Unification of $q-$exponential function and related $q-$numbers and polynomials

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Abstract

The main purpose of this paper is to introduce and investigate a class of generalized Bernoulli polynomials and Euler polynomials based on the generating function. we unify all forms of $q-$exponential functions by one more parameter. we study some conditions on this parameter to related this to some classical results for $q-$Bernoulli numbers and polynomials.

1 Introduction

In combinatorial mathematics, a $q$-exponential is a $q$-analog of the exponential function, namely the eigenfunction of a $q$-derivative. There are many $q$-derivatives, for example, the classical $q$-derivative, the Askey-Wilson operator, etc. Therefore, unlike the classical exponentials, $q$-exponentials are not unique. In the standard approach to the $q$-calculus, two exponential function are used. These $q$-exponentials are defined by

$$
E_q(z) = e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1 - (1 - q) q^k z)}, \quad 0 < |q| < 1, |z| < \frac{1}{|1 - q|},
$$

$$
E_{q^2}(z) = e_1/q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1 + (1 - q) q^k z), \quad 0 < |q| < 1, z \in \mathbb{C},
$$

In addition, The improved $q$-exponential function is defined by

$$
E_{q^2}(z) = e_1/q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \frac{(-1, q)_n}{2^n} = \prod_{k=0}^{\infty} \frac{(1 + (1 - q) q^k z^2)}{(1 - (1 - q) q^k z^2)}, \quad 0 < |q| < 1, |z| < \frac{2}{|1 - q|},
$$

The Bernoulli numbers $\{B_m\}_{m \geq 0}$ are rational numbers in a sequence defined by the binomial recurrence formula

$$
\sum_{k=0}^{m} \binom{m}{k} B_k - B_m = \begin{cases} 1, & m = 1, \\ 0, & m > 1, \end{cases}
$$
or equivalently, the generating function
\[ \sum_{k=0}^{\infty} \frac{B_k t^k}{k!} = \frac{t}{e^t - 1} \]

The q-binomial formula is known as
\[ (1 - a)_q^n = (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q q^{\frac{1}{2}k(k-1)} (-1)^k a^k. \]

The above q-standard notation can be found in [3].

Over 70 years ago, Carlitz extended the classical Bernoulli and Euler numbers and polynomials, and introduced the q-Bernoulli and q-Euler numbers and polynomials (see [7], [8] and [9]). There are numerous recent investigations on this subject. ([9], [10], [11], [13] and [12]), Srivastava [8], Srivastava et al. [7]. The main part of these generalizations is the definition of q-analogue of exponential function. By defining the suitable q-analogue of exponential function, they derive to the different definitions for q-Bernoulli numbers. In this case some interesting properties are discovered [14]. The unification of q-exponential is introduced in the next definition. This function depends on the parameter and by changing this parameter we can reach to the different versions of q-exponential function.

**Definition 1** we define unification of q-exponential function as follow

\[ E_{q,\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q^\alpha} \alpha(q, n) \]

where \( z \) is any complex number and \( \alpha(q, n) \) is a function of \( q \) and \( n \). In addition, \( \alpha(q, n) \) approaches to 1, where \( q \) tends one from the left side.In the special case where \( \alpha(q, n) = 1 \), and \( \alpha(q, n) = q^{\left( -1 \right) n} \) we reach to \( e_q(z) \) and \( E_q(z) \) respectively.

At the next lemma, we will discuss about the conditions that make \( E_{q,\alpha}(z) \) convergent.There are some restrictions, that has to be considered. since \( E_{q,\alpha}(z) \) is the q-analogue of exponential function, \( \alpha(q, n) \) approaches to 1, where \( q \) tends one from the left side. For the rest of the paper we will denote \( \alpha(q, n) \) by \( \alpha_n \), however we keep this in our mind that \( \alpha(q, n) \) is depend on \( q \) and \( n \).

**Lemma 2** If \( \lim_{n \to \infty} \frac{\alpha_{n+1} \alpha_n}{[n+1]_q \alpha_n} \) does exist as \( n \) tends infinity and is equal to \( l \), Then q–exponential function \( E_{q,\alpha}(z) \) is analytic in the disc \( |z| < (l)^{-1} \).

