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Remarks on approximate decompositions of the diagonal

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ABSTRACT
In this article, we investigate, for varieties over \( \mathbb{C} \) with trivial group of 0-cycles, the gap between essential CH_0-dimension 2 and essential CH_0-dimension 0. In particular, we present sufficient (and necessary) conditions for a variety with trivial group of 0-cycles and essential CH_0-dimension \( \leq 2 \) to have, in fact, essential CH_0-dimension 0.

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0. Introduction
The characterization of the complex smooth projective varieties of dimension \( n \geq 3 \) which are rational, i.e., birationally equivalent to \( \mathbb{P}^n_{\mathbb{C}} \), is a long standing problem in algebraic geometry. If such a variety \( X \) is rational then, by birational invariance of the group CH_0 of 0-cycles modulo rational equivalence, we have an isomorphism CH_0(\( X \)) = \( \mathbb{Z} \) for a (any) point \( x \in X(\mathbb{C}) \).

However, a more general class of varieties satisfy the isomorphism CH_0(\( X \)) \( \simeq \mathbb{Z} \). Among them, we can mention the class of rationally connected varieties, for which any pair of point is contained in a rational curve, or the (conjecturally) smaller class of unirational varieties, i.e., varieties admitting a dominant rational map from a projective space, which, in dimension \( \leq 2 \) coincide with the class of rational varieties. In dimension \( \geq 3 \), efficient obstructions to rationality for unirational varieties have been found since the early 1970s: Clemens and Griffiths [4] proved that the intermediate Jacobian of a cubic threefold \( X \), which is a unirational variety, is not isomorphic to a sum of Jacobian of curves which would be the case should \( X \) be rational; in a 1989 article [6], Colliot-Thélène and Ojanguren studied higher degree analog of the obstruction that Artin and Mumford used to provide an example of a unirational non stably rational threefold, namely the unramified cohomology groups, which are birational invariants trivial for stably rational varieties, and used them to prove non stable rationality of some unirational quadric bundles.

In fact, the birational invariance of the CH_0 group yields more for a rational variety \( X \); for any field extension \( L/\mathbb{C} \), we have, as for the projective space, CH_0(\( X \)) = \( \mathbb{Z} x_L \) where \( x \in X(\mathbb{C}) \) and \( x_L = x \times _{\mathbb{C}} \text{Spec}(L) \). The varieties satisfying this property are said to have universally trivial CH_0 group or to have their 0-cycles universally supported on a point. We can reformulate this property, after [1], saying that there exists a Chow-theoretic decomposition of the diagonal i.e. that we can write the diagonal of \( X \) as:
\[ \Delta_X = X \times x + Z \text{ in } \text{CH}^n(X \times X) \]  

where \( x \in X(\mathbb{C}) \) and \( Z \) is a cycle supported on \( D \times X \) for some proper closed algebraic subset \( D \) of \( X \). In [13], Voisin proved the existence of a Chow theoretic decomposition of the diagonal to be a strictly subtler birational invariant than Clemens-Griffiths and the non triviality of unramified cohomology groups criteria by exhibiting examples of unirational varieties whose intermediate Jacobian is isomorphic to a Jacobian of curve (Clemens-Griffiths criterion), having trivial unramified cohomology groups and which do not admit a Chow theoretic decomposition of the diagonal or equivalently, whose 0-cycles are not universally supported on a point.

For a variety \( X \) satisfying \( \text{CH}_0(X) = \mathbb{Z} \), we can also consider a weaker property than having the 0-cycles universally supported on a point: let us say that the \( \text{CH}_0 \) is universally supported on a subvariety \( Y \subseteq X \) if the natural morphism \( \text{CH}_0(Y) \rightarrow \text{CH}_0(X) \) is universally surjective, i.e. for any \( L/\mathbb{C}, \ \text{CH}_0(Y_L) \rightarrow \text{CH}_0(X_L) \) is surjective.

