Renormalization of the QED of self-interacting second order spin $\frac{1}{2}$ fermions.

Carlos A. Vaquera-Araujo, Mauro Napsuciale and René Ángeles-Martínez

Departamento de Física, Universidad de Guanajuato, Lomas del Campestre 103, Fraccionamiento Lomas del Campestre, León, Guanajuato México, 37150.

E-mail: vaquera@fisica.ugto.mx, mauro@fisica.ugto.mx, rene@fisica.ugto.mx

Abstract: We study the one-loop level renormalization of the electrodynamics of spin 1/2 fermions in the Poincaré projector formalism, in arbitrary covariant gauge and including fermion self-interactions, which are dimension four operators in this framework. We show that the model is renormalizable for arbitrary values of the tree level gyromagnetic factor $g$ within the validity region of the perturbative expansion, $\alpha g^2 \ll 1$. In the absence of tree level fermion self-interactions, we recover the pure QED of second order fermions, which is renormalizable only for $g = \pm 2$. Turning off the electromagnetic interaction we obtain a renormalizable Nambu-Jona-Lasinio-like model with second order fermions in four space-time dimensions.
1 Introduction

Second order spin 1/2 fermions were considered by Feynman for the first time in [1], following a seminal work by V. Fock [2]. The V-A structure of weak interactions proposed by Feynman and Gell-Mann was motivated by the existence of chiral irreducible representations of the Lorentz group for spin 1/2 fermions that naturally obey a second order equation of motion [3]. Feynman-Gell-Mann formalism is specially useful in the world-line formulation of perturbative quantum field theory (see [4] for a review and further references). The relativistic quantum mechanics and the quantum field theory of the Feynman-Gell-Mann equation were studied in [5–16]. The non-Abelian version was considered in [17]. In [18, 19], second order fermions were implemented in the lattice. Recent developments on the formalism can be found in [20, 21]. At one-loop level there are some partial results in [11, 13, 16, 18].

In [22], an alternative second order formalism for spin 1/2 fermions was presented, based on the projection onto irreducible representations of the Poincaré group [23–26], an idea that follows from previous attempts to solve the ancient problems of the quantum
description of interacting high spin fields. In this formalism, there exists a deep connection of the gyromagnetic factor of spin $3/2$ fields with their causal propagation in an electromagnetic background [26], and with the unitarity of the Compton scattering amplitude in the forward direction [27]. When applied to spin $1$ fields, a similar connection between unitarity of Compton scattering in the forward direction and the gyromagnetic factor is found. Besides, in that case the gyromagnetic factor is found to be intrinsically related to the the electric quadrupole moment of the field [28]. Furthermore, it was recently shown that in this formalism the multipole moments of particles with spin $1/2$, 1 and $3/2$ transforming in different representations of the Lorentz group, are fixed by the tree level value of the gyromagnetic factor [29] and the electric charge.

As shown in [22], the second order formalism for spin $1/2$ fermions yields the same results as Dirac QED for the tree level amplitude of Compton scattering, whenever $g = 2$ and the parity conserving solutions for the external fermions are used. However, unlike the spin 1 and $3/2$ cases, in the case of spin $1/2$ the value of $g$ is not constrained by unitarity arguments. This makes the formalism interesting in the formulation of effective field theories for the electromagnetic interactions of hadrons where the low energy constants are precisely the free parameters in the Lagrangian. Thus, the study of the renormalization properties of this formulation of QED with arbitrary values of the gyromagnetic factor is of paramount importance.

Beyond the tree level approximation, a first approach to the one-loop structure of the second order QED of $1/2$ fermions in the Poincaré projector formalism was presented in [30]. From the analysis of the superficial degree of divergence, only the 2-, 3- and 4-point vertex functions are superficially divergent. The complete 2- and 3-point vertex functions and the divergent piece of the only 4-point vertex function appearing at tree level ($ff\gamma\gamma$) where calculated in the Feynman gauge and shown to be renormalizable at one-loop level for arbitrary values of $g$. In this work we generalize the calculations done in [30] to an arbitrary gauge and complete the analysis of the renormalizability of the model, calculating in an arbitrary gauge the divergent piece of the remaining 4-point vertex functions generated at one-loop level. Furthermore, since the second order fermion fields have mass dimension 1 in four space-time dimensions, in addition to the interactions generated by minimal coupling arising from the gauge principle, point-like four-fermion interactions are also dimension-four and gauge-invariant operators. In this work we also take these potentially renormalizable Nambu-Jona-Lasinio-like terms [31, 32] into account and study the one-loop structure of the second order electrodynamics of spin $1/2$ self-interacting fermions in the Poincaré projector formalism and in an arbitrary covariant gauge.

This paper is organized as follows: In Section 2, we present the Feynman rules and Ward-Takahashi identities used in the paper. In Section 3 we calculate the renormalized 2- and 3-point vertex functions, and the divergent piece of all the 4-point vertex functions, closing with the derivation of the beta functions of the theory. We discuss our results in Section 4 and our conclusions are given in Section 5. Conventions, useful identities and some technical details of the calculations are presented in appendix A. Finally, the scalar functions arising in the calculation of the three point vertex function, at one-loop level, are given in appendix B.
Feynman rules and Ward-Takahashi identities

Fermion fields have dimension 1 \((d - 2)/2\) in \(d\) space-time dimensions in the second order formalism based on the Poincaré projectors \([22, 30]\), thus local \(U(1)\) gauge symmetry and naïve renormalization criteria allows for Nambu-Jona-Lasinio-like fermion self-interactions in the Lagrangian, which in this formalism are dimension-four operators. The most general dimension four \(U(1)\) gauge invariant Lagrangian in this framework is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_{\mu} \bar{\psi} T_{\mu\nu} D^{\nu} \psi - m^2 \bar{\psi} \psi + \frac{\lambda_1}{2} \left( \bar{\psi} \gamma^5 \psi \right)^2 + \frac{\lambda_2}{2} \left( \bar{\psi} \gamma^5 \psi \right) \left( \bar{\psi} \gamma^5 \psi \right) + \frac{\lambda_3}{2} \left( \bar{\psi} M_{\mu\nu} \psi \right) \left( \bar{\psi} M_{\mu\nu} \psi \right),
\]

where \(D_\mu = \partial_\mu + ieA_\mu\) (fermion charge \(-e\)) is the covariant derivative, \(\lambda_j\) \((j = 1, 2, 3)\) are the couplings of the three possible dimension four fermion self-interaction terms, and the space-time tensor \(T^{\mu\nu}\) is given by

\[
T^{\mu\nu} = g^{\mu\nu} - igM^{\mu\nu} - ig' \tilde{M}^{\mu\nu},
\]

with \(\tilde{M}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} M_{\alpha\beta}/2\) (see \([30]\) and appendix A for further conventions). In eq. (2.2), \(g'\) parameterizes parity violating electric dipole interactions, and \(g\) can be identified with the gyromagnetic factor. In the following, we restrict our analysis to parity conserving interactions setting \(g' = 0\).

Fixing the gauge through the non-gauge invariant contribution

\[
\mathcal{L}_{\text{gauge}} = -\frac{1}{2\xi} \left( \partial_\mu A_\mu \right)^2,
\]

allows us to write the Feynman rules shown in figure 1 (with explicit Latin spinor indices running form 1 to 4). For closed fermion loops, the familiar factor of \(-1\) given by Fermi statistics must be included. Here \(\Lambda_{ab}\) is the Kronecker delta function for spinor indices.

Concerning fermion self-interactions, in order to have control of the different fermionic label combinations, we adopt an unconventional graphical device, sometimes used in the literature for this purpose (see e.g. \([33]\)), which requires some comments. We unify the three kinds of self-interactions into a single diagram, depicted in figure 1. We remark that the dashed line in that diagram does not correspond to a particle exchange, it is only a notational convention that indicates the order in which the tensor product of the currents is done, and therefore, the diagram should not be seen as a reducible one \([33]\). The advantage of this notation is the automatic remotion of ambiguities when dealing with the contraction of spinor indices in diagrams involving self-interactions. For example, the self-interaction contribution to the tree level four fermion vertex function is given simply by the standard anti-symmetrization prescription for the fermion lines in the diagrams of figure 2, and yields \(i(\lambda_{abcd} - \lambda_{cbad})\).

In this work we study the renormalizability of the model as it is, and we do not include an additional 1/2 factor in closed fermion loops. This 1/2 factor is used to provide a connection between the second order formalism and Dirac theory \([20, 21]\), and we will see below that it is only justified in the case \(g = \pm 2, \lambda_j = 0\).
Figure 1. Feynman rules for the second order QED of self-interacting fermions for an arbitrary covariant gauge $\xi$.

Figure 2. Feynman diagrams for the self-interaction contribution to the tree level four fermion vertex function.

Gauge invariance impose relations among different Green functions. The relevant Ward-Takahashi identities were derived in \cite{30}. The first one is

\[ q^\mu \Gamma_\mu(p, q, p + q) = S'^{-1}(p + q) - S'^{-1}(p). \tag{2.4} \]

where $-ie\Gamma_\mu(p + q, p, q)$ is the $ff\gamma$ vertex function and $iS'^{-1}(p)$ stands for the inverse of
the fermion propagator. We will use also this relation in its differential form

$$\Gamma_\mu(p, 0, p) = \frac{\partial S'^{-1}(p)}{\partial p^\mu}. \quad (2.5)$$

The second Ward-Takahashi identity relates the $ff\gamma\gamma$ vertex to the $ff\gamma$ vertex

$$q^\mu \Gamma_{\mu\nu}(p, q, p', q') = \Gamma_\nu(p + q, q', p') - \Gamma_\nu(p, q', p' - q), \quad (2.6)$$

or in differential form

$$\Gamma_{\mu\nu}(p, 0, p', q') = \frac{\partial \Gamma_\nu(p, q', p')}{\partial p^\mu} + \frac{\partial \Gamma_\nu(p, q', p')}{\partial p'^\mu}. \quad (2.7)$$

The tree level vertices in figure 1 satisfy these relations.

### 3 Renormalization

In [30], the superficial degree of divergence for this theory was studied -in the absence of self-interactions- concluding that only vertex functions with at most four external legs can be ultraviolet divergent. That conclusion does not change when we introduce fermion self-interactions. Thus, the proof that the theory is renormalizable requires to work out all vertex functions up to four external legs. We will carry out this analysis at one-loop level for arbitrary covariant gauge in dimensional regularization, using the na"ive prescription for the chiral operator $\gamma^5$ (commuting with $M^{\mu\nu}$ in $d$ dimensions, see appendix A). In this work, $\gamma^5$ is present only in diagrams with self-interactions, and therefore, pure QED diagrams are free from possible dimensional regularization inconsistencies.

