Secret parameters in quantum bit commitment

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The no-go theorem of unconditionally secure quantum bit commitment depends crucially on the assumption that Alice knows in detail all the probability distributions generated by Bob. We show that if a protocol is concealing, then the cheating unitary transformation is independent of any parameters (including probability distributions) secretly chosen by Bob, so that Alice can calculate it without knowing Bob’s secret choices. Otherwise the protocol cannot be concealing. Our result shows that the original impossibility proof was based on an incorrect assumption, despite the fact that its conclusion remains valid within the adopted framework. Furthermore, our result eliminates a potential loophole in the no-go theorem.

The security of quantum bit commitment (QBC) is an important issue in quantum cryptography because QBC is a primitive which can be used as the building block of other important two-party cryptographic protocols [1].

A QBC protocol involves two parties customarily named Alice and Bob. Alice secretly commits to a bit \( b \) (0 or 1) which is to be revealed to Bob at a later time. In order to bind Alice to her commitment, the two parties execute a series of quantum and/or classical procedures, so that at the end of the commitment phase, Bob is in possession of a quantum mechanical state \( |\psi^{(b)}_B\rangle\). The idea is that, with additional classical information from Alice in the unveiling phase (when she unveils the value of \( b \)), Bob can use \( |\psi^{(b)}_B\rangle\) to check whether Alice is honest. A QBC protocol is said to be binding if Alice cannot change her commitment or Bob will find out. Furthermore it is concealing if Bob can obtain no information about the value of \( b \) before it is unveiled, which implies that the encoding density matrix \( \rho^{(b)}_B \) of the state \( |\psi^{(b)}_B\rangle \) is independent of the value of \( b \), i.e.,

\[
\rho^{(0)}_B = \rho^{(1)}_B.
\]

A QBC protocol is secure if and only if it is both binding and concealing. Moreover, if a protocol is secure even if Alice and Bob had unlimited computational power, then it is said to be unconditionally secure.

In 1997, Lo and Chau [2, 3] and Mayers [4, 5] proved that unconditionally secure QBC is impossible. In a nutshell, the proof goes as follows. It is observed that the commitment process, which may involves any number of rounds of quantum and classical exchange of information between Alice and Bob, can always be represented by an unitary transformation \( U^{(b)} \) on some initial state \( |\phi^{(b)}_{AB}\rangle \) in the combined Hilbert space \( H_A \otimes H_B \) of Alice and Bob:

\[
|\Psi^{(b)}_{AB}\rangle = U^{(b)}|\phi^{(b)}_{AB}\rangle.
\]

Without loss of generality, we can take \( |\phi^{(b)}_{AB}\rangle \) and \( |\Psi^{(b)}_{AB}\rangle \) to be pure states. In this approach, Alice and Bob do not fix their undisclosed classical parameters in the commitment phase, but leave them undetermined at the quantum level instead. This is called quantum purification. In general it requires that Alice and Bob have access to quantum computers with unrestricted capacities, which is consistent with the assumption that they have unlimited computational power.

Therefore instead of honestly following the original protocol, Alice can always follow a modified protocol as described above, so that at the end of the commitment phase, there exists a pure state \( |\Psi^{(b)}_{AB}\rangle \) in \( H_A \otimes H_B \). As long as the reduced density matrix on Bob’s side is unchanged, i.e.,

\[
\text{Tr}_A |\Psi^{(b)}_{AB}\rangle \langle \Psi^{(b)}_{AB}| = \rho^{(b)}_B,
\]

Bob has no way of knowing what Alice has actually done. Then it follows from Schmidt decomposition theorem [5, 6] that,

\[
|\Psi^{(0)}_{AB}\rangle = \sum_i \sqrt{\lambda_i} |\phi_A^i \rangle \otimes |\psi_B^i \rangle,
\]

and

\[
|\Psi^{(1)}_{AB}\rangle = \sum_i \sqrt{\lambda_i} |\phi_A^i \rangle \otimes |\psi_B^i \rangle,
\]
where \( \{|e_i^A\rangle, \{e_i^A\rangle, \{\psi^\omega_B\rangle \) are orthonormal bases in the respective Hilbert spaces as indicated, and \( \lambda \)'s are real coefficients.

