Supertranslation and superrotation from soldering transformations

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Abstract: When two spacetimes are stitched together across a null shell placed at the horizon of a black hole, BMS-supertranslation like soldering freedom arises if one demands that, the induced metric on the shell should remain invariant under the translations generated by the null generators of the shell. We revisit this phenomenon for null shells and demonstrate that not only BMS-supertranslation but also BMS-superrotation like symmetry can emerge if we solder two metrics across a null hypersurface for which the induced metric is invariant up to a conformal rescaling. We demonstrate the appearance of this new type of soldering symmetry for the case of BTZ and Schwarzschild black holes. We compute the expression for the conserved charge (Energy) corresponding to this symmetry for BTZ black hole and relate that with the existing results.
1 Introduction

Symmetry consideration has been proven to be a very powerful approach to study many physical systems, and General Relativity is not an exception. Recently there has been a growing interest in determining the structure of asymptotic symmetries in gravity. Many years ago Bondi, van der Burg, Metzner, and Sachs (BMS) studied the diffeomorphisms that preserve the asymptotic structure of any asymptotically flat spacetime at future null infinity $I^+$ and to their great surprise, the asymptotic symmetry group turned out to be infinite dimensional, a semi direct product of Lorentz group and supertranslations (angle dependent translations) [1, 2]. These supertranslations constitute an infinite dimensional abelian subgroup of the BMS group and they map one asymptotically flat solution of Einstein’s equation to another. Not long ago, BMS group has got extended and new symmetries like superrotations have emerged at the null infinities (both future and past) of asymptotically flat spacetimes [3–6].\(^1\) In simple terms this new symmetry rotates around each generator of $I^\pm$ separately. Due to superrotations the Lorentz part of the BMS group gets enlarged to full Virasoro group. Superrotation like symmetries have also emerged from diffeomorphisms preserving the near horizon geometry of black holes [8–10]. On the other hand it has been shown that, there is a deep connection between the infrared structure of gravity (and also few gauge theories) and the asymptotic symmetries of it. Certain subgroup of $\text{BMS}^+ \times \text{BMS}^-$ has emerged as an exact symmetry of quantum

\(^1\)Recently this has been extended to Conformal BMS group in [7].
gravitational S matrix \([11–21]\). For a more recent and comprehensive review on this subject, interested readers are referred to \([22]\) and also encouraged to consult various references therein. These interconnections have generated the intriguing possibility of resolving the black hole information paradox. The idea is that black holes are imparted by infinite number of soft hairs corresponding to diffeomorphisms that act non-trivially on the phase space \([23–26]\).\(^2\)

The supertranslation like transformations also arise in another way when one tries to stitch two spacetimes across a thin null shell assumed to be situated at the event horizon of a black hole \([29]\). It has been shown that there exists considerable amount of freedom to solder two metrics across a null shell for which the induced metric remains invariant under the translations along the null generators. In fact the group of soldering transformations turns out to be infinite dimensional and have identical structure- like supertranslations in BMS group.

In \([29]\), a detailed analyses has been presented to find the soldering group of Schwarzschild spacetime. First as a warm up exercise, we extend the results in \([29]\) in the case of rotating spacetime, particularly we have analysed a null shell in Kerr blackhole in slow rotating limit and rotating BTZ and demonstrate that BMS like symmetry (supertranslation) again emerges. The intrinsic properties of the shell are also studied.

Then the more important issue that we have addressed here and which is also the main focus of the paper is: Whether there is any way to obtain a superrotation type symmetry while joining two spacetimes across a null shell. Although people have already recovered this symmetry by relaxing some of the fall-off conditions either of asymptotic metric structure or of the near horizon metrics in 3 and 4 dimensional spacetimes \([4–6, 8, 9, 30]\) but here we want to reconsider those results from the point of view of soldering transformations. We want to see if we demand that the induced metric on the null shell across which we stitch the metrics is invariant up to conformal rescaling (thereby relaxing the condition on the induced metric of the shell to allow conformal isometries instead of only isometries) do we get something like superrotation type symmetries on the black hole horizon? Our analyses suggest the answer is affirmative. For example, in the case of a BTZ black hole we have found that we can patch two spacetimes across the horizon shell allowing the following freedom in the angular coordinate

\[
\phi \rightarrow \phi + T(\phi) \tag{1.1}
\]

if we consider that the induced metric on the shell allows conformal isometries. This is nothing but superrotation freedom that emerged in the analysis of asymptotic symmetries near \(I^\pm\) in recent past. We also show in a similar way superrotation

\(^2\)Recently there are some works which suggest that the soft charges do not play any role in resolving the information paradox. Interested readers are referred to \([27, 28]\).
arises in asymptotically flat spacetimes (Schwarzschild). The construction depicted here may appear closely related to Newman-Unti group of transformations that allow conformal isometries of the celestial spheres at null infinity [31, 32]. Our approach has also similarities with the Nutku-Penrose construction where two local vacuum solutions of Einstein equations are glued together across a null shell [33]. In this case conformal transformations ensure the continuity of the induced metric across the shell. We don’t know exactly how these constructions are related to our analyses but certainly there exist enough evidences for the relevance of conformal isometries in the context of horizon shells.

Role of conformal invariance in the near horizon physics of black hole has been explored quite extensively in recent past [34–37]. Much of the recent activities related to BMS algebra are reconsideration of asymptotic symmetries in relation to flat space holography [38–40]; that is, extending the AdS/CFT correspondence for asymptotically flat spacetimes. The emergence of superrotation as a new symmetry of the near horizon physics is an important outcome of that [8]. It is thus not quite unexpected thing to discover superrotation symmetry from the perspective of soldering freedom in a slightly modified version of Israel junction condition [41]. In the case of null shells allowing supertranslation like freedom to solder two metrics across it, it was shown in [29] that, the conserved charge associated with that symmetry has strikingly similar form as that of the BMS-supertranslation charge at \( I^+(I^-) \) or the near horizon counterpart of it. We also expect that superrotation like freedom will also bear such similarity with the corresponding charges at \( I^\pm \). In this paper we have shown that, when two supertranslated (by different amount) spacetimes with identical superrotations are soldered we get a conserved quantity that depends upon both the parameters of superrotation and supertranslation. This will make the direct relation between BMS-supertranslation and superrotation hair of a black hole and shell dynamics more apparent.

In section 2 we review the Israel junction condition briefly and in the next section emergence of BMS like transformations from soldering transformations is reviewed. In Section 4 we deduce BMS like transformations for slowly rotating Kerr and rotating BTZ spacetimes and obtain conserved quantities. Section 5 is devoted to explain how one can include conformal isometries in soldering freedom. In section 6 the off-shell extension of the set up depicted in section 5 has been demonstrated. Finally we conclude with discussions on our results and propose few scopes of further investigations.

2 Brief review of Israel junction condition

In this section we start by briefly reviewing all the essential features of Israel junction [41–43] conditions. Most commonly, in general relativity, the problem is to find the surface dynamics of a thin shell, where the surface is embedded in a spacetime (\( \mathcal{M} \))
of the form $\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_-$. $\mathcal{M}_+$ and $\mathcal{M}_-$ denote respectively the manifolds inside and outside of the shell together with the corresponding intrinsic metrics $g_{\mu\nu}^+(x^\mu_\pm)$ and $g_{\mu\nu}^-(x^\mu_\pm)$. Both the manifolds have boundaries, with $\mathcal{M}_+$ and $\mathcal{M}_-$ are defined to the future and past of null hypersurfaces $\Sigma_+$ and $\Sigma_-$ respectively. Now let us suppose we can define a set of intrinsic coordinates $\zeta^a$ on the surface of the shell $\Sigma$, across which the two manifolds will be joined. Now we can project both $g^\pm_{\mu\nu}$ on the surface from both sides. Then the junction condition is,

$$g_{ab} = g^+_{\mu\nu} e^+_a e^+_b |_{\Sigma_+} = g^-_{\mu\nu} e^-_a e^-_b |_{\Sigma_-}. \quad (2.1)$$

$e^\pm_\mu = \frac{\partial x^\pm_\mu}{\partial \zeta^a}$ are the tangent vectors to the surface. We will use the Greek alphabets for the spacetime indices and Latin ones for the hypersurface indices. The junction condition simply says that, the hypersurfaces are isometric, i.e $\Sigma_+ = \Sigma_- = \Sigma$. Then the condition (2.1) determines the functional dependence of the coordinates (although not uniquely) $x^\mu_\pm$ on $\zeta^a$. In the literature this junction condition is often written in the following form,

$$[g_{\alpha\beta}] = g^+_{\alpha\beta} |_{\Sigma_+} - g^-_{\alpha\beta} |_{\Sigma_-} = 0. \quad (2.2)$$

