GLOBAL GENERALIZED SOLUTIONS TO A CHEMOTAXIS MODEL OF CAPILLARY-SPROUT GROWTH DURING TUMOR ANGIOGENESIS

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Abstract. This paper considers a chemotaxis-convection model of capillary-sprout growth during tumor angiogenesis
\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla v) - \nabla \cdot (u \nabla w), & x \in \Omega, t > 0, \\
    v_t &= \Delta v + \nabla \cdot (v \nabla w) - v + u, & x \in \Omega, t > 0, \\
    w_t &= \Delta w - w + u, & x \in \Omega, t > 0,
\end{align*}
\]
under Neumann initial-boundary conditions in a smooth bounded domain. In the two-dimensional setting, introducing a generalized solution concept according to (Winkler, 2015 [35]) and constructing an appropriate regularized system, we prove the global existence of at least one such solution with suitably regular initial data by an approximation procedure. To overcome the difficulty in taking the limit to its regularized system, we establish some technical estimates related to several energy integrals with special structures like \( \int_0^T \int_{\Omega} u^{p+1} \ln^{k}(u_t + 1) \, dx \, dt, \) \( k \in (1, 2) \).

1. Introduction. Angiogenesis establishes linkage between avascular and vascular tumor growth. During the former growth stage the tumor remains in dormant growing to a few millimetres in diameter [26]. While during the latter stage, the tumor may secrete chemical substances known as tumor angiogenesis factors stimulating neighboring endothelial cells (ECs) to migrate into the extracellular matrix. Capillary sprouts are formed and grow towards the tumor cells by migration of the ECs, and later ECs begin to proliferate at certain distance from the tip to sprouts to aid with their migration [24]. Finally neighboring sprouts tend to incline towards one another to form loops or anastomoses at their tips, and new sprouts will emerge from the primary loops until the tumor is penetrated.

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In 1996, Orme and Chaplain developed a three species model including the density of endothelial cells $u = u(x, t)$, the density of adhesive sites $v = v(x, t)$ and the density of matrix $w = w(x, t)$ [21]:

$$\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \zeta \nabla \cdot (u \nabla w), \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v + \xi \nabla \cdot (v \nabla w) - \mu v + ku, \quad x \in \Omega, t > 0, \\
    w_t &= \Delta w - \delta w + ru, \quad x \in \Omega, t > 0,
\end{align*}$$

(1)

where $\chi, \zeta, \xi$ are positive constants, $k, \mu$ mean the rate of secretion and decay of adhesive sites, and $r, \delta$ represent the rate of secretion and decay of matrix. Based on the two facts that the matrix spreads out with ECs and adhesive sites on it and the ECs secrete matrix and adhesive sites [23], this model supposes the linear growth and decay of the matrix and adhesive sites describing growing and developing of the branching of capillary sprouts during angiogenesis. Meanwhile, Chaplain and Byrne proposed the similarities between tumor-induced angiogenesis and wound healing angiogenesis from a mathematical point of view further illustrating that the mathematical modeling of capillary-sprout growth during tumor angiogenesis provided new perspectives in understanding this highly complex and crucial step in the skin wound healing process [3].

When $r = 0$ and $w \equiv 0$, this is the classical Keller-Segel model [15] (see also [6, 9]). It is known that when $n = 1$, solutions are global in time and bounded [22]. When $n \geq 2$, there exist unbounded solutions within finite time [8, 10, 33]. When $k = 0$ and $v \equiv 0$, this is the classical repulsive chemotaxis model. A Lyapunov functional different from that of the attractive Keller-Segel model was constructed in [5], which detected that classical solutions are global in time when $n = 2$, and weak solutions globally exist when $n = 3, 4$. The convergence to steady states is proved in all these cases.

In a two-chemical substances chemotaxis system with PDE-mediated delay of signal production ($\zeta = \xi = 0$ in (1)), it is proved that if $n \leq 3$, solutions are globally bounded for reasonable initial data; if $n = 4$ and $\int_{\Omega} u_0 < \frac{(8\pi)^{2n}}{n}$, the radially symmetric solutions under homogeneous Neumann boundary conditions are globally defined and remain bounded [7, 1]. An ODE-type indirect signal production mechanism has also been studied in [28, 16, 11, 30]. A peculiar feature of this mechanism whenever $n = 2$ has uncovered in [28, 16] that while all solutions exist globally, a threshold phenomenon occurs in infinite time. However, an appropriate Lyapunov functional can not be expected in the three-component system of attraction-repulsion Keller-Segel model ($\xi = 0$ in (1)), which makes it more difficult to prove the global existence and boundedness of solutions (see e.g. [20] for a broader overview). In the one dimension, the stationary solutions and large time behavior of solutions were discovered in [14, 18]. Moser’s iteration, entropy inequality and other techniques were used in [27] to derive various behaviors of solutions of the system for various ranges of parameters in the multi-dimensional settings. Furthermore, if repulsion prevails meaning $\zeta r > \chi k$, this part of result in [27] was extended in [12] to confirm that the classical solution is globally defined if $n = 2$ and the weak solutions exist globally when $n = 3$. Further surveys in this direction we refer to [13, 19].

There are few results available to provide qualitative analysis on (1), and many problems still remain open. Recently, an analytical attempt has been made in [17] to assert the global solvability to (1) in the one dimension with zero-flux boundary conditions. We further consider the global solvability of the following initial-boundary
value problem:

\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u \nabla v) + \nabla \cdot (u \nabla w), & x \in \Omega, t > 0, \\
  v_t &= \Delta v + \nabla \cdot (v \nabla w) - v + u, & x \in \Omega, t > 0, \\
  0 &= \Delta w - w + u, & x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega,
\end{align*}
\]

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and where the initial data are a triple of functions $(u_0, v_0)$ such that

\[
\begin{align*}
  u_0 \in W^{1,\infty}(\Omega), & \quad u_0 \geq 0, u_0 \neq 0, \\
  v_0 \in W^{1,\infty}(\Omega), & \quad v_0 \geq 0, \quad \text{in } \bar{\Omega}.
\end{align*}
\]

Motivated by [35, 36], our result is enclosed in the following theorem.

**Theorem 1.1.** Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and the initial data comply with (3). Then the system (2) admits at least one global generalized solution $(u, v, w)$ in the sense of Definition 2.1.

In this paper, we will give a corresponding approximate system and introduce a global generalized solution concept of (2) in Section 2. Then we construct some important estimates of regularized system in Section 3. In Section 4 we proves the limit object $(v, w)$ solves (2) in the weak sense and $u$ is hereafter checked to comply with the generalized solution concept.

2. **Definition of generalized solutions.** Enlightened by [35], we will first introduce a generalized solution concept of great importance in this paper.

**Definition 2.1.** Let $p \in [1, \infty)$, $r \in (1, 2)$. A triple of nonnegative functions

\[
\begin{align*}
  u &\in L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \\
  v &\in L^p_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \cap L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)) \\
  w &\in L^r_{\text{loc}}([0, \infty); W^{1,r}(\Omega))
\end{align*}
\]

such that $\ln(u + 1) \in L^p_{\text{loc}}(\bar{\Omega} \times [0, \infty))$, $\nabla \ln(u + 1) \in L^2_{\text{loc}}(\bar{\Omega} \times [0, \infty))$ will be called a global generalized solution of (2) if for all $\varphi \in C^\infty_0(\bar{\Omega} \times [0, \infty))$ the following hold:

\[
\begin{align*}
  -\int_0^\infty \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) &= -\int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^\infty \int_\Omega \nabla w \cdot \nabla \varphi \\
  &\quad + \int_0^\infty \int_\Omega u \varphi - \int_0^\infty \int_\Omega \nabla \ln(u + 1) \cdot \nabla \varphi, & (4) \\
  \int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi + \int_0^\infty \int_\Omega w \varphi &= \int_0^\infty \int_\Omega u \varphi, & (5)
\end{align*}
\]

and if for each nonnegative $\varphi \in C^\infty_0(\bar{\Omega} \times [0, \infty))$ the inequality

\[
\begin{align*}
  -\int_0^\infty \int_\Omega \ln(u + 1) \varphi_t - \int_\Omega \ln(u_0 + 1) \varphi(\cdot, 0) \\
  \geq \int_0^\infty \int_\Omega (|\nabla \ln(u + 1)|^2 \varphi - \int_0^\infty \int_\Omega \nabla \ln(u + 1) \cdot \nabla \varphi \\
  -\int_0^\infty \int_\Omega \frac{u}{u + 1} (\nabla \ln(u + 1) \cdot \nabla v) \varphi + \int_0^\infty \int_\Omega \frac{u}{u + 1} \nabla v \cdot \nabla \varphi
\end{align*}
\]
\[-\int_0^\infty \int_\Omega \ln(u+1) \nabla w \cdot \nabla \varphi - \int_0^\infty \int_\Omega w \ln(u+1) \varphi + \int_0^\infty \int_\Omega u \ln(u+1) \varphi + \int_0^\infty \int_\Omega uv \varphi - \int_0^\infty \int_\Omega u^2 \varphi \]  

