Quantum Equivalence of Massive Antisymmetric Tensor Field Models in Curved Space

I.L. Buchbinder$^\dagger$ and E.N. Kirillova$^\ddagger$

Department of Theoretical Physics
Tomsk State Pedagogical University
Tomsk 634041, Russia.

N.G. Pletnev$^\ddagger$

Department of Theoretical Physics
Institute of Mathematics, Novosibirsk,
630090, Russia.
(Dated: July 11, 2008)

We study the effective actions in massive rank-2 and rank-3 antisymmetric tensor field models in curved space-time. These models are classically equivalent to massive vector field and massive scalar field with minimal coupling to gravity respectively. We prove that the effective action for massive rank-2 antisymmetric tensor field is exactly equal to that for massive vector field and the effective action for massive rank-3 antisymmetric tensor field is exactly equal to that for massive scalar field. The proof is based on an identity for mass-dependent zeta-functions associated with Laplacians acting on $p$-forms.

PACS numbers: 03.70.+k Theory of quantized fields; 04.62.+v Quantum fields in curved spacetime; 11.15.-q Gauge field theories

I. INTRODUCTION

Antisymmetric tensor fields or $p$-forms are the components of field content of all superstring models and hence can survive in the low-energy limit. Indeed, the attempts to relate Calabi-Yau compactifications of ten-dimensional superstring theories to the real world lead to low-energy supergravity theories with electric and magnetic fluxes of various $p$-form fields. Such theories drew a lot of attention due to observation that the resulting scalar potentials in the low-energy effective supergravity theories might lift the vacuum degeneracy [1], [2]. In particular, the magnetic charges should yield mass terms for $p$-forms [3] and, in fact, we get the supersymmetric models containing the massive antisymmetric tensor fields. Usually such fields belong to tensor supermultiplets. The massive $\mathcal{N} = 0, 1, 2$ tensor multiplets [4], [5] possess some interesting properties. However, the many aspects, especially the quantum ones, have not been studied so far.

In this paper, as the first step in studying the quantum aspects of massive supersymmetric theories we examine the structure of the effective action in bosonic massive antisymmetric tensor field models in four dimensional curved space-time. In four dimensions, a massive rank-2 antisymmetric tensor field is classically equivalent to a massive vector field. Similarly, a massive rank-3 antisymmetric tensor field is equivalent to a massive scalar field minimally coupled to gravity. In this paper we study the problem of quantum equivalence of these classically equivalent theories.

Some years ago there was a large work on studying the massless antisymmetric field models in curved space-time. In particular, in four dimensions, a massless rank-2 antisymmetric field [6] is classically equivalent to massless nonconformal scalar field and massless rank-3 antisymmetric field has no physical degrees of freedom [28]. Quantization aspects were discussed in [8], [9], [10], [11], [12], [13]. The problem of quantum equivalence of massless classically equivalent theories was considered in [9], [12], [13], [14], [15], [16]. For massive antisymmetric field models, the net of classical dualities in this case differs from the one for massless fields. Therefore one can expect that studying the problem of quantum equivalence for massive antisymmetric fields requires the other methods in comparison with massless antisymmetric fields. Some quantum aspects of massive antisymmetric field models have been considered in [17] on the base of the worldline approach to the effective action. It was pointed out in [17] that the unregulated effective actions have a topological mismatch between massive $p$-form and its dual massive $(D - p - 1)$-form.

$^\dagger$Electronic address: joseph@tspu.edu.ru
$^\ddagger$Electronic address: kirillovaen@tspu.edu.ru
$^\ddagger$Electronic address: pletnev@math.nsc.ru
In this paper, we study a generic structure of the effective actions for massive rank-2 and rank-3 antisymmetric fields in arbitrary curved space-time and prove, using the zeta-function technique, that they are exactly equal to those for massive vector field and massive scalar field minimally coupled to gravity respectively. The proof is essentially based on an identity for mass-dependent zeta-functions associated with p-forms. The paper is organized as follows. Section II is devoted to a brief description of massive antisymmetric field models in four-dimensional curved space-time. In Section III we discuss the definitions of the effective action for massive antisymmetric field models. Section IV is devoted to proving the quantum equivalence of dual massive field models under consideration. In Summary we formulate the results.

