The structure of the Mitchell order - I

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Abstract

We isolate here a wide class of well founded orders called tame orders, and show that each such order of cardinality at most $\kappa$ can be realized as the Mitchell order on a measurable cardinal $\kappa$, from the consistency assumption of $o(\kappa) < \kappa^+$.

1 Introduction

This paper is the first of two parts, of a study on the possible structure of Mitchell order. In this first part, we identify a large class of well founded orders with some very appealing properties, and proof that each of its members can be realize as Mitchell order on the set of normal measures on some measurable cardinal $\kappa$. In [13], Mitchell introduced the following relation: Given two normal measures $U, W$, then $U \preceq W$ denotes the fact that $U \in M_W \cong \text{Ult}(V,W)$. Mitchell proved that $\preceq$ is a well founded order, which is now known as the Mitchell ordering. The Mitchell ordering and its extension to arbitrary extenders (which may not be well-founded or even transitive anymore) have become a significant large cardinal characters, and a major ingredient in the study of consistency results, and inner model theory. Given a cardinal $\kappa$, we shall denote the restriction of $\preceq$ to the set of normal measures on $\kappa$, by $\preceq(\kappa)$, and the rank of the well founded ordering $\preceq(\kappa)$, by $o(\kappa)$. There have been numerous results regarding the structure of $\preceq$. Even the simpler question regarding the number of normal measures on a measurable cardinal,

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has been non trivial. Kunen [9] showed it is possible to have a single normal measure on $\kappa$, Kunen-Paris [10] showed it is possible to have $\left(2^\kappa\right)^+$ normal measures. Mitchell [13] introduced Mitchell order $\triangleleft$, to construct inner models $L[\mathcal{U}]$ in which a measurable cardinal $\kappa$ can have any number of normal measures $\lambda$ between 1 and $\kappa^{++}$. The large cardinal assumption used for this construction is larger than a single measurable cardinal and is now known as $o(\kappa) = \lambda$. Apter-Cummings-Hamkins [1] proved it is possible to have $\kappa^+$ many normal measures on $\kappa$ from the assumption of a single measurable cardinal. The question of the number of normal measures was fully resolved by Baldwin [2] obtains inner models in which $\triangleleft$ can realize any pre wellorder. Cummings [4],[5], and Witzany [18] studied the $\triangleleft$ ordering in various generic extensions, and obtained some very interesting $\triangleleft$-structures. In particular [18] shows that starting from $o(\kappa) = \lambda \leq \kappa^{++}$ in a Mitchell model $L[\mathcal{U}]$, then in a Kunen-Paris generic extension, the $\triangleleft$ structure on the set of normal measures on $\kappa$ is very large, and contains any well founded ordering $(S, <_S)$ of size $|S| \leq \kappa^{++}$ and rank $\text{rank}(<_S) \leq \lambda$, as a sub order. Steel [17] and Neeman [15] addressed the question of well-foundness of $\triangleleft$ for arbitrary extenders, showing that well founded breaks exactly at the level of a rank-to-rank extenders.

The general question, regarding the possible well founded relation which can be realized as $\triangleleft$ on normal measures remained open. This work is devoted to study the possible structure of $\triangleleft(\kappa)$. We gradually develop a series of techniques by which we obtain increasing variety of possible $\triangleleft$ structures, from an increasingly large cardinal (consistency) assumptions. Therefore each method presented is significant for being able to realize some well founded relations as $\triangleleft$ from different large cardinal assumptions.

- In this paper, we develop technique to realize a wide family of well founded orders, called tame orders, from assumptions below the existence of measurable cardinal $\kappa$ with $o(\kappa) \geq \kappa^+$.
- In part II, we build on the results obtain here, and extend it to incorporate ground model with overlapping extenders. We use these technique to prove that every well founded order can be consistently realized as $\triangleleft(\kappa)$ on some measurable cardinal $\kappa$.  

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Therefore, the results suggests that tame orders are significant for the possibility to realize them as $\triangleleft(\kappa)$, from relatively weak large cardinal assumptions. The author does not know if there is any non-tame order $(R, <_R)$ which can be realized as $\triangleleft(\kappa)$ from assumptions less than $0^\kappa$.

All results are obtained by forcing over a core model $V = \mathcal{K}(V)$, which is presented as an extender model $L[E]$ (as in [16]) or as a Mitchell model $L[U]$ (see [14] or [12]) with some large cardinal property. Furthermore, our results will always be obtained by first introducing a special intermediate extension $V'$ of $V$ so that

1. $\triangleleft(\kappa)^{V'}$ will be quite rich, depending on our large cardinal assumption.

2. The normal measures on $\kappa$ in $V'$ are separated by sets.

We say that the normal measures on $\kappa$, in some model of set theory $V'$, are separated by sets, if we can associate every normal measure $U$ on $\kappa$ a set $X_U \subset \kappa$ so that for any additional normal measure $U'$, then $X_U \in U'$ if and only if $U = U'$.

The properties of $V'$ will allow us to simply obtain forcing extensions in which $\triangleleft(\kappa)$ admits many suborders of $\triangleleft(\kappa)^{V'}$. More precisely, for any set of normal measures on $\kappa$ in $V'$. $\mathcal{W} \subset \text{dom}(\triangleleft(\kappa)^{V'})$, of cardinality $|\mathcal{W}| \leq \kappa$, then there exists a forcing extension $V''$ of $V'$ so that $\triangleleft(\kappa)^{V''} \cong \triangleleft(\kappa)^{V'} | \mathcal{W}$. We refer to this last step of our process as a final cut.

We can actually do more. By using the forcing technique of Friedman and Magidor we can will show how to modify the construction so that each normal measures in $V'$ has $\kappa$ equivalent copies. This will allow us to make a final cut so that every equivalent class in $\triangleleft(\kappa)^{V'}$ can have any number of measures between 0 and $\kappa$ in the final model. We note that if such “final cut” extension was possible to preform over the model in [18] then we could easily get models in which $\triangleleft(\kappa)$ realizes any well founded order $(R, <_R)$ of cardinality at most $\kappa$, from the minimal assumption of $o(\kappa) = \text{rank}(<_R)$. Unfortunately, the normal measures on $\kappa$ in this model are far from being separated.

Constructing the model intermediate model $V'$ is the main challenge, and will occupy most of the section in this work. Forcing over a core model $V$

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1By equivalent class of a relation $(R, <_R)$, we mean a class of elements of the order which agree on the sets of elements which are $<_R$ -greater and the elements which are $<_R$ -smaller.
grants us more tools to analyze the normal measures \( \kappa \) appearing in a forcing extension \( V' = V[G] \) by a poset \( \mathcal{P} \). If \( U \) is a normal measure on \( \kappa \) in \( V[G] \), and \( j_U : V[G] \to M_U \cong \text{Ult}(V[G], U) \) is the elementary embedding between transitive models, induced by the ultrapower of \( V[G] \) by \( U \), then \( j = j_U \upharpoonright V : V \to M \) is some iterated ultrapower by extenders of \( V \), and \( M_U = M[G_U] \), where \( G_U \subset j(\mathcal{P}) \) is generic over \( M \).

If \( U' \) is another normal measure on \( \kappa \) in \( V[G] \) with induced \( j_{U'} : V \to M'[G_{U'}] \) and \( j' = j_{U'} \upharpoonright V : V \to M' \), then, roughly speaking, \( U \triangleleft U' \) if and only if a large portion of \( M \) is an iterate ultrapower of \( M' \), and \( G_U \in M'[G_{U'}] \). So if we want to generate an elaborated \( \triangleleft(\kappa) \) structure in \( V' = V[G] \), we can potentially use forcings which can introduce many independent generics \( G_U \) or many complicated iterations \( j = j_U \upharpoonright V \). In [18] the large variety of normal measures is mostly due to the “wild” variety of generics and subgenerics added by the Cohen forcings which can be used for constructing \( G_U \) generics. On the one hand, this ingredient allows to obtain a very large \( \triangleleft(\kappa) \) structure which contain all well founded posets below certain cardinality. On the other hand, this is also the source to the fact that the normal measures in this model cannot be separated.

In order to avoid this the non-separative results, in this work, we restrict ourself to posets which produce a relatively small, and accurate variety of possible \( G_U \) generics, such as the Friedman-Magidor poset introduced in [9], and the Magidor iteration of Prikry forcings [11]. Therefore, our approach is to use the variety of possible iterated ultrapowers \( j = j_U \upharpoonright V : V \to M \) to obtain elaborated \( \triangleleft(\kappa) \) structure. As it turns out, constructing elaborated iterated ultrapowers which have the desired effect to \( \triangleleft(\kappa) \) in generic extensions, requires increasingly stronger large cardinal assumptions:

- Here, we start from an increasing \( \triangleleft- \)sequence \( \langle U_\alpha \mid \alpha < o(\kappa) \rangle \) in the ground model \( V \), with \( o(\kappa) < \kappa^+ \). The iterations which are the \( \triangleleft(\kappa) \) results, are of the form \( j_{U_\alpha}^{M_\beta} \circ j_{U_\beta} \), where \( \alpha < \beta < o(\kappa) < \kappa^+ \), \( j_{U_\beta} : V \to M_\beta \cong \text{Ult}(V, U_\beta) \), and \( j_{U_\alpha}^{M_\beta} \) is the iterated ultrapower of \( M_\beta \) taken by \( U_\alpha \in M_\beta \). In terms of the variety of \( \triangleleft(\kappa) \) structures, these iterations seems to be the best we can essentially get from using linear iterations. This suggests that realization of any non-tame order requires a significantly stronger large cardinal assumptions (at least \( 0^* \)).

- In part II, we move our large cardinal assumption to the realm of non-
linear iterations. We start with an increasing sequence of measurable cardinals \( \langle \theta_\alpha \mid \alpha < \rho < \kappa^+ \rangle \) so that each carries a normal measure \( U_{\theta_\alpha} \), and \( \kappa < \theta_0 \) which carries some \( (\kappa, \theta^+) \)-extenders, \( F \), with \( \theta = \bigcup_{\alpha < \rho} \theta_\alpha^+ \), and \( U_{\theta_0} \preceq F \) for each \( \alpha < \rho \). The iterations which are the \( \preceq \) results are of the form \( i^{MF} \circ j_F \), where \( j_F : V \to M_F \cong \text{Ult}(V, F) \), and \( i^{MF} \) is some iterated ultrapower by the measures \( U_{\theta_\alpha} \in M_F \).

Let \((S, <_S)\) be an order. For \( x \in S \) let \( D(x) = \{ z \in S \mid z <_S x \} \) and \( U(x) = \{ z \in S \mid x <_S z \} \). Two elements \( x, y \in S \) are said to be equivalent if \( D(x) = D(y) \) and \( U(x) = U(y) \), and we denote this by \( x \sim_S y \). The reduction of \((S, <_S)\) is the order denoted by \(((S), <_{[S]}))\), which consists of so that the elements of \([S]\) are the \( \sim_S \) equivalence classes \([x]_S\) of \( x \in S \), and \([x]_S < [y]_S\) if \( x <_S y \). Clearly, the order \(((S), <_{[S]}))\) is reduced. We say that \((S, <_S)\) is reduced if there are no distinct \( x, y \in S \) which are equivalent.

Suppose that \( o(\kappa) = \lambda < \kappa^+ \) in \( V \). Following the guidelines for constructing various \( \preceq(\kappa) \) orders, our study will include the following main tasks:

1. For an ordinal \( \lambda \), find a reduced order \((R_\lambda, <_{R_\lambda})\) which is as rich as possible, so that for some extension \( V' \), \( \preceq(\kappa)^{V'} \) is isomorphic to \((R_\lambda, <_{R_\lambda})\).

2. Characterize the class of orders \((S, <_S)\) which are isomorphic to a restriction of some \( <_{R_\lambda} \) to a subset of its domain.

The reduced order we will succeed to generate from \( o(\kappa) = \lambda \) is the order \((R_\lambda, <_{R_\lambda})\), \( \lambda < \kappa^+ \) defined by \( R_\lambda = \{ (\alpha, \beta) \in \lambda^2 \mid \alpha \leq \beta \} \) and \( (\alpha', \beta') <_{R_\lambda} (\alpha, \beta) \) if and only if \( \beta' < \alpha' \). Note that \( R_\lambda \) contains an increasing sequence of length \( \lambda \), \( \{ (\alpha, \alpha) \mid \alpha < \lambda \} \subset R_\lambda \).

By using a “final cut” extension, we will be able to proof the following theorem, which is the main theorem of the paper.

**Theorem 1.1.** Suppose that \( \kappa, \lambda < \kappa^+ \) are ordinals with \( o(\kappa) \geq \lambda \). Let \((S, <_S)\) be a well founded ordering, so that \( |S| \leq \kappa \), and the reduction \(((S), <_{[S]}))\) is isomorphic to a restriction suborder of \((R_\lambda, <_{R_\lambda})\), then there exists an model of ZFC + GCH so that \( \preceq(\kappa) \cong (S, <_S) \).

**Structure of this work**

\(^2\) the reader should relate to the iterated ultrapowers \( j^{M_\delta}_u \circ j_\beta \) mentioned above.
In section 2 we outline the main ideas leading to the results of this work.

In section 3 we will study the of the class of orders which are restriction of $R_\lambda$ for some ordinal $\lambda$. It turns out that the orders in this class have a natural description. Such orders will be called a tame orders.

In section 4 we will define the main posets $\mathcal{P}^0$, $\mathcal{P}^1$ used to construct models in which $\prec(\kappa)$ contains a copy of $R_\lambda$ assuming the $o(\kappa) = \lambda$ in $V$. $\mathcal{P}^0$ will be the Friedman Magidor iteration of Sacks forcing and coding, introduced in [6], and $\mathcal{P}^1$ is a Magidor iteration of Prikry type forcings which was introduced in [11]. A generic extension of $V$ by a $\mathcal{P}^0 \ast \mathcal{P}^1$ generic set, will be denoted $V^1$ (this extension plays the role of a $V'$ extension described above). At the last part of this section we define normal measures on $\kappa$, $U^1_{(\alpha, \beta)}$ for $\alpha, \beta < \lambda$, in a $\mathcal{P}^0 \ast \mathcal{P}^1$ generic extension (see Definition 4.6 for $\alpha \geq \beta$, and Definition 4.9 for $\alpha < \beta$). The section is concluded with some initial properties of these normal measures including the fact that these measures are separated by sets.

In section 5 we will define the restriction of the ultrapower embedding $j^1_{(\alpha, \beta)} : V^1 \to M^1_{(\alpha, \beta)} \cong \text{Ult}(V^1, U^1_{(\alpha, \beta)})$ to the ground model $V$. This description will be crucial to determined the $\prec$ order between the different normal measures $U^1_{(\alpha, \beta)}$, $\alpha, \beta < \lambda$.

In section 6 we use the results obtained so far to show that for couples $(\alpha, \beta), (\alpha', \beta') \in \lambda \times \lambda$ with $\alpha \leq \beta$ and $\alpha' \leq \beta'$, then $U^1_{(\alpha', \beta')} \prec U^1_{(\alpha, \beta)}$ is and only if $\beta' < \alpha$ (therefore establishing a copy of $R_\lambda$ in $\prec(\kappa)V^1$).

In section 7 we prove that in $V^1$ there are no other normal measures on $\kappa$ other then those in $\{U^1_{(\alpha, \beta)} \mid \alpha, \beta \in \lambda\}$.

In the final section 8 we will describe “final cut” forcing extension which will allow us to obtain models which can realize any restriction of $\prec(\kappa)V^1$ as $\prec(\kappa)$, and use these to prove Theorem 1.1.

**Notations** - A couple $(S, <_S)$ will be called an order, if $<_S \subseteq S \times S$ is a relation which defines a partial order (anti-symmetric and transitive relation) on $S$. We actually get that every $U^1_{(\alpha, \beta)}$ for $\alpha > \beta$ is $\prec(\kappa)$ equivalent to $U^1_{(\alpha, \alpha)}$, but this will not be important for the rest of the arguments.
When there is no danger of confusion, we will sometimes use $S$ to denote the entire order $(S, <_S)$.

For $X \subseteq S$ we denote $<_S \cap X \times X$ by $<_S| X$, and say that $(X, <_S| X)$ is a restriction suborder (or just suborder) of $(S, <_S)$. If $(S, <_S)$ is well founded, then $\text{rank}(<_S) = \sup(\{\text{otp}(C) + 1 \mid C \subseteq S \text{ is a } <_S-\text{increasing chain}\})$, and for every $x \in S$ then $\text{rank}_S(x) = \text{rank}(<_S| D(x))$.

Given a cardinal $\kappa$, then a set $U \subseteq \mathcal{P}(\kappa)$ is a measure on $\kappa$ if it is $\kappa$-complete ultrafilter on $\kappa$. $U$ is a normal measure if it is also closed to diagonal intersections of length $\kappa$. Given a measure $U$ on $\kappa$, we write $j : V \rightarrow M \cong \text{Ult}(V, U)$ to denote that $M$ is the transitive collapse of the ultrapower model of $V$ by $U$, and that $j : V \rightarrow M$ is the induced elementary embedding. Similarly, if $F$ is an extender on $\kappa$ (see [8] for a definition) then we similarly write $j : V \rightarrow M \cong \text{Ult}(V, F)$ when $\text{Ult}(V, F)$ is well founded.

We use the forcing notation in which for a forcing notion $(\mathcal{P}, \leq)$ and $p, q \in \mathcal{P}$, then $p \geq q$ indicates that $p$ is stronger than $q$ (i.e. $p$ is more informative than $q$). Furthermore, the trivial condition of $\mathcal{P}$ will be denoted by $0_\mathcal{P}$. A name of a set $x$ in a generic extension will be denoted by $\dot{x}$, and a canonical name for an element $x$ in the ground model $V$ will be denoted by $\check{x}$. A generic extension of a model $V$ by a generic filter for a poset $\mathcal{P} \in V$ will be sometimes denoted by $V^\mathcal{P}$.

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## 2 Outline of the Main Ideas

**More On Tame Orders** - Recall that tame orders are orderes which are isomorphic to restriction suborder of $R_\lambda$ for some ordinal $\lambda$. It turns out that the class of tame orders has a natural description, especially the family of finite such orders. For finite orders, we will show that being tame is equivalent to not having a restriction suborder which is isomorphic to the four elements order

$$
\begin{array}{ccc}
\bullet & \bullet \\
\downarrow & \\
\bullet & \bullet
\end{array}
$$

For example, we might say that $S$ is well founded when we mean that $(S, <_S)$ is.
which we denote by \((R_{2,2}, <_{R_{2,2}} \text{ (i.e., } R_{2,2} = \{x_0, x_1, y_0, y_1\} \text{ and } <_{R_{2,2}} = \{(x_0, x_1), (y_0, y_1)\})\). It is easy to see no \(R_\lambda\) contains such copy of this order as a restriction. It is less trivial to show that every finite reduced order \((S, <_S)\), which does not contains a copy of \(R_{2,2}\), can be realized as a restriction of \(R_n\) for some \(n < \omega\). The following observation will be the key to the analysis in Section 3.

Observation 2.1. Suppose that \((S, <_S)\) (finite or infinite) does not contain a copy of \(R_{2,2}\) as a restriction, then for every \(x, y \in S\) then \(\subset\) compares \(U(x)\) with \(U(y)\), and \(D(x)\) with \(D(y)\).

I.e., we have that

1. \(U(x) \subset U(y)\) or \(U(y) \subset U(x)\), and
2. \(D(x) \subset D(y)\) or \(D(y) \subset D(x)\).

Therefore, the sets \(D(S) = \{D(x) \mid x \in S\}\) and \(U(S) = \{U(x) \mid x \in S\}\), are both linearly ordered by \(\subset\). Note that it is possible that either \(D(x) = D(y)\), or \(U(x) = U(y)\), for distinct \(x, y \in S\). But since \((S, <_S)\) is reduced, then we can have both. We infer that the map \(x \mapsto (D(x), U(x))\) is injective.

Now suppose that \(S\) is finite. Then clearly both \(U(S), D(S)\) are finite as well. We will prove that if we list the linear ordered sets \(D(S)\) via \(\subset\), and \(U(S)\) via its reverse \(\supset\), then by replacing each \((D(x), U(x))\) by the appropriate positions on these list, we get an injective homomorphism from \((S, <_S)\) to 
\((R_n, <_{R_n})\) where \(n = |U(S)|\).

Example 2.2. Let \((R, <_R)\) be an order defined on five elements \(R = \{a, b, c, d, e\}\) with \(<_R = \{(b, d), (b, e), (d, e), (c, e)\}\), illustrated by

\[
\begin{array}{c}
\bullet & e & \\
\bullet & d & \\
\bullet & a & b & c
\end{array}
\]

The first table lists the members of \(R\) according to \((D(R), \subset)\) and enumerates their \(D\) classes it accordingly.

| \(x \in R\) | \(a, b, c\) | \(d\) | \(e\) |
| --- | --- | --- | --- |
| \(D(x)\) | \(\emptyset\) | \(\{b\}\) | \(\{d, b, c\}\) |
| 0 | 1 | 2 |
The second table lists the members of $R$ according to $(U(R), \varnothing)$ and enumerates their $U$ classes according to $\varnothing$.

| $x \in R$ | $a, e$ | $c, d$ | $b$ |
|-----------|--------|--------|-----|
| $U(x)$    | $\emptyset$ | $\{e\}$ | $\{b, e\}$ |
|           | $2$    | $1$    | $0$  |

Associating each $x \in R$ to its indices in the $D$ and $U$ tables, we get a function $\pi : R \to R_3$ with $\pi(a) = (0, 2)$, $\pi(b) = (0, 0)$, $\pi(c) = (0, 1)$, $\pi(d) = (1, 1)$, $\pi(e) = (2, 2)$, illustrated by

$$
(2, 2) \quad (0, 2) \quad (0, 0) \quad (0, 1) \quad (1, 1)
$$

It is easy to see that $\pi$ defines an embedding of $(R, <_R)$ in $(R_3, <_{R_3})$.

The analysis sketched above requires a small modification in the case $(S, <_S)$ is infinite. The analogue of the description above requires to associate each $(U(x), D(x))$ with a pair of ordinals. For this to work, we need that $(D(s), \varnothing)$ and $(U(S), \varnothing)$ are not only linear, but well founded as well. If this holds, then we can then associate $(D(x), U(x))$ with there rank in appropriate well ordered sets. Indeed, we will prove that an order $(S, <_S)$ is tame if and only if

1. $(S, <_S)$ does not contains a copy of $R_{2,2}$ as a restriction to a subset of $S$, and
2. $(D(s), \varnothing)$ and $(U(S), \varnothing)$ are well founded.

**More On $\mathcal{P}^0$ and $\mathcal{P}^1$** - Let us describe the main ideas leading to the realization of $R_\lambda$ as $\varnothing(\kappa)$, from $o(\kappa) = \lambda$. We start with a ground model $V = L[U]$ which is a Mitchell model, so that $U = \langle U_{\alpha, \tau} \alpha \leq \kappa, \tau < d^H(\kappa) \rangle$ is a coherent sequence with $d^H(\kappa) = \lambda$. For every $\tau < \lambda$ we abbreviate, and write $U_\tau$ for $U_{\kappa, \tau} \in U$. The goal of the main forcing is to construct a generic extension $V^1 = V[G]$ so that for every $\alpha \leq \beta < \lambda$, then $V^1$ has a normal measure $U^1_{\alpha, \beta}$ on $\kappa$. So that if $j_{U^1_{\alpha, \beta}} : V^1 \to M^1_{U^1_{\alpha, \beta}} \cong \Ult(V^1, U^1_{\alpha, \beta})$ is its induced ultrapower embedding, then is restriction to $V$, $\mathcal{P}_{\alpha, \beta} \upharpoonright V : V \to Z_{\alpha, \beta}$ satisfies that
1. $\pi^1_{(\alpha, \beta)}$ is of the form $k_{\alpha, \beta} \circ j_{U_{\alpha}} \circ j_{U_{\beta}}$, if $\alpha < \beta$, and

2. $\pi^1_{(\alpha, \beta)}$ is almost $k_{\alpha} \circ j_{U_{\alpha}}$ if $\alpha = \beta$,

where in both cases, the additional iterated ultrapower, $k_{\alpha, \beta}$ or $k_{\alpha}$, have critical points above $\kappa$, and will not have any effect on $\lhd(\kappa)$ in the final structure. Here, we have that $j_{U_{\beta}} : V \rightarrow M_\beta \cong \text{Ult}(V, U_{\beta})$, and if $\alpha < \beta$ then $j_{U_{\alpha}}$ is the iterated ultrapower of $M_\beta$ taken by $U_\alpha \in M_\beta$. The “almost” specification will be clarified below. For now there is no harm to think that $\pi^1_{(\alpha, \beta)}$ is equal to $j_{U_{\alpha}} \circ j_{U_{\beta}}$ or $j_{U_{\alpha}}$.

These properties of $j^1_{(\alpha, \beta)} \upharpoonright V : V \rightarrow Z_{\alpha, \beta}$ implies that the normal measures on $\kappa$ in $Z_{\alpha, \beta}$ are $\langle U_\tau \mid \tau < \alpha \rangle$. So if we want to take an additional ultrapowers from $Z_{\alpha, \beta}$ so that $\kappa$ is its critical points, we can only use $U_\tau$ with $\tau < \alpha$.

Let $(\alpha', \beta')$ be another couple with $\alpha' \leq \beta'$, and consider the possibility that $U^1_{(\alpha', \beta')} < U^1_{(\alpha, \beta)}$ in $V^1$. We will show that this holds if and only if $\beta' < \alpha$. The key is that if $U^1_{(\alpha', \beta')} < U^1_{(\alpha, \beta)}$ then the ultrapower of $M^1_{U^1_{(\alpha, \beta)}}$ by $U^1_{(\alpha', \beta')}$, has a restriction to $Z_{\alpha, \beta} = \mathcal{K}(M^1_{U^1_{(\alpha, \beta)}})$. By standard results in inner model theory, we will show that this restriction is an iterated ultrapower of $Z_{\alpha, \beta}$ with critical point $\kappa$, and it has to make use of $U_{\alpha'}, U_{\beta'}$, so we require that $U_{\alpha'}, U_{\beta'}$ appear on $Z_{\alpha, \beta}$. Since $\alpha' \leq \beta'$ then the last is equivalent to $\beta' < \alpha$.

