COPRODUCT FOR YANGIANS OF AFFINE KAC-MOODY ALGEBRAS

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Abstract. Given an affine Kac-Moody algebra $g$ and its associated Yangian $Y(g)$, we explain how to construct a coproduct for $Y(g)$. In order to prove that this coproduct is an algebra homomorphism, we obtain, in the first half of this paper, a minimalistic presentation of $Y(g)$ when $g$ is, more generally, a symmetrizable Kac-Moody algebra.

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1. Introduction

The quantized enveloping algebra $U_{\hbar}(g_0)$ of a simple Lie algebra $g_0$ is a Hopf algebra which provides a quantization of a certain Lie bialgebra structure on $g_0$. Being a Hopf algebra, it not only possesses an associative product, but is also equipped with a coproduct. This is what distinguishes it from the enveloping algebra $U(g_0)$ of $g_0$ because, as algebras, $U_{\hbar}(g_0)$ is actually a trivial deformation of $U(g_0)$. (This is a consequence of the vanishing of the second Hochschild cohomology group of $U(g_0)$ - see Theorem XVIII.4.1 in [Kas95].) The definition of the Drinfeld-Jimbo quantized enveloping algebra can be extended to any symmetrizable Kac-Moody algebra. Furthermore, using what is commonly referred to as Drinfeld’s second realization [Dri87], it is even possible to define affinizations of quantized Kac-Moody algebras [Her05]. These include, in particular, quantum toroidal algebras.

There are two families of quantized enveloping algebras of affine type: the Drinfeld-Jimbo quantum affine algebras $U_{\hbar}(g)$ and the Yangians $Y_{\hbar}(g_0)$. (Here, $g$ is the affine Lie algebra corresponding to $g_0$.) Although a priori quite different, there exist completions of these algebras which are in fact isomorphic [GTL13] (see also [GM12] for the proof of a weaker result). Furthermore, tensor equivalences between categories of representations of these two quantum groups have been established in [GTL16, GTL14]. It is also possible to associate Yangians to any symmetrizable Kac-Moody algebra, in particular to affine Lie algebras: one thus obtains...
the affine Yangians. Quantum toroidal algebras and affine Yangians are two of the main examples of quantized enveloping algebras of double affine type, a third example being provided by the deformed double current algebras \[ \text{Gua07, Gua09, GY16, TLY}. \]

For both quantum affine algebras and Yangians there is a standard coproduct: in the former case, it is the coproduct given in terms of the Kac-Moody generators (as in \[ \text{CP95, Definition-Prop. 6.5.1} \]), while in the latter case it is the coproduct given in terms of the generators \( \{X, J(X)\}_{X \in \mathfrak{g}_0} \) (as in \[ \text{CP95, Theorem 12.1.1} \]). There also exist non-standard coproducts on these two families which are originally due to V. Drinfeld - see Definition 3.2 in \[ \text{DF93} \] and Section 6 in \[ \text{DK00} \]. Actually, the authors of \[ \text{DK00} \] need to consider the double of the Yangian, but it is also possible to degenerate the non-standard coproduct on quantum affine algebras to obtain one on the Yangian itself - see \[ \text{GTL14} \]. These have also appeared in the recent work \[ \text{YZ16} \] via an isomorphism between the Yangian and a cohomological Hall algebra which turns out to be an isomorphism of bialgebras when the Yangian is equipped with the non-standard coproduct and the cohomological Hall algebra is equipped with the comultiplication constructed in \[ \text{loc. cit.} \]. Moreover, these non-standard coproducts are related by a meromorphic twist to the standard coproducts on these algebras - see \[ \text{GTL14} \]. They are not exactly genuine coproducts as they involve infinite sums and map into certain completed tensor products: see \[ \text{Her05} \] and \[ \text{GTL14} \]. Additionally, they cannot always be used to define a module structure on tensor products of two modules because of convergence issues. The definitions of these non-standard coproducts extend automatically to quantum toroidal algebras and affine Yangians. In this context, they were used in the work of D. Hernandez \[ \text{Her05, Her07} \] and of B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin \[ \text{FFJ+11a, FFJ+11b, FJMM13} \]. The papers \[ \text{FFJ+11a, FFJ+11b} \] are about the quantum toroidal algebra of \( \mathfrak{gl}_1 \) (which was however given the name quantum continuous \( \mathfrak{gl}_\infty \)). There is also an affine Yangian of type \( \mathfrak{gl}_1 \) studied, for instance, in \[ \text{Isy17, IB15} \]. This affine Yangian was shown in \[ \text{AS13} \] to be isomorphic to a certain algebra \( \text{SH}^c \) which is a sort of stable limit of spherical trigonometric Cherednik algebras of type \( \mathfrak{gl}_1 \) and was introduced in \[ \text{SV13} \] where it was used to prove a version of the AGT-conjecture. The algebra \( \text{SH}^c \), and thus the affine Yangian of \( \mathfrak{gl}_1 \), admits a topological coproduct which is close to the standard coproduct: see Theorem 7.9 in \[ \text{SV13} \]. It is not clear that the proof in \[ \text{SV13} \] that the coproduct is well-defined can be modified for general \( Y_\hbar(\mathfrak{g}) \). We will not consider the affine Yangian of type \( \mathfrak{gl}_1 \) in the present paper.

The goal of the present paper is to introduce a coproduct \( \Delta \) on affine Yangians which is a natural analog of the standard coproduct on Yangians of finite dimensional simple Lie algebras. We first define it via the action of the affine Yangian on the tensor product of two modules in the category \( \mathcal{O} \) (Definition 4.7) and prove that it is an algebra homomorphism (Theorem 4.11). (Our proof also works for \( Y(\mathfrak{g}_0) \); in this case, Theorem 4.11 is, of course, already known, but a proof has never appeared in the literature.) In the subsequent section (Section 5), we introduce a completion of the tensor product of the affine Yangian with itself and explain how \( \Delta \) can be viewed as an algebra homomorphism from the affine Yangian into that completion: see Proposition 5.17. That completion is defined using a grading which is not compatible with the algebra structure on \( Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g}) \), so an argument is needed to prove that the multiplication on \( Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g}) \) extends to it (see Proposition 5.13). One advantage of our coproduct is that it can be used to define a module structure on the tensor product of two modules in the category \( \mathcal{O} \) without any convergence issues.

It is natural to conjecture that this new coproduct \( \Delta \) is related to the coproduct alluded to two paragraphs above via a certain twist as in \[ \text{GTL14} \], but it is not at all clear that this is the case because the twist given in \[ \text{GTL14} \] is constructed using the lower triangular part
of the universal $R$-matrix of the Yangian and no universal $R$-matrix is known for the affine Yangians \[\text{[GTL]}.\]

In order to prove Theorem \[\text{[4.11]}\] we need to simplify the presentation of the affine Yangians: this is accomplished in Section \[\text{[2]}\] - see Theorem \[\text{[2.12]}\]. The results of this section are actually valid more generally for Yangians of symmetrizable Kac-Moody algebras which satisfy certain mild conditions. For affine Yangians, these conditions are equivalent to the assumption that $\mathfrak{g}$ is not of type $A^{(1)}_1$ or $A^{(2)}_2$. However, we expect that Theorem \[\text{[4.11]}\] holds more generally for all affine Lie algebras and even for any symmetrizable Kac-Moody algebra.

When $\mathfrak{g}$ is of affine type $A^{(1)}_{n-1}$, it is possible to introduce an extra parameter $\varepsilon$ in the definition of $Y_{\hbar}(\mathfrak{g})$ in order to obtain a two parameter Yangian $Y_{\hbar,\varepsilon}(\mathfrak{g})$ (see Definition \[\text{[6.1]}\]). All the main results of this paper hold in this greater generality: this is briefly explained in Section \[\text{[6]}\]. These two parameter Yangians have been studied by the first named author in \[\text{[Gua05, Gua07]}\]. (Quantum toroidal algebras of type $A$ can also depend on two parameters, see \[\text{[VV98]}\].)

When $\mathfrak{g}$ is symmetric (including the $\mathfrak{gl}_1$-case), there is a geometric construction of the Yangian using quiver varieties \[\text{[MO12]}\]. This construction gives a coproduct, as well as the universal $R$-matrix. By the construction in \[\text{[Var00]}\], we have a homomorphism from $Y_{\hbar}(\mathfrak{g})$ to the Yangian in \[\text{[MO12]}\]. Our formula \[\text{[4.8]}\] implies that it is compatible with the coproduct on both Yangians. Since we do not know that it is an isomorphism (or whether it is injective or surjective), \[\text{[MO12]}\] does not imply our main result, but it gives evidence that Theorem \[\text{[4.11]}\] is true in a more general setting.

In \[\text{[FKP16]}\], the authors define a coproduct on shifted Yangians which is related to the coproduct on $Y_{\hbar}(\mathfrak{g})$ via shift maps: see Subsection 4.6 in \textit{loc. cit.} Their Theorem 4.8 states that this coproduct is well-defined in the sense that it respects the defining relations of the shifted Yangians. The proof of that theorem depends on the main results of our present paper regarding the coproduct $\Delta$ on $Y_{\hbar}(\mathfrak{g})$.

It is natural to expect that a coproduct similar to the one constructed in the present paper exists for quantum toroidal algebras. It would also be interesting to obtain one for deformed double current algebras as it would certainly be useful to make progress in understanding their largely unexplored representation theory.

2. The Yangian of a Kac-Moody Lie algebra

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra associated with the indecomposable Cartan matrix $(a_{ij})_{i,j \in I}$ where $I$ is the set of vertices of the Dynkin diagram corresponding to $\mathfrak{g}$. We also fix an invariant inner product $(\ , \ )$ on $\mathfrak{g}$. We normalize the Chevalley generators $x_i^\pm$, $h_i$ so that $(x_i^+, x_i^-) = 1$ and $h_i = [x_i^+, x_i^-]$. Let $\Delta$ ($\Delta^{\text{re}}$, resp. $\Delta^{\text{im}}$) be the set of all roots of $\mathfrak{g}$ (of all real roots, resp. of all imaginary roots), and let $\Delta_{\pm}$ be the sets of positive and of negative roots. $\Delta^{\text{re}}_{\pm}$ is defined similarly. When $\mathfrak{g}$ is an affine Lie algebra, we let $\delta$ be the positive imaginary root such that $(\mathbb{Z} \setminus \{0\})\delta$ is the set of all imaginary roots of $\mathfrak{g}$ \[\text{[Kac90]}\]. Let $\mathfrak{g}'$ be the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$.

In the definition below, and consequently for the rest of this paper, we will assume that $\mathfrak{g}$ is not of type $A^{(1)}_1$; see the definition in Section 1.2 in \[\text{[TBI5]}\] and Definition 5.1 in \[\text{[Kod15]}\] for the correct definition of the Yangian in this case.

Definition 2.1. Let $\hbar \in \mathbb{C}$. The Yangian $Y_{\hbar}(\mathfrak{g})$ is the associative algebra over $\mathbb{C}$ with generators $x_{ir}^\pm$, $h_{ir}$ ($i \in I$, $r \in \mathbb{Z}_{\geq 0}$) subject to the following defining relations:

\[ [h_{ir}, h_{js}] = 0, \]
\[(h_{i0}, x_{js}^\pm) = \pm(\alpha_i, \alpha_j)x_{js}^\pm, \]
\[(x_{ir}^\pm, x_{js}^-) = \delta_{ij} h_{i,r+s}, \]
\[(h_{i,r+1}, x_{js}^\pm) - [h_{i,r}, x_{js}^\pm] = \pm h(\alpha_i, \alpha_j) \left( h_{i,r} x_{js}^\pm + x_{js}^- h_{i,r} \right), \]
\[(x_{i,r+1}, x_{js}^\pm) - [x_{i,r}, x_{js}^\pm] = \pm h(\alpha_i, \alpha_j) \left( x_{i,r} x_{js}^\pm + x_{js}^- x_{i,r} \right), \]
\[\sum_{\sigma \in S_b} [x_{ir}^\pm x_{is}^\pm, x_{js}^\pm] = 0 \text{ if } i \neq j, \]

where \( b = 1 - a_{ij}. \) The Yangian \( Y_h(g) \) is defined as the quotient of \( Y_h(g') \otimes_C U(h) \), where \( h \) is the Cartan subalgebra of \( g \), by the following relations:

\[ h_{i0} = \frac{(a_i, a_i)}{2} \alpha_i^\vee \text{ where the simple coroot } \alpha_i^\vee \text{ belongs to } h, \]
\[ [h, h_{ir}] = 0, \quad [h, x_{ir}^\pm] = (\alpha_i, h) x_{ir}^\pm \text{ for } h \in h. \]

Given two elements \( a, b \) of some algebra \( \mathcal{A} \), we set \( \{a, b\} = ab + ba. \) In particular, the right-hand sides of (2.3) and (2.5) could be written in terms of \( \{h_{ir}, x_{js}^-\} \) and \( \{x_{ir}^\pm, x_{js}^\pm\} \), respectively.

Observe that for any pair of non-zero complex numbers \( h_1, h_2 \in \mathbb{C}^\times \), we have \( Y_{h_1}(g) \cong Y_{h_2}(g). \) With this in mind, we set \( h = 1 \) and denote \( Y_h(g) \) simply by \( Y(g) \) hereafter (except in Section 3). Similarly, we denote \( Y_h(g') \) by \( Y(g') \).

Note that the assignment \( x_{ir}^\pm, h \mapsto x_{i0}^\pm, h \) gives an algebra homomorphism \( \iota: U(g) \rightarrow Y(g) \) which is injective. Indeed, if we start with an element \( \omega \) of the set \( X^+ \) of dominant, integral weights of \( g \) and a set of scalars \( A = \{a_{ir} \in \mathbb{C} \mid i \in I, r \geq 0\} \) such that \( a_{i0} = \omega(\iota(h_{i0})) \), we can define the Verma module \( V(\omega, A) \) over \( Y(g) \) by starting with a one dimensional space \( \mathbb{C} \cdot 1_\omega \) and making it into a representation of \( Y(g)^{\geq 0} \) via \( h_{ir}(1_\omega) = a_{ir} 1_\omega, x_{ir}^+(1_\omega) = 0 \) and inducing it to \( Y(g) \). (Here, \( Y(g)^{\geq 0} \) is the subalgebra generated by \( h_{ir} \) and \( x_{ir}^+ \) for all \( i \in I, r \geq 0. \)) Let \( V(\omega) \) be the \( U(g) \)-submodule of \( \iota^*(V(\omega, A)) \) generated by \( 1_\omega \). Then \( V(\omega) \) is a highest weight module; let \( L(\omega) \) be its unique irreducible quotient, which is integrable since \( \omega \) is integral and dominant. We have the following inclusions where \( \text{Ann} \) denotes the annihilator:

\[ \ker(\iota) \subseteq \text{Ann}(\iota^*(V(\omega, A))) \subseteq \text{Ann}(V(\omega)) \subseteq \text{Ann}(L(\omega)). \]

Since \( \bigcap_{\omega \in X^+} \text{Ann}(L(\omega)) = \{0\} \) (which can be proved using the ideas in Section 3.5 in [Lus10] and in Proposition 5.11 in [Jan96]), it follows that \( \ker(\iota) = \{0\} \), hence \( \iota \) is injective.

### 2(i). A minimalistic presentation of \( Y(g) \)

In this subsection, we state the first main result of this paper (Theorem 2.12), which we will prove in Subsection 2(ii) below.

From the defining relations, we can see that \( Y(g') \) is generated by \( x_{i0}^\pm, h_{i0} \) and \( h_{i1} \) with \( i \in I \) (see, for instance, [Lev93]). In fact, we can obtain \( x_{i,r}^\pm, h_{ir} \) inductively from the relations

\[ x_{i,r+1} = \pm(\alpha_i, \alpha_i)^{-1} [h_{i1} - \frac{1}{2} h_{i0}^2, x_{ir}^\pm], \]
\[ h_{ir} = [x_{ir}^+, x_{i0}^-]. \]

To simplify the first of these formulas as well as future computations, we introduce the auxiliary generators \( \tilde{h}_{i1} \), with \( i \in I \), by setting

\[ \tilde{h}_{i1} \overset{\text{def}}{=} h_{i1} - \frac{1}{2} h_{i0}^2. \]
Then \( x_{i,r+1}^{\pm} = \pm(\alpha_i, \alpha_i)^{-1}[\tilde{h}_{i1}, x_{i,r}^{\pm}] \) and (2.5) with \((r, s) = (0, 0)\) can be rewritten as

\[
(2.11) \quad [\tilde{h}_{i1}, x_{j0}^{\pm}] = \pm(\alpha_i, \alpha_j)x_{j1}^{\pm}.
\]

We want to reduce the number of relations to make it easier to check the compatibility of the coproduct \( \Delta \) to be introduced in Section 4(ii). Such work was done by S. Levendorskii for \( \mathfrak{g} \) finite dimensional in [Lev93]: See Theorem 1.2 therein.

**Theorem 2.12.** Suppose that, for any \( i, j \in I \) with \( i \neq j \), the matrix \( \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} \) is invertible. Moreover, assume also that there exists one pair of indices \( i, j \in I \) such that \( a_{ij} = -1 \). Then all the defining relations of \( Y(\mathfrak{g}') \) can be deduced from

\[
(2.13) \quad [h_{ir}, h_{js}] = 0 \quad (0 \leq r, s \leq 1),
\]

\[
(2.14) \quad [h_{i0}, x_{js}^{\pm}] = \pm(\alpha_i, \alpha_j)x_{js}^{\pm} \quad (s = 0, 1),
\]

\[
(2.15) \quad [x_{ir}^{\pm}, x_{js}^{-}] = \delta_{ij}h_{i,r+s} \quad (0 \leq r + s \leq 1),
\]

\[
(2.16) \quad [\tilde{h}_{i1}, x_{j0}^{\pm}] = \pm(\alpha_i, \alpha_j)x_{j1}^{\pm} \quad (\tilde{h}_{i1} \overset{\text{def}}{=} h_{i1} - \frac{1}{2}h_{i0}^2),
\]

\[
(2.17) \quad [x_{i1}^{\pm}, x_{j0}^{\pm}] - [x_{i0}^{\pm}, x_{j1}^{\pm}] = \pm \frac{(\alpha_i, \alpha_j)}{2} (x_{i0}^{\pm}x_{j0}^{\pm} + x_{j0}^{\pm}x_{i0}^{\pm}),
\]

\[
(2.18) \quad \text{ad}(x_{i0}^{\pm}) - a_{ij}(x_{j0}^{\pm}) = 0 \quad \text{if} \ i \neq j.
\]

**Remark 2.19.** If \( \mathfrak{g} \) is of affine type, then \( \mathfrak{g} \) satisfies the conditions of the previous theorem provided it is not of type \( A_1^{(1)} \) or \( A_2^{(2)} \). Indeed, that \( (a_{kl})_{k,l \in \{i,j\}} \) is invertible for \( i \neq j \) is due to [Kac90, Proposition 4.7(b)], and the existence of a pair \((i,j)\) such that \( a_{ij} = -1 \) can be seen by inspection of the corresponding Dynkin diagram (see Section 4.7 of [Kac90]).

