Ambiguities on the Hamiltonian Formulation
of the Free Falling Particle with
Quadratic Dissipation

G. V. López 1, P. López

Departamento de Física de la Universidad de Guadalajara
Blvd. Marcelino García Barragán 1421, esq. Calzada Olímpica
44420 Guadalajara, Jalisco, México

X. E. López

Facultad de Ciencias de la UNAM
Apartado Postal 70-348, Coyoacán, 04511 México D.F.

Abstract

For a free falling particle moving in a media which has quadratic
velocity force effect on the particle, two equivalent constants of motion,
with units of energy, two Lagrangians, and two Hamiltonians are de-
duced. These quantities describe the dynamics of the same classical
system. However, their quantization and the associated statistical me-
chanics (for an ensemble of particles) describe two completely different
quantum and statistical systems. This is shown at first order in the
dissipative parameter.

PACS: 03.20.+i, 03.30.+p, 03.65.-w

1 Introduction

It is well known that the Lagrangian (therefore the Hamiltonian) formulation
for some systems of more than one dimension may not exists (Douglas 1941).
Fortunately for our study of the nature up to now, most of our physical systems

1gulopez@cencar.udg.mx
have avoided this problem, and the whole quantum and statistical mechanics of non-dissipative systems can be given in terms of a Lagrangian or Hamiltonian formulation. Now, for dissipative systems there have been two main approaches. The first one consists of keeping the same Hamiltonian formalism for the whole system where the interacting background is included, as a result, one brings about a master equation with the dissipation and diffusion parameters appearing as part of the solution (Caldeira and Legget 1983, Unruh and Zurek 1989, and Hu et al 1992). This approach has its own merit, but it will not be followed in this paper. We will follow the second approach which consists in to obtain a phenomenological velocity depending Hamiltonian, representing a classical dissipative system, and to proceed to make the usual quantization (or statistical mechanics) with this Hamiltonian.

Within this last approach, one can, additionally, study the mathematical consistence of the the Hamiltonian formalism in quantum and statical mechanics. It is also known that even for one-dimensional systems, where the existence of their Lagrangian is guaranteed (Darboux 1894), the Lagrangian and Hamiltonian formulations are not free from problems (Havas 1973, Okubo 1980, Dodonov et al 1981, Marmon et al 1985, Glauber et al 1984, a L´opez 1998, and b L´opez 1999). One of the main problems is the implication on the quantization of the associated classical system when different Hamiltonians describe the same classical system (c L´opez 2002). This ambiguity has already been studied for the harmonic oscillator with dissipation and some general system (d L´opez 1996). In this paper, we want to show explicitly this ambiguity by studying the free falling particle within a dissipative medium. We will assume that this dissipation depends quadratically on the velocity of the particle. Firstly, two constants of motion are deduced for this system. Secondly, with these constants of motion two Lagrangian and two Hamiltonian are obtained using a known procedure (d L´opez 1996 and e L´opez and Hernández 1989). Finally, using the Hamiltonian expression at first order in the dissipation parameter and perturbation theory, the eigenvalues of their associated quantum Hamiltonian and their associated statistical mechanics properties (for an ensemble of particles) are shown.

2 Constants of Motion

The motion of the particle of mass \( m \) falling under a constant gravitational force, \( -mg \), where \( g \) is the constant acceleration due to gravity, and being inside a dissipative medium which has the effect of producing in the particle a
force proportional to the square of its velocity, $\alpha \dot{x}^2$ for $\dot{x} < 0$, can be described by the following autonomous dynamical system

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -g + \frac{\alpha}{m} v^2,$$

(1)

where the variable $x$ represents the vertical position of the particle, and $v$ represents its velocity. A constant of motion for this system is a function $K = K(x,v)$ such that $dK/dt = 0$, i.e. it satisfies the following equation (López 1999)

$$v \frac{\partial K}{\partial x} + \left( -g + \frac{\alpha}{m} v^2 \right) \frac{\partial K}{\partial v} = 0 .$$

(2)

