THE MUMFORD–TATE CONJECTURE IMPLIES
THE ALGEBRAIC SATO–TATE CONJECTURE
OF BANASZAK AND KEDLAYA

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Abstract. The algebraic Sato–Tate conjecture was initially introduced by
Serre and then discussed by Banaszak and Kedlaya in the papers [5] and [6].
This note shows that the Mumford–Tate conjecture for an abelian variety \( A \)
implies the algebraic Sato–Tate conjecture for \( A \).

The relevance of this result lies mainly in the fact that the list of known
cases of the Mumford–Tate conjecture was up to now a lot longer than the list
of known cases of the algebraic Sato–Tate conjecture.

1. Introduction

In 1966, Serre wrote a letter to Borel where he presented some remarkable
links between the Mumford–Tate conjecture and questions related to the equidistri-
bution of traces of Frobenius. Inspired by the previous work of Serre, the algebraic
Sato–Tate conjecture was introduced by Banaszak and Kedlaya as an attempt to
prove new instances of the generalized Sato–Tate conjecture [5, 6]. These articles
are considered as the theoretical motivation that allowed Fité, Kedlaya, Rotger and
Sutherland to provide the classification of all the possible Sato–Tate groups that
can appear for the Jacobian of a curve of genus 2 over a number field [15].

For more details about the generalized Sato–Tate conjecture and the conjectural
relation with the Mumford–Tate group we refer to the presentations of [14, 20, §13
of Serre’s paper [25], and [27].

Main result. In this paper we show that for abelian varieties (in fact, abelian
motives) the Mumford–Tate conjecture implies the algebraic Sato–Tate conjecture.

In the next section we introduce notations. The third section recalls the statements
of the conjectures and contains the proof of the main theorem. The last section
contains a short overview of known cases of the Mumford–Tate conjecture and
pointers to the literature.

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2. Notation

Let \( k \) be a field of characteristic 0, and fix a complex embedding \( \sigma : k \hookrightarrow \mathbb{C} \). Our results do not depend on the choice of \( \sigma \). We denote with \( \Gamma_k \) the absolute Galois group \( \text{Gal}(\bar{k}/k) \), where \( \bar{k} \) is the algebraic closure of \( k \) in \( \mathbb{C} \) along the embedding \( \sigma \).

Let \( \text{Mot}_k \) denote the category of motives in the sense of Yves André \([3]\). (To be precise, we use the category of smooth projective \( k \)-schemes as “base pieces”, and we use singular cohomology relative to \( \sigma \) as “reference cohomology”. See §2.1 of \([3]\).) Alternatively, one could use the theory of motives for absolute Hodge cycles \([13]\); this would not alter any of the following statements or proofs. Recall from théorème 0.4 of \([3]\) that \( \text{Mot}_k \) is a graded, polarisable, semisimple Tannakian category over \( \mathbb{Q} \).

The complex embedding \( \sigma : k \hookrightarrow \mathbb{C} \) induces a realisation functor \( r_\ell : \text{Mot}_k \to \text{HS}_\mathbb{Q} \). Every prime number \( \ell \) induces a realisation functor \( r_\ell : \text{Mot}_k \to \text{Rep}_{\mathbb{Q}_\ell}(\Gamma_k) \). We will denote with \( u_\sigma : \text{HS}_\mathbb{Q} \to \text{Vect}_{\mathbb{Q}} \) and \( u_\sigma : \text{Rep}_{\mathbb{Q}_\ell}(\Gamma_k) \to \text{Vect}_{\mathbb{Q}_\ell} \) the respective forgetful functors. By Artin’s comparison theorem (exposé IX of \([2]\)), we obtain a natural isomorphism of functors \( (\sigma \circ u_\sigma) \circ r_\sigma \cong u_\sigma \circ r_\ell \).

If \( T \) is a Tannakian category, and \( S \) is an object (or a collection of objects) of \( T \), then we denote with \( \langle S \rangle \subset T \) the smallest Tannakian subcategory of \( T \) that contains \( S \). In other words, \( \langle S \rangle \) is the full subcategory of \( T \) that is the closure of \( S \) under direct sums, tensor products, duals, and subquotients.

Let \( \omega \) be the fibre functor \( u_\sigma \circ r_\sigma : \text{Mot}_k \to \text{Vect}_{\mathbb{Q}} \). We write \( \mathcal{G}_k \) for the *motivic Galois group*, the group scheme \( \text{Aut}^\otimes(\omega) \) over \( \mathbb{Q} \). If \( M \) is a motive (or a collection of motives), then we denote with \( \mathcal{G}_k(M) \) the motivic Galois group of \( M \); it is the group scheme \( \text{Aut}^\otimes(\omega|_{\langle M \rangle}) \). We denote with \( G_\sigma(M) \) the group scheme \( \text{Aut}^\otimes(\omega|_{\langle r_\sigma(M) \rangle}) \) over \( \mathbb{Q} \); it is the *Mumford–Tate group* of the Hodge structure \( r_\sigma(M) \). Similarly, we denote with \( G_{\ell,k}(M) \) the group scheme \( \text{Aut}^\otimes(\omega|_{\langle r_\ell(M) \rangle}) \) over \( \mathbb{Q}_\ell \), also called the \( \ell \)-adic monodromy group; it is the Zariski closure of the image of \( \Gamma_k \) in \( \text{GL}(r_\ell(M)) \). Observe that there is an inclusion \( G_\sigma(M) \subset \mathcal{G}_k(M) \), and by Artin’s comparison isomorphism there is also a natural inclusion \( G_{\ell,k}(M) \subset \mathcal{G}_k(M)_{\mathbb{Q}_\ell} \).