Notice that apart from the sets of bases \( \{|e_i^A\rangle, \{e_i^A\rangle, \{\psi^\omega_B\rangle \) and \( \{\psi^\omega_B\rangle \) are identical. Since \( \{|e_i^A\rangle, \{\psi^\omega_B\rangle \) are related by an unitary transformation \( U_A \) acting on Alice’s Hilbert space \( H_A \), only, we have

\[
|\psi^{(1)}_{AB}\rangle = U_A|\psi^{(0)}_{AB}\rangle.
\]

The existence of \( U_A \) implies that Alice has a sure-win cheating strategy (called EPR attack): Alice always commits to \( b = 0 \) in the beginning. Later on, if she wants to keep her initial commitment, she unveils as prescribed. However if she wants to switch to \( b = 1 \) instead, she just needs to apply the unitary transformation \( U_A \) to the particles in her control, and then proceeds as if she had committed to \( b = 1 \) in the first place. The crucial point is that, because of Eq. (1), it is impossible for Bob to find out what Alice actually did, and he would conclude that she is honest in either case. Hence if a QBC protocol is concealing, it cannot be binding at the same time. This is the conclusion of the “no-go theorem” of unconditionally secure QBC.

Note that the no-go theorem only proves the existence of the cheating unitary transformation \( U_A \) in a QBC protocol which is concealing, but there is no proof that \( U_A \) is always known to Alice. The point is, at the end of the commitment phase, the overall state \( |\psi^{(b)}_{AB}(\omega)\rangle \) may depend on some unknown parameter \( \omega \) secretly chosen by Bob. If the reduced density matrix

\[
\rho^{(b)}_B(\omega) = \text{Tr}_A|\psi^{(b)}_{AB}(\omega)\rangle \langle \psi^{(b)}_{AB}(\omega)|
\]

is independent of \( b \), then in principle a cheating transformation \( U_A(\omega) \) exists, so that

\[
|\psi^{(1)}_{AB}(\omega)\rangle = U_A(\omega)|\psi^{(0)}_{AB}(\omega)\rangle.
\]

However without the knowledge of \( \omega \), Alice cannot calculate \( U_A(\omega) \) by herself. As a result unconditionally secure QBC may be possible. This is a potential loophole of the no-go theorem.

The no-go theorem emphasizes that one should purify all undisclosed classical variables in analyzing the security issues. Even so, the question remains: What if Bob is allowed to choose probability distributions secretly? To this question, the authors of the no-go theorem state that “In order that Alice and Bob can follow the procedures, they must know the exact forms of all unitary transformations involved” \[\text{[2]}\] and “It is a principle that we must assume that every participant knows every detail of the protocol, including the distribution of probability of a random variable generated by another participant” \[\text{[3]}\]. In other words the no-go theorem asserts without proof that in any QBC protocol the overall state \( |\psi^{(b)}_{AB}\rangle \) cannot contain any unknown parameters. This assertion is in fact not correct, and it has caused confusion among researchers. Without clarifying this issue, the impossibility proof is not complete and the no-go theorem will continue being challenged \[\text{[8]}\]. In any case, as long as it does not jeopardize the security of a protocol, there is no reason why a party has to disclose the values of any secret parameters he/she might have chosen in the commitment phase.

To settle this issue, we prove the following theorem. The secret parameter \( \omega \) will be taken to be a probability distribution, because in a fully quantum description, probability distributions are the only unknowns left. Except for the issue of secret parameters, we shall stay within the QBC framework adopted by the no-go theorem.

**Theorem 1** If a QBC protocol is concealing, then the cheating unitary transformation is independent of any probability distributions (\( \omega \)'s) secretly chosen by Bob.

**Proof** If Bob is allowed to choose \( \omega \) in secret, he can always postpone his choice with the help of a quantum computer. That means, instead of picking a particular \( \omega = \omega_i \) and keeping it secret, he can purify his choices with a probability distribution \( \pi = \{p_i\} \). The resulting overall state is given by

\[
|\psi^{(b)}_{AB}(\pi)\rangle = \sum_i \sqrt{p_i} |\psi^{(b)}_{AB}(\omega_i)\rangle |\chi_i\rangle,
\]

where \( \{|\chi_i\rangle\} \) is a set of orthonormal ancilla states in Bob’s Hilbert space \( H_B \). The new density matrix is given by

\[
\rho^{(b)}_B(\pi) = \text{Tr}_A|\psi^{(b)}_{AB}(\pi)\rangle \langle \psi^{(b)}_{AB}(\pi)|.
\]

Since the protocol is concealing, we have

\[
\rho^{(0)}_B(\pi) = \rho^{(1)}_B(\pi)
\]

for all possible \( \pi \). Consider the case where \( p_i \neq 0 \), for all \( i \). According to the no-go theorem there exists a cheating unitary transformation \( U'_A \), such that

\[
|\psi^{(1)}_{AB}(\pi)\rangle = U'_A |\psi^{(0)}_{AB}(\pi)\rangle.
\]

