Non-commutativity and non-inertial effects on a scalar field in a cosmic string space-time: I. Klein–Gordon oscillator

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Abstract

We analyse the Klein–Gordon oscillator in a cosmic string space-time and study the effects stemming from the rotating frame and non-commutativity in momentum space. We show that the latter mimics a constant magnetic field, imparting physical interpretation to the setup. The field equation for the scalar field is solved via separations of variables, and we obtain quantization of energy and angular momentum. The space-time metric is non-degenerate as long as the particle is confined within a hard-wall, whose position depends on the rotation frame velocity and the string mass parameter. We investigate the energy quantization both for a finite hard-wall (numerical evaluation) and in the limit of an infinite hard-wall (analytical treatment). We stress the effect of non-commutativity upon the energy quantization in each case.

Keywords: Klein–Gordon oscillator, cosmic string, non-commutative geometry, magnetic field, energy eigenvalues

(Some figures may appear in colour only in the online journal)
1. Introduction

The hypothetical cosmological objects known as cosmic strings are among the most investigated examples of topological defect space-times which have stimulated interest in theoretical physics for more than forty years [1–4]. In particular, the gravitational properties of cosmic strings are remarkably different from typical linear distributions of matter. For instance, the geometry of the space-time around a straight cosmic string is locally identical to a flat space-time, but globally conical rather than Euclidean, and with an azimuthal deficit angle related to the string tension [4, 5]. Naturally an interesting topic is the interaction between such gravitational fields and quantum fields. The Klein–Gordon (KG) scalar field is among the most basic cases to examine in that context. The KG equation in various types of curved space-times was discussed, e.g. in references [6–19]. In particular, the generalized KG oscillator has been recently studied in a Gödel-type space-time [17], Som–Raychaudhuri space-time [18] and in a global monopole space-time [19].

In this paper, we analyze the effects of non-commutativity on the relativistic dynamics of a KG oscillator in a rotating cosmic string space-time. In a recent paper, we studied the analogous problem for fermions by examining the non-inertial effects of a rotating frame on a Dirac oscillator in a cosmic string space-time with non-commutative geometry in the phase space [20]. The KG equation describes spin-zero fields, whether they are composite particles, like pions, or the Higgs boson, which is the first known elementary particle with spin-zero. The so-called ‘KG oscillator’ is defined in a way analogous to the ‘Dirac oscillator’, which was introduced in 1989 [21–23] and represents an exactly solvable model of many-particle models in relativistic quantum mechanics.

Hereafter we investigate the interaction between the KG oscillator and cosmic strings, which may be seen as spatial lines of trapped energy density, in analogy with vortex lines in superfluids and superconductors or line defects in crystals [3, 4, 24, 25]. Our main interest here is to evaluate the effects of non-commutativity in the phase space on the KG equation’s wave functions and energy eigenvalues. Non-commutative geometry has been applied recently to the quantum Hall effect [26–29], geometric phases [30, 31], the Dirac oscillator [32–36] and the relativistic Duffin–Kemmer–Petiau (DKP) oscillator [37–39]. The DKP approach aims at treating integer-spin fields with first-order field equations in a way that resembles the Dirac equation for half-integer spin [40–43]. The spin-zero DKP equation can be shown to the equivalent to KG equation in the case of free fields [44]. However, this equivalence does not necessarily hold when interactions come into play [45, 46]. In fact, gauge theory taught us long ago [47] that interactions can be introduced by means of a coupling prescription. The ‘DKP oscillator’ is built from a prescription that is different from the coupling prescription characterizing the KG oscillator. This fact will naturally lead to different physical consequences in each case. This paper explores the physical significance of KG oscillator in the presence of non-commutativity, rotating frame and a cosmic string ambient space-time; the companion paper [48] scrutinizes DKP oscillator for the spin-zero field in an analogous context.

Here, we consider a cosmic string space-time equipped with a rotating frame in cylindrical coordinates. We work with the cosmic string space-time with metric signature \( \text{diag}(-1, +1, +1, +1) \), and described in cylindrical coordinates \((\rho, \varphi, z)\) where \(\rho \in [0, \infty)\), \(\varphi \in [0, 2\pi]\), \(z \in (-\infty, +\infty)\) by the line element

\[
ds^2 = -(1 - \omega^2 \eta^2 \rho^2) \, dt^2 + 2 \omega \eta^2 \rho^2 \, d\varphi \, dt + \rho^2 \, d\varphi^2 + \eta^2 \rho^2 \, d\varphi^2 + dz^2. \tag{1}
\]

In equation (1), \(\rho = \sqrt{x^2 + y^2}\), \(\omega\) is the angular frequency of the rotating frame and \(\eta = 1 - 4\Lambda \in (0, 1]\) (with \(\Lambda\) the string’s linear mass density) is related to the deficit angle
\[ \theta = 2\pi (1 - \eta). \] We consider units such that the speed of light \( c = 1 \) and the Planck’s constant \( \hbar = 1 \). The line element in equation (1) corresponds geometrically to a Minkowski space-time with a conical singularity [49]. From the first term of equation (1) we deduce two natural intervals, delimited by

\[ \rho_0 \equiv \frac{1}{\omega \eta}, \quad 0 < \rho < \rho_0 \quad \text{and} \quad \rho > \rho_0. \] (2)

For the latter, the particle lies outside of the light-cone since its speed is greater than the speed of light, which implies that the wave function of the particle must vanish as \( \rho \) approaches \( \rho_0 \). Thus \( \omega \eta \) determines two classes of solutions: firstly, a finite wall \( \rho_0 \) with \( \omega \eta \) arbitrary but finite, and secondly, a wall \( \rho_0 \) at infinity when \( \omega \eta \ll 1 \) [50]. We will say more about this in section 3.

In this paper, we examine the effects of non-commutativity in a cosmic string space-time with a rotating frame in cylindrical coordinates by solving the KG equation in this curved space-time. The geometry of the spacetime can play the role of a hard-wall confining potential via non-inertial effects (see reference [50] and references therein). Hereafter, for simplicity, we consider non-commutativity in the momentum components only, and we restrict the non-commutativity parameters to be parallel to the cosmic string. Our purpose is to find the solutions of the field equations and the corresponding energies. In section 2, we solve the KG equation and find solutions restricted to the plane \( z = 0 \) as well as general solutions in terms of \( z \). In section 2.1 we show that the non-commutativity parameter can be interpreted as a magnetic field. In section 3, we obtain the energy eigenvalues which correspond to the wave functions and discuss the dissipative effects arising from the coupling prescription. Finally, in section 4, we discuss our results and propose potential applications.

2. Klein–Gordon oscillator in a non-inertial frame and cosmic string space-time with non-commutative geometry

The equation for the KG oscillator is obtained from the KG equation by applying the non-minimal coupling [21, 50, 51]:

\[ p^j \phi \rightarrow (p - im\omega_0 r) \phi, \quad p^j \phi^* \rightarrow (p + im\omega_0 r) \phi^*, \] (3)

where we carefully apply the appropriate sign for the complex conjugate field. An alternative possibility for the coupling prescription would be to keep the same sign for both \( p^j \phi \) and \( p^j \phi^* \); the motivation for doing that and its consequences is explored in part 2 of this work [48].

The KG field has a mass \( m \) and the oscillator’s frequency is denoted by \( \omega_0 \). Hereafter, we use cylindrical coordinates and select

\[ r = \rho \hat{\rho} + z \hat{z} = (\rho, 0, z). \] (4)

Since \( p_j = -i \partial_j \) \((j = 1, 2, 3, \) standing respectively for \( \rho, \varphi \) and \( z \)), we can express equation (3) as \(-i (\nabla \pm m\omega_0 r)\), where \(+\) (resp. \(-\)) applies to \( \phi \) (resp. \( \phi^* \)).

If we begin with the KG Lagrangian

\[ L = \sqrt{-g} \left( -g^{ij} \partial_i \phi^* \partial_j \phi - m^2 \phi^* \phi \right), \] (5)

and perform the non-minimal substitution in equation (3), we obtain the Lagrangian of the KG oscillator:
\( L_{osc} = -\sqrt{-g} \left[ g^{00} \partial_0 \phi^* \partial_0 \phi + g^{02} \partial_0 \phi^* \partial_2 \phi + g^{02} \partial_2 \phi^* \partial_0 \phi \right. \\
+ \left. g^{11} \left( \partial_1 \phi^* - m \omega_0 \rho \phi^* \right) \left( \partial_1 \phi + m \omega_0 \rho \phi \right) + g^{22} \partial_2 \phi^* \partial_2 \phi \right. \\
+ \left. g^{33} \left( \partial_3 \phi^* - m \omega_0 z \phi^* \right) \left( \partial_3 \phi + m \omega_0 z \phi \right) + m^2 \phi^* \phi \right] ,
\)

which, with 0, 1, 2, 3 standing for \( t, \rho, \varphi \) and \( z \), respectively, leads to the equation of motion

\[
-m^2 \phi + m^2 \omega_0^2 \rho^2 \phi + m^2 \omega_0^2 z^2 \phi - \partial_\phi^2 \phi + \left( 2m \omega_0 \rho + \frac{1}{\rho} \right) \partial_0 \phi + \partial_\phi^2 \phi \\
+ \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \partial_\phi^2 \phi + 2m \omega_0 \partial_\phi \partial_0 \phi + 2m \omega_0 z \partial_\phi \partial_z \phi + \partial_\phi^2 \phi + 3m \omega_0 \phi = 0.
\]

Next we introduce the non-commutative momentum space, which is a particular case from the general non-commutative phase space described by the operators \( \hat{r}_i \) and \( \hat{p}_i \), defined via the following generalized Bopp shift \([20, 52]\):

\[
\hat{r}_i = r_i - \frac{\Theta_{ij}}{2\hbar} p_j = r_i + \frac{(\Theta \times p)_j}{2\hbar},
\]

\[
\hat{p}_i = p_i + \frac{\Omega_{ij}}{2\hbar} r_j = p_i - \frac{(\Omega \times r)_j}{2\hbar},
\]

which satisfy the following commutation relations:

\[
[\hat{r}_i, \hat{r}_j] = i \Theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i \Omega_{ij}, \quad [\hat{r}_i, \hat{p}_j] = i \hbar \Delta_{ij},
\]

where \( \Theta_{ij} = \epsilon_{ijk} \Theta_k \), \( \Omega_{ij} = \epsilon_{ijk} \Omega_k \), with \( \Theta_i \) and \( \Omega_i \) \((i = 1, 2, 3)\) real parameters. The matrix \( \Delta_{ij} \) is given by

\[
\Delta_{ij} = \left( 1 + \frac{\Theta \cdot \Omega}{4\hbar^2} \right) \delta_{ij} - \frac{\Omega_i \Theta_j}{4\hbar^2}.
\]

