SPACES OF GEODESICS OF PSEUDO-RIEMANNIAN SPACE FORMS AND NORMAL CONGRUENCES OF HYPERSURFACES

HENRI ANCIAUX

Abstract. We describe natural Kähler or para-Kähler structures of the spaces of geodesics of pseudo-Riemannian space forms and relate the local geometry of hypersurfaces of space forms to that of their normal congruences, or Gauss maps, which are Lagrangian submanifolds.

The space of geodesics $L_{\pm}^{\pm}(S^{n+1}_{p,1})$ of a pseudo-Riemannian space form $S^{n+1}_{p,1}$ of non-vanishing curvature enjoys a Kähler or para-Kähler structure $(J, G)$ which is in addition Einstein. Moreover, in the three-dimensional case, $L_{\pm}^{\pm}(S^{n+1}_{p,1})$ enjoys another Kähler or para-Kähler structure $(J', G')$ which is scalar flat. The normal congruence of a hypersurface $S$ of $S^{n+1}_{p,1}$ is a Lagrangian submanifold $\bar{S}$ of $L_{\pm}^{\pm}(S^{n+1}_{p,1})$, and we relate the local geometries of $S$ and $\bar{S}$. In particular $\bar{S}$ is totally geodesic if and only if $S$ has parallel second fundamental form. In the three-dimensional case, we prove that $\bar{S}$ is minimal with respect to the Einstein metric $G$ (resp. with respect to the scalar flat metric $G'$) if and only if it is the normal congruence of a minimal surface $S$ (resp. of a surface $S$ with parallel second fundamental form); moreover $\bar{S}$ is flat if and only if $S$ is Weingarten.

Introduction

After the seminal paper of N. Hitchin [14] describing the natural complex structure of the space of oriented straight lines of Euclidean 3-space, several invariant structures on the space of geodesics of certain Riemannian manifolds and their submanifolds have recently been explored by different authors (see [4], [8], [11], [9], [10], [15], [16], [17], [23], [24]). In [1], a unified viewpoint has been given to this question, classifying all invariant Riemannian, symplectic, complex and para-complex structures that may exist on the space of geodesics of a number of spaces: the Euclidean and pseudo-Euclidean spaces, the Riemannian and pseudo-Riemannian space forms and the complex and quaternionic space forms. One of the interesting issues about the spaces of geodesics is that the normal congruence (or Gauss map) of a one-parameter family of parallel hypersurfaces in some space is a Lagrangian submanifold of the corresponding space of geodesics.

The purpose of this paper is twofold: first, to give a more precise picture of the structure of the space of geodesics of pseudo-Riemannian space forms, and second to study in detail the relationships between the pseudo-Riemannian geometry of a one-parameter family of parallel hypersurfaces and that of its normal congruence.

In particular, we describe the natural Kähler or para-Kähler structure of the space of geodesics of pseudo-Riemannian space forms of non-vanishing curvature...
and prove that the corresponding metric $G$ is Einstein (Theorem 2.1). The space of geodesics of pseudo-Riemannian three-dimensional space forms, which is four-dimensional, is specific since $(i)$ it is the only dimension for which the space of geodesics of flat pseudo-Euclidean spaces enjoys an invariant metric (see [23], [1]), and $(ii)$ in the non-flat case it enjoys another natural complex or para-complex structure, which in turn defines a neutral metric $G'$. We prove that $G'$ is scalar flat and locally conformally flat (Theorem 2.4).

Next we turn our attention to the relation between one-parameter families of parallel hypersurfaces in pseudo-Riemannian space forms and their normal congruences. We first check that an $n$-dimensional geodesic congruence $S$ is Lagrangian if and only if it orthogonally crosses a hypersurface $S$ (Theorem 2.10), and therefore all the hypersurfaces $S_t$ parallel to $S$ and to its polar. Given a one-parameter family of parallel hypersurfaces $(S_t)$ and its normal congruence $\bar{S}$, we relate the first and second fundamental forms of $(S_t)$ to those of $\bar{S}$ (Theorems 2.11 and 2.16). These formulas imply several interesting corollaries: $\bar{S}$ is totally geodesic (either with respect to $G$ or $G'$) if and only if the hypersurfaces $S_t$ have parallel second fundamental form; in the three-dimensional case, $\bar{S}$ is minimal with respect to $G$ if and only if one of the parallel surfaces $S_t$ is minimal (Corollary 2.14); $\bar{S}$ is minimal with respect to $G'$ if and only if the parallel surfaces $S_t$ are totally geodesic (Corollary 2.17); the induced metric on $\bar{S}$ is flat if and only if the surfaces $S_t$ are Weingarten (Corollary 2.18). We also exhibit three families of Lagrangian surfaces which are marginally trapped with respect to $G$ or $G'$ (Corollary 2.20).

The paper is organised as follows: Section 1 provides some useful preliminaries and Section 2 gives the precise statements of results; Section 3 deals with the geometry of the spaces of geodesics, while Section 4 is devoted to normal congruences of hypersurfaces.

1. Preliminaries

1.1. Hypersurfaces in pseudo-Riemannian space forms. Consider the real space $\mathbb{R}^{n+2}$ and endowed with the canonical pseudo-Riemannian metric of signature $(p, n + 2 - p)$, where $0 \leq p \leq n + 1$:

$$\langle \cdot, \cdot \rangle_p := -\sum_{i=1}^{p} dx_i^2 + \sum_{i=p+1}^{n+2} dx_i^2,$$

and the $(n+1)$-dimensional quadric

$$S^{n+1}_{p, \epsilon} = \{ x \in \mathbb{R}^{n+2} \mid \langle x, x \rangle_p = \epsilon \},$$

where $\epsilon = \pm 1$. The metric induced on $S^{n+1}_{p, \epsilon}$ by the canonical inclusion $S^{n+1}_{p, \epsilon} \rightarrow (\mathbb{R}^{n+2}, \langle \cdot, \cdot \rangle_p)$ has signature $(p, n + 1 - p)$ if $\epsilon = 1$ and $(p - 1, n + 2 - p)$ if $\epsilon = -1$, and has constant sectional curvature $K = \epsilon$. Conversely, it is known (see [18]) that any pseudo-Riemannian manifold with constant sectional curvature is, up to a scaling of the metric, locally isometric to one of these quadrics. The transformation

$$A : \mathbb{R}^{n+2} \rightarrow \mathbb{R}^{n+2}
\begin{array}{c}
(x_1, \ldots, x_p, x_{p+1}, \ldots, x_{n+2}) \\
(x_{p+1}, \ldots, x_{n+2}, x_1, \ldots, x_{p+1})
\end{array}
$$

defines an anti-isometry of $S^{n+1}_{p, \epsilon}$ onto $S^{n+1}_{n+2-p, -\epsilon}$. It is therefore sufficient to study the case $\epsilon = 1$. The two Riemannian space forms are $(i)$ the sphere $S^{n+1}_n := S^{n+1}_{0,1}$, which is the only compact quadric, and $(ii)$ the hyperbolic space $\mathbb{H}^{n+1} := \mathbb{R}^{n+1}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let \( \varphi : \mathcal{M}^n \to S_{p,1}^{n+1} \) be a smooth map from an orientable \( n \)-dimensional manifold \( \mathcal{M}^n \). We set \( g := \varphi^* \langle \cdot, \cdot \rangle_p \) for the induced metric on \( \mathcal{M}^n \). We shall always assume that \( \varphi \) is a pseudo-Riemannian immersion, i.e. \( g \) is non-degenerate. This is equivalent to the existence of a unit normal vector field along the immersed hypersurface \( S := \varphi(\mathcal{M}^n) \) that we will denote by \( N \). The curvature of \( S \) may be equivalently described by two tensors: the second fundamental form \( h \) with respect to \( N \), i.e. \( h(X,Y) = g(\nabla_X Y, N) \), where \( \nabla \) denotes the Levi-Civita connection of \( \langle \cdot, \cdot \rangle_p \); the shape operator defined by \( AX = -dN(X) \). They are related by the formula \( g(AX,Y) = h(X,Y) \). The shape operator \( A \) is not necessarily real diagonalizable since it is symmetric with respect to the possibly indefinite metric \( g \). More precisely, \( A \) may be of three types: real diagonalizable, complex diagonalizable, or not diagonalizable at all. In the two-dimensional case, we shall use the existence of a canonical form for \( A \), i.e. the existence of a frame \((e_1, e_2)\) such that the matrices of \( g \) and \( A \) take a simple form (see [20]):

- real diagonalizable case:
  \[
g = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},
\]
  with \( \epsilon_1, \epsilon_2 = \pm 1 \);
- complex diagonalizable case:
  \[
g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} H & \lambda \\ -\lambda & H \end{pmatrix},
\]
  with non-vanishing \( \lambda \);
- non-diagonalizable case:
  \[
g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} H & 1 \\ 0 & H \end{pmatrix}.
\]

1.2. Parallel hypersurfaces. It will be convenient to introduce some notation: we set \((\cos \epsilon, \sin \epsilon) := (\cos, \sin)\) if \( \epsilon = 1 \) and \((\cosh \epsilon, \sinh \epsilon) := (\cosh, \sinh)\) if \( \epsilon = -1 \). Given \( t \in \mathbb{R} \), the image of

\[\varphi_t := \cos t \varphi + \sin t \epsilon N,\]

when an immersion, is parallel to \( S \). When \( A \) is invertible, the map \( N : \mathcal{M}^n \to S_{p,1}^{n+1} \), where \( \epsilon := |N|^2_p \), is an immersion and its image \( S' := N(\mathcal{M}^n) \) is called the polar of \( S \). If \( \epsilon = 1 \), we have \( \varphi_{\pi/2} = N \); hence the polar of \( S \) is parallel to \( S \). If \( \epsilon = -1 \), \( S' \in S_{p,-1}^{n+1} = \mathcal{A}(S_{n+2-p,1}) \). In all cases, a unit normal vector field along \( S_t = \varphi_t(\mathcal{M}^n) \) is

\[N_t := \cos t N - \epsilon \sin t \varphi,
\]

which, when an immersion, is parallel to \( S' \).

**Lemma 1.1.** Let \( \varphi : \mathcal{M}^2 \to S_{p,1}^3 \) be an immersion with mean curvature \( H \) and Gaussian curvature \( K \), which satisfies the following linear Weingarten equation:

\[\frac{2H}{K - \epsilon} = C,\]
where \( C \in [0, \infty) \cup \{ \infty \} \) and \((\epsilon, C) \neq (-1, 1)\). Then there exists a minimal immersed hypersurface which is parallel to \( S := \varphi(M^2) \) or to its polar \( S' \).

