Boundary value problems governed by Helmholtz equation for anisotropic trigonometrically graded materials: A boundary element method solution

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Abstract. Trigonometrically graded media of anisotropic diffusion coefficient are under consideration. Boundary value problems (BVPs) of such kind of media, governed by a Helmholtz type equation, are solved numerically using a boundary element method (BEM). A technique of transforming the variable coefficient governing equation to a constant coefficient equation is utilized for deriving a boundary integral equation. Some particular problems are considered to illustrate the application of the BEM. The results show convergence, accuracy and consistency between the scattering and flow solutions. The results also show efficiency of the BEM procedure for producing the solutions in a short elapsed computation time length. Moreover the results indicate the effect of large wave number on the accuracy and the impact of the inhomogeneity and anisotropy of the material on the solutions.

1. Introduction

A functionally graded material (FGM) is usually defined as a material having particular properties that alter spatially in a continuous way. Today FGM has become a crucial topic and numerous works on FGM for a number of applications including wave propagation have been reported (see for example [1], [2]).

The boundary element method (BEM) has been widely used for solving many types of problems of either homogeneous or inhomogeneous, and either isotropic or anisotropic materials. Papers [3, 4] reported a work on transport problems in anisotropic homogeneous media. For FGMs there are two main techniques usually used. The first one uses a technique of deriving a relevant Green function or fundamental solution to the FGM problem. Cheng [5] had applied this technique. The second technique is by transforming the variable coefficient governing equation to a constant coefficient equation. Some progress on using the second technique has been made. For anisotropic inhomogeneous, papers [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] reported works on a different class of elliptic equation, diffusion convection reaction equation, elasticity problems, Helmholtz equation, and modified Helmholtz equation.

The Helmholtz equation is usually used to mathematically model physical processes such as wave propagation, electromagnetic radiation and seismology. The Helmholtz equation demonstrates the quality of linear wave problem to depict scattering phenomenon, which has
great importance in physics and engineering ([16]). In Helmholtz problems, large wave number has become an issue.

A number of studies dealing with the Helmholtz equation have been reported, however most of the studies deal with isotropic homogeneous materials. In [16] the Helmholtz with large wave number (wave numbers \( \leq 10 \) taken in the examples) was considered for isotropic homogeneous materials. The weak Galerkin mixed finite element method (FEM) used was found to be robust even when the wave number is large. The paper [17] utilized a Galerkin BEM to find numerical solution to exterior problems of isotropic homogeneous media governed by 2-D Helmholtz equation with arbitrary wave number (wave numbers \( \leq 5.1 \) taken in the examples) and found that the scheme is practical and effective. Paper [18] also deals with Helmholtz equation for isotropic homogeneous media in connection with finding analytical integration of the weakly singular integrals in a boundary element analysis. Moreover, in [19] Helmholtz problems of isotropic homogeneous media were solved using the BEM of direct radial basis function interpolation. In addition, papers [20, 21, 22, 23] also deal with Helmholtz equation for isotropic homogeneous media. Furthermore, in [24] interior and exterior problems of Helmholtz equation were solved using a spectral FEM for isotropic inhomogeneous materials, in [25] the Helmholtz equation was solved using a finite difference method (FDM) for anisotropic homogeneous media, and in [26] the Helmholtz equation was solved using BEM for anisotropic homogeneous media.

This paper discusses derivation of a boundary integral equation for numerically solving 2D interior boundary value problems governed by the Helmholtz type equation for anisotropic FGMs of the form

\[
\frac{\partial}{\partial x_i} \left[ \lambda_{ij} (x_1, x_2) \frac{\partial \phi (x_1, x_2)}{\partial x_j} \right] + \beta^2 (x_1, x_2) \phi (x_1, x_2) = 0
\]

(1)

where the coefficients \( \lambda_{ij} \) and \( \beta^2 \) depend on \( x_1 \) and \( x_2 \) and the repeated summation convention (summing from 1 to 2) is employed. As for the case \( \beta^2 = 0 \) a study has been done in [6], this paper will be restricted only for the case \( \beta^2 > 0 \).

The technique of transforming (1) to a constant coefficient equation will be used for obtaining a boundary integral equation for the solution of (1). Throughout the paper, the analysis used is purely mathematical; to develop a BEM for obtaining the numerical solution of BVPs of FGMs governed by equation (1) is the main purpose. Specific aims are to study the feasibility of BEM in solving such kind of problems for large wave numbers, and also to see the impact of anisotropy and inhomogeneity of materials on the scattering solutions.