#### 3.1 Counterterms

The parameters of the bare Lagrangian are the fermion mass $m_0$, the fermion charge $e_0$, the gyromagnetic factor $g_0$ and the self-interaction couplings $\lambda_{0j}$ ($j = 1, 2, 3$). The renormalized fields are related to the bare ones as

$$A_\mu^r = Z_1^{-\frac{1}{2}} A_\mu^0, \quad \psi_r = Z_2^{-\frac{1}{2}} \psi_0. \quad (3.1)$$

It is convenient to split the Lagrangian into its free and interacting parts

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_i \quad (3.2)$$

where

$$\mathcal{L}_0 = -\frac{1}{4} F_0^{\mu\nu} F_0^{\mu\nu} - \frac{1}{250} (\partial^\mu A_0^0)^2 + \partial^\mu \bar{\psi}_0 \partial_\mu \psi_0 - m_0^2 \bar{\psi}_0 \psi_0, \quad (3.3)$$

$$\mathcal{L}_i = -ie_0[\bar{\psi}_0 T_0^{\mu\nu} \partial^\mu \psi_0 - \partial^\mu \bar{\psi}_0 T_0^{\mu\nu} \psi_0] A_0^0 + e_0^2 (\bar{\psi}_0 A_0^0)^2 \bar{A}_0^0 A_0^0 + \frac{\lambda_{01}}{2} (\bar{\psi}_0 \psi_0)^2 + \frac{\lambda_{02}}{2} (\bar{\psi}_0 \gamma^5 \psi_0)^2 + \frac{\lambda_{03}}{2} (\bar{\psi}_0 M^{\mu\nu}_{\mu\nu} \psi_0)^2,$$

with

$$T_0^{\mu\nu} \equiv g^{\mu\nu} - ig_0 M^{\mu\nu}. \quad (3.4)$$
Writing the free Lagrangian in terms of the renormalized fields we get

$$\mathcal{L}_0 = -\frac{1}{4} F^\mu_\nu F^\nu_\mu - \frac{1}{2\xi_r} (\partial^\mu A^\mu_r)^2 - \frac{1}{4} F^\mu_\nu F^\nu_\mu \delta_1 + \partial^\mu \bar{\psi}_r \partial_\mu \psi_r - m^2_r \bar{\psi}_r \psi_r + [\partial^\mu \bar{\psi}_r \partial_\mu \psi_r - m^2_r \bar{\psi}_r \psi_r] \delta_2 - \delta_m m^2_r \bar{\psi}_r \psi_r.$$  \hspace{1cm} (3.5)

Here we used the following definitions:

$$\delta_1 \equiv Z_1 - 1, \quad \delta_2 \equiv Z_2 - 1, \quad \delta_m \equiv Z_m - Z_2, \quad Z_m \equiv \frac{m^2_0}{m^2_r} Z_2,$$ \hspace{1cm} (3.6)

and $\xi_r = Z_1^{-1} \xi_0$. Similarly, the interacting Lagrangian can be rewritten as

$$\mathcal{L}_{\text{int}} = -ie_r [\bar{\psi}_r T_r^\mu_\nu \partial^\mu \psi_r - \partial^\mu \bar{\psi}_r T_r^\mu_\nu \psi_r] A^\nu_r + e^2_r \bar{\psi}_r (M_\mu \psi_r) A^\mu_r \delta_e + \frac{\lambda_r}{2} (\bar{\psi}_r \gamma^5 \psi_r)^2 + \frac{\lambda_3}{2} (\bar{\psi}_r M_\mu \psi_r)^2$$

$$-ie_r [\bar{\psi}_r (M_\mu \psi_r) \partial^\mu \psi_r - \partial^\mu \bar{\psi}_r (M_\mu \psi_r)] A^\mu_r \delta_3 + \delta_e \equiv Z_e - 1, \quad \delta_3 \equiv Z_3 - 1, \quad \delta_3 \equiv Z_\lambda_j - 1, \quad \delta_3 \equiv Z_e g - Z_e,$$ \hspace{1cm} (3.8)

and

$$Z_e \equiv \frac{e_0}{e_r} Z_1^2 Z_2, \quad Z_3 \equiv \frac{e^2_r}{e_r} Z_1 Z_2, \quad Z_\lambda_j \equiv \frac{\lambda_0}{\lambda_r} Z_2^2, \quad Z_e g \equiv \frac{g_0}{g_r} Z_e.$$ \hspace{1cm} (3.9)

The renormalized space-time tensor $T_r^{\mu \nu}$ is defined in terms of the renormalized constant $g_r$ as

$$T_r^{\mu \nu} = g^{\mu \nu} - ig_r M^{\mu \nu}.$$ \hspace{1cm} (3.10)

In $d = 4 - 2\epsilon$ dimensions, the renormalized parameters must be modified as follows: $e_r \rightarrow \mu^r e_r, \lambda_r \rightarrow \mu^2 \lambda_r g_r \rightarrow g_r, m_r \rightarrow m_r$.

Notice that we have adopted a slightly different notation to the one used in [30] for the above definitions, in order to make our results easier to compare to Dirac and Scalar QED. Here the perturbative expansion is given around $e_r = 0, e_r g_r = 0$ and $\lambda_r g_r = 0$. The allowed values of $g_r$ by the perturbative expansion are discussed in section 4.

In the following, for the sake of clarity, we will drop the suffix $r$ in the renormalized parameters, keeping the suffix 0 for the bare quantities. In this notation, the Feynman rules for the renormalized fields are given in figure 1, and the corresponding rules for the counterterms are shown in figure 3.

### 3.2 Vacuum polarization

The vacuum polarization at one-loop level was calculated in [30] in Feynman gauge and in the absence of the fermion self-interacting terms. These calculations are not modified by the introduction of self-interactions nor by the consideration of an arbitrary covariant
\begin{equation}
-\imath \Pi^{\mu\nu} (q) = -\imath \Pi^{\nu\mu} (q) - \imath \delta_1 \left( g^{\mu\nu} q^2 - q^\mu q^\nu \right),
\end{equation}

where $-\imath \Pi^{\mu\nu} (q)$ stands for the contribution of the one-loop diagrams in Figure 4. These diagrams yield the polarization tensor

\begin{equation}
\Pi^{*\mu\nu} (q) = (g^{\mu\nu} q^2 - q^\mu q^\nu) \pi^* (q^2),
\end{equation}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Feynman rules for the counterterms in the second order QED of self-interacting fermions.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Feynman diagrams for the vacuum polarization in the QED of second order self-interacting fermions at one-loop.}
\end{figure}
with
\[
\pi^*(q^2) = \frac{e^2 \tau}{24\pi^2} \left\{ \frac{3}{8} g^2 - 4 B_0(q^2, m^2, m^2) + \frac{2m^2}{q^2} \left[ B_0(q^2, m^2, m^2) - B_0(0, m^2, m^2) \right] - \frac{1}{3} \right\}.
\] (3.13)

Here \( \tau = \text{Tr}[1] = 4 \). We will systematically leave this trace unevaluated until the very end. It will prove to be very useful to use \( \tau \) as a parameter to compare our findings with well established results below in Section 4.

We remark that in this work we do not attempt to correct this function with an additional \( 1/2 \) factor as in [30], where this was the only vertex function in which the extra \( 1/2 \) factor was used.

The Passarino-Veltman scalar integral \( B_0 \) in eq. (3.13) is defined as
\[
B_0(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int \frac{1}{[(l^2 - m_1^2)(l + p)^2 - m_2^2]},
\] (3.14)

Using \( d = 4 - 2\epsilon \) and the conventional Feynman parameterization, this function can be written as
\[
B_0(p^2, m_1^2, m_2^2) = \frac{1}{\epsilon} + \tilde{B}_0(p^2, m_1^2, m_2^2),
\] (3.15)

with
\[
\frac{1}{\epsilon} \equiv \frac{1}{\epsilon} - \gamma + \ln 4\pi
\] (3.16)

and
\[
\tilde{B}_0(p^2, m_1^2, m_2^2) \equiv - \int_0^1 dx \ln \left[ \frac{m_1^2(1 - x) + m_2^2x - p^2x(1 - x)}{\mu^2} \right].
\] (3.17)

For future purposes, our conventions for the Passarino-Veltman scalar integrals \( C_0 \) and \( D_0 \) are
\[
C_0(p_1^2, p_2^2; m_1^2, m_2^2, m_3^2) = \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int \frac{1}{[(l^2 - m_1^2)(l + p_1)^2 - m_2^2][l + p_2)^2 - m_3^2]},
\] (3.18)

\[
D_0(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2, m_4^2) = \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int \frac{1}{[(l^2 - m_1^2)(l + p_1)^2 - m_2^2][l + p_2)^2 - m_3^2][l + p_3)^2 - m_4^2]}.
\] (3.19)

From eq.(3.11), the total vacuum polarization tensor is given by
\[
\Pi^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\pi(q^2),
\] (3.20)

with
\[
\pi(q^2) = \pi^*(q^2) + \delta_1.
\] (3.21)

Using the on-shell renormalization scheme requires the form factor to satisfy
\[
\pi(q^2 \rightarrow 0) = 0,
\] (3.22)
which in turn fixes the value of the counterterm as

$$\delta_1 = -\pi^*(q^2 = 0) = -\frac{e^2 \tau}{(4\pi)^2} \left( \frac{q^2}{4} - \frac{1}{3} \right) \left[ \frac{1}{\varepsilon} - \ln \frac{m^2}{\mu^2} \right].$$  \hspace{1cm} (3.23)

Then, the physical form factor is given by

$$\pi(q^2) = \frac{e^2 \tau}{24\pi^2} \left\{ \left( \frac{3g^2 - 4}{8} + \frac{2m^2}{q^2} \right) \left[ B_0(q^2, m^2, m^2) - B_0(0, m^2, m^2) \right] - \frac{1}{3} \right\}. \hspace{1cm} (3.24)$$

### 3.3 Fermion self-energy

The fermion self-energy at one-loop level reads

$$-i\Sigma_{ab}(p^2) = -i\Sigma^*(p^2) + i(p^2 - m^2)\delta_2 I_{ab} - im^2\delta_m I_{ab}, \hspace{1cm} (3.25)$$

where $-i\Sigma^*(p^2)$ is computed with the one-loop diagrams contained in figure 5. The on-shell renormalization conditions for this Green function require the propagator

$$S(p) = \frac{1}{p^2 - m^2 - \Sigma(p) + i\varepsilon} \hspace{1cm} (3.26)$$

to have a simple pole at $p^2 = m^2$, which impose the following renormalization conditions

$$\Sigma(p^2 = m^2) = 0, \hspace{1cm} \frac{\partial \Sigma(p)}{\partial p^2} \bigg|_{p^2 = m^2} = 0. \hspace{1cm} (3.27)$$

These relations fix the counterterms in eq.(3.25) as

$$\delta_m = -\frac{\Sigma^*(p^2 = m^2)}{m^2}, \hspace{1cm} \delta_2 = \frac{\partial \Sigma^*(p^2)}{\partial p^2} \bigg|_{p^2 = m^2}, \hspace{1cm} (3.28)$$

and the renormalized fermion self-energy is given by

$$-i\Sigma(p^2) = -i \left[ \Sigma^*(p^2) - \Sigma^*(m^2) \right] + i(p^2 - m^2) \frac{\partial \Sigma^*(p^2)}{\partial p^2} \bigg|_{p^2 = m^2}. \hspace{1cm} (3.29)$$

The calculation of diagrams 1-4 in figure 5 yields

$$\Sigma^*(p^2) = \frac{e^2}{16\pi^2} \left[ 2 \left( p^2 + m^2 \right) B_0(p^2, m^2, m^2) + \frac{3g^2 - 4}{4} m^2 B_0(0, m^2, m^2) + \frac{g^2 - 4}{4} m^2 \right]$$

$$+ (1 - \xi) \left( p^2 - m^2 \right)^2 C_0(0, p^2, m^2, m^2) + (1 - \xi) \left( p^2 - m^2 \right) B_0(0, m^2, m^2) \right]$$

$$- \frac{m^2}{16\pi^2} \left\{ \left[ (\tau - 1)\lambda_1 - \lambda_2 - 3\lambda_3 \right] B_0(0, m^2, m^2) + (\tau - 1)\lambda_1 - \lambda_2 + \frac{\lambda_3}{2} \right\}, \hspace{1cm} (3.30)$$

and therefore, the counterterms in eq.(3.28) read

$$\delta_m = \frac{1}{(4\pi)^2} \left\{ \left[ \frac{1}{\varepsilon} - \ln \frac{m^2}{\mu^2} \right] \left( \tau - 1 \right)\lambda_1 - \lambda_2 - 3\lambda_3 - 3 \left( 1 + \frac{g^2}{4} \right) e^2 \right\}$$

$$+ (\tau - 1)\lambda_1 - \lambda_2 + \frac{\lambda_3}{2} - \left( 7 + \frac{g^2}{4} \right) e^2 \right\}, \hspace{1cm} (3.31)$$
\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.pdf}
\caption{Feynman diagrams for the fermion self-energy at one-loop.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.pdf}
\caption{Feynman diagrams for the $ff\gamma$ vertex at one-loop.}
\end{figure}

$$\delta_2 = \frac{e^2}{(4\pi)^2} (3 - \xi) \left( 1 - \ln \frac{m^2}{\mu^2} - \ln \frac{m^2}{m^2_\gamma} \right).$$

(3.32)

The renormalization constant of the fermion field $Z_2$ does not depend on the gyromagnetic factor nor on the self-interaction couplings. Here and in the following we use a small photon mass $m_\gamma$ to regulate infrared divergences. Finally, from eqs.(3.25,3.31,3.32) we get the renormalized fermion self-energy as

$$\Sigma(p^2) = \frac{e^2}{(4\pi)^2} \left\{ 2(p^2 + m^2) \left[ B_0(p^2, m^2, m^2_\gamma) - B_0(m^2, m^2, m^2_\gamma) \right] + 4(p^2 - m^2) \right. \right.$$  

$$+ 2(p^2 - m^2) \ln \frac{m^2_\gamma}{m^2} + (1 - \xi) \left( p^2 - m^2 \right)^2 C_0(0, p^2, m^2_\gamma, m^2_\gamma, m^2_\gamma) \right\}. \quad (3.33)$$

\newpage
3.4 $ff\gamma$ vertex

From the Feynman rules of figures 1 and 3, the $ff\gamma$ vertex at one-loop level is

$$-ie\Gamma^{\mu}_{ab}(p,q,p') = -ieV^{\mu}_{ab}(p,p') - ieV^{\mu}_{ab}(p,q,p') - ie|gM^{\mu\nu}_{ab}q_{\nu}|\delta_{e} - ie|gM^{\mu\nu}_{ab}q_{\nu}|\delta_{e} ,$$

(3.34)

where $-ie\Gamma^{\mu}_{ab}(p,q,p')$ stands for the contribution from the one-loop diagrams in figure 6.

