Analysis and approximations of an optimal control problem for the Allen–Cahn equation

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Abstract

The scope of this paper is the analysis and approximation of an optimal control problem related to the Allen–Cahn equation. A tracking functional is minimized subject to the Allen–Cahn equation using distributed controls that satisfy point-wise control constraints. First and second order necessary and sufficient conditions are proved. The lowest order discontinuous Galerkin—in time—scheme is considered for the approximation of the control to state and the state to adjoint mappings. Under a suitable restriction on maximum size of the temporal and spatial discretization parameters $k$, $h$ respectively in terms of the parameter $\epsilon$ that describes the thickness of the interface layer, a-priori estimates are proved with constants depending polynomially upon $1/\epsilon$. Unlike to previous works for the uncontrolled Allen–Cahn problem our approach does not rely on a construction of an approximation of the spectral estimate, and as a consequence our estimates are valid under low regularity assumptions imposed by the optimal control setting. These estimates are also valid in cases where the solution and its discrete approximation do not satisfy uniform space-time bounds independent of $\epsilon$. These estimates and a suitable localization technique, via the second order condition (see Arada et al. in Comput Optim Appl 23(2):201–229, 2002; Casas et al. in Comput Optim Appl 31(2): 193–219, 2005; Casas and Raymond in SIAM J Control Optim 45(5):1586–1611, 2006; Casas et al. in Control Optim 46(3):952–982, 2007), allows to prove error estimates for the difference between local optimal controls and their discrete approximations as well as between the associated state and adjoint state variables and their discrete approximations.

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1 Introduction

We consider the following distributed optimal control problem that is governed by the Allen–Cahn equation: Given data $y_0, y_d, y_\Omega$ and fixed parameters $\gamma \geq 0, \mu > 0$

minimize

$$J(u) = \frac{1}{2} \int_0^T \int_\Omega |y_u(t, x) - y_d(t, x)|^2 \, dx \, dt + \frac{\gamma}{2} \int_\Omega |y_u(T, x) - y_\Omega(x)|^2 \, dx$$

subject to

$$\begin{align*}
\frac{1}{\epsilon^2} \left( y_u^3 - y_u \right) = u & \quad \text{in } \Omega_T = \Omega \times (0, T), \\
\left. \frac{\partial y_u}{\partial n} \right|_{\Sigma_T} = 0 & \quad \text{on } \Sigma_T = \partial \Omega \times (0, T), \\
y_u(\cdot, 0) = y_0 & \quad \text{in } \Omega.
\end{align*}$$

The Allen–Cahn equation can be viewed as a prototype equation that describes a phase separation process. The physical meaning of (1.2) for the uncontrolled case with zero Neumann boundary data (i.e. when $u = 0$ and $\frac{\partial y}{\partial n} = 0$) is well understood. Indeed, denoting the nonlinear term by $F : \mathbb{R} \to \mathbb{R}$, $F(s) := \frac{1}{\epsilon^2} (s^3 - s)$, we observe that $F(s)$ represents the derivative of the classical double-well potential. Hence, at the uncontrolled case, it is well known that the solution (denoted by $y$) of (1.2) develops time-dependent interfaces $\Gamma_t := \{ x \in \Omega : y(t, x) = 0 \}$ that separate regions where $y \approx 1$ from regions where $y \approx -1$. Furthermore, it is well understood that the solution moves from one region to the other very fast, with typical length of “diffuse interface” $O(\epsilon)$ and that the Allen–Cahn equation satisfies a maximum principle in the sense that if the initial data satisfies $|y_0(x)| \leq 1$ then the solution also satisfies $|y(t, x)| \leq 1$. Furthermore, the “smallness” of the parameter $\epsilon < < 1$ should be taken into consideration in both the analysis as well as the numerical analysis of problems involving the Allen–Cahn equation. We refer the reader to [5] for an overview of various phase field models, their analysis and approximations.

Our work is devoted to the analysis and numerical approximation of an optimal control problem related to (1.2). In particular, our goal is to drive the solution $y_u$ of (1.2) as close as possible to a given target profile $y_d$ by using a control function $u$.

It is crucial to observe that in the optimal control setting the initial profile $y_0$ does not necessarily satisfy $|y_0(x)| \leq 1$ and/or does not possess additional smoothness properties. In addition, our possible target profile $y_d$ is not necessarily $\{-1, 1\}$ and/or satisfies additional smoothness properties.
In (1.1) $\mu > 0$ denotes the typical Tikhonov regularization term, while the inclusion of the terminal tracking term in the functional (with $\gamma \geq 0$) facilitates the effectiveness of approximations near the end point of the time interval. Throughout this work the set of admissible controls is defined as

$$U_{ad} = \left\{ u \in L_2(0, T; L_2(\Omega)) ; \ u_a \leq u(t, x) \leq u_b \text{ for a.e. } (t, x) \in [0, T] \times \Omega \right\},$$

and the optimal control problem is formulated, in the standard reduced functional form, as

$$\left\{ \begin{array}{l}
\min J(u) \\
u \in U_{ad}.
\end{array} \right.$$  (1.3)

Note that the above optimal control problem is non-convex. Thus, it is necessary to distinguish between local and global solutions. A control $\bar{u} \in U_{ad}$ is said to be a local optimal control of (1.3) in the sense of $L_2(0, T; L_2(\Omega))$, if there exists $\alpha > 0$ such that $J(\bar{u}) \leq J(u)$ for all $u \in U_{ad} \cap B_\alpha(\bar{u})$, where $B_\alpha(\bar{u})$ is the open ball of $L_2(0, T; L_2(\Omega))$ centered at $\bar{u}$ with radius $\alpha$. We shall use the notation $\bar{y} := y_{\bar{u}}$ and $\bar{\phi} := \phi_{\bar{u}}$ for the associated state and adjoint state solutions, respectively.

Optimal control problems having states constrained to semi-linear parabolic PDEs have been extensively studied. We refer the reader to [6] (see also references within) for an overview of optimal control problems related to classical elliptic and parabolic semi-linear PDEs. In fact, various issues such as existence, first and second order necessary and sufficient conditions have been considered even for nonstandard optimal control problems related to semi-linear parabolic PDEs: see for instance [7] (BV controls), [8] (control problems in absence of Tikhonov regularization term), [9–12] (and references within) (control problems with sparse controls), [13] (control problems for non smooth semilinear parabolic PDEs).

The numerical analysis of optimal control problems with semi-linear parabolic PDEs as constraints have been considered in [14] (control constraints, piecewise constants / linear controls / variational discretization approach), [15] (discontinuous in time schemes, no constraints), and [8] (error estimates in absence of Tikhonov term). In these works, error estimates of fully-discrete approximations have been presented under a monotonicity assumption on the semilinear term. In [16–19] the numerical analysis of optimal control problems related to the evolutionary Navier–Stokes equations, including error estimates, was considered. Finally, for related works for other nonlinear parabolic PDEs, we refer the reader to [20] (time-discrete two-phase flows), [21] (a-priori estimates for a coupled semilinear PDE-ODE system).

A common ingredient in the error analysis of fully-discrete schemes for optimal control problems related to nonlinear parabolic PDEs, under control constraints (such as [8, 14, 16–19, 21]) is the use of results regarding the Lipschitz continuity of the control to state and of state to adjoint mappings, the derivation of first and second order necessary and sufficient optimality conditions, and detailed error estimates of the corresponding control to state, and state to adjoint mappings that allow the classical localization argument of [1–4] (developed for error analysis of discretization schemes.
for semilinear elliptic PDE constrained optimization problems) to work under the prescribed regularity assumptions.

However, to our best knowledge none of the above key results are present in case of the Allen–Cahn equation. For instance, observe that the Allen–Cahn equation involves a nonmonotone nonlinearity that satisfies
\[\frac{1}{\varepsilon^2} F'(s) := \frac{1}{\varepsilon^2}(3s^2 - 1) \geq -\frac{1}{\varepsilon^2}.\]
As a consequence, for realistic values of ε the classical approaches for proving the Lipschitz continuity of the control to state mapping as well as its numerical analysis, fail since they introduce constants depending exponentially upon \(\frac{1}{\varepsilon^2}\). For the numerical analysis of the uncontrolled Allen–Cahn equation, i.e. for \(u = 0\), this difficulty was first circumvented in the seminal works of [22–26], where error estimates (a-priori and a-posteriori) were established for the homogeneous Allen–Cahn equation with constants that depend polynomially upon \(\frac{1}{\varepsilon}\) based on suitable discrete approximation of the spectral estimate and a nonstandard continuation argument of the form of a nonlinear Gronwall Lemma. Recall that the spectral estimate concerns the principal eigenvalue of the linearized Allen–Cahn operator about the solution \(y\) of the homogeneous (uncontrolled) Allen–Cahn equation. In particular, for the case of zero Neumann boundary data, it involves the quantity
\[\Lambda(t) := \inf_{v \in H^1(\Omega) \setminus \{0\}} \frac{\|\nabla v\|^2_{L^2(\Omega)} + \varepsilon^{-2} ((3y^2 - 1)v, v)}{\|v\|^2_{L^2(\Omega)}}. \tag{1.4}\]
The celebrated works [27–29] showed that for the homogeneous (uncontrolled) equation, \(\Lambda \in L^\infty(0, T)\) can be bounded independently of \(\varepsilon\) for the case of smooth, evolved interfaces. For the nonhomogeneous case, in Sect. 5, we show that a similar estimate holds, with a different bound that it is still independent of \(\varepsilon\), provided that the spectral estimate is valid in the homogeneous (uncontrolled) case with the same initial data. It is worth noting that (1.4) plays an important role in the analysis of the Allen–Cahn equation as well as on its limit behavior as \(\varepsilon \to 0\). We refer the reader to the important works of [30–33] for other approaches.

In addition, the numerical analysis of the homogeneous (uncontrolled) problem, cannot be directly employed in the optimal control setting since such analysis typically requires regularity assumptions on the state (as well as to its fully-discrete counterpart) that are not available within the optimal control context. For example, the assumption \(y \in L^\infty(0, T; H^2(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap H^2(0, T; L^2(\Omega))\) (used in the above mentioned works for a-priori estimates of the uncontrolled Allen–Cahn equation) is not present in our optimal control setting. In addition, similar difficulties arise when dealing with analysis and numerical analysis of the state to adjoint mapping. In fact, it turns out that the analysis can be even harder due to the absence of the cubic term that generates the \(L_4(0, T; L_4(\Omega))\) norm when testing (1.2) with \(y\).

Our work aims to present an error estimate for a fully-discrete scheme based on the discontinuous -in time- Galerkin dG(0) framework, (piecewise constant approximations in time, finite elements based on piecewise linear polynomials approximations in space). First, we present a detailed analysis of the control to state, and of the state to adjoint mappings and we derive first and second order optimality conditions. In particular, under a closeness assumption on the controls, we establish the Lipschitz
continuity of the control to state mapping, with Lipschitz constant that is independent of \( \epsilon \) in the \( L_\infty (0, T; L_2 (\Omega)) \) norm by exploiting the presence of \( L_4 (0, T; L_4 (\Omega)) \) critical term, the spectral estimate (for the nonhomogeneous case) and the nonlinear Gronwall Lemma. For the state to adjoint mapping, using similar technical tools we are able to obtain Lipschitz results with constants that depend polynomially upon \( \frac{1}{\epsilon} \) (see Sect. 3).

The numerical analysis of the control to state under low regularity assumptions is based on a slight but critical modification of the approach by [25, 26] that is the construction of a globally space-time projection as the \( dG(0) \) solution of a heat equation with right-hand side \( y_t - \Delta y \) similar to earlier works of [34] (uncontrolled Navier–Stokes equations), [16] (for controlled Navier–Stokes) and [35] (for the uncontrolled Allen–Cahn). In our approach we do not assume any point-wise space-time bound of the fully-discrete solution of the control to state mapping and most crucially we do not construct a discrete approximation of the spectral estimate, resulting an estimate that is valid under the limited regularity assumptions imposed by our optimal control setting.

For the numerical analysis of the discrete adjoint state mapping, the key difficulty involves the development of discrete stability and error bounds that depend polynomially upon \( \frac{1}{\epsilon} \). Indeed, we note that the spectral estimate is not longer valid if we replace \( y \) by its discretization and the direct application of the nonlinear Gronwall Lemma will lead to severe restrictions on size of \( \| y_u - y_d \|_{L_2 (0,T;L_2 (\Omega))} \) and \( \| y(T) - y_d \|_{L_2 (\Omega)} \) in terms of \( \epsilon \) that are not appropriate in the optimal control setting. Our approach is based on a pseudo duality argument that avoids the construction of a discrete approximation of the spectral estimate and the use of a nonlinear Gronwall Lemma resulting to discrete stability estimates. Then, for the derivation of error estimates for the discrete state to adjoint mapping, we employ a boot-strap argument. Note that the error analysis of the state to adjoint mapping is of independent interest since it concerns a linear singularly perturbed problem under low regularity assumptions on the given data. Combining these estimates, we are able to proceed similar to [16] to establish the desired estimates for the difference between local optimal controls and their discrete approximations as well as estimates for the differences between the corresponding state and adjoint state and their discrete approximations.

2 Preliminary setting

Let \( \Omega \subset \mathbb{R}^d \) be a convex, polygonal (\( d = 2 \)) or polyhedral (\( d = 3 \)) domain of the Euclidean space \( \mathbb{R}^d \) and by \( I := (0, T) \) the time interval where \( T \in \mathbb{R}^+ \) is the final time. We denote by \( \Omega_T \) the space-time cylinder. We use standard notation for \( L_p (\Omega) \) spaces with \( 1 \leq p \leq \infty \) and their norms. We set \( W^{k,p} (\Omega) \) for the \( k \)th order of Sobolev spaces based on \( L_p (\Omega) \), and we denote by \( H^k (\Omega) := W^{k,2} (\Omega), k \geq 0, \) for the Hilbertian space along with the corresponding norms \( \| \cdot \|_{W^{k,p} (\Omega)} \) and \( \| \cdot \|_{H^k (\Omega)} \), respectively. We denote \( H^1_0 (\Omega) := \{ v \in H^1 (\Omega) : v|_{\partial \Omega} = 0 \} \). For a given Banach space \( X \) we denote by \( X^* \) its dual. We also consider the Bochner spaces, \( L_p (I; X) \),
endowed with the norms:

\[ \|v\|_{L^p(I;X)} = \left( \int_I \|v\|^p_X \, dt \right)^{1/p}, \quad p \in [1, +\infty), \quad \|v\|_{L_\infty(I;X)} = \text{ess. sup}_{t \in I} \|v\|_X. \]

We will abbreviate \( \|\cdot\|_{L^\infty(I;L^\infty(\Omega))} := \|\cdot\|_{\text{ess. sup}}. \) We also consider the space

\[ W^{2,1}_p(\Omega_T) = \left\{ v \in L_p(\Omega_T) : \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j}, \frac{\partial v}{\partial t} \in L_p(\Omega_T), \ 1 \leq i, j \leq d \right\}, \]

with the respective norm

\[ \|v\|^p_{W^{2,1}_p(\Omega_T)} = \int_{\Omega_T} \left( |v|^p + \left| \frac{\partial v}{\partial t} \right|^p \right) \, dx \, dt + \sum_{i=1}^d \int_{\Omega_T} \left| \frac{\partial^2 v}{\partial x_i \partial x_j} \right|^p \, dx \, dt. \]

We adopt the notation: \( H^{2,1}(\Omega_T) := W^{2,1}_2(\Omega_T). \) Furthermore, we consider the set

\[ W(\mathcal{I}) := \left\{ v \in L_2(\mathcal{I}; H^1(\Omega)), \ v_t \in L_2(\mathcal{I}; (H^1(\Omega))^*) \right\}. \]

We will use extensively the classical interpolation inequality for all \( v \in L_4(\Omega), \)

\[ \|v\|^3_{L^3(\Omega)} \leq \|v\|_{L^2(\Omega)} \|v\|^2_{L^4(\Omega)}, \quad \text{for} \ d = 2, 3, \tag{2.1} \]

and Gagliardo-Nirenberg-Ladyzhenskaya inequalities (GNL) for all \( v \in H^1(\Omega) \)

\[ \|v\|_{L^4(\Omega)} \leq \tilde{c} \|v\|^1_{L^2(\Omega)} \|v\|^1_{H^1(\Omega)}, \quad \text{for} \ d = 2, \tag{2.2} \]

\[ \|v\|_{L^3(\Omega)} \leq \tilde{c} \|v\|^1_{L^2(\Omega)} \|v\|^1_{H^1(\Omega)}, \quad \text{for} \ d = 3, \tag{2.3} \]

\[ \|v\|_{L^4(\Omega)} \leq \tilde{c} \|v\|^1_{L^2(\Omega)} \|v\|^3_{H^1(\Omega)}, \quad \text{for} \ d = 3, \tag{2.4} \]

with \( \tilde{c} > 0, \) independent of \( v. \) Moreover, we consider Young’s inequality: For any \( \delta > 0, a, b \geq 0, p, q > 1 \) and for some \( C(p, q) > 0, \) it holds

\[ ab \leq \delta a^p + C(p, q)\delta^{-\frac{q}{p}} b^q, \quad \text{where} \quad 1/p + 1/q = 1. \]

Throughout this paper \( C \) denotes a positive constant depending only on \( \Omega_T. \)

Assume that \( u \in L_2(\mathcal{I}; L_2(\Omega)), \ y_0 \in L_2(\Omega). \) The weak formulation of the state equation (1.2) becomes: We seek \( y \in W(\mathcal{I}) \) such that for a.e. \( t \in I, \) for all \( v \in H^1(\Omega), \)

\[ \begin{cases} \langle y_t, v \rangle + (\nabla y, \nabla v) + \epsilon^{-2} \langle y^3 - y, v \rangle = (u, v), \\ (y(0), v) = (y_0, v). \end{cases} \tag{2.5} \]
Here, $(\cdot, \cdot)$ denotes the standard $L_2(\Omega)$ inner product, and $\langle \cdot, \cdot \rangle$ the associated duality pairing between $(H^1(\Omega))^*$ and $H^1(\Omega)$. For any $\epsilon > 0$, and $y_0 \in H^1(\Omega)$, it easy to show that $y \in H^{2, 1}(\Omega_T) \cap C(\bar{\Omega}; H^1(\Omega))$ (see [36]). We refer the readers to [36, 37] for straightforward techniques that prove the enhanced regularity results. The following lemma quantifies the dependence upon $\epsilon$ of various norms.

**Lemma 1** 1. Let $u \in L_2(I; L_2(\Omega))$ and $y_0 \in L_2(\Omega)$. Then, there exists a constant $C$ independent of $\epsilon$ such that:

\[
\|y\|_{L_2(I; L_2(\Omega))} + \|y\|_{L_2(I; L_4(\Omega))} \leq C \left( \left\| \Omega_T \right\|^{1/2} + \epsilon \|y_0\|_{L_2(\Omega)} + \epsilon^2 \|u\|_{L_2(I; L_2(\Omega))} \right) =: C_{st, 1},
\]
\[
\|y\|_{L_\infty(I; L_2(\Omega))} + \|y\|_{L_2(I; H^1(\Omega))} \leq C_{st, 1} \epsilon^{-1}.
\]

2. Let $u \in L_2(I; L_2(\Omega))$ and $y_0 \in H^1(\Omega)$. Then, there exists a constant $C$ independent of $\epsilon$, such that the following estimates hold:

\[
\|y\|_{L_\infty(I; H^1(\Omega))} + \|y\|_{L_2(I; L_2(\Omega))} + \frac{1}{2\epsilon} \|y_0\|^2_{L_2(\Omega)} + \epsilon \|\nabla y_0\|_{L_2(\Omega)} \leq C \left( \|\nabla y_0\|_{L_2(\Omega)} + \frac{1}{2\epsilon} \|y_0^2 - 1\|^2_{L_2(\Omega)} + \|u\|_{L_2(I; L_2(\Omega))} \right) := C_{st, 2},
\]
\[
\|y\|_{L_2(t; H^2(\Omega))} \leq C \left( T^{1/2} - C_{st, 2} \epsilon^{-1} \right) \|\nabla y_0\|_{L_2(\Omega)} + \|u\|_{L_2(I; L_2(\Omega))} \right) := C_{st, 3}.
\]

**Proof** The proof is given in the Appendix A. \(\square\)

**Remark 1** 1. If $\epsilon \|\nabla y_0\|_{L_2(\Omega)} + \|y_0\|^2_{L_4(\Omega)} \leq C$ (with $C$ bounded independent of $\epsilon$) then Lemma 1 implies that $\|y\|_{L_2(I; L_4(\Omega))}^2$ (and hence $\|y\|_{L_\infty(I; L_2(\Omega))}^2$) is bounded independent of $\epsilon$ by $C + \epsilon \|u\|_{L_2(I; L_2(\Omega))}$. In addition, $\|y\|_{L_\infty(I; H^1(\Omega))} \leq (C \epsilon^{-1} + \|u\|_{L_2(I; L_2(\Omega))})$, and $\|y\|_{H^2(\Omega)} \leq \tilde{C} \epsilon^{-2}$, where $\tilde{C}$ is a constant (independent of $\epsilon$), depending on $C, \|\nabla y_0\|_{L_2(\Omega)}$ and $T$.

2. If the more stringent condition $\|\nabla y(0)\|_{L_2(\Omega)} + \frac{1}{2\epsilon} \|y^2(0) - 1\|^2_{L_2(\Omega)} \leq D$ (with $D$ independent of $\epsilon$) holds, then $C_{st, 2}$ is bounded independent of $\epsilon$ and hence we deduce $\|y\|_{H^2(\Omega)} \leq \tilde{C} \epsilon^{-1} \left( D + \|u\|_{L_2(I; L_2(\Omega))} \right)$ where $\tilde{C}$ is an algebraic constant depending only on the domain. In addition we have $\|y\|_{L_\infty(I; H^1(\Omega))} \leq C \left( D + \|u\|_{L_2(I; L_2(\Omega))} \right)$.