2. The boundary value problem (BVP)

The BVP is restricted to interior two-dimensional (2D) problem with boundary conditions of type Dirichlet or Neumann. Referred to a Cartesian frame \( O x_1 x_2 \) a solution to (1) is sought which is valid in a region \( \Omega \) in \( R^2 \) with boundary \( \partial \Omega \) consisting of a number of piecewise continuous curves. On \( \partial \Omega \) either \( \phi (x) \) or \( P(x) \) is specified, where

\[
P(x) = \lambda_{ij} \left( \frac{\partial \phi}{\partial x_j} \right) n_i
\]

(2)

\( x = (x_1, x_2) \) and \( n = (n_1, n_2) \) is the normal vector pointing out on the boundary \( \partial \Omega \). For equation (1) to be an elliptic partial differential equation throughout \( \Omega \), the matrix of coefficients \( [\lambda_{ij}] \) is required to be a symmetric positive definite matrix. The coefficients \( \lambda_{ij} \) and \( \beta \) are also required to be twice differentiable functions.

For an exterior problem with an unbounded domain, one should consider to have artificial boundaries and apply absorbing boundary conditions on these boundaries so that to expect that the solution can tightly approximate the free space solution existing without these boundaries [27]. Some other necessary literature for further study on absorbing boundary conditions may be found in references [28, 29, 30].
3. The boundary integral equation

The boundary integral equation is derived by transforming the variable coefficient equation (1) to a constant coefficient equation. We restrict the coefficients $\lambda_{ij}$ and $\beta$ to be of the form

$$\lambda_{ij}(x) = \bar{\lambda}_{ij}g(x)$$  \hspace{1cm} (3)

$$\beta^2(x) = \bar{\beta}^2g(x)$$  \hspace{1cm} (4)

where $g(x)$ is a differentiable function and $\bar{\lambda}_{ij}$ and $\bar{\beta}$ are constant. Substitution of (3) and (4) into (1) gives

$$\bar{\lambda}_{ij}\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right) + \bar{\beta}^2 g \phi = 0$$  \hspace{1cm} (5)

Assume

$$\phi(x) = g^{-1/2}(x)\psi(x)$$  \hspace{1cm} (6)

therefore equation (5) can be written as

$$\bar{\lambda}_{ij}\frac{\partial}{\partial x_i}\left[g\frac{\partial (g^{-1/2}\psi)}{\partial x_j}\right] + \bar{\beta}^2 g^{1/2} \psi = 0$$  \hspace{1cm} (7)

which can be further written as

$$\bar{\lambda}_{ij}\left[\frac{1}{4}g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{1}{2}g^{-1/2}\frac{\partial^2 g}{\partial x_i\partial x_j}\right]\psi + g^{1/2}\frac{\partial^2 \psi}{\partial x_i\partial x_j} + \bar{\beta}^2 g^{1/2} \psi = 0$$  \hspace{1cm} (8)

Use of the identity

$$\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j} = -\frac{1}{4}g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} + \frac{1}{2}g^{-1/2}\frac{\partial^2 g}{\partial x_i\partial x_j}$$

allows equation (7) to be written in the form

$$g^{1/2}\bar{\lambda}_{ij}\frac{\partial^2 \psi}{\partial x_i\partial x_j} - \psi\bar{\lambda}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j} + \bar{\beta}^2 g^{1/2} \psi = 0$$  \hspace{1cm} (8)

If we further restrict the function $g(x)$ to take the trigonometrical form

$$g(x) = [A \cos(\alpha_m x_m) + B \sin(\alpha_m x_m)]^2$$  \hspace{1cm} (9)

where $\alpha_m$ and $k$ are constant, then

$$\bar{\lambda}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i\partial x_j} + kg^{1/2} = 0$$  \hspace{1cm} (10)

Substitution (10) into (8) implies a constant coefficients equation

$$\bar{\lambda}_{ij}\frac{\partial^2 \psi}{\partial x_i\partial x_j} + (k + \bar{\beta}^2) \psi = 0$$  \hspace{1cm} (11)

