ON ROTATING STAR SOLUTIONS TO NON-ISENTPROIC EULER-POISSON EQUATIONS

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Abstract. The rotating star solutions to the Euler-Poisson equations are considered with a non-isentropic equation of state. As a first step, the equation for gas density with a prescribed entropy and angular velocity distribution is studied. The resulting elliptic equation is solved either by the method of sub- and supersolutions or by a variational method, depending on the value of the polytropic index. The reversed problem of determining angular velocity given gas density is also considered.

1. Introduction

A Newtonian star is modelled by a body of fluids satisfying the Euler-Poisson equations in three spatial dimensions:

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p &= -\rho \nabla \phi
\end{align*}
\]

Here \(\rho, \mathbf{v}, p\) are the density, velocity, and pressure of the fluids. \(\phi\) is the Newtonian potential of \(\rho\) given by

\[
\phi(x) = -G \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy.
\]

Here \(G\) is Newton’s universal constant of gravitation.

There is a long history of investigation of stationary solutions to (1). As a first step, people considered non-rotating stars, which can be interpreted as spherically symmetric stationary solutions to (1). The theory of non-rotating stars culminated in the famous Lane-Emden equations, and is described in detail in Chandrasekhar’s classical work [1]. A more careful approximation to realistic stars involves rotation. The rotating stars, under the light of this model, are axisymmetric stationary solutions to (1), whose velocity field contains only the azimuthal component. In cylindrical coordinates \((r, \theta, z)\), the assumptions on rotating star solutions can be formulated as setting \(\rho, p, \mathbf{v}\) to be functions of \(r, z\) only, and \(\mathbf{v}\) to contain only the \(e_\theta\) component. Here \(e_r, e_\theta, e_z\) are the unit vectors in cylindrical coordinates. With these assumptions, the mass conservation equation in (1) is identically satisfied. Writing \(\mathbf{v}(r, z) = r\Omega(r, z)e_\theta\), the momentum conservation equation in (1) becomes

\[
\begin{align*}
p_r &= \rho(-\phi)_r + pr\Omega^2 \\
p_z &= \rho(-\phi)_z
\end{align*}
\]
or
\[
\nabla p \rho = -\nabla \phi + r\Omega^2 e_r.
\]

Auchmuty and Beals \cite{2} initiated the search for solutions to \( (3) \). They prescribed the angular velocity distribution and the total mass, to solve for the density function \( \rho \).

In order to close the underdetermined system \( (1) \), they assumed isentropic equation of state:
\[
(5) \quad p = g(\rho)
\]
for some given function \( g \) (Note that condition \( (5) \) is more often called barotropic equation of state in physics literature). Notice that if one writes \( \phi \) as \( (2) \), and \( p \) as \( (5) \), then \( (3) \) consists of two equations for one unknown function \( \rho \), thus appears to be overdetermined. In order to understand how this problem is well posed, let us take the curl of \( (4) \):
\[
(6) \quad \nabla \times \left( \frac{\nabla p}{\rho} \right) = \nabla \times (r\Omega^2 e_r),
\]
which then simplifies to
\[
(7) \quad \frac{\nabla p \times \nabla \rho}{\rho^2} = r \frac{\partial \Omega^2}{\partial z} e_\theta.
\]

\( (7) \) implies the following important proposition:

**Proposition 1.** Both sides of \( (4) \) are curl free if and only if \( \nabla p \times \nabla \rho = 0 \) if and only if \( \Omega^2 \) depends only on \( r \).

Proposition \( (1) \) shows that if \( p = g(\rho) \), then \( \Omega^2 \) can only depend on \( r \). On the other hand, if one prescribes \( \Omega^2 \) to be a function of \( r \) alone, and \( p = g(\rho) \) for some given \( g \), then the curl of \( (4) \) is identically satisfied. In this case, every term in \( (4) \) has zero curl, and is in fact a gradient. \( (4) \) can be written as
\[
(8) \quad \nabla \left( A(\rho) \right) = -\nabla \phi + \nabla J,
\]
where
\[
(9) \quad A(s) = \int_0^s \frac{g'(t)}{t} dt
\]
and
\[
(10) \quad J(r) = \int_0^r s\Omega^2(s) ds.
\]

Stripping off the gradient, one gets
\[
(11) \quad A(\rho) = -\phi + J(r) + C
\]
for some constant \( C \). With \( \Omega, g \) prescribed, and \( \phi \) given as \( (2) \), \( (11) \) is a single equation for \( \rho \). It is this equation that Auchmuty and Beals worked on with a variational method.

If we allow \( \Omega^2 \) to depend both on \( r \) and \( z \), proposition \( (1) \) forces us to use a non-isentropic equation of state:
\[
(12) \quad p = g(\rho, s)
\]
where $s$ is entropy. The full Euler-Poisson system has another equation for energy conservation, which we have been ignoring until now:

$$
\frac{1}{2} \rho |v|^2 + \rho e + \nabla \cdot \left( \frac{1}{2} \rho |v|^2 + \rho e v \right) = -\rho \nabla \phi \cdot v - \nabla \cdot (\rho v).
$$

Here $e$ is specific thermal energy. By the second law of thermodynamics,

$$
de = T(\rho, s) ds + \frac{p(\rho, s)}{\rho^2} d\rho,
$$

therefore

$$
e(\rho, s) = \int_0^\rho \frac{g(\xi, s)}{\xi^2} d\xi.
$$

By subtracting the mass and momentum conservation equations in (1) from (13) and simplifying, we get

$$
\rho e_t + \rho v \cdot \nabla e = -p \nabla \cdot v.
$$

By (14), this becomes

$$
\rho T \left( s_t + v \cdot \nabla s \right) + \frac{p}{\rho} (\rho_t + v \cdot \nabla \rho + \rho \nabla \cdot v) = 0
$$

$$
\rho T (s_t + v \cdot \nabla s) = 0
$$

$$
s_t + v \cdot \nabla s = 0.
$$

Notice that we have used the mass conservation equation again to get the penultimate step. We can combine (17) with (1) to get the full Euler-Poisson system

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho v) = 0 \\
(\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p = -\rho \nabla \phi \\
s_t + v \cdot \nabla s = 0
\end{cases}
$$

with Newtonian potential (2) and non-isentropic equation of state (12).

The study of rotating star solutions to (18) is a new problem and demands further investigation. As before, we assume $\rho, p, v, s$ to depend only on $r$ and $z$, and $v$ to contain only the $e_\theta$ component. Under these assumptions, the mass conservation equation and entropy transport equation in (13) are identically satisfied, therefore we again arrive at equations (3) or (4). The difference with the classical Auchmuty and Beals case is that, by proposition 1, neither sides of (4) has vanishing curl in general, and the individual terms in (11) are no longer gradients of other functions. It is not obvious how one can recast (4) as the Euler-Lagrange equation of some energy functional as in the classical Auchmuty and Beals case.

With the angular velocity distribution prescribed and $\phi$ given as (2), system (3) is a set of two equations for two unknowns $\rho$ and $p$. Different from the isentropic case, the introduction of the equation of state (12) does not decrease the number of unknowns, but merely transforms the unknowns to $\rho$ and $s$. For definiteness, from now on we will use the equation of state:

$$
p = g(\rho, s) = e^s \rho^\gamma.
$$
Here $\gamma$ is a constant, and $e = \exp(1)$ is base of natural log. Let us give an explicit warning that we have employed the same letter for specific energy, but the confusion should be minimal by watching the contexts they appear in. This equation of state is very general, it for instance incorporates arbitrary thermal processes of an ideal gas. For more motivation of this equation of state, see Courant and Friedrichs [7]. Inserting (19) into (3) or (4), we get

$$
\begin{cases}
(e^s \rho^\gamma)_r = \rho(\phi)_r + \rho r \Omega^2 \\
(e^s \rho^\gamma)_z = \rho(\phi)_z
\end{cases}
$$

or equivalently,

$$\frac{\nabla (e^s \rho^\gamma)}{\rho} = -\nabla \phi + r \Omega^2 e_r.$$  

This is a system of two equations for two unknowns $(\rho, s)$. The search for solutions to (20) with prescribed angular velocity distribution is still an open problem. Notice, however, if $\Omega^2$ is prescribed as a function of $r$ alone, by proposition 1 $\nabla p \times \nabla \rho = 0$, which implies $\nabla s \times \nabla \rho = 0$. This is a very peculiar condition, and is slightly weaker than the condition that the equation of state be isentropic. Since we have allowed non-isentropic equation of state, there is no reason why entropy and density should be related a priori. However this condition is forced by the equilibrium equation (20) if $\Omega^2$ depends on $r$ alone. Furthermore, in this case, all solutions to (4) with isentropic equation of state (5) can be regarded as a solution to (20) with some suitable entropy $s$. Hence one gets infinitely many solutions with the same total mass and $\Omega^2$ distribution. Both of these observations suggest that it might not be sensible to ask for a solution $(\rho, s)$ to (20) when $\Omega^2$ depends on $r$ alone. A better way of posing the question is to study solutions to (20) when $\Omega^2$ has both genuine $r$ and $z$ dependence.

As a first step to study such questions, I will treat two problems in this paper. One is to take the divergence of (21) and study solutions to

$$\nabla \cdot \left( \frac{1}{\rho} \nabla (e^s \rho^\gamma) \right) = -4\pi G \rho + \nabla \cdot (r \Omega^2 e_r)$$

with prescribed entropy. The other is to take the converse path, to consider what conditions on $\rho$ can one impose to solve (20) for some entropy $s$, and nonnegative angular velocity field $\Omega^2$.

2. Statement of Main Results

Following Luo and Smoller [8], let us make the change of variables

$$w = \frac{\gamma}{\gamma - 1} (e^{s/\gamma} \rho)^{\gamma - 1},$$

then (22) becomes

$$\nabla \cdot (e^{\alpha s} \nabla w) + Ke^{-\alpha s} |w|^q - f = 0,$$

where

$$q = \frac{1}{\gamma - 1}, \quad \alpha = \frac{1}{\gamma}, \quad K = 4\pi G (\frac{\gamma - 1}{\gamma})^{\gamma - 1}.$$
and

\[ f = 2\Omega^2 + r\frac{\partial\Omega^2}{\partial r} = 2\Omega \frac{\partial}{\partial r}(r\Omega). \]

Luo and Smoller [8] considered (24) and obtained some existence results when the entropy is assumed to be either constant or radially dependent, and a non-existence result when the entropy is not constant. In this work, I try to find some existence results for (24) with axisymmetric entropy.

The standard theory for elliptic equations (cf. for example, [12]) can solve the Dirichlet problem to (24) on bounded domains given suitable range of \( q \), but in order to conclude by maximum principle that \( w \) is nonnegative inside the domain, it is desirable that

\[ f \leq 0. \]

Because if (27) is true, we have

\[ \nabla \cdot (e^{\alpha s} \nabla w) = -Ke^{-\alpha s}|w|^q + f \leq 0, \]

which by the maximum principle implies that \( w \) has to be nonnegative. But unfortunately for most physically interesting \( \Omega^2 \), \( f \) is positive. For example, constant \( \Omega \) will induce positive \( f \).

We turn instead to the method of subsolution and supersolutions (see, for example [11]). Smoller and Wasserman in [10] have essentially considered the case with constant \( s \), constant \( f \) and \( 0 < q < 1 \) (\( \gamma > 2 \)). Their method is to exploit the spherical symmetry of the Laplacian and to work with the corresponding ODE. When the spherical symmetry in the coefficients is broken, a subsolution can still be constructed by solving an ODE, provided the entropy decreases radially. A supersolution may instead be constructed using a-priori estimates. The result is

**Theorem 2.1.** If \( 0 < q < 1 \) (\( \gamma > 2 \)), and provided \( x \cdot \nabla s \leq 0 \), there is a ball of radius \( R \) centered at the origin on which there exists an axisymmetric positive solution to (24) with zero boundary value on this ball.

The condition on entropy has the physical interpretation that the entropy is decreasing in the radial direction, so that the star is more thermally active the further one goes down the surface.

