Two-symmetric Lorentzian manifolds

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Abstract

We classify two-symmetric Lorentzian manifolds using methods of the theory of holonomy groups. These manifolds are exhausted by a special type of pp-waves and, like the symmetric Cahen-Wallach spaces, they have commutative holonomy.

Keywords: Two-symmetric Lorentzian manifold, pp-wave, holonomy algebra, curvature tensor, parallel Weyl conformal curvature tensor

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1. Introduction

Symmetric pseudo-Riemannian manifolds constitute an important class of spaces. A direct generalization of these manifolds is provided by the so-called $k$-symmetric pseudo-Riemannian spaces $(M,g)$ satisfying the condition

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0,$$

where $k \geq 1$ and $R$ is the curvature tensor of $(M,g)$. For Riemannian manifolds, the condition $\nabla^k R = 0$ implies $\nabla R = 0$ [17]. On the other hand, there exist pseudo-Riemannian $k$-symmetric spaces with $k \geq 2$, see e.g. [13, 15, 1].

The fundamental paper by J.M. Senovilla [15] is devoted to a detailed investigation of two-symmetric Lorentzian spaces. It contains many interesting results about such manifolds and their physical applications. In particular, it is proven there that any two-symmetric Lorentzian space admits a parallel null vector field. A classification of four-dimensional two-symmetric Lorentzian spaces is obtained in the paper [1], in which it is shown that these spaces are some special pp-waves. The result is based on the Petrov classification of the Weyl tensors.

In the present paper we generalize the result of [1] to any dimension. The main result can be stated as follows.

**Theorem 1.** Let $(M,g)$ be a locally indecomposable Lorentzian manifold of dimension $n + 2$. Then $(M,g)$ is two-symmetric if and only if locally there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + (H_{ij}u + F_{ij})x^ix^j(du)^2,$$

where $H_{ij}$ is a nonzero diagonal real matrix with the diagonal elements $\lambda_1 \leq \cdots \leq \lambda_n$, and $F_{ij}$ is a symmetric real matrix.

Any other metric of this form isometric to $g$ is given by the same $H_{ij}$ and by $\tilde{F}_{ij} = cH_{ij} + F_{kj}a^k_i a^j_i$, where $c \in \mathbb{R}$ and $a^i_j$ is an orthogonal matrix such that $H_{kl}a^k_i a^l_j = H_{ij}$.

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By the Wu Theorem [19], any Lorentzian manifold \((M, g)\) is either locally indecomposable, or it is locally a product of a Riemannian manifold \((M_1, g_1)\), and of a locally indecomposable Lorentzian manifold \((M_2, g_2)\). The manifold \((M, g)\) is two-symmetric if and only if \((M_1, g_1)\) is locally symmetric and \((M_2, g_2)\) is two-symmetric. Consequently, Theorem 11 provides the complete local classification of two-symmetric Lorentzian manifolds.

For the proof of Theorem 1 we use the methods of the theory of holonomy groups. The assumption that a Lorentzian manifold \((M, g)\) is two-symmetric implies that the holonomy algebra \(\mathfrak{hol}_m\) of \((M, g)\) at a point \(m \in M\) annihilates the tensor \(\nabla R_m \neq 0\). This cannot happen if the holonomy algebra is the whole Lorentzian algebra \(\mathfrak{so}(1, n + 1)\). Hence the holonomy algebra must preserve a null line and is contained in the similitude algebra, \(\mathfrak{hol}_m \subset \mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \oplus \mathbb{R}^n\), the maximal Lie algebra with this property [8]. It is sufficient to consider the following two cases: \(\mathfrak{hol}_m = \mathfrak{h} + \mathbb{R}^n\), where \(\mathfrak{h} \subset \mathfrak{so}(n)\) is an irreducible subalgebra, and \(\mathfrak{hol}_m = \mathbb{R}^n\).

We prove that the first case is impossible: for this we calculate \(\nabla R\) and \(\nabla \text{Ric}\), and show that the Weyl conformal tensor \(W\) is parallel (\(\nabla W = 0\)). Then, using the results of A. Derdzinski and W. Roter [4,5] and of [10], we get a contradiction.

The second case corresponds to pp-waves. The condition \(\nabla^2 R = 0\) and simple computations allow us to find the coordinate form of the metric.

### 2. Holonomy groups of Lorentzian manifolds

We recall some basic facts about holonomy groups of Lorentzian manifolds that can be found in [6, 8, 14].

Let \((M, g)\) be a Lorentzian \(d\)-dimensional manifold and \(\text{Hol}^0(M) = \text{Hol}^0(M)_m\) its connected holonomy group at a point \(m \in M\). It is a subgroup of the (connected) Lorentz group \(\text{SO}(V)\) which is the tangent space and it is determined by its Lie algebra \(\mathfrak{hol}(M) \subset \mathfrak{so}(V)\) which is called the holonomy algebra of \(M\).

The manifold \((M, g)\) is locally indecomposable (i.e., locally is not a direct product of two pseudo-Riemannian manifolds) if and only if the holonomy group \(\text{Hol}^0(M)\) (or the holonomy algebra \(\mathfrak{hol}(M)\)) is weakly irreducible, i.e. it does not preserve any proper nondegenerate subspace of \(V\). Any weakly irreducible holonomy group \(\text{Hol}(M)\) different from the Lorentz group \(\text{SO}(V)^0\) is a subgroup of the horospheric group \(\text{SO}(V)_{[p]}\), the subgroup of \(\text{SO}(V)\) which preserves a null line \([p] = \mathbb{R} p\). This group is identified with the group \(\text{Sim}_n = \mathbb{R} \cdot \text{SO}_n \cdot \mathbb{R}^n, n = d - 2\) of similarity transformations of the Euclidean space \(E = \mathbb{R}^n\) as follows (see [6, Sect. 2.3]).

