A FORMULA FOR THE LOCAL METRIC PRESSURE

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Abstract. In this note we present a formula for the local metric pressure that generalizes Brin-Katok result for the metric entropy. As an application, we give a straightforward proof of the fact that non-atomic weak-Gibbs invariant probability measures are equilibrium states.

1. Introduction

Let \((X, d)\) be a compact metric space, \(f: X \to X\) a continuous transformation and \(\varphi: X \to \mathbb{R}\) a continuous potential. The topological pressure \(P_{\text{top}}(f, \varphi)\) of \(f\) and \(\varphi\) is a topological invariant that generalizes the notion of topological entropy of \(f\), one denotes by \(h_{\text{top}}(f)\), in the sense that \(P_{\text{top}}(f, \varphi) = h_{\text{top}}(f)\) whenever \(\varphi \equiv 0\). We refer the reader to [3] for precise definitions and properties of these notions. A Borel \(f\)-invariant probability measure \(\mu\) is said to be an equilibrium state for \(f\) and the potential \(\varphi\) if

\[
P_{\mu}(f, \varphi) = \sup \left\{ P_{\nu}(f, \varphi) \mid \nu \text{ is } f\text{-invariant} \right\}
\]

where the \(P_{\nu}(f, \varphi)\) stands for the sum \(h_{\nu}(f) + \int \varphi \, d\nu\) and the supremum is taken over all the Borel \(f\)-invariant probability measures. According to the Variational Principle ([3, Theorem 9.10]), the previous least upper bound coincides with the supremum evaluated on the set of ergodic probability measures, and is equal to \(P_{\text{top}}(f, \varphi)\). We will show how to estimate the metric pressure of any continuous potential, thereby generalizing Brin-Katok formula for the metric entropy [1].

Theorem A. Let \(\mu\) be a Borel non-atomic \(f\)-invariant probability measure. Then there exists a \(\mu\)-integrable map \(x \in X \mapsto P_{\mu}(x, f, \varphi)\) which is \(f\)-invariant and satisfies

\[
\int P_{\mu}(x, f, \varphi) \, d\mu = P_{\mu}(f, \varphi).
\]

If, in addition, \(\mu\) is ergodic, then \(P_{\mu}(x, f, \varphi) = P_{\mu}(f, \varphi)\) for \(\mu\) almost every \(x \in X\).

We remark that, when \(\varphi \equiv 0\), the map \(x \in X \mapsto P_{\mu}(x, f, \varphi)\) is the local entropy as defined by Brin and Katok.

A Borel probability measure \(\mu\) is said to be weak-Gibbs for the dynamical system \(f\) with respect to a potential \(\varphi\) if there exists \(\varepsilon_0 > 0\) and a subset \(\Lambda \subset X\) with full \(\mu\)-measure such that,
for every $0 < \varepsilon < \varepsilon_0$ and every $x \in \Lambda$, there is a sequence of positive constants $(\delta_n(\varepsilon, x))_{n \in \mathbb{N}}$ satisfying
\[
\lim_{n \to +\infty} \frac{\log \delta_n(\varepsilon, x)}{n} = 0
\]
and, for every $n \in \mathbb{N},$
\[
\delta_n(\varepsilon, x)^{-1} \leq \frac{\mu(B_n^f(x, \varepsilon))}{\exp\left(-P_{\text{top}}(\varphi, f) n + S_n^f \varphi(x)\right)} \leq \delta_n(\varepsilon, x)
\]
where
\[
B_n^f(x, \varepsilon) = \{y \in X : d(f^i(x), f^i(y)) < \varepsilon, \ \forall 0 \leq i \leq n-1\}
\]
is the $n$th dynamical ball of $f$ at $x$ with radius $\varepsilon$ and $S_n^f \varphi(x)$ stands for the $n$th Birkhoff’s sum $\sum_{i=0}^{n-1} \varphi(f^i(x))$ at $x$ associated to the dynamics $f$ and the fixed potential $\varphi$. We say that a weak-Gibbs measure $\mu$ for $f$ with respect to $\varphi$ is Gibbs if the sequence $(\delta_n(\varepsilon, x))_{n \in \mathbb{N}}$ is independent of $n$ and $x$.

**Corollary I.** Let $f : X \to X$ be a continuous map on a compact metric space $(X, d)$ whose topological entropy $h_{\text{top}}(f)$ is finite and which preserves a Borel non-atomic probability measure $\mu$. Consider a continuous potential $\varphi : X \to \mathbb{R}$. If $\mu$ is a weak-Gibbs measure for $f$ with respect to $\varphi$, then $\mu$ is an equilibrium state for $f$ and $\varphi$.

2. Proof of Theorem A

In this section we will extend Brin-Katok local entropy formula to general continuous potentials (another generalization may be found in [4]). Brin-Katok’s result asserts that, given a compact metric space $X$, a continuous map $f : X \to X$ and Borel non-atomic $f$-invariant probability measure $\mu$, there exists a full $\mu$-measure set $\mathcal{B}K \subset X$ such that:

(a) For every $x \in \mathcal{B}K$,
\[
h_\mu(x, f) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \frac{-\log \mu(B_n^f(x, \varepsilon))}{n} = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \frac{-\log \mu(B_n^f(x, \varepsilon))}{n}
\]
is well defined.

(b) The map $x \in \mathcal{B}K \mapsto h_\mu(x, f)$ is $f$-invariant.

(c) $\int h_\mu(x, f) d\mu = h_\mu(f)$.

