PULLBACK ATTRACTORS AND IN Variant MEASURES FOR DISCRETE KLEIN-GORDON-SCHRÖDINGER EQUATIONS

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ABSTRACT. In this article, we first provide a sufficient and necessary condition for the existence of a pullback-$\mathcal{D}$ attractor for the process defined on a Hilbert space of infinite sequences. As an application, we investigate the non-autonomous discrete Klein-Gordon-Schrödinger system of equations, prove the existence of the pullback-$\mathcal{D}$ attractor and then the existence of a unique family of invariant Borel probability measures associated with the considered system.

1. Introduction. The global attractor is an object that captures the asymptotic behavior of autonomous systems (see e.g. Carvalho, Langa and Robinson [9], Chepyzhov and Vishik [11], Hale [21], Robinson [38], Sell and You [39], Temam [41], etc.). We know that a continuous semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor in a complete metric space $(X,d)$ if and only if $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set and is $\omega$-limit compact (or is asymptotically compact) in $X$. In [37, Theorem 3.11], Ma et al. established, using the measure of noncompactness, that a continuous semigroup $\{S(t)\}_{t\geq 0}$ possesses a global attractor in a Banach space if and only if $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set and satisfies the flattening condition (cf. Condition (C) in [37] for detailed definitions).

Similarly to the autonomous case, uniform attractor, kernel sections and pullback attractor are three different concepts used to capture the asymptotic behavior of non-autonomous systems. Among these three concepts, the latter seems to be the most proper one (cf. [9, P3]). There are several references discussing conditions for the existence of pullback attractors (including pullback-$\mathcal{D}$ attractor), see e.g. [5, 9, 18, 19, 30, 40, 45].

For a given Banach space $X$, by $\mathcal{P}(X)$ we denote the family of all nonempty subsets of $X$. Let $\mathcal{D}$ be a given nonempty class of parametrized families $\hat{D} = \{D(s)\mid s \in \mathbb{R}\} \subseteq \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$. Combining results for the existence of pullback-$\mathcal{D}$ attractors from the above references, we have

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Proposition 1.1. Let \( \{U(t,\tau)\}_{t \geq \tau} \) be a continuous process in a uniform convex Banach space \( X \), and \( D \) be a given universe. Then \( \{U(t,\tau)\}_{t \geq \tau} \) possesses a pullback-\( D \) attractor in \( X \) if and only if

(a) \( \{U(t,\tau)\}_{t \geq \tau} \) possesses a bounded pullback-\( D \) absorbing set \( \hat{D}_0 = \{D_0(s) \mid s \in \mathbb{R}\} \) in the following sense: for any \( t \in \mathbb{R} \) and any \( \hat{D} = \{D(s) \mid s \in \mathbb{R}\} \) in \( D \), with \( D(s) \subset X \) bounded for every \( s \in \mathbb{R} \), there exists some \( \tau_0(t,\hat{D}) \leq t \) such that \( U(t,\tau)D(\tau) \subseteq D_0(t) \) for all \( \tau \leq \tau_0(t,\hat{D}) \).

(b) For any \( t \in \mathbb{R} \), \( \hat{D} = \{D(s) \mid s \in \mathbb{R}\} \in D \) and any \( \epsilon > 0 \), there exists some \( \tau_0(t,\epsilon,\hat{D}) \) and a finite dimensional subspace \( X_1 \) of \( X \) such that

\[
\begin{align*}
\left\{ P\left( \bigcup_{\tau \leq \tau_0} U(t,\tau)D(\tau) \right) \right\} & \text{ is bounded in } X, \\
\| (I - P)\left( \bigcup_{\tau \leq \tau_0} U(t,\tau)D(\tau) \right) \|_X & \leq \epsilon,
\end{align*}
\]

where \( P : X \mapsto X_1 \) is a bounded projector.

The first goal of this article is to provide a sufficient and necessary condition for the existence of pullback-\( D \) attractors for process defined on a Banach space of infinite sequences. Our conditions are quite convenient for applications in many concrete lattice dynamical systems (LDSs) arising from spatial discretization of partial differential equations (PDEs) on unbounded domains.

LDSs are spatiotemporal systems with discretization in some variables including coupled ordinary differential equations (ODEs) and coupled map lattices and cellular automata [14]. In some cases, LDSs occur as spatial discretizations of PDEs on unbounded (or bounded) domains. LDSs arise in a wide variety of applications, ranging from electrical engineering [12] to image processing and pattern recognition [13], laser systems [15], biology [24], chemical reaction theory [26], etc. Nowadays, the asymptotic theory of general LDSs have been extensively studied (see e.g. [1, 2, 3, 7, 22, 23, 25, 43, 46, 51, 52, 53, 50, 54, 55]). Especially, Zhao and Zhou [50] proved a sufficient and necessary conditions for the existence of compact kernel sections for the process generated by general non-autonomous LDSs on Banach space \( \ell^p_\mu \) of infinite sequences.

Our result concerning the sufficient and necessary conditions for the existence of pullback-\( D \) attractors for process defined on a Banach space of infinite sequences can be regarded as a generalization of the result of Zhao and Zhou [50]. Different from the kernel sections discussed in [50], pullback-\( D \) attractors discussed here possess more general basins of attraction, which are referred to a given universe \( D \) rather than only fixed bounded sets. Different universes provide different basins of attraction and will give rise to different pullback attractor, reflecting different aspects of the dynamics. A particularly useful universe \( D \) is the collection of all families \( \hat{D} = \{D(s) \mid s \in \mathbb{R}\} \) such that for some \( \gamma > 0 \)

\[
\lim_{\tau \to -\infty} \sup_{x(\tau) \in \hat{D}(\tau)} e^{\gamma \tau \|x(\tau)\|_X} = 0. \tag{1.1}
\]

Obviously, all fixed bounded sets of \( X \) lie in the universe \( D \) satisfying (1.1).

As an application of our result, we investigate the following non-autonomous LDS

\[
\begin{align*}
iz_m - (Az)_m + i\alpha z_m + z_m u_m &= f_m(t), \quad m \in \mathbb{Z}, \quad t > \tau, \\
u\ddot{z}_m + (Au)_m + \mu u_m - \beta |z_m|^2 &= g_m(t), \quad m \in \mathbb{Z}, \quad t > \tau, \tag{1.2}
\end{align*}
\]

\[
\begin{align*}
iz_m - (Az)_m + i\alpha z_m + z_m u_m &= f_m(t), \quad m \in \mathbb{Z}, \quad t > \tau, \\
u\ddot{z}_m + (Au)_m + \mu u_m - \beta |z_m|^2 &= g_m(t), \quad m \in \mathbb{Z}, \quad t > \tau, \tag{1.3}
\end{align*}
\]
with initial conditions
\[ u_m(\tau) = u_{m,\tau}, \quad \dot{u}_m(\tau) = u_{1m,\tau}, \quad z_m(\tau) = z_{m,\tau}, \quad m \in \mathbb{Z}, \quad \tau \in \mathbb{R}, \quad (1.4) \]
where \( z_m(t) \in \mathbb{C}, \ u_m(t) \in \mathbb{R}, \) (\( \mathbb{C}, \mathbb{R} \) are the sets of complex and real numbers, respectively), \( \mathbb{Z} \) is the set of integers, \( i \) is the unit of the imaginary numbers such that \( i^2 = -1 \), \( \alpha, \beta, \nu, \mu \) are positive constants, and \( A \) is a linear operator defined as
\[ (Au)_m = 2u_m - u_{m+1} - u_{m-1}, \quad \forall u = (u_m)_{m \in \mathbb{Z}}. \quad (1.5) \]

Equations (1.2)-(1.3) can be regarded as a discrete analogue of the following non-autonomous Klein-Gordon-Schrödinger (KGS) equations on \( \mathbb{R} \):
\[ iz_t + z_{xx} + i\alpha z + zu = f(x, t), \quad (1.6) \]
\[ u_{tt} + \nu u_t - u_{xx} + \mu u - \beta |z|^2 = g(x, t). \quad (1.7) \]

Equations (1.6)-(1.7) describe the interaction of scalar nucleons with neutral scalar mesons through Yukawa coupling [17], where \( z \) denotes a complex scalar nucleon field and \( u \) represents a real meson field; the complex-valued function \( f(x, t) \) and the real-valued function \( g(x, t) \) are the time-dependent external sources. The dynamics of equations (1.6)-(1.7) was studied by [4, 20, 29, 32, 33, 42].

We want to point out that Zhao and Zhou [48] proved the existence of compact kernel sections for the process generated by problem (1.2)-(1.4). In [48], the external force, such as \( (g_m(t))_{m \in \mathbb{Z}} \), is assumed to be a continuous and bounded function from \( \mathbb{R} \) into \( \ell^2 = \{ u = (u_m)_{m \in \mathbb{Z}} \mid u_m \in \mathbb{R}, \sum_{m \in \mathbb{Z}} u_m^2 < +\infty \} \). Moreover, in reference [48] it is assumed that \( (g_m(t))_{m \in \mathbb{Z}} \) satisfies the following condition
\[ \left\{ \begin{array}{c}
\text{For each } \tau \in \mathbb{R} \text{ and } \forall \varepsilon > 0, \ \exists M(\varepsilon, \tau) \in \mathbb{N} \text{ such that } \\
\sum_{|m| \geq M(\varepsilon, \tau)} |f_m(s)|^2 \leq \varepsilon \text{ for any } s \leq \tau.
\end{array} \right. \quad (1.8) \]

Condition (1.8) was also imposed on the external forces in [47] when studying the pullback attractor (kernel sections) for second order non-autonomous LDSs.

The second result of this article is to prove the existence of a pullback-\( D \) attractor for the process generated by the solution operators of problem (1.2)-(1.4). Compared to the kernel sections discussed in [48], the pullback-\( D \) attractor has a more general basins of attraction in the phase space \( E_\mu \). Indeed, any fixed bounded sets lie in the universe \( D \). On the other hand, we remove condition (1.8) and the uniform boundedness imposed on the external force functions. Without these conditions, we shall perform some delicate estimations when verifying the pullback-\( D \) asymptotical nullness of the process.

The last purpose of this article is to investigate the existence of a family of invariant Borel probability measures associated with the considered system which are supported on the obtained pullback-\( D \) attractor. The invariant measures have proven to be very useful in the understanding of turbulence (see Foias et al. [16]). The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities. Later, the invariant measures and statistical properties of dissipative systems were studied in a series of references. For instance, Wang investigated the upper semi-continuity of stationary statistical properties of dissipative systems in [44]. Łukaszewicz, Real and Robinson [35] used the notion of a Generalized Banach limit to construct the invariant measures for
general continuous dynamical systems on metric spaces. Then, Chekroun and Glatt-Holtz [10] improved the results of [44] and [35] to construct invariant measures for a broad class of dissipative semigroups, which generalizing and simplifying the proofs.