The case \( q = 1 \) (\( \gamma = 2 \)) is linear. We can get a solution with zero boundary value of a smooth bounded domain if \( K \) does not coincide with an eigenvalue of the corresponding elliptic operator, but in general the solution is not positive definite. In the case when \( q > 1 \) (\( \gamma < 2 \)), there are other ways to find subsolutions to (24) using variational methods, but an Leray-Schauder estimate is lacking for a supersolution, as is realized by observing the simple model problem of an ODE \( u'' + \lambda u^q = 0 \). Suppose \( q > 1 \). In order for \( u \) to stay positive, symmetric about the origin, and be zero on the boundary of a symmetric domain, \( u(0) \) will be unbounded as \( \lambda \) varies between 0 and 1. However, if one is allowed to rescale the velocity field, the equation can still be solved. The results are as follows:

**Theorem 2.2.** Suppose \( f \) and \( s \) are smooth, and there exists \( c > 0 \) such that \( f \geq c \), and suppose \( 1 < q < 3 \) (\( \frac{4}{3} < \gamma < 2 \)), then for any \( R > 0 \), and sufficiently large
M > 0, there exists a non-negative axisymmetric function w in $H^1_0(B_R)$, and a \( \lambda > 0 \), such that

\[
\nabla \cdot (e^{\alpha s} \nabla w) + Ke^{-\alpha s}w^q - \lambda f = 0 \tag{29}
\]

on the set where $w$ is positive, and $w$ satisfies

\[
\int_{B_R} f w dx = M. \tag{30}
\]

It turns out that the positive set of $w$ is open, so there is no ambiguity in defining (29). Notice that (24) is solved with a rescaled the velocity field. Notice $M = \int_{B_R} f w dx$,

\[
\leq \int_{B_R} C w dx \leq \tilde{C} \int_{B_R} \rho^\gamma dx \leq \tilde{C} \left( \int_{B_R} \rho dx \right)^\gamma.
\]

Therefore the largeness of $M$ implies the largeness of the total mass in this case.

The method for deriving this result is variational. It is possible to extend the variational method to allow functions defined on the entire $\mathbb{R}^3$ and arrive at the following

**Theorem 2.3.** Assume $f$ and $s$ are smooth, $s$ is bounded and $f \geq c > 0$, and $1 < q < \frac{4}{3} < \gamma < 2$, then for sufficiently large $M > 0$, there exists a non-negative axisymmetric function $w$ in $H^1(\mathbb{R}^3)$, and a $\lambda > 0$, such that (29) is satisfied on the set where $w$ is positive, and $w$ satisfies

\[
\int_{\mathbb{R}^3} f w dx = M. \tag{31}
\]

As we will show in the following, one has to address the lost of compactness due to the unboundedness of the domain.

Another way of investigating solutions to (3) is by prescribing $\rho$ and solving for $p$ and $\Omega^2$. Apart from being suitably smooth, an obvious requirement for $p$ and $\Omega^2$ is that they be positive where $\rho$ is positive. Furthermore, $p$ should be zero on the boundary of the positive set of $\rho$. It is possible to develop conditions on $\rho$ that will guarantee the existence of such $p$ and $\Omega^2$. To find out what conditions on $\rho$ are natural, we look at the features of the classical Auchmuty and Beals solutions with isentropic equation of states. In [13], Caffarelli and Friedman studied the properties of the Auchmuty and Beals solutions $\rho$. Some of their results can be summarized as follows:

**Proposition 2.** Assume $\Omega^2$ is analytic, and the equation of state is given by

\[
p = c \rho^\gamma \tag{32}
\]
for some \( \frac{4}{3} < \gamma < 2 \) \((1 < q = \frac{1}{\gamma - 1} < 3)\). Then the axisymmetric Auchmuty and Beals solutions \( \rho \) to (3) have the following properties:

(1) Let \( D = \{ x \in \mathbb{R}^3 | \rho(x) > 0 \} \), then \( \bar{D} \) is compact, \( \partial D \) is smooth and \( D \) is ring shaped. i.e. \( D \) is a finite union of sets like \( \{ (r, z) | a < r < b, |z| < \psi(r) \} \), or \( \{ (r, z) | 0 \leq r < b, |z| < \psi(r) \} \), where \( \psi \) is a function vanishing at the end points except perhaps at \( a = 0 \). \( \rho \in C^{1,\beta}(\mathbb{R}^3) \cap C^\infty(D) \) for \( \beta < q - 1 \).

(2) \( \rho(r, z) = \rho(r, -z) \).

(3) \( \rho_z(r, z) > 0 \) for \( (r, z) \in D \) and \( r > 0, z < 0 \).

(4) \( \rho_{zz}(r, 0) < 0 \) for \( (r, 0) \in D \).

We make a note that when \( 2 < q < 3 \) \((\frac{4}{3} < \gamma < \frac{3}{2})\), \( \rho \) actually belongs to \( C^2(\bar{D}) \), whereas if \( 0 < q < 1 \) \((\gamma > 2)\), we may still conclude \( \rho \in C^{0,q}(\bar{D}) \). Assuming \( \rho \) have enough smoothness to the boundary, we can integrate the second equation in (3) to get

\[
(33) \quad \rho(r, z) = \int_{-\psi(r)}^{\psi(r)} \rho(r, \xi)(-\phi_\xi(r, \xi)) d\xi.
\]

When plugged into the the first equation in (3), we deduce

\[
(34) \quad r \rho \Omega^2 = \int_{-\psi(r)}^{\psi(r)} (\rho_r(-\phi) \xi - \rho_\xi(-\phi)_r) d\xi.
\]

Taking the \( z \) derivative in (34), and noticing that \( \Omega^2 \) has only \( r \) dependence for the Auchmuty and Beals solutions, we get

\[
(35) \quad \rho_r(-\phi)_z - \rho_z(-\phi)_r = r \Omega^2 \rho_z.
\]

From point 3 in proposition 2, we see that

\[
(36) \quad \rho_r(-\phi)_z - \rho_z(-\phi)_r \geq 0
\]

when \( (r, z) \in D \) and \( r > 0, z < 0 \).

We have mostly motivated the conditions in the following

**Theorem 2.4.** Let \( \rho \) be an axisymmetric nonnegative function such that

(1) \( \rho \in C^k(\bar{D}) \) \((k \geq 2)\), where \( D \) is a ring shaped domain, i.e. \( D \) is a finite union of sets like \( \{ (r, z) | a < r < b, |z| < \psi(r) \} \), or \( \{ (r, z) | 0 \leq r < b, |z| < \psi(r) \} \), where \( \psi \) is a function vanishing at the end points except perhaps at \( a = 0 \). Also assume \( \partial D \) is smooth, \( \rho > 0 \) on \( D \), \( \rho = 0 \) on \( \partial D \).

(2) \( \rho(r, z) = \rho(r, -z) \).

(3) \( \rho_r(-\phi)_z - \rho_z(-\phi)_r \geq 0 \) for \( z < 0 \), where \( \phi \) is given in (2).

(4) \( \rho_z > 0 \) for \( z < 0 \).

Also assume the following is satisfied:

(a) There is a \( c > 0 \), such that \( \rho_{zz} < -c \) on \( \{ z = 0 \} \cap \partial D \).

Then (3) is solvable for a nonnegative angular velocity function \( \Omega^2 \in C^{k-2}(\bar{D}) \cap C^0(\bar{D}) \) and a positive pressure \( p \in C^k(\bar{D}) \), such that \( p = 0 \) on \( \partial D \).

**Remark 1.** If \( \nabla \rho \) and \( \nabla(-\phi) \) both roughly point to the center of the star, condition 3 in theorem 2.4 means that the gradient of \( \rho \) is more inclined with respect to the
plane \{z = 0\} than the gravity force. Simple calculations with ellipsoids suggest that shapes that are wider at the equator more often satisfies condition 3.

It is desirable to relax the regularity assumptions on the boundary, since for some \(q\), the Auchmuty and Beals solutions are only Hölder continuous at the boundary. A similar result with weaker boundary regularity needs more control on the derivatives when close to the boundary. Here is one way of formulating the conditions:

**Theorem 2.5.** Let \(\rho\) be an axisymmetric nonnegative function such that

1. \(\rho \in C^2(D) \cap C^{0,\beta}(\bar{D})\), for some \(0 < \beta < 1\), where \(D\) is a ring shaped domain whose boundary is specified by the function \(z = \psi(r)\). Also assume \(\partial D\) is smooth, convex at \((0, \pm \psi(0)) \in \partial D\) (if there are such points), i.e., the interior of the segment \((0, \psi(0)) - (r, \psi(r))\) lies in \(D\) for \(r\) small enough.
2. \(\rho > 0\) on \(D\), \(\rho = 0\) on \(\partial D\).
3. \(\rho_r(\phi) z - \rho_z(\phi) r \geq 0\) for \(z < 0\), where \(\phi\) is given in (2).
4. \(\rho z > 0\) for \(z < 0\).
5. \(\forall \epsilon > 0, \exists C > 0\) such that on \(D \cap \{|z| \geq \epsilon\}\): 
   \[
   |\rho_r| \leq C|\rho_z|, \quad |\rho_{rr}| \leq C|\rho_z|, \quad |\rho_{rz}| \leq C|\rho_z|.
   \]

Also assume that one of the following is satisfied:

(a) \(\frac{\rho_{zz}}{\rho_{zz}}\) and \(\rho_r\) are bounded in a neighbourhood of \(\{z = 0\} \cap \partial D\).
(b) \(\rho_{zz} \leq 0\) in a neighbourhood of \(\{z = 0\} \cap \partial D\).
(c) \(\frac{\rho_z}{\rho_{rz}}\) is bounded on \(U \setminus \{z = 0\}\), where \(U\) is some neighbourhood of \(\{z = 0\} \cap \partial D\).

Then (33) is solvable for a nonnegative angular velocity function \(\Omega^2 \in C^0(D) \cap L^\infty(D)\) and a positive pressure \(p \in C^1(D) \cap C^0(\bar{D})\), such that \(p = 0\) on \(\partial D\).

**Remark 2.** If \(D\) has only one connected component containing the origin, \(\{z = 0\} \cap \partial D\) is the equator, and since \(\rho\) is zero on the equator and positive in the interior of \(D\), the condition (a') is most likely satisfied in this case. Condition (a'') is equivalently to \(\frac{z}{r} \frac{\rho_z}{\rho_r}\) being bounded on \(U \setminus \{z = 0\}\) and has the geometrical interpretation that the when \(x\) gets close to \(\{z = 0\} \cap \partial D\), the inclination of \(x\) to the horizontal plane is bounded by the inclination of \(\nabla \rho(x)\).

3. Existence of Solution when \(0 < q < 1\)

In this section we give a proof of theorem [24]. Here we are dealing with the equation

\[
\nabla \cdot \left(\frac{1}{\rho} \nabla (e^s \rho^q)\right) = f - 4\pi G \rho,
\]

or equivalently,

\[
\nabla \cdot (e^{\alpha s} \nabla w) + Ke^{-\alpha s} w^q - f = 0.
\]

We may absorb \(\alpha\) into \(s\), and without loss of generality, work with

\[
\nabla \cdot (e^s \nabla w) + Ke^{-s} w^q - f = 0.
\]

We assume that \(f\) and \(s\) are both smooth, bounded functions on \(\mathbb{R}^3\). One has the following lemma:
Lemma 1. If \( \mathbf{x} \cdot \nabla s \leq 0 \), there is a ball of radius \( R \), denoted by \( B_R \), centered at the origin, on which there is a smooth spherically symmetric positive function \( u \) with zero boundary value satisfying
\[
\nabla \cdot (e^s \nabla u) + Ke^{-s} u^q - f \geq 0.
\]

Proof. Let \( A, B \) be two positive constants such that \( Ke^{-2s} \geq A \), \( e^{-s}f \leq B \). We look for a positive function \( u \) on a ball which satisfies
\[
\Delta u + Au^q - B \geq 0.
\]
By lemma 3.1 in [10], we only need to check that the primitive of \( g(t) = At^q - B \), which is \( G(t) = \frac{A}{q+1} t^{q+1} - Bt \), satisfies \( G(t) > 0 \) for some \( t > 0 \). But this is certainly true for large enough \( t \). It follows that there is a ball of radius \( R \), and a spherically symmetric positive solution \( u \) of (41) on this ball with zero boundary value, and satisfying \( \mathbf{x} \cdot \nabla u < 0 \). By the definition of \( A \) and \( B \), we have
\[
\Delta u + Ke^{-2s} u^q - e^{-s}f \geq \Delta u + Au^q - B \geq 0.
\]
Furthermore, by (43)
\[
\nabla u = -\frac{|\nabla u|}{|\mathbf{x}|} \mathbf{x},
\]
we have
\[
\nabla s \cdot \nabla u = -(\mathbf{x} \cdot \nabla s) \frac{|\nabla u|}{|\mathbf{x}|} \geq 0.
\]
Therefore,
\[
\Delta u + \nabla s \cdot \nabla u + Ke^{-2s} u^q - e^{-s}f \geq 0,
\]
which differs from (40) only by a factor of \( e^{-s} \). Hence the assertion is proved. \( \square \)

