Wavelength Dependent Tunneling Delay Time

Xiaolei Hao\textsuperscript{1}, Zheng Shu\textsuperscript{2}, Weidong Li\textsuperscript{1} and Jing Chen\textsuperscript{2,3}\textsuperscript{*}

\textsuperscript{1}Institute of Theoretical Physics and Department of Physics, State Key Laboratory of Quantum Optics and Quantum Optics Devices, Collaborative Innovation Center of Extreme Optics, Shanxi University, Taiyuan 030006, China
\textsuperscript{2}Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China and \textsuperscript{3}HEDPS, Center for Applied Physics and Technology, Peking University, Beijing 100084, China

(Date textMarch 19, 2019; Received textMarch 19, 2019)

PACS numbers: 33.80.Rv, 34.50.Rk

A renewed urgency to clarify the long debated and fundamental question in quantum mechanics\textsuperscript{1}--\textsuperscript{11}: whether a point particle tunneling through an energetically forbidden region takes a finite time or is instantaneous, has been brought by the recently developed experimental capabilities in attosecond science\textsuperscript{12}--\textsuperscript{19}, which can encode the tunneling time in the strong-field ionization process and provides significant practical implications in exploring attosecond electron dynamics\textsuperscript{13}--\textsuperscript{17, 20, 21}. Especially, in the so called attoclock experiments\textsuperscript{13, 14, 20, 21}, which are based on the strong-field ionization of atoms in a laser field with close-to-circular polarization, the photoelectron momentum distribution at the detector is mapped into the time of the electron appearance in the continuum, and then the time the electron has spent to tunnel under the barrier can be extracted. Although it seems that the attoclock experiment makes the final solution of the long-standing tunneling time problem possible, there have been considerable controversy in the interpretation of the attoclock experiments\textsuperscript{21, 22, 23}. The controversy can be classified into two categories: i) Whether the tunneling delay time exists in principle, and how to define and understand it. ii) How to remove the influence of the laser field and the Coulomb field in the deconvolution procedure to extract the exact release time from the momentum distribution. Here we mainly focus on the problems in category i) which relate closely to the tunneling process.

In strong-field atomic physics, the majority of phenomena rely on the Keldysh tunneling time \(\tau_K = \sqrt{2\gamma/F} \) which can be obtained by calculating the classical time it takes a particle to cross the inverted triangular barrier\textsuperscript{32}, where \(I_p\) is the laser binding potential and \(F\) is the laser electric field. If the optical period \(2\pi/\omega\) is much longer than \(\tau_K\), i.e., Keldysh parameter \(\gamma = \omega \tau_K \ll 1\), the laser field can be treated as a static electric field. In this case, the bound electron is freed from the atom via tunneling (adiabatic process) through a potential barrier formed by the laser field and the atomic potential. Whereas in the case of \(\gamma \gg 1\), the electron sees a quickly oscillating electric field and absorbs many photons to transit to continuum. On the other hand, as a very popular tunneling time in the broader field of physics, B"uttiker-Landauer time \(\tau_{BL} = \int_{x_1}^{x_2} m/p(x)dx\) describes the time spent by the particle to travel from the entrance point \(x_1\) to the exit point \(x_2\) under the barrier \(V(x)\) with momentum \(p(x) = \sqrt{2m[V(x) - E]}\), where \(E\) is the energy of the particle. \(\tau_{BL}\) also characterizes the onset of transition from pure tunneling to tunneling while absorbing one or more energy quanta from the oscillating field\textsuperscript{2}. So the Keldysh time and the B"uttiker-Landauer time are very closely related not only in the definition, but also in distinguishing the process\textsuperscript{19}. However, these two well defined tunneling times with clear physical meaning are ruled out as a candidate for the tunneling delay time, since they \(\tau_{BL} \sim 600\) as are found to deviate far from the time measured by the attoclock experiments (less than 120 as)\textsuperscript{13, 31}.