The Lorentzian group \(\text{SO}(V)^0\) acts transitively on the celestial sphere \(S^n = PV^0\) (the space of null lines) which is the projectivization of the null cone \(V^0 \subset V\) with the stabilizer \(\text{SO}(V)_{[p]}\). The stabilizer has an open orbit \(S^n \setminus [p]\) which is identified via the stereographic projection with the Euclidean space \(E\). The group \(\text{SO}(V)_{[p]}\) acts in \(E\) as the full connected Lie group of similarity transformations. Having in mind this isomorphism, we will call the group \(\text{SO}(V)_{[p]}\) the similarity group and denote it by \(\text{Sim}_n\).

Using the metric \(\langle \cdot, \cdot \rangle = g_m\), we will identify the Lorentz Lie algebra \(\mathfrak{so}(V) \simeq \mathfrak{so}(1, n + 1)\) with the space \(\Lambda^2V\) of bivectors. Then the Lie algebra \(\text{sim}_n\) of the similarity group can be written as

\[
\text{sim}_n = \mathfrak{so}(V)_{[p]} = \mathbb{R} p \wedge q + p \wedge E + \mathfrak{so}(E)
\]

where \(p, q\) are isotropic vectors with \(\langle p, q \rangle = 1\) which span 2-dimensional Minkowski subspace \(U\) and \(E = U^\perp\) is its orthogonal complement. The commutative ideal \(p \wedge E\) generates the commutative normal subgroup \(T_E \subset \text{Sim}_n\) which acts on \(E\) by translations. This group is called the vector group. The one-dimensional subalgebra \(\mathbb{R} p \wedge q \subset \mathfrak{so}(U)\) generates the maximal diagonal subgroup \(\mathbf{A}\) of \(\text{Sim}_n\) which is the Lorentz group \(\text{SO}(U)^0\) and the maximal compact subalgebra \(\mathfrak{so}(E)\) generates the group \(\text{SO}(E)\) of orthogonal transformations of \(E\). The above decomposition of the Lie algebra \(\text{sim}_n\) defines the Iwasawa decomposition

\[
\text{Sim}_n = K \cdot \mathbf{A} \cdot N = \text{SO}(E) \cdot \text{SO}(U)^0 \cdot T_E
\]

of the group \(\text{Sim}_n\). The list of connected weakly irreducible holonomy groups \(\text{Hol}^0(M)\) of Lorentzian manifolds is known, see [14, 6]. Assume for simplicity that \(\text{Hol}^0(M)\) is an algebraic group. Then it contains the vector group \(T_E\) and has one of the following forms: 2
(type I) \( \text{Hol}^0(M) = K \cdot \text{SO}(U)^0 \cdot T_E \)
(type II) \( \text{Hol}^0(M) = K \cdot T_E \), where \( K \subset \text{SO}(E) \) is a connected holonomy group of a Riemannian \( n \)-dimensional manifold, i.e., a product of the Lie groups from the Berger list: \( \text{SO}_k, U_k, SU_k, \text{Sp}_1 \cdot \text{Sp}_k, \text{Sp}_k, \text{G}_2, \text{Spin}_7 \) and the isotropy groups of irreducible symmetric Riemannian manifolds.

If the holonomy group is not algebraic, it is obtained from one of the holonomy groups of type I or II by some twisting (holonomy groups of type III and IV). Note that all these holonomy groups act transitively on the Euclidean space \( E = PV^0 \setminus [p] \).

The Lorentzian holonomy algebras \( g \subset \text{sim}(n) \) are the following:

- (type I) \( \mathbb{R}p \wedge q + h + p \wedge E \),
- (type II) \( h + p \wedge E \),
- (type III) \( \{ \varphi(A)p \wedge q + A| A \in h \} + p \wedge E \),
- (type IV) \( \{ A + p \wedge \psi(A)| A \in h \} + p \wedge E \),

where \( h \subset \text{so}(E) \) is a Riemannian holonomy algebra; \( \varphi : h \to \mathbb{R} \) is a non-zero linear map that is zero on the commutant \([h,h] \); for the last algebra \( E = E_1 \oplus E_2 \) is an orthogonal decomposition, \( h \) annihilates \( E_2 \), i.e., \( h \subset \text{so}(E_1) \), and \( \psi : h \to E_2 \) is a surjective linear map that is zero on the commutant \([h,h] \). The subalgebra \( h \subset \text{so}(E) \), i.e., the \( \text{so}(E) \)-projection of \( g \) is called the \textit{orthogonal part} of \( g \).

A locally indecomposable simply connected Lorentzian manifold admits a parallel null vector field if and only if its holonomy group is of type II or IV.

3. The holonomy group of a two-symmetric Lorentzian manifold

**Definition 1.** A pseudo-Riemannian manifold \((M,g)\) with the curvature tensor \( R \) is called a \textit{k}-symmetric space if

\[ \nabla^k R = 0, \quad \nabla^{k-1} R \neq 0. \]

So, one-symmetric spaces are the same as nonflat locally symmetric spaces \((\nabla R = 0, R \neq 0)\). Recall that a complete simply connected locally symmetric space is a symmetric space, that is it admits a central symmetry \( S_m \) with center at any point \( m \), i.e., an involutive isometry \( S_m \) which has \( m \) as an isolated fixed point.

