On the Distribution of Zeros of a Ruelle Zeta-Function*

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Abstract

We study the limit distribution of zeros of a Ruelle ζ-function for the dynamical system \( z \mapsto z^2 + c \) when \( c \) is real and \( c \to -2 - 0 \) and apply the results to the correlation functions of this dynamical system.

Consider the dynamical system defined by the complex polynomial map \( f_c : z \mapsto z^2 + c \), where \( c < -2 \). We use the notions and results of the iteration theory of rational functions (see for example [5]). Denote by \( f_c^n \) the \( n \)-th iterate of the function \( f_c \). The Julia set \( J(f_c) \) is a Cantor set on the real line. So in particular all finite periodic points are real. This system is expanding (hyperbolic) on its Julia set. When \( c = -2 \) the Julia set is the segment \([-2, 2]\) and the map \( P = f_{-2} \) is not expanding anymore. We have the conjugation

\[
P \circ \phi = \phi \circ Q,
\]

where \( \phi : [0, 1] \to [-2, 2], \ t \mapsto 2 \cos \pi t \) and

\[
Q = \begin{cases} 
  t \mapsto 2t, & 0 \leq t \leq 1/2, \\
  t \mapsto 2 - 2t, & 1/2 \leq t \leq 1.
\end{cases}
\]

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Remark that the chaotic dynamic of $P$ on $[-2, 2]$ was investigated by J. von Neuman and S. Ulam on one of the first computers.

We are going to study the dynamics of $f_c$, $c < -2$ when $c \to -2$ and then compare it with the behavior of the limit system $P$. The chaotic dynamics of $f_c$ has to be described in probabilistic terms. This can be done by introducing an appropriate invariant probability measure $\sigma_c$ on the Julia set. We will show that the rate of asymptotic decrease of correlation functions of the system $(f_c, \nu_c)$ changes dramatically when we pass to the limit system as $c \to -2$.

Our tool is the Thermodynamic Formalism \cite{12, 13, 14, 15}. Let us introduce the main objects of this theory in our particular case. Consider the Fréchet space $C^\infty(U)$ of infinitely differentiable functions defined in some real neighborhood $U$ of the Julia set, such that $f_c^{-1}(U) \subset U$ and $U$ does not contain the critical point of $f_c$. We define the Ruelle operator $L_c$ acting on $C^\infty(U)$ by the formula

$$L_cg(x) = \sum_{\{y : f_c(y) = x\}} \frac{g(y)}{|f'_c(y)|^2}.$$ 

The weight $(f'_c)^{-2}$ is strictly positive on $U$. According to Ruelle’s extension of the Perron-Frobenius theorem $L_c$ has a simple maximal positive eigenvalue $\lambda_0^{-1}(c)$ such that the moduli of all other eigenvalues are strictly less then $|\lambda_0^{-1}(c)|$. Let $h_c$ and $\nu_c$ denote the eigenvectors of $L_c$ and the conjugate operator $L_c^*$ respectively, corresponding to the eigenvalue $\lambda_0^{-1}(c)$ ($h_c$ is a positive continuous function and $\nu_c$ is a Borel measure). Then $\sigma_c = h_c\nu_c$ is an $f_c$-invariant ergodic probability measure on the Julia set, called “The Gibbs state, corresponding to the weight $(f'_c)^{-2}$“. The operator $L_c$ can be also considered on the space $A$ of functions analytic in a complex neighborhood of the Julia set. Namely, for every complex neighborhood $W$ of the Julia set such that $U \subset W$, $f_c^{-1}(W) \subset W$ and $W$ does not contain the critical point of $f_c$, consider the Banach space $A(W)$ of functions analytic in $W$ with the supremum norm. Then $A$ is the union of all such $A(W)$. As the weight $(f'_c)^{-2}$ is analytic, the spectrum and eigenfunctions of $L_c$ in $A$ are the same as in $C^\infty(U)$ (see \cite{14}, Corollary 3.3(i)). This fact allows us to use the explicit expressions for eigenfunctions found in \cite{10} with the help of complex analysis. The following particular form of Ruelle’s zeta-function is connected to the
operator $L_c$:

