Speeding up the universe using dust with pressure

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We revise the cosmological standard model presuming that matter, i.e. baryons and cold dark matter, exhibits a non-vanishing pressure mimicking the cosmological constant effects. In particular, we propose a scalar field Lagrangian $\mathcal{L}_1$ for matter with the introduction of a Lagrange multiplier as constraint. We also add a symmetry breaking effective potential accounting for the classical cosmological constant problem, by adding a second Lagrangian $\mathcal{L}_2$. Investigating the Noether current due to the shift symmetry on the scalar field, $\varphi \rightarrow \varphi + \epsilon$, we show that $\mathcal{L}_1$ turns out to be independent from the scalar field $\varphi$. Further we find that a positive Helmotz free-energy naturally leads to a negative pressure without introducing by hand any dark energy term. To face out the fine-tuning problem, we investigate two phases: before and after transition due to the symmetry breaking. We propose that during transition dark matter cancels out the quantum field vacuum energy effects. This process leads to a negative and constant pressure whose magnitude is determined by baryons only. The numerical bounds over the pressure and matter densities are in agreement with current observations, alleviating the coincidence problem. Finally assuming a thermal equilibrium between the bath and our effective fluid, we estimate the mass of the dark matter candidate. Our numerical outcomes seem to be compatible with recent predictions on WIMP masses, for fixed spin and temperature. In particular, we predict possible candidates whose masses span in the range $0.5 - 1.7$ TeV.

I. INTRODUCTION

The $\Lambda$CDM concordance paradigm is described by the fewest number of assumptions possible. In particular, the universe is approximated at late times by two fluids: pressureless matter and a cosmological constant $\Lambda$. Both baryonic matter (BM) and cold dark matter (DM) are unable to push the universe to accelerate \cite{1}. Thus, besides dust-like fluids, one needs to include $\Lambda$ to account for the observed speed up. The simplicity of the concordance paradigm turns out to be the strong suit to admit its validity. However, the magnitude of $\Lambda$ predicted by quantum fluctuations of flat space-times leads to a severe fine-tuning problem with the observed value of $\Lambda$. Even considering a curved space-time one cannot remove the problem \cite{2}. Further, both baryonic and $\Lambda$ magnitudes are extremely close today, leading to the well-known coincidence problem \cite{3,4}. Under these aspects the $\Lambda$CDM model seems to be incomplete, whereas from a genuine observational point of view it well adapts to data.

In this work, we revise the cosmological standard model assuming an effective a scalar field $\varphi$ Lagrangian for baryons and cold DM. We require that matter provides a non-vanishing pressure term and we wonder whether it can accelerate the universe alone, i.e. without the need of $\Lambda$. To do so, we propose the most general Lagrangian, depending upon a kinetic term and Lagrange multiplier, with the inclusion of a potential term due to the vacuum energy cosmological constant, inducing a phase transition. During such an early-time phase transition the DM pressure counterbalances the $\Lambda$ pressure, leaving as unique contribute the pressure of baryons. In particular, the baryonic pressure turns out to be negative and guarantee a positive Helmotz free-energy for the whole system. In this picture, we find a Noether current, coinciding with the entropy density current and providing the Lagrangian to be independent from $\varphi$. We thus write up the thermodynamics associated to the model and we investigate small perturbations, finding the adiabatic and non-adiabatic sound speeds naturally vanish in analogy to the $\Lambda$CDM approach.

Our paradigm candidates as an alternative to the concordance model and predicts the existence of a single fluid, composed of baryons and cold DM with pressure. The fluid cancels out the quantum contribution due to $\Lambda$, driving the universe today with a constant and negative pressure. This process does not set $\Lambda$ to zero, but removes it naturally. This is possible if DM constituents lie on the mass interval $\sim 0.5–1.7$ TeV. To show this, we relate the predictions of our model to the thermal history of the primordial universe and to the expected DM relic abundance.

The paper is structured as follows. In Sec. II we propose the effective representation for matter with pressure. We thus write the equations of motion and we discuss the introduction of the potential term due to the vacuum energy cosmological constant. In Sec. III we describe the thermodynamics of our matter fluid, which naturally suggests an emergent negative pressure, and demonstrate that our Lagrangian does not depend upon $\varphi$. In Sec. IV we investigate the small perturbations and we find that

\footnote{\textsuperscript{1}At the end of transition.}
both the adiabatic and non-adiabatic sound speeds naturally vanish, leading to a constant pressure, in analogy to the ΛCDM approach. In Sec. IV we closely analyze the role of the effective potential $V^{\text{eff}}$. This term induces a first order transition phase during which the quantum field vacuum energy density mutually cancels with the DM pressure. Soon after the transition the emergent Λ field vacuum energy density naturally vanishes, leading to a constant pressure, in analogy to a curved space-time given by Lagrangian density $L = L_1 + L_2$, where

\begin{align*}
    L_1 & = K (X, \varphi) + \lambda Y [X, \nu (\varphi)] , \\
    L_2 & = -V^{\text{eff}} (X, \varphi),
\end{align*}

depend upon the scalar field $\varphi$ and its first covariant derivatives\footnote{Higher order derivatives are excluded because of the Ostrogradski’s theorem: systems characterized by a non-degenerate Lagrangian dependent on time derivatives of higher than the first leads to a linearly unstable Hamiltonian function.} in the form of the standard kinetic term

$$
X = \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi ,
$$

where $g^{\alpha\beta}$ is the metric tensor and $\nu (\varphi)$ plays the role of the specific inertial mass $m$.

The Lagrangian $L_1$ represents a dust component with pressure. It is written in the most generic form without indicating a priori the functional forms of the functions $Y$ and $K$, while the Lagrange multiplier $\lambda$ constrains the kinetic energy with the potential term in $\nu$. The physical motivation behind $L_1$ supports the idea of BM and DM with pressure \footnote{Higher order derivatives are excluded because of the Ostrogradski’s theorem: systems characterized by a non-degenerate Lagrangian dependent on time derivatives of higher than the first leads to a linearly unstable Hamiltonian function.}. It is important to stress that our fluid consists of BM and DM, so that in principle the Lagrangian $L_1$ can be written as

$$
L_1 = K_{BM} + K_{DM} + \lambda (Y_{BM} + Y_{DM}),
$$

where $K \equiv K_{BM} + K_{DM}$ and $Y \equiv Y_{BM} + Y_{DM}$.

