TOWARDS A SULLIVAN DICTIONARY IN DIMENSION TWO,
PART I: PURELY PARABOLIC COMPLEX KLEINIAN GROUPS

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Abstract. In this article we provide a full description of all the complex
Kleinian groups of $PSL(3, \mathbb{C})$ which contains only parabolic elements.

Introduction

In order to understand the structure of the limit set for groups of $PSL(3, \mathbb{C})$ it
is very important to understand the those subgroups of $PSL(3, \mathbb{C})$ without loxo-
dromic elements. The main purpose of this note is to provide a description of the
representation of such groups as well as a precise description of its limit set. More
precisely, in this article we show:

Theorem 0.1. Let $\Gamma_0 \subset PSL(3, \mathbb{C})$ be a complex Kleinian group without loxo-
dromic elements, then there exist a subgroup $\Gamma \subset \Gamma_0$ of finite index such that $\Gamma$
is conjugate to one of the following groups

1. The group:

$$W_\mu = \left\{ \begin{bmatrix} \mu(w) & \mu(w)w & 0 \\ 0 & \mu(w) & 0 \\ 0 & 0 & \mu(w)^{-2} \end{bmatrix} : w \in W \right\}.$$

where $W \subset \mathbb{C}$ is a discrete additive subgroup and $\mu : W \to (\mathbb{C}, +)$ is a group
morphism.

2. The group

$$W = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : (x, y) \in Span_\mathbb{Z}(W) \right\}.$$

where $W \subset \mathbb{C}^2$ is a set of $\mathbb{R}$-linearly independent points.

3. The group

$$WR_L = \left\{ \begin{bmatrix} 1 & x & L(x) + x^2/2 + w \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{R}, w \in W \right\}.$$

where $W \subset \mathbb{C}$ is an additive discrete subgroup, $R \subset \mathbb{C}$ is an additive group
and $L : R \to \mathbb{C}$ is an additive function, subject to the following conditions
(a) if $R$ is discrete, then $\text{rank}(W) + \text{rank}(R) \leq 4$, 

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(b) if $R$ is non-discrete, then $\text{rank}(W) \leq 1$, $\text{rank}(W) + \text{rank}(R) \leq 4$ and
\[
\lim_{n \to \infty} L(x^n) + w_n = \infty
\]
for every sequence $(w_n) \in W$ and any sequence $(x_n) \subset R$ converging to $0$.

(4) The group
\[
W^* = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (x, y) \in W \right\}
\]
where $W \subset \mathbb{C}^2$ is an additive discrete subgroup of rank at most two.

(5) $\Gamma_w = \left\{ \begin{bmatrix} 1 & k + lc + mx & ld + m(k + lc) + \left(\frac{m}{2}\right)x + my \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} : (k, l, m) \in \mathbb{Z} \right\}$
where $x, y \in \mathbb{C}$, $p, q, r \in \mathbb{Z}$ such that $p, q$ are co-primes, $q^2$ divides $r$ and $\Gamma$ is conjugated to $\Gamma = (x, y, p, q, r)$, $c = pq^{-1}$ and $d = r^{-1}$.

(6) $W_{a,b,c} = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{n} \begin{bmatrix} 1 & a + c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{m} : m, n \in \mathbb{Z}, w \in W \right\}$
here $W \subset \mathbb{C}$ is a discrete additive subgroup and $a, b, c \in \mathbb{C}$ are subject to one of the following conditions:
(a) $\{1, c\}$ is a $\mathbb{R}$-linearly dependent set but is a $\mathbb{Z}$-linearly independent set and $a \in W \setminus \{0\}$,
(b) $\{1, c\}$ is a $\mathbb{R}$-linearly independent set and $a \in W \setminus \{0\}$.

As we will see in the Section 3 all the possibilities described in the theorem are attained.

**Corollary 0.2.** Let $\Gamma \subset PSL(3, \mathbb{C})$ be a complex Kleinian group without parabolic elements, then

1. The equicontinuity set of $\Gamma$ coincides with its Kulkarni region of discontinuity i.e. $\text{Eq}(\Gamma) = \Omega_{Kul}(\Gamma)$, and is the largest open set on which the group $\Gamma$ acts properly discontinuously.
2. $\Lambda_{CG}(\Gamma^*)$ is either a point or an euclidean circle.

The paper is organized as follows: in Section 1 we review some general facts and introduce the notation used along the text. In Section 2 we provide a version of the $\lambda$- lemma which will be key to describe the dynamic of compact sets and in consequence get a description of the limit set. In Section 3 we provide a collection of example which depicts all the possible ways to construct purely parabolic Kleinian groups as well as their dynamic and algebraic properties. In Section 4 we show that every kleinian group without loxodromic elements is virtually triangularizable. Section 5 is devoted to the study of the problem of combining two parabolic elements cheating an invariant full flag with the additional condition that one is a screw parabolic element. In sections 6 and 7 we provide the full list of
possible representations for triangular groups without parabolic elements. Finally in Section 5 we provide a proof of the main theorem as well as its consequences.

1. Notations and Background

Through this section we are going to establish the terminology used along the text as well as some background on complex Kleinian groups, the intention of this is to make an self contained article as possible.

1.1. Projective Geometry. We recall that the complex projective plane $\mathbb{P}_C^2$ is

$$\mathbb{P}_C^2 := (\mathbb{C}^3 \setminus \{0\}) / \mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^3 \setminus \{0\}$ by the usual scalar multiplication. Let $[\ ] : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}_C^2$ be the quotient map. A set $\ell \subset \mathbb{P}_C^2$ is said to be a complex line if $[\ell]^{-1} \cup \{0\}$ is a complex linear subspace of dimension 2. Given $p, q \in \mathbb{P}_C^2$ distinct points, there is a unique complex line passing through $p$ and $q$, such line will be denoted by $\rightarrow \leftarrow p, q$.

The set of all the complex lines contained in $\mathbb{P}_C^2$, denoted $\text{Gr}(\mathbb{P}_C^2)$, endowed with the topology of the Hausdorff convergence, actually is diffeomorphic $\mathbb{P}_C^2$, given by duality: A point $x \in \mathbb{P}_C^2$ represents a line $\ell_x$ through the origin in $\mathbb{C}^3$; let $P_x$ be the complex orthogonal complement of $\ell_x$ in $\mathbb{C}^3$, then $[P_x \setminus \{0\}]$ is a line in $\mathbb{P}_C^2$. This identifies $\mathbb{P}_C^2$ with its dual $\mathbb{P}_C^2 \cong \text{Gr}(\mathbb{P}_C^2)$.

Consider the action of $\mathbb{Z}_3$ (viewed as the cubic roots of the unity) on $\text{SL}(3, \mathbb{C})$ given by the usual scalar multiplication. Then

$$\text{PSL}(3, \mathbb{C}) = \text{SL}(3, \mathbb{C}) / \mathbb{Z}_3,$$

is a Lie group whose elements are called projective transformations. Let $[[\ ] : \text{SL}(3, \mathbb{C}) \to \text{PSL}(3, \mathbb{C})$ be the quotient map, $\gamma \in \text{PSL}(3, \mathbb{C})$ and $\tilde{\gamma} \in \text{GL}(3, \mathbb{C})$. We say that $\tilde{\gamma}$ is a lift of $\gamma$ if there is a cubic root $\tau$ of $\text{Det}(\gamma)$ such that $[[\tau \tilde{\gamma}]] = \gamma$. We use the notation $(\gamma_{ij})$ to denote elements in $\text{SL}(3, \mathbb{C})$. One can show that $\text{PSL}(3, \mathbb{C})$ acts transitively, effectively and by biholomorphisms on $\mathbb{P}_C^2$ by $[[\gamma]](w) = [\gamma(w)]$, where $w \in \mathbb{C}^3 \setminus \{0\}$ and $\gamma \in \text{PSL}(3, \mathbb{C})$.

If $\gamma$ is an element in $\text{PSL}(3, \mathbb{C})$ and $\tilde{\gamma}$ is a lifting of $\gamma$ to $\text{SL}(3, \mathbb{C})$, following, see [5], we say that:

- $\gamma$ is elliptic if $\tilde{\gamma}$ is diagonalizable with unitary eigenvalues.
- $\gamma$ is parabolic if $\tilde{\gamma}$ is non-diagonalizable with unitary eigenvalues.
- $\gamma$ is loxodromic if $\tilde{\gamma}$ has some non-unitary eigenvalue.

There are two types of parabolic, depending on their Jordan form: Unipotent and ellipto-parabolic (see [5, Chapter 4]). By [5], up to conjugation, all parabolic elements have the following normal form; the first of these is for the unipotent elements while the latter is for the ellipto-parabolic elements:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix} \quad \text{with} \ |\lambda| = 1.$$

It is clear that given an upper triangular matrix $\gamma$ whose coefficients in the diagonal are unitary complex numbers induce either a Parabolic or a elliptic transformation in $\text{PSL}(3, \mathbb{C})$. 

Now let $M(3, \mathbb{C})$ be the group of all $3 \times 3$ matrices with complex coefficients. Define the space of pseudo-projective maps by:

$$SP(3, \mathbb{C}) = (M(3, \mathbb{C}) \setminus \{0\}) / \mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $M(3, \mathbb{C}) \setminus \{0\}$ by the usual scalar multiplication. In the sequel $[[\cdot]] : M(3, \mathbb{C}) \setminus \{0\} \to SP(3, \mathbb{C})$ will denote the quotient map and we equip $SP(3, \mathbb{C})$ with the quotient topology. Finally, given $\gamma \in SP(3, \mathbb{C})$ we define its kernel by:

$$\Ker(\gamma) = [\Ker(\tilde{\gamma}) \setminus \{0\}],$$

where $\tilde{\gamma} \in M(3, \mathbb{C})$ is a lift of $\gamma$. Clearly $PSL(3, \mathbb{C}) \subset SP(3, \mathbb{C})$ and $\gamma \in PSL(3, \mathbb{C})$ if and only if $\Ker(\gamma) = \emptyset$. Notice that $SP(3, \mathbb{C})$ is a manifold naturally diffeomorphic to $P^8_{\mathbb{C}}$ so it is compact. This maps provide us the following useful results, see [5]:

**Theorem 1.1** (See [5]). Let $\Gamma \subset PSL(3, \mathbb{C})$ be a discrete group, then the largest open set on which $\Gamma$ forms a normal family, aka the equicontinuity set, is given by:

$$Eq(\Gamma) = P^8_{\mathbb{C}} \setminus \{\Ker(\tau) \mid \tau \in PSL(3, \mathbb{C}) \setminus PSL(3, \mathbb{C}) : \exists(\tau_m) \subset \Gamma \text{ satisfying } \tau_m \xrightarrow{m \to \infty} \tau\}.$$

1.2. Kleinian groups of $PSL(3, \mathbb{C})$. The intention of this section is to provide a quick review of the result on 2-dimensional Kleinian groups which we are going to use along the text. Let’s start with an standard definition.

**Definition 1.2.** Let $\Gamma \subset PSL(3, \mathbb{C})$ be a discrete group. An open non-empty set $\Omega \subset P^2_{\mathbb{C}}$ is said to be a region where $\Gamma$ acts properly discontinuously if $\Gamma \Omega = \Omega$ and for each compact set $K \subset \Omega$ the set $\{\gamma \in \Gamma \mid \gamma(K) \cap K\}$ is finite.

**Example 1.3.** Let $\Gamma$ be a discrete group, then $\Gamma$ acts properly discontinuously on $Eq(\Gamma)$.

**Example 1.4.** Let $\Gamma \subset Sol^4_\mathbb{C}$, be the fundamental group of an Inoue Surface, then $Eq(\Gamma) = \emptyset$, but there is a non-empty open set on which the group acts properly discontinuously.

The following definition provide us another way to define an open set where the group acts properly discontinuously.

**Definition 1.5.** Given $\Gamma \subset PSL(3, \mathbb{C})$ a discrete group we define its Kulkarni limit set as follows (see [5]):

1. Let $L_0(\Gamma)$ be the closure of the points in $P^2_{\mathbb{C}}$ with infinite isotropy group.
2. Let $L_1(\Gamma)$ be the closure of the set of cluster points of $\Gamma z$ where $z$ runs over $P^2_{\mathbb{C}} \setminus L_0(\Gamma)$.
3. Let $L_2(\Gamma)$ be the closure of the set of cluster points of $\Gamma K$ where $K$ runs over all the compact sets in $P^2_{\mathbb{C}} \setminus (L_0(\Gamma) \cup L_1(\Gamma))$.
4. The Kulkarni limit Set for $\Gamma$ is defined as:

$$\Lambda_{Kul}(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma).$$

5. The Kulkarni ordinary set of $\Gamma$ is defined as:

$$\Omega_{Kul}(\Gamma) = P^2_{\mathbb{C}} \setminus \Lambda_{Kul}(\Gamma).$$

We have the following properties:

**Proposition 1.6** (See [5]). Let $\Gamma$ be a complex Kleinian group. Then:

1. The sets $\Lambda_{Kul}(\Gamma)$, $L_0(\Gamma)$, $L_1(\Gamma)$, $L_2(\Gamma)$ are $\Gamma$-invariant closed sets.
2. The group $\Gamma$ acts properly discontinuously on $\Omega_{Kul}(\Gamma)$. 

(3) Let $C \subset \mathbb{P}_C^n$ be a closed $\Gamma$-invariant set such that for every compact set $K \subset \mathbb{P}_C^n \setminus C$, the set of cluster points of $\Gamma K$ is contained in $\Lambda(\Gamma) \cap C$, then $\Lambda_{Kul}(\Gamma) \subset C$.

(4) The equicontinuity set of $\Gamma$ is contained in $\Omega_{Kul}(\Gamma)$.

(5) If $\Gamma_0 \subset \Gamma$ a subgroup with finite index, then $\Lambda_{Kul}(\Gamma) = \Lambda_{Kul}(\Gamma_0)$.

We now introduce another class of groups, we call these weakly semi-controllable groups. Consider $\Gamma \subset \text{PSL}(3, \mathbb{C})$ a subgroup which acts on $\mathbb{P}^2_C$ with a point $p$ which is fixed by $\Gamma$. Choose an arbitrary line $\ell \in \mathbb{P}^2_C \setminus \{p\}$, and notice that we have a canonical projection

$$
\pi = \pi_{p,\ell} : \mathbb{P}^2_C \setminus \{p\} \to \ell
$$

$$
\pi(x) = \frac{1}{\ell \cdot x} \cap \ell
$$

It is clear that this map is holomorphic and it allows us to define a group homomorphism:

$$
\Pi = \Pi_{p,\ell,\Gamma} : \Gamma \to \text{Bihol}(\ell) \cong \text{PSL}(2, \mathbb{C})
$$

$$
\Pi(\gamma)(x) = \pi(\gamma(x))
$$

If we choose another line, say $\ell'$, one gets similarly a projection $\pi' = \pi_{p,\ell'}$ and a group homomorphism $\Pi' = \Pi_{p,\ell',\Gamma}$. It is an exercise to see that $\Pi$ and $\Pi'$ are equivalent in the sense that there is a biholomorphism $h : \ell \to \ell'$ inducing an automorphism $H$ of $\text{PSL}(2, \mathbb{C})$ such that $H \circ \Pi = \Pi'$. As before, the line $\ell$ is called the horizon and $p$ the vanishing point. This leads to the following definition:

**Definition 1.7.** Let $\Gamma \subset \text{PSL}(3, \mathbb{C})$ be a discrete group. We say that $\Gamma$ is weakly semi-controllable if it acts with a fixed point in $\mathbb{P}^2_C$. In this case a choice of a horizon $\ell$ determines a control group $\Pi(\Gamma) \subset \text{PSL}(2, \mathbb{C})$, which is well defined and independent of $\ell$ up to an automorphism of $\text{PSL}(2, \mathbb{C})$.

It is clear that all suspension and all controllable groups are weakly semi-controllable, but this latter class is larger, to see this will be enough to consider the fundamental groups of Inoue surfaces. The following result about weakly semi-controllable subgroups will be useful along this section, see [5]

**Theorem 1.8.** Let $\Gamma \subset \text{PSL}(3, \mathbb{C})$ be a discrete weakly semi-controllable subgroups, with $p \in \mathbb{P}^1_C$ a $\Gamma$-invariant point and $\ell \subset \mathbb{P}^2_C$ be a complex line not containing $p$. Let $\Pi_{p,\ell} = \Pi$ is defined as in [5], if $\text{Ker}(\Pi)$ is finite and $\Pi(\Gamma)$ discrete, then $\Gamma$ acts properly discontinuously on

$$
\Omega = \left( \bigcup_{z \in \overline{\Pi(\tau)}} \mathbb{P}_C \right) \setminus \{p\}
$$

2. The $\lambda$-lemma

**Lemma 2.1.** Let $(\gamma_n) \subset \Gamma$ be a sequence of distinct elements, then there exist a subsequence $(\tau_n) \subset (\gamma_n)$ and $\alpha, \beta \in \text{QP}(3, \mathbb{C})$ satisfying:

1. We have $\tau_n \xrightarrow{m \to \infty} \alpha$ and $\tau_n^{-1} \xrightarrow{m \to \infty} \beta$. 
(2) We have the following relations:

\[ \text{Im}(\alpha) \subset \text{Ker}(\beta) \]
\[ \text{Im}(\beta) \subset \text{Ker}(\alpha) \]
\[ \dim(\text{Im}(\alpha)) + \dim(\text{Ker}(\alpha)) = 1 \]
\[ \dim(\text{Im}(\beta)) + \dim(\text{Ker}(\beta)) = 1 \]

(3) For every point \( x \in \text{Ker}(\alpha) \) we get

\[ \text{Ker}(\beta) = \bigcup_{x_n \to x} \{ \text{accumulation points of } (\gamma_n(x_n)) \} \]

(4) If \( \Omega \subset P^2 \) is an open set on which \( \Gamma \) acts properly discontinuously, then either \( \ker(\alpha) \subset P^2 \setminus \Omega \) or \( \ker(\beta) \subset P^2 \setminus \Omega \).

