Stability of nonlinear magnetic black holes

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We study the stability of static spherically symmetric exact solutions of Einstein equations coupled with nonlinear electrodynamics, in the magnetic sector. These solutions satisfy the heuristic model proposed by Ashtekar-Corichi-Sudarsky for hairy black holes, meaning that the horizon mass is related to their Arnowitt-Deser-Misner (ADM) mass and to the corresponding particle-like solution. We test the unstability conjecture that emerges for hairy black holes and it turned out that it becomes confirmed except for the Einstein-Born-Infeld solutions.

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I. INTRODUCTION

Remarkable properties of nonlinear electrodynamical black holes arise in the context of the isolated horizon formalism, recently put forward by Ashtekar et al. In this approach it is pointed out the unsatisfactory (un-complete) description of a black hole given by concepts such as ADM mass and event horizon, for instance, specially if one is dealing with hairy black holes. To remedy this uncompleteness, Ashtekar et al have proposed alternatively the isolated horizon formalism, that furnishes a more complete description of what happens in the neighborhood of the horizon of a hairy black hole.

In the isolated horizon formalism one considers space-times with an interior boundary, which satisfy quasi-local boundary conditions that insure that the horizon remains isolated. The boundary conditions imply that quasi-local charges can be defined at the horizon, which remain constant in time. In particular one can define a horizon mass, a horizon electric charge and a horizon magnetic charge.

Moreover, Ashtekar-Corichi-Sudarsky (ACS) conjecture about the relationship between the colored black holes and their solitonic analogs the Arnowitt-Deser-Misner (ADM) mass contains two contributions, one attributed to the black hole horizon and the other to the outside hair, captured by the solitonic residue. In this model, the hairy black hole can be regarded as a bound state of an ordinary black hole and a soliton.

For colored black holes the difference between the horizon and ADM masses can be seen as the energy that is available for radiation to fall both into the black hole and to infinity. In static solutions there is no radiation, thus a positive value of the difference $M^{ADM} - M_A = E$ means that a slightly perturbation on the initial data of the static solution will lead the system to a Schwarzschild black hole and the total radiated energy will be equal to $E$. For colored black holes a positive $E$ is a necessary condition for the solution to be unstable.

It is then natural to ask if those conjectures do apply to the case of (non colored) black hole solutions of Lagrangians more general than the Einstein-Maxwell one. In the present paper we address static spherically symmetric (SSS) solutions of nonlinear electrodynamics (NED) coupled with gravity, the corresponding action given by

$$S = \int d^4x \sqrt{-g}\{R(16\pi)^{-1} - L\},$$

where $R$ denotes the scalar curvature, $g := \det|g_{\mu\nu}|$ and $L$, the electromagnetic part, is assumed to depend in nonlinear way on the invariants of $P_{\mu\nu}$, the nonlinear generalization of the electromagnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$,

$$L = \frac{1}{2} p^{\mu\nu} F_{\mu\nu} - K(F, \tilde{F}),$$

where $F$ and $\tilde{F}$ are the invariants of $P_{\mu\nu}$ and $K(F, \tilde{F})$ is the structural function.

We verify the mass formula for static spherically symmetric black hole solutions and their soliton-like counterpart for three distinct nonlinear electrodynamics. We show that the masses relation is fulfilled and that the value of the “total energy” $E$ for SSS-NED solutions is positive.

Moreover we test the stability of the solitons and black holes with respect to nonspherically symmetric dynamical linear fluctuations using the results obtained by Moreno and Sarbach. In the dynamical stability of solutions in self-gravitating nonlinear electrodynamics is analyzed with respect to arbitrary linear fluctuations of the metric and electromagnetic field. Conditions are derived on the electromagnetic Lagrangian which imply linear stability. We show that these conditions hold for Einstein-Born-Infeld purely magnetic solutions but they fail for other solutions of NED.

The Born-Infeld example leads to the conclusion that for SSS-NED black holes and particle-like solutions the unstability conjecture derived from the positive difference between the horizon and ADM masses does not hold generically.

We shall restrict our attention to purely magnetic solutions because of two reasons: firstly, there exists no go
theorems \cite{4} that forbid the existence of purely electric or dyonic (electric and magnetic charge) solutions with regular center. However the prohibition does not concern purely magnetic solutions and there is a whole class of regular solutions with a nonzero magnetic charge. Since we want to test the ACS heuristic model we focus on NED theories possessing both solutions black hole like and its solitonic counterpart.

