Stability and instability of steady states for a branching random walk

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Abstract

We consider the time evolution of a lattice branching random walk with local perturbations. Under certain conditions, we prove the Carleman type estimation for the moments of a particle subpopulation number and show the existence of a steady state.

Keywords: Branching random walk; Local perturbation; Steady state; Limit theorems.

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1 Introduction

There are several models in population dynamics, in which steady states (statistical equilibrium) exist. Typical models in this area are based on the concept of branching random walk. Numerous variations and versions of branching random walk models have been studied and used to describe the population dynamics, see [1, 2, 3, 4, 8, 9, 10, 12, 13, 14, 15] and bibliography therein.

The simplest model of such a type, which we call the contact critical model, in the lattice case has the following structure. We consider a random field $n(t, \cdot)$ of particles on $\mathbb{Z}^d$, $d \geq 1$, where $n(t, y)$ is the number of particles at the point $y \in \mathbb{Z}^d$ at the time moment $t \geq 0$. At the moment $t = 0$, there is a single particle at each point $x \in \mathbb{Z}^d$, that is $n(0, x) \equiv 1$. Each of such initial particles, independently of others, generates its own subpopulation of the particle offsprings $n(t, y, x)$ at the point $y \in \mathbb{Z}^d$ at the moment $t$. Then the total population of the particle offsprings over $\mathbb{Z}^d$ is defined by $n(t, y) := \sum_{x \in \mathbb{Z}^d} n(t, y, x)$.

The evolution of the subpopulation $n(t, y, \cdot)$ includes the random walk of the particles until the first transformation. The underlying random walk is described by the generator

$$\mathcal{L}_a f(x) = \sum_{z \in \mathbb{Z}^d} (f(x+z) - f(z))a(z)$$

with $a(z) = a(-z)$, $\sum_{z \neq 0} a(z) = 1$ and $\text{span}\{z_j : a(z_j) > 0\} = \mathbb{Z}^d$. The transformation includes the splitting (i.e. the splitting of each particles into two particles at the same lattice point as parental one) with the rate $\beta > 0$ and the annihilation with the mortality rate $\mu$. The central assumption of criticality for such branching process at every lattice point is $\beta = \mu$.

Remark 1. A related result for the continuum $\mathbb{R}^d$ was obtained under some additional conditions in [5, 6]. It is based on the forward Kolmogorov equations for the correlation functions and is not applicable for the branching random walks on $\mathbb{Z}^d$.

The following theorem gives the final condition for the existence of the steady state, see, e.g. [9].
Theorem 1. If, for a BRW under consideration, the underlying random walk with the generator $L_a$ is transient, that is
\[ \int_{[\pi, \pi]} \frac{1}{1 - \hat{a}(k)} dk < \infty, \]
where $\hat{a}(k) = \sum_{z \neq 0} \cos(k, z)a(z)$, then
\[ n(t, y) \xrightarrow{\text{Law}} n(\infty, y), \]
where $n(\infty, y)$ is a steady state. If the random walk is recurrent, then
\[ n(t, y) \xrightarrow{\text{Law}} 0. \]

Remark 2. Note that for a recurrent random walk, the assertion of Theorem 1 corresponds to the phenomenon of clusterization: for large $t$, the population consists of “large islands” of particles the distance between which increases with increasing $t$.

Proof of Theorem 1 is based on the analysis of the moments and correction functions. First, it is necessary to prove that, for any positive integer $l$, the moment
\[ E n^l(t, y) \rightarrow M_l, \]
as $t \rightarrow \infty$, and then check the Carleman condition $M_l \leq l!C^l$ for some constant $C$.

The central point in this proof is the observation that
\[ n(t, y) = \sum_{x \in \mathbb{Z}^d} n(t, x, y), \]
where the subpopulations $n(t, x, y)$ for different $x \in \mathbb{Z}^d$ are independent. Instead of moments, we will use cumulants and the following properties of the cumulants: the $k$-th cumulant of a sum of independent random variables is just the sum of the $k$-th cumulants of the summands. For the calculation of the $k$-th cumulant of $n(t, x, y)$, we can use the backward Kolmogorov equations.

