CHARACTERIZATION OF THE 4-CANONICAL BIRATIONALITY OF ALGEBRAIC 3-FOLDS, II

MENG CHEN AND QI ZHANG

Abstract. For nonsingular projective 3-folds of general type with $p_g \geq 5$, the birationality of $\varphi_4$ was characterized by D.-Q. Zhang and the first author in 2008. This paper aims at characterizing the 4-canonical birationality for those with $p_g = 4$.

1. Introduction

We work over any algebraically closed field $k$ of characteristic 0. It is well-known that studying pluricanonical maps has been an important way of understanding birational geometry of projective varieties. Denote by $\varphi_{m,X}$ (or, in short, $\varphi_m$) the pluricanonical map of a given variety $X$ of dimension $n$ in question. A remarkable theorem of Hacon and M"{e}Kernan [12], Takayama [16] and Tsuji [17] shows that there exists a constant $c(n)$ ($n \geq 3$) so that $\varphi_{m,X}$ is birational for all $m \geq c(n)$ and for any general type $n$-fold $X$. However, $c(n)$ is non-explicit unless $n = 3$ (see Chen-Chen [3, 4]).

In this paper we are interested in the explicit aspect of birationally classifying minimal projective 3-folds of general type. In fact, the status concerning the behavior of $\varphi_m$ on minimal 3-folds $X$ can be briefly outlined by the following table, where $p_g$ denotes the geometric genus of $X$:

| $p_g$ | $\varphi_5$ is birational (see [6] Theorem 1.2(2)); $\varphi_4$ is not birational $\iff$ Theorem 0. |
|-------|----------------------------------------------------------------------------------|
| $p_g = 4$ | $\varphi_5$ is birational (see [6] Theorem 1.2(2)); $\exists$ examples s.t. $\varphi_4$ is not birational (see [6] Example 1.4) and Example [13]. |
| $p_g = 3$ | $\varphi_6$ is birational (see [6] Theorem 1.2(1)); $\exists$ examples s.t. $\varphi_5$ is not birational (see [10] or [11] p151, No.7). |
| $p_g = 2$ | $\varphi_8$ is birational (see [6] Section 4)); $\exists$ examples s.t. $\varphi_7$ is not birational (see [11] p151, No.12)]. |
| $p_g = 1$ | $\exists$ examples s.t. $\varphi_{13}$ is not birational (see [11] p151, No.19)]. |
| $p_g = 0$ | $\exists$ examples s.t. $\varphi_{26}$ is not birational (see [11] p151, No.23). |
| Any $X$ | $\varphi_m$ is birational for $m \geq 61$ (see [5]). |

Supported by National Natural Science Foundation of China (grants: #11171068, #11121101, #11231003).
In 2008, D.-Q. Zhang and the first author proved the following:

**Theorem 0.** ([8, Theorem 1.3]) Let $X$ be a minimal projective 3-fold of general type (admitting at worst canonical singularities) with the geometric genus $p_g(X) \geq 5$. Then:

1. $\varphi_4$ is not birational if and only if $X$ is birationally fibered by a family $\mathcal{C}$ of irreducible curves of geometric genus 2 with $(K_X \cdot C_0) = 1$ for a general member $C_0 \in \mathcal{C}$.
2. In (1) the family $\mathcal{C}$ is birationally, uniquely determined by the given 3-fold $X$.

The natural question aroused from Theorem 0 and the above table is whether it is possible to characterize the birationality of $\varphi_m$ when $m$ is small. Such kind of questions are worthwhile to study since they touch the explicit structures of many families of 3-folds in question.

In order to concisely formulate our main results. We need to set the following convention:

- Let $\iota : F \to \mathbb{P}^1$ be a fibration of genus 2 and $F$ a smooth projective surface of general type. Let $C$ be a general fiber of $\iota$. If $h^0(F, K_F - C) = 1$ and the horizontal part of $|K_F - C|$ is irreducible and reduced, we say that $F$ is $C$-horizontally (or $\iota$-horizontally) integral. Sometimes we abuse this definition on any birational model of $F$.
- Denote by $\mathbb{F}_2$ the Hirzebruch ruled surface and by $\overline{\mathbb{F}}_2$ the cone obtained by contracting the unique $(-2)$-curve section on $\mathbb{F}_2$. Denote by $l$ a general line in $\overline{\mathbb{F}}_2$ passing through the vertex.

The main purpose of this paper is to solve the extremal case of Theorem 0 and to prove the following:

**Theorem 1.1.** Let $X$ be a minimal projective 3-fold of general type with $p_g(X) = 4$. Then $\varphi_4$ is not birational if and only if $X$ has one of the following structures:

1. $K_X^3 = 2$ and the canonical map $\varphi_1$ is a generically double cover onto $\mathbb{P}^3$.
2. $X$ has a genus 2 curve family $\mathcal{C}$ of canonical degree 1, i.e. $(K_X \cdot C_0) = 1$ for a general element $C_0 \in \mathcal{C}$.
3. $X$ is canonically fibered by genus 2 curve family $\mathcal{C}$ of canonical degree $6/5$ over some cubic surface in $\mathbb{P}^3$.
4. $X$ is canonically fibered by genus 2 curve family $\mathcal{C}$ of canonical degree $4/3$ over the quadric cone $\overline{\mathbb{F}}_2 \subset \mathbb{P}^3$. Furthermore, $\hat{F}$ is $C$-horizontally integral, where $\hat{F}$ is a smooth model of the general irreducible component of $\varphi_1^{-1}(l)$ and $C$ is the general member in the restricted curve family $\mathcal{C}|_{\hat{F}}$.

The curve families $\mathcal{C}$ in Items (2),(3) and (4) are (birationally) uniquely determined by $X$.
The direct consequence is the following:

**Corollary 1.2.** Let $X$ be a minimal projective 3-fold of general type with $p_g(X) = 4$. Then $\varphi_4$ is either birational or generically finite of degree 2.

**Example 1.3.** (1) The general 3-fold hypersurface $X_{10} \subset \mathbb{P}(1,1,1,1,5)$ is a smooth canonical 3-fold with $p_g = 4$ and $K_X^3 = 2$. Clearly $\varphi_1$ is a finite morphism of degree 2 onto $\mathbb{P}^3$ and $\varphi_4$ of $X$ is a double cover.

(2) For any projective $\mathbb{Q}$-factorial terminal (QFT) 3-fold $X$, which is birationally fibered by $(1,2)$ surfaces, $X$ has a natural curve family of canonical degree 1. Clearly $\varphi_{4,X}$ is not birational by Bombieri’s theorem on $(1,2)$ surfaces.

**Remark 1.4.** (1) If a smooth projective surface $F$ is fibered by genus two curves $C$ and $F$ is $C$-horizontally integral, it is easy to see that either $p_g(F) = 2$ or $p_g(F) = 3$ and $|K_F|$ is not composed of any pencil of curves.

(2) It is unclear to the authors whether a minimal surface $S$ with $K_S^2 = 2$ and $p_g(S) = 3$ may admit a free pencil of curves of genus 2.

(3) Theorem 1.1(4) suggests that some 3-folds fibered by $(2,3)$-surfaces may have non-birational 4-canonical maps. Of course, it is clear that these 3-folds have non-birational 3-canonical maps by the Bombieri theorem on $(2,3)$-surfaces.

Throughout we are in favor of the following symbols:

- “$\sim$” denotes linear equivalence or $\mathbb{Q}$-linear equivalence;
- “$\equiv$” denotes numerical equivalence;
- “$|M_1| \geq |M_2|$” (or, equivalently, “$|M_2| \ll |M_1|$”) means, for linear systems $|M_1|$ and $|M_2|$ on a variety,
  $$|M_1| \supseteq |M_2| + \text{(fixed effective divisor)}.$$  

2. **Preliminaries**

Throughout $X$ will be a minimal projective QFT 3-fold of general type, on which $\omega_X = \mathcal{O}_X(K_X)$ is the canonical sheaf and $K_X$ a canonical divisor.

2.1. **Set up.** We assume $p_g(X) := h^0(X, \omega_X) \geq 2$. So we may study the birational structure of $X$ by considering the canonical map $\varphi_1: X \dashrightarrow \mathbb{P}^{p_g-1}$, which is a non-constant rational map.