The second reason to consider purely magnetic solutions is that for purely magnetic cases the horizon mass can be calculated in a straightforward way once the metric is known because the magnetic charge does not contribute as a global charge; the horizon mass is given by:

\[ M_\Delta = \frac{1}{2} \int_0^{\infty} [1 - 2M'(\hat{r})]d\hat{r}, \]

where \( M(r) \) is given by the metric function \( N(r) = 1 - 2M(r)/r \) in the SSS line element,

\[ ds^2 = -N(r)dt^2 + N(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

In contrast, for the electric case the expression for the horizon mass depends on the theory: it involves a potential, \( V(a_\Delta, P_\Delta, Q_\Delta) \), that depends on the horizon parameters and must be determined in consistence with the first law of black hole mechanics \cite{3}:

\[ M_\Delta = \frac{1}{4\pi} \kappa a_\Delta + \Psi Q_\Delta + V(a_\Delta, P_\Delta, Q_\Delta), \]

where \( \kappa \) is the surface gravity of the horizon, \( a_\Delta \) is the surface area, \( \Psi \) is the electric potential, \( Q_\Delta \) and \( P_\Delta \) are the electric and magnetic charges of the horizon, respectively.

On the other side, in NED there are cases in which the known solution is purely electric, however, for spherically symmetric solutions in nonlinear electrodynamics there exist a duality that connects purely electric solutions with purely magnetic ones. The electromagnetic field tensor \( F_{\mu\nu} \) compatible with spherical symmetry can have a radial electric field \( (F_{01}) \) and a magnetic field \( (P_{23}) \). The FP duality let us to pass from a solution with a radial electric field to a solution with a radial magnetic field and viceversa; the two solutions corresponding to the same line element (same \( g_{\mu\nu} \)) but to different Lagrangians. The FP duality is described in more detail in \cite{4}.

We shall address three solutions magnetically charged in nonlinear electrodynamics coupled to gravity, all of them having both types of solutions, particle-like and black hole ones. The studied cases are: 1) The purely magnetic Born-Infeld black hole and its corresponding EBI-on. 2) The magnetic solution obtained recently by Bronnikov \cite{4}, via FP duality, from a purely electric one derived by Ayon-Garcia \cite{6}. 3) A magnetic solution determined via FP duality from a regular electric solution recently presented in \cite{1}; this last solution avoided the non existence theorems by having a de Sitter center. Each case corresponds to a different nonlinear electrodynamical Lagrangian.

II. BORN-INFELD BLACK HOLE AND EBION

The EBI black hole is the solution for the field equations of the Lagrangian in \cite{2} with the Born-Infeld structural function given by

\[ K = b^2 \left( 1 - \sqrt{1 - 2F/b^2 + \hat{F}^2/b^4} \right), \]

where \( b \) is the maximum field strength and the relevant parameter of the BI theory. For the purely magnetic case one of the two electromagnetic invariants constructed from \( F_{\mu\nu} \) is zero \( (\hat{F} = 0) \).

The EBI solution for a SSS spacetime as \cite{1} is given by the metric function

\[ N(r) = 1 - 2m/r + \frac{2}{3}b^2(r^2 - \sqrt{r^4 + a^4}) + \frac{4g^2}{3r}G(r), \]

\[ G'(r) = -(r^4 + a^4)^{-\frac{1}{2}}, \]

where \( G'(r) \) denotes the derivative of \( G(r) \) with respect to the radial variable, \( m \) is the mass parameter, \( g \) is the magnetic charge (both in length units), \( a^4 = g^2/b^2 \) and \( b \) is the Born-Infeld parameter given in units of \( [\text{length}]^{-1} \). The nonvanishing components of the electromagnetic field are

\[ F_{rt} = g(r^4 + a^4)^{-\frac{1}{2}}, \quad P_{rt} = \frac{g}{r^2}. \]

The black hole solution given by García-Salazar-Plebański \cite{8} corresponds to

\[ G(r) = \int_r^{\infty} \frac{ds}{\sqrt{s^4 + a^4}} = \frac{1}{2a} \mathcal{F} \left[ \arccos \left( \frac{r^2 - a^2}{a^2 + r^2} \right), \frac{1}{\sqrt{2}} \right], \]

where \( \mathcal{F} \) is the elliptic integral of the first kind. On the other side, the particle-like solution given by Demianski \cite{9} is

\[ G(r) = \int_0^{r} \frac{-ds}{\sqrt{s^4 + a^4}} = -\frac{1}{2a} \mathcal{F} \left[ \arccos \left( \frac{a^2 - r^2}{a^2 + r^2} \right), \frac{1}{\sqrt{2}} \right]. \]