$$
\zeta_c(\lambda) = \exp \left( \sum_{m=1}^{\infty} \frac{\lambda^m}{m} \sum_{x \in \text{Fix}(f_{c}^{m})} \frac{1}{(f_{c}^{*m})'(x)} \right),
$$

where $\text{Fix}(f_{c}^{sm})$ is the set of fixed points of $f_{c}^{sm}$. (We chose the weight $\phi = (f_{c}')^{-1}$ in the definition of Ruelle $\zeta$-function. See section 8 of [14] and formula (3.3) with $\sigma = \infty$ in [10].) The function $\zeta_c$ can be expressed in terms of generalized Fredholm determinants ([14, Corollary 8.1]). In our particular case it coincides with the Fredholm determinant $D_{c}$ of $L_{c}$ [10]; this is an entire function of order zero and its zeros are reciprocal to the eigenvalues of $L_{c}$. There is an explicit formula found in [10] (see also [11]):

$$
\zeta_c(\lambda) = D_{c}(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n f_{c}(0) \ldots f_{c}^{sn}(0)}.
$$

In Appendix we will give a short direct proof of the fact that the eigenvalues of $L_{c}$ are reciprocal to the zeros of $D_{c}$.

Remark. Let us consider another extension of the operator $L_{c} : C^\infty(U) \to C^\infty(U)$ to the Fréchet space $C^\infty(W)$ of the $C^\infty$-functions of two real variables $u$ and $v$, $u + iv \in W$, given by the formula

$$
L_{c}^{R^2} g(x) = \sum_{\{y : f_{c}(y) = x\}} \frac{g(y)}{|f_{c}'(y)|^2}.
$$

(Note that $|f_{c}'(x)|^2$ is the Jacobian of the map $f_{c} : R^{2} \to R^{2}$ at the point $x$.) Then the eigenvalues and eigenfunctions of $L_{c}^{R^2}$ coincide with those for $L_{c}$. Really, every eigenfunction of $L_{c}^{R^2}$ restricted to $U = W \cap R$ is an eigenfunction of $L_{c}$. Conversely, the eigenfunctions of $L_{c}$ are analytic and, hence, belong to $C^\infty(W)$. In particular, $(\lambda_{0}(c))^{-1}$ is the leading eigenvalue of the operator $L_{c}^{R^2}$ and the value $\log \lambda_{0}(c)$ is the so-called “escape rate” [8].

One of the reasons why the study of eigenvalues of $L_{c}$ is important is their connection to correlation functions. For any two continuous $A$ and $B$ on the Julia set define the correlation function $\rho_{c,A,B}$ by

$$
\rho_{c,A,B}(m) = \sigma_{c}(A(f_{c}^{*m})).B - \sigma_{c}(A).\sigma_{c}(B),
$$

where $\sigma(A) = \int A \, d\sigma$. Let

$$
S_{c,A,B}(z) = \sum_{m=0}^{\infty} \rho_{c,A,B}(m)z^{m}
$$
be the corresponding generating function. If \( A \) and \( B \) are infinitely differentiable on the Julia set then \( S_{c,A,B} \) is meromorphic in \( C \) and its poles can be located only at the points \( \lambda \lambda_0^{-1} \), where \( \lambda^{-1} \) runs over the eigenvalues of \( L_c \) other than \( \lambda_0 \) [14, Proposition 5.3].

1. First we investigate the limit distribution of eigenvalues of \( L_c \) or, which is equivalent, zeros of \( D_c \). The following facts about distribution of zeros of \( D_c \) were established in [9]. For all \( c < -2 \) the zeros with moduli greater than 1000 are negative, and simple. There exists a constant \( c_0 = -2.85 \ldots \) such that for \( c \leq c_0 \) all zeros of \( D_c \) are real. If \( c < -2 \) is close to \(-2 \) then there are non-real zeros and their number tends to infinity as \( c \) tends to \(-2 \).

To study the asymptotic distribution of complex zeros we introduce the probability measures \( \mu_c \) which charge equally every zero whose modulus is less than 1000.

