Orthogonal labeling

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Abstract

Let $\Delta_G$ be the maximum degree of a simple connected graph $G(V,E)$. An injective mapping $P : V \rightarrow \mathbb{R}^{\Delta_G}$ is said to be an orthogonal labeling of $G$ if $uv, uw \in E$ implies $(P(v) - P(u)) \cdot (P(w) - P(u)) = 0$, where $\cdot$ is the usual dot product defined in Euclidean space. A graph $G$ which has an orthogonal labeling is called an orthogonal graph. This labeling is motivated by the existence of some labelings defined on some algebraic structure, i.e. harmonious labeling and group distance magic labeling. In this paper we study some preliminary results on orthogonal labeling. One of the early results is the fact that cycles with even number of vertices are orthogonal, while cycles with odd number of vertices are not. The main results in this paper state that any graph containing $K_3$ as a subgraph is non-orthogonal and that a graph $G'$ obtained from adding a pendant to a vertex in an orthogonal graph $G$ is orthogonal. Moreover, we show that any tree is orthogonal.

Keywords: separate, Euclidean space, orthogonal, orthogonal labeling
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1. Introduction

A general definition of labeling of a graph is a map that carries some set of graph elements to set of numbers such that a certain condition is fulfilled. For example, magic labeling is a one-to-one map onto the appropriate set of consecutive integers starting from 1, with some kind of “constant-sum” property. We can find other examples on [4]. Moreover, there are some labelings which map to a general set. An example that satisfies this condition is harmonious labeling, which

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maps the set of vertices to \( \mathbb{Z}_{|V|} \). Froncek in [3] and Cichacz in [2] worked on group distance magic labeling which maps vertices to a finite group.

Inspired by these examples, we define a labeling that maps the set of vertices to Euclidean space \( \mathbb{R}^n \) and call this labeling: orthogonal labeling. A graph \( G \) that has an orthogonal labeling is called an orthogonal graph. This labeling has some similarities with the \( n \)--cube graph which we show latter that it is indeed an orthogonal graph. Note that orthogonal labeling is also used in different meaning such as in connection with orthogonal double covers. See for example [5]. The main purpose of this paper is to introduce this labeling and some preliminaries results.

2. Orthogonal Labeling

Let \( \Delta_G \) be the maximum degree of a simple connected graph \( G(V, E) \). An orthogonal labeling is a mapping \( P : V \rightarrow \mathbb{R}^{\Delta_G} \) such that if \( uv, uw \in E \), then the images of \( u, v, w \), which are \( P(u), P(v), P(w) \), are vectors in \( \mathbb{R}^{\Delta_G} \) such that the vector \( P(v) - P(u) \) is orthogonal to \( P(w) - P(u) \). This leads us to a more formal definition.

**Definition 1.** Let \( G(V, E) \) be a simple connected graph. An injective mapping \( P : V \rightarrow \mathbb{R}^{\Delta_G} \) is said to be an orthogonal labeling of \( G \) if \( uv, uw \in E \) implies \((P(v) - P(u)) \cdot (P(w) - P(u)) = 0\), where \( \cdot \) is the usual dot product defined in the Euclidean space \( \mathbb{R}^{\Delta_G} \). A graph \( G \) which has an orthogonal labeling is said to be orthogonal. A non-orthogonal graph is a graph that does not have an orthogonal labeling.

For the sake of convenience, we say that a graph that consists of a single vertex is orthogonal.

It is obvious that the graph \( C_4 \) is orthogonal. Let \( V(C_4) = \{v_1, v_2, v_3, v_4\} \) and \( E(C_4) = \{v_iv_{i+1} : i = 1, 2, 3, 4\} \), where \( v_5 \) is defined to be \( v_1 \). Since \( \Delta_{C_4} = 2 \), then the injective mapping \( P : V(C_4) \rightarrow \mathbb{R}^2 \) defined by \( P(v_1) = (0, 0), P(v_2) = (1, 0), P(v_3) = (1, 1), \) and \( P(v_4) = (0, 1) \) is obviously an orthogonal labeling of \( C_4 \).