Moreover, substitution of (3) and (6) into (2) gives

$$P = -P_g \psi + P_\psi g^{1/2}$$  \hspace{1cm} (12)

where $P_g(x) = \bar{\lambda}_{ij}(\partial g^{1/2}/\partial x_j) n_i$ and $P_\psi(x) = \bar{\lambda}_{ij}(\partial \psi/\partial x_j) n_i$.
An integral equation for (11) is
\[ \eta(x_0) \psi(x_0) = \int_{\partial \Omega} [\Gamma(x, x_0) \psi(x) - \Phi(x, x_0) P_\psi(x)] \, ds(x) \]  
(13)
where \( x_0 = (a, b) \), \( \eta = 0 \) if \((a, b) \notin \Omega \cup \partial \Omega \), \( \eta = 1 \) if \((a, b) \) lies inside the domain \( \Omega \), \( \eta = \frac{1}{2} \) if \((a, b) \) is on the boundary \( \partial \Omega \) given that \( \partial \Omega \) has a continuously turning tangent at \((a, b) \). The function \( \Phi \) in (13) is called the fundamental solution, which is any solution of the equation \( \lambda_{ij} (\partial^2 \Phi/\partial x_i \partial x_j) + (k + \beta^2) \Phi = \delta(x - x_0) \) and the \( \Gamma \) is defined as \( \Gamma(x, x_0) = \sum_{i} \left[ \partial \Phi(x, x_0)/\partial x_j \right] n_i \) where \( \delta \) denotes the Dirac delta function. Following Azis [31], for 2-D problems \( \Phi \) and \( \Gamma \) are given by
\[ \Phi(x, x_0) = \frac{-iK}{4} \Omega \frac{(\omega R)}{H_0^{(2)}(\omega R)} \]  
(14)
\[ \Gamma(x, x_0) = \frac{-iK \omega}{4} \Omega \frac{(\omega R)}{H_1^{(2)}(\omega R)} \frac{\partial R}{\partial x_j} n_i \]  
(15)
where \( K = \frac{\dot{\tau}}{\zeta}, \omega = \sqrt{[(k + \beta^2)/\zeta]}, \zeta = \sqrt{[\lambda_{11} + 2\lambda_{12}\dot{\tau} + \lambda_{22} (\dot{\tau}^2 + \dot{\tau}^2)]}/2, \) \( R = \sqrt{(\dot{x}_1 - \dot{\omega})^2 + (\dot{x}_2 - \dot{\omega})^2} \), \( \dot{x}_1 = x_1 + \dot{\tau} x_2, \dot{a} = a + \dot{\tau} b, \dot{x}_2 = \dot{\tau} x_2 \) and \( \dot{b} = \dot{\tau} b \) where \( \dot{\tau} \) and \( \dot{\tau} \) are respectively the real and the positive imaginary parts of the complex root \( \tau \) of the quadratic \( \lambda_{11} + 2\lambda_{12} \tau + \lambda_{22} \tau^2 = 0 \) and \( H_0^{(2)}, H_1^{(2)} \) denote the Hankel function of second kind and order zero and order one respectively and \( i \) represents the square root of minus one. The derivatives \( \partial R/\partial x_j \) necessary for the calculation of the \( \Gamma \) in (15) are given by \( \partial R/\partial x_1 = (\dot{x}_1 - \dot{\omega})/R \) and \( \partial R/\partial x_2 = \left[ \dot{\tau} (\dot{x}_1 - \dot{\omega}) + \dot{\tau} (\dot{x}_2 - \dot{\omega}) \right]/R \). Use of (6) and (12) in (13) yields
\[ \eta(x_0) g^{1/2}(x_0) \phi(x_0) = \int_{\partial \Omega} \left\{ \left[ g^{1/2}(x) \Gamma(x, x_0) - P_g(x) \Phi(x, x_0) \right] \phi(x) - \left[ g^{-1/2}(x) \Phi(x, x_0) \right] P(x) \right\} \, ds(x) \]  
(16)
Equation (16) provides a boundary integral equation which is the starting point of BEM construction for determining the numerical solutions of \( \phi \) and its derivatives at all points of \( \Omega \).

The analysis is in general applicable for anisotropic media but it is not excepted to isotropic materials as a special case when \( \lambda_{11} = \lambda_{22} \) and \( \lambda_{12} = 0 \). Likewise the analysis is also valid for homogeneous media as a special case when \( g(x) \) is a constant function.

4. Numerical examples
Some examples of BVPs for FGMs governed by (1) will be considered. The coefficients \( \lambda_{ij} \) and \( \beta^2 \) in (1) are required to take the forms (3) and (4) respectively with \( g(x) \) taking the trigonometrical form (9). Hankel function in (14) and (15) are approximated by their series, and the integral in (16) is evaluated using Bode’s quadrature with 10 nodal points and error order \( O(h^{11}) \) (see Abramowitz and Stegun [32]). A FORTRAN script is developed to compute the solutions and a unique FORTRAN command is imposed to calculate the elapsed CPU computation time for obtaining the results.