3. The condition $\|\nabla y(0)\|^2_{L_2(\Omega)} + \frac{1}{4\epsilon^2} \|y^2(0) - 1\|^2_{L_2(\Omega)} \leq D^2$ relates to the assumption that the associated Ginzburg–Landau energy is bounded at 0 (see for instance [22]) and it is commonly used in the literature. Indeed, for the uncontrolled problem with zero Neumann boundary data and initial data $y_0 \in H^1(\Omega)$, the solution of (1.2) satisfies, for any $t \geq 0$,

\[
\frac{d}{dt} E(t) + \|y(t)\|_{L_2(\Omega)}^2 = 0 \tag{2.6}
\]

where $E(t)$ denotes the associated Ginzburg–Landau energy i.e., for a.e. $t \in (0, T]$,

\[
E(t) = \int_\Omega \left( \frac{1}{2} |\nabla y|^2 + \frac{1}{4\epsilon^2} (y^2 - 1)^2 \right) dx.
\]
As a consequence, (2.6), implies that $E(s) \leq E(0)$, $\forall s \in (0, T]$.

We conclude our preliminary section by stating three key results that are necessary for the analysis as well as the numerical analysis of the Allen–Cahn equation. A crucial idea is the spectral estimate of the principal eigenvalue of the linearized Allen–Cahn operator about the state solution, $y_u$,

$$-\lambda (t) := \inf_{v \in H^1(\Omega) \setminus \{0\}} \frac{\| \nabla v \|_{L^2(\Omega)}^2 + \epsilon^{-2} ((3y_u^2 - 1)v, v)}{\| v \|_{L^2(\Omega)}^2}.$$  \hspace{1cm} (2.7)

For the uncontrolled (homogenous) case, [27–29] showed that $\lambda \in L_\infty(I)$ can be bounded independently of $\epsilon$ for the case of smooth, evolved interfaces. In Sect. 5 we provide a short proof for the nonhomogeneous case. We state generalizations of the continuous and discrete Gronwall lemmas that are required for the upcoming analysis in Sect. 3 (see e.g. [25, Lemma 2.1 and Lemma 2.3], [38, Proposition 6.2, Chapter 6]).

**Lemma 2** Let the nonnegative functions $w_1 \in C(\bar{I})$, $w_2, w_3 \in L_1(I)$, $\alpha \in L_\infty(I)$ and the real number $A \geq 0$ satisfy, for all $t \in I$,

$$w_1(t) + \int_0^t w_2(s) \, ds \leq A + \int_0^t \alpha(s)w_1(s) \, ds + \int_0^t w_3(s) \, ds.$$

Assume also that for $B \geq 0$, $\beta > 0$ and for every $t \in I$, it holds,

$$\int_0^t w_3(s) \, ds \leq B \sup_{s \in [0, t]} w_1^\beta(s) \int_0^t (w_1(s) + w_2(s)) \, ds.$$

If in addition, $4AE \leq (4B(T + 1)E)^{-1/\beta}$, with $E := \exp\left( \int_I \alpha(t) \, dt \right)$, it holds,

$$\sup_{t \in I} w_1(t) + \int_I w_2(t) \, dt \leq 4AE.$$

**Lemma 3** Let $k > 0$ and suppose that the nonnegative real sequences $\{w_j^n\}_{n=0}^N$, $j = 1, 2, 3$, $\{\alpha^n\}_{n=0}^N$ and the real number $A \geq 0$ satisfy

$$w_1^m + k \sum_{n=1}^m w_2^n \leq A + k \sum_{n=1}^m \alpha^n w_1^n + k \sum_{n=1}^{m-1} w_3^n,$$

for all $m = 1, \ldots, N$. Let $\sup_{n=1, \ldots, N} k\alpha^n \leq 1/2$ and $Nk \leq T$. Assume also that for $B \geq 0$, $\beta > 0$ and for every $m = 1, \ldots, N$ we have,

$$k \sum_{n=1}^{m-1} w_3^n \leq B \sup_{n=1, \ldots, m-1} (w_1^n)^\beta k \sum_{n=1}^{m-1} (w_1^n + w_2^n).$$
If in addition, \(4AE \leq (4B(T + 1)E)^{-1/\beta}\) where \(E := \exp\left(2k \sum_{n=1}^{N} \alpha^n\right)\), then it holds,

\[
\sup_{n=1,...,N} w_1^n + k \sum_{n=1}^{N} w_2^n \leq 4AE.
\]

**Remark 2** In the remaining we will focus on the zero Neumann boundary data case, but all results also hold in case that zero Dirichlet boundary data are considered.

### 3 Optimality conditions

This section is devoted to the analysis of the optimal control problem (1.3). Since we are dealing with a non-convex problem, the first order necessary optimality conditions are no longer sufficient. Throughout the remaining of the paper we assume the following assumption on data.

**Assumption 1** Let \(\mu > 0, \gamma \geq 0\) be fixed constants. In addition, we assume that \(y_0 \in H^1(\Omega) \cap L_\infty(\Omega), y_d \in L_2(0, T; L_2(\Omega)),\) and \(y_\Omega \in H^1(\Omega)\).

We note that if \(y_0 \in H^1(\Omega)\) only, then it is clear that \(y_u \in H^{2,1}(\Omega_T)\). However, the enhanced \(H^{2,1}(\Omega_T)\) regularity does not imply that \(y_u \in L_\infty(\Omega_T)\) (see e.g. [39, Example 2.2, Sect. 2]). On the other hand, since the control space \(U_{ad}\) contains elements \(u \in L_\infty(I; L_\infty(\Omega))\), under Assumption 1, we deduce that \(y_u \in L_\infty(I; L_\infty(\Omega))\).

#### 3.1 Continuity

We begin by studying the continuity of the relation between the control and the state. The emphasis here is on quantifying the dependence of various Lipschitz constants upon \(\epsilon\).

**Definition 1** The mapping \(G : L_2(I; L_2(\Omega)) \to H^{2,1}(\Omega_T) \cap C(\bar{I}; H^1(\Omega))\) that assigns each control function \(u\) to the corresponding state \(y_u = y(u) = G(u)\), is called control to state operator.

Throughout the analysis we use the abbreviation: \(F(y) := y^3 - y\) and for \(u_i \in L_2(I; L_2(\Omega)),\) we denote by \(y_i = G(u_i) := y_{u_i}\). The Lipschitz continuity of the control to state mapping \(G\), with Lipschitz constant that depend exponentially upon \(1/\epsilon\), can be proved by standard techniques when \(y_0 \in H^1(\Omega)\). In the next Theorem, we prove an estimate with Lipschitz constant independent of \(\epsilon\), when data satisfy Assumption 1 and for control \(u \in U_{ad}\).

**Theorem 4** Suppose that Assumption 1 holds, and let \(u_1 \in U_{ad}\). For \(d = 2, 3\), assume that it holds,

\[
\|u_1 - u_2\|_{L_2(I; L_2(\Omega))} \leq C \epsilon^{d+1} \|y_1\|_\infty^{-1} (\tilde{c}(T + 1))^{-1} E_d^{-(d/2)}, \quad (3.1)
\]
where $E_d := \exp \left( \int_t^{2\lambda(t)} (1 - e^{-2}) + (6 - d) + 2e^{-2} \, dt \right)$, and $C$ is a constant depending only on $\Omega$. Then, with $L_1 := 2E_d^{1/2}$ the following estimate holds,
\[
\sup_{t \in I} \|y_1 - y_2\|_{L^2(\Omega)} + \epsilon \|y_1 - y_2\|_{L^2(I; H^1(\Omega))} + \epsilon^{-1} \|y_1 - y_2\|_{L^4(I; L^4(\Omega))}^2 \\
\leq L_1 \|u_1 - u_2\|_{L^4(I; L^4(\Omega))}.
\]

**Proof** Subtracting the equations satisfied by $y_1$ and $y_2$, it yields $\forall v \in H^1(\Omega)$,
\[
\begin{align*}
\langle (y_1 - y_2)_t, v \rangle + \langle \nabla (y_1 - y_2), \nabla v \rangle + \epsilon^{-2} \langle F(y_1) - F(y_2), v \rangle &= (u_1 - u_2, v) \\
y_1(0) - y_2(0) &= 0.
\end{align*}
\]
(3.2)

Rewriting the nonlinear part,
\[
F(y_1) - F(y_2) = y_1^3 - y_2^3 - (y_1 - y_2) = (y_1 - y_2)^3 + 3y_1y_2(y_1 - y_2) - (y_1 - y_2) \\
= (y_1 - y_2)^3 + (3y_1^2 - 1)(y_1 - y_2) - 3y_1(y_1 - y_2)^2.
\]

relation (3.2) becomes
\[
\begin{align*}
\langle (y_1 - y_2)_t, v \rangle + \langle \nabla (y_1 - y_2), \nabla v \rangle + \epsilon^{-2} \langle (y_1 - y_2)^3 + (3y_1^2 - 1)(y_1 - y_2), v \rangle &= (u_1 - u_2, v) + 3\epsilon^{-2}(y_1(y_1 - y_2)^2, v).
\end{align*}
\]
(3.3)

Testing with $y_1 - y_2$, and noting that $F'(y_1) = 3y_1^2 - 1$,
\[
\frac{1}{2} \frac{d}{dt} \|y_1 - y_2\|_{L^2(\Omega)}^2 + \|\nabla(y_1 - y_2)\|_{L^2(\Omega)}^2 + \epsilon^{-2} \|y_1 - y_2\|_{L^4(\Omega)}^4 \\
+ \epsilon^{-2} \langle F'(y_1)(y_1 - y_2), y_1 - y_2 \rangle &= (u_1 - u_2, y_1 - y_2) \\
+ 3\epsilon^{-2} \langle y_1(y_1 - y_2)^2, y_1 - y_2 \rangle.
\]

Recall the spectral estimate (2.7) of $y_{u_1}$ with $v = y_1 - y_2 \in H^1(\Omega)$ to deduce that
\[
\|\nabla(y_1 - y_2)\|_{L^2(\Omega)}^2 + \epsilon^{-2} \langle F'(y_1)(y_1 - y_2), y_1 - y_2 \rangle \\
\geq -\lambda(t)(1 - e^{-2}) \|y_1 - y_2\|_{L^2(\Omega)}^2 + \epsilon^2 \|\nabla(y_1 - y_2)\|_{L^2(\Omega)}^2 \\
+ \langle F'(y_1)(y_1 - y_2), y_1 - y_2 \rangle.
\]

Combining the last two relations, and using Hölder and Young’s inequalities it yields,
\[
\frac{1}{2} \frac{d}{dt} \|y_1 - y_2\|_{L^2(\Omega)}^2 + \epsilon^2 \|\nabla(y_1 - y_2)\|_{L^2(\Omega)}^2 + \epsilon^{-2} \|y_1 - y_2\|_{L^4(\Omega)}^4 \\
\leq \frac{1}{2} \|u_1 - u_2\|_{L^2(\Omega)}^2 + \left( \frac{\lambda(t)(1 - e^{-2}) + 3}{2} \right) \|y_1 - y_2\|_{L^2(\Omega)} + 3\epsilon^{-2} \|y_1\|_{L^\infty(\Omega)} \|y_1 - y_2\|_{L^2(\Omega)}^2 \|y_1 - y_2\|_{L^3(\Omega)}^3.
\]
Adding $\varepsilon^2 \| y_1 - y_2 \|_{L^2(\Omega)}^2$ on both sides and integrating over $t \in (0, \tau)$, we have

$$
\| (y_1 - y_2)(\tau) \|_{L^2(\Omega)}^2 + 2\varepsilon^2 \int_0^\tau \| y_1 - y_2 \|_{H^1(\Omega)}^2 \, dt + 2\varepsilon^2 \int_0^\tau \| y_1 - y_2 \|_{L^4(\Omega)}^4 \, dt
\leq \| (y_1 - y_2)(0) \|_{L^2(\Omega)}^2 + \int_0^\tau \| u_1 - u_2 \|_{L^2(\Omega)}^2 \, dt
\quad + \int_0^\tau (2\lambda(t)(1 - \varepsilon^2) + 3 + 2\varepsilon^2) \| y_1 - y_2 \|_{L^2(\Omega)}^2 \, dt
\quad + 6\varepsilon^{-2}\| y_1 \|_{\infty} \int_0^\tau \| y_1 - y_2 \|_{L^3(\Omega)}^3 \, dt.
$$

(3.4)

Note that the interpolation inequality (2.1) and the embedding $H^1(\Omega) \subset L^4(\Omega)$ imply $\| y_1 - y_2 \|_{L^3(\Omega)} \leq C \varepsilon^2 \| y_1 - y_2 \|_{H^1(\Omega)}^2$. Then, for

$$
w_1(\tau) = \| (y_1 - y_2)(\tau) \|_{L^2(\Omega)}, \quad w_2(t) = 2\varepsilon^2 \| y_1 - y_2 \|_{H^1(\Omega)}^2 + 2\varepsilon^2 \| y_1 - y_2 \|_{L^4(\Omega)}^4, \quad w_3(t) = \| y_1 - y_2 \|_{L^3(\Omega)}^3, \quad A = \| u_1 - u_2 \|_{L^2(\Omega)}^2, \quad B = 6C\varepsilon^{-4}\| y_1 \|_{\infty},
$$

$\beta = 1/2$, $\alpha(t) = 2\lambda(t)(1 - \varepsilon^2) + 3 + 2\varepsilon^2$.

Lemma 2 implies the result for $d = 3$. For $d = 2$, using (2.1) and the GNL inequality (2.2), we deduce, $\| y_1 - y_2 \|_{L^3(\Omega)}^3 \leq \tilde{c} \| y_1 - y_2 \|_{L^2(\Omega)}^2 \| y_1 \|_{L^4(\Omega)}^1$. Therefore, substituting the above inequality into (3.4), and using Young’s inequality,

$$
\| (y_1 - y_2)(\tau) \|_{L^2(\Omega)}^2 + 2\varepsilon^2 \int_0^\tau \| y_1 - y_2 \|_{H^1(\Omega)}^2 \, dt + 2\varepsilon^2 \int_0^\tau \| y_1 - y_2 \|_{L^4(\Omega)}^4 \, dt
\leq \| (y_1 - y_2)(0) \|_{L^2(\Omega)}^2 + \int_0^\tau \| u_1 - u_2 \|_{L^2(\Omega)}^2 \, dt
\quad + \int_0^\tau (2\lambda(t)(1 - \varepsilon^2) + 3 + 2\varepsilon^2) \| y_1 - y_2 \|_{L^2(\Omega)}^2 \, dt
\quad + \frac{9\tilde{c}^2}{\varepsilon^4} \| y_1 \|_{\infty}^2 \int_0^\tau \| y_1 - y_2 \|_{L^2(\Omega)}^2 \, dt.
$$

The result now follows upon choosing $\beta = 1$, $B = 9\tilde{c}^2 \varepsilon^{-6} \| y_1 \|_{\infty}^2$ and $\alpha(t) = 2\lambda(t)(1 - \varepsilon^2) + 4 + 2\varepsilon^2$ in Lemma 2. \hfill \Box

### 3.2 Differentiability

Next, we determine the first and second order derivatives of the control to state mapping $G$ of Definition 1.

**Theorem 5** Let $u, v \in L^2(I; L^2(\Omega))$. The mapping $G : L^2(I; L^2(\Omega)) \to H^{2,1}(\Omega_T) \cap C(\bar{I}; H^1(\Omega))$, such that $y_u := G(u)$, is of class $C^\infty$. We denote by
\[ z_v = G'(u)v \text{ and } z_{vv} = G''(u)v^2, \] the unique solutions to the following equations

\[
\begin{aligned}
&\begin{cases}
z_{v,t} - \Delta z_v + \epsilon^{-2} \left(3y_u^2 - 1\right) z_v = v & \text{in } \Omega_T, \\
\frac{\partial z_v}{\partial n} = 0 & \text{on } \Sigma_T, \ z_v(0) = 0 & \text{in } \Omega,
\end{cases} \\
&\begin{cases}
z_{v,v,t} - \Delta z_{vv} + \epsilon^{-2} \left(3y_u^2 - 1\right) z_{vv} = -6\epsilon^{-2}y_u z_v^2 & \text{in } \Omega_T, \\
\frac{\partial z_{vv}}{\partial n} = 0 & \text{on } \Sigma_T, \ z_{vv}(0) = 0 & \text{in } \Omega.
\end{cases}
\end{aligned}
\]

**Proof** The proof follows using similar arguments as in [16, 18].

The above differentiability properties of \( G \) imply that the reduced cost functional \( J : L_2(I; L_2(\Omega)) \to \mathbb{R} \) is of class \( C^\infty \), as well.

**Lemma 6** For any \( u, v \in L_2(I; L_2(\Omega)) \), it holds that

\[
\begin{aligned}
J'(u)v &= \int_I \int_{\Omega} (\varphi_u + \mu u) v \, dx \, dt, \\
J''(u)v^2 &= \int_I \int_{\Omega} |z_v|^2 \, dx \, dt + \gamma \int_I \int_{\Omega} |z_v(T)|^2 \, dx \, dt + \mu \int_I \int_{\Omega} |v|^2 \, dx \\
&\quad - 6\epsilon^{-2} \int_I \int_{\Omega} y_u z_v^2 \varphi_u \, dx \, dt,
\end{aligned}
\]

where \( z_v \) is the solution of (3.5) and \( \varphi_u \in H^{2,1}(\Omega_T) \cap C(\bar{I}; H^1(\Omega)) \), with \( \frac{\partial \varphi_u}{\partial n} = 0 \) on \( \Sigma_T \), is the unique solution of the adjoint state equation, i.e., \( \forall w \in H^1(\Omega) \), a.e. \( t \in I \)

\[
\begin{aligned}
&\begin{cases}
- \left( \varphi_{u,u}, w \right) + \left( \nabla \varphi_u, \nabla w \right) + \epsilon^{-2} \left( \left(3y_u^2 - 1\right) \varphi_u, w \right) = \left( y_u - y_d, w \right) \\
\left( \varphi_u(T), w \right) = \left( y(T) - y_\Omega, w \right).
\end{cases}
\end{aligned}
\]

**Remark 3** The assumption that \( y_\Omega \in H^1(\Omega) \) together with the regularity properties of the state solution \( y_u \) implies that \( \varphi_u(T) \in H^1(\Omega) \). Thus, we deduce that

\[
\| \varphi_u(T) \|_{H^1(\Omega)} = \| y(T) - y_\Omega \|_{H^1(\Omega)} \leq \gamma \left( \max_{t \in I} \| y_u(t) \|_{H^1(\Omega)} + \| y_\Omega \|_{H^1(\Omega)} \right).
\]

Lemma 1, implies \( \| \varphi_u(T) \|_{H^1(\Omega)} \leq \tilde{C} \epsilon^{-1} \) when \( \epsilon \| \nabla y_0 \|_{L_2(\Omega)} + \frac{1}{2} \| (y_0^2 - 1)^2 \|_{L_1(\Omega)} \leq C \). Similarly, we have that \( \| \varphi_u(T) \|_{H^1(\Omega)} \leq \tilde{C} \) when \( \| \nabla y_0 \|_{L_2(\Omega)} + \frac{1}{2\epsilon} \| (y_0^2 - 1)^2 \|_{L_1(\Omega)} \leq D \).

**Lemma 7** Let \( \varphi_u \) be the solution to (3.9), with \( u \in U_{ad}, \ y_d \in L_2(I; L_3(\Omega)) \) and \( \varphi_u(T) \in H^1(\Omega) \). Then, there exists a constant \( C > 0 \), depending on \( \Omega \) such that
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\[ \sup_{t \in I} \| \varphi_i(t) \|_{L^2(\Omega)} + \epsilon \| \varphi_i \|_{L^2(I; H^1(\Omega))) + \| \varphi_i y_i \|_{L^2(I; L^2(\Omega))} \]

\[ \leq C \varphi^{1/2} \left( \| \varphi_i(T) \|_{L^2(\Omega)} + \| y_i - y_d \|_{L^2(I; L^2(\Omega))} \right) \]

\[ := D_{st,1}, \quad (3.10) \]

\[ \| \varphi_i \|_{L^2(I; L^2(\Omega))} + \sup_{t \in I} \| \nabla \varphi_i(t) \|_{L^2(\Omega)} \leq C \left( \| y_i - y_d \|_{L^2(I; L^2(\Omega))} \right) \]

\[ + \| \nabla \varphi_i(T) \|_{L^2(\Omega)} + \epsilon^{-2} \left( \| y_i \|_\infty + T^{1/2} \right) D_{st,1} \]

\[ := D_{st,2}, \quad (3.11) \]

\[ \| \varphi_i \|_{L^2(I; H^2(\Omega))} \leq C \left( \| y_i - y_d \|_{L^2(I; L^2(\Omega))} + D_{st,2} + \epsilon^{-2} \right) \left( 1 + \| y_i \|_\infty \right) D_{st,1} \]

\[ (3.12) \]

where we denote by \( C_\varphi := \exp \left( \int_I (2\lambda(t)(1 - \epsilon^2) + 3 + 2\epsilon^2) \, dt \right) \).

