Fermion Masses and Bi-large Lepton Mixing from Singular Matrices

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ABSTRACT

We discuss conditions under which bi-large mixing in the lepton sector is achieved assuming that all Yukawa matrices and the right-handed neutrino mass matrix have the same singular form in the leading order. Due to obvious quark lepton symmetry, this approach can be embedded into grand unified theories. The right-handed neutrino mass scale can be identified with the GUT scale and, as a consequence of third generation Yukawa coupling unification, the mass of the lightest neutrino is given as $(m^2_{top}/M_{GUT})|U_{\tau 1}|^2$ in the leading order. This relation does not depend on the exact form of mass matrices.

1. Introduction

In order to keep quark-lepton symmetry obvious, let us assume that all mass matrices are given by the same universal matrix in the leading approximation, and all differences between up-type quarks, down-type quarks, charged leptons and neutrinos originate from small perturbations. Examples of such universal matrices are democratic mass matrix, a matrix with 3-3 element only, or, in general, a matrix formed by a product of two vectors:

\[ I_D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad I_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_{LR} = \vec{\mu}_L \cdot \vec{\mu}_R^T. \quad (1) \]

The democratic mass matrix $I_D$ does not distinguish between families, all three families are treated equally in the leading order, and yet it provides an explanation for the third generation being of order the weak scale and the first two generations being massless. If embedded into grand unified theories third generation Yukawa coupling unification is a generic feature (without necessity of distinguishing the third generation from the other two by family symmetries or in any other way) while masses of the first two generations of charged fermions depend on small perturbations. A matrix with non-zero 3-3 element only distinguishes the third generation from the beginning and it is the usual starting point of hierarchical models. I will mainly focus on the first example following Ref. [1] (for a related study, see Ref. [2]) and I will comment on similarities and differences in corresponding hierarchical models [3]. However, the discussion and many of the consequences are similar for any matrix which can be written as a product of two vectors, $I_{LR}$. (For a specific example with $\vec{\mu}_L = \vec{\mu}_R = (\lambda^2, \lambda, 1)^T$ see Ref. [4]). Such mass matrices can originate from an exchange of a heavy vector-like pair of fermions where $\vec{\mu}_L (\vec{\mu}_R)$ are couplings of left-handed (right-handed) fermions. The former two examples are clearly just special cases. Especially in the case $\vec{\mu}_L \sim \vec{\mu}_R \sim (1, 1, 1)^T$, in other words when all elements are random order one numbers, the results that follow are basically identical to the case of $I_D$. 
2. Bi-large lepton mixing in a democratic approach

Let us assume that Yukawa couplings are given as:

\[ Y_f \equiv \frac{1}{3} \lambda_f (I_D - \mathcal{E}_f), \quad f = u, d, e, \nu, \]  

(2)

where we parametrize departures from universality by matrices \( \mathcal{E}_f \). If Yukawa matrices were equal to \( I \lambda_f / 3 \) the mass eigenvalues are \( \{0, 0, \lambda_f\} \) and the diagonalization matrix is:

\[
U_I = \begin{pmatrix}
\cos \theta_I & \sin \theta_I & 0 \\
-\sin \theta_I & \cos \theta_I & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]  

(3)

As a consequence of degenerate zero eigenvalues the first two rows of this matrix are not uniquely specified and are model dependent (\( \mathcal{E} \) has to be taken into account). They can be replaced by any of their linear combinations and the corresponding orthogonal combination, which is accounted for by the first matrix which rotates the first two rows. As a result, the CKM matrix is not the identity matrix in the leading order as it was in the case of two families, but rather a unitary matrix with an arbitrary 1-2 element.

Let us parametrize the Majorana mass matrix for right-handed neutrinos in a similar way:

\[ M_{\nu_R} = \frac{1}{3} (I_D - R) M_0, \]  

(4)

where \( R \) represents small perturbations. The inverse of this matrix is given as:

\[ M_{\nu_R}^{-1} = \frac{1}{M_{\text{eff}}} (I + \hat{R}), \]  

(5)

where \( M_{\text{eff}} \equiv rM_0/3 \) with \( r \equiv \sum_{i,j=1}^{3} \hat{R}_{ij} \). The form of \( \hat{I} \) can found in Ref. [1] and \( \hat{R} \) contains higher order terms.

Due to the special form of \( I \) and \( \hat{I} \) we have these relations: \( I\hat{I} = 0, I\hat{R} = r \) and the usual see-saw formula for the left-handed neutrino mass matrix, \( M_{\nu_L} = -v_\nu^2 Y_\nu M_{\nu_R}^{-1} Y_\nu^T \), highly simplifies:

\[ M_{\nu_L} = -\frac{\lambda_{\nu}^2 v_\nu^2}{9 M_{\text{eff}}} \left[ \mathcal{M} + rI + O(\hat{R}_{ij} \epsilon_{\nu ij}) \right], \]  

(6)

where \( \mathcal{M} \equiv \mathcal{E}_\nu \hat{I} \mathcal{E}_\nu^T \) and we assume \( R_{ij} \) are much smaller than \( \mathcal{E}_{\nu ij} \) (so the terms \( O(\hat{R}_{ij} \epsilon_{\nu ij}) \) are negligible). If the second term in Eq. (6) dominates, the neutrino mass matrix resembles the charged lepton mass matrix and the lepton mixing matrix would be similar to the CKM matrix. In order to get large mixing in the lepton sector this term simply cannot dominate. On the other hand, the first term in Eq. (6), \( \mathcal{M} \), is given in terms of perturbations only. If this term dominates (this situation require strong hierarchy in masses of right handed neutrinos and negligible contribution of the heaviest one to the left-handed neutrino mass matrix, \( M_1 < M_2 < 10^{-4} M_3 \)) there is absolutely no reason why
the neutrino mass matrix should resemble the charge lepton mass matrix. It can be anything. In general, matrix $\mathcal{M}$ has one zero eigenvalue and the corresponding eigenvector, $\vec{v}_0$, is specified by $\mathcal{E}_\nu$ only.