**Theorem 1.** The measures \( \mu_c \) tend weakly to the uniform distribution on the circle \( \{ \lambda : |\lambda| = 4 \} \).

**Remarks.** Notice that 4 is the radius of convergence of the series \( D_{-2} = (4 - 2\lambda)/(4 - \lambda) \). Our proof is also applicable to the family of entire functions

\[
H_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{2n-1}}, \quad a > 1,
\]

whose distribution of zeros was studied by G. H. Hardy [6]. He proved that for fixed \( a \) all zeros with moduli greater than \( r_0(a) \) are negative. (In fact \( r_0(a) \) can be replaced by an absolute constant [9]). Our argument shows that the limit distribution of zeros of \( H_a \) when \( a \to 1 \) is the uniform distribution on the circle \( \{ z : |z| = 1 \} \). Theorem 1 should be compared with the following theorem of Jentzsch and Szegő: the limit distribution of zeros of partial sums of a power series \( \sum a_k z^k \) is the uniform distribution on \( \{ z : |z| = 1 \} \), provided that \( |a_k|^{1/k} \to 1 \). Our proof is based on the same idea as Beurling’s proof of the Jentzsch-Szegő theorem [3].

**Proof.** We assume that \( -3 < c < -2 \). It is convenient to introduce the variable \( z = \lambda/2 \) and set \( F_c(z) = D_c(2z) \) and 
\[
r_n(c) = f_c^{*n}(0).
\]

Thus \( F_c(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{r_1(c) \ldots r_n(c)} \) and

\[
r_{n+1}(c) = r_n^2(c) + c, \quad r_1(c) = c, \quad c < -2.
\]

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It is easy to see that all \( r_n \), except \( r_1 \), are positive, the sequence \( (r_n) \) is increasing and \( r_{n+1}(c)/r_n(c) \to \infty, n \to \infty, c < -2 \). Denote by \( k = k(c) \) the smallest natural \( k \) such that
\[
\frac{r_{k+1}(c)}{r_k(c)} \geq 36. \tag{3}
\]

It was proved in [9] that the number of zeros of \( F_c \) in any fixed disk \( \{ z : |z| < R \} \), \( R > 1000 \) is asymptotically equivalent to \( k(c) \) when \( c \to -2 \). This fact also follows from the estimates below (formula (7) plus Rouché theorem).

**Lemma 1.** If \( k = k(c) \) is as defined above, then

(i) \( 36 \leq |r_k(c)| \leq 1521 = 39^2 \).

(ii) \( k(c) \sim (\log |c + 2|^{-1})/\log 4, c \to -2 \).

(iii) \( (1/k(c)) \log |r_1(c) \ldots r_k(c)(c)| \to \log 2, c \to -2 \).

**Proof.** (i) From (3) we conclude that \( k = k(c) > 1 \). If \( r_k(c) < 36 \) then by (2)
\[
r_{k+1}(c)/r_k(c) = r_k(c) + c/r_k(c) < r_k(c) < 36,
\]
which contradicts the definition of \( k \). This proves the left inequality in (i). Now assume that \( |r_k| > 39^2 \). Then in view of (2) we have \( |r_{k+1}| > 39 \) and we obtain \( |r_k|^2 + c > |r_{k-1}|^2 - 3 \) and \( |r_k|/|r_{k-1}| > 36 \), which contradicts the definition of \( k \). This proves the right inequality in (i).

(ii) Set \( c = -2 - t, t > 0 \). An easy induction gives
\[
|r_n(c)| \geq 2 + (4^{n-1} - 1)t, \quad n = 1, 2, \ldots. \tag{4}
\]

To prove an inequality in the opposite direction we remark that \( r_{n+1}(c) = [r_n(c)]^2 - 2 - t \leq [r_n(c)]^2 - 2 = P(r_n(c)), \) so
\[
r_n(c) \leq P^{*(n-1)}(r_1(c)) \leq P^{*(n-1)}(2 + t).
\]