The election of \( G(r) \) as in Eq. \cite{10} or Eq. \cite{11} has as a consequence a different behavior of the solution at the origin. The metric function \( N(r) \) with \( G(r) \) given by Eq. \cite{10} diverges at \( r \to 0 \) (even when \( m = 0 \)), corresponding to the black hole solution. The other one, meaning \( N(r) \) with \( G(r) \) given by Eq. \cite{11} is the so called
EBIon solution that is finite at the origin (for \( m = 0 \)). The integrals of Eqs. (10) and (11) are related by

\[
\int_{r}^{\infty} \frac{ds}{\sqrt{s^4 + a^4}} + \int_{0}^{r} \frac{ds}{\sqrt{s^4 + a^4}} = \frac{1}{a} K\left[\frac{1}{2}\right], \tag{12}
\]

where \( K\left[\frac{1}{2}\right] \) is the complete elliptic integral of the first kind. In the limit of large distances, \( r \to \infty \), asymptotically the solution corresponds to the magnetic Reissner-Nordstrom (RN) solution, the SSS solution to Einstein-Maxwell equations. Also when the BI parameter goes to infinity, \( b \to \infty \), we recover the linear electromagnetic (Einstein-Maxwell) RN solution. In the uncharged limit, \( b = 0 \) (or \( g = 0 \)), it is recovered the Schwarzschild black hole.

### A. The mass relation

When the black hole is not completely determined by global charges defined at spatial infinity such as ADM mass, angular momentum or electric charge, but rather it possesses short range charges (hair) that vanish at infinity, then it is a hairy black hole. In a series of papers Ashtekar et al have proposed a more complete description to characterize a hairy black hole, based on quantities defined at the horizon. This formalism is intended to deal with situations more general than SSS hairy black holes and it involves the canonical formalism of gravity. Furthermore, they proposed a formula relating the horizon mass and the ADM mass of the colored black hole solution with the ADM mass of the soliton solution of the corresponding theory,

\[
M^{(n)}_{\text{sol}} = M^{(n)}_{\text{ADM}} - M^{(n)}_{\Delta}, \tag{13}
\]

where the superscript \( n \) indicates the colored version of the hole; in the papers of Ashtekar et al this \( n \) refers to the Yang-Mills hair, labeled by this parameter, corresponding to \( n = 0 \) the Schwarzschild limit (absence of YM charge). This relation has been proved numerically to work for the Einstein-Yang-Mills (EYM) black hole.

It was shown in \( \text{[10]} \) that the EBI black hole and the corresponding EBIon solution fulfill the relation between the masses as well as most of the properties of the model for the colored black hole. For the EBI solution the horizon and ADM masses as functions of the horizon radius \( r_{\Delta} \) are given, respectively, by

\[
M^{(b)}_{\Delta}(r_{\Delta}) = \frac{r_{\Delta}}{2} + \frac{b^2 r_{\Delta}}{3}(r_{\Delta}^2 - \sqrt{r_{\Delta}^4 + a^4}) - \frac{2g^2}{3} \int_{0}^{r_{\Delta}} \frac{ds}{\sqrt{a^4 + s^4}}, \tag{14}
\]

\[
M^{(b)}_{\text{ADM}}(r_{\Delta}) = \frac{r_{\Delta}}{2} + \frac{b^2 r_{\Delta}}{3}(r_{\Delta}^2 - \sqrt{r_{\Delta}^4 + a^4}) + \frac{2g^2}{3} \int_{r_{\Delta}}^{\infty} \frac{ds}{\sqrt{a^4 + s^4}}. \tag{15}
\]

The mass of the soliton can be obtained by letting \( r_{\Delta} \to 0 \) in the ADM mass, Eq. (13), obtaining \( M^{(b)}_{\text{sol}} = 2g \sqrt{3} K\left[\frac{1}{2}\right] / 3 \). From these expressions one can trivially check that they satisfy Eq. (13).

