Hamilton-Jacobi Equations for Nonholonomic Reducible Hamiltonian Systems on a Cotangent Bundle

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Abstract: In this paper, for a variety of nonholonomic (reducible) Hamiltonian systems, we first give to various distributional Hamiltonian systems, by analyzing carefully the dynamics and structures of the nonholonomic Hamiltonian systems. Secondly, we derive precisely the geometric constraint conditions of the induced distributional two-form for the nonholonomic dynamical vector field, which are called the Type I and Type II of Hamilton-Jacobi equations. Thirdly, we generalize the above results for the nonholonomic reducible Hamiltonian systems with symmetries, as well as with momentum maps, and prove two types of Hamilton-Jacobi theorems for various nonholonomic reduced distributional Hamiltonian systems. Finally, as an application, we give two examples to illustrate the theoretical results. These researches reveal the deeply internal relationships of the nonholonomic constraints, the induced (resp. reduced) distributional two-forms and the dynamical vector fields of the nonholonomic Hamiltonian system and its various distributional Hamiltonian systems.

Keywords: nonholonomic constraint, nonholonomic Hamiltonian system, distributional Hamiltonian system, nonholonomic reduction, momentum map.

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1 Introduction

It is well-known that Hamilton-Jacobi theory is an important research subject in mathematics and analytical mechanics. See Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [25], and the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it also plays an important role in the study of stochastic dynamical systems, see Woodhouse [41], Ge and Marsden [12], and Lázaro-Camí and Ortega [16]. Hamilton-Jacobi theory from the variational point of view is originally developed by Jacobi in 1866, which states that the integral of Lagrangian of a system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the generating function and the geometrical point of view is given by Abraham and Marsden in [1] as follows: Let $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ the cotangent bundle with the canonical symplectic form $\omega$, and the projection $\pi_Q: T^*Q \to Q$ induces the map $T\pi_Q: TT^*Q \to TQ$.

**Theorem 1.1** Assume that the triple $(T^*Q, \omega, H)$ is a Hamiltonian system with Hamiltonian vector field $X_H$, and $W: Q \to \mathbb{R}$ is a given generating function. Then the following two assertions are equivalent:

(i) For every curve $\sigma: \mathbb{R} \to Q$ satisfying $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$.

(ii) $W$ satisfies the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant.

From the proof of the above theorem given in Abraham and Marsden [1], we know that the assertion (i) with equivalent to Hamilton-Jacobi equation (ii) by the generating function, gives a geometric constraint condition of the canonical symplectic form on the cotangent bundle $T^*Q$ for Hamiltonian vector field of the system. Thus, the Hamilton-Jacobi equation reveals the deeply internal relationships of the generating function, the canonical symplectic form and the dynamical vector field of a Hamiltonian system.

On the other hand, we have known that, in mechanics, it is very often that many systems have constraints, and usually, the constraints in dynamics are restrictions on positions and velocities of the system. There are two types of constraints, and the first one is holonomic, which is that imposed on the configuration space of a system; and the second one is nonholonomic, which involves the conditions on the velocities of a system, such as rolling constraints. Thus, the nonholonomic mechanics describes the motion of systems constrained by nonintegrable constraints, i.e., constraints on the system velocities that do not arise from constraints on the configurations alone. For a nonholonomic Hamiltonian system, from Bates and Śniatycki [3], we know that, under the restriction given nonholonomic constraints, in general, we can derive a distributional Hamiltonian system, which is called a semi-Hamiltonian system in Patrick [32]. But, the leading distributional Hamiltonian system may not be a Hamiltonian system, and it has no generating function, then we cannot give the Hamilton-Jacobi theorem for the distributional Hamiltonian system just like same as the above Theorem 1.1. Moreover, for the nonholonomic Hamiltonian system with symmetry, the leading reduced distributional Hamiltonian system by nonholonomic reduction may not be yet...
a Hamiltonian system, and we cannot give the Hamilton-Jacobi theorem for the reduced distributional Hamiltonian system as the above Theorem 1.1. We have to look for a new way.

It is worthy of noting that the regular point symplectic reduction for the Hamiltonian system with symmetry and coadjoint equivariant momentum map was set up by famous professors Jerrold E. Marsden and Alan Weinstein, which is called Marsden-Weinstein reduction, and great developments have been obtained around the work in the theoretical study and applications of mathematics, mechanics and physics; see Abraham and Marsden [1], Arnold [2], Koiller [15], Libermann and Marle [19], Marsden [21], Marsden et al. [22, 23], Marsden and Perlmutter [24], Marsden and Ratiu [25], Marsden and Weinstein [27], Meyer [28], Nijmeijer and Van der Schaft [30] and Ortega and Ratiu [31] for more details and development. But, in Marsden et al. [26] and Wang [34], the authors found that the symplectic reduced space of a Hamiltonian system defined on the cotangent bundle of a configuration manifold may not be a cotangent bundle, and hence the set of Hamiltonian systems with symmetries on the cotangent bundle is not complete under the Marsden-Weinstein reduction. Thus, the symplectic reduced system of a Hamiltonian system with symmetry defined on the cotangent bundle may not be a Hamiltonian system on a cotangent bundle, then we cannot give the Hamilton-Jacobi theorem for the Marsden-Weinstein reduced system as the above Theorem 1.1.

Now, it is a natural problem how to generalize Theorem 1.1 to fit the above nonholonomic systems and their reduced systems. Note that if take that \( \gamma = dW \) in the above Theorem 1.1, then \( \gamma \) is a closed one-form on \( Q \), and the equation \( d(H \cdot dW) = 0 \) is equivalent to the Hamilton-Jacobi equation \( H(q^i, \frac{\partial W}{\partial q^i}) = E \), where \( E \) is a constant, which is called the classical Hamilton-Jacobi equation. This result is used the formulation of a geometric version of Hamilton-Jacobi theorem for Hamiltonian system, see Cariñena et al [7, 8]. Moreover, note that Theorem 1.1 is also generalized in the context of time-dependent Hamiltonian system by Marsden and Ratiu in [25], and the Hamilton-Jacobi equation may be regarded as a nonlinear partial differential equation for some generating function \( S \). Thus, the problem is become how to choose a time-dependent canonical transformation \( \Psi : T^*Q \times \mathbb{R} \rightarrow T^*Q \times \mathbb{R} \), which transforms the dynamical vector field of a time-dependent Hamiltonian system to equilibrium, such that the generating function \( S \) of \( \Psi \) satisfies the time-dependent Hamilton-Jacobi equation. In particular, for the time-independent Hamiltonian system, ones may look for a symplectic map as the canonical transformation. This work offers an important idea that one can use the dynamical vector field of a Hamiltonian system to describe Hamilton-Jacobi equation. In consequence, if assume that \( \gamma : Q \rightarrow T^*Q \) is a closed one-form on \( Q \), and define that \( X^\gamma_H = T_\pi \gamma \cdot X_H \cdot \gamma \), where \( X_H \) is the dynamical vector field of Hamiltonian system \((T^*Q, \omega, H)\), then the fact that \( X^\gamma_H \) and \( X_H \) are \( \gamma \)-related, that is, \( T\gamma \cdot X_H^\gamma = X_H \cdot \gamma \) is equivalent that \( d(H \cdot \gamma) = 0 \), which is given in Cariñena et al [7, 8]. Motivated by the above research work, Wang in [36] prove an important lemma, which is a modification for the corresponding result of Abraham and Marsden in [1], such that we can derive precisely the geometric constraint conditions of the regular reduced symplectic forms for the dynamical vector fields of a regular reducible Hamiltonian system on the cotangent bundle of a configuration manifold, which are called the Type I and Type II of Hamilton-Jacobi equation, because they are the development of the above classical Hamilton-Jacobi equation given by Theorem 1.1, see Abraham and Marsden [1] and Wang [36].

Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, it is natural idea to extend the Hamilton-Jacobi theory to the nonholonomic Hamiltonian systems, and they with symmetry and momentum map. Our idea is that to use a variety of dynamical vector fields of the (reduced) distributional Hamiltonian systems to describe a variety of Hamilton-Jacobi equations. In this paper, we first derive precisely the geometric constraint conditions of the induced distributional two-form and the reduced distributional two-form for the nonholonomic dynamical vector fields. Moreover, we consider the nonholonomic reductions compatible
with Marsden-Weinstein reduction and regular orbit reduction, and lead to the \( J \)-nonholonomic regular point and orbit reduced distributional Hamiltonian systems, by analyzing carefully the dynamics and structures of the nonholonomic Hamiltonian systems. These systems are not yet Hamiltonian, but, we can give their two types of Hamilton-Jacobi equations, as an extension of two types of Hamilton-Jacobi equations for the Marsden-Weinstein reduced Hamiltonian system and the regular orbit reduced Hamiltonian system given in [36] to the nonholonomic context.

The paper is organized as follows. In section 2 we first recall the main facts about the dynamics of a nonholonomic Hamiltonian system, including the influence of symmetries, which are helpful for us to understand the constructions of a distributional Hamiltonian system and its a variety of the nonholonomic reduced distributional Hamiltonian systems. In section 3, we first prove an important lemma, which is a tool for our research. Then derive precisely the geometric constraint conditions of the distributional two-form for the nonholonomic dynamical vector field, that is, the two types of Hamilton-Jacobi equation for the distributional Hamiltonian system. The nonholonomic reducible Hamiltonian systems with symmetries, as well as momentum maps, are considered respectively in section 4 and section 5, and derive precisely the geometric constraint conditions of a variety of nonholonomic reduced distributional two-forms for the nonholonomic reducible Hamiltonian vector fields, that is, the two types of Hamilton-Jacobi equations for a variety of nonholonomic reduced distributional Hamiltonian systems (in particular, when the Lie group is not Abelian). As the applications of the theoretical results, we consider the motions of the constrained particle in space \( \mathbb{R}^3 \) and the vertical rolling disk in section 6, and derive two types of Hamilton-Jacobi equations for the distributional Hamiltonian systems and their reduced distributional Hamiltonian systems corresponding to the two nonholonomic systems. These research work develop the nonholonomic reduction and Hamilton-Jacobi theory of the nonholonomic Hamiltonian systems with symmetries, as well as momentum maps, and make us have much deeper understanding and recognition for the structures of the nonholonomic Hamiltonian systems.

2 Dynamics of Nonholonomic Mechanical System

In this section, we first review briefly some basic facts about nonholonomic mechanical systems and give the descriptions of dynamics of a nonholonomic Hamiltonian system and the nonholonomic Hamiltonian system with symmetry, as well as momentum maps, which are helpful for us in subsequent sections to understand the constructions of distributional Hamiltonian system and the nonholonomic reduced distributional Hamiltonian system. We shall follow the notations and conventions introduced in Cantrijn et al. [6], Bates and Śniatycki in [3], Cushman et al. [10] and [11], Montgomery [29], de León et al. [17], Marsden et al. [26] and Wang [36].

In order to describe the dynamics of a nonholonomic mechanical system, we need some restriction conditions for nonholonomic constraints of the system. At first, we note that the set of Hamiltonian vector fields forms a Lie algebra with respect to the Lie bracket, since \( X_{\{f,g\}} = -[X_f, X_g] \). But, the Lie bracket operator, in general case, may not be closed on the restriction of a nonholonomic constraint. Thus, we have to give the following completeness condition for nonholonomic constraints of a system.

**D-completeness** Let \( Q \) be a smooth manifold and \( TQ \) its tangent bundle. A distribution \( \mathcal{D} \subset TQ \) is said to be completely nonholonomic (or bracket-generating) if \( \mathcal{D} \) along with all of its iterated Lie brackets \( [\mathcal{D}, \mathcal{D}], [\mathcal{D}, [\mathcal{D}, \mathcal{D}]], \cdots \), spans the tangent bundle \( TQ \). Moreover, we consider a mechanical system on \( Q \). Then nonholonomic constraints of the system are said to be completely nonholonomic if the distribution \( \mathcal{D} \subset TQ \) defined by the nonholonomic constraints is completely nonholonomic.
In this paper we consider that a nonholonomic mechanical system is given by a Lagrangian function \( L : TQ \to \mathbb{R} \) subject to constraints determined by a completely nonholonomic distribution \( \mathcal{D} \subset TQ \) on the configuration manifold \( Q \). We denote by \( D \) the total space of \( \mathcal{D} \) in \( TQ \), which is a constraint submanifold. For simplicity we always assume that \( \tau_Q(D) = Q \), where \( \tau_Q : TQ \to Q \) is the canonical projection, that is, the constraints are purely kinematical in the sense that they do not impose restrictions on the allowable positions. The motions of the system are forced to take place on \( D \) and this requires the introduction of some "reaction force". In order to describe the constraint submanifold in the phase space and the dynamics of system, we have to give the following regularity condition.

**\( \mathcal{D} \)-regularity** In the following we always assume that \( Q \) is a smooth manifold with coordinates \((q^i, p_j)\), and \( TQ \) its tangent bundle with coordinates \((q^i, \dot{q}^j, p_j)\), which are the canonical cotangent coordinates of \( T^*Q \) and \( \omega = dq^i \wedge dp_i \) is canonical symplectic form on \( T^*Q \). If the Lagrangian \( L : TQ \to \mathbb{R} \) is hyperregular, that is, the Hessian matrix \((\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)\) is nondegenerate everywhere, then the Legendre transformation \( F L : TQ \to T^*Q \) is a diffeomorphism. In this case the Hamiltonian \( H : T^*Q \to \mathbb{R} \) is given by \( H(q, p) = \dot{q} \cdot p - L(q, \dot{q}) \) with Hamiltonian vector field \( X_H \), which is defined by the Hamilton’s equation \( i_{X_H} \omega = dH \), and \( \mathcal{M} = F L(D) \) is a constraint submanifold in \( T^*Q \). In particular, for the nonholonomic constraint \( \mathcal{D} \), the Lagrangian \( L \) is said to be \( \mathcal{D} \)-regular, if the restriction of Hessian matrix \((\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)\) on \( \mathcal{D} \) is nondegenerate everywhere. Moreover, a nonholonomic system is said to be \( \mathcal{D} \)-regular, if its Lagrangian \( L \) is \( \mathcal{D} \)-regular. Note that the restriction of a positive definite symmetric bilinear form to a subspace is also positive definite, and hence nondegenerate. Thus, for a simple nonholonomic mechanical system, that is, whose Lagrangian is the total kinetic energy minus potential energy, it is \( \mathcal{D} \)-regular automatically, which is coincident with the sense of regularity of nonholonomic system given by de Léon and Martín de Diego [17].

A nonholonomic Hamiltonian system is a 4-tuple \((T^*Q, \omega, \mathcal{D}, H)\), which is a Hamiltonian system with a \( \mathcal{D} \)-completely and \( \mathcal{D} \)-regularly nonholonomic constraint \( \mathcal{D} \subset TQ \). In the following we shall describe the dynamics of the nonholonomic Hamiltonian system \((T^*Q, \omega, \mathcal{D}, H)\). We define the distribution \( \mathcal{F} \) as the pre-image of the nonholonomic constraints \( \mathcal{D} \) for the map \( T\tau_Q : TT^*Q \to TQ \), that is, \( \mathcal{F} = (T\tau_Q)^{-1}(\mathcal{D}) \subset TT^*Q \), which is a distribution along \( \mathcal{M} \), and \( \mathcal{F}^\circ := \{ \alpha \in T^*T^*Q \mid \alpha < \alpha \cdot v >= 0, \forall v \in TT^*Q \} \) is the annihilator of \( \mathcal{F} \) in \( T^*T^*Q \mid \mathcal{M} \). We consider the following nonholonomic constraints condition

\[
(i_X \omega - dH) \in \mathcal{F}^\circ, \quad X \in TM, \quad \tag{2.1}
\]

from Cantrijn et al. [6], we know that there exists an unique nonholonomic vector field \( X_n \) satisfying the above condition (2.1), if the admissibility condition \( \text{dim} \mathcal{M} = \text{rank} \mathcal{F} \) and the compatibility condition \( TM \cap F^\perp = \{ 0 \} \) hold, where \( F^\perp \) denotes the symplectic orthogonal of \( \mathcal{F} \) with respect to the canonical symplectic form \( \omega \) on \( T^*Q \). In particular, when we consider the Whitney sum decomposition \( T(T^*Q) \mid \mathcal{M} = TM \oplus F^\perp \) and the canonical projection \( P : T(T^*Q) \mid \mathcal{M} \to TM \), we have that \( X_n = P(X_H) \).

If the Lagrangian \( L : TQ \to \mathbb{R} \) is singular, in this case the Hessian matrix \((\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)\) is degenerate. By using the Gotay-Nester presymplectic constraint algorithm, see [13], we can find a final constraint submanifold \( \mathcal{M}_f \subset T^*Q \), such that on which there exists a nonholonomic vector field \( X_n \) satisfying the following nonholonomic constraints condition

\[
(i_X \omega - dH) \mid_{\mathcal{M}_f} \in \mathcal{F}^\circ, \quad X \mid_{\mathcal{M}_f} \in TM_f. \quad \tag{2.2}
\]
Therefore, without loss of generality, we shall henceforth always assume that there exists a nonholonomic vector field $X_n$ satisfying the nonholonomic constraints condition.

From the condition (2.1) we know that the nonholonomic vector field, in general case, may not be Hamiltonian, because of the restriction of nonholonomic constraints. But, we hope to study the dynamical vector field of nonholonomic Hamiltonian system by using the similar method of studying Hamiltonian vector field. On the other hand, we also note that Bates and Śniatycki in [3] give a method to study the nonholonomic Hamiltonian system and nonholonomic reduction. In fact, for a nonholonomic Hamiltonian system $(T^*Q, \omega, D, H)$, by using their method, we know that there exist a distribution $\mathcal{K} = \mathcal{F} \cap TM$, a non-degenerate distributional two-form $\omega_{\mathcal{K}}$, which is the restriction of the induced symplectic form $\omega_M$ on $TM$ fibrewise to the distribution $\mathcal{K}$, and a vector field $X_\mathcal{K}$ on the constraint submanifold $\mathcal{M} = FL(D) \subset T^*Q$, such that the distributional Hamiltonian equation $i_{X_\mathcal{K}} \omega_\mathcal{K} = dH_\mathcal{K}$ holds, where the function $H_\mathcal{K}$ satisfies $dH_\mathcal{K} = \tau_\mathcal{K} \cdot dH_\mathcal{M}$, and $H_\mathcal{M}$ is the restriction of the Hamiltonian function $H$ to the constraint submanifold $\mathcal{M}$, and $\tau_\mathcal{K}$ is the restriction map to distribution $\mathcal{K}$. Then the triple $(\mathcal{K}, \omega_\mathcal{K}, H_\mathcal{K})$ is called a distributional Hamiltonian system, and $X_\mathcal{K}$ is its nonholonomic vector field.

Moreover, we consider the nonholonomic Hamiltonian system with symmetry and nonholonomic reduction. Assume that Lie group $G$ acts smoothly by the left on $Q$, its tangent lifted acts on $TQ$ and its cotangent lifted acts on $T^*Q$, which is free, proper and symplectic. The orbit space $T^*Q/G$ is a smooth manifold and the canonical projection $\pi_G : T^*Q \to T^*Q/G$ is a surjective submersion. In the following we shall describe the dynamics of the nonholonomic Hamiltonian system with symmetry $(T^*Q, G, \omega, D, H)$, where $H : T^*Q \to \mathbb{R}$ is a $G$-invariant Hamiltonian, and the completely nonholonomic constraints $D \subset TQ$ is a $G$-invariant distribution, that is, the tangent of the group action maps $D_q \to D_g q$ for any $q \in Q$. Since the Legendre transformation $FL : TQ \to T^*Q$ is a fiber-preserving map, then $\mathcal{M} = FL(D) \subset T^*Q$ is $G$-invariant, and the quotient space $\mathcal{M} = \mathcal{M}/G$ of the $G$-orbit in $\mathcal{M}$ is a smooth manifold with projection $\pi_G : \mathcal{M} \to \mathcal{M}(\subset T^*Q/G)$ which is a surjective submersion. From Bates and Śniatycki [3], we know that there exists a distribution $\bar{\mathcal{K}}$, a non-degenerate distributional two-form $\omega_{\bar{\mathcal{K}}}$, and a vector field $X_{\bar{\mathcal{K}}}$ on $\mathcal{M}$ which takes values in the constraint distribution $\mathcal{K}$, such that the following equation holds, that is, $i_{X_{\bar{\mathcal{K}}} \omega_{\bar{\mathcal{K}}}} = d\bar{h}_{\mathcal{K}}$, where the function $h_{\bar{\mathcal{K}}}$ satisfies $d\bar{h}_{\mathcal{K}} = \tau_{\bar{\mathcal{K}}} \cdot d\bar{h}_{\mathcal{M}}$, and $h_{\bar{\mathcal{M}}} \cdot \pi_G = H_\mathcal{M}$. In this case, the triple $(\bar{\mathcal{K}}, \omega_{\bar{\mathcal{K}}}, h_{\bar{\mathcal{K}}})$ is called a nonholonomic reduced distributional Hamiltonian system, and $X_{\bar{\mathcal{K}}}$ is its nonholonomic reduced dynamical vector field.

In particular, we assume that the Lie group $G$ is not Abelian, and the cotangent lifted $G$-action on $T^*Q$ is free, proper and symplectic, and admits a $\text{Ad}^*$-equivariant momentum map $J : T^*Q \to \mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra of $G$ and $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$. Let $\mu \in \mathfrak{g}^*$ be a regular value of $J$ and denote by $G_\mu$ the isotropy subgroup of the coadjoint $G$-action at the point $\mu \in \mathfrak{g}^*$, which is defined by $G_\mu = \{ g \in G | \text{Ad}_{\mu}^* g = \mu \}$. Since $G_\mu(\subset G)$ acts freely and properly on $Q$ and on $T^*Q$, then $G_\mu$ acts also freely and properly on $J^{-1}(\mu)$, so that the space $(T^*Q)_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form $\omega_\mu$ uniquely characterized by the relation

$$
\pi_\mu^* \omega_\mu = i_\mu^* \omega.
$$

(2.3)

The map $i_\mu : J^{-1}(\mu) \to T^*Q$ is the inclusion and $\pi_\mu : J^{-1}(\mu) \to (T^*Q)_\mu$ is the projection. The pair $((T^*Q)_\mu, \omega_\mu)$ is the Marsden-Weinstein reduced space of $(T^*Q, \omega)$ at $\mu$, (see Marsden and Weinstein [27], Marsden [21], and Marsden et al. [22]). In the following we assume that for the regular value $\mu \in \mathfrak{g}^*$, the constraint submanifold $\mathcal{M}$ is clean intersection with $J^{-1}(\mu)$, that is, $\mathcal{M} \cap J^{-1}(\mu) \neq \emptyset$. Note that $\mathcal{M}$ is also $G_\mu(\subset G)$ action invariant, and so is $J^{-1}(\mu)$, because $J$ is $\text{Ad}^*$-equivariant. It follows that the quotient space $M_\mu = (\mathcal{M} \cap J^{-1}(\mu))/G_\mu \subset (T^*Q)_\mu$ of the $G_\mu$-orbit in $M \cap J^{-1}(\mu)$, is a smooth manifold with projection $\pi_\mu : \mathcal{M} \cap J^{-1}(\mu) \to M_\mu$ which is a
In the following we shall describe the dynamics of the nonholonomic Hamiltonian system with symmetry and momentum map \((T^*Q, G, \omega, J, D, H)\) by using the method given by Bates and Śniatycki in [3]. Assume that the distribution \(T(J^{-1}(\mu)) \cap F\) pushes down to a distribution \(F_\mu = T\pi_\mu(T(J^{-1}(\mu)) \cap F)\) on \((T^*Q)_\mu\) along \(M_\mu\), and \(h_\mu\) is the Marsden-Weinstein reduced Hamiltonian function \(h_\mu : (T^*Q)_\mu \to \mathbb{R}\) defined by \(h_\mu : \pi_\mu = H \cdot i_\mu\). We consider the following nonholonomic constraints condition

\[
(i_{X_\mu}\omega_\mu - d\mu_\mu)|_{M_\mu} \in F_{\mu}^\perp, \quad X_\mu \in T M_\mu.
\]

Thus, there exists an unique nonholonomic vector field \(X_\mu\) satisfying the above condition (2.4), if the admissibility condition \(\dim M_\mu = \text{rank} F_\mu\) and the compatibility condition \(T M_\mu \cap F_\mu^\perp = \{0\}\) hold, where \(F_\mu^\perp\) is denoted the symplectic orthogonal of \(F_\mu\) with respect to the Marsden-Weinstein reduced symplectic form \(\omega_\mu\). In consequence, we know that there exists a distribution \(K_\mu = F_\mu \cap T M_\mu\), a non-degenerate reduced distributional two-form \(\omega_{K_\mu}\), which is the restriction of the induced symplectic form \(\omega_{M_\mu}\) on \(T^*M_\mu\) fibrewise to the distribution \(K_\mu\), and a vector field \(X_{K_\mu}\) on the reduced constraint submanifold \(M_\mu = (M \cap J^{-1}(\mu))/G\), such that the equation \(i_{X_{K_\mu}}\omega_{K_\mu} = dh_{K_\mu}\) holds, where the function \(h_{K_\mu}\) satisfies \(dh_{K_\mu} = \tau_{K_\mu} \cdot dh_{\mu}\), that is, the restriction condition of the \(dh_{\mu}\) to the reduced distribution \(K_\mu\). Then the triple \((K_\mu, \omega_{K_\mu}, h_{K_\mu})\) is called a \(J\)-nonholonomic regular point reduced distributional Hamiltonian system, and \(X_{K_\mu}\) is its \(J\)-nonholonomic regular point reduced dynamical vector field.

It is worthy of noting that the orbit reduction of a Hamiltonian system is an alternative approach to symplectic reduction given by Marle [20] and Kazhdan, Kostant and Sternberg [14], which is different from the Marsden-Weinstein reduction. For the nonholonomic Hamiltonian system with symmetry and momentum map \((T^*Q, G, \omega, J, D, H)\), if \(\mu \in g^*\) is a regular value of the momentum map \(J\) and \(O_\mu = G \cdot \mu \subset g^*\) is the \(G\)-orbit of the coadjoint \(G\)-action through the point \(\mu\), by using the above method, we know that there exists a distribution \(K_{O_\mu}\), a non-degenerate reduced distributional two-form \(\omega_{K_{O_\mu}}\), and a vector field \(X_{K_{O_\mu}}\) on the regular orbit reduced constraint submanifold \(M_{O_\mu} = (M \cap J^{-1}(\mu))/G\), such that the equation \(i_{X_{K_{O_\mu}}}, \omega_{K_{O_\mu}} = dh_{K_{O_\mu}}\) holds. Here the regular orbit reduced space is \((T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G\), in which the symplectic form \(\omega_{O_\mu}\) uniquely characterized by the relation

\[
i^*_{O_\mu} \omega = \pi^*_{O_\mu} \omega_{O_\mu} + J^*_{O_\mu} \omega_{O_\mu}^+,
\]

where \(J^*_{O_\mu}\) is the restriction of the momentum map \(J\) to \(J^{-1}(O_\mu)\), that is, \(J^*_{O_\mu} = J \cdot i_{O_\mu}\) and \(\omega_{O_\mu}^+\) is the \(+\)-symplectic structure on the orbit \(O_\mu\) given by

\[
\omega_{O_\mu}^+(\nu)(\xi_\mu, \eta_\mu) := <\nu, [\xi, \eta]>, \quad \forall \nu \in O_\mu, \, \xi, \eta \in g.
\]

The maps \(i_{O_\mu} : J^{-1}(O_\mu) \to T^*Q\) and \(\pi_{O_\mu} : J^{-1}(O_\mu) \to (T^*Q)_{O_\mu}\) are natural injection and the projection, respectively. The distribution \(T(J^{-1}(O_\mu)) \cap F\) pushes down to a distribution \(F_{O_\mu} = T\pi_{O_\mu}((J^{-1}(O_\mu)) \cap F)\) on \((T^*Q)_{O_\mu}\) along \(M_{O_\mu}\), and \(K_{O_\mu} = F_{O_\mu} \cap T M_{O_\mu}\). Then the triple \((K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})\) is called a \(J\)-nonholonomic regular orbit reduced distributional Hamiltonian system, in which \(\omega_{K_{O_\mu}}\) is the restriction of the induced symplectic form \(\omega_{M_{O_\mu}}\) on \(T^*M_{O_\mu}\) fibrewise to the distribution \(K_{O_\mu}\), and the function \(h_{K_{O_\mu}}\) satisfies \(dh_{K_{O_\mu}} = \tau_{K_{O_\mu}} \cdot dh_{O_\mu}\), that is, the restriction condition of the \(dh_{O_\mu}\) to the reduced distribution \(K_{O_\mu}\), where \(h_{O_\mu}\) is regular orbit reduced Hamiltonian function \(h_{O_\mu} : (T^*Q)_{O_\mu} \to \mathbb{R}\) defined by \(h_{O_\mu} \cdot \pi_{O_\mu} = H \cdot i_{O_\mu}\), and \(X_{K_{O_\mu}}\) is the \(J\)-nonholonomic regular orbit reduced dynamical vector field.
In the following we shall derive precisely the geometric constraint conditions of the induced distributional two-form and the reduced distributional two-forms for a variety of nonholonomic dynamical vector fields, that is, the two types of Hamilton-Jacobi equations for the various distributional Hamiltonian systems.

3 Hamilton-Jacobi Equations for a Distributional Hamiltonian System

In this section, for a nonholonomic Hamiltonian system \((T^*Q, \omega, D, H)\), where \(\omega\) is the canonical symplectic form on \(T^*Q\), and \(D \subset TQ\) is a \(D\)-completely and \(D\)-regularly nonholonomic constraint of the system, we first give its distribution \(K\), an associated non-degenerate distributional two-form \(\omega_K\) induced by the canonical symplectic form and a distributional Hamiltonian system, then derive precisely the geometric constraint conditions of the distributional two-form for the nonholonomic dynamical vector field, that is, the two types of Hamilton-Jacobi equation for the distributional Hamiltonian system. In order to do this, we need first to analyze carefully the dynamics and structure of the nonholonomic Hamiltonian system following the results given by Bates and Šniatycki in [3], (see also Cushman et al. [10] and [11] for more details).

