Simultaneous ordinary and type A $\mathcal{N}$-fold supersymmetries in Schrödinger, Pauli, and Dirac equations

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We investigate physical models which possess simultaneous ordinary and type A $\mathcal{N}$-fold supersymmetries, which we call type A $(\mathcal{N},1)$-fold supersymmetry. Inequivalent type A $(\mathcal{N},1)$-fold supersymmetric models with real-valued potentials are completely classified. Among them, we find that a trigonometric Rosen–Morse type and its elliptic version are of physical interest. We investigate various aspects of these models, namely, dynamical breaking and interrelation between ordinary and $\mathcal{N}$-fold supersymmetries, shape invariance, quasi-solvability, and an associated algebra which is composed of one bosonic and four fermionic operators and dubbed type A $(\mathcal{N},1)$-fold superalgebra. As realistic physical applications, we demonstrate how these systems can be embedded into Pauli and Dirac equations in external electromagnetic fields.

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I. INTRODUCTION

Ever since the formulation of supersymmetric quantum mechanics [1], a much deeper understanding of structure of various solvable potentials in non-relativistic quantum mechanics has been gained through the ideas of supersymmetry (SUSY) (for a good review, see, e.g., [2, 3]). For instance, seemingly distinct potentials, e.g., an infinite square-well and a cosec$^2$ potential, are in fact supersymmetric partners, and their respective energy levels are connected by supercharges. The methods of SUSY have also been applied to obtaining exact spectra of quantum systems with multi-component wave functions, such as Pauli and Dirac equations. It is based on the fact that, for some electromagnetic field configurations, Pauli and Dirac equations possess intrinsic SUSY structure, and different components of a wave function are related by some supercharges [4, 5]. More recently, the method of SUSY has been employed to construct quasi-exactly solvable potentials in the Pauli and Dirac equations [6, 7]. For quasi-exactly solvable systems, it is only possible to determine algebraically a part of the spectrum but not the whole spectrum [8–12].

Recently, the concepts of SUSY and quasi-exact solvability were combined within the framework of so-called $\mathcal{N}$-fold supersymmetry [13], which is a natural generalization [14] of ordinary supersymmetric quantum mechanics. Among various nonlinear extensions of ordinary SUSY such as parasupersymmetry [15–17], fractional supersymmetry [18], and so on, $\mathcal{N}$-fold SUSY is characterized by the fact that anti-commutators of fermionic operators are polynomials of degree (at most) $\mathcal{N}$ in bosonic operators. It has been proved in a generic way that $\mathcal{N}$-fold SUSY is essentially equivalent to (weak) quasi-solvability (this latter term, in contrast to quasi-exact solvability, is used to include the case where the system admits non-normalizable solutions in closed form) [13]. Up to now, three different families of $\mathcal{N}$-fold supersymmetric systems have been found for arbitrary finite integer $\mathcal{N}$, namely, type A [19, 20], type B [21], and type C [22], which have correspondence with the classification of second-order linear differential operators preserving a monomial-type vector space [23]. Of the three types, type A and type C under a specific condition have been completely classified [20, 22].

While there are still a lot to be done in the mathematical developments of $\mathcal{N}$-fold SUSY, it is also interesting that one looks for physical systems which possess such generalized SUSY. In view of the fact that so far all the $\mathcal{N}$-fold SUSY potentials are only one-dimensional, it is natural that one should look for physical models which are effectively one-dimensional. Experience gained in the work in [6, 7] suggests that Pauli and Dirac equations are good candidates to start with. In this respect we note that the authors of [24] found that if the Pauli equation is generalized such that the gyromagnetic ratio $g = 2$ of the electron is changed to some unphysical values $g = 2\mathcal{N}$ ($\mathcal{N} \geq 2$), then for certain forms of magnetic fields, the generalized Pauli equation could possess type A $\mathcal{N}$-fold SUSY. The result is interesting in a purely mathematical view point, but unfortunately it would not describe any existing physical systems.

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Therefore, we would like to extend the realistic systems in [6, 7] to include \( \mathcal{N} \)-fold SUSY. Since the Pauli and Dirac equations considered there all possess ordinary SUSY, it is therefore natural that we look for an ordinary supersymmetric system which has \( \mathcal{N} \)-fold supersymmetry as well, as a starting point of the aforementioned purpose. This naturally led us further to consider a more general situation in which a system has simultaneous \( \mathcal{N} \)-fold supersymmetry with two different values of \( \mathcal{N} \). In our previous paper [25], we have succeeded in constructing a family of such systems, which we have called type A \((\mathcal{N}_1, \mathcal{N}_2)\)-fold SUSY. These systems possess simultaneous type A \(\mathcal{N}_1\) and \(\mathcal{N}_2\)-fold SUSYs. Hence, the aim mentioned at the beginning of this paragraph is equivalent to finding Pauli and Dirac systems with type A \((\mathcal{N}, 1)\)-fold SUSY.

In this paper, we shall present a detailed study of Schrödinger, Pauli, and Dirac systems which possess ordinary and type A SUSYs simultaneously. It is amazing to realize that some well-known shape-invariant potentials, namely, the infinite square-well and the cosec\(^2\) potentials with specific strengths of the coupling constant, are indeed potentials of such kind. We also find an elliptic generalization of these potentials which breaks shape invariance but preserves the structure of type A \((\mathcal{N}, 1)\)-fold SUSY. We study various aspects of these models such as dynamical breaking and interrelation between ordinary and \(\mathcal{N}\)-fold supersymmetries, shape invariance, quasi-solvability, and an associated algebra which is composed of one bosonic and four fermionic operators and dubbed type A \((\mathcal{N}, 1)\)-fold superalgebra.

The organization of the paper is as follows. In Sect. II some of the main ideas of \(\mathcal{N}\)-fold SUSY are reviewed. We then discuss in Sect. III the idea of simultaneous type A \((\mathcal{N}, 1)\)-fold SUSY. Section IV gives a detailed discussion of trigonometric Rosen–Morse type potentials in Schrödinger equation. In particular, we show that shape invariance in the case enables us to establish a relation between ordinary and \(\mathcal{N}\)-fold supercharges, and to derive type A \((\mathcal{N}, 1)\)-fold superalgebra in closed form for arbitrary \(\mathcal{N}\). Similar discussion of elliptic potentials is given in Sect. V. Extensions to the Pauli and the Dirac equations are presented in Sect. VI. Section VII concludes the paper. In the Appendix A, we list the complete classification of real type A \((\mathcal{N}, 1)\)-fold SUSY. Appendix B gives a summary of the generalized Bender–Dunne polynomials (GBDPs), and present some explicit forms of them for the shape-invariant and elliptic potentials investigated in Sections IV and V.

II. \(\mathcal{N}\)-FOLD SUPERSYMMETRY

A. Definition of \(\mathcal{N}\)-fold Supersymmetry

We begin with defining \(\mathcal{N}\)-fold supersymmetry in one-dimensional quantum mechanics. Consider a matrix Hamiltonian \(H_{\mathcal{N}}\) given by

\[
H_{\mathcal{N}} = \begin{pmatrix} H^+_{\mathcal{N}} & 0 \\ 0 & H^-_{\mathcal{N}} \end{pmatrix},
\]

(2.1)

where \(H^\pm_{\mathcal{N}}\) are ordinary scalar Hamiltonians

\[
H^\pm_{\mathcal{N}} = \frac{1}{2} p^2 + V^\pm_{\mathcal{N}}(q),
\]

(2.2)

with \(p = -i\partial/\partial q\). \(\mathcal{N}\)-fold supercharges \(Q^\pm_{\mathcal{N}}\) are defined by,

\[
Q^-_{\mathcal{N}} = \begin{pmatrix} 0 & P^-_{\mathcal{N}} \\ 0 & 0 \end{pmatrix}, \quad Q^+_{\mathcal{N}} = \begin{pmatrix} 0 & 0 \\ 0 & P^+_{\mathcal{N}} \end{pmatrix},
\]

(2.3)

where the components \(P^\pm_{\mathcal{N}}\) are defined by,

\[
P^-_{\mathcal{N}} = P_{\mathcal{N}}, \quad P^+_{\mathcal{N}} = P^t_{\mathcal{N}},
\]

(2.4)

in terms of a monic \(\mathcal{N}\)th-order linear differential operator \(P_{\mathcal{N}}\) of the form,

\[
P_{\mathcal{N}} = \frac{d^\mathcal{N}}{dq^{\mathcal{N}}} + w_{\mathcal{N}-1}(q) \frac{d^{\mathcal{N}-1}}{dq^{\mathcal{N}-1}} + \cdots + w_1(q) \frac{d}{dq} + w_0(q).
\]

(2.5)

In Eq. (2.4), the superscript \(t\) denotes the formal transposition of operators in a linear space of functions of \(q\) defined by \((d/dq)^t = -d/dq\). Note that when all the functions \(w_k(q)\) appearing in Eq. (2.5) are real-valued, the operator \(P^+_{\mathcal{N}}\) defined by Eq. (2.4) is identical with the formal adjoint of \(P_{\mathcal{N}}\): \(P^+_{\mathcal{N}} = P^t_{\mathcal{N}}\).
A system (2.1) is said to be $\mathcal{N}$-fold supersymmetric with respect to $Q^+_{\mathcal{N}}$ if it commutes with them:

$$[Q^+_{\mathcal{N}}, H_{\mathcal{N}}] = [Q^+_{\mathcal{N}}, H_{\mathcal{N}}] = 0. \quad (2.6)$$

In components, the commutation relations (2.6) are expressed as the following intertwining relations:

$$P^-_N H^-_N - H^+_N P^+_N = 0, \quad P^+_N H^-_N - H^-_N P^+_N = 0. \quad (2.7)$$

Therefore, the relations in (2.7) give the conditions for the system $H_{\mathcal{N}}$ to be $\mathcal{N}$-fold supersymmetric. Note that the Hamiltonians (2.2) are always symmetric under the formal transposition even when they are not Hermitian, and thus each of the relations in Eq. (2.7) actually implies the other.

It was investigated in Ref. [13] that the $\mathcal{N}$-fold supersymmetric models defined above have several significant properties similar to those of the ordinary supersymmetric models. In the following, we shall summarize some of the most important aspects of $\mathcal{N}$-fold supersymmetry.

### B. Quasi-solvability, mother Hamiltonian, and generalized Witten index

One of the most important aspects of $\mathcal{N}$-fold supersymmetry is that the component Hamiltonians $H^-$ and $H^+$ are always weakly quasi-solvable with respect to the operators $P^-_N$ and $P^+_N$, respectively [13, 20]. That is, $H^\pm$ leave the kernels of $P^\pm_N$ invariant:

$$H^\pm V^\pm_N \subset V^\pm_N, \quad V^\pm_N = \ker P^\pm_N. \quad (2.8)$$

As a consequence, we can in principle diagonalize algebraically the Hamiltonians $H^\pm$ in the finite $\mathcal{N}$-dimensional vector spaces $V^\pm_N$, which are thus called *solvable sectors* of $H^\pm$. If the space $V^{\pm(-)}_N$ is a subspace of a Hilbert space $L^2$ on which the Hamiltonian $H^{\pm(-)}$ is defined, the elements of $V^{\pm(-)}_N$ provide a part of the exact eigenfunctions and thus $H^{\pm(-)}$ is called *quasi-exactly solvable*.

In ordinary SUSY, the anti-commutator of the supercharges corresponds to the original Hamiltonian $H_{\mathcal{N}}$. However, it is not generally the case with $\mathcal{N}$-fold SUSY. This is because $\{Q^-_{\mathcal{N}}, Q^+_{\mathcal{N}}\}$ is now a $2\mathcal{N}$th-order differential operator. The half of the anti-commutator of $Q^-_{\mathcal{N}}$ and $Q^+_{\mathcal{N}}$ is called *mother Hamiltonian* and is denoted by $H_{\mathcal{N}}$:

$$H_{\mathcal{N}} = \frac{1}{2} (Q^-_{\mathcal{N}}, Q^+_{\mathcal{N}}) = \frac{1}{2} \begin{pmatrix} P^-_N P^+_N & 0 \\ 0 & P^+_N P^-_N \end{pmatrix}. \quad (2.9)$$

An immediate consequence of the above definition is that the mother Hamiltonian always commutes with the $\mathcal{N}$-fold supercharges, that is, it is $\mathcal{N}$-fold supersymmetric:

$$[Q^-_{\mathcal{N}}, H_{\mathcal{N}}] = [Q^+_{\mathcal{N}}, H_{\mathcal{N}}] = 0. \quad (2.10)$$

Furthermore, if the original Hamiltonian $H_{\mathcal{N}}$ is $\mathcal{N}$-fold supersymmetric, the mother Hamiltonian also commutes with $H_{\mathcal{N}}$ due to the relation (2.6):

$$[H_{\mathcal{N}}, H_{\mathcal{N}}] = 0. \quad (2.11)$$

We now come to discuss the characteristics of the spectrum of an $\mathcal{N}$-fold supersymmetric system. As with ordinary SUSY, the bosonic (lower) and the fermionic (upper) states of such a system are in one-to-one correspondence provided that these states are eigenstates of the mother Hamiltonian with non-zero eigenvalues. This can be seen as follows. Consider a normalized bosonic state $\Phi^-$ satisfying

$$H_{\mathcal{N}} \Phi^- = E^- \Phi^- \quad (2.12)$$

Since $H_{\mathcal{N}}$ commutes with $H_{\mathcal{N}}$, $\Phi^-$ is also an eigenstate of $H_{\mathcal{N}}$:

$$H_{\mathcal{N}} \begin{pmatrix} 0 \\ \Phi^- \end{pmatrix} = \mathcal{E} \begin{pmatrix} 0 \\ \Phi^- \end{pmatrix}. \quad (2.13)$$

If $\mathcal{E}$ is non-zero, then there exists the following normalized state,

$$\Phi^+ = \frac{1}{\sqrt{2\mathcal{E}}} P^+_N \Phi^-. \quad (2.14)$$
From Eq. (2.7), we have
\[ H_N^+ \Phi^+ = E^− \Phi^+, \] (2.15)
which shows that \( \Phi^+ \) is also an eigenstate of \( H_N^+ \) with the same eigenvalue \( E^− \). Furthermore, this state is also the eigenstate of the mother Hamiltonian with the same \( E \):
\[ \mathcal{H}_N \left( \Phi^+ \right)_0 = E \left( \Phi^+ \right)_0, \] (2.16)
since \( \mathcal{H}_N \) commutes with \( Q_N^+ \) and
\[ \left( \Phi^+ \right)_0 = \frac{Q_N}{\sqrt{2E}} \left( \begin{array}{c} 0 \\ \Phi^- \end{array} \right). \] (2.17)
Similarly, one can show that a bosonic state is obtainable from a fermionic one at each energy level unless \( E = 0 \):
\[ \Phi^− = \frac{1}{\sqrt{2E}} P_N^+ \Phi^+. \] (2.18)

One can generalize the Witten index of ordinary SUSY to \( N \)-fold SUSY by
\[ \text{Tr}(-1)^F = \dim \ker \mathcal{Q}^+_N - \dim \ker \mathcal{Q}_N^−, \] (2.19)
where \( \ker \mathcal{Q}_N^\pm \) are subspaces of a Hilbert space on which the super-Hamiltonian \( H_N \) is naturally defined. We note that only the physical states with energy \( E = 0 \) of the mother Hamiltonian \( \mathcal{H}_N \) contribute to the index. The index thus takes an integer value as the number of these zero energy states is finite (\( 2N \) at most). When the generalized Witten index is non-zero, \( N \)-fold supersymmetry is not broken dynamically.