There are several ways to construct a generic extension $V^1$ of $V$ which has normal measures of the form $U^1_{(\alpha, \beta)}$ with the properties above. For this we use the Magidor iteration of single point Prikry forcing. Such forcing adds (almost) every measurable cardinal $\alpha < \kappa$ a single Prikry points $d(\alpha) < \alpha$ instead of an entire Prikry cofinal sequence $\langle d_n(\alpha) \mid n < \omega \rangle$. Let us explain its purpose in our construction.

Let $\alpha < \beta$ be ordinals below $\lambda$. First note that in $V = L[U]$ the embedding $j_{U_{\alpha}} = j_{U_{\alpha}} \circ j_{U_{\beta}} : V \rightarrow M_{\alpha, \beta}$ cannot be described as an ultrapower by a single normal measure, as not all ordinals in $M_{\alpha, \beta}$ are of the form $j_{U_{\alpha}}(h)(\kappa)$ for some $h : \kappa \rightarrow \text{On}$. More precisely, $\kappa' = j_{U_{\alpha}}(\kappa) < j_{U_{\beta}}(\kappa)$, which is the generator of the second ultrapowers of $M_\beta$ by $U_\alpha$, is such an ordinal, and every ordinal in $M_{\alpha, \beta}$ is of the form $j_{U_{\alpha}}(h)(\kappa, \kappa')$, for some $h : \kappa^2 \rightarrow \text{On}$. Our goal is therefore to construct a generic extension in which adds a function $f : \kappa \rightarrow \kappa$, and an extension of $j_{U_{\alpha}}$ to some $j^1_{U_{\alpha}}$ so that $\kappa' = j^1_{U_{\alpha}}(f)(\kappa)$. The most standard way to obtain such generic extension is by iterating Cohen forcings which adds functions $f_\alpha : \alpha \rightarrow \alpha$ at many $\alpha < \kappa$, and at $\alpha = \kappa$. The problem is that the $\lhd(\kappa)$ structure in such generic extension is “wild” (as in the model of $\mathbb{L}$), and the normal measures and are not separated
by sets. Instead we use a more “accurate” forcing. We force with Magidor iteration of Prikry type forcings up to \( \kappa \), i.e., for every measurable cardinal \( \alpha < \kappa \), we do not force a cofinal Prikry sequence in \( \alpha \), but rather choose one point \( d(\alpha) < \alpha \). Therefore this adds a regressive and almost injective function \( d : \Delta \to \kappa \), where \( \Delta \) denotes the set of measurable cardinals below \( \kappa \). The upside of this forcing \( P^1 \) is that is we can arrange that \( 0^{\kappa_0}_{\alpha, \beta}(P^1) \) forces that \( j_{\alpha, \beta}(\dot{d})(\kappa') = \kappa \). Since \( d \) is almost injective then the induced \( d^{-1} \) can be used as \( f \), for mapping \( \kappa \) to \( \kappa' \). The downside to this forcing is that the embedding \( j_{\alpha, \beta} \) does not directly extend. The problem is that there are many other measurable cardinals \( \mu \in (\kappa, j_{\alpha, \beta}(\kappa)) \) beside \( \kappa' = j_{U_\alpha}(\kappa) \), and there are no possible candidates for suitable generic Prikry points \( d(\mu) \) for such \( \mu \). In order to fix this, we take a additional ultrapower at each such \( \mu \), moving \( \mu \) to \( \mu' > \mu \) which now becomes measurable, and has \( \mu \) as a possible generic Prikry point (i.e. we will have \( d(\mu') = \mu \)). We will denote this suitable iterated ultrapower by \( k_{\alpha, \beta} : M_{\alpha, \beta} \to Z_{\alpha, \beta} \) and show that the composition \( \pi_{\alpha, \beta} : V \to Z_{\alpha, \beta} \) extends to \( \pi^1_{\alpha, \beta} : V^1 \to Z^1_{\alpha, \beta} \) which satisfies the required properties, i.e. every ordinal is of the form \( \pi^1_{\alpha, \beta}(h)(\kappa) \) for some \( h : \kappa \to \text{On} \) in \( V^1 \), and therefore \( \pi^1_{\alpha, \beta} \) are the embeddings of the form \( j_{U^1_{\alpha, \beta}} : V^1 \to M_{U^1_{\alpha, \beta}} \cong \text{Ult}(V^1, U^1_{\alpha, \beta}) \) mentioned above. Luckily, the additional iteration \( k_{\alpha, \beta} \) has no effect on \( \triangleleft(\kappa) \) in \( V^1 \). This is due to both the fact that \( cp(k_{\alpha, \beta}) > \kappa \), and the fact that all “new” measurable cardinals \( \mu \in (\kappa, j_{\alpha, \beta}(\kappa)) \) and their normal measures used to construct \( k_{\alpha, \beta} \), are all solely determined by \( U_\alpha \) and \( U_\beta \).

The second ingredient in our forcing construction of \( V^1 \) is the Friedman Magidor poset in \( [6] \). The purpose of this poset is to make enough room to deal with many \( \alpha < \beta < \lambda \) in a single Magidor iteration of Priky type forcings. Fix some \( \beta < \lambda \), and denote \( \Delta_{\beta} = \{ \nu < \kappa \mid o(U_\nu) = \beta \} \). For every \( \alpha < \beta \) then if we want the above description of \( U^1_{\alpha, \beta} \) to work, we need to make a specific choice of the Prikry forcing at \( \nu \in \Delta_{\beta} \). (Basically, we need to choose a measure \( U^*_{\nu} \) on \( \nu \) so that \( o(U^*_{\nu}) = \alpha \)). So different \( \alpha \) values dictates different forcing choices at \( \nu \in \Delta_{\beta} \) points, and so we will have to choose only a single value \( \alpha < \beta \) for which a measure of the form \( U^1_{\alpha, \beta} \) exists in \( V^1 \). We use the Friedman Magidor poset to overcome this. We apply it before the Magidor iteration, i.e. we first force over \( V \) with a Friedman Magidor forcing, which we denote by \( P^0 \). In a generic extension \( V^0 = V[G^0], G^0 \subset P^0 \), we have pairwise disjoint sets \( \langle \Delta(\gamma) \mid \gamma < \kappa \rangle \), and for every \( \beta < \lambda \) then \( U_\beta \) extends in \( V^0 \). Its extensions are \( \langle U_\beta(\gamma) \mid \gamma < \kappa \rangle \) so
that $\Delta(\gamma) \in U_\alpha(\gamma)$. The idea is to now use the different separated extensions of $U_\beta$ to deal with different $\alpha < \beta$, i.e. We will define a Magidor iteration of single point Prikry forcings, denoted $P^1$ so that the choice of forcings at points $\nu \in \Delta(\alpha) = \Delta_\beta \cap \Delta(\alpha)$ will guarantee the existence of $U_{(\alpha, \beta)}^1$ in a $V^0$ generic extension $V^1 = V^0[G^1] = V[G^0 * G^1]$, where $G^1 \subset P^1$ is $P^1$ generic over $V^0$.

3 Analysis of Tame Orders

For every ordinal $\lambda \in On$ we define an order $(R_\lambda, <_{R_\lambda})$, $\lambda < \kappa^+$, and use these to define the class of tame orders.

**Definition 3.1.**
1. $R_\lambda = \{(\alpha, \beta) \in \lambda^2 \mid \alpha \leq \beta\}$, and for every $\alpha \leq \beta < \lambda$ and $\alpha' \leq \beta' < \lambda$, then $(\alpha', \beta') <_{R_\lambda} (\alpha, \beta)$ if and only if $\beta' < \alpha$.
2. An order $(S, <_S)$ is tame, if it is isomorphic to a restriction of $<_{R_\lambda}$ to a sub domain, i.e. , if there exists some $X \subset R_\lambda$ so that $(S, <_S) \cong (X, R_\lambda \upharpoonright X)$.

Given two orders $(S, <_S)$ and $(R, <_R)$ we say that a map $\pi : S \to R$ is an embedding of $(S, <_S)$ in $(R, <_R)$, if and only if its trivial extension to $\pi : S \times S \to R \times R$ induces an isomorphism of $(S, <_S)$ with $(X, <_R \upharpoonright X)$ for $X = \text{rng}(\pi) \subset R$ We say that $(R, <_R)$ embeds $(S, <_S)$, or that $(S, <_S)$ is embedded in $(R, <_R)$ if there exists such an embedding $\pi$. When there is no danger of confusion, we abbreviate, and write that $R$ embeds $S$. Therefore, we clearly have that for an order $(S, <_S)$, then it is tame if and only if there it is embedded in $R_\lambda$ for some ordinal $\lambda$.

For every $(S, <_S)$ which is embedded in some $R_\lambda$, then the main theorem (Theorem 1.1) implies that the consistency of $o(\kappa) \geq \lambda$ for $\lambda < \kappa$, implies there exists a model of set theory with $<_(\kappa) \cong (S, <_S)$.

The main of this section is to study the class of tame orders. It turns out that tame orders have a natural (internal) description. The key order in this study is the (non tame) order $(R_{2,2}, <_{R_{2,2}})$, defined on a set of four elements $R_{2,2} = \{x_0, x_1, y_0, y_1\}$, by $<_{R_{2,2}} = \{(x_0, x_1), (y_0, y_1)\}$ (illustrated in diagram 2 above). We claim that for every $\lambda$, then $R_{2,2}$ does not embed into $R_\lambda$. Indeed, note that if $(n_0, n_1), (N_0, N_1), (m_0, m_1), (M_0, M_1)$ are four points in $R_\lambda$ so that $(n_0, n_1) <_{R_\lambda} (N_0, N_1)$ and $(m_0, m_1) <_{R_\lambda} (M_0, M_1)$, then $n_0, n_1 < N$ where $N = \max(N_0, M_0)$. 

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We turn to study arbitrary orders \((R, <_R)\) which do not embed \(R_{2,2}\). We begin with the following key observation.

**Observation 3.2.** Suppose that \((R, <_R)\) does not embed \(R_{2,2}\). Then for any \(x, x' \in R\) the relation \(\subset\) compares \(U(x)\) with \(U(y)\), and \(D(x)\) with \(D(y)\).

**Proof.** Fix two distinct elements \(x, x' \in R\), then either \(U(x) \subset U(x')\) or \(U(x') \subset U(x)\). Suppose otherwise, then there are \(y, y'\) so that

1. \(x <_R y, x' \not<_R y, \) and
2. \(x' <_R y', x \not<_R y'\).

This is impossible as it follows that \(<_R\{x, x', y, y'\}\) is isomorphic to \(R_{2,2}\).

Similarly, we get that for every \(x, x' \in R\) with \(x \neq x'\), then either \(D(x) \subset D(x')\) or \(D(x') \subset D(x)\).

\[\square\]

If \((S, <_S)\) does not embed \(R_{2,2}\), and define sets \(D(S) = \{D(x) | x \in S\}\) and \(U(S) = \{U(x) | x \in S\}\). It follows that \(D(S), U(S)\) are both linearly ordered by \(\subseteq\). We will be interested in the linear orders \((D(S), \subseteq)\) and \((U(S), \supseteq)\) (i.e. the reverse subset order on \(U(S)\)). We add the following notations,

**Definition 3.3.** For every \(x \in S\) let \(CU(x) = S \setminus U(x)\) be the complement of \(U(x)\). For \(x, y \in S\) then clearly \(U(x) \supseteq U(y)\) if and only if \(CU(x) \subseteq CU(y)\) so if \(CU(S) = \{CU(x) | x \in S\}\), then \((CU(S), \subseteq) \cong (U(S), \supseteq)\). So whenever comfortable, we will work with \((CU(S), \subseteq)\) instead of the equivalent \((U(S), \supseteq)\).

For every \(x \in S\), define sets \(<_D (x) \subset D(S)\) and \(<_{CU} (x) \subset CU(S)\) by \(<_D (x) = \{D(y) \in D(S) | D(y) \not\subseteq D(x)\}\), and \(<_{CU} (x) = \{CU(y) \in CU(S) | CU(y) \not\subseteq CU(x)\}\).

By the observation above, if \((S, <_S)\) does not embed \(R_{2,2}\), then for every \(x \in S\), \(<_D (x), \subseteq)\) is an initial segment of the linear \((D(S), \subseteq)\), and \(<_{CU} (x), \subseteq)\) is an initial segment of the linear \((CU(S), \subseteq)\).

We will see below that every finite order \((R, <_R)\) which does not embed \(R_{2,2}\) is tame, i.e., there exists an embedding from \((R, <_R)\) to \((R_n, <_{R_n})\) for some \(n < \omega\). Let us describe this construction. We have that both \(D(R), CU(R)\) are finite, so we can use the linear finite orders \((D(R), \subseteq)\), \((CU(R), \subseteq)\) to associate every \(x \in R\) a pair of integers \((m(x), M(x)) \in |D(R)| \times |CU(R)|\) so that \(m(x)\) is the position (rank) of \(D(x)\) in \((D(R), \subseteq)\),
and $M(x)$ is the position of $CU(x)$ in $(CU(R), \subset)$. In our notations, we can write this by $m(x) = |<_D(x)|$ and $M(x) = |<_{CU}(x)|$. Lemma 3.6 below implies that the map $\pi : R \to R_n$ defined by $\pi(x) = (m(x), M(x))$ is an embedding of $(R, <_R)$ in $(R_n, <_{R_n})$ where $n = |CU(R)|$. By the definition of $<_{R_n}$, for this to work we will need to verify the following three properties:

1. $m(x) \leq M(x)$ for all $x \in R$,
2. if $x <_R y$ then $M(x) < m(y)$, and
3. if $x \not<_R y$ then $M(x) \geq m(y)$.

Note that for an infinite order $(R, <_R)$ the construction above may not work to produce a map $\pi : R \to R_\lambda$ for some ordinal $\lambda$ since the linear orders $(D(R), \subset)$ and $(CU(R), \subset)$ may not be well founded and so we cannot associate ordinals to $m(x), M(x)$. It therefore makes sense to add the well founded as a required property.

**Definition 3.4.** A order $(R, <_R)$ is called a pseudo-tame order if it does not embed a copy of $R_{2,2}$ and $(D(R), \subset)$ and $(CU(R), \subset)$ are well founded.

**Remark 3.5.** We note that the well foundness of $(D(R), \subset)$ implies that $(R, <_R)$ is well founded, so we can omit it from the definition of pseudo-tame orders. Furthermore, it is clear that every finite order which do not embed $R_{2,2}$ is pseudo-tame. Finally, we note that if $(R, <_R)$ is pseudo-tame then every restriction suborder of $(R, <_R)$ is also pseudo-tame.

It is not difficult to verify that $(R_\lambda, <_{R_\lambda})$ is pseudo-tame for every ordinal $\lambda$. We leave this the reader. It follows by Remark 3.5 that every tame order is pseudo-tame. The rest of this section is devoted to prove that every pseudo-tame order is tame. The following Lemma will establish this to to finite orders.

**Lemma 3.6.** Let $(R, <_R)$ be a pseudo tame order, then

1. for every $x \in R$, then there exists an injection $f : <_D(x) \to <_{CU}(x)$ which is $\subset$ order preserving,
2. for every $x <_R y$ then there exists an injection $g : <_{CU}(x) \to <_D(y)$ which is $\subset$ order preserving, and $\emptyset \not\in \text{rng}(g)$, and

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3. for every $x \not<_R y$, then there exists an injection $h : <_D (y) \rightarrow <_{CU} (x)$ which is $\subseteq$ order preserving.

Proof.

1. Let $z \in R$ so that $D(z) \subseteq D(x)$. Since $(D(R), \subseteq)$ is well founded, then $D(z)$ has an immediate successor $D(z^+)$ in $(D(R), \subseteq)$, and clearly, $D(z) \subseteq D(z^+) \subset D(x)$ (note that this is not necessarily true that $z <_R z^+$). So if $y \in D(z^+) \setminus D(z)$, we get that for every $z'$ with $D(z) \subset D(z')$ then $y \in D(z')$. For each such $z$, pick an elements $y_z \in D(z^+) \setminus D(z)$. Define $f : <_D (x) \rightarrow <_{CU} (x)$ by $f(D(z)) = CU(y_z)$. Note that $z \in f(D(z))$ and if $z'$ satisfies that with $D(z) \subset D(z')$ then $z' \notin f(D(z))$. Since $CU(R)$ is linearly ordered by $\subseteq$, it follows that $f(D(z)) \subseteq f(D(z'))$ whenever $D(z) \subset D(z')$. Therefore $f$ is injective, and $\subseteq$ order preserving. Looking at the construction of $y_z$, it can be easily seen to be depend only on the set $D(z)$ and not on the specific value of $z$. So the function $f$ is well defined.

2. Suppose that $x <_R y$. Let $z \in R$ so that $CU(z) \subseteq CU(x)$. Since $(CU(R), \subseteq)$ is well a well order, then $CU(z)$ has an immediate successor in $(CU(R), \subseteq)$ and $CU(z) \subseteq CU(z^+) \subset CU(x)$. Of course $U(z^+) \subseteq U(z)$. We claim that for every $w \in U(z) \setminus U(z^+)$ then $D(w) \subseteq D(y)$. Indeed, we have that $U(x) \subset U(z^+)$, hence $w \notin U(x)$ i.e., $x \not<_R w$. Since for every $w' <_R w$ then $<_R | \{x, y, w, w'\}$ is not isomorphic to $R_{2,2}$, we must have that $w' <_R y$, thus $D(w) \subset D(y)$. The two sets are not equal as $x \in D(y) \setminus D(w)$.

It is clear from the choice of $w$ that $z <_R w$ and if $z'$ is any element of $R$ with $CU(z) \subseteq CU(z') \subset CU(x)$, then $z \in U(z) \setminus U(z')$ and so $z \in D(w)$ but $z' \notin D(w)$. It follows that if we pick $w_z \in U(z) \setminus U(z^+)$ for any suitable $z$, then the map $g : <_{CU} (x) \rightarrow <_D (y)$ define by $g(CU(z)) = D(w_z)$, is an injection and $\subseteq$ preserving. Furthermore, note that $z \in g(D(z))$ whenever $CU(z) \subset CU(x)$, hence $\emptyset \notin <_D (y) \setminus \text{rng}(g)$. Similarly to the construction of $f$ above, the values $w_z$ depends on $D(z)$ and not on a specific value $z$, so $g$ is well defined. For a reason which will become clear later (i.e., in the proof of Proposition 3.9), we would like our choice of the values $w_z$ to be more careful. For each $z$ with $D(z) \subset D(x)$, we picked $w_z$ from the set $D(z) \setminus D(z^+)$. Consider the family $\Gamma_z = \{D(w) \mid w \in U(z) \setminus U(z^+)\}$. Since $\subseteq$ well orders $D(R)$,
then there exists a $\in\subset$ minimal element $D \in \Gamma$. Let us pick $w_z$ to be a value so that $D(w_z) = D$.  

3. Suppose that $x \neq_R y$. Let $z \in R$ with $D(z) \subset D(y)$, and let $z^+$ be so that $D(z^+)$ is an immediate successor of $D(z)$ in $(D(R), \subset)$. In particular $D(z) \subset D(z^+) \subset D(y)$. We claim that for every $w \in D(z^+) \setminus D(z)$ then $U(x) \subset U(w)$. Take $x' \in R$ with $x <_R x'$ and look at $<_R \{x, y, w, y\}$. Since this is not isomorphic to $R_2$, and $x \neq_R y$ then we must have that $w <_R x'$. So $U(x) \subset U(w)$ and $U(x) \neq U(w)$ since $y \in U(w) \setminus U(x)$. Equivalently, we get that $CU(w) \subset CU(x)$. For every such $z \in R$, choose $w_z \in D(z^+) \setminus D(z)$, and define $h : <_D(y) \rightarrow <_{CU}(x)$ by $h(D(z)) = CU(w_z)$. Finally, if $z' \in R$ satisfies that $D(z) \subset D(z')$ then $D(z^+) \subset D(z')$ so $w_z <_R z'$ so $z' \in CU(w_z) \setminus CU(w_z)$. It follows that $h$ is $\subset$ preserving. \hfill \Box

Before proving the main result of this section we list some immediate corollaries of the Lemma.

**Corollary 3.7.** Let $(R, <_R)$ be a pseudo-tame order. Since it is well-founded it has a rank denoted rank$(R, <_R)$. Note that for every $x < R y$ then $D(x) \subset D(y)$. It follows that rank$(R, <_R) \leq$ rank$(D(R), \subset) = otp(D(R), \subset)$. By the first statement of Lemma 3.6 we find that otp$(D(R), \subset) \leq$ otp$(CU(R), \subset)$. We conclude that  

$$\text{rank}(R, <_R) \leq \text{otp}(D(R), \subset) \leq \text{otp}(CU(R), \subset).$$

**Corollary 3.8** (Finite Tame orders). Lemma 3.6 implies that every finite order which does not embed $R_2$ is tame. Indeed, recall the map $\pi(x) = (m(x), M(x))$ described above. By the previous corollary, we have that for every $x \in R$ then $m(x), M(x) \leq \max(|D(R)|, |CU(R)|) = |CU(R)|$.

So $\pi : R \rightarrow R_n, n = |CU(R)|$. By the first statement of Lemma 3.6 we conclude that for $x \in R$ then $m(x) = |<_D(x)| \leq |<_CU(x)| = M(x)$.

Using the second statement of the Lemma, we get that for $x <_R y$, then $M(x) = |<_CU(x)| < |<_D(y)| = m(y)$. Finally, by the third statement of the Lemma, we conclude that for $x \neq_R y$ then $m(y) = |<_D(y)| \leq |<_CU(x)| = M(x)$. It follows that $\pi$ is an embedding of $(R, <_R)$ in $(R_n, <_{R_n})$.

We state the main result of this section.

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5 With this last distinction, we get that the construction of $g$ does not require any use of the axiom of choice. The same can be shown for more careful constructions of $f$ and $h$. 

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Proposition 3.9. Let \((R, <_R)\) be a pseudo-tame order then \((R, <_R)\) is tame. Furthermore, if \(\lambda = \text{otp}(CU(R), \subset)\), then \((R, <_R)\) can be embedded in \((R_\lambda, <_{R_\lambda})\).

The next example illustrates why the analogue of the function \(\pi : R \to R_n\) defined for finite pseudo tame orders does not work for arbitrary infinite pseudo tame orders. However we will see in the proof of Proposition 3.9 that a simple modification of this construction does work in general.

Example 3.10. The trivial generalization of the construction \(\pi(x) = (m(x), M(x))\) used in the finite case will be to define \(m(x) = \text{otp}(<_D (x))\) and \(M(x) = \text{otp}(<_CU (x))\). Consider the reduced pseudo tame order \((R, <_R)\) defined on a set \(R = \{y\} \cup \{x_n \mid n < \omega\} \cup \{x_\omega\}\). The relation \(<_R\) linearly orders the set \(\{x_i \mid i \leq \omega\}\) so that \(x_i <_R x_j\) iff \(i < j\). \(<_R\) compares the additional element \(y\) with \(x_\omega\) by \(y <_R x_\omega\). I.e. \((R, <_R)\) is illustrated by

![Diagram](image)

The following tables express the \(\subset\) separates the elements \(x \in R\) into common \(D(x)\) \((CU(x))\) classes, and list them according to the order types of \(<_D(x)\) \((<_CU(x))\).

| \(x \in R\) | \(y, x_0\) | \(x_1\) | \(\ldots\) | \(x_n\) | \(\ldots\) | \(x_\omega\) |
|-------------|-------------|---------|----------|--------|--------|--------|
| \(D(x)\)    | \(\emptyset\) | \(\{x_0\}\) | \(\ldots\) | \(\{x_i\}_{i<n}\) | \(\ldots\) | \(\{x_n\}_{n<\omega} \cup \{y\}\) |
| \(\text{otp}(<_D(x), \subset)\) | 0            | 1       | \(\ldots\) | \(n\)   | \(\ldots\) | \(\omega\) |

| \(x \in R\) | \(x_0\) | \(x_1\) | \(\ldots\) | \(x_n\) | \(\ldots\) | \(y\) | \(\ldots\) | \(x_\omega\) |
|-------------|--------|---------|----------|--------|--------|------|--------|--------|
| \(CU(x)\)  | \(\{y\}\) | \(\{y, x_0\}\) | \(\ldots\) | \(\{y\} \cup \{x_i\}_{i<n}\) | \(\ldots\) | \(\{y\} \cup \{x_n\}_{n<\omega}\) | \(R \setminus \{x_\omega\}\) |
| \(\text{otp}(<_CU(x), \subset)\) | 0            | 1       | \(\ldots\) | \(n\)   | \(\ldots\) | \(\omega\) | \(\omega+1\) |

The following illustrates the function \(\pi(x) = \langle \text{otp}(<_D(x)), \text{otp}(<_CU(x)) \rangle\) for \(x \in R\).
Looking at the ordinal pair \( \pi(x) = (\upsilon_D(x), \upsilon(CU(x))) \) associated to each \( x \in R \), we find that this fails to described \( <_R \) in terms of \( <_{R^{\omega+1}} \) in a single place. We have that \( \pi(y) = (0, \omega) \) while \( \pi(x_\omega) = (\omega, \omega + 1) \), which fails to express that \( y <_R x_\omega \).

This can easily solved. If we only replace \( \pi(x_\omega) = (\omega, \omega + 1) \) by \( (\omega + 1, \omega + 1) \) then we would get a valid embedding of \( R \) in \( R^{\lambda} \). Furthermore, this modification does not require any change in the \( <_{CU} \) values, which implies that the modified map still gives values in \( R^{\upsilon(CU(R), \subseteq)} = R^{\omega+1} \).