(6)

is true, as well as

\[ \int_\Omega u(t, \cdot) = \int_\Omega u_0 \quad \text{for a.e. } t > 0. \]  

(7)

By reason of Lemma 2.1 [35], we can deduce from Definition 2.1 that if \((u, v, w)\) is a global generalized solution of (2) satisfying \(C^0 \left( \bar{\Omega} \times [0, \infty) \right) \cap C^{2,1} \left( \bar{\Omega} \times (0, \infty) \right)\), \((u, v, w)\) is also a classical solution of (2) in \(\Omega \times (0, \infty)\). Furthermore, an appropriate regularized system corresponding to (2) will be as follows:

\[
\begin{aligned}
    u_{\varepsilon t} &= \Delta u_{\varepsilon} - \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) + \nabla \cdot (u_{\varepsilon} \nabla w_{\varepsilon}), \quad x \in \Omega, t > 0, \\
    v_{\varepsilon t} &= \Delta v_{\varepsilon} + \nabla \cdot (w_{\varepsilon} \nabla w_{\varepsilon}) - v_{\varepsilon} + \frac{w_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}}, \quad x \in \Omega, t > 0, \\
    0 &= \Delta w_{\varepsilon} - w_{\varepsilon} + \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}}, \quad x \in \Omega, t > 0, \\
    \frac{\partial u_{\varepsilon}}{\partial \nu} &= \frac{\partial v_{\varepsilon}}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
    u_{\varepsilon}(x, 0) &= u_0(x), \quad v_{\varepsilon}(x, 0) = v_0(x), \quad x \in \Omega, \end{aligned}
\]

(8)

for \(\varepsilon \in (0, 1)\). We then confirm the global existence and boundedness of solutions of (8) in the classical sense, thus global generalized solutions of (2) can be obtained by an approximation procedure. The local existence and uniqueness result can also be consulted in [2, 29].

**Lemma 2.2.** Assume that \(\Omega \subset \mathbb{R}^2\) is a bounded domain with smooth boundary and \((u_0, v_0)\) fulfills (3), then for each \(\varepsilon \in (0, 1)\) there exist nonnegative functions

\[(u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon}) \in \left( \bigcap_{q > n} C^{2,1}(\bar{\Omega} \times (0, \infty)) \right) \cap C^0 \left( [0, \infty); W^{1,q}(\Omega) \right) \]

which globally solve (2) in the classical sense.

**Proof.** We claim that there exists a unique triple of classical solutions \((u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon})\) in \(\Omega \times (0, T_{\max, \varepsilon})\) for \(T_{\max, \varepsilon} \in (0, \infty)\). Moreover, if \(T_{\max, \varepsilon} < \infty\), then

\[
\limsup_{t_{\varepsilon} \to T_{\max, \varepsilon}} \left\{ \| u_{\varepsilon}(\cdot, t) \|_{L^\infty(\Omega)} + \| v_{\varepsilon}(\cdot, t) \|_{W^{1,q}(\Omega)} + \| w_{\varepsilon}(\cdot, t) \|_{W^{1,q}(\Omega)} \right\} = \infty, \]

(9)

for any \(q > n\).

Let \(\| u_0 \|_{L^\infty(\Omega)} = M\). We consider the closed convex set

\[ S := \{ \bar{u}_{\varepsilon} \in X \mid \| \bar{u}_{\varepsilon} \|_{L^\infty(\bar{\Omega} \times [0, T])} \leq 2M \} \]

in the Banach space \(X := C(\bar{\Omega} \times [0, T])\) with \(T < 1\) to be fixed below. Let \(\alpha \in (0, 1)\), the mapping \(F : S \to C^{0, \frac{n}{1+2M\varepsilon}}(\bar{\Omega} \times [0, T])\) is defined by \(F(\bar{u}_{\varepsilon}) = u_{\varepsilon}\), where \(u_{\varepsilon}\) is the first component of the solution \((u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon})\) to the following initial-boundary value problem

\[
\begin{aligned}
    u_{\varepsilon t} &= \Delta u_{\varepsilon} - \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) + \nabla w_{\varepsilon} \cdot \nabla u_{\varepsilon} + (w_{\varepsilon} - \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}}), \\
    v_{\varepsilon t} &= \Delta v_{\varepsilon} + \nabla w_{\varepsilon} \cdot \nabla v_{\varepsilon} + (w_{\varepsilon} - \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}} - 1) v_{\varepsilon} + \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}}, \\
    0 &= \Delta w_{\varepsilon} - w_{\varepsilon} + \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}} \\
    \frac{\partial u_{\varepsilon}}{\partial \nu} &= \frac{\partial v_{\varepsilon}}{\partial \nu} = 0, \\
    u_{\varepsilon}(x, 0) &= u_0(x), \quad v_{\varepsilon}(x, 0) = v_0(x). \end{aligned}
\]

(10)

Since \(\bar{u}_{\varepsilon} \in C(\bar{\Omega} \times [0, T])\), we obtain \(w_{\varepsilon} \in C^{1,\alpha, \frac{n}{1+2M\varepsilon}}(\bar{\Omega} \times [0, T])\) and \(0 \leq w_{\varepsilon} \leq \frac{2M}{1+2M\varepsilon}\). Then

\[ \nabla w_{\varepsilon} \in C^{0, \frac{n}{1+2M\varepsilon}}(\bar{\Omega} \times [0, T]), \]
and thus
\[ v_\varepsilon \in C^{1+\alpha,\frac{\alpha}{2}}(\bar{\Omega} \times [0,T]). \]
Standard parabolic theory imply that \( u_\varepsilon \) is uniformly bounded in \( C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega} \times [0,T]) \), and \( \|u_0\|_{L^\infty(\Omega)} = M \) entails that there exists
\[ T(M) := \min \{1, \sup\{T \in (0,T_{\text{max},\varepsilon}] \mid u_\varepsilon \leq 2M \text{ on } [0,T]\} \} > 0 \]
such that \( u_\varepsilon \leq 2M \) in \( \Omega \times [0,T] \). Moreover, \( F \) is continuous because of the uniqueness of the solution, and \( C^{\alpha,\frac{\alpha}{2}}(\Omega \times [0,T]) \subset C(\bar{\Omega} \times [0,T]) \) is a compact embedding. Applying the Schauder fixed point theorem we gain the existence of at least one fixed point \( u_\varepsilon \in S \) which solves (8) classically in \( \Omega \times (0,T) \). By a standard argument, \( (u_\varepsilon, v_\varepsilon, w_\varepsilon) \) can be extended up to \( T_{\text{max},\varepsilon} \leq \infty \) and (9) holds. The positivity of the solution in \( \Omega \times (0,T_{\text{max},\varepsilon}) \) is a consequence of the maximum principle.

Next we claim that \( T_{\text{max},\varepsilon} = \infty \). Otherwise if \( T_{\text{max},\varepsilon} \) is finite, define
\[ X_p := L^\infty((0,T_{\text{max},\varepsilon});W^{1,\infty}(\Omega)) \cap L^p((0,T_{\text{max},\varepsilon});W^{2,p}(\Omega)) \]
for all \( p \in (1,\infty) \), an elliptic gradient estimate implies that \( w_\varepsilon \in X_p \). Denoting \( k_1(x,t) := \nabla w_\varepsilon, k_2(x,t) := \Delta w_\varepsilon \), we apply regularity properties to the equation \( w_{\varepsilon t} = \Delta w_\varepsilon + k_1(x,t) \nabla w_\varepsilon + k_2(x,t) v_\varepsilon - v_\varepsilon + \frac{u_\varepsilon}{1+u_\varepsilon} \) to see \( v_\varepsilon \in X_p \). Following the similar argument as above, we find \( u_\varepsilon \in X_p \) for all \( p \in (1,\infty) \), in contradiction to (9). This verifies the assertion. \( \square \)

3. Uniform estimates of regularized system. Let us begin with the following basic boundedness properties involving \( u_\varepsilon, v_\varepsilon \) and \( w_\varepsilon \).

Lemma 3.1. The solution of (8) fulfills
\[ \int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \] (11)
and
\[ \int_\Omega w_\varepsilon(\cdot,t) \leq \int_\Omega u_0 \] (12)

and
\[ \int_\Omega v_\varepsilon(\cdot,t) \leq \int_\Omega u_0 \] (13)

for all \( t > 0 \).

Proof. Notice that (11) and (12) are obvious, similarly (13) is directly from the second equation in (8) and the ODE comparison argument. \( \square \)

The next three lemmata will provide some easily obtained properties on three components of the solution, which more or less rely on (8) and the comparison principle.