Very recently, Lukaszewicz and Robinson [36] used the techniques developed in the articles Lukaszewicz [34] and Lukaszewicz et al. [35], which were in turn based on works of Foias et al. [16] and Wang [44], to provide a construction of invariant measures for non-autonomous systems with minimal assumptions on the underlying dynamical process. The results of Lukaszewicz and Robinson ([36, Theorem 3.1]) show that a continuous process \{U(t, \tau)\}_{t \geq \tau} on a complete metric space \(X\) possesses a unique family of invariant Borel probability measures in \(X\) if \{U(t, \tau)\}_{t \geq \tau} satisfies the following conditions,

(i) the process \{U(t, \tau)\}_{t \geq \tau} possesses a pullback attractor in \(X\); and

(ii) for every \(x_0 \in X\) and every \(t \in \mathbb{R}\), the \(X\)-valued function \(\tau \mapsto U(t, \tau)x_0\) is continuous and bounded on \((-\infty, t]\).

We will use this result to obtain the existence of a unique family of invariant Borel probability measures which are supported on the obtained pullback-\(D\) attractor.

Invariant measures and statistical solutions of some PDEs, such as Navier-Stokes equations, globally modified Navier-Stokes equation and Navier-Stokes-Voigt equations, were studied by [6, 16, 27, 28, 34, 49]. However, as far as we know, up to now there are no references discussing the invariant measures for lattice system.

The rest of this article is arranged as follows. In the next section we provide a sufficient and necessary condition for the existence of pullback-\(D\) attractors for process defined on a Banach space of infinite sequences. In Section 3 we apply our result to the non-autonomous KGS equations on infinite lattices and establish the existence of the pullback-\(D\) attractor for the associated process. In Section 4 we prove that there exists a unique family of invariant Borel probability measures for the considered system. We end the article with a brief summary and some remarks in Section 5.

2. \textbf{Sufficient and necessary conditions for the existence of pullback-\(D\) attractor for dissipative lattice systems.} In this section, we present a sufficient and necessary condition for the existence of a pullback-\(D\) attractor for the process \{U(t, \tau)\}_{t \geq \tau} defined by general lattice systems in the following space of infinite sequences

\[
\ell^2 = \left\{ u = (u_m)_{m \in \mathbb{Z}}, \ u_m \in \mathbb{R} \ \middle| \ \sum_{m \in \mathbb{Z}} u_m^2 < +\infty \right\}. \tag{2.1}
\]

We must remark that our result is also valid in a general Banach space of infinite sequences, cf. Remark 5.1. In the sequel we consider only the simplest case of the \(\ell^2\) space just for the clarity of exposition.

We equip \(\ell^2\) with the inner product and norm as

\[
(u, v) = \sum_{m \in \mathbb{Z}} u_m v_m, \quad ||u||^2 = (u, u), \quad \forall u = (u_m)_{m \in \mathbb{Z}}, \quad v = (v_m)_{m \in \mathbb{Z}} \in \ell^2.
\]

Clearly, \(\ell^2 = (\ell^2, (\cdot, \cdot), ||\cdot||)\) is a Hilbert space. In this section, by \(\mathcal{P}(\ell^2)\) we denote the family of all nonempty subsets of \(\ell^2\). Let \(D\) be a given nonempty class of parameterized families \(\hat{D} = \{D(s) | s \in \mathbb{R}\} \subseteq \mathcal{P}(\ell^2)\). The class \(D\) will be called a universe in \(\mathcal{P}(\ell^2)\). We next introduce two definitions. We can refer to [9] for general definitions of pullback \(D\)-attractor for the process in general metric space.
Definition 2.1. It is said that a family of subsets $\tilde{D}_0 = \{D_0(s) | s \in \mathbb{R}\} \subseteq \mathcal{P}(\ell^2)$ is bounded pullback-$\mathcal{D}$ absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$ in $\ell^2$ if for any $t \in \mathbb{R}$ and any $\tilde{D} = \{D(s) | s \in \mathbb{R}\} \in \mathcal{D}$, with $D(s) \subseteq \ell^2$ bounded for every $s \in \mathbb{R}$, there exists a $\tau_0(t, \tilde{D}) \leq t$ such that $U(t, \tau)D(\tau) \subseteq D_0(t)$ for all $\tau \leq \tau_0(t, \tilde{D})$.

Definition 2.2. A family $\tilde{A}_D = \{A_D(s) | s \in \mathbb{R}\} \subseteq \mathcal{P}(\ell^2)$ is said to be a pullback-$\mathcal{D}$ attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ in $\ell^2$ if it has the following properties:

(i) Compactness: for every $t \in \mathbb{R}$, $A_D(t)$ is a nonempty compact subset of $\ell^2$;

(ii) Invariance: $U(t, \tau)A_D(\tau) = A_D(t)$, $\forall \tau \leq t$;

(iii) Pullback attraction: $\tilde{A}_D$ is pullback-$\mathcal{D}$ attracting in the following sense

$$\lim_{\tau \to -\infty} \text{dist}_{\ell^2}(U(t, \tau)D(\tau), A_D(t)) = 0, \ \forall \tilde{D} = \{D(s) | s \in \mathbb{R}\} \in \mathcal{D}, t \in \mathbb{R}.$$

Consider a general lattice system with initial-value condition in $\ell^2$:

$$\begin{cases}
    \dot{u} = g(u, t), & u = (u_m)_{m \in \mathbb{Z}}, \ t > \tau,
    \\
    \left.u\right|_{t=\tau} = u_\tau = (u_m(\tau))_{m \in \mathbb{Z}} \in \ell^2,
\end{cases}
$$

where $g : \ell^2 \times \mathbb{R} \mapsto \ell^2$ satisfies some smoothness and dissipative conditions. We assume that a unique solution $u(t) = (u_m(t))_{m \in \mathbb{Z}}$ of (2.2) exists globally on $[\tau, +\infty)$ for any given $\tau \in \mathbb{R}$, and the maps $U(t, \tau) : u(\tau) \mapsto u(t) = U(t, \tau)u(\tau)$ generate a continuous process $\{U(t, \tau)\}_{t \geq \tau}$ on $\ell^2$. Then the existence of a pullback-$\mathcal{D}$ attractor of the process $\{U(t, \tau)\}_{t \geq \tau}$ on $\ell^2$ is guaranteed by the following

Theorem 2.1. The continuous process $\{U(t, \tau)\}_{t \geq \tau}$ has a pullback-$\mathcal{D}$ attractor $\tilde{A}_D = \{A_D(s) | s \in \mathbb{R}\}$ in $\ell^2$ if and only if the following two conditions hold:

(1) The process $\{U(t, \tau)\}_{t \geq \tau}$ has a bounded pullback-$\mathcal{D}$ absorbing set $\tilde{D}_0 = \{D_0(s) | s \in \mathbb{R}\}$ in $\ell^2$;

(2) (Pullback-$\mathcal{D}$ asymptotical nullness)

For any given $t \in \mathbb{R}$, $\epsilon > 0$ and $\tilde{D} = \{D(t) | t \in \mathbb{R}\} \in \mathcal{D}$, there exist some $M_0 = M_0(t, \epsilon, \tilde{D}) \in \mathbb{N}$ and $\tau_0 = \tau_0(t, \epsilon, \tilde{D}) \leq t$ such that

$$\sup_{u_\tau \in D(\tau)} \sum_{|m| \geq M_0} |(U(t, \tau)u_\tau)_m|^2 \leq \epsilon^2, \ \forall \tau \leq \tau_0. \ (2.3)$$

Proof. The proof is based on Proposition 1.1 and the idea comes from [54, Theorem 1].

Firstly, we prove the sufficiency. Let conditions (1), (2) of Theorem 2.1 hold. To prove that the process $\{U(t, \tau)\}_{t \geq \tau}$ possesses a pullback-$\mathcal{D}$ attractor in $\ell^2$, we only need to show that conditions (a) and (b) in Proposition 1.1 are satisfied. Indeed, condition (a) is directly obtained from condition (1). Let us consider condition (b).

For any given $t \in \mathbb{R}$, $\epsilon > 0$ and $\tilde{D} = \{D(s) | s \in \mathbb{R}\} \in \mathcal{D}$, by condition (1) there exists some $\tau_1 = \tau_1(t, \tilde{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subseteq D_0(t), \ \forall \tau \leq \tau_1. \ (2.4)$$

In view of condition (2) there exist some $M_0 = M_0(t, \epsilon, \tilde{D}) \in \mathbb{N}$ and $\tau_0 = \tau_0(t, \epsilon, \tilde{D}) \leq \tau_1$ such that for any solution $u(t) = U(t, \tau)u_\tau \in \ell^2$ of problem (2.2) with initial condition $u_\tau \in D(\tau)$,

$$\sup_{u_\tau \in D(\tau)} \sum_{|m| \geq M_0} |(U(t, \tau)u_\tau)_m|^2 \leq \epsilon^2, \ \forall \tau \leq \tau_0. \ (2.5)$$
For such $M_0 \in \mathbb{N}$ we set

$$X_1 = \left\{ v = (v_m)_{m \in \mathbb{Z}} \in \ell^2 \mid v_m = \begin{cases} u_m, & |m| \leq M_0 \\ 0, & |m| > M_0, \end{cases} \quad u = (u_m)_{m \in \mathbb{Z}} \in \ell^2 \right\},$$

$$X_2 = \left\{ w = (w_m)_{m \in \mathbb{Z}} \in \ell^2 \mid w_m = \begin{cases} 0, & |m| \leq M_0 \\ u_m, & |m| > M_0, \end{cases} \quad u = (u_m)_{m \in \mathbb{Z}} \in \ell^2 \right\}.$$  

Clearly, $X_1$ and $X_2$ are two subspaces of $\ell^2$ and $\dim (X_1) = 2M_0 + 1 < +\infty$. Moreover, we have $X_1 \perp X_2$ and $\ell^2 = X_1 \oplus X_2$.

Hence, for any $u \in \ell^2$ there exist unique $v \in X_1$ and $w \in X_2$ such that

$$u = v + w \quad \text{with} \quad u_m = v_m + w_m, \quad m \in \mathbb{Z}. \quad (2.6)$$

Thus, we can define the operator $P : \ell^2 \to X_1$ as

$$Pu = v, \quad (I - P)u = w, \quad \forall u \in \ell^2,$$

where $I$ is the identity operator in $\ell^2$ and $P$ is a linear and bounded projector from $\ell^2$ to $X_1$. Since $\tau_0 \leq \tau_1$, we get by (2.4) and condition (1) that

$$\sup_{u \in D(\tau)} \|P \mathcal{U}(t, \tau)u\| \leq \sup_{u \in D_0(t)} \|P u(t)\| < +\infty, \quad \forall \tau \leq \tau_0. \quad (2.7)$$

Thus $P\left( \bigcup_{\tau < \tau_0} \mathcal{U}(t, \tau)D(\tau) \right)$ is bounded in $\ell^2$. At the same time, by (2.4) and (2.5), we see that

$$\sup_{u \in D(\tau)} \| (I - P) \mathcal{U}(t, \tau)u \| \leq \sup_{u \in D_0(t)} \sum_{|m| \geq M_0} |u_m(t)|^2 \leq \epsilon^2, \quad \forall \tau \leq \tau_0. \quad (2.8)$$

From (2.8) it follows that $\left\| (I - P) \left( \bigcup_{\tau < \tau_0} \mathcal{U}(t, \tau) D(\tau) \right) \right\| \leq \epsilon$ and condition (b) holds. By Proposition 1.1, the continuous process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a pullback-$D$ attractor in $\ell^2$.

