THE LOEWNER-NIRENBERG PROBLEM IN CONES

QING HAN, XUMIN JIANG, AND WEIMING SHEN

Abstract. We study asymptotic behaviors of solutions to the Loewner-Nirenberg problem in finite cones and establish optimal asymptotic expansions in terms of the corresponding solutions in infinite cones. The spherical domains over which cones are formed are allowed to have singularities. An elliptic operator on such spherical domains with coefficients singular on boundary play an important role. Due to the singularity of the spherical domains, extra cares are needed for the study of the global regularity of the eigenfunctions and solutions of the associated singular Dirichlet problem.

1. Introduction

In a pioneering work, Loewner and Nirenberg [17] studied the following problem which bears their names:

\begin{align}
\Delta u &= \frac{1}{4} n(n - 2) u^{\frac{n+2}{n-2}} \text{ in } \Omega, \\
u &= \infty \quad \text{on } \partial \Omega,
\end{align}

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), for \( n \geq 3 \). Under the condition that \( \Omega \) is a \( C^2 \)-domain, they proved the existence of a unique positive solution \( u \) of (1.1)-(1.2) and established an estimate of \( d_{\Omega}^{-\frac{n-2}{2}} u \), where \( d_{\Omega} \) is the distance function in \( \Omega \) to \( \partial \Omega \). Specifically, they proved, for \( d_{\Omega} \) sufficiently small,

\begin{equation}
|d_{\Omega}^{-\frac{n-2}{2}} u - 1| \leq C d_{\Omega},
\end{equation}

where \( C \) is a positive constant depending only on certain geometric quantities of \( \partial \Omega \).

The Loewner-Nirenberg problem has a geometric interpretation. Its solution implies the existence of a complete conformal metric on \( \Omega \) with the constant scalar curvature \( -n(n-1) \).

The Loewner-Nirenberg problem has many generalizations. Aviles and McOwen [2] proved the existence of complete conformal metrics with a negative constant scalar curvature on compact manifolds with boundary. Guan [5] and Gursky, Streets, and Warren [6] proved the existence of complete conformal metrics of negative Ricci curvature on compact manifolds with boundary. Del Mar Gonzalez, Li, and Nguyen [4] studied the existence and uniqueness to a fully nonlinear version of the Loewner-Nirenberg problem.

The estimate (1.3) stimulated more studies of asymptotic behaviors of solutions \( u \) of (1.1)-(1.2). Kichenassamy [13] expanded further if \( \Omega \) has a \( C^{2,\alpha} \)-boundary. If \( \Omega \) has a smooth boundary, Andersson, Chruściel and Friedrich [1] and Mazzeo [18] established a polyhomogeneous expansion, an estimate up to an arbitrarily finite order. All these
results require $\partial \Omega$ to have some degree of regularity. If $\Omega$ is a Lipschitz domain, Han and Shen [9] studied asymptotic behaviors of solutions near singular points on $\partial \Omega$, and proved an estimate similar as (1.3), under appropriate conditions of the domain near singular points.

In this paper, we study a more basic question and investigate asymptotic behaviors of solutions of (1.1)-(1.2) if $\Omega$ is a finite cone. Let $V$ be an infinite cone over some spherical domain $\Sigma \subseteq S^{n-1}$. We always assume that the boundary $\partial \Sigma$ is $(n-2)$-dimensional. We consider (1.1)-(1.2) in truncated cones. Let $u$ be a positive solution of
\begin{equation}
\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{1}{4} n(n-2) u^{\frac{n+2}{n-2}} \quad \text{in } V \cap B_1, \tag{1.4}
\end{equation}
\begin{equation}
u = \infty \quad \text{on } \partial V \cap B_1. \tag{1.5}
\end{equation}
We will study asymptotic behaviors of $u$ near the vertex of the cone $V$. To this end, we introduce the corresponding solution in the infinite cone. Consider
\begin{equation}
\sum_{i,j=1}^{n} \frac{\partial^2 u_V}{\partial x_i \partial x_j} = \frac{1}{4} n(n-2) u_V^{\frac{n+2}{n-2}} \quad \text{in } V, \tag{1.6}
\end{equation}
\begin{equation}
u = \infty \quad \text{on } \partial V. \tag{1.7}
\end{equation}
According to [9], under appropriate assumptions on $\Sigma$, there exists a unique positive solution $u_V$ of (1.6)-(1.7). Moreover, in the polar coordinates $x = r\theta$ with $r = |x|$ and $\theta = x/|x|$, $u_V$ has the form
\begin{equation}
u_V(x) = |x|^{-\frac{n-2}{2}} \xi_\Sigma(\theta), \tag{1.8}
\end{equation}
where $\xi_\Sigma$ is a smooth function on $\Sigma$, with $\xi_\Sigma = \infty$ on $\partial \Sigma$.

Our primary goal in this paper is to investigate how a solution $u$ of (1.4)-(1.5) in $V \cap B_1$ is approximated by the corresponding solution $u_V$ of (1.6)-(1.7) restricted to $V \cap B_1$. We point out that both $u$ and $u_V$ are infinite on $\partial V \cap B_1$. It is not clear whether the quotient $u/u_V$ should remain bounded. In the first result, we prove that $u/u_V$ is not only bounded, but has an expansion near $\partial V \cap B_1/2$ with the constant leading term 1.

**Theorem 1.1.** For $n \geq 3$, let $V$ be an infinite Euclidean cone over some Lipschitz domain $\Sigma \subseteq S^{n-1}$, and $u_V \in C^\infty(V)$ be the positive solution of (1.6)-(1.7). Then, there exist a positive constant $\tau$ and an increasing sequence of positive constants $\{\mu_i\}$, with $\mu_i \to \infty$, such that, for any positive solution $u \in C^\infty(V \cap B_1)$ of (1.4)-(1.5) and any integer $m \geq 0$,
\begin{equation}
|u_V^{-1}u(x) - 1 - \sum_{i=1}^{m} \sum_{j=0}^{i-1} c_{ij}(\theta)|x|^\mu_i (-\ln |x|)^j| \leq Cd_\Sigma^m|x|^{\mu_{m+1}}(-\ln |x|)^m, \tag{1.9}
\end{equation}
where $C$ is a positive constant, $d_\Sigma$ is the distance function on $\Sigma$ to $\partial \Sigma$, and each $c_{ij}$ is a bounded smooth function on $\Sigma$, with $c_{ij} = O(d_\Sigma^m)$.

We point out that the sequence $\{\mu_i\}$ in Theorem 1.1 is determined only by the cone $V$ or the spherical domain $\Sigma$, independent of the specific solution $u$ of (1.4)-(1.5). The growth rate $\tau$ near $\partial \Sigma$ is also determined only by $\Sigma$ and will be proved to satisfy $\tau > \frac{n-2}{2}$. We can improve $\tau$ under appropriate enhanced assumptions of $\Sigma$. 

Theorem 1.2. For $n \geq 3$, let $V$ be an infinite Euclidean cone over some domain $\Sigma \subset \mathbb{S}^{n-1}$. Assume that $\Sigma$ is either a $C^{1,\alpha}$-domain for some $\alpha \in (0, 1)$ or a Lipschitz domain which is the union of an increasing sequence of $C^3$-domains $\{\Sigma_i\} \subset \mathbb{S}^{n-1}$ such that each $\Sigma_i$ has a nonnegative mean curvature with respect to the inner unit normal. Then, $\tau$ in Theorem 1.1 is given by $\tau = n$.

We point out that $\tau = n$ provides an optimal estimate near $\partial \Sigma$.

For $m = 0$ and $m = 1$, (1.9) reduces to
\[
| (u_V^{-1}u)(x) - 1 | \leq C d_{\Sigma}^2 |x|^\mu_1,
\]
and, for some $\mu > \mu_1$,
\[
| (u_V^{-1}u)(x) - 1 - c_1(\theta)| \leq C |x|^\mu,
\]
where $\mu_1$ is related to the first eigenvalue of a singular elliptic operator on $\Sigma$ associated with the function $\xi_\Sigma$ in (1.8), and can be computed explicitly. The estimate (1.10) with $\tau = n$ is optimal, in terms of both powers $\mu_1$ and $n$. In the case $V = \mathbb{R}_+^n$ (i.e., $\Sigma = \mathbb{S}^{n-1}_+$), we can prove that $\mu_1 = n$. Then, (1.10) with $\tau = n$ reduces to a more familiar form
\[
| x_n^{-\frac{2}{n}} u(x) - 1 | \leq C x_n^n.
\]

Jiang [12] proved Theorem 1.1 with $\tau = n$ for the case that $\Sigma$ is a star-shaped smooth spherical domain, and also discussed asymptotic expansions near $\partial \Sigma$. If $\Sigma$ is at least $C^2$, the distance function $d_\Sigma$ is at least $C^2$ near $\partial \Sigma$ and hence can be used to construct various barrier functions. This is the strategy adopted in [12].

Under the weakened assumption that $\Sigma$ is Lipschitz, the distance function $d_\Sigma$ is not $C^2$ near $\partial \Sigma$ anymore and cannot be used to construct barrier functions. We will construct barrier functions with the help of $\xi_\Sigma$, the smooth function introduced in (1.8).

To prove Theorems 1.1 and 1.2, consider
\[
v = |x|^{\frac{n-2}{2}} (u - u_V) = |x|^{\frac{n-2}{2}} u - \xi_\Sigma(\theta),
\]
where $\xi_\Sigma$ is the function on $\Sigma$ as in (1.8). We will linearize the equation of $|x|^{\frac{n-2}{2}} u$ at $|x|^{\frac{n-2}{2}} u_V$ and study its kernels. The corresponding linearized operator $\mathcal{L}$ is given by
\[
\mathcal{L}w = r^2 \partial_{rr} w + r \partial_r w + \Delta_\theta w - \frac{1}{4} n(n+2) \xi_\Sigma^{-\frac{2}{n-2}} w - \frac{1}{4} (n-2)^2 w.
\]

We need to discuss the eigenvalue problem for the linear operator
\[
L_\Sigma w = \Delta_\theta w - \frac{1}{4} n(n+2) \xi_\Sigma^{-\frac{4}{n-2}} w \quad \text{in } \Sigma.
\]

Since $\xi_\Sigma$ blows up on boundary $\partial \Sigma$, $L_\Sigma$ has a singularity on $\partial \Sigma$. The singularity of some coefficient of $L_\Sigma$, coupled with the singularity of boundary of Lipschitz domains, makes the study of the asymptotic expansions a tricky issue.

If $\Sigma$ is merely Lipschitz, solutions related to the operator $\mathcal{L}$ in (1.12) which are continuous up to boundary are proved to be Hölder continuous, but with a small Hölder index. Such smallness causes two serious issues. It does not allow us to perform some important computations such as integration by parts, and it causes some singular terms
to possess less desirable integrability, especially in low dimensions. To overcome the first difficulty, we prove that derivatives of solutions with respect to \( r \) are also Hölder continuous, and then demonstrate that solutions restricted to each slice are in appropriate Sobolev spaces. For the second issue, we derive a universal lower bound estimate of the Hölder index in the case \( n = 3 \), and hence improve the integrability to the desired level. Refer to Corollaries 5.3 and 5.4 for details, respectively.

We point out that Theorem 1.1 is not a prerequisite of Theorem 1.2, which can be proved directly and more easily. Moreover, both Theorem 1.1 and Theorem 1.2 hold for \( V = \mathbb{R}^{n-k} \times V_k \), where \( V_k \) is a cone over an appropriate domain \( \Sigma_{k-1} \subseteq S^{k-1} \) in \( \mathbb{R}^k \).

We now compare results in this paper with corresponding results for the singular Yamabe equation corresponding to positive scalar curvatures. We consider positive solutions of the Yamabe equation of the form

\[
- \Delta u = \frac{1}{4} n(n-2) u^{\frac{n+2}{n-2}},
\]

which are defined in the punctured ball \( B_1 \setminus \{0\} \), with a nonremovable singularity at the origin.

Let \( u \) be a positive solution of (1.14) in \( B_1 \setminus \{0\} \), with a nonremovable singularity at the origin. In a pioneering paper \[3\], Caffarelli, Gidas, and Spruck proved that \( u \) is asymptotic to a radial singular solution of (1.14) in \( \mathbb{R}^n \setminus \{0\} \); namely,

\[
|x|^{\frac{n-2}{2}} u(x) - \psi(-\ln |x|) \to 0 \quad \text{as} \quad x \to 0,
\]

where \( |x|^{\frac{n-2}{2}} \psi(-\ln |x|) \) is a positive radial solution of (1.14) in \( \mathbb{R}^n \setminus \{0\} \), with a nonremovable singularity at the origin. In fact, \( \psi \) is a positive periodic function in \( \mathbb{R} \). Subsequently in \[14\], Korevaar, Mazzeo, Pacard, and Schoen studied refined asymptotics and expanded solutions to the next order in the following form: for some constant \( \alpha \in (1,2] \),

\[
||x|^{\frac{n-2}{2}} u(x) - \psi(-\ln |x|) - \phi(-\ln |x|) P_1(x)| \leq C|x|^\alpha \quad \text{in} \quad B_{1/2},
\]

where \( \psi \) is the function as in (1.15), \( P_1 \) is a linear function, and \( \phi \) is a function given by

\[
\phi = -\psi' + \frac{n - 2}{2} \psi.
\]

In \[7\], Han, Li, and Li established an expansion of \( |x|^{\frac{n-2}{2}} u(x) \) up to arbitrary orders. Specifically, there exists a positive sequence \( \{\mu_i\} \), strictly increasing, divergent to \( \infty \) and with \( \mu_1 = 1 \), such that, for any positive integer \( m \) and any \( x \in B_{1/2} \setminus \{0\} \),

\[
||x|^{\frac{n-2}{2}} u(x) - \psi(-\ln |x|) - \sum_{i=1}^m \sum_{j=0}^{i-1} c_{ij}(x) |x|^\mu_i (-\ln |x|)^j | \leq C|x|^\mu_{m+1} (-\ln |x|)^m,
\]

where \( \psi \) is the function as in (1.15), and \( c_{ij} \) is a bounded smooth function in \( B_{1/2} \setminus \{0\} \), for each \( i = 1, \ldots , m \) and \( j = 0, \ldots , i - 1 \). We point out that the sequence \( \{\mu_i\} \) here is determined by the leading term \( \psi \), and hence by the solution \( u \).

Compare (1.10), (1.11), and (1.9) with (1.15), (1.16), and (1.17), respectively.
The paper is organized as follows. In Section 2 we discuss some basic properties of solutions in cones. In Section 3 we derive some necessary gradient estimates. In Section 4, we study the eigenvalue problem of the elliptic operator $-L_\Sigma$ introduced in (1.13). In Section 5 we discuss the regularity of solutions of the Yamabe equation near boundary of cylinders. In Section 6, we derive an optimal estimate along the $t$-direction. In Section 7 we discuss asymptotic expansions for large $t$ and prove Theorems 1.1 and 1.2.

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2. Solutions in Cones

In this section, we discuss the existence of solutions of the Loewner-Nirenberg problem in infinite cones, and compare solutions in infinite cones and in finite cones.

First, we quote a well-known result.

Theorem 2.1. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, there exists a unique positive solution $u \in C^\infty(\Omega)$ of (1.1) - (1.2).

Loewner and Nirenberg [17] proved the existence and uniqueness for $C^2$-domains. Refer to [15] for the general case.

Now, we state a basic result which will be needed later.

Lemma 2.2. Let $\Omega$ be a domain in $\mathbb{R}^n$, and $u$ and $v$ be two nonnegative solutions of (1.1). Then, $u + v$ is a nonnegative supersolution of (1.1).

We omit the proof as it is based on a straightforward calculation.

Next, we discuss (1.1)-(1.2) in infinite cones. Throughout this paper, cones are always solid. Let $(r, \theta)$ be the polar coordinates in $\mathbb{R}^n$. Then,

\[ \Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_\theta, \]

where $\Delta_\theta$ is the Laplace-Beltrami operator on the unit sphere $S^{n-1}$.

Suppose $u$ is a positive function and set

\[ \hat{u}(x) = |x|^{\frac{n-2}{2}} u(x). \]

Then, $u$ is a solution of (1.1) if and only if

\[ r \partial_r (r \partial_r \hat{u}) + \Delta_\theta \hat{u} - \frac{1}{4} (n-2)^2 \hat{u} = \frac{1}{4} n(n-2) \hat{u}^{\frac{n+2}{n-2}}. \]

Theorem 2.3. Let $V$ be an infinite cone in $\mathbb{R}^n$ over a Lipschitz domain $\Sigma \subset S^{n-1}$. Then, there exists a unique positive solution $u_V \in C^\infty(V)$ of (1.6) - (1.7).

Proof. The proof consists of three steps.

Step 1. We prove the existence. Set

\[ u_V(x) = |x|^{\frac{n-2}{2}} \xi(\theta). \]
By (2.2), $u_V$ satisfies (1.6)-(1.7) if
\begin{align}
\Delta \phi \xi - \frac{1}{4}(n - 2)^2 \xi &= \frac{1}{4} n(n - 2)\xi^{\frac{n+2}{n-2}} \quad \text{in } \Sigma, \\
\xi &= \infty \quad \text{on } \partial \Sigma. \tag{2.5}
\end{align}

By a similar method as the proof of Theorem 2.1, there exists a unique positive solution $\xi$ of (2.4)-(2.5). Hence, there exists a positive solution $u$ of (1.6)-(1.7). In fact, for each $i \geq 1$, we consider
\begin{align}
\Delta \phi \xi_i - \frac{1}{4}(n - 2)^2 \xi_i &= \frac{1}{4} n(n - 2)\xi_i^{\frac{n+2}{n-2}} \quad \text{in } \Sigma, \tag{2.6} \\
\xi &= i \quad \text{on } \partial \Sigma. \tag{2.7}
\end{align}

It is standard to prove that there exists a solution $\xi_i \in C^\infty(\Sigma) \cap C(\bar{\Sigma})$ of (2.6)-(2.7) and that $\{\xi_i\}$ is a monotone increasing sequence, and hence converges to some $\xi$, a solution of (2.4)-(2.5). We point out that the uniqueness of $\xi$ shows that the solution of (1.6)-(1.7) in the form (2.3) is unique.

**Step 2.** We introduce some notations. Let $x_0 \in \mathbb{R}^n$ be a point and $r, R > 0$ be constants. Set, for any $x \in B_R(x_0)$,
\begin{align}
 u_{R,x_0}(x) = \left(\frac{2R}{R^2 - |x - x_0|^2}\right)^{\frac{n-2}{2}}.
\end{align}

Then, $u_{R,x_0}$ is the solution of (1.6)-(1.7) in $B_R(x_0)$. Set, for any $x \in \mathbb{R}^n \setminus B_r(x_0)$,
\begin{align}
 v_{r,x_0}(x) = \left(\frac{2r}{|x - x_0|^2 - r^2}\right)^{\frac{n-2}{2}}.
\end{align}

Then, $v_{r,x_0}$ is a solution of (1.6)-(1.7) in $\mathbb{R}^n \setminus B_r(x_0)$.

For any fixed $x$, we have $x \in B_R(x_0)$ for all large $R$, and $u_{R,x_0}(x) \to 0$ as $R \to \infty$. Similarly, for any fixed $x \neq x_0$, we have $x \in \mathbb{R}^n \setminus B_r(x_0)$ for all small $r$ and $v_{r,x_0}(x) \to 0$ as $r \to 0$.

**Step 3.** We now prove the uniqueness. Let $u_V$ be the solution of (1.6)-(1.7) established in Step 1, given by (2.3), and $u$ be an any other solution of (1.6)-(1.7).