**Proof** The proof is given in Appendix B.

Let \( u_1, u_2 \in L^2(I; L^2(\Omega)) \) be two control functions. Then, we denote by \( y_i = y_{u_i} \) and \( \varphi_i = \varphi_{\varphi_i} \) the associated state and adjoint state solutions for \( i = 1, 2 \), respectively. Below, we state the Lipschitz continuity of the adjoint state mapping, which is based on the estimates of Theorem 4, and the structure of the linearized operator. A detailed proof is presented in the Appendix C.

**Lemma 8** Suppose that the assumptions of Theorem 4 hold. Then, for \( d = 2 \), there exists a constant \( C_T \) depending only on the domain \( \Omega_T \) such that,

\[ \sup_{t \in I} \| \varphi_1 - \varphi_2 \|_{L^2(\Omega)} + \epsilon \| \varphi_1 - \varphi_2 \|_{L^2(I; H^1(\Omega))} \]

\[ \leq C_T \epsilon^{1/2} L_1 \left( 1 + C_{\infty} C_D \epsilon^{-7/2} \right) \| u_1 - u_2 \|_{L^2(I; L^2(\Omega))}. \]

\[ (3.13) \]

For \( d = 3 \), there exists a constant \( C_T \) depending only on the domain \( \Omega_T \) such that,

\[ \sup_{t \in I} \| \varphi_1 - \varphi_2 \|_{L^2(\Omega)} + \epsilon \| \varphi_1 - \varphi_2 \|_{L^2(I; H^1(\Omega))} \]

\[ \leq C_T \epsilon^{1/2} L_1 \left( 1 + C_{\infty} C_D \epsilon^{-15/4} \right) \| u_1 - u_2 \|_{L^2(I; L^2(\Omega))}. \]

\[ (3.14) \]

Here, we denote by \( C_{\infty} := \left( C \left( \| y_1 \|_\infty^2 + \| y_2 \|_\infty^2 \right) \right)^{1/2} \) with \( C \) depending on \( |\Omega| \), by \( E_\varphi := \int_I (2\lambda(t)(1 - \epsilon^2) + 4 + 2\epsilon^2) \, dt \), and by \( L_1, D_{st,1} \) the constants of Theorem 4, and Lemma 7 respectively.

**Proof** The proof is given in Appendix C.

3.3 Necessary and sufficient conditions

We are now ready to state the optimality conditions. We refer the reader to [16, Theorems 3.4 and 3.3] for the related proofs.
Theorem 9 Every locally optimal control $\bar{u}$ for problem (1.3), satisfies, together with its associated state $\bar{y} \in H^{2,1}(\Omega_T)$ and adjoint state $\bar{\varphi} \in H^{2,1}(\Omega_T)$

\[
\begin{aligned}
\left\{
\begin{array}{l}
\langle \bar{y}, v \rangle + (\nabla \bar{y}, \nabla v) + \varepsilon^{-2} \langle \bar{y}^3 - \bar{y}, v \rangle = \langle \bar{u}, v \rangle \quad \forall v \in H^1(\Omega) \\
\bar{y}(0) = y_0,
\end{array}
\right.
\end{aligned}
\] (3.15)

\[
\begin{aligned}
\left\{
\begin{array}{l}
-\langle \bar{\varphi}, w \rangle + (\nabla \bar{\varphi}, \nabla w) + \varepsilon^{-2}\{3 \bar{y}^2 - 1\} \langle \bar{\varphi}, w \rangle = \langle \bar{y} - y_d, w \rangle \quad \forall w \in H^1(\Omega),
\end{array}
\right.
\end{aligned}
\] (3.16)

and the variational inequality (optimality condition)

\[
\int_I \int_\Omega (\bar{\varphi} + \mu \bar{u}) (u - \bar{u}) \, dx \, dt \geq 0 \quad \forall u \in U_{ad},
\] (3.17)

where $\bar{u} \in L^2(I; W^{1,p}(\Omega)) \cap C(\bar{I}; H^1(\Omega)) \cap H^1(\Omega_T)$, for any $1 \leq p < \infty$.

Furthermore, let $\varepsilon \|\nabla y_0\|_{L_2(\Omega)} + \frac{1}{2} \|\gamma_0^2 - 1\|_{L_1(\Omega)}^{1/2} \leq C$. Then,

\[
\|\bar{y}\|_{H^{2,1}(\Omega_T)} \leq \tilde{C} \varepsilon^{-2}, \quad \text{and} \quad \|\bar{\varphi}\|_{H^{2,1}(\Omega_T)} \leq \tilde{D} \|\bar{y}\|_{\infty} \varepsilon^{-2}.
\]

If in addition, $\|\nabla y_0\|_{L_2(\Omega)} + \frac{1}{2\varepsilon} \|\gamma_0^2 - 1\|_{L_1(\Omega)}^{1/2} \leq D$ then,

\[
\|\bar{y}\|_{H^{2,1}(\Omega_T)} \leq \tilde{C} \varepsilon^{-1}, \quad \text{and} \quad \|\bar{\varphi}\|_{H^{2,1}(\Omega_T)} \leq \tilde{D} \|\bar{y}\|_{\infty} \varepsilon^{-2}.
\]

Here, the constants $\tilde{C}$, $\tilde{D}$ are independent of $\varepsilon$ and depend only on data.

Proof Note that every local optimal solution satisfies $J'(\bar{u})(u - \bar{u}) \geq 0$, $\forall u \in U_{ad}$. The optimality system (3.15), (3.16) and (3.17) follows from (3.7). Inequality (3.17) implies the standard projection formula

\[
\bar{u}(t, x) = \text{Proj}_{[u_a, u_b]}\left(-\frac{1}{\mu} \bar{\varphi}(t, x)\right) \quad \text{for a.e.} \quad (t, x) \in \Omega_T,
\] (3.18)

from which we deduce $\bar{u} \in H^1(\Omega_T) \cap C(\bar{I}; H^1(\Omega)) \cap L_2(I; W^{1,p}(\Omega))$, for all $1 \leq p < \infty$. Therefore, if $\varepsilon \|\nabla y_0\|_{L_2(\Omega)} + \frac{1}{2} \|\gamma_0^2 - 1\|_{L_1(\Omega)}^{1/2} \leq C$, (independent of $\varepsilon$) we observe from Lemma 1 and Remark 1 that $\|\bar{y}\|_{H^{2,1}(\Omega_T)} \leq \tilde{C} \varepsilon^{-2}$. In addition, we have that $\|\bar{\varphi}(T)\|_{L_2(\Omega)} = \gamma \|\bar{y}(T) - y_\Omega\|_{L_2(\Omega)}$ is bounded independent of $\varepsilon$ which implies that the constant $D_{st, 1}$ of Lemma 7 is bounded independent of $\varepsilon$. Hence, using Remark 3 and the estimates of Lemma 7 we obtain $\|\bar{\varphi}\|_{H^{2,1}(\Omega_T)} \leq \tilde{D} \|\bar{y}\|_{\infty} \varepsilon^{-2}$. Under the assumption $\|\nabla y_0\|_{L_2(\Omega)} + \frac{1}{2\varepsilon} \|\gamma_0^2 - 1\|_{L_1(\Omega)}^{1/2} \leq D$, a similar boot-strap argument, imply that $\|\bar{y}\|_{H^{2,1}(\Omega_T)} \leq \tilde{C} \varepsilon^{-1}$ and $\|\bar{\varphi}\|_{H^{2,1}(\Omega_T)} \leq \tilde{D} \|\bar{y}\|_{\infty} \varepsilon^{-2}$.  \(\square\)

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In the usual manner, we deduce from (3.17) that for a.e. \((t, x) \in \Omega_T\),
\[
\begin{align*}
\bar{u}(t, x) &= u_a \Rightarrow \bar{\varphi}(t, x) + \mu \bar{u}(t, x) \geq 0, \\
\bar{u}(t, x) &= u_b \Rightarrow \bar{\varphi}(t, x) + \mu \bar{u}(t, x) \leq 0, \\
\bar{u}(t, x) \in (u_a, u_b) \Rightarrow \bar{\varphi}(t, x) + \mu \bar{u}(t, x) = 0, \\
\bar{\varphi}(t, x) + \mu \bar{u}(t, x) > 0 &\Rightarrow \bar{u}(t, x) = u_a, \\
\bar{\varphi}(t, x) + \mu \bar{u}(t, x) < 0 &\Rightarrow \bar{u}(t, x) = u_b.
\end{align*}
\]

We introduce the cone of critical directions that is necessary to state the second order conditions.
\[
C_{\bar{u}} = \{ \nu \in L_2(I; L_2(\Omega)) : \nu \text{ satisfies } (3.20) \},
\]
\[
\begin{align*}
&\nu(t, x) = 0 \text{ if } \bar{\varphi}(t, x) + \mu \bar{u}(t, x) \neq 0, \\
&\nu(t, x) \geq 0 \text{ if } \bar{u}(t, x) = u_a, \\
&\nu(t, x) \leq 0 \text{ if } \bar{u}(t, x) = u_b.
\end{align*}
\]

Let us notice that
\[
J'(\bar{u}) \nu = \int_I \int_\Omega (\bar{\varphi}(t, x) + \mu \bar{u}(t, x)) \nu dx dt,
\]
\[
(\bar{\varphi}(t, x) + \mu \bar{u}(t, x)) \nu(t, x) = 0 \text{ for a.e. } (t, x) \in \Omega_T \text{ and } \forall \nu \in C_{\bar{u}}.
\]

**Theorem 10** Let \(\bar{u}\) be a local solution of problem (1.3). Then, it holds that \(J''(\bar{u})\nu^2 \geq 0\), \(\forall \nu \in C_{\bar{u}}\). Conversely, if \(\bar{u} \in U_{ad}\) satisfies
\[
J'(\bar{u})(u - \bar{u}) \geq 0 \text{ } \forall u \in U_{ad},
\]
\[
J''(\bar{u})\nu^2 > 0 \text{ } \forall \nu \in C_{\bar{u}} \setminus \{0\},
\]
then there exist \(\alpha > 0\) and \(\delta > 0\) such that
\[
J(\bar{u}) + \frac{\delta}{2} \|u - \bar{u}\|_{L_2(I; L_2(\Omega))^2}^2 \leq J(u) \text{ } \forall u \in U_{ad} \cap B_\alpha(\bar{u}),
\]

where \(B_\alpha(\bar{u})\) is the open ball of \(L_2(I; L_2(\Omega))\) centered at \(\bar{u}\) with radius \(\alpha\).

**Proof** The proof follows identically to [16, 18] based on arguments of [4] (see also references within). We note that \(\|y_\epsilon\|_{L_2(I; L_2(\Omega))}\) is bounded independent of \(\epsilon\) for any \(u \in U_{ad}\) (see Lemma 1). In the remaining, we point out that the constant \(\delta > 0\) will not appear at any exponential. \(\Box\)

**Remark 4** The second order sufficient condition (3.23) is equivalent to
\[
J''(\bar{u})\nu^2 \geq \delta \|\nu\|_{L_2(I; L_2(\Omega))^2}^2 \text{ } \forall \nu \in C_{\bar{u}}.
\]
4 Approximation of the control problem

Let \( \{T_h\}_{h>0} \) be a family of triangulations of \( \tilde{\Omega} \). We consider each \( T_h \) to be a conforming and quasi-uniform subdivision such that \( \cup_{\tau \in T_h} \tau = \tilde{\Omega} \). With each element \( \tau \in T_h \) we associate two parameters \( h_{\tau} \) and \( \rho_{\tau} \), where \( h_{\tau} \) is the diameter of \( \tau \) while \( \rho_{\tau} \) is the diameter of the largest ball contained in \( \tau \). Then, we define the meshsize parameter as \( h := \max_{\tau \in T_h} h_{\tau} \). To each \( T_h \) we associate the finite element space:

\[
Y_h := \{ y_h \in C(\tilde{\Omega}); \ y_h|_{\tau} \in \mathbb{P}_{1}(\tau), \ \forall \tau \in T_h \} \subset H^1(\Omega),
\]

with \( \mathbb{P}_{1} \) denoting the \( d \)-variate space of linear polynomials. Recall the following classical inverse estimates (see e.g. [40, Chapter 4]):

\[
\|v_h\|_{L^1(\Omega)} \leq C_{\text{inv}} h^{-1/2} \|v_h\|_{L^2(\Omega)}, \quad \text{and} \quad \|v_h\|_{H^1(\Omega)} \leq C_{\text{inv}} h^{-1} \|v_h\|_{L^2(\Omega)}. \tag{4.1}
\]

Furthermore, we set

\[
U_h = \{ u_h \in L_2(\Omega); \ u_h|_{\tau} \equiv u_{\tau} \in \mathbb{R} \}.
\]

Let \( 0 = t_0 < t_1 < \ldots < t_N = T \). We partition the time interval \( I \) into subintervals \( J_n := (t_{n-1}, t_n) \) with \( k_n := t_n - t_{n-1}, n = 1, \ldots, N \) each time step. We assume that

\[
\exists C_0 > 0 \text{ s.t. } k_n \leq C_0 k_0, \quad \forall 1 \leq n \leq N \text{ and } \forall k > 0. \tag{4.2}
\]

Setting \( \sigma = (k, h) \), we consider the following fully discrete spaces:

\[
Y_{\sigma} := \{ y_{\sigma} \in L_2(I; H^1(\Omega)); \ y_{\sigma}|_{J_n} \in Y_h, \ 1 \leq n \leq N \},
\]

\[
U_{\sigma} := \{ u_{\sigma} \in L_2(I; L_2(\Omega)); \ u_{\sigma}|_{J_n} \in U_h, \ 1 \leq n \leq N \}.
\]

The functions of \( Y_{\sigma} \) and \( U_{\sigma} \) are piecewise constant in time. We seek discrete controls in \( U_{\sigma} \) that are written as: \( u_{\sigma} = \sum_{n=1}^{N} \sum_{\tau \in T_h} u_{n,\tau} \chi_n \chi_{\tau} \), with \( u_{n,\tau} \in \mathbb{R} \), where \( \chi_n, \chi_{\tau} \) are the characteristic functions over \((t_{n-1}, t_n)\) and \( \tau \), respectively. We consider the convex subset of \( U_{\sigma} \),

\[
U_{\sigma,ad} = U_{\sigma} \cap U_{ad} = \{ u_{\sigma} \in U_{\sigma}; \ u_{n,\tau} \in [a, b] \}.
\]

Every element of \( Y_{\sigma} \) can be written as, \( y_{\sigma} = \sum_{n=1}^{N} y_{n,h} \chi_n \), with \( y_{n,h} \in Y_h \). In the above notation, we note that \( y_{\sigma}(t_n) = y_{n,h} \) in order \( y_{\sigma} \) to be continuous from the left. Thus, we have \( y_{\sigma}(T) = y_{\sigma}(t_N) = y_{N,h} \). The following projections will be needed.

**Definition 2** We define the projection operator \( P_h : L_2(\Omega) \rightarrow Y_h \) through \( (P_h y, w_h) = (y, w_h) \ \forall w_h \in Y_h \). Also, we define \( P_{\sigma} : C(I; L_2(\Omega)) \rightarrow Y_{\sigma} \) by \( (P_{\sigma} y)_{n,h} = P_h y(t_n) \), for each \( 1 \leq n \leq N \).
To introduce the discrete control problem, we need to define the fully discrete scheme of the state equation (1.2). For any \( u \in L_2(I; L_2(\Omega)) \), the backward Euler-finite element method (discontinuous in time Galerkin dG(0)) reads: For each \( n = 1, \ldots, N \) and for all \( w_h \in Y_h \),

\[
\left( \frac{y_{n,h} - y_{n-1,h}}{k_n}, w_h \right) + (\nabla y_{n,h}, \nabla w_h) + \epsilon^{-2} (F(y_{n,h}), w_h) = (u_n, w_h),
\]

(4.3)

with \( y_{0,h} := P_h y_0 \). Here, we define by,

\[
(u_n, w_h) := \frac{1}{k_n} \int_{t_{n-1}}^{t_n} (u(t), w_h) \, dt,
\]

(4.4)

and we note that the discrete initial data \( y_{0,h} \) satisfy

\[
y_{0,h} \in Y_h \text{ with } \|y_0 - y_{0,h}\|_{L_2(\Omega)} \leq C h \text{ and } \|y_{0,h}\|_{H^1(\Omega)} \leq C \forall h > 0.
\]

(4.5)

Then, we define the discrete control problem as follows,

\[
\left\{ \begin{array}{ll}
\min_{u_\sigma} J_\sigma(u_\sigma) \\
u_\sigma \in U_{\sigma,ad},
\end{array} \right.
\]

(4.6)

where

\[
J_\sigma(u_\sigma) = \frac{1}{2} \int_I \int_{\Omega} |y_\sigma(u_\sigma) - y_d|^2 \, dx \, dt + \frac{\gamma}{2} \int_\Omega |y_\sigma(T) - y_{\Omega,h}|^2 \, dx + \frac{\mu}{2} \int_I \int_{\Omega} |u_\sigma|^2 \, dx \, dt.
\]

(4.7)

Here, we note that \( y_\sigma(u_\sigma) \) denotes the solution of (4.3) with right hand side \( u_\sigma \). We also define by \( y_{\Omega,h} := P_h y_\Omega \in Y_h \) and we note that

\[
\|y_\Omega - y_{\Omega,h}\|_{L_2(\Omega)} \leq C h \text{ and } \|y_{\Omega,h}\|_{H^1(\Omega)} \leq C \forall h > 0.
\]

(4.8)

We refer the reader to [41] for the stability of \( P_h \) in \( H^1(\Omega) \).

The study of the control problem consists of four steps. We begin with the analysis and error estimation of the discrete state equation. The choice of the dG(0) method is due to the low regularity imposed by the optimal control setting. Other approaches for discretization of the Allen–Cahn can be found in [42, 43] (BDF and extrapolated Runge-Kutta methods via an auxiliary variable formulation), [44] (symmetric interior penalty discontinuous Galerkin methods), [45, 46] (scalar auxiliary variable / Crank-Nickolson scheme) [47, 48] (second order semi-implicit scheme). Results regarding discrete maximum principles can be found in [49–51] (see also references within).
4.1 Analysis of the discrete state equation

Let \( y = y_u = G(u) \) and \( y_\sigma = y_\sigma(u) \in Y_\sigma \) be a solution to (4.3). We begin by presenting stability estimates.

Lemma 11 Let \( y_\sigma \) be a solution to (4.3) corresponding to the control function \( u \in L_2(I; L_2(\Omega)) \) and \( y_{0h} := P_h y_0 \). Then, there exists a constant \( C > 0 \) independent of \( \sigma = (k, h), \epsilon \) and \( \|y_u\|_\infty \), such that

\[
\|y_\sigma\|_{L_2(I; L_2(\Omega))} + \|y_\sigma\|_{L_4(I; L_4(\Omega))}^2 \\
\leq C (|\Omega_T|^1 + \|y_0\|_{L_2(\Omega)} + \|u\|_{L_2(I; L_2(\Omega))}) := C_{st,1}^{dG},
\]

\[
\|y_\sigma\|_{L_\infty(I; L_2(\Omega))} + \|y_\sigma\|_{L_4(I; L_4(\Omega))}^2 \\
+ \left( \sum_{n=1}^N \|y_{n,h} - y_{n-1,h}\|_{L_2(\Omega)}^2 \right)^{1/2} \leq C_{st,1}^{dG} \epsilon^{-1}.
\]

(4.9)

(4.10)

If in addition, \( k \leq \frac{3C_0 \epsilon^2}{2} \), with \( C_0 \) defined by (4.2), then the following estimate holds,

\[
\|y_\sigma\|_{L_\infty(I; \mathcal{H}^1(\Omega))} + \epsilon^{-1} \|y_\sigma\|_{L_\infty(I; L_2(\Omega))}^2 \\
\leq C \left( \|\nabla y_0\|_{L_2(\Omega)} + \epsilon^{-1} \|y_0\|_{L_4(\Omega)}^2 + |\Omega|^{1/2} \epsilon^{-1} + \|u\|_{L_2(I; L_2(\Omega))} \right) := C_{st,2}^{dG}.
\]

(4.11)

Proof The first two stability estimates can be derived by setting \( w_h = y_\sigma \) into (4.3) (see also [35, Sect. 3]). For the proof of the third, we proceed as follows: We choose \( w_h = (y_{n,h} - y_{n-1,h})/k_n \) in (4.3), to get

\[
\left\| \frac{y_{n,h} - y_{n-1,h}}{k_n} \right\|_{L_2(\Omega)}^2 + \frac{1}{k_n} (\nabla y_{n,h}, \nabla (y_{n,h} - y_{n-1,h})) + \frac{1}{\epsilon^2 k_n} (y_3, y_{n,h} - y_{n-1,h})
\]

\[
= \frac{1}{k_n} (u_n, y_{n,h} - y_{n-1,h}) + \frac{1}{\epsilon^2 k_n} (y_{n,h}, y_{n,h} - y_{n-1,h}).
\]

Young’s inequality yields \( \int_\Omega \|y_{n,h} \|^2 \|y_{n-1,h}\| \, dx \leq \frac{3}{4} \|y_{n,h}\|_{L_4(\Omega)}^4 + \frac{1}{4} \|y_{n-1,h}\|_{L_4(\Omega)}^4 \), for \( p = \frac{4}{3} \) and \( q = 4 \). Hence, using Hölder and Young’s inequalities and after some standard algebra, we deduce that

\[
\frac{3}{4} \left( \frac{y_{n,h} - y_{n-1,h}}{k_n} \right)^2_{L_2(\Omega)} + \frac{1}{2k_n} \|\nabla y_{n,h}\|_{L_2(\Omega)}^2 + \frac{1}{2k_n} \|\nabla (y_{n,h} - y_{n-1,h})\|_{L_2(\Omega)}^2
\]

\[
+ \frac{1}{4 \epsilon^2 k_n} \|y_{n,h}\|_{L_4(\Omega)}^4 + \frac{1}{2 \epsilon^2 k_n} \|y_{n-1,h}\|_{L_2(\Omega)}^2
\]

\[
\leq \|u_n\|_{L_2(\Omega)}^2 + \frac{1}{2k_n} \|\nabla y_{n-1,h}\|_{L_2(\Omega)}^2.
\]
There exists a constant $C$