Since orthogonality with respect to usual dot product in Euclidean space is the same as perpendicularity whenever the space is \( \mathbb{R}^n \), \( n = 1, 2, 3 \), then we can induce an orthogonal labeling from drawing the graph in \( \mathbb{R}^n \) such that each pair of edges that incident to a same vertex is perpendicular to each other. Thus, by drawing \( C_4 \) as a square in \( \mathbb{R}^2 \), it is obvious that \( C_4 \) is orthogonal.

For simplicity we will use \( P_v \) to denote \( P(v) \).

3. Some Results on Orthogonal Graphs

The graph \( C_4 \) can also be seen as a special case of \( n \)--cube graph \( Q_n \), in this case \( n = 2 \). The hypercube or \( n \)--cube \( Q_n \) is defined as \( K_2 \) if \( n = 1 \), and

\[
Q_n = Q_{n-1} \times K_2 \quad \text{if } n \geq 2.
\]

It can be easily seen that \( Q_n \) is a regular graph of degree \( n \) and has \( 2^n \) vertices.

The cubes \( Q_1, Q_2, \) and \( Q_3 \) can simply be drawn as a line segment, a square, and a cube [1]. From earlier discussion we know that those drawings show that \( Q_1, Q_2, \) and \( Q_3 \) are orthogonal. The cube \( Q_4 \) is shown in Figure 1.
A better way to describe the \( n \)-cube is to represent its vertex set by the collection of all \( n \)-tuples, where each coordinate of the \( n \)-tuple is 0 or 1, and where two vertices of \( Q_n \) are adjacent if and only if corresponding \( n \)-tuples differ in exactly one coordinate. Figure 1 above shows the labeling of the vertices of \( Q_n \) for \( n = 4 \). The way of representing the vertices of \( Q_n \) as \( n \)-tuples as described above induces an orthogonal labeling. We argue that it is indeed an orthogonal labeling.

Since each vertex in \( Q_n \) has a unique representation in \( n \)-tuples, we know that this mapping is injective. Let \( u, v, w \in V(Q_n) \) such that \( uv, uw \in E(Q_n) \) and let \( p_u, p_v, p_w \) be the \( n \)-tuple representations of \( u, v, w \) respectively. Since two vertices of \( Q_n \) are adjacent if and only if corresponding \( n \)-tuples differ in exactly one coordinate, then \((p_v - p_u), (p_w - p_u)\) must be unit vectors that have value 1 or -1 in exactly one coordinate. This implies \((p_v - p_u) \cdot (p_w - p_u) = 0\). Thus we have proved the following theorem.

**Theorem 3.1.** The \( n \)-cube graph \( Q_n \) is orthogonal for all \( n \in \mathbb{N} \).

We have shown that \( C_4 \) is orthogonal. In general, \( C_{2k} \) is orthogonal for any natural number \( k \geq 2 \).

**Theorem 3.2.** \( C_{2k} \) is orthogonal, \( k \in \mathbb{N}, k \geq 2 \).

**Proof.** Suppose that \( V(C_{2k}) = \{v_i : i = 0, 1, 2, \ldots, 2k - 1\} \) and \( E(C_{2k}) = \{v_iv_{i+1} : i = 0, 1, 2, \ldots, 2k-1\} \), where \( v_{2k} = v_0 \). We know that \( \Delta C_{2k} = 2 \). Define a mapping \( P : V(C_{2k}) \rightarrow \mathbb{R}^2 \) as follows:

\[
P(v_{2r+1}) = p_{2r+1} = (r, k - r - 1), \text{ for } r = 0, 1, 2, \ldots, k - 1
\]

\[
P(v_{2r}) = p_{2r} = (r, k - r), \text{ for } r = 1, 2, \ldots, k - 1
\]

\[
p_0 = (0, 0)
\]