The general solution of this equation is given by (John 1974)

$$K_{\alpha}(x,v) = G(C(x,v)) ,$$

(3)

where $G$ is an arbitrary function of the characteristic curve $C(x,v)$. This characteristic curve can be given in two different ways as

$$C_1 = -\frac{mg}{2\alpha} \ln \left( 1 - \frac{\alpha}{mg} v^2 \right) + gx ,$$

(4)

or

$$C_2 = \left( 1 - \frac{\alpha}{mg} v^2 \right) e^{-2\alpha x/m} .$$

(5)

Considering that one must obtain the usual constant of motion (Energy) expression for $\alpha$ equal to zero, the functionality of $G$ in Eq. (3) is determined for each above characteristic ($G(C_1) = mC_1$ and $G(C_2) = -(mg/2\alpha)C_2 - m^2 g/2\alpha$), and the following constants of motion are gotten

$$K_{\alpha}^{(1)}(x,v) = -\frac{m^2 g}{2\alpha} \ln \left( 1 - \frac{\alpha}{mg} v^2 \right) + mgx ,$$

(6)

and

$$K_{\alpha}^{(2)}(x,v) = \frac{m^2}{2\alpha} \left( -g + \frac{\alpha}{m} v^2 \right) e^{-2\alpha x/m} + \frac{m^2 g}{2\alpha} .$$

(7)

We need to point out that the following limit is gotten for the parameter $\alpha$ going to zero

$$\lim_{\alpha \to 0} K_{\alpha}^{(i)} = \frac{1}{2} m v^2 + mgx \quad i = 1, 2 .$$

(8)
3 Lagrangians and Hamiltonians

Using the known expression (Kobussen 1979, Leuber 1987, Yan 1981, and López 1996), to get the Lagrangian once the constant of motion is given,

\[ L(x, v) = v \int \frac{K(x, v)}{v^2} \, dv, \] (9)

the Lagrangian associated to Eq. (6) and Eq. (7) are

\[ L^{(1)}_\alpha(x, v) = m \sqrt{\frac{mg}{\alpha}} v \arctanh \left( \sqrt{\frac{\alpha}{mg}} \, v \right) + \frac{m^2 g}{2\alpha} \ln \left( 1 - \frac{\alpha}{mg} v^2 \right) - mgx \] (10)

and

\[ L^{(2)}_\alpha(x, v) = \frac{m^2}{2\alpha} \left( g + \frac{\alpha}{m} v^2 \right) e^{-2\alpha x/m} - \frac{m^2 g}{2\alpha}. \] (11)

Their generalized linear momenta \( p = \partial L/\partial v \) are then given by the following expressions

\[ p^{(1)}_\alpha = m \sqrt{\frac{mg}{\alpha}} \arctanh \left( \sqrt{\frac{\alpha}{mg}} \, v \right) \] (12)

and

\[ p^{(2)}_\alpha = mv e^{-2\alpha x/m}. \] (13)

Thus, their associated Hamiltonians, \( H(x, p) = K(x, v(x, p)) \), can be calculated and determined as

\[ H^{(1)}_\alpha(x, p) = -\frac{m^2 g}{2\alpha} \ln \left[ 1 - \tanh^2 \left( \sqrt{\frac{\alpha}{mg}} \, \frac{p}{m} \right) \right] + mgx \] (14)

and

\[ H^{(2)}_\alpha(x, p) = \frac{m^2}{2\alpha} \left( -g + \frac{\alpha p^2}{m^3} e^{4\alpha x/m} \right) e^{-2\alpha x/m} + \frac{m^2 g}{2\alpha}, \] (15)
where one has made the substitution of $p_\alpha^{(1)}$ and $p_\alpha^{(2)}$ by just $p$. One must note that the following limits are gotten

\[
\lim_{\alpha \to 0} L_\alpha^{(i)}(x, v) = \frac{1}{2}mv^2 - mgx ,
\]

\[
\lim_{\alpha \to 0} p_\alpha^{(i)} = mv ,
\]

and

\[
\lim_{\alpha \to 0} H_\alpha^{(i)}(x, p) = \frac{p^2}{2m} + mgx \quad \text{for} \quad i = 1, 2 .
\]