Secondly, we prove the necessity, that is, if the continuous process $\{\mathcal{U}(t, \tau)\}_{t \geq \tau}$ possesses a pullback-$D$ attractor in $\ell^2$, then conditions (a) and (b) of Proposition 1.1 hold. We shall prove that conditions (1) and (2) of Theorem 2.1 are satisfied. Obviously, by (a) of Proposition 1.1, condition (1) holds. We next show that condition (2) holds. For any given $t \in \mathbb{R}$, $\epsilon > 0$ and $\hat{D} = \{D(s) \mid s \in \mathbb{R}\} \in D$, by condition (b) of Proposition 1.1, there exist some $\tau_0 = \tau(t, \epsilon, \hat{D}) \leq t$ and a finite dimensional subspace $X_1$ of $\ell^2$ such that

$$\sup_{u \in D(\tau)} \|P \mathcal{U}(t, \tau)u\| \leq C, \quad \forall \tau \leq \tau_0, \quad (2.9)$$

and

$$\sup_{u \in D(\tau)} \|(I - P) \mathcal{U}(t, \tau)u\| \leq \frac{\epsilon}{2}, \quad \forall \tau \leq \tau_0, \quad (2.10)$$

where $C$ is a constant depending only on $D_0(t)$ and $P : \ell^2 \to X_1$ is a bounded projector.

Let $u(t) = u(t; \tau, u_\tau) = \mathcal{U}(t, \tau)u_\tau \in \ell^2$ be a solution of the initial value problem (2.2) with initial condition $u(\tau) = u_\tau \in D(\tau)$. Then with the use of the bounded projector $P : \ell^2 \to X_1$, the solution $u(t)$ can be decomposed as

$$u(t) = v(t) + w(t), \quad (2.11)$$

where
\[
\begin{align*}
\left\{ \begin{array}{l}
v(t) = Pu(t) = Pu(t, \tau)u_\tau \in X_1 \text{ with } v_m(t) = (Pu(t, \tau)u_\tau)_m \\
w(t) = (I - P)u(t) = (I - P)u(t, \tau)u_\tau \text{ with } w_m(t) = ((I - P)u(t, \tau)u_\tau)_m.
\end{array} \right.
\end{align*}
\]
Note that \(X_1\) is a finite-dimensional subspace of \(\ell^2\) and its dimension \(\text{dim}(X_1)\) is independent of \(\tau\). Set \(\text{dim}(X_1) = M_1 = M_1(t, \epsilon, \tilde{D}) < +\infty\). We can choose a normal basis of \(X_1\) as follows: \(e_1, e_2, \cdots, e_{M_1} \in X_1 \subset \ell^2\), where \((e_j, e_k) = 0\) for \(j \neq k\), and \(\|e_j\| = 1\). Then \(v(t)\) in (2.11) can be expressed as
\[
v(t) = \sum_{j=1}^{M_1} \xi_j(t)e_j, \text{ where } \xi_j(t) \in \mathbb{R}, \ j = 1, 2, \cdots, M_1.
\]
Moreover,
\[
\|v(t)\|^2 = \sum_{j=1}^{M_1} \xi_j^2(t) \text{ and } \xi_j^2(t) \leq \|v(t)\|^2 \text{ for } j = 1, 2, \cdots, M_1.
\] (2.12)

By (2.9)-(2.10) and (2.12), we have
\[
\abs{\xi_j(t)} \leq \|v(t)\| \leq \sup_{u_\tau \in D(\tau)} \|Pu(t, \tau)u_\tau\| \leq C, \ \forall \tau \leq \tau_0,
\] (2.13)
\[
\|w(t)\|^2 \leq \sup_{u_\tau \in D(\tau)} \|((I - P)u(t, \tau)u_\tau)\|^2 \leq \frac{\epsilon^2}{4}, \ \forall \tau \leq \tau_0.
\] (2.14)

Then (2.12) and (2.13) give \(\|v(t)\|^2 \leq C^2 M_1\). Thus there exists some positive integer \(M_0 = M_0(\epsilon, M_1)\) (that depends on \(t, \epsilon\) and \(\tilde{D}\)), denoted by \(M_0 = M_0(t, \epsilon, \tilde{D})\), such that
\[
\sup_{u_\tau \in D(\tau) \mid \|u\| \geq M_0} \sum_{|m| \geq M_0} \abs{(Pu(t, \tau)u_\tau)_m}^2 \leq \frac{\epsilon^2}{4}.
\] (2.15)

From (2.14) and (2.15) it follows that
\[
\sup_{u_\tau \in D(\tau) \mid |m| \geq M_0} \abs{\langle Pu(t, \tau)u_\tau \rangle_m}^2
\]
\[
= \sup_{u_\tau \in D(\tau) \mid |m| \geq M_0} \sum_{|m| \geq M_0} \abs{(Pu(t, \tau)u_\tau)_m + ((I - P)u(t, \tau)u_\tau)_m}^2
\]
\[
\leq 2 \sup_{u_\tau \in D(\tau) \mid |m| \geq M_0} \sum_{|m| \geq M_0} \abs{(Pu(t, \tau)u_\tau)_m}^2 + 2 \sup_{u_\tau \in D(\tau) \mid |m| \geq M_0} \sum_{|m| \geq M_0} \abs{((I - P)u(t, \tau)u_\tau)_m}^2
\]
\[
\leq \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2, \ \forall \tau \leq \tau_0,
\]
that is, condition (2) holds. This ends the proof of Theorem 2.1. \( \square \)

3. Pullback attractor for non-autonomous Klein-Gordon-Schrödinger equations on infinite lattices. In this section, we will employ Theorem 2.1 to prove that the process associated with the initial value problem (1.2)-(1.4) possesses a pullback-\(D_\tau\) attractor.

We first introduce the mathematical setting of our problem. Besides the Hilbert space \(\ell^2\) defined by (2.1), we set
\[
L^2 = \left\{ u = (u_m)_{m \in \mathbb{Z}} \mid u_m \in \mathbb{C}, \ \sum_{m \in \mathbb{Z}} |u_m|^2 < +\infty \right\},
\] (3.1)
and equip $L^2$ with the inner product (without confusion with that of $\ell^2$) and norm
\[
(u, v) = \sum_{m \in \mathbb{Z}} u_m \overline{v}_m, \quad \|u\|^2 = (u, u),
\]
where $\overline{v}_m$ denotes the conjugate of $v_m$.

In the sequel, $\alpha$, $\mu$, and $\nu$ denote the parameters of system (1.2)-(1.3).

For any two elements $u, v \in \ell^2$ (or $L^2$) we define a bilinear form on $\ell^2$ (or $L^2$) as
\[
(u, v)_\mu = (Bu, Bv) + \mu(u, v),
\]
where $\bar{B}$ is a linear operator from $\ell^2$ to $\ell^2$ (alternatively, from $L^2$ to $L^2$) defined as
\[
(Bu)_m = u_{m+1} - u_m, \quad \forall m \in \mathbb{Z}, \quad \forall u = (u_m)_{m \in \mathbb{Z}} \in \ell^2 \text{ or } L^2.
\]

We also define a linear operator $B^*$ from $\ell^2$ to $\ell^2$ (or from $L^2$ to $L^2$) as
\[
(B^*u)_m = u_{m-1} - u_m, \quad \forall m \in \mathbb{Z}, \quad \forall u = (u_m)_{m \in \mathbb{Z}} \in \ell^2 \text{ or } L^2.
\]

In fact, $B^*$ is the adjoint operator of $B$ and one can easily check that
\[
(Au, v) = (B^*Bu, v) = (Bu, Bv), \quad (Bu, v) = (u, B^*v), \quad \forall u, v \in \ell^2 \text{ or } L^2. \quad (3.3)
\]

Clearly, the bilinear form $(\cdot, \cdot)_\mu$ defined by (3.2) is also an inner product in $\ell^2$ (or $L^2$). Since
\[
\mu\|u\|^2 \leq \|Bu\|^2 + \mu\|u\|^2 \leq \|u\|^2 \leq (4 + \mu)\|u\|^2, \quad \forall u = (u_m)_{m \in \mathbb{Z}} \in \ell^2 \text{ or } L^2,
\]
the norm $\|\cdot\|_\mu$ induced by $(\cdot, \cdot)_\mu$ is equivalent to the norm $\|\cdot\|$. Let
\[
\ell^2_\mu = (\ell^2, (\cdot, \cdot)_\mu), \quad L^2 = (L^2, (\cdot, \cdot)),
\]
then both $\ell^2_\mu$ and $L^2$ are Hilbert spaces. Set
\[
E_\mu = \ell^2_\mu \times \ell^2 \times L^2, \quad E = \ell^2 \times \ell^2 \times L^2,
\]
and equip these spaces with the inner products and norms as follows: for any two elements $(\psi^{(k)}) = (u^{(k)}, v^{(k)}, z^{(k)})^T \in E_\mu$ (or $E$), $k = 1, 2$,
\[
(\psi^{(1)}, \psi^{(2)})_{E_\mu} = (u^{(1)}, u^{(2)})_\mu + (v^{(1)}, v^{(2)}) + (z^{(1)}, z^{(2)})
\]
\[
= \sum_{m \in \mathbb{Z}} \left( (Bu^{(1)})_m(Bu^{(2)})_m + \mu u^{(1)}_m u^{(2)}_m + v^{(1)}_m v^{(2)}_m + z^{(1)}_m \overline{z}^{(2)}_m \right), \quad (3.4)
\]
\[
\|\psi\|_{E_\mu} = \sqrt{(\psi, \psi)_{E_\mu}}, \quad \forall \psi \in E_\mu,
\]
\[
(\psi^{(1)}, \psi^{(2)})_E = (u^{(1)}, u^{(2)}) + (v^{(1)}, v^{(2)}) + (z^{(1)}, z^{(2)})
\]
\[
= \sum_{m \in \mathbb{Z}} \left( u^{(1)}_m u^{(2)}_m + v^{(1)}_m v^{(2)}_m + z^{(1)}_m \overline{z}^{(2)}_m \right),
\]
\[
\|\psi\|_E = \sqrt{(\psi, \psi)_E}, \quad \forall \psi \in E,
\]
where $\overline{z}^{(2)}_m$ stands for the conjugate of $z^{(2)}_m$. We have
\[
\min\{\mu, 1\}\|\psi\|^2 \leq \|\psi\|^2_{E_\mu} \leq (4 + \mu)\|\psi\|^2_E, \quad \forall \psi = (\psi_m)_{m \in \mathbb{Z}} \in E. \quad (3.5)
\]

Now, let us come back to the initial value problem (1.2)-(1.4). For convenience, we express equations (1.2)-(1.3) as an abstract non-autonomous first-order ODE with respect to time $t$ in $E_\mu$. To this end, we set $u = (u_m)_{m \in \mathbb{Z}}, \ z = (z_m)_{m \in \mathbb{Z}}, \ \psi ^{(1)} = (\psi ^{(1)} _m)_{m \in \mathbb{Z}}$, \( \psi ^{(2)} = (\psi ^{(2)} _m)_{m \in \mathbb{Z}} \).
\[ f(t) = (f_m(t))_{m \in \mathbb{Z}}, \quad g(t) = (g_m(t))_{m \in \mathbb{Z}}, \quad u_\tau = (u_{m, \tau})_{m \in \mathbb{Z}}, \quad u_{1\tau} = (u_{1m, \tau})_{m \in \mathbb{Z}}, \text{ and} \\
\quad z_\tau = (z_{m, \tau})_{m \in \mathbb{Z}}, \text{ and take} \\
v = \dot{u} + \delta u, \quad \text{where} \quad \delta \equiv \frac{\mu \nu}{\nu^2 + 4\mu} > 0. \quad (3.6)
\]

