Synchronization Methods for the Degn-Harrison Reaction-Diffusion Systems

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\section*{ABSTRACT}
The paper concerns the problem of synchronization-control in nonlinear bacterial cultures reaction-diffusion model, linear and nonlinear controllers have been proposed to study the complete synchronization of couples of the Degn-Harrison system with identical and non-identical coefficients. Throughout the paper, we use numerical simulation to show the effectiveness of the proposed results.

\section*{INDEX TERMS}
Degn-Harrison, complete synchronization, Lyapunov method, reaction-diffusion.

\section*{I. INTRODUCTION}
Synchronization is a process of controlling the output of the response system (slave system) to force its behavior to follow that of the corresponding drive system (master system) asymptotically. Since the pioneering study of Pecora and Carroll [1], various control schemes have been introduced to synchronize dynamical systems due to its applications in image processing, cryptography, ecological system, combinatorial optimization, lasers technology, and secure communications [2].

Considerable research has been devoted to study the synchronization in low dimensional systems represented by unidimensional ordinary differential equations or maps [3]–[6]. Nevertheless, synchronizing in high-dimensional systems modeled in the spatial-temporal domain and described by nonlinear reaction-diffusion systems which state variables depend on the time and spatial position stills in its initial stage.

Reaction-diffusion systems models act a central role in describing the phenomena that exist in neuronal networks, chemical reaction systems, image processing and ecosystems. Due to the spatial component, this kind of model is extensively used to understand a wide range of complex dynamical structures and spatiotemporal patterns as well as rotating spirals, circulating pulses on a ring, target waves and oscillating spots [7]–[10]. For this reason, the study of synchronization in this kind of model is important to our comprehension of a wide variety of phenomena in the real-life.

Recently, significant efforts are made to investigate the synchronization in reaction-diffusion systems. For instance, the backstopping synchronization method [11], the hybrid adaptive synchronization approach [12], the graph-theoretic synchronization technique [13], impulsive type synchronization approach [14] and pinning impulsive synchronization [15] for PDEs have been proposed. Moreover, novel control synchronization schemes have been designed to achieve synchronization of the FitzHugh-Nagumo model [16], a three-component autocatalytic model [17], multi-layered natural and media networks [18], [19], the Newton-Leipnik chaotic system [20]. Also, linear and nonlinear control are suggested to synchronize a class of reaction diffusion systems [20], [21].

The present paper deals with the analysis of control synchronization for bacterial culture model introduced by Degn and Harrison [23] as

$$
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + a - u_1 - u_2 g_k(u_1), \quad (x, t) \in \Omega \times \mathbb{R}^+, \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + b - u_2 g_k(u_1), \quad (x, t) \in \Omega \times \mathbb{R}^+, \\
\frac{\partial u_i}{\partial \eta} &= 0, \quad i = 1, 2, \quad (x, t) \in \partial \Omega \times \mathbb{R}^+, \\
0 &\leq u_i(x, 0) = u_{i0}(x), \quad i = 1, 2, \quad x \in \Omega,
\end{align*}
$$

(1)
where \( u_1(x, t), u_2(x, t) \) represent the oxygen and the nutrient respectively, \( a, b, d_1, d_2, k \) are positive constants and 
\[
g_k(u_1) = \frac{u_1}{1 + ku_1}, \quad \Omega \subset \mathbb{R}^n, (n \geq 1)
\]
is a bounded domain with smooth boundary \( \partial \Omega \), \( \eta \) is unit vector normal to \( \partial \Omega \).

The Degn-Harrison model (1) is used to describe the effect of the oxygen concentration in the Klebsiella aerogenes bacteria

culture, for a more detailed background of chemical reaction scheme and significance of system (1) we refer interested reader to [24], [25].

The Degn-Harrison system (1) has been studied extensively in the literature, but most of the research focuses on

the dynamics of this model including the local and global asymptotic stability of the steady-state solutions [25], [26],

Turing instability [27], [28] and Hopf bifurcation [29], [30]. However, as far as we know, this is the first work deal with

control synchronization of the model (1).

The contribution of this paper is the development of novel methods for synchronization of Degn-Harrison

reaction-diffusion system with identical or non-identical coefficients.