Proof. We first compute
\[
d\varphi_t = \cos(t) d\varphi + \sin(t) dN = (\cos(t) Id - \sin(t) A) \circ d\varphi.
\]
Observe that \( \varphi_t \) is an immersion if and only if \( \cos(t) Id - \sin(t) A \) is invertible. When this is the case, we have
\[
A_t = -dN_t = (\cos(t) A + \epsilon \sin(t) Id) \circ (\cos(t) Id - \sin(t) A)^{-1}.
\]
A straightforward calculation shows that the mean curvature \( H_t = trA_t \) of \( \varphi_t \) vanishes if and only if \( \cos(2t) 2H + \sin(2t)(\epsilon - K) \) vanishes as well. If \( \epsilon = 1 \) we get the vanishing of \( H_{t_0} \) setting \( t_0 := 1/2 \tan^{-1}\left(\frac{2H}{K-1}\right) \). If \( \epsilon = -1 \) and \( \left| \frac{2H}{K+1} \right| > 1 \), the same occurs with \( t_0 := 1/2 \tanh^{-1}\left(\frac{2H}{K+1}\right) \). Finally, if \( \epsilon = -1 \) and \( \left| \frac{2H}{K+1} \right| < 1 \), we easily check that \( N_{t_0} := \cosh(t_0) N - \epsilon \sinh(t_0) \varphi \), where \( t_0 := 1/2 \coth^{-1}\left(\frac{2H}{K+1}\right) \), is minimal.

We shall denote by \( \arctane \) the integral of the map \( \frac{1}{1+\epsilon t^2} \), i.e.
\[
\arctane(t) = \begin{cases} 
\tan^{-1}(t) & \text{if } \epsilon = 1, \\
\tanh^{-1}(t) & \text{if } \epsilon = -1, |t| < 1, \\
\coth^{-1}(t) & \text{if } \epsilon = -1, |t| > 1.
\end{cases}
\]

1.3. Lagrangian submanifolds. We first recall the definition of a Lagrangian submanifold:

**Definition 1.2.** Let \((N, \omega)\) be a \(2n\)-dimensional symplectic manifold. An immersion \( \varphi : M^n \to N \) is said to be Lagrangian if \( \varphi^* \omega = 0 \).

We refer the reader to [2] or [7] for material about para-complex geometry (sometimes referred to as split-complex or bi-Lagrangian geometry). By a pseudo-Kähler or a para-Kähler manifold, we mean a manifold equipped with a complex or para-complex structure \( J \) and a compatible pseudo-Riemannian metric \( G \), i.e. such that \( G(J_\cdot, J_\cdot) = \epsilon G(\cdot, \cdot) \). Here, \( \epsilon = 1 \) in the complex case and \( \epsilon = -1 \) in the para-complex case. In other words \( J \) is an isometry in the complex case and an anti-isometry in the para-complex case. It is furthermore required that the symplectic form \( \omega := \epsilon G(J_\cdot, J_\cdot) \) be closed.\(^1\)

Observe that the metric \( G \) is determined by the pair \((J, \omega)\) via the equation \( G := \omega(J_\cdot, J_\cdot) \).

It is well known that the extrinsic curvature of a Lagrangian submanifold in a Kähler manifold \((N, J, G)\) is described by the tri-symmetric tensor \( h(X, Y, Z) := G(D_X Y, JZ) \), where \( D \) denotes the Levi-Civita connection of \( G \) (see [3]). It turns out that the same fact holds in the para-Kähler case. Since the proof is similar, it is omitted.

**Lemma 1.3.** Let \( \mathcal{L} \) be a non-degenerate, Lagrangian submanifold of a pseudo-Kähler or para-Kähler manifold \((N, J, G, \omega)\). Denote by \( D \) the Levi-Civita connection of \( G \). Then the map \( h(X, Y, Z) := G(D_X Y, JZ) \) is tensorial and tri-symmetric,

\(^1\)Of course the factor \( \epsilon \) is unessential here and is put in order to simplify further exposition. In particular, this convention allows us to recover, in the case of \( \mathbb{R}^2 \), the “natural” objects \( G := dx^2 + \epsilon dy^2, J(\partial_x, \partial_y) := (\partial_y, -\epsilon \partial_x) \) and \( \omega := dx \wedge dy \).
It is straightforward that
\[ h(X, Y, Z) = h(Y, X, Z) = h(X, Z, Y). \]

2. Statement of results

2.1. Structures of the space of geodesics of pseudo-Riemannian spaceforms. Let \( x \) be a point of \( S^{n+1}_{p,1} \) and \( v \in T_x S^{n+1}_{p,1} = x^\perp \) a unit vector tangent to \( x \).

Setting \( \epsilon := |v|^2 \), the unique geodesic \( \gamma \) of \( S^{n+1}_{p,1} \) passing through \( x \) with velocity \( v \) is the periodic curve parametrized by \( \gamma(t) = \cos(t)x + \sin(t)v \).

The set \( L^+(S^{n+1}_{p,1}) \) of positive oriented geodesics of \( S^{n+1}_{p,1} \) identifies with the Grassmannian \( Gr^+(n + 2, 2) \) of oriented two-planes of \( \mathbb{R}^{n+2} \) with positive induced metric, while the set \( L^-(S^{n+1}_{p,1}) \) of negative oriented geodesics of \( S^{n+1}_{p,1} \) identifies with the Grassmannian \( Gr^-(n + 2, 2) \) of oriented two-planes of \( \mathbb{R}^{n+2} \) with indefinite induced metric:

\[
L^+(S^{n+1}_{p,1}) \simeq Gr^+_p(n + 2, 2) \simeq \{ x \wedge y \in \Lambda^2(\mathbb{R}^{n+2}) \mid (x, y) \in TS^{n+1}_{p,1}, \langle y, y \rangle_p = 1 \},
\]

\[
L^-(S^{n+1}_{p,1}) \simeq Gr^-_p(n + 2, 2) \simeq \{ x \wedge y \in \Lambda^2(\mathbb{R}^{n+2}) \mid (x, y) \in TS^{n+1}_{p,1}, \langle y, y \rangle_p = -1 \}.
\]

Observe that the anti-isometry \( \Lambda \) induces a canonical one-to-one correspondence between \( Gr^-_p(n + 2, 2) \) and \( Gr^-_{n+2-p}(n+2,2) \), hence between \( L^-(S^{n+1}_{p,1}) \) and \( L^-(S^{n+1}_{n+2-p,1}) = L^-(\Lambda(S^{n+1}_{p,-1})). \)

We may regard \( L^+(S^{n+1}_{p,1}) \) and \( L^-(S^{n+1}_{p,1}) \) as two submanifolds of the pseudo-Euclidean space

\[
\Lambda^2(\mathbb{R}^{n+2}) := \text{Span} \{ e_i \wedge e_j, 1 \leq i < j \leq n + 2 \} \simeq \mathbb{R}^{\binom{n+2}{2} + \binom{n+2}{2}},
\]

where \( (e_1, \ldots, e_{n+2}) \) denotes the canonical basis of \( \mathbb{R}^{n+2} \). This viewpoint allows us to define in a natural way several structures on \( L^\pm(S^{n+1}_{p,1}) \): first, we use the fact that \( \Lambda^2(\mathbb{R}^{n+2}) \) is equipped with the flat pseudo-Riemannian metric

\[
\langle \langle x \wedge y, x' \wedge y' \rangle \rangle := \langle x, x' \rangle_p \langle y, y' \rangle_p - \langle x, y' \rangle_p \langle y, x' \rangle_p;
\]

we shall denote by \( \mathbb{G} \) the induced metric on \( L^\pm(S^{n+1}_{p,1}) \), i.e. \( \mathbb{G} = \iota^* \langle \langle ., . \rangle \rangle \), where \( \iota : L^\pm(S^{n+1}_{p,1}) \rightarrow \Lambda^2(\mathbb{R}^{n+2}) \) is the canonical inclusion. Second, observe that a positive (resp. indefinite) oriented plane is equipped with a canonical complex (resp. para-complex) structure \( J \). Explicitly, given \( \bar{x} = x \wedge y \in Gr^+_{p}(n + 2, 2), \) with \( |x|^2_p = 1 \) and \( |y|^2_p = \epsilon \), we set \( Jx = y \) and \( Jy = -\epsilon x \). In particular, \( J^2 = \epsilon Id \). On the other hand, a tangent vector to \( \iota(L^+(S^{n+1}_{p,1})) \) at the point \( \bar{x} \) takes the form \( x \wedge X + y \wedge Y \), where \( X, Y \in x^\perp \). We then define:

\[
\mathbb{J}(x \wedge X + y \wedge Y) := (Jx) \wedge X + (Jy) \wedge Y = y \wedge X - \epsilon x \wedge Y.
\]

It is straightforward that \( \mathbb{J}^2 = \epsilon Id \big|_{\bar{x}} \), i.e. \( \mathbb{J} \) is an almost complex or para-complex structure.

Theorem 2.1. \( (L^+(S^{n+1}_{p,1}), \mathbb{J}, \mathbb{G}) \) is a 2n-dimensional pseudo-Kähler manifold with signature \((2p, 2(n-p))\) and \((L^-(S^{n+1}_{p,1}), \mathbb{J}, \mathbb{G}) \) is a 2n-dimensional para-Kähler manifold, hence with neutral signature \((n, n)\). In both cases, the metric \( \mathbb{G} \) is Einstein, with scalar curvature \( S = \epsilon 2n^2 \), and is never conformally flat.
Remark 2.2. It is not difficult to check that $G$ and $\mathbb{J}$ are invariant under the natural action of the group $SO(n+2-p,p)$ of isometries of $\mathbb{S}^{n+1}_{p,1}$. Such invariant structures have been studied with the Lie algebra formalism in [1], where in particular it is proved that such an invariant pseudo-Riemannian metric and complex or para-complex structure are unique on $L^{\pm}(\mathbb{S}^{n+1}_{p,1})$, for $n \geq 3$. The fact that $G$ is Einstein has been proved in [19] in the spherical case.

