with a branch
cut running from 0 to $+\infty$, and on shell there is a phase change as we go
around the origin once:

$$\psi(\theta = 2\pi) = \eta\psi(\theta = 0), \quad \eta = \exp(2i\pi\alpha), \quad \psi(z = \bar{z} = 0) = 0 \quad (2.7)$$

and subject to conditions of finiteness at infinity on shell. In the following
we will mainly be concerned with the eigenvalue equation. An elementary
solution of Eq.(2.6) is

$$\exp(zt - E\bar{z}/t) \quad (2.8)$$

where $t$ and $\bar{t}$ are complex momenta $t$. The general solution of Eq.(2.6)
that satisfies the boundary conditions may be built up as a superposition of
elementary solutions

$$\psi = \int \exp(zt - \bar{z}/t)f(t)dt \quad (2.9)$$
with an appropriate choice of the function $f(t)$ and the integration path in such a way as to satisfy Eqs.(2.7) on shell. We recognize the familiar integral representation of Bessel functions.

Now assume for the time being $0 < \alpha < 1$, and let

$$f(t) = t^{-\alpha-n-1}, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (2.10)

and choose the contour $C$ of integration to be

$$C : (U(\Re(zt) < 0) \cap \mathbb{R}, (0+)),$$  \hspace{1cm} (2.11)

which goes around 0 starting from, and ending at, infinity in the direction $U$ such that the integral is convergent. This means that, as the phase of $z$ rotates by $2\pi$, the contour integration will have to make a counter-rotation by $-2\pi$, so the factor $f(t)$ will yield a phase factor $e^{2i\pi\alpha}$ as is required by Eq.(2.7). Furthermore, $\psi'$ stays finite on shell when $z$ and $\bar{z}$ go to infinity, and also at zero since

$$\int t^{-\alpha-1-n}dt = 0, \quad n + \alpha > 0$$  \hspace{1cm} (2.12)

These properties can be made more explicit by a change of integration variable, $t \rightarrow E/z$

$$\psi = z^{n+\alpha} \int_C \exp(\bar{E}t - z\bar{z}/t)t^{-\alpha-n-1}dt$$  \hspace{1cm} (2.13)

up to a constant factor, and where the contour $C$ is now $(-\infty, (0+))$. So Eq.(2.13) gives a desired set of solutions for any $\alpha$ and integer $n = 0, 1, 2, \ldots$ that satisfy $n + \alpha > 0$. In the case $n + \alpha < 0$, we can choose a new contour around 0:

$$C' : (0 \times U(\Re(1/t) > 0, (0+)),$$  \hspace{1cm} (2.14)

i.e. one that starts from the origin in the direction $U$, and comes back after encircling it clockwise. This is equivalent to the conjugate form ($\bar{z}$-type) of $\psi$ according to Eq.(2.4), which can be converted to the original contour by the substitution in Eq.(2.10): $t \rightarrow E/t$, resulting in a new $f(t)' = t^{\alpha+n-1}$. Eq.(2.13) then becomes, up to a constant factor,

$$\psi = z^{-\alpha-n} \int_C \exp(-Et + z\bar{z}/t)t^{\alpha+n-1}dt, \quad n + \alpha < 0$$  \hspace{1cm} (2.15)

In terms of Bessel functions, the $z$- and $\bar{z}$-type solutions lead to solutions in the original gauge:

$$n + \alpha > 0, \quad z\text{-type} : \quad \Psi = (z/\bar{z})^{n/2}J_{\alpha+n}(2(Ez\bar{z})^{1/2}),$$

$$n + \alpha < 0, \quad \bar{z}\text{-type} : \quad \Psi = (z/\bar{z})^{n/2}J_{\alpha+n}(2(Ez\bar{z})^{1/2})$$  \hspace{1cm} (2.16)
Thus the physical states are labeled by $|\alpha + n|$ and an integer angular momentum $n = 0, \pm 1, \pm 2, \cdots$. The angular momentum operator $L$ is given by $z\partial_z - \bar{z}\partial_{\bar{z}}$. Eqs.(2.13) and (2.15) show that the nontrivial phase information in $\psi$ is carried respectively by $z$ and $\bar{z}$. Applying the operators $\partial_z$ and $\partial_{\bar{z}}$ on a $\psi$ serve to change $n$ by $\mp 1$. This may be repeated any number of times as long as the wave function remains finite, thereby generating a complete set of states.

Some remarks are in order here about the problems associated with non-single valued wave functions and the limitations in taking derivatives. The first problem poses a difficulty in defining a complete set of states. Although scalar products are independent of the choice of (unitary) gauges, it does not make sense to expand a function with a branch point at $z = a$ in terms of functions with a branch point at $z = b$. Similarly, the meaning of going to momentum space by Fourier transforms is not clear since it will depend on where a cut is made, and is difficult to evaluate. We should, therefore, go back to the original non-singular gauges to build a Hilbert space of physical states. We may, however, work with non-single valued functions in enumerating the states since there is a 1-1 correspondence between the two sets.

The second problem, on the other hand, causes a problem in defining higher powers of momentum as observables, e.g., the finite translation operators in power series. For convenience, however, we may pretend to ignore these problems, and use such formal expressions as $(i\hat{k})^\alpha$ to mean $\partial_z^\alpha = t^\alpha$ when applied to Eq.(2.8) in momentum space.

3 Scattering

Since the solutions Eqs.(2.16) are Bessel functions of non integer order, we expect that the scattering amplitude of an electron by the flux can be obtained by a superposition of all solutions labeled by $n$, which would form a complete set. However, since they all vanish at the origin, they form a complete set only in the punctured plane $R^2 - 0$, hence it would not be possible to construct a plane wave in $R^2$ with this set of functions. What is missing is an equivalent of the $s$ wave component that remains finite at the origin, a situation analogous to the case of hard sphere scattering. But it is also different because of the long range nature of the gauge potential as in the case of the Coulomb scattering.
A proper scattering amplitude can be constructed in the following way. Consider the integral.

\[ \psi = \int_{C_1 + C_2} \exp(Ezt - \bar{z}/t)f(t)dt, \]

\[ f(t) = (t/t_{in})^\alpha/(t - t_{in}), \quad t_{in} = iE^{1/2} \quad (3.1) \]

\( t_{in} \) is the momentum of the incoming plane wave moving in the \( x \) direction:

\[ \psi_{in} = \exp(iE^{1/2}(z + \bar{z})) = \exp(2iE^{1/2}x) \quad (3.2) \]

The contours \( C_1 \) and \( C_2 \) are defined as follows. [Fig.1] Denote by \( R_0 \) the reference Riemann sheet, and by \( R_n \) the sheet reached by going counter-clockwise around the origin \( n \) times. For a given position \( z = r\exp(i\theta) = rU \), \( C_1 \) starts from \( \infty \times U \), goes counterclockwise around the origin and \( t_{in} \), to end up at \( \infty \times U^1 \) on \( S_1 \). \( C_2 \) starts from \( 0 \times U^{-1} \), goes clockwise around the origin, but avoiding \( t_{in} \), to end up at \( 0 \times U^1 \).

Some properties of Eq.(3.1) are obvious:
1) The integral converges for \( z = \bar{z} = 0 \), giving \( \psi = 0 \).
2) When \( \alpha \) is an integer, the two contours can be detached from their limits to form a closed circle around \( t_{in} \), which then picks up the incoming wave \( \psi_{in} \) only, so there is no scattering.