Remark 0.1. An equivalent formulation is the existence of a decomposition of the diagonal \( \Delta_X = \Gamma_1 + \Gamma_2 \) in \( \text{CH}^n(X \times X) \) where \( \Gamma_1 \) is supported on \( D \subseteq X \) and \( \Gamma_2 \) is supported \( X \setminus Y \). Indeed, applying universal surjectivity to the field \( L = \mathbb{C}(X) \), we see that the diagonal point \( \delta_X \in \text{CH}_0(X(\mathbb{C}(X))) \) can be written \( i_L \cdot z \) in \( \text{CH}_0(X(\mathbb{C}(X))) \cong \text{CH}^n(X(\mathbb{C}(X))) \), where \( z \in \text{CH}_0(Y_L) \) and \( i : Y, \rightarrow X \). This implies (2), using Bloch identity \( \text{CH}^n(X(\mathbb{C}(X))) = \lim_{U \subseteq X, \text{open}} \text{CH}^n(U \times X) \) and the localization exact sequence.

Voisin made in [14] the following definition:

Definition 0.2. ([14, Definition 1.2]) The essential \( \text{CH}_0 \)-dimension of a variety is the minimal integer \( k \) such that there is a closed subscheme \( Y \subseteq X \) of dimension \( k \) such that the \( \text{CH}_0 \) group of \( X \) is universally supported on \( Y \).

In the case of cubic hypersurfaces, the following is proved in [14]:

Theorem 0.3. ([14, Theorem 1.4]) The essential \( \text{CH}_0 \)-dimension of a very general \( n \)-dimensional cubic hypersurface over \( \mathbb{C} \) is either \( n \) or \( 0 \), for \( n = 4 \) or \( n \) odd.

In [6], Colliot-Thélène proved the following proposition concerning varieties of essential \( \text{CH}_0 \)-dimension \( \leq 1 \):

Theorem 0.4. ([6, Proposition 2.5]) Let \( X \) a smooth projective variety over \( \mathbb{C} \) satisfying \( \text{CH}_0(X) = \mathbb{Z} \). Assume \( X \) has essential \( \text{CH}_0 \)-dimension \( \leq 1 \). Then the essential \( \text{CH}_0 \)-dimension of \( X \) is \( 0 \) (i.e. \( X \) admits a Chow-theoretic decomposition of the diagonal).

The goal of this article is to investigate how far a variety with trivial \( \text{CH}_0 \), which has essential \( \text{CH}_0 \) dimension \( \leq 2 \) is from having a universally trivial \( \text{CH}_0 \) group i.e. essential \( \text{CH}_0 \) dimension 0. A first result in this direction is the following:

Theorem 0.5. ([14, Corollary 2.2], [1, Proposition 1.9]) Let \( \Sigma \) be a smooth complex projective surface. Assume \( \text{CH}_0(\Sigma) = \mathbb{Z} \) and \( \text{Tors}(H^*(\Sigma, \mathbb{Z})) = 0 \). Then \( \Sigma \) has \( \text{CH}_0 \) universally trivial.

We show the following generalizations:

Theorem 0.6. Let \( X \) be a smooth complex projective variety of dimension \( n \) such that \( \text{CH}_0(X) = \mathbb{Z} \). Assume \( X \) has essential \( \text{CH}_0 \) dimension \( \leq 2 \). If

1. \( \text{Tors}(H^2(X, \mathbb{Z})) = 0 \) and
2. \( H^3(X, \mathbb{Z}) = 0 \),

then \( X \) has universally trivial \( \text{CH}_0 \) group i.e. has essential \( \text{CH}_0 \) dimension 0.
In another direction, we have the following result. Let us introduce first the following condition: 

(*) there is a smooth projective variety $\tilde{Y}$ of dimension $(\dim(X)-1)$ and a morphism $j : \tilde{Y} \to X$ such that 

$$j_* : \text{Pic}^0(\tilde{Y}) \to \text{CH}^2(X)_{\text{alg}}$$

is universally surjective.

In more geometric terms, the condition means that any family of algebraically trivial codimension 2 cycles factors generically through $j_*$. Indeed, let $Z \in \text{CH}^2(B \times X)$ be such a family, parametrized by a smooth projective base $B$. Applying condition (*) to the field $\mathbb{C}(B)$, gives that the cycle $Z_{\mathbb{C}(B)} \in \text{CH}^2(X_{\mathbb{C}(B)})_{\text{alg}} = \lim_{U \subset B \text{ open}} \text{CH}^2(U \times X)_{\text{alg}}$ has a preimage $D \in \text{Pic}^0(Y)(\mathbb{C}(B))$ by $j_{\mathbb{C}(B),*}$. The $\mathbb{C}(B)$-point $D$ corresponds naturally to a rational map (thus a morphism since $\text{Pic}^0(Y)$ is an abelian variety) $D : B \to \text{Pic}^0(Y)$. The identity $j_{\mathbb{C}(B),*}(D) = Z_{\mathbb{C}(B)}$ says that the applications $B(\mathbb{C}) \to \text{CH}^2(X)_{\text{alg}}$, given by $b \to Z_b$ and $B(\mathbb{C}) \to \text{Pic}^0(Y)(\mathbb{C}) \to \text{CH}^2(X)_{\text{alg}} \sim \text{J}^3(X)$ coincide on a dense open set of $B$ hence everywhere since targets are abelian varieties.