Remark 2.3. The complex structure of $L^{+}(\mathbb{S}^{n+1}_{p,1})$ may be alternatively described by identifying $L^{+}(\mathbb{S}^{n+1}_{p,1})$ with the hyperquadric

$$\left\{ [z_1 : \ldots : z_{n+2}] \mid -\sum_{i=1}^{p} z_i^2 + \sum_{i=p+1}^{n+2} z_i^2 = 0 \right\}$$

of the pseudo-complex projective space $\mathbb{C}P^{n+1}_p$ (see [21]).

In the three-dimensional case, $L^{\pm}(\mathbb{S}^{3}_{p,1})$ enjoys other natural structures, which may be defined as follows: since the orthogonal two-plane $\bar{x}^{\perp}$ admits a canonical orientation (that orientation compatible with the orientations of $\bar{x}$ and $\mathbb{R}^{4}$), it enjoys a canonical complex or para-complex structure $J'$ (depending on whether the induced metric on $\bar{x}^{\perp}$ is positive or indefinite). Hence we set

$$J'(x \wedge x + y \wedge Y) := x \wedge (J'X) + y \wedge (J'Y).$$

We therefore get another almost complex or para-complex structure on $L^{\pm}(\mathbb{S}^{3}_{p,1})$. Finally, we introduce one more tensor: we want to define a pseudo-Riemannian structure $G'$ on $L^{\pm}(\mathbb{S}^{3}_{p,1})$ with the requirement that the pair $(J',G')$ induces the same symplectic structure, up to sign, than that of $(J,G)$. In other words, we require that $\omega(.,.) := \epsilon'G'(J',.,.)$ be the same as $\omega(.,.) := \epsilon G(J,.,.)$. Hence, we must have:

$$G' = \omega(J',.) = \epsilon G(J,.,.) = -\epsilon G(.,J').$$

It turns out that this defines another Kähler or para-Kähler structure:

**Theorem 2.4.** The two-form $G' := -\epsilon G(.,J' \circ J.)$ is symmetric and therefore defines a pseudo-Riemannian metric on $L^{\pm}(\mathbb{S}^{3}_{p,1})$. The Levi-Civita connection of $G'$ is the same as that of $G$, and the structures $(J,G)$ and $(J',G')$ share the same symplectic form $\omega$. Moreover, $(L^{+}(\mathbb{S}^{3}),J',G'), (L^{+}(\mathbb{H}^{3}),J',G'), (L^{-}(dS^{3}),J',G')$ and $(L^{+}(AdS^{3}),J',G')$ are pseudo-Kähler manifolds while $(L^{+}(dS^{3}),J',G')$ and $(L^{-}(AdS^{3}),J',G')$ are para-Kähler manifolds. In all cases, the metric $G'$ has neutral signature $(2,2)$, is scalar flat and locally conformally flat.

**Remark 2.5.** The properties of $G'$ have been derived in [9] in the case of hyperbolic space.

The fact that $(J,G)$ and $(J',G')$ share both the same Levi-Civita connection and the symplectic form implies that they also share some distinguished classes of submanifolds:

**Corollary 2.6.** Lagrangian surfaces, flat and totally geodesic submanifolds in $L^{\pm}(\mathbb{S}^{3}_{p,1})$, are the same for $(J,G)$ and $(J',G')$.

**Remark 2.7.** In some cases, these invariant structures may be defined in a more intuitive way. For example, using the direct sum of self-dual and anti-self-dual
bivectors in \((\Lambda^2(\mathbb{R}^4, \langle \cdot, \cdot \rangle_0), \langle \cdot, \cdot \rangle_0)\) \(\simeq (\mathbb{R}^6, \langle \cdot, \cdot \rangle_0)\), one can prove that \(L^+(S^3) \simeq S^2 \times S^2\) and that

\[
\mathcal{G} = \langle \cdot, \cdot \rangle_0 \oplus \langle \cdot, \cdot \rangle_0, \quad \mathcal{G}' = \langle \cdot, \cdot \rangle_0 \oplus -\langle \cdot, \cdot \rangle_0,
\]

\[
\mathbb{J} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad \mathbb{J}' = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix},
\]

where \((j, \langle \cdot, \cdot \rangle_0)\) is the canonical Kähler structure of \(S^2\) (see [5]). Analogously, in \((\Lambda^2(\mathbb{R}^4, \langle \cdot, \cdot \rangle_2), \langle \cdot, \cdot \rangle)\) \(\simeq (\mathbb{R}^6, \langle \cdot, \cdot \rangle_3)\) the Hodge operator is para-complex and we still have a direct sum of self-dual and anti-self-dual bivectors. A computation then shows that \(L^+(AdS^3) \simeq \mathbb{H}^2 \times \mathbb{H}^2\) and \(L^-(AdS^3) \simeq dS^2 \times dS^2\), and again we could describe \((\mathbb{J}, G)\) and \((\mathbb{J}', G')\) as product structures built from the canonical Kähler and para-Kähler structures of \(\mathbb{H}^2\) and \(dS^2\) respectively. On the other hand, the Hodge operator being complex in \((\Lambda^2(\mathbb{R}^4, \langle \cdot, \cdot \rangle_1), \langle \cdot, \cdot \rangle)\) \(\simeq (\mathbb{R}^6, \langle \cdot, \cdot \rangle_2)\), there is no natural direct sum of it into eigenspaces, and it does not seem possible a priori to describe \(L^\pm(dS^3)\) and \(L^+(\mathbb{H}^3)\) as a Cartesian product of two surfaces.

**Remark 2.8.** Since the two complex or para-complex structures \(\mathbb{J}\) and \(\mathbb{J}'\) commute, their composition \(\mathbb{J}'' := \mathbb{J} \circ \mathbb{J}'\) defines one more invariant structure: if \(\mathbb{J}\) and \(\mathbb{J}'\) are both complex or both para-complex, then \(\mathbb{J}''\) is complex, and if \(\mathbb{J}\) and \(\mathbb{J}'\) are of different types, \(\mathbb{J}''\) is para-complex. The two-form \(G'' := \omega(\cdot, \cdot)\) is not symmetric, so there is no pseudo- or para-Kähler structure associated to \(\mathbb{J}''\).

Observe also that the triple \((\mathbb{J}, \mathbb{J}', \mathbb{J}'')\) is not a para-quaternionic structure, since \(\mathbb{J}\) and \(\mathbb{J}'\) commute rather than anti-commute. The case \(L^-(AdS^3)\) excepted, this triple is what is called an *almost product bi-complex* structure in [6].

### Table 1. Structures on \(L^\pm(S^3_{p,1})\)

| Space form | Space of geodesics | \((\epsilon, \epsilon')\) | Signature of \(G\) | \(\mathbb{J}\) | \(\mathbb{J}'\) | \(\mathbb{J}''\) |
|------------|-------------------|-----------------|-----------------|---------------|---------------|---------------|
| \(S^3_{0,1} = S^3\) | \(L(S^3)\) | \((1, 1)\) | \((+, +, +, +)\) | complex | complex | para |
| \(S^3_{1,1} = dS^3\) | \(L^+(dS^3)\) | \((1, -1)\) | \((+, - ,+ , -)\) | complex | para | complex |
| \(L^-(dS^3) \simeq L^-(\mathbb{H}^3)\) | \((-1, -1)\) | \((+, + , - , -)\) | para | complex | complex |
| \(S^3_{2,1} \simeq AdS^3\) | \(L^+(AdS^3)\) | \((1, 1)\) | \((- , - , - , -)\) | complex | complex | para |
| \(L^-(AdS^3)\) | \((-1, -1)\) | \((+, - , - , +)\) | para | para | para |
| \(S^3_{3,1} \simeq \mathbb{H}^3\) | \(L^-(\mathbb{H}^3) \simeq L^-(dS^3)\) | \((-1, 1)\) | \((- , - , + , +)\) | para | complex | complex |

2.2. Normal congruences of immersed hypersurfaces as Lagrangian submanifolds.

**Definition 2.9.** Let \(\mathcal{S}\) be an immersed surface of pseudo-Riemannian space form \(S^p_{p+1}\) with unit normal vector \(N\). The *normal congruence* (or *Gauss map*) \(\mathcal{S}\) of \(\mathcal{S}\) is the set of geodesics crossing \(\mathcal{S}\) orthogonally in the direction \(N\).
Theorem 2.10. Let \( \varphi \) be a pseudo-Riemannian immersion of an orientable manifold \( M^n \) in pseudo-Riemannian space form \( S^{n+1}_{p,1} \) with unit normal vector \( N \). Then the normal congruence of \( S := \varphi(M^n) \), i.e. the image of the immersion \( \bar{\varphi} : M^n \to L^\pm(S^{n+1}_{p,1}) \) defined by \( \bar{\varphi} = \varphi \wedge N \), is Lagrangian with respect to \( \omega \). In this case, \( \bar{S} \) is also the normal congruence of the hypersurfaces parallel to \( S \) and to its polar \( S' \). Conversely, let \( \bar{\varphi} : M^n \to L^\pm(S^{n+1}_{p,1}) \) be an immersion of a simply connected \( n \)-manifold. Then \( \bar{S} \) is the normal congruence of an immersed hypersurface of \( S^{n+1}_{p,1} \) if and only if \( \bar{\varphi} \) is Lagrangian.

In view of this result, it is natural to relate the geometry of a Lagrangian submanifold to that of the corresponding hypersurface of \( S^{n+1}_{p,1} \).

Theorem 2.11. Let \( \varphi \) be a pseudo-Riemannian immersion of an orientable manifold \( M^n \) in pseudo-Riemannian space form \( S^{n+1}_{p,1} \) with unit normal vector \( N \). Set \( |N|^2_p := \epsilon \), and denote by \( A \) the shape operator of \( \varphi \) with respect to \( N \) and by \( \nabla^g \) the Levi-Civita connection of \( g \). Then the induced metric \( \bar{g} := \bar{\varphi}^* G \), with \( \bar{\varphi} = \varphi \wedge N \), is given by the following formula:

\[
\bar{g} = \epsilon g + g(A., A.).
\]

In particular, \( \bar{g} \) is non-degenerate if and only if \( \epsilon \text{Id} + A^2 \) is invertible.

Moreover, the extrinsic curvatures \( h \) of \( S := \varphi(M^n) \) and of \( \bar{h} \) of \( \bar{S} := \bar{\varphi}(M^n) \) are related by the formula

\[
\bar{h} = \epsilon \nabla^g h.
\]

In particular, the normal congruence \( \bar{S} \) is totally geodesic if and only if \( S \) has parallel second fundamental form.

Remark 2.12. The fact that the tensor \( \bar{h} \) of \( \bar{S} \) is tri-symmetric is equivalent to the Codazzi equation for the hypersurface \( S \).

