Nice colourings and the 4-colour theorem

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Abstract

Proving for triangulations an extended version of the 4-colour theorem by induction, we manage to exclude the case which led to the failure of Kempe’s attempted proof. The new idea is to claim the existence of a “nice” 4-colouring, in which existing Kempe chains satisfy a special condition, and to show that the assumption of its non-existence by a counterexample always leads to a contradiction: hence there is such a colouring.

Keywords: planar graph, triangulation, chromatic number, Kempe chain, Jordan’s curve theorem.

1 Introduction

In response to Kempe’s incorrect proof of the 4-colour conjecture in 1879 [1], Heawood [2] published a proof of the 5-colour theorem for planar graphs, together with an example graph which exposed an essential error in Kempe’s approach. The 4-colour theorem was first proved by Appel and Haken (see e.g. [3]) in 1976 with the help of extensive computer calculations. Improved and independent versions of this type of proof do exist since 1996 [4]. The search for “old-fashioned” proofs which can be checked “by hand” can still be considered as a worthwhile enterprise, at least for a posterior verification of a computer-based proof. Hopes that list colouring ideas could be helpful to find such a proof have not been fulfilled to date. In what follows a proof “by hand” without the use of list colouring ideas is presented.

2 Theorems for the colouring of planar graphs

The standard statement of the 4-colour theorem is expressed in the vertex-colouring context with the usual assumptions, i.e. a coloured map in the plane or on the sphere is represented by its dual simple graph $G$ with coloured vertices. When two vertices $v_k, v_l \in V(G)$ are connected by an edge $v_kv_l \in E(G)$ (i.e. when countries have a common line-shaped border), a (proper) colouring requires $c(v_k) \neq c(v_l)$ for the colours of the endpoints of the edge.
**Theorem 1.** The chromatic number of a planar graph is at most four.

Without loss of generality graphs to be studied for the proof can be restricted to triangulations. A planar graph $G$ is called a triangulation if it is connected, without loops, and every interior face is (bounded by) a triangle (3-cycle), as well as the (in the plane infinite) exterior face. It follows from Euler’s polyhedral formula that a planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges, and the triangulations are the edge-maximal planar graphs. Every planar graph $H$ (e.g. a proper near-triangulation) can be generated from a triangulation $G$ by removing edges and disconnected vertices. As removal of edges reduces the number of restrictions for colouring, the chromatic number of $H$ is not greater than that of $G$.

**Definition 1.** Let $H_{\alpha\beta} \subseteq G$ denote the subgraph of $G$, induced by the vertices coloured with distinct colours $\alpha$ and $\beta$, and let $H_{\alpha\beta}(v) \subseteq H_{\alpha\beta}$ be the component of $H_{\alpha\beta}$ that has $v$ in its vertex set. A bi-coloured path $P_{\alpha\beta} = P_{\beta\alpha} \subseteq H_{\alpha\beta}$ in which consecutive vertices are alternatingly coloured with colours $\alpha$ and $\beta$ is called a Kempe chain.

**Definition 2.** A chord of a cycle $C$ is an edge not in $E(C)$ between two vertices in $V(C)$, the endpoints of the chord. A virtual chord of a cycle $C$ is a not necessarily existing chord which is used (as a tool) to identify two non-adjacent vertices in $V(C)$ by its endpoints.

Then instead of Theorem 1, it is convenient to prove the following sharper version.

**Theorem 2 (extended 4-colour theorem).** Let $G$ be a triangulation. Then $G$ is 4-colourable, and for every arbitrarily chosen 5-cycle $C_5 \subset G$ without vertices in its interior, there is a nice colouring of $G$ and $C_5$: a colouring such that for every arbitrarily chosen pair of non-crossing virtual chords of $C_5$, a Kempe chain condition holds: at most one Kempe chain $P_{\alpha\beta}$ exists between the endpoints of one of these chords in the exterior of $C_5$, coloured with two distinct colours $\alpha, \beta \in [4]$ occurring on $C_5$ only once.

3 **Proof of Theorem 2**

Triangulations with $n \in \{3, 4\}$ have no 5-cycles and are trivially 4-colourable. Then we perform induction with respect to the number $n$ of vertices in $G$.