After all, the main purpose of this section is to verify the validity of analysis used to derive the boundary integral equation in the previous sections and to examine the developed FORTRAN script.
The 3rd International Conference On Science

Journal of Physics: Conference Series 1341 (2019) 062007 doi:10.1088/1742-6596/1341/6/062007

The 3rd International Conference On Science

Journal of Physics: Conference Series 1341 (2019) 062007 doi:10.1088/1742-6596/1341/6/062007

5

6

x2

D(0,1) φ given C(1,1)

P given

A(0,0) φ given B(1,0)

x1

Figure 1. The domain Ω

Table 1. The values of \( \bar{\beta}^2 \) and corresponding analytical solutions

| \( \bar{\beta}^2 \) | Analytical solution |
|-----------------|-------------------|
| 15              | \( \phi(x) = \frac{0.5\cos(x_1 + 2.256x_2) + 3\sin(x_1 + 2.256x_2)}{2\cos(0.5x_1 + 0.183x_2) + 3\sin(0.5x_1 + 0.183x_2)} \) |
| 8               | \( \phi(x) = \frac{0.3\cos(x_1 + 1.5x_2) + 3\sin(x_1 + 1.5x_2)}{2\cos(0.5x_1 + 0.183x_2) + 3\sin(0.5x_1 + 0.183x_2)} \) |
| 3               | \( \phi(x) = \frac{0.5\cos(x_1 + 0.7247x_2) + 3\sin(x_1 + 0.7247x_2)}{2\cos(0.5x_1 + 0.183x_2) + 3\sin(0.5x_1 + 0.183x_2)} \) |

4.1. A test problem

The aim of this problem is to show the convergence, accuracy, efficiency and consistency of the BEM solutions. In addition to this, the impact of the large wave number \( \beta(x) \) on the accuracy will also be examined. For a simplicity, the domain Ω is taken to be a unit square and the boundary conditions are as depicted in Figure 1. A number of elements of equal length on each side of the unit square are used. We take the constant diffusion coefficients \( \lambda_{ij} \) and the parameter \( k = 0.5 \)

\[
\lambda_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
\]

The inhomogeneity function \( g(x) \) taking trigonometrical form (9) and satisfying (10) is assumed to be

\[
g(x) = [2\cos(0.5x_1 + 0.183x_2) + 3\sin(0.5x_1 + 0.183x_2)]^2
\]

To show the impact of the large wave number \( \beta(x) \) on the solution, we take several values of \( \bar{\beta}^2 \) namely \( \bar{\beta}^2 = 3, 8, 15 \). Table 1 shows analytical solutions \( \phi \) associated with the values of \( \bar{\beta}^2 \). Table 2 shows the solutions \( \phi \) when \( \bar{\beta}^2 = 3 \) at some points inside the domain. As expected, the BEM solution converges to the analytical solution as the number of elements increases. Table 3 shows the efficiency of BEM for obtaining solutions at 19×19 interior points. Figure 2 shows consistency between the scattering and flow solutions. Figure 3 shows the absolute errors of BEM \( \phi \) solutions for \( \bar{\beta}^2 = 3, 8, 15 \). It indicates the accuracy and the impact of the large wave number.
Table 2. Convergence of solutions $\phi$ when $\beta^2 = 3$

| $(x_1, x_2)$ | 160 elements | 320 elements | 640 elements | Analytical |
|---------------|--------------|--------------|--------------|------------|
| (0.5, 0.1)    | 0.3528       | 0.3530       | 0.3531       | 0.3532     |
| (0.5, 0.3)    | 0.3679       | 0.3681       | 0.3682       | 0.3683     |
| (0.5, 0.5)    | 0.3751       | 0.3753       | 0.3753       | 0.3754     |
| (0.5, 0.7)    | 0.3748       | 0.3749       | 0.3750       | 0.3750     |
| (0.5, 0.9)    | 0.3673       | 0.3674       | 0.3675       | 0.3675     |

Table 3. Elapsed computation time (in seconds) when $\beta^2 = 3$

|                | 160 elements | 320 elements | 640 elements |
|----------------|--------------|--------------|--------------|
|                | 29.453125    | 65.375       | 154.390625   |

Figure 2. The scattering $\phi$ and flow vector $\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)$ solutions when $\beta^2 = 3$

Figure 3. BEM $\phi$ absolute errors along the line $x_1 = 0.5$
4.2. A problem without simple analytical solutions

A layered material consisting of eight layers of the same size as depicted in Figure 4 is under consideration. Each layer is supposed to be homogeneous, but from layer to layer the coefficients $\lambda_{ij}$ and $\beta^2$ vary as smoothly as the variability can be fitted to a trigonometrical function

$$g(x) = [A \cos(\alpha_2 x) + B \sin(\alpha_2 x)]^2$$

As an illustration, suppose that we have a set of values of the coefficients $\lambda_{ij}$ and $\beta^2$ at center point of each layer as shown in Table 4. And we also have reference values of constant coefficients

$$\bar{\lambda}_{ij} = \begin{bmatrix} 0.25 & 0 \\ 0 & 1.75 \end{bmatrix} \quad \bar{\beta}^2 = 2$$

