IMPULSIVE HEMIVARIATIONAL INEQUALITY FOR A CLASS OF HISTORY-DEPENDENT QUASISTATIC FRICIONAL CONTACT PROBLEMS

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Abstract. This paper deals with a class of history-dependent frictional contact problem with the surface traction affected by the impulsive differential equation. The weak formulation of the contact problem is a history-dependent hemivariational inequality with the impulsive differential equation. By virtue of the surjectivity of multivalued pseudomonotone operator theorem and the Rothe method, existence and uniqueness results on the abstract impulsive differential hemivariational inequalities is established. In addition, we consider the stability of the solution to impulsive differential hemivariational inequalities in relation to perturbation data. Finally, the existence and uniqueness of weak solution to the contact problem is proved by means of abstract results.

1. Introduction. Contact mechanics plays an important role in people’s daily lives. For example, locomotive wheel-rail contact, braking systems, internal combustion engines, metalworking, metal forming, etc. So more and more people are interested in frictional mechanics. As a result the general mathematical theory of contact mechanics is currently emerging. The theory of hemivariational inequalities plays an important role in the study of the contact problem of various states and materials in [19, 23, 24, 15, 35, 36, 5]. Some classical results about contact problem can be found in several research papers and monographs, see [1, 8, 28, 16, 14]

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and the reference therein. Various classes of quasistatic contact problems for viscoelastic materials have been considered, we can refer to [34, 30, 17, 20, 18] and the references therein. Recently, variational inequalities with history-dependent operators were widely studied and applied to quasistatic contact problems, see [9, 29, 31, 32, 33, 21, 25, 13, 4]. In addition, there are many papers considering the viscoelastic constitutive law with long memory, see [7, 26, 11, 22]. Specially, Migórski et al. [22] studied a class of frictional contact problem for viscoelastic materials with long memory by using pseudomonotone operator and fixed point theory. Han et al. [11] considered a class of contact problem and existence of a unique weak solution was obtained by abstract results on hemivariational inequalities.

Motivated by above works, we are interested to study the quasistatic contact problem for a viscoelastic body. Notice that the corresponding constitutive law is viscoelastic constitutive law with long memory, i.e.,

\[ \sigma(t) = \mathcal{A} \varepsilon(u(t)) + \int_{0}^{t} \mathcal{B}(t-s)\varepsilon(u(s))ds, \]

where \( \mathcal{A} \) and \( \mathcal{B} \) represent the elasticity operator and relaxation tensor, \( \sigma(t) \) and \( u(t) \) stand for the stress field and the displacement field respectively. In addition, we consider the contact problem with the surface traction driven by the impulsive differential equation as follows

\[
\begin{align*}
\mathbf{f}_N(t) &= F(t, \mathbf{f}_N(t), u(t)), \forall t \in [0, T], t \neq t_k, k = 1, 2, \ldots, m, \\
\Delta \mathbf{f}_N(t_k) &= I_k(\mathbf{f}_N(t_k^+)), \\
\mathbf{f}_N(0) &= \mathbf{f}_N^0 \in X,
\end{align*}
\]

where \( I_k \) is impulsive function, impulsive time sequences \( t_k \) satisfy \( 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T \). The function \( F : [0, T] \times Z \times Y \to Z \), \( Z \) and \( Y \) are Banach space. The symbol \( \Delta \mathbf{f}_N(t_k) = \mathbf{f}_N(t_k^+)-\mathbf{f}_N(t_k^-) \) with \( \mathbf{f}_N(t_k^+) = \lim_{\epsilon \to 0^+} \mathbf{f}_N(t_k^+ + \epsilon) \) and \( \mathbf{f}_N(t_k^-) = \lim_{\epsilon \to 0^-} \mathbf{f}_N(t_k^- + \epsilon) \) represent the right and left limits of \( \mathbf{f}_N(t) \) at \( t = t_k \).

As far as we know, no one has studied the contact problem with long memory and impulsive differential equation.

The paper is organized as follows. In Section 2, we consider the model of frictional contact between a viscoelastic body and a rigid foundation. In Section 3, we provide a result on the existence of solutions for impulsive hemivariational inequality. In Section 4, we consider a stability result of the solution with respect to the perturbation of data. Finally, we obtain the weak solvability to the contact problem.

2. The frictional contact problem. Before introducing the physical model, we give some notations and preliminary results.

Let \( X, Y \) be reflexive Banach space, \( X^* \) and \( Y^* \) is the dual space of \( X \) and \( Y \), respectively. Let \( \mathcal{X} = L^2(I; X), \mathcal{Y} = L^2(I; Y) \) with \( I = [0, T] \). And \( \mathcal{L}(X, Y) \) denotes the space of all linear and bounded operators from a Banach space \( X \) to another Banach space \( Y \) with the usual norm \( \| \cdot \| \). The \( C(I; X) \) denotes the space of continuous functions on \( I \). The \( PC(I; X) \) denotes the space of all functions \( x(t) : I \to X \) such that \( x(t) : I \setminus U_{k=1,\ldots,m}\{t_k\} \to X \) is continuous and \( x(t_k^+) \) and \( x(t_k^-) \) exist with \( x(t_k^-) = x(t_k^+) \). Moreover, we give the norm of the above space respectively,
Next, let’s briefly review some physical knowledge and introduce our contact model.

Figure 1. A deformable body in contact with a foundation.

We assume a viscoelastic body which occupied an open and bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with the Lipschitz continuous boundary $\Sigma = \partial \Omega$ and it is decomposed into three disjoint measurable parts $\Sigma_D, \Sigma_N$ and $\Sigma_C$ with $m(\Sigma_D) > 0$. $\mathbb{S}^d$ is the space second order symmetric tensors on $\mathbb{R}^d$. The canonical inter products and corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$u \cdot v = u_i v_i, \quad \|u\|_{\mathbb{R}^d} = (u \cdot u)^{\frac{1}{2}} \quad \text{for all } u = (u_i), \quad v = (v_i) \in \mathbb{R}^n,$$

$$\sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\sigma\|_{\mathbb{S}^d} = (\sigma : \sigma)^{\frac{1}{2}} \quad \text{for all } \sigma \in (\sigma_{ij}), \quad \tau = (\tau_{ij}) \in \mathbb{S}^d.$$ 

The outward unit normal exists a.e. on $\Sigma$ and is denoted by $\nu$. For any $v \in \mathbb{R}^d$, we denote by $v_\nu = v \cdot \nu$ and $v_{\tau} = v - v_\nu \nu$ the normal and tangential traces of $v$ on $\Sigma$, respectively. Let $\sigma = \sigma(x, t)$ and $u = u(x, t)$ be the stress field and the displacement field. We denote by $\varepsilon(u) = (\varepsilon_{ij}(u))$ the small strain tensor on the displacement field $u$, with

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad (\varepsilon_{ij}(u)) = \frac{1}{2} (u_{i,j} + u_{j,i}),$$

where the indices $i, j$ run from 1 to $d$, and the summation convention over repeated indices is used.

Problem 1. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$ and a surface traction density $f_N : \Sigma_N \times [0, T] \to \mathbb{R}^d$ such that

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u(t))) + \int_0^t \mathcal{B}(t - s) \varepsilon(u(s))ds \quad \text{in } \Omega \times [0, T], \quad (1)$$

$$\text{Div } \sigma(t) + f_D(t) = 0 \quad \text{in } \Omega \times [0, T], \quad (2)$$

$$u(t) = 0 \quad \text{on } \Sigma_D \times [0, T], \quad (3)$$

$$\sigma(t) \nu = f_N(t) \quad \text{on } \Sigma_N \times [0, T], \quad (4)$$

$$f'_N(t) = f(t, f_N(t), u(t)) \quad \text{on } \Sigma_N \times [0, T], \quad (5)$$

$$\forall t \in I, t \neq t_k, k = 1, 2, \cdots, m$$

$$\Delta f_N(t_k) = I_k(f_N(t_k)), k = 1, 2, \cdots, m \quad \text{on } \Sigma_N \times [0, T], \quad (6)$$

$$f_N(0) = f_N^0 \quad \text{on } \Sigma_N \times [0, T], \quad (7)$$

$$-\sigma_\nu(t) \in \partial \mu_\nu(u(t)) \quad \text{on } \Sigma_C \times [0, T], \quad (8)$$

$$-\sigma_\tau(t) \in \partial \mu_\tau(u(t)) \quad \text{on } \Sigma_C \times [0, T]. \quad (9)$$
First, let’s give a brief explanation of Problem 1. The constitutive law with long memory \( \int_0^t B(t - s)\varepsilon(u(s))ds \) for viscoelastic body is given by equation (1). Here, \( A \) and \( B \) denote the elasticity and relaxation operators, respectively. Since contact problem is assumed to be quasistatic, the inertia of mechanical system is negligible. Thus, the stress field satisfies the equation (2), where \( f_D \) represents density of volume forces. Relation (3) is the displacement boundary condition, i.e., the body is clamped on \( \Sigma_D \). Equations (4), (5), (6), (7) represent the surface traction of density \( \Sigma_N \) is driven by an impulsive differential equation. We need to explain (5) and (6). At present, many people have studied the contact problem, which the contact surface is not affected by the impact, see [23, 24, 36, 17] and the reference therein. Notice that few people studied the effect of impulse about the surface force \( f_N \). However, in real life, the surface \( \Sigma_N \) of body \( \Omega \) is vulnerable to impact. In addition, we consider the influence of roughness and deformation of objects on surface \( \Sigma_N \). This surface force can be described by an impulse differential equation (5) and (6). Relation (7) is traction boundary condition. The multivalued relations (8) and (9) are the contact and friction conditions, in which \( \partial_j \nu \) and \( \partial_j \tau \) denote the Clarke generalized gradients of functions \( \partial_j \nu \) and \( \partial_j \tau \), respectively. For a detailed discussion on the contact and friction laws of the form (8) and (9), we refer to [20]. We can describe Problem 1 by Figure 1.

