On Chiral Symmetry Breaking in a Constant Magnetic Field in Higher Dimension

E. V. Gorbar

Instituto de Fisica Teorica, 01405-900 Sao Paulo, Brazil †

Abstract

Chiral symmetry breaking in the Nambu–Jona-Lasinio model in a constant magnetic field is studied in spacetimes of dimension $D > 4$. It is shown that a constant magnetic field can be characterized by $\lfloor (D-1)/2 \rfloor$ parameters. For the maximal number of nonzero field parameters, we show that there is an effective reduction of the spacetime dimension for fermions in the infrared region $D \to 1 + 1$ for even-dimensional spacetimes and $D \to 0 + 1$ for odd-dimensional spacetimes. Explicit solutions of the gap equation confirm our conclusions.

PACS 11.10.Kk, 11.30.Qc, 11.30.Rd

†On leave of absence from Bogolyubov Institute for Theoretical Physics, 252143, Kiev, Ukraine
1 Introduction

It was discovered recently [1, 2] that a constant magnetic field in 3+1 and 2+1 dimensions is a strong catalyst of dynamical chiral symmetry breaking leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions. The essence of this effect is the dimensional reduction of spacetime for fermions in the infrared region, which is $3 + 1 \rightarrow 1 + 1$ for $D = 3 + 1$ and $2 + 1 \rightarrow 0 + 1$ for $D = 2 + 1$. The dimensional reduction can be understood as follows. The motion of charged fermions along the direction of magnetic field is free, therefore, the spectrum is continuous. On the other hand, the motion in directions perpendicular to the magnetic field is restricted and the spectrum is discrete (fermions fill the Landau levels). Thus, the dynamics of fermions in a constant magnetic field effectively corresponds in the infrared region to the dynamics of fermions in (1 + 1)- and (0 + 1)-dimensional spacetimes in the cases of (3+1)- and (2+1)-dimensional spacetimes, respectively. (In this paper we consider chiral symmetry breaking in flat spacetimes of higher dimension $D > 4$ with trivial topology (if topology of spacetime is not trivial, then there can be an additional reduction of the spacetime dimension [3, 4]). To study this problem, we are motivated in addition to purely academic interest also by recent activity in studying models with extra dimensions [5, 6] and the availability of string solutions with constant magnetic field (see, e.g., [7]).

In Sect.2 we specify the Nambu–Jona-Lasinio (NJL) model [8] in $D > 4$ and discuss the number of parameters which characterize a constant magnetic field in $D > 4$. We show in Sect.3 that, for the maximal number of field parameters, the effective reduction of the spacetime dimension for fermions in the infrared region is $D \rightarrow 1 + 1$ for even-dimensional spacetimes and $D \rightarrow 0 + 1$ for odd-dimensional
spacetimes. We find the corresponding solutions of the gap equation in the NJL model. Our conclusions are given in Sect.4.

2 The NJL model in a constant magnetic field

To study dynamical chiral symmetry breaking in a constant magnetic field, we first need to classify constant magnetic fields in $D > 4$, i.e., to define the number of independent parameters which specify a constant magnetic field. Mathematically, the problem is the following: A constant electromagnetic field is completely characterized by the field strength $F_{\mu\nu}$. Elements $F_{0\nu}$ and $F_{\mu 0}$ characterize the electric field. Elements $F_{ij}$, where $i$ and $j$ take values $1,..., n$ ($D = n + 1$), define a constant magnetic field. By using orthogonal rotations, one can set some elements of $F_{ij}$ to zero. It is a well-known fact of linear algebra that the number of independent parameters, which define an arbitrary $F_{ij}$ up to orthogonal rotations, is $\left[\frac{n}{2}\right]$.