Lemma 3.2. Let \( n = 2 \) and \( r \in (1,2), q \in (1,\infty) \). Then there are constants \( k_1, k_2 > 0 \) such that for each \( \varepsilon \in (0,1) \), the third component \( w_\varepsilon \) of the solution of (8) satisfies
\[ \|w_\varepsilon(\cdot,t)\|_{W^{1,r}(\Omega)} \leq k_1 \] (14)
and
\[ \|w_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq k_2 \] (15)

Proof. (14) can be consulted to Lemma 1.5 [32], while (15) is a direct consequence of (12), (14) and the Gagliardo-Nirenberg inequality in the case \( n = 2 \). \( \square \)
Lemma 3.3. Let $n = 2$, $p \in (1, \infty)$. There exists $k_3 > 0$ such that for all $\varepsilon \in (0, 1)$ the second component $v_\varepsilon$ of the solution of (8) fulfills

$$\|v_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq k_3 \quad \text{for all } t > 0.$$  \hfill (16)

Moreover, for any $T > 0$ there exists $k_4(T) > 0$ such that for all $\varepsilon \in (0, 1)$

$$\int_0^T \int_{\Omega} |\nabla v_\varepsilon|^2 \leq k_4(T)$$  \hfill (17)

and

$$\int_0^T \int_{\Omega} v_\varepsilon^p |\nabla v_\varepsilon|^2 \leq k_4(T)$$  \hfill (18)

as well as

$$\int_0^T \int_{\Omega} \frac{u_\varepsilon v_\varepsilon^p}{1 + \varepsilon u_\varepsilon} \leq k_4(T).$$  \hfill (19)

Proof. Testing the second equation in (8) by $v_\varepsilon^{p-1}$, integrating over $\Omega$ and using Young’s inequality we obtain

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} v_\varepsilon^p + \int_{\Omega} v_\varepsilon^p$$

$$\leq - (p - 1) \int_{\Omega} v_\varepsilon^{p-2} |\nabla v_\varepsilon|^2 + \frac{p - 1}{p} \int_{\Omega} v_\varepsilon^p w_\varepsilon - \frac{p - 1}{p} \int_{\Omega} v_\varepsilon^p \cdot \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}$$

$$+ \int_{\Omega} v_\varepsilon^{p-1} \cdot \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}$$

$$\leq - \frac{4(p - 1)}{p^2} \int_{\Omega} |\nabla v_\varepsilon|^2 + \frac{p - 1}{p} \int_{\Omega} v_\varepsilon^p w_\varepsilon - \frac{p - 1}{2p} \int_{\Omega} v_\varepsilon^p \cdot \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}$$

$$+ \frac{2p - 1}{p} \int_{\Omega} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}$$  \hfill (20)

for all $t > 0$. According to Hölder’s inequality and (15), there exists a number $\theta > 1$ such that

$$\int_{\Omega} v_\varepsilon^p w_\varepsilon \leq \left( \int_{\Omega} v_\varepsilon^{p\theta} \right)^\frac{1}{\theta} \cdot \left( \int_{\Omega} w_\varepsilon^{\theta'} \right)^\frac{1}{\theta'} \leq k_2 \left( \int_{\Omega} v_\varepsilon^{p\theta} \right)^\frac{1}{\theta}$$

with $k_2 = k_2(\theta) > 0$ and $\theta' := \frac{\theta}{p - 1}$. Next apply the Gagliardo-Nirenberg inequality and Young’s inequality to find $c_1 = c_1(p, \theta, \Omega) > 0$ and $c_2 := c_1 + c_1^\theta \left( \frac{2(p - 1)}{p^2} \right)^{-(p - 1)}$ satisfying

$$k_2 \left( \int_{\Omega} v_\varepsilon^{p\theta} \right)^\frac{1}{\theta} \leq k_2 \left\| v_\varepsilon^\frac{1}{2} \right\|_{L^{2\theta}(\Omega)}^2 \leq c_1 \left\| \nabla v_\varepsilon^\frac{1}{2} \right\|_{L^{2\theta}(\Omega)}^{\frac{2(p - 1)}{p^2}} + c_1 \left\| v_\varepsilon^\frac{1}{2} \right\|_{L^{2\theta}(\Omega)}^2.$$

Combining the above inequalities we may estimate

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} v_\varepsilon^p + \int_{\Omega} v_\varepsilon^p + \frac{p - 1}{2p} \int_{\Omega} v_\varepsilon^p \cdot \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} + \frac{2(p - 1)}{p^2} \int_{\Omega} |\nabla v_\varepsilon|^2$$

$$\leq \frac{2p - 1}{p} \int_{\Omega} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} + c_2 \int_{\Omega} v_\varepsilon^p \leq 2p - 1 \int_{\Omega} u_0 + c_2 k_0^p$$  \hfill (21)

for all $t > 0$. A simple ODE comparison deduces (16) with

$$k_3 := \max \{ \int_{\Omega} v_0, 2p - 1 \int_{\Omega} u_0 + c_2 k_0^p \}. $$
In particular, (21) after time-integration shows that (17), (18) and (19) are valid. □

**Lemma 3.4.** Let $n = 2$. For any $T > 0$ there exists a constant $k_5(T) > 0$ with the property that for all $\varepsilon \in (0, 1)$

$$
\int_0^T \int_{\Omega} |\nabla u_\varepsilon|^2 \leq k_5(T) \quad (22)
$$

and

$$
\int_0^T \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} \ln(u_\varepsilon + 1) \leq k_5(T). \quad (23)
$$

**Proof.** Multiplying both sides of the first equation in (8) by $\frac{1}{u_\varepsilon + 1}$ and then integrating by parts yield

$$
\frac{d}{dt} \int_{\Omega} \ln(u_\varepsilon + 1) = \int_{\Omega} \frac{1}{u_\varepsilon + 1} \cdot (\Delta u_\varepsilon - \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \nabla \cdot (u_\varepsilon \nabla w_\varepsilon))
$$

$$
= \int_{\Omega} \frac{\nabla u_\varepsilon}{(u_\varepsilon + 1)^2} \cdot (\nabla u_\varepsilon - u_\varepsilon \nabla v_\varepsilon) + \int_{\Omega} \frac{\nabla u_\varepsilon \cdot \nabla w_\varepsilon}{u_\varepsilon + 1}
$$

$$
+ \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} \cdot (w_\varepsilon - \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon})
$$

$$
= \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} - \int_{\Omega} \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(u_\varepsilon + 1)^2} - \int_{\Omega} w_\varepsilon \ln(u_\varepsilon + 1)
$$

$$
+ \int_{\Omega} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln(u_\varepsilon + 1) + \int_{\Omega} \frac{u_\varepsilon w_\varepsilon}{u_\varepsilon + 1} - \int_{\Omega} (u_\varepsilon + 1)(1 + \varepsilon u_\varepsilon)
$$

for all $t > 0$. Due to Young’s inequality and (15), there is a number $\mu > 2$ fulfilling

$$
\left| - \int_{\Omega} \frac{u_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon}{(u_\varepsilon + 1)^2} \right| \leq \frac{1}{2} \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} + \frac{1}{2} \int_{\Omega} |\nabla v_\varepsilon|^2
$$

and

$$
\int_{\Omega} w_\varepsilon \ln(u_\varepsilon + 1) \leq \frac{1}{\mu} \int_{\Omega} u_\varepsilon^\mu + \frac{1}{\mu} \int_{\Omega} \ln^\mu(u_\varepsilon + 1) \leq \frac{k_2}{\mu} \int_{\Omega} \ln^\mu(u_\varepsilon + 1) \quad (24)
$$

where $k_2 = k_2(\mu) > 0$ and $\mu' = \frac{\mu}{\mu - 1}$. Since $\lim_{s \to \infty} \ln^\mu(s + 1) = 0$, there exists $M(\mu) > 0$ such that for all $s \geq M$, $\ln^\mu(s + 1) \leq s + 1$, then

$$
\int_{\Omega} \ln^\mu(u_\varepsilon + 1) = \int_{\Omega \cap (u_\varepsilon < M)} \ln^\mu(u_\varepsilon + 1) + \int_{\Omega \cap (u_\varepsilon \geq M)} \ln^\mu(u_\varepsilon + 1)
$$

$$
\leq |\Omega| \cdot \ln^\mu(M + 1) + (\int_{\Omega} u_0 + |\Omega|) := M_0. \quad (25)
$$

Insert (25) into (24) to see

$$
\int_{\Omega} w_\varepsilon \ln(u_\varepsilon + 1) \leq \frac{k_2}{\mu'} + \frac{1}{\mu} M_0 := c_1 \quad (26)
$$

for all $t > 0$. In virtue of $0 \leq \ln(s + 1) \leq s$, for $s \geq 0$, we gain

$$
\frac{1}{2} \int_0^T \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} + \int_0^T \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} \ln(u_\varepsilon + 1)
$$

$$
\leq \int_{\Omega} \ln(u_\varepsilon(\cdot, T) + 1) + \frac{1}{2} \int_0^T \int_{\Omega} |\nabla v_\varepsilon|^2 + \int_0^T \int_{\Omega} w_\varepsilon \ln(u_\varepsilon + 1)
$$

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\[
+ \int_0^T \int_\Omega \frac{u_e^2}{(u_e + 1)(1 + \varepsilon u_e)} \\leq (1 + T) \int_\Omega u_0 + \frac{1}{2} k_4 + c_1 T
\]

(27)

for all \( T > 0 \) and \( \varepsilon \in (0, 1) \) with \( k_4 = k_4(T) > 0 \) being the constant from (17), which completes the proof. \( \square \)

In addition, the time derivatives of \( \ln(u_e + 1) \) and \( v_e \) are necessary in preparation for their pointwise convergence a.e. in \( \Omega \times (0, T) \).