Remark that for a Riemannian manifold the condition \( \nabla^k R = 0 \) implies \( \nabla R = 0 \) [17].

All indecomposable simply connected Lorentzian symmetric spaces are exhausted by the De Sitter and the anti De Sitter spaces and the Cahen-Wallach spaces, which have the vector holonomy group \( T_E \).

The following result is proven, using so-called casual tensors and the super-energy techniques, in [15].

**Theorem 2.** [13] Any two-symmetric Lorentzian manifold \((M,g)\) admits a parallel null vector field.

This implies that the holonomy group can be only of type II or IV. To make the exposition complete, we will sketch a proof of Theorem 2 using the holonomy theory.

The corner stone of the paper is the following statement.

**Theorem 3.** The holonomy group \( \text{Hol}^0(M) \) of an \((n+2)\)-dimensional locally indecomposable two-symmetric Lorentzian manifold \((M,g)\) is the vector group \( T_E \) with the Lie algebra \( p \wedge E \subset \text{so}(V) \).

It is known that any \((n+2)\)-dimensional Lorentzian manifold with the holonomy algebra \( p \wedge E \) is a pp-wave (see e.g. [8], Sect 5.4), i.e., locally there exist coordinates \( v, x^1, ..., x^n, u \) such that the metric \( g \) can be written in the form

\[ g = 2dvdu + \delta_{ij}dx^i dx^j + H(du)^2, \quad \partial_v H = 0. \]

We will need only to decide which functions \( H \) correspond to two-symmetric spaces.
3.1. Algebraic curvature tensors

For a subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ define the space of algebraic curvature tensors of type $\mathfrak{g}$.

$$\mathcal{R}(\mathfrak{g}) = \{ R \in \Lambda^2 V^* \otimes \mathfrak{g} | R(u,v)w + R(v,w)u + R(w,u)v = 0 \text{ for all } u,v,w \in V \}. $$

If $\mathfrak{g} \subset \mathfrak{so}(V)$ is the holonomy algebra of a manifold $(M,g)$, where $V = T_m M$ is tangent space at some point $m \in M$, then the curvature tensor $R_m$ of $(M,g)$ belongs to $\mathcal{R}(\mathfrak{g})$. The spaces $\mathcal{R}(\mathfrak{g})$ for holonomy algebras of Lorentzian manifolds are found in [7, 9]. For example, let $\mathfrak{g} = \mathbb{R}p \wedge q + h \wedge p \wedge E$. For a subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ define the space

$$\mathcal{P}(\mathfrak{h}) = \{ P \in E^* \otimes \mathfrak{h} | g(P(x)y,z) + g(P(y)z,x) + g(P(z)x,y) = 0 \text{ for all } x,y,z \in E \}. $$

Any $R \in \mathcal{R}(\mathfrak{g})$ is uniquely determined by the data $(\lambda, e, P, R^0, T)$, where

$$\lambda \in \mathbb{R}, \ e \in E, \ P \in \mathcal{P}(\mathfrak{h}), \ R^0 \in \mathcal{R}(\mathfrak{h}), \ T \in S^2 E,$$

i.e. $T$ is a symmetric tensor considered as an endomorphism of $E$. The tensor $R$ is defined by

$$R(p, q) = -\lambda p \wedge q - p \wedge e, \quad R(X, Y) = R^0(X, Y) - p \wedge (P(Y)X - P(X)Y),$$

$$R(X, q) = -g(e, X)p \wedge q + P(X) - p \wedge T(X), \quad R(p, X) = 0, \quad \forall X, Y \in E.$$

We will write

$$R = R^{(\lambda, e, P, R^0, T)}.$$ 

If some of these elements are zero, we omit them. For example, if $R$ is defined only by $T$, then we write $R = R^T$. Note that

$$R^T = \sum_{i,j} T_{ij} p \wedge e_i \vee p \wedge e_j, \quad T_{ij} = g(Te_i, e_j), \quad \mathcal{R}(p \wedge E) = \{ R^T | T \in S^2 E \} \simeq S^2 E,$$

where $e_1, \ldots, e_m$ is an orthonormal basis of $E$, and $\vee$ denotes the symmetric product. Similarly,

$$\mathcal{R}(h + p \wedge E) = \{ R^{(P, R^0, T)} | P \in \mathcal{P}(\mathfrak{h}), \ R^0 \in \mathcal{R}(\mathfrak{h}), \ T \in S^2 E \}.$$ 

Now we define the space of covariant derivatives of the curvature tensor

$$\nabla \mathcal{R}(\mathfrak{g}) = \{ S \in \text{Hom}(V, \mathcal{R}(\mathfrak{g})) \} = V^* \otimes \mathcal{R}(\mathfrak{g}) | S_u(v, w) + S_v(w, u) + S_w(u, v) = 0 \text{ for all } u, v, w \in V \}. $$

If $\mathfrak{g} \subset \mathfrak{so}(V)$ is the holonomy algebra of a manifold $(M,g)$ at a point $m \in M$, then $\nabla R_m \in \nabla \mathcal{R}(\mathfrak{g})$. The decomposition of the space $\nabla \mathcal{R}(\mathfrak{so}(r,s))$ into irreducible $\mathfrak{so}(r,s)$-modules is found in [10], see also [12].