From the very beginning we fix an effective Weil divisor $K_1 \sim K_X$. Take successive blow-ups $\pi : X' \rightarrow X$, which exists by Hironaka’s big theorem, such that:

(i) $X'$ is nonsingular and projective;
(ii) the moving part of $|K_{X'}|$ is base point free;
(iii) the union of supports of both $\pi^*(K_1)$ and exceptional divisors of $\pi$ is simple normal crossing.
Denote by $\tilde{g}$ the composition $\varphi_1 \circ \pi$. So $\tilde{g} : X' \to \Sigma \subseteq \mathbb{P}^{p_g(X) - 1}$ is a morphism by the above assumption. Let $X' \xrightarrow{f} \Gamma \xrightarrow{s} \Sigma$ be the Stein factorization of $\tilde{g}$. We get the following commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & \Gamma \\
\downarrow{\pi} & & \downarrow{s} \\
X & \xrightarrow{\varphi_1} & \Sigma
\end{array}
\]

We may write $K_{X'} = \pi^*(K_X) + E_\pi \sim M_1 + Z_1$, where $|M_1|$ is the moving part of $|K_{X'}|$, $Z_1$ the fixed part and $E_\pi$ an effective $\mathbb{Q}$-divisor which is a sum of distinct exceptional divisors with positive rational coefficients. Since $h^0(X', \mathcal{O}_{X'}(M_1)) = h^0(\omega_X)$, we may also write $\pi^*(K_X) \sim Q M_1 + E'_1$ where $E'_1 = Z_1 - E_\pi$ is an effective $\mathbb{Q}$-divisor. Set $d_1 := \dim \varphi_1(X) = \dim(\Gamma)$. Clearly one has $1 \leq d_1 \leq 3$.

If $d_1 = 2$, a general fiber of $f$ is a smooth projective curve of genus $\geq 2$. We say that $X$ is canonically fibred by curves.

If $d_1 = 1$, a general fiber $F$ of $f$ is a smooth projective surface of general type. We say that $X$ is canonically fibred by surfaces with invariants $(c_2(F_0), p_g(F_0))$, where $F_0$ is the minimal model of $F$ via the contraction morphism $\sigma : F \to F_0$. We may write $M \equiv p_1 F$ where $p_1 := \deg f_* \mathcal{O}_{X'}(M_1) \geq p_g(X) - 1$. Denote $b := g(\Gamma)$.

Just to fix the convention, a generic irreducible element $S$ of $|M_1|$ means either a general member of $|M_1|$ in the case of $d_1 \geq 2$ or, otherwise, a general fiber $F$ of $f$.

For any integer $m > 0$, $|M_m|$ denotes the moving part of $|mK_{X'}|$. Let $S_m$ be a general member of $|M_m|$ whenever $m > 1$. Set

$$p = \begin{cases} 
1, & \text{if } d_1 \geq 2; \\
 p_1, & \text{if } d_1 = 1.
\end{cases}$$

We always have

$$\pi^*(K_X) \equiv pS + E'_1$$

for the effective $\mathbb{Q}$-divisor $E'_1$ on $X'$.

2.2. Technical inequalities. We refer to Chen-Zhang [8, Section 3] for birationality principles (see [8, Lemma 3.1, Lemma 3.2]). For the convenience of readers, we briefly recall the technical, however useful, theorem as follows.

Pick a generic irreducible element $S$ of $|M|$. Assume we have a base point free linear system $|G|$ on $S$. Denote by $C$ a generic irreducible element of $|G|$. Since $\pi^*(K_X)|_S$ is nef and big, Kodaira’s lemma implies that there is a positive rational number $\beta$ so that $\pi^*(K_X)|_S - \beta C \geq 0$. 
Set $\xi := (\pi^*(K_X) \cdot C)$ and, given any positive integer $m,$

$$\alpha_m := (m - 1 - \frac{1}{p} - \frac{1}{\beta})\xi.$$ 

We will frequently use the following theorem:

**Theorem 2.1.** (Chen-Zhang [8, Theorem 3.6]) Keep the above setting and notation. Let $m > 0$ be an integer. Then

1. $m\xi \geq \deg(K_C) + \lceil \alpha_m \rceil$ provided that $\alpha_m > 1$;
2. $\varphi_m$ is birational provided that $|mK_X'| |_S$ distinguishes different generic irreducible elements of $|G|$ and that $\alpha_m > 2$.

Note, however, that Theorem 2.1 (1) implies

$$\xi \geq \frac{p\beta}{p\beta + p + \beta} \cdot \deg(K_C)$$

by taking a sufficiently large $m$ so that $\alpha_m > 1$.

**Definition 2.2.** Let $|N|$ be a moving linear system on a normal projective variety $Z.$ We say that the rational map $\Phi|_N$ distinguishes sub-varieties $W_1, W_2 \subset Z$ if, set theoretically, $\Phi|_N(W_1) \not\subseteq \Phi|_N(W_2)$ and $\Phi|_N(W_2) \not\subseteq \Phi|_N(W_1).$ We say that $\Phi|_N$ separates points $P, Q \in Z$ (for $P, Q \not\in \text{Bs}|N|$), if $\Phi|_N(P) \neq \Phi|_N(Q)$.

2.3. Other required results. We recall the following:

**Theorem 2.3.** ([7, Theorem 1.5 (2)]) Let $X$ be a minimal projective 3-fold of general type with $p_g(X) \geq 4.$ Then $K_X^3 \geq 2,$ which is optimal.

Kawamata’s extension theorem will be used in our proof.

**Theorem 2.4.** (Kawamata [14, Theorem A]) Let $V$ be a smooth algebraic variety on which $D$ is a smooth divisor. Let $\tau : V \to W$ be a projective morphism onto a germ of an algebraic variety $W.$ Assume $K_V + D$ is $\tau$-big for the pair $(V, D).$ Then the natural homomorphism $	au_*\mathcal{O}_V(mK_V + mD) \to \tau_*\mathcal{O}_D(mK_D)$ is surjective for any integer $m > 1.$

**Corollary 2.5.** Under the setting of 2.3, if $d_1 = 1$ and $g(\Gamma) = 0,$ then

$$\pi^*(K_X)|_F \sim_Q \frac{p}{p + 1} \sigma^*(K_{F_0}) + Q'$$

where $\sigma : F \to F_0$ is the birational contraction onto the minimal model $F_0$ and $Q'$ is an effective $Q$-divisor on $F.$

**Proof.** In order to apply Theorem 2.4 we set $V := X', D := F$ and $\tau$ to be the trivial map onto a point. Then, for any sufficiently large and divisible integer $m > 0,$ one has $|(p + 1)mK_{X'}| \geq |mp(K_{X'} + F)|$ and the surjective map:

$$H^0(X', mp(K_{X'} + F)) \to H^0(F, mpK_F).$$ (2.2)
Since $\text{Mov}|mpK_F| = |mp\sigma^*(K_{F_0})|$ and $m(p+1)\pi^*(K_X) \geq M_{m(p+1)}$, we clearly have the following:

$$m(p + 1)\pi^*(K_X)|_F$$

$$\geq M_{m(p+1)}|_F$$

$$\geq \text{Mov}(mp(K_{X'} + F))|_F \geq mp\sigma^*(K_{F_0}).$$

Thus the statement follows. \qed

3. Proof of the main theorem

Let $X$ be a minimal projective 3-fold of general type with $p_g(X) = 4$. Keep the same setting as in 2.1.

3.1. Part one. $d_1 = 1$.

We have an induced fibration $f : X' \to \Gamma$ with the general fiber $F$. Since $p_g(X) > 0$ and, for the general fiber $F$, the map $H^0(K_{X'} \cdot F) \to H^0(F)$ can't be surjective and so that $p_g(F) > 0$. We say that $F$ is an "$(a, b)$" surface if $K_{F_0}^2 = a$ and $p_g(F_0) = b$ where $F_0$ is the minimal model of $F$.

By Chen-Zhang [8, 4.8], we have the following:

**Proposition 3.1.** Assume $b = g(\Gamma) > 0$. Then $\varphi_4$ is birational if and only if $F$ is not a (1,2) surface.