Observe that the statement of Theorem 1.2 in [Lev93] is precisely that \( Y(\mathfrak{g}) \) (where \( \mathfrak{g} \) is finite-dimensional and simple) is isomorphic to the unital associative algebra generated by the elements \( x_{i0}^{\pm}, h_{i0} \) and \( h_{i1} \), with \( i \in I \), subject to the defining relations (2.13)-(2.18) together with the relation

\[
(2.20) \quad [[\tilde{h}_{i1}, x_{i1}^{\pm}], x_{i1}^{-}] + [x_{i1}^{\pm}, [\tilde{h}_{i1}, x_{i1}^{-}]] = 0,
\]

where \( x_{i1}^{\pm} \) is defined by the first formula in (2.9) with \( r = 0 \). Moreover, Levendorskii’s argument also applies in the case where \( \mathfrak{g} \) is a symmetrizable Kac-moody Lie algebra satisfying the conditions of Theorem 2.12.

Unfortunately, for our purposes, the relation (2.20) is still difficult to work with. In the case where \( \mathfrak{g} \) is of type \( \mathfrak{sl}_{n+1} \) or \( \mathfrak{sl}_{n+1} \) with \( n \geq 3 \), this difficulty was addressed by the first named author in [Gua07], where it was shown that the relation (2.20) can be deduced from those given in the statement of Theorem 2.12.

2(ii). **Proof of Theorem 2.12** As consequence of the remarks made at the end of the previous subsection, to prove Theorem 2.12 it suffices to show that the relation (2.20) can be derived from (2.13)-(2.18). To prove this, we will proceed as follows: First, we establish that some of the relations (2.2)-(2.7) can be derived from (2.13)-(2.18) for certain values of \( i, j \) and \( r, s \). Then, following Levendorskii’s argument, we use these relations to establish a sequence of lemmas and propositions which allow us to conclude that (2.20) is indeed satisfied for all \( i \in I \).

Our first goal is the construction of an element \( \tilde{h}_{i2} \) such that \([\tilde{h}_{i2}, x_{i0}^{\pm}] = \pm(\alpha_i, \alpha_i)x_{i2}^{\pm} \pm (\alpha_i, \alpha_i)^3x_{i0}^{\pm}/12 \). This was done in [Lev93, Cor. 1.5]. We reproduce the proof in order to point
Lemma 2.21. The following relations are satisfied for all \(i, j \in I\) and \(r \in \mathbb{Z}_{\geq 0}\):
\[
[h_{i0}, x^\pm_{jr}] = \pm (\alpha_i, \alpha_j) x^\pm_{jr}, \quad [h_{i1}, x^\pm_{jr}] = \pm (\alpha_i, \alpha_j) x^\pm_{j,r+1}.
\]

Proof. One can show the second equality by induction on \(r\). If \(r = 0\), it is nothing but (2.16). The general case follows by using (2.23), \([\hat{h}_{i1}, \hat{h}_{i1}] = 0\) (which follows immediately from (2.13)) and the inductive assumption. The first equality can be proven in the same way. □

Lemma 2.22. The relation (2.6) holds when \(i = j\), \((r, s) = (1, 0)\), i.e.,
\[
(2.23) \quad [x^\pm_{i2}, x^\pm_{i0}] = \pm \frac{(\alpha_i, \alpha_i)}{2} (x^\pm_{i2} x^\pm_{i0} + x^\pm_{i0} x^\pm_{i1}).
\]

Proof. This follows immediately by applying \([\hat{h}_{i1}, \cdot]\) to (2.17) with \(i = j\). □

Lemma 2.24. The relation (2.5) holds when \(i = j\), \((r, s) = (1, 0)\), i.e.,
\[
(2.25) \quad [h^\pm_{i2}, x^\pm_{i0}] - [h^\pm_{i1}, x^\pm_{i1}] = \pm \frac{(\alpha_i, \alpha_i)}{2} (h^\pm_{i1} x^\pm_{i0} + x^\pm_{i0} h^\pm_{i1}).
\]

Proof. We rewrite the second equality in Lemma 2.21 with \(i = j\), \(r = 1\) as
\[
(2.25) \quad [h^\pm_{i1}, x^\pm_{i1}] - [h^\pm_{i0}, x^\pm_{i2}] = \pm \frac{(\alpha_i, \alpha_i)}{2} (h^\pm_{i0} x^\pm_{i1} + x^\pm_{i1} h^\pm_{i0}).
\]

We apply \([\cdot, x^\pm_{i0}]\) to (2.23) and combine the resulting relation with (2.25) to obtain the desired conclusion. □

Lemma 2.26. Suppose that \(i, j \in I\) and \(i \neq j\). The relations (2.4) and (2.6) hold for any \(r\) and \(s\).

Proof. We prove (2.6) by induction on \(r\) and \(s\). The same argument applies also to (2.4). The initial case \(r = s = 0\) is our assumption.

Let \(X^\pm(r, s)\) be the result of subtracting the right-hand side of (2.6) from the left-hand side. We apply \([\hat{h}_{i1}, \cdot]\) and \([\hat{h}_{j1}, \cdot]\) to (2.6) to get
\[
0 = (\alpha_i, \alpha_i) X^\pm(r + 1, s) + (\alpha_i, \alpha_j) X^\pm(r, s + 1),
\]
\[
0 = (\alpha_i, \alpha_j) X^\pm(r + 1, s) + (\alpha_j, \alpha_j) X^\pm(r, s + 1).
\]

Since the determinant of
\[
\begin{pmatrix}
(\alpha_i, \alpha_i) & (\alpha_i, \alpha_j) \\
(\alpha_j, \alpha_i) & (\alpha_j, \alpha_j)
\end{pmatrix}
\]
is nonzero by assumption, we have \(X^\pm(r + 1, s) = 0 = X^\pm(r, s + 1)\). Therefore, the assertion is true by induction. □

Lemma 2.27. Suppose that \(i, j \in I\) and \(i \neq j\). Then (2.5) holds for any \(r\) and \(s\).

Proof. We prove the + case, the − case can be proved in the same way. We simply use Lemma 2.26
\[
[h_{i,r+1}, x^+_{js}] = [[x^+_{i,r+1}, x^+_{i0}], x^+_{js}] = [[x^+_{i,r+1}, x^+_{js}], x^+_{i0}]
\]
\[
= [x^+_{ir}, x^+_{j,s+1}], x^+_{i0}] + \frac{(\alpha_i, \alpha_j)}{2} [x^+_{ir} x^+_{js} + x^+_{js} x^+_{ir}, x^+_{i0}]
\]
\[
= [h_{ir}, x^+_{j,s+1}] + \frac{(\alpha_i, \alpha_j)}{2} (h_{ir} x^+_{js} + x^+_{js} h_{ir}).
\]
We are now prepared to introduce the element $\tilde{h}_{i_2}$. For each $i \in I$, we define $\tilde{h}_{i_2}$ by

$$\tag{2.28} \tilde{h}_{i_2} = h_{i_2} - h_{i_0} h_{i_1} + \frac{1}{3} h_{i_0}^3.$$

The next proposition is a special case of Lemma 1.4 in [Lev93].

**Proposition 2.29.** For any $i, j \in I$, the following identity holds:

$$\tag{2.30} [\tilde{h}_{i_2}, x_{j_0}^\pm] = \pm (\alpha_i, \alpha_j) x_{j_2}^\pm \pm \left(\frac{1}{12} (\alpha_i, \alpha_j)^3 x_{j_0}^\pm.\right.$$

**Proof.** This follows from Lemma 2.24 and Lemma 2.27. Here are the details for the sake of the reader.

$$[\tilde{h}_{i_2}, x_{j_0}^\pm] = [h_{i_2}, x_{j_0}^\pm] - [h_{i_0} h_{i_1}, x_{j_0}^\pm] + \frac{1}{3} [h_{i_0}^3, x_{j_0}^\pm]$$

$$= [h_{i_1}, x_{j_1}^\pm] \pm (\alpha_i, \alpha_j) \left[ \frac{1}{2} (h_{i_1} x_{j_0}^\pm + x_{j_0}^\pm h_{i_1}) - (x_{j_0}^\pm h_{i_1} + h_{i_0} x_{j_1}^\pm) \right.$$

$$+ \frac{1}{3} (x_{j_0}^\pm h_{i_0}^2 + h_{i_0} x_{j_0}^\pm h_{i_0} + h_{i_0}^2 x_{j_0}^\pm) \right]$$

$$= [h_{i_0}, x_{j_2}^\pm] \pm (\alpha_i, \alpha_j) \left[ \frac{1}{2} (h_{i_0} x_{j_1}^\pm + x_{j_1}^\pm h_{i_0}) + \frac{1}{2} [h_{i_1}, x_{j_0}^\pm] - h_{i_0} x_{j_1}^\pm \right.$$

$$+ \frac{1}{6} (2 x_{j_0}^\pm h_{i_0}^2 - h_{i_0} x_{j_0}^\pm h_{i_0} - h_{i_0}^2 x_{j_0}^\pm) \right]$$

$$= \pm (\alpha_i, \alpha_j) \left[ x_{j_2}^\pm + \frac{1}{2} ( [h_{i_1}, x_{j_0}^\pm] - [h_{i_0}, x_{j_1}^\pm] ) + \frac{1}{6} ( [x_{j_0}^\pm, h_{i_0}] h_{i_0} + [x_{j_0}^\pm, h_{i_0}^2] ) \right.$$}

$$= \pm (\alpha_i, \alpha_j) \left[ x_{j_2}^\pm \pm \frac{(\alpha_i, \alpha_j)}{4} (h_{i_0} x_{j_0}^\pm + x_{j_0}^\pm h_{i_0}) + \frac{(\alpha_i, \alpha_j)}{6} (2 x_{j_0}^\pm h_{i_0} + h_{i_0} x_{j_0}^\pm) \right]$$

$$= \pm (\alpha_i, \alpha_j) \left[ x_{j_2}^\pm \pm \frac{(\alpha_i, \alpha_j)}{12} [h_{i_0}, x_{j_0}^\pm] \right] = \pm (\alpha_i, \alpha_j) x_{j_2}^\pm \pm \frac{(\alpha_i, \alpha_j)}{12} x_{j_0}^\pm. \quad \square$$

Now we are ready to check several cases of (2.7).

**Lemma 2.31.** The relation (2.7) holds for the following cases:

1. $r_1 = \cdots = r_b = 0$, $s \in \mathbb{Z}_{\geq 0}$,
2. $r_1 = 1$, $r_2 = \cdots = r_b = 0$, $s \in \mathbb{Z}_{\geq 0}$,
3. $r_1 = 2$, $r_2 = \cdots = r_b = 0$, $s \in \mathbb{Z}_{\geq 0}$,
4. ($b \geq 2$ and) $r_1 = r_2 = 1$, $r_3 = \cdots = r_b = 0$, $s \in \mathbb{Z}_{\geq 0}$.

**Proof.** Let $\vec{r} = (r_1, \ldots, r_b)$ and denote the left hand side of (2.7) by $X^\pm(\vec{r}, s)$. We first show $X^\pm(\vec{0}, s) = 0$ by induction on $s \geq 0$. If $s = 0$, this is just (2.18). Suppose that $X^\pm(\vec{0}, s) = 0$
Lemma 2.32. We have

\[
\frac{(\alpha_i, \alpha_j)}{(b-1)!} X^\pm((1,0,\ldots,0), s) + \frac{(\alpha_j, \alpha_j)}{b!} X^\pm(0, s+1) = 0,
\]

\[
\frac{(\alpha_i, \alpha_j)}{(b-1)!} X^\pm((1,0,\ldots,0), s) + \frac{(\alpha_j, \alpha_j)}{b!} X^\pm(0, s+1) = 0.
\]

Since the determinant of \( \begin{pmatrix} (\alpha_i, \alpha_i) & (\alpha_i, \alpha_j) \\ (\alpha_j, \alpha_i) & (\alpha_j, \alpha_j) \end{pmatrix} \) is nonzero by hypothesis, we obtain that

\[
X^\pm((1,0,\ldots,0), s) = 0 = X^\pm(0, s+1).
\]

Therefore, by induction we have \( X^\pm(0, s) = 0 \) for all \( s \geq 0 \). We simultaneously have proven that \( X^\pm((1,0,\ldots,0), s) = 0 \) for all \( s \geq 0 \).

Next, we apply \([\tilde{h}_{i2}, \cdot]\) to \( X^\pm(0, s) = 0 \). By (2.30) we have

\[
0 = b(\alpha_i, \alpha_i) X^\pm((2,0,\ldots,0), s) + \frac{b(\alpha_i, \alpha_i)^3}{12} X^\pm(0, s) + (\alpha_i, \alpha_j) X^\pm(0, s+2) + \frac{(\alpha_i, \alpha_j)^3}{12} X^\pm(0, s).
\]

Since the last three terms vanish, we have \( X^\pm((2,0,\ldots,0), s) = 0 \).

In order to prove (4), we apply \([\tilde{h}_{i1}, \cdot]\) to \( X^\pm((1,0,\ldots,0), s) = 0 \). We have

\[
0 = \frac{(\alpha_i, \alpha_i)}{(b-1)!} X^\pm((2,0,\ldots,0), s)
\]

\[
+ \frac{(\alpha_i, \alpha_j)}{(b-2)!} X^\pm((1,1,0,\ldots,0), s) + \frac{(\alpha_i, \alpha_j)}{(b-1)!} X^\pm((1,0,\ldots,0), s+1).
\]

Since the first and third terms vanish, we have \( X^\pm((1,1,0,\ldots,0), s) = 0 \). \( \square \)

We move on to proving the relation (2.22) with \( i = j, (r,s) = (1,2) \) from Lemma 2.31(4): see Proposition 2.33 and Proposition 2.38. A few intermediary lemmas will be necessary. The argument was originally noticed in [Gua07, one paragraph after the proof of Prop. 2.1] for type \( \tilde{sl}_{n+1} \). Since the proof was omitted there, we reproduce it here.

Lemma 2.32. We have

\[
[h_{j1}, x_{i1}^\pm] = \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} [h_{i1}, x_{i1}^\pm] \pm \frac{(\alpha_i, \alpha_j)}{2} \{h_{j0}, x_{i1}^\pm\} - \{h_{i0}, x_{i1}^\pm\}\]

for all \( i, j \in I \).

Proof. The left-hand side is equal to

\[
\pm \frac{1}{(\alpha_i, \alpha_i)} [h_{j1}, [h_{i1}, x_{i0}^\pm]] - \frac{1}{2} [h_{j1}, \{h_{i0}, x_{i0}^\pm\}]
\]

\[
= \pm \frac{1}{(\alpha_i, \alpha_i)} [h_{i1}, [h_{j1}, x_{i0}^\pm]] - \frac{(\alpha_i, \alpha_j)}{2} \left( \pm \{h_{i0}, x_{i1}^\pm\} \pm \frac{1}{2} \{h_{i0}, x_{i0}^\pm, h_{j0}\} \right)
\]

\[
= \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} [h_{i1}, x_{i1}^\pm] + \frac{1}{2} \{h_{j0}, x_{i0}^\pm\} \mp \frac{(\alpha_i, \alpha_j)}{2} \left( \{h_{i0}, x_{i1}^\pm\} \pm \frac{1}{2} \{h_{i0}, x_{i0}^\pm, h_{j0}\} \right).
\]

Using

\[
\{h_{j0}, \{h_{i0}, x_{i0}^\pm\}\} = \{h_{i0}, \{x_{i0}^\pm, h_{j0}\}\},
\]

we find that this is equal to the right-hand side. \( \square \)
Lemma 2.33. The relation \((2.4)\) holds when \(i = j, r + s \leq 2\).

Proof. From \((2.13)\) with \(r, s \leq 1\) and \((2.15)\) with \(i = j, (r, s) = (1, 0)\), we have

\[
0 = [h_{i_1}, h_{i_1}] = [[x^+_{i_1}, x^-_{i_0}], h_{i_1}] = -(\alpha_i, \alpha_i) \left( [x^+_{i_2}, x^-_{i_0}] - [x^+_{i_1}, x^-_{i_1}] \right),
\]

where we have used Lemma \((2.21)\) in the last equality. Therefore

\[
[x^+_{i_1}, x^-_{i_1}] = [x^+_{i_2}, x^-_{i_0}] = h_{i_2}.
\]

Similarly we use \((2.15)\) with \((r, s) = (0, 1)\) instead to get \(h_{i_2} = [x^+_{i_1}, x^-_{i_1}] = [x^+_{i_0}, x^-_{i_2}]\). \(\square\)

Lemma 2.34. For all \(i, j \in I\), we have

\[
[h_{i_2}, h_{j_0}] = 0.
\]

Proof. We have

\[
[h_{i_2}, h_{j_0}] = [[x^+_{i_2}, x^-_{i_0}], h_{j_0}] = [[x^+_{i_2}, h_{j_0}], x^-_{i_0}] + [x^+_{i_2}, [x^-_{i_0}, h_{j_0}]].
\]

Employing the first relation in Lemma \((2.21)\) we see that this expression is equal to zero. \(\square\)

Proposition 2.35. Let \(i, j \in I\) be such that \(a_{ij} = -1\). Then

\[
[h_{i_1}, h_{i_2}] = [h_{i_1}, [x^+_{i_1}, x^-_{i_1}]] = 0.
\]

Proof. For brevity, we suppose \((\alpha_i, \alpha_i) = 2\) and \((\alpha_i, \alpha_j) = -1\).

The first equality follows from Lemma \((2.33)\) so we prove the second equality.