**Lemma 12** There exists a constant $C > 0$ independent of $\sigma$ such that for every $y \in H^{2,1}(\Omega_T) \cap C(\overline{I}; H^{1}(\Omega))$ it holds that

$$
\|y - P_\sigma y\|_{L_2(I; L_2(\Omega)))} \leq C \left( k \|y_t\|_{L_2(I; L_2(\Omega)))} + h^2 \|y\|_{L_2(I; H^2(\Omega)))} \right),
$$

(4.13)

$$
\|y - P_\sigma y\|_{L_2(I; H^1(\Omega)))} \leq C \left( \sqrt{k} \|y_t\|_{L_2(I; L_2(\Omega)))} + (\sqrt{k} + h) \|y\|_{L_2(I; H^2(\Omega)))} \right).
$$

(4.14)

**Proof** The proof is standard (see e.g. [52, Lemma 3.4]).

Another technical tool is the definition of a global space-time projection onto $Y_\sigma$ as a discrete solution to the following auxiliary linear parabolic problem. Let $y \in H^{2,1}(\Omega_T) \cap C(\overline{I}; H^{1}(\Omega))$ be the solution to (1.2) and $y_{0,h} := P_h y_0$. We define $\hat{y}_\sigma \in Y_\sigma$ that satisfies: For each $n = 1, \ldots, N$ and $\forall w_h \in Y_h$

$$
\left( \frac{\hat{y}_{n,h} - \hat{y}_{n-1,h}}{k_n}, w_h \right) + (\nabla \hat{y}_{n,h}, \nabla w_h) = (\hat{f}_n, w_h) \quad \text{and} \quad \hat{y}_{0,h} = y_{0,h}.
$$

(4.15)

The right hand side is defined as

$$
(f_n, w_h) = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \left( y_t(t) - \Delta y(t), \nabla w_h \right) \, dt
$$

$$
= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \left( \nabla y(t), \nabla w_h \right) \, dt + \left( \frac{y(t_n) - y(t_{n-1})}{k_n}, w_h \right),
$$

(4.16)
where we have used integration by parts in space and the fact that $w_h \in Y_h$ is independent of $t$. It is clear (see for instance [17, Lemma 4.6]) that (4.15) has a unique solution $\hat{y}_\sigma \in Y_\sigma$. We split the error as follows: $\hat{e} = y - \hat{y}_\sigma = (y - P_\sigma y) + (P_\sigma y - \hat{y}_\sigma)$. Substituting (4.16) into (4.15), we note that $\hat{e}$ satisfies the following orthogonality condition, for each $w_h \in Y_h$, $\forall n = 1, \ldots, N$,

$$\left(\hat{e}(t_n) - \hat{e}(t_{n-1}), w_h\right) + \int_{t_{n-1}}^{t_n} \left(\nabla \hat{e}, \nabla w_h\right) dt = 0. \quad (4.17)$$

The following Lemma collects various stability and error estimates.

**Lemma 13** Suppose that $\hat{y}_\sigma \in Y_\sigma$ is the solution to (4.15). Then, there exists $C > 0$ independent of $\sigma$ and $\epsilon$, such that,

$$\begin{align*}
\|\hat{e}\|_{L_\infty(I; L_2(\Omega))} + \|\hat{e}\|_{L_2(I; H^1(\Omega))} &
\leq C \left(\sqrt{k} \|y_t\|_{L_2(I; L_2(\Omega))} + h \|y\|_{L_2(I; H^2(\Omega))} + h \|y_0\|_{H^1(\Omega)}\right), \\
\|\hat{e}\|_{L_2(I; L_2(\Omega))} &
\leq C \left(k \|y_t\|_{L_2(I; L_2(\Omega))} + h^2 \|y\|_{L_2(I; H^2(\Omega))}\right), \\
\|\hat{y}_\sigma\|_{L_\infty(I; H^1(\Omega))} &
\leq C \|y\|_{H^2(\Omega_T)}. \quad (4.18)
\end{align*}$$

**Proof** Recall that we split the error as $\hat{e} = y - \hat{y}_\sigma = (y - P_\sigma y) + (P_\sigma y - \hat{y}_\sigma)$. In view of Lemma 12, the first term is bounded immediately. As for the estimation of the second one, that belongs on $Y_\sigma$, we refer the readers to [17, 34, 52, 53]. □

The following result states the main error estimate for the control to state mapping. We emphasise that, unlike previous works for the uncontrolled Allen–Cahn equation, we do not exceed the $H^{2,1}(\Omega_T)$ regularity. Our technique employs the spectral estimate at the “continuous level”, and hence it avoids the construction of discrete approximations, which typically lead to higher regularity requirements. In addition, we will use the structure of the nonlinear term and in particular presence of $\|\cdot\|_{L_4(t_n; L_4(\Omega))}$ norm, as well as the construction of a ”global” projection $\hat{y}_\sigma$ through (4.15) and (4.16).

**Theorem 14** Let $u \in U_{ad}$, $y \in H^{2,1}(\Omega_T) \cap C(\bar{\Omega}; H^1(\Omega))$ and $y_\sigma \in Y_\sigma$ satisfy (1.2) and (4.3) respectively. Suppose that (2.7) holds with $\lambda \|L_\infty(I) \leq C$.

1. For $d = 2$, if in addition

$$\left(\sqrt{k} + h\right) \|y\|_{H^{2,1}(\Omega_T)} \max \left\{C_\infty \epsilon^{-1}, \|y\|_{H^{2,1}(\Omega_T)}\right\}^{1/2} \|y\|_{L_\infty(\Omega)}^{1/2} \leq \epsilon^2 CE^{-1/2}, \quad (4.19)$$

then, the following estimates hold:

$$\|y - y_\sigma\|_{L_\infty(I; L_2(\Omega))} + \epsilon^{-1} \|y - y_\sigma\|_{L_4(I; L_4(\Omega))} \leq \max \left\{C_1 \epsilon^{-2} \left(\sqrt{k} + h\right), C_{II} \epsilon^{-1} \left(\sqrt{k} + h\right) \|y\|_{H^{2,1}(\Omega_T)}, C \left(\sqrt{k} + h\right) \|y\|_{H^{2,1}(\Omega_T)}\right\}, \quad (4.20)$$

$$\|y - y_\sigma\|_{L_2(I; H^1(\Omega))}$$

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\[
\leq \max \left\{ C I \epsilon^{-3} (\sqrt{k} + h), C I I \epsilon^{-2} (\sqrt{k} + h) \| y \|_{H^{2,1}(\Omega_T)}, C \left( \sqrt{k} + h \right) \| y \|_{H^{2,1}(\Omega_T)} \right\}.
\]

(4.21)

2. For \( d = 3 \), if in addition

\[
(\sqrt{k} + h) \| y \|_{H^{2,1}(\Omega_T)}^{4/3} \| y \|_{\infty}^{2/3} \leq \epsilon^{3} C E^{-1},
\]

then, the following estimates hold:

\[
\| y - y_\sigma \|_{L_\infty(I;L_2(\Omega))} + \epsilon^{-1} \| y - y_\sigma \|_{L_2(I;L_4(\Omega))}^2 \\
\leq \max \left\{ C I I \epsilon^{-1} (\sqrt{k} + h)^{1/2} \| y \|_{H^{2,1}(\Omega_T)}, C \left( \sqrt{k} + h \right) \| y \|_{H^{2,1}(\Omega_T)} \right\},
\]

(4.23)

\[
\| y - y_\sigma \|_{L_2(I;H^1(\Omega))} \leq \max \left\{ C I I \epsilon^{-2} (\sqrt{k} + h)^{1/2} \| y \|_{H^{2,1}(\Omega_T)}, C \left( \sqrt{k} + h \right) \| y \|_{H^{2,1}(\Omega_T)} \right\}.
\]

(4.24)

Here, we denote by \( C I := 2 \epsilon^{1/2} C_\infty \), \( C I I := C E^{1/2} \) and \( E := \exp(2T\alpha) \) where

\[
C_\infty := C \left( 1 + (1 + \epsilon^2) \| y \|_{\infty}^2 \right)^{1/2}, \quad \alpha := \sup_{t \in I} (2 \lambda(t)(1 - \epsilon^2) + 4 + 2 \epsilon^2),
\]

and \( C > 0 \) an algebraic constant (that might be different in each occurrence) but independent of \( \sigma, \epsilon, \) and \( \| y \|_{\infty} \).

None of the above constants depend on \( \| y \|_{\infty} \) exponentially. Indeed, \( C I \) depends on the \( \| y \|_{\infty} \) polynomially while \( C > 0 \) is an algebraic constant independent of \( \epsilon \). We mainly focus on the case where \( \| y \|_{\infty} \) is bounded independent of \( 1/\epsilon \) (see Remark 5 for a detailed discussion). However, our results hold, without any exponential dependence upon \( 1/\epsilon \) even in more general cases.

**Proof** We begin by splitting the total error:

\[
e = y - y_\sigma = (y - \hat{y}_\sigma) + (\hat{y}_\sigma - y_\sigma) = \hat{e} + e_\sigma.
\]

(4.25)

The aim of our analysis is to bound the term \( e_\sigma \) in terms of \( \hat{e} \), whose bounds are known from Lemma 13. From relations (2.5) and (4.3), for all \( w_h \in Y_h, \forall n = 1, \ldots, N \) it holds that

\[
(e(t_n) - e(t_{n-1}), w_h) + \int_{t_{n-1}}^{t_n} (\nabla e(t), \nabla w_h) \, dt + \epsilon^{-2} \int_{t_{n-1}}^{t_n} \left( F(y) - F(y_{n,h}), w_h \right) \, dt = 0.
\]

Using (4.25), the orthogonality condition (4.16) and choosing \( w_h = e_{n,h} \) we obtain,

\[
(e_{n,h} - e_{n-1,h}, e_{n,h}) + \int_{t_{n-1}}^{t_n} \|\nabla e_{n,h}\|_{L_2(\Omega)}^2 \, dt + \epsilon^{-2} \int_{t_{n-1}}^{t_n} \left( F(y) - F(y_{n,h}), e_{n,h} \right) \, dt = 0.
\]
A standard algebraic manipulation implies that
\[
F(y) - F(y_{n,h}) = (3y^2 - 1)(y - y_{n,h}) - 3y(y - y_{n,h})^2 + (y - y_{n,h})^3.
\]

Using the decomposition (4.25) and elementary identities, we deduce that
\[
\begin{align*}
\frac{1}{2} \|e_{n,h}\|^2_{L^2(\Omega)} & = \frac{1}{2} \|e_{n-1,h}\|^2_{L^2(\Omega)} + \frac{1}{2} \|e_{n,h} - e_{n-1,h}\|^2_{L^2(\Omega)} + \epsilon^{-2} \int_{t_{n-1}}^{t_n} \|e_{n,h}\|^2_{H^1(\Omega)} dt \\
& \quad + \int_{t_{n-1}}^{t_n} (\|\nabla e_{n,h}\|^2_{L^2(\Omega)} + \epsilon^{-2} (F'(y)e_{n,h}, e_{n,h})) dt + 3\epsilon^{-2} \int_{t_{n-1}}^{t_n} (\hat{e}, e_{n,h}) dt \\
& \quad - \epsilon^{-2} \int_{t_{n-1}}^{t_n} (\hat{e}', e_{n,h}) dt - 3\epsilon^{-2} \int_{t_{n-1}}^{t_n} (\hat{e}', e_{n,h}) dt + 3\epsilon^{-2} \int_{t_{n-1}}^{t_n} (y, e_{n,h}^3) dt =: \sum_{j=1}^{6} I_j.
\end{align*}
\]

First, we recover additional coercivity at the right-hand side based on (2.7). Indeed,
\[
\begin{align*}
I_1 & \leq \int_{t_{n-1}}^{t_n} \|\nabla e_{n,h}\|^2_{L^2(\Omega)} dt + \frac{1}{4\epsilon^4} \int_{t_{n-1}}^{t_n} (9\|y\|^2_{L^\infty(\Omega)} + 1) \|\hat{e}\|^2_{L^2(\Omega)} dt \\
& \quad + \int_{t_{n-1}}^{t_n} \|e_{n,h}\|^2_{L^2(\Omega)} dt, \\
I_2 & \leq \frac{1}{2\epsilon^2} \int_{t_{n-1}}^{t_n} \|\hat{e}e_{n,h}\|^2_{L^2(\Omega)} dt + \frac{9}{2\epsilon^2} \int_{t_{n-1}}^{t_n} \|y\|^2_{L^\infty(\Omega)} \|\hat{e}\|^2_{L^2(\Omega)} dt, \\
I_3 & \leq \frac{36}{\epsilon^2} \int_{t_{n-1}}^{t_n} \|y\|^2_{L^\infty(\Omega)} \|\hat{e}\|^2_{L^2(\Omega)} dt + \frac{1}{4\epsilon^2} \int_{t_{n-1}}^{t_n} \|e_{n,h}\|^4_{L^4(\Omega)} dt, \\
I_4 & \leq \frac{3}{4\epsilon^2} \int_{t_{n-1}}^{t_n} \|\hat{e}\|^4_{L^4(\Omega)} dt + \frac{1}{4\epsilon^2} \int_{t_{n-1}}^{t_n} \|e_{n,h}\|^4_{L^4(\Omega)} dt, \\
I_5 & \leq \frac{37}{4\epsilon^2} \int_{t_{n-1}}^{t_n} \|\hat{e}\|^4_{L^4(\Omega)} dt + \frac{1}{4\epsilon^2} \int_{t_{n-1}}^{t_n} \|e_{n,h}\|^4_{L^4(\Omega)} dt.
\end{align*}
\]
The bound of $I_6$ for $d = 3$ differs from one for $d = 2$. We begin with $d = 3$:

$$I_6 \leq \frac{3}{\epsilon^2} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n,h}\|_{L_3(\Omega)}^3 \, dt$$

$$\leq \frac{12}{\epsilon^2} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n,h} - e_{n-1,h}\|_{L_3(\Omega)}^3 \, dt$$

$$+ \frac{12}{\epsilon^2} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n-1,h}\|_{L_3(\Omega)}^3 \, dt.$$  

We want to ensure that the first term of $I_6$ can be absorbed by the term $\frac{1}{2} \|e_{n,h} - e_{n-1,h}\|_{L_2(\Omega)}^2$ of the left-hand side of (4.26). Using (2.3) and (4.1) it yields that $\|e_{n,h} - e_{n-1,h}\|_{L_3(\Omega)}^3 \leq h^{-1/2} C_{inv} \epsilon^3 \|e_{n,h} - e_{n-1,h}\|_{L_2(\Omega)}^2 \|e_{n,h} - e_{n-1,h}\|_{H^1(\Omega)}$. We can bound the quantity $\|e_{n,h} - e_{n-1,h}\|_{H^1(\Omega)}$ through (4.11) and (4.18). Therefore, it is enough to assume that for all $n = 1, \ldots, N$ it holds

$$12 \epsilon^3 C_{inv} \|y\|_{L_\infty(t_{n-1}, t_n; L_\infty(\Omega))} \frac{k_n}{\epsilon^2 \sqrt{h}} \left( C_{st,2} + \|y\|_{H^2(\Omega_T)} \right) \leq 1/4. \quad (4.27)$$

Recalling (2.7) for $v = e_{n,h}$, inserting the bounds of $I_i$ into (4.26) and adding $\epsilon^2 \int_{t_{n-1}}^{t_n} \|e_{n,h}\|_{L_2(\Omega)}^2 \, dt$ to both sides we deduce,

$$\|e_{n,h}\|_{L_2(\Omega)}^2 - \|e_{n-1,h}\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon^2} \int_{t_{n-1}}^{t_n} \|e_{n,h}\|_{L_4(\Omega)}^4 \, dt + 2\epsilon^2 \int_{t_{n-1}}^{t_n} \|e_{n,h}\|_{H^1(\Omega)}^2 \, dt$$

$$+ \frac{5}{\epsilon^2} \int_{t_{n-1}}^{t_n} \|\hat{e}_{n,h}\|_{L_2(\Omega)}^2 \, dt + 4 \int_{t_{n-1}}^{t_n} \|\hat{y}_{n,h}\|_{L_2(\Omega)}^2 \, dt + \frac{1}{2} \|e_{n,h} - e_{n-1,h}\|_{L_2(\Omega)}^2$$

$$\leq \int_{t_{n-1}}^{t_n} \left( \left( \frac{9}{2\epsilon^4} + \frac{81}{\epsilon^2} \right) \|y\|_{L_\infty(\Omega)}^2 + \frac{1}{2\epsilon^4} \right) \|\hat{e}\|_{L_2(\Omega)}^2 \, dt + \int_{t_{n-1}}^{t_n} \left( \frac{3}{2\epsilon^2} + \frac{3}{2\epsilon^2} \right) \|\hat{e}\|_{L_2(\Omega)}^2 \, dt$$

$$+ \int_{t_{n-1}}^{t_n} (2\lambda(t)(1 - \epsilon^2) + 4 + 2\epsilon^2) \|e_{n,h}\|_{L_2(\Omega)}^2 \, dt + \frac{24}{\epsilon^2} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n-1,h}\|_{L_3(\Omega)}^3 \, dt.$$  

Summing from $n = 1$ up to $m$ where $1 \leq m \leq N$, and dropping positive terms, we obtain

$$\|e_{m,h}\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon^2} \sum_{n=1}^{m} k_n \|e_{n,h}\|_{L_4(\Omega)}^4 + 2\epsilon^2 \sum_{n=1}^{m} k_n \|e_{n,h}\|_{H^1(\Omega)}^2$$

$$\leq A + \sum_{n=1}^{m} \|e_{n,h}\|_{L_2(\Omega)}^2 + \frac{24}{\epsilon^2} \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n,h}\|_{L_3(\Omega)}^3 \, dt, \quad (4.28)$$

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where we denote by \( \alpha = \sup_{t \in I} (2\lambda(t)(1 - \epsilon^2) + 4 + 2\epsilon^2) \) and by

\[
A =: \left( (9/2)\epsilon^{-4} + 81\epsilon^{-2} + (1/2)\epsilon^{-4} \right) \|y\|_\infty^2 + (3/2)\epsilon^{-2} + (3^7/2)\epsilon^{-2} \right) \|\hat{e}\|_{L_4(I; L_4(\Omega))}^4.
\]

Assume that \( \sup_{n=1, \ldots, N} k_n \alpha \leq 1/2 \). Note that for every \( m = 1, \ldots, N \), (2.3) and Young’s inequalities (with \( p = 4 \) and \( q = 4/3 \)) and standard algebra imply that

\[
\frac{24}{\epsilon^2} \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n,h}\|_{L_2(\Omega)}^3 \, dt \\
\leq \frac{24\tilde{c}^3}{\epsilon^2} \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n,h}\|_{L_2(\Omega)}^{3/2} \|e_{n,h}\|_{H^1(\Omega)}^{3/2} \, dt \\
= \frac{24\tilde{c}^3}{\epsilon^2} \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \|y\|_{L_\infty(\Omega)} \|e_{n,h}\|_{L_2(\Omega)} \left( \|e_{n,h}\|_{L_2(\Omega)}^{1/2} \|e_{n,h}\|_{H^1(\Omega)}^{3/2} \right) \, dt \\
\leq \frac{24\tilde{c}^3}{\epsilon^2} \|y\|_\infty \sup_{n=1, \ldots, m-1} \|e_{n,h}\|_{L_2(\Omega)} \sum_{n=1}^{m-1} k_n \left( \|e_{n,h}\|_{L_2(\Omega)}^2 + \epsilon^2 \|e_{n,h}\|_{H^1(\Omega)}^2 \right) .
\]

Then, applying Lemma 3 we deduce that

\[
\sup_{n=1, \ldots, N} \|e_{n,h}\|_{L_2(\Omega)}^2 + 2\epsilon^2 \sum_{n=1}^{N} k_n \|e_{n,h}\|_{H^1(\Omega)}^2 + \frac{2}{\epsilon^2} \sum_{n=1}^{N} k_n \|e_{n,h}\|_{L_4(\Omega)}^4 \leq 4AE.
\]

(4.29)

The above estimate holds, upon setting \( \beta = 1/2 \) and \( B = 24\epsilon^{-7/2}\tilde{c}\|y\|_\infty \) in Lemma 3 as long as

\[
A \leq \epsilon^7 \left( 192\tilde{c}\|y\|_\infty (T + 1) E^{3/2} \right)^{-2} := \epsilon^7 C_T E^{-3\|y\|_\infty^{-2}}.
\]

(4.30)

To quantify the relation between \( k \), \( h \) and \( \epsilon \) from (4.30), observe that Lemma 13 and the embedding \( H^{2,1}(\Omega_T) \subset C(\tilde{I}; H^4(\Omega)) \) imply \( \|y - \tilde{y}\|_{L_\infty(I; H^4(\Omega))} \leq C\|y\|_{H^{2,1}(\Omega_T)} \).