It can be shown that the one-loop contribution satisfy the Ward-Takahashi identity

$$q^{\mu}\Gamma^{*}_{\mu}(p,q,p') = -\Sigma^{*}(p^{2}) + \Sigma^{*}(p^{2}),$$

(3.35)

It was pointed out in [30] that the contribution of diagrams 1,2 of figure 5 and 1-3 of figure 6 do indeed satisfy this equation. Diagrams 4,5 of figure 6 are proportional to $M^{\mu\nu}q_{\nu}$ and vanish upon contraction with $q^{\mu}$, while the related contribution of diagrams 3,4 of figure 5 is constant and does not modify the right hand side of eq.(3.35). Writing eq.(3.35) in its differential form

$$\Gamma^{*}_{\mu}(p,0,p) = -\frac{\partial\Sigma^{*}(p^{2})}{\partial p_{\mu}} ,$$

(3.36)

and using eqs.(2.5,3.34) we get

$$\delta_{e} = \delta_{2} = \frac{e^{2}}{(4\pi)^{2}} (3 - \xi) \left[ \frac{1}{\xi} - \ln \frac{m^{2}}{\mu^{2}} - \ln \frac{m^{2}}{\gamma m^{2}} \right] .$$

(3.37)

From eqs.(3.8,3.9), this relation also fixes the counterterm for the $ff\gamma\gamma$ vertex function

$$\delta_{3} = \delta_{2} .$$

(3.38)

The total contribution of diagrams 1-5 in figure 6 can be written as

$$\Gamma^{\star\mu}(p,q,p') = E^{*}q^{\mu} + F^{*}r^{\mu} + G^{*}igM^{\mu\nu}q_{\nu} + H^{*}igM^{\mu\nu}r_{\nu} + \Gamma^{*}igM^{\alpha\beta r_{\alpha}q_{\beta} r^{\mu}} + J^{*}igM^{\mu\nu}r_{\alpha}q_{\beta} q^{\mu} ,$$

(3.39)

where $E^{*}$-$J^{*}$ are scalar form factors and $r^{\mu} = (p+p')^{\mu}$. A convenient decomposition of the form factors is the following:

$$O^{*} = \sum_{i=0}^{9} O_{i} PV_{i} ,$$

(3.40)

with $O^{*} = E^{*}, F^{*}, G^{*}, H^{*}, \Gamma^{*}, J^{*}$. Here $O_{i} (i = 0, ..., 9)$ are scalar functions and $PV_{i}$ denote
the following Passarino-Veltman scalar integrals:

\[ PV_0 = 1, \]
\[ PV_1 = D_0(0, p^2, p'^2; m_\gamma^2, m_\gamma^2, m^2, m^2), \]
\[ PV_2 = C_0(0, p^2; m_\gamma^2, m_\gamma^2, m^2), \]
\[ PV_3 = C_0(0, p^2; m_\gamma^2, m_\gamma^2, m^2), \]
\[ PV_4 = C_0(p^2, p'^2; m_\gamma^2, m_\gamma^2, m^2), \]
\[ PV_5 = B_0(q^2, m_\gamma^2, m^2), \]
\[ PV_6 = B_0(p^2, m_\gamma^2, m^2), \]
\[ PV_7 = B_0(p'^2, m_\gamma^2, m^2), \]
\[ PV_8 = B_0(0, m_\gamma^2, m^2), \]
\[ PV_9 = B_0(0, m_\gamma^2, m^2). \]

The explicit form of the scalar functions \( \Omega_i \) is presented in appendix B. The form factors \( E^*, H^*, I^* \) and \( J^* \) turn out to be finite. Furthermore, \( E^*, H^*, J^* \) vanish on-shell \( (p^2 = p'^2 = m^2, q^2 = (p' - p)^2 = m_\gamma^2 \to 0) \). The remaining form factors take the following values in this limit:

\[
F_{OS}^* = -\frac{e^2}{(4\pi)^2} (3 - \xi) \left[ \frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} - \ln \frac{m_\gamma^2}{m^2} \right],
\]
\[
G_{OS}^* = F_{OS}^* + \frac{1}{(4\pi)^2} \left\{ \left( \frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} \right) \left[ \left( 1 - \frac{g^2}{4} \right) e^2 + \lambda_1 + \lambda_2 - \left( 1 + \frac{\tau}{2} \right) \lambda_3 \right] + 2e^2 + \frac{\lambda_3}{2} \right\},
\]
\[
I_{OS}^* = -\frac{e^2}{2(4\pi)^2 m^2}.
\]

There is a misprint in [30] for \( I_{OS}^* \) which does not affect any other result. The on-shell renormalized vertex function in eq.\((3.34)\) reads

\[-ie\Gamma_{OS}^\mu = -ie (1 + \delta_e + F_{OS}^*) \Gamma^{\mu} - ie (1 + \delta_e + \delta_g + G_{OS}^*) igM^{\mu\nu} q_\nu + I_{OS}^* gM^{\alpha\beta} r_\alpha q_\beta r^{\mu}.\]

The counterterm in eq. \((3.37)\) cancels the divergence of the charge form factor and yields \( F_{OS} = 1 \) for the corresponding on-shell renormalized form factor. This choice for \( \delta_e \) also cancels one of the divergences of the magnetic form factor. In fact, the coefficient of the \( egM^{\mu\nu} q_\nu \) term in eq.\((3.44)\) reads

\[ 1 + \delta_e + \delta_g + G_{OS}^* = \]
\[ 1 + \delta_g + \frac{1}{(4\pi)^2} \left\{ \left( \frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} \right) \left[ \left( 1 - \frac{g^2}{4} \right) e^2 + \lambda_1 + \lambda_2 - \left( 1 + \frac{\tau}{2} \right) \lambda_3 \right] + 2e^2 + \frac{\lambda_3}{2} \right\}.
\]

We choose the following value for the \( \delta_g \) counterterm to remove the remaining divergence

\[ \delta_g = -\frac{1}{(4\pi)^2} \left( \frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} \right) \left[ \left( 1 - \frac{g^2}{4} \right) e^2 + \lambda_1 + \lambda_2 - \left( 1 + \frac{\tau}{2} \right) \lambda_3 \right]. \]
According to this choice, the on-shell value of the renormalized magnetic moment form factor is given by

$$g_{\text{OS}} = g \left( 1 + \frac{\alpha}{2\pi} + \frac{\lambda_3}{32\pi^2} \right), \quad (3.47)$$

with $\alpha = e^2/(4\pi)$.

In summary, the renormalized vertex function in eq.(3.34) reads

$$\Gamma^\mu = E q^\mu + F r^\mu + G i g M^{\mu\nu} q_\nu + H i g M^{\mu\nu} r_\nu + i g M^{\alpha\beta} r_\alpha q_\beta q^\mu, \quad (3.48)$$

with the finite form factors

$$E = E^*, \quad H = H^*, \quad I = I^*, \quad J = J^*, \quad (3.49)$$

and the renormalized form factors

$$F = 1 + F^* - F^*_{\text{OS}}, \quad G = 1 + \frac{\alpha}{2\pi} + \frac{\lambda_3}{32\pi^2} + G^* - G^*_{\text{OS}}. \quad (3.50)$$

### 3.5 $ff\gamma\gamma$ vertex

The $ff\gamma\gamma$ vertex function at one-loop level is obtained from the Feynman rules of figures 1 and 3 as

$$ie^2 \Gamma_{ab}^{\mu\nu}(p, q, p', q') = ie^2 V_{ab}^{\mu\nu} + ie^2 \Gamma_{ab}^{\mu\nu}(p, q, p', q') + ie^2 V_{ab}^{\mu\nu} \delta_3, \quad (3.52)$$

where the one-loop corrections $ie^2 \Gamma_{ab}^{\mu\nu}(p, q, p', q')$ are given by the diagrams of figure 7. The counterterm $\delta_3$ has been already fixed in eqs.(3.32,3.38).

In [30], it was pointed out that the contribution of diagrams 1-3 of figure 6 and 1-9 of figure 7 satisfy

$$q_\mu \Gamma^{\mu\nu}(p, q, p', q') = \Gamma^{\mu\nu}(p + q, q', p') - \Gamma^{\mu\nu}(p, q', p' - q). \quad (3.53)$$

It can be easily shown that this relation is unmodified with the inclusion of the remaining diagrams in figures 6 and 7. Indeed, diagrams 4,5 of figure 6 depend only on $q^\mu = (p' - p)^\mu$ and their contribution to the right hand side of eq.(2.6) vanishes. On the other hand, the contribution of diagrams 10-15 in figure 7 satisfy $q_\mu \Gamma^{\mu\nu}(p, q, p', q') |_{10-15} = 0$. Therefore, eq.(3.53) holds for the full set of one-loop diagrams in figures 6 and 7. Using now eqs.(3.38,3.53) in eqs.(3.34,3.52) it can be explicitly shown that the second Ward-Takahashi identity in eq.(2.6) holds for the renormalized vertex functions.

The divergent pieces of the loop contributions to the $\Gamma^{\mu\nu}(p, q, p', q')$ vertex function can be isolated taking the zero external momentum limit. In this limit, the sum of the first two diagrams can be written as

$$ie^2 \Gamma_{ab}^{\mu\nu}(0, 0, 0, 0)|^{1+2} = -\frac{ie^4}{(4\pi)^2} 2 g^{\mu\nu} [\xi + \frac{3g^2}{4}] \frac{1}{\epsilon} F_{ab} + O(\epsilon^0). \quad (3.54)$$
Adding up eqs. (3.54), (3.55), (3.56), (3.57), and (3.58), we obtain the divergent contribution of the first three diagrams. The calculation of the next pure QED contributions is straightforward:

\[
\frac{i e^2 \Gamma_{ab}^{\mu\nu}(0,0,0)}{2 g_{\mu\nu}} = \frac{i e^4}{(4\pi)^2} 2g_{\mu\nu} \left[ \xi + \frac{3g^2}{4}\right] \frac{1}{\varepsilon} \mathbb{I}_{ab} + O(\varepsilon^0). \tag{3.55}
\]

Figure 7. Feynman diagrams for the \(ff\gamma\gamma\) vertex at one-loop.

