Finite rank kernel varieties: A variant of Hilbert’s Nullstellensatz for graphons and applications to Hadamard matrices

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Abstract

Graphons are symmetric measurable functions that arise from a sequence of graphs. A graphon variety is the a set of all graphons defined by a condition of the form \( t(g, W) = 0 \) for a fixed quantum graph \( g \), where \( t(., .) \) is the homomorphism density and a quantum graph is a formal linear combination of multigraphs. Using a method of representing graphs as polynomials, we construct an epimorphism from the space of quantum graphs to a subring of the complex polynomial ring that is invariant under permutations of variables. When graphons are of finite rank, we demonstrate that an analog of the “ideal” inverse in Algebraic Geometry is an ideal in our polynomial representation. Defining an algebraic kernel set using kernel varieties, we demonstrate that we can call such sets closed under the Zariski Topology. We determine several ties to Algebraic Geometry as a result of utilizing finite rank kernels and discover that a weaker version of Hilbert’s Nullstellensatz applies to kernel zero-sets with respect to homomorphism density. Throughout, we examine the connection between Algebraic Geometry and Graphon Theory.

1 Introduction

Graphons naturally arise as limiting objects for convergent sequences of graphs (Borgs et al., 2008; Glasscock, 2015, 2016; Lovász, 2012). There have been several studies examining the theoretical properties of graphons in recent years. A natural line of inquiry is when sequences of graphs converge trivially and when they converge to a non-zero limiting object; this consideration was studies by Borgs, Chayes, Cohn, and Holden in (Borgs et al.,
Graphons may be considered as symmetric measurable functions on probability spaces (Borgs et al., 2010; Lovász, 2012). From this probabilistic perspective, they may also be used to study σ-finite measure spaces and compact metric spaces as the space of all graphons is a compact metric space and satisfies certain probability properties; for more details we refer the reader to (Borgs et al., 2010; Borgs et al., 2018; Glasscock, 2015; Janson, 2016).

Graphons also appear in a variety of applications. Graphons are used in network classification and training algorithms; we refer the reader to consider (Hu et al., 2021; Maskey et al., 2021; Ruiz et al., 2020; Ruiz, Wang, et al., 2021; Sabanayagam et al., 2021; Xu et al., 2021). A graphon Fourier transform has applications to signal processing; see (Ghandehari et al., 2021; Ruiz, Chamon, et al., 2021). Graphon mean field games are used to model the interaction of particles systems through graphon mean field games; see (Athreya et al., 2019; Aurell et al., 2021; Bassino et al., 2021; Bayraktar et al., 2021; Bayraktar & Wu, 2021; Gao, Caines, et al., 2021; Gao, Tchuendom, et al., 2021; Vasal et al., 2020).

In comparison to the probabilistic study and algorithmic applications of graphons, the algebraic study of graphons is still relatively unexplored; some discussion of the current state of the algebraic study is in (Lovász, 2012). The semi-group structure of the set of all graphons is discussed in (Nagy, 2021). In this paper, we will further address the algebraic study of graphons by connecting graphons to well-studied objects in algebraic geometry, namely zero loci and polynomial varieties (Hartshorne, 2013).

We begin by introducing the concept of a graphon and its generalized version, the kernel, in section 2. We then introduce quantum graphs and use a method of representing them as polynomials, from (Lovász, 2012; Schrijver, 2009). Through a brief commentary on the category theory structure of this representation in section 3, we set the stage to utilize algebraic geometry tools in building understanding about kernel varieties, which are homomorphism density zero-sets of kernels. We discuss these objects further and introduce relevant notation in section 4. A series of new notation is introduced to simplify the understanding of the new material.

In section 5, we build upon the previous sections to prove the Zariski Topology on the space of all finite rank kernels and we establish a weaker version of Hilbert’s Nullstellensatz. We touch on resulting implications for varieties and ideals in our new setting throughout section 5. We conclude with natural follow-up questions and conjectures for future work.
2 Background

Although graphons originally arose as limiting objects for convergent sequences of graphs (Borgs et al., 2008; Glasscock, 2015, 2016; Lovász, 2012), we will use a more general definition of a graphon.

**Definition 2.1.** [[Lovász, 2012]] Let \( W \) be the space of all bounded symmetric measurable functions \( W : [0, 1]^2 \to \mathbb{R} \). We call the elements *kernels*. Let \( W_0 \) denote the set of all kernels \( W \in W \) such that \( 0 \leq W \leq 1 \). We call these elements *graphons*.

Graphons only exist non-trivially when considering graphs that are sufficiently dense. For a deeper understanding of what it means for a graph to be “sufficiently” dense, we refer the reader to (Borgs et al., 2008; Borgs et al., 2018).

In order to use an algebraic approach to study graphons, we introduce the concept of homomorphism density, which allows us to measure the similarity of two graphs with respect to edge adjacencies. When considering two graphs, the homomorphism density is the probability that a randomly chosen map between vertex sets of the two graphs will preserve edge adjacency, i.e. the probability that a randomly chosen map is a graph homomorphism; see (Borgs et al., 2008; Lovász, 2012). As we see in the following definition, we may extend homomorphism density of two graphs to consider the homomorphism density between a kernel and a multigraph without loops, a graph allowing multiple edges between the same nodes.

**Definition 2.2.** For every \( W \in W \) and multigraph \( F = (V, E) \) (without loops), define the homomorphism density

\[
\tau(F, W) = \int_{[0,1]^V} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V} dx_i.
\]

We may further generalize multigraphs into formal linear combinations with the idea of a quantum graph.