Take any $0 < r < R$. By Lemma 2.2 and the maximum principle, we have
\begin{align}
 |x|^{-\frac{n-2}{2}} \xi_i \leq u + u_{R,0} + v_{r,0} \quad \text{in } V \cap (B_R \setminus \bar{B}_r),
\end{align}

where $\xi_i$ is the solution of (2.6)-(2.7), and $u_{R,0}$ and $v_{r,0}$ are given by (2.8) and (2.9), respectively, for $x_0 = 0$. Letting $i \to \infty$, we get
\begin{align}
 u_V \leq u + u_{R,0} + v_{r,0} \quad \text{in } V \cap (B_R \setminus \bar{B}_r). \tag{2.10}
\end{align}

Next, we consider a sequence of Lipschitz domains $\Sigma_k \subset \subset \Sigma$ such that $\Sigma_k \to \Sigma$, and denote by $V_k$ the cone over $\Sigma_k$. Then, $V_k \subset V$. For each $k$, let $\xi^{(k)}$ be the solution of (2.4)-(2.5) in $\Sigma_k$. Then, $\xi^{(k)} \to \xi$ uniformly locally in $\Sigma$. Similarly, by Lemma 2.2 and the maximum principle, we have
\begin{align}
 u \leq |x|^{-\frac{n-2}{2}} \xi^{(k)} + u_{R,0} + v_{r,0} \quad \text{in } V_k \cap (B_R \setminus \bar{B}_r).
\end{align}
Letting \( k \to \infty \), we get
\[
(2.11) \quad u \leq u_V + u_{R,0} + v_{r,0} \quad \text{in } V \cap (B_R \setminus \bar{B}_r).
\]
By combining (2.10) and (2.11), we obtain
\[
(2.12) \quad |u - u_V| \leq u_{R,0} + v_{r,0} \quad \text{in } V \cap (B_R \setminus \bar{B}_r).
\]
For any fixed \( x \in V \), take \( r < |x| < R \), and then let \( R \to \infty \) and \( r \to 0 \). We conclude that \( u = u_V \) in \( V \).

The proof is modified from [9], with a new proof of the uniqueness. In the present version, we removed the requirement in [9] that \( \Sigma \) is star-shaped. The function \( \xi \) introduced in (2.3) plays an important role in this paper. We now present some of its properties.

**Lemma 2.4.** Let \( \Sigma \subset \mathbb{S}^{n-1} \) be a Lipschitz domain. Then, there exists a unique positive solution \( \xi \in C^\infty(\Sigma) \) of (2.4)-(2.5). Moreover,
\[
(2.13) \quad c_1 \leq d^{\frac{n-2}{2}} \xi \leq c_2 \quad \text{in } \Sigma,
\]
and, for any \( k \geq 0 \),
\[
(2.14) \quad d^{\frac{n-2}{2}+k} |\nabla^k \xi| \leq C \quad \text{in } \Sigma,
\]
where \( c_1, c_2, \) and \( C \) are positive constants depending only on \( n, k \) and \( \Sigma \). If, in addition, \( \Sigma \) is a \( C^{1,\alpha} \)-domain for some \( \alpha \in (0,1] \), then
\[
(2.15) \quad |d^{\frac{n-2}{2}} \xi - 1| \leq Cd^\alpha \quad \text{in } \Sigma.
\]

The existence and the uniqueness of the solution \( \xi \) are proved in the proof of Theorem 2.3. Estimates similar as (2.13)-(2.15) for solutions \( u \) of (1.1)-(1.2) are well-known, and proofs are standard. (Refer to [9].) These proofs can be modified easily to yield (2.13)-(2.15) for solutions \( \xi \) of (2.4)-(2.5).

For later purposes, we present an equivalent form of (2.4)-(2.5). Consider
\[
(2.16) \quad \rho \Delta_\rho + S \rho^2 = \frac{n}{2} (|\nabla_\rho \rho|^2 - 1) \quad \text{in } \Sigma,
\]
\[
(2.17) \quad \rho = 0 \quad \text{on } \partial \Sigma,
\]
where \( S \) is the constant given by
\[
(2.18) \quad S = \frac{1}{2} (n - 2).
\]
In the equation (2.16), we purposely introduce the constant \( S \), which is related to the scalar curvature of \( \mathbb{S}^{n-1} \). The equation in the Euclidean space corresponds to \( S = 0 \).

**Lemma 2.5.** Let \( \Sigma \subset \mathbb{S}^{n-1} \) be a Lipschitz domain. Then, there exists a unique \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \), positive in \( \Sigma \) and satisfying (2.16)-(2.17). Moreover,
\[
(2.19) \quad c_1 \leq \rho \leq c_2 \quad \text{in } \Sigma,
\]
where \( c_1 \) and \( c_2 \) are positive constants depending only on \( n \) and \( \Sigma \).
Proof. Let \( \xi \in C^\infty(\Sigma) \) be the positive solution of (2.4)-(2.5) as in Lemma 2.4. Set (2.20)
\[
\rho = \xi - \frac{2}{n}.
\]
Then, \( \rho \in C^\infty(\Sigma) \cap C(\overline{\Sigma}) \) is a positive function in \( \Sigma \) and satisfies (2.16)-(2.17). By (2.13), \( \rho \) satisfies (2.19) for some positive constants \( c_1 \) and \( c_2 \) depending only on \( n \) and \( \Sigma \). Next, by (2.14) with \( k = 1 \), we have \( |\nabla \theta \rho| \leq C \). Hence, \( \rho \) is Lipschitz in \( \Sigma \). \( \square \)

We now return to the equation (1.1) and prove a simple lemma comparing solutions in infinite cones and in truncated cones.

Lemma 2.6. Let \( V \) be an infinite cone in \( \mathbb{R}^n \) over some Lipschitz domain \( \Sigma \subseteq \mathbb{S}^{n-1} \).

Suppose \( u \in C^\infty(V \cap B_1) \) is a positive solution of (1.4)-(1.5) and \( u_V \in C^\infty(V) \) is the positive solution of (1.6)-(1.7). Then,
\[
(2.21) \quad |u - u_V| \leq C \quad \text{in} \quad V \cap B_{1/2},
\]
where \( C \) is a positive constant depending only on \( n \).

Proof. Proceeding similarly as in Step 3 in the proof of Theorem 2.3, we have, for \( r < 1 \),
\[
|u - u_V| \leq u_{1,0} + v_{r,0} \quad \text{in} \quad V \cap (B_1 \setminus \overline{B}_r).
\]
This is (2.12) with \( R = 1 \). For any fixed \( x \in V \cap B_1 \), take \( r < |x| \), and then let \( r \to 0 \). We conclude
\[
|u - u_V| < u_{1,0} \quad \text{in} \quad V \cap B_1.
\]
This yields the desired result if we restrict to \( V \cap B_{1/2} \). \( \square \)

Our goal in this paper is to improve the estimate (2.21). Let \( u \in C^\infty(V \cap B_1) \) and \( u_V \in C^\infty(V) \) be as in Lemma 2.6. Set
\[
(2.22) \quad v = |x|^{\frac{2}{n-2}} (u - u_V) = |x|^{\frac{2}{n-2}} u - \xi,
\]
where \( \xi \) satisfies (2.4)-(2.5). Since both \( |x|^{\frac{2}{n-2}} u \) and \( |x|^{\frac{2}{n-2}} u_V \) satisfy the same equation (2.2), we can take a difference of these two equations and obtain
\[
r \partial_r (r \partial_r v) + \Delta v - \frac{1}{4} (n-2)^2 v = \frac{1}{4} n(n-2) \left[ (v + \xi)^{\frac{n+2}{n-2}} - \xi^{\frac{n+2}{n-2}} \right].
\]
For the right-hand side, a simple computation yields
\[
\text{RHS} = \frac{1}{4} n(n-2) \xi^{\frac{n+2}{n-2}} \left[ (1 + \xi^{-1} v)^{\frac{n+2}{n-2}} - 1 \right]
= \frac{1}{4} n(n-2) \xi^{\frac{n+2}{n-2}} \left[ (1 + \xi^{-1} v)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2} \xi^{-1} v + \frac{n+2}{n-2} \xi^{-1} v \right]
= \xi^{-\frac{n+2}{n-2}} v^2 h(\xi^{-1} v) + \frac{1}{4} n(n+2) \xi^{-\frac{1}{n-2}} v,
\]
where
\[
h(s) = \frac{1}{4} n(n-2) s^{-2} \left[ (1 + s)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2} s \right].
\]
Hence,
\[ r \partial_r (r \partial_r v) + \Delta_\theta v - \frac{1}{4} n(n+2) \xi^\frac{4}{n-2} v - \frac{1}{4} (n-2)^2 v = \xi^{-\frac{n-6}{n-2}} v^2 h(\xi^{-1} v). \]

Let \( \rho \) be given by (2.20). Then,
\[ (2.23) \quad r \partial_r (r \partial_r v) + \Delta_\theta v - \frac{1}{4} n(n+2) \frac{v}{\rho^2} - \frac{1}{4} (n-2)^2 v = \rho^{-\frac{n-6}{n-2}} v^2 h(\rho^{-\frac{n-2}{n-2}} v). \]

We point out that the left-hand side of (2.23) is linear in \( v \) and the right-hand side is nonlinear. We also note that \( h \) is a smooth function on \((-1, 1)\) and \( h(0) \neq 0 \).

In cylindrical coordinates, we write (2.22) as
\[ (2.25) \quad v(t, \theta) = \frac{|x|}{|x|^{n-2}} (u(x) - u_V(x)) = |x|^{\frac{n-2}{2}} u(x) - \xi(\theta) = |x|^{\frac{n-2}{2}} u(x) - \rho^{-\frac{n-2}{2}} (\theta). \]

We have,
\[ (2.26) \quad L v = F(v), \]
where
\[ (2.27) \quad L v = \partial_{tt} v + \Delta_\theta v - \frac{1}{4} n(n+2) \frac{v}{\rho^2} - \frac{1}{4} (n-2)^2 v, \]
and
\[ (2.28) \quad F(v) = \rho^{-\frac{n-6}{n-2}} v^2 h(\rho^{-\frac{n-2}{n-2}} v). \]

By Lemma 2.6 and (2.25), we have, for any \((t, \theta) \in (1, \infty) \times \Sigma, \)
\[ (2.29) \quad |v(t, \theta)| \leq C e^{-\frac{n-2}{2}} \]
where \( C \) is a positive constant, depending on \( n \) and \( \Sigma \). By (2.29), \( v \) is bounded and decays to zero exponentially as \( t \to \infty \).

3. Gradient Estimates

Gradient estimates of solutions \( \rho \) of (2.16)-(2.17) play an important role in this paper. In this section, we present some of these estimates which will be needed later on.

In general, a gradient bound of \( \rho \) depends on the dimension \( n \) and the size of the exterior cone at each boundary point, the opening angle and the height. In the case \( n = 3 \), there is a universal upper bound. For \( n = 3 \), (2.16) and (2.17) reduce to
\[ (3.1) \quad \rho \Delta_{S^2} \rho + \frac{1}{2} \rho^2 = \frac{3}{2} (|\nabla_{S^2} \rho|^2 - 1) \quad \text{in} \ \Sigma, \]
\[ (3.2) \quad \rho = 0 \quad \text{on} \ \partial \Sigma. \]

**Lemma 3.1.** Let \( \Sigma \subseteq S^2 \) be a Lipschitz domain, and \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \) be the positive solution of (3.1)-(3.2) in \( \Sigma \). Then, \( \rho < 4.2 \) in \( \Sigma \).
Proof. Let $\xi$ be given by (2.20), i.e.,

$$\rho = \xi^{-2}. \quad (3.3)$$

Then, (2.4) and (2.5) reduce to

$$\Delta S^2 \xi - \frac{1}{4} \xi = \frac{3}{4} \xi^5 \text{ in } \Sigma, \quad (3.4)$$

$$\xi = \infty \text{ on } \partial \Sigma. \quad (3.5)$$

We denote by $(\theta, \varphi)$ the spherical coordinates on $S^2$, with $\theta = 0$ and $\theta = \pi$ corresponding to the north pole and the south pole, respectively. We first construct a subsolution of (3.4) in the compliment of the north pole. For any function $\eta = \eta(\theta)$, we have

$$\Delta S^2 \eta = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta h).$$

Hence, for any constant $\tau$, we get

$$\Delta S^2 \left( \sin \frac{\theta}{2} \right)^{\tau} = \frac{1}{4} \tau^2 \left( \sin \frac{\theta}{2} \right)^{\tau - 2} - \frac{1}{4} \tau (\tau + 2) \left( \sin \frac{\theta}{2} \right)^{\tau}. \quad (3.6)$$

With $c = (12)^{-\frac{1}{4}}$, set

$$\eta_1(\theta) = c \left( \sin \frac{\theta}{2} \right)^{-\frac{1}{2}}, \quad \eta_2(\theta) = c \left( \sin \frac{\theta}{2} \right)^{\frac{1}{2}}. \quad (3.7)$$

Take a positive constant $\alpha$ to be determined. Then,

$$\Delta S^2 (\eta_1 - \alpha \eta_2) = \frac{c}{16} \left[ (\sin \frac{\theta}{2})^{-\frac{5}{2}} - \alpha (\sin \frac{\theta}{2})^{-\frac{3}{2}} + 3 (\sin \frac{\theta}{2})^{-\frac{3}{2}} + 5 \alpha (\sin \theta \frac{1}{2} \right]^2]. \quad (3.6)$$

We will find $\alpha$ such that

$$\Delta S^2 (\eta_1 - \alpha \eta_2) > \frac{3}{4} (\eta_1 - \alpha \eta_2)^5 + \frac{1}{4} (\eta_1 - \alpha \eta_2). \quad (3.7)$$

To this end, we first note

$$\langle \eta_1 - \alpha \eta_2 \rangle^5 = c^5 \left( \sin \frac{\theta}{2} \right)^{-\frac{5}{2}} \left[ 1 - \alpha \sin \frac{\theta}{2} \right]^5$$

$$< c^5 \left( \sin \frac{\theta}{2} \right)^{-\frac{5}{2}} \left[ 1 - 5 \alpha \sin \theta \frac{2}{2} + 10 \alpha^2 \sin \frac{\theta}{2} \right]^2$$

$$= c^5 \left( \sin \frac{\theta}{2} \right)^{-\frac{5}{2}} - 5 c^5 \alpha \sin \theta \frac{1}{2} + 10 c^5 \alpha^2 \sin \theta \frac{1}{2}.$$
or
\[4\alpha - (10\alpha^2 + 1) \sin \frac{\theta}{2} + 9\alpha (\sin \frac{\theta}{2})^2 \geq 0.\]
To this end, we require
\[10\alpha^2 + 1 < 12\alpha.
We can take \(\alpha = 1/11\).
In summary, set, for \(\theta \neq 0\),
\[(3.9) \quad \eta(\theta) = \frac{1}{\sqrt[12]{12}} \left[ (\sin \frac{\theta}{2})^{-\frac{1}{12}} - \frac{1}{11} (\sin \frac{\theta}{2})^{\frac{1}{12}} \right] = \frac{1}{\sqrt[12]{12}} (\sin \frac{\theta}{2})^{-\frac{1}{12}} \left[ 1 - \frac{1}{11} \sin \frac{\theta}{2} \right].\]
Then,
\[\Delta_{S^2} \eta > \frac{3}{4} \eta^5 + \frac{1}{4} \eta.\]

Let \(\rho\) and \(\xi\) be the solution of \((3.1)-(3.2)\) and \((3.4)-(3.5)\) on \(\Sigma\), respectively. Denoting by \(N\) the north pole on \(S^2\), we assume \(N \not\in \Sigma\). (Here, we allow \(\Sigma = S^2 \setminus \{N\}\).) Consider a sequence of increasing domains \(\{\Sigma_i\} \subset S^2\) with \(\cup \Sigma_i = \Sigma\), and let \(\xi_i\) be the solution of \((3.4)-(3.5)\) on \(\Sigma_i\). Then, \(\xi_i \to \xi\) uniformly in any compact subset of \(\Sigma\). By the maximum principle, we have \(\xi_i \geq \eta\) on \(\Sigma_i\), and hence \(\xi \geq \eta\) on \(\Sigma\). With \((3.3)\), we have
\[\rho \leq \eta^{-2}\] on \(\Sigma\).
By \((3.9)\), the maximum of \(\eta^{-2}\) is attained at \(\theta = \pi\) with a value \(\sqrt{12}(11/10)^2 < 4.2\). □

**Lemma 3.2.** Let \(\Sigma \subseteq S^2\) be a Lipschitz domain, and \(\rho \in C^\infty(\Sigma) \cap \text{Lip}(\bar{\Sigma})\) be the positive solution of \((3.1)-(3.2)\) in \(\Sigma\). Then, \(|\nabla \rho| < 2.8\) in \(\Sigma\).

**Proof.** For brevity, we write \(\nabla\) and \(\Delta\), instead of \(\nabla_{S^2}\) and \(\Delta_{S^2}\). We only consider the case that \(\Sigma\) has a \(C^2\)-boundary, and obtain the general case by a simple approximation.
Set \(\bar{\rho} = 4.2\), the universal upper bound established in Lemma 3.1

**Step 1.** We claim
\[(3.10) \quad |\nabla \rho|^2 + \frac{3 + \bar{\rho}^2}{2\bar{\rho}^2} \rho^2 \leq \frac{3 + \bar{\rho}^2}{2} \quad \text{on} \quad \Sigma.
For some constant \(A \in [0,1]\) to be determined, set
\[(3.11) \quad w = |\nabla \rho|^2 + A \rho^2 \quad \text{on} \quad \Sigma.
It is obvious that \(w = 1\) on \(\partial \Sigma\). Assume \(w\) attains its maximum at some \(\theta_0 \in \Sigma\). Then, \(\nabla w = 0\) and \(\Delta w \leq 0\) at \(\theta_0\), i.e.,
\[(3.12) \quad \rho_i \rho_{ij} + A \rho \rho_j = 0 \quad \text{at} \quad \theta_0,
and
\[(3.13) \quad |\nabla^2 \rho|^2 + \nabla \rho \cdot \nabla \Delta \rho + \text{Ric}(\nabla \rho, \nabla \rho) + A |\nabla \rho|^2 + A \rho \Delta \rho \leq 0 \quad \text{at} \quad \theta_0.
Here and hereafter, we use normal coordinates \((\theta_1, \theta_2)\) at \(\theta_0\). We consider two cases.