Similarly, the divergent piece of the third diagram is

\[
\frac{i e^2 \Gamma_{ab}^{\mu\nu}(0,0,0)}{2 g_{\mu\nu}} = -\frac{i e^4}{(4\pi)^2} 2 \left(3 + \xi\right) g_{\mu\nu} \frac{1}{\varepsilon} \mathbb{I}_{ab} + O(\varepsilon^0), \tag{3.56}
\]

Notice that this divergence cancels the one coming from the first two diagrams in eq. (3.54), yielding a finite contribution of the first three diagrams. The calculation of the next pure QED contributions is straightforward:

\[
\frac{i e^2 \Gamma_{ab}^{\mu\nu}(0,0,0)}{2 g_{\mu\nu}} = \frac{i e^4}{(4\pi)^2} 4\xi g_{\mu\nu} \frac{1}{\varepsilon} \mathbb{I}_{ab} + O(\varepsilon^0). \tag{3.57}
\]

Adding up eqs. (3.54,3.55,3.56,3.57), we obtain the divergent part of the pure QED loop divergent contributions to the \(\gamma\gamma ff\) vertex function as

\[
\frac{i e^2 \Gamma_{ab}^{\mu\nu}(0,0,0)}{2 g_{\mu\nu}} = -\frac{e^2}{(4\pi)^2} \left(3 - \xi\right) \frac{1}{\varepsilon} [2ie^2 g_{\mu\nu}] \mathbb{I}_{ab} + O(\varepsilon^0). \tag{3.58}
\]
There is no room for further contributions to $\delta_3$ due to the fermion self-interactions. Therefore, the renormalizability of the model requires the finiteness of diagrams involving self-interactions in figure 7. For these diagrams one has

$$
ie^2 \Gamma^{\nu\mu}_{ab}(p, q, q', p')|^{10-12} = e^2 \lambda_{abcd} \mu^2 e \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{V^{\nu}(l, l - q') V^{\nu}(l - q', l)}{[l - q][l][l - q']} + \frac{V^{\nu}(l, l + q) V^{\nu}(l + q', l)}{[l + q][l][l + q] - \frac{2g^{\mu\nu} \square}{[l - q][l - q]}} \right\},$$

(3.59)

$$
ie^2 \Gamma^{*\nu\mu}_{ab}(p, q, p', q')|^{13-15} = - e^2 \lambda_{abcd} \mu^2 e \int \frac{d^4 l}{(2\pi)^4} \left\{ - \frac{2g^{\mu\nu} \square}{[l + p][l + p']} + \frac{[V^{\nu}(p' - q, l) V^{\nu}(p + l, p' - q + l)]_{dc}}{[p + l][p' + l][p' - q + l]} \right\}.$$

(3.60)

Here and in the following, we adopt the shorthand notation $\square [p] \equiv p^2 - m^2$ and $\square [q] \equiv q^2 - m^2$. Again, the possibly divergent part of this contribution can be isolated taking vanishing external momenta and is given by

$$
ie^2 \Gamma^{\nu\mu}_{ab}(0, 0, 0, 0)|^{10-15} = 2e^2 \mu^2 e \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{4l_{\mu} \mu^{\nu}}{[l][l]} - \frac{g^{\mu\nu}}{[l][l]} \right\} \square_{dc}[\lambda_{abcd} - \lambda_{abcd}]$$

$$= 2e^2 g^{\mu\nu} \mu^2 e \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{4l^2}{[l][l]^3} - \frac{1}{[l][l]^2} \right\} \left[ (\tau - 1) \lambda_1 - \lambda_2 - \frac{d(d - 1)}{4} \lambda_3 \right] \square_{ab}.$$

(3.61)

This integral is finite. As a consequence, the sum of the diagrams involving fermion self-interactions in figure 7 is free of divergences. Thus, the divergent part of the full set of one-loop diagrams in figure 7 is

$$
ie^2 \Gamma^{*\nu\mu}_{ab}(0, 0, 0, 0) = - \frac{e^2}{(4\pi)^7} (3 - \xi) \frac{1}{\epsilon} [2ie^2 g^{\mu\nu}] \square_{ab} + \mathcal{O}(\epsilon^0).$$

(3.62)

This divergent piece is proportional to $V^{\mu\nu}_{ab}$ and is exactly of the same magnitude and opposite sign to the divergent piece in $\delta_3$ in eq.(3.38), which has been already fixed from the Ward-Takahashi identities. Inserting eq.(3.38) in eq.(3.52) we get a renormalized $ff\gamma\gamma$ vertex function which is free of ultraviolet divergences.

The calculations so far presented generalize the results obtained in [30] to an arbitrary covariant gauge and the inclusion of fermion self-interactions. It is easy to show that all the above results reduce to those of [30] in Feynman gauge $\xi = 1$ and vanishing fermion self-interactions $\lambda_j = 0$ ($j = 1, 2, 3$). Now we turn our attention to the 4-point vertex functions not considered in [30].

3.6 4$\gamma$ vertex function

The four gamma vertex function is absent at tree level and is generated at one-loop level by the diagrams in figure 8. Notice that the fermion self-interactions do not contribute to these diagrams and there is no counterterm for this vertex function. Thus, if the model
Figure 8. Feynman diagrams for the $4\gamma$ vertex at one-loop. There are 9 additional diagrams obtained from diagrams 1-9 reversing the charge flow in the loop. We denote these diagrams with a prime, e.g., $1'$ stands for diagram 1 with arrows pointing in the opposite direction.

is renormalizable, the sum of all these pure QED diagrams must be finite. Again we are only interested in the study of the renormalizability of the theory and we will focus on the possible divergent parts of these diagrams. We can isolate the divergent pieces setting all the external momenta to zero. In this limit, the triangle diagrams $1 - 6$ and $1' - 6'$ of figure 8 (see figure caption for an explanation of the primed notation used here) yield

$$-ie^2\Gamma^{\mu\nu\alpha\beta}(0,0,0,0)^{1-6, 1'-6'} = 8\tau e^4 \frac{4}{d}\mu^{2e} \int \frac{d^d l}{(2\pi)^d} \left[ \frac{l^2}{\Box[l]^2} \right] T^{\mu\nu\alpha\beta},$$

with

$$T^{\mu\nu\alpha\beta} = g^{\mu\nu} g^{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha}.$$  \hspace{1cm} (3.64)$$

The divergent piece of the box diagrams $7 - 9$ and $7' - 9'$ of figure 8 is

$$-ie^2\Gamma^{\mu\nu\alpha\beta}(0,0,0,0)^{7-9, 7'-9'} = -4\tau e^4 \frac{24}{d(d+2)}\mu^{2e} \int \frac{d^d l}{(2\pi)^d} \left[ \frac{(l)^2}{\Box[l]^4} \right] T^{\mu\nu\alpha\beta},$$

and for the remaining diagrams we have

$$-ie^2\Gamma^{\mu\nu\alpha\beta}(0,0,0,0)^{10-12} = -4\tau e^4 \mu^{2e} \int \frac{d^d l}{(2\pi)^d} \left[ \frac{1}{\Box[l]^2} \right] T^{\mu\nu\alpha\beta}.$$ \hspace{1cm} (3.66)$$
Adding up the contributions in eqs. (3.63, 3.65, 3.66), all the divergences cancel. Therefore, the four-gamma vertex function is finite, as expected.

3.7 4f vertex function

The last potentially divergent vertex (according to the analysis of superficial degree of divergence), is the four-fermion vertex function (three photon vertex function vanishes because of charge conjugation). From figures 1 and 3, this function is $i\Lambda_{abcd} (p_1, p_2, p_1', p_2') - i\Lambda_{cbad} (p_1, p_2, p'_2, p'_1)$, with

$$i\Lambda_{abcd} (p_1, p_2, p_1', p_2') = i\lambda_{abcd} + i\Lambda^*_{abcd} (p_1, p_2, p_1', p_2')$$

$$+ i \left[ \delta_{\lambda_1} \lambda_1 \delta_{\lambda_2} \lambda_2 \gamma_{\mu\nu} \gamma_{\rho\sigma} \delta_{\lambda_3} \delta_{\lambda_4} \lambda_3 M^\mu_{\rho\sigma} M^\nu_{\mu\rho} \right]. \tag{3.67}$$

Here $i\Lambda^*_{abcd}$ is obtained from the loop diagrams in figure 9.

We use the same technique to isolate the possible divergent terms in these diagrams. We start with the pure QED diagrams. In the zero external momenta limit, diagrams 1, 2
of figure 9 yield

\[ i\Lambda_{abcd}^*(0,0,0,0) \bigl|^{1+2} = 2e^4 \left[ \xi^2 + \frac{(d-1)g^4}{16} \right] \mu^{4\epsilon} \int \frac{d^dl}{(2\pi)^d} \left[ \frac{(l^2)^2}{\Box[l^2] \triangle[l^2]} \right] \mathds{1}_{ab} \mathds{1}_{cd}. \]  

(3.68)

Similarly, the divergent piece of the triangle diagrams 3,4 is

\[ i\Lambda_{abcd}^*(0,0,0,0) \bigl|^{3+4} = -4e^4 \left[ \xi^2 + \frac{(d-1)g^2}{4} \right] \mu^{4\epsilon} \int \frac{d^dl}{(2\pi)^d} \left[ \frac{l^2}{\Box[l^2] \triangle[l^2]} \right] \mathds{1}_{ab} \mathds{1}_{cd}. \]  

(3.69)

Finally, for diagram 5 we get

\[ i\Lambda_{abcd}^*(0,0,0,0) \bigl|^{5} = \left( \frac{1}{2} \right) 4e^4 \left[ \xi^2 + (d-1) \right] \mu^{4\epsilon} \int \frac{d^dl}{(2\pi)^d} \left[ \frac{1}{\triangle[l^2]} \right] \mathds{1}_{ab} \mathds{1}_{cd}. \]  

(3.70)

Notice an additional symmetry factor 1/2 in this last contribution, the same that appears in Scalar QED. This factor is not related to the extra 1/2 factor in fermion loops used in \[20, 21\] (and should not be confused with it). Instead, this 1/2 is the appropriate factor required to correct the double-counting of internal photon configurations that comes from the factor 2 contained in each \( V_{\mu\nu} \) vertex. In appendix A, we give a more detailed calculation of the above results.

The divergent pieces of the integrals in eqs.(3.68,3.69,3.70) are alike. Integrating and adding up these contributions we get

\[ i\Lambda_{abcd}^*(0,0,0,0) \bigl|^{1-5} = i e^4 \frac{1}{(4\pi)^2} \left( 1 - \frac{g^2}{4} \right) \frac{1}{\epsilon} \mathds{1}_{ab} \mathds{1}_{cd} + \mathcal{O}(\epsilon^0). \]  

(3.71)

In addition to the 4\(\gamma\) vertex function, which turns out to be finite and dictated by pure electrodynamics (no influence of the fermion self-interactions), the 4\(f\) vertex function must be finite in order to conclude that the pure QED studied in \[30\] is a renormalizable theory. eq.(3.71) is a gauge independent result, and besides, is no affected by the \(\gamma^5\) issues related to dimensional regularization. Thus, from eq.(3.71), we conclude that pure electrodynamics of second order fermions is a renormalizable theory only in the case \( g = \pm 2 \). As we will discuss in the next section, these values of \( g \) lie in the range where the perturbative expansion is valid. This sharp prediction for the electrodynamics of second order fermions in the Poincaré projector formalism is modified by the inclusion of fermion self-interactions, which relax this condition and provide the adequate counter terms to have a renormalizable theory for arbitrary values of \( g \). This is an analogous situation to the case of scalar QED, where a similar divergence appear at one loop level in the 4-scalar vertex function. The quartic \(- \frac{1}{4} \lambda \phi^4 \) coupling is necessary in scalar QED to remove this divergent dynamical interaction.