Having produced a subsolution to (39), we only need a supersolution to arrive at a genuine solution. And that is given by

Lemma 2. If \( 0 < q < 1 \), there is a smooth positive function \( \bar{u} \) on \( B_R \), such that \( \bar{u} \geq u \) on \( B_R \), and satisfies
\[
\nabla \cdot (e^s \nabla \bar{u}) + Ke^{-s} \bar{u}^q - f \leq 0.
\]

Proof. Let \( C = \|u\|_{L^\infty(B_R)} \), and \( M = \|f\|_{L^\infty(B_R)} \). Let \( g(t) \geq 0 \) be a smooth function on \( \mathbb{R} \) such that
\[
g(t) = \begin{cases} t^q & \text{if } t \geq C \\ 0 & \text{if } t \leq 0 \end{cases}
\]
and \( 0 \leq g'(t) \leq 2Cq^{-1} \) when \( 0 < t < C \). We look for a solution to the equation:
\[
\nabla \cdot (e^s \nabla u) + Ke^{-s} g(u + C) + M = 0
\]
by the standard Leray-Schauder estimate. For that we define
\[
A : H^1_0(B_R) \rightarrow H^1_0(B_R)
\]
\[
u \mapsto v
\]
by

\[ \nabla \cdot (e^s \nabla v) + Ke^{-s}g(u + C) + M = 0 \quad \text{on } B_R \]
\[ v = 0 \quad \text{on } \partial B_R \]

By the definition of \( g(t) \) we have

\[ (g(u + C))^2 \leq C^{2q} + (u + C)^{2q}, \]
\[ \leq B(1 + u^2) \]

where \( B \) is a constant which maybe enlarged appropriately in the following. Therefore \( A(u) \in H^2(B_R) \), and

(49) \[ \|A(u)\|_{H^2(B_R)} \leq B(1 + \|u\|_{H^1_0(B_R)}^q). \]

It now follows easily that \( A \) is continuous and compact. Furthermore if \( u = tA(u) \), for \( 0 \leq t \leq 1 \), we have

(50) \[ \nabla \cdot (e^s \nabla u) + t(Ke^{-s}g(u + C) + M) = 0 \]

weakly. Therefore for some \( c > 0 \)

\[ c \int_{B_R} |\nabla u|^2 \]
\[ \leq \int_{B_R} e^s |\nabla u|^2 \]
\[ = t \int_{B_R} Ke^{-s}g(u + C)u + Mu. \]

Notice now that \( g(u + C) \leq C^q + B(u^q + C^q) \),

\[ c \int_{B_R} |\nabla u|^2 \]
\[ \leq B(1 + \int_{B_R} u^{q+1} + BMu) \]
\[ \leq B(1 + C(\epsilon) + \epsilon \int_{B_R} u^2) \]
\[ \leq B(C(\epsilon) + \epsilon \|u\|_{H^1_0(B_R)}^2), \]

where the constant \( B \) and \( C(\epsilon) \) are enlarged appropriately from line to line. Now let us choose \( \epsilon \) so small that \( Be < \frac{c}{2} \), it follows that \( \{u|u = tA(u), 0 \leq t \leq 1\} \) is bounded in \( H^1_0(B_R) \). Therefore there exists a \( u \in H^1_0(B_R) \) solving (48). By Sobolev imbedding, \( u \in H^2(B_R) \subset W^{1,6}(B_R) \subset C^{0,\frac{1}{2}}(\overline{B_R}) \). Since

\[ |g(u(x) + C) - g(u(y) + C)| \]
\[ \leq |g'(\theta)||u(x) - u(y)| \]
\[ \leq \max(2C^{q-1}, g(C + \|u\|_{C^0(\overline{B_R})}^{q-1})[u]_{0,\frac{1}{2}; B_R}^q) \]

where \( \theta \) is between \( u(x) + C \) and \( u(y) + C \), it follows that \( g(u + C) \in C^{0,\frac{1}{2}}(\overline{B_R}) \). Elliptic regularity estimates imply \( u \in C^{2,\frac{1}{2}}(\overline{B_R}) \), and an iteration of the regularity
estimates imply that \( u \) is smooth. Now by the classical maximum principle, \( u \geq 0 \) on \( B_R \), therefore \( u \) solves
\[
\nabla \cdot (e^s \nabla u) + Ke^{-s}(u + C)^q + M = 0.
\]
Hence
\[
\nabla \cdot (e^s \nabla (u + C)) + Ke^{-s}(u + C)^q - f \leq 0.
\]
Let \( \bar{u} = u + C \), the proof is complete. □

Now it follows from a standard construction in [11] that there is a solution to (39). And it follows directly from the construction that the resulting solution is axisymmetric if \( u \) is. This completes the proof of theorem 2.1.

4. Existence of Minimizer

In this section we show the existence of minimizer of the following energy functional:
\[
E(w) = \int_{\mathbb{R}^3} \left( \frac{e^{\alpha s}}{2} |\nabla w|^2 - \frac{K}{q + 1} w^{q+1} e^{-s} \right) dx
\]
subject to the constraint:
\[
N(w) = \int_{\mathbb{R}^3} f w dx = M
\]
where \( f \) is assumed to be locally bounded, and
\[
f \geq c > 0.
\]
We take the set \( W_M \) of admissible functions to be
\[
H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \cap \{ w : \mathbb{R}^3 \to \mathbb{R} | w \geq 0 \text{ a.e.}, w \text{ is axisymmetric}, N(w) = M \}.
\]
We will apply this to construct solutions to (3) when \( q > 1 \) and the domain is infinite.

The proof of the existence of minimizer will depend on a bound of the \( L^{q+1} \) norm by the \( L^p \) norm and the \( L^2 \) norm of the derivative. We will only concern the case in \( \mathbb{R}^3 \). This is given by the following inequality (see, for example [14]).

**Proposition 3.** (Gagliard-Nirenberg inequality): If \( w \in L^p(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \), then \( \exists C > 0 \), such that
\[
\|w\|_{L^{q+1}(\mathbb{R}^3)} \leq C \|\nabla w\|_{L^2(\mathbb{R}^3)} \|w\|_{L^p(\mathbb{R}^3)}^{1-a}.
\]
If \( w \in L^p(\mathbb{R}^3 \setminus B_R) \cap H^1(\mathbb{R}^3 \setminus B_R) \), where \( B_R \) is the closed ball centered at the origin with radius \( R \), and assume \( R > R_0 \), then \( \exists C(R_0) > 0 \), such that
\[
\|w\|_{L^{q+1}(\mathbb{R}^3 \setminus B_R)} \leq C \|\nabla w\|_{L^2(\mathbb{R}^3 \setminus B_R)} \|w\|_{L^p(\mathbb{R}^3 \setminus B_R)}^{1-a}.
\]
In both of these inequalities,
\[
a = \frac{1}{p} - \frac{1}{q+1}.
\]
Notice when $q \leq 5$, $0 < a \leq 1$. This is the most useful range of exponents for us.

In the following, we always assume $s$ to be bounded. With the Gagliardo-Nirenberg inequality, we can show $E(w)$ is bounded from below. In particular the following inequality is true.

**Lemma 3.** If $w \in L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, and $N(w) = M$, and if $q < 3$, then $\exists$ a constant $C(M)$ depending only on $M$, such that

$$E(w) \geq \frac{1}{2} \int_{\mathbb{R}^3} \frac{e^{\alpha s}}{2} |\nabla w|^2 dx - C(M).$$

**Proof.** Since $s$ is bounded,

$$\int w^{q+1} e^{-\alpha s} dx \leq C \int w^{q+1} dx = C \|w\|_{q+1}^{q+1}.$$

By the Gagliardo-Nirenberg inequality, we have

$$C \|w\|_{q+1}^{q+1} \leq C \|
abla w\|_{L^2}^{a(q+1)} \|w\|_{L^1}^{(1-a)(q+1)} \leq C(M) \|
abla w\|_{L^2}^{a(q+1)}.$$

The last inequality is because of the boundedness of $s$ and (54), (55).

Now since $q < 3$, we can easily show $a(q + 1) < 2$. By an elementary inequality we have

$$C(M) \|
abla w\|_{L^2}^{a(q+1)} \leq \hat{C}(M, \epsilon) + \epsilon \|
abla w\|_{L^2}^2 \leq \hat{C}(M, \epsilon) + \epsilon \int |\nabla w|^2 dx \leq \hat{C}(M, \epsilon) + C' \epsilon \int \frac{e^{\alpha s}}{2} |\nabla w|^2 dx.$$

Therefore,

$$E(w) \geq (1 - C' \epsilon) \int \frac{e^{\alpha s}}{2} |\nabla w|^2 dx - \hat{C}(M, \epsilon).$$

Choose $\epsilon$ so small that $(1 - C' \epsilon) > \frac{1}{2}$, the assertion is established. \qed

We see from the above inequality that $E(w)$ is bounded from below. Let us define:

$$I_M = \inf_{w \in W_M} \{E(w) | N(w) = M\}$$

We can quickly find a few scaling inequalities about $I_M$, summarized in the following

**Lemma 4.** Suppose $q > 1$. Given $s$ and $f$, $I_M < 0$ for $M$ sufficiently large. If $M' > M > 0$, then $I_{M'} \leq (\frac{M'}{M})^{q+1} I_M$. 

Proof. Noticing in (54) \( N(w) \) is linear in \( w \), we have for \( \theta > 1 \),
\[
I_{\theta M} = \inf \{ E(w) | N(w) = \theta M \} \\
= \inf \{ E(\theta w) | N(w) = M \} \\
= \inf \{ \int \frac{e^{\alpha s}}{2} \theta^2 |\nabla w|^2 - \frac{K}{q+1} \theta^{q+1} w^{q+1} e^{-\alpha s} |N(w) = M| \}.
\]
Now observe that
\[
\int w^{q+1} e^{-\alpha s} > 0
\]
and the term with the coefficient \( \theta^{q+1} \) will dominate as \( \theta \) increases, we can conclude
that \( I_{\theta M} < 0 \) if \( \theta \) is sufficiently large.

Following the same line of reasoning,
\[
I_{M'} = \inf \{ E(w) | N(w) = M' \} \\
= \inf \{ E\left(\frac{M'}{M} w\right) | N(w) = M \} \\
= \inf \{ \int \frac{e^{\alpha s}}{2} \left(\frac{M'}{M}\right)^2 |\nabla w|^2 - \frac{K}{q+1} \left(\frac{M'}{M}\right)^{q+1} w^{q+1} e^{-\alpha s} |N(w) = M \} \\
= \left(\frac{M'}{M}\right)^{q+1} \inf \{ \int \frac{e^{\alpha s}}{2} |\nabla w|^2 - \frac{K}{q+1} w^{q+1} e^{-\alpha s} |N(w) = M \} \\
\leq \left(\frac{M'}{M}\right)^{q+1} \inf \{ \int \frac{e^{\alpha s}}{2} |\nabla w|^2 - \frac{K}{q+1} w^{q+1} e^{-\alpha s} |N(w) = M \} \\
= \left(\frac{M'}{M}\right)^{q+1} I_{M'}.
\]
We get the inequality because \( M' > M \) and \( q > 1 \). \( \square \)

Now we are ready to introduce a concentration-compactness principle due to P.L. Lions [5]. This is the starting point of the existence proof.

**Lemma 5.** Let \( \{w_n\} \) be a sequence in \( L^1(\mathbb{R}^n) \) such that \( w_n \geq 0 \ a.e. \), the \( w_n \)'s are axisymmetric, and \( \int_{\mathbb{R}^3} f w_n \, dx = M \). Then \( \exists \) a subsequence \( \{w_{n_k}\} \) such that one of the following is true:

1. \( \exists a_k \in \mathbb{R} \) such that \( \forall \epsilon > 0, \exists R > 0, N > 0 \) such that \( \forall k > N \)
\[
M \geq \int_{a_k e_3 + B_R} f w_{n_k} \, dx \geq M - \epsilon.
\]
2. \( \forall R > 0 \)
\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^3} \int_{y + B_R} f w_{n_k} \, dx = 0.
\]
3. \( \exists \lambda \in (0, M), \forall \epsilon > 0, \exists R_0 > 0, a_k \in \mathbb{R}, \forall R > R_0, \exists k_0 > 0, \forall k > k_0: \)
\[
\int_{a_k e_3 + B_R} f w_{n_k} \, dx > \lambda - \epsilon, \\
\int_{a_k e_3 + B_{2R}} f w_{n_k} \, dx < \lambda + \epsilon.
\]
Proof. Denote \( fw_n \) by \( \rho_n \). Let \( Q_n(t) = \sup_{y \in \mathbb{R}^3} \int_{y + B_t} \rho_n dx \).