### B. Stability of the EBI solutions

For hairy black holes the positivity of the difference between the ADM mass and the horizon mass, \( M_{\text{ADM}} - M_{\Delta} = E > 0 \), indicates that there exists an energy \( E \) available to be radiated. For static black holes this result can be interpreted as a potential unstability, i.e. a slightly perturbation in the initial data will lead the solution to decay to a Schwarzschild black hole.

Stability properties in self-gravitating nonlinear electrodynamics were studied by Moreno and Sarbach \( \text{[3]} \). They derived sufficient conditions for linear stability with respect to arbitrary linear fluctuations in the metric and in the gauge potential, \( \delta g_{\mu
u} \) and \( \delta A_{\mu} \), respectively; the conditions were obtained in the form of inequalities to be fulfilled by the nonlinear electromagnetic Lagrangian \( L(F) \) and its derivatives.

The application of this criterion is restricted to static, spherically symmetric solutions of NED coupled to gravity, that are purely electric or purely magnetic. For these systems a gauge invariant perturbation formalism was used obtaining that linear fluctuations around a SSS purely electric (or purely magnetic) solution are governed by a wavelike equation with symmetric potential of the form:

\[
(-P \nabla^a P^{-1} \nabla_a P + S)u = 0, \tag{16}
\]

where \( \nabla \) denotes covariant derivative of the metric \( \hat{g} = -N dt^2 + N^{-1} d\sigma^2 \), \( P \) is a positive-definite symmetric matrix and \( S \) is a symmetric matrix; \( u \), a vector-valued function, is a gauge-invariant combination of the perturbed metric and perturbed electromagnetic field. Linear stability follows if the potential \( S \) is positive-definite. In
terms of the Lagrangian $L(F)$ and metric function $N$, the positiveness of $S$ is accomplished if

$$L(y) > 0, \quad L(y)_y > 0, \quad L(y)_{yy} > 0, \quad (17)$$

where $y = \sqrt{2g^2F}$ ($g$ is the magnetic charge and $F$ is one of the invariants of the electromagnetic field.

Besides, there are more inequalities to be fulfilled, that arise from the pulsations equations for $l \geq 2$ in the even-parity sector. The potential $S$ is positive definite if $\kappa > 0$, $N \leq 1$ and also $l(l+1) - 2N\kappa > 0$. While for $l = 1$ the corresponding potential is positive if $\kappa > 0$, where $\kappa$ is defined as

$$\kappa = 1 + 2L^{-1}_{FF}L_{FF}F,$$  \hspace{1cm} (18)

$L_{FF}$ denotes the derivative of the Lagrangian with respect to the invariant $F$. In terms of the variable $y = \sqrt{2g^2F}$ these additional inequalities are equivalent to

$$f(y) = yL_{yy}/L_y > 0, \quad f(y)N(y) < 3, \quad (19)$$

We shall apply this criterion to test the stability of the particle-like and black hole solutions; in the former case the boundary point is the origin, $r = 0$, while for the black hole case the conditions must be held in the domain of outer communication (DOC), i.e. positions outside the horizon, $r > r_h$, $r_h$ being the radius of the horizon of the black hole.

The BI Lagrangian fulfill the stability conditions; in terms of the variable $y$, the BI Lagrangian, Eq. (2), is given by

$$L(y) = b^2\sqrt{1 + \frac{y^2}{b^2g^2} - 1} > 0, \quad (20)$$

and the rest of the inequalities read as:

$$L_y = \frac{y}{g^2}(1 + \frac{y^2}{b^2g^2})^{-\frac{3}{2}} > 0,$$

$$L_{yy} = \frac{1}{g^2}(1 + \frac{y^2}{b^2g^2})^{-\frac{3}{2}} > 0,$$

$$f(y) = y \frac{L_{yy}}{L_y} = (1 + \frac{y^2}{b^2g^2})^{-1} > 0, \quad (21)$$

Conditions (21) are fulfilled in all the range of $y$. Moreover, $f(y)$ is monotonically decreasing with $f(y = 0) = 1$, $0 < f(y) \leq 1$: then the last stability condition $f(y)N(y) < 3$ reduces to prove that $N(y) < 3$, for the black hole it must be fulfilled in DOC, while for the particle-like solution the domain to be considered is $0 \leq r < \infty$.

