CONSTRUCTION OF SPINES OF TWO-BRIDGE LINK
COMPLEMENTS AND UPPER BOUNDS OF THEIR
MATVEEV COMPLEXITIES

MASAHARU ISHIKAWA AND KEISUKE NEMOTO

Abstract. We give upper bounds of the Matveev complexities of two-
bridge link complements by constructing their spines explicitly. In
particular, we determine the complexities for an infinite sequence of
two-bridge links corresponding to the continued fractions of the form
\[ [2, 1, \ldots, 1, 2]. \]
We also give upper bounds for the 3-manifolds obtained
as meridian-cyclic branched coverings of the 3-sphere along two-bridge
links.

1. Introduction

Let \( M \) be a compact connected 3-manifold possibly with boundary. If \( M \)
has nonempty boundary then a polyhedron \( P \subset M \) to which \( M \)
collapses is called a spine of \( M \). If \( M \) is closed then a spine of \( M \)
means that of \( M \setminus B^3 \), where \( B^3 \) is a 3-ball in \( M \). A spine \( P \) of \( M \) is said to be almost-simple if the
link of any point can be embedded into the complete graph \( K_4 \) with four
vertices. A point of almost-simple spine whose link is \( K_4 \) is called a true
vertex. The minimal number \( c(M) \) of true vertices among all almost-simple
spines of \( M \) is called the complexity of \( M \).

The notion of the complexity was introduced by S. Matveev in [8]. The
complexity gives an efficient measure on the set of all compact 3-manifolds
\( \mathcal{M} \), because it has the following properties: the complexity is additive under
connected sum, that is, \( c(M_1 \# M_2) = c(M_1) + c(M_2) \), and it has a finiteness
property, that is, for any \( n \in \mathbb{Z}_{\geq 0} \), there exists finitely many closed irreducible manifolds
\( M \in \mathcal{M} \) with \( c(M) = n \). Note that if \( M \) is closed, irreducible and other than \( S^3, \mathbb{RP}^3 \) and \( L(3, 1) \) then \( c(M) \) coincides with the minimal number of ideal tetrahedra of all triangulations of \( M \).

Determining the complexity \( c(M) \) of a given 3-manifold \( M \) is very difficult in general. For the complexity of the lens space \( L(p, q) \), Matveev proved
the upper inequality \( c(L(p, q)) \leq S(p, q) - 3 \), where \( S(p, q) \) is the sum of all partial quotients in the expansion of \( p/q \) as a regular continued fraction with positive entries, and conjectured that the equality holds (see also [7]). In recent studies, Jaco, Rubinstein and Tillmann solved this conjecture positively
for some infinite sequences of lens spaces [5]. Petronio and Vesnin studied
the complexity of closed 3-manifolds which are obtained as meridian-cyclic branched coverings of \( S^3 \) along two-bridge links \([10]\). In the case of compact manifolds with nonempty boundary, Fominkh and Wiest obtained sharp upper bounds on the complexity of torus link complements \([2]\). A certain lower bound of the complexity of a two-bridge link complement is given in \([10]\). There are several related studies, see for instance \([3, 4, 1, 6, 12, 13]\).

In this paper, we give upper bounds of the complexities of two-bridge link complements.

Let \( K(p, q) \) be a two-bridge link in the 3-sphere \( S^3 \), where \( p, q \) are coprime integers with \( p \geq 2 \) and \( q \neq 0 \). We may represent it by using Conway’s notation as \( C(a_1, \ldots, a_n) \), where the integers \( a_i \) are the partial quotients of a regular continued fraction of \( p/q \). We represent the regular continued fraction of \( p/q \) as \( p/q = [a_1, \ldots, a_n] \). For each continued fraction, \( K(p, q) \) has a diagram as shown in Figure 1. By taking the mirror image if necessary, we may assume that \( p/q > 0 \). In this paper we only consider the continued fraction of \( p/q \) such that each \( a_i \) is positive and \( a_1, a_n > 1 \). Let \( N(K(p, q)) \) be a compact tubular neighborhood of \( K(p, q) \) in \( S^3 \) and \( \text{int} N(K(p, q)) \) its interior.