**Remark 1.** It is convenient for readers to better understand the \( f_N(t) \) affected by the impulse. We discuss a special case of (5), (6) combined with (7) as follows

\[
\begin{align*}
&f'_N(t) = f_N(t) + u(t), \forall t \in I, t \neq t_k, k = 1, 2, \cdots, m, \\
&\Delta f_N(t_k) = C_k > 0, k = 1, 2, \cdots, m, \\
&f_N(0) = f'_N.
\end{align*}
\]

Then, the solution of (10) has the following form

\[
f_N(t) = \int_0^t u(s)e^{t-s}ds + C_k e^{t-t_k}e^{\sum_{i=1}^{k-1}t_i} + e^tf'_N, t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m.
\]

We give the sketch figure of \( f_N \), see Figure 2.

**Figure 2.** The surface traction \( f_N \) with impact influence.

In order to study the Problem 1, we consider the following function spaces that will be used throughout the paper.

\[
\begin{align*}
H_1 &= \{v = (v_1, \cdots, v_d)^T : v_i \in H^1(\Omega), 1 \leq i \leq d\} = H^1(\Omega; \mathbb{R}^d), \\
X &= L^2(\Sigma_C; \mathbb{R}^d), \\
Y &= \{v \in H_1 : v = 0 \text{ on } \Sigma_D\}, \\
Z &= L^2(\Sigma_N; \mathbb{R}^d), \\
Q &= \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega), 1 \leq i, j \leq d\} = L^2(\Omega; \mathbb{S}^d),
\end{align*}
\]
For all \( \eta \) and \( \nu \) for all \( \xi \) and \( \omega \), \( \lambda \)

To study Problem 1, we need to give some conclusions and conditions about the correlation parameters.

**Definition 2.1.** [2] Assume \( J : E \to \mathbb{R} \) is a locally Lipschitz function. Then, the generalized (Clarke) directional derivative of \( J \) at the point \( u \in E \) in the direction \( v \in E \) defined by

\[
J^0(u; v) = \lim_{\lambda \to 0^+, \omega \to u} \frac{J(\omega + \lambda v) - J(\omega)}{\lambda},
\]

where \( E \) is a Banach space.

The generalized gradient of \( J \) is defined by the set

\[
\partial J(u) = \{ \xi \in E^* \mid J^0(u; v) \geq \langle \xi, v \rangle \text{ for all } v \in E \},
\]

where \( E^* \) is the dual space of \( E \).

For \( u \in X \) denote the function \( J : X \to \mathbb{R} \) by

\[
J(u) = \int_{\Sigma_C} (j_{\nu}(x, u_{\nu}) + j_{\nu}(x, u_{\nu}))d\Gamma.
\]

Furthermore, from [20], we get the following equality

\[
J^0(u) = \int_{\Sigma_C} (j_{\nu}^0(x, u_{\nu}) + j_{\nu}^0(x, u_{\nu}))d\Gamma.
\]

\((H_a)\): The elasticity tensor \( a : \Omega \times I \times \mathbb{S}^d \to \mathbb{S}^d \), \( a = (a_{ijkl}) \in Q_\infty \), satisfies the following conditions

(i) \( a(\cdot, \cdot, \xi) \) is measurable on \( \Omega \times I \) for all \( \xi \in \mathbb{S}^d \);

(ii) \( a(x, t, \cdot) \) is continuous on \( \mathbb{S}^d \) for a.e. \( (x, t) \in \Omega \times I \);

(iii) There exists \( L_a \) such that \( \|a \tau \cdot \tau\| \leq L_a \|\tau\|_{\mathbb{S}^d}^2 \) for all \( \tau \in \mathbb{S}^d \), for a.e. \( (x, t) \in \Omega \times I \).

\((H_b)\): The relaxation tensor \( b : \Omega \times I \times \mathbb{S}^d \to \mathbb{S}^d \), \( b = (b_{ijkl}) \in Q_\infty \), and \( b \) is Lipschitz continuous with Lipschitz constant \( L_b > 0 \).

\((H_{j\nu})\): \( j_{\nu} : \Sigma_C \times \mathbb{R} \to \mathbb{R} \) with the following conditions

(i) \( j_{\nu}(\cdot, \mu) \) is measurable on \( \Sigma_C \) for all \( \mu \in \mathbb{R} \);

(ii) \( j_{\nu}(x, \cdot) \) is locally Lipschitz a.e. \( x \in \Sigma_C \);

(iii) \( j_{\nu}(x, \cdot) \) or \( -j_{\nu}(x, \cdot) \) is regular a.e. \( x \in \Sigma_C \);

(iv) There exists \( c_{\nu} > 0 \) such that \( |\partial j_{\nu}(x, \mu)| \leq c_{\nu}(1 + |\mu|) \) for all \( \mu \in \mathbb{R} \) a.e. \( x \in \Sigma_C \);

(v) There exists \( m_{\nu} > 0 \) such that

\[
(\eta_1 - \eta_2)(\mu_1 - \mu_2) \geq -m_{\nu}|\mu_1 - \mu_2|^2
\]

for all \( \eta_i \in \partial j_{\nu}(x, \mu_i), \mu_i \in \mathbb{R}, i = 1, 2 \) a.e. \( x \in \Sigma_C \).

\((H_{j_{\nu}})\): \( j_{\nu} : \Sigma_C \times \mathbb{R}^d \to \mathbb{R} \) such that

(i) \( j_{\nu}(\cdot, \mu) \) is measurable on \( \Sigma_C \) for all \( \mu \in \mathbb{R}^d \);

(ii) \( j_{\nu}(x, \cdot) \) is locally Lipschitz a.e. \( x \in \Sigma_C \);

(iii) \( j_{\nu}(x, \cdot) \) or \( -j_{\nu}(x, \cdot) \) is regular a.e. \( x \in \Sigma_C \);

(iv) There exists \( c_\nu > 0 \) such that \( |\partial j_{\nu}(x, \mu)| \leq c_\nu(1 + ||\mu||_{\mathbb{R}^d}) \) for all \( \mu \in \mathbb{R}^d \) a.e. \( x \in \Sigma_C \);

(v) There exists \( m_\nu > 0 \) such that

\[
(\eta_1 - \eta_2)(\mu_1 - \mu_2) \geq -m_\nu||\mu_1 - \mu_2||_{\mathbb{R}^d}^2
\]

for all \( \eta_i \in \partial j_{\nu}(x, \mu_i), \mu_i \in \mathbb{R}^d, i = 1, 2 \) a.e. \( x \in \Sigma_C \).
Subsequently, we will discuss the hemivariational formulation of Problem 1. On the space $Y$, we define the inner product by
\[ \langle u, v \rangle_Y = \langle \varepsilon(u), \varepsilon(v) \rangle_Q \text{ for all } u, v \in Y. \]

On the other hand, we know that $Q$ is Hilbert space with the inner product
\[ \langle \sigma, \tau \rangle_Q = \int_K \sigma_{ij}(t)\tau_{ij}(t)dt \text{ for all } \sigma, \tau \in Q. \]

The force and traction densities satisfy $f_D \in C(I, L^2(\Omega; \mathbb{R}^d))$, $f_N \in C(I, L^2(\Sigma_N; \mathbb{R}^d))$. Surface traction of density $f_N$ act on $\Sigma_N$ and volume forces of density $f_D$ act in $\Omega$. We denote by $g$ the element of $Y^*$ given by
\[ (g(f_N), v)_{Y^* \times Y} = \int_{\Omega} f_D \cdot v\ dV + \int_{\Sigma_N} f_N \cdot v\ d\Gamma \forall v \in Y. \tag{12} \]

In order to obtain the hemivariational formulation of Problem 1. We assume the pair of functions $(u, f_N)$ is sufficiently smooth. For any $u \in Y$ and $t \in [0, T]$, by virtue of the Green formula
\[ \int_{\Omega} \sigma \cdot \varepsilon(u)\ dV + \int_{\Omega} \text{Div} \sigma \cdot u\ dV = \int_{\partial\Omega} \sigma \nu \cdot u\ d\Gamma, \]
we know that
\[ \langle \sigma(t), \varepsilon(v) \rangle_Q = \langle f_D(t), v \rangle_Y + \int_{\Sigma} \sigma(t)\nu \cdot v\ d\Gamma \text{ for } v \in Y. \tag{13} \]

Further
\[ \langle \sigma(t), \varepsilon(v) \rangle_Q = \langle f_D(t), v \rangle_Y + \int_{\Sigma_N} \sigma(t)\nu \cdot v\ d\Gamma + \int_{\Sigma_C} \sigma(t)\nu \cdot v\ d\Gamma. \tag{14} \]

From the boundary condition (3), (12), (13) and equality (14), we obtain
\[ \langle \sigma(t), \varepsilon(v) \rangle_Q = \langle f_D(t), v \rangle_Y + \int_{\Sigma_N} \sigma(t)\nu \cdot v\ d\Gamma + \int_{\Sigma_C} \sigma(t)\nu \cdot v\ d\Gamma, \tag{15} \]
and
\[ \langle \sigma(t), \varepsilon(v) \rangle_Q - \int_{\Sigma_C} \sigma(t)\nu \cdot v\ d\Gamma = \langle g(f_N), v \rangle. \tag{16} \]