**Definition 2.3.** A *quantum graph* is defined to be a formal linear combination of a finite number of unweighted multigraphs with real coefficients. They form an infinite-dimensional linear space which we will denote \( Q_0 \). In the formal linear combination, the multigraphs that occur with non-zero coefficient are called the constituents of \( x \). Any quantum graph \( x \) can be represented as

\[
x = \sum_{i=1}^{n} \alpha_i F_i
\]
for some weights $\alpha_i$ and multigraphs $F_i$.

In this paper, we will use a lowercase $g$ to refer to a quantum graph and a capital $G$ to refer to a multigraph. Unless otherwise stated, we assume that definitions pertaining to a graph are defined for multigraphs and we will state when they can be extended linearly to quantum graphs. Homomorphism density, definition 2.2, linearly extends to quantum graphs.

When considering quantum graphs in relation to kernels and graphons, we will utilize the polynomial method which was originally developed in (Schrijver, 2009) and altered to the following version in (Lovász, 2012). Greater discussion about the origins of this method and how it was developed can be found in (Lovász, 2012, p. 6.6).

We treat the edge weights of a target graph $H$ as variables, then homomorphism numbers into $H$ will be polynomials in these variables. Let $H$ be a weighted graph on $[q]$, this terminology is explicitly discussed after definition 3.1, with node weights 1 and edge weights different variables $x_{ij}$ (where $x_{ij} = x_{ji}$). We arrange these variables into a symmetric $q \times q$ matrix $X$. Every substitution of real (complex) numbers for the variables gives a real (complex) valued homomorphism function into an edge-weighted graph on $[q]$ and vice versa. The homomorphism number

$$\text{hom}(G, H) = \sum_{\phi: V(G) \rightarrow [q]} \prod_{ij \in E(G)} x_{\phi(i)\phi(j)}$$

is a polynomial in the $\binom{q}{2} + q$ variables which we denote $\text{hom}(G, X)$. The polynomials $\text{hom}(g, X)$, where $g \in Q_0$, form the space $\mathbb{C}[X]^{S_q}$ which is the space of polynomials in $\mathbb{C}[X]$ that is invariant under the permutations of $[q]$. To be precise, if $\sigma \in S_q$, we may define $X^\sigma = (x_{\sigma(i)\sigma(j)} : i, j \in [q])$ and $\text{hom}(., X^\sigma) = \text{hom}(., X^\sigma)$ by (Lovász, 2012, Lemma 6.49).

**Definition 2.4.** For a fixed $q$ defining $X$ and a quantum graph $g \in Q_0$, we will call $\text{hom}(g, X)$ the *homomorphism polynomial of $g$ with respect to $q$*, or *homomorphism polynomial* for short.

The notations $\text{hom}(G, H)$ and $\text{hom}(G, X)$ both have the same meaning; the latter notation simply replaces $H$ with the constructed edge weight matrix. The notation $\text{hom}(g, X)$ refers to the linear extension to quantum graphs.

Now that we’ve introduced some background, let’s build our understanding of the polynomial representation method. In order to more accurately classify the nature of this representation, we utilize tools from category theory.
3 Categories

We’d like to further consider the mapping defined by the homomorphism polynomials. To do so, we must establish a few facts from category theory. To further explore the intricacies of category theory, (Wisbauer, 2018) provides a fantastic starting point. We also recommend reading (Adámek et al., 1990; Kashiwara & Schapira, 2005; MacLane, 1971). We assume that the reader is familiar with the definition of a category as well as various types of morphisms.

As we will use the homomorphism polynomial concept extensively in the following results, we want to be precise about what this mapping is. By understanding the nature of the structures in which the homomorphism polynomial interacts, we may understand why we have a weaker version of Hilbert’s Nullstellensatz in theorem 5.12 and why we must define our notation in later sections so carefully. In this section, we use a category theory approach to demonstrate that the mapping defined by the homomorphism polynomial is a commutative ring morphism that is both an epimorphism and a retraction.

In order to establish the precise nature of the homomorphism polynomial mapping, we begin by considering the image and establishing that it is, in fact, a commutative ring morphism.

We introduce the concept of labeling a graph to prepare the reader for considering a set of quantum graphs as a commutative ring.

**Definition 3.1.** A $k$-labeled graph is a graph in which $k$ of the nodes are labeled by $1, \ldots, k$, allowing any number of unlabeled nodes. We call a graph partially labeled if it is $T$-labeled for some finite set $T \subseteq \mathbb{N}$.

We denote by $Q_k$ the infinite-dimensional linear space of quantum graphs whose constituents are all $k$-labeled graphs. We may sometimes use the notation that a graph $G$ is on $[n]$ which means that it has $n$ nodes and is $n$-labeled. We may use the gluing product to turn this linear space into an algebra and allow it to exist in the category of rings.

**Definition 3.2.** For two $k$-labeled graphs $F_1, F_2$, the gluing product $F_1 F_2$ is defined by taking the disjoint union of $F_1$ and $F_2$, adding edges between nodes with the same label and contracting the new edges.

Using the gluing product $F_1 F_2$ as the product of two generators, we extend this multiplication to the other elements of the gluing algebra $Q_k$ by linearity. Thus $Q_k$ is associative and commutative. The fully labeled graph $O_k$ on $[k]$ with no edges is the multiplicative unit in $Q_k$. 
**Example 3.3.** We may illustrate $K_2$, the unlabeled complete graph on 2 nodes as 1 and we can illustrate $K_2^\bullet$, the complete graph on 2 nodes with one label as 1. The gluing product $K_2K_2$ is 11 as there are no labels to identify together. Similarly, the gluing product $K_2^\bullet K_2^\bullet$ is 11 as there are no labeled nodes in $K_2$. The gluing product $K_2^\bullet K_2^\bullet$ does have a shared label so it may be illustrated as $11 = \mathcal{V}$.

