New phases of thermal SYM and LST from Kaluza-Klein black holes

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We review the recently found map that takes any static and neutral Kaluza-Klein black hole, i.e. any static and neutral black hole on Minkowski-space times a circle \(M^d \times S^1\), and maps it to a corresponding solution for a non- and near-extremal brane on a circle. This gives a precise connection between phases of Kaluza-Klein black holes and the thermodynamic behavior of the non-gravitational theories dual to near-extremal branes on a circle. In particular, for the thermodynamics of strongly-coupled supersymmetric Yang-Mills theories on a circle we predict the existence of a new non-uniform phase and find new information about the localized phase. We also find evidence for the existence of a new stable phase of \((2,0)\) Little String Theory in the canonical ensemble for temperatures above its Hagedorn temperature.

1 Introduction

It has been established in recent years that near-extremal branes in string theory and M-theory provide a link between black hole phenomena in gravity and the thermal physics of non-gravitational theories. Among the most prominent examples is the duality between near-extremal D3-branes and \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory [1]. More generally, for all the supersymmetric branes in string/M-theory one has dualities between the near-extremal limit of the brane and a non-gravitational theory [2,3].

In this talk we briefly review the recently found map [4] from static and neutral Kaluza-Klein black holes to near-extremal branes on a transverse circle (see also [5,6] for related work). This gives a precise connection between phases of Kaluza-Klein black holes and the thermodynamic behavior of certain non-gravitational theories. The non-gravitational theories include \((p+1)\)-dimensional supersymmetric Yang-Mills theories with 16 supercharges compactified on a circle, the uncompactified \((2+1)\)-dimensional supersymmetric Yang-Mills theory with 16 supercharges, and \((2,0)\) Little String Theory.

2 Review of phases of Kaluza-Klein black holes

We start with a short review of the current knowledge on phases of static and neutral Kaluza-Klein black holes (see [8,9] for recent reviews). These are static solutions of the vacuum Einstein equations (i.e. pure gravity) that have an event horizon, and that asymptote to Minkowski-space times a circle \(M^d \times S^1\), i.e. Kaluza-Klein space, with \(d \geq 4\). The asymptotic behavior of these solutions is characterized by the mass
$M$ and the tension $T$ associated to the compact direction \cite{10–12}. Together with the circumference $L$ of the circle at infinity, we can construct two dimensionless parameters which we take as the reduced mass $\mu = 16\pi G_N/L^{d-2}$ and relative tension $n = LT/M$. The possible phases of Kaluza-Klein black holes can then be drawn in a $(\mu, n)$ phase diagram, where all physically sensible solutions fall in the range $\mu > 0$ and $0 \leq n \leq d - 2$ \cite{10}.

The phase diagram appears to be divided in two separate regions \cite{8,13}: (i) The region $0 \leq n \leq 1/(d-2)$, which contains solutions without Kaluza-Klein bubbles. The solutions in this region have a local $SO(d-1)$ symmetry, and hence two types of event horizon topologies, namely $S^{d-1}$ and $S^{d-2} \times S^1$ for the black hole and string on a cylinder respectively. (ii) The region $1/(d-2) < n \leq d-2$ contains solutions with Kaluza-Klein bubbles, which is the subject of \cite{13}.

In this talk, we focus on the phases in region (i) and their mapping to phases of non- and near-extremal branes. The mapping of the region (ii) into phases of non- and near-extremal branes will be considered in \cite{14}.

At present, three branches are known for $0 \leq n \leq 1/(d-2)$:

- **The uniform black string branch.** The metric is that of a Schwarzschild black hole in $d$ dimensions times a compact $S^1$ direction. This branch has $n = 1/(d-2)$ and exists for all $\mu$. It is classically stable for $\mu > \mu_{GL}$ and classically unstable for $\mu < \mu_{GL}$ where the Gregory-Laflamme mass $\mu_{GL}$ can be obtained numerically for each dimension $d$ \cite{15,16}. See also \cite{17,18} for analysis of the $d$-dependence of $\mu_{GL}$.

- **The non-uniform black string branch.** This branch was discovered in \cite{19,20}. It starts at $\mu = \mu_{GL}$ with $n = 1/(d-2)$ in the uniform string branch. The approximate behavior near the Gregory-Laflamme point $\mu = \mu_{GL}$ is studied in \cite{17,20,21}. For $4 \leq d \leq 9$ the results are that the branch moves away from the Gregory-Laflamme point according to

$$n(\mu) = \frac{1}{d-2} - \gamma(\mu - \mu_{GL}) + \mathcal{O}((\mu - \mu_{GL})^2), \quad 0 \leq \mu - \mu_{GL} \ll 1. \quad (1)$$

where $\gamma > 0$, so that the branch has decreasing $n$ and increasing $\mu$. As shown in \cite{10} this means that the uniform string branch has higher entropy than the non-uniform string branch for a given mass. (See for example the tables in \cite{4} for the numerical values of $\mu_{GL}$ and $\gamma$). For $d = 5$ a large piece of the branch was found numerically in \cite{21} thus providing detailed knowledge of the behavior of the branch away from $\mu = \mu_{GL}$.