Now we assume $b = g(\Gamma) = 0$. By definition we have $p = 3$ and $K_{X'} \geq 3F$. By Relation (2.2), one has the surjective map

$$H^0(X', 3(K_{X'} + F)) \to H^0(F, 3K_F),$$

which means that $\varphi_4$ is birational as long as $F$ is neither a (1,2) surface nor a (2,3) surface. Besides, it is clear that $\varphi_4$ is not birational when $F$ is a (1,2) surface.

**Claim 3.2.** If $F$ is a (2,3) surface, then $\varphi_4$ is birational.

**Proof.** By Bombieri’s theorem (cf. [2,1]), $|3\sigma^*(K_{F_0})|$ is base point free. Thus Relation (3.1) implies

$$4\pi^*(K_X)|_F \geq M_4|_F \geq \text{Mov}(3(K_{X'} + F))|_F \geq 3\sigma^*(K_{F_0}),$$

which gives $\pi^*(K_X)|_F \geq \frac{3}{4}\sigma^*(K_{F_0})$.

Take $|G| := |\sigma^*(K_{F_0})|$, which is base point free (see [1, p227]). A generic irreducible element $C$ is a smooth curve of genus 3. Relation (3.1) also implies $|4K_{X'}|_F \geq |3\sigma^*(K_{F_0})|$, which distinguishes different generic $C$.

We have $p = 3$, $\beta \geq \frac{3}{4}$ and $\xi = (\pi^*(K_X) \cdot C) \geq \frac{3}{4}C^2 = \frac{3}{4}$. Since $\alpha_5 \geq \frac{\xi}{2} > 3$, Theorem [2.1] (1) implies $\xi \geq \frac{\beta}{2}$. Now since $\alpha_4 \geq \frac{\xi}{3} > 2$, Theorem [2.1] (2) implies that $\varphi_4$ is birational. \qed

Thus we have the following:
Corollary 3.3. Let $X$ be a minimal projective 3-fold of general type with $p_g(X) = 4$. Keep the same notation as above. Assume $d_1 = 1$. Then $\varphi_4$ is not birational if and only if $F$ is a $(1,2)$ surface. When $\varphi_4$ is not birational, $X$ has a natural genus 2 curve family $\mathcal{C}$ of canonical degree 1.

Proof. The first part is due to Proposition 3.1, Relation (3.1) and Claim 3.2.

When $\varphi_4$ is not birational, we have an induced fibration $f : X' \to \Gamma$ with the general fiber a $(1,2)$ surface. We may consider the relative canonical map $\Psi : X'/\varphi_1 \to \mathbb{P}(f_*\omega_{X'/\Gamma})$ over $\Gamma$. By taking further birational modifications we may assume that $\Psi$ is a morphism over $\Gamma$. So we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\Psi} & \mathbb{P}(f_*\omega_{X'/\Gamma}) \\
\downarrow{\pi} & & \downarrow{f} \\
X & \xrightarrow{\varphi_1} & \Gamma
\end{array}
\]

Clearly a general fiber of $\Psi$ is a smooth curve of genus 2. Set $\mathcal{C}$ to be the set of fibers of $\Psi$. As been proved in Chen-Zhang [8, 4.10], we know $(\pi^*(K_X) \cdot C) = 1$ for a general element $\tilde{C} \in \mathcal{C}$. The $\pi$-image of $\mathcal{C}$ is what we have claimed on $X$. \qed

3.2. Part two. $d_1 = 2$.

We have an induced fibration $f : X' \to \Gamma$ onto a normal surface $\Gamma$. Pick a general member $S \in |M_1|$. We have $p = 1$ by definition. Set $|G| := |M_1|$. Let $C$ be a generic irreducible element of $|G|$. Clearly $C$ is a smooth curve of genus $g(C) \geq 2$. We may write

\[
\pi^*(K_X)|_S \equiv \nu_2 C + E'_1|_S
\]  

(3.2)

where

\[
\nu_2 = \text{deg}(s) \deg \tilde{g}(X') \geq p_g(X) - 2 = 2.
\]  

(3.3)

Claim 3.4. For the general member $S \in |M_1|$, $|4K_{X'}||_S$ distinguishes different generic irreducible elements of $|G|$.  

Proof. We have $|4K_{X'}| \not\equiv |K_{X'} + [2\pi^*(K_X)] + M_1|$. On the other hand, the Kawamata-Viehweg vanishing theorem \[13, 18\] implies:

\[
|K_{X'} + [2\pi^*(K_X)] + M_1||_S = |K_S + [2\pi^*(K_X)]|_S \not\equiv |K_S + [2L]| \quad \text{(3.4)}
\]

where $L := \pi^*(K_X)|_S$ is an effective nef and big $\mathbb{Q}$-divisor on $S$. 

For arbitrary two different generic irreducible elements \( C_1 \) and \( C_2 \), since \( 2L - C_1 - C_2 - \frac{2}{\nu_2}E'_1|_S \equiv (2 - \frac{2}{\nu_2})L \) is nef and big, the Kawamata-Viehweg vanishing theorem gives the surjective map:

\[
H^0(S, K_S + [2L - \frac{2}{\nu_2}E'_1|_S]) 
\rightarrow H^0(C_1, K_{C_1} + D_1) \oplus H^0(C_2, K_{C_2} + D_2)
\]

where \( D_i := ([2L - \frac{2}{\nu_2}E'_1|_S] - C_i)|_{C_i} \) with

\[
\deg(D_i) \geq (2 - \frac{2}{\nu_2})\xi > 0
\]

for \( i = 1, 2 \). Since \( h^0(C_i, K_{C_i} + D_i) > 0, |K_S + 2L| \) clearly distinguishes \( C_1 \) and \( C_2 \).

**Claim 3.5.** If \( g(C) \geq 3 \), \( \varphi_4 \) is birational.

**Proof.** Since \( \beta = \nu_2 \geq 2 \). It follows from Inequality (2.1) that \( \xi \geq \frac{\deg(K_{C_1})}{2 + \frac{1}{\beta}} \geq \frac{8}{5} \). Then \( \alpha_4 \geq (4 - 2 - \frac{1}{2})\xi \geq \frac{12}{5} > 2 \). By Theorem 2.1(2), \( \varphi_4 \) is birational. \( \square \)

**Claim 3.6.** If \( g(C) = 2 \), \( \nu_2 \geq 3 \) and \( \varphi_4 \) is not birational, then either \( \xi = 1 \) or \( \xi = \frac{6}{5} \) and \( \deg g(X') = 3 \).

**Proof.** If \( g(C) = 2 \), one gets from Inequality (2.1) that \( \xi \geq \frac{2}{1+1+\frac{1}{\beta}} = \frac{4}{5} \). Then, since \( \alpha_4 \geq (2 - \frac{1}{2})\xi \geq \frac{6}{5} > 1 \), Theorem 2.1(1) implies \( \xi \geq 1 \).

Assume \( \xi > 1 \). Find an integer \( l_0 > 5 \) such that \( \xi \geq \frac{l_0+1}{l_0} \). Set \( m' = l_0 - 1 \) and then we have

\[
\alpha_{m'} = (l_0 - 1 - 2 - \frac{1}{\nu_2})\xi \geq (l_0 - \frac{10}{3})\frac{l_0+1}{l_0} > l_0 - 3 > 1.
\]

By Theorem 2.1(1), one gets \( \xi \geq \frac{l_0}{l_0-1} \). Recursively running this procedure as long as \( m' \geq 5 \), we shall finally see \( \xi \geq \frac{8}{5} \). Clearly, if \( \nu_2 > 3 \), one gets \( \xi > \frac{6}{5} \). When \( \xi > \frac{6}{5} \), since \( \alpha_4 = (2 - \frac{1}{\nu_2})\xi \geq \frac{8}{5} \xi > 2 \), \( \varphi_4 \) is birational by Theorem 2.1(2). In other words, if \( \varphi_4 \) is not birational, then \( \xi = \frac{6}{5} \) and \( \nu_2 = 3 \). By Inequality (3.3), this means \( \deg \tilde{g}(X') = 3 \). \( \square \)

**Proposition 3.7.** Assume \( g(C) = 2 \), \( \nu_2 = 2 \) and \( \varphi_4 \) is not birational. Then either \( \xi = 1 \) or \( \xi = \frac{4}{3} \), \( \deg \tilde{g}(X') = 2 \), \( \tilde{g}(X') = \tilde{F}_2 \) and the general irreducible component in \( \tilde{g}^{-1}(l) \) is \( C \)-horizontally integral, where \( l \) is the general line in \( \tilde{F}_2 \) passing through the vertex.