We will restrict our study to

\[
\Theta_j = 0, \quad j = 1, 2, 3,
\]

so that we allow for a non-commutative momentum space only, whereas the configuration space for the cosmic string remains commutative. The reason for this particular choice will be addressed at the end of section 2.1. Therefore, the non-commutative phase-space components are

\[
\hat{r}_i = r_i, \\
\hat{p}_i = p_i + \frac{\Omega_{ij}}{2\hbar} r_j = p_i - \frac{(\Omega \times r)_j}{2\hbar},
\]

where

\[
[\hat{r}_i, \hat{r}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = i \Omega_{ij}, \quad [\hat{r}_i, \hat{p}_j] = i \hbar \delta_{ij}.
\]

In cylindrical coordinates, the second term of equation (13) is

\[
\frac{\Omega_{ij}}{2} \delta_j = \frac{1}{2} \left( \Omega_{13} z, \Omega_{21} \rho + \Omega_{23} z, \Omega_{31} \rho \right) = \frac{1}{2} \left( -\Omega_{22} z, -\Omega_{33} \rho + \Omega_{11} z, \Omega_{22} \rho \right).
\]
We perform this prescription at the level of the Lagrangian (differently of the approach in [14]). This leads to:

\[
\text{cosmic string space-time equipped with a rotating frame in the presence of a magnetic field,}
\]

Then, equation (6) is generalized to the non-commutative oscillator Lagrangian:

\[
L_{\text{NC-osc}} = -\sqrt{-g} \left[ g^{00} \partial_0 \phi^* \partial_0 \phi + g^{02} \partial_0 \phi^* \left( \partial_2 \phi + i \frac{1}{2} \Omega_3 \rho \phi - \frac{1}{2} i \Omega_1 \phi \right) \right. \\
\left. + g^{02} \left( \partial_2 \phi^* - \frac{1}{2} i \Omega_3 \rho \phi^* + \frac{1}{2} i \Omega_1 \phi^* \right) \partial_0 \phi \right. \\
\left. + g^{11} \left( \partial_1 \phi^* - m \omega_0 \rho \phi^* - \frac{1}{2} i \Omega_2 \phi \right) \left( \partial_1 \phi + m \omega_0 \rho \phi + \frac{1}{2} i \Omega_2 \phi \right) \right. \\
\left. + g^{32} \left( \partial_2 \phi^* - \frac{1}{2} i \Omega_3 \rho \phi^* + \frac{1}{2} i \Omega_1 \phi^* \right) \times \left( \partial_2 \phi + i \frac{1}{2} \Omega_3 \rho \phi - \frac{1}{2} i \Omega_1 \phi \right) \right. \\
\left. + g^{33} \left( \partial_3 \phi^* - m \omega_0 \rho \phi^* + \frac{1}{2} i \Omega_2 \phi \right) \left( \partial_3 \phi + m \omega_0 \rho \phi - \frac{1}{2} i \Omega_2 \rho \phi \right) + m^2 \phi^* \phi \right]
\]

which modifies equation (7) as follows:

\[
-m^2 \phi + 3m \omega_0 \phi + \left( m \omega_0 \rho + i \frac{1}{2} \Omega_2 \rho \right) \phi + \left( m \omega_0 \rho - i \frac{1}{2} \Omega_2 \rho \right) \phi^2 \\
+ \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \left( -i \frac{1}{2} \Omega_3 \rho + i \frac{1}{2} \Omega_1 \rho \right) \phi^2 + \left( i \frac{1}{2} \Omega_2 \rho \right) \phi \\
- 2 \omega \left( -i \frac{1}{2} \Omega_3 \rho + i \frac{1}{2} \Omega_1 \rho \right) \partial_0 \phi + 2 \left( m \omega_0 \rho + i \frac{1}{2} \Omega_2 \rho \right) \partial_1 \phi \\
- 2 \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \left( -i \frac{1}{2} \Omega_3 \rho + i \frac{1}{2} \Omega_1 \rho \right) \partial_2 \phi \\
+ 2 \left( m \omega_0 \rho - i \frac{1}{2} \Omega_2 \rho \right) \partial_3 \phi - \partial_0^2 \phi + 2 \omega \partial_0 \partial_2 \phi + \partial_1^2 \phi + \frac{1}{\rho} \partial_1 \phi \\
+ \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \partial_2^2 \phi + \partial_3^2 \phi = 0.
\]

The solution for this equation will be dealt with in section 2.2. Before that, we would like to address the mapping existing between non-commutativity and a magnetic field.

2.1. Non-commutativity as a magnetic field

The non-commutativity in momentum space, such as considering hereafter, can be interpreted as a magnetic field. In this subsection, we first analyze the commutative KG oscillator in a cosmic string space-time equipped with a rotating frame in the presence of a magnetic field, and then map the equations on our non-commutative model.

As usual we consider the minimal coupling between the KG and electromagnetic fields, i.e.

\[
p \rightarrow p \mp eA.
\]

We perform this prescription at the level of the Lagrangian (differently of the approach in [14]). This leads to:
\[ L_{B-\text{osc}} = -\sqrt{-g} \left[ \epsilon^{0i} \partial_i \phi^* \partial_j \phi + g^{i2} \partial_i \phi^* \right] (\partial_2 \phi - i e A_2 \phi) + g^{i2} (\partial_2 \phi^* + i e A_2 \phi^*) \partial_i \phi \\
+ g^{11} (\partial_1 \phi^* - m \omega_0 \rho \phi^* + i e A_1 \phi^*) (\partial_1 \phi + m \omega_0 \phi \rho - i e A_1 \phi) + g^{22} (\partial_2 \phi^* + i e A_2 \phi^*) (\partial_2 \phi - i e A_2 \phi) \\
+ g^{33} (\partial_3 \phi^* - m \omega_0 \z \phi^* + i e A_3 \phi^*) (\partial_3 \phi + m \omega_0 \z \phi - i e A_3 \phi) \\
+ m^2 \phi^* \phi \right] . \tag{19} \]

From this Lagrangian we readily obtain the field equation:
\[-m^2 \phi + 3 m \omega_0 \phi + (m \omega_0 \rho - i e A_1)^2 \phi + (m \omega_0 \z - i e A_3)^2 \phi \\
+ \left( \frac{1}{\eta \rho^2} - \omega^2 \right) (i e A_2)^2 \phi - (i e A_1) \frac{1}{\rho} \phi \\
- 2 \omega (i e A_2) \partial_0 \phi + 2 (m \omega_0 \rho - i e A_1) \partial_1 \phi \\
- 2 \left( \frac{1}{\eta \rho^2} - \omega^2 \right) (i e A_2) \partial_2 \phi + 2 (m \omega_0 \z - i e A_3) \partial_3 \phi \\
- \partial_0^2 \phi + 2 \omega \partial_0 \partial_2 \phi + \partial_2^2 \phi + \frac{1}{\rho} \partial_1 \phi + \left( \frac{1}{\eta \rho^2} - \omega^2 \right) \partial_2^2 \phi + \partial_3^2 \phi \\
- i e \left[ \omega \partial_0 A_2 + \partial_1 A_1 + \left( \frac{1}{\eta \rho^2} - \omega^2 \right) \partial_2 A_2 + \partial_3 A_3 \right] \phi = 0. \tag{20} \]

The comparison between the Lagrangians \( L_{B-\text{osc}} \) and \( L_{NC-\text{osc}} \) and the comparison between the field equations in equation (17) and (20) exhibit the following mapping between the non-commutativity parameters and the components of vector potential:
\[ e A_1 = - \frac{1}{2} \Omega_2 \z, \quad e A_2 = - \frac{1}{2} \Omega_3 \rho + \frac{1}{2} \Omega_1 \z, \quad e A_3 = \frac{1}{2} \Omega_2 \rho, \tag{21} \]
so that \( \partial_1 A_1 = \partial_2 A_2 = \partial_3 A_3 = \partial_3 A_1 = 0. \) In particular, if we consider \( \Omega_1 = \Omega_2 = 0, \) we are left with
\[ -e \nabla \times \mathbf{A} = \Omega_3 \hat{\z}, \tag{22} \]
which shows that our non-commutative parameter \( \Omega_3 \) represents an external constant magnetic field along the \( \z \)-direction as in the Landau problem [53]. (This is also typical of a magnetic field within an infinite solenoid.) As a second possibility, had we considered \( \Omega_1 = \Omega_3 = 0, \) we would have obtained:
\[ -e \nabla \times \mathbf{A} = \Omega_2 \hat{\varphi} \tag{23} \]
which is a constant magnetic field in the \( \varphi \) direction, representing a field that circles around the \( \z \)-axis. (This might describe the magnetic field inside a toroid whose inner radius is much smaller than the outer radius.)

We will now analyze if non-commutativity in the coordinate space could also be related to a magnetic field. For this, we take \( \Omega_j = 0 \) but keep \( \Theta_j \neq 0, \) \( j = 1, 2, 3, \) so that the relations in equation (13) are replaced by
\[ \dot{r}_i = r_i - \frac{\Theta_i}{2\hbar} p_i = r_i + \frac{(\Theta \times p)_i}{2\hbar}. \]

\[ \dot{p}_i = p_i. \]  

(24)

From this and equation (3), the oscillator prescription takes on the form

\[ \mathbf{p} \rightarrow \mathbf{p} + i m \omega_0 \mathbf{r} \rightarrow -i \left( \nabla \pm m \omega_0 \mathbf{r} \pm m \omega_0 \frac{(\Theta \times \mathbf{p})}{2\hbar} \right), \]

and the Lagrangian for the scalar field KG oscillator in the presence of coordinate non-commutativity reads:

\[ L_{\text{NC-osc}} = \sqrt{-g} \left[ g^{00} \partial_0 \phi^* \partial_0 \phi + g^{02} \partial_0 \phi^* \times \left( \partial_2 \phi + \frac{1}{2} i m \rho \omega_0 \left[ \Theta_3 \partial_3 \phi - \Theta_1 \partial_1 \phi \right] \right) \right. \]

\[ + g^{02} \left( \partial_2 \phi^* - \frac{1}{2} i m \rho \omega_0 \left[ \Theta_3 \partial_3 \phi^* - \Theta_1 \partial_1 \phi^* \right] \right) \partial_0 \phi \]

\[ + g^{11} \left( \partial_1 \phi^* - m \omega_0 \rho \phi^* - \frac{1}{2} i m \omega_0 \left[ -\Theta_1 \frac{1}{\rho} \partial_1 \phi + \Theta_2 \partial_2 \phi \right] \right) \]

\[ \times \left( \partial_1 + m \omega_0 \rho \partial_1 + \frac{1}{2} i m \omega_0 \left[ -\Theta_1 \frac{1}{\rho} \partial_1 \phi + \Theta_2 \partial_2 \phi \right] \right) \]

\[ + g^{22} \left( \partial_2 \phi^* - \frac{1}{2} i m \rho \omega_0 \left[ \Theta_3 \partial_3 \phi^* - \Theta_1 \partial_1 \phi^* \right] \right) \left( \partial_2 \phi + \frac{1}{2} i m \rho \omega_0 \left[ \Theta_3 \partial_3 \phi - \Theta_1 \partial_1 \phi \right] \right) \]

\[ + g^{33} \left( \partial_3 \phi^* - m \omega_0 \rho \phi^* - \frac{1}{2} i m \omega_0 \left[ -\Theta_2 \partial_2 \phi^* + \Theta_1 \frac{1}{\rho} \partial_1 \phi^* \right] \right) \]

\[ \left. \times \left( \partial_3 + m \omega_0 \rho \partial_3 + \frac{1}{2} i m \omega_0 \left[ -\Theta_2 \partial_2 \phi + \Theta_1 \frac{1}{\rho} \partial_1 \phi \right] \right) + m^2 \phi^* \phi \right]. \]