At first order in the dissipation parameter $\alpha$, these Hamiltonians have the following expressions

\[
H^{(1)}(x, p) = \frac{p^2}{2m} + mgx - \frac{\alpha}{12m^4g^2}p^4
\]

and

\[
H^{(2)}(x, p) = \frac{p^2}{2m} + mgx + \alpha \left(\frac{xp^2}{m^2} - gx^2\right).
\]

4 Quantization at first order in perturbation theory

The evolution of the quantum system characterized by the Hamiltonian $\hat{H}^{(i)}$, being initially in the state $|\Psi(0)\rangle$, is given by $|\Psi(t)\rangle = U_i(t)|\Psi(0)\rangle$, where $U_i$ is the unitary evolution operator, $U_i(t) = e^{-it\hat{H}^{(i)}/\hbar}$, which is determined once we solve the eigenvalue problem

\[
\hat{H}^{(i)}|n_i\rangle = E_{n_i}|n_i\rangle.
\]

Given the same initial state, the evolution is then written as

\[
|\Psi_i(t)\rangle = \sum_{n_i} c_{n_i} e^{-iE_{n_i}t/\hbar}|n_i\rangle , \quad \text{with} \quad c_{n_i} = \langle n_i|\Psi(0)\rangle , i = 1, 2.
\]
In this way, if the eigenvalues of both Hamiltonian are different, the evolution and the dynamic of the systems will be different. The Hamiltonians $\hat{H}^{(i)}$ for $i = 1, 2$ are the Hermitian operators associated to Eq. (19) and Eq. (20), and this can be done through Weyl quantization method (Weyl 1927) which is applied basically on the term $x\hat{p}^2$ appearing in Eq. (20), $x\hat{p}^2 = (x\hat{p}^2 + \hat{p}^2 x + \hat{x}\hat{p})/3$. Of course, this represents a well known ambiguity about selection of proper Hermitian operator since Weyl quantization method is not the only way to do that, one also could assign the Hermitian operator $(x\hat{p}^2 + \hat{p}^2 x)/2$ to the function $x\hat{p}^2$. Probably, the best guide we can have is the experiment. On the other hand, it is well known in classical and quantum physics that in order to incorporate the electromagnetic field interaction into the Hamiltonian formalism, one uses the so called "minimal substitution" where one makes the change $\hat{p} \rightarrow \hat{p} + \frac{q}{c} \hat{A}$, being $q$ the charge of the particle, $c$ is the speed of light, and $\hat{A}$ is the vector potential, and in this way, to the function $\hat{p} \cdot \hat{A}$ is assigned the operator $(\hat{p} \cdot \hat{A} + \hat{A} \cdot \hat{p})/2$, which is consistent with Weyl quantization method. The experimental evident supporting this procedure are overwhelm. This procedure can be generalized in the following way: Let $\hat{A}$ and $\hat{B}$ be two Hermitian operators, then, the expression $(\hat{A} + \hat{B})^3$ represents also an Hermitian operator which is given by $\hat{A}^3 + \hat{B}^3 + \hat{A}^2 \hat{B} + \hat{B} \hat{A}^2 + \hat{A} \hat{B}^2 + \hat{B} \hat{A} \hat{B} + \hat{B}^2 \hat{A}$. Identifying the power terms with the resulting power of numbers $(A + B)^3 = A^3 + B^3 + 3A^2B + 3BA^2$, one gets the association:

$$
\begin{align*}
A^3 & \rightarrow \hat{A}^3 \\
B^3 & \rightarrow \hat{B}^3, \\
3A^2B & \rightarrow \hat{A}^2 \hat{B} + \hat{A} \hat{B} \hat{A} + \hat{B} \hat{A}^2, \\
3AB^2 & \rightarrow \hat{B}^2 \hat{A} + \hat{B} \hat{A} \hat{B} + \hat{A} \hat{B}^2.
\end{align*}
$$

In fact the associated Hermitian operator of any monomial $A^r B^s$ such that $r + s = n$ can be deduced from the Hermitian operator $(\hat{A} + \hat{B})^n$. Then, the associated Hermitian operator to any polynomial in the variables $A$ and $B$ can be done straight forward. Although we are confident that this procedure is right, experimental confirmation would be required any way.