- **The black hole on cylinder branch.** This branch has been studied analytically in \cite{11,22–25} (see also \cite{10}) and numerically for $d = 4$ in \cite{26,27} and for $d = 5$ in \cite{27,28}. The branch starts in $(\mu, n) = (0, 0)$ and then has increasing $n$ and $\mu$. In the limit of vanishing mass (or, equivalently, very large $L$) this branch approaches the solution of a Schwarzschild black hole in $d + 1$ dimensions. The first correction to the Schwarzschild black hole metric has been found analytically in \cite{23,25}. In particular, the branch starts off as \cite{23}

$$n(\mu) = \lambda_d \mu + \mathcal{O}(\mu^2), \quad \lambda_d = \frac{(d-2)d(\zeta(d-2) - 1)}{2(d-1)d^{-1}} \quad (2)$$

For $d = 4$, the second order correction to the metric has also been studied \cite{29}. Moreover, recent numerical analysis \cite{27} for $d = 5$ shows that the black hole branch meets the non-uniform black string branch, indicating a topology changing transition point \cite{30,31}.

Some further useful facts are the existence of a Smarr formula \cite{10,11}, $\nu_s = \frac{d^2 - d - 1}{2} \mu_s$, where $t$, $s$ are the rescaled temperature $t = LT$ and entropy $s = \frac{16\pi G_N}{LT}$ respectively. In particular, together with the first law of thermodynamics $\delta t = t \delta s$, this means that, given a curve $n(\mu)$, the entire thermodynamics can be obtained. We also note that, as originally proposed in \cite{22,7}, there exists a consistent ansatz \cite{24,32} that describes the solutions with $0 \leq n \leq 1/(d-2)$. Finally we mention that for any solution in this ansatz one can generate an infinite number of copies \cite{24,33}.

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3 Phases of non- and near-extremal branes

Any Kaluza-Klein black hole in \( d+1 \) dimensions (\( 4 \leq d \leq 9 \)), can be mapped [4] to a corresponding brane solution of Type IIA/B String Theory and M-theory, following the method originally conceived in [34]. In particular, we obtain in this way a non-extremal \( p \)-brane on a circle. These are thermal excitations of singly-charged extremal \( 1/2 \) BPS branes in String/M-theory with transverse space \( \mathbb{R}^{d-1} \times S^1 \), i.e. with a transverse circle. A precise definition of this class of branes, including a detailed discussion of the physical parameters, thermodynamical relations and other properties is given in [4]. These branes are characterized by three independent dimensionless quantities, the dimensionless mass \( \mu \), the relative tension \( \tilde{n} \) and the charge \( q \).

To obtain the map from Kaluza-Klein black holes on \( M^d \times S^1 \) to non-extremal \( p \)-branes on a circle one first uplifts the neutral solution to 11 dimensions, then performs a Lorentz boost and subsequently uses a sequence of U-dualities. The relation between \( d \) and \( p \) is \( d + p + 1 = D \) where \( D = 10 \) or 11 for String theory or M-theory. Since the focus in this talk is on near-extremal branes, we refer to [4] for the resulting expression of the background obtained in this way as well as the associated map for the physical quantities.

One can apply the map in particular to the ansatz describing the class of Kaluza-Klein black holes with a local \( SO(d-1) \) symmetry. If the neutral solution is a black hole (black string) then the topology of the horizon of the non-extremal \( p \)-brane solution is \( \mathbb{R}^p \times S^{d-1} \). For this class of solutions the map was already discovered in [22] at the level of equations of motion. The boost/U-duality derivation thus provides us with a physical understanding of this correspondence. Some of these maps have also been examined in [5, 6].

We now wish to take the near-extremal limit of any non-extremal brane on a circle, which gives us a corresponding near-extremal brane on a circle. This is a non-extremal brane with infinitesimally small temperature, or, equivalently, a non-extremal brane with infinitely high charge. More precisely, we want to take a near-extremal limit such that the size of the circle has the same scale as the excitations of the energy above extremality. This is because we want to keep the non-trivial physics related to the presence of the circle.