Using the semiconjugacy
\[
2 \cosh 2z = [2 \cosh z]^2 - 2 = P(2 \cosh z),
\]
(it is more convenient to use \( \cosh \) rather then \( \cos \) here) we obtain \( r_n(c) \leq 2 \cosh (2^{n-1}y), \) where \( y \) is the smallest positive solution of the equation such that \( 2 \cosh y = 2 + t \). There exists an absolute constant \( C_0 = 30 \) such that \( 2 \cosh x \leq C_0 x^2 + 2 \) whenever \( 2 \cosh x \leq 1521, x \in \mathbb{R} \). Thus we obtain
\[
r_n(c) \leq 2 + 4^{n-1}C_0t, \quad n = 1, 2, \ldots, k(c). \tag{5}
\]
The statement (ii) follows from (4) and (5).

(iii) From (ii) follows

\[ t \leq C_1 4^{-k}. \]  

(6)

In view of (4), (5) (6) we have

\[ \left| \left( \frac{1}{k} \sum_{n=1}^{k} \log |r_n(c)| \right) - \log 2 \right| \leq \frac{1}{k} \sum_{n=1}^{k} \log(1 + 4^{n-1} C_0 t) \leq \frac{1}{k} \sum_{n=1}^{k} C_0 C_1 4^n - k \leq \frac{1}{k} \sum_{n=0}^{\infty} C_0 C_1 4^{-n} \rightarrow 0, \quad k \rightarrow \infty. \]

This finishes the proof of the Lemma 1.

Denote by \( A(t_1, t_2) \) the annulus \( \{ z : t_1 < |z| < t_2 \} \) and set \( A(c) = A(4r_k(c), 9r_k(c)) \), where \( k = k(c) \). Put \( M_c(z) = z^k/(r_1(c) \ldots r_k(c)) \). If \( z \in A(c) \) we have

\[ \left| 1 - \frac{F_c(z)}{M_c(z)} \right| \leq \sum_{j=1}^{k} \frac{r_k \ldots r_{k-j+1}}{|z|^j} + \sum_{j=1}^{\infty} \frac{|z|^j}{r_{k+1} \ldots r_{k+j}} \leq \sum_{j=1}^{\infty} 4^{-j} + \sum_{j=1}^{\infty} 4^{-j} = \frac{2}{3}. \]

(7)

Thus if we denote \( u_c(z) = (k(c))^{-1} \log |F_c(z)| \) then by (iii) of the Lemma 1

\[ u_c(z) = (k(c)^{-1}) \log |M_c(z)| + o(1) = \log |z/2| + o(1), \quad c \rightarrow -2, \]

(8)

uniformly when \( z \in A(c) \). We are going to prove that

\[ u_c(z) \rightarrow \log^+ |z/2|, \quad |z| \leq 324, \]

(9)

where the convergence holds in \( L^1 \) with respect to the Lebesgue measure (area) in \( \{ z : |z| \leq 324 \} \).

From the definition of \( A(c) \) and Lemma 1, (i) follows that \( A_c \subset A(144, 13689) \). So from any sequence \( c_m \rightarrow -2, \ c_m < -2 \) we can chose a subsequence (which we again denote by \( c_m \)) such that the annuli \( A(c_m) \) contain a fixed annulus \( A(q_1, q_2), \ q_1 < q_2, \ q_2 > 324. \) Then in view of (8) we have

\[ u_{c_m}(z) \rightarrow \log |z/2| \quad \text{uniformly in } A(q_1, q_2). \]