We introduced here the box notation which means for any tensor $A^\mu$,

$$[A^\mu] = A^\mu |_{\Sigma_+} - A^\mu |_{\Sigma_-}. \quad (2.3)$$

In the rest of the paper we will be only considering null shell. Next we define the normal vectors $n^\pm_\alpha = \chi \partial_\alpha (\Sigma_\pm)$ for both $\Sigma_\pm$ which are also generators for the null congruences orthogonal to both the hypersurfaces. We always work with future directing normal vectors for each side of the shell. $\chi$ is an arbitrary normalization. To complete the basis we also have to define the auxiliary vector $N^\pm$ such that , $N.N |_{\pm} = 0, n.N = -1 |_{\pm}$. So together with $e^\pm_\mu$ they define a complete basis. From the continuity of the null congruence it follows,

$$[n^\mu] = 0. \quad (2.4)$$

The normal vectors must satisfy $n.e_\alpha |_{\pm} = 0$. Also if the induced metric is unique,

$$[e^\mu_\alpha] = 0. \quad (2.5)$$

Now we can use the distributional tensor calculus to derive the form of the stress tensor for thin shell such that the Einstein equations will be satisfied.\(^3\) Suppose we define a coordinate chart $(x^\mu)$ such that it covers both sides of the shell and on the

\(^3\)We have only quoted the important results. For detail derivations interested readers are referred to [41–43].
hypersurface coincide with $\zeta^a$, then we express the metric covering both sides as a distribution valued tensor,

$$g_{\mu\nu} = g^+_{\mu\nu} \theta(\Sigma) + g^-_{\mu\nu} \theta(-\Sigma),$$  \hspace{1cm} (2.6)

where, both $g^+$ and $g^-$ has been expressed in terms of the single coordinate chart $x^\mu$. Now if we compute the derivative of (2.6),

$$\partial_\alpha g_{\mu\nu} = \partial_\alpha g^+_{\mu\nu} \theta(\Sigma) + \partial_\alpha g^-_{\mu\nu} \theta(-\Sigma) + [g_{\mu\nu}](\partial_\alpha \Sigma) \delta(\Sigma).$$  \hspace{1cm} (2.7)

Using (2.1) and (2.5) the last term in (2.7) becomes zero. So we end up with the following form for the Christoffel symbol,

$$\Gamma^\mu_{\alpha\beta} = \Gamma^+_{\alpha\beta} \theta(\Sigma) + \Gamma^-_{\alpha\beta} \theta(-\Sigma).$$  \hspace{1cm} (2.8)

and the Riemann tensor takes the following form,

$$R^{\alpha\beta\gamma\delta} = R^+_{\alpha\beta\gamma\delta} \theta(\Sigma) + R^-_{\alpha\beta\gamma\delta} \theta(-\Sigma) + \delta(\Sigma)Q_{\alpha\beta\gamma\delta},$$  \hspace{1cm} (2.9)

where, $Q_{\alpha\beta\gamma\delta} = -\left(\left[\Gamma^\alpha_{\beta\delta}\right] n_\gamma - \left[\Gamma^\alpha_{\beta\gamma}\right] n_\delta\right)$. To satisfy the Einstein equation we start with the following form for the stress tensor,

$$T_{\alpha\beta} = T^+_{\alpha\beta} \theta(\Sigma) + T^-_{\alpha\beta} \theta(-\Sigma) + S_{\alpha\beta} \delta(\Sigma),$$  \hspace{1cm} (2.10)

where,

$$8\pi S_{\alpha\beta} = Q_{\alpha\beta} - \frac{1}{2} Q g_{\alpha\beta}.\hspace{1cm} (2.11)$$

(2.11) follows from satisfying the $\delta(\Sigma)$ part of the Einstein equations. Then the stress tensor of the shell ($S_{ab}$) can be found by projecting (2.11) to the surface,

$$S_{ab} = S_{\alpha\beta} e^A_\alpha e^B_\beta.\hspace{1cm} (2.12)$$

We further observe that because of the junction conditions there is no discontinuity in the tangential derivatives of the metric, only there is a jump in the normal direction.

$$[\partial_\alpha g_{\mu\nu}] = -\gamma_{\mu\nu} n_\alpha.\hspace{1cm} (2.13)$$

Projecting $\gamma_{\mu\nu}$ to the surface gives us a unique induced metric on the surface of the shell. So using this we can finally recast $S_{\alpha\beta}$ after some manipulation [41–43] in the following form,

$$S_{ab} = \mu n^a n^b + J^A \left( k^A e^B + e^A k^B \right) + p \sigma^{AB} e^A e^B,\hspace{1cm} (2.14)$$

where, $\sigma_{AB}$ is the non degenerate metric of the spatial slice of the surface of the null shell. Capital indices $A, B$ denote the spatial indices of the null surface. $\mu, J^A$ and $p$ can be interpreted as the surface energy density, current and pressure of shell respectively. These quantities are defined as follows,

$$\mu = \frac{1}{16\pi} (\sigma^{AB}) \gamma_{AB}, \hspace{0.5cm} J^A = \frac{1}{16\pi} (\sigma^{AB} \gamma_B), \hspace{0.5cm} p = -\frac{1}{16\pi} \gamma_{\alpha\beta} n^\alpha n^\beta,\hspace{1cm} (2.15)$$

where, $\gamma_{AB} = \gamma_{\alpha\beta} e^A_\alpha e^B_\beta, \gamma_B = \gamma_{\alpha\beta} e^\alpha_B n^\beta$. 

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3 Soldering freedom and BMS like transformations

It is a well known fact that, there exist considerable freedom in the choice of intrinsic coordinate on $\Sigma$ in the null direction. This is often termed as “soldering freedom.” Recently in [29], this fact has been utilized to show that it generates BMS (super-translation) type transformations on the null shell horizon. In this section we will briefly review all the essential points of this development. To make things concrete, throughout the paper we will work in Kruskal coordinates. On the surface $\Sigma$ of the null shell we define a coordinate chart such that $\zeta^a = \{V, x^A\}$. $V$ is the parameter along the hypersurface generating null congruences. In this coordinate system the normal vector takes the following form,

$$ n = \partial_V. $$ (3.1)

On $\Sigma$ the metric will take the form $g_{ab}d\zeta^a d\zeta^b = g_{AB}dx^A dx^B$. One then adopts a single chart $x = \{U, x^a\}$ such that $x^a|\Sigma = \zeta^a$, such that not only the conditions (2.1), (2.4) and (2.5) hold but all the components of the metric $g_{\mu\nu}$ become continuous across the junction. Equipped with this coordinate system, one can now ask, what are the possible allowed coordinate transformations on either side of the shell preserving the junction condition (2.1). In our adapted coordinate system this boils down to solve for the Killing vectors ($Z^a$) of $g_{ab}$, the metric on $\Sigma$ i.e

$$ \mathcal{L}_Z g_{ab} = 0. $$ (3.2)

From (3.2) we get,

$$ Z^c \partial_c g_{ab} + (\partial_a Z^c) g_{cb} + (\partial_b Z^c) g_{ca} = 0. $$ (3.3)

In [29] the authors have worked with the metric where $g_{aV} = 0$. From the $aV$ components of (3.3) we can easily conclude,

$$ \partial_V Z^A = 0. $$ (3.4)

This implies that the isometry transformations along the spatial directions of null surface is independent of $V$. Next if we consider the spatial components $AB$ of (3.3) we get,

$$ Z^V \partial_V g_{AB} + Z^C \partial_C g_{AB} + (\partial_A Z^C) g_{CB} + (\partial_B Z^C) g_{CA} = 0. $$ (3.5)

Now if the metric is generically dependent on $V$ then setting $Z^V = 0$ in (3.5) we get the isometry generators of the spatial slice. But most importantly when the metric is independent of $V$ we are free to choose $Z^V$ as an arbitrary function of $(V, x^A)$.

$$ Z^V = F(V, X^A). $$ (3.6)

\footnote{For an exceptional case interested readers are referred to [33].}
Next one can investigate the effect of this isometry transformation on the normal vector $n^a$. Following [29] $\mathcal{L}_An^a \sim n^a$. If one consider only those transformations such that $\mathcal{L}_Zn^a = 0$, then it gives,

$$\partial_V Z^V = 0. \quad (3.7)$$

(3.7) implies,

$$Z^V = F(x^A). \quad (3.8)$$

(3.8) implies that it generates the following transformation,

$$V \rightarrow V + F(X^A). \quad (3.9)$$

This generates supertranslation like soldering transformation. One can extend this soldering transformation off the shell surface $\Sigma$. We follow then the convention and notations of [29]. One starts with,

$$Z = Z^V \partial_V + Uz^a \partial_a. \quad (3.10)$$

Demanding the $\mathcal{L}_Z(g_{U\alpha}) = 0$ we can arrive at the following form

$$Z = \omega (V \partial_V - U \partial_V) + U(z^A \partial_A - V(\partial_V \omega \partial_U)). \quad (3.11)$$