That the formula gives the correct scattering amplitude can be shown by computing its asymptotic form by the saddle point method. First take the general formula Eq.(2.9). Its saddle points are determined by the extrema \( t_0 \).
of \( S(t) \), and performing a Gaussian integration around them in the direction of the steepest descent. In the case of scattering,

\[
S' = z + E\bar{z}/t_0^2 + \alpha/t_0 - 1/(t_0 - t_{in}) = 0,
S'' = -2E\bar{z}/t_0^3 - \alpha/t_0 + 1/(t_0 - t_{in})^2
\]

(3.3)

For large \( E^{1/2}|z| = E^{1/2}r >> 1 \), there are three saddle points on a circle of radius \( E^{1/2} \): one near the pole \( t_{in} \) representing the plane wave, and a conjugate pair \( t_+, t_- \) representing radially outgoing and incoming waves respectively. (\( t_{in} \) and \( t_\pm \) do not coincide except when \( \theta = 0 \) or \( \pi \).) But the contours. \( C_1 \) and \( C_2 \) are such that the incoming component is canceled, leaving only the outgoing one. For the radial components,

\[
t_\pm \sim E^{1/2} \exp(\pm i\theta),
e^S \sim -iE^{-1/2} \exp(\pm 2iE^{1/2}R - i\alpha \theta \pm i\pi/4)/(\pm e^{-i\theta} - 1)
\]

(3.4)

The Gaussian integration around \( t_\pm \) yields

\[
\psi_{+-} \sim -\exp(\pm 2iE^{1/2}R - i(1/2 + \alpha) \theta \pm i\pi/4))/R^{1/2}/\sin(\theta/2) \quad \text{for } \psi_+,
\psi_- \sim -\exp(\pm 2iE^{1/2}R - i(1/2 + \alpha) \theta \pm i\pi/4))/R^{1/2}/\cos(\theta/2) \quad \text{for } \psi_-
\]

(3.5)

These should also be multiplied respectively by an additional factor depending on the contour: \( -\eta(\eta^{-1}) \) and \( -1(1) \) for \( C_1(C_2) \). So when the two contributions are added, and after reverting to the original gauge, the scattering amplitude becomes

\[
\psi_{scat} \sim \sin(\pi \alpha) \exp(2iE^{1/2}R - i(1/2 + \alpha) \theta - i\pi/4))/R^{1/2}\sin(\theta/2))
\]

(3.6)

which reproduces the result given by [1]. The formula is valid except for a near forward region where the width of the Gaussian integration \( \sim R^{1/2}E^{1/4} \) becomes larger than \(|t_{in} - t_+|\). (see [Ruijsenaars][12]Sakoda).

4 Presence of a magnetic field

4.1 Basic formulas

In this section we allow for the presence of a real magnetic field in addition to the fluxes. The vector potential for a constant magnetic field \( \gamma \) is

\[
A_z = \gamma \bar{z}/2, \ A_\bar{z} = -\gamma/z/2
\]

(4.1)
which is to be added to the Aharonov-Bohm potential. After gauging away the latter, we have
\[ H \Psi = E \Psi, \quad H = -\{(\partial_z - \gamma \bar{z}/2), (\partial_{\bar{z}} + \gamma z/2)\}/2, o \quad (4.2) \]

Assuming \( \gamma > 0 \), make a further transformation
\[ \Psi = \exp(-\gamma z \bar{z}/2) \psi, \quad H' \psi = E \psi, \quad H = (\partial_z \gamma \bar{z}) \partial_{\bar{z}} + \gamma/2 \quad (4.3) \]

Since \( \partial_z \) commutes with \( H \), an elementary solution is then
\[ \psi = \exp(zt)(t - \gamma \bar{z})^{E/\gamma - 1/2}, \quad (4.4) \]

where \( t \) is a complex constant. If there are no fluxes, \( \psi \) must be single-valued and finite everywhere, i.e., the second factor must be a polynomial, which leads to the familiar Landau spectrum:
\[ E/\gamma - 1/2 = n = 1, 2 \cdots \quad (4.5) \]

so that
\[ \psi_n = \exp(-\gamma z \bar{z}/2 + tz)(t - \gamma \bar{z})^n \quad (4.6) \]

For \( n \geq 0 \), \( \psi \) vanishes at \( \gamma \bar{z} = t \), and goes to zero at large \( |z| \). The states are labeled by the level number \( n \), and a continuous complex parameter \( t \) which is the analog of the continuous real parameter in the Landau gauge. (The true energy is \( E/4m \) in our convention.) Here \( t/\gamma \) corresponds to the center of the classical Larmor orbit with angular momentum \( -n \leq 0 \). Note that \( (t - \gamma \bar{z}) \) is a raising operator for the level number \( n \), hence \( < t > = < \gamma \bar{z} > \).

We will conveniently refer to this factor as raising factor.

The space of all \( n \)'s and all complex \( t \)'s is overcomplete on the physical shell, unlike in the Landau gauge. (In the circular gauge, the radius \(|t|\) and \( n \) label the states \([14]\).) But by expanding Eq.(4.4) in powers of \( t \), or by the Fourier integral
\[ \psi_{n,l} = \int \psi_n t^{-l-1} dt, \quad l \geq 0 \quad (4.7) \]

we get a complete set centered at zero and labeled by two integers \( n \) and \( l \), giving a total angular momentum \( j = l - n \). Each solution contains a factor \( z' \) times a polynomial of order \( n \) in \( z \) and \( \bar{z} \). Eq.(4.7) allows an intuitive interpretation that these states are an epicycle-like superposition of Larmor
orbits, with ‘orbital’ angular momentum \( l \), along another circle of a certain radius (see below). When \( \gamma < 0 \), clearly we may take the conjugate solutions to the above \( z \leftrightarrow \bar{z}, \gamma \rightarrow -\gamma, j \rightarrow -j \), and the energy levels are given in general by \( E/|\gamma| = n + 1/2 \). It is also instructive to examine the limit \( \gamma \rightarrow 0 \), where we should recover the original free field results. Since the energy spectrum is proportional to \( \gamma \), however, in general it will be driven to zero in this limit, as may be seen from the elementary solution Eq.(4.6) which does not have a limit \( E = n/\gamma \) except for \( n = 0, E \rightarrow 0 \). But supplying a factor \( t^{-n} \), i.e. dividing the raising factor by \( t \), we get
\[
\lim \exp(-\gamma \bar{z}^2/2 + zt)(1 - \gamma \bar{z}/t)^{E/\gamma - 1/2} = \exp(zt - \bar{z}E/t) \tag{4.8}
\]
which reproduces the free field exponent.

### 4.2 A flux in a magnetic field

There are two ways to introduce a flux of strength \( \alpha \). One is to change \( n \) to \( n + \alpha \) in Eq.(4.6) (provided that it is \( \geq 0 \)), and to let it be the gauge transformed \( \psi \). It represents a flux located at \( t/\gamma \), and the energy is shifted:
\[
\psi_{n+\alpha,t} = \exp(zt)(t - \gamma \bar{z})^{n+\alpha},
\quad E/\gamma - 1/2 = n + \alpha \geq 0 \tag{4.9}
\]
By expanding Eq.(4.9) in powers of \( t \), or taking the moments
\[
\psi_{n,t} = \oint \psi_{n+\alpha,t} t^{-l-1} dt, \quad 0 \leq l \leq l_0 = [n + \alpha] \tag{4.10}
\]
we get \( l_0 \) states with the same shifted energy. The restriction \( l \leq l_0 \) arises from the condition \( \psi(0) = 0 \). This is a \( \bar{z} \) type representation. Both Eq.(4.9) and Eq.(4.10) are now nonpolynomial functions in \( \bar{z} \).

For \( l \geq l_0 \), the proper formula is given by
\[
\psi_{n,t} = \int_C \psi_{n,t} t^{l-\alpha-1} dt,
\quad E/\gamma - 1/2 = n, \quad C = (\infty, (0+, \gamma \bar{z}+)) \tag{4.11}
\]
\(^2\text{It can be turned into a } z\text{-type representation if we change the sign of } \gamma, \text{ and let the path be } C = (\infty, (0+)). \text{ The energy } \sim E/\gamma \text{ remains positive. See section 6.3}\)
which is a $z$-type representation. The energy does not get shifted for them. For $n = 0$, in particular, $\psi$ is the same as the zero energy limit of the field-free case, i.e., a product of $z_i$’s, apart from an overall Gaussian factor. The reason why a non-polynomial raising factor in the integrand is not allowed is that, at large distances, $\psi$ becomes $\sim \exp(+\gamma z \bar{z}/2)$, as may be seen by finding the saddle points. It is not possible to avoid the exponential blowup by changing the sign of $\gamma$ or energy. The problem is avoided only if the factor is a polynomial. (According to the previous subsection, it can also be seen that only unshifted levels have zero field limits.) Obviously these same results will hold for $\gamma < 0$, for which the conjugate representations are used.