\( Q_n(t) \) is a sequence of nondecreasing, nonnegative, uniformly bounded functions on \( \mathbb{R}^+ \), and \( \lim_{t \to +\infty} Q_n(t) = M \). By the Helly selection theorem, there exists a subsequence \( Q_{n_k}(t) \), and a function \( Q(t) \), such that \( Q_{n_k}(t) \to Q(t) \) pointwise on \( \mathbb{R}^+ \). \( Q(t) \) is hence nondecreasing and nonnegative.

Let \( \lambda = \lim_{n \to \infty} Q(t) \in [0, M] \).

(1) If \( \lambda = M \), then

\[ \forall \epsilon > 0, \exists R(\epsilon) > 0 \text{ such that } Q(R) > M - \frac{\epsilon}{2}. \]

Since \( \lim_{k \to \infty} Q_{n_k}(R) = Q(R) \),

\[ \exists N(\epsilon) > 0, \forall k > N(\epsilon): Q_{n_k}(R) > M - \frac{\epsilon}{2}. \]

Hence, \( \exists y_k(\epsilon) \in \mathbb{R}^3 \) such that \( \int_{y_k(\epsilon) + B_R} \rho_{n_k} dx > M - \frac{\epsilon}{2} \).

Take \( y_k = y_k(\frac{M}{2}) \). We claim that \( |y_k(\epsilon) - y_k| < R(\frac{M}{2}) + R(\epsilon) \) for \( \epsilon \) small.

If not,

\[ \int_{\mathbb{R}^3} \rho_{n_k} dx \geq \int_{y_k + R(\frac{M}{2})} \rho_{n_k} dx + \int_{y_k(\epsilon) + R(\epsilon)} \rho_{n_k} dx \]

\[ > M - \frac{M}{2} + M - \frac{\epsilon}{2} \]

\[ = 3M - \frac{\epsilon}{2} > M \text{ if } \epsilon \text{ is small.} \]

Take \( R'(\epsilon) = 2R(\epsilon) + R(\frac{M}{2}) \). By the previous inequality, we have

\[ y_k + R'(\epsilon) \supset y_k(\epsilon) + R(\epsilon). \]

Therefore,

\[ \int_{y_k + B_{R'(\epsilon)}} \rho_{n_k} dx > M - \frac{\epsilon}{2}. \]

Take \( a_k = y_k \cdot e_3 \), and let \( \eta(y) \) be the distance of \( y \) to the \( e_3 \) axis. There must exist an \( \eta_0 \) such that \( \eta(y_k) \leq \eta_0 \). Otherwise the integral of \( \rho_{n_k} \) on the torus obtained from revolving \( y_k + B_{R(\frac{M}{2})} \) around the \( e_3 \) axis will give

\[ \int_{T_k} \rho_{n_k} dx \geq C(M - \frac{M}{2}) \eta(y_k) \]

for some constant \( C \). This integral will be unbounded if the \( \eta(y_k) \)'s are.

Let \( R''(\epsilon) = R'(\epsilon) + \eta_0 \), then

\[ \int_{a_k e_3 + B_{R''(\epsilon)}} \rho_{n_k} dx > M - \epsilon. \]

(2) If \( \lambda = 0 \), then \( \lim_{R \to \infty} Q(R) = 0 \), which implies \( Q(R) \equiv 0 \). The result follows immediately.
If $\lambda \in (0, M)$, since $\lim_{t \to \infty} Q(t) = \lambda$, $\lim_{k \to \infty} Q_{n_k}(t) = Q(t)$, we know:

$\forall \epsilon > 0, \exists R(\epsilon) > 0, K > 0, \forall k > K, R \geq R(\epsilon)$:

$$Q_n(R) = \sup_{y \in \mathbb{R}^3} \int_{y + B_R} \rho_{n_k} dx > \lambda - \epsilon.$$  

Let $f_k(y) = \int_{y + B_R} \rho_{n_k} dx$. It is easy to verify that $f_k(y)$ is a continuous function. Consider the set \{\{y|f_k(y) \geq \lambda - \epsilon\}. This set is nonempty because $\sup_{y \in \mathbb{R}^3} f_k(y) > \lambda - \epsilon$, is closed by the continuity of $f_k$, and is bounded because the contrary will indicate that $\rho_{n_k}$ has infinite mass. Therefore, there exists $y_k \in \mathbb{R}^3$ such that $f_k(y_k) = \int_{y_k + B_R} \rho_{n_k} dx = \sup_{y \in \mathbb{R}^3} \int_{y + B_R} \rho_{n_k} dx > \lambda - \epsilon$.

Also for any $R \geq R(\epsilon)$, we have

$$\int_{y_k + B_R} \rho_{n_k} dx > \lambda - \epsilon.$$  

For the same reason as in case 1, there must be an $\eta_0 = \eta_0(\epsilon)$ such that $\eta(y_k) \leq \eta_0$. Let $a_k = y_k \cdot e_3$, and $R_0 = R(\epsilon) + \eta_0$, $\forall R > R_0, k > K$,

$$\int_{a_k e_3 + B_R} \rho_{n_k} dx > \lambda - \epsilon.$$  

On the other hand, because $\lim_{k \to \infty} Q_{n_k}(2R) \leq \lambda$, there must be a $k_0 > K$ such that $\forall k > k_0$:

$$Q_{n_k}(2R) < \lambda + \epsilon,$$

which implies

$$\int_{a_k e_3 + B_{2R}} \rho_{n_k} dx < \lambda + \epsilon.$$  

This concludes the proof of the lemma. \qed

Intuitively, lemma 5 says that if we have a sequence of densities with fixed total mass, then the densities can either be concentrated in a ball of radius $R$, or gradually vanish, or split up into at least two parts (with masses roughly $\lambda$ and $M - \lambda$) that run infinitely apart from each other. Our analysis in the following will show that case 2 and case 3 cannot happen, provided that the scaling inequalities hold. On the other hand, case 1 will force the existence of a minimizer.

**Lemma 6.** Let $1 < q < 3$. If $w_n$ is bounded in $L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, $w_n \geq 0$ a.e., and if

$$\exists R > 0, \lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{y + B_R} w_n dx \to 0,$$

Then $\int_{\mathbb{R}^3} w_n^{q+1} dx \to 0$. 


Proof. Fix $\alpha \in (\max\left(\frac{3}{2}, \frac{2(q+1)}{3}\right), q+1)$, and let $\beta = \frac{q+1}{\alpha}$. We get $1 < \beta < \frac{3}{2}$.

For any $w \in L^1(R^3) \cap H^1(R^3)$, by the Sobolev embedding $L^\beta \subset W^{1,1}$,

$$
\int_{y+B_R} w^{q+1} dx \\
= \int_{y+B_R} w^{\alpha \beta} dx \\
\leq C(R) \left( \int_{y+B_R} (w^\alpha + \alpha w^{\alpha-1} |\nabla w|) dx \right)^\beta \\
\leq C(R) \left( \int_{y+B_R} w^\alpha dx + \alpha \left[ \int_{y+B_R} w^{2(\alpha-1)} dx \right]^{\frac{1}{\alpha}} \left[ \int_{y+B_R} |\nabla w|^2 \right] \right)^\beta \\
(58) = C(R) \left( \|w\|_{L^\alpha(y+B_R)}^2 + \alpha \|\nabla w\|_{L^2(y+B_R)} \cdot \|w\|_{L^{2\alpha-2}(y+B_R)} \right)^\beta.
$$

By the Gagliardo-Nirenberg inequality,

$$
\|w\|_{L^\alpha(y+B_R)} \leq C(R) \|\nabla w\|_{L^2(y+B_R)} \|w\|_{L^1(y+B_R)}^{\frac{1-a}{\alpha}} \\
\|w\|_{L^{2\alpha-2}(y+B_R)} \leq C(R) \|\nabla w\|_{L^2(y+B_R)} \|w\|_{L^1(y+B_R)}^{\frac{1-b}{\alpha}},
$$

where

$$
a = \frac{1 - \frac{1}{\alpha}}{1 - \frac{1}{6}}, \\
b = \frac{1 - \frac{2\alpha-2}{\alpha}}{1 - \frac{1}{6}}.
$$

$a, b \in (0, 1)$ if $1 < \alpha < 6, 1 < 2\alpha - 2 < 6$, or $\frac{3}{2} < \alpha < 4$, which is guaranteed by the choice of $\alpha$. Hence given the hypotheses for $\bar{w}_n$, we can easily see that

$$
\|\bar{w}_n\|_{L^\alpha(y+B_R)} \to 0, \\
\|\bar{w}_n\|_{L^{2\alpha-2}(y+B_R)} \to 0,
$$
as $n \to \infty$. By (58)

$$
\int_{y+B_R} \bar{w}_n^{q+1} dx \\
\leq C(R) \left( \int_{y+B_R} (\bar{w}_n^\alpha + \alpha \bar{w}_n^{\alpha-1} |\nabla \bar{w}_n|) dx \right)^\beta \\
= C(R) \epsilon_n^\beta \\
\leq C(R) \epsilon_n^{\beta-1} \left( \int_{y+B_R} (\bar{w}_n^\alpha + \alpha \bar{w}_n^{\alpha-1} |\nabla \bar{w}_n|) dx \right),
$$

where

$$
\epsilon_n = \int_{y+B_R} (\bar{w}_n^\alpha + \alpha \bar{w}_n^{\alpha-1} |\nabla \bar{w}_n|) dx \\
\leq \|\bar{w}_n\|_{L^\alpha(y+B_R)}^2 + \alpha \|\nabla \bar{w}_n\|_{L^2(y+B_R)} \cdot \|\bar{w}_n\|_{L^{2\alpha-2}(y+B_R)}^{\alpha-1} \to 0.
$$
Cover \( \mathbb{R}^3 \) with these balls of radius \( R \) in such a way that each point in \( \mathbb{R}^3 \) is contained in an overlap of at most \( K \) balls. Then,

\[
\int_{\mathbb{R}^3} w_n^{q+1} dx \leq C(R)K^{\beta-1}\int_{\mathbb{R}^3} (w_n^\alpha + \alpha w_n^{\alpha-1}|\nabla w|)dx.
\] (59)

Just as in (58), we have

\[
\int_{\mathbb{R}^3} (w_n^\alpha + \alpha w_n^{\alpha-1}|\nabla w|)dx \\
\leq \|w_n\|_{L^\alpha(\mathbb{R}^3)}^\alpha + \alpha \|\nabla w_n\|_{L^2(\mathbb{R}^3)} \cdot \|w_n\|_{L^{2\alpha-2}(\mathbb{R}^3)}^{\alpha-1}.
\]

Similarly by the Gagliardo-Nirenberg inequality,

\[
\|w_n\|_{L^\alpha(\mathbb{R}^3)} \leq C\|\nabla w_n\|_{L^2(\mathbb{R}^3)} \|w_n\|_{L^1(\mathbb{R}^3)}^{1-a} \\
\|w_n\|_{L^{2\alpha-2}(\mathbb{R}^3)} \leq C\|\nabla w_n\|_{L^2(\mathbb{R}^3)} \|w_n\|_{L^1(\mathbb{R}^3)}^{1-b}.
\]

By the boundedness of \( w_n \) in \( L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \), we can conclude from (59) that

\[
\int_{\mathbb{R}^3} w_n^{q+1} dx \to 0
\]
as \( n \to \infty \).

\[ \square \]

**Corollary.** If \( \{w_n\} \) is a minimizing sequence of \( E(w) \) in \( L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \) with \( N(w) = M \), and if \( I_M < 0 \), then case 2 in lemma 5 cannot happen.

**Proof.** If case 2 in lemma 5 happens, there will be a subsequence \( \{w_{n_k}\} \) such that \( \forall R > 0 \),

\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^3} \int_{y+B_R} f w_{n_k} dx = 0
\]

Since \( f \geq c > 0 \), this implies

\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^3} \int_{y+B_R} w_{n_k} dx = 0
\]

By lemma 6, this implies \( \lim_{k \to \infty} \int_{\mathbb{R}^3} \frac{K}{q+1} e^{-\alpha s} w_{n_k}^{q+1} dx = 0 \) since \( s \) is bounded, which then implies \( I_M \geq 0 \).

\[ \square \]

For our purpose of eliminating the possibility of case 3 in lemma 5, we need an elementary inequality.