C. Type A \( N \)-fold supersymmetry

As previously mentioned, at present three different families of \( N \)-fold supersymmetric systems have been found for arbitrary finite integer \( N \). We shall be concerned only with \( N \)-fold supersymmetry of type A.

The component of the type A \( N \)-fold supercharge is defined by
\[ P_N = \left( \frac{d}{dq} + W(q) - \frac{N−1}{2} E(q) \right) \left( \frac{d}{dq} + W(q) - \frac{N−3}{2} E(q) \right) \times \cdots \times \left( \frac{d}{dq} + W(q) + \frac{N−3}{2} E(q) \right) \left( \frac{d}{dq} + W(q) + \frac{N−1}{2} E(q) \right), \] (2.20)
According to Ref. [20], the necessary and sufficient condition for type A \( N \)-fold supersymmetry is the following:
\[ V_N^\pm(q) = \frac{1}{2} W(q)^2 - \frac{N^2−1}{24} (2E′(q) − E(q)^2) ± \frac{N}{2} W′(q) − R, \] (2.21)
where \( R \) is a constant, and the functions \( W(q) \) and \( E(q) \) satisfy
\[ \left( \frac{d}{dq} - E(q) \right) \frac{d}{dq} \left( \frac{d}{dq} + E(q) \right) W(q) = 0 \quad \text{for} \quad N \geq 2, \] (2.22)
\[ \left( \frac{d}{dq} - 2E(q) \right) \left( \frac{d}{dq} - E(q) \right) \frac{d}{dq} \left( \frac{d}{dq} + E(q) \right) E(q) = 0 \quad \text{for} \quad N \geq 3. \] (2.23)
Construction of potentials possessing type A \( N \)-fold supersymmetry has been investigated by analytic calculations of the intertwining relations, and by \( \mathfrak{sl}(2) \) construction based on quasi-solvability. We refer the readers to [20] for the complete classification of the potentials.

The solvable sectors \( \mathcal{V}_N^\pm \) of the type A \( N \)-fold supersymmetric Hamiltonians, which are the vector spaces preserved by \( H_N^\pm \), are given by the kernel of the type A \( N \)-fold supercharges:
\[ \mathcal{V}_N^\pm = \ker P_N^\pm = e^{-\mathcal{W}_N^\pm (1,z,...,z^{N−1})}, \] (2.24)
where the gauge functions $W^\pm_{\mathcal{N}}$ are given by

$$W^\pm_{\mathcal{N}} = \frac{\mathcal{N} - 1}{2} \int dq E \mp \int dq W.$$  \hfill (2.25)

Finally, we note here that, when $\mathcal{N} = 1$, 1-fold SUSY is just ordinary SUSY, and $P^-_{\mathcal{N}}$ and $P^+_{\mathcal{N}}$ are simply ordinary supercharges $A^- = \frac{d}{dq} + W$ and $A^+ = -\frac{d}{dq} + W$, respectively.

### III. SIMULTANEOUS ORDINARY AND $\mathcal{N}$-FOLD SUSYS

Since various quantum systems of physical interest possess ordinary SUSY, an interesting question is therefore whether some of these also possess $\mathcal{N}$-fold SUSY. This question has led us to investigate a more general problem, namely, to classify systems which possess type A $\mathcal{N}$-fold SUSY with two different values of $\mathcal{N}$ [25]. We have called such systems as type A $(\mathcal{N}_1, \mathcal{N}_2)$-fold supersymmetric.

A system $H$ is said to have $(\mathcal{N}_1, \mathcal{N}_2)$-fold supersymmetry if it commutes with two different $\mathcal{N}_i$-fold supercharges ($i = 1, 2$) simultaneously, namely,

$$[Q^{(1)\pm}_{\mathcal{N}_1}, H] = [Q^{(2)\pm}_{\mathcal{N}_2}, H] = 0.$$  \hfill (3.1)

Without loss of generality, we can assume $\mathcal{N}_1 \geq \mathcal{N}_2$. In this case, it is evident that the components $H^\pm$ of the system $H$ preserve two vector spaces $\mathcal{V}^{(1)\pm}_{\mathcal{N}_1} = \ker P^{(1)\pm}_{\mathcal{N}_1}$ and $\mathcal{V}^{(2)\pm}_{\mathcal{N}_2} = \ker P^{(2)\pm}_{\mathcal{N}_2}$ separately, where $P^{(i)\pm}_{\mathcal{N}_i}$ are components of $Q^{(i)\pm}_{\mathcal{N}_i}$ ($i = 1, 2$). Hence, the solvable sectors $\mathcal{V}^{\pm}_{\mathcal{N}_1, \mathcal{N}_2}$ of $(\mathcal{N}_1, \mathcal{N}_2)$-fold supersymmetric Hamiltonians $H^\pm$ are generally given by

$$\mathcal{V}^{\pm}_{\mathcal{N}_1, \mathcal{N}_2} = \mathcal{V}^{(1)\pm}_{\mathcal{N}_1} \cup \mathcal{V}^{(2)\pm}_{\mathcal{N}_2}. $$  \hfill (3.2)

Classification of type A $(\mathcal{N}_1, \mathcal{N}_2)$-fold supersymmetry has been presented in [25]. Here we shall focus on type A $(\mathcal{N}, 1)$-fold supersymmetric system, i.e., system possessing simultaneous ordinary and $\mathcal{N}$-fold SUSYs. For such a system, its potential must satisfy

$$\mathcal{V}^{(\mathcal{N})\pm}_{\mathcal{N}}(q) = \frac{1}{2} W(q)^2 - \frac{\mathcal{N}^2 - 1}{24} (2E'(q) - E(q))^2 \pm \frac{\mathcal{N}}{2} W'(q) - R$$
$$= \frac{1}{2} \left( W^{(\mathcal{N})\pm}_{\mathcal{N}}(q)^2 \pm W^\prime_{\mathcal{N}}(q) \right),$$  \hfill (3.3)

where $R$ is a constant, and $W^{(\mathcal{N})\pm}_{\mathcal{N}}$ is the first derivative of a superpotential of ordinary SUSY, with the subscript $\mathcal{N}$ signifying that it is also related to $\mathcal{N}$-fold SUSY.

From Eq. (2.21) we immediately have

$$\mathcal{N} W' = W^{(\mathcal{N})}_\mathcal{N}, $$  \hfill (3.4)
$$W^2 - \frac{\mathcal{N}^2 - 1}{12} (2E' - E^2) - 2R = W^2_{\mathcal{N}}.$$  \hfill (3.5)

The first condition (3.3) is easily integrated as

$$W^{(\mathcal{N})}_\mathcal{N} = \mathcal{N} W + C,$$  \hfill (3.6)

where $C$ is a constant. Substituting Eq. (3.6) into Eq. (3.5), we obtain

$$(\mathcal{N}^2 - 1) W^2 + 2NCW + \frac{\mathcal{N}^2 - 1}{12} (2E' - E^2) + C^2 + 2R = 0.$$  \hfill (3.7)

To investigate Eq. (3.7), it is more convenient to gauge-transform the type A Hamiltonians to [20]

$$H^\pm = e^{-W^{(\mathcal{N})\pm}_{\mathcal{N}}} \left[ -A(z) \frac{d^2}{dz^2} + \left( \frac{\mathcal{N} - 2}{2} A'(z) \pm Q(z) \right) \frac{d}{dz} - \frac{(\mathcal{N} - 1)(\mathcal{N} - 2)}{12} A''(z) \pm \frac{\mathcal{N} - 1}{2} Q'(z) - R \right] e^{W^{(\mathcal{N})\pm}_{\mathcal{N}}},$$  \hfill (3.8)
where the gauge potentials \( W^\pm_N \) are given by Eq. (2.25), and \( A(z) \) and \( Q(z) \) are polynomials of at most fourth- and second-degree, respectively, and related to \( E(q) \) and \( W(q) \) by

\[
A(z) = \frac{1}{2}(z')^2 = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, \\
A'(z) = z'' = E z', \\
Q(z) = -W z' = b_2 z^2 + b_1 z + b_0.
\]

(3.9) (3.10) (3.11)

With the aid of these relations, we can show that the condition (3.7) is satisfied if

\[
\left( Q^2 + \frac{1}{3}H[A] + \frac{2(C^2 + 2R)}{N^2 - 1} \right)^2 = \frac{8N^2C^2}{(N^2 - 1)^2} A^2,
\]

(3.12)

where

\[
H[A] = AA'' - \frac{3}{4}(A')^2,
\]

(3.13)

is an algebraic covariant called the Hessian of \( A \) [26, 27]. The solvable sectors \( \mathcal{V}_{N,1}^\pm \) of the type \( A \) \((N,1)\)-fold supersymmetric Hamiltonians are

\[
\mathcal{V}_{N,1}^\pm = \mathcal{V}_N^\pm \cup \mathcal{V}_N^1,
\]

(3.14)

where \( \mathcal{V}_N^\pm \) are given by Eq. (2.24), and \( \mathcal{V}_N^1 \) consists of the states

\[
\psi_0^{\pm} \propto e^{\pm C q} \exp \left( \pm N \int dq W \right).
\]

(3.15)

We note here that the condition (3.12) is expressed as an algebraically covariant form under the projective transformations \( GL(2, K) \) \((K = \mathbb{R} \text{ or } \mathbb{C})\) on \( A(z) \) and \( Q(z) \) (cf. Refs. [11, 20]):

\[
A(z) \mapsto \tilde{A}(z) = \Delta^{-2}(\gamma z + \delta)^4 A \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right), \\
Q(z) \mapsto \tilde{Q}(z) = \Delta^{-1}(\gamma z + \delta)^2 Q \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right),
\]

(3.16) (3.17)

where \( \alpha, \beta, \gamma, \delta \in K \) and \( \Delta = \alpha \delta - \beta \gamma \neq 0 \). We can easily show that the polynomial \( A(z) \) of at most fourth-degree is transformed to one of the five canonical forms listed in Table I by the \( GL(2, \mathbb{C}) \) transformation. The complete exhibition of each model for more general type \( A \) \((N_1, N_2)\)-fold SUSY in this complex classification scheme is found in Ref. [25].

| Case | Canonical Form | \( H[A] \) |
|------|---------------|-------------|
| I    | 1/2           | 0           |
| II   | 2z            | -3          |
| III  | 2\( \mu z^2 \) | -4\( \mu^2 z^2 \) |
| IV   | 2\( \mu (z^2 - 1) \) | -4\( \mu^2 (z^2 + 2) \) |
| V    | 2\( z^3 - g_2 z^2 / 2 - g_3 z^2 / 2 - 6g_3^2 / 2 - 6g_3 z - 3g_2^2 / 16 \) | |

TABLE I: Canonical forms of \( A(z) \) and their corresponding \( H[A] \). The parameters \( \mu, g_2, g_3 \in \mathbb{C} \) satisfy \( \mu \neq 0 \) and \( g_2^3 - 27 g_3^2 \neq 0 \).

However, for physical applications where the Hamiltonians should be real and Hermitian, it is more convenient and suitable to express potentials in terms of real functions. For this purpose, we should use the real classification scheme under the \( GL(2, \mathbb{R}) \) transformation. In Appendix A, we list all the inequivalent models of type \( A \) \((N, 1)\)-fold SUSY in terms of real functions. Among these models, we find that only the Rosen–Morse type and some of the elliptic models can have physical interest, which we shall discuss in the next two sections.
IV. SHAPE-ININVARIANT POTENTIAL

In this section, we concentrate on the trigonometric version of Case IVa, Eqs. (A8)–(A10). The hyperbolic version is not of physical interest, as any element of the solvable sectors $V^\pm_N$ and $\psi_0^\pm$ is non-normalizable and thus unphysical. For convenience, we shall hereafter adopt a unit system in which $\hbar = e = 2m = c = 1$, so that the results obtained here for the Schrödinger equation can be readily carried over to the Pauli and Dirac equations. As such, the Hamiltonians $H_N^\pm$ and potentials $V_N^\pm$ in this and the next two sections differ from those in the previous sections and in Appendix A by a multiplicative factor 2.

The trigonometric version of Case IVa can be obtained by applying a scale transformation Eqs. (A1) and (A2) with $\nu < 0$ to Eqs. (8)–(10). We take $\nu = -1/4$ without loss of generality. With this choice, the functions which characterize the $N$-fold ($N \geq 2$) and the ordinary supercharges are given by (we hereafter change the variable $q$ to $x$)

$$E(x) = \cot x, \quad W(x) = -\frac{1}{2}\cot x, \quad W_N(x) = -\frac{N}{2}\cot x.$$  \hspace{1cm} (4.1)

The corresponding Hamiltonians read

$$H_N^\pm = -\frac{d^2}{dx^2} + V_N^\pm(x), \quad V_N^\pm(x) = \frac{1}{4} [N(N \pm 2) \csc^2 x - N^2].$$ \hspace{1cm} (4.2)

These potentials are periodic in $x$, but tend to positive infinity at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Hence, the particle is confined only in any one of the infinite $\csc^2 x$ wells with width $\pi$ in $x$. Without loss of generality, we shall take the physical domain of interest to be $0 < x < \pi$. The solvable sectors are given by

$$V_N^\pm = (\sin x)^{\frac{N-1}{2}} (1, \cos x, \ldots, (\cos x)^{N-1}), \quad \psi_0^\pm \propto (\sin x)^{\frac{N}{2}}. \hspace{1cm} (4.3)$$

In what follows, we will investigate and discuss various aspects of simultaneous ordinary and $N$-fold supersymmetries, interplay of them, and their physical consequences.