The main result of this paper is the following:

**Theorem 1.1.** Let \( K(p, q) \) be a two-bridge link with \( p/q = [a_1, \ldots, a_n] \), \( a_i > 0 \) and \( a_1, a_n > 1 \). Then,

\[
c(S^3 \setminus \text{int} N(K(p, q))) \leq \sum_{i=1}^{n} a_i + 2(n - 3) - \sharp \{a_i = 1\},
\]

where \( \sharp \{a_i = 1\} \) is the number of indices \( i \) such that \( a_i = 1 \).

As a corollary, we determine the complexities of an infinite sequence of two-bridge links.

**Figure 1.** Two-bridge link \( C(a_1, \ldots, a_n) \).
Corollary 1.1. Let $K(p, q)$ be a two-bridge link with $p/q = [2, 1, \ldots, 1, 2]$, $n \geq 2$. Then,

$$c(S^3 \setminus \text{int}N(K(p, q))) = 2n - 2.$$ 

As an application of Theorem 1.1 we obtain the following upper bounds on the complexity of meridian-cyclic $d$-fold branched coverings of $S^3$, which is sharper than the upper bounds given in [10].

Corollary 1.2. Let $M_d(K(p, q))$ be the meridian-cyclic branched covering of $S^3$ along $K(p, q)$ of degree $d$. Then,

$$c(M_d(K(p, q))) \leq d\left(\sum_{i=1}^{n} a_i + 2(n - 3) - \sharp\{a_i = 1\}\right) + rd,$$

where, $r$ is one if $p$ is odd and three if $p$ is even.

We are deeply grateful to Yuya Koda for valuable information and instructive discussion on the results. The first author is supported by the Grant-in-Aid for Scientific Research (C), JSPS KAKENHI Grant Number 25400078.

2. Proof of Theorem 1.1

We will show the upper bound of the complexity in Theorem 1.1 by constructing an almost-simple spine of the two-bridge link complement explicitly.

Let $K(p, q)$ be a two-bridge link in the 3-sphere $S^3$, where $p, q$ are coprime integers with $p \geq 2$ and $q > 0$. We may represent it as $C(a_1, \ldots, a_n)$, where the integers $a_i$ are the partial quotients of a regular continued fraction of $p/q$. For each $i = 1, \ldots, n$, let $T_i$ be the tangle containing the $a_i$ twists in Figure 1 which is a 3-ball with two tunnels mutually twisted $a_i$-times. For each $i = 1, \ldots, n$, let $A_i$ be the union of a sphere with four holes and a disk which is placed inside the sphere twisted $a_i$-times as shown in Figure 2. Each $A_i$ is a spine of $T_i$ obtained by collapsing it from $\partial N(K(p, q))$ with keeping the holed sphere on the boundary of $T_i$. In this paper, we call $A_i$ a pillowcase. We denote the disk lying in the middle of $A_i$ by $D_i$. The pillowcase $A_i$ has four boundary components. We denote these components by $\partial A_{i}^{NW}$, $\partial A_{i}^{NE}$, $\partial A_{i}^{SW}$, $\partial A_{i}^{SE}$, see Figure 2. We will make a spine of $S^3 \setminus K(p, q)$ by gluing these pillowcases.

We first prove the following weaker version of Theorem 1.1

Lemma 2.1. Let $C(a_1, \ldots, a_n)$ be a two-bridge link, where each $a_i$ is positive and $a_1, a_n > 1$. Then,

$$c(S^3 \setminus \text{int}N(C(a_1, \ldots, a_n))) \leq \sum_{i=1}^{n} a_i + 2(n - 3).$$
Proof. We first construct a spine $P$ of $S^3 \setminus C(a_1, \ldots, a_n)$ by gluing the pillowcases $A_1, \ldots, A_n$ together as follows:

- For each adjacent pair of pillowcases $A_i$ and $A_j$, $1 \leq i, j \leq n - 1$, we attach two tubes in order to pass the knot strands and then glue a disk for each region bounded by tubes and pillowcases as shown in Figure 3.
- We attach three tubes and two disks to the pillowcase $A_n$ as shown in Figure 4.

Next, we make a spine $P_0$ by collapsing $P$ from $\partial A_1$ and $\partial A_n$, which decreases the number of true vertices. Let $x_i$ be the number of true vertices on $\partial D_i$ in the spine $P_0$. Since any true vertex of $P$ lies on $\partial D_i$ for some $1 \leq i \leq n$, there exists $\sum_{i=1}^{n} x_i$ true vertices on $P_0$.

We calculate the number of true vertices in the spine $P_0$.