The Lagrangian $L_2$ models the coupling with the standard cosmological constant through an interacting potential $V^{\text{eff}}$ used to investigate the phase transition. We write down the simplest form of $V^{\text{eff}}$ by

$$
V (\varphi, \psi) = V_0 + \frac{\chi}{4} (\varphi^2 - \varphi_0^2)^2 + \frac{\bar{g}^2}{2} \psi^2 ,
$$

in which the first two terms describe the self-interacting potential, with a dimensionless coupling constant $\chi$ of the scalar field $\varphi$, and the last one the interacting potential, with a dimensionless coupling constant $\bar{g}$, between $\varphi$ and another scalar field $\psi$. The quantity $V_0$ denotes the classical off-set, while $\varphi_0^2$ is the value of $\varphi$ at the minimum of its potential without interactions with $\psi$. We can thus assume that $\psi$ is in thermal equilibrium. In such a case, $\psi^2$ can be replaced through its average in a thermal state. So that, we can have: $\langle \psi^2 \rangle_T \propto T^2 \bar{\Omega}$. After cumbersome manipulations, we simple have

$$
V^{\text{eff}} (X, \varphi) = V_0 + \frac{\chi}{4} (\varphi^2 - \varphi_0^2)^2 + \frac{\bar{g}^2}{2} \psi^2 \frac{T^2 (X)}{T_c^2},
$$

where $T_c = \varphi_0 \sqrt{\chi / \bar{g}}$ is the critical temperature, discriminating as transition starts. Before the transition when $T > T_c$, the minimum of $V^{\text{eff}}$ is located at $\varphi = 0$ and the corresponding value is $V_0 + \chi \varphi_0^4 / 4$. After the transition, when $T < T_c$ the minimum is at $\varphi = \varphi_0$ with a value $V_0$. From Eqs. (2.1) – (2.2) we define the action $S = \int L \sqrt{-g^4} dx$, where $g$ is the determinant of $g^{\alpha\beta}$. Assuming a standard minimal coupling with gravity, from the variation of the action with respect to $\lambda$, $\varphi$ and the metric tensor we obtain a constraint and a dynamical
equation and the energy-momentum tensor respectively (details of calculations are reported in Appendix A)

\[ Y = 0 , \tag{2.7} \]
\[ \mathcal{L}_\varphi - \nabla_\alpha (\mathcal{L}_X \nabla^\alpha \varphi) = 0 , \tag{2.8} \]
\[ T_{\alpha\beta} = \mathcal{L}_X \nabla_\alpha \varphi \nabla_\beta \varphi - (K - V^{\text{eff}}) g_{\alpha\beta} , \tag{2.9} \]
where the subscripts label the partial derivatives, so that \( \mathcal{L}_X = K_X - V^{\text{eff}} + 2 \lambda X \) and \( \mathcal{L}_\varphi = K_\varphi - V^{\text{eff}} + \lambda \varphi \nabla_\varphi \).

For time-like derivatives it holds \( X > 0 \) and, from Eq. (2.8), we can introduce an effective 4-velocity

\[ u_\alpha = \frac{\nabla_\alpha \varphi}{\sqrt{2X}} , \tag{2.10} \]

while the 4-acceleration identically vanishes

\[ a_\beta = \dot{v}_\beta = v_\gamma \nabla^\gamma v_\beta = 0 , \tag{2.11} \]
where \( \dot{y} = v^\alpha \nabla_\alpha y \) is the Lie derivative of \( y \) along \( v^\alpha \), which is tangent to time-like geodesics. Using Eq. (2.10), the energy-momentum tensor can be written as

\[ T_{\alpha\beta} = 2X \mathcal{L}_X u_\alpha v_\beta - (K - V^{\text{eff}}) g_{\alpha\beta} , \tag{2.12} \]

which is of the perfect fluid form for an energy density and a pressure, respectively

\[ \rho (\lambda, X, \varphi) = 2X \mathcal{L}_X - (K - V^{\text{eff}}) , \tag{2.13} \]
\[ P (X, \varphi) = K - V^{\text{eff}} . \tag{2.14} \]

Thus, from the above definitions one wonders whether it is possible to fulfill the weak energy conditions \( T_{\alpha\beta}k^\alpha k^\beta \geq 0, \rho \geq 0, \rho + P \geq 0 \), where \( k^\alpha \) is a time-like vector field. From the above conditions and the fact that \( X > 0 \), it follows that

\[ 2X \mathcal{L}_X \geq K - V^{\text{eff}} , \quad \mathcal{L}_X \geq 0 . \]

The energy-momentum tensor conservation gives

\[ \nabla_\alpha T^{\alpha\beta} = [\rho + \theta (\rho + P)] \dot{v}_\beta = 0 , \tag{2.15} \]
leading to the energy conservation \( \rho + \theta (P + \rho) = 0 \) where we defined the expansion

\[ \theta = \nabla_\alpha v^\alpha = \frac{\nabla_\alpha \nabla^\alpha \varphi - X \varphi}{\sqrt{2X}} . \tag{2.16} \]

The energy flux \( T^{\alpha\beta} v_\beta = \rho v^\alpha \) always follows time-like geodesics, as for the perfect fluid with no pressure. By means of this position

\[ \eta_\varphi = 2X (\mathcal{L}_XX \varphi + \mathcal{L}_X \varphi) + \mathcal{L}_X \varphi - \mathcal{L}_\varphi , \tag{2.17} \]
the constraint in Eq. (2.8) can be written as follows

\[ \lambda = - \frac{1}{2XY \mathcal{L}_X \eta_\varphi + \theta (P + \rho)} . \tag{2.18} \]

Eqs. (2.7) and (2.18) represent the equations of motion for a perfect fluid ruled by two first-order ordinary differential equations for the scalar fields \( \varphi \) and \( \lambda \)

### III. THERMODYNAMICS OF MATTER WITH PRESSURE

Now we address the thermodynamics of the perfect fluid described by the effective Lagrangian in Sec. II. Non-dissipative fluids are described by virtue of the pull-back formalism through Carter’s covariant formulation in a relativistic effective field theory. Generally fluids are framed with four scalar fields, namely \( \phi^a \). In this puzzle three scalar fields, corresponding to \( a = 1, 2, 3 \), become fluid comoving coordinates as they propagate in space, whereas \( \phi^0 \) is interpreted as an internal time coordinate. These scalars can be viewed as St"uckelberg fields that allow to restore broken diffeomorphisms in four-dimensional spacetimes. So that, the fluid physical properties are encoded within a set of symmetries of the scalar field action.