Hence, denoting by \( C_\infty^2 := C(1 + (1 + \epsilon^2)\|y\|_\infty^2) \) where \( C \) is an algebraic constant, (2.4) and Lemma 13 imply that

\[
A \sim C_\infty^2 \epsilon^{-4} \|\hat{e}\|_{L_2(I; L_2(\Omega))}^2 + C\tilde{c}^4 \|y\|_{H^{2,1}(\Omega_T)} \epsilon^{-2} \|\hat{e}\|_{L_\infty(I; L_2(\Omega))} \|\hat{e}\|_{L_2(I; H^1(\Omega))}^2 \\
\sim C_\infty^2 \epsilon^{-4} (k + h^2)^2 \|y\|_{H^{2,1}(\Omega_T)}^2 + C\tilde{c}^4 \epsilon^{-2} (\sqrt{k} + h)^3 \|y\|_{H^{2,1}(\Omega_T)}^4 .
\]

It is clear that if

\[
\sqrt{k} + h \leq \epsilon^2 \|y\|_{H^{2,1}(\Omega_T)} C^{-2}_\infty
\]

(4.31)
then the second term of (4.1) dominates the above quantity, and hence (4.1) and (4.30) result to the following restriction:

$$(\sqrt{k} + h)^3 \| y \|_{H^2,1}(\Omega_T)^2 C \leq \epsilon^9 C_T E^{-3} \| y \|_{\infty}^{-2},$$

where $C_T > 0$ is independent of $k$, $h$, $\epsilon$, and $\| y \|_{\infty}$, $\| y \|_{H^2,1}(\Omega_T)$. The estimates (4.23) and (4.24) follow by triangle inequality and the estimate of Lemma 13. Note that if (4.22) is satisfied then both conditions (4.27) and (4.31) are also satisfied.

For $d = 2$, we proceed in a similar way. Using Hölder, (2.3), and Young’s inequalities we deduce,

$$I_6 \leq \frac{3\tilde{c}}{\epsilon^2} \int_{t_{n-1}}^{t_n} \| y \|_{L^\infty(\Omega)} \| e_{n,h} \|_{L^2(\Omega)}^2 \| e_{n,h} \|_{H^1(\Omega)} \, dt$$

$$\leq \frac{6\tilde{c}}{\epsilon^2} \int_{t_{n-1}}^{t_n} \left( \| e_{n,h} - e_{n-1,h} \|_{L^2(\Omega)}^4 + \| e_{n-1,h} \|_{L^2(\Omega)}^2 \right) \| e_{n,h} \|_{H^1(\Omega)} \, dt$$

$$\leq \frac{6\tilde{c}}{\epsilon^2} \int_{t_{n-1}}^{t_n} \| y \|_{L^\infty(\Omega)} \| e_{n,h} - e_{n-1,h} \|_{L^2(\Omega)}^4 \| e_{n,h} \|_{H^1(\Omega)} \, dt$$

$$+ \frac{36\tilde{c}^2}{\epsilon^6} \int_{t_{n-1}}^{t_n} \| y \|_{L^\infty(\Omega)} \| e_{n-1,h} \|_{L^2(\Omega)}^4 \, dt + \frac{\epsilon^2}{4} \int_{t_{n-1}}^{t_n} \| e_{n,h} \|_{H^1(\Omega)}^2 \, dt.$$  

Then, through (4.11) and (4.18), we conclude to an analogous assumption to (4.27),

$$6\tilde{c}\| y \|_{L^\infty(t_{n-1},t_n; L^\infty(\Omega))} k_n \epsilon^{-2} \left( C_{st,2} + \| y \|_{H^2,1}(\Omega_T) \right) \leq 1/4. \quad (4.32)$$

Working exactly as in $d = 3$-case, we have that

$$\| e_{n,h} \|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon^2} \sum_{n=1}^{m} k_n \| e_{n,h} \|_{L^4(\Omega)}^4 + \frac{3}{2} \epsilon^2 \sum_{n=1}^{m} k_n \| e_{n,h} \|_{H^1(\Omega)}^2$$

$$\leq A + \sum_{n=1}^{m} k_n \alpha \| e_{n,h} \|_{L^2(\Omega)}^2 + \frac{36\tilde{c}^2}{\epsilon^6} \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \| y \|_{L^\infty(\Omega)} \| e_{n,h} \|_{L^2(\Omega)}^4 \, dt, \quad (4.33)$$

where $\sup_{n=1,\ldots,N} k_n \alpha \leq 1/2$. Note that for every $m = 1, \ldots, N$

$$\frac{36\tilde{c}^2}{\epsilon^6} \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_n} \| y \|_{L^\infty(\Omega)} \| e_{n,h} \|_{L^2(\Omega)}^4 \, dt$$

$$\leq \frac{36\tilde{c}^2}{\epsilon^6} \| y \|_{L^\infty}^2 \sup_{n=1,\ldots,m-1} \| e_{n,h} \|_{L^2(\Omega)} \sum_{n=1}^{m-1} k_n \| e_{n,h} \|_{L^2(\Omega)}^2.$$
From Lemma 3 for $\beta = 1$ and $B = 36e^{-6}\hat{c}^2\|y\|_\infty^2$, there holds an identical result to (4.29) as long the assumption

$$A \leq e^6 \left( 16 \cdot 36 \hat{c}^4 \|y\|_\infty^2 (T + 1) E^2 \right)^{-1} := e^6 C_T E^{-2} \|y\|_\infty^{-2},$$

(4.34)
is satisfied. To quantify the dependence of $k, h$ upon $\epsilon$ through (4.34) observe that (2.2) and Lemma 13, implies

$$A \sim \frac{C^2}{\epsilon^4} \|\hat{e}\|_{L_2(I; L_2(\Omega))}^2 + \frac{C^2}{\epsilon^4} \|\hat{e}\|_{L_\infty(I; L_2(\Omega))}^2 \|\hat{e}\|_{L_2(I; H^1(\Omega))}^2$$

$$\sim \frac{C^2}{\epsilon^4} (k + h^2)^2 \|y\|_{H^{2,1}(\Omega_T)}^2 + \frac{C^2}{\epsilon^2} (k + h^2)^2 \|y\|_{H^{2,1}(\Omega_T)}^4,$$

that along with (4.30) imply

$$(k + h^2)^2 \|y\|_{H^{2,1}(\Omega_T)}^2 \max \left\{ \frac{C^2}{\epsilon^2}, C \hat{c}^4 \|y\|_{H^{2,1}(\Omega_T)}^2 \right\} \leq e^8 C_T E^{-2} \|y\|_\infty^{-2}.$$

Observe that if the above estimate is satisfied then (4.32) also holds. The estimate follows by triangle inequality and the estimate of Lemma 13. □

**Remark 5** Recall that $C_I \sim \|y\|_\infty$ and $\|y\|_{H^{2,1}(\Omega_T)} \sim \epsilon^{-r}$, where $r \in \{1, 2\}$. Here, the notation states that $r$ can be chosen 1 or 2 depending on the smoothness of data (due to Lemma 1 and Remark 1). Assume that there exists $C > 0$ independent of $\sigma, \epsilon$ such that: $\|y\|_\infty \leq C$. Then, even for $r = 2$, the conditions (4.32), (4.27) are less restrictive than (4.19), (4.22). Now, we shall indicate the dominant term in the bounds of Theorem 14. Indeed, for $d = 2$, (4.19) becomes

$$(\sqrt{k} + h) \|y\|_{H^{2,1}(\Omega_T)} \leq Ce^2 \implies \sqrt{k} + h \leq Ce^{2 + r},$$

(4.35)

where $C$ is an algebraic constant depending only on the domain, and $\|\lambda\|_{L_\infty(I)}$ and hence replacing (4.35) into the estimates (4.20) and (4.21) we obtain,

$$\|y - y_\sigma\|_{L_\infty(I; L_2(\Omega))} + \epsilon^{-1} \|y - y_\sigma\|_{L_4(I; L_2(\Omega))} \leq C(\sqrt{k} + h) \epsilon^{-r},$$

$$\|y - y_\sigma\|_{L_2(I; H^1(\Omega))} \leq C(\sqrt{k} + h) \epsilon^{-r}.$$

For $d = 3$, we deduce that (4.22) becomes

$$(\sqrt{k} + h) \|y\|_{H^{2,1}(\Omega_T)}^{4/3} \leq Ce^3 \implies \sqrt{k} + h \leq Ce^{3 + (4r/3)},$$

(4.36)
and the estimates (4.23) and (4.24) can be written as

\[
\|y - y_\sigma\|_{L^\infty(t_i;L^2(\Omega))} + \epsilon^{-1}\|y - y_\sigma\|_{L^4(t_i;L^4(\Omega))}^2 \leq C \max\{\epsilon^{(1/2)-(r/3)}, 1\}(\sqrt{k} + h)\epsilon^{-r},
\]

\[
\begin{cases} 
C(\sqrt{k} + h)\epsilon^{-1} & \text{when } r = 1, \\
C(\sqrt{k} + h)\epsilon^{-13/6} & \text{when } r = 2,
\end{cases}
\]

\[
\|y - y_\sigma\|_{L^2(t_i;H^1(\Omega))} \leq C(\sqrt{k} + h)\epsilon^{-(7r-1)/6},
\]

\[
\|y - y_\sigma\|_{L^2(t_i;H^1(\Omega))} \leq C(\sqrt{k} + h)\epsilon^{-(8r+3)/6}.
\]

**Remark 6** If additional regularity is present, i.e., if \(\|y\|_{W^{2,1}_4(\Omega_T)} \sim \epsilon^{-r}\), then observe that using the maximal parabolic regularity results of [54] we may bound \(\|\hat{e}\|_{L^4(t_i;L^4(\Omega))} \sim (\kappa + h)^2\|y\|_{W^{2,1}_4(\Omega_T)}^2 \sim (\kappa + h)^2\epsilon^{-4r}\). Therefore, we easily deduce that \(A \sim \frac{C^2}{\epsilon^2}(k + h)^2\|y\|_{W^{2,1}_4(\Omega_T)}^2\) in both \(d = 2, 3\). As a consequence the conditionality reads as \(\sqrt{k} + h \sim \epsilon^{2+(r/2)}\) for \(d = 2\) and \(\sqrt{k} + h \sim \epsilon^{(11/4)+(r/2)}\) for \(d = 3\).

**Remark 7** We note that the proof of Theorem exploits in a crucial way the presence of the \(e_{n,h}\) terms on the left hand side. Consequently, we should emphasize that the structure of the nonlinear term plays an important role in the derivation of the error estimate.

Theorem 14 and Remark 5 imply the following result which will be used subsequently for the derivation of error estimates for the control problem.

**Corollary 15** Let \(u, v \in U_{ad}\), \(y_\sigma \in H^{2,1}(\Omega_T) \cap C(\bar{J}; H^1(\Omega))\) be the solution of (1.2) while \(y_\sigma(v) \in Y_\sigma\) the solution of (4.3) corresponding to the control \(v\). Suppose that the assumptions of Theorems 4 and 14 hold. In addition, let \(\|y_v\|_{L^\infty(\Omega_T)} \leq C\) and \(\|y_v\|_{H^{2,1}(\Omega_T)} \leq C\epsilon^{-r}\) with \(r \in \{1, 2\}\), where \(C\) denotes a constant that depends only on data and it is independent of \(\epsilon\) and that (4.35) or (4.36) hold for \(d = 2\) or 3, respectively. Then, for \(d = 2, 3\) there holds

\[
\|y_u - y_\sigma(v)\|_{L^\infty(t;L^2(\Omega))} \leq L_1\|u - v\|_{L^2(t;L^2(\Omega))} + C\epsilon^{-s_1}(\sqrt{k} + h),
\]

(4.37)

\[
\|y_u - y_\sigma(v)\|_{L^2(t;H^1(\Omega))} \leq L_1\epsilon^{-1}\|u - v\|_{L^2(t;L^2(\Omega))} + C\epsilon^{-s_2}(\sqrt{k} + h),
\]

(4.38)

where \(s_1 = s_2 = r\) when \(d = 2\) and \(s_1 = (7r - 1)/6, s_2 = (8r + 3)/6\) when \(d = 3\). Let \(u_\sigma \in U_\sigma\). If \(u_\sigma \rightarrow u\) weakly in \(L^2(t;L^2(\Omega))\) for every \(\sigma\), then it holds

\[
\begin{cases} 
\|y_u - y_\sigma(u_\sigma)\|_{L^2(t;H^1(\Omega))} \rightarrow 0, \\
\|y_u - y_\sigma(u_\sigma)\|_{L^p(t;L^2(\Omega))} \rightarrow 0
\end{cases}
\]

(4.39)

for all \(1 \leq p < \infty\). Here \(L_1\) denotes the constant of Theorem 4.

**Proof** Inequalities (4.37), (4.38), follow from Theorem 4 and Remark 5 using triangle inequality. For (4.39) we split, \(y_u - y_\sigma(u_\sigma) = (y_u - y_{u_\sigma}) + (y_{u_\sigma} - y_\sigma(u_\sigma))\).
According to Lemma 1 and the boundedness of \( \{u_\sigma\}_\sigma \) in \( L_2(I; L_2(\Omega)) \) we have that \( \|y_{u_\sigma}\|_{H^{2,1}(\Omega_T)} \leq C_\mu \epsilon^{-r}, \ r \in \{1, 2\} \). Then, any weakly convergent subsequence of \( \{y_{u_\sigma}\}_\sigma \) in \( H^{2,1}(\Omega_T) \), converges to \( y_u \). Note that the compact embeddings \( H^{2,1}(\Omega_T) \subset L_2(I; H^1(\Omega)), \ H^{2,1}(\Omega_T) \subset L_p(I; L_2(\Omega)) \) for \( 1 \leq p < \infty \) and \( H^{2,1}(\Omega_T) \hookrightarrow L_2(\partial \Omega_T) \) imply that

\[
\|y_u - y_{u_\sigma}\|_{L_2(I; H^1(\Omega))} + \|y_u - y_{u_\sigma}\|_{L_p(I; L_2(\Omega))} + \|y_u(T) - y_{u_\sigma}(T)\|_{L_2(\Omega)} \to 0.
\]

For every fixed \( \epsilon \) as \( \sigma \to 0, \) with \( \kappa, h \) under the assumptions of Theorem 14 we get,

\[
\|y_{u_\sigma} - y_{\sigma}(u_\sigma)\|_{L_\infty(I; L_2(\Omega))} + \|y_{u_\sigma}(T) - y_{\sigma}(u_\sigma)(T)\|_{L_2(\Omega)}
+ \|y_{u_\sigma} - y_{\sigma}(u_\sigma)\|_{L_2(I; H^1(\Omega))} \to 0,
\]

which completes the proof. \( \square \)

The next Theorem studies the differentiability of the relation between control and discrete state. The proof follows well known techniques and it is omitted (see for instance [16, Theorem 4.10]).

**Theorem 16** Let \( u, v \in L_2(I; L_2(\Omega)) \). The mapping \( G_{\sigma} : L_2(I; L_2(\Omega)) \to Y_{\sigma}, \) such that \( y_{\sigma} = y_{\sigma}(u) = G_{\sigma}(u) \), is of class \( C^\infty \). We denote by \( z_{\sigma}(v) = G'_{\sigma}(u)v \) the unique solution to the problem: For \( n = 1, \ldots, N \) and for all \( w_h \in Y_h \),

\[
\left( \frac{z_{n,h} - z_{n-1,h}}{k_n}, w_h \right) + \left( \nabla z_{n,h}, \nabla w_h \right) + \epsilon^{-2} \left( 3y_{n,h}^2 - 1 \right)z_{n,h}, w_h \right)
= \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \left( v(t), w_h \right) dt, \ \text{with} \ z_{0,h} = 0.
\]

**Remark 8** We note that existence and uniqueness of (4.40) can be proved by standard techniques for any fixed \( \epsilon \). The main difficulty is to prove bounds with constants that do not depend exponentially upon \( 1/\epsilon \). Let us mention that the absence of the cubic term and the fact that we cannot use the spectral estimate (2.7) for \( y_{n,h} \) significantly complicates the recovery of stability bounds that are independent of the exponential of \( 1/\epsilon \). A key part of the remaining of our work is to circumvent this difficulty. Our approach is demonstrated in Sect. 4.2 for the discrete adjoint-state equation, but can be applied to (4.40) in a straightforward manner.

**4.2 Analysis of the discrete adjoint state equation**

The differentiability properties of \( G_{\sigma} : L_2(I; L_2(\Omega)) \to Y_{\sigma} \) imply that the reduced cost functional \( J_{\sigma} : L_2(I; L_2(\Omega)) \to \mathbb{R} \) is of class \( C^\infty \), as well. Applying the chain rule, we get
\[ J_\sigma'(u)v = \int_I \int_\Omega (y_\sigma - y_d)z_\sigma \, dx \, dt + \gamma \int_\Omega (y_\sigma(T) - y_{\Omega,h}(T))z_\sigma(T) \, dx + \mu \int_I \int_\Omega uv \, dx \, dt, \]

(4.41)

To eliminate \( z_\sigma \) from (4.41) we consider the corresponding discrete scheme of the adjoint state equation (3.9), reading: For each \( n = N, \ldots, 1 \) and for all \( w_h \in Y_h \),

\[
\begin{align*}
(\varphi_{n,h} - \varphi_{n+1,h}, w_h) + \int_{t_{n-1}}^{t_n} \left( (\nabla \varphi_{n,h}, \nabla w_h) + \epsilon^{-2} \left( (3\gamma^2_{n,h} - 1)\varphi_{n,h}, w_h \right) \right) \, dt \\
= \int_{t_{n-1}}^{t_n} (y_{n,h} - y_d(t), w_h) \, dt, \quad \text{with} \quad \varphi_{N+1,h} = \gamma \left( y_{N,h} - y_{\Omega,h} \right).
\end{align*}
\]

(4.42)

The above (backwards in time) fully-discrete equation is understood as follows: We begin by computing \( \varphi_{N,h} \) using \( \varphi_{N+1,h} \) and then we descent from \( n = N \) until \( n = 1 \).

Note that we set \( \varphi_{n,h} = \varphi_{n}((t_{n-1}), 1 \leq n \leq N. \) Following identically [16, Sect. 4.2], the expression (4.41) can be written as

\[
J_\sigma'(u)v = \int_I \int_\Omega (\varphi_\sigma + \mu u) \, v \, dx \, dt, \quad \forall v \in L_2(I; L_2(\Omega)).
\]

(4.43)

The following projection operator \( R_\sigma \) is the analogue of projection operator \( P_\sigma \) suitably modified to handle the backwards in time problem. Let \( R_\sigma : C(I; L_2(\Omega)) \rightarrow Y_\sigma \) defined through \( (R_\sigma w)_{n,h} = (R_\sigma w)(t_{n-1}) = P_h w(t_{n-1}), 1 \leq n \leq N. \) If \( w \in H^{2,1}(\Omega_T) \cap C(\hat{I}; H^1(\Omega)) \) there exists \( C > 0 \) such that:

\[
\begin{align*}
\| w - R_\sigma w \|_{L_2(I; L_2(\Omega))} &\leq C \left( k \| w_t \|_{L_2(I; L_2(\Omega))} + h^2 \| w \|_{L_2(I; H^2(\Omega))} \right), \\
\| w - R_\sigma w \|_{L_2(I; H^1(\Omega))} &\leq C \left( \sqrt{k} \| w_t \|_{L_2(I; L_2(\Omega))} + h \| w \|_{L_2(I; H^2(\Omega))} \right), \\
\| w - R_\sigma w \|_{L_\infty(I; L_2(\Omega))} &\leq C \left( \sqrt{k} \| w_t \|_{L_2(I; L_2(\Omega))} + h \| w \|_{L_\infty(I; H^1(\Omega))} \right).
\end{align*}
\]

(4.44)-(4.46)

The following Lemma provides the basic stability estimates for (4.42). We note that our approach avoids assumptions regarding the construction of a discrete approximation of the spectral estimate.

**Lemma 17** Let \( u \in U_{ad}, y_\sigma \in H^{2,1}(\Omega_T) \cap C(\hat{I}; H^1(\Omega)) \) be the solution of (1.2) while \( y_\sigma(u) := y_\sigma \in Y_\sigma \) the solution of (4.3) corresponding to the control \( u \). Suppose that the assumptions of Theorem 4 hold. Let \( C_\infty \sim \| y_u \|_\infty \), and \( \| y_u \|_{H^{2,1}(\Omega_T)} \leq Ce^{-r} \) with \( r \in \{1, 2\} \), where \( C \) denotes a constant that depends only on data and it is independent of \( \epsilon \) and \( C_\zeta := \exp \left( \int f \lambda(x)(1 - \epsilon^2) + 3 \, dt \right) \), and that (4.35) or (4.36) hold for \( d = 2 \) or 3, respectively. If in addition,

\[
\sqrt{k} + h \leq \frac{Ce^{q_d}}{C_\infty C_\zeta}
\]

(4.47)
where \( q_2 = 2 + r \) when \( d = 2 \) and \( q_3 = 3 + (4r/3) \) when \( d = 3 \), then there exists \( D_{st,1}^{dG} > 0 \) (independent of \( \sigma = (k, h) \) and \( \epsilon \)) such that:

\[
\| \varphi_\sigma \|_{L^2(I; H^1(\Omega))} + \epsilon^{-1} \| Y_\sigma \varphi_\sigma \|_{L^2(I; L^2(\Omega))} \leq D_{st,1}^{dG} \epsilon^{-1}.
\]

Here, \( D_{st,1}^{dG} := C \left( \gamma \| y_{N,h} - y_{\Omega,h} \|_{L^2(\Omega)} + \| y_\sigma - y_d \|_{L^2(I; L^2(\Omega))} \right) \) with \( C \) denoting an algebraic constant independent of \( \epsilon \).