(26)

The corresponding Euler–Lagrange equation is:

\[ 0 = -m^2 \phi + 3 m \omega_0 \phi + (m \omega_0)^2 \phi + (m \omega_0)^2 \phi + \frac{1}{2} i \Theta_2 m^2 \omega_0^2 \frac{z}{\rho} \phi \]

\[ + 2 m \omega_0 \rho \partial_1 + 2 m \omega_0 \rho \partial_2 - i m \omega_0 \Theta_3 \partial_3 \phi + i m \omega_0 \Theta_3 \partial_3 \phi + m \omega_0 \frac{1}{\rho} \Theta_2 \partial_1 \phi \]

\[ + \frac{1}{4} m^2 \omega_0^2 \left[ \Theta_2^2 \frac{\rho^2}{r^2} - \left( 3 \omega^2 - \frac{1}{\eta^2 \rho^2} \right) \Theta_3^2 \right] \rho \partial_1 \phi + \frac{1}{2} m^2 \omega_0^2 \left( 3 \omega^2 - \frac{1}{\eta^2 \rho^2} \right) \left[ \Theta_1 \Theta_1 \right] \rho \partial_1 \phi \]

\[ - \partial_1^2 \phi + 2 \omega \partial_0 \partial_2 \phi + \partial_1^2 \phi + \frac{1}{\rho} \partial_1 \phi + \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \partial_1^2 \phi + \partial_1^2 \phi \]

\[ - \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \left( \frac{1}{2} i m \rho \omega_0 \right)^2 \left[ \Theta_3^2 \partial_1^2 \phi - 2 \left( \Theta_1 \Theta_3 \right) \partial_3 \partial_1 \phi + \Theta_2^2 \partial_1^2 \phi \right] \]

\[ + \left( \frac{1}{2} i m \omega_0 \right)^2 \left[ -\Theta_3^2 \partial_1^2 \phi + 2 \Theta_1 \Theta_3 \frac{1}{\rho} \partial_1 \partial_2 \phi - \left( \Theta_1^2 + \Theta_3^2 \right) \frac{1}{\rho^2} \partial_1^2 \phi + 2 \Theta_2 \Theta_3 \frac{1}{\rho} \partial_3 \partial_2 \phi - \Theta_2 \partial_2 \phi \right]. \]

(27)

Comparison of this equation with the field equation containing the contribution of the magnetic vector potential does not reveal a manifest mapping from \( A = (A_\mu, A_\nu, A_\lambda) \) to
By substituting this ansatz into equation (2), it becomes

\[ \Phi = \Theta \]  

such that a magnetic field could be described by non-commutativity among space coordinates. This is the physical reason for our choice to take into account only non-commutativity in the momentum space.

### 2.2. Solution to the non-commutative-momentum-space field equation

In this subsection, we discuss the solutions of equation (2). Since we are interested in the time-independent solutions, we define the field \( \psi \) by

\[ \psi (\rho, \varphi, z, t) = e^{-i \mathcal{E}_0 t} \psi (\rho, \varphi, z) , \]  

(28)

As usual, \( \mathcal{E} \) is interpreted as the energy and we perform separation of variables in the space part:

\[ \psi (\rho, \varphi, z) = R (\rho) \Phi (\varphi) Z (z) . \]  

(29)

By substituting this ansatz into equation (2), it becomes

\[
2 \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \left( \frac{1}{2} \Omega_3 \rho - \frac{1}{2} \Omega_1 z \right) \frac{\partial_{\rho} \Phi (\varphi)}{\Phi (\varphi)} - 2i \mathcal{E} \frac{\partial_{\varphi} \Phi (\varphi)}{\Phi (\varphi)} \\
+ \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \frac{\partial^2 \Phi (\varphi)}{\Phi (\varphi)} + 2 \left( m_{\omega \rho} z - \frac{1}{2} i \Omega_2 \rho \right) \frac{\partial_\rho Z (z)}{Z (z)} \\
+ \frac{\partial^2 Z (z)}{Z (z)} + \frac{\partial^2 R (\rho)}{R (\rho)} + 1 \frac{\partial_\rho R (\rho)}{R (\rho)} + 2 \left( m_{\omega \rho} + \frac{1}{2} i \Omega_2 \rho \right) \frac{\partial_\rho R (\rho)}{R (\rho)} \\
+ \mathcal{E}^2 - m^2 + 3 m_{\omega \rho} + \left( m_{\omega \rho} + \frac{1}{2} i \Omega_2 \rho \right)^2 + \left( m_{\omega \rho} - \frac{1}{2} i \Omega_2 \rho \right)^2 + \left( \frac{1}{\eta^2 \rho^2} - \omega^2 \right) \\
\times \left( \frac{1}{2} \Omega_3 \rho - \frac{1}{2} \Omega_1 z \right)^2 + \frac{1}{2} i \Omega_2 \rho \frac{\partial_\rho Z (z)}{Z (z)} - 2i \mathcal{E} \omega \times \left( \frac{1}{2} \Omega_3 \rho - \frac{1}{2} \Omega_1 z \right) = 0 . \]  

(30)

The outcome of equation (29) is not directly effective because several terms involve the product of coordinates \( \rho \) and \( z \). In order to make further progress, we choose

\[ \Omega_1 = \Omega_2 = 0 . \]  

(31)

This particular choice is not only convenient from the computational standpoint, but also physically meaningful as discussed in section 2.1 because of the mapping between the momentum-space non-commutativity with \( \Omega_3 \neq 0 \) and a constant magnetic field pointing in the \( z \)-direction [54]. The remaining non-commutativity parameter couples to the frame angular velocity \( \omega \) but not to the oscillator frequency \( \omega_0 \).

The solution of equation (30) is built by further imposing the ansatz

\[ \Phi = e^{i L \varphi} , \]  

(32)

where

\[ L = 0, \pm 1, \pm 2, \pm 3, \ldots . \]  

(33)

The quantization of \( L \) stems from the familiar periodic boundary condition upon the azimuthal function \( \Phi (\varphi) \). From equation (32), we see that equation (30) decouples into a \( \rho \)-dependent part,
\[
\begin{align*}
\frac{\partial^2 R(\rho)}{R(\rho)} & + \frac{1}{\rho} \frac{\partial R(\rho)}{R(\rho)} + 2m\omega_0 \frac{\partial R(\rho)}{R(\rho)} \\
& + \mathcal{E}^2 - m^2 + 3m\omega_0 + (m\omega_0 \rho)^2 - \left( \frac{1}{\eta \rho} - \omega^2 \right) \left( \frac{1}{2} \Omega_3 \eta \right)^2 + \mathcal{E} \omega \Omega_3 \rho \\
& - 2L \left( \frac{1}{\eta \rho} - \omega^2 \right) \frac{1}{2} \Omega_3 \rho + 2L \mathcal{E} \omega - L^2 \left( \frac{1}{\eta \rho} - \omega^2 \right) = 2km
\end{align*}
\] (34)

and a \(z\)-dependent sector,

\[
\frac{\partial^2 Z(z)}{Z(z)} + 2m\omega_0 \frac{\partial Z(z)}{Z(z)} + (m\omega_0 z)^2 = -2mk,
\] (35)

where \(k\) is a separation constant with units of energy. The solution of (35) is

\[
Z(\zeta) = C_1 e^{-\frac{\zeta^2}{2} - \zeta \sqrt{1 - K}} + C_2 e^{-\frac{\zeta^2}{2} + \zeta \sqrt{1 - K}}
\] (36)

with the definitions

\[
\zeta \equiv \sqrt{m\omega_0} \quad \text{and} \quad K \equiv \frac{2k}{\omega_0}
\] (37)

Both \(\zeta\) and \(K\) are dimensionless constants.

The radial equation (34) may be cast into the form (organized in powers of \(\rho\)):

\[
0 = \frac{\partial^2 R(\rho)}{R(\rho)} + \frac{1}{\rho} \frac{\partial R(\rho)}{R(\rho)} + 2m\omega_0 \frac{\partial R(\rho)}{R(\rho)} \\
+ \left( 1 + \epsilon^2 \right) (m\omega_0)^2 \rho^2 + 2\epsilon (\mathcal{E} + L\omega) (m\omega_0) \rho \\
+ \left[ (\mathcal{E} + L\omega)^2 - m^2 + (3 - K) m\omega_0 \right] \left( \frac{1}{2} \Omega_3 \eta \right)^2 \\
- 2 \left( \frac{1}{2} \Omega_3 \eta \right) \left( \frac{L}{\eta} \right) \frac{1}{\rho} - \left( \frac{L}{\eta} \right)^2 \frac{1}{\rho^2},
\] (38)

where the new parameter \(\epsilon\),

\[
\epsilon \equiv \left( \frac{1}{2} \Omega_3 \omega \right) \left( \frac{m\omega_0}{2} \right),
\] (39)

contains the non-commutativity parameter \(\Omega_3\). It is convenient to perform a change of variables

\[
\xi \equiv S \rho,
\] (40)

where \(S\) could assume two forms: either

\[
S = S(m, \omega_0) \equiv \sqrt{m\omega_0}
\] (41)

or

\[
S = S(m, \omega_0, \omega, \Omega_3) \equiv \sqrt{m\omega_0} \left( 1 + \epsilon^2 \right)^{\frac{1}{2}}.
\] (42)
The form of \( \mathcal{S} \) in equation (40) resembles the one in the definition of the coordinate \( \zeta \), equation (37). The functional form of \( \mathcal{S} \) in equation (42) includes the non-commutativity parameter \( \Omega_3 \).

The solution of the radial equation by using equation (42) leads to interesting results, such as dissipative quantized energy, but it also causes problems, which we will explore in appendix A.

Henceforth, we will work with the choice in equation (40). In terms of \( \zeta \), the radial equation reads:

\[
0 = \frac{\partial^2 R(\zeta)}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial R(\zeta)}{\partial \zeta} + 2\zeta \frac{\partial R(\zeta)}{\partial \zeta} + (1 + \epsilon^2) \zeta^2 R(\zeta) + 2 \frac{(E + L\omega)}{\sqrt{m\omega_0}} (\epsilon) \zeta R(\zeta) \\
+ \frac{1}{(m\omega_0)} \left[ (E + L\omega)^2 - m^2 + (3 - K) m\omega_0 - \left( \frac{1}{2} \frac{\Omega_3}{\eta} \right)^2 \right] R(\zeta) \\
- \frac{2}{\sqrt{m\omega_0}} \left( \frac{1}{2} \frac{\Omega_3}{\eta} \right) \left( \frac{L}{\eta} \right) \frac{1}{\zeta} R(\zeta) - \left( \frac{L}{\eta} \right)^2 \frac{1}{\zeta^2} R(\zeta). \tag{43}
\]

At this point, we proceed to analyze two broad possibilities: the commutative case and the non-commutative scenario.