So far, all the renormalized vertex functions are finite at one-loop level for self-interacting fermions. The proof of the renormalizability of the QED of self-interacting fermions requires the calculation of the diagrams involving fermion self-interactions in fig-
From eq.(3.67) it is clear that, in the case of self-interacting fermions, we still have at our disposal the $\delta_{\lambda_j}$ counterterms to absorb this divergence. The choice

$$
\delta_{\lambda_j} = - \frac{1}{(4\pi)^2} \frac{1}{\epsilon} \left\{ 6 \left( 1 - \frac{g^2}{4} \right)^2 e^4 + e^2 \left[ \frac{3g^2}{2} (\lambda_1 + \lambda_3) + 2\xi \lambda_1 \right] + (4 - \tau) \lambda_1 \lambda_2 + 3\lambda_2^2 + 2\lambda_1 \lambda_2 + 6\lambda_1 \lambda_3 \right\} \Gamma_{ab} \Gamma_{cd} \\
+ \frac{i}{(4\pi)^2} \frac{1}{\epsilon} \left\{ e^2 \left[ \frac{3g^2}{2} (\lambda_2 + \lambda_3) + 2\xi \lambda_2 \right] + (2 - \tau) \lambda_2 \lambda_3 + 3\lambda_3^2 + 6\lambda_1 \lambda_2 + 6\lambda_2 \lambda_3 \right\} \gamma^5_{ab} \gamma^5_{cd} \\
+ \frac{i}{(4\pi)^2} \frac{1}{\epsilon} \left\{ e^2 \left[ \frac{g^2}{2} (2\lambda_1 + 2\lambda_2 - \lambda_3) + 2\xi \lambda_3 \right] - \left( \frac{4 + \tau}{2} \right) \lambda_3^2 + 6\lambda_1 \lambda_3 + 6\lambda_2 \lambda_3 \right\} M^{\mu\nu}_{ab} M^{\mu\nu}_{cd} + O(\epsilon^0).
$$

(3.74)

From eq.(3.67) it is clear that, in the case of self-interacting fermions, we still have at our disposal the $\delta_{\lambda_j}$ counterterms to absorb this divergence. The choice

$$
\delta_{\lambda_1} = - \frac{1}{(4\pi)^2} \frac{1}{\epsilon} \left\{ 6 \left( 1 - \frac{g^2}{4} \right)^2 e^4 + e^2 \left[ \frac{3g^2}{2} \left( 1 + \frac{\lambda_3}{\lambda_1} \right) + 2\xi \right] + (4 - \tau) \lambda_1 \lambda_2 + 2\lambda_1^2 + 3\lambda_2^2 + 2\lambda_2 + 6\lambda_3 \right\},
$$

(3.75)
\[ \delta \lambda_3 = - \frac{1}{(4\pi)^2} \frac{1}{\epsilon} \left\{ e^2 \left[ \frac{3g^2}{2} \left( 1 + \frac{\lambda_3}{\lambda_2} \right) + 2\xi \right] + 3 \frac{\lambda_3^2}{\lambda_2} + 6\lambda_1 + (2 - \tau)\lambda_2 + 6\lambda_3 \right\}, \]  
(3.76)

\[ \delta \lambda_3 = - \frac{1}{(4\pi)^2} \frac{1}{\epsilon} \left\{ e^2 \left[ \frac{g^2}{2} \left( 2 \frac{\lambda_1}{\lambda_3} + 2 \frac{\lambda_2}{\lambda_3} - 1 \right) + 2\xi \right] + 6\lambda_1 + 6\lambda_2 - \left( \frac{4 + \tau}{2} \right) \lambda_3 \right\}, \]  
(3.77)

yields a finite four-fermion vertex function and completes our analysis the renormalizability of the QED of second order self-interacting fermions at one-loop level.

### 3.8 Beta Functions

Summarizing, form the results obtained in eqs.(3.23,3.31,3.32,3.37,3.38,3.46,3.75,3.76,3.77) and the definition of the counterterms in eqs.(3.6,3.8,3.9), the relation between the bare and renormalized parameters of the theory is given by

\[ e_0 = Z_1^{-1} Z_2^{-1} Z_3^{-1} \mu^\epsilon e, \quad e_0^2 = Z_1^{-1} Z_2^{-1} Z_3^2 \mu^2 e^2, \quad \lambda_{0j} = Z_2^{-2} Z_3 \mu^2 \lambda_j, \]

\[ g_0 = Z_3^{-1} Z_3^e g, \quad m_0^2 = Z_2^{-1} Z_m m^2, \]

with the renormalization constants (defined all the MS subtraction scheme, for consistency)

\[ Z_1^{\text{MS}} = 1 - \frac{e^2}{(4\pi)^2} \left( \frac{g^2}{4} - \frac{1}{3} \right) \frac{1}{\epsilon}, \]  
(3.79)

\[ Z_2^{\text{MS}} = Z_3^{\text{MS}} = Z_3^{\text{MS}} = 1 + \frac{e^2}{(4\pi)^2} (3 - \xi) \frac{1}{\epsilon}, \]  
(3.80)

\[ Z_{\lambda_1}^{\text{MS}} = 1 - \frac{1}{(4\pi)^2} \left\{ 6 \left( 1 - \frac{g^2}{4} \right) \frac{e^2}{\lambda_1} + e^2 \left[ \frac{3g^2}{2} \left( 1 + \frac{\lambda_3}{\lambda_1} \right) + 2\xi \right] \right\} \frac{1}{\epsilon}, \]  
(3.81)

\[ Z_{\lambda_2}^{\text{MS}} = 1 - \frac{1}{(4\pi)^2} \left\{ e^2 \left[ \frac{3g^2}{2} \left( 1 + \frac{\lambda_3}{\lambda_2} \right) + 2\xi \right] + 3 \frac{\lambda_3^2}{\lambda_2} + 6\lambda_1 + (2 - \tau)\lambda_2 + 6\lambda_3 \right\} \frac{1}{\epsilon}, \]  
(3.82)

\[ Z_{\lambda_3}^{\text{MS}} = 1 - \frac{1}{(4\pi)^2} \left\{ e^2 \left[ \frac{g^2}{2} \left( 2 \frac{\lambda_1}{\lambda_3} + 2 \frac{\lambda_2}{\lambda_3} - 1 \right) + 2\xi \right] + 6\lambda_1 + 6\lambda_2 - \left( \frac{4 + \tau}{2} \right) \lambda_3 \right\} \frac{1}{\epsilon}, \]  
(3.83)

\[ Z_{eg}^{\text{MS}} = Z_3^{\text{MS}} + \delta g^{\text{MS}} = 1 + \frac{1}{(4\pi)^2} \left[ (2 - \xi + \frac{g^2}{4}) e^2 - \lambda_1 - \lambda_2 + \left( \frac{4 + \tau}{2} \right) \lambda_3 \right] \frac{1}{\epsilon}, \]  
(3.84)

\[ Z_m^{\text{MS}} = Z_2^{\text{MS}} + \delta m^{\text{MS}} = 1 + \frac{1}{(4\pi)^2} \left[ (\tau - 1)\lambda_1 - \lambda_2 - 3\lambda_3 - \left( \xi + \frac{3g^2}{4} \right) e^2 \right] \frac{1}{\epsilon}, \]  
(3.85)
According to these constants, the two different relations between \( e_0 \) and \( \epsilon \) in eq. (3.78) collapse to
\[
e_0 = Z_1^{-1/2} \mu \epsilon, \tag{3.86}
\]
just as in Dirac and Scalar QED.

From eqs. (3.78-3.85) one can extract the following beta functions \( \beta_\eta \equiv \mu \frac{\partial \eta}{\partial \mu} \) and anomalous dimensions \( \gamma_m \equiv \frac{\mu}{m} \frac{\partial m}{\partial \mu} \) in the \( \epsilon \to 0 \) limit:
\[
\beta_\epsilon = \frac{e^3 \tau}{48 \pi^2} \left( \frac{3}{4} g^2 - 1 \right), \tag{3.87}
\]
\[
\beta_g = \frac{g}{32 \pi^2} \left[ e^2 (g^2 - 4) - 4(\lambda_1 + \lambda_2) \right] + 4 \left( 1 + \frac{\tau}{2} \right) \lambda_3, \tag{3.88}
\]
\[
\beta_{\lambda_1} = -\frac{1}{16 \pi^2} \left\{ 3 e^2 (g^2 - 4)^2 + 6e^2 \left[(4 + g^2) \lambda_1 + g^2 \lambda_3 \right] + 2(4 - \tau) \lambda_1^2 + 4\lambda_2 (\lambda_1 + \lambda_2) + 6\lambda_3 (2\lambda_1 + \lambda_3) \right\} \tag{3.89}
\]
\[
\beta_{\lambda_2} = -\frac{1}{16 \pi^2} \left\{ 3 e^2 \left[(4 + g^2) \lambda_2 + g^2 \lambda_3 \right] + 12 \lambda_2 \lambda_1 + 2(2 - \tau) \lambda_2^2 + 6\lambda_3 (2\lambda_2 + \lambda_3) \right\}, \tag{3.90}
\]
\[
\beta_{\lambda_3} = -\frac{1}{16 \pi^2} \left\{ e^2 \left[(12 - g^2) \lambda_3 + 2g^2 (\lambda_1 + \lambda_2) \right] + 12\lambda_3 (\lambda_1 + \lambda_2) - (4 + \tau) \lambda_3^2 \right\}, \tag{3.91}
\]
\[
\gamma_m = \frac{1}{64 \pi^2} \left\{ -3e^2 (g^2 + 4) + 4((\tau - 1)\lambda_1 - \lambda_2 - 3\lambda_3) \right\}. \tag{3.92}
\]

As expected, all \( \xi \) dependence drops out and the beta functions are gauge invariant. Finally, taking \( \tau = \text{Tr}[1] = 4 \) in eqs. (3.87-3.92), we obtain the definitive one-loop beta functions and anomalous dimensions of the theory:
\[
\beta_\epsilon = \frac{e^3}{12 \pi^2} \left( \frac{3}{4} g^2 - 1 \right), \tag{3.93}
\]
\[
\beta_g = \frac{g}{32 \pi^2} \left[ e^2 (g^2 - 4) - 4(\lambda_1 + \lambda_2 - 3\lambda_3) \right], \tag{3.94}
\]
\[
\beta_{\lambda_1} = -\frac{1}{16 \pi^2} \left\{ 3 e^2 (g^2 - 4)^2 + 6e^2 \left[(4 + g^2) \lambda_1 + g^2 \lambda_3 \right] + 4\lambda_2 (\lambda_1 + \lambda_2) + 6\lambda_3 (2\lambda_1 + \lambda_3) \right\}, \tag{3.95}
\]
\[
\beta_{\lambda_2} = -\frac{1}{16 \pi^2} \left\{ 3 e^2 \left[(4 + g^2) \lambda_2 + g^2 \lambda_3 \right] + 4\lambda_2 (3\lambda_1 - \lambda_2) + 6\lambda_3 (2\lambda_2 + \lambda_3) \right\}, \tag{3.96}
\]
\[
\beta_{\lambda_3} = -\frac{1}{16 \pi^2} \left\{ e^2 \left[(12 - g^2) \lambda_3 + 2g^2 (\lambda_1 + \lambda_2) \right] + 4\lambda_3 [3(\lambda_1 + \lambda_2) - 2\lambda_3] \right\}, \tag{3.97}
\]
\[
\gamma_m = \frac{1}{64 \pi^2} \left\{ -3e^2 (g^2 + 4) + 4[3\lambda_1 - \lambda_2 - 3\lambda_3] \right\}. \tag{3.98}
\]

Notice that \( \beta_g \) in eq. (3.94) vanishes for \( \{ g = \pm 2, \lambda_1 = \lambda_2 = \lambda_3 = 0 \} \), and for \( g = 0 \) with arbitrary \( \lambda_j \). The analysis of the evolution of the coupling constants is beyond the scope...
of this paper and here we will just relate these special fixed points of $\beta_g$ to the well known Dirac and Scalar QED, and the simplest model of pure self interactions.