Corollary 2.13. If the shape operator \( A \) of \( S \) is real diagonalizable (this is always the case if \( \epsilon = 1 \)), the mean curvature vector of \( \bar{S} \) with respect to \( G \) is

\[
\bar{H} = -\frac{\epsilon}{n} \nabla \left( \sum_{i=1}^{n} \arctan(\kappa_i) \right),
\]

where \( \kappa_1, ..., \kappa_n \) are the principal curvatures of \( S \) and \( \nabla \) is the gradient with respect to the induced metric \( \bar{g} \). In particular, if \( S \) is isoparametric (i.e. its principal curvatures are constant) or austere (i.e. the set of its principal curvatures is symmetric with respect to 0), then its normal congruence \( \bar{S} \) is \( G \)-minimal.

Corollary 2.14. If \( n = 2 \), the mean curvature vector of \( \bar{S} \) with respect to \( G \) is

\[
\bar{H} = -\frac{\epsilon}{2} \nabla \arctan \left( \frac{2H}{1 - \epsilon K} \right),
\]

where \( H \) and \( K \) denote the mean curvature and the Gaussian curvature of \( S \) respectively. In particular, \( \bar{S} \) is \( G \)-minimal if and only if it is the normal congruence of a minimal surface.

Remark 2.15. Corollaries 2.13 and 2.14 have been proved in [22] in the spherical case. The fact that the mean curvature vector takes the form \( \bar{H} = \frac{\epsilon}{n} \nabla \beta \), where \( \beta \) is an \( S^1 \)-valued map, is due to the fact that the metric \( G \) is Einstein (cf [13]). The map \( \beta \) is called the Lagrangian angle of the submanifold \( \bar{S} \).
In the three-dimensional case, it is natural to study the pseudo-Riemannian geometry of Lagrangian surfaces of \( L^\pm(S^3_{p,1}) \) with respect to the metric \( G' \) described in Theorem 2.4.

**Theorem 2.16.** Let \( \varphi \) be a pseudo-Riemannian immersion of an orientable surface \( \mathcal{M}^2 \) in pseudo-Riemannian space form \( S^3_{p,1} \) with shape operator \( A \) and unit normal vector \( N \). Then the induced metric \( \bar{g}' := \varphi^*G' \), with \( \bar{\varphi} = \varphi \wedge N \), is given by the following formula:

\[
\bar{g}' = g(\cdot, (AJ' - J'A)\cdot).
\]

Moreover,
- If \( A \) is real diagonalizable, the metric \( \bar{g}' \) is degenerate at umbilic points of \( S := \varphi(\mathcal{M}^2) \) and indefinite elsewhere. The null directions of \( \bar{g}' \) are the principal directions of \( S \).
- If \( A \) is complex diagonalizable, the metric \( \bar{g}' \) is everywhere definite.
- If \( A \) is not diagonalizable, the metric \( \bar{g}' \) is everywhere degenerate.

When \( \bar{g}' \) is not degenerate, the extrinsic curvatures \( h \) and \( \bar{h} \) of \( S \) and \( \bar{S} := \bar{\varphi}(\mathcal{M}^2) \) are related by the formula

\[
\bar{h} = \epsilon \nabla^g h.
\]

In particular, the normal congruence \( \bar{S} \) of \( S \) is totally geodesic if and only if \( S \) has parallel second fundamental form.

**Corollary 2.17.** \( \bar{S} \) is \( G' \)-minimal if and only if it is totally geodesic, i.e. \( S \) has parallel second fundamental form. In addition \( A \) is real diagonalizable, \( S \) is the set of equidistant points to a geodesic of \( S^3_{p,1} \).

**Corollary 2.18.** The induced metric \( \bar{g}' \) is flat (and the metric \( \bar{g} \) as well by Corollary 2.6) if and only if the surface \( S \) is Weingarten, i.e. there exists a functional relation \( f(H, K) = 0 \) satisfied by the mean curvature and the Gaussian curvature of \( S \).

**Remark 2.19.** Corollaries 2.17 and 2.18 have been proved in the case of hyperbolic space in [8] and [10] respectively. Corollary 2.18 has been proved in the case of Euclidean space in [12].

**Corollary 2.20.** If the shape operator \( A \) of \( S \) is not diagonalizable, then its normal congruence \( \bar{S} \) is a \( G' \)-marginally trapped surface, i.e. the mean curvature vector of \( \bar{S} \) with respect to \( G \) is null. If \( S \) is a tube (i.e. the set of equidistant points to an arbitrary curve of \( S^3_{p,1} \)) or a surface of revolution, then its normal congruence \( \bar{S} \) is a \( G' \)-marginally trapped surface.

## 3. The geometry of the space of geodesics

### 3.1. The Einstein metric \( G \) (Proof of Theorem 2.1).

#### 3.1.1. The second fundamental form of \( h' \) and the complex structure \( J \). Let \( \bar{x} := x \wedge y \in L^\pm(S^3_{p,1}) \) with \( |x|^2_p = 1 \) and \( |y|^2_p = \epsilon \) and let \( (e_1, \ldots, e_n) \) be an orthonormal basis of the orthogonal complement of \( x \wedge y \). We set \( e_i := |e_i|^2_p \) and \( \epsilon_{n+1} := \epsilon e_i \). Then an orthonormal basis \( (E_a)_{1 \leq a \leq 2n} \) of \( T_x L^\pm(S^3_{p,1}) \), with \( G(E_a, E_a) = \epsilon_a \), is given by

\[
E_i := x \wedge e_i \quad \text{and} \quad E_{n+i} := y \wedge e_i.
\]
Fix the index $i$ and introduce the curve
\[
\gamma_i(t) := x \wedge y_i(t) := x \wedge (\cos \epsilon_{n+i}(t) y + \sin \epsilon_{n+i}(t) e_i).
\]
In particular, $\gamma_i(0) = \bar{x}$ and $\gamma_i'(0) = E_i$. Introduce furthermore the following orthonormal frame $\bar{V} = (\bar{v}_1, \ldots, \bar{v}_{2n})$ along $\gamma_i$:
\[
\bar{v}_j(t) := x \wedge e_j, \quad \bar{v}_{n+j}(t) := y_i(t) \wedge e_j, \quad \text{if } j \neq i,
\]
\[
\bar{v}_i(t) := x \wedge y'_i(t), \quad \bar{v}_{n+i}(t) := y_i(t) \wedge y'_i(t).
\]
Observe that $\bar{v}_a(0) = E_a, \forall a, 1 \leq a \leq 2n$, and that the frame $\bar{V}$ is parallel along $\gamma_i$. On the other hand, it is easily checked that $D_{E_i} \mathbb{J} = \mathbb{J} D_{E_i}$, so $\mathbb{J}$ is parallel, and therefore integrable.

We now proceed to compute the second fundamental form of the immersion $\iota$.

**Proposition 3.1.** The second fundamental form of the embedding $\iota : L^\pm(S_p^{n+1}) \to \Lambda^2(\mathbb{R}^{n+2})$ is given by the formula
\[
h^i(v \wedge V, w \wedge W) = -\langle v, w \rangle_p \langle V, W \rangle_p \bar{x} + \varpi(v, w)V \wedge W,
\]
where $\varpi$ is the symplectic form of the plane $\bar{x}$ defined by $\varpi(\cdot, \cdot) = \epsilon(J \cdot, \cdot)_p$.

**Proof.** We have
\[
h^i(E_i, E_j) = (D_{\gamma_i'} \bar{v}_j)\perp = \bar{v}'_j(0) = -\epsilon_{n+i} \delta_{ij} \bar{x},
\]
\[
h^i(E_i, E_{n+j}) = (D_{\gamma_i'} \bar{v}_{n+j})\perp = \bar{v}_{n+j}'(0) = \epsilon_i \wedge e_j.
\]
An analogous computation, using the curve $\gamma_{n+i}(t) = (\cos \epsilon_{i}(t) x - \sin \epsilon_{i}(t) e_i) \wedge y$, implies that
\[
h^i(E_{n+i}, E_{n+j}) = -\epsilon_i \delta_{ij} \bar{x}.
\]
The claimed formula follows from the bi-linearity of $h^i$. \hfill \square

3.1.2. The curvature of $\mathbb{G}$. We use the Gauss equation and Proposition 3.1 in order to compute the curvature tensor $\bar{R}$ of $\mathbb{G}$: for $1 \leq a, b, c, d \leq 2n$, we have
\[
\mathbb{G}(\bar{R}(E_a, E_b)E_c, E_d) = \langle \langle h^i(E_a, E_c), h^j(E_b, E_d) \rangle \rangle - \langle \langle h^i(E_a, E_d), h^j(E_b, E_c) \rangle \rangle.
\]
For example, we calculate
\[
\mathbb{G}(\bar{R}(E_i, E_j)E_k, E_l) = \epsilon_{n+i} \epsilon_{n+j} \langle \bar{x}, \bar{x} \rangle (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}) = \epsilon \epsilon_i \epsilon_j (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{kj}).
\]
This expression vanishes unless $\{k, l\} = \{i, j\}$ and $i \neq j$, in which case it becomes
\[
\mathbb{G}(\bar{R}(E_i, E_j)E_i, E_j) = -\mathbb{G}(\bar{R}(E_i, E_j)E_j, E_i) = \epsilon \epsilon_i \epsilon_j.
\]
It is now easy to calculate the Ricci curvature of $\bar{R}$:
\[
\bar{R}ic(E_i, E_j) = \sum_{a=1}^{2n} \mathbb{G}^{aa} \mathbb{G}(\bar{R}(E_i, E_a)E_j, E_a)
\]
\[
= \sum_{k=1}^{n} \left( \mathbb{G}^{kk} \mathbb{G}(\bar{R}(E_i, E_k)E_j, E_k) + \mathbb{G}^{n+k,n+k} \mathbb{G}(\bar{R}(E_i, E_{n+k})E_j, E_{n+k}) \right)
\]
\[
= \sum_{k=1, k \neq i}^{n} \epsilon_k (\delta_{ij} \epsilon_k \epsilon_i) + \sum_{k=1}^{n} \epsilon_{n+k} (\delta_{ik} \delta_{jk} \epsilon_i \epsilon_k)
\]
\[
= \epsilon n \mathbb{G}_{ij}.
\]
Analogous calculations show that \( \overline{Ric}(E_{n+i}, E_{n+j}) = \varepsilon n \mathcal{G}_{n+i,n+j} \) and that \( \overline{Ric}(E_i, E_{n+j}) = 0 \). Hence the metric \( \mathcal{G} \) is Einstein, with constant scalar curvature \( \bar{S} = \varepsilon 2n^2 \).