It is not difficult to find the space $\nabla \mathcal{R}(\mathfrak{g})$ for each Lorentzian holonomy algebra $\mathfrak{g} \subset \mathfrak{sim}(n)$. It consists of tensors

$$S \in \text{Hom}(V, \mathcal{R}(\mathfrak{g})), \quad S : u \in V \mapsto S_u = R^{(\lambda_u, e_u, P_u, R^0_u, T_u)} \in \mathcal{R}(\mathfrak{g})$$

satisfying the second Bianchi identity. For example,

$$\nabla \mathcal{R}(p \wedge E) = \{ S = q' \otimes R^T | T \in S^2 E \} \oplus \{ S = R^{Q_x} | Q \in S^2 E \} \simeq S^2 E \oplus S^3 E,$$

where $q' = g(p, \cdot)$ is the 1-form $g$-dual to $p$, the tensor $S = R^{Q_x}$ is defined by $S_p = S_q = 0$, $S_x = R^{Q_x}$, $x \in E$, $Q_x \in S^2 E$ (since $Q \in S^3 E$).
3.2. Adapted coordinates and reduction lemma

Let \((M, g)\) be an \((n + 2)\)-dimensional locally indecomposable (hence with weakly irreducible holonomy algebra \(\mathfrak{g}\)) two-symmetric Lorentz manifold, i.e. the tensor \(\nabla R\) is nonzero, parallel and annihilated by the holonomy algebra. The space \(\nabla R(\mathfrak{so}(1, n + 1))\) does not contain nonzero elements annihilated by \(\mathfrak{so}(1, n + 1)\), see e.g. [10]. Since \(\mathfrak{so}(1, n + 1)\) is the only irreducible holonomy algebra [2], it follows that \(\mathfrak{g} \subset \mathfrak{sim}(n)\).

Let \((M, g)\) be a Lorentzian manifold with the holonomy algebra \(\mathfrak{g} \subset \mathfrak{sim}(n)\). Then \((M, g)\) admits a parallel distribution of null lines. According to [18], locally there exist so called Walker coordinates \(v, x^1, \ldots, x^n, u\) such that the metric \(g\) has the form

\[
g = 2dvdu + h + 2Adu + H(du)^2,
\]

where \(h = h_{ij}(x^1, \ldots, x^n, u)dx^idx^j\) is an \(u\)-dependent family of Riemannian metrics, \(A = A_i(x^1, \ldots, x^n, u)dx^i\) is an \(u\)-dependent family of one-forms, and \(H = H(v, x^1, \ldots, x^n, u)\) is a local function on \(M\). Consider the local frame

\[
p = \partial_v, \quad X_i = \partial_i - A_i\partial_v, \quad q = \partial_u - \frac{1}{2}H\partial_v.
\]

Let \(E\) be the distribution generated by the vector fields \(X_1, \ldots, X_n\). Clearly, the vector fields \(p, q\) are isotropic, \(g(p, q) = 1\), the restriction of \(g\) to \(E\) is positive definite, and \(E\) is orthogonal to \(p\) and \(q\). The vector field \(p\) defines the parallel distribution of null lines and it is recurrent, i.e. \(\nabla p = \theta \otimes p\), where \(\theta = \frac{1}{2}\partial_vHdu\).

Since the manifold is locally indecomposable, any other recurrent vector field is proportional to \(p\). Next, \(p\) is proportional to a parallel vector field if and only if \(dh = 0\), which is equivalent to \(\partial_v^2H = \partial_v\partial_uH = 0\). In the last case the coordinates can be chosen in such a way that \(\partial_vH = 0\) and \(\nabla p = \nabla \partial_u = 0\), see e.g. [4].

Let \(\mathfrak{h} \subset \mathfrak{sim}(n)\) be the holonomy algebra of the Lorentzian manifold \((M, g)\) and \(\mathfrak{h} \subset \mathfrak{so(E)}\) be its orthogonal part. Then there exist the decompositions

\[
E = E_0 \oplus E_1 \oplus \cdots \oplus E_r, \quad \mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r
\]

such that \(\mathfrak{h}\) annihilates \(E_0\), \(\mathfrak{h}_i(E_j) = 0\) for \(i \neq j\), and \(\mathfrak{h}_i \subset \mathfrak{so(E}_i)\) is an irreducible subalgebra for \(1 \leq i \leq s\). Ch. Boubel [2] proved that there exist Walker coordinates

\[
v, \quad x_0 = (x_0^1, \ldots, x_0^n), \quad \ldots, \quad x_r = (x_r^1, \ldots, x_r^n), \quad u
\]

adapted to the decomposition (3.2). This means that

\[
h = h_0 + h_1 + \cdots + h_r, \quad h_0 = \sum_{i=1}^{n_0}(dx_0^i)^2, \quad h_\alpha = \sum_{i,j=1}^{n_\alpha} h_{\alpha ij}dx_i^jdx_j, \quad (3.3)
\]

\[
A = \sum_{\alpha=1}^{r} A_\alpha, \quad A_0 = 0, \quad A_\alpha = \sum_{k=1}^{n_\alpha} A_\alpha^k dx^k, \quad (3.4)
\]

and one has

\[
\frac{\partial}{\partial x^k_\beta}h_{\alpha ij} = \frac{\partial}{\partial x^k_\beta}A_\alpha^0 = 0, \quad \text{if} \ \beta \neq \alpha.
\]

The coordinates can be chosen so that \(A = 0\), see [11]. Thus we will assume that \(g\) is given by (3.1) with \(A = 0\), and with \(h\) satisfying (3.3) and (3.4).