Notice that the first coordinates of \( p_{2i} \) and \( p_{2j+1} \) are equal if and only if \( i = j \). Despite the equality of first coordinates of \( p_{2r} \) and \( p_{2r+1} \), we can see that the odd-even parity of the second coordinates must be different or differ by \( k \). Since the first coordinates of \( p_{2r} \) and \( p_{2r+1} \) are strictly increasing functions of \( r \), we can deduce that \( P \) is an injective function.
We see that
\[(p_{2k-1} - p_0) \cdot (p_1 - p_0) = p_{2k-1} \cdot p_1 = (k - 1, 0) \cdot (0, k - 1) = 0,
\]
\[(p_0 - p_1) \cdot (p_2 - p_1) = (0, -k + 1) \cdot (1, 0) = 0,
\]
\[(p_0 - p_{2k-1}) \cdot (p_{2k-2} - p_{2k-1}) = (-k + 1, 0) \cdot (0, 1) = 0,
\]
\[(p_{2r+1} - p_{2r}) \cdot (p_{2(r-1)+1} - p_{2r}) = (0, -1) \cdot (1, 0) = 0, \text{for } r = 1, 2, \ldots, k - 1,
\]
and
\[(p_{2(r+1)} - p_{2r+1}) \cdot (p_{2r} - p_{2r+1}) = (1, 0) \cdot (0, 1) = 0, \text{for } r = 2, 3, \ldots, k - 2
\]
Since all the dot products are zero, we conclude that \(P\) is indeed an orthogonal labeling on \(C_{2k}\). Therefore \(C_{2k}\) is orthogonal.

Intuitively we know that \(C_{2k+1}\) is non-orthogonal for all natural number \(k\). It is true and we can show it using the fact that a simple connected graph with \(\Delta \leq 3\) is orthogonal if and only if it can be drawn such that all edges sharing an endpoint are pairwise perpendicular in \(\mathbb{R}^\Delta\).

**Theorem 3.3.** \(C_{2k+1}\) is non-orthogonal for all natural number \(k\).

**Proof.** Suppose on the contrary that \(C_{2k+1}\) is orthogonal, then we can draw it on \(\mathbb{R}^2\) such that all edges sharing an endpoints are pairwise perpendicular. Without loss of generality assume that the first edge is drawn horizontally. In order to maintain the perpendicularity we need to have the second edge and the \((2k + 1)st\) edge to be drawn vertically. Since the second edge is drawn vertically, we need the third edge to be drawn horizontally. Continuing this argument will lead us to a conclusion that \((2r + 1)st\) edge must be drawn horizontally for \(r = 0, 1, 2, \ldots, k\). But this contradicts the fact that the \((2k + 1)st\) edge is drawn vertically. Therefore \(C_{2k+1}\) is non-orthogonal. \(\square\)

On the theory of planar graphs, we know that if a graph \(G\) has a subgraph that is not planar then neither is \(G\). However, if a simple connected graph \(G\) has a non-orthogonal simple connected subgraph \(G'\), then it is not necessarily true that \(G\) must be orthogonal.