Then the initial value problem (1.2)-(1.4) can be written as
\[ \dot{\psi} + H\psi = F(\psi, t), \quad t > \tau, \quad (3.7) \]
where \[ \psi = (u, v, z)^T, \quad \tau \equiv (u_\tau, v_\tau, z_\tau)^T = (u_\tau, u_{1\tau} + \delta u_\tau, z_\tau)^T, \quad \tau \in \mathbb{R}, \]
\[ \psi(\tau) = (u_\tau, v_\tau, z_\tau)^T = (u_\tau, u_{1\tau} + \delta u_\tau, z_\tau)^T, \quad \tau \in \mathbb{R}, \]
\[ \text{and} \quad z_\tau = (z_{m, \tau})_{m \in \mathbb{Z}}. \]

Let \[ C(\mathbb{R}, \ell^2) \] and \[ C(\mathbb{R}, L^2) \] be the spaces of continuous functions from \( \mathbb{R} \) into \( \ell^2 \) and \( L^2 \), respectively. Then for every \( f(\cdot) \in C(\mathbb{R}, \ell^2) \) (or \( C(\mathbb{R}, L^2) \)) and each \( t \in \mathbb{R}, \)
\[ \|f(t)\|^2 = \sum_{m \in \mathbb{Z}} \|f_m(t)\|^2 < +\infty. \quad (3.10) \]

We have, cf. [48, Lemma 2.2],

**Lemma 3.1.** Let \( f(t) = (f_m(t))_{m \in \mathbb{Z}} \in C(\mathbb{R}, L^2) \) and \( g(t) = (g_m(t))_{m \in \mathbb{Z}} \in C(\mathbb{R}, \ell^2) \). Then for any initial value \( \psi \equiv (u_\tau, v_\tau, z_\tau) \in E_\mu \), there is a unique local solution \( \psi(t) \equiv (u(t), v(t), z(t))^T \in E_\mu \) of problem (3.7)-(3.8) such that \( \psi(\cdot) \in C([\tau, T_0], E_\mu) \cap C^1([\tau, T_0], E_\mu) \) for some \( T_0 > \tau \). Moreover, if \( T_0 < +\infty \), then \( \lim_{t \to T_0^-} \|\psi(t)\|_{E_\mu} = +\infty. \)

We next prove the main estimates of the solutions.

**Lemma 3.2.** Let \( f(t) = (f_m(t))_{m \in \mathbb{Z}} \in C(\mathbb{R}, L^2) \) and \( g(t) = (g_m(t))_{m \in \mathbb{Z}} \in C(\mathbb{R}, \ell^2) \). Let \( \psi(t) = (u(t), v(t), z(t))^T \in E_\mu \) be the solution of problem (3.7)-(3.8) corresponding to initial condition \( \psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in E_\mu \). Then
\[ \|z(t)\|^2 \leq\|z_\tau\|^2 e^{-\alpha(t-\tau)} + \frac{e^{-\alpha t}}{\alpha} \int_{\tau}^{t} e^{\alpha s} \|f(s)\|^2 ds, \quad \forall t \geq \tau. \quad (3.11) \]

**Proof.** Indeed, equation (1.2) can be written in the vector form as
\[ i \dot{z} - Az + i\alpha z + zu = f(t), \quad t > \tau. \quad (3.12) \]
Taking the imaginary part of the inner product \( (L^2, (\cdot, \cdot)) \) of (3.12) with \( z \) yields
\[ \frac{1}{2} \frac{d}{dt} \|z(t)\|^2 + \alpha \|z(t)\|^2 = \Im f(t, z(t)) \leq \frac{\alpha}{2} \|z(t)\|^2 + \frac{1}{2\alpha} \|f(t)\|^2, \quad \forall t \geq \tau, \]
that is
\[ \frac{d}{dt} \|z(t)\|^2 + \alpha \|z(t)\|^2 \leq \frac{1}{\alpha} \|f(t)\|^2, \quad \forall t \geq \tau. \quad (3.13) \]
Applying Gronwall’s inequality to (3.13) we obtain (3.11). The proof is complete.

To estimate the solution \( \psi(t) \), we shall use the coercivity of the operator \( H \) defined by (3.9).
Lemma 3.3. ([48]) For any \( \psi = (u, v, z)^T \in E_\mu \), there holds
\[
\Re(H\psi, \psi)_{E_\mu} \geq \vartheta(\|u\|_\mu^2 + \|v\|^2) + \frac{\nu}{2}\|v\|^2 + \alpha\|z\|^2,
\] (3.14)
where
\[
\vartheta \triangleq \frac{\mu\nu}{\sqrt{\nu^2 + 4\mu(\sqrt{\nu^2 + 4\mu + \nu})}} \in (0, \frac{\nu}{2}).
\] (3.15)

Lemma 3.4. Let \( f(t) = (f_m(t))_{m \in \mathbb{Z}} \in \mathcal{C}(\mathbb{R}, L^2) \) and \( g(t) = (g_m(t))_{m \in \mathbb{Z}} \in \mathcal{C}(\mathbb{R}, L^2) \). Then the solution \( \psi(t) = (u(t), v(t), z(t))^T \in E_\mu \) of problem (3.7)-(3.8) corresponding to initial condition \( \psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in E_\mu \) satisfies
\[
\|\psi(t)\|^2_{E_\mu} \leq\|\psi_\tau\|^2_{E_\mu} e^{-\sigma(t-\tau)} + c_1 e^{-\sigma t} \int_{\tau}^{t} e^{\sigma s} \left(\|f(s)\|^2 + \|g(s)\|^2\right) ds
\]
\[+ c_1 e^{-\sigma t} \int_{\tau}^{t} e^{\sigma s} \|z(s)\|^4 ds, \quad \forall t \geq \tau,
\] (3.16)
where
\[
c_1 = \{4/\nu, \alpha\}, \quad \sigma \triangleq \min\{\vartheta, \alpha/4\}.
\] (3.17)

Proof. We shall prove (3.16). Taking the real part of the inner product \((\cdot, \cdot)_{E_\mu}\) of equation (3.7) with \( \psi(t) \) gives
\[
\frac{1}{2} \frac{d}{dt}\|\psi(t)\|^2_{E_\mu} + \Re(H\psi(t), \psi(t))_{E_\mu} = \Re(F(\psi(t), t), \psi(t))_{E_\mu}, \quad \forall t > \tau.
\] (3.18)
We need to estimate the term \( \Re(F(\psi(t), t), \psi(t))_{E_\mu} \). Indeed,
\[
\Re(F(\psi(t), t), \psi(t))_{E_\mu} = \left(\beta z(t)^2, v(t)\right) + \left(g(t), v(t)\right) - \text{Im}(f(t), z(t)),
\] (3.19)
\[
\beta z(t)^2, v(t) \leq \frac{\nu\|v(t)\|^2}{4} + \frac{\nu^2}{4}\|z(t)\|^4,
\] (3.20)
\[
g(t), v(t) \leq \frac{\nu\|v(t)\|^2}{4} + \frac{\nu}{\nu}\|g(t)\|^2,
\] (3.21)
\[
\text{Im}(f(t), z(t)) \leq \frac{\alpha}{2}\|z(t)\|^2 + \frac{1}{\alpha}\|f(t)\|^2,
\] (3.22)
Combining (3.14), (3.18) and the above estimates, we obtain, for any \( t \geq \tau \),
\[
\frac{d}{dt}\|\psi(t)\|^2_{E_\mu} + \sigma\|\psi(t)\|^2_{E_\mu} \leq c_1\|f(t)\|^2 + c_1\|g(t)\|^2 + c_1\|z(t)\|^4
\] (3.23)
with \( c_1 \) and \( \sigma \) defined in (3.17). Applying Gronwall’s inequality to (3.23) we obtain (3.16). This ends the proof.

From estimate (3.11) and (3.16) we conclude that for any \( \psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in E_\mu \), the corresponding solution \( \psi(t) = (u(t), v(t), z(t))^T \in E_\mu \) of problem (3.7)-(3.8) exists globally on \([\tau, +\infty)\). Moreover, from Lemma 3.1 we can see that
\[
\psi(\cdot) \in \mathcal{C}([\tau, +\infty), E_\mu) \cap C^1((\tau, +\infty), E_\mu).
\] (3.24)
Therefore, in view of Lemma 3.1, the maps
\[
U(t, \tau): \psi_\tau = (u_\tau, v_\tau, z_\tau)^T \in E_\mu \mapsto \psi(t) = (u(t), v(t), z(t))^T \in E_\mu, \quad \forall t \geq \tau,
\] (3.25)
generate a continuous process \( \{U(t, \tau)\}_{t \geq \tau} \) on \( E_\mu \), where \( v = \dot{u} + \delta u \).
We shall now analyze that under what assumptions on the data there exists a bounded pullback absorbing set for the process \( \{U(t, \tau)\}_{t \geq t} \). From (3.16) it follows that the following assumptions

\[
\lim_{\tau \to -\infty} \|\psi_\tau\|_E^2 e^{\alpha \tau} = 0, \tag{3.26}
\]

\[
\int_{-\infty}^t e^{\sigma s} \left( \|f(s)\|^2 + \|g(s)\|^2 \right) ds < +\infty, \quad \text{for each } t \in \mathbb{R}, \tag{3.27}
\]

are needed. The third term on the right hand side of (3.16) should tend to zero as \( \tau \to -\infty \), the bound should not depend on initial conditions, and this produces additional assumptions on \( f \). Observe that the only assumption we need for \( g \) is

\[
\int_{-\infty}^t e^{\sigma s} \|g(s)\|^2 ds < +\infty, \quad \text{for each } t \in \mathbb{R}. \tag{3.28}
\]

By (3.11) and (3.16) we have

\[
\int_{\tau}^t e^{\sigma s} \|z(s)\|^4 ds \leq I_1(t, \tau) + I_2(t, \tau) + I_3(t, \tau), \tag{3.29}
\]

where

\[
I_1(t, \tau) = \int_{\tau}^t e^{\sigma s} \|z_\tau\|^4 e^{-2\alpha (s-\tau)} ds,
\]

\[
I_2(t, \tau) = 2 \int_{\tau}^t e^{\sigma s} \|z_\tau\|^2 e^{\alpha \tau} e^{-2\alpha s} \int_{\tau}^s e^{\alpha \eta} \|f(\eta)\|^2 d\eta ds,
\]

\[
I_3(t, \tau) = \int_{\tau}^t e^{(\sigma-2\alpha)s} \left( \int_{\tau}^s e^{\alpha \eta} \|f(\eta)\|^2 d\eta \right)^2 ds.
\]