Conversely, any cycle $Z_L \in \text{CH}^2(X_L)_{\text{alg}}$ has a model $Z$ which is a family of algebraically trivial codimension 2 cycles of $X$ parametrized by a smooth quasi-projective model $B$ of $L$. The factorization of that family through $j_*$ gives rise to a morphism $D : B \to \text{Pic}^0(Y)$ which, passing to the limit over the open sets of $B$ yields a $\mathbb{C}(B) = L$ point of $\text{Pic}^0(Y)$ mapped by $j_{L,*}$ to $Z_L$.

Now, let us state the second theorem of the article:

**Theorem 0.7.** Let $X$ be a smooth complex projective variety of dimension $n$ such that $\text{CH}_0(X) = \mathbb{Z}$. Assume $X$ has essential $\text{CH}_0$ dimension $\leq 2$. If $X$ satisfies the condition (*), then $X$ has universally trivial $\text{CH}_0$ group i.e. has essential $\text{CH}_0$ dimension 0.

**Remark 0.8.** We observe, conversely, that if $X$ has universally trivial $\text{CH}_0$ group, then the conditions (1) of Theorem 0.6 and (*) are satisfied (see Lemma 1.1).

The key condition (*) appearing in the theorem is expressed in terms of universal generation, a notion introduced by Shen in [9], where he uses universal generation of 1-cycles on cubic hypersurfaces (of dimension $\geq 3$) to relate the existence of a decomposition of the diagonal for cubic threefolds and fourfolds to the algebraicity of some cohomological classes on their Fano varieties of lines associated to the pairing in the middle cohomology of cubic hypersurfaces.

In the case of threefolds, another relation between essential $\text{CH}_0$-dimension and condition (*) is presented in Section 3. Combined with Theorem 0.7, it yields the following result:

**Theorem 0.9.** The essential $\text{CH}_0$-dimension of a very general Fano complete intersection threefold is 0 or 3. In particular, the essential $\text{CH}_0$-dimension of the very general quartic threefold is equal to 3.

The article is organized as follows: the first section is devoted to the proofs of Theorems 0.6 and 0.7. The second section is devoted to the analysis of condition (*); we try to relate it, at least in some special case to a more geometric condition. The third section is devoted to the proof of Theorem 0.9.

**1. Main theorems**

Let us begin this section by a lemma which proves that the conditions (1) of Theorem 0.6 and (*) are also necessary.
Lemma 1.1. Let $X$ be a smooth projective variety of dimension $n$ whose essential $CH_0$ dimension is 0. Then

1. $\text{Tors}(H^2(X, \mathbb{Z})) = 0 = \text{Tors}(H^3(X, \mathbb{Z}))$ and
2. $X$ satisfies $(\ast)$.  

Proof. Saying that $X$ has essential $CH_0$-dimension 0 is equivalent to the existence of a Chow-theoretic decomposition of the diagonal of $X$. We have:

$$\Delta_X = X \times X + Z \text{ in } CH^n(X \times X) \quad (3)$$

where $x \in X(\mathbb{C})$ and $Z$ is a cycle supported on $D \times X$ for some proper closed algebraic subset $D$ of $X$. We can choose $D$ such that, denoting $\tilde{D}$ a desingularization of $D$ and $j : \tilde{D} \to X$ the composition of the desingularization followed by the inclusion, the cycle $Z$ lifts to a cycle $\tilde{Z} \in CH^{n-1}(\tilde{D} \times X)$. Item (1) is proved in [13, Theorem 4.4]. Let us prove (2). Let $L/\mathbb{C}$ be a field extension and $\gamma \in CH^2(X_L)_{\text{alg}}$. Letting both sides of the extension of (3) to $L$ act on $\gamma$, we get the equality:

$$\gamma = \Delta^*_L \gamma = j_{L,*} \left( Z^*_L \gamma \right) \text{ in } CH^2(X_L)$$

with $Z^*_L \gamma \in CH^1(\tilde{D}_L)_{\text{alg}} = \text{Pic}^0(\tilde{D}_L) = \text{Pic}^0(\tilde{D})_L$, which proves the universal surjectivity of $j_*$.