In the black hole case, the metric function $N(r)$ has a minimum in the extreme case ($g = m$) for $bm = 0.5224$ at $r_h = 0.346m$; DOC is considered for distances larger than the radius of the horizon, $r > r_h = 0.346m$. In terms of $y$, considering that $F = g^2/2r^4$ then $y = g^2/r^2$, the metric function $N(y)$ is

$$N(y) = 1 - \frac{2m\sqrt{y}}{g} + \frac{2b^2g^2}{3y}[1 - \sqrt{1 + \frac{y^2}{b^2g^2}}] + \frac{2\sqrt{gb^2y}}{3}\left[\arccos\left(\frac{gb - y}{gb + y}\right), \frac{1}{\sqrt{2}}\right], \quad (22)$$

In the range $0 < y < y_h = 8.35$ it turns out that $0 < N(y) \leq 1$ with $N(0) = 1$ therefore, $0 < N(y) < 1 < 3$, fulfilling the last inequality required as sufficient conditions for stability of the EBI black hole.

For the particle-like solution of the EBI equations, the metric function $N(r)$ in terms of $y$ is

$$N(y) = 1 - \frac{2m\sqrt{y}}{g} + \frac{2b^2g^2}{3y}[1 - \sqrt{1 + \frac{y^2}{b^2g^2}}] - \frac{2\sqrt{gb^2y}}{3}\left[\arccos\left(\frac{y - gb}{gb + y}\right), \frac{1}{\sqrt{2}}\right], \quad (23)$$

The last stability condition $N < 3$ in fact occurs since $N(y = 0) = 1$ and the function is monotonically decreasing, having $N(y) \leq 1$, for $bg \neq 0$; the finiteness in the origin of $N(r)$ is valid when $m = 0$. Therefore, as far as this analysis proves, the EBI solutions, both black hole and particle-like one, are stable, becoming in this case non valid the unstability conjecture. However, this situation is far from being generic in NED as the next two examples will show.

### III. BRONNIKOV SOLUTION

In [4] were derived magnetic black hole and soliton-like solutions with NED coupled to gravity. The Lagrangian in this case is of the form

$$L = F/\cosh^2(a|F/2|^{1/4}), \quad (24)$$

where $a$ is a constant, $F = 2g^2/r^4$, $g$ is the magnetic charge. The use of $|F|$ violates analyticity of $L$ at $F = 0$, though, in the range of interest, $F > 0$, $L(F)$ is well behaved. The corresponding function $M(r)$ in the line element [4] is given by

$$M(r) = \frac{g}{2a}[1 - \tanh(\sqrt{\frac{g}{r}})], \quad (25)$$

We also have, from the Komar integral, that

$$M_{ADM}(r) = \frac{g^{3/2}}{2a}[1 - \tanh(\sqrt{\frac{g}{r}})] - \frac{g^2}{2r}\sech^2(\sqrt{\frac{g}{r}}), \quad (26)$$
Moreover, the minimum value of \( N(r) = 1 - 2M(r)/r \) depends on the ratio \( \xi = m/g \) (we consider \( g > 0 \)), so that \( N_{\text{min}} \) is negative for \( \xi > \xi_0 \approx 0.96 \) (we deal with a black hole with two horizons), zero for \( \xi = \xi_0 \) (an extremal black hole with one double horizon) and positive for \( \xi < \xi_0 \) (a regular particle-like system). Given any specific value of the constant \( a \) in Eq. 21 we can obtain all three types of solutions depending on the charge value; we have a nonextremal or extremal black hole if \( g \leq 4a^2 \xi_0^2 \), or we have a particle-like solution (a monopole) otherwise.

As function of the horizon radius \( r_\Delta \) the ADM mass and the horizon mass are, respectively,

\[
M_{\text{ADM}}(r_\Delta) = \frac{r_\Delta}{2} - \frac{g^3/2}{2a} \left[ 1 - \tanh \left( \frac{\sqrt{g}}{r_\Delta} \right) \right] + \frac{g^{3/2}}{2a},
\]

\[
M_D(r_\Delta) = \frac{r_\Delta}{2} - \frac{g^{3/2}}{2a} \left[ 1 - \tanh \left( \frac{\sqrt{g}}{r_\Delta} \right) \right], \quad \text{(28)}
\]

The limits when the horizon radius go to zero, \( r_\Delta \to 0 \), are

\[
\lim_{r_\Delta \to 0} M_D(r_\Delta) = 0, \quad \lim_{r_\Delta \to 0} M_{\text{ADM}}(r_\Delta) = \frac{g^{3/2}}{2a}. \quad \text{(29)}
\]

From expressions (23) it can be shown that the relation holds: \( M_{\text{ADM}}(r_\Delta) - M_D(r_\Delta) = g^{3/2}/2a \), obtaining the soliton mass \( g^{3/2}/2a \). According to the hairy black hole heuristic model, the solution may be unstable since \( M_{\text{ADM}} > M_D \). This turns out to be the case as will be shown below.