Proof. Let us show part (1). By the Singular Value decomposition theorem there are sequences \((\alpha_m), (\beta_m), (\gamma_m) \in \mathbb{R}, (\kappa_m), (\tilde{\kappa}_m) \in U(n+1)\), such that

\[ \gamma = \begin{bmatrix} \kappa_m \begin{pmatrix} e^{\alpha_m} & e^{\beta_m} & e^{\gamma_m} \end{pmatrix} \tilde{\kappa}_m \end{bmatrix} \]

After taking a subsequence, if necessary, we can assume that

\[ \alpha_m \geq \beta_m \geq \gamma_m \]
\[ \kappa_m \xrightarrow{\to\infty} \kappa_1 \in U(3) \]
\[ \tilde{\kappa}_m \xrightarrow{\to\infty} \kappa_2 \in U(3) \]
\[ \alpha_m \xrightarrow{\to-\infty} \infty \]
\[ \beta_m - \alpha_m \xrightarrow{\to\infty} a \in [-\infty, 0] \]
\[ \gamma_m - \alpha_m \xrightarrow{\to\infty} -\infty \]
\[ \gamma_m - \beta_m \xrightarrow{\to\infty} b \in [-\infty, 0] \]

Clearly \( a, b \) can not be both finite. Now, Equation (2.1) shows that

\[ \gamma_m \xrightarrow{\to\infty} \tau = \begin{bmatrix} \kappa_1 \begin{pmatrix} 1 & e^a & 0 \end{pmatrix} \kappa_2 \end{bmatrix} \]

(2.2)

\[ \gamma_m^{-1} \xrightarrow{\to\infty} \vartheta = \begin{bmatrix} \kappa_2^{-1} \begin{pmatrix} 0 & e^b & 1 \end{pmatrix} \kappa_1^{-1} \end{bmatrix} \]

and it shows part (1).

Let us show part (2). Observe that equation (2.2) yields

\[ \text{Im}(\tau) = \begin{cases} \kappa_1(e_1) & \text{if } a = -\infty \\ \kappa_1(e_1, e_2) & \text{if } a \neq -\infty \end{cases} \]
\[ \text{Im}(\vartheta) = \begin{cases} \kappa_2^{-1}(e_3) & \text{if } b = -\infty \\ \kappa_2^{-1}(\tilde{e}_3, e_2) & \text{if } b \neq -\infty \end{cases} \]
\[ \text{Ker}(\tau) = \begin{cases} \kappa_2^{-1}(\tilde{e}_2, e_3) & \text{if } a = -\infty \\ \kappa_2^{-1}(e_4) & \text{if } a \neq -\infty \end{cases} \]
\[ \text{Ker}(\vartheta) = \begin{cases} \kappa_1(e_1, e_2) & \text{if } b = -\infty \\ \kappa_1(e_1) & \text{if } b \neq -\infty \end{cases} \]
which shows part (2).

Let us show part (3). Let \( w \in \text{Ker}(\alpha) \setminus \text{Im}(\beta) \), in this case \( a = b = -\infty \), now let \((x_m) \subset \mathbb{P}_2^\mathbb{C}\) such that \( x_m \rightarrow \infty \rightarrow w \), then \( \kappa_2(x_m) \rightarrow \kappa_2(w) \), where \( \kappa_2(w) \in \vec{e}_2, e_3 \setminus \{e_3\} \), \( \kappa_2(x_m) = [a_m, b_m, c_m] \), \( \kappa_2(w) = [0, b, c] \), \( |b| \neq 0 \), \( a_m \rightarrow \infty \rightarrow 0 \), \( b_m \rightarrow \infty \rightarrow b \), \( c_m \rightarrow \infty \rightarrow c \). Define \( w_m = [e^{\alpha_m - \beta_m}a_m, b_m, e^{\gamma_m}c_m] \) thus the accumulation points of \((w_m)\) lie on the line \( \vec{e}_1, e_2 \), in consequence the accumulation points of \((\gamma_m(x_m))\) lie on the line \( \kappa_1(\vec{e}_1, e_2) = \text{Ker}(\beta) \), which concludes this part of proof.

To conclude observe that part (4) is now trivial. \( \square \)

3. Examples of groups containing only parabolic elements

Through this section we are going provide examples of such s of groups containing only parabolic elements, later we will prove that these are essentially the “only” examples in \( PSL(3, \mathbb{C}) \), moreover contrary to the one dimensional case we will see these groups might not be abelian and its limit set is not “trivial” in some cases. The following is the simplest example of a purely parabolic group.

**Example 3.1.** The cyclic groups generated by parabolic elements. It is well known, see [5], that this kind of groups are discrete subgroups satisfying that the equicontinuity set agrees with the kulkarni’s discontinuity region, moreover the Kulkarni’s discontinuity regions is the largest open set where the group acts properly discontinuously and the limit set is a complex line.

Our second example is also quite simple and its properties can be deduced by means of Theorem 1.8.

**Example 3.2.** Let \( W \subset \mathbb{C}^2 \) be an \( \mathbb{R} \)-linearly independent set of points and \( \tilde{\eta} : \text{Span}_\mathbb{Z}(W) \rightarrow (S^1, \cdot) \) a group morphism, define

\[
\eta = \left\{ \begin{bmatrix} \eta^{-2}(w) & 0 & 0 \\ 0 & \eta(w) & \eta(w)w \\ 0 & 0 & \eta(w) \end{bmatrix} : w \in \text{Span}_\mathbb{Z}(W) \right\}
\]

Clearly the group induced \( \eta \), satisfies:

(1) The group \( \eta \) is a discrete weakly semi-controllable whose kernel is trivial.
(2) The control group of \( \eta \) is isomorphic to \( \text{Span}_\mathbb{Z}(W) \).
(3) The Kukarni’s discontinuity set of \( \eta \) is the largest open set on which \( \eta \) acts properly and \( \text{Eq}(\eta) = \Omega_{Ku}(\eta) = \mathbb{P}^2 \setminus \vec{e}_1, \vec{e}_2 \).

**Example 3.3.** Let \( W \subset \mathbb{C}^2 \) be an \( \mathbb{R} \)-linearly independent set of points. Let us define

\[
W = \left\{ \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : (x, y) \in \text{Span}_\mathbb{Z}(W) \right\}
\]

then

(1) The group \( W \) is weakly semi-controllable.
(2) The control group of \( W \) is given by \( \pi_2(\text{Span}_\mathbb{Z}(W)) \).
(3) The Kulkarni’s discontinuity set of $W$ is the largest open set on which $W$ acts properly and satisfies

$$\Omega_{Kul}(W) = Eq(W) = \mathbb{P}^2 \setminus \ell_1, \ell_2.$$ 

Proof. Clearly the group $W$ is a weakly semi-controllable, discrete, its Kernel is trivial and the control group is non-discrete and given by $\pi_2(Span_{\mathbb{Z}}(W))$. Since $Eq(W) \subset \Omega_{Kul}(W)$ and $\ell_1, \ell_2 \subset L_0(W)$, to conclude the proof will be enough to show $Eq(W) = \mathbb{P}^2 \setminus \ell_1, \ell_2$. Let $(\gamma_m)_{m \in \mathbb{N}}$ be a sequence of distinct elements, then there is sequence $w_m = (x_m, y_m) \in Span_{\mathbb{Z}}(W)$ of distinct elements such that $\gamma_m$ has a lift $\tilde{\gamma}_m \in SL(3, \mathbb{C})$ given by

$$\tilde{\gamma}_m = \begin{pmatrix} 1 & 0 & x_m \\ 0 & 1 & y_m \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Then $w_m \xrightarrow{m \to \infty} \infty$, that is $r_m = \max\{|x_m|, |y_m|\} \xrightarrow{m \to \infty} \infty$. So we can assume that there is $w = (x, y) \in \mathbb{C}^2 \setminus \{0\}$ such that $r_m^{-1} w_m \xrightarrow{m \to \infty} w$, then

$$\gamma_m \xrightarrow{m \to \infty} \gamma = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

in the sense of pseudo projective transformations. Clearly $Ker(\gamma) = \ell_1, \ell_2$, which concludes the proof. \hfill \Box

Now, let us consider the “dual” example of our previous example and see how changes the Klulkarni’s limit set. But before we present the example a technical lemma will be needed.

**Lemma 3.4.** Let $W \subset \mathbb{C}^2$ be a non-empty and $\mathbb{R}$-linearly independent set, then $\ell = [\text{Span}_{\mathbb{Z}}(W) \setminus \{0\}]$ is:

1. if $W$ has exactly one point, then $\ell$ is a single point.
2. if $W$ contains exactly two points which are $\mathbb{C}$-linearly dependent, then $\ell$ is a single point.
3. if $W$ contains exactly two points which are $\mathbb{C}$-linearly independent, then $\ell$ is a real line in $\mathbb{P}^1_{\mathbb{C}}$.
4. if $W$ contains more than two points then $\ell = \mathbb{P}^1_{\mathbb{C}}$.

Proof. If $W$ contains a single point, then trivially $\ell$ is a single point. If $W$ contains two points which are $\mathbb{C}$-linearly dependent then $\ell$ is trivially a single point. If $W$ contains two points which are $\mathbb{C}$-linearly independent then we can assume that $W = \{(0, 1), (1, 0)\}$, therefore $\ell = \{[1, m/n] : m, n \in \mathbb{Z}\}$ which is a line in $\mathbb{P}^1_{\mathbb{C}}$. Finally, if $W$ contains at least 3 points, then we can pick up two elements in $W$ which are $\mathbb{C}$-linearly independent, then we can assume that $(1, 0)$ and $(0, 1)$ are in $W$, let $p = (w_1, w_2)$ be the other point in $W$, then

$$\ell = \{[k + nw_1 : l + nw_2] : k, l, n \in \mathbb{Z}\} = \{[r + sw_1 : t + sw_2] : r, s, t \in \mathbb{R}\}.$$ 

Let $z \in \mathbb{C}$ such that $\text{Im}(z) \neq 0$ and define

$$s_0 = 1, \ r_0 = \frac{\text{Im}(w_2) - \text{Re}(z) \text{Im}(w_1)}{\text{Im}(z)} - \text{Re}(w_1), \ t_0 = z(r + w_1) - w_2.$$ 

Then a straightforward calculation shows that $[1 : z] = [r_0 + s_0w_1 : t_0 + s_0w_2]$, which concludes the proof. \hfill \Box
Example 3.5. Let $W \subset \mathbb{C}^2$ be a $\mathbb{R}$-linearly independent set with at most two points. Set

$$W^* = \left\{ \gamma(x,y) = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (x,y) \in \text{Span}_\mathbb{Z}(W) \right\}.$$ 

we have:

1. The group $W^*$ is discrete and weakly semi-controllable.
2. The control group of $W^*$ is trivial.
3. For each $w = (x, y) \in \text{Span}(W) \setminus \{0\}$, one has:

$$\text{Fix}([\gamma_w]) = \begin{pmatrix} \hat{e}_1, [0 : y : -x] \end{pmatrix}.$$ 

4. The Kukarni’s discontinuity set of $W^*$ is the largest open set on which $W^*$ acts properly and satisfies

$$\Lambda_K(W^*) = \mathbb{P}^2 \setminus E_0(W^*) = \bigcup_{p \in W^*} e_1, p.$$ 

Here $W^* = \overline{\text{Span}_\mathbb{Z}\{(y, -x) : (x, y) \in W \setminus \{0\}\}}$. In particular, Lemma 3.3 is saying us that $W^*$ is Kleinain if and only $\Lambda_K(W^*)$ is either a line or a pencil of lines over a circle.

Proof. Clearly the group $W^*$ is a weakly semi-controllable, discrete and the control group trivial. Let $w = (x, y) \in W$ then

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & y \\ -x \end{pmatrix} = \begin{pmatrix} 0 & y \\ -x \end{pmatrix}.$$ 

Therefore $\text{Fix}([\gamma_w]) = e_1, [0 : y : -x]$. Finally, since $E_0(W^*) \subset \Omega_K(W^*)$ and

$$\bigcup_{p \in W^*} e_1, p \subset L_0(W^*),$$

we need to show $\mathbb{P}^2 \setminus E_0(W^*) = \mathbb{P}^2 \setminus \bigcup_{p \in W^*} e_1, p$. Let $(\gamma_m)_{m \in \mathbb{N}}$ be a sequence of distinct elements, then there is sequence $w_m = (x_m, y_m) \in \text{Span}_\mathbb{Z}(W)$ of distinct elements such that $\gamma_m$ has a lift $\tilde{\gamma}_m \in SL(3, \mathbb{C})$ given by

$$\tilde{\gamma}_m = \begin{pmatrix} 1 & x_m & y_m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

therefore $w_m \xrightarrow{m \to \infty} \infty$, in consequence $r_m = \max\{|x_m|, |y_m|\} \xrightarrow{m \to \infty} \infty$. So we can assume there is $w = (x, y) \in \mathbb{C}^2 \setminus \{0\}$ satisfying $r_m^{-1} w_m \xrightarrow{m \to \infty} w$, therefore $p = [y, -x] \in W^t$ and

$$\gamma_m \xrightarrow{m \to \infty} \gamma = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in the sense of pseudo projective transformations. Clearly $\text{Ker}(\gamma) = \tilde{e}_1, \tilde{p}$, which concludes the proof. $\square$
Example 3.6. Let $W \subset \mathbb{C}$ is a discrete additive subgroup with rank 2 and $x, b, c \in \mathbb{C}$ such that $1 \in W$ and $c \notin \mathbb{R}$, consider the group define by:

$$W_{x,b,c} = \left\{ \left[ \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right], \left[ \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right], \left[ \begin{array}{cc} 1 & c \\ 0 & 1 \end{array} \right] : m \in \mathbb{Z}, (x, y) \in W \right\}$$

Then $W_{x,b,c}$ is a purely parabolic group, weakly semi controllable, discrete and a non-abelian group which is a semi direct product of the form $\mathbb{Z}^{\text{rank}(W) + 1} \times \mathbb{Z}$.

Moreover

1. The kernel of $W_{x,b,c}$ is isomorphic to $\mathbb{Z}^{	ext{Card}(W)}$ and its control group is discrete and isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
2. The discontinuity region in the sense of kulkarni for $W_{x,b,c}$ is the largest open set on which the group acts properly discontinuously and agrees with the equicontinuity set. Moreover, the Kulkarni’s limit set is given by

$$\Lambda_{Kul}(W_{x,b,c}) = \hat{e}_1, e_2$$

Example 3.7. Let $x, y \in \mathbb{C}$ and $p, q, r \in \mathbb{Z}$ such that $p, q$ are co-primes and $q^2$ divides $r$, define

$$\Gamma_w = \left\{ \left[ \begin{array}{ccc} 1 & k + lc + mx & ld + m(k + lc) + \left( \frac{m}{2} \right) x + my \\ 0 & 1 & m \\ 0 & 0 & 1 \end{array} \right] : k, l, m \in \mathbb{Z} \right\}$$

where $w = (x, y, p, q, r)$, $c = pq^{-1}$ and $d = r^{-1}$. Then $\Gamma_w$ is a purely parabolic group, weakly semi controllable, discrete and a non-abelian group which is a semi direct product of the form $\mathbb{Z}^2 \times \mathbb{Z}$. Moreover

1. $\Gamma_w$ is a discrete group, weakly semi-contrrollable, has infinite kernel and its control group is given by $\{ z + n : n \in \mathbb{Z} \}$.
2. The Kulkarni’s discontinuity set of $\Gamma_w$ is the largest open set on which $\Gamma_w$ acts properly and satisfies

$$\Lambda_{Kul}(\Gamma_w) = \mathbb{P}^2 \setminus \text{Eq}(\Gamma_w) = \hat{e}_1, e_2 \cup \bigcup_{r \in \mathbb{R}} \hat{e}_1, [0 : 1 : r].$$

Proof: Clearly $\Gamma_w$ is a weakly semi-controllable, discrete, has infinite kernel and the control group is given by $\{ z + n : n \in \mathbb{Z} \}$. Let $k, l \in \mathbb{Z}$ then a straightforward calculation shows $\gamma_{x,y,0}[0 : -ld : k + lc] = [0 : -ld : k + lc]$, in consequence

$$\hat{e}_1, e_2 \cup \bigcup_{r \in \mathbb{R}} \hat{e}_1, [0 : 1 : r] \subset L_0(\Gamma_w).$$

So to conclude the proof will be enough to show

$$\mathbb{P}^2 \setminus \text{Eq}(\Gamma_w) = \hat{e}_1, e_2 \cup \bigcup_{r \in \mathbb{R}} \hat{e}_1, [0 : 1 : r].$$

Let $(\gamma_m)_{m \in \mathbb{N}} \subset \Gamma_w$ be a sequence of distinct elements, then there is sequence $u_m = (k_m, l_m, n_m) \in \mathbb{Z}^3$ of distinct elements such that

$$\gamma_m = \left[ \begin{array}{ccc} 1 & k_m + l_mc + n_mx & ld + n_m(k_m + l_mc) + \left( \frac{n_m}{2} \right) x + n_my \\ 0 & 1 & n_m \\ 0 & 0 & 1 \end{array} \right].$$
Since $\Gamma_w$ is discrete we get $r_m = \max\{|k_m|, |l_m|, |n_m|\} \xrightarrow{m \to \infty} \infty$. Now we can assume that there is $u = (x, y, z) \in \mathbb{R}^3 \setminus \{0\}$ such that $r_m^{-1}u_m \xrightarrow{m \to \infty} u$, thus

$$
\gamma_m \xrightarrow{m \to \infty} \gamma = \begin{bmatrix}
0 & k_0 + l_0c + n_0x & l_0d + n_0(k_0 + l_0c) + \left(\frac{n_0}{2}\right)x + n_0y \\
0 & 0 & n_0 \\
0 & 0 & 0
\end{bmatrix},
$$

as pseudo projective transformations. Clearly

$$
\text{Ker}(\gamma) = \begin{cases}
\hat{e}_1, e_2 & \text{if } k_0 + l_0c + n_0x = 0 \\
\hat{e}_1, [0 : -l_0d : k_0 + l_0c] & \text{if } k_0 + l_0c \neq 0, n_0 = 0 \\
e_1 & \text{in other case}
\end{cases}
$$

Which concludes the proof. □

**Example 3.8.** Let $W = \{1, \sqrt{2}, \sqrt{i}, \sqrt{2i}\}$ and $L : W \to \mathbb{C}$ given by $\kappa(1) = 2^{-1}$, $\kappa(\sqrt{2}) = \sqrt{2} - 1$, $\kappa(\sqrt{i}) = i + 2^{-1}$, $\kappa(\sqrt{2i}) = \sqrt{2i} + 1$. If $L : \text{Span}_\mathbb{Z}(W) \to \mathbb{C}$ is the additive extension of $L$, then

$$L = \left\{ \begin{bmatrix} 1 & x & L(x) + x^22^{-1} \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x \in \text{Span}_\mathbb{Z}(W) \right\}
$$

is an abelian discrete group of $SL(3, \mathbb{C})$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Moreover, one has:

1. The group $L$ is a purely parabolic weakly semi controllable group of $PSL(3, \mathbb{C})$.
2. The kernel of $L$ is trivial and its control group is
   $\{z + k + l\sqrt{2} + m\sqrt{i} + n\sqrt{2i} : k, l, m, n \in \mathbb{Z}\}$
   which is a dense subgroup of $\{z + b : b \in \mathbb{C}\}$.
3. The Kulkarni’s limit set is given by
   $$\Lambda_{kul}(L) = \hat{e}_1, e_2$$
4. The discontinuity region in the sense of kulkarni for $L$ is the largest open set on which the group acts properly discontinuously and agrees with the equicontinuity set.