**Proof** Standard arguments imply existence and uniqueness of solution \( \varphi_\sigma \) of (4.42). As usual, we need to develop stability bounds with constants independent of \( 1/\epsilon \). For this purpose, we employ a duality argument. Given right-hand side \( \varphi_\sigma \), we define \( \zeta_\sigma \in Y_\sigma \) such that for \( n = 1, \ldots, N \), and for \( w_h \in Y_\sigma \),

\[
(\zeta_{n,h} - \zeta_{n-1,h}, w_h) + \int_{t_{n-1}}^{t_n} \left( (\nabla \zeta_{n,h}, \nabla w_h) + \epsilon^{-2} \left( (3y^2 - 1) \zeta_{n,h}, w_h \right) \right) dt
= \int_{t_{n-1}}^{t_n} (\varphi_{n,h}, w_h) dt, \quad \text{with} \quad \zeta_{0,h} = 0.
\]  

(4.48)

Setting \( w_h = \zeta_{n,h} \) and using (2.7), we easily deduce that

\[
\| \varphi_\sigma \|_{L^\infty(I; L^2(\Omega))} + \| \zeta_\sigma \|_{L^2(I; H^1(\Omega))} + \| Y_\sigma \zeta_\sigma \|_{L^2(I; L^2(\Omega))} \leq C_\zeta \| \varphi_\sigma \|_{L^2(I; L^2(\Omega))},
\]

(4.49)

where \( C_\zeta := C \exp \left( \int_I 2\lambda(t)(1 - \epsilon^2) + 3 + 2\epsilon^2 \, dt \right) \) (with \( C \) an algebraic constant depending on the domain). Setting now \( w_h = (\zeta_{n,h} - \zeta_{n-1,h})/k_n \) into (4.48), and using Hölder and Young’s inequalities, we obtain:

\[
\begin{aligned}
\frac{1}{k_n} \| \zeta_{n,h} - \zeta_{n-1,h} \|^2_{L^2(\Omega)} &+ \| \nabla \zeta_{n,h} \|^2_{L^2(\Omega)} - \| \nabla \zeta_{n-1,h} \|^2_{L^2(\Omega)} + \| \nabla (\zeta_{n,h} - \zeta_{n-1,h}) \|^2_{L^2(\Omega)} \\
&\leq \left( \frac{3\| y \|_{L^2(\Omega)}^2 + 1}{\epsilon^2} \right) \left( \| \zeta_{n,h} \|_{L^2(\Omega)} + \| \varphi_{n,h} \|_{L^2(\Omega)} \right) \| \zeta_{n,h} - \zeta_{n-1,h} \|_{L^2(\Omega)} \\
&\leq \frac{1}{2k_n} \| \zeta_{n,h} - \zeta_{n-1,h} \|^2_{L^2(\Omega)} + \frac{(3\| y \|_{L^2(\Omega)}^2 + 1)^2}{\epsilon^4} k_n \| \zeta_{n,h} \|^2_{L^2(\Omega)} + k_n \| \varphi_{n,h} \|^2_{L^2(\Omega)}.
\end{aligned}
\]

Summing the above inequalities and using (4.49) we derive the estimate,

\[
\| \zeta_\sigma \|_{L^\infty(I; H^1(\Omega))} \leq C_\zeta \left( \frac{C_\infty^2 + 1}{\epsilon^2} + 1 \right) \| \varphi_\sigma \|_{L^2(I; L^2(\Omega))},
\]

(4.50)

upon setting \( C_\infty^2 := 3\| y \|_{L^2(\Omega)}^2 \) for brevity. Now we proceed with the duality argument. Setting \( w_h = \zeta_{n,h} \) into (4.42) we obtain that
\[
(\varphi_{n,h} - \varphi_{n+1,h}, \zeta_{n,h}) + \int_{t_{n-1}}^{t_n} \left( (\nabla \varphi_{n,h}, \nabla \zeta_{n,h}) + \epsilon^{-2} \left( 3y_{n,h}^2 - 1 \right) \varphi_{n,h}, \zeta_{n,h} \right) dt
\]
\[
= \int_{t_{n-1}}^{t_n} (y_{n,h} - y_d(t), \zeta_{n,h}) dt.
\]

Similarly setting \( w_h = \varphi_{n,h} \) into (4.48) respectively
\[
(\zeta_{n,h} - \zeta_{n-1,h}, \varphi_{n,h}) + \int_{t_{n-1}}^{t_n} \left( (\nabla \varphi_{n,h}, \nabla \zeta_{n,h}) + \epsilon^{-2} \left( 3y^2 - 1 \right) \zeta_{n,h}, \varphi_{n,h} \right) dt
\]
\[
= \int_{t_{n-1}}^{t_n} \| \varphi_{n,h} \|^2_{L^2(\Omega)} dt.
\]

Subtracting the last two equalities and summing them from 1 up to \( N \), we deduce,
\[
\| \varphi_{\sigma} \|^2_{L^2(I; L^2(\Omega))} = (\xi_{N,h}, \varphi_{N+1,h}) + \int_I (y_{\sigma} - y_d, \zeta_{\sigma}) dt + \frac{3}{\epsilon^2} \int_I \left( (y^2 - y_{\sigma}^2) \varphi_{\sigma}, \zeta_{\sigma} \right) dt,
\]
where we have used the definition \( \zeta_{0,h} = 0 \) and the calculation
\[
\sum_{n=1}^{N} \left( (\zeta_{n,h} - \zeta_{n-1,h}, \varphi_{n,h}) - (\varphi_{n,h} - \varphi_{n+1,h}, \zeta_{n,h}) \right) = (\xi_{N,h}, \varphi_{N+1,h}).
\]

Therefore, we may bound the terms of the right-hand side of (4.51), as follows: Using the estimates of (4.49), and Young’s inequality,
\[
\int_I (y_{\sigma} - y_d, \zeta_{\sigma}) dt + (\xi_{N,h}, \varphi_{N+1,h})
\]
\[
\leq \| y_{\sigma} - y_d \|_{L^2(I; L^2(\Omega))} \| \zeta_{\sigma} \|_{L^2(I; L^2(\Omega))} + \| \xi_{N,h} \|_{L^2(\Omega)} \| \varphi_{N+1,h} \|_{L^2(\Omega)}
\]
\[
\leq \frac{1}{4} \| \varphi_{\sigma} \|^2_{L^2(I; L^2(\Omega))} + C_\xi \left( \| y_{\sigma} - y_d \|^2_{L^2(I; L^2(\Omega))} + \| \varphi_{N+1,h} \|^2_{L^2(\Omega)} \right).
\]

For the third term of (4.51), the identity \( y^2 - y_{\sigma}^2 = 2y(y - y_{\sigma}) - (y - y_{\sigma})^2 \) yields,
\[
\frac{3}{\epsilon^2} \int_I ((y^2 - y_{\sigma}^2) \varphi_{\sigma}, \zeta_{\sigma}) dt = \frac{6}{\epsilon^2} \int_I (y(y - y_{\sigma}) \varphi_{\sigma}, \zeta_{\sigma}) dt
\]
\[
- \frac{3}{\epsilon^2} \int_I ((y - y_{\sigma})^2 y_{\sigma} \varphi_{\sigma}, \zeta_{\sigma}) dt.
\]

Once again we need to distinguish the cases \( d = 2 \) and \( d = 3 \). For \( d = 3 \) we work as follows: Using Hölder and Young’s inequalities (with an appropriate \( \delta_1 > 0 \) to be
Choosing \( k, h \) such that
\[
\frac{3C_2\xi C_\xi^2 (k + h^2)}{2\delta_1 \epsilon^{6+8r/3}} \leq \frac{1}{8},
\]  
we obtain,
\[
\frac{6}{\epsilon^2} \int_I ((y - y_\sigma)y \varphi_\sigma \cdot \zeta_\sigma) \, dt \leq \frac{1}{4} \| \varphi_\sigma \|_{L^2(I;L^2(\Omega))}^2 + 2\epsilon^2 \delta_1^2 \| \varphi_\sigma \|_{L^2(I;H^1(\Omega))}^2.
\]  
For the second term of (4.53), we proceed as follows: For a suitable chosen \( \delta_2 > 0 \) (to be determined later) we use Hölder and Young’s inequalities, the Gagliardo-Nirenberg inequality (2.3), estimates (4.49), (4.50) and the estimates of Theorem 14–Remark 5 for \( y - y_\sigma \) to get,
\[
\frac{3}{\epsilon^2} \int_I (y - y_\sigma)^2 \varphi_\sigma \cdot \zeta_\sigma \, dt \leq \frac{3}{\epsilon^2} \int_I \| y - y_\sigma \|_{L^4(\Omega)}^2 \| \varphi_\sigma \|_{L^6(\Omega)} \| \zeta_\sigma \|_{L^3(\Omega)} \, dt
\]
\[
\leq \frac{9\epsilon^2}{\delta_2 \epsilon^6} \| \zeta_\sigma \|_{L^4(I;H^1(\Omega))} \| \varphi_\sigma \|_{L^6(I;L^2(\Omega))} \| y - y_\sigma \|_{L^4(I;L^4(\Omega))}^4
\]
\[
+ \frac{\delta_2 \epsilon^2}{2} \| \varphi_\sigma \|_{L^2(I;H^1(\Omega))}^2
\]
\[
\leq \frac{9\epsilon^2}{\delta_2 \epsilon^6} C_\xi^2 \left( \frac{C_\infty^2 + 1}{\epsilon^2} + 1 \right) \epsilon^2 (k + h^2) \frac{\| \varphi_\sigma \|_{L^2(I;L^2(\Omega))}^2 + \| \varphi_\sigma \|_{L^2(I;H^1(\Omega))}^2}{\epsilon^{(7r-1)/3}}.
\]
Choosing \( \kappa, h \) such that
\[
\left( \frac{9}{\delta_2} \right) \epsilon^2 C_\xi^2 \left( \frac{C_\infty^2 + 1}{\epsilon^2} + 1 \right) \frac{(k + h^2)}{\epsilon^{(7r-1)/3}} \leq \frac{1}{4},
\]  
there holds that
\[
\frac{3}{\epsilon^2} \int_I (y - y_\sigma)^2 y \varphi_\sigma \cdot \zeta_\sigma \, dt \leq \frac{1}{4} \| \varphi_\sigma \|_{L^2(I;L^2(\Omega))}^2 + \frac{\delta_2 \epsilon^2}{2} \| \varphi_\sigma \|_{L^2(I;H^1(\Omega))}^2.
\]
Substituting (4.55), (4.57) into (4.53) we get,
\[
\frac{3}{\epsilon^2} \int_I \left( (y^2 - y_\sigma^2) \varphi_\sigma, \zeta_\sigma \right) \, dt \leq \frac{1}{2} \| \varphi_\sigma \|_{L^2_2(I;L^2_2(\Omega))}^2 + (2\epsilon^2 \delta_1^2 + \delta_2)\epsilon^2 \| \varphi_\sigma \|_{L^2_2(I;H^1(\Omega))}^2.
\]
(4.58)

Thus, returning back to (4.51), to substitute (4.52) and (4.58), it yields,
\[
\frac{1}{4} \| \varphi_\sigma \|_{L^2_2(I;L^2(\Omega))}^2 \leq (2\epsilon^2 \delta_1^2 + \delta_2)\epsilon^2 \| \varphi_\sigma \|_{L^2_2(I;H^1(\Omega))}^2
\]
\[
+ C_\epsilon^2 \left( \| y_\sigma - y_d \|_{L^2_2(I;L^2_2(\Omega))}^2 + \| \varphi_{N+1.\,h} \|_{L^2_2(\Omega)}^2 \right).
\]
(4.59)

Next we employ a boot-strap argument. First, we return to (4.42) and set \( w_h = \varphi_{n,h} \) to deduce, after adding to both sides \( \int_{t_n}^{t_{n+1}} \| \varphi_{n,h} \|_{L^2_2(\Omega)}^2 \, dt \) summing the resulting equalities from 1 to \( N \), and standard algebra,
\[
(1/2) \| \varphi_{0,h} \|_{L^2_2(\Omega)}^2 + \| \varphi_\sigma \|_{L^2_2(I;H^1(\Omega))}^2 + 3\epsilon^{-2} \| y_\sigma \|_{L^2_2(I;L^2_2(\Omega))}^2
\]
\[
\leq (3/2) \epsilon^{-2} \| \varphi_\sigma \|_{L^2_2(I;L^2_2(\Omega))}^2 + (1/2) \| y_\sigma - y_d \|_{L^2_2(I;L^2_2(\Omega))}^2 + (1/2) \| \varphi_{N+1.\,h} \|_{L^2_2(\Omega)}^2.
\]

We apply (4.59) to substitute the first term of the right-hand side,
\[
\| \varphi_{0,h} \|_{L^2_2(\Omega)}^2 + 2\| \varphi_\sigma \|_{L^2_2(I;H^1(\Omega))}^2 + 6\epsilon^{-2} \| y_\sigma \|_{L^2_2(I;L^2_2(\Omega))}^2
\]
\[
\leq (3 + 2\epsilon^{-2}) 4(2\epsilon^2 \delta_1^2 + \delta_2)\epsilon^2 \| \varphi_\sigma \|_{L^2_2(I;H^1(\Omega))}^2
\]
\[
+ \left( (3 + 2\epsilon^{-2}) 4C_\epsilon^2 + 1 \right) \left( \| y_\sigma - y_d \|_{L^2_2(I;L^2_2(\Omega))}^2 + \| \varphi_{N+1.\,h} \|_{L^2_2(\Omega)}^2 \right).
\]

Choosing \( \delta_1, \delta_2 > 0 \) such that \( (3 + \frac{2}{\epsilon^2}) 4(2\epsilon^2 \delta_1^2 + \delta_2)\epsilon^2 \leq 1 \), we finally deduce the desired estimate. For \( d = 2 \), we work analogously. \( \square \)

**Remark 9** We note that under the condition (4.47), the constant \( D^d_{st,1} \) is bounded independent of \( \epsilon \). Indeed, adding and subtracting \( y(T) \), and using the estimate of Theorem 14 and Remark 5 and triangle inequality,
\[
D^d_{st,1} \leq C \left( \gamma \| y_{N,h} - y(T) \|_{L^2_2(\Omega)} + \gamma \| y(T) - y_{\Omega,h} \|_{L^2_2(\Omega)} + \| y_\sigma - y_d \|_{L^2_2(I;L^2_2(\Omega))} \right)
\]
\[
\leq C \left( e^{-((7r-1)/6)(\sqrt{h} + h)}(\gamma \| y(T) - y_{\Omega,h} \|_{L^2_2(\Omega)} + \| y_\sigma - y_d \|_{L^2_2(I;L^2_2(\Omega))}) \right).
\]

Substituting now (4.47) we obtain that the first term can be bounded independent of \( 1/\epsilon \), from Lemma 1 we have that the second term is bounded independent of \( \epsilon \) when \( \epsilon \| \nabla y_0 \|_{L^2_2(\Omega)} + \| y_0 \|_{L^4_4(\Omega)} \) is bounded independent of \( \epsilon \). For the third term the estimate (4.9) implies the desired bound.
Theorem 18 Let $u \in U_{ad}$ and $\varphi, \psi \in H^{2,1}(\Omega_T) \cap C(\overline{I}; H^1(\Omega))$ be the associated state solution of (1.2) and the adjoint state solution of (3.9), respectively. Let, $y_\alpha$ be the associated discrete state solution of (4.3) while $\varphi_\alpha$ the associated discrete adjoint state solution of (4.42). Then, under the assumptions of Theorem 14 and Lemma 17 there exists $\hat{C}_d>0$ such that for $r \in \{1, 2\}$ there holds that:

$$
\|\varphi - \varphi_\alpha\|_{L_\infty(I; L_2(\Omega))} + \varepsilon\|\varphi - \varphi_\alpha\|_{L_2(I; H^1(\Omega))} \leq \hat{C}_d\varepsilon^{-\rho_d}(\sqrt{k} + h) \tag{4.60}
$$

where $\rho_2 = 4 + r$ when $d = 2$ and $\rho_3 = 4 + (7r - 1)/6$ when $d = 3$. Here, we denote by $E := \exp(T\alpha)$ where $\alpha := \sup_{t \in I} (2\lambda(t)(1 - \varepsilon^2) + 5 + \varepsilon^2)$ and by $\hat{C}_d$.

$$
\hat{C}_3 := CE^{1/2}(C_{st,2}^dG + C_{st,2}D_{st,1}^dG), \quad \hat{C}_2 := CE^{1/2}(\varepsilon(C_{st,2}^dG + C_{st,2})D_{st,1}^dG).
$$

In the above $C_{st,2}^dG, C_{st,2}, D_{st,1}^dG$ denote the constants of Lemmas 1, 11 and 17 respectively.

Proof We split the total error as follows:

$$
e := \varphi - \varphi_\alpha = (\varphi - R_\alpha \varphi) + (R_\alpha \varphi - \varphi_\alpha) = \eta + \xi_\alpha. \tag{4.61}
$$

For each $n = 0, \ldots, N - 1$ we have,

$$
\eta(t_n) = \varphi(t_n) - (R_\alpha \varphi)(t_n) = \varphi(t_n) - (R_\alpha \varphi)_{n+1,h} = \varphi(t_n) - P_h \varphi(t_n)
$$

$$
\xi_\alpha(t_n) = (R_\alpha \varphi)(t_n) - \varphi_\alpha(t_n) = (R_\alpha \varphi)_{n+1,h} - \varphi_{n+1,h} = \xi_{n+1,h}.
$$

For $n = N$, note that $(R_\alpha \varphi)_{N+1,h} = P_h \varphi(T)$ and $\varphi_{N+1,h} = \gamma(y_{N,h} - y_{\Omega,h})$. From (3.9) and (4.42), we deduce for each $n = N, \ldots, 1$ and for all $w_h \in Y_h$ that

$$
(e(t_{n-1}) - e(t_n), w_h) + \int_{t_{n-1}}^{t_n} \left(\nabla e(t), \nabla w_h\right) dt + \varepsilon^2 \int_{t_{n-1}}^{t_n} \left((3y^2 - 1)\varphi, w_h\right) dt
$$

$$
- \varepsilon^2 \int_{t_{n-1}}^{t_n} \left((3y_n^2 - 1)\varphi_{n,h}, w_h\right) dt = \int_{t_{n-1}}^{t_n} \left(y(t) - y_{n,h}, w_h\right) dt. \tag{4.62}
$$

Setting $w_h = \xi_{n,h}$, using (4.61) and $(\eta(t_n), \xi_{n,h}) = 0$, and adding and subtracting appropriate terms, (4.62) yields,

$$
\left(\xi_{n,h} - \xi_{n+1,h}, \xi_{n,h}\right) + \int_{t_{n-1}}^{t_n} \left\|\nabla \xi_{n,h}\right\|_{L_2(\Omega)}^2 dt + \varepsilon^2 \int_{t_{n-1}}^{t_n} \left((3y^2 - 1)(\varphi - \varphi_{n,h}), \xi_{n,h}\right) dt
$$

$$
+ 3\varepsilon^2 \int_{t_{n-1}}^{t_n} \left((y^2 - y_{\sigma}^2)\varphi_{n,h}, \xi_{n,h}\right) dt = \int_{t_{n-1}}^{t_n} \left(y - y_{\sigma}, \xi_{n,h}\right) dt = \int_{t_{n-1}}^{t_n} \left(\nabla \eta, \nabla \xi_{n,h}\right) dt.
$$

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Relation (4.61), and standard algebraic manipulations imply that

\[
\frac{1}{2} \| \xi_{n,h} \|^2_{L^2(\Omega)} - \frac{1}{2} \| \xi_{n+1,h} \|^2_{L^2(\Omega)} + \frac{1}{2} \| \xi_{n,h} - \xi_{n+1,h} \|^2_{L^2(\Omega)} + \int_{t_{n-1}}^{t_n} \left( \| \nabla \xi_{n,h} \|^2_{L^2(\Omega)} + \epsilon^{-2} \left( (3\epsilon^2 - 1) \xi_{n,h}, \xi_{n,h} \right) \right) dt
\]

\[
= \int_{t_{n-1}}^{t_n} (y - y_\sigma, \xi_{n,h}) dt - \int_{t_{n-1}}^{t_n} (\nabla \eta, \nabla \xi_{n,h}) dt - \epsilon^{-2} \int_{t_{n-1}}^{t_n} \left( (3\epsilon^2 - 1) \eta, \xi_{n,h} \right) dt
\]

\[- 3\epsilon^{-2} \int_{t_{n-1}}^{t_n} ((y - y_\sigma)(y_\sigma + y) \varphi_{n,h}, \xi_{n,h}) dt := \sum_{j=1}^{4} T_j.
\]

Applying Hölder and Young’s inequalities, we easily get

\[
T_1 \leq \frac{1}{2} \int_{t_{n-1}}^{t_n} \| y - y_\sigma \|^2_{L^2(\Omega)} dt + \frac{1}{2} \int_{t_{n-1}}^{t_n} \| \xi_{n,h} \|^2_{L^2(\Omega)} dt,
\]

\[
T_2 \leq \frac{1}{\epsilon^2} \int_{t_{n-1}}^{t_n} \| \nabla \eta \|^2_{L^2(\Omega)} dt + \frac{\epsilon^2}{4} \int_{t_{n-1}}^{t_n} \| \nabla \xi_{n,h} \|^2_{L^2(\Omega)} dt,
\]

\[
T_3 \leq \int_{t_{n-1}}^{t_n} \| y \xi_{n,h} \|^2_{L^2(\Omega)} dt + \frac{1}{4\epsilon^4} \int_{t_{n-1}}^{t_n} \left( 9\| y \|^2_{L^\infty(\Omega)} + 1 \right) \| \eta \|^2_{L^2(\Omega)} dt
\]

\[+ \int_{t_{n-1}}^{t_n} \| \xi_{n,h} \|^2_{L^2(\Omega)} dt.
\]