- For the pillowcase $A_1$.
  True vertices which lie on $\partial D_1$ are $y_1^{(1)}, \ldots, y_{a_1}^{(1)}, y_{a_1+1}^{(1)}$ shown in Figure 5. Since the true vertices $y_1^{(1)}, y_2^{(1)}$ are removed by the collapsing from $\partial A_1^{NW}$, we get $x_1 = a_1 - 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Pillowcase $A_i$ and its boundary components.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Gluing a pair of adjacent pillowcases.}
\end{figure}
For pillowcases $A_i$, where $i = 2, \ldots, n - 1$.
True vertices which lie on $\partial D_i$ are $y_1^{(i)}, \ldots, y_{a_i}^{(i)}, y_{a_i+1}^{(i)}, y_{a_i+2}^{(i)}$ shown in Figure 6. Therefore, we get $x_i = a_i + 2$.

- For the pillowcase $A_n$.
True vertices which lie on $\partial D_n$ are $y_1^{(n)}, \ldots, y_{a_n}^{(n)}, y_{a_n+1}^{(n)}$ shown in Figure 7. Since the true vertices $y_{a_n-1}^{(n)}, y_{a_n}^{(n)}$ are removed by the collapsing from $\partial A_n^{SW}$, we get $x_n = a_n - 1$.

In summary, the number of true vertices in each pillowcase is

\[
\begin{cases}
x_1 = a_1 - 1 \\
x_i = a_i + 2 & (i = 2, \ldots, n - 1) \\
x_n = a_n - 1.
\end{cases}
\]
Therefore, the number of true vertices in the spine $P_0$ is

$$
\sum_{i=1}^{n} x_i = a_1 - 1 + \sum_{i=2}^{n-1} (a_i + 2) + a_n - 1
$$

$$
= \sum_{i=1}^{n} a_i + 2(n-3).
$$

\[\square\]

**Proof of Theorem 1.1.** We will make a new spine $P'$ from $P_0$ constructed in Lemma 2.1 by collapsing it as follows (An example of $P'$ is given in Example 2.1 below): Let $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ be the set of indices with $a_{i_j} = 1$, $j = 0, \ldots, r - 1$. For each $j$, let $P_{j+1}$ be a spine obtained from $P_j$ by applying the replacement shown in Figure 8. The left figure represents the replacement in the case of $a_{i_j-1} > 1$ and the right one is in the case of $a_{i_j-1} = 1$. Applying this replacement inductively for $j = 0, \ldots, r - 1$, we get a new spine $P_r$ of $S^3 \setminus C(a_1, \ldots, a_n)$.

Let $P'_j$ be the spine obtained from $P_j$ by collapsing it from the boundary. In the following we fix $j$, $0 \leq j \leq r$, and denote $i_j$ by $i$ for simplicity. Let $x_k^{(j)}$
Figure 8. Gluing pillowcases together in the case of $a_{i_j-1} > 1$ (shown in the left hand side) and $a_{i_j-1} = 1$ (shown in the right hand side).

be the number of true vertices on $\partial D_k$ in the spine $P'_j$, where $k = 1, \ldots, n$. All true vertices of $P_j$ which do not lie on $\partial D_k$ are removed in $P'_j$ by the collapsing. Therefore, there exists $\sum_{k=1}^n x^{(j)}_k$ true vertices in $P'_j$.

We now calculate the difference of the numbers of true vertices between $P'_j$ and $P'_{j+1}$.

(i) In the case of $a_{i-1} > 1$.

- For the pillowcase $A_{i-1}$.
  
  True vertices which lie on $\partial D_{i-1}$ in $P'_j$ are $y_1^{(i-1)}, \ldots, y_{a_{i-1}}^{(i-1)}, y_{a_{i-1}+1}^{(i-1)}$, shown on the left hand side in Figure 8. Hence, we get $x^{(j)}_{i-1} = a_{i-1} + 2$. On the other hand, in $P'_{j+1}$, true vertices which lie on $\partial D_{i-1}$ are $z_1^{(i-1)}, \ldots, z_{a_{i-1}}^{(i-1)}, z_{a_{i-1}+1}^{(i-1)}$, shown on the right hand side, and hence $x^{(j+1)}_{i-1} = a_{i-1} + 3$. Thus $x^{(j+1)}_{i-1} = x^{(j)}_{i-1} + 1$.

- For the pillowcase $A_i$.
  