The “correct” way to think the problem in the original \( \pi \) map, is that the original collection \( D(R) = \{\emptyset, \{x_0\}, \{x_0, x_1\}, \ldots, \{x_i\}_{i<n}, \ldots, \{y\} \cup \{x_i\}_{i<\omega}\} \) is not \( \subseteq \) complete. The missing “hole” is the set \( \{x_i\}_{i<\omega} \) which is the union of the sets \( \{x_i\}_{i<n} \mid n < \omega \} \subset D(R) \). In order to fix this, we take the completion \( \bar{D}(R) = D(R) \cup \{x_i\}_{i<\omega} \). of \( D(R) \). One can see that the induced map \( \bar{\pi} \) by using \( \bar{D}(R) \) and \( CU(R) \), has an effect only on \( x_\omega \) and indeed \( \bar{\pi}(x_\omega) = (\omega + 1, \omega + 1) \). In proposition 3.9 we will show that by using a similar completion \( \bar{D}(R) \) of \( D(R) \) for an arbitrary pseudo tame order \( (R, <_R) \), we get an embedding of \( (R, <_R) \) in \( (R^{\lambda}, <_{R^{\lambda}}) \) for \( \lambda = \upsilon(CU(R), \subseteq) \).

**Definition 3.11.** Let \( (R, <_R) \) be a pseudo-tame order. We define \( \bar{D}(R) \) to be the completion of \( D(R) \) under increasing sequences in \( \subseteq \). Elements \( D \in \bar{D}(R) \setminus D(R) \) are sets \( D = \bigcup C \) so that \( C \subset D(R) \) satisfies that

1. \( C \) is \( \subseteq \) downwards closed, i.e. for all \( D_1 \subset C \) and \( D_2 \in D(R) \) so that \( D_2 \subset D_1 \), then \( D_1 \in C \).
2. There is no \( x \in R \) so that \( \bigcup C = D(x) \).

Since \( \bar{D}(R) \) adds only \( \subseteq \) limit elements to \( D(R) \), then it is easy to verify that \( \subseteq \) still well orders \( \bar{D}(R) \).
Proof. (Proposition 3.9)
Let \((R, <)\) be a pseudo tame order. For every \(x \in R\), define \(\bar{\pi}(x) = (\bar{m}(x), M(x))\) where \(\bar{m}(x), M(x)\) defined by \(\bar{m}(x) = \text{otp}(<_D (x), \subseteq)\), and \(M(x) = \text{otp}(<_\mathcal{CU} (x), \subseteq)\). Let \(\lambda = \text{otp}(\mathcal{CU}(R), \subseteq)\) We claim that \(\bar{m}\) is an embedding of \((R, <)\) in \((R_\lambda, <_{R_\lambda})\).

Fix some \(x \in R\). Recall the function \(f : <_D (x) \to <_{\mathcal{CU}} (x)\) defined in Lemma 3.6. In order to show that \(\text{rng}(\bar{\pi}) \subset R_\lambda\), it is sufficient to show that \(f\) extends to a \(\subseteq\) -preserving function \(\bar{f} : <_{\bar{D}} (x) \to <_{\mathcal{CU}} (x)\). For \(D \in <_{\bar{D}} (x) \cap D(R)\), let \(D^+\) be its \(\subseteq\) immediate successor in \(\bar{D}(R)\). We must have that \(D^+ = D(z^+)\) for some \(z^+ \in R\), so \(D \subset D(z^+) \subset D(x)\). Choose any \(w_D \in D(z^+) \cap D\), then \(w_D <_R x\), therefore \(U(x) \not\subseteq U(w_D)\) and \(\mathcal{CU}(w_D) <_{\mathcal{CU}} (x)\). Define \(\bar{f}(D) = \mathcal{CU}(w_D)\). To show that \(\bar{f}\) is still \(\subseteq\) preserving, it is sufficient to show that \(\bar{f}(D) \subseteq \bar{f}(D(z'))\) whenever \(D \subset D(z) \subset D(x)\). By the definition of \(D^+ = D(z^+)\), we must have that \(D(z^+) \subset D(z')\), thus \(w_D <_R z'\). It follows that \(z' \not\in \mathcal{CU}(w_D) = \bar{f}(D)\), however the construction of \(\bar{f}(D(z'))\) in Lemma 3.6 guarantees that \(z' \in \bar{f}(D(z'))\). Since \(\subseteq\) linearly orders \(\mathcal{CU}(R)\), it follows that \(\bar{f}(D) \not\subseteq \bar{f}(D(z'))\). Next, let \(x, y \in R\) so that \(x <_R y\). In Lemma 3.6 we defined a function \(g : <_{\mathcal{CU}(x)} \to <_{\bar{D}} (y)\) and showed it is \(\subseteq\) preserving. Clearly, \(g\) is also \(g \subseteq\) preserving function to \(<_{\bar{D}} (y)\). In order to prove that \(M(x) < \bar{m}(y)\), we verify that \(\text{otp}(\text{rng}(g), \subseteq) < \text{otp}(<_D (y))\). We note that \(D(x) \in <_{\bar{D}} (y)\) so \(\text{otp}(<_D (y))\) is not empty. Let \(\text{otp}(<_D (y)) = \rho + k\) where \(\rho\) is a limit ordinal and \(k < \omega\) is finite. Let \(\langle D_i \mid i < \rho + k \rangle\) be a \(\subseteq\) order preserving enumeration of \(<_{\bar{D}} (y)\). Since \(\rho\) is limit, and \(\bar{D}(R)\) is \(\subseteq\) complete, then \(D_\rho = \bigcup_{i < \rho} D_i\). We claim that \(D_\rho \not\subseteq \text{rng}(\bar{\pi})\). We consider three possible options:

1. \(\rho = 0\). Then \(D_\rho = \emptyset \not\subseteq \text{rng}(g)\) by Lemma 3.6
2. \(D_\rho \in \bar{D}(R) \setminus D(R)\). Then \(D_\rho \not\subseteq \text{rng}(g)\) since \(\text{rng}(g) \subset D(R)\).
3. \(D_\rho = D(w')\) for some \(w' \in R\). Then the fact that \(D_\rho \not\subseteq \text{rng}(g)\) follows from our specific choice of the value \(w_z\) used in the definition of \(g(\mathcal{CU}(z)) = D(w_z)\) given in the proof of Lemma 3.6. The value \(w_z\) was chosen so that \(D(w_z)\) is the \(\subseteq\) minimal in \(\Gamma_z = \{D(w) \mid w \in U(z) \setminus U(z^+)\}\), where \(z^+ \in R\) is so that \(\mathcal{CU}(z^+)\) is the \(\mathcal{CU}(z)\) immediate extension in \(\subseteq\). It is sufficient to show that if \(D_\rho = D(w') \in \Gamma_z\) then it is not its \(\subseteq\) minimal. If \(D(w') \in \Gamma_z\) then \(z <_R w'\) so \(z \in D(w')\). Since \(D(w') = D_\rho = \bigcup_{i < \rho} D_i\), then there exists a successor ordinal \(i < \rho\) so that \(z \in D_i\). Let \(w_i \in R\) be an element so that \(D_i = D(w_i)\),
then $z <_R w_i$ and $z^+ \not<_R w_i$. It follows that $D(w_i) \in \Gamma_z$ and that $D_\rho = D(w')$ is not $\not<_R$ minimal.

Finally, suppose that $x, y \in R$ so that $x \not<_R y$. In Lemma 3.6 we defined a $\not<_R$ preserving function $h : <_D (y) \rightarrow <_{CU} (x)$. In order to prove that $\bar{m}(y) \leq M(x)$ it is sufficient to show that the definition of $h(D)$ for $D = D(z) \in <_D (y)$ naturally extends to $D \in <_D (y)$ and induces a $\not<_R$ preserving function $h : <_D (y) \rightarrow <_{CU} (x)$. Let $D \in <_D (y) \setminus D(R)$ and let $z^+ \in R$ so that $D(z^+)$ is the immediate $\not<_R$ successor of $D$. Therefore $D \not<_R D(z^+) \subset D(y)$. Choose any $w_D \in D(z^+) \setminus D$. Finally, if $z' \in R$ satisfies that $D \not<_R D(z')$, then $D(z^+) \subset D(z')$ thus $w_D <_R z'$ so $z' \in h(D(z')) \setminus CU(w_D)$. The proposition follows. \hfill $\square$ Proposition 3.9

4 The Posets $\mathcal{P}^0$ and $\mathcal{P}^1$

The purpose of this section is to introduce the posets $\mathcal{P}^0$ and $\mathcal{P}^1$ which are the main ingredient in the construction of $\triangleleft \langle \kappa \rangle \equiv R_\lambda$ from the assumption $o(\kappa) \geq \lambda$ in $V$. Both posets $\mathcal{P}^0, \mathcal{P}^1$ are variants of well known forcings:

- $\mathcal{P}^0$ will one of the posets introduced by Friedman and Magidor in [6]. We will survey its definition and main feature below.

- $\mathcal{P}^1$ is a Magidor iteration of Prikry type forcings. The Madigor iteration of Prikry type forcings was introduced by Magidor in [11] (See also [8] for an extensive survey). The behavior of $\triangleleft \langle \kappa \rangle$ in some specific generic extension by Magidor iteration of Prikry forcings was studied in [3]. We will rely on the results of this work in some of our arguments.

In general, the definition of $\mathcal{P}^0, \mathcal{P}^1$ depends on the value $\lambda = o(\kappa)$ in the ground model $V$. To simplify the notations in this section, we will first present a version of $\mathcal{P}^0, \mathcal{P}^1$ which fits cases where $\lambda \leq \kappa$. The general construction, used to realized $R_\lambda$ for arbitrary values $\lambda < \kappa^+$, will be given in the section 8.

Our assumptions of the ground model $V$ is a Mitchell model $V = K(V) = L[U]$ (14) so that $U = \langle U_{\kappa, \tau} : \nu \leq \kappa, \tau < o(\alpha) \rangle$ is a coherent sequence, with $o(\kappa) = \lambda \leq \kappa$. For $\nu = \kappa$, we abbreviate and write $U_\tau$ for $U_{\kappa, \tau}$, for every $\tau < \lambda$. For every $\tau < \lambda$ set $\Delta_\tau = \{\nu < \kappa : o(\nu) = \tau\}$. Since $\lambda \leq \kappa$ then the sets $\{\Delta_\tau : \tau < \lambda\}$ are pairwise disjoint.
For every $\alpha < \lambda$, let $j_\alpha : V \to M_\alpha \cong \text{Ult}(V, U_\alpha)$ be the induced ultrapower embeddings. If $\alpha < \beta < \lambda$, then $U_\alpha \in M_\beta$. Let $j_{\alpha,\beta}^M : M_\beta \to M_{\alpha,\beta} \cong \text{Ult}(M_\beta, U_\alpha)$, and $j_\alpha,\beta = j_{\alpha,\beta}^M \circ j_\alpha : V \to M_{\alpha,\beta}$. There is an equivalent way to the embedding $j_{\alpha,\beta}$: Trivially, $j_\alpha(U_\beta) \in M_\alpha$. Let $i_{\alpha,\beta}^M : M_\alpha \to M_{\alpha,\beta} \cong \text{Ult}(M_\alpha, j_\alpha(U_\beta))$ denote the induced ultrapower embedding of $M_\alpha$ by $j_\alpha(U_\beta)$, then $j_{\alpha,\beta} = i_{\alpha,\beta}^M \circ j_\alpha$. Note that the critical points of the iteration $i_{\alpha,\beta}^M \circ j_\alpha$ are $\kappa < j_\alpha(\kappa)$ (i.e. the iteration is normal). Furthermore, note that since $j_\beta(\kappa) > \kappa$ is inaccessible in $M_\beta$, then $j_{\alpha,\beta}(\kappa) = j_{\alpha,\beta}^M(j_\beta(\kappa)) = j_\beta(\kappa)$.

### 4.1 The Poset $\mathcal{P}^0$

For $\mathcal{P}^0$, we use a poset introduced by Friedman and Magidor in [6]. $\mathcal{P}^0 = \mathcal{P}^0_\kappa = \langle \mathcal{P}_\nu, Q_\nu \mid \nu \leq \kappa \rangle$ is a non-stationary support iteration. Conditions $p \in \mathcal{P}^0_\nu$ are denoted by $p = \langle p_{\mu} \mid \mu < \nu \rangle$. Non-stationary support means that for every limit $\nu \leq \kappa$, then every $p \in \mathcal{P}^0_\nu$ belongs to the inverse limit of the posets $\langle \mathcal{P}^0_\mu \mid \mu < \nu \rangle$ with the restriction that if $\nu$ is inaccessible, then the set of $\mu < \nu$ such that $p_\mu$ is nontrivial is a non stationary subset of $\nu$. For every $\nu \leq \kappa$, if $\nu$ is non-inaccessible then $\Vdash_{\mathcal{P}^0_\nu} \dot{Q}^1_\nu = \emptyset$. Otherwise, $\Vdash_{\mathcal{P}^0_\nu} \dot{Q}^1_\nu = \emptyset$. For every $\nu < \kappa$, let $\lambda(\nu) = \begin{cases} \lambda & \text{if } \lambda < \kappa, \\ \nu & \text{if } \nu = \kappa. \end{cases}$

Conditions in $\text{Sacks}_\nu(\nu)$ are trees $T \subset \text{Sacks}_\nu(\nu)$ so that for some club $C \subset \nu$, then for every $s \in T$, if $\text{len}(s) \in C$, then $s \sim (i) \in T$ for every $i < \lambda(\text{len}(s))$, and $\text{Code}(\nu)$ a coding posets which adds a club set to $\nu$ which codes the Sacks$_{\lambda(\nu)}(\nu)$ generic Sacks function $s_\nu : \nu \to \nu$, by destroying stationary sets of $\text{Cof}(\nu) \cap \nu^+$ in $V$ (see [6] for more details). Let $G^0 \subset \mathcal{P}^0$ generic filter over $V$. For every $\nu \leq \kappa$ $G^0(\mathcal{Q}^0_\nu) = \bigcup \{ p_{\nu[G^0]^\kappa} \mid p \in G^0 \}$ be the $G^0$ induced $\mathcal{Q}^0_\nu$ generic filter over $V[G^0] \upharpoonright \nu$. For every non trivial stage $\nu \leq \kappa$ in the iteration $\mathcal{P}^0$, let $s_\nu : \nu \to \lambda(\nu)$ denote the generic Sacks function specified by $G^0(\mathcal{Q}^0_\nu)$. For every $\eta < \lambda$ let

$$\Delta(\eta) = \{ \nu < \kappa \mid s_\nu = s_\kappa \upharpoonright \nu, s_\kappa(\nu) = \eta \}.$$  

Clearly, the sets in $\{ \Delta(\eta) \mid \eta < \lambda \}$ are pairwise disjoint. By the results in [6], then the following holds in $V[G^0]$.

**Fact 4.1.** 1. $V[G^0]$ agrees with $V$ on all cardinals and cofinalities.

2. For every normal measures $U$ on $\kappa$, in $V$, has exactly $\kappa$ many extensions $\langle U(\eta) \mid \eta < \kappa \rangle$ in $V[G^0]$, so that $\Delta(\eta) \in U(\eta)$.
3. For every $\eta < \lambda$, let $j_{U(\eta)} : V[G^0] \to M^0_{U(\eta)} \cong \text{Ult}(V[G^0], U(\eta))$ be the induced ultrapower embedding of $V[G^0]$ by $U(\eta)$, then

(a) $j_{U(\eta)} \restriction V = j_U : V \to M_U \cong \text{Ult}(V, U)$ is the induced ultrapower embedding of $V$ by $U$.

(b) $M^0_{U(\eta)} = M_U[G^0_{U(\eta)}]$, where $G^0_{U(\eta)} \subset j_U(\mathcal{P}^0)$ is $M_U$-generic.

(c) $G^0_{U(\eta)} \restriction \kappa + 1 = G^0$.

(d) $\bigcup j_{U(\eta)}^{-1}(G^0 \restriction \kappa)$ totally determines $G^0_{U(\eta)} \restriction j_U(\mathcal{P}^0) \restriction j_U(\kappa)$. In particular $G^0_{U(\eta)} \restriction j_U(\kappa)$ is independent of $\eta < \lambda$.

(e) The $G^0_{U(\eta)} \restriction j_U(\kappa)$ interpretation of the $j_U(\mathcal{P}^0) \restriction j_U(\kappa)$—name $\bigcap j_{U(\eta)}^{-1}(G^0(\check{Q}_\kappa))$ (i.e., the intersection of all the trees of $j_U(\kappa) < j_U(\kappa)$ of the form $(j_U(\hat{p})_{j_U(\kappa)}(G^0_{U(\eta)})))$ is a tuning fork which consists of a single branch up to level $\kappa$, which totally splits to $\lambda$ extensions, each followed by the same single branch of length $j(\kappa)$, and for every $\eta < \lambda$, the generic $G^0_{U(\eta)}$ is determined by the branch which takes the value $\eta$ at level $\kappa$, i.e., forces that $s_{j_U(\kappa)}(\kappa) = \eta$.

(f) For every $p \in G^0$ let $j_U(p)(\eta)$ be the condition obtained by shrinking the (name of level $\kappa$ of the tree $T = (j_U(p))_\kappa$ so that for each $s \in \text{Lev}_{\kappa+1}(T)$ then $s(\kappa) = \eta$. Then, for every $D \subset j_U(\mathcal{P}^0)$ dense open set, there exists some $p \in G^0$ so that $j_U(p)(\eta) \in D$.

Every normal measure $U$ on $\kappa$ in $V$ is of the form $U = U_\alpha$ for some $\alpha < \lambda$. For every $\eta < \lambda$, we add the following definitions.

**Definition 4.2.** 1. Let $U^0_{(\alpha, \eta)}$ denote the $V[G^0]$ extension $U(\eta)$ described above, for $U = U_\alpha$.

2. let $j^0_{(\alpha, \eta)} : V[G^0] \to M^0_{\alpha}(G^0_{\alpha}(\eta))$ denote the ultrapower embedding of $V[G^0]$ by $U^0_{(\alpha, \eta)}$.

By the above, it is clear that $j^0_{(\alpha, \eta)} \restriction V = j_\alpha$. $G^0_{\alpha}(\eta) = j^0_{(\alpha, \eta)}(G^0)$ denotes the $j_\alpha(\mathcal{P}^0)$ generic filter $G^0_{U(\eta)}$ described above, with $U = U_\alpha$.

For $\alpha < o(\kappa) = \lambda$ and $\eta < \lambda$, let $\Delta_\alpha(\eta) = \Delta(\eta) \cap \Delta_\alpha$. We get that

1. $\{\Delta_\alpha(\eta) \mid \alpha, \eta < \lambda\}$ are pairwise disjoint.

2. $\Delta_\alpha(\eta) \in U^0_{(\alpha, \eta)}$ for every $\alpha, \eta < \lambda$. 

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The description of the ultrapower embeddings by the measures $U^0_{(\alpha, \eta)}$ easily extends to iterated ultrapowers by these measures. By the results in [6], we conclude that $j_{\alpha, \beta} \cdot G^0$ determines a unique generic filter in $j_{\alpha, \beta}(P_0^0)$, up to two tuning forks. The first tuning fork determines the value of $s_{j_{\alpha}(\kappa)}(\kappa) < \lambda$, and the second determines the value of $s_{j_{\alpha, \beta}(\kappa)}(j_{\alpha}(\kappa)) < j_{\alpha}(\lambda)$. For every $\eta_\alpha < \lambda$, and $\eta_\beta < j_{\alpha}(\lambda)$, then there exists a unique extension of $j_{\alpha, \beta}$ to an embedding of $V[G^0]$ which determines the generic values $s_{j_{\alpha}(\kappa)}(\kappa) = \eta_\alpha$, and $s_{j_{\alpha, \beta}(\kappa)}(j_{\alpha}(\kappa)) = \eta_\beta$. For our generic construction, we will be only interested in values $\eta_\alpha, \eta_\beta$ which are both below $\lambda$.

**Notation 4.3.** Let us add the following notations:

1. Denote $V[G^0]$ by $V^0$.
2. For every $\alpha < \omega(\kappa) = \lambda$ and $\eta < \lambda$, let
   \[ j^0_{(\alpha, \eta)} : V^0 \rightarrow M^0_{(\alpha, \eta)} \]
   be an abbreviation for $j^0_{(\alpha, \eta)} : V[G^0] \rightarrow M_{\alpha}[G^0_\alpha(\eta)] \cong \text{Ult}(V^0, U^0_{(\alpha, \beta)})$.
3. For every $\eta_\alpha, \eta_\beta < \lambda$, let
   \[ j^0_{(\alpha, \eta_\alpha), (\beta, \eta_\beta)} : M^0_{(\alpha, \eta_\alpha)} \rightarrow M^0_{(\alpha, \eta_\alpha), (\beta, \eta_\beta)} \cong \text{Ult} \left( M^0_{(\alpha, \eta_\alpha)}, j^0_{(\alpha, \eta_\alpha)}(U^0_{(\beta, \eta_\beta)}) \right) \]
   denote the ultrapower of $M^0_{(\alpha, \eta_\alpha)}$ by $j^0_{(\alpha, \eta_\alpha)}(U^0_{(\beta, \eta_\beta)})$, and let
   \[ j^0_{(\alpha, \eta_\alpha), (\beta, \eta_\beta)} = i^{0, M^0_{(\alpha, \eta_\alpha)}}_{(\beta, \eta_\beta)} \circ j^0_{(\alpha, \eta_\alpha)} : V[G^0] \rightarrow M^0_{(\alpha, \eta_\alpha), (\beta, \eta_\beta)} \]

The following summarizes the connections between the different iterated ultrapowers of $V$ and $V^0$:

1. $j^0_{(\alpha, \eta_\alpha)} \upharpoonright V = j_\alpha$,
2. $i^{0, M^0_{(\alpha, \eta_\alpha)}}_{(\beta, \eta_\beta)} \upharpoonright M_\alpha = i^M_\beta$,
3. $j^0_{(\alpha, \eta_\alpha), (\beta, \eta_\beta)} \upharpoonright V = j_{\alpha, \beta}$,
4. $s^0_{j_\alpha(\kappa)}(\kappa) = \eta_\alpha$. \[ ^6 \]
5. \( s_{j_\alpha,\beta}(j_\alpha(\kappa)) = \eta_\beta. \)

By the results of Friedman Magidor ([6] listed above in 4.1, it is clear that for every \( \alpha < o(\kappa) = \lambda \) and \( \eta < \lambda \), then in order to construct \( U^0_{(\alpha,\eta)} \) it is sufficient to know \( U_\alpha \) and \( G^0 \). For every \( \beta > \alpha \), and \( \eta' < \lambda \) then \( U_\alpha \in M_\beta \subseteq M^0_{(\beta,\eta')} \) and \( G^0 = G^0_{\beta}(\eta')\uparrow (\kappa + 1) \in M^0_{(\beta,\eta')} \). We therefore conclude that

**Corollary 4.4.** For every \( \alpha < \beta < o(\kappa) \), and every \( \eta, \eta' < \lambda \), then \( U^0_{(\alpha,\eta)} \triangleleft U^0_{(\beta,\eta')} \).

One can actually show that \( U^0_{(\alpha,\eta)} \triangleleft U^0_{(\beta,\eta')} \) if and only if \( \alpha < \beta \), however we do not need this result here.

For all measurable cardinal \( \nu < \kappa \), the normal measures on \( \nu \) in \( V \) similarly extends in \( V[G^0] \). Let \( \bar{U}_\nu = \langle U_{\nu,\alpha} \mid \alpha < o(\nu) \rangle \) be an \( \triangleleft \)-increasing sequence of the normal measures on \( \nu \) in the coherent sequence \( \bar{U} \in V = L[\bar{U}] \). By the same arguments describing the normal measures above, for \( \alpha < o(\kappa) \) and \( \eta < \lambda \), we conclude that each normal measure \( U^0_{\nu,\alpha} \in V \) has \( \lambda(\nu) \) many extensions in \( V[G^0 \uparrow (\nu + 1)] \) denoted \( \{ U^0_{\nu,\alpha} \mid \eta < \lambda(\nu) \} \). No measures on \( \nu \) are added or removed by the rest of the iteration since \( \mathcal{P}^0 \setminus (\nu + 2) \) is \( 2^{(2^{\nu})} \)-distributive, and \( U^0_{\nu,\alpha} \triangleleft U^0_{\nu,(\beta,\eta')} \) for every \( \alpha < \beta < o(\nu) \) and \( \eta, \eta' < \lambda(\nu) \).

### 4.2 The Poset \( \mathcal{P}^1 \)

The poset \( \mathcal{P}^1 = < \mathcal{P}^1_\nu, \mathcal{Q}^1_\nu \mid \nu < \kappa > \) is a Magidor iteration of one-point Prikry forcings. See [8] for a comprehensive survey of Magidor iteration of Prikry type forcings. One-point Prikry forcing is a simplified version of the well known Prikry forcing which associate a measurable cardinal \( \nu \) a single (indiscernible) ordinal \( d(\nu) < \nu \), instead of a cofinal \( \omega \) sequence. For a measurable cardinal \( \nu < \kappa \) and a normal measure \( U_\nu \) on \( \nu \) then the one-point Prikry forcing \( Q(U_\nu) \) consists of condition \( q \in Q(U_\nu) \) which are either ordinals \( \mu < \nu \), or sets \( X \in U_\nu \). For \( q, q' \in Q(U_\nu) \) we say that \( q \) is a direct extension of \( q' \), denoted \( q >^*_{Q(U_\nu)} q' \) if \( q, q' \in U_\nu \) and \( q \subseteq q' \). For \( q, q' \in Q(U_\nu) \) we say that \( q \) is a end extension of \( q' \), denoted \( q >^1_{Q(U_\nu)} q' \) if \( q' \in U_\nu \), \( q < \nu \) and \( q \in q' \). Let \( >_{Q(U_\nu)} = >_{Q(U_\nu)}^* \cup <_{Q(U_\nu)}^1 \). It is shown in [8] that \( (Q(U_\nu), \geq_{Q(U_\nu)}, >_{Q(U_\nu)}) \) is a Prikry type forcing notion. The posets \( \mathcal{Q}^1_\nu \) will be determined by recursion on \( \nu < \kappa \) and the following guideline:
1. Suppose that \( \nu < \kappa \) is a measurable cardinal in \( V \), and \( \vec{U}_\nu^1 = \langle U_{\nu, \langle \beta, \alpha \rangle}^1 \mid \alpha < o(\nu), \beta < \rho(\nu) \rangle \) is a given collection of normal measures on \( \nu \) in a \( \mathcal{P}_\nu^1 \upharpoonright \nu \) generic extension of \( V^0 \), which depends only on the \( \mathcal{P}_\nu^1 \) generic set, and on the normal measures on \( \kappa \), \( \langle U_{\nu, \langle \alpha, \eta \rangle}^0 \mid \alpha < o(\nu), \eta < \rho(\nu) \rangle \). We will choose a normal measures \( U_\nu^* \) from the measures in \( \vec{U}_\nu^1 \) according to the different \( \alpha, \eta \) values for which \( \nu \in \Delta(\eta) \), and define \( \mathcal{Q}_\nu^1 \) to be the one-point Prikry forcing \( Q(U_\nu^*) \).

