Landau superfluids as non equilibrium stationary states

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Abstract

Following B. Kuckert, we introduce a class of quantum states, called semipassive. They include Galilean-covariant, space-translation invariant states of Bosons in uniform translational motion, as well as rotationally invariant states of Bosons in a rotating cylindrical bucket which, at positive temperature, are shown to be non equilibrium stationary states (NESS). Restricting us to translational motion, we define a superfluid state to be a NESS, which, at zero temperature, satisfies certain metastability conditions, which physically express that there should be a sufficiently small energy-momentum transfer between the particles of the fluid and the surroundings (e.g., pipe). It is shown that two models, the Girardeau model and the Huang-Yang-Luttinger (HYL) model describe superfluids in this sense and, moreover, that, in the case of the HYL model, the metastability condition is directly related to Nozières’ conjecture that, due to the repulsive interaction, the condensate does not suffer fragmentation into two (or more) parts, thereby assuring its quantum coherence. The examples, both at positive and zero temperature, seem to be the first rigorous examples of NESS in which the system is not finite, but rather a many-body system.

Dedicated to the memory of Bernd Kuckert.
1 Introduction and Summary

Superfluidity of a Bose fluid (e.g. Helium IV) remains an outstanding and fascinating theoretical problem (see the complementary reviews [Kad13] and [Leg99], as well as the book [Leg06]). In particular, Kadanoff in [Kad13] (see also [BP12]) recently rather sharply questioned the relevance of Landau’s criterion of superfluidity to the superfluid property. This indicates that the concept of superfluidity still lacks a clear and precise theoretical foundation. The present paper is an attempt to bridge this gap.

In this paper, we shall be concerned with homogeneous systems, but some preliminary remarks on trapped gases may be helpful. The superfluidity of dilute trapped Bose gases has been rigorously proved in a pioneer work [EHLY02], in the dilute limit in which the Gross-Pitaevski equation is exact, which we refer to briefly as the GP model. This brilliantly confirmed the existing experimental results on rotational superfluidity ([DS03], Fig. 7a). We refer to [SW09] for a somewhat different viewpoint, viz. that rotational superfluidity is a consequence of off-diagonal long range order (ODLRO) [PO56] — which defines [Yan62] Bose Einstein condensation (BEC) precisely in the present of interactions — and has been proved in the ground state of the GP model by Lieb and Seiringer in a seminal paper [LS02].

In the GP model in dimension $d$, the quantity $N \frac{a}{L}$ is kept fixed, where $N$ is the particle number, $L$ the side of the cubic box enclosing the system, and $a$ the scattering length; in the GP limit, $L$ is fixed and $a \to 0$, which is achieved by a scaling of the potential [EHLY05]. Alternatively, one may consider a fixed potential, which is best for comparison with the usual setting for the Bose fluid, take $L \to \infty$ and scale the density as $\rho \approx L^{1-d}$ as $L \to \infty$. For $d = 2, 3$ this leads to zero density and zero sound velocity, the latter signifying that by the Landau criterion of linear dispersion of the elementary excitations ([LL67],[HDCZ09],[WdSJ05]) there is no translational superfluidity. This criterion may be roughly stated in the following way: by the flow of a fluid along a pipe, momentum may be lost to the walls, if the modulus of the velocity $|\vec{v}|$ is greater than

$$v_c \equiv \min_{\vec{p}} \frac{\epsilon(\vec{p})}{|\vec{p}|},$$

(1.1)

where $\epsilon(\vec{p})$ are the energies of the “elementary excitations” generated by friction.
The fact that no experimental results have been reported on translational superfluidity for dilute trapped Bose gases may be an indication that the GP regime of dilution (now rigorously analysed in [AEY06]), which has been shown to describe the experiments extremely well, may indeed be too strong to account for that property, in spite of allowing for rotational superfluidity, a property shared by the free Bose gas [Le75]. This conjecture would also be supported by the analysis of [SW09], which suggests that the origins of both types of superfluidity are entirely different. The criterion used in [EHL02] indicates otherwise, but the authors of [EHL02] emphasize that their use of this criterion does not imply advocacy of any particular approach to the superfluidity question.

In general, now regarding the homogeneous case, the issue persists, and, indeed, in [Kad13], Kadanoff argues that “given the many mechanisms for broadening the distributions of both energy and momentum, it seems very implausible that a condition like (1.1) can begin to account for the very long-lived nature of the flow of superfluid Helium” — see also [BP12]. This argument is essentially supported by the results of the present paper, see Remark 3.4. In the sequel, however, Kadanoff suggests [Kad13] that superfluidity is brought about by the existence of a coherent, “macroscopic” wave-function of type

$$\Psi(\vec{x}) = \exp \left[ \frac{i\chi(\vec{x})}{\hbar} \right]$$

with $\chi$ possibly complex; see also the more complete analysis of ([Leg99], [Leg06]). This macroscopic wave-function is precisely the complex classical field occurring in the definition of ODLRO, and explains the two-fluid model and the London-Landau superfluid hydrodynamics ([Lon54], [Kha65], [Lan41]; see also [MR04] for a nice textbook treatment, and [SW09], pg. 7, for a related discussion). Thus, ODLRO and the associated coherence properties of the condensate wave-function would suffice as the basis of the phenomenon of translational superfluidity, which would, then, not even require an interaction, at least conceptually, since the free Bose gas also exhibits ODLRO (see, e.g., [MR04], pg. 119).

It seems to us, however, that a crucial element is missing in the above discussion, namely, the stability of the condensate, a point brought up emphatically by Nozières in [Ner03]. Consider the situation in which one asks whether, instead of having at $T = 0$ a condensate strictly in the lowest energy state, one could fragment it into two states 1 and 2, of arbitrarily close energies in the thermodynamic limit, with populations $N_1$ and $N_2 = N - N_1$. 3
Naturally, only the potential energy is able to distinguish between these two choices. Adopting a Hartree (mean-field) approximation with interaction Hamiltonian

$$H_{N,V} = \frac{UN^2}{2V}, \quad (1.3)$$

with $U > 0$, he suggests that the energy costs

$$U\left[\left(N_1 + N_2\right)^2 - N_1^2 - N_2^2\right] = UN_1N_2,$$

i.e., extensive in both $N_1$ and $N_2$ in the thermodynamic limit, where $U > 0$ is the strength of the interaction, assumed repulsive, and thus that it is the Coulomb interaction which interdicts the fragmentation, thereby assuring the condensate’s quantum coherence, with the corresponding wave-function becoming a macroscopic observable. His argument in the above form is not directly relevant to superfluidity, because it is easy to see that the mean-field interaction Hamiltonian (1.3) does not exhibit superfluidity (in Landau’s sense), see Section 3. We shall show in that section, however, that an effective Hamiltonian for dilute systems — the Huang-Yang-Luttinger (HYL) model ([KHL57], [Hua87], [MvdBP]) — does reproduce Nozières’ heuristics in a precise sense.

The homogeneous Bose gas in a cubic box with periodic boundary conditions (b.c.) at zero temperature has been recently studied in an interesting paper by Cornean, Derezinski and Zin [HDCZ09]: in particular, various new rigorous results, as well as an extensive review of known results, together with important conjectures for further research, were presented there. Their discussion suggests a further sharpening of Landau’s criterion — that the crucial quantity which determines the existence or non-existence of translational superfluidity is the infimum of the excitation spectrum, which the authors define precisely in the thermodynamic limit. In the present paper we propose a more general framework, which adds what we believe is a significant input to the analysis of [HDCZ09], and which is at the same time related to Nozières’ observation on stability.

In order that the theory does not depend on the details of finite systems in a box, but only on quantities which remain fixed in the thermodynamic limit, we formulate our framework taking this into account in Section 2. It turns out that superfluids fit into a modification of a broader concept, that of semipassivity, introduced by Kuckert in [Kuc02], based on an idea of Bros and Buchholz in [BB94], which includes Galilean-covariant, space-translation invariant states of Bosons in uniform translational motion, as well as rotationally invariant states of Bosons in a rotating cylindrical bucket. At
positive temperature, they are shown quite generally to be non-equilibrium stationary states (NESS) (Theorem 2.9), while, for \( T = 0 \), some uniformity conditions are required for NESS (Theorem and Corollary 2.12). Restricting us to translational motion, we define a superfluid state to be a NESS, which, at zero temperature, satisfies certain metastability conditions (Definition 2.8), which physically express that there should be a sufficiently small energy-momentum transfer between the particles of the fluid and the surroundings (e.g., the pipe). In Section 3 it is shown that two models, the Girardeau model [Gir60] and the Huang-Yang-Luttinger (HYL) model [KHL57] describe superfluids in this sense and, moreover, that, in the case of the HYL model, the metastability condition is directly related to the previously mentioned Nozières’ conjecture.

The examples, both at positive and zero temperature, seem to be the first rigorous examples of NESS in which the system is not finite, but rather a many-body system.

As a last remark, our results for \( T = 0 \) do not rely on any assumptions on states of infinite systems. That is not the case regarding the results for \( T > 0 \), which rely on standard assumptions [Sew82], see also [SW09], which hold for the free Bose gas but have not been verified for any interacting system. In fact, even for the simplest model, considered in section 3, the one-dimensional system of impenetrable Bosons, results on time-dependent correlation functions - Green functions - which are essential to describe equilibrium properties - are very scarce: Lenard’s on the Fourier transform of the one-particle density matrix, implying the absence of ODLRO [Len64], and a few results on higher order correlations, restricted, however, to Dirichlet and Neumann boundary conditions [PJFG03]. One basic reason for this difficulty is that, for continuous Bose systems, time evolution does not define an automorphism of the algebra of quasi-local observables (see section 2), a fact first proved and analysed by Dubin and Sewell [DS70] (see also the comments on this paper and subsequent ones in [BR97], second edition, pg. 460): this already occurs for the free Bose gas, which, as a consequence, is not a C*, but rather a W* dynamical system (see [DS70] as well as well as [Pil06], Def. 4.13 and Example 4.17); in contrast, systems with finite propagation speed (or finite group velocity) such as relativistic systems and quantum spin systems (as a consequence of the Lieb-Robinson bounds, see [NS06] and references given there), are C* dynamical systems and time evolution is an automorphism of the observable algebra.
2 The general framework

2.1 General considerations

As a preliminary, we briefly describe in this section the usual framework of C* or W* dynamical systems, referring to [Pil06] and [Sew86] for comprehensive surveys, [Hug72] and [MW13], Section 6.4 for shorter expositions, and [BR87], [BR97] for a full account. The infinite quantum system is characterized by a quasi-local C* algebra of observables \( A = \bigcup \mathcal{A}(\mathcal{O}) \) with \( \mathcal{A}(\mathcal{O}) \) the C* algebra associated to a finite region \( \mathcal{O} \) of \( \mathbb{R}^d \) with \( d \) the dimension and the bar denoting closure in the norm topology of \( A \). We shall assume that \( A \) has an identity \( 1 \).

Time evolution \( \alpha_t \) with \( t \in \mathbb{R} \) is assumed to be a norm continuous automorphism group of \( A \) (see later how this is achieved for continuous quantum systems). Each state of the system is described by a linear functional \( \omega \) on \( A \) which associates to each \( A \in A \) a number \( \omega(A) \in \mathbb{C} \) such that \( \omega(A^*A) \geq 0 \) for all \( A \in A \), and \( \omega(1) = 1 \). By the GNS construction \( \omega \) is induced by a cyclic vector \( \Omega_\omega \), a (physical) separable Hilbert space \( H_\omega \), and a representation \( \pi_\omega(A) \) of \( A \) by bounded operators on \( H_\omega \), such that \( \omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \) and \( \pi_\omega(A)\Omega_\omega = H_\omega \). We assume that \( \omega \) is invariant under \( \alpha_t \) for all \( t \in \mathbb{R} \). It follows that there exists a strongly continuous unitary group \( U_t, t \in \mathbb{R} \), implementing \( \alpha_t \) and leaving \( \Omega_\omega \) invariant. By Stone’s theorem, \( U_t = \exp(itH_\omega) \), where the self-adjoint operator \( H_\omega \) is called the physical Hamiltonian.