Summarizing, a number $0 \leq l_0 = [n + \alpha]$ of states at level $n$ have their energy (more precisely $E/|\gamma| - 1/2$) shifted to $n + \alpha \geq 0$, and the rest is uninfluenced by the flux. Consider in particular the ground states $n = 0$ with $|a| < 1$. If $\alpha > 0, \gamma > 0$ (or if $\alpha < 0, \gamma < 0$), i.e., the magnetic field and the flux are parallel, then the energy of one state is shifted upwards by $\alpha$. If, on the other hand, the field and the flux are antiparallel, none of the ground states can change energy; the energy can only go up, but not down below the zero point minimum.

The following semiclassical argument explains the difference between shifted and unshifted levels (see also [14]). The state $\psi_{n,t}$ around $t$ occupies a disk of radius $R \sim (n/\gamma)^{1/2}$. The orbital radius of the ‘epicycle’ state $\psi_{n,l}$ is $r \sim l/E^{1/2} \sim l/(\gamma n)^{1/2}$. So the latter state will cover the flux at the origin only if $R < r$. As for the asymmetry in the ground state, write the Hamiltonian as

$$H = p^2 - L(\alpha/r^2 + \gamma) + (\alpha/r + \gamma r)^2 \quad (4.12)$$

For $L = 0$, the minimum of the last term is $\alpha \gamma$ if $\alpha \gamma \geq 0$, and 0 if $\alpha \gamma < 0$.

5 The multi-flux problem

5.1 Multiple integral representation

At zero energy, the general solution of the Schroedinger equation is either analytic or antianalytic. The one vertex solution reduces to $\psi \sim z^\alpha$ or $\bar{z}^\alpha$. More than one vertices can be easily accommodated by taking products of $n$ such $z$-type or $\bar{z}$-type factors, but we cannot form general mixed products.
When $E > 0$, the solutions will involve both $z$ and $\bar{z}$, but by continuity this property will persist. In the following we assume nonzero energy. Our goal is to find a solution which has proper monodromy properties around each flux: $M_i \psi = \eta_i \psi$, where $M_i$ denotes the monodromy operation around flux $i$, giving a phase $\eta_i = \exp(i\theta_i)$. Since, however, the radial second order differential equation obtained after removing $G$ has singularities only at 0 and $\infty$, it is not clear how to introduce more flux singularities. So let us first try a perturbative solution in energy for two fluxes. Denote their location and strength by $(b_1, \alpha_1), (b_2, \alpha_2)$, and write $z - b_1 = z_1, z_2 = z_2$ for short: $\psi = \sum \psi^{(n)}(z, \bar{z}), \psi^{(0)} = (z_1)^{\alpha_1}(z_2)^{\alpha_2}$. Then

$$E \psi^{(n+1)} = -\partial_z \partial_{\bar{z}} \psi^{(n)}$$  \hspace{1cm} (5.1)$$

$\psi^{(1)}$ can be obtained by integrating $\psi^{(0)}$ over $z$ from $b_1$ ($b_2$), and over $\bar{z}$ from $\bar{b}_1$ ($\bar{b}_1$). This $\psi^{(1)}$ vanishes at both sites, but it does not have a proper monodromy at the second (first) site. We can make $\psi^{(1)}$ behave correctly at both sites by taking $\psi^{(1)} = (z_1)^{\alpha_1+n_1}(z_2)^{\alpha_2+n_2}z$ with integer $n_i \geq 1$. $\psi^{(0)}$ is then obtained by differentiating $\psi^{(1)}$. But to generate a series to the $N$th order, we have to start from an $N$th order term of sufficiently high power $n_i \geq N$, and we lose control of the perturbation expansion.

Next we look for a nonperturbative solution in terms of a double integral representation for two fluxes. The gauge potential and gauge function are respectively a sum and a product of two components:

$$A_z = \alpha_1/z_1 + \alpha_2/z_2, \quad A_{\bar{z}} = \bar{A}_z, \quad G(z) = (\bar{z}_1/z_1)^{\alpha_1/2}(\bar{z}_2/z_2)^{\alpha_2/2}$$  \hspace{1cm} (5.2)$$

Consider a solution of the Schrödinger equation of the form

$$e^S = \exp[z_1 t_1 + z_2 t_2 - E^{1/2}(\bar{z}_1 s_1 + \bar{z}_2 s_2)]f(t_1, t_2, s_1, s_2),
(t_1 + t_2)(s_1 + s_2) = E$$  \hspace{1cm} (5.3)$$

The idea is to integrate over one of $t_1, s_1$ and one of $t_2, s_2$ independently to satisfy proper monodromy, but the constraint precludes mixed types involving $t$ and $s$, and it will be sufficient to examine the $z$ type only. Since the third variable turns out superfluous, we are led to the integrand

$$e^S = \exp[z_1 t_1 + z_2 t_2 - (\bar{z} - \bar{b})E/(t_1 + t_2)]t_1^{\alpha_1} t_2^{-\alpha_2 - 1}, \quad (5.4)$$
with an arbitrary \( \vec{b} \). The integration contours \((C_1, C_2)\) are formally taken respectively around \( 0 \) to \( U(\theta_1) \) and \( U(\theta_2) \). It is in general not possible, however, to avoid the vanishing of \( t_1 + t_2 \) by a proper choice of the contours, and this would invalidate the formula. As \( C_1 \) or \( C_2 \) makes a \( 2\pi \) rotation, there always occur crossings of \(-C_1\) and \( C_2\), and we have to set up a convention for one of the \( t \)'s to make a detour around the other. A change of the crossing point leads to a change in the integral, and after making a \( 2\pi \) rotation, it will cause an extra change in \( \psi \) beyond a phase factor. A detailed study shows that the extra piece is a regular function of \( z \) and spoils the monodromy properties. Therefore it is not possible to satisfy the two monodromy conditions simultaneously. Later an alternative form of single integral representation for dynamical fluxes will be found to solve the problem. But we will first show next that the multiple integral representation can be used without difficulty if a magnetic field is present, although the solutions obtained are of the type that do not have free field limits.

### 5.2 Solutions in a magnetic field

Unlike the field-free case, it is easy to generalize the \( z \)-type representation for unshifted levels, Eq.(4.11), to \( N \) fluxes with the multiple integral method. The solution is given by

\[
\psi = \exp(-\gamma z\bar{z}/2) \int \cdots \int_C \exp(\sum z_i t_i)(T - \gamma \bar{z})^n \prod (t^-_{\alpha'_i} - \alpha_{i-1}^- n dt_i),
\]

\[
T = \sum t_i, \quad \alpha'_i = \alpha_i + l_i > 0, \quad i = 1...N
\]

(5.5)

The factor \((\sum (z_it_i - \gamma \bar{z}))^n\) can become zero, but this does not cause the earlier problem since it is a polynomial in the \( t \)'s containing powers up to \( N \). The condition on the \( \alpha_i \)'s insures that the overall power of each \( t_i \) is \(< 1\). For the ground state \( n = 0 \) we have

\[
\psi_{(\alpha'),0} = \prod z_{\alpha'}/(\alpha')!, \quad \Psi_0 = \exp(-\gamma z\bar{z}/2)\psi_0
\]

(5.6)

The higher levels are generated by applying the raising operator \( \partial_z - \bar{z}\gamma \) \( n \) times to \( \psi_{(\alpha'+n),0} \). Thus for this class of solutions the Landau levels are not affected by the fluxes. The wave function is a homogeneous polynomial of order \( n \) made up of \( \bar{z}\gamma \) and the \( z_i \)'s.
5.3 Zero field limit