Proof of Theorem 0.6. Let us assume that $X$ satisfies conditions (1) and (2) of the Theorem 0.6. By Remark 0.1, the diagonal of $X$ can be written:

$$\Delta_X = \Gamma_1 + \Gamma_2 \text{ in } CH^n(X \times X) \quad (4)$$

where $\Gamma_1$ is supported on $D \times X$ for some proper closed subset $D \subset X$ and $\Gamma_2$ is supported $X \times \Sigma$. Let $\tau : \Sigma \to \Sigma$ be a desingularization of $\Sigma$. Enlarging $\Sigma$ if necessary we can find $\Gamma_2 \in CH^2(X \times \Sigma)$ such that $(\text{id}_X, i \circ \tau)_* \Gamma_2 = \Gamma_2$ in $CH^n(X \times X)$, where $i : \Sigma, \to X$ is the inclusion. To get a Chow-theoretic decomposition of the diagonal, it is sufficient to prove that $\Gamma_2$ (hence $\Gamma_2$ in $CH^n(X \times X)$) can be decomposed as $X \times x + Z$ in $CH^2(X \times \Sigma)$ for a cycle $Z$ supported on $D' \times \Sigma$, $D'$ being a proper closed subset of $X$. We have the following proposition:

Proposition 1.2. ([14, Proposition 2.1]) Let $Y$ be a smooth projective variety. If $Y$ admits a decomposition of the diagonal modulo algebraic equivalence, that is

$$\Delta_Y = Y \times y + Z \text{ in } CH^{\dim(Y)}(Y \times Y)/_{\text{alg}}$$

with $Z$ supported on $D \times Y$ for some proper closed algebraic subset $D$ of $Y$, then $Y$ admits a Chow-theoretic decomposition of the diagonal.

We include the proof for the sake of completeness:

Sketch of proof. The result is obtained as a consequence of a nilpotence result of Voevodsky [10] and Voisin [11] asserting that given a self-correspondence $\Gamma \in CH^n(Y \times Y)$ that is algebraically trivial, there is an integer $N$ such that $\Gamma^{\circ N} = 0$ in $CH^n(Y \times Y)$.

Applying the nilpotence result to the algebraically trivial self-correspondence $(\Delta_Y - (Y \times y) + Z)$ yields the result since, writing down the different terms, using the fact that $Z \circ (Y \times y) = 0$ and $(\Delta_Y - Y \times y) \circ (\Delta_Y - Y \times y) = \Delta_Y - Y \times y$, we see that any power of $(\Delta_Y - (Y \times y) + Z)$ is of the form $\Delta_Y - (Y \times y) + Z'$ for a cycle $Z'$ supported on $D \times Y$.

We conclude from this proposition that in order to get the decomposition of the diagonal of $X$, it suffices to decompose $\Gamma_2$ as $X \times x + Z$ for a cycle $Z$ supported on $D' \times \Sigma$, $D'$ being a proper
closed subset of $X$, in $\text{CH}^2(X \times \Sigma)/\text{alg}$. Indeed, this will decompose $\Gamma_2$ as $X \times x + Z$, with $Z$ supported on $D \times X$ in $\text{CH}^n(X \times X)/\text{alg}$.

Now, since $\text{CH}_0(X) = \mathbb{Z}$, we have $\text{CH}_0(X \times \Sigma) \cong \text{CH}_0(\Sigma)$ i.e. the group of 0-cycles of $X \times \Sigma$ supported on a 2-dimensional subscheme of $X \times \Sigma$. Then, by work of Bloch and Srinivas ([3, Theorem 1 (iii)]), algebraic and homological equivalences coincide on $\text{CH}^2(X \times \Sigma)$ so that, in order to get the equality $\Gamma_2 = X \times x + Z$ in $\text{CH}^2(X \times \Sigma)/\text{alg}$, it is sufficient to prove the corresponding cohomological decomposition, that is to prove that $[\Gamma_2]$ can be written $[X \times x] + [Z]$ in $H^n(X \times \Sigma)$ for a cycle $Z$ supported on $D' \times \Sigma$, $D'$ being a proper closed subset of $X$.