Thermal equilibrium states are characterized by the KMS condition. Let \( 0 < \beta < \infty \) denote the inverse temperature.

Definition 2.1. A state \( \omega \) is a \((\alpha_t, \beta)\) - KMS state if, \( \forall A, B \in A \), there exists a function \( F_{A,B} \) analytic inside the strip \( \mathcal{D}_\beta \equiv \{ z \mid 0 < \Im z < \beta \} \), bounded and continuous on its closure \( \overline{\mathcal{D}_\beta} \), and satisfying the KMS boundary condition

\[
F_{A,B}(t) = \omega(A\alpha_t(B)) \text{ and } F_{A,B}(t + i\beta) = \omega(\alpha_t(B)A) \text{ for all } t \in \mathbb{R}.
\]

(2.1)

A KMS state is \( \alpha_t \)-invariant.

The C* dynamical system \((A, \alpha_t, \omega)\), where \( \omega \) is a \((\alpha_t, \beta)\) KMS state, is said to describe a physical system in thermal equilibrium at temperature \( 1/\beta \), i.e., a temperature state. This is motivated by the fact that (2.1) is satisfied by the thermodynamic limit of Gibbs states [Hug72]. In a similar way, one may define a ground state as a thermodynamic limit of ground states [Hug72]. It is mathematically convenient to consider ground states as
\[ \beta = \infty \] KMS states, whence the notation \( \omega_\infty \). An alternative definition (see \cite{BR97}, Definition 5.3.18, and \cite{Hug72}, Section 3), which is satisfied by the thermodynamic limit of ground states and is thus similarly motivated, is the following one.

**Definition 2.2.** \( \omega_\infty \) is a \( \alpha_t \)- ground state if

\[ -i\omega_\infty (A^*\delta(A)) \geq 0 \quad \forall A \in D(\delta), \tag{2.2} \]

where \( \delta \) is the infinitesimal generator of \( \alpha_t \), i.e., the derivation

\[ \delta(A) \equiv \lim_{t \to 0} \frac{d\alpha_t(A)}{dt}, \quad A \in D(\delta), \]

and \( D(\delta) \) denotes the domain of the derivation \( \delta \); in the representation \( \pi_{\omega_\infty} \) determined by \( \omega_\infty \) (in the following we omit the suffix \( \infty \) for brevity),

\[ \delta(\pi_{\omega}(A))\Psi = i[H_{\omega}, \pi_{\omega}(A)]\Psi \quad \forall \Psi \in H_{\omega} \text{ with } [H_{\omega}, \pi_{\omega}(A)]\Psi \in H_{\omega}. \]

We have \cite[Proposition 5.3.19 or Hug72, Theorem 3.4]{BR97}:

**Proposition 2.3.** A state is a ground state if it is \( \alpha_t \)- invariant and satisfies the spectrum condition

\[ H_{\omega_\infty} \geq 0. \tag{2.3} \]

Both temperature states and ground states will be referred to as equilibrium states. In \cite{PW78} the important notion of passivity was introduced:

**Definition 2.4.** The state \( \omega \) is said to be a passive state if

\[ -i\omega(U^*\delta(U)) \leq 0 \tag{2.4} \]

for any

\[ U \in U_0(A) \cap D(\delta), \tag{2.5} \]

where \( U_0(A) \) denotes the connected component of the identity of the group of unitaries of \( A \) with the uniform topology.

The passivity condition (2.5) is related to entropy production and Carnot’s version of the second law of thermodynamics for equilibrium states, see, e.g., \cite{JP02}. It was proved \cite[see also BR97, Theorem 5.3.22]{PW78} that,
roughly speaking, passive states are equilibrium states (an additional requirement, that of complete passivity, is required — see the references above, [Kuc02] and [Wre05]).

Our states will be assumed to be functionals of the local observables which on $\mathcal{A}(\mathcal{O})$ reduce to normal states, i.e., states of the form

$$\omega(A) = \operatorname{Tr}_{\mathcal{H}_\mathcal{O}}(\rho_\mathcal{O}A) \quad \forall A \in \mathcal{A}(\mathcal{O}),$$

where $\rho_\mathcal{O}$ is a density matrix, i.e., a positive, normalized trace-class operator on $\mathcal{H}_\mathcal{O}$; for the ground state, $\rho_\mathcal{O} = |\Omega_\mathcal{O}\rangle\langle\Omega_\mathcal{O}|$, where $|\Omega_\mathcal{O}\rangle$ is a normalized vector in $\mathcal{H}_\mathcal{O}$. Such states are called \textit{locally finite} or \textit{locally normal}.

For Bosons in translational motion, we shall take as our finite regions $\mathcal{O}$ cubes $\Lambda$ of side $L$ with periodic boundary conditions (b.c.). The restriction of a ground state to $\mathcal{A}_\Lambda$ will be denoted by

$$\omega_{\infty,\Lambda} = \langle \Omega_\Lambda, \cdot \Omega_\Lambda \rangle,$$

where we shall omit the subscript infinity when there is no risk of confusion with a temperature state

$$\omega_{\beta,\Lambda} = \frac{\operatorname{Tr}_{\mathcal{H}_\Lambda}(\exp(-\beta H_\Lambda)\cdot)}{Z_\Lambda(\beta)}, \quad \beta > 0,$$

where

$$Z_\Lambda(\beta) \equiv \operatorname{Tr}_{\mathcal{H}_\Lambda} \exp(-\beta H_\Lambda).$$

Above, $\mathcal{H}_\Lambda = \mathcal{F}_{\Lambda,s}$, the (symmetrical) Fock space over $L^2(\Lambda)$. The generator of space translations on $\mathcal{H}_\Lambda$ — the total momentum — will be denoted by $\vec{P}_\Lambda$. Above, $\Omega_\Lambda$ is a ground state of $H_\Lambda$ on $\mathcal{H}_\Lambda$:

$$H_\Lambda \Omega_\Lambda = E_\Lambda \Omega_\Lambda,$$

where

$$E_\Lambda \equiv \inf \operatorname{spec}(H_\Lambda).$$

In addition,

$$\vec{P}_\Lambda \Omega_\Lambda = 0.$$

By thermodynamic stability,

$$E_\Lambda \geq -c|\Lambda|,$$
where $|\Lambda|$ is the volume of $\Lambda$, and $c$ is a positive constant. In general, $E_\Lambda$ is of order of $O(-d|\Lambda|)$ for some $d > 0$, and in order to obtain a physical Hamiltonian satisfying (2.3) it is necessary to perform a renormalization (infinite in the thermodynamic limit)

$$H_\Lambda \rightarrow \tilde{H}_\Lambda \equiv H_\Lambda - E_\Lambda.$$  (2.9.3)

For Bosons in a rotating cylindrical bucket, $\Lambda$ will denote a cylindrical region, with Neumann b.c. on the boundary (see [Le75], [SW09]).

Define

$$\alpha_{\Lambda,t}(A) \equiv \exp(itH_\Lambda)A \exp(-itH_\Lambda)$$

and the Green’s function for the finite system as

$$G_\Lambda(A, B; t, \bar{x}) \equiv \omega_\Lambda(\alpha_{\Lambda,t}(\sigma_{\Lambda,\bar{x}}(B)))$$

$$\forall A, B \in \mathcal{A}_L \text{ and } t \in \mathbb{R}.$$  (2.10)

and the Green’s function for the finite system as

$$G_\Lambda(A, B; t, \bar{x}) = \lim_{\alpha} G_{\Lambda,\alpha}(A, B; t, \bar{x})$$

exist $\forall A, B \in \mathcal{A} \forall t \in \mathbb{R} \forall \bar{x} \in \mathbb{R}^d$
Thus a net \( \{\Lambda_\alpha\} \) such that the thermodynamic limit (2.11.5) exists can always be found but is not unique: such nonuniqueness is associated with phase transitions, and is thus expected, e.g., for the Bose fluid at sufficiently low temperatures.

Clearly \( \omega \), defined by
\[
\omega(A) = G(A, 1; 0, \vec{0})
\]
is a state over \( \mathcal{A} \). In section 2.3 we shall need

**Assumption A** \( \omega \) is time- and space-translation invariant, i.e.,
\[
\omega(\alpha_t(A)) = \omega(A)
\]
and
\[
\omega(\sigma_{\vec{x}}(A)) = \omega(A)
\]
where \( \alpha_t \) and \( \sigma_{\vec{x}} \) are commuting automorphisms of \( \pi_\omega(A)'' \), i.e., i.e., \( \alpha_t \circ \sigma_{\vec{x}} = \sigma_{\vec{x}} \circ \alpha_t \), for all \( t \in \mathbb{R} \) and \( \vec{x} \in \mathbb{R}^d \). Further,
\[
G(A, B; t, \vec{x}) = \omega(A(\alpha_t \circ \sigma_{\vec{x}})(B))
\]
\( \forall A, B \in \pi_\omega(A)'' \)

There is a significant literature supporting assumption A, see [Sew82] and references given there, as well as section 2 of [SW09] particularly the paragraph Physical states, representations and dynamics, and it has been verified in the free Boson gas. The KMS condition definition 2.1, as well as the ground state condition proposition 2.3 concern assumption A, see also [SW09], (2.13),(2.19) and the ground state condition stated there.

We also formulate a weaker assumption:

**Assumption B** In case (2.12.1,2) hold,
\[
G(A, B; t, \vec{x}) =
(\Omega_\omega, \pi_\omega(A) \exp[i(tH_\omega + \vec{x} \cdot \vec{P}_\omega)] \pi_\omega(B) \Omega_\omega)
\]

where \( \sigma_{\vec{\varphi}} \) is the automorphism corresponding to rotations of angle \( \vec{\varphi} \).

There is a significant literature supporting assumption A, see [Sew82] and references given there, as well as section 2 of [SW09] particularly the paragraph Physical states, representations and dynamics, and it has been verified in the free Boson gas. The KMS condition definition 2.1, as well as the ground state condition proposition 2.3 concern assumption A, see also [SW09], (2.13),(2.19) and the ground state condition stated there.
where $\Omega_\omega$ and $\pi_\omega$ are the GNS vector and representation associated to $\omega$, and $H_\omega$ and $\vec{P}_\omega$ are the commuting (in the sense of spectral projections) self-adjoint generators of time and space translations in the representation $\pi$. In case (2.12.4) holds,

$$G(A, B; t, \vec{\omega}) = (\Omega_\omega, \pi_\omega(A) \exp[i(tH_\omega + \vec{\omega} \cdot \vec{J}_\omega)] \pi_\omega(B) \Omega_\omega)$$

(2.12.6)

where $\vec{J}_\omega$ is the self-adjoint generator of rotations.

It is well-known that assumption B follows from assumption A [Hug72, BR87]; the requirement of existence of generators is indeed a mild requirement, which is almost universally assumed to be a characteristic of the infinite system. For continuous Boson systems, however, even this requirement is not easy to prove: see [BR97], second edition, theorem 6.3.27 (ii). The $t-$ continuity of $G(A, B; t, \vec{x})$ was shown for a Bose gas with repulsive interactions in [BR80], but only for strictly negative chemical potential. We are, however, interested in the case of a low-temperature Bose fluid, with BEC.