Earlier we found that the multiple integral representation failed in the field-free case, so it is an interesting problem to study the limit of letting $\gamma$ go to zero, $\gamma = E/(n + 1/2) \to 0$, keeping energy fixed, or $n \to \infty$. We will evaluate the formula Eq.(5.5) for $N$ fluxes in the $n \to \infty$ limit by finding the saddle points with respect to each $t_i$:

$$t_i = (\alpha_i' + n)/(z_i + n/(T - E\bar{z}/n)) \sim T, T = \sum t_i \sim NT \quad (5.7)$$

which is a contradiction for $N > 1$. A more careful analysis shows $t_i \sim O(n)$. The wave function is then $O(n^{(1-N)n})$, and it is driven to zero. For $N = 1$, on the other hand, a consistent limit exists as was observed in section 4. By writing

$$(\alpha_i' + n)/t_1 = (z_1 + n/t_1 + E\bar{z}/t_1^2), t_1 = \alpha_1'/z_1 + E\bar{z}/t_1^2 \quad (5.8)$$

we get the same equation derived from the zero field solution Eqs.(2.8),(2.9). The origin of the problem is traced back to the fact that each $t_i$ factor in Eq.(4.16) had a power $-\alpha' - n$ in order to give negative powers of $t_i$ for all the terms $\sim T^m(\gamma\bar{z})^{n-m}$ in $(T - \gamma\bar{z})^n$, and this contributed an overall power of $Nn$.

6 Dynamical fluxes

6.1 Wave functions for the charge-flux system

The difficulties of the general $N$-flux problem can be resolved if we treat the fluxes themselves as dynamical objects with their own kinetic energies, for reasons which are not immediately clear. First we note that, when the fluxes are made dynamical objects, they are under the influence of a magnetic (dual) vector potentials produced by the charges which, however, are of the same form as the electric vector potentials produced by the fluxes and acting on the charges. This is obvious because, for the wave function $\psi$ as a function of the coordinates of a charge and a flux, a rotation of the charge around a flux should be equivalent to a rotation of the flux around the charge. (A general description of charges and fluxes as dual objects is given in the Appendix 2.)
Thus the gauge transformation $G$ of Eq.(2.5) will remove the potentials from both the charge and the flux, and we will end up with a free Hamiltonian

$$H = H_z + H_b = -\partial_z \bar{\partial}_z - (1/\mu) \partial_b \bar{\partial}_b$$

(6.1)

where the flux is represented by coordinates $b, \bar{b}$ and mass $\mu$ (the mass of the charge = 1). This generalizes to any number of fluxes as well as charges if we consider only charge-flux interactions, and ignore charge-charge and flux-flux interactions.

The center of mass commutes with the free Hamiltonian, hence it is a constant of motion even with the boundary condition imposed by the flux singularities since they depend on relative coordinates. So the wave function can be separated into that for a free center of mass motion and that for the relative motion, and we may deal with only the relative part. We will not do this here because it will complicate the formulation.

Suppose there are $N$ fluxes with masses $\mu_i, i = 1..N$. We relabel the $2N + 2$ coordinates of the charge and the dynamical fluxes $z, \bar{z}, \{b_i, \bar{b}_i\}$ uniformly as $\{u_i, \bar{u}_i\}, i = 0..N$ and similarly the masses as $\{\mu_i\}, \mu_0 = 1$. The fluxes will eventually be set to their classical positions $b_i^0 = u_i^0, i \neq 0$. The coordinates relative to them are denoted with primes: $u_i' = u_i - u_i^0, i > 0)$. The total Hamiltonian $H$ is the sum of $N$ individual ones

$$H_i = -\partial_u u_i \partial_{\bar{u}_i} \sum H_i = H$$

(6.2)

Further define

$$I_i = \exp(-\mu_i u_i \bar{u}_i / t), I = \prod I_i$$

(6.3)

The $I_i$'s are the heat kernels satisfying the relations

$$H_i I_i = (-\mu_i u_i \bar{u}_i / t^2 + 1/t) I_i = -\partial_t I_i, H I = -\partial_t I, (t \neq 0)$$

(6.4)

Here $t$ may be regarded either as an imaginary time in a time-dependent Schrödinger equation, or simply as a parameter to be integrated over.

With the first interpretation, there are time-dependent solutions for non-dynamical fluxes of the following form, using only the kernel: $I_0(z) : \psi = I_0 \phi(z/t), \phi = \prod_{i=1}^N (z/t - iv_i)^{\alpha_i}$. The flux coordinates $b_i = v_i t$ are moving with velocities $v_i$. But since the $b_i$'s are fixed, it does not correspond to physically moving fluxes. (However, this form is of the same type as the function $\phi$ given below.) Solutions with dynamical fluxes of the simple form $\exp(iPX + i\bar{P}X) \prod (z - b_i)^{\alpha_i}$, where $P$ and $X$ refer to the center of mass, are not admissible since they are not bounded at infinity.
The ansatz for $\psi$ is now,

$$\psi \equiv \int_C \exp(Et)I\phi(u_i, t)dt,$$

with some contour $C$. Now apply $H$ to it, making use of Eq.(6.3):

$$H\psi = \int \exp(Et)[(-\partial_t I)\phi - \sum 1/\mu_i \partial_{u_i} I \partial_{u_i} \phi]dt = E\psi + \int (-\exp(Et)[\partial_t I]\phi + I \sum (u_i \partial_{u_i} \phi)]dt = E\psi + \int \exp(Et)DI\phi dt,$$

$$D \equiv t\partial_t + \sum u_i \partial_{u_i}$$

(6.6)

So $\psi$ will be a solution provided that $D\phi = 0$, i.e. $\phi(u_i, t)$ is a homogeneous function of degree zero. We will choose

$$\phi = \prod_{i=1}^{N} ((u_0 - u_i)/t)^{\alpha_i}'$$

(6.7)

with a contour $C = (-\infty, (0+))$. Thus

$$\psi = \int \exp(Et - \sum u_i \bar{u}_i'/t) \prod((z - b_i)/t)^{\alpha_i'} dt/t^{N+1} = J_{\sum \alpha_i' + N}(2RE^{1/2}) R^{-N} \prod((z - b_i)/R)^{\alpha_i'},$$

$$R = (\sum \mu_i u_i \bar{u}_i')^{1/2}$$

(6.8)

up to a numerical factor. Its conjugate representation is obtained by replacing the $u_i, \alpha'$ by $\bar{u}_i, -\alpha_i'$, and changing the contour to $(0, (0+))$ (or equivalently $t \rightarrow 1/t$ with the original contour). The above $\psi$ obviously has the right monodromy properties. When the flux masses become large, $\mu_i \rightarrow \infty$, the Bessel function will become small and oscillate rapidly except at their intended ‘classical’ positions where $E^{1/2}\mu_i >> |u_i'|$ ($i \neq 0$), $R^2 \rightarrow z(\tilde{z})$. For large values of $|z|$, $\psi$ behaves like a single-flux solution with strength $\sum \alpha' + N$.

In momentum space, $\psi$ has a simple but symbolic form

$$\psi(k, \bar{k}) = \delta(\sum k_i \bar{k}_i/\mu_i - E) \prod(i(k_0 - k_i))^{\alpha_i'}$$

(6.9)

4There is a freedom of including an optional factor $z/t$ representing a nondynamical flux at the origin. When no other fluxes are present, this is equivalent to Eq.(2.13). We also note that even a single flux, not located at the origin of coordinates, does not have a static representation.

5For $|z| >> |b_i|$, $\phi$ satisfies the truncated form of Eq.(6.6): $D'\phi \equiv (z\partial_z + t\partial_t)\phi = 0$ at the saddle points, which assures, as may be seen easily, that $\psi$ is a semiclassical (Hamilton-Jacobi type) solution of $H_z$. 