Let \(G\) be the graph \(C_5\) with one additional vertex \(v_0\) that is adjacent to exactly one vertex of \(C_5\). We have \(V(G) = \{v_i : i = 0, 1, \ldots, 5\}\) and \(E(G) = \{v_iv_{i+1} : i = 0, 1, \ldots, 5\}\), where \(v_6 = v_1\). It is clear that \(\Delta_G = 3\). Defined \(P : V(G) \to \mathbb{R}^3\) by \(p_0 = (0, 0, 1), p_1 = (0, 0, 0), p_2 = (1, 0, 0), p_3 = (1, 1, 0), p_4 = (1, 1, 1),\) and \(p_5 = (0, 0, 1)\), where \(P(v_i) = p_i, i = 0, 1, \ldots, 5\). It is easily seen that \(P\) is injective. Notice that
\[(p_0 - p_1) \cdot (p_2 - p_1) = (0, 0, 1) \cdot (1, 0, 0) = 0,
\]
\[(p_0 - p_1) \cdot (p_5 - p_1) = (0, 0, 1) \cdot (0, 0, 1) = 0,
\]
\[(p_2 - p_1) \cdot (p_5 - p_1) = (0, 1, 0) \cdot (0, 0, 1) = 0,
\]
\[(p_1 - p_2) \cdot (p_3 - p_2) = (-1, 0, 0) \cdot (0, 1, 0) = 0,
\]
\[(p_4 - p_3) \cdot (p_2 - p_3) = (0, 0, 1) \cdot (0, -1, 0) = 0,
\]
Proof. Let \( \Delta \) be the orthogonal labeling of \( G \). We can generally prove this result if the \( C_5 \) is substituted by \( C_n \), \( n > 3 \) in a similar fashion.

**Theorem 3.4.** Let \( G \) be a graph obtained from adding a pendant to \( C_n \). Then \( G \) is orthogonal.

**Proof.** Let \( V(G) = \{v_i : i = 0, 1, \ldots, n\} \) and \( E(G) = \{v_i v_{i+1} : i = 0, 1, \ldots, n\} \), where \( v_{n+1} = v_1 \). It is obvious that \( \Delta_G = 3 \). Define a mapping \( P : V(G) \to \mathbb{R}^3 \) by

\[
\begin{align*}
p_0 &= (0, 0, 1), \quad p_1 = (0, 0, 0),
p_{2k} &= p_{2k-1} + (1, 0, 0), 1 \leq k \leq \frac{n-1}{2},
p_{2k+1} &= p_{2k} + (0, 1, 0), 1 \leq k \leq \frac{n-1}{2},
p_n &= p_{n-1} + (0, 0, 1),
\end{align*}
\]

where \( p_i = P(v_i), i = 0, 1, \ldots, n \). It can easily be seen that \( v_0 \) is the only vertex that is mapped to \( (0, 0, 1) \) and that \( p_i \) is an increasing function in \( i \) for \( i \in \{1, 2, \ldots, n\} \), hence \( P \) is injective. We can easily check that

\[
(p_{i+1} - p_i) \cdot (p_{i-1} - p_i) = 0, \text{ for } i = 1, 2, \ldots, n.
\]

Thus \( P \) is an orthogonal labeling on \( G \).

We will now prove some simple results but truly are useful to determine if a graph is non-orthogonal.

**Theorem 3.5.** Let \( G \) be a simple connected graph and \( G' \) be a connected subgraph of \( G \), such that \( \Delta_{G'} = \Delta_G = \Delta \). If \( G \) is orthogonal, then so is \( G' \).

**Proof.** Let \( P : V(G) \to \mathbb{R}^\Delta \) be the orthogonal labeling of \( G \). Since \( \Delta_{G'} = \Delta \), we can defined a labeling for \( V(G') \) by restricting \( P \) to \( V(G') \). Since \( P \) is injective, so is its restriction. Suppose \( u, v, w \in V(G') \) such that \( uv, uw \in E(G') \). Since \( E(G') \subseteq E(G) \), we have

\[
(P_{|V(G')} (v) - P_{|V(G')} (u)) \cdot (P_{|V(G')} (w) - P_{|V(G')} (u)) =

(P(v) - P(u)) \cdot (P(w) - P(u)) = 0.
\]

Thus proving that \( P_{|V(G')} \) is an orthogonal labeling of \( G' \).
Corollary 3.1. Let $G$ be a simple connected graph and $G'$ be a connected subgraph of $G$, such that $\Delta_{G'} = \Delta_G$. If $G'$ is non-orthogonal, then so is $G$.

We now prove a stronger result.

Theorem 3.6. Let $G$ be a simple connected graph. If $K_3$ is a subgraph of $G$, then $G$ is non-orthogonal.