To interpret the attoclock experiments\textsuperscript{13, 14, 20, 22}, the complementary aspects of the tunneling ionization
dynamics in strong laser field have been extensively explored and various associated times have been suggested \cite{8,10,13,21}. The time spent by the initial ground state in a time varying laser field to develop the under barrier wave function components necessary for reaching a static field ionization rate was suggested as tunneling time in Ref. \cite{3}. Based on the Feynman path integral approach, authors in Refs. \cite{19,21} calculated the probability distribution of tunneling times and found that the common tunneling time definitions can be viewed as averaged quantities rather than deterministic values. By numerically solving the time-dependent Schrödinger equation (TDSE) and employing a virtual detector at the tunnel exit, a finite positive time delay between the electric field maximum and the instant of ionization was identified in Ref. \cite{10}. And the following attoclock measurement on two atomic species with slightly deviating atomic potentials \cite{29} supported this nonzero tunneling delay time.

Considering that momentum operator is well defined in quantum mechanics, here we introduce a naive definition of quantum travel time by analogy with the classical travel time

$$\tau_t = \frac{m|\mathbf{x}_2 - \mathbf{x}_1|}{||\mathbf{p}||},$$  

(1) 

where \(m\) is the mass of the particle, ||\(\cdots||\) means making the modula of “\(\cdots\)”, and \(\mathbf{p}\) is the average momentum of the particle during its staying within the region \((x_1, x_2)\) and can be obtained by calculating the expected value of the momentum operator \(\mathbf{p} = -i\hbar \nabla\psi\)

$$\bar{\mathbf{p}} = \frac{\int_{x_1}^{x_2} \psi^*(x) \mathbf{p} \psi(x) dx}{\int_{x_1}^{x_2} \psi^*(x) \psi(x) dx}$$  

(2) 

with \(\psi(x)\) the wave function of the particle within the region \((x_1, x_2)\). The quantum travel time in Eq. \(\text{(1)}\) tells us how long it takes a particle to travel through two spatially separated positions \(x_1\) and \(x_2\) in quantum mechanics, and can be considered as tunneling time if \(\psi(x)\) is the wave function under the barrier. Note that \(\tau_t\) is also an average quantity rather than a deterministic one. In this letter, first we check the validity and rationality of the definition about \(\tau_t\) in the rectangular barrier tunneling problem. Then we apply it to the strong-field tunneling process to interpret the tunneling delay time measured by attoclock experiment. Finally, as an example of applying attoclock technique to explore attosecond electron dynamics, we identify a coherent tunneling process by studying the wavelength dependence of tunneling delay time.

**Rectangular barrier tunneling** Considering a particle with kinetic energy \(E = \hbar^2 k^2/2m\) moving along the \(x\) axis and interacting with a rectangular barrier of height \(V_0\) and width \(d\) centered at \(x = 0\), the average momentum of the particle within the barrier is (see textbook on quantum mechanics, for example, Ref. \cite{32})

$$\bar{\mathbf{p}} = R(r, \Delta \theta) \hbar k + i I(r, \Delta \theta) \hbar \kappa$$  

(3) 

with

$$R(r, \Delta \theta) = \frac{2\kappa^2}{\kappa^2 - k^2} \frac{(\kappa^2 + k^2) \kappa d}{\kappa^2 + k^2 + 2k \cot(\Delta \theta)}$$  

(4) 

and

$$I(r, \Delta \theta) = \frac{2k\kappa}{\kappa^2 - k^2} \frac{2k}{\kappa^2 + k^2 + 2k \cot(\Delta \theta)}$$  

(5) 

where \(\kappa = \sqrt{2m|V_0 - E|}/\hbar\), \(r\) is the ratio of transmission probability to reflection probability, and \(\Delta \theta\) is the phase increased across the barrier. In the limit of the very thin barrier \((\kappa d \ll 1)\), we have \(R(r, \Delta \theta) \approx 1\) and \(I(r, \Delta \theta) \approx \kappa^2 (\kappa^2 + k^2) \kappa d \ll 1\), which means that the movement of the tunneling particle can be treated as a free one with \(\bar{\mathbf{p}} = \hbar k\). While in the opposite limit of opaque barrier \((\kappa d \gg 1)\), we simply have \(R(r, \Delta \theta) \approx 0\) and \(I(r, \Delta \theta) \approx 1\). The momentum of the tunneling particle is imaginary and approaches the effective Buttiker-Landauer momentum \(\hbar k\) \cite{2}.