**Lemma 7.** If \( 0 \leq \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 = 1, q > 1 \), then

\[ 1 - \lambda_1^{q+1} - \lambda_2^{q+1} \geq 2\lambda_1\lambda_2. \]

**Proof.** Since \( q > 1, q + 1 > 2 \). Hence

\[
1 - \lambda_1^{q+1} - \lambda_2^{q+1} \geq 1 - \lambda_1^2 - \lambda_2^2 \\
= (\lambda_1 + \lambda_2)^2 - \lambda_1^2 - \lambda_2^2 \\
= 2\lambda_1\lambda_2.
\]

\[ \square \]

Now we are ready to eliminate case 3 in lemma 5.
Lemma 8. If \( \{w_n\} \) is a minimizing sequence of \( E(w) \) in \( L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \), with \( N(w) = M \), and if \( I_M < 0 \), \( \forall M > M_1 > 0 \), \( I_{M_2} \leq \left( \frac{M_2}{M_1} \right)^{q+1} I_{M_1} \), then case 3 in lemma 7 cannot happen.

Proof. Assume the contrary. Then there exists a subsequence \( \{w_{n_k}\} \) such that \( \exists \lambda \in (0, M), \forall \epsilon > 0, \exists R_0 > 0, a_k \in \mathbb{R}, \forall R > R_0, \exists k_0 > 0, \forall k > k_0: \)

\[
\int_{a_k e_3 + B_R} f w_{n_k} \, dx > \lambda - \epsilon,
\]

\[\int_{a_k e_3 + B_{2R}} f w_{n_k} \, dx < \lambda + \epsilon.
\]

(60)

Let \( \varphi : \mathbb{R}^+ \to [0, 1] \) be a smooth cut off function, such that

\[
\varphi(t) = 1 \text{ when } |t| \leq 1,
\]

\[
\varphi(t) = 0 \text{ when } |t| \geq 2,
\]

\[
|\nabla \varphi(t)| \leq 2 \text{ for all } t.
\]

Let us now define

\[
\varphi_{k,1}(x) = \varphi\left( \frac{|x - a_k e_3|}{R} \right),
\]

\[
\varphi_{k,2}(x) = 1 - \varphi_{k,1}(x),
\]

\[
w_{k,1}(x) = \varphi_{k,1}(x) w_{n_k}(x),
\]

\[
w_{k,2}(x) = \varphi_{k,2}(x) w_{n_k}(x),
\]

\[
M_{k,1} = \int_{\mathbb{R}^3} f w_{k,1} \, dx,
\]

\[
M_{k,2} = \int_{\mathbb{R}^3} f w_{k,2} \, dx.
\]
Obviously \( w_{k,1} \in W_{M_{k,1}}, w_{k,2} \in W_{M_{k,2}}, |\nabla \varphi_{k,1}| \leq \frac{2}{R}, |\nabla \varphi_{k,2}| \leq \frac{2}{R}, \) also \( M = M_{k,1} + M_{k,2}. \) We now estimate

\[
E(w_{nk}) = \int_{\mathbb{R}^3} \left( \frac{e^\alpha}{2} |\nabla w_{nk}|^2 - \frac{K}{q+1} w_{nk}^{q+1} e^{-\alpha s} \right) dx
\]

\[
= \int_{\mathbb{R}^3} \left( \frac{e^\alpha}{2} |\nabla w_{k,1} + \nabla w_{k,2}|^2 - \frac{K}{q+1} (w_{nk}^{q+1} - w_{k,1}^{q+1} - w_{k,2}^{q+1}) e^{-\alpha s} \right. \\
- \left. \frac{K}{q+1} (w_{k,1}^{q+1} + w_{k,2}^{q+1}) e^{-\alpha s} \right) dx
\]

\[
= \int_{\mathbb{R}^3} \left( \frac{e^\alpha}{2} |\nabla w_{k,1}|^2 - \frac{K}{q+1} w_{k,1}^{q+1} e^{-\alpha s} \right) dx \\
+ \int_{\mathbb{R}^3} \left( \frac{e^\alpha}{2} |\nabla w_{k,2}|^2 - \frac{K}{q+1} w_{k,2}^{q+1} e^{-\alpha s} \right) dx \\
+ \int_{\mathbb{R}^3} \left( e^\alpha \nabla w_{k,1} \cdot \nabla w_{k,2} - \frac{K}{q+1} w_{nk}^{q+1} (1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1}) e^{-\alpha s} \right) dx
\]

\[
\geq I_{M_{k,1}} + I_{M_{k,2}} + \int_{\mathbb{R}^3} \left( e^\alpha \nabla w_{k,1} \cdot \nabla w_{k,2} - \frac{K}{q+1} w_{nk}^{q+1} (1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1}) e^{-\alpha s} \right) dx
\]

\[
\geq \left[ \left( \frac{M_{k,1}}{M} \right)^{q+1} + \left( \frac{M_{k,2}}{M} \right)^{q+1} \right] I_M
\]

\[
+ \int_{\mathbb{R}^3} \left( e^\alpha \nabla w_{k,1} \cdot \nabla w_{k,2} - \frac{K}{q+1} w_{nk}^{q+1} (1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1}) e^{-\alpha s} \right) dx.
\]

The last inequality follows from the hypothesis in the lemma. If we denote

\[
Re = \int_{\mathbb{R}^3} \left( e^\alpha \nabla w_{k,1} \cdot \nabla w_{k,2} - \frac{K}{q+1} w_{nk}^{q+1} (1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1}) e^{-\alpha s} \right) dx,
\]

then the above estimate gives us

\[
I_M - E(w_{nk}) \leq \left[ 1 - \left( \frac{M_{k,1}}{M} \right)^{q+1} - \left( \frac{M_{k,2}}{M} \right)^{q+1} \right] I_M - Re.
\]

Since \( M = M_{k,1} + M_{k,2}, \) and \( I_M < 0, \) by lemma \( \boxed{7} \) we get

\[
I_M - E(w_{nk}) \leq \left[ 1 - \left( \frac{M_{k,1}}{M} \right)^{q+1} - \left( \frac{M_{k,2}}{M} \right)^{q+1} \right] I_M - Re
\]

\[
\leq 2 \frac{M_{k,1} M_{k,2}}{M^2} I_M - Re,
\]

or

\[
(61) \quad - \frac{2}{M^2} I_M M_{k,1} M_{k,2} \leq E(w_{nk}) - I_M - Re.
\]

Let us now estimate \( Re: \)

\[
- Re = - \int_{\mathbb{R}^3} \left( e^\alpha \nabla w_{k,1} \cdot \nabla w_{k,2} - \frac{K}{q+1} w_{nk}^{q+1} (1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1}) e^{-\alpha s} \right) dx.
\]
By the definition of $\varphi_{k,1}$ and $\varphi_{k,2}$, we know $1 - \varphi_{k,1}^{q+1} - \varphi_{k,2}^{q+1} \in [0, 1]$, and is nonzero only when $R \leq |x - a_k e_3| \leq 2R$. Therefore

$$-Re \leq -\int_{\mathbb{R}^3} e^{ax} \nabla w_{k,1} \cdot \nabla w_{k,2} dx + C(q, K, \alpha, s) \int_{R \leq |x - a_k e_3| \leq 2R} w_{n_k}^{q+1} dx = L_1 + L_2.$$ 

We estimate $L_1$ and $L_2$ separately.

$$L_1 = -\int_{\mathbb{R}^3} e^{ax} \nabla w_{k,1} \cdot \nabla w_{k,2} dx$$

$$= -\int_{\mathbb{R}^3} e^{ax} \nabla (w_{n_k} \varphi_{k,1}) \cdot \nabla (w_{n_k} \varphi_{k,2}) dx$$

$$= -\int_{\mathbb{R}^3} e^{ax} \nabla \varphi_{k,1} \cdot \nabla \varphi_{k,2} |w_{n_k}|^2 dx - \int_{\mathbb{R}^3} e^{ax} w_{n_k} \varphi_{k,2} \nabla \varphi_{k,1} \cdot \nabla w_{n_k} dx$$

$$- \int_{\mathbb{R}^3} e^{ax} w_{n_k} \varphi_{k,1} \nabla \varphi_{k,2} \cdot \nabla w_{n_k} dx - \int_{\mathbb{R}^3} e^{ax} (\varphi_{k,1} \varphi_{k,2}) |\nabla w_{n_k}|^2 dx$$

$$\leq -\int_{\mathbb{R}^3} e^{ax} \nabla \varphi_{k,1} \cdot \nabla \varphi_{k,2} |w_{n_k}|^2 dx - \int_{\mathbb{R}^3} e^{ax} w_{n_k} \varphi_{k,2} \nabla \varphi_{k,1} \cdot \nabla w_{n_k} dx$$

$$- \int_{\mathbb{R}^3} e^{ax} w_{n_k} \varphi_{k,1} \nabla \varphi_{k,2} \cdot \nabla w_{n_k} dx$$

$$\leq C(s) \frac{R}{R}.$$ 

The last inequality is because $|\nabla \varphi_{k,1}| \leq \frac{2}{R}, |\nabla \varphi_{k,2}| \leq \frac{2}{R}$, and that $\{w_{n_k}\}$ is bounded in $H^1(\mathbb{R}^3)$. On the other hand, by the Gagliardo-Nirenberg inequality,

$$L_2 \leq C(q, K, \alpha, s) ||w_{n_k}||^{q+1}_{L^{q+1}(R \leq |x - a_k e_3| \leq 2R)}$$

$$\leq C(q, K, \alpha, s) ||\nabla w_{n_k}||^{q+1}_{L^2(R \leq |x - a_k e_3| \leq 2R)} ||w_{n_k}||(1-a)(q+1)$$

$$\leq C(q, K, \alpha, s) ((\lambda + \epsilon) - (\lambda - \epsilon))^{1-a)(q+1)}$$

The constant $C(q, K, \alpha, s)$ maybe different in different lines. The last inequality follows from (60), and the fact that $\{w_{n_k}\}$ is bounded in $H^1(\mathbb{R}^3)$.

In summary, we have

$$-Re \leq \frac{C(s)}{R} + C(q, K, \alpha, s)(2\epsilon)^{(1-a)(q+1)}.$$ 

From the range of $q$, we deduce that $a \in (0, 1)$. Choose $R > R_0$ so big that

$$-Re \leq C(q, K, \alpha, s)\epsilon^{(1-a)(q+1)}.$$ 

By the definition of $w_{k,1}$, we have

$$M_{k,1} \geq \int_{|x - a_k e_3| \leq R} f w_{n_k} dx > \lambda - \epsilon.$$ 

By (61), and the estimates on $Re$, we have

$$M_{k,2} \leq C(M, I_M, \lambda, q, K, \alpha, s)\epsilon^{(1-a)(q+1)}.$$
However,

\[ M_{k,2} = \int_{\mathbb{R}^3} f w_{k,2} dx \]

\[ \geq \int_{\mathbb{R}^3 \setminus a_k e_3 + B_2 R} f w_{n_k} dx. \]

Hence,

\[ \int_{\mathbb{R}^3 \setminus a_k e_3 + B_2 R} f w_{n_k} dx \leq C(M, I_M, \lambda, q, K, \alpha, s)(\epsilon + \epsilon^{(1-a)(q+1)}). \]

On the other hand,

\[ \int_{a_k e_3 + B_2 R} f w_{n_k} dx < \lambda + \epsilon. \]

This implies

\[ M = \int_{\mathbb{R}^3} f w_{n_k} dx \]

\[ < \lambda + \epsilon + C(M, I_M, \lambda, q, K, \alpha, s)(\epsilon + \epsilon^{(1-a)(q+1)}). \]

If we have initially chosen \( \epsilon \) so small that

\[ \lambda + \epsilon + C(M, I_M, \lambda, q, K, \alpha, s)(\epsilon + \epsilon^{(1-a)(q+1)}) < M, \]

a contradiction will therefore be obtained. \( \square \)

With these preparations, we are ready to prove the existence of a minimizer.