**Proof.** First of all, if \( \xi > \frac{4}{3} \), then \( \varphi_4 \) is birational since \( \alpha_4 = (2 - \frac{1}{2})\xi > 2 \). So we may and do assume \( 1 < \xi \leq \frac{4}{3} \) from now on.

By Inequality (3.3), the image surface \( \Sigma := \tilde{g}(X') \subset \mathbb{P}^3 \) has degree 2. Classical surface theory (cf. Reid [15, p30, Ex.19]) says that \( \Sigma \) must be either of the following surfaces:

1. \( \Sigma = \mathbb{P}^1 \times \mathbb{P}^1 \).
(II) \( \Sigma \) is the cone \( \overline{\mathbb{F}_2} \) obtained by blowing-down the unique \((-2)\) curve section on Hirzebruch surface \( \mathbb{F}_2 \).

In both cases, \( \Sigma \) is normal. Modulo further birational modifications, we may and do assume that \( \Gamma \) dominates the minimal resolution of singularities (if any) of \( \overline{\mathbb{F}_2} \) (i.e. \( \Gamma \) is over \( \mathbb{F}_2 \) in the second case). By pulling back the hyperplane section of \( \Sigma \) to \( \Gamma \), we have a base point free divisor \( H_\Gamma = s^*(\mathcal{O}_\Sigma(1)) \) so that \( M_1 \sim f^*(H_\Gamma) \). We now analyze the structure of \( H_\Gamma \) in details.

**Case (I).** We consider the morphism \( g := s \circ f : X' \rightarrow \Sigma \). Since \( \mathcal{O}_\Sigma(1) \sim L_1 + L_2 \) with \( (L_1 \cdot L_2) = 1 \), the pull backs of \( L_1 \) and \( L_2 \) form two fiber structures on \( X' \). Set \( F_1 := g^*(L_1) \) and \( F_2 := g^*(L_2) \). Then \( S \geq F_1 + F_2 \). We see that both \( F_1 \) and \( F_2 \) are irreducible for general \( L_1 \) and \( L_2 \) since \( h^0(X', S) = 4 \). Now the vanishing theorem gives

\[
|K_{X'} + [2\pi^*(K_X)] + F_1 + F_2|_{F_1} \geq |K_{F_1} + [2\pi^*(K_X)]|_{F_1} + C|
\]

and

\[
|K_{F_1} + [2\pi^*(K_X)]|_{F_1} + C|_C = |K_C + \tilde{D}_1|
\]

with \( \text{deg}(\tilde{D}_1) \geq 2 \xi > 2 \). This simply implies the birationality of \( \varphi_4 \) (a contradiction!) and thus \( \Sigma \neq \mathbb{P}^1 \times \mathbb{P}^1 \).

**Case (II).** Denote by \( \nu : \mathbb{F}_2 \rightarrow \Sigma = \overline{\mathbb{F}_2} \) the blow up at the singularity of \( \mathbb{F}_2 \). Denote \( H_2 := \nu^*(\mathcal{O}_\Sigma(1)) \). Then \( h^0(\mathbb{F}_2, H_2) = 4 \). Noting that \( H_2 \) is a nef and big divisor on \( \mathbb{F}_2 \), we can write

\[ H_2 \sim \mu G_0 + nT \]

where \( G_0 \) is the unique section of the ruling structure with \( G_0^2 = -2 \), \( T \) is the general fiber of the ruling of \( \mathbb{F}_2 \), \( \mu \) and \( n \) are integers. Necessarily we get \( n = 2 \) and \( \mu = 1 \). Let \( \theta_0 : \mathbb{F}_2 \rightarrow \mathbb{P}^1 \) be the \( \mathbb{P}^1 \)-bundle fibration and \( \eta_2 : \Gamma \rightarrow \mathbb{F}_2 \) the birational morphism. Let \( f_0 : X' \rightarrow \mathbb{P}^1 \) be the composition, i.e. \( f_0 := \theta_0 \circ \eta_2 \circ f \). Let \( \hat{F} \) be a general fiber of \( f_0 \). Clearly, we see \( S \sim M_1 \sim 2\hat{F} + N_0 \) where \( N_0 = f^*\eta_2^*(G_0) \). Observing that \( |S|_S \) is composed of a pencil of curves and \( \hat{F} \cap S \equiv (\eta_2 \circ f)^*(H_2 \cap T) \), we have \( S|_F \sim N_0|_F \sim C \). Since \( K_\hat{F} \geq S|_\hat{F} \), we see \( p_g(\hat{F}) \geq 2 \).

Denote by \( \hat{\sigma} : \hat{F} \rightarrow \hat{F}_0 \) the contraction onto the minimal model. Since \( (\hat{\sigma}^*(K_{\hat{F}_0}) \cdot C) = \xi > 1 \), we see that \( \hat{F}_0 \) is not a \((1,2)\) surface. We may write \( \pi^*(K_X) \sim 2\hat{F} + \hat{E}_1 \) for some effective \( \mathbb{Q} \)-divisor \( \hat{E}_1 \) on \( X' \).

Consider the pencil \( |2\hat{F}| \ll |K_{X'}| \) and the morphism \( \Phi_{|2\hat{F}|} \). Clearly \( f_0 \) is the induced fibration of \( \Phi_{|2\hat{F}|} \). Since \( K_{X'} \geq 2\hat{F} \), the relation (2.22) in the proof of Crollary [2.5] implies \( \pi^*(K_X)|_{\hat{F}} \geq \frac{2}{3} \hat{\sigma}^*(K_{\hat{F}_0}) \) and, for a smooth fiber \( C \) of \( f \) contained in a general surface \( \hat{F} \),

\[
\xi = \xi_{\hat{F}} := (\pi^*(K_X)|_{\hat{F}} \cdot C) \geq \frac{2}{3} (\hat{\sigma}^*(K_{\hat{F}_0}) \cdot C) \geq \frac{4}{3}
\]
as $\hat{F}_0$ is not a $(1, 2)$ surface. As we mention at the very beginning, we get $\xi = \frac{4}{3}$. We shall analyze this very special case more explicitly as follows.

Since the Kawamata-Viehweg vanishing theorem implies
\[ H^1(X', K_{X'} + [3\pi^*(K_X) - \hat{F} - \frac{1}{2}\hat{E}_1]) = 0, \]
we have the surjective map
\[ H^0(X', K_{X'} + [3\pi^*(K_X) - \frac{1}{2}\hat{E}_1]) \rightarrow H^0(\hat{F}, K_{\hat{F}} + [3\pi^*(K_X) - \hat{F} - \frac{1}{2}\hat{E}_1]|_{\hat{F}}). \]
Noting that
\[ [3\pi^*(K_X) - \hat{F} - \frac{1}{2}\hat{E}_1]|_{\hat{F}} \geq [(3\pi^*(K_X) - \hat{F} - \frac{1}{2}\hat{E}_1)|_{\hat{F}}] = [Q_{5/2}], \]
The birationality of $\varphi_4$ follows from that of $\Phi_{[K_{\hat{F}} + Q_{5/2}]}$. By definition, we have $Q_{5/2} = \frac{5}{2}\hat{E}_1|_{\hat{F}} = \frac{5}{2}\pi^*(K_X)|_{\hat{F}}$ since $\hat{F}$ is trivial. Denote by $\iota := \eta_2 \circ f|_{\hat{F}}$. Then $\iota : \hat{F} \rightarrow T \cong \mathbb{P}^1$ is a fibration with the general fibre $C \sim S \cap \hat{F}$. Since $\pi^*(K_X) \geq S$, we may write
\[ \pi^*(K_X)|_{\hat{F}} = \hat{E}_1|_{\hat{F}} = C_0 + \sum_{i=1}^{t} k_i H_i + \hat{E}_v \]
where $C_0 \sim C$, $k_i \in \mathbb{Q}^+$, $H_i$ is horizontal with respect to $\iota$ for each $i$ and $\hat{E}_v$ is an $\iota$-vertical effective $\mathbb{Q}$-divisor on $\hat{F}$. Clearly we have $\sum_{i} k_i(H_i \cdot C) = \frac{4}{3}$. Since $H^1(\hat{F}, K_{\hat{F}} + [\frac{3}{2}\hat{E}_1|_{\hat{F}}]) = 0$ by the Kawamata-Viehweg vanishing theorem, we have the surjective map:
\[ H^0(\hat{F}, K_{\hat{F}} + [\frac{3}{2}\hat{E}_1|_{\hat{F}}] + C_0) \rightarrow H^0(C, K_C + D_{3/2}) \quad (3.5) \]
where $D_{3/2} := ([\frac{3}{2}\hat{E}_1|_{\hat{F}}] + C_0 - C)|_C \sim [\sum_{i=1}^{t}(\frac{3}{2}k_i)H_i]|_C$.

