Array enhanced stochastic resonance and spatially correlated noises

P. F. Góra

M. Smoluchowski Institute of Physics and Complex Systems Research Center, Jagellonian University, Reymonta 4, 30-059 Kraków, Poland

(Dated: March 22, 2022)

We discuss the role of spatial correlations of the noise in the array enhanced stochastic resonance. We show numerically that the noises with negative correlations between different sites lead to significantly larger values of the signal-to-noise ratio than the uncorrelated noises or noises with positive correlations. If the noise is global, the system displays only the conventional stochastic resonance, without any array enhancement.

PACS numbers: 05.40.Ca

I. INTRODUCTION

Stochastic resonance (SR) [1] is the best-known example of the constructive role of noise. The SR is a phenomenon in which the response of a system is optimized by the presence of a specific level of noise. It has been detected in so many seemingly different systems that it has been claimed to be “an inherent property of rate-modulated series of events” [2]. It has been recently suggested that the functioning of important natural devices, e.g., communication and information processing in neural systems or subthreshold signal detection in biological receptors, rely on phase synchronization rather than stochastic resonance [3], but this does not exclude the possibility that some natural devices may rely on the SR or that effective artificial detectors that use this feature may be constructed and operated.

It has been observed that the SR gets enhanced if an array of similar nonlinear elements collectively responds to the same signal. This phenomenon has been termed the array enhanced stochastic resonance (AESR). It was first observed in chains of nonlinear oscillators [4] and later, mostly without explicitly using the term AESR, in arrays of FritzHugh-Nagumo model neurons [5], in ion channels [6], in ensembles of nondynamical elements, first introduced in Ref. [12], and later discussed in Refs. [13].

II. UNCOUPLED THRESHOLD ELEMENTS

A single threshold device fires whenever the signal on its input exceeds the threshold. Here we consider an array of N such devices, acting in parallel in response to a common signal; the average (or integrated) output of

*Electronic address: gora@if.uj.edu.pl
all individual devices is taken as the output of the whole array:

\[
g_i(t) = \begin{cases} 
1 & \text{if } A \sin(\omega t + \phi) + \eta_i(t) \geq 1 \\
0 & \text{otherwise}
\end{cases} \quad (1a)
\]

\[
g(t) = \frac{1}{N} \sum_{i=1}^{N} g_i(t). \quad (1b)
\]

Here \(\omega\) is the frequency of the signal (we take \(\omega = 2\pi\)), \(\phi\) is the initial phase, and \(A\) is the amplitude; we take \(A = 0.8\) to make the signal subthreshold. \(g(t)\) is the output of the array and \(\eta_i(t)\) are the noises. We take them to be zero-mean Gaussian white noises (GWNs), possibly spatially correlated:

\[
\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t) \eta_j(t') \rangle = \sigma^2 \delta_{ij} \delta(t - t'). \quad (2)
\]

The matrix \(C = [C_{ij}]\) represents spatial correlations of the noises. It must be symmetric and positively definite. \(\sigma\) is a parameter controlling the intensity of the noises. This system is similar to, but different from that discussed in the first part of Ref. [7], where the local noise was added to the output of each element after the element had decided whether to fire in response to the signal contaminated by a global noise. Thus, the output of a single element was not binary. Rather than that, it could assume, in principle, any value, positive or negative. Moreover, as all elements received identical inputs, they all fired or not fired in unison. After an appropriate scaling, the collective output of the whole array was equivalent to a binary series contaminated by a GWN. The local noises in Ref. [7] were, by assumption, spatially uncorrelated. In our approach, the internal noise is added to the signal at each site before the threshold elements make their decisions whether to fire or not. The internal noise represents fluctuations in the connections or environmentally induced random perturbations, but the output of every threshold element remains binary.

We also start with uncorrelated, or local, noises, \(C_{ij} = \delta_{ij}\). We calculate the signal and the noises with a time step \(h = 1/32\) and calculate the output of the nondynamical system [11]. We use Marsaglia algorithm [18] to generate the GWNs; we use the famous Mersenne Twister [19] as the underlying uniform generator. For each array, we collect a time series of 4096 elements, calculate its power spectrum and calculate the signal-to-noise ratio (SNR):

\[
\text{SNR} = 10 \log_{10} \frac{\text{power density at the signal frequency}}{\text{background power density}}. \quad (3)
\]

For each \(N\) and \(c_1\) we average the results over 512 realizations of the noises and initial phases of the signal. Final results are presented in Fig. 1 and their interpretation is clear: The SNR increases significantly for all

![](image)

**FIG. 1: The AESR for the nondynamical system** [11]. The curves correspond, bottom to top, to the arrays of lengths 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, and 1024, respectively. The noises acting on different elements of the arrays are not correlated, \(C_{ij} = \delta_{ij}\).

noise strengths as the array size doubles. We can see that the AESR is not a result of any specific dynamics but is present also in arrays of uncoupled elements. This, together with previous results for coupled systems, shows that the AESR is a generic feature of arrays of elements that individually display the SR.