4 Discussion

As stated in the previous section, $\beta_g$ vanishes for $g = \pm 2$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In this case, the theory has a simple connection to Dirac QED, as expected. It amounts to make the replacements $\tau \rightarrow (1/2)\text{Tr}[\Gamma] = 2$, $g \rightarrow \pm 2$ and $\lambda_j \rightarrow 0$ in eq. (3.87) to obtain the well known Dirac-QED beta function and the corresponding anomalous dimension of the mass

$$\beta_e^D = \frac{e^3}{12\pi^2}, \quad \gamma_m^D = -\frac{3e^2}{8\pi^2}. \tag{4.1}$$

Here the additional $1/2$ factor in $\tau$ is the same used in [20, 21], which can be traced to the relation between the Dirac operator and the Poincaré projector operator for $g = 2$:

$$\ln \det[\gamma^5(i\gamma^\mu D_\mu - m)] = \frac{1}{2} \ln \det[D^2 + eM_{\mu\nu}F^{\mu\nu} + m^2]. \tag{4.2}$$

The recovery of Dirac QED shows that the perturbative expansion for $g$ is justified in the $g = \pm 2$ case, and therefore, for $-2 < g < 2$. This observation is important because $g$ is a running coupling constant in this theory. One could be tempted to assume that the perturbative expansion is done around $e = 0$ and $g = 0$, however, as stated in Section 3.1, the true couplings in the Lagrangian are $e$ and $eg$, thus, the perturbative expansion is given by powers of $e^2/(4\pi) \equiv \alpha$, and $(eg)^2/(4\pi) = \alpha g^2$. This double expansion may lead to terms of order $\alpha g^2$, as -for example- the one-loop corrections to the gyromagnetic factor $\Delta g = \alpha g/2\pi$ in eq. (3.47). In general, the validity of the perturbative expansion will be driven by the conditions $\alpha \ll 1$, $\alpha g^2 \ll 1$. Hence, even if we consider values like e.g. $g = 5$, at low energies we will have a perturbative expansion in powers of $\alpha g^2 \approx 25/137 = 0.18$, which is still a reliable parameter. Notice, however, that for $g \neq \pm 2$ the $\lambda_j$ terms are generated at one loop level. Said in other words, if we put $\lambda_j = 0$ at some energy scale (say atomic scale, for example), the pure QED dynamics generates self-interactions at one-loop level at a different energy scale, as shown in eqs. (3.93-3.98). Therefore, the evolution of $g$ itself is influenced by its implicit self-interaction coupling dependence, and this dependence must be taken into account even in explicitly $\lambda_j$ independent results, like the case of $\beta_e$. In this theory, all couplings $e$, $g$ and $\lambda_j$ run with the energy in an intricate pattern, and the naive estimation of the perturbative regime must be modified. These modifications are expected to be small for small values of $\lambda_j$ at low energies, but certainly, the elucidation of the energy scale where the perturbative expansion is valid requires to solve the renormalization group equations.

Similarly, we should be able to recover results of scalar QED under the appropriate considerations. The vanishing of $\beta_g$ at $g = 0$ points to this possibility, and it is compatible with the fact that for this special value of $g$, there is no spin dynamics and the spin degrees of freedom must collapse to a multiplicative factor. This is indeed the case whenever we also change the Fermi statistics of our fermion degrees of freedom to the Bose statistics of conventional scalars. Formally, the theory described by eq. (2.1) can be turned into scalar...
QED if one changes the statistics and rescales the degrees of freedom and couplings accordingly. At the beta-function-level, this is accomplished with the following replacements in eq. (3.87): \( g \rightarrow 0, \lambda_2, \lambda_3 \rightarrow 0, \tau \rightarrow -1 \) (Fermi→Bose and reduction of degrees of freedom) and \( \lambda_1 \rightarrow -\frac{\lambda_1}{\tau} \) (rescaling and sign rectification of quartic coupling in scalar QED). This procedure yields the desired scalar QED beta and gamma functions:

\[
\beta_e^S = \frac{e^3}{48\pi^2},
\]

\[
\beta_{\lambda_0}^S = -2\beta_{\lambda_1}|_{\lambda_1 \rightarrow -\lambda_0/2} = \frac{1}{16\pi^2} (24e^4 - 12e^2\lambda_0 + 5\lambda_0^2),
\]

\[
\gamma_m^S = -\frac{1}{16\pi^2} (3e^2 - \lambda_\phi).
\]

Finally, we remark that this result does not mean that, in the case \( g = 0 \), the theory describes just several copies of scalar QED. Statistics play an important role here. Taking \( g = 0 \) in eq. (3.93) changes completely the nature of the theory, since the sign for \( \beta_e \) flips:

\[
\beta_e|_{g=0} = -\frac{e^3}{12\pi^2}.
\]

Thus, in this case (and in general for \( g^2 < 4/3 \)), our results point to an asymptotically free theory. Conclusive results require to solve the renormalization group equations in eq. (3.93), in order to understand better the contents of the theory and the new effects produced by arbitrary values of \( g \) and \( \lambda_j \). However, it is interesting that these fixed points of \( \beta_e \) -in the second order formalism for spin 1/2 based on the Poincaré projector- correspond with the two simplest realizations of renormalizable theories.

While this paper was under preparation, a work with interesting results for second order QED [35], was announced. In that work, a non-perturbative derivation of \( \beta_e \) is given, based on a generalization of the minimally-coupled Klein-Gordon equation to include a Pauli term with arbitrary \( g \). The \( \beta_e \) derived in [35] coincides with eq. (3.93) up to a constant factor (related to \( \tau \) and the 1/2 additional factor) for \(-2 < g < 2\). Also there, the possibility of asymptotic freedom behavior for fermions was recognized.

Regarding fermion self-interactions, many results involving the \( \lambda_j \) couplings are subject to the issues of \( \gamma_5 \) in dimensional regularization. The naïve prescription used in this work is known to be internally inconsistent, but it has proven to give the correct results in anomaly-free theories (see for example [34] and references therein), like ours. Previous works on the renormalization of two dimensional fermion models (which have deep similarities to our model) show that the possible problems related to the evanescent operators, that arise when the Clifford algebra is continued to \( d \) dimensions, start only at two-loop level [33], thus, it is reasonable to expect that the results involving fermion self-interactions obtained in this paper are reliable. Admittedly, verifications of the regularization procedure are necessary to draw definitive conclusions for the renormalizability of the QED of self-interacting fermions.

Finally, another attractive feature of the theory is the case \( e = 0 \), i.e., when we switch off the electromagnetic interactions. In this limit, we are left with a theory of self-interacting fermions, similar to the one proposed long ago by Nambu and Jona-Lasinio, but which in our formalism turns out to be renormalizable. There is a special class of models of this kind
in which dimensional regularization also gives unambiguous results. Taking, for example, 
\( e = 0, \lambda_1 = \lambda, \lambda_2 = \lambda_3 = 0 \), we obtain the following two-parameter model:

\[
\mathcal{L} = \partial^\mu \bar{\psi} \partial_\mu \psi - m^2 \bar{\psi} \psi + \frac{\lambda}{2} (\bar{\psi} \psi)^2, \tag{4.5}
\]

with one-loop beta and gamma functions

\[
\beta_\lambda = 0, \quad \gamma_m = \frac{3\lambda}{16\pi^2}, \tag{4.6}
\]

which is free from \( \gamma_5 \) inconsistencies. Besides, the massless limit of this model is indeed finite in dimensional regularization at one-loop for arbitrary values of \( \lambda \).

5 Summary and conclusions

In this work, we studied the one-loop level renormalization of the electrodynamics of second order fermions in the Poincaré projector formalism, in an arbitrary covariant gauge and including fermion self-interactions. In contrast to Dirac fermions, the second order ones have dimension mass dimension 1 (\( \frac{d-2}{2} \) in \( d \) space-time dimensions). Thus, four-fermion interactions are dimension-four operators and -according to the superficial degree of divergence- they must be included at tree level. There are three main conclusions of this work. First, if we start with vanishing tree level self-interactions (pure QED), at one-loop level all the Green functions with up to four legs turn out to be renormalizable, except for the four-fermion interaction generated by the loops, which is renormalizable only in the case \( g = \pm 2 \). Hence, pure QED of second order fermions in the Poincaré projector formalism is renormalizable only for these values of \( g \). This sharp prediction is obtained using dimensional regularization but it is not affected by the known inconsistencies of this regularization method related to the definition of \( \gamma_5 \) in \( d \) space-time dimensions. Second, the introduction of fermion self-interactions at tree level modify this behavior and render a renormalizable electrodynamics of second order self-interacting fermions for arbitrary values of the gyromagnetic factor. In this case, \( g \) becomes a true coupling constant, running with the energy. This theory may be used for composite particles like baryons, where the gyromagnetic factor is one of the low energy constants and its physical value is a suitable starting point for the formulation of the corresponding effective field theories. Third, if we turn off the electromagnetic interaction we are left with a four-dimensional renormalizable model of second order self-interacting fermions, which may be relevant in the formulation of effective theories in the strong regime, along the proposal by Nambu and Jona-Lasinio, but with second order fermions. In general, the last two main conclusions involve the \( \gamma_5 \) issues of dimensional regularization method, and further verifications by alternative regularization methods would be desirable. In the case of the third conclusion, however, there is a class of models with scalar-scalar self-interactions which is also free of possible inconsistencies of dimensional regularization.

Acknowledgments

This work was supported by CONACyT under project 156618 and by DAIP-UG.
A Notation and useful identities

Lorentz generators satisfy the following algebra in 4 dimensions:

\[
[M^{\alpha\beta}, M^{\mu\nu}] = -i(g^{\alpha\mu} M^{\beta\nu} - g^{\alpha\nu} M^{\beta\mu} + g^{\beta\mu} M^{\alpha\nu} - g^{\beta\nu} M^{\alpha\mu}),
\]
(A.1)

\[
\{M^{\mu\nu}, M^{\alpha\beta}\} = \frac{1}{2}(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) \mathbb{1} + \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} \gamma^5,
\]
(A.2)

\[\epsilon^{0123} = -\epsilon_{0123} = 1.\] The chirality operator \(\gamma^5\) is defined as

\[\gamma^5 = \frac{i}{3} \tilde{M}_{\mu\nu} M^{\mu\nu}, \quad \tilde{M}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} M^{\alpha\beta},\]
(A.3)

and satisfies

\[
[\gamma^5, M^{\mu\nu}] = 0, \quad (\gamma^5)^2 = 1. \quad (A.4)
\]

Higher products of the generators can be calculated using recursively these relations. We also need to calculate the trace of the product of generators. The simplest one is

\[\text{Tr} [M^{\mu\nu}] = 0, \quad (A.5)\]
as required from Lorentz covariance. Similarly, for \(\gamma^5\) we have

\[\text{Tr} [\gamma^5] = 0, \quad \text{Tr} [\gamma^5 M^{\mu\nu}] = 0. \quad (A.6)\]

In \(d\) dimensions we assume that the generators still satisfy the Lorentz algebra in eq. (A.1) and the anti-commutator relation in eq. (A.2), but now with \(g^{\mu\mu} = d\). Also in \(d\) dimensions eq. (A.5) and \(\text{Tr}[\mathbb{1}] = 4\) hold. The well known issues of \(\gamma^5\) in dimensional regularization come from the impossibility of satisfying simultaneously both eq. (A.4) and

\[\text{Tr} [\gamma^5 M^{\alpha\beta} M^{\mu\nu}] \neq 0. \quad (A.7)\]

As \(\text{Tr} [\gamma^5 M^{\alpha\beta} M^{\mu\nu}]\) is nowhere needed in our calculations, we use the naïve dimensional regularization prescription, which amounts to keep eqs. (A.4, A.6) unmodified in \(d\) dimensions.

The following reduction formulas for tensor integrals are useful in the determination of divergences:

\[
\int \frac{d^d l}{(2\pi)^d} \frac{l^{\mu\nu} l^{\alpha\beta}}{\Delta[l]^{m} \square[l]^{n}} = \frac{T^{\mu\nu\alpha\beta}}{d(d+2)} \int \frac{d^d l}{(2\pi)^d} \frac{(l^2)^2}{\Delta[l]^{m} \square[l]^{n}}; \quad (A.8)
\]

\[
\int \frac{d^d l}{(2\pi)^d} \frac{l^{\mu} l^{\nu}}{\Delta[l]^{m} \square[l]^{n}} = \frac{g^{\mu\nu}}{d} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{\Delta[l]^{m} \square[l]^{n}}; \quad (A.9)
\]

with \(T^{\mu\nu\alpha\beta}\) given by eq. (3.64).