The quantum travel (or tunneling) time \(\tau_t = m d/\hbar \sqrt{2\kappa^2 + I^2 \kappa^2}\) is inversely proportional to the average momentum, and its dependence on \(\kappa d\) is shown in Fig. \(\text{1}\). The Buttiker-Landauer time \(\tau_{BL}\) and the free travel time \(\tau_c\) which denotes the time a free particle spends traveling through the same distance but without potential barrier, are also shown for comparison. Interesting thing is that \(\tau_t\) is always bounded by two limit cases, free travel time \(\tau_c\) for very thin barrier and Buttiker-Landauer time \(\tau_{BL}\) for opaque barrier. Therefore, \(\tau_t\) not only can retrieve the Buttiker-Landauer time \(\tau_{BL}\) in the case of opaque barrier, but also has a clear meaning even in the case of very thin barrier, wherein \(\tau_{BL}\) cannot be well defined \cite{33,34}.

**Tunneling delay time in attoclock measurement** The interpretations of the attoclock experiments are mainly based on the following assumptions \cite{33}: i) the highest probability for the electron to tunnel is at the peak of the electric field; ii) ionization is completed once the electron emerges from the barrier; iii) after the barrier exit, the electron dynamics is classical. The first two assumptions

![FIG. 1: (Color online) The quantum travel time \(\tau_t\), the Böttiker-Landauer time \(\tau_{BL}\) and the free travel time \(\tau_c\) as functions of \(\kappa d\) with (a) \(E = 1.8\) a.u., \(V_0 = 2.0\) a.u. and (b) \(E = 0.2\) a.u., \(V_0 = 2.0\) a.u.. The atomic units are used with \(e = m = \hbar = 1\).](image-url)
Therefore, the tunneling process in linearly polarized
tunneling process via introducing parabolic coordinates [33].

\[ \tau \]

\[ \tau \text{ describing the time spent by electron to travel from } x_{\text{in}} \text{ to } x_{\text{exit}}, \text{ while } \tau_{\text{A}} \text{ corresponds to the travel region from } x_F \text{ to } x_{\text{exit}}. \]

The laser parameters is \( F_0 = 0.04 \) a.u. and the wavelength is 5\( \mu \)m.

are closely related to the tunneling process, and imply that the experimentally measured delay time \( \tau_{\text{A}} \), in principle, is the time interval between the instant of the peak of the laser field (as a reference instant \( t_F \)) and the instant at which the ionized electron wave packet appears at the tunnel exit. Theoretically, the instant the electron leaves the barrier can be read from the probability current, so \( \tau_{\text{A}} \) can be directly extracted from our calculation. By comparing it with the quantum travel time \( \tau_t \) defined in Eq. [1] as well as the Büttiker-Landauer time \( \tau_{\text{BL}} \), the physical meaning of \( \tau_{\text{A}} \) can be clarified.

Tunneling of an initially bounded electron through the 3D Coulomb potential bent by a linearly polarized intense laser field, can be reduced to an effective 1D tunneling process via introducing parabolic coordinates [33]. Therefore, the tunneling process in linearly polarized intense laser field can be well described by 1D TDSE (atomic units are used with \( e = m = \hbar = 1 \))

\[ i \frac{\partial \psi(x, t)}{\partial t} = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{\sqrt{x^2 + \alpha}} - F(t)x \right) \psi(x, t), \tag{6} \]

where \( F(t) = F_0 \sin(\omega t) \) is the laser electric field with frequency \( \omega \), \( \alpha \) is a soften parameter which is chosen to be 2.0 for a hydrogen atom.

In our calculation, the instant at which the tunneling electron arrives at a fixed position \( x_0 \) is the instant at which the probability current \( j(x_0, t) \) reaches its maximum. As shown in Fig. 2, the escape instant can be extracted by reading the maximum of \( j(x_{\text{exit}}, t) \), and the tunneling delay time \( \tau_{\text{A}} \) is the time interval between the escape instant \( t_e \) and the reference instant \( t_F \) [10]. On the other hand, we can also determine the location of tunneling electron at \( t_F \) by scanning \( j(x, t) \) at different positions. As shown in Fig. 2 we can find a position \( x_F \) under the barrier, at which the peak of \( j(x_F, t) \) coincides with the peak of the laser field \( F(t) \).