Let $Q_N$ denote the vector space of formal linear combinations of partially labeled graphs. Considering unlabeled graphs to be $\emptyset$-labeled, it is clear that this vector space contains all (simply labeled) quantum graphs. Following (Lovász, 2012), we turn $Q_N$ into an algebra by using the gluing product $G_1G_2$ (gluing along the labeled nodes) as the product of two generators, and then extending this multiplication to the other elements linearly. $Q_N$ is associative and commutative, and the empty graph is a unit element.

We show both our graph algebra and our subset of the polynomial ring are commutative rings with unity so that we may say they are in the same category.

**Proposition 3.4.** The graph algebra $Q_N$ is a commutative ring with unity.

*Proof.* By (Lovász, 2012), $Q_N$ is an infinite-dimensional vector space. By being a vector space, we can say that $(Q_N, +)$ is an abelian group, letting $+$ be linear combination as in the definition of quantum graphs. Lovász shows that the gluing product is associative, commutative, and $Q_N$ contains a multiplicative unit element. Linear extension of the gluing product to quantum graphs gives us distributivity.  

**Proposition 3.5.** The set $\mathbb{C}[X]^{S_q}$ is a commutative ring with unity.

*Proof.* The desired set is a subset of the polynomial ring $\mathbb{C}[X]$ on $(\binom{q}{2}) + q$ variables. Hence the only properties we must verify are closure, existence of unit/identity elements, and existence of additive inverses.

For any polynomials $p_1, p_2 \in \mathbb{C}[X]^{S_q}$, if $p_1, p_2$ are both invariant under permutations of $[q]$ then so will be $-p_1, -p_2, p_1 + p_2$, and $p_1p_2$ (inverses and closure). The identity elements 0 and 1 are both invariant under permutations of $[q]$ (as are all elements of $\mathbb{C}$).

Now that we have shown both $\mathbb{C}[X]^{S_q}$ and $Q_N$ are commutative rings with unity, we may conclude that they are in the same category and consider whether the mapping between them is a morphism in the category of rings.

**Proposition 3.6.** Fix $q$. Let $\vartheta_q : Q_N \rightarrow \mathbb{C}[X]^{S_q}$ be the mapping defined by $\vartheta_q : g \mapsto \text{hom}(g, X)$ for $g \in Q_N$. The mapping $\vartheta_q$ is a strong epimorphism and a retraction.
Proof. First, we show $\vartheta_q$ is a ring homomorphism. By Lovász, the definition of the polynomial method asserts that $\text{hom}(a + b, X) = \text{hom}(a, X) + \text{hom}(b, X)$ and $\text{hom}(ab, X) = \text{hom}(a, X) \text{hom}(b, X)$ for all quantum graphs $a, b \in \mathbb{Q}_N$. The multiplicative identity element that is an empty graph $K_0$ gives us $\text{hom}(K_0, X) = 1$ which is the multiplicative identity in $\mathbb{C}[X]^{S_q}$. Hence $\vartheta_q$ is a ring homomorphism.

As noted in the discussion leading up to definition 2.4, $\vartheta_q$ is surjective; further, it must be a strong epimorphism in the category-theoretic sense as it is surjective.

Define $g : \mathbb{C}[X]^{S_q} \to \mathbb{Q}_N$ to be the map described in the proof of Lemma 6.49 in (Lovász, 2012), breaking ties arbitrarily so that each polynomial maps to exactly one quantum graph, we may see that $g\vartheta_q = \text{id}_{\mathbb{C}[X]^{S_q}}$ and hence $\vartheta_q$ is a retraction.

Now that we have established some information about the mapping from quantum graphs to complex polynomials defined by the homomorphism polynomial notation, we may shift our focus to introducing and discussing kernel varieties and ideals in the next section. In section 5, we will use the information established in this section to understand why certain results come about.

4 Varieties and Ideals

In this section, we introduce some useful notation regarding zero-sets and ideals. This notation serves to simplify expressions and harkens to notation in (Hartshorne, 2013).

Before we introduce notation, we will define the three known types of varieties discussed in (Lovász, 2012).

Definition 4.1. Let $g$ be a quantum graph. A set of kernels $K \subseteq \mathcal{W}$ such that $t(g, W) = 0$ for all $W \in K$ will be called a kernel variety. If $g$ is simple, then $K$ is a simple variety. A graphon variety $K'$ is the intersection of a kernel variety with $\mathcal{W}_0$, i.e., $K' = K \cap \mathcal{W}_0$ for some kernel variety $K$.

Trying to be more general and considering the common solutions of a system of constraints $t(f_1, W) = 0, \ldots, t(f_m, W) = 0$ is equivalent to the single condition $t(f_1^2 + \ldots + f_m^2, W) = 0$.

We briefly recall that $\mathcal{W}_0$ is the set of all kernels $W \in \mathcal{W}$ such that $0 \leq W \leq 1$ and $\mathcal{W}_1$ is the set of all kernels $W \in \mathcal{W}$ such that $-1 \leq W \leq 1$.