We here are interested in cosmological perfect fluids describing the matter sector only. Such a framework may be seen as in Sec. II. Moreover we only deal with the temporal St"uckelberg field, hereafter renamed \( \varphi \). In other words, providing the cosmological principle, it is licit to take into account that our model is well motivated if one fluid is accounted, say \( \phi^0 \equiv \varphi \). Without considering the spatial fields implies that the corresponding Lagrangian respects the global shift symmetry

\[ \varphi \to \varphi + c^0 , \tag{3.1} \]

with \( c^0 \) an arbitrary constant.

The scenario defined in Sec. II turns out to describe a barotropic matter fluid, i.e., its pressure is completely defined by knowing its energy density and viceversa, see Eqs. (2.8) and (2.9). To provide its thermodynamic interpretation, we choose the particle number density \( n \) and the temperature \( T \) of the fluid as thermodynamic variables to find correspondence with the field \( X \).

To account for perfect fluid thermodynamics we use the first principle, the Gibbs-Duhem relation and the Helmholtz free-energy density \( f = p - Ts \), respectively

\[ dp = T ds + \mu dn , \tag{3.2} \]
\[ dP = s dT + n d\mu , \tag{3.3} \]
\[ df = \mu dn - s dT . \tag{3.4} \]

where \( s \) is the entropy density and \( \mu \) the chemical potential. Combining Eqs. (3.2)–(3.3) and (2.13)–(2.14) with the definitions of \( f \) we get

\[ d(\mu n) = d (2X \mathcal{L}_X) - d (Ts) , \tag{3.5} \]
\[ df = d (2X \mathcal{L}_X) - d (K - V^{\text{eff}}) - d (Ts) . \tag{3.6} \]

This name commonly designates a field that makes explicit a (spontaneously broken) gauge symmetry.
Keeping in mind that in view of Eq. (2.7) \( d \left( K - V^{\text{eff}} \right) \equiv dL \), the above two relations admit as solutions:

\[
\begin{align*}
    f &= -L, \\
    s &= \sqrt{2X}L_X, \\
    T &= \sqrt{2X}, \\
    \mu &= 0.
\end{align*}
\]

Equation (3.7) fulfills the thermodynamic relations:

\[
\begin{align*}
    \frac{\partial (fV)}{\partial T} \bigg|_V &= -\sqrt{2X}L_X V = -sV, \\
    \frac{\partial (fV)}{\partial V} \bigg|_T &= -L = -(K - V^{\text{eff}}) = -P.
\end{align*}
\]

We deduce that \( P \) turns out to be naturally negative as we guarantee the Helmholtz free energy is positive-definite:

\[
f > 0 \Rightarrow P < 0.
\]

This fact implies that dust-like matter having pressure naturally fixes the sign of \( P \) to be negative. This ensures no need of putting by hand the sign of \( P \) inside Einstein’s equations to guarantee the universe speed up.

Now we define the densities of internal energy \( u \), enthalpy \( h \), and Gibbs free-energy \( g \) respectively by:

\[
\begin{align*}
    u &= \rho = 2X L_X - (K - V^{\text{eff}}), \\
    h &= u + P = 2X L_X, \\
    g &= f + P = 0.
\end{align*}
\]

Invoking the Noether’s theorem we notice the global shift symmetry changes the matter Lagrangian density \( L_1 \) mostly by a total divergence. We explicitly get:

\[
\begin{align*}
    L_1 (X', \varphi') &= L_1 \left( \frac{1}{2} \nabla_\alpha \varphi \nabla^\alpha \varphi + c^0 \right) = \\
    L_1 (X, \varphi) + c^0 \left[ \frac{\partial L_1}{\partial \varphi} - \nabla_\alpha \frac{\partial L_1}{\partial (\nabla_\alpha \varphi)} \right] + c^0 \nabla_\alpha \frac{\partial L_1}{\partial (\nabla_\alpha \varphi)} = \\
    L_1 (X, \varphi) + c^0 \nabla_\alpha (L_1 X \nabla_\alpha \varphi),
\end{align*}
\]

where, in the second line of Eq. (3.14), the quantity in the brackets identically vanishes in view of the Euler–Lagrange equation. The conserved current \( J_1^\alpha \) corresponds to the total divergence of Eq. (3.14), i.e.,

\[
J_1^\alpha = \sqrt{2X} \left( K_X + \lambda Y \right) v^\alpha.
\]

At this point it behooves us to discuss how to deal with \( V^{\text{eff}} \). Its behavior during the phase transition is not trivial, since it depends on both \( \varphi \) and its covariant derivatives via the kinetic term \( X \). Instead, \( V^{\text{eff}} \) is well defined in its minima, namely before the transition (BT, with \( V^{\text{eff}} = V_0 + \chi \varphi_0^2/4 \) at \( \varphi = 0 \)), and after the transition (AT, with \( V^{\text{eff}} = V_0 \) at \( \varphi = \varphi_0 \)), when it is a function of \( X \), i.e., \( V^{\text{eff}} = V^{\text{eff}}(X) \). For the above reasons, in the following we limit our investigation to the meaningful BT and AT cases, without assessing the intermediate cases, i.e. during transition. Therefore, during the BT and the AT phases, the Noether’s theorem implies that

\[
L_2 (X') = -V^{\text{eff}}(X) - c^0 \nabla_\alpha (V_X^{\text{eff}} \nabla_\alpha \varphi),
\]

where we can define another conserved current from the total divergence of Eq. (3.17), i.e.,

\[
J_2^\alpha = -\sqrt{2X} V^{\text{eff}}(X) v^\alpha.
\]

Hence the total conserved current \( J^\alpha \) is given by combining Eqs. (3.16) and (3.17), i.e.,

\[
J^\alpha = J_1^\alpha + J_2^\alpha = \sqrt{2X} L_X v^\alpha = s^\alpha,
\]

and coincides with the entropy density current \( s^\alpha = sv_\alpha \).