For the term $T_4$, we distinguish two cases. For $d = 3$, we note that using Hölder and Young’s inequalities, and the stability estimates of $y$, $y_\sigma$ and $\varphi_\sigma$, we deduce,

\[
T_4 \leq \frac{3}{\epsilon^2} \int_{t_{n-1}}^{t_n} \| y - y_\sigma \|^2_{L^2(\Omega)} \| \varphi_{n,h} \|_{L^6(\Omega)} \left( \| y_\sigma \|_{L^6(\Omega)} + \| y \|_{L^6(\Omega)} \right) \| \xi_{n,h} \|_{L^6(\Omega)} dt
\]

\[\leq \frac{9C(C_{\text{st},2} + C_{\text{st},2}^d)^2}{\epsilon^6} \int_{t_{n-1}}^{t_n} \| y - y_\sigma \|^2_{L^2(\Omega)} \| \varphi_{n,h} \|^2_{L^6(\Omega)} dt + \frac{\epsilon^2}{4} \int_{t_{n-1}}^{t_n} \| \xi_{n,h} \|^2_{H^1(\Omega)} dt,
\]

\[\leq \frac{9C(C_{\text{st},2} + C_{\text{st},2}^d)^2}{\epsilon^6} \| y - y_\sigma \|^2_{L^\infty(I;L^2(\Omega))} \int_{t_{n-1}}^{t_n} \| \varphi_{n,h} \|^2_{L^6(\Omega)} dt + \frac{\epsilon^2}{4} \int_{t_{n-1}}^{t_n} \| \xi_{n,h} \|^2_{H^1(\Omega)} dt,
\]

where $C$ is a constant depending only on the domain. Using the spectral estimate (2.7) for $v = \xi_{n,h}$, collecting the above bounds, adding to both sides $(\epsilon^2/2) \int_{t_{n-1}}^{t_n} \| \xi_{n,h} \|^2_{L^2(\Omega)} dt$, summing from $n = m$ up to $N$ where $1 \leq m \leq N$,
using a standard (linear) Gronwall Lemma, for sup_{n=1,...,N} \alpha k_n < 1, where \alpha := sup_{t \in I} (2\lambda(t)(1 - \epsilon^2) + 5 + \epsilon^2), we deduce that

\[ \|\xi_{m,h}\|^2_{L_2(\Omega)} + \sum_{n=m}^N \|\xi_{n,h} - \xi_{n+1,h}\|^2_{L_2(\Omega)} + \epsilon^2 \sum_{n=m}^N k_n \|\xi_{n,h}\|^2_{H^1(\Omega)} \leq E \left( \|\xi_{N+1,h}\|^2_{L_2(\Omega)} + \int_{I_m}^{T} \left( \frac{9\|y\|^2_{L_\infty(\Omega)} + 1}{2\epsilon^4} + \|\eta\|^2_{L_2(\Omega)} + \frac{2\|\nabla\eta\|^2_{L_2(\Omega)}}{\epsilon^2} \right) dt \right) + \int_{I_m}^{T} \|y - y_\sigma\|^2_{L_2(\Omega)} dt + \frac{C(C_{st,2} + C_{dG})^2}{\epsilon^6} \|y - y_\sigma\|^2_{L_\infty(I;L_2(\Omega))} \int_{I_m}^{T} \|\phi_\sigma\|^2_{L_6(\Omega)} dt \]

where \( E := \exp(T\alpha) \). The estimate using the embedding \( H^1(\Omega) \subset L_6(\Omega) \) and Lemma 17 to bound \( \int_{I_m}^{T} \|\phi_\sigma\|^2_{L_6(\Omega)} dt \leq (D_{st,1}/\epsilon)^2 \) and Remark 5 to bound \( \|y - y_\sigma\|^2_{L_\infty(I;L_2(\Omega))} \leq \frac{C_{dG}}{\epsilon^2} (\kappa + h^2) \) for \( d = 2 \) and \( \|y - y_\sigma\|^2_{L_\infty(I;L_2(\Omega))} \leq \frac{C_{dG}}{\epsilon^2} (\kappa + h^2) \) for \( d = 3 \). Note that the term \( \frac{2\epsilon^2}{\epsilon^2} \int_{I_m}^{T} \|\nabla\eta\|^2_{L_2(\Omega)} dt \leq \frac{C_{dG}}{\epsilon^2} (\kappa + h^2) \|\phi_\sigma\|^2_{H^2(\Omega)} \) dominates all \( \eta \) terms. Although, the last term of the right-hand side dominates all terms. For the two dimensional case, in order to use that bound (independent of \( \epsilon \)) of \( \|y\|_{L_\infty(I;L_4(\Omega))} \), we split \( T_4 \) as follows:

\[ T_4 \leq \frac{3}{\epsilon^2} \int_{I_{n-1}}^{I_n} \|y - y_\sigma\|_{L_2(\Omega)} \|\phi_{n,h}\|_{L_8(\Omega)} \left( \|y_\sigma\|_{L_4(\Omega)} + \|y\|_{L_4(\Omega)} \right) \|\xi_{n,h}\|_{L_8(\Omega)} dt \]

\[ \leq \frac{9C(C_{st,2} + C_{dG})^2}{\epsilon^6} \|y - y_\sigma\|^2_{L_\infty(I;L_2(\Omega))} \int_{I_{n-1}}^{I_n} \|\phi_{n,h}\|^2_{H^1(\Omega)} dt + \frac{\epsilon^2}{4} \int_{I_{n-1}}^{I_n} \|\xi_{n,h}\|^2_{H^1(\Omega)} dt, \]

where we have used the embedding \( H^1(\Omega) \subset L_8(\Omega) \), and \( C \) is a constant depending only on the domain. The remaining of the proof follows identical to the three dimensional case.

\[ \square \]

The following result is analogous to Corollary 15 and an immediate consequence of the Theorem 18, Lemma 8 and triangle inequality.

**Corollary 19** Let \( u, v \in U_{ad} \) and \( \varphi_u \in H^{2,1}(\Omega_T) \cap C(\bar{I}; H^1(\Omega)) \) be the solution of (3.9) while \( \varphi_\sigma(v) \in Y_\sigma \) the solution of (4.42) corresponding to the control \( v \). Assume that Lemmas 8 and 17 and Theorem 18 hold and let \( r \in \{1, 2\} \). Then, for \( d = 2, 3 \) there holds

\[ \|\varphi_u - \varphi_\sigma(v)\|_{L_\infty(I;L_2(\Omega))} + \epsilon \|\varphi_u - \varphi_\sigma(v)\|_{L_2(I;H^1(\Omega))} \leq L_d \epsilon^{-\frac{3}{4}} \|u - v\|_{L_2(I;L_2(\Omega))} + \hat{C}_d \epsilon^{-\beta_d}(\sqrt{k} + h), \]
where \( s_3 = 7/2 \) when \( d = 2 \) and \( s_3 = 15/4 \) when \( d = 3 \), and \( \rho_d, \hat{C}_d \) are defined as in Theorem 18. Here, according to the notation of Lemma 8 we denote by

\[
L_2 := L_1 \left( e^{7/2} C_T E_\varphi^{1/2} + \hat{c} C^{1/2} D_{st,1} \right), \quad L_3 := L_1 \left( e^{15/4} C_T E_\varphi^{1/2} + \hat{c} C^{1/2} D_{st,1} \right).
\]

Here, \( L_1, E_\varphi, D_{st,1} \) are defined in Lemmas 1, 7 and 8 respectively.

**Remark 10** To quantify the dependence upon \( 1/\epsilon \), in the worst case scenario where \( \epsilon \parallel y_0 \parallel_{L_2(\Omega)} + \| y_0 \|_{L_4(\Omega)}^2 \leq C \) (which corresponds the case \( r = 2 \)) observe first that \( L_2 \) and \( L_3 \) of Corollary 19 are bounded independent of \( \epsilon \) since the constant \( D_{st,1}^{dG} \) of Lemma 17 is bounded independent of \( \epsilon \) (see Remark 9) and \( L_1 \) is also bounded independent of \( \epsilon \). Note also that for \( d = 2 \), \( \hat{C}_2 \) is bounded independent of \( \epsilon \). Indeed, from (4.9) of Lemma 11, we deduce that \( \| y_\sigma \|_{L_\infty(I; L_4(\Omega))} \) is bounded independent of \( \epsilon \). From the definition of \( C_{st,2} \) (Lemma 1, see also Remark 1), and of \( C_{st,2}^{dG} \) (Lemma 11), imply that \( C_{st,2} + C_{st,2}^{dG} \approx \epsilon^{-1} \) and hence the desired estimate for \( \hat{C}_2 \) follows. Therefore, the estimate (4.63), when \( d = 2 \) and \( r = 2 \), is dominated by

\[
C \left( \epsilon^{-7/2} \| u - v \|_{L_2(I; L_2(\Omega))} + \epsilon^{-6} (\sqrt{k} + h) \right).
\]

where \( C \) denotes a constant that depends only on data. Similarly for \( d = 3 \), \( r = 2 \) we have \( s_3 = 15/4 \) and \( \rho_3 = 4 + 13/6 \), while \( \hat{C}_3 \) is bounded by \( 1/\epsilon \), hence the estimate takes the form

\[
C \left( \epsilon^{-15/4} \| u - v \|_{L_2(I; L_2(\Omega))} + \epsilon^{-(5+(13/6))} (\sqrt{k} + h) \right).
\]

### 4.3 Convergence of the discrete control problem

In this section we study the convergence of the solutions of the discrete control problem (4.6) towards the solutions of the continuous problem (1.3). Every discrete problem (4.6) has at least one solution because the minimisation function is continuous and coercive on a nonempty closed subset of a finite dimensional space.

**Theorem 20** For every \( \sigma = (k, h) \), let \( \tilde{u}_\sigma \) be a global solution of problem (4.6). Then, the sequence \( \{ \tilde{u}_\sigma \}_\sigma \) is bounded in \( L_2(I; L_2(\Omega)) \) and there exist subsequences denoted in the same way, converging to a point \( \tilde{u} \) weakly in \( L_2(I; L_2(\Omega)) \). Any of these limit points is a solution of problem (4.6). Moreover, we have

\[
\lim_{\sigma \to 0} \| \tilde{u} - \tilde{u}_\sigma \|_{L_2(I; L_2(\Omega))} = 0 \quad \text{and} \quad \lim_{\sigma \to 0} J_\sigma (\tilde{u}_\sigma) = J(\tilde{u}). \tag{4.65}
\]

**Theorem 21** Let \( \tilde{u} \) be a strict local minimum of (1.3). Then, there exists a sequence of local minima of problems (4.6) such that (4.65) holds.

The proofs of the Theorems are presented in [16, Theorems 4.15 and 4.17]. In our case, the above convergence results are proved for each fixed interface length, \( \epsilon \).
4.4 Error estimates

In this section we denote by $\tilde{u}$ a local solution of (1.3) and by $\tilde{u}_\sigma$ a local solution of (4.6) $\forall \sigma$. From Theorems 20 and 21, we deduce that $\|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I;L^2(\Omega))} \to 0$. Moreover, let $\tilde{y}$ and $\tilde{\varphi}$ be the state and adjoint state associated to $\tilde{u}$ while $\tilde{y}_\sigma$ and $\tilde{\varphi}_\sigma$ the discrete state and adjoint state associated to $\tilde{u}_\sigma$.

Definition 3 We denote by $u_\sigma$ the projection of $\tilde{u}$ into $U_\sigma$ defined through,

$$u_\sigma = \sum_{n=1}^N \sum_{\tau \in T_h} u_{n,\tau} x_n x_{\tau} \quad \text{where} \quad u_{n,\tau} = \frac{1}{k_n |\tau|} \int_{t_{n-1}}^{t_n} |\tilde{u}(t, x)| \, dx \, dt.$$ (4.66)

Lemma 22 Let $\tilde{u} \in H^1(\Omega_T)$. Then, there exists a constant $C > 0$ independent of $\sigma = (k, h)$ and $\epsilon$ such that,

$$\|\tilde{u} - u_\sigma\|_{L^2(I;L^2(\Omega))} \leq C(\sqrt{k} + h)\|\tilde{u}\|_{H^1(\Omega_T)},$$ (4.67)

$$\|\tilde{u} - u_\sigma\|_{(H^1(\Omega_T))^*} \leq C(k + h^2)\|\tilde{u}\|_{H^1(\Omega_T)},$$ (4.68)

Proof We refer the reader to [16, Lemma 4.17].

We are ready to present the main result. Once again we have to distinguish between the two dimensional and three dimensional cases.

Theorem 23 Let the assumptions of Theorems 9 and 10 and Corollaries 15 and 19 hold. Then, there exist $C_1 := \sqrt{T} L_2$, $\tilde{C}_1 := L_1 C_1$ such that for $d = 2$ and $r = 1$

$$\|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I;L^2(\Omega))} \leq \frac{C_1}{\epsilon^{7/2}}(\sqrt{k} + h)\|\tilde{u}\|_{H^1(\Omega_T)},$$ (4.69)

$$\|\tilde{y} - \tilde{y}_\sigma\|_{L^\infty(I;L^2(\Omega))} + \epsilon\|\tilde{y} - \tilde{y}_\sigma\|_{L^2(I;H^1(\Omega))} \leq \frac{\tilde{C}_1}{\epsilon^{7/2}}(\sqrt{k} + h)\|\tilde{u}\|_{H^1(\Omega_T)},$$ (4.70)

while for $r = 2$ there exist $C_2 := \sqrt{T} \tilde{C}_2$, where $\tilde{C}_2$ is defined by $\tilde{C}_2 := L_1 C_2$ such that

$$\|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I;L^2(\Omega))} \leq \frac{C_2}{\epsilon^6}(\sqrt{k} + h),$$ (4.71)

$$\|\tilde{y} - \tilde{y}_\sigma\|_{L^\infty(I;L^2(\Omega))} + \epsilon\|\tilde{y} - \tilde{y}_\sigma\|_{L^2(I;H^1(\Omega))} \leq \frac{\tilde{C}_2}{\epsilon^6}(\sqrt{k} + h).$$ (4.72)

In addition, for $d = 3$ and $r = 1$ there exist $C_3 := \sqrt{T} L_3$, $\tilde{C}_3 := L_1 C_3$ such that

$$\|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I;L^2(\Omega))} \leq \frac{C_3}{\epsilon^{15/4}}(\sqrt{k} + h)\|\tilde{u}\|_{H^1(\Omega_T)},$$ (4.73)

$$\|\tilde{y} - \tilde{y}_\sigma\|_{L^\infty(I;L^2(\Omega))} + \epsilon\|\tilde{y} - \tilde{y}_\sigma\|_{L^2(I;H^1(\Omega))} \leq \frac{\tilde{C}_3}{\epsilon^{15/4}}(\sqrt{k} + h)\|\tilde{u}\|_{H^1(\Omega_T)},$$ (4.74)

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Finally, for $d = 3$ and $r = 2$ there exist $C_4 = \sqrt{T} \hat{C}_3$, $\tilde{C}_4 := L_1 C_4$ such that

$$
\| \bar{u} - \bar{u}_\sigma \|_{L_2(I; L_2(\Omega_1))} \leq \frac{C_4}{\epsilon^{4+13/6}} (\sqrt{k} + h),
$$

(4.75)

$$
\| \bar{y} - \bar{y}_\sigma \|_{L_\infty(I; L_2(\Omega_1))} + \epsilon \| \bar{y} - \bar{y}_\sigma \|_{L_2(I; H^1(\Omega_1))} \leq \frac{\tilde{C}_4}{\epsilon^{4+13/6}} (\sqrt{k} + h).
$$

(4.76)

In the above $\hat{C}_j$ are defined in Theorem 18 (see also Remark 10), where $\hat{C}_2$ is bounded independent of $\epsilon$, and $\tilde{C}_3 \approx 1/\epsilon$ and $L_1$, $L_2$, $L_3$ are defined in Lemma 1 and (4.64) respectively.

**Proof (Sketch):** The proof follows [16, Sect. 4]. We begin with $d = 3$ and $r = 1$. Note that (4.74) is a consequence of (4.73) combined with (4.37). Thus, the main subject is the proof of (4.73). Suppose that (4.73) is false. More specifically, this assumption implies that

$$
\lim_{\sigma \to 0} \sup \frac{\epsilon^{15/4} \| \bar{u} \|^{-1}_{H^1(\Omega_{\tau})}}{C_3(\sqrt{k} + h)} \| \bar{u} - \bar{u}_\sigma \|_{L_2(I; L_2(\Omega_1))} = +\infty.
$$

This means that there exists a sequence of $\sigma$ such that

$$
\lim_{\sigma \to 0} \frac{\epsilon^{15/4} \| \bar{u} \|^{-1}_{H^1(\Omega_{\tau})}}{C_3(\sqrt{k} + h)} \| \bar{u} - \bar{u}_\sigma \|_{L_2(I; L_2(\Omega_1))} = +\infty.
$$

(4.77)

We aim to conclude to a contradiction. First of all, the fact that $\bar{u}_\sigma$ is a local minimum of (4.6) combined with the property that $J_\sigma$ is of class $C^\infty$ and $u_\sigma \in U_{ad}$ imply that $J'_\sigma(\bar{u}_\sigma)(u_\sigma - \bar{u}_\sigma) \geq 0$. After basic manipulations, we have that

$$
\begin{align*}
J'(\bar{u}_\sigma)(\bar{u} - \bar{u}_\sigma) &+ [J'_\sigma(\bar{u}_\sigma) - J'(\bar{u}_\sigma)](\bar{u} - \bar{u}_\sigma) \\
+ [J'_\sigma(\bar{u}_\sigma) - J'(\bar{u})](u_\sigma - \bar{u}) + J'(\bar{u})(u_\sigma - \bar{u}) &\geq 0.
\end{align*}
$$

Since $\bar{u}$ is an local minimum of (1.3) and $\bar{u}_\sigma \in U_{ad}$, then $J'(\bar{u})(\bar{u}_\sigma - \bar{u}) \geq 0$. Adding this nonnegative term to the above inequality, it yields that

$$
\begin{align*}
[&J'(\bar{u}_\sigma) - J'(\bar{u})](\bar{u}_\sigma - \bar{u}) \leq [J'_\sigma(\bar{u}_\sigma) - J'(\bar{u}_\sigma)](\bar{u} - \bar{u}_\sigma) \\
+ [J'_\sigma(\bar{u}_\sigma) - J'(\bar{u})](u_\sigma - \bar{u}) + J'(\bar{u})(u_\sigma - \bar{u}) &\geq 0.
\end{align*}
$$

(4.78)

Using the second order necessary and sufficient condition of Theorem 10 we derive an estimate from below for the left-hand side and then an upper bound for the three terms on the right-hand side through the estimates of Sects. 3.1, 3.2, 4.1 and 4.2. Indeed, working identically to [16, Lemma 4.18] we obtain, for a suitable $\sigma_0$ independent of $\epsilon$,

$$
\frac{1}{2} \min \{ \delta, \mu \} \| \bar{u}_\sigma - \bar{u} \|_{L_2(I; L_2(\Omega_1))}^2 \leq [J'(\bar{u}_\sigma) - J'(\bar{u})](\bar{u}_\sigma - \bar{u}) \quad \text{for} \quad |\sigma| \leq |\sigma_0|.
$$
Inserting the above lower bound in (4.78) we obtain that,
\[
\frac{1}{2} \min\{\delta, \lambda\} \|\tilde{u}_\sigma - \tilde{u}\|^2_{L^2(I; L^2(\Omega))} \leq [J'_\sigma(\tilde{u}_\sigma) - J'(\tilde{u}_\sigma)](\tilde{u} - \tilde{u}_\sigma) + [J'_\sigma(\tilde{u}_\sigma) - J'(\tilde{u})](u_\sigma - \tilde{u}) + J'(\tilde{u})(u_\sigma - \tilde{u}).
\]
(4.79)

We shall estimate from above the three terms on the right-hand side. More specifically, from (3.7) and (4.43)

\[
[J'_\sigma(\tilde{u}_\sigma) - J'(\tilde{u})](u_\sigma - \tilde{u}) \leq \|\tilde{\varphi}_\sigma - \varphi\|_{L^2(I; L^2(\Omega))} \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))} + \mu \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))}
\]
\[
\leq \sqrt{T} \|\tilde{\varphi}_\sigma - \varphi\|_{L^2(I; L^2(\Omega))} \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))} + \mu \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))}
\]
\[
\leq C \left( \frac{\sqrt{T}L_3}{\epsilon^{15/4}} + \mu \right) \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))} + \sqrt{T}\tilde{C}_3 \left( \frac{\sqrt{k} + h}{\epsilon^{4+(7r-1)/6}} \right) \|\tilde{u}\|_{H^1(\Omega_T)}.
\]
(4.80)

For the second term on the right-hand side of (4.79), we recall again (4.63) for \(u = \tilde{u}\) and \(\nu = \tilde{u}_\sigma\) and (4.67)

\[
[J'_\sigma(\tilde{u}_\sigma) - J'(\tilde{u})](u_\sigma - \tilde{u}) \leq \|\tilde{\varphi}_\sigma - \varphi_{\tilde{u}_\sigma}\|_{L^2(I; L^2(\Omega))} \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))} + \mu \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))}
\]
\[
\leq \sqrt{T} \|\tilde{\varphi}_\sigma - \varphi_{\tilde{u}_\sigma}\|_{L^2(I; L^2(\Omega))} \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))} + \mu \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))}
\]
\[
\leq C \left( \frac{\sqrt{T}L_3}{\epsilon^{15/4}} + \mu \right) \|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))} + \sqrt{T}\tilde{C}_3 \left( \frac{\sqrt{k} + h}{\epsilon^{4+(7r-1)/6}} \right) \|\tilde{u}\|_{H^1(\Omega_T)}.
\]
(4.81)

