RIESZ PROJECTIONS FOR A NON-HYPONORMAL OPERATOR

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ABSTRACT. J. G. Stampfli proved that if a bounded linear operator T on a Hilbert space H satisfies (G1) property, then the Riesz projection Pλ associated with λ ∈ spec(T) is self-adjoint and PλH = (T − λ)⁻¹(0) = (T* − λ)⁻¹(0).

In this note we show that Stampfli’s result is generalized to an nilpotent extension of an operator having (G1) property.

1. Introduction

Let L(H) denote the algebra of bounded linear operators on a Hilbert space H. Recall that an operator T ∈ L(H) is said to be hyponormal if T*T ≥ TT* and normaloid if ||T|| = r(T), the spectral radius of T. It is well known that a hyponormal operator is normaloid. Recall that a projection P ∈ L(H) is called an orthogonal projection if the range of P, denoted by ran(P), and the kernel of P, denoted by ker(P), are orthogonal complements. It is well known [7, Proposition 63.1] that a projection is orthogonal if and only if it is self-adjoint. For an operator T ∈ L(H), if λ is an isolated point of the spectrum of T, λ ∈ spec(T), the Riesz projection Pλ associated with λ is defined by

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Cauchy integral

\begin{equation}
P_{\lambda} = \frac{1}{2\pi i} \int_{\partial D} (T - z)^{-1} \, dz,
\end{equation}

where $D$ is a closed disk centered at $\lambda$ and $D \cap \sigma(T) = \{\lambda\}$. The Riesz projection $P_{\lambda}$ for $\lambda$ is generally not orthogonal, that’s not self-adjoint.

Stampfli ([11, Theorem 2]) proved that if $T$ is hyponormal, then $P_{\lambda}$ is self-adjoint and

\begin{equation}
P_{\lambda} \mathcal{H} = (T - \lambda)^{-1}(0) = (T^{*} - \bar{\lambda})^{-1}(0) \text{ for } \lambda \in \text{iso}(T).
\end{equation}

This result has since been generalized by many mathematicians ([1], [6], [4], [13]). In particular we should recall Duggal’s result ([2]; [3]) for an extended class of non-hyponormal operators.

A part of an operator is its restriction to an invariant subspace. We say that $T \in \mathcal{L}(\mathcal{H})$ is \textit{totally hereditarily normaloid}, denoted $T \in \mathcal{JHN}$, if every part of $T$, and (also) invertible part of $T$, is normaloid.

\textbf{Proposition 1.1}. [2, Theorem 1.1] Suppose that an operator $T \in \mathcal{L}(\mathcal{H})$ has a representation

\begin{equation}
T = \begin{bmatrix} T_{1} & T_{2} \\ 0 & T_{3} \end{bmatrix} \left( \begin{array}{c} \mathcal{H}_{1} \\ \mathcal{H}_{2} \end{array} \right)
\end{equation}

such that $T_{3}$ is nilpotent and $\sigma(T_{1}) \subset \sigma(T) \subset \sigma(T_{1}) \cup \{0\}$. If $T_{1} \in \mathcal{JHN}$, non-zero isolated eigenvalues of $T_{1}$ are normal and $(T_{1} - \lambda)^{-1}(0) \oplus 0 \subseteq (T^{*} - \bar{\lambda})^{-1}(0)$, then the Riesz projection $P_{\lambda}$ associated with $\lambda$ is self-adjoint and $P_{\lambda} \mathcal{H} = (T - \lambda)^{-1}(0) = (T^{*} - \bar{\lambda})^{-1}(0)$ for every non-zero $\lambda \in \text{iso}(T)$.

We say that $T \in \mathcal{L}(\mathcal{H})$ has (G$_1$) property if

\[ ||(T - \lambda)^{-1}|| = r((T - \lambda)^{-1}) \text{ for } \lambda \notin \sigma(T). \]

It is well known that a hyponormal operator satisfies (G$_1$) property, but an operator satisfying (G$_1$) property is generally not normaloid.