A. Structure of ordinary SUSY

First of all, due to ordinary SUSY, the Hamiltonians $H_N^\pm$ are factorizable as

$$H_N^- = A^-_N A_N^-, \quad H_N^+ = A_N^+ A_N^+.$$ \hspace{1cm} (4.4)

with the first-order operators

$$A^-_N \equiv \frac{d}{dx} + W_N, \quad A^+_N \equiv -\frac{d}{dx} + W_N.$$ \hspace{1cm} (4.5)

It follows that

$$A_N^- H_N^- = H_N^+ A_N^- \quad A_N^+ H_N^+ = H_N^- A_N^+,$$ \hspace{1cm} (4.6)

which corresponds to Eq. (2.7) in the case of 1-fold SUSY. If we introduce the supercharges $A_{\pm N}$ in a matrix representation by

$$A^-_N = \begin{pmatrix} 0 & A^-_N \\ 0 & 0 \end{pmatrix}, \quad A^+_N = \begin{pmatrix} 0 & 0 \\ A^+_N & 0 \end{pmatrix},$$ \hspace{1cm} (4.7)

the relations (4.4) and (4.6) are expressed as a Lie superalgebra:

$$[A^\pm_{\pm N}, H_N] = 0, \quad \{A^-_{\pm N}, A^+_N\} = H_N.$$ \hspace{1cm} (4.8)

Let us denote the normalized eigenfunctions of the Hamiltonians $H_N^\pm$ by $\psi^{(N)}_n^\pm$ with eigenvalues $E^{(N)}_n^\pm$, respectively. Here the subscript $n = 0, 1, 2, \ldots$ denotes the number of nodes of the wave function. It is easily proved that $V_N^-$ and $V_N^+$, being partners of ordinary SUSY, have the same energy spectrum except for the ground state of $V_N^-$ with $E^{(N)}_0^- = 0$, which has no corresponding level for $V_N^+$. More explicitly, we have the following supersymmetric relations:

$$E^{(N)}_n^+ = E^{(N)}_{n+1}^-.$$ \hspace{1cm} (4.9)
\[ \psi_{n}^{(N)} = (E_{n+1}^{(N)})^{-1/2} A_{N+1}^{+} \psi_{n+1}^{(N)}, \]
\[ \psi_{n+1}^{(N)} = (E_{n}^{(N)})^{-1/2} A_{N}^{+} \psi_{n}^{(N)}. \]  

(4.10) \hspace{1cm} (4.11)

Hence \( A_{N}^{+} \) converts an eigenfunction of \( H_{N}^{-} \) into an eigenfunction of \( H_{N}^{+} \) with the same energy, but with one less number of nodes, while \( A_{N}^{-} \) does the reverse.

The supersymmetric element \( \psi_{0}^{-} \) of the solvable sectors in Eq. (4.3) is normalizable in the physical domain \( 0 < x < \pi \) we have considered, and corresponds to the ground state wave function \( \psi_{0}^{(N)}^{-} \) of \( V_{N}^{-} \):

\[ \psi_{0}^{(N)}^{-} = \psi_{0}^{-} \propto (\sin x)^{N/2}, \quad A_{N}^{-} \psi_{0}^{(N)}^{-} = 0. \]  

(4.12)

Hence, ordinary SUSY in this system is unbroken.

### B. Two chains of shape-invariant potentials

Next, we note that the potentials \( V_{N}^{\pm} \) satisfy

\[ V_{N}^{\pm}(x) = V_{N+2}^{\mp}(x) + R_{N}, \quad R_{N} = N + 1. \]  

(4.13)

This implies

\[ A_{N}^{-} A_{N}^{+} = A_{N+1}^{+} A_{N+2}^{+} + R_{N}, \]  

(4.14)

or, equivalently, \( H_{N}^{+} = H_{N+2}^{-} + R_{N} \). Potentials satisfying Eq. (4.13) are called shape-invariant, and are (exactly) solvable [2, 3]. Indeed, it turns out that \( V_{N}^{\pm} \) belong to one of the well known shape-invariant potentials, namely, the Rosen–Morse-I potentials [2, 3], only with parameters being related to the integer \( N \). In particular, the potential \( V_{N}^{-} \) corresponding to \( N = 2 \) is simply an infinite square-well in the domain \( 0 < x < \pi \). Below we give a brief discussion of the relation between shape invariance and exact solvability, specializing to the potentials \( V_{N}^{\pm} \) considered here.

The wave function and energy of the ground state of \( V_{N+2}^{\pm} \) are \( \psi_{0}^{(N+2)^{-}} \propto (\sin x)^{N/2} \) (cf. Eq. (4.12)) and \( E_{0}^{(N+2)^{-}} = 0 \), respectively. The shape invariance (4.13) indicates that the ground state wave function \( \psi_{0}^{(N)^{+}} \) of \( V_{N}^{+} \) is given by \( \psi_{0}^{(N)^{+}} = \psi_{0}^{(N+2)^{-}} \) with the ground state energy \( E_{0}^{(N)^{+}} = R_{N} \). On the other hand, from the supersymmetric relations (4.9)–(4.11), the first excited state \( \psi_{1}^{(N)^{-}} \) of \( V_{N}^{-} \) is related to the ground state \( \psi_{0}^{(N)^{+}} \) of \( V_{N}^{+} \) by \( \psi_{1}^{(N)^{-}} \propto A_{N}^{+} \psi_{0}^{(N)^{+}} \), with the same energy \( E_{1}^{(N)^{-}} = E_{0}^{(N)^{+}} = R_{N} \). Thus, we have \( \psi_{1}^{(N)^{-}} \propto A_{N}^{+} \psi_{0}^{(N+2)^{-}} \). By iterating this process, we can obtain the wave function of the \( n \)th excited state of \( V_{N}^{-} \) as

\[ \psi_{n}^{(N)^{-}} \propto A_{N}^{+} A_{N+2}^{+} \cdots A_{N+2(n-2)+1}^{+} A_{N+2(n-1)+1}^{+} \psi_{0}^{(N+2n)^{-}}, \]  

(4.15)

and energy eigenvalues are given by

\[ E_{n}^{(N)^{-}} = \sum_{k=0}^{n-1} R_{N+2k} = n(N + n), \quad n = 0, 1, 2, \ldots. \]  

(4.16)

With these formulas, the energy and wave functions for all \( V_{N+2k}^{\pm} \) are also exactly calculable in closed form. We note here that since Eq. (4.13) requires the integers \( N \) in \( V_{N}^{\pm} \) to be differed by two, we therefore have two different chains of shape-invariant cosec\(^2\) potentials in which two neighboring potentials possess type A \( (N, 1) \)-fold SUSY, one for even \( N \)'s, and the other for odd \( N \)'s.

To make discussion of \( N \)-fold SUSY structure of \( V_{N}^{\pm} \) transparent in the next subsection, it convenient to rewrite the symbols of the potentials in the same chain of shape invariance related by supercharges \( A_{N+2k}^{-} \) and \( A_{N+2k}^{+} \), \((k = 0, 1, 2, \ldots \text{ and } N + 2k \geq 2)\). Let us write

\[ V^{(N+2k)}(x) = V_{N+2k}^{-}(x) + \sum_{l=0}^{k-1} R_{N+2l}, \quad k = 0, 1, 2, \ldots. \]  

(4.17)

The potentials \( V^{(m)} \) and \( V^{(m+2)} \) possess type A \( (m, 1) \)-fold SUSY. Now let us denote by \( \psi_{n}^{(m)} \) and \( E_{n}^{(m)} \) \((m = N + 2k, \ n, k = 0, 1, 2, \ldots \) the wave function and energy eigenvalue, respectively, of the \( n \)th energy level of the
potential $V^{(m)}$. According to Eq. (4.17), the ground state of $V^{(m)}$ is also the ground state of $V_n^m$ (with energy $E_0^{(m)} = 0$), hence we have $\psi_0^{(m)} \propto (\sin x)^N$, and $E_0^{(m)} = (m^2 - N^2)/4$ from Eqs. (4.12) and (4.16). Then the supersymmetric relations (4.4), (4.9)–(4.11) can be rewritten in terms of the eigenfunctions $\psi_n^{(m)}$ and eigenvalues $E_n^{(m)}$ of the potential $V^{(m)}$ defined by Eq. (4.17) as follows:

$$A^+_m A^-_m \psi_n^{(m)} = E_n^{(m)} \psi_n^{(m)}, \quad E_n^{(m)} = E_{n+2}^{(m)},$$

$$A^-_m \psi_n^{(m)} \propto \psi_{n-1}^{(m+2)}, \quad A^+_m \psi_n^{(m)} \propto \psi_{n+1}^{(m-2)}.$$  

The relations (4.20) and (4.21) are represented diagrammatically in Fig. 1 for odd $N$.

C. Structure of type A $\mathcal{N}$-fold SUSY

Now we examine how type A $\mathcal{N}$-fold SUSY manifests itself in the two chains of shape-invariant potentials. Essentially three issues need be addressed, namely, (1) whether $\mathcal{N}$-fold SUSY is dynamically broken or not; (2) how the supercharges $P^\pm_N$ and $A^\pm_N$ are related; and (3) what kind of superalgebra the five operators $H_N, A^\pm_N, Q^\pm_N$ which characterize the system satisfy. We shall address these questions below for the two chains of shape-invariant potentials.

1. Dynamical $\mathcal{N}$-fold SUSY breaking

If $\mathcal{N}$-fold SUSY is not broken, then the ground state, and perhaps some excited states, must belong to

$$V^-_N = (\sin x)^{-\frac{N}{2}+1} \langle 1, \cos x, \ldots, (\cos x)^{N-1} \rangle.$$  

(4.22)

In our present case, the ground state of $V^-_N$, given by Eq. (4.12), is arranged as

$$\psi_0^{(N)} \propto (\sin x)^\frac{N}{2} = (\sin x)^{-\frac{N}{2}+1} (\sin x)^{N-1}.$$  

(4.23)

Now it is not hard to check that for odd $\mathcal{N}$, the factor $(\sin x)^{N-1}$ belongs to $\langle 1, \cos x, \ldots, (\cos x)^{N-1} \rangle$, while for even $\mathcal{N}$ it does not.

Next, we shall consider the first excited states $\psi_1^{(N)}$. From the relation (4.21), we find that $\psi_1^{(N)}$ can be obtained by applying the supercharge $A^+_N$ on $\psi_0^{(N+2)}$, i.e.,

$$\psi_1^{(N)} \propto A^+_N \psi_0^{(N+2)} = \left(-\frac{d}{dx} - \frac{N}{2} \cot x\right) (\sin x)^{\frac{N+2}{2}} \propto (\sin x)^\frac{N}{2} \cos x = (\sin x)^{-\frac{N}{2}+1} (\sin x)^{N-1} \cos x.$$  

(4.24)

We easily see that $\psi_1^{(N)}$ does not belong to $V^-_N$ for any $\mathcal{N}$. Similarly, one can check that no higher excited states belong to $V^-_N$, as they pick up higher powers in sin and cos terms as $\mathcal{N}$ increases.

In the same way, one can check that all the physical states of $V^+_N = V^{(N+2)}$ do not belong to $V^+_N$ for any $\mathcal{N}$. Hence, from the generalized Witten index, we conclude that $\mathcal{N}$-fold SUSY is preserved barely by the ground state of $V^-_N = V^{(N)}$ for odd $\mathcal{N}$, and is completely broken for even $\mathcal{N}$.

2. Actions of $P^\pm_N$ on eigenstates

As we have seen in Eq. (4.1), we have $E = -2W = \cot x$ in our case. This significant property enables us to express the $\mathcal{N}$-fold supercharges $P^\pm_N$ in terms of the ordinary supercharges $A^\pm_N$. In fact, the type A $\mathcal{N}$-fold supercharge defined by Eq. (2.20) in this case becomes
\[ P_N = \left( \frac{d}{dx} - \frac{N}{2} \cot x \right) \left( \frac{d}{dx} - \frac{N-2}{2} \cot x \right) \cdots \left( \frac{d}{dx} + \frac{N-4}{2} \cot x \right) \left( \frac{d}{dx} + \frac{N-2}{2} \cot x \right). \]  

Comparing each factor in \( P_N \) to \( A_N^\pm \) defined by Eq. (4.5) with \( W_N = -N(\cot x)/2 \), one has for odd \( N = 2M + 1 \) \((M = 1, 2, 3, \ldots)\)

\begin{align*}
P_{2M+1}^- &= (-1)^M A_{2M+1}^- A_{2M-1}^- \cdots A_1^- A_1^+ A_2^+ \cdots A_{2M-3}^+ A_{2M-1}^+; \\
P_{2M+1}^+ &= (-1)^M A_{2M+1}^- A_{2M-1}^- \cdots A_1^- A_1^+ A_2^+ \cdots A_{2M-3}^+ A_{2M-1}^+; 
\end{align*}

and for even \( N = 2M \) \((M = 1, 2, 3, \ldots)\)

\begin{align*}
P_{2M}^- &= (-1)^M A_{2M}^- A_{2M-2}^- \cdots A_1^- A_1^+ A_2^+ \cdots A_{2M-3}^+ A_{2M-2}^+; \\
P_{2M}^+ &= (-1)^M A_{2M}^- A_{2M-2}^- \cdots A_1^- A_1^+ A_2^+ \cdots A_{2M-3}^+ A_{2M-2}^+; 
\end{align*}

We first discuss the action of \( P_N \) on the eigenstates of \( V(N) \) for odd \( N \). As shown in the last section, for odd \( N = 2M + 1 \) the ground state \( \psi_0^{(2M+1)} \), which is the solution of \( A_{2M+1}^- \psi_0^{(2M+1)} = 0 \), belongs to the kernel \( V_{2M+1}^- \) of \( P_{2M+1}^- \); namely, \( P_{2M+1}^- \psi_0^{(2M+1)} = 0 \). Hence, the ground state \( \psi_0^{(2M+1)} \) is annihilated by both supercharges \( A_{2M+1}^- \) and \( P_{2M+1}^- \). For the excited states \( \psi_n^{(2M+1)} \) \((n > 0)\) we have

\[ P_{2M+1}^- \psi_n^{(2M+1)} \propto A_{2M+1}^- (A_{2M+1}^- A_{2M-3}^- \cdots A_3^-)(A_1^- A_1^+) (A_3^+ \cdots A_{2M-3}^+ A_{2M-1}^-) \psi_n^{(2M+1)}. \]  

Applying Eq. (4.21) repeatedly, we have

\[ (A_3^+ \cdots A_{2M-3}^+ A_{2M-1}^+) \psi_n^{(2M+1)} \propto \psi_{n+M-1}^{(3)}, \]

since each \( A^+ \) maps the \( n \)th excited state of \( V(m) \) to the \((n + 1)\)-th state of \( V(m-2) \). The action of the \( A_1^- A_1^+ \) on \( \psi_{n+M-1}^{(3)} \) is calculated using Eqs. (4.14) and (4.18) as

\[ A_1^- A_1^+ \psi_{n+M-1}^{(3)} = (A_3^- A_3^+ + R_1) \psi_{n+M-1}^{(3)} = \left( E_{n+M-1}^{(3)} + R_1 \right) \psi_{n+M-1}^{(3)}. \]

Hence \( A_1^- A_1^+ \psi_{n+M-1}^{(3)} \propto \psi_{n+M-1}^{(3)} \). Finally, we note that all the \( A^- \)'s grouped in the first bracket in the r.h.s. of Eq. (4.28) simply reverse the actions of the \( A^+ \)'s grouped in the third bracket, and from Eq. (4.20) we have

\[ (A_{2M-1}^- A_{2M-3}^- \cdots A_3^-) \psi_{n+M-1}^{(3)} \propto \psi_{n}^{(2M+1)}. \]

Taken together, Eq. (4.28) implies

\[ P_{2M+1}^- \psi_n^{(2M+1)} \propto A_{2M+1}^- \psi_n^{(2M+1)} \propto \psi_{n}^{(2M+3)}. \]

Equations (4.28) and (4.32) can be visualized, as shown in Fig. 1, as a series of arrows of \( A^+ \)'s (starting with \( A_{2M-1}^+ \)) pointing to the left, turning around with \( A_1^- A_1^+ \), and then pointing to the right with a series of \( A^- \)'s (ending with \( A_{2M-1}^- \)).