First, assume \(\rho_1 = \rho_2 = 0\) at \(\theta_0\). Then, \(w \leq A \bar{\rho}^2\) at \(\theta_0\), and hence
\[(3.14) \quad w \leq \max \{1, A \bar{\rho}^2\} \quad \text{on} \quad \Sigma.
Second, assume $\rho_1 \neq 0$ and $\rho_2 = 0$ at $\theta_0$. Then, \( (3.12) \) implies
\begin{equation}
\rho_{11} = -A \rho, \quad \rho_{12} = 0 \quad \text{at } \theta_0.
\end{equation}
A simple evaluation of the equation \( (3.11) \) at $\theta_0$ yields
\begin{equation}
\rho_{22} = \frac{3}{2} (\rho_1^2 - 1) + \left( A - \frac{1}{2} \right) \rho^2 \quad \text{at } \theta_0.
\end{equation}
By differentiating \( (3.11) \) with respect to $\theta_1$ and substituting \( (3.15) \), we have
\begin{equation}
(\Delta \rho)_1 = -\rho_1 \rho_{22} - (2A + 1) \rho \rho_1 \quad \text{at } \theta_0.
\end{equation}
By substituting \( (3.15), (3.16) \), and \( (3.17) \) in \( (3.13) \) and by a straightforward computation, we obtain
\begin{equation}
\left[ \rho_1^2 - 1 + \frac{1}{3} (4A - 1) \rho^2 \right] \left[ \rho_1^2 - 3 + (2A - 1) \rho^2 \right] \leq 0 \quad \text{at } \theta_0.
\end{equation}
By a simple rearrangement, we get
\begin{equation}
\left[ \rho_1^2 + A \rho^2 - \frac{1}{3} [3 + (1 - A) \rho^2] \right] \left[ \rho_1^2 + A \rho^2 - [3 + (1 - A) \rho^2] \right] \leq 0 \quad \text{at } \theta_0.
\end{equation}
Therefore,
\begin{equation}
\rho_1^2 + A \rho^2 \leq 3 + (1 - A) \rho^2 \quad \text{at } \theta_0,
\end{equation}
and hence
\begin{equation}
w \leq 3 + (1 - A) \rho^2 \quad \text{on } \Sigma.
\end{equation}
By combining \( (3.14) \) and \( (3.19) \), we obtain
\begin{equation}
w \leq \max \{ A \rho^2, 3 + (1 - A) \rho^2 \} \quad \text{on } \Sigma.
\end{equation}
We now take $A$ such that $A \rho^2 = 3 + (1 - A) \rho^2$. Then,
\begin{equation}
A = \frac{3 + \rho^2}{2 \rho^2},
\end{equation}
and hence
\begin{equation}
w \leq \frac{3 + \rho^2}{2} \quad \text{on } \Sigma.
\end{equation}
This is \( (3.10) \).

Step 2. We claim
\begin{equation}
|\nabla \rho| \leq \frac{3 + \rho^2}{\sqrt{3(1 + \rho^2)}} \quad \text{on } \Sigma.
\end{equation}
Consider $w = |\nabla \rho|^2$; namely, we take $A = 0$ in \( (3.11) \). Assume $w$ attains its maximum at some $\theta_0 \in \Sigma$. By repeating arguments in Step 1, we obtain, instead of \( (3.18) \),
\begin{equation}
\rho_1^2 \leq 3 + \rho^2 \quad \text{at } \theta_0.
\end{equation}
Evaluating \( (3.10) \) at $\theta_0$, we have
\begin{equation}
\rho_1^2 + \frac{3 + \rho^2}{2 \rho^2} \rho^2 \leq \frac{3 + \rho^2}{2} \quad \text{at } \theta_0.
\end{equation}
Adding the \( \frac{3\bar{\rho}^2}{2\rho^2} \) multiple of (3.21) to (3.22), we get
\[
\rho_1^2 \leq \frac{(3 + \bar{\rho}^2)^2}{3(1 + \bar{\rho}^2)} \quad \text{at } \theta_0.
\]
This implies (3.20).

With \( \bar{\rho} = 4.2 \), the expression in the right-hand side of (3.20) is less than 2.8.  

The bound 2.8 in Lemma 3.2 is by no means optimal, but is sufficient for applications later on. We also point out that the universal gradient bound is a special property for 2-dimensional spherical domains and do not hold for higher dimensions.

In the next two results, we improve gradient bounds under strengthened assumptions on \( \Sigma \) for arbitrary dimensions.

**Lemma 3.3.** Let \( \Sigma \subseteq S^{n-1} \) be a \( C^{1,\alpha} \)-domain, for some \( \alpha \in (0, 1) \), and \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \) be the positive solution of (2.16)-(2.17) in \( \Sigma \). Then, \( \rho \in C^{1,\alpha}(\bar{\Sigma}) \), and \( |\nabla_\theta \rho| = 1 \) on \( \partial \Sigma \).

**Proof.** Let \( d \) be the distance function in \( \Sigma \) to \( \partial \Sigma \). Then, \( d \) is \( C^{1,\alpha} \) near \( \partial \Sigma \). By (2.15), we have
\[
|\rho - d| \leq C d^{1+\alpha}.
\]
Take a function \( \eta \in C^{1,\alpha}(\Sigma) \cap C^2(\Sigma) \) such that \( \eta > 0 \) in \( \Sigma \), \( \eta = 0 \) and \( |\nabla_\theta \eta| = 1 \) on \( \partial \Sigma \), and
\[
\eta^{1-\alpha} |\nabla_\theta^2 \eta| \leq C \quad \text{in } \Sigma,
\]
where \( C \) is a positive constant depending only on \( \Sigma \) and \( \alpha \). Then,
\[
c_1 \leq \frac{\eta}{d} \leq c_2, \quad |d - \eta| \leq C \eta^{1+\alpha} \quad \text{in } \Sigma,
\]
and hence
\[
|\rho - \eta| \leq C \eta^{1+\alpha} \quad \text{in } \Sigma.
\]

With (2.10), a simple computation yields
\[
\Delta_\theta (\rho - \eta) + b \cdot \nabla_\theta (\rho - \eta) + S(\rho - \eta) = f \quad \text{in } \Sigma,
\]
\[
\rho - \eta = 0 \quad \text{on } \partial \Sigma,
\]
where
\[
b = -\frac{n}{2\rho} \nabla_\theta (\rho + \eta),
\]
and
\[
f = \frac{n}{2\rho} (|\nabla_\theta \eta|^2 - 1) - \Delta_\theta \eta - S \eta.
\]
It is easy to see that \( \eta b \) and \( \eta^2 S \) are bounded in \( \Sigma \) and that, by (3.23),
\[
|\eta^2 f| \leq C \eta^{1+\alpha} \quad \text{in } \Sigma.
\]
For any $\theta \in \Sigma$, consider $B_{d(\theta)/2}(\theta) \subset \Sigma$. By the interior $C^{1,\alpha}$-estimate, we have

$$\eta(\theta)|\nabla_{\theta}(\rho - \eta)(\theta)| + \eta(\theta)^{1+\alpha}[\nabla_{\theta}(\rho - \eta)]_{C^{\alpha}(B_{d(\theta)/4}(\theta))} \leq C\{|\rho - \eta|_{L^\infty(B_{d(\theta)/2}(\theta))} + \eta(\theta)^2|\nabla_{\theta}(\rho - \eta)|_{L^\infty(B_{d(\theta)/2}(\theta))}\} \leq C\eta(\theta)^{1+\alpha}. $$

Hence,

$$|\nabla_{\theta}(\rho - \eta)| \leq C\eta^\alpha \text{ in } \Sigma,$$

and, for any $\theta \in \Sigma$,

$$[\nabla_{\theta}(\rho - \eta)]_{C^{\alpha}(B_{d(\theta)/4}(\theta))} \leq C.$$ 

Since $\nabla_{\theta}\eta \in C^{\alpha}(\bar{\Sigma})$ with $|\nabla_{\theta}\eta| = 1$ on $\partial\Sigma$, we conclude $\nabla_{\theta}\rho \in C^{\alpha}(\bar{\Sigma})$ with $|\nabla_{\theta}\rho| = 1$ on $\partial\Sigma$. \hfill \Box

For the next result, we denote by $H_{\partial\Sigma}$ the mean curvature of $\partial\Sigma$ with respect to the inner unit normal vector.

**Lemma 3.4.** Let $\Sigma \subseteq S^{n-1}$ be a Lipschitz domain and $\rho \in C^{\infty}(\Sigma) \cap \text{Lip}(\bar{\Sigma})$ be the positive solution of (2.16)-(2.17) of $\Sigma$. Assume that $\Sigma$ is the union of an increasing sequence of $C^3$-domains $\{\Sigma_i\}$ such that $H_{\partial\Sigma_i} \geq -m$ for some nonnegative constant $m$. Then, $|\nabla_{\theta}\rho| \leq 1 + cp$ in $\Sigma$, for some constant $c \geq 0$.

**Proof.** We first consider a special case that $\Sigma$ is a $C^3$-bounded domain such that $H_{\partial\Sigma} \geq -m$ for some nonnegative constant $m$. The Bochner identity yields

$$\frac{1}{2}\Delta_{\theta}(\nabla_{\theta}\rho)^2 = \nabla_{\theta}\rho \cdot \nabla_{\theta}\Delta_{\theta}\rho + \text{Ric}(\nabla_{\theta}\rho, \nabla_{\theta}\rho).$$

On $S^{n-1}$, $R_{ij} = (n - 2)g_{ij}$. Hence,

$$\frac{1}{2}\Delta_{\theta}(\nabla_{\theta}\rho)^2 = \nabla_{\theta}\rho \cdot \nabla_{\theta}\Delta_{\theta}\rho + (n - 2)|\nabla_{\theta}\rho|^2.$$ 

We write (2.16) as

$$\rho(\Delta_{\theta}\rho + S\rho) = \frac{n}{2}(|\nabla_{\theta}\rho|^2 - 1).$$

By applying the Laplacian operator, we get

$$\rho\Delta_{\theta}(\Delta_{\theta}\rho + S\rho) + 2\nabla_{\theta}\rho \cdot \nabla_{\theta}(\Delta_{\theta}\rho + S\rho) + \Delta_{\theta}\rho(\Delta_{\theta}\rho + S\rho)$$

$$= n(\nabla_{\theta}\rho)^2 + \nabla_{\theta}\rho \cdot \nabla_{\theta}\Delta_{\theta}\rho + 2S|\nabla_{\theta}\rho|^2).$$

For the last term in the left-hand side and the last two terms in the right-hand side, we write

$$\Delta_{\theta}(\Delta_{\theta}\rho + S\rho) = (\Delta_{\theta}\rho)^2 + S\rho\Delta_{\theta}\rho$$

$$= (\Delta_{\theta}\rho)^2 + S\left[\frac{n}{2}|\nabla_{\theta}\rho|^2 - S\rho^2 - \frac{n}{2}\right],$$

and

$$\nabla_{\theta}\rho \cdot \nabla_{\theta}\Delta_{\theta}\rho + 2S|\nabla_{\theta}\rho|^2 = \nabla_{\theta}\rho \cdot \nabla_{\theta}(\Delta_{\theta}\rho + S\rho) + S|\nabla_{\theta}\rho|^2.$$


By simple substitutions and rearrangements, we have
\[ \rho \Delta \theta + S \rho^2 - 2 \nabla \rho \cdot (\Delta \theta + S \rho) \]
\[ = n |\nabla \rho|^2 - (\Delta \theta)^2 + \frac{1}{2} S \left[ n |\nabla \rho|^2 + 2 S \rho^2 + n \right] \geq 0. \]

Since \( \Sigma \) has a \( C^3 \)-boundary, we have
\[ \rho = d - \frac{1}{2(n-1)} H_{\partial \Sigma} d^2 + O(d^3), \]
and hence
\[ \Delta \theta = - H_{\partial \Sigma} - \frac{1}{n-1} H_{\partial \Sigma} + O(d) = - \frac{n}{n-1} H_{\partial \Sigma} + O(d). \]

By the assumption \( H_{\partial \Sigma} \geq -m \) on \( \partial \Sigma \), we have
\[ \Delta \theta + S \rho \leq \frac{mn}{n-1} \quad \text{on} \quad \partial \Sigma. \]

By the strong maximum principle, we obtain
\[ \Delta \theta + S \rho < \frac{mn}{n-1} \quad \text{in} \quad \Sigma. \]

Therefore, by (2.16),
\[ |\nabla \rho|^2 < 1 + \frac{2m}{n-1} \rho \quad \text{in} \quad \Sigma. \]

We next consider the general case that \( \Sigma \) is the union of an increasing sequence of \( C^3 \)-domains \( \{ \Sigma_i \} \) with \( H_{\partial \Sigma_i} \geq -m \). Let \( \rho_i \) be the positive solution of (2.16), (2.17) in \( \Sigma_i \). By what we just proved in the special case, we have
\[ |\nabla \rho_i|^2 \leq 1 + \frac{2m}{n-1} \rho_i \quad \text{in} \quad \Sigma_i. \]

Note that \( \rho_i \to \rho \) in \( C^1(\Sigma') \) for any subdomain \( \Sigma' \subset \subset \Sigma \). The desired result follows by a simple approximation. \( \square \)

The proof of Lemma 3.3 is modified from [10].

4. Spherical Elliptic Operators

In this section, we discuss a class of elliptic operators over spherical domains with a certain singularity on the boundary. Our main focus is the regularity of solutions near boundary. Throughout this section, differentiation is always with respect to \( \theta \in S^{n-1} \).

Let \( \Sigma \subset S^{n-1} \) be a Lipschitz domain and \( d \) be the distance function in \( \Sigma \) to \( \partial \Sigma \). It is well-known that \( d \) may not be \( C^1 \) even when \( \Sigma \) has a smooth boundary. Take a function \( \rho \in C^\infty(\Sigma) \cap C(\Sigma) \) such that
\[ c_1 \leq \frac{\rho}{d} \leq c_2 \quad \text{on} \quad \Sigma, \]
for some positive constants \( c_1 \) and \( c_2 \). In particular, \( \rho \) is positive in \( \Sigma \) and \( \rho = 0 \) on \( \partial \Sigma \).

For a given positive constant \( \kappa \), set, for any \( u \in C^2(\Sigma) \),
\[ Lu = \Delta_{\theta} u - \frac{\kappa}{\rho^2} u. \]
This is a linear operator on $\Sigma$ with a singular coefficient on $\partial \Sigma$, since $\rho = 0$ on $\partial \Sigma$.

We first establish the existence of weak solutions of $-Lu = f$.

**Lemma 4.1.** Let $\Sigma \subsetneq S^{n-1}$ be a Lipschitz domain and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma)$ be a positive function in $\Sigma$ satisfying (4.1). Then, for any $f \in L^2(\Sigma)$, there exists a unique weak solution $u \in H^1_0(\Sigma)$ of $Lu = -f$, and

$$\|\nabla \theta u\|_{L^2(\Sigma)} + \|\rho^{-1}u\|_{L^2(\Sigma)} \leq C\|f\|_{L^2(\Sigma)},$$

where $C$ is a positive constant depending only on $n$ and $\Sigma$.

**Proof.** By Hardy’s inequality and (4.1), we have, for any $u \in H^1_0(\Sigma)$,

$$\|\rho^{-1}u\|_{L^2(\Sigma)} \leq C\|\nabla \theta u\|_{L^2(\Sigma)},$$

where $C$ is a positive constant depending only on $n$ and $\Sigma$. Set, for any $u,v \in H^1_0(\Sigma)$,

$$(u,v)_\rho = \int_\Omega (\nabla \theta u \cdot \nabla \theta v + \kappa \rho^{-2}uv) d\theta.$$

Denote by $\| \cdot \|_{\rho}$ the induced norm, i.e., for any $u \in H^1_0(\Sigma)$,

$$\|u\|_{\rho} = \left( \int_\Sigma (|\nabla \theta u|^2 + \kappa \rho^{-2}u^2) d\theta \right)^{1/2}.$$

Then, $\| \cdot \|_{\rho}$ is equivalent to the standard norm in $H^1_0(\Sigma)$ and hence, $(H^1_0(\Sigma),(\cdot ,\cdot )_{\rho})$ is a Hilbert space. We can apply the Riesz representation theorem to conclude the desired result. \hfill $\Box$

With the help of the compact embedding of $H^1_0(\Sigma)$ in $L^2(\Sigma)$, we have the following result concerning the eigenvalue problem for $L$.

**Theorem 4.2.** Let $\Sigma \subsetneq S^{n-1}$ be a Lipschitz domain and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma)$ be a positive function in $\Sigma$ satisfying (4.1). Then, there exist an increasing sequence of positive constants $\{\lambda_i\}_{i \geq 1}$, divergent to $\infty$, and an $L^2(\Sigma)$-orthonormal basis $\{\phi_i\}_{i \geq 1}$ such that, for $i \geq 1$, $\phi_i \in C^\infty(\Sigma) \cap H^1_0(\Sigma)$ and $L\phi_i = -\lambda_i\phi_i$ weakly.

We point out that the first eigenvalue $\lambda_1$ is of multiplicity 1. We also have the following Fredholm alternative.

**Theorem 4.3.** Let $\Sigma \subsetneq S^{n-1}$ be a Lipschitz domain and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma)$ be a positive function in $\Sigma$ satisfying (4.1). Assume $\{\lambda_i\}_{i \geq 1}$ and $\{\phi_i\}_{i \geq 1}$ are sequences of eigenvalues and eigenfunctions as in Theorem 4.2, respectively.

(i) For any $\lambda \notin \{\lambda_i\}$ and any $f \in L^2(\Sigma)$, there exists a unique weak solution $u \in H^1_0(\Sigma)$ of the equation

$$(4.3) \quad Lu + \lambda u = f \quad \text{in} \ \Sigma.$$

Moreover,

$$(4.4) \quad \|u\|_{H^1_0(\Sigma)} \leq C\|f\|_{L^2(\Sigma)},$$

where $C$ is a positive constant depending only on $n$, $\lambda$, and $\Sigma$. 
For any \( \lambda = \lambda_i \) for some \( i \) and any \( f \in L^2(\Sigma) \) with \( (f, \phi_k)_{L^2(\Sigma)} = 0 \) for any eigenfunction \( \phi_k \) corresponding to the eigenvalue \( \lambda_i \), there exists a unique weak solution \( u \in H^1_0(\Sigma) \) of the equation (4.3), satisfying \( (u, \phi_k)_{L^2(\Sigma)} = 0 \) for any eigenfunction \( \phi_k \) corresponding to the eigenvalue \( \lambda_i \). Moreover, (4.4) holds.

Next, we study boundary regularity of eigenfunctions \( \phi_i \) in Theorem 4.2 and solutions \( u \) in Theorem 4.3. In the following, we consider only a special function \( \rho \), the unique positive solution of (2.16)-(2.17), and focus on the linear operator \( L \) defined by (4.2) for such \( \rho \).

We prove a more general result for later purposes.

**Theorem 4.4.** Let \( \Sigma \subset S^{n-1} \) be a Lipschitz domain, and \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\overline{\Sigma}) \) be the positive solution of (2.16)-(2.17). Assume that for some \( \lambda \in \mathbb{R} \) and \( f \in C^\infty(\Sigma) \), \( u \in H^1_0(\Sigma) \) is a weak solution of (4.3). Then, there exists a constant \( \nu > 0 \) depending only on \( n \) and \( \Sigma \) such that if, for some constant \( A > 0 \),

\[
|f| \leq A \rho^{\nu-2} \text{ in } \Sigma,
\]

then

\[
|u| \leq C (\|u\|_{L^2(\Sigma)} + A) \rho^\nu \text{ in } \Sigma,
\]

where \( C \) is a positive constant depending only on \( n, \kappa, \lambda, \) and \( \Sigma \). Moreover, \( u \in C^\nu(\Sigma) \) if \( \nu < 1 \) and \( u \in \text{Lip}(\overline{\Sigma}) \) if \( \nu \geq 1 \). In particular, let \( \phi_i \) be an eigenfunction as in Theorem 4.2. Then,

\[
|\phi_i| \leq C \rho^\nu \text{ in } \Sigma,
\]

where \( C \) is a positive constant depending only on \( n, \kappa, \lambda_i, \) and \( \Sigma \). For \( n = 3 \), if \( \kappa \geq 15/4 \), then \( \nu \) can be taken to satisfy \( \nu > 1/2 \).

**Proof.** We write (4.3) as

\[
L_\lambda u \equiv \Delta_\theta u - \frac{1}{\rho^2} (\kappa - \lambda \rho^2) u = f \text{ in } \Sigma.
\]

Take a small positive constant \( \rho_0 \) such that \( \kappa > \lambda \rho_0^2 \) and set

\[
\Sigma_0 \equiv \{ 0 < \rho < \rho_0 \}.
\]

For some domain \( \Sigma' \) with \( \Sigma \setminus \Sigma_0 \subset \subset \Sigma' \subset \subset \Sigma \), by the interior \( L^\infty \)-estimates, we have

\[
\sup_{\Sigma \setminus \Sigma_0} |u| \leq C \left\{ \|u\|_{L^2(\Sigma')} + \|f\|_{L^\infty(\Sigma')} \right\} \leq C' \left\{ \|u\|_{L^2(\Sigma)} + A \right\},
\]

where \( C \) is a positive constant depending only on \( n, \kappa, \lambda, \Sigma_0, \Sigma', \) and \( \Sigma \).