Here we have reparametrized $Z^V$ as $Z^V = V \omega (V, X^A)$ and $z^A = -g_{UV}g^{AB}\partial_B Z^V$. $g_{UV}$ is evaluated on $\Sigma$. Here we have written the soldering transformations as local killing transformations. Now one can demonstrate if one start with the following form of the coordinate transformations,

$$V_+ = F(x^a) + UA(x^a), x^A_+ = X^A + UB^A(x^a), U_+ = UC(x^a), \quad (3.12)$$

it will give us the finite transformation corresponding to (3.11). Here we have assumed without loss of any generality that, we can choose the coordinates $(x^a)$ of $\mathcal{M}_-$ as the intrinsic coordinates on the shell and which can be used to cover both sides of the shell $(x^a = x^a)$. The undetermined functions in (3.12) can be determined by demanding the continuity of the metric $g_{U\alpha}$ across the shell. One important consequence of this off-shell extension is that we can now read of the intrinsic quantities of the shell using the transformations defined in (3.12). Before that, we remind ourselves, that the $\gamma_{\mu\nu}$ defined in (2.13) behaves as three tensor and after suitably projecting it on the surface $\Sigma$, $\gamma_{ab} = \gamma_{\mu\nu}e_\mu^a e_\nu^b$ gives all the informations about the intrinsic properties of the shell as defined in (2.15). Now,

$$\gamma_{ab} = N^a[\partial_a g_{ab}], \quad (3.13)$$

where we have used $n.N = -1$. To determine this $\gamma_{ab}$ we will extend the tangential components of the of the metric on $\Sigma$ off the shell, in both direction of $\mathcal{M}_\pm$ but using the same coordinate chart. Then we end up with,

$$g^{\pm}_{ab} = (g^{0}_{ab} + U g^{(1)}_{ab}) dx^a dx^b \quad (3.14)$$
From this,
\[ \gamma_{ab} = \frac{1}{\chi} \left( g^{+1} - g^{-1} \right) |_{\Sigma}, \]
where, \( \chi = g_{UV}|_{\Sigma}. \) Also, \( n = \partial U \) and \( N = \frac{1}{\chi} \partial V \) in terms of Kruskal coordinates. Then the quantities defined in (2.15) will be functions of \( F(V, X^A) \) and its derivatives. It has been demonstrated in [29] that, to get the supertranslation like transformation coming from the soldering freedom, one needs to put certain conditions on the intrinsic properties of the shell, for example, for the Schwarzschild case, the pressure \((p)\) or current \((J^A)\) or both of them has to be zero. At this point we would like to emphasize that, we only get supertranslation like transformations following this construction. But the full BMS group (in three and four dimensions) consists of both supertranslation and superrotation (apart from the usual Poincare transformations) [4]. The main goal of this paper as explained before to explore whether it is possible to generate superrotation like transformations on the horizon shell by appropriately modifying the junction condition. Before that, we do a simpler exercise, we extend the above construction to rotating spacetime.

4 Rotating spacetime and soldering transformation

In this section we first try to generalize all the previous arguments for the metrics where \( g_{aV} \neq 0. \) We take two specific examples: Kerr spacetime in slow rotation limit and rotating BTZ. As before, we will use the coordinates of \( \mathcal{M}_- \) as the intrinsic coordinates covering both sides, \( x^a_- = x^a. \) Also \( x^a|_{\Sigma} = \zeta^a. \) The horizon in the Kruskal coordinates is identified by setting \( U = 0 \) for both \( \mathcal{M}_\pm. \) The two spacetimes will be isometrically soldered at \( U = 0, U_+ = 0. \) The components \( g_{aV} \) is proportional to \( U \) such that at \( U = 0, g_{aV} \) will go to zero. So it can be easily checked that (3.3), (3.5) and (3.7) remains the same. On the horizon we have \( Z^V = F(V, X^B), \) which corresponds to the soldering freedom along the null direction. This will again give the supertranslation like transformations. We then extend this construction off the shell inside a small neighborhood of \( U = 0. \) There will be corrections to the finite transformations due to the presence of rotation compared to the non-rotating case considered in [29]. Then we compute the stress energy tensor of the shell and comment on the physical constraints needed to be imposed to obtain supertranslation like soldering transformation.
Kerr spacetime: In Slow rotation limit

We consider first, the Kerr metric in slow rotation limit. In Kruskal coordinates the metric takes the following form,

\[
\begin{align*}
    ds^2 &= r^2(d\theta^2 + \sin(\theta)^2 d\tilde{\phi}^2) - \left(\frac{32m^3}{r}\right)e^{-r/2m}dUdV \\
    &+ \frac{2a}{r}\sin(\theta)^2e^{-r/2m}(r^2 + 2mr + 4m^2)d\tilde{\phi}(UdV - VdU).
\end{align*}
\]

\[ (4.1) \]

\( a \) is the rotation parameter and we expand all the quantities in small \( a \) and keep only term linear in \( a \). On horizon (\( \Sigma \)) the induced metric takes the following form,

\[
    ds^2|_{\Sigma} = r^2(d\theta^2 + \sin(\theta)^2 d\tilde{\phi}^2)
\]

\[ (4.2) \]

As mentioned before we take the \( \{U_-, V_-, \theta_-, \tilde{\phi}_-\} \) as the intrinsic coordinates, \( \{U, V, \theta, \tilde{\phi}\} \). On the horizon because of the soldering freedom once again we get, from the killing equations (3.3), \( Z^V = F(V, \theta, \tilde{\phi}) \). The angular components of the killing equations give the usual isometry transformations on the 2-sphere. Then we extend this soldering transformations off the shell such that, the entire metric across the junction remains continuous i.e \([g_{\alpha\beta}] = 0\). So following [29] we take the ansatz for the off shell soldering transformation,

\[
    Z_+ = Z^V \partial_V + Uz^\alpha \partial_\alpha.
\]

\[ (4.3) \]

On the horizon,

\[
    Z_+|_\pm = Z^V \partial_V, \quad \partial_\alpha Z_+|_\pm = \partial_\alpha Z^V \partial_V.
\]

\[ (4.4) \]

Since on the horizon ,

\[
    L_{Z_+} g_{ab}|_\pm = 0,
\]

\[ (4.5) \]

from (3.3), We need to just impose the following for extending it off the shell,

\[
    L_{Z_+} g_{U\alpha}|_\mathcal{N} = 0.
\]

\[ (4.6) \]

Below we write down all the components explicitly.

\[
\begin{align*}
    Z^V \partial_V g_{UU} + 2\tilde{\phi}_U g_{\tilde{\phi}U} + 2V g_{UV} &= 0, \\
    Z^V \partial_V g_{UV} + \left(z^U + (\partial_V Z^V)\right)g_{VV} &= 0, \\
    Z^V \partial_V g_{UA} + z^\alpha g_{\alpha A} + (\partial_A Z^V)g_{UV} &= 0.
\end{align*}
\]

\[ (4.7) \]

\[ ^5 \text{Virtue of the slow rotation limit is that, we will have more analytical control. It will be an interesting future problem to repeat this analysis without taking the slow rotation limit.} \]
Solving (4.1) upto $O(a)$,

\[ z^V = -z^\phi \frac{g_{\phi V}}{g_{\phi V}} = -z^\phi \frac{3aV}{2m} \sin(\theta)^2 = -(\partial_\phi Z^V) \frac{3aV}{em}, \]
\[ z^U = -\partial_\phi Z^V, \]
\[ z^\phi = \left(Z^V - (\partial_\phi Z^V) V\right) \frac{3a}{em} + (\partial_\phi Z^V) \frac{2}{e \sin(\theta)^2}, \]
\[ z^\theta = \frac{2}{e}(\partial_\phi Z^V). \]  

(4.8)

Then using $Z^V = V \omega(V, \theta, \tilde{\phi})$ we get the required Killing vectors for off-shell transformation,

\[ Z_+ = \omega(V \partial_V - U \partial_U) + UV \left( \frac{2}{e \sin(\theta)^2} (\partial_\phi \omega) \partial_\phi + \frac{2}{e} (\partial_\phi \omega) \partial_\theta - (\partial_\phi \omega) \partial_U \right) \]
\[ - 3a UV^2 \left( (\partial_\phi \omega) \partial_V + (\partial_\phi \omega) \partial_\theta \right) \]  

(4.9)

We now consider the following ansatz to obtain the finite counterparts of the infinitesimal transformations given in (4.9),

\[ V_+ = F(V, \theta, \tilde{\phi}) + U A(V, \theta, \tilde{\phi}), U_+ = UC(V, \theta, \tilde{\phi}), \]
\[ \theta_+ = \theta + UB^\theta(V, \theta, \tilde{\phi}), \tilde{\phi}_+ = \tilde{\phi} + UB^\phi(V, \theta, \tilde{\phi}). \]  

(4.10)