We have the Künneth exact sequence:

$$0 \to \bigoplus_{i+j=4} H^i(X, \mathbb{Z}) \otimes H^j(\Sigma, \mathbb{Z}) \to H^4(X \times \Sigma, \mathbb{Z}) \to \bigoplus_{i+j=5} \text{Tor}_1\left(H^i(X, \mathbb{Z}), H^j(\Sigma, \mathbb{Z})\right) \to 0$$

from which, we see, using the fact that the groups $H^{p \leq 1}(\ast, \mathbb{Z})$ are always torsion-free and the assumption $\text{Tors}(H^2(X, \mathbb{Z})) = 0 = \text{Tors}(H^3(X, \mathbb{Z}))$, that $H^4(X \times \Sigma, \mathbb{Z})$ admits a Künneth decomposition.

Let us denote $\delta^{ij} \in H^i(X, \mathbb{Z}) \otimes H^j(\Sigma, \mathbb{Z})$ the Künneth components of $[\Gamma_2]$. They are Hodge classes since the projection on Künneth types are morphism of Hodge structures.

The component $\delta^{4,1}$ is 0 since by assumption (2), $H^3(X, \mathbb{Z}) = 0$. The component $\delta^{0,4}$ is of the form $[X \times z]$ for a 0-cycle $z$ on $\Sigma$.

Let us write $\Sigma = \bigsqcup \Sigma_i$ where the $\Sigma_i$ are smooth connected surfaces. Since $H^0(\Sigma, \mathbb{Z}) = \mathbb{Z}[\Sigma_i]$, for each $i$, the component $\delta^{ij}_i \in H^i(X, \mathbb{Z}) \otimes H^j(\Sigma_i, \mathbb{Z})$ can be written $pr^*_i \alpha_i$ for a cohomology class $\alpha_i \in H^i(X, \mathbb{Z})$ and by projection formula $pr_{1,*}(\delta^{ij}_i \otimes [X \times x_i]) = pr_{1,*}(pr^*_i \alpha_i \otimes [X \times x_i]) = \alpha_i$ for a (any) point $x_i \in \Sigma_i$. Now, we have $[\Gamma_2] \otimes [X \times x_i] = [\Gamma_2 \cdot (X \times x_i)] = \delta^{0,4}_i \otimes [X \times x_i]$; applying $pr_{1,*}$ to these equalities yields $[pr_{1,*}(\Gamma_2 \cdot (X \times x_i))] = \alpha_i$. Let $\delta^{4,0}_i = pr^*_i \alpha_i$ is algebraic and supported on $pr_{1,*}(\Gamma_2 \cdot (X \times x_i)) \times \Sigma_i$ and $pr_{1,*}(\Gamma_2 \cdot (X \times x_i))$ has codimension 2 in $X$, in particular it does not dominate $X$. So the component $\delta^{4,0} = \sum \delta^{4,0}_i$ is algebraic and represented by a cycle which does not dominate $X$ by the first projection.

As $\text{CH}_0(X) \cong \mathbb{Z}$, by [3, Proposition 1], there is an integer $N \neq 0$ such that

$$N\Delta_X = N(X \times x) + Z \text{ in } \text{CH}^n(X \times X)$$

where $Z$ is supported on $D' \times X$ for some proper closed subset $D'$ of $X$. Looking at the action on $H^1(X, \mathbb{Z})$, we see that $H^1(X, \mathbb{Z})$ is a torsion group annihilated by $N$ but since $H^1(X, \mathbb{Z})$ is torsion-free, $H^1(X, \mathbb{Z}) = 0$. Hence, $\delta^{1,3} = 0$. Letting the correspondences of (5) act on the complex vector space $H^1(X, \Omega^2_X)$ we see that it is annihilated by $N$ i.e. it is 0, so that by Lefschetz theorem on $(1, 1)$ classes, $H^2(X, \mathbb{Z})$ is algebraic. So the Hodge class $\delta^{2,2}$ belongs to $H^{1,1}(X) \otimes H^{1,1}(\Sigma)$; it is thus algebraic and of the form $[\sum D_i \times C_i]$ where the $D_i$ are divisors on $X$ and the $C_i$ are curves on $\Sigma$, in particular it does not dominate $X$ by the first projection. \[\square\]

Proof of Theorem 0.7. Let us assume that $X$ satisfies condition (*) and has essential $\text{CH}_0$-dimension $\leq 2$. We thus have:

$$\Delta_X = \Gamma_1 + \Gamma_2 \text{ in } \text{CH}^n(X \times X)$$

where $\Gamma_1$ is supported on $D \times X$ for some proper closed subset $D \subset X$ and $\Gamma_2$ is supported $X \times \Sigma$.