**Theorem 4.1.** If \( 1 < q < 3 \), there exists a minimizer in \( W_M \) of the energy functional \( E(w) \).

**Proof.** By lemma 4, the scaling inequalities are true in this \( q \) range. Therefore lemma 5 and lemma 6 apply. For any minimizing sequence \( \{w_n\} \), there exists a subsequence \( \{w_{n_k}\} \) such that case 1 in lemma 6 is true. Without loss of generality, we assume \( \{w_{n_k}\} \) is already shifted, and satisfies that condition. In other words, \( \forall \epsilon > 0, \exists R > 0, N > 0, \forall n > N: \)

\[ M \geq \int_{B_R} f w_{n} dx \geq M - \epsilon. \]

By lemma 6 \( \{w_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). The Banach-Alaoglu theorem implies there exists a subsequence of \( \{w_n\} \) which converges weakly in \( H^1(\mathbb{R}^3) \) to \( \tilde{w} \). Without loss of generality, we call this subsequence \( \{w_n\} \) again. We claim that \( \tilde{w} \) is a minimizer of \( E(w) \) in \( W_M \).

We first show \( \tilde{w} \in W_M \). Obviously \( \tilde{w} \in H^1(\mathbb{R}^3) \). Notice for any \( R > 0 \), we have \( w_n \rightharpoonup w \) weakly in \( H^1(B_R) \). By the Rellich-Kondrachov theorem, \( H^1(B_R) \) is compactly embedded in \( L^p(B_R) \) for \( 1 \leq p < 6 \). This implies \( \forall R > 0, w_n \rightharpoonup \tilde{w} \) in \( L^q(B_R) \) for \( 1 \leq q < 6 \). The facts that \( w \geq 0 \) a.e. and \( w \) is axisymmetric are now easily established if we integrate the \( w_n \)'s against positive smooth test functions with compact supports and take the limit. Let us now show \( N(\tilde{w}) = M \). For that we observe \( \forall \epsilon > 0, \exists R > 0, N > 0, \forall n > N: \)

\[ \int_{B_R} f w_n dx \geq M - \epsilon. \]
Since $w_n \to \tilde{w}$ in $L^q(B_R)$ for all $R > 0$, and $f$ is locally bounded, we have
\[ \int_{B_R} f \tilde{w} dx \geq M - \epsilon. \]

Therefore for any $\epsilon > 0$
\[ (62) \quad \int_{\mathbb{R}^3} f \tilde{w} dx \geq M - \epsilon. \]

On the other hand, for any $R > 0$,
\[ M \geq \int_{B_R} f w_n dx, \]
which implies
\[ M \geq \int_{B_R} f \tilde{w} dx, \]
which implies
\[ (63) \quad M \geq \int_{\mathbb{R}^3} f \tilde{w} dx. \]

Combine (62) and (63), we get
\[ \int_{\mathbb{R}^3} f \tilde{w} dx = M. \]

This also shows $\tilde{w} \in L^1(\mathbb{R}^3)$. We have shown $\tilde{w} \in W_M$, it remains to establish that $E(w)$ is weakly lower-semicontinuous. We can treat the first term in $E(w)$ by the standard method. In particular, it can be shown that
\[ F_c = \left\{ w \left| \int_{\mathbb{R}^3} e^{\alpha s} |\nabla w|^2 dx \leq c \right\} \right. \]
is a convex norm closed set in $H^1(\mathbb{R}^3)$, therefore is weakly closed. For that purpose, let us observe that \( \int_{\mathbb{R}^3} e^{\alpha s} |\nabla w|^2 dx \) is a continuous functional on $H^1(\mathbb{R}^3)$, therefore $F_c$ is closed. Now pick $t \in [0, 1]$. If
\[ \int_{\mathbb{R}^3} e^{\alpha s} |\nabla w_1|^2 dx \leq c \]
\[ \int_{\mathbb{R}^3} e^{\alpha s} |\nabla w_2|^2 dx \leq c. \]

Then
\[
\int_{\mathbb{R}^3} \frac{e^{\alpha s}}{2} |\nabla(t w_1 + (1 - t) w_2)|^2 dx \\
\leq \int_{\mathbb{R}^3} \frac{e^{\alpha s}}{2} (t^2 |\nabla w_1|^2 + (1 - t)^2 |\nabla w_2|^2 + 2t(1 - t) |\nabla w_1| \cdot |\nabla w_2|) dx \\
\leq \int_{\mathbb{R}^3} \frac{e^{\alpha s}}{2} (t^2 |\nabla w_1|^2 + (1 - t)^2 |\nabla w_2|^2 + t(1 - t) (|\nabla w_1|^2 + |\nabla w_2|^2)) dx \\
\leq c(t^2 + (1 - t)^2 + 2t(1 - t)) \\
= c.
\]
Therefore $F_c$ is convex.

For the second term
\[-\int_{\mathbb{R}^3} \frac{K}{q+1} w^{q+1} e^{-\alpha s} \, dx\]
we recall, $\forall \epsilon > 0$, $\exists R > 0$, $\forall n, n' > N$:
\[
\int_{\mathbb{R}^3 \setminus B_R} f w_n \, dx \leq \epsilon \\
\int_{\mathbb{R}^3 \setminus B_R} f w_n' \, dx \leq \epsilon \\
\|w_n - w_n'\|_{L^{q+1}(B_R)} < \epsilon.
\]
Therefore,
\[
\|w_n - w_n'\|_{L^{q+1}(\mathbb{R}^3)} \leq \|w_n - w_n'\|_{L^{q+1}(B_R)} + \|w_n - w_n'\|_{L^{q+1}(\mathbb{R}^3 \setminus B_R)}
\leq \epsilon + C(a^{\frac{1}{q+1}} + a^{\frac{1}{q+1}})^{1-a}
\leq \epsilon + C' \epsilon.
\]

The second inequality above follows from the Gagliardo-Nirenberg inequality. Hence $\{w_n\}$ converges in $L^{q+1}(\mathbb{R}^3)$. But $w_n \to \tilde{w}$ in $L^{q+1}(B_R)$ for any $R > 0$. This implies $w_n \to \tilde{w}$ in $L^{q+1}(\mathbb{R}^3)$. Therefore,
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} w_n^{q+1} e^{-\alpha s} \, dx = \int_{\mathbb{R}^3} \tilde{w}^{q+1} e^{-\alpha s} \, dx.
\]
Combine the results, we have
\[
\lim_{n \to \infty} \inf E(w_n) \geq E(\tilde{w}).
\]
This shows $\tilde{w}$ is a minimizer.

\[
5. \text{ Existence of Solution in } H^1 \text{ when } 1 < q < 3
\]

In this section we provide proofs of theorem 2.2 and theorem 2.3. Let us first treat the finite case, the infinite case will be similar. We write
\[
E(w) = \int_{B_R} \left( \frac{e^{\alpha s}}{2} |\nabla w|^2 - \frac{K}{q+1} w^{q+1} e^{-\alpha s} \right) \, dx,
\]
and
\[
N(w) = \int_{B_R} f w \, dx = M.
\]
Let $W_M$ be
\[
H^1_0(B_R) \cap \{ w : B_R \to \mathbb{R} | w \geq 0 \ \text{a.e.,} \ w \text{ is axisymmetric,} \ N(w) = M \},
\]
and let $W$ be
\[
H^1_0(B_R) \cap \{ w : B_R \to \mathbb{R} | w \geq 0 \ \text{a.e.,} \ w \text{ is axisymmetric,} \ N(w) < \infty \}.
\]

We need the existence of minimizer of $E(w)$ in $W_M$ subject to the constraint $N(w) = M$. A similar calculation as lemma 3 shows that $E(w)$ is bounded from below on $W_M$. We likewise define
\[
I_M = \inf_{w \in W_M} \{ E(w) | N(w) = M \},
\]
and can get similar scaling inequalities as in lemma 4. In particular, $I_M < 0$ if $M$ is large enough. The lower-semicontinuity of $E(w)$ is standard in this case. We summarize these by the following

**Lemma 9.** There exists a minimizer $\tilde{w}$ of $E(w)$ in the class $W_M$:

\[ I_M = \inf_{w \in W_M} \{ E(w) | N(w) = M \} = E(\tilde{w}) \]

and $I_M < 0$ when $M$ is sufficiently large.

**Proof.** A similar inequality as in lemma 3 holds, hence $I_M > -\infty$. The first term

\[ \int_{B_R} e^{\alpha s} |\nabla w|^2 dx \]

is convex in $w$ and norm continuous on $H^1_0(B_R)$, hence is weakly lower-semicontinuous. Now if $w_n$ is a minimizing sequence, by lemma 3, it is bounded in $H^1_0(B_R)$. By the Banach-Alaoglu theorem, it contains a weakly convergent subsequence, and by the Rellich-Kondrachov theorem, it is norm convergent in $L^p(B_R)$ for $p < 6$. Hence the second term

\[- \int_{B_R} \frac{K}{q+1} w_n^{q+1} e^{-\alpha s} dx \]

has a subsequence that converges to

\[- \int_{B_R} \frac{K}{q+1} \tilde{w}^{q+1} e^{-\alpha s} dx. \]

The fact that the limit $\tilde{w}$ belongs to $W_M$ is easy to establish. \qed

We now study the Euler-Lagrange equation of the energy.

**Lemma 10.** $\exists \lambda \in \mathbb{R}, \forall u \in W$:

\[ \int_{B_R} (e^{\alpha s} \nabla \tilde{w} \cdot \nabla (u - \tilde{w}) - Ke^{-\alpha s} \tilde{w}^q (u - \tilde{w})) dx \geq -\lambda \int_{B_R} f(u - \tilde{w}) dx \]

**Proof.** Given $u \in W$, when $t > 0$ is small enough,

\[ \tilde{w} + t[(u - \tilde{w}) - \frac{N(u - \tilde{w})}{N(\tilde{w})} \tilde{w}] \in W_M, \]

therefore,

\[ \frac{d}{dt} E(\tilde{w} + t[(u - \tilde{w}) - \frac{N(u - \tilde{w})}{N(\tilde{w})} \tilde{w}])|_{t=0^+} \geq 0. \]

Denote $(u - \tilde{w}) - \frac{N(u - \tilde{w})}{N(\tilde{w})} \tilde{w}$ by $\sigma$, we have

\[ \frac{E(\tilde{w} + t\sigma) - E(\tilde{w})}{t} \]

\[ = \int_{B_R} (e^{\alpha s} \nabla \tilde{w} \cdot \nabla \sigma - \frac{K}{q+1} e^{-\alpha s} (q+1)(\tilde{w} + t\sigma)^q (\tilde{w} + t\sigma) dx + O(t), \]

where $\theta$ is between 0 and $t$, and depends on $x$. Take the limit as $t \to 0^+$, and by the dominated convergence theorem, we get

\[ \lim_{t \to 0^+} \frac{E(\tilde{w} + t\sigma) - E(\tilde{w})}{t} = \int_{B_R} (e^{\alpha s} \nabla \tilde{w} \cdot \nabla \sigma - Ke^{-\alpha s} \tilde{w}^q \sigma) dx. \]
Denote this by $E'_w(\sigma)$, we easily get
\[
0 \leq E'_w(\sigma) = E'_w(u - \tilde{w}) - \frac{E'_w(\tilde{w})}{N(\tilde{w})} N(u - \tilde{w}).
\]
Let $-\lambda = \frac{E'_w(\tilde{w})}{N(\tilde{w})}$, the proof is complete. \□

Lemma 11. If $I_M < 0, q > 1$, then $\lambda > 0$

Proof. Observe that $2 \tilde{w} \in W$, therefore we may plug in $u = 2 \tilde{w}$ to find
\[
-\lambda M = -\lambda \int_{B_R} f(2\tilde{w} - \tilde{w}) dx
\]
\[
\leq \int_{B_R} (e^{\alpha s} |\nabla \tilde{w}|^2 - Ke^{-\alpha s} \tilde{w}^{q+1}) dx
\]
\[
= \int_{B_R} \left(\frac{e^{\alpha s}}{2} |\nabla \tilde{w}|^2 - \frac{K}{q+1} e^{-\alpha s} \tilde{w}^{q+1}\right) dx + \int_{B_R} \left(\frac{e^{\alpha s}}{2} |\nabla \tilde{w}|^2 - \frac{qK}{q+1} e^{-\alpha s} \tilde{w}^{q+1}\right) dx
\]
\[
\leq 2I_M < 0.
\]

Let us write $\tilde{w} = w_1 + w_2$, where $w_1 \in H_0^1(B_R)$ weakly solves
\[
- \nabla \cdot (e^{\alpha s} \nabla w_1) - Ke^{-\alpha s} \tilde{w}^q + \lambda f = 0,
\]
and $w_2 \in H_0^1(B_R)$ weakly solves
\[
- \nabla \cdot (e^{\alpha s} \nabla w_2) = d\mu.
\]
Now from the range of $q$ and the fact that $\tilde{w} \in H^1_0(B_R) \subset L^6(B_R)$, we see $\tilde{w}^q \in L^2(B_R)$. By standard elliptic regularity theory, we conclude that $w_1$ is continuous. We next show $w_2$ is lower semicontinuous following Lewy and Stampacchia [15].