(10)
Furthermore we have
\[ u_{cm}(z) \to 0, \quad |z| < 2 \quad (11) \]
(convergence in $L^1$ on compacts in \{z : |z| < 2\}), because $F_c(z) \to F_{-2}(z) = 1 - z/(2 - z)$, $c \to -2$ uniformly on compacts in \{z : |z| < 2\}. Now we use the following fact (see for example [7], Theorem 4.1.9): if a sequence of subharmonic functions $u_m$ is bounded from above on \{z : |z| = R\} and their values at the point 0 are bounded from below then there is a subsequence which converges in $L^1$ on every compact in \{z : |z| < R\} to a subharmonic function $u$. Applying this statement to our functions $u_{cm}$ and $R = q_2$, we obtain a subsequence (which we again denote by $u_{cm}$) which converges to a subharmonic function $u$. This function $u$ has the properties:
\[ u(z) = 0, \quad |z| < 2 \quad (12) \]
and
\[ u(z) = \log |z/2|, \quad q_1 < |z| < q_2, \quad (13) \]
which follow from (11) and (10) respectively. Remark that $u(z) \leq 0, \quad |z| = 2$. This follows from (12) and the following theorem of M. Brelot [4]: if $u$ is a subharmonic function and $u(z_0) = a$ then for every $\epsilon > 0$ there exists a sequence of circles centered at $z_0$ and radii tending to zero such that $u(z) \geq a - \epsilon$ on these circles. (It follows from the upper semi-continuity of $u$ that $u(z) \geq 0, \quad |z| = 2$, but we do not need this.) Now $\log |z/2|$ is a harmonic majorant of $u$ in the annulus $A(2, q_2)$, but $u(z) = \log |z/2|$ at some points in this annulus, for example for $|z| = q_1$. It follows from the Maximum Principle that $u(z) = \log^+ |z/2|, \quad |z| < q_2$.

Thus we have proved that from every sequence $u_{cm}$ we can select a subsequence tending to $\log^+ |z/2|$. This means that (9) is true. In fact our proof shows that $u_c$ converge to $\log^+ |z/2|$ in $L^1$ on every compact in the plane.

Now we conclude from the general results on convergence of subharmonic functions [1, 2, 7] that the Riesz measures $\rho_c$ of $u_c$ converge weakly to the Riesz measure of $u$, which is the uniform measure on the circle $|\lambda| = 2|z| = 4$. This proves the theorem.

2. Now we consider the application of Theorem 1 to the dynamical system $(f_c, \sigma_c)$ where $\sigma_c$ is the Gibbs state defined in the introduction. We have
\[ \zeta_c(\lambda) \to 1 - \frac{\lambda}{4 - \lambda}, \quad c \to -2, \]
uniformly on compacts in \( \{ \lambda : |\lambda| < 4 \} \). So \( \lambda_0(c) \to 2 \) and

\[
\inf \{ \lambda : \zeta_c(\lambda) = 0, \ \lambda \neq \lambda_0 \} \to 4, \quad c \to -2.
\]

Thus by Theorem 1 and by Ruelle’s theorem mentioned in introduction we have the following asymptotic behavior of correlation functions:

\[
\limsup_{m \to \infty} |\rho_{c,A,B}(m)|^{1/m} = r(c),
\]

where \( r(c) \to 1/2 \) as \( c \to -2 \).

We want to compare this result with the behavior of the limiting dynamical system when \( c \to -2 \). First we have to understand what the limit invariant measure is. Recall the conjugation (1). The Lebesgue measure \( l_1 \) on \([0,1]\) is invariant with respect to \( Q \) thus its image \( \sigma_{-2} = \phi_* l_1 \) is invariant with respect to \( P = f_{-2} \). The measure \( \sigma_{-2} \) is absolutely continuous with the density

\[
\frac{1}{\pi \sqrt{4 - x^2}}
\]

on the interval \([-2, 2]\).

**Proposition 1.** \( \sigma_c \to \sigma_{-2} \) weakly as \( c \to -2 \).

**Proof.** We will use the explicit expressions for the eigenfunction \( h_c \) of \( L_c \) and for the Cauchy transform

\[
H_c(z) = \int \frac{d\nu_c(x)}{x - z}
\]

of the eigenmeasure \( \nu_c \) of \( L_c^* \), corresponding to the greatest eigenvalue \( \lambda_0^{-1} \) (see [16, 10]). Using the notation \( r_n(c) = f_c^{-n}(0) \) we have

\[
h_c(x) = \sum_{n=0}^{\infty} \frac{\lambda_0^n(c)}{2^n r_1(c) \ldots r_n(c) [r_{n+1}(c) - x]}
\]

