Amenability and weak amenability of the Fourier algebra

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Abstract

Let $G$ be a locally compact group. We show that its Fourier algebra $A(G)$ is amenable if and only if $G$ has an abelian subgroup of finite index, and that its Fourier–Stieltjes algebra $B(G)$ is amenable if and only if $G$ has a compact, abelian subgroup of finite index. We then show that $A(G)$ is weakly amenable if the component of the identity of $G$ is abelian, and we prove some partial results towards the converse.

Keywords: amenability, weak amenability, locally compact groups, coset ring, piecewise affine maps, Banach algebras, Fourier algebra, Fourier–Stieltjes algebra, operator spaces, completely bounded maps.

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Introduction

In [Joh 1], B. E. Johnson initiated the theory of amenable Banach algebras. He proved that the amenable locally compact groups $G$ can be characterized through a cohomological triviality condition for the group algebra $L^1(G)$. This triviality condition can be formulated for any Banach algebra and is used to define the class of amenable Banach algebras.

The Fourier algebra $A(G)$ of a locally compact group $G$ was introduced by P. Eymard in [Eym]. Since an amenable Banach algebra always has a bounded approximate identity, Leptin’s theorem ([Lep]) immediately yields that $A(G)$ can be amenable only for amenable $G$.

The first to systematically investigate which locally compact groups have amenable Fourier algebras was Johnson in [Joh 3]. For any locally compact group $G$, let $\hat{G}$ denote its dual object, i.e. the collection of all equivalence classes of irreducible unitary representations of $G$. For $\pi \in \hat{G}$, let $d_\pi$ denote its degree, i.e. the dimension of the corresponding Hilbert space. For compact $G$, Johnson showed: If $\sup\{d_\pi : \pi \in \hat{G}\} < \infty$, then $A(G)$ is amenable, whereas, for infinite $G$ such that $\{\pi \in \hat{G} : d_\pi = n\}$ is finite for each $n \in \mathbb{N}$, the

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Fourier algebra cannot be amenable. Hence, for example, $A(\text{SO}(3))$ is not amenable; in fact, it is not even weakly amenable (\cite{Joh}, Corollary 7.3).

Soon after Johnson, A. T.-M. Lau, R. J. Loy, and G. A. Willis (\cite{L-L-W}) — and, independently, the first-named author in \cite{For} — extended one of Johnson’s results by showing that any locally compact group $G$ having an abelian subgroup of finite index has an amenable Fourier algebra (this condition is equivalent to $\sup\{d_\pi : \pi \in \hat{G}\} < \infty$; see \cite{Mod}).

As the predual of the group von Neumann algebra $VN(G)$, the Fourier algebra $A(G)$ has a canonical operator space structure with respect to which multiplication is completely contractive (see \cite{E-R}). Adding operator space overtones to Johnson’s definition of an amenable Banach algebra, Z.-J. Ruan (\cite{Rua}) introduced a variant of amenability — called operator amenability — for which an analog of Johnson’s theorem for $A(G)$ is true: A locally compact group $G$ is amenable if and only if $A(G)$ is operator amenable (\cite{Rua}, Theorem 3.6). There are further results that suggest that in order to deal with amenability and its variants for the Fourier algebra, one has to take the canonical operator space structure into account. For example, N. Spronk proved in \cite{Spr} that $A(G)$ is operator weakly amenable for every locally compact group $G$.

Nevertheless, the question for which locally compact groups $G$ precisely the Fourier algebra $A(G)$ is amenable (\cite{Run}, Problem 14) or weakly amenable remains an intriguing open problem. No locally compact groups besides those with an abelian subgroup of finite index are known to have an amenable Fourier algebra. In view of \cite{Joh} and \cite{Los}, it is plausible to conjecture that these are the only ones. In the present paper, we prove this conjecture.

As far as the weak amenability of the Fourier algebra is concerned, we prove that, if the component of the identity of a locally compact group $G$ is abelian, then $A(G)$ is weakly amenable. We believe, but have been unable to prove, that the converse holds as well. We introduce the formally stronger notion of hereditary weak amenability, and then show for $[\text{SIN}]$-groups, that the hereditary weak amenability of $A(G)$ forces the component of the identity of $G$ to be abelian.

Part of the material in this paper has been taken from the unpublished manuscripts \cite{For} and \cite{Run} which it supersedes.