As eq. (3.71) constitutes a crucial result, here we work out in detail the divergent part of its constituent diagrams. Thus, defining \(P^{\alpha\beta}(q, \xi) \equiv g^{\alpha\beta} q^2 - (1 - \xi) q^2 \frac{\partial}{\partial q^2}\), for diagrams
In this way, eq. (A.8) follows:

\[ QED \text{ divergent parts, are unaffected by the } \{ M_{\mu\rho}, M_{\nu\sigma}\} \text{ contributions always appear contracted either by } l^\mu l^\nu \text{ or by } g^\rho\sigma, \text{ giving unambiguous results when continued to } d \text{ dimensions. This fact is more evident when the generators are written in terms of the conventional Dirac matrices. The evaluation of the product of generators in eq. (A.10) can be performed with the aid of eqs. (A.1, A.2) as follows:} \]

\[
T_{\alpha\beta}^{\rho\sigma} \left( M^{\mu\alpha} M^{\nu\beta} \right)_{ab} \{ M_{\mu\rho}, M_{\nu\sigma} \}_{cd} = \\
(M^{\mu\alpha} M^{\nu\beta})_{ab} \{ M_{\mu\alpha}, M_{\nu\beta} \}_{cd} + (M^{\mu\alpha} M^{\nu\beta})_{ab} \{ M_{\mu\beta}, M_{\nu\alpha} \}_{cd} \\
= (M^{\mu\alpha} M^{\nu\beta})_{ab} \left[ \frac{1}{2} (d-1) g_{\mu\nu} \mathbb{1}_{cd} \right] + \left( M^{\mu\alpha} M^{\nu\beta} \right)_{ab} \left[ g_{\mu\nu} g_{\alpha\beta} - \frac{1}{2} (g_{\mu\nu} g_{\alpha\beta} + g_{\mu\beta} g_{\nu\alpha}) \right] \mathbb{1}_{cd} \\
= \left[ \frac{d(d-1)^2}{8} + \frac{3d(d-1)}{8} \right] \mathbb{1}_{ab} \mathbb{1}_{cd} = \frac{d(d+2)(d-1)}{8} \mathbb{1}_{ab} \mathbb{1}_{cd}. \tag{A.11} \]

In this way, eq. (3.68) is obtained from eqs. (A.11, A.10). Similarly, for diagrams 3, 4 of figure 9, the use of eqs. (A.1, A.2, A.9) gives

\[
i \Lambda_{abcd}^* (0, 0, 0, 0) \left[ V^\mu(l, 0) V^\nu(0, l) \right]_{ab} \left[ V^\rho(-l, 0) V^\sigma(0, l) \right]_{cd} = \frac{e^4 \mu^4}{\Delta[l \triangle[l]^2} \\
= -2e^4 \mu^4 \int \frac{d\xi^l}{(2\pi)^d} \left[ (M^{\mu\alpha} M^{\nu\beta})_{ab} \{ M_{\mu\alpha}, M_{\nu\beta} \}_{cd} + (M^{\mu\alpha} M^{\nu\beta})_{ab} \{ M_{\mu\beta}, M_{\nu\alpha} \}_{cd} \right] \\
= -2e^4 \mu^4 \int \frac{d\xi^l}{(2\pi)^d} \left[ \frac{d(d-1)^2}{8} + \frac{3d(d-1)}{8} \right] \mathbb{1}_{ab} \mathbb{1}_{cd} = \frac{d(d+2)(d-1)}{8} \mathbb{1}_{ab} \mathbb{1}_{cd}, \tag{A.12} \]

in agreement with eq. (3.69). Finally, the contribution of diagram 5 of figure 9 is

\[
i \Lambda_{abcd}^* (0, 0, 0, 0) = \left( \frac{1}{2} \right) e^4 \mu^4 \int \frac{d\xi^l}{(2\pi)^d} \frac{P_{\mu\rho}(l, \xi) P_{\nu\sigma}(l, \xi) [2g^{\mu\nu} \mathbb{1}_{ab}] [2g^{\rho\sigma} \mathbb{1}_{cd}]}{\Delta[l \triangle[l]^2}, \tag{A.13} \]
and reduces to eq. (3.70) upon contraction of Lorentz indices.

We close this appendix with a list of some useful products needed in the evaluation of eqs. (3.72, 3.73):

\[
[M^\mu \nu M^\alpha \beta]_{ab} [M^\alpha \beta M^\mu \nu]_{cd} = \frac{3}{2} I_{ab} I_{cd} + \frac{3}{2} \gamma^5 \gamma^5 - 2 M^\mu \nu M^\alpha \beta + O(\epsilon) \quad (A.14)
\]

\[
[M^\mu \nu M^\alpha \beta]_{ab} [M^\alpha \beta M^\mu \nu]_{cd} = \frac{3}{2} I_{ab} I_{cd} + \frac{3}{2} \gamma^5 \gamma^5 M^\mu \nu M^\alpha \beta + O(\epsilon)
\]

\[
M^\mu \nu M^\alpha \beta M^\mu \nu = -M^\alpha \beta + O(\epsilon)
\]

\[
[\gamma^5 M^\mu \nu]_{ab} [\gamma^5 M^\mu \nu]_{cd} = M^\mu \nu M^\mu \nu + O(\epsilon).
\]

B Scalar functions for the decomposition of the form factors of the three point function $f f f \gamma$

The scalar functions $O_i = E_i, F_i, G_i, H_i, J_i, I_i$ from the decomposition of the form factors in eq. (3.40) are the following:

\[
E_0 = 0,
\]

\[
E_1 = 4 \zeta (1 - \xi) \left( p^2 - p'^2 \right) \left( p^2 - m^2 \right) \left( p \cdot p' + m^2 \right),
\]

\[
E_2 = 4 \zeta (1 - \xi) \left( p^2 - m^2 \right) \left( p \cdot p' + p^2 \right),
\]

\[
E_3 = -4 \zeta (1 - \xi) \left( p^2 - m^2 \right) \left( p \cdot p' + p^2 \right),
\]

\[
E_4 = \zeta \left( p^2 - p'^2 \right) \left\{ (g^2 - 4) \left[ m^2 q^2 + p \cdot p' \left( p^2 + p'^2 \right) - 2p^2 p'^2 \right] + 8 \left[ m^4 + 2p \cdot p' \left( m^2 + (p \cdot p') \right) - p^2 p'^2 \right] \right\} - 4 \zeta (1 - \xi) \left( p^2 - p'^2 \right) \left( p^2 - m^2 \right) \left( p \cdot p' - p'^2 \right),
\]

\[
E_5 = -\zeta \left( p^2 - p'^2 \right) \left[ (g^2 - 4) q^2 + 8 \left( p \cdot p' + m^2 \right) \right],
\]

\[
E_6 = \zeta \left( g^2 - 4 \right) \left( p^2 - p'^2 \right) \left( p^2 - m^2 \right) \left( p \cdot p' \right) - 8 p^2 \left( p^2 + m^2 \right) \left( p \cdot p' \right) - 8 p^2 \left( p^2 + m^2 \right) \left( p \cdot p' \right),
\]

\[
E_7 = \zeta \left( g^2 - 4 \right) \left( p^2 - p'^2 \right) \left( p^2 - m^2 \right) \left( p \cdot p' \right) - 8 p^2 \left( p^2 + m^2 \right) \left( p \cdot p' \right) - 8 p^2 \left( p^2 + m^2 \right) \left( p \cdot p' \right),
\]

\[
E_8 = 0,
\]

\[
E_9 = 0.
\]

\[
E_0 = 0,
\]

\[
F_1 = 4 \zeta (1 - \xi) q^2 \left( p^2 - m^2 \right) \left( p^2 - m^2 \right) \left( p \cdot p' + m^2 \right),
\]

\[
F_2 = -4 \zeta (1 - \xi) \left( p^2 - m^2 \right) \left( p \cdot p' \left( g^2 - p^2 - m^2 \right) + p^2 \left( p^2 + m^2 \right) \right),
\]

\[
F_3 = -4 \zeta (1 - \xi) \left( p^2 - m^2 \right) \left( p \cdot p' \left( g^2 - p^2 - m^2 \right) + p^2 \left( p^2 + m^2 \right) \right),
\]

\[
F_4 = \zeta q^2 \left\{ (g^2 - 4) \left[ m^2 q^2 + p \cdot p' \left( p^2 + p'^2 \right) - 2p^2 p'^2 \right] + 8 \left[ m^4 + 2p \cdot p' \left( m^2 + (p \cdot p') \right) - p^2 p'^2 \right] \right\},
\]

\[
F_5 = -\zeta q^2 \left[ (g^2 - 4) q^2 + 8 \left( p \cdot p' + m^2 \right) \right] - 4 \zeta (1 - \xi) q^2 \left( p^2 - m^2 \right) \left( p^2 - m^2 \right),
\]

\[
F_6 = \zeta \left( g^2 - 4 \right) \left( p^2 - m^2 \right) \left( p \cdot p' \right) \left( q^2 + 8 p \cdot p' \left( p^2 - m^2 \right) - 8 p^2 \left( p^2 - m^2 \right) \right),
\]