and

\[
H_c(z) = \sum_{n=0}^{\infty} \frac{\lambda_0^n(c)}{2^n z f_c(z) \ldots f_c^{-n}(z)}.
\]
The function $z \mapsto H_c(z)$ is holomorphic in the complement of the Julia set $J(f_c)$. We have
\[ h_c(x) \to -\left(\frac{1}{2+x} + \frac{1}{2-x}\right), \quad c \to -2 \]
in $\mathbb{C}\setminus((\infty, -2] \cup [2, \infty))$ and
\[ H_c(z) \to H_{-2}(z) = \sum_{n=0}^{\infty} \frac{1}{z P(z) \ldots P^n(z)}, \quad c \to -2 \]
in $\mathbb{C}\setminus[-2, 2]$.

Consider the measure $\nu_{-2}$ on $[-2, 2]$ with the density $\sqrt{4-x^2}$. We claim that $H_{-2}(z)$ is proportional to the Caushy transform of $\nu_{-2}$. This follows from the fact that they both satisfy the same functional equation
\[ H(z) = \frac{H(P(z))}{z} + \text{const} \frac{1}{z}, \quad z \in \mathbb{C}\setminus[-2, 2]. \]

Now Proposition 1 follows from the identity
\[ \left(\frac{1}{2+x} + \frac{1}{2-x}\right) \sqrt{4-x^2} = \frac{4}{\sqrt{4-x^2}}. \]

So the dynamical system $(P, \sigma_{-2})$ is the limit of $(f_c, \sigma_c)$ when $c \to -2$. We will show that the asymptotic behavior of correlations changes drastically when we pass to the limit as $c \to -2$.

**Proposition 2.** Let $A$ and $B$ be holomorphic functions on $[-2, 2]$. Then there exists a constant $a = a(A, B) > 1$ such that
\[ \rho_{-2,A,B}(m) \sim a^{-2m}, \quad m \to \infty. \]

**Proof.** In view of Cauchy formula is enough to prove the proposition for the set of functions
\[ A_z(x) = \frac{1}{z-x}, \quad x \in [-2, 2], \quad z \in \mathbb{C}\setminus[-2, 2]. \]

After the pullback to the segment $[0, 1]$ via the conjugation (1) we have to consider the correlations
\[ \rho_{A,B}(m) = l_1(A(Q^m).B) - l_1(A).l_1(B) \]
with \( A \) and \( B \) of the form

\[
\frac{1}{z - 2 \cos \pi t}
\]

If we introduce the operator

\[
G : g(t) \mapsto \frac{1}{2} \sum_{y: Q(y)=t} g(y) = \frac{1}{2} (g(t/2) + g(1-t/2))
\]  

(14)
then

\[
\rho_{A,B}(m) = l_1(A.G^m(B)) - l_1(A).l_1(B).
\]  

(15)

Now we notice that

\[
G \left( \frac{1}{z - 2 \cos \pi t} \right) = \frac{P'(z)}{2(P(z) - 2 \cos \pi t)},
\]

which implies

\[
G^m \left( \frac{1}{z - 2 \cos \pi t} \right) = \frac{(P^m)'(z)}{2^m (P^m(z) - 2 \cos \pi t)} = S(z) + \frac{\cos \pi t + o(1)}{2^{m-1}(P^m(z))^2},
\]  

(16)

where \( S \) is a function depending only on \( z \). Combining (15) and (16) we get the statement of Proposition 2.

**Remark.** The analyticity assumption in Proposition 2 is crucial. Indeed consider the operator \( G \) defined in (14) in the space of infinitely differentiable functions on \([0,1]\). Its eigenvalues are \( 4^{-m} \quad m = 0, 1, 2, \ldots \), and to each eigenvalue \( 4^{-m} \) corresponds one (up to a constant multiple) eigenfunction \( p_m \) which is a polynomial of degree \( 2m \). Now if \( A \) and \( B \) belong to the subspace of \( L^2([0,1], l_1) \) generated by \( \{p_m : m = 0, 1, 2, \ldots\} \) then we have

\[
\rho_{A,B}(m) \sim \text{const.} \cdot 4^{-km}, \quad m \to \infty,
\]

where \( \text{const} \neq 0 \) and \( k \) depend on \( A \) and \( B \).

