A CRITERION RELATED TO THE RIEMANN HYPOTHESIS

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ABSTRACT. A crucial role in the Nyman-Beurling-Báez-Duarte approach to the Riemann Hypothesis is played by the distance

$$d^2_N := \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta A_N \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{\frac{1}{4} + t^2},$$

where the infimum is over all Dirichlet polynomials

$$A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$$

of length $N$.

In this paper we investigate $d^2_N$ under the assumption that the Riemann zeta function has four non-trivial zeros off the critical line. Thus we obtain a criterion for the non validity of the Riemann Hypothesis.

Key words: Riemann hypothesis, Riemann zeta function, Nyman-Beurling-Báez-Duarte criterion.

2000 Mathematics Subject Classification: 30C15, 11M26

1. INTRODUCTION

The Nyman-Beurling-Báez-Duarte approach to the Riemann hypothesis asserts that the Riemann hypothesis is true, if and only if

$$\lim_{N \to \infty} d^2_N = 0,$$

where

$$d^2_N := \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta A_N \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{\frac{1}{4} + t^2}$$

and the infimum is over all Dirichlet polynomials

$$A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$$

of length $N$ (see [3]).

Burnol [4], improving on work of Báez-Duarte, Balazard, Landreau and Saias [1], [2] showed that

$$\liminf_{N \to \infty} d^2_N \log N \geq \sum_{\text{Re}(\rho) = \frac{1}{2}} \frac{m(\rho)^2}{|\rho|^2},$$

Date: May 30, 2017.
where $m(\rho)$ denotes the multiplicity of the zero $\rho$.
This lower bound is believed to be optimal and one expects that

\[ (*) \quad d_N^2 \sim \frac{1}{\log N} \sum_{\text{Re}(\rho)=\frac{1}{2}} \frac{m(\rho)^2}{|\rho|^2}. \]

Under the Riemann hypothesis one has

\[ (1.2) \quad \sum_{\text{Re}(\rho)=\frac{1}{2}} \frac{m(\rho)}{|\rho|^2} = 2 + \gamma - \log 4\pi, \]

where $\gamma$ is the Euler-Mascheroni constant.
S. Bettin, J. B. Conrey and D. W. Farmer \cite{3} prove (*) under an additional assumption and also identify the Dirichlet polynomials $A_N$, for which the expected infimum in (1.1) is assumed. They prove (Theorem 1 of \cite{3}):

Let

\[ V_N(s) := \sum_{n=1}^{N} \left( 1 - \frac{\log n}{\log N} \right) \frac{\mu(n)}{n^s}. \]

If the Riemann hypothesis is true and if

\[ \sum_{|\text{Im}(\rho)| \leq T} |\zeta(\rho)|^2 \ll T^{2-\delta} \]

for some $\delta > 0$, then

\[ (1.3) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta V_N \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{\frac{1}{4} + t^2} \sim \frac{2 + \gamma - \log 4\pi}{\log N}. \]

In this paper we investigate the expression (1.3) under an assumption contrary to the Riemann hypothesis: There are exactly four nontrivial zeros off the critical line. We observe that nontrivial zeros off the critical line always appear as quadruplets. Indeed, if $\zeta(\rho) = 0$ for $\rho = \sigma + i\gamma$ with $1 > \sigma > \frac{1}{2}, \gamma > 0$, then from the functional equation

\[ \Lambda(s) = \Lambda(1-s), \]

where

\[ \Lambda(s) := \pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) \]

and the trivial relation $\zeta(s) = \zeta(s)$, we obtain that

\[ \zeta(\sigma + i\gamma) = \zeta(1 - \sigma + i\gamma) = \zeta(\sigma - i\gamma) = \zeta(1 - \sigma - i\gamma) = 0. \]

We prove the following:

**Theorem 1.1.** Let $\sigma_0 > 1/2, \gamma_0 > 0$, $\zeta(\sigma_0 \pm i\gamma_0) = \zeta(1 - \sigma_0 \pm i\gamma_0) = 0$ and $\zeta(\sigma + i\gamma) \neq 0$ for all other $\sigma + i\gamma$ with $\sigma > 1/2$. Assume that

\[ \sum_{|\text{Im}(\rho)| \leq T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{\frac{3}{2}-\delta} \quad (T \to \infty), \]

for some $\delta > 0$. Then, there are constants $A = A(\sigma_0, \gamma_0)$ and $B = B(\sigma_0, \gamma_0) \in \mathbb{R}$, such that for all $\epsilon > 0$:

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| 1 - \zeta V_N \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{\frac{1}{4} + t^2} \]
\[
\frac{1}{\log^2 N} \left( AN^{2\gamma_0 - 1} \cos(2\gamma_0 \log N) + BN^{2\gamma_0 - 1} \right) \left( 1 + O(N^{\frac{1}{2} - \sigma_0 + \epsilon}) \right),
\]
for some \( \epsilon > 0 \).