Note that we have observed the AESR for a local noise. If the noise were global, \(\forall i, j: \eta_i(t) \equiv \eta_j(t)\) (or \(C_{ij} \equiv 1\)), no array enhancement would be possible. Indeed, for a global noise, each element of the array receives identical input, and the collective output of the whole array is identical with that of a single element. The array does not display the AESR, but only the conventional SR. This simple argument shows that it is the differences between the noises at different sites that cause the AESR. This observation leads to the question of the role of spatial correlation between the noises. Qualitatively speaking, noises with large correlations are “nearly identical” and the enhancement of the SR should be small. The enhancement of the SR should increase as the correlations decrease. We will now see that this is indeed the case.

Let \(C = GG^T\) be the Cholesky decomposition [20] of the correlation matrix \(C\) and let \(\bar{\eta}(t)\) be a vector of spatially uncorrelated, zero mean GWNs, \((\bar{\eta}_i(t)\bar{\eta}_j(t')) = \sigma^2 \delta_{ij} \delta(t - t').\) Then \(\eta = GG\) has correlations of the form [21]. In the following, we will consider only correlations between the nearest neighbors in the array of threshold elements. Specifically,

\[
C_{ij} = \begin{cases} 
1 & \text{if } i = j \\
c_1 & \text{if } |i - j| = 1 \\
0 & \text{otherwise}
\end{cases} \quad (4)
\]

where \(|c_1| \leq 0.5\) in order to keep the matrix \(C\) positively definite. For each specific value of \(c_1\), the Cholesky de-
The curves correspond, top to bottom, to $c_1 = -1/2$, $-1/4$, 0, 1/4, and 1/2, respectively. The array consists of 128 threshold devices.

Composition needs to be performed only once; later, during the simulation, vectors of independent Gaussian variables are generated and multiplied by the Cholesky factor $\mathbf{G}$. Since for a tridiagonal correlation matrix $\mathbf{C}$, the Cholesky factor has only two non-zero elements per row, generating GWNs with nearest-neighbor correlations is only twice as computationally expensive as generating independent GWNs, or even less than that given the fact that generating the independent Gaussian variables is the most computationally intensive part of the procedure.

We now simulate the system (4) in the manner described above for various values of $c_1$. Results for an array of $N = 128$ elements are plotted in Fig. 2. We can see that noises with positive spatial correlations lead to a smaller enhancement of SR than the independent noises. On the other hand, the SNR increases as $c_1$ becomes negative and approaches $-0.5$. The effect of negative correlations (anticorrelations) becomes larger as the noise strength increases. Similar results have been observed for arrays of different lengths.

Clean signals are rare in nature. Suppose that the incoming signal is contaminated by a noise that cannot be controlled. To improve detection of this signal, we pass it through an array of threshold elements and we apply additional noise that we can control to each of the elements. The external noise acts here as a global noise; the additional (local) noise is not correlated to the global noise. The above discussion suggests that the additional noise should have negative spatial correlations. If the global noise is weak (below the peak of the ordinary SR), the additional anticorrelated noise can significantly enhance the SNR, Fig. 3. However, if the global noise is large, the enhancement provided by the array is only marginal. This is similar to the result reported in Ref. 4, where the presence of a strong global noise markedly deteriorated the performance of the system studied.

It is important to understand the “microscopic” mechanism responsible for the AESR. In the conventional SR, the threshold element may occasionally misfire or miss some peaks of the signal, cf. Fig. 4. Also the shape of the incoming signal is not resolved by the outgoing binary signal. When an array of such elements acts in parallel, if one of the elements makes a mistake (fires when the signal is low or fails to fire when the signal is strong), other elements that are not positively correlated with it are not likely to repeat the mistake (Figs. 4a and c). This effect is even stronger when the other elements are negatively correlated with the one that makes the mistake: negative correlations between noises at different sites tend to correct the mistakes, while positive correlations tend to repeat them. As a result, the shape of the incoming signal is resolved much better. This does not lead to a visible increase in the height of the signal peak in the power spectrum, but it does lead to a significant decrease of the noise background, Fig. 4 bottom row. Thus, lowering of the flat noise background is primarily responsible for the increase in the SNR.