Take any positive constant \( \nu \). By a straightforward computation and (2.16), we have

\[
L_\lambda \rho^\nu = \rho^{\nu-2} \left[ \nu (\nu-1) |\nabla_\theta \rho|^2 - \kappa + \nu \rho \Delta_\theta \rho + \lambda \rho^2 \right]
= \rho^{\nu-2} \left[ \nu (\nu-1 + \frac{n}{2}) |\nabla_\theta \rho|^2 - \kappa - \frac{1}{2} n \nu + (\lambda - \nu S) \rho^2 \right].
\]
Note that the coefficient of $|\nabla \rho|^2$ is positive. Since $\rho$ is Lipschitz in $\Sigma$, we fix a positive constant $\nu$ such that

$$
\nu (\nu - 1 + \frac{1}{2} n) \sup_{\{0 < \rho < \rho_0\}} |\nabla \rho|^2 - \frac{1}{2} \nu n \nu < \kappa.
$$

By taking $\rho_0$ sufficiently small, we have

$$
L_\lambda \rho^\nu \leq -c_0 \kappa \rho^{\nu - 2} \quad \text{in } \Sigma_0,
$$

for some small positive constant $c_0$. In the following, we take

$$
w = M \rho^\nu.
$$

For $M$ sufficiently large, we have

$$
L_\lambda w \leq L_\lambda u \quad \text{in } \Sigma_0.
$$

By (4.11), we choose $M > 0$ large further such that $u < w$ on $\partial \Sigma_0 \cap \Sigma$.

We first discuss the case that $u = w = 0$ on $\partial \Sigma$. Note that $u < w$ on $\partial \Sigma_0 \cap \Sigma$. By the maximum principle, we have $u < w$ in $\Sigma_0$. Similarly, we have $u \geq -w$ in $\Sigma_0$, and hence $|u| \leq w$ in $\Sigma_0$. We obtain the desired estimate of $u$ by combining with (4.9). We note that $M = C(\|u\|_{L^2(\Sigma)} + A)$ for sufficiently large $C$.

We now consider the general case that $u \in H^1_0(\Sigma)$. We note that $\rho^\nu$ may not be in $H^1(\Omega)$ if $\nu$ is small. We take $p$ large so that $[(u - w)^+]^p \in H^1_0(\Sigma_0)$. By multiplying $L_\lambda (u - w) \geq 0$ by $[(u - w)^+]^{2p - 1}$ and integrating by parts, we have

$$
\int_{\Sigma_0} \left( \frac{2p - 1}{p^2} |\nabla (u - w)^+|^2 + \rho^{-2} (\kappa - \lambda \rho^2) [(u - w)^+]^{2p} \right) d\theta \leq 0.
$$

We point out that all terms in the above integral make sense. Then, $(u - w)^+ = 0$ in $\Sigma_0$, and hence $u \leq w$ in $\Sigma_0$. Similarly, we have $u \geq -w$ in $\Sigma_0$, and hence $|u| \leq w$ in $\Sigma_0$. We obtain the desired estimate of $u$ by combining with (4.9).

For any $\theta \in \Sigma$, consider $B_{d/2}(\theta) \subset \Sigma$. By the interior $C^1$-estimate, we have, for any $\alpha \in (0, 1)$ and with $d = d(\theta)$,

$$
d^\alpha \rho^{C^\nu(B_{d/2}(\theta))} + d|\nabla \rho| \leq C \{ \rho^{C^\nu(B_{d/2}(\theta))} + d^2 |f|_{L^\infty(B_{d/2}(\theta))} \} \leq C d^\nu.
$$

If $\nu \in (0, 1)$, we take $\alpha = \nu$ and get $[\rho]_{C^\nu(B_{d/2}(\theta))} \leq C$. Hence, $\rho \in C^\nu(\Sigma)$, with the help of (4.6). If $\nu \geq 1$, we have $|\nabla \rho| \leq C$. Hence, $\rho \in \text{Lip}(\Sigma)$.

For $n = 3$, in view of Lemma 3.2, (4.10) reduces to

$$
(2.8)^2 \nu (\nu + \frac{1}{2}) - \frac{3}{2} \nu < \kappa.
$$

If $\kappa \geq 15/4$, we can take some $\nu > 1/2$ satisfying (4.11). \qed

We now make several remarks. First, if $\Sigma$ is a $C^2$-domain and $\rho \in C^2(\Sigma)$, it is not necessary to assume that $\rho$ is a solution of (2.16)-(2.17), since there is no need to substitute $\Delta \rho$, which is already bounded. Second, the function $f$ in Theorem 4.4 may not be bounded nor $L^2$ in $\Sigma$. This is clear from the assumption (4.3), if $\nu < 2$. It is important for later applications that the estimate (4.6) does not depend on the integrals.
of $f$. Third, the assertion $\nu > 1/2$ for $n = 3$ plays an important role and allows us to improve the integrability of some singular terms to desired levels. Refer to Corollary 5.4.

In Theorem 4.4 we proved that weak solutions are continuous up to boundary. In the next result, we prove the converse; namely, solutions continuous up to boundary are weak solutions.

**Corollary 4.5.** Let $\Sigma \subsetneq S^{n-1}$ be a Lipschitz domain, and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\bar{\Sigma})$ be the positive solution of (2.16) - (2.17). Assume that for some $f \in C^\infty(\Sigma) \cap L^2(\Sigma)$ satisfying (4.5), $u \in C(\Sigma) \cap C^\infty(\Sigma)$ is a solution of $-Lu = f$ in $\Sigma$ and $u = 0$ on $\partial \Sigma$. Then, $u \in H^1_0(\Sigma)$.

**Proof.** By Lemma 4.1, there exists a unique weak solution $w \in H^1_0(\Sigma)$ of $Lw = -f$. By Theorem 4.4, we have $w \in C(\bar{\Sigma})$ with $w = 0$ on $\partial \Sigma$. Then, the maximum principle implies that $u = w$ and hence $u \in H^1_0(\Sigma)$. $\square$

In Corollary 4.5, $f$ is not necessarily bounded in $\Sigma$.

Next, we improve Theorem 4.4. The constant $\nu$ in (4.10) is small in general, even with an explicit lower bound for $n = 3$. In order to improve $\nu$, we need to get a more precise estimate of $|\nabla \theta \rho|^2$ near the boundary $\partial \Sigma$. If $|\nabla \theta \rho|^2$ is close to 1 near $\partial \Sigma$, then we can take $\nu$ such that

$$\nu(\nu - 1) < \kappa.$$ 

The corresponding equality will provide the optimal $\nu$.

**Theorem 4.6.** Let $\Sigma \subsetneq S^{n-1}$ be a Lipschitz domain, and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\bar{\Sigma})$ be the positive solution of (2.16) - (2.17) satisfying

$$|\nabla \theta \rho| \leq 1 + c_0 \rho^\alpha \quad \text{in } \Sigma,$$

for some constants $c_0 \geq 0$ and $\alpha \in (0, 1]$. Assume that $s$ is the positive constant satisfying

$$s(s - 1) = \kappa,$$

and that, for some $\lambda \in \mathbb{R}$ and $f \in C^\infty(\Sigma)$, $u \in H^1_0(\Sigma)$ is a solution of (4.3). If, for some $A > 0$ and $\alpha > 0$ with $\alpha \neq s$,

$$|f| \leq A \rho^{\alpha-2} \quad \text{in } \Sigma,$$

then

$$|u| + \rho|\nabla \theta u| \leq C(\|u\|_{L^2(\Sigma)} + A) \rho^b \quad \text{in } \Sigma,$$

where $b = \min\{\alpha, s\}$, and $C$ is a positive constant depending only on $n$, $\alpha$, $c_0$, $\kappa$, $\alpha$, $\lambda$, and $\Sigma$. Moreover, $u \in C^b(\Sigma)$ if $b < 1$ and $u \in \text{Lip}(\Sigma)$ if $b \geq 1$. In particular, let $\phi_i$ be an eigenfunction as in Theorem 4.2. Then, $\phi_i \in \text{Lip}(\Sigma)$ and

$$|\phi_i| + \rho|\nabla \theta \phi_i| \leq C \rho^s \quad \text{in } \Sigma,$$

where $C$ is a positive constant depending only on $n$, $\kappa$, $\alpha$, $c_0$, $\lambda_i$, and $\Sigma$. 


Proof. We will prove the estimate of \( u \) itself in (4.14). The estimate of the gradient \( \nabla u \) follows from the interior \( C^1 \)-estimate, as in the proof of Theorem 4.4. Let \( L_\lambda \) be the operator in (4.8). By Theorem 4.4, \( u \in C^\nu(\Sigma) \) with \( u = 0 \) on \( \partial \Sigma \), for some \( \nu > 0 \).

We modify the proof of Theorem 4.4 and construct appropriate supersolutions. Take any constant \( a \). By a straightforward computation and (2.10), we have

\[
L_\lambda \rho^a = \rho^{a-2} \left[ a(a-1 + \frac{n}{2}) |\nabla \rho|^2 - \kappa - \frac{na}{2} + (\lambda - aS)\rho^2 \right].
\]

Note that the coefficient of \( |\nabla \rho|^2 \) is positive. By (4.12), we get

\[
L_\lambda \rho^a \leq \rho^{a-2} \left[ a(a-1 + \frac{n}{2})(1 + c_0 \rho^a) - \kappa - \frac{na}{2} + (\lambda - aS)\rho^2 \right]
\]

\[
= \rho^{a-2} \left[ a(a-1) - \kappa + c_0(a-1 + \frac{n}{2}) \rho^a + (\lambda - aS)\rho^2 \right].
\]

If \( a < s \), then \( a(a-1) < \kappa \). By taking \( \rho_0 \) sufficiently small, we have

\[
L_\lambda \rho^a \leq \frac{1}{2}(\kappa - a(a-1)) \rho^{a-2} \quad \text{in } \Sigma_0.
\]

For \( a > s \), we take some fixed \( \tau > s \). Then,

\[
L_\lambda (\rho^s - \rho^\tau) = \rho^{s-2} \left[ s(s-1 + \frac{n}{2}) - \tau \rho^{\tau-s}(\tau-1 + \frac{n}{2}) \right] |\nabla \rho|^2
\]

\[
+ \rho^{s-2} \left[ -\kappa - \frac{ns}{2} + (\lambda - sS)\rho^2 \right] - \rho^{\tau-2} \left[ -\kappa - \frac{n\tau}{2} + (\lambda - \tau S)\rho^2 \right].
\]

For \( \rho \) sufficiently small, the coefficient of \( |\nabla \rho|^2 \) is nonnegative. By replacing \( |\nabla \rho|^2 \) by its upper bound \( 1 + c_0 \rho^a \) due to (4.12), we have

\[
L_\lambda (\rho^s - \rho^\tau) \leq \rho^{s-2} \left[ - (\tau(\tau-1) - \kappa) + (\lambda - sS)\rho^{s+2-\tau} - (\lambda - \tau S)\rho^{\tau-2} \right]
\]

\[
+ c_0 \rho^{s+a-\tau} \left[ (s(s-1) + \frac{ns}{2}) - \rho^{\tau-s}(\tau(\tau-1) + \frac{n\tau}{2}) \right].
\]

where we used \( s(s-1) = \kappa \). Take \( \tau \in (s, s + \alpha) \) with \( \tau \leq a \). Since \( \tau > s \), we have \( \tau(\tau-1) > \kappa \). (The requirement \( a \geq \tau > s \) prohibits \( a = s \)!) By taking \( \rho_0 \) sufficiently small, we have

\[
L_\lambda (\rho^s - \rho^\tau) \leq -\frac{1}{2}(\tau(\tau-1) - \kappa) \rho^{a-2} \quad \text{in } \Sigma_0.
\]

In the following, we take, for \( a \in (0, s) \),

\[
w = M \rho^a,
\]

and, for \( a > s \),

\[
w = M(\rho^s - \rho^\tau).
\]

For \( M \) sufficiently large, we have

\[
L_\lambda w \leq -A \rho^{a-2} \leq L_\lambda (\pm u) \quad \text{in } \Sigma_0.
\]

By taking \( \rho_0 \) small further, we assume that \( w > 0 \) on \( \partial \Sigma_0 \cap \Sigma \). By (4.9), we choose \( M > 0 \) large further such that \( u < w \) on \( \partial \Sigma_0 \cap \Sigma \). Note that \( u = w = 0 \) on \( \partial \Sigma \). By the maximum principle, we get \( \pm u \leq w \) in \( \Sigma_0 \), and hence \( |u| \leq w \) in \( \Sigma_0 \). We obtain the desired estimate of \( u \) by combining with (4.9). \( \square \)
By Lemma 3.3 and Lemma 3.4, we conclude that Theorem 4.6 holds for domains as in Lemma 3.3 and Lemma 3.4. We point out that the power \( b = \min\{a, s\} \) in (4.14) is optimal for both cases \( a < s \) and \( a > s \).

5. Estimates near Cylindrical Boundaries

In this section, we discuss the regularity of solutions of the Yamabe equation on cylinders and focus on the regularity near the boundary.

Let \( \Sigma \subseteq S^{n-1} \) be a Lipschitz domain, and \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \) be the positive solution of (2.16)-(2.17). For some positive constants \( \kappa, \beta, \) and \( T \), consider the equation

\[
(5.1) \quad L v = F(v) \quad \text{in} \ (T, \infty) \times \Sigma,
\]

where \( L \) and \( F \) are given by

\[
(5.2) \quad L v = \partial_t v + \Delta_\theta v - \frac{\kappa}{\rho^2} v - \beta^2 v,
\]

and, for some smooth function \( h \) on \((-1, 1)\) with \( h(0) \neq 0 \),

\[
(5.3) \quad F(v) = \rho^{\beta-2} v^2 h(\rho^\beta v).
\]

Here and hereafter, we always take

\[
(5.4) \quad \kappa = \frac{1}{4} n(n + 2), \quad \beta = \frac{1}{2} (n - 2).
\]

Then, (5.1), (5.2), and (5.3) are (2.26), (2.27), and (2.28), respectively. We note that \( \kappa = 15/4 \) for \( n = 3 \). This fact is used in Theorem 4.4.

For some positive constants \( \gamma \) and \( A > 0 \), let \( v \) be a smooth solution of (5.1) in \((T, \infty) \times \Sigma\) satisfying

\[
(5.5) \quad |v| \leq Ae^{-\gamma t} \quad \text{in} \ (T, \infty) \times \Sigma.
\]

We note that (2.29) implies (5.5) for \( \gamma = \beta \) and some universal constant \( A \) depending only on \( n \). In particular, \( v \) is bounded. Now, we derive a decay estimate near boundary.

**Lemma 5.1.** Let \( \Sigma \subseteq S^{n-1} \) be a Lipschitz domain, and \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \) be the positive solution of (2.16)-(2.17). Assume that \( v \) is a smooth solution of (5.1) in \((T, \infty) \times \Sigma\) satisfying (5.5), for some constants \( \gamma \geq \beta \) and \( A > 0 \). Then,

\[
(5.6) \quad |v| + \rho |\nabla (v, \theta) v| \leq Ce^{-\gamma t} \rho^\nu \quad \text{in} \ (T + 1, \infty) \times \Sigma,
\]

where \( \nu \) is the positive constant as in Theorem 4.4, and \( C \) is a positive constant depending only on \( n, \gamma, A, \) and \( \Sigma \). Moreover, \( v \in C^\nu((T, \infty) \times \Sigma) \) if \( \nu < 1 \) and \( v \in \text{Lip}((T, \infty) \times \Sigma) \) if \( \nu \geq 1 \).

**Proof.** By (5.3) and (5.5), we have

\[
(5.7) \quad |F(v)| \leq Ce^{-2\gamma t} \rho^{\beta-2} \leq Ce^{-\gamma t} \rho^{\beta-2}.
\]
We point out that the assumption on $v$ in (5.5) implies that $v$ is bounded. It is not known whether $v$ is continuous in $(T, \infty) \times \bar{\Sigma}$, with $v = 0$ on $(T, \infty) \times \partial \Sigma$. For a remedy, we consider $\rho^\mu v$, for some $\mu > 0$. A straightforward computation yields

$$L^*_\rho (\rho^\mu v) = \rho^\mu F,$$

where

$$L^*_\rho u = \partial_t u + \Delta u - \frac{2\mu}{\rho} \nabla \rho \cdot \nabla u + \frac{c}{\rho^2} u - \beta^2 u,$$

and

$$c = -\kappa + \mu \left(-\left(\frac{n}{\rho} - \frac{n}{\rho + 1}\right) |\nabla \rho|^2 + \frac{n}{2} + S \rho^2\right).$$

Here, $S$ is the constant given by (2.18). Fix a $t_0 > T + 1$, and set

$$\Omega_0 = \{(t, \theta); |t - t_0| < 1, 0 < \rho < \rho_0\}.$$

Note that

$$c \leq -\kappa + \mu \left(\frac{n}{2} + S \rho^2\right) \leq -\frac{1}{2} \kappa$$

in $\Omega_0$, if $\mu + 1 \leq n/2$, $n\mu < \kappa/2$, and $\rho_0$ is small.

Consider

$$w = e^{-\gamma t} \left[\varepsilon (t - t_0)^2 + \rho^b\right].$$

A straightforward calculation, with the help of (2.16), yields

$$L^*_\rho w = I_1 + I_2,$$

where

$$I_1 = \varepsilon e^{-\gamma t} \left\{ \left((\gamma^2 - \beta^2)(t - t_0)^2 - 4\gamma (t - t_0) + 2\right) + c \rho^{-2}(t - t_0)^2 \right\},$$

and

$$I_2 = e^{-\gamma t} \rho^b \left\{ (\gamma^2 - \beta^2) \rho^2 - \kappa + (b - \mu) \left[ (b - \mu - 1 + \frac{1}{2} n) |\nabla \rho|^2 - \frac{1}{2} n - S \rho^2 \right] \right\}.$$

Choose $b > \mu$. Then, the coefficient of $|\nabla \rho|^2$ in $I_2$ is positive. By $c \leq 0$, we have

$$I_1 \leq C_1 \varepsilon e^{-\gamma t},$$

and

$$I_2 \leq e^{-\gamma t} \rho^b \left\{ C_2 \rho^2 - \kappa + (b - \mu) \left( b - \mu - 1 + \frac{1}{2} n \right) \sup_{\Omega_0} |\nabla \rho|^2 - \frac{1}{2} n (b - \mu) \right\}.$$

By taking $b$, $\varepsilon$, and $\rho_0$ small, with $b \leq 2$ in particular, we obtain

$$L^*_\rho w = I_1 + I_2 \leq -c_0 \kappa e^{-\gamma t} \rho^b \leq - Ae^{-\gamma t} \rho^{b+\mu-2} \leq L^*_\rho (\pm \rho^\mu v)$$

in $\Omega_0$, for some small positive constant $c_0$. By (5.7), we have

$$|\rho^\mu F| \leq Ce^{-\gamma t} \rho^{b+\mu-2}.$$

By taking $M$ large, we obtain

$$L^*_\rho (M w) \leq -c_0 M \kappa e^{-\gamma t} \rho^b \leq -A e^{-\gamma t} \rho^{b+\mu-2} \leq L^*_\rho (\pm \rho^\mu v)$$

in $\Omega_0$. 
if \( b \leq \beta + \mu \). Next, we claim, by choosing \( M \) large further if necessary,

\[
(5.10) \quad \pm \rho^\mu v \leq Mw \quad \text{on } \partial \Omega.
\]

First, \( \rho^\mu v = 0 \) on \( (T, \infty) \times \partial \Sigma \). (Here, we used \( \mu > 0 \).) Next, by (5.5), we have,

\[
\pm \rho^\mu v \leq Ce^{-\gamma t} \quad \text{on } |t - t_0| = 1 \quad \text{or} \quad \rho = \rho_0.
\]

On the other hand, \( w = 0 \) on \( (T, \infty) \times \partial \Sigma \), \( w \geq e^{-\gamma t} \rho_0 \) on \( \rho = \rho_0 \). This proves (5.10) for some \( M \) sufficiently large. We emphasize that \( M \) is independent of \( t_0 \). By the maximum principle, we obtain

\[
\pm (\rho^\mu v) \leq Mw \quad \text{in } \Omega_0,
\]

and in particular at \( t = t_0 \),

\[
|v(t_0, \theta)| \leq Me^{-\gamma t_0} \rho b - \mu.
\]

This holds for any \( t_0 > T + 1 \) and hence proves the estimate of \( v \) in (5.6) for \( \nu = b - \mu > 0 \).