We start with

\[
0 = [x^+_{i_1}, [x^+_{i_1}, x^-_{j_0}]],
\]

which holds by Lemma \((2.31)(4)\). We apply \([\cdot, x^-_{j_1}]\) and use Lemmas \((2.26)\) and \((2.33)\) to get

\[
0 = [h_{i_1}, [x^+_{i_1}, h_{j_1}]] + [x^+_{i_1}, [h_{i_1}, h_{j_1}]] + [x^+_{i_1}, [x^+_{i_1}, x^-_{i_1}]] + \frac{1}{2} [x^+_{i_1}, \{h_{j_0}, x^-_{i_0}\}].
\]

Here we have used that \([h_{i_1}, h_{j_1}] = 0\). We apply \([\cdot, x^-_{i_0}]\) again to obtain

\[
0 = -2 [x^-_{i_1}, [x^+_{i_1}, h_{j_1}]] - \{h_{j_0}, x^-_{i_0}\}, [x^+_{i_1}, h_{j_1}]] + [h_{i_1}, [x^+_{i_1}, x^-_{i_1}]]
+ \frac{1}{2} [h_{i_1}, [x^+_{i_1}, \{x^-_{i_0}, h_{j_0}\}]] + [h_{i_1}, [x^+_{i_1}, x^-_{i_1}]] + [x^+_{i_1}, [h_{i_1}, x^-_{i_1}]]
\]

\[
- [x^+_{i_1}, [x^-_{i_0}, (x^+_{i_1})^2]] + \frac{1}{2} [[x^+_{i_1}, \{x^-_{i_1}, x^-_{i_0}\} + \{h_{j_0}, h_{i_1}\}], x^-_{i_0}]
\]

\[
= -2 [x^-_{i_1}, [x^+_{i_1}, h_{j_1}]] + 2 [h_{i_1}, [x^+_{i_1}, x^-_{i_1}]] + [x^+_{i_1}, [h_{i_1}, x^-_{i_1}]]
- \{h_{j_0}, x^-_{i_0}\}, [x^+_{i_1}, h_{j_1}]] + [h_{i_1}, \{x^+_{i_1}, x^-_{i_0}\}] + \frac{1}{2} [x^+_{i_1}, \{h_{j_0}, [h_{i_1}, x^-_{i_0}]\}].
\]

From Lemma \((2.32)\) we have

\[
-2 [x^-_{i_1}, [x^+_{i_1}, h_{j_1}]] = [x^-_{i_1}, [x^+_{i_1}, h_{i_1}]] - [x^-_{i_1}, \{h_{j_0}, x^+_{i_1}\}] - \{h_{j_0}, x^+_{i_1}\}
\]

\[
= [x^-_{i_1}, [x^+_{i_1}, h_{i_1}]] + 3 \{x^+_{i_1}, x^-_{i_1}\} - \{h_{j_0}, [x^-_{i_1}, x^+_{i_1}]\} + \{h_{j_0}, [x^-_{i_1}, x^+_{i_1}]\},
\]
and also

\[-\{h_{i0}, x_{i0}^-, [x_{i1}^+, h_{j1}]\} = - \frac{1}{2} \{h_{i0}, x_{i0}^-, [h_{i1}, x_{i1}^+] + \{h_{j0}, x_{i1}^+\} - \{h_{i0}, x_{i1}^+\}\] 

\[= - \{h_{i1}, x_{i1}^+, x_{i0}^-\} - \frac{1}{2} \{h_{i0}, [x_{i0}^-, h_{i1}, x_{i1}^+]\} - \{\{h_{j0}, h_{i1}\} + \{x_{i0}^-, x_{i1}^+\}\} + \{\{h_{i0}, x_{i1}^+\}, x_{i0}^-\}

\[+ \frac{1}{2} \{h_{i0}, \{h_{j0}, h_{i1}\} + \{x_{i0}^-, x_{i1}^+\}\} + \{\{h_{i0}, x_{i1}^+\}, x_{i0}^-\}

- 2h_{i1}h_{i0}^2 + \{h_{i0}, \{x_{i0}^-, x_{i1}^+\}\}.

We substitute these into (2.36) to get

\[-3[h_{i1}, [x_{i1}^+, x_{i0}^-]] = 3\{x_{i1}^+, x_{i1}^-\} - \{h_{j0}, [x_{i1}^+, x_{i1}^-]\} + \{h_{i0}, [x_{i1}^+, x_{i1}^-]\}

\[\text{ (2.37)}\]

\[+ \frac{1}{2} \{h_{i0}, \{h_{j0}, h_{i1}\} + \{x_{i0}^-, x_{i1}^+\}\} + \{\{h_{i0}, x_{i1}^+\}, x_{i0}^-\}

- 2h_{i1}h_{i0}^2 + \{h_{i0}, \{x_{i0}^-, x_{i1}^+\}\} + \{[h_{i1}, x_{i1}^+]\}, x_{i0}^-

+ \{x_{i1}^+, [h_{i1}, x_{i0}^-]\} + \frac{1}{2} \{h_{j0}, [x_{i1}^+, [h_{i1}, x_{i0}^-]]\} + \frac{1}{2} \{x_{i1}^+, [h_{i1}, x_{i0}^-]\}.

We then substitute

\[-\frac{1}{2} \{h_{i0}, [x_{i0}^-, [h_{i1}, x_{i1}^+]\}]\] 

\[= - \{h_{i0}, [x_{i1}^+, h_{i1}^+]\} - \frac{1}{2} \{h_{i0}, [x_{i0}^-, h_{i1}^+]\}

\[= - \{h_{i0}, [x_{i1}^+, h_{i1}^+]\} + \frac{1}{2} \{h_{i0}, [x_{i0}^-, h_{i1}^+]\} + \frac{3}{2} \{x_{i1}^+, [h_{i0}, x_{i0}^-]\},

\[\frac{3}{2} \{x_{i1}^+, [h_{i1}, x_{i0}^-]\} = - 3\{x_{i1}^+, x_{i1}^-\} - \frac{3}{2} \{x_{i1}^+, [h_{i0}, x_{i0}^-]\},

\[\frac{1}{2} \{h_{j0}, [x_{i1}^+, [h_{i1}, x_{i0}^-]\}]\] 

\[= - \{h_{j0}, [x_{i1}^+, h_{i1}^+]\} - \frac{1}{2} \{h_{j0}, [x_{i1}^+, h_{i1}^+]\} - \frac{1}{2} \{h_{j0}, \{h_{i0}, h_{i1}\}\}

\[= - \{h_{j0}, [x_{i1}^+, h_{i1}^+]\} + \{h_{j0}, [x_{i1}^+, h_{i0}^-]\} - \frac{1}{2} \{h_{j0}, \{h_{i0}, h_{i1}\}\}

\text{ into (2.37) to get}

\[-3[h_{i1}, [x_{i1}^+, x_{i0}^-]] = - \{h_{j0}, [x_{i1}^+, h_{i1}^+]\} + \frac{1}{2} \{h_{i0}, [x_{i0}^-, h_{i1}^+]\} + \{h_{i0}, x_{i1}^+\}, x_{i0}^-

\[+ \frac{3}{2} \{x_{i1}^+, \{h_{i0}, x_{i0}^-\}\} + \{h_{j0}, \{x_{i1}^+, x_{i0}^-\}\}

\[= \{h_{i0}, x_{i1}^+\}, [x_{i0}^-, h_{j0}] + \frac{3}{2} \{h_{i0}, [x_{i1}^+, x_{i0}^-]\} + \frac{1}{2} \{x_{i1}^+, [x_{i0}^-, h_{i0}]\} = 0.

This is nothing but the assertion. □

**Proposition 2.38.** Assume that \([h_{i1}, h_{i2}] = 0\) and \((\alpha_i, \alpha_j) \neq 0\). Then we have

\([h_{j1}, h_{j2}] = 0\).

Together with Proposition 2.35 this gives (2.20) for any \(i, j\) because the Dynkin diagram of \(g\) is connected, this being a consequence of the assumption that the Cartan matrix of \(g\) is indecomposable. We are thus able to conclude the proof of Theorem 2.12.
Proof. By the assumption and Lemma 2.33 we have $[\tilde{h}_{i1}, h_{i2}] = 0$. Therefore

$$0 = [\tilde{h}_{i1}, h_{i2}] = [\tilde{h}_{i1}, [x_{i1}^+, x_{i1}^-]]$$

by Lemma 2.33

$$= ([\tilde{h}_{i1}, x_{i1}^+], x_{i1}^-) + [x_{i1}^+,[\tilde{h}_{i1}, x_{i1}^-]]$$

by Lemma 2.32

$$= (\alpha_i, \alpha_i) \left( \left([\tilde{h}_{j1}, x_{i1}^+], x_{i1}^-\right) + \left[x_{i1}^+, [\tilde{h}_{j1}, x_{i1}^-]\right] \right)$$

by Lemma 2.33

We take $\tilde{h}_{i2}$ as in (2.28). Then we have $[\tilde{h}_{j1}, \tilde{h}_{i2}] = 0$ and we apply $[\cdot, x_{j0}^+]$ to this to get:

$$0 = (\alpha_j, \alpha_j)[x_{j1}^+, \tilde{h}_{i2}] + (\alpha_i, \alpha_j)[\tilde{h}_{j1}, x_{j2}^+] + \frac{(\alpha_i, \alpha_j)^3}{12} [\tilde{h}_{j1}, x_{j0}^+]$$

where we have used (2.30).

We next apply $[\cdot, x_{j0}^-]$ to this and, using again (2.30), we obtain:

$$0 = (\alpha_j, \alpha_j)[x_{j1}^+, \tilde{h}_{i2}] - (\alpha_j, \alpha_j)(\alpha_i, \alpha_j)[x_{j1}^+, x_{j2}^-] - \frac{(\alpha_j, \alpha_j)(\alpha_i, \alpha_j)^3}{12} [x_{j1}^+, x_{j0}^-]$$

$$- (\alpha_i, \alpha_j)(\alpha_i, \alpha_j)[x_{j1}^+, x_{j2}^-] + (\alpha_i, \alpha_j)[\tilde{h}_{j1}, [x_{j2}^+, x_{j0}^-]]$$

$$- \frac{(\alpha_j, \alpha_j)(\alpha_i, \alpha_j)^3}{12} [x_{j1}^-, x_{j0}^-]$$

(2.39)

$$= - (\alpha_i, \alpha_j)(\alpha_i, \alpha_j) \left( [x_{j1}^+, x_{j2}^-] + [x_{j1}^-, x_{j2}^+] \right) + (\alpha_i, \alpha_j)[\tilde{h}_{j1}, h_{j2}].$$

We simplify the last expression as follows. Start with $[x_{j1}^+, x_{j1}^-] = h_{j2}$ from Lemma 2.33 and apply $[\tilde{h}_{j1}, \cdot]$ to it to obtain

$$(\alpha_i, \alpha_j) \left( [x_{j2}^+, x_{j1}^-] - [x_{j1}^+, x_{j2}^-] \right) = [\tilde{h}_{j1}, h_{j2}].$$

Therefore the right-hand side of (2.39) is $2(\alpha_i, \alpha_j)[\tilde{h}_{j1}, h_{j2}]$, so $[\tilde{h}_{j1}, h_{j2}] = 0$. \qed

3. Operators on modules in the category $\mathcal{O}$

3(i). Category $\mathcal{O}$. The category $\mathcal{O}$ of modules over a finite dimensional simple Lie algebra has been studied extensively over the past forty years [Hum08]. The definition of this category generalizes naturally for all Kac-Moody algebras (see for instance [Kac90 §9.1]). It is also possible to extend the notion of category $\mathcal{O}$ to quantum toroidal algebras and affine Yangians: see [Her05, GTL16].

Definition 3.1. The category $\mathcal{O}$ of modules over the Yangian $Y(\mathfrak{g})$ consists of all the modules $V$ such that:

(1) $V$ is diagonalizable with respect to $\mathfrak{h}$.
(2) Each $\mathfrak{h}$-weight space $V_\mu$ is finite dimensional ($\mu \in \mathfrak{h}^*$).
(3) There exist $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ such that if $V_\mu \neq 0$, then $\lambda_i - \mu \in \sum_{j \in I} \mathbb{Z}_{\geq 0} \alpha_j$ for some $1 \leq i \leq k$.

One consequence of this definition which we will use implicitly is that if $V$ is a module in $\mathcal{O}$, $\alpha \in \Delta_+$ and $\mu \in \mathfrak{h}^*$, then there exists $N \in \mathbb{Z}_{\geq 0}$ such that $V_{\mu + ra} = 0$ for all $r \geq N$. Moreover, $V$ is said to be integrable and in the category $\mathcal{O}$ if, in addition, such an $N$ can be chosen so that $V_{\mu \pm ra} = 0$ for all $r \geq N$. 
Another presentation of the Yangian and operators on category $\mathcal{O}$. When $\mathfrak{g}$ is finite dimensional, Drinfeld gave another presentation of $Y(\mathfrak{g})$ as an associative algebra generated by elements $x$ and $J(x)$ for $x \in \mathfrak{g}$ with the defining relations:

\[
(3.2)
xy - yx = [x, y] \text{ for all } x, y \in \mathfrak{g}, \quad J \text{ is linear in } x \in \mathfrak{g}, \quad J([x, y]) = [x, J(y)],
\]

\[
[J(x), J([y, z])] + [J(z), J([x, y])] + [J(y), J([z, x])] = \sum_{a, b, c \in \mathbb{A}} ([x, \xi_a], [y, \xi_b], [z, \xi_c]) \{\xi_a, \xi_b, \xi_c\},
\]

\[
[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] = \sum_{a, b, c \in \mathbb{A}} \left( ([x, \xi_a], [y, \xi_b], [z, w], \xi_c]) + ([z, \xi_a], [w, \xi_b], [x, y], \xi_c) \right) \{\xi_a, \xi_b, J(\xi_c)\}
\]

where $\{\xi_a\}_{a \in \mathbb{A}}$ is an orthonormal basis of $\mathfrak{g}$, $A$ being a fixed indexing set of size dim $\mathfrak{g}$, and $\{\xi_a, \xi_b, \xi_c\} = \frac{1}{24} \sum_{i \in G} \xi_{\pi(a)}(\xi_{\pi(b)}(\xi_{\pi(c)}))$, $G$ being the group of permutations of $\{a, b, c\}$.

The isomorphism between this presentation and the one provided in Definition 2.1 is given by

\[
x_i^\pm = x_{i,0}^\pm, \quad h_i = h_{i,0}
\]

\[
J(h_i) = h_{i,1} + v_i, \quad v_i \text{ def.} = \frac{1}{4} \sum_{a \in \Delta_+} (\alpha, \alpha_i)(x_a^+ x_{a}^- + x_{a}^- x_a^+) - \frac{1}{2} h_i^2,
\]

\[
(3.3)
J(x_i^+) = x_{i,1}^+ + w_i^+, \quad w_i^+ \text{ def.} = \pm \frac{1}{4} \sum_{a \in \Delta_+} \left( [x_a^+, x_a^+] x_a^- + x_a^- [x_a^+, x_a^-] \right) - \frac{1}{4} (x_a^+ h_i + h_i x_a^+),
\]

where $x_a^+$ is a root vector corresponding to a positive root $\alpha$ and $x_{a,1}^+$ coincides with the previous $x_i^\pm$. These are normalized so that $(x_a^+, x_a^-) = 1$.

The right hand sides of (3.2) and (3.3) do not make sense unless $\mathfrak{g}$ is finite dimensional. However, we can change the definition of $v_i$ (and thus of $w_i^+$) so that it gives a well-defined operator on representations in the category $\mathcal{O}$ as follows. First observe that

\[
\sum_{\alpha \in \Delta_+} (\alpha, \alpha_i)[x_a^+, x_a^-] = \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i)\nu^{-1}(\alpha)
\]

where $\nu: \mathfrak{h} \to \mathfrak{h}^*$ is the linear isomorphism given by $\nu(h_1)(h_2) = (h_1, h_2)$ for all $h_1, h_2 \in \mathfrak{h}$. Assuming that $(, )$ is normalized so that $(\alpha, \alpha) = 2$ for a long root $\alpha$, we claim that this is equal to $h^\nu h_i$, where $h^\nu$ is the dual Coxeter number. In fact,

\[
\sum_{\alpha \in \Delta_+} (\alpha, \alpha_i)\alpha_j(\nu^{-1}(\alpha)) = \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i)(\alpha, \alpha_j)
\]

and the right-hand side is equal to the half of the Killing form $B(h_i, h_j)$ [Kna02, Corollary 2.24]. By [Kac90, Ex. 6.2], it is also equal to $h^\nu(h_i, h_j)$. Therefore we have

\[
v_i = \frac{1}{4} h^\nu h_i + \frac{1}{2} \sum_{a \in \Delta_+} (\alpha, \alpha_i)x_a^- x_a^+ - \frac{1}{2} h_i^2.
\]

We now want to obtain a similar formula which makes sense for arbitrary Kac-Moody Lie algebra $\mathfrak{g}$ and gives a well-defined operator on representations in the category $\mathcal{O}$. For each $\alpha \in \Delta_+$, choose a basis $\{x_{a,1}^{(k)}\}$ of $\mathfrak{g}_a$ and a dual basis $\{x_{a,1}^{(l)}\}$ of $\mathfrak{g}_{-a}$ so that $(x_{a,1}^{(k)}, x_{a,1}^{(l)}) = \delta_{kl}$. 


Then the formula

\[(3.4)\]

\[v_i = \frac{1}{4} h^\vee h_i + \frac{1}{2} \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i) \sum_{k=1}^{\dim g_\alpha} x_{-\alpha}^{(k)} x_{\alpha}^{(k)} - \frac{1}{2} h_i^2,\]

gives a well-defined operator on representations in the category \(\mathcal{O}\) as \(x_{\alpha}^{(k)}\) kills a given vector if \(\alpha\) is sufficiently large.

The definition of the operators \(w_i^{\pm}\) can then be determined from (2.3) and (2.5) together with the requirement that \(J([h_i, x_i^{\pm}]) = [J(h_i), x_i^{\pm}]\). We obtain

\[w_i^{\pm} = \pm \frac{1}{(\alpha_i, \alpha_i)} [v_i, x_i^{\pm}] + \frac{1}{2} \left(h_i x_i^{\pm} + x_i^{\pm} h_i\right)\]

which, using [Kac90] Corollary 2.4, can be rewritten as

\[w_i^{+} = \frac{1}{4} h^\vee x_i^{+} + \frac{1}{2} \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim g_\alpha} x_{-\alpha}^{(k)} [x_{\alpha}^{+}, x_{\alpha}^{(k)}] - \frac{1}{2} h_i x_i^{+},\]

\[w_i^{-} = \frac{1}{4} h^\vee x_i^{-} - \frac{1}{2} \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim g_\alpha} [x_{\alpha}^{-}, x_{\alpha}^{(k)}] x_{\alpha}^{(k)} - \frac{1}{2} x_i^{-} h_i.\]

These can also be viewed as well-defined operator on modules in the category \(\mathcal{O}\). Let’s see briefly how to obtain \(w_i^{+}\). We have

\[(3.5)\]

\[v_i, x_i^{+} = (\alpha_i, \alpha_i) \left(\frac{1}{4} h^\vee x_i^{+} - \frac{1}{2} \{h_i, x_i^{+}\} - \frac{1}{2} h_i x_i^{+}\right) + \frac{1}{2} \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (\alpha, \alpha_i) \sum_{k=1}^{\dim g_\alpha} (x_{-\alpha}^{(k)} [x_{\alpha}^{+}, x_{\alpha}^{(k)}] + [x_{-\alpha}^{+}, x_{\alpha}^{(k)}] x_{\alpha}^{(k)})\]

and

\[\sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (\alpha, \alpha_i) \sum_{k=1}^{\dim g_\alpha} (x_{-\alpha}^{(k)} [x_{\alpha}^{+}, x_{\alpha}^{(k)}] + [x_{-\alpha}^{+}, x_{\alpha}^{(k)}] x_{\alpha}^{(k)})\]

\[= \sum_{\beta \in \Delta_+ \setminus \{\alpha_i\}} (\alpha_i - \beta, \alpha_i) \sum_{k=1}^{\dim g_{\beta - \alpha_i}} x_{\alpha_i - \beta}^{(k)} [x_{\alpha_i}^{+}, x_{\beta - \alpha_i}^{(k)}] + \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (\alpha, \alpha_i) \sum_{k=1}^{\dim g_\alpha} [x_{\alpha}^{(k)} x_{\alpha}^{+}, x_{\alpha}^{(k)}]\]

after setting \(\beta = \alpha + \alpha_i\) and using Lemma 1.3 in [Kac90];

\[= \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \sum_{k=1}^{\dim g_{\beta - \alpha_i}} (\alpha_i - \alpha, \alpha_i) x_{\alpha_i - \alpha}^{(k)} [x_{\alpha_i}^{+}, x_{\alpha_i - \alpha}^{(k)}] + (\alpha, \alpha_i) x_{\alpha_i - \alpha}^{(k)} [x_{\alpha_i}^{+}, x_{\alpha_i - \alpha}^{(k)}]\]

by Corollary 2.4 in [Kac90];

\[= (\alpha_i, \alpha_i) \sum_{\beta \in \Delta_+} \sum_{k=1}^{\dim g_{\beta}} x_{\beta}^{(k)} [x_{\beta}^{+}, x_{\beta}^{(k)}].\]

Combining this with (3.5), we obtain the desired expression for \(w_i^{+}\).