For \(\alpha = 0, \ldots, r\), consider the submanifolds \(M_\alpha \subset M\) defined by \(x_\beta = c_\beta, \ \alpha \neq \beta\), where \(c_\beta\) are constant vectors. Then the induced metric is given by

\[
g_\alpha = 2dvdu + h_\alpha + H_\alpha(du)^2.
\]

**Lemma 1.** The submanifold \(M_\alpha \subset M\) is totally geodesic. The orthogonal part of the holonomy algebra \(\mathfrak{g}_\alpha\) of the metrics \(g_\alpha\) coincides with \(\mathfrak{h}_\alpha \subset \mathfrak{so}(E_\alpha)\), which is irreducible for \(\alpha = 1, \ldots, r\). If the metric \(g\) is two-symmetric, then the curvature tensor of each metric \(g_\alpha\) satisfies \(\nabla^2 R = 0\).
Proof. The non-zero Christoffel symbols of the metric (3.1) with \( A = 0 \) are the following:

\[
\Gamma^u_{uu} = \frac{1}{2} H_u, \quad \Gamma^u_{uv} = \frac{1}{2} H_v, \quad \Gamma^v_{uu} = \frac{1}{2} H_v, \quad \Gamma^i_{ju} = \frac{1}{2} h_{ik} h_{jk,u},
\]

where the comma denotes the partial derivative and \( \Gamma^i_{jk}(h) \) are the Christoffel symbols of the metric \( h \). This shows that the Christoffel symbols of the metric \( g \), where the corresponding Christoffel symbols of the metric \( g \), i.e. each submanifold \( M_\alpha \subset M \) is totally geodesic. This implies that if \( \nabla^2 R = 0 \), then each \( g_\alpha \) satisfies the same condition. Finally, the statement about the orthogonal parts follows from the fact that the orthogonal part of any Walker metric \( g \) coincides with the holonomy algebra of the induced connection on the vector bundle with the fibers \( p_m/\mathbb{R}p_m \simeq E_m \), and this connection does not depend on the function \( H \). \[ \square \]

3.3. Sketch of the proof of Theorem 2 using the holonomy theory

We may assume that the metric \( g \) is locally given by (3.1) with \( A = 0 \), and with \( h \) satisfying (3.3) and (3.4). As it is noted above, it is enough to prove that \( \partial^2 v H = \partial_i \partial_v h = 0 \). Clearly, this will be true if it is true for each metric \( g_\alpha \).

Lemma 2. If \( \mathfrak{g} \) is the holonomy algebra of type I (with any orthogonal part \( \mathfrak{h} \subset \mathfrak{so}(E) \)), or \( \mathfrak{g} \) is the holonomy algebra of type III with an irreducible orthogonal part \( \mathfrak{h} \subset \mathfrak{so}(E) \), then the subspace \( \nabla R(\mathfrak{g})^0 \subset \nabla R(\mathfrak{g}) \), consisting of tensors annihilated by \( \mathfrak{g} \), is trivial.

Proof. If \( \mathfrak{g} \) is of type I, then it contains \( A = p \wedge q \). If \( \mathfrak{g} \) is of type III, then \( \mathfrak{h} \subset \mathfrak{u}(E) \subset \mathfrak{so}(E) \) and for some \( \alpha \in \mathbb{R} \), the element \( A = p \wedge q + \alpha J \) belongs to \( \mathfrak{g} \). The lemma follows from the consideration of the tensors in \( \nabla R(\mathfrak{g}) \) annihilated by the operator \( A \) and the second Bianchi identity as in Lemma 3 below. \[ \square \]

The lemma shows that the holonomy algebra of each metric \( g_\alpha \) cannot be of type I or III, i.e. it is of type II or IV. Thus, \( \partial^2 v H = \partial_i \partial_v h = 0 \) holds. \[ \square \]

3.4. Proof of Theorem 3

Consider the decomposition (3.2). If \( E = E_0 \), then \( \mathfrak{h} = 0 \) and there is nothing to prove. If \( E_1 \neq 0 \), then the metric \( g_1 \) satisfies \( \nabla^2 R = 0 \) and the orthogonal part of its holonomy algebra \( \mathfrak{h}_1 \subset \mathfrak{so}(E) \) is irreducible. We will show that this is not possible. Thus, suppose that \( (M,g) \) satisfies \( \nabla^2 R = 0 \) and its holonomy algebra equals to \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{e}(E) \), where \( \mathfrak{h} \subset \mathfrak{so}(E) \) is irreducible.

Lemma 3. Let \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{e}(E) \), where \( \mathfrak{h} \subset \mathfrak{so}(E) \) is irreducible. Then the subspace \( \nabla R(\mathfrak{g})^0 \subset \nabla R(\mathfrak{g}) \) of \( \mathfrak{g} \)-annihilated tensors is the one-dimensional subspace given by

\[
\nabla R(\mathfrak{g})^0 = \mathbb{R}S, \quad S = q' \otimes R^{14e}, \quad q' = g(p, \cdot).
\]

Proof. Let \( S \in \nabla R(\mathfrak{g})^0 \). For any \( v \in V \), the element \( S_v \in \mathcal{R}(\mathfrak{g}) \) can be written as \( S_v = R(R_0, v, T_v) \) as it is explained above. Since \( S(p, \cdot) = 0 \), by the second Bianchi identity \( S_v = 0 \). The fact that \( \mathfrak{g} \) annihilates \( S \) can be expressed as

\[
[A, S_v](v_1, v_2)] - S_{Av_2}(v_1, v_2) - S_{v_2}(Av_1, v_2) - S_{v_2}(v_1, Av_2) = 0
\]

for all \( A \in \mathfrak{g} \) and \( v_1, v_2, v_3 \in V \). Let \( U, X, Y, Z \in E \). We have

\[
[p \wedge X, S_U(Y, Z)] = 0.
\]