**Lemma 12.** Let $\tilde{B}$ be any ball contained in $B_R$. Let $G(x, y)$ be the Dirichlet Green’s function of $\tilde{B}$ with respect to the operator $-\nabla \cdot (e^{\alpha s} \nabla)$, that is

$$-\nabla_x (e^{\alpha s} \nabla G(x, y)) = \delta_y \quad \text{on } \tilde{B}
$$

$$G(x, y) = 0 \quad \text{on } \partial \tilde{B}$$

then

$$(77) \quad w_2(x) = \int_{\tilde{B}} G(x, y) d\mu(y) - \int_{\partial \tilde{B}} e^{\alpha s(x)} \frac{\partial G(x, y)}{\partial n(y)} w_2(y) d\sigma(y)$$

in $\tilde{B}$, where $\sigma$ is the standard surface measure on $\partial \tilde{B}$.

**Proof.** Pick any ball $B$ contained in $\tilde{B}$. $\forall \varphi \in C^\infty_0(B)$, $\varphi \geq 0$, $\exists u$ solving

$$-\nabla \cdot (e^{\alpha s} \nabla u) = \varphi \quad \text{on } \tilde{B}
$$

$$u = 0 \quad \text{on } \partial \tilde{B}$$

It follows from the Green’s theorem that

$$(78) \quad u(x) = \int_{\tilde{B}} G(x, y) \varphi(y) dy.$$ 

Now

$$\int_{\tilde{B}} w_2(x) \varphi(x) dx
$$

$$= \int_{\tilde{B}} w_2(x) \varphi(x) dx
$$

$$- \int_{\tilde{B}} w_2 \nabla \cdot (e^{\alpha s} \nabla u) dx
$$

$$= \int_{\tilde{B}} e^{\alpha s} \nabla w_2 \cdot \nabla u dx - \int_{\partial \tilde{B}} e^{\alpha s} w_2 \frac{\partial u}{\partial n} d\sigma
$$

$$= \int_{\tilde{B}} u d\mu - \int_{\partial \tilde{B}} e^{\alpha s} w_2 \frac{\partial u}{\partial n} d\sigma
$$

$$= \int_{\tilde{B}} ( \int_{\tilde{B}} G(x, y) \varphi(y) dy) d\mu(x) - \int_{\partial \tilde{B}} e^{\alpha s(x)} w_2(x) ( \int_{\tilde{B}} \frac{\partial G(x, y)}{\partial n(x)} \varphi(y) dy) d\sigma(x)
$$

$$= \int_{\tilde{B}} ( \int_{\tilde{B}} G(x, y) d\mu(x) ) \varphi(x) dy - \int_{\tilde{B}} ( \int_{\tilde{B}} e^{\alpha s(x)} w_2(x) \frac{\partial G(x, y)}{\partial n(x)} d\sigma(x) ) \varphi(y) dy.
$$

The last equality follows from Fubini’s theorem and the fact that $G(x, y) > 0$ when $x \neq y$. 

When $x$ is in a compact subset of $\tilde{B}$, and $y$ on $\partial \tilde{B}$, $\frac{\partial G(x, y)}{\partial n(y)}$ is a smooth function in $x$ and $y$. Hence the integral on $\tilde{B}$ in (77) is continuous in $x$. Also notice that $G(x, y)$ is a pointwise limit of

$$G_\alpha(x, y) = \begin{cases} G(x, y) & \text{if } G(x, y) \leq a \\ a & \text{if } G(x, y) > a \end{cases}$$
and that
\[(79) \quad \int_{\tilde{B}} G_n(x,y)d\mu(y)\]
is continuous in \(x\) on \(\tilde{B}\). By the monotone convergence theorem,
\[\int_{\tilde{B}} G(x,y)d\mu(y)\]
is an increasing pointwise limit of \((79)\), and hence is lower semicontinuous. We can now conclude that \(w_2\), and \(\tilde{w}\) also, are lower semicontinuous. This implies that the set \(U_+ = \{x \in B_R | \tilde{w}(x) > 0\}\) is open. If \(\varphi \in C_0^\infty(U_+)\), then \(\tilde{w} + tS(\varphi) \in W\) for \(|t|\) sufficiently small. A similar calculation as \((72)\) will show that
\[(80) \quad \int_{B_R} e^{\alpha s} \nabla \tilde{w} \cdot \nabla \varphi - Ke^{-\alpha s} \tilde{w}^q \varphi + \lambda f \varphi = 0.\]
In other words, \(\tilde{w}\) solves
\[(81) \quad \nabla \cdot (e^{\alpha s} \nabla w) + Ke^{-\alpha s} w^q - \lambda f = 0\]
weakly on \(U_+\). This completes the proof of theorem 2.2.

Having dealt with the case of a ball of finite radius, we may treat the case with the whole \(\mathbb{R}^3\) in an analogous fashion. Define \(E(w)\) by \((53)\), \(N(w)\) by \((54)\), \(W_M\) by \((56)\) and \(W\) by \(H_1^0(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \cap \{w : B_R \rightarrow \mathbb{R} | w \geq 0 \text{ a.e., } w \text{ is axisymmetric, } N(w) < \infty\}\).

The existence of minimizer in this case is given by theorem 4.1, and the proof for theorem 2.2 works verbatim except that one changes the various integrals from being performed on \(B_R\) to being performed on \(\mathbb{R}^3\). This completes the proof of theorem 2.3.

6. Solution for Given \(\rho\)

In this section we give proofs to theorem 2.4 and theorem 2.5. Again the rotating star equations written out componentwise in the cylindrical coordinates are given by
\[(82) \quad \begin{cases} p_r = \rho(-\phi)_r + \rho r \Omega^2 \\ p_z = \rho(-\phi)_z \end{cases} \]
where \(\phi\) is the Newtonian potential of \(\rho\):
\[(83) \quad \phi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} dy.\]
To prepare ourselves for higher regularity at \(r = 0\), let us prove the following lemma.

**Lemma 13.** Let \(f : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}\) be such that \(f(-r,z) = -f(r,z)\), and assume that \(f \in C^k, k \geq 1\), then the function
\[(84) \quad g(r,z) = \begin{cases} f(r,z) & r \neq 0 \\ f_r(0,z) & r = 0 \end{cases} \]
is in \(C^{k-1}\).
Proof. Obviously \( f(0, z) = 0 \), hence for \( r \neq 0 \),
\[
g(r, z) = \frac{1}{r}(f(r, z) - f(0, z))
\]
\[
= \frac{1}{r} \int_0^r f_s(s, z) ds
\]
\[
= \frac{1}{r} \int_0^1 f_s(rs, z) r ds
\]
\[
= \int_0^1 f_s(rs, z) ds
\]
(85)

Apparently the same equation is true for \( r = 0 \), and the assertion is clear from this formula. \( \Box \)

Now we can give a proof of theorem 2.4.

Proof. From the definition of \( \phi \), we easily get
\[
\phi_z(x) = -\int_D \frac{\rho_z(y)}{|x - y|} dy.
\]
Therefore \( \phi_z(r, -z) = -\phi_z(r, z) \) and \( \phi_z(r, z) < 0 \) when \( z < 0 \), by hypothesis 4 and the symmetry of \( \rho \) and \( D \). Let
\[
p(r, z) = \int_{-\psi(r)}^z \rho(r, \xi)(-\phi_z(r, \xi)) d\xi.
\]
(87)

From now on we allow \( r \) to take negative values by evenly extending all the relevant functions across \( r = 0 \). It is easily seen that \( p > 0 \) in \( D \), \( p = 0 \) on \( \partial D \) and that \( p \) satisfies the second equation in (82). Since \( \rho \in C^k(\bar{D}) \) and \( \partial D \) is smooth, we have at least \( \phi \in C^{k+1}(\bar{D}) \). It is not difficult to see that \( p \in C^k(\bar{D}) \). Differentiate (87) under the integral sign, we get
\[
p_r(r, z) = \int_{-\psi(r)}^z (\rho_r(r, \xi)(-\phi_z(r, \xi)) + \rho(r, \xi)(-\phi_z(r, \xi))) d\xi.
\]
(88)

By the first equation in (82), when \( r > 0 \), \( \Omega^2 \) has to have the form:
\[
\Omega^2 = \frac{1}{r \rho}(p_r - \rho(-\phi)_r)
\]
\[
= \frac{1}{r \rho} \left( \int_{-\psi(r)}^z (\rho_r(-\phi_\xi) + \rho(-\phi_\xi)) d\xi - \rho(-\phi)_r \right)
\]
\[
= \frac{1}{r \rho} \left( \int_{-\psi(r)}^z (\rho_r(-\phi_\xi) + \rho(-\phi_\xi)) d\xi - \int_{-\psi(r)}^z (\rho(-\phi)_r) \xi d\xi \right)
\]
\[
= \frac{1}{r \rho} \int_{-\psi(r)}^z (\rho_r(-\phi_\xi) - \rho(\phi)_r) d\xi.
\]
(89)

(89) is indeed nonnegative on \( D \) by hypothesis 3. Define \( \Omega^2 \) by (89), when \( r > 0 \), and if \( D \) contains points at \( r = 0 \), by
\[
\Omega^2(0, z) = \frac{1}{\rho}(p_r - \rho(-\phi)_r)(0, z)
\]
(90)
Notice \( p_r - \rho(-\phi)_r \) is odd in \( r \). By lemma \([13]\) \( \rho \Omega^2 \in C^{k-2}(D) \), hence so is \( \Omega^2 \). Such \( p \) and \( \Omega^2 \) obviously satisfy \([82]\). It remains to show that \( \Omega^2 \) extends to a continuous function on \( \bar{D} \). Let us consider the following three cases:

1. Let \( r_0 \) be a nonzero radius such that \((r_0, -\psi(r_0)) \in \partial D \) and \( \psi(r_0) > 0 \).

\[
\lim_{(r,z) \to (r_0, -\psi(r_0))} \Omega^2(r, z) = \lim_{(r,z) \to (r_0, -\psi(r_0))} \frac{1}{r \rho} \int_{-\psi(r)}^{z} (\rho_r(-\phi \xi) - \rho \xi(-\phi_r)) d\xi
= \lim_{(r,z) \to (r_0, -\psi(r_0))} \frac{1}{r \rho z} \left( \rho_r(-\phi z - \rho z(-\phi_r)) \right)
= \lim_{r \to 0} \frac{1}{\rho z(0, -\psi(0))} \left( \rho_r(-\phi z - \rho z(-\phi_r)) \right)
(91)
\]

Here we have used the mean value theorem and hypothesis 4.

2. If \( \partial D \) contains points at \( r = 0 \), since \( \partial D \) is smooth and symmetric about \( z = 0 \) and \( r = 0 \), we have \( \psi(0) > 0 \), \( \psi'(0) = 0 \). Hence

\[
\lim_{(r,z) \to (0, -\psi(0))} \Omega^2(r, z) = \lim_{r \to 0} \frac{1}{r \rho} \int_{-\psi(r)}^{z} (\rho_r(-\phi \xi) - \rho \xi(-\phi_r)) d\xi
= \lim_{r \to 0} \frac{1}{r \rho z} \left( \rho_r(-\phi z - \rho z(-\phi_r)) \right)
= \lim_{r \to 0} \frac{1}{\rho z(0, -\psi(0))} \left( \rho_r(-\phi z - \rho z(-\phi_r)) \right),
(92)
\]

and

\[
\lim_{z \to -\psi(0)} \Omega^2(0, z) = \lim_{z \to -\psi(0)} \frac{1}{\rho(0, z)} \left( \rho_r - \rho(-\phi)_r \right)_r(0, z)
= \lim_{z \to -\psi(0)} \frac{1}{\rho(0, z)} \left( \int_{-\psi(0)}^{z} (\rho_r(-\phi \xi) - \rho \xi(-\phi_r)) d\xi
+ \psi'(0) \left( \rho_r(-\phi z - \rho z(-\phi_r))_r(0, -\psi(0)) \right) \right)
= \frac{1}{\rho z(0, -\psi(0))} \left( \rho_r(-\phi z - \rho z(-\phi_r))_r(0, -\psi(0)) \right).
(93)
\]
(3) Let \( r_0 \) be such that \( \psi(r_0) = 0 \). When \((r, z)\) gets close to \((r_0, 0)\) and \(z \leq 0\), we observe by the mean value theorem that

\[
\Omega^2(r, z) = \frac{1}{r\rho} \int_{-\psi(r)}^{z} \left( \rho_r(-\phi)\xi - \rho_\xi(-\phi)_r d\xi \right)
\]

\[
= \frac{1}{r\rho_z} \left( \rho_r(-\phi)_z - \rho_z(-\phi)_r \right) (r, z')
\]

\[
= \frac{1}{r\rho_{zz}} \left( \rho_r(-\phi)_z - \rho_z(-\phi)_r \right)_z (r, z''),
\]

where \( z' \) is between \(-\psi(r)\) and \( z \), and \( z'' \) is between \( z' \) and \( 0 \). Therefore

\[
\lim_{(r, z) \to (r_0, 0)} \Omega^2(r, z) = \frac{1}{r\rho_{zz}} \left( \rho_r(-\phi)_z - \rho_z(-\phi)_r \right)_z (r_0, 0).
\]

Here we use hypothesis (a) to conclude that the limit is finite.