From now on, we assume that \(L : TQ \to \mathbb{R}\) is a hyperregular Lagrangian, and the Legendre transformation \(FL : TQ \to T^*Q\) is a diffeomorphism. As above, our nonholonomic constraint \(D \subset TQ\) is \(D\)-completely and \(D\)-regularly, and let \(D^0 \subset T^*Q\) its annihilator. From §2, we can define the constraint submanifold \(M = FL(D) \subset T^*Q\), \(i_M : M \to T^*Q\), and \(\omega_M = i_M^*\omega\), that is, the symplectic form \(\omega_M\) is induced from the canonical symplectic form \(\omega\) on \(T^*Q\), where \(i_M^* : T^*T^*Q \to T^*M\). For the distribution \(\mathcal{F} = (T\pi_Q)^{-1}(D) \subset TT^*Q\), we define the distribution \(K = \mathcal{F} \cap TM\). Note that \(K^\perp = F^\perp \cap TM\), where \(K^\perp\) denotes the symplectic orthogonal of \(K\) with respect to the canonical symplectic form \(\omega\), and the admissibility condition \(\dim M = \text{rank} \mathcal{F}\) and the compatibility condition \(TM \cap F^\perp = \{0\}\) hold, then we know that the restriction of the symplectic form \(\omega_M\) on \(T^*M\) fibrewise to the distribution \(K\), that is, \(\omega_K = \tau_K \cdot \omega_M\) is non-degenerate, where \(\tau_K\) is the restriction map to distribution \(K\). It is worthy of noting that \(\omega_K\) is not a true two-form on a manifold, so it does not make sense to speak about it being closed. We call \(\omega_K\) as a distributional two-form to avoid any confusion. Because \(\omega_K\) is non-degenerate as a bilinear form on each fibre of \(K\), there exists a vector field \(X_K\) on \(M\) which takes values in the constraint distribution \(K\), such that the following nonholonomic constraints condition holds, that is,

\[
i_{X_K}\omega_K = dH_K \tag{3.1}
\]

where \(dH_K\) is the restriction of \(dH_M\) to \(K\), and the function \(H_K\) satisfies \(dH_K = \tau_K \cdot dH_M\), and \(H_M = \tau_M \cdot H\) is the restriction of \(H\) to \(M\). Then (3.1) is called the distributional Hamiltonian equation, see Bates and Šniatycki [3]. Thus, the geometric formulation of a distributional Hamiltonian system may be summarized as follows.

**Definition 3.1** (Distributional Hamiltonian System) Assume that the 4-tuple \((T^*Q, \omega, D, H)\) is a nonholonomic Hamiltonian system, where \(\omega\) is the canonical symplectic form on \(T^*Q\), and \(D \subset TQ\) is a \(D\)-completely and \(D\)-regularly nonholonomic constraint of the system. If there exist a distribution \(K\), an associated non-degenerate distributional two-form \(\omega_K\) induced by the canonical symplectic form and a vector field \(X_K\) on the constraint submanifold \(M = FL(D) \subset T^*Q\), such that the distributional Hamiltonian equation \(i_{X_K}\omega_K = dH_K\) holds, where \(dH_K\) is the restriction of \(dH_M\) to \(K\), and the function \(H_K\) satisfies \(dH_K = \tau_K \cdot dH_M\) as defined above, then the triple \((K, \omega_K, H_K)\) is called a distributional Hamiltonian system of the nonholonomic Hamiltonian system \((T^*Q, \omega, D, H)\), and \(X_K\) is called a nonholonomic dynamical vector field of the distributional Hamiltonian system.
Under the above circumstances, we refer to \((T^*Q, \omega, D, H)\) as a nonholonomic Hamiltonian system with an associated distributional Hamiltonian system \((K, \omega_K, H_K)\).

Since the non-degenerate distributional two-form \(\omega_K\) is not symplectic, and the distributional Hamiltonian system \((K, \omega_K, H_K)\) is not yet a Hamiltonian system, and has no yet generating function, and hence we can not describe the Hamilton-Jacobi equation for a distributional Hamiltonian system just like as in Theorem 1.1. But, for a given nonholonomic Hamiltonian system \((T^*Q, \omega, D, H)\) with an associated distributional Hamiltonian system \((K, \omega_K, H_K)\), we can derive precisely the geometric constraint conditions of the non-degenerate distributional two-form \(\omega_K\) for the nonholonomic dynamical vector field \(X_K\), that is, the two types of Hamilton-Jacobi equation for the distributional Hamiltonian system \((K, \omega_K, H_K)\). In order to do this, we need first give two important notions and a key lemma, (see also Wang [36]), which is obtained by a careful modification for the corresponding results of Abraham and Marsden in [1]. This lemma offers also an important tool for the proofs of the two types of Hamilton-Jacobi theorems for the distributional Hamiltonian system and the nonholonomic reduced distributional Hamiltonian system.

Let \(Q\) be a smooth manifold and \(TQ\) its tangent bundle, \(T^*Q\) its cotangent bundle with the canonical symplectic form \(\omega\), and \(D \subset TQ\) is a \(D\)-regularly nonholonomic constraint, and the projection \(\pi_Q : T^*Q \rightarrow Q\) induces the map \(T\pi_Q : T^*T^*Q \rightarrow TQ\). Assume that \(\gamma : Q \rightarrow T^*Q\) is an one-form on \(Q\), if \(\gamma\) is closed, then \(d\gamma(x, y) = 0, \forall \ x, y \in TQ\). In the following we introduce two weaker notions.

**Definition 3.2** (i) The one-form \(\gamma\) is called to be closed with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\), if for any \(v, w \in T^*T^*Q\), we have \(d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0\);

(ii) The one-form \(\gamma\) is called to be closed on \(D\) with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\), if for any \(v, w \in T^*T^*Q\), and \(T\pi_Q(v), T\pi_Q(w) \in D\), we have \(d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0\).

From the above definition we know that, the notion that \(\gamma\) is closed on \(D\) with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\), is weaker than the notion that \(\gamma\) is closed with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\). From Wang [36] we also know that the latter, that is, \(\gamma\) is closed with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\), is weaker than the notion that \(\gamma\) is closed. Thus, the notion that \(\gamma\) is closed on \(D\) with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\), is weaker than that \(\gamma\) is closed on \(D\), that is, \(d\gamma(x, y) = 0, \forall \ x, y \in D\). In fact, if \(\gamma\) is a closed one-form on \(D\), then it must be closed on \(D\) with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\). Conversely, if \(\gamma\) is closed on \(D\) with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\), then it may not be closed on \(D\). We can prove a general result as follows.

**Proposition 3.3** Assume that \(\gamma : Q \rightarrow T^*Q\) is an one-form on \(Q\) and it is not closed on \(D\). We define the set \(N\), which is a subset of \(TQ\), such that the one-form \(\gamma\) on \(N\) satisfies the condition that for any \(x, y \in N\), \(d\gamma(x, y) \neq 0\). Denote \(\text{Ker}(T\pi_Q) = \{u \in T^*Q | T\pi_Q(u) = 0\}\), and \(T\gamma : TQ \rightarrow T^*Q\). If \(T\gamma(N) \subset \text{Ker}(T\pi_Q)\), then \(\gamma\) is closed with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\), and hence \(\gamma\) is closed on \(D\) with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\).

**Proof:** In fact, for any \(v, w \in T^*T^*Q\), if \(T\pi_Q(v) \notin N\), or \(T\pi_Q(w) \notin N\), then by the definition of \(N\), we know that \(d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0\); If \(T\pi_Q(v) \in N\), and \(T\pi_Q(w) \in N\), from the condition \(T\gamma(N) \subset \text{Ker}(T\pi_Q)\), we know that \(T\pi_Q \cdot T\gamma \cdot T\pi_Q(v) = T\pi_Q(v) = 0\), and \(T\pi_Q \cdot T\gamma \cdot T\pi_Q(w) = T\pi_Q(w) = 0\), where we have used the relation \(\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q\), and hence \(d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0\). Thus, for any \(v, w \in T^*T^*Q\), we always have that \(d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0\). In particular, for any \(v, w \in T^*T^*Q\), and \(T\pi_Q(v), T\pi_Q(w) \in D\), we have \(d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0\), that is, \(\gamma\) is closed on \(D\) with respect to \(T\pi_Q : T^*T^*Q \rightarrow TQ\).
Now, we prove the following Lemma 3.4. It is worthy of noting that this lemma is an extension of Lemma 2.4 given in Wang [36] to the nonholonomic context.

**Lemma 3.4** Assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q \), and \( \lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q \). Then we have that

(i) for any \( x, y \in TQ \), \( \gamma^\ast \omega(x, y) = -d\gamma(x, y) \), and for any \( v, w \in TT^*Q \), \( \lambda^\ast \omega(v, w) = -d\gamma(T\pi_Q(v), T\pi_Q(w)) \), since \( \omega \) is the canonical symplectic form on \( T^*Q \);

(ii) for any \( v, w \in TT^*Q \), \( \omega(T\lambda \cdot v, v) = \omega(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) \);

(iii) If \( L \) is \( D \)-regular, and \( \text{Im}(\gamma) \subset M = FL(D) \), then we have that \( X^H \cdot \gamma \in F \) along \( \gamma \), and \( X^H \cdot \lambda \in F \) along \( \lambda \), that is, \( T\pi_Q(X^H \cdot \gamma(q)) \in D_q \), \( \forall q \in Q \), and \( T\pi_Q(X^H \cdot \lambda(q, p)) \in D_q \), \( \forall q \in Q \), \( (q, p) \in T^*Q \).

**Proof:** The proofs of (i) and (ii) are given in Wang [36]. Now, we prove (iii). For any \( q \in Q \), \( (q, p) \in T^*Q \), we have that

\[
X^H \cdot \gamma(q) = \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) \gamma(q).
\]

and

\[
X^H \cdot \lambda(q, p) = \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) \gamma \cdot \pi_Q(q, p).
\]

Then,

\[
T\pi_Q(X^H \cdot \gamma(q)) = T\pi_Q(X^H \cdot \lambda(q, p)) = \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right) \gamma(q) = \gamma^\ast \left( \frac{\partial H(q, p)}{\partial p_i} \right) \frac{\partial}{\partial q^i},
\]

where \( \gamma^\ast : T^*T^*Q \to T^*Q \). Since \( \text{Im}(\gamma) \subset M \), and \( \gamma^\ast \left( \frac{\partial H(q, p)}{\partial p_i} \right) \in M_{(q, p)} = FL(D_q) \), from \( L \) is \( D \)-regular, \( FL \) is a diffeomorphism, then there exists a \( v_q \in D_q \), such that \( FL(v_q) = \gamma^\ast \left( \frac{\partial H(q, p)}{\partial p_i} \right) \). Thus,

\[
T\pi_Q(X^H \cdot \gamma(q)) = T\pi_Q(X^H \cdot \lambda(q, p)) = FL(v_q) \frac{\partial}{\partial q^i} \in D,
\]

it follows that \( X^H \cdot \gamma \in F \) along \( \gamma \), and \( X^H \cdot \lambda \in F \) along \( \lambda \). \( \blacksquare \)

By using the above Lemma 3.4, we can derive precisely the geometric constraint conditions of the non-degenerate distributional two-form \( \omega_K \) for the nonholonomic dynamical vector field \( X_K \), that is, the following two types of Hamilton-Jacobi equation for the distributional Hamiltonian system \((K, \omega_K, H_K)\). At first, by using the fact that the one-form \( \gamma : Q \to T^*Q \) is closed on \( D \) with respect to \( T\pi_Q : TT^*Q \to TQ \), \( \text{Im}(\gamma) \subset M \), and \( \text{Im}(T\gamma) \subset K \), we can prove the Type I of Hamilton-Jacobi theorem for the distributional Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-1.

![Diagram-1](image)

**Theorem 3.5** (Type I of Hamilton-Jacobi Theorem for a Distributional Hamiltonian System) For the nonholonomic Hamiltonian system \((T^*Q, \omega, D, H)\) with an associated distributional Hamiltonian system \((K, \omega_K, H_K)\), assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q \), and \( X^\gamma = T\pi_Q \cdot X^H \cdot \gamma \), where \( X^H \) is the dynamical vector field of the corresponding unconstrained Hamiltonian system.
(T^*Q, ω, H). Moreover, assume that \( \text{Im}(\gamma) \subset M = FL(D) \), and \( \text{Im}(T\gamma) \subset K \). If the one-form \( \gamma : Q \rightarrow T^*Q \) is closed on \( D \) with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), then \( \gamma \) is a solution of the equation \( T\gamma \cdot X^\gamma = X_K \cdot \gamma \). Here \( X_K \) is the dynamical vector field of the distributional Hamiltonian system \((K, \omega_K, H_K)\). The equation \( T\gamma \cdot X^\gamma = X_K \cdot \gamma \) is called the Type I of Hamilton-Jacobi equation for the distributional Hamiltonian system \((K, \omega_K, H_K)\).

**Proof:** At first, we note that \( \text{Im}(\gamma) \subset M \), and \( \text{Im}(T\gamma) \subset K \), in this case, \( \omega_K \cdot \tau_K = \tau_K \cdot \omega_M = \tau_K \cdot i_M \cdot \omega \), along \( \text{Im}(T\gamma) \). Thus, using the non-degenerate distributional two-form \( \omega_K \), from Lemma 3.4 (ii) and (iii), if we take that \( v = X_H \cdot \gamma \in F \), and for any \( w \in F \), \( T\lambda(w) \neq 0 \), and \( \tau_K \cdot w \neq 0 \), then we have that

\[
\omega_K(T\gamma \cdot X^\gamma, \tau_K \cdot w) = \omega_K(\tau_K \cdot T\gamma \cdot X^\gamma, \tau_K \cdot w) = \tau_K \cdot i_M \cdot \omega(T\gamma \cdot X^\gamma, w) = \tau_K \cdot i_M \cdot \omega(X_H \cdot \gamma, -d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w))) = \omega_K(\tau_K \cdot X_H \cdot \gamma, \tau_K \cdot w) - \omega_K(\tau_K \cdot X_H \cdot \gamma, \tau_K \cdot T\gamma \cdot \pi_Q(w)) = \omega_K(X_K \cdot \gamma, \tau_K \cdot w) - \omega_K(X_K \cdot \gamma, \tau_K \cdot T\gamma \cdot \pi_Q(w)) = \tau_K \cdot i_M \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)),
\]

where we have used that \( \tau_K \cdot T\gamma = T\gamma \) and \( \tau_K \cdot X_H \cdot \gamma = X_K \cdot \gamma \), since \( \text{Im}(T\gamma) \subset K \). If the one-form \( \gamma : Q \rightarrow T^*Q \) is closed on \( D \) with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), then we have that \( d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0 \), since \( X_H \cdot \gamma, w \in F \), and \( T\pi_Q(X_H \cdot \gamma), T\pi_Q(w) \in D \), and hence

\[
\tau_K \cdot i_M \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0,
\]

and

\[
\omega_K(T\gamma \cdot X^\gamma, \tau_K \cdot w) - \omega_K(X_K \cdot \gamma, \tau_K \cdot w) = -\omega_K(\tau_K \cdot X_H \cdot \gamma, \tau_K \cdot T\gamma \cdot T\pi_Q(w)). \quad (3.2)
\]

If \( \gamma \) satisfies the equation \( T\gamma \cdot X^\gamma = X_K \cdot \gamma \), from Lemma 3.4(i) we deduce that

\[
-\omega_K(X_K \cdot \gamma, \tau_K \cdot T\gamma \cdot T\pi_Q(w)) = -\omega_K(T\gamma \cdot X^\gamma, \tau_K \cdot T\gamma \cdot T\pi_Q(w)) = -\omega_K(\tau_K \cdot T\gamma \cdot X^\gamma, \tau_K \cdot T\gamma \cdot T\pi_Q(w)) = -\tau_K \cdot i_M \cdot \omega(T\gamma \cdot T\pi_Q(X_H \cdot \gamma), T\gamma \cdot T\pi_Q(w)) = -\tau_K \cdot i_M \cdot i^* \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = \tau_K \cdot i_M \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0.
\]

Because the distributional two-form \( \omega_K \) is non-degenerate, the left side of (3.2) equals zero, only when \( \gamma \) satisfies the equation \( T\gamma \cdot X^\gamma = X_K \cdot \gamma \). Thus, if the one-form \( \gamma : Q \rightarrow T^*Q \) is closed on \( D \) with respect to \( T\pi_Q : TT^*Q \rightarrow TQ \), then \( \gamma \) must be a solution of the Type I of Hamilton-Jacobi equation \( T\gamma \cdot X^\gamma = X_K \cdot \gamma \). ■

Next, for any symplectic map \( \varepsilon : T^*Q \rightarrow T^*Q \), we can prove the following Type II of Hamilton-Jacobi theorem for the distributional Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-2.
Theorem 3.6 (Type II of Hamilton-Jacobi Theorem for a Distributional Hamiltonian System) For the nonholonomic Hamiltonian system \((T^*Q, \omega, D, H)\) with an associated distributional Hamiltonian system \((K, \omega_K, H_K)\), assume that \(\gamma : Q \to T^*Q\) is an one-form on \(Q\), and \(\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q\), and for any symplectic map \(\varepsilon : T^*Q \to T^*Q\), denote by \(X^\varepsilon = T\pi_Q \cdot X_H \cdot \varepsilon\), where \(X_H\) is the dynamical vector field of the corresponding unconstrained Hamiltonian system \((T^*Q, \omega, H)\). Moreover, assume that \(\text{Im}(\gamma) \subset M = \mathcal{FL}(D)\), and \(\text{Im}(T\gamma) \subset K\). If \(\varepsilon\) is a solution of the equation \(\tau_K \cdot T\varepsilon(X_H \cdot \varepsilon) = T\lambda \cdot X_H \cdot \varepsilon\), if and only if it is a solution of the equation \(T\gamma \cdot X^\varepsilon = X_K \cdot \varepsilon\). Here \(X_H \cdot \varepsilon\) is the Hamiltonian vector field of the function \(H \cdot \varepsilon : T^*Q \to \mathbb{R}\), and \(X_K\) is the dynamical vector field of the distributional Hamiltonian system \((K, \omega_K, H_K)\). The equation \(T\gamma \cdot X^\varepsilon = X_K \cdot \varepsilon\), is called the Type II of Hamilton-Jacobi equation for the distributional Hamiltonian system \((K, \omega_K, H_K)\).

Proof: In the same way, we note that \(\text{Im}(\gamma) \subset M\), and \(\text{Im}(T\gamma) \subset K\), in this case, \(\omega_K \cdot \tau_K = \omega_K \cdot \varepsilon M = \tau_K \cdot \varepsilon M \cdot \omega\), along \(\text{Im}(T\gamma)\). Thus, using the non-degenerate distributional two-form \(\omega_K\), from Lemma 3.4, if we take that \(v = \tau_K \cdot X_H \cdot \varepsilon = X_K \cdot \varepsilon \in K(\subset F)\), and for any \(w \in F\), \(T\lambda(w) \neq 0\), and \(\tau_K \cdot w \neq 0\), then we have that

\[
\omega_K(T\gamma \cdot X^\varepsilon, \tau_K \cdot w) = \omega_K(\tau_K \cdot T\gamma \cdot X^\varepsilon, \tau_K \cdot w)
= \tau_K \cdot i_M^* \cdot \omega(T\gamma \cdot X^\varepsilon, w)
= \tau_K \cdot i_M^* \cdot \omega(T\gamma \cdot \pi_Q \cdot X_H \cdot \varepsilon, w)
= \tau_K \cdot i_M^* \cdot (\omega(X_H \cdot \varepsilon, w - T(\gamma \cdot \pi_Q) \cdot w) - \partial_\gamma(T\pi_Q(X_H \cdot \varepsilon), T\pi_Q(w)))
= \tau_K \cdot i_M^* \cdot \omega(X_H \cdot \varepsilon, w) - \tau_K \cdot i_M^* \cdot \omega(X_H \cdot \varepsilon, T\lambda \cdot w)
- \tau_K \cdot i_M^* \cdot \partial_\gamma(T\pi_Q(X_H \cdot \varepsilon), T\pi_Q(w))
= \omega_K(\tau_K \cdot X_H \cdot \varepsilon, \tau_K \cdot w) - \omega_K(\tau_K \cdot X_H \cdot \varepsilon, T\lambda \cdot w)
+ \tau_K \cdot i_M^* \cdot \lambda^* \omega(X_H \cdot \varepsilon, w)
= \omega_K(\tau_K \cdot X_H \cdot \varepsilon, \tau_K \cdot w) - \omega_K(\tau_K \cdot X_H \cdot \varepsilon, T\lambda \cdot w) + \omega_K(T\lambda \cdot X_H \cdot \varepsilon, T\lambda \cdot w),
\]

where we have used that \(\tau_K \cdot T\gamma = T\gamma\), \(\tau_K \cdot T\lambda = T\lambda\), and \(\tau_K \cdot X_H \cdot \varepsilon = X_K \cdot \varepsilon\), since \(\text{Im}(T\gamma) \subset K\). Note that \(\varepsilon : T^*Q \to T^*Q\) is symplectic, and \(X_H \cdot \varepsilon = T\varepsilon \cdot X_H \cdot \varepsilon\), along \(\varepsilon\), and hence \(\tau_K \cdot X_H \cdot \varepsilon = \tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon\), along \(\varepsilon\). Then we have that

\[
\omega_K(T\gamma \cdot X^\varepsilon, \tau_K \cdot w) - \omega_K(\tau_K \cdot X_H \cdot \varepsilon, \tau_K \cdot w)
= -\omega_K(\tau_K \cdot X_H \cdot \varepsilon, T\lambda \cdot w) + \omega_K(T\lambda \cdot X_H \cdot \varepsilon, T\lambda \cdot w)
= \omega_K(T\lambda \cdot X_H \cdot \varepsilon - \tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon, T\lambda \cdot w).
\]

Because the distributional two-form \(\omega_K\) is non-degenerate, it follows that the equation \(T\gamma \cdot X^\varepsilon = X_K \cdot \varepsilon\), is equivalent to the equation \(\tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon = T\lambda \cdot X_H \cdot \varepsilon\). Thus, \(\varepsilon\) is a solution of the equation \(\tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon = T\lambda \cdot X_H \cdot \varepsilon\), if and only if it is a solution of the Type II of Hamilton-Jacobi equation \(T\gamma \cdot X^\varepsilon = X_K \cdot \varepsilon\). ■

Remark 3.7 If the nonholonomic Hamiltonian system we considered has not any constrains, in this case, the distributional Hamiltonian system is just the Hamiltonian system itself. From the above Type I and Type II of Hamilton-Jacobi theorems, that is, Theorem 3.5 and Theorem 3.6, we can get the Theorem 2.5 and Theorem 2.6 in Wang [36]. It shows that Theorem 3.5 and Theorem 3.6 can be regarded as an extension of two types of geometric Hamilton-Jacobi theorem for Hamiltonian system given in [36] to the nonholonomic context. In particular, in this case, if the one-form \(\gamma\) is given by a generating function of a symplectic map, then the classical Hamilton-Jacobi equation \(H(q, \gamma(q)) = E, \) (constant in \(t\)), or equivalently, \(d(H \cdot \gamma) = 0\), as well as the Type I of Hamilton-Jacobi equation \(T\gamma \cdot X^\gamma = X_H \cdot \gamma\), and the Type II of Hamilton-Jacobi theorem, all of them hold, see Wang [36].
Remark 3.8 It is worthy of note that the formulations of Type I and Type II of Hamilton-Jacobi equation for a distributional Hamiltonian system, given by Theorem 3.5 and Theorem 3.6, have more extensive sense, because, in general, the one-form $\gamma$ is not given by a generating function of a symplectic map. When $\gamma$ is a solution of the classical Hamilton-Jacobi equation, that is, $X_H \cdot \gamma = 0$, which is equivalent to the equation $d(H \cdot \gamma) = 0$, or $H(q, \gamma(q)) = E$, $q \in Q$, and $E$ is a constant, in this case, $X^q_H = T\pi_Q \cdot X_H \cdot \gamma = 0$, and hence from the Type I of Hamilton-Jacobi equation, we have that $X_K \cdot \gamma = T_{\gamma} \cdot X^q_H = 0$. Since the classical Hamilton-Jacobi equation $X_H \cdot \gamma = 0$, shows that the dynamical vector field of the corresponding unconstrained Hamiltonian system $(T^*Q,\omega, H)$ is degenerate along $\gamma$, then the equation $X_K \cdot \gamma = 0$, shows that the dynamical vector field of the distributional Hamiltonian system $(\mathcal{K}, \omega_\mathcal{K}, H_\mathcal{K})$ is degenerate along $\gamma$. The equation $X_K \cdot \gamma = 0$ is called the classical Hamilton-Jacobi equation for the distributional Hamiltonian system $(\mathcal{K}, \omega_\mathcal{K}, H_\mathcal{K})$. In addition, for a symplectic map $\varepsilon : T^*Q \to T^*Q$, if $X_H \cdot \varepsilon = 0$, then from the Type II of Hamilton-Jacobi equation, we have that $X_K \cdot \varepsilon = T_{\gamma} \cdot X^q_H \cdot \varepsilon = 0$. But, from the equation $\tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon = T\lambda \cdot X_H \cdot \varepsilon$, we know that $X_K \cdot \varepsilon = 0$ is not equivalent to $X_H \cdot \varepsilon = 0$.

4 Hamilton-Jacobi Equations for a Reduced Distributional Hamiltonian System

It is well-known that the reduction of nonholonomically constrained mechanical systems is very important subject in geometric mechanics, and it is also regarded as a useful tool for simplifying and studying concrete nonholonomic systems, see Koiller [15], Bates and Śniatycki [3], Cantrijn et al. [5,6], Cushman et al. [10] and [11], Cendra et al. [9], Bloch et al. [4] and de León and Rodrigues [18] and so on.

In this section, for a nonholonomic Hamiltonian system with symmetry $(T^*Q, G, \omega, D, H)$, where $\omega$ is the canonical symplectic form on $T^*Q$, and $D \subset TQ$ is a $D$-completely and $D$-regularly nonholonomic constraint of the system, and $D$ and $H$ are both $G$-invariant, we first give the nonholonomic reduction of the system, and a reduced distribution $\mathcal{K}$, an associated non-degenerate and nonholonomic reduced distributional two-form $\omega_\mathcal{K}$ induced by the canonical symplectic form $\omega$, and a nonholonomic reduced distributional Hamiltonian system. Then we derive precisely the geometric constraint conditions of the reduced distributional two-form $\omega_\mathcal{K}$ for the nonholonomic dynamical vector field, that is, the two types of Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system, which are an extension of the above two types of Hamilton-Jacobi equation for the distributional Hamiltonian system under nonholonomic reduction. In order to do this, we need first to describe the nonholonomic reduction, and analyze carefully the dynamics and structure of the nonholonomic Hamiltonian system with symmetry, following the results given by Bates and Śniatycki in [3], see also, Cushman et al. [10] and [11].

Now, we assume that the 5-tuple $(T^*Q, G, \omega, D, H)$ is a $D$-completely and $D$-regularly nonholonomic Hamiltonian system with symmetry, and the Lie group $G$ acts smoothly on $Q$ by the left, and we also consider the natural lifted actions on $TQ$ and $T^*Q$, and assume that the cotangent lifted action on $T^*Q$ is free, proper and symplectic. The orbit space $T^*Q/G$ is a smooth manifold and the canonical projection $\pi_G : T^*Q \to T^*Q/G$ is a surjective submersion.

Assume that $H : T^*Q \to \mathbb{R}$ is a $G$-invariant Hamiltonian, and that the $D$-completely and $D$-regularly nonholonomic constraints $D \subset TQ$ is a $G$-invariant distribution, that is, the tangent of group action maps $D_q$ to $D_{gq}$ for any $q \in Q$. Note that the Legendre transformation $FL : TQ \to T^*Q$ is a fiber-preserving map, from §2 we know that $M = FL(D) \subset T^*Q$ is $G$-invariant, and the quotient space $M = M/G$ of the $G$-orbit in $M$ is a smooth manifold with
projection $\pi_{/G} : \mathcal{M} \to \mathcal{M}(\subset T^*Q/G)$, which is a surjective submersion.