Now, in relation to our problem (21), it is obvious that it is not necessary to solve completely the problem to see whether or not the quantum dynamics involved are different or not. To see this, it is enough to calculate the eigenvalues $E_{n_i}$ at first order in perturbation theory. Therefore, we will do a simple first order perturbation theory calculation of these eigenvalues.
Our Hamiltonians Eq. (19) and Eq. (20) can be written as the following Hermitian operators

\[ \hat{H}^{(i)}(x, p) = \hat{H}_0(x, p) + \hat{W}^{(i)}(x, p) \quad \text{for} \quad i = 1, 2, \] (23)

where \( \hat{H}_0 \) represents the non dissipative part of the Hamiltonian,

\[ \hat{H}_0(x, p) = \frac{\hat{p}^2}{2m} + mgx, \] (24)

and \( \hat{W}^{(i)} \) represents the contribution of the dissipation at first order in \( \alpha \),

\[ \hat{W}^{(1)}(x, p) = -\frac{\alpha \hat{p}^4}{12gm^4} \] (25)

and

\[ \hat{W}^{(2)}(x, p) = \alpha \left[ \frac{1}{3m^2} (xp^2 + \hat{p}^2x + \hat{p}x\hat{p}) - gx^2 \right]. \] (26)

The Hamiltonians are time independent, and \( \hat{H}^{(i)} \) is the associated Hermitian operator of Eq. (23). Of course, one must not allow the particle to go beyond down the surface level. Thus, Eq. (24) is representing the Hamiltonian of the quantum bouncer \((x \geq 0)\) (Gean-Banacloche 1999), where the eigenvalue problem

\[ \hat{H}_0(\hat{x}, \hat{p})\psi_n^{(0)}(x) = E_n^{(0)}\psi_n^{(0)}(x) \] (27)

has the eigenvectors and eigenvalues solution given by

\[ \psi_n^{(0)}(x) = \frac{Ai(z - z_n)}{|Ai'(-z_n)|}, \] (28)

and

\[ E_n^{(0)} = mglz_n, \] (29)

where \( \psi_n^{(0)}(x) = \langle x | n \rangle \) is the zero order solution \((\alpha = 0)\). The functions \( Ai \) and \( Ai' \) represent the Airy function and its differentiation respect to \( z \). The
variable $z$ is defined as $z = x/l_g$, where $l_g$ is given by $l_g = (\hbar^2/2m^2g)^{1/3}$, and $z_n$ is the $n$th-zero of the Airy function $(Ai(-z_n) = 0)$. In fact, the bouncing problem has already been studied for linear and quadratic dissipation (G. López 2004). For the later, the correction given to the eigenvalue problem using Eq. (26) at first order in perturbation theory is

$$\langle n|\hat{W}^{(2)}|n\rangle = \alpha \frac{4g^2l_g^2z_n^2}{15}. \quad (30)$$

Now, using the relation $\langle n|d^4/dz^4|n\rangle = z_n^2/5$, the correction at first order in perturbation due to Eq. (25) is given by

$$\langle n|\hat{W}^{(1)}|n\rangle = -\alpha \frac{h^4z_n^2}{60gm^4l_g^4}. \quad (31)$$

Therefore, for the same classical dynamical system we have two different associated quantum systems which have completely different quantum dynamics, which is shown through the eigenvalues

$$E_n^{(1)} = E_n^{(0)} + \alpha \frac{4g^2l_g^2z_n^2}{15} \quad (32)$$

and

$$E_n^{(2)} = E_n^{(0)} - \alpha \frac{h^4z_n^2}{60gm^4l_g^4}. \quad (33)$$

We must note that in order to say something about the quasi-classical limit ($n_i \gg 1$), we would need to solve the full problem Eq. (22).