Finally, since \( \mathcal{G} \) is Einstein, the Weyl tensor is given by the formula
\[
W^\mathcal{G} = \mathcal{G} (\bar{\mathcal{R}},.) - \frac{\bar{S}}{4n(2n-1)} \mathcal{G} \circ \mathcal{G} = \mathcal{G} (\bar{\mathcal{R}},.) - \frac{\varepsilon n}{2(2n-1)} \mathcal{G} \circ \mathcal{G}.
\]
It is easily seen, for example, that \( \mathcal{G} \circ \mathcal{G} (E_i, E_j, E_{n+i}, E_{n+j}) \) vanishes. On the other hand, if \( i \neq j \), \( \mathcal{G} (\bar{\mathcal{R}}(E_i, E_j), E_{n+i}, E_{n+j}) = \varepsilon_i \varepsilon_j \), so \( W^\mathcal{G} \) does not vanish and therefore \( \mathcal{G} \) is never conformally flat.

3.2. The scalar flat metric \( \mathcal{G}' \) in dimension \( n = 2 \) (Proof of Theorem 2.4). We are going to express all the relevant tensors in the orthonormal basis \( (E_1, E_2, E_3, E_4) \) of \( T^*_\mathcal{L}(\mathbb{S}^{n+1}) \). Observe first that the matrix of \( \mathcal{G} \) in this basis is
\[
\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2, \varepsilon_1 \varepsilon_2) \text{ and } J = \begin{pmatrix} 0 & 0 & -\varepsilon & 0 \\ 0 & 0 & 0 & -\varepsilon \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } J' = \begin{pmatrix} 0 & -\varepsilon' & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon' \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]
It follows that
\[
\varepsilon J J' = \varepsilon J' J = \begin{pmatrix} 0 & 0 & 0 & \varepsilon' \\ 0 & 0 & -1 & 0 \\ 0 & -\varepsilon' & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \end{pmatrix}.
\]
Hence, taking into account that \( \varepsilon' = \varepsilon_1 \varepsilon_2 \), the matrix of the bilinear form \( \mathcal{G}' := -\varepsilon \mathcal{G}(.,J \circ J'.,) \) in the basis \( (E_a)_{1 \leq a \leq 4} \) is
\[
\mathcal{G}' = \begin{pmatrix} 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & -\varepsilon_2 & 0 \\ 0 & -\varepsilon_2 & 0 & 0 \\ \varepsilon_2 & 0 & 0 & 0 \end{pmatrix}.
\]

The fact that \( \mathcal{G} \) and \( \mathcal{G}' \) have the same Levi-Civita connection follows from the next lemma:

**Lemma 3.2.** Let \( (\mathcal{N}, \mathcal{G}) \) be a pseudo-Riemannian manifold with Levi-Civita connection \( D \) and \( T \) a symmetric, \( D \)-parallel \((1,1)\) tensor. Then the Levi-Civita connection of the pseudo-Riemannian metric \( \mathcal{G}'(.,.) := \mathcal{G}(.,T.) \) is \( D \).

**Proof.** Elementary using local coordinates and the explicit formula for the Christoffel symbols. \( \square \)

Since \( \mathcal{G} \) and \( \mathcal{G}' \) have the same Levi-Civita connection, they have the same curvature tensor \( \bar{\mathcal{R}} \). Therefore,
\[
\mathcal{G}'(\bar{\mathcal{R}}(.,.) , .) := -\varepsilon \mathcal{G}(\bar{\mathcal{R}}(.,.) , J \circ J'.).)
\]
Then by an elementary calculation we obtain
\[
-\overline{Ric}'(X,Y) = \overline{Ric}(X,Y) = \epsilon 2 \mathcal{G}(X,Y).
\]
It follows that the scalar curvature of $\mathbb{G}'$ vanishes:

$$S' = \sum_{a,b=1}^{4} (\mathbb{G}')^{ab} R_{ab} = -2e \sum_{a,b=1}^{4} (\mathbb{G}')^{ab} G_{ab} = 0.$$ 

It may be interesting to point out that the Ricci curvature of $\mathbb{G}'$ is non-negative in the case of $L(S^3)$, non-positive in the case of $L^+(AdS^3)$, and indefinite in the other cases.

Finally, since $\mathbb{G}'$ is scalar flat, its Weyl tensor is given by the formula

$$W^{\mathbb{G}'} = \mathbb{G}'(\bar{R}, \cdot) - \frac{1}{2} \bar{Ric} \circ \mathbb{G}' = G(\bar{R}, e \mathbb{J} \circ \mathbb{J}', \cdot) - eG \circ \mathbb{G}' + L.$$ 

We may calculate, for example, that

$$W^{\mathbb{G}'}(E_1, E_2, E_3, E_4) = \mathbb{G}(\bar{R}(E_1, E_2)E_1, e \mathbb{J} \circ \mathbb{J}' E_4) - e \mathbb{G} \circ \mathbb{G}'(E_1, E_2, E_3, E_4) = -e' \epsilon_1 \epsilon_2 + e \mathbb{G}(E_2, E_3) \mathbb{G}'(E_1, E_4) = -e + e \epsilon_2 \epsilon_2 = 0.$$ 

It is easily checked in the same manner that the other components of the Weyl tensor vanish. The metric $\mathbb{G}'$ is therefore locally conformally flat.

4. Normal congruences of hypersurfaces and Lagrangian submanifolds

4.1. Lagrangian submanifolds are normal congruences (Proof of Theorem 2.10). Let $\varphi: \mathcal{M}^n \to \mathbb{S}^{n+1}_{p,1}$ be an immersed, orientable hypersurface with non-degenerate metric and unit normal vector $N$ and introduce the map

$$\bar{\varphi}: \mathcal{M}^n \to L^\pm(\mathbb{S}^{n+1}_{p,1}), \quad \bar{x} \mapsto \varphi(x) \wedge N(x).$$

In the following, we shall often allow the abuse of notation in identifying a tangent vector $X$ to $\mathcal{M}^n$ with its image $d\varphi(X)$, a vector tangent to $\mathbb{S}^{n+1}_{p,1}$, and therefore an element of $\mathbb{R}^{n+2}$. We furthermore set $\bar{X} := d\bar{\varphi}(X)$, so that

$$\bar{X} := d\bar{\varphi}(X) = d(\varphi \wedge N)(X) = d\varphi(X) \wedge N + \varphi \wedge dN(X) = X \wedge N + AX \wedge \varphi.$$ 

It follows that $\bar{\varphi}$ is Lagrangian since

$$\omega(\bar{X}, \bar{Y}) = e\mathbb{G}(X \wedge (JN) + AX \wedge (J\varphi), Y \wedge N + AY \wedge \varphi) = (\langle X, AY \rangle_p (JN, \varphi)_p + \langle AX, Y \rangle_p (J\varphi, N)_p) = -\langle X, AY \rangle_p + \langle AX, Y \rangle_p = 0.$$ 

Conversely, let $\bar{S}$ be an $n$-dimensional geodesic congruence, i.e. the image of an immersion $\bar{\varphi}: \mathcal{M}^n \to L^\pm(\mathbb{S}^{n+1}_{p,1})$. We shall investigate under which condition there exists a hypersurface $S$ of $\mathbb{S}^{n+1}_{p,1}$ which intersects orthogonally the geodesics $\bar{\varphi}(x)$, $\forall x \in \mathcal{M}^n$. For this purpose set $\bar{\varphi}(x) := e_1(x) \wedge e_2(x)$ with $|e_1|_p^2 = 1$ and $|e_2|_p^2 = e$. Let $\varphi: \mathcal{M}^n \to \mathbb{S}^{n+1}_{p,1}$ such that $\varphi(x) \in \bar{\varphi}(x)$, $\forall x \in \mathcal{M}^n$. Therefore there exists $t: \mathcal{M}^n \to \mathbb{S}^1$, such that $\varphi(x) = e_1(x) \cos \epsilon(t(x)) + e_2(x) \sin \epsilon(t(x))$. Remember that $J$ denotes the complex or para-complex structure on $\bar{\varphi}(x)$, in particular $J\varphi = e_2 \cos \epsilon(t) - e_1 \sin \epsilon(t)$. It is easily seen that $S$ intersects the geodesic $\bar{\varphi}(x) = $
We now discuss the degeneracy of $\bar{\varphi}$ such that $\epsilon \text{Id}$ vanishes. Hence, $\bar{S}$ is the normal congruence of $S$ if and only if there exists $t : \mathcal{M}^n \to \mathbb{S}^1$ such that $\langle de_1, e_2 \rangle_p = -\epsilon dt$. Since $\mathcal{M}^n$ is simply connected, it is sufficient to have $d\langle de_1, e_2 \rangle_p = 0$. On the other hand,

$$\mathbb{J}d\bar{\varphi} = \mathbb{J}(de_1 \wedge e_2 + e_1 \wedge de_2) = -de_1 \wedge e_1 + e_2 \wedge de_2,$$

so that

$$\omega(d\bar{\varphi}(X), d\bar{\varphi}(Y)) = \langle \langle de_1(X) \wedge e_2 + e_1 \wedge de_2(X), -de_1(Y) \wedge e_1 + e_2 \wedge de_2(Y) \rangle \rangle = -\langle de_1(X), de_2(Y) \rangle_p + \langle de_1(Y), de_2(X) \rangle_p = -d\langle de_1, e_2 \rangle_p(X, Y).$$

We conclude that $t$, and thus $\varphi$ as well, exists if and only if $\bar{\varphi}$ is Lagrangian. Of course, the choice of different constants of integration when solving $t$ corresponds to different, parallel hypersurfaces.