  True vertices which lie on $\partial D_i$ in $P'_j$ are $y_1^{(i)}; y_{a_{i-1}}^{(i)}; y_{a_{i-1}+1}^{(i)}$, shown on the left hand side in Figure 8 and hence we get $x^{(j)}_i = a_i + 2 = 3$. On the other hand, true vertices which lie on $\partial D_i$ in the $P'_{j+1}$ are $z_1^{(i)}; \ldots, z_{a_{i-1}}^{(i)}$ shown on the right hand side. By collapsing from $\partial A_{i+1}^{SE}$, the true vertices $z_1^{(i)}, z_2^{(i)}$ are removed in $P'_{j+1}$. Hence $x^{(j+1)}_i = 2$. Therefore we get $x^{(j+1)}_i = x^{(j)}_i - 1$.

- For the pillowcase $A_{i+1}$.
  
  True vertices which lie on $\partial D_{i+1}$ in $P'_j$ are $y_1^{(i+1)}, \ldots, y_{a_{i+1}}^{(i+1)}, y_{a_{i+1}+1}^{(i+1)}$, shown on the left hand side in Figure 8 and hence we get $x^{(j)}_{i+1} = a_{i+1} + 2$. On the other hand, true vertices which lie on $\partial D_{i+1}$ in
\( P_{j+1} \) are \( z^{(i+1)}_{a_{i-1}+1}, \ldots, z^{(i+1)}_{a_{i-2}+1}, z^{(i+1)}_{a_{i-2}+2}, z^{(i+1)}_{a_{i-2}+3} \) shown on the right hand side. By collapsing from \( \partial A_{i-1}^{SW} \), the true vertices \( z^{(i+1)}_{a_{i-2}+2}, z^{(i+1)}_{a_{i-2}+3} \) are removed in \( P_{j+1}' \). Hence \( x^{(j+1)}_{i+1} = a_{i+1} + 1 \). Thus we get \( x^{(j+1)}_{i+1} = x^{(j)}_{i+1} - 1 \).

- For the other pillowcases \( A_k \), where \( k \neq i-1, i, i+1 \).
  
  Since the replacement \( P_{j} \to P_{j+1} \) does not change the true vertices in \( A_k \), we get \( x^{(j+1)}_k = x^{(j)}_k \).

\[ \begin{array}{c}
\text{Figure 9. True vertices in the spine } P_j \text{ (shown in the left hand side) and in the spine } P_{j+1} \text{ (shown in the right hand side) in the case of } a_{i-1} > 1. \\
\end{array} \]

(ii) In the case of \( a_{i-1} = 1 \).

- For the pillowcase \( A_{i-1} \).
  
  By the replacement \( P_{j} \to P_{j+1} \), the true vertex \( z^{(i-1)} \) shown in Figure 10 appears and this is not removed in \( P_{j+1}' \). Therefore, we get \( x^{(j+1)}_{i-1} = x^{(j)}_{i-1} + 1 \).

- For the pillowcase \( A_i \).
  
  True vertices which lie on \( \partial D_i \) in \( P_j \) are \( y^{(i)}_{a_i}, y^{(i)}_{a_i+1}, y^{(i)}_{a_i+2}, y^{(i)}_{a_i+3} \) shown on the left hand side in Figure 10. By collapsing \( P_j \) from \( \partial A_{i-2}^{NE} \), the true vertices \( y^{(i)}_{a_i+2}, y^{(i)}_{a_i+3} \) are removed in \( P_{j+1}' \). Hence \( x^{(j)}_i = a_{i} + 1 = 2 \). On the other hand, true vertices which lie on \( \partial D_i \) in \( P_{j+1} \) are \( z^{(i)}_{a_i}, \ldots, z^{(i)}_{a_i+4} \) shown on the right hand side. By collapsing from \( \partial A_{i-2}^{NE} \) and \( \partial A_{i}^{SE} \), the true vertices \( z^{(i)}_{a_i+2}, z^{(i)}_{a_i+3} \) and \( z^{(i)}_{a_i+1}, z^{(i)}_{a_i+4} \) are removed in \( P_{j+1}' \), respectively, and hence \( x^{(j+1)}_i = 1 \). Thus we get \( x^{(j+1)}_i = x^{(j)}_i - 1 \).
• For the pillowcase $A_{i+1}$.
  Applying the same argument as in case (i), we get $x^{(j+1)}_{i+1} = x^{(j)}_{i+1} - 1$.