The groups \( \mathcal{G}_k(M) \), \( G_\sigma(M) \), and \( G_{\ell,k}(M) \) are naturally endowed with a character that comes from the weight structure on \( \text{Mot}_k \). We denote with \( \mathcal{G}_{1,k}(M) \), \( G_{1,\sigma}(M) \), and \( G_{1,\ell,k}(M) \) the kernel of this character. We write \( \mathcal{G}^\sigma_k(M) \) (resp. \( G^\sigma_{\ell,k}(M) \)) for the identity component of \( \mathcal{G}_k(M) \) (resp. \( G_{\ell,k}(M) \)). Note that \( G_\sigma(M) \) is always a connected algebraic group. For any algebraic group \( G \), we denote with \( \pi_0 G \) the component group of \( G \).

A motive is called an *Artin motive* if it is isomorphic to an object of the Tannakian subcategory of \( \text{Mot}_k \) generated by the motives \( H(X) \), where \( X \) ranges over all finite étale \( k \)-schemes. (See example (ii) after §4.5 of \([3]\).)

A motive is called an *abelian* motive if it is isomorphic to an object of the Tannakian subcategory of \( \text{Mot}_k \) generated by Artin motives and the motives \( H(X) \), where \( X \) ranges over all abelian varieties over \( k \). (See §6.1 of \([3]\).)
3. Main result

Conjectures. Assume that the field $k$ is finitely generated as field, and let $M$ be a motive over $k$. We recall the following conjectures.

(1) A motivic analogue of the Tate conjecture:

$$\text{TC}'_{\ell}(M): \quad G_{\ell,k}(M) = G_{k}(M)_{\mathbb{Q}_{\ell}}, \quad \forall \ell, \text{TC}'_{\ell}(M)$$

(N.b.: The “classical” $\ell$-adic Tate conjecture for $\ell$-adic cohomology of a smooth projective variety $X/k$ does not formally imply $\text{TC}'_{\ell}(H(X))$, nor is the converse implication a formal fact.)

(2) The following is called the motivic Sato–Tate conjecture in conj. 10.7 of [6]:

$$\text{MST}_{\ell}(M): \quad G_{1,\ell,k}(M) = G_{1,k}(M)_{\mathbb{Q}_{\ell}}, \quad \forall \ell, \text{MST}_{\ell}(M)$$

(3) A motivic version of the Mumford–Tate conjecture:

$$\text{MTC}_{\ell}(M): \quad G_{\sigma,\ell,k}(M) = G_{\sigma}(M)_{\mathbb{Q}_{\ell}}, \quad \forall \ell, \text{MTC}_{\ell}(M)$$

(4) The algebraic Sato–Tate conjecture (conj. 5.1(a,b) of [6]):

For every prime $\ell$, there exists a natural-in-$k$ reductive algebraic group $\text{AST}_{k}(M) \subset \text{GL}(\omega(M))$ over $\mathbb{Q}$ and a natural-in-$k$ isomorphism of group schemes $G_{1,\ell,k}(M) \cong \text{AST}_{k}(M)_{\mathbb{Q}_{\ell}}$.

Note that MST$(M)$ is a more precise version of the algebraic Sato–Tate conjecture: it predicts that $\text{AST}_{k}(M) = G_{1,k}(M)$.

Theorem. Let $k$ be a finitely generated field of characteristic 0, and fix a complex embedding $\sigma: k \hookrightarrow \mathbb{C}$. Let $M$ be an abelian motive over $k$, and let $\ell$ be a prime number. The following assertions are equivalent.

$$(i) \quad \text{TC}'(M) \quad (ii) \quad \text{MST}(M) \quad (iii) \quad \text{MTC}(M)$$

$$(i)_{\ell} \quad \text{TC}'_{\ell}(M) \quad (ii)_{\ell} \quad \text{MST}_{\ell}(M) \quad (iii)_{\ell} \quad \text{MTC}_{\ell}(M)$$

In particular, $\text{MTC}(M)$ implies the algebraic Sato–Tate conjecture for $M$.