To understand the notion of measuring the distance between two graphons, we will introduce the cut distance between two graphons. From (Glasscock,
we may define the cut distance between graphons $W$ and $U$ to be

$$\inf_{\phi,\psi} \sup_{S,T} \left| \int_{S \times T} W(\phi(x), \phi(y)) - U(\psi(x), \psi(y)) \right| \, dx \, dy,$$

noting that the infimum is taken over all re-labelings $\phi$ of $W$ and $\psi$ of $U$ and the supremum is taken over all measurable subsets $S$ and $T$ of $[0,1]$.

Every simple graphon variety is closed in the cut distance and hence a compact subset of the graphon space ($\mathcal{W}_0, \delta$). The union and intersection of two [simple] graphon varieties are [simple] graphon varieties. It is well known that the space of all graphons with the $\delta$ metric is a compact metric space (Glasscock, 2015), (Lovász, 2012), (Lovász, 2019).

We use the following notational definition in the remainder of this paper to make things clearer.

**Definition 4.2.** Given a quantum graph $g$, we denote the correspondence $V_W(g)$ to be the set of kernels $K \subseteq \mathcal{W}$ such that $t(g, W) = 0$ for all $W \in K$. That is, we let

$$V_W(g) = K = \{ W \in \mathcal{W} | t(g, W) = 0 \}.$$ (2)

Note that $V_W(g)$ is a kernel variety defined by the quantum graph $g$. If $g$ is simple, then $K = V_W(g)$ is a simple variety.

Throughout this paper, we restrict our choice of kernels to allow for utilization of a finite spectral decomposition in our proofs. That is, we restrict to kernels that are finite rank and have a finite number of eigenfunctions and eigenvalues as $L_2$ operators.

**Definition 4.3.** ([Lovász, 2012]) For every $W \in \mathcal{W}$, define the operator $T_W : L_1[0,1] \to L_\infty[0,1]$ by

$$(T_W f)(x) = \int_0^1 W(x,y) f(y) \, dy.$$ If $T_W$ has a finite number of eigenvalues and eigenfunctions, then we call $W$ a finite rank kernel.

For any subset $Y \subseteq \mathcal{W}$ where every $W \in Y$ is of finite rank, we first define the pre-ideal of $Y$ in $\mathcal{Q}_0$ to be

$$I_{\mathcal{Q}_0}(Y) = \{ g \in \mathcal{Q}_0 | t(g, W) = 0 \text{ for all } W \in Y \}.$$ We then can define the ideal of $Y$ in $\mathbb{C}[X]^{S_q}$ by

$$I_q(Y) = \{ \text{hom}(g, X) | g \in I_{\mathcal{Q}_0}(Y) \}.$$
We define a mapping $V_q$ such that for a set $Q \subseteq Q_0$ and an ideal $a = I_q(Q)$, we have

$$V_q(Q, a) = V_W(Q)$$

With this notation under our belts, we may approach the results of this paper.

## 5 Results

We begin by establishing some important facts about finite rank kernel varieties in relation to their affine variety counterparts in algebraic geometry. For the remainder of this paper, we assume all kernels are finite rank if unspecified. Many of these results are much trickier to attempt to prove for all kernels and we leave that as an exercise for future researchers.

Being closed under the gluing product is equivalent to corresponding to an ideal of homomorphism polynomials.

**Proposition 5.1.** Let $W$ be a kernel of any rank and $Q = I_{Q_0}(W)$ the pre-ideal of $W$. Let $S = I_q(W) = \{\text{hom}(q, X)|q \in Q\}$. The following are equivalent:

1. $S$ is an ideal.
2. For any $q \in Q$ and $q' \in Q_0$, $t(qq', W) = 0$.

In the next proposition, we demonstrate that any finite rank kernel has an ideal of homomorphism polynomials corresponding to graphons with zero homomorphism density with respect to the kernel.

**Proposition 5.2.** Let $W$ be a finite rank kernel. The set $S = I_q(W)$ is an ideal.

**Proof.** Let $W$ have rank $r$. Consider the set $S = I_q(W) = \{\text{hom}(q, X)|q \in Q_0, t(q, W) = 0\}$. Let $q = \sum_i \gamma_i F_i \in S$ and $q' = \sum_j \beta_j G_j \in Q_0$. The density of a graph $F$ in $W$ becomes finite since $W$ is of finite rank, giving us:

$$t(q, W) = \sum_i \gamma_i \sum_{\chi: E(F_i) \to [r]} \prod_{e \in E(F_i)} \lambda_{\chi(e)} \prod_{v \in V(F_i)} M_{\chi}(v) = 0$$

where the finite spectral decomposition of $W$ is

$$W(x, y) = \sum_{k=1}^r \lambda_k f_k(x)f_k(y)$$
with

\[ M_\chi(v) = \int_0^1 \prod_{u:v \in E} f_{\chi(uv)}(x) \, dx \]

Then \( t(qq', W) \) is

\[ t(qq', W) = \sum_{i,j} \gamma_i \beta_j \sum_{\chi: E(F_i) \cup E(G_j) \to [r]} \prod_{e \in E(F_i) \cup E(G_j)} \lambda_\chi(e) \prod_{v \in V(F_i \cup G_j)} M_\chi(v) \]

For multigraphs or simple graphs \( F, G \), we can see that (following Lovasz 6.6 reasoning about the polynomial method)

\[ \sum_{\chi: E(FG) \to [r]} \prod_{e \in E(FG)} \lambda_\chi(e) \prod_{v \in V(FG)} M_\chi(v) = \left( \sum_{\chi: E(F) \to [r]} \prod_{e \in E(F)} \lambda_\chi(e) \prod_{v \in V(F)} M_\chi(v) \right) \left( \sum_{\chi: E(G) \to [r]} \prod_{e \in E(G)} \lambda_\chi(e) \prod_{v \in V(G)} M_\chi(v) \right) \]

i.e. \( t(FG, W) = t(F, W)t(G, W) \) for finite rank \( W \). This allows us to rewrite \( t(qq', W) \) as

\[ t(qq', W) = \sum_j \beta_j t(G_j, W) \left( \sum_i \gamma_i t(F_i, W) \right) \]

and since the inner summation is zero then we have that \( t(qq', W) = 0 \). \( \square \)

We may generalize the previous proposition to show that multiple finite rank kernels induce an ideal of homomorphism polynomials.