Proof. Suppose on the contrary $G$ is orthogonal, then there exists an orthogonal labeling $P$ of $G$. Let $v_i, i = 1, 2, 3$ be the vertices of the subgraph $K_3$ of $G$ and $p_i = P(v_i), i = 1, 2, 3$. We will have

$$(p_2 - p_1) \cdot (p_3 - p_1) = 0$$

and

$$(p_1 - p_2) \cdot (p_3 - p_2) = 0 \Rightarrow (p_2 - p_1) \cdot (p_2 - p_3) = 0.$$ 

If we add these equations, we will have

$$(p_2 - p_1) \cdot (p_2 - p_1) = 0 \Rightarrow p_2 - p_1 = 0 \Rightarrow p_2 = p_1.$$ 

This contradicts the fact that $P$ is an orthogonal labeling, that is an injective function. Therefore $G$ must be non-orthogonal.

Theorem 3.6 works in a way such that many classes of graph can be determined to be non-orthogonal. We state some of those classes below.

Corollary 3.2. The graphs $K_n$ and $W_n$ are non-orthogonal for $n > 2$.

We also have another corollary.

Corollary 3.3. If $G$ is an orthogonal simple graph with cycle, then the girth of $G$ is at least 4.

We have seen some properties that ensure the non-orthogonality of some classes of graph. Thus we proceed to a strong lemma concerning orthogonal graph construction. This lemma is a general result of Theorem 3.4.

Lemma 3.1. Let $G$ be an orthogonal graph. If $G'$ is obtained from adding a pendant to a vertex in $G$, then $G'$ is orthogonal.

Proof. Since $G'$ is obtain form adding a pendant to $G$ so that the addition vertex is adjacent to exactly one vertex in $G$, then $\Delta_{G'} = \Delta_G$ or $\Delta_{G'} = \Delta_G + 1$. Let $P : V(G) \to \mathbb{R}^{\Delta_G}$ be the orthogonal labeling on $G$. We will define a mapping $P' : V(G') \to \mathbb{R}^{\Delta_{G'}}$ so that $P'$ is an orthogonal labeling on $G'$. First we will define $P'$ on $V(G) \subseteq V(G')$. If $\Delta_{G'} = \Delta_G$, we let $P'(v) = P(v), v \in V(G)$, and $\Delta_{G'} = \Delta_G + 1$, we let $P'(v) = (P(v), 0), v \in V(G)$.

Let $u$ be the vertex that is adjacent to the pendant vertex $u'$. It is clear that $\delta_{G'}(u) \leq \Delta_{G'}$. Let $\{v_1, v_2, \ldots, v_{\delta_{G'}(u) - 1}\}$ be the set of vertices to which $u$ is adjacent to in $G$. Let $p_i = P'(v_i), i = 1, 2, \ldots, \delta_{G'}(u) - 1$ and $p_0 = P'(u)$. It suffices to assign the value of $P'(u')$ satisfying $(P'(u') - p_0) \cdot (p_i - p_0) = 0, \text{ for } i = 1, 2, \ldots, \delta_{G'}(u) - 1$, so that $P'$ is an orthogonal mapping on $G'$. Since $P$ is an orthogonal mapping on $G$, we have $S = \{p_i - p_0 : i = 1, 2, \ldots, \delta_{G'}(u) - 1\}$ be a set of orthogonal vectors in $\mathbb{R}^{\Delta_{G'}}$. In view of Gram-Schmidt theorem we can have a vector $p'$, such
that $S \cup \{p'\}$ is orthogonal. However, $S \cup \{tp'\}$ is still orthogonal for any non-zero real number $t$. Since $P'(V(G'))$ is finite, we can choose a non-zero real number $t$ such that $tp' + p_0 \notin P'(V(G'))$. In view of our earlier discussion above we conclude that if $P'(u')$ is defined to be $tp' + p_0$, we will have $P'$ be an orthogonal labeling on $G'$.