This work is organized as follows. In section 2 and 3 along with the proposed control laws and proofs of their

convergence based on the Lyapunov approach and Green identity. In section 4, we consider numerical applications

to illustrate the effectiveness of the development schemes. Finally, In section 5 we give the conclusion of our work.

II. IDENTICAL SYSTEMS

To analyze the synchronization between two identical Degn-Harrison models, we use the master-slave (drive-response)

formalism, where the two Degn-Harrison reaction-diffusion systems are coupled, in such a manner that the output of

the second (slave) system tracks the output of the first (master) system asymptotically. In this case, we design appropriate

functions called controllers to force the difference of states of synchronized systems converge to zero. This process is called

complete synchronization.

The following result provides the existence, uniqueness and the boundedness of the solution of the Degn-Harrison

system (1).

Lemma 1: [25] Suppose that \( b < a \) and \( u_{00}(x) \in C(\overline{\Omega}) \cap C^2(\Omega) \) then

1) The model (1) possesses a global unique solution 

\( u_i(x, t) \in C^{1,2}(\Omega \times \mathbb{R}_+^+) \cap C(\Omega \times \mathbb{R}_+^+) \), this solution is positive for all \((x, t) \in \Omega \times \mathbb{R}_+^+\).

2) \( \mathcal{R} = [\tilde{u}_1, a] \times [2b\sqrt{k}, \tilde{u}_2] \) where \( \tilde{u}_1 = \frac{b(a-b)}{a(1+a^2k)} \) and 

\( \tilde{u}_2 = \frac{a-2a}{\sqrt{g_k(u_1)}} \) is an invariant rectangle for the system (1).

First of all, we assume that the master and the slave systems are identical in all coefficients except in the associate

initial condition. Therefore, the slave system associated with the master system (1) can be written as

\[
\begin{align*}
\frac{\partial v_1}{\partial t} &= d_1 \Delta v_1 + a - v_1 - v_2 g_k(v_1) + L_1, \\
\frac{\partial v_2}{\partial t} &= d_2 \Delta v_2 + b - v_2 g_k(v_1) + L_2,
\end{align*}
\]

where \( v_i = v_i(x, t), (i = 1, 2) \) are states of the slave system (2) and \( L_i \) are controllers to be designed.

The aim this section is to determine a suitable controls \( L_i \) to force the synchronization errors \( e(x, t) = (e_1(x, t), e_2(x, t)) \), defined by

\[
e(x, t) = v(x, t) - u(x, t),
\]

where \( u(x, t) = (u_1(x, t), u_2(x, t)) \) and \( v(x, t) = (v_1(x, t), v_2(x, t)) \) are the solutions of systems (1) and (2)

respectively, to converge towards zero as \( t \) goes to infinity.

Definition 2: The master system (1) and the slave system (2) are said to be completely synchronized if

\[
\lim_{t \to \infty} \|e(x, t)\| = 0.
\]

Differentiate the errors given (3) with respect to time to get

\[
\begin{align*}
\frac{\partial e_1}{\partial t} &= d_1 \Delta e_1 - e_1 + u_2 g_k(u_1) - v_2 g_k(v_1) + L_1, \\
\frac{\partial e_2}{\partial t} &= d_2 \Delta e_2 + u_2 g_k(u_1) - v_2 g_k(v_1) + L_2.
\end{align*}
\]

One can observe that the error system (4) satisfies the homogeneous Neumann boundary conditions

\[
\frac{\partial e_1}{\partial n} = \frac{\partial e_2}{\partial n} = 0, \quad \text{for all} \quad (x, t) \in \partial \Omega \times \mathbb{R}_+^+.
\]

The following lemma is needed in the proofs of results of this paper.