• For the other pillowcases $A_k$, where $k \neq i - 1, i, i + 1$.
  Since the replacement $P'_j \rightarrow P_{j+1}$ does not change the true vertices in $A_k$, we get $x^{(j+1)}_k = x^{(j)}_k$.

By the above arguments (i) and (ii), the number of true vertices in each pillowcase changes as

$$
\begin{align*}
  x^{(j+1)}_{i-1} &= x^{(j)}_{i-1} + 1 \\
  x^{(j+1)}_i &= x^{(j)}_i - 1 \\
  x^{(j+1)}_{i+1} &= x^{(j)}_{i+1} - 1 \\
  x^{(j+1)}_k &= x^{(j)}_k & (k \neq i - 1, i, i + 1).
\end{align*}
$$

Therefore, we get

$$
\sum_{k=1}^{n} x^{(j+1)}_k = \sum_{k=1}^{n} x^{(j)}_k - 1.
$$

Since $P'_j$ has $\sum_{k=1}^{n} x^{(j)}_k$ true vertices, the replacement $P'_j \rightarrow P'_{j+1}$ decrease the number of true vertices in $P'_{j+1}$ by one. Hence, by the inductive sequence $P'_0 \rightarrow P'_1 \rightarrow \cdots \rightarrow P'_r = P'$, the number of true vertices decreases by $r$. Now
Figure 11. The spine $P_r$ of $S^3 \setminus \text{int}N(C(3, 2, 1, 3, 3))$.

we apply Lemma 2.1. The number of true vertices in the spine $P'$ is

$$\sum_{i=1}^{n} a_i + 2(n - 3) - r$$

$$= \sum_{i=1}^{n} a_i + 2(n - 3) - \# \{a_i = 1\}.$$

This completes the proof. \qed

**Example 2.1.** The spine $P_r$ of $S^3 \setminus \text{int}N(C(3, 2, 1, 3, 3))$ is as shown in Figure 11. The spine $P'$ is obtained by collapsing this from the boundaries.

**Remark 2.1.** Let $P(a_1, \ldots, a_n)$ be a pretzel link, where $|a_i| > 0$, $|a_1|, |a_n| > 1$. We can construct a spine $P$ of the complement of $P(a_1, \ldots, a_n)$, by attaching $2(n - 1) + 1$ tubes and $n$ disks to the $n$ tangles with $a_i$-twists, $i = 1, \ldots, n$ by a way similar to what we did in the proof of Lemma 2.1. Then the number of true vertices of the spine obtained from $P$ by collapsing becomes

$$|a_1| + 2 \sum_{i=1}^{n} |a_i| + |a_n| + n - 4,$$

which is an upper bound of the complexity $c(S^3 \setminus P(a_1, \ldots, a_n))$. See [9] for precise discussion.
3. Proof of Corollaries 1.1 and 1.2

To prove Corollary 1.1 we need to have a lower bound on the complexity of \( S^3 \setminus C(a_1, \ldots, a_n) \). Let \( \text{vol}(M) \) denote the hyperbolic volume of a hyperbolic 3-manifold \( M \), and \( v_3 \) denote the hyperbolic volume of the regular ideal tetrahedron in the hyperbolic 3-space \( \mathbb{H}^3 \), that is, \( v_3 = 1.01494 \ldots \).

In order to prove Corollary 1.1 we will use the following theorems.

**Theorem 3.1** ([10]). Let \( K(p, q) \) be a hyperbolic two-bridge link with \( p/q = [a_1, \ldots, a_n] \), \( a_i > 0 \) and \( a_1, a_n > 1 \). Then,

\[
(1) \quad \text{vol}(S^3 \setminus \text{int} N(K(p, q))) \geq v_3 \cdot \max\{2, 2n - 2.6667 \ldots \}.
\]

A spine is called *simple* if the link of each point is either a circle, a theta-graph, or \( K_4 \), and it is called *special* if each 2-strata of the simple spine is an open disk. Remark that if \( P \) is a special spine of a link complement \( M \), then its dual is a topological ideal triangulation of \( M \), and vice versa.

**Theorem 3.2** ([7]). Let \( M \) be a compact, irreducible and boundary-irreducible 3-manifold which differs from a 3-ball, \( S^3 \), \( \mathbb{RP}^3 \), \( L(3,1) \) and suppose that all proper annuli in \( M \) are inessential. Then, for any almost-simple spine of \( M \), there exists a special spine of \( M \) which has the same or a fewer number of true vertices.