It is easy to see that this same \( U'_A \) also transforms \( |\psi^{(b)}_{AB}(\omega_i)\rangle \) to \( |\psi^{(1)}_{AB}(\omega_i)\rangle \) for all possible \( \omega_i \), i.e.,

\[
|\psi^{(1)}_{AB}(\omega_i)\rangle = U'_A |\psi^{(0)}_{AB}(\omega_i)\rangle, \quad \forall \omega_i.
\]
Suppose Bob chooses an unitary operator in \( \{ V_k \} \) with a probability distribution \( \omega_i = \{ q_{ik} \} \) and applies it to a state \( |\phi_{AB}\rangle \), such that
\[
|\psi_{AB}\rangle = V_k |\phi_{AB}\rangle.
\] (15)

As is well known, if \( V_k \) is not disclosed, Bob can postpone (or purify) his decision by entangling with a set of orthonormal ancilla states \( \{|\xi_k\rangle\} \), so that instead of \( |\psi_{AB}\rangle \), he generates
\[
|\Psi_{AB}(\omega_i)\rangle = \sum_k \sqrt{q_{ik}} |\xi_k\rangle V_k |\phi_{AB}\rangle.
\] (16)

Likewise, if \( \omega_i \) is not disclosed, Bob can also purify his choices with another probability distribution \( \pi = \{ p_i \} \), such that
\[
|\Psi'_{AB}(\pi)\rangle = \sum_i \sqrt{p_i} |\chi_i\rangle |\Psi_{AB}(\omega_i)\rangle,
\] (17)
where \( \{|\chi_i\rangle\} \) is another set of orthonormal ancilla states.

**Theorem 2** Purifying a probability distribution [in Eq. (17)] is equivalent to picking a new effective one [in Eq. (16)].

**Proof** Define
\[
q'_k = \sum_i p_i q_{ik},
\] (19)
so that
\[
\sum_k q'_k = 1.
\] (20)

On the right hand side of Eq. (18), we write
\[
\sum_i \sqrt{p_i} q_{ik} |\chi_i\rangle = \sqrt{q'_k} \sum_i \sqrt{p_i q_{ik}} q'_k |\chi_i\rangle = \sqrt{q'_k} |\chi'_k\rangle,
\] (21)
where
\[
|\chi'_k\rangle = \sum_i \sqrt{p_i q_{ik}/q'_k} |\chi_i\rangle.
\] (22)

Note that the \( |\chi'_k\rangle \)'s are normalized but not necessarily orthogonal.

Substituting Eq. (21) into Eq. (18), we get
\[
|\Psi'_{AB}(\pi)\rangle = \sum_k \sqrt{q'_k} |\chi'_k\rangle |\xi_k\rangle V_k |\phi_{AB}\rangle,
\] (23)

In summary, we find that there is nothing wrong with secret parameters in QBC. We prove that in a concealing protocol, the cheating unitary transformation is independent of any parameters secretly chosen by Bob. Our result shows that the original proof of the no-go theorem \( 2 \times 2 \times 4 \times 5 \) was based on an incorrect assumption. Nevertheless, even with secret parameters, unconditional security remains impossible within the framework adopted by the no-go theorem.

**Appendix**
\[ |\xi'_{k}\rangle \equiv |\chi'_{k}\rangle |\xi_{k}\rangle \] (24)

are the new orthonormal ancilla states. Comparing with Eq. (16), we obtain the desired result
\[ |\Psi'_{AB}(\pi)\rangle = |\Psi_{AB}(\omega_{j}(\pi))\rangle, \] (25)

where \( \omega_{j}(\pi) \equiv \{q'_{k}\} \) is the new effective probability distribution.

Using the above result, we can prove the following corollary:

**Corollary** It is in general not meaningful to specify a probability distribution to an untrustful party in any quantum protocol, because he/she can always cheat.

**Proof** Suppose the protocol specifies that Bob should take certain action \( V_{k} \) on each qubit (or group of qubits) in his possession according to a probability distribution \( \omega_{j} = \{q'_{k}\} \). According to the theorem just proven, he can always generate a superposition of distributions with appropriately chosen \( p_{i} \)’s, such that the effective distribution is \( \omega_{j} \) [see Eqs. (17, 20)]. Obviously Bob would have no problem passing any checks concerning \( \omega_{j} \). In general some qubits are measured and discarded in the checking procedure. For each of the remaining qubits, Bob could either stay with \( \omega_{j} \), or he could measure the ancilla states \( \{|\chi_{i}\}\} \) in Eq. (17) to obtain a new distribution \( \omega_{i} \) which is not necessarily equal to \( \omega_{j} \). For a large number of qubits, the probability that \( \omega_{j} \) is obtained for every qubit is exponentially small. Hence it is not meaningful to specify a probability distribution to an untrustful Bob, because one can never be sure that he is honest.

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