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The Bessel function in Eq. (6.8) corresponds to a solution of the wave equation in a \((2N+2)\)-dimensional space in the so-called hyperspherical harmonics expansion. In the absence of the factor \(\phi\), the general solutions in this space are spanned by Bessel functions times the \((2N+2)\)-dimensional spherical harmonics. They can be generated by applying \(\partial_{u_i}\) and \(\bar{\partial}_{\bar{u}_i}\), which commute with one another and with \(H\), repeatedly to the \(s\)-wave solution \(J_N/R^N\). Obviously each operation respectively generates a negative or positive angular momentum around the origin in the \(2\)-dimensional subspaces spanned by \(u_i, \bar{u}_i\). In the integral representation of Eq. (6.8) without \(\phi\), application of \(\partial_{u_i}\) \(\partial_{\bar{u}_i}\) will bring down a factor \(\bar{u}_i/t, (u_i/t)\). Application of a monomial of these operators will generate a polynomial in the \(u\)'s and \(\bar{u}\)'s, and this way a complete set of states labeled by the \(N+1\) ‘magnetic’ quantum numbers of \(O(2N+2)\) and a Casimir invariant (the degree of the monomial) will be generated. Fractional factors in \(\phi\) are of a similar nature, but the power of angular momentum around each flux are correlated to be all positive for \(\psi\), or all negative for its conjugate representation. Nevertheless, by applying these same operators to the \(\psi\) of Eq. (6.8), new admissible states can be created as long as the lowest power of each factor \((z-b_i)\) is positive. Together with their conjugates, these will presumably span a complete set of states.

The solutions of the type given by Eq. (6.8), however, are not the natural ones when considered as wave functions for each of the individual particles. Because of the factor \(1/t^{N+1}\) characteristic of the \((2N+2)\)-dimensional space, they go like \(1/R^{N+1/2} \sim 1/|u_i|^{N+1/2}\) when one of the \(u_i\)'s becomes large, contrary to the expectation that they should go like \(1/|u_i|^{1/2}\) in the physical \(2D\) space. The extra power of \(1/t\) which is responsible for this is a quantum effect associated with the dynamical degrees of freedom of each flux. If a flux is to be pinned to a fixed position, its degrees of freedom will have to be frozen. By the same token, if two fluxes are to be merged, those of relative motion will have to be suppressed. But the flux velocities \(\partial_{b_i}/\mu, \partial_{\bar{b}_i}/\mu\) (unless evaluated at the origin) are independent of \(\mu\), which was necessary for construction of the factor \(\phi\). So the derivatives, i.e. the dynamics of the fluxes cannot be ignored at the level of the Schroedinger equation even for large masses. Herein lies the need for treating the fluxes as dynamical, and the difficulties of pinning them down. 

\[\begin{array}{c}
\text{6} \text{However, in the opposite limit of the single charge becoming infinitely heavy, the wave function of the fluxes would reduce to a product of the individual ones as in the case of}
\end{array}\]
A sensible way to recover the proper 2D asymptotic behavior may be to produce highly polarized states in the direction of the particular pair of coordinates $u_i, \bar{u}_i$, i.e., those consisting of high powers of homogenious monomials $u_i/t$ and $\bar{u}_i/t$. But this cannot be a finite series since they increase powers of $1/t$ rather than reducing them. Consider then applying the Green’s function

$$P = 1/(E_z - H_z) = 1/(\partial_z \partial_{\bar{z}} + E_z), \ E_z < E$$

(6.10)
to the $\psi$ of Eq.(6.8). $P$ commutes with $H$ so $P\psi$ is again a solution, and should go like $\cos(2E^{1/2}/|z| + \delta)/|z|^{1/2}$. Since the source $\psi$ is concentrated near the origin ($\sim 1/|z|^{N+1/2}$), it is clear without explicit computation that $P\psi$ will go as $1/|z|^{1/2}$. In order for this to be an exact solution for all $z$, however, $P$ must be a Green’s function in which the flux singularities are already incorporated. Such a Green’s function can in principle be constructed from a complete set of states, but it is not practical to carry it out.

Instead, we can define an effective Hamiltonian for which solutions with the desired properties can be written down easily:

$$H_{\text{eff}} = H_z - z^{-1} \partial_z \sum_0^N b_i \partial_{b_i} - \bar{z}^{-1} \partial_{\bar{z}} \sum_1^N \bar{b}_i \partial_{\bar{b}_i}$$

$$= -z^{-1} \partial_z \sum_0^N u_i \partial_{u_i} - \bar{z}^{-1} \partial_{\bar{z}} \sum_1^N \bar{u}_i \partial_{\bar{u}_i}$$

$$= -\bar{z}^{-1} \partial_{\bar{z}} \sum_0^N \bar{u}_i \partial_{\bar{u}_i} - z^{-1} \partial_z \sum_1^N u_i \partial_{u_i}$$

(6.11)

with the stipulation that elementary solutions depend only on either the $b$’s or the $\bar{b}$’s, and satisfy proper monodromy and boundary conditions. Indeed,

$$\psi = \int (-z\bar{z}/t + Et) \prod (z - b_i)/t)^{\alpha_i} dt/t$$

$$= J \sum_{\alpha_i} (2E^{1/2}/|z|) \prod (z - b_i)/|z|^{\alpha_i}$$

(6.12)

and its conjugate form clearly satisfy Eq.(6.11), and have the right asymptotic behavior. Being linear in the derivatives, we may say that the flux coordinates retain only partially their dynamical status. The general properties of $H_{\text{eff}}$ will not be examined here, but Eq.(6.12) will be adopted in subsequent sections.

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\[\text{many charges in the presence of a single static flux.}\]
6.2 Solutions in a magnetic field

The Hamiltonian for a charge in a magnetic field is

$$H_z = -(\partial_z - \bar{z}\gamma)\partial_{\bar{z}}$$  \hspace{1cm} (6.13)

and a solution which has a zero-field limit is, according to Eqs.(4.8),

$$\psi = \int \exp(zt)(1 - \bar{z}\gamma/t)^n dt/t$$

$$= \int \exp(-z\bar{z}/t)(1 + \gamma t)^n dt/t = \int I_z dt/t$$

$$I_z = \exp(-z\bar{z}/t)(1 + \gamma t)^n/t$$  \hspace{1cm} (6.14)

The last expression was obtained by a change of variable $t \rightarrow -\bar{z}/t$. It satisfies the relation

$$(H_z - n\gamma)I_z = -\partial_t((1 + \gamma t)I_z)$$  \hspace{1cm} (6.15)

By substituting the new $I_z$, the analog of Eqs.(6.4-5) becomes

$$\psi = \int I\phi dt,$$

$$(H - E)\psi = \int -(\partial_t(1 + \gamma t)I)\phi + \sum(u_i/t)\partial_{u_i}\phi,$$

$$= \int ID'\phi dt/t, D' = (1 + \gamma t)t\partial_t + \sum u_i\partial_{u_i}$$  \hspace{1cm} (6.16)

The solution for $\phi$ is then obtained by substituting $t/(1 + \gamma t)$ for $t$ in Eq.(6.5), and $\psi$ becomes (dropping $\exp(ET)$ in favor of $(1 + \gamma t)^n$)

$$\psi = \int \exp(-\sum u_i\bar{u}_i/t)(1/t + \gamma))^N(1 + \gamma t)^n$$

$$\prod((z - b_i)(1/t + \gamma))^{\alpha'_i} dt/t$$  \hspace{1cm} (6.17)

The integrand has a branch point at $t = 0$ of order $-N - \sum \alpha'$ and one at $t = -1/\gamma$ of order $N + n + \sum \alpha'$. We can adopt a closed contour $(-1/\gamma+, 0+)$ around them. When the integral is evaluated by expanding the exponential, we get a series of Beta functions

$$(\gamma\gamma')^m B(-N - \sum \alpha' - n, N + n + \sum \alpha' + 1)/m!$$  \hspace{1cm} (6.18)

which is zero if $m \geq n + 1$, so it is a polynomial of order $n$. In the limit

$$\gamma \rightarrow 0, n = E/\gamma \rightarrow \infty$$

for fixed $E$, $(1 + \gamma t)^n$ becomes $\exp(ET)$, the contour becomes $(-\infty, (0+))$, and Eq.(6.17) reduces to that for the field-free case.
6.3 Shifted Landau levels