**Proof.** In order to obtain the radius of convergence, we compute

\[ \lim_{n \to \infty} \left| \frac{z^{n+1} \alpha_{n+1}}{[n+1]_q \alpha_n} \right| = \lim_{n \to \infty} \left| \frac{\alpha_{n+1}}{[n+1]_q \alpha_n} \frac{[n]_q}{z^n} \right| = \lim_{n \to \infty} \left| \frac{\alpha_{n+1}}{[n+1]_q \alpha_n} \right| \frac{\alpha_n}{z^n} = \lim_{n \to \infty} \frac{\alpha_{n+1} \alpha_n}{[n+1]_q \alpha_n} \frac{\alpha_n}{z^n} \]

Then, using d’Alembert’s test, we get (for \( q \neq 1 \)) the radius of convergence.

**Example 3** Let \( \alpha_n \) is equal to 1, \( q^{\left( -1 \right) n} \), then we reach to \( e_q(z) \), \( E_q(z) \) and improved q–exponential function \( E_q(z) \) [29] repectively. Then the radius of convergence becomes \( \frac{1}{1-q} \), infinity and \( \frac{2}{1-q} \) repectively where \( 0 < |q| < 1 \).

With this q–exponential function, we define the new class of q–Bernoulli numbers and polynomials. Next definition denotes a general class of these new q–numbers and polynomials.
Definition 4 Let \(q \in \mathbb{C}, \; 0 < |q| < 1\). The \(q\)-Bernoulli numbers \(B_{n,q,\alpha}\) and polynomials \(B_{n,q,\alpha}(x, y)\) and \(q\)-Euler numbers \(E_{n,q,\alpha}\) and polynomials \(E_{n,q,\alpha}(x, y)\) and The \(q\)-Genocchi numbers \(G_{n,q,\alpha}\) and polynomials \(G_{n,q,\alpha}(x, y)\) in two variables \(x, y\) respectively are defined by the means of the generating functions:

\[
\mathfrak{B} (t) = \frac{t}{\xi_{q,\alpha} (t) - 1} = \sum_{n=0}^{\infty} B_{n,q,\alpha} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,
\]

\[
\frac{t}{\xi_{q,\alpha} (t) - 1} E_{q,\alpha} (tx) E_{q,\alpha} (ty) = \sum_{n=0}^{\infty} B_{n,q,\alpha} (x, y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,
\]

\[
\frac{2}{\xi_{q,\alpha} (t) + 1} = \sum_{n=0}^{\infty} E_{n,q,\alpha} \frac{t^n}{[n]_q!}, \quad |t| < \pi,
\]

\[
\frac{2}{\xi_{q,\alpha} (t) + 1} E_{q,\alpha} (tx) E_{q,\alpha} (ty) = \sum_{n=0}^{\infty} E_{n,q,\alpha} (x, y) \frac{t^n}{[n]_q!}, \quad |t| < \pi,
\]

\[
\frac{2t}{\xi_{q,\alpha} (t) + 1} E_{q,\alpha} (tx) E_{q,\alpha} (ty) = \sum_{n=0}^{\infty} G_{n,q,\alpha} (x, y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.
\]

If the convergence conditions are hold for \(q\)-exponential function, It is obvious that by tending \(q\) to 1 from the left side, we lead to the classic definition of these polynomials, we mention that \(\alpha(q, \alpha)\) is respect to \(q\) and \(n\). In addition by tending \(q\) to \(1^-\), \(E_{q,\alpha} (z)\) approach to the ordinary exponential function. that means:

\[
B_{n,q,\alpha} = B_{n,q,\alpha} (0), \quad \lim_{q \rightarrow 1^-} B_{n,q} (x + y) = B_n (x + y), \quad \lim_{q \rightarrow 1^-} B_{n,q} = B_n,
\]

\[
E_{n,q,\alpha} = E_{n,q,\alpha} (0), \quad \lim_{q \rightarrow 1^-} E_{n,q} (x + y) = E_n (x + y), \quad \lim_{q \rightarrow 1^-} E_{n,q} = E_n,
\]

\[
G_{n,q,\alpha} = G_{n,q,\alpha} (0), \quad \lim_{q \rightarrow 1^-} G_{n,q} (x + y), \quad \lim_{q \rightarrow 1^-} G_{n,q} = G_n.
\]