We set

\[(3.6)\]

\[J(h_i) = h_{i1} + v_i\] and \(J(x_i^{\pm}) = x_{i1}^{\pm} + w_i^{\pm}\).
We view \(J(h_i)\) and \(J(x_i^\pm)\) as operators on modules in \(\mathcal{O}\). Later, we will see how to view these also as elements in a completion of the Yangian (Section 5).

3(iii). **Commutation relations and reflection operators.** The goal of this subsection is to obtain relations (see Proposition 3.15 and Corollary 3.17) which will be useful in the next section to verify that the coproduct on \(Y(g)\) respects the defining relations of the Yangian.

In this subsection, we fix a module \(V\) in the category \(\mathcal{O}\) and view the generators \(x_i^\pm, h_{ir}\) along with \(v_i, w_i^\pm\) as operators on \(V\).

With the relation (2.10) in mind, we set \(\tilde{v}_i = v_i + h_i^2/2\). We will continue to use the notation \(x_i^\pm\) for \(x_{i0}^\pm\) and \(h_i\) for \(h_{i0}\).

**Lemma 3.7.** The following relations hold.

\[
\begin{align*}
(3.8) & \quad [h_i, v_j] = 0, \quad [h_i, w_j^\pm] = \pm(\alpha_i, \alpha_j)w_j^\pm, \\
(3.9) & \quad \tilde{v}_i, x_j^\pm = \pm(\alpha_i, \alpha_j)w_j^\pm, \\
(3.10) & \quad [w_i^+, x_j^-] = \delta_{ij}v_i = [x_i^+, w_j^-], \\
(3.11) & \quad [w_i^+, x_j^+] = [x_i^+, w_j^+] = \pm \frac{(\alpha_i, \alpha_j)}{2}(x_i^+x_j^+ + x_j^+x_i^+). 
\end{align*}
\]

**Proof.** (3.8) is straightforward to check, (3.9) was shown above in the + case (see (3.5)), and (3.10), (3.11) follow from [Kac90] Lemma 1.3, Corollary 2.4 as above.

The previous lemma implies the following equivalences:

\[
\begin{align*}
[h_i, J(x_j^\pm)] = J([h_i, x_j^\pm]) & \iff [h_{i0}, x_j^{\pm1}] = \pm(\alpha_i, \alpha_j)x_j^{\pm1}, \\
[J(h_i), x_j^\pm] = J([h_i, x_j^\pm]) & \iff [h_{i1}, x_j^{\pm0}] = \pm(\alpha_i, \alpha_j)x_j^{\pm0}, \\
[J(x_i^\pm), x_j^-] = J([x_i^\pm, x_j^-]) = [x_i^\pm, J(x_j^-)] & \iff [x_i^{\pm1}, x_j^-] = \delta_{ij}h_{i1} = [x_i^{\pm0}, x_j^{\pm1}], \\
[J(x_i^\pm), x_j^+] = [x_i^\pm, J(x_j^\pm)] & \iff [x_i^{\pm1}, x_j^{\pm0}] = [x_i^{\pm0}, x_j^{\pm1}] = \pm \frac{(\alpha_i, \alpha_j)}{2}(x_i^{\pm0}x_j^{\pm1} + x_j^{\pm1}x_i^{\pm0}).
\end{align*}
\]

If \(\alpha\) is a simple root \(\alpha_i\), then \(J(x_i^\pm)\) has already been defined, and now we want to obtain such operators for any positive real root \(\alpha\). To achieve this, we need to introduce automorphisms \(\tau_i\) of the Yangian. Let \(s_i\) be the simple reflection corresponding to a simple root \(\alpha_i\). Following [Kac90] §3.8, we consider the operator on \(Y(g)\) given by

\[
\tau_i \defeq \exp(\text{ad}(e_i)) \exp(-\text{ad}(f_i)) \exp(\text{ad}(e_i)),
\]

where \(e_i = \sqrt{2/(\alpha_i, \alpha_i)}x_i^{\pm0}\), \(f_i = \sqrt{2/(\alpha_i, \alpha_i)}x_i^{\pm0}\). Since \(\text{ad}(e_i)\), \(\text{ad}(f_i)\) are locally nilpotent derivations on \(Y(g)\), \(\tau_i\) provides an algebra automorphism of \(Y(g)\). Since \(\tau^2\) may be different from the identity, these automorphisms do not provide a representation of the Weyl group of \(g\) on \(Y(g)\).

We would like to apply the \(\tau_i\) to the operators \(J(h_i)\) on \(V\). We can view also \(e_i\) and \(f_i\) as elements of \(\text{End}_C(V)\). Moreover, we can view \(\text{ad}(e_i)\) and \(\text{ad}(f_i)\) as derivations on \(\text{End}_C(V)\) which are locally nilpotent when restricted to the image of \(Y(g)\) inside \(\text{End}_C(V)\). This implies that \(\tau_i\) can also be interpreted as an automorphism of this image, hence we can apply it to \(J(h_i)\) and \(J(x_i^\pm)\) \(\forall i \in I\).

**Lemma 3.13.** We have

\[
\tau_i(J(h_j)) = J(h_j) - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}J(h_i).
\]
Proof. We consider the subalgebra \( \mathfrak{s} \mathfrak{l}_2^{(i)} \) of \( Y(\mathfrak{g}) \) spanned by \( e_i, f_i, h_i \). The space \( \text{End}_C(V) \) is a representation of \( \mathfrak{s} \mathfrak{l}_2^{(i)} \) via the adjoint action. Let us prove the lemma first when \( j = i \).

Consider the subspace \( \mathbb{C}J(x_i^+) + \mathbb{C}J(h_i) + \mathbb{C}J(x_i^-) \) of \( \text{End}_C(V) \), which is stable under the adjoint action of \( \mathfrak{s} \mathfrak{l}_2^{(i)} \) of \( Y(\mathfrak{g}) \) by Lemma 3.7 and (3.12). There are two cases: either \( J(x_i^+) = 0 = J(h_i) = J(x_i^-) \), in which case the lemma is trivial, or that subspace is a three-dimensional irreducible representation of \( \mathfrak{s} \mathfrak{l}_2^{(i)} \). In the latter case, one can check directly that \( \tau_i(J(h_i)) = -J(h_i) \).

Now assume that \( j \neq i \). The operator

\[
J(h_j) - \frac{(\alpha_i, \alpha_j)}{\langle \alpha_i, \alpha_i \rangle} J(h_i)
\]

is killed by \( \text{ad}(e_i) \) and \( \text{ad}(f_i) \) by (3.12). Therefore this vector is fixed by \( \tau_i \), hence the lemma holds also when \( j \neq i \).

Let \( \alpha \) be a positive real root. By definition, there is an element \( w \) of the Weyl group of \( \mathfrak{g} \) and a simple root \( \alpha_j \) such that \( \alpha = w(\alpha_j) \). Then we define a corresponding (real) root vector by

\[
x_\alpha^\pm = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_{p-1}}(x_{i_p}^\pm),
\]

where \((i_1, \ldots, i_{p-1})\) is a reduced expression of \( w \) and \( i_p = j \). This is independent of the choice of sequence \( i_1, i_2, \ldots, i_p \) up to a constant multiple. This ambiguity will not be important in the following discussion.

We define

\[
J(x_\alpha^\pm) \overset{\text{def}}{=} \tau_{i_1} \tau_{i_2} \cdots \tau_{i_{p-1}}(J(x_{i_p}^\pm)).
\]

Proposition 3.15. Suppose \( \alpha \) is a positive real root. Then

\[
[J(h_i), x_\alpha^\pm] = [h_i, J(x_\alpha^\pm)] = \pm(\alpha_i, \alpha) J(x_\alpha^\pm) \quad \text{for all } i \in I.
\]

Proof. We prove the proposition by induction on \( p \). If \( p = 1 \), then \( x_\alpha^\pm = x_j^\pm \), and the assertion is a direct consequence of Lemma 3.7 and (3.12). Suppose the statement of the proposition holds for \( x_\beta^\pm \) with \( \beta = s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}) \). Then

\[
[J(h_i), x_\alpha^\pm] = \tau_{i_1} \left( [\tau_{i_1}^{-1} J(h_i), x_\alpha^\pm] \right) = \tau_{i_1} \left( [J(h_i) - \frac{2(\alpha_i, \alpha_i)}{\langle \alpha_i, \alpha_i \rangle} J(h_i), x_\beta^\pm] \right)
\]

\[
= \pm \left( \alpha_i, \beta - \frac{2(\alpha_i, \alpha_i)}{\langle \alpha_i, \alpha_i \rangle} \right) \tau_{i_1} J(x_\beta^\pm) = \pm(\alpha_i, \alpha) J(x_\alpha^\pm) = \pm(\alpha_i, \alpha) J(x_\alpha^\pm),
\]

where we have used Lemma 3.12 in the second equality, and the induction assumption in the third. Similarly, the second equality and the relation \([J(h_j), x_\alpha^\pm] = [h_j, J(x_\alpha^\pm)]\), for all \( j \in I \), imply that

\[
[J(h_i), x_\alpha^\pm] = \tau_{i_1} \left( [h_i - \frac{2(\alpha_i, \alpha_i)}{\langle \alpha_i, \alpha_i \rangle} h_i, J(x_\beta^\pm)] \right) = \tau_{i_1} \left( [\tau_{i_1}^{-1} h_i, J(x_\beta^\pm)] \right) = [h_i, J(x_\alpha^\pm)].
\]

Therefore, by induction, the assertion is true for all \( \alpha \in \Delta^\text{re}_+ \).

For the rest of this subsection, we restrict to the special case where \( \mathfrak{g} \) is of affine type. In this case, the set of imaginary roots \( \Delta^\text{im} \) is equal to \((\mathbb{Z} \setminus \{0\})\delta \) ([Kac90, Th. 5.6]). As \((\delta, \alpha_i) = 0 \ \forall \ i \in I \) ([Kac90, (6.2.4)]), the summation which appears in the definition of the operator \( v_i \) (see (3.3)) needs only to be taken over the set of real positive roots \( \Delta^\text{re}_+ \). Since the multiplicity of a real root is 1, we can change the notation \( x_{k \alpha} \) to \( x_\alpha^\pm \) for each \( \alpha \in \Delta^\text{re}_+ \).
The same applies (trivially) to \( \mathfrak{g} \) of finite type.

**Proposition 3.16.** \([J(h_i), v_j] = [J(h_j), v_i]\) for all \( i, j \in I \).

**Proof.** Fix \( i, j \in I \). As a consequence of Lemma 3.17 \([J(h_i), h_j] = 0\). This together with (2.2) implies that

\[
[J(h_i), v_j] = \frac{1}{2} \sum_{\alpha \in \Delta^+_{\text{re}}} (\alpha, \alpha_i) \left( [J(h_i), x^+_\alpha] x^+_\alpha + x^-_\alpha [J(h_i), x^+_\alpha] \right)
\]

By Proposition 3.15 this is equal to

\[
\frac{1}{2} \sum_{\alpha \in \Delta^+_{\text{re}}} (\alpha, \alpha_i)(\alpha, \alpha_i) \left( -J(x^-_\alpha)x^+_\alpha + x^-_\alpha J(x^+_\alpha) \right).
\]

As this expression is symmetric with respect to \( i \) and \( j \), it is also equal to \([J(h_j), v_i]\). \( \Box \)

**Corollary 3.17.** The equality \([h_i, h_j] = 0\) of operators on \( V \) is equivalent to

\[
[J(h_i), J(h_j)] = -[v_i, v_j].
\]

4. **Coproduct and modules in the category \( \mathcal{O} \)**

4(i). **Casimir operators.** Fix a basis \( \{u_k\} \) of \( \mathfrak{h} \), and let \( \{u^k\} \) denote its dual basis with respect to the invariant inner product \((\cdot, \cdot)\). Given a positive root \( \alpha \) we choose a base \( \{x^{(k)}_{\alpha}\} \) of \( \mathfrak{g}_\alpha \) and the dual base \( \{x^{(l)}_{-\alpha}\} \) of \( \mathfrak{g}_{-\alpha} \) so that \((x^{(k)}_{\alpha}, x^{(l)}_{-\alpha}) = \delta_{kl}\). For a simple root \( \alpha_i \), we suppose \( x^{(k)}_{\alpha_i} = x_i^+, x^{(l)}_{-\alpha} = x_i^- \) with \( k = 1 \). Recall the dual bases \( \{x^{(k)}_{\alpha}\} \) and \( \{x^{(l)}_{-\alpha}\} \) defined just above (3.4).

Let us fix modules \( V_1 \) and \( V_2 \) in \( \mathcal{O} \). We define an operator \( \Omega_+ \) on \( V_1 \otimes V_2 \) by:

\[
\Omega_+ \overset{\text{def}}{=} \sum_{k=1}^{\dim \mathfrak{h}} u^k \otimes u_k + \sum_{\alpha \in \Delta^+_{\text{re}}} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} x^{(k)}_{-\alpha} \otimes x^{(k)}_{\alpha}.
\]

The definition of \( \Omega_+ \) is independent of the choice of bases.

Note that \( \Omega_+ \) does not coincide with the usual Casimir operator when \( \mathfrak{g} \) is finite-dimensional as it does not contain the term \( \sum_{\alpha \in \Delta} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} x^{(k)}_{\alpha} \otimes x^{(k)}_{-\alpha} \). We call it the half Casimir operator.

In the general case, the Casimir operator \( \Omega \) is replaced with the generalized Casimir operator (denoted \( \Omega^\text{gen} \)) which is given by

\[
\Omega^\text{gen} = 2\nu^{-1}(\rho) + \sum_{k=1}^{\dim \mathfrak{h}} u^k \otimes u_k + 2 \sum_{\alpha \in \Delta^+_{\text{re}}} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} x^{(k)}_{-\alpha} \otimes x^{(k)}_{\alpha}
\]

(see §2.5 of [Kac90]). \( \Omega^\text{gen} \) coincides with the usual Casimir element when \( \mathfrak{g} \) is finite-dimensional.

The half Casimir operator \( \Omega_+ \) does not commute with coproducts of the generators \( x_i^\pm \) or \( h_i \). It does however satisfy the following simple commutation relations:

**Lemma 4.2.** We have

\[
\square(h), \Omega_+] = 0 \quad \text{for } h \in \mathfrak{h},
\]

\[
\square(x_i^+), \Omega_+] = -x_i^+ \otimes h_i,
\]

\[
\square(x_i^-), \Omega_+] = h_i \otimes x_i^-,
\]

for all \( i \in I \), where \( \square(X) = X \otimes 1 + 1 \otimes X \).
Proof. These relations can be proven using the same techniques as used to prove Lemma 3.7. The first formula is a simple consequence of the definition. The second and third formulas follow from [Kac90, Lemmas 1.3, 2.4]. For example,

\[ [1 \otimes x_i^-, \Omega_+] = \sum_{k=1}^{\dim h} u^k \otimes [x_i^-, u_k] + \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} x_{-\alpha}^{(k)} \otimes [x_i^-, x_{\alpha}^{(k)}] \]

\[ = \sum_{k=1}^{\dim h} u^k \otimes (h_i, u_k) x_i^- - x_i^- \otimes h_i + \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} x_{-\alpha}^{(k)} \otimes [x_i^-, x_{\alpha}^{(k)}] \]

\[ = h_i \otimes x_i^- - x_i^- \otimes h_i - \sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} [x_i^-, x_{-\alpha}^{(k)}] \otimes x_{\alpha}^{(k)} \]

\[ = h_i \otimes x_i^- - [x_i^- \otimes 1, \Omega_+] \square \]

For a later purpose, we introduce the ordinary Casimir operator \( \Omega \) on \( V_1 \otimes V_2 \) and a related operator \( \Omega_- \):

\[ \Omega \overset{\text{def}}{=} \sum_{k=1}^{\dim h} u^k \otimes u_k + \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} \left( x_{\alpha}^{(k)} \otimes x_{-\alpha}^{(k)} + x_{-\alpha}^{(k)} \otimes x_{\alpha}^{(k)} \right) \]

\[ \Omega_- \overset{\text{def}}{=} \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} x_{\alpha}^{(k)} \otimes x_{-\alpha}^{(k)} = \Omega - \Omega_+ \]

4(ii). The coproduct and statement of the main theorem. Let \( \square \) be the operator defined by \( \square(X) = X \otimes 1 + 1 \otimes X \), as in Lemma 4.2. It is not an algebra homomorphism, but satisfies \( \square([X,Y]) = [\square(X), \square(Y)] \) for all \( X,Y \in \mathfrak{y}(\mathfrak{g}) \).

We want to define an algebra homomorphism \( \Delta_{V_1, V_2} : Y(\mathfrak{g}) \rightarrow \text{End}_C(V_1 \otimes V_2) \), so we first specify it on the generators of \( Y(\mathfrak{g}) \) and then prove afterwards that this assignment does indeed extend to an algebra homomorphism (see Theorem 4.11). When \( V_1 \) and \( V_2 \) are fixed, we simply write \( \Delta \).