**Basis case.** (This paragraph is not a necessary part of the proof, but serves illustrative purposes.) For $n = 5$ the only existing triangulation (isomorphism class) has degree sequence 44433 and is derived from the non-planar complete graph $K_5$ by removing one of the two crossing edges. This graph is 4-colourable, has a 5-cycle without vertex in its interior, and can be coloured in such a way that there is at most one Kempe chain in the exterior of $C_5$ (see Fig. 1) with colours occurring on $C_5$ only once, namely $P_{42}$ between $v_3$ and $v_1$ (and not $P_{41}$ between $v_3$ and $v$). Hence this colouring is nice.
Let $n \geq 5$ and the theorem be true for up to $n$ vertices. Then consider in the induction step a triangulation $G$ with $n+1$ vertices. Let $G' = G - v$ denote the triangulation with one vertex $v$ removed (and zero, one, or two edges temporarily inserted such that $G'$ is again a triangulation), and let $C_k$ denote the $k$-cycle of neighbours $N(v)$ of $v$.

In the first step of the proof we extend the colouring of $G'$ to $G$. Euler’s polyhedral formula for planar graphs with $n > 3$ vertices assures that there is a $k$-valent vertex $v \in V(G)$, $k \in \{3, 4, 5\}$, as every triangulation has $3n - 6$ edges, which implies an average degree less than 6. In the second step it is proved that for every 5-cycle $C_5 \subseteq G$ with vertex-free interior, and every arbitrarily chosen pair of virtual chords, the Kempe chain condition holds.

We begin with the first step. First assume that $v$ is a 3-valent vertex. After colouring $G'$, add $v$ again and colour it with the colour not used by its neighbours.

Otherwise assume that $v$ is a 4-valent vertex. Then we follow the correct part of Kempe’s work [1]. If the 4-cycle $C_4 = v_1 \cdots v_4$ of the neighbours $N(v)$ is not yet 3-coloured, assume a colour sequence 1-2-3-4. By Jordan’s curve theorem, there cannot be Kempe chains $P_{13}$ (between $v_1$ and $v_3$) and $P_{24}$ (between $v_2$ and $v_4$) simultaneously. If $P_{13}$ does not exist, re-colour in $H_{13}(v_1)$. Otherwise it follows that $P_{24}$ does not exist, then re-colour in $H_{24}(v_2)$. Hence either colour 1 or 2 becomes free and can be used for $v$ to obtain a colouring of $G$.

Without 3- or 4-valent vertices in $V(G)$, there is a 5-valent vertex $v$, and $G' = G - v$. Then consider the cycle of its neighbours $C_5 = v_1 \cdots v_5$ as the chosen 5-cycle. All possible colourings of $C_5 \subseteq G'$ obtained from applying the induction hypothesis to $G'$ are (isomorphic to the colourings) shown in Fig. 2a. Among them there must be a nice colouring of $G'$ and $C_5$, such that for every chosen pair of virtual chords, the Kempe chain condition holds.

If this colouring is a colouring of $G'$ in which $C_5$ receives a 3-colouring (colour sequence e.g. 1-4-3-4-3), no Kempe chain exists with colours occurring on $C_5$ once, and $G$ is nicely 4-colourable without changing the colouring of $G'$.

Next assume that the nice colouring of $G'$ leads to a colour sequence 1-2-3-4-2 for $v_1, \ldots, v_5 \in V(C_5)$ (see Fig. 2b), i.e. both chords $v_1v_3$ and $v_1v_4$ have endpoint colours occurring

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Fig. 1: Two drawings of the triangulation $G$ with $n = 5$ and a 5-cycle $C_5$ with vertex-free interior.
once on $C_5$. Then choose a pair of virtual chords such that the temporarily inserted edges correspond to the virtual chords. By the induction hypothesis, at least one of the possible Kempe chains $P_{13}$ between $v_1$ and $v_3$, or $P_{14}$ between $v_1$ and $v_4$, does not exist in the exterior of $C_5 \subset G'$, say $P_{14}$ does not exist. Hence re-colour in $H_{14}(v_4)$ such that now $c(v_4) = 1$, and colour 4 can be used for $v$ to obtain a 4-colouring of $G$.

Figure 2a, 2b, 2c: A 5-cycle $C_5 \subset G'$ with $v_1v_3$ and $v_1v_4$ as temporary edges and all possible 4-colourings (Fig. 2a). Figs. 2b and 2c show special cases. Virtual chords are dashed.