Since $G$ is the symmetry group of the system, all intrinsically defined vector fields and distributions push down to $\mathcal{M}$. In particular, the vector field $X_{/\mathcal{M}}$ on $\mathcal{M}$ pushes down to a vector field $X_{\mathcal{M}} = T\pi_{/G} \cdot X_{/\mathcal{M}}$, and the distribution $\mathcal{K}$ pushes down to a distribution $T\pi_{/G} \cdot \mathcal{K}$ on $\mathcal{M}$, and the Hamiltonian $H$ pushes down to $h_{/\mathcal{M}}$, such that $h_{/\mathcal{M}} \cdot \pi_{/G} = \tau_{/\mathcal{M}} \cdot H$. However, $\omega_{/\mathcal{K}}$ need not push down to a distributional two-form defined on $T\pi_{/G} \cdot \mathcal{K}$, despite of the fact that $\omega_{/\mathcal{K}}$ is $G$-invariant. This is because there may be infinitesimal symmetry $\eta_{/\mathcal{K}}$ that lies in $\mathcal{M}$, such that $i_{\eta_{/\mathcal{K}}} \omega_{/\mathcal{K}} \neq 0$. From Bates and Śniatycki [3], we know that to eliminate this difficulty, $\omega_{/\mathcal{K}}$ is restricted to a sub-distribution $\mathcal{U}$ of $\mathcal{K}$ defined by

$$\mathcal{U} = \{ u \in \mathcal{K} \mid \omega_{/\mathcal{K}}(u,v) = 0, \ \forall v \in \mathcal{V} \cap \mathcal{K} \},$$

where $\mathcal{V}$ is the distribution on $\mathcal{M}$ tangent to the orbits of $G$ in $\mathcal{M}$ and it is spanned by the infinitesimal symmetries. Clearly, $\mathcal{U}$ and $\mathcal{V}$ are both $G$-invariant, project down to $\mathcal{M}$ and $T\pi_{/G} \cdot \mathcal{V} = 0$, and define the distribution $\bar{\mathcal{K}}$ by $\bar{\mathcal{K}} = T\pi_{/G} \cdot \mathcal{U}$. Moreover, we take that $\omega_{/\mathcal{U}} = \tau_{/\mathcal{U}} \cdot \omega_{/\mathcal{M}}$ is the restriction of the induced symplectic form $\omega_{/\mathcal{M}}$ on $T^*\mathcal{M}$ fibrewise to the distribution $\mathcal{U}$, where $\tau_{/\mathcal{U}}$ is the restriction map to distribution $\mathcal{U}$, and the $\omega_{/\mathcal{U}}$ pushes down to a non-degenerate distributional two-form $\omega_{/\mathcal{U}}$ on $\bar{\mathcal{K}}$, such that $\pi_{/G}^* \omega_{/\mathcal{U}} = \omega_{/\mathcal{U}}$. Because $\omega_{/\mathcal{U}}$ is non-degenerate as a bilinear form on each fibre of $\bar{\mathcal{K}}$, there exists a vector field $X_{\bar{\mathcal{K}}}$ on $\mathcal{M}$ which takes values in the constraint distribution $\bar{\mathcal{K}}$, such that the reduced distributional Hamiltonian equation holds, that is, $i_{X_{\bar{\mathcal{K}}}} \omega_{/\mathcal{U}} = dh_{/\mathcal{U}}$, where $dh_{/\mathcal{U}}$ is the restriction of $dh_{/\mathcal{M}}$ to $\bar{\mathcal{K}}$ and the function $h_{/\mathcal{U}}$ satisfies $dh_{/\mathcal{U}} = \tau_{/\mathcal{U}} \cdot dh_{/\mathcal{M}}$, and $h_{/\mathcal{M}} \cdot \pi_{/G} = H_{/\mathcal{M}}$ and $H_{/\mathcal{M}}$ is the restriction of the Hamiltonian function $H$ to $\mathcal{M}$. In addition, the vector fields $X_{\bar{\mathcal{K}}}$ and $X_{\mathcal{K}}$ are $\pi_{/G}$-related. Thus, the geometrical formulation of a nonholonomic reduced distributional Hamiltonian system may be summarized as follows.

**Definition 4.1 (Nonholonomic Reduced Distributional Hamiltonian System)** Assume that the 5-tuple $(T^*Q,G,\omega,D,H)$ is a nonholonomic Hamiltonian system with symmetry, where $\omega$ is the canonical symplectic form on $T^*Q$, and $D \subset TQ$ is a $D$-completely and $D$-regularly nonholonomic constraint of the system, and $D$ and $H$ are both $G$-invariant. If there exists a nonholonomic reduced distribution $\mathcal{K}$, an associated non-degenerate and nonholonomic reduced distributional two-form $\omega_{/\mathcal{K}}$ and a vector field $X_{/\mathcal{K}}$ on the reduced constraint submanifold $\mathcal{M} = \mathcal{M}/G$, where $\mathcal{M} = \mathcal{F}(D) \subset T^*Q$, such that the nonholonomic reduced distributional Hamiltonian equation $i_{X_{/\mathcal{K}}} \omega_{/\mathcal{U}} = dh_{/\mathcal{U}}$ holds, where $dh_{/\mathcal{U}}$ is the restriction of $dh_{/\mathcal{M}}$ to $\bar{\mathcal{K}}$ and the function $h_{/\mathcal{U}}$ satisfies $dh_{/\mathcal{U}} = \tau_{/\mathcal{U}} \cdot dh_{/\mathcal{M}}$ and $h_{/\mathcal{M}} \cdot \pi_{/G} = H_{/\mathcal{M}}$ as defined above. Then the triple $(\mathcal{K},\omega_{/\mathcal{K}},h_{/\mathcal{K}})$ is called a nonholonomic reduced distributional Hamiltonian system of the nonholonomic Hamiltonian system with symmetry $(T^*Q,G,\omega,D,H)$, and $X_{/\mathcal{K}}$ is called a dynamical vector field of the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K},\omega_{/\mathcal{K}},h_{/\mathcal{K}})$. Under the above circumstances, we refer to $(T^*Q,G,\omega,D,H)$ as a nonholonomic reducible Hamiltonian system with an associated nonholonomic reduced distributional Hamiltonian system $(\mathcal{K},\omega_{/\mathcal{K}},h_{/\mathcal{K}})$.

Since the non-degenerate and nonholonomic reduced distributional two-form $\omega_{/\mathcal{K}}$ is not symplectic, and the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K},\omega_{/\mathcal{K}},h_{/\mathcal{K}})$ is not yet a Hamiltonian system, and has no yet generating function, and hence we can not describe the Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system just like as in Theorem 1.1. But, for a given nonholonomic reducible Hamiltonian system $(T^*Q,G,\omega,D,H)$ with an associated nonholonomic reduced distributional Hamiltonian system $(\mathcal{K},\omega_{/\mathcal{K}},h_{/\mathcal{K}})$, by using Lemma 3.4, we can derive precisely the geometric constraint conditions of the nonholonomic reduced distributional two-form $\omega_{/\mathcal{K}}$ for the dynamical vector field $X_{/\mathcal{K}}$, that is, the two types of Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K},\omega_{/\mathcal{K}},h_{/\mathcal{K}})$. At first, using the fact that the one-form $\gamma : Q \to T^*Q$ is closed on $D$ with respect to $T\pi_Q : TT^*Q \to TQ$, $\text{Im}(\gamma) \subset \mathcal{M}$, and it is $G$-invariant, as well as $\text{Im}(T\gamma) \subset K$, we can
proves the Type I of Hamilton-Jacobi theorem for the nonholonomic reduced distributional Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-3.

\[
\begin{array}{c}
\xrightarrow{i_M} \\
\xrightarrow{\pi_Q} \\
\xrightarrow{\tau} \\
\xrightarrow{T^*Q}
\end{array}
\]

**Theorem 4.2 (Type I of Hamilton-Jacobi Theorem for a Nonholonomic Reduced Distributional Hamiltonian System)** For a given nonholonomic reducible Hamiltonian system \((T^*Q, G, \omega, D, H)\) with an associated nonholonomic reduced distributional Hamiltonian system \((K, \omega_K, h_K)\), assume that \(\gamma : Q \to T^*Q\) is an one-form on \(Q\), and \(X_H^\gamma = T\pi_Q \cdot X_H \cdot \gamma\), where \(X_H\) is the dynamical vector field of the corresponding unconstrained Hamiltonian system with symmetry \((T^*Q, G, \omega, H)\). Moreover, assume that \(\text{Im}(\gamma) \subset M\), and it is \(G\)-invariant, \(\text{Im}(T\gamma) \subset K\), and \(\tilde{\gamma} = \pi_G(\gamma) : Q \to T^*Q/G\). If the one-form \(\gamma : Q \to T^*Q\) is closed on \(D\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\tilde{\gamma}\) is a solution of the equation \(T\tilde{\gamma} \cdot X_H^\gamma = X_K \cdot \tilde{\gamma}\). Here \(X_K\) is the dynamical vector field of the nonholonomic reduced distributional Hamiltonian system \((K, \omega_K, h_K)\). The equation \(T\tilde{\gamma} \cdot X_H^\gamma = X_K \cdot \tilde{\gamma}\) is called the Type I of Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system \((K, \omega_K, h_K)\).

**Proof:** At first, from Theorem 3.5, we know that \(\gamma\) is a solution of the Hamilton-Jacobi equation \(T\gamma \cdot X_H^\gamma = X_K \cdot \gamma\). Next, we note that \(\text{Im}(\gamma) \subset M\), and it is \(G\)-invariant, \(\text{Im}(T\gamma) \subset K\), and hence \(\text{Im}(T\gamma) \subset K\) in this case, \(\pi_G \cdot \omega_K \cdot \tilde{\gamma} = \pi_M \cdot \omega_M = \pi_M \cdot \omega_M \cdot \omega\), along \(\text{Im}(\tilde{\gamma})\). Thus, using the non-degenerate distributional two-form \(\omega_K\), from Lemma 3.4, if we take that \(\nu = \pi_K \cdot T\pi_G \cdot X_H \cdot \gamma = X_K \cdot \gamma \in K\), and for any \(w \in F\), \(T\lambda(w) \neq 0\), and \(\nu_K \cdot T\pi_G \cdot w \neq 0\), then we have that

\[
\omega_K(T\tilde{\gamma} \cdot X_H^\gamma, \tau_K \cdot T\pi_G \cdot w) = \omega_K(\tau_K \cdot T\pi_G \cdot \gamma \cdot X_H^\gamma, \tau_K \cdot T\pi_G \cdot w) \\
= \pi_G \cdot \omega_K \cdot \tau_K(T\gamma \cdot X_H^\gamma, w) = \pi_M \cdot \omega(T\gamma \cdot X_H^\gamma, w) \\
= \pi_M \cdot \omega(T\gamma \cdot \pi_Q \cdot X_H \cdot \gamma, w) \\
= \pi_M \cdot \omega(X_H \cdot \gamma, \nu \cdot T\pi_Q \cdot \gamma, w) - d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w))) \\
= \pi_M \cdot \omega(X_H \cdot \gamma, \nu) - \pi_M \cdot \omega(X_H \cdot \gamma, \nu \cdot \pi_Q \cdot \gamma) \\
- \pi_M \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) \\
= \pi_G \cdot \omega_K \cdot \tau_K(X_H \cdot \gamma, w) - \pi_G \cdot \omega_K \cdot \tau_K(X_H \cdot \gamma, T\pi_Q(w)) \\
- \pi_M \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) \\
= \omega_K(T\pi_G(X_H \cdot \gamma), \nu \cdot T\pi_G \cdot w) - \omega_K(T\nu \cdot T\pi_G \cdot X_H \cdot \gamma, \nu \cdot T\pi_G \cdot \gamma) \\
- \pi_M \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) \\
= \omega_K(X_K \cdot \tilde{\gamma}, \nu \cdot T\pi_G \cdot w) - \omega_K(X_K \cdot \gamma, T\tilde{\gamma} \cdot T\pi_Q(w)) - \pi_M \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) \\
= \tau_K \cdot T\pi_G \cdot X_H \cdot \gamma = \tau_K \cdot T\pi_G \cdot \gamma = X_K \cdot \tilde{\gamma}, \text{and } \tau_K \cdot T\tilde{\gamma} = T\tilde{\gamma}\), since \(\text{Im}(T\gamma) \subset K\). If the one-form \(\gamma : Q \to T^*Q\) is closed on \(D\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then we have that \(d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0\), since \(X_H \cdot \gamma, w \in F\), and \(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w) \in D\), and hence

\[
\nu_M \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0,
\]
If $\bar{\gamma}$ satisfies the equation $T\bar{\gamma} \cdot X^\gamma_H = X^\gamma_{\bar{K}}$, from Lemma 3.4(i) we know that the right side of (4.1) becomes

$$-\omega_{\bar{K}}(X^\gamma_{\bar{K}} \cdot \bar{\gamma}, T\bar{\gamma} \cdot T\pi_Q(w)) = -\omega_{\bar{K}}(X^\gamma_{\bar{K}} \cdot \bar{\gamma}, T\bar{\gamma} \cdot T\pi_Q(w))$$

$$= -\gamma^* \pi^*_{/G} \cdot \omega_{\bar{K}}(T\pi_Q \cdot X_H \cdot \gamma, T\pi_Q(w))$$

$$= -\gamma^* \pi^*_{/G} \cdot \omega_{\bar{K}}(T\pi_Q \cdot X_H \cdot \gamma, T\pi_Q(w))$$

$$= -\gamma^* \cdot i^*_{\bar{M}} \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w))$$

$$= -\gamma^* \cdot i^*_{\bar{M}} \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w))$$

$$= -\gamma^* \cdot i^*_{\bar{M}} \cdot \omega(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0,$$

where $\gamma^* \cdot i^*_{\bar{M}} \cdot \omega = \gamma^* \cdot i^*_{\bar{M}} \cdot \omega$, because $\text{Im}(\gamma) \subset M$. But, since the nonholonomic reduced distributional two-form $\omega_{\bar{K}}$ is non-degenerate, the left side of (4.1) equals zero, only when $\bar{\gamma}$ satisfies the equation $T\bar{\gamma} \cdot X^\gamma_H = X^\gamma_{\bar{K}} \cdot \bar{\gamma}$. Thus, if the one-form $\gamma : Q \to T^*Q$ is closed on $D$ with respect to $T\pi_Q : T^*Q \to TQ$, then $\bar{\gamma}$ must be a solution of the Type I of Hamilton-Jacobi equation $T\bar{\gamma} \cdot X^\gamma_H = X^\gamma_{\bar{K}} \cdot \bar{\gamma}$. 

Next, for any $G$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$, we can prove the following Type II of Hamilton-Jacobi theorem for the nonholonomic reduced distributional Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-4.

**Theorem 4.3** (Type II of Hamilton-Jacobi Theorem for a Nonholonomic Reduced Distributional Hamiltonian System) For a given nonholonomic reducible Hamiltonian system $(T^*Q, G, \omega, D, H)$ with an associated nonholonomic reduced distributional Hamiltonian system $(\bar{K}, \omega_{\bar{K}}, h_{\bar{K}})$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and for any $G$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$, denote by $X^\gamma_H = T\pi_Q \cdot X_H \cdot \varepsilon$, where $X_H$ is the dynamical vector field of the corresponding unconstrained Hamiltonian system with symmetry $(T^*Q, G, \omega, H)$. Moreover, assume that $\text{Im}(\gamma) \subset M$, and it is $G$-invariant, $\text{Im}(T\gamma) \subset K$, and $\bar{\gamma} = \pi_{/G}(\gamma) : Q \to T^*Q/G$, and $\bar{\lambda} = \pi_{/G}(\lambda) : T^*Q \to T^*Q/G$, and $\bar{\varepsilon} = \pi_{/G}(\varepsilon) : T^*Q \to T^*Q/G$. Then $\varepsilon$ and $\bar{\varepsilon}$ satisfy the equation $\tau_K \cdot T\bar{\varepsilon} \cdot X_{h_{\bar{K}}} = T\lambda \cdot X_H \cdot \varepsilon$, and if only they satisfy the equation $T\bar{\gamma} \cdot X^\varepsilon_H = X^\varepsilon_{\bar{K}} \cdot \bar{\varepsilon}$. Here $X_{h_{\bar{K}}} \cdot \varepsilon$ is the Hamiltonian vector field of the function $h_{\bar{K}} \cdot \varepsilon : T^*Q \to \mathbb{R}$, and $X_{\bar{K}}$ is the dynamical vector field of the nonholonomic reduced distributional Hamiltonian system $(\bar{K}, \omega_{\bar{K}}, h_{\bar{K}})$. The equation $T\bar{\gamma} \cdot X^\varepsilon_H = X^\varepsilon_{\bar{K}} \cdot \bar{\varepsilon}$, is called the Type II of Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system $(\bar{K}, \omega_{\bar{K}}, h_{\bar{K}})$.

**Proof:** In the same way, we note that $\text{Im}(\gamma) \subset M$, and it is $G$-invariant, $\text{Im}(T\gamma) \subset K$, and hence $\text{Im}(T\bar{\gamma}) \subset \bar{K}$, in this case, $\tau_K \cdot T\pi_{/G} \cdot X_H \cdot \varepsilon = X^\varepsilon_{\bar{K}} \cdot \bar{\varepsilon} \in \bar{K}$, and for any $w \in F$, $T\lambda(w) \neq 0$, and $\bar{\tau}_{\bar{K}} \cdot T\pi_{/G} \cdot w \neq 0$. 

and

$$\omega_{\bar{K}}(T\bar{\gamma} \cdot X^\gamma_H, \tau_{\bar{K}} \cdot T\pi_{/G} \cdot w) = \omega_{\bar{K}}(X^\gamma_{\bar{K}} \cdot \bar{\gamma}, \tau_{\bar{K}} \cdot T\pi_{/G} \cdot w) = -\omega_{\bar{K}}(X^\gamma_{\bar{K}} \cdot \bar{\gamma}, T\bar{\gamma} \cdot T\pi_Q(w)).$$
then we have that
\[
\begin{align*}
\omega_{K}(T\gamma \cdot X_{H}^{\varepsilon}, \tau_{K} \cdot T\pi_{G} \cdot w) &= \omega_{K}(\tau_{K} \cdot T(\pi_{G} \cdot \gamma) \cdot X_{H}^{\varepsilon}, \tau_{K} \cdot T\pi_{G} \cdot w) \\
&= \pi_{G}^{*} \cdot \omega_{K} \cdot \tau_{K}(T\gamma \cdot X_{H}^{\varepsilon}, w) = \pi_{U} \cdot i_{M}^{*} \cdot \omega(T\gamma \cdot X_{H}^{\varepsilon}, w) \\
&= \pi_{U} \cdot i_{M}^{*} \cdot \omega(T(\gamma \cdot \pi_{Q}) \cdot X_{H}^{\varepsilon}, w) \\
&= \pi_{U} \cdot i_{M}^{*} \cdot \omega(X_{H}^{\varepsilon}, w - T(\gamma \cdot \pi_{Q}) \cdot w) - d\gamma(T\pi_{Q}(X_{H}^{\varepsilon}, T\pi_{Q}(w))) \\
&= \pi_{U} \cdot i_{M}^{*} \cdot \omega(X_{H}^{\varepsilon}, w) - \pi_{U} \cdot i_{M}^{*} \cdot \omega(X_{H}^{\varepsilon}, T\lambda \cdot w) + \pi_{U} \cdot i_{M}^{*} \cdot \lambda^{*} \omega(X_{H}^{\varepsilon}, w) \\
&= \omega_{K}(\tau_{K} \cdot T\pi_{G}(X_{H}^{\varepsilon}, T\lambda \cdot w) - \omega_{K}(\tau_{K} \cdot T\pi_{G}(\pi_{G}(\varepsilon), T\lambda \cdot w) \\
&\quad + \pi_{U} \cdot \omega_{K}(T\lambda \cdot X_{H}^{\varepsilon}, T\lambda \cdot w) \\
&= \omega_{K}(X_{K}^{\varepsilon}, T\pi_{G} \cdot w) - \omega_{K}(\tau_{K} \cdot X_{h_{K}^{\varepsilon}} \cdot T\lambda \cdot w) + \omega_{K}(T\lambda \cdot X_{H}^{\varepsilon}, T\lambda \cdot w),
\end{align*}
\]
where we have used that \(\tau_{K} \cdot T\pi_{G}(X_{H}^{\varepsilon}) = \tilde{\varepsilon} = \tau_{K}(X_{h_{K}^{\varepsilon}}) = \tilde{\varepsilon} = \tilde{X}_{K} \cdot \tilde{\varepsilon}, \) and \(\tau_{K} \cdot T\pi_{G} \cdot T\lambda = T\lambda, \) since \(\text{Im}(T\gamma) \subset \tilde{K}.)

Note that \(\varepsilon : T^{*}Q \rightarrow T^{*}Q\) is symplectic, and \(\tilde{\varepsilon} = \varepsilon^{*} \cdot \pi_{G}^{*} : T^{*}(T^{*}Q/G) \rightarrow T^{*}T^{*}Q\) is also symplectic along \(\tilde{\varepsilon}\), and hence \(X_{h_{K}^{\varepsilon}} \cdot \tilde{\varepsilon} = T\tilde{\varepsilon} \cdot X_{h_{K}^{\varepsilon}},\) along \(\tilde{\varepsilon}\), and hence \(\tau_{K} \cdot X_{h_{K}^{\varepsilon}} \cdot \tilde{\varepsilon} = \tau_{K} \cdot T\tilde{\varepsilon} \cdot X_{h_{K}^{\varepsilon}},\) along \(\tilde{\varepsilon}\). Then we have that
\[
\omega_{K}(T\gamma \cdot X_{H}^{\varepsilon}, \tau_{K} \cdot T\pi_{G} \cdot w) = \omega_{K}(X_{K}^{\varepsilon}, \tau_{K} \cdot T\pi_{G} \cdot w) - \omega_{K}(\tau_{K} \cdot X_{h_{K}^{\varepsilon}} \cdot T\lambda \cdot w) + \omega_{K}(T\lambda \cdot X_{H}^{\varepsilon}, T\lambda \cdot w).
\]

Because the nonholonomic reduced distributional two-form \(\omega_{K}\) is non-degenerate, it follows that the equation \(T\gamma \cdot X_{H}^{\varepsilon} = X_{K}^{\varepsilon}\) is equivalent to the equation \(T\lambda \cdot X_{H}^{\varepsilon} = \varepsilon = \tau_{K} \cdot T\tilde{\varepsilon} \cdot X_{h_{K}^{\varepsilon}}\). Thus, \(\varepsilon\) and \(\tilde{\varepsilon}\) satisfy the equation \(T\lambda \cdot X_{H}^{\varepsilon} = \tau_{K} \cdot T\tilde{\varepsilon} \cdot X_{h_{K}^{\varepsilon}}\), if and only if they satisfy the Type II of Hamilton-Jacobi equation \(T\gamma \cdot X_{H}^{\varepsilon} = X_{K}^{\varepsilon}\). \(\blacksquare\)

For a given nonholonomic reducible Hamiltonian system \((T^{*}Q, G, \omega, D, H)\) with an associated nonholonomic reduced distributional Hamiltonian system \((\tilde{K}, \omega_{K}, h_{K})\), we know that the nonholonomic dynamical vector field \(X_{K}\) and the nonholonomic reduced dynamical vector field \(X_{K}^{\varepsilon}\) are \(\pi_{G}\)-related, that is, \(X_{K}^{\varepsilon} \cdot \pi_{G} = T\pi_{G} \cdot X_{K}\). Then we can prove the following Theorem 4.4, which states the relationship between the solutions of Type II of Hamilton-Jacobi equations and nonholonomic reduction.

**Theorem 4.4** For a given nonholonomic reducible Hamiltonian system \((T^{*}Q, G, \omega, D, H)\) with an associated nonholonomic reduced distributional Hamiltonian system \((\tilde{K}, \omega_{K}, h_{K})\), assume that \(\gamma : Q \rightarrow T^{*}Q\) is an one-form on \(Q\), and \(\lambda = \gamma \cdot \pi_{Q} : T^{*}Q \rightarrow T^{*}Q\), and \(\varepsilon : T^{*}Q \rightarrow T^{*}Q\) is a \(G\)-invariant symplectic map. Moreover, assume that \(\text{Im}(\gamma) \subset M\), and it is \(G\)-invariant, \(\text{Im}(T\gamma) \subset K\), and \(\tilde{\gamma} = \pi_{G}(\gamma) : Q \rightarrow T^{*}Q/G\), and \(\tilde{\lambda} = \pi_{G}(\lambda) : T^{*}Q \rightarrow T^{*}Q/G\), and \(\varepsilon = \pi_{G}(\varepsilon) : T^{*}Q \rightarrow T^{*}Q/G\). Then \(\varepsilon\) is a solution of the Type II of Hamilton-Jacobi equation, \(T\gamma \cdot X_{H}^{\varepsilon} = X_{K}^{\varepsilon}\), for the distributional Hamiltonian system \((K, \omega_{K}, H_{K})\), if and only if \(\varepsilon\) and \(\tilde{\varepsilon}\) satisfy the Type II of Hamilton-Jacobi equation \(T\tilde{\gamma} \cdot X_{H}^{\varepsilon} = X_{K}^{\varepsilon}\), for the nonholonomic reduced distributional Hamiltonian system \((\tilde{K}, \omega_{K}, h_{K})\).

**Proof:** Note that \(\text{Im}(\gamma) \subset M\), and it is \(G\)-invariant, \(\text{Im}(T\gamma) \subset K\), and hence \(\text{Im}(T\tilde{\gamma}) \subset \tilde{K}\), in this case, \(\pi_{G}^{*} \cdot \omega_{K} \cdot \tau_{K} = \pi_{U} \cdot \omega_{M} = \pi_{U} \cdot i_{M}^{*} \cdot \omega,\) along \(\text{Im}(T\tilde{\gamma})\), and \(\tau_{K} \cdot T\tilde{\gamma} = T\tilde{\gamma}, \tau_{K} \cdot X_{K} = X_{K}\). Since
nonholonomic vector field $X_K$ and the vector field $X_\bar{K}$ are $\pi/G$-related, that is, $X_{\bar{K}} \cdot \pi/G = T_{\pi/G} \cdot X_K$, using the non-degenerate and nonholonomic reduced distributional two-form $\omega_{\bar{K}}$, we have that

$$\omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}} - X_{\bar{K}} \cdot \bar{\varepsilon}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w) = \omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w) - \omega_{\bar{K}} (X_{\bar{K}} \cdot \bar{\varepsilon}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w) = \omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w) - \omega_{\bar{K}} (X_{\bar{K}} \cdot \bar{\varepsilon}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w)$$

$$= \omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w) - \omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w) - \omega_{\bar{K}} (X_{\bar{K}} \cdot \bar{\varepsilon}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w)$$

$$= \omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}}, w) - \omega_{\bar{K}} (\pi_{\gamma/G} \cdot \omega_{\bar{K}} (X_{\bar{K}} \cdot \bar{\varepsilon}, w) = \tau_\ell \cdot i_M^* \cdot \omega (T\bar{\gamma} \cdot X_{\bar{H}}, w) - \tau_\ell \cdot i_M^* \cdot \omega (X_{\bar{K}} \cdot \bar{\varepsilon}, w).$$

In the case we considered that $\tau_\ell \cdot i_M^* \cdot \omega = \tau_{\bar{K}} \cdot i_M^* \cdot \omega = \omega_{\bar{K}} \cdot \tau_{\bar{K}}$, and $\tau_{\bar{K}} \cdot T_\gamma = T_{\gamma}, \tau_{\bar{K}} \cdot X_{\bar{K}} = X_{\bar{K}}$, since $\text{Im}(\gamma) \subseteq \mathcal{M}$, and $\text{Im}(T_\gamma) \subseteq \mathcal{K}$. Thus, we have that

$$\omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}} - X_{\bar{K}} \cdot \bar{\varepsilon}, \tau_{\bar{K}} \cdot T_{\pi/G} \cdot w) = \omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}}, w) - \omega_{\bar{K}} (X_{\bar{K}} \cdot \bar{\varepsilon}, \tau_{\bar{K}} \cdot w) = \omega_{\bar{K}} (T\bar{\gamma} \cdot X_{\bar{H}}, w) - \omega_{\bar{K}} (X_{\bar{K}} \cdot \bar{\varepsilon}, \tau_{\bar{K}} \cdot w).$$

Because the distributional two-form $\omega_{\bar{K}}$ and the nonholonomic reduced distributional two-form $\omega_{\bar{K}}$ are both non-degenerate, it follows that the equation $T\bar{\gamma} \cdot X_{\bar{H}} = X_{\bar{K}} \cdot \bar{\varepsilon}$, is equivalent to the equation $T\gamma \cdot X_{\bar{H}} = X_{\bar{K}} \cdot \bar{\varepsilon}$, for the distributional Hamiltonian system $(\mathcal{K}, \omega_{\bar{K}}, H_{\bar{K}})$, if and only if $\bar{\varepsilon}$ and $\bar{\varepsilon}$ satisfy the Type II of Hamilton-Jacobi equation $T\bar{\gamma} \cdot X_{\bar{H}} = X_{\bar{K}} \cdot \bar{\varepsilon}$, for the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K}, \omega_{\bar{K}}, h_{\bar{K}})$.

**Remark 4.5** It is worthy of noting that the formulations of Type I and Type II of Hamilton-Jacobi equation for a nonholonomic reduced distributional Hamiltonian system, given by Theorem 4.2 and Theorem 4.3, have more extensive sense, because in general, the one-form $\gamma$ is not given by a generating function of a symplectic map. When $\gamma$ is a solution of the classical Hamilton-Jacobi equation, that is, $X_H \cdot \gamma = 0$, then $X_H \cdot \tau = 0$, and hence from the Type I of Hamilton-Jacobi equation, we have that $X_{\bar{K}} \cdot \bar{\gamma} = T\bar{\gamma} \cdot X_{\bar{H}} = 0$, which shows that the dynamical vector field of the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K}, \omega_{\bar{K}}, h_{\bar{K}})$ is degenerate along $\bar{\gamma}$. The equation $X_{\bar{K}} \cdot \bar{\gamma} = 0$ is called the classical Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K}, \omega_{\bar{K}}, h_{\bar{K}})$. In addition, for a symplectic map $\varepsilon : T^*Q \rightarrow T^*Q$, if $X_{\bar{K}} \cdot \varepsilon = 0$, then from the Type II of Hamilton-Jacobi equation, we have that $X_{\bar{K}} \cdot \bar{\varepsilon} = T\bar{\gamma} \cdot X_{\bar{H}} = 0$. But, from the equation $T\bar{\gamma} \cdot X_{\bar{H}} \cdot \varepsilon = \bar{\tau} \cdot i_M^* \cdot \omega (X_{\bar{K}} \cdot \bar{\varepsilon}, \bar{\varepsilon})$, we know that the equation $X_{\bar{K}} \cdot \bar{\varepsilon} = 0$ is not equivalent to the equation $X_{h_{\bar{K}}} \cdot \bar{\varepsilon} = 0$.