Eq. (3.10) simplifies Eq. (2.5) into \( \nabla_\alpha v^\alpha = 0 \), implying that the Lagrangian does not depend upon \( \varphi \). We thus have:

\[
L (\lambda, X, \nu) = K (X) - V^{\text{eff}} (X) + \lambda Y (X, \nu).
\]

By combining Eqs. (3.2)–(3.3) we get the Euler relation

\[
P + \rho = Ts + \mu n,
\]

and recast the energy-momentum tensor as

\[
T_{\alpha\beta} = (Ts_\alpha + \mu n_\alpha) v_\beta + Pg_{\alpha\beta},
\]

where \( n_\alpha = n v_\alpha \) is the particle number density current. The projection of the energy-momentum tensor conservation along \( v^\alpha \), i.e., \( v^\alpha \nabla_\beta T_{\alpha\beta} = 0 \), leads to

\[
T \nabla^\alpha s_\alpha + \mu \nabla^\alpha n_\alpha = 0,
\]

and by virtue of the existence of \( J^\alpha \), it reduces to

\[
\mu \nabla^\alpha n_\alpha = 0,
\]

which represents and identity, since \( \mu = 0 \). However, one can also safely assume that the particle number density current is also conserved, i.e., \( \nabla^\alpha n_\alpha = 0 \).

The conservation of the energy-momentum tensor can be recast as the Carter-Lichnerowicz equations

\[
n \nabla_{\alpha \nu} v_\nu = nT \nabla_\alpha \sigma - \zeta_\alpha \nabla^\nu n_\nu,
\]

where \( \zeta_\alpha = \nabla_\nu s_\alpha - \nabla_\alpha s_\nu \) is the vorticity tensor, \( \zeta_\alpha = h/\nu v_\alpha \) the current of the enthalpy per particle, and \( \sigma = s/\nu \) the entropy per particle.

Since the 4-velocity is the derivative of the scalar field \( \varphi \) and \( \nabla^\alpha n_\alpha = 0 \), we infer from Eq. (3.25) that:

\[
\begin{align*}
    \nabla_\alpha \varsigma_\alpha &= 0 \Rightarrow \text{the fluid is irrotational}, \\
    \nabla_\alpha \sigma &= 0 \Rightarrow \text{the fluid is isentropic},
\end{align*}
\]

respectively from the first and second conditions.
IV. COSMOLOGICAL PERTURBATIONS AND THE ROLE OF SOUND SPEED

In the previous sections we demonstrated that our matter fluid is irrotational and insentropic. We now discuss the cosmological perturbations taking into account our Lagrangian, as in Eq. (5.20). We thus unveil additional features characterizing our matter fluid concerning the magnitudes of the pressure $P$ and the sound speed.

In the conformal Newtonian gauge, in absence of any anisotropic stress, we consider

$$ds^2 = a(t)^2 \left[ (1 + 2 \Phi)^2 - (1 - 2 \Phi) dx^2 \right] ,$$  \hspace{1cm} (4.1)

where $\Phi$ is the Newtonian potential, $\tau = a(t)t$ the conformal time and $a(t)$ the scale factor. The first order $(0,0)$, $(0,i)$ and $(i,j)$ components of Einstein’s equations in the Friedmann-Robertson-Walker model are respectively,

$$\nabla^2 \Phi - 3 \mathcal{H} (\Phi' + \mathcal{H} \Phi) = 4 \pi a^2 G \delta \rho ,$$  \hspace{1cm} (4.2)

$$\nabla_i (\Phi' + \mathcal{H} \Phi) = 4 \pi a^2 \left( \rho + \mathcal{P} \right) \delta v_i ,$$  \hspace{1cm} (4.3)

$$\Phi'' + 3 \mathcal{H} \Phi' + \left( 2 \mathcal{H'} + \mathcal{H}^2 \right) \Phi = 4 \pi a^2 G \delta P ,$$  \hspace{1cm} (4.4)

where the prime denotes the derivatives with respect to the conformal time and $\mathcal{H} = a'/a$. The perturbation of the 3-velocity $\delta v_i = \nabla_i \delta \varphi / (a \varphi')$ depends upon the perturbation of the scalar field $\delta \varphi$ which depends also on the spatial coordinates. The density perturbations depend on the kinetic term and on the Lagrange multiplier perturbations, respectively $\delta \lambda$ and $\delta \lambda$, whereas the pressure perturbations depend only upon $\delta X$, so that:

$$\delta \rho = A(X) \delta X + 2 X \lambda Y X \nu \delta \nu + 2 X Y X \lambda \delta \lambda ,$$  \hspace{1cm} (4.5)

$$\delta P = B(X) \delta X .$$  \hspace{1cm} (4.6)

To infer the explicit expressions of $A(X)$ and $B(X)$, we discriminate between two regimes [27]:

- BT, when the minimum is at $\varphi = 0$ and the potential is $V_{\text{eff}} = V_0 + \lambda \varphi_0^4/4$;

- AT, when the minimum is at $\varphi = \varphi_0$ and the potential is $V_{\text{eff}} = V_0$.

Hence, the pressure and density become

$$P(X) = \left\{ \begin{array}{l} K - V_0 - \lambda \varphi_0^2/4 \hspace{1cm} \text{(BT)} \\ K - V_0 \hspace{1cm} \text{(AT)} \end{array} \right. ,$$  \hspace{1cm} (4.7)

$$\rho(X) = \left\{ \begin{array}{l} 2X \mathcal{L}_X - K + V_0 + \lambda \varphi_0^2/4 \hspace{1cm} \text{(BT)} \\ 2X \mathcal{L}_X - K + V_0 \hspace{1cm} \text{(AT)} \end{array} \right. ,$$  \hspace{1cm} (4.8)

From the above definitions it follows that

$$A(X) = (2X \mathcal{L}_X)_X - K_X ,$$  \hspace{1cm} (4.9)

$$B(X) = K_X .$$  \hspace{1cm} (4.10)

Combining Eqs. (4.2) and (4.4) we get

$$\Phi'' + 3 \mathcal{H} (1 + c_X^2) \Phi' + \left[ 2 \mathcal{H}' + (1 + 3 c_X^2) \mathcal{H}^2 \right] \Phi +$$

$$- c_X^2 \nabla^2 \Phi = 4 \pi a^2 G \left[ D(X) \delta \nu + E(X) \delta \lambda \right] ,$$  \hspace{1cm} (4.11)

in which

$$c_X^2 \equiv B(X)/A(X) ,$$  \hspace{1cm} (4.12)

$$D(X) \equiv - 2X \lambda Y X \nu c_X^2 ,$$  \hspace{1cm} (4.13)

$$E(X) \equiv - 2X Y X c_X^2 .$$  \hspace{1cm} (4.14)