The first integral \( I_1(t, \tau) \) should go to zero as \( \tau \to -\infty \), and for that we need

\[
\lim_{\tau \to -\infty} \|z_\tau\|^2 e^{\tau} = 0, \tag{3.30}
\]

which is a stronger assumption for \( z_\tau \) than that from (3.26). Knowing (3.30) we write the second integral \( I_2(t, \tau) \) in the form

\[
2 \|z_\tau\|^2 e^{\frac{\sigma}{2} \tau} \int_{\tau}^t e^{\sigma s} e^{(\alpha - \frac{\sigma}{2}) \tau} e^{-2\alpha s} \int_{\tau}^s e^{\alpha \eta} \|f(\eta)\|^2 d\eta ds. \tag{3.31}
\]

Thus, \( I_2(t, \tau) \to 0 \) as \( \tau \to -\infty \) if the integral in (3.31) stays bounded as \( \tau \to -\infty \). For that we can assume

\[
e^{(\frac{\sigma}{2} - \alpha) s} \int_{-\infty}^s e^{\alpha \eta} \|f(\eta)\|^2 d\eta \leq K(s), \tag{3.32}
\]

where \( K(s) \) is a continuous function on the real line which is bounded on every interval of the form \((-\infty, t)\). Quite similarly, to have \( I_3(t, \tau) \to 0 \) as \( \tau \to -\infty \) we assume

\[
e^{(\frac{\sigma}{2} - \alpha - \gamma) s} \int_{-\infty}^s e^{\alpha \eta} \|f(\eta)\|^2 d\eta \leq \tilde{K}(s), \tag{3.33}
\]

for some \( \gamma \in (0, \sigma/2) \), where the function \( \tilde{K}(\cdot) \) has the same properties as the function \( K(\cdot) \) above.
Assumption (3.33) is stronger than assumption (3.32). Thus, for the existence of a bounded absorbing set for our process, we have two conditions for \( f \), namely
\[
\int_{-\infty}^{s} e^{\sigma \eta} \| f(\eta) \|^2 \, d\eta < +\infty, \quad \text{for each } s \in \mathbb{R}. \tag{3.34}
\]
These two assumptions are not independent. Assuming
\[
\int_{-\infty}^{s} e^{\sigma \eta} \| f(\eta) \|^2 \, d\eta < e^{(\frac{\sigma}{2} + \gamma) s} K(s) \tag{3.35}
\]
for some function \( K(\cdot) \) as above, we have also, cf. (3.33),
\[
\int_{-\infty}^{s} e^{\alpha \eta} \| f(\eta) \|^2 \, d\eta \leq e^{(\alpha - \sigma) s} \int_{-\infty}^{s} e^{\sigma \eta} \| f(\eta) \|^2 \, d\eta \leq e^{(\alpha - \frac{\sigma}{2} + \gamma) s} K(s). \tag{3.36}
\]
By above analysis, we see that if \( f(t) = (f_m(t))_{m \in \mathbb{Z}} \in \mathcal{C}(\mathbb{R}, L^2) \) and satisfies (3.35), then
\[
\lim_{\tau \to -\infty} c_1 e^{-\sigma t} \int_{\tau}^{t} e^{\sigma s} \| z(s) \|^4 \, ds = 0. \tag{3.37}
\]
We now summarize our assumptions on the external sources \( f \) and \( g \) leading to the existence of a bounded pullback absorbing set for the process \( \{ U(t, \tau) \}_{t \geq \tau} \).

(H) Assume \( f(t) = (f_m(t))_{m \in \mathbb{Z}} \in \mathcal{C}(\mathbb{R}, L^2) \) and \( g(t) = (g_m(t))_{m \in \mathbb{Z}} \in \mathcal{C}(\mathbb{R}, \ell^2) \). Moreover, let
\[
\int_{-\infty}^{s} e^{\sigma \eta} \| f(\eta) \|^2 \, d\eta < e^{(\frac{\sigma}{2} + \gamma) s} K(s), \tag{3.38}
\]
for some continuous function \( K(\cdot) \) on the real line, bounded on intervals of the form \((-\infty, t)\), with \( 0 < \gamma < \frac{\sigma}{2} \), and let
\[
\int_{-\infty}^{s} e^{\sigma \eta} \| g(\eta) \|^2 \, d\eta < +\infty, \quad \text{for each } s \in \mathbb{R}. \tag{3.39}
\]

Example 3.1. Let \( \| f(\eta) \|^2 \leq M e^{\kappa \eta} \) for all \( \eta \), with constant \( M > 0 \) and \( \kappa \geq \gamma - \frac{\sigma}{2} \). Then
\[
\int_{-\infty}^{s} e^{\sigma \eta} \| f(\eta) \|^2 \, d\eta < e^{(\frac{\sigma}{2} + \gamma) s} K(s), \tag{3.40}
\]
with
\[
K(s) = \frac{M}{\sigma + \kappa} e^{(\frac{\sigma}{2} + \kappa - \gamma) s}. \tag{3.41}
\]
Thus, choosing \( \kappa < 0, \kappa = 0 \) or \( \kappa > 0 \), we allow different behavior of \( f \) near infinities.

From now on, by \( \mathcal{P}(E_\mu) \) we denote the family of all nonempty subsets of \( E_\mu \) and by \( \mathcal{D}_\sigma \) the class of families of nonempty subsets \( \bar{D} = \{ D(s) | s \in \mathbb{R} \} \subseteq \mathcal{P}(E_\mu) \) satisfying
\[
\lim_{s \to -\infty} \left( e^{\frac{\sigma}{2}} \sup_{\psi \in D(s)} \| \psi \|^2_{L^2_{E_\mu}} \right) = 0. \tag{3.42}
\]
The class \( \mathcal{D}_\sigma \) will be called a universe in \( \mathcal{P}(E_\mu) \). Obviously, all fixed bounded subsets of \( E_\mu \) lie in \( \mathcal{D}_\sigma \).
Lemma 3.5. Let assumption (H) hold. Then the process \( \{U(t, \tau)\}_{t \geq \tau} \) corresponding to problem (3.7)-(3.8) possesses a bounded pullback-\( \mathcal{D}_\sigma \) absorbing set \( \tilde{B}_0 = \{B_0(s) \mid s \in \mathbb{R}\} \in \mathcal{P}(E_\mu) \), where \( B_0(s) = B(0, r_\sigma(s)) \) is the ball of radius \( r_\sigma(s) \) and centered at 0 in \( E_\mu \).

Proof. Let \( r_\sigma(t) = R_\sigma^{1/2}(t) \), where

\[
R_\sigma(t) = 1 + c_1 e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma s} \left( \|f(s)\|^2 + \|g(s)\|^2 \right) ds.
\]

(3.43)

Then from (3.16) and above analysis we conclude that the family \( \tilde{B}_0 \) constitutes the desired bounded pullback-\( \mathcal{D}_\sigma \) absorbing set for \( \{U(t, \tau)\}_{t \geq \tau} \) in \( E_\mu \).

We next prove that solutions of problem (3.7)-(3.8) satisfy condition (2) of Theorem 2.1, that is all solutions have pullback-\( \mathcal{D}_\sigma \) asymptotical nullness.

Lemma 3.6. Let assumption (H) hold. Then for any given \( t \in \mathbb{R}, \epsilon > 0 \) and \( \tilde{D} = \{D(s) \mid s \in \mathbb{R}\} \in \mathcal{D}_\tau \), there exist \( M_* = M_*(t, \epsilon, \tilde{D}) \in \mathbb{N} \) and \( \tau_* = \tau_*(t, \epsilon, \tilde{D}) \leq t \) such that

\[
\sup_{\tau \in \mathcal{D}(\tau)} \sum_{|m| \geq M_*} |(U(t, \tau)\psi_\tau)_m|^2_{E_\mu} \leq \epsilon^2, \quad \forall \tau \leq \tau_*,
\]

(3.44)

where

\[
|\psi_m|^2_{E_\mu} \triangleq \|(Bu)_m\|^2 + \mu u_m^2 + v_m^2 + |z_m|^2, \quad \psi_m = (u_m, v_m, z_m)^T.
\]

(3.45)

Proof. Choose a smooth function \( \chi(x) \in C^1(\mathbb{R}^+, \mathbb{R}^+) \) such that

\[
\left\{
\begin{array}{ll}
\chi(x) = 0 & \text{for } 0 \leq x \leq 1, \\
0 \leq \chi(x) \leq 1 & \text{for } 1 \leq x \leq 2, \\
\chi(x) = 1 & \text{for } x \geq 2, \\
|\chi'(x)| \leq \chi_0 \text{ (positive constant)} & \text{for } x > 0.
\end{array}
\right.
\]

(3.46)

Consider any \( \tilde{D} = \{D(s) \mid s \in \mathbb{R}\} \in \mathcal{D}_\tau \). For any \( t, \tau \in \mathbb{R} \) with \( t \geq \tau \), let

\( \psi(t) = \psi(t; \tau, \psi_\tau) = U(t, \tau)\psi_\tau = (u(t), v(t), z(t))^T = (u_m(t), v_m(t), z_m(t))^T \in E_\mu \)

be a solution of problem (3.7)-(3.8) with \( \psi_\tau \in \mathcal{D}(\tau) \). Let \( M \) be some positive integer and set

\[
p_m = \chi\left(\frac{|m|}{M}\right)u_m, \quad q_m = \chi\left(\frac{|m|}{M}\right)v_m, \quad w_m = \chi\left(\frac{|m|}{M}\right)z_m, \quad \forall m \in \mathbb{Z}
\]

and

\[
\phi = (\phi_m)_{m \in \mathbb{Z}} \text{ with } \phi_m = (p_m, q_m, z_m)^T.
\]

Taking the real part of the inner product \( \langle \cdot, \cdot \rangle_{E_\mu} \) of equation (3.7) with \( \phi(t) \) yields

\[
\text{Re}(\dot{\psi}(t), \phi(t))_{E_\mu} + \text{Re}(H\psi(t), \phi(t))_{E_\mu} = \text{Re}(F(\psi(t), t), \phi(t))_{E_\mu}, \quad \forall t > \tau. \quad (3.47)
\]

We shall estimate the terms in (3.47) one by one. Firstly, direct computation gives

\[
\text{Re}(\dot{\psi}, \phi)_{E_\mu} = \sum_{m \in \mathbb{Z}} (B\dot{u})_m(Bp)_m + \mu(\dot{u}, p) + (\dot{v}, q) + (\dot{z}, w)
\]

\[
= \frac{1}{2} \frac{d}{dt} \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right)|\psi_m|^2_{E_\mu}
\]

\[
+ \sum_{m \in \mathbb{Z}} \left[\chi\left(\frac{|m+1|}{M}\right) - \chi\left(\frac{|m|}{M}\right)\right] (v_{m+1} - \delta u_{m+1} - v_m + \delta u_m) u_m.
\]
By Lemma 3.5 and (3.46) there exists some $\tau(t, \hat{D}) \leq t$ such that
\[
\Re\left(\psi(t, \phi(t))_{E^\nu} - \frac{1}{2} \frac{d}{dt} \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right)|\psi_m(t)|^2_{E^\nu}\right)
= \sum_{m \in \mathbb{Z}} \left(\chi\left(\frac{|m+1|}{M}\right) - \chi\left(\frac{|m|}{M}\right)\right) (v_m + 1 - \delta u_m + v_m + \delta u_m) u_m
= \sum_{m \in \mathbb{Z}} \chi'\left(\frac{\tilde{m}}{M}\right) \frac{1}{M} (v_m + 1 - \delta u_m + v_m + \delta u_m) u_m
\geq - \frac{2\chi_0}{M鲁(\mu + \delta + 1)} R_\sigma(t), \quad \forall \tau \leq \tau(t, \hat{D}),
\]
hereinafter $\tilde{m}$ stands for some constant located between $|m+1|$ and $|m|$, and $|\psi_m|_{E^\nu}$ is defined as in (3.45).