Let us write $\Sigma = \bigsqcup \Sigma_i$ where the $\Sigma_i$ are smooth connected surfaces. Choose a point $\sigma_i \in \Sigma_i$. For each $i$, the cycle $\Gamma_{2,i} := \Gamma_{2,i} \Sigma_i \in \text{CH}^2(X \times \Sigma_i)$ can be written as $Z_i \times \Sigma_i + \Gamma_{2,i,\text{alg}}$ where $Z_i \in \text{CH}^2(X)$ is defined as $\Gamma_{2,i,\text{alg}}(\sigma_i)$ and $\Gamma_{2,i,\text{alg}}$ is a family of cycles algebraically equivalent to 0 on $X$ parametrized by $\Sigma_i$. By condition (*) applied to each field $\text{Pic}(\Sigma_i)$, we get a cycle $Z_i \in \text{Pic}(Y \times \Sigma_i)$ such that $Z_i - \Gamma_{2,i,\text{alg}}$ vanishes in $\text{CH}^2(X \times U_i)$ where $U_i \subset \Sigma_i$ is a dense open subset. By the localization exact sequence, we conclude that the cycle $Z_i - \Gamma_{2,i,\text{alg}}$ is supported on $\bigcup_i X \times C_i$ where $\bigcup_i C_i = \Sigma \bigcup_i U_i$. Putting everything together, we conclude that $\Delta_X = Z' + Z''$ where $Z'$ is...
supported on \( D' \times X \) and \( Z'' \) is supported on \( X \times C \) where \( C = \bigcup_{i,j} C_{i,j} \). We thus conclude that the essential \( \text{CH}_0 \)-dimension of \( X \) is \( \leq 1 \) and the proof is concluded by applying Theorem 0.4. \( \square \)

2. Universal generation of codimension 2 cycles

In this section, we discuss the relation of condition (*) to the existence of a universal codimension 2 cycle.

Let \( X \) be a smooth projective complex variety satisfying \( \text{CH}_0(X) = \mathbb{Z} \). Then by a theorem of Roitman \( H^0(X) = 0 \) for any \( i > 0 \), so that the Hodge structure on \( H^3(X, \mathbb{Z}) \) has level 1 and \( H^{2n-1}(X, \mathbb{Q}) = 0 \). So \( H^3(X, \mathbb{Z})_{\text{prim}} := \text{Ker}(c_1(\mathcal{O}_X(1)))^{n-3+1} \cup : H^3(X, \mathbb{Z})_{/\text{tors}} \to H^{2n-1}(X, \mathbb{Z})_{/\text{tors}} \) is the whole of \( H^3(X, \mathbb{Z})_{/\text{tors}} \) so that the bilinear form defined on \( H^3(X, \mathbb{Z})_{/\text{tors}} \) using a polarization \( \mathcal{O}_X(1) \), polarizes the intermediate Jacobian for codimension 2 cycles \( J^3(X) \) is an abelian variety. By work of Bloch and Srinivas, we have in our setting \( \text{CH}_2(X)_{\text{alg}} = \text{CH}_2(X)_{\text{hom}} \simeq J^3(X)(\mathbb{C}) \). The condition (*) is related to codimension 2 cycles. In [15, Theorem 2.1], Voisin exhibited a (birationally invariant) necessary condition for stable rationality, namely the existence of a universal codimension 2 cycle i.e. the existence of a correspondence \( Z \in \text{CH}^2(J^3(X) \times X) \) such that the induced Abel-Jacobi morphism \( \Phi_Z : J^3(X) \to J^3(X) \), given by \( t \mapsto p(Z_t - Z_0) \), where \( p : \text{CH}^2(X)_{\text{alg}} \to J^3(X) \) is the natural regular morphism in the sense of Murre [8], is the identity. We have the following relation with condition (*):

**Proposition 2.1.** Let \( X \) be a smooth projective complex variety satisfying \( \text{CH}_0(X) = \mathbb{Z} \) and condition (*). Assume moreover that \( j_* : \text{Pic}^0(Y) \to J^3(X) \simeq \text{CH}^2(X)_{\text{alg}} \) is split. Then there is a universal codimension 2 cycle \( Z \in \text{CH}^2(J^3(X) \times X) \).