### 2.2.1. Commutative case

In this subsection, we obtain the solution for the commutative case, which will serve as our reference when taking the commutative limit. By taking \( \Omega_3 = 0 \) (which implies \( \epsilon = 0 \)) in equation (43), we obtain

\[
0 = \frac{\partial^2 R(\zeta)}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial R(\zeta)}{\partial \zeta} + 2\zeta \frac{\partial R(\zeta)}{\partial \zeta} + \zeta^2 R(\zeta) \\
+ \frac{1}{(m\omega_0)} \left[ (E + L\omega)^2 - m^2 + (3 - K) m\omega_0 \right] R(\zeta) - \left( \frac{L}{\eta} \right)^2 \frac{1}{\zeta^2} R(\zeta). \tag{44}
\]

The solution of this equation is in terms of the first-kind Bessel function \( J_\nu(x) \):

\[
R(\zeta) = C_c e^{\frac{i}{2} \frac{\zeta}{\eta} J_\frac{\nu}{\eta} (\alpha_\zeta \zeta)}, \quad \alpha_\zeta \equiv \sqrt{-m^2 + (1 - K) m\omega_0 + (E + L\omega)^2 \eta}. \tag{45}
\]

The quantity \( C_c \) is an integration constant. In principle, the differential equation also admits another solution in terms of the Bessel \( Y_\nu \) \((\alpha_\zeta \zeta)\), but this one diverges in the limit \( \zeta \to 0 \) and is thus discarded.

The Bessel functions of the first kind \( J_\nu \) and of the second kind \( Y_\nu \) are linearly independent solutions to the ordinary second-order linear differential equation [55]

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2) y = 0, \tag{46}
\]
called Bessel equation. This equation appears when solving the Laplace equation and the Helmholtz equation in cylindrical or spherical coordinates via separation of variables. Its applications span from quantum mechanics to thermodynamics, hydrodynamics and acoustics. In fact, the Bessel functions are common place as solutions of the Schrödinger equation in spherical and cylindrical coordinates for the free particle; in the description of heat conduction in cylindrical bars; in the description of motion of floating bodies, and for modelling the
vibration of thin circular membranes, among others [56]. The series expansion of \( J_\alpha (x) \) around \( x = 0 \) is found by the Frobenius method and gives

\[
J_\alpha (x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma (m + \alpha + 1)} \left( \frac{x}{2} \right)^{2m+\alpha},
\]

where \( \Gamma (z) \) is the gamma function [55].

As it happens, in quantum mechanics, for a particle confined in an infinite spherical well [57], the energy quantization is enforced via a boundary condition where the wave function should be zero. In our context the boundary is set by the hard-wall condition at \( \rho = 1/\omega \eta \). The roots of the Bessel functions do not have an analytical closed form, but we will calculate them numerically in section 3.2.

2.2.2. Non-commutative case. We will take a progressive approach, meaning that we shall seek solutions of equation (43) with different orders of the parameter \( \epsilon \), defined in equation (39), by keeping terms up to order \( \epsilon^2 \) in the first place, then accounting for orders of \( \epsilon^1 \), and, finally, considering only terms of \( \epsilon^0 \) in the radial differential equation. The reason for this will become clear below.

\( \epsilon^2 \)-order. Here we keep all powers of \( \epsilon \) in the differential equation (43), which exhibits terms up to order \( \epsilon^2 \). The solution—given by the Maple Software [58]—is in terms of the biconfluent Heun solution [59]:

\[
R (\xi) = C_{\text{nc}} \xi^{L/\eta} \exp \left\{ -\frac{\xi^2}{2} (1 - i\epsilon) + i \frac{(\xi \eta + \omega L)}{\sqrt{m \omega_0}} \xi \right\} \text{HeunB} (\alpha, \beta, \gamma, \delta, \alpha_{\text{nc}} \xi),
\]

(48)

where \( C_{\text{nc}} \) is a constant,

\[
\alpha \equiv \frac{2L}{\eta}, \quad \beta \equiv i \frac{2^{3/2} (\Omega \omega) (m \omega_0)^2 (\xi \eta + \omega L)}{\sqrt{m \omega_0} \left[-(\Omega \omega m \omega_0)^2\right]^{3/4}},
\]

\[
\gamma \equiv -2 \frac{(\Omega \omega)^2 (m \omega_0)^3 \left[(m^2 + m \omega_0 (K - 1)) \eta^2 + \frac{Q_i^2}{\xi^2} \right]}{\eta^2 \left[-(\Omega \omega m \omega_0)^2\right]^{3/2}},
\]

\[
\delta \equiv i \frac{2^{3/2} (\Omega L) (m \omega_0)}{\eta^2 \sqrt{m \omega_0} \left[-(\Omega \omega m \omega_0)^2\right]^{1/4}},
\]

and

\[
\alpha_{\text{nc}} = -i \frac{\sqrt{2} \left[-(\Omega \omega m \omega_0)^2\right]^{1/4}}{2m \omega_0}.
\]

(49)

(50)

Heun’s equation is the following second-order linear ordinary differential equation [60, 61]:

\[
\frac{d^2 w}{dz^2} + \left[ \frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\epsilon}{z - a} \right] \frac{dw}{dz} + \frac{\alpha \beta z - q}{z(z - 1)(z - a)} w = 0,
\]

(51)
where $\varepsilon = \alpha + \beta - \gamma - \delta + 1$. The accessory parameter $q$ is a complex number. In general, the above equation presents four regular singular points, namely: 0, 1, $a$ and $\infty$. Its importance is connected to the fact that every second-order ordinary differential equation extended to the complex plane with four singular points can be transformed to Heun’s equation by a suitable change of variables. This includes the Lamé equation (appearing when solving Laplace equation in elliptic coordinates) and the hypergeometric equation [55]. Hypergeometric functions are special functions represented by a series and encompass many other special functions (such as Legendre polynomials); the hypergeometric functions are general solutions of second-order linear ordinary differential equations with three singular points. Two or more regular singularities of Heun’s equation may coalesce into irregular singularities; this phenomenon gives rise to distinct confluent types of Heun’s equation. (An analogous process takes place for the hypergeometric differential equation.) The biconfluent Heun equation showcases two irregular (rank 2) singularities at 0 and $\infty$; moreover, it is characterized by the differential equation [20]

$$\frac{d^2w}{dz^2} + (\gamma + z) \frac{dw}{dz} + (\alpha z - q) w = 0.$$  \hspace{1cm} (52)

(Recall that $\alpha$, $\beta$, $\gamma$ and $\delta$ are related through the parameter $\varepsilon$).

As a solution, equation (48) is problematic with respect to the energy quantization. In fact, the quantization condition for the biconfluent Heun function [59] is $\gamma - 2 - \alpha = 2n$ ($n = 1, 2, 3, \ldots$). It does not lead to the energy quantization since $\gamma$ does not contain $E$. At best, this could be interpreted as a quantization condition on $K$ defined in equation (37). Another attempt to quantify the energy would be to look for the roots of the biconfluent Heun function. The problem with this is that the argument, or variable, $\alpha_\infty \xi$ of the Heun function also does not depend on the energy. Thus, even the numerical analysis does not help us to quantify $E$.

In addition, the commutative limit is not well defined: in the limit $\Omega_3 \to 0$, the variable $\alpha_\infty \xi$ is ill defined, and the parameters $\beta$ and $\gamma$ diverge—see equations (49) and (50). We are thus led to disregard this case.

$\varepsilon^1$-order. In the face of the difficulties with the $\varepsilon^2$-order, we try and keep the terms scaling with $\varepsilon^1$ in equation (43) but neglect the term depending on $\varepsilon^2$. This is justified in the context where $\Omega_3 \ll \omega_3$, i.e. the combination of non-commutativity and frame rotation is subdominant with respect to the rest mass and the oscillator’s frequency. That is the standpoint we adopt henceforth.

Accordingly, we set the term $\varepsilon^2$ equals zero in equation (43) and input the resulting differential equation in a computer algebra system to check for possible solutions. In this case, neither Maple nor Mathematica can return a solution to the differential equation. As an alternative, we resort to the Frobenius method for building a solution. In this manner, it is possible to obtain a quantization condition for the energy. However, we end up with the caveat of fixing a large number of conditions on the coefficients of the series: at least four coefficients have to be fixed, as discussed in appendix B. This is troublesome since we have a limited number of physical parameters to deal with. Thus, we will abandon this possibility and consider the even more simple case where terms scaling with $\varepsilon$ and $\varepsilon^2$ are neglected and only explicit term on the non-commutative parameter $\Omega_3$ is kept in equation (43).

$\varepsilon^0$-order. We can rewrite the differential equation (43) as:
\[ 0 = \frac{\partial^2 R(\xi)}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial R(\xi)}{\partial \xi} + 2 \xi \frac{\partial^2 R(\xi)}{\partial \xi^2} + (1 + \epsilon^2) \xi^2 R(\xi) + \frac{1}{(m \omega_0)} \left\{ \left( E + L \omega \right)^2 \left[ 1 + 2 \frac{1}{(E + L \omega)} \eta \omega \frac{\xi}{\xi_0} \right] - m^2 + (3 - K) m \omega_0 - \left( \frac{1}{2} \frac{\Omega_3}{\eta} \right)^2 \right\} R(\xi) \]

\[ - \frac{2}{\sqrt{m \omega_0}} \left( \frac{1}{2} \frac{\Omega_3}{\eta} \right) \left( \frac{L}{\eta} \right) \frac{1}{\xi} R(\xi) - \left( \frac{L}{\eta} \right)^2 \frac{1}{\xi^2} R(\xi), \quad (53) \]

where

\[ \xi_0 = S \rho_0 = \frac{\sqrt{m \omega_0}}{\eta \omega} \quad (54) \]

is the hard-wall in the variable \( \xi \) defined in terms of \( \rho \) in equation (40). In order to neglect the terms scaling with \( \epsilon^2 \) and \( \epsilon^1 \) in equation (53) there are two conditions that must be matched. The first is to assume \( \Omega_3 \omega \ll m \omega_0 \); this enables us to discard the term with \( \epsilon^2 \) in accordance with what was done in the previous case. The second is to take

\[ \epsilon \ll \frac{1}{2} \frac{\eta \omega}{m \omega_0} (E + L \omega) \Rightarrow \frac{\Omega_3}{\eta} \ll (E + L \omega). \quad (55) \]

In the face of this condition, we should also consider the term \( \left( \frac{\Omega_3}{2 \eta} \right)^2 \) as negligibly small when compared to \( (E + L \omega)^2 \).