The evolution of $\Phi$ in terms of $\rho$ and $\sigma$ perturbations can be written as [24]

$$\Phi'' + 3 \mathcal{H} (1 + c_X^2) \Phi' + \left[ 2 \mathcal{H}' + (1 + 3 c_X^2) \mathcal{H}^2 \right] \Phi +$$

$$- c_X^2 \nabla^2 \Phi = 4 \pi a^2 G \delta \sigma \delta \lambda ,$$  \hspace{1cm} (4.15)

where

$$c_X^2 \equiv \partial P/\partial \rho|_\sigma .$$  \hspace{1cm} (4.16)

is the square of the adiabatic speed of sound and $\zeta \equiv \partial P/\partial \sigma|_\rho$. From the above considerations, one can define the entropy perturbation shift, $\Delta$, which quantifies how much $\delta P/\delta \rho$ departs from $c_X^2$. It can be written as:

$$\Delta = \left( \frac{\delta P}{\delta \rho} - c_X^2 \right) \frac{\delta \rho}{P} = - \frac{D(X) \delta \nu + E(X) \delta \lambda}{P} .$$  \hspace{1cm} (4.17)

For isentropic fluids (see Sec. III), it immediately follows that $\zeta = 0$. This is in agreement with our previous outcomes, since from Eq. (4.11) we require $Y_X \neq 0$ and so $c_X^2 \equiv 0$ as one assumes $P = \text{const}$ and vice-versa [28,31].

Taking into account that $P = \text{const}$, we may draw relevant consequences on our fluid temperature. Indeed, according to Eq. (4.11) we find our fluid to lie on the minimum of the Gibbs energy, i.e. at an equilibrium state. By combining Eqs. (3.3), (3.4), and (3.10) we get

$$dg = dP - s dT = 0 .$$  \hspace{1cm} (4.18)

Since $P = \text{const}$, one necessarily has $T = \text{const}$ in the proximity of each minimum of the effective potential.

Last but not least, it is worth noticing that an isentropic fluid can be even attained from Eq. (4.11) by setting $\lambda \to 0$. However, this would represent a particular case for which the Lagrangian term $Y$ has no longer relevance. For the sake of generality, this case is thus excluded into our picture.

V. CONSIDERATIONS ON QUANTUM VACUUM ENERGY

We now analyze in more details the role played by the effective potential $V_{\text{eff}}$. In particular, we wonder whether the two possible choices of the off-set $V_0$ provide different physical considerations. Hence, to alleviate the degeneracy between the two approaches, we need to fix the magnitude associated to $K$. Our target is to bound $K$ in order to heal the fine-tuning issue associated to the cosmological constant $\Lambda$.

We thus explore two possibilities:

\footnote{That is requested to guarantee the validity of Eq. (2.18).}
1) $V_0 = -\chi \varphi_0^4/4$, so BT we have $V^{\text{eff}} = 0$ and hence

$$P_1 = \begin{cases} K & \text{(BT)} \\ K + \chi \varphi_0^4/4 & \text{(AT)} \end{cases},$$

$$\rho_1 = \begin{cases} 2X \lambda Y - K & \text{(BT)} \\ 2X \lambda Y - \chi \varphi_0^4/4 & \text{(AT)} \end{cases},$$ (5.1)

and since $f > 0$, then $K < -\chi \varphi_0^4/4$.

2) $V_0 = 0$, so AT we have $V^{\text{eff}} = 0$ and hence

$$P_2 = \begin{cases} K - \chi \varphi_0^4/4 & \text{(BT)} \\ K & \text{(AT)} \end{cases},$$

$$\rho_2 = \begin{cases} 2X \lambda Y - K + \chi \varphi_0^4/4 & \text{(BT)} \\ 2X \lambda Y - K & \text{(AT)} \end{cases},$$ (5.2)

and since $f > 0$, then $K < 0$.

In both cases $K < 0$, but with different magnitudes.

1. The case $V_0 = -\chi \varphi_0^4/4$

Since $X_\varphi = 0$ and $P = \text{const}$, from Eq. (2.17) we get that $\eta_\varphi = 0$, and, therefore, Eq. (2.18) reduces to

$$\dot{\lambda} = -\theta \lambda,$$ (5.5)

In the Friedmann-Robertson-Walker spacetime, $\varphi$ is a function of the time only, thus it is easy to demonstrate that $X = \dot{\varphi}^2/2$ and $\theta = 3\dot{a}/a$. Finally, the solution of Eq. (5.5) becomes

$$\lambda = \lambda_0 a^{-3},$$ (5.6)

where $\lambda_0$ is a constant. Further, in the Friedmann-Robertson-Walker scenario the simplest choice for the (adiabatic) volume may be $V = V_0 a^3$, where $V_0$ the initial volume. Recalling that our fluid is isentropic, with constant $P$ and $T$, by using Eqs. (5.8) and (5.6), we get

$$sV = \sqrt{2X \lambda_0 V_0 Y} = \text{const},$$ (5.7)

from which it follows that $Y_X = Y_{BM,X} + Y_{DM,X} = \text{const}$.

We propose the following assumptions:

$$K_{\text{DM}} \approx -\chi \varphi_0^4/4,$$ (5.8)

$$K_{\text{BM}} \ll K_{\text{DM}}.$$ (5.9)

These positions and the fact that $a = (1 + z)^{-1}$ (where $z$ is the redshift) allow us to rewrite Eqs. (5.1)–(5.2) as

$$P_1 \approx \begin{cases} K_{\text{DM}} & \text{(BT)} \\ K_{\text{BM}} & \text{(AT)} \end{cases},$$

$$\rho_1 \approx \begin{cases} (\rho_{\text{DM}} + \rho_{\text{BM}}) (1 + z)^3 - K_{\text{DM}} & \text{(BT)} \\ (\rho_{\text{DM}} + \rho_{\text{BM}}) (1 + z)^3 - K_{\text{BM}} & \text{(AT)} \end{cases},$$ (5.10)

where $\rho_{\text{BM}} = 2X \lambda_0 Y_{BM,X}$ and $\rho_{\text{DM}} = 2X \lambda_0 Y_{DM,X}$ are constants.