Let us present a simple inductive proof of this fact. Since $F_{ij}$ is antisymmetric, its diagonal elements are zero. Obviously, $F_{ij}$ is characterized in general by $\frac{n(n-1)}{2}$ parameters, which we choose to be, e.g., the elements above the diagonal. On the other hand, the orthogonal group in n-dimensional space has also $\frac{n(n-1)}{2}$ independent parameters because there are $\frac{n(n-1)}{2}$ independent rotations in n-dimensional space. Does it mean that we can set to zero all elements of $F_{ij}$ by using appropriate orthogonal rotations? Of course, not. Explicit calculations show that an orthogonal rotation in the plane $mn$ leaves unchanged the $F_{mn}$ element. Another fact is that if we perform a rotation in the plane $kl$, where $kl$ takes value on the antidiagonal, then it leaves also all other antidiagonal elements unchanged. It can be shown inductively for any $n \geq 3$ that one can set all elements of $F_{ij}$ to zero (except the elements on
the antidiagonal) by using orthogonal rotations in planes \( pq \), where \( pq \) take all values except those on the antidiagonal. Since the remaining rotations in planes \( kl \), where \( kl \) take values on the antidiagonal, do not change antidiagonal elements, the number of independent elements of \( F_{ij} \) is exactly the number of elements on the antidiagonal, which is obviously \[ \left[ \frac{n}{2} \right] \].

We can assume without loss of generality that the magnetic part of the field strength \( F_{\mu\nu} \) in a convenient reference frame is given by

\[
F_{ij} = \sum_{k=1}^{[\frac{D}{2}]} H_k (\delta^k_i \delta^m_{j+1-k} - \delta^k_j \delta^m_{i+1-k})
\]

(1)

and the corresponding vector potential is

\[
A_i = -H_i x_{n+1-i}.
\]

Let us now consider dynamical chiral symmetry breaking in the NJL model in a constant magnetic field in \( D > 4 \). We first discuss what we mean by chiral symmetry in spacetimes of arbitrary dimension. As well known, the notion of chiral symmetry is connected with properties of representations of the Clifford algebra (for a very clear discussion see, e.g., [10]). The Clifford algebra for spacetimes of even dimension has only one complex irreducible representation in the \( 2^{D/2} \)-dimensional spinor space. These spinors are reducible with respect to the even subalgebra (generated by products of an even number of Dirac matrices) and split in a pair of \( 2^{D/2-1} \)-component irreducible Weyl spinors (\( \gamma_D = \gamma_0 \ldots \gamma_{D-1} \) is an analog of the \( \gamma_5 \) matrix in D-dimensional spacetime and \( \frac{1+\gamma_0}{2} \) are the corresponding chiral projectors). In odd-dimensional spacetimes, there are two different representations of the Clifford algebra (they differ by the sign of the \( \gamma \)-matrices) and chiral symmetry is not defined because \( \gamma_D \) is proportional to the unity. In order to define an analog of chiral symmetry in odd-dimensional spacetimes, it is the usual practice to assume that fermion fields are in a reducible representation of the Clifford algebra so that we can define an
analog of chiral symmetry (for an explicit example in (2 + 1)-dimensional spacetime see, e.g., [1]). In what follows we understand chiral symmetry in odd-dimensional spacetimes in this sense.

For our aims it is enough to consider the following generalization of the NJL model with $U_L(1) \times U_R(1)$ chiral symmetry to $D > 4$:

$$\mathcal{L} = \bar{\psi}i\gamma^\mu D_\mu \psi + \frac{G}{2}[[\bar{\psi}\psi]^2 + (\bar{\psi}i\gamma_D\psi)^2] , \quad (3)$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative and fermion fields carry an additional 'flavor' index $i = 1, \ldots, N$.

3 Dynamical chiral symmetry breaking

By introducing auxiliary fields, we can rewrite Lagrangian (3) in the following way:

$$\mathcal{L} = \bar{\psi}i\gamma^\mu D_\mu \psi - \bar{\psi}(\sigma + i\gamma_D\pi)\psi - \frac{1}{2G}(\sigma^2 + \pi^2) . \quad (4)$$

Indeed, the Euler-Lagrange equations for the auxiliary fields $\sigma$ and $\pi$ are

$$\sigma = -G(\bar{\psi}\psi) , \quad \pi = -G(\bar{\psi}i\gamma_D\psi) , \quad (5)$$

and Lagrangian (4) gives Lagrangian (3) if we use the equation of motion (5).