Concerning Galilean transformations, let $\Psi(\vec{x})$ be the quantized field operator whose smeared version satisfies the CCR:

$$[\Psi(f), \Psi(g)^\dagger] = (f, g)$$

(2.13)

In terms of the basic destruction operator $\Psi$ the group of Galilei transformations is given by

$$\Psi(\vec{x}, t) \rightarrow \Psi(\vec{x} - \vec{v}t, t) \exp[i(\vec{v} \cdot \vec{x} + (\vec{v})^2 t/2)]$$

(2.14)

(the mass $m = 1$). As explained, e.g., in [WdSJ05], we may restrict the transformation (2.14) to $t = 0$, in which case

$$\Psi(\vec{x}) \rightarrow \Psi(\vec{x})e^{i\vec{v} \cdot \vec{x}}.$$ 

(2.15)

In order to see how Galilean transformations act on finite systems, we shall have to consider the latter in greater detail. In $\Lambda$ we consider a generic conservative system of $N$ identical particles of mass $m$. In units in which $\hbar = m = 1$, $H_\Lambda$ and $\vec{P}_\Lambda$ take the standard forms

$$H_\Lambda = -\sum_{r=1}^{N} \Delta_r + V(\vec{x}_1, \ldots, \vec{x}_N)$$

(2.16)
with \( V \) a potential satisfying
\[
V \geq 0,
\]
i.e., only repulsive interactions will be considered, and
\[
\vec{P}_\Lambda = -i \sum_{r=1}^{N} \nabla_r
\]
with usual notations for the Laplacean \( \Delta_r \) and the gradient \( \nabla_r \) acting on the coordinates of the \( r \)-th particle. We assume that \( \mathcal{H}_\Lambda \) and \( \vec{P}_\Lambda \) are self-adjoint operators acting on \( \mathcal{H}_\Lambda^{\otimes N} = L^2(\Lambda)^{\otimes N} \), the symmetrized tensor product of \( \mathcal{H}_\Lambda \) corresponding to \( N \) particles, with domains \( D(\mathcal{H}_\Lambda) \) and \( D(\vec{P}_\Lambda) \), and
\[
D(\vec{P}_\Lambda) \supset D(\mathcal{H}_\Lambda).
\]
Let
\[
S^d_\Lambda \equiv \{ \frac{2\pi \vec{n}}{L} \mid \vec{n} \in \mathbb{Z}^d \}
\]
and, given \( \vec{v} \in \mathbb{R}^d \), let \( \vec{v}_{\vec{n},L} = \vec{k}_{\vec{n},L} \) such that
\[
|\vec{k}_{\vec{n},L} - \vec{v}| = \inf_{\vec{k} \in S^d_\Lambda} |\vec{k} - \vec{v}|
\]
and \( |\vec{k}_{\vec{n},L}| \leq |\vec{v}| \).

If there is more than one \( \vec{k}_{\vec{n},L} \) satisfying (2.19.2), we pick any one of them. We have:
\[
\lim_{N,L \to \infty} \vec{v}_{\vec{n},L} = \vec{v},
\]
where \( N, L \to \infty \) will be always taken to mean the thermodynamic limit, whereby
\[
N \to \infty, \quad L \to \infty, \quad \frac{N}{L^d} = \rho \quad \text{with} \quad 0 < \rho < \infty,
\]
where \( \rho \) is a fixed density. The unitary operator of Galilei transformations appropriate to velocity \( \vec{v}_{\vec{n},L} \) follows from (2.15) (upon restriction to the \( N \)-particle subspace of symmetric Fock space):
\[
U_\Lambda^{\vec{v}} \equiv \exp(i \vec{v}_{\vec{n},L} \cdot (\vec{x}_1 + \ldots + \vec{x}_N)).
\]
We shall assume (2.18) and that $U^\vec{v}_\Lambda$ maps $D(H_\Lambda)$ into $D(H_\Lambda)$. From now on we shall write $\vec{v}$ for $\vec{v}_{\bar{\imath}_L}$, $\bar{\imath}_L$. It follows from (2.15), (2.16) and (2.19.3) that, on $D(H_\Lambda)$,

\[(U^\vec{v}_\Lambda)^\dagger \tilde{H}_\Lambda U^\vec{v}_\Lambda = \tilde{H}_\Lambda + \vec{v} \cdot \vec{P}_\Lambda + \frac{N(\vec{v})^2}{2},\]

which is the expression of Galilean covariance. In addition to the standard model (2.16), we shall also consider effective Hamiltonians, for dilute Bose systems, which we define as follows. Let

\[e(\rho) \equiv \lim_{N,L \to \infty} L^{-d} E_\Lambda,\]

where $E_\Lambda$ is the ground state energy defined in (2.8), and $a$ denotes the scattering length corresponding to the potential $V$ in (2.16), which is also a measure of the interaction range; the mean particle distance is $\rho^{-1/3}$, now with $d = 3$. If $a \ll \rho^{-1/3}$, we say that we have a dilute system. It has been conjectured that $e(\rho)$ has an asymptotic expansion

\[e(\rho) = \sum_{j \geq 1} e^j(\rho),\]

where $(j)$ denotes the order of the approximation;

\[e^1(\rho) = 4\pi a \rho^2\]
\[e^2(\rho) = e^1(\rho) \frac{128(\rho a^3)^{1/2}}{15\sqrt{\pi}}.\]

Several rigorous results support the first of (2.21.2), it was rigorously proved by Lieb and Yngvason in a seminal paper [LY98] that $e(\rho) = 4\pi a(\rho)^2 + o(a(\rho)^2)$, and various rigorous additional results are given and reviewed in [Yin12], to which we also refer for the references. The famous second order correction, the second of (2.21.2), was first conjectured by Lee and Yang in 1957 [LY57], on the basis of the pseudopotential approximation of Lee, Huang and Yang [TDLY57], and the binary-collision expansion method [Hua87].
**Definition 2.5.** We call a Hamiltonian $H_{\Lambda}$ an *effective Hamiltonian* to order $j$ for the dilute Boson system, if it satisfies both (2.21.1) up to order $j$ and (2.20) (Galilean covariance).

As we shall see in Section 3, the Bogoliubov approximation does not define an effective Hamiltonian according to the above definition, because it lacks Galilean covariance. We have included this requirement both because it plays a key role in our framework and because of the obvious physical content attached to it. The pseudopotential approximation does define an effective Hamiltonian: for $j = 1$, see Section 3, for $j = 2$, see the conclusion.

From (2.20) we may be led, as in [SW09], to ask whether the Hamiltonian

$$
H_{\vec{v},\Lambda} = \tilde{H}_{\Lambda} + \vec{v} \cdot \vec{P}_{\Lambda},
$$

is the one appropriate to describe the Bose fluid in uniform motion with velocity $\vec{v}$. By (2.20),

$$
H_{\vec{v},\Lambda} \geq -\Delta E_{\vec{v}}(\Lambda) = -\frac{N(\vec{v})^2}{2}.
$$

On the other hand, by (2.20),

$$
H_{\vec{v},\Lambda} = (U_{\Lambda}^{\vec{v}})^\dagger \tilde{H}_{\Lambda} U_{\Lambda}^{\vec{v}} - \Delta E_{\vec{v}}(\Lambda),
$$

from which

$$
\text{spec} (H_{\vec{v},\Lambda}) = \text{spec} (\tilde{H}_{\Lambda}) - \Delta E_{\vec{v}}(\Lambda)
$$

follows. By (2.22.3), $-\Delta E_{\vec{v}}(\Lambda) \in \text{spec} (\tilde{H}_{\Lambda})$, which, together with (2.22.1), implies that

$$
-\Delta E_{\vec{v}}(\Lambda) = \inf \text{spec} (H_{\vec{v},\Lambda}).
$$

Incidently, this proves that the Bogoliubov approximation (BA) is not Galilean covariant: this is a consequence of the fact that, in the BA, the operator $H_{\vec{v},\Lambda}$ is non-negative for $|\vec{v}|$ sufficiently small (see [ZB01]), contradicting (2.23), which was derived on the basis of Galilean covariance.
By (2.23), one might be led to consider the vector $\Psi_{\vec{v},\Lambda}$ corresponding to the lowest eigenvalue (2.23), and the corresponding state $\langle \Psi_{\vec{v},\Lambda}, \cdot, \Psi_{\vec{v},\Lambda} \rangle$, with the renormalization $H_{\vec{v},\Lambda} \rightarrow \tilde{H}_{\vec{v},\Lambda} + \Delta E_{\vec{v}}(\Lambda) \geq 0$ as describing the ground state of the Bose fluid in motion with velocity $\vec{v}$, which would yield an equilibrium state in the thermodynamic limit $N, L \rightarrow \infty$. We have, however, the following result.

**Lemma 2.6.** $U_{\Lambda}^{-\vec{v}} |\Omega_\Lambda\rangle$ is the eigenvector of $H_{\vec{v},\Lambda}$ corresponding to the eigenvalue $-\Delta E_{\vec{v}}(\Lambda)$, unique if the potential $V$ is “sufficiently regular”.

*Proof.* Applying (2.22.2) to $(U_{\Lambda}^{\vec{v}})^\dag |\Omega_\Lambda\rangle = U_{\Lambda}^{-\vec{v}} |\Omega_\Lambda\rangle$ we obtain the first assertion. For the connection between unicity and a rough description of “sufficiently regular”, see [MR04], and for precise conditions, [RS78], Chapter XIII.12: these are valid for $\tilde{H}_{\Lambda}$, but, by (2.22.3), they remain valid for $H_{\vec{v},\Lambda}$.

Lemma 2.6 shows that $\Psi_{\vec{v},\Lambda} = U_{\Lambda}^{-\vec{v}} |\Omega_\Lambda\rangle$ represents a state of the Bose fluid in uniform motion with velocity $-\vec{v}$, rather than $\vec{v}$, being therefore not the state we are looking for. On the other hand, friction or dissipation is detectable in the static frame of reference whenever it is able to induce a transition from the initial state of the system, assumed to be the ground state of $\tilde{H}_{\Lambda}$, to an eigenstate of the same operator with the lowest energy above the ground state, i.e., containing an “elementary excitation”. Let it correspond to an eigenvalue $\epsilon_{\Lambda}(\vec{k})$, say, of $\tilde{H}_{\Lambda}$, corresponding to momentum $\vec{k}$. In a static frame of reference, the initial energy of the system is, by (27)-(2.9.1) and (2.23), equal to $\Delta E_{\vec{v}}(\Lambda)$, and the final one, $\epsilon_{\Lambda}(\vec{k}) + \vec{v} \cdot \vec{k} + \Delta E_{\vec{v}}(\Lambda)$, and thus the condition for frictionless motion — expected to hold for $|\vec{v}| \leq v_c$, where $v_c$ is a critical velocity — is $\epsilon_{\Lambda}(\vec{k}) + \vec{v} \cdot \vec{k} \geq 0$, which is Landau’s criterion ([LL67] [WdSJ05], [HDCZ09]). In the above argument the identification of the (sums of) energies of elementary excitations as the eigenvalues of $\tilde{H}_{\Lambda}$ (expected to be true up to corrections $O(1/N)$ see [Lie63]) is implicit. We are, thus, led to investigate the eigenvalues of the operator (2.19.4) and their positivity, and adopt (2.19.4) as the Hamiltonian describing the Bose fluid in uniform motion with velocity $\vec{v}$.
2.2 A general class of candidates for NESS: semipassive states

In order to describe a simple nonequilibrium situation, one may be led to study passivity in the reference frame of a moving observer (with respect to the rest frame of the state ([Kuc02], see also [BB94], (6.2), and [Wre05]). The energy

$$\Delta E \equiv i \frac{d}{dt} \omega(U^* \alpha_t(U))|_{t=0}$$  \hspace{1cm} (2.24)

may be interpreted as the energy gained in a cyclic process between the initial state $\omega(\cdot)$ and the final state $\omega(U \cdot U)$, with the unitary $U$ satisfying (2.5). By (2.4) and (2.24) this energy is negative, i.e., the work performed on the system is positive:

$$\Delta E \leq 0.$$  \hspace{1cm} (2.25)

This is adequate as an expression of the second law of thermodynamics for an observer in the rest frame of the state $\omega$, but, in the case of moving observers, one has to take into account the energy necessary to maintain the motion of the observer.