**Appendix.** Here we indicate a direct proof of the fact that the eigenvalues of \( L_c \) are reciprocal to the zeros of \( D_c, \ c < -2 \) (see also [11]). Let us look at the eigenvalues of the adjoint operator \( L_c^* \). The dual space \( A^* \) is the space of functions \( g \) analytic in the complement of the Julia set \( J(f_c) \) and equal to zero at infinity. To every such function corresponds a linear functional given by

\[
h \mapsto \frac{1}{2\pi i} \int gh,
\]  

10
where the integral is taken along some countur surrounding $J(f_c)$. Now a change of the variable in this integral shows that $\lambda^{-1}$ is an eigenvalue iff for every function $h$ holomorphic in a neighborhood of $J(f_c)$

$$\int \left( g - \lambda \frac{g \circ f_c}{f_c'} \right) h = 0.$$ 

Thus $w = g - \lambda g \circ f_c/f_c'$ is holomorphic on $J_c$. It is also holomorphic in $\mathcal{C}\setminus(J(f_c) \cup \{0\})$ because $f_c'(z) = 2z$. We conclude that $w(z) = \text{const}/z$ and after the normalization of $g$ we get the functional equation

$$g(z) = \frac{\lambda}{2z} g(f_c(z)) + \frac{1}{z},$$

from which follows that

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n z f_c(z) \ldots f_c^n(0)}.$$ 

Now $g$ is holomorphic at 0 so the residue of the series in the right side should vanish that is

$$D_c(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n f_c(0) \ldots f_c^n(0)} = 0.$$ 

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References

[1] Azarin, V.: On the asymptotic behavior of subharmonic functions of finite order. Math. USSR Sbornik, 36, 135-134(1980).

[2] Anderson, J., Baernstein, A.: The size of the set on which a meromorphic function is large. Proc. London Math. Soc., 36, 518-539(1978).

[3] Beurling, A.: Some remarks on entire functions of exponential type. Collected Works, v.1, p. 386; Boston:Birkhäuser 1989.

[4] Brelot, M.: On topologies and boundaries in potential theory. Lect. Notes Math., 175, New York:Springer Verlag 1971.
[5] Eremenko, A., Lyubich, M.: The dynamics of analytic transformations. Leningrad Math. J., Vol. 1, 563-634 (1990).

[6] Hardy, G.H.: On the zeroes of a class of integral functions. Mess. Math., 34, 97-101 (1905); Collected Papers, v. IV, 95-99.

[7] Hörmander, L.: The analysis of linear partial differential operators, Volume 1. Berlin: Springer Verlag 1983.

[8] Kadanoff, L., Tang, C.: Escape from strange repellers. Proc. Nat. Acad. Sci. USA, 81, 1276-1279 (1984).

[9] Levin, G.: On Mayer’s conjecture and zeros of entire functions. Preprint 17 (1991/92), Hebrew University, Jerusalem, 1992.

[10] Levin, G., Sodin, M., Yuditskii, P.: A Ruelle operator for a real Julia set. Comm. Math. Phys., 141, 119-132 (1991).

[11] Levin, G., Sodin, M., Yuditskii, P.: Ruelle operators with rational weights for Julia sets. To appear in J. d’Analyse Math.

[12] Ruelle, D.: Zeta-Functions for Expanding Maps and Anosov Flows. Invent. math., 34, 231-242 (1976).

[13] Ruelle, D.: Repellers for real analytic maps. Ergod. Th. & Dynam. Sys. 2, 99-107 (1982).

[14] Ruelle, D.: The thermodynamic formalism for expanding maps. Comm. Math. Phys., 125, 239-262 (1989).

[15] Ruelle, D.: Spectral properties of a class of operators associated with conformal maps in two dimension. Comm. Math. Phys., 144, 537-556 (1992).

[16] Sodin, M., Yuditskii, P.: The limit-periodic finite-difference operator on $l^2(Z)$ associated with iterations of quadratic polynomials. J. Stat. Phys., Vol. 60, 853-873 (1990).

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