1 Preliminaries

Let $\mathcal{A}$ be a Banach algebra, and let $E$ be a Banach $\mathcal{A}$-bimodule. A derivation $D: \mathcal{A} \to E$ is a bounded, linear map satisfying

$$D(ab) = a \cdot Db + (Da) \cdot b \quad (a, b \in \mathcal{A}).$$
A derivation $D$ is called inner if there is $x \in E$ such that

$$Da = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

The dual space $E^*$ of $E$ becomes a Banach $\mathcal{A}$-bimodule via

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \quad \text{and} \quad \langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle \quad (a \in \mathcal{A}, \phi \in E^*, x \in E).$$

Modules of this kind are called dual Banach $\mathcal{A}$-bimodules.

We recall the definition of an amenable Banach algebra from ([Joh 1]):

**Definition 1.1** A Banach algebra $\mathcal{A}$ is said to be amenable if every derivation from $\mathcal{A}$ into a dual Banach $\mathcal{A}$-bimodule is inner.

For the theory of amenable Banach algebra, see [Joh 1] and [Run 1].

For many purposes, it is convenient and even necessary not to use Definition 1.1 but equivalent, more intrinsic characterizations. The following is also due to Johnson.

Let $\hat{\otimes}$ denote the projective tensor product of Banach spaces. If $\mathcal{A}$ is a Banach algebra, $\mathcal{A} \hat{\otimes} \mathcal{A}$ becomes a Banach $\mathcal{A}$-bimodule via

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya \quad (a, x, y \in \mathcal{A}).$$

Multiplication induces a bounded linear map $\Delta : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}$. An approximate diagonal for $\mathcal{A}$ is a bounded net $(d_\alpha)_\alpha$ in $\mathcal{A} \hat{\otimes} \mathcal{A}$ such that

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \quad (a \in \mathcal{A})$$

and

$$a \Delta d_\alpha \to a \quad (a \in \mathcal{A}).$$

The existence of an approximate diagonal characterizes the amenable Banach algebras ([Joh 2]):

**Theorem 1.2** The following are equivalent for a Banach algebra $\mathcal{A}$:

(i) $\mathcal{A}$ is amenable.

(ii) $\mathcal{A}$ has an approximate diagonal.

This characterization allows to introduce a quantitative aspect into the notion of an amenable Banach algebra: The amenability constant of a Banach algebra $\mathcal{A}$ is the smallest $C \geq 1$ such that $\mathcal{A}$ has an approximate diagonal bounded by $C$.

We also require another characterization of amenable Banach algebras. For a Banach algebra $\mathcal{A}$, let $\mathcal{A}^{\text{op}}$ denote the same algebra with reversed multiplication. Then $\ker \Delta$ becomes a left ideal in $\mathcal{A} \hat{\otimes} \mathcal{A}^{\text{op}}$.

The following is [Hel, Proposition VII.2.15]:
**Proposition 1.3** The following are equivalent for a Banach algebra $\mathcal{A}$:

(i) $\mathcal{A}$ is amenable.

(ii) $\mathcal{A}$ has a bounded approximate identity, and the left ideal $\ker \Delta$ of $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$ has a bounded right approximate identity.

Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule $E$ is called *symmetric* if

$$a \cdot x = x \cdot a \quad (a \in \mathcal{A}, \, x \in E).$$

The following definition is from [B–C–D]:

**Definition 1.4** A commutative Banach algebra $\mathcal{A}$ is said to be *weakly amenable* if there is no non-zero derivation from $\mathcal{A}$ into a symmetric Banach $\mathcal{A}$-bimodule.

It is easy to see that weak amenability is indeed weaker than amenability.

Definition [LJ] is of little use for non-commutative $\mathcal{A}$. Even though it is possible to extend the notion of weak amenability meaningfully to non-commutative Banach algebras, we content ourselves with this definition: all the algebras we consider are commutative. The hereditary properties of weak amenability for commutative Banach algebras — which parallel those of amenability — are discussed in [Gre].

The algebras we are concerned with in this paper were introduced in [Eym]. Let $G$ be a locally compact group. Then the *Fourier algebra* $A(G)$ of $G$ is the collection of all functions

$$G \to \mathbb{C}, \quad x \mapsto \langle \lambda(x)\xi, \eta \rangle,$$

where $\xi, \eta \in L^2(G)$ and $\lambda$ is the left regular representation of $G$ on $L^2(G)$. The *Fourier–Stieltjes algebra* $B(G)$ of $G$ consists of all functions

$$G \to \mathbb{C}, \quad x \mapsto \langle \pi(x)\xi, \eta \rangle,$$

where $\pi$ is some unitary representation of $G$ on a Hilbert space $\mathcal{H}$ — always presumed to be continuous with respect to the given topology on $G$ and the weak operator topology on $B(\mathcal{H})$ — and $\xi, \eta \in \mathcal{H}$. For more information on $A(G)$ and $B(G)$, in particular for proofs that they are indeed Banach algebras, we refer to [Eym].