Across the junction, demanding that the full spacetime metric is continuous at the leading order in $U$ upto $O(a)$ we get,

\[ C = \frac{1}{\partial_V F} \]
\[ B^\theta = \frac{2}{e \partial_V F} \]
\[ B^\phi = \frac{2}{e \sin(\theta)^2 \partial_V F} \frac{1}{2e m} (\frac{F}{\partial_V F} - V) \]
\[ A = \frac{e}{4} (\frac{\partial_V F}{e \partial_V F})^2 + \sin(\theta)^2 (\frac{2}{e \sin(\theta)^2 \partial_V F})^2 \]
\[ - 3a \frac{\partial_\phi F V}{2em}. \]  

(4.11)

Now we want to evaluate $\gamma_{ab}$ and the conserved quantities are as defined in (2.13) and (2.15). On $M_-$ we have,

\[ r(UV)^2 = 4m^2 - \frac{8m^2}{e} UV + \cdots. \]  

(4.12)

and on $M_+$ we have,

\[ r(U_+ V_+)^2 = 4m^2 - \frac{8m^2}{e} U_+ V_+ + \cdots = 4m^2 - \frac{8m^2}{e} UF. \]  

(4.13)
Also
\[\sin(\theta_+)^2 = \sin(\theta)^2 + 2UB^2 \sin(\theta) \cos(\theta) + \cdots \] (4.14)

We now expand the tangential components of the metric off the shell towards both the side up to linear order in \(a\) and \(U\). For \(\mathcal{M}_-\)

\[g_{ab}^+ d^a d^b = g_{AB}^0 d^A d^B - U \frac{8m^2}{e} \left( - \frac{3a}{2m} \sin(\theta)^2 d\phi \cdot d\phi + V d\theta^2 + V \sin(\theta)^2 d\tilde{\phi}^2 \right) \] (4.15)

For \(\mathcal{M}_+\) we have,

\[g_{ab}^+ d^a d^b = g_{AB}^0 d^A d^B + 8m^2 U \left( - \frac{2}{e} dCdF + \sigma_{AB} d^A (dB^B - (F/eF_V) d^B) + \sin(\theta) \cos(\theta) \Theta d\tilde{\phi}^2 + \frac{6a \sin(\theta)^2}{4em} d\phi (CdF - FdC) \right) \] (4.16)

Now,
\[\gamma_{ab} = N^a [\partial_a g_{ab}] = N^U [\partial_U g_{ab}] + N^\phi [\partial_\phi g_{ab}] \] (4.17)
and
\[N^U = \frac{e}{8m^2}, N^\phi = \frac{3aV}{16m^3}. \] (4.18)

Using (4.11) we get, on \(\mathcal{M}_+\),

\[g_{ab}^+ d^a d^b = g_{AB}^0 d^A d^B + 8m^2 U \left( \frac{2}{e} \left( \frac{\partial_V \partial_a F}{\partial_V F} dV d^a \right) + \frac{2}{e} \frac{\partial_A \partial_B F}{\partial_V F} d^A d^B - \frac{4}{e} \cot(\theta) \frac{\partial_\phi F}{\partial_V F} d\phi d\tilde{\phi} \right. \]
\[\left. - \sigma_{AB} d^A (F/eF_V) d^B + \frac{2}{e} \sin(\theta) \cos(\theta) \frac{\partial_B F}{\partial_V F} d\tilde{\phi}^2 + \frac{3a}{2em} \sin(\theta)^2 d\tilde{\phi} \left( 2 \frac{\partial_B F}{\partial_V F} d^B + dV \right) \right) \] (4.19)

For \(\mathcal{M}_-\) we have,

\[g_{ab}^- d^a d^b = g_{AB}^0 d^A d^B - U \frac{8m^2}{e} \left( - \frac{3a}{2m} \sin(\theta)^2 d\phi \cdot d\phi + V d\theta^2 + V \sin(\theta)^2 d\tilde{\phi}^2 \right) \] (4.20)

So we will have up to \(O(a)\),
\[\gamma_{Va} = 2 \frac{\partial_V \partial_a F}{\partial_V F}, \]
\[\gamma_{\theta\theta} = 2 \left( \nabla_\theta^{(2)} \frac{\partial_a F}{\partial_V F} - \frac{1}{2} \left( \frac{F}{\partial_V F} - V \right) \right), \]
\[\gamma_{\theta\tilde{\phi}} = 2 \left( \nabla_\theta^{(2)} \frac{\partial_\phi F}{\partial_V F} + \frac{3a \sin(\theta)^2}{2m} \frac{\partial_B F}{\partial_V F} \right), \]
\[\gamma_{\phi\phi} = 2 \left( \nabla_\phi^{(2)} \frac{\partial_\phi F}{\partial_V F} - \frac{1}{2} \sin(\theta)^2 \left( \frac{F}{\partial_V F} - V \right) + \frac{3a \sin(\theta)^2}{2m} \frac{\partial_B F}{\partial_V F} \right). \] (4.21)
Using (4.21) we can write down the intrinsic energy momentum tensor of the shell. From that we get,

\[ p = -\frac{1}{16\pi} \gamma_{VV} = -\frac{1}{8\pi} \frac{\partial\nu^2}{\partial\nu F}, \]

\[ j^A = \frac{1}{32m^2\pi} \sigma^{AB} \frac{\partial B}{\partial\nu F}, \]

\[ \mu = -\frac{1}{32m^2\pi} \frac{\partial}{\partial\nu F} \left( \nabla^{(2)} F - F + V \partial\nu F + \frac{3a}{2m} \partial\phi F \right). \]

(4.22)

Now when, \( p = 0, \partial\nu^2 F = 0 \), which gives us,

\[ F = V + T(\theta, \tilde{\phi}), \]

(4.23)

where \( T(\theta, \tilde{\phi}) \) is an arbitrary function of \( \theta \) and \( \phi \). Also when \( J^A = 0 \) we have, \( \partial_B \partial\nu F = 0 \), which will imply,

\[ F = aV + B(\theta, \tilde{\phi}), \]

(4.24)

where, \( a \) is a scale factor and \( B(\theta, \tilde{\phi}) \) is an arbitrary function of angular coordinates. So in general upto a rescaling factor we have,

\[ F(V, \theta, \tilde{\phi}) = V + T(\theta, \tilde{\phi}). \]

(4.25)

The corresponding energy density of this kind of the shell is,

\[ \mu = -\frac{1}{32m^2\pi} \left( \nabla^{(2)} T - T + \frac{3a}{2m} \partial\phi T \right), \]

(4.26)

which will imply,

\[ \partial\nu \mu = 0. \]

(4.27)

The total energy is given by integrating \( \mu \) over the spacelike cross-section of the horizon.

\[ E = \frac{1}{32m^2\pi} \int d\theta d\tilde{\phi} \sqrt{g} T, \]

(4.28)

where, \( \sqrt{g} \) is the determinant of the induced metric of the spacelike cross-section of the horizon of the null shell. Here we have thrown away the \( \nabla^{(2)} T \) and \( \partial\phi T \) terms by doing an integration by parts. So at least up to \( \mathcal{O}(a) \) we do not get any corrections coming from the rotation parameter to (4.28). Next we consider another example of rotating metric when the ambient spacetime is BTZ.
Rotating BTZ

The matching conditions are insensitive to the asymptotic structure of spacetime. In view of this, here we study three dimensional BTZ black hole which is not asymptotically flat but the supertranslation like soldering freedom still appears. Unlike the Kerr metric we do not have to consider slow rotation limit for this case. We get an exact analytic answer for arbitrary rotation. So it will be interesting to see whether we get any qualitative changes because of the presence of rotation. The metric takes the following form,

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left[N^\phi dt + d\phi\right]^2, \]  (4.29)

where, \( f(r) = -M + \left(\frac{r}{l}\right)^2 + \frac{J^2}{r^2} \) and \( N^\phi(r) = \frac{4r^2-r^2_h}{2r^2} \). Also,

\[ r_h^2 = \frac{1}{2}\left(Ml^2 + \sqrt{(Ml^2)^2 - J^2l^2}\right), \tilde{r}_h = \frac{1}{2}\left(Ml^2 - \sqrt{(Ml^2)^2 - J^2l^2}\right). \]  (4.30)

We express all the quantities in terms of \( r_h \) and \( \tilde{r}_h \).

\[ M = \frac{r_h^2 + \tilde{r}_h^2}{l^2}, \quad J = \frac{2r_h\tilde{r}_h}{l}. \]  (4.31)

Then we have,

\[ N^\phi(r) = \frac{\tilde{r}_h r^2 - r_h^2}{r_h^2} \quad f(r) = \frac{(r^2 - r_h^2)(r^2 - \tilde{r}_h^2)}{l^2r^2}. \]  (4.32)

Then we change to Kruskal coordinates \([44]\).