**Lemma 3.5.** Suppose \( n = 2 \) and \( m \in \mathbb{N} \) such that \( m > 1 \). Then we can find constants \( k_6, k_7 > 0 \) such that for each \( \varepsilon \in (0, 1) \), the solution of (8) satisfies

\[
\| \partial_t \ln (u_e(\cdot, t) + 1) \|_{L^1((0,T);(W_0^{m,2}(\Omega))^*)} \leq k_6(T + 1) \quad \text{for all } T > 0
\]

(28)

and

\[
\| v_e(\cdot, t) \|_{L^1((0,T);(W_0^{m,2}(\Omega))^*)} \leq k_7(T + 1) \quad \text{for all } T > 0.
\]

(29)

**Proof.** Let \( p \in (1, \infty) \) and \( r \in (1, 2) \). We pick \( 1 < m \in \mathbb{N} \) large enough fulfilling

\[
W_0^{m,2}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{and} \quad W_0^{m,2}(\Omega) \hookrightarrow W^{1,\gamma}(\Omega)
\]

(30)

where \( \gamma := \max\{2, \frac{pr}{pr - r - p}\} \). Then for fixed \( t > 0 \) and arbitrary \( \psi \in W_0^{m,2}(\Omega) \) we infer from the first equation in (8) that

\[
\int_\Omega \partial_t \ln (u_e(\cdot, t) + 1) \cdot \psi
= \int_\Omega \frac{\psi}{u_e + 1} \cdot (\Delta u_e - \nabla \cdot (u_e \nabla v_e) + \nabla \cdot (u_e \nabla w_e))
\]

\[
\leq \| \psi \|_{L^\infty(\Omega)} \cdot \int_\Omega \frac{\| \nabla u_e \|}{(u_e + 1)^2} + \| \psi \|_{L^\infty(\Omega)} \cdot \left( \int_\Omega \frac{\| \nabla u_e \|}{(u_e + 1)^2} \right)^{\frac{1}{2}} \cdot \left( \int_\Omega \frac{\| \nabla v_e \|}{(u_e + 1)^2} \right)^{\frac{1}{2}}
\]

\[
+ \left( \int_\Omega \frac{\| \nabla u_e \|}{(u_e + 1)^2} \right)^{\frac{1}{2}} \cdot \left( \int_\Omega \| \nabla v_e \|^2 \right)^{\frac{1}{2}} + \left( \int_\Omega \| \nabla v_e \|^2 \right)^{\frac{1}{2}} \cdot \left( \int_\Omega \| \nabla \psi \|^{\frac{pr}{pr - r - p}} \right)^{\frac{pr - r - p}{pr}}
\]

\[
+ \| \psi \|_{L^\infty(\Omega)} \cdot \int_\Omega u_e \ln(u_e + 1) + \| \psi \|_{L^\infty(\Omega)} \cdot \int_\Omega \frac{u_e}{1 + \varepsilon u_e} \ln(u_e + 1)
\]

\[
+ \| \psi \|_{L^\infty(\Omega)} \cdot \int_\Omega w_e + \| \psi \|_{L^\infty(\Omega)} \cdot \int_\Omega u_e
\]

for all \( \varepsilon \in (0, 1) \). Since (30) entails the existence of a \( c_1 > 0 \) such that

\[
\| \partial_t \ln (u_e(\cdot, t) + 1) \|_{(W_0^{m,2}(\Omega))^*}
\]
\[ \leq c_1 \left( \int _\Omega \frac{\vert \nabla u_\varepsilon \vert ^2}{(u_\varepsilon + 1)^2} + \int _\Omega \vert \nabla v_\varepsilon \vert ^2 + \left( \int _\Omega \ln ^p (u_\varepsilon + 1) \right) \frac{1}{p} \cdot \left( \int _\Omega \vert \nabla w_\varepsilon \vert ^r \right) ^\frac{1}{r} \right. \\
\left. + \int _\Omega w_\varepsilon \ln (u_\varepsilon + 1) + \int _\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln (u_\varepsilon + 1) + 2 \int _\Omega u_0 + 1 \right) \]  

(31)

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \), (28) can be obtained upon integration over \( t \in (0, T) \) on both sides of (31) combined with (14), (17), (22), (23), (25) and (26).

By the same token we note that for \( t > 0 \) and \( \psi \in W^{m,2}_0 (\Omega) \),

\[ \left\vert \int _\Omega v_\varepsilon t \cdot \psi \right\vert = \left\vert - \int _\Omega \nabla v_\varepsilon \cdot \nabla \psi - \int _\Omega v_\varepsilon \nabla w_\varepsilon \cdot \nabla \psi - \int _\Omega v_\varepsilon \psi + \int _\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \psi \right\vert \]

\[ \leq \left( \int _\Omega v_\varepsilon ^p \right) ^\frac{1}{p} \cdot \left( \int _\Omega \vert \nabla w_\varepsilon \vert ^r \right) ^\frac{1}{r} \cdot \left( \int _\Omega \frac{\ln ^p (u_\varepsilon + 1)}{p} \right) ^\frac{1}{p} \]

\[ + \left( \int _\Omega \vert \nabla v_\varepsilon \vert ^2 \right) ^\frac{1}{2} \cdot \left( \int _\Omega \vert \nabla \psi \vert ^2 \right) ^\frac{1}{2} \]

\[ + \left\| \psi \right\| _{L^\infty (\Omega)} \cdot \int _\Omega v_\varepsilon + \left\| \psi \right\| _{L^\infty (\Omega)} \cdot \int _\Omega u_\varepsilon \]

for all \( \varepsilon \in (0, 1) \). Recalling (30), for all \( t > 0 \) we can find a constant \( c_2 > 0 \) satisfying

\[ \left\| v_\varepsilon t (\cdot, t) \right\| _{L^p (\Omega)} \leq c_2 \left( \int _\Omega \vert \nabla v_\varepsilon \vert ^2 + \left( \int _\Omega v_\varepsilon ^p \right) ^\frac{1}{p} \cdot \left( \int _\Omega \frac{\ln ^p (u_\varepsilon + 1)}{p} \right) ^\frac{1}{p} + k_0 + \int _\Omega u_0 + 1 \right) \]

(32)

for all \( \varepsilon \in (0, 1) \). Thereupon (29) results from (32) together with (14), (16) and (17).

The bounds that have been gained so far can thereafter reach the following estimates essential to prove strong precompactness of \( \{ \nabla v_\varepsilon \}_{\varepsilon \in (0, 1)} \).

**Lemma 3.6.** Let \( n = 2 \), then for each \( T > 0 \), the family \( \{ \frac{u_\varepsilon v_\varepsilon ^2}{1 + \varepsilon u_\varepsilon} \}_{\varepsilon \in (0, 1)} \) is equi-integrable over \( \Omega \times (0, T) \).