Lemma 3: There exists a positive constant \( K \) depending on \( a, b \) and \( k \) such that

\[
|u_2 g_k(u_1) - v_2 g_k(v_1)| \leq K (|v_1 - u_1| + |v_2 - u_2|).
\]

Proof:

\[
|u_2 g_k(u_1) - v_2 g_k(v_1)| \leq |u_2 g_k(u_1) - v_2 g_k(u_1)| + |v_2 g_k(u_1) - v_2 g_k(v_1)|
\leq |v_2 - u_2| |g_k(u_1)| + |v_2| |u_1 - v_1| + k |v_2| |u_1 - v_1| |g_k(u_1)| |g_k(v_1)|.
\]
The function $g_k(.)$ has a maximum $\frac{1}{2\sqrt{k}}$, then

$$|u_2g_k(u_1) - v_2g_k(v_1)| \leq \frac{5}{4} |v_2| |u_1 - v_1| + \frac{1}{2\sqrt{k}} |v_2 - u_2|.$$  

Due to Lemma 1 $|v_2| < \hat{v}_2$, thus, we can choose the constant $K$ as

$$K \geq \max\left\{ \frac{5}{4} \hat{v}_2, \frac{1}{2\sqrt{k}} \right\}.$$  

The synchronization error defined in (3) goes to 0, as $t$ goes to $+\infty$ if and only if the zero solution of the synchronization error system (4) is globally asymptotically stable. That is, in the following Theorem, we determine the controllers $L_1$ and $L_2$, in linear forms to achieve synchronization between systems given in Eq. (1) and Eq. (2).

**Theorem 4:** The master system (1) and the slave system (2) are completely synchronized under the following linear control law

$$L_1 = (1 - 2K)(v_1 - u_1),$$  
$$L_2 = -2K(v_2 - u_2).$$  

**Proof:** Merging the Eq. (7) and Eq. (4), we obtain

$$\begin{cases} \frac{\partial e_1(x,t)}{\partial t} = d_1 \Delta e_1 - 2K e_1 + u_2g_k(u_1) - v_2g_k(v_1), \\ \frac{\partial e_2(x,t)}{\partial t} = d_2 \Delta e_2 - 2K e_2 + u_2g_k(u_1) - v_2g_k(v_1). \end{cases} \tag{8}$$  

Now, we construct a Lyapunov functional as

$$V = \frac{1}{2} \int_{\Omega} \left( e_1^2 + e_2^2 \right) dx,$$

then

$$\frac{\partial V}{\partial t} = \int_{\Omega} \left( e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t} \right) dx$$  
$$= \int_{\Omega} e_1 \left( d_1 \Delta e_1 - 2K e_1 + u_2g_k(u_1) - v_2g_k(v_1) \right) dx$$  
$$+ \int_{\Omega} e_2 \left( d_2 \Delta e_2 - 2K e_2 + u_2g_k(u_1) - v_2g_k(v_1) \right) dx$$  
$$= \int_{\Omega} (d_1 e_1 \Delta e_1) dx + \int_{\Omega} (d_2 e_2 \Delta e_2) dx$$  
$$- 2K \int_{\Omega} \left( e_1^2 + e_2^2 \right) dx$$  
$$+ \int_{\Omega} [(u_2g_k(u_1) - v_2g_k(v_1))(e_1 + e_2)] dx.$$  

Using Green identity, we can deduce

$$\frac{\partial V}{\partial t} = -\int_{\Omega} d_1 |\nabla e_1|^2 dx + \int_{\partial \Omega} d_1 e_1 \frac{\partial e_1}{\partial n} d\sigma - \int_{\Omega} d_2 |\nabla e_2|^2 dx$$  
$$+ \int_{\partial \Omega} d_2 e_2 \frac{\partial e_2}{\partial n} d\sigma - 2K \int_{\Omega} \left( e_1^2 + e_2^2 \right) dx$$  
$$+ \int_{\Omega} [(u_2g_k(u_1) - v_2g_k(v_1))(e_1 + e_2)] dx.$$  