II. MODELS OF MASSIVE ANTISYMMETRIC TENSOR FIELDS

We consider a model of massive antisymmetric second rank tensor field $B_2 = (B_{\mu\nu})$ in curved 4D space. The model is described by the action

$$S[B_2] = \int d^4x \sqrt{-g(x)} \left\{ \frac{1}{12} F^{\mu\nu\lambda}(B) F_{\mu\nu\lambda}(B) + \frac{1}{4} m^2 B^{\mu\nu} B_{\mu\nu} \right\},$$

(1)

where

$$F_{\mu\nu\lambda}(B) = \nabla_\mu B_{\nu\lambda} + \nabla_\nu B_{\lambda\mu} + \nabla_\lambda B_{\mu\nu}.$$  

(2)

It is easy to see that the kinetic part of action (1) is gauge invariant under the transformations

$$B_{\mu\nu} \rightarrow B_{\mu\nu}^\xi = B_{\mu\nu} + \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu,$$

(3)

with a vector gauge parameter $\xi_\mu$ defined up to a transformation $\xi_\mu' = \xi_\mu + \nabla_\mu \xi$ with scalar parameter $\xi$. This means that the gauge generators are linearly dependent. The massive term in the action (1) violates this symmetry.

For quantization of the theory and evaluation of the effective action it is convenient [18] to restore the gauge invariance under (3) in massive theory (1) with help of the Stückelberg procedure. We introduce the vector field $C_1 = (C_\mu)$ and consider the following action

$$S[B_2, C_1] = \int d^4x \sqrt{-g(x)} \left\{ -\frac{1}{12} F^{\mu\nu\lambda}(B) F_{\mu\nu\lambda}(B) + \frac{1}{4} m^2 (B^{\mu\nu} + \frac{1}{m} F^{\mu\nu}(C))^2 \right\},$$

(4)

where $F_{\mu\nu}(C) = \nabla_\mu C_\nu - \nabla_\nu C_\mu$. The action (4) is invariant under the gauge transformations (3) of the field $B_{\mu\nu}$ and under shift of the field $C_\mu$

$$C_\mu \rightarrow C_\mu^\xi = C_\mu - m \xi_\mu,$$

(5)

and also under the gauge transformations of the Stückelberg vector field:

$$C_\mu \rightarrow C_\mu^\lambda = C_\mu + \nabla_\mu \lambda, \quad B_{\mu\nu} \rightarrow B_{\mu\nu}^\lambda = B_{\mu\nu},$$

(6)

with a scalar gauge parameter $\lambda$. In four dimensions, a massive rank-2 antisymmetric tensor field can be interpreted as a massive pseudovector field. The physical component of $B_{\mu\nu}$ then corresponds to the longitudinal mode of a massive pseudovector field, while the two physical components of $C_\mu$ correspond to its transverse modes. The Lagrangian (4) can simply be expressed as

$$\mathcal{L} = -\frac{1}{12} \bar{F}_{\mu\nu\lambda} \bar{F}^{\mu\nu\lambda} + \frac{1}{4} m^2 \bar{B}_{\mu\nu} \bar{B}^{\mu\nu},$$

(7)

where $\bar{B}_{\mu\nu} \equiv B_{\mu\nu} + \frac{1}{m} F_{\mu\nu}(C)$ and $\bar{F}_{\mu\nu\lambda}$ is as in (2). Since $\bar{B}_{\mu\nu}$ is gauge invariant the gauge invariance of $\mathcal{L}$ is evident. Then it is obvious that $\mathcal{L}$ describes on Abelian rank-2 antisymmetric tensor field with mass $m$. In four dimensions the theory (1) is classically equivalent to the theory of massive vector field $A_\mu$ with action

$$S[A, C] = \int d^4x \sqrt{-g(x)} \left\{ -\frac{1}{4} F^{\mu\nu\alpha}(A) F_{\mu\nu\alpha}(A) + \frac{1}{2} m^2 (A^{\mu} - \frac{1}{m} \partial^{\mu} C) (A_{\mu} - \frac{1}{m} \partial_{\mu} C) \right\}.$$  

(8)