**Proof.** For fixed \( T > 0 \), thanks to (19) and (23), we have constants \( c_1, c_2 > 0 \) with the property that for all \( \varepsilon \in (0, 1) \),

\[ \int _0 ^T \int _\Omega \frac{u_\varepsilon v_\varepsilon ^2}{1 + \varepsilon u_\varepsilon} (v_\varepsilon + 1) \leq c_1 \]

(33)

\[ \int _0 ^T \int _\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln (u_\varepsilon + 1) \leq c_2. \]

(34)

For arbitrary \( \eta > 0 \), we choose \( M > 0 \) sufficiently large such that

\[ \frac{c_1}{(M + 1)} \leq \frac{\eta}{3} \]

(35)

and then take \( N > 0 \) large enough contended with

\[ \frac{c_2 M ^2 \cdot \ln (N + 1)}{N} \leq \frac{\eta}{3} \]

(36)

and we can proceed to pick \( \delta > 0 \) suitably small fulfilling

\[ NM ^2 \delta \leq \frac{\eta}{3}. \]

(37)
Then for any given measurable set $E \subset \Omega \times (0, T)$ satisfying $|E| < \delta$, for all $\varepsilon \in (0, 1)$ we find
\[
\int \int_E \frac{u_\varepsilon v_\varepsilon^2}{1 + \varepsilon u_\varepsilon} = \int \int_{E \cap \{v_\varepsilon \geq M\}} \frac{u_\varepsilon v_\varepsilon^2}{1 + \varepsilon u_\varepsilon} + \int \int_{E \cap \{v_\varepsilon < M\} \cap \{u_\varepsilon \geq N\}} \frac{u_\varepsilon v_\varepsilon^2}{1 + \varepsilon u_\varepsilon} + \int \int_{E \cap \{v_\varepsilon < M\} \cap \{u_\varepsilon < N\}} \frac{u_\varepsilon v_\varepsilon^2}{1 + \varepsilon u_\varepsilon} := I_1 + I_2 + I_3. \tag{38}
\]
Following (33) and (34) we estimate
\[
I_1 \leq \frac{1}{(M + 1)} \int \int_{E \cap \{v_\varepsilon \geq M\}} \frac{u_\varepsilon v_\varepsilon^2}{1 + \varepsilon u_\varepsilon} (v_\varepsilon + 1) \leq \frac{c_1}{(M + 1)},
\]
\[
I_2 \leq M^2 \int \int_{E \cap \{v_\varepsilon < M\} \cap \{u_\varepsilon \geq N\}} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \leq \frac{M^2}{\ln(N + 1)} \int \int_{E \cap \{v_\varepsilon < M\} \cap \{u_\varepsilon \geq N\}} \frac{u_\varepsilon \ln(u_\varepsilon + 1)}{1 + \varepsilon u_\varepsilon} \leq \frac{c_2 M^2}{\ln(N + 1)},
\]
\[
I_3 \leq \int \int_{E \cap \{v_\varepsilon < M\} \cap \{u_\varepsilon < N\} \cap \{\varepsilon_0 \leq \varepsilon \leq \varepsilon_1\}} \leq N M^2 |E| \leq N M^2 \delta,
\]
for all $\varepsilon \in (0, 1)$. Combine (35)-(37) with the above estimates to see
\[
\int \int_E \frac{u_\varepsilon v_\varepsilon^2}{1 + \varepsilon u_\varepsilon} \leq \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta
\]
for all $\varepsilon \in (0, 1)$. Since $\eta > 0$ is arbitrary, the proof is completed. \hfill \Box

Moreover, we obtain from Lemma 3.6 that:

**Corollary 1.** Let $n = 2$, then for all $T > 0$, the family $\{\frac{u_\varepsilon v_\varepsilon^2}{1 + \varepsilon u_\varepsilon}\} \subset (0, 1)$ is equi-integrable over $\Omega \times (0, T)$.

Our final preparation will give an integrability property of $\{\frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}, \ln(u_\varepsilon + 1)\} \subset (0, 1)$.

**Lemma 3.7.** Take $n = 2$ and $k \in (1, 2)$, then for each $T > 0$ we find a constant $k_8(T) > 0$ satisfying
\[
\int_0^T \int_{\Omega} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln^k(u_\varepsilon + 1) \leq k_8(T) \quad \text{for all } \varepsilon \in (0, 1). \tag{39}
\]
Particularly, $\{\frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}, \ln(u_\varepsilon + 1)\} \subset (0, 1)$ is equi-integrable over $\Omega \times (0, T)$.

**Proof.** Firstly we claim that for any $T > 0$, there is some $c_1(T) > 0$ fulfilling
\[
\int_0^T \int_{\Omega} \frac{|
abla v_\varepsilon|^2}{(v_\varepsilon + 1)^2} \ln(u_\varepsilon + 1) \leq c_1(T),
\]
for all $\varepsilon \in (0, 1)$. In view of the first and second equations in (8), we write
\[
\frac{d}{dt} \int_{\Omega} \ln(u_\varepsilon + 1) \cdot \ln(v_\varepsilon + 1)
\]
\[
= \int_{\Omega} \frac{\ln(v_\varepsilon + 1)}{u_\varepsilon + 1} \cdot (\Delta u_\varepsilon - \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \nabla \cdot (u_\varepsilon 
abla w_\varepsilon))
\]
\[
+ \int_{\Omega} \frac{\ln(u_\varepsilon + 1)}{v_\varepsilon + 1} \cdot (\Delta v_\varepsilon + \nabla \cdot (v_\varepsilon \nabla w_\varepsilon) - v_\varepsilon + \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon})
\]
\[
= \int_{\Omega} \frac{|
abla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \ln(v_\varepsilon + 1) - 2 \int_{\Omega} \frac{\nabla v_\varepsilon \cdot \nabla u_\varepsilon}{(v_\varepsilon + 1)(u_\varepsilon + 1)}
\]
for all $t > 0$. Upon a time integration, the above inequality becomes

$$\int_0^T \int_\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln(u_\varepsilon + 1) \cdot \ln(v_\varepsilon + 1) + \int_0^T \int_\Omega \frac{|\nabla v_\varepsilon|^2}{(v_\varepsilon + 1)^2} \ln(u_\varepsilon + 1)$$

$$\leq \int_\Omega v_\varepsilon(\cdot, T) \ln(u_\varepsilon(\cdot, T) + 1) + \frac{3}{2} \int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} + \int_0^T \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon + 1}$$

$$+ \frac{1}{2} \int_0^T \int_\Omega \frac{v_\varepsilon^2}{1 + \varepsilon u_\varepsilon} \nabla v_\varepsilon)^2 + \int_0^T \int_\Omega v_\varepsilon w_\varepsilon \ln(u_\varepsilon + 1) + \int_0^T \int_\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln(v_\varepsilon + 1)$$

$$+ \int_0^T \int_\Omega w_\varepsilon \ln(u_\varepsilon + 1) + T \cdot \int_\Omega u_0$$

(40)

for all $T > 0$. An application of (15), (16), (25) and Hölder’s inequality entails the existence of some $p \in (1, 2)$ and some $\mu > \frac{p}{p - 1} > 2$ such that

$$\int_\Omega v_\varepsilon \ln(u_\varepsilon + 1) \leq \left( \int_\Omega v_\varepsilon^p \right)^{\frac{1}{p}} \cdot \left( \int_\Omega \ln^p(u_\varepsilon + 1) \right)^{\frac{1}{p}} \cdot |\Omega|^{\frac{\mu - p - \mu}{\mu - p - 2}} \leq M_0 \frac{1}{2} k_3 |\Omega|^{\frac{\mu - p - 2}{\mu - p - 2}}$$

and

$$\int_\Omega v_\varepsilon w_\varepsilon \ln(u_\varepsilon + 1) \leq \left( \int_\Omega v_\varepsilon^p \right)^{\frac{1}{p}} \cdot \left( \int_\Omega w_\varepsilon^{\frac{\mu - p - 2}{\mu - p - 2}} \right)^{\frac{\mu - p - 2}{\mu - p - 2}} \cdot \left( \int_\Omega \ln^p(u_\varepsilon + 1) \right)^{\frac{1}{p}}$$

$$\leq M_0 \frac{1}{2} k_2 k_3$$

with $k_2 = k_2(\mu, p) > 0$ and $\mu' = \frac{\mu - p}{\mu - 1}$. Thus combining (22), (23), (17), (18) and (19) with (40), the estimation holds.

It can then turn to prove (39). We split

$$\int_\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln^k(u_\varepsilon + 1)$$

$$= \int_{\Omega \cap \{u_\varepsilon \geq 1\}} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln^k(u_\varepsilon + 1) + \int_{\Omega \cap \{u_\varepsilon < 1\}} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} \ln^k(u_\varepsilon + 1)$$

$$:= I_4 + I_5$$

(41)
Lemma 4.1. Let \( p \in (1, \infty) \) and \( r \in (1, 2) \). For each \( \varepsilon_j \in (0, 1) \), suppose that \((u_{\varepsilon_j}, v_{\varepsilon_j}, w_{\varepsilon_j})\) is a global classical solution of (8) and (3) holds, then there is a

for all \( \varepsilon \in (0, 1) \). Since the boundedness of \( I_5 \) is obvious, it hereafter suffices to estimate \( I_4 \). Define \( \Omega_0 := \Omega \cap \{ u_{\varepsilon} \geq 1 \} \), from the first equation in (8) we compute

\[
\frac{1}{k} \frac{d}{dt} \int_{\Omega_0} \ln^k(u_{\varepsilon} + 1)
= \int_{\Omega_0} \ln^{k-1}(u_{\varepsilon} + 1) \cdot (\Delta u_{\varepsilon} - \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) + \nabla \cdot (u_{\varepsilon} \nabla w_{\varepsilon}))
\]