By considering the aforementioned approximations in equation (53), together with the definition of auxiliary parameters

\[ B \equiv \frac{1}{(m \omega_0)} \left[ (E + L \omega)^2 - m^2 + (3 - K) m \omega_0 \right], \]

\[ C \equiv \frac{2}{\sqrt{m \omega_0}} \left( \frac{1}{2} \frac{\Omega_3}{\eta} \right) \left( \frac{L}{\eta} \right), \quad (57) \]

and the ansatz

\[ R(\xi) = F(\xi) e^{-\xi^2/2} \xi^{-1/2}, \quad (58) \]

which is motivated by the asymptotic limits imposed on equation (53), then this differential equation is reduced to

\[ 0 = F''(\xi) + \left\{ (B - 2) - \frac{C}{\xi} + \frac{\frac{1}{2} - \left( \frac{L}{\eta} \right)^2}{\xi^2} \right\} F(\xi). \quad (59) \]
We used the notation $F''(\xi) \equiv \frac{d^2 F(\xi)}{d\xi^2}$ in equation (59). We can see that its solution is given in terms of Whittaker functions [55]:

$$F(\xi) = C_n M_{\kappa, \mu}(\xi),$$

(60)

The second independent solution allowed for $F(\xi)$ is in terms of the Whittaker $U_{\kappa, \mu}(x)$ function. However, it presents divergence issues and is not physically acceptable. It should be emphasized that the quantum number $L$ has to be restricted to positive values so as to avoid divergences in the form of equation (60) near $\xi = 0$.

The Whittaker functions $M_{\kappa, \mu}(x)$ and $U_{\kappa, \mu}(x)$ are two linear independent solutions of Whittaker’s equation,

$$\frac{d^2 w}{dz^2} + \left( -\frac{1}{4} + \frac{\kappa}{z} - \frac{1/4 - \mu^2}{z^2} \right) w = 0,$$

(61)

which presents a singular points at 0 (regular) and at $\infty$ (irregular). The solutions read

$$M_{\kappa, \mu}(z) = \exp\left(-z/2\right) z^{\mu+\frac{1}{2}} M\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu, z\right),$$

$$U_{\kappa, \mu}(z) = \exp\left(-z/2\right) z^{\mu+\frac{1}{2}} U\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu, z\right),$$

(62)

and are given in terms of Kummer’s confluent hypergeometric function (or confluent hypergeometric function of the first kind) [55]

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} = {}_1 F_1(a; b; z),$$

(63)

with the Pochhammer symbol defined by

$$(a)_0 = 1,$$

$$(a)_n = a(a + 1)(a + 2)\ldots(a + n - 1),$$

(64)

and Tricomi’s confluent hypergeometric function (or confluent hypergeometric function of the second kind)

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)} M(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a + 1 - b, 2 - b, z).$$

(65)

Therein, the parameters $(a, b)$ are the ones appearing in the (Kummer’s form of the) confluent hypergeometric equation

$$z^2 \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0.$$

(66)

The function $U$ has a singularity at zero, unlike Kummer’s function $M(a, b, z)$. 
3. Energy quantization

The energy eigenvalues are sensitive to the interplay between the Gaussian function and the Whittaker function $M_{\kappa,\mu}(x)$ appearing in the functional form of the solution $R(\xi)$ in equation (58). In fact, this is related to the location of hard-wall condition $\xi_0$ in equation (54). The method for analyzing energy quantization depends on whether $\xi_0$ lies at a finite distance from the origin, or is at infinity. These two possibilities will be explored in the next subsections: hard-wall at infinity in section 3.1 and finite hard-wall in section 3.2.

3.1. Energy for the hard-wall in the limit $\xi_0 \to \infty$: asymptotic limits

If the hard-wall is very far from the origin, the behavior of the radial solution will depend on the asymptotic limits of the Whittaker function $M_{\kappa,\mu}(x)$. The functional form of these limits are determined by the nature of the variable $x = 2\sqrt{2 - \xi}$, which could be real-valued or complex-valued.

3.1.1. Complex-valued argument for the Whittaker function $M_{\kappa,\mu}(x)$. The variable $x$ will be a pure complex number if $2 - B < 0$, so that equation (57) implies

$$2 - \frac{1}{(m\omega_0)}[(E + L\omega)^2 - m^2 + (3 - K) m\omega_0] < 0.$$  

(67)

In this case, $x = iy$, where $y$ is real, and the asymptotic behavior for $M_{\kappa,\mu}(x)$ is [61]:

$$M_{\kappa,\mu}(iy) \sim \frac{\Gamma (1 + 2\mu)}{\Gamma (1 + \mu - \kappa)} e^{\frac{iy}{2}} \sum_{s=0}^{\infty} \left( \frac{1}{2} - \mu + \kappa \right) \frac{1}{s!} \left( \frac{1}{2} + \mu + \kappa \right) (-iy)^s \quad s \in \mathbb{N}.$$  

(68)

where $(a)_s$ is the Pochhammer symbol of equation (64).

Then, $M_{\kappa,\mu}(x)$ will assume a polynomial form under the condition [55]

$$\mu - \kappa + \frac{1}{2} = -n \quad (n = 0, 1, 2, 3, \ldots).$$  

(69)

This also leads to the quantization of energy. In fact, if we substitute the values of the constants $B$ and $C$, from equation (57), in the above condition, we obtain the energy eigenvalues:

$$E_n = -L\omega \pm \sqrt{m^2 + (K - 1) m\omega_0 - \left( \frac{1 + \Omega_1 \omega_0}{2} \right)^2 \left( \frac{1 + \Omega_1 \omega_0}{2} \right)^2 \left( \frac{1 + \Omega_1 \omega_0}{2} + n \right)^2}. $$  

(70)

The energy eigenvalues scale as $1/n$. It is of particular interest that the energy quantization is lost either in the commutative case, when $\Omega_1 = 0$, or when the angular momentum quantum number is null, $L = 0$. We observe that the contribution of the non-commutative parameter is to decrease the modulus of the energy of the system.

Equation (70) also shows that the angular momentum quantum number couples with both the non-commutative parameter and the angular velocity of the rotating frame. It is interesting
to note that the quantization is lost in the limit as \( L/\eta \gg n \) when the contribution of the non-commutative parameter freezes out.

The sign \( \pm \) in equation (70) could be interpreted as corresponding to the energy spectrum of particle (plus sign) and anti-particles (negative sign). This interpretation can be applied to other similar expressions later in this paper.

The energy \( E_n \) increases as the mass \( m \) increases, as expected. In particular, the large-mass regime leads to the non-relativistic limit

\[
E_n \approx -L\omega \pm m \pm \frac{1}{2} (K - 1) \omega_0 \pm \frac{1}{2m} \left( \frac{L\omega}{\eta} \right)^2 \left( \frac{\omega_0}{\eta} \right)^2 \left( \frac{1}{\eta} + n \right)^2.
\]  

Equation (71) also reveals that the quantization is lost when \( m \to \infty \), and that in this limit, the non-commutativity is irrelevant.

3.1.2. Real-valued argument for \( M_{\kappa, \mu}(x) \). Unlike the section 3.1.1, here we assume \( 2 - B > 0 \), and equation (57) leads to

\[
2 - \frac{1}{(m\omega_0)} \left[ (E + L\omega)^2 - m^2 + (3 - K) m\omega_0 \right] > 0.
\]  

Then the argument of \( M_{\kappa, \mu}(x) \) is real and its asymptotic behavior is:

\[
M_{\kappa, \mu}(x) \sim \frac{\Gamma(1 + 2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} e^{x^2 - \kappa \sum_{s=0}^{\infty} \left( \frac{1}{2} - \mu + \kappa \right) \frac{(\frac{1}{2} + \mu + \kappa)^s}{s!} x^{-s}}, \quad \mu - \kappa \neq -\frac{1}{2}, -\frac{3}{2}, \ldots
\]  

By considering the dominant term in the series above, the radial solution becomes

\[
R(\xi) \sim C_1 \xi^{-\frac{1}{2} - \kappa} e^{-\frac{1}{2}(\xi - \xi_m)^2},
\]  

where

\[
\xi_m = \sqrt{\frac{m^2 - (E + L\omega)^2 + (K - 1) m\omega_0}{m\omega_0}}.
\]  

We have expressed the radial function as the product of a polynomial by a Gaussian function with standard deviation \( \sigma = 1 \) and mean \( \xi_m \).

Now let us impose that the wave function be zero at the hard-wall position. We know that the Gaussian does not vanish at any point in its domain. However, it quickly tends to zero as we move away from \( \xi_m \). If the hard-wall is larger than \( k_G\sigma \), where \( \sigma \) denotes the standard deviation and \( k_G \) is a real number sufficiently large (e.g. \( k_G \gtrsim 6 \)), then the radial function can be considered small enough so that its contribution to the probability distribution is negligible near the hard-wall. If this condition is satisfied, then the quantization condition is completely analogous to the case discussed in section 3.1.1 with the energy eigenvalues given precisely by equation (70). Otherwise, if the condition \( (\xi_0 - \xi_m) > k_G\sigma \) is not satisfied, then we have to enforce the boundary condition \( R(\xi_0) = 0 \), and a quantization condition can only be obtained by numerical calculations. This is the case when the hard-wall is located in a definite finite value of \( \xi \). We shall analyze this possibility in section 3.2.
3.2. Energy for a finite hard-wall: numerical results

This section deals with the energy eigenvalues admissible by the wave function for a finite hard-wall. The roots of the special functions entering the radial part of the wave function do not exhibit a closed form in terms of the physical parameters and an integer number. This is the reason why we need to utilize a numerical method to compute the roots of the wave function.
Table 1. Values for the energies $E_{n\pm}$ of the Whittaker function $M_{\kappa,\mu}(x) = 0$ with the fixed values $m = 1$, $\omega_0 = 0.1$ and $\omega = 0.5$.