Definition 4.7. \( \Delta \) assigns to the generators of \( Y(\mathfrak{g}) \) the following operators in \( \text{End}_C(V_1 \otimes V_2) \): by

\[ \Delta(h) = \square(h) \quad \text{(for } h \in \mathfrak{h} \text{)}, \quad \Delta(x_{i0}^\pm) = \square(x_{i0}^\pm) \]

\[ \Delta(h_{i1}) = \square(h_{i1}) + h_{i0} \otimes h_{i0} + [h_{i0} \otimes 1, \Omega_+] \]

\[ = h_{i1} \otimes 1 + 1 \otimes h_{i1} + h_{i0} \otimes h_{i0} - \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim \mathfrak{g}_\alpha} x_{-\alpha}^{(k)} \otimes x_{\alpha}^{(k)} \]

It follows that

\[ \Delta(\tilde{v}_i) = \square(\tilde{v}_i) + [h_{i0} \otimes 1, \Omega_+] \]

It will also be useful to define \( \Delta(\tilde{v}_i) \) to be:

\[ \Delta(\tilde{v}_i) = \square(\tilde{v}_i) + \frac{1}{2} \sum_{\alpha \in \Delta_+^{re}} (\alpha, \alpha_i) \left( x_{-\alpha}^- \otimes x_{\alpha}^+ + x_{\alpha}^- \otimes x_{\alpha}^+ \right) = \square(\tilde{v}_i) - \frac{1}{2} [h_{i0} \otimes 1, \Omega_+ - \Omega_-] \]

When \( \mathfrak{g} \) is finite dimensional, this is the formula obtained by applying the coproduct on \( U(\mathfrak{g}) \) to \( \tilde{v}_i \).
Since \( J(h_i) = \hat{h}_{i1} + \tilde{v}_i \), it is natural to define \( \Delta(J(h_i)) \) in the following way:

\[
\Delta(J(h_i)) = \Delta(\hat{h}_{i1}) + \Delta(\tilde{v}_i)
\]

(4.10)

\[
= \Box(J(h_i)) + \frac{1}{2} \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i) \sum_{k=1}^{\dim g_\alpha} (x^{(k)}_{\alpha^{-}} \otimes x^{(k)}_{\alpha^-} - x^{(k)}_{\alpha} \otimes x^{(k)}_{\alpha})
\]

\[
= \Box(J(h_i)) + \frac{1}{2}[h_{i0} \otimes 1, \Omega],
\]

where \( \Omega \) is as in (4.6). When \( g \) is finite dimensional, this shows that the formulas in Definition 4.7 coincide with the formulas for the coproduct of \( Y(g) \) which is compatible with the presentation (3.2) [Dri85].

**Theorem 4.11.** Assume \( g \) is either finite dimensional (but not \( sl_2 \)) or of affine type (but not of type \( A_1^{(1)} \) or \( A_2^{(2)} \)). Then the assignment \( \Delta \) defines an algebra homomorphism \( \Delta: Y(g) \to \text{End}_C(V_1 \otimes V_2) \).

**Remark 4.12.** When \( g \) is finite dimensional (including when \( g \cong sl_2 \)), this theorem is already known (see [Dri85]) but a proof has never appeared in the literature. We have excluded \( Y(sl_2) \) simply because the proof below would have to be modified in this case. As for the case when \( g \) is of type \( A_1^{(1)} \), a formula for a coproduct identical to ours is given in [BL94], but it is not clear if their definition of the Yangian of the affine Lie algebra \( \widehat{sl}_2 \) is equivalent to the one which can be found in [TB15] and in [Kod15] (to this effect, see also Remark 5.2 in [Kod15]).

**Remark 4.13.** In Section 3 we will explain how to replace \( \text{End}_C(V_1 \otimes V_2) \) with a completion of the tensor product \( Y(g) \otimes Y(g) \).

The rest of this section is devoted to the proof of this theorem. We will be able to use Theorem 2.12 because we will be working under the same assumptions in the finite or affine setting - see Remark 2.19. Note also that if we check that the restriction of \( \Delta \) to \( Y(g') \) is an algebra homomorphism, the compatibility for the extra relations (2.8) is straightforward.

Therefore, it is enough to check the compatibility of the relations listed in Theorem 2.12.

In what follows, it will be useful to have formulas for \( \Delta(x_{i1}^+) \) for all \( i \in I \). From (2.9) with \( r = 0 \), we obtain:

\[
\Delta(x_{i1}^+) = \pm(\alpha_i, \alpha_i)^{-1} \Delta([\hat{h}_{i1}, x_{i0}^+])
\]

\[
= \pm(\alpha_i, \alpha_i)^{-1} [\Box(\hat{h}_{i1}) + [h_{i0} \otimes 1, \Omega_+], \Box(x_{i0}^+)]
\]

\[
= \Box(x_{i1}^+) \pm (\alpha_i, \alpha_i)^{-1}[[h_{i0} \otimes 1, \Omega_+], \Box(x_{i0}^+)].
\]

We consider the + case first. Note that

\[
[h_{i0} \otimes 1, \Omega_+, \Box(x_{i0}^+)] = -[[1 \otimes h_{i0}, \Omega_+], \Box(x_{i0}^+)]
\]

\[
= -[[1 \otimes h_{i0}, \Box(x_{i0}^+)], \Omega_+] - [1 \otimes h_{i0}, [\Omega_+, \Box(x_{i0}^+)]]
\]

\[
= -(\alpha_i, \alpha_i)[1 \otimes x_{i0}^+, \Omega_+] - [1 \otimes h_{i0}, x_{i0}^+ \otimes h_{i0}] = -(\alpha_i, \alpha_i)[1 \otimes x_{i0}^+, \Omega_+],
\]

where we have used (4.3) in the first equality and (4.4) in the third. Therefore we have

(4.14)

\[
\Delta(x_{i1}^+) = \Box(x_{i1}^+) - [1 \otimes x_{i0}^+, \Omega_+].
\]
More explicitly
\[ \Delta(x_i^+) = x_i^+ \otimes 1 + 1 \otimes x_i^+ + h_{i0} \otimes x_i^+ - \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim g_\alpha} x_{-\alpha}^{(k)} \otimes [x_{i0}^+, x_{\alpha}^{(k)}]. \]

Similarly we have
\[ \Delta(x_i^-) = \Box(x_i^-) - [x_{i0}^- \otimes 1, \Omega_+] \]
\[ = x_i^- \otimes 1 + 1 \otimes x_i^- + x_{i0}^- \otimes h_{i0} + \sum_{\alpha \in \Delta_+} \sum_{k=1}^{\dim g_\alpha} [x_{i0}^-, x_{-\alpha}^{(k)}] \otimes x_{\alpha}^{(k)}.

4(iii). Proof of Theorem 4.11. Part I. We begin by checking that all the defining relations except (2.13) when \( r = 1 = s \) are compatible with \( \Delta \). Computations for these relations work for any Kac-Moody Lie algebra \( g \) satisfying the assumptions of Theorem 2.12. We do not need to check the relations involving only \( h_{i0} \) or \( x_i^\pm \).

We first check (2.13) with \( (r, s) = (0, 1) \):
\[ [\Delta h_{i0}, \Delta(h_{j1})] = [\Box h_{i0}, \Box(h_{j1}) + [h_{i0} \otimes 1, \Omega_+] = [[\Box(h_{i0}), h_{i0} \otimes 1], \Omega_+] + [h_{i0} \otimes 1, [\Box(h_{i0}), \Omega_+]]. \]
This vanishes thanks to (4.3).

Next let us check (2.15) with \( (r, s) = (1, 0) \):
\[ [\Delta(x_i^+), \Delta(x_{j0}^-)] = [\Box(x_i^+) - [1 \otimes x_{i0}^+, \Omega_+], \Box(x_{j0}^-)] \text{ by (2.14)} \]
\[ = \Box((x_i^+, x_{j0}^-) - [1 \otimes x_{i0}^+, \Box(x_{j0}^-), \Omega_+] - [1 \otimes x_{i0}^+, [\Omega_+, \Box(x_{j0}^-)]]) \]
\[ = \delta_{ij} \Box(h_{i1}) - \delta_{ij} [1 \otimes h_{i0}, \Omega_+] + [1 \otimes x_{i0}^+, h_{j0} \otimes x_{j0}^-] \]
\[ = \delta_{ij} (\Box(h_{i1}) + [h_{i0} \otimes 1, \Omega_+] + h_{i0} \otimes h_{i0}) = \delta_{ij} \Delta(h_{i1}), \]

where we have used (4.5) in the third equality and (4.3) in the fourth.

The relation (2.15) with \( (r, s) = (0, 1) \) can be checked in a similar way.

Next we check (2.16):
\[ [\Delta(h_{i1}), \Delta(x_{j0}^\pm)] = [\Box(h_{i1}) + [h_{i0} \otimes 1, \Omega_+], \Box(x_{j0}^\pm)] \text{ by (4.9)} \]
\[ = \Box((h_{i1}^\pm, x_{j0}^\pm) + [h_{i0} \otimes 1, \Box(x_{j0}^\pm), \Omega_+] + [h_{i0} \otimes 1, [\Omega_+, \Box(x_{j0}^\pm)]]) \]
\[ = \pm (\alpha_i, \alpha_j) (\Box(x_{j0}^\pm + 1, \Omega_+) + [h_{i0} \otimes 1, \Omega_+, \Box(x_{j0}^\pm)]). \]

In the + case, the above is equal to
\[ (\alpha_i, \alpha_j) (\Box(x_{j0}^+) + [x_{j0}^+ \otimes 1, \Omega_+] + h_{i0} \otimes 1, x_{j0}^+ \otimes h_{j0}) \]
\[ = (\alpha_i, \alpha_j) (\Box(x_{j0}^+) + [x_{j0}^+ \otimes 1, \Omega_+] + x_{j0}^+ \otimes h_{j0}) = (\alpha_i, \alpha_j) (\Box(x_{j0}^+) - [1 \otimes x_{j0}^+, \Omega_+]), \]

thanks to (4.4). By (1.14), this is precisely \((\alpha_i, \alpha_j) \Delta(x_{j0}^+)\). The − case can be proved in a similar way. Thus, \( \Delta \) preserves the relation (2.16).

Let us check that (2.17) is compatible with \( \Delta \). We have
\[ [\Delta(x_{i1}^+), \Delta(x_{j0}^+) = [\Box(x_{i1}^+) - [1 \otimes x_{i0}^+, \Omega_+], \Box(x_{j0}^+) \text{ by (4.14)} \]
\[ = \Box((x_{i1}^+, x_{j0}^+)) - [[1 \otimes x_{i0}^+, \Box(x_{j0}^+), \Omega_+] - [1 \otimes x_{i0}^+, [\Omega_+, \Box(x_{j0}^+)])] \]
\[ = \Box((x_{i1}^+, x_{j0}^+)) - [1 \otimes x_{i0}^+, x_{j0}^+ \otimes \Omega_+] + [1 \otimes x_{i0}^+, x_{j0}^+ \otimes h_{j0]} \text{ by (4.3)} \]
\[ = \Box((x_{i1}^+, x_{j0}^+)) - [1 \otimes x_{i0}^+, x_{j0}^+ \otimes \Omega_+] + (\alpha_i, \alpha_j)x_{j0}^+ \otimes x_{i0}^+. \]
Exchanging $i$ and $j$, we also obtain an expression for $\Delta(x_{j_1}^+, \Delta(x_{j_0}^+))$. Adding this to the above expression for $\Delta(x_{i_1}^+, \Delta(x_{i_0}^+))$ yields
\begin{equation}
\Delta(x_{i_1}^+, \Delta(x_{j_0}^+)) + \Delta(x_{j_1}^+, \Delta(x_{i_0}^+)) = \Box \left( [x_{i_1}^+, x_{j_0}^+] - [x_{i_0}^+, x_{j_1}^+] \right) + \left( \alpha_i, \alpha_j \right) \left( x_{i_0}^+ \otimes x_{j_0}^+ + x_{i_0}^+ \otimes x_{j_0}^+ \right).
\end{equation}

On the other hand, applying $\Delta$ to the right hand side of (2.17), we have
\begin{equation}
\frac{1}{2} \Box(x_{i_0}^+ \Box(x_{j_0}^+) + \Box(x_{j_0}^+) \Box(x_{i_0}^+)) = \frac{1}{2} \left( \Box \left[ x_{i_0}^+ \otimes x_{j_0}^+ \right] + 2 \left( x_{i_0}^+ \otimes x_{j_0}^+ + x_{i_0}^+ \otimes x_{j_0}^+ \right) \right).
\end{equation}

This is equal to (4.15) thanks to (2.6) in $Y(g')$. This proves the compatibility of $\Delta$ with (2.17) when $\pm = \pm$. The same proof works for the $-\pm$ case.

4(iv). Proof of Theorem 4.11. Part II. It remains to verify that $\Delta$ preserves the relation $[\hat{h}_{i_1}, \hat{h}_{j_1}] = 0$ for all $i, j \in I$. To accomplish this, we will need to make use of the assumption that $\hat{g}$ is of finite or affine type. We start with:
\begin{equation}
\Delta(h_{i_1}), \Delta(h_{j_1}) = \Delta(J(h_i)) - \Delta(v_i), \Delta(J(h_j)) - \Delta(v_j) ] \text{ by (4.10)}
\end{equation}
\begin{equation}
\Delta(J(h_i)), \Delta(J(h_j)) + \Delta(v_i), \Delta(v_j) - \Delta(J(h_i)), \Delta(J(h_j)) - \Delta(v_i), \Delta(J(h_j)) = 0.
\end{equation}

We will give a detailed proof that
\begin{equation}
\Delta(J(h_i)), \Delta(J(h_j)) + \Delta(v_i), \Delta(v_j) = 0
\end{equation}
and we will explain briefly how a similar argument shows that
\begin{equation}
\Delta(J(h_i)), \Delta(v_j) + \Delta(v_i), \Delta(J(h_j)) = 0.
\end{equation}

These results are very similar to Proposition 3.16 and Corollary 3.17 and we will need Proposition 3.16 to establish those analogs.

We now turn to the proof of (4.17). Using (4.10) we get
\begin{equation}
\Delta(J(h_i)), \Delta(J(h_j)) = \Box(\left[ J(h_i), J(h_j) \right])
+ \frac{1}{2} \left( \Box(\left[ J(h_i), h_{j_0} \otimes 1, \Omega \right]) - \Box(\left[ J(h_j), h_{i_0} \otimes 1, \Omega \right]) \right) + \frac{1}{4} \left[ h_{i_0} \otimes 1, \Omega, \left[ h_{j_0} \otimes 1, \Omega \right] \right].
\end{equation}

We have
\begin{equation}
\Box(\left[ J(h_i), h_{j_0} \otimes 1, \Omega \right]) = \sum_{\alpha \in \Delta^+} (\alpha, \alpha_j) \Box(\left[ J(h_i), x_{\alpha}^+ \otimes x_{\alpha}^- - x_{\alpha}^- \otimes x_{\alpha}^+ \right])
\end{equation}
\begin{equation}
\sum_{\alpha \in \Delta^+} (\alpha, \alpha_j) (\alpha, \alpha_i) \left( J(x_{\alpha}^+) \otimes x_{\alpha}^- - x_{\alpha}^+ \otimes J(x_{\alpha}^-) + J(x_{\alpha}^-) \otimes x_{\alpha}^+ - x_{\alpha}^- \otimes J(x_{\alpha}^+) \right);
\end{equation}

where we have used Proposition 3.16 This is symmetric with respect to $i$ and $j$. Therefore the sum of the second and third terms of the right hand side of (4.19) vanishes.

Using (4.10), we obtain:
\begin{equation}
\Delta(v_i), \Delta(v_j) = \Box(\left[ v_i, v_j \right]) + \frac{1}{2} \left( \Box(\left[ v_i, h_{j_0} \otimes 1, \Omega_+ - \Omega_- \right]) - \Box(\left[ v_j, h_{i_0} \otimes 1, \Omega_+ - \Omega_- \right]) \right)
+ \frac{1}{4} \left[ h_{i_0} \otimes 1, \Omega_+ - \Omega_- \right], \left[ h_{j_0} \otimes 1, \Omega_+ - \Omega_- \right].
\end{equation}
Using this along with $\Omega = \Omega_+ + \Omega_- \text{ and } (4.19)$, we arrive at the equality
\begin{equation}
(4.20) \quad \left[ \Delta(J(h_i)), \Delta(J(h_j)) \right] + [\Delta(\tilde{v}_i), \Delta(\tilde{v}_j)] = \Box \left[ \left[ J(h_i), J(h_j) \right] + [\tilde{v}_i, \tilde{v}_j] \right]
\end{equation}
\begin{equation}
(4.21) \quad + \frac{1}{2} \left( -[\Box(\tilde{v}_i), [h_{j0} \otimes 1, \Omega_+ - \Omega_-]] + [\Box(\tilde{v}_j), [h_{j0} \otimes 1, \Omega_+ - \Omega_-]] \right)
\end{equation}
\begin{equation}
(4.22) \quad + \frac{1}{2} \left( [[h_{j0} \otimes 1, \Omega_+], [h_{j0} \otimes 1, \Omega_+]] + [[h_{j0} \otimes 1, \Omega_-], [h_{j0} \otimes 1, \Omega_-]] \right).
\end{equation}
By Corollary 3.17, the expression on the right-hand side of (4.20) vanishes. Therefore it is enough to check that the last two lines cancel out. We have
\begin{equation}
[[h_{j0} \otimes 1, \Omega_+], [h_{j0} \otimes 1, \Omega_-]] = \sum_{\alpha, \beta \in \Delta^e} (\alpha_i, \alpha)(\alpha_j, \beta)[x^+_\alpha \otimes x^+_{\beta}, x^+_\alpha \otimes x^+_{\beta}]
\end{equation}
\begin{equation}
= \frac{1}{2} \sum_{\alpha, \beta \in \Delta^e} (\alpha_i, \alpha)(\alpha_j, \beta) \left( [x^+_\alpha, x^+_{\beta}] \otimes \{x^+_{\alpha}, x^+_{\beta}\} + \{x^+_{\alpha}, x^+_{\beta}\} \otimes [x^+_{\alpha}, x^+_{\beta}] \right).
\end{equation}
Adding this with the same expression with $\alpha, \beta$ exchanged, and then dividing by 2, we change this to
\begin{equation}
\frac{1}{4} \sum_{\alpha, \beta \in \Delta^e} ((\alpha_i, \alpha)(\alpha_j, \beta) - (\alpha_j, \alpha)(\alpha_i, \beta)) \left( [x^+_\alpha, x^+_{\beta}] \otimes \{x^+_{\alpha}, x^+_{\beta}\} + \{x^+_{\alpha}, x^+_{\beta}\} \otimes [x^+_{\alpha}, x^+_{\beta}] \right).
\end{equation}
This implies that
\begin{equation}
(4.23) \quad [[h_{j0} \otimes 1, \Omega_+], [h_{j0} \otimes 1, \Omega_-]] + [[h_{j0} \otimes 1, \Omega_+], [h_{j0} \otimes 1, \Omega_-]]
\end{equation}
\begin{equation}
= \frac{1}{4} \sum_{\alpha, \beta \in \Delta^e} A_{i, j, \alpha, \beta} \sum_{\sigma = +, -} \left( [x^\sigma_{\alpha}, x^\sigma_{\beta}] \otimes \{x^{-\sigma}_{\alpha}, x^{-\sigma}_{\beta}\} + \{x^{-\sigma}_{\alpha}, x^{-\sigma}_{\beta}\} \otimes [x^\sigma_{\alpha}, x^\sigma_{\beta}] \right),
\end{equation}
where $A_{i, j, \alpha, \beta} = (\alpha_i, \alpha)(\alpha_j, \beta) - (\alpha_j, \alpha)(\alpha_i, \beta)$.