\[ U = -e^{-\kappa u}, \quad V = e^{\kappa v}, \]  (4.33)

where, \( u, v = t \pm r_* \). Also,

\[ r^* = \frac{1}{2\kappa} \log\left(\frac{\sqrt{r^2 - r_h^2} - \sqrt{r^2 - \tilde{r}_h^2}}{\sqrt{r^2 - r_h^2} + \sqrt{r^2 - \tilde{r}_h^2}}\right) \]  (4.34)

\[ \kappa = \frac{r_h^2 - \tilde{r}_h^2}{l^2 r_h}. \]  (4.35)

So finally we get \([44]\),

\[ ds^2 = \frac{1}{(1+UV)^2}\left(-4l^2dUdV - 4l\tilde{r}_h(UdV - VdU)d\phi + [(1-UV)^2r_h^2 + 4UV\tilde{r}_h^2]d\phi^2\right). \]  (4.36)
Like the Kerr metric we will again get $Z^V = F(V, \phi)$ on the horizon. So there is again an arbitrary soldering freedom in the $V$ direction. Next extending this off the shell and proceeding as before, we find the generators for the off-shell transformation,

$$Z_+ = \omega(V \partial_V - U \partial_U) + UV \left(- (\partial_V \omega) \partial_U + \frac{4l^2}{r_h^2} [\partial_\phi \omega] + \frac{\tilde{r}_h V}{l} (\partial_V \omega) \partial_U \right) + UV^2 \left( \frac{4l \tilde{r}_h}{r_h^2} [\partial_\phi \omega] + \frac{\tilde{r}_h V}{l} (\partial_V \omega) \partial_V \right),$$

(4.37)

where $F(V, \phi) = V \omega(V, \phi)$. The finite transformations will be generated by the following,

$$V_+ = F(V, \phi) + U A(V, \phi), \quad \phi_+ = \phi + U B^\phi(V, \phi), \quad U_+ = U C.$$  

(4.38)

Then demanding that the entire metric is continuous across the shell we get,

$$C = \frac{1}{\partial_V F}, \quad B^\phi = \frac{2(l^2 \partial_\phi F - l \tilde{r}_h F)}{r_h^2 \partial_V F} + \frac{2l \tilde{r}_h V}{r_h^2}, \quad A = \frac{(l \partial_\phi F - \tilde{r}_h F + \tilde{r}_h V \partial_V F)(l \partial_\phi F + \tilde{r}_h F + \tilde{r}_h V \partial_V F)}{\partial_V F r_h^2}.$$  

(4.39)

Using the following expressions,

$$N^V = \frac{1}{2l^2}, \quad N^\phi = -\frac{V^2 \tilde{r}_h^2}{r_h^2 l^2}, \quad N^\phi = -\frac{V \tilde{r}_h}{r_h^2 l}.$$  

(4.40)

we compute $\gamma_{ab}$ across the junction defined in (2.13) as,

$$\gamma_{VV} = \frac{\partial^2 F}{\partial_V F}, \quad \gamma_{V\phi} = \frac{2 \partial_V \partial_\phi F}{\partial_V F}, \quad \gamma_{\phi\phi} = \left( \frac{l^2 \partial_\phi^2 F - 2l \tilde{r}_h \partial_\phi F - (r_h^2 - \tilde{r}_h^2)(F - V \partial_V F)}{\partial_V F l^2} \right).$$  

(4.41)

Using (4.41), the expression for the pressure and the current will be given by,

$$p = -\frac{1}{8\pi} \frac{\partial^2 F}{\partial_V F}, \quad J^\phi = \frac{1}{4\pi} \frac{\partial_V \partial_\phi F}{r_h^2 \partial_V F}, \quad \mu = -\frac{1}{8\pi} \left( \frac{l^2 \partial_\phi^2 F - 2l \tilde{r}_h \partial_\phi F - (r_h^2 - \tilde{r}_h^2)(F - V \partial_V F)}{r_h^2 l^2 \partial_V F} \right).$$  

(4.42)

Again setting $p = 0$ we get,

$$F(V, \phi) = V + T(\phi),$$  

(4.43)

where $T(\phi)$ is an arbitrary function of angular coordinate. Also, if we want to set $J^\phi = 0$ we will get,

$$F(V, \phi) = a V + B(\phi).$$  

(4.44)
\( B(\phi) \) is an arbitrary function of angular coordinate.

Also for \( p = 0 \) (or \( J^\phi = 0 \)) case we again retrieve the fact that, \( \partial_V \mu = 0 \), where,

\[
\mu = -\frac{1}{8\pi l^2} \left( \frac{r_h^2 T''(\phi) - 2l_r h T'(\phi) - (r_h^2 - r_{\tilde{r}}^2) T(\phi)}{r_h^2} \right).
\] (4.45)

The total energy once again is obtained by integrating \( \mu \) on the spatial slice of the horizon.

\[
E = \frac{1}{8\pi} \int d\phi \left( \kappa T(\phi) \right),
\] (4.46)

where we have thrown away both \( \partial^2 \phi \) and \( \partial_\phi T \) term as we have performed an integration by part. This matches with the supertranslation charge derived in the literature, for example readers are referred to ([4, 8]).

5 Conformal isometries and modified soldering conditions

Till now we have used the usual Israel-junction [41–43] condition which says,

\[
\Sigma_+ (x_+) = \Sigma_-(x_-) = \Sigma(\zeta),
\] (5.1)
as advocated in the section (2). To remind ourselves, this condition implies that the induced metric from both sides of the shell is isometrically matched with the intrinsic metric of the shell. So the line elements of \( \Sigma_\pm \) takes the same form as that of \( \Sigma \). But let us consider the following situation,

\[
g^+_c d|_{\Sigma_+} \left( \frac{\partial x^c_+}{\partial \zeta^a} \frac{\partial x^d_+}{\partial \zeta^b} \right) = g^-_c d|_{\Sigma_-} \left( \frac{\partial x^c_-}{\partial \zeta^a} \frac{\partial x^d_-}{\partial \zeta^b} \right) = \Omega(\zeta)^2 g_{ab}(\zeta)|_{\Sigma}. \] (5.2)

This basically means, that the induced metric from both sides of the shell will be equal to the intrinsic metric of the shell up to a conformal factor. This particular situation may not be very common but still this type of deviations are perfectly allowed in the Israel’s junction condition formalism [41]. In fact any jump in the first junction condition may be taken care of by switching to a new coordinate system and having a jump in the first derivatives so that continuity (\( C^1 \)) of the induced metric is maintained [45]. As a consequence of (5.1), any infinitesimal coordinate transformation of the form, \( x^a_\pm + Z^a_\pm \), generated by \( Z^a_\pm \) preserving (5.2) obeys,

\[
Z^c_\pm \partial_c g_{ab} + (\partial_a Z^c_\pm) g_{cb} + (\partial_b Z^c_\pm) g_{ac} = (1 - \Omega(\zeta)^2) g_{ab}. \] (5.3)

(5.3) is nothing but the conformal Killing equation.

We define a common coordinate system \( x \) such that, we can express the both sides of the shell using this same coordinate system. Then we again use the distributional method to analyze the Einstein equation as defined in section (2).

\[
g_{\alpha\beta}(x) = g^+_{\alpha\beta}(x) \Theta(\Sigma) + g^-_{\alpha\beta}(x) \Theta(-\Sigma). \] (5.4)
From this we get,
\[ \partial_\mu g_{\alpha\beta}(x) = \partial_\mu g^+_{\alpha\beta}(x)\Theta(\Sigma) + \partial_\mu g^-_{\alpha\beta}(x)\Theta(-\Sigma) + [g^+_{\alpha\beta}|_{\Sigma_+} - g^-_{\alpha\beta}|_{\Sigma_-}]n_\rho \delta(\Sigma). \] (5.5)

The third term again vanishes because of (5.2). Armed with this result we can safely conclude that the the distributional method can be applied as before and there will be only jump in the normal direction. Consequently we can apply the results of the section (2) for computing the conserved quantities. Then the question of soldering freedom in this case boils down to the fact that, what are possible coordinate transformations one can do say on \( \mathcal{M}_+ \) (or \( \mathcal{M}_- \)) such that the condition (5.1) will hold. This says that we need to solve the conformal killing equation (5.2) on the horizon. At this point let us take some concrete examples.