| $\eta$ | $n$ | $L = 0$ | $\Omega_3 = 0$ | $\Omega_3 = 0.1$ |
|--------|-----|---------|----------------|-----------------|
|        |     | $K = 0$ | $K = 2$ | $K = 0$ | $K = 2$ |
| 0.9    | 1   | 1.43913 | 1.50702 | 1.43913 | 1.50702 |
|        | 2   | 2.65903 | 2.69637 | 2.65903 | 2.69637 |
|        | 3   | 4.00807 | 4.03294 | 4.00807 | 4.03294 |
| 0.5    | 1   | 1.12314 | 1.2089 | 1.12314 | 1.2089 |
|        | 2   | 1.67465 | 1.73334 | 1.67465 | 1.73334 |
|        | 3   | 2.3623 | 2.40425 | 2.3623 | 2.40425 |
| 0.1    | 1   | 0.956273 | 1.05568 | 0.956273 | 1.05568 |
|        | 2   | 0.988017 | 1.08452 | 0.988017 | 1.08452 |
|        | 3   | 1.0427 | 1.13456 | 1.0427 | 1.13456 |



| $\eta$ | $n$ | $L = 1$ | $\Omega_3 = 0$ | $\Omega_3 = 0.1$ |
|--------|-----|---------|----------------|-----------------|
|        |     | $K = 0$ | $K = 2$ | $K = 0$ | $K = 2$ |
| 0.9    | 1   | 1.52706 | 1.57581 | −2.52706 | −2.57581 |
|        | 2   | 2.86526 | 2.89484 | −3.86526 | −3.89484 |
|        | 3   | 4.24752 | 4.26854 | −5.24752 | −5.26854 |
| 0.5    | 1   | 1.09638 | 1.15783 | −2.09638 | −2.15783 |
|        | 2   | 1.80827 | 1.8512 | −2.80827 | −2.8512 |
|        | 3   | 2.55594 | 2.58849 | −3.55594 | −3.58849 |
| 0.1    | 1   | 0.693252 | 0.774304 | −1.693252 | −1.774304 |
|        | 2   | 0.988017 | 1.08452 | −2.988017 | −2.08452 |
|        | 3   | 1.0427 | 1.13456 | −2.0427 | −1.13456 |

Notice that the roots $r_{n\pm}$ are a function of the non-commutative parameter $\Omega_3$. The commutative case admits a radial function as given in equation (45). The quantization of the energy is obtained when $r_c = \alpha_c \xi_0$ is a root of the Bessel function. If we evaluate the first $n$ roots $r_{nc,n}$, we can find the corresponding energies $E_{n\pm}$:

$$E_{n\pm} = -L\omega \pm \sqrt{m^2 + (K - 1)m\omega_0 + \left(\eta\omega r_{nc,n}\right)^2}, \quad n = 1, 2, 3, \ldots \tag{76}$$

The non-commutative case is described by the radial function in equations (58) and (60). It is thus given in terms of a Whittaker function, whose zeros at the hard-wall location determine the energy eigenvalues. The roots are $r_{nc} = r_{nc} (\Omega_3) = 2\sqrt{2 - B}\xi_0$ and the corresponding energies $E_{n\pm}$ are:

$$E_{n\pm} = -L\omega \pm \sqrt{m^2 + (K - 1)m\omega_0 - \frac{1}{4}\left(\eta\omega r_{nc,n}\right)^2}, \quad n = 1, 2, 3, \ldots \tag{77}$$
Table 2. Values for the energies $E_{nL}$ of the Whittaker function $M_{\kappa,\mu}(x) = 0$ with the fixed values $m = 1$, $\omega_0 = 0.1$ and $\omega = 1$.

| $L = 0$ |
| --- |
| $\Omega_3 = 0$ | $\Omega_3 = 0.1$ |
| $|E_+|$ | $|E_+|$ |
| $K = 0$ | $K = 2$ | $K = 0$ | $K = 2$ |
| $n$ | $\eta$ | $K = 0$ | $K = 2$ | $K = 0$ | $K = 2$ |
| 0.9 | 1 | 2.36313 | 2.40507 | 2.36313 | 2.40507 |
| 2 | 2.05784 | 2.07757 | 2.05784 | 2.07757 |
| 3 | 7.84592 | 7.85866 | 7.84592 | 7.85866 |
| 0.5 | 1 | 1.5316 | 1.59556 | 1.5316 | 1.59556 |
| 2 | 2.91853 | 2.95259 | 2.91853 | 2.95259 |
| 3 | 4.42964 | 4.45216 | 4.42964 | 4.45216 |
| 0.1 | 1 | 0.978689 | 1.07603 | 0.978689 | 1.07603 |
| 2 | 1.09759 | 1.18521 | 1.09759 | 1.18521 |
| 3 | 1.28408 | 1.35973 | 1.28408 | 1.35973 |

$L = 1$

| $\Omega_3 = 0$ | $\Omega_3 = 0.1$ |
| --- |
| $E_1$ | $E_1$ | $E_1$ | $E_1$ |
| $K = 0$ | $K = 2$ | $K = 0$ | $K = 2$ |
| $n$ | $\eta$ | $K = 0$ | $K = 2$ | $K = 0$ | $K = 2$ |
| 0.9 | 1 | 2.7062 | -4.7062 | 2.73601 | -4.73601 |
| 2 | 5.52685 | -7.52685 | 5.54853 | -7.54853 |
| 3 | 8.35179 | -10.3518 | 8.36903 | -10.369 |
| 0.5 | 1 | 1.77374 | -3.77374 | 1.79951 | -3.79951 |
| 2 | 3.31422 | -5.31422 | 3.36409 | -5.36409 |
| 3 | 4.88686 | -6.88686 | 4.92865 | -6.92865 |
| 0.1 | 1 | 0.730723 | -2.73072 | 0.7253 | -2.7253 |
| 2 | 1.07314 | -3.07314 | 1.10834 | -3.10834 |
| 3 | 1.40014 | -3.40014 | 1.43554 | -3.43554 |

The presence of $\Omega_3$ displaces the zeros of the wave function as can be seen in figures 1 and 2.

The displacement of the roots of the wave function due to non-commutativity implies that different values of the energy have to be chosen for the commutative and non-commutative cases, cf equations (76) and (77). We use Mathematica [62] to determine the energy eigenvalues for both cases. These values are given in tables 1 and 2 for particular values of the physical parameters $\{m, \omega_0, \omega, \eta, \Omega_3\}$, of the quantum numbers $\{n, L\}$, and of $K$.

Tables 1 and 2 show that, for $L = 0$, non-commutativity has no effect on the energy eigenvalues, because of the coupling between $\Omega_3$ and $L$ in the second-to-last term in equation (53). For $L \neq 0$, non-commutativity comes into effect such that, as a general trend, the absolute value of $E_{nL}$ increases in the presence of non-commutativity. We also observe that the absolute value of the energy is larger for a larger angular velocity $\omega$, and mainly for a larger cosmic string parameter $\eta$.

The quantum number $n$ has the effect of increasing the energies $E_{\pm}$ in absolute values. Although the majority of the values of $E_{\pm}$ increases with the increase of $L$, a few exceptions...
Figure 3. Effects of the parameters \( \{ \eta, n, \omega \} \) on the energy eigenvalues \( \mathcal{E}_\pm \). (a) Plot of \( \mathcal{E}_\pm (\eta) \) for \( n = 3 \) and \( \omega = 0.5 \); (b) plot of \( \mathcal{E}_\pm (n) \) for \( \eta = 0.5 \) and \( \omega = 0.5 \); and (c) plot of \( \mathcal{E}_\pm (n) \) for \( \eta = 0.5 \) and \( \omega = 1.0 \). The absolute value of the energy increases for larger values of \( \eta \), from part (a); \( |\mathcal{E}_\pm| \) increases as \( n \) increases, from part (b); and the comparison of parts (b) and (c) shows that \( |\mathcal{E}_\pm| \) grows with increasing \( \omega \). The several lines in each part correspond to different values of the parameters \( L = 0, 1, \Omega_3 = 0.0, 0.1 \) and \( K = 0, 2 \).

In tables 1 and 2 show that there is no global pattern for the energy related to the quantum number \( L \).

Figure 3 helps to visualize the effects mentioned in the last two paragraphs.

In figure 3(a) we set \( n = 3 \) (the third energy level), take \( \omega = 0.5 \) (which selects the results in table 1) and take \( \eta = 0.1, 0.5, 0.9 \). The plot clearly shows that \( |\mathcal{E}_\pm| \) increases with increasing values of \( \eta \). Figure 3(b) assumes \( \omega = 0.5 \) (as in table 1 again), string parameter \( \eta = 0.5 \), and we let the principal quantum number \( n = 1, 2, 3 \). The graph shows that the values of \( |\mathcal{E}_\pm| \) increase as \( n \) scales up. Furthermore, we can conclude that \( |\mathcal{E}_\pm| \) increases as \( \omega \) increases by comparing the values on the y-axis of parts (b) and (c) of figure 3, which display \( \mathcal{E}_\pm (n, \eta = 0.5) \) for \( \omega = 0.5 \) and 1.0, respectively.

The effect of non-commutativity is displayed in figure 4.
Figure 4. Effect of the non-commutativity parameter $\Omega_3$ on the energy eigenvalues $E_{\pm}$.
(a) Plot of $\Delta E_{\pm} \equiv E_{\pm} (\Omega_3 = 0.1) - E_{\pm} (\Omega_3 = 0)$ as a function of the string parameter $\eta$ for a fixed energy level $n = 2$. (b) Plot of $\Delta E_{\pm}$ as a function of the principal quantum number $n$ for a fixed string parameter $\eta = 0.9$. Both parts (a) and (b) show that the effect of non-commutativity decreases for increasing values of $\eta$ and $n$. The various lines in each part correspond to different values of the parameters $\omega = 0.5, 1.0$, $L = 0, 1$, and $K = 0, 2$.

We define the energy difference $\Delta E_{\pm} \equiv E_{\pm} (\Omega_3 = 0.1) - E_{\pm} (\Omega_3 = 0)$ which vanishes if the non-commutativity parameter $\Omega_3$ has no impact on the energy levels. We evaluate $\Delta E_{\pm}$ as a function of $\eta$ (for a constant value of the principal quantum number $n$) in figure 4(a): it shows that the effect of non-commutativity on the values of $|E_{\pm}|$ increases as the string parameter value decreases. Analogously, figure 4(b) shows that the difference between energy levels decreases as $n$ increases (for a fixed value of the string parameter). The pile-up effect of the levels in the energy spectrum as $n$ increases is also observed in the hydrogen atom, where the energy levels become closer for higher values of the principal quantum number $n$.

We already mentioned in section 2.1 that non-commutativity can be physically interpreted as a magnetic field. In the present section, we discussed that the non-commutative term engenders the energy difference $\Delta E_{\pm}$. In analogy with the quantum mechanics of the hydrogen atom, it is known that the energy spectrum is affected by the presence of an external magnetic field through the Zeeman effect [57], which is responsible for lifting some degeneracies. In this sense, with regards to the effect on the energy levels, there exists a parallel between
non-commutativity for the KG oscillator and the Zeeman effect for the hydrogen atom. Also regarding the effects on the energy levels, one could affirm that there is an analogy between the coupling $\omega L$ observed for the KG oscillator in equations (70), (76) and (77), and the spin–orbit coupling $\langle S \cdot L \rangle$ of the hydrogen atom.

We emphasize that the above analogies pertain exclusively to changes in energy levels. The system that we deal with in this work (i.e. a KG oscillator in a rotating frame with momentum non-commutativity in a cosmic-string background space-time) is not of a physical nature similar to the proton-electron system forming the hydrogen atom.

4. Concluding remarks

This paper presents the study of a KG oscillator in a cosmic-string background space endowed with a rotating frame and non-commutativity.