In section 4.2 we showed that there are shifted Landau levels for a single static flux. Eq.(6.17) serves a basis for incorporating $N$ dynamical fluxes. Assume this time that $\gamma < 0$, replace $n$ in Eq.(6.16) by $-n - \sum \alpha' - 1$ so that the energy $E/\gamma = (n + \sum \alpha' + 1/2)/|\gamma|$ is still positive. The singularity at $-1/\gamma$ becomes a pole of order $N - n$. With a change of variable $t = s - 1/\gamma$ the full wave function $\Psi$ can be turned into

$$
\Psi = \exp(-\gamma z \bar{z}/2) \psi \\
= \int_0^+ \exp(-r'^2/(s - 1/\gamma) - \gamma z \bar{z}/2) \gamma^{N - n - 1} \prod((z - b_i)') \left(s - 1/\gamma)^{-N - \sum \alpha' - 1}s^{N - n - 1}ds \right)
$$

(6.19)

which is nonzero if $n \geq N$, or if $n \geq 0$ after $N$ is eliminated. $\Psi$ converges at $\infty$ since $\gamma < 0$ and $r' = z \bar{z}$ effectively. The energy is $n + \alpha'$. This, then, is the generalization of the shifted Landau levels, Eqs.(4.9), (4.10). For a single static flux plus $N$ dynamical fluxes, the energy gets shifted by all the flux strengths simultaneously. We do no know if partially shifted states by some of the $\alpha'$s can be found. When $\gamma \to 0, n \to \infty$, the integral is pushed to $\infty$ and disappears, meaning that the shifted states have no free field limits.

7 Large N limit and flux medium

According to the results of Section 6.1, we will effectively regard the fluxes as fixed objects and deal with wave functions of the type given in Eq.(6.12). The order of the Bessel function representing $\psi$ then goes up with the number of fluxes as $\sum \alpha'$, with all $\alpha$'s being of the same sign $> 0$ and adding up. Suppose the fluxes of equal strength $\alpha$ are distributed with a density $1/a^2$ up to a large radius $R_0$. The total flux is $\alpha \pi (R/a)^2$. The angular momentum $L$ of the wave function at radius $R < R_0$ ($R > R_0$) is $\alpha \pi (R/a)^2$ ($\alpha \pi (R_0/a)^2$). Its azimuthal momentum is $L/R = \alpha \pi R/a^2$ ($\alpha \pi (R_0/a)^2/R$), so the radial momentum for large $R$ : $\alpha \pi R^2E >> 1$, is $p_R \sim \pm i \alpha \pi R/a^2$, and the radial “phase” is $\int ip_Rdr \sim \pm \alpha \pi (R/a)^2/2$ ($\pm (R_0/a)^2 \ln R$). For $R < R_0$, the wave function will behave like Gaussian or anti-Gaussian. Below we will investigate it in more detail.

Assume that all fluxes have the same strength $\alpha$, and form a regular square lattice of unit size at sites $b_i, i = m + in$, although these assumptions
are not essential. The gauge potential $A_z, A_{\bar{z}}$ and the gauge function $G$ are

\begin{align*}
A_z &= (\alpha/2) \sum \frac{1}{(z - b_i)} , \\
A_{\bar{z}} &= -(\alpha/2) \sum \frac{1}{(\bar{z} - \bar{b}_i)} , \\
G &= \prod \left( \frac{(\bar{z} - \bar{b}_i)/(z - b_i)}{\alpha/2} \right)^{\alpha/2} \tag{7.1}
\end{align*}

As the sum goes to infinity, they fail to converge, hence an extra pure gauge subtraction terms become necessary (except for $i = (0,0)$). The modified forms are then

\begin{align*}
A_z &= (\alpha/2) \zeta(z), \\
A_{\bar{z}} &= -(\alpha/2) \zeta(\bar{z}), \\
\zeta(z) &= \sum \left( \frac{1}{(z - b_i)} + \frac{1}{z + z/b_i^2} \right), \\
G &= \left[ \frac{\sigma(z)}{\sigma(\bar{z})} \right]^{\alpha/2}, \\
\sigma(z) &= z \prod_{i \neq 0} ((1 - z/b_i) \exp(z/b_i + z^2/2b_i^2)) \tag{7.2}
\end{align*}

Here $\zeta$ and $\sigma = \exp(\int \zeta dz)$ and are the elliptic functions of Weierstrass with full periods $1, i$. Because of the subtractions, they are not periodic but satisfy

\begin{align*}
\zeta(z + m + in) &= \zeta(z) + \pi(m - in), \\
\sigma(z + m + in) &= \sigma(z) \exp[\pi(m^2 + n^2)/2 + i\pi mn/4 + z\pi(m - in)] \tag{7.3}
\end{align*}

The leading asymptotic factors are insensitive to the regular lattice structure, and may be derived by a continuum approximation. If the flux density is $1/a^2$,

\begin{align*}
\sum \frac{1}{(z - b_i)} &\to \int \frac{1}{(z - r) \exp(i\theta) r dr d\theta/a^2} \\
&= \pi|z|^2/za^2 = \pi \bar{z}/a^2 \tag{7.4}
\end{align*}

The contributions to the integral over $b$ from $r > |z|$ vanish because the phases of the contributions from points at radius $R > |z|$ from the origin cancel. Integrating this from 0 to $z$ along the radius, we get

\begin{align*}
z \prod_{i} (z - b_i)/b_i \equiv \sigma_R \sim \exp(\pi(|z|/a)^2/2) \tag{7.5}
\end{align*}

These non-analytic expressions are a result of confusing $z$ and the cell coordinates $(m,n)a$. We can also use this method to define a cutoff version of $\zeta$.
and $\sigma$:

\[
\begin{align*}
\zeta_{R_0}(z/a) & \sim \pi z/a \quad \text{if} \quad |z| < R_0, \\
& = 0 \quad \text{if} \quad |z| > R_0; \\
\sigma_{R_0}(z/a) & \sim \exp(N/2) \quad \text{if} \quad |z| < R_0, \\
& \sim \exp(N/2) N = \pi R_0^2 \quad \text{if} \quad |z| < R_0, \\
& \sim \exp(N/2)(z/R_0)^N \quad \text{if} \quad |z| > R_0, 
\end{align*}
\]

(7.6)

We can now substitute them in Eq.(6.12), taking the origin of coordinates at the center of the lattice. Then $\phi = \sigma_R/t^{\alpha N}$. First, the value of $\psi$ at the origin can be exactly calculated, and yields $\psi(0) = 1/N!$. For the asymptotic behavior, we can use the the saddle points to get

\[
\psi \sim C(t_0) \exp(-E|z|^2/t_0 + t_0) t_0^{-N\alpha} \sigma_{R_0}(z/a)^{\alpha}
\]

(7.7)

where $C(t_0)$ is the contribution from the Gaussian integral around $t_0$, and

\[
\begin{align*}
t_0 &= \pm i E^{1/2}|z| \quad \text{if} \quad N >> E|z|^2, \\
\psi &\sim \exp(\pm 2iE^{1/2}|z| + N\alpha/2)(z/\pm iaE^{1/2}|z|)^{N\alpha}/(\pm i E^{1/2}|z|)^{1/2}; \\
t_0 &= N\alpha \quad \text{if} \quad ER_0^2 < E|z|^2 < N, \\
\psi &\sim N^{1/2} \exp(3N\alpha/2)(z/aN\alpha)^{N\alpha}/N! \\
&= \exp(N\alpha/2)(z/a)^{N\alpha}\psi(0); \\
t_0 &= N\alpha \quad \text{if} \quad 1 < E|z|^2 < E R_0^2, \\
\psi &\sim \exp(\alpha \pi |z|^2/2)(z/|z|)^{\alpha\pi|z|^2}\psi(0)
\end{align*}
\]

(7.8)

The first form is the true asymptotic limit. The second is valid in the near outside region $|z| > R_0$. The third applies to the inside of the flux medium. These are all consistent with the physical argument given at the beginning of the section. From above we see that the wave function inside the flux area grows with the distance in an anti-Gaussian way like $\exp(\alpha \pi |z|^2)$ inside

\footnote{A possible alternative choice for $\phi$ is $\phi = \sigma_{R_0}(z/a)/\sigma_{R_0}(lt)$ with some $l$ and the contour running outside of the fluxes: $lt > R_0$. But this leads to the same asymptotic results as those given below, up to a normalization.}

\footnote{In the exterior region, $\phi$ satisfies $D'\phi = z\partial_z \phi = -t\partial_t \phi = \alpha N$, so $\psi$ is a semiclassical solution of $H_z$ (footnote 5), whereas $D'\phi$ is not zero in the interior region. (Of course the full $D\phi = 0$ is always guaranteed).}
the medium, then with a power like $|z|^{N\alpha}$ to distances $E^{1/2}|z| > N\alpha$ where
the true asymptotic behavior sets in. In other words, the wave function is
pushed out of the flux region and beyond (unless the energy is so large as to
skip the intermediate regimes).