## 5 Nonholonomic Hamiltonian System with Symmetry and Momentum Map

As it is well known that momentum map is a very important notion in modern study of geometric mechanics, and it is a geometric generalization of the classical linear and angular momentum. A fundamental fact about momentum map is that if the Hamiltonian $H$ is invariant under the action of a Lie group $G$, then the vector valued function $J$ is a constant of the motion for the dynamics of the Hamiltonian vector field $X_H$ associated to $H$, that is, all momentum maps are conserved quantities. Moreover, momentum map has infinitesimal equivariance, such that it plays an important role in the study of reduction theory of Hamiltonian systems with symmetries, see
Marsden [21] and Marsden et al. [22, 23]. Now, it is a natural problem what and how we could do, when the Hamiltonian system we considered has nonholonomic constrains, and the Lie group $G$ is not Abelian, and $G_\mu \neq G$, where $G_\mu$ is the isotropy subgroup of coadjoint $G$-action at the point $\mu \in \mathfrak{g}^*$, and hence the above procedure of nonholonomic reduction given in §4 does not work or is not efficient enough. In this section, we shall consider a nonholonomic Hamiltonian system with symmetry and momentum map, and give two types of Hamilton-Jacobi theorems of the nonholonomic point and orbit reduced distributional Hamiltonian systems with respect to momentum map.

5.1 Hamilton-Jacobi equations in the case compatible with Marsden-Weinstein reduction

In this subsection, for a nonholonomic Hamiltonian system with symmetry and momentum map

\[
(T^*Q, G, \omega, J, D, H), \quad \text{where } \omega \text{ is the canonical symplectic form on } T^*Q, \quad \text{and } D \subset TQ \text{ is a } D\text{-completely and } D\text{-regularly nonholonomic constraint of the system},
\]

and $D$ and $H$ are both $G$-invariant, we first give the $J$-nonholonomic regular point reduction of the system compatible with Marsden-Weinstein reduction, and a $J$-nonholonomic $R_p$-reduced distribution $\mathcal{K}_\mu$, an associated non-degenerate and $J$-nonholonomic $R_p$-reduced distributional two-form $\omega_{K_\mu}$ which is induced by the canonical symplectic form $\omega$ on $T^*Q$, and a $J$-nonholonomic $R_p$-reduced distributional Hamiltonian system, where the "regular point reduced" is simply written as $R_p$-reduced. Then we derive precisely the geometric constraint conditions of the $J$-nonholonomic $R_p$-reduced distributional two-form $\omega_{K_\mu}$ for the nonholonomic reducible dynamical vector field, that is, the two types of Hamilton-Jacobi equation for the $J$-nonholonomic $R_p$-reduced distributional Hamiltonian system, which are an extension of the above two types of Hamilton-Jacobi equation for the distributional Hamiltonian system under $J$-nonholonomic regular point reduction.

At first, we need to give carefully a geometric formulation of the $J$-nonholonomic $R_p$-reduced distributional Hamiltonian system, by using momentum map and the nonholonomic reduction compatible with Marsden-Weinstein reduction. Now, we assume that the 6-tuple $(T^*Q, G, \omega, J, D, H)$ is a $D$-completely and $D$-regularly nonholonomic Hamiltonian system with symmetry and momentum map, and the Lie group $G$, which may not be Abelian, acts smoothly by the left on $Q$, its tangent lifted action on $TQ$ and its cotangent lifted action on $T^*Q$, and we assume that the action on $T^*Q$ is free, proper and symplectic, and admits an $\text{Ad}^*$-equivariant momentum map $J : T^*Q \rightarrow \mathfrak{g}^*$, where $\mathfrak{g}$ is a Lie algebra of $G$ and $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$. Let $\mu \in \mathfrak{g}^*$ be a regular value of $J$ and denote by $G_\mu$ the isotropy subgroup of the coadjoint $G$-action at the point $\mu \in \mathfrak{g}^*$, which is defined by $G_\mu = \{ g \in G | \text{Ad}^*_g \mu = \mu \}$. Since $G_\mu(\subset G)$ acts freely and properly on $Q$ and on $T^*Q$, then $G_\mu$ acts also freely and properly on $J^{-1}(\mu)$, so that the space $(T^*Q)_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form $\omega_\mu$ uniquely characterized by the relation

\[
\pi_\mu^* \omega_\mu = i_\mu^* \omega. \tag{5.1}
\]

The map $i_\mu : J^{-1}(\mu) \rightarrow T^*Q$ is the inclusion and $\pi_\mu : J^{-1}(\mu) \rightarrow (T^*Q)_\mu$ is the projection. The pair $((T^*Q)_\mu, \omega_\mu)$ is the Marsden-Weinstein reduced space of $(T^*Q, \omega)$ at $\mu$.

Assume that $H : T^*Q \rightarrow \mathbb{R}$ is a $G$-invariant Hamiltonian, and the $D$-completely and $D$-regularly nonholonomic constraints $D \subset TQ$ is a $G$-invariant distribution. From §2, we know that, by using the Legendre transformation $\mathcal{F}L : TQ \rightarrow T^*Q$, we can define the constraint submanifold $\mathcal{M} = \mathcal{F}L(D) \subset T^*Q$ and the distribution $\mathcal{F}$ which is the pre-image of the nonholonomic constraints $D$ for the map $T\pi_Q : TT^*Q \rightarrow TQ$, that is, $\mathcal{F} = (T\pi_Q)^{-1}(D)$, and $\mathcal{K} = \mathcal{F} \cap T\mathcal{M}$. Moreover, we can also define the distributional two-form $\omega_\mathcal{K}$, a vector field $X_\mathcal{K}$ and the function
For a regular value $M$ of the momentum map $J : T^*Q \to \mathfrak{g}^*$, the Marsden-Weinstein reduced symplectic form $F$ on $M$ is defined by $F = i_{X_{\mu}} \omega_K = dh_K$. Since $\mathcal{D} \subset TQ$ is a $G$-invariant distribution, and the Legendre transformation $\mathcal{F}L : TQ \to T^*Q$ is a fiber-preserving map, then $\mathcal{M} = \mathcal{F}L(\mathcal{D}) \subset T^*Q$ is $G$-invariant. For a regular value $\mu \in \mathfrak{g}^*$ of the momentum map $J : T^*Q \to \mathfrak{g}^*$, we shall assume that the constraint submanifold $\mathcal{M}$ is clean intersection with $J^{-1}(\mu)$, that is, $\mathcal{M} \cap J^{-1}(\mu) \neq \emptyset$. Note that $\mathcal{M}$ is also $G_{\mu}(\subset G)$ action invariant, and so is $J^{-1}(\mu)$, because $J$ is Ad$^*$-equivariant. It follows that the quotient space $\mathcal{M}_\mu = (\mathcal{M} \cap J^{-1}(\mu))/G_{\mu} \subset (T^*Q)_\mu$ of the $G_{\mu}$-orbit in $\mathcal{M} \cap J^{-1}(\mu)$, is a smooth manifold with projection $\pi_{\mu} : \mathcal{M} \cap J^{-1}(\mu) \to \mathcal{M}_\mu$ which is a surjective submersion. Denote $i_{\mathcal{M}_\mu} : \mathcal{M}_\mu \to (T^*Q)_\mu$, and $\omega_{M_\mu} = i_{\mathcal{M}_\mu}^* \omega_{\mu}$, that is, the symplectic form $\omega_{M_\mu}$ is induced from the Marsden-Weinstein reduced symplectic form $\omega_{\mu}$ on $(T^*Q)_\mu$, where $i_{\mathcal{M}_\mu} : T^*(T^*Q)_\mu \to T^*\mathcal{M}_\mu$. Moreover, the distribution $\mathcal{F}$ pushes down to a distribution $\mathcal{F}_\mu = T\pi_{\mu} \cdot \mathcal{F}$ on $(T^*Q)_\mu$, and we define $\mathcal{K}_\mu = \mathcal{F}_\mu \cap T\mathcal{M}_\mu$. Assume that $\omega_{\mathcal{K}_\mu} = \tau_{\mathcal{K}_\mu} \cdot \omega_{M_\mu}$ is the restriction of the symplectic form $\omega_{M_\mu}$ on $T^*\mathcal{M}_\mu$ fibrewise to the distribution $\mathcal{K}_\mu$, where $\tau_{\mathcal{K}_\mu}$ is the restriction map to distribution $\mathcal{K}_\mu$.

From the above construction, we know that $\omega_{\mathcal{K}_\mu}$ is non-degenerate, and is called as a J-nonholonomic $R_p$-reduced distributional two-form to avoid any confusion. Because $\omega_{\mathcal{K}_\mu}$ is non-degenerate as a bilinear form on each fibre of $K_\mu$, there exists a vector field $X_{\mathcal{K}_\mu}$ on $M_\mu$, which takes values in the constraint distribution $K_\mu$, such that the J-nonholonomic $R_p$-reduced distributional Hamiltonian equation holds, that is, $i_{X_{\mathcal{K}_\mu}} \omega_{\mathcal{K}_\mu} = dh_{\mathcal{K}_\mu}$, if the admissibility condition $\dim \mathcal{M}_\mu = \text{rank} \mathcal{F}_\mu$ and the compatibility condition $T\mathcal{M}_\mu \cap \mathcal{F}_\mu = \{0\}$ hold, where $\mathcal{F}_\mu$ denotes the symplectic orthogonal of $\mathcal{F}_\mu$ with respect to the J-nonholonomic $R_p$-reduced symplectic form $\omega_{\mu}$, and $dh_{\mathcal{K}_\mu}$ is the restriction of $dh_{M_\mu}$ to $K_\mu$, and the function $h_{\mathcal{K}_\mu}$ satisfies $dh_{\mathcal{K}_\mu} = \tau_{\mathcal{K}_\mu} \cdot dh_{M_\mu}$, and $h_{M_\mu} = \tau_{M_\mu} : h_\mu$ is the restriction of $h_\mu$ to $M_\mu$, and $h_\mu$ is the Marsden-Weinstein point reduced Hamiltonian function $h_\mu : (T^*Q)_\mu \to \mathbb{R}$ defined by $h_\mu \cdot \pi_{\mu} = H \cdot i_\mu$. Thus, the geometrical formulation of the J-nonholonomic $R_p$-reduced distributional Hamiltonian system may be summarized as follows.

**Definition 5.1 (J-Nonholonomic $R_p$-reduced Distributional Hamiltonian System)** Assume that the 6-tuple $(T^*Q, G, \omega, J, \mathcal{D}, H)$ is a nonholonomic Hamiltonian system with symmetry and momentum map, where $\omega$ is the canonical symplectic form on $T^*Q$, and $\mathcal{D} \subset TQ$ is a $G$-invariant and $\mathcal{D}$-regularly nonholonomic constraint of the system, and $\mathcal{D}$ and $H$ are both $G$-invariant. For a regular value $\mu \in \mathfrak{g}^*$ of the momentum map $J : T^*Q \to \mathfrak{g}^*$, assume that there exists a J-nonholonomic $R_p$-reduced distribution $K_\mu$, an associated non-degenerate and J-nonholonomic $R_p$-reduced distributional two-form $\omega_{\mathcal{K}_\mu}$ and a vector field $X_{\mathcal{K}_\mu}$ on the J-nonholonomic $R_p$-reduced constraint submanifold $M_\mu = (\mathcal{M} \cap J^{-1}(\mu))/G_{\mu}$, where $\mathcal{M} = \mathcal{F}L(\mathcal{D})$, and $\mathcal{M} \cap J^{-1}(\mu) \neq \emptyset$, and $G_{\mu} = \{g \in G \mid \text{Ad}^*_g \mu = \mu\}$, such that the J-nonholonomic $R_p$-reduced distributional Hamiltonian equation $i_{X_{\mathcal{K}_\mu}} \omega_{\mathcal{K}_\mu} = dh_{\mathcal{K}_\mu}$ holds, where $dh_{\mathcal{K}_\mu}$ is the restriction of $dh_{M_\mu}$ to $K_\mu$, and the function $h_{\mathcal{K}_\mu}$ is defined above. Then the triple $(K_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$ is called a J-nonholonomic $R_p$-reduced distributional Hamiltonian system of the system $(T^*Q, G, \omega, J, \mathcal{D}, H)$, and $X_{\mathcal{K}_\mu}$ is the dynamical vector field of the J-nonholonomic $R_p$-reduced distributional Hamiltonian system $(K_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$. Under the above circumstances, we refer to $(T^*Q, G, \omega, J, \mathcal{D}, H)$ as a J-nonholonomic point reducible Hamiltonian system with an associated J-nonholonomic $R_p$-reduced distributional Hamiltonian system $(K_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$, by using Lemma 3.4, we can derive precisely the
geometric constraint conditions of the J-nonholonomic $R_p$-reduced distributional two-form $\omega_{K_\mu}$ for the J-nonholonomic regular point reducible dynamical vector field, that is, the two types of Hamilton-Jacobi equation for the J-nonholonomic $R_p$-reduced distributional Hamiltonian system $(K_\mu, \omega_{K_\mu}, h_{K_\mu})$. At first, from Theorem 3.5, we know that $(\mu, \omega, J, T \gamma)$ is a solution of the Hamilton-Jacobi equation for the J-nonholonomic $R_p$-reduced distributional Hamiltonian system $(K_\mu, \omega_{K_\mu}, h_{K_\mu})$. Moreover, assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $J$, and $\text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(\mu)$, and it is $G_\mu$-invariant, as well as $\text{Im}(T \gamma_\mu) \subset K_\mu$. If the one-form $\gamma : Q \to T^*Q$ is closed on $D$ with respect to $T \pi_Q : TT^*Q \to TQ$, and $\text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(\mu)$, and it is $G_\mu$-invariant, as well as $\text{Im}(T \gamma_\mu) \subset K_\mu$, we can prove the Type I of Hamilton-Jacobi theorem for the J-nonholonomic $R_p$-reduced distributional Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-5.

![Diagram-5](image)

**Theorem 5.2 (Type I of Hamilton-Jacobi Theorem for a J-Nonholonomic $R_p$-reduced Distributional Hamiltonian System)** For a given J-nonholonomic point reducible Hamiltonian system $(T^*Q, G, \omega, J, D, H)$ with an associated J-nonholonomic $R_p$-reduced distributional Hamiltonian system $(K_\mu, \omega_{K_\mu}, h_{K_\mu})$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $X_H^\gamma = T \pi_Q \cdot X_H \cdot \gamma$, where $X_H$ is the dynamical vector field of the corresponding unconstrained Hamiltonian system with symmetry and momentum map $(T^*Q, G, \omega, J, H)$. Moreover, assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $J$, and $\text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(\mu)$, and it is $G_\mu$-invariant, and $\gamma_\mu = \pi_\mu(\gamma) : Q \to \mathcal{M}_\mu$, and $\text{Im}(T \gamma_\mu) \subset K_\mu$. If the one-form $\gamma : Q \to T^*Q$ is closed on $D$ with respect to $T \pi_Q : TT^*Q \to TQ$, then $\gamma_\mu$ is a solution of the equation $T \gamma_\mu \cdot X^\gamma_H = X_{K_\mu} \cdot \gamma_\mu$. Here $X_{K_\mu}$ is the dynamical vector field of the reduced system $(K_\mu, \omega_{K_\mu}, h_{K_\mu})$. The equation $T \gamma_\mu \cdot X^\gamma_H = X_{K_\mu} \cdot \gamma_\mu$, is called the Type I of Hamilton-Jacobi equation for the J-nonholonomic $R_p$-reduced distributional Hamiltonian system $(K_\mu, \omega_{K_\mu}, h_{K_\mu})$.

**Proof:** At first, from Theorem 3.5, we know that $\gamma$ is a solution of the Hamilton-Jacobi equation $T \gamma \cdot X^\gamma_H = X_K \cdot \gamma$. Next, we note that $\text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(\mu)$, and it is $G_\mu$-invariant, in this case, $\pi_\mu \omega = i_\mu \omega = \omega$, along $\text{Im}(\gamma)$. On the other hand, because $\text{Im}(T \gamma_\mu) \subset K_\mu$, then $\omega_{K_\mu} \cdot \tau_{K_\mu} = \tau_{K_\mu} \cdot \omega_{K_\mu} = \pi_{K_\mu} (\mu, \omega, J)$. Thus, using the J-nonholonomic $R_p$-reduced distributional two-form $\omega_{K_\mu}$, from Lemma 3.4, if we take that $v = \tau_{K_\mu} \cdot T \gamma_\mu \cdot X_H \cdot \gamma = X_{K_\mu} \cdot \gamma_\mu \in K_\mu$, and for
any \( w \in \mathcal{F} \), \( T\lambda(w) \neq 0 \), and \( \tau_{K_{\mu}} \cdot T\bar{\pi}_{\mu} \cdot w \neq 0 \), then we have that

\[
\omega_{K_{\mu}}(T\bar{\gamma}_{\mu} \cdot X_H^\gamma, \tau_{K_{\mu}} \cdot T\bar{\pi}_{\mu} \cdot w) = \omega_{K_{\mu}}(\tau_{K_{\mu}} \cdot T\bar{\gamma}_{\mu} \cdot X_H^\gamma, \tau_{K_{\mu}} \cdot T\bar{\pi}_{\mu} \cdot w)
\]

\[
= \tau_{K_{\mu}} \cdot \omega_{M_{\mu}}(T(\bar{\pi}_{\mu} \cdot \gamma) \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w) - \tau_{K_{\mu}} \cdot i^{*}_{M_{\mu}} \cdot \omega(T\bar{\pi}_{\mu} \cdot \gamma \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w)
\]

\[
= \tau_{K_{\mu}} \cdot \omega_{M_{\mu}}(T(\bar{\pi}_{\mu} \cdot \gamma) \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w) - \tau_{K_{\mu}} \cdot i^{*}_{M_{\mu}} \cdot \omega(T(\bar{\pi}_{\mu} \cdot \gamma) \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w)
\]

\[
= \tau_{K_{\mu}} \cdot i^{*}_{M_{\mu}} \cdot \omega(T\bar{\pi}_{\mu} \cdot \gamma \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w) - \tau_{K_{\mu}} \cdot i^{*}_{M_{\mu}} \cdot \omega(T\bar{\pi}_{\mu} \cdot \gamma \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w)
\]

\[
= \tau_{K_{\mu}} \cdot i^{*}_{M_{\mu}} \cdot \omega(T\bar{\pi}_{\mu} \cdot \gamma \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w) - \tau_{K_{\mu}} \cdot i^{*}_{M_{\mu}} \cdot \omega(T\bar{\pi}_{\mu} \cdot \gamma \cdot X_H^\gamma, T\bar{\pi}_{\mu} \cdot w)
\]

_at this point there seems to be a typographical error or some missing content._
Theorem 5.3 (Type II of Hamilton-Jacobi Theorem for a J-Nonholonomic \( R_p \)-reduced Distributional Hamiltonian System) For a given J-nonholonomic point reducible Hamiltonian system \((T^*Q, G, \omega, J, D, H)\) with an associated J-nonholonomic \( R_p \)-reduced distributional Hamiltonian system \((K_\mu, \omega_{K_\mu}, h_{K_\mu})\), assume that \( \gamma : Q \to T^*Q \) is an one-form on \( Q \), and \( \lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q \), and for any symplectic map \( \varepsilon : T^*Q \to T^*Q \), denote \( X^\varepsilon_H = T\pi_Q \cdot X_H \cdot \varepsilon \), where \( X_H \) is the dynamical vector field of the corresponding unconstrained Hamiltonian system with symmetry and momentum map \((T^*Q, G, \omega, J, H)\). Moreover, assume that \( \mu \in g^* \) is a regular value of the momentum map \( J \), and \( \text{Im}(\gamma) \subset M \cap J^{-1}(\mu) \), and that it is \( G_\mu \)-invariant, and \( \varepsilon \) is \( G_\mu \)-invariant and \( \varepsilon(J^{-1}(\mu)) \subset J^{-1}(\mu) \). Denote \( \bar{\gamma}_\mu = \pi_\mu(\gamma) : Q \to M_\mu \), and \( \text{Im}(\bar{\gamma}_\mu) \subset K_\mu \), and \( \bar{\lambda}_\mu = \pi_\mu(\lambda) : J^{-1}(\mu)(\subset T^*Q) \to M_\mu \), and \( \bar{\varepsilon}_\mu = \pi_\mu(\varepsilon) : J^{-1}(\mu)(\subset T^*Q) \to M_\mu \). Then \( \varepsilon \) and \( \bar{\varepsilon}_\mu \) satisfy the equation \( \tau_{K_\mu} \cdot T\bar{\varepsilon}(X_{h_{K_\mu}} \cdot \bar{\varepsilon}_\mu) = \bar{T}\bar{\lambda}_\mu \cdot X_H \cdot \varepsilon \), if and only if they satisfy the equation \( T\bar{\gamma}_\mu \cdot X^\varepsilon_H = X_{K_\mu} \cdot \bar{\varepsilon}_\mu \).

Here \( X_{h_{K_\mu}} \cdot \varepsilon \) is the Hamiltonian vector field of the function \( h_{K_\mu} \cdot \varepsilon : T^*Q \to \mathbb{R} \), and \( K_\mu \) is the dynamical vector field of the reduced system \((K_\mu, \omega_{K_\mu}, h_{K_\mu})\). The equation \( T\bar{\gamma}_\mu \cdot X^\varepsilon_H = X_{K_\mu} \cdot \bar{\varepsilon}_\mu \), is called the Type II of Hamilton-Jacobi equation for the J-nonholonomic \( R_p \)-reduced distributional Hamiltonian system \((K_\mu, \omega_{K_\mu}, h_{K_\mu})\).

Proof: At first, we note that \( \text{Im}(\gamma) \subset M \cap J^{-1}(\mu) \), and it is \( G_\mu \)-invariant, in this case, \( \pi^*_\mu \omega_\mu = i^*_\mu \omega = \omega \), along \( \text{Im}(\gamma) \). On the other hand, because \( \text{Im}(T\bar{\gamma}_\mu) \subset K_\mu \), then \( \omega_{K_\mu} \cdot \tau_{K_\mu} = \tau_{K_\mu} \cdot \omega_{M_\mu} = \tau_{K_\mu} \cdot i^*_{\mathcal{L}_{\mu}} \cdot \omega_\mu \), along \( \text{Im}(T\bar{\gamma}_\mu) \). Thus, using the J-nonholonomic \( R_p \)-reduced distributional two-form \( \omega_{K_\mu} \), from Lemma 3.4, if we take that \( v = \tau_{K_\mu} \cdot T\pi_\mu \cdot X_H \cdot \varepsilon = X_{K_\mu} \cdot \bar{\varepsilon}_\mu \in \mathcal{K}_\mu \), and for any
$w \in \mathcal{F}$, $T\lambda(w) \neq 0$, and $\tau_{K_\mu} \cdot T\pi_\mu \cdot w \neq 0$, then we have that

$$
\omega_{K_\mu}(T\gamma_\mu \cdot X^H, \tau_{K_\mu} \cdot T\pi_\mu \cdot w) = \omega_{K_\mu}(\tau_{K_\mu} \cdot T\gamma_\mu \cdot X^H, \tau_{K_\mu} \cdot T\pi_\mu \cdot w)
$$

$$
= \tau_{K_\mu} \cdot \omega_{M_\mu}(T(\tau_\mu \cdot \gamma) \cdot X^H, T\pi_\mu \cdot w) = \tau_{K_\mu} \cdot i^{*}_{M_\mu} \cdot \omega(T\pi_\mu \cdot T\gamma \cdot X^H, T\pi_\mu \cdot w)
$$

$$
= \tau_{K_\mu} \cdot i^{*}_{M_\mu} \cdot \pi^*_\mu \omega(\gamma \cdot \pi_Q \cdot X_H \cdot \varepsilon, w) = \tau_{K_\mu} \cdot i^{*}_{M_\mu} \cdot \omega(\gamma \cdot \pi_Q \cdot X_H \cdot \varepsilon, w)
$$

$$
= \tau_{K_\mu} \cdot i^{*}_{M_\mu} \cdot (\omega(X_H \cdot \varepsilon, w - T(\gamma \cdot \pi_Q \cdot w) - \delta(\pi_Q(X_H \cdot \varepsilon), T\pi_Q(w)))
$$

$$
= \tau_{K_\mu} \cdot i^{*}_{M_\mu} \cdot \omega(X_H \cdot \varepsilon, w) - \tau_{K_\mu} \cdot i^{*}_{M_\mu} \cdot \omega(X_H \cdot \varepsilon, T\lambda \cdot w)
$$

$$
- \tau_{K_\mu} \cdot i^{*}_{M_\mu} \cdot \delta(\pi_Q(X_H \cdot \varepsilon), T\pi_Q(w))
$$

where we have used that $\tau_{K_\mu} \cdot T\gamma_\mu = T\gamma_\mu$, and $\tau_{K_\mu} \cdot X_{h_{K_\mu} \cdot \bar{\varepsilon}} = X_{K_\mu} \cdot \bar{\varepsilon}_\mu$, since $\text{Im}(T\gamma_\mu) \subset K_\mu$.

Note that $\varepsilon : T^*Q \rightarrow T^*Q$ is symplectic, and $\pi^*_\mu \omega = i^{*}_{\mu} \omega = \omega$, along $\text{Im}(\gamma)$, and hence $\bar{\varepsilon}_\mu = \pi(\varepsilon) : T^*Q \rightarrow (T^*Q)_\mu$ is also symplectic along $\text{Im}(\gamma)$, and hence $X_{h_{K_\mu} \cdot \bar{\varepsilon}} = T\bar{\varepsilon}_\mu \cdot X_{h_{K_\mu} \cdot \bar{\varepsilon}}$, along $\bar{\varepsilon}_\mu$, and hence $\tau_{K_\mu} \cdot X_{h_{K_\mu} \cdot \bar{\varepsilon}} = \tau_{K_\mu} \cdot T\bar{\varepsilon}_\mu \cdot X_{h_{K_\mu} \cdot \bar{\varepsilon}}$, along $\bar{\varepsilon}_\mu$, because $\text{Im}(T\gamma_\mu) \subset K_\mu$. Then we have that

$$
\omega_{K_\mu}(T\gamma_\mu \cdot X^H, \tau_{K_\mu} \cdot T\pi_\mu \cdot w) - \omega_{K_\mu}(X_{K_\mu} \cdot \bar{\varepsilon}_\mu, \tau_{K_\mu} \cdot T\pi_\mu \cdot w)
$$

$$
= -\omega_{K_\mu}(\tau_{K_\mu} \cdot X_{h_{K_\mu} \cdot \bar{\varepsilon}} \cdot T\lambda_\mu \cdot w) + \omega_{K_\mu}(T\bar{\varepsilon}_\mu \cdot X_H \cdot \varepsilon, T\lambda_\mu \cdot w)
$$

$$
= \omega_{K_\mu}(T\lambda_\mu \cdot X_H \cdot \varepsilon - \tau_{K_\mu} \cdot T\bar{\varepsilon}_\mu \cdot X_{h_{K_\mu} \cdot \bar{\varepsilon}_\mu}, T\lambda_\mu \cdot w).
$$

Because the J-nonholonomic $R_p$-reduced distributional two-form $\omega_{K_\mu}$ is non-degenerate, it follows that the equation $T\gamma_\mu \cdot X^H = X_{K_\mu} \cdot \bar{\varepsilon}_\mu$, is equivalent to the equation $T\lambda_\mu \cdot X_H \cdot \varepsilon = X_{K_\mu} \cdot \bar{\varepsilon}_\mu$. Thus, $\varepsilon$ and $\bar{\varepsilon}_\mu$ satisfy the equation $T\lambda_\mu \cdot X_H \cdot \varepsilon = \tau_{K_\mu} \cdot T\bar{\varepsilon}_\mu \cdot X_{h_{K_\mu} \cdot \bar{\varepsilon}}$, if and only if they satisfy the Type II of Hamilton-Jacobi equation $T\gamma_\mu \cdot X^H = X_{K_\mu} \cdot \bar{\varepsilon}_\mu$. ■

**Remark 5.4** If a J-nonholonomic point reducible Hamiltonian system we considered has not any constrains, in this case, the J-nonholonomic $R_p$-reduced distributional Hamiltonian system is just the Marsden-Weinstein reduced Hamiltonian system itself. From the above Type I and Type II of Hamilton-Jacobi theorems, that is, Theorem 5.2 and Theorem 5.3, we can get the Theorem 3.3 and Theorem 3.4 in Wang [36]. It shows that Theorem 5.2 and Theorem 5.3 can be regarded as an extension of two types of Hamilton-Jacobi theorem for the Marsden-Weinstein reduced Hamiltonian system given in [36] to the nonholonomic context.

**Remark 5.5** It is worthy of note that the formulations of Type I and Type II of Hamilton-Jacobi equation for a J-nonholonomic $R_p$-reduced distributional Hamiltonian system, given by Theorem
5.2 and Theorem 5.3, have more extensive sense, because, in general, the one-form $\gamma$ is not given by a generating function of a symplectic map. When $\gamma$ is a solution of the classical Hamilton-Jacobi equation, that is, $X_H \cdot \gamma = 0$, then $X_H^\gamma = T^*Q \cdot X_H \cdot \gamma = 0$, and hence from the Type I of Hamilton-Jacobi equation, we have that $X_{\mu} \cdot \gamma \mu = \gamma (X_H) = 0$, which shows that the dynamical vector field of the $\mathcal{J}$-nonholonomic $R^p$-reduced distributional Hamiltonian system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$ is degenerate along $\gamma \mu$. The equation $X_{\mathcal{K}_\mu} \cdot \gamma \mu = 0$ is called the classical Hamilton-Jacobi equation for the $\mathcal{J}$-nonholonomic $R^p$-reduced distributional Hamiltonian system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$. In addition, for a symplectic map $\varepsilon : T^*Q \to T^*Q$, if $X_H \cdot \varepsilon = 0$, then from the Type II of Hamilton-Jacobi equation, we have that $X_{\mathcal{K}_\mu} \cdot \varepsilon \mu = \varepsilon (X_H) = 0$. But, from the equation $T \lambda \mu \cdot X_H \cdot \varepsilon = \tau \mu \cdot T \varepsilon \mu \cdot X_{h_{\mathcal{K}_\mu}} \cdot \varepsilon \mu$, we know that the equation $X_{\mathcal{K}_\mu} \cdot \varepsilon \mu = 0$ is not equivalent to the equation $X_{h_{\mathcal{K}_\mu}} \cdot \varepsilon \mu = 0$.