Hence, \( R^0_U(Y, Z)X = 0 \), i.e. \( R^0_U = 0 \). Next,

\[
[p \wedge X, S_Z(Y, q)] - S_Z(Y, X) = 0.
\]
Consequently,
\[-p \wedge P_Z(Y)X - p \wedge (P_Z(Y)X - P_Z(X)Y) = 0,
\]
i.e. $2P_Z(Y)X = P_Z(X)Y$. Since this equality holds for any $X, Y \in E$, we conclude $P_Z = 0$. We have got $S_Z(X, Y) = 0$. Similarly,
\[[p \wedge X, S_q(Y, Z)] = 0,
\]
i.e. $R_q^0 = 0$. The equality
\[[p \wedge X, S_q(Y, q)] - S_X(Y, q) - S_q(Y, X) = 0
\]
implies
\[T_X(Y) = 2P_q(Y)X - P_q(X)Y.
\]
From the second Bianchi identity
\[S_q(X, Y) + S_X(Y, q) + S_Y(q, X) = 0
\]
it follows that
\[T_X(Y) - T_Y(X) = P_q(X)Y - P_q(Y)X.
\]
We conclude $P_q(Y)X - P_q(X)Y = 0$. This and the definition of the space $\mathcal{P}(h)$ imply $P_q = 0$. Consequently, $T_X = 0$. Finally, let $A \in h$, then
\[[A, S_q(X, q)] - S_q(AX, q) = 0.
\]
This implies $AT_q(X) = T_q(AX)$, i.e. $T_q$ commutes with $h$. Since $T_q$ is a symmetric endomorphism of $E$ and $h \subset \mathfrak{so}(E)$ is irreducible, by the Schur Lemma, $T_q$ is proportional to the identity. This proves the lemma. \(\square\)

We write the metric $g$ in the form (3.1). Then $\partial_t$ is parallel and $\partial_t H = 0$.

By Lemma 3 $\nabla R$ has the form
\[\nabla U R = fg(p, U)R^{Id\times}, \quad \forall U \in TM,
\]
for some smooth function $f$. It is clear that
\[R^{Id\times}(U_1, U_2) = p \wedge ((U_1 \wedge U_2)p), \quad \forall U_1, U_2 \in TM.
\]

**Lemma 4.** Under the above assumptions, the conformal Weyl curvature tensor $W$ is parallel, i.e. $\nabla W = 0$.

**Proof.** It is known that
\[W = R + L \wedge g,
\]
where
\[L = \frac{1}{d-2}\left(\text{Ric} - \frac{s}{2(d-1)}\text{Id}\right)
\]
is the Schouten tensor, $\text{Ric}$ is the Ricci operator, and $s$ is the scalar curvature. Recall that by definition,
\[(L \wedge g)(U_1, U_2) = LU_1 \wedge U_2 + U_1 \wedge LU_2, \quad U_1, U_2 \in TM.
\]

For any vector field $U$ it holds
\[\nabla U W = \nabla U R + (\nabla U L) \wedge g.
\]
Let the indexes $a, b$ run from 0 to $n + 1$, and let $X_0 = p$, $X_{n+1} = q$. The covariant derivative of the Ricci operator is given by
\[
\begin{align*}
(\nabla_{U_1}\text{Ric})U_2 &= g^{ab}\nabla_{U_1}R(U_2, X_a)X_b = g^{ab}fg(p, U_1)R^{Id\times}(U_2, X_a)X_b = g^{ab}fg(p, U_1)(p \wedge ((U_2 \wedge X_a)p))X_b \\
&= fg(p, U_1)(g^{ab}g(p, X_b)(U_2 \wedge X_a)p - g^{ab}g((U_2 \wedge X_a)p, X_b)p) \\
&= fg(p, U_1)((U_2 \wedge p)p - g^{ab}g(U_2, p)X_a - g(X_a, p)U_2, X_b)p) = (2 - d)fg(p, U_1)g(p, U_2)p.
\end{align*}
\]
Thus, \((\nabla_{U_1}Ric)U_2 = -ng(f(p, U_1)g(p, U_2)p).\) The gradient of the scalar curvature is given by
\[
g(\text{grads}, U_1) = g^{ab}g((\nabla_{U_1}Ric)X_a, X_b) = 0,
\]
i.e. \(\text{grads} = 0.\) Hence,
\[
(\nabla_{U_1}L)U_2 = -fg(p, U_1)g(p, U_2)p.
\]
Consequently,
\[
(\nabla_{U_1}L)U_2 \wedge U_3 + U_2 \wedge (\nabla_{U_1}L)U_3 = -fg(p, U_1)g(p, U_2)p \wedge U_3 - U_2 \wedge f(g(p, U_1)g(p, U_3)p
\]
\[
= fg(p, U_1)(p \wedge g(p, U_3)U_2 - p \wedge g(p, U_2)U_3) = -fg(p, U_1)p \wedge ((U_2 \wedge U_3)p) = -\nabla_{U_1}R(U_2, U_3).
\]
Thus, \((\nabla_{U_1}L) \wedge g = -\nabla_{U_1}R\) and \(\nabla W = 0.\)
\[\square\]

**4. Lorentzian manifolds with vector holonomy group \(T_E\) (pp-waves)**

In this section we derive formulas for the curvature tensor and its covariant derivatives for an \((n + 2)\)-dimensional Lorentzian manifold with the vector holonomy group \(\text{Hol}(M) = T_E\) (or, equivalently, the holonomy algebra \(\mathfrak{so}(M) = p \wedge E\)).