- **Rotating BTZ**

We consider rotating BTZ metric in Kruskal coordinates [44],
\[ ds^2_\pm = \frac{1}{(1 + U_\pm V_\pm)^2}\left(-4l^2dU_\pm dV_\pm - 4\tilde{h}(U_\pm dV_\pm - V_\pm dU_\pm) + [(1 - U_\pm V_\pm)^2r_\pm^2 + 4U_\pm V_\pm \tilde{r}_\pm^2]d\phi_\pm^2\right). \] (5.6)

\( \pm \) denote the two spacetimes inside and outside the shell. \( U_\pm = 0 \) denote the horizon \( (\Sigma_\pm) \) for these two metrics. Now we want to identify these two surfaces in a way such that, (5.1) holds. For this case, it amounts to the fact that, on the surface of the shell from both the side we will have,
\[ r_\pm^2 d\phi_\pm^2 = r^2_\Omega(\phi)^2 d\phi^2, \]
\[ r_\pm^2 d\phi_\pm^2 = r^2_\Omega(\phi)^2 d\phi^2. \] (5.7)

This will gives us \( \phi_\pm \) as function \( \phi \). So we have,
\[ \phi_+ = \Omega(\phi) + c_1, \phi_- = \Omega(\phi) + c_2. \] (5.8)

So \( \phi_+ \) and \( \phi_- \) differ upto a constant factor, so that \( d\phi_+ = d\phi_- \). On \( \Sigma \) the coordinates are \( \{V, \phi\} \), which we can set as the intrinsic coordinates of the shell. The killing vector takes the following form,
\[ Z|_N = \left(Z^V \partial_V + Z^\phi \partial_\phi \right)|_N. \] (5.9)

First we check \( \phi V \) component of (5.3). This gives,
\[ \partial_V Z^\phi = 0. \] (5.10)
This implies, that there cannot be any \( V \) dependent transformation for the spatial coordinate. Then lets look at the \( \phi \phi \) component of the Killing equations.

\[
Z^\lambda \partial_\lambda g_{\phi\phi} + (\partial_\phi Z^\lambda)g_{\phi\lambda} + (\partial_\lambda Z^\phi)g_{\lambda\phi} = (1 - \Omega(V, \phi)^2)g_{\phi\phi}.
\] (5.11)

Further simplifying we get,

\[
2(\partial_\phi Z^\phi) = (1 - \Omega(V, \phi)^2).
\] (5.12)

Using (5.10) we get, \( \partial_V \Omega(V, \phi) = 0 \). This will imply, \( \Omega(V, \phi) = \Omega(\phi) \). So we get,

\[
\begin{align*}
Z^V &= F(V, \phi), \\
Z^\phi &= \frac{1}{2} \int_1^\phi (1 - \Omega(\tilde{\phi})^2)d\tilde{\phi} + C_1 = T(\phi) + C_1,
\end{align*}
\] (5.13)

both are arbitrary functions. Second of (5.13) actually consistent with the condition (5.7). Let us now look at the normal vectors generating the null congruence orthogonal to the hypersurface. As \( \Sigma \) is identified by both \( U_+ = U_- = U = 0 \), so \( [n^a] = 0 \). This implies the continuity of the null congruence on both sides. Now if we start from the following,

\[
g_{ab}n^b = 0,
\] (5.14)

we get again like in the section (3),

\[
\mathcal{L}_Z n^a \sim n^a.
\] (5.15)

Then demanding that it exactly preserves the normal vector, we get,

\[
F(V, \phi) = F(\phi).
\] (5.16)

Therefore on the horizon we get the following two transformations,

\[
V \rightarrow V + F(\phi), \phi \rightarrow T(\phi) + C_1,
\] (5.17)

which are nothing but the supertranslation and superrotation like transformations.

We would like to make one more observation, if we are working with the usual Israel junction condition, then we can choose either of the coordinates of \( M_\pm \) as the intrinsic coordinates for \( \Sigma \) without loss of any generality. It seems that for our new condition (5.1) it is not possible. If we want to do that, it will mean the following,

\[
\Sigma_+(x_+) = \Omega(x_-)^2 \Sigma_-(x_-) = \Sigma(x_-).
\] (5.18)

For this case of BTZ we will have the following,

\[
r^2_h d\phi^2_+ = \Omega(\phi_-)^2 r^2_h d\phi^2_-.
\] (5.19)
We can set, $\phi_- = \phi$. It will give us the required coordinate transformation. $\phi_+ = \Omega(\phi) + c_1$. To achieve this condition consistently (by projecting the bulk metric to the surface) we need to rescale at least the $\phi \phi$ component of the metric of $\mathcal{M}_-$ by the conformal factor $\Omega(\phi)$. Then again we can infer that, the third term in (5.5) goes to zero. But there will be a jump in the tangent vector as,

$$
\bar{e}_\phi^+ \phi_+ = \Omega(\phi), \bar{e}_\phi^- \phi_- = 1.
$$

So there will be a discontinuity in the tangential derivative of the metric. We can avoid this discontinuity by using the condition as shown in (5.2).

Henceforth we will work with the condition (5.2). But we emphasize one more time here, utilizing the rescaling freedom we can indeed obtain both the supertranslational and superrotation like transformations by using the appropriate soldering conditions.

**Schwarzschild**

We now look at the Schwarzschild case. We follow the same strategy as in previous section and proceed to solve the conformal Killing equations on the horizon shell.

First we write down the Schwarzschild metric in double null coordinate,

$$
ds^2 = -2G(r)dUdV + r^2(d\theta^2 + \sin(\theta)^2d\phi^2).\tag{5.21}
$$

We do the following coordinate transformations,

$$
z = e^{i\phi} \cot\left(\frac{\theta}{2}\right), \bar{z} = e^{-i\phi} \cot\left(\frac{\theta}{2}\right).\tag{5.22}
$$

Then in terms of these complex variables we get,

$$
ds^2 = -2G(r)dUdV + r^2 \frac{4dz \, d\bar{z}}{(1 + z\bar{z})^2}.\tag{5.23}
$$

On the horizon the Killing vectors takes the following form,

$$
Z|_{\mathcal{N}} = \left(Z^V \partial_V + Z^{\bar{z}} \partial_{\bar{z}} + Z^z \partial_z\right).\tag{5.24}
$$

The we proceed to solve the conformal killing vector equations,

$$
\nabla_a Z_b + \nabla_b Z_a - (\nabla_a Z^a) g_{ab} = 0.\tag{5.25}
$$

From $V A$ components of these equations we again obtain,

$$
\partial_V Z^C = 0.\tag{5.26}
$$
Remaining components give,
\[
\partial_z Z^\bar{z} = 0, \\
\partial_{\bar{z}} Z^z = 0.
\]  
(5.27)

(5.27) gives,
\[
Z^\bar{z} = \bar{T}(\bar{z}) + c_2, Z^z = T(z) + c_3,
\]  
(5.28)

Where, \(T(\bar{z})\) and \(T(z)\) are arbitrary functions of only \(z\) (or \(\bar{z}\)).

So any transformations of the following is permitted
\[
z \to T(z) + c_2, \quad \bar{z} \to T(\bar{z}) + c_3.
\]  
(5.29)

A point to be noted here, \(T(z)\) and \(\bar{T}(\bar{z})\) can be in general singular at several points on the 2-sphere. If we require these transformations to be globally well defined on the 2-sphere, then it will only pick out the appropriate spherical harmonics. Also we will have the following transformation,
\[
V \to V + B(V, z, \bar{z}).
\]  
(5.30)

By demanding again that it preserves the normal vector, we can set \(B(V, z, \bar{z}) = B(z, \bar{z})\). So we will have both supertranslation and superrotation type soldering on the horizon shell.

6 Off-shell soldering for modified junction condition

We now want to extend the above conformal soldering transformations off the shell such that the junction condition remains same. The infinitesimal superrotation generators which act on the celestial sphere at null infinity are singular and it is not very clear whether there exist finite version of these transformations\(^6\). But in our context (at least for the BTZ) there seems to be no obstacle to generalize the infinitesimal transformations to finite ones.

- Rotating BTZ

We start with the seed rotating BTZ metric for both the side. Then we introduce a coordinate system such that all the components of the metric remains continuous across the junction. At the surface the intrinsic coordinates are \(\zeta^a\) as before. Also we introduce a set of common coordinates \(x^\alpha\) such that \(x^\alpha|_\Sigma = \zeta^a\). We can extend the soldering transformations towards both \(\mathcal{M}_+\) and \(\mathcal{M}_-\) direction and it will be of

\(^6\)This issue has been addressed in [46] and a physical interpretation is given for finite superrotation.
the same form for both the cases. So without loss of any generality we extend this
in the direction of $\mathcal{M}_+$. We now start with the following form of the killing vectors,
\[ Z_+ = Z^V \partial_V + Z^\phi \partial_\phi + U z^a \partial_a. \] (6.1)

On the junction,
\[ Z_+|_N = Z^V \partial_V + Z^\phi \partial_\phi, \quad \partial_a Z_+|_N = \partial_a Z^V \partial_V + \partial_a Z^\phi \partial_\phi. \] (6.2)

Since $U = 0$ on the horizon, $Z_+$ satisfy the conformal Killing vector equations
with respect to $g_{ab}$, we just need to impose that the $Z_+$ satisfy the conformal killing
equations with respect to the $g_{U\alpha}$. This will give,
\[ z^\phi g_{U\phi} + z^V g_{UV} = 0, \]
\[ z^U + (\partial_V Z^V) = (1 - \Omega(\phi)^2), \]
\[ Z^V \partial_V g_{U\phi} + z^U g_{U\phi} + z^\phi g_{\phi\phi} + (\partial_\phi Z^V) g_{UV} = 0. \] (6.3)