In addition to (2.9.1), in order to incorporate rotations, we assume that there exists a rotationally invariant state, whose restriction

$$\omega_\Lambda = (\Omega_\Lambda, \cdot | \Omega_\Lambda)$$  \hspace{1cm} (2.26)

to $\mathcal{A}_\Lambda$ satisfies

$$\vec{J}_\Lambda \Omega_\Lambda = \vec{0},$$  \hspace{1cm} (2.9.3)

where $\vec{J}_\Lambda$ is the total angular momentum associated to the cylindrical region $\Lambda$.

In the case $\vec{u} = \vec{v}$, the Hamiltonian for the region $\Lambda$ is (2.19.4): and, in case $\vec{u} = \vec{w}$ (see, e.g., [Le75] or ([SW09], (4.1)):

$$H_{\vec{w}}(\Lambda) = \tilde{H}_\Lambda + \vec{w} \cdot \vec{J}_\Lambda,$$  \hspace{1cm} (2.19.5)

with the automorphisms

$$\alpha_{t,\vec{v},\Lambda}(A) = e^{itH_{\vec{v}}(\Lambda)} A e^{-itH_{\vec{v}}(\Lambda)} \forall A \in \mathcal{A}_\Lambda,$$

$$\alpha_{t,\vec{w},\Lambda}(A) = e^{itH_{\vec{w}}(\Lambda)} A e^{-itH_{\vec{w}}(\Lambda)} \forall A \in \mathcal{A}_\Lambda.$$

where $\mathcal{A}_\Lambda$ is given by (2.11.2); note that

$$\alpha_{t,\vec{u},\Lambda} = \alpha_{t,\Lambda} \circ \sigma_{\vec{u},\Lambda}$$
\[ \alpha_{t,\vec{u},\Lambda} = \alpha_{t,\Lambda} \circ \sigma_{\vec{u},\Lambda} \]

by (2.19.4), (2.19.5), where \( \sigma_{\vec{u},\Lambda} \) denotes the finite-volume automorphism corresponding to rotations. In the case of a state \( \omega \) of the infinite system satisfying assumption \( \Lambda \), we let \( \alpha_{t,\vec{v}} \) or \( \alpha_{t,\omega} \) denote the automorphism groups of \( \mathcal{A} \) in the presence of matter in uniform translational motion with velocity \( \vec{v} \) or in rotational motion (e.g. in a cylinder) with angular velocity \( \vec{\omega} \) (two typical situations in the study of superfluidity): we write generically \( \alpha_{t,\vec{u}} \) with \( \vec{u} = \vec{v} \) or \( \vec{u} = \vec{\omega} \).

By local normality of \( \omega \) and (2.4), the restriction \( \omega_{\Lambda} \) of \( \omega \) to \( \mathcal{A}_{\Lambda} \) might be expected to change to

\[ i \frac{d}{dt} \omega_{\Lambda}(U^* \alpha_{t,\vec{u},\Lambda} U) \big|_{t=0} \leq \Delta E_{\vec{u}}(\Lambda), \quad (2.27.1) \]

where \( U \) is a unitary element of \( \mathcal{A}_{\Lambda} \) in the domain of the derivation \( \delta_{\vec{u},\Lambda} \) associated to \( \alpha_{t,\vec{u},\Lambda} \). By local finiteness it is expected that

\[ 0 < \Delta E_{\vec{u}}(\Lambda) = O(|\Lambda|). \quad (2.27.2) \]

**Definition 2.6** A locally normal state which satisfies (2.27) is said to be *semipassive* (a modification of the definition in [Kuc02]).

(2.27.1) is equivalent to the statement that for all unitaries in the norm connected component of \( \mathcal{A}_{\Lambda} \) which contains the identity, such that \( [H_{\vec{u}}(\Lambda), U] \in \mathcal{A}_{\Lambda} \),

\[ -\langle U\Omega_{\Lambda}, [H_{\vec{u}}(\Lambda), U]\Omega_{\Lambda} \rangle \leq \Delta E_{\vec{u}}(\Lambda). \quad (2.27.3) \]

(2.27.3) is, in the cases \( \vec{u} = \vec{v} \) and \( \vec{u} = \vec{\omega} \), a consequence of the operator inequalities

\[
\begin{align*}
\tilde{H}_{\Lambda} + \vec{v} \cdot \vec{P}_{\Lambda} + \Delta E_{\vec{v}}^{\text{op}}(\Lambda) & \geq 0 \\
\tilde{H}_{\Lambda} + \vec{\omega} \cdot \vec{J}_{\Lambda} + \Delta E_{\vec{\omega}}^{\text{op}}(\Lambda) & \geq 0,
\end{align*}
\]

(2.27.4)

where \( \Delta E_{\vec{v},\vec{\omega}}^{\text{op}}(\Lambda) \) are nonnegative operators. Defining as a consequence of Galilean covariance,

\[ \Delta E_{\vec{v}}^{\text{op}}(\Lambda) = \frac{N_{\text{op}} \vec{v}^2}{2} \quad (2.28) \]
where $N^{\text{op}}$ is the usual number operator

$$N^{\text{op}} = \int_\Lambda d\vec{x} \, \Psi^\dagger(\vec{x})\Psi(\vec{x}) \quad (2.29)$$

and $\Psi(\vec{x})$ is the quantized field operator whose smeared version is (2.13) i.e.,

$$\Psi(f) = \int_\Lambda d\vec{x} \, f(\vec{x})\Psi(\vec{x}) \text{ for } f \in \mathcal{H}_\Lambda = L^2(\Lambda) .$$

we see that, in first quantized form, which amounts to the restriction $N^{\text{op}} = N$, (2.27.1) follows directly from the first inequality in (2.27.4). Further, the second inequality in (2.27.4) may be seen to hold (see [SW09], (4.1)), with

$$\Delta E^{\text{op}}(\Lambda) = \frac{\int_\Lambda d\vec{x} \, (\vec{A}(\vec{x}))^2\Psi^\dagger(\vec{x})\Psi(\vec{x})}{2} ,$$

where

$$\vec{A}(\vec{x}) \equiv \vec{\omega} \times \vec{x} ,$$

with $\times$ denoting the vector product. In the rotating case above, $\Lambda$ is a cylinder and, if $\vec{\omega}$ is directed along the positive $z$ axis, assumed to be the rotation axis, $\Delta E^{\text{op}}(\Lambda)$ is the operator corresponding to the moment of inertia perpendicular to the rotation axis. The expectation value of $\Delta E^{\text{op}}_{\vec{v}}$ in the state $\omega_\Lambda$ is

$$\frac{\omega_\Lambda(\Delta E^{\text{op}}_{\vec{v}}(\Lambda))}{|\Lambda|} = \frac{\vec{v}^2\omega_\Lambda(N^{\text{op}})}{2|\Lambda|} = \frac{N(\vec{v})^2}{2|\Lambda|} = \rho\vec{v}^2 > 0 ,$$

where $\rho > 0$ is the density, fixed in the thermodynamic limit (T.L.), and

$$\frac{\omega_\Lambda(\Delta E^{\text{op}}_{\vec{v}}(\Lambda))}{|\Lambda|} = \frac{|\vec{\omega}|\omega_\Lambda(J_{\vec{z}})}{2|\Lambda|} \rightarrow \frac{1}{2}|\vec{\omega}|c > 0 \text{ (T.D.)} ,$$

18
whereby density of particles and density of angular momentum are kept fixed and positive in the T.L. (see [Le75]), leading to the r.h.s. of (2.31); (2.30) and (2.31) yield finally (2.27.2), showing the following result.

**Proposition 2.7.** Galilean covariant space translation invariant states of Bosons in uniform translational motion, as well as rotationally invariant states of Bosons in a rotating cylindrical bucket satisfy (2.27.1) and (2.27.2) with the special choices (2.30) and (2.31), respectively, and thus are special cases of semipassive systems as defined in Definition 2.6.

**Remark 2.8.** The extensive character (2.27.2) of the r.h.s. of (2.27.1) is the difference between our definition and Kuckert’s in [Kuc02] and is the main reason for the instability, i.e., the non-equilibrium character of semipassive states. Closely related to this, e.g. in the translational case, is the fact that, due to the fact that the group parameter \( \vec{v} \) is multiplied by the number operator in (2.28) (leading to (2.30)), the Galilean transformation, to quote Swieca [Swi67] in his pioneering paper, “produces an infinite change in an infinitely extended system and can therefore not be unitarily implementable”. Indeed, by Lemma 2.6 the eigenvalue \(-\frac{N(\vec{v})^2}{2}\) which lies at the bottom of the spectrum of \( H_{\vec{v},\Lambda} = \tilde{H}_\Lambda + \vec{v} \cdot \tilde{P}_\Lambda \) is \( U_{\vec{v}}^{-} |\Omega_\Lambda\rangle \), which corresponds to velocity \(-\vec{v}\) and, for \( \vec{v} \neq 0 \), is not connected to the system with velocity \( \vec{v} \) by a unitary transformation: the latter is incompatible with a macroscopic inversion of the velocities of a macroscopic number of particles. One also understands the connection to metastability: the original ground state connects with such states only by excitations of very large energy and momentum.

The above feature contrasts with the Lorentz group, and, for this reason, in the case of relativistic systems, passivity (which is a special case of semipassivity in the case of locally normal states) is the adequate concept, e.g., to study vacuum states with respect to a moving observer, and, indeed, in this case equilibrium states arise ([Kuc02], Proposition 5.1).

We predict additional relevant applications of semipassivity in the context of superconductive electrodynamics, see Part III, Chapters 8 and 9 of [Sew02] for a comprehensive survey.

Remark 2.8 suggests that one might hope to prove that semipassive states are NESS directly from their definition, leading to a general characterization of their equilibrium properties. Unfortunately, we were not able to achieve this aim, which remains an open problem. One problem is the exclusion of the
passive states in definition 2.6 which, for the first example of proposition 2.7 is achieved by (2.23). Indeed, for the special cases mentioned in Proposition 2.7 we shall now show that they are NESS for $T > 0$, under assumption A.

### 2.3 Bosons in translational and rotational motion: a special class of semipassive states which are NESS at positive temperature

Let, now, $\omega^1$ denote a state of a Boson system satisfying assumption A and $\omega^2$ its corresponding version for rotations $\omega^{1,2} = (\Omega^{1,2}, \cdot \Omega^{1,2})$, with $\omega^1$ invariant under space translations and $\omega^2$ invariant under the rotation group (see (2.12.1) and (2.12.2)):

\[
    H_{\omega^{1,2}} \Omega^{1,2} = 0
\]
\[
    \vec{P}_{\omega^1,\omega^1} = \vec{0} \quad \text{and} \quad \vec{J}_{\omega^2,\omega^2} = \vec{0}
\]

(2.31.1)

where $\vec{J}$ denotes the total angular momentum operator for the infinite system. Define the self-adjoint operators

\[
    H_{\vec{v},\omega^1} \equiv H_{\omega^1} + \vec{v} \cdot \vec{P}_{\omega^1},
\]

(2.31.2)

\[
    H_{\vec{\omega},\omega^2} \equiv H_{\omega^2} + \vec{\omega} \cdot \vec{J}_{\omega^2},
\]

(2.31.3)

for $\vec{v} \in \mathbb{R}^d$, $\vec{\omega} \in \mathbb{R}^d$, and let $\alpha_{t,\vec{v}}^1$ and $\alpha_{t,\vec{\omega}}^2$ denote the corresponding automorphism groups of $\mathcal{A}$.