2. Suppose that \( \nu < \kappa \) is a measurable cardinal in \( V \), so that the iteration up to \( \nu \), \( \mathcal{P}_\nu^1 \) has been defined. We define an appropriate sequence of of normal measures on \( \nu \), \( \vec{U}_\nu^1 = \langle U_{\nu, \langle \alpha, \beta \rangle}^1 \mid \beta < o(\nu), \alpha < \lambda(\nu) \rangle \).

Suppose that \( \vec{U}_\nu^1 = \langle U_{\nu, \langle \alpha, \beta \rangle}^1 \mid \beta < o(\nu), \alpha < \lambda(\nu) \rangle \) is given in a \( \mathcal{P}_\nu^1 \) generic extension of \( V^0 \). The following motivates the choice of the posets \( \mathcal{Q}_\nu^1 \) for \( \nu < \kappa \). We want that for every \( U_{\nu, \langle \alpha, \beta \rangle}^0 \) in \( V^0 \), then \( \kappa \) will be a non-trivial forcing stage in \( j_{\langle \alpha, \beta \rangle}(\mathcal{P}_\nu^1) \) if and only if \( \beta < \alpha \) and the one-point used at is by a normal measures in a \( \mathcal{M}_{\langle \alpha, \beta \rangle}^0 \) generic extension by \( \mathcal{P}_\nu^0 \) which extends \( U_{\langle \beta, \alpha \rangle}^0 \).

**Definition 4.5.** Given \( \mathcal{P}_\nu^1 \) and \( \vec{U}_\nu^1 \) as above, then \( \mathcal{Q}_\nu^1 \) is taken to be non-trivial if and only if there are \( \alpha < \beta < \lambda \) so that \( \nu \in \Delta(\beta) \) (i.e. \( o(\nu) = \alpha \), and \( s(\nu) = \beta \)). If so, then we choose \( U_\nu^* = U_{\nu, \langle \alpha, \beta \rangle}^1 \in \vec{U}_\nu^1 \). Let us add some auxiliary definitions.

1. Let \( \Delta' = \{ \nu \in \Delta \mid \mathcal{P}_\nu^1 \models \mathcal{Q}_\nu^1 \text{ is not trivial} \} \).

2. Let \( \dot{d} : \Delta' \to \kappa \), be the \( \mathcal{P}_\nu^1 \) name for the generic Prikry function, so that for every \( V^0 \) generic filter \( G^1 \subset \mathcal{P}_\nu^1 \) then \( d(\nu) < \nu \) is the \( \mathcal{Q}_\nu^1 = Q(U_\nu^*) \) generic point, given by \( G^1 \).

Next, suppose that \( G^1_\nu \subset \mathcal{P}_\nu^1 \) is generic over \( V^0 \). We define the normal measures

\[
\vec{U}_\nu^1 = \langle U_{\nu, \langle \alpha, \beta \rangle}^1 \mid \alpha < \lambda(\nu), \beta < o(\nu) \rangle
\]

in the generic extension \( V^0[G^1_\nu] \). To simplify the notations, let us assume that \( \nu = \kappa \) where we write \( U_{\nu, \langle \alpha, \beta \rangle}^0 \) for \( U_{\kappa, \langle \alpha, \beta \rangle}^0 \), and we use the notations \( j_{\langle \alpha, \eta \rangle}^0, i_{\langle \beta, \alpha \rangle}^0, \), and \( j_{\langle \alpha, \eta \rangle}^0, i_{\langle \beta, \alpha \rangle}^0, \), introduced above. We also abbreviate here

\[\text{Note that this makes sense as } o_{\langle \alpha, \beta \rangle}(\kappa) = \alpha \text{ and } U_{\langle \beta, \alpha \rangle}^0 \in \mathcal{M}_{\langle \alpha, \beta \rangle}^0.\]
by writing $U^1_{(\alpha,\beta)}$ for $U^1_{\kappa, (\alpha, \beta)}$. We define the measures in $\bar{U}^1_\kappa$ in two steps. We first define $U^1_{(\alpha,\beta)}$ for $\beta \leq \alpha$, and then use it to define $U^1_{(\alpha,\beta)}$ for $\alpha < \beta$.

**Definition 4.6** ($U^1_{(\alpha,\beta)}$ for $\alpha \geq \beta$). Let $\beta \leq \alpha$ so that $\beta < o(\kappa)$ and $\alpha < \lambda(\kappa) = \lambda$. Define $U^1_{(\alpha,\beta)}$ in $V^0[G^1]$ as follows. For $X \subset \kappa$ in $V^0[G^1]$ and a $P^1$ name for $\hat{X}$ for $X$, then $X \in U^1_{(\alpha,\beta)}$ iff there exists some $p \in G^1$ and $q \geq^* j^0_{(\beta,\alpha)}(p) \setminus \kappa$ so that $p^\frown q \geq^* j^0_{(\beta,\alpha)}(p)$ is a condition in $j^0_{(\beta,\alpha)}(P^1)$ and

$$p^\frown q \Vdash_{j^0_{(\beta,\alpha)}(P^1)} \bar{k} \in j^0_{(\beta,\alpha)}(\hat{X}).$$

(1)

There are three main properties which contribute to the fact that $U^1_{(\alpha,\beta)}$ is a normal measure on $\kappa$ ($\kappa$ complete ultrafilter) in $V^0[G^1]$: The fact that every two direct extensions of $j^0_{(\beta,\alpha)}(p)$ are $\geq^*$ compatible for every $p \in P^1$ is the main property which implies that $U^1_{(\alpha,\beta)}$ is a filter. The fact that $j^0_{(\beta,\alpha)}(P^1) \setminus \kappa$ satisfies Prikry property, implies that we can always take direct extension of this part to decide statements of the form $\exists \bar{k} \in j^0_{(\beta,\alpha)}(\hat{X})$, which in turn, implies that $U^1_{(\alpha,\beta)}$ is an ultrafilter. Finally, the fact that $\geq^*$ of $j^0_{(\beta,\alpha)}(P^1) \setminus \kappa$ is $(2^\kappa)^+$ closed guarantees that we can decide $\kappa$ many statements, which establishes that $U^1_{(\alpha,\beta)}$ is $\kappa$-complete, and normal. We need to justify the third fact. If we consider the closure rate of $\geq^*$ in $j^0_{(\beta,\alpha)}(P^1) \setminus (\kappa + 1)$ then this would be trivial as the next non trivial forcing is at stage $\mu > \kappa^+$. For this to hold true in $j^0_{(\beta,\alpha)}(P^1) \setminus \kappa$ we need to verify that stage $\kappa$ of this forcing is trivial. Indeed, we note that $\kappa \in j^0_{(\beta,\alpha)}(\Delta^\beta(\alpha))$, i.e., $\mathcal{O}^M_{(\beta,\alpha)}(\kappa) = \beta$ and $s^M_{j^0_{(\beta,\alpha)}(\kappa)}(\kappa) = \alpha$. Since $\beta \leq \alpha$, then by definition 4.5 the forcing $\hat{Q}^M_{(\beta,\alpha)}$ is trivial. Using these three properties, it is routine to show that $U^1_{(\alpha,\beta)}$ is a normal measures on $\kappa$ (for a detailed proof, see [8], or the description of $U^*_{0}$ in [3]). We note that definition 4.6 clearly implies that $U^1_{(\alpha,\beta)}$ extends $U^0_{(\beta,\alpha)}$ in $V^0$, i.e. $U^1_{(\alpha,\beta)} \cap V^0 = U^0_{(\beta,\alpha)}$.

The following remark motivates the definition of $U^1_{(\alpha,\beta)}$ for $\alpha < \beta$, given below.

**Remark 4.7.** Let us consider the outcome of applying the last definition 4.6 for the case $\alpha < \beta$. As noted above, the measure defined in 4.6 clearly extends $U^0_{(\beta,\alpha)}$ which concentrates on the ordinals of $\Delta^\beta(\alpha)$. The main difference between the two cases is that the ordinals $\nu \in \Delta^\beta(\alpha)$ are trivial forcing stage of the iteration $P^1$ if $\beta \leq \alpha$, and are not trivial if $\alpha < \beta$. So if $\alpha < \beta$ then the
generic $G^1 \subseteq \mathcal{P}^1$ choose a Prikry points $d(\nu) < \nu$, and the Magidor iteration style (i.e. the fact that we take inverse limit in the $\succeq^*$ order) implies that $d : \Delta_\beta(\alpha) \to \kappa$ is injective outside of a finite set (see the argument regarding $U^*_1$ in [3] for a proof of this fact). Therefore, when $\alpha < \beta$ there cannot be a normal measure extending $U^0_{(\beta,\alpha)}$.

For every $\alpha < \beta$, let $W^1_{(\alpha,\beta)}$ be the the subset of $\mathcal{P}(\kappa)$ given by definition 4.6. By its definition, $W^1_{(\alpha,\beta)}$ extends $U^0_{(\beta,\alpha)}$. By the previous remark, we get that cannot be a normal. Nevertheless, $W^1_{(\alpha,\beta)}$ is still a $\kappa$ complete ultrafilter. The difference between the two cases ($\alpha \geq \beta$ and $\alpha < \beta$) is that for $\alpha < \beta$, then stage $\kappa$ of $j^0_{(\beta,\alpha)}(\mathcal{P}^1)$ is not trivial, and therefore $\geq^*$ of $j^0_{(\beta,\alpha)}(\mathcal{P}^1) \setminus \kappa$ is $\kappa$ closed (unlike the case $\alpha \geq \beta$ where this order is $(2^\kappa)^+-$closed) which is sufficient to guarantee the $\kappa$ completeness of $W^1_{(\alpha,\beta)}$. Instead, when $\alpha < \beta$ our choice of $U^1_{(\alpha,\beta)}$ will be the normal projection of $W^1_{(\alpha,\beta)}$. This fact is not seen immediately from definition 4.9 below, and will be explain later. It turns out that the the Prikry generic function $d$, induces a projection to a the normal measure $\mu$. Let us add the following definitions.

**Definition 4.8.** 1. For $\alpha < \beta < \lambda = \sigma(\kappa)$, let $k^0_{\alpha,\beta} : V^0 \to N^0_{\alpha,\beta}$ denote the iterated ultrapower $k^0_{\alpha,\beta} = j^0_{(\alpha,\beta),(\beta,\alpha)}$, and $N^0_{\alpha,\beta} = M^0_{(\alpha,\beta),(\beta,\alpha)}$.

2. For every condition $p \in \mathcal{P}^1$, $\nu < \kappa$ so that $p \upharpoonright \nu \models \check{p}_\nu \in Q(U^*_\nu)$, and $\mu < \nu$, let $p^{+(\mu,\nu)}$ denote the condition obtained from $p$ by replacing $p_\nu$ with $\{\check{\mu}\}$. I.e., $p^{+(\mu,\nu)} \models \check{\mu} = \check{d}(\check{\nu})$.

Note that $p^{+(\mu,\nu)}$ is not necessarily an extension of $p$. If $p \upharpoonright \nu \models \check{\mu} \in \check{p}_\nu$, then $p^{+(\mu,\nu)}$ is an extension of $p$. We say that $\mu$ is available for $p$ at $\nu$.

**Definition 4.9** ($U^1_{(\alpha,\beta)}$ for $\alpha < \beta$). Let $\beta < \alpha$ for $\alpha < \sigma(\kappa)$ and $\beta < \lambda$. For $X \subseteq \kappa$ in $V^0[G^1]$ and a $\mathcal{P}^1$ name for $\dot{X}$ for $X$, then $X \in U^1_{(\alpha,\beta)}$ iff there exists some $p \in G^1$ and $q \succeq^* k^0_{\alpha,\beta}(p) \setminus \kappa$ so that $(p^- q)^{+(\kappa,j^0_{(\alpha,\beta)}(\kappa))} \succeq p^- q$, and

\[(p^- q)^{+(\kappa,j^0_{(\alpha,\beta)}(\kappa))} \models \check{\kappa} \in k^0_{\alpha,\beta}(\check{X}).\]  \hspace{1cm} (2)

See the arguments for $U^\succeq_1$ in [3] for a proof that $U^1_{(\alpha,\beta)}$ is a normal measures on $\kappa$, when $\alpha < \beta$. The above concludes the definition of the iteration $\mathcal{P}^1$ and of the normal measures $U^1_{(\alpha,\beta)}$ for every $\alpha, \beta < \lambda$.
We mentioned above that $U^1_{(\alpha, \beta)}$ is the normal projection of $W^1_{(\alpha, \beta)}$ by $d$. Let us prove this statement.

Lemma 4.10. $U^1_{(\alpha, \beta)}$ is the projection of $W^1_{(\alpha, \beta)}$ by the Prikry generic function $d$.

Proof. Since $d_* W^1_{(\alpha, \beta)}$ is an ultrafilter, it is sufficient to show that $d^* W^1_{(\alpha, \beta)} \subseteq U^1_{(\alpha, \beta)}$. Let $X \subseteq \kappa$ belong to $d_* W_{(\alpha, \beta)}$, the projection of $W^1_{(\alpha, \beta)}$ by the Prikry function $d$. The by the definition 4.6 there is some $t \in j^{0\kappa}_{(\beta, \alpha)}(P^1)$ so that $p = t \upharpoonright \kappa \in G^1$ and $t \setminus \kappa \geq_* j^{0\kappa}_{(\beta, \alpha)}(p) \setminus \kappa$.

$\exists \hspace{1cm} t \upharpoonright \kappa \models j^{0\kappa}_{(\beta, \alpha)}(P^1) \hspace{1cm} (\kappa \subseteq \kappa)$

Since $Q(\beta)$ belongs to $\kappa$ forces that $\exists \hspace{1cm} \kappa \subseteq \kappa$. Let $\kappa \subseteq \kappa$.

Let $\kappa \subseteq \kappa$. Then $\kappa \subseteq \kappa$.

1. $r^-(t' \setminus j^{0\kappa}_{(\beta, \alpha)}(\kappa)) \geq t'$,

2. $t' \upharpoonright j^{0\kappa}_{(\beta, \alpha)}(\kappa) = j^{0\kappa}_{(\beta, \alpha)}(t \upharpoonright \kappa) \models j^{0\kappa}_{(\beta, \alpha)}(P^1) \hspace{1cm} k^{0\kappa}_{(\alpha, \beta)}(d)(j^{0\kappa}_{(\beta, \alpha)}(\kappa)) \in k^{0\kappa}_{(\alpha, \beta)}(\kappa)$,

3. $j^{0\kappa}_{(\alpha, \beta)}(\kappa) = j^{0\kappa}_{(\alpha, \beta)}(t(\kappa))$ and therefore $\exists \hspace{1cm} k \subseteq \kappa$. Let $\kappa \subseteq \kappa$. Then $\kappa \subseteq \kappa$.

Let $s = r^-(t' \setminus j^{0\kappa}_{(\beta, \alpha)}(\kappa))$, then $\kappa$ is available at $j^{0\kappa}_{(\beta, \alpha)}(\kappa)$ for $s$, so $s^+(\kappa j^{0\kappa}_{(\beta, \alpha)}) \geq s$ forces that $k^{0\kappa}_{(\alpha, \beta)}(d)(j^{0\kappa}_{(\beta, \alpha)}) = \kappa$. Furthermore, we conclude that $s \in k^{0\kappa}_{(\alpha, \beta)}(P^1)$ satisfies that $p' = s \upharpoonright \kappa \in G^1$, $\kappa \setminus \kappa \geq_* k^{0\kappa}_{(\alpha, \beta)}(p') \setminus \kappa$, and $s^+(\kappa j^{0\kappa}_{(\beta, \alpha)}(\kappa)) \models \kappa = k^{0\kappa}_{(\alpha, \beta)}(d)(j^{0\kappa}_{(\beta, \alpha)}) \in k^{0\kappa}_{(\alpha, \beta)}(\kappa)$. It follows that $X \in U^1_{(\alpha, \beta)}$.

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8see also a similar discussion for $U^1_1$ and $U^1_1$ in [3].
4.3 Separation of $U^1_{(\alpha,\beta)}$

Let $G^1 \subset P^1$ be a generic filter over $V^0 V[G^0]$. Let us denote $V^0[G^1]$ by $V^1$. The normal measures $\{U^1_{(\alpha,\beta)} \mid \alpha \leq \beta < \lambda\}$ are the normal measures on $\kappa$ which will be used to prove the main result of the paper (Theorem 1.1).

As discussed in the introduction, this requires these normal measures to be separated by sets. In the last part of this section (4), we introduce some sets which have a significant role in the analysis of the normal measures $\{U^1_{(\alpha,\beta)} \mid \alpha, \beta < \lambda\}$, and prove that these measures are separated by sets.

Definition 4.11. We define sets $\Gamma$, $\Sigma$, $\Pi \subset \kappa$.

1. $\Gamma = d^\nu \Delta'$ is the set of Prikry generic points,
2. $\Sigma = \{\nu < \kappa : \text{if } \mu > \nu \text{ is measurable then } d(\mu) \geq \nu \}$, i.e.,
3. $\Pi = \{\alpha : |d^{-1}(\alpha)| = 1 \text{ and } \forall \mu \in \Delta. (\mu > d^{-1}(\alpha)) \rightarrow (d(\mu) \notin [\alpha, d^{-1}(\alpha)])\}$.

It is shown in [3] that $\Sigma$ is a club in $V^1$, and $\Gamma \setminus \Pi$ is bounded in $\kappa$. It follows that $\Sigma \in U^1_{(\alpha,\beta)}$ for every $\alpha, \beta < \lambda$. By definitions 4.6 and 4.9 we see that $\Pi \notin U^1_{(\alpha,\beta)}$ if and only if $\alpha < \beta$. Recall that in $V^0$, the normal measures $U^0_{(\alpha,\eta)}$ where separated by by pairwise disjoint sets $\Delta_\alpha(\eta)$ where $\Delta_\alpha(\eta) \in U^0_{(\alpha,\eta)}$. Let us show that $U^1_{(\alpha,\beta)}, \alpha, \beta < \lambda = o(\kappa)$, are similarly separated.

Definition 4.12. For every $\alpha, \beta < \lambda$, we define $X_{(\alpha,\beta)}$ by,

$$X_{(\alpha,\beta)} = \begin{cases} 
\Delta_\beta(\alpha) \setminus \Pi & \text{if } \alpha \geq \beta \\
\Delta_\alpha(\beta) \cap \Pi & \text{if } \alpha < \beta
\end{cases}$$

It is clear that the sets $X_{(\alpha,\beta)}, \alpha, \beta < \lambda = \kappa$, are pairwise disjoint. It is also easy to verify that for $\alpha \geq \beta$, then $U^1_{(\alpha,\beta)} \cap V^0 = U^0_{(\beta,\alpha)},$ and $\Pi \notin U^1_{(\alpha,\beta)}$. It follows that $X_{(\alpha,\beta)} \in U^1_{(\alpha,\beta)}$ when $\alpha \geq \beta$. Similarly, one concludes that $X_{(\alpha,\beta)} \in U^1_{(\alpha,\beta)}$ when $\alpha < \beta$. Let us conclude the last in a single statement.

Corollary 4.13. The sets in the collection $\{X_{(\alpha,\beta)} \mid \alpha, \beta < o(\kappa)\}$ are pairwise disjoint, and $X_{(\alpha,\beta)} \in U^1_{(\alpha,\beta)}$ for all $\alpha, \beta < \lambda$.

\(^9\text{see also the paragraph proceeding Definition 2.2 in [3].}\)
5 The restriction of $j^1_{(\alpha,\beta)}$

For every $\alpha, \beta < \lambda$, let $j^1_{(\alpha,\beta)} : V^1 \to M^1_{(\alpha,\beta)} \cong \text{Ult}(V^1, U^1_{(\alpha,\beta)})$. The goal of this section is to describe the restriction of embeddings $j^1_{(\alpha,\beta)} : V^1 \to M^1_{(\alpha,\beta)}$, $\alpha, \beta < \lambda$ to $V^0$ and $V$, as an iterated ultrapower of the measures of $V^0$ ($V$ respectively). We start this section by defining an iterated ultrapower of $V^0$, $\pi^0_{\alpha,\beta} : V^0 \to Z^0_{\alpha,\beta}$, for every $\alpha, \beta < \lambda$. Proposition 5.3 states that $\pi^0_{\alpha,\beta} = j^1_{(\alpha,\beta)} \upharpoonright V^0$. Proving this proposition requires several preliminary technical results. The reader who wish to avoid this can read the beginning of this section, up to formulation of the proposition. The description of $j^1_{(\alpha,\beta)} \upharpoonright V$ which we denote by $\pi_{\alpha,\beta}$, will easily follow from the description of $\pi^0_{\alpha,\beta}$.

Our assumption is that the ground model $V$ is a Mitchell model $V = L[U]$ with $V = \mathcal{K}(V)$, where $U$ is the coherent sequence of normal measures which witnesses that $o(\kappa) = \lambda$. In particular $V$ does not contain overlapping extender. Let $\alpha, \beta < o(\kappa) = \lambda$. The fact that $V = \mathcal{K}(V)$ and $V^1 = V[G^0 \ast G^1]$ is a set generic extension of $V$ implies the following properties.

1. $V = \mathcal{K}(V^1)$,
2. $j^1_{(\alpha,\beta)} \upharpoonright V : V \to Z_{\alpha,\beta}$ is an iterated ultrapower of $V$,
3. if $j^1_{(\alpha,\beta)} : V^1 \to M^1_{(\alpha,\beta)}$ then $M^1_{(\alpha,\beta)} = Z_{\alpha,\beta}[G^0_{(\alpha,\beta)} \ast G^1_{(\alpha,\beta)}]$ where $G^0_{(\alpha,\beta)} \ast G^1_{(\alpha,\beta)} \subset j^1_{(\alpha,\beta)}(P^0 \ast P^1)$ is generic over $Z_{\alpha,\beta}$.

We refer to [19] for these results. The definition of $\pi^0_{\alpha,\beta}$ for $\alpha, \beta < \lambda$ makes use of the ultrapower embedding $j^0_{(\alpha,\beta)} : V^0 \to M^0_{(\alpha,\beta)}$ defined in 4.2 and the two steps iterated ultrapower $k^0_{\alpha,\beta} : V^0 \to N^0_{\alpha,\beta}$ defined in 4.8. Let $\tilde{\Delta} = \langle \Delta_\alpha(\eta) \mid \alpha < o(\kappa), \eta < \lambda \rangle$.

**Definition 5.1 (\(\pi^0_{\alpha,\beta}\)).**

$\pi^0_{\alpha,\beta}$ is the embedding which results form a linear iteration $T^0 = \langle Z_i, \sigma^0_{i,j} \mid 0 \leq i < j \leq \theta \rangle$, with critical points $\nu_i = \text{cp}(\sigma^0_{i,i+1})$ of length $\theta$. Here $Z_i$ are the intermediate models (iterands) of the iteration, and $\sigma^0_{i,j} : Z_i \to Z_j$ are the connecting iterations. The definition of $T^0$ is given by recursion on ordinals. For every $i < \theta$ we denote the image of the $i$–th critical point $\nu_i$, $\sigma^0_{i,i+1}(\nu_i)$, by $\nu^1_i$. We set $Z^0_0 = V^0$, $\sigma^0_{0,0} = \text{id}_{Z^0_0}$, $Z^0_1 = N^0_{\alpha,\beta}$, and $\sigma^0_{0,1} = \begin{cases} j^0_{(\beta,\alpha)} & \text{if } \beta \leq \alpha \\ k^0_{(\alpha,\beta)} & \text{if } \alpha \geq \beta \end{cases}$.
We define $\nu_0 = \kappa$, and set $\nu_0^1 = j_{(\alpha, \beta)}^0(\kappa) < \sigma_{0, 1}^0(\kappa)$ if $\alpha < \beta$, and leave $\nu_0^1$ undefined otherwise. Given the iteration $T^0 \upharpoonright i$ up to stage $i \geq 1$ for some $i < \theta$, we define $Z_{i+1}$, and $\sigma_{i, i+1}$ as follows. First, let $\nu_i^*$ be the supremum of all critical points $\{\nu_j \mid j < i\}$ of $T^0 \upharpoonright i$. We then take $\nu_i$ to be the minimal ordinal $\nu \geq \nu_i^*$ which satisfies

1. The forcing of $\sigma_{0,i}^0(\mathcal{P}^1)$ at stage $\nu$ is not trivial, i.e. $\nu \in \sigma_{0,i}^0(\Delta^i)$, and

2. $\nu$ does not belong to $\sigma_{0,i}^0 \{\nu_j^1 \mid j < i\}$.

i.e., $\nu_i$ is the minimal non-trivial stage in the iteration $\sigma_{0,i}^0(\mathcal{P}^1)$, which is not below $\nu_i^*$, and is not included in $\sigma_{0,i}^0 \{\nu_j^1 \mid j < i\}$. The two requirements above imply that the critical points of the iteration $T^0$ are strictly increasing (i.e. the iteration is normal). Since $\nu_i \in \sigma_{0,i}^0(\Delta^i)$ then there are unique $\beta_i < \alpha_i$ so that $\nu_i \in \sigma^0_i(\Delta^i)_{\alpha_i}(\beta_i)$. We define $\sigma_{i+1, i+1}^0 = j_{\nu_i, (\beta_i, \alpha_i)}^0 : Z_i \rightarrow Z_{i+1}$, and set $\nu_i^1 = j_{\nu_i, (\beta_i, \alpha_i)}^0(\nu_i)$. If $\delta < \theta$ is a limit ordinal, then we take $Z_\delta^0$ to be the direct limit of $T^0 \upharpoonright \delta$, and set $\sigma_{i, \delta}^0 : Z_i^0 \rightarrow Z_\delta^0$ to be the limit embedding. The iteration terminates at stage $\theta$ where $\sigma_{0, \theta}^0(\Delta^i) \subset \{\nu_i^1 \mid i < \theta\} \cup \nu_\theta^*$. Note that

$$\sigma^0_{0, 1}(\kappa) = \begin{cases} j_{(\beta, \alpha)}^0(\kappa) & \text{if } \beta \leq \alpha, \\ k_{(\alpha, \beta)}^0(\kappa) & \text{if } \alpha > \beta, \end{cases}$$

It is not difficult to verify by induction on the ordinals $i$, $\sigma^0_{0, 1}(\kappa)$ is a fixed point of $\sigma_{1,i}^0$. Since $\nu_i < \sigma^0_{1,i}(\sigma^0_{0, 1}(\kappa)) = \sigma^0_{0, 1}(\kappa)$ and the iteration is normal, it follows that it must terminate after at most $\sigma^0_{0, 1}(\kappa)$ many steps.