The sufficient conditions Eqs. 17 and 18 for the solution to be stable are fulfilled for the black hole in the domain of outer communication, however the particle-like solution fails to hold most of the stability conditions. In terms of the variable \( y = \sqrt{2g^2}F \), the Lagrangian (24) is

\[
L(y) = \frac{y^2}{2g^2} \text{sech}^2 x > 0,
\]

\[
L_y = \frac{y}{g^2} \text{sech}^2 x \left[ 1 - \frac{x}{2} \tanh x \right] > 0,
\]

\[
L_{yy} = \frac{\text{sech}^2 x}{g^2} \left[ 1 - \frac{x^2}{4} - \frac{7}{4} x \tanh x + \frac{3x^2}{4} \tanh^2 x \right] > 0,
\]

\[
f(y) = y L_{yy} / L_y > 0. \quad \text{(30)}
\]

where \( x = a \sqrt{y/2g} \). The condition \( L_y > 0 \) is fulfilled if \( 2 - xtanh x > 0 \), or \( x < 2.065 \). That in terms of \( y \) amounts to \( y < 2(2.065)^2 g/a^2 \). If we consider the black hole case, for which \( g/a^2 < 4\xi_0^2 \), then this condition is fulfilled if \( y < 2(2.065)^2 4\xi_0^2 = 31.439 \). In terms of \( r \), \( L_y > 0 \) is held if \( r_h > r > g/(2(2.065)\xi_0) = 0.252 g \), with \( r_h \) being the horizon radius. To define if this range is contained in DOC, we analyze the metric function \( N(r) \); in terms of \( y \) it is

\[
N(y) = 1 - \sqrt{\frac{yg}{2a^2}} \left[ 1 - \tanh (a \sqrt{\frac{y}{2g}}) \right]. \quad \text{(31)}
\]

Or in terms of the variable \( x = a \sqrt{y/2g} \), \( N(x) = 1 - x(1 - \tanh x)g/a^2 \); independently of the nonvanishing value of \( a \) and \( g \), \( N(x) \) has a single minimum (double horizon) for \( x_h = 0.64 \) that is the only positive solution for \( x \) in \( \partial_x N(x) = 0 \). This value amounts to \( g/a^2 = 3.571 \). The corresponding value is \( y_h = 2.93 \), or since \( y = 2g^2/r^2 \), \( r_h = 0.827 g \). Putting it together then, \( r_h = 0.827 g > r > 0.252 g \), so \( L_y > 0 \) in DOC.

In relation to the stability condition \( L_{yy} > 0 \), independently of the values of the constants, it holds in the range \( x < 0.886 \) and \( x > 2.827 \). In other words, the condition fails to hold in the range \( 0.886 < x < 2.827 \).

The inequality \( x < 0.886 \) is equivalent, in terms of \( r \), to \( r > 0.886 \sqrt{a^2/g} \) that for the black hole can be casted as \( r > g/(0.8862\xi_0) = 0.587 g \); then for the black hole \( L_{yy} > 0 \) since \( r_h = 0.827 g > r > 0.587 g \).

There is a fourth condition still to be fulfilled, \( 3 > N(y)f(y) \). Since \( 0 < \tanh x \leq 1 \), \( x > 0 \) then \( 0 < N(y) < 1 \) in DOC; also \( N(y)f(y) > 0 \). Moreover, in the range \( y < y_h = 2.93 \), \( N(y)f(y) \leq 1 \), satisfying the inequality \( 3 > N(y)f(y) > 0 \); holding then all the sufficient conditions for the black hole stability.

However, for the particle-like solution it can be shown that the stability condition \( L_y > 0 \) fails to hold. If \( y > 2(2.065)^2 g/a^2 \) then \( L_y < 0 \), or in terms of \( r \) if \( r < g/(2(2.065)\xi_0) \), that evidently can occur when \( r \) approaches the origin. The condition for \( L_{yy} < 0 \), \( x > 0.886 \) that amounts to \( r < g/(1.772\xi_0) \) can also be attained when \( r \) approaches the origin.