Finally, using (3.18) to bound \(\|\tilde{\varphi} + \mu \tilde{u}\|_{H^1(\Omega_T)} \leq C \|\tilde{\varphi}\|_{H^1(\Omega_T)}\) and (4.68) we have

\[
J'(\tilde{u})(u_\sigma - \tilde{u}) \leq \|\tilde{\varphi} + \mu \tilde{u}\|_{H^1(\Omega_T)} \|u_\sigma - \tilde{u}\|_{H^1(\Omega_T)}^r
\]
\[
\leq C \left( k + h^2 \right) \|\tilde{u}\|_{H^1(\Omega_T)} \|\tilde{\varphi}\|_{H^1(\Omega_T)}.
\]
(4.82)

Applying Young’s inequality on the right-hand side of (4.80), (4.81), (4.82) and collecting the terms properly in (4.79), we derive

\[
\|\tilde{u} - \tilde{u}_\sigma\|_{L^2(I; L^2(\Omega))} \leq \max \left\{ \frac{\sqrt{T}L_3}{\epsilon^{15/4}} \|\tilde{u}\|_{H^1(\Omega_T)}^r, \sqrt{T}\tilde{C}_3 \frac{\sqrt{\tilde{k}}}{\epsilon^{4+(7r-1)/6}} \right\} (\sqrt{k} + h).
\]

We note that from (3.18) and Theorem 9 we have that \(\|\tilde{u}\|_{H^1(\Omega_T)} \sim \epsilon^{-2}\) hence for \(r = 1\) the first term dominates the maximum. Hence, we deduce the desired estimate for \(C_3 = \sqrt{T}L_3\) (4.73). For the proof of (4.74), we recall (4.37) and (4.38) for \(u = \tilde{u}\), \(\nu = \tilde{u}_\sigma\) combined with (4.73). Working identically for \(d = 3\) and \(r = 2\), we deduce (4.75) and (4.76), respectively. We proceed similarly for \(d = 2\)-case. \(\square\)
5 The spectral estimate for the nonhomogeneous Allen–Cahn equation

We discuss a simple generalization of the spectral estimate in case of the nonhomogeneous Allen–Cahn equation. We observe that in case of our optimal control setting, since the right hand side \( u \in U_{ad} \) we deduce from the regularity estimates of Lemma 2.1 that \( y_u \in H^{2,1}(\Omega_T) \) with bounds that depend polynomially upon \( 1/\epsilon \). In particular, we note that the stability bounds exhibit similar dependence upon \( 1/\epsilon \) to homogeneous case (see e.g. [22], [25]). Throughout this work, and similar to the above mentioned works, we assume that (a uniform) bound for the spectral estimate holds for the linearized operator associated to the solution \( v \) of the homogeneous (uncontrolled) Allen–Cahn equation. Indeed, given initial data \( v_0 \), and with zero forcing term, \( v \) is the solution of, the following problem:

\[
\begin{align*}
    v_t - \Delta v + \frac{1}{\epsilon^2} (v^3 - v) &= 0 & \text{in } \Omega_T = \Omega \times (0, T), \\
    \frac{\partial v}{\partial n} &= 0 \text{ or } v = 0 & \text{on } \Sigma_T = \partial \Omega \times (0, T), \\
    v(\cdot, 0) &= v_0 & \text{in } \Omega.
\end{align*}
\]  

(5.1)

The spectral estimate of the associated linearized operator concerns the quantity:

\[
- \Lambda(t) := \inf_{\nu \in H^1(\Omega) \setminus \{0\}} \frac{\|\nabla \nu\|_{L^2(\Omega)}^2 + \epsilon^{-2} \left( (3v^2 - 1)\nu, \nu \right)}{\|\nu\|_{L^2(\Omega)}^2}.
\]  

(5.2)

In [27–29] it is showed that \( \Lambda \in L_\infty(I) \) can be bounded independently of \( \epsilon \) for the case of smooth, evolved interfaces. We note that a careful inspection of the proof of [27] reveals that the uniform bound is a property related to the interface profile generated by the initial data.

**Lemma 24** Suppose that the initial data profile \( v_0 \) is such that the spectral estimate (5.2) for the linearized operator associated to the solution \( v \) of (5.1), with \( \|\Lambda\|_{L_\infty(I)} \leq C \) holds, where \( C \) denotes a constant independent of \( \epsilon \). Then, for \( y_0 \) such that \( \|y_0\|_{L_\infty(\Omega)} \leq C\epsilon^{-1} \) the spectral estimate for the linearized operator related to the solution \( y_u \) of (1.2) with \( u \in U_{ad} \) holds with uniform in time bound, i.e.,

\[
- \lambda(t) := \inf_{\nu \in H^1(\Omega) \setminus \{0\}} \frac{\|\nabla \nu\|_{L^2(\Omega)}^2 + \epsilon^{-2} \left( (3y_u^2 - 1)\nu, \nu \right)}{\|\nu\|_{L^2(\Omega)}^2}.
\]  

(5.3)

where \( \|\lambda\|_{L_\infty(I)} \leq \tilde{C} \), with \( \tilde{C} \) a constant depending on \( C \) and independent of \( \epsilon \).
Proof Given any control $u \in U_{ad}$ and its corresponding state $y_u$, we define $w$ as the solution of the following auxiliary problem:

$$w_t - \Delta w + \frac{1}{\epsilon^2} (w^3 - w) = \frac{1}{\epsilon^2} (3y_u w^2 - 3wy_u^2) - u \quad \text{in } \Omega_T = \Omega \times (0, T),$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{or} \quad w = 0 \quad \text{on } \Sigma_T = \partial\Omega \times (0, T), \quad (5.4)$$

$$w(\cdot, 0) = (1 - \epsilon^2)y_0 \quad \text{in } \Omega.$$

Due to the regularity of $y_u$, it is evident that there exists solution $w \in W(I)$ (at least) of (5.4). We note that $v := y_u - w$ satisfies (5.1), with $v_0 = \epsilon^2 y_0$. In addition, a straightforward application of the maximum principle implies that $\|v\|_\infty \leq \epsilon^2 \|y_0\|_{L_\infty(\Omega)}$. Let $\varphi \in H^1(\Omega)$, with $\varphi \neq 0$. Then, substituting $y_u = v + w$, using the (5.2) for $v$, and the inequality $ab \geq -\frac{1}{2}(a^2 + b^2)$

$$\|\nabla \varphi\|_{L_2(\Omega)}^2 + \frac{1}{\epsilon^2} \left((3y_u^2 - 1)\varphi, \varphi\right) = \|\nabla \varphi\|_{L_2(\Omega)}^2 + \frac{1}{\epsilon^2} \left((3v^2 - 1)\varphi, \varphi\right)$$

$$+ \frac{1}{\epsilon^2} \left(3w^2 \varphi, \varphi\right) + \frac{1}{\epsilon^2} \left(6vw \varphi, \varphi\right)$$

$$\geq -\Lambda(t) \|\varphi\|_{L_2(\Omega)}^2 - \frac{1}{\epsilon^2} \left(3v^2 \varphi, \varphi\right) \geq -\Lambda(t) \|\varphi\|_{L_2(\Omega)}^2 - \frac{3}{\epsilon^2} \|v\|_\infty^2 \|\varphi\|_{L_2(\Omega)}^2$$

$$\geq - \left(\Lambda(t) + \epsilon^2 \|y_0\|_{L_\infty(\Omega)}\right) \|\varphi\|_{L_2(\Omega)}^2 \overset{\text{5.4}}{=} -\lambda(t) \|\varphi\|_{L_2(\Omega)}^2,$$

where at the last step we have used the estimate $\|v\|_\infty \leq \epsilon^2 \|y_0\|_{L_\infty(\Omega)}$. Note that if $\|y_0\|_{L_\infty(\Omega)} \leq C\epsilon^{-1}$, then $\|\lambda\|_{L_\infty(I)}$ is bounded independently of $\epsilon$. \qed

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Appendix A: Proof of Lemma 1

These estimates are standard (see e.g. [22, 35, 38]). For completeness we state the main steps that lead to the estimates. For the first result testing (2.5) with $y$, and using Young’s inequalities, we obtain

$$\frac{1}{2} \frac{d}{dt} \|y\|_{L_2(\Omega)}^2 + \|\nabla y\|_{L_2(\Omega)}^2 + \frac{1}{\epsilon^2} \|y\|_{L_4(\Omega)}^4 \leq \frac{2}{\epsilon^2} \|y\|_{L_2(\Omega)}^2 + \frac{\epsilon^2}{4} \|u\|_{L_2(\Omega)}^2$$

$$\leq \frac{1}{2\epsilon^2} \|y\|_{L_4(\Omega)}^4 + \frac{2}{\epsilon^2} \|\Omega\| + \frac{\epsilon^2}{4} \|u\|_{L_2(\Omega)}^2.$$

Integrating from 0 to $T$ and using standard algebra we get the estimate on $\|\cdot\|_{L_4(I; L_4(\Omega))}$. Integrating from 0 to $t$, and using the bound on $\|\cdot\|_{L_4(I; L_4(\Omega))}$, we
deduce the estimate on $\| \cdot \|_{L^\infty(I;L^2(\Omega))}$. Note that Hölder and Young’s inequalities yield,

$$\| y \|_{L^2(I;L^2(\Omega))} \leq |\Omega|^{1/4} T^{1/4} \| y \|_{L^4(I;L^4(\Omega))} \leq C \left( |\Omega|^{1/2} T^{1/2} + \| y \|_{L^4(I;L^4(\Omega))} \right).$$

The estimate for $\| y \|_{L^2(0,T;H^1(\Omega))}$ follows by triangle inequality. Now we test (2.5) with $y_t$, and we use Young’s inequality to get

$$\| y_t \|_{L^2(\Omega)}^2 + \frac{d}{dt} \left( \frac{1}{2} \| \nabla y \|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon^2} (y^2 - 1)^2 \| L^1(\Omega) \right) \leq \frac{1}{2} \| y_t \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u \|_{L^2(\Omega)}^2.$$

Integrating with respect to $t \in (0,\tau)$,

$$\begin{align*}
\| y_t \|_{L^2(0,\tau;L^2(\Omega))} + \| \nabla y(\tau) \|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon^2} (y^2(\tau) - 1)^2 \| L^1(\Omega) \right) &\leq \frac{1}{2} \| u \|_{L^2(0,\tau;L^2(\Omega))}^2 + \| \nabla y(0) \|_{L^2(\Omega)}^2 + \frac{1}{2\epsilon^2} (y^2(0) - 1)^2 \| L^1(\Omega) \right), \quad (A1)
\end{align*}$$

which implies the desired estimate. Multiplying (1.2) with $-\Delta y$, integrating by parts with respect to space, and Young’s inequality yield,

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \nabla y \|_{L^2(\Omega)}^2 + \| \Delta y \|_{L^2(\Omega)}^2 &+ \frac{3}{\epsilon^2} \| \nabla y \|_{L^2(\Omega)}^2 \\
&\leq \frac{1}{2} \left( \| \Delta y \|_{L^2(\Omega)}^2 + \| u \|_{L^2(\Omega)}^2 \right) + \frac{1}{\epsilon^2} \| \nabla y \|_{L^2(\Omega)}^2.
\end{align*}$$

The estimate is completed after integrating with respect to time and substituting the bounds of $\| \nabla y \|_{L^\infty(I;L^2(\Omega))}^2$.

**Appendix B: Proof of Lemma 7**

To derive the first stability estimate, we test (3.9) with $w = \varphi_u$.

$$-\frac{1}{2} \frac{d}{dt} \| \varphi_u \|_{L^2(\Omega)}^2 + \| \nabla \varphi_u \|_{L^2(\Omega)}^2 + \epsilon^{-2} (F'(y_u) \varphi_u, \varphi_u) = (y_u - y_d, \varphi_u).$$

Recalling (2.7) for $v = \varphi_u$ about state solution $y_u$, adding $\epsilon^2 \| \varphi \|_{L^2(\Omega)}^2$ on both sides applying Cauchy-Schwarz and Young’s inequalities on the right-hand side and integrating with respect to $t \in (\tau, T)$ we obtain

$$\begin{align*}
\frac{1}{2} \| \varphi_u(\tau) \|_{L^2(\Omega)}^2 + \epsilon^2 \int^{\tau}_T \| \varphi_u \|_{H^1(\Omega)}^2 \, dt + 3 \int^{\tau}_T \| \varphi_u y_u \|_{L^2(\Omega)} \, dt &\leq \frac{1}{2} \| \varphi_u(T) \|_{L^2(\Omega)}^2 + \int^{\tau}_T \left( \lambda(t)(1 - \epsilon^2) + \frac{3}{2} + \epsilon^2 \right) \| \varphi_u \|_{L^2(\Omega)}^2 \, dt
\end{align*}$$
\[ + \frac{1}{2} \int_{\tau}^{T} \| y_u - y_d \|_{L_2(\Omega)}^2 \, dt. \]

The (linear) Gronwall inequality yields the result. Setting \( w = -\varphi_{u,t} \) into (3.9) we have

\[ \| \varphi_{u,t} \|_{L_2(\Omega)}^2 \leq \frac{1}{2} \frac{d}{dt} \| \nabla \varphi_u \|_{L_2(\Omega)}^2 = (y_u - y_d, \varphi_{u,t}) + \epsilon^{-2} (F'(y_u) \varphi_u, \varphi_{u,t}). \quad \text{(B1)} \]

Note that using Young’s inequalities, we may bound the last two terms as follows:

\[ |(y_u - y_d, \varphi_{u,t})| \leq (1/4) \| \varphi_{u,t} \|_{L_2(\Omega)}^2 + \| y_u - y_d \|_{L_2(\Omega)}^2, \]
\[ \epsilon^{-2} |(F'(y_u) \varphi_u, \varphi_{u,t})| = \epsilon^{-2} |(3y_u^2 - 1) \varphi_{u,t}| \leq (1/4) \| \varphi_{u,t} \|_{L_2(\Omega)}^2 + 18 \epsilon^{-4} \| y_u \|_{L_\infty(\Omega)}^2 \| \varphi_u y_u \|_{L_2(\Omega)}^2 + 2 \epsilon^{-4} \| \varphi_u \|_{L_2(\Omega)}^2. \]

Substituting the last two inequalities into (B1) and integrating from \( \tau \) to \( T \) it yields,

\[ \int_{\tau}^{T} \| \varphi_{u,t} \|_{L_2(\Omega)}^2 \, dt + \| \nabla \varphi_u (\tau) \|_{L_2(\Omega)}^2 \leq 2 \int_{\tau}^{T} \| y_u - y_d \|_{L_2(\Omega)}^2 \, dt + \| \nabla \varphi_u (T) \|_{L_2(\Omega)}^2 + 36 \epsilon^{-4} \| y_u \|_{L_\infty(\Omega)}^2 \int_{\tau}^{T} \| \varphi_u y_u \|_{L_2(\Omega)}^2 \, dt + 4 \epsilon^{-4} \int_{\tau}^{T} \| \varphi_u \|_{L_2(\Omega)}^2 \, dt. \]

Using the stability bound (3.10) we obtain (3.11). The third estimate follows using similar techniques by setting \( w = -\Delta \varphi_u \) into (3.9) and using the previous bounds to estimate \( \| \varphi_{u,t} \|_{L_2(\Omega)} \), \( \| \nabla \varphi_u \|_{L_2(\Omega)} \) and \( \| y_u \varphi_u \|_{L_2(\Omega)} \).

**Appendix C: Proof of Lemma 8**

Subtracting the equations satisfied by \( \varphi_1 \) and \( \varphi_2 \), it yields for \( w \in H^1(\Omega) \)

\[
\begin{cases}
-(\varphi_{1,t} - \varphi_{2,t}, w) + (\nabla \varphi_1 - \nabla \varphi_2, \nabla w) + \epsilon^{-2} (F'(y_1) \varphi_1 - F'(y_2) \varphi_2, w) = (y_1 - y_2, w) \\
(\varphi_1 - \varphi_2) (T) = \gamma (y_1 - y_2) (T).
\end{cases}

\]

Inserting the identity, \( F'(y_1) \varphi_1 - F'(y_2) \varphi_2 = (3y_1^2 - 1) \varphi_1 - \varphi_2 + 3 (y_1^2 - y_2^2) \varphi_2 \) and testing with \( w = \varphi_1 - \varphi_2 \), we deduce that

\[
-\frac{1}{2} \frac{d}{dt} \| \varphi_1 - \varphi_2 \|_{L_2(\Omega)}^2 + \| \nabla (\varphi_1 - \varphi_2) \|_{L_2(\Omega)}^2 + \epsilon^{-2} (F'(y_1) (\varphi_1 - \varphi_2), \varphi_1 - \varphi_2) = (y_1 - y_2, \varphi_1 - \varphi_2) - 3 \epsilon^{-2} (y_1^2 - y_2^2) \varphi_2, \varphi_1 - \varphi_2) := K_1 + K_2. \quad \text{(C1)}
\]
Cauchy-Schwarz and Young’s inequalities yield,
\[ K_1 \leq \frac{1}{2} \| \varphi_1 - \varphi_2 \|^2_{L^2(\Omega)} + \frac{1}{2} \| y_1 - y_2 \|^2_{L^2(\Omega)}. \]

Applying Hölder inequality, the embedding \( H^1(\Omega) \subset L^4(\Omega) \) and Young’s inequality on the second term, we obtain
\[ K_2 \leq 3\epsilon^{-2} \| y_1 - y_2 \|^2_{L^2(\Omega)} \left( \| y_1 \|_{L^\infty(\Omega)} + \| y_2 \|_{L^\infty(\Omega)} \right) \| \varphi_2 \|_{L^4(\Omega)} \| \varphi_1 - \varphi_2 \|_{L^4(\Omega)} \]
\[ \leq C_2^2 \epsilon^{-6} \| y_1 - y_2 \|^2_{L^2(\Omega)} \| \varphi_2 \|^2_{L^4(\Omega)} + \left( 1/2 \right) \epsilon^2 \| \varphi_1 - \varphi_2 \|^2_{H^1(\Omega)}. \]

Substituting the bounds on \( K_i \), the spectral estimate (2.7) for \( u = \varphi_1 - \varphi_2 \), \( y = y_1 \), adding \( \epsilon^2 \| \varphi_1 - \varphi_2 \|^2_{L^2(\Omega)} \) and integrating from \( \tau \) to \( T \), (C1) yields,
\[ \| (\varphi_1 - \varphi_2)(\tau) \|^2_{L^2(\Omega)} + \epsilon^2 \int_{\tau}^{T} \| \varphi_1 - \varphi_2 \|^2_{H^1(\Omega)} \, dt \]
\[ \leq \| y(y_1 - y_2)(T) \|^2_{L^2(\Omega)} + \int_{\tau}^{T} \| y_1 - y_2 \|^2_{L^2(\Omega)} \, dt \]
\[ + \int_{\tau}^{T} \left( 2\lambda(t)(1 - \epsilon^2) + 4 + 2\epsilon^2 \right) \| \varphi_1 - \varphi_2 \|^2_{L^2(\Omega)} \, dt \]
\[ + C_2^2 \epsilon^{-6} \| y_1 - y_2 \|^2_{L^\infty(I;L^2(\Omega))} \int_{\tau}^{T} \| \varphi_2 \|^2_{L^4(\Omega)} \, dt. \]

For \( d = 3 \), (2.4) and Hölder’s inequalities with \( s_1 = 4 \) and \( s_2 = 4/3 \), and (3.10) imply,
\[ \int_{\tau}^{T} \| \varphi_2 \|^2_{L^4(\Omega)} \, dt \leq \tilde{c}^2 \int_{\tau}^{T} \| \varphi_2 \|_{L^2(\Omega)}^{1/2} \| \varphi_2 \|_{H^1(\Omega)}^{3/2} \, dt \]
\[ \leq \tilde{c}^2 \| \varphi_2 \|_{L^2(I;L^2(\Omega))}^{1/2} \| \varphi_2 \|_{L^2(I;H^1(\Omega))}^{3/2} \leq T^{1/4} \tilde{c}^2 D_{st,1}^2 \epsilon^{-3/2}. \]

Similarly, for \( d = 2 \), we have
\[ \int_{\tau}^{T} \| \varphi_2 \|^2_{L^4(\Omega)} \, dt \leq c^2 \| \varphi_2 \|_{L^2(I;L^2(\Omega))} \| \varphi_2 \|_{L^2(I;H^1(\Omega))} \leq T^{1/2} c^2 D_{st,1}^2 \epsilon^{-1}. \]

Then, for \( d = 3 \), the (linear) Gronwall inequality implies that
\[ \sup_{t \in [0, T]} \| (\varphi_1 - \varphi_2)(t) \|^2_{L^2(\Omega)} + \epsilon^2 \| \varphi_1 - \varphi_2 \|^2_{L^2(I;H^1(\Omega))} \]
\[ \leq E_\varphi \left( \| y(y_1 - y_2)(T) \|^2_{L^2(\Omega)} + \left( T + T^{1/4} C_{\infty} D_{st,1}^2 \epsilon^{-15/2} \right) \sup_{t \in I} \| y_1 - y_2 \|^2_{L^2(\Omega)} \right). \]

The estimate now follows by using the Lipschitz continuity estimate of Theorem 4. Working in an identical way, we deduce the estimate for \( d = 2 \).
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