**Proof.** A straightforward calculation shows $\hat{k}$ is an abelian group isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Now, let us assume that $\hat{k}$ is non-discrete, then there is a sequence $(x_n, x_n)_{n \in \mathbb{N}} \subset \text{Span}_\mathbb{Z}(W)$ such that $x_n \xrightarrow{m \to \infty} 0$ and $\tilde{k}(x_n, x_n) \xrightarrow{m \to \infty} 0$. Since $x_n(1, 1) \in \text{Span}_\mathbb{Z}(W)$, there are $k_n, l_n, m_n, \tilde{n}_n \in \mathbb{Z}$ such that

$$x_n = k_n + l_n\sqrt{2} + m_n\sqrt{i} + \tilde{n}_n\sqrt{2i}.$$ 

Therefore $k_n + l_n\sqrt{2} \xrightarrow{m \to \infty} 0$ and $m_n + \tilde{n}_n\sqrt{2} \xrightarrow{m \to \infty} 0$. Thus

$$|k_n|, |l_n|, |m_n|, |\tilde{n}_n| \xrightarrow{m \to \infty} \infty.$$

On the other hand, observe the following three facts

$$
\tilde{k}(x_n, x_n) = 2^{-1} \left((k_n + \sqrt{2}l_n + 1)^2 - (k_n + 2l_n + 1)\right) + 2^{-1}i \left((m_n + \sqrt{2}\tilde{n}_n + 1)^2 - (m_n + 2\tilde{n}_n + 1)\right) + \sqrt{i}(k_n + l_n\sqrt{2})(m_n + \tilde{n}_n\sqrt{2})
$$

$$2\text{Re}(\tilde{k}(x_n, x_n)) = (k_n + \sqrt{2}l_n + 1)^2 - (k_n + 2l_n + 1) + 2(k_n + l_n\sqrt{2})(m_n + \tilde{n}_n\sqrt{2})\text{Re}(\sqrt{i}).$$
|k_n + 2l_n| = |k_n + \sqrt{2}l_n + (2 - \sqrt{2})l_n| \xrightarrow{m \to \infty} \infty

Therefore \( \bar{\kappa}(x_n, x_n) \xrightarrow{m \to \infty} \infty \), which is a contradiction. The rest of the proof is trivial. \( \square \)

Through similar arguments we can verify the validity of the following example.

**Example 3.9.** Let us consider \( W_1, W_2 \subset \mathbb{C} \) linearly independent sets and an additive function \( L : \text{Span}_Z(W_1) \to \text{Span}_Z(W_1) \)

\[
\Gamma_{W_1, W_2, L} = \left\{ \begin{array}{ccc}
1 & x & L(x) + x^2/2 + w \\
0 & 1 & x \\
0 & 0 & 1
\end{array} : w \in \text{Span}_Z(W_2), x \in \text{Span}_Z(W_1) \right\}
\]

Then \( \Gamma_{W_1, W_2, L} \) is purely parabolic group, weakly semi controllable, discrete and isomorphic to \( \mathbb{Z}^{\text{Card}(W_1)} \oplus \mathbb{Z}^{\text{Card}(W_2)} \). Moreover

1. The kernel of \( \Gamma_{W_1, W_2, L} \) is isomorphic to \( \mathbb{Z}^{\text{Card}(W_2)} \) and its control group is discrete and isomorphic to \( \mathbb{Z}^{\text{Card}(W_2)} \).
2. The discontinuity region in the sense of kulkarni for \( \Gamma_{W_1, W_2, L} \) is the largest open set on which the group acts properly discontinuously and agrees with the equicontinuity set. Moreover, the Kulkarni’s limit set is given by

\[ \Lambda_{\text{Kul}}(\Gamma_{W_1, W_2, L}) = \mathbb{Z}, \mathbb{Z} \]

**Example 3.10.** Let \( W \subset \mathbb{C} \) be an additive discrete subgroup and \( x, a, b, c \in \mathbb{C} \) such that \( a - c \in \text{Span}_Z(W) \) and \( \{1, c\} \) is a \( \mathbb{R} \)-linearly independent set. Define

\[
\gamma_{m, n, w} = \begin{bmatrix}
am + n & mb + ac & cmn + w + nx + \left( \frac{n}{2} \right) \\
0 & 1 & cm + n \\
0 & 0 & 1
\end{bmatrix}
\]

and \( \Xi = \{ \gamma_{m, n, w} : m, n \in \mathbb{Z}, w \in W \} \). Then \( \Xi \) is a purely parabolic group, weakly semi controllable, discrete and a non-abelian group which is a semi direct product of the form \( (\mathbb{Z}^{\text{rank}(W)} \rtimes \mathbb{Z}) \rtimes \mathbb{Z} = \mathbb{Z} \). Moreover

1. The kernel of \( \Xi \) is isomorphic to \( \mathbb{Z}^{\text{Card}(W)} \) and its control group is discrete and isomorphic to \( \mathbb{Z} \).
2. The discontinuity region in the sense of kulkarni for \( \Xi \) is the largest open set on which the group acts properly discontinuously and agrees with the equicontinuity set. Moreover, the Kulkarni’s limit set is given by

\[ \Lambda_{\text{Kul}}(\Xi) = \mathbb{Z}, \mathbb{Z} \]

**Proof.** A straightforward calculation shows that for every \( k, l, m, n \in \mathbb{Z} \) and \( u, w \in W \) we have

\[
\gamma_{m, n, w}^{-1} \gamma_{k, l, v}^{-1} = \gamma_{m-k, n-l, w-v+(c-a)(m-k)}
\]

Thus \( \Xi \) is a group iff \( c - a \in W \). \( \square \)

**Example 3.11.** Let \( r \in \mathbb{R} \) be an irrational number. Also, let \( W_r \subset PSL(3, \mathbb{C}) \) be the group given by:

\[
W_r = \left\{ \begin{array}{ccc}
1 & n + m(r + 1) & w + \frac{n(n-1)}{2} + mnr + \frac{m(m-1)r(r+1)}{2} \\
0 & 1 & n + mr \\
0 & 0 & 1
\end{array} : m, n, w \in \mathbb{Z} \right\}
\]
Then \( W_r \) is a non-abelian discrete group of \( \text{PSL}(3, \mathbb{C}) \) which is a semi direct product of the form \((\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}\). Moreover, one has:

1. The group \( W_r \) is a purely parabolic weakly semi controllable group of \( \text{PSL}(3, \mathbb{C}) \).
2. The kernel of \( W_r \) is
   \[
   \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : w \in \mathbb{Z} \right\}
   \]
3. Its control group is
   \[
   \{ z + n + mr : m, n \in \mathbb{Z} \},
   \]
   which is a dense subgroup of \( \{ z + b : b \in \mathbb{R} \} \).
4. The Kulkarni’s limit set is given by
   \[
   \Lambda_{Kul}(L) = \leftarrow e_1, e_2 \cup \bigcup_{r \in \mathbb{R}} e_1, [0, r, 1]
   \]
5. The discontinuity region in the sense of kulkarni for \( L \) is the largest open set on which the group acts properly discontinuously and agrees with the equicontinuity set.

**Proof.** It is clear that \( W_r \) is a discrete subgroup which can be described as \((\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}\). Also, it is straightforward to check that \( W_r \) satisfies properties (1) to (3), so in the following we will focus on the characterization of its Kulkarni limit set. Let’s start with some useful observations, first

\[
\Omega = \mathbb{P}^2_{\mathbb{C}} \setminus \left( e_1, e_2 \cup \bigcup_{r \in \mathbb{R}} e_1, [0, r, 1] \right) \subset \text{Eq}(\Gamma).
\]

This observation is more less trivial since for every sequence \( (\gamma_n) \subset \Gamma \) of distinct elements for which there is \( \gamma \in \text{SP}(3, \mathbb{C}) \) we get

\[
\gamma = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}
\]

where \( a, b, c \in \mathbb{R} \). In virtue of the \( \lambda \)-lemma the following claim concludes the proof.

Claim.- There is a sequence of distinct elements \( (\gamma_n) \subset W_r, \gamma, \tau \in \text{PS}(3, \mathbb{C}) \) and \( r, s \in \mathbb{R} \) such that

\[
\gamma_n \xrightarrow{m \to \infty} \gamma \\
\gamma_n^{-1} \xrightarrow{m \to \infty} \tau \\
\text{Ker}(\gamma) = e_1, [0, r, 1] \\
\text{Ker}(\tau) = e_1, [0, s, 1]
\]

Since \( r \in \mathbb{R} \setminus \mathbb{Q} \) we deduce that there are sequences \( (k_n), (m_n) \in \mathbb{Z} \) such that

\[
c_n = k_n + m_n r \xrightarrow{m \to \infty} 0
\]

let us define

\[
b_n = \frac{k_n(k_n - 1)}{2} + m_n k_n r + \frac{m_n(m_n - 1)r(r + 1)}{2}
\]
and \( w_n = -[b_n] \) thus

\[
\gamma_n = \begin{bmatrix}
1 & c_n + m_n & w_n + b_n \\
0 & 1 & c_n \\
0 & 0 & 1
\end{bmatrix} \xrightarrow{n \to \infty} \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\gamma_n^{-1} = \begin{bmatrix}
1 & -c_n - m_n & -w_n - b_n + c_n^2 + m_n c_n \\
0 & 1 & -c_n \\
0 & 0 & 1
\end{bmatrix} \xrightarrow{n \to \infty} \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[\square\]

4. Purely parabolic groups are virtually triangularizable

As a part of the path to provide classification of discrete subgroups in \( PSL(3, \mathbb{C}) \) without loxodromic elements, in this section we are going to show that this kind of groups have a full flag with finite orbit, in first instance we need to provide sufficient conditions that ensure a group in \( PSL(3, \mathbb{C}) \) contains loxodromic elements. The following lemma is essentially proved in [3], here we include a proof for sake of completeness.

**Lemma 4.1** (See proposition 4.10 in [3]). Let \( \Gamma \subset PSL(3, \mathbb{C}) \) be a discrete group acting strongly irreducible on \( \mathbb{P}^2_\mathbb{C} \), then \( \Gamma \) contains a loxodromic elements.

**Proof.** Since \( \Gamma \) is discrete, we can ensure that there is an element \( \Gamma \) that is either loxodromic or parabolic. Let \( (m_n) \subset \mathbb{N} \) be a strictly increasing sequence and \( \gamma_0 \in PS(3, \mathbb{C}) \setminus PSL(n+1, \mathbb{C}) \) such that \( \gamma_n^n \xrightarrow{m \to \infty} \gamma_0 \), then \( \ell = Ker(\gamma_0) \) is a line and \( p_0 = Im(\gamma_0) \) is a point contained in \( \ell \). Since \( \Gamma \) acts strongly irreducible on \( \mathbb{P}^2_\mathbb{C} \), there are \( \gamma_1, \gamma_2 \in \Gamma \) such that \( \{ \ell_1 = \gamma_1(\ell_0), \ell_2 = \gamma_2(\ell_0), \ell_3 = \gamma_3(\ell_0) \} \) are in general position. Now is not hard to show that there are \( i, j \in \{1, 2, 3\} \) such that

\[
p_i = \gamma_i(p_0) \notin \ell_j \\
p_j = \gamma_j(p_0) \notin \ell_i
\]

Let us define \( \tau_m = \gamma_j \gamma_i \gamma_j^{-1} \gamma_i \gamma_j^{-1} \gamma_i \). Now, if \( \tilde{\gamma}_i, \tilde{\gamma}_j, \tilde{\gamma}_0 \in M(3, \mathbb{C}) \) are such that \( \tilde{\gamma}_i = \gamma_i, \tilde{\gamma}_j = \gamma_j, \tilde{\gamma}_0 = \gamma_0, \gamma = \gamma \) and \( \tilde{\gamma}_i^n \xrightarrow{m \to \infty} \tilde{\gamma}_0 \), then

\[
\gamma_i \xrightarrow{m \to \infty} \tau
\]

where \( \tau = [\tilde{\gamma}_j \tilde{\gamma}_0 \tilde{\gamma}_j^{-1} \tilde{\gamma}_i \tilde{\gamma}_0 \tilde{\gamma}_i^{-1}] \). A straightforward calculation shows \( Ker(\tau) = \ell_i \) and \( Im(\tau) = p_j \). Let \( W \) be an open neighborhood of \( p_j \) such that \( \overline{W} \cap \ell_i = \emptyset \), by equation \[4.1\] there is \( m \in \mathbb{N} \) such that \( \gamma_m(\overline{W}) \subset W \), therefore \( \gamma_m \) is loxodromic which concludes the proof. \[\square\]

**Lemma 4.2.** Let \( \Gamma \subset PSL(3, \mathbb{C}) \) be a discrete group without loxodromic elements then \( \Gamma \) is either affine or weakly semi-controllable .

**Proof.** Since discrete groups in \( PSL(3, \mathbb{C}) \) acting strongly-irreducible on \( \mathbb{P}^2_\mathbb{C} \) contains loxodromic elements, we can assume that there is a non-empty proper subspace \( l \subset \mathbb{P}^2_\mathbb{C} \) such that \( l \) has a finite orbit under the action of \( \Gamma \), observe that by a duality we can assume \( l \) is a point, let \( l_1, \ldots, l_k \) be the orbit of \( l \) under \( \Gamma \). Now we claim:

Claim 1.- We know \( Span(\{l_1, \ldots, l_k\}) \neq \mathbb{P}^2_\mathbb{C} \). On the contrary let us assume that \( Span(\{l_1, \ldots, l_k\}) = \mathbb{P}^2_\mathbb{C} \). Let \( \gamma \in \Gamma \) be a parabolic element, then there exist
s ∈ {1, . . . , k} such that \( l_s \notin \Lambda_{Kal}(\langle \gamma \rangle) \) then \( l_s \) has infinite orbit under the cyclic group \( \langle \gamma \rangle \), which is a contradiction.

To conclude observe that is enough to take \( \ell = \text{Span}(\{l_1, \ldots, l_k\}) \).

Now, let us proceed to understand how is the group acting either on the invariant space or on its dual.

**Lemma 4.3.** Let \( \Gamma \subset PSL(3, \mathbb{C}) \) be a discrete group without loxodromic elements. Then

1. If \( \Gamma \) is affine then the action of \( \Gamma \) on the invariant line does not contains a subgroup conjugate to a dense subgroup of \( SO(3) \).
2. If \( \Gamma \) is weakly semi-controllable then the control group of \( \Gamma \) does not contains a subgroup conjugate to a dense subgroup of \( SO(3) \).

**Proof.** Let us show only the case when \( \Gamma \) is weakly semi-controllable, the proof in the affine case will be similar. Let proceed by contradiction, we claim:

**Claim 1.** The group \( \Pi(\Gamma) \) contains an element with infinite order. Let \( (\tau_n)_{n\in\mathbb{N}} \) be an enumeration of \( \Gamma \). Let \( H_m = \Pi(\langle \tau_1, \ldots, \tau_m \rangle) \), if each group a finite set, then by the classification of subgroups in \( PSL(2, \mathbb{C}) \) with finite order, we conclude that for \( m \) large \( H_m \) is either cyclic or dihedral, therefore the control group \( \Pi(\Gamma) \) is either infinite or dihedral, which is not possible. Let \( m_0 \in \mathbb{N} \) such that the group \( H = H_{m_0} \) is infinite, by Tits alternative \( H \) either contains a solvable subgroup \( S \) of finite index or a non-commutative free group \( F \). If \( H \) contains a free subgroup \( F \) the work is done, so we may assume that \( H \) contains a solvable subgroup \( S \) of finite index. Since \( S \subset SO(3) \) is a solvable group and in view of the Borel fixed point theorem we conclude that \( \text{Fix}(s_1) = \text{Fix}(s_2) \) for every \( s_1, s_2 \in S \), therefore there is \( \vartheta \in PSL(2, \mathbb{C}) \) and \( S \) an infinite, finitely generated subgroup of \( S \) such that

\[
\vartheta S \vartheta^{-1} = \{az : a \in S\}.
\]

So we conclude that \( S \) contains an element with infinite order.

**Claim 2.** The group \( \Gamma \) contains a subgroup \( \Gamma_0 \) such that \( \Pi(\Gamma_0) \) is non-commutative and torsion free group. Let \( \rho_1, \rho_2 \in \Pi(\Gamma) \) be such that \( \rho_1 \) has infinite order, \( o(\rho_2) \neq 2 \) and \( \text{Fix}(\rho_1) \cap \text{Fix}(\rho_2) = \emptyset \), then there is \( n \in \mathbb{N} \) such that \( \text{Fix}(\rho_1) \cap \text{Fix}(\rho_3 = \rho_n \rho_1 \rho_2^{-n}) = \emptyset \). By Selberg's lemma there is a natural number \( k \) such that \( \Gamma_1 = \langle \rho_1^k, \rho_2^k \rangle \) is torsion free, clearly \( \Gamma_1 \) is non abelian. To conclude define \( \Gamma_0 = \Pi^{-1}(\Gamma_1) \).

Finally, let \( \gamma_1, \gamma_2 \in \Gamma_0 \) be such that \( \Pi(\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}) \) has infinite order, then \( \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} \) has a lift \( \rho \in SL(3, \mathbb{C}) \), given by

\[
\rho = \begin{pmatrix} 1 & b \\ 0 & B \end{pmatrix}
\]

where \( B \in SO(3) \) has infinite order and \( b \in \mathbb{C}^2 \). Clearly \( \rho \) is non-diagonalizable, with unitary proper values and infinite order. Therefore \( \Gamma \) is non-discrete. \( \square \)

Now we are in position to deal with the following lemma.
Theorem 4.4. Let \( \Gamma \subset PSL(3, \mathbb{C}) \) be a discrete group without loxodromic elements, then there is a normal subgroup \( \Gamma_0 \) of \( \Gamma \) with finite index such that \( \Gamma_0 \) leaves invariant a full flag in \( \mathbb{P}^2_\mathbb{C} \), i.e. the group \( \Gamma_0 \) is simultaneously triangularizable.