2. Preliminary Lemmas and Definitions

**Lemma 2.1.** Let \( \epsilon > 0 \) be fixed but arbitrarily small. Under the assumptions of Theorem 1.1 we have

\[
\zeta(\sigma + it) \ll |t|^{3\epsilon} \quad (|t| \to \infty),
\]
for

\[
\frac{1}{2} - \epsilon \leq \sigma \leq \frac{1}{2} + \epsilon.
\]

**Proof:** The estimate (2.1) is well known as the Lindelöf hypothesis, which is a consequence of the Riemann hypothesis. In [5], the Lindelöf hypothesis is proven on the assumption of the Riemann hypothesis. This proof may be adapted to the new situation by slight modifications.

The function \( \log \zeta(s) \) is holomorphic in the domain

\[
G := \left\{ \sigma + it : \sigma > \frac{1}{2} \right\} \setminus \left\{ \left[ \frac{1}{2}, 1 \right] \cup \left[ \frac{1}{2} + i\gamma_0, \sigma + i\gamma_0 \right] \cup \left[ \frac{1}{2} - i\gamma_0, \sigma - i\gamma_0 \right] \right\}.
\]

Let now \( \frac{1}{2} < \sigma^* \leq \sigma \leq 1 \). As in [3], let \( z = \sigma + it \), but now \( |t| \) sufficiently large. We apply the Borel-Carathéodory theorem to the function \( \log \zeta(z) \) and the circles with centre \( 2 + it \) and radius \( \frac{3}{2} - \frac{5}{2} \delta \) and \( \frac{3}{2} - \delta \), \( (0 < \delta < \frac{1}{2}) \).

On the larger circle

\[
\Re(\log \zeta(z)) = \log |\zeta(z)| < A \log t
\]
for a fixed positive constant \( A \). Hence on the smaller circle

\[
|\log \zeta(z)| \leq \frac{3 - 2\delta}{2\delta} A \log t + \frac{3 - \frac{3}{2} \delta}{2\delta} |\log |\zeta(2 + it)|| < A \delta^{-1} \log t.
\]

We now apply Hadamard’s three circle theorem as in [3]. The proof there can be taken over without change, to obtain

\[
\zeta(z) = O(t^\epsilon), \quad \text{for every } \sigma > \frac{1}{2},
\]
which is (14.2.5) of [3].

By the functional equation (1.4) we obtain

\[
\left| \zeta \left( \frac{1}{2} - \epsilon + it \right) \right| = O \left( |t|^{3\epsilon} \right).
\]

The claim (2.1) now follows from (2.2), (2.3) and the theorem of Phragmén-Lindelöf. \( \square \)

**Definition 2.2.** For \( \rho \) a non-trivial zero of \( \zeta(s) \) let

\[
R_N(\rho, s) := \text{Res}_{z=\rho} \frac{N^{z-s}}{\zeta(z)(z-s)^2}
\]

and

\[
F_s(z) := \pi z^s \sum_{n=1}^{\infty} (-1)^n \frac{(2\pi)^{2n+1} z^{2n}}{(2n)! \zeta(2n+1)(2n+s)^2}
\]
Lemma 2.3. If \( 0 < \text{Re}(s) < 1 \), then
\[
V_N(s) = \frac{1}{\zeta(s)} \left( 1 - \frac{1}{\log N} \zeta'(s) \right) + \frac{1}{\log N} \sum \rho R_N(\rho, s) + \frac{1}{\log N} F_s \left( \frac{1}{N} \right),
\]
where the sum is over all distinct non-trivial zeros of \( \zeta(s) \).

Proof. This is Lemma 2 of [3]. \( \square \)

Lemma 2.4. Let \( \epsilon > 0 \). Under the assumptions of Theorem 1.1 we have
\[
\sum_{\rho, \text{Re}(\rho) = \frac{1}{2}} R_N(\rho, s) \ll N^{3 \epsilon} |s|^\frac{3}{4} - \frac{5}{2} + \epsilon.
\]

Proof. The proof is identical to the proof of Lemma 3 of [3]. There the summation condition \( \text{Re}(s) = 1/2 \) is not needed, since the Riemann hypothesis is assumed. \( \square \)

Lemma 2.5.
\[
N^{\pm \epsilon} \sum_{\rho, |\rho| = \frac{1}{2}} R_N(\rho, s) \ll \sum_{|\rho - s| < \epsilon} \frac{1}{|\rho - s|^2} + 1.
\]

Proof. This is (5) of [3]. \( \square \)