By using the homogeneous Neumann boundary conditions (5) and Lemma 3, we get

$$\frac{\partial V}{\partial t} = -\int_{\Omega} \left[ d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2 \right] dx - 2K \int_{\Omega} \left( e_1^2 + e_2^2 \right) dx$$  
$$+ \int_{\partial \Omega} \left[ (u_2g_k(u_1) - v_2g_k(v_1))(e_1 + e_2) \right] d\sigma \leq -\int_{\Omega} \left[ d_1 (|\nabla e_1|^2 + d_2 (|\nabla e_2|^2) \right] dx - 2K \int_{\Omega} \left( e_1^2 + e_2^2 \right) dx$$  
$$+ \int_{\partial \Omega} \left[ (u_2g_k(u_1) - v_2g_k(v_1))(e_1 + e_2) \right] d\sigma \leq -\int_{\Omega} \left[ d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2 \right] dx - 2K \int_{\Omega} \left( e_1^2 + e_2^2 \right) dx$$  
$$+ \int_{\partial \Omega} K(e_1 + e_2)^2 d\sigma \leq -\int_{\Omega} \left[ d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2 \right] dx - 2K \int_{\Omega} \left( e_1^2 + e_2^2 \right) dx$$  
$$+ \int_{\partial \Omega} K(e_1 + e_2)^2 d\sigma$$  
$$= -\int_{\Omega} \left[ d_1 |\nabla e_1|^2 + d_2 |\nabla e_2|^2 \right] dx - K \int_{\Omega} (e_1^2 + e_2^2) dx < 0.$$  

Based on the Lyapunov stability theory, we conclude that the zero solution of the synchronization error system (4) is globally asymptotically stable. Therefore, the master-slave systems (1) and (2) are completely synchronized.

### III. NON IDENTICAL SYSTEMS

In this section, we consider master-slave of the Degn-Harrison reaction-diffusion systems which are not identical. In this case, the slave system associated with the master system (1) is given by

$$\begin{cases} \frac{\partial v_i}{\partial t} = \hat{d}_1 \Delta v_i + \hat{a} - v_1 - v_2g_k(v_1) + U_1, \\ (t, x) \in \Omega \times \mathbb{R}^+ \\ \frac{\partial v_i}{\partial t} = \hat{d}_2 \Delta v_i + \hat{b} - v_2g_k(v_1) + U_2, \end{cases} \tag{9}$$  

where $v_i = v_i(x,t), \ i = 1, 2$, are states of the slave system (9), $d_1, \ d_2, \ \hat{a}, \ \hat{b}$ are positive constants and $U_i$ are controllers to be designed.

The synchronization error system for $(x, t) \in \Omega \times \mathbb{R}^+$, can be derived as