Proof. Let \( \Gamma \subset PSL(3, \mathbb{C}) \) be a discrete group without loxodromic elements, then by Lemma 4.2, we know that \( \Gamma \) has a proper, non-empty projective subspace \( p \) invariant under \( \Gamma \). Let us assume that \( p \) is a point, the arguments in the other case will be dual. Since \( \Gamma \) does not contain loxodromic elements we conclude that \( \mathbb{P}^1_\mathbb{C} \setminus Eq(\Pi(\Gamma)) \), see [5], is either empty or contains a single point. If \( \mathbb{P}^1_\mathbb{C} \setminus Eq(\Pi(\Gamma)) \) contains a single point is clear that \( \Gamma \) is simultaneously triangularizable. So let us deal the case when \( \mathbb{P}^1_\mathbb{C} = Eq(\Pi(\Gamma)) \), in this case by Lemma 4.3 we know that \( \Pi(\Gamma) \) is either finite or we can assume that is a subgroup of the infinite dihedral group \( Dih_\infty \). If \( \Pi(\Gamma) \) is finite, it will be enough to consider \( \Gamma_0 = Ker(\Pi) \), in the case \( \Pi(\Gamma) \subset Dih_\infty \) the proof is concluded by considering \( \Gamma_0 = \{ \gamma \in \Gamma : \Pi(\gamma) \in Rot_\infty \} \), where \( Rot_\infty \) is the group of all rotations about the origin. \( \square \)

The following result is essentially proved in [2].

Theorem 4.5. Let \( \Gamma \subset PSL(n, \mathbb{C}) \) be a discrete nilpotent group, then \( \Gamma \) is finitely generated.

As a corollary we get

Corollary 4.6. Let \( \Gamma \subset PSL(3, \mathbb{C}) \) be a discrete group without loxodromic elements, then \( \Gamma \) is finitely generated.

Proof. Let \( \Gamma_0 \) be a subgroup of \( \Gamma \) with finite index, such that \( \Gamma_0 \) is triangularizable. A straightforward calculation shows that \( \Gamma_0 \) is nilpotent, from Theorem 4.5 we conclude the proof. \( \square \)

5. Triangulable groups containing screw parabolic elements

In this part of the article we are going to characterize those discrete groups containing a parabolic element which behaves as a rotation of infinite order on some invariant line. As we will see through this section the knowledge of this groups is very useful.

Lemma 5.1. Let \( \alpha \in S^1 \) be an element with infinite order and \( a, b, c, x, y, z \in \mathbb{C} \). If \( |x| + |y| \neq 0 \), then the group

\[
\Gamma = \langle \tau_1 = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \tau_2 = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} \rangle
\]

is non-discrete.

Proof. Let \( \tau \in PSL(3, \mathbb{C}) \) be given by:

\[
\tau = \begin{bmatrix} 1 & a(1 - \alpha^3)^{-1} & b \\ 0 & 1 & -c(1 - \alpha^3)^{-1} \\ 0 & 0 & 1 \end{bmatrix}
\]

then a straightforward calculation shows:

\[
\tau \tau_2 \tau^{-1} = \begin{bmatrix} 1 & 0 & \tilde{b} \\ 0 & \alpha^3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tau \tau_1 \tau^{-1} = \begin{bmatrix} 1 & x & z + cx(1 - \alpha^3)^{-1} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}
\]


So we may assume that $a = c = 0$. Observe that
\[
\gamma_n = \tau_2^n \tau_1 \tau_2^{-n} = \begin{bmatrix}
1 & x\alpha^{-3n} & z \\
0 & 1 & y\alpha^{3n} \\
0 & 0 & 1
\end{bmatrix},
\]
clearly $(\gamma_n)$ contains a convergent sequence of distinct elements, which concludes the proof. \hfill \Box

Before we prove the next result we must develop some technical lemmas.

**Lemma 5.2.** Let $\vartheta \in S^1 \setminus \{\pm 1\}$, then
\[
\beta_1 = \{(0, 1), \vartheta(1, 1), \vartheta^2(2, 1), \vartheta^3(3, 1)\} \subset \mathbb{C}^2
\]
is $\mathbb{R}$-linearly independent.

**Proof.** If this is not the case, then there are real numbers $r_0, r_1, r_2 \in \mathbb{R}$ such that
\[
\vartheta^3(3, 1) = r_0(0, 1) + r_1\vartheta(1, 1) + r_2\vartheta^2(2, 1),
\]
thus $\vartheta$ is a common root of the polynomials
\[
q_1(x) = x^2 - 2r_2x - \frac{r_1}{4}, \\
q_2(x) = x^3 - r_2x^2 - r_1x - r_0.
\]
Since $q_1, q_2$ are monic polynomials in $\mathbb{R}[x]$ we deduce that we can write $q_1(x)$ and $q_2(x)$ in terms of $\vartheta$ as follows:
\[
q_1(x) = x^2 - 2\Re(\vartheta)x + 1 \\
q_2(x) = x^3 - (2\Re(\vartheta) + r)x^2 + (1 + 2r\Re(\vartheta))x - r
\]
By comparing coefficients we get
\[
\begin{align*}
    r_1 &= -3 \\
    r_2 &= 3\Re(\vartheta) \\
    r &= r_0 \\
    r_1 &= -(1 + 2r\Re(\vartheta)) \\
    r_2 &= (2\Re(\vartheta) + r)
\end{align*}
\]
which has solution if and only if $\Re(\vartheta) = r_0 = \pm 1$. \hfill \Box

**Definition 5.3.** Let $\eta(x), \nu(x), P(x) \in \mathbb{Z}[x]$ be defined by
\[
\eta(x) = 2 + 63x^2 + 72x^3 - 16x^4 \\
\nu(x) = 192x^7 - 64x^6 + 496x^5 + 288x^4 + 510x^3 + 209x + 8 \\
P(x) = \eta(x)\nu(x)
\]

**Lemma 5.4.** Let $\vartheta \in S^1 \setminus \{\pm 1\}$ such that $P(\Re(\vartheta))$ and
\[
\beta_2 = \{(0, 1), \vartheta(1, 1), \vartheta^2(2, 1), \vartheta^3(3, 1), \vartheta^4(4, 1)\} \subset \mathbb{C}^2
\]
is $\mathbb{Q}$-linearly dependent, then there is $\alpha \in \mathbb{C}^*$ such that:
\[
\alpha e_1 \in \text{Span}_\mathbb{Z}\{\vartheta^j(j, 1) : j \in \{0, \ldots, 5\}\}.
\]
Proof. Since \( \beta_2 \) is a \( \mathbb{Q} \)-linearly dependent set, there are \( m_0, m_1, m_2, m_3, m_4 \in \mathbb{Z} \) such that

\[
m_4 \vartheta^4(4,1) = m_3 \vartheta^3(3,1) + m_2 \vartheta^2(2,1) + m_1 \vartheta(1,1) + m_0(0,1)
\]

and \( m_4 \neq 0 \). Thus we get the following equations

\[
4m_4 \vartheta^3 = 3m_3 \vartheta^2 + 2m_2 \vartheta + m_1
\]

\[
m_4 \vartheta^4 = m_3 \vartheta^3 + m_2 \vartheta^2 + m_1 \vartheta + m_0
\]

Let us consider \( p_1(x), p_2(x) \in \mathbb{Z}[x] \) be given by

\[
p_1(x) = -4m_4x^3 + 3m_3x^2 + 2m_2x + m_1
\]

\[
p_2(x) = -4m_4x^3 + m_3x^3 + m_2x^2 + m_1x + m_0
\]

clearly \( p_1(\vartheta) = p_2(\vartheta) = 0 \). Thus there are \( r_1, r_2, r_3 \in \mathbb{R} \) such that

\[
p_1(x) = -4m_4(x - \vartheta)(x - \vartheta^{-1})(x - r_1)
\]

\[
= -4m_4x^3 + 4m_4(2Re(\vartheta) + r_1)x^2 - 4m_4(1 + 2r_1Re(\vartheta))x + 4m_4r_1
\]

\[
p_2(x) = -4m_4(x - \vartheta)(x - \vartheta^{-1})(x^2 + r_2x + r_3)
\]

\[
= -4m_4x^4 - 4m_4(-2Re(\vartheta) + r_2)x^3 - 4m_4(1 - 2r_2Re(\vartheta) + r_3)x^2
\]

\[
-4m_4(r_2 - 2Re(\vartheta)r_3)x - 4m_4r_3
\]

By comparing the coefficients of \( p_1 \) and \( p_2 \) with the previous equations we get

\[
m_1 = 4m_4r_1
\]

\[
2m_2 = -4m_4(1 + 2r_1Re(\vartheta))
\]

\[
3m_3 = 4m_4(2Re(\vartheta) + r_1)
\]

\[
m_0 = -4m_4r_3
\]

\[
m_1 = -4m_4(r_2 - 2r_3Re(\vartheta))
\]

\[
m_2 = -4m_4(1 - 2r_2Re(\vartheta) + r_3)
\]

\[
m_3 = -4m_4(r_2 - 2Re(\vartheta))
\]

Which induces the following linear system

\[
r_1 + r_2 - 2r_3Re(\vartheta) = 0
\]

\[
2r_1Re(\vartheta) + 4r_2Re(\vartheta) - 2r_3 = 1
\]

\[
r_1 + 3r_3 = 4Re(\vartheta)
\]

Solving the system by Cramer’s rule we get:

\[
r_1 = \frac{Re(\vartheta)(16Re^2(\vartheta) - 7)}{2(1 - Re^2(\vartheta))}; r_2 = \frac{-Re(\vartheta)(8Re^2(\vartheta) + 3)}{2(1 - Re^2(\vartheta))}; r_3 = \frac{4Re^2(\vartheta) - 1}{2(1 - Re^2(\vartheta))}.
\]

Is straightforward that \( Re(\vartheta) = pq^{-1} \), where \( p, q \in \mathbb{Z} \) are co primes, let us define:

\[
n_0 = (4Re^2(\vartheta) - 1)q^4
\]

\[
n_1 = (-Re(\vartheta) - 16Re^3(\vartheta))q^4
\]

\[
n_2 = (1 + 8Re^2(\vartheta) + 16Re^4(\vartheta))q^4
\]

\[
n_3 = (-7Re(\vartheta) - 4Re^3(\vartheta))q^4
\]

\[
n_4 = (2 - 2Re^2(\vartheta))q^4
\]

By construction \( n_0, n_1, n_2, n_3, n_4 \in \mathbb{Z} \) and

\[
n_4 \vartheta^4(4,1) = n_3 \vartheta^3(3,1) + n_2 \vartheta^2(2,1) + n_1 \vartheta(1,1) + n_0(0,1)
\]

From this equation we are able to deduce

\[
\vartheta^5 = n_4^2(n_0n_3 + (n_0n_4 + n_1n_3)\vartheta + (n_1n_4 + n_2n_3)\vartheta^2 + (n_2n_4 + n_3^2)\vartheta^3)
\]
Thus
\[ n_1^2\theta^5(5, 1) = n_0 n_3(0, 1) + (n_0 n_4 + n_1 n_3)\theta(1, 1) + (n_1 n_4 + n_2 n_3)\theta^2(2, 1) + (n_2 n_4 + n_3^2)\theta^3(3, 1) + (5n_0 n_3 + 4n_0 n_4 + n_1 n_3)\theta + 3(n_1 n_4 + n_2 n_3)\theta^2 + 2(n_2 n_4 + n_3^2)\theta^3)(1, 0) \]

In order to conclude the proof will be enough to show that
\[ 5n_0 n_3 + 4(n_0 n_4 + n_1 n_3)\theta + 3(n_1 n_4 + n_2 n_3)\theta^2 + 2(n_2 n_4 + n_3^2)\theta^3 \neq 0 \]

Let \( \xi = 2(n_2 n_4 + n_3^2) \), then
\[ \xi = 2\eta^8(2 + 63Re^2(\vartheta) + 72Re^4(\vartheta) - 16Re^6(\vartheta)) = 2\eta^8\eta(Re(\vartheta)) \neq 0 \]

Thus we conclude that \( p_3(x) = 5n_0 n_3 + 4(n_0 n_4 + n_1 n_3)x + 3(n_1 n_4 + n_2 n_3)x^2 + 2(n_2 n_4 + n_3^2)x^3 \) is a cubic polynomial with coefficients in \( \mathbb{Z} \). Let us assume that \( \vartheta \) is a root of \( p_3(x) \). Then there is \( r_0 \in \mathbb{R} \) such that
\[ p_3(x) = 2\eta(x - \vartheta)(x - \vartheta^{-1})(x - r_0) = 2\eta x^3 - 2\eta r_0 + 2Re(\vartheta)x^2 + 2\eta(1 + 2r_0 Re(\vartheta))x - 2\eta r_0 \]

By comparing the quadratic coefficient of \( p_3 \) with the respective in the previous equation we obtain
\[ -2\eta(2Re(\vartheta) + r_0) = 3(n_1 n_4 + n_2 n_3) \]

Straightforward calculations show
\[ 3(n_1 n_4 + n_2 n_3) = \eta^8(27Re(\vartheta) + 270Re^3(\vartheta) + 336Re^5(\vartheta) + 192Re^7(\vartheta)) - 2\eta(2Re(\vartheta) + r_0) = 2(5n_0 n_3 - 2\eta Re(\vartheta)) = \eta^8(-8 - 182Re(\vartheta) - 240Re^3(\vartheta) - 288Re^4(\vartheta) - 160Re^5(\vartheta) + 64Re^6(\vartheta)) \]

Thus \( \nu(Re(\vartheta)) = 0 \), which is not possible, therefore \( p_3(\vartheta) \neq 0 \), which concludes the proof.

\[ \square \]

**Lemma 5.5.** Let \( \alpha \in S^1 \) be an element with infinite order and \( x, y, z, \beta, \mu, \nu \in \mathbb{C} \). If \( |x| + |y| \neq 0 \), the group
\[ \Gamma = \left\langle \tau_1 = \begin{bmatrix} 1 & z & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} , \tau_2 = \begin{bmatrix} 1 & \beta & \mu \\ 0 & 1 & \nu \\ 0 & 0 & \alpha^{-3} \end{bmatrix} \right\rangle \]
is non-discrete.

**Proof.** Let us start observing that \( \beta = 0 \) implies \( \tau_2 \) is an elliptic element with infinite order, which makes \( \Gamma \) non-discrete, so in the following we will assume that \( \beta \neq 0 \). Let us assume that \( \Gamma \) is a discrete group, then there is a non trivial additive and discrete subgroup \( W \subset \mathbb{C}^2 \) such that the commutator subgroup of \( \Gamma \) is given by:
\[ G = [\Gamma, \Gamma] = \left\{ \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : (a, b) \in W \right\} \]

Clearly \( G \) is a normal subgroup of \( \Gamma \). Consider \( [\gamma_{ij}] = [\tau_2, [\tau_2, \tau_1]] \), thus \( \gamma \in G \) and \( \gamma_{13}\gamma_{23} \neq e \), so after conjugating with an upper triangular element, if necessary, we can assume that \((1, 1) \in W \). A straightforward calculation shows:
\[
\tau_2 \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} \tau_2^{-n} = \begin{bmatrix}
1 & 0 & \alpha^{3n}(n\beta + 1) \\
0 & 1 & \alpha^{3n} \\
0 & 0 & 1
\end{bmatrix}
\]

thus \(\{\alpha^{3n}(n\beta + 1, 1) : n \in \mathbb{Z}\} \subset W\). Now we claim:

Claim 1.- The set \(A = \{\alpha^{3n}(n\beta + 1, 1) : n = 0, 1, 2, 3\}\) is \(\mathbb{R}\)-linearly independent. On the contrary assume that there are \(r_0, \ldots, r_3 \in \mathbb{R}\) not all equal to 0 such that:

\[
0 = 3 \sum_{j=0}^{3} r_j \alpha^{3j} \left(\frac{j\beta + 1}{1}\right) = \left(\sum_{j=0}^{3} r_j \alpha^{3j}\right) + 3 \sum_{j=0}^{3} \beta r_j \alpha^{3j} \left(\frac{j}{1}\right) = \sum_{j=0}^{3} \beta r_j \alpha^{3j} \left(\frac{j}{1}\right)
\]

thus the set \(\{\alpha^{3j}(j, 1) : j = 0, 1, 2, 3\}\) is \(\mathbb{R}\)-linearly dependent, which is contradiction.

Claim 2.- The set \(A = \{\alpha^{3j}(j, 1) : j = 0, 1, 2, 3, 4\}\) is \(\mathbb{Q}\)-linearly dependent. Observe the set \(B = \{\alpha^{3j}(j\beta + 1, 1) : j = 0, 1, 2, 3, 4\}\) is \(\mathbb{Q}\)-linearly dependent, now by similar arguments used in the previous claim, we conclude that \(A\) is \(\mathbb{Q}\)-linearly dependent.

Claim 3.- There is \(d \in \mathbb{C}^*\) such that \((d, 0) \in W\). We know by Lemma 5.4 that there is \(c \in \mathbb{C}^*\) and \(m_0, \ldots, m_5 \in \mathbb{Z}\) such that

\[
\begin{pmatrix} c \\ 0 \end{pmatrix} = \sum_{j=0}^{5} m_j \alpha^{3j} \left(\frac{j}{1}\right)
\]

Now a simple calculation shows

\[
\begin{pmatrix} c\beta \\ 0 \end{pmatrix} = \sum_{j=0}^{5} m_j \alpha^{3j} \left(\frac{j\beta + 1}{1}\right).
\]

which concludes the proof of the claim.