Fitting the data in Table 4 to the function $g(x) = [A \cos(\alpha_2 x) + B \sin(\alpha_2 x)]^2$ we will get the values of the parameters $\alpha_0$ and $\alpha_2$

$$A = 0.75 \quad B = 0.5 \quad \alpha_2 = 0.378$$

Therefore we can then approximate the layered material as a sole material with continuously varying coefficients. So we may now use the analysis in Section 2 to solve the problem numerically.

| Layer | $\lambda_{11}$ | $\lambda_{12}$ | $\lambda_{22}$ | $\beta^2$ |
|-------|-----------------|-----------------|-----------------|-----------|
| 1     | 0.14501         | 0               | 1.01507         | 1.16008   |
| 2     | 0.15348         | 0               | 1.07434         | 1.22782   |
| 3     | 0.16148         | 0               | 1.13038         | 1.29186   |
| 4     | 0.16895         | 0               | 1.18267         | 1.35163   |
| 5     | 0.17582         | 0               | 1.23076         | 1.40658   |
| 6     | 0.18203         | 0               | 1.27420         | 1.45623   |
| 7     | 0.18752         | 0               | 1.31262         | 1.50014   |
| 8     | 0.19224         | 0               | 1.34567         | 1.53791   |
The boundary conditions are depicted in Figure 4. We take the parameter $k = 0.25$ and the inhomogeneity function $g(x)$ takes the trigonometrical form in equation (9). Again, a number of 640 elements of equal length, namely 160 elements on each side of the unit square, are used. As shown in Figure 5 for the given constant orthotropic coefficient $\lambda_{ij}$ above the solution $\phi$ resembles the nature of the considered medium as a layered material.

If, however, we change the constant orthotropic coefficient $\lambda_{ij}$ to an anisotropic coefficient

$$\bar{\lambda}_{ij} = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1.75 \end{bmatrix}$$

therefore the values of $\lambda_{12}$ in Table 4 are not appropriate anymore) by keeping the values of $k, \beta^2, A, B, \alpha_2$ then we will obtain a significantly different solution $\phi$ as shown in Figure 6. This means that the anisotropy of the medium gives an impact on the solution. Therefore in application it is necessary for the anisotropy to be taken into account.

Now, if we assume that the medium is anisotropic homogeneous with

$$\bar{\lambda}_{ij} = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1.75 \end{bmatrix} \quad \alpha_2 = 0 \quad k = 0$$

then we will obtain solutions $\phi$ as shown in Figure 7 which are different with those for the previous case of anisotropic inhomogeneous as shown in Figure 6. It indicates that the impact of material’s inhomogeneity is also evident. This suggests to put the inhomogeneity in consideration for any application.

Figure 5. The BEM scattering $\phi$ and flow vector $\left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)$ solutions of the orthotropic inhomogeneous medium

Figure 6. The BEM scattering $\phi$ and flow vector $\left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right)$ solutions of the anisotropic inhomogeneous medium
5. Conclusion

It is certainly possible to use a standard BEM for solving BVPs governed by an equation of variable coefficients such as the Helmholtz type equation (1). One way to do it, which is adopted in this work, is by transforming the variable coefficient equation into a constant coefficient equation from which a boundary integral equation can be derived. The boundary integral equation becomes a starting point for constructing a BEM. Implementation of the standard BEM is rather easy and the numerical solution resulted from it is sufficiently accurate and timely efficient.

A variable coefficients governing equation such as (1) is usually used for modeling physical application for an anisotropic FGM. In this paper, types of FGMs covered are trigonometrically graded materials.

In addition to its accuracy (even with relatively larger wave numbers) and efficiency (short computation time), consistency between the flow vectors and scattering solutions of the BEM have been shown so as to say the BEM has been working appropriately. Moreover, it is also observed that the anisotropy and inhomogeneity of materials effect the results. This suggests both anisotropy and inhomogeneity should be taken into account in applications.

Acknowledgements

The authors acknowledge the research grants provided by The Ministry of Higher Education of Indonesia (KEMRISTEKDIKTI) under the contract numbered as 007/SP2H/PTNBH/DRPM/2019 and by The Hasanuddin University under the Hasanuddin University’s Rector decrees numbered as 2006/UN4.1/KEP/2019 and 641/UN4.1/KEP/2019.

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