Definition 2.6. We set
\[
\Sigma^{(1)}(N, s) := \frac{1}{\log N} \sum_{\rho: \text{Re}(\rho) = \frac{1}{2}} R_N(\rho, s)
\]
and
\[
\Sigma^{(2)}(N, s) := \frac{1}{\log N} \sum_{\rho \in \{\sigma_0 \pm i\gamma_0\}} R_N(\rho, s)
\]

3. Proof of Theorem 1.1

We closely follow the proof of Theorem 1 in [3]. We have
\[
\frac{1}{2 \pi} \int_{-\infty}^{\infty} \left| 1 - \zeta V_N \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{1 + t^2} = \frac{1}{2 \pi i} \int_{(\frac{1}{2})} (1 - \zeta V_N(s))(1 - \zeta V_N(1 - s)) \frac{ds}{s(1 - s)}
\]
\[
= \frac{1}{2 \pi i} \int_{(\frac{1}{2} - \epsilon)} (1 - \zeta V_N(s))(1 - \zeta V_N(1 - s)) \frac{ds}{s(1 - s)}
\]
By Lemma 2.3 and Definition 2.6 this is
\[
(3.1) \frac{1}{\log N} \frac{1}{2 \pi i} \int_{(\frac{1}{2} - \epsilon)} \left( \frac{\zeta'}{\zeta^2}(s) - \Sigma^{(1)}(N, s) - \Sigma^{(2)}(N, s) - F_s \left( \frac{1}{N} \right) \right) \times \left( \frac{\zeta'}{\zeta^2}(1 - s) - \Sigma^{(1)}(N, 1 - s) - \Sigma^{(2)}(N, 1 - s) - F_{1-s} \left( \frac{1}{N} \right) \right) \frac{\zeta(s)\zeta(1 - s)}{s(1 - s)} ds.
\]
We now expand the product in (3.1) and separately estimate the products that do not contain terms \( \Sigma^{(2)} \) and the products consisting of a term \( \Sigma^{(2)} \) and another term.
The asymptotic finally is obtained by asymptotically evaluating the products consisting only of factors $\Sigma^{(2)}$.

We closely follow [3]. It follows from Lemmas 2.1, 2.3, 2.4 that

$$\frac{1}{\log^2 N} \int \frac{1}{2\pi i} \left( \sum_{\rho_1, \rho_2} R_N(\rho_1, s)R_N(\rho_2, 1-s) \frac{\zeta(s)\zeta(1-s)}{s(1-s)} \right) ds$$

$$\ll \frac{1}{\log^2 N} \int \frac{1}{2\pi i} \left( \sum_{|\rho-s|<\epsilon} \frac{1}{|\zeta'(\rho)||\rho-s|^2 |s|^{-\frac{1}{2}+\frac{5}{2}\epsilon}} + O\left( \frac{1}{\log^2 N} \right) \right).$$

Now by Lemma 2.4 and the trivial estimate

$$F_s\left( \frac{1}{N} \right) = O(N^{-5/2}),$$

all the other terms in (3.1) not containing factors $\Sigma^{(2)}$ are trivially $O(1/\log^2 N)$ apart from

$$-\frac{1}{\log^2 N} \int \frac{1}{2\pi i} \left( (1-s)\Sigma^{(1)}(N, s) \frac{\zeta(s)\zeta(1-s)}{s(1-s)} \right) ds = \log N - \frac{1}{2} \frac{\zeta''}{\zeta'}(\rho) + \frac{\chi'}{\chi}(\rho) + \frac{1-2\rho}{|\rho|^2}$$

$$= \frac{\log N}{|\rho|^2} + O\left( \frac{1}{|\rho|^{2-\epsilon} |\zeta'(\rho)|} + \frac{1}{|\rho|^2} \right),$$

where we set

$$\chi(s) := \pi^{-s/2} \Gamma\left( \frac{s}{2} \right)$$

and use the bound

$$\zeta'' \left( \frac{1}{2} + it \right) \ll |t|^\epsilon,$$

which follows from Lemma 2.1 by the well-known estimate for the derivatives of a holomorphic function. By moving the line of integration to $\Re(s) = \frac{1}{2} + \epsilon$, we get that the contribution from the products not containing $\Sigma^{(2)}$ is

$$\frac{1}{\log N} \sum_{\rho: \Re(\rho)=\frac{1}{2}} \frac{1}{|\rho|^2} + O\left( \frac{1}{\log^2 N} \right).$$

We now come to the products that contain the factor $\Sigma^{(2)}(N, s)$.

They may be handled by adding the factor $N^{\sigma_0 - \frac{1}{2} + \epsilon}$ stemming from $N^{z-s}$ in Definition 2.2.

These estimates yield the error-term in Theorem 1.1. The main term finally is obtained by evaluating the contribution with two factors $\Sigma^{(2)}$ and by observing that the integral in (1.3) is real.

□

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