On a more formal level, let $p$ be the probability that a single element fires. This probability depends on the current phase of the incoming signal, on the signal’s amplitude, and on the noise level. If exactly $k$ out of $N$ elements fire, the array’s output equals $g = k/N$. If the noises at different sites are mutually independent (uncorrelated), the probability of such an event is given by the binomial distribution:

$$P_N(g(t) = k/N) = \binom{N}{k} p^k (1-p)^{N-k}. \quad (5a)$$

In an array twice as large with other parameters the same, exactly $2k$ elements should fire to produce the same output. The probability of this event is

$$P_{2N}(g(t) = k/N) = \binom{2N}{2k} p^{2k} (1-p)^{2(N-k)}. \quad (5b)$$
The distributions (5a) have the same expectation values $\bar{k}/N$, but for all values of $0 < p < 1$ the distribution (5b) is narrower than the distribution (5a). More importantly, the distribution (5b), corresponding to the larger array, allows for a more dense output, with values $(k \pm 1)/N$ being more probable than $(k\pm1)/N$ etc. Consequently, in larger arrays, wildly “wrong” outputs are less probable. This leads to lowering of the noise background and to an increase of the SNR.

If the noises are spatially correlated, the probability that exactly $k$ elements fire is no longer given by the binomial distribution. We have not been able to derive an exact formula for this probability, but the general mechanism of the SNR increase with the array size appears to be similar to that for the uncorrelated noises. Note that for the case presented in Fig. 4, introducing the maximal negative correlations between the nearest neighbors lowers the background by a factor of the order of 2, increasing the SNR by $10 \log_{10} 2 \approx 3$ dB, or about 10% of the total.

### III. A COUPLED SYSTEM

In order to verify whether similar effects are present in coupled systems, we consider the same system that was discussed in Ref. [4] where the AESR was first observed. Namely, we consider a chain of overdamped, coupled, nonlinear (double–well) oscillators

\[
\dot{x}_n = \frac{k}{2} x_n - k' x_n^3 + \varepsilon(x_{n-1} - 2x_n + x_{n+1}) + A(\sin \omega t + \phi) + \eta_n(t),
\]  

where $k = 2.1078$, $k' = 1.4706$, $A = 1.3039$ (these are the values used in Ref. [4]), $\omega = 2\pi$, $\phi$ is the initial phase of the signal, and $\eta_n(t)$ are GWNs, possibly spatially correlated according to (2).

If the system responds to the external periodic stimulation, the central oscillator switches between the two wells in synchrony with the stimulation. We show results for the “extreme” cases of a global noise, a local (uncorrelated) noise, and noises maximally correlated and anticorrelated between the nearest neighbors. The noises are generated by the algorithm presented in the preceding Section. We choose a chain of a modest length of $N = 33$, integrate the equations (6) numerically using the Heun scheme [21] with a time step $h = 1/64$ and analyze the behavior of the central oscillator. We filter the analog time series to generate the time series of $\pm 1$, reflecting which well the oscillator is in. From the power spectrum of the binary time series we calculate the SNR and average over 64 realizations of the noises and the initial phases of the signal. Results are presented in Fig. 5.

As we can see, for low noises the SNR curves display rather wild oscillations, but for larger noises the effect is much the same as for the nondynamical system discussed above: As the correlations decrease, the SNR maximum
shifts towards higher noise levels. The maximal value, as well as values for large noise levels, are the largest for the noise maximally anticorrelated between the nearest oscillators and the smallest for the global noise. This effect grows as the coupling strength increases. A more thorough analysis shows that for large noise intensities, and increase in the SNR is again achieved mainly by lowering the noise background. Note that in case of the global noise, the response practically does not change with the coupling strength. In this case, all the oscillators receive identical inputs and there is no AESR, but only the ordinary SR, exactly as in the nondynamical system.