**Subcase II-1.** Now if for certain $i$ so that $\frac{3}{2}k_i$ is not an integer, then we have $\deg(D_{3/2}) > \frac{3}{2} \xi = 2$ and the birationality principle implies that $\varphi_4$ is birational, a contradiction.

**Subcase II-2.** Assume that $\frac{3}{2}k_i$ is integral for each $i$. Clearly one of the following situations occurs:
\[ \circ t = 2, \ k_1 = k_2 = \frac{2}{3} \quad \text{and} \quad (H_1 \cdot C_0) = (H_2 \cdot C_0) = 1, \ H_1 \neq H_2; \]
\[ \circ t = 1, \ k_1 = \frac{4}{3} \quad \text{and} \quad (H_1 \cdot C_0) = 1; \]
\[ \circ t = 1, \ k_1 = \frac{5}{3} \quad \text{and} \quad (H_1 \cdot C_0) = 2. \]
Since $C_0$ is not necessarily irreducible nor reduced, there exists an irreducible and reduced curve $A_0 \leq C_0$ such that either
\[ (1) \ (A_0 \cdot H_1) = 1 \quad \text{and} \quad ((C_0 - A_0) \cdot H_1) = 0; \text{ or} \]
Characterization of the 4-canonical birationality 11

(2) \((A_0 \cdot H_1) = 2\) and \(((C_0 - A_0) \cdot H_1) = 0\); or
(3) \((A_0 \cdot H_1) = 1\) and \(2A_0 \leq C_0\).

For cases (1) and (2), we claim that \(\varphi_{4,X}\) is birational. In fact, since we have already had the vanishing group \(H^1(F, K_F + [\frac{3}{2} \hat{E}_1|_F]) = 0\), \(H^1(F, K_F + [\frac{3}{2} \hat{E}_1|_F] + H_1) = 0\) follows, suppose we can prove \(H^1(H_1, K_{H_1} + D_H) = 0\), where \(D_H := [\frac{3}{2} \hat{E}_1|_F]|_{H_1}\). One has
\[
\deg(D_H) \geq \frac{3}{2} (\hat{E}_1|_F \cdot H_1) + (\frac{3}{2} A_0) - \frac{3}{2} A_0 \cdot H_1 \geq \frac{3}{2} (\hat{E}_1|_F \cdot H_1) = \frac{3}{2} (\pi^*(K_X)|_F \cdot H_1) \geq 0.
\]
As \(H_1\) is an irreducible curve on \(F\), we clearly have \(H^1(H_1, K_{H_1} + D_H) = 0\). Now we have
\[
K_F + [\frac{3}{2} \hat{E}_1|_F] + C_0 + H_1 \leq K_F + [\frac{5}{2} \hat{E}_1|_F]
\]
and, for a general fiber \(C\) of \(\iota\), the vanishing of \(H^1\) gives the surjective map
\[
H^0(F, K_F + [\frac{3}{2} \hat{E}_1|_F] + C_0 + H_1) \rightarrow H^0(C, K_C + \hat{D})
\]
where \(\hat{D} := ([\frac{3}{2} \hat{E}_1|_F] + C_0 - C + H_1)|_C\) with\[
\deg(\hat{D}) \geq \frac{3}{2} \xi + (H_1 \cdot C) \geq 3.
\]
Thus we see that \(\varphi_4\) is birational, which means that neither Case (1) nor Case (2) can happen.

For Case (3), \(H_1\) is clearly the unique \(\iota\)-horizontal component in \(\pi^*(K_X)|_F\). First we consider the case that \(|K_F|\) is composed with a pencil. Then \(\iota : \hat{F} \rightarrow \mathbb{P}^1\) must be the induced fibration from \(\Phi_{|K_F|}\). Assume \(p_g(\hat{F}) \geq 3\). Then we have \(\sigma^*(K_{F_0}) \sim 2C + E_0\) for an effective divisor \(E_0\). Then we get\[
\pi^*(K_X)|_\hat{F} \sim \frac{4}{3} C + \hat{E}_1
\]
for certain effective \(\mathbb{Q}\)-divisor \(\hat{E}_1\). Since\[
Q_{5/2} - C - \frac{3}{4} \hat{E}_1 \equiv \frac{7}{4} \pi^*(K_X)|_\hat{F}
\]
is nef and big, the vanishing theorem again implies:\[
|K_F + [Q_{5/2} - \frac{3}{4} \hat{E}_1]|_C = |K_C + \hat{D}|
\]
where \(\hat{D} := [Q_{5/2} - C - \frac{3}{4} \hat{E}_1]|_C\) and \(\deg(\hat{D}) \geq \frac{2}{7} \xi > 2\), which means \(\varphi_{4,X'}\) is birational (a contradiction). Thus \(p_g(\hat{F}) = 2\). Clearly we have \(h^0(K_F - C) = 1\). Next, we consider the case that \(|K_F|\) is not composed.
of a pencil. Let us assume $p_g(\hat{F}) \geq 4$. Modulo further birational modifications, we may and do assume that the moving part $|\hat{C}|$ of $|K_{\hat{F}}|$ is base point free. Pick a general curve $\hat{C}$. Then $\sigma^*(K_{\hat{F}_0})^2 \geq \hat{C}^2 \geq 2p_g(\hat{F}) - 4 \geq 4$ and
\[
(p^*(K_X)|_F \cdot \hat{C}) \geq \frac{2}{3} \sqrt{\sigma^*(K_{\hat{F}_0})^2 \cdot \hat{C}^2} \geq \frac{8}{3}.
\]
Write $p^*(K_X)|_F \sim \frac{3}{2} \hat{C} + \hat{E}_{00}$ for an effective $\mathbb{Q}$-divisor $\hat{E}_{00}$ on $F$. Since $Q_{5/2} - \hat{C} - \frac{3}{2} \hat{E}_{00} \equiv p^*(K_X)|_F$ is nef and big, the vanishing theorem again implies:
\[
|K_F + [Q_{5/2} - \frac{3}{2} \hat{E}_{00}]|_C = |K_{\hat{C}} + \hat{D}_0|
\]
where $\hat{D}_0 := [Q_{5/2} - \hat{C} - \frac{3}{2} \hat{E}_{00}]|_C$ and $\deg(\hat{D}_0) \geq \frac{8}{3} > 2$, which means $\varphi_{4,X'}$ is birational (a contradiction). Thus $p_g(\hat{F}) = 3$. Let us consider the exact sequence:
\[
0 \to H^0(\hat{F}, K_{\hat{F}} - C) \to H^0(\hat{F}, K_{\hat{F}}) \to H^0(C, K_C) \to \cdots
\]
Clearly, since $\dim \text{Im}(j) = 2$, we have $h^0(\hat{F}, K_{\hat{F}} - C) = 1$. In both cases, we have $C \leq p^*(K_X)|_F \leq K_{\hat{F}}$. Since the horizontal part of $p^*(K_X)|_F$ is $\frac{2}{3} H_1$ and $(H_1 \cdot C) = 2$, $K_{\hat{F}}$ has the unique irreducible and reduced horizontal part $H_1$. In a word, we have shown that $\hat{F}$ is $C$-horizontally integral. 