5 Classical Statistical model for dissipation

Consider a system of $N = N_1 + N_2$ particles, where $N_1$ particles are small of mass $m_1$, and $N_2$ particles are are big of mass $m_2$ ($m_2 \gg m_1$). The small particles move under the action of an external force with components $(0, 0, -mg)$ and suffer collisions with the walls of the container which consists in a narrow-square shape pipe of cross sectional area $L^2$. In addition, each small particle can have occasional (stochastic) collision with the big particles, when they are added, establishes the dissipative medium where the big particles will move.
The big particles move in this dissipative medium, and it is assumed that, since
this type of collision does not occur frequently, its average effect may have ne-
glected contribution on the dynamical macroscopic variables of the system.

Newton’s equations of motion for this system can be written as

\[ m_1 \ddot{q}_{1ij} = 0 \quad j = 1, \ldots, N_1; \quad i = x, y \tag{34} \]

\[ m_1 \ddot{q}_{1zj} = -m_1 g \quad j = 1, \ldots, N_1 \tag{35} \]

\[ m_2 \ddot{q}_{2ik} = \alpha (\dot{q}_{2ik})^2 \quad k = 1, \ldots, N_2; \quad i = x, y \tag{36} \]

\[ m_2 \ddot{q}_{2zk} = \alpha (\dot{q}_{2zk})^2 - m_2 g \quad k = 1, \ldots, N_2 \tag{37} \]

where \( q_{aij}, \dot{q}_{aij} \) and \( \ddot{q}_{aij} \) are the generalized coordinates, velocities and accel-
erations of the light-small \( (a = 1) \) and heavy-gross \( (a = 2) \) particles, and the
parameter \( \alpha \) characterizes the dissipative medium. The Hamiltonian associ-
ated the the motion of 1-particle, Eq. (34) and Eq. (35), is given by \(^{1}\)López
et al 1997

\[ H_{1;\dot{x},\dot{y},\dot{z}} = \sum_{j=1}^{N_1} \left[ 3 \sum_{i=1}^{3} \frac{p_{1ij}^2}{2m_1} + m_1 g \dot{q}_{1zj} \right], \tag{38} \]

The Hamiltonian associated to Eq. (36) is written as

\[ H_{2;\dot{x},\dot{y}} = \sum_{k=1}^{N_2} \sum_{i=1}^{2} \frac{p_{2ik}^2}{2m_2} \exp \left( \frac{2\alpha \dot{q}_{2ik}}{m_2} \right), \tag{39} \]

and, as we have seen in section 3, there are at least two Hamiltonians associated
to Eq. (37) which are given by

\[ H_{2;z}^{(1)} = \sum_{k=1}^{N_2} \left\{ -\frac{gm_2^2}{2\alpha} \ln \left[ 1 - \tanh^2 \left( \sqrt{\frac{\alpha}{m_2 g}} \frac{p_{2zk}}{m_2} \right) \right] + m_2 g \dot{q}_{2zk} \right\} \tag{40} \]
and

$$H^{(2)}_{2;z} = \sum_{k=1}^{N_2} \left\{ \frac{p_{2zk}^2}{2m_2} \exp \left( \frac{2\alpha q_{2zk}}{m_2} \right) - \frac{m_2^2 g}{2\alpha} \left[ \exp \left( -\frac{2\alpha q_{2zk}}{m_2} \right) - 1 \right] \right\}.$$

(41)

Therefore, one has two different Hamiltonians to describe the same system, $H^{(1)} = H_{1;x,y,z} + H_{2;x,y} + H^{(2)}_{2;z}$ and $H^{(2)} = H_{1;x,y,z} + H_{2;x,y} + H^{(1)}_{2;z}$, which are written as

$$H^{(1)} = \sum_{j=1}^{N_1} \left[ \sum_{i=1}^{3} \frac{p_{1ij}^2}{2m_1} + m_1 g q_{1zj} \right] + \sum_{k=1}^{N_2} \sum_{i=1}^{3} \frac{p_{2ik}^2}{2m_2} \exp \left( \frac{2\alpha q_{2ik}}{m_2} \right)$$

$$+ \sum_{k=1}^{N_2} \left\{ \frac{p_{2zk}^2}{2m_2} \exp \left( \frac{2\alpha q_{2zk}}{m_2} \right) - \frac{m_2^2 g}{2\alpha} \left[ \exp \left( -\frac{2\alpha q_{2zk}}{m_2} \right) - 1 \right] \right\}$$