In general, even for non-constant $\beta(x)$ and $\mu(x)$, let $V(x) = \beta(x) - \mu(x)$, all moment equations include the basic operator
\[ H_a = L_a + V(x). \]

In fact, it is easy to see that the moment generation function of $n(t, x, y)$, which is $\phi_z(t, x, y) := E_x z^n(t, x, y)$, is the solution of the KPP type non-linear equation:
\[ \frac{\partial \phi_z}{\partial t} = L_a \phi_z + \beta(x)\phi_z^2 - (\beta(x) + \mu(x))\phi_z + \mu(x), \]
\[ \phi_z(0, x, y) = \begin{cases} z, & x = y, \\ 1, & x \neq y. \end{cases} \]

Denote $m_l(t, x, y) := \frac{\partial \phi_z}{\partial x^l}|_{z=1}$, then the first moment satisfies the following equation:
\[ \frac{\partial m_1}{\partial t} = H_a m_1, \]
\[ m_1(0, x, y) = \delta_x(y). \]

In a similar way, for all $l \geq 2$, we can find the equation for the $l$-th factorial moment
\[ m_l(t, x, y) := E(n(t, x, y)(n(t, x, y) - 1) \cdots (n(t, x, y) - l + 1)) \]
\[ \frac{\partial m_l}{\partial t} = H_a m_l + 2\beta(x) \sum_{i=1}^{l-1} \binom{l-1}{i} m_i m_{l-i}. \]
The first moment equation gives

\[ m_1(t, x, y) = q(t, x, y), \]

where \( q(t, x, y) \) is the fundamental solution of the equation

\[ \frac{\partial q(t, x, y)}{\partial t} = H_0 q(t, x, y). \]

If \( \beta = \mu \), then \( V(x) \equiv 0 \) and \( m_1(t, y) \equiv 1 \). The density of the global population is constant in time. One can solve one by one the moment equations and finally prove Theorem 1. For more details, please refer to [9].

But the assumption of criticality \( \beta = \mu \) is not realistic. Any model in population dynamics measured, at least on the qualitative level, on the similarity with nature bio-systems must be stable with respect to small random perturbation of the form

\[ \beta(x) = \beta_0 + \sigma I_{\{x=0\}}, \quad \sigma > 0 \]

and

\[ \mu(x) = \beta_0 \]

for all \( x \in \mathbb{Z}^d \), i.e. \( V(x) = \sigma \delta_0(x) \).

2 The case of a single-source perturbation

Let us now consider the local perturbations and concentrate, in this section, on the simplest case:

\[ \beta(x) = \beta_0 + \sigma I_{\{x=0\}}, \quad \sigma > 0 \]

and

\[ \mu(x) = \beta_0 \]

for all \( x \in \mathbb{Z}^d \), i.e. \( V(x) = \sigma \delta_0(x) \).
2.1 Transition probability and the Green function for the operator $L_a$

**Lemma 1.** The fundamental solution of

\[
\begin{cases}
\frac{\partial p(t,x,y)}{\partial t} = L_a p(t,x,y), \\
p(0,x,y) = \delta_x(y),
\end{cases}
\]

is

\[
p(t,x,y) = \frac{1}{(2\pi)^d} \int_{T^d} e^{i\mathcal{L}_a(k)} e^{-ik(y-x)} dk;
\]

where $k \in T^d := [-\pi,\pi]^d$, $\mathcal{L}_a(k) = -(1 - \hat{a}(k))$ and $\hat{a}(k) = \sum_{z \neq 0} \cos(k,z) a(z)$.

Note that

\[
p(t,x,y) \leq p(t,x,x) = p(t,0,0) = \frac{1}{(2\pi)^d} \int_{T^d} e^{i\mathcal{L}_a(k)} dk.
\]

Let us consider, for the operator $H_a := L_a + \sigma \delta_0(x)$, the following equation

\[
H_a \psi = \lambda \psi, \psi \in L^2(\mathbb{R}^d), \quad \lambda > 0,
\]

that is

\[
L_a \psi + \sigma \delta_0(x) \psi = \lambda \psi,
\]

The Fourier transform gives

\[
-(1 - \hat{a}(k)) \hat{\psi}(k) + \sigma \psi(0) = \lambda \hat{\psi}(k),
\]

therefore,

\[
\frac{1}{(2\pi)^d} \int_{T^d} \hat{\psi}(k) dk = \frac{1}{(2\pi)^d} \int_{T^d} \frac{\sigma \psi(0)}{\lambda + (1 - \hat{a}(k))} dk.
\]