On the other hand, by the definition of $\tilde{v}_i$ and of $\Omega_+ - \Omega_-$, we have the relations
\begin{equation}
- [\tilde{v}_i \otimes 1, [h_{j0} \otimes 1, \Omega_+ - \Omega_-]] = \sum_{\beta \in \Delta^e} (\alpha_i, \beta) \left( [\tilde{v}_i, x^+_\beta] \otimes x^+_\beta + [\tilde{v}_i, x^-_\beta] \otimes x^-_\beta \right),
\end{equation}
\begin{equation}
[\tilde{v}_i, x^\pm_\beta] = \pm \frac{1}{4} h^\vee (\alpha_i, \beta) x^\pm_\beta + \frac{1}{2} \sum_{\alpha \in \Delta^e} (\alpha_i, \alpha) \left( [x^\pm_\alpha, x^+_{\beta}] x^\pm_\alpha + [x^-_\alpha, x^-_{\beta}] x^\pm_\alpha \right).
\end{equation}
Thus, we find that $- [\tilde{v}_i \otimes 1, [h_{j0} \otimes 1, \Omega_+ - \Omega_-]]$ is equal to
\begin{equation}
\frac{1}{4} h^\vee \sum_{\beta \in \Delta^e} (\alpha_j, \beta)(\alpha_i, \beta)(x^+_\beta \otimes x^-_\beta - x^-_\beta \otimes x^+_\beta)
\end{equation}
\begin{equation}
+ \frac{1}{2} \sum_{\alpha, \beta \in \Delta^e} (\alpha_j, \beta)(\alpha_i, \alpha) \left( [x^-_{\alpha}, x^-_{\beta}] x^+_\alpha + [x^-_{\alpha}, x^-_{\beta}] x^-_\alpha \right) \otimes x^-_\beta + (x^-_{\alpha}, x^+_\beta) \otimes x^-_\beta + (x^+_{\alpha}, x^+_\beta) \otimes x^+_\beta).
\end{equation}
Multiplying by $-1$ and switching $i$ and $j$ we obtain an expression for $[\tilde{v}_j \otimes 1, [h_{j0} \otimes 1, \Omega_+ - \Omega_-]]$. Hence, $- [\tilde{v}_i \otimes 1, [h_{j0} \otimes 1, \Omega_+ - \Omega_-]] + [\tilde{v}_j \otimes 1, [h_{j0} \otimes 1, \Omega_+ - \Omega_-]]$ can be written as
\begin{equation}
\frac{1}{2} \sum_{\alpha, \beta \in \Delta^e} A_{i, j, \alpha, \beta} \left( ([x^-_{\alpha}, x^-_{\beta}] x^+_\alpha + [x^-_{\alpha}, x^-_{\beta}] x^-_\alpha) \otimes x^+_\beta + ([x^-_{\alpha}, x^-_{\beta}] x^+_\alpha + [x^-_{\alpha}, x^-_{\beta}] x^-_\alpha) \otimes x^-_\beta \right),
\end{equation}
where $A_{i, j, \alpha, \beta}$ has been defined below (4.23).
Consider the second term \( x^-_\alpha [x^-_\gamma, x^-_\beta] \otimes x^+_\beta \) of the previous expression. Assume that \( \alpha - \beta \) is positive, and let \( \gamma = \alpha - \beta \). If \( \gamma \) is an imaginary root, the coefficient \((\alpha_i, \alpha)(\alpha_j, \beta) - (\alpha_j, \alpha)(\alpha_i, \beta)\) vanishes. Therefore we may assume \( \gamma \) is a real root. Then
\[
x^-_\alpha [x^-_\gamma, x^-_\beta] \otimes x^+_\beta = ([x^-_\gamma, x^-_\gamma], x^-_\gamma) x^-_\alpha x^-_\gamma \otimes x^+_\gamma = x^-_\alpha x^-_\gamma \otimes [x^-_\gamma, x^-_\alpha].
\]
Next suppose \( \alpha - \beta \) is negative, and let \( \gamma = \beta - \alpha \). Then, similarly,
\[
x^-_\alpha [x^-_\gamma, x^-_\beta] \otimes x^+_\beta = x^-_\alpha x^-_\gamma \otimes [x^-_\gamma, x^-_\alpha].
\]
Therefore, \( \frac{1}{2} \sum_{\alpha, \beta \in \Delta^+_re} A_{i,j,\alpha,\beta} x^-_\alpha x^-_\gamma \otimes [x^-_\gamma, x^-_\alpha] \) is equal to

\[
\frac{1}{2} \sum_{\alpha, \beta \in \Delta^+_re} A_{i,j,\alpha,\beta} x^-_\alpha x^-_\gamma \otimes [x^-_\gamma, x^-_\alpha] + \frac{1}{2} \sum_{\alpha, \gamma \in \Delta^+_re} A_{i,j,\alpha,\gamma} x^-_\alpha x^-_\gamma \otimes [x^-_\gamma, x^-_\alpha].
\]

Consider now the first term of (4.24). In this case we set \( \gamma = \alpha + \beta \) and a similar computation yields
\[
\frac{1}{2} \sum_{\alpha, \beta \in \Delta^+_re} A_{i,j,\alpha,\beta} x^-_\alpha x^-_\gamma \otimes [x^-_\gamma, x^-_\alpha] = \frac{1}{2} \sum_{\alpha, \gamma \in \Delta^+_re} A_{i,j,\alpha,\gamma} x^-_\alpha x^-_\gamma \otimes [x^-_\gamma, x^-_\alpha].
\]
Exchanging \( \alpha \) and \( \gamma \), we find that this cancels with the first sum of (4.25). We can deal with the third and fourth terms in (4.24) in the same way. These calculations allow us to deduce that (4.24) is equal to

\[
\frac{1}{2} \sum_{\alpha, \beta \in \Delta^+_re} A_{i,j,\alpha,\beta} \left( x^-_\alpha x^-_\gamma [x^-_\gamma, x^-_\alpha] + x^+_\beta x^+_\alpha \otimes [x^-_\gamma, x^-_\alpha] \right)
\]

Repeating this argument with \( \bar{v}_i \otimes 1 \) and \( \bar{v}_j \otimes 1 \) replaced by \( 1 \otimes \bar{v}_i \) and \( 1 \otimes \bar{v}_j \), respectively, we find that
\[
-\left[ 1 \otimes \bar{v}_i, [h_{j0} \otimes 1, \Omega_+ - \Omega_-] \right] + \left[ 1 \otimes \bar{v}_j, [h_{i0} \otimes 1, \Omega_+ - \Omega_-] \right]
\]
\[
= \frac{1}{2} \sum_{\alpha, \gamma \in \Delta^+_re} A_{i,j,\alpha,\gamma} \left( [x^-_\gamma, x^-_\alpha] \otimes [x^+_\gamma, x^+_\alpha] + [x^-_\gamma, x^-_\alpha] \otimes x^-_\alpha x^-_\gamma \right).
\]
Finally, combining this with (4.26) we obtain that
\[
-\left[ [\square(\bar{v}_i), [h_{j0} \otimes 1, \Omega_+ - \Omega_-]] + [\square(\bar{v}_j), [h_{i0} \otimes 1, \Omega_+ - \Omega_-]] \right)
\]
(which is twice (4.21)) is equal to
\[
\frac{1}{2} \sum_{\alpha, \gamma \in \Delta^+_re} A_{i,j,\alpha,\gamma} \left( x^-_\alpha x^-_\gamma \otimes [x^+_\gamma, x^+_\alpha] + x^+_\beta x^+_\alpha \otimes [x^-_\gamma, x^-_\alpha] \right.
\]
\[
\left. + [x^-_\gamma, x^-_\alpha] \otimes x^-_\alpha x^-_\gamma \right)
\]
which is exactly the right-hand side of (4.23) multiplied by \(-1\). The left-hand side of (4.23) is twice (4.22), so (4.21) and (4.22) cancel out. Therefore, we may conclude that \( [\Delta J(h_i), \Delta J(h_j)] = -[\Delta(\bar{v}_i), \Delta(\bar{v}_j)] \), so (4.17) holds.

The proof that (4.18) holds is very similar. We have
\[
[\Delta(J(h_i)), \Delta(\bar{v}_i)] = \left[ \square(J(h_i)) + \frac{1}{2} [h_{i0} \otimes 1, \Omega], \square(\bar{v}_j) - \frac{1}{2} [h_{j0} \otimes 1, \Omega_+ - \Omega_-] \right]
\]
\[
= \left[ \square(J(h_i)), \square(\bar{v}_j) - \frac{1}{2} [\square(J(h_i)), [h_{j0} \otimes 1, \Omega_+ - \Omega_-]] \right]
\]
\[
+ \frac{1}{2} [[h_{i0} \otimes 1, \Omega], \square(\bar{v}_j)] - \frac{1}{4} [[h_{i0} \otimes 1, \Omega], [h_{j0} \otimes 1, \Omega_+ - \Omega_-]]
\]
By Proposition 3.16, \([\Box(J(h_i)), \Box(v_j)]\) is symmetric in \(i\) and \(j\), and applying the same argument as in the proof of that proposition, we can also conclude that \([\Box(J(h_i)), [h_{j0} \otimes 1, \Omega_+ - \Omega_-]]\) is symmetric in \(i\) and \(j\). Hence, since \(\Omega = \Omega_+ + \Omega_-\), we find that

\[
\Delta(J(h_i)) \cdot \Delta(v_i) - \Delta(J(h_j)) \cdot \Delta(v_j) = \frac{1}{2} (-[\Box(v_j), [h_{i0} \otimes 1, \Omega]] + [\Box(v_i), [h_{j0} \otimes 1, \Omega]])
\]

\[
+ \frac{1}{2} ([[h_{i0} \otimes 1, \Omega_-], [h_{j0} \otimes 1, \Omega_-]] - [[h_{i0} \otimes 1, \Omega_+], [h_{j0} \otimes 1, \Omega_+]]) .
\]

It can be shown that the right-hand side of this equality vanishes using the same argument as employed to show (4.21) and (4.22) add to zero. Therefore, the equality (4.18) holds and, by (4.10), it follows that \(\Delta\) preserves the relation \([\tilde{h}_{i1}, \tilde{h}_{j1}] = 0\) for all \(i, j \in I\). This completes the proof of Theorem 4.11.

4(v). Coassociativity. It follows from Theorem 4.11 that we can turn the tensor product \(V_1 \otimes V_2\) of two representations in the category \(\mathcal{O}\) into a representation of \(Y(\mathfrak{g})\). It is very desirable that the coproduct be compatible with the associativity of the tensor product.

**Proposition 4.27.** Let \(V_1, V_2\) and \(V_3\) be \(Y(\mathfrak{g})\)-modules in the category \(\mathcal{O}\). Then the natural isomorphism of vector spaces

\[a_{V_1, V_2, V_3} : (V_1 \otimes V_2) \otimes V_3 \rightarrow V_1 \otimes (V_2 \otimes V_3)\]

is an isomorphism of \(Y(\mathfrak{g})\)-modules.

**Proof.** We need to show that, after identifying the spaces \(\text{End}_C((V_1 \otimes V_2) \otimes V_3)\) and \(\text{End}_C(V_1 \otimes (V_2 \otimes V_3))\) (via \(a_{V_1, V_2, V_3}\)), we have \(\Delta_{V_1 \otimes V_2, V_3} = \Delta_{V_1, V_2 \otimes V_3}\). Since \(Y(\mathfrak{g})\) is generated by \(\mathfrak{g}\) and \(\tilde{h}_{i1}\) (for all \(i \in I\)), we need only establish this equality when both sides are applied to \(\tilde{h}_{i1}\). By (4.9), we have

\[
\Delta_{V_1 \otimes V_2, V_3}(\tilde{h}_{i1}) = (\tilde{h}_{i1} \otimes 1) \otimes 1 + (1 \otimes 1) \otimes \tilde{h}_{i1} + (1 \otimes \tilde{h}_{i1}) \otimes 1
\]

\[
- \sum_{\alpha \in \Delta^e_i} (\alpha_i, \alpha) \left((x^+_{\alpha} \otimes x^-_{\alpha}) \otimes 1 + (x^-_{\alpha} \otimes 1 + 1 \otimes x^-_{\alpha}) \otimes x^+_{\alpha}\right),
\]

\[
\Delta_{V_1, V_2 \otimes V_3}(\tilde{h}_{i1}) = \tilde{h}_{i1} \otimes (1 \otimes 1) + 1 \otimes (\tilde{h}_{i1} \otimes 1) + 1 \otimes (1 \otimes \tilde{h}_{i1})
\]

\[
- \sum_{\alpha \in \Delta^e_i} (\alpha_i, \alpha) \left((1 \otimes (x^-_{\alpha} \otimes x^+_{\alpha}) + x^-_{\alpha} \otimes (x^+_{\alpha} \otimes 1 + 1 \otimes x^+_{\alpha}))\right).
\]

Hence, \(\Delta_{V_1 \otimes V_2, V_3}(\tilde{h}_{i1}) = \Delta_{V_1, V_2 \otimes V_3}(\tilde{h}_{i1})\) and consequently \(a_{V_1, V_2, V_3}\) is an isomorphism of \(Y(\mathfrak{g})\)-modules.

5. COPRODUCT AND COMPLETIONS OF YANGIANS

The collection of algebra homomorphisms \(\Delta_{V_1, V_2}\), which are defined on generators by (1.8), can be viewed together as a sort of comultiplication on \(Y(\mathfrak{g})\) which is coassociative in the sense of Proposition 4.27. Our present goal is to improve on this by showing that each morphism \(\Delta_{V_1, V_2} : Y(\mathfrak{g}) \rightarrow \text{End}_C(V_1 \otimes V_2)\) can be recovered from a single homomorphism \(\Delta : Y(\mathfrak{g}) \rightarrow Y(\mathfrak{g}) \otimes Y(\mathfrak{g})\), where \(Y(\mathfrak{g}) \otimes Y(\mathfrak{g})\) is a suitable completion of \(Y(\mathfrak{g}) \otimes Y(\mathfrak{g})\).

Our first step is to define a completion \(\widehat{Y}(\mathfrak{g})\) of \(Y(\mathfrak{g})\) which behaves nicely with respect to modules in the category \(\mathcal{O}\), and from which the definition of \(Y(\mathfrak{g}) \otimes Y(\mathfrak{g})\) can be obtained as a special case.
5(i). **The completion** \( \hat{Y}(\mathfrak{g}) \). Let \( \mathfrak{g} \) be a symmetrizable Kac-Moody algebra as in Section 2, except that we no longer require the Cartan matrix \( (a_{ij})_{i,j \in I} \) to be indecomposable (because we want to consider the Yangian of \( \mathfrak{g} \oplus \mathfrak{g} \)). Note that in this case we can still define the Yangian \( Y_\mathfrak{g}(\mathfrak{g}) \), and thus \( Y(\mathfrak{g}) \), using Definition 2.1. For the purpose of introducing the completion \( \hat{Y}(\mathfrak{g}) \) we need to impose two mild conditions on \( Y(\mathfrak{g}) \):

A. We suppose that \( Y(\mathfrak{g}) \) admits the multiplicative triangular decomposition

\[
Y(\mathfrak{g}) \cong Y^- \otimes Y^0 \otimes Y^+,
\]

where \( Y^\pm \) (resp. \( Y^0 \)) denotes the subalgebra of \( Y(\mathfrak{g}) \) generated \( x^\pm_{ir} \) (resp. \( h_{ir} \) and \( h \in \mathfrak{h} \)) with \( i \in I \) and \( r \geq 0 \).

B. We also assume that \( Y^\pm \) is isomorphic to the quotient of the free algebra on the generators \( x^\pm_{ir} \) for all \( i \in I \), \( r \geq 0 \) by the ideal corresponding to the relations (2.6) and (2.7).

It is very plausible that these assumptions on \( Y(\mathfrak{g}) \) are always satisfied (even when \( \mathfrak{g} \) is not affine). Indeed, for affinizations of quantum Kac-Moody algebras such a result was obtained by D. Hernandez in [Her03, Theorem 3.2], and the corresponding result for Yangians could most likely be proven using exactly the same technique.

Set \( \deg x^\pm_{ir} = 1 \) for all \( i \in I \) and \( r \geq 0 \). The assumption \([\mathbf{B}]\) implies that we have \( Y^+ = \bigoplus_{k=0}^{\infty} Y^+[k] \), where \( Y^+[k] \) is the span of all monomials of degree \( k \) in \( Y^+ \), and this grading is compatible with the algebra structure on \( Y^+ \). This together with the assumption \([\mathbf{A}]\) imply that we have the vector space grading

\[
Y(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} Y(\mathfrak{g})[k] = \bigoplus_{k=0}^{\infty} Y^{\leq 0} \otimes Y^+[k],
\]

where \( Y^{\leq 0} \) is the subalgebra of \( Y(\mathfrak{g}) \) generated by \( x^-_{ir} \) and \( h_{ir} \) for all \( i \in I \), \( r \geq 0 \) along with all \( h \in \mathfrak{h} \). Note that \( Y(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} Y(\mathfrak{g})[k] \) is not a grading of algebras.

For each \( n \in \mathbb{Z}_{\geq 0} \), let \( Y(\mathfrak{g}) \) denote the subspace \( \bigoplus_{k=n}^{\infty} Y^+[k] \) of \( Y^+ \), and let \((A_n, q_n)\) consist of the (left) \( Y(\mathfrak{g})\)-module \( A_n \) and natural quotient map \( q_n \) which are given by

\[
A_n = Y(\mathfrak{g})/Y(\mathfrak{g})Y_{\geq n+1}, \quad q_n: Y(\mathfrak{g}) \to A_n.
\]

For each \( n \geq 0 \), \( q_{n-1} \) factors through \( A_n \) to yield a \( Y(\mathfrak{g})\)-module homomorphism \( p_n: A_n \to A_{n-1} \) such that \( p_n \circ q_n = q_{n-1} \). Therefore, \((A_n, p_n)_{n \geq 0}\) forms an inverse system of \( Y(\mathfrak{g})\)-modules.

Following [CP95, 10.1.D], we introduce \( \hat{Y}(\mathfrak{g}) \) as the inverse limit of this system:

**Definition 5.1.** We define \( \hat{Y}(\mathfrak{g}) \) to be the \( Y(\mathfrak{g})\)-module obtained by taking the inverse limit of the system \((A_n, p_n)_{n \geq 0}\):

\[
\hat{Y}(\mathfrak{g}) = \varprojlim_n A_n = \varprojlim_n (Y(\mathfrak{g})/Y(\mathfrak{g})Y_{\geq (n+1)}).
\]

Let \( \iota: Y(\mathfrak{g}) \to \hat{Y}(\mathfrak{g}) \) be the homomorphism (of \( Y(\mathfrak{g})\)-modules) given by \( X \mapsto (q_n(X))_{n \geq 0} \) for all \( X \in Y(\mathfrak{g}) \). Note that \( \iota \) is injective: if \( X \in \text{Ker}(\iota) \) then \( X \in \cap_{n \geq 0} Y(\mathfrak{g})Y_{n+1}^+ = \{0\} \).