Since hairy-cycle graphs can be obtained from adding pendants, as described in Lemma 3.10, to graph $G$ which is described in Theorem 3.4, we have this following theorem.

**Theorem 3.7.** Any hairy-cycle graph of $C_n$, $n > 3$ is orthogonal.

Further, we see that Lemma 3.1 is indeed a strong lemma. Since any tree $T$ has at least two vertices of degree one, we can remove one of these vertices and the edge incident to it and have a smaller tree. Continuing this process will yield to a tree of the form $K_2$. Retracing this process gives us an algorithm to construct $T$ from $K_2$ by adding pendants successively. In view of this argument and repeated application of Lemma 3.1, we have just proved Theorem 3.8.

**Theorem 3.8.** Let $G$ be an orthogonal graph and $T$ be a tree. If $G'$ is obtained from attaching the vertex $v_0$ of $T$, where $\delta_T(v_0) = 1$, to $G$ so that the $v_0$ is adjacent to exactly one vertex in $G$, then $G'$ is orthogonal.

Let $G$ be an orthogonal graph and $T$ be a tree such that $|V(G) \cap V(T)| = 1$. Consider the set $V(G) \cap V(T) = \{v_0\}$ and $N_T(v_0) = \{v_1, v_2, \ldots, v_d\}$, where $N_T(v_0)$ is the neighborhood of $v_0$ in $T$ and $d = \delta_T(v_0)$. Let $T_i$ be the largest subgraph of $T$ containing $v_i$ but not containing $v_0$. It is easy to see that $\delta_{T_i}(v_i) = 1$. By attaching $v_1 \in V(T_1)$ to $G$ as described in Theorem 3.8, we have another orthogonal graph. Attaching $v_2 \in V(T_2)$ to the resulting graph yield us another orthogonal graph. By continuing the process we will have the resulting orthogonal graph be $G \cup T$. Thus we have proved Theorem 3.9

**Theorem 3.9.** If $G$ is an orthogonal graph and $T$ is a tree such that $|V(G) \cap V(T)| = 1$, then $G \cup T$ is orthogonal.

We conclude our discussion by stating a direct implication of Theorem 3.8.

**Theorem 3.10.** Any tree is orthogonal.

4. Conclusion

In this paper we have some main results stating that any graph containing $K_3$ as its subgraph is non-orthogonal and that a graph $G'$ obtained from adding a pendant to a vertex in orthogonal graph $G$ is orthogonal. As a corollary of the latter we have that any tree is orthogonal. For further study we can generalize the result for orthogonal labeling on inner product space $F^n$, where $F$ is a field. However, we need to make sure that the inner product to be consistent in $F^n$ for any natural number $n$.

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Figure 2. This is the caption of figure

Table 1. This is the table caption

| Graph              | Notation | Lower Bound | Upper Bound |
|--------------------|----------|-------------|-------------|
| Cycle              | $C_n$    | $n + 2$     | $n^2 + 2$   |
| Path               | $P_n$    | $n + 1$     | $n^2 + 1$   |
| Complete graph     | $K_n$    | 2           | $n + 4$     |
| Complete bipartite graph | $K_{m,n}$ | $2n + 2$   | $3n^2 + 2$ |

References

[1] Chartrand, G. and Oellermann, O., *Applied and Algorithmic Graph Theory*, McGraw-Hill (1993).

[2] Cichacz, S., Note on Group Distance Magic Graphs $G[C_4]$, *Graphs Combin.*, 30, no. 3 (2014), 565-571.

[3] Froncek, D., Group Distance Magic Labeling of Cartesian Product of Cycles, *Aust. J. Combinatorics* 55 (2013), 167-174.

[4] Gallian, J. A., A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics* 19 (2015).

[5] Gronau, H-D.O.F., Mullin, R. C. and Rosa, A., Orthogonal double covers of complete graphs by trees, *Graphs Combin.* 13, no. 3 (1997), 251-262.