Since
\[ \int_{\Sigma_C} \sigma(t)\nu \cdot v\ d\Gamma = \int_{\Sigma_C} \gamma(t)\nu + \sigma_\tau(t)\cdot \nu\ d\Gamma. \tag{17} \]

The friction boundary conditions (8) and (9) are equivalent to the following formulas
\[ -\sigma_\nu(t)\mu \leq j^0_0(u_\nu; \mu) \text{ for all } \mu \in \mathbb{R}, \quad -\sigma_\tau(t)\cdot \mu \leq j^0_0(u_\tau; \mu) \text{ for all } \mu \in \mathbb{R}^d. \tag{18} \]

According to (1), (15), (16), (17) and (18), we have the following inequality
\[ \langle A\varepsilon(u), \varepsilon(v) \rangle_Q + \left\{ \int_{0}^{T} B(t-s)\varepsilon(u(s))ds, \varepsilon(v) \right\}_Q + \int_{\Sigma_N} (j^0_0(u_\nu(t); v_\nu) + j^0_0(u_\tau(t); v_\tau))d\Gamma \geq \langle g(f_N(t)), v \rangle \tag{19} \]

From (19), Problem 1 can be described as the following hemivariational formulation.
Problem 2. Find a displacement field $u : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ and a surface traction density $f_N : \Sigma_N \times [0, T] \rightarrow \mathbb{R}^d$ such that

\[
\begin{aligned}
\left\langle \mathcal{A}(\varepsilon(u), \varepsilon(v)) \right\rangle_Q + \left\langle \int_0^1 \mathcal{B}(t-s)\varepsilon(u(s))ds, \varepsilon(v) \right\rangle_Q \\
+ \int_{\Sigma_N} \{g(u_r(t); v_r) + J_0(u_r(t); v_r)\}d\Gamma \geq (g(f_N(t)), v) \forall (t, v) \in I \times Y,
\end{aligned}
\]

\[
\begin{aligned}
f_N'(t) = F(t, f_N(t), u(t)), \forall t \in I, t \neq t_k, k = 1, 2, \cdots, m,
\end{aligned}
\]

\[
\begin{aligned}
\Delta f_N(t_k) = I_k(f_N(t_k^+)), k = 1, 2, \cdots, m,
\end{aligned}
\]

\[
\begin{aligned}
f_N(0) = f_N^0, \quad x(0) = x_0 \in Z.
\end{aligned}
\]

The existence and uniqueness of Problem 2 will be provided in Section 5. In the next section, we will give the abstract results of Problem 2.

3. Existence and uniqueness for abstract problem. To study the Problem 2, we firstly consider the following impulsive hemivariational inequality.

Problem 3. Find $x : I \rightarrow Z$, and $\eta : I \rightarrow Y$ satisfy the following equation

\[
\begin{aligned}
A(q(t)) + (S\eta(t)) + M^*J(M(\eta(t))) &\equiv f(t, x(t)), \forall t \in I,
\end{aligned}
\]

\[
\begin{aligned}
x'(t) = F(t, x(t), \eta(t)), \forall t \in I, t \neq t_k, k = 1, 2, \cdots, m,
\end{aligned}
\]

\[
\begin{aligned}
\Delta x(t_k) = I_k(x(t_k^+)), k = 1, 2, \cdots, m,
\end{aligned}
\]

\[
\begin{aligned}
x(0) = x_0 \in Z.
\end{aligned}
\]

To study of Problem 3, we need the following assumption about the data of Problem 3.

(H0): $L_A = \max\{m_B\|B\| + L_B\|M\|, Tm_q\|B\| + m_f\|M\|^2\}$.

(HA): The operator $A \in \mathcal{L}(Y, Y^*)$ and $\langle Av, v \rangle_{Y, Y^*} \geq L_A\|v\|_Y^2$ for all $v \in Y$ with $L_A > 0$.

(HB): The operator $B$ is linear, bounded, i.e. $B \in \mathcal{L}(Y, Y^*)$.

(HF): $F : I \times Z \rightarrow Z^*$ is such that

(i) The mapping $t \mapsto F(t, x, \eta)$ is continuous for all $x \in Z, \eta \in Y$.

(ii) For any compact set $P \subset I$ there exists $L_F > 0$, such that

\[
\|F(t, x_1, \eta_1) - F(t, x_2, \eta_2)\|_Z \leq L_F(\|x_1 - x_2\|_Z + (\|\eta_1 - \eta_2\|_Y)\)
\]

for all $x_1, x_2 \in Z, \eta_1, \eta_2 \in Y, t \in I'$.

(iii) There exists $\theta \in L^p[0, T]([p \geq 2]$ satisfying

\[
\|F(t, x, \eta)\|_{Z^*} \leq \theta(t), \forall (t, x, \eta) \in I \times Z \times Y.
\]

(HJ): $f : I \times Z \rightarrow Y^*$ is a function such that

(i) For any $x \in Z, f(\cdot, x)$ is continuous;

(ii) There exists $L_f > 0$ such that

\[
\|f(t, x_1) - f(t, x_2)\|_{Y^*} \leq L_f\|x_1 - x_2\|_Z;
\]

(HI): $I_k : Z \rightarrow Z$ is bounded and exists $d_k > 0$ such that

\[
|I_k(x_1) - I_k(x_2)| \leq d_k\|x_1 - x_2\|_Z, \forall x_1, x_2 \in Z.
\]

(HJ): The function $J : X \rightarrow \mathbb{R}$ satisfies the following conditions

(i) $J$ is locally Lipschitz;

(ii) $\|\partial J(u)\|_{X^*} \leq L_J(1 + \|u\|_X)$ for all $u \in X$ with $L_J > 0$;

(iii) There exists $m_J \geq 0$ such that

\[
\langle \xi - \zeta, u - v \rangle_{X^* \times X} \geq -m_J\|u - v\|_X^2,
\]

for all $u, v \in X$ and $\xi \in \partial J(u), \zeta \in \partial J(v)$.
(H_M): $M : Y \to X$ is linear, continuous, and compact operator.
(H_q): $q \in C(I \times I, \mathcal{L}(Y, Y^*))$, there exists $L_q > 0$ such that
$$\|q(t_1, s) - q(t_2, s)\| \leq L_q |t_1 - t_2| \text{ for all } s, t_1, t_2 \in I,$$
and $m_q = \max_{(t, s) \in I \times I} \|q(t, s)\|$.

(H_S): The operator $S \in \mathcal{L}(Y, Y^*)$ is defined by
$$(S \eta)(t) = B \left( \int_0^t q(t, s) \eta(s) ds + \beta \right),$$
and there exists constant $L_S > 0$ such that
$$\|(S \eta_1)(t) - (S \eta_2)(t)\| \leq L_S \int_0^t \|\eta_1(s) - \eta_2(s)\|_Y ds$$
for all $\eta_1(t), \eta_2(t) \in C(I; Y)$, where $\beta \in Y$, $L_S = m_q \|B\|$.

(H_Jδ): For any $\delta > 0$, the function $J_\delta : X \to R$ satisfies the following conditions
(i) $J_\delta$ is locally Lipschitz;
(ii) $\|\partial J_\delta(u)\|_X \leq L_{J_\delta}(1 + \|u\|_X)$ for all $u \in X$ with $L_{J_\delta} > 0$;
(iii) There exists $m_{J_\delta} \geq 0$ such that
$$J_\delta^0(u; v - u) + J_\delta^0(v; u - v) \leq m_{J_\delta}\|u - v\|_X^2,$$
for all $u, v \in X$.

(H_Gδ): There exists a function $G : R^+ \to R^+$ satisfying $\|\phi - \phi_\delta\| \leq G(\delta)$, for any $y \in Y$ and $\delta > 0$, for all $(\phi, \phi_\delta) \in \partial J(My(t)) \times \partial J_\delta(My(t))$, and $\lim_{\delta \to 0} G(\delta) = 0$.

Next we will give an example of function which satisfies condition (H_Jδ).

Example 1. Define a function $J_\delta : \mathbb{R} \to \mathbb{R}$ by
$$J_\delta(s) = \int_0^s p(r) dr + s\delta, \quad s \in \mathbb{R},$$
where
$$p(r) = \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } 0 \leq r < \frac{3}{2}, \\ 3 - r & \text{if } \frac{3}{2} \leq r < 3, \\ \sqrt{r - 3} + r - 3 & \text{if } 3 \leq r < 12, \\ r & \text{if } r \geq 12. \end{cases}$$

It’s easy to know that $p$ is a continuous function, but it is neither monotone nor Lipschitz continuous. Furthermore, we know that $J_\delta$ is not a convex function and $J_\delta'(s) = p(s) + \delta$ for all $s \in \mathbb{R}$. Thus, the function $J_\delta$ is locally Lipschitz. From $|p(s)| \leq |s|$ for all $s \in \mathbb{R}$, we conclude that the function $J_\delta$ satisfies (H_Jδ)(i). It is easy to see that $J_\delta''(s_1; s_2) = p(s_1)s_2 + \delta s_2$ for all $s_1, s_2 \in \mathbb{R}$. So we have
$$J_\delta''(s_1; s_2 - s_1) + J_\delta''(s_2; s_1 - s_2) = (p(s_1) - p(s_2))(s_2 - s_1) \leq m_{J_\delta}(s_1 - s_2)^2$$
for all $s_1, s_2 \in \mathbb{R}$ with $m_{J_\delta} \geq 0$. Therefore, we deduce that $J_\delta$ satisfies the conditions (H_Jδ). Further, set $\phi = p(s) \in \partial J(My(t))$ and $\phi_\delta = (p(s) + \delta) \in \partial J_\delta(My(t))$, then $\|\phi - \phi_\delta\|_Y \leq \delta$ and $\lim_{\delta \to 0} \delta = 0$.