In fact, we can choose \( \nu \) to satisfy \( \nu \leq \min\{2, \beta\} \) and (4.10).

The assertion concerning the Hölder continuity or the Lipschitz continuity of \( v \) follows from the interior \( C^1 \)-estimate, as in the proof of Theorem 4.4. □

If \( v \) is continuous in \( (T, \infty) \times \Sigma \) with \( v = 0 \) on \( (T, \infty) \times \partial \Sigma \), there is no need to introduce \( \mathcal{L}_e \) in (5.8). We can simply apply \( \mathcal{L} \) to \( w \) in (5.9). In other words, we can take \( \mu = 0 \) in the proof above.

In the next result, we discuss derivatives of solutions with respect to \( t \). As we see in the proof, it is more helpful to view \( F(v) \) in the equation (5.1) as a perturbative term of the operator \( \mathcal{L}v \), instead of a nonhomogeneous term.

**Lemma 5.2.** Let \( \Sigma \subset \mathbb{S}^{n-1} \) be a Lipschitz domain, and \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \) be the positive solution of (2.16)–(2.17). Assume that \( v \) is a smooth solution of (5.1) in \( (T, \infty) \times \Sigma \) satisfying (5.5), for some constants \( \gamma \geq \beta \) and \( A > 0 \). Then,

\[
(5.11) \quad |\partial_t v| + |\partial_{tt} v| \leq Ce^{-\gamma t} \rho^\nu \quad \text{in } (T + 1, \infty) \times \Sigma,
\]

where \( \nu \) is the constant as in Theorem 4.4 and \( C \) is a positive constant depending only on \( n, \gamma, A, \) and \( \Sigma \). In particular, \( \partial_t v, \partial_{tt} v \in C((T + 1, \infty) \times \Sigma) \) with \( \partial_t v = \partial_{tt} v = 0 \) on \( (T + 1, \infty) \times \partial \Sigma \).

**Proof.** We first derive the estimate of \( \partial_t v \). Set

\[
(5.12) \quad \epsilon(s) = sh(s).
\]

Then, \( \epsilon \) is a smooth function on \( (-1, 1) \) with \( \epsilon(0) = 0 \), and

\[
F(v) = \rho^{-2}v\epsilon(\rho^\beta v).
\]

We now write (5.11) as

\[
(5.13) \quad \partial_{tt} v + \Delta_\theta v - \frac{1}{\rho^2}(\kappa + \epsilon(\rho^\beta v))v - \beta^2 v = 0.
\]

Then, (5.5) implies

\[
|\epsilon(\rho^\beta v)| \leq C \rho^\beta |v| \leq C.
\]
Consider any \( \Sigma' \subset \subset \Sigma \). For any \( t > T + 1 \) and \( \theta \in \Sigma' \), by the interior \( C^1 \)-estimates and (5.5), we have, with \( r = \min\{d(\theta)/2, 1\} \),

\[
|\partial_t v(t, \theta)| \leq C \sup_{(t-r,t+r) \times B_r(\theta)} |v| \leq C e^{-\gamma t}.
\]

To estimate \( \partial_t v \) near \( \partial \Sigma \), we fix an arbitrary \( t_0 > T + 1 \) and consider, for some \( \rho_0 > 0 \) to be determined,

\[
\Omega_0 = \{(t, \theta); t < t_0 + 1, \ 0 < \rho(\theta) < \rho_0\}.
\]

Set

\[
v_1(t) = v(t) - v(2t_0 - t).
\]

We now evaluate the equation (5.13) at \( t \) and \( 2t_0 - t \), respectively, and take a difference. A straightforward computation yields

\[
\partial_t v_1 + \Delta_\theta v_1 - \frac{1}{\rho^2} (\kappa + \tilde{\varepsilon}) v_1 - \beta^2 v_1 = 0,
\]

where \( \tilde{\varepsilon} \) is given by

\[
\tilde{\varepsilon}(t, \theta) = \epsilon(\rho^3 v(t)) + \rho^3 v(2t_0 - t) \int_0^1 \epsilon'(s\rho^3 v(t) + (1 - s)\rho^3 v(2t_0 - t)) ds.
\]

For any \( t_0 < t < t_0 + 1 \), we have

\[
|\tilde{\varepsilon}(t, \theta)| \leq C \rho^3 (|v(t)| + |v(2t_0 - t)|) \leq C e^{-\gamma t_0} \rho^3.
\]

Set

\[
\mathcal{L}_1 w = \partial_t w + \Delta_\theta w - \frac{1}{\rho^2} (\kappa + \tilde{\varepsilon}) w - \beta^2 w.
\]

Then, \( \mathcal{L}_1 v_1 = 0 \). To construct a supersolution, set

\[
(5.16) \quad w = (t - t_0) e^{-\gamma t} \rho^\nu.
\]

A straightforward computation yields

\[
\mathcal{L}_1 w = (t - t_0) e^{-\gamma t} \rho^\nu [\gamma^2 - \beta^2 - \kappa + \nu(\nu - 1 + \frac{n}{2})|\nabla_\theta \rho|^2 - \frac{n}{2} \nu \rho^2 - \nu S \rho^2 + \tilde{\varepsilon}] - 2\gamma e^{-\gamma t} \rho^\nu.
\]

We now take \( \nu \) to satisfy (4.10). By taking \( \rho_0 \) sufficiently small, we have

\[
\mathcal{L}_1 w \leq -c_0 \kappa(t - t_0) e^{-\gamma t} \rho^\nu - 2 \leq 0 \quad \text{in } \Omega_0,
\]

for some small positive constant \( c_0 \). Next, we compare \( \pm v_1 \) and \( w \) on \( \partial \Omega_0 \). First, \( v_1 = 0 \) on \( (t_0, t_0 + 1) \times \partial \Sigma \) and on \( \{ t = t_0 \} \times \Sigma \). Next, we get, by (5.6),

\[
|v_1| \leq C e^{-\gamma t_0} \rho^\nu \quad \text{on } \{ t_0 + 1 \} \times \Sigma,
\]

and, by (5.14),

\[
|v_1| \leq C(t - t_0) \sup_{(t_0-1,t_0+1) \times \{ \rho = \rho_0 \}} |\partial_t v| \leq C(t - t_0) e^{-\gamma t} \quad \text{on } (t_0, t_0 + 1) \times \{ \rho = \rho_0 \}.
\]

For \( w \), we have \( w = 0 \) on \( (t_0, t_0 + 1) \times \partial \Sigma \) and on \( \{ t = t_0 \} \times \Sigma \), \( w = e^{-\gamma t} e^{-\gamma t_0} \rho^\nu \) on \( \{ t = t_0 + 1 \} \times \Sigma \) and \( w = (t - t_0) e^{-\gamma t_0} \rho^\nu \) on \( (t_0, t_0 + 1) \times \{ \rho = \rho_0 \} \). By choosing \( M \).
sufficiently large, independent of \( t_0 \), we have \( \pm v_1 \leq Mw \) on \( \partial \Omega_0 \). By the maximum principle, we obtain \( \pm v_1 \leq Mw \) in \( \Omega_0 \), and hence
\[
|v(t) - v(2t_0 - t)| \leq M(t - t_0)e^{-\gamma t} \rho^\nu \text{ in } \Omega_0.
\]
By dividing by \( t - t_0 \) and taking the limit as \( t \to t_0^+ \), we get, for any \( \rho \in (0, \rho_0) \),
\[
|\partial_t v(t_0, \theta)| \leq M e^{-\gamma t_0} \rho^\nu.
\]
This holds for any \( t_0 > T + 2 \). By combining with (5.14) for \( \Sigma' = \{ \rho > \rho_0 \} \), we obtain (5.17)
\[
|\partial_t v| \leq C e^{-\gamma t} \rho^\nu \text{ in } (T + 2, \infty) \times \Sigma.
\]
This is the estimate of \( \partial_t v \) in (5.11).

The derivation of the estimate of \( \partial_{tt} v \) is similar. We point out some key steps. By differentiating (5.13) with respect to \( t \), we have
\[
\partial_{tt} v_t + \Delta_\theta v_t - \frac{1}{\rho^2} (\kappa + \epsilon_1(\rho^\beta v)) v_t - \beta^2 v_t = 0,
\]
where
\[
\epsilon_1(s) = \epsilon(s) + s \epsilon'(s).
\]
Note that \( \epsilon_1 \) is a smooth function on \((-1, 1)\) with \( \epsilon_1(0) = 0 \). Similar as (5.14), we have, for any \( t > T + 3 \) and any \( \theta \in \Sigma' \subset \subset \Sigma \),
\[
|\partial_{tt} v(t, \theta)| \leq C e^{-\gamma t}.
\]
Set
\[
v_2(t) = v_t(t) - v_t(2t_0 - t).
\]
Similarly, we have
\[
\partial_{tt} v_2 + \Delta_\theta v_2 - \frac{1}{\rho^2} (\kappa + \epsilon_1(\rho^\beta v)) v_2 - \beta^2 v_2 = f_1,
\]
where \( f_1 \) is given by
\[
f_1 = \rho^{-2} [\epsilon_1(\rho^\beta v(t)) - \epsilon_1(\rho^\beta v(2t_0 - t))] v_t(2t_0 - t).
\]
By (5.17), we have
\[
|f_1| \leq C \rho^{-2} |v(t) - v(2t_0 - t)||v_t(2t_0 - t)|
\leq C (t - t_0) e^{-2\gamma t} \rho^{2 + 2\nu - 2} \leq C (t - t_0) e^{-\gamma t} \rho^{\nu - 2}.
\]
The rest of the proof is almost identical as the derivation of \( \partial_t v \), and hence will be omitted. We point out that the equation for \( v_1 \) in (5.15) can be regarded as a homogeneous equation and that the equation for \( v_2 \) in (5.20) is not homogeneous. The function \( w \) in (5.16) still serves as a supersolution.

We can derive estimates similar as (5.11) for higher derivatives of \( v \) with respect to \( t \). However, the estimate of the second derivative is sufficient for our applications.
Proof. We write the equation (5.1) or (5.13) as

\[ \Delta_\theta v - \frac{1}{\rho^2} (\kappa + \epsilon(\rho^\beta v)) v = -\partial_t v + \beta^2 v. \]  

By \( \epsilon(0) = 0 \) and (5.5), we take \( T_* \) large such that \( |\epsilon(\rho^\beta v)| < \kappa/2 \) for \( t > T_* \). By (5.11), the right-hand side of (5.22) is a bounded function. By restricting (5.22) on \( \{t\} \times \Sigma \) for each \( t > T_* \), we conclude \( v(t, \cdot) \in H^1_0(\Sigma) \), by Corollary 4.5. By multiplying (5.22) by \(-v\) and integrating over \( \{t\} \times \Sigma \), we have

\[ \int_{\{t\} \times \Sigma} [|\nabla_\theta v|^2 + (\kappa + \epsilon(\rho^\beta v)) \rho^{-2} v^2] d\theta = \int_{\{t\} \times \Sigma} (\partial_t v - \beta^2 v) v d\theta. \]  

By (5.5) and (5.11), we get

\[ \int_{\Sigma} (|\nabla_\theta v(t, \cdot)|^2 + \rho^{-2} v^2(t, \cdot)) d\theta \leq C e^{-\gamma t} \|v(t, \cdot)\|_{L^2(\Sigma)}. \]  

This implies the desired result.

The property \( v(t, \cdot) \in H^1_0(\Sigma) \) plays an essential role and permits us to integrate by parts on each slice \( \{t\} \times \Sigma \), in deriving (5.23).

Corollary 5.4. Let \( \Sigma \subseteq \mathbb{S}^{n-1} \) be a Lipschitz domain, and \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \) be the positive solution of (2.16)–(2.17). Assume that \( v \) is a smooth solution of (5.1) in \((T, \infty) \times \Sigma\) satisfying (5.5). Then for any \( t > T \), \( F(v)(t, \cdot) \in L^p(\Sigma) \) for some \( p > \max\{2, n/2\} \), and, for any \( q \in [1, p] \),

\[ \|F(v)(t, \cdot)\|_{L^q(\Sigma)} \leq C e^{-\gamma t}, \]  

where \( C \) is a positive constant depending only on \( n \), \( \gamma \), \( A \), and \( \Sigma \).

Proof. Recall that \( \beta = \frac{1}{2}(n - 2) \) by (5.4). By (5.3), we have

\[ |F(v)| \leq C \rho^{\frac{n-6}{2}} v^2 = C \rho^{\frac{n-2}{2}} (\rho^{-1} v)^2. \]  

Then, \( F(v)(t, \cdot) \in L^1(\Sigma) \) and (5.25) holds for \( q = 1 \), by (5.21).

If \( n \geq 6 \), then \( F(v) \) is bounded and \( |F(v)| \leq C v^2 \). Hence, for each \( t > T \),

\[ \|F(v)(t, \cdot)\|_{L^\infty(\Sigma)} \leq C \|v(t, \cdot)\|_{L^\infty(\Sigma)}^2. \]  

For \( n = 5 \), we have \( \beta = 3/2 \) and then

\[ |F(v)| \leq C \rho^{-\frac{1}{2}} v^2 = C v^2 (\rho^{-1} v)^{\frac{1}{2}}. \]
Hence, for each $t > T$,
\[
\|F(v)(t, \cdot)\|_{L^1(\Sigma)} \leq C\|v(t, \cdot)\|_{L^2(\Sigma)}^{\frac{3}{n}} \|\rho^{-1}v(t, \cdot)\|_{L^2(\Sigma)}^{\frac{1}{n}}.
\]
We obtain the desired estimate for $n \geq 5$ by (5.5) and (5.21).

For $n = 4$, we have $\beta = 1$. If $\nu < 1/2$, then,
\[
|F(v)| \leq C\rho^{-1}v^2 = C(\rho^{-\nu}v)^{\frac{1}{1-\nu}}(\rho^{-1}v)^{\frac{1-2\nu}{1-\nu}},
\]
and hence, by (5.6) and (5.21),
\[
\|F(v)(t, \cdot)\|_{L^2(\Sigma)}^{\frac{2(1-\nu)}{1-2\nu}} \leq C(e^{-\gamma t})^{\frac{1}{1-\nu}}\|\rho^{-1}v(t, \cdot)\|_{L^2(\Sigma)}^{\frac{1-2\nu}{1-\nu}} \leq Ce^{-2\gamma t}.
\]
Note that $\frac{2(1-\nu)}{1-2\nu} > 2$. If $\nu \geq 1$, we get a similar estimate of the $L^\infty$-norm.

For $n = 3$, we have $\beta = 1/2$. If $\nu < 3/4$, then,
\[
|F(v)| \leq C\rho^{-\frac{3}{2}}v^2 = C(\rho^{-\nu}v)^{\frac{1}{2(1-\nu)}}(\rho^{-1}v)^{\frac{3-4\nu}{2(1-\nu)}},
\]
and hence, by (5.6) and (5.21),
\[
\|F(v)(t, \cdot)\|_{L^2(\Sigma)}^{\frac{4(1-\nu)}{3-4\nu}} \leq C(e^{-\gamma t})^{\frac{1}{2(1-\nu)}}\|\rho^{-1}v(t, \cdot)\|_{L^2(\Sigma)}^{\frac{3-4\nu}{2(1-\nu)}} \leq Ce^{-2\gamma t}.
\]
Note that $\frac{4(1-\nu)}{3-4\nu} > 2$, since $\nu > 1/2$ by Theorem 4.4. If $\nu \geq 3/4$, we get a similar estimate of the $L^\infty$-norm.

As we see in the proof, we can take $p = \infty$ for $n \geq 6$, $p = 4$ for $n = 5$, and some universal $p > 2$ for $n = 3$. These choices of $p$ are universal, independent of specific domains $\Sigma$. The discussion of the case $n = 3$ is quite subtle. It depends essentially on the estimate of the Hölder index that $\nu > 1/2$.

In the rest of this section, we discuss optimal regularity near boundary. Recall that $\kappa$ and $\beta$ are given by (5.4). Then, the positive $s$ satisfying (4.13) is in fact given by
\[
(5.26)
\]
\[
s = \frac{1}{2}(n + 2).
\]
We now improve Lemma 5.1

**Lemma 5.5.** Let $\Sigma \subseteq \mathbb{S}^{n-1}$ be a Lipschitz domain, and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma)$ be the positive solution of (2.10) - (2.17), satisfying (4.12), for some constants $c_0 \geq 0$ and $\alpha \in (0, 1]$. Assume that $v$ is a smooth solution of (5.1) in $(T, \infty) \times \Sigma$ satisfying (5.5), for some constants $\gamma \geq \beta$ and $A > 0$. Then,
\[
(5.27)
\]
\[
|v| + \rho|\nabla_{(t, \theta)}v| \leq Ce^{-\gamma t}\rho^s \quad \text{in} \ (T+1, \infty) \times \Sigma,
\]
where $C$ is a positive constant depending only on $n$, $\gamma$, $A$ and $\Sigma$. Moreover, $v$ is Lipschitz in $(T, \infty) \times \Sigma$. 

Proof. For some \( a > 0 \), assume
\[
|v| \leq Ce^{-\gamma t} \rho^a.
\]
Then, by (5.23), we have
\[
|F(v)| \leq Ce^{-2\gamma t} \rho^{\beta+2a-2} \leq Ce^{-\gamma t} \rho^{a+\alpha-2},
\]
where \( \alpha \) above is the minimum of \( \beta \) and \( \alpha \) in (4.12). Obviously, \( \alpha \in (0, 1] \). It will be apparent soon why we replace the power \( \beta + 2a \) by a smaller one \( a + \alpha \). We now proceed to prove, for some \( b > a \),
\[
|v| \leq Ce^{-\gamma t} \rho^b \quad \text{in } (T+1, \infty) \times \Sigma.
\]
We consider two cases:
- Case 1: If \( 0 < a < a + \alpha < s \), we can take \( b = a + \alpha \);
- Case 2: If \( a < s < a + \alpha \), we can take \( b = s \).
Suppose this is already done. We note that (5.28) holds for \( a = \nu \) by Lemma 5.1. We can start to iterate. After finitely many steps and by adjusting the initial \( a \) if necessary, we obtain the estimate of \( v \) in (5.27). Then, the estimate of the gradient \( \nabla (t, \theta)v \) in (5.27) follows from the interior \( C^1 \)-estimate, as in the proof of Theorem 4.4.