For a given $\mathcal{J}$-nonholonomic regular point reducible Hamiltonian system $(T^*Q, G, \omega, \mathcal{J}, D, H)$ with an associated $\mathcal{J}$-nonholonomic $R^p$-reduced distributional Hamiltonian system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$, we know that the Hamiltonian vector field $X_H$ and the $\mathcal{J}$-nonholonomic $R^p$-reduced dynamical vector field $X_{h_{\mathcal{K}_\mu}}$ are $\pi_{\mu}$-related, that is, $X_{h_{\mathcal{K}_\mu}} \cdot \pi_{\mu} = T^{\pi_{\mu}} \cdot X_H \cdot i_{\mu}$. Then we can prove the following Theorem 5.6, which states the relationship between the solutions of Type II of Hamilton-Jacobi equations and $\mathcal{J}$-nonholonomic regular point reduction.

**Theorem 5.6** For a given $\mathcal{J}$-nonholonomic regular point reducible Hamiltonian system $(T^*Q, G, \omega, \mathcal{J}, D, H)$ with an associated $\mathcal{J}$-nonholonomic $R^p$-reduced distributional Hamiltonian system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and $\varepsilon : T^*Q \to T^*Q$ is a symplectic map. Moreover, assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $\mathcal{J}$, and $\text{Im}(\gamma) \subset \mathcal{M} \cap \mathcal{J}^{-1}(\mu)$, and it is $G_{\mu}$-invariant, and $\varepsilon$ is $G_{\mu}$-invariant and $\varepsilon(\mathcal{J}^{-1}(\mu)) \subset \mathcal{J}^{-1}(\mu)$. Denote $\gamma \mu = \pi_{\mu}(\gamma) : Q \to \mathcal{M}_\mu$, and $\text{Im}(T \gamma \mu) \subset \mathcal{K}_\mu$, and $\lambda \mu = \pi_{\mu}(\lambda) : \mathcal{J}^{-1}(\mu)(\subset T^*Q) \to (T^*Q)_\mu$, and $\varepsilon \mu = \pi_{\mu}(\varepsilon) : \mathcal{J}^{-1}(\mu)(\subset T^*Q) \to (T^*Q)_\mu$. Then $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T \gamma \cdot X^\varepsilon = X_{\mathcal{K}_\mu} \cdot \varepsilon \mu$, for the distributional Hamiltonian system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$, if and only if $\varepsilon$ and $\varepsilon \mu$ satisfy the Type II of Hamilton-Jacobi equation $T \gamma \mu \cdot X^\mu = X_{\mathcal{K}_\mu} \cdot \varepsilon \mu$, for the $\mathcal{J}$-nonholonomic $R^p$-reduced distributional Hamiltonian system $(\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu})$.

**Proof:** Note that $\text{Im}(\gamma) \subset \mathcal{M} \cap \mathcal{J}^{-1}(\mu)$, and it is $G_{\mu}$-invariant, as well as $\text{Im}(T \gamma \mu) \subset \mathcal{K}_\mu$, in this case, $\omega_{\mathcal{K}_\mu} \cdot \gamma \mu = \pi_{\mu} \cdot \omega_{\mathcal{M}_\mu} = \pi_{\mu} \cdot (T \pi_{\mu} \cdot w_{\mathcal{M}_\mu})$, along $\text{Im}(T \gamma \mu)$, and $\pi_{\mu} \cdot \omega_{\mu} = \pi_{\mu} \cdot \omega_{\mathcal{M}_\mu} = \pi_{\mu} \cdot \omega_{\mathcal{K}_\mu}$, along $\text{Im}(\gamma)$, and $\tau \mu \cdot T \gamma \mu = T \gamma \mu$, and $\tau \mu \cdot X_{h_{\mathcal{K}_\mu}} = X_{\mathcal{K}_\mu}$. Since the Hamiltonian vector field $X_H$ and the $\mathcal{J}$-nonholonomic $R^p$-reduced dynamical vector field $X_{h_{\mathcal{K}_\mu}}$ are $\pi_{\mu}$-related, that is, $X_{h_{\mathcal{K}_\mu}} \cdot \pi_{\mu} = T^{\pi_{\mu}} \cdot X_H \cdot i_{\mu}$, using the $\mathcal{J}$-nonholonomic $R^p$-reduced distributional two-form $\omega_{\mathcal{K}_\mu}$, we have that

\[
\omega_{\mathcal{K}_\mu}(T \gamma \mu \cdot X^\mu = X_{\mathcal{K}_\mu} \cdot \varepsilon \mu, \tau \mu \cdot T \pi_{\mu} \cdot w) = \omega_{\mathcal{K}_\mu}(T \gamma \mu \cdot X_H^\mu, \tau \mu \cdot T \pi_{\mu} \cdot w) - \omega_{\mathcal{K}_\mu}(X_{\mathcal{K}_\mu} \cdot \varepsilon \mu, \tau \mu \cdot T \pi_{\mu} \cdot w)
\]

\[
= \omega_{\mathcal{K}_\mu}(T \gamma \mu \cdot X_H^\mu, \tau \mu \cdot T \pi_{\mu} \cdot w) - \omega_{\mathcal{K}_\mu}(X_{\mathcal{K}_\mu} \cdot \varepsilon \mu, \tau \mu \cdot T \pi_{\mu} \cdot w)
\]

\[
= \omega_{\mathcal{K}_\mu} \cdot \tau \mu \cdot (T \pi_{\mu} \cdot T \gamma \cdot X^\mu, T \pi_{\mu} \cdot w) - \omega_{\mathcal{K}_\mu} \cdot \tau \mu \cdot (T \pi_{\mu} \cdot X_H \cdot \varepsilon, T \pi_{\mu} \cdot w)
\]

\[
= \tau \mu \cdot i_{\mathcal{M}_\mu} \cdot \omega_{\mu}(T \pi_{\mu} \cdot T \gamma \cdot X_H^\mu, T \pi_{\mu} \cdot w) - \tau \mu \cdot i_{\mathcal{M}_\mu} \cdot \omega_{\mu}(T \pi_{\mu} \cdot X_H \cdot \varepsilon, T \pi_{\mu} \cdot w)
\]

\[
= \tau \mu \cdot i_{\mathcal{M}_\mu} \cdot \omega_{\mu}(T \gamma \cdot X_H^\mu, \varepsilon, w) - \tau \mu \cdot i_{\mathcal{M}_\mu} \cdot \omega_{\mu}(X_H \cdot \varepsilon, w)
\]

In the case we considered that $\tau \mu \cdot i_{\mathcal{M}_\mu} \cdot \omega = \tau \mathcal{K} \cdot i_{\mathcal{M}_\mu} \cdot \omega = \omega_{\mathcal{K}} \cdot \tau \mathcal{K}$, and $\tau \mathcal{K} \cdot T \gamma = T \gamma$, $\tau \mathcal{K} \cdot X_H = X_{\mathcal{K}}$. 

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since \( \text{Im}(\gamma) \subset \mathcal{M} \), and \( \text{Im}(T\gamma) \subset \mathcal{K} \). Thus, we have that

\[
\omega_{\mathcal{K}_\mu}(T\tilde{\gamma}_\mu \cdot X^\varepsilon_H - X_{\mathcal{K}_\mu} \cdot \tilde{\varepsilon}_\mu, \tau_{\mathcal{K}_\mu} \cdot T\varepsilon_{\mathcal{K}_\mu} \cdot w)
= \omega_{\mathcal{K}} \cdot \tau_{\mathcal{K}}(T\gamma \cdot X^\varepsilon_H, w) - \omega_{\mathcal{K}} \cdot \tau_{\mathcal{K}}(X_H \cdot \varepsilon, w)
= \omega_{\mathcal{K}}(\tau_{\mathcal{K}} \cdot T\gamma \cdot X^\varepsilon_H, \tau_{\mathcal{K}} \cdot w) - \omega_{\mathcal{K}}(\tau_{\mathcal{K}} \cdot X_H \cdot \varepsilon, \tau_{\mathcal{K}} \cdot w)
= \omega_{\mathcal{K}}(T\gamma \cdot X^\varepsilon_H, \tau_{\mathcal{K}} \cdot w) - \omega_{\mathcal{K}}(X_K \cdot \varepsilon, \tau_{\mathcal{K}} \cdot w)
= \omega_{\mathcal{K}}(T\gamma \cdot X^\varepsilon_H - X_K \cdot \varepsilon, \tau_{\mathcal{K}} \cdot w).
\]

Because the distributional two-form \( \omega_{\mathcal{K}} \) and the \( J \)-nonholonomic \( R_o \)-reduced distributional two-form \( \omega_{\mathcal{K}_\mu} \) are both non-degenerate, it follows that the equation \( T\tilde{\gamma}_\mu \cdot X^\varepsilon_H = X_{\mathcal{K}_\mu} \cdot \tilde{\varepsilon}_\mu \), is equivalent to the equation \( T\gamma \cdot X^\varepsilon_H = X_K \cdot \varepsilon \). Thus, \( \varepsilon \) is a solution of the Type II of Hamilton-Jacobi equation \( T\gamma \cdot X^\varepsilon_H = X_K \cdot \varepsilon \), for the distributional Hamiltonian system \( (\mathcal{K}, \omega), \) if and only if \( \varepsilon \) and \( \tilde{\varepsilon}_\mu \) satisfy the Type II of Hamilton-Jacobi equation \( T\tilde{\gamma}_\mu \cdot X^\varepsilon_H = X_{\mathcal{K}_\mu} \cdot \tilde{\varepsilon}_\mu \), for the \( J \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian system \( (\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu}) \).

**Remark 5.7** If \( (T^*Q, \omega) \) is a connected symplectic manifold, and \( J : T^*Q \to \mathfrak{g}^* \) is a non-equivariant momentum map with a non-equivariance group one-cocycle \( \sigma : G \to \mathfrak{g}^* \), which is defined by \( \sigma(g) := J(g \cdot z) - \text{Ad}_{g^{-1}}^* J(z) \), where \( g \in G \) and \( z \in T^*Q \). Then we know that \( \sigma \) produces a new affine action \( \Theta : G \times \mathfrak{g}^* \to \mathfrak{g}^* \) defined by \( \Theta(g, \mu) := \text{Ad}_{g^{-1}}^* \mu + \sigma(g) \), where \( \mu \in \mathfrak{g}^* \), with respect to which the given momentum map \( J \) is equivariant. Assume that \( G \) acts freely and properly on \( T^*Q \), and \( \tilde{G}_\mu \) denotes the isotropy subgroup of \( \mu \in \mathfrak{g}^* \) relative to this affine action \( \Theta \) and \( \mu \) is a regular value of \( J \). Then the quotient space \( (T^*Q)_\mu = J^{-1}(\mu)/\tilde{G}_\mu \) is also a symplectic manifold with symplectic form \( \omega_\mu \) uniquely characterized by (5.1), see Ortega and Ratiu [31]. In this case, we can also define the \( \mathcal{J} \)-nonholonomic regular point reducible Hamiltonian system \( (T^*Q, G, \omega, J, D, H) \) with an associated \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian system \( (\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu}) \), and prove the Type I and Type II of the Hamilton-Jacobi theorem for the \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian system \( (\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}, h_{\mathcal{K}_\mu}) \) by using the above similar way, in which the \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced space \( (\mathcal{K}_\mu, \omega_{\mathcal{K}_\mu}) \) is determined by the affine action and \( \mathcal{J} \)-nonholonomic regular point reduction.

### 5.2 Hamilton-Jacobi equations in the case compatible with regular orbit reduction

In this subsection, for a nonholonomic Hamiltonian system with symmetry and momentum map \( (T^*Q, G, \omega, J, D, H) \), where \( \omega \) is the canonical symplectic form on \( T^*Q \), and \( D \subset TQ \) is a \( D \)-completely and \( D \)-regularly nonholonomic constraint of the system, and \( D \) and \( H \) are both \( G \)-invariant, we first give the \( \mathcal{J} \)-nonholonomic regular orbit reduction of the system compatible with regular orbit reduction, and a \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distribution \( \mathcal{K}_{\mathcal{O}_\mu} \), an associated non-degenerate and \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distributional two-form \( \omega_{\mathcal{K}_{\mathcal{O}_\mu}} \), which is induced by the canonical symplectic form \( \omega \) on \( T^*Q \), and a \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian system, where the "regular orbit reduced" is simply written as \( R_o \)-reduced. Then we derive precisely the geometric constraint conditions of the \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distributional two-form \( \omega_{\mathcal{K}_{\mathcal{O}_\mu}} \) for the nonholonomic reducible dynamical vector field, that is, the two types of Hamilton-Jacobi equation for the \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian system, which are an extension of the above two types of Hamilton-Jacobi equation for the distributional Hamiltonian system under \( \mathcal{J} \)-nonholonomic regular orbit reduction.

At first, we need to give carefully a geometric formulation of the \( \mathcal{J} \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian system, by using momentum map and the nonholonomic reduction compatible with regular orbit reduction. Now, we assume that the 6-tuple \( (T^*Q, G, \omega, J, D, H) \) is a
\( D \)-completely and \( D \)-regularly nonholonomic Hamiltonian system with symmetry and momentum map, and the Lie group \( G \) acts smoothly by the left on \( Q \). For the cotangent lifted left action \( \Phi^T_t : G \times T^*Q \to T^*Q \), assume that it is free, proper and symplectic, and the action admits an \( \Ad^* \)-equivariant momentum map \( J : T^*Q \to \mathfrak{g}^* \). Let \( \mu \in \mathfrak{g}^* \) be a regular value of the momentum map \( J \) and \( \mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^* \) be the \( G \)-orbit of the coadjoint \( G \)-action through the point \( \mu \). Since \( G \) acts freely, properly and symplectically on \( T^*Q \), then the quotient space \( (T^*Q)_{\mathcal{O}_\mu} = J^{-1}(\mathcal{O}_\mu)/G \) is a regular quotient symplectic manifold with the symplectic form \( \omega_{\mathcal{O}_\mu} \), uniquely characterized by the relation

\[
i^*_{\mathcal{O}_\mu} \omega = \pi^*_{\mathcal{O}_\mu} \omega_{\mathcal{O}_\mu} + J^*_{\mathcal{O}_\mu} \omega^+_{\mathcal{O}_\mu}, \quad (5.3)
\]

where \( J_{\mathcal{O}_\mu} \) is the reduction of the momentum map \( J \) to \( J^{-1}(\mathcal{O}_\mu) \), that is, \( J_{\mathcal{O}_\mu} = J \cdot i_{\mathcal{O}_\mu} \) and \( \omega^+_{\mathcal{O}_\mu} \) is the \( + \)-symplectic structure on the orbit \( \mathcal{O}_\mu \) given by

\[
\omega^+_{\mathcal{O}_\mu}(\nu)(\xi, \eta) = \langle \nu, [\xi, \eta] \rangle, \quad \forall \nu \in \mathcal{O}_\mu, \quad \xi, \eta \in \mathfrak{g}.
\]

The maps \( i_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \to T^*Q \) and \( \pi_{\mathcal{O}_\mu} : J^{-1}(\mathcal{O}_\mu) \to (T^*Q)_{\mathcal{O}_\mu} \) are natural injection and the projection, respectively. The pair \( ((T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu}) \) is called the regular orbit reduced symplectic space of \( (T^*Q, \omega) \) at \( \mu \).

Let \( H : T^*Q \to \mathbb{R} \) be a \( G \)-invariant Hamiltonian, the flow \( F_t \) of the Hamiltonian vector field \( X_H \) leaves the connected components of \( J^{-1}(\mathcal{O}_\mu) \) invariant and commutes with the \( G \)-action, so it induces a flow \( f^\mu_t \) on \( (T^*Q)_{\mathcal{O}_\mu} \) defined by \( f^\mu_t \pi_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu} \cdot F_t \cdot i_{\mathcal{O}_\mu} \), and the vector field \( X_{h_{\mathcal{O}_\mu}} \) generated by the flow \( f^\mu_t \) on \( (T^*Q)_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu} \) is Hamiltonian with the associated \( R_o \)-reduced Hamiltonian function \( h_{\mathcal{O}_\mu} : (T^*Q)_{\mathcal{O}_\mu} \to \mathbb{R} \) defined by \( h_{\mathcal{O}_\mu} \cdot \pi_{\mathcal{O}_\mu} = H \cdot i_{\mathcal{O}_\mu} \), and the Hamiltonian vector fields \( X_H \) and \( X_{h_{\mathcal{O}_\mu}} \) are \( \pi_{\mathcal{O}_\mu} \)-related.

In the same way, using the Legendre transformation \( \mathcal{F}L : TQ \to T^*Q \), we can define the constraint submanifold \( \mathcal{M} = \mathcal{F}L(D) \subset T^*Q \) and the distribution \( \mathcal{F} = (\mathcal{T}\pi_Q)^{-1}(D) \), and \( \mathcal{K} = \mathcal{F} \cap T\mathcal{M} \). Moreover, we can also define the distributional two-form \( \omega_{\mathcal{K}} \), a vector field \( X_{\mathcal{K}} \) and the function \( H_{\mathcal{K}} \), such that \( i_{X_{\mathcal{K}}} \omega_{\mathcal{K}} = dH_{\mathcal{K}} \). Since \( D \subset TQ \) is a \( G \)-invariant distribution, and the Legendre transformation \( \mathcal{F}L : TQ \to T^*Q \) is a fiber-preserving map, then \( \mathcal{M} = \mathcal{F}L(D) \subset TQ \) is \( G \)-invariant. For a regular value \( \mu \in \mathfrak{g}^* \) of the momentum map \( J \), \( \mathcal{O}_\mu = G \cdot \mu \subset \mathfrak{g}^* \) is the \( G \)-orbit of the coadjoint \( G \)-action through the point \( \mu \), we shall assume that the constraint submanifold \( \mathcal{M} \) is clean intersection with \( J^{-1}(\mathcal{O}_\mu) \), that is, \( \mathcal{M} \cap J^{-1}(\mathcal{O}_\mu) \neq \emptyset \). It follows that the quotient space \( \mathcal{M}_{\mathcal{O}_\mu} = (\mathcal{M} \cap J^{-1}(\mathcal{O}_\mu))/G \subset (T^*Q)_{\mathcal{O}_\mu} \), of the \( G \)-orbit in \( \mathcal{M} \cap J^{-1}(\mathcal{O}_\mu) \), is a smooth manifold with projection \( \pi_{\mathcal{O}_\mu} : \mathcal{M} \cap J^{-1}(\mathcal{O}_\mu) \to \mathcal{M}_{\mathcal{O}_\mu} \), which is a surjective submersion. Denote \( i_{\mathcal{M}_{\mathcal{O}_\mu}} : \mathcal{M}_{\mathcal{O}_\mu} \to T^*Q_{\mathcal{O}_\mu} \), and \( \omega_{\mathcal{M}_{\mathcal{O}_\mu}} = i^*_{\mathcal{M}_{\mathcal{O}_\mu}} \omega_{\mathcal{O}_\mu} \), that is, the symplectic form \( \omega_{\mathcal{M}_{\mathcal{O}_\mu}} \) is induced from the \( R_o \)-reduced symplectic form \( \omega_{\mathcal{O}_\mu} \) on \( (T^*Q)_{\mathcal{O}_\mu} \), where \( i^*_{\mathcal{M}_{\mathcal{O}_\mu}} : T^*(T^*Q)_{\mathcal{O}_\mu} \to T^*\mathcal{M}_{\mathcal{O}_\mu} \). Moreover, the distribution \( \mathcal{F} \) pushes down to a distribution \( \mathcal{F}_{\mathcal{O}_\mu} = T\pi_{\mathcal{O}_\mu} \cdot \mathcal{F} \) on \( (T^*Q)_{\mathcal{O}_\mu} \), and we define \( \mathcal{K}_{\mathcal{O}_\mu} = \mathcal{F}_{\mathcal{O}_\mu} \cap T\mathcal{M}_{\mathcal{O}_\mu} \). Assume that \( \omega_{\mathcal{K}_{\mathcal{O}_\mu}} = \pi_{\mathcal{K}_{\mathcal{O}_\mu}} \cdot \omega_{\mathcal{M}_{\mathcal{O}_\mu}} \) is the restriction of the symplectic form \( \omega_{\mathcal{M}_{\mathcal{O}_\mu}} \) on \( T^*\mathcal{M}_{\mathcal{O}_\mu} \) fibrewise to the distribution \( \mathcal{K}_{\mathcal{O}_\mu} \), where \( \pi_{\mathcal{K}_{\mathcal{O}_\mu}} \) is the restriction map to distribution \( \mathcal{K}_{\mathcal{O}_\mu} \).

From the above construction, we know that \( \omega_{\mathcal{K}_{\mathcal{O}_\mu}} \) is non-degenerate, and is called as a \( J \)-nonholonomic \( R_o \)-reduced distributional two-form to avoid any confusion. Because \( \omega_{\mathcal{K}_{\mathcal{O}_\mu}} \) is non-degenerate as a bilinear form on each fibre of \( \mathcal{K}_{\mathcal{O}_\mu} \), there exists a vector field \( X_{\mathcal{K}_{\mathcal{O}_\mu}} \) on \( \mathcal{M}_{\mathcal{O}_\mu} \), which takes values in the constraint distribution \( \mathcal{K}_{\mathcal{O}_\mu} \), such that the \( J \)-nonholonomic \( R_o \)-reduced distributional Hamiltonian equation holds, that is, \( i_{X_{\mathcal{K}_{\mathcal{O}_\mu}}} \omega_{\mathcal{K}_{\mathcal{O}_\mu}} = dH_{\mathcal{K}_{\mathcal{O}_\mu}} \), if the admissibility condition \( \dim \mathcal{M}_{\mathcal{O}_\mu} = \text{rank} \mathcal{F}_{\mathcal{O}_\mu} \) and the compatibility condition \( T\mathcal{M}_{\mathcal{O}_\mu} \cap \mathcal{F}_{\mathcal{O}_\mu} = \{0\} \) hold, where \( \mathcal{F}_{\mathcal{O}_\mu} \) denotes the symplectic orthogonal of \( \mathcal{F}_{\mathcal{O}_\mu} \) with respect to the \( R_o \)-reduced symplectic form \( \omega_{\mathcal{O}_\mu} \), and \( dH_{\mathcal{K}_{\mathcal{O}_\mu}} \).
is the restriction of $dh_{MO_\mu} \to K_{O_\mu}$, and the function $h_{K_{O_\mu}}$ satisfies $dh_{K_{O_\mu}} = \tau_{K_{O_\mu}} \cdot dh_{MO_\mu}$, and $h_{MO_\mu} = \tau_{MO_\mu} \cdot h_{O_\mu}$ is the restriction of $h_{O_\mu}$ to $MO_\mu$, and $h_{O_\mu}$ is the $R_0$-reduced Hamiltonian function $h_{O_\mu} : (T^*Q)_{O_\mu} \to \mathbb{R}$ defined by $h_{O_\mu} \cdot \pi_{O_\mu} = H \cdot T_{O_\mu}$. Thus, the geometrical formulation of the $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system may be summarized as follows.

**Definition 5.8** (J-Nonholonomic $R_0$-reduced Distributional Hamiltonian System) Assume that the $6$-tuple $(T^*Q, G, \omega, J, D, H)$ is a nonholonomic Hamiltonian system with symmetry and momentum map, where $\omega$ is the canonical symplectic form on $T^*Q$, and $D \subset TQ$ is a $D$-completely and $D$-regularly nonholonomic constraint of the system, and $D$ and $H$ are both $G$-invariant. For a regular value $\mu \in g^*$ of the momentum map $J$, $O_\mu = G \cdot \mu \subset g^*$ is the $G$-orbit of the coadjoint $G$-action through the point $\mu$, assume that there exists a $J$-nonholonomic $R_0$-reduced distribution $K_{O_\mu}$, an associated non-degenerate and $J$-nonholonomic $R_0$-reduced distributional two-form $\omega_{K_{O_\mu}}$ and a vector field $X_{K_{O_\mu}}$ on the $J$-nonholonomic $R_0$-reduced constraint submanifold $MO_\mu = (M \cap J^{-1}(O_\mu))/G$, where $M = FL(D)$, and $M \cap J^{-1}(O_\mu) \neq \emptyset$, such that the $J$-nonholonomic $R_0$-reduced distributional Hamiltonian equation $i_{X_{K_{O_\mu}}} \omega_{K_{O_\mu}} = dh_{K_{O_\mu}}$ holds, where $dh_{K_{O_\mu}}$ is the restriction of $dh_{MO_\mu}$ to $K_{O_\mu}$, and the function $h_{K_{O_\mu}}$ is defined above. Then the triple $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$ is called a $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system of the system $(T^*Q, G, \omega, J, D, H)$, and $X_{K_{O_\mu}}$ is the dynamical vector field of the $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$. Under the above circumstances, we refer to $(T^*Q, G, \omega, J, D, H)$ as a $J$-nonholonomic regular orbit reducible Hamiltonian system with an associated $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$.

Since the non-degenerate and $J$-nonholonomic $R_0$-reduced distributional two-form $\omega_{K_{O_\mu}}$ may not be symplectic, and the $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$ may not be yet a Hamiltonian system, and has no yet generating function, and hence we can not describe the Hamilton-Jacobi equation for a $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system just like as in Theorem 1.1. But, for a given $J$-nonholonomic regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, J, D, H)$ with an associated $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$, by using Lemma 3.4, we can derive precisely the geometric constraint conditions of the $J$-nonholonomic $R_0$-reduced distributional two-form $\omega_{K_{O_\mu}}$ for the nonholonomic reducible dynamical vector field that is, the two types of Hamilton-Jacobi equation for the $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$. At first, using the fact that the one-form $\gamma : Q \to T^*Q$ is closed on $D$ with respect to $T\pi_Q : TT^*Q \to TQ$, and $\mathrm{Im}(\gamma) \subset M \cap J^{-1}(O_\mu)$, and it is $G$-invariant, as well as $\mathrm{Im}(T_{\gamma}O_\mu) \subset K_{O_\mu}$, we can prove the Type I of Hamilton-Jacobi theorem for the $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram 7.

**Theorem 5.9** (Type I of Hamilton-Jacobi Theorem for a $J$-Nonholonomic $R_0$-reduced Distributional Hamiltonian System) For a given $J$-nonholonomic regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, J, D, H)$ with an associated $J$-nonholonomic $R_0$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $X_H = T\pi_Q \cdot X_H \cdot \gamma$;
where $X_H$ is the Hamiltonian vector field of the corresponding unconstrained Hamiltonian system with symmetry and momentum map $(T^*Q, G, \omega, J, H)$. Moreover, assume that $\mu \in g^*$ is a regular value of the momentum map $J$, and $\text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(O\mu)$, and it is $G$-invariant, and $\gamma_{O\mu} = \pi_{O\mu}(\gamma) : Q \to M_{O\mu}$, and $\text{Im}(T\gamma_{O\mu}) \subset K_{O\mu}$. If the one-form $\gamma : Q \to T^*Q$ is closed on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \to TQ$, then $\gamma_{O\mu}$ is a solution of the equation $T\gamma_{O\mu} \cdot X_H^\gamma = X_{K_{O\mu}} \cdot \gamma_{O\mu}$. Here $X_{K_{O\mu}}$ is the dynamical vector field of the $J$-nonholonomic $R_o$-reduced system $(K_{O\mu}, \omega_{K_{O\mu}}, h_{K_{O\mu}})$. The equation $T\gamma_{O\mu} \cdot X_H^\gamma = X_{K_{O\mu}} \cdot \gamma_{O\mu}$ is called the Type I of Hamilton-Jacobi equation for the $J$-nonholonomic $R_o$-reduced distributional Hamiltonian system $(K_{O\mu}, \omega_{K_{O\mu}}, h_{K_{O\mu}})$.