(42)

and
\[ H^{(2)} = \sum_{j=1}^{N_1} \left[ \sum_{i=1}^{3} \frac{p_{1ij}^2}{2m_1} + m_1 q_{1zj} \right] + \sum_{k=1}^{N_2} \sum_{i=1}^{3} \frac{p_{2ik}^2}{2m_2} \exp \left( \frac{2\alpha q_{2ik}}{m_2} \right) + \sum_{k=1}^{N_2} \left\{ -\frac{gm_2^2}{2\alpha} \ln \left[ 1 - \tanh^2 \left( \sqrt{\frac{\alpha}{m_2 g}} \frac{p_{2z}}{m_2} \right) \right] + m_2 g q_{2z} \right\} . \] (43)

Then, one can calculate for each Hamiltonian the canonical partition function (Toda et al 1998) which is associated to the same statistical system,

\[ Z^{(i)} = \frac{1}{N_1!N_2!h^{3N}} \int \exp (-\beta H^{(i)}) \ dq \ dp \ i = 1, 2, \] (44)

where \( \beta \) is defined as \( \beta = 1/kT \) with \( k \) being the Boltzman’s constant and \( T \) being the temperature, and the integration is carried out over all the coordinates and linear momenta of the two particles. The integration of momenta is carried out in the interval \((-\infty, +\infty)\). The integration on the transverse coordinates \((x, y)\) is carried out in the interval \([0, L]\), and the integration of the vertical coordinate is carried out in the interval \([0, z]\). The partition functions for both cases are given by

\[ Z^{(1)} = \frac{L^{2N_1}}{N_1!N_2!h^{3N}} \left( \frac{2\pi m_1}{\beta} \right)^{3N_1/2} \left( \frac{1 - e^{-\beta m_1 g z}}{\beta m_1 g} \right)^{N_1} \left( \frac{2\pi m_2}{\beta} \right)^{3N_2/2} \left( \frac{m_2}{\alpha} \right)^{2N_2} \times \left( e^{-\frac{\alpha L}{m_2}} - 1 \right)^{2N_2} \left[ \sqrt{\frac{\pi}{2\beta \alpha g}} e^{-\frac{\beta m_2^2 g z}{2\alpha}} \left( Erfi \left( \sqrt{\frac{\beta g m_2^2}{2\alpha}} e^{-\frac{\alpha z}{m_2}} \right) - Erfi \left( \sqrt{\frac{\beta g m_2^2}{2\alpha}} \right) \right) \right]^{N_2} \] (45)

and
The system has two internal energies, \( U^{(i)} = -\frac{\partial \ln Z^{(i)}}{\partial \beta} \),

\[
U^{(1)} = \left( \frac{5N_1}{2} + 2N_2 \right) \frac{1}{\beta} - \frac{N_1 m_1 g z e^{-\beta m_1 g z}}{1 - e^{-\beta m_1 g z}} - \frac{N_2 m_2 g z e^{-\beta m_2 g z}}{2\alpha} - \frac{N_2 f'(\beta)}{f(\beta)} \tag{47}
\]

and

\[
U^{(2)} = \left( \frac{5N_1}{2} + 2N_2 \right) \frac{1}{\beta} - \frac{N_1 m_1 g z e^{-\beta m_1 g z}}{1 - e^{-\beta m_1 g z}} - \frac{N_2 m_2 g z e^{-\beta m_2 g z}}{1 - e^{-\beta m_2 g z}} - \frac{N_2 m_2 g z}{2\alpha} \left[ \psi \left( \frac{\beta m_2^2 g z}{2\alpha} \right) - \psi \left( \frac{\beta m_2^2 g z}{2\alpha} + \frac{1}{2} \right) \right] \tag{48}
\]

where the function \( f(\beta) \) \((f'(\beta) = df(\beta)/d\beta)\) has been defined as