\[
= \int_{\Omega_0} \ln^{k-1}(u_{\varepsilon} + 1) \cdot \frac{|\nabla u_{\varepsilon}|^2}{(u_{\varepsilon} + 1)^2} - \int_{\Omega_0} \ln^{k-1}(u_{\varepsilon} + 1) \cdot \frac{u_{\varepsilon} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}}{(u_{\varepsilon} + 1)^2} - (k - 1) \int_{\Omega_0} \ln^{k-2}(u_{\varepsilon} + 1) \cdot \frac{|\nabla u_{\varepsilon}|^2}{(u_{\varepsilon} + 1)^2} + (k - 1) \int_{\Omega_0} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}
\]

\[
- \frac{1}{k} \int_{\Omega_0} w_{\varepsilon} \ln^k(u_{\varepsilon} + 1) + \frac{1}{k} \int_{\Omega_0} \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}} \ln^k(u_{\varepsilon} + 1)
\]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \). Denote by \( k_4 = k_4(T) > 0 \) the constant provided by Lemma 3.3 and by \( k_5 = k_5(T) > 0 \) that of Lemma 3.4. Due to \( k \in (1, 2) \) and Young’s inequality, the inequality above then provides

\[
\frac{1}{k} \int_0^T \int_{\Omega_0} \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}} \ln^k(u_{\varepsilon} + 1) + \int_0^T \int_{\Omega_0} \frac{v_{\varepsilon}}{(u_{\varepsilon} + 1)^2} \ln(u_{\varepsilon} + 1)
\]

\[
+ \int_0^T \int_{\Omega_0} (v_{\varepsilon} + 1) \left( \frac{k - 1}{u_{\varepsilon} + 1} \right) |\nabla v_{\varepsilon}|^2 + \frac{3}{2} (\ln 2)^{k-2} (k - 1) \int_0^T \int_{\Omega_0} \frac{|\nabla u_{\varepsilon}|^2}{(u_{\varepsilon} + 1)^2}
\]

\[
+ \frac{1}{k} (\ln 2)^{k-2} (k - 1) \int_0^T \int_{\Omega_0} |\nabla v_{\varepsilon}|^2 + \int_0^T \int_{\Omega_0} w_{\varepsilon} \ln^k(u_{\varepsilon} + 1)
\]

\[
+ (k - 1) \int_0^T \int_{\Omega_0} \ln(u_{\varepsilon} + 1) \cdot \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}} + (2 - k) \int_0^T \int_{\Omega_0} \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}}
\]

\[
\leq \frac{1}{k} \int_{\Omega_0} \ln^k(u_{\varepsilon} + 1) + c_1 + k_4 + \frac{3}{2} (\ln 2)^{k-2} (k - 1) k_5 + \frac{1}{2} (\ln 2)^{k-2} (k - 1) k_4
\]

for all \( T > 0 \) and \( \varepsilon \in (0, 1) \). Employing Hölder’s inequality once again to the last term on the right side of the above inequality and utilizing (25), the estimate arrives.

4. Convergence of solutions of regularized system. Collecting the estimates in Section 3 and extracting appropriately convergent subsequences, we find the second and third components in accordance with Definition 2.1 by passing to the limit.

**Lemma 4.1.** Let \( p \in (1, \infty) \) and \( r \in (1, 2) \). For each \( \varepsilon_j \in (0, 1) \), suppose that \((u_{\varepsilon_j}, v_{\varepsilon_j}, w_{\varepsilon_j})\) is a global classical solution of (8) and (3) holds, then there is a
sequence \( \{ \varepsilon_j \}_{j \in \mathbb{N}} \subset (0, 1) \) such that \( \varepsilon_j \to 0 \) as \( j \to \infty \) and with some \( p_0 > 2 \) we have

\[
\begin{align*}
    u_{\varepsilon_j} &\to u \text{ a.e. in } \Omega \times (0, \infty) \quad (42) \\
    \ln(u_{\varepsilon_j} + 1) &\to \ln(u + 1) \text{ in } L^2_{\text{loc}} ([0, \infty); W^{1,2}(\Omega)) \quad (43) \\
    \ln(u_{\varepsilon_j} + 1) &\to \ln(u + 1) \text{ in } L^p_{\text{loc}} (\Omega \times [0, \infty)) \quad (44) \\
    \frac{u_{\varepsilon_j}}{1 + \varepsilon_j u_{\varepsilon_j}} &\to u \text{ in } L^1_{\text{loc}} (\Omega \times [0, \infty)) \quad (45) \\
    v_{\varepsilon_j} &\to v \text{ a.e. in } \Omega \times (0, \infty) \quad (46) \\
    \nabla v_{\varepsilon_j} &\to \nabla v \text{ in } L^2_{\text{loc}} (\Omega \times [0, \infty)) \quad (47) \\
    \nabla w_{\varepsilon_j} &\to \nabla w \text{ in } L^2_{\text{loc}} ([0, \infty); W^{1,2}(\Omega)) \quad (48) \\
    \frac{u_{\varepsilon_j}}{1 + \varepsilon_j u_{\varepsilon_j}} - \ln(u_{\varepsilon_j} + 1) &\to u \ln(u + 1) \text{ in } L^1_{\text{loc}} (\Omega \times [0, \infty)) \quad (49) \\
    \frac{u_{\varepsilon_j} v_{\varepsilon_j}}{1 + \varepsilon_j u_{\varepsilon_j}} &\to uv \text{ in } L^1_{\text{loc}} (\Omega \times [0, \infty)) \quad (50) \\
\end{align*}
\]

as \( \varepsilon_j \to 0 \) with certain nonnegative functions \( u, v \) and \( w \) defined in \( \Omega \times (0, \infty) \). Moreover, \( (v, w) \) is a global weak solution of (2) in the sense of Definition 2.1.

**Proof.** Let \( p \in (1, \infty) \) and \( r \in (1, 2) \). For all \( T > 0 \), we get from Lemma 3.4 and Lemma 3.5 that there exists some \( c_1 > 0 \) satisfying

\[
\| \ln(u_{\varepsilon} + 1) \|_{L^2((0, T); W^{1,2}(\Omega))} + \| \partial_t \ln(u_{\varepsilon} + 1) \|_{L^1((0, T); W^{0,2}_0(\Omega))} \leq c_1
\]

with some \( m \in \mathbb{N} \) such that \( m > 1 \). Adopting a variant of Aubin-Lions Lemma [4], \( \{ \ln(u_{\varepsilon} + 1) \}_{\varepsilon \in (0, 1)} \) is strongly precompact in \( L^2 (\Omega \times (0, T)) \). Thereby (42) can be found upon a subsequence, and (43) holds. (25) entails that there exists \( p_0 \in (2, p) \) such that

\[
\ln(u_{\varepsilon_j} + 1) \to \ln(u + 1) \quad \text{in } L^{p_0}(\Omega \times (0, T)) \quad (54)
\]

as \( \varepsilon_j \to 0 \). Again we obtain from (25) that \( \ln^{p_0}(u_{\varepsilon_j} + 1) \to w \text{ in } L^{s_0}(\Omega \times (0, T)) \) with \( s_0 = \frac{p}{p_0} > 1 \). By (42), \( w = \ln^{p_0}(u + 1) \) along another subsequence. In particular,

\[
\int_0^T \int_{\Omega} \ln^{p_0}(u_{\varepsilon_j} + 1) \to \int_0^T \int_{\Omega} \ln^{p_0}(u + 1)
\]

as \( \varepsilon_j \to 0 \). Together with (54), this ensures the validity of (44).

According to Lemma 3.3 and Lemma 3.5, there is some \( c_2 > 0 \) such that

\[
\| v_{\varepsilon} \|_{L^2((0, T); W^{1,2}(\Omega))} + \| \partial_t v_{\varepsilon} \|_{L^1((0, T); W^{0,2}_0(\Omega))} \leq c_2
\]

for all \( T > 0 \) with some \( m \in \mathbb{N} \) such that \( m > 1 \). Thus the strong precompactness of \( \{ v_{\varepsilon} \}_{\varepsilon \in (0, 1)} \) in \( L^2 (\Omega \times (0, T)) \) is checked by the Aubin-Lions Lemma, and (46) holds along a subsequence for some \( v \in L^2 (\Omega \times (0, T)) \). The validity of (48) follows at once from Lemma 3.3. Let \( p_1 \in (1, p) \), (16) indicates that there exists a weakly
convergent subsequence \( v_{p_1}^{\varepsilon_j} \to z \) in \( L^{s_1} (\Omega \times (0, T)) \) with \( s_1 = \frac{p}{p_1} > 1 \). By another subsequence, (46) performs \( z = v^{p_1} \). Testing \( \varphi = 1 \in (L^{p_1} (\Omega \times (0, T)))^* \) yields
\[
\int_0^T \int_\Omega v_{p_1}^{\varepsilon_j} \to \int_0^T \int_\Omega v^{p_1}
\]
as \( \varepsilon_j \searrow 0 \). Along with the weak convergence \( v_{\varepsilon_j} \rightharpoonup v \) in \( L^{p_1} (\Omega \times (0, T)) \), \( \{v_{\varepsilon_j}\}_{\varepsilon_j \in (0, 1)} \) is strongly precompact in \( L^{p_1} (\Omega \times (0, T)) \) for each \( p_1 \in (1, \infty) \). By Lemma 3.2, for any \( T > 0 \) \( \{w_\varepsilon\}_{\varepsilon \in (0, 1)} \) is bounded in
\[
L^\infty (0, T); W^{1,r}(\Omega) \hookrightarrow L^r (0, T); W^{1,r}(\Omega),
\]
extracting a weakly convergent subsequence of \( \{w_\varepsilon\}_{\varepsilon \in (0, 1)} \) shows that (49) is valid.