**Proof:** At first, from Theorem 3.5, we know that $\gamma$ is a solution of the Hamilton-Jacobi equation $T\gamma \cdot X_H^\gamma = X_{K} \cdot \gamma$. Next, we note that the $R_o$-reduced symplectic space $(T^*Q)_{O\mu} = J^{-1}(O\mu)/G$ is $\mathcal{J}^{-1}(\mu)/G \times \mathcal{O}_\mu$, with the symplectic form $\omega_{O\mu}$ uniquely characterized by the relation $i^*_{O\mu}\omega = \pi_{O\mu}^\ast \omega_{O\mu} + J_{O\mu}^\ast \omega_{O\mu}$. Since $\text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(O\mu)$, and it is $G$-invariant, in this case for any $V \in TQ$, and $w \in TT^*Q$, we have that $J_{O\mu}^\ast \omega_{O\mu}(T\gamma \cdot V, w) = 0$, and hence $\pi_{O\mu}^\ast \omega_{O\mu} = i_{O\mu}^\ast \omega = \omega$, along $\text{Im}(\gamma)$. On the other hand, because $\text{Im}(T\gamma_{O\mu}) \subset K_{O\mu}$, then $\omega_{K_{O\mu}} \cdot \tau_{K_{O\mu}} = \tau_{K_{O\mu}} \cdot \omega_{M_{O\mu}} = \tau_{K_{O\mu}} \cdot i_{M_{O\mu}}^\ast \omega_{O\mu}$, along $\text{Im}(T\gamma_{O\mu})$. Thus, using the $J$-nonholonomic $R_o$-reduced distributional two-form $\omega_{K_{O\mu}}$, from Lemma 3.4, if we take that $v = \tau_{K_{O\mu}} \cdot T\pi_{O\mu} \cdot X_H \cdot \gamma = X_{K_{O\mu}} \cdot \gamma_{O\mu} \in K_{O\mu}$, and for any $w \in \mathcal{F}$, $T\chi(w) \neq 0$, and $\tau_{K_{O\mu}} \cdot T\pi_{O\mu} \cdot w \neq 0$, then we have that

$$
\omega_{K_{O\mu}}(T\gamma_{O\mu} \cdot X_H^\gamma, \tau_{K_{O\mu}} \cdot T\pi_{O\mu} \cdot w) = \omega_{K_{O\mu}}(\tau_{K_{O\mu}} \cdot T\gamma_{O\mu} \cdot X_H^\gamma, \tau_{K_{O\mu}} \cdot T\pi_{O\mu} \cdot w)
$$

$$
= \tau_{K_{O\mu}} \cdot \omega_{M_{O\mu}}(T(\pi_{O\mu} \cdot \gamma) \cdot X_H^\gamma, T\pi_{O\mu} \cdot w) = \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \omega_{O\mu}(T(\pi_{O\mu} \cdot T\gamma \cdot X_H^\gamma, T\pi_{O\mu} \cdot w)
$$

$$
= \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot \pi^\ast_{O\mu} \omega_{O}(T\gamma \cdot T\pi_Q \cdot X_H \cdot \gamma, w) = \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot \omega(T(\gamma \cdot T\pi_Q) \cdot X_H \cdot \gamma, w)
$$

$$
= \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot (\omega(X_H \cdot \gamma, w - T(\gamma \cdot T\pi_Q) \cdot w - d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)))
$$

$$
= \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot \omega_{O}(X_H \cdot \gamma, w) - \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot \pi^\ast_{O\mu} \omega_{O}(X_H \cdot \gamma, T(\gamma \cdot T\pi_Q) \cdot w)
$$

$$
- \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w))
$$

$$
= \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot \omega_{O}(T\pi_{O\mu} \cdot (X_H \cdot \gamma), T(\pi_{O\mu} \cdot \gamma) \cdot T\pi_Q(w))
$$

$$
= \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot \omega_{O}(X_H_{K_{O\mu}} \cdot \gamma_{O\mu}, T\pi_{O\mu} \cdot w - \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot \omega_{O}(X_H_{K_{O\mu}} \cdot \gamma_{O\mu}, T\gamma_{O\mu} \cdot T\pi_Q(w))
$$

$$
- \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w))
$$

$$
= \omega_{K_{O\mu}}(\tau_{K_{O\mu}} \cdot X_{K_{O\mu}} \cdot \gamma_{O\mu}, T\pi_{O\mu} \cdot w - \omega_{K_{O\mu}}(X_{K_{O\mu}} \cdot \gamma_{O\mu}, T\gamma_{O\mu} \cdot T\pi_Q(w))
$$

$$
- \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w))
$$

$$
= \omega_{K_{O\mu}}(X_H_{K_{O\mu}} \cdot \gamma_{O\mu}, \tau_{K_{O\mu}} \cdot T\pi_{O\mu} \cdot w - \omega_{K_{O\mu}}(X_{K_{O\mu}} \cdot \gamma_{O\mu}, T\gamma_{O\mu} \cdot T\pi_Q(w))
$$

$$
- \tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)),
$$

where we have used that $\tau_{K_{O\mu}} \cdot T\gamma_{O\mu} = T\gamma_{O\mu}$, and $\tau_{K_{O\mu}} \cdot X_{K_{O\mu}} \cdot \gamma_{O\mu} = X_{K_{O\mu}} \cdot \gamma_{O\mu}$, since $\text{Im}(T\gamma_{O\mu}) \subset K_{O\mu}$. If the one-form $\gamma : Q \to T^*Q$ is closed on $\mathcal{D}$ with respect to $T\pi_Q : TT^*Q \to TQ$, then we have that $d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0$, since $X_H \cdot \gamma, w \in \mathcal{F}$, and $T\pi_Q(X_H \cdot \gamma), T\pi_Q(w) \in \mathcal{D}$, and hence

$$
\tau_{K_{O\mu}} \cdot i^\ast_{M_{O\mu}} \cdot d\gamma(T\pi_Q(X_H \cdot \gamma), T\pi_Q(w)) = 0,
$$

and

$$
\omega_{K_{O\mu}}(T\gamma_{O\mu} \cdot X_H^\gamma, \tau_{K_{O\mu}} \cdot T\pi_{O\mu} \cdot w) - \omega_{K_{O\mu}}(X_{K_{O\mu}} \cdot \gamma_{O\mu}, \tau_{K_{O\mu}} \cdot T\pi_{O\mu} \cdot w)
$$

$$
= -\omega_{K_{O\mu}}(X_{K_{O\mu}} \cdot \gamma_{O\mu}, T\gamma_{O\mu} \cdot T\pi_Q(w)).
$$

(5.5)
If $\bar{\gamma}_{\mathcal{O}_\mu}$ satisfies the equation $T\gamma_{\mathcal{O}_\mu} \cdot \gamma_{\mathcal{O}_\mu} = T \mathcal{K}_{\mathcal{O}_\mu} \cdot \bar{\gamma}_{\mathcal{O}_\mu}$, from Lemma 3.4(i) we know that the right side of (5.5) becomes

$$\begin{align*}
- \omega_{\mathcal{K}_{\mathcal{O}_\mu}}(X_{\mathcal{K}_{\mathcal{O}_\mu}} \cdot \bar{\gamma}_{\mathcal{O}_\mu}, \tau_{\mathcal{O}_\mu} \cdot T\gamma_{\mathcal{O}_\mu} \cdot T\pi_Q(w)) \\
= \omega_{\mathcal{K}_{\mathcal{O}_\mu}}(T\gamma_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}}^\gamma, T\gamma_{\mathcal{O}_\mu} \cdot T\pi_Q(w)) \\
= \omega_{\mathcal{K}_{\mathcal{O}_\mu}}(\tau_{\mathcal{O}_\mu} \cdot T\gamma_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}}^\gamma, \tau_{\mathcal{O}_\mu} \cdot T\gamma_{\mathcal{O}_\mu} \cdot T\pi_Q(w)) \\
= -\tau_{\mathcal{O}_\mu} \cdot i_{\mathcal{M}_{\mathcal{O}_\mu}}^* \cdot \bar{\gamma}_{\mathcal{O}_\mu} \cdot \omega_{\mathcal{O}_\mu}(T\gamma_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}}^\gamma, T\gamma_{\mathcal{O}_\mu} \cdot T\pi_Q(w)) \\
= -\tau_{\mathcal{O}_\mu} \cdot i_{\mathcal{M}_{\mathcal{O}_\mu}}^* \cdot \bar{\gamma}_{\mathcal{O}_\mu} \cdot \omega_{\mathcal{O}_\mu}(T\pi_Q \cdot X_{\mathcal{H}} \cdot \gamma, T\pi_Q(w)) \\
= -\tau_{\mathcal{O}_\mu} \cdot i_{\mathcal{M}_{\mathcal{O}_\mu}}^* \cdot \bar{\gamma}_{\mathcal{O}_\mu} \cdot \gamma^* \cdot \pi_{\mathcal{O}_\mu}^* \cdot \omega_{\mathcal{O}_\mu}(T\pi_Q \cdot X_{\mathcal{H}} \cdot \gamma, T\pi_Q(w)) \\
= -\tau_{\mathcal{O}_\mu} \cdot i_{\mathcal{M}_{\mathcal{O}_\mu}}^* \cdot \bar{\gamma}_{\mathcal{O}_\mu} \cdot \gamma^* \omega(T\pi_Q(X_{\mathcal{H}} \cdot \gamma), T\pi_Q(w)) \\
= -\tau_{\mathcal{O}_\mu} \cdot i_{\mathcal{M}_{\mathcal{O}_\mu}}^* \cdot \bar{\gamma}_{\mathcal{O}_\mu} \cdot \omega(T\pi_Q(X_{\mathcal{H}} \cdot \gamma), T\pi_Q(w)) = 0.
\end{align*}$$

But, because the $J$-nonholonomic $R_\mu$-reduced distributional two-form $\omega_{\mathcal{K}_{\mathcal{O}_\mu}}$ is non-degenerate, the left side of (5.5) equals zero, only when $\bar{\gamma}_{\mathcal{O}_\mu}$ satisfies the equation $T\gamma_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}}^\gamma = X_{\mathcal{K}_{\mathcal{O}_\mu}} \cdot \bar{\gamma}_{\mathcal{O}_\mu}$. Thus, if the one-form $\gamma : Q \to T^*Q$ is closed on $\mathcal{D}$ with respect to $T\pi_Q : T^*Q \to TQ$, then $\bar{\gamma}_{\mathcal{O}_\mu}$ must be a solution of the Type I of Hamilton-Jacobi equation $T\gamma_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}}^\gamma = X_{\mathcal{K}_{\mathcal{O}_\mu}} \cdot \bar{\gamma}_{\mathcal{O}_\mu}$. ■

Next, for any $G$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$, we can prove the following Type II of geometric Hamilton-Jacobi theorem for the $J$-nonholonomic $R_\mu$-reduced distributional Hamiltonian system. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-8.

$$\begin{align*}
\text{Diagram-8}
\end{align*}$$

**Theorem 5.10 (Type II of Hamilton-Jacobi Theorem for a J-Nonholonomic $R_\mu$-Reduced Distributional Hamiltonian System)** For a given $J$-nonholonomic regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, J, \mathcal{D}, H)$ with an associated $J$-nonholonomic $R_\mu$-reduced distributional Hamiltonian system $(\mathcal{K}_{\mathcal{O}_\mu}, \omega_{\mathcal{K}_{\mathcal{O}_\mu}}, h_{\mathcal{K}_{\mathcal{O}_\mu}})$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and for any symplectic map $\varepsilon : T^*Q \to T^*Q$, denote $X_{\mathcal{H}}^\gamma = T\pi_Q \cdot X_{\mathcal{H}} \cdot \varepsilon$, where $X_{\mathcal{H}}$ is the Hamiltonian vector field of the corresponding unconstrained Hamiltonian system with symmetry and momentum map $(T^*Q, G, \omega, J, H)$. Moreover, assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum map $J$, and $\text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(\mathcal{O}_\mu)$, and it is $G$-invariant, and $\varepsilon$ is also $G$-invariant and $\varepsilon(J^{-1}(\mathcal{O}_\mu)) \subset J^{-1}(\mathcal{O}_\mu)$. Denote $\bar{\gamma}_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu}(\gamma) : Q \to \mathcal{M}_{\mathcal{O}_\mu}$, and $\text{Im}(T\bar{\gamma}_{\mathcal{O}_\mu}) \subset \mathcal{K}_{\mathcal{O}_\mu}$, and $\bar{\lambda}_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu}(\lambda) : J^{-1}(\mathcal{O}_\mu)(\subset T^*Q) \to \mathcal{M}_{\mathcal{O}_\mu}$, and $\varepsilon_{\mathcal{O}_\mu} = \pi_{\mathcal{O}_\mu}(\varepsilon) : J^{-1}(\mathcal{O}_\mu)(\subset T^*Q) \to \mathcal{M}_{\mathcal{O}_\mu}$.

Then $\varepsilon$ and $\varepsilon_{\mathcal{O}_\mu}$ satisfy the equation $\tau_{\mathcal{O}_\mu} \cdot T\varepsilon(X_{\mathcal{H}} \cdot \varepsilon_{\mathcal{O}_\mu}) = T\bar{\lambda}_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}} \cdot \varepsilon$, if and only if they satisfy the equation $T\gamma_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}}^\varepsilon = X_{\mathcal{K}_{\mathcal{O}_\mu}} \cdot \varepsilon_{\mathcal{O}_\mu}$. Here $X_{h_{\mathcal{K}_{\mathcal{O}_\mu}}, \varepsilon_{\mathcal{O}_\mu}}$ is the Hamiltonian vector field of the function $h_{\mathcal{K}_{\mathcal{O}_\mu}} \cdot \varepsilon_{\mathcal{O}_\mu} : T^*Q \to \mathbb{R}$, and $X_{h_{\mathcal{K}_{\mathcal{O}_\mu}}, \varepsilon_{\mathcal{O}_\mu}}$ is the dynamical vector field of the $J$-nonholonomic $R_\mu$-reduced system $(\mathcal{K}_{\mathcal{O}_\mu}, \omega_{\mathcal{K}_{\mathcal{O}_\mu}}, h_{\mathcal{K}_{\mathcal{O}_\mu}})$. The equation $T\gamma_{\mathcal{O}_\mu} \cdot X_{\mathcal{H}}^\varepsilon = X_{\mathcal{K}_{\mathcal{O}_\mu}} \cdot \varepsilon_{\mathcal{O}_\mu}$, is called the Type II of Hamilton-Jacobi equation for the $J$-nonholonomic $R_\mu$-reduced distributional system $(\mathcal{K}_{\mathcal{O}_\mu}, \omega_{\mathcal{K}_{\mathcal{O}_\mu}}, h_{\mathcal{K}_{\mathcal{O}_\mu}})$.

**Proof:** At first, we note that the $R_\mu$-reduced symplectic space $(T^*Q)_{\mathcal{O}_\mu} = J^{-1}(\mathcal{O}_\mu)/G \cong (J^{-1}(\mu)/G) \times \mathcal{O}_\mu$, with the $R_\mu$-reduced symplectic form $\omega_{\mathcal{O}_\mu}$ uniquely characterized by the relation

$$\begin{align*}
\text{Diagram-8}
\end{align*}$$
\[ i^*_{O_{\mu}} \omega = \pi^*_{O_{\mu}} \omega_{O_{\mu}} + J^*_{O_{\mu}} \omega^+_{O_{\mu}}. \]

Since \( \text{Im}(\gamma) \subset \mathcal{M} \cap J^{-1}(O_{\mu}) \), and it is \( G \)-invariant, in this case for any \( V \in TQ \) and \( w \in TT^*Q \), we have that \( J^*_{O_{\mu}} \omega_{O_{\mu}}(T\gamma \cdot V, w) = 0 \), and hence \( \pi^*_{O_{\mu}} \omega_{O_{\mu}} = i^*_{O_{\mu}} \omega = \omega \), along \( \text{Im}(\gamma) \). On the other hand, because \( \text{Im}(T\tilde{\gamma}_{O_{\mu}}) \subset K_{O_{\mu}} \), then \( \omega_{K_{O_{\mu}}} \cdot \tau_{K_{O_{\mu}}} = \pi^*_{K_{O_{\mu}}} \omega_{M_{O_{\mu}}} = \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \omega_{O_{\mu}} \), along \( \text{Im}(T\tilde{\gamma}_{O_{\mu}}) \). Thus, using the \( J \)-nonholonomic \( R_{\mu} \)-reduced distributional two-form \( \omega_{K_{O_{\mu}}} \), from Lemma 3.4, if we take that \( v = \tau_{K_{O_{\mu}}} \cdot T\pi_{O_{\mu}} \cdot X_H \cdot \varepsilon = X_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}} \in K_{O_{\mu}} \), and for any \( w \in F \), \( T\lambda(w) = 0 \), and \( \tau_{K_{O_{\mu}}} \cdot T\pi_{O_{\mu}} \cdot w \neq 0 \), then we have that

\[
\omega_{K_{O_{\mu}}}(T\tilde{\gamma}_{O_{\mu}} \cdot X^\varepsilon_H, \tau_{K_{O_{\mu}}} \cdot T\pi_{O_{\mu}} \cdot w) = \omega_{K_{O_{\mu}}}(\tau_{K_{O_{\mu}}} \cdot T\tilde{\gamma}_{O_{\mu}} \cdot X^\varepsilon_H, \tau_{K_{O_{\mu}}} \cdot T\pi_{O_{\mu}} \cdot w)
= \tau_{K_{O_{\mu}}} \cdot \omega_{M_{O_{\mu}}}(T(\pi_{O_{\mu}} \cdot \gamma) \cdot X^\varepsilon_H, T\pi_{O_{\mu}} \cdot w) = \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \omega_{O_{\mu}}(T\pi_{O_{\mu}} \cdot T\gamma \cdot X^\varepsilon_H, T\pi_{O_{\mu}} \cdot w)
= \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \pi^*_{O_{\mu}} \omega_{O_{\mu}}(T\gamma \cdot T\pi_{O_{\mu}} \cdot X^\varepsilon_H, \cdot w) = \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \cdot \omega(T(\gamma \cdot T\pi_{O_{\mu}}) \cdot X^\varepsilon_H \cdot w)
= \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \cdot (\omega(X^\varepsilon_H, \cdot w) - \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \cdot \omega(X_H \cdot \varepsilon, T\lambda(w))
= \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \omega(X_H \cdot \varepsilon, \cdot w) - \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \omega(X_H \cdot \varepsilon, T\lambda(w))
= \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \cdot \omega(X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, \cdot T\pi_{O_{\mu}} \cdot w) - \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \cdot \omega(X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, \cdot T\lambda_{O_{\mu}} \cdot w)
= \tau_{K_{O_{\mu}}} \cdot i^*_{M_{O_{\mu}}} \cdot \omega(X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, \cdot T\pi_{O_{\mu}} \cdot w) - \omega_{K_{O_{\mu}}}(\tau_{K_{O_{\mu}}} \cdot X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, \cdot T\lambda_{O_{\mu}} \cdot w)
\]

where we have used that \( \tau_{K_{O_{\mu}}} \cdot T\tilde{\gamma}_{O_{\mu}} = T\tilde{\gamma}_{O_{\mu}}, and \( \tau_{K_{O_{\mu}}} \cdot X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}} = X_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, since Im(T\tilde{\gamma}_{O_{\mu}}) \subset K_{O_{\mu}}. Note that \( \varepsilon : T^*Q \rightarrow T^*Q \) is symplectic, and \( \pi^*_{O_{\mu}} \omega_{O_{\mu}} = i^*_{O_{\mu}} \omega = \omega \), along \( \text{Im}(\gamma) \), and hence \( \tilde{\varepsilon}_{O_{\mu}} = \pi_{O_{\mu}}(\varepsilon) : T^*Q \rightarrow (T^*Q)_{O_{\mu}} \) is also symplectic along \( \text{Im}(\gamma) \), and hence \( X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}} = T\tilde{\varepsilon}_{O_{\mu}} \cdot X_{h_{K_{O_{\mu}}} \cdot \varepsilon_{O_{\mu}}}, along \tilde{\varepsilon}_{O_{\mu}} \), and hence \( \tau_{K_{O_{\mu}}} \cdot X_{h_{K_{O_{\mu}}} \cdot \varepsilon_{O_{\mu}} = \tau_{K_{O_{\mu}}} \cdot T\lambda_{O_{\mu}} \cdot X_{h_{K_{O_{\mu}}} \cdot \varepsilon_{O_{\mu}}} \), along \( \varepsilon_{O_{\mu}} \), because \( \text{Im}(T\tilde{\gamma}_{O_{\mu}}) \subset K_{O_{\mu}}. Then we have that

\[
\omega_{K_{O_{\mu}}}(T\tilde{\gamma}_{O_{\mu}} \cdot X^\varepsilon_H, \tau_{K_{O_{\mu}}} \cdot T\pi_{O_{\mu}} \cdot w) - \omega_{K_{O_{\mu}}}(X_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, \tau_{K_{O_{\mu}}} \cdot T\pi_{O_{\mu}} \cdot w)
= -\omega_{K_{O_{\mu}}}(\tau_{K_{O_{\mu}}} \cdot X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, \cdot T\lambda_{O_{\mu}} \cdot w) + \omega_{K_{O_{\mu}}}(T\lambda_{O_{\mu}} \cdot X_H \cdot \varepsilon, T\lambda_{O_{\mu}} \cdot w)
= \omega_{K_{O_{\mu}}}(T\lambda_{O_{\mu}} \cdot X_H \cdot \varepsilon - \tau_{K_{O_{\mu}}} \cdot T\varepsilon_{O_{\mu}} \cdot X_{h_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}}, T\lambda_{O_{\mu}} \cdot w).
\]

Because the \( J \)-nonholonomic \( R_{\mu} \)-reduced distributional two-form \( \omega_{K_{O_{\mu}}} \) is non-degenerate, it follows that the equation \( T\gamma_{O_{\mu}} \cdot X^\varepsilon_H = X_{K_{O_{\mu}}} \cdot \tilde{\varepsilon}_{O_{\mu}} \), is equivalent to the equation \( T\lambda_{O_{\mu}} \cdot X_H \cdot \varepsilon =
Remark 5.11 If a J-nonholonomic regular orbit reducible Hamiltonian system we considered has not any constrains, in this case, the J-nonholonomic $R_o$-reduced distributional Hamiltonian system is just the regular orbit reduced Hamiltonian system itself. From the above Type I and Type II of Hamilton-Jacobi theorems, that is, Theorem 5.9 and Theorem 5.10, we can get the Theorem 4.3 and Theorem 4.4 in Wang [36]. It shows that Theorem 5.9 and Theorem 5.10 can be regarded as an extension of two types of Hamilton-Jacobi theorem for the $R_o$-reduced Hamiltonian system given in [36] to the nonholonomic context.

Remark 5.12 It is worthy of note that the formulations of Type I and Type II of Hamilton-Jacobi equation for a J-nonholonomic $R_o$-reduced distributional Hamiltonian system, given by Theorem 5.9 and Theorem 5.10, have more extensive sense, because, in general, the one-form $\gamma$ is not given by a generating function of a symplectic map. When $\gamma$ is a solution of the classical Hamilton-Jacobi equation, that is, $X_H \cdot \gamma = 0$, then $X_H = T\pi_Q \cdot X_H = 0$, and hence from the Type I of Hamilton-Jacobi equation, we have that $X_{K_{O_\mu}} \cdot \gamma_{O_\mu} = T\gamma_{O_\mu} \cdot X_H^\tau = 0$, which shows that the dynamical vector field of the J-nonholonomic $R_o$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$ is degenerate along $\gamma_{O_\mu}$. The equation $X_{K_{O_\mu}} \cdot \gamma_{O_\mu} = 0$ is called the classical Hamilton-Jacobi equation for the J-nonholonomic $R_o$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$. In addition, for a symplectic map $\varepsilon : \mathbb{T} Q \to \mathbb{T} Q$, if $X_H \cdot \varepsilon = 0$, then from the Type II of Hamilton-Jacobi equation, we have that $X_{K_{O_\mu}} \cdot \varepsilon_{O_\mu} = T\gamma_{O_\mu} \cdot X_H^\varepsilon = 0$. But, from the equation $T\lambda_{O_\mu} \cdot X_H \cdot \varepsilon = T\pi_Q \cdot X_{h_{K_{O_\mu}}} \cdot \varepsilon_{O_\mu}$, we know that the equation $X_{K_{O_\mu}} \cdot \varepsilon_{O_\mu} = 0$ is not equivalent to the equation $X_{h_{K_{O_\mu}}} \cdot \varepsilon_{O_\mu} = 0$.

For a given J-nonholonomic regular orbit reducible Hamiltonian system $(\mathbb{T} Q, G, \omega, J, D, H)$ with an associated J- nonholonomic $R_o$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$, we know that the Hamiltonian vector field $X_H$ and the J-nonholonomic $R_o$-reduced dynamical vector field $X_{h_{K_{O_\mu}}}$ are $\pi_{O_\mu}$-related, that is, $X_{h_{K_{O_\mu}}} = T\pi_Q \cdot X_H = i_{O_\mu}$. Then we can prove the following Theorem 5.13, which states the relationship between the solutions of Type II of Hamilton-Jacobi equations and J-nonholonomic regular orbit reduction.

Theorem 5.13 For a given J-nonholonomic regular orbit reducible Hamiltonian system $(\mathbb{T} Q, G, \omega, J, D, H)$ with an associated J-nonholonomic $R_o$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$, assume that $\gamma : Q \to \mathbb{T} Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : \mathbb{T} Q \to \mathbb{T} Q$, and $\varepsilon : \mathbb{T} Q \to \mathbb{T} Q$ is a symplectic map. Moreover, assume that $\mu \in g^*$ is a regular value of the momentum map $J$, and $\text{Im}(\gamma) \subset M \cap J^{-1}(O_\mu)$, and it is G-invariant, and $\varepsilon$ is also G-invariant and $\varepsilon(J^{-1}(O_\mu)) \subset J^{-1}(O_\mu)$. Denote $\gamma_{O_\mu} = \pi_{O_\mu}(\gamma) : Q \to M_{O_\mu}$, and $\text{Im}(\gamma_{O_\mu}) \subset K_{O_\mu}$, and $\lambda_{O_\mu} = \pi_{O_\mu}(\lambda) : J^{-1}(O_\mu) \subset \mathbb{T} Q \to \mathbb{T} Q \subset \mathbb{T} Q_{O_\mu}$, and $\varepsilon_{O_\mu} = \pi_{O_\mu}(\varepsilon) : J^{-1}(O_\mu) \subset \mathbb{T} Q \to \mathbb{T} Q_{O_\mu}$. Then $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T\gamma \cdot X_H^\varepsilon = X_{K_{O_\mu}} \cdot \varepsilon_{O_\mu}$, for the J-nonholonomic $R_o$-reduced distributional Hamiltonian system $(K_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$.

Proof: Note that $\text{Im}(\gamma) \subset M \cap J^{-1}(O_\mu)$, and it is $G$-invariant, as well as $\text{Im}(T\gamma_{O_\mu}) \subset K_{O_\mu}$, in this case, $\omega_{K_{O_\mu}} \cdot \tau_{K_{O_\mu}} = \tau_{K_{O_\mu}} \cdot \omega_{M_{O_\mu}} = \tau_{K_{O_\mu}} \cdot i_{M_{O_\mu}} \cdot \omega_{O_\mu}$, along $\text{Im}(T\gamma_{O_\mu})$, and $\tau_{K_{O_\mu}} \cdot \omega_{O_\mu} = i_{O_\mu} \cdot \omega = \omega$, along $\text{Im}(\gamma)$, and $\tau_{K_{O_\mu}} \cdot X_{h_{K_{O_\mu}}} = X_{K_{O_\mu}} \cdot \varepsilon_{O_\mu}$. Since the dynamical vector fields $X_H$ and $X_{h_{K_{O_\mu}}}$ are $\pi_{O_\mu}$-related, that is, $X_{h_{K_{O_\mu}}} \cdot \pi_{O_\mu} = T\pi_Q \cdot X_H = i_{O_\mu}$, using the J-nonholonomic
$R_o$-reduced distributional two-form $\omega_{K_{O_\mu}}$, we have that

\[
\omega_{K_{O_\mu}}(T\gamma_{O_\mu} \cdot X_H - X_{K_{O_\mu}} \cdot \bar{\varepsilon}_{O_\mu}, \tau_{K_{O_\mu}} \cdot T\pi_{O_\mu} \cdot w) = \omega_{K_{O_\mu}}(T\gamma_{O_\mu} \cdot X_H, \tau_{K_{O_\mu}} \cdot T\pi_{O_\mu} \cdot w) - \omega_{K_{O_\mu}}(X_{K_{O_\mu}} \cdot \bar{\varepsilon}_{O_\mu}, \tau_{K_{O_\mu}} \cdot T\pi_{O_\mu} \cdot w)
\]

In the case we considered that $T\gamma \cdot X_H = X_K \cdot \varepsilon$, and $T\gamma \cdot T\pi_{O_\mu} \cdot w = T\gamma \cdot X_H = X_K \cdot \varepsilon$, since $\text{Im}(\gamma) \subset \mathcal{M}$, and $\text{Im}(T\gamma) \subset \mathcal{K}$. Thus, we have that

\[
\omega_{K_{O_\mu}}(T\gamma_{O_\mu} \cdot X_H - X_{K_{O_\mu}} \cdot \bar{\varepsilon}_{O_\mu}, \tau_{K_{O_\mu}} \cdot T\pi_{O_\mu} \cdot w) = \omega_{K_{O_\mu}}(T\gamma_{O_\mu} \cdot X_H, \tau_{K_{O_\mu}} \cdot T\pi_{O_\mu} \cdot w) - \omega_{K_{O_\mu}}(X_{K_{O_\mu}} \cdot \bar{\varepsilon}_{O_\mu}, \tau_{K_{O_\mu}} \cdot T\pi_{O_\mu} \cdot w)
\]

Because the distributional two-form $\omega_{K}$ and the $J$-nonholonomic $R_o$-reduced distributional two-form $\omega_{K_{O_\mu}}$ are both non-degenerate, it follows that the equation $T\gamma_{O_\mu} \cdot X_H = X_{K_{O_\mu}} \cdot \bar{\varepsilon}_{O_\mu}$, is equivalent to the equation $T\gamma \cdot X_H = X_K \cdot \varepsilon$. Thus, $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T\gamma \cdot X_H = X_K \cdot \varepsilon$, for the distributional Hamiltonian system $(\mathcal{K}, \omega_K, H_K)$, if and only if $\varepsilon$ and $\bar{\varepsilon}_{O_\mu}$ satisfy the Type II of Hamilton-Jacobi equation $T\gamma_{O_\mu} \cdot X_H = X_{K_{O_\mu}} \cdot \bar{\varepsilon}_{O_\mu}$, for the $J$-nonholonomic $R_o$-reduced distributional Hamiltonian system $(\mathcal{K}_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$.

**Remark 5.14** If $(T^*Q, \omega)$ is a connected symplectic manifold, and $J : T^*Q \to g^*$ is a non-equivariant group one-cocycle $\sigma : G \to g^*$, which is defined by $\sigma(g) := J(g \cdot z) - \text{Ad}_g \sigma(z)$, where $g \in G$ and $z \in T^*Q$. Then we know that $\sigma$ produces a new affine action $\Theta : G \times g^* \to g^*$ defined by $\Theta(g, \mu) := \text{Ad}_g^{-1} \mu + \sigma(g)$, where $\mu \in g^*$, with respect to which the given momentum map $J$ is equivariant. Assume that $G$ acts freely and properly on $T^*Q$, and $O_\mu = G \cdot \mu \subset g^*$ denotes the $G$-orbit of the point $\mu \in g^*$ with respect to this affine action $\Theta$, and $\mu$ is a regular value of $J$. Then the quotient space $(T^*Q)_{O_\mu} = J^{-1}(O_\mu)/G$ is also a symplectic manifold with symplectic form $\omega_{O_\mu}$ uniquely characterized by (6.3), see Ortega and Ratiu [31]. In this case, we can also define the $J$-nonholonomic regular orbit reducible Hamiltonian system $(T^*Q, G, \omega, J, D, H)$ with an associated $J$-nonholonomic $R_o$-reduced distributional Hamiltonian system $(\mathcal{K}_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$, and prove the Type I and Type II of the Hamilton-Jacobi theorem for the $J$-nonholonomic $R_o$-reduced distributional Hamiltonian system $(\mathcal{K}_{O_\mu}, \omega_{K_{O_\mu}}, h_{K_{O_\mu}})$ by using the above similar way, in which the $J$-nonholonomic $R_o$-reduced space $(\mathcal{K}_{O_\mu}, \omega_{K_{O_\mu}})$ is determined by the affine action and $J$-nonholonomic regular orbit reduction.