Hence, the state \( \psi_{n+M-1}^{(2M+3)} \) is obtainable from the excited state \( \psi_n^{(2M+1)} \) \((n > 0)\) either by the ordinary supercharge \( A_{2M+1}^- \) or by the \( N \)-fold supercharge \( P_{2M+1}^- \). By the same argument, it can be shown that the actions of \( A_{2M+1}^+ \) and \( P_{2M+1}^+ \) are equivalent on the state \( \psi_{n-1}^{(2M+3)} \). That all states in the pair of potential \( V(N) \) and \( V(N+2) \) are related by both sets of supercharges is a manifestation of the underlying unbroken SUSY structures inherent in the \( \cot x \) potential with odd \( N = 2M + 1 \).

For even \( N = 2M \), type A \( N \)-fold SUSY is broken. Also, the presence of \( A_0^- \) prevents the \( A_m^- \) \((m = 2, 4, \ldots, N-2)\) to reverse the actions of the corresponding \( A_m^+ \) on \( \psi_{n}^{(N)} \). Hence, the states in \( V(N) \) and \( V(N+2) \) are only related by the supercharges of unbroken ordinary SUSY, and not by those of broken type A \( N \)-fold SUSY.
Suppose the latter formula holds for a given integer \( M \) which is again factorizable as a product of \( A \). Now by repeated use of the shape invariance relation (4.14), one can check that the 5-fold supercharge \( P \), where we have made use of the property of shape invariance of the potentials, namely, Eq. (4.14). This shows that \( H \) of solid arrows of \( A \)'s (starting with \( A^- \)) pointing to the left, turning around with \( A^+ \), and then pointing to the right with a series of \( A^- \)'s (ending with \( A^+ \)). Hence \( P^{-3}_{2M+1} \psi_{n+1}^{(2M+1)} \propto A_{2M+1}^- \psi_{n+1}^{(2M+1)} \propto \psi_{n-1}^{(2M+3)} \).

3. Operator relations among \( P_N^\pm \) and \( A_N^\pm \) for odd \( N \)

The discussion in the previous subsection indicates that, for odd \( N \), the supercharge \( P_N^\pm \) are proportional to \( A_N^\pm \) when acting on the eigenstates of \( H_N^- \), respectively. In this section, we will derive the operator relations between them which show that the \( N \)-fold supercharges \( P_N^\pm \) are indeed factorizable into the product of \( A_N^\pm \) and a polynomial in \( H_N^- \) or \( H_N^+ \) for odd \( N \).

Let us begin with the simplest case. For \( N = 3 \), the supercharges are given by (4.26a):

\[
P_3^- = -A_3^- (A_1^- A_1^+) = -A_3^- (H_3^- + R_1) = -(H_3^+ + R_1) A_3^-, \tag{4.33}
\]

where we have made use of the property of shape invariance of the potentials, namely, Eq. (4.14). This shows that \( P_3^- \) is factorizable as a product of \( A_3^- \) and a polynomial in \( H_3^\pm \). The form of \( P_3^+ \) is obtained by taking transposition. Now by repeated use of the shape invariance relation (4.14), one can check that the 5-fold supercharge \( P_5^- \) is given by

\[
P_5^- = A_5^- (A_3^- A_1^- A_1^+ A_3^+) = A_5^- (H_5^- + R_3) (H_5^+ + R_3 + R_1)
= (H_5^+ + R_3) (H_5^- + R_3 + R_1) A_5^-, \tag{4.34}
\]

which is again factorizable as a product of \( A_5^- \) and a polynomial in \( H_5^\pm \).

We shall now prove the following formula by induction:

\[
P_{2M+1}^- = (-1)^M A_{2M+1}^+ \prod_{n=0}^{M-1} \left( H_{2M+1}^- + \sum_{k=n}^{M-1} R_{2k+1} \right)
= (-1)^M \prod_{n=0}^{M-1} \left( H_{2M+1}^+ + \sum_{k=n}^{M-1} R_{2k+1} \right) \cdot A_{2M+1}^-. \tag{4.35}
\]

Suppose the latter formula holds for a given integer \( M \). From Eq. (4.26a) and the assumption, we have

\[
P_{2M+3}^- = -A_{2M+3}^- \cdot P_{2M+1}^- A_{2M+1}^+
= (-1)^{M+1} A_{2M+3}^- A_{2M+1}^- \cdot \prod_{n=0}^{M-1} \left( H_{2M+1}^- + \sum_{k=n}^{M-1} R_{2k+1} \right) \cdot A_{2M+1}^+. \tag{4.36}
\]
Using Eqs. (4.4), (4.6), and (4.14), we can derive

\[
P_{2M+3}^- = (-1)^{M+1} A_{2M+3}^+ A_{2M+1}^+ A_{2M+1}^- \prod_{n=0}^{M-1} \left( H_{2M+1}^+ + \sum_{k=n}^{M-1} R_{2k+1} \right)
\]

\[
= (-1)^{M+1} A_{2M+3}^- (H_{2M+3}^- + R_{2M+1}) \prod_{n=0}^{M-1} \left( H_{2M+3}^- + \sum_{k=n}^{M} R_{2k+1} \right)
\]

\[
= (-1)^{M+1} A_{2M+3}^- \prod_{n=0}^{M} \left( H_{2M+3}^- + \sum_{k=n}^{M} R_{2k+1} \right).
\]

(4.37)

The last expression is nothing but the formula (4.35) with \( M \) replaced by \( M + 1 \). Since we have already shown that Eq. (4.35) holds for \( N = 3 \), Eq. (4.33), we complete the proof of the formula (4.35) for arbitrary odd integer \( N = 2M + 1 (\geq 3) \). Equation (4.35) naturally explains the equivalent actions of \( P_{2M+1}^- \) and \( A_{2M+1}^- \) on the eigenstates, i.e., Eq. (4.32).

4. Type A \((N, 1)\)-fold Superalgebra

It was proved [13, 28] that the anti-commutator of \( N \)-fold supercharges is an \( N \)th-degree polynomial in the super-Hamiltonian \( H_N \), the polynomial being (proportional to) the characteristic polynomial of the component Hamiltonians \( H_N^\pm \) restricted in the invariant subspaces \( V_N^\pm \):

\[
\{ Q_N^-, Q_N^+ \} \propto \det \left( H_N^\pm | V_N^\pm - H_N \right).
\]

(4.38)

However, the direct calculation of the characteristic polynomial becomes a much harder task as \( N \) increases. For type A and C \( N \)-fold supersymmetry, it was shown [20, 22] that the characteristic polynomials are given by the critical generalized Bender–Dunne polynomials [29] and are systematically calculated through a recursion relation. Hence, in the type A \((N, 1)\)-fold supersymmetric case, we can also calculate them systematically using a recursion relation in the same way. We present some explicit forms of the critical generalized Bender–Dunne polynomials in Appendix B. Interestingly, for our present shape-invariant potential, four significant properties, namely, \( N \)-fold SUSY (2.7), ordinary SUSY (4.6), shape invariance (4.14), and the relation between ordinary and \( N \)-fold supercharges (4.26)–(4.27) enable us to calculate directly not only the anti-commutator of the \( N \)-fold supercharges but also all the anti-commutators among the ordinary and \( N \)-fold supercharges. This will be discussed in what follows.

Let us first begin with the calculation of the anti-commutator of the \( N \)-fold supercharges:

\[
\{ Q_N^-, Q_N^+ \} = \begin{pmatrix} P_N^- P_N^+ & 0 \\ 0 & P_N^+ P_N^- \end{pmatrix},
\]

(4.39)

for odd \( N = 2M + 1 \). The calculation of the each component in this case is facilitated by the use of Eq. (4.35). In fact, from Eq. (4.35), its transposition and Eq. (4.4) we have

\[
P_{2M+1}^+ P_{2M+1}^- = \prod_{m=0}^{M-1} \left( H_{2M+1}^- + \sum_{k=m}^{M-1} R_{2k+1} \right) \cdot (A_{2M+1}^+ A_{2M+1}^-) \cdot \prod_{n=0}^{M-1} \left( H_{2M+1}^- + \sum_{k=n}^{M-1} R_{2k+1} \right)
\]

\[
= H_{2M+1}^- \prod_{n=0}^{M-1} \left( H_{2M+1}^- + \sum_{k=n}^{M-1} R_{2k+1} \right)^2 \equiv P_{2M+1}(H_{2M+1}^-),
\]

(4.40)

where \( P_{2M+1} \) is a monic polynomial of degree \( 2M + 1 \). In a similar way, we can easily show that \( P_{2M+1}^- P_{2M+1}^+ = P_{2M+1}(H_{2M+1}^-) \). Therefore, the anti-commutator for odd \( N = 2M + 1 \) reads,

\[
\{ Q_{2M+1}^-, Q_{2M+1}^+ \} = P_{2M+1}(H_{2M+1}) = H_{2M+1} \cdot \prod_{n=0}^{M-1} \left( H_{2M+1} + \sum_{k=n}^{M-1} R_{2k+1} \right)^2.
\]

(4.41)

For instance, the anti-commutators for \( N = 3 \) and 5 are

\[
\{ Q_3^-, Q_3^+ \} = P_3(H_3) = H_3(H_3 + R_1)^2,
\]

(4.42)
\[ \{Q_5^-, Q_5^+\} = \mathcal{P}_5(H_5) = H_5(H_5 + R_3)^2(H_5 + R_3 + R_1)^2. \] (4.43)

For even \(N = 2M\), the supercharge \(P_N^- (P_N^+)\) cannot be factorized as a product of \(A_N^- (A_N^+)\) and a polynomial in \(H_N\) as in Eq. (4.35). So the previous proof cannot be carried over straightforwardly. Let us begin with the simplest case, i.e., \(N = 2\). From Eq. (4.27), we have \(P_2^- = -A_2^+ A_0^+\). Hence, \(P_2^+ P_2^+\) can be rewritten, with the use of the relation \(A_0^- A_0^+ = A_0^+ A_0^-\) and Eqs. (4.4) and (4.14), as

\[
P_2^+ P_2^- = A_0^- (A_2^+ A_2^-) A_0^+ = A_0^- (A_0^+ - R_0) A_0^+ = (A_0^- A_0^+ - R_0) A_0^+ = H_2^- (H_2^+ + R_0).
\] (4.44)

Similarly, we can calculate the component \(P_4^+ P_4^-\) as

\[
P_4^+ P_4^- = A_2^- A_0^- A_4^+ A_4^- A_2^+ A_0^+ A_2^+ = A_2^- A_0^- (A_2^+ A_2^- - R_2) A_2^+ A_2^+ A_0^+ A_2^+ = (A_2^- A_2^+ - R_2) A_2^- A_2^+ A_2^+ (A_2^- A_2^+ + R_0) = H_4^- (H_4^+ + R_2)^2(H_4^- + R_2 + R_0).
\] (4.45)

We shall now prove the following formula by induction:

\[
P_{2M}^+ P_{2M}^- = H_{2M}^- \cdot \prod_{n=1}^{M-1} \left( H_{2M}^- + \sum_{k=n}^{M-1} R_{2k} \right)^2 \left( H_{2M}^- + \sum_{k=0}^{M-1} R_{2k} \right) = \mathcal{P}_{2M}(H_{2M}^-),
\] (4.46)

where \(\mathcal{P}_{2M}\) is a monic polynomial of degree \(2M\). Suppose the latter formula holds for a given integer \(M\). Using the relations (2.7), (4.4), (4.14), and (4.27), we have

\[
P_{2M+2}^+ P_{2M+2}^- = A_{2M}^- P_{2M}^+ A_{2M+2} A_{2M+2}^+ P_{2M}^- A_{2M}^+ = A_{2M}^- P_{2M}^+ (H_{2M}^- - R_{2M}) P_{2M}^- A_{2M}^+ = A_{2M}^- P_{2M}^+ P_{2M}^- (H_{2M}^- - R_{2M}) A_{2M}^+.
\] (4.47)

From the assumption and the relations (4.4), (4.6), and (4.14), we obtain

\[
P_{2M+2}^+ P_{2M+2}^- = A_{2M}^- H_{2M}^- \cdot \prod_{n=1}^{M-1} \left( H_{2M}^- + \sum_{k=n}^{M-1} R_{2k} \right)^2 \left( H_{2M}^- + \sum_{k=0}^{M-1} R_{2k} \right) (H_{2M}^- - R_{2M}) A_{2M}^+ = \left( H_{2M}^- \right)^2 \cdot \prod_{n=1}^{M-1} \left( H_{2M}^- + \sum_{k=n}^{M-1} R_{2k} \right)^2 \left( H_{2M}^- + \sum_{k=0}^{M-1} R_{2k} \right) H_{2M}^- - R_{2M}) \] (4.48)

The last expression is nothing but the formula (4.46) with \(M\) replaced by \(M + 1\). Since we have already shown that Eq. (4.46) holds for \(N = 2\), Eq. (4.44), we complete the proof of the formula (4.46) for arbitrary even integer \(N = 2M(\geq 2)\). In a similar way, we can easily show that \(P_{2M}^+ P_{2M}^- = \mathcal{P}_{2M}(H_{2M}^+).\) Therefore, for even \(N = 2M\) we have,

\[
\{Q_{2M}^-, Q_{2M}^+\} = \mathcal{P}_{2M}(H_{2M}) = H_{2M}^+ \cdot \prod_{n=1}^{M-1} \left( H_{2M}^+ + \sum_{k=n}^{M-1} R_{2k} \right)^2 \left( H_{2M}^+ + \sum_{k=0}^{M-1} R_{2k} \right).
\] (4.49)