In [12, Theorem C], Stampfli also proved that if $T$ has (G$_1$) property, then for $\lambda \in \text{iso}(T)$

\begin{equation}
P_{\lambda} \text{ is self-adjoint and } P_{\lambda} \mathcal{H} = (T - \lambda)^{-1}(0) = (T^{*} - \bar{\lambda})^{-1}(0).
\end{equation}

In this note we show that Stampfli’s result is generalized to an nilpotent extension of an operator having (G$_1$) property.
2. Main results

In [8] M. Mbekhta introduced two important subspaces of $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, the quasi-nilpotent part of $T$ is the set

$$H_0(T) = \{ x \in \mathcal{H} : \lim_{n \to \infty} ||T^n x||^{\frac{1}{n}} = 0 \}$$

and the analytic core of $T$ is the set

$$K(T) = \{ x \in \mathcal{H} : \text{there exist a sequence} \{ x_n \} \subset \mathcal{H} \text{ and} \delta > 0$$

$$\text{for which} \ x = x_0, T x_{n+1} = x_n \text{ and} ||x_n|| \leq \delta^n ||x||$$

$$\text{for all} \ n = 1, 2, \cdots \}$$,

which are generally not closed subspaces of $\mathcal{H}$ such that

$$(T)^{-n}(0) \subseteq H_0(T) \text{ and} T K(T) = K(T).$$

It is well known ([8], [9], [10]) that

$$\lambda \in \text{iso}(T) \iff \mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda),$$

where $H_0(T - \lambda)$ and $K(T - \lambda)$ are closed subspaces. Moreover, (2.2) and (2.3) implies that if $H_0(T - \lambda) = (T - \lambda)^{-d}(0)$, then $\lambda \in \text{iso}(T)$ is a pole of the resolvent of $T$ of order $d$.

**Lemma 2.1.** If $T \in \mathcal{L}(\mathcal{H})$ has $(G_1)$ property, then $\lambda \in \text{iso}(T)$ is a pole of the resolvent of $T$ of order 1.

**Proof.** From (1.4) we observe that

$$P_\lambda \mathcal{H} = H_0(T - \lambda) = (T - \lambda)^{-1}(0) \text{ for} \lambda \in \text{iso}(T).$$

Thus, from the above arguments, $\lambda \in \text{iso}(T)$ is a pole of the resolvent of $T$ of order 1.

To prove the following Lemmas we fully adopt Duggal’s arguments ([2], [3]).

**Lemma 2.2.** Suppose that an operator $T \in \mathcal{L}(\mathcal{H})$ has a representation

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}$$

such that $T_1 \in \mathcal{L}(\mathcal{H}_1)$ has $(G_1)$ property and $T_3$ is nilpotent. Then every non-zero $\lambda \in \text{iso}(T)$ is a simple pole(i.e., order one pole) of the resolvent of $T$. 

Proof. Assume that \( \lambda(\neq 0) \in \text{iso}\sigma(T) \). Then \( \lambda(\neq 0) \in \text{iso}\sigma(T_1) \) because \( \sigma(T) = \sigma(T_1) \cup \{0\} \) by [5, Corollary 8]. Since, by Stampfli’s result (1.4), \( (T_1 - \lambda)^{-1}(0) \) reduces \( T \), it follows that

\[
T_1 - \lambda = \begin{bmatrix} 0 & 0 \\ 0 & T_{11} - \lambda \end{bmatrix}
\]
on \( \mathcal{H}_1 = (T_1 - \lambda)^{-1}(0) \oplus (T_1 - \lambda)\mathcal{H} \).