### 6 Applications

In this section, in order to illustrate the Hamilton-Jacobi theory for the nonholonomic reducible Hamiltonian system with symmetry, we shall discuss the following two examples: (1) the motion of constrained particle in space $\mathbb{R}^3$; (2) the motion of vertical rolling disk. These two examples are classical in the theory of nonholonomic mechanical systems. We shall follow the notations and conventions introduced in Bates and Śniatycki [3], Bloch et al. [4] and Wang [36].
6.1 The constrained particle in $\mathbb{R}^3$

In this subsection, we consider the motion of constrained particle in space $\mathbb{R}^3$, and give explicitly the motion equations of this problem and derive precisely the geometric constraint conditions of the induced distributional two-forms for the nonholonomic dynamical vector fields, that is, the Type I and Type II of Hamilton-Jacobi equations of this problem. At first, the configuration space of motion of the constrained particle in space is $Q = \mathbb{R}^3$, whose coordinates are denoted by $q = (x, y, z)$, its velocity space is $T\mathbb{R}^3$, and the phase space is $T^*\mathbb{R}^3$ with canonical symplectic form

$$\omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z.$$ The constraint set on the velocities is given by

$$\mathcal{D} = \{(x, y, z, v_x, v_y, v_z) \in T\mathbb{R}^3 | v_z = \sigma(y)v_x\},$$

where $\sigma(y)$ is a smooth function. For any $q \in Q$, $\mathcal{D}(q) = \text{Span}\{\partial_x + \sigma(y)\partial_z, \partial_y\}$. Note that

$$[\partial_x + \sigma(y)\partial_z, \partial_y] = [\partial_x, \partial_y] + [\sigma(y)\partial_z, \partial_y] = \sigma'(y)\partial_z,$$

which is nonzero everywhere if $\sigma'(y) \neq 0$, then $\mathcal{D}$ is nonholonomic and it is completely nonholonomic, that is, $\mathcal{D}$ along with all of its iterated Lie brackets $[\mathcal{D}, \mathcal{D}], [\mathcal{D}, [\mathcal{D}, \mathcal{D}]], \cdots$ spans the tangent bundle $TQ$. The Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$ is the kinetic energy of the Euclidean metric of $\mathbb{R}^3$, that is,

$$L = \frac{1}{2}(v_x^2 + v_y^2 + v_z^2),$$

which is simple and it is hyperregular, and hence the system is $\mathcal{D}$-regular automatically. The momenta are $p_x = \frac{\partial L}{\partial v_x} = v_x$, $p_y = \frac{\partial L}{\partial v_y} = v_y$, and $p_z = \frac{\partial L}{\partial v_z} = v_z = \sigma(y)p_x$, and the Hamiltonian $H : T^*\mathbb{R}^3 \to \mathbb{R}$ is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2).$$

The unconstrained Hamiltonian vector field is given by

$$X_H = p_x\partial_x + p_y\partial_y + p_z\partial_z.$$

By using the Legendre transformation

$$\mathcal{FL} : T\mathbb{R}^3 \to T^*\mathbb{R}^3, \quad \mathcal{FL}(x, y, z, v_x, v_y, v_z) = (x, y, z, p_x, p_y, p_z),$$

the constraint submanifold $\mathcal{M} = \mathcal{FL}(\mathcal{D})$ is given by

$$\mathcal{M} = \{(x, y, z, p_x, p_y, p_z) \in T^*Q | p_z = \sigma(y)p_x\}.$$ Moreover, define $\mathcal{F} = (T\pi_Q)^{-1}(\mathcal{D})$, and the compatibility condition $T\mathcal{M} \cap \mathcal{F}^\perp = \{0\}$ holds, where $\mathcal{F}^\perp$ denotes the symplectic orthogonal of $\mathcal{F}$ with respect to the canonical symplectic form $\omega$. Then the distribution is given by

$$\mathcal{K} = \mathcal{F} \cap T\mathcal{M} = \text{Span}\{\partial_x + \sigma(y)\partial_z, \partial_y, \partial_{p_x}, \partial_{p_y}\}.$$ The induced two-form $\omega_{\mathcal{M}} = i_{\mathcal{M}}^* \omega$ is given by

$$\omega_{\mathcal{M}} = dx \wedge dp_x + dy \wedge dp_y + dz \wedge (p_x\sigma'(y)dy + \sigma(y)dp_x),$$

and the non-degenerate distributional two-form is given by

$$\omega_{\mathcal{K}} = \tau_{\mathcal{K}} \cdot \omega_{\mathcal{M}}.$$
A direct computation yields
\[ i_{\partial_x + \sigma(y)\partial_z} \omega = (1 + \sigma^2(y)) dp_x + \sigma(y)\sigma'(y)p_x dy, \quad i_{\partial_y} \omega = dp_y - \sigma'(y)p_x dz, \]
\[ i_{\partial_z} \omega = -\sigma(y) dz - dx, \quad i_{\partial_y} \omega = -dy, \]
and
\[ dH = p_x dp_x + p_y dp_y + \sigma(y)\sigma'(y)p_z^2 dy + \sigma^2(y)p_x dp_x \]
\[ = \sigma(y)\sigma'(y)p_z^2 dy + (1 + \sigma^2(y))p_x dp_x + p_y dp_y. \]

Assume that \( X = X_1(\partial_x + \sigma(y)\partial_z) + X_2\partial_y + X_3\partial_{p_x} + X_4\partial_{p_y}, \) then we have that
\[ i_{X} \omega = X_1((1 + \sigma^2(y)) dp_x + \sigma(y)\sigma'(y)p_x dy) \]
\[ + X_2(dp_y - \sigma'(y)p_x dz) + X_3(-\sigma(y) dz - dx) + X_4(-dy) \]
\[ = (-X_3) dx + (\sigma(y)\sigma'(y)p_x X_1 - X_4) dy \]
\[ + (-\sigma'(y)p_x X_2 - \sigma(y)X_3) dz + (1 + \sigma^2(y))X_1 dp_x + X_2 dp_y. \]

From the equation of distributional Hamiltonian system \( i_{X} \omega = dH, \) we have that
\[ X_1 = p_x, \quad X_2 = p_y, \quad X_3 = 0, \quad X_4 = 0. \]

Hence, the nonholonomic dynamical vector field is given by
\[ X = p_x(\partial_x + \sigma(y)\partial_z) + p_y\partial_y, \]
and the motion equations of the distributional Hamiltonian system \((\mathcal{K}, \omega, H)\) are expressed by
\[ \dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{z} = \sigma(y)p_x, \quad \dot{p}_x = 0, \quad \dot{p}_y = 0. \]

In the following we shall derive precisely the geometric constraint conditions of the induced distributional two-form for the nonholonomic dynamical vector field, that is, the Type I and Type II of Hamilton-Jacobi equations for the distributional Hamiltonian system \((\mathcal{K}, \omega, H)\). Assume that
\[ \gamma : \mathbb{R}^3 \to T^*\mathbb{R}^3, \quad \gamma(x, y, z) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6), \]
and \( \lambda = \gamma \cdot \pi_Q : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3 \) given by
\[ \lambda(x, y, z, p_x, p_y, p_z) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \]
\[ = \gamma \cdot \pi_Q(x, y, z, p_x, p_y, p_z) = \gamma(x, y, z) \]
\[ = (\gamma_1 \cdot \pi_Q, \gamma_2 \cdot \pi_Q, \gamma_3 \cdot \pi_Q, \gamma_4 \cdot \pi_Q, \gamma_5 \cdot \pi_Q, \gamma_6 \cdot \pi_Q), \]
that is, \( \lambda_i = \gamma_i \cdot \pi_Q, \ i = 1, \cdots, 6, \) where \( \lambda_i, \ i = 1, \cdots, 6, \) are functions on \( T^*\mathbb{R}^3, \) and \( \gamma_i, \ i = 1, \cdots, 6, \) are functions on \( \mathbb{R}^3. \) We may choose \( q = (x, y, z) \in \mathbb{R}^3, \) such that \( \gamma_1(q) = x, \ \gamma_2(q) = y, \ \gamma_3(q) = z, \)
and \( \gamma(q) = \gamma(q)(dx + \gamma_5(q) dy + \gamma_6(q) dz). \) Note that \( D(q) = \text{Span}\{\partial_x + \sigma(y)\partial_z, \ \partial_y\}, \) take that \( \alpha = \partial_x + \sigma(y)\partial_z \) and \( \beta = \partial_y, \) then we have that
\[ d\gamma(\alpha, \beta) = \alpha(\gamma(\beta)) - \beta(\gamma(\alpha)) - \gamma([\alpha, \beta]) \]
\[ = (\partial_{\gamma_5} - \partial_{\gamma_4}) dy - \sigma(y)(\partial_{\gamma_6} - \partial_{\gamma_5}) dz - 2\sigma'(y)\gamma_6. \]
Thus, when \( d\gamma(\alpha, \beta) = 0, \) we know that for any \( v, w \in \mathcal{F}, \) and \( T\pi_Q(v), T\pi_Q(w) \in D, \) then \( d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0, \) that is, \( \gamma \) is closed on \( D \) with respect to \( T\pi_Q : TT^*\mathbb{R}^3 \to T\mathbb{R}^3. \) Note that \( \text{Im}(\gamma) \subset \mathcal{M}, \) then \( p_x = \gamma_4, \ p_y = \gamma_5, \ p_z = \gamma_6 \) and \( \gamma_6 = \sigma(y)\gamma_4, \) and hence
\[ H \cdot \gamma = \frac{1}{2}(1 + \sigma^2(y))\gamma_4^2 + \gamma_5^2, \]
$X_H \cdot \gamma = \gamma^4 \partial_x + \gamma^5 \partial_y + \sigma(y) \gamma^4 \partial_z = X_K \cdot \gamma$,

$X_H^\gamma = T \pi_Q \cdot X_H \cdot \gamma = \gamma^4 \partial_x + \gamma^5 \partial_y + \sigma(y) \gamma^4 \partial_z$.

Thus, $T \gamma \cdot X_H^\gamma = X_K \cdot \gamma$, that is, the Type I of Hamilton-Jacobi equation for the distributional Hamiltonian system $(K, \omega_K, H)$ holds trivially.

Now, for any symplectic map $\epsilon : T^* \mathbb{R}^3 \to T^* \mathbb{R}^3$, from $\omega = \epsilon^* \omega = \omega \cdot \epsilon = (\partial_x \epsilon \cdot \partial_p \epsilon) dx \wedge dp_x + (\partial_y \epsilon \cdot \partial_p \epsilon) dy \wedge dp_y + (\partial_z \epsilon \cdot \partial_p \epsilon) dz \wedge dp_z$, we have that

$$\partial_x \epsilon \cdot \partial_p \epsilon = 1, \quad \partial_y \epsilon \cdot \partial_p \epsilon = 1, \quad \partial_z \epsilon \cdot \partial_p \epsilon = 1.$$  

Denote by $\epsilon(x, y, p_x, p_y, p_z) = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6)$, then we have that

$$H \cdot \epsilon = \frac{1}{2}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2),$$

and

$$X_H \cdot \epsilon = \epsilon_4 \partial_x + \epsilon_5 \partial_y + \epsilon_6 \partial_z,$$

and hence

$$X_H^\epsilon = T \pi_Q \cdot X_H \cdot \epsilon = \epsilon_4 \partial_x + \epsilon_5 \partial_y + \epsilon_6 \partial_z.$$

Since $Im(\gamma) \subset \mathcal{M}$, then

$$T \gamma \cdot X_H^\epsilon = \epsilon_4 \partial_x + \epsilon_5 \partial_y + \sigma(y) \epsilon_4 \partial_z = \epsilon_4 (\partial_x + \sigma(y) \partial_z) + \epsilon_5 \partial_y = X_K \cdot \epsilon,$$

because $\epsilon_6 = \sigma(y) \epsilon_4$. In the same way, note that $\lambda = \gamma \cdot \pi_Q$, and $Im(\lambda) \subset \mathcal{M}$, then

$$T \lambda \cdot X_H \cdot \epsilon = \epsilon_4 \partial_x + \epsilon_5 \partial_y + \sigma(y) \epsilon_4 \partial_z = X_K \cdot \epsilon.$$  

On the other hand, since $\epsilon : T^* Q \to T^* Q$ is symplectic, we have that

$$T \gamma \cdot T \epsilon \cdot X_{H, \epsilon} = \tau_K \cdot T \epsilon \cdot X_{H, \epsilon} = \tau_K \cdot T \epsilon \cdot X_{H, \epsilon} = \tau_K \cdot T \epsilon \cdot X_{H, \epsilon}.$$

Thus, $T \gamma \cdot X_H^\epsilon = X_K \cdot \epsilon = T \lambda \cdot X_H \cdot \epsilon = \tau_K \cdot T \epsilon \cdot X_{H, \epsilon}$. In this case, we must have that $\epsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T \gamma \cdot X_H^\epsilon = X_K \cdot \epsilon$, for the distributional Hamiltonian system $(K, \omega_K, H)$, if and only if it is a solution of the equation $T \lambda \cdot X_H \cdot \epsilon = \tau_K \cdot T \epsilon \cdot X_{H, \epsilon}$.

Next, we consider the action of Lie group $G = \mathbb{R}^2$ on $\mathbb{R}^3$, and derive precisely the motion equations and the geometric constraint conditions of the reduced distributional two-form for the nonholonomic reduced dynamical vector field, that is, the Type I and Type II of Hamilton-Jacobi equations of the nonholonomic reduced distributional Hamiltonian system. At first, the action of Lie group $G = \mathbb{R}^2$ on $\mathbb{R}^3$ is given by

$$\Phi : G \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \Phi((r, s), (x, y, z)) = (x + r, y, z + s),$$

and we have the cotangent lifted $G$-action on $T^* \mathbb{R}^3$, such that the Hamiltonian $H : T^* \mathbb{R}^3 \to \mathbb{R}$ is $G$-invariant. Therefore,

$$\tilde{\mathcal{M}} = \{(y, p_x, p_y, p_z) \in T^* \mathbb{R}^3 | p_z = \sigma(y) p_x\},$$

and the reduced distribution is given by

$$\tilde{K} = \text{span}\{(1 + \sigma^2(y)) \partial_y - \sigma(y) \sigma'(y) p_x \partial_{p_x}, \partial_{p_y}\},$$

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and the non-degenerate and the reduced distributional two-form $\omega_{\overline{K}}$ is given by

$$\omega_{\overline{K}} = dx \wedge dp_x + dy \wedge dp_y + dz \wedge (p_x \sigma'(y)dy + \sigma(y)dp_x).$$

A direct computation yields

$$i_{(1+\sigma^2(y))\partial_y - \sigma(y)\sigma'(y)p_x \partial_{p_x}} \omega_{\overline{K}} = \sigma(y)\sigma'(y)p_x dx - \sigma'(y)p_x dz + (1 + \sigma^2(y))dp_y,$$

$$i_{\partial_{p_y}} \omega_{\overline{K}} = -dy$$

and

$$dh_{\overline{K}} = dH_{\overline{K}} = \sigma(y)\sigma'(y)p_x^2 dy + (1 + \sigma^2(y))p_x dp_x + p_y dp_y.$$ Assume that $X_{\overline{K}} = X_1((1 + \sigma^2(y))\partial_y - \sigma(y)\sigma'(y)p_x \partial_{p_x}) + X_2\partial_{p_y}$, then we have that

$$i_{X_{\overline{K}}} \omega_{\overline{K}} = X_1(\sigma(y)\sigma'(y)p_x dx - \sigma'(y)p_x dz + (1 + \sigma^2(y))dp_y) - X_2 dz$$

$$= (X_1\sigma(y)\sigma'(y)p_x dx + (-X_2)dy + (-X_1\sigma'(y)p_x)dz + (X_1(1 + \sigma^2(y)))dp_y).$$

From the nonholonomic reduced distributional Hamiltonian equation $i_{X_{\overline{K}}} \omega_{\overline{K}} = dh_{\overline{K}}$, we have that $X_1 = 0$, $X_2 = -\sigma(y)\sigma'(y)p_x^2$. Hence, we get that the nonholonomic reduced dynamical vector field is given by

$$X_{\overline{K}} = -\sigma(y)\sigma'(y)p_x^2 \partial_{p_y},$$

and the motion equations of the nonholonomic reduced distributional Hamiltonian system $(\overline{K}, \omega_{\overline{K}}, h)$ are expressed by

$$\dot{y} = 0, \quad \dot{p}_x = 0, \quad \dot{p}_y = -\sigma(y)\sigma'(y)p_x^2.$$

In the following we shall derive precisely the Type I and Type II of Hamilton-Jacobi equations of the nonholonomic reduced distributional Hamiltonian system $(\overline{K}, \omega_{\overline{K}}, h)$. Assume that $\gamma : \mathbb{R}^3 \to T^*\mathbb{R}^3$, and $\lambda = \gamma \cdot \pi_Q : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$, and $\text{Im}(\gamma) \subset K$, and it is $G$-invariant, $\text{Im}(T\gamma) \subset \overline{K}$, then we have that $\bar{\gamma} = \pi/G(\gamma) : \mathbb{R}^3 \to T^*\mathbb{R}^3/G$, $\bar{\gamma}(x, y, z) = (\bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)$, and $\bar{\lambda} = \pi/G(\lambda) : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3/G$, $\lambda(x, y, z, p_x, p_y, p_z) = (\bar{\lambda}_0, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$, that is, $\lambda_i = \bar{\gamma}_i \cdot \pi_Q$, $i = 0, \cdots, 3$, where $\bar{\lambda}_i$, $i = 0, \cdots, 3$, are functions on $T^*\mathbb{R}^3$, and $\bar{\gamma}_i$, $i = 0, \cdots, 3$, are functions on $\mathbb{R}^3$. Note that $h \cdot \pi_G = \tau_M \cdot H$, since $\text{Im}(\gamma) \subset \mathcal{M}$, and it is $G$-invariant, we have that $\text{Im}(\bar{\gamma}) \subset \mathcal{M}$, and $\bar{\gamma}_3 = \sigma(y)\bar{\gamma}_1$, and hence

$$h \cdot \bar{\gamma} = \frac{1}{2}((1 + \sigma^2(y))\bar{\gamma}_1^2 + \bar{\gamma}_2^2),$$

and

$$X_h \cdot \bar{\gamma} = \bar{\gamma}_2 \partial_y - \sigma(y)\sigma'(y)\bar{\gamma}_1^2 \partial_{p_y}.$$ When $d\gamma(\alpha, \beta) = 0$, that is, $\gamma$ is closed on $D$ with respect to $T\pi_Q : TT^*\mathbb{R}^3 \to T\mathbb{R}^3$, we have that

$$T\bar{\gamma} \cdot X_h = \tau_{\overline{K}} \cdot X_h \cdot \bar{\gamma} = -\sigma(y)\sigma'(y)\bar{\gamma}_1^2 \partial_{p_y} = X_{\overline{K}} \cdot \bar{\gamma},$$

that is, the Type I of Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system $(\overline{K}, \omega_{\overline{K}}, h)$ holds.

Now, for any $G$-invariant symplectic map $\varepsilon : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$, $\bar{\varepsilon} = \pi/G(\varepsilon) : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3/G$, is given by $\varepsilon(x, y, z, p_x, p_y, p_z) = (\bar{\varepsilon}_0, \bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3)$, then we have that

$$h \cdot \bar{\varepsilon} = \frac{1}{2}((1 + \sigma^2(y))\bar{\varepsilon}_1^2 + \bar{\varepsilon}_2^2),$$

and

$$X_h \cdot \bar{\varepsilon} = \bar{\varepsilon}_2 \partial_y - \sigma(y)\sigma'(y)\bar{\varepsilon}_1^2 \partial_{p_y}.$$
Because $\text{Im}(\tilde{\gamma}) \subset \tilde{\mathcal{M}}$, and $\text{Im}(T\tilde{\gamma}) \subset \tilde{\mathcal{K}}$, and hence

$$T\tilde{\gamma} \cdot X_H = \tau_{\tilde{\mathcal{K}}} \cdot X_h \cdot \tilde{\varepsilon} = -\sigma(y)\sigma'(y)\tilde{\varepsilon}^2\partial_{p_y} = X_{\tilde{\mathcal{K}}} \cdot \tilde{\varepsilon}. $$

Note that $\tilde{\lambda} = \tilde{\gamma} \cdot \pi_Q$, and $\text{Im}(\tilde{\lambda}) \subset \tilde{\mathcal{M}}$, and $\text{Im}(T\tilde{\lambda}) \subset \tilde{\mathcal{K}}$, then we have that

$$T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_{\tilde{\mathcal{K}}} \cdot X_h \cdot \tilde{\varepsilon} = X_{\tilde{\mathcal{K}}} \cdot \tilde{\varepsilon}. $$

On the other hand, since $\varepsilon : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3$ is symplectic, and $\tilde{\varepsilon}^* = \varepsilon^* \cdot \pi^*_G : T^*(T^*\mathbb{R}^3)/G \to T^*T^*\mathbb{R}^3$ is also symplectic along $\tilde{\varepsilon}$, then we have that

$$\tau_{\tilde{\mathcal{K}}} \cdot T\tilde{\varepsilon} \cdot X_{h \cdot \varepsilon} = \tau_{\tilde{\mathcal{K}}} \cdot X_h \cdot \tilde{\varepsilon} = -\sigma(y)\sigma'(y)\tilde{\varepsilon}^2\partial_{p_y} = X_{\tilde{\mathcal{K}}} \cdot \tilde{\varepsilon}. $$

Thus, $T\tilde{\gamma} \cdot X_H^* = X_{\tilde{\mathcal{K}}} \cdot \tilde{\varepsilon} = T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_{\tilde{\mathcal{K}}} \cdot T\tilde{\varepsilon} \cdot X_{h \cdot \varepsilon}$. In this case, we must have that $\varepsilon$ and $\tilde{\varepsilon}$ are the solution of the Type II of Hamilton-Jacobi equation $T\tilde{\gamma} \cdot X_H^* = X_{\tilde{\mathcal{K}}} \cdot \tilde{\varepsilon}$, for the nonholonomic reduced distributional Hamiltonian system $(\tilde{\mathcal{K}}, \omega_{\tilde{\mathcal{K}}}, \tilde{\varepsilon})$, if and only if they satisfy the equation $T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_{\tilde{\mathcal{K}}} \cdot T\tilde{\varepsilon} \cdot X_{h \cdot \varepsilon}$.

### 6.2 The vertical rolling disk

In this subsection, we consider the motion of a vertical rolling disk, and give explicitly the motion equations of this problem and derive precisely the geometric constraint conditions of the induced distributional two-forms for the nonholonomic dynamical vector fields, that is, the Type I and Type II of Hamilton-Jacobi equations of this problem. Assume that a vertical disk of zero width rolls without slipping on a horizontal plane and it rotates freely about its vertical axis. Let $x$ and $y$ denote the position of contact point of the disk in the plane, and the variables $\theta$ and $\varphi$ denote the orientations of a chosen material point with respect to the vertical plane and the "heading angle" of the disk, see [4]. Thus, the configuration space of motion for the vertical rolling disk is $Q = \mathbb{R}^2 \times S^1 \times S^1$ whose coordinates are denoted by $q = (x, y, \theta, \varphi)$, and its velocity space is $TQ$, and the phase space is $T^*Q$ with canonical symplectic form

$$\omega = dx \wedge dp_x + dy \wedge dp_y + d\theta \wedge dp_\theta + d\varphi \wedge dp_\varphi. $$

The rolling constraint set on the velocities is given by

$$\mathcal{D} = \{(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}) \in TQ | \dot{x} = R\dot{\theta} \cos \varphi, \dot{y} = R\dot{\theta} \sin \varphi\},$$

where $R$ denotes the radius of the disk. For any $q \in Q$, we have that

$$\mathcal{D}(q) = \text{Span}\{R \cos \varphi \partial_x + R \sin \varphi \partial_y + \partial_\theta, \partial_\varphi\}. $$

Note that

$$[R \cos \varphi \partial_x + R \sin \varphi \partial_y + \partial_\theta, \partial_\varphi] = [R \cos \varphi \partial_x, \partial_\varphi] + [R \sin \varphi \partial_y, \partial_\varphi] + [\partial_\theta, \partial_\varphi] = -R \sin \varphi \partial_x + R \cos \varphi \partial_y,$$

which is nonzero everywhere and it is not in $\mathcal{D}$, then $\mathcal{D}$ is nonholonomic and it is completely nonholonomic, that is, $\mathcal{D}$ along with all of its iterated Lie brackets $[\mathcal{D}, \mathcal{D}], [\mathcal{D}, [\mathcal{D}, \mathcal{D}]], \cdots$ spans the tangent bundle $TQ$. The Lagrangian $L : TQ \to \mathbb{R}$ is the kinetic energy, that is,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2,$$
where \( m \) is the mass of the disk, and \( I \) and \( J \) are its moments of inertia. Note that \( L \) is simple and it is hyperregular, and hence the system is \( \mathcal{D} \)-regular automatically. The momenta are \( p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \ p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}, \ p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = J\dot{\varphi}, \) and the Hamiltonian \( H : T^*Q \to \mathbb{R} \) is given by

\[
H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2I}p_\theta^2 + \frac{1}{2J}p_\varphi^2.
\]

The unconstrained Hamiltonian vector field is given by

\[
X_H = \frac{1}{m} p_x \partial_x + \frac{1}{m} p_y \partial_y + \frac{1}{I} p_\theta \partial_\theta + \frac{1}{J} p_\varphi \partial_\varphi.
\]

By using the Legendre transformation

\[
\mathcal{F}L : TQ \to T^*Q, \quad \mathcal{F}L(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = (x, y, \theta, \phi, p_x, p_y, p_\theta, p_\varphi),
\]

we obtain the constraint submanifold \( \mathcal{M} = \mathcal{F}L(\mathcal{D}) \) given by

\[
\mathcal{M} = \{(x, y, \theta, \phi, p_x, p_y, p_\theta, p_\varphi) \in T^*Q | p_x = \frac{mR}{I} p_\theta \cos \phi, \ p_y = \frac{mR}{I} p_\theta \sin \phi \}.
\]

Moreover, if we define \( \mathcal{F} = (T\pi_Q)^{-1}(\mathcal{D}) \), then the compatibility condition \( T\mathcal{M} \cap \mathcal{F}^\perp = \{0\} \) holds, where \( \mathcal{F}^\perp \) denotes the symplectic orthogonal of \( \mathcal{F} \) with respect to the canonical symplectic form \( \omega \). Thus, the distribution is given by

\[
\mathcal{K} = \mathcal{F} \cap T\mathcal{M} = \text{span}\{\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y, \partial_x, \partial_y, \partial_\phi, \partial_\varphi\}.
\]

The induced two-form \( \omega_\mathcal{M} = i_{\mathcal{M}}^* \omega \) is given by

\[
\omega_\mathcal{M} = dx \wedge \left( \frac{mR \cos \phi}{I} dp_\theta - \frac{mR \sin \phi}{I} p_\theta d\phi \right) + dy \wedge \left( \frac{mR \sin \phi}{I} dp_\theta + \frac{mR \cos \phi}{I} p_\theta d\phi \right) + d\theta \wedge dp_\theta + d\phi \wedge dp_\varphi,
\]

and hence we have the non-degenerate distributional two-form \( \omega_\mathcal{K} = \tau_\mathcal{K}^* \omega_\mathcal{M} \). A direct computation yields

\[
\begin{align*}
\mathbf{i}_{\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y} \omega_\mathcal{K} &= (1 + \frac{mR^2}{I}) dp_\theta, \\
\mathbf{i}_{\partial_x} \omega_\mathcal{K} &= \frac{mR \sin \phi}{I} dp_\theta dx - \frac{mR \cos \phi}{I} p_\theta dy, \\
\mathbf{i}_{\partial_y} \omega_\mathcal{K} &= -\frac{mR \cos \phi}{I} dx - \frac{mR \sin \phi}{I} dy - d\theta, \\
\mathbf{i}_{\partial_\phi} \omega_\mathcal{K} &= -d\varphi,
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{d}H_\mathcal{K} &= \frac{1}{m} \frac{mR}{I} p_\theta \cos \phi \left( \frac{mR \cos \phi}{I} dp_\theta - \frac{mR \sin \phi}{I} p_\theta d\phi \right) \\
&\quad + \frac{1}{m} \frac{mR}{I} p_\theta \sin \phi \left( \frac{mR \sin \phi}{I} dp_\theta + \frac{mR \cos \phi}{I} p_\theta d\phi \right) + \frac{1}{I} p_\theta dp_\theta + \frac{1}{J} p_\varphi dp_\varphi \\
&= \frac{1}{I} (1 + \frac{mR^2}{I}) p_\theta dp_\theta + \frac{1}{J} p_\varphi dp_\varphi.
\end{align*}
\]