To conclude let \((m_n) \subset \mathbb{Z}\) such that \((\alpha^{3m_n})\) is a sequence of distinct elements which converge to 1 and \(d \in \mathbb{C}^*\) such that \((d, 0) \in W\), thus

\[
\tau_m = \tau_1^{m_n} \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tau_1^{-m_n} = \begin{bmatrix} 1 & 0 & \alpha^{3m_n}d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tau_1^{-m_n} \xrightarrow{n \to \infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

which is a contradiction, thus \(\Gamma\) is non-discrete.

\[\square\]

Lemma 5.6. Let \(\alpha \in \mathbb{S}^1\) be an element with infinite order and \(x, y, z, \beta, \mu, \nu \in \mathbb{C}\). If \(|x| + |y| \neq 0\), the group

\[
\Gamma = \left\langle \tau_1 = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \tau_2 = \begin{bmatrix} \alpha^{-3} & \beta & \mu \\ 0 & 1 & \nu \\ 0 & 0 & 1 \end{bmatrix} \right\rangle
\]

is non-discrete.
Proof. Consider the group morphism $\rho : PSL(3, \mathbb{C}) \rightarrow PSL(3, \mathbb{C})$ given by $\rho([M]) = (M^*)^{-1}$, clearly will be enough to show that $\rho(\Gamma)$ is non-discrete. By Lemma 5.5 the following calculations conclude the proof.

$$
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x & y & 0 \\
0 & 1 & z \\
0 & 0 & 1 \\
\end{bmatrix}^t
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & -z & zz-y \\
0 & 1 & -x \\
0 & 0 & 1 \\
\end{bmatrix}.
$$

Lemma 5.8. Let $\lambda \in GL(3, \mathbb{C})$, then $\pi_{ij}(\lambda)$ will denote the $ij$-th element of the matrix $\lambda$. Let us define

**Definition 5.7.** Let us define the following group

$$
U_+ = \left\{ \begin{pmatrix}
\alpha & \beta & \gamma \\
0 & \nu & \eta \\
0 & 0 & \mu \\
\end{pmatrix} : \alpha\mu\nu = 1, \beta, \gamma, \eta \in \mathbb{C} \right\}
$$

and the group morphisms $D : U_+ \rightarrow Mob(\mathbb{C})$ and $\lambda_{12}, \lambda_{23}, \lambda_{13} : U_+ \rightarrow \mathbb{C}^*$, which are given by:

$$
D([\gamma_{ij}])z = \gamma_{11}\gamma_{22}^{-1}z + \gamma_{12}\gamma_{22}^{-1}
$$

$$
\lambda_{12}([\gamma_{ij}]) = \gamma_{11}\gamma_{22}^{-1}
$$

$$
\lambda_{23}([\gamma_{ij}]) = \gamma_{22}\gamma_{33}^{-1}
$$

$$
\lambda_{13}([\gamma_{ij}]) = \gamma_{11}\gamma_{33}^{-1}
$$

**Lemma 5.8.** Let $\Gamma \subset U_+$ be a discrete group, then $\Gamma$ contains a subgroup $\Gamma_{nd}$ of finite index such that the groups $\Gamma_{nd}$, $\lambda_{12}(\Gamma_{nd}), \lambda_{23}(\Gamma_{nd}), D(\Gamma_{nd}), \Pi(\Gamma_{nd})$ are torsion free and finitely generated.

Proof. Clearly $\Gamma$ is nilpotent, thus by Theorem 4.4 the group $\Gamma$ is finitely generated. In consequence $\lambda_{12}(\Gamma)$ is finitely generated, by Selberg’s lemma we can find a subgroup $\Gamma_1 \subset \lambda_{12}(\Gamma)$ of finite index such that $\Gamma_1$ is torsion free. Observe that $\Gamma_1 = \lambda_{12}^{-1}(\Gamma_1)$ is a subgroup of $\Gamma$. By repeating this procedure with $\Gamma_1$ and $\lambda_{23}$, we deduce that there is a subgroup $\Gamma_2$ of $\Gamma_1$ such that $\lambda_{23}(\Gamma_2)$ is torsion free. Now is clear the existence of the subgroup $\Gamma_{nd}$. □

**Lemma 5.9.** Let $\Gamma \subset U_+$ be a discrete group such that the groups $\lambda_{12}(\Gamma)$, $\lambda_{23}(\Gamma)$, $\lambda_{13}(\Gamma)$ are torsion free and there is a parabolic element $\gamma \in \Gamma$ satisfying

$$
\max\{o(\lambda_{12}(\gamma)), o(\lambda_{23}(\gamma))\} = \infty,
$$

then every element $\Gamma$ commutes with $\gamma$. 
Proof. On the contrary, let us assume that there is an element \( \tau = [\tau_{ij}] \in \Gamma \) such that \( [\gamma, \tau] \neq Id \). A straightforward calculation shows that there are \( x, y, z \in \mathbb{C} \) such that

\[
[\gamma, \tau] = \begin{bmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1
\end{bmatrix}.
\]

Now consider the following cases:

**Case 1.** We have \( o(\lambda_{12}(\gamma)) = o(\lambda_{23}(\gamma)) = \infty \). Since \( \Gamma \) is discrete we deduce that \( \lambda_{13}(\gamma) = 1 \), moreover, in virtue of Lemma 5.1 we deduce \( x = z = 0 \) but \( y \neq 0 \). In consequence we conclude \( D[\gamma, \tau] = \Pi[\gamma, \tau] = Id, \gamma_{13} \neq 0 \) and \( o(\lambda_{13}\tau) = \infty \), therefore a simple calculation shows:

\[
\tau^n \gamma \tau^{-n} = \begin{bmatrix}
\gamma_{11} & \gamma_{12} & \lambda_{13}^n(\tau)\gamma_{13} \\
0 & \gamma_{11} & \gamma_{23} \\
0 & 0 & \gamma_{11}
\end{bmatrix}
\]

thus \( (\tau^n \gamma \tau^{-n}) \) contains a convergent sequence of distinct elements.

**Case 2.** We have \( o(\lambda_{12}(\gamma)) = \infty \) and \( \lambda_{23}(\gamma) = 1 \). Since \( \Gamma \) is discrete we deduce that \( \gamma_{23} \neq 0 \), moreover, in virtue or Lemma 5.6 we deduce \( x = y = 0 \) but \( z \neq 0 \). In consequence \( D[\gamma, \tau] = Id \) and \( o(\lambda_{23}\tau) = \infty \), finally it is simple to show:

\[
\tau^n \gamma \tau^{-n} = \begin{bmatrix}
\gamma_{11}^{-2} & \gamma_{12} & \gamma_{13} \\
0 & \gamma_{11} & \lambda_{23}^n(\tau)\gamma_{23} \\
0 & 0 & \gamma_{11}
\end{bmatrix}
\]

thus \( (\tau^n \gamma \tau^{-n}) \) contains a convergent sequence of distinct elements.

**Case 3.** We have \( o(\lambda_{23}(\gamma)) = \infty \) and \( \lambda_{12}(\gamma) = 1 \). Again, since \( \Gamma \) is discrete we conclude \( \gamma_{12} \neq 0 \), also from Lemma 5.5 we deduce \( x = z = 0 \) but \( y \neq 0 \). In consequence \( \Pi[\gamma, \tau] = Id \) and \( o(\lambda_{12}\tau) = \infty \), as in the previous cases we get:

\[
\tau^n \gamma \tau^{-n} = \begin{bmatrix}
\gamma_{11} & \lambda_{23}^n(\tau)\gamma_{12} & \gamma_{13} \\
0 & \gamma_{11} & \gamma_{23} \\
0 & 0 & \gamma_{11}^{-2}
\end{bmatrix}
\]

thus \( (\tau^n \gamma \tau^{-n}) \) contains a convergent sequence of distinct elements.

Thus we have shown that under the assumption \( \Gamma \) not being commutative we get \( \Gamma \) is non-discrete. Which is a contradiction. \( \square \)

**Lemma 5.10.** Let \( \Gamma \subset U_+ \) be a discrete group such that the groups \( \lambda_{12}(\Gamma) \), \( \lambda_{23}(\Gamma) \), \( \lambda_{13}(\gamma) \) are torsion free and there is a parabolic element \( \gamma \in \Gamma \) satisfying \( \max\{o(\lambda_{12}(\gamma)), o(\lambda_{23}(\gamma))\} = \infty \), then \( \Gamma \) is abelian.

**Proof.** Let us consider the case:

**Case 1.** We have \( o(\lambda_{12}(\gamma)) = o(\lambda_{23}(\gamma)) = \infty \). Since \( \Gamma \) is discrete we deduce that \( \lambda_{13}(\gamma) = 1 \). Is straightforward that we can find a transformation \( \tau \in U_+ \) such that

\[
\tau \gamma \tau^{-1} = \begin{bmatrix}
\gamma_{11} & 0 & a \\
0 & \gamma_{11}^{-2} & 0 \\
0 & 0 & \gamma_{11}
\end{bmatrix}.
\]
where $a \neq 0$. Since every element $\beta \in \Gamma$ commutes with $\gamma$ we conclude

$$\tau \beta \tau^{-1} = \begin{bmatrix} \beta_{11} & 0 & b \\ 0 & \beta_{11}^{-2} & 0 \\ 0 & 0 & \beta_{11} \end{bmatrix}.$$ 

Clearly this shows $\Gamma$ is commutative.

Trough similar arguments one can show that in any other case $\Gamma$ is commutative. 

□

Lemma 5.11. Let $\Gamma \subset U_+^*$ be a commutative group, then there is a matrix $\tau \in SL(3, \mathbb{C})$ one of the following facts occurs:

1. each element in $\tau \Gamma_0 \tau^{-1}$ has the form

$$\begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{bmatrix}.$$ 

2. each element in $\tau \Gamma_0 \tau^{-1}$ has the form

$$\begin{bmatrix} \alpha & 0 & \beta \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{bmatrix}.$$ 

where $\alpha$ is a cubic root of the unity.

3. each element in $\tau \Gamma_0 \tau^{-1}$ has the form

$$\begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \mu \\ 0 & 0 & \alpha \end{bmatrix}.$$ 

where $\alpha$ is a cubic root of the unity.

4. each element in $\tau \Gamma_0 \tau^{-1}$ has the form

$$\begin{bmatrix} \alpha & 0 & \mu \\ 0 & \alpha^{-2} & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$ 

5. The group $\tau \Gamma \tau^{-1}$ is diagonal.

6. each element in $\tau \Gamma_0 \tau^{-1}$ has the form

$$\begin{bmatrix} \alpha & \mu & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{bmatrix}.$$ 

7. each element in $\tau \Gamma_0 \tau^{-1}$ has the form

$$\begin{bmatrix} \alpha & \mu & \nu \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$ 

where $\alpha$ is a cubic root.

Proof. Since $\Gamma$ is commutative we deduce that $D(\Gamma)$ and $\Pi(\Gamma)$ are abelian groups.

Now consider the following cases:
Case 1.- The groups $D(\Gamma)$ and $\Pi(\Gamma)$ contains a parabolic element. In this case, since $D(\Gamma), \Pi(\Gamma) \subset Mob(\mathbb{C})$ are abelian, we deduce $D(\Gamma)$ and $\Pi(\Gamma)$ are purely parabolic, that is $\Gamma \subset Ker(\lambda_{12}) \cap Ker(\lambda_{12})$, which concludes the proof in this case.

Case 2.- The group $D(\Gamma)$ contains a parabolic element but $\Pi(\Gamma)$ does not. Under this assumption, we deduce $D(\Gamma)$ is purely parabolic and there is $w \in \mathbb{C}$ such that $\Pi(\Gamma)w = w$. Clearly $\Gamma \subset Ker(\lambda_{12})$, in order to conclude define

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}$$

an straightforward calculation shows that for every $\gamma \in \Gamma$, there is $c_\gamma \in \mathbb{C}$ such that:

$$\tau \gamma \tau^{-1} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & c_\gamma \\ 0 & \gamma_{11} & 0 \\ 0 & 0 & \gamma_{11}^{-2} \end{pmatrix}$$

Observe that $\Gamma_1 = \tau \Gamma \tau^{-1}$ leaves invariant $e_1, e_3$, so $\Pi_1 : \Gamma_1 \to Mob(\mathbb{C})$, given by $\Pi_1([\gamma_{ij}]) = \gamma_{11}\gamma_{33}z + \gamma_{13}\gamma_{33}^2$ is a well defined group morphism. Now we only need to consider the following sub-cases:

Sub case 1.- The group $\Pi_1(\Gamma_1)$ contains a parabolic element. Thus $\Pi_1(\Gamma_1)$ is purely parabolic, which shows that $\Gamma_1 \subset Ker(\lambda_{13})$, this concludes the proof in this sub case.

Sub case 2.- The group $\Pi_1(\Gamma_1)$ does not contains a parabolic element. Thus there is $p \in \mathbb{C}$ such that $\Pi_1(\Gamma_1)p = p$, define

$$\tau_1 = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

an straightforward calculation shows that for every $\gamma \in \Gamma_1$ there is $c_\gamma$ such that:

$$\tau_1 \gamma \tau_1^{-1} \gamma^{-1} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ 0 & \gamma_{11} & 0 \\ 0 & 0 & \gamma_{11}^{-2} \end{pmatrix}$$

which concludes the proof in this sub case.

Case 3.- The group $D(\Gamma)$ does not contains a parabolic element but $\Pi(\Gamma)$ does. We deduce that $\Pi(\Gamma)$ is purely parabolic and there is $z \in \mathbb{C}$ such that $D(\Gamma)z = z$. Clearly $\Gamma \subset Ker(\lambda_{23})$, define

$$\tau = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

an straightforward calculation shows that for every $\gamma \in \Gamma$ there is $c_\gamma$ such that:

$$\tau \gamma \tau^{-1} = \begin{pmatrix} \gamma_{11}^{-2} & 0 & c_\gamma \\ 0 & \gamma_{11} & \gamma_{13} \\ 0 & 0 & \gamma_{11} \end{pmatrix}$$.
Now we can consider $\Pi_2 = \Pi_{\tau_1, \tau_3}$, thus $\Pi_2(\Gamma) \subset Mob(\mathbb{C})$ is an abelian group. So we must consider the following sub cases:

Sub case 1.- The group $\Pi_2(\Gamma)$ contains a parabolic element. We get $\Pi_2(\Gamma)$ is purposely parabolic, which shows that $\Gamma \subset Ker(\lambda_{13})$, which conclude the proof in this sub case.

Sub-case 2.- The group $\Pi_2(\Gamma)$ does not contains a parabolic element. Again there is $p \in \mathbb{C}$ such that $\Pi_2(\Gamma)p = p$, define

$$\tau_1 = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

an straightforward calculation shows that for every $\gamma \in \Gamma$ there is $c_{\gamma}$ such that:

$$\tau_1 \gamma \tau_1^{-1} \tau_1^{-1} = \begin{pmatrix} \gamma_{11}^2 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{13} \\ 0 & 0 & \gamma_{11} \end{pmatrix},$$

which concludes the proof in this sub case.

Case 4.- The groups $\mathcal{D}(\Gamma)$ and $\Pi(\Gamma)$ does not contains parabolic elements. In this setting, there are $z, w \in \mathbb{C}$ such that $\mathcal{D}(\Gamma)z = z$ and $\Pi(\Gamma)w = w$, define

$$\tau = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}$$

a simple calculation shows that for every $\gamma \in \Gamma$ there is $c_{\gamma} \in \mathbb{C}$ such that:

$$\tau \gamma \tau^{-1} = \begin{pmatrix} \gamma_{11} & 0 & c_{\gamma} \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix}$$

Consider the following sub-cases:

Sub case 1.- The group $\Pi_2(\Gamma)$ contains a parabolic element. Thus $\Pi_2(\Gamma)$ is purely parabolic, which shows that $\Gamma \subset Ker(\lambda_{13})$, which conclude the proof in this sub case.

Sub case 2.- The group $\Pi_2(\Gamma)$ does not contains a parabolic element. We know there is $p \in \mathbb{C}$ such that $\Pi_2(\Gamma)p = p$, set

$$\tau_1 = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a simple calculation shows $\tau_1 \tau \tau_1^{-1} \tau_1^{-1}$ contains only diagonal elements. which concludes the proof. \qed
Lemma 5.12. Let $\Gamma \subset U_+^*$ be a discrete torsion free group such that the group $\text{Ker}(\Pi) \cap \Gamma$ is trivial and each element in $\gamma \in \Gamma$ has the form
\[
\begin{pmatrix}
\alpha^{-2} & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha
\end{pmatrix}
\]
then there exist $W \subset \mathbb{C}$ a discrete additive subgroup a group morphism $\eta : W \to \mathbb{C}^*$ such that:
\[
\Gamma = \left\{ \begin{pmatrix} \eta(w)^{-2} & 0 & 0 \\
0 & \eta(w) & \eta(w)w \\
0 & 0 & \eta(w) \end{pmatrix} : w \in W \right\}.
\]

Proof. Let us define $\zeta : \Gamma \to \mathbb{C}$ which is given by $\zeta([\gamma_{ij}]) = \gamma_{23}^{-1}\gamma_{33}$. A simple calculation shows that $\zeta$ is a group morphism and $\text{Ker}(\zeta)$ is trivial, then the following is a well defined group morphism
\[
\eta : \zeta(\Gamma) \to S^1 \\
x \mapsto \pi_{22}\left(\zeta^{-1}(x)\right).
\]
Clearly
\[
\Gamma = \left\{ \begin{pmatrix} \eta(w)^{-2} & 0 & 0 \\
0 & \eta(w) & \eta(w)w \\
0 & 0 & \eta(w) \end{pmatrix} : w \in \zeta(\Gamma) \right\}.
\]
The following claim concludes the proof:

Claim 1.- The group $\zeta(\Gamma)$ is discrete. On the contrary let us assume that $\zeta(\Gamma)$ is discrete in this case there is a sequence $(\gamma_n)_{n \in \mathbb{N}} \subset \Gamma$ of distinct elements such that $(\zeta(\gamma_n))_{n \in \mathbb{N}}$ is a sequence of distinct element such that $\zeta(\gamma_n) \overset{m \to \infty}{\longrightarrow} 0$, then
\[
\gamma_n = \begin{pmatrix}
\eta(\zeta(\gamma_n))^{-2} & 0 & 0 \\
0 & \eta(\zeta(\gamma_n)) & \eta(\zeta(\gamma_n))\zeta(\gamma_n) \\
0 & 0 & \eta(\zeta(\gamma_n))
\end{pmatrix}
\]
Since $\eta(\Gamma) \subset S^1$, we conclude that $(\gamma_n)$ contains a convergent subsequence, which is a contradiction. Therefore $\zeta(\Gamma)$ is discrete. \qed

Lemma 5.13. Let $\Gamma \subset \text{PSL}(3, \mathbb{C})$ be a discrete group where each element has the form:
\[
\gamma = \begin{pmatrix}
a^{-2} & a \\
a & a
\end{pmatrix}.
\]
Then $\Gamma$ is virtually cyclic.

Proof. Define $\rho_{12} : \Gamma \to \mathbb{R}$ by $\rho_{12}(\gamma) = \log(|\lambda_{12}|)$, clearly $\rho_{12}$ is a well defined group morphism, $\text{Ker}(\rho_{12}) = \{ \gamma \in \gamma : |\lambda_{12}| = 1 \}$ and $\rho_{12}(\Gamma)$ is discrete. Let $\Gamma_{nd} \subset \Gamma$ be a torsion free subgroup of $\Gamma$ with finite index. Clearly $\rho_{12}|_{\Gamma_{nd}}$ is injective and $\rho_{12}(\Gamma_{nd})$ is cyclic, which concludes the proof. \qed

Now the following result is trivial.
Theorem 5.14. Let $\Gamma \subset U_+$ be a discrete group such that there is a parabolic element $\gamma \in \Gamma$ satisfying $\max \{o(\lambda_{12}(\gamma)), o(\lambda_{23}(\gamma))\} = \infty$, then $\Gamma$ contains a subgroup of finite index which is conjugate to:

$$G = \left\{ \begin{bmatrix} 1 & w & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^2\eta(w) \end{bmatrix} : w \in W, n \in \mathbb{Z} \right\}$$

where $W$ is a discrete additive subgroup of $\mathbb{C}^2$, $\eta : W \to \mathbb{C}^*$ is a group morphism and $b$ is either 1 or $|b| \neq 1$.