This mechanism elides the vacuum energy cosmological constant contribution through the use of DM. As $\chi > 0$, the sign of $K_{\text{DM}}$ is opposite to the vacuum energy term. Hence, from the one hand the DM fluid pushes the universe up to accelerate, while on the other hand vacuum energy provides the opposite contribution in the net pressure.

Then, AT the universe accelerates because of the presence of a negative baryonic pressure. This plays the role of emergent cosmological constant, which is is negligible with respect to the vacuum energy BT, whereas becomes dominant AT. Since its magnitude is due to the baryon pressure, this alleviates the coincidence problem. In addition the fine-tuning problem is clearly removed because the high value of the predicted vacuum energy density is suppressed and does not enter our framework AT.

2. The case $V_0 = 0$

Eqs. (5.5)–(5.7) still hold and retain the same form. However, in this case the only needed assumption to get the measured cosmological constant is that $K \ll \chi \varphi_0^4/4$. Therefore we obtain

$$P_2 \approx \begin{cases} -\chi \varphi_0^4/4 & \text{(BT)} \\ K & \text{(AT)} \end{cases},$$ (5.12)

$$\rho_2 \approx \begin{cases} (\rho_{\text{DM}} + \rho_{\text{BM}}) (1 + z)^3 + \chi \varphi_0^4/4 & \text{(BT)} \\ (\rho_{\text{DM}} + \rho_{\text{BM}}) (1 + z)^3 - K & \text{(AT)} \end{cases},$$ (5.13)

where the BM can be considered even pressureless.

In this case the vacuum energy density cancels without the effect of any matter component. This occurrence is due to the discontinuity of the effective potential introduced by the phase transition only. The emergent cosmological constant appears soon after the transition as related to the DM sector of the universe and holds the ad hoc value to justify the observed acceleration of the universe. Therefore, this case still suffers from the coincidence problem, which affects the $\Lambda$CDM model. Moreover, differently from the previous case and in analogy with the concordance model, the baryons do not play a significant role in speeding up the universe. Indeed, they can be viewed as pressureless particles.

VI. TEMPERATURE AND MASS OF THE DM CANDIDATE

As discussed above, the $V_0 = -\chi \varphi_0^4/4$ case is preferred over $V_0 = 0$, to avoid discontinuities in the pressure contribution. In so doing, one may break the degeneracy between the two approaches, choosing the case $V_0 = -\chi \varphi_0^4/4$ which corresponds to a dark fluid defined by matter with pressure. Thus, limiting on $V_0 = -\chi \varphi_0^4/4$ we draw in the thermal universe the bounds over the DM constituent as particle candidate for DM enabling
the process for that the DM pressure elides the vacuum energy contribution. The energy and number densities, together with the pressure of each particle, can be computed as

$$
\epsilon = g_s (k_B T)^3 \frac{3}{2\pi^2 \hbar^3 c^4} \int_0^\infty \frac{\xi^2 \sqrt{\xi^2 + A^2}}{e^{\sqrt{\xi^2 + A^2}}} + 1 \, d\xi, \quad (6.1)
$$

$$
N = g_s (k_B T)^3 \frac{3}{2\pi^2 \hbar^3 c^4} \int_0^\infty \frac{\xi^2}{e^{\sqrt{\xi^2 + A^2}}} + 1 \, d\xi, \quad (6.2)
$$

$$
P = g_s (k_B T)^4 \frac{3}{2\pi^2 \hbar^3 c^5} \int_0^\infty \frac{\xi^4 \sqrt{\xi^2 + A^2}}{3e^{\sqrt{\xi^2 + A^2}}} + 1 \, d\xi, \quad (6.3)
$$

where $\xi = p c / (k_B T)$, $A = m c^2 / (k_B T)$, $g_s = 2s + 1$ is the spin degeneracy parameter, $e$ the speed of light, $\hbar$ the reduced Planck constant and $k_B$ the Boltzmann constant. The current value of the universe critical temperature is obtained for each of them, i.e., $\epsilon_{\text{BT}}$, $N_{\text{BT}}$, and $P_{\text{BT}}$. Independently from the offset on $\rho_0$, the BT total energy density in Eqs. (6.10)-(6.13) is given by

$$
\epsilon_{\text{BT}} = \frac{\pi^2 (k_B T p)^4}{30 (hc)^3}, \quad (6.5)
$$

where $g_s$ is the sum of the standard term $g^{\text{ST}}_s = \sum_b g_b + \frac{3}{2} \sum_f g_f \approx 100.75$ and our DM particle term $g_{\text{DM}} = 2s_{\text{DM}} + 1$ with spin $s_{\text{DM}}$. Independently from the offset on $\rho_0$, the BT total energy density in Eqs. (6.10)-(6.13) is given by

$$
\epsilon_{\text{BT}} = \left[ \Omega_r \left( \frac{T_p}{T_0} \right)^4 + \Omega_m \left( \frac{T_p}{T_0} \right)^3 + \Omega_\Lambda \right] \epsilon_c + \epsilon_v, \quad (6.6)
$$

where $\Omega_m = 0.3089 \pm 0.0062$, $\Omega_\Lambda = 0.6911 \pm 0.0062$, and $\Omega_r = (9.16 \pm 0.19) \times 10^{-5}$ are, respectively, the BM+DM, $\Lambda$, and the radiation density parameters, and $\epsilon_c = \rho_c c^2$, where $\rho_c = 3H^2 / (8\pi G)$ is the universe critical density, in which $H$ is the Hubble parameter and $G$ the gravitational constant. The current value of the universe critical density and $\epsilon_{\text{BT}}$, $\rho_{c,0} = (8.62 \pm 0.12) \times 10^{-30} \, \text{g/cm}^3$. With respect to Eqs. (6.10)-(6.13), we include the radiation, which is not negligible at early times, and use the relation $T_p / T_0 = (1 + z)$, in which $T_p$ is the cosmic plasma temperature and $T_0 = 2.725$ K the current

Cosmic Microwave Background temperature.