### 4.2. Geometry of Lagrangian submanifolds with respect to the Einstein metric $\mathbb{G}$

#### 4.2.1. The induced metric $\bar{g} = \bar{\varphi}^* \mathbb{G}$ and the second fundamental form $\bar{h}$ (Proof of Theorem 2.11)

Using the description of the metric $\mathbb{G}$ given in Section 3.1, we have:

$$\bar{g}(X, Y) = \mathbb{G}(X \wedge N + AX \wedge \varphi, Y \wedge N + AY \wedge \varphi) = \langle X, Y \rangle_p(N, N)_p + \langle \varphi, Y \rangle_p(AX, N)_p - \langle \varphi, N \rangle_p(AX, Y)_p + \langle X, \varphi \rangle_p(N, AY)_p - \langle X, AY \rangle_p(N, \varphi)_p + \langle AX, AY \rangle_p(\varphi, \varphi)_p - \langle AY, \varphi \rangle_p(AX, \varphi)_p = \epsilon g(X, Y) + g(AX, AY).$$

We now discuss the degeneracy of $\bar{g}$: suppose there exist $X$ such that

$$\bar{g}(X, Y) = \epsilon g(X, Y) + g(AX, AY) = g(\epsilon X + A^2 X, Y)$$

vanishes $\forall Y \in TM$. Since the metric $g$ is non-degenerate, it follows that $\epsilon X + A^2 X$ vanishes. Hence $\epsilon Id + A^2$ is not invertible. If $A$ is diagonalizable, the eigenvalues of $A^2$ are non-negative, so we must have $\epsilon = -1$.

Next, denoting by $\nabla$ (resp. $D$) the flat connection of $\mathbb{R}^{n+2}$ (resp. $\Lambda^2(\mathbb{R}^{n+2})$), we have

$$D_X \bar{Y} = (\nabla_X Y) \wedge N + (\nabla_X AY) \wedge \varphi,$$

so

$$\bar{h}(X, Y, Z) = \mathbb{G}(\nabla_X Y \wedge N + \nabla_X AY \wedge \varphi, Z \wedge (JN) + AZ \wedge (J\varphi)) = \langle \nabla_X Y, AZ \rangle_p(N, N)_p - \epsilon \langle \nabla_X AY, Z \rangle_p(\varphi, \varphi)_p = \epsilon \left( h(\nabla_X Y, Z) - (X(\langle AY, Z \rangle_p) - \langle AY, \nabla_X Z \rangle_p) \right) = \epsilon (\nabla_X h)(Y, Z).$$
2.13. The mean curvature vector in the diagonalizable case (Proof of Corollary 4.2.2). Assume that \( A \) is real diagonalizable and let \((e_1, \ldots, e_n)\) be an orthonormal frame \((e_1, \ldots, e_n)\) on \((T\mathcal{M}, g)\), with \(e_i := g(e_i, e_i)\) and such that \(Ae_i = \kappa_ie_i\), where \(\kappa_1, \ldots, \kappa_n\) are the principal curvatures of \(\mathcal{S}\).

We introduce the notation \(\omega^i_{jk} := g(\nabla_{e_i} e_j, e_k)\). In particular, \(\omega^i_{jk}\) is anti-symmetric in its lower indices. It follows that

\[
\bar{g}(e_i, e_j) = 0 \text{ if } i \neq j, \quad \text{and} \quad \bar{g}(e_i, e_i) = ee_i + \epsilon_i \kappa_i^2 = \epsilon_i (\epsilon + \kappa_i^2).
\]

Moreover, if \(j \neq k\),

\[
\bar{h}(e_i, e_j, e_k) = \epsilon \left( h(\nabla_{e_i} e_j, e_k) + h(e_j, \nabla_{e_i} e_k) - e_i h(e_j, e_k) \right) = \epsilon (\kappa_k - \kappa_j) \omega^i_{jk}
\]

and

\[
\bar{h}(e_i, e_j, e_j) = \epsilon \left( 2h(\nabla_{e_i} e_j, e_j) - e_i h(e_j, e_j) \right) = -\epsilon \kappa_i (\kappa_j).
\]

For further use, observe that the tri-symmetry of \(\bar{h}\), or equivalently the Codazzi equation of the immersion \(\varphi\), implies

\[
(\kappa_j - \kappa_i) \omega^i_{ij} = \epsilon \kappa_i (\kappa_j).
\]

Since the basis \((e_1, \ldots, e_n)\) is orthogonal with respect to the metric \(\bar{g}\), we have

\[
\mathcal{G}(n\bar{H}, Jd\bar{\varphi}(e_i)) = \sum_{j=1}^{n} \frac{\bar{h}(e_j, e_j, e_i)}{\bar{g}(e_j, e_j)} = -\sum_{j=1}^{n} \frac{e_i (\kappa_j)}{1 + \epsilon \kappa_j^2} = e_i (\beta),
\]

where \(\beta := -\sum_{j=1}^{n} \arctan \epsilon (\kappa_j)\). Hence \(\bar{H} = \frac{\beta}{n} J\nabla \beta\).

Clearly the immersion \(\bar{\varphi}\) is \(\mathcal{G}\)-minimal if and only if the map \(\beta\) is constant. This happens of course if the principal curvatures of \(\mathcal{S}\) are constant, i.e. it is isoparametric. Moreover, if \(\mathcal{S}\) is austere, i.e. the set of the principal curvatures is symmetric with respect to 0, the Lagrangian angle \(\beta\) vanishes because the function \(\arctan \epsilon\) is odd. This completes the proof of Corollary 2.13.

4.2.3. The mean curvature vector in the two-dimensional case (Proof of Corollary 2.13). Here and in the next section, we shall make use of canonical form of \(A\) (see Section 1.1 and [20]).

The real diagonalizable case

By the computation of the previous section:

\[
\beta = -(\arctan \epsilon (\kappa_1) + \arctan \epsilon (\kappa_2)) = \arctan \left( \frac{2H}{1 - \epsilon K} \right)
\]

which is the required expression of the Lagrangian angle \(\beta\). We now prove that if \(\beta\) is constant, the assumptions of Lemma 1.1 are satisfied. Assume by contradiction that \((\epsilon, \frac{2H}{K - \epsilon}) = (-1, \pm 1)\). It follows that \(\frac{\kappa_1 + \kappa_2}{\kappa_1 \kappa_2 + 1} = \pm 1\), which in turn implies that \(|\kappa_1| = |\kappa_2| = 1\). Therefore, \(-Id + A^2\) is not invertible, and the metric \(\bar{g}\) is degenerate by Theorem 2.11. Since this situation is excluded a priori, we may use Lemma 1.1 and conclude that there exists a minimal hypersurface parallel to \(\mathcal{S}\) or its polar \(\mathcal{S}'\), and therefore whose normal congruence is \(\mathcal{S}\).

The complex diagonalizable case

Using the normal form of \(A\) (Section 1.1), a quick computation shows that

\[
h = \begin{pmatrix} -H & -\lambda \\ -\lambda & H \end{pmatrix} \quad \text{and} \quad \bar{g} = \begin{pmatrix} -\epsilon - H^2 + \lambda^2 & -2H \lambda \\ -2H \lambda & \epsilon + H^2 - \lambda^2 \end{pmatrix}.
\]
Hence, using the fact that
\[ \nabla e_1 e_1 = \omega^1 e_2, \quad \nabla e_1 e_2 = \omega^2 e_1, \]
\[ \nabla e_2 e_1 = \omega^2 e_2, \quad \nabla e_2 e_2 = \omega^1 e_1, \]
we calculate\(^2\)
\[ \bar{h}_{111} = \epsilon(-2\lambda \omega^1 + e_1(H)), \quad \bar{h}_{112} = \epsilon(-2\lambda \omega^2 + e_2(H)) = \epsilon e_1(\lambda), \]
\[ \bar{h}_{122} = -\epsilon(2\lambda \omega^1 + e_1(H)) = \epsilon e_2(\lambda), \quad \bar{h}_{222} = -\epsilon(2\lambda \omega^2 + e_2(H)). \]
Hence
\[ G(2\tilde{H}, \mathbb{J}d\tilde{\varphi}(e_1)) = \frac{(\epsilon + H^2 - \lambda^2)(\bar{h}_{111} - \bar{h}_{122}) + 4H\lambda \bar{h}_{112}}{-(\epsilon + H^2 - \lambda^2)^2 - 4H^2 \lambda^2} \]
\[ = -\frac{2\epsilon(\epsilon + H^2 - \lambda^2)e_1(H) + \epsilon 4H\lambda e_1(\lambda)}{1 + H^4 + \lambda^4 + 2\epsilon H^2 - 2\epsilon \lambda^2 - 2H^2 \lambda^2 + 4H^2 \lambda^2} \]
\[ = \frac{-2(1 + \epsilon H^2 - \epsilon \lambda^2)e_1(H) + \epsilon 4H\lambda e_1(\lambda)}{1 + H^4 + \lambda^4 + 2\epsilon (H^2 - \lambda^2) + 2H^2 \lambda^2}. \]
In the same way, we get
\[ G(2\tilde{H}, \mathbb{J}d\tilde{\varphi}(e_2)) = \frac{-2(1 + \epsilon H^2 - \epsilon \lambda^2)e_2(H) + \epsilon 4H\lambda e_2(\lambda)}{1 + H^4 + \lambda^4 + 2\epsilon (H^2 - \lambda^2) + 2H^2 \lambda^2}. \]
On the other hand, using the fact that \( K = H^2 + \lambda^2, \)
\[ d\beta = \arctan \left( \frac{2H}{1 - \epsilon H^2 - \epsilon \lambda^2} \right). \]
It follows that \( G(2\tilde{H}, \mathbb{J}) = d\beta, \) which is equivalent to \( 2\tilde{H} = \epsilon \mathbb{J} \nabla \beta, \) the required formula. If \( \epsilon = -1 \) we have, using the fact that \( \lambda \neq 0, \)
\[ \left| \frac{2H}{K + 1} \right| = \frac{2|H|}{H^2 + \lambda^2 + 1} < \frac{2|H|}{H^2 + 1} \leq 1. \]
Therefore, if \( \hat{S} \) is \( \mathbb{G} \)-minimal, i.e. \( \beta \) is constant, we may again use Lemma\(^1\) to conclude that there exists a minimal surface parallel to \( \varphi \) or \( N. \) Hence we have proved Corollary\(^2\) in this complex diagonalizable case.

The non-diagonalizable case

By a calculation similar to that of the previous cases we have:
\[ G(2\tilde{H}, \mathbb{J}d\tilde{\varphi}(e_1)) = 0 \quad \text{and} \quad G(2\tilde{H}, \mathbb{J}d\tilde{\varphi}(e_2)) = 2\frac{e_2(H)}{1 + \epsilon H^2}. \]
On the other hand, we have \( d\beta = \frac{2dH}{1 + \epsilon H^2}. \) Taking into account that \( e_1(H) \) vanishes, we deduce that \( G(2\tilde{H}, \mathbb{J}) = d\beta, \) which is equivalent to the required formula. If \( \epsilon = -1 \) we have, using the fact that \( |H| \neq 1, \)
\[ \left| \frac{2H}{K + 1} \right| = \frac{2|H|}{1 + H^2} < 1. \]
Therefore, we may use Lemma\(^1\) again and complete the proof of Corollary\(^2\).
4.3. Geometry of Lagrangian surfaces with respect to the scalar flat metric $G'$.