In addition, for any \( T > 0 \), (23) guarantees that \( \{\frac{\varphi}{\varphi + \varepsilon_j}\}_{\varepsilon_j \in (0, 1)} \) is equi-integrable, and therefore weakly sequentially precompact in \( L^1 (\Omega \times (0, T)) \). Using (42) we have
\[
\frac{u_{\varepsilon_j}}{1 + \varepsilon_j u_{\varepsilon_j}} \to u \quad \text{a.e. in } \Omega \times (0, T)
\]
as \( \varepsilon_j \searrow 0 \), so adopting the Vitali convergence theorem proves (45). Following the same argument as above, thanks to Lemma 3.6, Corollary 1 and Lemma 3.7, (50), (51) and (52) conclude. A combination of (47) and (49) asserts (53). For any \( \varphi \in C^\infty_0 (\Omega \times [0, \infty)) \), use (45) and (49) to see that (5) holds. Finally test the second equation in (8) against \( \varphi \) as specified in Definition 2.1, the convergence properties gained in Lemma 4.1 imply (4).

Following a related procedure presented in [35], we proceed to prove the strong \( L^2 \) convergence of \( \{\nabla v_\varepsilon\}_{\varepsilon \in (0, 1)} \).

**Lemma 4.2.** Let \( \{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0, 1) \) and the limit function \( v \) be as in Lemma 4.1. Then for all \( T > 0 \),
\[
\nabla v_{\varepsilon_j} \to \nabla v \quad \text{in } L^2 (\Omega \times (0, T)) \quad \text{as } \varepsilon_j \searrow 0.
\]

**Proof.** Take the truncations
\[
\eta_\delta (t) := \begin{cases} 
1, & t \in [0, t_0], \\
1 - \frac{t_0 - t}{\delta}, & t \in (t_0, t_0 + \delta), \\
0, & t \geq t_0 + \delta,
\end{cases}
\]
for each \( t_0 \in (0, \infty) \) and \( \delta \in (0, 1) \), let \( p \in (1, \infty) \) and let
\[
\tilde{v}_k (x, t) := \begin{cases} 
v(x, t), & x \in \Omega, t > 0, \\
v_{0k} (x), & x \in \Omega, t \in (-1, 0],
\end{cases}
\]
where \( \{v_{0k}\}_{k \in \mathbb{N}} \subset C^1 (\bar{\Omega}) \) converges to \( v_0 \) in \( L^p (\Omega) \) as \( k \to \infty \). Then for \( k \in \mathbb{N} \) and \( h \in (0, 1) \), we take
\[
\varphi (x, t) := \eta_\delta (t) \cdot (R_h \tilde{v}_k) (x, t), \quad x \in \Omega, t > 0
\]
with
\[
(R_h \tilde{v}_k) (x, t) := \frac{1}{h} \int_{t-h}^t \tilde{v}_k (x, s) \, ds, \quad x \in \Omega, t > 0.
\]
Since Lemma 4.1 implies that
\[
v \in L^p_{loc} (\Omega \times [0, \infty)) \cap L^2_{loc} (0, \infty) \cap W^{1,2} (\Omega)
\]
for each \( p \in [1, \infty) \), then \( \varphi \in L^\infty (\Omega \times (0, \infty)) \cap L^2 (\Omega \times (0, \infty); W^{1,2} (\Omega)) \) with compact support in \( \bar{\Omega} \times [0, t_0 + 1] \). Furthermore,
\[
\varphi_t (x, t) = \eta'_\delta (t) \cdot (R_h \tilde{v}_k) (x, t) + \eta_\delta (t) \frac{\tilde{v}_k (x, t - h) - \tilde{v}_k (x, t)}{h}, \quad x \in \Omega, t > 0
\]
entails that \( \varphi_t \in L^p(\Omega \times (0, \infty)) \) for \( p \in (1, \infty) \). By a completion argument, we insert \( \varphi \) into (4) to write

\[
- \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)(R_h \tilde{v}_k)(x, t)dxdt - \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)\tilde{v}_k(x, t)dxdt - \int_\Omega v_0(x)v_{0k}(x)dx = - \int_0^\infty \int_\Omega \eta_\delta(t)\nabla v(x, t) \cdot \nabla (R_h \tilde{v}_k)(x, t)dxdt
- \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)\nabla w(x, t) \cdot \nabla (R_h \tilde{v}_k)(x, t)dxdt + \int_0^\infty \int_\Omega \eta_\delta(t)u(x, t)(R_h \tilde{v}_k)(x, t)dxdt
- \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)(R_h \tilde{v}_k)(x, t)dxdt
\]

for all \( \delta \in (0, 1), k \in \mathbb{N} \) and \( h \in (0, 1) \). Taking into account Young’s inequality to the second term on the left side of (56), we calculate

\[
- \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)\tilde{v}_k(x, t)dxdt - \int_\Omega \frac{\varphi(t)}{h}v_0(x)v_{0k}(x)dx
= - \frac{1}{\delta} \int_0^\infty \int_\Omega \eta_\delta(t)\tilde{v}_k^2(x, t)dxdt + \frac{1}{\delta} \int_0^\infty \int_\Omega \eta_\delta(t)\tilde{v}_k(x, t)\tilde{v}_k(x, t - h)dxdt
\]

for all \( \delta \in (0, 1), k \in \mathbb{N} \) and \( h \in (0, 1) \). Inserting the above inequality into (56) leads to

\[
- \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)(R_h \tilde{v}_k)(x, t)dxdt + \frac{1}{\delta} \int_0^\infty \int_\Omega \eta_\delta(t)(h) - \eta_\delta(t)\tilde{v}_k^2(x, t)dxdt
+ \frac{1}{\delta} \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)\tilde{v}_k(x, t)dxdt - \int_\Omega v_0(x)v_{0k}(x)dx
\geq - \int_0^\infty \int_\Omega \eta_\delta(t)\nabla v(x, t) \cdot \nabla (R_h \tilde{v}_k)(x, t)dxdt
- \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)\nabla w(x, t) \cdot \nabla (R_h \tilde{v}_k)(x, t)dxdt + \int_0^\infty \int_\Omega \eta_\delta(t)u(x, t)(R_h \tilde{v}_k)(x, t)dxdt
- \int_0^\infty \int_\Omega \eta_\delta(t)v(x, t)(R_h \tilde{v}_k)(x, t)dxdt
\]

for all \( \delta \in (0, 1), k \in \mathbb{N} \) and \( h \in (0, 1) \).

Now for fixed \( T > 0 \), we pick \( t_0 > T \) fulfilling with \( t_0 \) a Lebesgue point of

\[
0 < t \mapsto \int_0^t w^2(\cdot, t) dt,
\]

which indicate that

\[
\lim_{\delta \to 0} \frac{1}{\delta} \int_{t_0}^{t_0 + \delta} \int_\Omega v^2(x, t)dxdt = \int_\Omega v^2(x, t_0)dx
\]
for any such $t_0 > T$. Next utilize Lemma A.2 [35],

$$\nabla(R_h \tilde{v}_k) = R_h(\nabla \tilde{v}_k) \to \nabla \tilde{v}_k = \nabla v, \quad \text{in } L^2(\Omega \times (0, t_0 + 1))$$

and

$$R_h \tilde{v}_k \to \tilde{v}_k = v, \quad \text{in } L^p(\Omega \times (0, t_0 + 1))$$
as $h \to 0$ for each $p \in (1, \infty)$. Invoke the dominated convergence theorem,

$$\lim_{h \to 0} \int_0^\infty \int_\Omega \frac{\eta_h(t + h) - \eta_h(t)}{h} v_k^2(x, t) dx dt = \int_0^\infty \int_\Omega \eta_h'(t) v_k^2(x, t) dx dt.$$

Thereupon passing to the limit $h \to 0$ and $k \to 0$, (57) provides

$$-\frac{1}{2} \int_0^\infty \int_\Omega \eta_h'(t) v^2(x, t) dx dt - \frac{1}{2} \int_\Omega v_0^2(x) dx$$

$$\geq - \int_0^\infty \int_\Omega \eta_h(t) |\nabla v(x, t)|^2 dx dt - \int_0^\infty \int_\Omega \eta_h(t) v(x, t) \nabla v(x, t) \cdot \nabla w(x, t) dx dt$$

$$+ \int_0^\infty \int_\Omega \eta_h(t) u(x, t) v(x, t) dx dt - \int_0^\infty \int_\Omega \eta_h(t) v(x, t) dx dt$$