Secondly, we compute
\[
\Re(\bar{H}\psi, \phi)_{E^\nu} = \delta(Bu, Bp) + \mu\delta(u, p) - (Bv, Bp) - \mu(v, p) + (Bu, Bq) + \mu(u, q) + \delta(\delta - \nu)(u, q) + (\nu - \delta)(v, q) - \Im(Az, w) + \alpha(z, w),
\]
where
\[
(Bu, Bp) = \sum_{m \in \mathbb{Z}} (Bu)_m(Bp)_m = \sum_{m \in \mathbb{Z}} (Bu)_m \left(\chi\left(\frac{|m+1|}{M}\right)u_{m+1} - \chi\left(\frac{|m|}{M}\right)u_m\right)
\geq \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right)|u_{m+1} - u_m|^2 - \frac{2\chi_0 R_\sigma(t)}{M鲁}, \quad \forall \tau \leq \tau(t, \hat{D}),
\]
and
\[
(Bu, Bq) - (Bv, Bp) = \sum_{m \in \mathbb{Z}} (Bu)_m(Bq)_m - \sum_{m \in \mathbb{Z}} (Bv)_m(Bp)_m
= \sum_{m \in \mathbb{Z}} (u_{m+1} - u_m) \left(\chi\left(\frac{|m+1|}{M}\right)v_{m+1} - \chi\left(\frac{|m|}{M}\right)v_m\right)
- \sum_{m \in \mathbb{Z}} (v_{m+1} - v_m) \left(\chi\left(\frac{|m+1|}{M}\right)u_{m+1} - \chi\left(\frac{|m|}{M}\right)u_m\right)
= \sum_{m \in \mathbb{Z}} \chi'\left(\frac{\tilde{m}}{M}\right) \frac{1}{M} (u_{m+1}v_m - u_m v_{m+1})
\geq - \frac{2\chi_0(\mu + 1)}{M鲁} R_\sigma(t), \quad \forall \tau \leq \tau(t, \hat{D}),
\]
and
\[
\Im(Az, w) = - \Im(Bz, Bw)
= - \Im\left(\sum_{m \in \mathbb{Z}} (z_{m+1} - z_m) \left(\chi\left(\frac{|m+1|}{M}\right)e_{m+1} - \chi\left(\frac{|m|}{M}\right)e_m\right)\right)
\geq - \frac{\chi_0}{M鲁} R_\sigma(t), \quad \forall \tau \leq \tau(t, \hat{D}).
\]
Since $4\mu(\delta - \vartheta)(\nu - \delta - \theta) = \delta^2 \nu^2$, we have for any $m \in \mathbb{Z}$ that

$$(\delta - \vartheta)((u_{m+1} - u_m)^2 + \mu u_m^2) + (\nu - \delta - \vartheta)v_m^2 + \delta(\delta - \nu)u_m v_m$$

$$\geq \mu(\delta - \vartheta)u_m^2 + (\nu - \delta - \vartheta)v_m^2 - \delta\nu|u_m v_m| \geq 0. \quad (3.55)$$

It then follows from (3.49)-(3.55) that

$$\text{Re}(H\psi, \phi)_{\mu} - \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right) \left(\vartheta((u_{m+1} - u_m)^2 + \mu u_m^2 + v_m^2) + \left|u_m \nu + \alpha |z_m|\right|\right)$$

$$\geq \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right) \left((\delta - \vartheta)((u_{m+1} - u_m)^2 + \mu u_m^2) + (\nu - \delta - \vartheta)v_m^2 + \delta(\delta - \nu)u_m v_m\right)$$

$$- \chi_0 R_0(t)^2(2\delta + 3\mu + 2)$$

$$\geq - \frac{\chi_0(2\delta + 3\mu + 2) R_0(t)}{M\mu}, \quad \forall \tau \leq \tau(t, \tilde{D}). \quad (3.56)$$

Finally,

$$\text{Re}(F(\psi, t), \phi)_{\mu} = \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right) \left(\beta|z_m|^2 v_m + v_m g_m\right) - \text{Im} \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right) \bar{z}_m f_m. \quad (3.57)$$

For any $\tau \leq \tau(t, \tilde{D}) < t$, we have

$$\beta \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right)|z_m(t)|^2 v_m(t)$$

$$= \beta \sum_{|m| \geq M} \chi\left(\frac{|m|}{M}\right)|z_m(t)|^2 v_m(t)$$

$$\leq \frac{\nu}{4} \sum_{|m| \geq M} \chi\left(\frac{|m|}{M}\right)v_m^2(t) + \frac{2\beta^2 R_0(t)}{\nu} \sum_{|m| \geq M} \chi\left(\frac{|m|}{M}\right)|z_m(t)|^2, \quad (3.58)$$

$$\sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right)v_m(t)g_m(t) \leq \frac{\nu}{4} \sum_{|m| \geq M} \chi\left(\frac{|m|}{M}\right)v_m^2(t) + \frac{2}{\nu} \sum_{|m| \geq M} g_m^2(t), \quad (3.59)$$

$$- \text{Im} \sum_{m \in \mathbb{Z}} \chi\left(\frac{|m|}{M}\right)\bar{z}_m(t) f_m(t) \leq \frac{\alpha}{2} \sum_{|m| \geq M} \chi\left(\frac{|m|}{M}\right)|z_m(t)|^2 + \frac{1}{\alpha} \sum_{|m| \geq M} |f_m(t)|^2. \quad (3.60)$$

We now estimate the underlined term $I$ in (3.58). Taking the imaginary part of the inner product $(\cdot, \cdot)$ of equation (3.12) with $w = (w_m)_{m \in \mathbb{Z}} = (\chi\left(\frac{|m|}{M}\right)z_m)_{m \in \mathbb{Z}}$ and using (3.54) yield

$$\frac{1}{2} \frac{d}{dt} \sum_{|m| \geq M} \chi\left(\frac{|m|}{M}\right)|z_m(t)|^2 + \alpha \sum_{|m| \geq M} \chi\left(\frac{|m|}{M}\right)|z_m(t)|^2$$
\[
\text{Im}\left( \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) z_m(t) f_m(t) \right) + \text{Im}(A z(t), v(t)) \\
\leq \frac{\alpha}{2} \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(t)|^2 + \frac{1}{2\alpha} \sum_{|m| \geq M} |f_m(t)|^2 + \frac{\chi_0}{M} R_\sigma^2(t), \quad \forall \tau \leq \tau(\hat{\tau}, \hat{D}) < t,
\]
that is
\[
\frac{d}{dt} \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(t)|^2 + \alpha \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(t)|^2 \\
\leq \frac{1}{\alpha} \sum_{|m| \geq M} |f_m(t)|^2 + \frac{2\chi_0}{M} R_\sigma(t), \quad \forall \tau \leq \tau(\hat{\tau}, \hat{D}) < t. \quad (3.61)
\]
Applying Gronwall inequality to (3.61), we obtain
\[
\sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(t)|^2 \leq \frac{1}{\alpha} \int^t_\tau \left( \sum_{|m| \geq M} |f_m(s)|^2 + \frac{2\chi_0 R_\sigma(s)}{\alpha M} \right) e^{-\alpha(t-s)} ds \\
+ e^{-\alpha(t-\tau)} \left( \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(\tau)|^2 \right), \quad \forall \tau \leq \tau(\hat{\tau}, \hat{D}) < t. \quad (3.62)
\]
It follows from (3.57)-(3.62) that
\[
\text{Re}(F(\psi, t), \phi(t))_{E_\mu} \\
\leq \frac{\nu}{2} \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) v_m^2(t) + \frac{\alpha}{2} \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(t)|^2 + \frac{1}{\nu} \sum_{|m| \geq M} g_m^2(t) \\
+ \frac{1}{2\alpha} \sum_{|m| \geq M} |f_m(t)|^2 + \frac{\beta^2 R_\sigma(t)}{\alpha \nu} \int^t_\tau \left( \sum_{|m| \geq M} |f_m(s)|^2 + \frac{2\chi_0 R_\sigma(s)}{\alpha M} \right) e^{-\alpha(t-s)} ds \\
+ \frac{\beta^2 R_\sigma^2(t)}{\alpha \nu} e^{-\alpha(t-\tau)} \left( \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(\tau)|^2 \right), \quad \forall \tau \leq \tau(\hat{\tau}, \hat{D}) < t. \quad (3.63)
\]
Inserting (3.48), (3.56) and (3.63) into (3.47) gives
\[
\frac{d}{dt} \sum_{m \in \mathbb{Z}} \chi\left( \frac{|m|}{M} \right) |\psi_m|^2_{E_\mu} + \sigma \sum_{m \in \mathbb{Z}} \chi\left( \frac{|m|}{M} \right) |\psi_m|^2_{E_\mu} \\
\leq \frac{\nu}{2} \sum_{|m| \geq M} g_m^2(t) + \frac{1}{\alpha} \sum_{|m| \geq M} |f_m(t)|^2 + \frac{2\chi_0 (4\delta + 5\mu + 4) R_\sigma(t)}{M\mu} \\
+ \frac{2\beta^2 R_\sigma(t)}{\alpha \nu} \int^t_\tau \left( \sum_{|m| \geq M} |f_m(s)|^2 + \frac{2\chi_0 R_\sigma(s)}{\alpha M} \right) e^{-\alpha(t-s)} ds \\
+ \frac{2\beta^2 R_\sigma^2(t)}{\alpha \nu} e^{-\alpha(t-\tau)} \left( \sum_{|m| \geq M} \chi\left( \frac{|m|}{M} \right) |z_m(\tau)|^2 \right), \quad \forall \tau \leq \tau(\hat{\tau}, \hat{D}) < t. \quad (3.64)
\]
For any \( \epsilon > 0 \) and the given \( t \in \mathbb{R} \), we see from (3.27) and (3.43) that there is some \( M_1 = M_1(t, \epsilon) \in \mathbb{N} \) such that
\[
\frac{2\chi_0 (4\delta + 5\mu + 4) R_\sigma(t)}{M\mu} \leq \epsilon \sigma^2/12, \quad \forall M > M_1. \quad (3.65)
\]
At the same time,
\[
\frac{2\beta^2 R_\sigma(t)}{\alpha \nu} \int_t^1 e^{-\alpha(t-s)} \sum_{|m| \geq M} |f_m(s)|^2 ds = \frac{2\beta^2 R_\sigma(t)e^{-\alpha t}}{\alpha \nu} \int_t^1 e^{\alpha s} \sum_{|m| \geq M} |f_m(s)|^2 ds.
\]
By (3.27) and (3.43), \(2\beta^2 R_\sigma(t)e^{-\alpha t}\) is a constant depending only on \(t\). We also see from (3.36) that
\[
\int_{-\infty}^{t} e^{\alpha \theta} \|f(s)\|^2 ds < +\infty, \quad \text{for each } t \in \mathbb{R}.
\]
Thus there is some \(M_2 = M_2(t, \epsilon) \in \mathbb{N}\) such that
\[
\frac{2\beta^2 R_\sigma(t)e^{-\alpha t}}{\alpha \nu} \int_t^1 e^{\alpha s} \sum_{|m| \geq M} |f_m(s)|^2 ds \leq \frac{2\beta^2 R_\sigma(t)e^{-\alpha t}}{\alpha \nu} \sum_{|m| \geq M} \int_{-\infty}^{t} e^{\alpha s} |f_m(s)|^2 ds \leq \sigma \epsilon^2/12, \quad \forall M > M_2.
\] (3.66)