**4.1. Adapted local coordinates and associated pseudo-group of transformations**

It is well known that the connected holonomy group of a Lorentzian manifold \((M, g)\) is a subgroup of \(T_E\) if and only if in a neighborhood of any point \(x \in M\) with respect to some local coordinates \(v, x^1, \ldots, x^n, u\) (called adapted coordinates) the metric is given by
\[
g = 2dudv + \delta_{ij}dx^idx^j + Hud^2,
\]
where \(H\) is a function of \(x^j\) and \(u\), see e.g. [3, Sect. 5.4]. Such Lorentzian manifolds are called pp-waves.

**Lemma 5.** Any two adapted coordinate systems with the same \(\partial_v\) are related by
\[
\tilde{v} = v - \sum a^i_j \frac{d\phi^i(u)}{du}x^i + d(u), \quad \tilde{x}^i = a^i_j x^j + b^i(u), \quad \tilde{u} = u + c,
\]
where \(c \in \mathbb{R}\), \(a^i_j\) is an orthogonal matrix, and \(b^i(u), d(u)\) are arbitrary functions of \(u\).

**Proof.** In [11] it is shown that two Walker systems of coordinates with the same \(\partial_v\) are related by
\[
\tilde{v} = v + h(x^1, \ldots, x^n, u), \quad \tilde{x}^i = \phi^i(x^1, \ldots, x^n, u), \quad \tilde{u} = u + c.
\]

Since \(h = \delta_{ij}dx^idx^j\) must be preserved, \(\phi^i(x^1, \ldots, x^n, u)\) must define an \(u\)-dependent family of isometries of \(\mathbb{R}^n\), i.e.
\[
\phi^i(x^1, \ldots, x^n, u) = a^i_j(u)x^j + b^i(u),
\]
where \(a^i_j(u)\) is a family of orthogonal matrices. Next, the equalities \(g(\partial_i, \partial_u) = g(\partial_i, \partial_u) = 0\) imply
\[
\partial_i f + \sum a^k_j(u) \frac{d}{du}(a^k_j(u)x^r + b^k(u)) = 0.
\]
This shows that \(\sum a^k_j(u)\frac{d}{du}a^k_j(u) = 0\), i.e. \(\frac{d}{du}a^k_j(u) = 0\). Finally, we easily find the function \(f\). \[\square\]
4.2. Levi-Civita connection

We associate with an adapted coordinate system \((u, x^i, v)\) of a pp-wave space \((M, g)\) with a potential \(H = H(x^i, u)\) a standard field of frames

\[
p = \partial_v, \quad e_i = \partial_i, \quad q = \partial_u - \frac{1}{2} H \partial_v
\]

and the dual field of coframes

\[
p' = dv + \frac{1}{2} H du, \quad e^i = dx^i, \quad q' = du.
\]

The Gram matrix of these bases is given by

\[
G = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

We will consider coordinates of all tensor fields with respect to these non-holonomic frame and coframe.

Then the covariant derivative of a vector \(Y = Y^p p + Y^i e_i + Y^q q\) and a covector \(\omega = \omega^p p' + \omega^i e^i + \omega^q q'\) in direction of a vector field \(X\) can be written as

\[
\nabla_X Y = \partial_X Y + A_X Y, \quad \nabla_X \omega = \partial_X \omega - A^T_X \omega
\]

where \(\partial_X\) is the coordinate derivative in direction of \(X\) and \(A_X\) is a matrix and \(A^T_X\) is the transposed matrix.

Lemma 6. The matrices \(A_u, A_i, A_v\) of the connection which correspond to the coordinate vector fields \(\partial_u, \partial_i, \partial_v\) and their transposes are given by

\[
A_u = \begin{pmatrix}
0 & -\frac{1}{2} H_{i}\ \\
0 & 0 \\
-\frac{1}{2} H_{i} & 0
\end{pmatrix}, \quad A^T_u = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{1}{2} H_i & 0 \\
\frac{1}{2} H_i & 0 & 0
\end{pmatrix}, \quad A_i = A^T_i = A_v = A^T_v = 0.
\]

In particular, \(\nabla p = \nabla p' = 0\).

Proof: The only non zero Christoffel symbols are

\[
\Gamma^u_{uu} = \frac{1}{2} H_{u}, \quad \Gamma^i_{uu} = -\frac{1}{2} H_{i}, \quad \Gamma^v_{uu} = \frac{1}{2} H_{v}
\]

where the commas stand for the partial derivatives. Then we calculate

\[
\nabla_\partial_u = \nabla p = 0, \quad \nabla_u \partial_i = \frac{1}{2} H_{i} p, \quad \nabla_u q = \nabla_u (\partial_u - \frac{1}{2} H \partial_v) = \frac{1}{2} H_{u} p - \frac{1}{2} H_{i} e_i - \frac{1}{2} H_{u} p = -\frac{1}{2} H_{i} e_i,
\]

\[
\nabla_i \partial_j = 0, \quad \nabla_i \partial_u = \frac{1}{2} H_{i} p, \quad \nabla_i q = \nabla_i (\partial_u - \frac{1}{2} H \partial_v) = 0, \quad \nabla_v \partial_u = \nabla_v \partial_i = \nabla_v \partial_v = 0.
\]

\[\square\]

Corollary 1. A Lorentzian manifold \(M\) with vector holonomy group \(\text{Hol}(M) = T_E\) has the (globally defined) parallel vector field \(p = \partial_v\) and parallel 1-form \(q' = du\).
4.3. The curvature tensor of a pp-wave space