(59)

for all $\delta \in (0, 1)$. Moreover, (58) guarantees that

$$-\frac{1}{2} \int_0^\infty \int_\Omega \eta_h'(t) v^2(x, t) dx dt = \frac{1}{2} \int_0^{t_0 + \delta} \int_\Omega v^2(x, t) dx dt - \frac{1}{2} \int_\Omega v^2(x, t_0) dx$$
as $\delta \to 0$. Hence seeking the monotone convergence theorem to the both sides of (59) and then applying Lemma 4.1 produce

$$\int_0^{t_0} \int_\Omega |\nabla v|^2$$

$$\leq -\frac{1}{2} \int_\Omega v^2(\cdot, t_0) + \frac{1}{2} \int_0^{t_0} v^2 dx - \int_\Omega v \cdot \nabla w + \int_\Omega uv - \int_\Omega v^2$$

$$= -\frac{1}{2} \int_0^{t_0} v^2(\cdot, t_0) + \frac{1}{2} \int_\Omega v^2 dx + \frac{1}{2} \int_\Omega v^2 \Delta w + \int_\Omega uv - \int_\Omega v^2$$

$$= \lim_{\varepsilon_j \to 0} \left\{ -\frac{1}{2} \int_\Omega v_{\varepsilon_j}^2 (\cdot, t_0) + \frac{1}{2} \int_\Omega v_{\varepsilon_j}^2 + \frac{1}{2} \int_\Omega v_{\varepsilon_j}^2 \Delta w - \frac{1}{2} \int_\Omega v_{\varepsilon_j}^2 - \frac{1}{2} \int_\Omega \frac{u_{\varepsilon_j}}{1 + \varepsilon_j u_{\varepsilon_j}} v_{\varepsilon_j}^2 \right\}$$

$$+ \lim_{\varepsilon_j \to 0} \int_\Omega |\nabla v_{\varepsilon_j}|^2.$$  

(60)

Since (48) deduces that

$$\int_0^{t_0} \int_\Omega |\nabla v|^2 \leq \lim_{\varepsilon_j \to 0} \int_0^{t_0} \int_\Omega |\nabla v_{\varepsilon_j}|^2,$$

(61)
a combination of (60) and (61) along with our choice of $t_0 > T$, (55) is proved. \(\square\)

Now we could apply (55) and Lemma 4.1 to find that $v$ coincides with (6) and (7) by passing to the limit. On the basis of Moser-Trudinger inequality in the two-dimensional setting, strong $L^1$ convergence of the sequence \(\{u_\varepsilon\}_{\varepsilon \in (0, 1)}\) has been analyzed in [34].
Lemma 4.3. Take \( n = 2 \) and let a triple of limit functions \((u, v, w)\) be as in Lemma 4.1. Then (6) is true for all nonnegative \( \varphi \in C^0_0(\Omega \times [0, \infty)) \) as well as

\[
\int_\Omega u(\cdot, t) = \int_\Omega u_0 \quad \text{for a.e.} \ t > 0. \tag{62}
\]

Proof. By Lemma 4.2 \([34]\), (22) and (42) imply that

\[
u_{\varepsilon_j} \to u \quad \text{in} \ L^1_{\text{loc}}(\Omega \times [0, \infty)) \tag{63}
\]
as \( \varepsilon_j \searrow 0 \). Note that \( \int_\Omega u_{\varepsilon} (\cdot, t) = \int_\Omega u_0 \) for all \( \varepsilon \in (0, 1) \) and \( t > 0 \), so (62) follows from (63).

For fixed nonnegative \( \varphi \in C^0_0(\Omega \times [0, \infty)) \), we can see from the first equation in (8) that

\[
\int_0^\infty \int_\Omega |\nabla \ln(u_{\varepsilon} + 1)|^2 \varphi
\]

\[
= -\int_0^\infty \int_\Omega \ln(u_{\varepsilon} + 1) \varphi_t - \int_\Omega \ln(u_0 + 1) \varphi(\cdot, 0)
\]

\[
+ \int_0^\infty \int_\Omega \nabla \ln(u_{\varepsilon} + 1) \cdot \nabla \varphi + \int_0^\infty \int_\Omega \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} (\nabla \ln(u_{\varepsilon} + 1) \cdot \nabla v_{\varepsilon}) \varphi
\]

\[
- \int_0^\infty \int_\Omega \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} \nabla v_{\varepsilon} \cdot \nabla \varphi + \int_0^\infty \int_\Omega \ln(u_{\varepsilon} + 1) \nabla w_{\varepsilon} \cdot \nabla \varphi
\]

\[
+ \int_0^\infty \int_\Omega \frac{w_{\varepsilon}}{1 + \varepsilon_{\varepsilon}} \ln(u_{\varepsilon} + 1) \varphi \to \int_0^\infty \int_\Omega \frac{w_{\varepsilon}}{1 + \varepsilon_{\varepsilon}} \cdot \frac{u_{\varepsilon}}{1 + u_{\varepsilon}} \varphi
\]  \( \varphi \)

(64)

for all \( \varepsilon \in (0, 1) \). Upon passing to a subsequence, (43) and (50) indicate

\[
- \int_0^\infty \int_\Omega \ln(u_{\varepsilon_j} + 1) \varphi_t \to -\int_\Omega \ln(u + 1) \varphi_t,
\]

\[
\int_0^\infty \int_\Omega \nabla \ln(u_{\varepsilon_j} + 1) \cdot \nabla \varphi \to \int_0^\infty \int_\Omega \nabla \ln(u + 1) \cdot \nabla \varphi
\]

and

\[
- \int_0^\infty \int_\Omega \frac{u_{\varepsilon_j}}{1 + \varepsilon_{\varepsilon_j} u_{\varepsilon_j}} \ln(u_{\varepsilon_j} + 1) \varphi \to -\int_0^\infty \int_\Omega u \ln(u + 1) \varphi
\]
as \( \varepsilon_j \searrow 0 \). Since we can pick \( p_0 > 2 \) and \( r > 1 \) close to 2 such that \( \frac{1}{p_0} + \frac{1}{r} \leq 1 \), (44) and (49) entail that

\[
\int_0^\infty \int_\Omega \ln(u_{\varepsilon_j} + 1) \nabla w_{\varepsilon_j} \cdot \nabla \varphi \to \int_0^\infty \int_\Omega \ln(u + 1) \nabla w \cdot \nabla \varphi
\]

and

\[
\int_0^\infty \int_\Omega w_{\varepsilon_j} \ln(u_{\varepsilon_j} + 1) \varphi \to \int_0^\infty \int_\Omega w \ln(u + 1) \varphi
\]
as \( \varepsilon_j \searrow 0 \). Moreover, in view of (45) and (49), (42) and the dominated convergence theorem result in

\[
- \int_0^\infty \int_\Omega \frac{u_{\varepsilon_j} w_{\varepsilon_j}}{1 + u_{\varepsilon_j}} \varphi \to -\int_0^\infty \int_\Omega \frac{u}{1 + u} w \varphi
\]

and

\[
\int_0^\infty \int_\Omega \frac{u_{\varepsilon_j}}{1 + \varepsilon_{\varepsilon_j} u_{\varepsilon_j}} \cdot \frac{w_{\varepsilon_j}}{1 + u_{\varepsilon_j}} \varphi \to \int_0^\infty \int_\Omega \frac{u^2}{1 + u} \varphi
\]
as $\varepsilon_j \downarrow 0$. Due to (43) and $u_{\varepsilon_j} \rightarrow \frac{u}{u+1}$ a.e. in $\Omega \times (0, \infty)$ as $\varepsilon_j \rightarrow 0$, (55) and Lemma A.4 [35] ensure that
\[-\int_0^\infty \int_{\Omega} \frac{u_{\varepsilon_j}}{u_{\varepsilon_j}+1} \nabla v_{\varepsilon_j} \cdot \nabla \varphi \rightarrow -\int_0^\infty \int_{\Omega} \frac{u}{u+1} \nabla v \cdot \nabla \varphi \]
and
\[\int_0^\infty \int_{\Omega} \frac{u_{\varepsilon_j}}{u_{\varepsilon_j}+1} (\nabla v_{\varepsilon_j} \cdot \nabla \ln(u_{\varepsilon_j}+1)) \varphi \rightarrow \int_0^\infty \int_{\Omega} \frac{u}{u+1} (\nabla v \cdot \nabla \ln(u+1)) \varphi \]
as $\varepsilon_j \downarrow 0$. Upon a lower semicontinuity argument, we collect the above convergence results to identify (6) as required.

Proof of Theorem 1.1. This result is directly from Lemma 4.1 and Lemma 4.3. ∎

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