Since two of the conditions for stability are not fulfilled, in particular when \( L_y < 0 \) and \( L_{yy} < 0 \) the consequence is that the potential in the pulsation equations becomes negative and then perturbations can grow without limit. We conclude that the particle-like solution is unstable. A slightly perturbation of the soliton will lead the system to a Schwarzschild black hole and the total radiated energy will be equal to the difference between the ADM mass of the black hole minus its horizon mass. Therefore we have shown that in this case the unstability conjecture is valid.

IV. DYMNIKOVA SOLUTION

In 5 it was presented a SSS solution of NED coupled to gravity that satisfies the weak energy condition (WEC) and is an electrically charged regular structure. Discarding the requirement of a weak field limit at the center, with a de Sitter center, the no go theorems for electrically charged regular structures can be avoided. In
NED coupled to general relativity each electric solution has its magnetic counterpart. We shall address here the magnetic structure associated with Dynnikova’s electric solution, obtained via FP duality.

The Lagrangian for the magnetic case is of the form

$$L = F/(1 + \alpha \sqrt{F})^2$$

(32)

where $\alpha = r_0^2/g \sqrt{2}$, $r_0 = \pi g^2/8m$, $F = 2g^2/r^4$ and $g$ is the magnetic charge. The Lagrangian has stress-energy tensor with the algebraic structure $T^t_t = T^r_r$; WEC leads to de Sitter asymptotic at approaching a regular center. The corresponding function $M(r)$ in the SSS line element is

$$M(r) = \frac{2m}{\pi} \left[ \text{arctan} \left( \frac{r}{r_0} \right) - \frac{rr_0}{r^2 + r_0^2} \right] - \frac{q^2 r^2}{2(r^2 + r_0^2)^3}.$$  

(33)

As $r \to \infty$ it behaves as a Reissner-Nordström (RN) solution; for $r < r_0$ it is asymptotically de Sitter with a cosmological constant $\Lambda = g^2/2r_0^4$, $N(r \to 0) = 1 - \Lambda r^2/3$. The magnetic field is given by

$$B^2 = \frac{g^2 r^8 y^8}{(r^2 + r_0^2)^8},$$

(34)

In terms of $g/2m$ black hole exists for $g/2m \leq 0.536\sqrt{2}$ while if $g/2m > 0.536\sqrt{2}$ we have an electrically charged self-gravitating particle-like NED structure. The metric function $N(r)$ has a double zero (double horizon) for $r_\pm = 1.825r_0$. We also have, from the Komar integral, the ADM mass and the soliton mass $m$ given by

$$M_{ADM}(r) = \frac{2m}{\pi} \left[ \text{arctan} \left( \frac{r}{r_0} \right) - \frac{rr_0}{r^2 + r_0^2} \right] - \frac{q^2 r^2}{2(r^2 + r_0^2)^3}.$$  

(35)

$$\lim_{r \to \infty} M_{ADM} = m,$$  

(36)

For the magnetic version of Dynnikova’s solution, the ADM mass and the horizon mass, as functions of the horizon radius $r_\Delta$, are, respectively,

$$M_{ADM}(r_\Delta) = \frac{r_\Delta^2}{2} - \frac{2m}{\pi} \left[ \text{arctan} \left( \frac{r_\Delta}{r_0} \right) - \frac{rr_0}{r^2 + r_0^2} \right] + m,$$

$$M_\Delta(r_\Delta) = \frac{r_\Delta^2}{2} - \frac{2m}{\pi} \left[ \text{arctan} \left( \frac{r_\Delta}{r_0} \right) - \frac{rr_0}{r^2 + r_0^2} \right].$$

(37)

The limits when the horizon radius goes to zero are

$$\lim_{r_\Delta \to 0} M_\Delta(r_\Delta) = 0, \quad \lim_{r_\Delta \to 0} M_{ADM}(r_\Delta) = m,$$  

(38)

From the previous expressions the difference between the ADM mass and the horizon mass is positive, corresponding to the particle-like mass $m$, $M_{ADM}(r_\Delta) - M_\Delta = m$. Satisfying in this way the ACS mass relation conjecture; the question arises if it is related with black hole unstability. In terms of the variable $y = \sqrt{2g^2F}$, according to the analysis in \( \text{Eq. 37} \) the conditions on the electromagnetic Lagrangian which imply linear stability are

$$L(y) = \frac{y^2}{(g \sqrt{2 + \alpha g})^2} > 0,$$

(39)

$$L(y)_{,yy} = \frac{2\sqrt{2}gy}{(g \sqrt{2 + \alpha g})^3} > 0,$$

$$L(y)_{,yy} = \frac{4g(y - \sqrt{2\alpha y})}{(g \sqrt{2 + \alpha g})^4} > 0,$$

$$f(y) = yL_{,yy}/L_{,y} = \frac{\sqrt{2}(g - \sqrt{2}\alpha y)}{\sqrt{2g + \alpha g}} > 0.$$