In order to discuss the Problem 3, we review some definitions and Lemmas.
Definition 3.1. [6] Let $V$ be a reflexive Banach space. The operator $A : V \to V^*$ is pseudomonotone if for every $\{y_n\} \in V$ converging weakly to $y \in V$ such that \( \limsup_{n \to \infty} \langle Ay_n, y_n - y \rangle \leq 0 \), then, we have
\[
\langle Ay, y - x \rangle \leq \liminf_{n \to \infty} \langle Ay_n, y_n - x \rangle \quad \text{for all } x \in V.
\]

Lemma 3.2. [6] Assume multivalued operator $T : V \to 2^{V^*}$ satisfies the following conditions
(i) The operator $T$ is bounded;
(ii) For every $v \in V$, the set $Tv \in V^*$ is a nonempty, closed, and convex;
(iii) If $v_n \rightharpoonup v$ weakly in $V$ and $v_n^* \rightharpoonup v^*$ weakly in $V^*$ with $v_n^* \in Tv_n$ and
\[
\limsup_{n \to \infty} (v_n^*, v_n - v) \leq 0,
\]
then $v^* \in Tv$ and $(v_n^*, v_n) \to (v^*, v)$, then the operator $T$ is pseudomonotone.

Lemma 3.3. [20] Let $E$ be a Banach space and $A : E \to E^*$
(i) If the operator $A$ is a bounded, hemicontinuous, and monotone, then $A$ is pseudomonotone;
(ii) If $A$ is pseudomonotone, then $A$ is demicontinuous.

Lemma 3.4. [20] Let $V$ be a reflexive Banach space.
(i) If $A : V \to 2^{V^*}$ is a maximal monotone operator with $D(A) = A$ then $A$ is pseudomonotone;
(ii) If $A_1, A_2 : V \to 2^{V^*}$ are pseudomonotone operators, then $A_1 + A_2$ is pseudomonotone.

Lemma 3.5. [6] Let $V$ be a reflexive Banach space and $T : V \to 2^{V^*}$ be pseudomonotone and coercive. Then, $T$ is surjective, i.e., for every $\pi \in V^*$, there is $u \in V$ such that $Tu \ni \pi$.

Lemma 3.6. [28] Let $V$ and $V_1$ be reflexive Banach spaces. The operator $M : V_1 \to V$ is a linear, bounded, and compact. Denoting by $M^* : V^* \to V_1^*$ the adjoint operator of $M$. Let $J : V \to \mathbb{R}$ be a locally Lipschitz function such that
\[
\|\partial J(u)\|_{V^*} \leq m(1 + \|u\|_V)
\]
for all $u \in V$ with $m > 0$. Then, the multivalued operator $\Theta : V_1 \to 2^{V_1^*}$ defined by $\Theta(u) = M^* \partial J(Mu)$ for $u \in V_1$ is pseudomonotone.

Lemma 3.7. [27] (Generalized discrete Gronwall inequality) Let $\{u_n\}, \{v_n\}$, and $\{w_n\}$ be nonnegative sequence satisfying
\[
u_n \leq v_n + \sum_{k=1}^{n-1} w_n u_n \quad \text{for } n \geq 1.
\]
Then, we have
\[
u_n \leq v_n + \sum_{k=1}^{n-1} w_k v_k \exp\left(\sum_{j=k+1}^{n-1} w_j\right) \quad \text{for } n \geq 1.
\]
Moreover, if $\{u_n\}$ and $\{w_n\}$ are such that
\[
u_n \leq \alpha + \sum_{k=1}^{n-1} w_k u_k \quad \text{for } n \geq 1,
\]
where $\alpha > 0$ is a constant. Then, for all $n \geq 1$, it holds
\[ u_n \leq \alpha \exp\left(\sum_{k=1}^{n-1} u_k\right). \]

**Lemma 3.8.** [14] (Gronwall inequality) Assume that $h, z \in C[a, b]$, satisfy
\[ h(t) \leq z(t) + c \int_a^t h(s)ds, \quad t \in [a, b], \]
where $c > 0$ is a constant. Then
\[ h(t) \leq z(t) + c \int_a^t z(s)e^{c(t-s)}ds, \quad t \in [a, b]. \]
Moreover, if $z$ is nondecreasing, then
\[ h(t) \leq z(s)e^{c(t-s)}, \quad t \in [a, b]. \]

**Lemma 3.9.** [10] For any $\eta \in Y$, a function $x \in Z$ is a solution of Cauchy problem:
\[
\begin{align*}
\{ x'(t) &= F(t, x(t), \eta(t)), \forall t \in I, t \neq t_k, k = 1, 2, \ldots, m, \\
\Delta x(t_k) &= I_k(x(t_k^-)), k = 1, 2, \ldots, m, \\
x(0) &= x_0 \in Z.
\end{align*}
\]  
if and only if $x$ is a solution of the following integral equation
\[ x(t) = x_0 + \sum_{i=1}^k I_i(x(t_i^-)) + \int_0^t F(s, x(s), \eta(s))ds, \text{ for } t \in (t_{k-1}, t_k]. \]  

**Problem 4.** For any given $x \in Z$, find $\eta \in Y$ such that
\[ \langle A(\eta(t)) + (S\eta)(t), v \rangle + J^0(M(\eta(t)) ; Mv) \geq \langle f(t, x(t)), v \rangle \text{ for all } v \in Y, \text{ a.e. } t \in [0, T]. \]

We know that Problem 4 is equivalent to another operator inclusion formulation of the form
\[ A(\eta(t)) + (S\eta)(t) + M^*\partial J(M(\eta(t))) \ni f(t, x(t)), \text{ a.e. } t \in [0, T]. \]  

**Definition 3.10.** Assume that $\eta \in Y$ is called a solution to Problem 4, if and only if there exists $\xi$ such that
\[ A(\eta(t)) + (S\eta)(t) + M^*\xi(t) = f(t, x(t)) \]  
with $\xi(t) \in \partial J(M(\eta(t)))$ a.e. $t \in [0, T]$. 

In order to discuss the Problem 4, we will consider the following discretized Problem 5, which called the Rothe Problem. Let $N \in N^+$ be fixed and $\varsigma = \frac{N}{T}$. Moreover, we denote $f_k^\varsigma = \frac{1}{\varsigma} \int_{t_{k-1}}^{t_k} f(s, x(s))ds$ for $k = 1, 2, \ldots, N$, with $t_k = k\varsigma$.

**Problem 5.** Find $\{\eta_k^\varsigma\}_{k=0}^N \subset Y$ such that
\[ A\eta_k^\varsigma + u_k^\varsigma + M^*\partial J(M\eta_k^\varsigma) \ni f_k^\varsigma, \]
where $u_k^\varsigma \in Y^*$ is defined by
\[ u_k^\varsigma = B \left( \beta + \sum_{i=1}^{k-1} \int_{t_i}^{t_i} q(t_k, s)\eta_i^\varsigma ds + \int_{t_{k-1}}^{t_k} q(t_k, s)\eta_k^\varsigma ds \right). \]

Next, we prove the existence and uniqueness of the solution to Problem 5.
Lemma 3.11. Assume that $(H_A), (H_B), (H_J), (H_M)$ and $(H_q)$ hold. Then, there exists $\eta_0 > 0$ such that for all $\varsigma \in (0, \eta_0)$, Problem 5 has a unique solution.

Proof. Suppose $\eta^0, \eta^1, ..., \eta^{k-1}$ are given. We find a unique element $\eta^k \in Y$, which satisfies the equation (25). To this end, we prove the multivalued operator $T : Y \rightarrow 2^Y$ denoted by

$$Tv = Av + B \left( \beta + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} q(t_k, s)\eta^i_k ds \right) + M^* \partial J(Mv)$$

for all $v \in Y$ is surjective. By virtue of Lemma 3.5, we need to prove that $T$ is pseudomonotone and coercive.

First, we show that $T$ is pseudomonotone. From conditions $(H_A), (H_B)$ and $(H_q)$, we know that the operator

$$v \mapsto Av + B \left( \beta + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} q(t_k, s)\eta^i_k ds \right)$$

is continuous, bounded and monotone. Using Lemma 3.3, we know that the operator defined by (26) is pseudomonotone. On the other hand, from the hypotheses $(H_J), (H_M)$ and Lemma 3.6, we obtain the operator $v \mapsto M^* \partial J(Mv)$ is pseudomonotone as well. So, by virtue of Lemma 3.4, we prove $T$ is pseudomonotone.

Subsequently, we check the operator $T$ is coercive. We define a constant $K_0 = 0$

$$K_0 = \|\beta\| + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \|q(t_k, s)\|\|\eta^i_k\|ds.$$

For $v \in Y$, by $(H_A), (H_B), (H_J)$ and $(H_q)$, for all $\xi \in X^*$, we have

$$\left\langle Av + B \left( \beta + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} q(t_k, s)\eta^i_k ds \right), v \right\rangle + \langle \xi, Mv \rangle$$

$$\geq L_A \|v\|^2 - \|B\||\beta||\|v|| - \|B\||\|v\| \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \|q(t_k, s)\|\|\eta^i_k\|ds$$

$$- \|B\| \int_{t_{k-1}}^{t_k} \|q(t_k, s)\|\|v\|^2 ds - L_J \left( \|M\||\|v|| + \|M\|^2 \|v\|^2 \right)$$

$$= \left( L_A - \varsigma m_\varsigma \|B\| - L_J \|M\|^2 \right) \|v\|^2 - K_0 \|B\||\|v\| - L_J \|M\||\|v\|.$$

Let $\varsigma_0 = \frac{L_A - L_J \|M\|^2}{m_\varsigma \|B\|} > 0$, and $\varsigma \in (0, \varsigma_0)$. We conclude that $T$ is a coercive operator. Thus, according to Lemma 3.5, we conclude that $T$ is surjective. Then, Problem 5 has at least one solution.