Here $C$ is the Stückelberg scalar field. The equivalence is resulted from the analysis of the equations of motion in both theories. The duality relation looks like $m \bar{B}_{\mu\nu} \sim \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}(A)$.
Also we consider a model of massive totally antisymmetric third rank tensor field $B_3 = (B_{\mu\nu\rho})$ in curved space. Such a model is described by the action

$$S[B_3] = \frac{1}{2} \int d^4 x \sqrt{-g(x)} \left\{ -\frac{1}{4!} F_{\mu\nu\rho\sigma} (B) F_{\mu\nu\rho\sigma} (B) + \frac{m^2}{3!} B_{\mu\nu\rho} B_{\mu\nu\rho} \right\},$$  

where

$$F_{\mu\nu\rho\sigma} (B) = \nabla_{[\mu} B_{\nu\rho\sigma]} - \nabla_{[\nu} B_{\rho\sigma]\mu} + \nabla_{\rho} B_{\sigma\mu\nu} - \nabla_{\sigma} B_{\mu\nu\rho},$$  

$$B_{\nu\rho\sigma} = -B_{\rho\nu\sigma} = -B_{\sigma\rho\nu}.$$  

The kinetic term of the action (9) is gauge invariant under the transformations

$$B_{\mu\nu\rho} \to B'_{\mu\nu\rho} = B_{\mu\nu\rho} + \nabla_{\mu} \xi_{\nu\rho} + \nabla_{\nu} \xi_{\rho\mu} + \nabla_{\rho} \xi_{\mu\nu},$$  

with a tensor gauge parameter $\xi_{\mu\nu} = -\xi_{\nu\mu}$. This parameter is defined up to a gauge transformation $\xi'_{\mu\nu} = \xi_{\mu\nu} + \nabla_{\mu} \xi_{\nu} - \nabla_{\nu} \xi_{\mu}$ with vector a gauge parameter $\xi$. In its turn, the parameter $\xi$ is defined up to gauge transformation $\xi' = \xi + \nabla \xi$ with scalar gauge parameter $\xi$. This means that the gauge generators are linearly dependent. As in the previous case, we restore the gauge invariance under (11) in massive theory (9) with help of the Stückelberg procedure. We introduce the second rank antisymmetric tensor field $C_2 = (C_{\mu\nu})$ and consider the following action

$$S[B_3, C_2] = \frac{1}{2} \int d^4 x \sqrt{-g(x)} \left\{ -\frac{1}{4!} F_{\mu\nu\rho\sigma} (B) F_{\mu\nu\rho\sigma} (B) + \frac{m^2}{3!} (B_{\mu\nu\rho} + \frac{1}{m} F_{\mu\nu\rho} (C))^2 \right\},$$  

where

$$F_{\mu\nu\rho} (C) = \nabla_{\mu} C_{\nu\rho} + \nabla_{\nu} C_{\rho\mu} + \nabla_{\rho} C_{\mu\nu}. \tag{13}$$

The action (12) is invariant under the gauge transformations of the fields $B_{\mu\nu\rho}$, $C_{\mu\nu}$:

$$B_{\mu\nu\rho} \to B'_{\mu\nu\rho} = B_{\mu\nu\rho} + \nabla_{\mu} \xi_{\nu\rho} + \nabla_{\nu} \xi_{\rho\mu} + \nabla_{\rho} \xi_{\mu\nu}, \quad C_{\mu\nu} \to C'_{\mu\nu} = C_{\mu\nu} - m \xi_{\mu\nu} \tag{14}$$

and also under the Stückelberg gauge transformations

$$C_{\mu\nu} \to C'_{\mu\nu} = C_{\mu\nu} + \nabla_{\mu} \Lambda_{\nu} - \nabla_{\nu} \Lambda_{\mu}, \quad B_{\mu\nu\rho} \to B'_{\mu\nu\rho} = B_{\mu\nu\rho}, \tag{15}$$

because $F_{\mu\nu\rho} = F_{\mu\nu\rho}$, where $\Lambda$ is a vector gauge parameter, defined up to a gauge transformation $\Lambda' = \Lambda + \nabla \Lambda$ with a scalar parameter $\Lambda$. This means that the corresponding gauge generators are linearly dependent. The equations of motion in the theory (12) are equivalent to those in the theory (9). In particular, the action (12) coincides with the action (9) in the gauge $C_{\mu\nu} = 0$. One can prove that in four dimensions the theory of a massive third rank antisymmetric tensor field is classically equivalent to the theory of a real massive scalar field $\phi$ minimally coupled to gravity. The corresponding duality relation has the form $m B_{\mu\nu\alpha\beta} \sim \epsilon_{\mu\nu\alpha\beta} \partial^\beta \phi$.