Having fixed a continuous potential $\varphi$ whose pressure $P_{\text{top}}(f, \varphi)$ finite, consider the Birkhoff’s sums $S_n^f \varphi$ and, for $x \in X$, define the local pressure of $\varphi$ at $x$ by
\[
P_\mu(x, f, \varphi) = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \frac{-\log \mu(B_n^f(x, \varepsilon)) + S_n^f \varphi(x)}{n} = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \frac{-\log \mu(B_n^f(x, \varepsilon)) + S_n^f \varphi(x)}{n}
\]
if these limits exist and are equal.

**Lemma 2.1.** The following properties are valid for $\mu$:
(a) \( P_\mu(x, f, \varphi) \) is well defined at \( \mu \) almost every \( x \in X \).

(b) The map \( x \mapsto P_\mu(x, f, \varphi) \) is \( f \)-invariant.

(c) \( \int P_\mu(x, f, \varphi) \, d\mu = h_\mu(f) + \int \varphi \, d\mu \).

(d) If, in addition, \( \mu \) is ergodic, then \( P_\mu(x, f, \varphi) = h_\mu(f) + \int \varphi \, d\mu \) at \( \mu \) almost every \( x \in X \).

Proof. Birkhoff’s Theorem provides a full \( \mu \)-measure set \( \mathcal{B}_\varphi \) and an \( f \)-invariant map

\[
x \in \mathcal{B}_\varphi \quad \mapsto \quad \tilde{\varphi}(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x))
\]

satisfying \( \int \tilde{\varphi} \, d\mu = \int \varphi \, d\mu \). Therefore, for every \( x \in \mathcal{B}_\mathcal{K} \cap \mathcal{B}_\varphi \), we have

\[
\liminf_{n \to +\infty} \frac{- \log \mu(B_f^i(x, \varepsilon))}{n} + \tilde{\varphi}(x) = \liminf_{n \to +\infty} \frac{- \log \mu(B_f^i(x, \varepsilon))}{n} + \mathcal{S}_n^f \varphi(x)
\]

\[
\leq \limsup_{n \to +\infty} \frac{- \log \mu(B_f^i(x, \varepsilon))}{n} + \mathcal{S}_n^f \varphi(x) = \limsup_{n \to +\infty} \frac{- \log \mu(B_f^i(x, \varepsilon))}{n} + \tilde{\varphi}(x)
\]

Taking the limit when \( \varepsilon \) goes to 0 at the last inequality, we conclude that

\[
P_\mu(x, f, \varphi) = h_\mu(x, f) + \tilde{\varphi}(x)
\]

exists for every \( x \in \mathcal{B}_\mathcal{K} \cap \mathcal{B}_\varphi \). Items (b), (c) and (d) are immediate after (a).

\[\square\]

3. Proof of Corollary I

Firstly recall that, given a compact metric space \( X \) and a continuous map \( f : X \to X \), the pressure map \( P_{\text{top}}(f, \cdot) : C^0(X, \mathbb{R}) \to \mathbb{R} \cup \{+\infty\} \), defined on the space \( C^0(X, \mathbb{R}) \) of continuous potentials, is either finite valued or constantly \( +\infty \) (cf. [3, §9.2]).

Consider a Gibbs measure \( \mu \) for the dynamics \( f \) and a continuous potential \( \varphi \), and gather the corresponding \( \varepsilon_0, \Lambda \) and \( \left( \delta_n(x, \varepsilon) \right)_{n \in \mathbb{N}} \) satisfying equations (1) and (2) for every \( x \in \Lambda \) and every \( n \in \mathbb{N} \). As we are assuming that \( h_{\text{top}}(f) < +\infty \), we know that \( P_{\text{top}}(f, \varphi) \) is finite. Rewriting (2), we obtain, for every \( x \in \mathcal{B}_\mathcal{K} \cap \mathcal{B}_\varphi \cap \Lambda \),

\[
P_{\text{top}}(f, \varphi) - \frac{\log \delta_n(x, \varepsilon)}{n} \leq - \frac{\log \mu(B_f^i(x, \varepsilon)) + \mathcal{S}_n^f \varphi(x)}{n} \leq P_{\text{top}}(\varphi, f) - \frac{\log \delta_n(x, \varepsilon)^{-1}}{n}.
\]

Taking \( \limsup_{n \to +\infty} \) (or \( \liminf_{n \to +\infty} \)) and afterwards the limit as \( \varepsilon \) goes to 0, we get

\[
P_\mu(x, f, \varphi) = P_{\text{top}}(f, \varphi) \quad \text{at } \mu - a.e. \ x \in X.
\]

Thus, applying Lemma 2.1, we conclude that

\[
h_\mu(f) + \int \varphi \, d\mu = \int P_\mu(x, f, \varphi) \, d\mu = \int P_{\text{top}}(f, \varphi) \, d\mu = P_{\text{top}}(f, \varphi).
\]

Therefore \( \mu \) is an equilibrium state for \( f \) with respect to \( \varphi \).
4. Open questions

Is a Gibbs measure for a dynamical system $f$ with respect to a potential $\varphi$ always $f$-invariant? We may also wonder whether the existence of a Gibbs measure for $f$ and $\varphi$ prompts the existence of an equilibrium state for $\varphi$. Or we may ask under what additional conditions, other than $f$-invariance, is a Gibbs measure for $f$ and $\varphi$ an equilibrium state for these dynamics and potential, or else an equilibrium state for another natural potential somehow related to $\varphi$. For instance, under a stronger definition of the Gibbs property and assuming that $f$ is a homeomorphism satisfying expansiveness and specification, the answer is positive (cf. [2]). As far as we know, these are still open questions.

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