By integrating over fermions, we get the following effective action for the composite fields:

$$\Gamma(\sigma, \pi) = -i\text{Tr} \ln[i\hat{D} - (\sigma + i\gamma_D\pi)] - \frac{1}{2G} \int d^Dx(\sigma^2 + \pi^2) , \quad (6)$$

where $\hat{D} = \gamma^\mu D_\mu$. As usual in calculation of the effective potential, it is enough to set $\sigma = \text{const}$ and $\pi = \text{const}$. Since the effective potential $V$ depends only on the
$U_L(1) \times U_R(1)$-invariant $\rho^2 = \sigma^2 + \pi^2$, it is sufficient to consider a configuration with $\pi = 0$ and $\sigma = \text{const}$. Since

$$\text{Det}(i\hat{D} - \sigma) = \text{Det}(\gamma_D(i\hat{D} - \sigma)\gamma_D) = \text{Det}(-i\hat{D} - \sigma) ,$$

we find that

$$\text{Tr} \ln(i\hat{D} - \sigma) = -\frac{i}{2} \text{Tr} [\ln(i\hat{D} - \sigma) + \ln(-i\hat{D} - \sigma)] = -\frac{i}{2} \text{Tr} \ln(\hat{D}^2 + \sigma^2) .$$

By using the method of proper time, we have

$$-\frac{i}{2} \text{Tr} \ln(\hat{D}^2 + \sigma^2) = \frac{i}{2} \int d^Dx \int_0^\infty \frac{ds}{s} \text{tr} \langle x | e^{-is(\hat{D}^2 + \sigma^2)} | x \rangle ,$$

where

$$\hat{D}^2 = D_\mu D^\mu - \frac{i}{2} \gamma_\mu \gamma^\nu F_{\mu\nu} .$$

For $A_\mu$ given by Eq.(2), it is obvious that the problem of calculation of the matrix element $\langle x | e^{-is(\hat{D}^2 + \sigma^2)} | x \rangle$ is reduced to the calculation of the corresponding matrix element for every $H_k$, i.e. for $x_k$ and $x_{n+1-k}$ components. By using [12], we obtain the following effective potential:

$$V(\rho) = \frac{\rho^2}{2G} + \frac{2^{D+1}}{2(4\pi)^{D/2}} \int_{\Lambda^2}^\infty ds \frac{e^{-s\rho^2}[(D-1)/2]}{s^{D/2-[(D-1)/2]}+1} e^{eH_k \coth(eH_k s)} ,$$

where $\Lambda$ is a ultraviolet cutoff. The gap equation $\frac{dV}{d\rho} = 0$ takes the form

$$\frac{1}{G} = \frac{2^{D+1}}{(4\pi)^{D/2}} \int_{\Lambda^2}^\infty \frac{ds}{s^{D/2-[(D-1)/2]}} e^{-s\rho^2} \prod_{k=1}^{D-1/2} e^{eH_k \coth(eH_k s)} .$$

If the magnetic field is absent, then the right-hand side of the gap equation is

$$\frac{2^{D+1}}{(4\pi)^{D/2}} \int_{\Lambda^2}^\infty \frac{ds}{s^{D/2}} e^{-s\rho^2} ,$$

where the integrand is exactly the heat kernel of the Dirac operator squared in $D$-dimensional spacetime. Since $\coth x \to 1$ as $x \to \infty$, it follows from Eq.(12)
that every independent parameter of magnetic field $H_k$, which is not equal to zero, effectively reduces the spacetime dimension by 2 units in the infrared region (for $s \to \infty$ only the lowest part of the spectrum of the Dirac operator squared gives contribution). Consequently, for the maximal number of field parameters $[\frac{D-1}{2}]$, we obtain that the effective reduction of the spacetime dimension in the infrared region for fermions is $D \to 1 + 1$ for even-dimensional spacetimes and $D \to 0 + 1$ for odd-dimensional spacetimes. Thus, we expect that for the maximal number of field parameters the critical coupling constant is zero for even-dimensional spacetimes and the gap analytically depends on coupling constant in odd-dimensional spacetimes, which are the characteristic features of solutions of the gap equation in two- and one-dimensional spacetimes, respectively (see [1, 2]).