Next we discuss the uniqueness solution of the Problem 5. Assuming that $\eta^k_\varsigma$ and $\tilde{\eta}^k_\varsigma$ are solutions of (25), we have

$$\langle A\eta^k_\varsigma + \pi^k_\varsigma + M^* \partial J(M\eta^k_\varsigma), v \rangle \geq \langle f^k_\varsigma, v \rangle$$

for all $v \in Y$ (27)

and

$$\langle A\tilde{\eta}^k_\varsigma + \tilde{\eta}^k_\varsigma + M^* \partial J(M\tilde{\eta}^k_\varsigma), v \rangle \geq \langle f^k_\varsigma, v \rangle$$

for all $v \in Y$. (28)
where \( \overline{\eta}^k \) and \( \tilde{\eta}^k \) are defined by

\[
\overline{\eta}^k = B \left( \beta + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} q(t_k, s) \eta_i^k ds + \int_{t_{k-1}}^{t_k} q(t_k, s) \tilde{\eta}_k^k ds \right)
\]

and

\[
\tilde{\eta}^k = B \left( \beta + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} q(t_k, s) \eta_i^k ds + \int_{t_{k-1}}^{t_k} q(t_k, s) \tilde{\eta}_k^k ds \right).
\]

By taking \( v = \overline{u}^k - u^k \) and \( v = \overline{\eta}^k - \tilde{\eta}^k \) in (27) and (28), we deduce that

\[
\langle A\overline{\eta}^k + B\overline{\eta}^k + M^* \partial J(M\overline{\eta}^k), \overline{u}^k - \overline{\eta}^k \rangle \geq \langle f^k, \overline{\eta}^k - \overline{\eta}^k \rangle \quad \text{for all} \quad v \in V
\]

and

\[
\langle A\overline{\eta}^k + B\overline{\eta}^k + M^* \partial J(M\overline{\eta}^k), \overline{\eta}^k - \overline{\eta}^k \rangle \geq \langle f^k, \overline{\eta}^k - \overline{\eta}^k \rangle \quad \text{for all} \quad v \in V.
\]

According to (29) and (30), we have

\[
\langle A\overline{\eta}^k - A\overline{\eta}^k, \eta^k - \overline{\eta}^k \rangle + \langle \overline{\eta}^k - \overline{\eta}^k, \eta^k - \overline{\eta}^k \rangle + \langle \partial J(M\overline{\eta}^k) - \partial J(M\overline{\eta}^k), M\overline{\eta}^k - M\overline{\eta}^k \rangle \leq 0.
\]

Hence,

\[
(L_A - \varsigma m_q \|B\| - L_J \|M\|^2)\|\eta^k - \overline{\eta}^k\| \leq 0.
\]

Using condition (H_0), we get \( \eta^k = \overline{\eta}^k \). Thus, the proof is completed.

In the sequel, we estimate the boundedness of the solution to Problem 5.

**Lemma 3.12.** Assume that (H_A),(H_B),(H_J),(H_M) and (H_0) hold. Then, there exist \( \varsigma_0 > 0 \) and \( C_1, C_2, C_3 > 0 \), such that for all \( \varsigma \in (0, \varsigma_0) \), the solutions of Problem 5 satisfy

\[
\max_{k=1,2,\ldots,N} \|\eta^k_x\| \leq C_1; \quad \max_{k=1,2,\ldots,N} \|u^k_x\| \leq C_2; \quad \max_{k=1,2,\ldots,N} \|\xi^k_x\| \leq C_3;
\]

where \( \xi^k_x \in \partial J(M\eta^k_x) \) with

\[
A\eta^k_x + u^k_x + M^* \xi^k_x = f^k_x \quad (k = 1, 2, \ldots, N).
\]

**Proof.** Firstly, we have

\[
A\eta^k_x + u^k_x + M^* \xi^k_x = f^k_x,
\]

where \( \xi^k_x \in \partial J(M\eta^k_x) \). According to (H_A) and (H_B), we have

\[
\langle f^k_x, \eta^k_x \rangle = \langle A\eta^k_x + u^k_x + M^* \xi^k_x, \eta^k_x \rangle + \langle u^k_x + \eta^k_x + \xi^k_x, M\eta^k_x \rangle \geq L_A \|\eta^k_x\|^2 - \|\eta^k_x\|\|B\| \left( \|\beta\| + \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \|q(t_i, s)\| \|\eta^k_x\| ds \right)
\]

\[
+ \int_{t_{n-1}}^{t_n} \|q(t_n, s)\| \|\eta^k_x\| ds \right) - L_J (1 + \|M\|\|\eta^k_x\|) \|M\|\|\eta^k_x\|.
\]

Furthermore, we have

\[
\|f^k_x\| + \|B\| \left( \|\beta\| + \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} \|q(t_i, s)\| \|\eta^k_x\| ds \right) + L_J \|M\|
\]
\[
\|u^n_\varsigma\| \leq L_0 \exp \left( \frac{2m_q\|B\|}{L_A} \sum_{i=1}^{n-1} \int_{t_{i-1}}^{t_i} ds \right) \leq L_0 \exp \left( \frac{2Tm_q\|B\|}{L_A} \right) = C_1, \quad (34)
\]
which implies (31). Next, by the definition of the \( u^n_\varsigma \), we have
\[
\|u^n_\varsigma\| = \|B\| \left( \|\beta\| + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|q(t,s)\|\|\eta^i_\varsigma\| ds \right) \\
\leq \|B\|\|\beta\| + C_1m_qT\|B\| = C_2.
\]
Finally, according to \((H_f)\), we have
\[
\|\xi^n_\varsigma\| \leq L_J(1 + \|M\|\|\eta^n_\varsigma\|) \leq L_J(1 + C_1\|M\|) = C_3.
\]
Thus, this completes the proof. \( \square \)

Next, we will discuss the convergence.

Define \( \eta^k, \tau^k : I \to Y \), \( f^k : I \to Y^* \) and \( \xi^k : I \to X^* \) by
\[
\eta^k(t) = \eta^k_\varsigma, \quad \tau^k(t) = u^k_\varsigma, \quad f^k(t) = f^k_\varsigma, \quad \xi^k(t) = \xi^k_\varsigma,
\]
where \( \eta^k, \tau^k, f^k \) and \( \xi^k \) are piecewise constant interpolant functions \( t \in [t_{k-1}, t_k] \), \( k = 1, 2, \ldots, N \).

**Theorem 3.13.** Assume that \((H_A),(H_B),(H_f),(H_M),(H_S)\) and \((H_S)\) hold. The Problem 4 has a unique solution. Moreover, for all \( x_1, x_2 \in PC(I; Z) \), we have
\[
\|\eta - \eta_2\|_Y \leq \frac{L_J}{L_A - m_J\|M\|^2 - TL_q\|B\|} \|x_1 - x_2\|_X.
\]