Assume that \( X_\mathcal{K} = X_1(\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y) + X_2 \partial_\varphi + X_3 \partial_\theta + X_4 \partial_\varphi \), then

\[
\begin{align*}
\mathbf{i}_{X_\mathcal{K}} \omega_\mathcal{K} &= X_1((1 + \frac{mR^2}{I}) dp_\theta) + X_2(dp_\varphi + \frac{mR \sin \phi}{I} p_\theta dx - \frac{mR \cos \phi}{I} p_\theta dy) \\
&\quad + X_3(-\frac{mR \cos \phi}{I} dx - \frac{mR \sin \phi}{I} dy - d\theta) + X_4(-d\varphi) \\
&= (X_2 \frac{mR \sin \phi}{I} p_\theta - X_3 \frac{mR \cos \phi}{I} dx + (X_2 \frac{mR \cos \phi}{I} p_\theta - X_3 \frac{mR \sin \phi}{I} dy) \\
&\quad + (X_3 \frac{mR \cos \phi}{I} p_\theta - X_4 \frac{mR \sin \phi}{I} dx) + (X_4 \frac{mR \cos \phi}{I} p_\theta - X_3 \frac{mR \sin \phi}{I} dy) + (X_1(1 + \frac{mR^2}{I}) dp_\theta + (X_2 dp_\varphi).
\end{align*}
\]
From the distributional Hamiltonian equation $i_{X_K} \omega_K = dH$, we have that

$$X_1 = \frac{1}{J} p_\theta, \quad X_2 = \frac{1}{J} p_\varphi, \quad X_3 = 0, \quad X_4 = 0.$$ 

Hence, we get that the nonholonomic dynamical vector field is given by

$$X_K = \frac{1}{J} p_\theta (\partial_\theta + R \cos \varphi \partial_x + R \sin \varphi \partial_y) + \frac{1}{J} p_\varphi \partial_\varphi,$$ 

and the motion equations of the distributional Hamiltonian system $(\mathcal{K}, \omega_K, H)$ are given by

$$\dot{x} = \frac{R \cos \varphi}{J} p_\theta, \quad \dot{y} = \frac{R \sin \varphi}{J} p_\theta, \quad \dot{\theta} = \frac{1}{J} p_\theta, \quad \dot{\varphi} = \frac{1}{J} p_\varphi, \quad \dot{p}_\theta = 0, \quad \dot{p}_\varphi = 0.$$

In the following we shall derive precisely the geometric constraint conditions of the induced distributional two-forms for the nonholonomic dynamical vector fields, that is, the Type I and Type II of Hamilton-Jacobi equations for the distributional Hamiltonian system $(\mathcal{K}, \omega_K, H)$. Assume that

$$\gamma : Q \to T^* Q, \quad \gamma(x, y, \theta, \varphi) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8),$$

then $\lambda = \gamma \cdot \pi_Q : T^* Q \to T^* Q$ given by

$$\lambda(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$$

$$= \gamma \cdot \pi_Q(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi) = \gamma(x, y, \theta, \varphi)$$

$$= (\gamma_1 \cdot \pi_Q, \gamma_2 \cdot \pi_Q, \gamma_3 \cdot \pi_Q, \gamma_4 \cdot \pi_Q, \gamma_5 \cdot \pi_Q, \gamma_6 \cdot \pi_Q, \gamma_7 \cdot \pi_Q, \gamma_8 \cdot \pi_Q),$$

that is, $\lambda_i = \gamma_i \cdot \pi_Q, \ i = 1, \ldots, 8$, where $\lambda_i, \ i = 1, \ldots, 8$, are functions on $T^* Q$, and $\gamma_i, \ i = 1, \ldots, 8$, are functions on $Q$. We may choose $q = (x, y, \theta, \varphi) \in Q$, such that $\gamma_1(q) = x$, $\gamma_2(q) = y$, $\gamma_3(q) = \theta$, $\gamma_4(q) = \varphi$, and $\gamma(q) = \gamma_5(q) dx + \gamma_6(q) dy + \gamma_7(q) d\theta + \gamma_8(q) d\varphi$. Note that $D(q) = \text{Span} \{R \cos \varphi \partial_x + R \sin \varphi \partial_y + \partial_\theta, \partial_\varphi\}$, take that $\alpha = R \cos \varphi \partial_x + R \sin \varphi \partial_y + \partial_\theta$ and $\beta = \partial_\varphi$, then we have that

$$d\gamma(\alpha, \beta) = \alpha(\gamma(\beta)) - \beta(\gamma(\alpha)) - \gamma([\alpha, \beta])$$

$$= R \cos \varphi \left(\frac{\partial \gamma_8}{\partial x} - \frac{\partial \gamma_5}{\partial \varphi}\right) + R \sin \varphi \left(\frac{\partial \gamma_8}{\partial y} - \frac{\partial \gamma_6}{\partial \varphi}\right)$$

$$+ \left(\frac{\partial \gamma_8}{\partial \theta} - \frac{\partial \gamma_7}{\partial \varphi}\right) + 2(R \sin \varphi \gamma_7 - R \cos \varphi \gamma_6).$$

Thus, when $d\gamma(\alpha, \beta) = 0$, we know that for any $v, w \in F$, and $T\pi_Q(v), T\pi_Q(w) \in D$, then $d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0$, that is, $\gamma$ is closed on $D$ with respect to $T\pi_Q : TT^* Q \to TQ$. Note that $\text{Im}(\gamma) \subseteq \mathcal{M}$, then we have that $p_x = \gamma_5, \ p_y = \gamma_6, \ p_\theta = \gamma_7, \ p_\varphi = \gamma_8$, and $\gamma_5 = \frac{mR}{I} \gamma_7 \cos \varphi, \ \gamma_6 = \frac{mR}{I} \gamma_7 \sin \varphi$. Hence,

$$H \cdot \gamma = \frac{1}{2m} \left(\frac{m^2 R^2}{I^2} \gamma_7^2 \right) + \frac{1}{2I} \gamma_7^2 + \frac{1}{2J} \gamma_8^2 = \frac{1}{2I} \left(1 + \frac{mR^2}{I}\right) \gamma_7^2 + \frac{1}{2J} \gamma_8^2,$$

and

$$X_H \cdot \gamma = \frac{R}{I} \gamma_7 \cos \varphi \partial_x + \frac{R}{I} \gamma_7 \sin \varphi \partial_y + \frac{1}{I} \gamma_7 \partial_\theta + \frac{1}{J} \gamma_8 \partial_\varphi$$

$$= \frac{\gamma_7}{I}(R \cos \varphi \partial_x + R \sin \varphi \partial_y + \partial_\theta) + \frac{\gamma_8}{J} \partial_\varphi = X_K \cdot \gamma,$$

$$X_H^\gamma = T\pi_Q \cdot X_H \cdot \gamma = \frac{\gamma_7}{I}(R \cos \varphi \partial_x + R \sin \varphi \partial_y + \partial_\theta) + \frac{\gamma_8}{J} \partial_\varphi.$$
Thus, \( T\gamma \cdot X_H^\gamma = X_K \cdot \gamma \), that is, the Type I of Hamilton-Jacobi equation for the distributional Hamiltonian system \((K, \omega_K, H)\) holds trivially.

Now, for any symplectic map \( \varepsilon : T^*Q \to T^*Q \), from \( \omega = \varepsilon^*\omega = \omega = (\partial_x \varepsilon \cdot \partial_{p_x} \varepsilon) dx \wedge dp_x + (\partial_y \varepsilon \cdot \partial_{p_y} \varepsilon) dy \wedge dp_y + (\partial_\theta \varepsilon \cdot \partial_{p_\theta} \varepsilon) d\theta \wedge dp_\theta + (\partial_\phi \varepsilon \cdot \partial_{p_\phi} \varepsilon) d\phi \wedge dp_\phi \), we have that
\[
\partial_x \varepsilon \cdot \partial_{p_x} \varepsilon = 1, \quad \partial_y \varepsilon \cdot \partial_{p_y} \varepsilon = 1, \quad \partial_\theta \varepsilon \cdot \partial_{p_\theta} \varepsilon = 1, \quad \partial_\phi \varepsilon \cdot \partial_{p_\phi} \varepsilon = 1.
\]
Denote by \( \varepsilon(x, y, \theta, \phi, p_x, p_y, p_\theta, p_\phi) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8) \), then we have that
\[
H \cdot \varepsilon = \frac{1}{2m} (\varepsilon_5^2 + \varepsilon_6^2) + \frac{1}{2I} \varepsilon_7^2 + \frac{1}{2J} \varepsilon_8^2,
\]
and
\[
X_H \cdot \varepsilon = \frac{1}{m} \varepsilon_5 \partial_x + \frac{1}{m} \varepsilon_6 \partial_y + \frac{1}{I} \varepsilon_7 \partial_\theta + \frac{1}{J} \varepsilon_8 \partial_\phi,
\]
and hence
\[
X_H^\varepsilon = T\pi_Q \cdot X_H \cdot \varepsilon = \frac{1}{m} \varepsilon_5 \partial_x + \frac{1}{m} \varepsilon_6 \partial_y + \frac{1}{I} \varepsilon_7 \partial_\theta + \frac{1}{J} \varepsilon_8 \partial_\phi.
\]
Since \( \text{Im}(\gamma) \subset M \), then
\[
T\gamma \cdot X_H^\gamma = T\gamma \cdot (\frac{1}{m} \varepsilon_5 \partial_x + \frac{1}{m} \varepsilon_6 \partial_y + \frac{1}{I} \varepsilon_7 \partial_\theta + \frac{1}{J} \varepsilon_8 \partial_\phi)
= \frac{R}{I} \varepsilon_7 \cos \phi \partial_x + \frac{R}{I} \varepsilon_7 \sin \phi \partial_y + \frac{1}{I} \varepsilon_7 \partial_\theta + \frac{1}{J} \varepsilon_8 \partial_\phi
= \frac{\varepsilon_7}{I} (R \cos \phi \partial_x + R \sin \phi \partial_y + \partial_\theta) + \frac{\varepsilon_8}{J} \partial_\phi
= X_K \cdot \varepsilon,
\]
because \( \varepsilon_5 = \frac{mR}{I} \varepsilon_7 \cos \phi, \varepsilon_6 = \frac{mR}{I} \varepsilon_7 \sin \phi \). In the same way, note that \( \lambda = \gamma \cdot \pi_Q \), and \( \text{Im}(\lambda) \subset M \), then
\[
T\lambda \cdot X_H \cdot \varepsilon = \frac{R}{I} \varepsilon_7 \cos \phi \partial_x + \frac{R}{I} \varepsilon_7 \sin \phi \partial_y + \frac{1}{I} \varepsilon_7 \partial_\theta + \frac{1}{J} \varepsilon_8 \partial_\phi
= \frac{\varepsilon_7}{I} (R \cos \phi \partial_x + R \sin \phi \partial_y + \partial_\theta) + \frac{\varepsilon_8}{J} \partial_\phi = X_K \cdot \varepsilon.
\]
On the other hand, since \( \varepsilon : T^*Q \to T^*Q \) is symplectic, we have that
\[
\tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon = \tau_K \cdot X_H \cdot \varepsilon
= \frac{R}{I} \varepsilon_7 \cos \phi \partial_x + \frac{R}{I} \varepsilon_7 \sin \phi \partial_y + \frac{1}{I} \varepsilon_7 \partial_\theta + \frac{1}{J} \varepsilon_8 \partial_\phi
= \frac{\varepsilon_7}{I} (R \cos \phi \partial_x + R \sin \phi \partial_y + \partial_\theta) + \frac{\varepsilon_8}{J} \partial_\phi = X_K \cdot \varepsilon.
\]
Thus, \( T\gamma \cdot X_H^\gamma = X_K \cdot \varepsilon = T\lambda \cdot X_H \cdot \varepsilon = \tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon \). In this case, we must have that \( \varepsilon \) is a solution of the Type II of Hamilton-Jacobi equation \( T\gamma \cdot X_H^\gamma = X_K \cdot \varepsilon \), for the distributional Hamiltonian system \((K, \omega_K, H)\), if and only if it is a solution of the equation \( T\lambda \cdot X_H \cdot \varepsilon = \tau_K \cdot T\varepsilon \cdot X_H \cdot \varepsilon \).

In the following we consider respectively the actions of two Lie groups \( G = \mathbb{R}^2 \) and \( G = SE(2) \) on \( Q \), and derive precisely the motion equations and the geometric constraint conditions of the reduced distributional two-forms for the nonholonomic reduced dynamical vector fields, that is, the Type I and Type II of Hamilton-Jacobi equations of the nonholonomic reduced distributional Hamiltonian systems. Firstly, we consider the action of Lie group \( G = \mathbb{R}^2 \) on \( Q \), which is given by
\[
\Phi : G \times Q \to Q, \; \Phi((r, s), (x, y, \theta, \phi)) = (x + r, y + s, \theta, \phi),
\]
and we have the cotangent lifted $G$-action on $T^*Q$, and the Hamiltonian $H : T^*Q \to \mathbb{R}$ is $G$-invariant. In this case we have that
\[
\mathcal{M} = \{ (\theta, \varphi, p_x, p_y, p_\theta, p_\varphi) \in T^*Q/G \mid p_x = \frac{mR}{I} p_y \cos \varphi, \ p_y = \frac{mR}{I} p_\theta \sin \varphi \},
\]
and the reduced distribution is given by
\[
\mathcal{K} = \text{span}\{ \partial_\theta, \partial_\varphi, \partial p_\theta, \partial p_\varphi \},
\]
and the non-degenerate and the reduced distributional two-form $\omega_\mathcal{K}$ is given by
\[
\omega_\mathcal{K} = (1 + \frac{mR^2}{I})d\theta \wedge dp_\theta + d\varphi \wedge dp_\varphi.
\]
A direct computation yields
\[
\iota_{\partial_\theta} \omega_\mathcal{K} = (1 + \frac{mR^2}{I})dp_\theta, \quad \iota_{\partial_\varphi} \omega_\mathcal{K} = dp_\varphi, \quad \iota_{\partial p_\theta} \omega_\mathcal{K} = -(1 + \frac{mR^2}{I})d\theta, \quad \iota_{\partial p_\varphi} \omega_\mathcal{K} = -d\varphi,
\]
and
\[
dh_\mathcal{K} = dH_\mathcal{K} = \frac{1}{I} (1 + \frac{mR^2}{I}) p_\theta dp_\theta + \frac{1}{J} p_\varphi dp_\varphi.
\]
Assume that $X_\mathcal{K} = X_1 \partial_\theta + X_2 \partial_\varphi + X_3 \partial p_\theta + X_4 \partial p_\varphi$, then
\[
\iota_{X_\mathcal{K}} \omega_\mathcal{K} = X_1 ((1 + \frac{mR^2}{I}) dp_\theta) + X_2 dp_\varphi - X_3 (1 + \frac{mR^2}{I}) d\theta - X_4 d\varphi
\]
\[
= -X_3 (1 + \frac{mR^2}{I}) d\theta - X_4 d\varphi + X_1 (1 + \frac{mR^2}{I}) dp_\theta + X_2 dp_\varphi.
\]
From the nonholonomic reduced distributional Hamiltonian equation $\iota_{X_\mathcal{K}} \omega_\mathcal{K} = dh_\mathcal{K}$, we have that
\[
X_1 = \frac{1}{I} p_\theta, \quad X_2 = \frac{1}{J} p_\varphi, \quad X_3 = 0, \quad X_4 = 0.
\]
Hence, the nonholonomic reduced dynamical vector field is given by
\[
X_\mathcal{K} = \frac{1}{I} p_\theta \partial_\theta + \frac{1}{J} p_\varphi \partial_\varphi,
\]
and the motion equations of the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K}, \omega_\mathcal{K}, h)$ are expressed by
\[
\dot{\theta} = \frac{1}{I} p_\theta, \quad \dot{\varphi} = \frac{1}{J} p_\varphi, \quad \dot{p}_\theta = 0, \quad \dot{p}_\varphi = 0.
\]
In the following we shall derive precisely the Type I and Type II of Hamilton-Jacobi equations for the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K}, \omega_\mathcal{K}, h)$. As above $\gamma : Q \to T^*Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and assume that $\text{Im}(\gamma) \subset \mathcal{M}$, and it is $G$-invariant, $\text{Im}(\gamma \cdot \pi_Q) \subset \mathcal{K}$, then we have that $\tilde{\gamma} = \pi_{/G}(\gamma) : Q \to T^*Q/G$, $\tilde{\gamma}(x, y, \theta, \varphi) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6)$, and $\lambda = \pi_{/G}(\lambda) : T^*Q \to T^*Q/G$, $\lambda(x, y, \theta, \varphi, p_\theta, p_\varphi, p_\theta, p_\varphi) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6)$, that is, $\lambda_i = \tilde{\gamma}_i \cdot \pi_Q$, $i = 1, \cdots, 6$, where $\tilde{\gamma}_i$, $i = 1, \cdots, 6$, are functions on $T^*Q$, and $\tilde{x}_i$, $i = 1, \cdots, 6$, are functions on $Q$.

Note that $h \cdot \pi_{/G} = \tau_M \cdot H$, and the reduced distribution $\mathcal{K} = \text{span}\{ \partial_\theta, \partial_\varphi, \partial p_\theta, \partial p_\varphi \}$, then we have that
\[
h \cdot \tilde{\gamma} = \frac{1}{I} \tilde{\gamma}_1^2 + \frac{1}{2J} \tilde{\gamma}_2^2, \quad X_h \cdot \tilde{\gamma} = \frac{1}{I} \tilde{\gamma}_5 \partial_\theta + \frac{1}{J} \tilde{\gamma}_6 \partial_\varphi.
\]
When $\mathbf{d}\gamma(\alpha, \beta) = 0$, that is, $\gamma$ is closed on $\mathcal{D}$ with respect to $T \pi_Q : TT^*Q \to TQ$, we have that
\[
T\tilde{\gamma} \cdot X_h^\mathcal{K} = \tilde{\gamma} \cdot X_h \cdot \tilde{\gamma} = X_h \cdot \tilde{\gamma},
\]
that is, the Type I of Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}, h)\) holds.

Now, for any \(G\)-invariant symplectic map \(\varepsilon : T^*Q \to T^*Q\), \(\varepsilon = \pi/G(\varepsilon) : T^*Q \to T^*Q/G\), is given by \(\tilde{\varepsilon}(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi) = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \tilde{\varepsilon}_3, \tilde{\varepsilon}_4, \tilde{\varepsilon}_5, \tilde{\varepsilon}_6)\), then we have that

\[
h \cdot \tilde{\varepsilon} = \frac{1}{2I}\tilde{\varepsilon}_5^2 + \frac{1}{2J}\tilde{\varepsilon}_6^2, \quad X_h \cdot \tilde{\varepsilon} = \frac{1}{I}\tilde{\varepsilon}_5\partial_\theta + \frac{1}{J}\tilde{\varepsilon}_6\partial_\varphi.
\]

Since \(\text{Im}(\gamma) \subset \mathcal{M}\), and \(\text{Im}(T\gamma) \subset \mathcal{K}\), and hence \(\text{Im}(\tilde{\gamma}) \subset \mathcal{M}\), \(\text{Im}(T\tilde{\gamma}) \subset \mathcal{K}\). Thus,

\[
T\tilde{\gamma} \cdot X^\varepsilon_H = \tau_\mathcal{K} \cdot X_h \cdot \tilde{\varepsilon} = \frac{1}{I}\tilde{\varepsilon}_5\partial_\theta + \frac{1}{J}\tilde{\varepsilon}_6\partial_\varphi = X_\mathcal{K} \cdot \tilde{\varepsilon}.
\]

Note that \(\tilde{\lambda} = \tilde{\gamma} \cdot \pi_Q\), and \(\text{Im}(\tilde{\lambda}) \subset \mathcal{M}\), \(\text{Im}(T\tilde{\lambda}) \subset \mathcal{K}\), then we have that

\[
T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_\mathcal{K} \cdot X_h \cdot \tilde{\varepsilon} = X_\mathcal{K} \cdot \tilde{\varepsilon}.
\]

On the other hand, since \(\varepsilon : T^*Q \to T^*Q\) is symplectic, and \(\varepsilon^* = \varepsilon \cdot \pi^*/G : T^*(T^*Q)/G \to T^*T^*Q\) is also symplectic along \(\tilde{\varepsilon}\), then we have that

\[
\tau_\mathcal{K} \cdot T\tilde{\varepsilon} \cdot X_h \cdot \tilde{\varepsilon} = \tau_\mathcal{K} \cdot X_h \cdot \varepsilon
\]

Thus, \(T\tilde{\gamma} \cdot X^\varepsilon_H = X_\mathcal{K} \cdot \tilde{\varepsilon} = T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_\mathcal{K} \cdot T\tilde{\varepsilon} \cdot X_h \cdot \tilde{\varepsilon}\). In this case, we must have that \(\varepsilon\) and \(\tilde{\varepsilon}\) are the solution of the Type II of Hamilton-Jacobi equation \(T\tilde{\gamma} \cdot X^\varepsilon_H = X_\mathcal{K} \cdot \tilde{\varepsilon}\), for the nonholonomic reduced distributional Hamiltonian system \((\mathcal{K}, \omega_{\mathcal{K}}), h)\), if and only if they satisfy the equation \(T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_\mathcal{K} \cdot T\tilde{\varepsilon} \cdot X_h \cdot \tilde{\varepsilon}\).

Next, we consider the action of the Lie group \(G = SE(2) \cong SO(2) \oplus \mathbb{R}^2\) on \(Q\), which is given by

\[
\Phi : G \times Q \to Q, \Phi((\alpha, r, s), (x, y, \theta, \varphi)) = (x \cos \alpha - y \sin \alpha + r, x \sin \alpha + y \cos \alpha + s, \theta, \varphi + \alpha),
\]

and we have the cotangent lifted \(G\)-action on \(T^*Q\); and the Hamiltonian \(H : T^*Q \to \mathbb{R}\) is \(G\)-invariant. In this case we have that

\[
\tilde{\mathcal{M}} = \{(\theta, p_x, p_y, p_\theta) \in T^*Q/G | p_x = \frac{mR}{I}p_\theta \cos \varphi, p_y = \frac{mR}{I}p_\theta \sin \varphi\},
\]

and the reduced distribution is given by \(\tilde{\mathcal{K}} = \text{span}\{\partial_\theta, \partial_{p_\theta}\}\), and the non-degenerate and the reduced distributional two-form \(\omega_{\mathcal{K}}\) is given by

\[
\omega_{\mathcal{K}} = (1 + \frac{mR^2}{I})d\theta \wedge dp_\theta.
\]

A direct computation yields

\[
i_{\partial_\theta} \omega_{\mathcal{K}} = (1 + \frac{mR^2}{I})dp_\theta, \quad i_{\partial_{p_\theta}} \omega_{\mathcal{K}} = -(1 + \frac{mR^2}{I})d\theta,
\]

and

\[
dh_{\mathcal{K}} = dH_{\mathcal{K}} = \frac{1}{I}(1 + \frac{mR^2}{I})p_\theta dp_\theta + \frac{1}{J}p_\varphi dp_\varphi.
\]

Assume that \(X_{\mathcal{K}} = X_1 \partial_\theta + X_2 \partial_{p_\theta}\), then we have that

\[
i_{X_{\mathcal{K}}} \omega_{\mathcal{K}} = X_1((1 + \frac{mR^2}{I})dp_\theta) + X_2(-(1 + \frac{mR^2}{I})d\theta).
\]
From the nonholonomic reduced distributional Hamiltonian equation $i_{X_{\mathcal{K}}\omega_{\mathcal{K}}} = \mathcal{D}h_{\mathcal{K}}$, we have that $X_1 = \frac{1}{I}p_\theta$, $X_2 = 0$. Hence, the nonholonomic reduced dynamical vector field is given by

$$X_{\mathcal{K}} = \frac{1}{I} p_\theta \partial_\theta,$$

and the motion equations of the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, h)$ are expressed by

$$\dot{\theta} = \frac{1}{I} p_\theta, \quad \dot{p}_\theta = 0.$$

In the following we shall derive precisely the Type I and Type II of Hamilton-Jacobi equations for the nonholonomic reduced distributional Hamiltonian system $(\mathcal{K}, \omega_{\mathcal{K}}, h)$. As above $\gamma : Q \to T^*Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and assume that $\text{Im}(\gamma) \subset \mathcal{M}$, and it is $G$-invariant, $\text{Im}(T\gamma) \subset \mathcal{K}$, then we have that $\tilde{\gamma} = \pi_G(\gamma) : Q \to T^*Q/G$, $\bar{\gamma}(x, y, \theta, \varphi) = (\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4)$, and $\bar{\lambda} = \pi_{G}(\lambda) : T^*Q \to T^*Q/G$, $\bar{\lambda}(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi) = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \bar{\lambda}_4)$, that is, $\bar{\lambda}_i = \bar{\gamma}_i \cdot \pi_Q$, $i = 1, \ldots, 4$, are functions on $T^*Q$, and $\bar{\gamma}_i$, $i = 1, \ldots, 4$, are functions on $Q$. Note that $h \cdot \pi_G = \tau_M \cdot H$, and the reduced distribution is given by $\check{K} = \text{span}\{\partial_\theta, \partial_{p_\theta}\}$, then we have that

$$h \cdot \bar{\gamma} = \frac{1}{2(I^2 \gamma_3)}, \quad X_h \cdot \bar{\gamma} = \frac{1}{I} \gamma_3 \partial_\theta.$$

When $\mathcal{D} \gamma(\alpha, \beta) = 0$, that is, $\gamma$ is closed on $D$ with respect to $T\pi_Q : TT^*Q \to TQ$, we have that

$$T\bar{\gamma} \cdot X_H^\epsilon = \tau_{\check{K}} \cdot X_h \cdot \bar{\gamma} = X_{\mathcal{K}} \cdot \bar{\gamma},$$

that is, the Type I of Hamilton-Jacobi equation for the nonholonomic reduced distributional Hamiltonian system $(\check{K}, \omega_{\check{K}}, h)$ holds.

Now, for any $G$-invariant symplectic map $\varepsilon : T^*Q \to T^*Q$, $\bar{\varepsilon} = \pi_G(\varepsilon) : T^*Q \to T^*Q/G$, is given by $\bar{\varepsilon}(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi) = (\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3, \bar{\varepsilon}_4)$, then we have that

$$h \cdot \bar{\varepsilon} = \frac{1}{2(I^2 \varepsilon_3)}, \quad X_h \cdot \bar{\varepsilon} = \frac{1}{I} \varepsilon_3 \partial_\theta.$$

Since $\text{Im}(\gamma) \subset \mathcal{M}$, and $\text{Im}(T\gamma) \subset \mathcal{K}$, and hence $\text{Im}(\gamma) \subset \check{M}$, $\text{Im}(T\gamma) \subset \check{K}$. Thus,

$$T\bar{\gamma} \cdot X_H^\varepsilon = \tau_{\check{K}} \cdot X_h \cdot \bar{\varepsilon} = \frac{1}{I} \varepsilon_3 \partial_\theta = X_{\mathcal{K}} \cdot \bar{\varepsilon}.$$

Note that $\tilde{\lambda} = \bar{\gamma} \cdot \pi_Q$, and $\text{Im}(\tilde{\lambda}) \subset \check{M}$, and $\text{Im}(T\tilde{\lambda}) \subset \check{K}$, then we have that

$$T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_{\check{K}} \cdot X_h \cdot \bar{\varepsilon} = X_{\mathcal{K}} \cdot \bar{\varepsilon}.$$

On the other hand, since $\varepsilon : T^*Q \to T^*Q$ is symplectic, and $\varepsilon^* = \varepsilon^* \cdot \pi_G : T^*(T^*Q)/G \to T^*T^*Q$ is also symplectic along $\bar{\varepsilon}$, then we have that

$$\tau_{\check{K}} \cdot T\bar{\varepsilon} \cdot X_{h, \varepsilon} = \tau_{\check{K}} \cdot X_h \cdot \bar{\varepsilon} = \frac{1}{I} \varepsilon_3 \partial_\theta = X_{\mathcal{K}} \cdot \bar{\varepsilon}.$$

Thus, $T\tilde{\lambda} \cdot X_H^\varepsilon = X_{\mathcal{K}} \cdot \bar{\varepsilon} = T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_{\check{K}} \cdot T\bar{\varepsilon} \cdot X_{h, \varepsilon}$. In this case, we must have that $\varepsilon$ and $\bar{\varepsilon}$ are the solution of the Type II of Hamilton-Jacobi equation $T\bar{\gamma} \cdot X_H^\varepsilon = X_{\mathcal{K}} \cdot \bar{\varepsilon}$, for the nonholonomic reduced distributional Hamiltonian system $(\check{K}, \omega_{\check{K}}, h)$, if and only if they satisfy the equation $T\tilde{\lambda} \cdot X_H \cdot \varepsilon = \tau_{\check{K}} \cdot T\bar{\varepsilon} \cdot X_{h, \varepsilon}$. 


In this paper, we study the Hamilton-Jacobi theory for the nonholonomic Hamiltonian system and the nonholonomic reducible Hamiltonian system on a cotangent bundle, by using the distributional Hamiltonian system and the reduced distributional Hamiltonian system. These researches reveal from the geometrical point of view the internal relationships of nonholonomic constraints, distributional two forms and nonholonomic dynamical vector fields of a mechanical system and its nonholonomic reduced systems. It is well known that the theory of controlled mechanical systems became an important subject in recent years. Its research gathers together some separate areas of research such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system, in a way that helps both for analysis and design. Thus, it is natural to study controlled mechanical systems by combining with the analysis of dynamic systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. In particular, Marsden et al. in [26] set up the regular reduction theory of a regular controlled Hamiltonian system on a symplectic fiber bundle, by using momentum map and the associated reduced symplectic form, and from the viewpoint of completeness of Marsden-Weinstein symplectic reduction, and some developments around the above work are given in Wang and Zhang [40], Ratiu and Wang [33], and Wang [35]. Since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, it is natural idea to extend the Hamilton-Jacobi theory to the (regular) controlled Hamiltonian system and its a variety of reduced systems, and it is also possible to describe the relationship between the CH-equivalence for the controlled Hamiltonian systems and the solutions of corresponding Hamilton-Jacobi equations, see Wang [37–39] for more details. In particular, it is the key thought of the researches of geometrical mechanics of the Professor Jerrold E. Marsden to explore and reveal the deeply internal relationship between the geometrical structure of phase space and the dynamical vector field of a mechanical system. It is also our goal of pursuing and inheriting.

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