6. Triangular purely parabolic groups with trivial kernel

In this subsection we study purely parabolic groups with an invariant full flag and finite kernel.

Lemma 6.1. Let $W \subset \mathbb{C}^2$ be an additive subgroup such that for each $x, y \in W$ we have $\pi_1(x)\pi_2(y) = \pi_1(y)\pi_2(x)$, then:

1. If $\ker(\pi_1) \cap W$ and $\ker(\pi_2) \cap W$ are trivial, then there is $\mu \in \mathbb{C}^* \setminus R$ an additive group of $\mathbb{C}$ such that $W = \{r(1, \mu) : r \in R\}$.

2. If $\ker(\pi_1) \cap W$ is non-trivial, then there and $R$ an additive group of $\mathbb{C}$ such that $W = \{(r, 0) : r \in R\}$.

3. If $\ker(\pi_2) \cap W$ is non-trivial, then there is $R$ an additive group of $\mathbb{C}$ such that $W = \{0, r) \in R\}$.

Proof. Let show 1. Let $x \in W \setminus \{0\}$ and define $\mu = \pi_2(x)/\pi_1(x)$, clearly $\mu$ is well defined and does not depend on the choice of $x$. Then $W = \{(\pi_1(x), \pi_1(x)\mu) : x \in W\}$. To conclude this part we will be enough to take $R = \pi_1(W)$.

In order to proof 2 will be enough to show that $\pi_2(W)$ is trivial. On the contrary let us assume that there is $B \in W$ such that $\pi_2(B) \neq 0$, consider an element $A \in \ker(\pi_2) \cap W \setminus \{0\}$, thus $0 = \pi_1(A)\pi_2(B) = \pi_1(B)\pi_2(A) \neq 0$, which is a contradiction. The proof of part 3 is similar to this one so we will omit here.

Definition 6.2. We define

$$\text{Heis}(3, \mathbb{C}) = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{C} \right\}$$

Clearly $\text{Heis}(3, \mathbb{C})$ is six dimensional lie group.

Lemma 6.3. Let $\Gamma \subset \text{Heis}(3, \mathbb{C})$ be a commutative discrete group. If $\ker(D) \cap \Gamma$ and $\ker(\Pi) \cap \Gamma$ are trivial then there exist $R \subset \mathbb{C}$ an additive subgroup and $L : W \to \mathbb{C}$ an additive function such that:

1. The group $\Gamma$ is conjugate to:

$$L = \left\{ \begin{bmatrix} 1 & x & L(x) + 2^{-1}x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x \in R \right\}.$$
The Kulkarni's limit set is given by
\[ \Lambda_M(\mathcal{L}) = \Lambda_{Kul}(\mathcal{L}) = \overrightarrow{e_1}, \overrightarrow{e_2}. \]

Moreover, the ordinary region in the sense of Kulkarni for \( \mathcal{L} \) is the largest open set on which the group acts properly discontinuously.

The group \( \mathcal{L} \) is a free abelian group with rank at most four.

If \( R \) is discrete then \( \mathcal{L} \) admits a linear extension to \( \text{Span}_2(R) \).

**Proof.** Let us show part (1). Consider the following auxiliary function
\[ \zeta : \Gamma \to \mathbb{C}^2 \]

\[ \gamma \mapsto (\pi_1(\gamma), \pi_3(\gamma)), \]

then \( \zeta \) is a group morphism. Since \( \text{Ker}(\zeta) \subset \text{Ker}(D) \cap \text{Ker}(\Pi) \cap \Gamma \), the following is a well defined function
\[ \kappa : R = \zeta(\Gamma) \to \mathbb{C} \]

\[ x \mapsto \pi_1(\zeta^{-1}(x)). \]

Now is trivial that
\[ \Gamma = \left\{ \begin{bmatrix} 1 & x & \kappa(x, y) \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : (x, y) \in \text{Span}_2(W) \right\}. \]

Now let \( x, y \in R \), then \( A = B \) where
\[
A = \begin{pmatrix}
1 & \pi_1(x) & \kappa(x) \\
0 & 1 & \pi_2(x) \\
0 & 0 & 1
\end{pmatrix}
\quad \quad \quad
B = \begin{pmatrix}
1 & \pi_1(y) & \kappa(y) \\
0 & 1 & \pi_2(y) \\
0 & 0 & 1
\end{pmatrix}
\]

\[
A^{-1} = \begin{pmatrix}
1 & \pi_1(x+y) & \kappa(x) + \pi_1(x)\pi_2(y) + \kappa(y) \\
0 & 1 & \pi_2(x+y) \\
0 & 0 & 1
\end{pmatrix}
\quad \quad \quad
B^{-1} = \begin{pmatrix}
1 & \pi_1(x+y) & \kappa(x) + \pi_1(x)\pi_2(x) + \kappa(y) \\
0 & 1 & \pi_2(x+y) \\
0 & 0 & 1
\end{pmatrix}
\]

Therefore \( \kappa(x+y) = \kappa(x) + \kappa(y) + \pi_1(x)\pi_2(y) \) and \( \pi_1(x)\pi_2(y) = \pi_1(y)\pi_2(x) \), for every \( x, y \in W \). Thus by Lemma 6.1 there is \( R \subset \mathbb{C} \) an additive subgroup and \( \mu \in \mathbb{C}^* \) such that \( W = R(1, \mu) \), let us consider
\[ \tau = \begin{bmatrix} 1 & 0 & 0 \\
0 & \mu^{-1/2} & 0 \\
0 & 0 & \mu^{1/2} \end{bmatrix} \]

and observe that:
\[ \tau \Gamma \tau^{-1} = \left\{ \begin{bmatrix} 1 & x & \bar{\kappa}(x) \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x \in R \right\}, \]

where \( \bar{\kappa} : R \to \mathbb{C} \) satisfies \( \bar{\kappa}(x+y) = \bar{\kappa}(x) + \bar{\kappa}(y) + xy \). Define \( L : R \to \mathbb{C} \) by \( L(x) = \bar{\kappa}(x) - 2^{-1}x^2 \), thus
\[ L(x+y) = \bar{\kappa}(x+y) - 2^{-1}(x+y)^2 = \bar{\kappa}(x) + \bar{\kappa}(y) - 2^{-1}x^2 - 2^{-1}y^2 = L(x) + L(y) \]

which concludes this part of the proof.

The proof of part (2) goes as follows. Let \( (\gamma_m) \subset \Gamma \) be a sequence of distinct elements of \( \Gamma \), then there is a \( (x_m) \subset R \) a sequence of distinct elements such that
\[
\gamma_m = \begin{bmatrix}
k_m^{-1} & x_mk_m^{-1} & k_m^{-1}(L(x_m) + x^2/2) \\
k_m^{-1} & 0 & x_mk_m^{-1} \\
0 & 0 & k_m^{-1}
\end{bmatrix}
\]
where \( k_m = \max\{|x_m|, |L(x_m) + x^2/2|\} \). If \( (\gamma_{n_m}) \) is a subsequence of \( (\gamma_m) \) such that \( (\gamma_{n_m}) \) converges to every point in the equality \( \tau \in QP(3, \mathbb{C}) \setminus PSL(3, \mathbb{C}) \), then there are \( a, b, c \in \mathbb{C} \) such that \( |a| + |b| + |c| \neq 0 \) and
\[
\tau = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}
\]
this shows that \( Eq(\Gamma) = P^2_{\mathbb{C}} \setminus \mathbb{C}^2 \), since \( Eq(\Gamma) \subset \Omega_{Ku}(\Gamma) \) and \( \Lambda_{Ku}(\Gamma) \) always contains a line, then \( Eq(\Gamma) = \Omega_{Ku}(\Gamma) \). If \( \Omega \subset P^2_{\mathbb{C}} \) is any open set on which \( \Gamma \) acts properly discontinuously, then \( P^2_{\mathbb{C}} \setminus \Omega \) contains a complex line, say \( \ell \). If \( \gamma \in \Gamma \setminus \{Id\} \), then \( \gamma^m \ell \to m \to \infty \), which concludes this part of the proof.

In order to prove part \( \mathcal{B} \) we need to observe that \( \Gamma \) is an abelian group acting properly discontinuously and freely on \( \mathbb{C}^2 \), thus the rank of \( \Gamma \) must be at most four, see \( \mathcal{B} \). Last part of the theorem is trivial. \( \square \)

As a consequence we get the following result.

**Lemma 6.4.** Let \( \Gamma \subset Heis(3, \mathbb{C}) \) be an commutative discrete group , then

1. If \( Ker(D) \) is non-trivial, then there is a discrete additive subgroup \( W \subset \mathbb{C}^2 \) with rank at most four, such that:
\[
\Gamma = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : (y, z) \in W \right\}
\]

2. If \( Ker(\Pi) \cap \Gamma \) is non trivial, then there exist discrete additive subgroup \( W \subset \mathbb{C}^2 \) such that the group \( \Gamma \) is given by:
\[
\tilde{\Gamma} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in W \right\}.
\]

Moreover, if \( \Gamma \) is complex Kleinian, then \( W \) has rank at most 2.

To conclude this subsection let us present the main result of this part.

7. **Weakly semi-controllable purely groups with infinite Kernel**

In this section we are going to describe those purely parabolic groups with a fixed point and infinite kernel. The following definition will be helpful:

**Definition 7.1.** Let \( \Gamma \subset Heis(3, \mathbb{C}) \) be a discrete group, we define
\[
B(\Gamma) = \{ (\gamma_n) \subset \Gamma : \Pi(\gamma_n) \text{ converges to some projective transformation} \}
\]
\[
L(\Gamma) = \{ \gamma \in QP(3, \mathbb{C}) : \text{there is } (\gamma_n) \in B(\Gamma) \text{ converging to } \gamma \}
\]
\[
\mathcal{L}(\Gamma) = \{ \ell \in P^2_{\mathbb{C}} : \exists \gamma \in L(\Gamma) \text{ satisfying } Ker(\gamma) = \ell \}
\]
\[
\Omega(\Gamma) = P^2_{\mathbb{C}} \setminus \bigcup_{\ell \in \mathcal{L}(\Gamma)} \ell
\]

**Lemma 7.2.** Let \((a_n) \subset \mathbb{C}\) be a sequence converging to \( \infty \), then there exist a subsequence \((k_n) \subset (n)\) such that \((a_n a^{-1}_{k_n}) \) converges to 0.

**Lemma 7.3.** Let \( \Gamma \subset Heis(3, \mathbb{C}) \) be a discrete group. Then for each \( \ell \in \mathcal{L}(\Gamma) \), there is a sequence \((\gamma_n) \subset \Gamma\) of distinct elements and \( \gamma \in QP(3, \mathbb{C})\) such that \( \pi(\gamma_n) \) converges to \( Id \), \( \gamma_n \) converges to \( \gamma \) and \( Ker(\gamma) = \ell \).
Proof. Let \( \ell \in \mathcal{L}(\Gamma) \), then there is a sequence \((\tau_n) \subset \Gamma\) of distinct elements and \( \gamma \in QP(3, \mathbb{C}) \) such that \( \pi(\tau_n) \) converges, \( \tau_n \) converges to \( \gamma \) and \( \text{Ker}(\gamma) = \ell \). So we can assume that:

\[
\tau_n = \begin{pmatrix}
1 & x_n & y_n \\
0 & 1 & z_n \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\gamma = \begin{pmatrix}
0 & x & y \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Set \( (a_n = \max\{|x_n|, |y_n|\}) \) and \((k_n) \subset (n)\) the sequence given by lemma 7.2, thus

\[
\gamma_n = \tau^{-1}_{n}k_n = \begin{pmatrix}
1 & x_{k_n} - x_n & -y_n + x_nz_n + y_k - x_nz_k \\
0 & 1 & z_k - z_n \\
0 & 0 & 1
\end{pmatrix}.
\]

Clearly \( \pi(\gamma_n) \) converges to \( Id \) and \( \gamma_n \xrightarrow{n \to \infty} \gamma \), which concludes the proof. \( \square \)

**Lemma 7.4.** Let \( \Gamma \subset \text{Heis}(3, \mathbb{C}) \) be a discrete group such that \( \text{Ker}(\Gamma) \) is infinite, then:

1. \( \Gamma \) acts properly discontinuously on \( \Omega(\Gamma) \);
2. \( \Omega(\Gamma) \) is the largest open set on which \( \Gamma \) acts properly discontinuously.
3. Each connected component of \( \Omega(\Gamma) \) is contractible.

Proof. Let us start showing part (1). It is clear that \( \Omega(\Gamma) \) is non-empty, open and \( \Gamma \)-invariant. Now, let \( K \subset \Omega(\Gamma) \) be a compact set and define \( K(\Gamma) = \{ \gamma \in \Gamma : \gamma(K) \cap K \neq \emptyset \} \). Assume that \( K(\Gamma) \) is infinite. Let \( \gamma_n \) be an enumeration of \( K(\Gamma) \), then there exists a subsequence of \( \gamma_n \), still denoted \( \gamma_n \), such that either \( \pi(\gamma_n) \) converges to a projective transformation or \( \pi(\gamma_n) \xrightarrow{n \to \infty} [e_2] \) uniformly on \( e_2, e_3 \).

If \( \pi(\gamma_n) \) converges to a projective transformation, we can find a subsequence \( \tau_n \subset \gamma_n \) and \( \alpha \in \text{LimB}(\Gamma) \) such that \( \tau_n \) converges to \( \alpha \). Thus \( \text{Ker}(\alpha) \in \mathcal{L} \) and \( \text{Im}(\alpha) = \{e_1\} \), therefore the accumulation set of \( \{\tau_n(K) : n \in \mathbb{N}\} \) is \( \{e_1\} \). Now, in case \( \pi(\gamma_n) \xrightarrow{n \to \infty} [e_2] \) uniformly on \( e_2, e_3 \)

\[
\{\gamma_n : n \in \mathbb{N}\} \subset \{\gamma \in \Pi(\Gamma) : \gamma(\pi(K)) \cap \pi(K) \neq \emptyset\},
\]

which is not possible, this part of the proof.

Now let us present the proof of part (2). Let \( \Omega \subset \mathbb{P}^2_\mathbb{C} \) such that \( \Omega \) is open, non-empty, \( \Gamma \)-invariant on which \( \Gamma \) acts properly discontinuously on \( \Omega \) and \( \ell \in \mathcal{L}(\Gamma) \). Then there are \( \gamma_n \in B(\Gamma) \) and \( \alpha \in \text{LimB} \) such that \( \text{Ker}(\alpha) = \ell \) and \( \gamma_n \) converges to \( \alpha \). For each \( n \in \mathbb{N} \), \( \gamma_n \) is given by:

\[
\gamma_n = \begin{pmatrix}
1 & x_n & y_n \\
0 & 1 & z_n \\
0 & 0 & 1
\end{pmatrix}
\]

By Lemma 7.3 we can assume that \( z_n \) converges to 0. So we get

\[
\alpha = \begin{pmatrix}
0 & x & y \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
On the other hand, observe
\[
\gamma^{-1}_n = \begin{bmatrix} 1 & -x_n & x_n z_n - y_n \\ 0 & 1 & -z_n \\ 0 & 0 & 1 \end{bmatrix} \quad n \to \infty \alpha
\]

By Lemma 2.1 we conclude \( \ell \cap \Omega = \emptyset \). This concludes this part of the proof.

To conclude let us see the proof of part (3). Define
\[
C(\Gamma) = \hat{e}_2, e_3^2 \cap \bigcup_{\ell \in C(\Gamma)} \ell
\]
then \( C(\Gamma) \) is a closed \( \Pi(\Gamma) \)-invariant set on \( \hat{e}_2, e_3^2 \). In consequence \( C(\Gamma) \) is a closed \( \Pi(\Gamma) \)-invariant set. It is clear that there is an additive Lie subgroup \( G \subset \mathbb{C} \) such that \( \Pi(1) = \{z + b : b \in G\} \), therefore there is a family \( P \) of 1-dimensional real projective spaces in \( \hat{e}_2, e_3^2 \) satisfying \( C(\Gamma) = \bigcup P \) and \( \ell_1 \cap \ell_2 = e_2 \) for every pair of distinct \( \ell_1, \ell_2 \in P \). In consequence each connected component of \( \Omega(\Gamma) \) is homeomorphic to \( \mathbb{R}^4 \).

**Definition 7.5.** We will say the sequences \( (a_n), (b_n) \in \mathbb{C}^2 \) are co-bounded if both sequences converge to \( \infty \) and the \( ||a_n||, |b_n|^{-1} \) is bounded and bounded away from \( 0 \).

**Lemma 7.6.** Let \( (a_n), (b_n), (c_n), (x_n), (y_n), (z_n) \in \mathbb{C} \) be sequences of distinct elements such that:
1. \( (c_n) \) and \( (z_n) \) converge to \( 0 \),
2. \( (a_n, b_n), (x_n, y_n) \) are co-bounded:
3. \( [a_n, b_n] \xrightarrow{n \to \infty} [a, b] \);
4. \( [x_n, y_n] \xrightarrow{n \to \infty} [x, y] \);
5. \( [a, b] \neq [x, y] \).

then there is \( w \in \mathbb{C} \setminus \{0\} \) such that for each \( k, m \in \mathbb{N} \setminus \{0\} \) we get:
\[
\gamma_{nkm} = \begin{bmatrix} 1 & a_n & b_k \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_n & y_n \\ 0 & 1 & z_n \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{n \to \infty} \gamma_{kmw} = \begin{bmatrix} 0 & ka + mx_0w & kb + myw \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

**Proof.** Define \( r_n = \max\{|a_n|, |b_n|\} \), \( s_n = \max\{|x_n|, |y_n|\} \) and \( t_n = \max\{s_n, r_n\} \). Since \( (a_n, b_n), (x_n, y_n) \) are co-bounded we can assume there are \( r, s \in \mathbb{R} \setminus \{0\} \) such that \( r_n t_n^{-1} \xrightarrow{n \to \infty} r \) and \( s_n t_n^{-1} \xrightarrow{n \to \infty} s \). In addition, since \( [a_n, b_n] \xrightarrow{n \to \infty} [a, b] \) and \( [x_n, y_n] \xrightarrow{n \to \infty} [x, y] \), we deduce that there are \( u, v \in \mathbb{C}^* \) such that
\[
r_n^{-1}(a_n, b_n) \xrightarrow{n \to \infty} u(a, b)
\]
\[
s_n^{-1}(x_n, y_n) \xrightarrow{n \to \infty} v(x, y)
\]
A simple calculation shows:
\[
\gamma_{nkm} = \begin{bmatrix} t_n^{-1} & \frac{k a_n + mx_n}{t_n} & \frac{k b_n + ax_n}{t_n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{n \to \infty} \gamma_{kmw}
\]
where \( w = vs(ur)^{-1} \), which concludes the proof. \( \square \)

**Lemma 7.7.** Let \( \Gamma \subset \text{Heis}(3, \mathbb{C}) \) be a Kleinian group such that \( \Pi(\Gamma) \) non-discrete and \( \mathbb{P}_E^2 \setminus \Omega(\Gamma) \) contains more than a line, then \( \overline{\Pi(\Gamma)} \) is isomorphic to \( \mathbb{R} \).