The next lemma gives a more familiar presentation of \( \hat{Y}(\mathfrak{g}) \).

**Lemma 5.3.** The embedding \( \iota \) extends to a linear isomorphism

\[
\Phi: \prod_{k=0}^{\infty} Y(\mathfrak{g})[k] \to \hat{Y}(\mathfrak{g}), \quad \sum_{k=0}^{\infty} X_k \mapsto \left( \sum_{k=0}^{n} q_n(X_k) \right)_{n \geq 0}.
\]
Henceforth, we will always identify \( \widehat{Y}(\mathfrak{g}) \) and \( \prod_{k=0}^{\infty} Y(\mathfrak{g})[k] \), and we shall especially view the elements of \( \widehat{Y}(\mathfrak{g}) \) as infinite series \( \sum_{k=0}^{\infty} X_k \) with \( X_k \in Y(\mathfrak{g})[k] \) for all \( k \geq 0 \).

The main goal for the rest of this section is to prove that \( \widehat{Y}(\mathfrak{g}) \) can be naturally made into a \( \mathbb{C} \)-algebra with structure compatible with that of \( Y(\mathfrak{g}) \). We begin by naively defining what the multiplication should be.

Given \( X^o = \sum_{k=0}^{\infty} X_k^o \) and \( X^* = \sum_{r=0}^{\infty} X_r^* \) in \( \widehat{Y}(\mathfrak{g}) \), define

\[
X^o \cdot X^* = \sum_{m=0}^{\infty} (X^o X^*)_m,
\]

where \((X^o X^*)_m = \sum_{k, \ell=0}^{\infty} (X_k^o X_\ell^*)_m \) and \((X_k^o X_\ell^*)_m \) is the component of the product \( X_k^o X_\ell^* \) which belongs to \( Y(\mathfrak{g})[m] \) (note that the product \( X_k^o X_\ell^* \) is inside \( Y(\mathfrak{g}) \)). To see that the right-hand side of (5.5) is a well-defined element of \( \widehat{Y}(\mathfrak{g}) \), we have to show that \( \sum_{k, \ell=0}^{\infty} (X_k^o X_\ell^*)_m \) reduces to a finite sum. This will be established in the proof of Proposition 5.13, however first we will need Proposition 5.9 below whose proof depends on the next lemma.

**Lemma 5.6.** For each \( k \geq 0, i \in I \) and \( r \geq 0 \) we have the inclusions

\[
(5.7) \quad Y^+[k] h_{ir} \subset Y(\mathfrak{g})[k],
\]

\[
(5.8) \quad Y^+[k] x_{ir}^+ \subset Y(\mathfrak{g})[k] \oplus Y(\mathfrak{g})[k-1].
\]

**Proof.** Let’s prove the first inclusion. Let \( X = x_{i_1 r_1}^+ \cdots x_{i_k r_k}^+ \) be a monomial in \( Y^+[k] \). The proof will be by double induction on \( r \) and on \( k \). When \( k = 0 = r \), (5.7) is true since \( Y^+[0] = \mathbb{C} \).

We proceed with induction on \( k \), so assume that \( k \geq 1 \) and write \( X = X_1 x_{i_k r_k}^+ \). Then, we have

\[
X h_{i_r+1} = X_1 h_{i_r+1} x_{i_k r_k}^+ + X_1 [x_{i_k r_k}^+, h_{i_r+1}]
\]

\[
= X_1 h_{i_r+1} x_{i_k r_k}^+ - X_1 \left( \frac{\alpha_i}{2} (h_{i_r} x_{i_k r_k}^+ + x_{i_k r_k}^+ h_{i_r} + h_{i_r} x_{i_k r_k}^+ - x_{i_k r_k}^+ h_{i_r}) \right).
\]

Since \( X_1 \) has length \( k - 1 \), \( X_1 h_{i_r+1} x_{i_k r_k}^+ \in Y(\mathfrak{g})[k] \), and, by induction on \( r \), the rest of the terms on the right-hand side of the above expression also belong to \( Y(\mathfrak{g})[k] \). Hence, by double induction, (5.7) holds for any \( k, r \geq 0 \).

The inclusion (5.8) can be proved similarly using induction on \( k \) and (5.7). \( \square \)

Note that, since \( Y^+[k] h \subset Y(\mathfrak{g})[k] \) for all \( h \in \mathfrak{h} \) and \( k \geq 0 \), the relation (5.7) of Lemma 5.6 implies that \( Y^+[k] \cdot Y^0 \subset Y(\mathfrak{g})[k] \) for all \( k \geq 0 \). We shall use this fact in the next Proposition.

**Proposition 5.9.** Let \( Z \in Y^{\leq 0} \). Then, for every non-negative integer \( m \geq 0 \) there exists \( N_m^Z \geq 0 \) such that

\[
(5.10) \quad [Y^+[k], Z] \in \bigoplus_{a=m+1}^{k} Y(\mathfrak{g})[a] \quad \text{for all} \quad k \geq N_m^Z.
\]

**Proof.** Without loss of generality, we may assume that \( Z \) is a monomial in the generators of \( Y^{\leq 0} \), say

\[
Z = x_{j_1 s_1}^- \cdots x_{j_r s_r}^- H,
\]

where
where $H$ is a monomial in the generators of $Y^0$. Since $Y^+[k]$ is spanned by homogeneous monomials of degree $k$, it suffices to prove the existence of $N_m^Z \geq 0$ such that

$$[X_k, Z] \in \bigoplus_{a=m+1}^{k} Y(g)[a] \quad \text{for all } k \geq N_m^Z$$

for every monomial $X_k$ of degree $k$. We will prove the stronger result that, for $Z$ as in (5.11), $N_m^Z$ can be taken to be precisely $m + \ell + 1$. Set $Z_0 = x_{j_1,s_1}^{-} \cdots x_{j_{\ell},s_{\ell}}^{-}$ for each $0 \leq b \leq \ell$, where $Z_0$ is understood to equal 1. By relation (5.7) of Lemma 5.6 we have for every monomial $X_d$ by Lemma 5.6 yields that $Z_0$ is a monomial in the generators of $Y(g)[a]$ whenever $k \geq m + 1$. These observations together with the fact that

$$[X_k, Z] = [X_k, Z_\ell] H + [X_k, H]$$

imply that it suffices to show that $[X_k, Z_\ell] \in \bigoplus_{a=m+1}^{k} Y(g)[a]$ for all $k \geq m + \ell + 1$. We will prove this statement by induction on $\ell \geq 0$. The base of the induction is immediate since $Z_0 = 1$. Next, fix $d > 0$ and assume inductively that the statement holds when $\ell$ is replaced by $d - 1$. If now $\ell$ is replaced instead by $d$, then we may rewrite $[X_k, Z_d]$ as

$$[X_k, Z_d] = Z_{d-1}[X_k, x_{j_d,s_d}^{-}] + [X_k, Z_{d-1}]x_{j_d,s_d}^{-}.$$

If $d = 1$, the second term on the right-hand side of the above equation vanishes and (5.8) of Lemma 5.6 yields that $Z_{d-1}[X_k, x_{j_d,s_d}^{-}]$ belongs to $\bigoplus_{a=m+1}^{k} Y(g)[a]$ for all $k \geq m + 2$, as desired. If instead $d > 1$, then the latter statement of the previous sentence still holds. Moreover, the inductive hypothesis implies that $[X_k, Z_{d-1}] \in \bigoplus_{a=m+1}^{k} Y(g)[a]$ for $k \geq m + d + 1$, and thus $[X_k, Z_{d-1}]x_{j_d,s_d}^{-} \in \bigoplus_{a=m+1}^{k} Y(g)[a]$ for all such values of $k$ as a consequence of (5.8). Since $m + d + 1 \geq m + 2$, we may conclude from (5.12) that $[X_k, Z_d] \in \bigoplus_{a=m+1}^{k} Y(g)[a]$ whenever $k \geq m + d + 1$.

**Proposition 5.13.** The operation given in (5.9) is a well-defined product equipping $\hat{Y}(g)$ with the structure of an associative algebra.

**Proof.** We have to see that $\sum_{k, \ell=0}^{\infty} (X_k^0 X_\ell^\bullet)_m$ reduces to a finite sum. The product $X_k^0 X_\ell^\bullet$ is in $\bigoplus_{r=\ell}^{\infty} Y(g)[r]$, so if $(X_k^0 X_\ell^\bullet)_m \neq 0$, then $\ell \leq m$.

For each pair $k, \ell \in \mathbb{Z}_{\geq 0}$ write $X_k^0 = \sum_{i \in I_k} y_{k,i} x_{k,i}^0$ and $X_\ell^\bullet = \sum_{j \in J_\ell} z_{\ell,j} x_{\ell,j}^\bullet$, where $y_{k,i}, z_{\ell,j} \in Y^0, x_{k,i}^0 \in Y^+[k]$ and $x_{\ell,j}^\bullet \in Y^+[\ell]$ for all $i \in I_k, j \in J_\ell$; $I_k$ and $J_\ell$ being finite sets. We then have

$$\sum_{k, \ell=0}^{\infty} (X_k^0 X_\ell^\bullet)_m = \sum_{k, \ell=0}^{\infty} \sum_{i \in I_k, j \in J_\ell} (y_{k,i} z_{\ell,j} x_{k,i}^0 x_{\ell,j}^\bullet)_m + \sum_{k, \ell=0}^{\infty} \sum_{i \in I_k, j \in J_\ell} (y_{k,i} [x_{k,i}^0, z_{\ell,j}] x_{\ell,j}^\bullet)_m$$

$$= \sum_{k+\ell=m} \sum_{i \in I_k, j \in J_\ell} y_{k,i} z_{\ell,j} x_{k,i}^0 x_{\ell,j}^\bullet + \sum_{k, \ell=0}^{\infty} \sum_{i \in I_k, j \in J_\ell} (y_{k,i} [x_{k,i}^0, z_{\ell,j}] x_{\ell,j}^\bullet)_m.$$  

The first sum is finite, so we need only show that the second summation is also finite. Set

$$N = \max_{0 \leq \ell \leq m} \max_{j \in J_\ell} N_{m-\ell}^{z_{\ell,j}}.$$
If \( k \geq N \), then Proposition 5.9 implies that \( [x_{k,i}^n, z_{\ell,j}] \) is a sum of homogeneous elements of degree \( \geq m - \ell + 1 \) for all \( 0 \leq \ell \leq m \). Therefore,

\[
\sum_{k,\ell \geq 0, i \in I_k, j \in J_\ell} (y_{k,i} [x_{k,i}^n, z_{\ell,j}] x_{\ell,j}^*)_m = \sum_{k=0}^{N} \sum_{\ell=0}^{m} \sum_{i \in I_k, j \in J_\ell} (y_{k,i} [x_{k,i}^n, z_{\ell,j}] x_{\ell,j}^*)_m
\]

and the sum on the right-hand side is a finite sum, which completes the proof. \( \square \)

The last result of this subsection illustrates that \( \hat{Y}(\mathfrak{g}) \) is particularly well-behaved with respect to the category \( \mathcal{O} \) of \( Y(\mathfrak{g}) \).

**Proposition 5.14.** The completion \( \hat{Y}(\mathfrak{g}) \) has the following properties:

1. For each \( i \in I, J(x_i^\pm) \) and \( J(h_i) \) (see (3.0)) can be viewed as elements of \( \hat{Y}(\mathfrak{g}) \);
2. Every module \( V \) of \( Y(\mathfrak{g}) \) in the category \( \mathcal{O} \) extends to a module over \( \hat{Y}(\mathfrak{g}) \).

**Proof.** To prove (1), it suffices to show that the infinite sum \( \sum_{\alpha \in \Delta_+} (\alpha, \alpha) \sum_{k=1}^{\text{dim} \mathfrak{g}_\alpha} x_{-\alpha}^k x_{\alpha}^k \) is contained in \( \hat{Y}(\mathfrak{g}) \), which can be verified directly. As for (2), given \( \mathfrak{g} \), given \( X = \bigoplus_{k=0}^{\infty} X_k \in \hat{Y}(\mathfrak{g}) \), the operator \( X_V \) given by \( X_V(v) = \sum_{k=0}^{\infty} X_k v \) for all \( v \in V \) is a well-defined element of \( \text{End}_C V \) because \( Y[k]v = 0 \) for all \( k \gg 0 \). \( \square \)

**Remark 5.15.** The definition of \( \hat{Y}(\mathfrak{g}) \) as the inverse limit \( \prod \) has been motivated by Section 10.1.D of [CP95]. Therein, an analogous completion \( \hat{U}_q(\mathfrak{g}) \) of the quantum enveloping algebra \( U_q(\mathfrak{g}) \) (where \( \mathfrak{g} \) is finite-dimensional) was introduced in order to study the universal \( R \)-matrix of \( U_q(\mathfrak{g}) \). A similar completion for more general quantized Kac-Moody algebras was constructed in [Jos99] 4.1.

In [CI84, Section 2], the authors defined an algebra \( \mathfrak{U}(R, \mathcal{C}) \) which can be associated to any ring \( R \) and a full subcategory \( \mathcal{C} \) of the category of \( R \)-modules. Of specific interest in [CI84] was the case where \( R = U(\mathfrak{g}) \) and \( \mathcal{C} \) is taken to be the category \( \mathcal{O} \) for the Kac-Moody algebra \( \mathfrak{g} \) (to this effect, see also [Kum86]). However, when one takes instead \( R = Y(\mathfrak{g}) \) and \( \mathcal{C} \) to be the category \( \mathcal{O} \) for \( Y(\mathfrak{g}) \), one arrives at an algebra which is closely related to \( \hat{Y}(\mathfrak{g}) \) as a left \( Y(\mathfrak{g}) \)-module, but has a different multiplicity. This construction has also served as a source of motivation for our definition of \( \hat{Y}(\mathfrak{g}) \).

5(ii). **The coproduct** \( \Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) \). Let \( \mathfrak{g} \) be as in the previous subsection with \( Y(\mathfrak{g}) \) satisfying the assumptions \( \mathbb{A} \) and \( \mathbb{B} \). Consider the Yangian \( Y(\mathfrak{g} \oplus \mathfrak{g}) \). As algebras, we have the isomorphism \( Y(\mathfrak{g} \oplus \mathfrak{g}) \cong Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) \) (see for instance Proposition II.4.2 of [Kas95]). In particular, \( Y(\mathfrak{g} \oplus \mathfrak{g}) \) also satisfies the assumptions \( \mathbb{A} \) and \( \mathbb{B} \), and therefore we can define \( Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g}) \) using Definition 5.11

\[
Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g}) = \hat{Y}(\mathfrak{g} \oplus \mathfrak{g}).
\]

More generally, we define the completed \( n \)-th tensor power \( Y(\mathfrak{g}) \hat{\otimes}^n = Y(\mathfrak{g}) \otimes \cdots \otimes Y(\mathfrak{g}) \) (where \( Y(\mathfrak{g}) \) appears \( n \)-times) as \( \hat{Y}(\mathfrak{g} \hat{\otimes}^n) \). Using Lemma 5.3 we can identify \( Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g}) \) with the direct product

\[
\prod_{k=0}^{\infty} (Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[k] = \prod_{k=0}^{\infty} \left( \bigoplus_{r+s=k} Y(\mathfrak{g})[r] \otimes Y(\mathfrak{g})[s] \right).
\]

We now return to the setting where \( \mathfrak{g} \) is an affine Lie algebra with indecomposable Cartan matrix \( (a_{ij})_{i,j \in I} \), which is of not of type \( A_1^{(1)} \) or \( A_2^{(2)} \).
Note that the half Casimir $\Omega_+$ from (4.1) can be viewed as an element of $Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$, and therefore we may define the assignment
\[
\Delta : \{ x_{i, h}^\pm, h_{i, h} : i \in I, h \in \mathfrak{h} \} \to Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})
\]
extactly as $\Delta_{V_1, V_2}$ has been defined in (4.8), except that $\text{End}_\mathbb{C}(V_1 \otimes V_2)$ should be replaced by $Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$.

**Proposition 5.17.** Assume that $\mathfrak{g}$ is of affine type, but not $A_1^{(1)}$ or $A_2^{(2)}$. The assignment $\Delta$ extends to an algebra homomorphism $\Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$. Additionally, if $V_1, V_2$ belong to the category $\mathcal{O}$ and $\rho_1, \rho_2$ are the corresponding homomorphisms $Y(\mathfrak{g}) \to \text{End}_\mathbb{C}(V_1)$ and $Y(\mathfrak{g}) \to \text{End}_\mathbb{C}(V_2)$, respectively, then $\rho_1 \otimes \rho_2$ extends to $\rho_1 \hat{\otimes} \rho_2 : Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g}) \to \text{End}_\mathbb{C}(V_1 \otimes V_2)$, and we have
\[
\Delta_{V_1, V_2} = (\rho_1 \hat{\otimes} \rho_2) \circ \Delta.
\]

*Proof.* Part I of the proof of Theorem [1.11] can be carried out without modification when $\text{End}_\mathbb{C}(V_1 \otimes V_2)$ is replaced by $Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$. Since $Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$ also contains the Casimir operators $\Omega$ and $\Omega_-$, Part II of the proof also proceeds without alteration when $\text{End}_\mathbb{C}(V_1 \otimes V_2)$ is replaced by $Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$. Therefore, $\Delta$ extends to an algebra homomorphism $\Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$.

If $(\rho_1, V_1)$ and $(\rho_2, V_2)$ are two representations in the category $\mathcal{O}$, then $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$ belongs to the category of $\mathcal{O}$ for the Yangian $Y(\mathfrak{g} \oplus \mathfrak{g})$. Therefore, by Proposition [5.14], $\rho_1 \otimes \rho_2$ extends to a homomorphism
\[
\rho_1 \hat{\otimes} \rho_2 : \hat{Y}(\mathfrak{g} \oplus \mathfrak{g}) = Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g}) \to \text{End}_\mathbb{C}(V_1 \otimes V_2).
\]
The equality $\Delta_{V_1, V_2} = (\rho_1 \hat{\otimes} \rho_2) \circ \Delta$ is now immediate since both sides agree on generators of $Y(\mathfrak{g})$. \qed

An alternative to working with the completed tensor $Y(\mathfrak{g}) \hat{\otimes} Y(\mathfrak{g})$ is to replace $\Delta$ by a family of linear maps $\Delta_{\lambda, \mu, \lambda_1, \mu_2}$ as suggested, for instance, in Chapter 23 of [Lus10]. This alternative should be closer to the geometric construction in [MaOk]; see [Nak13] also. We assume again that $\mathfrak{g}$ is affine and not of type $A_1^{(1)}$ or $A_2^{(2)}$. In particular, $\mathfrak{h} = \text{span} \mathbb{B}$ where $\mathbb{B} = \{ h_i, d \mid i \in I \}$ and $d$ is the derivation. Given two elements $\lambda, \mu$ of the weight lattice of $\mathfrak{g}$, set
\[
\lambda Y(\mathfrak{g})_\mu = Y(\mathfrak{g}) \big( \sum_{h \in \mathbb{B}} (h - \lambda(h)) Y(\mathfrak{g}) + Y(\mathfrak{g}) \sum_{h \in \mathbb{B}} (h - \mu(h)) \big)
\]
and let $\pi_{\lambda, \mu} : Y(\mathfrak{g}) \to \lambda Y(\mathfrak{g})_\mu$ be the projection map. Following [Lus10], the non-unital algebra $\bigoplus_{\lambda, \mu} \lambda Y(\mathfrak{g})_\mu$ could be called the modified Yangian. We will denote it $\hat{Y}(\mathfrak{g})$. Its algebra structure is defined similarly to the one on the modified quantized enveloping algebra in loc. cit.