To understand the mechanism that is responsible for the AESR in this case, observe that the oscillators exchange energy via the elastic coupling. We calculate the change in the elastic energy between two neighboring oscillators that occurs during a time interval \( h \gtrsim 0 \), \( \Delta E = E_n(t + h) - E_n(t) \), where \( E_n(t) = \frac{1}{2} \varepsilon (x_{n-1}(t) - x_n(t))^2 \).

To the lowest order in \( h \), \( x_n(t + h) \simeq x_n(t) + h\phi_n(t) \), where \( \phi \) has contributions from the elastic interactions, the nonlinear part of the potential, the external signal, and the noise. If either the coupling constant \( \varepsilon \) is large, or the oscillator happens to be in the vicinity of the barrier between the wells, the nonlinear part may be neglected. Straightforward calculations show that the noises contribute to the expectation value \( \langle \Delta E \rangle \) a term equal to \( \varepsilon\sigma^2(1-c_1) \). We can see that noises with negative correlations maximize this contribution: Noises with negative correlations tend to pull the neighboring oscillators in the opposite directions, thus maximizing the energy transfer between the oscillators and providing one of the oscillators with the extra energy needed to cross the barrier. This simple argument explains why the AESR grows when the coupling strength increases, why the AESR is larger for anticorrelated noises, and why the system with a global noise, corresponding to \( \langle \eta_n\eta_{n\pm1} \rangle = \langle \eta_n^2 \rangle \), does not display the AESR. Note that if the coupling were repulsive, the situation would be the opposite.

The effects that the correlations have on the spatiotemporal synchronization of the system [10] will be discussed separately.

IV. DISCUSSION

We have discussed the AESR in arrays of nondynamical threshold elements. We have not observed any saturation of the SNR curves that was reported previously in Ref. [5]. However, the input signal in that reference was aperiodic and the output SNR is not a natural measure for such signals [22]. This problem has been already discussed [23]. We have shown in the present paper that for arrays of nondynamical elements, noises with negative spatial correlations lead to an enhancement of the AESR. In case of positive spatial correlations the AESR is weaker than for the independent noises, and arrays with a global noise do not display the AESR, but only the ordinary SR. This happens because detectors with negative correlations tend to correct each other’s mistakes, while positively correlated detectors tend to repeat the mistakes. As a result, negatively correlated detectors better resolve the shape of the incoming signal. The mechanism of enhancing the SNR relies on lowering the noise background, not on elevating the signal peak. Note that we have analyzed these facts for spatial correlations between the nearest neighbors only. We expect that noises with long-ranging negative correlations would resolve the shape of the incoming signal even better; this point will be discussed elsewhere. It should be noted, though, that long-ranging correlations are more costly to generate.

These results are, superficially, in disagreement with those of Ref. [3], where it has been claimed that spatial correlations between the noises diminished the positive effect of passing the signal through an array of model neurons. This discrepancy is easily solved: In Ref. [3], each neuron was subjected to a superposition of a subthreshold periodic signal, a local GWN noise, and a global Orstein-Uhlenbeck noise. The local noises were mutually uncorrelated and the spatial correlations resulted solely from the presence of the global noise. It was the strong global noise that was responsible for the deterioration of the output signal. We have observed a similar effect — see Fig. 3 above and the subsequent discussion. The
beneficial effects of negative spatial correlations reported here result from the correlations between the local noises.

Our results suggest that from a technological point of view, not only the additive noises should be incorporated into the design of multi-component signal-detection systems, as was already suggested in Ref. [3], but also that these noises should have, if possible, negative spatial correlations to further improve the system’s ability to detect weak signals, even with a weak global noise present.

We have also shown that spatial correlations of the noise act similarly in the AESR in a coupled system. While we have discussed this for the specific system (6) only, we have shown that the AESR is enhanced by noises with negative spatial correlations due to the nature of the attracting harmonic interactions between the individual oscillators, regardless of the properties of the nonlinear part, provided the nonlinear part admits the conventional SR. For such interactions, positive spatial correlations of the noise reduce the AESR and a global noise eliminates it altogether, leaving only the ordinary SR, just like in the case of the AESR in arrays of nondynamical elements. As harmonic interactions between different particles are ubiquitous in many physical models, we expect similar phenomena to happen in a variety of situations. This analysis has also some interesting consequences for the interpretation of experimental results with interacting agents (particles, oscillators, detectors etc.) and a global noise, like those reported in Ref. [1]. Even though one cannot examine a single agent and has to excite a group of interacting ones, with only a global noise added to the signal, the SNR response of the whole array is the same as that of a single agent.