Now we give a few notations. Suppose that \( K(p, q) \) is hyperbolic. Let \( T \) denote a topological ideal triangulation of \( S^3 \setminus \text{int} N(K(p, q)) \), \( n(T) \) denote the number of ideal tetrahedra of \( T \) and \( \sigma_{\text{ideal}}(S^3 \setminus \text{int} N(K(p, q))) \) denote the minimal number of \( n(T) \). By inequality \( 1 \) we have

\[
(2) \quad \sigma_{\text{ideal}}(S^3 \setminus \text{int} N(K(p, q))) \geq \frac{\text{vol}(S^3 \setminus \text{int} N(K(p, q)))}{v_3} \geq \max\{2, 2n - 2.6667 \ldots \}.
\]

The next proposition suggests that we can replace the left hand side of inequality \( 2 \) by the complexity \( c(S^3 \setminus \text{int} N(K(p, q))) \).

**Proposition 3.1.** If \( L \) is a hyperbolic link then,

\[
\sigma_{\text{ideal}}(S^3 \setminus \text{int} N(L)) = c(S^3 \setminus \text{int} N(L)).
\]

**Proof.** Let us prove the inequality \( \leq \). Since \( S^3 \setminus \text{int} N(L) \) is hyperbolic, it is irreducible and boundary-irreducible, and contains no essential annuli. By Theorem 3.2 we can deform any almost-simple spines of \( S^3 \setminus \text{int} N(L) \) into a special one such that it has the same or a fewer number of true vertices. Since the dual of a special spine is a topological ideal triangulation of \( S^3 \setminus \text{int} N(L) \), we have \( \sigma_{\text{ideal}}(S^3 \setminus \text{int} N(L)) \leq c(S^3 \setminus \text{int} N(L)) \). The inverse inequality \( \geq \) is obvious, since \( \{ \text{Special spine} \} \subset \{ \text{Almost-simple spine} \} \).

\( \square \)
From Theorem 1.1, Theorem 3.1 and Proposition 3.1 we can determine
the exact values of the complexities for an infinite sequence of two-bridge
links as mentioned in Corollary 1.1.

Proof of Corollary 1.1. Let $K(p, q)$ be hyperbolic. By Theorem 3.1 and
Proposition 3.1 the following inequality holds:

$$2n - 2.66 \leq \sigma_{\text{ideal}}(S^3 \setminus \text{int}\, \mathcal{N}(K(p, q))) = c(S^3 \setminus \text{int}\, \mathcal{N}(K(p, q)))$$

$$\leq \sum_{i=1}^{n} a_i + 2(n - 3) - \sharp\{a_i = 1\}.$$ 

In particular, if $K(p, q) = C(2, 1, \ldots, 1, 2)$ then

$$2n - 2.66 \leq \sigma_{\text{ideal}}(S^3 \setminus \text{int}\, \mathcal{N}(K(p, q))) = c(S^3 \setminus \text{int}\, \mathcal{N}(K(p, q)))$$

$$\leq 2n - 2.$$ 

\[\square\]

Remark 3.1. Sakuma and Weeks constructed canonical decompositions of
hyperbolic two-bridge link complements explicitly in [11]. Calculating the
number of ideal tetrahedra in their ideal triangulation, we get the upper
bound

$$(3) \quad \sigma_{\text{ideal}}(S^3 \setminus \text{int}\, \mathcal{N}(K(p, q))) \leq 2 \sum_{i=1}^{n} a_i - 6.$$ 

Let $c$ be the number of true vertices of the spine constructed in Theorem
1.1. We can obtain a special spine with the same as or a fewer number
of vertices than $c$ by applying Theorem 3.2. Hence by considering its dual, we
can obtain a topological ideal triangulation of $S^3 \setminus K(p, q)$ consisting of at
most $c$ ideal tetrahedra. If $p/q = [2, 1, \ldots, 1, 2]$ then the upper bound $c$ of
the number of ideal tetrahedra constructed in Theorem 1.1 coincides with
the upper bound in inequality (3). In general, the upper bounds obtained
by our construction are better than those obtained in [11].

Finally we give a proof of Corollary 1.2.