For instance, the anti-commutators for \(N = 2\) and 4 are

\[
\{Q_2^-, Q_2^+\} = \mathcal{P}_2(H_2) = H_2(H_2 + R_0),
\] (4.50)

\[
\{Q_4^-, Q_4^+\} = \mathcal{P}_4(H_4) = H_4^2(H_4 + R_2)^2(H_4 + R_2 + R_0).
\] (4.51)

It is interesting to note that the polynomial system \(\{\mathcal{P}_N(E)\}\) for both odd and even \(N\) satisfies the following recursion relation with \(\mathcal{P}_1(E) \equiv E\) and \(\mathcal{P}_0(E) \equiv 1:\)

\[
\mathcal{P}_{N+2}(E) = E(E + R_N)\mathcal{P}_N(E + R_N).
\] (4.52)
Next, we shall consider the anti-commutators between the ordinary and $\mathcal{N}$-fold supercharges:

$$\{A^+_N, Q^-_N\} = \begin{pmatrix} P^+_N A^+_N & 0 \\ 0 & A^+_N P^-_N \end{pmatrix}, \quad \{A^-_N, Q^+_N\} = \begin{pmatrix} A^-_N P^+_N & 0 \\ 0 & P^-_N A^-_N \end{pmatrix}. \quad (4.53)$$

For odd $\mathcal{N} = 2M + 1$, we can easily calculate each component of the anti-commutators. For instance, using Eqs. (4.4) and (4.35), we immediately have

$$A^+_{2M+1} P^-_{2M+1} = (-1)^M A^+_{2M+1} A^-_{2M+1} \prod_{n=0}^{M-1} \left( H^-_{2M+1} + \sum_{k=n}^{M-1} R_{2k+1} \right) = (-1)^M \prod_{n=0}^{M} \left( H^-_{2M+1} + \sum_{k=n}^{M-1} R_{2k+1} \right) \equiv (-1)^M \mathcal{R}_{M+1}(H^-_{2M+1}), \quad (4.54)$$

where $\mathcal{R}_{M+1}$ is a monic polynomial of degree $M + 1$ (we shall adopt the convention that a summation is zero if the upper index is smaller than the lower one, i.e., $\sum_{n=a}^{b} f_n = 0$ if $b < a$). Similarly, we obtain $A^+_{2M+1} P^+_{2M+1} = (-1)^M \mathcal{R}_{M+1}(H^+_{2M+1})$ and $P^+_P A^+_{2M+1} = (-1)^M \mathcal{R}_{M+1}(H^+_{2M+1})$. Therefore, the anti-commutators for odd $\mathcal{N} = 2M + 1$ reads,

$$\{A^+_N, Q^-_{2M+1}\} = \{A^-_N, Q^+_{2M+1}\} = (-1)^M \mathcal{R}_{M+1}(H^+_{2M+1}) = (-1)^M \prod_{n=0}^{M} \left( H^+_{2M+1} + \sum_{k=n}^{M-1} R_{2k+1} \right). \quad (4.55)$$

For even $\mathcal{N} = 2M$, each component in the anti-commutators cannot be expressed as a polynomial in $H^-_N$ or $H^+_N$. However, with the aid of Eqs. (2.7), (4.4), (4.6), and (4.49) we have

$$A^+_N P^-_N A^+_N P^-_N = A^+_N \cdot P^-_N (H^+_N) A^+_N = H^-_N \cdot P^-_N (H^+_N), \quad (4.56a)$$

$$P^+_N A^-_N A^+_N P^-_N = P^+_N H^+_N P^-_N = H^-_N \cdot P^-_N (H^+_N). \quad (4.56b)$$

In a similar way, we can show $P^-_N A^+_N A^-_N P^-_N = A^+_N P^-_N A^-_N P^-_N = H^+_N \cdot P^-_N (H^+_N)$. Therefore, combining the results, we obtain the following algebraic relation:

$$\{A^+_N, Q^-_N\} \cdot \{A^-_N, Q^+_N\} = \{A^+_N, Q^+_N\} \cdot \{A^-_N, Q^-_N\} = H^+_N \cdot P^-_N (H^+_N). \quad (4.57)$$

In particular, for even $\mathcal{N} = 2M$ we have

$$\{A^+_N, Q^-_{2M}\} \cdot \{A^+_N, Q^+_{2M}\} = \prod_{n=1}^{M} \left( H^+_{2M} + \sum_{k=n}^{M-1} R_{2k} \right)^2 \left( H^-_{2M} + \sum_{k=0}^{M-1} R_{2k} \right). \quad (4.58)$$

The existence of the last factor in the r.h.s. naturally explains why the anti-commutator of $A^+_N$ and $Q^+_N$ for even $\mathcal{N}$ cannot be expressed as a polynomial in $H^+_N$. From Eqs. (4.55) and (4.57), or directly from the expressions of $\mathcal{P}_{2M+1}$ in Eq. (4.40) and $\mathcal{R}_{M+1}$ in Eq. (4.55), we see the following relation:

$$\mathcal{R}_{M+1}(E)^2 = E \cdot P_{2M+1}(E). \quad (4.59)$$

Finally, we summarize the complete type A ($\mathcal{N}, 1$)-fold superalgebra composed of one bosonic and four fermionic operators $\{H^+_N, A^+_N, Q^+_N, Q^-_N\}$ for both odd and even $\mathcal{N}$:

**Type A ($\mathcal{N}, 1$)-fold superalgebra:**

$$[A^+_N, H^+_N] = [Q^+_N, H^+_N] = 0, \quad (4.60a)$$

$$\{A^+_N, A^+_N\} = \{A^+_N, Q^+_N\} = \{Q^+_N, Q^+_N\} = 0, \quad (4.60b)$$

$$\{A^-_N, A^+_N\} = H^+_N, \quad \{Q^-_N, Q^+_N\} = \mathcal{P}_N(H^+_N), \quad (4.60c)$$

$$\{A^+_N, Q^+_N\} \cdot \{A^+_N, Q^+_N\} = H^+_N \cdot \mathcal{P}_N(H^+_N). \quad (4.60d)$$

For odd $\mathcal{N} = 2M + 1$, the last relation can be decomposed as

$$\{A^+_N, Q^+_N\} \cdot \{A^+_N, Q^+_N\} = (-1)^M \mathcal{R}_{M+1}(H^+_{2M+1}). \quad (4.60e)$$
Using the explicit value of $R_N$ in Eq. (4.13), the monic polynomials $\mathcal{P}_N$ and $\mathcal{R}_{M+1}$ are calculated as

$$
\mathcal{P}_N(E) = \begin{cases} 
E \cdot \prod_{n=1}^{M-1} (E + M^2 - n^2)^2 \cdot (E + M^2) & \text{for } N = 2M, \\
E \cdot \prod_{n=0}^{M-1} (E + (M - n)(M + n + 1))^2 & \text{for } N = 2M + 1,
\end{cases} 
$$

(4.61)

$$
\mathcal{R}_{M+1}(E) = \prod_{n=0}^{M} (E + (M - n)(M + n + 1)).
$$

(4.62)

V. ELLIPTIC POTENTIAL

Next, we consider an example from Case V, i.e., the elliptic case. For physical systems, it is more relevant to consider this case in terms of the real Jacobi elliptic functions instead of the Weierstrass functions discussed in [25].

A case of physical interest, which corresponds to Case V, is given by the potentials

$$
V^\pm_N(x) = \frac{\mathcal{N}^2\text{cn}^2 x}{4\text{sn}^2 x \text{dn}^2 x} \pm \frac{\mathcal{N}}{2} \left( \frac{1}{\text{sn}^2 x \text{dn}^2 x} - \frac{k^2 - k'^2}{\text{dn}^2 x} - 1 \right),
$$

(5.1)

where $0 < k < 1$, and $k'^2 = 1 - k^2$. The functions in the $\mathcal{N}$-fold and ordinary supercharges are

$$
E(x) = \frac{\text{cn} x}{\text{sn} x \text{dn} x} \left( 2\text{dn}^2 x - 1 \right), \quad W_N(x) = \mathcal{N}W(x) = -\frac{\mathcal{N}\text{cn} x}{2\text{sn} x \text{dn} x}.
$$

(5.2)

The solvable sectors are given by

$$
V^\pm_N = (\text{sn} x)^{-\frac{N-1+1}{2}} (\text{dn} x)^{-\frac{N-1+1}{2}} \langle 1, \text{cn} x, \ldots, (\text{cn} x)^{\mathcal{N}-1} \rangle, \\
\psi^\pm_0 \propto \frac{\text{sn} x}{(\text{dn} x)^{\frac{\mathcal{N}}{2}}}. 
$$

(5.3a, b)

These potentials are periodic in $x$ with period $2K$, where $K$ is the complete elliptic integral of the first kind satisfying $\text{sn} K = 1$. As in the case of cosec$^2$ potentials, the particle is confined in one of these periodic intervals bounded by infinite walls. Hence, we shall take the physical domain to be $0 < x < 2K$ without loss of generality. We mention here that, in the limit $k \rightarrow 0$, the Jacobi functions $\text{sn} x$, $\text{cn} x$, and $\text{dn} x$ reduce to $\sin x$, $\cos x$, and 1, respectively. We easily see that, in this limit, the elliptic system given by Eqs. (5.1)–(5.3) exactly reduces to the trigonometric system (4.1)–(4.3) investigated in Sect. IV. The other limit, $k \rightarrow 1$, gives a hyperbolic case, which is not of physical interest as mentioned previously. In the following, we shall discuss what kinds of properties remain unchanged and what kinds of new features arise in comparison with the trigonometric case in Sect. IV.

A. Structure of ordinary SUSY

Let us begin with the structure of ordinary SUSY. The supersymmetric relations (4.4)–(4.11) are completely preserved in the elliptic case with

$$
A^\pm_N = \mp \frac{d}{dx} - \frac{\mathcal{N}\text{cn} x}{2\text{sn} x \text{dn} x}.
$$

(5.4)

The supersymmetric element $\psi_0^-$ of the solvable sectors in Eq. (5.3b) remains normalizable in the physical domain $0 < x < 2K$ we have considered, and thus corresponds to the ground state wave function $\psi_0^{(N)-}$ of $V^-_N$:

$$
\psi_0^{(N)-} = \psi_0^- \propto \left( \frac{\text{sn} x}{(\text{dn} x)^{\frac{\mathcal{N}}{2}}} \right)^{\frac{\mathcal{N}}{2}}, \quad A^-_N\psi_0^{(N)-} = 0.
$$

(5.5)

Hence, ordinary SUSY is preserved in the elliptic case, too.

Unlike the cosec$^2$-potentials discussed in Sect. IV, however, the potentials (5.1) are not shape-invariant, as they do not satisfy Eq. (4.13) with $x$-independent $R_N$. Thus, all the remarkable properties of the trigonometric potentials discussed in Sect. IV B are lost. In particular, the Hamiltonians with the elliptic potentials (5.1) are only quasi-solvable but are not (exactly) solvable, which is also guaranteed by the fact $a_4 \neq 0$ (cf. Refs. [9, 10]).
B. Structure of $\mathcal{N}$-fold SUSY

Next, we shall examine the aspect of $\mathcal{N}$-fold SUSY. The ground state wave function (5.5) can be arranged as

$$\psi_0^{(\mathcal{N})} - \propto (\text{sn} \, x)^{-\frac{\pi}{2} + 1}(\text{dn} \, x)^{-\frac{\pi}{2}}(\text{sn} \, x)^{\mathcal{N}-1}. \quad (5.6)$$

Thus, as in the trigonometric case, we easily see that for odd $\mathcal{N}$ the ground state wave function $\psi_0^{(\mathcal{N})}$ is an element of $V_\mathcal{N}^-$ given in Eq. (5.3a), but not for even $\mathcal{N}$. For the excited states, we cannot proceed the investigation as in the trigonometric case due to the lack of shape invariance. However, we can check by using the method of Bethe ansatz equations [6, 7, 12] that no elements of $V_\mathcal{N}^\pm$ can be normalizable physical eigenstates except for the ground state $\psi_0^{(\mathcal{N})}$− for odd $\mathcal{N}$.

This means $\mathcal{N}$-fold SUSY is preserved barely by the ground state of $V_\mathcal{N}^-$ for odd $\mathcal{N}$ while it is completely broken for even $\mathcal{N}$. Therefore, the dynamical aspect of both ordinary and $\mathcal{N}$-fold SUSY breaking in the trigonometric model is unchanged in the elliptic model although some characteristic features in the former, such as shape invariance and exact solvability, are lost in the latter.

Type A ($\mathcal{N},1$)-fold superalgebra for arbitrary $\mathcal{N}$ cannot be obtained explicitly in closed form in the elliptic case. However, the anti-commutator of the $\mathcal{N}$-fold supercharges in this case is still given by the critical GBDP of degree $\mathcal{N}$ in $H_\mathcal{N}$. In Appendix B, we show the recursion relation which generates the GBDPs and the first several critical GBDPs $\pi_\mathcal{N}^{[\mathcal{N}]}$ for the elliptic model in Case V'a. For the anti-commutators between the ordinary and $\mathcal{N}$-fold supercharges, we can derive

$$\{A_\mathcal{N}, Q_\mathcal{N}^{\pm}\} \cdot \{A_\mathcal{N}, Q_\mathcal{N}^{\pm}\} = H_\mathcal{N} \cdot \pi_\mathcal{N}^{[\mathcal{N}]}(H_\mathcal{N}), \quad (5.7)$$

using the ordinary and $\mathcal{N}$-fold SUSY relations (2.7), (4.4), (4.6), and (B5). Hence, type A ($\mathcal{N},1$)-fold superalgebra for the elliptic case is simply given by Eq. (4.60) with $P_\mathcal{N}$ generalized to $\pi_\mathcal{N}^{[\mathcal{N}]}$ (note that $\pi_\mathcal{N}^{[\mathcal{N}]} \to P_\mathcal{N}$ as $k \to 0$). Unfortunately, we cannot obtain a closed formula for the anti-commutators between $A_\mathcal{N}^{\pm}$ and $Q_\mathcal{N}^{\pm}$ in terms of $H_\mathcal{N}$ for both odd and even $\mathcal{N}$. However, direct calculations for smaller $\mathcal{N}$ indicate that the $\mathcal{N}$-fold supercharges are factorizable for odd $\mathcal{N}$ in the elliptic case, too. Suppose, for instance, $P_3^{\pm}$ is factorizable as

$$P_3^{\pm} = -A_3^{\pm}(H_3^{\pm} + R_c) = -A_3^{\pm}(A_3^{\pm} A_3^{\pm} + R_c), \quad (5.8)$$

where $R_c$ is a constant. Substituting Eqs. (2.20) and (4.5) with $W_\mathcal{N} = NW$ in the latter equation, we find that $E$ and $W$ must satisfy

$$2E' - E^2 + 12W^2 + R_c = 0, \quad (5.9)$$

$$E'' - 2W'' + (2W - E)E' + 12WW' - WE^2 + 28W^3 + 3R_c W = 0. \quad (5.10)$$

The first condition (5.9) is identical with Eq. (3.7) for $\mathcal{N} = 3$ with $C = 0$ and $R = R_c/3$, and thus automatically satisfied. Eliminating the constant $R_c$ in Eqs. (5.9) and (5.10), we obtain

$$E'' - 2W'' - (4W + E)E' + 12WW' + 2WE^2 - 8W^3 = 0. \quad (5.11)$$

In our elliptic case, Case V'a, $E$ and $W$ are given by Eq. (5.2). Using these formulas, we can easily check that the factorization condition (5.11) is indeed satisfied. Another evidence of the factorization comes from the form of the critical GBDPs for odd $\mathcal{N}$. From Eq. (B9), we easily see that all the critical GBDPs of odd degree (up to $\mathcal{N} = 9$) have the following form:

$$\pi_{2M+1}^{[2M+1]}(E) = E \tilde{R}_M(E)^2, \quad (5.12)$$

where $\tilde{R}_M$ is a monic polynomial of degree $M$. The latter form is in fact a necessary condition for the factorization of a $(2M+1)$-fold supercharge as $P_{2M+1}^{-} = (-1)^M A_{2M+1}^{\pm} \tilde{R}_M(H_{2M+1}^{\pm})$; if the factorization is the case, we can derive in a similar way of the derivation of Eq. (4.41)

$$\{Q_{2M+1}^{\pm}, Q_{2M+1}^{\pm}\} = H_{2M+1} \cdot \tilde{R}_M(H_{2M+1})^2, \quad (5.13)$$

and thus Eq. (5.12) follows. Therefore, we conjecture that a decomposed relation for odd $\mathcal{N}$ similar to Eq. (4.55) would hold in the elliptic case, too (see also the discussion in Section VII).
VI. PAULI AND DIRAC EQUATIONS