### III. THE EFFECTIVE ACTION

The models of massive antisymmetric fields in their initial formulations (11), (12) contain gauge invariant kinetic terms and non-gauge invariant massive terms. In this case the effective actions are given by the functional determinants of the differential operators with degenerate matrices at higher derivative terms but non-degenerate matrices at mass terms. Then, the calculations of the effective actions becomes very complicated and problematic[29]. To avoid such a problem we reformulated the models under consideration with help of the Stückelberg procedure as the gauge theories. Therefore, to construct the corresponding effective actions we can apply now the quantization methods of gauge theories. It is especially worth pointing out that the above models belong to a class of gauge theories with linearly dependent generators. Their quantization is very nontrivial and differs from quantization of the Yang-Mills type theories where the gauge generators are independent. General quantization procedure for theories with dependent generators in the Lagrangian formalism is given by BV-method [14]. However, quantization of simple theories with quadratic actions and Abelian dependent gauge generators can be carried out by successive multi-step applications of the Faddeev-Popov procedure (see e.g. [8], [9], [13], [20].
To quantize the theories under consideration we use a rather simple procedure developed in [13] (see also [20]). Omitting the calculations we formulate only the final results for effective actions.

First, the effective action $\Gamma^{(m)}_2[g_{\mu\nu}]$ of the massive second rank antisymmetric field model is given by the relation

$$\Gamma^{(m)}_2[g_{\mu\nu}] = \frac{i}{2} [\text{Tr} \ln(\Box_2 + m^2) - \text{Tr} \ln(\Box_1 + m^2) + \text{Tr} \ln(\Box_0 + m^2)] .$$

(16)

Second, the effective action $\Gamma^{(m)}_1[g_{\mu\nu}]$ of the massive vector field $A_\mu$ is given by the relation

$$\Gamma^{(m)}_1[g_{\mu\nu}] = \frac{i}{2} [\text{Tr} \ln(\Box_1 + m^2) - \text{Tr} \ln(\Box_0 + m^2)] .$$

(17)

Third, the effective action $\Gamma^{(m)}_3[g_{\mu\nu}]$ of the massive third rank antisymmetric tensor field is given by the relation

$$\Gamma^{(m)}_3[g_{\mu\nu}] = \frac{i}{2} [\text{Tr} \ln(\Box_3 + m^2) - \text{Tr} \ln(\Box_2 + m^2) + \text{Tr} \ln(\Box_1 + m^2) - \text{Tr} \ln(\Box_0 + m^2)] .$$

(18)

Fourth, the effective action $\Gamma^{(m)}_0[g_{\mu\nu}]$ of the massive scalar field $\phi$ with minimal coupling to gravity is given by the relation

$$\Gamma^{(m)}_0[g_{\mu\nu}] = \frac{i}{2} \text{Tr} \ln(\Box_0 + m^2) .$$

(19)

Here the $\Box_3, \Box_2, \Box_1$ and $\Box_0$ are the d’Alembertians acting on rank-$p$ antisymmetric tensor fields and

$$\text{Tr}(...) = \int d^4x \sqrt{-g(x)} \text{tr}(...) ,$$

(20)

where tr(... is taken over tensor indices.