4.3.1. The metric $g'$ and the second fundamental form $\tilde{h}$ (Proof of Theorem 2.16).

Using the description of the metric $G'$ given in Section 3.2 and by a calculation analogous to that of Section 4.2.1, we obtain

$$ g'(X, Y) = -\epsilon G(X \wedge N + AX \wedge \phi, J'J(Y \wedge N + AY \wedge \phi)) = g(X, (-J'A + AJ')Y), $$

which gives the claimed formula for $g'$. We now discuss the degeneracy and the signature of $g'$, which depend on the type of the shape operator $A$.

**The real diagonalizable case**

Write $g$ and $A$ in canonical form, with $(e_1, e_2)$ an oriented, orthonormal local frame. It follows that $J'e_1 = e_2, J'e_2 = -\epsilon' e_1$. We easily get

$$ g' = \begin{pmatrix} 0 & \epsilon_2(\kappa_2 - \kappa_1) \\ \epsilon_2(\kappa_2 - \kappa_1) & 0 \end{pmatrix}. $$

In particular, $g'$ is degenerate at umbilic points and indefinite otherwise.

**The complex diagonalizable case**

Write $g$ and $A$ in canonical form. It follows that $J'e_1 = e_2, J'e_2 = e_1$ (here $\epsilon' = -1$ since the metric $g$ is indefinite). Hence

$$ g' = \begin{pmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix}, $$

which shows that the metric $g'$ is everywhere definite.

**The non-diagonalizable case**

Writing $g$ and $A$ in canonical form and observing that $J'e_1 = e_1, J'e_2 = -e_2$, we get

$$ g' = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, $$

which shows that the metric $g'$ is everywhere degenerate. In particular, we do not need to take into consideration the case of $A$ being non-diagonalizable in the proofs of Corollaries 2.17 and 2.18.

4.3.2. The mean curvature vector and the Proof of Corollary 2.17. Again we split the proof into two cases:

**The real diagonalizable case**

It has been seen in Section 4.2.2 that $\tilde{h}_{ijj} := \tilde{h}(e_i, e_j, e_j) = -\epsilon e_j e_i(\kappa_j)$. It follows that

$$ G'(2\tilde{H}', J'd\bar{\varphi}(e_1)) = \frac{\tilde{h}_{112}}{g'(e_1, e_2)} = \frac{-\epsilon e_1 e_2(\kappa_1)}{\epsilon_2(\kappa_2 - \kappa_1)} = \frac{-\epsilon' e_2(\kappa_1)}{\kappa_2 - \kappa_1}, $$

and

$$ G'(2\tilde{H}', J'd\bar{\varphi}(e_2)) = \frac{\tilde{h}_{122}}{g'(e_1, e_2)} = \frac{-\epsilon e_2 e_1(\kappa_2)}{\epsilon_2(\kappa_2 - \kappa_1)} = \frac{-\epsilon e_1(\kappa_2)}{\kappa_2 - \kappa_1}. $$

Hence

$$ \tilde{H}' = \frac{-\epsilon}{2(\kappa_2 - \kappa_1)^2} \left( e_1 e_1(\kappa_2) J'd\bar{\varphi}(e_1) + e_2 e_2(\kappa_1) J'd\bar{\varphi}(e_2) \right). $$

In particular, we see that if $S$ is $G'$-minimal, both $e_1(\kappa_2)$ and $e_2(\kappa_1)$ vanish. We now use the Codazzi equation derived in Section 4.2.2

$$ \begin{cases} (\kappa_2 - \kappa_1) \omega_{12} = e_2 e_1(\kappa_2), \\ (\kappa_2 - \kappa_1) \omega_{22} = e_1 e_2(\kappa_1). \end{cases} $$


Since we assume that the metric \( \bar{g}' \) is not degenerate, \( \kappa_2 - \kappa_1 \) does not vanish. Therefore the \( G' \)-minimality condition implies the vanishing of \( \omega_{12}^1 \) and \( \omega_{12}^2 \), i.e. the flatness of \( g \). The next step consists of using the Gauss equation with respect to the immersion \( \varphi : \mathcal{M}^2 \to \mathbb{S}^3_{p,1} \), giving

\[
g(R^g(e_1, e_2)e_1, e_2) = \epsilon h(e_1, e_1)h(e_2, e_2) - h(e_1, e_2)h(e_1, e_2) + K_{\mathbb{S}_{p,1}^3} = \epsilon' \kappa_1 \kappa_2 + 1.
\]

Hence \( \kappa_1 \kappa_2 = -\epsilon \epsilon' \). Taking into account the vanishing of \( e_1(\kappa_2) \) and \( e_2(\kappa_1) \), it implies that both principal curvatures are constant, non-vanishing and different from \( \pm 1 \). In particular, \( \bar{S} \) has parallel second fundamental form and \( \bar{S} \) is totally geodesic.

In the real diagonalizable case we are able to give a more precise characterization of surfaces with parallel second fundamental form: introducing the map \( \varphi_t := \cos(\epsilon t)\varphi + \sin(\epsilon t)N \) and differentiating, we get

\[
d\varphi_t(e_2) = \cos(\epsilon t)d\varphi(e_2) + \sin(\epsilon t)N e_2 = (\cos(\epsilon t) - \kappa_2 \sin(\epsilon t))d\varphi(e_2).
\]

Hence, choosing \( t_0 \) such that \( \frac{\cos(t_0)}{\sin(t_0)} = \kappa_2 = -\epsilon(\kappa_1)^{-1} \) yields the vanishing of \( d\varphi_{t_0}(e_2) \). Defining local coordinates \( (s_1, s_2) \) on \( \mathcal{M}^2 \) such that \( \partial_s_1 = e_1 \) and \( \partial_s_2 = e_2 \), we claim that the curve \( \gamma(s_1) := \varphi_{t_0}(s_1, s_2) \) is a geodesic of \( \mathbb{S}_{p,1}^3 \). To see this, we calculate the acceleration of \( \gamma \) in \( \mathbb{R}^4 \):

\[
\gamma'' = \frac{\cos(t_0) - \kappa_1 \sin(t_0)}{\epsilon \cos(t_0)}(-\sin(t_0)N - \cos(t_0)\varphi),
\]

which is collinear to \( \gamma \). Hence \( \gamma \) is a geodesic and \( \varphi(\mathcal{M}^2) \) is a tube over \( \gamma \).

**The complex diagonalizable case**

Since the basis \( (e_1, e_2) \) is orthogonal with respect to \( \bar{g}' \), the \( G' \)-minimality of \( \bar{S} \) is equivalent to the vanishing of

\[
\bar{h}_{111} + \bar{h}_{122} = -4\epsilon \lambda \omega_{12}^1 \quad \text{and} \quad \bar{h}_{112} + \bar{h}_{222} = -4\epsilon \lambda \omega_{12}^2
\]

(the coefficients \( \bar{h}_{ijk} \) have been determined in Section [4.2.3]. Hence \( \omega_{12}^1 \) and \( \omega_{12}^2 \) vanish and \( g \) is flat. Again we use the Gauss equation with respect to the immersion \( \varphi : \mathcal{M}^2 \to \mathbb{S}_{p,1}^3 \), obtaining

\[
g(R^\bar{g}(e_1, e_2)e_1, e_2) = \epsilon(h(e_1, e_1)h(e_2, e_2) - h(e_1, e_2)h(e_1, e_2)) + K_{\mathbb{S}_{p,1}^3}
\]

\[
= -\epsilon(H^2 + \lambda^2) + 1;
\]

hence \( H^2 + \lambda^2 = \epsilon \). On the other hand, the Codazzi equation becomes a Cauchy-Riemann system satisfied by the pair \( (H, \lambda) \):

\[
\begin{aligned}
e_1(H) &= -e_2(\lambda), \\
e_2(H) &= e_1(\lambda),
\end{aligned}
\]

so by the Liouville theorem, \( H \) and \( \lambda \) are constant, which implies that \( \bar{h} \) vanishes.

**4.3.3. Flat Lagrangian surfaces: Proof of Corollary 2.18** Again we consider two cases:

**The real diagonalizable case**

In order to characterize the flatness of \( \bar{g}' := \bar{\varphi}^*G' \), we shall use the Gauss equation twice, first with respect to the immersion \( \bar{\varphi} : \mathcal{M}^2 \to L^\pm(\mathbb{S}_{p,1}^3) \), and then with respect to the embedding \( t : L^\pm(\mathbb{S}_{p,1}^3) \to \Lambda^2(\mathbb{R}^4) \).
First, using the principal frame \((e_1, e_2)\) introduced in the previous section, we have

\[
K^{\hat{g}} = \hat{g}'(R^{\hat{g}}(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2)
\]

\[
= \mathcal{G}'(\hat{\bar{h}}(\bar{e}_1, \bar{e}_2), \hat{\bar{h}}(\bar{e}_1, \bar{e}_2)) - \mathcal{G}'(\hat{\bar{h}}(\bar{e}_1, \bar{e}_1), \hat{\bar{h}}(\bar{e}_2, \bar{e}_2)) + \mathcal{G}'(\bar{R}(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2),
\]

where \(\bar{h} : TS \times T\bar{S} \to NS\) denotes the second fundamental form of the immersion \(\bar{\varphi}\) with respect to the metric \(\mathcal{G}'\). In other words, \(\mathcal{G}'(\hat{\bar{h}}(X, Y), JZ) = \bar{h}(X, Y, Z)\). We have

\[
\hat{h}(\bar{e}_i, \bar{e}_j) = \frac{\bar{h}_{ij}N_1 + \bar{h}_{i1}N_2}{\epsilon_1(\kappa_2 - \kappa_1)},
\]

so that

\[
\mathcal{G}'(\hat{\bar{h}}(\bar{e}_1, \bar{e}_2), \hat{\bar{h}}(\bar{e}_1, \bar{e}_2)) = 2\epsilon_1 \frac{\bar{h}_{112}\bar{h}_{122}}{\kappa_2 - \kappa_1}
\]

and

\[
\mathcal{G}'(\hat{\bar{h}}(\bar{e}_1, \bar{e}_1), \hat{\bar{h}}(\bar{e}_2, \bar{e}_2)) = \epsilon_1 \frac{\bar{h}_{111}\bar{h}_{222} + \bar{h}_{112}\bar{h}_{122}}{\kappa_1 - \kappa_2}.
\]