The following concludes the construction of $\pi_{0,\alpha,\beta}^0$. All non trivial results can be simply verified by induction on $1 \leq i < \theta$.

**Corollary 5.2.**

1. Suppose that $\alpha \geq \beta$ then,

- $\sigma^0_{0, 1} = j_{(\beta, \alpha)}^0$, $\nu_0 = \kappa$, and $\nu_0^1$ is not defined,
- for every $1 \leq i \leq \theta$, then both $\kappa = \nu_0$ and $j_{(\beta, \alpha)}^0(\kappa)$ are not moved by $\sigma^0_{1,i}$. In particular $\nu_i \in (\kappa, j_{(\beta, \alpha)}^0(\kappa))$ for every $1 \leq i < \theta$,
- $\sigma^0_{0, i}(\Delta^i) \cap [\kappa, \nu_i) = \{\nu_j^1 \mid j < i\}$ for every $1 \leq i < \theta$.

2. Suppose that $\alpha < \beta$ then,

- $\sigma^0_{0, 1} = k_{(\alpha, \beta)}^0$, $\nu_0 = \kappa$, $\nu_0^1 = j_{(\alpha, \beta)}^0 < k_{(\alpha, \beta)}^0(\kappa)$,
• for every $1 \leq i \leq \theta$, then all $\nu_0$, $\nu'_0$, and $k^0_{\alpha,\beta}(\kappa)$, are not moved by $\sigma^0_{1,i}$. In particular the iteration can be naturally decompose to $\nu_i \in (\nu_0, \nu_1)$, and $\nu_i \in (\nu'_0, k^0_{\alpha,\beta}(\kappa))$.

• For every $i < \theta$ so that $\nu_i \in (\nu_0, \nu_1)$, then $\sigma^0_{1,i}(\Delta') \cap [\kappa, \nu_i) = \{\nu^1_j \mid 1 \leq j < i\}$.

• For every $i < \theta$ so that $\nu_i \in (\nu'_0, k^0_{\alpha,\beta})$, then $\sigma^0_{1,i}(\Delta') \cap [\kappa, \nu_i) = \{\nu^1_j \mid 0 \leq j < i\}$.

Proposition 5.3. There exists a set $G^1_{\alpha,\beta} \subset \pi^0_{\alpha,\beta}(\mathcal{P}^0)$ in $V^1$, so that $G^1_{\alpha,\beta}$ is generic over $Z^0_{\alpha,\beta}$, and the induced embedding extension $\pi^1_{\alpha,\beta} : V^0[G^1] \to Z^0_{\alpha,\beta}[G^1_{\alpha,\beta}]$ coincides with the ultrapower of $V^0[G^1]$ by $U^1_{(\alpha,\beta)}$.

In order to proof Proposition 5.3 we first need to establish some technical results regarding the poset $\mathcal{P}^1$, and introduce auxiliary notions of structural iterations and compatible conditions.

5.1 Structural results for dense open sets in $\mathcal{P}^1$

Definition 5.4 (structural function, and structural extension). For every $n < \omega$, we introduce notions of structural function $f$ of degree $n$, avoiding support $b \subset \kappa$. For $n = 0$, a structural function of degree $0$, is the trivial function $f^0 = \emptyset$. A function $f = f^{n+1}$ is a structural function of degree $n + 1$ avoiding support $b$, if there is a unique ordinal $\nu_f \in \kappa \setminus b$, and a $\mathcal{P}^1_{\nu_f}$ name $\dot{X}_f$ so that the following holds:

1. $\nu_f \in \Delta'$.
2. $0_{\mathcal{P}^1} \forces \dot{X}_f \in U^*_{\nu_f}$.
3. $\text{dom}(f)$ is the set of all $\mathcal{P}^1_{\nu_f}$ names for ordinals in $\dot{X}_f$.
4. For every name $\tau \in \text{dom}(f)$, then $f(\tau)$ is a structural function $g^n$ of degree $n$, and $\nu_g < \nu_f$.

We say that $f$ is a structural function if there exists some $n < \omega$ so that $f$ is a structural function of degree $n$.

Let $p$ be a condition, and $f$ be a structural function avoiding $\text{supp}(p)$. We say that a condition $q$ is a structural extension of $p$ by $f$, if the following holds:
1. If $f$ has degree 0 then $q$ is a structural extension of $p$ by $f$ if $q \geq^* p$.

2. If $f$ has degree $n + 1$, then $q$ is a structural extension of $p$ by $f$ if there are $r \geq^* p \upharpoonright \nu_f$, $\tau \in \text{dom}(f)$, so that $r \forces \tau \in p_{\nu_f}$, and $q$ is a structural extension of $r^-(p \setminus \nu_f)^{(\tau,\nu_f)}$ the degree $n$ structural function, $f(\tau)$. Note that $r^-(p \setminus \nu_f)^{(\tau,\nu_f)} \geq p$.

**Lemma 5.5.** For every $D \subset P^1$ open dense set, and $p \in P^1$, then there exists a structural function $f$ avoiding $\text{supp}(p)$, so that every structural extension of $p$ by $f$ has a direct extension in $D$.

**Proof.** By induction of $\nu \leq \kappa$, we prove that the claim holds for the Magidor iteration $P^1_{\nu}$. For $\nu = 0$, the result is trivial. Suppose that the results is true for every dense open set $D \subset P^1_{\nu}$, and $p \in P^1_{\nu}$. $P^1_{\nu+1} = P^1_{\nu} * \dot{Q}_{\nu}$. If $\nu \notin \Delta'$ then $\dot{Q}_{\nu}$ it trivial and the claim clearly holds in $P^1_{\nu+1}$. If $\nu \in \Delta'$ then $\dot{Q}_{\nu} = Q(\dot{U}_{\nu}^*)$. Let $D \subset P^1_{\nu+1}$ be a open dense set, and $p = p \upharpoonright \nu \dot{\forces} p_{\nu} \in P^1_{\nu+1}$. If $\nu \in \text{supp}(p)$ then the forcing $P^1_{\nu}$ over $p$ is equivalent to $P^1_{\nu}$, and therefore the claim holds by the induction hypothesis.

Suppose that $\nu \notin \text{supp}(p)$, then $p_{\nu}$ is a $P^1_{\nu}$ name of a set in $U_{\nu}^*$. For every generic $G_{\nu} \subset P^1_{\nu}$, then $D(G_{\nu}) = \{(q_{\nu})_{G_{\nu}} \mid q \in D\}$ is a dense open set of $Q(U_{\nu}^*)$ in $V[G_{\nu}]$. It is then clear that for some subset $Y_{\nu} \subset (p_{\nu})_{G_{\nu}}$, $Y_{\nu} \in U_{\nu}^*$, then $\mu \in D(G_{\nu})$ for every $\mu \in Y_{\nu}$. Let $\dot{Y}_{\nu}$ be a name for $Y_{\nu}$ in $P^1_{\nu}$, then for every name $\tau$ of an ordinal in $Y_{\nu}$, then the set $D_{\tau} = \{q \geq^* p \upharpoonright \nu \mid q \dot{\forces} (\tau) \in D\}$ is dense open in $P^1_{\nu}$. By the induction hypothesis, then for some $n(\tau) < \omega$, there exists some structural function $f(\tau)$ of degree $n(\tau)$, so that every structural extension of $p \upharpoonright \nu$ by $f(\tau)$, has a direct extension in $D_{\tau}$. For every $n < \omega$, let $\dot{X}_{\nu}^n$ be the $P^1_{\nu}$ name of the set $\{\tau \in Y_{\nu} \mid n(\tau) = n\}$, and let $\sigma^0_n$ be the $P^1_{\nu}$ statement

$$\sigma^0_n : \dot{X}_{\nu}^n \in U_{\nu}^*.$$

Then since $P^1_{\nu}$ satisfies Prikry property, then there exists a unique $n < \omega$ and some $r \geq^* p \upharpoonright \nu$ so that $r \forces \dot{X}_{\nu}^n \in U_{\nu}^*$. Let us denote $X_{\nu}^n$ by $X_{\nu}$. We conclude that for every $\tau$, a name of an ordinal in $X_{\nu}$, then there exists a structural tree $f(\tau)$ of degree $n$, such that every $f(\tau)$ structural extension of $r^{+(\tau,\nu)}$, has a direct extension in $D$. We conclude the the function $f$ mapping every such name $\tau$ to $f(\tau)$, is a structural function of degree $n + 1$, and the claim of the Lemma holds with respect to $p, f$.

Let $\delta \leq \kappa$ be a limit ordinal, and suppose that the claim holds in every $P^1_{\nu}$ for all $\nu < \delta$. Fix some $p \in P^1_{\delta}$, and $D \subset P^1_{\delta}$ dense open. In order to prove the
result, it is sufficient to show that for some \( \nu < \delta \), there is a \( \mathcal{P}_\nu \) name \( \dot{t} \) with \( p \restriction \nu \forces \dot{t} \geq^* p \setminus \nu \), so that the set \( D_i = \{ r \geq p \restriction \nu \mid r \setminus i \in D \} \) is dense open in \( \mathcal{P}_\nu \). Suppose otherwise, and let us construct a condition \( p^* = \langle p_\nu^* \mid \nu < \delta \rangle \) so that \( p^* \geq^* p \), and for every \( \nu < \delta \), \( p^* \restriction \nu \forces \sigma_{\nu}^0 \), where

\[
\sigma_{\nu}^0 : \forall t \geq^* (p \setminus \nu). t \notin D(G_\nu).
\]  

We construct \( p^* \nu \) by recursion on \( \nu < \delta \). Suppose that \( p^* \restriction \nu = \langle p_\nu^* \mid \mu < \nu \rangle \) have been defined, and satisfy 9. Fix some \( G_\nu \subseteq \mathcal{P}_\nu \) generic, with \( p^* \restriction \nu \setminus \nu \in G_\nu \), and consider the forcing \( \mathcal{P}_1 \nu = Q_{\nu+1} \mathcal{P}_1 \nu (\nu+1) \). Since \( Q_{\nu+1} \) satisfies Prikry property, then there exists some \( r_\nu \geq^* (p_\nu)_G \) which decides \( \sigma_{\nu+1}^0 \). If \( r_\nu \forces \neg \sigma_{\nu+1}^0 \), then there would exist some \( q_\nu \geq^* p \setminus (\nu + 1) \), so that \( r_\nu \forces q_\nu \subseteq D(G_\nu) \). This is impossible as \( r_\nu \setminus q_\nu \geq^* p \setminus \nu \), but \( \sigma_{\nu}^0 \) holds in \( V[G_\nu] \) as \( p^* \restriction \nu \in G_\nu \).

We conclude that in \( V[G_\nu] \) then \( r_\nu \forces q_\nu \sigma_{\nu+1}^0 \). Back in \( V \), let \( p_\nu^* \) be a \( \mathcal{P}_\nu \) name for \( r_\nu \), so \( p^* \restriction \nu \forces p_\nu^* \geq^* p_\nu \), and \( p^* \restriction \nu \setminus p_\nu^* \forces \sigma_{\nu+1}^0 \).

Now suppose that \( \delta' < \delta \) is a limit ordinal, and we have constructed the sequence \( \langle p_\mu^* \mid \mu < \delta' \rangle \), so that for every \( \nu < \delta' \), then \( p^* \restriction \nu = \langle p_\mu^* \mid \mu < \nu \rangle \) satisfies \( \sigma_{\nu}^0 \). Let us verify that \( p^* \restriction \delta' \forces \sigma_{\delta'}^0 \). If this is not true, then there exists some \( t \in \mathcal{P}_1 \delta' \) so that \( p^* \restriction \delta' \forces t \geq^* p \setminus \delta' \), and some \( r \geq p^* \restriction \delta' \) so that \( r \forces t \in D(G_{\delta'}) \). By the definition of \( r \forces t \in D(G_{\delta'}) \) there exists some \( r' \geq r \) so that \( r' \setminus t \in D \). As \( \delta' \) is a limit ordinal, and \( \text{supp}(r') \) is finite, then there exists some \( \nu < \delta' \) so that \( r' \setminus \nu \geq^* p \setminus \nu \). Let \( s = r' \restriction \nu \), then \( (r' \setminus \nu) \setminus t \geq^* p \setminus \nu \) and \( s \forces (r' \setminus \nu) \setminus t \in D(G_\nu) \). This is absurd as \( s \geq p^* \restriction \nu \) and therefore forces \( \sigma_{\nu}^0 \). \( \square \)

Next, for every \( \alpha, \beta < \lambda \), we introduce a family of finite iterated ultrapowers of \( V^0 \) connected to \( \pi_{\alpha, \beta}^0 \) consider some related structural functions and structural extensions.

**Definition 5.6** (structural iteration and compatible conditions). For \( \alpha, \beta < \lambda \) a structural iteration with respect to \( U_{\alpha, \beta}^1 \) is a finite iterated ultrapower \( \bar{M} = \langle M_{m, j_{k, m}} \mid k < m \leq n \rangle \) of length \( n < \omega \) which is constructed as follows:

\[
M_0 = V_0, j_{0,1} = \begin{cases} j_{0, \beta, \alpha}^0 : V_0 \rightarrow M_{0, \beta, \alpha}^0 & \text{if } \beta \leq \alpha \\ k_{0, \alpha, \beta}^0 : V_0 \rightarrow M_{0, \alpha, \beta}^0 & \text{if } \alpha > \beta \end{cases}
\]

\(^{10}\)Considered now as a \( Q_\nu \) statement.

\(^{11}\)Here \( D(G(Q_\nu)) \) is the name for a \( Q_\nu \) generic filter and \( D(G(Q_\nu)) \) is the induced dense set in \( \mathcal{P}_1 \setminus (\nu + 1) \).
and set $\nu_0^1 = j_0^0(\alpha, \beta)(\kappa) < k_0^0(\kappa)$ if $\alpha < \beta$, and leave $\nu_0^1$ undefined otherwise.

For every $1 \leq k < n$, suppose that $\tilde{M} \upharpoonright k+1 = \langle M_m, j_{i,m} \mid i < m \leq k \rangle$ and $\langle (\nu_i, \nu_i^1) \mid i < k \rangle$ have been defined, then there exists some $\nu_k < j_{0,k}(\kappa)$ so $\nu_k \in j_{0,k}(\Delta') \setminus (\kappa \cup \{i \mid i < k\})$. Therefore, there are unique $\alpha_k, \beta_k$ with $\nu \in j_{0,k}(\Delta)_{\alpha_k}(\beta_k)$. We then define

1. $j_{k,k+1} : M_k \to M_{k+1} \cong \text{Ult}(M_k, U_0^{0, \nu_k}((\beta_k, \alpha_k)))$.
2. $\nu_1^1 = j_{k,k+1}(\nu_k)$.

We say that a condition $p \in j_{0,n}(\mathcal{P}^1)$ is compatible with the structural iteration $\tilde{M}$, if there exists a sequence $\bar{p} = \langle p^k \mid k \leq n \rangle$ so that the following holds:

1. $p^0 \in \mathcal{P}^1$, in $V^0$.
2. $p^1 = \begin{cases} j_{0,1}(p^0) = j_{0,1}(\beta, \alpha)(p^0) & \text{if } \beta \leq \alpha \\ (p^1)^{\circ (\kappa, j_{0,1}(\Delta)(\kappa))} & \text{as in Definition 4.9 if } \alpha < \beta. \end{cases}$
3. For every $1 \leq k < n$, then $p^{k+1} = (p^k)^{\circ (\kappa, j_{k,k+1}(p) \setminus \nu_k^1)}(\nu_k, \nu_1^1)$, where
   - $q \geq^* j_{k,k+1}(p) \upharpoonright [\nu_k, \nu_1^1]$,
   - $p^k \models \nu_k \in j_{k,k+1}(p_{\nu_k}) = j_{k,k+1}(p)^{\circ (\kappa, j_{k,k+1}(\Delta)(\kappa))}$. Note that the existence of such $q$ is guaranteed by Definition 4.6
4. $p \geq^* p^n$.

**Remark 5.7.** We point out some simple facts.

1. If $\tilde{M}$ is a structural iteration with respect to $U^1_{(\alpha, \beta)}$, then $j_{0,1}$ is exactly the same (iterated) ultrapower which is used in the definition of $\pi^0_{(\alpha, \beta)}$, which is also the embedding used to define $U^1_{(\alpha, \beta)}$ in definitions 4.6 and 4.9.
2. For every $k < n - 1$, both $\nu_k, \nu_1^1$ are not moved in the rest of the iteration, $j_{k+1,n}$ of $\tilde{M}$. 

Lemma 5.8. Let $\vec{M}$ be a structural iteration of length $n$. Furthermore, since the measure $U^0_{(\beta,\alpha)}$ which generates $j_{k,k+1}$ does not include $j_{0,k}(\Delta') \cap \nu_k$, then $\nu_k \notin j_{0,k+1}(\Delta')$. Similarly, note that $\kappa$ does not belong to $k^0_{\alpha,\beta}(\Delta')$ if $\alpha < \beta$, nor to $j_{(\beta,\alpha)}(\Delta')$ if $\alpha \geq \beta$. We conclude that for every $k < n$, then $\nu_k \notin j_{0,n}(\Delta')$, and $\nu_k \in j_{0,n}(\Delta')$ whenever defined.$^{12}$

4. If $p \in j_{0,n}(P^1)$ is compatible with the iteration $\bar{M}$, witnessed by a sequence $\langle p \mid i \leq n \rangle$ as in the definition, we get that for every $k < n$, then $p^{k+1} \models j_{0,k+1}(d)(\nu_k^1) = \nu_k$. Since both $\nu_k, \nu_k^1$ are not moved by the rest of the iteration, we conclude that

$$p \models j_{0,n}(d)(\nu_k^1) = \nu_k$$

whenever $\nu_k^1$ is defined.

5. Let $p_0, p_1 \in j_{0,n}(P^1)$ be two compatible with $\bar{M}$, as witnessed by sequences $\langle p^k \mid k \leq n \rangle$, and $\langle p^k \mid k \leq n \rangle$ respectively. Suppose that $p^0_0, p^0_1 \in P^1$ are compatible in the $P^1$ ordering, then $p_0, p_1$ are compatible in the $j_{0,n}(P^1)$ ordering. For this, just note that by induction on $k \leq n$, we have that $p^k_0, p^k_1$ are compatible in $j_{0,k}(P^1)$ for every $0 \leq k \leq n$. The inductive step holds since $p^k_0, p^k_1$ are compatible, then so are $j_{k,k+1}(p^0_k), j_{k,k+1}(p^1_k)$, and for every $q_0, q_1$ with $q_0 \geq j_{k,k+1}(p^0_k) \upharpoonright [\nu_k, \nu_k^1]$ $q_1 \geq j_{k,k+1}(p^1_k) \upharpoonright [\nu_k, \nu_k^1]$, then $q_0, q_1$ are compatible.

Lemma 5.8. Let $\bar{M}_0 = \langle M_m, j_{k,m} \mid k < m \leq n_0 \rangle$ be a structural iteration of length $n_0$, and $p \in j_{0,n_0}(P^1)$ compatible with $\bar{M}_0$. Then for every structural function $f = f^n$ of degree $n$, which avoids supp($p$) $\cup \kappa$, then there exists a structural iteration $\bar{M} = \langle M_m, j_{k,m} \mid k < m \leq n_0 + n \rangle$ of length $n_0 + n$, extending $\bar{M}_0$, and $q \in j_{0,n_0+n}(P^1)$ so that

1. $q$ is compatible with $\bar{M}$, and
2. $q$ is a structural extension of $j_{n_0,n_0+n}(p)$ by $j_{n_0,n_0+n}(f)$.

Proof. The construction of $q$ is straight forward. Let us denote $n_0 + n$ by $n^*$. For every $k$, $n_0 \leq k < n^*$, we define simultaneously $j_{k,k+1} : M_k \to M_{k+1}$ extending the iteration, $p^{k+1} \in j_{0,k+1}(P^1)$ which is compatible with

$^{12}$It may be non-defined for $k = 0$, i.e. $\nu_k = \kappa$. 36
\( \bar{M} \upharpoonright (k + 1) \), and a structural function \( g^{n^*-k} \) of degree \( n^* - k \) which avoids \( \text{supp}(p^{k+1}) \cup \kappa \). \( M_{n_0} \) is given by \( \bar{M}_0 \), \( p^{n_0} = p \) is compatible with \( \bar{M}_0 \), and \( g^{n^*-n_0} = g^n = f \). Suppose that \( \bar{M} \upharpoonright k + 1, p^k, g^{n^*-k} \) have been defined. Let \( \nu_k = \nu_{g^{n^*-k}}, \) then \( \nu_k \in j_{0,k}(\Delta^k) \setminus (\kappa \cup \text{supp}(p^k)) \). Note that \( \{ \nu_i^1 \mid i < k \} \subset \text{supp}(p^k) \) since \( p^k \) is compatible with \( \bar{M} \upharpoonright (k + 1) \). Let \( j_{k,k+1} : M_k \to M_{k+1} \cong \text{Ult}(M_k, U^0_{\nu_k(\beta_0, \alpha_k)}) \) as in Definition 5.6. We have that \( X_{g^{n^*-k}} \) is a \( j_{0,k}(P^1) \upharpoonright \nu_k \) name of a set in \( U^1_{\nu_k, \alpha, \beta} \). Let \( Y_k \) be the name of \( X_{g^{n^*-k}} \cap p^k_{\nu_k} \), then

\[
p^k \upharpoonright \nu_k \models Y_k \in U^1_{\alpha, \beta}.
\]

By the definition of \( U^1_{\alpha, \beta} \) there is some \( q \geq^* j_{k,k+1}(p^k) \upharpoonright [\nu_k, \nu_k^1] \), so that \( p^k \upharpoonright q \models \nu_k \in j_{k,k+1}(Y_k) \). We define \( p^{k+1} = (p^k \upharpoonright q)(j_{k,k+1}(p^k) \setminus \nu_k^1))^{(\nu_k, \nu_k^1)} \).

Let \( \nu_k \) be a name which is interpreted as \( \nu_k \) by every condition \( r \) with \( r \models \nu_k \in j_{k,k+1}(X^{g^{n^*-k}}) \), and is interpreted as \( \min(j_{k,k+1}(Y_k)) \) otherwise. Clearly \( \nu_k \in \text{dom}(j_{k,k+1}(g^{n^*-k})) \), and we set \( g^{n^*-(k+1)} = j_{k,k+1}(g^{n^*-k})(\nu_k) \). By the induction hypothesis and the definition of \( p^{k+1} \) and \( g^{n^*-(k+1)} \), it is simple to verify that \( p^{k+1} \) is compatible with \( \bar{M} \upharpoonright k + 2 \), and that \( g^{n^*-(k+1)} \) is of degree \( n^* - (k + 1) \), and avoids \( \kappa \cup \text{supp}(p^{k+1}) \).

Also, note that the definition of \( p^{k+1} \) suits the recursive condition which appear in the definition of structural extension of \( j_{k,k+1}(p^k) \) by \( j_{k,k+1}(g^{n^*-k}) \).

The construction terminates after \( n = n^* - n_0 \) steps, and obtains \( \bar{M} = \langle M_m, j_{k,m} \mid k < m \leq n^* \rangle, \langle p^i \mid i \leq n^* \rangle, \) and \( g^0 = g^{n^*-n^*} = 0 \), a structural function of degree 0. Since for every \( k \leq n^* \), \( \nu_k, \nu_k^1 \) are not moved after stage \( k \) of the construction, i.e., are not moved by \( j_{k+1,n} \), then we see that \( g^0 \) is the \( n \)-th image of \( j_{n_0,n_0+n}(p) \), which can described by

1. \( h^0 = j_{n_0,n^*}(f) \) is of degree \( n \),
2. \( h^{i+1} = h^i(\nu_{n^*+i}) \) is of degree \( n - i - 1 \) for all \( i < n \),
3. \( g^0 = h^n \).

By the definition of \( p^{k+1} \) for every \( n_0 \leq k < n^* \), we conclude that the sequence \( \langle j_{k,n^*}(p^k) \rangle \in j_{n_0,n^*}(P^1) \rangle \) is an increasing sequence, starting at \( j_{n_0,n^*}(p^{n_0}) = j_{n_0,n_0+n}(p) \), and witnesses that \( q = p^{n^*} = j_{n^*,n^*}(p^{n^*}) \) is a structural extension of \( j_{n_0,n_0+n}(p) \) by \( j_{n_0,n_0+n}(f) \). \( \square \)

By combining the results of Lemma 5.3 and Lemma 5.8 we conclude the following.