The proof of both Case 1 and Case 2 is similar as that of Lemma 5.1. Fix a \( t_0 > T + 1 \) and set, for some \( \rho_0 \),
\[
\Omega_0 = \{(t, \theta); |t-t_0| < 1, 0 < \rho < \rho_0\}.
\]
We first consider Case 1, \( 0 < a < a + \alpha < s \) For some \( b > a \) and some \( \varepsilon > 0 \) to be determined, we set
\[
w = e^{-\gamma t} [\varepsilon(t-t_0)^2 \rho^a + \rho^b].
\]
If \( a(a-1) < \kappa \) and \( b(b-1) < \kappa \), a similar computation as in the proof of Lemma 5.1 yields
\[
\mathcal{L}w \leq C_1 \varepsilon e^{-\gamma t} \rho^{a+\alpha-2} + e^{-\gamma t} \rho^{b-2} \left[ - [\kappa - b(b-1)] + C_2 \rho^a \right].
\]
We take \( b = a + \alpha \). By taking \( \varepsilon \) and \( \rho_0 \) small, we obtain
\[
\mathcal{L}w \leq -\frac{1}{2} [\kappa - b(b-1)] e^{-\gamma t} \rho^{b-2} \quad \text{in } \Omega_0.
\]
By (5.29), we get
\[
|F| \leq Ae^{-\gamma t} \rho^{b-2}.
\]
By taking \( M \) large, we obtain
\[
\mathcal{L}(Mw) \leq -Ae^{-\gamma t} \rho^{b-2} \leq \mathcal{L}(\pm v) \quad \text{in } \Omega_0.
\]
Next, similarly as in (5.10), we have, for \( M \) sufficiently large,
\[
\pm v \leq Mw \quad \text{on } \partial \Omega_0.
\]
By the maximum principle, we obtain \( \pm v \leq Mw \) in \( \Omega_0 \), and in particular at \( t = t_0 \),
\[
|v(t_0, \theta)| \leq Me^{-\gamma t_0} \rho^b.
\]
This holds for any \( t_0 > T + 1 \) and hence proves the estimate (5.30) for any \( t > T + 1 \).
Next, we consider Case 2, $a < s < a + \alpha$. Instead of (5.31), we consider, for some $\tau > s$

$$w = e^{-\gamma t} \left[ \varepsilon(t-t_0)^2 \rho^\alpha + \rho^\beta - \rho^\tau \right].$$

The proof of the estimate (5.30) for $b = s$ can be modified easily, with a combination of some calculations in the proofs of Case 1 and Theorem 4.6.

In Lemma 5.5, the best decay estimate near $\partial \Sigma$ is given by $\rho^s$. This is consistent with the decay estimate of eigenfunctions near $\partial \Sigma$ as in Theorem 4.6. Lemma 5.5 simply says that Lemma 5.1 holds for $\nu = s$ under the additional assumption (4.12).

We now make some remarks. There are two main issues in this section, integration by parts on each slice $\{t\} \times \Sigma$ and the boundedness or the integrability of the nonlinear term $F(v)$, which are essentially needed in the proof of Theorem 6.1 and Lemma 7.2 later on. If $\Sigma$ is merely Lipschitz, Lemma 5.1 asserts that $v$ is globally Hölder continuous, with possibly a small Hölder index $\nu$. Such smallness does not allow us to integrate by parts. In Corollary 5.3, we proved that $v$ is $H^1_0$ when restricted to each slice $\{t\} \times \Sigma$, and hence we are able to perform integration by parts. Next, concerning the function $F(v)$, by (5.3) and (5.6), we have

$$|F(v)| \leq C \rho^{\beta-2} v^2 \leq C e^{-2\gamma t} \rho^{\beta-2+2\nu}.$$  

If $n \geq 6$, then $\beta \geq 2$, and hence $F(v)$ is bounded. In Corollary 5.3, we proved that $F(v)$ is actually $L^p$-integrable, for some $p > \max\{2, n/2\}$ in the case $3 \leq n \leq 5$. The assertion $p > n/2$ allows us to apply a well-known $L^{\infty}$-estimates.

We next examine the case when (4.12) is assumed. First, Lemma 5.5 asserts that $v$ is Lipschitz, and then allows us to perform integration by parts. Next, with $\nu = s$ as in Lemma 5.5 we have $\beta + 2\nu - 2 > 0$, and hence $F(v)$ is always bounded by (5.33). In conclusion, when (4.12) is assumed, Lemma 5.2 and Corollaries 5.3,5.4 are not needed.

6. An Optimal Estimate in the $t$-direction

In this section and the next, we continue to study the Yamabe equation in cylinders as introduced in the previous section, and concentrate on expansions as $t \to \infty$. We adopt notations in the previous section.

Let $v$ be a smooth solution of (5.1) in $(T, \infty) \times \Sigma$ satisfying

$$|v| \leq C_0 e^{-\beta t} \quad \text{in } (T, \infty) \times \Sigma,$$

for some positive constant $C_0$. In other words, (5.5) holds for $\gamma = \beta$. Hence, all results proved in the previous section hold for $\gamma = \beta$. Recall that $\beta = \frac{n-2}{2}$ by (5.4). The assumption (6.1) demonstrates that $v$ decays exponentially at the rate $\beta$ as $t \to \infty$. Next, we improve the exponential decay in the $t$-direction and derive an optimal decay rate. In the following result, $\lambda_1$ is the first eigenvalue as in Theorem 4.2.

**Theorem 6.1.** Let $\Sigma \subseteq S^{n-1}$ be a Lipschitz domain, and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma)$ be the positive solution of (2.16)-(2.17). Assume that $v$ is a smooth solution of (5.1) in

$$w = e^{-\gamma t} \left[ \varepsilon(t-t_0)^2 \rho^\alpha + \rho^\beta - \rho^\tau \right].$$
(T, ∞) × Σ satisfying (6.1). Then,

\[ |v| \leq C e^{-\gamma_1 t} \rho'' \text{ in } (T_*, \infty) \times \Sigma, \]

where \( \gamma_1 = \sqrt{\lambda_1 + \beta^2} \), \( \nu \) is the positive constant as in Theorem 4.4, and \( T_* \) and \( C \) are positive constants depending only on \( n \), \( C_0 \), and \( \Sigma \).

**Proof.** We adopt notations in the proof of Lemma 5.2 and Corollary 5.3, and divide the proof into several steps.

**Step 1.** We write (5.1) as (5.13), i.e.,

\[ \partial_t v + \Delta \theta + \frac{1}{\rho^2} \left( \kappa + \epsilon(\rho^3 v) \right) v - \beta^2 v = 0. \]

For simplicity, we write \( v(t) = v(t,) \). For any \( t \in (T, \infty) \), we have \( v(t) \in H^1_0(\Sigma) \) by Corollary 5.3. We write (5.23) as

\[ \hat{\Sigma} \left[ v_{tt}(t) v(t) - \beta^2 v^2(t) \right] d\theta = \hat{\Sigma} \left[ \nabla_{\theta} v(t)^2 + \left( \kappa + \epsilon(\rho^3 v(t)) \right) \rho^{-2} v^2(t) \right] d\theta. \]

**Step 2.** We claim that there exists a \( T_* > T \) such that, for any \( t > T_* \),

\[ \|v(t,\cdot)\|_{L^2(\Sigma)} \leq C e^{-\gamma_1 t}. \]

To prove (6.5), set

\[ y(t) = \left( \int_\Sigma v^2(t) d\theta \right)^{1/2}. \]

Then,

\[ y(t)y'(t) = \int_\Sigma v(t) v_t(t) d\theta, \]

and

\[ y(t)y''(t) + [y'(t)]^2 = \int_\Sigma \left[ v(t) v_{tt}(t) + v_t^2(t) \right] d\theta. \]

The Cauchy inequality implies that, if \( y(t) > 0 \), then

\[ [y'(t)]^2 \leq \int_\Sigma v_t^2(t) d\theta, \]

and hence

\[ y(t)y''(t) \geq \int_\Sigma v(t) v_{tt}(t) d\theta. \]

By (5.12) and (6.1), we have

\[ |\epsilon(\rho^3 v(t))| \leq C \rho^2 |v(t)| \leq \kappa C_* e^{-\beta t}. \]
We take $T_*$ so that $C_* e^{-\beta T_*} < 1/2$. Since $v(t) \in H^1_0(\Sigma)$, we have, for any $t > T_*$,
\[
\int_\Sigma \left( |\nabla_\theta v(t)|^2 + (\kappa + \epsilon(\rho^\beta v(t))) \rho^{-2} v^2(t) \right) d\theta \\
\geq (1 - C_* e^{-\beta t}) \int_\Sigma \left( |\nabla_\theta v(t)|^2 + \kappa \rho^{-2} v^2(t) \right) d\theta \\
\geq \lambda_1 (1 - C_* e^{-\beta t}) \int_\Sigma v^2(t) d\theta.
\]
By simple substitutions in (6.4), we get
\[y(t) y''(t) - (\lambda_1 + \beta^2) y^2(t) + \lambda_1 C_* e^{-\beta t} y^2(t) \geq 0.\]
With $\gamma_1 = \sqrt{\lambda_1 + \beta^2}$, set
\[L_* w = w'' - (\gamma_1^2 - \lambda_1 C_* e^{-\beta t}) w.\]
If $y(t) > 0$, we obtain
\[L_* y \geq 0.
\]
Note that $\lambda_1 C_* e^{-\beta T_*} < \gamma_1^2/2$.

Set
\[z(t) = C e^{-\gamma_1 t} \arctan t.\]
A straightforward computation yields
\[L_* z = C e^{-\gamma_1 t} \left[ - \frac{2t}{(1 + t^2)^2} - \frac{2\gamma_1}{1 + t^2} + \lambda_1 C_* e^{-\beta t} \arctan t \right] \]
\[\leq C e^{-\gamma_1 t} \left[ - 2\gamma_1 + \lambda_1 C_* (1 + t^2)e^{-\beta t} \arctan t \right] < 0,
\]
if $t > T_*$ for $T_*$ large further. Then, for some constant $C$ sufficiently large, we have $z(T_*) \geq y(T_*)$ and, if $y(t) > 0$,
\[L_*(z - y)(t) \leq 0.
\]
Note that $y(t) \to 0$ as $t \to \infty$. If $z - y$ is negative somewhere on $(T_*, \infty)$, then the minimum of $z - y$ is negative and attained at some $t_* \in (T_*, \infty)$. Hence, $y(t_*) > z(t_*) > 0$ and (6.7) holds at $t_*$. However, $(z - y)'(t_*) < 0$, $(z - y)(t_*) < 0$, and $(z - y)''(t_*) \geq 0$, contradicting (6.7). Therefore, for any $t > T_*$, $y(t) \leq z(t)$, and hence (6.5) holds.

Step 3. We now prove (6.2) for any $t > T_* + 1$. First, by applying the De Giorgi-Moser $L^\infty$-estimates to the equation (6.3), we obtain, for any $t > T_* + 1$,
\[\sup_{ \{t\} \times \Sigma} |v| \leq C \|v\|_{L^2((t-1,t+1)\times \Sigma)} \leq C e^{-\gamma_1 t}.
\]
To get estimate (6.8), we only need to introduce cutoff functions along the $t$-direction, since $v(t) \in H^1_0(\Sigma)$ for any $t > T_*$. Also note that the coefficient of the term $\rho^{-2} v$ in the equation (6.3) has a good sign. Hence, the corresponding term can be simply dropped in the estimate (6.8). With (6.8), we have the desired results by applying Lemma 5.1 with $\gamma = \gamma_1$. \qed
In the proof of Theorem 6.1, we performed integration by parts on each slice \{t\} \times \Sigma, the justification of which is provided by the assertion that \(v(t, \cdot) \in H^1_0(\Sigma)\) by Corollary 5.3. We also point out that the exponential decay rate \(\gamma_1\) in \(t\) in (6.2) is optimal.

Under the additional assumption that \(\rho\) also satisfies (4.12), Theorem 6.1 holds for \(\nu = s\). In this case, (6.2) has the form

\[
|v| \leq Ce^{-\gamma_1 t} \rho^s \quad \text{in } (T_*, \infty) \times \Sigma.
\]

This is an optimal estimate both in \(t\) and near \(\partial \Sigma\).

7. Asymptotic Expansions

In this section, we continue to study the Yamabe equation in cylinders. We first discuss the corresponding linear equations and then study the nonlinear equation (5.1).

Let \(L\) be the operator given by (4.2). By Theorem 4.2, there exist an increasing sequence of positive constants \(\{\lambda_i\}_{i \geq 1}\), divergent to \(\infty\), and an \(L^2(\Sigma)\)-orthonormal basis \(\{\phi_i\}_{i \geq 1}\) such that, for \(i \geq 1\), \(\phi_i \in C^\infty(\Sigma) \cap H^1_0(\Sigma)\) and \(L\phi_i = -\lambda_i \phi_i\).

By Theorem 4.4, \(\phi_i \in C^{\nu}(\bar{\Sigma})\) for some \(\nu > 0\). In the following, we fix such a sequence \(\{\phi_i\}\).

Let \(L\) be the operator given by (5.2). Then,

\[
L v = \partial_{tt} v + L v - \beta^2 v.
\]

For each \(i \geq 1\) and any \(\psi = \psi(t) \in C^2(\mathbb{R})\), we write

\[
L(\psi \phi_i) = (L_i \psi) \phi_i,
\]

where \(L_i\) is given by

\[
L_i \psi = \psi_{tt} - (\lambda_i + \beta^2) \psi.
\]

Then, the kernel \(\text{Ker}(L_i)\) has a basis \(e^{-\gamma_i t}\) and \(e^{\gamma_i t}\), with

\[
\gamma_i = \sqrt{\lambda_i + \beta^2}.
\]

We note that \(e^{-\gamma_i t}\) decays exponentially and \(e^{\gamma_i t}\) grows exponentially as \(t \to \infty\), for each \(i \geq 1\), and that \(\{\gamma_i\}_{i \geq 1}\) is increasing and divergent to \(\infty\).

For each fixed \(i \geq 1\), we now analyze the linear equation

\[
L_i \psi = f \quad \text{on } (T, \infty).
\]

We study a class of solutions of (7.5), which converge to zero as \(t \to \infty\).

**Lemma 7.1.** Let \(\gamma > 0\) be a constant, \(m \geq 0\) be an integer, and \(f\) be a smooth function on \((T, \infty)\), satisfying, for any \(t > T\),

\[
|f(t)| \leq Ct^m e^{-\gamma t}.
\]

Let \(\psi\) be a smooth solution of (7.5) on \((T, \infty)\) such that \(\psi(t) \to 0\) as \(t \to \infty\). Then, for any \(t > T\),

\[
|\psi(t)| \leq \begin{cases} 
Ct^m e^{-\gamma t} & \text{if } \gamma < \gamma_i, \\
Ct^{m+1} e^{-\gamma t} & \text{if } \gamma = \gamma_i,
\end{cases}
\]
and there exists a constant \( c \) such that

\[
|\psi(t) - ce^{-\gamma t}| \leq Ct^m e^{-\gamma t} \quad \text{if} \quad \gamma > \gamma_i.
\]

**Proof.** We first construct a particular solution with a desired decay rate as \( t \to \infty \). We claim that there exists a smooth solution \( \psi \) of (7.5) on \((T, \infty)\) such that, for any \( t > T \),

\[
|\psi(t)| \leq \begin{cases} 
  Ct^m e^{-\gamma t} & \text{if} \quad \gamma \neq \gamma_i, \\
  Ct^{m+1} e^{-\gamma t} & \text{if} \quad \gamma = \gamma_i.
\end{cases}
\]

The proof is standard. Since the kernel \( \text{Ker}(L_i) \) has a basis \( e^{-\gamma t} \) and \( e^{\gamma t} \), a particular solution \( \psi \) of (7.5) can be given by the following: for \( \gamma \leq \gamma_i \),

\[
\psi(t) = \frac{e^{-\gamma t}}{2\gamma_i} \int_t^\infty e^{\gamma s} f(s) ds - \frac{e^{\gamma t}}{2\gamma_i} \int_t^\infty e^{-\gamma s} f(s) ds,
\]

and, for \( \gamma > \gamma_i \),

\[
\psi(t) = \frac{e^{-\gamma t}}{2\gamma_i} \int_t^\infty e^{\gamma s} f(s) ds - \frac{e^{\gamma t}}{2\gamma_i} \int_t^\infty e^{-\gamma s} f(s) ds.
\]

This particular solution satisfies the desired estimate.

Next, any solution \( \psi \) of (7.5) can be written as

\[
\psi = c_1 e^{-\gamma t} + c_2 e^{\gamma t} + \psi,
\]

for some constants \( c_1 \) and \( c_2 \), and the function \( \psi \) just constructed. Since \( \psi(t) \to 0 \) as \( t \to \infty \), then \( c_2 = 0 \). Therefore, we have the desired estimate. \( \square \)

Now, we study the linear equation

\[
Lv = f \quad \text{in} \quad (T, \infty) \times \Sigma.
\]

We discuss asymptotic expansions of solutions of (7.6) as \( t \to \infty \).

**Lemma 7.2.** Let \( \Sigma \subseteq \mathbb{R}^{n-1} \) be a Lipschitz domain, \( \rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma) \) be the positive solution of (2.16)-(2.17), and \( p \) be a constant such that \( p > \max\{2, n/2\} \). Let \( \nu \) be the constant in Theorem 4.4, and \( f \) be a smooth function in \((T, \infty) \times \Sigma\), satisfying, for some positive constants \( \gamma, A, \) and some integer \( m \geq 0 \),

\[
|f| \leq At^m e^{-\gamma t} \rho^{\nu-2} \quad \text{in} \quad (T, \infty) \times \Sigma,
\]

and, for any \( t \in (T, \infty) \),

\[
\|f(t, \cdot)||_{L^2(\Sigma)} + \|f(t, \cdot)||_{L^p(\Sigma)} \leq A t^m e^{-\gamma t}.
\]

Let \( v \in C^\infty((T, \infty) \times \Sigma) \cap C^\nu((T, \infty) \times \Sigma) \) be a solution of (7.6) in \((T, \infty) \times \Sigma\) such that \( v = 0 \) on \((T, \infty) \times \partial \Sigma\), \( v(t, \theta) \to 0 \) as \( t \to \infty \) uniformly for \( \theta \in \Sigma \), and \( v(t, \cdot) \in H_0^1(\Sigma) \) and \((Lv)(t, \cdot) \in L^2(\Sigma)\), for any \( t > T \).

(i) If \( \gamma \leq \gamma_1 \), then, for any \( (t, \theta) \in (T + 1, \infty) \times \Sigma \),

\[
|v| \leq \begin{cases} 
  Ct^m e^{-\gamma t} \rho^{\nu} & \text{if} \quad \gamma < \gamma_1, \\
  Ct^{m+1} e^{-\gamma t} \rho^{\nu} & \text{if} \quad \gamma = \gamma_1.
\end{cases}
\]
(ii) If $\gamma_l < \gamma \leq \gamma_{l+1}$ for some positive integer $l$, then, for any $(t, \theta) \in (T + 1, \infty) \times \Sigma$,

$$|v - \sum_{i=1}^{l} c_i e^{-\gamma_i t} \phi_i| \leq \left\{
\begin{aligned}
&Cl^m e^{-\gamma t} \rho^\nu & \text{if } \rho_l < \gamma < \rho_{l+1}, \\
&Cl^{m+1} e^{-\gamma t} \rho^\nu & \text{if } \gamma = \rho_{l+1},
\end{aligned}\right.$$

for some constant $c_i$, for $i = 1, \ldots, l$.

**Proof.** The proof is similar as that of Theorem 6.1. We consider (ii) only, $\gamma_l < \gamma \leq \gamma_{l+1}$ for some positive integer $l$. The proof consists of several steps.