\[
F_7 = \zeta \left( g^2 - 4 \right) \left( p^2 - m^2 \right) \left( p \cdot p' \right) \left( q^2 + 8 p \cdot p' \left( p^2 - m^2 \right) - 8 p^2 \left( p^2 - m^2 \right) \right),
\]
\[ F_8 = 0, \]
\[ F_9 = 0. \]
\[ G_0 = \frac{1}{(4\pi)^2} \left( -e^2 + \frac{\lambda_3}{2} \right), \]
\[ G_1 = -8\zeta (1 - \xi) \left( p^2 - m^2 \right) \left( p^2 - \xi^2 \right) \left( (p \cdot p')^2 - p^2 \xi^2 \right), \]
\[ G_2 = 8\zeta (1 - \xi) \left( p^2 - m^2 \right) \left( (p \cdot p')^2 - p^2 \xi^2 \right), \]
\[ G_3 = 8\zeta (1 - \xi) \left( p^2 - m^2 \right) \left( (p \cdot p')^2 - p^2 \xi^2 \right), \]
\[ G_4 = 2\zeta \left\{ 2m^4 q^2 + 2m^2 (p^2 - \xi^2)^2 + 4p \cdot p' \left\{ (m^2 + p \cdot p') q^2 - 2 \left( (p \cdot p')^2 - p^2 \xi^2 \right) \right\} \right. \]
\[ + 2p^4 (p \cdot p' - \xi^2) + 2p^4 (p \cdot p' - \xi^2) + (g - 2) (m^2 + p \cdot p') (p^2 - \xi^2)^2 \}, \]
\[ G_5 = -2\zeta \left( 2 (m^2 + p \cdot p') q^2 + g (p^2 - \xi^2)^2 + (g^2 + 4) \left( p^2 q^2 - (p \cdot p')^2 \right) \right) \]
\[ + \frac{1}{(4\pi)^2} \left[ \lambda_1 + \lambda_2 - \left( 1 + \frac{1}{2} \right) \lambda_3 \right], \]
\[ G_6 = \frac{2\zeta}{p^2} \left\{ 2p^2 (m^2 + p \cdot p') (p^2 - p \cdot p') + 2m^2 \left( p^2 q^2 - (p \cdot p')^2 \right) \right. \iffalse \]
\[ + gp^2 (p^2 - \xi^2) (p^2 + p \cdot p') \right\}, \]
\[ G_7 = \frac{2\zeta}{p^2} \left\{ 2p^2 (m^2 + p \cdot p') (p^2 - p \cdot p') + 2m^2 \left( p^2 q^2 - (p \cdot p')^2 \right) \right. \iffalse \]
\[ + gp^2 (p^2 - \xi^2) (p^2 + p \cdot p') \right\}, \]
\[ G_8 = \zeta \left\{ \frac{4 m^2}{p^2 p^2} (p^2 + \xi^2) \left( (p \cdot p')^2 - p^2 \xi^2 \right) \right\}, \]
\[ G_9 = -8\zeta (1 - \xi) \left( (p \cdot p')^2 - p^2 \xi^2 \right). \]
\[ H_0 = 0, \]
\[ H_1 = 0, \]
\[ H_2 = 0, \]
\[ H_3 = 0, \]
\[ H_4 = 2\zeta g (p^2 - \xi^2) (m^2 + p \cdot p') q^2, \]
\[ H_5 = -2\zeta g (p^2 - \xi^2) q^2, \]
\[ H_6 = \frac{2\zeta}{p^2} \left\{ -2 (p^2 - m^2) \left( (p \cdot p')^2 - p^2 \xi^2 \right) + gp^2 q^2 (p^2 + p \cdot p') \right\}, \]
\[ H_7 = -\frac{2\zeta}{p^2} \left\{ -2 (p^2 - m^2) \left( (p \cdot p')^2 - p^2 \xi^2 \right) + gp^2 q^2 (p^2 + p \cdot p') \right\}, \]
\[ H_8 = \frac{4 \zeta m^2 (p^2 - \xi^2)}{p^2 p^2} \left( (p \cdot p')^2 - p^2 \xi^2 \right), \]
\[ H_9 = 0. \]
\[ I_0 = 2\zeta q^2, \]
\[ I_1 = 0, \]
\[ I_2 = 0, \]
\[ I_3 = 0, \]
\[ I_4 = \frac{\zeta}{(p \cdot p')^2 - p^2 p'^2} \left\{ 3m^4 q^2 + 6q^2 p \cdot p' \left[ m^2 \left( p^2 + p'^2 \right) - p^2 p'^2 \right] \\
+ 2 \left( p \cdot p' \right)^2 \left( p^2 + p'^2 - 2m^2 \right) + p^2 p'^2 \left( p^2 + p'^2 - 8m^2 \right) \right\} \]
\[ I_5 = - \frac{\zeta q^2}{(p \cdot p')^2 - p^2 p'^2} \left\{ 3 \left( p^2 + p'^2 \right) \left( m^2 + p \cdot p' \right) - 2 \left( p \cdot p' \right) \left( p \cdot p' + 3m^2 \right) - 4p^2 p'^2 \right\} , \]
\[ I_6 = \frac{\zeta}{p^2 \left( (p \cdot p')^2 - p^2 p'^2 \right)} \left\{ 3p^6 \left( m^2 + p \cdot p' - p'^2 \right) \\
+ p^4 \left[ -9m^2 p \cdot p' + p'^2 \left( 5m^2 + 7p \cdot p' \right) - 6 \left( p \cdot p' \right)^2 - p'^4 \right] \\
+ p^2 \left\{ 4m^2 \left( p \cdot p' \right)^2 + p'^2 \left[ -9m^2 p \cdot p' + 2 \left( p \cdot p' \right)^2 \right] + 2 \left( p \cdot p' \right)^3 \right\} + 2m^2 \left( p \cdot p' \right)^3 \right\} , \]
\[ I_7 = \frac{\zeta}{p^2 \left( (p \cdot p')^2 - p^2 p'^2 \right)} \left\{ 3p^6 \left( m^2 + p \cdot p' - p'^2 \right) \\
+ p^4 \left[ -9m^2 p \cdot p' + p'^2 \left( 5m^2 + 7p \cdot p' \right) - 6 \left( p \cdot p' \right)^2 - p'^4 \right] \\
+ p^2 \left\{ 4m^2 \left( p \cdot p' \right)^2 + p'^2 \left[ -9m^2 p \cdot p' - 2 \left( p \cdot p' \right)^2 \right] + 2 \left( p \cdot p' \right)^3 \right\} + 2m^2 \left( p \cdot p' \right)^3 \right\} , \]
\[ I_8 = - \frac{2\zeta m^2}{p^2 p'^2} \left( (p^2 + p'^2) p \cdot p' - 2p^2 p'^2 \right) , \]
\[ I_9 = 0 . \]
\[ J_0 = 2\zeta \left( p^2 - p'^2 \right) , \]
\[ J_1 = 0 , \]
\[ J_2 = 0 , \]
\[ J_3 = 0 , \]
\[ J_4 = \frac{\zeta \left( p^2 - p'^2 \right)}{(p \cdot p')^2 - p^2 p'^2} \left\{ 2g \left( p \cdot p' + m^2 \right) \left[ (p \cdot p')^2 - p^2 p'^2 \right] + q^2 \left( 6m^2 p \cdot p' + 3m^4 + p^2 p'^2 \right) \\
+ 8m^2 \left[ (p \cdot p')^2 - p^2 p'^2 \right] + 2 \left( p \cdot p' \right)^2 \left( p^2 + p'^2 \right) - 4p^2 p'^2 p \cdot p' \right\} , \]
\[ J_5 = \frac{\zeta \left( p^2 - p'^2 \right)}{(p \cdot p')^2 - p^2 p'^2} \left\{ - 2g \left[ (p \cdot p')^2 - p^2 p'^2 \right] - 3m^2 q^2 \\
- 3p \cdot p' \left( p^2 + p'^2 \right) + 4p^2 p'^2 + 2 \left( p \cdot p' \right)^2 \right\} , \]
\[ J_6 = \frac{\zeta}{p^2 \left( (p \cdot p')^2 - p^2 p'^2 \right)} \left\{ 2g p^2 \left( p^2 + p' \right) \left[ (p \cdot p')^2 - p^2 p'^2 \right] \\
- p^2 \left[ 3p^4 - p^2 p'^2 - 2 \left( p \cdot p' \right)^2 \right] \left( p^2 - m^2 \right) - p \cdot p' \left[ 5p^2 p'^2 - 3p^4 - 2 \left( p \cdot p' \right)^2 \right] \left( p^2 - m^2 \right) \right\} , \]
\[ J_7 = \frac{\zeta}{p^2 \left( (p \cdot p')^2 - p^2 p'^2 \right)} \left\{ - 2g p^2 \left( p^2 + p \cdot p' \right) \left[ (p \cdot p')^2 - p^2 p'^2 \right] \\
+ p^2 \left[ 3p^4 - p^2 p'^2 - 2 \left( p \cdot p' \right)^2 \right] \left( p^2 - m^2 \right) + p \cdot p' \left[ 5p^2 p'^2 - 3p^4 - 2 \left( p \cdot p' \right)^2 \right] \left( p^2 - m^2 \right) \right\} , \]
\[ J_8 = - \frac{2\zeta m^2}{p^2 p'^2} \left( p^2 - p'^2 \right) \left( p \cdot p' \right) , \]
\[ J_9 = 0 . \]
Here, the global factor $\zeta$ stands for

$$\zeta = -\frac{e^2}{128\pi^2 \left[(p \cdot p')^2 - p^2 p'^2\right]^2}.$$

References

[1] R. P. Feynman, *An Operator calculus having applications in quantum electrodynamics*, Phys. Rev. 84 (1951) 108.

[2] V. Fock, *Proper time in classical and quantum mechanics*, Physik. Z. Sowjetunion 12 (1937) 404.

[3] R. P. Feynman and M. Gell-Mann, *Theory of Fermi interaction*, Phys. Rev. 109 (1958) 193.

[4] C. Schubert, *Perturbative quantum field theory in the string inspired formalism*, Phys. Rept. 355 (2001) 73 [hep-th/0101036].

[5] L. C. Biedenharn, M. Y. Han and H. Van Dam, *Two-component alternative to Dirac’s equation*, Phys. Rev. D 6 (1972) 500.

[6] N. Cufaro Petroni, P. Gueret, J. P. Vigier and A. Kyprianidis, *Second order wave equation for spin 1/2 fields*, Phys. Rev. D 31 (1985) 3157.

[7] N. Cufaro Petroni, P. Gueret, J. P. Vigier and A. Kyprianidis, *Second order wave equation for spin 1/2 fields. 2. the Hilbert space of the states*, Phys. Rev. D 33 (1986) 1674.

[8] N. Cufaro Petroni, P. Gueret and J. P. Vigier, *Form of a spin dependent quantum potential*, Phys. Rev. D 30 (1984) 1179.

[9] L. Hostler, *An SL(2, C) invariant representation of the Dirac equation*, J. Math. Phys. 23 (1982) 2366.

[10] L. Hostler, *An SL(2, C) Invariant representation of the Dirac equation. 2. Coulomb Green’s function*, J. Math. Phys. 24 (1983) 2366.

[11] L. M. Brown, *Two-component Fermion theory*, Phys. Rev. 111 (1958) 957.

[12] M. Tonin, *Quantization of the two-component fermion theory*, Nuovo Cimento 14 (1959) 1108.

[13] Herbert Pietschmann, *Zur Renormierung der zweikomponentigen Quantenelektrodynamik*, Acta Phys. Austriaca 14 (1961) 63.

[14] A. O. Barut and G. H. Mullen, *Quantization of two-component higher order spinor equations*, Annals of Physics 20 (1962) 184.

[15] R. Y. Volkovskii, *On the two-component theory of fermions* (in Russian), Izv. Vuz. Fiz. 5 (1971) 53 [Soviet Phys. J. 14 (1973) 611].

[16] L. Hostler, *Scalar formalism for quantum electrodynamics*, J. Math. Phys. 26 (1985) 1348.

[17] L. Hostler, *Scalar formalism for nonabelian gauge theory*, J. Math. Phys. 27 (1986) 2423.

[18] A. C. Longhitano and B. Svetitsky, *Second order lattice fermions*, Phys. Lett. B 126 (1983) 259.

[19] F. Palumbo, *Second order formalism for fermions and lattice regularization*, Nuovo Cim. A 104 (1991) 1851.
A. G. Morgan, *Second order fermions in gauge theories*, *Phys. Lett.* B 351 (1995) 249 [hep-ph/9502230].

M. J. G. Veltman, *Two component theory and electron magnetic moment*, *Acta Phys. Polon.* B 29 (1998) 783 [hep-th/9712216].

E. G. Delgado-Acosta, M. Napsuciale, S. Rodriguez, *Second order formalism for spin 1/2 fermions and Compton scattering*, *Phys. Rev.* D 83 (2011) 073001 [arXiv:1012.4130 [hep-ph]].

K. Johnson, E. C. Sudarshan, *Inconsistency of the local field theory of charged spin 3/2 particles*, *Annals Phys.* 13 (1961) 126.

G. Velo, D. Zwanziger, *Propagation and quantization of Rarita-Schwinger waves in an external electromagnetic potential*, *Phys. Rev.* 186 (1969) 1337.

G. Velo, D. Zwanziger, *Noncausality and other defects of interaction lagrangians for particles with spin one and higher*, *Phys. Rev.* 188 (1969) 2218.

M. Napsuciale, M. Kirchbach, S. Rodriguez, *Spin 3/2 beyond the Rarita-Schwinger framework*, *Eur. Phys. J.* A 29 (2006) 289 [hep-ph/0606308].

E. G. Delgado-Acosta, M. Napsuciale, *Compton scattering off elementary spin 3/2 particles*, *Phys. Rev.* D 80 (2009) 054002 [arXiv:0907.1124].

M. Napsuciale, S. Rodriguez, E. G. Delgado-Acosta, M. Kirchbach, *Electromagnetic couplings of elementary vector particles*, *Phys. Rev.* D 77 (2008) 014009 [arXiv:0711.4162].

E. G. Delgado-Acosta, M. Kirchbach, M. Napsuciale and S. Rodriguez, *Electromagnetic multipole moments of elementary spin-1/2, 1, and 3/2 particles*, *Phys. Rev.* D 85 (2012) 116006 [arXiv:1204.5337].

R. Angeles-Martinez and M. Napsuciale, *Renormalization of the QED of second order spin 1/2 fermions*, *Phys. Rev.* D 85 (2012) 076004 [arXiv:1112.1134].

Y. Nambu and G. Jona-Lasinio, *Dynamical model of elementary particles based on an analogy with superconductivity. I*, *Phys. Rev.* 122 (1961) 345.

Y. Nambu and G. Jona-Lasinio, *Dynamical model of elementary particles based on an analogy with superconductivity. II*, *Phys. Rev.* 124 (1961) 246.

A. Bondi, G. Curci, G. Paffuti and P. Rossi, *Metric and central charge in the perturbative approach to two-dimensional fermionic models*, *Annals Phys.* 199 (1990) 268.

F. Jegerlehner, *Facts of life with gamma(5)*, *Eur. Phys. J.* C 18 (2001) 673 [hep-th/0005255].

J. Rafelski and L. Labun, *A cusp in QED at g=2*, arXiv:1205.1835.