It is well known that, under certain field configurations, the Pauli and the Dirac equation possess ordinary SUSY structures [2, 5–7]. And this fact has been well made use of in the studies of exact and quasi-exact solvability of these equations. The results in the last two sections now allow us to construct Pauli and Dirac systems with simultaneous ordinary and \( \mathcal{N} \)-fold SUSYs.

A. Pauli equation

The two-dimensional Pauli equation describes the nonrelativistic motion of a charged spin \( \frac{1}{2} \) particle in an external magnetic field in a plane. The Hamiltonian is given by

\[
H = (p_x + A_x)^2 + (p_y + A_y)^2 + \frac{g}{2} (\nabla \times A)_z \sigma_z,
\]

(6.1)

where \( p_x \) and \( p_y \) are the momentum operators, \( g = 2 \) is the gyromagnetic ratio, \( A \) is the vector potential of the electromagnetic field, and \( \sigma_z \) is the Pauli matrix. For uniform magnetic field, \( B_x = B_y = 0 \) and \( B_z = B \), the system is exactly solvable, giving the well-known Landau levels. On the other hand, the Aharonov–Casher theorem guarantees that for any general magnetic field \( B_z = B(x, y) \) perpendicular to the \( xy \) plane, the ground state is exactly calculable, owing to the existence of supersymmetry in Eq. (6.1) [2, 4].

Consider a magnetic field in the asymmetric gauge given by the vector potential

\[
A_x(x, y) = 0, \quad A_y(x, y) = -\bar{W}(x),
\]

(6.2)

where \( \bar{W}(x) \) is an arbitrary function of \( x \). The magnetic field \( B \) has components \( B_z = B_y = 0 \) and \( B_z = -\bar{W}'(x) \). The Pauli Hamiltonian is then given by

\[
H = p_x^2 + (p_y - \bar{W}(x))^2 - \bar{W}'(x) \sigma_z.
\]

(6.3)

Due to the translational invariance along \( y \) direction of the latter system, \([p_y, H] = 0\), eigenfunctions \( \psi \) of \( H \) can be factorized as

\[
\psi(x, y) = e^{i k_y y} \tilde{\psi}(x).
\]

(6.4)

Here \( k_y \) (\( -\infty < k_y < \infty \)) is an eigenvalue of \( p_y \), and \( \tilde{\psi}(x) \) is a two-component function of \( x \). The upper and lower components of \( \tilde{\psi} \) are then governed by the Hamiltonians \( H_- \) and \( H_+ \) respectively, where

\[
H = \begin{pmatrix}
H_- & 0 \\
0 & H_+
\end{pmatrix}, \quad H_\mp = -\frac{d^2}{dx^2} + (\bar{W}(x) - k_y)^2 \mp \bar{W}'(x).
\]

(6.5)

In this form the SUSY structure of the Pauli equation is manifest, with \( W(x) = \bar{W}(x) - k_y \) playing the rôles of the first derivative of a superpotential. Once the upper component of \( \tilde{\psi} \) is analytically solved for a nonzero energy, the lower component can be obtained by applying an appropriate supercharge on the upper component and vice versa [2, 3].

The SUSY structure of the Pauli equation can be made use of in several ways. First, from the knowledge of shape-invariant SUSY potentials, it was found that there are four allowed forms of shape-invariant \( W(x) \) for which the spectrum of the Pauli equation can be algebraically written down [2]. One of the four forms gives rise to a uniform magnetic field. Second, from the close connection of ordinary SUSY with quasi-exact solvability, several field configurations were constructed giving rise to quasi-exactly solvable Pauli Hamiltonians which have underlying \( sl(2) \) Lie-algebraic structures [6].

Now, we could further construct field configurations so that the Pauli equation describes a system possessing type \( A (\mathcal{N}, 1) \)-fold SUSY. One simply takes \( W(x) \) to be \( W_N(x) = -\mathcal{N} \text{cn} x / (2 \text{sn} x \text{dn} x) \) \( (N \geq 2) \). The magnetic field is given by

\[
B_z(x) = -\bar{W}'(x) = \frac{N}{2} \left( \frac{1}{\text{sn}^2 x \text{dn}^2 x} + \frac{k^2 - k_y^2}{\text{dn}^2 x} - 1 \right).
\]

(6.6)

Hence, the system (6.5) describes a quantum particle moving along the \( y \) direction with the momentum \( k_y \) and confined in one of the potential wells in the strips \( x \in (2nK, 2(n + 1)K) \) \( (n \in \mathbb{Z}) \) formed by the magnetic field (6.6).
The discussions in Section V tell us that the system is quasi-solvable and only the ground state with zero energy eigenvalue can be obtained in closed form, Eq. (5.5). For the Pauli Hamiltonian, the two-component ground state wave function \( \bar{\psi}_0 \) in this case is apparently given by

\[
\bar{\psi}_0(x) = \begin{pmatrix} \psi_{0\downarrow}^{(N)} \\ 0 \end{pmatrix} \propto \begin{pmatrix} (\frac{\sin x}{\cos x})^{N} \\ 0 \end{pmatrix}.
\]  

(6.7)

All the other two-component functions \( \bar{\psi}(x) = (\phi^-(x), \phi^+(x))^t \) with \( \phi^\mp(x) \in \mathcal{V}^\mp_N \), which are preserved by the Pauli Hamiltonian (6.5), are not normalizable and thus unphysical.

In the limit \( k \to 0 \), the system reduces to the trigonometric model in Section IV, which is shape-invariant and exactly solvable. For the Pauli Hamiltonian (6.5) in this limit, we immediately have

\[
H \bar{\psi}_n(x) = E_n^{(N)} \bar{\psi}_n(x), \quad \bar{\psi}_n(x) = \begin{pmatrix} \psi_{n\downarrow}^{(N)}(x) \\ \psi_{n-1\uparrow}^{(N)}(x) \end{pmatrix}, \quad n = 0, 1, 2, \ldots.
\]  

(6.8)

where \( \psi_{-1\uparrow}^{(N)}(x) \equiv 0 \), the component wave functions \( \psi_{n\downarrow}^{(N)} \) and \( \psi_{n-1\uparrow}^{(N)} \) are calculated using Eqs. (4.10), (4.12), and (4.15), and the energy eigenvalue \( E_n^{(N)} \) is given by Eq. (4.16). It is interesting to note that if we regard the Pauli Hamiltonian (6.5) as a super-Hamiltonian in the matrix representation (2.1), \( \mathbf{H}_N = H \), the system satisfies type A \((N,1)\)-fold superlagebra (4.60) with (note that the upper and lower components of the Hamiltonians are reversed)

\[
\mathbf{A}_N = \begin{pmatrix} 0 & 0 \\ A_N & 0 \end{pmatrix}, \quad \mathbf{A}_N^+ = \begin{pmatrix} 0 & A_N^+ \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Q}_N^- = \begin{pmatrix} 0 & 0 \\ P_N^- & 0 \end{pmatrix}, \quad \mathbf{Q}_N^+ = \begin{pmatrix} 0 & P_N^- \\ 0 & 0 \end{pmatrix}.
\]  

(6.9)

The other cases in Appendix A can be treated in the same manner, which results in different configurations of magnetic field.

### B. Dirac equations

Dirac equation which couples minimally to a stationary vector potential can be treated as in the Pauli equation described in the last section, since the square of the Dirac Hamiltonian in this form is proportional to the Pauli equation up to an additive constant. We shall therefore not repeat the discussions. Instead, we will give an example of Dirac system coupled non-minimally to an electric field, which is described by the Dirac–Pauli equation [7].

Consider the motion of a neutral fermion of spin \( \frac{1}{2} \) with mass \( m \) and an anomalous magnetic moment \( \mu \), in an external static electric field \( E \). The fermion is described by a time-independent four-component spinor \( \psi \) which obeys the Dirac–Pauli equation

\[
H \psi = \mathcal{E} \psi, \quad H = \mathbf{p} + i \mu \mathbf{\alpha} \cdot \mathbf{E} + \beta m,
\]  

(6.10)

where \( \mathbf{p} = -i \nabla \) and the Dirac matrices are given, in the standard representation, by

\[
\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]  

(6.11)

where \( \sigma \) are the Pauli matrices. We introduce two-component spinors \( \chi \) and \( \varphi \) by \( \psi = (\chi, \varphi)^t \). Then the Dirac–Pauli equation (6.10) becomes

\[
\sigma \cdot (p - i \mu E) \chi = (\mathcal{E} + m) \varphi,
\]  

(6.12a)

\[
\sigma \cdot (p + i \mu E) \varphi = (\mathcal{E} - m) \chi.
\]  

(6.12b)

When the electric field is radial, \( E = E(r) \hat{r} \), we have a set of commuting operators \{\( H, J^2, J_z, S^2 = 3/4, \tilde{K} \)\} where \( J \) is the total angular momentum \( \mathbf{J} = \mathbf{L} + \mathbf{S} \), \( \mathbf{L} \) is the orbital angular momentum, \( \mathbf{S} = \frac{1}{2} \Sigma \) is the spin operator, and \( \tilde{K} \) is defined as \( \tilde{K} = \beta (\Sigma \cdot L + 1) \). Simultaneous eigenfunctions of these commuting operators can be written as

\[
\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = \frac{1}{r} \begin{pmatrix} f_-(r) Y_{jm}^-(\theta, \phi) \\ i f_+(r) Y_{jm}^+(\theta, \phi) \end{pmatrix},
\]  

(6.13)
where $Y_{j m_j}^\kappa (\theta, \phi)$ is the spin harmonics satisfying
\begin{align}
J^2 Y_{j m_j}^\kappa &= j(j+1)Y_{j m_j}^\kappa, \quad j = \frac{1}{2}, \frac{3}{2}, \ldots, \tag{6.14} \\
J_y Y_{j m_j}^\kappa &= m_j Y_{j m_j}^\kappa, \quad |m_j| \leq j, \tag{6.15} \\
K Y_{j m_j}^\kappa &= -\kappa Y_{j m_j}^\kappa, \quad \kappa = \pm \left(j + \frac{1}{2}\right), \tag{6.16} \\
(\sigma \cdot \hat{r}) Y_{j m_j}^\kappa &= -\gamma^0 Y_{j m_j}^\kappa. \tag{6.17}
\end{align}

With these relations, Eq. (6.12) reduces to
\begin{align}
\left( \frac{d}{dr} + \frac{\kappa}{r} + \mu E(r) \right) f_-(r) &= (\mathcal{E} + m) f_-(r), \tag{6.18a} \\
\left( -\frac{d}{dr} + \frac{\kappa}{r} + \mu E(r) \right) f_+(r) &= (\mathcal{E} - m) f_+(r). \tag{6.18b}
\end{align}

This shows that $f_-$ and $f_+$ form a one-dimensional SUSY pair, with the first derivative of a superpotential being
\begin{equation}
W(r) = \frac{\kappa}{r} + \mu E(r). \tag{6.19}
\end{equation}

Now if we choose $W \equiv W_N = -N \text{cn} r/(2 \text{sn} r \text{dn} r)$ ($N \geq 2$), the Dirac–Pauli equation (6.18) becomes a system possessing type $A$ ($N, 1$)-fold SUSY in its lower and upper components, just like the situation in the Pauli equation discussed before. The required electric field configuration is given by
\begin{equation}
\mu E(r) = -\frac{\kappa}{r} - \frac{N \text{cn} r}{2 \text{sn} r \text{dn} r}. \tag{6.20}
\end{equation}

In this case, the system (6.18) describes a Dirac particle confined in a sphere $r < 2K$ or between two spherical surfaces $2nK < r < 2(n+1)K$ ($n \in \mathbb{N}$) by the electric field (6.20). From Eq. (6.18), the components $f_-$ and $f_+$ satisfy
\begin{equation}
\mathcal{H}_N^- f_-(r) = A_N^+ A_N^- f_+(r) = (\mathcal{E}^2 - m^2) f_+(r), \tag{6.21}
\end{equation}

where $A_N^\pm$ are defined as Eq. (4.5) with $x$ replaced by $r$. Hence, the lowest eigenfunction $\psi_{0,\kappa}$ with energy eigenvalue $\mathcal{E}_{0,\kappa} = m$ for the given integer value of $\kappa$ is exactly computed as
\begin{equation}
\psi_{0,\kappa} = \frac{1}{r} \left( \psi_0^{(N)}(r) Y_{j m_j}^\kappa (\theta, \phi) \right), \tag{6.22}
\end{equation}

where $\psi_0^{(N)}$ is defined by Eq. (4.12). Due to $N$-fold supersymmetry, the system also admits local analytic solutions with
\begin{equation}
f_\mp(r) \in \mathcal{V}_N^\mp = \text{sn} \left( \frac{\mathcal{E}^2 - m^2}{2} \text{dn} \right)^{\frac{N-1}{2}} r(1, \text{cn} r, \ldots, (\text{cn} r)^{N-1}). \tag{6.23}
\end{equation}

As discussed in Section IV C 1, they are not normalizable except for the configuration of $\psi_{0,\kappa}$ in Eq. (6.22).