Combining with the knowledge of travel region and wave function under the barrier, the quantum travel time can be obtained by Eq. [1]. In Fig. 2 there are two quantum travel times \( \tau_t \) and \( \tau_{\text{A}} \) corresponding to the travel regions \((x_{\text{in}}, x_{\text{exit}})\) and \((x_F, x_{\text{exit}})\), respectively. Apparently, it is \( \tau_{\text{A}} \), which is much shorter than \( \tau_t \), that relates closely to the tunneling delay time \( \tau_{\text{A}} \). Since both of the Büttiker-Landauer time \( \tau_{\text{BL}} \) and the Keldysh time \( \tau_K \) are defined in the same travel region as \( \tau_t \), they are found to deviate far from \( \tau_{\text{A}} \). To further illuminate the properties of these times, we have performed 1D TDSE calculations for two different intensities of laser field with a relative wide wavelength range, which lie in the deep tunneling regime with Keldysh parameter \( \gamma < 0.6 \). Three points deserve to be emphasized. i) The travel time \( \tau_t \) depends only on the intensity of laser field, but not on the wavelength, the same as the Büttiker-Landauer time \( \tau_{\text{BL}} \) does. ii) \( \tau_t \) is shorter than \( \tau_{\text{BL}} \). For example, \( \tau_{\text{BL}} = 650 \) as & \( \tau_t = 400 \) as for \( F_0 = 0.04 \) a.u., and \( \tau_{\text{BL}} = 560 \) as & \( \tau_t = 305 \) as for \( F_0 = 0.05 \) a.u.. iii) There is a well consistency between the quantum travel time \( \tau_{\text{A}} \) and the tunneling delay time \( \tau_{\text{A}} \), both of which are much shorter than \( \tau_t \) and \( \tau_{\text{BL}} \), in a wide wavelength range as can be seen in Fig. 3. Therefore, the tunneling delay time measured by attoclock experiment can be interpreted as the time spent by the electron to tunnel from a point under barrier \( (x_F) \) to the tunnel exit, which would be much shorter than the Büttiker-Landauer time \( \tau_{\text{BL}} \) and the Keldysh time \( \tau_K \).

Wavelength dependent tunneling delay time According to the adiabatic tunneling theory, the tunneling delay time will be determined only by the intensity of the laser field. This has partially inspired a series of experiments to measure the dependence of the tunneling delay time on laser intensity [21]. However, we find that the wavelength dependence of tunneling delay time \( \tau_{\text{A}} \) exhibits a peculiar oscillation even in the deep tunneling regime (\( \gamma < 0.6 \)), which is beyond the scope of the adiabatic tunneling picture. As shown in Fig. 3, when the amplitude of the laser field is fixed to \( F_0 = 0.04 \) a.u., \( \tau_{\text{A}} \) shows a quick oscillation with decreasing amplitude, and finally approaches to a constant. While increasing the laser field to \( F_0 = 0.05 \) a.u., the oscillation in short wavelength region becomes weak and \( \tau_{\text{A}} \) decreases monotonously with wavelength. Even a negative tunneling delay time \( \tau_{\text{A}} \) can be found when wavelength is over 10 \( \mu \)m (not shown here). The decrease of \( \tau_{\text{A}} \) can be attributed to the depletion of the ground state: a loss of population before the peak of the field would enhance the relative contribution of early ionization events [30]. The travel time \( \tau_{\text{A}} \) is also shown in Fig. 3 for comparison and is found in coincidence with \( \tau_{\text{A}} \) for all laser parameters considered here.

On close inspection of Fig. 3(a), it is surprised that
the oscillation keeps a constant period around 850 nm. We will see below that the oscillation is actually a result of interference between the ground state tunneling channel and the excited states tunneling channels. First we perform a calculation in which the component of the first excited state in the total wave function is removed at every step of temporal evolution. After doing so, as is shown in Fig. 4(a), the oscillation disappears and an almost constant tunneling delay time \( \tau_A \) is found, which is larger than \( \tau_A \) at long wavelength. This is a clear evidence that the first excited state plays an important role in the construction of the oscillation. Then we investigate the temporal evolution of the population of the first excited state at different wavelengths. As shown in Fig. 4(b), the population shows a series of steps with equal time intervals close to \( 2\pi/\Delta E \), where \( \Delta E = 0.2671 \) a.u. is the energy gap between the ground state and the first excited state in our model. And the position of each step in Fig. 4(b) is independent of wavelength. This kind of dynamical feature can be well understood by considering a two-level system with dynamical coupling when the photon energy of laser is much smaller than the energy gap (see supplementary for details). Since the first step is usually the most prominent, at time \( t_s \) when the first step emerges, a considerable amount of wave function is excited to the first excited state and then evolves until tunneling at \( T/4 \) (\( T \) is the optical cycle). Tunneling current through the above channel will interfere with that direct from the ground state. The phase difference between these two channels can be simply estimated by \( \Delta \varphi = \Delta E (T/4 - t_s) \). Since \( \Delta E \) and \( t_s \) are independent of the optical cycle \( T \), a phase shift \( \Delta \varphi \) of \( 2\pi \) corresponds to change of the optical cycle as \( 8\pi/\Delta E \), which is nothing else but the period of the interference shown in Fig. 4(a). After converting optical cycle to wavelength, the estimated value of the oscillation period is determined as 682 nm which is smaller than that read from Fig. 4(a). This difference is not surprising since the contributions of the higher excited states are not included in the above analysis.