**Step 1.** For each $i = 1, \ldots, l$, and $t > T$, set

$$v_i(t) = \int_{\Sigma} v(t, \theta) \phi_i(\theta) d\theta, \quad f_i(t) = \int_{\Sigma} f(t, \theta) \phi_i(\theta) d\theta.$$

Then, by (7.8),

$$|f_i(t)| \leq A_i t^m e^{-\gamma t}. \quad (7.9)$$

By multiplying (7.6) by $\phi_i$ and integrating over \( \{t\} \times \Sigma \) for any $t > T$, we obtain

$$L_i v_i = f_i \quad \text{on } (T, \infty).$$

We point out that the integration is performed on $\{t\} \times \Sigma$ for each $t > T$ and that the boundary integrals vanish by $v(t, \cdot), \phi_i \in H^1_0(\Sigma)$. Since $v(t, \theta) \to 0$ as $t \to \infty$ uniformly for $\theta \in \Sigma$, then $v_i(t) \to 0$ as $t \to \infty$. Note that $\gamma > \gamma_i$ for $i = 1, \ldots, l$. By applying Lemma 7.1 to $L_i$, we conclude that there exists a constant $c_i$ for $i = 1, \ldots, l$ such that, for any $t > T$,

$$|v_i(t) - c_i e^{-\gamma_i t}| \leq C_i t^m e^{-\gamma t}. \quad (7.10)$$

Set

$$\tilde{v}(t, \theta) = v(t, \theta) - \sum_{i=1}^{l} v_i(t) \phi_i(\theta), \quad \tilde{f}(t, \theta) = f(t, \theta) - \sum_{i=1}^{l} f_i(t) \phi_i(\theta).$$

A simple calculation yields

$$L \tilde{v} = \tilde{f} \quad \text{in } (T, \infty) \times \Sigma. \quad (7.12)$$

For simplicity, we write $\tilde{v}(t) = \tilde{v}(t, \cdot)$ and $\tilde{f}(t) = \tilde{f}(t, \cdot)$.

**Step 2.** Set $m_* = m$ if $\gamma_l < \gamma < \gamma_{l+1}$ and $m_* = m + 1$ if $\gamma = \gamma_{l+1}$. We proceed to prove that there exists a $T_* > T$ such that, for any $t > T_*$,

$$\|\tilde{v}(t, \cdot)\|_{L^2(\Sigma)} \leq C t^{m_*} e^{-\gamma t}. \quad (7.13)$$

First, we note that $(L \tilde{v})(t) \in L^2(\Sigma)$ and $\tilde{v}(t) \in H^1_0(\Sigma)$ for each $t > T$. By multiplying (7.12) by $\tilde{v}(t)$ and integrating over $\Sigma$, we obtain

$$\int_{\Sigma} \left[ \tilde{v}_t(t) \tilde{v}(t) - \beta^2 \tilde{v}^2(t) \right] d\theta - \int_{\Sigma} \left[ |\nabla \tilde{v}(t)|^2 + \kappa \rho^{-2} \tilde{v}^3(t) \right] d\theta = \int_{\Sigma} \tilde{f}(t) \tilde{v}(t) d\theta.$$
Set
\[ y(t) = \left[ \int_{\Sigma} \hat{\nu}^2(t) \, d\theta \right]^{1/2}. \]
Then,
\[ y(t)y'(t) = \int_{\Sigma} \hat{\nu}(t) \hat{\nu}_t(t) \, d\theta, \]
and
\[ y(t)y''(t) + [y'(t)]^2 = \int_{\Sigma} \left[ \hat{\nu}(t) \hat{\nu}_{tt}(t) + \hat{\nu}_t^2(t) \right] \, d\theta. \]
The Cauchy inequality implies that, if \( y(t) > 0 \), then
\[ [y'(t)]^2 \leq \int_{\Sigma} \hat{\nu}_t^2(t) \, d\theta, \]
and hence
\[ y(t)y''(t) \geq \int_{\Sigma} \hat{\nu}(t) \hat{\nu}_{tt}(t) \, d\theta. \]
Since \( \hat{\nu}(t) \perp \phi_i \) in \( L^2(\Sigma) \) for each \( i = 1, \ldots, l \) and any \( t > T \), then
\[ \int_{\Sigma} \left[ \| \nabla_{\phi_i} \hat{\nu}(t) \|^2 + \kappa \rho^{-2} \hat{\nu}^2(t) \right] \, d\theta \geq \lambda_{l+1} \int_{\Sigma} \hat{\nu}^2(t) \, d\theta. \]
Moreover, by (7.14),
\[ \left| \int_{\Sigma} \hat{f}(t) \hat{\nu}(t) \, d\theta \right| = \left| \int_{\Sigma} f(t) \hat{\nu}(t) \, d\theta \right| \leq C_0 t^m e^{-\gamma t} \| \hat{\nu}(t) \|_{L^2(\Sigma)}. \]
A simple substitution yields
\[ y(t)y''(t) - (\lambda_{l+1} + \beta^2) y^2(t) \geq -C_0 t^m e^{-\gamma t} y(t). \]
If \( y(t) > 0 \), we obtain
\[ L_{l+1} y \geq -C_0 t^m e^{-\gamma t}. \]
Set
\[ (7.14) \quad z(t) = Ct^m e^{-\gamma t}. \]
If \( \gamma_l < \gamma < \gamma_{l+1} \), then
\[ L_{l+1}(t^m e^{-\gamma t}) = t^m e^{-\gamma t} \left[ \gamma^2 - \gamma_{l+1}^2 - mt^{-1}(2\gamma - (m-1)t^{-1}) \right] \leq -\frac{1}{2} (\gamma_{l+1}^2 - \gamma^2) t^m e^{-\gamma t}, \]
if \( t > T_\ast \) for some \( T_\ast \) large. If \( \gamma = \gamma_{l+1} \), then
\[ L_{l+1}(t^{m+1} e^{-\gamma t}) = t^m e^{-\gamma t} \left[ -(m+1)(2\gamma - mt^{-1}) \right] \leq -(m+1)\gamma t^m e^{-\gamma t}, \]
if \( t > T_\ast \) for some \( T_\ast \) large. Hence, for some constant \( C \) sufficiently large, we have \( z(T_\ast) \geq y(T_\ast) \) and, if \( y(t) > 0 \),
\[ L_{l+1}(z - y)(t) \leq 0. \]
(7.15) Note that \( y(t) \to 0 \) as \( t \to \infty \). If \( z - y \) is negative somewhere on \( (T_\ast, \infty) \), then the minimum of \( z - y \) is negative and attained at some \( t_* \in (T_\ast, \infty) \). Hence, \( y(t_*) > z(t_*) > 0 \) and (7.14) holds at \( t_* \). However, \((z - y)(t_*) < 0, (z - y)'(t_*) = 0, \) and \((z - y)''(t_*) \geq 0, \) contradicting (7.15). Therefore, for any \( t > T_\ast \), \( y(t) \leq z(t) \), and hence (7.13) holds.
Step 3. We now claim that, for any \((t, \theta) \in (T_\ast + 2, \infty) \times \Sigma\),

\[
\hat{v}(t, \theta) \leq C t^m e^{-\gamma t} \rho^\nu.
\]

By the definition of \(\hat{f}\), we have, for any \((t, \theta) \in (T_\ast, \infty) \times \Sigma\),

\[
\hat{f}(t, \theta) \leq C t^m e^{-\gamma t} \rho^{\nu - 2},
\]

and

\[
\|\hat{f}(t, \cdot)\|_{L^p(\Sigma)} \leq C t^m e^{-\gamma t}.
\]

By \(p > n/2\) and applying the \(L^\infty\)-estimates to the equation (7.12), we obtain, for any \(t > T_\ast + 1\),

\[
\sup_{\{t\} \times \Sigma} |\hat{v}| \leq C \left\{ \|\hat{v}\|_{L^2((t-1, t+1) \times \Sigma)} + \|\hat{f}\|_{L^p((t-1, t+1) \times \Sigma)} \right\} \leq C t^m e^{-\gamma t}.
\]

With (7.19), we have (7.16) by proceeding similarly as in the proof of Lemma 5.1.

Step 4. We now finish the proof. We write

\[
v(t, \theta) = \sum_{i=1}^l c_i e^{-\gamma_i t} \phi_i(\theta) = \sum_{i=1}^l \left( v_i(t) - c_i e^{-\gamma_i t} \right) \phi_i(\theta) + \hat{v}(t, \theta).
\]

By combining (7.10) and (7.16), we have the desired result. \(\square\)

We point out that the stated assumption \(p > n/2\) is essentially needed in order to apply the \(L^\infty\)-estimates to get (7.16) from the \(L^2\)-estimate (7.13). Later on, we will apply Lemma 7.2 to functions \(f = F(v)\) as in Corollary 5.4.

Next, we return to the nonlinear equation (5.1) and discuss higher order expansions of its solutions along the \(t\)-direction. We first describe our strategy. Let \(v\) be a smooth solution of (5.1) in \((T, \infty) \times \Sigma\) satisfying (6.1). We will apply Lemma 7.2 to the equation \(Lv = F(v)\). Since \(F(v)\) is nonlinear in \(v\), we need to apply Lemma 7.2 successively. In each step, we aim to get a decay estimate of \(F(v)\), with a decay rate better than that of \(v\). Then, we can subtract expressions with the lower decay rates generated by Lemma 7.2 to improve the decay rate of \(v\). To carry out this process, we make two preparations.

As the first preparation, we introduce the index set \(I\) by defining

\[
I = \left\{ \sum_{i \geq 1} m_i \gamma_i; m_i \in \mathbb{Z}_+ \text{ with finitely many } m_i > 0 \right\}.
\]

In other words, \(I\) is the collection of linear combinations of finitely many \(\gamma_1, \gamma_2, \ldots\) with positive integer coefficients. It is possible that some \(\gamma_i\) can be written as a linear combination of some of \(\gamma_1, \ldots, \gamma_{i-1}\) with positive integer coefficients, whose sum is at least two.

For the second preparation, we need to construct particular solutions.
Lemma 7.3. Let \( \nu \) be as in Theorem 4.4, \( m \) be a nonnegative integer, \( \gamma \) be a positive constant, and \( h_0, h_1, \ldots, h_m \in L^2(\Sigma) \). Then, there exist \( w_0, w_1, \ldots, w_m, w_{m+1} \in H^1_0(\Sigma) \) such that

\[
(7.22) \quad \mathcal{L} \left( \sum_{j=0}^{m+1} t^j e^{-\gamma t} w_j \right) = \sum_{j=0}^{m} t^j e^{-\gamma t} h_j \quad \text{in} \quad \mathbb{R} \times \Sigma.
\]

Moreover, if \( |h_j| \leq A \rho^{-2} \) for any \( j = 0, 1, \ldots, m \) and some constant \( A > 0 \), then \( w_j \in C^\nu(\Sigma) \) with \( w_j = 0 \) on \( \partial \Sigma \) for \( j = 0, 1, \ldots, m+1 \).

Proof. For any nonnegative integer \( j \), a straightforward calculation yields

\[
\mathcal{L}(t^j e^{-\gamma t} w_j) = t^j e^{-\gamma t} (Lw_j + (\gamma^2 - \beta^2)w_j) - 2\gamma j t^j e^{-\gamma t} w_j + j(j-1) t^{j-2} e^{-\gamma t} w_j.
\]

We consider two cases.

Case 1: \( \gamma^2 - \beta^2 \neq \lambda_l \) for any \( l \). For \( w_0, w_1, \ldots, w_m \) to be determined, we consider

\[
\mathcal{L} \left( \sum_{j=0}^{m} t^j e^{-\gamma t} w_j \right) = \sum_{j=0}^{m} t^j e^{-\gamma t} h_j \quad \text{in} \quad \mathbb{R} \times \Sigma.
\]

Note that

\[
\mathcal{L} \left( \sum_{j=0}^{m} t^j e^{-\gamma t} w_j \right) = \sum_{j=0}^{m} t^j e^{-\gamma t} (Lw_j + (\gamma^2 - \beta^2)w_j) - \sum_{j=0}^{m-1} 2(j+1) \gamma t^j e^{-\gamma t} w_{j+1} + \sum_{j=0}^{m-2} (j+2)(j+1) t^{j-2} e^{-\gamma t} w_{j+2}.
\]

Hence, we take \( w_{m+2} = w_{m+1} = 0 \) and solve for \( w_m, \ldots, w_1, w_0 \) inductively, such that, for \( j = m, \ldots, 1, 0, \)

\[
Lw_j + (\gamma^2 - \beta^2)w_j - 2(j+1) \gamma w_{j+1} + (j+2)(j+1) w_{j+2} = h_j.
\]

Theorem 4.3(i) implies the existence of a unique \( w_j \in H^1_0(\Sigma) \).

Case 2: \( \gamma^2 - \beta^2 = \lambda_l \) for some \( l \). Let \( \mathcal{E} \) be the eigenspace corresponding to \( \lambda_l \). For each \( j = 0, 1, \ldots, m \), write

\[
h_j = \tilde{h}_j + \bar{h}_j,
\]

with \( \tilde{h}_j \in \mathcal{E} \) and \( \bar{h}_j \in \mathcal{E}^\perp \). First, for \( \tilde{w}_0, \tilde{w}_1, \cdots, \tilde{w}_m \) to be determined, we consider

\[
\mathcal{L} \left( \sum_{j=0}^{m} t^j e^{-\gamma t} \tilde{w}_j \right) = \sum_{j=0}^{m} t^j e^{-\gamma t} \tilde{h}_j \quad \text{in} \quad \mathbb{R} \times \Sigma.
\]

Since \( \bar{h}_j \in \mathcal{E}^\perp \), we can find such \( \bar{w}_j \in \mathcal{E}^\perp \) by Theorem 4.3(ii), similarly as in Case 1. Next, for \( \bar{w}_1, \cdots, \bar{w}_{m+1} \) to be determined, we consider

\[
\mathcal{L} \left( \sum_{j=1}^{m+1} t^j e^{-\gamma t} \bar{w}_j \right) = \sum_{j=0}^{m} t^j e^{-\gamma t} \bar{h}_j \quad \text{in} \quad \mathbb{R} \times \Sigma.
\]
To this end, we take $\hat{w}_{m+2} = 0$ and solve for $\hat{w}_{m+1}, \ldots, \hat{w}_1$ inductively, such that, for $j = m, \ldots, 0$,

$$-2(j + 1)\gamma \hat{w}_{j+1} + (j + 2)(j + 1)\hat{w}_{j+2} = \tilde{h}_j.$$  

Then, for each $j = 1, \ldots, m + 1$, $\hat{w}_j \in \mathcal{E}$, and hence

$$L\hat{w}_j + (\gamma^2 - \beta^2)\hat{w}_j = 0.$$  

In summary, we have

$$L\left( \sum_{j=0}^{m} t^j e^{\gamma t} \hat{w}_j + \sum_{j=1}^{m+1} t^j e^{-\gamma t} \hat{w}_j \right) = \sum_{j=0}^{m} t^j e^{-\gamma t} \tilde{h}_j.$$  

By combining the two cases, we have the existence of the desired $w_j \in H^1_0(\Sigma)$. Theorem 4.4 implies $w_j \in C^\nu(\overline{\Sigma})$ and $w_j = 0$ on $\partial \Sigma$, for $j = 0, 1, \ldots, m + 1$. \hfill $\Box$

Here, particular solutions were constructed as a simple application of the Fredholm alternative. In [12], similar particular solutions were constructed as infinite series and the convergence of such series is based on a growth estimate of eigenvalues.

The main result in this section is the following expansions up to arbitrary orders.

**Theorem 7.4.** Let $\Sigma \subset S^{n-1}$ be a Lipschitz domain, and $\rho \in C^\infty(\Sigma) \cap \text{Lip}(\Sigma)$ be the positive solution of (2.16) - (2.17). Assume that $v$ is a smooth solution of (5.1) in $(T, \infty) \times \Sigma$ satisfying (6.1). Then, there exist functions $c_{ij} \in C^\nu(\Sigma)$ with $c_{ij} = 0$ on $\partial \Sigma$ such that, for any $m \geq 1$,

$$\left| v - \sum_{i=1}^{m} \sum_{j=0}^{i-1} c_{ij} t^j e^{-\mu_i t} \right| \leq C t^m e^{-\mu_m t} \rho^\nu \text{ in } (T+1, \infty) \times \Sigma,$$

where $\nu$ is the positive constant as in Theorem 4.4, and $C$ is a positive constant depending only on $n$, $m$, $C_0$, and $\Sigma$.

We point out that there are two indices in the summation, $i$ for the decay rate in $e^{-\mu_i t}$ and $j$ for the polynomial growth rate in $t^j$.

**Proof.** Throughout the proof, we adopt the following notation: $f = O(h)$ if $|f| \leq Ch$, for some positive constant $C$. All estimates in the following hold for any $t > T + 1$ and any $\theta \in S^{n-1}$. Take $p$ to be the constant as in Corollary 5.4.

Let $\{\phi_i\}$ be an orthonormal basis of $L^2(\Sigma)$, formed by eigenfunctions of $-L$, and $\{\lambda_i\}$ be the sequence of corresponding eigenvalues, arranged in an increasing order. Let $L$ be the linear operator given by (7.11), and $L_i$ be the projection of $L$ given by (7.3). For each $i \geq 1$, there is an exponentially decaying solution $e^{-\gamma_i t}$ in $\text{Ker}(L_i)$. Write

$$F(v) = \rho^{-2-\beta} \sum_{i=2}^{\infty} b_i (\rho^\beta v)^i,$$

where $b_i$ is a constant, for each $i \geq 2$. We point out that we write the infinite sum just for convenience. We do not need the convergence of the infinite series and we always expand up to finite orders.
Let $\mathcal{I}$ be the index set defined in (7.21). We decompose $\mathcal{I}$ by setting

$$\mathcal{I}_\gamma = \{\gamma_j : j \geq 1\},$$

and

$$\mathcal{I}_{\tilde{\gamma}} = \left\{ \sum_{i=1}^{k} n_i \gamma_i : n_i \in \mathbb{Z}_+, \sum_{i=1}^{k} n_i \geq 2 \right\}.$$

We assume $\mathcal{I}_{\tilde{\gamma}}$ is given by a strictly increasing sequence $\{\tilde{\gamma}_i\}_{i \geq 1}$, with $\tilde{\gamma}_1 = 2\gamma_1$.

We first consider the case that

(7.24) $\mathcal{I}_\gamma \cap \mathcal{I}_{\tilde{\gamma}} = \emptyset$.

In other words, no $\gamma_i$ can be written as a linear combination of some of $\gamma_1, \ldots, \gamma_{i-1}$ with positive integer coefficients, except a single $\gamma_i'$ which is equal to $\gamma_i$. In this case, we arrange $\mathcal{I}$ as follows:

(7.25) $(\beta <) \gamma_1 \leq \cdots \leq \gamma_{k_1} < \tilde{\gamma}_1 < \cdots < \tilde{\gamma}_{l_1} < \gamma_{k_1+1} \leq \cdots \leq \gamma_{k_2} < \tilde{\gamma}_{l_1+1} < \cdots$.

For each $\tilde{\gamma}_i$, by the definition of $\mathcal{I}_{\tilde{\gamma}}$, there are finitely many collections of nonnegative integers $n_1, \ldots, n_{k_1}$ satisfying

(7.26) $n_1 + \cdots + n_{k_1} \geq 2, \quad n_1 \gamma_1 + \cdots + n_{k_1} \gamma_{k_1} = \tilde{\gamma}_i$.

By Theorem 6.1, we have

(7.27) $v = O(e^{-\gamma_1 t} \rho^\nu)$.

We divide the proof for the case (7.24) into several steps.