Let $\text{VN}(G)$ denote the *group von Neumann algebra* of the locally compact group $G$, i.e. the von Neumann algebra acting on $L^2(G)$ that is generated by $\lambda(G)$. As the unique predual of $\text{VN}(G)$, the Fourier algebra $A(G)$ carries a canonical operator space structure (see [ER] for the necessary background from operator space theory; for more on the operator space $A(G)$, see [FW]). Similarly, $B(G)$ — as the dual space of the full group $C^*$-algebra $C^*(G)$ — is an operator space in a canonical manner.
For any function $f$ on a group $G$, we define a function $\tilde{f}$ by letting $\tilde{f}(x) := f(x^{-1})$ for $x \in G$. The following is a compilation of known facts which show that the canonical operator space structure of the Fourier algebra is considerably finer than the mere Banach space structure:

**Proposition 1.5** Let $G$ be a locally compact group. Then the algebra homomorphism

$$\theta_* : A(G) \to A(G), \quad f \mapsto \tilde{f}$$

(i) is an isometry,

(ii) but is completely bounded if and only if $G$ has an abelian subgroup of finite index.

**Proof** (i) is well known (see [Eym]), and the “if” part of (ii) follows from [F–W, Theorem 4.5].

For any Banach space $E$ over $\mathbb{C}$, let $\overline{E}$ denote its complex conjugate space, i.e. the complex Banach space for which scalar multiplication is defined via $\lambda \odot x := \overline{\lambda} x$ for $\lambda \in \mathbb{C}$ and $x \in E$. It is easy to see that pointwise conjugation is a complete isometry from $A(G)$ to $\overline{A(G)}$. Consequently, if $\theta_*$ is completely bounded, so is $\overline{\theta}_* : A(G) \to A(G)$, $f \mapsto \overline{\tilde{f}}$ with $\|\overline{\theta}_*\|_{cb} = \|\theta_*\|_{cb}$. The adjoint of $\overline{\theta}_*$, however, is nothing but forming the Hilbert space adjoint

$$\overline{\theta} : VN(G) \to VN(G), \quad T \mapsto T^*,$$

which then also has to be completely bounded (with the same cb-norm) by [E–R, Proposition 3.2.2].

We claim that this forces $VN(G)$ to be subhomogeneous, i.e. the irreducible representations of $VN(G)$ are finite-dimensional, and their degrees are bounded. Assume otherwise. The structure theory of von Neumann algebras then entails that $VN(G)$ contains the full matrix algebra $M_n$ as a $\ast$-subalgebra for each $n \in \mathbb{N}$ (see, e.g., [Run 1, Chapter 6] for details). For $n \in \mathbb{N}$, let $\theta_n : M_n \to M_n$ stand for taking the transpose of an $n \times n$-matrix. Since entrywise conjugation of matrices is a complete isometry, it follows from [E–R, Proposition 2.2.7] that

$$n = \|\theta_n\|_{cb} = \|\overline{\theta}_n\|_{cb} \leq \|\theta\|_{cb} = \|\overline{\theta}_*\|_{cb} = \|\theta_*\|_{cb} \quad (n \in \mathbb{N}),$$

which is impossible. Hence, $VN(G)$ is subhomogeneous.

Let $m \in \mathbb{N}$ be such that every irreducible representation of $VN(G)$ has degree $m$ or less. Then $VN(G)$ satisfies the (non-commutative) polynomial identity $S_{2m} = 0$ (see [A–D] for details), as does its subalgebra $L^1(G)$. By [A–D, Lemma 3.4(i)], the full group $C^*$-algebra $C^*(G)$ then also satisfies $S_{2m} = 0$, which, by [A–D, Lemma 3.4(ii)], means
that the degrees of the irreducible representations of $C^*(G)$ are bounded by $m$ as well. This, in turn, entails that $\sup\{d_\pi : \pi \in \hat{G}\} \leq m$, so that $G$ must have an abelian subgroup of finite index by [Moo, Theorem 1].

This proves the “only if” part of (ii). $\square$

2 (Non-)amenability of $A(G)$ and $B(G)$

To prove that every locally compact group with amenable Fourier algebra has an abelian subgroup of finite index, we proceed rather indirectly.

The anti-diagonal of a (discrete) group $G$ is defined as

$$\Gamma_G := \{(x, x^{-1}) : x \in G\}.$$

It is clear that $\Gamma_G$ is a subgroup of $G \times G$ if and only if $G$ is abelian.