Proof. We start by proving $\text{MST}_{\ell}(M) \iff \text{TC}'_{\ell}(M)$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & G_{1,\ell,k}(M) & \longrightarrow & G_{\ell,k}(M) & \longrightarrow & G_{m} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G_{1,k}(M)_{\mathbb{Q}_{\ell}} & \longrightarrow & G_{k}(M)_{\mathbb{Q}_{\ell}} & \longrightarrow & G_{m} \\
\end{array}$$

A formal argument shows that the inclusion on the left is an isomorphism if and only if the inclusion in the middle is an isomorphism.
Let us now prove $\text{MTC}_\ell(M) \iff \text{TC}_\ell'(M)$. Consider the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \longrightarrow & G^{\sigma}_{\ell,k}(M) & \longrightarrow & G_{\ell,k}(M) & \longrightarrow & \pi_0 G_{\ell,k}(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^\sigma_{k}(M)_{\mathbb{Q}_\ell} & \longrightarrow & G_{k}(M)_{\mathbb{Q}_\ell} & \longrightarrow & \pi_0 G_{k}(M)_{\mathbb{Q}_\ell} & \longrightarrow & 0 \\
\end{array}
$$

Now we argue as follows:

(1) Observe that if the inclusion in the middle is an isomorphism, then the inclusion on the left is an isomorphism, by definition of the identity component.

(2) Claim: The arrow $\pi_0 G_{\ell,k}(M) \to \pi_0 G_{k}(M)_{\mathbb{Q}_\ell}$ is surjective. Indeed, since $\pi_0 G_{k}(M)$ is a finite group, we may view it as the motivic Galois group of some Artin motive $N$. The image of $\pi_0 G_{\ell,k}(M)$ in $\pi_0 G_{k}(M)_{\mathbb{Q}_\ell}$ is exactly $G_{\ell,k}(N)$. The equality $G_{\ell,k}(N) = G_{k}(N)_{\mathbb{Q}_\ell}$ is well-known; see for example remark 6.18 on page 211 of [12].

(3) Now a formal argument shows that if the inclusion on the left is an isomorphism and the right arrow is a surjection, then the inclusion in the middle is an isomorphism.

(4) To finish the proof, we recall that the canonical inclusion $G_\sigma(M) \hookrightarrow G^{\sigma}_{k}(M)$ is an isomorphism. Indeed, by théorème 1.6.2 of [3] we know that $G_\sigma(M) \cong G_C(M_C)$. We also have an isomorphism $G_{C}(M_C) \cong G_{k}^{\sigma}(M_k)$, see théorème 0.6.1 and remarque (ii) after théorème 5.2 of [3]. Thus it suffices to show $G_{k}^{\sigma}(M_k) = G_{k}^{\sigma}(M)$.

By example (ii) in §4.6 of [1] we know that the quotient $G_{k}(M)/G_{k}(M_k)$ is a quotient of $\Gamma_k$. Since it is also a quotient of an algebraic group of finite type, this quotient is a finite group, and therefore isomorphic to $\pi_0 G_{k}(M)$. We conclude that $\text{MTC}_\ell(M) \iff \text{TC}_\ell'(M)$. Finally, corollary 7.6 of [11] shows that the Mumford–Tate conjecture is independant of $\ell$: it proves the implication $\text{MTC}_\ell(M) \implies \text{MTC}(M)$. Altogether, this proves the theorem.

4. NEW INSTANCES OF THE ALGEBRAIC SATO–TATE CONJECTURE

The main theorem of our paper asserts that for abelian varieties the Mumford–Tate conjecture is equivalent to the motivic Sato–Tate conjecture and therefore implies the algebraic Sato–Tate conjecture. In this section we will give a non-exhaustive presentation of some known results on the Mumford–Tate conjecture for abelian varieties. We refer the reader to the survey paper of Moonen for further details [19].

For instance, one of the first examples where the Mumford–Tate conjectures is useful for proving the algebraic Sato–Tate conjecture is the case of abelian varieties of CM type [22]. Serre proved that the Mumford–Tate conjecture is true for elliptic curves [24]. Moreover, this conjecture is known to be true for abelian varieties.
of dimension less than or equal to 3, and for simple abelian varieties of prime dimension \( \leq 3 \).

Further results are known if we impose some conditions on the endomorphism algebra of the abelian variety \( A \). Indeed, if the endomorphism algebra of \( A \) is trivial, and the dimension of \( A \) is an odd number, Serre proved the Mumford–Tate conjecture \([26]\). Several generalizations of this result were done afterwards by Chi \([9]\) for abelian varieties with larger endomorphism algebra. More results in this direction were proven by Pink \([21]\).

Banaszak, Gajda and Krasoń \([3,4]\) proved the Mumford–Tate conjecture for some classes of abelian varieties of type I, II and III in the sense of Albert’s classification. Hindry and Ratazzi \([16]\) proved new instances of the Mumford–Tate conjecture for certain classes of abelian varieties of type I and II. In \([7]\) the first author extends those results to a larger class of abelian varieties of type III.

Ichikawa \([17]\) (resp. Lombardo \([18]\)) proved that under suitable conditions the Hodge group (resp. \( \ell \)-adic Hodge group) of a product of abelian varieties is the product of the Hodge groups (resp. \( \ell \)-adic Hodge groups). Lombardo \([18]\) used these results to prove the Mumford–Tate conjecture for arbitrary products of abelian varieties of dimension \( \leq 3 \). Inspired by these results, the second author \([10]\) proved that if two arbitrary abelian varieties satisfy the Mumford–Tate conjecture, then their product also satisfies this conjecture.

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