**Proposition 5.3.** Let \( V \) be a variety containing only finite rank kernels [graphons]. Then the set \( S = \{ \text{hom}(q, X) | t(q, W) = 0, q \in Q_0, W \in V \} \) is an ideal.

**Proof.** For every \( W_i \in V \), let \( S_i \) be the ideal defined in proposition 5.2. The set \( S \) is the intersection \( S = \bigcap_i S_i \) and hence is an ideal. \( \square \)

If we limit the rank of the kernels, then we have an induced ideal.

**Corollary 5.4.** Kernels of rank at most \( m \) create a homomorphism polynomial ideal.

**Proof.** By Lovasz Proposition 14.47 (b), kernels of rank at most \( m \) form a simple variety. Application of proposition 5.3 gives us the desired result. \( \square \)
Now, we may define an algebraic kernel set and verify that, considering algebraic kernel sets as closed, the algebraic kernel sets satisfy the Zariski topology on all finite rank graphons.

**Definition 5.5.** A subset $Y$ of finite rank kernels in $\mathcal{W}$ is an algebraic kernel set if there exists a subset $Q \subseteq Q_0$ such that $Y = V_{\mathcal{W}}(Q)$.

**Proposition 5.6.** The union of two algebraic kernel sets is an algebraic kernel set. The intersection of any family of algebraic kernel sets is an algebraic kernel set. The empty set and the whole space of finite rank kernels are algebraic kernel sets.

**Proof.** Our proof follows that of Proposition 1.1 in (Hartshorne, 2013). If $Y_1 = V_{q}(Q_1, T_1)$ and $Y_2 = V_{q}(Q_2, T_2)$, then $Y_1 \cup Y_2 = V_{q}(Q_1 Q_2, T_1 T_2)$, where $T_1 T_2$ denotes the set of all products of an element of $T_1$ by an element of $T_2$. If $W \in Y_1 \cup Y_2$, then either $W \in Y_1$ or $W \in Y_2$, so $W$ is a kernel that has a vanishing homomorphism density for every polynomial in $T_1 T_2$. Conversely, if $W \in V_{q}(Q_1 Q_2, T_1 T_2)$, and $W \notin Y_1$, then there is a homomorphism polynomial $\text{hom}(g, X) \in T_1$ such that $t(g, W) \neq 0$. Now for any homomorphism $t(h, X) \in T_2$, $t(gh, W) = t(g, W)t(h, W) = 0$ implies that $t(h, W) = 0$, so then $W \in Y_2$.

If $Y_\alpha = V_{q}(Q_\alpha, T_\alpha)$ is any family of algebraic kernel sets, then $\bigcap Y_\alpha = Z(\bigcup T_\alpha)$, so $\bigcap Y_\alpha$ is also an algebraic kernel set. Finally, the empty set $\emptyset = V_{q}(\{K_1\}, q) = V_{\mathcal{W}}(K_1)$ and the whole space $\mathcal{W} = V_{q}(\{K_0\}, 1) = V_{\mathcal{W}}(K_0)$.

We define the Zariski Topology on the subset of $\mathcal{W}$ that is finite rank by taking the open subsets to be the complements of the algebraic kernel sets. By the above proposition, the intersection of two open sets is open and the union of any family of open sets is open. The empty set and the whole space are both open, hence this is a topology.

**Definition 5.7.** We define a nonempty subset $Y$ of $\mathcal{W}$ to be irreducible in the classic sense in (Hartshorne, 2013), i.e. it cannot be expressed as the union $Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in $Y$. The empty set is not considered to be irreducible

We can now summarize properties of the function $I_q$ which maps finite rank subsets of $\mathcal{W}$ to ideals and the function $V_{\mathcal{W}}$ which maps subsets of $Q_0$ to varieties.

**Proposition 5.8.** (a) If $A_1 \subseteq A_2$ are subsets of $Q_0$, then $\{\text{hom}(g, X) | g \in A_1\} \subseteq \{\text{hom}(g, X) | g \in A_2\}$ and $V_{\mathcal{W}}(A_1) \supseteq V_{\mathcal{W}}(A_2)$.
(b) If $Y_1 \subseteq Y_2$ are subsets of finite rank of $\mathcal{W}$, then $I_q(Y_1) \supseteq I_q(Y_2)$ and $I_{Q_0}(Y_1) \supseteq I_{Q_0}(Y_2)$.

(c) For any two subsets of finite rank $Y_1, Y_2$ of $\mathcal{W}$, we have $I_q(Y_1 \cup Y_2) = I_q(Y_1) \cap I_q(Y_2)$.

(d) For any set of quantum graphs $Q \subseteq Q_0$ such that the set of corresponding homomorphism polynomials is an ideal $a = \{\text{hom}(g, X)|g \in Q\} \subseteq \mathbb{C}[X]^{S_q}$, $I_q(V_{\mathcal{W}}(Q)) = I_q(V_q(Q, a)) \supseteq \sqrt{a}$, the radical of $a$.