For certain, possibly non-abelian $G$, the anti-diagonal is at least not very far from being a subgroup:

**Example** Let $G$ be a group with an abelian subgroup, say $A$, of finite index. Let $x_1, \ldots, x_n \in G$ be representatives of the left cosets of $A$. For $j = 1, \ldots, n$, define

$$A_j := \{(a, x_j a^{-1} x_j^{-1}) : a \in A\},$$

so that

$$\Gamma_G = (x_1, x_1^{-1}) A_1 \cup \cdots \cup (x_n, x_n^{-1}) A_n.$$

Hence, $\Gamma_G$ is at least a finite union of left cosets in $G \times G$.

The coset ring $\Omega(G)$ of a group $G$ is the ring of subsets generated by all left cosets of subgroups of $G$. The example shows that $\Gamma_G \in \Omega(G \times G)$ if $G$ has an abelian subgroup of finite index. Our first goal in this section, is to prove the converse, namely that $\Gamma_G$ lies in the coset ring of $G \times G$ only if $G$ has such a subgroup.

The coset ring of a group is crucial in the definition of piecewise affine maps (see, [Hos], for example): Let $G$ and $H$ be groups, and let $S \in \Omega(H)$. Then a map $\alpha : S \to G$ is called piecewise affine if:

(a) there are pairwise disjoint $S_1, \ldots, S_n \in \Omega(H)$ with $S = \bigcup_{j=1}^n S_j$;

(b) for each $j = 1, \ldots, n$, there is a subgroup $H_j$ of $H$ and an element $x_j \in H$ such that $S_j \subset x_j H_j$;

(c) for each $j = 1, \ldots, n$ and with $x_j$ and $H_j$ as before, there is a group homomorphism $\beta_j : H_j \to G$ and an element $y_j \in G$ such that $\alpha(y) = y_j \beta_j(x_j^{-1} y)$ for $y \in S_j$. 

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The following lemma is contained in [LS Proposition 3.1] (and originally from M. Ilie’s thesis [III]). We quote it here for convenience:

**Lemma 2.1** Let $G$ and $H$ be discrete groups, let $S \in \Omega(H)$, let $\alpha : S \to G$ be piecewise affine, and let $\theta : A(G) \to B(H)$ be given by

\[
\theta(f)(x) := \begin{cases} 
  f(\alpha(x)), & x \in S, \\
  0, & x \notin S,
\end{cases}
\]

for $f \in A(G)$ and $x \in H$. Then $\theta$ is a completely bounded algebra homomorphism.

With the help of this lemma, we can now prove:

**Proposition 2.2** The following are equivalent for a (discrete) group $G$:

(i) $\Gamma \in \Omega(G \times G)$.

(ii) $G$ has an abelian subgroup of finite index.

**Proof** In view of the example at the beginning of this section, only (i) $\implies$ (ii) needs proof.

Suppose that $\Gamma \in \Omega(G \times G)$. From [Rud 4.3.1, Theorem], it follows that $G \to G, \ x \mapsto x^{-1}$ is piecewise affine. (Even though [Rud 4.3.1, Theorem] is stated and proved for abelian groups only, it is true for arbitrary $G$; see [LS Lemma 1.2(ii)].) Lemma 2.1 then yields that $\theta_* : A(G) \to A(G)$ in Proposition 1.5 is completely bounded, so that $G$ must have an abelian subgroup of finite index by Proposition 1.5(ii).

**Remark** Since the sets in the coset ring of a group allow for a rather explicit description (see [For 2]), a more elementary proof of Proposition 2.2 should be possible.

We shall now characterize those locally compact groups with an amenable Fourier algebra:

**Theorem 2.3** The following are equivalent for a locally compact group $G$:

(i) $A(G)$ is amenable.

(ii) $G$ has an abelian subgroup of finite index.

**Proof** In view of [L-L-W Theorem 4.1], only (i) $\implies$ (ii) needs proof.
By Proposition 1.3, the kernel of $\Delta : A(G) \hat{\otimes} A(G) \to A(G)$ has a bounded approximate identity, say $(u_\alpha)_{\alpha \in \mathbb{A}}$. For each $\alpha \in \mathbb{A}$, there are sequences $(f_k^{(\alpha)})_{k=1}^\infty$ and $(g_k^{(\alpha)})_{k=1}^\infty$ in $A(G)$ such that

$$\sum_{k=1}^\infty \|f_k^{(\alpha)}\| \|g_k^{(\alpha)}\| < \infty \quad \text{and} \quad u_\alpha = \sum_{k=1}^\infty f_k^{(\alpha)} \otimes g_k^{(\alpha)}.$$  

Thanks to Proposition 1.5(i), we may define a bounded net $(v_\alpha)_{\alpha \in \mathbb{A}}$ in $A(G) \hat{\otimes} A(G)$ by letting

$$v_\alpha = \sum_{k=1}^\infty f_k^{(\alpha)} \otimes g_k^{(\alpha)} \quad (\alpha \in \mathbb{A}).$$

It is immediate that $(v_\alpha)_{\alpha \in \mathbb{A}}$ is a bounded approximate identity for the kernel of $\Gamma : A(G) \hat{\otimes} A(G) \to A(G)$, $f \otimes g \mapsto \hat{f} \hat{g}$.