Finally, $\epsilon_v = 7.74 \times 10^{16}$ erg/cm$^3$ is the vacuum energy density. By equating Eq. (6.6) and Eq. (6.5) and solving numerically, we get the plasma temperature

$$
T_p = (6.6559 \pm 0.0019) \times 10^{14} h(s_{\text{DM}}) K, \quad (6.7)
$$

where $h(0) = 1$, $h(1) = 0.995$, and $h(2) = 0.991$.

The primordial DM interactions can be viewed as the annihilation of a heavier DM particle $Q$ and its antiparticle $\bar{Q}$, both with masses $M$, to produce two lighter particles $q$ and $\bar{q}$. Assuming no initial asymmetry between the particles $Q$ and $\bar{Q}$, their comoving density must be the same, i.e., $n_Q = n_{\bar{Q}} \equiv n$; on the other hand $q$ and $\bar{q}$ are tightly coupled to the cosmic plasma. Therefore the Boltzmann equation for the evolution of $n$ writes as

$$
\frac{1}{a^3} \frac{d}{dt} (a^3 n) = - \langle \epsilon v \rangle (n^2 - n_{\text{eq}}^2), \quad (6.8)
$$

where $n_{\text{eq}}$ is the equilibrium number density and $\langle \epsilon v \rangle$ the thermally averaged cross-section. From the entropy conservation, we write the number density as an adimensional quantity $N = n k_B / s$. Then, we note that the comoving time $t$ is related to $A$ by: $dA = HA dt$. Before neutrino decoupling, the entropy density degeneracy parameters is $g_s^{\text{DM}} \approx 32$, therefore the total entropy density is given by $s = 2\pi^2 k_B^2 T^4 / [45(hc)^3]$. From the identity $\epsilon_{\text{BT}} \equiv \rho_c c^2$, we obtain

$$
H \equiv (\frac{\dot{a}}{a}) = \sqrt{\frac{4\pi^3 c^3 g_s G}{45 h^3}} \left( \frac{M}{A} \right)^2. \quad (6.9)
$$

From the above definitions, we can recast the Boltzmann equation to obtain a Riccati-like equation

$$
\frac{dN}{dA} = - \frac{\Gamma}{A^2} (N^2 - N_{\text{eq}}^2), \quad (6.10)
$$

where we defined the interaction rate

$$
\Gamma \equiv \sqrt{\frac{g_s \pi c^5}{45 \hbar G}} \langle \epsilon v \rangle M. \quad (6.11)
$$

Fig. 4 shows $N(A)$ for $\Gamma = 10^5, 10^6, 10^{11}$, and $10^{14}$. The value of the DM relic abundance is given by

$$
N_\infty \approx A_t / \Gamma, \quad (6.12)
$$

where $A_t$ marks the transition from the relativistic regime to the non-relativistic one. For the above wide range of $\Gamma$, we can safely assume that the non-relativistic regime is attained for $A_t = 10^{-30}$. We assume that at this stage the temperature is approximately the above

5 For the sake of clearness, we hereafter restore the usual physical constants, previously set to 1.
The mass range of the DM boson particle candidate depending on the spin particle. Columns list DM spin $s_{DM}$, the function $h(s_{DM})$, and the mass range of the DM particle.

| $s_{DM}$ | $h(s_{DM})$ | $M$ (TeV) |
|---------|------------|---------|
| 0       | 1.000      | 0.574–1.723 |
| 1       | 0.995      | 0.572–1.715  |
| 2       | 0.991      | 0.569–1.708  |

The freeze-out occurs at $A_f = 11, 17, 25, 32$, respectively.

VII. PREDICTIONS OF OUR PARADIGM

We here sum up the main results of our paradigm. We revise the concordance model, assuming the most general Lagrangian for matter with pressure. To do so, we consider a transition phase induced by the effective potential of a vacuum energy cosmological constant, with a mechanism in which the DM pressure elides the vacuum energy pressure itself. So that we obtain:

\[ P = \text{const (always)} \Rightarrow c_s^\text{DM} = c_s^\text{BM} = 0, \tag{7.1} \]
\[ P < 0 \text{ (from thermodynamics)}, \tag{7.2} \]
\[ T = \text{const (during the transition)}, \tag{7.3} \]
\[ P_{DM} \gg P_{BM}, \quad P_{DM} \approx \epsilon_v, \tag{7.4} \]
\[ \rho_A \equiv P_{DM} (BT), \tag{7.5} \]
\[ \rho_A \equiv P_{BM} (AT), \tag{7.6} \]
\[ 0.5 \lesssim M^2 / \text{TeV} \lesssim 1.7 \text{ (Cold Dark Matter)}, \tag{7.7} \]
\[ 0.81 \lesssim \langle x v \rangle / (10^{-26} \text{cm}^3 \text{s}^{-1}) \leq 2.42. \tag{7.8} \]

Hence, in our scheme there exists only one perfect, irrotational, and isentropic fluid, composed of BM and DM. $\Lambda$ is coupled with the matter. The thermodynamics of such a fluid naturally suggests an emergent negative pressure. The effective potential $V_{\text{eff}}$ induces a transition phase during which the quantum vacuum energy density mutually cancels with the DM pressure. Soon after the transition the emergent cosmological constant is given by the (negative) pressure of baryons. This overcomes the fine-tuning problem between the predicted and observed values of $\Lambda$ and the coincidence problem, due to the fact that it is the matter which induces the effective cosmological constant at late times and, therefore, it is natural that their magnitudes are extremely close today. The model mimics the $\Lambda$CDM effects, without departing from observations made at both late and early stages of universe's evolution.

As principal responsible for DM in the universe, our predictions on the mass constituents, i.e. $0.5 \lesssim M \lesssim 1.7$ TeV, leave open the possibility to detect in laboratory additional heavier bosons, e.g. for example additional $Z'$ or $W'$ bosons or Leptoquarks as potentially predicted by extensions of the particle standard model.