**Theorem 2.9.** Let $\omega^{1,2} = \omega_{\beta}^{1,2}$ satisfy the KMS condition (2.1) with respect to $\alpha_{t,\vec{v}}^{1,2}$ for a given $0 < \beta < \infty$, where $\alpha_{t,\vec{v}}^1 = \alpha_{t,\vec{0}}^1$ and $\alpha_{t,\vec{v}}^2 = \alpha_{t,\vec{0}}^2$. Then, the quantum dynamical systems $(\omega_{\beta}^1, \alpha_{t,\vec{v}}^1, \mathcal{A})$ (resp. $(\omega_{\beta}^2, \alpha_{t,\vec{\omega}}^2, \mathcal{A})$) both define NESS whenever $\vec{v} \neq \vec{0}$ (resp. $\vec{\omega} \neq \vec{0}$).

**Proof.** We must show that $\omega_{\beta}^1$ (resp. $\omega_{\beta}^2$) does not satisfy the KMS condition (2.1) with respect to the dynamics defined by $\alpha_{t,\vec{v}}^1$ (resp. $\alpha_{t,\vec{\omega}}^2$). We consider only $\omega_{\beta}^1$, the other case being obviously identical.

Assume $\omega_{\beta}^1 \equiv \omega_{\beta}$ satisfies the KMS condition with respect to $\alpha_{t}^1 \equiv \alpha_{t,\vec{v}}^1$. Since it also satisfies the KMS condition with respect to $\alpha_{t}$ by hypothesis,
applying the KMS condition first w.r.t. $\alpha_t$ and then w.r.t. $\alpha'_t$, we find, for $A \in A_{\alpha_t}$, $B \in A_{\alpha'_t}$, where $A_\tau$ is a norm-dense $*$-subalgebra of $A$ consisting of entire analytic elements for $\tau$ (see [BR87], Definition 2.5.20 and Proposition 2.5.22),

$$F_{A,B}(t) \equiv \omega_\beta(\alpha_t(A)\alpha'_t(B)) = \omega_\beta(\alpha'_t(B)\alpha_{t+i\beta}(A))$$

$$= \omega_\beta(\alpha_{t+i\beta}(A)\alpha'_{t+i\beta}(B))$$

$$= F_{A,B}(t+i\beta).$$

(2.32)

The function $F_{A,B}(z)$ is, for $A \in A_{\alpha_t}$, $B \in A_{\alpha'_t}$, analytic in $D_\beta = \{z \in \mathbb{C} \mid 0 < \Im z < \beta\}$ and continuous on the closure $\overline{D}_\beta$. By the three-line lemma (see, e.g., [BR97], Proposition 5.3.5), it is uniformly bounded in $\overline{D}_\beta$ by $\|A\||B\|$. Choosing sequences $\{A_n\}_{n \geq 1}, \{B_n\}_{n \geq 1}$ with $A_n \in A_{\alpha_t}$, $B_n \in A_{\alpha'_t}$, such that

$$\|A_n\| \leq \|A\|; \quad \lim_{n \to \infty} \|A_n - A\| = 0,$$

$$\|B_n\| \leq \|B\|; \quad \lim_{n \to \infty} \|B_n - B\| = 0,$$

(2.33) (2.34)

one obtains that $F_{A_n,B_n}(z) \to F_{A,B}(z)$ uniformly in $\overline{D}_\beta$ (see [BR97], pg. 82), the limit function being therefore continuous and bounded on $\overline{D}_\beta$, analytic in $D_\beta$, and satisfying the periodicity condition (2.38):

$$F_{A,B}(t) = F_{A,B}(t + i\beta) \quad \forall t \in \mathbb{R}, \quad \forall A, B \in A.$$  

(2.35)

Furthermore, the uniform bound holds as a consequence of the first inequalities in (2.33)-(2.34), which themselves follow from Kaplansky's density theorem, Theorem 2.4.16 of [BR87]:

$$|F_{A,B}(z)| \leq \|A\||B\| \quad \forall z \in D_\beta.$$  

(2.36)

Hence, by the Schwarz reflection principle, the function $F_{A,B}$ extends uniquely to an analytic function in the whole of $\mathbb{C}$, satisfying the bound (2.36), and is therefore a constant. From (2.32), it follows that $F_{A,B}(z) = 1$ for all $z \in \mathbb{C}$, whence the unitaries $U_{t,\omega_\beta}$ implementing the automorphisms $\alpha_t^{1,2}$ in the GNS representation of $\omega_\beta$, satisfy

$$(U_{t,\omega_\beta})^{-1}U'_{t,\omega_\beta} = 1,$$  

(2.37)

upon using the cyclicity of $\Omega_\beta$. By our choices of $\alpha_t, \alpha'_t$, and (2.37), this implies that $\bar{\nu} = \bar{0}$. q.e.d.
Remark 2.10. Assuming the previously described connection to the finite systems, Theorem 2.9 holds under the usual regularity assumptions of self-adjointness and semiboundedness of $H^\Lambda$; in particular, no restriction appears otherwise on the sign of the interaction potential. Note also that the proof of Theorem 2.9 is elementary, not requiring the extension of $\omega_\beta$ to the von Neumann algebra $(\pi_{\omega_\beta}(A))^\prime$ essentially, we use only part of a proof in [Hug72]. Theorem 4.11, but not the hypothesis of the statement of the theorem, which itself is patterned after the original paper of Sirugue and Winnink [SW70].

Remark 2.11. The study of quantum systems carrying a current was initiated by Sewell in a pioneer paper [Sew80]. In [SW09], Theorem 5.1, it was shown, under quite general conditions, that Bosons in uniform translational motion cannot be locally thermodynamically stable (LTS) (see [Sew80] pg. 323 for the definition) at $T = 0$. For quantum continuous systems, the equivalence between LTS and the ground state condition (Proposition 2.3) has not been proved, but for lattice systems this equivalence has been demonstrated both for $T = 0$ and $T > 0$ [AS77]; using this equivalence, it has been shown in [Sew80] that a class of translationally invariant current-carrying states on a lattice cannot satisfy the equilibrium conditions, either at zero or nonzero temperature.

A metastability condition in terms of the canonical free energy, characterizing a superfluid, might be introduced at this point. Because of the restriction to a subspace (see the forthcoming Definition 2.8), this condition would be very difficult to verify even in the simplest examples (see Remark 3.3). We therefore adopt the usual procedure (also followed by Landau) of formulating a metastability criterion only for zero temperature in the next section, hoping that the expected continuity in the temperature will render the system metastable for sufficiently low temperatures.

2.4 The case of Bosons in uniform translational motion at zero temperature

We now restrict ourselves to the translational case $\vec{u} = \vec{v}$. Equations (2.19.4) and (2.22.3) suggest that, “in normal circumstances”, $H_{\vec{u},\Lambda}$ has a spectrum contained in $[-\Delta E_{\vec{u}}(\Lambda), \infty)$, which tends to the whole real line as $N, L \to \infty$. In order to see what may go wrong, assume that the spectrum $\text{spec}(H_{\vec{u},\Lambda}) = [-\Delta E_{\vec{u}}(\Lambda), -\lambda_{N,L}] \cup [0, \infty)$, with $\lambda_{N,L} \to \infty$ as $N, L \to \infty$. In this case, the negative part of the spectrum “disappears” in the thermodynamic limit, and
the assertion that the spectrum of the physical Hamiltonian (for the infinite system) contains a non-empty set in the negative real axis does not follow. What we need is a certain uniformity in the thermodynamic limit.

Recalling (2.7)-(2.9) and (2.19.4), we define the energy-momentum spectrum (emsp) of the finite system as the set of pairs \( (E_{i,N,L}(\vec{v}), \vec{k}_{j,L})_{i,j} \), where \( \{E_{i,N,L}(\vec{v})\}_i \) denotes the set of eigenvalues of \( H_{\vec{v},A} = H_{\vec{v},N,L} \), and \( \{\vec{k}_{j,L}\}_j = S_{\Lambda}^d \) (given by (2.19.1)) the set of eigenvalues of the momentum \( \vec{P}_{N,L} \). Above, when \( \vec{v} \) occurs in the context of the finite system, (2.19.2) is understood.

**Theorem 2.12**

Let \( \text{emsp}_{\infty}(\vec{v}) \equiv \text{support of the Fourier transform of} \)
\[
G(A^*, A; t, \vec{x} + \vec{v}) \text{ as a tempered distribution}
\]

(2.38)

where \( G(A^*, A; t, \vec{x}) \) is defined in (2.11.5), agrees with the support of the joint spectral family \( E(\lambda, \vec{k}) \) of \( (H_{\vec{v},\omega}, \vec{P}_\omega) \) (defined in (2.37.2)) in case assumption B holds true for the state \( \omega \) defined by (2.11.6): note that the Fourier transform in (2.44) is assumed defined in terms of the variables \( t, \vec{x} \).

**Proof**

\[
G(A^*, A; t, \vec{x} + \vec{v}) = \left< \Omega_{N_\alpha,L_\alpha}, A^* \exp(it(\vec{H}_{N_\alpha,L_\alpha} + \vec{v}_{n_\alpha,L_\alpha} \cdot \vec{P}_{N_\alpha,L_\alpha}))) \exp(i\vec{x} \cdot \vec{P}_{N_\alpha,L_\alpha}) \right> \Omega_{N_\alpha,L_\alpha} \]

which is uniformly bounded

\[
|G_{N_\alpha,L_\alpha}(A^*, A; t, \vec{x} + \vec{v})| \leq \|A\|^2.
\]

Hence, the \( G_{N_\alpha,L_\alpha} \) converge to \( G \) as tempered distributions, and thus the Fourier transforms \( \hat{G}_{N_\alpha,L_\alpha} \) also converge to \( \hat{G} \) as tempered distributions, and therefore the supports of \( \hat{G}_{N_\alpha,L_\alpha} \), i.e., the complements of the largest open sets on which \( \hat{G}_{N_\alpha,L_\alpha} \) vanish, also converge to the support of \( \hat{G} \). In case
assumption B holds, we have

\[ G(A^*, A; t, \vec{x} + \vec{v}) = \]

\[ = (\pi_\omega(A)\Omega_\omega, e^{itH_{\vec{v},\omega}} e^{i\vec{x} \cdot \vec{P}_\omega} \pi_\omega(A)\Omega_\omega) \]

\[ = \int e^{it\lambda(\vec{k})} e^{i\vec{x} \cdot \vec{k}} d\|E(\lambda, \vec{k})\pi_\omega(A)\Omega_\omega\|^2 \]

\[ \forall t \in \mathbb{R}, \forall \vec{x} \in \mathbb{R}^d. \]

where \( \{E(\lambda)E(\vec{k}) \equiv E(\lambda, \vec{k})\} \) denote the spectral family associated to \((H_{\vec{v},\omega}, \vec{P}_\omega)\), i.e., \((H_{\vec{v},\omega}, \vec{P}_\omega) = \int (\lambda, \vec{k}) dE(\lambda, \vec{k})\). The latter is the energy-momentum spectrum of \((H_{\vec{v},\omega}, \vec{P}_\omega)\) q.e.d.

Theorem 2.12 justifies the

**Definition 2.7** \(emsp\infty(\vec{v})\) will be called the energy-momentum spectrum of the infinite system. If \(\exists (\lambda, \vec{k}) \in emsp_\infty(\vec{v})\) such that \(\lambda < 0\), the system describes a NESS.

The last statement of definition 2.7 corresponds to the fact that, when generators exist, i.e., assumption B holds, a zero-temperature state is a NESS iff \(spec(H_\omega) \cap (-\infty, 0) \neq \emptyset\) by proposition 2.3.

**Corollary 2.12** The zero temperature state obtained from (2.11.5) and (2.11.6) (with \(\vec{x} = \vec{v}\)) is a NESS if \(emsp_\infty(\vec{v})\) contains a point \((\lambda(\vec{k}_0), \vec{k}_0)\) with \(\lambda(\vec{k}_0) < 0\). Assume, now, that

\[ (\lambda_{N,L}(\vec{k}_{N,L}), \vec{k}_{N,L}) \]

\[ \in \{E_{i,N,L}(\vec{v}), \vec{k}_{j,L}\}_{i,j} \]

(2.39.1)

i.e., belongs to the energy-momentum spectrum of the finite system, as defined before theorem 2.12, and is such that

\[ \lim_{N,L \to \infty} (\lambda_{N,L}(\vec{k}_{N,L}), \vec{k}_{N,L}) = (\lambda(\vec{k}_0), \vec{k}_0) \]

(2.39.2)

for some \(\vec{k}_0 \in \mathbb{R}^d\). Then

\[ (\lambda(\vec{k}_0), \vec{k}_0) \in emsp_\infty(\vec{v}) \]
and, in particular, if \( \lambda(k_0) < 0 \), the zero temperature state is a NESS.