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Corollary 5.9. Let $\vec{M}_0$ be a structural iteration of length $n_0$, and let $D$ be a $\mathcal{P}^1$–name of a dense open set in $j_{0,n_0}(\mathcal{P}^1) \setminus \kappa$. Then for every $\vec{M}_0$ compatible condition $p \in j_{0,n_0}(\mathcal{P}^1)$, there exists a structural iteration $\vec{M}$ extending $\vec{M}_0$, of length $n^*$, and a $\vec{M}$ compatible condition $q \in j_{0,n^*}(\mathcal{P}^1)$, so that

1. $q \geq j_{n_0,n^*}(p)$,
2. $q \upharpoonright \kappa = j_{n_0,n^*}(p) \upharpoonright \kappa = p \upharpoonright \kappa$, and
3. $q \upharpoonright \kappa \models (q \setminus \kappa) \in D$.

5.2 A proof for Proposition 5.3

Let $\alpha, \beta < \lambda$. We return to the $V^0$ iterated ultrapower $T^0 = \langle Z^0_i, \sigma^0_{i,j} \mid 0 \leq i < j < \theta \rangle$ associated to $U^{1}_{(\alpha, \beta)}$ defined in 5.1, and the resulting $\pi^{0}_{\alpha, \beta} : V^0 \rightarrow Z^0_{\alpha, \beta}$. Comparing the definition of the iteration $T^0$, with the definition of structural iterations with respect to $U^{1}_{(\alpha, \beta)}$ (Definition 5.6), we get that $\pi^{0}_{\alpha, \beta}$ is the direct limit of all the finite iterations $\vec{M} = \langle M_k, j_{k,m} \mid k \leq m \leq n \rangle$ which are structural iteration with respect to $U^{1}_{(\alpha, \beta)}$. Before turning to prove Proposition 5.3 we add the following notations.

Definition 5.10. 1. For every structural iteration $\vec{M} = \langle M_k, j_{k,m} \mid k \leq m \leq n \rangle$ with respect to $U^{1}_{(\alpha, \beta)}$, let $j_{\vec{M}} : V^0 \rightarrow M_n$ denote $j_{0,n}$, and let $k_{\vec{M}} : M_n \rightarrow Z^0_{\alpha, \beta}$ denote then embedding of $M_n$ in the directed limit.

2. Let $G^1 \subset \mathcal{P}^1$ be a $V^0$ generic. We say that a condition $p \in j_{\vec{M}}(\mathcal{P}^1)$ is compatible with both $\vec{M}$ and $G_1$ if there exists a sequence $\langle p^k \mid k \leq n \rangle$ witnessing that $p$ is compatible with $\vec{M}$ so that $p^0 \in G_1$.

3. Let $F_{\vec{M}, G^1} \subset j_{\vec{M}}(\mathcal{P}^1)$ denote the set of all the conditions $p \in j_{\vec{M}}(\mathcal{P}^1)$ which are compatible with both $\vec{M}$ and $G^1$.

(Proof. Proposition 5.3). We define $G^1_{\alpha, \beta}$ by

$$G^1_{\alpha, \beta} = \bigcup \{ k_{\vec{M}}^\frown F_{\vec{M}, G^1} \mid \vec{M} \text{ is a structural iteration } \}.$$ 

It is clear that $\pi^0_{\alpha, \beta}^\frown G^1 \subset G^1_{\alpha, \beta}$. Furthermore, by Remark 5.7, every two conditions in $G^1_{\alpha, \beta}$ are compatible in the $\pi^0_{\alpha, \beta}(\mathcal{P}^1)$ ordering.

Let us show that $G^1_{\alpha, \beta}$ is $\pi^0_{\alpha, \beta}(\mathcal{P}^1)$ generic over $Z^0_{\alpha, \beta}$. It is clear that $G^1_{\alpha, \beta} \upharpoonright$
\[ \kappa = G_1 \text{ is } \mathcal{P}_1 = \pi_{\alpha,\beta}^0(\mathcal{P}_1) \upharpoonright \kappa \text{ generic over } Z_{\alpha,\beta}^0. \] Let \( D' \) be a \( \mathcal{P}_1 \) name for a dense open set in \( \pi_{\alpha,\beta}^0(\mathcal{P}_1) \setminus \kappa \). Then \( D' \) can be represented in some finite structural sub-iteration of \( Z_{\alpha,\beta}^0 \). Let \( \bar{M}_0 \) be a structural iteration, and \( D \subset j_{\bar{M}_0}(\mathcal{P}_1) \) dense open, so that \( k_{\bar{M}}(D) = D' \). Take any \( p \in F_{\bar{M}_0,G}, i.e., \) \( p \) is compatible with both \( \bar{M}_0 \) and \( G_1 \), then by Corollary 5.9 is a structural iteration \( \bar{M} \) extending \( \bar{M}_0 \), and a compatible condition \( q \in j_{\bar{M}}(\mathcal{P}_1) \) so that \( q \upharpoonright \kappa = p \upharpoonright \kappa \), and \( q \upharpoonright \kappa \models q \setminus \kappa \in D \). The last implies that \( q \in F_{\bar{M},G} \), which in turn, implies that \( q' = k_{\bar{M}}(q) \in D' \cap G_1^{\alpha,\beta} \). This conclude the genericity of \( G_1^{\alpha,\beta} \).

If it follows we can naturally extend the embedding \( \pi_{\alpha,\beta}^0 : V^0 \to Z_{\alpha,\beta}^0 \) to an elementary embedding, \( \mathcal{P}_{\alpha,\beta}^1 : V^0[G^1] \to Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1] \), defined by \( \pi_{\alpha,\beta}^0(\bar{Y}) = (\pi_{\alpha,\beta}^0(\bar{X}))_{G_{\alpha,\beta}^1} \), for every \( \mathcal{P}_1 \) name \( \bar{X} \). It remains to verify that this embedding coincides with the ultrapower embedding by \( U_{\alpha,\beta}^1 \). Let \( U \) denote the the normal measure on \( \kappa \) in \( V^0[G^1] \) defined by \( X \in U \iff \kappa \in \pi_{\alpha,\beta}^0(X) \). We claim that \( U_{\alpha,\beta}^1 = U \). It is sufficient to show that \( U_{\alpha,\beta}^1 \subset U \). For this, note that in both Definitions 4.6 and Definition 4.9 for \( \alpha < \beta \), then the iterations used to define \( U_{\alpha,\beta}^1 \) are in fact structural iterations with respect to \( U_{\alpha,\beta}^1 \). Furthermore, if \( X \in U_{\alpha,\beta}^1 \), and let \( t \) a condition in \( j_{0,1}(\mathcal{P}_1) \) which forces \( \bar{\kappa} \in j_{0,1}(\bar{X}) \) according to Definitions 4.6 and 4.9 then \( t \) is compatible with \( G_1 \). We get that for every \( X \in U_{\alpha,\beta}^1 \) and \( \bar{X} \) a \( \mathcal{P}_1 \) name for \( X \), then there exists a structural iteration \( \bar{M} \) with respect to \( U_{\alpha,\beta}^1 \), and a compatible condition \( t \in F_{\bar{M},G} \), so that \( t \models \bar{\kappa} \in j_{\bar{M}}(\bar{X}) \). Applying \( k_{\bar{M}} \), we conclude that \( k_{\bar{M}}(t) \models k_{\bar{M}}(\bar{\kappa}) \in \pi_{\alpha,\beta}^0(\bar{X}) \). As \( cp(k_{\bar{M}}) \) \( \kappa \), and \( k_{\bar{M}}(t) \in G_{\alpha,\beta}^0 \), then \( X \in U \).

Let \( j_{\alpha,\beta}^1 : V^0[G^1] \to M_{\alpha,\beta}^1 \simeq \text{Ult}(V^0[G^1],U_{\alpha,\beta}^1) \). Since \( U_{\alpha,\beta}^1 = U \) then \( \pi_{\alpha,\beta}^1 \) can be factorized by \( \pi_{\alpha,\beta}^1 \) \( \alpha e_{\alpha,\beta}^1 \circ j_{\alpha,\beta}^1 \) so that \( e_{\alpha,\beta}^1 : M_{\alpha,\beta}^1 \to Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1] \) defined by \( [f]_{U_{\alpha,\beta}^1} \mapsto \pi_{\alpha,\beta}^1(f)(\kappa) \). It is standard to verify that \( e_{\alpha,\beta}^1 \) is an elementary embedding. Therefore, in order to show that \( j_{\alpha,\beta}^1, M_{\alpha,\beta}^1 \) equals to \( \pi_{\alpha,\beta}^1, Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1] \) it suffice to show that \( e_{\alpha,\beta}^1 \) is surjective. Take \( x \in Z_{\alpha,\beta}^0[G_{\alpha,\beta}^1] \), and let \( \tilde{x} \) a \( \pi_{\alpha,\beta}^1(\mathcal{P}_1) \) name for \( x \). Then \( \tilde{x} \in Z_{\alpha,\beta}^0 \) means that there exists some structural iteration \( \bar{M} \) and a \( j_{\bar{M}}(\mathcal{P}_1) \) name \( \tilde{y} \), so that \( \tilde{x} = k_{\bar{M}}(\tilde{y}) \).

Suppose that \( \bar{M} = \langle M_k, j_{k,m} \mid k < m \leq n \rangle \) has critical points \( \langle \nu_k \mid k < n \rangle \). Therefore, for every \( k < n \), there exists some function \( h : \kappa^n \to \kappa \) so that \( \nu_k = j_{0,k}(h)(\nu_0, \ldots, \nu_{k-1}) \). It follows that every \( y \in M_n \) of the form \( j_{0,n}(h)(\nu_0, \ldots, \nu_{n-1}) \).

For every \( k < n \), let \( \nu_i = k_{\bar{M}}(\nu_k) \), where \( i < \theta \). By applying \( k_{\bar{M}} \) we get
that \( \bar{x} = \bar{\pi}_{\alpha,\beta}(h)(\nu_0, \ldots, \nu_k) \) for some \( h : \kappa \to V^0 \) in \( V^0 \). Note that if \( x \in Z^0_{\alpha,\beta}[G^1_{\alpha,\beta}] \) is of the form \( \pi^1_{\alpha,\beta}(\gamma_0, \ldots, \gamma_k) \) for some \( k < \omega \), and every \( \gamma_i \) is of the form \( \gamma_i = \pi^1_{\alpha,\beta}(h_i)(\kappa) \), then by replacing the functions \( h_i \) in \( h(\nu_0, \ldots, \nu_k) \) we get a function \( h' \) for which \( x = \pi^1_{\alpha,\beta}(h')(\kappa) \). Therefore, if we show that for every \( k < n \) then \( \nu_k = \pi^1_{\alpha,\beta}(h)(\nu_0, \ldots, \nu_{k-1}) \) for some \( h \), then we can conclude by induction on \( 0 < k < \omega \) that there exists a function \( h' \) so that \( \nu_k = \pi^1_{\alpha,\beta}(h')(\kappa) \). For \( k = 0 \), this is trivial as \( \nu_0 = \kappa \) and \( k_M(\kappa) = \kappa \). Fix \( 0 < k < n \). By the above, \( \nu_k = j_{0,k}(\nu_0, \ldots, \nu_{k-1}) \) in \( M_n \), and therefore \( \nu^1_k = j_{0,k+1}(\nu_0, \ldots, \nu_{k-1}) \) in \( M_{k+1} \). Since \( j_{k+1,n} \) does not move any of \( \nu_0, \ldots, \nu_k \), then \( \nu^1_k = j_{0,n}(h)(\nu_0, \ldots, \nu_{k-1}) \) in \( M_n \). Let \( p \in F_{M,G^1} \), then \( p \Vdash \check{\nu}_k = j_{0,n}(\check{d})(\check{\nu}^1_k) = j_{0,n}(\check{h}')(\nu_0, \ldots, \nu_{k-1}) \), where \( \check{d} \) is the name of the generic Priikry function, and \( h' = d \circ h \). Let \( q = k_M(p) \), then \( q \in G^1_{\alpha,\beta} \), and \( q \Vdash \check{\nu}_k = \pi^0_{\alpha,\beta}(h')(\nu_0, \ldots, \nu_{k-1}) \). \[ \square \]

**Corollary 5.11 (Description Of \( j_{(a,\beta)} \upharpoonright V \)).** Given a description of the \( V^0 \) iteration, \( T^0 = \langle Z^0_i, \sigma^0_{i,j} \mid 0 \leq i < j < \theta \rangle \) leading to the definition of \( \pi^0_{\alpha,\beta} : V^0 \to Z^0_{\alpha,\beta} \), it is straight forward to deduce its restriction to \( V \). We shall denote the restriction of \( T^0 \) to \( V \) by \( T = \langle Z_i, \sigma_{i,j} \mid 0 \leq i < j < \theta \rangle \), and its resulting limit by \( \pi_{\alpha,\beta} : V \to Z_{\alpha,\beta} \), which is just \( \pi^0_{\alpha,\beta} \upharpoonright V \).

As noted in Section [4.1](#), the restriction of the embeddings \( j^0_{(a,\beta)} \) and \( k^0_{a,\beta} \) to \( V \), are given by

1. \( j^0_{(a,\gamma_0)} \upharpoonright V = j_\alpha \),
2. \( i^1_{(\beta,\gamma_0)} \upharpoonright M_a = i^{M_a}_\beta \),
3. \( j^0_{(a,\gamma_a),(\beta,\gamma_0)} \upharpoonright V = j_{a,\beta} \),

We can therefore describe \( T = \langle Z_i, \sigma_{i,j} \mid 0 \leq i < j < \theta \rangle \) as follows:

1. We have \( Z_0 = V \). For \( \sigma_{0,1} = \sigma^0_{0,1} \upharpoonright V : Z_0 \to Z_1 \), we have that
   \[
   \sigma_{0,1} = \begin{cases}
   j^0_{(\beta,\alpha)} \upharpoonright V = j_\beta & \text{if } \alpha \geq \beta \\
   k^0_{a,\beta} \upharpoonright V = j^0_{(a,\beta),(\beta,\alpha)} \upharpoonright V = j_{a,\beta} & \text{if } \alpha > \beta
   \end{cases}
   \]
2. Given \( T \upharpoonright i \) and embeddings \( \sigma_{j,i} : Z_j \to Z_i \) for \( j < i \), so that \( Z_j = K(Z^0_j) \) (i.e., the core of \( Z_j \)), and \( \sigma_{j,i} = \sigma^0_{j,i} \upharpoonright Z_j \), we have that
   \[
   \sigma_{i,i+1} = j^0_{\nu_{i,(\beta_1,\alpha_1)}} \upharpoonright Z_i = j^0_{\nu_{i,(\beta_1,\alpha_1)}} : Z_i \to Z_{i+1}.
   \]

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6 The structure of $\triangleleft(\kappa)$ in $V^1$

In this section we prove that the $\triangleleft$ order on the measures $\{U^1_{(\alpha,\beta)} \mid \alpha \leq \beta < \lambda\}$ is isomorphic to $R_\lambda$. The following is the main result of this section.

**Proposition 6.1.** Let $\alpha' \leq \beta'$, $\alpha \leq \beta$ be ordinals below $\lambda = o(\kappa)$, then $U^1_{(\alpha',\beta')} \triangleleft U^1_{(\alpha,\beta)}$ in $V^0[G^1]$ if and only if $\beta' < \alpha$.

Note that the Proposition refers only to $U^1_{(\alpha,\beta)}$ for $\alpha \leq \beta < \lambda$, though we know that there are more normal measures in $V^1$, i.e. $U^1_{(\alpha,\beta)}$ for $\alpha > \beta$.

We turn to the first claim.

**Claim 6.2.** If $\beta' < \alpha$ then $U^1_{(\alpha',\beta')} \triangleleft U^1_{(\alpha,\beta)}$.

**Proof.** By Definitions 4.6 and 4.9 it is clear that for $\alpha' \geq \beta'$, then $U^1_{(\alpha',\beta')}$ is defined in every inner model of $V^1$ which contains $G^0, G^1$ and $U^0_{(\beta',\alpha')}$. Similarly, if $\alpha' < \beta'$ then $U^1_{(\alpha',\beta')}$ is defined in every inner model of $V^1$ which contains $G^0, G^1$, $U^0_{(\beta',\alpha')}$, and $U^0_{(\beta',\beta')}$. We know that for every $\alpha, \beta$ then $\text{Ult}(V^1, U^1_{(\alpha,\beta)}) \cong Z^0_{\alpha,\beta}[G^1_{\alpha,\beta}] = Z_{\alpha,\beta}[G^0_{\alpha,\beta} * G^1_{\alpha,\beta}]$, where $Z^0_{\alpha,\beta}$ is the iterated ultrapower of $V^0$ by $\pi^0_{\alpha,\beta}$. Furthermore, we have that $G^0 = G^0_{\alpha,\beta} \upharpoonright (\kappa + 1)$ and $G^1 = G^1_{\alpha,\beta} \upharpoonright \kappa$.

We will conclude the result of the claim by the following observations, and a simple case by case inspection. First, by Corollary 4.4 we have that for every $\alpha, \beta, \alpha', \beta' < \lambda$ then $U^0_{(\alpha',\beta')} \triangleleft U^0_{(\alpha,\beta)}$ whenever $\alpha' < \alpha$. Second, by Definition 5.4 we have that $\alpha, \beta < \lambda$ then $\pi^0_{\alpha,\beta}$ is of the form $\sigma^0_{\alpha,\beta} \circ \sigma^0_{\alpha',\beta'}$, where
\[
\sigma_{0,1}^0 = \begin{cases}
  \beta < \alpha & \text{if } \beta \leq \alpha \text{ and } \operatorname{cp}(\sigma_{1,0}^0) > \kappa. \text{ Therefore if } U_{0}(\alpha',\beta') \in M_{0}(\alpha,\beta) \text{ then } \\
  k_{0,1}^0 & \text{if } \alpha \geq \beta, \\
\end{cases}
\]
\[
U_{0}(\alpha',\beta') \in Z_{0,1,\beta}. \text{ We therefore conclude the following results:}
\]

1. For \( \alpha' = \beta' \) and \( \alpha = \beta \), then \( U_{0}(\beta',\beta') \in Z_{0,\alpha,\beta} \) if \( U_{0}(\beta',\beta') \in M_{0}(\alpha,\alpha) \) which holds when \( \beta' < \alpha \).

2. For \( \alpha' = \beta' \) and \( \alpha < \beta \), then \( U_{0}(\beta',\beta') \in Z_{0,\alpha,\beta} \) if \( U_{0}(\beta',\beta') \in M_{0}(\alpha,\alpha,\beta) \). The last holds if \( U_{0}(\beta',\beta') \in M_{0}(\alpha,\beta) \) which occurs when \( \beta' < \alpha \).

3. For \( \alpha' < \beta' \) and \( \alpha = \beta \), then \( U_{0}(\beta',\beta'), U_{0}(\beta',\alpha') \in Z_{0,\alpha,\beta} \) if \( U_{0}(\beta',\beta'), U_{0}(\beta',\alpha') \in M_{0}(\alpha,\alpha) \), which holds when \( \beta' < \alpha \).

4. For \( \alpha' < \beta' \) and \( \alpha < \beta \), then \( U_{0}(\alpha',\beta'), U_{0}(\beta',\alpha') \in Z_{0,\alpha,\beta} \) if \( U_{0}(\alpha',\beta'), U_{0}(\beta',\alpha') \in M_{0}(\alpha,\alpha,\beta) \). This occurs when \( U_{0}(\alpha',\beta'), U_{0}(\beta',\alpha') \in M_{0}(\alpha,\beta) \) which holds when \( \beta' < \alpha \).

We next show that for \( \alpha' < \beta' \) and \( \alpha < \beta \), then \( U_{1}(\alpha',\beta') \land U_{1}(\alpha,\beta) \) implies that \( \beta' < \alpha \). For this we need to survey some basic corollaries of inner model theory. All of these can be found in [19]. Suppose that \( U_{1}(\alpha',\beta') \in M_{0}(\alpha,\beta) \). By the uniqueness of the core model \( K(M) = Z_{0,\alpha,\beta} \) (described in Corollary 5.11). Let \( j_{\alpha',\beta'} : M_{0}(\alpha,\beta) \rightarrow M' \cong \operatorname{Ult}(M_{0}(\alpha,\beta), U_{0}(\alpha',\beta')) \), we have that \( \pi_{\alpha',\beta'} = j_{\alpha',\beta'} \downarrow Z_{0,\alpha,\beta} : Z_{0,\alpha,\beta} \rightarrow Z_{0,\alpha',\beta'} \) is an iteration of \( Z_{0,\alpha,\beta} \), and that \( Z_{0,\alpha',\beta'} = K(M') \). We claim that the ultrapowers \( M'_{0}(\alpha',\beta') \cong \operatorname{Ult}(M_{0}(\alpha,\alpha',\beta') \land U_{0}(\alpha',\beta')) \) and \( M_{1}(\alpha',\beta') \cong \operatorname{Ult}(V_{0}, U_{1}(\alpha',\beta')) \) agree on \( V_{1}(\alpha',\beta')(\kappa) \). For this, note first that \( M_{1}(\alpha,\beta) \cap V_{\kappa+1} = V_{\kappa+1} \). Taking an ultrapower by \( U_{0}(\alpha',\beta') \) of both models, which results in \( M'_{0}(\alpha',\beta') \) and \( M_{1}(\alpha',\beta') \) respectively, we conclude that \( j_{0}(\alpha',\beta') \downarrow (\kappa+1) = j_{0}(\alpha',\beta') \downarrow (\kappa+1) \), and that \( M'_{0}(\alpha',\beta') \cap V_{1}(\alpha,\beta')(\kappa) = M_{1}(\alpha',\beta') \cap V_{1}(\alpha,\beta')(\kappa) \). This agreement between \( M'_{0}(\alpha',\beta') \) and \( M_{1}(\alpha',\beta') \), also applies to the the initial segments of the embeddings \( j_{0}(\alpha',\beta') \) and \( j_{1}(\alpha,\beta') \) and to their restrictions to their core models. Namely, \( \pi_{\alpha',\beta'} \downarrow \kappa = \pi_{\alpha',\beta'} \downarrow \kappa \) and \( Z_{\alpha',\beta'} \downarrow j_{0}(\alpha',\beta')(\kappa) = Z_{\alpha',\beta'} \downarrow j_{1}(\alpha,\beta')(\kappa) \).

By results of inner model theory, we also know that \( j_{0}(\alpha',\beta') : M_{0}(\alpha,\beta) \rightarrow M'_{0}(\alpha',\beta') \) is an extension of \( \pi_{\alpha',\beta'} : Z_{\alpha,\beta} \rightarrow Z_{\alpha',\beta'} \). I.e., denote \( G_{0}(\alpha,\beta) \land G_{1}(\alpha,\beta) \subseteq \pi_{\alpha,\beta}(\mathcal{P}^0 \land \mathcal{P}_1)\) by \( G' \), then we get that

1. \( G' = j_{0}(\alpha',\beta') \downarrow \pi_{\alpha,\beta}(\mathcal{P}^0 \land \mathcal{P}_1) \) is \( \pi_{\alpha',\beta'}(\mathcal{P}^0 \land \mathcal{P}_1) \) generic over \( Z_{\alpha',\beta'} \).
2. \( M'_{\alpha',\beta'} = Z'_{\alpha',\beta'}[G'] \), and

3. for every \( x \in M^1_{(\alpha,\beta)} = Z_{\alpha,\beta}[G] \) and \( x = (\hat{x})_G \), then \( j'_{\alpha',\beta'}(x) = \pi'_{\alpha',\beta'}(\hat{x})G' \).

Finally, we note that both \( M^1_{(\alpha,\beta)} \) and \( M'_{\alpha',\beta'} \) are closed to \( \kappa \)-sequence in \( V^1 \). This holds since \( M'_{\alpha',\beta'} \cong \text{Ult}(M^1_{(\alpha,\beta)},U^1_{(\alpha',\beta')}) \) is closed to \( \kappa \)-sequences of \( M^1_{(\alpha,\beta)} \), while \( M^1_{(\alpha,\beta)} \cong \text{Ult}(V^1,U^1_{(\alpha,\beta)}) \) is closed to \( \kappa \)-sequences of \( V^1 \).

**Claim 6.3.** Let \( \alpha' \leq \beta' \), \( \alpha \leq \beta \) be ordinals below \( \lambda = o(\kappa) \), then \( U^1_{(\alpha',\beta')} \prec U^1_{(\alpha,\beta)} \) in \( V^1 \) implies that \( \beta' < \alpha \).

**Proof.** Suppose that \( U^1_{(\alpha',\beta')} \in M^1_{(\alpha,\beta)} \), and let \( j'_{\alpha',\beta'} : M^1_{(\alpha,\beta)} \to M'_{\alpha',\beta'} \cong \text{Ult}(M^1_{(\alpha,\beta)},U^1_{(\alpha',\beta')}) \), and \( \pi'_\alpha,\beta' = j' | Z_{\alpha,\beta} : Z_{\alpha,\beta} \to Z'_{\alpha',\beta'} \) as described above.

We know that \( \text{cp}(\pi'_\alpha,\beta') = \text{cp}(j'_\alpha,\beta') = \kappa \), and that \( \pi'_\alpha,\beta' \) is an embedding induced by a normal iteration of \( Z_{\alpha,\beta} \) which results in a model \( Z'_{\alpha',\beta'} \). By the argument given in the paragraph above, we know that \( Z'_{\alpha',\beta'} \) which agrees with \( Z_{\alpha',\beta'} \) up to \( j^1_{(\alpha',\beta')}(\kappa) \). In particular, for every cardinal \( \nu < j^1_{(\alpha',\beta')}(\kappa) \) then \( o^{Z_{\alpha',\beta'}}(\nu) = o^{Z'_{\alpha',\beta'}}(\nu) \).

We first compare \( o^{Z_{\alpha',\beta'}}(\kappa) = o^{Z'_{\alpha',\beta'}}(\kappa) \). For \( \alpha \leq \beta \), the description of \( Z_{\alpha,\beta} \) implies that \( o^{Z_{\alpha,\beta}}(\kappa) = \alpha \). Since \( \text{cp}(\pi'_\alpha,\beta') = \kappa \) then the normal iteration of \( Z_{\alpha,\beta} \) which induces \( \pi'_\alpha,\beta' \), begins with an ultrapower by a measures \( U_\gamma \) on \( \kappa \), for some \( \gamma < \alpha \). It follows that \( o^{Z_{\alpha',\beta'}}(\kappa) = \gamma < \alpha \).