**Proof.** We have that \( \Pi(\Gamma) \) is a non-discrete subgroup of \( \mathbb{C} \), thus \( \overline{\Pi(\Gamma)} \) is isomorphic to \( \mathbb{C} \), \( \mathbb{R} \oplus \mathbb{Z} \) or \( \mathbb{R} \). Since \( \Gamma \) is Kleinian we deduce \( \Pi(\Gamma) \) is isomorphic either to \( \mathbb{R} \oplus \mathbb{Z} \) or \( \mathbb{R} \). Let us assume that \( \Pi(\Gamma) \) is isomorphic to \( \mathbb{R} \oplus \mathbb{Z} \). After conjugation, if necessary, we can assume that there is \( s > 0 \) such that \( \Pi(\Gamma) = \{ r + ms : r \in \mathbb{R}, m \in \mathbb{Z} \} \).

On the other hand, since \( \mathcal{L}(\Gamma) \) contains more than a line, we can find a line \( \ell \) containing \( e_1 \) such that \( \ell \neq \overline{e_1, e_2} \). By Lemma 7.3 we can find \((\gamma_n) \subset \Gamma \) and \( \gamma \in QP(3, \mathbb{C}) \) such that \( \Pi(\gamma_n) \xrightarrow{n \to \infty} Id \), \( \gamma_n \xrightarrow{n \to \infty} \gamma \) and \( \ell = \ker(\gamma) \). Thus there are sequences \((a_n), (b_n), (c_n) \subset \mathbb{C} \) such that \( \max\{ |a_n|, |b_n| \} \xrightarrow{n \to \infty} \infty \), \( c_n \xrightarrow{n \to \infty} 0 \), \( [a_n, b_n] \xrightarrow{n \to \infty} [a, b] \) and

\[
\gamma_n = \begin{pmatrix} 1 & a_n & b_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{pmatrix}; \quad \gamma_n = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

a simple calculation shows \( \ell = \overline{[0, b, -a], e_1} \). Let us consider the following claim.

Claim 1.- There are \( f : \mathbb{Z} \to \mathbb{C} \) and \( g : \mathbb{Z} \to \{ \text{real projective subspaces of } \mathbb{C} \} \) such that \( \text{Im}(f(n)) \xrightarrow{n \to \infty} \pm \infty \), \( [b, -a], [f(n), 1] \in g(n) \) and for each \( c \in g(n) \setminus \{ \infty \} \) we have \( [e_1, [0, c, 1]] \subset \mathbb{P}_E^2 \setminus \Omega(\Gamma) \). Let \( x, y \in \mathbb{C} \) such that

\[
\tau = \begin{bmatrix} 1 & x & y \\ 0 & 1 & is \end{bmatrix} \in \Gamma
\]

Let \( m \in \mathbb{Z} \), then a straightforward calculation shows:

\[
\gamma_{nm} = \tau^{-m} \gamma_n \tau^m = \begin{pmatrix} 1 & a_n & b_n - c_nmx + isma_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{n \to \infty} \begin{pmatrix} 0 & a & b + isma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Applying Lemma 7.6 to the sequences \((\gamma_n)_{n \in \mathbb{N}}, (\gamma_{nm})_{n \in \mathbb{N}} \) we conclude there is \( w_n \in \mathbb{C}^* \) such that \( \ell_{klm} = e_1, [0, kbw_n + l(b + isma), -kaw_n - la] \in \mathcal{L}(\Gamma) \), for each \( k, l \in \mathbb{Z} \). Now, by Lemma 3.3 \( C_m = \mathcal{S} \text{span}_\mathbb{Z}(\{ w_m(b, -a), (b + isma, -a) \}) \setminus \{ 0 \} \) is a real projective space containing \([b,-a]\) and \([b+isma,-a]\). To conclude define \( f(m) = -(b + isma)a^{-1}, g(m) = C_m \) and observe that for each \( c \in g(n) \setminus \{ \infty \} \)

\[
[e_1, [0, c, 1]] \subset \bigcup_{k,l \in \mathbb{Z}} \ell_{klm} \subset \mathbb{P}_E^2 \setminus \Omega(\Gamma).
\]

Since

\[
\Pi(\Gamma)[w, 1] = \bigcup_{r \in \mathbb{R}, n \in \mathbb{N}} [w + r + isn, 1],
\]

the claim yield \( \Omega(\Gamma) = \emptyset \), which is a contradiction. \( \square \)

**Lemma 7.8.** Let \( \alpha, \beta, \gamma \in \text{Heis}(3, \mathbb{C}) \) such that \( \alpha, \beta, \gamma \notin \text{Ker}(\Pi) \). If \( [\alpha, \beta] = Id \) and \( [\beta, \gamma] = Id \) then \( [\alpha, \gamma] = Id \).
Proof. Assume that  
\[
\alpha = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}
\]
then  
\[
[\alpha, \beta] = \begin{pmatrix} 1 & 0 & cx - az \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
that is \([\alpha, \beta] = Id\) if and only if \(xz^{-1} = ac^{-1}\), which concludes the proof. \(\Box\)

Lemma 7.9. Let \(\Gamma \subset Heis(3, \mathbb{C})\) be a discrete group, then there are \(B_1, \ldots, B_n\) subgroups of \(\Gamma\) such that \(\Gamma = \text{Core}(\Gamma) \rtimes B_1 \times \cdots \times B_n\) and each \(B_i\) is isomorphic to \(\mathbb{Z}^{k_i}\). Moreover if \(\Gamma\) is Kleinian, then \(\text{rank}(\text{Core}(\Gamma)) + \sum_{i=1}^{n} k_i \leq 4\).

Proof. We know that \(\Gamma\) is finitely generated, therefore the group \(\Pi(\Gamma)\) is finitely group. If \(n = \text{rank}(\Pi(\Gamma))\), let \(G = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma\) such that \(\Pi(\Gamma)\) is generated by \(\Pi(G)\). Let us consider the following equivalence relation in \(G\), let’s say that \(a \sim b\) if and only if \([a, b] = Id\). If \(A_1, \ldots, A_n\) are the equivalence classes in \(G\) induced by \(\sim\), then define \(B_0 = \text{Ker}(\text{Core}(\Gamma))\) and \(B_i = \langle A_i \rangle\). Now it is clear that \(\Gamma = B_0 \rtimes \cdots \rtimes B_n\).

On the other hand, by Lemma 7.7 the group \(\Gamma\) leaves invariant each connected component of \(\Omega(\Gamma)\). Since each component of \(\Omega(\Gamma)\) is simply connected (see Lemma 7.4), the obstruction dimension of \(\Gamma\), see Theorem 1 in [4], satisfies \(\text{obdim}(\Gamma) \leq 4\). Moreover, since each \(B_i\) is finitely generated and torsion free abelian group we deduce each group is semi-hyperbolic, see [1]. Therefore, Corollary 27 in [4] yields \(\sum_{i=0}^{n} \text{obdim}(B_i) \leq 4\). To conclude observe that \(\text{obdim}(B_i) = \text{rank}(B_i)\). \(\Box\)

Definition 7.10. Let \(\Gamma \subset U_+\) be a discrete subgroup, then we define:
\[
\text{Core}(\Gamma) = \text{Ker}(\lambda_{12}) \cap \text{Ker}(\lambda_{23}) \cap \text{Ker}(\Gamma)
\]
It is clear that \(\text{Core}(\Gamma)\) is a normal subgroup of \(\Gamma\) were each element has a lift in \(SL(3, \mathbb{C})\) given by:
\[
\begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Lemma 7.11. Let \(\Gamma \subset Heis(3, \mathbb{C})\) be a complex Kleinian group with infinite kernel. Then:
(1) The group \(\text{Core}(\Gamma)\) is non-trivial.
(2) \(\text{Core}(\Gamma) = \mathbb{Z}^k\) where \(k \leq 2\).
(3) \(\Lambda_{Kul}(\text{Core}(\Gamma)) = L_0(\text{Core}(\Gamma))\) is either a line or a pencil of lines over a circle.
(4) If the group \(\Lambda_{Kul}(\text{Core}(\Gamma))\) is a line, then there is a discrete additive subgroup \(W\) of \(\mathbb{C}\) such that \(\Gamma\) is conjugate \n\[
\Gamma_W = \left\{ \begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : w \in W \right\}
\]
and \(\text{rank}(\text{Core}(\Gamma)) \leq 2\).
If $\Lambda_{Kul}(\text{Core}(\Gamma))$ is a pencil of lines over a circle, then the groups $\mathcal{D}(\text{Core}(\Gamma))$ and $\pi_{23}(\text{Core}(\Gamma))$ are non-trivial and $\text{rank}(\text{Core}(\Gamma)) = 2$.

If the group $\Pi(\Gamma)$ is non-trivial. Then the group $\mathcal{D}(\text{Core}(\Gamma))$ is non-trivial if and only if $\Lambda_{Kul}(\text{Core}(\Gamma))$ is a pencil of lines over a circle.

Proof. The proof of parts (1), (2) and (3) are trivial in view of example 3.5, so we will omit it here. The proof of part (4), goes as follows: It is clear that there is $W \subset \mathbb{C}^2$ a $\mathbb{R}$-linearly independent set such that $\Gamma = W^*$, where $W^*$ is given as in example 3.5. We know

$$\Lambda_{Kul}(\text{Core}(\Gamma)) = \bigcup_{p \in W^*} \tilde{e}_1, \tilde{p}.$$ 

where $W^* = \text{Span}_\mathbb{Z}\{(y, -x) : (x, y) \in W \setminus \{0\}\}$. Since $\Lambda_{Kul}(\text{Core}(\Gamma))$ is a single line, from Lemma 3.4 we deduce $W$ is either a single point or contains exactly two $\mathbb{C}$-linearly dependent vectors. Let us workout the later case, thus there is $\alpha \in \mathbb{C}$ and $w \in W$, such that one has:

$$\Gamma = \left\{ \begin{pmatrix} 1 & (n + m\alpha)x & (n + m\alpha)y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n, m \in \mathbb{Z} \right\}$$

Let $r \in \mathbb{R}^*$ be such that $x \neq yr$, it is clear the following calculation concludes this part of the proof

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 1 \\ 0 & x & y \end{pmatrix} \begin{pmatrix} 1 & (n + m\alpha)x & (n + m\alpha)y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 1 \\ 0 & x & y \end{pmatrix}^{-1} = \begin{pmatrix} 1 & m + n\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The proof of part (5), is simple, we just need to remember that $\mathcal{D}(\text{Core}(\Gamma))$ trivial or $\pi_{13}(\text{Core}(\Gamma))$ implies $\Lambda_{Kul}(\text{Core}(\Gamma))$ is a single line.

To conclude let us show part (6). Since $\mathcal{D}(\text{Core}(\Gamma))$ and $\Pi(\Gamma)$ are both non-trivial, we deduce that there are $a, b, x, y, z \in \mathbb{C}$ and $\gamma, \tau \in \Gamma$ such that $az \neq 0$ and

$$\gamma = \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

A straightforward calculation shows

$$\tau \gamma \tau^{-1} = \begin{pmatrix} 1 & a & b - az \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to conclude the proof we only need to observe that $(a, b)$ and $(a, b-az)$ are $\mathbb{C}$-linearly independent vectors. \(\square\)

In the following will say a subgroup $\Gamma$ of $U^+(3, \mathbb{C})$ is irreducible if $\Lambda_{Kul}(\text{Core}(\Gamma))$ is a pencil of lines over a circle and reducible if it is a line.

**Lemma 7.12.** Let $\Gamma \subset \text{Heis}(3, \mathbb{C})$ be an irreducible complex Kleinian such that $\Pi(\Gamma)$ is non-elementary, then

1. $\Pi(\Gamma)$ is discrete
2. $\text{rank}(\Pi(\Gamma)) = 1$
Proof. Let us show $\Pi(\Gamma)$ is discrete, if this is not the case, then we can assume there is a sequence $(\gamma_n) \subset \Gamma$ such that $\Pi(\gamma_n)$ is a sequence of distinct elements converging to $Id$. On the other hand, since $Core(\Gamma)$ is irreducible we conclude there $\tau \in Core(\Gamma)$ such that $D(\tau) \neq Id$. If $\gamma_n$ and $\tau$ are given respectively by

$$\gamma_n = \begin{bmatrix} 1 & a_n & b_n \\ 0 & 1 & c_n \\ 0 & 0 & 1 \end{bmatrix}; \tau = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A straightforward calculation shows

$$\gamma_n \tau \gamma_n^{-1} = \begin{bmatrix} 1 & x & ye^{-c_n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{n \to \infty} \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

which concludes the proof of this part.

Now let us show that $\Pi(\Gamma)$ has rank 1. On the contrary let us assume that $\Pi(\Gamma)$ has rank 2. Let $\gamma_1, \gamma_2, \tau \in \Gamma$ such that $\langle \Pi(\gamma_1), \Pi(\gamma_2) \rangle = \Pi(\Gamma)$, $\gamma \in Core(\Gamma)$ and $D(\tau) \neq Id$. f $\gamma_1, \gamma_2, \tau$ are given respectively by

$$\gamma_1 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}; \gamma_2 = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}; \tau = \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A straightforward calculation shows

$$\tau^{-1} \gamma_1 \tau \gamma_1^{-1} = \begin{bmatrix} 1 & 0 & -uc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tau^{-1} \gamma_2 \tau \gamma_2^{-1} = \begin{bmatrix} 1 & 0 & -uz \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

To conclude the proof observe $\{(u, v), (0, -uc), (0, -uz)\}$ is a $\mathbb{R}$-linearly independent set. 

Lemma 7.13. Let $W = \{(1, 0), (c, d)\} \subset \mathbb{C}^2$ be an $\mathbb{R}$-linearly independent set. Then $(0, 1), (0, B) \in Span_\mathbb{Z}(W)$ if and only if there are $p, q, r \in \mathbb{N}$ such that $p, q$ are co-primes, $q^2$ divides $r$, $c = pq^{-1}$, and $d = r^{-1}$.

Proof. Since $(0, a), (0, c) \in Span_\mathbb{Z}(W)$ we deduce there are $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ such that

$$k_1 + k_2c = 0$$
$$k_3d = 1$$
$$k_3 + k_4c = 0$$
$$k_4d = c$$

From the first two equations, we conclude $d = k_2^{-1}$, $c = -k_1 k_2^{-1}$. Let $p, q \in \mathbb{N}$ be co-primes such that $c = pq^{-1}$, substituting in the last two inequalities we get

$$k_3q + k_4p = 0$$
$$k_4q = pk_2$$

form the first equation we get $q$ divides $k_4$, thus there is $m \in \mathbb{Z}$ such that $k_4 = qm$, substituting in the last equality we get:

$$mq^2 = pk_2$$

From which we conclude $q^2$ divides $k_2$. 


Finally, let us assume that \( p, q \) are co-primes and \( r = q^2 n \), then:

\[
-p^2 n(1, 0) + qpn(pq^{-1}, (q^2 n)^{-1}) = (0, pq^{-1})
\]
\[
-pqn(1, 0) + qqn(pq^{-1}, (q^2 n)^{-1}) = (0, 1)
\]

\( \square \)

**Proposition 7.14.** If \( \Gamma \subset \text{Heis}(3, \mathbb{C}) \) is an irreducible complex Kleinian such that \( \Pi(\Gamma) \) is non-elementary. Then there exist \( x, y \in \mathbb{C}, p, q, r \in \mathbb{Z} \) such that \( p, q \) are co-primes, \( q^2 \) divides \( r \) and \( \Gamma \) is conjugated to

\[
\Gamma_w = \left\{ \begin{bmatrix} 1 & k + lc + mx & ld + m(k + lc) + \left( \frac{m}{2} \right) x + my \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} : (k, l, m) \in \mathbb{Z} \right\}.
\]

where \( w = (x, y, p, q, r) \), \( c = pq^{-1} \) and \( d = r^{-1} \).

**Proof.** By Lemma 7.12 we know \( \Pi(\Gamma) \) is discrete and has rank equal to 1 and by Lemma 7.11 we have \( \text{rank}(\text{Core}(\Gamma)) = 2 \) and \( \text{D}(\text{Core}(\Gamma)) \) is non-trivial, thus by Lemma 7.9 there is \( W = \{(a, b), (c, d)\} \) a \( \mathbb{C} \)-linearly independent set and \( u, v, w \in \mathbb{C} \) such that

\[
(7.1) \quad \Gamma = \left\{ \begin{bmatrix} 1 & ka + lc & kb + ld \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n : k, l, n \in \mathbb{Z} \right\},
\]

\( aw \neq 0 \). A straightforward calculation shows

\[
\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \gamma_1
\]

\[
\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{c}{a} & \frac{d}{a} - \frac{bc}{a^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \gamma_2
\]

\[
\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} \\ 0 & 0 & w \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{u}{a} & \frac{v}{w} - \frac{bu}{aw} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \gamma_3
\]

Now by expression 7.1 we deduce

\[
\Gamma_1 = \{ (\gamma_1)^{k,l,n} : k,l,n \in \mathbb{Z} \}
\]

is a group conjugated to \( \Gamma \). On the other hand, in order \( \Gamma_1 \) forms a group it is necessary and sufficient that for \( i = 1, 2 \)

\[
\gamma_3 \gamma_i \gamma_3^{-1} \in \langle \langle \gamma_1, \gamma_2 \rangle \rangle.
\]

A straightforward calculation shows this is equivalent to

\[
(0, 1), (0, ca^{-1}) \in \text{Span}_x(\{(1, 0), (ca^{-1}, (aw)^{-1}(d - bca^{-1}))\}).
\]

Now the conclusion follows easily from Lemma 7.13

\( \square \)

**Proposition 7.15.** Let \( \Gamma \subset \text{Heis}(3, \mathbb{C}) \) be complex Kleinian such that \( \text{Core}(\Gamma) \) is reducible and \( \Pi(\Gamma) \) is discrete, then \( \Gamma \) is conjugated to one of the following groups
Proof. Since \( \Pi(\Gamma) \) is discrete and in virtue of Lemma 7.4 we will assume that 
\( \text{rank} \, \Pi(\Gamma) \) are torsion free abelian groups with rank less than 2. For simplicity in the 
rest of this proof we will assume that 
\( W \subset \mathbb{C}^2 \) is an additive subgroup such that \( \pi_2(W) \) is discrete.