Otherwise assume a colour sequence 1-2-3-4-3 (see Fig. 2c, or the analogous case 1-4-3-4-2 not shown graphically) as part of the nice colouring of $G'$. Then choose $v_4v_1$ and $v_4v_2$ as virtual chords. By the induction hypothesis it follows that possible Kempe chains $P_{14}$ and $P_{24}$ cannot exist both. If $P_{14}$ does not exist, continue as in the previous case. Otherwise assume $P_{14}$ exists, which excludes the existence of $P_{24}$ between $v_2$ and $v_4$. Then re-colour in $H_{24}(v_2)$ to obtain $c(v_2) = 4$, and colour 2 can be used for $v$ to obtain a 4-colouring of $G$.

The second step of the proof requires the extension of the nice colouring to $G$, i.e. to show the validity of the Kempe chain condition for every arbitrarily chosen 5-cycle $C_5 \subset G$ with vertex-free (but not edge-free) interior, and every arbitrarily chosen pair of virtual chords of $C_5$.

Assume a counterexample for a contradiction, i.e. there is a 5-cycle $C_5$ and there exist only colourings (at least one) of $G$ and $C_5$ such that for an arbitrarily chosen pair of virtual chords, two Kempe chains exist, each with both endpoint colours occurring once on $C_5$. Then $C_5$ must have a 4-colouring, say $c(v_i) = i$ for $v_i \in V(C_5)$, $i \in [4]$, and $c(v_5) = 2$ is the colour occurring twice (see Fig. 3a). Furthermore, as the only possible pair of virtual chords with endpoint colours occurring once on $C_5$ is $v_1v_3$ and $v_1v_4$, there must be two Kempe chains $P_{13}$ (between $v_1$ and $v_3$) and $P_{14}$ (between $v_1$ and $v_4$). Now we have found a 5-cycle $C_5$ and a pair of virtual chords with an invalid Kempe chain condition. However, it is easy to show that with the choice of virtual chords fixed, we always run into a contradiction.

As there is especially a Kempe chain $P_{13}$ between $v_1$ and $v_3$, it follows by Jordan’s curve theorem that there cannot be a Kempe chain $P_{24}$ between $v_2$ and $v_4$ in the exterior of $C_5$. 

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Case A. We first consider the case that a (real) chord $v_2v_4 \in E(G)$ does not exist. Then we may re-colour in $H_{24}(v_2)$ such that the colour 4 of $P_{14}$ (if this Kempe chain still exists) now occurs twice on $C_5$, as $c(v_4) = 4$ remains. Note that re-colouring in $H_{24}(v_2)$ can alternatively destroy $P_{14}$ and create a Kempe chain $P_{23}$ between $v_5$ and $v_3$ (this can be considered as the reason why Kempe’s proof attempt failed). Here we notice that $v_5$ and $v_3$ are not endpoints of a chord from the pair of chosen virtual chords (to be considered as fixed). In total we have $P_{13}$ together with either $P_{14}$ (colour 4 occurs twice) or no $P_{14}$ at all. Hence there is another colouring of $C_5$ and $G$ in which the Kempe chain condition holds for the chosen pair of virtual chords, a contradiction.

![Fig. 3a](image1.png) ![Fig. 3b](image2.png)

Figs. 3a-b. A 5-cycle $C_5$ with a chosen pair of virtual chords with endpoint colours occurring once on $C_5$. In Fig. 3b, edge $v_2v_4$ is a real chord.

Case B. Now suppose there is a (real) chord $v_2v_4$ (see Fig. 3b). With two existing Kempe chains, there is especially a Kempe chain $P_{14}$ between $v_1$ and $v_4$. It follows analogously that a Kempe chain $P_{23}$ between $v_3$ and $v_5$ cannot exist in the exterior of $C_5$. Hence re-colour in $H_{23}(v_5)$ such that the colour 3 of $P_{13}$ (if this Kempe chain still exists) occurs twice on $C_5$. This re-colouring is not possible in case there is a real chord $v_3v_5$. With this case excluded, we again obtain a contradiction, even if alternatively, $P_{13}$ is destroyed and a Kempe chain $P_{24}$ between $v_2$ and $v_4$ comes into existence, as it has other endpoints than the chosen pair of virtual chords. Note that here the aim of re-colouring in a component of $G$ is not to find a nice colouring, but to create a contradiction.

Case C. However, a chord $v_3v_5$ of $C_5$ cannot co-exist with a chord $v_2v_4$ of $C_5$ in a planar graph $G$. Hence there is no counterexample.

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