(e) For any subset $Y \subseteq \mathcal{W}$ of finite rank, $V_{\mathcal{W}}(I_{Q_0}(Y)) = \overline{Y}$, the closure of $Y$.

Proof. (a), (b), and (c) follow from the definitions prior. (d) is a direct consequence of Hilbert’s Nullstellensatz applied to kernels (HNAK) theorem 5.12, stated below, and proposition 5.9.

To prove (e), we note that $Y \subseteq V_{\mathcal{W}}(I_{Q_0}(Y))$, which is a closed set by the Zariski Topology, so clearly $\overline{Y} \subseteq V_{\mathcal{W}}(I_{Q_0}(Y))$. Let $A$ be any closed set containing $Y$. Then $A = V_{\mathcal{W}}(Q)$ for some set of quantum graphs $Q$. So $V_{\mathcal{W}}(Q) \supseteq Y$, and by (b), $I_{Q_0}(V_{\mathcal{W}}(Q)) \subseteq I_{Q_0}(Y)$. But clearly $Q \subseteq I_{Q_0}(V_{\mathcal{W}}(Q))$, so (a) gives us $A = V_{\mathcal{W}}(Q) \supseteq V_{\mathcal{W}}(I_{Q_0}(Y))$. Thus $V_{\mathcal{W}}(I_{Q_0}(Y)) = \overline{Y}$. □

Proposition 5.9. For an ideal $a \subseteq \mathbb{C}[X]^{S_q}$, $I_q(V_q(\overline{a}) \supseteq \sqrt{a}$.

Proof. To prove this statement, it suffices to find an example of strict inclusion and an example of strict equality.

Consider the ideal $a$ generated by $(\text{hom}((K_2^4 - C_4)^2) + (P_3 - 2K_3)^2)$. Solving gives us the simple graphon variety $\{W_{1/2} = \frac{1}{2}\}$ only containing the constant $\frac{1}{2}$ function. Consider $g = \frac{1}{2}K_3^3 - C_4$. We compute that when $q = 2$, $\text{hom}(g, X)$ is not in $\sqrt{a}$, thus demonstrating a case of strict inclusion.

Consider the ideal $b = (X)$. We can solve to find that $V_q(b) = \{0\}$. Note, however that the quantum graph $h = K_0 - K_1$ has homomorphism polynomial $\text{hom}(h, X) = 1 - q$ which is not in the radical of the ideal but $t(h, 0) = t(K_0, 0) - t(K_1, 0) = 0$. This is another example of inclusion rather than equality.

As a trivial example of equality, consider $\mathbb{C}[X]^{S_q}$. Then we know that $V_q(\mathbb{C}[X]^{S_q}) = \emptyset$, and $I_q(\emptyset) = \mathbb{C}[X]^{S_q}$. The radical of the entire space is itself, hence $I_q(V_q(\mathbb{C}[X]^{S_q})) = \sqrt{\mathbb{C}[X]^{S_q}}$. □

We restate Hilbert’s Nullstellensatz to illustrate the differences we find in our new version.
Theorem 5.10. (Hilbert’s Nullstellensatz) (Hartshorne, 2013) Let \( k \) be an algebraically closed field, let \( a \) be an ideal in \( A = k[x_1, \ldots, x_n] \), and let \( f \in A \) be a polynomial which vanishes at all points of \( Z(a) \). Then \( f^r \in a \) for some integer \( r > 0 \).

Building off of Hilbert’s Nullstellensatz, we are not able to entirely replicate the results for the zero-set of polynomials using the zero-sets of kernels. Recall from proposition 3.6 that the homomorphism polynomial does not define an isomorphism. Rather, it is an epimorphism so we may easily shift from quantum graphs to polynomials. As it is a retraction, we may invert it in a sense to create quantum graphs from polynomials but a composition of the homomorphism polynomial and any sort of inversion does not give us the identity map. In fact, this interaction is the reason why we cannot exactly apply Hilbert’s Nullstellensatz as in (Hartshorne, 2013).

Theorem 5.11. (HNAK.1) Let \( a \) be an ideal in \( C[X]^{S_q} \) and let \( q \in Q_0 \) be a quantum graph such that \( \text{hom}(q, X) \in C[X]^{S_q} \) vanishes at all points of \( Z(a) \). Then \( f^r = \text{hom}(q^r, X) \in a \) for some integer \( r > 0 \).

Proof. Since \( a \) is an ideal in \( C[X]^{S_q} \), it is certainly an ideal in \( C[x_1, \ldots, x_{q^2}] \) which is a polynomial ring over an algebraically closed field. Hence Hilbert’s Nullstellensatz gives us the desired result.

Theorem 5.12. (HNAK.2) Let \( a \) be an ideal in \( C[X]^{S_q} \) and let \( q \in Q_0 \) be a quantum graph such that \( \text{hom}(q, X) \in C[X]^{S_q} \) vanishes at all points of \( Z(a) \). Then \( t(q, W) = 0 \) for all kernels \( W \) in \( V = V_W\{q \in Q_0| \text{hom}(q, X) \in a\}\). That is, \( q \in I_q(V) \).

Proof. Theorem 5.11 asserts that there exists some integer \( r > 0 \) such that \( f^r = \text{hom}(q^r, X) \in a \). Then by definition of \( V \), we must have \( t(q^r, W) = 0 \) for all \( W \in V \). Linearity allows us to expand this to \( t(q, W)^r = 0 \) which gives us the desired result of \( t(q, W) = 0 \).