Let $I(\lambda)$ be the function

\[
I(\lambda) := \frac{1}{(2\pi)^d} \int_{T^d} \frac{1}{\lambda + (1 - \hat{a}(k))} dk.
\]

Using the fact that $\psi(0) = \frac{1}{(2\pi)^d} \int_{T^d} \hat{\psi}(k) dk$ we have

\[
\frac{1}{\sigma} = I(\lambda).
\]

Let $G_\lambda(x,y)$ denote the Green function of the operator $L_a$, which is defined as the Laplace transform of the transition probability $p(t,x,y)$:

\[
G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t,x,y) dt
\]

for $\lambda \geq 0$. We have

\[
G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y) dt = \frac{1}{(2\pi)^d} \int_{T^d} \frac{e^{-ik(y-x)}}{\lambda + 1 - \hat{a}(k)} dk
\]
and
\[ G_0(0,0) = \int_0^\infty p(t,0,0)\,dt = \frac{1}{(2\pi)^d} \int_{T^d} \frac{1}{1 - \hat{a}(k)}\,dk = I(0). \]

For the random walk, \( G_0(0,0) \) denotes the total time that the random walk stays in the origin if it start from 0. If \( \sum_{z \in \mathbb{Z}^d} a(z)|z|^2 < \infty \), then \( G_0(0,0) < \infty \) for \( d \geq 3 \). Please refer to [15] for more discussion of \( G_0(0,0) \).

In the following, we assume that \( \sum_{z \in \mathbb{Z}^d} a(z)|z|^2 < \infty \). It means that the underlying random walk has a finite variance of jumps. The asymptotic of the Green function is studied in [10, 11]. We recall the following lemma from [10].

**Lemma 2.** Suppose that \( d \geq 3 \), assume \( \sum_{z \in \mathbb{Z}^d} a(z)|z|^2 < \infty \), then
\[ G_0(x,y) \sim C_d \, \frac{1}{|y - x|^{d-2}}, \quad \text{as} \quad |y - x| \to \infty, \]
where \( C_d \) is a positive constant.

### 2.2 The First moment

From the previous section, for the first moment \( m_1(t,x,y) = E_n(t,x,y) \), we have the following equation
\[ \frac{\partial m_1(t,x,y)}{\partial t} = L_a m_1(t,x,y) + \sigma \delta_0(x)m_1(t,x,y) \]
\[ m_1(0,x,y) = \delta_x(y). \]

From the Kac-Feyman formula, we have the solution for \( m_1(t,x,y) \):
\[ m_1(t,x,y) = p(t,x,y)E_x \left[ e^{\sigma \int_0^t \delta_0(X_s)\,ds} |X_t = y \right] \]

Here \( X_t \) is the underlying random walk with the generator \( L_a \) and \( E_x \) is the expectation under the condition that the random walk starts at the point \( x \).

Let \( m_1(t,y) = E_n(t,y) = \sum_{x \in \mathbb{Z}^d} m_1(t,x,y) \), then
\[ \frac{\partial m_1(t,y)}{\partial t} = L_a m_1(t,y) + \sigma \delta_0(x)m_1(t,y), \]
\[ m_1(0,y) = 1. \]

Then
\[ m_1(t,0) = E_0 \left[ e^{\sigma \int_0^t \delta_0(X_s)\,ds} \right] \]

As \( t \to \infty \), we find \( E_0 \left[ e^{\sigma \int_0^\infty \delta_0(X_s)\,ds} \right] \) in the following explicit form:
\[ E_0 \left[ e^{\sigma \int_0^\infty \delta_0(X_s)\,ds} \right] = \frac{1}{1 - \sigma G_0(0,0)}. \tag{1} \]

From the last formula we obtain the following lemma.