Here $g_{\phi\phi}, g_{U\phi}$ and $g_{UV}$ are evaluated on the horizon. (6.3) gives the following
solutions,
\[ z^V = -z^\phi \frac{g_{U\phi}}{g_{UV}} = z^\phi \frac{\hat{r}_h V}{l}, \]
\[ z^U = (1 - \Omega(\phi)^2) - (\partial_V Z^V), \]
\[ z^\phi = \frac{4l^2}{r_h^2} (\partial_\phi Z^V) + \frac{4l \hat{r}_h}{r_h^2} (V(\partial_V Z^V) - Z^V). \] (6.4)

Setting $Z^V = V \omega(V, \phi)$ we get,
\[ Z_+ = \omega(V \partial_V - U \partial_U) + U \left( (1 - \Omega(\phi)^2 - V(\partial_V \omega)) \partial_U + \frac{4l^2 V}{r_h^2} [\partial_\phi \omega] + \frac{\hat{r}_h V}{l} (\partial_V \omega) \right) \partial_\phi \]
\[ + U V^2 \left( \frac{4l \hat{r}_h}{r_h^2} [\partial_\phi \omega] + \frac{\hat{r}_h V}{l} (\partial_V \omega) \right) \partial_V + Z^\phi \partial_\phi. \] (6.5)

Similar expression can easily be obtained for $\mathcal{M}_-$. The finite transformations generated by $Z_+$ can be achieved by the following set
of transformations,
\[ V_+ = F(V, \phi) + U A(V, \phi), \phi_+ = \int_1^\phi \Omega(\phi) d\phi + U B^\phi(V, \phi), U_+ = U C(V, \phi). \] (6.6)

Similarly for the offshell extension towards $\mathcal{M}_-$,
\[ V_- = \hat{F}(V, \phi) + U \hat{A}(V, \phi), \phi_- = \int_1^\phi \Omega(\phi) d\phi + U \hat{B}^\phi(V, \phi), U_+ = U \hat{C}(V, \phi). \] (6.7)
Then the conserved quantities of the shell will be related to the difference between \( F(V, \phi) \) and \( \tilde{F}(V, \phi) \) and there derivatives. The intrinsic coordinates covering both sides of the shell can be denoted by, \( x^\alpha = \{ U, V, \zeta^A \} \). At the surface,

\[
g_{\alpha\beta}|_{\Sigma^+} = g_{\alpha\beta}|_{\Sigma^-} = \Omega^2(\zeta)g_{ab}(\zeta)|_{\Sigma}. \tag{6.8}
\]

We also demand that in our new coordinate chart \( x \) both the metrics of \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) will be of the following form,

\[
ds_+^2(x) = g_{\alpha\beta}^+(x)dx^\alpha dx^\beta \equiv \Omega(\phi)^2g_{\alpha\beta}(x)dx^\alpha dx^\beta,
\]

and

\[
ds_-^2(x) = g_{\alpha\beta}^-(x)dx^\alpha dx^\beta \equiv \Omega(\phi)^2g_{\alpha\beta}(x)dx^\alpha dx^\beta. \tag{6.9}
\]

We get the following solutions,

\[
C = \frac{\Omega(\phi)^2}{\partial_V F}, \quad \tilde{C} = \frac{\Omega(\phi)^2}{\partial_V \tilde{F}},
\]

\[
B^\phi = \frac{2\Omega(\phi)(l^2\partial_\phi F - l\tilde{\gamma}_h F\Omega(\phi) + l\tilde{\gamma}_h V\partial_V F)}{\partial_V \tilde{F}r_h^2}, \quad \tilde{B}^\phi = \frac{2\Omega(\phi)(l^2\partial_\phi \tilde{F} - l\tilde{\gamma}_h F\Omega(\phi) + l\tilde{\gamma}_h V\partial_V \tilde{F})}{r_h^2\partial_V \tilde{F}},
\]

\[
A = \frac{l^2((\partial_\phi F)^2 - \tilde{\gamma}_h^2\Omega^2 F^2 + 2l\tilde{\gamma}_h V\partial_\phi F\partial_V F + \tilde{\gamma}_h^2V^2(\partial_V F)^2)}{\partial_V \tilde{F}r_h^2},
\]

\[
\tilde{A} = \frac{l^2((\partial_\phi \tilde{F})^2 - \tilde{\gamma}_h^2\Omega^2 \tilde{F}^2 + 2l\tilde{\gamma}_h \tilde{V}\partial_\phi \tilde{F}\partial_V \tilde{F} + \tilde{\gamma}_h^2\tilde{V}^2(\partial_V \tilde{F})^2)}{\partial_V \tilde{F}r_h^2}. \tag{6.11}
\]

Now the virtue of the condition (6.8) is that, we can use the usual distributional calculus as (2.13) still holds. Hence the expressions for conserved quantities of the shell still takes the form as shown in (2.15). So following the prescription outlined in section (3) we get,

\[
\gamma_{VV} = 2\left(\frac{\partial_\phi^2 F}{\partial_V F} - \frac{\partial_\phi \tilde{F}}{\partial_V F}\right),
\]

\[
\gamma_{V\phi} = 4\left(\frac{\partial_\phi \partial_{\phi} F}{\partial_V F} - \frac{\partial_V \partial_\phi F}{\partial_V F}\right),
\]

\[
\gamma_{\phi\phi} = 2\left(\frac{\partial_{\phi}^2 F}{\partial_V F} - \frac{\partial_{\phi} \tilde{F}}{\partial_V F} - \frac{\partial_{\phi} F\Omega(\phi)}{\partial_V F\Omega(\phi)} + \frac{\partial_{\phi} \tilde{F}\Omega(\phi)}{\partial_V F\Omega(\phi)} - \frac{F\Omega(\phi)^2\tilde{\gamma}_h^2}{\partial_V Fl^2} + \frac{F\Omega(\phi)^2\tilde{\gamma}_h^2}{\partial_V Fl^2} + \frac{2\partial_{\phi} F\Omega(\phi)\tilde{\gamma}_h}{\partial_V Fl} - \frac{2\partial_{\phi} F\Omega(\phi)\tilde{\gamma}_h}{\partial_V Fl}\right). \tag{6.12}
\]
From this we get,

\[ p = -\frac{1}{8\pi} \left( \frac{\partial^2_F}{\partial_V F} - \frac{\partial^2_{\tilde{F}}}{\partial_V \tilde{F}} \right), \]

\[ J^\phi = \frac{1}{4\pi \Omega(\phi)^2 r_h^2} \left( \frac{\partial_F}{\partial_V F} \frac{\partial \tilde{F}}{\partial_V \tilde{F}} - \frac{\partial \tilde{F}}{\partial_V F} \frac{\partial F}{\partial_V \tilde{F}} \right), \]

\[ \mu = -\frac{1}{8\pi \Omega(\phi)^2 r_h^2} \left( \frac{\partial^2_F}{\partial_V F} - \frac{\partial^2_{\tilde{F}}}{\partial_V \tilde{F}} - \frac{\partial \tilde{F}}{\partial_V F} \frac{\partial F}{\partial_V \tilde{F}} \right) + \frac{\partial \tilde{F}}{\partial_V F} \frac{\partial F}{\partial_V \tilde{F}} - \frac{\partial \tilde{F}}{\partial_V F} \frac{\partial F}{\partial_V \tilde{F}}. \]

(6.13)

Now if we try to set \( p = 0 \) as before, that will leave us with several possibilities. One possibility is to simply set \( F = \tilde{F} \). This will not only make \( p = 0 \) but all also make \( \mu = 0 \) and \( J^\phi = 0 \). This is expected as we are soldering two exactly same spacetime which are obtained by the same amount of supertranslation from the seed BTZ metric.

Second possibility is to set both \( \partial_V F = \partial_V \tilde{F} = 0 \) individually, but still \( F(V, \phi) \neq \tilde{F}(V, \phi) \). This will give us, \( F = V + T(\phi) \) and \( \tilde{F} = V + \tilde{T}(\phi) \). We can set \( J^\phi = 0 \) and this will give, upto a factor, \( F = aV + T(\phi) \) and \( \tilde{F} = bV + \tilde{T}(\phi) \). As \( V \) is the intrinsic coordinate covering both sides of the shell we set \( a = b \), so that we can rescale \( V \) in the same way everywhere. Both \( p = 0 \) and \( J^\phi = 0 \) will give, upto a overall factor, supertranslation like transformation which is already expected from the work [29].