We have a root grading on $Y(\mathfrak{g})$ given by $\deg(x_i^\pm) = \pm \alpha_i$, $\deg(h_i) = 0$ for all $i \in I$, $r \geq 0$ and $\deg(d) = 0$, which leads to direct sum decompositions into graded pieces
\[
Y(\mathfrak{g}) = \bigoplus_{\nu \in \mathbb{Z} \Delta} Y(\mathfrak{g})\{\nu\} \text{ and } \hat{Y}(\mathfrak{g}) = \bigoplus_{\nu \in \mathbb{Z} \Delta} \bigoplus_{\lambda, \mu} \pi_{\lambda, \mu}(Y(\mathfrak{g})\{\nu\}).
\]
Moreover, $\pi_{\lambda, \mu}(Y(\mathfrak{g})\{\nu\}) \neq 0$ only if $\lambda - \mu = \nu$.

Now let $\lambda_1, \mu_1, \lambda_2, \mu_2$ be elements of the weight lattice of $\mathfrak{g}$. The map $\pi_{\lambda_1, \mu_1} \otimes \pi_{\lambda_2, \mu_2} : Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) \to \lambda_1 Y(\mathfrak{g})_{\mu_1} \otimes \lambda_2 Y(\mathfrak{g})_{\mu_2}$ can be restricted to $(Y(\mathfrak{g}) \otimes Y(\mathfrak{g}))[k]$ for any $k$ and we denote
its restriction by \((\pi_{\lambda_1,\mu_1} \otimes \pi_{\lambda_2,\mu_2})|_k\). Set \((\lambda_1 Y(g)_{\mu_1} \otimes \lambda_2 Y(g)_{\mu_2})[k] = (\pi_{\lambda_1,\mu_1} \otimes \pi_{\lambda_2,\mu_2})|_k((Y(g) \otimes Y(g))[k])\). It can also be extended to a map

\[
\pi_{\lambda_1,\mu_1} \otimes \pi_{\lambda_2,\mu_2} : Y(g) \otimes Y(g) \rightarrow \prod_{k=0}^{\infty} (\lambda_1 Y(g)_{\mu_1} \otimes \lambda_2 Y(g)_{\mu_2})[k]
\]

by setting

\[
\pi_{\lambda_1,\mu_1} \otimes \pi_{\lambda_2,\mu_2} = \prod_{k=0}^{\infty} (\pi_{\lambda_1,\mu_1} \otimes \pi_{\lambda_2,\mu_2})|_k.
\]

Following [Lus10], we define the linear map

\[
\Delta_{\lambda_1,\mu_1,\lambda_2,\mu_2} : \lambda_1 + \lambda_2 Y(g)_{\mu_1+\mu_2} \rightarrow \prod_{k=0}^{\infty} (\lambda_1 Y(g)_{\mu_1} \otimes \lambda_2 Y(g)_{\mu_2})[k]
\]

by

\[
\Delta_{\lambda_1,\mu_1,\lambda_2,\mu_2}(\pi_{\lambda_1+\lambda_2,\mu_1+\mu_2}(x)) = (\pi_{\lambda_1,\mu_1} \otimes \pi_{\lambda_2,\mu_2})(\Delta(x)).
\]

It turns out that the image of \(\Delta_{\lambda_1,\mu_1,\lambda_2,\mu_2}\) is actually contained in \(\bigoplus_{k=0}^{\infty}(\lambda_1 Y(g)_{\mu_1} \otimes \lambda_2 Y(g)_{\mu_2})[k] \cong \lambda_1 Y(g)_{\mu_1} \otimes \lambda_2 Y(g)_{\mu_2}\): to see this, observe that, for any fixed \(\lambda_2, \mu_2\), there are only finitely many terms of \(\Delta(h_{\nu})\) which are contained in \(Y(g)\{\lambda_1 - \mu_1\} \otimes Y(g)\{\lambda_2 - \mu_2\}\) and the same is true consequently for \(\Delta(x)\) for any \(x \in Y(g)\).

6. Two Parameter Yangian in Type \(A_{n-1}^{(1)}\)

In this section, we assume that \(g\) is of type \(A_{n-1}^{(1)}\) and \(n \geq 3\). (Definition 6.1 below is not the correct one when \(n = 2\): in this case, see the definition in Section 1.2 in [TB15] and Definition 5.1 in [Kod15].) We identify the index set \(I\) with \(\mathbb{Z}/n\mathbb{Z}\) and normalize \((\cdot,\cdot)\) so that \((\alpha_i,\alpha_i) = 2\) for all \(i \in I\). In this case, the definition of the Yangian \(Y(h)(g)\) can be generalized by introducing a second parameter \(\varepsilon\) (see [Gua07]: for quantum toroidal algebras, see [VV98]).

**Definition 6.1.** Let \(h, \varepsilon \in \mathbb{C}\). The Yangian \(Y_{h,\varepsilon}(g)\) is the associative algebra over \(\mathbb{C}\) with generators \(x_{ir}^{\pm}\), \(h_{ir}\) \((i \in I, r \in \mathbb{Z}_{\geq 0})\) subject to the defining relations of \(Y_h(g)\) given in Definition 2.1 with the modification that, when \(j = i + 1\) or \(j = i - 1\), (2.3) and (2.6) are replaced with the relations:

\[
(h_{i,r+1}, x_{i+1,s}^{\pm}) - (h_{i,r}, x_{i+1,s+1}^{\pm}) = \frac{\varepsilon}{2} [h_{i,r}, x_{i+1,s}^{\pm}] + \frac{h}{2} (x_{i+1,s}^{\pm} h_{i,r} + h_{i,r} x_{i+1,s}^{\pm}),
\]

\[
(h_{i,r+1}, x_{i-1,s}^{\pm}) - (h_{i,r}, x_{i-1,s+1}^{\pm}) = \frac{\varepsilon}{2} [h_{i,r}, x_{i-1,s}^{\pm}] - \frac{h}{2} (x_{i-1,s}^{\pm} h_{i,r} + h_{i,r} x_{i-1,s}^{\pm}),
\]

\[
(x_{i,r+1}^{\pm}, x_{i+1,s}^{\pm}) - (x_{i,r+1}, x_{i+1,s+1}^{\pm}) = \frac{\varepsilon}{2} [x_{i,r+1}^{\pm}, x_{i+1,s}^{\pm}] + \frac{h}{2} (x_{i+1,s}^{\pm} x_{i,r+1}^{\pm} + x_{i,r+1}^{\pm} x_{i+1,s}^{\pm}) + \frac{h}{2} (x_{i,r+1}^{\pm}, x_{i+1,s}^{\pm}).
\]

Then Yangian \(Y_{h,\varepsilon}(g)\) is then defined in the same manner as \(Y_h(g)\): it is the quotient of \(Y_{h,\varepsilon}(g) \otimes_{\mathbb{C}} U(\mathfrak{h})\) by the ideal generated by the relations (2.8).

The defining relations for \(Y_{h,\varepsilon}(g)\) given above are slightly different from those which appear in [Gua07, Def. 2.3] (where \(Y_{h,\varepsilon}(g)\) is denoted by \(\hat{Y}_{\beta,\lambda}\)). One advantage of the relations in Definition 6.1 is that they are invariant under the rotational symmetry \(0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow\)
Both definitions are equivalent as follows. We set
\[
\begin{align*}
 h'_{ir} &\equiv \sum_{s=0}^{r} \binom{r}{s} e^r-s \left( \frac{i}{2} - \frac{n}{4} \right) h_{is}, \\
x'_{ir} &\equiv \sum_{s=0}^{r} \binom{r}{s} e^r-s \left( \frac{i}{2} - \frac{n}{4} \right) x_{is}^{\pm}
\end{align*}
\]
for \(i = 1, 2, \ldots, n-1\) and \(x'_{0r} = x_{0r}, h'_{0r} = h_{0r}\). Then \(x'_{ir}\) and \(h'_{ir}\) satisfy the relations in [Gua07, Def. 2.3] with \(\lambda = \hbar, \beta = -\frac{e\hbar}{4} + \frac{\hbar}{2}\).

When \(\varepsilon \neq 0\), \(Y_{h=0,\varepsilon}(g')\) is isomorphic to the enveloping algebra of the universal central extension of the Lie algebra of \(n \times n\) matrices with entries in the ring of differential operators on \(\mathbb{C}^n\): see Section 5 in [Gua07]. Otherwise, if \(h_1 \neq 0\) and \(h_2 \neq 0\), then \(Y_{h_1,\varepsilon_1}(g) \cong Y_{h=1,\varepsilon/h_1}(g) \cong Y_{h_2,\varepsilon_2/h_1}(g)\), so it is enough to focus on \(Y_{h=1,\varepsilon}(g)\) for any \(\varepsilon \in \mathbb{C}\).

Our goal for the rest of this paper is to explain how the main results established in the previous sections also hold for \(Y_{h=1,\varepsilon}(g)\) after making only a few minor adjustments.

We begin by noting that it has already been proven in [Gua07] that Theorem 2.12 holds for \(Y_{h=1,\varepsilon}(g)\) with (2.16) and (2.17) replaced by (6.2), (6.3) and (6.4) with \(r = s = 0\) when \(j = i + 1\) or \(j = i - 1\); see Proposition 2.1 in loc. cit.

It is also the case that Theorem 4.11 holds for \(Y_{h=1,\varepsilon}(g)\) with \(\Delta\) given by the same formula (4.8). The proof of Theorem 4.11 in this case follows the same steps as before, except that some new terms appear due to the presence of the second parameter \(\varepsilon\). The remainder of this section will be devoted to explaining the key differences and necessary modifications. We start by introducing operators \(J(h_i)\) and \(J(x_i^\pm)\) on modules in the category \(O\) exactly as in (3.6).

As a consequence of Lemma 6.8, these operators still satisfy the equivalences (3.12), however, the second and fourth equivalences should be altered when \(j = i + 1\) or \(j = i - 1\) to account for the modified relations of Definition 6.11. It is straightforward to verify that (6.2), (6.3) and (6.4) with \((r, s) = (0, 0)\) are equivalent to the relations
\[
\begin{align*}
 [J(h_i), x_{i+1}^\pm] &= \pm (\alpha_i, \alpha_{i+1}) (J(x_{i+1}^\pm) - \frac{\varepsilon}{2} x_{i+1}^\pm), \\
 [J(h_i), x_{i-1}^\pm] &= \pm (\alpha_i, \alpha_{i-1}) (J(x_{i-1}^\pm) + \frac{\varepsilon}{2} x_{i-1}^\pm), \\
 [J(x_i^\pm), x_{i+1}^\pm] &= [x_{i+1}^\pm, J(x_{i+1}^\pm) + \frac{\varepsilon}{2} x_{i+1}^\pm],
\end{align*}
\]
respectively. To account for these changes, Lemma 3.13 has to be slightly modified as follows.

**Lemma 6.8.** We have
\[
\tau_i(J(h_j)) = J(s_i(h_j)) + \frac{\varepsilon}{2} (\delta_{i+1,j} - \delta_{i-1,j}) h_i = J(h_j) - \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} J(h_i) + \frac{\varepsilon}{2} (\delta_{i+1,j} - \delta_{i-1,j}) h_i
\]
for all \(i, j \in I\).

The proof of this lemma is the same as for Lemma 3.13. The operators \(J(x_{i}^\pm)\) are also defined as before (see (5.14)), but Proposition 3.15 has to be modified to account for the second parameter \(\varepsilon\).

**Proposition 6.9.** For every positive real root \(\alpha\) and every \(i \in I\), there exists an integer \(c_{\alpha, i}\) such that
\[
[\alpha_i, J(h_j)] = \pm (\alpha, \alpha_j) J(x_{\alpha, j}^\pm) \pm \frac{\varepsilon}{2} c_{\alpha, i} x_{\alpha, i}^\pm = [h_i, J(x_{\alpha, j}^\pm)] \pm \frac{\varepsilon}{2} c_{\alpha, i} x_{\alpha, i}^\pm.
\]
Proof. We employ the same strategy as was used in the proof of Proposition 3.15 and argue by induction on \( p \), where \( x_{\pm}^i = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_p-1} (x_{\pm}^{i_p}) \) with \( x_{\pm}^{i_j} = x_{j}^{\pm} \) for some \( j \in I \). If \( p = 1 \), then \( x_{\pm}^i = x_{j}^{\pm} \) and the equivalences \((6.3),(6.6)\) and \((3.12)\) imply that we have

\[
(6.11) \quad [J(h_i), x_{\pm}^i] = \pm (\alpha, \alpha) J(x_{\pm}^i) \pm (\delta_{i+1,j} - \delta_{i-1,j}) \frac{\varepsilon}{2} x_{j}^{\pm} = [h_i, J(x_{\pm}^i)] \pm (\delta_{i+1,j} - \delta_{i-1,j}) \frac{\varepsilon}{2} x_{j}^{\pm},
\]

and hence we may take \( c_{\alpha,j,i} = \delta_{i+1,j} - \delta_{i-1,j} \). Suppose that the proposition holds for \( x_{\pm}^j \) with \( \beta = s_{i_2} \cdots s_{i_p-1} (\alpha_{i_p}) \). Then, since \([J(h_i), x_{\pm}^i] = \tau_{i_1} (\tau_{i_1}^\dagger J(h_i), x_{\pm}^i)\), Lemma 6.8 gives

\[
[J(h_i), x_{\pm}^i] = \tau_{i_1} \left( [J(s_{i_1}(h_i)) + \frac{\varepsilon}{2} (\delta_{i_1+1,i} - \delta_{i_1-1,i}) x_{i_1}^i, x_{\pm}^i] \right)
\]

\[
= \tau_{i_1} (\tau_{i_1} - (\alpha, \alpha) J(h_i), x_{\pm}^i) \pm (\alpha, \beta) (\delta_{i_1+1,i} - \delta_{i_1-1,i}) \frac{\varepsilon}{2} \tau_{i_1} (x_{\pm}^i)
\]

\[
= \pm (s_{i_1}(\alpha, \beta) \tau_{i_1} J(x_{\pm}^i)) \pm \frac{\varepsilon}{2} (\delta_{i_1+1,i} - \delta_{i_1-1,i}) \tau_{i_1} (x_{\pm}^i)
\]

\[
= \pm (\alpha, \alpha) J(x_{\pm}^i) \pm \frac{\varepsilon}{2} (\delta_{i_1+1,i} - \delta_{i_1-1,i}) \tau_{i_1} (x_{\pm}^i).
\]

where the third equality uses the induction assumption and the fourth equality uses that \((s_{i_1}(\alpha, \beta) = (\alpha, \alpha), (\alpha, \beta) = (\alpha, \alpha))\). Setting \( c_{\alpha,i} = c_{\beta,i} - (\alpha, \alpha) c_{\beta,i} + (\alpha, \beta) c_{\alpha,i} \) we obtain that \([J(h_i), x_{\pm}^i] = \pm (\alpha, \alpha) J(x_{\pm}^i) + \frac{\varepsilon}{2} c_{\alpha,i} x_{\pm}^i \). Moreover, by the induction assumption we have the equivalences \([J(h_i), x_{\pm}^i] = [h_i, J(x_{\pm}^i)] = \frac{\varepsilon}{2} c_{\beta,i} x_{\pm}^i \) and \([J(h_i), x_{\pm}^i] = [h_i, J(x_{\pm}^i)] = \frac{\varepsilon}{2} c_{\alpha,i} x_{\pm}^i \). These together with the first equality in the expansion of \([J(h_i), x_{\pm}^i] \) above also imply that

\[
[J(h_i), x_{\pm}^i] = \tau_{i_1} (\tau_{i_1}^\dagger J(h_i), x_{\pm}^i)) \pm \frac{\varepsilon}{2} (\delta_{i_1+1,i} - \delta_{i_1-1,i}) \tau_{i_1} (x_{\pm}^i)
\]

\[
= [h_{i_0}, J(x_{\pm}^i)] \pm \frac{\varepsilon}{2} c_{\alpha,i} x_{\pm}^i.
\]

\[\square\]

Proposition 3.15 was used to prove Proposition 3.16 from which Corollary 3.17 can be deduced. These two results also hold for \( Y_{h=1,\varepsilon}(\mathfrak{g}) \) without modification, using this time Proposition 6.9. Part I of the proof of the theorem of 4.11 is the same as before except for new terms involving \( \varepsilon \) which appear when computing \([\Delta(h_{i_1}), \Delta(x_{\pm}^j)]\) and \([\Delta(x_{\pm}^i), \Delta(x_{\pm}^j)]\) when \( j = \pm 1 \). A careful inspection of the second part of the proof of Theorem 4.11 shows that the same arguments will apply provided that brackets of the form \([\square(J(h_i)), [h_{j0} \otimes 1, \Omega_\pm]]\) have the same expansions as they did for \( Y(\mathfrak{g}) \) \( (= Y_{h=1,\varepsilon}(\mathfrak{g})) \). Let us verify this.

\[
[\square(J(h_i)), [h_{j0} \otimes 1, \Omega_\pm]] = [\square(J(h_i)), \sum_{\alpha \in \Delta^\vee} \pm (\alpha, \alpha) x_{\alpha}^{\mp} \otimes x_{\alpha}^{\pm}]
\]

\[
= \sum_{\alpha \in \Delta^\vee} \pm (\alpha, \alpha) \left( \pm J(x_{\alpha}^{\mp}) \pm \frac{\varepsilon}{2} c_{\alpha,i} x_{\alpha}^{\pm} \otimes x_{\alpha}^{\pm} \right)
\]

\[
= \sum_{\alpha \in \Delta^\vee} (\alpha, \alpha) \left( J(x_{\alpha}^{\mp}) \otimes x_{\alpha}^{\pm} - J(x_{\alpha}^{\mp}) \otimes x_{\alpha}^{\pm} \right).
\]

Observe that, indeed, the last line does not depend on \( \varepsilon \). Therefore we may conclude that Theorem 4.11 holds for \( Y_{h=1,\varepsilon}(\mathfrak{g}) \).

Finally, the results on completions in Section 5 also hold for \( Y_{h=1,\varepsilon}(\mathfrak{g}) \), so in particular we may view \( \Delta \) as an algebra homomorphism \( Y_{h=1,\varepsilon}(\mathfrak{g}) \rightarrow Y_{h=1,\varepsilon}(\mathfrak{g}) \otimes Y_{h=1,\varepsilon}(\mathfrak{g}) \).
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