Note that
\[
\int_t^1 \frac{2\chi_0 R_\sigma(s)}{\alpha M} e^{-\alpha(t-s)} ds = \frac{2\chi_0}{\alpha M} \int_t^1 R_\sigma(s) e^{-\alpha(t-s)} ds
\leq \frac{2\chi_0}{\alpha M} \int_t^1 e^{-\alpha(t-s)} ds + \frac{2c_1 \chi_0}{\alpha M} \int_t^1 e^{(\alpha-\sigma)s} \int_{-\infty}^{t} e^{\sigma \theta} (\|f(\theta)\|^2 + \|g(\theta)\|^2) d\theta ds
\leq \frac{2\chi_0}{\alpha^2 M} + \frac{2c_1 \chi_0}{\alpha M} \int_t^1 e^{(\alpha-\sigma)s} \int_{-\infty}^{t} e^{\sigma \theta} (\|f(\theta)\|^2 + \|g(\theta)\|^2) d\theta.
\] (3.67)

Hence, for above \(t\) and \(\epsilon\), there exists some \(M_3 = M_3(t, \epsilon) \in \mathbb{N}\) such that
\[
\frac{2\beta^2 R_\sigma(t)}{\alpha \nu} \int_t^1 \frac{2\chi_0 R_\sigma(s)}{\alpha M} e^{-\alpha(t-s)} ds \leq \sigma \epsilon^2/12, \quad \forall M > M_3.
\] (3.68)

It is obvious that
\[
\frac{2\beta^2 R_\sigma(t)}{\alpha \nu} e^{-\alpha(t-\tau)} \left( \sum_{|m| \geq M} \chi \left( \frac{|m|}{M} \right) \left| z_m(\tau) \right|^2 \right) \leq \frac{2\beta^2 R_\sigma(t)}{\alpha \nu} e^{-\alpha(t-\tau)} \left| z_\tau \right|^2.
\]
So we get from (3.27), (3.30) and (3.43) that for above \(t\) and \(\epsilon\), there exists some \(\tau_1 = \tau_1(t, \epsilon) \in \mathbb{N}\) yielding
\[
\frac{2\beta^2 R_\sigma(t)}{\alpha \nu} e^{-\alpha(t-\tau)} \left( \sum_{|m| \geq M} \chi \left( \frac{|m|}{M} \right) \left| z_m(\tau) \right|^2 \right) \leq \sigma \epsilon^2/12, \quad \forall \tau \leq \tau_1 \leq \tau(t, \tilde{D}) < t.
\] (3.69)

Inserting (3.65), (3.66), (3.68), (3.69) into (3.64) gives for \(\forall \tau \leq \tau_1 \leq \tau(t, \tilde{D}) < t\) and \(M > \max\{M_1, M_2, M_3\}\) that
\[
\frac{d}{dt} \sum_{m \in Z} \chi \left( \frac{|m|}{M} \right) |\psi_m|_{E_\nu}^2 + \sigma \sum_{m \in Z} \chi \left( \frac{|m|}{M} \right) |\psi_m|_{E_\nu}^2
\leq \frac{2}{\nu} \sum_{|m| \geq M} \gamma^2_m(t) + \frac{1}{\alpha} \sum_{|m| \geq M} |f_m(t)|^2 + \sigma \epsilon^2/3.
\] (3.70)
Applying Gronwall inequality to (3.70) yields
\[
\sum_{m \in \mathbb{Z}} \chi(|m|) |\psi_m(t)|_E^2 \leq \|\psi\|_{E^\nu} e^{-\sigma(t-\tau)} + \frac{2e^{-\sigma t}}{\nu} \int_\tau^t e^{\sigma \theta} \sum_{|m| \geq M} g_m^2(\theta) d\theta \\
+ \frac{e^{-\sigma t}}{\alpha} \int_\tau^t e^{\sigma \theta} \sum_{|m| \geq M} |f_m(\theta)|^2 d\theta.
\]

(3.71)

Again from (3.27) we see that there is some \( M_4 = M_4(t, \epsilon) \in \mathbb{N} \) such that
\[
\frac{2e^{-\sigma t}}{\nu} \int_\tau^t e^{\sigma \theta} \sum_{|m| \geq M} g_m^2(\theta) d\theta + \frac{e^{-\sigma t}}{\alpha} \int_\tau^t e^{\sigma \theta} \sum_{|m| \geq M} |f_m(\theta)|^2 d\theta \\
\leq \frac{2e^{-\sigma t}}{\nu} \sum_{|m| \geq M} \int_\tau^t e^{\sigma \theta} g_m^2(\theta) d\theta + \frac{e^{-\sigma t}}{\alpha} \sum_{|m| \geq M} \int_\tau^t e^{\sigma \theta} |f_m(\theta)|^2 d\theta \\
\leq \epsilon^2/3, \quad \forall \, M > M_4.
\]

(3.72)

Now, by (3.30) we conclude that for above \( \epsilon \) and \( t \), there exists some \( \tau_2 = \tau_2(t, \epsilon, \hat{D}) \) such that
\[
e^{-\sigma t} \cdot e^{\sigma \tau} \sup_{\psi(t) \in D(\tau)} \|\psi(t)\|_{E^\nu}^2 \leq \frac{\epsilon^2}{3}, \quad \forall \, \tau \leq \tau_2.
\]

(3.73)

Choosing
\[
M_* = \max\{M_1, M_2, M_3, M_4\}, \quad \tau_* = \min\{\tau(t, \hat{D}), \tau_1, \tau_2\}
\]

(3.74)

and inserting the inequalities (3.72) and (3.73) into (3.71) yield
\[
\sup_{\psi(t) \in D(\tau)} \sum_{|m| \geq 2M_*} \|U(t, \tau) \psi_m\|^2_{E^\nu} = \sup_{\psi(t) \in D(\tau)} \sum_{|m| \geq 2M_*} \|\psi_m(t)\|^2_{E^\nu} \leq \epsilon^2, \quad \forall \, \tau \leq \tau_*.
\]

The proof of Lemma 3.6 is complete.

Combining Lemmas 3.5-3.6 (cf. also [18, Theorem 3.11, Corollary 3.13]) we get the main result of this section, namely

Theorem 3.1. Let assumption (H) hold. Then there exists a pullback-\( D_* \) attractor \( \hat{A}_{D_*} = \{A_{D_*(t)}\mid t \in \mathbb{R}\} \) for the process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( E^\mu \) associated to the solution operators of problem (3.7)-(3.8).

4. Invariant measures on the pullback attractor. The aim of this section is to apply the results of Lukaszewicz and Robinson [36] to prove the existence of a unique family of invariant Borel probability measures associated with the process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( E^\mu \) form Theorem 3.1. We first recall two definitions. For the basic properties of generalized Banach limits we refer the reader to [10, 16, 35].

Definition 4.1. ([16]) A generalized Banach limit is any linear functional, which we denote by \( \text{LIM}_{T \to \infty} \), defined on the space of all bounded real-valued functions on \( [0, +\infty) \) that satisfies
(i) \( \text{LIM}_{T \to \infty} \varphi(T) \geq 0 \) for nonnegative functions \( \varphi() \);
(ii) \( \text{LIM}_{T \to \infty} \varphi(T) = \lim_{T \to \infty} \varphi(T) \) if the usual limit \( \lim_{T \to \infty} \varphi(T) \) exists.

Definition 4.2. ([36]) A process \( \{\mathcal{U}(t, \tau)\}_{t \geq \tau} \) is said to be \( \tau \)-continuous on a metric space \( X \) if for every \( x_0 \in X \) and every \( t \in \mathbb{R} \), the \( X \)-valued function \( \tau \mapsto \mathcal{U}(t, \tau)x_0 \) is continuous and bounded on \( (-\infty, t] \).
Remark 4.1. Notice that we consider the “pullback” asymptotic behavior and we require generalized limits as \( \tau \to -\infty \). For a given real-valued function \( \varphi \) defined on \((-\infty, 0]\) and a given Banach limit \( \operatorname{LIM}_{t \to -\infty} \), we define
\[
\operatorname{LIM}_{t \to -\infty} \varphi(t) = \operatorname{LIM}_{t \to \infty} \varphi(-t). \tag{4.1}
\]

The following result was proved by Łukaszewicz and Robinson in [36].

Proposition 4.1. ([36]) Let \( \{U(t, \tau)\}_{t \geq \tau} \) be a \( \tau \)-continuous evolutionary process in a complete metric space \( X \) that has a pullback-\( \mathcal{D} \) attractor \( \mathcal{A}(\cdot) \). Fix a generalized Banach limit \( \operatorname{LIM}_{t \to -\infty} \) and let \( \gamma : \mathbb{R} \to X \) be a continuous map such that \( \gamma(\cdot) \in \mathcal{D} \).

Then there exists a unique family of Borel probability measures \( \{\mu_t\}_{t \in \mathbb{R}} \) in \( X \) such that the support of the measure \( \mu_t \) is contained in \( \mathcal{A}(t) \) and
\[
\operatorname{LIM}_{t \to -\infty} \frac{1}{t - \tau} \int_{\tau}^{t} \varphi(U(t, s)\gamma(s))ds = \int_{\mathcal{A}(t)} \varphi(v)d\mu_t(v) \tag{4.2}
\]
for any real-valued continuous functional \( \varphi \) on \( X \). In addition, \( \mu_t \) is invariant in the sense that
\[
\int_{\mathcal{A}(t)} \varphi(v)d\mu_t(v) = \int_{\mathcal{A}(\tau)} \varphi(U(t, \tau)v)d\mu_{\tau}(v), \quad t \geq \tau.
\]

In order to employ the above result to the pullback-\( \mathcal{D}_\tau \) attractor \( \hat{\mathcal{A}}_{\mathcal{D}_\tau} \) obtained in Theorem 3.1, we need to check the \( \tau \)-continuous property of the process \( \{U(t, \tau)\}_{t \geq \tau} \)
in the space \( E_\mu \).