For $f, g \in A(G)$, the function

$$G \times G \to \mathbb{C}, (x, y) \mapsto f(x)g(y)$$

lies in $A(G \times G)$: this induces a canonical contraction from $A(G) \hat{\otimes} A(G)$ to $A(G \times G)$ (note that, by [Los], we cannot a priori suppose that $A(G) \hat{\otimes} A(G) \cong A(G \times G)$). Nevertheless, we can use this canonical inclusion to identify the elements of $A(G) \hat{\otimes} A(G)$ with elements of $A(G \times G)$. Define

$$I := \left\{ f \in A(G \times G) : \lim_{\alpha} f v_\alpha = f \right\}.$$

Then $I$ is a closed ideal of $A(G \times G)$ which contains (the canonical image of) ker $\Gamma$ and, by definition, has a bounded approximate identity. Since $G$ — and thus $G \times G$ — is amenable, [F–K–L–S, Theorem 2.3] applies. Hence, $I$ must be of the form

$$I(F) := \{ f \in A(G \times G) : f|_F \equiv 0 \},$$

where $F$ is the hull of $I$ in $G \times G$; furthermore, this hull must be contained in $\Omega(G \times G)$. Since $(v_\alpha)_\alpha$ lies in ker $\Gamma$, it is easy to see that $\Gamma_G \subset F$, and since $(v_\alpha)_\alpha$ converges pointwise to the indicator function of $(G \times G) \setminus \Gamma_G$, the converse inclusion holds as well. Hence, $\Gamma_G = F \in \Omega(G \times G)$ holds. From Proposition 2.2 we conclude that $G$ has an abelian subgroup of finite index. □

Remarks 1. The results from [Joh 3] indicate that the amenability constant of $A(G)$ and the maximum degree of the irreducible representations of $G$ are likely to be closely related. It would be interesting to further investigate this possible relation.

2. In [Run 2], the second-named author introduced another operator space structure, denoted by $OA(G)$, over the Fourier algebra $A(G)$ which, in general, is different.
from the canonical one — even though both have the same underlying Banach space. The operator space $OA(G)$ is a completely contractive Banach algebra in the sense of [Rua]. At the end of [Run 2], the author conjectured that $OA(G)$ is operator amenable if and only if $G$ is amenable. Thanks to [Run 2 Proposition 2.4], however, the proof of Theorem 2.3 can easily be adapted to yield that $OA(G)$ is operator amenable if and only if $G$ has an abelian subgroup of finite index.

With Theorem 2.3 proven, the corresponding assertion for Fourier–Stieltjes algebras is easy to obtain:

**Corollary 2.4** The following are equivalent for a locally compact group $G$:

(i) $B(G)$ is amenable.

(ii) $G$ has a compact, abelian subgroup of finite index.

**Proof** (i) $\implies$ (ii): Since $A(G)$ is a complemented ideal in $B(G)$, the hereditary properties of amenability immediately yield the amenability of $A(G)$ ([Run 1 Theorem 2.3.7]), so that $G$ has an abelian subgroup $A$ of finite index by Theorem 2.3. We may replace $A$ by its closure and thus suppose that $A$ is closed. Since $A$ is then automatically open, the restriction map from $B(G)$ to $B(A)$ is surjective. Hence, $B(A)$ is also amenable by [Run 1 Proposition 2.3.1]. Let $\hat{A}$ denote the dual group of $A$. The Fourier–Stieltjes transform yields an isometric algebra isomorphism of $B(A)$ and the measure algebra $M(\hat{A})$, so that $M(\hat{A})$ is amenable. Amenability of $M(\hat{A})$, however, forces $\hat{A}$ to be discrete ([B–M]; see also [D–G–H] for a more general result). Hence, $A$ must be compact.

(ii) $\implies$ (i): If $G$ has a compact, abelian subgroup of finite index, then $G$ itself is compact, so that $B(G) = A(G)$ is amenable by Theorem 2.3. $\square$

**Remark** Corollary 2.4 characterizes the locally compact groups $G$ for which $B(G)$ is amenable. The corresponding — and in a certain sense more natural — characterization for operator amenability seems to by still open. A partial result can be found in [R–S].