We encourage the reader to carefully re-read the previous theorems and notice that we still require vanishing using a zero-locus of polynomials rather than a zero-locus of kernels.

Corollary 5.13. An algebraic kernel set \( V \) is irreducible if and only if \( I_q(V) \) is a prime ideal.

Proof. This proof follows exactly that of Corollary 1.4 in (Hartshorne, 2013).
In this section, we established an understanding of an algebraic kernel set that satisfies the Zariski Topology. We demonstrated that when working with finite rank kernels, we may construct ideals of homomorphism polynomials. We also proved that a slightly weaker version of Hilbert’s Nullstellensatz, the HNAK, exists for finite rank kernels and their homomorphism polynomial ideals. These results naturally leave us with a number of questions. In the next section, we discuss applications to Hadamard matrices and compute a few examples of the preceding concepts.

6 Hadamard Matrices

Definition 6.1. (Ferber et al., 2018) A Hadamard matrix of order $n$ is a square matrix $H$ of order $n$ whose entries are $\{\pm 1\}$ with pairwise orthogonal rows, i.e. $HH^T = nI_n$. They are named after Jacque Hadamard who studied them in connection with his maximal determinant problem.

Hadamard matrices are widely studied. While we acknowledge the broad impacts of their applications, we will not touch on this subject. For greater discussion of their applications and Hadamard’s studies, see (Banica, 2021; Breen et al., 2020; Ferber et al., 2018; Li et al., 2020; Lofano & Lutz, 2021). Throughout this paper, we will use the example of the $2 \times 2$ Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ to better represent the ideas within in the lens of studying Hadamard matrices.

We would like to represent our example of the Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ as a graphon. To do so, we must introduce a type of function that allows for steps to occur in a piecewise manner as introduce in (Lovász, 2012).

Definition 6.2. A kernel $W \in W$ is called a stepfunction if:

(i) we have a partition $\{T_1, \ldots, T_k\}$ of $[0, 1]$ into measurable sets;

(ii) $W$ is constant on every product set $T_i \times T_j$, i.e. $W(x_1, y_1) = W(x_2, y_2)$ if $x_1, x_2 \in T_i$ and $y_1, y_2 \in T_j$ for some $i, j \in \{1, \ldots, k\}$.

We call the sets $T_i$ the steps of $W$.

If we have a symmetric $n \times n$ matrix $P$ with entries $p_{ij}$, we may construct a stepfunction kernel $W_P$. Set $T = \{[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, 1]\}$ to be the steps, letting $T_i = [\frac{i-1}{n}, \frac{i}{n})$. We define the stepfunction kernel $W_P$ to be $W_P(x, y) = p_{ij}$ for $x \in T_i$ and $y \in T_j$. 

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Definition 6.3. (Lovász, 2012, Example 14.35 pp.251) A symmetric $n \times n$ Hadamard matrix $B$ gives rise to a stepfunction kernel $W_B$, which we alter a little to get a graphon $U_B = (W_B + 1)/2$. We call $U_B$ an Hadamard graphon.

Example 6.4. Let $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ be our Hadamard matrix. We may construct the stepfunction kernel $W_H : [0, 1] \to \mathbb{R}$ to be

$$W_H(x, y) = \begin{cases} 
1, & x, y \in [0, \frac{1}{2}) \\
1, & x \in [0, \frac{1}{2}) \text{ and } y \in [\frac{1}{2}, 1] \\
1, & x \in [\frac{1}{2}, 1] \text{ and } y \in [0, \frac{1}{2}) \\
-1, & x, y \in [\frac{1}{2}, 1] 
\end{cases}.$$ 

From this stepfunction kernel, we may construct the Hadamard graphon $U_H$ to be

$$U_H(x, y) = \frac{1}{2}(1 + W_H(x, y)) = \begin{cases} 
1, & x, y \in [0, \frac{1}{2}) \\
1, & x \in [0, \frac{1}{2}) \text{ and } y \in [\frac{1}{2}, 1] \\
1, & x \in [\frac{1}{2}, 1] \text{ and } y \in [0, \frac{1}{2}) \\
0, & x, y \in [\frac{1}{2}, 1] 
\end{cases}.$$ 

The set of all hadamard graphons and the constant $1/2$ graphon form a simple variety by (Lovász, 2012).

Definition 6.5. Given a symmetric $n \times n$ Hadamard matrix $B$ and a graph $F$ on $v$ vertices, we define $P(B, F)$ as follows. For $m \times n$ matrices $X, Y$, we let $X \odot Y$ be the Hadamard Product of $X, Y$, i.e. $X \odot Y$ is a $m \times n$ matrix with element-wise multiplication of $X, Y$.

Let $A$ be the $v \times v$ adjacency matrix of $F$ for an arbitrary labeling (the labeling does not matter: if we have a labeling and a map, there will be a map that is the same as changing to a different labeling and using the original map. We will be looking at all possible maps so the start labeling is arbitrary). We assume that $F$ is unweighted so that $A$ is entirely 1s and 0s. $P(B, F)$ is the probability that a randomly chosen map $\varphi : [v] \to [n]$ (allowing repeats) will satisfy $B \circ \varphi(A) = \varphi(A)$ where $\varphi(A)$ is defined to be the adjacency matrix of $F$ under the new labelings from the map $\varphi$ where we simply have value 1 at entry $ij$ if $i, j$ are connected by at least one edge.