**Proof.** Let \( \{\varsigma_n\} \) be a sequence such that \( \varsigma_n \to 0 \) as \( n \to \infty \). For a subsequence, still denoted by \( \varsigma \). Using the boundedness of \( \{\eta^n_\varsigma\} \) and the definition of \( \eta^k_\varsigma \). As \( \varsigma \to 0 \), there exists \( \eta \in Y \) such that \( \eta^k_\varsigma \to \eta \) weakly in \( Y \). In addition, by virtue of the boundedness of \( \{\xi^n_\varsigma\} \), there exists \( \xi \in Y^* \) such that \( \xi^k_\varsigma \to \xi \) weakly in \( Y^* \), as \( \varsigma \to 0 \). Using the condition \((H_M)\), we deduce \( M\eta^k_\varsigma(t) \to M\eta_t(t) \) strongly in \( X \) for a.e. \( t \in (0, T) \). Furthermore, since \( \xi^k(t) \in \partial J(M\eta^k_\varsigma(t)) \), we deduce
\[
\xi(t) \in \partial J(M\eta_t(t)) \text{ for a.e. } t \in [0, T].
\]
Now we discuss the Nemytskii operator \( M : Y \to X \) that is defined by \( (Mv)(t) = M(v(t)) \) for all \( v \in Y \) and a.e. \( t \in (0, T) \). Therefore,
\[
\lim_{\varsigma \to 0} (\xi^k_\varsigma, Mv) = (\xi, Mv)
\]
for all \( v \in \mathcal{Y} \). Moreover, from [3] implies that
\[
f_\zeta \to f \text{ strongly in } v, \text{ as } \zeta \to 0. \tag{36}
\]
From boundedness of sequence \( \{ \eta_k \} \) and the hypothesis \((H_q)\), there exists some constant \( C_0 \) for \( t \in [t_{k-1}, t_k] \) such that
\[
\left\| \int_0^t q(t, s)\eta_k(t)ds - \int_0^{t_k} q(t, s)\eta(t)ds \right\|
\leq \int_t^{t_k} \|q(t, s)\eta_k(t)ds + \int_0^t \|q(t, s) - q(t, s)\eta(t)\|ds \leq C_0\zeta.
\]
Next, we will prove \( \eta(t) \) is a solution of Problem 3. To this end, we introduce a Nemytskii operator \( A : \mathcal{Y} \to \mathcal{Y}^* \), which defined by \( (Av)(t) = A(v(t)) \) for all \( v \in \mathcal{Y} \) and a.e. \( t \in [0, T] \). Using \((H_A)\) and \((H_B)\), we know that \( A \) is linear and bounded. Thus, \( A \) is weakly continuous. From (31), it’s easy to find \( A\eta_k \rightharpoonup A\eta \) weakly in \( \mathcal{Y}^* \), as \( \zeta \to 0 \), i.e.,
\[
\lim_{\zeta \to 0} \langle A\eta_k, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} = \langle A\eta, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}}. \tag{37}
\]
Further, we consider the Nemytskii operators \( B_1, B_2 : \mathcal{Y} \to \mathcal{Y}^* \) by
\[
(B_1 v)(t) = B \left( \int_0^t q(t, s)v(s)ds \right) \quad \text{and} \quad (B_2 v)(t) = B v(t)
\]
for all \( v \in \mathcal{Y} \) and a.e. \( t \in [0, T] \). With \( \eta_k \rightharpoonup \eta \) weakly in \( \mathcal{Y} \), as \( \zeta \to 0 \), we have
\[
\lim_{\zeta \to 0} \langle B_1 \eta_k, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} = \langle B_1 \eta, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}}
\]
for all \( v \in \mathcal{Y} \). Let
\[
\omega_k(t) = \beta + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} q(t, s)\eta_k^i ds, t \in (t_{k-1}, t_k].
\]
According to \((H_B)\) and \((H_q)\), we have
\[
B_2(\omega_k - \beta) - B_1(\eta_k) \to 0 \text{ strongly in } \mathcal{Y}^*, \text{ as } \zeta \to 0,
\]
which implies
\[
\begin{align*}
\lim_{\zeta \to 0} \langle B_2 \omega_k, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} &= \lim_{\zeta \to 0} \left( \langle B_2(\omega_k - \beta) - B_1(\eta_k), v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} + \langle B_1(\eta_k), v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} + \langle B_2(\beta), v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} \right) \\
&= \langle B_1 \eta, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} + \langle B_2(\beta), v \rangle_{\mathcal{Y}^* \times \mathcal{Y}}. \tag{38}
\end{align*}
\]
From (35), (36), (37) and (38), we deduce
\[
0 \leq \limsup_{\zeta \to 0} \langle A\eta_k, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} + \limsup_{\zeta \to 0} \langle B_2 \omega_k, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}}
+ \limsup_{\zeta \to 0} \langle \xi, M v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} - \liminf_{\zeta \to 0} \langle f_\zeta, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}}
\]
for all \( v \in \mathcal{Y} \).
Therefore, we know that
\[
\langle A\eta + (S\eta)(t) + M^* \xi, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}} \geq \langle f, v \rangle_{\mathcal{Y}^* \times \mathcal{Y}}
\]
for all \( \eta \in \mathcal{Y} \), where \( \xi(t) \in \partial J(M\eta(t)) \) for all \( t \in [0, T] \). This implies that \( \eta \in \mathcal{Y} \) is a solution of Problem 4. Subsequently, we will prove that the solution of Problem 4 is unique. Assume that \( \eta_1, \eta_2 \) are solutions of Problem 4, we get
\[
\langle A(\eta_1(t)) + (S\eta_1)(t), v \rangle + J^0(M(\eta_1(t)), M v) \geq \langle f(t, x(t)), v \rangle \tag{39}
\]
and
\[
\langle A(\eta_2(t)) + (S\eta_2)(t), v \rangle + J^0(M(\eta_2(t)); Mv) \geq \langle f(t, x(t)), v \rangle \tag{40}
\]
for all \( v \in V \).

From (39) and (40), we have
\[
\langle A\eta_1(t) - A\eta_2(t), \eta_1(t) - \eta_2(t) \rangle + J^0(M\eta_1(t); M(\eta_1(t) - \eta_2(t))) - J^0(M\eta_2(t); M(\eta_1(t) - \eta_2(t))) \leq \langle (S\eta_1)(t) - (S\eta_2)(t), \eta_2(t) - \eta_1(t) \rangle. \tag{41}
\]
According to (HA), (HB), (HS) and (HF), we have
\[
\langle A\eta_1(t) - A\eta_2(t), \eta_1(t) - \eta_2(t) \rangle \geq L_A \| \eta_1(t) - \eta_2(t) \|^2_Y. \tag{42}
\]
\[
J^0(M\eta_1(t); M(\eta_1(t) - \eta_2(t))) - J^0(M\eta_2(t); M(\eta_1(t) - \eta_2(t))) \geq -m_J \| M \|^2 \| \eta_1(t) - \eta_2(t) \|^2_Y. \tag{43}
\]
and
\[
\langle (S\eta_1)(t) - (S\eta_2)(t), \eta_2(t) - \eta_1(t) \rangle \leq \| (S\eta_1)(t) - (S\eta_2)(t) \|_Y \cdot \| \eta_1(t) - \eta_2(t) \|_Y \leq m_q \| B \| \int_0^t \| \eta_1(s) - \eta_2(s) \|_Y \| \eta_1(t) - \eta_2(t) \|_Y ds. \tag{44}
\]
From (41), (42), (43) and (44), one can obtain
\[
L_A \| \eta_1(t) - \eta_2(t) \|^2_Y - m_J \| M \|^2 \| \eta_1(t) - \eta_2(t) \|^2_Y = (L_A - m_J \| M \|^2) \| \eta_1(t) - \eta_2(t) \|^2_Y \leq m_q \| B \| \int_0^t \| \eta_1(s) - \eta_2(s) \|_Y \| \eta_1(t) - \eta_2(t) \|_Y ds. \tag{45}
\]
Then, we have
\[
\| \eta_1(t) - \eta_2(t) \|_Y \leq \frac{m_q \| B \|}{L_A - m_J \| M \|^2} \int_0^t \| \eta_1(s) - \eta_2(s) \|_Y ds.
\]
Using Gronwall inequality, we know that \( \eta_1(t) = \eta_2(t) \), which implies that Problem 4 has a unique solution.

Furthermore, assume \( x_i(t) \in Z \) with \( i = 1, 2 \), and \( \eta_1, \eta_2 \in Y \) are the solutions of (24), then, we obtain
\[
A(\eta_1) + (S\eta_1) + \zeta_1 = f_1, \; A(\eta_2) + (S\eta_2) + \zeta_2 = f_2,
\]
where \( \zeta_1 \in M^* \partial J(M\eta_1) \) and \( \zeta_2 \in M^* \partial J(M\eta_2) \).

Further, one can easily achieve
\[
\langle A(\eta_1) - A(\eta_2), \eta_1 - \eta_2 \rangle + \langle \zeta_1 - \zeta_2, \eta_1 - \eta_2 \rangle + \langle S\eta_1 - S\eta_2, \eta_1 - \eta_2 \rangle = \langle f_1 - f_2, \eta_1 - \eta_2 \rangle.
\]
From (HA), (HF), (HS) and (HF), we have
\[
(L_A - m_J \| M \|^2 - Tm_q \| B \|) \| \eta_1 - \eta_2 \|^2_Y \leq \| f_1 - f_2 \|_Y \cdot \| \eta_1 - \eta_2 \|_Y \leq L_f \| x_1 - x_2 \|_Z \| \eta_1 - \eta_2 \|_Y.
\]
From (H0), we obtain that
\[
\| \eta_1 - \eta_2 \|_Y \leq \frac{L_f}{L_A - m_J \| M \|^2 - Tm_q \| B \|} \| x_1 - x_2 \|_Z. \tag{46}
\]
This completes the proof. \( \square \)
Theorem 3.14. Assume that \((H_A), (H_B), (H_I), (H_F), (H_J), (H_M), (H_0)\) and
\[
\max \left\{ L_T, \frac{L_f T L_f}{L_A - m_f M^2 - T m_q B} \right\} \sum_{k=1}^{n} d_k < \frac{1}{3}
\]
hold. Then Problem 3 has a unique solution \((x(t), \eta(t)) \in PC(I; Z) \times PC(I; Y)\).

Proof. For any given \(x \in PC(I; Z)\), Problem 4 has a unique solution \(\eta_x\). We define an operator \(\Lambda : PC(I; Z) \to PC(I; Z)\) as follows:
\[
(\Lambda x)(t) = x_0 + \sum_{k=1}^{n} I_k(x(t_k^-)) + \int_{0}^{t} F(s, x(s), \eta_x(s))ds, \text{ for } t \in (t_{k-1}, t_k).
\]

Step 1. We prove the operator \(\Lambda\) maps \(PC(I; Z)\) into \(PC(I; Z)\). To this end, let \(t_k < t_l < t_l + \Delta l < t_{l+1}, k = 0, 1, 2, \cdots, m, \Delta l > 0\) be given. By using condition \((H_F)\) and Hölder inequality, we get
\[
\| (\Lambda x)(t_l) - (\Lambda x)(t_l + \Delta l) \| = \left| \int_{t_l}^{t_l + \Delta l} F(s, x(s), \eta_x(s))ds - \int_{t_l}^{t_l + \Delta l} F(s, x(s), \eta_x(s))ds \right|
\leq \int_{t_l}^{t_l + \Delta l} \| F(s, x(s), \eta_x(s)) \| ds
\leq \int_{t_l}^{t_l + \Delta l} \theta(s)ds
\leq \left( \int_{t_l}^{t_l + \Delta l} (\theta(s))^p ds \right)^{\frac{1}{p}} \left( \int_{t_l}^{t_l + \Delta l} ds \right)^{\frac{p-1}{p}}
\leq M_2 (\Delta l)^{\frac{p-1}{p}},
\]
which implies that \(\| (\Lambda x)(t_l) - (\Lambda x)(t_l + \Delta l) \| \to 0\) as \(\Delta l \to 0\), where \(M_2 = \| \theta \|_{L_p[0, T]}\).