Another consequence of Theorem 2.3 — slightly extending it — is:

**Corollary 2.5** The following are equivalent for a locally compact group $G$ with at least two elements:

(i) $G$ has an abelian subgroup of finite index.

(ii) $A(G)$ is amenable.

(iii) $I(H)$ is amenable for each closed proper subgroup $H$ of $G$.

(iv) $I(\{e\})$ is amenable.
(v) $I(H)$ is amenable for some closed proper subgroup $H$ of $G$.

Proof (i) $\iff$ (ii) is Theorem 2.3

(ii) $\implies$ (iii): By [FKL, Theorem 1.5], $I(H)$ has a bounded approximate identity and thus is amenable by [Run 1, Theorem 2.3.7].

(iii) $\implies$ (iv) $\implies$ (v) are trivial.

(iv) $\implies$ (ii): Let $x \in G \setminus H$, so that $xH \cap H = \emptyset$. Then $I(xH) \cong I(H)$ is amenable. Restriction to $H$, maps $I(xH)$ onto a dense subalgebra of $A(H)$, so that $A(H)$ is amenable. Since $I(H)$ is just the kernel of the restriction map, the amenability of $A(G)$ follows from [Run 1, Theorem 2.3.10].

\[\square\]

3 Weak amenability of $A(G)$

In [Joh 3], Johnson showed that $A(G)$ is weakly amenable whenever $G$ is a totally disconnected, compact group, and soon thereafter, the first-named author extended this result to arbitrary totally disconnected, locally compact groups ([For 3, Theorem 5.3]): this shows already that weak amenability of the Fourier algebra imposes considerably weaker constraints on the underlying group than amenability. In this section, we shall present a sufficient condition — which we believe to be necessary as well — which forces a locally compact group to have a weakly amenable Fourier algebra and which subsumes [For 3, Theorem 5.3].

We begin with an elementary hereditary lemma:

Lemma 3.1 Let $G$ be a locally compact group, and let $H$ be a closed subgroup of $G$. Then:

(i) If $A(G)$ is weakly amenable, then so is $A(H)$.

(ii) If $H$ is open, and if $A(H)$ is weakly amenable, then so is $A(G)$.

Proof Since $A(H)$ is a quotient of $A(G)$ by [For 1, Lemma 3.8], (i) follows immediately from the basic hereditary properties of weak amenability for commutative Banach algebras ([Gel]).

For (ii), let $E$ be a symmetric Banach $A(G)$-bimodule, and let $D : A(G) \to E$ be a derivation. Let $f \in A(G)$ have compact support, and let $x_1, \ldots, x_n \in G$ be such that $\text{supp}(f) \subset x_1H \cup \cdots \cup x_nH$. For $j = 1, \ldots, n$, let $\chi_j \in B(G)$ be the indicator function of $x_jH$. Then $\mathfrak{A}_j := \chi_j A(G)$ is a commutative subalgebra of $A(G)$ isometrically isomorphic to $A(H)$. Since $A(H)$ is weakly amenable, $D|_{\mathfrak{A}_j} = 0$ for $j = 1, \ldots, n$ follows. In particular, $D(\chi_j f) = 0$ holds for $j = 1, \ldots, n$; since $f = \chi_1 f + \cdots + \chi_n f$, it follows that $Df = 0$. Since the elements with compact support are dense in $A(G)$, we obtain that $D$ is zero on all of $A(G)$. \[\square\]
For every locally compact group $G$, we shall denote the component of $G$ containing
the identity element by $G_e$. It is well known that $G_e$ is a closed, normal subgroup of $G$. Our
main result in this section (Theorem 3.3 below) is that that $A(G)$ is weakly amenable whenever $G_e$ is abelian.

We first prove a lemma:

**Lemma 3.2** Let $G$ be a locally compact group such that $G_e$ is abelian, and let $K$ be a
compact, normal subgroup of $G$ such that $G/K$ is a Lie group. Then $A(G/K)$ is weakly amenable.

**Proof** The component of the identity of $G/K$ is $\pi(G_e)^{-1}$, where $\pi : G \to G/K$ is the
quotient map ([H-E-R (7.12) Theorem]). Thus, if $G_e$ abelian, so is $(G/K)_e$; in particular,
$A((G/K)_e)$ is weakly amenable. Since $G/K$ is a Lie group, $(G/K)_e$ is open. It follows
from Lemma 3.1(ii) that $A(G/K)$ is weakly amenable. $\Box$

**Theorem 3.3** Let $G$ be a locally compact group such that $G_e$ is abelian. Then $A(G)$ is
weakly amenable.

**Proof** We shall first treat the case where $G$ is almost connected, i.e. where $G/G_e$ is compact.

Let $E$ be a symmetric Banach $A(G)$-bimodule, let $D : A(G) \to E$ be a derivation, let $f \in A(G)$ be arbitrary, and let $\epsilon > 0$. We claim that $\|Df\| \leq \epsilon$: since $\epsilon > 0$ is arbitrary,
this is enough to conclude that $Df = 0$.