The matrix diagonalizing the charged fermion mass matrix is given (up to small corrections) by $U_T$ in Eq. (3). Since it already contains large mixing angles, in order to avoid any exact relations between elements of $\mathcal{E}_e$, $\mathcal{E}_\nu$ and $\mathcal{R}_i$, the simplest way is to assume that the perturbation matrix $\mathcal{E}_\nu$ introduces the minimal amount of mixing into the lepton mixing matrix. This corresponds to a situation when the eigenvector corresponding to the zero eigenvalue is dominated by one element. The most general form of the lepton mixing matrix in this case can be written as:

$$U = \begin{pmatrix}
\cos \theta_e & \sin \theta_e & 0 \\
-\sin \theta_e & \cos \theta_e & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_\nu & \sin \theta_\nu \\
0 & -\sin \theta_\nu & \cos \theta_\nu
\end{pmatrix}. \quad (7)$$

where $\cos \theta_e$ and $\cos \theta_\nu$ are free parameters. Plots of their allowed values that satisfy $3\sigma$ experimental bounds of $\sin^2 \theta_{23}$ and $\sin^2 \theta_{12}$ can be found in Ref. [1]. Since the lepton mixing matrix is determined by only two parameters in this minimal case the value of the remaining mixing angle is a prediction. The predicted values of $\sin^2 \theta_{13}$ are either $0.008 \leq \sin^2 \theta_{13} \leq 0.14$ or $0.22 \leq \sin^2 \theta_{13} \leq 0.66$. Therefore this framework naturally leads to either all three mixing angles large or at most one small mixing angle. Note that the minimal value of $\sin^2 \theta_{13} = 0.008$ corresponds to the maximal allowed values of $\sin^2 \theta_{23}$ and $\sin^2 \theta_{12}$. On the other hand, the central values of $\sin^2 \theta_{23}$ and $\sin^2 \theta_{12}$ correspond to $\sin^2 \theta_{13}$ near its present experimental upper bound.

### 3. Mass of the lightest neutrino

The mass of the lightest neutrino is lifted when the second term in Eq. (6) is taken into account. Since we assume it is just a small correction to the first two terms it can be treated as a perturbation. Adding this perturbation does not significantly affect the two heavy eigenvalues and the diagonalization matrix, but it is crucial for the lightest eigenvalue which is exactly zero in the limit when this term is ignored. In the case of non-degenerate eigenvalues, corrections to eigenvalues $m_i$ of a matrix $M$ generated by a matrix $\delta M$ are given as $\delta m_i = u_i^\dagger \delta M u_i$, where $u_i$ are normalized eigenvectors. Due to the universal form of the perturbation matrix we have $\vec{v}_0^\dagger r \vec{v}_0 = r |\xi|^2$, where $\xi = \sum_{i=1}^3 v_{0i}$. Therefore, the mass of the lightest neutrino is given as:

$$m_{\nu_1} = \frac{\lambda_\nu^2 v_\nu^2}{3M_0} |\xi|^2. \quad (8)$$

The parameter $\xi$ is however related to the 3-1 element of the lepton mixing matrix, $U_{\tau 1} = (U_e U_{\nu e}^\dagger)_{31} = (1, 1, 1) \vec{v}_0 / \sqrt{3} = \xi / \sqrt{3}$, and so we get

$$m_{\nu_1} = \frac{\lambda_\nu^2 v_\nu^2}{M_0} |U_{\tau 1}|^2. \quad (9)$$

Note, the 3rd row in $U_e$ is not model dependent unlike the first two rows are! It can receive only small corrections from the perturbation matrix.
In simple SO(10) models $\lambda_u = \lambda_\nu$, in which case the lightest and the heaviest fermion of the standard model are connected through the relation above where $\lambda_u^2 \nu_\alpha^2$ is replaced by $m_{\text{top}}^2$. This is a very pleasant feature since we can further identify $M_0$ with the GUT scale, $M_{\text{GUT}} \sim 2 \times 10^{16}$ GeV, in which case we get

$$m_{\nu_1} = \frac{m_{\text{top}}^2}{M_{\text{GUT}}} |U_{\tau 1}|^2,$$

and predict the mass of the lightest neutrino to be between $5 \times 10^{-5}$ eV and $5 \times 10^{-4}$ eV depending on the value of $U_{\tau 1}$ (a global analysis of neutrino oscillation data [5] gives the $3\sigma$ range: $0.20 \leq |U_{\tau 1}| \leq 0.58$). This prediction does not depend on details of a model and is well motivated possible consequence of Yukawa coupling unification. It adds to predictions of Yukawa coupling unification in quark and charged lepton sector [6].

4. Conclusions

The scenario we discussed, when embedded into GUTs, is very compact and has many virtues: obvious quark-lepton symmetry, 3rd generation Yukawa coupling unification, bi-large lepton mixing with a prediction for $\sin \theta_{13}$ in the minimal case, and more importantly, no need for an intermediate right-handed neutrino scale and with that associated prediction for the mass of the lightest neutrino.

Many features of this scenario are similar to those in the hierarchical framework discussed in Ref. [3]. The third generation Yukawa coupling unification is obvious in that picture. This can be understood from two possible ways permutation symmetry can be used in model building. A matrix with 3-3 element only can be also motivated by permutation symmetry under which the first two families transform as a doublet [7]. Both approaches require strong hierarchy in masses of right handed neutrinos and negligible contribution of the heaviest one to the left-handed neutrino mass matrix.

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6. References

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