VIII. FINAL OUTLOOKS AND PERSPECTIVES

In this work, we proposed an alternative model to the standard $\Lambda$CDM paradigm. We assumed the existence of a single fluid composed by matter only, i.e. baryons and cold DM. The fluid pushes the universe up, canceling the quantum contribution due to the cosmological constant through the assumption that matter shows a non-vanishing pressure. In particular, we proposed that both DM and BM are collisional, through a generalized scalar field $\phi$ representation of the matter fluid Lagrangian $L_1$ depending upon a kinetic term, $X$, and a Lagrange multi-

\[ \phi \approx \phi_0 \Rightarrow \Omega_{\phi} \approx \Omega_{\phi} / \rho_\phi \Rightarrow \rho_{\phi} \approx \rho_{\phi}, \tag{5.11} \]
\[ \Omega_{\phi} \approx \Omega_{\phi} + \Omega_{\phi} \approx \Omega_{\phi} \tag{6.18} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.11} \]
\[ \Omega_{\phi} \approx \Omega_{\phi} + \Omega_{\phi} \approx \Omega_{\phi}, \tag{7.12} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.13} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.14} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.15} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.16} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.17} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.18} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.19} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.20} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.21} \]
\[ \rho_{\phi} = \rho_{\phi} + \rho_{\phi} = 0.2589 \pm 0.0057 \tag{7.22} \]

For completeness, one may also deal with the case $V_0 = 0$, which corresponds to the $\Lambda$CDM case. By looking at Eq. (5.12), this time we are forced to impose $\Omega_{\phi} = \Omega_{\phi} / \rho_{\phi} \Rightarrow \rho_{\phi} = \rho_{\phi}$ in Eq. (6.18). This position gives as range $2.20 \lesssim \langle x v \rangle / (10^{-27} \text{cm}^3 \text{s}^{-1}) \leq 6.61$. This case however, albeit degenerating with the previous one, is not favored for the requests we made in the previous section.
plier, $\lambda$. We even included a potential, $V_{\text{eff}}$, which models the coupling with the standard cosmological constant and induces a phase transition. We described the thermodynamics of our matter fluid, showing that it is perfect, irrotational, and isentropic. Moreover, we demonstrated that the positiveness of the Helmotz energy naturally suggests a negative pressure. We showed the existence of a Noether current due to the shift symmetry, which coincided with the entropy density current $s^\alpha$, making the Lagrangian independent from $\phi$. Thus, we assumed a homogeneous and isotropic space-time to investigate small perturbations and we found that the adiabatic sound speed naturally vanishes, leading to a constant pressure, but with an evolving energy density, differently from the standard $\Lambda$CDM paradigm. The so-obtained density and BM can be viewed as pressureless. Hence, their magnitudes are extremely close today.

Constant at late times. Thus, it is natural to presume that the positiveness of the Helmotz energy is canceled out because it is the matter which induces the effective cosmological constant, able to accelerate the universe today, naturally arising from Eqs. (5.10)–(5.11), through our hypothesis of matter with non-vanishing pressure. We will better analyze also additional symmetries of our Lagrangians and we will put more stringent constraints on the DM particle candidate. These estimates are quite independent from the spin of DM particles.

In future works, we will study inflationary scenarios, naturally arising from Eqs. (5.10)–(5.11), through our hypothesis of matter with non-vanishing pressure. We will better analyze also additional symmetries of our Lagrangians and we will put more stringent constraints on the DM particle, bounding the cross-section from current DM experiments.

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Appendix A: Derivation of the equations in Sec. II

From the variation of the action in Sec. II, we get

$$
\delta S = \int \left[ (K_\phi - V_{\text{eff}} + \lambda Y_\nu \nu_\nu) \delta \phi + \mathcal{L}_X \nabla_\alpha \phi \nabla^\alpha \delta \phi + Y \delta \lambda + \frac{1}{2} g_{\alpha \beta} \left( \mathcal{L}_X \nabla_\alpha \phi \nabla^\alpha \phi - K + V_{\text{eff}} - \lambda Y \right) \delta g^{\alpha \beta} \right] \sqrt{-g} d^4 x = 
$$

$$
\int \left\{ \left[ L_\phi - \nabla_\alpha (\mathcal{L}_X \nabla^\alpha \phi) \right] \delta \phi + Y \delta \lambda + \frac{1}{2} g_{\alpha \beta} \left( \mathcal{L}_X \nabla_\alpha \phi \nabla^\alpha \phi - K + V_{\text{eff}} - \lambda Y \right) \delta g^{\alpha \beta} \right\} \sqrt{-g} d^4 x .
$$

(A1)

The variations with respect to $\lambda$, $\phi$ and $g_{\alpha \beta}$, respectively, lead to Eqs. (2.7), (2.8), and (2.9).

Eq. (2.15), instead, is obtained through simple calculations including the vanishing acceleration in Eq. (2.11)

$$
\nabla_\alpha T^{\alpha \beta} = \dot{\rho} v^\beta + \nabla_\alpha P v^\alpha v^\beta + \theta (\rho + P) v^\beta - \nabla^\beta P = 
$$

$$
= [\dot{\rho} + \theta (\rho + P)] v^\beta ,
$$

(A2)
and the expansion $\theta$ in Eq. (2.16) is obtained as

$$\theta = \frac{\nabla_\alpha \nabla^\alpha \varphi}{\sqrt{2X}} - \frac{X_\varphi \nabla_\alpha \varphi \nabla^\alpha \varphi}{2X} = \frac{\nabla_\alpha \nabla^\alpha \varphi - X_\varphi}{\sqrt{2X}}. \tag{A3}$$

By using Eqs. (2.7) and (2.17), we can recast Eq. (2.8) to obtain Eq. (2.18)

$$K_\varphi = \lambda Y_\nu L_\varphi - \nabla_\alpha \left( K_X - V_{\text{eff}}^X + \lambda Y_X \right) \nabla^\alpha \varphi - (K_X - V_{\text{eff}}^X + \lambda Y_X) \nabla_\alpha \nabla^\alpha \varphi =
\mathcal{L}_\varphi - \left[ K_{XX} X_\varphi + K_{X^2} X_\varphi - V_{\text{eff}}^X X_\varphi + \lambda Y_{XX} X_\varphi + \lambda Y_{X^2} \varphi \right] \nabla_\alpha \varphi \nabla^\alpha \varphi - \lambda X \nabla_\alpha \nabla^\alpha \varphi - \nabla_\alpha \lambda Y_X \nabla^\alpha \varphi =
\mathcal{L}_\varphi - 2X \left( \mathcal{L}_{XX} X_\varphi + \mathcal{L}_{X^2} \varphi \right) - \left( \sqrt{2X} \theta + X_\varphi \right) \mathcal{L}_X - \sqrt{2X} \lambda Y_X = -\eta_\varphi - \frac{\theta}{2X} (\rho + P) - \sqrt{2X} \lambda Y_X = 0. \tag{A4}$$