Hence

\[
\mathcal{G}'(\hat{\bar{h}}(\bar{e}_1, \bar{e}_2), \hat{\bar{h}}(\bar{e}_1, \bar{e}_2)) - \mathcal{G}'(\hat{\bar{h}}(\bar{e}_1, \bar{e}_1), \hat{\bar{h}}(\bar{e}_2, \bar{e}_2)) = \epsilon_1 \frac{2\bar{h}_{112}\bar{h}_{122} - \bar{h}_{111}\bar{h}_{222} + \bar{h}_{112}\bar{h}_{122}}{\kappa_1 - \kappa_2}
\]

\[
= \epsilon_2 \frac{\bar{e}_2(\kappa_1e_1 - e_1\kappa_2) - e_1(\kappa_1e_2 - e_2\kappa_2)}{\kappa_1 - \kappa_2}
\]

\[
= -\epsilon_2 \frac{(d\kappa_1 \wedge d\kappa_2)(e_1, e_2)}{\kappa_1 - \kappa_2}.
\]

We now proceed to calculate \(\mathcal{G}'(R(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2)\). We have

\[
\bar{e}_i = d\varphi(e_i) \wedge N + \varphi \wedge dN(e_i) = -E_{2+i} - \kappa_i E_i.
\]

Then we easily get that \(h'(\bar{e}_1, \bar{e}_1) = -e_1(\epsilon + \kappa_1)\bar{x}\) and \(h'(\bar{e}_1, \bar{e}_2) = (\kappa_1 - \kappa_2)e_1 \wedge e_2\). Analogously we may check that \(h'(\bar{e}_2, e_1J\circ J\bar{e}_2)\) is collinear to \(\bar{x}\), while \(h'(\bar{e}_2, e_1J\circ J\bar{e}_2)\) is collinear to \(e_1 \wedge e_2\).

It follows that, again using the Gauss equation and the fact that the metric \(\langle (\cdot, \cdot) \rangle\) is flat,

\[
\mathcal{G}'(R(\bar{e}_1, \bar{e}_2)\bar{e}_1, \bar{e}_2) = -\mathcal{G}'(R(e_1, e_2)\bar{e}_1, e_1J\circ J\bar{e}_2)
\]

\[
= -\left(\langle h'(\bar{e}_1, J\circ J\bar{e}_2, h'(\bar{e}_2, \bar{e}_1)) \rangle - \langle h'(\bar{e}_1, \bar{e}_1), h'(\bar{e}_2, J\circ J\bar{e}_2) \rangle \right)
\]

\[
= 0.
\]

We conclude that the metric \(\hat{g}'\) (and therefore \(\hat{g}\) as well) is flat if and only if \(d\kappa_1 \wedge d\kappa_2\) vanishes, i.e., \(S\) is Weingarten.

**The complex diagonalizable case**

The calculations are analogous to the real diagonalizable case and left to the reader.

### 4.4. Marginally trapped Lagrangian surfaces: Proof of Corollary 4.20

#### 4.4.1. \(G\)-marginally trapped Lagrangian surfaces

We have seen in Section 4.2.3 that if the shape operator \(A\) of \(\varphi\) is not diagonalizable, then \(\hat{g}(e_1, e_1)\) vanishes. It follows that \(d\hat{\varphi}(e_1)\), and therefore \(Jd\hat{\varphi}(e_1)\) as well, is a \(G\)-null vector. We have also seen that \(G(2\bar{H}, Jd\hat{\varphi}(e_1))\) vanishes, so \(\bar{H}\), a vector of the plane \(NS\) spanned by \(Jd\hat{\varphi}(e_1)\) and \(Jd\hat{\varphi}(e_2)\), must be collinear to \(Jd\hat{\varphi}(e_1)\). Hence it is a \(G\)-null vector as well.
4.4.2. $G'$-marginally trapped Lagrangian surfaces. We start from the expression of the mean curvature vector of $\mathcal{S}$ with respect to $G'$ obtained in Section 4.3.2

$$\bar{H}' = -\frac{e}{2(\kappa_2 - \kappa_1)^2} \left( \epsilon_1 e_1(\kappa_2) \mathcal{J}'d\bar{\varphi}(e_1) + \epsilon_2 e_2(\kappa_1) \mathcal{J}'d\bar{\varphi}(e_2) \right).$$

Since $\bar{g}'(e_1, e_1)$ and $\bar{g}'(e_1, e_1)$ vanish, the pair $(\mathcal{J}'d\bar{\varphi}(e_1), \mathcal{J}'d\bar{\varphi}(e_2))$ is a $G'$-null basis of the normal space $NS$. Therefore, the mean curvature vector $\bar{H}'$ is $G'$-null if and only if it is collinear to one of the two vectors $\mathcal{J}'d\bar{\varphi}(e_1)$, i.e. if and only if either $e_1(\kappa_2)$ or $e_2(\kappa_1)$ vanishes. This occurs in at least the following two cases:

- If $\mathcal{S}$ is a tube, i.e. the set of equidistant points to a given curve of $S^3_{p,1}$, then one of its principal curvatures is constant;
- If $\mathcal{S}$ is a surface of revolution, i.e. a surface invariant by the action of a subgroup $SO(2)$ or $SO(1, 1)$ of $SO(4-p, p)$, then both principal curvatures are constant along the orbits of the action, which are in addition tangent to one of the principal directions (cf. [3]). Therefore, $e_1(\kappa_2)$ or $e_2(\kappa_1)$ vanishes.

Acknowledgement

The author thanks Nikos Georgiou for interesting observations about the early version of this manuscript.

References

1. D. V. Alekseevsky, B. Guilfoyle and W. Klingenberg, On the Geometry of Spaces of Oriented Geodesics, Diff. Geom. and its Applications 28 (2010) no. 4, 454–468.
2. D. V. Alekseevsky, C. Medori and A. Tomassini, Homogeneous para-Kähler Einstein manifolds, Russian Mathematical Surveys 64 (2009) 1–43. MR2503094 (2010k:53068)
3. H. Anciaux, Minimal submanifolds in Pseudo-Riemannian geometry, World Scientific, 2010. MR2722116 (2012g:53071)
4. H. Anciaux, B. Guilfoyle and P. Romon, Minimal Lagrangian surfaces in the tangent bundle of a Riemannian surface, J. of Geom. and Physics 61 (2011) 237–247. MR2746995 (2012a:53152)
5. I. Castro and F. Urbano, Minimal Lagrangian surfaces in $S^2 \times S^2$, Communications in Anal. and Geom. 15 (2007) 217–248. MR2344322 (2008j:53106)
6. V. Cruceanu, Almost product bi-complex structures on manifolds, An. Stiint. Univ. Al. I. Cuza Iasi 51 (2005), no. 1, 99–118. MR2187361 (2006h:53020)
7. V. Cruceanu, P. Fortuny and P. M. Gadea, A survey on paracomplex geometry, Rocky Mountain J. 26 (1996), 83–115. MR1386154 (97c:53112)
8. N. Georgiou, On maximal surfaces in the space of oriented geodesics of hyperbolic 3-space, Math. Scand. 111 (2012), no. 2, 187–209.
9. N. Georgiou and B. Guilfoyle, On the space of oriented geodesics of hyperbolic 3-space, Rocky Mountain J. Math. 40 (2010) no. 4, 1183–1219. MR2718810 (2011e:53116)
10. N. Georgiou and B. Guifoloye, A characterization of Weingarten surfaces in hyperbolic 3-space, Abh. Math. Semin. Univ. Hamburg 80 (2010) no. 2, 233–253. MR2734689 (2011j:51017)
11. B. Guifoloye and W. Klingenberg, An indefinite Kähler metric on the space of oriented lines, J. London Math. Soc. 72 (2005) 497–509. MR2156666 (2006e:53125)
12. B. Guifoloye and W. Klingenberg, On Weingarten surfaces in Euclidean and Lorentzian 3-space, Diff. Geom. and its Applications 28 (2010) no. 4, 454–468. MR2651535 (2011d:53173)
13. F. Hélein and P. Romon (2002), Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces, in Differential geometry and integrable systems (Tokyo, 2000), M. Guest, R. Miyaoka and Y. Ohnita, Eds., Contemp. Math. 308, Amer. Math. Soc., Providence, RI, 2002, 161–178. MR1953633 (2004j:53103)
14. N. Hitchin, Monopoles and geodesics, Comm. Math. Phys. 83 (1982), 579–602. MR649818 (84e:53071)
15. A. Honda, *Isometric Immersions of the Hyperbolic Plane into the Hyperbolic Space*, Tohoku Math. J. **64** (2012) no. 2, 171–193.
16. B. Khesin and S. Tabachnikov, *Pseudo-Riemannian geodesics and billiards*, Adv. Math. **221** (2009), 1364–1396. MR2518642 (2010g:53163)
17. M. Kimura, *Space of geodesics in hyperbolic spaces and Lorentz numbers*, Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. **36** (2003), 61–67. MR1976998 (2004a:53045)
18. M. Kriele, *Spacetime, Foundations of General Relativity and Differential Geometry*, Springer (1999). MR1726656 (2001g:53126)
19. K. Leichtweiss, *Zur Riemannschen geometrie in Grassmannschen manigfaltigkeiten*, Math. Zeit. **76** (1961) 334–366. MR0126808 (23:A4102)
20. M. A. Magid, *Lorentzian Isothermic Surfaces in $\mathbb{R}^n_j$*, Rocky Mountain J. Math. **35** (2005) 627–640. MR2135589 (2006e:53031)
21. R. Osserman, *A survey of minimal surfaces*, Van Nostrand Reinhold Co., New York-London-Melbourne (1969). MR0256278 (41:934)
22. B. Palmer, *Hamiltonian minimality and Hamiltonian stability of Gauss maps*, Diff. Geom. Appl. **7** (1997) no. 1, 51–58. MR1441918 (97m:53106)
23. M. Salvai, *On the geometry of the space of oriented lines of Euclidean space*, Manuscripta Math. **118** (2005) no. 2, 181–189. MR2177684 (2006h:53043)
24. M. Salvai, *On the geometry of the space of oriented lines of the hyperbolic space*, Glasg. Math. J. **49** (2007) no. 2, 357–366. MR2347266 (2008h:53062)