**Lemma 3.** Suppose that \( d \geq 3 \), then \( E_0 \left[ e^{\sigma \int_0^\infty \delta_0(X_s)\,ds} \right] < \infty \) if and only if \( \sigma < G_0^{-1}(0,0) \).
Theorem 2. If \( d \geq 3 \) and \( \sigma < G_0^{-1}(0,0) \) then we obtain
\[
m_1(t, 0) \xrightarrow{t \to \infty} m_1(\infty, 0) = E_0 \left[ e^{\sigma \int_0^\infty \delta_0(X_s) \, ds} \right].
\]

From the Kac-Feyman formula, we have the solution for \( m_1(t, x, y) \):
\[
m_1(t, x, y) = p(t, x, y) E_x \left[ e^{\sigma \int_0^t \delta_0(X_s) \, ds} | X_t = y \right].
\]

First we prove the following lemma.

Lemma 4. The following inequalities are valid: \( m_1(t, x, 0) \leq m_1(t, 0, 0) \) and \( m_1(t, x, y) \leq m_1(t, 0, 0) \).

Proof. Denote
\[
\tau_{x, 0} := \inf \{ t \mid X_t = 0 | X_0 = x \}.
\]

Then
\[
m_1(t, x, 0) = p(t, x, 0) E_x \left[ e^{\sigma \int_0^\tau \delta_0(X_s) \, ds} | X_t = 0 \right] \\
= p(t, x, 0) E_x \left[ e^{\sigma \int_0^\tau \delta_0(X_s) \, ds} e^{\sigma \int_{\tau}^t \delta_0(X_s) \, ds} | X_t = 0 \right] \\
= p(t, x, 0) E_0 \left[ e^{\sigma \int_0^{\tau_{x, 0}} \delta_0(X_s) \, ds} | X_t = 0 \right] \\
\leq p(t, 0, 0) E_0 \left[ e^{\sigma \int_0^{\tau_{x, 0}} \delta_0(X_s) \, ds} | X_t = 0 \right] \\
= m_1(t, 0, 0).
\]

Here the proof of the second equality uses the fact that before the moment \( \tau_{0,x} \), the random walk does not reach zero, thus \( \int_0^\tau \delta_0(X_s) \, ds = 0 \). The third equality is based on the Markov property of the random walk and the second last equality uses the definition of \( m(t, x, y) \). In a similar way, we can prove that
\[
m_1(t, x, y) \leq p(t, x, y) E_0 \left[ e^{\sigma \int_0^\tau \delta_0(X_s) \, ds} \right]
\]

Denote \( A_t := E_0 \left[ e^{\sigma \int_0^\tau \delta_0(X_s) \, ds} \right] \), that is
\[
m_1(t, x, y) \leq p(t, x, y) A_t
\]

Due to the fact that \( p(t, x, y) \leq p(t, 0, 0) \), we have
\[
m_1(t, x, y) \leq m_1(t, 0, 0).
\]

Note that \( m_1(t, x, y) = m_1(t, y, x) \). The estimation \( m_1(t, x, y) \leq m_1(t, 0, 0) \) gives the possibility to extend the method from [2] to a local perturbation situation.
2.3 Higher moments

As we know, for $l \geq 2$, the $l$-th factorial moment satisfies:

$$\frac{\partial m_l}{\partial t} = L_a m_l + \sigma \delta_0(x)m_l + 2(\mu + \sigma \delta_0(x)) \sum_{i=1}^{l-1} \binom{l-1}{i} m_im_{l-i}$$

$$m_l(0, x, y) = 0.$$

We will prove the estimates given below for the factorial moment.

**Theorem 3.** For $d \geq 3$, we have

$$m_l(t, x, y) \leq A^{l-1} B^l! p(t, x, y),$$

where $A := E_0 \left[ e^\sigma \int_0^\infty \delta_0(X_s) ds \right]$ and $B := 2(\mu + \sigma) G_0(0, 0)$.