If \( \alpha' = \beta' \) then \( o^{Z_{\alpha',\beta'}}(\kappa) = \beta' \) and therefore \( \beta' = \gamma < \alpha \), as claimed.

Suppose now that \( \alpha' < \beta' \), then \( o^{Z_{\alpha',\beta'}}(\kappa) = \alpha' \), so we may factor \( \pi'_\alpha,\beta' \) as \( \pi'_\alpha,\beta' = k' \circ j_{\alpha'} \). The fact \( \pi'_\alpha,\beta' | \kappa^+ = \pi'_\alpha,\beta' | \kappa^+ \) implies that \( \pi'_\alpha,\beta'(\kappa) = j_{\alpha'}(\kappa) \). Since \( j_{\alpha'}(\kappa) = k'(j_{\alpha'}(\kappa)) \) we infer that \( k'(j_{\alpha'}(\kappa)) > j_{\alpha'}(\kappa) \).

**subclaim 6.3.1.** \( j_{\alpha'}(\kappa) \) is a critical point of the iteration inducing \( k' \).

We have that \( k'(j_{\alpha'}(\kappa)) > j_{\alpha'}(\kappa) \). If \( j_{\alpha'}(\kappa) \) is not a critical point of \( k' \) then this can only occur if the length of the iteration generating \( k' \) is longer than \( j_{\alpha'}(\kappa) \). Each ultrapower in the iteration which generates \( k' \) is an ultrapower by a normal measure of ordinals \( \nu \neq j_{\alpha'}(\kappa) \), which cannot move \( j_{\alpha'}(\kappa) \). So, in order to obtain \( k'(j_{\alpha'}(\kappa)) > j_{\alpha'}(\kappa) \) the iteration \( k' \) must apply the same measure more than \( \omega \) many times. By iterating a single measure more than \( \omega \) many times, we get some \( \mu < \pi'_\alpha,\beta'(\kappa) \) which is inaccessible in \( Z'_{\alpha',\beta'} \), but has a cofinal \( \omega \) sequence in \( V^1 \). This is impossible since \( M'_{\alpha',\beta'} \) is a generic
extension of $\mathcal{Z}^{'\alpha,\beta'}$ by $\pi'_\alpha,\beta' \circ \pi_{\alpha,\beta}(\mathcal{P}^0 \ast \mathcal{P}^1)$, but by [23] and Lemma 7.1, both posets do not introduce new $\omega$-sequence. The sub claim follows.

Denote $j_{\alpha'}(\kappa)$ by $\mu$. Knowing that $\mu$ is a critical point of the iteration inducing $k'$, we get that $\mu < \pi'_\alpha,\beta'(\kappa)$, and $\mathcal{O}^{Z'_\alpha,\beta'}(\mu) = \mathcal{O}^{Z'_\alpha,\beta'}(\mu) = j_{\alpha'}(\beta')$. Furthermore, we have that $\mathcal{O}^{Z_{\alpha,\beta}}(\kappa) = \alpha$, and that $\pi' = k' \circ j_{\alpha'}$, where $j_{\alpha'} : Z_{\alpha,\beta} \to Z^*$, and $k' : Z^* \to Z'_\alpha,\beta'$. We get that $\mathcal{O}^{Z^*}(\mu) = j_{\alpha'}(\alpha)$. Since $\mu$ is a critical point of $k'$, then in some stage of the iteration of $Z^*$, defining $k'$, we apply an ultrapower of $\nu$ by a normal measure $U_{\nu,\gamma}$ on $\nu$, for some $\gamma < j_{\alpha'}(\alpha)$. It follows that $\mathcal{O}^{Z^*}(\mu) = \gamma$. We conclude that $j_{\alpha'}(\beta) = \mathcal{O}^{Z^*}(\mu) = \gamma < j_{\alpha'}(\alpha)$ which implies that $\beta < \alpha$. 

\section{The Normal Measures on $\kappa$ in $V^1$}

We assumed that the ground model $V$ is an extender model $V = L[E]$ so that $V = K(V)$. As our large cardinal usage does not exceed the assumption of $o(\kappa) = \kappa$, then we may assume that all extenders on the sequence of $V = L[E]$ are all equivalent to normal measures. In particular, $E$ does not include overlapping extenders (i.e., below $0^\sharp$). For description of the core model below $0^\sharp$, see [19].

Given generic filters, $G^0 \subset \mathcal{P}^0 V$, and $G^1 \subset \mathcal{P}^1 V^0$, then we denote $V^0 = V[G^0]$, and $V^1 = V^0[G^1] = V[G^0 \ast G^1]$.

The purpose of this section is to prove that the measures $U^1_{(\alpha,\beta)}$ are the only normal measures on $\kappa$ in $V^1$.

We first introduce some notations from [3], and play an important role in the analysis of normal measures on $\kappa$ in $V^1$.

**Definition 7.1.** For every $p \in \mathcal{P}1$ and and ordinal $\mu$, let $p^{-\mu} \supseteq p$ be the condition, obtained form $p$ by reducing the measure one set $p_\nu \in U^\mu_\nu$ for every $\nu \in \Delta \setminus (\mu + 1) \cup \text{supp}(p)$ to $p_\nu^{-\mu} = p_\nu \setminus (\mu + 1)$.

Also, for $\mu < \nu < \kappa$ with $\nu \in \Delta'$ We say that $\mu$ is available at $p_\nu$ if $p \upharpoonright \nu \models p_\nu \in p_\nu$. We then define an extension $p^{(\mu,\nu)} \supseteq p$, obtained by replacing $p_\nu$ with (the name of) $\mu$. I.e, $(p^{(\mu,\nu)})_\tau = p_\tau$ for every $\tau \neq \nu$, and $p^{(\mu,\nu)} \upharpoonright \nu \models p_\nu$, $p^{(\mu,\nu)}_\nu = \tilde{\mu}$.

**Remark 7.2.** Recall the sets $\Gamma, \Sigma, \Pi$ which were defined in Section 4.2, and which was discussed at the beginning of Section ??.
of [3], we get that for every \( p \in G^1 \), and \( \mu \in \Sigma \setminus \Gamma \) then \( p^{-\mu} \in G^1 \) is well.
Similarly, if \( \mu = d(\nu) \) and \( \mu \in \Pi \cap \Sigma \), then \( p^{(\mu,\nu)} - \mu - \nu \in G^1 \) as well.

We turn to the main claim of this section.

**Proposition 7.3.** The measures in \( \bar{U}^I = \{U^I_{(\alpha,\beta)} \mid \alpha, \beta < \lambda \} \) are the only measures on \( \kappa \) in \( V^1 \).

**Proof.** The proof heavily relies on the arguments and results of [3].

For \( \beta < \alpha < \lambda \), recall the notations from Section 4.1, by which \( j_\alpha : V \rightarrow M_\alpha \cong \text{Ult}(V, U_\alpha), i^M_{\alpha,\beta} : M_\alpha \rightarrow M_{\alpha,\beta} \), and \( j_{\alpha,\beta} = i^M_{\beta} \circ j_\alpha : V \rightarrow M_{\alpha,\beta} \).

Let \( W \) be a normal measure on \( \kappa \) in \( V^1 \), and \( j_W : V^1 \rightarrow M_W \cong \text{Ult}(V^1, W) \). Then by the results of inner model theory, there exists a normal iteration \( \pi : V \rightarrow M \), so that

1. \( \pi = j_W \upharpoonright V \), and a \( M \) generic filter,
2. \( G_W = j_W(G) \), then \( G_W = G^0_W \ast G^1_W \subset \pi(\mathcal{P}^0 \ast \mathcal{P}^1) \) is generic over \( M \), and
3. for every \( x \in V^1 \), \( x = (\dot{x})_G \), then \( j_W(x) = (\pi(\dot{x}))_G \).

The arguments in [?] show that the generic \( G^0 \subset \mathcal{P}^0 \), determines \( G^0_W \) except the values \( s_{G^0_W}(\gamma) \), where \( s^G_\tau \) is the generic Sacks real specified in \( G \) to a cardinal \( \tau \), \( \gamma \) is a generator of the iteration \( \pi \) at some stage \( i \), i.e., \( \gamma = cp(\pi_{i,i+1}) \), and \( \gamma^1 = \pi_{i,i+1}(\gamma) \).

Since \( \text{cp}(\pi) = \kappa \), and the iteration of \( \pi \) is normal, then we can write \( \pi = k \circ j_\beta \) for some \( \beta < o(\kappa) \), and \( \text{cp}(k) > \kappa \). Let \( s_{\pi(\kappa)} = j_W(s^G_{\kappa}) \) be the \( G^0_W \) induced \( \pi(\kappa) \) generic Sacks function, and let \( \gamma = s_{\pi(\kappa)}(\kappa) < \kappa \).

Following the description of the ultrapower embeddings by \( U^0_{(\beta,\gamma)} \), let \( G^0_{V^1_{(\beta,\gamma)}} = j^0_{(\beta,\gamma)}(G^0) \) be the induced \( j_\beta(\mathcal{P}^1) \) generic filter over \( M_\beta \). We have that \( j^0_{(\beta,\gamma)}(G^0) \) and \( s^0_{j^0_{(\beta,\gamma)}(\kappa)} \) agree \( G^0_W \), hence \( k^{(\beta,\gamma)}(G^0_W) \subset G^0_W \).

We claim that \( \gamma \geq \beta \). First note that since
1. \( s_{j_W(\kappa)}^W(\kappa) = \gamma \),

2. \( \sigma^M(\kappa) = \beta < \kappa \), and

3. \( j_W \restriction V = k \circ j_\beta : V \to M \) with \( cp(k) > \kappa \),

then \( \sigma^M(\kappa) = \beta \), so \( \kappa \in j_W(\Delta_\beta(\gamma)) \). Now, suppose that \( \gamma > \beta \), then \( \Delta_\beta(\gamma) \subset \Delta' \), therefore \( W \) concentrates on the set of non trivial forcing points in the iteration \( \mathcal{P}_1 \). This is impossible since the \( G^1 \) induced Prikry function \( d : \Delta' \to \kappa \) is regressive and almost injective (except of a bounded set in \( \kappa \)), which contradicts the normality of \( W \).

Next, we claim that if \( \Gamma \notin W \) then \( W = U^1_{(\gamma, \beta)} \). If \( \Gamma \notin W \) then \( \Sigma \setminus \Gamma \in W \), hence for every \( p \in G^1 \) then \( j_W(p)^{-\kappa} \) is a member of \( G^1_W \).

By the argument of Lemma 1.5 of [3], we have that for every \( X \in U^*_\beta(\gamma) \) then there exists some \( p \in G^1 \) so that

\[
j_0^0(\beta, \gamma)(p)^{-\kappa} \forces \kappa \in j_0^0(\beta, \gamma)(X).
\]

(4)

We have that \( j_0^0(\beta, \gamma)(p)^{-\kappa} \in M^0_{(\beta, \gamma)} = M_\beta[G^0_\beta] \), hence the of [4] is forced by some \( q^0 \in G^0_{U^0_{(\beta, \gamma)}} \). As \( k(q^0) \in G_W^0 \) we conclude that in \( M[G^0_W] \) then

\[
j_W(p)^{-\kappa} \forces k(\kappa) \in j_W(\hat{X}).
\]

As \( j_W(p)^{-\kappa} \in G^1_W \), we conclude that \( X \in W \). It follows that \( U^1_{(\gamma, \beta)} \subset W \) and the two measures coincide.

Suppose now that \( \Gamma \in W \). Since \( \Gamma \setminus \Pi \) is bounded in \( \kappa \), and \( \Sigma \subset \kappa \) is a club, then \( \Sigma \cap \Pi \in W \). By the definition of \( \Pi \), there exists a unique \( \mu \in (\kappa, \pi(\kappa)) \) for which \( \kappa = j_W(d)(\mu) \). Since \( \Sigma \cap \Pi \in W \), then \( \kappa \in j_W(\Sigma \cap \Pi) \). The results mentioned in Remark 7.2 implies that \( q^{+(\kappa, \mu)} - \kappa - \mu \in G^1_W \) for every \( q \in G^1_W \). In particular, \( j_W(p)^{+(\kappa, \mu)} - \kappa - \mu \in G^1_W \) for every \( p \in G^1 \).

By the proof of Proposition 3.2 in [3], the ordinal \( \mu \) is a key generator in the sense of Definition 3.2 in [3], namely, there is a finite sub-iteration \( j' : V \to M' \) of \( M \), and some \( \mu' \in M' \), so that

1. \( \pi = e' \circ j' \) for the appropriate \( e' : M' \to M \),

2. \( \mu' = j'(\kappa) \), and
3. $\mu = k'(\mu')$.

Furthermore, since $\mu = k(\mu') < \pi'(\kappa)$ then $\mu$ must be a critical point of the rest of the iteration of $\epsilon'$. We can therefore assume that $\epsilon' : M' \to M$ is of the form $\epsilon' = k' \circ j_{U'}$, where

1. $U'_{\mu'}$ is a normal measures on $\mu'$ in $M'$,
2. $j_{U'} : M' \to N' = \text{Ult}(M', U'_{\mu'})$ is the induced ultrapower embedding.
3. $k' : N' \to M$ is induced by the rest of the iteration.

By the proof of Lemma 3.6 in [3], the measure $U^* = \{ X \subset \kappa \mid \mu' \in j_{U'} \circ j'(X) \}$ is a normal measure on $\kappa$ in $V$. Hence, there exists some $\beta' < o(\kappa)$ so that $U'_{\mu'} = j'(U_{\beta'})$. Back to, $M[G_{W}^{0}]$ let $\gamma' = s_{\pi(\kappa)}^{\beta'}(\mu) < \mu$.

We claim that $\beta = \gamma'$ and $\gamma = \beta'$. For this note that $o^{\beta'}(\mu') = \beta'$ and $\beta' < \kappa < cp(k')$, so $\sigma^{M}(\mu) = \beta'$. We infer that $\mu \in \Delta^{M}_{\beta'}(\gamma')$, where $\Delta^{M}_{\beta'}(\gamma)$ is the appropriate subset of $\pi(\kappa)$ which relates to the forcing $j_{W}(\mathcal{P}^{1})$.

The fact that $\kappa = j_{W}(d)(\mu)$ implies that stage $\mu$ of the iteration $j_{W}(\mathcal{P}^{1})$ is not trivial, which in turn, implies that $\gamma' < \beta'$, and the conditions used at stage $\mu$ belong to the single point Prikry forcing $Q(U_{\mu(\gamma', \beta')})$, and are above (stronger than) the condition $\Delta^{M[G_{W}^{0}]}_{\gamma'}(\beta') \cap \mu$. As $\kappa = j_{W}(d)(\mu)$, we get that $\kappa \in \Delta^{M[G_{W}^{0}]}_{\gamma'}(\beta')$. However, we already know that $\kappa \in j_{W}(\Delta(\beta)) = \Delta^{M[G_{W}^{0}]}_{\beta}(\gamma))$. We conclude that $\beta = \gamma'$ and $\gamma = \beta'$. In particular, we see that $\beta < \gamma$.

Note that in the description of $j' : V \to M'$, we may assume that $j' = i' \circ j_{\beta}$. Also, we now know that $U'_{\mu'} = j'([U_{\gamma})]$. Let $i^{M_{\beta}}_{\gamma} : M_{\beta} \to M_{\beta, \gamma}$ denote the ultrapower embedding of $M_{\beta}$ by the normal measure $j_{\beta}(U_{\gamma})$ on $j_{\beta}(\kappa)$, and $j_{\beta, \gamma} = i^{M_{\beta}}_{\gamma} \circ j_{\beta} : V \to M_{\beta, \gamma}$.

Since $V$ does not include overlapping extenders then $M_{\beta, \gamma}$ is a sub-iterand of the iteration of $j_{U'_{\mu'}} \circ j' : V \to N'$, i.e., there exists some $r' : M_{\beta, \gamma} \to N'$ so that $j_{U'_{\mu'}} \circ j' = r' \circ j_{\beta, \gamma}$ and $r'(j_{\beta}(\kappa)) = j_{\beta}(\kappa)$. In particular $\mu' = r'(j_{\beta}(\kappa))$. Let $k' = k' \circ r' : M_{\beta, \gamma} \to M$.

We can now show that $W = U_{(\beta, \gamma)}^{1}$. Recall that $j_{\beta, \gamma}$ is the restriction of $k^{0}_{\beta, \gamma}$ to $V$. Let $G_{\beta, \gamma}^{0} = j_{(\beta, \gamma)}^{1}(G^{0})$ be the generic associated to the ultrapower.
of $V^1$ by $U^1_{(\beta, \gamma)}$. Using our observations on $G_W$, and that $\gamma = \beta', \gamma' = \beta$, it is straightforward to verify that $k^* \mathord{\text{“}} G^0_{(\beta, \gamma)} \subseteq G^0_W$.

Let $X \in U^1_{(\beta, \gamma)}$, and let $\check{X}$ be a $\mathcal{P}^1$ name for $X$. By Definition 4.9 and the argument of Lemma 2.12 in [7], there exists some $p \in G^1$ so that

$$k^0_{\beta, \gamma}(p)^{(\kappa, j_\beta(\kappa)) - \kappa - j_\beta(\kappa)} \models \check{\kappa} \in k^0_{\beta, \gamma}(\check{X}).$$

The last statement holds in the model $M^0_{(\beta, \gamma), (\gamma, \beta)} = M_{\beta, \gamma}[G^0_{(\beta, \gamma)}]$. Therefore, there is some $t \in G^0_{\beta, \gamma}$ which forces this statement over $M_{\beta, \gamma}$. We now apply $k^*$ to get a condition $k^*(t)$ which forces in $\pi(\mathcal{P}^0)$ over $M$, that

$$j_W(p)^{(\kappa, \mu) - \kappa - \mu} \models \check{\kappa} \in j_W(\check{X})$$

where $p$ is the $\mathcal{P}^1$-name which belongs to $G^1$. Since we already established that $j_W(p)^{(\kappa, \mu) - \kappa - \mu} \in G^1_W$ we conclude that $X \in W$. We proved that $U^1_{(\beta, \gamma)} \subseteq W$, hence $U^1_{(\beta, \gamma)} = W$. \qed

8 The Final Cut

In [4], Friedman and Magidor introduced the forcing $\text{Sacks}^*(\nu)$, which we call a degenerated Sacks forcing. Conditions in $\text{Sacks}^*(\nu)$ are called $\ast$–perfect $\nu$–trees, are trees $T \subseteq 2^{<\nu}$, so that $T$ is closed under initial segments, closed under increasing sequences of length less than $\nu$, and for some closed unbounded set $C \subseteq \nu$, then every singular ordinal in $C$ is a splitting node in $T$. Let $\mathcal{P}_{\kappa+1} = \langle \mathcal{P}_\nu, Q_\nu \mid \nu \leq \kappa \rangle$ be a non-stationary support iteration (i.e. the same support used in $\mathcal{P}^0$) of posets $Q_\nu = \text{Sacks}^*(\nu) \ast \text{Code}(\nu)$ at inaccessible $\nu \leq \kappa$, where $\text{Code}(\nu)$ is a club forcing poset at $\nu^+$ which codes the Sacks$^*(\nu)$ generic function (as in the Code($\nu$) used in $\mathcal{P}^0$).

The arguments of [4], show that if we force with $\mathcal{P}_{\kappa+1}$ over $V$, then in a generic extension $V[G]$, every normal measure $U$ on $\kappa$ has a unique extension $U^*$ in $V[G]$. Furthermore, these arguments easily show that every elementary embedding $j : V \to M$ of $V$, with $cp(j) = \kappa$, then the conditions $(j^*)^\mathord{\text{“}} G \subset j^*(\mathcal{P}_{\kappa+1})$ determine a unique generic $G^*$ over $M$, which belongs to $V[G]$. It follows that $j : V \to M$ has a unique extension $j^* : V[G] \to M[G^*]$, with $j^* \upharpoonright V = j$ and $j^*(G) = G^*$.

We introduce a simple variant of the forcing $\mathcal{P}_{\kappa+1}$. Given a set $X \subseteq \kappa$, we define a poset $\mathcal{P}^X$ which is an iteration of length $\kappa$, $\mathcal{P}^X = \mathcal{P}^X_\kappa = \langle \mathcal{P}_\nu^X, Q_\nu^X \mid \nu \leq \kappa \rangle$.
then there is some generic over $M$, it follows that for every dense open set $D$.

Lemma 9 in [6], it follows that for every dense open set $D$

\[ \nu < \kappa \]

where we force with $Q^X_\nu = \text{Sacks}^*(\nu) * \text{Code}(\nu)$ if and only if $\nu \in X$ is inaccessible. Otherwise we take $Q^X_\nu$ to be the trivial poset. We further specify that in $\text{Code}(\nu)$ uses an enumeration of stationary sets on $\nu^+$, which are definable over $H(\nu^+)^{V^1}$ and are minimal in a canonical well ordering $<_{\nu^+}$ on $H(\nu^+)^{V^1}$ (induced from the canonical well ordering on $H(\nu^+)^V$). It is important to note that for every ultrapower $M'$ of $V^1$ with critical point $\nu$, then $H(\nu^+)^{V^1} = H(\nu^+)^{M'}$ and agree on the canonical well ordering $<_{\nu^+}$. It follows that $M'$ computes the same stationary sets for the $\text{Code}(\nu)$ forcing.

Let $G^X \subset P^X$ be a $V$-generic filter, It is not hard to verify that in $V[G^X]$, then for every normal measure $U$ on $\kappa$ in $V$, $U$ extends to a normal measure on $\kappa$ in $V[G^X]$ if and only $X \not\in U$, and in this case, then there exists a unique extension $U^*$ of $U$. It is also possible to show that every normal the measure $\nu$ in $V[G^X]$ is of the form $U^*$ for $U \in V$ as above, and that for $U_0, U_1 \in V$, then $U_0 \mathrel{\triangleleft} U_1^*$ if and only if $U_0 \mathrel{\triangleleft} U_1$ in $V$.

Our intension is to show that the same results holds when we force over $V^1$ instead of $V$.

Lemma 8.1. Let $X \subset \kappa$ be a set in $V^1$, and let $G^X \subset P^X$ be a generic filter over $V^1$, then

1. For every normal measure $U$ on $\kappa$ in $V^1$, then $U$ extends to a normal measure on $\kappa$ in $V^1[G^X]$ if and only if $X \not\in U$, and then $U$ has a unique normal extension, $U^* \in V^1[G^X]$.

2. Every normal measure in $V^1[G^X]$ is of the form $U^*$ for some normal measure $U \in V^1$ with $X \not\in U$.

Proof. Suppose that $U \in V^1$ is a normal measure on $\kappa$, so that $X \not\in U^1$, let $j : V^1 \rightarrow M^1 \cong \text{Ult}(V^1, U)$ be the induced ultrapower embedding. By the definition of $P^X$ we get that $j(P^X) \upharpoonright \kappa = P^X$. Furthermore, since $X \not\in X$, then $\kappa \not\in j(X)$, so stage $\kappa$ of the poset $j(P^X)$ is trivial. We conclude that $j(P^X) \upharpoonright (\kappa + 1)$ is equivalent to $P^X$. By the arguments of Lemma Lemma 9 in [6], it follows that for every dense open set $D \subset j(P^X)$ then there is some $g \in G^X$ so that $j(g)$ reduces $D$ to a dense open set in $j(P^X) \upharpoonright (\kappa + 1)$. As $j(P^X) \upharpoonright (\kappa + 1)$ is equivalent to $P^X$, and $G^X \subset P^X$ is generic over $M^1$, it follows that $j"G^X$ determines a unique generic filter $H^X$ over $M^1[G^X]$. Hence, letting $G^* = G^X * H^X$ we conclude that $G^* \subset j(P^X)$ is the unique generic filter over $M^1$ so that $j"G^X \subset G^*$. Since the structure of $P^X$ ensures that there are no other $j(P^X)$ generics in $M[G^*]$ other than $G^*$.
(see Lemma 17 in [3]), we infer that there is a unique elementary embedding $j^*: V^1[G^X] \to M^1[G^*]$ extending $j: V^1 \to M^1$. In $V^1[G^X]$ we defined

Let $U^X = \{ Y \subset \kappa \mid \kappa \in j^*(Y) \}$, then $U^X$ is a normal measure on $\kappa$ in $V^1[G^X]$ which extends $U$, i.e., $U^X \cap V^1 = U$. For every $\alpha, \beta < o(\kappa)$ so that $X \notin U^1_{(\alpha, \beta)} = U$, let $U^X_{(\alpha, \beta)}$ denote the $V^1[G^X]$ extension, $U^X$, described above.

Next, suppose that $W \in V^1[G^X]$ is a normal measure on $\kappa$. We claim that $X \not \in W$. Let $j_W: V^1[G^X] \to M_W$, then $j = j_W \upharpoonright V: V \to M$ is an iteration of $V$. Furthermore, writing $V^1 = V[G^0 * G^1]$ and $G = G^0 * G^1 * G^X$, then $G_W = j_W(G) \subset j(P^0 * P^1 * P^X)$ is generic over $M$. If $X \in W$, then $\kappa \in j_W(X)$. It follows that $\kappa$ is a non trivial forcing stage in the iteration $j_W(P^X)$, with $Q^X_1 = Sacks^*(\kappa) * Code(\kappa)$. The construction of $Code(\kappa)$ specified above, ensures that the generic $G_W$ adds a $Code(\kappa)$ set $C \subset \kappa^+$, which is club set in $M_W$, and is disjoint from $S \subset \kappa^+$, for some $S$ which is stationary in $V^1$. As $M_W$ is closed under $\kappa$ sequences, it follows that $C$ is also a club in $V^1[G^X]$, however this is absurd, as $P^X$ has cardinality $\kappa$ and preserves all stationary sets in $\kappa^+$.