In the limit $k \to 0$, the Dirac–Pauli system (6.18) gets shape invariance discussed in Section IV B. Owing to it, infinite number of eigenfunctions $\psi_{n,\kappa}$ and corresponding eigenvalues $\mathcal{E}_{n,\kappa}$ ($n = 0, 1, 2, \ldots$) for the given integer value of $\kappa$ can be computed exactly as
\begin{equation}
\psi_{n,\kappa} = \frac{1}{r} \left( \psi_0^{(N)}(r) Y_{j m_j}^\kappa (\theta, \phi) \right), \quad \mathcal{E}_{n,\kappa} = \sqrt{m^2 + E_n^{(N)}}. \tag{6.24}
\end{equation}

where $\psi_{-1}^{(N)}(r) \equiv 0$, $\psi_{n-1}^{(N)}$ and $\psi_{n-1}^{(N)}$ are obtained from Eqs. (4.10), (4.12), and (4.15), and $E_n^{(N)}$ is given by Eq. (4.16).

The other cases in Appendix A can be treated in the same manner, which results in different configurations of electric field.
VII. DISCUSSION AND SUMMARY

In this paper, we have discussed the interesting issue of simultaneous ordinary and type \( \mathcal{N} \)-fold SUSYs, which we have called type \( \mathcal{A} (\mathcal{N}, 1) \)-fold SUSY. For realistic systems with real-valued potentials, there are essentially eight inequivalent type \( \mathcal{A} (\mathcal{N}, 1) \)-fold supersymmetric models: one is conformal (Case II), three of them are hyperbolic (trigonometric) including the Rosen–Morse type (Cases IVa, IVb, IV′a), and the other four cases are elliptic (Cases Va, Vb, V′a, V′′a). Of these models, the trigonometric Rosen–Morse type and some of the elliptic models turn out to be of physical interest.

We have fully investigated the trigonometric Rosen–Morse type, namely, cosec\(^2\) potentials and found that when they have type \( \mathcal{A} (\mathcal{N}, 1) \)-fold SUSY, they constitute two chains of shape-invariant potentials, one is the even \( \mathcal{N} \) chain and the other is the odd \( \mathcal{N} \) chain. For both of the chains, ordinary SUSY is always unbroken. On the other hand, \( \mathcal{N} \)-fold SUSY is completely broken for the even \( \mathcal{N} \) chain while it is unbroken barely by the ground state for the odd \( \mathcal{N} \) chain. We have further shown that type \( \mathcal{A} (\mathcal{N}, 1) \)-fold SUSY together with shape invariance enable us to obtain the complete \( \mathcal{A} (\mathcal{N}, 1) \)-fold superalgebra, which is composed of one bosonic and four fermionic operators \( \{ H_{\mathcal{N}}, A^\pm_{\mathcal{N}}, Q^\pm_{\mathcal{N}} \} \), in closed form for arbitrary \( \mathcal{N} \).

For the elliptic models, we have examined the one which reduces to the cosec\(^2\) potential in the limit \( k \to 0 \). We have found that the dynamical aspects of both ordinary and \( \mathcal{N} \)-fold SUSYs are completely the same as the trigonometric case although shape invariance and exact solvability are lost in the elliptic case. We have also shown that the structure of type \( \mathcal{A} (\mathcal{N}, 1) \)-fold superalgebra is essentially preserved and that it is obtained through a recursion relation though we cannot obtain a closed formula for arbitrary \( \mathcal{N} \).

As physical applications, we have presented how type \( \mathcal{A} (\mathcal{N}, 1) \)-fold SUSY can be embedded in the Pauli and Dirac systems, namely, a nonrelativistic spin \( \frac{1}{2} \) charged particle in an external static magnetic field in 2 space dimensions and a Dirac particle coupled non-minimally to an external static electric field in 3 space dimensions. Some consequences of the symmetries have been also discussed briefly.

Relation to some mathematical theorems in Ref. [28] also gets clearer. Applying the theorem on (in)dependence of supercharges to \( \mathcal{A} (\mathcal{N}, 1) \)-fold SUSY, for instance, the necessary and sufficient condition for \( \mathcal{N} \)-fold supercharges to be factorizable as a product of a polynomial in Hamiltonians \( H^\pm_{\mathcal{N}} \) and ordinary supercharges \( A^\pm_{\mathcal{N}} \) is

\[
\{ A^+_{\mathcal{N}}, Q^-_{\mathcal{N}} \} = \{ A^-_{\mathcal{N}}, Q^+_{\mathcal{N}} \}. \tag{7.1}
\]

In our present cases, the latter condition holds for the shape-invariant potentials in Section IV with odd \( \mathcal{N} \) (cf. Eq. (4.55)), and the \( \mathcal{N} \)-fold supercharges are indeed factorized as Eq. (4.35) in the case. On the other hand, for the shape-invariant potentials with even \( \mathcal{N} \), we have from Eqs. (4.6), (4.27), and \( A^0_0 = -A^-_0 \)

\[
\{ A^+_{2M}, Q^-_{2M} \} = -\{ A^-_{2M}, Q^+_{2M} \}, \tag{7.2}
\]

and thus the condition (7.1) is not satisfied. It could be possible that the \( \mathcal{N} \)-fold supercharges for even \( \mathcal{N} \) are factorizable as a product of a polynomial in the Hamiltonians and a differential operator of even order \( 2k \) \((0 < k < M)\) which has no factor proportional to \( A^0_0 \). But direct calculations for smaller even \( \mathcal{N} \) indicate that it would not be the case. In other words, \( A^0_{\mathcal{N}} \) and \( P^0_{\mathcal{N}} \) for even \( \mathcal{N} \) would be an optimal set [28]. For the elliptic models in Section V, although we have not been able to (dis)prove the relation (7.1) for arbitrary odd and even \( \mathcal{N} \), direct calculations for smaller \( \mathcal{N} \) also indicate that factorizability of the \( \mathcal{N} \)-fold supercharges would hold for arbitrary odd \( \mathcal{N} \).

To the best of our knowledge, it is the first time that both ordinary and \( \mathcal{N} \)-fold SUSYs are discussed not only in Schrödinger equations but also in Pauli and Dirac equations. Hence, we are expecting that further studies in this direction would reveal much more novel properties and rich structures in various physical systems. In what follows, we would like to mention a couple of interesting future issues as examples.

Recently, the so-called \( PT \)-symmetric quantum theories [30] have attracted much attention. Applications of type \( \mathcal{A} (\mathcal{N}_1, \mathcal{N}_2) \)-fold SUSY to \( PT \)-symmetric theories would be straightforward since the framework of \( \mathcal{N} \)-fold SUSY does not rely on the concepts of (pseudo-)Hermiticity and so on. For this aim, one should simply employ another suitable classification scheme different from the real classification in this paper.

The present approaches to embed \( \mathcal{N} \)-fold SUSY into the Pauli and Dirac systems essentially rely on the intrinsic structure of ordinary SUSY. On the other hand, there exists a Dirac system which has no intrinsic SUSY structure but is quasi-(exactly) solvable [31]. One of the peculiar aspects of the system is that the underlying algebraic structure is the Lie superalgebra \( \mathfrak{osp}(2|2) \) instead of \( \mathfrak{s}(2) \) [32]. Hence, it is interesting to clarify whether or not we are able to embed \( \mathcal{N} \)-fold SUSY into such a system, too.
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APPENDIX A: CLASSIFICATION OF REAL TYPE A ($\mathcal{N}, 1$)-FOLD SUPERSYMMETRIC MODELS

In this appendix, we shall present a detailed classification of real type A ($\mathcal{N}, 1$)-fold supersymmetric models. For this purpose, we must use the classification scheme of real polynomial of fourth-degree under the real projective transformation $GL(2, \mathbb{R})$. In Table II, we show the real canonical forms of $A(z)$ (second column) and the complex projective transformations which convert the real canonical form to the corresponding complex one in Table I (third column).

| Case | Real Canonical Form | Transformations |
|------|--------------------|----------------|
| I    | $1/2$              | Identity       |
| II   | $2z$               | Identity       |
| III  | $\pm 2\nu z^2$    | Identity with $\mu = \pm \nu$ |
| IV   | $\pm 2\nu(z^2 - 1)$ | Identity with $\mu = \pm \nu$ |
| IV'  | $\pm 2\nu(z^2 + 1)$ | $\beta = \gamma = 0, \delta = i\alpha$ with $\mu = \pm \nu$ |
| V    | $\pm \nu(1 - z^2)(1 - k^2z^2)$ | $\gamma = \alpha, \delta = (4k^2 - k^2)\beta/(4 + k^2)$ |
| V'   | $\pm \nu(1 - z^2)(k^2 + k^2z^2)$ | $\gamma = \alpha, \delta = -(1 + 4k^2)\beta/(1 + 4k^2)$ |
| V''  | $\pm \nu(1 + k^2z^2)$ | $\gamma = i\alpha, \delta = i(4k^2 - k^2)\beta/(4 + k^2)$ |

TABLE II: Real canonical forms of $A(z)$ and complex projective transformations which convert them to the corresponding complex ones in Table I. The parameters $\nu, k, k' \in \mathbb{R}$ satisfy $\nu > 0, 0 < k < 1$ and $k^2 + k^2 = 1$.

We note that the choice of the canonical forms is not unique, and we select them so that the change of variable determined by Eq. (3.9) is as simple as possible; this is justified by the fact that the final form of the potentials does not depend on the choice of the canonical forms, cf. Ref. [20], Section 4. As a consequence, our canonical forms are different from, e.g., those in Ref. [26].

Furthermore, we note that by Eq. (3.9) a rescaling of the parameters $a_i, b_i, R$ in the models by an overall nonzero constant factor $\nu$ has the following effect on the change of variable:

$$z(q; \nu a_i, \nu b_i, \nu R) = z(\sqrt{\nu}q; a_i, b_i, R). \quad (A1)$$

From this equation and Eqs. (2.25), (3.8), (3.10), and (3.11), we easily obtain the scaling relations:

$$E(q; \nu a_i, \nu b_i, \nu R) = \sqrt{\nu}E(\sqrt{\nu}q; a_i, b_i, R), \quad (A2a)$$

$$W(q; \nu a_i, \nu b_i, \nu R) = \sqrt{\nu}W(\sqrt{\nu}q; a_i, b_i, R), \quad (A2b)$$

$$W(q; \nu a_i, \nu b_i, \nu R) = W(\sqrt{\nu}q; a_i, b_i, R), \quad (A2c)$$

$$V^{\pm}(q; \nu a_i, \nu b_i, \nu R) = \nu V^{\pm}(\sqrt{\nu}q; a_i, b_i, R). \quad (A2d)$$

Therefore, we can set $\nu = 1$ in the canonical forms III–V'' without loss of generality; the models corresponding to an arbitrary value of $\nu$ follow easily from Eqs. (A1) and (A2).

In all the cases, the condition (3.12) can be satisfied if and only if $C = 0$. In Case IV, the condition (3.12) requires a further separate analysis depending on whether $b_0 = 0$ or $b_1 = 0$. Similarly, in the real elliptic cases, namely, Cases V–V'', we must make a further separate analysis depending on whether $b_2 = b_0 = 0$ or $b_1 = 0$. Some of them however have no real solutions of Eq. (3.12), for which we shall omit the presentation of the models. We also neglect Cases I and III since both cases lead to a trivial model, as has been already shown in Ref. [25].

1. **Case II**: $A(z) = 2z$, $z(q) = q^2$

Parameters:

$$b_2 = b_1 = 0, \quad b_0 = 1, \quad R = 0. \quad (A3)$$
Supercharge:

\[ E(q) = \frac{1}{q}, \quad W(q) = -\frac{1}{2q}, \]  
\[ \text{(A4)} \]

Potentials:

\[ V^{\pm}_{N,1}(q) = \frac{N(N \pm 2)}{8q^2}. \]  
\[ \text{(A5)} \]

Solvable sectors:

\[ V^{(A)\pm}_{N} = q^{\frac{N-4}{2}} \langle 1, q^2, \ldots, q^{2(N-1)} \rangle, \quad \psi^\pm_0(q) \propto q^{\pm \frac{N}{2}}. \]  
\[ \text{(A6)} \]

2. Case IV: \( A(z) = 2(z^2 - 1), z(q) = \cosh 2q \)

a. Case IVa: \( b_0 = 0 \)

Parameters:

\[ b_2 = b_0 = 0, \quad b_1 = 2, \quad R = -\frac{N^2 - 1}{3}. \]  
\[ \text{(A7)} \]

Supercharge:

\[ E(q) = \frac{2 \cosh 2q}{\sinh 2q}, \quad W(q) = -\frac{\cosh 2q}{\sinh 2q}. \]  
\[ \text{(A8)} \]

Potentials:

\[ V^{\pm}_{N,1} = \frac{N(N \pm 2)}{2 \sinh^2 2q} + \frac{N^2}{2}. \]  
\[ \text{(A9)} \]

Solvable sectors:

\[ V^{(A)\pm}_{N} = (\sinh 2q)^{-\frac{N-4}{2}} \langle 1, \cosh 2q, \ldots, (\cosh 2q)^{N-1} \rangle, \quad \psi^\pm_0(q) \propto (\sinh 2q)^{\mp \frac{N}{2}}. \]  
\[ \text{(A10a)} \]

b. Case IVb: \( b_1 = 0 \)

Parameters:

\[ b_2 = b_1 = 0, \quad b_0 = 2, \quad R = \frac{N^2 - 1}{6}. \]  
\[ \text{(A11)} \]

Supercharge:

\[ E(q) = \frac{2 \cosh 2q}{\sinh 2q}, \quad W(q) = -\frac{1}{\sinh 2q}. \]  
\[ \text{(A12)} \]

Potentials:

\[ V^{\pm}_{N,1} = \frac{N(N \mp 2)}{2 \sinh^2 2q} \pm \frac{N^2}{2 \sinh^2 q}. \]  
\[ \text{(A13)} \]

Solvable sectors:

\[ V^{(A)\pm}_{N} = (\sinh 2q)^{-\frac{N-4}{2}} (\tanh q)^{\mp \frac{1}{2}} \langle 1, \cosh 2q, \ldots, (\cosh 2q)^{N-1} \rangle, \quad \psi^\pm_0(q) \propto (\tanh q)^{\mp \frac{N}{2}}. \]  
\[ \text{(A14a)} \]
3. Case IV': \( A(z) = 2(z^2 + 1), \ z(q) = \sinh 2q \)

\[\begin{align*}
\text{a. Case IV'a: } b_0 &= 0
\end{align*}\]

Parameters:

\[\begin{align*}
b_2 &= b_0 = 0, \quad b_1 = 2, \quad R = \frac{N^2 - 1}{3}.
\end{align*}\]  

Supercharge:

\[\begin{align*}
E(q) &= \frac{2\sinh 2q}{\cosh 2q}, \\
W(q) &= -\frac{\sinh 2q}{\cosh 2q}.
\end{align*}\]  

Potentials:

\[\begin{align*}
V_{N,1}^\pm &= \frac{N(N \pm 2)}{2 \cosh^2 2q} - \frac{N^2}{2}.
\end{align*}\]  

Solvable sectors:

\[\begin{align*}
V_{N}^{(A)\pm} = (\cosh 2q)^{-\frac{N-1}{2}} (1, \sinh 2q, \ldots, (\sinh 2q)^{N-1}), \\
\psi_0^\pm(q) &\propto (\cosh 2q)^{+\frac{N}{2}}.
\end{align*}\]  

b. Case IV'b: \( b_1 = 0 \)

In this case, there are no real solutions of Eq. (3.12) for \( b_0 \).