The relations (16), (17), (18), (19) can also be understood in terms of dimensional reduction [21]. Let us consider massless antisymmetric tensor fields $B_{MN}$ and $B_{MNK}$ in five-dimensional space of the topology $R_4 \times S^1$ where $R_4$ is the four-dimensional Riemannian space. In D5, the massless field $B_{MN}$ has three physical degrees of freedom equivalent to a massless vector, the massless field $B_{MNK}$ has one physical degree of freedom equivalent to a massless scalar. General form of effective actions of massless antisymmetric tensor field models has been obtained in [10], [13], [14], [17]. The results for the corresponding five-dimensional effective actions are

$$\Gamma^{(D5)}_2 = \frac{i}{2} [\text{Tr} \ln(\Box_2) - 2\text{Tr} \ln(\Box_1) + 3\text{Tr} \ln(\Box_0)] ,$$

$$\Gamma^{(D5)}_3 = \frac{i}{2} [\text{Tr} \ln(\Box_3) - 2\text{Tr} \ln(\Box_2) + 3\text{Tr} \ln(\Box_1) - 4\text{Tr} \ln(\Box_0)] .$$

Here all Tr-operations are defined in D5. Then, following [21] we identify $B_{MN}$ as $B_{\mu\nu}, C_\mu$ and $B_{MNK}$ as $B_{\mu\nu\rho}, C_{\mu\nu}$ and specify the $x_5$ dependence by setting $e^{imx_5}$. After that, $\Gamma^{(D5)}_2$ and $\Gamma^{(D5)}_3$ exactly reproduce (16) and (18) respectively [31].

IV. QUANTUM EQUIVALENCE

To study the quantum equivalence of the classically equivalent theories one considers the differences of the effective actions (13) and (17) and also (18) and (19). Denoting

$$\Delta \Gamma^{(1)} = \Gamma^{(m)}_2[g_{\mu\nu}] - \Gamma^{(m)}_1[g_{\mu\nu}]$$

(21)

and

$$\Delta \Gamma^{(2)} = \Gamma^{(m)}_3[g_{\mu\nu}] - \Gamma^{(m)}_0[g_{\mu\nu}]$$

(22)

and using the relations (16), (17), (18), and (19) ones get

$$\Delta \Gamma^{(1)} = \frac{i}{2} [\text{Tr} \ln(\Box_2 + m^2) - 2\text{Tr} \ln(\Box_1 + m^2) + 2\text{Tr} \ln(\Box_0 + m^2)] ,$$

(23)
It is easy to see, taking into account the numbers of independent components of the antisymmetric third and second rank tensors, that in flat space the relation (23) is exactly zero. Similarly, the relation (24) is also zero. Now we will study the structure of the relations (23) and (24) in curved space and prove, using the definition of functional determinants in terms of generalized zeta-function, that they are still zero. This means that the above classically equivalent theories are quantum equivalent. Before presenting a proof, we point out that the relations (23), (24) contain mass m and, therefore, a proof of quantum equivalence should include some new issues in comparison with the corresponding massless cases.

It is convenient to consider the fields \( \phi, A_\mu, B_2 \) and \( B_3 \) as the corresponding p-forms \((p = 0, 1, 2, 3)\). In Euclidean formulation, the operators \( \Box \) become the corresponding Laplacians, acting on p-forms (see the details in [22], [23]). We define the effective actions in terms of generalized zeta-functions \( \zeta_p(s, m) \) associated with the operators \( -\Box_p + m^2 \)

\[
\zeta_p(s, m) = \sum_{\lambda_i \neq 0} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} e^{-tm^2} \text{Tr}(e^{t\Box_p} - P_p) ,
\]

where \( P_p \) is the projector onto the space of the zero modes of the operator \( \Box_p \). According to this definition the zeta-function is analytic at \( s = 0 \) together with its derivative. In these terms, the effective action associated with the operator \( (-\Box_p + m^2) \) is given by

\[
\ln\text{Det}(-\Box_p + m^2) = -(\zeta'_1(0, m) + \ln(\mu^2)\zeta_0(0, m)) .
\]

Here \( \mu \) is an arbitrary mass scale parameter. By definition, the expression (26) is finite for any \( p \). Using the relation (20) we rewrite (23) and (24) in the form

\[
\Delta\Gamma^{(1)} = (\zeta'_2(0, m) - 2\zeta'_1(0, m) + 2\zeta'_0(0, m)) + \ln(\mu^2)(\zeta_2(0, m) - 2\zeta_1(0, m) + 2\zeta_0(0, m))
\]

and

\[
\Delta\Gamma^{(2)} = (\zeta'_3(0, m) - 2\zeta'_2(0, m) + \zeta'_0(0, m) - 2\zeta'_1(0, m)) + \ln(\mu^2)(\zeta_3(0, m) - 2\zeta_2(0, m) + \zeta_1(0, m) - 2\zeta_0(0, m)) .
\]