$$\begin{cases} \frac{\partial e_1(x,t)}{\partial t} = \frac{\partial v_1(x,t)}{\partial t} - \frac{\partial u_1(x,t)}{\partial t} = \hat{d}_1 \Delta v_1 - \hat{d}_1 \Delta u_1 + (\hat{a} - a) - e_1 + (u_2g_k(u_1) - v_2g_k(v_1)) + U_1, \\ \frac{\partial e_2(x,t)}{\partial t} = \frac{\partial v_2(x,t)}{\partial t} - \frac{\partial u_2(x,t)}{\partial t} = \hat{d}_2 \Delta v_2 - \hat{d}_2 \Delta u_2 + (\hat{b} - b) + (u_2g_k(u_1) - v_2g_k(v_1)) + U_2. \end{cases} \tag{10}$$
Differentiate the above function with respect to time, yields
\begin{equation}
\begin{aligned}
\frac{\partial}{\partial t} (\hat{d}_1 - d_1 - l_1) &\geq 0, \\
\frac{\partial}{\partial t} (\hat{d}_2 - d_2 - l_3) &\geq 0, \\
\hat{l}_2 &= \hat{a} - a, \\
\hat{l}_4 &= \hat{b} - b,
\end{aligned}
\end{equation}
then, the non identical master-slave systems given in (1) and (9) are completely synchronized under the following nonlinear control low
\begin{equation}
\begin{aligned}
U_1 &= -\left(\frac{\partial}{\partial t} (\hat{d}_1 - 2d_1) \Delta u_1 + d_1 \Delta v_1\right) \\
&\quad + (1 - 2K) e_1 - l_1 \Delta e_1 - l_2, \\
U_2 &= -\left(\frac{\partial}{\partial t} (\hat{d}_2 - 2d_2) \Delta u_2 + d_2 \Delta v_2\right) - 2Ke_2 - l_3 \Delta e_2 - l_4.
\end{aligned}
\end{equation}
Proof: By using (12) the error system (10) can be written as
\begin{equation}
\begin{aligned}
\frac{\partial e_1(x, t)}{\partial t} &= \left(\frac{\partial}{\partial t} (d_1 - d_1 - l_1) \right) \Delta e_1 + \left(\frac{\partial}{\partial t} (\hat{a} - a - l_2) \right) e_1 \\
&\quad - 2Ke_1 + (u_2g_k(u_1) - v_2g_k(v_1)), \\
\frac{\partial e_2(x, t)}{\partial t} &= \left(\frac{\partial}{\partial t} (d_2 - d_2 - l_3) \right) \Delta e_2 + \left(\frac{\partial}{\partial t} (\hat{b} - b - l_4) \right) e_2 \\
&\quad - 2Ke_2 + (u_2g_k(u_1) - v_2g_k(v_1)).
\end{aligned}
\end{equation}
Let us introduce the following Lyapunov functional
\begin{equation}
V = \frac{1}{2} \int_\Omega (e_1^2 + e_2^2) \, dx.
\end{equation}
Differentiate the above function with respect to time, yields
\begin{equation}
\begin{aligned}
\frac{\partial V}{\partial t} &= \int_\Omega \left(e_1 \frac{\partial e_1}{\partial t} + e_2 \frac{\partial e_2}{\partial t}\right) \, dx \\
&= \int_\Omega \left(\frac{\partial}{\partial t} (d_1 - d_1 - l_1) \right) e_1 \Delta e_1 \, dx \\
&\quad + \int_\Omega \left(\frac{\partial}{\partial t} (d_2 - d_2 - l_3) \right) e_2 \Delta e_2 \, dx \\
&\quad + \int_\Omega \left[(\hat{a} - a - l_2) e_1 + (\hat{b} - b - l_4) e_2ight. \\
&\quad \left. - 2K(e_1^2 + e_2^2)\right] \, dx \\
&\quad + \int_\Omega [(u_2g_k(u_1) - v_2g_k(v_1))(e_1 + e_2)] \, dx \\
&\leq - \int_\Omega \left(\frac{\partial}{\partial t} (d_1 - d_1 - l_1) \right) |\nabla e_1|^2 \, dx \\
&\quad - \int_\Omega \left(\frac{\partial}{\partial t} (d_2 - d_2 - l_3) \right) |\nabla e_2|^2 \, dx \\
&\quad + \int_\Omega \left[(\hat{a} - a - l_2) e_1 + (\hat{b} - b - l_4) e_2ight. \\
&\quad \left. - 2K(e_1^2 + e_2^2)\right] \, dx \\
&\quad + \int_\Omega K \left(e_1^2 + 2|e_1||e_2| + e_2^2\right) \, dx \\
&= - \int_\Omega \left(\frac{\partial}{\partial t} (d_1 - d_1 - l_1) \right) |\nabla e_1|^2 \, dx.
\end{aligned}
\end{equation}
Using the conditions given in (11), we obtain
\begin{equation}
\begin{aligned}
\frac{\partial V}{\partial t} &= - \int_\Omega \left(\frac{\partial}{\partial t} (d_1 - d_1 - l_1) \right) |\nabla e_1|^2 \, dx \\
&\quad - \int_\Omega \left(\frac{\partial}{\partial t} (d_2 - d_2 - l_3) \right) |\nabla e_2|^2 \, dx \\
&\quad - K \int_\Omega (|e_1| - |e_2|)^2 \, dx < 0.
\end{aligned}
\end{equation}
Based on the Lyapunov stability theory, we conclude that the zero solution of the synchronization error system (13) is globally asymptotically stable. Hence, the master system (1) and the slave system (9) are globally completely synchronized.