**Proof.** We will use the mathematical induction to prove the theorem. For $l = 2$, the second moment satisfies

$$\frac{\partial m_2}{\partial t} = L_a m_2 + \sigma \delta_0(x)m_2 + 2(\mu + \sigma \delta_0(x))m_1^2$$

$$m_2(0, x, y) = 0.$$

From Duhamel’s formula, we have

$$m_2(t, x, y) = 2(\mu + \sigma \delta_0(x)) \int_0^t \sum_{v \in \mathbb{Z}^d} m_1(t-s, x, v)m_1^2(s, v, y) ds$$

$$\leq 2(\mu + \sigma) \int_0^t \sum_{v \in \mathbb{Z}^d} p(t-s, x, v)A_{t-s} p^2(s, v, y)A_s ds$$

$$\leq 2(\mu + \sigma) \int_0^t \sum_{v \in \mathbb{Z}^d} p(t-s, x, v)A_t p^2(s, v, y)A_s ds$$

$$\leq 2(\mu + \sigma) A^2 \int_0^t \sum_{v \in \mathbb{Z}^d} p(t-s, x, v)p^2(s, v, y) ds$$

$$\leq 2(\mu + \sigma) A^2 \int_0^t p(s, 0, 0) \sum_{v \in \mathbb{Z}^d} p(t-s, x, v)p(s, v, y) ds$$

$$\leq 2(\mu + \sigma) A^2 p(t, x, y) \int_0^t p(s, 0, 0) ds$$

$$\leq 2(\mu + \sigma) A^2 p(t, x, y) G_0(0, 0)$$

$$\leq 2BA^2 p(t, x, y).$$

To prove this chain of inequalities we use the following facts.

1. $A_t \leq A$ and $A_{t-s}A_s \leq A$ because

$$A_t = E_0 \left[ e^\sigma \int_0^t \delta_0(X_s) ds \right]$$

and $A = E_0 \left[ e^\sigma \int_0^\infty \delta_0(X_s) ds \right]$,

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2. \( p(s, v, y) \leq p(s, 0, 0) \),
3. \( \sum_{v \in \mathbb{Z}^d} p(t - s, x, v) p(s, v, y) = p(t, x, y) \).

We will prove that

\[
m_{t-1}(t, x, y) \leq A^l B^{l-1} D_l p(t, x, y),
\]

where the sequence \( D_l \) is recurrently defined as

\[
D_1 = 1,
D_l = \sum_{i=1}^{l-1} \binom{l-1}{i} D_i D_{l-i}.
\]

Note that \( [2, 3] \) defines the sequence of the Catalan numbers, thus, the exponential generating function for this sequence \( D(z) := \sum_{l=1}^{\infty} D_l \frac{z^l}{l!} \) from \( [2, 3] \) satisfies \( D^2(z) + z = D(z) \) or \( D(z) = \frac{1 - \sqrt{1 - 4z}}{2} \). The last equality means that the \( l \)-th coefficient of \( D(z) \) grows no faster than \( 4^l \) or, equivalently, \( D_l < 4^l l! \).

For \( l \geq 2 \), the \( l \)-th factorial moment satisfies:

\[
\frac{\partial m_l}{\partial t} = L_a m_l + \sigma \delta_0(x) m_l + 2(\mu + \sigma \delta_0(x)) \sum_{i=1}^{l-1} \binom{l-1}{i} m_i m_{l-i},
\]

\( m_l(0, x, y) = 0 \).

After applying Duhamel’s formula, we have

\[
m_l(t, x, y) = 2(\mu + \sigma \delta_0(x)) \int_0^t \sum_{v \in \mathbb{Z}^d} m_1(t - s, x, v) \times \sum_{i=1}^{l-1} \binom{l-1}{i} m_i(s, v, y) m_{l-i}(s, v, y) ds
\]

\[
\leq 2(\mu + \sigma) D_l \int_0^t \sum_{v \in \mathbb{Z}^d} p(t - s, x, v) A_{t-s} A^l B^{l-1} \times p(s, v, y) A^{l-1} B^{l-1-i} p(s, v, y) ds
\]

\[
\leq 2(\mu + \sigma) A^l B^{l-2} D_l \int_0^t \sum_{v \in \mathbb{Z}^d} p(t - s, x, v) A_{t-s} p^2(s, v, y) ds
\]

\[
\leq 2(\mu + \sigma) A^{l+1} B^{l-2} D_l \int_0^t \sum_{v \in \mathbb{Z}^d} p(t - s, x, v) p^2(s, v, y) ds
\]

\[
\leq 2(\mu + \sigma) A^{l+1} B^{l-2} D_l \int_0^t p(s, 0, 0) \sum_{v \in \mathbb{Z}^d} p(t - s, x, v) p(s, v, y) ds
\]

\[
\leq 2(\mu + \sigma) A^{l+1} B^{l-2} D_l p(t, x, y) \int_0^t p(s, 0, 0) ds
\]

\[
\leq A^{l+1} B^{l-1} D_l p(t, x, y).
\]