**Proposition 3.8.** Assume $g(C) = 2$, $\nu_2 = 2$, $\xi = \frac{4}{3}$, $\deg \hat{g}(X') = 2$, $\hat{g}(X') = \hat{F}_2$ and the general surface $\hat{F}$ on $X'$ in the family induced from the ruling of $\mathbb{F}_2$ is $C$-horizontally integral. Then $\varphi_4$ is not birational.

**Proof.** Naturally we are in the situation of Case (II) in the proof of Proposition 3.7. We keep the same setting as there. Pick a general fiber $\hat{F}$ of $\iota$. Since $\hat{F}$ is $C$-horizontally integral and $p^*(K_X)|_F = \hat{E}_1|_F \leq K_{\hat{F}}$, the $C$-horizontal part of $\hat{E}_1|_F$ is irreducible and reduced. Thus the horizontal part of $p^*(K_X)|_F$ is exactly $\frac{2}{3} H_1$ with $(H_1 \cdot C) = 2$.

Since, for a general fiber $\hat{F}$ of $f_0$, we have
\[
M_4|_{\hat{F}} \leq \lfloor 4p^*(K_X)|_F \rfloor
\leq \lfloor \frac{8}{3} H_1 + (\text{vertical divisors}) \rfloor
\leq 2 H_1 + (\text{vertical divisors with respect to } \iota).
\]
Thus, for a general fiber $C$ of $\iota$, $(M_4|_C \cdot C) \leq 4$. Note that $H_1|_C$ gives a $g^1_2$ of the involution on $C$. Thus $2H_1|_C$ gives a finite map of degree 2. On the other hand, Relation (3.3) implies $(M_4 \cdot C) \geq 4$ and $|M_4||_C$ is base point free since $\deg(D_{3/2}) \geq 2$. This simply implies that $\varphi_4|_C$ is finite.
of degree 2. Thus \( \varphi_{4,\mathcal{X}} \) is generically finite of degree 2. In particular, \( \varphi_4 \) is not birational.

**Claim 3.9.** Assume \( g(C) = 2 \). If either \( \xi = 1 \) or \( \xi = \frac{g}{3} \) and \( \deg \tilde{g}(X') = 3 \), then \( \varphi_4 \) is not birational.

**Proof.** The proof is similar in the spirit to that of Chen-Zhang [8, Proposition 4.6].

**Notation.** Recall that we have \( K_{X'} = \pi^*(K_X) + E_\pi \). On \( X \), we set \( Z := \pi_*(Z_1) \) and \( N := \pi_*(M_1) \). Clearly \( K_X \sim N + Z \). Then there is an effective \( \mathbb{Q} \)-divisor \( E_1 \), which is supported by some exceptional divisors, such that \( \pi^*(N) = M_1 + E_1 \). Therefore \( E_1' = \pi^*(Z) + E_1 \).

For a general member \( S \) of \( |M_1| \), we have \( K_{X'}|_S = \pi^*(K_X)|_S + E_\pi|_S = (M_1|_S + E_1'|_S) + E_{\pi}|_S \). One knows that \( E_{\pi} \) is composed of all those exceptional divisors of \( \pi \). Also it is clear that \( \text{Supp}(E_1) \subset \text{Supp}(E_{\pi}) \).

**Further modifications to \( \pi \).** We now need slightly detailed assumptions on the map \( \pi \). We may take \( \pi \) to be the composition:

\[
X' \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X
\]

where \( \pi_0 \) is the resolution of the indeterminacy of the moving part of \( |K_{X'}| \), \( \pi_1 \) is the resolution of those isolated singularities on \( X_1 \) which are away from all exceptional locus of \( \pi_0 \), and finally \( \pi_2 \) is the minimal further modification such that \( \pi^*(K_1) \) has simple normal crossing support (recall here that \( K_1 \sim K_X \) is a fixed Weil divisor as in [2.1]). Set \( \pi_3 := \pi_0 \circ \pi_1 \). By abuse of notations we will have a set of divisors for \( \pi_3 \) similar to that for \( \pi \). For example we may write \( K_{X_2} = \pi_3^*(K_X) + E_{\pi_3} \), where \( E_{\pi_3} \) is an effective \( \mathbb{Q} \)-divisor. The moving part \( |M_{\pi_3}| \) of \( |K_{X_2}| \) is already base point free. Write \( \pi_3^*(N) = M_{\pi_3} + E_{1,\pi_3} \) and \( \pi_3^*(K_X) = M_{\pi_3} + E_{1,\pi_3} + E_{1',\pi_3} \) where \( E_{1,\pi_3} \) and \( E_{1',\pi_3} \) are both effective \( \mathbb{Q} \)-divisors. Clearly \( E_{1,\pi_3} = \pi_3^*(Z) + E_{1,\pi_3} \). By the definition of \( \pi_3 \), \( E_{\pi_3} \) is the sum of two parts \( E_{1',\pi_3} + E_{\pi_3} \) where \( E_{1',\pi_3} \) consists of all those components over the indeterminacy of \( \varphi_1 \) while \( E_{\pi_3} \) is totally disjoint from \( E_{1',\pi_3} \). Denote by \( S_{\pi_3} \) a general member of \( |M_{\pi_3}| \). Then \( |M_{\pi_3}| \circ S_{\pi_3} \) is a free pencil of genus 2 with a general member \( C_{\pi_3} \). As we have seen \( \text{Supp}(E_{1',\pi_3}|_{S_{\pi_3}}) = 0 \) and so \( \text{Supp}(E_{\pi_3}|_{S_{\pi_3}}) = \text{Supp}(E_{1,\pi_3}|_{S_{\pi_3}}) \).

Now we have \( t = (\pi^*(K_X) \cdot C) = (\pi_3^*(K_X) \cdot \pi_{2*}(C)) = (\pi_3^*(K_X) \cdot C_{\pi_3}) \), where \( t = 1 \) or \( \frac{g}{3} \). Since \( 2 = \deg(K_{C_{\pi_3}}) = (\pi_3^*(K_X) + E_{\pi_3}) \cdot C_{\pi_3} \) and \( (\pi_3^*(K_X) \cdot C_{\pi_3}) = (\pi_3^*(K_X)|_{S_{\pi_3}} \cdot C_{\pi_3}) = t \), we get \( (E_{\pi_3}|_{S_{\pi_3}} \cdot C_{\pi_3}) = (E_{1,\pi_3} \cdot C_{\pi_3}) = 2 - t > 0 \). Therefore \( (E_{1,\pi_3} \cdot C_{\pi_3}) = (E_{1,\pi_3}|_{S_{\pi_3}} \cdot C_{\pi_3}) > 0 \). Noting that \( \pi_3^*(K_X) \leq E_1 \), one has

\[
(E_1|_{S} \cdot C) \geq (\pi_3^*(E_{1,\pi_3})|_{S} \cdot C) = (E_{1,\pi_3}|_{S_{\pi_3}} \cdot C_{\pi_3}) > 0.
\]  

**Reduction of birationality.** As we have known, \( \Phi_4 \) is birational if and only if \( \Phi_4|_{S} \) is birational for a general \( S \). Now on a general surface \( S \), we have a pencil \( |M_1| \circ S \) and \( \Phi_4|_S \) distinguishes different generic irreducible elements of \( |M_1| \circ S \). So \( \Phi_4|_S \) is birational if and only if \( \Phi_4|_C \) is birational. We will show that \( \Phi_4|_C = \Phi_{2|K_C|} \), which is, however, not birational.
Step 1. By the Kawamata-Viehweg vanishing theorem, we have the surjective map:

$$H^0(X', K_{X'} + [2\pi^*(K_X)] + S) \longrightarrow H^0(S, K_S + [2\pi^*(K_X)]|S). \quad (3.7)$$