Taking into account the Hodge duality between p-form and \((4 - p)\)-form and the corresponding properties of the operators \( \Box_p \), where \( p = 0, 1, 2, 3, 4 \), ones get \( \zeta_p(s, m) = \zeta_{(4-p)}(s, m) \) for any \( m \). Then it is evident, that \( \Delta\Gamma^{(2)} = -\Delta\Gamma^{(1)} \).

Hence, it is sufficient to study only \( \Delta\Gamma^{(1)} \) (27).

Let us apply the evident identity

\[
\sum_{p=0}^{4} (-1)^p p \zeta_p(s, m) = 2(\zeta_2(s, m) - 2\zeta_1(s, m) + 2\zeta_0(s, m))
\]

to the expression (27). Then one obtains

\[
\Delta\Gamma^{(1)} = \frac{1}{2} \sum_{p=0}^{4} (-1)^p p \zeta_p'(0, m) + \ln(\mu^2) \sum_{p=0}^{4} (-1)^p p \zeta_p(0, m) .
\]

Expansion of the zeta-function at non-zero mass [25] in power series in mass allows us to get the mass-dependent zeta-function as a power series in massless zeta-functions in the form

\[
\zeta_p(s, m) = \sum_{n=0}^{\infty} \frac{(-m^2)^n \Gamma(n + s)}{n! \Gamma(s)} \zeta_p(s + n, 0) .
\]

It allows us to write

\[
\sum_{p=0}^{4} (-1)^p p \zeta_p(s, m) = \sum_{n=0}^{\infty} \frac{(-m^2)^n \Gamma(n + s)}{n! \Gamma(s)} \sum_{p=0}^{4} (-1)^p p \zeta_p(s + n, 0) .
\]
However, the zeta-functions at zero mass \( \zeta_p(s + n) = \zeta_p(s + n, 0) \) satisfy for any \( n \) the identity \(^{26}\) (see also \(^{23}\) for a review)

\[
\sum_{p=0}^{4} (-1)^p \zeta_p(s) = 0 .
\]

(34)

The equations \(^{33}\) and \(^{34}\) mean that the analogous identity is also valid for mass-dependent zeta-functions

\[
\sum_{p=0}^{4} (-1)^p \zeta_p(s, m) = 0 .
\]

(35)

Using this relation in \(^{31}\) ones get

\[
\Delta \Gamma^{(1)} = 0 .
\]

(36)

The last relation means that the effective action of a massive second rank antisymmetric tensor field coincides with the effective action of a massive vector field. That is these theories are quantum equivalent. Taking into account \(^{29}\) one concludes that the effective action of a massive third rank antisymmetric tensor field coincides with the effective action of a massive scalar field. That is these theories are also quantum equivalent.

Usually, the (one-loop) effective actions, associated with differential operators, are defined in quantum field theory with help of the Schwinger-De Wit representation. It is equivalent to using another zeta-function, which includes the zero modes. Such zeta-function is defined by \(^{25}\) where now the term with the projector \( P_p \) is omitted (see e.g. \(^{24}\), \(^{25}\)). Then the relation \(^{29}\) still holds. However, the identity \(^{34}\) is not valid now and, hence, the identity \(^{35}\) is also not valid. Therefore, the final relation \(^{36}\) would be violated. First of all, we point out that the relation \(^{29}\) will still be valid if we define the effective actions on the base of the zeta-function including the zero modes. Relation \(^{23}\) can be identically rewritten in terms of the zeta-functions including the zero modes. The difference of the two zeta-functions is given by \((m^2)^s \text{Tr} P_p\), which is exactly calculated in terms of De Wit coefficients at the coincident limit, associated with the operators \( \Box_p \). As a result one obtains

\[
\Delta \tilde{\Gamma}^{(1)} = \ln\left(\frac{\mu^2}{m^2}\right) \left\{ \int d^4 x \sqrt{-g(x)} \left[ b_2 - m^2 b_1 + \frac{m^4}{2} b_0 \right] \right\},
\]