\(\square\)
Theorem 4. Let \( N(t, y), y \in \mathbb{Z}^d \), be a random field as described above, and consider the single-source perturbation case, that is, \( \beta - \mu = \sigma \delta_0(x) \) with \( \sigma > 0 \). If \( d \geq 3 \) and \( \sigma < G_0^{-1}(0,0) \), then for all \( y \in \mathbb{Z}^d \) we have

\[
N(t, y) \xrightarrow{\text{Law}} N(\infty, y).
\]

The proof of Theorem 4 is based on Lemma 4 and is completely identical to the proof of the main result from [9], which also uses the inequality \( p(t, x, y) \leq p(t, 0, 0) \).

3 The case of multiple-source perturbations

Now, let us consider a more general case of perturbations. Assume that each particle during the time interval \((t, t + dt)\) can die with probability \( \mu(x) \, dt \) or split, at the same point \( x \in \mathbb{Z}^d \), into two particles with probability \( \beta(x) \, dt \). We assume that \( \mu(x) \equiv \mu \) and \( \beta(x) = \mu + \sum_{i=1}^{k} \sigma_i \delta_{x_i}(x) \) with \( \sigma_i > 0 \). In this case the first moment satisfies the following equation

\[
\frac{\partial m_1}{\partial t} = \mathcal{L}_a m_1 + \sum_{i=1}^{k} \sigma_i \delta_{x_i}(x)m_1,
\]

\[
m_1(0, x, y) = \delta_x(y).
\]

In a similar way, for the \( l \)-th factorial moment

\[
m_l(t, x, y) := E \left( n(t, x, y) \, (n(t, x, y) - 1) \cdots (n(t, x, y) - l + 1) \right), \quad l \geq 2,
\]

we get the equation

\[
\frac{\partial m_l}{\partial t} = \mathcal{L}_a m_l + \sum_{i=1}^{k} \sigma_i \delta_{x_i}(x)m_l + 2\beta(x) \sum_{i=1}^{l-1} \binom{l-1}{i} m_i m_{l-i},
\]

\[
m_l(0, x, y) = 0.
\]

To prove the following lemma we use the definition

\[
C := E_0 \left[ e^{\sum_{i=1}^{k} \sigma_i \int_0^\infty \delta_{X_s}(x) \, ds} \right].
\]

Lemma 5. The following inequalities are valid: \( m_1(t, x, 0) \leq C \, p(t, x, 0) \) and \( m_1(t, x, y) \leq C \, p(t, x, y) \).

Proof. From the Kac-Feyman formula, we have the solution for \( m_1(t, x, y) \):

\[
m_1(t, x, 0) = p(t, x, 0) E_x \left[ e^{\sum_{i=1}^{k} \sigma_i \int_0^t \delta_{X_s}(x) \, ds} \big| X_t = 0 \right]
\]

\[
= p(t, x, 0) E_x \left[ e^{\sigma_1 \int_0^t \delta_{X_s}(x) \, ds} \cdots e^{\sigma_k \int_0^t \delta_{X_s}(x) \, ds} \big| X_t = 0 \right].
\]

By the Hölder inequality we get

\[
E \left[ X_1 X_2 \cdots X_k \right] \leq (E|X_1|^{p_1})^{\frac{1}{p_1}} \cdot (E|X_2|^{p_2})^{\frac{1}{p_2}} \cdots (E|X_k|^{p_k})^{\frac{1}{p_k}},
\]
where
\[ \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = 1. \]