**Proof.** As \( j_* \) is split, the splitting morphism gives an imbedding \( s : J^3(X) \to \text{Pic}^0(Y) \). Denoting \( P \) the Poincaré divisor on \( \text{Pic}^0(Y) \times Y \), set \( Z = (\text{id}_{J^3(X)}, j)_* (s, \text{id}_Y)^* P \) in \( \text{CH}^2(J^3(X) \times X) \). Then, by construction, \( Z \) is a universal codimension 2 cycle. Indeed, the Abel-Jacobi morphism \( \Phi_Z : J^3(X) \to J^3(X) \) is just given by \( j_* \circ s_* \) which is the identity by definition of \( s \).

We have this other proposition relating condition (*) to the existence of a universal codimension 2 cycle:

**Proposition 2.2.** Let \( X \) be a smooth projective complex variety satisfying \( \text{CH}_0(X) = \mathbb{Z} \). Assume \( J^3(X) \) is a direct factor of a sum of Jacobian of curves \( \oplus_1 J(C_i) \). Then the existence a universal codimension 2 cycle \( Z \in \text{CH}^2(J^3(X) \times X) \) implies condition (*).

**Proof.** Let us denote \( p : \oplus_1 J(C_i) \to J^3(X) \) the projection to \( J^3(X) \) and \( s : J^3(X) \to \oplus_1 J(C_i) \) a section. Using the morphisms

\[
C_i \xrightarrow{\text{id}} J(C_i) \xrightarrow{p} J^3(X)
\]

we get a correspondence \( Y \in \text{CH}^2(\bigcup_i C_i \times X) \), defined on each \( C_i \times X \) as \( (p \circ j_i, \text{id}_X)^* Z \), such that the induced Abel-Jacobi morphism \( \Phi_Y : \bigcup_i C_i \to J^3(X) \) coincides with \( \oplus_1 p \circ j_i \). Let us denote \( D = \text{pr}_{2,*}(Y) \in \text{CH}^1(X) \) and \( D \) a desingularization of (a divisor in the class) \( D \). We have a morphism \( j : \text{Pic}^0(D) \to J^3(X) \) and Abel-Jacobi morphism \( \Phi_Y : \bigcup_i C_i \to J^3(X) \) naturally factors through the morphism \( j \) so that, by the universal property of the Albanese variety, the morphism \( p : \oplus_1 J(C_i) \to J^3(X) \) also factors through \( j \).

Now, let \( \mathcal{K} \in \text{CH}^2(W \times X) \) be a family of codimension 2 cycles of \( X \) parametrized by a smooth quasi-projective base \( W \). Let us consider the Abel-Jacobi morphism \( \Phi_W : W \to J^3(X) \); we have the equality \( p \circ s \circ \Phi_W = \Phi_W \) but \( p \) factors through \( j : \text{Pic}^0(D) \to J^3(X) \).

**Remark 2.3.** Instead of the existence of a universal codimension 2 cycle, we can consider weaker conditions. Let us introduce the following conditions:
(1) there exist a smooth projective variety $W$ and a cycle $Z \in CH^2(W \times X)$ such that the 
induced morphism $Z_* : Alb(W) \to J^3(X)$ is an isomorphism;
(2) there exist a universally generating cycle of codimension 2, i.e., there exist a quasi-projective 
variety $W$ and a cycle $Z \in CH^2(W \times X)$ such that 
$Z_* : CH_0(W) \to CH^2(X)_{alg}$
is universally surjective.
But the hypotheses we have to add to get condition ($*$) are not clear.