To analyze the gap equation, we assume for simplicity that all $H_k$ are equal, i.e. $H_1 = H_2 = \ldots = H_{[(D-1)/2]} = H$. Since we are mainly interested only in qualitative results, we split the interval of integration in two parts from $\frac{1}{\Lambda^2}$ to $\frac{1}{eH}$ and from $\frac{1}{eH}$ to $\infty$ and approximate coth $x$ by $1/x$ on the first interval and by 1 on the second (we assume also that $\rho^2 \ll eH$ and approximate $e^{-s\rho^2}$ by 1 on the first interval). One can check that this approximation for $D=3$ and $D=4$ gives the same result for the gap (the same dependence on $eH$, $\Lambda^2$, and $G$) as the exact result [1, 2] up to a numerical constant of order $O(1)$. For even $D$, we obtain the following gap equation:

$$\frac{(2\pi)^{D/2}}{G N \Lambda^{D-2}} = 1 - \frac{(eH/\Lambda^2)^{D/2-1}}{D/2 - 1} + (eH/\Lambda^2)^{D/2-1} \int_{\rho^2/eH}^{\infty} \frac{ds}{s} e^{-s}. \quad (14)$$

By using [3]

$$\Gamma(\alpha, x) = \int_x^{\infty} e^{-t} t^{\alpha-1}$$

and an expansion of the incomplete Gamma-function for small $x$

$$\Gamma(0, x) = -C - \ln x - \sum_{k=1}^{\infty} \frac{(-x)^k}{k \cdot k!}, \quad (15)$$
where $C$ is the Euler constant, we obtain the solution

$$
\rho^2 = eH \exp \frac{(2 \pi) \Gamma(1-y)}{G N(eH)^{y-1}},
$$

(16)

where $g = \frac{G N A^{D-2}}{(D/2-1)(2\pi)^{D}}$. For odd $D$, the gap equation is

$$
\frac{(2\pi)^{D/2}}{GN \sqrt{2}} = \frac{\Lambda^{D-2}(1-(eH/\Lambda^2)^{D/2-1})}{D/2-1} + \frac{(eH)^{(D-1)/2}}{\rho} \int_{\rho^2/eH}^{\infty} ds \frac{1}{s^{1/2}} e^{-s}.
$$

(17)

By using \[13\]

$$
\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{k=0}^{\infty} \frac{(-1)^k x^{\alpha+k}}{k!(\alpha+k)} \quad (\alpha \neq 0, -1, -2, \ldots),
$$

(18)

we obtain the solution

$$
\rho = \frac{(eH)^{\frac{D-1}{2}} G N}{(2\pi)^{\frac{D-1}{2}}}.
$$

(19)

(Note that $\rho^2 \ll eH$ for sufficiently small $G$ and the approximation of $e^{-s\rho^2}$ by 1 on the interval $[\frac{1}{\Lambda^2}, \frac{1}{eH}]$ is consistent.) Thus, for the maximal number of field parameters, the critical coupling constant is zero in even-dimensional spacetimes and the gap has an essential singularity at zero value of coupling constant. In odd-dimensional spacetimes the gap depends analytically on coupling constant. These results confirm our conclusions about the effective reduction of the spacetime dimension for fermions in the infrared region because our solutions are characteristic for solutions of the gap equation in $(1+1)$- and $(0+1)$-dimensional spacetimes, respectively \[1, 2\].

Let us consider briefly the case where only $m < \frac{D-1}{2}$ field parameters are not equal to zero. As follows from the gap equation (12) the critical coupling constant is not equal to zero in this case. The most interesting for us is the dependence of the critical coupling constant on the number of field parameters and the value of magnetic field. For simplicity we again assume that all $H_k$ are equal, i.e. $H_1 = H_2 = \ldots = H_m = H$. By approximating $\coth x$ by 1 in the interval $[\frac{1}{\Lambda^2}, \infty]$ and $1/x + x/3$ in the interval $[\frac{1}{\Lambda^2}, \frac{1}{eH}]$, we find that the critical coupling constant is equal
to (due to the approximations made, the coefficients near terms \((\frac{eH}{\Lambda^2})^2\), \((\frac{eH}{\Lambda})^2\), and \((\frac{eH}{\Lambda^2})^2 \ln \frac{\Lambda^2}{eH}\) are actually defined up to a constant of order 1)

\[
g_{cr} = \frac{1}{1 + 2(\frac{eH}{\Lambda^2})^2}
\]  