For a non-extremal \( p \)-brane with volume \( V_p \), circumference \( L \) and rescaled charge \( q \), the near-extremal limit is then [4]

\[
q \to \infty, \quad L \to 0, \quad g \equiv \frac{16\pi G_D}{V_p L^{d-2}} \quad \text{fixed}, \quad \ell \equiv L\sqrt{q} \quad \text{fixed}, \quad x^a \quad \text{fixed}.
\]

(3)

where \( x^a \) are dimensionless transverse coordinates (obtained by scaling with \( L \)). A near-extremal brane on a circle asympotes at infinity to a non-flat background which is the near-horizon limit of the solution of extremal branes on a circle. The latter is of course exactly known using the superposition principle of BPS branes. This space is also the one that we use as the reference space when computing the physical quantities, which are the energy [35] \( E \) above extremality and the tension [12] \( \tilde{T} \) in the circle direction.

In this way we have, in analogy with the \((\mu, n)\) phase diagram of neutral Kaluza-Klein black holes, a two-dimensional phase diagram for near-extremal branes. The quantities we use are the rescaled energy \( \epsilon = gE \) and tension \( r = 2\pi\tilde{T}/E \), in terms of which we have the \((\epsilon, r)\) phase diagram. It is also useful to define the dimensionless versions of the temperature \( t = l\tilde{T} \) and entropy \( \delta = \frac{q}{2}S \) since \( l \) and \( g \) in (3) have dimension length. In terms of these the near-extremal Smarr formula takes the form \( \delta\epsilon = 2\delta - 2\epsilon \) and the first law of thermodynamics is \( \delta\epsilon = \delta\delta \). As a consequence, knowing a curve \( r(\epsilon) \) in the near-extremal phase diagram determines the entire thermodynamics.

We can apply now the near-extremal limit to the non-extremal branes obtained via boost and U-duality from neutral Kaluza-Klein black holes. The resulting background is of the form [4]

\[
ds^2 = \hat{H}^{\frac{d-2}{2}} \left( -U dt^2 + \sum_{i=1}^p (du^i)^2 + \hat{H} V_{ab} dx^a dx^b \right),
\]

(4)
where \( \hat{H} \propto 1 - U \) and \( U \) and \( V_{ab} \) are functions determining the metric of the Kaluza-Klein black hole one started with. The corresponding map from the \((\mu, n)\) phase diagram of Kaluza-Klein black holes to the \((\epsilon, r)\) phase diagram of near-extremal branes on a circle, then takes the simple form

\[
\epsilon = \frac{d + n}{2(d - 1)} \mu, \quad r = 2 \frac{(d - 1)n}{d + n}, \quad \hat{t} = t \sqrt{\frac{\mu}{s}}, \quad \hat{s} = \frac{s}{\sqrt{ts}}
\]

We also note the expression

\[
p = -\hat{t} = \frac{d - 4 - 2r}{d} \epsilon
\]

for the dimensionless pressure in the world-volume directions of the near-extremal brane. Here \( \hat{t} \) is the rescaled free energy. In order to make physically sense from the dual field theory point of view, the pressure should be positive. Using (7) this implies the interesting bound \( r \leq (d - 4)/2 \) or, equivalently, \( n \leq (d - 4)/3 \), which is further examined in [4].

The three phases of Kaluza-Klein black holes reviewed in Section 2 thus immediately imply the existence of three corresponding phases of non- and near-extremal branes on a circle:

- **Uniform phase.** This is obtained by applying the map to the uniform black string branch, thereby generating a non- or near-extremal \( p \)-brane smeared on a circle. This solution, which is really a \( (p + 1) \)-brane since it is uniformly distributed on the transverse circle, is of course well known.

- **Non-uniform phase.** By applying the map to the non-uniform black string branch we find a new branch of solutions for non- and near-extremal branes on a circle that are non-uniformly distributed on the circle. The physics of the neutral non-uniform string branch near \( \mu = \mu_{GL} \) is captured by the formula (1). Using the map from the neutral case to the near-extremal case one finds that the new non-extremal branch emerges out of the uniform phase (with \( \bar{n} = 1/(d - 2) \)) at some critical mass \( \bar{m}_c(\mu, q) \), which has the natural interpretation as being a Gregory-Laflamme critical mass of non-extremal branes uniformly distributed on a circle.

For the near-extremal case we find likewise from (1) and (6) that the Gregory-Laflamme point is mapped to the point \((\epsilon_c, 2/(d - 1))\) in the \((\epsilon, r)\) phase diagram with critical energy \( \epsilon_c = \frac{d - 1}{2(d - 2)} \mu_{GL} \). Moreover, the first part of the non-uniform near-extremal branch is described by

\[
r(\epsilon) = \frac{2}{d - 1} - \hat{\gamma}(\epsilon - \epsilon_c) + \mathcal{O}((\epsilon - \epsilon_c)^2), \quad 0 \leq \epsilon - \epsilon_c \ll 1,
\]

with \( \hat{\gamma} \) a \( d \)-dependent constant determined by \( \gamma \) and \( \mu_{GL} \). From (8) all other physical information, such as the entropy and free energy near the critical point can be derived [4].