For $x \in G$, define the right translate $R_x f$ of $f$ by $x$ as $(R_x f)(y) := f(yx)$ for $y \in G$. Since $G \ni x \mapsto R_x f \in A(G)$ is continuous with respect to the norm topology on $A(G)$
([E-Y-M]), there is a neighborhood $U$ of the identity in $G$ such that $\|f - R_x f\| < \frac{\epsilon}{1 + \|D\|}$ for all $x \in U$. By [Pal] 12.2.15 Theorem, there is a compact, normal subgroup $K \subset U$ of $G$
such that $G/K$ is a Lie group. Define $P_K : A(G) \to A(G)$ by letting

$$P_K f := \int_K R_x f \, dx \quad (f \in A(G)),$$

where the integral is a Bochner integral with respect to normalized Haar measure on $K$. It follows that

$$\|f - P_K f\| \leq \int_K \|f - R_x f\| \, dx \leq \frac{\epsilon}{1 + \|D\|} \quad (f \in A(G)).$$

It is easy to see that $P_K$ is a projection onto $A(G : K)$, the subalgebra of $A(G)$ consisting
of those functions that are constant on cosets of $K$; this algebra is isometrically isomorphic to $A(G/K)$ ([E-Y-M]). Since $A(G : K) \cong A(G/K)$ is weakly amenable by Lemma 3.2, we have $D|_{A(G : K)} \equiv 0$ and, consequently,

$$\|Df\| = \|Df - D(P_K f)\| \leq \|D\|\|f - P_K f\| \leq \epsilon.$$
This proves our claim.

For arbitrary locally compact $G$, note that $G$ has an open, almost connected subgroup $H$ (this follows, for instance, from \([\text{H–R}, (7.7)\) Theorem\]). Since $A(G)$ is weakly amenable if and only if this is true for $A(H)$ by Lemma \([3.1]\)(ii), the claim for general $G$ follows from the almost connected case. \(\square\)

As we already stated at the beginning of this section, we believe that the sufficient condition of Theorem \([3.3]\) for the weak amenability of the Fourier algebra is also necessary, but we have not been able to prove it. If this conjecture is correct, it implies that no non-abelian, connected group can have a weakly amenable Fourier algebra.

The following proposition is a small step into this converse direction:

**Proposition 3.4** Let $G$ be a Lie group such that $A(G)$ is weakly amenable. Then every compact subgroup of $G$ has an abelian subgroup of finite index.

**Proof** Let $K$ be a compact subgroup of $G$. If $A(G)$ is weakly amenable, then Lemma \([3.1]\)(i) implies that $A(K)$ is also weakly amenable. For the same reason, $A(K_e)$ is weakly amenable. But $K_e$ is a compact connected Lie group. It follows from the remarks following Corollary 7.3 that $K_e$ is abelian. Finally, $K_e$ is open in the compact Lie group $K$ and is therefore of finite index. \(\square\)

The equivalence of (i) and (iii) in the next corollary can be already be found in \([\text{L–L–W}, \text{Proposition 4.4}]\):

**Corollary 3.5** Let $G$ be an almost connected, semisimple Lie group. Then the following are equivalent:

(i) $A(G)$ is amenable.

(ii) $G$ is amenable, and $A(G)$ is weakly amenable.

(iii) $G$ is finite.

**Proof** (i) $\implies$ (ii) is obvious.

For (ii) $\implies$ (iii), simply note that, if $G$ is amenable, then it is already compact (\([\text{Pat}, \text{Theorem 3.8}]\)). It follows from Proposition \([3.4]\) $G_e$ is both semisimple and abelian. This means that $G_e$ is trivial and, in turn, that $G$ is finite.

(iii) $\implies$ (i) is trivial. \(\square\)

Let $K$ be a compact normal subgroup of the locally compact group $G$. In the proof of Theorem \([3.8]\) we made crucial use of the fact that $A(G)$ contains a closed subalgebra $A(G : K)$ which is isometrically isomorphic to $A(G/K)$. There ought to be a strong
connection between the weak amenability of \( A(G) \) and that of \( A(G/K) \): otherwise, it would be extremely unlikely that any reasonable structural characterization exists of those \( G \) for which \( A(G) \) is weakly amenable. However, since subalgebras generally do not inherit weak amenability, we cannot conclude (yet) that, if \( A(G) \) is weakly amenable, then so is \( A(G/K) \).