8 Summary and discussion

In terms of physical concepts, the Aharonov-Bohm problem for many fluxes
is a straightforward extension of the single flux case, and is not expected to
contain anything conceptually new. Yet in an attempt to construct explicit
solutions, we have found some unexpected features. In the idealized A-B
problem of an infinitely thin flux, the wave function of a charge in its vicinity
is described by Bessel functions whose order is shifted by the flux strength $n$
to $|\alpha + n|$. Since the latter has to vanish at the flux, the component which
would be an $s$-wave in free space is pushed out irrespective of the sign of $\alpha$.
This effect will magnify in a medium made up of fluxes.

To handle the general case of many fluxes, the gauge potential may be
removed by a singular gauge transformation. The Hamiltonian is reduced to
that of a free particle except for boundary conditions. Around each flux $i$ the
transformed wave function must develop a branch point in one of the complex
coordinates, with a positive fractional power so that it has the form $z^{\alpha_i + n_i}$, or
$\bar{z}^{-\alpha_i + n_i}$, times a holonomic function in $z$ and $\bar{z}$. In particular, at zero energy
the general solution is an arbitrary sum of a function of $z$ and a function
of $\bar{z}$, and each part must carry all the branch points. In other words, the
fractional angular momentum around each flux, as opposed to the intrinsic
flux strengths $\alpha_i$, must all be either positive for the analytic, and negative
for the anti-analytic solution. At nonzero energies, the wave function of
either type contains both variables, but by continuity the above qualitative
features do not change. (This turns out to be true even in a magnetic field
where the spectrum is discrete.) The total angular momentum of a wave
function around a large circle containing a cluster of fluxes is then a sum
of positive or negative numbers which will grow with the number of fluxes,
resulting in the expulsion of a charge from the interior of the cluster. This
is because the wave function has to vanish at each flux site, whether the
fluxes add up to an integer or not. Only if an individual flux is integral, will
it not contribute to the effect. This conclusion does not require an explicit
construction of solutions.
These general results do not seem to be due to our reliance on the singular gauge and analyticity, or the simplifying assumption of infinitely thin fluxes. To see this, first consider a flux $\alpha_1$ at the origin in the original gauge so a solution may be expanded in Bessel functions $J_{|\alpha_1+n|}$ of positive fractional order. If there is a second flux $\alpha_2$ at point $P$, they have to be re-expanded in Bessel functions $J_{|\alpha_2+m|}$ around $P$. But the complete set of eigenfunctions for the latter covers only the space $R^2 - P$. Therefore we have to start from the beginning with a function that has the proper behavior at both sites. (The general addition formulas for Bessel functions show that all positive and negative powers and with the same $\alpha$ are involved in the re-expansion around $P$ if $\alpha$ is not an integer. The integer case is special since $J_{-n} = (-1)^n J_n$.)

It is also instructive to compare the A-B problem with that of hard core potentials. Here the similarity is that the eigenfunctions do not form a complete set in $R^2$. The difference is that in the latter the eigenfunctions are Bessel functions of integer order, the $s$-wave included. If the radius of the potential is finite, the solutions involve both regular functions $J_n$ and singular ones $N_n$. This is equivalent to having both positive and negative orders for noninteger $n$, or the Hankel functions $H^{1,2}_{\alpha+n}$. In a realistic A-B problem where the fluxes have a finite size, the solution around a flux will be a superposition of functions $J_{\alpha+n}$ with all $n$, or, after gauging away the potential, it will have all powers $z^{\alpha+n}$ or $z^{\alpha-n}$. This, however, would not alter the difficulties with many fluxes since the quasi-free wave function has to satisfy not only the condition that it vanishes at the boundaries but also that it has the right fractional angular momenta in the neighborhood of the fluxes, in contrast to the hard core case where only the boundary condition matters.

The explicit construction of wave functions for many fluxes has also turned out to be possible, with or without a magnetic field. To construct a solution in free space, however, the fluxes have to be treated as dynamical objects with their own Hamiltonians. Technically we may say that it is because kinematic correlations are induced between fluxes indirectly through those between the charge and the fluxes, and are independent of the masses. But the role of the magnetic field in this context is unclear. These are puzzling features that remain to be understood in simple physical terms. Explicit solutions with proper asymptotic behavior can be written down for an effective Hamiltonian which partially retains the dynamical nature of the fluxes. With a magnetic field, it is noteworthy that some of the Landau states get the energy shifted by the fluxes in proportion to the flux strength while others
remain unaffected. The effect is asymmetric for the lowest level with respect to the relative sign of the field and the flux. This nonlinear behavior may lead to some interesting consequences.

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Appendices

A.1 Complex metric

Let a covariant and a contravariant vector be denoted in general by $v_c$ and $v_v$ respectively, and let the contravariant vector $v_v = (x, y)$ be mapped to $(z, \bar{z}) = (x + iy, x - iy)$. The metric for the complex pair is then off-diagonal:

$$g_{ik} = (1/2)[[0, 1], [1, 0]], \quad g^{ik} = 2[[0, 1], [1, 0]], \quad (A.1)$$

The rule for forming a vector $A$ in the complex metric is

$$A_c = (A^x, A^\bar{z}) = (A^x + iA^y, A^x - iA^y), \quad (A.2)$$
$$A_v = (A_x, A_{\bar{z}}) = (A^x - iA^y, A^x + iA^y)/2$$

Since the gradient operator is

$$\text{grad} = (\partial/\partial z, \partial/\partial \bar{z}), \quad (A.3)$$

and the Laplace operator becomes

$$\nabla^2 = 2\{\partial/\partial z, \partial/\partial \bar{z}\} \quad (A.4)$$

Rotation $R$ by an angle $\theta$ is represented by a diagonal matrix

$$R_c = (\eta^{-1}, \eta), \quad R_v = (\eta, \eta^{-1}), \quad \eta = \exp(i\theta) \quad (A.5)$$
so the outer product of two covariant or two contravariant vectors is given by

\[ v \times u = i(v_z u_z, -v_z u_z), \]
\[ v' \times u' = -i(v^z u^z, -v^z u^z) \]  
(A.6)

It follows also that the divergence and curl \((= F_{z\bar{z}})\) of \(A\) are given respectively by

\[ \text{div} \, A = (\partial_z A_z + \partial_{\bar{z}} A_{\bar{z}})/2, \]
\[ \text{curl} \, A = -\text{curl} \, A' = i(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)/2 \]  
(A.7)

The Gauss’ and Stokes’ formulas read (on shell)

\[ \int v_n ds = -i \int (v_z dz - v_{\bar{z}} d\bar{z}), \]
\[ \int v_s ds = \int (v_z dz + v_{\bar{z}} d\bar{z}) \]  
(A.8)

respectively.