Step 1. Note $\gamma_{k_1} < \tilde{\gamma}_1 = 2\gamma_1$. We claim that there exists a function $\eta_1$ such that

$$v = \eta_1 + O(e^{-\tilde{\gamma}_1 t} \rho^\nu).$$

To prove this, we note that, by (7.27), the explicit expression of $F$, and Corollary 5.4

(7.28) $\mathcal{L}v = O(e^{-2\gamma_1 t} \rho^{3+2\nu-2}) = O(e^{-\tilde{\gamma}_1 t} \rho^{\beta+2\nu-2})$

and

(7.29) $\|\mathcal{L}v(t, \cdot)\|_{L^p(\Sigma)} = O(e^{-\tilde{\gamma}_1 t})$.

We point out that we need to verify (7.29) only for $3 \leq n \leq 5$. For $n \geq 6$, $\beta \geq 2$ and hence (7.29) is implied by (7.28). By (7.28), (7.29), and Lemma 7.2(ii), we can take

(7.30) $\eta_1(t, \theta) = \sum_{i=1}^{k_1} c_i e^{-\gamma_i t} \phi_i(\theta)$

for appropriate constant $c_i$. Set

(7.31) $v_1 = v - \eta_1$.

Then, $\mathcal{L}\eta_1 = 0$, $\mathcal{L}v_1 = F(v)$, and

(7.32) $v_1 = O(e^{-\tilde{\gamma}_1 t} \rho^\nu)$.

Note that (7.32) improves (7.27).
Step 2. We claim there exists an \( \tilde{\eta}_1 \) such that, with

\[
\tilde{v}_1 = v_1 - \tilde{\eta}_1 = v - \eta_1 - \tilde{\eta}_1, \tag{7.33}
\]
we have

\[
L \tilde{v}_1 = O(e^{-\tilde{\gamma}_1 t} \rho^{\beta+2\nu-2}), \tag{7.34}
\]
and

\[
\| (L \tilde{v}_1)(t, \cdot) \|_{L^\nu(\Sigma)} = O(e^{-\tilde{\gamma}_1 t}). \tag{7.35}
\]

We will prove that \( \tilde{\eta}_1 \) has the form

\[
\tilde{\eta}_1(t, \theta) = \sum_{i_1}^t e^{-\tilde{\gamma}_1 t} w_i(\theta), \tag{7.36}
\]
where \( w_i \in C^\infty(\Sigma) \cap C^\nu(\Sigma) \cap H^1_0(\Sigma) \) with \( w_i = 0 \) on \( \partial \Sigma \). Note that \( (7.34) \) improves \( (7.28) \).

To prove this, we take some function \( \tilde{\eta}_1 \) to be determined, and then set \( \tilde{v}_1 \) by \( (7.33) \). Then,

\[
L \tilde{v}_1 = F(v) - L \tilde{\eta}_1. \tag{7.37}
\]

Note 3. \( \gamma_1 \in I_\gamma \). We discuss this step in several cases.

Case 1. We assume \( \gamma_{k_1+1} < 3\gamma_1 \). Then, \( \tilde{\gamma}_1 < \gamma_{k_1+1} < 3\gamma_1 \) and \( \tilde{\gamma}_{l_1+1} \le 3\gamma_1 \). We now analyze \( F(v) \) in \( (7.37) \). Note

\[
F(v) = F(v_1 + \eta_1) = \rho^{-2-\beta} \sum_{i=2}^\infty b_i \rho^{i\beta} (v_1 + \eta_1)^i. \]

It is worth mentioning again that we write the infinite sum just for convenience and we always expand up to finite orders. For terms involving \( v_1 \), we have, by \( (7.32) \),

\[
v_1^2 \le Ce^{-4\gamma_1 t} \rho^{2\nu}, \quad |v_1 \eta_1| \le Ce^{-3\gamma_1 t} \rho^{2\nu}. \]

Note that \( \eta_1 \) is given by \( (7.30) \). We write

\[
\sum_{i=2}^\infty b_i \rho^{i\beta} \eta_1^i = \sum_{n_1 + \cdots + n_{k_1} \ge 2} a_{n_1 \cdots n_{k_1}} e^{-\gamma_{k_1} \rho (n_{n_1+\cdots+n_{k_1}+\gamma_{k_1})} \phi_{n_1} \cdots \phi_{k_1} \rho (n_{n_1+\cdots+n_{k_1}+\gamma_{k_1} \beta},
\]
where \( n_1, \ldots, n_{k_1} \) are nonnegative integers, and \( a_{n_1 \cdots n_{k_1}} \) is a constant. By the definition of \( I_\gamma \), \( n_1 \gamma_1 + \cdots + n_{k_1} \gamma_{k_1} \) is some \( \tilde{\gamma}_i \). Hence, we can write

\[
\rho^{-2-\beta} \sum_{i=2}^\infty b_i \rho^{i\beta} \eta_1^i = \sum_{i=1}^\infty e^{-\tilde{\gamma}_i t} \tilde{h}_i, \tag{7.38}
\]
where \( \tilde{h}_i \) is given by

\[
\tilde{h}_i = \rho^{-2-\beta} \sum_{n_1 \cdots n_{k_1} \in N_{\tilde{\gamma}_i}} a_{n_1 \cdots n_{k_1}} \phi_{n_1} \cdots \phi_{k_1} \rho (n_{n_1+\cdots+n_{k_1}+\gamma_{k_1} \beta).
Here, we denote by $\mathcal{N}_{\tilde{\gamma}_i}$ the collection of all $(n_1, \cdots, n_k)$ satisfying (7.26). Then,

$$|h_i| \leq C \rho^{\beta+2\nu-2},$$

and $h_i$ has the same integrability as $F(v)$ in Corollary 5.1, i.e., $h_i \in L^p(\Sigma)$ for some $p > n/2$. We now take the finite sum up to $l_1$ in the right-hand side of (7.38) and denote it by $I_1$.

$$I_1 = \sum_{i=1}^{l_1} e^{-\tilde{\gamma}_i t} h_i.$$  

Then,

$$F(v) = I_1 + O(e^{-\tilde{\gamma}_{l_1+1} t} \rho^{\beta+2\nu-2}),$$

and hence, by (7.37),

$$\mathcal{L} \tilde{v}_1 = I_1 - \mathcal{L} \tilde{\eta}_1 + O(e^{-\tilde{\gamma}_{l_1+1} t} \rho^{\beta+2\nu-2}).$$

Similar estimates for $L^p$-norms also hold. We now solve

$$\mathcal{L} \tilde{\eta}_1 = I_1.$$  

Note that $\gamma_m \neq \tilde{\gamma}_i$ for any $m$ and $i$. By Lemma 7.3 with $m = 0$ and $\gamma = \tilde{\gamma}_i$ for $i = 1, \cdots, l_1$, (7.40) admits a solution $\tilde{\eta}_1$ of the form (7.36). In conclusion, we obtain a function $\tilde{\eta}_1$ in the form (7.36), and $\tilde{v}_1$ defined by (7.33) satisfies (7.34). By (7.32) and (7.36), we have

$$\tilde{v}_1 = O(e^{-\tilde{\gamma}_1 t} \rho^{\nu}).$$

Case 2: We now assume $\gamma_{k_1+1} > 3\gamma_1$. Then, $\tilde{\gamma}_{l_1} \geq 3\gamma_1$.

Let $n_1$ be the largest integer such that $\tilde{\gamma}_{n_1} < 3\gamma_1$. Then, $\tilde{\gamma}_{n_1+1} = 3\gamma_1$. We can repeat the argument in Case 1 with $n_1$ replacing $l_1$. In defining $I_1$ in (7.39), the summation is from $i = 1$ to $n_1$. Similarly for $\tilde{\eta}_1$ in (7.36), we define

$$\tilde{\eta}_{11}(t, \theta) = \sum_{i=1}^{n_1} e^{-\tilde{\gamma}_i t} w_i(\theta),$$

for appropriate functions $w_i$, and then set

$$\tilde{v}_{11} = v_1 - \tilde{\eta}_{11}.$$  

A similar arguments yields

$$\mathcal{L} \tilde{v}_{11} = O(e^{-\tilde{\gamma}_{n_1+1} t} \rho^{\beta+2\nu-2}) = O(e^{-3\gamma_1 t} \rho^{\beta+2\nu-2}),$$

and a similar estimate for the $L^p$-norm. Moreover, by (7.32) and (7.42),

$$\tilde{v}_{11} = O(e^{-\tilde{\gamma}_1 t} \rho^{\nu}) = O(e^{-2\gamma_1 t} \rho^{\nu}).$$

We point out there is no $\gamma_i$ between $\tilde{\gamma}_1$ and $\tilde{\gamma}_{n_1+1}$. Hence, by Lemma 7.2(ii), we have

$$\tilde{v}_{11} = O(e^{-3\gamma_1 t} \rho^{\nu}).$$

Note that (7.45) improves (7.41) and hence (7.32).
Now, we are in a similar situation as at the beginning of Step 2, with \( \tilde{\gamma}_{n+1} = 3\gamma_1 \) replacing \( \gamma_1 = 2\gamma_1 \). If \( \gamma_{k_1+1} < 4\gamma_1 \), we proceed as in Case 1. If \( \gamma_{k_1+1} > 4\gamma_1 \), we proceed as at the beginning of Case 2 by taking the largest integer \( n_2 \) such that \( \tilde{\gamma}_{n_2} < 4\gamma_1 \). After finitely many steps, we reach \( \tilde{\gamma}_l \).

In summary, we have \( \tilde{\eta}_1 \) as in (7.36) and, by defining \( \tilde{v}_1 \) by (7.33), we conclude (7.34) and (7.35), as well as (7.41). This finishes the discussion of Step 2.

Step 3. Now we are in the same situation as in Step 1, with \( \tilde{\gamma}_{l_1+1} \) replacing \( \gamma_1 \). We repeat the argument there with \( k_1+1, k_2 \) and \( l_1+1 \) replacing 1, \( k_1 \) and 1, respectively.

Note \( \gamma_{k_2} < \tilde{\gamma}_{l_1+1} \). By (7.34), (7.35), and Lemma 7.2(ii), we obtain

\[
\tilde{v}_1(t, \theta) = \sum_{i=k_1+1}^{k_2} c_i e^{-\gamma_i t} \phi_i(\theta) + O(e^{-\tilde{\gamma}_{l_1+1} t} \rho'),
\]

where \( c_i \) is a constant, for \( i = k_1+1, \ldots, k_2 \). By (7.41), there is no need to adjust by terms involving \( e^{-\gamma_i t} \) corresponding to \( i = 1, \ldots, k_1 \). Set

\[
(7.46) \quad \eta_2(t, \theta) = \sum_{i=k_1+1}^{k_2} c_i e^{-\gamma_i t} \phi_i(\theta),
\]

and

\[
(7.47) \quad v_2 = \tilde{v}_1 - \eta_2.
\]

Then, \( \mathcal{L}\eta_2 = 0, \, v_2 = v - \eta_1 - \tilde{\eta}_1 - \eta_2 \), and

\[
(7.48) \quad v_2 = O(e^{-\tilde{\gamma}_{l_1+1} t} \rho').
\]

Step 4. The discussion is similar as that in Step 2. For some \( \tilde{\eta}_2 \) to be determined, set

\[
(7.49) \quad \tilde{v}_2 = v_2 - \tilde{\eta}_2.
\]

Then,

\[
\mathcal{L}\tilde{v}_2 = F(v) - \mathcal{L}\eta_1 - \mathcal{L}\eta_2.
\]

Note

\[
F(v) = F(v_2 + \eta_1 + \tilde{\eta}_1 + \eta_2) = \rho^{-2-\beta} \sum_{i=2}^{\infty} b_i \rho^{i\beta} (v_2 + \eta_1 + \tilde{\eta}_1 + \eta_2)^i.
\]

As in Step 2, we need to analyze

\[
\sum_{i=2}^{\infty} b_i \rho^{i\beta} (\eta_1 + \tilde{\eta}_1 + \eta_2)^i.
\]

In Step 2, by choosing \( \tilde{\eta}_1 \) as in (7.36) appropriately, we used \( \mathcal{L}\tilde{\eta}_1 \) to cancel the terms \( e^{-\tilde{\gamma}_i t} \) in \( F(v) \), for \( i = 1, \ldots, l_1 \). Proceeding similarly, we can find \( \tilde{\eta}_2 \) in the form

\[
(7.50) \quad \tilde{\eta}_2(t, \theta) = \sum_{i=l_1+1}^{l_2} e^{-\tilde{\gamma}_i t} w_i(\theta)
\]
to cancel the terms \(e^{-\tilde{\gamma}_1 t}\) in \(F(v)\), for \(i = l_1 + 1, \cdots, l_2\). By defining \(\tilde{v}_2\) by (7.49), we conclude

\[
(7.51) \quad \mathcal{L}\tilde{v}_2 = O(e^{-\tilde{\gamma}_2 + 1 t} \rho^{\beta + 2 \nu - 2}),
\]

and

\[
(7.52) \quad \|(\mathcal{L}\tilde{v}_2)(t, \cdot)\|_{L^p(\Sigma)} = O(e^{-\tilde{\gamma}_2 + 1 t}).
\]

We can continue these steps indefinitely and hence finish the proof for the case (7.24).

Next, we consider the general case; namely, some \(\gamma_i\) can be written as a linear combination of some of \(\gamma_1, \cdots, \gamma_{i-1}\) with positive integer coefficients. We will modify discussion above to treat the general case. Whenever some \(\gamma_i\) coincides some \(\tilde{\gamma}_\nu\), an extra power of \(t\) appears when solving the linear equation \(\mathcal{L}v = f\), according to Lemma 7.3 and such a power of \(t\) will generate more powers of \(t\) upon iteration.

For an illustration, we consider \(\gamma_{k_1} = \tilde{\gamma}_1\) instead of the strict inequality in (7.25). This is the first time that some \(\gamma_i\) may coincide some \(\tilde{\gamma}_\nu\). We start with (7.27) and (7.28), and proceed similarly as in Step 1. Take \(k_1 = \{1, \cdots, k_1 - 1\}\) such that

\[
\gamma_{k_1} < \gamma_{k_1 + 1} = \cdots = \gamma_{k_1} = \tilde{\gamma}_1 = 2\gamma_1.
\]

By Lemma 7.2(ii), we obtain

\[
v(t, \theta) = \sum_{i=1}^{k_1} c_i e^{-\gamma_i t \phi_i(\theta)} + O(t e^{-\tilde{\gamma}_1 t \rho^{\nu}}),
\]

where \(c_i\) is a constant, for \(i = 1, \cdots, k_1\). Instead of (7.30), we define

\[
(7.53) \quad \eta_1(t, \theta) = \sum_{i=1}^{k_1} c_i e^{-\gamma_i t \phi_i(\theta)},
\]

and then define \(v_1\) as in (7.31). Then,

\[
(7.54) \quad v_1 = O(t e^{-\tilde{\gamma}_1 t \rho^{\nu}}).
\]

Next, we proceed similarly as in Step 2. In the discussion of Case 1 in Step 2, we need to solve (7.40) and find \(\tilde{\eta}_1\), which is a linear combination of \(e^{-\tilde{\gamma}_1 t}, \cdots, e^{-\tilde{\gamma}_1 t}\). For \(i = 1, \cdots, l_1\), the part corresponding to \(e^{-\tilde{\gamma}_1 t}\) is the same, still given by \(e^{-\tilde{\gamma}_1 t} w_1(\theta)\). For \(i = 1\), the part corresponding to \(e^{-\tilde{\gamma}_1 t}\) is given by

\[
(7.55) \quad t e^{-\tilde{\gamma}_1 t} w_{11}(\theta) + e^{-\tilde{\gamma}_1 t} \phi_{k_1}(\theta),
\]

where \(w_{11} \in C^\infty(\Sigma) \cap C^\nu(\Sigma) \cap H^1_0(\Sigma)\) with \(w_{11} = 0\) on \(\partial \Sigma\). Then, by defining \(\tilde{\eta}_1\) by (7.36), with the new expression given by (7.55) for \(e^{-\tilde{\gamma}_1 t}\), and defining \(\tilde{v}_1\) by (7.33), we have (7.34). We can modify the rest of the proof similarly.

According to the proof, the summation in (7.29) has two sources, the kernel of the linearized equation and the nonlinearity. The kernel part is a linear combination of \(e^{-\gamma_i t} \phi_i\) as in Lemma 7.2(ii), with constant coefficients. The nonlinear part consists of solutions constructed in Lemma 7.3 to eliminate nonlinear combinations of lower order terms in \(F(v)\).
Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $\xi \in C^\infty(\Sigma)$ be the solution of (2.4)-(2.5). By (7.23), we have
\[
\left|\xi^{-1}v - \sum_{i,j=0}^{m,n-1} \xi^{-1}c_{ij}t^i e^{-\mu t}\right| \leq Cd^{m}e^{-\mu t}e^{-\nu t} \quad \text{in} \quad (T + 1, \infty) \times \Sigma.
\]
By (2.13), we get
\[
|\xi^{-1}c_{ij}| + |\xi^{-1}e^{\nu t}| \leq d^{\beta + \nu} \quad \text{in} \quad \Sigma,
\]
where $d$ is the distance function in $\Sigma$ to $\partial \Sigma$. We now have the desired result with $\tau = \beta + \nu$ by the definition of $v$ in (2.25) and the change of coordinates (2.24). \hfill $\Box$

Under the additional assumption that $\rho$ also satisfies (4.12), Theorem 7.4 holds for $\tau = s$. Hence, we have Theorem 1.2.

To end this paper, we make one final remark. Let $\Sigma$ be a smooth domain and $\rho$ be the positive solution of (2.10)-(2.17). Recall that $d$ is the distance function in $\Sigma$ to $\partial \Sigma$. Then, for any $m \geq n + 1, \alpha \in (0,1)$, and any $\theta \in \Sigma$ near $\partial \Sigma$,
\[
(7.56) \quad \left|\rho(\theta) - \left[\sum_{i=1}^{n} c_i(\theta')d^i + \sum_{i=n+1}^{m} \sum_{j=0}^{N_i} c_{i,j}(\theta')d^i(\log d)^j\right]\right| \leq Cd^{\alpha + 1},
\]
where $d = d(\theta), \theta' \in \partial \Sigma$ is the unique point with $d(\theta) = \text{dist}(\theta, \theta')$, $N_i$ is a positive integer depending only $n$ and $i$, $C$ is a positive constant depending only on $n$, $m$ and $\alpha$, and $c_i$ and $c_{i,j}$ are smooth functions on $\partial \Sigma$. Refer to [11] and [18] for details. Similarly, let $\phi_i$ be the eigenfunction established in Theorem 1.2. Then, for any $m \geq n + 1, \alpha \in (0,1)$, and any $\theta \in \Sigma$ near $\partial \Sigma$,
\[
(7.57) \quad \left|\phi_i(\theta) - d^i\left[\sum_{i=0}^{n} c_{i,\theta}(\theta')d^i + \sum_{i=n+1}^{m} \sum_{j=0}^{N_i} c_{i,\theta,j}(\theta')d^i(\log d)^j\right]\right| \leq Cd^{n+s+\alpha}.
\]
Similar expansions hold for the coefficients $c_{ij}$ in (7.23) near $\partial \Sigma$. As a consequence, we can expand $v$ as a series in terms of $t^i e^{-\mu t} t^k t^l (\log t)^j$ with coefficients defined on $\partial \Sigma$, for positive integer $i$ and nonnegative integers $j, k$ and $l$. Refer to [12] for details.

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Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
Email address: qhan@nd.edu

Department of Mathematics, Fordham University, Bronx, NY 10458, USA
Email address: xjiang77@fordham.edu

School of Mathematical Sciences, Capital Normal University, Beijing, 100048, China
Email address: wmshen@pku.edu.cn