We set

\[
\mathcal{H}^{c,d}_{N,L} = \text{the subspace of } \mathcal{H}_{N,L} \text{ such that } \\
E_{i,N,L}(\vec{v} = 0) \leq c \forall i \text{ and } |\vec{k}_{j,L}| \leq d
\]

(2.40)

On \( \mathcal{H}_{N,L} \) let \( \mathcal{n}_{\vec{k}} \) with \( \vec{k} \in S^d_{N,L} \) be the number operators associated to momentum \( \vec{k} \), and define the set of states

\[
\mathcal{E}_{\rho_{\text{max}}} \equiv \{ \omega_{N,L} | \\
\lim \sup_{N,L \to \infty} (\rho - \omega_{N,L}(\mathcal{n}_{\vec{v}})) \leq \rho_{\text{max}} \}
\]

for some \( \rho_{\text{max}} \).

(2.41)

(2.40) and (2.41) represent two different restrictions on the magnitude of the energy-momentum around the ground state, (2.41) on the number of nonzero momentum modes involved. Physically, they express that there should be a sufficiently small energy-momentum transfer between the particles of the fluid and the surroundings (pipe). As remarked by Baym [Bay69], footnote on pg. 132, there is implicit the assumption that the energy of the excitations (in our case: the spectrum of \( \mathcal{H}_{\vec{v}} \)) does not depend on the velocity of the walls, an assumption which should be valid, but only for slow relative motion of the superfluid and walls. Accordingly, we pose:

**Definition 2.8** The system defines a superfluid at \( T = 0 \) if \( \exists 0 < v_c < \infty \) such that, whenever \( |\vec{v}| \leq v_c \), one of the following holds a.) there exists a subspace \( \mathcal{R}^{c,d}_{N,L} \) of \( \mathcal{H}^{c,d}_{N,L} \) such that the eigenvalues \( \{E_{j,N,L}(\vec{v})\}' \) of the restriction of \( \tilde{H}_{N,L} + \vec{v}_{\tilde{v}} \cdot \vec{P}_{N,L} = H_{\tilde{v},N,L} \) to \( \mathcal{R}^{c,d}_{N,L} \) satisfy

\[
\epsilon^c_{j}(\vec{v}) \equiv \lim_{N,L \to \infty} E_{j,N,L}(\vec{v}) \geq 0 \forall j
\]

and that

\[
\{\epsilon^c_{j}(\vec{v})\}_j \neq \{0\}
\]

(2.42.1)

(2.42.2)
for some $0 < c < \infty$, $0 < d < \infty$; b.) the eigenvalues $\{E_{j,N,L}(\vec{v})''\}$ of the restriction of $H_{\vec{v},N,L}$ to the subspace defined by (2.41) satisfy

$$E_{j,N,L}(\vec{v})'' \geq 0$$  \hspace{1cm} (2.43)

We refer to (2.42) and (2.43) as metastability conditions.

The basic feature of the above definition which guarantees that it does not depend on the parameters of the finite systems is the $(N, L)$-independence of $c,d$ in (2.40) and $\rho_{\text{max}}$ in (2.41); (2.42.2) guarantees that the subspace $\mathcal{R}_{N,L}^{c,d}$ does not shrink to the empty set as $N, L \to \infty$. In case assumption B holds, we may define

$$\mathcal{H}_{\omega}^{c,d} \equiv \int_{\lambda \leq c, |\vec{k}| \leq d} dE(\lambda, \vec{k}) \mathcal{H}_{\omega}.$$  \hspace{1cm} (2.44)

and are led to the alternative

**Definition 2.9** The state $\omega$ defines a superfluid at $T = 0$ if $\exists 0 < v_c < \infty$ such that either: there exists a subspace $\mathcal{H}_{\omega}^{c,d,'}$ of $\mathcal{H}_{\omega}^{c,d}$ such that

$$H_{\vec{v},\omega} \text{ when restricted to } \mathcal{H}_{\omega}^{c,d,'} \geq 0$$  \hspace{1cm} (2.45)

for some $0 < c < \infty$, $0 < d < \infty$; or

$$H_{\vec{v},\omega_0} \geq 0 \quad \forall \omega_0 \in \mathcal{E}_{\text{max}},$$  \hspace{1cm} (2.46)

whenever $|\vec{v}| \leq v_c$.

The metastability conditions comprise a precise way of characterizing “stability under creation of a few elementary excitations”, and, when they are satisfied, the model of independent quasiparticles introduced in [SW09] may be sometimes shown to be effective, as we shall demonstrate in the case of the first model treated in Section 3.

### 3 Examples

We apply the theory of Section 2 to two examples; in each of them, the system is a NESS in a sense stronger than that stated in Theorem 2.12: the energy spectrum will be shown to be unbounded from below.
3.1 The Girardeau model

We start with the Lieb-Liniger model [LL63] of \( N \) particles (for simplicity odd) in one dimension with repulsive delta function interactions, whose formal Hamiltonian is given by

\[
H_{N,L} = -\sum_{i=1}^{N} \frac{\partial^2}{(\partial x)^2} + 2c(\sum_{i,j=1}^{N})'\delta(x_i - x_j) \text{ with } 0 \leq x_i, x_j \leq L . \tag{3.1}
\]

The prime over the second sum indicates that the sum is confined to nearest neighbors. For the (standard) rigorous definition corresponding to (3.1), see [Dor93].

The limit, as \( c \to \infty \), of equation (3.1), yields Girardeau’s model [Gir60]. It is the free particle Hamiltonian with Dirichlet b.c. on the lines \( x_i = x_j \), with quadratic form domain given in ([Dor93], pg. 353, (2.18)-(2.19)). It is straightforward that \( U^q_\Lambda \), given by (2.19.3), leaves this domain invariant and (2.20) holds. Our considerations should also apply to the much richer Lieb-Liniger model, but, since details are much simpler in the case of the Girardeau model, the essence of the following argument becomes clearer and more concise.

For \( c \to \infty \), the b.c. on the wave-functions reduces to

\[
\Psi(x_1, \ldots, x_N) = 0 \text{ if } x_j = x_i, \ 1 \leq x_i, x_j \leq N , \tag{3.2}
\]

and the (Bose) eigenfunctions \( \Psi^B \) satisfying equation (3.2) simplify to

\[
\Psi^B(x_1, \ldots, x_N) = \Psi^F(x_1, \ldots, x_N)A(x_1, \ldots, x_N) , \tag{3.3}
\]

where \( \Psi^F \) is the Fermi wave-function for the free system of \( N \) particles confined to the region \( 0 \leq x_i < L, i = 1, \ldots, N \), with periodic b.c., and

\[
A(x_1, \ldots, x_N) = \prod_{j<i} \text{sign}(x_j - x_i) . \tag{3.4}
\]

Note that \( \Psi^F \) automatically satisfies equation (3.2) by the exclusion principle. Indicating the Bose and Fermi ground states by the subscript 0, it follows from equation (3.3) and the non-negativity of \( \Psi^B_0 \) that

\[
\Psi^B_0 = |\Psi^F_0| . \tag{3.5}
\]
Since $A^2 = 1$, by equation \((3.4)\), the correspondence between $\Psi^B$ and $\Psi^F$ given by equation \((3.3)\) preserves all scalar products, and therefore the energy spectrum of the Bose system is the same as that of the free Fermi gas. Furthermore, $\Psi^F$ is a Slater determinant of plane-wave functions labelled by wave vectors $k_i, i = 1, \ldots, N$, equally spaced over the range $[-k_F, k_F]$, where $k_F$ is the Fermi momentum. Hence $k_F = \pi \frac{N-1}{L}$, which, in the thermodynamic limit reduces to

$$k_F = \pi \rho .$$

The simplest excitation is obtained by moving a particle from $k_F$ to $q > k_F$ (or from $-k_F$ to $q < -k_F$) thereby leaving a hole at $k_F$ (or $-k_F$). This excitation has momentum $k = q - k_F$ (or $-(q - k_F)$), and energy $\epsilon(k) = (k^2 - k_F^2)/2$, i.e.,

$$\epsilon^1(k) = k^2/2 + k_F |k| .$$

This type of excitation must be supplemented by the umklapp excitations, which we consider in a more general form than [Lie63]. They consist in taking a particle from $(-k_F - p)$ to $(k_F + q)$ or $(k_F - q)$ to $(-k_F - p)$. We call the corresponding eigenvalues $\epsilon^{2,1}(k)$ and $\epsilon^{2,2}(k)$ and consider just the latter in detail; for them

$$0 \leq q \leq 2\pi (N - 1) \text{ and } \frac{2\pi}{L} \leq p \quad (3.7)$$

and their momentum

$$k = -2k_F - (p - q) \quad (3.8)$$

and their energy $\epsilon^{2,2}(k)$ are given by

$$2\epsilon^{2,2}(k) = [(-k_F - p)^2 - (k_F - q)^2] = [2k_F + (p - q)](p + q) . \quad (3.9)$$

A different, equivalent choice involving holes may be made [Lie63]. The complete set of eigenvalues of $\tilde{H}_{N,L} = H_{N,L} - E^0_{N,L}$ is

$$E_{n_i,m_i}^{m,\tilde{F}}(\vec{k}, \vec{l}) = \sum_{n_i,m_i=0,1; j=1,2} \left( n_i \epsilon^1(\vec{k}_i) + +m_i \epsilon^{2,j}(\vec{l}_i) \right) , \quad (3.10)$$

with $m = 1, \ldots, N$ and $\vec{k}_i \in \Lambda_\ast = \{ \frac{2\pi n}{L} \mid n \in \mathbb{Z}^\ast \}$ and $\vec{l}_i$ belongs to the set specified by \((3.7)\) and \((3.8)\) for $j = 2$.

In \((3.10)\), $\epsilon^1$ and $\epsilon^2$ have different meanings: $\epsilon^1$ are elementary excitations as the sum involving $\epsilon^1(\vec{k}_i)$ in \((3.10)\) is only an eigenvalue of $\tilde{H}_{N,L}$ up to
For the umklapp excitations, the same would follow if we considered, as in [Lie63], just the first one in (3.7)-(3.9), i.e., corresponding to $q = 0, p = \frac{2\pi}{L}$. By considering the general case (3.7)-(3.9), we have no corrections ($O(N^{-1})$ or otherwise) as far as the energies of the umklapp excitations are concerned. The momenta are, however, strictly additive in all cases [Lie63]: the momentum $\vec{P}$ associated to $E_{n,m}(\vec{k},\vec{l})$ in (3.10) is

$$\vec{P} = \vec{k} + \vec{l} \text{ where } \vec{k} = \sum_{i=1}^{r} \vec{k}_i \text{ and } \vec{l} = \sum_{j=1}^{s} \vec{l}_j,$$

with $r, s$ integers. We keep the vector index in the momenta to emphasize the (crucial) point that there are, of course, two directions, along the positive and negative axes. Without loss of generality we assume that $\vec{v}$ is directed along the positive axis, and denote $v = |\vec{v}|$. When momenta appear without a vector sign, their sign will define their orientation.