**Lemma 7.** With respect to the standard frame $p = \partial_v, e_i = \partial_i, q = \partial_u - \frac{1}{2} H \partial_v$ and the dual coframe $p', e^i, q'$, the curvature tensor of a pp-wave with potential $H(u,x^i)$ is given by

$$R = \sum_{i,j} \frac{1}{2} H_{ij}(p \wedge e_i \vee p \wedge e_j) \quad \text{(the contravariant curvature tensor)}$$

$$\bar{R} = \frac{1}{2} H_{ij}(q' \wedge e^i \vee q' \wedge e^j) \quad \text{(the covariant curvature tensor)}.$$

**Proof:** It follows from the formula $R(X,Y) = \partial_X A_Y - \partial_Y A_X - A_{[X,Y]}$. 

**Corollary 2.** The Ricci tensor of $M$ is given by

$$\text{ric} = \frac{1}{2} \Delta H q' \otimes q' = \frac{1}{2} \Delta H du^2$$

where $\Delta$ is the Laplacian in $\mathbb{R}^n$.

4.4. The covariant derivatives of the curvature tensor

Note that for any $i,j$, the covariant tensor $q' \wedge e^i \vee q' \wedge e^j$ and the contravariant tensor $p \wedge e_i \vee p \wedge e_j$ are parallel. Hence the first covariant derivative of the curvature tensor is given by

$$\nabla \bar{R} = \frac{1}{2} H_{ijk} e^k \otimes (q' \wedge e^i \vee q' \wedge e^j) + \frac{1}{2} H_{ij} q' \otimes (q' \wedge e^i \vee q' \wedge e^j).$$

(4.3)

**Corollary 3.** The manifold $(M,g)$ is a locally symmetric space if and only if the Hessian $H_{ij}$ of the potential $H$ is a constant, that is $H = H_{ij} x^i x^j + G_i(u)x^i + K(u)$.

It can be shown that in the last case the coordinates can be chosen in such a way that $H = \lambda_1(x^1)^2 + \cdots + \lambda_n(x^n)^2$ for some non-zero real numbers $\lambda_i$ such that $\lambda_1 \leq \cdots \leq \lambda_n$.

The second covariant derivative of the curvature tensor is given by

$$\nabla^2 \bar{R} = \left( \frac{1}{2} H_{ijk} - \frac{1}{4} \sum_k H_{ik} H_{ijk} \right) q'^2 \otimes (q' \wedge e^i \vee q' \wedge e^j)$$

$$+ \frac{1}{2} H_{ijk} q' \otimes (q' \wedge e^i \vee q' \wedge e^j) + \frac{1}{2} H_{ij} e^k \otimes (e^k \otimes e^j) \otimes (q' \wedge e^i \vee q' \wedge e^j).$$

(4.4)

This implies the following.

**Theorem 4.** A pp-wave with the metric $(4.1)$ is two-symmetric if and only if

$$H = (uH_{ij} + F_{ij})x^i x^j + G_i(u)x^i + K(u),$$

where $H_{ij}$ and $F_{ij}$ are symmetric real matrices, the matrix $H_{ij}$ is non-zero, $G_i(u)$ and $K(u)$ are functions of $u$.

5. Proof of Theorem 1

To prove the theorem we start with the metric $(4.1)$ and $H$ as in Theorem 1 and use transformation $(4.2)$ in order to write the metric as in Theorem 1. Let $v, \tilde{x}^1, \ldots, \tilde{x}^n, \tilde{u}$ be a new coordinate system. We may assume that the inverse transformation is given by

$$u = \tilde{u} + c, \quad x^i = a^i_j \tilde{x}^j + b^i(\tilde{u}), \quad v = \tilde{v} - \sum_j a^i_j \frac{db^j(\tilde{u})}{d\tilde{u}} \tilde{x}^i + d(\tilde{u}).$$

(5.1)
For the new function $\tilde{H}$ written as in Theorem 1 we get

$$\tilde{H}_{kl} = H_{ij}a^i_ka^j_l,$$

$$\tilde{F}_{kl} = (cH_{ij} + F_{ij})a^i_ka^j_l,$$  \hspace{1cm} (5.2)

$$\tilde{G}_k(\tilde{u}) = -2\sum_j a^i_k \frac{dd^b_j}{d\tilde{u}} + 2((\tilde{u} + c)H_{ij} + F_{ij})b^i_j + G_iG^i,$$  \hspace{1cm} (5.3)

$$\tilde{K}(\tilde{u}) = 2\frac{dd(\tilde{u})}{d\tilde{u}} + \sum_j \left(\frac{dd^b_j}{d\tilde{u}}\right)^2 + ((\tilde{u} + c)H_{ij} + F_{ij})b^i_j + G_i b^j + K.$$  \hspace{1cm} (5.4)

Equation (5.4) implies the existence of $b^j(\tilde{u})$ such that $\tilde{G}_k = 0$. Using the last equation, we can chose $d(\tilde{u})$ such that $K = 0$. Equation (5.2) implies the existence of an orthogonal matrix $a^i_k$ such that $\tilde{H}_{kl}$ is a diagonal matrix with the diagonal elements $\lambda_1, ..., \lambda_n$ such that $\lambda_1 \leq \cdots \leq \lambda_n$.

Since $\nabla R \neq 0$, Corollary 2 shows that $H_{ij}$ is not zero.

The transformation (5.1) does not change the form of the metric from Theorem 1 if and only if $H_{kl}a^k_i a^l_j = H_{ij}$ and $b^j(\tilde{u}), d(\tilde{u})$ satisfy certain conditions. This and (5.3) prove the last claim of the theorem. \hfill $\Box$

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