Lemma 4.1. Let \( \psi^{(1)}(t) \) and \( \psi^{(2)}(t) \) be two solutions of problem (3.7)-(3.8) corresponding to the initial conditions \( \psi_{1\tau} \) and \( \psi_{2\tau} \in E_\mu \), respectively. Then
\[
\|\psi^{(1)}(t) - \psi^{(2)}(t)\|_{E_\mu}^2 \leq \|\psi_{1\tau} - \psi_{2\tau}\|_{E_\mu}^2 \exp\left\{ \frac{4(\beta^2 + 1)}{\sigma} \int_\tau^t (\|z^{(1)}(s)\|^2 + \|z^{(2)}(s)\|^2 + \|u^{(2)}(s)\|^2)ds \right\}. \tag{4.3}
\]

Proof. Let \( \psi^{(k)}(t) = \phi^{(k)}(t; \tau, \psi_{\tau}) = (u^{(k)}(t), v^{(k)}(t), z^{(k)}(t))^T \) \( k = 1,2 \), be two solutions of (3.7)-(3.8) corresponding to the initial data \( \psi_{1\tau}^{(1)}, \psi_{2\tau}^{(2)} \in E_\mu \), respectively. Set
\[
\begin{align*}
\tilde{u}(t) &= u^{(1)}(t) - u^{(2)}(t), \\
\tilde{v}(t) &= v^{(1)}(t) - v^{(2)}(t) = \tilde{u}^{(1)}(t) - \tilde{u}^{(1)}(t) + \delta(u^{(1)}(t) - u^{(2)}(t)), \\
\tilde{z}(t) &= z^{(1)}(t) - z^{(2)}(t), \\
\tilde{\psi}(t) &= \psi^{(1)}(t) - \psi^{(2)}(t).
\end{align*}
\]

Then \( \tilde{\psi} \) satisfies
\[
\begin{align*}
\frac{d}{dt} \tilde{\psi}(t) + H \tilde{\psi}(t) &= F(\psi^{(1)}(t), t) - F(\psi^{(2)}(t), t), \quad \forall t > \tau, \\
\tilde{\psi}|_{t=\tau} &= \tilde{\psi}(\tau) = \psi_{1\tau} - \psi_{2\tau}.
\end{align*}
\] \tag{4.4}

From (3.14), we conclude that
\[
\Re(H \tilde{\psi}, \tilde{\psi})_{E_\mu} \geq \theta(\|\tilde{u}\|_{\mu}^2 + \|\tilde{v}\|^2) + \frac{\nu}{2}\|\tilde{v}\|^2 + \alpha\|\tilde{z}\|^2, \quad \forall t \geq \tau. \tag{4.6}
\]

By direct computation,
\[
\|F(\psi^{(1)}(t), t) - F(\psi^{(2)}(t), t)\|_{E_\mu}^2
\]
This ends the proof.

Lemma 4.2. Let \( f(t) \) and \( g(t) \) satisfy the conditions of (H). Then for every \( \psi_s \in E_\mu \) and every \( t \in \mathbb{R} \), the \( E_\mu \)-valued function \( \tau \mapsto U(t, \tau)\psi_s \) is continuous and bounded on \( (-\infty, t) \).

Proof. Consider any \( \psi_s = (u_s, v_s, z_s)^T \in E_\mu \) and \( t \in \mathbb{R} \). We shall prove that for any \( \epsilon > 0 \) there exists some \( \delta = \delta(\epsilon) > 0 \), such that if \( r < t, s < t \) and \( |r - s| < \delta \), then \( \|U(t, r)\psi_s - U(t, s)\psi_s\|_{E_\mu} < \epsilon \). We assume \( r < s \) without loss of generality. Set

\[
\begin{align*}
U(\cdot, s) &= (u_s, v_s, z_s)^T, \\
U(\cdot, r) &= (u_r, v_r, z_r)^T.
\end{align*}
\]

Employing Lemma 4.1 and the continuity property of the process, we have

\[
\|U(t, r)\psi_s - U(t, s)\psi_s\|_{E_\mu}^2 \leq \|U(t, s)U(s, r)\psi_s - U(t, s)U(r, r)\psi_s\|_{E_\mu}^2 \leq \|U(s, r)\psi_s - U(r, r)\psi_s\|_{E_\mu}^2 \times \exp\left\{\frac{4(\beta^2 + 1)}{\sigma} \int_s^t (\|z^{(1)}(\theta)\|^2 + \|z^{(2)}(\theta)\|^2 + \|u^{(2)}(\theta)\|^2) d\theta\right\}.
\]

Now, by (3.24) we can see that solutions of problem (3.7)-(3.8) belong to the space \( \mathcal{C}(\mathbb{R}, E_\mu) \), hence for any \( s \in \mathbb{R} \) with \( s \leq t \),

\[
\exp\left\{\frac{4(\beta^2 + 1)}{\sigma} \int_s^t (\|z^{(1)}(\theta)\|^2 + \|z^{(2)}(\theta)\|^2 + \|u^{(2)}(\theta)\|^2) d\theta\right\} < +\infty.
\]

Therefore, from (3.24) and (4.11) we conclude that the right hand side of inequality (4.10) is as small as needed if \( |r - s| \) is small enough. Therefore, the \( E_\mu \)-valued function \( \tau \mapsto U(t, \tau)\psi_s \) is continuous with respect to \( \tau \in (-\infty, t] \) in the space \( E_\mu \).
Finally, for above $\psi_\ast \in E_\mu$ and $t \in \mathbb{R}$, we see from (3.16), (3.27), (3.30) and (3.37) that
\[
\lim_{\tau \to -\infty} \|U(t, \tau)\psi_\ast\|^2_{E_\mu} \\
\leq \lim_{\tau \to -\infty} \|\psi_\tau\|^2_{E_\mu} e^{-\sigma(t-\tau)} + \lim_{\tau \to -\infty} c_1 e^{-\sigma t} \int_{\tau}^{t} e^{\sigma \theta} \left( \|f(\theta)\|^2 + \|g(\theta)\|^2 \right) d\theta \\
+ \lim_{\tau \to -\infty} c_1 e^{-\sigma t} \int_{\tau}^{t} e^{\sigma \theta} \left\| z(\theta) \right\|^4 d\theta \\
= c_1 e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma \theta} \left( \|f(\theta)\|^2 + \|g(\theta)\|^2 \right) d\theta < +\infty,
\] (4.12)

where $c_1 e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma \theta} \left( \|f(\theta)\|^2 + \|g(\theta)\|^2 \right) d\theta$ is independent of $\tau$. Notice that the $E_\mu$-valued function $\tau \mapsto U(t, \tau)\psi_\ast$ is continuous with respect to $\tau \in (-\infty, t]$ in the space $E_\mu$. Thus the $E_\mu$-valued function $\tau \mapsto U(t, \tau)\psi_\ast$ is bounded on $(-\infty, t]$. The proof is complete.

Combining Theorem 3.1, Proposition 4.1 and Lemma 4.2 we obtain the following

**Theorem 4.1.** Let $f(t)$ and $g(t)$ satisfy the conditions of (H). Let $\{U(t, \tau)\}_{t \geq \tau}$ be the process associated to the solution operators of problem (3.7)-(3.8) and $A_{D_\sigma} = \{A_{D_\sigma}(t) \mid t \in \mathbb{R}\}$ be the pullback $D_\sigma$-attractor obtained in Theorem 3.1. Fix a generalized Banach limit $\text{LIM}_{\tau \to \infty}$ and let $\psi : \mathbb{R} \to E_\mu$ be a continuous map such that $\psi(\cdot) \in D_\sigma$. Then there exists a unique family of Borel probability measures $\{m_t\}_{t \in \mathbb{R}}$ in the space $E_\mu$ such that the support of the measure $m_t$ is contained in $A_{D_\sigma}(t)$ and
\[
\text{LIM}_{\tau \to \infty} \frac{1}{t-\tau} \int_{\tau}^{t} \lambda(U(t, s)\psi(s))ds = \int_{A_{D_\sigma}(t)} \lambda(\phi)d m_t(\phi)
\]
for any real-valued continuous functional $\lambda$ on $E_\mu$. In addition, $m_t$ is invariant in the sense that
\[
\int_{A_{D_\sigma}(t)} \lambda(\phi)d m_t(\phi) = \int_{A_{D_\sigma}(\tau)} \lambda(U(t, \tau)\psi(\cdot))d m_t(\phi), \quad t \geq \tau.
\]

5. **Conclusions and remarks.** In this article, we extend the results of Zhao and Zhou [50] and Zhou and Shi [54] to the scenario of pullback asymptotic behavior for general non-autonomous LDSs, and provide a sufficient and necessary condition for the existence of pullback-$D$ attractor for the process defined on a Hilbert space of infinite sequences. In comparison to previous results,

(1) The basins of attraction are extended: compared to the kernel sections discussed in [48], the pullback-$D_\sigma$ attractor discussed in this article has more general basins of attraction, which are referred to a given universe $D_\sigma$ rather than only fixed bounded subsets in the phase space $E_\mu$. In fact, any fixed bounded subsets of the phase space $E_\mu$ lie in the universe $D_\sigma$.

(2) The conditions imposed to the external forces are weaken: in [47] and [48], the external forces are assumed to be continuous and bounded functions from $\mathbb{R}$ into $ell^2$ (or $L^2$), and also to satisfy condition (1.8). In this article, we remove condition (1.8) and the boundedness imposed on the external forces.
(3) The abstract result of [36] (Proposition 4.1) on the existence of invariant Borel probability measures was applied directly thanks to our using the theory of the pullback attractors, while it was not possible to apply it in the frame of the theory of kernel sections discussed in [48].

We end this article with the following remark on a possible extension of our results.

Remark 5.1. The validity of Theorem 2.1 could be examined for general lattice system (2.2) considered in $\mathbb{Z}^k$ for some positive integer $k \geq 2$,

$$
\begin{align*}
\dot{u} &= \varrho(u, t), \quad u = (u_m)_{m \in \mathbb{Z}^k}, \quad t > \tau, \\
u|_{t=\tau} &= u_\tau = (u_m(\tau))_{m \in \mathbb{Z}^k} \in \ell^2,
\end{align*}
$$

(5.1)

where

$$
\ell^2 = \left\{ u = (u_m)_{m \in \mathbb{Z}^k} \mid u_m \in \mathbb{R}, \sum_{m \in \mathbb{Z}^k} u_m^2 < +\infty \right\}.
$$

Also, the validity of Theorem 3.1 and Theorem 4.3 could be examined for the following infinite lattice systems:

$$
\begin{align*}
i \dot{z}_m - (Az)_m + i \alpha z_m + z_m u_m &= f_m(t), \quad m = (m_1, m_2, \cdots, m_k) \in \mathbb{Z}^k, \\
u_m + \nu u_m + (Au)_m + \mu u_m - \beta |z_m|^2 &= g_m(t), \quad m = (m_1, m_2, \cdots, m_k) \in \mathbb{Z}^k,
\end{align*}
$$

(5.2)-(5.3)

where the operator $A$ is assumed to satisfied some properties. For example, if the operator $A$ is defined as

$$
(Au)_m = (Au)_{(m_1, m_2, \cdots, m_k)}
= 2ku_m - u_{(m_1+1, m_2, \cdots, m_k)} - u_{(m_1, m_2+1, \cdots, m_k)} - \cdots - u_{(m_1, m_2, \cdots, m_k+1)}
- u_{(m_1-1, m_2, \cdots, m_k)} - u_{(m_1, m_2-1, \cdots, m_k)} - \cdots - u_{(m_1, m_2, \cdots, m_k-1)},
$$

then similar results of Theorem 3.1 and Theorem 4.1 are still valid for equations (5.2)-(5.3). At this case, equations (5.2)-(5.3) can be regarded as a discrete analogue of the non-autonomous KGS equations (1.3)-(1.4) in $\mathbb{R}^k$.

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