In order to exclude the effects of the higher excited states, we employ a quantum dot potential with only two bounded states with energy gap \( \Delta E = 0.4328 \) a.u. (see supplementary material for more details). As shown in Fig. 4(c), similar to the case of Coulomb potential, obvious oscillation exists in the wavelength dependence of \( \tau_A \) and almost disappears if the component of the first excited state in the wave function is removed at every step of temporal evolution. The population of the first excited state also shows a series of steps with constant time interval determined by the energy gap. The period of the oscillation is estimated as 421 nm which is very close to the period (430 nm) read from Fig. 4(c). Successful prediction of the oscillation period provides a strong support to the interpretation of the oscillation as a result of the interference between the ground state and the excited states tunneling channels.

It is noteworthy that, when the wavelength or the intensity of the laser field increases, the step structure in the population of the first excited state becomes weaker (see Figs. 4(b),(d) and the supplementary materials), which results in a decreasing amplitude of the oscillation with increasing wavelength (Figs. 4(a)) and intensity of the laser field (Fig. 4(b)).

In conclusion, based on a newly introduced quantum travel time, the tunneling delay time measured by attoclock experiment can be interpreted as the travel time spent by the electron to tunnel from a point under barrier to the tunnel exit, which is actually a part of the Büttiker-Landauer time. Our interpretation may bridge the gap between the conventional tunneling time (Büttiker-Landauer time and Keldysh time) and the measured tunneling delay time. In addition, a peculiar oscillation structure with constant period in the wavelength dependence of tunneling delay time is observed in deep tunneling regime, which is beyond the scope of the adia-
atic tunneling picture. This oscillation structure can be attributed to the interference between the ground state tunneling channel and the excited states tunneling channels. Our results reveal the important role of the excited states in strong-field tunneling process, which is usually ignored. On the other hand, the identified coherent tunneling process paves the way towards probing and imaging of the tunneling dynamics of excited states.

X. Hao and Z. Shu contributed equally to this work. This work was supported by the National Key Research and Development program (No. 2016YFA0401100) and NNSFC (Nos. 11334009, 11425414, 11504215, and 11874246).