(40)

The last two inequalities hold if $y < g/\sqrt{2}\alpha$. It is the case for the black hole solution that $y < g/\sqrt{2}\alpha$ in DOC. The double horizon occurs when $N(r) = 0$, with two possible roots, $r_\pm = 1.825r_0$; DOC involves distances $r > r_+$, or in terms of $y$,

$$y < \frac{2g^2}{r_+^2} = \frac{2g^2}{(1.825r_0)^2} = 0.6g/\sqrt{2\alpha},$$

(41)

Since $0.6g/\sqrt{2\alpha} < g/\sqrt{2}\alpha$ all the inequalities in \( \text{Eq. 37} \) are held. To complete the stability test for the black hole stability it must be shown that $3 > N(y)f(y)$. The metric function $N(y)$ is

$$N(y) = 1 - \frac{g}{2r_0} \sqrt{\frac{y}{2}} \arctan \left( \frac{g}{r_0} \sqrt{\frac{y}{2}} \right) + \frac{y}{2} \left[ 2 + y(y/r_0)^2 \right]^{-1}.$$  

(42)

If one considers that the extreme black hole occurs if $g/\sqrt{2\alpha} = g^2/\sqrt{2}/r_0^3 = \sqrt{2}(2.37)^2$, then one obtains $y_b = 3.37$. In the interval $0 < y < y_b = 3.37$, the range of $N(y)$ is $0 < N(y) < 1$. Moreover, since $N(0) = 1$ and $f(0) = 1$, the following inequality holds $0 < N(y)f(y) < 1$. In such a manner that this magnetic black hole has passed the stability test.

However, for the particle-like solution a simple analysis shows that neither $L_{,yy} > 0$ nor $f(y) > 0$ since $y < g/\sqrt{2\alpha}$ amounts to $r > r_0(2.036)^2/2$ that fails to hold ($g \neq 0$) when $r$ approaches the origin $(r \to 0)$. We can check that the total ADM mass of the black hole in Eq. \( \text{Eq. 37} \) is less than the sum of the ADM mass of the soliton $(m)$ and the horizon mass of the Schwarzschild black hole $(r_\Delta/2)$. If the NED soliton is unstable, then, if perturbed, its mass can be radiated away and one ends up with a bare black hole. In this case the NED black hole is considered as unstable, confirming the ACS unstability conjecture.
V. FINAL REMARKS

We have shown for three magnetic NED (coupled to gravity) SSS solutions that the ADM mass is greater than the horizon mass, \( M_{\text{ADM}}^{(b)} > M_{\text{hor}}^{(b)} \). Since the difference between the hamiltonian horizon mass and the ADM mass can be seen as the energy that is available for radiation to fall both into the black hole and to infinity, then the nonzero value of the Hamiltonian could be an indication of unstability of the NED studied solutions. On this basis, one can conjecture that NED black holes are unstable. We have shown that this conjecture is valid for some NED solutions, failing to be true in the case of the Einstein-Born-Infeld black hole and its corresponding particle-like solution, since the EBI solutions fulfill the sufficient stability conditions concluding then that the unstability conjecture does not apply generically for NED solutions.

We must mention that in relation to Born-Infeld black hole stability recently was presented in [11] the analysis of quasinormal modes for the gravitational perturbations, deriving a one dimensional Schrodinger type wave equation for the axial perturbations. From the behavior of the potentials it was concluded in [11] that the EBI black holes are classically stable.

The stability problem deserves a deeper analysis in relation to the energy conditions satisfied by the energy-momentum tensor since for the NED unstable solutions their corresponding energy-momentum tensor fulfills the weak energy condition only, while in the stable (EBI) case, the dominant energy condition is satisfied by the EBI energy-momentum tensor. Another feature worth to be mentioned is that the EBI black hole is a singular one, while the unstable cases (Bronnikov and Dymnikova solutions) correspond to regular black holes and the behavior of the solution as black hole or as particle-like depends on the value given to one parameter.

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