If $\Lambda_\varphi \subseteq [n] \times [n]$ is the set of index pairs $(i, j)$ such that element $a_{i,j}$ in $\varphi(A)$ is 1, then the equality $B \circ \varphi(A) = \varphi(A)$ is the same as saying that all $(i, j) \in \Lambda_\varphi$ have $b_{ij} = 1$ in $B$.

For a quantum graph $q = \sum \alpha_i F_i$ and Hadamard graphon $U_B$, the homomorphism density $t(q, U_B)$ will be

$$t(q, U_B) = \frac{1}{2} \sum \alpha_i P(B, F_i) |E(F_i)|$$
where $E(F_i)$ are the edges of $F_i$ and $P(B, F_i)$ is the probability defined above. Let’s consider a few examples to illustrate this point.

**Example 6.6.** Let $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. We let $U_B$ be the corresponding Hadamard graphon. By definition 6.3, we first construct the stepfunction kernel $W_B$. It has steps $\{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}$ and image

$$W_B(x, y) = \begin{cases} 1, & x, y < \frac{1}{2} \\ -1, & \text{otherwise} \end{cases}$$

Then $U_B = (W_B + 1)/2$. We will consider what quantum graphs $q = \sum_i \alpha_i F_i$ satisfy $t(q, U_B) = 0$. We first consider quantum graphs with a single constituent, i.e. $q = \alpha F$. If $t(\alpha F, U_B) = 0$, then by linearity this is equal to $\alpha t(F, U_B)$ and, since $\alpha \neq 0$ as a constituent has a non-zero coefficient, we have $t(F, U_B) = 0$. Letting $F$ have vertices $V(F)$ and edges $E(F)$, we may expand the homomorphism density to be

$$t(F, U_B) = \int_{[0,1]|V(F)|} \prod_{ij \in E(F)} U_B(x_i, x_j) \prod_{i \in V(F)} dx_i$$

$$= \frac{1}{2} \int_{[0,1]|V(F)|} \prod_{ij \in E(F)} (1 + W_B(x_i, x_j)) \prod_{i \in V(F)} dx_i$$

Since $W_B$ is a stepfunction kernel, we solve this by breaking each integral up into the steps and evaluating as $W_B$ is either $-1$ or $1$. Consider $F = K_3$, the complete graph on three vertices.

$$t(K_3, U_B) = \frac{1}{2} \left( \int_0^1 \int_0^1 \int_0^1 (1 + W_B(x_1, x_2))(1 + W_B(x_1, x_3))(1 + W_B(x_2, x_3)) dx_1 dx_2 dx_3 \right)$$

In the above integral, we note that when any of $x_1, x_2, x_3$ fall into the step $[\frac{1}{2}, 1]$, one of the $W_B$ values will be $-1$ giving us a zero integrand. Hence, this integral only has two non-zero integrands and we can solve it to be

$$t(K_3, U_B) = \frac{1}{2} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 2^3 dx_1 dx_2 dx_3 = \frac{1}{2} \frac{1}{2^3} \cdot 2^3 = \frac{1}{2}.$$

For this value of $B$, there are no quantum graphs of a single constituent that have a zero homomorphism density. Consider any graph $F$. In the expression for the homomorphism density, the portion when all $x_i$ are considered in the interval from 0 to $\frac{1}{2}$ will be non-zero.
In fact, any symmetric $n \times n$ Hadamard matrix $B$ with any positive entries on the diagonal will have $t(q, U_B) > 0$ for any single-constituent quantum graphs $q$. The proof follows the same logic as in the example above. If $b_{ii}$ is a positive entry, consider the portion of the homomorphism density integral where all $x_j$ are in the step $T_i$.

**Example 6.7.** Let $B = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. We let $U_B$ be the corresponding Hadamard graphon. By definition 6.3, we first construct the stepfunction kernel $W_B$. It has steps $\{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}$ and image

$$W_B(x, y) = \begin{cases} 1, & x < \frac{1}{2}, y \geq \frac{1}{2} \\ -1, & \text{otherwise} \end{cases}$$

Then $U_B = (W_B + 1)/2$. We will consider what quantum graphs $q = \sum_i \alpha_i F_i$ satisfy $t(q, U_B) = 0$.

## 7 Future Work

This work leaves a number of unanswered questions. We propose the following series of conjectures.

We have restricted our consideration to finite rank kernels. We conjecture that corollary 5.4 holds for kernels of infinite rank as do the other results in this paper. Since we are unable to take a finite spectral decomposition of infinite rank kernels, we must consider other methods of approach to prove this conjecture.

We propose extending the homomorphism ideal concept as follows. We conjecture that we may generalize the idea of a homomorphism polynomial ideal to construct a functor from the category of metric spaces to the category of rings and we may generalize the idea of a graphon variety to construct a functor in the opposite direction such that the composition of the graphon variety functor with the homomorphism ideal functor is the identity functor on the category of rings.

If we have distinct or even disjoint sets $Q_1, Q_2 \subseteq Q_0$ such that they define the same homomorphism polynomial ideal, i.e. $\{\text{hom}(g, X) | g \in Q_1\} = \{\text{hom}(g, X) | g \in Q_2\}$, we conjecture that the kernel varieties defined by each respective set are the same, i.e. $V_W(Q_1) = V_W(Q_2)$.

We pose the following question related to Hadamard matrices building on definition 6.5.

**Conjecture 7.1.** Given a Hadamard graphon $U_B$ defined for a symmetric $n \times n$ Hadamard matrix $B$, we conjecture that the set of quantum graphs $q$ satisfying $P(B, q) = 0$ is the quantum graph ideal of $U_B$. 

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