[1] L. A. MacColl, Phys. Rev. 40, 621 (1932).
[2] M. Büttiker and R. Landauer, Phys. Rev. Lett. 49, 1739 (1982).
[3] R. Landauer, Nature (London) 341, 567 (1989).
[4] M. Büttiker, Phys. Rev. B 27, 6178 (1983).
[5] E. H. Hauge, J.A. Støvneng, Rev. Modern Phys. 61, 917 (1989).
[6] R. Landauer, T. Martin, Rev. Modern Phys. 66 (1994) 217.
[7] N. Yamada, Phys. Rev. Lett. 93, 170401 (2004).
[8] C. R. McDonald, G. Orlando, G. Vampa, and T. Brabec, Phys. Rev. Lett. 111, 090405 (2013).
[9] T. Zimmermann, S. Mishra, B. R. Doran, D. F. Gordon, and A. S. Landsman, Phys. Rev. Lett. 116, 233603 (2016).
[10] N. Teeny, E. Yakaboylu, H. Bauke, and C. H. Keitel, Phys. Rev. Lett. 116, 063003 (2016).
[11] H. Ni, U. Saalmann, and J.-M. Rost, Phys. Rev. Lett. 117, 023002 (2016).
[12] M. Uiberacker, T. Uphues, M. Schultz, A. J. Verhoef, V. Yakovlev, M. F. Kling, J. Rauschenberger, N. M. Kabachnik, H. Schröder, M. Lezius, K. L. Kompa, H.-G. Muller, M. J. J. Vrakking, S. Hendel, U. Kleineberg, U. Heinzmann, M. Drescher and F. Krausz, Nature (London) 446, 627 (2007).
[13] P. Eckle, A. N. Pfeiffer, C. Cirelli, A. Staudte, R. Dörner, H. G. Muller, M. Büttiker, and U. Keller, Science 322, 1525 (2008).
[14] P. Eckle, M. Smolarski, P. Schlup, J. Biegert, A. Staudte, M. Schöffler, H. G. Muller, R. Dörner, and U. Keller, Nat. Phys. 4, 565 (2008).
[15] M. Schultz, M. Fieß, N. Karpowicz, J. Gagnon, M. Korbut, F. Kmet, M. Holstetter, S. Nepp, A. L. Cavaleri, Y. Kominos, T. Mercouris, C. A. Nicolaides, R. Pazourek, S. Nagele, J. Feist, J. Burgdörfer, A. M. Azzeer, R. Ernstorfer, R. Kienberger, U. Kleineberg, E. Goulielmakis, F. Krausz and V. S. Yakovlev, Science 328, 1658 (2010).
[16] E. Goulielmakis, Z.-H. Loh, A. Wirth, R. Santra, N. Rohringer, V. S. Yakovlev, S. Zherebtsov, T. Pfeifer, A. M. Azzeer, M. F. Kling, S. R. Leone and F. Krausz, Nature 466, 739 (2010).
[17] L. Arissian, C. Smeenk, F. Turner, C. Trallero, A.V. Sokolov, D. M. Villeneuve, A. Staudte, and P. B. Corkum, Phys. Rev. Lett. 105, 133002 (2010).
[18] D. Shafir, H. Soifer, B. D. Bruner, M. Dagan, Y. Mairesse, S. Patchkovskii, M. Y. Ivanov, O. Smirnova and N. Dudovich, Nature 485, 343 (2012).
[19] A. S. Landsman and U. Keller, Phys. Rep. 547, 1 (2015).
[20] A. N. Pfeiffer, C. Cirelli, M. Smolarski, D. Dimitrovski, M. A.-Samha, L. B. Madsen and U. Keller, Nature Phys. 8, 76 (2012).
[21] A. S. Landsman, M. Weger, J. Maurer, R. Boge, A. Ludwig, S. Heuser, C. Cirelli, L. Gallmann, and U. Keller, Optica 1, 343 (2014).
[22] I. Barth and O. Smirnova, Phys. Rev. A 84, 063415 (2011).
[23] A. N. Pfeiffer, C. Cirelli, A. S. Landsman, M. Smolarski, D. Dimitrovski, L. B. Madsen, and U. Keller, Phys. Rev. Lett. 109, 083002 (2012).
[24] M. Klaiber, E. Yakaboylu, H. Bauke, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. 110, 155004 (2013).
[25] M. Li, Y. Liu, H. Liu, Q. Ning, L. Fu, J. Liu, Y. Deng, C. Wu, L.-Y. Peng, and Q. Gong, Phys. Rev. Lett. 111, 023006 (2013).
[26] E. Yakaboylu, M. Klaiber, and K. Z. Hatsagortsyan, Phys. Rev. A 90, 012116 (2014).
[27] M. Klaiber, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rev. Lett. 114, 083001 (2015).
[28] M. Li, J.-W. Geng, M. Han, M.-M. Liu, L.-Y. Peng, Q. Gong, and Y. Liu, Phys. Rev. A 93, 013402 (2016).
[29] N. Camus, E. Yakaboylu, L. Fechner, M. Klaiber, M. Laux, Y. H. Mi, K. Z. Hatsagortsyan, T. Pfeifer, C. H. Keitel, and R. Moshammer, Phys. Rev. Lett. 119, 023201 (2017).
[30] L. Torlina, F. Morales, J. Kaushal, I. Ivanov, A. Kheifets, D. Corkum, Phys. Rev. Lett. 105, 133002 (2010).
[31] A. W. Bray, S. Eckart and A. S. Kheifets, Phys. Rev. Lett. 121, 123201 (2018).
[32] L. V. Keldysh, Sov. Phys. JETP 20, 1307 (1965).
[33] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon Press, New York, 1977).