Previous research on systems that display the AESR in our present results let us conclude that the following features appear not to depend on the details of the dynamics: (i) For periodic subthreshold inputs, the SNR is systematically enhanced as the size of the array grows; (ii) Negative spatial correlations between the local noises provide further enhancement of the SNR; (iii) The SNR enhancement is mainly due to lowering the noise background, not due to increasing the signal peaks; (iv) Positive spatial correlations reduce the enhancement; in particular, a purely global noise eliminates the array enhancement altogether. The detailed shape of the SNR curves, their slope, locations of the maxima, depend on particulars of the system studied and on properties of the input signals.

[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A 14 L453 (1981); for a review see L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. 70, 233 (1998).
[2] S. M. Bezrukov, Phys. Lett. A 248, 29 (1998).
[3] J. A. Freund, L. Schimansky-Geier, and P. Hänggi, Chaos 13, 225 (2003).
[4] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. R. Bulsara, Phys. Rev. Lett. 75, 3 (1995).
[5] J. J. Collins, C. C. Chow and T. T. Imhoff, Nature (London) 376, 236 (1995); See also F. Moss and Xing Pei, Nature (London), 376, 211 (1995).
[6] S. M. Bezrukov and I. Voydanoy, Nature (London) 378, 362 (1995).
[7] P. C. Gailey, A. Neiman, J. J. Collins, and F. Moss, Phys. Rev. Lett. 79, 4701 (1997).
[8] N. G. Stocks, Phys. Rev. Lett. 84, 2310 (2000).
[9] Feng Liu, Bambi Hu, and Wei Wang, Phys. Rev. E 63 031907 (2001).
[10] Gun Sang Jeon, Hyun Jin Kim, M. Y. Choi, Beom Jun Kim, and P. Minnhagen, Phys. Rev. B 65, 184510, (2002); Gun Sang Jeon, Jong Soo Lim, Hyun Jin Kim, and M. Y. Choi, Phys. Rev. B 66, 024511 (2002); Gun Sang Jeon and M. Y. Choi, Phys. Rev. B 66, 064514 (2002).
[11] B. J. Gluckman, T. I. Netoff, E. J. Neel, W. L. Ditto, M. L. Spano, and S. J. Schiff, Phys. Rev. Lett. 77, 4098 (1996).
[12] Z. Gingl, L. B. Kiss, and F. Moss, Europhys. Lett. 29, 191 (1995).
[13] K. Wiesenfeld and F. Moss, Nature (London) 373, 33 (1995); S. M. Bezrukov and I. Voydanoy, Nature (London) 385, 319 (1997); S. M. Bezrukov and I. Voydanoy, Nature (London) 386, 738 (1997); F. Chapeau-Blondeau and X. Godivier, Phys. Rev. E 55, 1478 (1997); J. M. G. Vilar, G. Gomila, and J. M. Rubi, Phys. Rev. Lett. 81, 14 (1998).
[14] A. Fulinski and T. Telejko, Phys. Lett. A 152, 11 (1991).
[15] A. J. R. Madureira, P. Hänggi, and H. S. Wio, Phys. Lett. A 217, 248 (1996).
[16] Jing-hui Li, J. Luczka, and P. Hänggi, Phys. Rev. E 64, 011113 (2001).
[17] K. P. Singh, G. Ropars, M. Brunel, and A. Le Floch, Phys. Rev. Lett. 90, 073901 (2003).
[18] G. Marsaglia and T. A. Bray, SIAM Review 6, 260 (1964); A. J. Kinderman and J. G. Ramage, J. Amer. Statist. Assoc. 71, 893 (1976); R. Wieczorkowski and R. Zielinski, Komputerowe generatorzy liczb losowych (WNT, Warszawa, 1997) (in Polish).
[19] M. Matsumoto and T. Nishimura, ACM Trans. on Modeling and Computer Simulation, 8, 3 (1998).
[20] G. H. Golub and C. F. Van Loan, Matrix Computations (The John Hopkins University Press, Baltimore, 1983).
[21] R. Manella, Int. J. Mod. Phys. C 13, 1177 (2002) and references quoted therein.
[22] J. J. Collins, C. C. Chow, and T. T. Imhoff, Phys. Rev. E 52, R3321 (1995).
[23] A. J. Noest, J. J. Collins, C. C. Chow, and T. T. Imhoff, Nature (London), 378, 341 (1995).