Since

$$2\pi^*(K_X)|_S - C - \frac{1}{\nu_2}E'_1|_S \equiv (2 - \frac{1}{\nu_2})\pi^*(K_X)|_S$$

is nef and big, the vanishing theorem gives the surjective map:

$$H^0(S, K_S + [2\pi^*(K_X)]|_S - \frac{1}{\nu_2}E'_1|_S) \longrightarrow H^0(C, K_C + \hat{D}), \quad (3.8)$$

where

$$\hat{D} := [2\pi^*(K_X)]|_S - \frac{1}{\nu_2}E'_1|_S|_{C} = [(2 - \frac{1}{\nu_2})E'_1|_S]|_{C}$$

and \(\deg(\hat{D}) \geq (2 - \frac{1}{\nu_2})\xi \geq 2 - \frac{1}{2} > 1\), noting that \((E'_1 \cdot C) = (\pi^*(K_X) \cdot C) = \xi = t\). So \(|K_C + \hat{D}|\) is base point free. Denote by \(M'_4, N'_4\) the moving parts of \([K_{X'} + [2\pi^*(K_X)] + S], |K_S + [2\pi^*(K_X)]|_S - \frac{1}{\nu_2}E'_1|_S|\) respectively. Then one has

$$4\pi^*(K_X)|_{C} \geq M_4|_{C} \geq (M'_4|_{S})|_{C} \geq N'_4|_{C} \geq K_C + \hat{D}.$$ 

So \(5 > 4t = 4\pi^*(K_X)|_{S} \cdot C \geq M_4 \cdot C = \deg(K_C + \hat{D}) \geq 4\). This means \(M_4|_{C} \sim K_C + \hat{D}\) and \(\deg(\hat{D}) = 2\). On the other hand, we have shown \(|M_4|_{C} \gg |K_C + \hat{D}|\). Clearly \(|M_4|_{C} = |K_C + \hat{D}|\). Since \(\deg(\Phi_{|K_C|}) = 2\), we have \(\deg(\Phi_4) \leq 2\). So \(\Phi_4\) is either birational or a generically double cover.

Step 2. We have:

$$K_C \sim (K_{X'}|_S + S|_S)|_{C} = (\pi^*(Z)|_S|_{C} + (E_1|_S)|_{C} + (E_{\pi}|_S)|_{C}. \quad (3.9)$$

Since \(2 = \deg(K_C) = (\pi^*K_X + E_{\pi}) \cdot C\) and \((E'_1|S\cdot C) = (\pi^*(K_X)|_S \cdot C) = t\), we get \((E_{\pi}|_S \cdot C) = (E_{\pi} \cdot C) = 2 - t > 0\). As a sub-divisor of \(K_C\), \((E_{\pi}|_S)|_C\) has its support \(\text{Supp}(E_{\pi}|_S)|_C\) be one of the following situations:

Case A. a single point \(P\);

Case B. two different points \(P\) and \(Q\) on \(C\).

We consider Case A and Case B separately and note that \((E'_1|_S = \pi^*(Z)|_S + E_1|_S\) and \(\text{Supp}(E_1|_S) \subset \text{Supp}(E_{\pi}|_S)\).

Suppose we are in Case A. Then \((E_{\pi}|_S)|_{C} = (2 - t)P\). First, if \(\text{Supp}((\pi^*(Z)|_S)|_{C} + (E_1|_S)|_{C})\) contains a point other than \(P\) (say a point \(R\)), then \((\pi^*(Z)|_S)|_{C} + (E_1|_S)|_{C} + (E_{\pi}|_S)|_{C} = P + R\) and \(R\) is not contained in \(\text{Supp}(E_1|_C)\) since, otherwise, \(R\) is in \(\text{Supp}(E_{\pi}|_C)\), a contradiction. Thus \(R \leq (\pi^*(Z)|_S)|_{C}\) as an integral part because \((\pi^*(Z)|_S)|_{C} + (E_1|_S)|_{C} + (E_{\pi}|_S)|_{C}\) is an integral divisor. This says \(\hat{D} = P + R \sim K_C\). If \(\text{Supp}((\pi^*(Z)|_S + E_1|_S)|_{C}\) only contains a single point, \((\pi^*(Z)|_S)|_{C} + (E_1|_S)|_{C} = tP\) and \(K_C \sim 2P\). In this case, we
have \( \hat{D} = 2P \). In a word, we always have \( \varphi_4|_C = \Phi_{|2K_C|} \), which is not birational. So \( \varphi_4 \) is not birational onto its image.

Suppose we are in Case B. The right hand side of \( (3.9) \) must be \( P + Q \) and \( K_C \sim P + Q \). We also know that \( \hat{D} = P + Q \). Thus \( \varphi_4|_C = \Phi_{|2K_C|} \) is not birational either.

So far, we have actually proved the following:

Theorem 3.10. Let \( X \) be a minimal projective 3-fold of general type with \( p_g(X) = 4 \). Keep the same notation as in \( (2.7) \). Assume \( d_1 = 2 \). Then \( \varphi_4 \) is not birational if and only if \( g(C) = 2 \) and one of the following holds:

\begin{enumerate}
  \item \( (\pi^*(K_X) \cdot C) = 1 \);
  \item \( (\pi^*(K_X) \cdot C) = \frac{5}{3} \) and \( \tilde{g}(X') \) is a cubic surface in \( \mathbb{P}^3 \).
  \item \( (\pi^*(K_X) \cdot C) = \frac{4}{3} \). \( \tilde{g}(X') \) is the quadric cone \( \mathbb{P}_2 \) in \( \mathbb{P}^3 \) and \( \tilde{F} \) is \( C \)-horizontally integral, where \( \tilde{F} \) on \( X' \) is the general irreducible component of the \( \tilde{g}^{-1}(l) \) and \( l \) is the line in the ruling of \( \mathbb{P}_2 \) passing through the vertex.
\end{enumerate}

3.3. Part III. \( d_1 = 3 \).

We provide a concise proof for the following theorem to make this paper as self-contained as possible, though relevant statements have been partially presented in another preprint of the first author.

Theorem 3.11. Let \( X \) be a minimal projective 3-fold of general type. Assume \( p_g(X) = 4 \) and \( \varphi_4 \) is generically finite. Then \( \varphi_4 \) is birational if and only if \( K_X = 2 \) and \( \deg(\varphi_4) = 2 \).

Keep the same setting and notation as in \( (2.1) \). Pick a general member \( S \in |M_1| \). Consider the linear system \( |4K_X| \) and its sub-system \( |K_X' + 2\pi^*(K_X)| + M_1| \). Clearly \( \varphi_4 \) distinguishes different general members of \( |M_1| \). By the Kawamata-Viehweg vanishing theorem, we have the following relation:

\[
|K_X' + 2\pi^*(K_X)| + M_1|_s = |K_S + 2\pi^*(K_X)|_s \not\succ |K_S + 2L| \tag{3.10}
\]

where \( L := \pi^*(K_X)|_s \) is an effective nef and big \( \mathbb{Q} \)-divisor on \( S \). Set \( |G| = |M_1|_s | \). Pick a generic irreducible element \( C \) of \( |G| \). Then, since \( p_g(S) > 0 \), \( |K_S + 2L| \) distinguishes different general curves \( C \). Thus it is sufficient to prove the birationality (or non-birationality) of \( \varphi_4|_C \). In fact, the Kawamata-Viehweg vanishing theorem gives

\[
|K_F + 2L - E'|_F|_C = |K_C + D_3|
\]

where \( D_3 := [2L - E'|_F - C]|_C \) with \( \deg(D_3) \geq (L \cdot C) = \xi \).

Claim 3.12. \( K_X^3 > 2 \) if and only if \( \xi > 2 \).

Proof. Pick a general member \( S \in |M_1| \). We have

\[
\pi^*(K_X)|_s \sim S|_s + E'_1|_s
\]
and so
\[ K_X^3 = (\pi^*(K_X))^3 \geq (\pi^*(K_X)^2 \cdot S) = \xi. \]
On \( S \), since \(|C|\) is not composed of a pencil of curves, \( C^2 \geq 2 \). Thus
\[ \xi = (\pi^*(K_X) \cdot S^2) \geq C^2 \geq 2. \]

On the other hand, by choosing a sufficiently large and divisible integer \( n > 0 \) so that \([n\pi^*(K_X)]\) is base point free, one applies the Hodge Index Theorem on the general member \( S_{[n]} \) to get the inequality:
\[ \xi = (\pi^*(K_X) \cdot S^2) = \frac{1}{n}(\pi^*(K_X)|_{S_{[n]} \cdot S}|_{S_{[n]}}) \geq \sqrt{K_X^3} \cdot \xi. \]
By Theorem 2.3 one has \( K_X^3 \geq 2 \). Thus it follows that \( \xi = 2 \) if and only \( K_X^3 = 2 \). The claim is proved. \( \square \)

**Claim 3.13.** \( \varphi_4 \) is generically finite of degree \( \leq 2 \); \( \varphi_4 \) is birational if and only if \( K_X^3 > 2 \).