(20)

for D=5,

\[
g_{cr} = \frac{1}{1 + (\frac{eH}{\Lambda^2})^2 + \frac{2}{3}(\frac{eH}{\Lambda})^2 \ln \frac{\Lambda^2}{eH}}
\]  

(21)

for D=6, and

\[
g_{cr} = \frac{1}{1 + \frac{m(D-1)}{3(D-3)}(\frac{eH}{\Lambda^2})^2}
\]  

(22)

for \(D > 6\). For \(D = 5\) and \(D = 6\), a constant magnetic field is characterized by only two independent field parameters, therefore, \(m\) can be equal only to 1 in the case under consideration. As follows from Eq.\( (22)\) for \(D > 6\) the more the number of field parameters \(m\), the less the critical coupling constant in agreement with expectations. Eqs.\( (20)-(22)\) imply that the critical coupling constant is very close to 1 in the realistic case \(eH \ll \Lambda^2\).

### 4 Conclusions

We considered chiral symmetry breaking in the NJL model in a constant magnetic field in \(D > 4\). We showed that for the maximal number of field parameters the effective reduction of the spacetime dimension for fermions in the infrared region is \(D \rightarrow 1 + 1\) for even-dimensional spacetimes and \(D \rightarrow 0 + 1\) for odd-dimensional spacetimes. We studied the gap equation of the NJL model and found that for the maximal number of field parameters the gap is analytic in coupling constant for odd-dimensional spacetimes and the gap has an essential singularity at zero value.
of coupling constant for even-dimensional spacetimes, which are characteristic features of solutions of the gap equation in \(0 + 1\) and \(1 + 1\) dimension, respectively, that confirms the dimensional reduction. We would like to note also that our results can be relevant in the context of certain string solutions [7] in the low-energy domain, where we can have a constant magnetic field in spacetimes with dimension \(D > 4\).

The author thanks I.L. Buchbinder, S.P. Gavrilov, D.M. Gitman, A.A. Natale, and F. Toppan for useful discussions and valuable remarks. I am grateful to V.P. Gusynin for reading the manuscript and useful suggestions. This work was supported in part by FAPESP grant No. 98/06452-9.

References

[1] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys.Rev.Lett. 73 (1994) 3499; Phys.Lett. B349 (1995) 477; Phys.Rev. D52 (1995) 4718.

[2] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys.Rev. D52 (1995) 4747; Nucl.Phys. B462 (1996) 249.

[3] D.J. O’Connor, C.R. Stephens, and B.L. Hu, Ann. Phys. 190 (1989) 310.

[4] E.J. Ferrer, V.P. Gusynin, and V. de la Incera, Phys.Lett. B455 (1999) 217.

[5] N. Arkani-Hamed, S. Dimopolous, and G. Dvali, Phys.Lett. B429 (1998) 263; Phys.Rev. D59 (1999) 0860.

[6] L. Randall and R. Sundrum, Phys.Rev.Lett. 83 (1999) 3370; Phys.Rev.Lett. 83 (1999) 4690.

[7] J.G. Russo and A.A. Tseytlin, Nucl.Phys. B461 (1996) 131.

[8] Y. Nambu and G. Jona–Lasinio, Phys.Rev. 122 (1961) 345.
[9] S.P. Gavrilov and D.M. Gitman, Phys.Rev. D53 (1996) 7162.

[10] P. West, [hep-th/9811101].

[11] T. Appelquist, M.J. Bowick, D. Karabali, and L.C.R. Wijewardhana, Phys.Rev. D33 (1986) 3774.

[12] J. Schwinger, Phys.Rev. 82 (1951) 664.

[13] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, Orlando, 1980).