Consider again the iteration $j = j_W \upharpoonright V: V \to M$ given above. We claim that $U^1_{(\alpha, \beta)} \subset W$ for some $\alpha, \beta < o(\kappa)$. For this we can just run the argument of Proposition [7.3]. We note that the proof of this proposition requires to know only the values $\alpha = o^M(\kappa)$ and $\beta = j_W(s)_{j(\kappa)}(\kappa)$, and the arguments of the proof are straightforward, and cannot be effected by any possible additional information on $j, G_W, G^1_W, s$ which can (theoretically) depend on the generic data in $G^X$.

Let us show that $W = U^X_{(\alpha, \beta)}$. It is sufficient to verify that $U^X_{(\alpha, \beta)} \subset W$. Fix some $Y \in U^X_{(\alpha, \beta)}$, and let $Y$ be a $P^X$ name. By the description of $U^X_{(\alpha, \beta)}$ given above, then there is some $g \in G^X$ so that $j^1_{(\alpha, \beta)}(g) \models \kappa \in j_{(\alpha, \beta)}(Y)$.

In $V^1$ define $Y' = \{ \nu < \kappa \mid g \models \nu \in \check{Y} \}$. It follows that $\kappa \in j^1_{(\alpha, \beta)}$, hence $Y' \in U^1_{(\alpha, \beta)}$ so $Y' \in W$. Finally, since $g \in G$ and $Y = (\check{Y})_G$, then $Y' \subset Y$ and $Y \in W$.

**Lemma 8.2.** For every $U^X_{(\alpha', \beta')} \subset U^X_{(\alpha, \beta)} \subset V^1$ with $\alpha' \leq \beta'$ and $\alpha \leq \beta$, then $U^X_{(\alpha', \beta')} \triangleleft U^X_{(\alpha, \beta)}$ if and only if $U^1_{(\alpha', \beta')} \triangleleft U^1_{(\alpha, \beta)}$ in $V^1$.

**Proof.** First note that the definition of $U^X_{(\alpha, \beta)}$ essentially requires $U^1_{(\alpha, \beta)}$. Therefore, suppose that $j^*: V^1[G^X] \to M^1[G^*]$ is an elementary embedding of $V^1[G^X]$ which extends the $V^1$ embedding $j = j^* \upharpoonright V^1: V^1 \to M^1$ so that
For the other direction, we claim that $U^X_{\alpha, \beta}$ trivially follows that $U^X_{\alpha', \beta'}$ so that $U^X_{\alpha, \beta} \triangleleft U^X_{\alpha', \beta'}$ only if $\beta' < \alpha$. For every $U^X_{\alpha, \beta} \in V^1$, let $j^X_{\alpha, \beta} : V^1[G^X] \to M^X_{\alpha, \beta} \cong \text{Ult}(V^1[G^X], U^X_{\alpha, \beta})$, be the induced elementary embedding. By the definition of the measures $U^X_{\alpha, \beta}$, we get that $j^X_{\alpha, \beta} \upharpoonright V^1 = j_{\alpha, \beta}^1$, and therefore $j^X_{\alpha, \beta} \upharpoonright V = \pi_{\alpha, \beta} : V \to Z_{\alpha, \beta}$. We can now use the appropriate argument used in the proof Proposition 6.1 to show that $\beta' < \alpha$ when $U^1_{\alpha', \beta'} \triangleleft U^1_{\alpha, \beta}$, as this argument is only based on basic inner model results and a comparison between the models $Z_{\alpha, \beta}$ and $Z_{\alpha', \beta'}$. Both apply here. \qed

8.1 The main theorem

By using a final cut forcing of the form $\mathcal{P}^X$ for some $X \subset \kappa$ we can easily include the following result.

**Corollary 8.3.** Let $(R, <_R)$ be a tame order so that $\text{otp}(CU(R), \emptyset) \leq \lambda = o(\kappa)$ in $V = L[U]$, and $\lambda \leq \kappa$, then there exists a generic extension $V[G]$ of $V$ so that $\triangleleft (V[G] \cong (R, < R)$

**Proof.** Let $G^0 \subset \mathcal{P}^0$ be generic over $V$, and $G^1 \subset \mathcal{P}^1$ be generic over $V^0 = V[G^0]$. Then in $V^1 = V^0[G^1]$, we have by Proposition 7.3 that the every normal measure on $\kappa$ is of the form $U^1_{\alpha, \beta}$ for some $\alpha, \beta < \lambda$. By Proposition 6.1 we know that for every $\alpha \leq \beta < \lambda$ and $\alpha' \leq \beta' < \lambda$ then $U^1_{\alpha', \beta'} \triangleleft U^1_{\alpha, \beta}$ if and only if $\beta' < \alpha$. Let $\mathcal{M}^1 = \{U^1_{\alpha, \beta} \mid \alpha \leq \beta\}$, then $\triangleleft (V[G^1] \cong (R, <_R)$.

By Corollary 4.13 we know that the normal measures $\{U^1_{\alpha, \beta} \mid \alpha, \beta < \lambda\}$ are separated by sets $\{X_{\alpha, \beta} \mid \alpha, \beta < o(\kappa)\}$ so that $X_{\alpha, \beta} \in U^1_{\alpha', \beta'}$ if and only if $\alpha = \alpha'$ and $\beta = \beta'$. Since $(R, <_R)$ is tame, and $\text{otp}(CU(R), \emptyset) \leq \lambda$, then $(R, <_R)$ can be embedded in $R_\lambda$. Choose some $\mathcal{M} \subset \mathcal{M}^1$ so that $\triangleleft (V[G^1] \cong (R, <_R)$. Take some enumeration $\{U^1_{\alpha, \beta_i} \mid i < \eta\}$ of $\mathcal{M}$, with $\eta \leq \kappa$. Let $X = \Delta_{i<\eta}(\kappa \setminus X_{\alpha_i, \beta_i})$ where $\Delta$ is the standard intersection if $\eta < \kappa$, and $\Delta$ is the diagonal intersection otherwise. If follows that for every $U^1_{\alpha, \beta} \in \mathcal{M}^1$, then $X \in U^1_{\alpha, \beta}$ if and only if $U^1_{\alpha, \beta} \notin \mathcal{M}$. Let $G^X \subset \mathcal{P}^X$ be generic over $V^1$, and $V^X = V^1[G^X]$ then by Lemma 5.2 $\triangleleft (V[G^X] \cong (R, < R)$. \qed

Next, we show how to simply modify the definitions of $\mathcal{P}^0$ and $\mathcal{P}^1$ to realize arbitrary orders $(R, <_R)$ of cardinality at most $\kappa$, such that its reduction is tame.
Lemma 8.4. Suppose that \((R, <_R)\) is an order so that \(|R| \leq \kappa\), and \(\text{otp}(CU(R), \varnothing) \leq \lambda = o(\kappa)\) and \(\lambda \leq \kappa\), then there exists a generic extension \(V[G]\) of \(V\) so that \(\triangleleft(\kappa)^{V[G]} \cong (R, <_R)\).

Proof. We compare this to previous Corollary 8.3. By the arguments of Lemma 8.2, the result will be immediate once we obtain a model \(V_1\) so that the normal measures on \(\kappa\) belong to a collection \(\{U_{(\alpha, \beta)}(\eta) \mid \alpha, \beta < \lambda, \gamma < \kappa\}\) which satisfy that

1. There are pairwise disjoint sets \(\{X_{(\alpha, \beta)}(\gamma) \mid \alpha, \beta < \lambda, \gamma < \kappa\}\) which separates \(\{U_{(\alpha, \beta)}(\eta) \mid \alpha, \beta < \lambda, \gamma < \kappa\}\).

2. For every \(\alpha \leq \beta < \lambda\), \(\alpha' \leq \beta' < \lambda\), and \(\gamma, \gamma' < \kappa\), then \(U_{(\alpha', \beta')}(\gamma') \triangleleft U_{(\alpha, \beta)}(\gamma)\) if and only if \(\beta' < \alpha\).

Let us modify \(P_0\) and \(P_1\) to obtain such collection. In \(P_0\) we replace the Sacks forcings \(\text{Sacks}_{\lambda(\nu)}(\nu)\) with \(\text{Sacks}_{\lambda(\nu)}(\nu) \times \nu(\nu)\) which consists of trees \(T \subset \nu \times \nu\) so that for some club \(C \subset \nu\), then for every \(s \in T\), if \(\text{len}(s) \in C\), then \(s^\frown (i, j) \in T\) for every \(i < \lambda(\text{len}(s))\) and \(j < \text{len}(s)\). Such forcing generates a generic function \(s_\nu : \nu \rightarrow \lambda(\nu) \times \nu\). In particular, in \(\kappa\) the generic function if \(s_\kappa : \kappa \rightarrow \lambda \times \kappa\). We then modify the coding poset \(\text{Code}(\nu)\) to appropriately code the new Sacks function \(s_\nu\). Let \(V^0 = V[G^0]\) be a \(P_0\) generic extension. We replace the previous collection of pairwise disjoint sets \(\{\Delta(\eta) \mid \eta < \lambda\}\) with sets \(\Delta(\eta, \gamma)\) for \(\tau < \lambda\) and \(\gamma < \kappa\), defined by

\[
\Delta(\eta) = \{\nu < \kappa \mid s_\nu = s_\kappa \upharpoonright \nu \times \nu, s_\kappa(\nu) = (\eta, \gamma)\}.
\]

Clearly, the sets in \(\{\Delta(\eta, \gamma) \mid \eta < \lambda, \gamma < \kappa\}\) are pairwise disjoint.

By the exact same arguments, we conclude that the the normal measures on \(\kappa\) in \(V^0\) are of the form \(\{U_{(\alpha, \eta)}(\gamma) \mid \alpha, \eta < \lambda, \gamma < \kappa\}\) so that for \(\alpha < o(\kappa) = \lambda\), \(\eta < \lambda\), and \(\gamma < \kappa\) then \(U_{(\alpha, \eta)}^0(\gamma)\) is a normal measure which extends \(U_\alpha\) and contains the set \(\Delta_\alpha(\eta, \gamma) = \Delta_\alpha \cap \Delta(\eta, \gamma)\). The sets \(\{\Delta_\alpha(\eta, \gamma) \mid \alpha, \eta < \lambda, \gamma < \kappa\}\) are clearly pairwise disjoint.

Next, we modify \(P_1\) accordingly, \(P_1\) is a Magidor iteration of one-point Prikry forcings which is a straightforward modification of the previous \(P_1\). We point out the significant differences between the the current and the previous construction of \(P_1\) and leave the details to the reader. Given an ordinal \(\nu < \kappa\) which is measurable in \(V\) then \(Q^\nu\) is not trivial if and only \(\nu \in \Delta_\beta(\alpha, \gamma)\) for some \(\beta < \alpha\). We then use the normal measure \(U_{(\beta, \alpha)}^1(\gamma)\) for the one point
Prikry forcing at stage $\nu$. Where $U^1_{(\beta,\alpha)}(\gamma)$ is a normal measure on $\nu$ in $V^0[G^1_{\nu}]$ which extends $U^0_{\nu, (\alpha, \beta)}(\gamma)$ in $V^0$. The definition of the normal measures $\{U^1_{(\alpha, \beta)}(\gamma) \mid \alpha, \beta < \lambda, \gamma < \kappa\}$ is given by the obvious modification in Definitions 4.6 and 4.9. For $\alpha \geq \beta$ and any $\gamma < \kappa$, we use the $U^0_{(\beta, \alpha)}(\gamma)$ ultrapower embedding $j^0_{(\beta, \alpha), \gamma}$ to define $U^1_{(\alpha, \beta)}(\gamma)$. For $\alpha < \beta$ and any $\gamma < \kappa$, we use the obvious alternative of Definition 4.9 using the iterated ultrapower $j^0_{(\alpha, \beta), (\beta, \alpha), \gamma} : V^0 \to M^0_{(\alpha, \beta), (\beta, \alpha), \gamma}$ which is formed by $U^0_{(\alpha, \beta)}(\gamma)$ an $U^0_{(\beta, \alpha)}(\gamma)$ exactly as $j^0_{(\alpha, \beta), (\beta, \alpha)}$ was formed by $U^0_{(\alpha, \beta)}$ and $U^0_{(\beta, \alpha)}$.

The arguments of Sections 5, 6 and ?? can be trivially adjusted to fit the modified version. One concludes the desired results specified in the beginning of the proof.

We turn to complete the details of the main result.

(Proof. Theorem 4.1). Lemma 8.4 above, establish the result for $\lambda \leq \kappa$, i.e., for tame orders $(R, <_R)$ so that $\text{otp}(CU(R), \varnothing) \leq \kappa$. Therefore, to complete the proof, we need to extend these results to tame orders $(R, <_R)$ so that $\text{otp}(CU(R), \varnothing) = \lambda$ for $\kappa < \lambda < \kappa^*$. For this, we follow the construction of the original $\mathcal{P}^0$ and $\mathcal{P}^1$ (not the modified version of Lemma 8.4) and suggest some modifications. Let $\langle \rho_i \mid i < \kappa^* \rangle$ be a sequence of canonical functions on $\kappa$, which satisfies the following properties:

1. $\rho_i : \kappa \to \kappa$ for all $i < \kappa^*$.
2. For every $i < j < \kappa^*$ then $\{\nu < \kappa \mid \rho_i(\nu) < \rho_j(\nu)\}$ is bounded in $\kappa$.
3. For every normal measure $U$ on $\kappa$ in $V$, and $j_U : V \to M_U \cong \text{Ult}(V, U)$, then $j_U(\rho_i)(\kappa) = i$ for every $i < \kappa$.

Our ground model assumptions are similar to the previous construction. Suppose that $o(\kappa) = \lambda < \kappa^*$ in $V = L[U]$, $\langle U_{\nu, \tau} \mid \nu \leq \kappa, \tau < o(\nu) \rangle$ is a coherent sequence so that every normal measure in $V$ appears on $U$. For every $\alpha < \kappa^*$, let $\Delta_\alpha = \{\nu < \kappa \mid o(\nu) = \rho_\alpha(\nu)\}$. Then $\Delta_\alpha \in U_\alpha$ for every $\alpha < \lambda$, Since the functions $\rho_i$ are pairwise distinct on a co-bounded sets in $\kappa$, the sets $\{\Delta_\alpha \mid \alpha < \lambda\}$ are almost disjoint. It follows that $\alpha$ is the only value $i < \lambda$ so that $\Delta_\alpha \in U_i$. We modify these sets to obtain pairwise disjoint collection. The set $C^0 = \{\nu < \kappa \mid \rho_\lambda \restriction \nu : \nu \to \nu\}$ is clearly club in $\kappa$. Let $\Delta^0 \subset C^0$ be the set of all $\nu \in C^0$ so that either $\nu$ is not measurable, or $\nu$ is measurable, and for every $\tau < o(\nu)$, then $\rho_\lambda(\nu) = j_{U_{\nu, \tau}}(\rho_\lambda \restriction \nu)(\nu)$, where
\[ j_{U,\tau} : V \to M_{U,\tau} \cong \text{Ult}(V, U, \tau). \] By the properties of canonical functions listed above, we get that \( \Delta^0 \in \bigcap_{\alpha < \lambda} U_\alpha. \)

We now introduce the modified posets \( P^{0,\alpha} \) and \( P^{1,\alpha} \) which replace \( P^0 \) and \( P^1. \)

\[ P^{0,\alpha} = \langle P^{0,\lambda}, Q^{0,\lambda} \mid \nu \leq \kappa \rangle \] is a non-stationary iteration so that \( Q^{0,\alpha} \) is non-trivial if and only if \( \nu \) is inaccessible in \( V \), and \( Q^{0,\alpha} = \text{Sacks}_{\rho_\lambda(\nu)} * \text{Code}(\nu) \) where conditions in \( \text{Sacks}_{\rho_\lambda(\nu)} \) are trees \( T \subset <\nu \) so that for some club \( C \subset \nu \), then for every \( s \in T \), if \( \text{len}(s) \in C \), then \( s \prec (i) \in T \) for every \( i < \rho_\lambda(\text{len}(s)) \). Such forcing generates a generic function \( s_\nu : \nu \to \nu \). Then we take \( \text{Code}(\nu) \) to be the appropriate club shooting poset which code the Sacks generic function \( s_\nu \). This can be done exactly as in [6], where we only need to specify the way by which translate \( \rho_\lambda(\nu) \) to \( \nu \). Let \( e_{\rho_\lambda(\nu)} : \nu \to \rho_\lambda(\nu) \) be a surjection which is \( <_{H(\nu^+)} \) minimal, in the canonical well order of \( H(\nu^+) \).

Let \( G^0 \subset P^{0,\alpha} \) be a generic over \( V \). For every \( \eta < \lambda \) define \( \Delta(\eta) = \{ \nu \in \Delta^0 \mid s_\nu = s_\kappa \upharpoonright \nu \) and \( s_\kappa(\nu) = \rho_\eta(\nu) \} \). It follows that \( \{ \Delta(\eta) \mid \eta < \lambda \} \) are pairwise almost disjoint. The arguments in [6] imply that for every normal measure \( U \) on \( \kappa \) in \( V \), then one can list all the normal measures in \( V[G^0] \) extending \( U \) by \( \langle U(\eta) \mid \eta < \lambda \rangle \) so that

1. \( \Delta(\eta) \in U(\eta) \) for all \( \eta < \lambda \).
2. \( j_{U(\eta)} : V[G^0] \to M_U[G^0, U(\eta)] \cong \text{Ult}(V[G^0], U(\eta)) \) extends the embedding \( j_U : V \to M_U \cong \text{Ult}(V, U) \).
3. The set \( j_U "G^0 \) determines the generic \( G^0 \subset j_U(P^0) \) besides the value \( s_{j_U(\kappa)}(\kappa) < \lambda \), which is forced to be \( \eta \) by \( G^0 \).

For every \( \alpha < \text{o}(\kappa) = \lambda \) and \( \eta < \lambda \), let \( U^0_{(\alpha, \eta)} = U_\alpha(\eta) \) and \( \Delta_\alpha(\eta) = \Delta_\alpha \cap \Delta(\eta) \).

It follows that the sets \( \{ \Delta_\alpha(\eta) \mid \alpha < \text{o}(\kappa), \eta < \lambda \} \) are pairwise almost disjoint, and \( \Delta_\alpha(\eta) \in U^0_{(\alpha, \eta)} \) for all \( \alpha < \text{o}(\kappa), \eta < \lambda \). We also get that \( U^0_{(\alpha', \eta')} \subset U^0_{(\alpha, \eta)} \) whenever \( \alpha' < \alpha \). Similarly for every \( \nu \in \Delta^0 \), then the normal extension of each \( U_{\nu, \tau} \), \( \tau < \text{o}(\nu) \) in \( V \), are \( \{ U^0_{\nu, (\tau, \eta)} \mid \eta < \rho_\lambda(\nu) \} \).

In order to define the modified Magidor iteration \( P^{1,\alpha} \) we need to replace the sets \( \{ \Delta_\alpha(\eta) \mid \alpha < \text{o}(\kappa), \eta > \lambda \} \) with a collection of pairwise disjoint sets. Fix a surjection \( e_\lambda : \kappa \to \text{o}(\kappa) \times \kappa. \) For every \( \nu < \kappa \) we write \( e_\lambda(\nu) = (\bar{\alpha}(\nu), \bar{\eta}(\nu)) \), then \( \bar{\alpha} : \kappa \to \text{o}(\kappa) \) and \( \bar{\eta} : \kappa \to \lambda. \) For every \( \alpha < \text{o}(\kappa) \) and \( \eta < \lambda \), let \( \nu_{\alpha, \eta} < \kappa \) with \( (\alpha, \eta) = e_\lambda(\nu_{\alpha, \eta}) \), and define

\[ \Delta^*_\alpha(\eta) = \{ \nu \in \Delta_\alpha(\eta) \mid \rho_\alpha(\nu) < \rho_\lambda(\nu) \} \setminus \bigcup_{\nu < \nu_{\alpha}} \Delta_{\bar{\alpha}(\nu)}(\bar{\eta}(\nu)). \]
It follows that for every \( \alpha < o(\kappa) \) and \( \eta < \lambda \), then \( \Delta_{\alpha}(\eta) \setminus \Delta^*_\alpha(\eta) \) is bounded in \( \kappa \), and the sets \( \{ \Delta^*_\alpha(\eta) \mid \alpha < o(\kappa), \eta < \lambda \} \) are pairwise disjoint. Let \( \Delta^1 \subset \Delta^0 \) consists of the ordinals \( \nu \in \Delta^0 \) so that for every \( \alpha < o(\nu) \) and \( \eta < \rho_\lambda(\nu) \) then \( \Delta^*_\alpha(\eta) \cap \nu \in U_{\nu, (\alpha, \eta)}^0 \). By a simple reflection argument, we get that \( \Delta^1 \in U_{(\alpha, \eta)}^0 \) for every \( 1 \leq \alpha < o(\kappa) \) and \( \eta < \lambda \).

For \( P^1, \lambda = \langle P^1_\nu, \lambda \nu \mid \nu < \kappa \rangle \) we use the same definitions of section 4 where we take the forcing to be non-trivial at stages \( \nu \in \Delta^1 \), and replace \( \Delta^*_\alpha(\eta) \) in Definitions 4.5, 4.6, and 4.9 with \( \Delta^*_\alpha(\eta) \). We generate normal measures \( U^1_{(\alpha, \beta)} \) for every \( \alpha < o(\kappa) = \lambda \), and \( \beta < \lambda \). Furthermore, the sets \( X_{(\alpha, \beta)} \), \( \alpha, \beta < \lambda \) defined in 4.12 are pairwise disjoint in our construction, and satisfy that \( X_{(\alpha, \beta)} \in U^1_{(\alpha, \beta)} \) for all \( \alpha, \beta < \lambda \). The results in the sections following section 4 are still valid. The arguments in these sections do not make any use of the fact that previous assumption of \( o(\kappa) \leq \kappa \). Let \( G^1 \subset P^{1, \lambda} \) be a generic filter over \( V^0 = V[G^0] \). It follows that all the normal measures on \( \kappa \) in \( V^1 \) are \( \{ U^1_{(\alpha, \beta)} \mid \alpha, \beta < \lambda \} \), they are separated by sets. Furthermore the restriction of \( \triangleleft(\kappa)\upharpoonright V^1 \) to \( \mathfrak{W}^1 = \{ U^1_{(\alpha, \beta)} \mid \alpha \leq \beta \} \) is isomorphic to \( (R_\lambda, <_{R_\lambda}) \). By applying a final cut forcings \( P^X \) to \( V^1 \), we conclude that every tame order \( (R, <_R) \) with \( \text{otp}(CU(R), \varnothing) \leq \lambda < \kappa^+ \) can be realized as \( \triangleleft(\kappa)\upharpoonright V^1 \) in a generic extension of \( V^1 \).

Finally, to realize orders \( (R, <_R) \) so that \( |R| \leq \kappa \), and the reduction of \( (R, <_R) \) is tame with \( \text{otp}(CU(R), \varnothing) \leq \lambda \) we need to combine the modifications to \( P^0, P^1 \) suggested in this proof, and the modifications of Lemma 8.4. These modification are independent and can be easily combined together to obtain the claim of Theorem 1.1. We leave the details to the reader.

\[ \square \]

References

[1] A. Apter, J. Cummings, and J. Hamkins, Large cardinals with few measures, Proceedings of the American Mathematical Society, vol. 135 (7) (2007), 2291-2300.

[2] Stewart Baldwin, The \( \triangleleft \)-Ordering on Normal ultrafilters, The Journal of Symbolic Logic, 50 (1985), 936-952.

[3] Omer Ben-Neria, Forcing Magidor iteration over a core model below \( 0^\dagger \), to appear in the Archive for Mathematical Logic.
[4] James Cummings, *Possible behaviours for the Mitchell ordering*, Annals of Pure and Applied Logic, Volume 65 (2) (1993), 107-123.

[5] James Cummings, *Possible Behaviours for the Mitchell Ordering II*, Journal of Symbolic Logic, Volume 59 (4) (1994), 1196-1209.

[6] Sy-David Friedman and Menachem Magidor, *The number of normal measures*, The Journal of Symbolic Logic, 74 (2009), 1069-1080.

[7] Sy-David Friedman and Katherine Thompson, *Perfect trees and elementary embeddings*, The Journal of Symbolic Logic, 73 (2008), 729-1096.

[8] Moti Gitik, *Prikry Type Forcings*, Handbook of set theory (Foreman, Kanamori editors), Volume 2, 1351-1448.

[9] Kenneth Kunen, *Some application of iterated ultrapowers in set theory*, Annals of Mathematical Logic 1 (1970), 179-227.

[10] K. Kunen and J. B. Paris, *Boolean extensions and measurable cardinals*, Annals of Mathematical Logic 2 (1970/71), 359-377.

[11] Menachem Magidor, *How large is the first strongly compact cardinal? or A study on identity crisis*, Annals of Mathematical Logic, 10 (1976), 33-57.

[12] William J. Mitchell, *Beginning Inner Model Theory*, Handbook of set theory (Foreman, Kanamori editors), Volume 3, 1449-1495.

[13] William J. Mitchell, *Sets Constructed from Sequences of Measures: Revisited*, The Journal of Symbolic Logic, Volume 39 (1) (1974), 57-66.

[14] William J. Mitchell, *Sets Constructed from Sequences of Measures: Revisited*, The Journal of Symbolic Logic, Volume 48 (3) (1983), 600-609.

[15] Itay Neeman, *The Mitchell order below rank-to-rank*, The Journal of Symbolic Logic, Volume 69 (4) (2004), 1143-1162.

[16] John R. Steel, *An Outline of Inner Model Theory*, Handbook of set theory (Foreman, Kanamori editors), Volume 3, 1601-1690.
[17] John R. Steel, *The Well-Foundedness of the Mitchell Order*, The Journal of Symbolic Logic, Volume 58 (3) (1993), 931-940.

[18] Jiri Witzany, *Any Behaviour of the Mitchell Ordering of Normal Measures is Possible*, Proceedings of the American Mathematical Society, Volume 124 (1) (1996), 291-297.

[19] Martin Zeman, *Inner Models and Large Cardinals*. de Gruyter series in Mathematical Logic, vol 5, 2002.