**Remark 3.** It is possible to establish higher regularity for \( \Omega^2 \) at the first two types of boundary points. However, at the third type of boundary points, \( \psi'(r_0) = \infty \), in order to get higher regularity, we need conditions on how fast \( \psi'(r) \) grows at around \( r_0 \), which we do not employ ourselves doing here.

With relaxed regularity conditions at the boundary, the same computation works if further growth conditions are imposed on the derivatives of \( \rho \) when close to the boundary, as is done in theorem \( \ref{thm:2.5} \). Here is a proof of that theorem:

**Proof.** The proof will follow the lines of the proof to theorem \( \ref{thm:2.4} \). As before, we define

\[
p(r, z) = \int_{-\psi(r)}^{z} \rho(r, \xi)(-\phi_\xi(r, \xi)) d\xi.
\]

It follows from hypothesis 1, 2, 4 that \( p > 0 \) in \( D \), \( p = 0 \) on \( \partial D \) and that \( p \) satisfies the second equation in \( \ref{eq:82} \). Since \( \rho \in C^{\beta}(\bar{D}) \), we have \( \phi \in C^{2,\beta}(\bar{D}) \), hence \( p \in C^0(D) \). Now let us calculate the \( r \) partial derivative of \( p \). In the following, let \( b = \max_{r \leq s \leq r + h} (-\psi(s)) \).

\[
\frac{1}{h} \left( \int_{-\psi(r+h)}^{z} \rho(-\phi)_\xi(r + h, \xi) d\xi - \int_{-\psi(r)}^{z} \rho(-\phi)_\xi(r, \xi) d\xi \right)
\]

\[
= \frac{1}{h} \left( \int_{-\psi(r+h)}^{b} \rho(-\phi)_\xi d\xi - \int_{-\psi(r)}^{b} \rho(-\phi)_\xi d\xi + \int_{b}^{z} (\rho(-\phi)_\xi(r + h, \xi) - \rho(-\phi)_\xi(r, \xi)) d\xi \right).
\]

It is easily seen that the first two terms converge to 0 as \( h \) goes to 0. Let us focus on the last term:

\[
\frac{1}{h} \int_{b}^{z} (\rho(-\phi)_\xi(r + h, \xi) - \rho(-\phi)_\xi(r, \xi)) d\xi
\]

\[
= \int_{b}^{z} (\rho(-\phi)_\xi)_r(r', \xi) d\xi,
\]
where \( r' \) is between \( r \) and \( r + h \). We will use the dominated convergence theorem to compute the limit of \( (98) \). For that purpose we need an estimate on \( \rho(r, \xi) \). For the moment let us assume \( z < 0 \). By hypothesis 5, there is a \( C > 0 \) such that \( |\rho_r| \leq C|\rho|, |\rho_{rr}| \leq C|\rho|, |\rho_{r\xi}| \leq C|\rho| \) for \( \xi < z \). Also \( \rho_{\xi} > 0 \). Therefore

\[
|\rho(r, \xi)| \\
\leq C|\rho(r, \xi)| + C_2|\rho(r', \xi)| \\
\leq C_1|\rho_\xi(r, \xi) + \int_r^{r'} \rho_{ss}(s, \xi) ds| + C_2|\rho(r, \xi) + \int_r^{r'} \rho_s(s, \xi) ds| \\
\leq (C_1 + C_2)\left( |\rho_\xi(r, \xi)| + |\rho(r, \xi)| + C \int_{r-h_0}^{r+h_0} \rho_{\xi}(s, \xi) ds \right) \\
(99) \\
\leq C_3 \left( \rho_\xi(r, \xi) + \rho(r, \xi) + \int_{r-h_0}^{r+h_0} \rho_{\xi}(s, \xi) ds \right),
\]

for some fixed \( h_0 > h \). In the last integral term, if \( (s, \xi) \) lies outside of \( D \), then extend \( \rho_{\xi} \) to be 0. The fact that the integral of \( (99) \) is finite is manifested by the following:

\[
(100) \\
\int_{-\psi(r)}^{z} \rho_{\xi}(r, \xi) d\xi = \rho(r, z)
\]

\[
(101) \\
\int_{-\psi(r)}^{z} \rho(r, \xi) d\xi < \infty
\]

\[
\int_{-\psi(r)}^{z} \int_{r-h_0}^{r+h_0} \rho_{\xi}(s, \xi) ds d\xi \\
\leq \int_{r-h_0}^{r+h_0} \int_{-\psi(s)}^{z} \rho_{\xi}(s, \xi) d\xi ds \\
\leq \int_{r-h_0}^{r+h_0} \rho(s, z) ds
\]

\[
(102) \\
< \infty.
\]

Therefore, by the dominated convergence theorem,

\[
(103) \\
p_r(r, z) = \int_{-\psi(r)}^{z} (\rho(-\phi)_{\xi}) d\xi.
\]

Now if \( z \geq 0 \), the integral in \( (98) \) can be broken into two pieces: one from \( b \) to \( z' \) and the other from \( z' \) to \( z \), for some \( z' < 0 \). Notice that the second piece lies completely inside \( D \), where \( \rho \) is \( C^2 \), so the limit is the same as before. We have proved \( p \in C^1(D) \). Now define \( \Omega^2 \) by

\[
(104) \\
\frac{1}{r} \int_{-\psi(r)}^{z} (\rho_r(-\phi_{\xi}) - \rho_{\xi}(-\phi_{r})) d\xi.
\]

when \( r > 0 \), and if \( D \) contains points at \( r = 0 \), by

\[
(105) \\
\Omega^2(0, z) = \frac{1}{\rho} \int_{-\psi(0)}^{z} (\rho_r(-\phi_{\xi}) - \rho_{\xi}(-\phi_{r})) d\xi
\]
The convergence of these integrals are guaranteed by hypothesis 5. It is easy to verify that such \( p \) and \( \Omega^2 \) satisfy (82). Let us now show that \( \Omega^2 \) is continuous on \( D \). Since \( \partial D \) is smooth and convex at \((0, -\psi(0))\), \(-\psi(r) = \max_{0 \leq s \leq r} (-\psi(s)) \) for \( r \) small enough. Therefore,

\[
\frac{1}{r} \int_{-\psi(r)}^{z} \left( \rho_r(-\phi_\xi) - \rho_\xi(-\phi_r) \right) d\xi
\]

(106)

where \( r' \) is between 0 and \( r \). As before we assume \( z < 0 \) and estimate the integrand,

\[
| \left( \rho_r(-\phi_\xi) - \rho_\xi(-\phi_r) \right) z (r', \xi) |
\]

\[
\leq C_1 (|\rho_{rr}(r', \xi)| + |\rho_{r\xi}(r', \xi)| + |\rho_r(r', \xi)| + |\rho_\xi(r', \xi)|)
\]

\[
\leq \tilde{C} \rho_\xi(r', \xi)
\]

(107)

where \( r_0 > r \) is small and fixed. As before, (107) has a finite \( \xi \) integral. By the dominated convergence theorem,

\[
\lim_{(r, z) \to (0, z_0)} \Omega^2(r, z) = \Omega^2(0, z).
\]

(108)

Again by splitting the integral (104) into boundary and interior parts, (108) continues to be true when \( z_0 > 0 \), and \( \Omega^2(0, z) \) is evidently continuous in \( z \). This establishes the continuity of \( \Omega^2 \) at \( D \cap \{ r = 0 \} \). The continuity of \( \Omega^2 \) away from the \( z \) axis is obvious. It remains to show that \( \Omega^2 \in L^\infty(D) \). Let us consider the following three cases:

1. Let \( r_0 \) be a nonzero radius such that \((r_0, -\psi(r_0)) \in \partial D \) and \( \psi(r_0) > 0 \).

When \( z < -\frac{1}{2} \psi(r_0) \),

\[
\Omega^2(r_0, z) = \frac{1}{r_0 \rho} \int_{-\psi(r_0)}^{z} \left( \rho_r(-\phi_\xi) - \rho_\xi(-\phi_r) \right) d\xi
\]

\[
= \frac{1}{r_0 \rho z} (\rho_r(-\phi_z) - \rho_z(-\phi_r))(r_0, z')
\]

\[
\leq \frac{1}{r_0} (C |(-\phi)_z| + |(-\phi)_r|)
\]

(109)

where \( C \) is given by hypothesis 5.
(2) If \( \partial D \) contains points at \( r = 0 \), as \((r, z) \) gets close to \((0, -\psi(0)), r \neq 0, \)
\[
\Omega^2(r, z) = \frac{1}{r \rho z} \int_{-\psi(r)}^{z} (\rho_r(-\phi_r) - \rho_z(-\phi_r)) d\xi
\]
\[
= \frac{1}{r \rho z} (\rho_r(-\phi_z) - \rho_z(-\phi_r))(r, z')
\]
\[
= \frac{(\rho_r(-\phi)_z - \rho_z(-\phi)r)}{\rho_z + r \rho_r z}(r', z')
\]
\[
\leq \frac{\tilde{C}}{1 + r'^2 C}
\]
where \( z' \) is between \(-\psi(r) \) and \( z \), and \( r' \) is between 0 and \( r \). In this process we have used the mean value theorem several times, the justification being that the convexity of \( \partial D \) at \((0, -\psi(0)) \) guarantees that all the relevant segments lie inside \( D \). On the other hand if \( r = 0, \)
\[
\Omega^2(0, z) = \frac{1}{\rho z} \int_{-\psi(0)}^{z} (\rho_r(-\phi_r) - \rho_z(-\phi_r)) d\xi
\]
\[
= \frac{1}{\rho z} (\rho_r(-\phi)_z - \rho_z(-\phi)r)(0, z')
\]
\[
\leq \tilde{C}.
\]
(3) Let \( r_0 \) be such that \( \psi(r_0) = 0 \). When \((r, z) \) gets close to \((r_0, 0) \) and \( z \leq 0, \)
\[
\Omega^2(r, z) = \frac{1}{r \rho z} \int_{-\psi(r)}^{z} (\rho_r(-\phi_r) - \rho_z(-\phi_r)) d\xi
\]
\[
= \frac{1}{r \rho z} (\rho_r(-\phi)_z - \rho_z(-\phi)r)(r, z')
\]
\[
= \frac{1}{r \rho z} (\rho_r(-\phi)_z - \frac{(-\phi)_r}{r})(r, z')
\]
\[
= \frac{1}{r \rho z} (\rho_r(-\phi)_z)(r, z'' - \frac{(-\phi)_z}{r})(r, z'),
\]
where \( z' \) is between \(-\psi(r) \) and \( z \), and \( z'' \) is between \( z' \) and \( 0 \). If hypothesis (a) is satisfied, by \([114] \), \( \Omega^2(r, z) \) is bounded. If hypothesis (a') is satisfied, by \([112] \) and the fact that \( \rho_z > 0, (-\phi)_z > 0 \) when \( z < 0, \)
\[
0 \leq \Omega^2(r, z) \leq -\frac{1}{r} (-\phi)_r(r, z').
\]
Therefore \( \Omega^2(r, z) \) is bounded. If hypothesis (a'') is satisfied, by \([113] \) and the fact that \( |(-\phi)_z| < C|z| \), we again get the boundedness of \( \Omega^2(r, z) \).

\[\Box\]

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