The existence of this new non-uniform phase suggests that near-extremal branes smeared on a circle have a critical energy below which they are classically unstable (see also [5, 6]). The specific heat of the uniform near-extremal branch is, however, positive. It would be interesting to study this further in connection with the Gubser-Mitra conjecture\(^1\) [36–39] (see also the recent paper [40]).

- **Localized phase.** Finally, we can apply the map to the neutral black hole branch for which the first correction to the metric and the thermodynamics is known, generating non- and near-extremal \( p \)-branes localized on a circle. This phase is not new, as it is expected on general physical grounds. However, we are now able to compute the first correction to the metric and thermodynamics of this background.

\(^1\) For non-extremal branes that are near extremality (without the near-extremal limit taken) the Gubser-Mitra conjecture is satisfied since, although the specific heat is positive, the quantity \((\partial \nu / \partial Q)_T\) is negative and hence the brane is not thermodynamically stable.
For a given charge $q$, the branch of non-extremal branes localized on a circle starts in the extremal point $(\bar{\mu}, \bar{n}) = (q, 0)$ and goes up with a slope $2(d - 1)\lambda d/d$. Thus, a non-extremal $p$-brane localized on a circle becomes point-like in the extremal limit, i.e. for small temperatures.

For the near-extremal case we find from (2) and (6) the leading behavior of the localized branch as

$$r(\epsilon) = 4(d - 1)^2 \frac{\lambda d}{d^2} \epsilon + \mathcal{O}(\epsilon^2),$$

from which the entropy and free energy can be computed [4].

For the particular case of the M5-brane on a circle we can go even further and use the numerically obtained $d = 5$ non-uniform branch of Wiseman [21] to obtain the corresponding near-extremal non-uniform phase. Similarly, the new numerical results of [27] can be used to also determine the complete near-extremal localized phase. This is currently in progress.

## 4 Phases of thermal SYM and LST

The results for the non-uniform and localized phase of near-extremal branes are particularly interesting, since they provide us with new information about the dual non-gravitational theories at finite temperature\(^2\). In particular, we have studied [4]:

- The M5-brane on a circle, which is dual to thermal (2,0) Little String Theory (LST).
- The $D(p-1)$-brane on circle, which is dual to thermal $(p+1)$-dimensional supersymmetric Yang-Mills (SYM) theory on $\mathbb{R}^{p-1} \times S^1$.
- The M2-brane on a circle, which is dual to thermal $(2+1)$-dimensional SYM theory on $\mathbb{R}^2$.

In these dual non-gravitational theories, the localized phase corresponds to the low temperature/low energy regime of the dual theory, whereas the uniform phase corresponds to the high temperature/high energy regime of the theory. The non-uniform phase appears instead for intermediate temperatures/energies.

In particular, by translating the results for the thermodynamics of the localized and non-uniform phase in terms of the dual non-gravitational theories we find:

- The first correction to the thermodynamics for the localized phase of the SYM theories. The dimensionless expansion parameter in the localized phase is $\hat{T}/T_0$ with $T_0 = \sqrt{2\pi(\lambda L^{3-p})^{-1/2}}L^{-1}$, with $\lambda$ the ’t Hooft coupling of the gauge theory and $L$ the circumference of the field theory circle.
- A prediction of a new non-uniform phase of the SYM theories including the first correction around the point where the non-uniform phase emanates from the uniform phase. The critical temperature that characterizes the emergence of the non-uniform phase is $\hat{T}_c = \hat{T}_0 \hat{t}_c$, with $\hat{t}_c$ a numerically determined constant that depends on $p$. In [6] evidence for the existence of this non-uniform phase at weak ’t Hooft coupling was presented by considering (1+1)-dimensional SYM on $T^2$. Moreover, it was shown that the critical temperature behaves as $\hat{T}_c \sim 1/\lambda$ as opposed to the predicted strong coupling behavior $\hat{T}_c \sim 1/\sqrt{\lambda}$.
- Using the numerical data of Wiseman [21] for the $d = 5$ non-uniform branch we have numerically computed the corresponding thermodynamics in (2,0) LST. This gives a new stable phase of (2,0) LST in the canonical ensemble, for temperatures above its Hagedorn temperature $\hat{T}_c = T_{hg}$. We furthermore computed the first correction to the thermodynamics in the infrared region, when moving away from the infrared fixed point, which is superconformal (2,0) theory.

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\(^2\) See also [6, 41–46] and references therein.
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