(37)

where \( \tilde{\Gamma} \) means the effective action defined in terms of zeta-function, including the zero modes. Here

\[
b_n = a_n^{(2)} - 2a_n^{(1)} + 2a_n^{(0)}, \quad n = 0, 1, 2 .
\]

(38)

The De Witt coefficients at coincident limit \( a_n^{(p)}, n = 0, 1, 2; p = 0, 1, 2 \) are known in literature (see e.g. \(^{27}\), \(^{17}\) and reference therein). Using the results of these calculations ones get: (i). The coefficients \( b_0 = 0 \) owing to the elementary balance of degrees of freedom for antisymmetric tensor fields. (ii). The coefficients \( b_1 = 0 \). This means that the difference of the effective action for classically equivalent massive theories is mass independent. (iii). Taking into account the explicit expressions for the coefficients \( a_2^{(p)}, p = 0, 1, 2 \), ones obtain

\[
b_2 = \frac{1}{2} [R^2_{\mu \nu, \lambda \rho} - 4 R^2_{\mu \nu} + R^2] .
\]

(39)

As a result we get

\[
\Delta \tilde{\Gamma}^{(1)} = \ln\left(\frac{\mu^2}{m^2}\right) \chi,
\]

(40)

where \( \chi \) is the Gauss-Bonnet topological invariant. The effective actions, defined in terms of the zeta-function \(^{25}\) and in terms of the zeta-function with the zero modes are equal to each other up to the topological invariant. The corresponding energy-momentum tensors coincide. If we define a quantum equivalence of the theories as equality of their currents (energy-momentum tensors in the given case), the definitions of the effective actions both in terms the zeta-function \(^{25}\) and in terms of the zeta-function with zero modes, lead to the same conclusion on the quantum equivalence.
V. SUMMARY

We have studied the structure of the effective actions in massive second rank and third rank antisymmetric tensor field models in four-dimensional curved space-time. The effective actions are defined in terms of the generalized zeta-function associated with the corresponding d’Alembertians (25). We have proven that the effective action for a massive second rank antisymmetric tensor field is exactly equal to the effective action for a massive vector field. Similarly, we have shown that the effective action for a massive third rank antisymmetric tensor field is exactly equal to the effective action for a massive scalar field minimally coupled to gravity. The proof is essentially based on the identity (35) for mass-dependent zeta-functions (25). Our general statement is analogous to the one of [17] although our method is quite different.

We would like to point out that the zeta-function \( \zeta_p(s, m) \) (25) in the basic identity (35), does not contain the zero modes of the operators \( \Box_p \) unlike another zeta-function, which is used often for definitions of the effective actions. Two these definitions of the zeta-function differ by the quantity \( (m^2)^s \mathrm{Tr}P_p \). We have shown that if the effective action is defined in terms of the zeta-function with the zero modes the difference between the effective actions of classically equivalent theories under consideration is the Gauss-Bonnet topological invariant. This means that the effective energy momentum tensors for these two theories coincide. Treating the quantum equivalence of two theories as the equality of their effective energy-momentum tensors, we conclude that the given classically equivalent theories are quantum equivalent both if the effective action is defined in terms of the zeta-function (25) and if it is defined in terms of the zeta-function with the zero modes.

Acknowledgments

We are grateful to F. Bastianelli for bringing to results of the paper [17] to our attention and to E. Elizalde for useful comments. The research was partially supported by LRSS grant, project No. 2553.2008.2. Work of I.L.B and N.G.P was partially sponsored by RFBR grant, project No. 06-02-16346, No. 08-02-00334-a and INTAS grant, project No. 05-100008-7928. Work of I.L.B was also partially supported by joint RFBR-Ukraine grant, project No. 08-02-90490.