**Proposition 3.1.** For the Girardeau model, if

$$0 < v < 2\pi \rho,$$

the spectrum $\text{spec}(H_{v,N,L})$ of $H_{v,N,L}$ contains the set $\mathcal{E}$, defined by

$$\mathcal{E} = \left\{ -2k_{F}v + o(L) \geq \cdots \geq \sum_{j=1}^{m} e^{2.2}(k_{j}) \geq \cdots - \frac{Nv^{2}}{2} + o(L) \right\},$$

with $m = 1, \ldots, \left\lfloor \frac{L}{2\pi v} \right\rfloor; k_{j} = -2k_{F} - (p_{j} - q_{j})$,

where

$$p_{1} = \frac{2\pi i}{L}; \quad q_{i} = \frac{2\pi (i - 1)}{L} \text{ with } i = 1, \ldots, m,$$

and the quantities

$$\lambda_{j} \equiv -2k_{F}vj \text{ with } j = 1, 2, \ldots$$

belong to the energy spectrum in the sense of definition 2.7, which is thus unbounded from below and the system describes a NESS. Furthermore, if

$$c = 2(\pi \rho)^{2}$$

and

$$d = \pi \rho,$$

then the subspace (2.40) satisfies (2.39.1) and (2.39.2), and thus the system describes a superfluid at $T = 0$. 

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Proof. By (3.7)-(3.9), (3.10) and (3.11), with the choice $j = 2, n_i = 0, m_i = 1$, $m$ as in (3.13), we obtain (3.13), with

$$
\sum_{j=1}^{m} \epsilon^{2,2}(k_j) = \frac{2k_F m (m+1)\pi}{L} - \frac{2k_F \pi m}{L} + \frac{2(\pi)^2}{L^2} (m-1) - 2k_F m v - \frac{2\pi m v}{L}.
$$

(3.15)

The minimum of the right hand side of (3.15) coincides approximately (for large enough $L$) with that of the function

$$
f(m) = \alpha m^2 - \beta m \quad \text{with} \quad \alpha = \frac{2\pi^2 \rho}{L} \quad \text{and} \quad \beta = 2\pi \rho v,
$$

(3.16)

which is situated at $m = \left\lfloor \frac{\beta}{2\alpha} \right\rfloor = \left\lfloor \frac{Lv}{2\pi} \right\rfloor$, for $L$ large enough, and is given by $\frac{-(\beta)^2}{4\alpha} = -\frac{N(v)^2}{2}$; we have $\left\lfloor \frac{Lv}{2\pi} \right\rfloor \sim \frac{Lv}{2\pi} - 1 \leq N$ or $v \leq \frac{2\pi(N+1)}{L}$, which is satisfied under assumption (3.12). By (3.15), for $m = 1$, we obtain for the r.h.s.

$$
\sum_{j=1}^{m} \epsilon^{2,2}(k_j) - \sum_{j=1}^{m+1} \epsilon^{2,2}(k_j) = -2k_F v + o(L).
$$

(3.17)

Thus, the points of the set $\mathcal{E}$ are $\{\epsilon_{j,L} = -2k_F v j + o(L)\}_{j=1}^{m}$ with $-2k_F v m = \frac{-Nv^2}{2} + o(L)$, and it follows that any $\lambda_j$ given by (3.14) is such that $(\lambda(k_0), k_0)$, with $\lambda(k_0) = -2k_F v j; k_0 = -2k_F j$, by (3.13), is a limit point of the sequence $(\lambda_{N,L}^j, k_{N,L}^j = -2k_F (j + \frac{\pi}{k_F j}))$, with $\mathcal{E}_{j,L} \geq \lambda_{N,L}^j \geq \mathcal{E}_{j+1,L}$, as $N, L \to \infty$. By Corollary 2.12 the assertion connected to (3.14) follows.

Let, now, $c, d$ be as in (3.15), (3.16). If we restrict $|\vec{P}|$ in (3.11) to satisfy

$$
|\vec{P}| < \pi \rho
$$

(3.18)

and

$$
|\vec{k}| < \pi \rho
$$

(3.19)

then

$$
|\vec{l}| = |\vec{P} - \vec{k}| < \pi \rho + \pi \rho = 2\pi \rho,
$$

(3.20)

contradicting (3.8). Thus, in (3.10), the umklapp excitations are absent.
We now show that (3.19) is satisfied when $H_{N,L}$ and $\vec{P}_{N,L}$ are restricted to the subspace

$$\mathcal{R}_{N,L}^{c,d} \equiv \text{subspace of } \mathcal{H}_{N,L}^{c,d}$$

corresponding to a fixed $r$ independent of $N, L$

of elementary excitations $\epsilon^1(\cdot)$ in (3.10)

(3.21)

Under the assumption

$$H_{N,L} - E_0^{\theta} \text{ restricted to } \mathcal{R}_{N,L}^{c,d} \leq c \quad (3.22)$$

we have that, by (3.6) and (3.10), since $\epsilon^1(\vec{k}) \geq 0$, and $\epsilon^{2,j}(\vec{l}) \geq 0$, (3.22) implies

$$2\pi \rho (|\vec{k}_1| + \ldots + |\vec{k}_r|) \leq c \text{ for } \vec{k} = \vec{k}_1 + \ldots + \vec{k}_r \quad (3.23.1)$$

for $N$ sufficiently large and fixed $r$ due to the $O(N^{-1})$ corrections, hence

$$|\vec{k}| = |\vec{k}_1 + \ldots + \vec{k}_r| \leq |\vec{k}_1| + \ldots + |\vec{k}_r|$$

$$\leq \frac{c}{2\pi \rho} = \pi \rho \quad (3.23.2)$$

by (3.15), proving (3.19).

In order to prove (2.42.1), we consider $H_{v,N,L}$ when restricted to the subspace defined by (3.21) where, by (3.19),

$$|\vec{P}_{N,L}| \leq d = \pi \rho \quad (3.23.3)$$

By (3.10), (3.19.4) and what was shown above, the spectrum of $H_{\vec{v},N,L}$ restricted to $\mathcal{R}_{N,L}^{c,d}$ consists of eigenvalues

$$\epsilon_{r,N,L}^{c,d} \equiv \{ \sum_{i=1}^{r} (\epsilon^1(\vec{k}_i) + \vec{v} \cdot \vec{k}_i) \}

\text{with } |\vec{k}_1 + \ldots + \vec{k}_r| \leq d; \sum_{i=1}^{r} \epsilon^1(\vec{k}_i) \leq c \}$$
up to corrections $O(N^{-1})$ when $r$ is a fixed number (independent of $N$ and $L$).

By (3.6) and (3.12), $\epsilon^1(\vec{k}_i) + \vec{v} \cdot \vec{k}_i \geq 0$ for all $i = 1, \ldots, r$, and thus, by (3.23.4), (2.42.1) holds. Moreover, defining $\epsilon^{c,d}_r(\vec{v}) \equiv \lim_{N,L \to \infty} \epsilon^{c,d}_{r,N,L}$ we have that $\{\epsilon^{c,d}_r(\vec{v})\}_r \neq \{0\}$ for any $r \geq 1$ by (3.23.4), so that (2.42.2) also holds.

q.e.d.

**Remark 3.2.** In case assumption B holds, defining by (2.44)

$$H^{c,d,\prime}_\omega = \int dE(\lambda, \vec{k}) \ H^{c,\prime}_\omega$$

with

$$(\lambda, \vec{k}) = (\epsilon^1(\vec{k}_1) + \vec{v} \cdot \vec{k}_1 + \ldots + \epsilon^1(\vec{k}_r) + \vec{v} \cdot \vec{k}_r, \vec{k})$$

(3.24.2)

we have $H^{c,d,\prime}_\omega$ when restricted to $H^{c,d,\prime}_\omega \geq 0$ (3.24.4)

**Remark 3.3.** Due to the corrections $O(N^{-1})$ we do not know whether $H^{c,d,\prime}_\omega$ in the above remark 3.2 comprises the whole of $H^{c,\prime}_\omega$, although we conjecture that to be so. In particular, nothing can be said about the canonical free energy restricted to the subspace (3.24.1) in the thermodynamic limit. For other conjectures associated to the (supposedly subadditive) excitation spectrum of Bose gases, see ([HDCZ09], (1.10) et seq.), and ([HDCZ09], Appendix B). On $H^{c,d,\prime}_\omega$ the general model of independent elementary excitations of [SW09] is fully justified. We believe that a qualitatively similar assertion applies to a wide variety of models in condensed matter physics.

**Remark 3.4.** (3.12) corresponds to the original Landau assumption (1.1). The restriction on energy and momentum necessary in Proposition 3.1 seems to confirm Kadanoff’s remarks cited in the introduction, as well as some observations in Remark 2.8.
3.2 The Huang-Yang-Luttinger (HYL) model

Our second example is the Huang-Yang-Luttinger (HYL) model [KHL57], whose pressure was rigorously derived by van den Berg, Dorlas, Lewis and Pulé in [MvdBP]. Let

\[ H_0^\Lambda \equiv \sum_{\vec{k} \in S^d_{\Lambda=3}} \frac{(\vec{k})^2 n_{\vec{k}}}{2} \]

(3.25)

denote the free Hamiltonian, and

\[ H_{\Lambda}^{HYL} \equiv H_0^\Lambda + \frac{\tilde{a}(2N^2 - \sum_{\vec{k}} n_{\vec{k}}^2)}{2V}, \]

(3.26)

where \( n_{\vec{k}} = a_{\vec{k}}^\dagger a_{\vec{k}} \) denotes the number operator for the \( \vec{k} \)-th mode, \( \tilde{a} \) is a positive number and \( V = L^3 \) denotes the volume of the cubic region. We have

\[ a_{\vec{k}} = \int d\vec{x} \exp(-i\vec{k} \cdot \vec{x}) \Psi(\vec{x}) \sqrt{(V)}, \]

(3.27)

while, in terms of the basic destruction operator \( \Psi \) (see after (2.29)) the group of Galilean transformations is given by (2.15). The free Hamiltonian (3.25) may be written [AW63]

\[ H_0^\Lambda = \int_V d\vec{x} \nabla \Psi^\dagger(\vec{x}) \cdot \nabla \Psi(\vec{x}), \]

(3.28)

which transforms under (2.15) to

\[ H_\Lambda \rightarrow H_0^\Lambda + \vec{v} \cdot \vec{P}_\Lambda + \frac{N(\vec{v})^2}{2}, \]

(3.29)

with

\[ \vec{P}_\Lambda = \frac{1}{2} \int d\vec{x} [\Psi^\dagger(\vec{x})(\nabla \Psi)(\vec{x}) - (\nabla \Psi^\dagger(\vec{x})\Psi(\vec{x})] \]

(3.30)
the momentum operator; by (3.27), $a_{\vec{k}}^\dagger \to a_{\vec{k} + \vec{v}}^\dagger$, $a_{\vec{k}} \to a_{\vec{k} + \vec{v}}$ under (2.15) and thus $\sum_{\vec{k} \in S_\Lambda^3} n_{\vec{k}}^2$ remains invariant under (2.15) (recall our choice (2.19.2)). Therefore (3.29) becomes

$$H_\Lambda \to H_\Lambda + \vec{v} \cdot \vec{P}_\Lambda + \frac{N(\vec{v})^2}{2},$$

(3.31)

where we omit the suffix HYL from now on. By ([Hua87], Section 10.5), the correspondence

$$\tilde{a} = 8\pi a$$

(3.32)

is valid, where $a > 0$ is the scattering length of the given repulsive potential, which is approximated by a poitwise (delta) repulsion in each order $j$ of perturbation theory (and only in a fixed order, since, globally, such a potential is not stable in the sense of (2.9.2) — see [SAH88] — it exhibits, however, non-trivial scattering [SAH88], in spite of contrary assertions in many books). In first order, we obtain (2.21),(2.22), with the correspondence (3.32), see ([Hua87], (10.124) and (A36)).