We introduced the oscillation through a non-minimal coupling with the radial coordinate and the complex number, which introduces a change in sign when the prescription operates onto the complex-conjugate of the spin-zero field. This sign change between $p\phi$ and $p\phi^*$ is key for the physical interpretation of the non-commutativity parameters. In part 2 of this paper [48], we shall explore a different coupling that leads to interpreting the spin-zero case as the scalar sector of the DKP field in the ways explored by reference [63] in the commutative case.

We introduced non-commutativity via the generalized Bopp shift and began with non-commutativity both in the coordinate space and in the momentum space. Then we specified the Lagrangian of the system and its equation of motion for non-commutativity restricted to momentum space. This particular choice is based on firm physical grounds for, as we have shown, the momentum-space non-commutativity is equivalent to the presence of a magnetic field. Conversely, coordinate-space non-commutativity does not map to a magnetic field. That is why we decided to consider non-commutativity in momentum-space only.

Our study of the scalar field continued with the solution to the field equations by applying the separation of variables to build the time-dependent wave function. The space part of the latter was determined after we restricted ourselves to non-commutativity in the momentum coordinate pointing in the direction of the linear topological defect, i.e. by assuming a uniform magnetic field directed along the string.

The time-independent wave function $\psi$ contains the ordinary periodic angular solution with its associated quantum number $L = 0, \pm 1, \pm 2, \ldots$. The $z$-dependent part of $\psi$ has a Gaussian form, shaped by the mass $m$ of the oscillator and its natural frequency $\omega_0$; the separation constant $K$ for this part of the solution is not constrained or quantized by boundary conditions. The radial $\rho$-dependent part of $\psi$ is not so simple to compute because the non-commutativity parameter $\Omega_3$ appears, up to power two, in several terms of the differential equation for $R(\rho)$. Our strategy was to define the parameter $\epsilon = \frac{\Omega_3}{m \omega_0}$ and analyze the instances where it is small, i.e. where non-commutativity ($\Omega_3$) and frame rotation ($\omega$) are sub-dominant effects when compared to $m \omega_0$. Accordingly, we have studied extensively only the solution obtained from keeping linear terms in $\Omega_3$ in the differential equation for $R(\rho)$ and neglecting terms scaling with $\epsilon$ and $\epsilon^2$. In this way, we have obtained $R(\rho)$ in terms of Whittaker functions multiplied by a Gaussian-type term $e^{-\xi^2/2}$ and the moderator factor $\xi^{-1/2}$, where $\xi = \sqrt{m \omega_0 \rho}$.

An important distance scale is the hard-wall position, at $\rho_0 = 1/\eta \omega$; it sets the interval $[0, \rho_0]$ within which the particle should be confined. This establishes a boundary condition leading to energy quantization. In fact, the energy eigenvalues were shown to scale as the inverse of the principal quantum number $n$ in the case of a hard-wall position at infinity, $\rho_0 \rightarrow \infty$, a condition that is attained, for example, by slowing down the frame rotation to
the limit $\omega \to 0$. Here the paramount role of $\Omega_3$ is evident: the quantization is lost in the commutative limit. It is also noteworthy that the non-relativistic limit is duly obtained.

In the circumstances where the hard-wall takes on finite values, it is necessary to perform a numerical analysis of the zeros of the Wittaker function to determine the particle energy levels. Here, the effect of non-commutativity is to increase the absolute values of energy. We emphasize the coupling between $\Omega_3$ and $\omega$; for $L = 0$, the eigenvalues reduce to their commutative counterparts.

All the above remarks regarding the time-independent wave function $\psi$ and its energy eigenvalues $E$ are valid when we use the distance variable $\xi = \sqrt{m\omega_0 \rho}$ to integrate the differential equation for the radial part. The alternative definition $\xi = \left((m\omega_0)^2 + (\Omega_3 \omega/2)^2\right)^{1/4}$ includes non-commutativity in the very definition of distances and leads to different conclusions; with this latter definition, non-commutativity introduces energy dissipation. However, the importance of this interesting result is questionable since there is no meaningful non-relativistic limit in this approach, as it is shown in appendix A.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Energy eigenvalues in a dissipative context

As mentioned after equation (42), the solution obtained with this definition leads to interesting results, like dissipative quantized energy, but it also causes problems, which we will discuss in this appendix. First, let us return to and analyze equation (34) which covers the radial dependence of the wave function utilizing a different definition of variable that will encompass the non-commutative parameter $\Omega_3$. If we introduce the following parameters with dimension of energy,

$$S = S(m, \omega_0, \omega, \Omega_3) \equiv \left((m\omega_0)^2 + \left(\frac{\Omega_3 \omega}{2}\right)^2\right)^{1/4},$$

(A.1)
\[ T = T(E, m, \omega_0, \omega, \Omega_3) \equiv \left( (E + L\omega)^2 - m^2 + 3m\omega_0 - Km\omega_0 - \frac{1}{\eta} \left( \frac{1}{2} \Omega_3 \right)^2 \right)^{1/2}, \]  

(A.2)

and the dimensionless variable

\[ \xi \equiv \frac{S}{\rho}, \]  

(A.3)

we rewrite equation (34) in the form:

\[ 0 = \left( \frac{1}{\Omega_3} \right) \frac{\xi}{S^3} R(\xi) + \frac{1}{\xi} R(\xi) + \frac{2m\omega_0}{S^2} \omega \xi R(\xi) + \xi^2 R(\xi) \]

\[ + 2 \left( \frac{1}{2} \Omega_3 \right) \frac{(E + L\omega)}{S^3} \xi R(\xi) \]

\[ + \left( \frac{T}{S} \right)^2 R(\xi) - \frac{\Omega_3}{\eta^2} \frac{L}{S^2} R(\xi) - \frac{L^2}{\eta^2} \xi^2 R(\xi). \]  

(A.4)

The solution to this differential equation is

\[ R(\xi) = e^{-\frac{\xi}{2b} \omega \xi} H_B \left[ \frac{2L}{\eta} - \frac{2a}{b^2} - \left( \frac{T}{S} \right)^2 + \left( \frac{2L}{\eta} + 4 \right) \frac{m\omega_0}{S^2} \right] \frac{1}{b^2} - 4i \left( \frac{1}{2} \Omega_3 \right) \frac{L}{\eta^2} S \frac{1}{b} n \xi, \]  

(A.5)

where \( H_B \) is the biconfluent Heun function with the definitions

\[ a = \left( \frac{1}{2} \Omega_3 \right) \frac{(E + L\omega)}{S^3} \left[ \frac{m\omega_0}{S^2} \right], \quad \text{and} \quad \frac{1}{b^2} = \frac{m\omega_0}{S^2} \pm \sqrt{\left( \frac{m\omega_0}{S^2} \right)^2 - 1}. \]  

(A.6)

In order to characterize our physical system, besides the wave function, we need to study the energy and its eventual quantization. Our goal is to treat this problem analytically although in an approximate approach. Accordingly, let us consider the case where the non-commutativity and the rotation of the frame are subdominant effects with respect to the oscillator’s mass and the oscillator’s frequency, i.e. \( \Omega_3 \omega \ll m\omega_0 \). In this case,

\[ b \simeq (-1)^j \left\{ 1 + \frac{i}{2\sqrt{2}} \left( \frac{1}{2} \Omega_3 \omega \right) + \frac{1}{2} \left( \frac{1}{2} \Omega_3 \omega \right)^2 \right\}, \quad j = 1, 2. \]  

(A.7)

Note that the parameter \( b \) is a complex number that exhibits real and imaginary parts. This leads to a complex variable for the KG-oscillator described by the Heun function. Moreover, under the approximation \( \Omega_3 \omega \ll m\omega_0 \):

\[ a \simeq \pm 2^{1/2} \left( \frac{(E + L\omega)}{m\omega_0} \right)^{1/2} \left[ 1 + 2^{1/2} \left( \frac{1}{2} \Omega_3 \omega \right) + \frac{13}{8} \left( \frac{1}{2} \Omega_3 \omega \right)^2 + \cdots \right]. \]  

(A.8)

We are interested in the polynomial version of the Heun function in the radial solution. This will be addressed in the following.
A.1. Energy for the hard-wall in the limit $\rho_0 \to \infty$

Heun functions $H_{\nu}(\alpha, \beta, \gamma, \delta, \lambda)$ can be cast in a polynomial form under the conditions [59]:

$$\gamma - 2 - \alpha = 2n, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad A_{n+1} = 0. \quad (A.9)$$

From a physical perspective, this is necessary in order to avoid divergences. The numbers $A_{n+1}$ are the series coefficients which obey the recurrence relation:

$$A_{l+2} - A_{l+1} \left\{ (l + 1) \beta + \frac{1}{2} \{ \delta + \beta (1 + \alpha) \} \right\}$$

$$+ A_{l} (\gamma - 2 - \alpha - 2l) (l + 1) (l + 1 - \alpha) = 0, \quad l \geq 0, \quad (A.10)$$

with

$$A_0 = 1 \quad \text{and} \quad A_1 = \frac{1}{2} [\delta + \beta (1 + \alpha)]. \quad (A.11)$$

Our equation (A.9) leads to

$$\mathcal{E}_\pm = \pm \left\{ m \omega_0 \left( 2n + 2 + \frac{2L}{\eta} \right) \left[ 1 - (8 \pm 5) \frac{i}{\sqrt{2}} \left( \frac{1}{2} \Omega_{3\omega} \right) \right] \right.$$

$$+ (29 \pm 28) \left( \frac{1}{2} \Omega_{3\omega} \right)^2 \left( \frac{m \omega_0}{\omega \eta} \right)^2 - \left( \frac{m \omega_0}{\omega \eta} \right)^2 \left( \frac{1}{2} \Omega_{3\omega} \right)^2$$

$$- \left[ m^2 + m \omega_0 \left( 1 + K + \frac{2L}{\eta} \right) \right] \left[ 1 - \sqrt{2i} (4 \pm 2) \left( \frac{1}{2} \Omega_{3\omega} \right) \right]$$

$$+ \frac{1}{2} (61 \pm 64) \left( \frac{1}{2} \Omega_{3\omega} \right)^2 \right\}^{1/2} - L \omega. \quad (A.12)$$

For consistency, we expand this expression in powers of $\frac{1}{2} \Omega_{3\omega}$ up to second order, which results in

$$\mathcal{E}_\pm \approx -\omega L \pm \sqrt{m \omega_0 (2n + 1 - K) - m^2}$$

$$\pm \frac{1}{2} \sqrt{2i} \frac{m \omega_0}{\sqrt{m \omega_0 (2n + 1 - K) - m^2}} \left( \frac{1}{2} \Omega_{3\omega} \right)^2$$

$$\pm \frac{1}{2} \left\{ m \omega_0 \left[ n (58 \pm 56) + \left( \frac{32}{\eta} \pm 24 \right) - K \left( \frac{32}{\eta} \pm 32 \right) - \frac{8}{\eta} (3 \pm 8) \right] - \left( \frac{m \omega_0}{\omega \eta} \right)^2 - m^2 \left( \frac{32}{\eta} \pm 32 \right) \right\}$$

$$+ \frac{1}{2} \frac{m \omega_0}{\sqrt{m \omega_0 (2n + 1 - K) - m^2}} \left[ \frac{1}{2} \Omega_{3\omega} \right]^2$$

$$+ \frac{1}{2} \frac{1}{\sqrt{m \omega_0 (2n + 1 - K) - m^2}}. \quad (A.13)$$
The term in the second line contains a complex number, provided the argument of the square root in its denominator is positive; that is, if \( m\omega_0(2n + 1 - K) - m^2 > 0 \). The presence of the complex number in equation (A.13) can be interpreted as a dissipative term; in fact, 
\[
\phi = e^{-iE_0t}\psi = e^{-i\text{Re}(E)t}e^{i\text{Im}(E)t}\psi
\]
. The positive sign in the factor containing \( \text{Im}(E) \) leads to the divergence of the wave function; it is thus an unacceptable choice. The negative sign in the factor containing \( \text{Im}(E) \) attenuates the wave function as the time increases. Therefore, this implies that the half-life of the particle associated to the wave-function is related to the expansion parameter \( \frac{1}{2}\Omega_3 \omega_m \omega_0 \).