IV. NUMERICAL SIMULATIONS
To test the theoretical findings and to clarify the feasibility of the synchronization schemes introduced in the previous sections, two numerical examples are presented. The first one is to the case of identical coefficients, while the second one is to the nonidentical coefficients case. The numerical simulations are performed in one (two) dimensional space using a three (five) central difference scheme.
Example 1:
Let \((a, b, k, d_1, d_2) = (1.2371, 0.34, 19.974, 3, 2)\) and
\begin{equation}
\begin{aligned}
u_1 (0, x) &= 0.8 (1 + 0.3 \sin (0.2x)), \\
u_2 (0, x) &= 0.4 (1 + 0.3 \cos (0.2x)).
\end{aligned}
\end{equation}
and the slave system (2) equipped with the initial conditions
\begin{equation}
\begin{aligned}v_1 (0, x) &= \sin (0.3x), \\
v_2 (0, x) &= \cos (0.3x).
\end{aligned}
\end{equation}
The spatio-temporal solution of the system (1) (i.e., the system (2) for \(L_1 = L_2 = 0\)) with zero Neumann boundary conditions are shown in Figures 1 and 2.
According to the Theorem 4, if we choose \(K = 158, \) then the controller \(L_1, L_2\) can be designed as
\begin{equation}
\begin{aligned}
L_1 &= -315(v_1 - u_1), \\
L_2 &= -316(v_2 - u_2).
\end{aligned}
\end{equation}
As a result from the performed numerical simulations, we can observe that with the addition of appropriate linear controllers given by (7), the dynamics of the systems, given in (1) and (2) become synchronized and the zero steady-state of the synchronization error system given in (8) becomes asymptotically stable. Hence, the errors defined in (3) goes to 0, as \(t\) goes to +\(\infty, \) see Figures 3 and 4. In addition, Figures 5, 6, 7 show the pattern formations in two-dimensional space, of synchronization error system (8) in
Example 2: In order to put Theorem 5 to the test, let us consider the parameter set

\[
(a, b, k, d_1, d_2) = (10, 6, 1, 5, 3), \quad (\hat{a}, \hat{b}, k, \hat{d}_1, \hat{d}_2) = (6, 5, 1, 3, 2),
\]

when the initial conditions associated with the system (1) are given by

\[
u_1 (0, x) = 0.8 (1 + 0.3 \sin (0.2 x)),
\]
\[
u_2 (0, x) = 0.4 (1 + 0.3 \cos (0.2 x)).
\]

Then the solutions \(u_1\) and \(u_2\) of the system (1) with zero Neumann boundary conditions are shown in Figure 8 and 9. For the slave system (9) with \((U_1 = 0, U_2 = 0)\), if the initial conditions given by

\[
\begin{align*}
v_1 (0, x) &= 0.8(1 + 16.3 \cos x), \\
v_2 (0, x) &= 0.4(1 + \cos x).
\end{align*}
\]

The solutions \(v_1, v_2\) are shown in Figures 10 and 11.
According to the Theorem 5, if we choose the control constants as
\[(l_1, l_2, l_3, l_4) = (-3, -4, -2, -1)\],
and \(K = 55\), then the controllers \(U_1, U_2\) can be designed as
\[
U_1 = 7\Delta u_1 - 5\Delta v_1 - 109e_1 + 3\Delta e_1 + 4.
\]
\[
U_2 = 4\Delta u_2 - 3\Delta v_2 - 110e_2 + 2\Delta e_2 + 1. \quad (23)
\]
so, from the Theorem 5, with the parameter set (17-18), the master system (1) and the slave (9) are globally synchronized. The distribution of the controllers \(U_1, U_2\) and the spatio-temporal evolution of the synchronization error system (13) states are shown in Figures 12-13 and 14-15 respectively.

Moreover, Figures 16, 17, and 18 show the pattern formations in of error systems in 2D spatial domain for \(t = 0\), \(t = 5\), and \(t = 20\), respectively, this figures indicate that the zero steady
conclude that the synchronization in nonidentical coefficients in Fig. 7 and 18, for the two-dimensional space, we can investigate the synchronization behaviors in many types of spatiotemporal models, including the lattice maps and stochastic reaction-diffusion systems.

V. CONCLUSION
In the present work, we have established novel methods to investigate the synchronization in nonlinear bacterial cultures spatiotemporal model. First, a spatial-time coupling process for the complete synchronization was introduced. Next, suitable linear and nonlinear control schemes are proposed to realize the synchronization for identical and nonidentical cases. The synchronization results are derived based on the Lyapunov theory and master-slave formulation. Numerical simulations, consisting of displaying synchronously behaviors of identical and nonidentical Degn-Harrison systems, are given to show the effectiveness and applicability of the proposed synchronization schemes. Simultaneously, comparing time evolutions of the synchronization errors displayed in Fig. 7 and 18, for the two-dimensional space, we can conclude that the synchronization in nonidentical coefficients case (converges to zero as $t = 20$) is slower than the case of identical coefficients (converges to zero as $t = 10$). In our future research, we plan to study the synchronization behaviors in many types of spatiotemporal models, including the lattice maps and stochastic reaction-diffusion systems.

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