Since $H_\Lambda$ depends only on the number operators and not on the creation and destruction operators individually, we may work on the $N$- particle Fock space $F(H_\Lambda)$, the symmetrized $N$- fold tensor product of $H_\Lambda = L^2(\Lambda)$ (with periodic b.c.). The ground state energy $E_{N,L}^0$ of $H_\Lambda = H_{N,L}$, given by (3.26), obtains upon setting $n_0 = N$, whereby

$$E_{N,L}^0 = \frac{\tilde{a}(2N^2 - N^2)}{2V} = \frac{\tilde{a}N^2}{2V}$$

and the ground state energy per unit volume in the limit $N, L \to \infty$ is

$$e_0(\rho) = E_{N,L}^0 / V = \frac{\tilde{a}\rho^2}{2} = 4\pi a\rho^2$$

(3.33)

by (3.32). By (3.31), (3.33) and Definition 2.5 we arrive at the following result.

**Lemma 3.5.** The HYL Hamiltonian (3.26) is an effective Hamiltonian to order $j = 1$ for the dilute Boson system.

One reason that it is believed that $H_\Lambda$ is a good approximation (to order one) for the dilute system, besides being an effective Hamiltonian to order one, is that it may be regarded as a correction to the mean-field Hamiltonian

$$H_\Lambda^{mf} \equiv H_\Lambda^0 + \frac{\tilde{a}N^2}{V}$$

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by incorporating a local repulsion. Indeed, the mean-field Hamiltonian would also be an effective Hamiltonian to order one for the dilute Boson system (with a slightly different choice of parameters), but it does not exhibit superfluidity (in the Landau sense) for the simple reason that

\[ H_{\Lambda}^{mf} - E_{\Lambda}^{mf,0} = H_{\Lambda}^{0} = H_{\Lambda}^{0} - E_{\Lambda}^{0,0}, \tag{3.34} \]

where

\[ E_{\Lambda}^{mf,0} = \frac{\tilde{a}N^{2}}{V} = \tilde{a}\rho^{2}V \tag{3.35} \]

and \( E_{\Lambda}^{0,0} = 0 \) are the ground state energies of the mean field and the free Hamiltonian. Thus the eigenvalues and/or elementary excitations of \( H_{\Lambda}^{mf} - E_{\Lambda}^{mf,0} \) are the same as those of the free Bose gas, which is not a Landau superfluid. The same remark in somewhat different language was made in (Ver11, pg. 85), where it is also remarked (pp. 82-84) that, for the mean-field model, however, the canonical and grand-canonical states coincide, in contrast to the free Bose gas. It may be added that, due to (3.35), \( e_{0}(\rho) = \tilde{a}\rho^{2} \) and, therefore, the compressibility is nonzero: \( e''_{0}(\rho) = 2\tilde{a} > 0 \). The formula \( c = \left(\frac{1}{e''_{0}(\rho)}\right)^{1/2} \) for the sound velocity (see, e.g., WdSJ05) is not, however, true for the mean field model, because the latter is zero as in the case of the free Boson gas by (3.34).

We now consider the Hamiltonian \( H_{\bar{v},N,L} \) given by (2.19.4). Our main result for the HYL model is the following one.

**Proposition 3.6.** If

\[ \bar{v}^{2} < 2\tilde{a}\rho, \tag{3.36} \]

then the HYL model describes a NESS at \( T = 0 \) with energy spectrum according to definition 2.7 unbounded from below. Moreover, if (2.41) holds, with

\[ \rho_{\max} \equiv \rho - \frac{v^{2}}{2\tilde{a}}, \tag{3.37} \]

then it describes a \( T = 0 \) superfluid.

**Proof.** We have that

\[
H_{\bar{v},N,L} = \sum_{\vec{k} \in S^{3}_{\Lambda}} \frac{(\vec{k} - \bar{v})^{2}n_{\vec{k}}}{2} - \frac{Nv^{2}}{2} + \tilde{a}(N^{2} - \sum_{\vec{k}}n_{\vec{k}}^{2})
\]

\[
- \frac{2V}{2}
\]

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(with \( \vec{v} = \vec{\nu}_{L,L} \) defined by (2.19.2). The lowest eigenvalue \(-\frac{N_v^2}{2}\) of \( H_{\vec{v},N,L} \) obtains for \( n_v = N \). In general, the energy is minimized by choosing, first, \( n_\vec{\nu} = N - n \) together with \( n_v = n \), and then finding the minimum with respect to \( n \). Indeed, a splitting of \( \sum l n_{\vec{k}_l} = n \) with \( l > 1 \) entails a relative increase in energy of \( \frac{2\bar{a}\sum l n_{\vec{k}_l}}{V} \).

The corresponding eigenvalues \( E_{\vec{v},N,L} \) of \( H_{\vec{v},N,L} \) are

\[
E_{\vec{v},N,L} = -\frac{n v^2}{2} + \bar{a}(Nn - n^2)
\]

\[
= -\frac{n v^2}{2} + (\bar{a}\rho - \bar{a}n^2) \frac{V}{V}.
\]

(3.38)

We now assume (3.36). The function

\[
f_{\vec{v}}(n) \equiv (\bar{a}\rho - v^2/2)n - \bar{a}n^2
\]

has a maximum at

\[
n_{\text{max}} \equiv (\bar{a}\rho - v^2/2) = N/2 - \frac{v^2V}{4\bar{a}},
\]

and minima at \( n_{\text{min}}^1 = 0 \) and

\[
n_{\text{min}}^2 = \frac{V(\bar{a}\rho - v^2/2)}{\bar{a}} = N - \frac{v^2V}{2\bar{a}}
\]

(3.39)

(we ignore the corrections due to the fact that these numbers are integer, which are easily shown to be negligible for large enough \( V, N \)). For

\[
n \leq n_{\text{min}}^2
\]

(3.40)

the eigenvalues are positive: this assertion is, of course, always true for \( \vec{v} = \vec{0} \) as it should be. By (3.39), (3.40), this yields the last assertion of the proposition.

Let, now, \( k \) be an integer such that

\[
V(\rho - \frac{v^2}{2\bar{a}}) + k \leq N
\]

(3.41.1)
or
\[
(\rho - \frac{v^2}{2a}) + k/V \leq \rho,
\]
which always holds in the thermodynamic limit, for any fixed integer \( k > 0 \).
Note that for \( \vec{v} = \vec{0} \) the density interval (3.41) becomes empty, as it should; for \( \vec{v} \neq \vec{0} \),
\[
f_{\vec{v}}(V(\rho - \frac{v^2}{2a}) + k) = -k(\bar{a}\rho - v^2/2) - \bar{a}/V
\]
yielding, by Corollary 2.12, the first assertion of the proposition; moreover, since the integer \( k \) is arbitrary, the energy spectrum in the sense of definition 2.7 is unbounded from below. q.e.d.

Remark 3.7. Proposition 3.6 may be regarded as a precise statement corresponding to Nozières’ intuition, mentioned in the introduction [er03]: under (3.36), with \( \rho_{\max} \) given by (3.37), a fragmentation of the condensate costs a macroscopic amount of energy, rendering it stable. Further, (3.38) for small \( n \) and \( \vec{v} = \vec{0} \), exhibits his conjectured macroscopic stability, when \( n = \bar{n}^\vec{k} \) with \( \bar{k} \) arbitrarily close to the zero vector.

4 Conclusion

We have seen that the concept of semipassivity (Definition 2.6), which is a variation of that introduced by Kuckert [Kuc02] and Bros-Buchholz ([BB94], (6.2)) is a natural one to describe quantum systems carrying a (classical) current. For Bosons in uniform translational motion, semipassivity is a consequence of Galilean covariance, the classical field is the velocity field \( \vec{v} \) — when uniform, the phase in (1.2) is \( \chi(\vec{x}) = \vec{v} \cdot \vec{x} \), otherwise \( \vec{v} = (\nabla \chi)(\vec{x}) \) and the motion is, quite generally, irrotational — see [MR04], pp. 161-171 for a nice textbook discussion of the London macroscopic hydrodynamics, and [Sew02], Part III, Chapters 8 and 9 for a deep discussion of superconductive electrodynamics, with references.

For \( T > 0 \), the states of both translational and rotational superfluids are NESS under assumption A (Theorem 2.10). For \( T = 0 \), some restrictions for the existence of NESS are necessary (Theorem 2.12), which have been shown to hold for the two examples in Section 3 under no assumptions. The main point of theorem 2.10, i.e, that the same state cannot be an equilibrium state under two different dynamics, may be conjectured to be true independently
of assumption A; moreover, for sufficiently low temperature, it is expected to go over to the results for \( T = 0 \).

Since local stability conditions are not expected to be valid for NESS (because, under certain conditions, they imply that the state is an equilibrium state \([[\text{HKTP}74],[\text{BKR}78]]\) - this fact was actually proved by Ogata \([\text{Oga}04]\) for a special model -, it is natural to characterize a superfluid at \( T = 0 \) by certain metastability conditions (Definition 2.8), which physically express that there should be a sufficiently small energy-momentum transfer between the particles of the fluid and the surroundings (e.g., the pipe). They have been verified for the models of Section 3. In the case of the second model (HYL model) in Section 3, the metastability condition is directly related to Nozières’ conjecture \([\text{Noz03}]\) that it is the repulsive interaction which interdicts the fragmentation of the condensate, thereby assuring its quantum coherence, with the wave-function becoming a macroscopic observable by the condition of ODLRO.

The instability responsible for the state becoming a non-equilibrium state may be expected to be due to certain excitations inherent to the system, when their energy and/or momenta exceed certain values. Although excitations such as the umklapp excitations, imparting large momenta and low energy, may be special to singular repulsive one-dimensional interactions, the remarkable result of Seiringer and Yin \([\text{SY}08]\), showing that the Lieb-Liniger model is a suitable limit of dilute Bosons in three dimensions, indicates that a qualitatively similar picture may take place for more realistic systems. In general, for realistic systems, it is conjectured that that the instability in the case of a rotating bucket is caused by vortices (see, e.g. \([\text{MR}04]\, pg. 235), which, for systems with nonzero density, have been seen to exist in the free Boson gas in \([\text{Le75}]\). For translational superfluids, the instability is conjectured to be caused by rotons, which arise as local minima of the infimum of the excitation spectrum, see \([\text{MR}04]\, pg. 246. These should also take place only for sufficiently large momentum, as a consequence of the properties of the liquid structure factor, if one assumes the Feynman variational wave function, see \([\text{HDCZ}09]\, pg. 256. For the Feynman variational wave function, see \([\text{WdSJ}05]\). In both models of section 3 the ultimate cause of instability is seen to be the local repulsion.

Unfortunately, the HYL model of Section 3 displays a gap and is, therefore, an unrealistic model of a Boson superfluid. It has been conjectured, however, that that the gap disappears in second order \(([\text{Hua}87], pg. 330)\), and it would be very nice to prove this, together with the other properties
characterizing a superfluid. This would provide a theory of superfluidity at
the level of effective Hamiltonians, which would be very interesting, with
applications to real systems, see [GK81] and [BCCR83], but still very far
from the full theory of dilute Boson systems at order $j = 2$. Even the lat-

ter would be, as Leggett remarked about Bogoliubov’s theory [Leg99], “ex-
tremely suggestive but rather far from real-life Helium II”, because the latter
is a strongly correlated system, for which, in addition, the attractive part
of the interaction is expected to play a significant role (see also the discussion,
with references, in [HDCZ09]).

As a final remark, the present framework seems to be the first rigorous
example of a NESS in which the system is a many-body system, not a finite
system; for the latter see [JP02] [JP], as well as references given there, and
[JFU03]). Although the stationarity arises naturally from the physical input
for our systems, without having to rely on a somewhat unphysical time-
average as in [JP02] [JP] (see, however, [JFU03] for an alternative approach),
and the metastability assumption justifies the long-lived nature of the NESS,
with a potentially wide domain of applications (see Remark 3.3), the beautiful
proof of entropy production in [JP02] is an open problem for the class of
systems treated here - indeed, even the definition of that quantity poses a
difficult problem.

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