If
\[ \frac{1}{p_i} = \frac{\sigma_i}{\sigma_1 + \sigma_2 + \cdots + \sigma_k}, \quad i = 1, 2, \ldots, k, \]
and
\[ \sum_{i=0}^{k} \sigma_i < G_0^{-1}(0, 0) \]
then we have
\[
m_1(t, x, 0) \leq p(t, x, 0)E_x \left[ e^{\sigma_1 p_1 \int_0^t \delta_{x_1}(X_s)ds} \big| X_t = 0 \right]^{\frac{1}{p_1}}
\]
\[
\cdots E_x \left[ e^{\sigma_k p_k \int_0^t \delta_{x_k}(X_s)ds} \big| X_t = 0 \right]^{\frac{1}{p_k}}
\]
\[
\leq p(t, 0, 0)E_0 \left[ e^{\sigma_1 p_1 \int_0^t \delta_0(X_s)ds} \big| X_t = 0 \right]^{\frac{1}{p_1}}
\]
\[
\cdots E_0 \left[ e^{\sigma_k p_k \int_0^t \delta_0(X_s)ds} \big| X_t = 0 \right]^{\frac{1}{p_k}}
\]
\[
= p(t, 0, 0)E_0 \left[ e^{\sum_{i=1}^{k} \sigma_i \int_0^t \delta_0(X_s)ds} \big| X_t = 0 \right].
\]

In a similar way, one can prove that
\[ m_1(t, x, y) \leq p(t, x, y)C. \]

**Theorem 5.** For \( d \geq 3, \)
\[
m_l(t, x, y) \leq C^{d-1}B^{l}l!p(t, x, y)
\]
where \( C := E_0 \left[ e^{\sum_{i=1}^{k} \sigma_i \int_0^t \delta_0(X_s)ds} \right], \ C_t := E_0 \left[ e^{\sum_{i=1}^{k} \sigma_i \int_0^t \delta_0(X_s)ds} \right] \) and \( B = 2(\mu + \sum_{i=1}^{k} \sigma_i)G_0(0, 0). \)

**Proof.** The basic idea is similar to that of the proof of Theorem 3. We will use the mathematical induction. For \( l = 2 \) the second moment satisfies the following equation
\[
\frac{\partial m_2}{\partial t} = L_m m_2 + \sum_{i=1}^{k} \sigma_i \delta_{x_i}(x)m_2 + 2(\mu + \sum_{i=1}^{k} \sigma_i \delta_{x_i}(x))m_1^2,
\]
\[ m_2(0, x, y) = 0. \]
From Duhamel’s formula, we have

\[
m_2(t, x, y) = 2(\mu + \sum_{i=1}^{k} \sigma_i \delta_{x_i}(x)) \int_0^t \sum_{v \in \mathbb{Z}^d} m_1(t - s, x, v)m^2_1(s, v, y) \, ds
\]

\[
\leq 2(\mu + \sum_{i=1}^{k} \sigma_i) \int_0^t \sum_{v \in \mathbb{Z}^d} p(t - s, x, v)C_{t-s}p^2(s, v, y)C_s^2 \, ds
\]

\[
\leq 2(\mu + \sum_{i=1}^{k} \sigma_i) \int_0^t \sum_{v \in \mathbb{Z}^d} p(t - s, x, v)C_{t-s}p^2(s, v, y)C_s \, ds
\]

\[
\leq 2(\mu + \sum_{i=1}^{k} \sigma_i)C \int_0^t \sum_{v \in \mathbb{Z}^d} p(s, 0, 0) \sum_{v \in \mathbb{Z}^d} p(t - s, x, v)p(s, v, y) \, ds
\]

\[
\leq 2(\mu + \sum_{i=1}^{k} \sigma_i)C^2 p(t, x, y) \int_0^t p(s, 0, 0) \, ds
\]

\[
\leq 2(\mu + \sum_{i=1}^{k} \sigma_i)C^2 p(t, x, y)G_0(0, 0)
\]

\[
\leq 2BC^2 p(t, x, y)
\]

The rest of the proof is similar to the proof of Theorem 3, so we omit it here. \(\square\)

**Theorem 6.** Let \(N(t, y), y \in \mathbb{Z}^d\), be a random field as described above, and consider the multiple-source perturbations case, i.e. \(\beta - \mu = \sum_{i=1}^{k} \sigma_i \delta_{x_i}(x)\), for all \(i = 1, \ldots, k\) and \(\sigma_i > 0\). If \(d \geq 3\) and \(\sum_{i=1}^{k} \sigma_i < G_{0}^{-1}(0, 0)\) then for all \(y \in \mathbb{Z}^d\) we have

\[N(t, y) \xrightarrow{\text{Law}} N(\infty, y).\]

The proof of the theorem is based on the scheme of the proof of Theorem 4, see details in [9].

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