We now analyze the form of the energy density \( \mu \).

\[ \mu = -\frac{1}{8\pi \Omega(\phi)^2 r_h^2} \left( T''(\phi) - \frac{\Omega'(\phi)}{\Omega(\phi)} T'(\phi) - \frac{\Omega(\phi)^2 (r_h^2 - \hat{r}_h^2)}{l^2} T(\phi) - \frac{2\Omega(\phi) \hat{r}_h}{l} T'(\phi) \right). \]

Where we have defined \( T(\phi) = T(\phi) - \tilde{T}(\phi) \). From this it is evident that it satisfy \( \partial_V \mu = 0 \). Using this we can define the energy of the shell,

\[ E = \frac{1}{8\pi} \int d\phi \left( -\partial_\phi \left( \frac{T'(\phi)}{\Omega(\phi)} \right) + \frac{\Omega(\phi)(r_h^2 - \hat{r}_h^2)}{l^2} T(\phi) + \frac{2\hat{r}_h}{l} T'(\phi) \right). \]  

(6.14)

Dropping the first and last term by doing an integration by parts we get,

\[ E = \frac{1}{8\pi} \int d\phi \left( \kappa \Omega(\phi) T(\phi) \right). \]  

(6.15)

When \( \Omega(\phi) = 1 \), (6.14) recovers the usual result [8]. Next we set, \( \Omega(\phi) = 1 + \epsilon \tilde{\Omega}(\phi) \), where \( \epsilon \) is a small parameter. We then expand (6.15) in \( \epsilon \) and keep only term upto \( O(\epsilon) \).

\[ E = \frac{1}{8\pi} \int d\phi \left( \kappa T(\phi) + \epsilon \kappa \tilde{\Omega}(\phi) T(\phi) \right). \]  

(6.16)
(6.16) shows that the total energy gets contribution from both the supertranslation and superrotation like soldering transformations on the horizon. We compare the expression for the energy (6.16) with the conserved charge calculated in [8]. The first term matches with that of the supertranslation contribution to the conserved charge computed in [8]. At the leading order in $\epsilon$ there is no contribution from superrotation type soldering transformation. It appears only at the $O(\epsilon)$ order. Although it contains the superrotation factor but this $O(\epsilon)$ term is not quite similar to the superrotation charge as computed in [6, 8]. In general there is no a priori reason to expect that (although the supertranslation part indeed match). The conserved charge computed in [6, 8] has been done using Noether procedure from the gravitational action, but for our case we are just computing the intrinsic quantities of the shell which comes from the stress energy tensor required for satisfying the Einstein equations. Although we have recovered an infinite dimensional symmetry which closely resembles superrotation but from the perspective of charge algebra our findings do not exactly match with the existing results [4–6, 8, 9]. At this point we leave this issue for a detail study in a future publication.

\section*{Schwarzschild}

For this case, proceeding similarly as before, we get the following equations,

\begin{align}
    z^V g_{VU} &= 0, \\
    z^U + (\partial_V Z^V) &= (1 - \Omega(z, \bar{z}))^2, \\
    z_A + (\partial_A Z^V) g_{UV} &= 0.
\end{align}

This gives the following solutions,

\begin{align}
    z^V &= 0, \\
    z^U &= (1 - \Omega(z, \bar{z}))^2 - (\partial_V Z^V), \\
    z^A &= 4l^2 g^{AB}(\partial_B Z^V).
\end{align}

\begin{align}
    Z_+ = \omega \left( V \partial_V - U \partial_U \right) + U \left( (1 - \Omega(z, \bar{z}))^2 - V (\partial_V \omega) \right) \partial_U + 4l^2 V g^{AB} (\partial_B \omega) \partial_A + Z^A \partial_A.
\end{align}

The corresponding finite transformations may again be generated analogously as what we have seen for BTZ case before. But one has to check this more carefully as the finite superrotation is in general not globally well defined and hence the interpretation for the corresponding conserved charge is subtle [47]. We leave this as a future problem.
Before we end this section we briefly apply our construction for non stationary spacetimes. From (3.5) it is evident that if we consider Vaidya type spacetime we will get, $Z^V = 0$. But now we will consider our modified junction condition where we are patching two spacetimes by allowing a conformal rescaling freedom. In the following we look at both three and four dimensional Vaidya spacetime. For Vaidya spacetime,

$$ds^2 = -f(U,V) dU dV + r(U,V)^2 d\Omega^2_{D-2},$$  \hspace{1cm} (6.20)

where the $U = 0$ is the horizon of the collapsing shell. $d\Omega^2_{D-2}$ is the metric of the $D-2$ sphere expressed in the standard spherical polar coordinate. The functions $f(U,V)$ and $r(U,V)$ are such that even at $U = 0$ they are nontrivial functions of $V$. So for this case we will have $\partial^V g_{\lambda\beta} \neq 0$ even at $U = 0$. Next we consider briefly two specific examples.

- **BTZ-Vaidya spacetime**

Form (5.3) it follows,

$$\partial^V Z^\phi = 0.$$ \hspace{1cm} (6.21)

and

$$Z^\lambda \partial_\lambda g_{\phi\phi} + (\partial_\phi Z^\lambda) g_{\lambda\phi} + (\partial_\phi Z^\lambda) g_{\lambda\phi} = (1 - \Omega(V,\phi)^2) g_{\phi\phi}.$$ \hspace{1cm} (6.22)

Now we will have ,

$$Z^V \partial^V g_{\phi\phi} + 2(\partial_\phi Z^\phi) g_{\phi\phi} = (1 - \Omega(V,\phi)^2) g_{\phi\phi}.$$ \hspace{1cm} (6.23)

If $\Omega(V,\phi)$ is an arbitrary function of $V$ and $\phi$ we get,

$$Z^\phi = C_3$$ \hspace{1cm} (6.24)

and

$$Z^V = \frac{g_{\phi\phi}}{\partial^V g_{\phi\phi}} (1 - \Omega(V,\phi)^2).$$ \hspace{1cm} (6.25)

So we generate a supertranslation type soldering transformation.
Schwarzschild-Vaidya spacetime

For this case we will have,
\[ \partial_r Z^A = 0. \]  \hspace{1cm} (6.26)

Other equations are,
\[ \frac{2\partial_r Z^V}{r} + 2(\partial_\theta Z^\theta) = (1 - \Omega(V, \theta, \phi)^2), \]
\[ \frac{2\partial_r Z^V}{r} + 2(\partial_\phi Z^\phi) + 2Z^\theta \cot(\theta) = (1 - \Omega(V, \theta, \phi)^2), \]  \hspace{1cm} (6.27)
\[ (\partial_\theta Z^\phi) \sin(\theta)^2 + (\partial_\phi Z^\theta) = 0. \]

Again, \( \Omega(V, \theta, \phi) \) is an arbitrary function of \( V, \theta \) and \( \phi \), we get the Killing vectors on 2-sphere satisfying the following equations,
\[ (\partial_\theta Z^\theta) = 0, \]
\[ (\partial_\phi Z^\phi) + Z^\theta \cot(\theta) = 0, \]  \hspace{1cm} (6.28)
\[ (\partial_\phi Z^\phi) \sin(\theta)^2 + (\partial_\theta Z^\theta) = 0. \]

Apart from that we get,
\[ \frac{2\partial_r Z^V}{r} = (1 - \Omega(V, \theta, \phi)^2), \quad Z^V = \frac{r}{2\partial_r Z^V} (1 - \Omega(V, \theta, \phi)^2). \]  \hspace{1cm} (6.29)

We get supertranslation type transformation. So we can conclude this section by observing that we can get supertranslation type soldering transformation on the horizon even for the nonstationary spacetime. Next logical step would be to compute the intrinsic stress energy tensor of the shell, which we leave for future study.

7 Summary and Discussion

We have revisited the dynamics of thin null shell situated at the horizon of black holes in general relativity and explored the consequences when the induced metric at the shell is invariant up to a conformal rescaling. In the process we have prescribed a slightly modified version of the Israel junction condition to include the effect of conformal transformations. Using that fact we have demonstrated that we can have superrotation type soldering transformation along with supertranslation. For the BTZ, superrotation being generated by an arbitrary conformal factor. We extended these soldering transformations off the shell and evaluated the shell’s intrinsic stress energy tensor. When the pressure (and the current ) of the shell vanishes there exists a supertranslation like transformation. We can get an expression for total energy of
the shell with contributions coming from both superrotation and supertranslation generators. Taking the conformal factor infinitesimal, we can demonstrate that although the supertranslation contribution to the shell energy matches precisely with the results of the existing literature [8, 9], the superrotation contribution seems to be different. We leave this for a future study to understand better the implications of these results. Also, the matching conditions that we have used to generate superrotation type symmetries is locally well defined but we don’t know whether there can be any global inconsistency arising from the effects of these conformal isometries on horizon topology. This needs further investigations.

We anticipate that the results obtained here can have far reaching consequences specially in the context of black hole information paradox. It will be interesting to investigate the soldering symmetries for collapsing shells. At late time we expect the emerging symmetry will be close to the BMS like soldering transformations [48]. We hope to get back to this issue in near future. Also it would be interesting to study any possible connection between black hole membrane paradigm and supertranslation-superrotation transformations and how the BMS like soldering transformations affect the properties of the horizon fluid [49, 50].

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