Then, \(\Lambda \in PC(I, Z)\).

Step 2. We show that the operator \(\Lambda\) is a contractive map. From the condition of Theorem 3.14, we have
\[
M_0 = \max \left\{ L_T, \frac{L_f T L_f}{L_A - m_f M^2 - T m_q B} \right\} \sum_{k=1}^{n} d_k < \frac{1}{3}.
\]
For any \(x_1, x_2 \in Z\). By \((H_F)\) and \((46)\), we have
\[
\| (\Lambda x_1)(t) - (\Lambda x_2)(t) \|
\leq \int_{0}^{t} \| F(s, x_1(s), \eta_{x_1}(s)) - F(s, x_2(s), \eta_{x_2}(s)) \| ds
\leq L_f T \| x_1 - x_2 \| Z + \| \eta_{x_1} - \eta_{x_2} \| Y + \sum_{k=1}^{n} d_k \| x_1(t_k^-) - x_2(t_k^-) \| Z
\leq L_f T \left( \| x_1 - x_2 \| Z + \frac{L_f}{L_A - m_f M^2 - T m_q B} \| x_1 - x_2 \| Z \right)
Theorem 4.1. Under $(H_A),(H_f),(H_J),(H_{J^1}),(H_o)$ and
\[
\frac{L_1}{1 - L_2 - L_3} > 0
\]
where $L_1 = \frac{TL_F\|M\|}{L_A-\sum_{j=1}^n M_j^2-L_T}$, $L_2 = TL_F\left(\frac{L_j}{L_A-\sum_{j=1}^n M_j^2-L_T} + 1\right)$, and $L_3 = \sum_{k=1}^n d_k$.
Then, we have the following conclusions:
(i) The Problem 6 has a unique solution $(x_\delta(t), \eta_\delta(t)) \in PC(I; Z) \times PC(I; Y)$;
(ii) $(x_\delta(t), \eta_\delta(t)) \to (x(t), \eta(t))$ as $\delta \to 0$, $\forall t \in I$, where $(x(t), \eta(t))$ is the solution of Problem 3.

Proof. Firstly, from the Theorem 3.14, we easily know that the Problem 6 has a unique solution $(x_\delta(t), \eta_\delta(t)) \in PC(I; Z) \times PC(I; Y)$. Next, we consider
\[
\begin{cases}
x'_\delta(t) = F(t, x_\delta(t), \eta_\delta(t)), \forall t \in I, t \neq t_k, k = 1, 2, \ldots, m, \\
\Delta x_\delta(t_k) = I_k(x_\delta(t_k^-)), k = 1, 2, \ldots, m, \\
x(0) = x_0 \in Z, \\
A(\eta_\delta(t)) + (S\eta_\delta)(t) + M^*\zeta_\delta \ni f(t, x_\delta(t)), \\
\zeta_\delta \in \partial J_\delta(M(\eta_\delta(t))) a.e. t \in [0, T].
\end{cases}
\]
We have
\[
\begin{align*}
&\langle A(\eta(t)) - A(\eta_\delta(t)), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y} + \langle \zeta - \zeta_\delta, M(\eta(t) - \eta_\delta(t)) \rangle_{Y^* \times Y} \\
&\quad + \langle (S\eta)(t) - (S\eta_\delta)(t), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y} \\
&= \langle f(t, x(t)) - f(t, x_\delta(t)), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y}.
\end{align*}
\]
Let $\lambda_\delta \in \partial J(M(\eta_\delta(t)))$, one has
\[
\begin{align*}
&\langle A(\eta(t)) - A(\eta_\delta(t)), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y} + \langle \zeta - \lambda_\delta, M(\eta(t) - \eta_\delta(t)) \rangle_{Y^* \times Y} \\
&\quad + \langle (S\eta)(t) - (S\eta_\delta)(t), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y} \\
&= \langle \zeta_\delta - \lambda_\delta, M(\eta(t) - \eta_\delta(t)) \rangle_{Y^* \times Y} \\
&\quad + \langle f(t, x(t)) - f(t, x_\delta(t)), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y}.
\end{align*}
\]
From $(H_A), (H_J)$ and $(H_{J^1})$, we have
\[
\begin{align*}
&\langle A(\eta(t)) - A(\eta_\delta(t)), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y} \geq L_A\|\eta(t) - \eta_\delta(t)\|^2, \\
&\langle \zeta - \lambda_\delta, M(\eta(t) - \eta_\delta(t)) \rangle_{Y^* \times Y} \geq -m_j\|M\|^2\|\eta(t) - \eta_\delta(t)\|^2.
\end{align*}
\]
\[
(\zeta_\delta - \lambda_\delta, M\eta(t) - M\eta_\delta(t))_{Y^* \times Y} \leq \|\zeta_\delta - \lambda_\delta\|_{Y^*} \|M\|\|\eta(t) - \eta_\delta(t)\|_Y
\]
\[
\leq G(\delta)\|M\|\|\eta(t) - \eta_\delta(t)\|_Y.
\] 

(51)

By virtue of \((H_f)\), we deduce
\[
\begin{align*}
\langle f(t,x(t)) - f(t,x_\delta(t)), \eta(t) - \eta_\delta(t) \rangle_{Y^* \times Y} & \leq \|f(t,x(t)) - f(t,x_\delta(t))\|_{Y^*} \|\eta(t) - \eta_\delta(t)\|_Y \\
& \leq L_f \|x(t) - x_\delta(t)\|_Z \|\eta(t) - \eta_\delta(t)\|_Y.
\end{align*}
\] 

(52)

Using \((H_S)\), we have
\[
\begin{align*}
\|(S\eta)(t) - (S\eta_\delta)(t), \eta(t) - \eta_\delta(t)\|_{Y^* \times Y} & \leq \|(S\eta)(t) - (S\eta_\delta)(t)\|_{Y^*} \|\eta(t) - \eta_\delta(t)\|_Y \\
& \leq L_T \|\eta(t) - \eta_\delta(t)\|_Y^2.
\end{align*}
\] 

(53)

From \((49),(50),(51),(52)\) and \((53)\), we achieve following inequality
\[
\begin{align*}
(L_A - m_J\|M\|^2 - LT)\|\eta(t) - \eta_\delta(t)\|_Y^2 & \leq G(\delta)\|M\|\|\eta(t) - \eta_\delta(t)\|_Y + L_f \|x(t) - x_\delta(t)\|_Z \|\eta(t) - \eta_\delta(t)\|_Y.
\end{align*}
\]

With the assumption \((H_0)\), we have
\[
\|\eta(t) - \eta_\delta(t)\|_Y \leq \frac{L_f}{L_A - m_J\|M\|^2 - LT} \|x(t) - x_\delta(t)\|_Z + \frac{G(\delta)\|M\|}{L_A - m_J\|M\|^2 - LT}
\] 

(54)

On the other hand, under the assumption \((H_f)\) we have
\[
\begin{align*}
\|x_\delta(t) - x(t)\|_Z & \leq \sum_{k=1}^n \|I_k(x_\delta(t_k^-)) - I_k(x(t_k^-))\|_Z + \int_0^t \|F(s,x_\delta(s),\eta_\delta(s)) - F(s,x(s),\eta(s))\|_Z ds \\
& \leq \sum_{k=1}^n d_k \|x_\delta(t_k^-) - x(t_k^-)\|_Z + L_F \int_0^t (\|x_\delta(s) - x(s)\|_Z + \|\eta(s) - \eta_\delta(s)\|_Y) ds \\
& \leq TL_F \left[ G(\delta)\|M\| \frac{L_f}{L_A - m_J\|M\|^2 - LT} + \left( 1 + \frac{L_f}{L_A - m_J\|M\|^2 - LT} \right) \|x_\delta(t) - x(t)\|_Z \right] \\
& + \sum_{k=1}^n d_k \|x_\delta(t) - x(t)\|_Z.
\end{align*}
\] 

(55)

By the conditions of Theorem 4.1 and \((55)\), we have
\[
\|x_\delta(t) - x(t)\|_Z \leq \frac{L_1}{1 - L_2 - L_3} G(\delta), \text{ with } \frac{L_1}{1 - L_2 - L_3} > 0.
\] 

(56)

From \((H_{f^*})\), as \(\delta \to 0\), we know that
\[
\|x_\delta(t) - x(t)\|_Z \to 0.
\] 

(57)

From \((54),(57)\) and \((H_{f^*})\), one has \(\|\eta_\delta(t) - \eta(t)\|_Y \to 0\) as \(\delta \to 0\). Thus, as \(\delta \to 0\), we get
\[
(x_\delta(t), \eta_\delta(t)) \to (x(t), \eta(t)).
\]

We complete the proof. \(\square\)
5. Existence and uniqueness results of contact problem. In this section, we will discuss existence and uniqueness of Problem 1 by using Theorem 3.14. To this end, we introduce the following Lemma.

Lemma 5.1. \[20\] Suppose that (H_{j_{0}}) and (H_{j_{r}}) hold, then the function J defined by (11) has the following properties:

(i) J is Lipschitz continuous on X;
(ii) \(\|\partial J(u)\|_{X^{*}} \leq c_{0}(1 + \|u\|_{X})\) for all \(u \in X\);
(iii) For all \(u, v \in X, \xi \in \partial J(u), \eta \in \partial J(v)\), then
\[
\langle \eta_{1} - \eta_{2}, u - v \rangle_{X^{*} \times X} \geq -m_{0}\|u - v\|^{2}_{X};
\]
(iv) \(J_{0}(u; v) = \int_{\Sigma_{N}}(j_{0}^{0}(u_{\nu}; u_{\nu}) + j_{r}^{0}(u_{\nu}; v_{\nu}))d\Gamma\) for all \(u, v \in X\) where \(J_{0}(u; v)\) is the directional derivative of \(J\) at a point \(u\) in direction \(v\).