[1] J. Louis and A. Micu, Nucl. Phys. B 635, 395 (2002).
[2] R. D’Auria, L. Sommovigo and S. Vaula, JHEP 0411, 028 (2004).
[3] R. D’Auria and S. Ferrara, Phys. Lett. B 606, 211 (2005).
[4] J. Louis and W. Schulgin, Fortsch. Phys. 53, 235 (2005).
[5] S.M. Kuzenko, JHEP 0501, 041 (2005).
[6] V.I. Ogievetsky and I.V. Polyubarinov, Sov. J. Nucl. Phys. 4, 156 (1967); M. Kalb and P. Ramond, Phys. Rev. D 9, 2273 (1974); Y. Nambu, Phys. Rep. 23, 250 (1976).
[7] E. Harikumar and M. Sivakumar, Phys. Rev. D 57, 3794 (1998).
[8] A.S. Schwarz, Lett. Math. Phys. 2, 247 (1978); Commun. Math. Phys. 67, 1 (1979).
[9] A.S. Schwarz and Yu.S. Tyupkin, Nucl. Phys. B 242, 436 (1984).
[10] W. Siegel, Phys. Lett. B 93, 170 (1980).
[11] E. Sezgin and P. van Nieuwenhuizen, Phys.Lett. B 94, 179 (1980).
[12] T. Kimura, Progr. Theor. Phys. 65, 338 (1981); H. Hata, T. Kugo and N. Ohta, Nucl. Phys. B 178, 527 (1981).
[13] I.L. Buchbinder and S.M. Kuzenko, Nucl. Phys. B 308, 162 (1988).
[14] M.J. Duff and P. van Nieuwenhuizen, Phys. Lett. B 94, 179 (1980).
[15] M.T. Grisaru, N.K. Nielsen, W. Siegel and D. Zanon, Nucl. Phys. B 247, 157 (1984).
[16] E.S. Fradkin and A.A. Tseytlin, Annals Phys. 162, 31 (1985).
[17] F. Bastianelli, P. Benincasa and S. Giombi, JHEP 0504, 010 (2005); JHEP 0510, 114 (2005).
[18] I.L. Buchbinder, G. de Berredo-Peixoto and I.L. Shapiro, Phys. Lett. B 649, 454 (2007).
[19] I.A. Batalin and G.A. Vilkovisky, Phys. Rev. D 28 (1983) 2567; Erratum, Phys. Rev. D 30, 508 (1984).
[20] I.L. Buchbinder and S.M. Kuzenko. Ideas and Methods of Supersymmetry and Supergravity. IOP Publishing Ltd 1995, 1998.
[21] J. Scherk and J.H. Schwarz, Phys. Lett. B 82, 60 (1979); Nucl. Phys. B 153, 61 (1979).
[22] M. Nakahara, Geometry, Topology and Physics, IOP Publ., 1990.
[23] S. Rosenberg, Laplacian on Riemannian Manifold, Cambridge Univ. Press, 1998.
[24] S.W. Hawking, Commun. Math. Phys. 55, 133 (1977).
[25] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, Zeta regularization techniques with applications, Singapore: World Scientific (1994).
[26] D. Ray and I. Singer, Advan. Math. 7, 145 (1971); Ann. Math, 98, 154 (1973).

[27] B S DeWitt, Dynamical Theory of Groups and Fields. Gordon and Breach NY 1965. B.S. DeWitt, The Global Approach to Quantum Field Theory, The International Series of Monographs on Physics, 114 (2003), Oxford University Press, Oxford, 2003.

[28] See discussion of the classical equivalence in [27] and references herein.

[29] See discussion of this point in [18].

[30] The authors of [19] emphasized that "To apply the methods of this paper to the simple linear theories .... is like cracking nuts by a sledgehammer."

[31] Reduction to D4 yields to \( \text{Tr} \ln(\Box_3) = \text{Tr} \ln(\Box_3 + m^2) + \text{Tr} \ln(\Box_2 + m^2) \), \( \text{Tr} \ln(\Box_2) = \text{Tr} \ln(\Box_2 + m^2) + \text{Tr} \ln(\Box_1 + m^2) \) and \( \text{Tr} \ln(\Box_1) = \text{Tr} \ln(\Box_1 + m^2) + \text{Tr} \ln(\Box_0 + m^2) \). Here left hand sides correspond to D5 and right hand sides to D4.

[32] See the various applications of zeta-function techniques in [25].

[33] See proof of this identity for \( m = 0 \) in [24].