This prompts us to formulate the following definition:

**Definition 3.6** Let \( G \) be a locally compact group. We say that \( A(G) \) is **hereditarily weakly amenable** if \( A(G/K) \) is weakly amenable for every compact normal subgroup \( K \) of \( G \).

**Remark** If \( G_c \) is abelian, then so is \( (G/K)_c \) for each compact normal subgroup \( K \) of \( G \). Hence, the only examples of locally compact groups \( G \) for which we positively know that \( A(G) \) is weakly amenable, also satisfy the formally stronger Definition 3.6.

Recall that a locally compact group \( G \) is called a [SIN]-group if it has a basis of neighborhoods of the identity consisting of sets that are invariant under conjugation.

With Definition 3.6 in place of mere weak amenability, we can prove a converse of Theorem 3.3 for [SIN]-groups:

**Theorem 3.7** Let \( G \) be a [SIN]-group. Then \( A(G) \) is hereditarily weakly amenable if and only if \( G_c \) is abelian.

**Proof** In view of the remark following Definition 3.6, it is clear that \( A(G) \) is hereditarily weakly amenable whenever \( G_c \) is abelian (this does not require \( G \) to be a [SIN]-group).

Conversely, suppose that \( A(G) \) is hereditarily weakly amenable. Since \( G \) is a [SIN]-group, there are a vector group \( V \) and a compact group \( K \) such that

\[
G_c \cong V \times K
\]

(see [Pal] 12.4.48 Theorem). If \( K \) is not abelian, we can find \( x, y \in K \) and a neighborhood \( U \) in \( G \) of the identity such that the commutator \([x,y]\) does not lie in \( U \). Since \( G \) is a [SIN]-group, \( U \) contains a compact normal subgroup \( K_U \) such that \( G/K_U \) is a Lie group (see [Pal] 12.6.12). Let \( \pi_U : G \to G/K_U \) be the quotient map. Then \( \pi_U(K) \) is a compact connected Lie group. From the choice of \( U \), it is clear that \( \pi_U(K) \) is not abelian in \( G/K_U \). We know that \( A(G/K_U) \) is weakly amenable because \( A(G) \) is hereditarily weakly amenable, and Lemma 3.1(i) implies that \( A(\pi_U(K)) \) is also weakly amenable. Since \( \pi_U(K) \) is a non-abelian, compact, connected Lie group, this is impossible by Proposition 3.4. We conclude that \( K \) is abelian, so that \( G_c \) is also abelian. \( \square \)

Recall that a locally compact group \( G \) is called an [IN]-group if the identity of \( G \) has a compact neighborhood which is invariant under conjugation.

We conclude this paper with a corollary of Theorem 3.7:

(see [Pal] 12.4.48 Theorem). If \( K \) is not abelian, we can find \( x, y \in K \) and a neighborhood \( U \) in \( G \) of the identity such that the commutator \([x,y]\) does not lie in \( U \). Since \( G \) is a [SIN]-group, \( U \) contains a compact normal subgroup \( K_U \) such that \( G/K_U \) is a Lie group (see [Pal] 12.6.12). Let \( \pi_U : G \to G/K_U \) be the quotient map. Then \( \pi_U(K) \) is a compact connected Lie group. From the choice of \( U \), it is clear that \( \pi_U(K) \) is not abelian in \( G/K_U \). We know that \( A(G/K_U) \) is weakly amenable because \( A(G) \) is hereditarily weakly amenable, and Lemma 3.1(i) implies that \( A(\pi_U(K)) \) is also weakly amenable. Since \( \pi_U(K) \) is a non-abelian, compact, connected Lie group, this is impossible by Proposition 3.4. We conclude that \( K \) is abelian, so that \( G_c \) is also abelian. \( \square \)

Recall that a locally compact group \( G \) is called an [IN]-group if the identity of \( G \) has a compact neighborhood which is invariant under conjugation.

We conclude this paper with a corollary of Theorem 3.7.
**Corollary 3.8** Let $G$ be a locally compact [IN]-group such that $A(G)$ is hereditarily weakly amenable. Then there exists a compact normal subgroup $K$ of $G_e$ such that $G_e/K$ is abelian.

**Proof** Since $G$ is an [IN]-group, there is a compact normal subgroup $N$ of $G$ such that $G/N$ is a [SIN]-group (see [Pal, 12.1.31 Theorem]). It is easy to see that the hereditary weak amenability of $A(G)$ forces $A(G/N)$ to be hereditarily weakly amenable as well. By Theorem 3.7, this means that $(G/N)_e$ is abelian. Letting $K := G_e \cap N$ and observing that $(G/N)_e \cong G_e/K$ yields the result. □

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