(2) \[
\Gamma_{W_1, W_2, L} = \left\{ \begin{bmatrix} 1 & x & L(x) + x^2/2 + w \end{bmatrix} : w \in \text{Span}_\mathbb{Z}(W_2), x \in \text{Span}_\mathbb{Z}(W_1) \right\}
\]

where \( W_1, W_2 \subset \mathbb{C} \) are \( \mathbb{R} \)-linearly independent sets and \( L : \text{Span}_\mathbb{Z}(W_1) \to \mathbb{C} \) 
is an additive function.

(3)

(4)

(7.5) \[
\Gamma = \left\{ \begin{bmatrix} 1 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & x & y \end{bmatrix}^n \begin{bmatrix} 1 & a & b \end{bmatrix}^m \begin{bmatrix} 1 & u & v \end{bmatrix}^n : m, n \in \mathbb{Z}, w \in \text{Span}_\mathbb{Z}(W) \right\},
\]

where \( W \subset \mathbb{C} \) is a \( \mathbb{R} \)-linearly independent set, \( a, c \in \text{Span}_\mathbb{Z}(W) \) and \( \{1, c\} \) 
is a \( \mathbb{R} \)-linearly independent set.

Proof. Since \( \Pi(\Gamma) \) is discrete and in virtue of Lemma ?? we deduce \( \text{Core}(\Gamma) \) and 
\( \Pi(\Gamma) \) are torsion free abelian groups with rank less than 2. For simplicity in the 
rest of this proof we will assume that \( \text{rank}(\text{Core}(\Gamma)) = \text{rank}(\Pi(\Gamma)) = 2 \), as we will 
see in the proof any other possibility will be covered by this case. Now by Lemma 7.9 there is \( W \subset \mathbb{C} \) 
an additive discrete subgroup with rank 2 and \( x, y, z, u, v, \tilde{w} \in \mathbb{C} \) such that

\[
\Gamma = \left\{ \begin{bmatrix} 1 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & x & y \end{bmatrix}^n \begin{bmatrix} 1 & a & b \end{bmatrix}^m \begin{bmatrix} 1 & u & v \end{bmatrix}^n : m, n \in \mathbb{Z}, w \in \text{Span}_\mathbb{Z}(W) \right\},
\]

and \( z\tilde{w}^{-1} \notin \mathbb{R} \). To simplify the things let us consider the following cases:

Case 1.- \( xw - zu = 0 \). Once again, in order to make calculations easier, we should split this case into the following sub-cases:

Sub-case 1.- \( x = u = 0 \). In virtue of Equation (7.5) is trivial that \( \Gamma \) is conjugate 
to the group given by Equation (7.5)

Sub-case 2.- \( xu \neq 0 \). Let us consider the following calculations

\[
\gamma_w = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & w \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & w(xz)^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\gamma = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & \frac{y}{x} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\]

\[
\tau = \begin{bmatrix} \frac{1}{z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & u & v \end{bmatrix} \begin{bmatrix} \frac{1}{z} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{u}{z} & \frac{v}{z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
By Lemma 6.3 there is an additive function \( L : W_1 = \text{Span}_\mathbb{Z}(\{1, ux^{-1}\}) \rightarrow \mathbb{C} \) such that
\[
\langle \tau, \gamma \rangle = \left\{ \begin{array}{l}
\left[ \begin{array}{ccc}
1 & r & L(r) + 2^{-1}r^2 \\
0 & 1 & r \\
0 & 0 & 1
\end{array} \right] \quad r \in W_1
\end{array} \right\}
\]
Now by a simple inspection of equation 7.5 we conclude \( \Gamma \) is conjugate to a group of the form expressed in equation 7.3.

From Lemma 7.11 it is clear that the previous cases cover all the possibilities for the case \( xw - zu \).

Case 2. \( xw - zu \neq 0 \).

Sub-case 1. \( x = 0, u \neq 0 \). Trivially the following calculations plus equation 7.5 yields \( \Gamma \) is conjugate to a subgroup of the form given by equation 7.4.

\[
\gamma_w = \left[ \begin{array}{ccc}
\frac{1}{w} & 0 & 0 \\
0 & 1 & w \\
0 & 0 & z
\end{array} \right] \left[ \begin{array}{ccc}
1 & 0 & w \\
0 & 1 & 0 \\
0 & 0 & z
\end{array} \right]^{-1} = \left[ \begin{array}{ccc}
1 & 0 & w(uz)^{-1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right]
\]
\[
\gamma = \left[ \begin{array}{ccc}
\frac{1}{u} & 0 & 0 \\
0 & 1 & y \\
0 & 0 & z
\end{array} \right] \left[ \begin{array}{ccc}
1 & 0 & y \\
0 & 1 & z \\
0 & 0 & 1
\end{array} \right]^{-1} = \left[ \begin{array}{ccc}
1 & 0 & \frac{0}{uz} \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array} \right]
\]
\[
\tau = \left[ \begin{array}{ccc}
\frac{1}{u} & 0 & 0 \\
0 & 1 & v \\
0 & 0 & z
\end{array} \right] \left[ \begin{array}{ccc}
1 & 0 & v \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right]^{-1} = \left[ \begin{array}{ccc}
1 & 0 & \frac{0}{uz} \\
0 & 1 & \frac{0}{uz} \\
0 & 0 & 1
\end{array} \right]
\]

Sub-case 2. \( xu \neq 0 \). Through similar arguments one can show in this sub-case \( \Gamma \) is conjugate to a subgroup of the form given by equation 7.4. \( \square \)

Through similar arguments one can show:

**Proposition 7.16.** Let \( \Gamma \subset \text{Heis}(3, \mathbb{C}) \) be a commutative complex kleinian group such that \( \text{Ker}(\Gamma) \) is infinite, \( \Pi(\Gamma) \) non-discrete, then:

1. If \( D(\Gamma) \) is trivial, then there is \( W \subset \mathbb{C}^2 \) an additive discrete subgroup of rank equal to four, such that \( \pi_2(W) \) is non-discrete and \( \Gamma \) is conjugate to

\[
W = \left\{ \left[ \begin{array}{ccc}
1 & 0 & a \\
0 & 1 & b \\
0 & 0 & 1
\end{array} \right] : (a, b) \in W \right\}
\]

2. If \( D(\Gamma) \) is non trivial then there are \( W_1, W_2 \subset \mathbb{C} \) additive subgroups and \( L : W_1 \rightarrow \mathbb{C} \) an additive function such that \( W_1 \) is non-discrete and \( W_2 \) has rank 1 and \( \Gamma \) is conjugate to:

\[
\Gamma_{W_1, W_2, L} = \left\{ \left[ \begin{array}{ccc}
1 & x & L(x) + x^2/2 + w \\
0 & 1 & x \\
0 & 0 & 1
\end{array} \right] : w \in \text{Span}_\mathbb{Z}(W_2), x \in \text{Span}_\mathbb{Z}(W_1) \right\}
\]

**Lemma 7.17.** Let \( r \in \mathbb{R} \setminus \mathbb{Q} \), and \((m_n), (k_n) \subset \mathbb{Z}\) sequences such that \( m_n + k_nr \xrightarrow{n \to \infty} 0 \), then:

1. \( |m_n|, |k_n| \xrightarrow{n \to \infty} \infty \)
2. \( m_n k_n^{-1} \xrightarrow{n \to \infty} -r \)
Lemma 7.18. Let $a, b, c \in \mathbb{C}$ and $r \in \mathbb{R} \setminus \mathbb{Q}$, and let $\Gamma \subset Heis(3, \mathbb{C})$ be the group given by

$$\Gamma = \langle \begin{array}{c}
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix},
B = \begin{bmatrix}
1 & a + r & 0 \\
0 & 1 & r \\
0 & 0 & 1
\end{bmatrix}
\end{array} \rangle$$

then

1. $\Gamma$ is commutative if and only if $a = 0$.
2. If $a \neq 0$, we get $P^2 \setminus \Omega(\Gamma)$ is a cone of lines over a circle.

Proof. The proof of part (1) is straightforward so we will omit it here. In order to proof part (2), observe that by Lemma 7.9 we have:

$$\Gamma = \left\{ \gamma_{mnk} = \begin{bmatrix}
1 & m + nr + na & ak + \frac{m}{2} + mnr + \frac{n}{2}(r + a) \\
0 & 1 & m + nr \\
0 & 0 & 1
\end{bmatrix} : k, m, n \in \mathbb{Z} \right\}$$

By simple algebraic manipulations we get:

$$\frac{a(2k + n^2r) + rn(-a - r + 1) + (m + nr)^2 - (m + nr)}{2} = ak + \frac{m}{2} + mnr + \frac{n}{2}(r + a).$$

Claim 1. $P^2 \setminus \Omega(\Gamma)$ contains more than one line. In order to proof the claim will be enough to show that $P^2 \setminus \Omega(\Gamma)$ contains a line different from $\vec{e}_1, \vec{e}_2$. Let $(a_n), (b_n) \in \mathbb{Z}$ be sequences such that $a_n + b_nr \xrightarrow{n \to \infty} 0$, let us assume that all the elements of the sequence $(a_n)$ are either odd or even. Let $k_0 \in \mathbb{N}$ an even number such that

$$k_0 | a | > | r(-a - r + 1)|.$$ 

Let us define the following sequence

$$c_n = \begin{cases}
2^{-1}b_n(a_n + k_0 + 1) & \text{if } a_n \text{ is odd} \\
2^{-1}b_n(a_n + k_0) & \text{if } a_n \text{ is even}
\end{cases}$$

clearly $(c_n) \subset \mathbb{Z}$ and

$$\gamma_{a_n, b_n, c_n} \xrightarrow{n \to \infty} \gamma = \begin{bmatrix}
0 & 2a & w_0 + r(-a - r + 1) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

where $w_0$ is either $k_0$ or $k_0 + 1$. Trivially $Ker(\gamma)$ is a complex line distinct from $\vec{e}_1, \vec{e}_2$.

Claim 2. $P^2 \setminus \Omega(\Gamma)$ is contained in a pencil of lines over an euclidean circle. Let $(a_n), (b_n), (c_n) \in \mathbb{Z}$ be sequences such that $a_n + b_nr \xrightarrow{n \to \infty} 0$. Let us assume that

$$\gamma_{a_n, b_n, c_n} \xrightarrow{n \to \infty} \gamma = \begin{bmatrix}
0 & x & y \\
0 & 0 & z \\
0 & 0 & 0
\end{bmatrix}.$$
By Lemma 7.17 we have $z = 0$. On the other hand if $x \neq 0$, we get
\[
\begin{align*}
x &= \lim_{n \to \infty} 2(a_n + b_n r + b_n a) b_n^{-1} = 2a, \\
y &= \lim_{n \to \infty} (a(2a_n + b_n^2 r) + rb_n(-a - r + 1) + (a_n + b_n r)^2 - (a_n + b_n r)) b_n^{-1}, \\
&= r(-a - r + 1) + a \lim_{n \to \infty} (2c_n b_n^{-1} + b_n r).
\end{align*}
\]
In consequence $y = sa + 1 - r$ for some $s \in \mathbb{R}$. Therefore
\[
P_\infty \setminus \Omega(\Gamma) \subset \text{Span}_\mathbb{C} \{e_1, e_2\} \cup \bigcup_{s \in \mathbb{R}} \{0 : sa + 1 - r : -2a\}
\]
Finally, since $\Pi(\Gamma)$ is conjugate to a dense subgroup of $\mathbb{R}$ and $P_\infty \setminus \Omega(\Gamma)$ has more than two lines we conclude $P_\infty \setminus \Omega(\Gamma)$ contains a pencil of lines over an euclidean circle, which concludes the proof.

**Lemma 7.19.** Let $a, b \in \mathbb{C}$ and $r, s \in \mathbb{R}$ such that $\{1, r, s\}$ is a $\mathbb{Z}$-linearly independent set and $|a| + |b| \neq 0$, then $W = \text{Span}_\mathbb{Z}\{a, b, ra - sb\}$ is non-discrete.

**Proof.** Since $\{1, r, s\}$ is a $\mathbb{Z}$-linearly independent set, there are sequences $(k_n), (l_n) \subset \mathbb{Z}$ satisfying $k_n + l_n r \to 0$ as $n \to \infty$, define $m_n = \text{floor}(l_n s)$, thus it is clear that $(m_n - l_n s)$ is bounded. Therefore $(k_n a + l_n (ra - sb) + m_n b) \subset W$ contains a convergent sequence.

**Lemma 7.20.** Let $\Gamma \subset \text{Heis}(3, \mathbb{C})$ be a non-abelian kleinian group such that $\text{Ker}(\Gamma)$ is infinite and $\Pi(\Gamma)$ non-discrete, then

1. $\text{Core}(\Gamma)$ is reducible,
2. $P_\infty \setminus \Omega(\Gamma)$ contains more than a line,
3. $\Pi(\Gamma)$ is conjugate to a subgroup of $\mathbb{R}$
4. $\text{rank}(\Pi(\Gamma)) = 2$

**Proof.** The proof of part (1) is straightforward from Lemma 7.12.

The proof of (2) is as follows. Since $\Gamma$ is commutative there are $x, y, z, a, b, c \in \mathbb{C}$ such that $\{z, c\}$ is a $\mathbb{R}$-linearly dependent set but is a $\mathbb{Z}$-linearly independent set, $xc - az \neq 0$ and
\[
\gamma = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \tau = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \Gamma
\]
Since $[\gamma, \tau] \neq \text{Id}$ we can assume $x \neq 0$. A simple computation shows
\[
\gamma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x/z \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]
\[
\tau_1 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y/x \\ 0 & 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & b - ay/xz \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}
\]
The proof in this part conclude applying Lemma 7.18 to $(\gamma_1, \tau_1)$.

Now the proof of (3) is a simple consequence of (2).

To conclude let us present a proof of part (4). On the contrary, let us assume that $\Pi(\Gamma)$ has rank equal to 3. Since $\Gamma$ is not commutative, by Theorem 7.9 the
previous parts of this lemma and after conjugation, if necessary, we can find an
additive discrete subgroup \( W \subset \mathbb{C} \), \( a, b, c, r, s, t \in \mathbb{C} \) such that \( \{1, t, c\} \) is a \( \mathbb{R} \)-linearly dependent set and a \( \mathbb{Z} \)-linearly independent set \( a \neq 0 \) and

\[
\Gamma = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a + c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & r + t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} : k, m, n \in \mathbb{Z}, w \in W \right\}
\]

Since \( \Gamma \) is a group with the previous representation implies \( a, r, rc - at \in W \), by
Lemma \( \ref{lem:group_representation} \) we conclude \( W \) is non-discrete, which is a contradiction. \( \square \)

Now the proof of the following result is simple.

**Proposition 7.21.** Let \( \Gamma \subset \text{Heis}(3, \mathbb{C}) \) be a non-abelian kleinian group such that
\( \ker(\Pi(\Gamma)) \) is infinite and \( \Pi(\Gamma) \) non-discrete, then we can find an additive discrete
subgroup \( W \subset \mathbb{C} \), \( a, b, c \in \mathbb{C} \) such that \( \{1, c\} \) is a \( \mathbb{R} \)-linearly dependent set but a \( \mathbb{Z} \)-linearly independent set \( a \neq 0 \) and up to conjugation we have:

\[
\Gamma = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a + c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & r + t & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} : m, n \in \mathbb{Z}, w \in W \right\}
\]

8. **Proof of the main theorem**

*Proof.* Let \( \Gamma_1 \) be a subgroup of \( \Gamma_0 \) which is triangularizable. Let \( \Gamma_{nd} \subset \Gamma_1 \) be a
subgroup of finite index, such that \( \Gamma_{nd}, \lambda_{12}(\Gamma_{nd}), \lambda_{23}(\Gamma_{nd}), \Pi(\Gamma_{nd}) \)
are torsion free, see \( \ref{sec:main_theorem} \). If \( \Gamma_{nd} \) contains a parabolic element \( \gamma \) satisfying

\[
\max\{o(\lambda_{12}(\gamma)), o(\lambda_{23}(\gamma))\} = \infty,
\]

thus the result follows from Theorem \( \ref{thm:main_theorem} \). So have that each element in \( \Gamma \) can be
written as

\[
\begin{bmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1
\end{bmatrix}
\]

If \( \ker(\Gamma_{nd}) \) is trivial we deduce that \( \Gamma_{nd} \) is commutative. The proof in this case is
done in view of Lemmas \( \ref{lem:commutative} \), \( \ref{lem:finite_index} \) and \( \ref{lem:trivial} \). If \( \ker(\Gamma_{nd}) \) is non-trivial the result
follows from Lemma \( \ref{lem:non_trivial} \) and Propositions \( \ref{prop:non_trivial} \), \( \ref{prop:finite_index} \), \( \ref{prop:trivial} \) and \( \ref{prop:commutative} \). \( \square \)

The proof of Corollary \( \ref{cor:main_theorem} \) is straightforward from Theorem \( \ref{thm:main_theorem} \) and Section \( \ref{sec:main_theorem} \)
so we will omit here.

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References

1. J. M. Alonso, M. R. Bridson, *Semihyperbolic groups*, Proc. London Math. Soc. (3) 70 (1995), no. 1, 56–114.
2. L. Auslander, *Discrete solvable matrix groups*, Proc. Amer. Math. Soc. 11 (1960), 687-688.
3. W. Barrera, A. Cano, J. P. Navarrete, *On the number of lines in the limit set for discrete subgroups of PSL(3, C)*, Vol. 281 (2016), No. 1, 17–49.
4. M. Bestvina, M. Kapovich, B. Kleiner, *Van Kampen’s embedding obstruction for discrete groups*, Invent. math. 150, 219–235 (2002).
5. A. Cano, J. P. Navarrete, J. Seade, *Complex Kleinian Groups*, Birkhäuser, Progress in Mathematics, Vol. 303, 2013.
6. A. I. Mal’tsev, *On a class of homogeneous spaces*, Izv. Akad. Nauk SSSR Ser. Mat., 1949, Volume 13, Issue 1, 9–32.

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