Set

\[
(T_1 - \lambda)^{-1}(0) = \mathcal{H}_1', \quad \mathcal{H}_1' \oplus \mathcal{H}_3' = \mathcal{H}_3' \quad \text{and} \quad \mathcal{H}_3' \oplus \mathcal{H}_2 = \mathcal{H}_2'.
\]

Then it follows that

\[
T - \lambda = \begin{bmatrix} 0 & 0 & T_{21} \\ 0 & T_{11} - \lambda & T_{22} \\ 0 & 0 & T_3 - \lambda \end{bmatrix}
\]

\[
\begin{bmatrix} \mathcal{H}_1' \\ \mathcal{H}_3' \\ \mathcal{H}_2' \end{bmatrix}
= \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix} \begin{bmatrix} \mathcal{H}_1' \\ \mathcal{H}_2' \end{bmatrix},
\]

where \( A = [0 \ T_{21}] \), and where

\[
B = \begin{bmatrix} T_{11} - \lambda & T_{22} \\ 0 & T_3 - \lambda \end{bmatrix}
\]
is invertible. Since

\[
H_0(T - \lambda) = \left\{ x \in \mathcal{H} : \lim_{n \to \infty} \|(T - \lambda)^n x\|^1/n = 0 \right\}
\]

\[
= \left\{ x = x_1 \oplus x_2 \in \mathcal{H} : \lim_{n \to \infty} \left\| \begin{bmatrix} A B^{n-1} x_2 \\ B^n x_2 \end{bmatrix} \right\|^1/n = 0 \right\},
\]

the invertibility of \( B \) implies that

\[
\|x_2\|^1/n \leq \|B^{-1}\|\|B^n x_2\|^1/n \to 0 \text{ as } n \to \infty.
\]

Hence, \( x_2 = 0 \), and

\[
H_0(T - \lambda) = (T_1 - \lambda)^{-1}(0) \oplus \{0\} = (T - \lambda)^{-1}(0).
\]

Therefore we have that

\[
\mathcal{H} = (T - \lambda)^{-1}(0) \oplus (T - \lambda)\mathcal{H} \quad \text{for} \quad \lambda(\neq 0) \in \text{iso}\sigma(T).
\]

\[\square\]

The following result is a slight improvement of [3, Theorem 2.7].

Lemma 2.3. If \( \lambda \in \text{iso}\sigma(T) \) is a simple pole of the resolvent of \( T \), then the Riesz projection \( P_\lambda \) is self-adjoint if and only if

\[
(T - \lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0).
\]

(2.5)
Proof. Since \( \lambda \in \text{iso}(T) \) is a simple pole of the resolvent of \( T \),
\[
(2.6) \quad \mathcal{H} = (T - \lambda)^{-1}(0) \oplus (T - \lambda)\mathcal{H}.
\]
Observe that
\[
P_\lambda\mathcal{H} = H_0(T - \lambda) = (T - \lambda)^{-1}(0) \text{ and } P_\lambda^{-1}(0) = (T - \lambda)\mathcal{H}.
\]
If \( P_\lambda \) is self-adjoint, then
\[
[P_\lambda^{-1}(0)]^\perp = P_\lambda\mathcal{H}.
\]
Since
\[
[P_\lambda^{-1}(0)]^\perp = [(T - \lambda)\mathcal{H}]^\perp = (T^* - \bar{\lambda})^{-1}(0),
\]
it immediately implies that \((T - \lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0)\). Conversely, assuming that
\[
(T - \lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0),
\]
\( P_\lambda\mathcal{H} = H_0(T - \lambda) = (T - \lambda)^{-1}(0) \) is a reducing subspace of \( T \). From (2.6), we have
\[
[P_\lambda\mathcal{H}]^\perp = [(T - \lambda)^{-1}(0)]^\perp = (T - \lambda)\mathcal{H} = P_\lambda^{-1}(0).
\]
Therefore \( P_\lambda \) is self-adjoint.

Theorem 2.4. Suppose that an operator \( T \in \mathcal{L}(\mathcal{H}) \) has a representation
\[
(2.7) \quad T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}
\]
such that \( T_1 \in \mathcal{L}(\mathcal{H}_1) \) has \((G_1)\) property and \( T_3 \) is nilpotent. Then \( \lambda(\neq 0) \in \text{iso}(T) \) is a simple pole of the resolvent of \( T \) and the Riesz projection \( P_\lambda \) is self-adjoint if and only if
\[
(2.8) \quad (T - \lambda)^{-1}(0) \subseteq (T^* - \bar{\lambda})^{-1}(0).
\]

Proof. Combining Lemma 2.2 and Lemma 2.3 completes the proof. \( \square \)
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