Note that the condition \( m\omega_0(2n + 1 - K) - m^2 > 0 \), implies an upper bound on the dimensionless constant \( K \) defined in equation (37), namely,

\[
K < 2n + 1 - \frac{m}{\omega_0}
\]  
(A.14)

This constraint is not arbitrary; in fact, it is mandatory if we expect no dissipation to be observed in the commutative limit.

We notice that the complex number is associated to a coupling between \( \Omega_3 \) and \( \omega \). We could eliminate the complex dissipative term in two ways: (1) with \( \Omega_3 = 0 \), which leads to discard all the information on non-commutativity, or (2) with \( \omega = 0 \), which eliminates frame rotation but keeps the effect of non-commutativity in the last term of equation (A.13).

The global sign in the term of the second line of equation (A.13) depends on the signs of the several physical parameters in it. In order to see that more clearly, we write down equation (A.13) in its two possible forms.

A.2. Energy eigenvalue \( E_+ \)

With the upper sign in equation (A.13), a physical solution requires the second line of equation (A.13) to satisfy

\[
\left[ \omega_0 \left( 6K - 7 - 13n - \frac{L}{\eta} \right) + 6m \right] \left( \frac{1}{2} \Omega_3 \omega \right) \leq 0 \quad \text{(convergence condition)}.
\]  
(A.15)

If the term in the second line of (A.13) is zero, we obtain

\[
\frac{m\omega_0}{\sqrt{m\omega_0(2n + 1 - K) - m^2}} \left( \frac{1}{2} \Omega_3 \omega \right) = 0,
\]  
(A.16)

then the energy dissipation does not occur. The trivial ways to achieve this condition, as pointed out above, are (1) in the commutative context, \( \Omega_3 = 0 \), and (2) for a non-rotating frame, \( \omega = 0 \). Alternatively, this imposes the condition

\[
K = \frac{1}{6} \left( 7 + 13n + \frac{L}{\eta} \right) - \frac{m}{\omega_0} \quad \text{(conservative system)}
\]  
(A.17)

upon the separation constant \( K \). For this particular value of \( K \), the contribution of non-commutativity to the energy is of the second order only—see the three last lines in (A.13). Moreover, this value of \( K \) constrains \( L \) to assume negative values with \( |L/\eta| > n + 1 \) in order to guarantee that the condition \( m\omega_0(2n + 1 - K) - m^2 > 0 \) is satisfied.

The condition for convergence of the solution, equation (A.15), allows for the following possibilities: (1) \( \Omega_3 \omega > 0 \) (non-commutativity and frame rotation in the same direction) and (2)
\( \Omega_3 \omega < 0 \) (non-commutativity and frame rotation in opposite directions). The first possibility implies:

\[
K < \left( 2n + 1 - \frac{m}{\omega_0} \right) + \frac{1}{6} \left( n + 1 + \frac{L}{\eta} \right), \quad (\Omega_3 \omega > 0).
\] (A.18)

This condition is already satisfied for \( n + 1 + L/\eta > 0 \). Otherwise, it is more restrictive than condition (A.14). The second possibility requires

\[
K > \left( 2n + 1 - \frac{m}{\omega_0} \right) + \frac{1}{6} \left( n + 1 + \frac{L}{\eta} \right), \quad (\Omega_3 \omega < 0),
\] (A.19)

which, together with (A.14), demands \( (L/\eta) < - (n + 1) \).

A.3. Energy eigenvalue \( E_- \)

If we select the lower sign in equation (A.13), the second line in \( E_- \) should comply with

\[
\left[ m \omega_0 \left( 2K - 1 - 3n + \frac{L}{\eta} \right) + 2m^2 \right] \left( \frac{1}{2} \frac{\Omega_3 \omega}{m \omega_0} \right) \geq 0, \quad \text{(convergence condition)},
\] (A.20)

in order to avoid divergences of the wave function. If \( \Omega_3 \neq 0 \) and \( \omega \neq 0 \), the equality is attained for

\[
K = - \frac{1}{2} \left( n + 1 + \frac{L}{\eta} \right) + 2n + 1 - \frac{m}{\omega_0}, \quad \text{(conservative system)},
\] (A.21)

which concurs with (A.14) under the requirement \( \frac{L}{\eta} > - (n + 1) \). This would correspond to a system without dissipation of energy. When we admit the possibility for dissipation, the convergence condition opens two possibilities: first, for \( \Omega_3 \omega > 0 \), the relation (A.20) leads to

\[
K > - \frac{1}{2} \left( n + 1 + \frac{L}{\eta} \right) + 2n + 1 - \frac{m}{\omega_0}, \quad (\Omega_3 \omega > 0)
\] (A.22)

which, together with equation (A.14), leads to the same conclusion as before: \( \frac{L}{\eta} > - (n + 1) \).

Second, for \( \Omega_3 \omega < 0 \), the inequality (A.20) gives

\[
K < - \frac{1}{2} \left( n + 1 + \frac{L}{\eta} \right) + 2n + 1 - \frac{m}{\omega_0}, \quad (\Omega_3 \omega < 0).
\] (A.23)

This is automatically satisfied under equation (A.14) for \( \frac{L}{\eta} < - (n + 1) \). Otherwise, the inequality above is more restrictive than (A.14).

When we study the limits of \( E \) related to particular regimes of the physical parameters \( \{\omega_0, \omega, \Omega_3\} \) there appears a caveat, namely, that the non-relativistic limit leads to an imaginary rest mass. The root of this problem resides in the definition of the radial variable in
equations (A.3) and (A.1), which includes the non-commutative parameter $\Omega_3$. This is the reason why we preferred the radial variable $\xi = \sqrt{m\omega_0}$ defined in equations (40) and (41) of section 2.2.

Appendix B. Solution of the radial equation up to order $\epsilon^1$

In this appendix we discuss in details the difficulties encountered when we keep terms up to first power in $\epsilon = \Omega_3 \frac{\omega}{m\omega_0}$ in the radial part of the field equation, equation (43).

We write equation (53) back in terms of $\Omega_3$ and neglect the terms scaling with $(\Omega_3)^2$:

$$0 = R''(\xi) + \frac{1}{\xi} R'(\xi) + 2\xi R'(\xi) + \xi^2 R(\xi) + A\xi R(\xi) + BR(\xi)
- C\xi R(\xi) - D\xi^2 R(\xi),$$

where

$$A \equiv 2 \frac{(E + L\omega)}{\sqrt{m\omega_0}} \left( \frac{1}{2} \Omega_3 \frac{\omega}{m\omega_0} \right),$$

$$B \equiv \frac{1}{(m\omega_0)} \left[ (E + L\omega)^2 - m^2 + (3 - K) m\omega_0 \right],$$

$$C \equiv \frac{2}{\sqrt{m\omega_0}} \left( \frac{1}{2} \Omega_3 \frac{\omega}{\eta} \right) \left( \frac{L}{\eta} \right),$$

$$D \equiv \left( \frac{L}{\eta} \right)^2.$$  

In order to try and solve the previous equation via the Frobenius method, we propose the ansatz

$$R(\xi) = F(\xi) e^{-\frac{1}{2} \xi^2} e^{-\sqrt{2m\omega_0} \xi}. $$

Now we replace this result in equation (B.1) and search solutions by means of the following series expansion for $F(\xi)$:

$$F(\xi) = \sum_{j=0}^{\infty} c_j \xi^{j+r}. $$

Then by substitution into equation (B.1) and rearrangement according to powers of $\xi$, it follows that
\[ c_0 \left( r + \frac{2L}{\eta} \right) = 0, \]
\[ c_1 \left( 1 + r \right) \left( 1 + r + \frac{2L}{\eta} \right) + c_0 \left[ 2\sqrt{2 - B} (1 + r) + \left( \frac{2L}{\eta} + 1 \right) \right] \sqrt{2 - B} - C = 0, \]
\[ c_2 \left( 2 + r \right) \left( 2 + r + \frac{2L}{\eta} \right) + c_1 \left[ 2\sqrt{2 - B} (1 + r) + \left( \frac{2L}{\eta} + 1 \right) \right] \sqrt{2 - B} - C = 0, \]
\[ c_{j+3} \left( j + 3 + r \right) \left( j + 3 + r + \frac{2L}{\eta} \right) + c_{j+2} \left[ 2\sqrt{2 - B} (j + 2 + r) + \left( \frac{2L}{\eta} + 1 \right) \right] \sqrt{2 - B} - C + Ac_j = 0. \]

(B.5)

The series is truncated if there is a maximum value \( n \) for the index \( j \) such that the coefficient multiplying \( c_{j+2} \) vanishes, which leads to

\[ \mathcal{E} = -L\omega \pm \sqrt{m^2 - (1 - K) m \omega_0 - \left( \frac{1}{2} \Omega_3 \right)^2 \left( \frac{L}{\eta} \right)^2 \left[ n + 2 + r + \frac{L}{\eta} + \frac{2}{3} \right]^2}. \]

(B.6)

Moreover, additional constraints upon other coefficients must be imposed: (1) in order to get \( c_{n+3} = 0 \), it is necessary to demand that \( c_n = 0 \); (2) for \( c_{n+4} = 0 \), we need \( c_{n+1} = 0 \); and (3) a \( c_{n+5} = 0 \) already guarantees the truncation of the series but at the cost of requiring \( c_{n+2} = 0 \). However, if \( c_{n+2} = 0 \), then we do not need to kill its coefficient in the series, which ultimately eliminates the energy quantization requirement (B.6).

We could still argue that the energy quantization is valid together with the imposition \( c_{n+2} = 0 \). Then, we would still have to demand that \( c_n = c_{n+1} = 0 \); this amounts to at least four conditions to impose. However, we have only three separation constants \( \{ \mathcal{E}, L, K \} \) to be constrained. On top of this, the consistency of the series truncation should be assessed; this could lead to the undesirable demand for a greater number of constraints, an additional possible caveat that we will not investigate any further.

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29
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