The Laplace operator applied to the Green’s function

\[ G = \ln |z| = (\ln z + \ln \bar{z})/2 \]  
(A.9)

is zero, but the delta function singularity at the origin of the on-shell space can be seen from the Gauss’ formula

\[ \int \text{grad} \, G_n ds = (-i/2) \int (dz/z - d\bar{z}/\bar{z}) = 2\pi \]  
(A.10)

### A.2 Field theory description of charges and fluxes

The fluxes can be regarded as dynamic magnetic particles. They interact with the charges through the Aharonov-Bohm effect, but not directly among themselves. A flux \(\Phi\) is a concentration \(\Phi = Ba^2\) of the magnetic field density \(B\) over an infinitesimal area \(a^2\), its energy (per unit \(z\) direction) is a part of the energy density \(F_{xy}^2/2\) of the magnetic field plus an amount necessary to keep it concentrated. We will assume uniformity and ignore the dynamics in the \(z\) direction. When a flux moves, the magnetic field inside will acquire
electric field components of order $v/c$, so $(B_z, -E_y, E_x)$ will be associated with a moving flux. The vector potential associated with $E$ will be present in the medium, even though it may be a shielded potential.

Let $A$ be the usual vector potential, $G$ and $F$ two vector fields, $J$ and $K$ the electric and magnetic currents in $2 + 1$ dimensions, so they have mass dimension 2. $J$ is then not the usual charge density since the charge is not extended in the $z$ direction, but may be thought of as a kink soliton $\partial_z \sigma$ of an extended field $\sigma$. This is the only place where the $z$ direction enters. Let the Lagrangian $L$ be

$$L = -\sum G_i \cdot F_i / 2 - iG \cdot \text{curl}A + eA \cdot J + gG \cdot K + L_{\text{mat}},$$

$$G_1 = D_2, \ G_2 = -D_1, \ G_3 = H_3, \ F = \Lambda G,$$

$$\Lambda = \text{diagonal matrix} [\lambda, \lambda, \kappa] \quad \text{(A.11)}$$

$L_{\text{mat}}$ refers to the kinetic part of ‘matter’ (charge and flux) Lagrangian. Euclidean metric is used for simplicity, with $x_3 = ix_0$, etc. The electric and magnetic charges $e$ and $g$ are dimensionless; $G$ and $F$ have mass dimensions 1 and 2 respectively. $B, H, E, D$ have the usual meaning in the Maxwell theory except for dimensionality: the parameters $\kappa, \lambda$ are respectively the analog of magnetic permeability and inverse dielectric constant with the dimensions of mass. Eq.(A11) shows that the magnetic “vector potential” is invariant under the usual gauge transformation of the electric vector potential $A$. Varying $G$ and $A$ as independent fields, we get the field equations

$$F + i\text{curl}A = gK, \quad i\text{curl}G = eJ, \quad \text{(A.12)}$$

If the magnetic current $K$ is absent, Eq.(A12) takes the usual Maxwell form. If, on the other hand, $\kappa$ and $\lambda$ are set to zero, the equations show a symmetry under duality between $J, A$ and $K, G$:

$$i\text{curl}A = gK, \quad i\text{curl}G = eJ, \quad \text{(A.13)}$$

and an invariance under both an electric and a magnetic gauge transformation:

$$A \rightarrow A + \text{grad} \phi, \ G \rightarrow G + \text{grad} \chi \quad \text{(A.14)}$$

(Boundary contributions are assumed to vanish under these gauge transformations.) The left-hand side of Eq.(A13) are the Coulomb-Lorentz forces acting respectively on charge and flux, which are seen to come directly from the currents of their opposite numbers.
For static electric and magnetic sources located respectively at \(x_j, y_j\) and \(x_k, y_k\), Eq.(A13) yields

\[
A_1, A_2 = g(-v_2, v_1)/(\pi |v|^2), \quad v = (x - x_k, y - y_k)
\]
\[
G_1, G_2 = e(-u_2, u_1)/(\pi |u|^2), \quad u = (x - x_j, y - y_j)
\]

\[
A_+ = -iA_1 + A_2 = g/z, \quad A^- = iA_1 + A_2 = g/z, \quad z = v_1 + iv_2,
\]
\[
G_+ = iG_1 - G_2 = e/w, \quad G_- = iG_1 - G_2 = -e/w, \quad w = u_1 + iu_2 \quad (A.15)
\]

Hence the vector fields \(eA\) and \(gG\) acting on charge and on flux are the same, and proportional to \(eg\), and the same form of gauge transformation removes the potential from the charge and the flux Hamiltonian.

If \(\Lambda \neq 0\), there is no duality symmetry. Conservation of the magnetic charge requires

\[
\text{div} \, \Lambda \, G = 0 \quad (A.16)
\]

which is the same as in the Maxwell case. The magnetic gauge invariance still holds if the \(\chi\) in Eq.(A14) satisfies

\[
\text{div} \, \Lambda \, \text{grad} \, \chi = 0 \quad (A.17)
\]

By taking the Coulomb gauge \(\partial_1 A_1 + \partial_2 A_2 = 0\), Eq.(A13) leads to

\[
\nabla^2 A_3 = -ig\lambda^{-1}(\partial \times K)_{12} - \lambda eJ_3,
\]
\[
\lambda^{-1}\nabla^2 A_1 + \kappa^{-1}\partial_3 F_{31} = -ig(\lambda^{-1}\nabla_2 K_3 + \kappa^{-1}\partial_3 K_2) - ieJ_1,
\]
\[
\lambda^{-1}\nabla^2 A_2 - \kappa^{-1}\partial_3 F_{32} = -ig(\lambda^{-1}\nabla_1 K_3 - \kappa^{-1}\partial_3 K_1) - ieJ_2 \quad (A.18)
\]

If static, \(\partial_3 = 0\), or if \(\kappa >> \lambda\), the contribution to \(A\) from \(K\), and one to \(G\) from \(J\) become

\[
\nabla^2 A_3 = -\lambda eJ_3 - ig\lambda^{-1}(\partial \times K)_{12},
\]
\[
\nabla^2 A_1 = -ig\partial_2 K_3, \quad \nabla^2 A_2 = ig\partial_1 K_3,
\]
\[
G_3 = i\kappa^{-1}g(\partial \times A)_{12},
\]
\[
G_1 = i\lambda^{-1}\partial_2 A_3, \quad G_2 = -i\lambda^{-1}\partial_1 A_3 \quad (A.19)
\]

Since the Green’s kernel is \(\ln(r)/2\pi, r = \sqrt{(x^2 + y^2)}\), the potentials \(A_{1,2}\) and \(G_{1,2}\) are then found to be the same as Eq.(A13):

\[
e A_3 = ieg \ln(r)/\pi, \quad eA_{1,2} = -gG_{1,2} = eg(\partial_2, -\partial_1) \ln(r)/\pi \quad (A.20)
\]
Eq. (A18) shows that the magnetic ‘Coulomb’ potential $G_3$ acting on magnetic charge $gK_0 = -igK_3$ is, apart from self interaction, indeed the usual magnetic field $G_{12} = H_3 = B_3/\kappa$, where $\kappa$ is interpreted as an effective magnetic permeability. The term $G_{12}F_2/2 = \kappa H_3^2/2$ is the magnetic field energy. When an external magnetic field $H_{ext}$ is imposed, we should change it to $\kappa(H + H_{ext})^2/2$, which means that the flux will feel a magnetic Coulomb potential $G_{ext}$. By the same token, if an external electric field is present without suffering shielding, the flux will feel an electric Lorentz force. On the other hand, the potential $A_3$ acting on a charge at $(x, y)$ and generated by another charge at 0 is $-\lambda \ln(r)/(4\pi r)$. Since this should be the 3-D Coulomb potential $1/(4\pi r)$, we must interpret $\lambda$ to be $\sim \partial_z$, i.e. the derivative in the hidden direction.

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As was mentioned above, charge is like a kink soliton, or a monopole with a string, that is sitting in the 2-D space. If charges were extended in the $z$ direction like the fluxes, the logarithmic potential would arise by integrating the 3-D Coulomb potential over $z$: $\int dz/(r^2 + z^2) = \ln((z + (r^2 + z^2)^{1/2})/r)$ (after renormalizing away infinities). We get back the 3-D potential if we take this as the 2-D potential and differentiate it by $z$ at $z = 0$. 

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