**Proof.** By definition, we have \( p = 1 \) and \( \beta = 1 \). Then \( \alpha_4 = \xi. \)

Assume \( K_X^3 > 2 \). Claim 3.12 implies \( \xi > 2 \) and Theorem 2.1 (2) implies the birationality of \( \varphi_4 \).

Assume \( K_X^3 = 2 \). Note that \( g : X' \rightarrow \mathbb{P}^3 \) can not be birational. We have
\[ 2 = K_X^3 \geq S^3 \geq \deg(\varphi_1) \geq 2, \quad (3.11) \]
it follows that \( \varphi_1 \) is generically finite of degree 2. This means \( \varphi_1|_C \) is a double cover onto \( \mathbb{P}^1 \). In particular, \( C \) is hyperelliptic and \( S|_C \) is exactly a \( g_2^1 \) of \( C \). Note that \( C \) is a curve of genus \( \geq 4 \) since \( K_SC + C^2 \geq 6 \). We have
\[ |K_F + 2L|_C \geq |K_F + 2S|_C = |K_C + S||_C \]
by the vanishing theorem. This, together with the relation (3.10), implies \( |M_4||_C \geq |K_C + S||_C \), where the last one is base point free with \( \deg(K_C + S)|_C \geq 8 \). Since \( 4\pi^*(K_X) \cdot C = 4\xi = 8 \), we see \( |M_4||_C = |K_C + S||_C \), which gives exactly a double cover. Clearly, since \( |M_4| \) distinguishes different curves \( C \), \( \varphi_4 \) is generically a double cover. We are done. \( \square \)

Theorem 3.11 automatically follows from Claim 3.13

**Proof of Theorem 1.1.** Assume \( \varphi_4 \) is not birational. Then \( X \) has the listed 4 structures by Corollary 3.3, Theorem 3.10 and Theorem 3.11.

Contrarily, if \( X \) has structures (1), (3) and (4), then \( \varphi_4 \) is not birational by Theorem 3.11, Theorem 3.10 (ii) and Theorem 3.10 (iii). Assume \( X \) has structure (2). Then automatically \( d_1 \leq 2 \) since, otherwise, \( (K_X \cdot C_0) = (\pi^*(K_X) \cdot \hat{C}) \geq 2 \) where we assume \( \pi(\hat{C}) = C_0 \) and \( \hat{C} \) is a moving curve on \( X' \). In the case \( d_1 = 2 \), \( \varphi_4 \) is not birational by Theorem 3.11 (i). Finally let us consider the case \( d_1 = 1 \). We have an induced fibration \( f : X' \rightarrow \Gamma \) with the general fiber \( F \). By [8], Lemma
4.7] and Corollary 2.5, we have $\pi^*(K_X)|_F \geq \frac{3}{4}\sigma^*(K_{F_0})$. Still consider the curve $\hat{C}$ on $X'$ with $\pi(\hat{C}) = C_0$. If $\hat{C}$ is not vertical with respect to $f$, then $\Phi_{|M_1|}(\hat{C}) = \Gamma$. In particular, we have $(F \cdot \hat{C}) \geq 1$. Then $(\pi^*(K_X) \cdot \hat{C}) \geq p(F \cdot \hat{C}) \geq 3$, a contradiction. Therefore we see $\hat{C} \subset F$ for some smooth fiber $F$ if we choose a general curve $\hat{C}$. But then

$$1 = (\pi^*(K_X) \cdot \hat{C}) = (\pi^*(K_X)|_F \cdot \hat{C}) \geq \frac{3}{4}(\sigma^*(K_{F_0}) \cdot \hat{C})$$

implies that $(\sigma^*(K_{F_0}) \cdot \hat{C}) = 1$. Since $C$ is a smooth genus 2 curve, we have $K_{F_0}^2 = 1$ by the Hodge Index Theorem. Besides, $|C|$ must be a rational pencil on $F$ and $K_F \geq C$. All these clearly imply that $F$ is a $(1,2)$ surface. Therefore $\varphi_4$ is not birational by Corollary 3.3.

Finally we would like to ask the following very interesting, but challenging question:

**Problem 3.14.** (1) Is it possible to characterize the birationality of $\varphi_m$ ($m = 4,5$) of minimal projective 3-folds $X$ of general type with $p_g = 3$?

(2) Is it possible to characterize the birationality of $\varphi_m$ ($m = 4,5,6$) of minimal projective 3-folds $X$ of general type with $p_g = 2$?

**Acknowledgment.** This article was schemed while Chen was visiting Universität Bayreuth in February of 2012. Chen would like to thank Ingrid Bauer and Fabrizio Catanese for their hospitality and the generous support. Chen feels indebted to Fabrizio Catanese for many stimulating discussions. Zhang would like to thank LMNS of Fudan University for the Visiting Fellow support in 2012 and 2013.

**References**

[1] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, 1984.

[2] E. Bombieri, *Canonical models of surfaces of general type*, Inst. Hautes Études Sci. Publ. Math. 42 (1973), 171–219.

[3] J. A. Chen, M. Chen, *Explicit birational geometry of threefolds of general type, I*, Ann. Sci. Éc. Norm. Sup. (43) 2010, 365–394.

[4] J. A. Chen, M. Chen, *Explicit birational geometry of threefolds of general type, II*, J. of Differential Geometry 86 (2010), 237–271.

[5] J. A. Chen, M. Chen, *Explicit birational geometry for 3-folds and 4-folds of general type, III*, ArXiv: 1302.0374.

[6] M. Chen, *Canonical stability of 3-folds of general type with $p_g \geq 3$*, Internat J. Math. 14 (2003), 515–528.

[7] M. Chen, *A sharp lower bound for the canonical volume of 3-folds of general type*, Math. Ann. 337 (2007), no. 4, 887–908.

[8] M. Chen, D.-Q. Zhang, *Characterization of the 4-canonical birationality of algebraic threefolds*, Math. Zeit. 258 (2008), 565–585.

[9] M. Chen, *On an efficient induction step with $N_{klt}(X,D)$–notes to Todorov*, Comm. Anal. Geom. 20 (2012), no. 4, 765–779.

[10] S. Chiaruttini, R. Gattazzo, *Examples of birationality of pluricanonical maps*, Rend. Sem. Mat. Univ. Padova 107 (2002), 81–94.
[11] A. R. Iano-Fletcher. Contributions to Riemann-Roch on Projective 3-folds with Only Canonical Singularities and Applications. Proceedings of Symposia in Pure Mathematics 46(1987), 221-231.

[12] C. D. Hacon and J. McKernan, Boundedness of pluricanonical maps of varieties of general type, Invent. Math. 166 (2006), 1–25.

[13] Y. Kawamata, A generalization of Kodaira-Ramanujam’s vanishing theorem, Math. Ann. 261 (1982), 43–46.

[14] Y. Kawamata, On the extension problem of pluricanonical forms. Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 193–207, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999.

[15] M. Reid, Chapters on algebraic surfaces. Complex algebraic geometry (Park City, UT, 1993), 3–159, IAS/Park City Math. Ser., 3, Amer. Math. Soc., Providence, RI, 1997.

[16] S. Takayama, Pluricanonical systems on algebraic varieties of general type, Invent. Math. 165 (2006), 551–587.

[17] H. Tsuji, Pluricanonical systems of projective varieties of general type. I. Osaka J. Math. 43 (2006), no. 4, 967–995

[18] E. Viehweg, Vanishing theorems, J. reine angew. Math. 335 (1982), 1–8.

Institute of Mathematics & LMNS, Fudan University, Shanghai 200433, China
E-mail address: mchen@fudan.edu.cn

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: qi@math.missouri.edu