\[
f(\beta) = \text{Erfi} \left( \sqrt{\frac{\beta g m_2^2}{2\alpha}} \right) e^{-\alpha z/m_2} - \text{Erfi} \left( \sqrt{\frac{\beta g m_2^2}{2\alpha}} \right),
\]

and \( \text{Erfi}(x) = -i \text{Erf}(ix) \) is the complex error function which can be expressed in the form of the Dawson’s integral, \( \text{Erfi}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \text{Dawson}(x) \), with \( \text{Dawson}(x) = e^{-x^2} \int_0^x e^{t^2} dt \). The function \( \psi \) is the so called digamma function, \( \psi(x) = d\ln \Gamma(x)/dx \). Thus, one can have two heat capacity expressions for the system, \( C_V^{(i)} = \frac{\partial U^{(i)}}{\partial T} = -k\beta^2 \frac{\partial U^{(i)}}{\partial \beta} \),

\[
C_V^{(1)} = \left( \frac{5N_1}{2} + 2N_2 \right) k - \frac{N_1 k (m_1 g z)^2 e^{-m_1 g z} \beta}{(1 - e^{-m_1 g z} \beta)^2} + N_2 k \beta^2 \left( \frac{f''(\beta)}{f(\beta)} - \frac{(f'(\beta))^2}{(f(\beta))^2} \right) \tag{50}
\]
and

\[
C_V^{(2)} = \left( \frac{5N_1}{2} + 2N_2 \right) k - \frac{N_1 k (m_1 g z \beta)^2 e^{-m_1 g z \beta}}{(1 - e^{-m_1 g z \beta})^2} - \frac{N_2 k (m_2 g z \beta)^2 e^{-m_2 g z \beta}}{(1 - e^{-m_2 g z \beta})^2} \\
+ N_2 k \left( \frac{m_2^2 g \beta}{2\alpha} \right)^2 \left[ \psi^{(1)} \left( \frac{m_2^2 g \beta}{2\alpha} \right) - \psi^{(1)} \left( \frac{m_2^2 g \beta}{2\alpha} + \frac{1}{2} \right) \right],
\]

where \( \psi^{(1)} \) is the trigamma function, \( \psi^{(1)}(x) = d^2 \Gamma(x)/dx^2 \). Figure 1 shows the difference \(|C_V^{(1)} - C_V^{(2)}|\) as a function of \( \beta = 1/kT \). As one can see, this difference is not small at low temperatures (high \( \beta \) values). From \( \beta \) lower than about 2100, \( C_{V2} \) is higher than \( C_{V1} \), and the situation is reversed for higher values. This difference seems to have an important implication related with the ergodic hypothesis (Toda et al. 1998). Assuming the validity of the hypothesis, one would expect not difference at all on the calculated heat capacities (or internal energies) since averaging over the time variable must bring about the same value for both Hamiltonians (they represent the same dynamical system). However, averaging over the canonical ensemble must be different if the Hamiltonians are different. Even more, this ambiguity will remain when quantum canonical ensemble is considered (using Eq. (32) and Eq. (33)) for the quantum statical analysis of the system.

6 Conclusions

We have shown two constants of motion, two Lagrangians, and two Hamiltonians for a free falling particle moving in a media with quadratic velocity dissipative force. These quantities describe the same classical dynamics of the dissipative system. However, we have showed, at first order in the dissipative parameter and at first order in perturbation theory, that their quantization and the associated statistical mechanics (for an ensemble of particles) describe two different quantum and statistical dynamics. We think that this ambiguity is not an exception for this particular dissipative mode, and that this ambiguity can only be solve through experimental analysis. On the other hand, one point which remains to be studied is the quasi-classical limit. The question here is whether or not both quantum Hamiltonians, Eq. (23), describes the same quasi-classical dynamics and coincides with the classical dynamics within the limit \( \hbar \to 0 \) (or \( n \gg 1 \)).
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Received: November, 2010
Figure 1: Heat capacities difference: $\beta = 1/kT$, $\alpha = 0.01$, $g = 1$, and $m_1/m_2 = 0.1$