Proof of Corollary 1.2. Since $a_1 > 1$, there exists a tube connecting $A_1$ and
$A_2$ such that the union of the meridian-disk $D$ of this tube and the spine
$P'$ constructed in the proof of Theorem 1.1 has only one true vertex on the
boundary of $D$. Let $P_d$ be a spine of the $d$-fold cyclic covering space $\tilde{M}$ of
$S^3 \setminus K(p, q)$ induced by $P'$.

Suppose that $p$ is odd, that is, $K(p, q)$ is a knot. Recall that the complexity
of a closed 3-manifold is by definition the complexity of that manifold
minus an open ball. Therefore, the complexity of $M_d(K(p, q))$ is at most
the number of true vertices of the spine obtained from $P_d$ by attaching a
meridian-disk along the preimage of the boundary of $D$. Thus we have
\[
c(M_d(K(p, q))) \leq d \left( \sum_{i=1}^{n} a_i + 2(n - 3) - \sharp \{ a_i = 1 \} \right) + d.
\]

Suppose that \( p \) is even, that is, \( K(p, q) \) is a link. We attach one more meridian-disk \( D' \) to the other boundary component of \( \tilde{M} \) such that the union of \( P_d, D \) and \( D' \) has two true vertices on the boundary of \( D' \). It is known in \([7]\) that the complexity does not change even if we remove several open balls. Therefore, the complexity of \( M_d(K(p, q)) \) is bounded above by the number of true vertices of this union. Thus we have

\[
c(M_d(K(p, q))) \leq d \left( \sum_{i=1}^{n} a_i + 2(n - 3) - \sharp \{ a_i = 1 \} \right) + 3d.
\]

This completes the proof. \( \square \)

**Remark 3.2.** The complexities of 3-manifolds obtained as meridian-cyclic branched coverings along two-bridge links had been studied in \([10]\). We can easily check that the upper bound in Corollary \([12]\) is better than theirs.

**References**

[1] S. Anisov, *Exact values of complexity for an infinite number of 3-manifolds*, Mosc. Math. J. 5 (2005), no. 2, 305-310.

[2] E. A. Fominykh, B. Wiest, *Upper bounds for the complexity of torus knot complements*, J. Knot Theory Ramif., 22 (2013), no. 10, 1350053.

[3] R. Frigerio, B. Martelli, C. Petronio, *Complexity and Heegaard genus of an infinite class of compact 3-manifolds*, Pacific J. Math. 210 (2003), no. 2, 283-297.

[4] R. Frigerio, B. Martelli, C. Petronio, *Dehn filling of cusped hyperbolic 3-manifolds with geodesic boundary*, J. Differential Geom. 64 (2003), no. 3, 425-455.

[5] W. Jaco, H. Rubinstein, S. Tillmann, *Minimal triangulations for an infinite family of lens spaces*, J. Topol. 2 (2009), no. 1, 157-180.

[6] W. Jaco, H. Rubinstein, S. Tillmann, *Coverings and minimal triangulations of 3-manifolds*, Algebr. Geom. Topol. 11 (2011), no. 3, 1257-1265.

[7] S. Matveev, *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics, 9, Springer-Verlag, Berlin, 2003.

[8] S. Matveev, *Complexity theory of three-dimensional manifolds*, Acta Appl. Math. 19 (1990), no. 2, 101-130.

[9] K. Nemoto, *Construction of spines of two-bridge link complements and their Matveev complexities*, in Japanese, master’s thesis, Tohoku univ., 2015.

[10] C. Petronio, A. Vesnin, *Two-sided bounds for the complexity of cyclic branched coverings of two-bridge links*, Osaka J. Math. 46 (2009), no. 4, 1077-1095.

[11] M. Sakuma, J. Weeks, *Examples of canonical decompositions of hyperbolic link complements*, Japan. J. Math. (N.S.) 21 (1995), no. 2, 393-439.

[12] A.Yu. Vesnin, E.A. Fominykh, *Exact values of complexity for Paoluzzi-Zimmermann manifolds*, Doklady Math., 84 (2011), 542-544.

[13] A.Yu. Vesnin, E.A. Fominykh, *On complexity three-dimensional hyperbolic manifolds with geodesic boundary*, Siberian Math. J., 53 (2012), 625-634.
Mathematical Institute, Tohoku University, Sendai, 980-8578, Japan
E-mail address: ishikawa@math.tohoku.ac.jp

Mathematical Institute, Tohoku University, Sendai, 980-8578, Japan
E-mail address: keisuke.nem0121@gmail.com