We give the operator \(A \in \mathcal{L}(Y, Y^{*})\) by
\[
(Au, v)_{Y^{*} \times Y} = \langle \mathcal{A}(u), \mathcal{A}(v) \rangle_{Q}
\] (58)
and the operator \(S \in \mathcal{L}(Y, Y^{*})\) by
\[
((Su)(t), v)_{Y^{*} \times Y} = \left\langle \int_{0}^{t} \mathcal{B}(t - s)\mathcal{E}(u(s))ds, \mathcal{E}(v) \right\rangle_{Q}
\]
for all \(u, v \in Y\). Moreover, let \((g(t, f_{N})\) be defined by
\[
g(t, f_{N}(t)) = f(t, x(t))\text{ for all } f_{N}(t), x(t) \in PC(I; L^{2}(\Sigma_{N}; \mathbb{R}^{d})).
\]
Further, let \(M = \lambda \in \mathcal{L}(Y, X)\), where \(\lambda\) is the trace operator. Using above notation, Problem 2 can be reformulated as the following impulsive differential hemivariational inequality.

Problem 7. Find \((u(t), f_{N}(t)) \in PC(I; Y) \times PC(I; L^{2}(\Sigma_{N}; \mathbb{R}^{d}))\), such that

\[
\begin{cases}
(Au(t) + (Su)(t), v)_{Y^{*} \times Y} + J_{0}(\lambda u(t); v) \geq \langle g(f_{N}(t)), v \rangle_{Y^{*} \times Y} \\
\text{for all } v \in Y \text{ and a.e. } t \in [0, T], \\
f_{N}(t) = F(t, f_{N}(t), u(t)), \forall t \in I, t \neq t_{k}, k = 1, 2, \ldots, m, \\
\Delta f_{N}(t_{k}) = I_{k}(f_{N}(t_{k})), k = 1, 2, \ldots, m, \\
f_{N}(0) = f_{N}^{0} \in Z.
\end{cases}
\]

Next, we will study the existence and uniqueness of solutions to Problem 7 by using Theorem 3.14.

Theorem 5.2. If the conditions (H_{\alpha}), (H_{\phi}), (H_{\gamma}), (H_{F}), (H_{j_{0}}), (H_{j_{r}}) hold, then Problem 7 has a unique solution \((u(t), f_{N}(t)) \in PC(I; Y) \times PC(I; L^{2}(\Sigma_{N}; \mathbb{R}^{d}))\).

Proof. Now, we check Problem 7 has unique solution by using Theorem 3.14. To this end, we need to verify the operators \(A, S\), and function \(J\) satisfied with the Theorem 3.14. Under the assumption (H_{\alpha}), the operator \(A\) given by (58) satisfies hypothesis (H_{A}). The conditions (H_{\phi}) and (H_{\gamma}) satisfy with \(B = I\) and \(q = \phi\) under the hypothesis (H_{\phi}) and definition of \(S\). With the conditions (H_{j_{0}}) and (H_{j_{r}}), the function \(J\) defined by (11) satisfies (H_{J}) with \(L_{J} = \max\{c_{\nu}, c_{r}\}, m_{J} = \max\{m_{\nu}, m_{r}\}\). Using the properties of the trace operator, we know that (H_{M}) is satisfied. Thus, the proof of Theorem 5.2 is completed. \(\square\)
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REFERENCES

[1] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems*, John Wiley, New York, 1984.

[2] S. Carl, V. K. Le and D. Motreanu, *Nonsmooth Variational Problems and Their Inequalities: Comparison Principles and Applications*, Springer, New York, 2007.

[3] C. Carstensen and J. Gwinner, *A theory of discretization for nonlinear evolution inequalities applied to parabolic Signorini problems*, *Ann. Mat. Pura Appl.*, 177 (1999), 363–394.

[4] X. Cheng, S. Migórski, A. Ochal and M. Sofonea, *Analysis of two quasistatic history-dependent contact models*, *Discrete Contin. Dyn. Syst. Ser. B*, 19 (2014), 2425–2445.

[5] Z. Denkowski and S. Migórski, *Hemivariational inequalities in thermoviscoelasticity*, *Nonlinear Anal.*, 63 (2005), 87–97.

[6] Z. Denkowski, S. Migórski and N. S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic Plenum Publishers, Boston, 2003.

[7] A. D. Drozdov, *Finite Elasticity and Viscoelasticity: A Course in the Nonlinear Mechanics of Solids*, World Scientific, Singapore, 1996.

[8] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer, Berlin, 1976.

[9] A. Farcas, F. Patrulescu and M. Sofonea, *A history-dependent contact problem with unilateral constraint*, *Ann. Acad. Rom. Sci. Ser. Math. Appl.*, 4 (2012), 90–96.

[10] M. Frigon and D. O'Regan, *Existence results for first-order impulsive differential equations*, *J. Math. Anal. Appl.*, 193 (1995), 96–113.

[11] J. Han, Y. Li and S. Migórski, *Analysis of an adhesive contact problem for viscoelastic materials with long memory*, *J. Math. Anal. Appl.*, 427 (2015), 646–668.

[12] W. Han, S. Migórski and M. Sofonea, *Advances in Variational and Hemivariational Inequalities with Applications: Theory, Numerical Analysis, and Applications*, Advances in Mechanics and Mathematics, Springer, 2015.

[13] W. Han, S. Migórski and M. Sofonea, *Analysis of a general dynamic history-dependent variational hemivariational inequality*, *Nonlinear Anal. Real World Appl.*, 36 (2017), 69–88.

[14] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics, vol. 30. Americal Mathematical Society, Providence, International Press, Somerville, 2002.

[15] S. Migórski, *Evolution hemivariational inequality for a class of dynamic viscoelastic nonmonotone frictional contact problems*, *Comput. Math. Appl.*, 52 (2006), 677–698.

[16] S. Migórski, *Dynamic hemivariational inequality modeling viscoelastic contact problem with normal damped response and friction*, *Appl. Anal.*, 84 (2005), 669–699.

[17] S. Migórski and P. Gamorski, *A new class of quasistatic frictional contact problems governed by a variational-hemivariational inequality*, *Nonlinear Anal. Real World Appl.*, 50 (2019), 583–602.

[18] S. Migórski and A. Ochal, *Quasi-static hemivariational inequality via vanishing acceleration approach*, *SIAM J. Math. Anal.*, 41 (2009), 1415–1435.

[19] S. Migórski, A. Ochal and M. Sofonea, *Integrodifferential hemivariational inequalities with applications to viscoelastic frictional contact*, *Math. Models Methods Appl. Sci.*, 18 (2008), 271–290.

[20] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics, vol. 26. Springer, New York, 2013.

[21] S. Migórski, A. Ochal and M. Sofonea, *History-dependent variational-hemivariational inequalities in contact mechanics*, *Nonlinear Anal. Real World Appl.*, 22 (2015), 604–618.

[22] S. Migórski, A. Ochal and M. Sofonea, *Analysis of frictional contact problem for viscoelastic materials with long memory*, *Discrete Contin. Dyn. Syst. Ser. B*, 15 (2011), 687–705.
[23] S. Migórski and S. Zeng, Mixed variational inequalities driven by fractional evolutionary equations, Acta Math. Sci. Ser. B, 39 (2019), 461–468.

[24] S. Migórski and S. Zeng, A class of differential hemivariational inequalities in Banach spaces, J. Global Optim., 72 (2018), 761–779.

[25] S. Migórski and S. Zeng, Rothe method and numerical analysis for history-dependent hemivariational inequalities with applications to contact mechanics, Numer. Algorithms, 82 (2019), 423–450.

[26] A. C. Pipkin, Lectures on Viscoelasticity Theory, Applied Mathematical Sciences, Springer, New York, 1972.

[27] S. Shen, F. Liu, J. Chen, I. Turner and V. Anh, Numerical techniques for the variable order time fractional diffusion equation, Appl. Math. Comput., 218 (2012), 10861–10870.

[28] M. Shillor, M. Sofonea and J. J. Telega, Models and Analysis of Quasistatic Contact: Variational Methods, Springer, Berlin, 2004.

[29] M. Sofonea and A. Farcaș, Analysis of a history-dependent frictional contact problem, Appl. Anal., 93 (2014), 428–444.

[30] M. Sofonea and A. Matei, Mathematical Models in Contact Mechanics: Preliminaries on Functional Analysis, London Mathematical Society Lecture Note Series, vol. 398. Cambridge University Press, 2012.

[31] M. Sofonea and A. Matei, History-dependent quasi-variational inequalities arising in contact mechanics, European J. Appl. Math., 22 (2011), 471–491.

[32] M. Sofonea and F. Pătrulescu, Analysis of a history-dependent frictionless contact problem, Math. Mech. Solids, 18 (2012), 409–430.

[33] M. Sofonea, F. Pătrulescu and A. Farcaș, A viscoplastic contact problem with normal compliance, unilateral constraint and memory term, Appl. Math. Optim., 69 (2014), 175–198.

[34] G. Xue, F. Lin and B. Qin, Solvability and optimal control of fractional differential hemivariational inequalities, Optimization, 3 (2020), 1–32.

[35] S. Zeng and S. Migórski, A class of time-fractional hemivariational inequalities with application to frictional contact problem, Commun. Nonlinear Sci. Numer. Simul., 56 (2018), 34–48.

[36] S. Zeng and S. Migórski, Noncoercive hyperbolic variational inequalities with applications to contact mechanics, J. Math. Anal. Appl., 455 (2017), 619–637.

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