3. Applications to threefolds

As we have seen in Remark 0.1, when $X$ has dimension 3, the fact that the $CH_0$ group of $X$ is 
universally supported on a surface $\Sigma$ can be written in $CH^3(X \times X)$, as
$$\Delta_X = \Gamma_1 + \Gamma_2 \quad (7)$$
where $\Gamma_2$ is supported $X \times \Sigma$ and $\Gamma_1$ is supported on $D \times X$ for some proper closed subset $D \subset X$, in particular $D$ is a 2-dimensional closed subscheme of $X$. We thus have a symmetric situation in term of dimension of the supports of the $\Gamma_i$. We recall that in this setting, $J^3(X)$ is a 
 principally polarized abelian variety, whose polarization is given by the unimodular intersection form $\langle \cdot, \cdot \rangle_{J^3(X)}$ induced by the intersection form on $H^3(X, \mathbb{Z})$ via the isomorphism $H_2(J^3(X, \mathbb{Z}) \cong H^3(X, \mathbb{Z})/\text{Tors}$. We have the following result:

Theorem 3.1. Let $X$ be a smooth threefold satisfying $CH_0(X) = \mathbb{Z}$ and whose essential $CH_0$-dimension is $\leq 2$. Assume that any endomorphism of $J^3(X)$ is self-adjoint for $\langle \cdot, \cdot \rangle_{J^3(X)}$. Then $X$ satisfies condition ($*$). So that, by Theorem 0.7, the essential $CH_0$-dimension of $X$ is 0.

Proof. Let us consider the decomposition (7). For any $z \in CH^2(X)_{alg} \cong J^3(X)(\mathbb{C})$, letting both sides act on $z$, we get
$$z = \Gamma_1^*(z) + \Gamma_2^*(z).$$

By construction, denoting $\tilde{D} \to X$ a desingularization (followed by the inclusion) of $D$, $\Gamma_1^*$ factorizes through $j_* : Pic^0(\tilde{D}) \to J^3(X)$. Moreover, as $\Gamma_2^*$ is an endomorphism of $J^3(X)$, it is self-
adjoint so that $\Gamma_2^*(z) = \Gamma_{2s}(z)$ and the last term factorizes through $i_* : Pic^0(\Sigma) \to J^3(X)$ where $i : \Sigma \to X$ is the desingularization (followed by the inclusion) of $\Sigma$. Thus, we see that $j_* + i_* : Pic^0(\tilde{D} \cup \Sigma) \to J^3(X)$ is surjective. Now, let $Z$ be a family of algebraically trivial codimension 2 cycles parametrized by a smooth quasi-projective variety $T$. Since the identity on $CH^2(X)_{alg}$ factors as $\Gamma_1^* + \Gamma_{2s}$, the map $T \to CH^2(X)_{alg}$ factors through $Pic^0(\tilde{D} \cup \Sigma)$. So $X$ satisfies ($*$). We conclude applying Theorem 0.7.

As an application of Theorem 3.1, let us state the following result:

Corollary 3.2. The essential $CH_0$-dimension of a very general Fano complete intersection of dimension 3 is either 0 or 3. In particular a very general quartic threefold has essential $CH_0$-dimension equal to 3.

Proof. Let us begin by the following lemma:

Lemma 3.3. For a very general complete intersection $X$ of dimension 3, we have $\text{End}_{HS}(H^3(X, \mathbb{Q})) = \mathbb{Q}Id$ where $\text{End}_{HS}(H^3(X, \mathbb{Q}))$ denote the space of endomorphisms of Hodge structure of $H^3(X, \mathbb{Q})$.

Proof. The proof works as in [14, Lemma 5.1]. Indeed, by [2], the monodromy group of a smooth complete intersection of dimension 3 is Zariski dense in the symplectic group of
$H^3(X, \mathbb{Q}) = \hat{H}^3(X, \mathbb{Q})_{\text{prim}}$. By [12], the Mumford-Tate group of the Hodge structure on $H^3(X, \mathbb{Q})$ contains a finite index sub-group of the monodromy group so that the Mumford-Tate group of $H^3(X, \mathbb{Q})$ in the above cases is the symplectic group. □

Let $X$ be a very general Fano complete intersection of dimension 3, then the endomorphisms of $J^3(X)$ are all self-adjoint for $\langle \cdot, \cdot \rangle_{p(X)}$. Suppose $X$ has essential CH0-dimension $<3$. Then by Theorem 3.1, $X$ has essential CH0-dimension 0. So we have the alternative.

For quartic threefolds, it was proved in [7] that a very general quartic threefold does not admit a Chow-theoretic decomposition of the diagonal, so its essential CH0-dimension is 3. □

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