4. Case V: \( A(z) = \frac{1}{2}(1 - z^2)(1 - k^2 z^2), \ z(q) = \text{sn} q \)

\[\begin{align*}
\text{a. Case Va: } b_2 &= b_0 = 0
\end{align*}\]

Parameters:

\[\begin{align*}
b_2 &= b_0 = 0, \quad b_1 = \frac{k^2}{2}, \quad R = \frac{N^2 - 1}{12}(1 + k^2).
\end{align*}\]  

Supercharge:

\[\begin{align*}
E(q) &= -\frac{\text{sn} q(2 \text{dn}^2 q - k^2)}{\text{cn} q \text{dn} q}, \\
W(q) &= -\frac{k^2}{2} \frac{\text{sn} q}{\text{cn} q \text{dn} q}.
\end{align*}\]  

Potentials:

\[\begin{align*}
V_{N,1}^\pm &= \frac{N^2 k^4 \text{sn}^2 q}{8 \text{cn}^2 q \text{dn}^2 q} \mp \frac{Nk^2}{4} \left( \frac{1 + k^2}{\text{dn}^2 q} + \frac{k^2}{\text{cn}^2 q \text{dn}^2 q} - 1 \right).
\end{align*}\]  

Solvable sectors:

\[\begin{align*}
V_{N}^{(A)\pm} = (\text{en} q)^{-\frac{N-1}{2}} (\text{dn} q)^{-\frac{N-1}{2}} (1, \text{sn} q, \ldots, (\text{sn} q)^{N-1}), \\
\psi_0^\pm &\propto \left( \frac{\text{dn} q}{\text{cn} q} \right)^{+\frac{N}{2}}.
\end{align*}\]
b. Case Vb: $b_1 = 0$

In this case, we have two sets of real solutions of Eq. (3.12), namely, $b_2 = \frac{k}{2}(1 \pm k)$ and $b_0 = -\frac{k}{2}(k \pm 1)$. Here we only show the result corresponding to the upper-sign solutions.

Parameters:

\[
b_2 = \frac{k(1+k)}{2}, \quad b_1 = 0, \quad b_0 = -\frac{1+k}{2}, \quad R = -\frac{N^2 - 1}{24}(1 + 6k + k^2).
\]  

Supercharge:

\[
E(q) = -\frac{\text{sn} \, q(2 \, \text{dn} \, q - k'^2)}{\text{cn} \, q \, \text{dn} \, q} \quad \text{and} \quad W(q) = -\frac{k(1+k) \, \text{sn} \, q - k - 1}{2 \, \text{cn} \, q \, \text{dn} \, q}.
\]  

Potentials:

\[
V_{N,1}^\pm = \frac{N^2 k'^4 \, \text{sn}^2 q}{8 \, \text{cn}^2 q \, \text{dn}^2 q} + \frac{N^2(1+k)^2}{8} \pm \frac{Nk'^2}{2} \left( \frac{k'^2 \, \text{sn} \, q}{2 \, \text{cn}^2 q \, \text{dn}^2 q} - \frac{1-k \, \text{sn} \, q}{2 \, \text{dn}^2 q} \right).
\]  

Solvable sectors:

\[
\psi_0^\pm(q) \propto \left( \frac{\text{cn} \, q \, \text{dn} \, q}{(1+k \, \text{sn} \, q) \, (1+\text{sn} \, q)} \right)^{\frac{N}{2}}.
\]  

5. Case V': $A(z) = \frac{1}{2}(1-z^2)(k'^2-k^2z^2)$, $z(q) = \text{cn} \, q$

a. Case Va: $b_2 = b_0 = 0$

Parameters:

\[
b_2 = b_0 = 0, \quad b_1 = -\frac{1}{2}, \quad R = \frac{N^2 - 1}{12}(k'^2 - k^2).
\]  

Supercharge:

\[
E(q) = \frac{\text{cn} \, q(2 \, \text{dn}^2 q - 1)}{\text{sn} \, q \, \text{dn} \, q}, \quad W(q) = -\frac{\text{cn} \, q}{2 \, \text{sn} \, q \, \text{dn} \, q}.
\]  

Potentials:

\[
V_{N,1}^\pm = \frac{N^2 \, \text{cn}^2 q}{8 \, \text{sn}^2 q \, \text{dn}^2 q} \pm \frac{N}{4} \left( \frac{1}{\text{sn}^2 q \, \text{dn}^2 q} - \frac{k'^2 - k^2}{\text{dn}^2 q} - 1 \right).
\]  

Solvable sectors:

\[
\psi_0^\pm(q) \propto \left( \frac{\text{sn} \, q}{\text{dn} \, q} \right)^{\frac{N}{2}}.
\]  

b. Case Vb: $b_1 = 0$

In this case, there are no real solutions of Eq. (3.12) for $b_2$ and $b_0$. 
6. Case V′′: \( A(z) = \frac{1}{2}(1 + z^2)(1 + k^2 z^2) \), \( z(q) = t_n q \)

a. Case V′′a: \( b_2 = b_0 = 0 \)

**Parameters:**

\[
\begin{align*}
b_2 &= b_0 = 0, & b_1 &= \frac{k^2}{2}, & R &= -\frac{N^2 - 1}{12}(1 + k^2).
\end{align*}
\]

**Supercharge:**

\[
\begin{align*}
E(q) &= \frac{\text{sn} q(k^2 \text{cn}^2 q + 2k^2)}{\text{cn} q \text{dn} q}, & W(q) &= -\frac{k^2 \text{sn} q \text{cn} q}{2 \text{dn} q}.
\end{align*}
\]

**Potentials:**

\[
V_{N,1}^\pm = -\frac{N^2}{8} \left( \text{dn}^2 q + \frac{k^2}{\text{dn}^2 q} - 1 - k^2 \right) \mp \frac{N}{4} \left( \text{dn}^2 q - \frac{k^2}{\text{dn}^2 q} \right).
\]

**Solvable sectors:**

\[
\begin{align*}
\psi_n^{\pm}(q) \propto (\text{dn} q)^{\pm N}.
\end{align*}
\]

In this case, there are no real solutions of Eq. (3.12) for \( b_2 \) and \( b_0 \).

**APPENDIX B: GENERALIZED BENDER–DUNNE POLYNOMIALS**

In this appendix, we summarize the generalized Bender–Dunne polynomials (GBDPs) and present some explicit forms of them for the shape-invariant and elliptic potentials investigated in Sections IV and V. They are a natural generalization of the ones first constructed for the quasi-exactly solvable sextic anharmonic oscillator by Bender and Dunne [29] to all the type A \( N \)-fold supersymmetric models. For more details, see Ref. [20], Section 6.2. In the unit system adopted in Sections IV and V, the GBDPs \( \pi_{n[N]}(E) \) are characterized by the following four-term recursion relation:

\[
\pi_{n+1}[N](E) = (E + 2A_n^{[N]})\pi_n^{[N]}(E) - 4n(n - N)B_n^{[N]}\pi_{n-1}^{[N]}(E) + 8n(n - 1)(n - N)(n - N - 1)C_n^{[N]}\pi_{n-2}^{[N]}(E),
\]

where \( A_n^{[N]} \), \( B_n^{[N]} \), and \( C_n^{[N]} \) are given by

\[
\begin{align*}
A_n^{[N]} &= \left[ n(n - N + 1) + \frac{(N - 1)(N - 2)}{6} \right] a_2 \mp \left( n - \frac{N - 1}{2} \right)b_1 + R, \\
B_n^{[N]} &= \left[ \left( n - \frac{N}{2} \right)a_1 \mp b_0 \right] \left[ \left( n - \frac{N}{2} \right)a_3 \mp b_2 \right], \\
C_n^{[N]} &= a_4 \left[ \left( n - \frac{N}{2} \right)a_1 \mp b_0 \right] \left[ \left( n - \frac{N + 2}{2} \right)a_3 \mp b_2 \right],
\end{align*}
\]

and the parameters \( a_i \) and \( b_i \) correspond to the coefficients of the polynomials \( A(z) \) in Eq. (3.9) and \( Q(z) \) in Eq. (3.11), respectively. The degree \( N \) polynomial \( \pi_{n[N]}(E) \) in each polynomial system \( \{ \pi_{n[N]}(E) \}_{n=0}^\infty \) is called \( N \)th critical GBDP. It was shown in Ref. [20] that the anti-commutator of the type A \( N \)-fold supercharges is (proportional to) the \( N \)th critical GBDP in the type A Hamiltonian:

\[
\{Q_N, Q_N^+\} = \pi_{n[N]}^{[N]}(H_N).
\]
To obtain the recursion relation (B1) for a specific case, we must first make a $GL(2, \mathbb{C})$ transformation so that the coefficient $a_0$ in $A(z)$ vanishes. For the elliptic potential in Section V, Case V'a, the canonical form is $A(z) = \frac{1}{2}(1 - z^2)(k^2 - k^2 z^2)$. Thus, we transform $A(z)$ as

$$\hat{A}(z) = \Delta^{-2}(z + 1)^4 A \left( \frac{z - 1}{z + 1} \right) = \frac{1}{2} z \left[ 2(z^2 + 2(k^2 - k^2)z + 1) \right].$$

(B6)

In our Case V'a, $Q(z) = -z/2$ from Eq. (A27) and it is transformed simultaneously as

$$\hat{Q}(z) = \Delta^{-1}(z + 1)^2 Q \left( \frac{z - 1}{z + 1} \right) = -\frac{1}{4}(z^2 - 1).$$

(B7)

Hence, the values of the parameters which should be substituted into the recursion relation (B1) are as follows:

$$a_3 = \frac{1}{2}, \quad a_2 = k^2 - k^2, \quad a_1 = \frac{1}{2}, \quad a_4 = a_0 = 0,$$

and $R = (k^2 - k^2)(\mathcal{N}^2 - 1)/12$ (cf. Eq. (A27)). With these parameter values, the recursion relation (B1) reduces to

$$\pi_{n+1}^{[\mathcal{N}]}(E) = \left[ E + \frac{k^2 - k^2}{2}(2n - \mathcal{N} + 1)^2 \right] \pi_n^{[\mathcal{N}]}(E) - n(n - \mathcal{N}) \left( n - \frac{\mathcal{N} + 1}{2} \right) \left( n - \frac{\mathcal{N} - 1}{2} \right) \pi_{n-1}^{[\mathcal{N}]}(E).$$

(B8)

Using this three-term recursion relation, we can calculate, in principle, the critical polynomial $\pi_{\mathcal{N}}^{[\mathcal{N}]}(E)$ for an arbitrary value of $\mathcal{N}$. For instance, we have

$$\pi_1^{[1]}(E) = E,$$

(B9a)

$$\pi_2^{[2]}(E) = (E - k^2)(E + k^2),$$

(B9b)

$$\pi_3^{[3]}(E) = E \left[ E + 2(k^2 - k^2) \right]^2,$$

(B9c)

$$\pi_4^{[4]}(E) = \left[ E^2 + 2(2k^2 - 3k^2)E + 9k^4 \right] \left[ E^2 + 2(3k^2 - 2k^2)E + 9k^4 \right],$$

(B9d)

$$\pi_5^{[5]}(E) = E \left[ E^2 + 10(k^2 - k^2)E + 8(3k^4 - 2k^2 k^2 + 3k^4) \right]^2,$$

(B9e)

$$\pi_6^{[6]}(E) = \left[ E^2 + 16(k^2 - 16k^2)E^2 + (64k^4 - 80k^2 k^2 + 115k^4)E - 225k^6 \right]$$

$$\times \left[ E^2 + 16(k^2 - 16k^2)E^2 + (115k^4 - 80k^2 k^2 + 64k^4)E + 225k^6 \right],$$

(B9f)

$$\pi_7^{[7]}(E) = E \left[ E^2 + 28(k^2 - k^2)E^2 + 28(9k^4 - 10k^2 k^2 + 9k^4)E \right]$$

$$+ 144(k^2 - k^2)(5k^4 + 2k^2 k^2 + 5k^4)^2,$$

(B9g)

$$\pi_8^{[8]}(E) = \left[ E^4 + 4(10k^2 - 11k^2)E^3 + 2(264k^4 - 376k^2 k^2 + 347k^4)E^2 \right.$$

$$+ 4(576k^6 - 672k^4 k^2 + 826k^2 k^4 - 1155k^6)E + 11025k^8 \left] \right.$$

$$\times \left[ E^4 + 4(11k^2 - 10k^2)E^3 + 2(347k^4 - 376k^2 k^2 + 264k^4)E^2 \right.$$

$$+ 4(1155k^6 - 826k^4 k^2 + 672k^2 k^4 - 576k^6)E + 11025k^8 \left] \right.$$

$$\times \left[ E^4 + 4(10k^2 - 11k^2)E^3 + 2(264k^4 - 376k^2 k^2 + 347k^4)E^2 \right.$$

$$+ 4(576k^6 - 672k^4 k^2 + 826k^2 k^4 - 1155k^6)E + 11025k^8 \right],$$

(B9h)

$$\pi_9^{[9]}(E) = E \left[ E^4 + 60(k^2 - k^2)E^3 + 12(109k^4 - 146k^2 k^2 + 109k^4)E^2 \right.$$

$$+ 16(k^2 - k^2)(761k^4 - 118k^2 k^2 + 761k^4)E$$

$$+ 1152(35k^6 - 20k^6 k^2 + 18k^4 k^4 - 20k^2 k^6 + 35k^6))^2.$$
where $\bar{R}^{(o)}_M$ and $\bar{R}^{(e)}_M$ are monic polynomials of degree $M$ in the first variable, though we have not been able to prove them for arbitrary $M$.

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