Here \(B_n (x)\), \(E_n (x)\) and \(G_n (x)\) denote the classical Bernoulli, Euler and Genocchi polynomials which are defined by

\[
\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n (x) \frac{t^n}{n!}, \quad \frac{2}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} E_n (x) \frac{t^n}{n!} \quad \text{and} \quad \frac{2t}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} G_n (x) \frac{t^n}{n!}
\]

The aim of the present paper is to obtain some results for the above newly defined \(q\)-Bernoulli and \(q\)-Euler polynomials. In the next section we will discuss about some restriction for \(\alpha(q, \alpha)\), such that the familiar results discovered. we will focus on two main properties of \(q\)-exponential function, first in which situation \(E_{q,\alpha} (z) = E_{q-1,\alpha} (z)\), second we investigate the conditions for \(\alpha(q, \alpha)\) such that \(E_{q,\alpha} (-z) = (E_{q,\alpha} (z))^{-1}\). A lot of classical results are found by these two properties. The form of new type of \(q\)-exponential function, motivate us to define a new \(q\)-addition and \(q\)-subtraction like a Daehee formula as follow

\[
(x \oplus_q y)^n := \sum_{k=0}^{n} \binom{n}{k}_q \alpha(q, k)x^k y^{n-k}, \quad n = 0, 1, 2, ...
\]

\[
(x \boxplus_q y)^n := \sum_{k=0}^{n} \binom{n}{k}_q \alpha(q, k)x^k (-y)^{n-k}, \quad n = 0, 1, 2, ...
\]

2 New exponential function and its properties

In this section we shall provide some conditions on \(\alpha(q, \alpha)\) to reach two main properties. first we try to find out, in which situation \(E_{q,\alpha} (z) = E_{q-1,\alpha} (z)\). This condition make \(q\)-exponential symmetry to \(q\) factor and
the properties of related \( q \)-numbers will be preserved even if we change \( q \) to \( q^{-1} \). Second property is the condition on \( q \)-exponential to reach multiplicative inverse i.e. \( \mathcal{E}_{q,a} (-z) = (\mathcal{E}_{q,a} (z))^{-1} \). This property make the odd coefficient of \( q \)-Bernoulli numbers exactly zero and related them to \( q \)-trigonometric functions.

**Lemma 5** The new \( q \)-exponential function \( \mathcal{E}_{q,a} (z) \) satisfy \( \mathcal{E}_{q,a(q)} (z) = \mathcal{E}_{q^{-1},a(q^{-1})} (z) \), if and only if \( q(z) \alpha(q^{-1}, n) = \alpha(q, n) \).

**Proof.** The proof is based on the fact that \([n]_{q^{-1}} = q^{-\binom{n}{2}} [n]_q!\), therefore

\[
\mathcal{E}_{q^{-1},a(q^{-1})} (z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}} \alpha(q^{-1}, n) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q} \alpha(q, n) = \mathcal{E}_{q,a} (z)
\]

The proof is based on the fact that \([n]_{q^{-1}} = q^{-\binom{n}{2}} [n]_q!\), therefore

\[
\mathcal{E}_{q^{-1},a(q^{-1})} (z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}} \alpha(q^{-1}, n) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q} \alpha(q, n) = \mathcal{E}_{q,a} (z)
\]

**Proof.** On the another hand, the another side of statement can be found by equating the coefficient of above summation.

**Corollary 6** If \( \alpha(q, n) \) is in a form of polynomial that means \( \alpha(q, n) = \sum_{i=0}^{m} a_i q^i \), to satisfy \( \mathcal{E}_{q,a(q)} (z) = \mathcal{E}_{q^{-1},a(q^{-1})} (z) \), we have

\[
\deg(\alpha(q, n)) = m = \left( \begin{array}{c} n \\ 2 \end{array} \right) - j \leq \left( \begin{array}{c} n \\ 2 \end{array} \right) , \text{ and } a_{j+k} = a_{m+k} \text{ where } k = 0, 1, \ldots, m - j
\]

where \( j \) is the leading index, such that \( a_j \neq 0 \) and for \( 0 \leq k < j, a_k = 0 \).

**Proof.** First, we want to mention that \( \sum_{i=0}^{m} a_i = 1 \), because \( \alpha(q, n) \) approaches to 1, where \( q \) tends one from the left side. In addition as we assumed \( \alpha(q, n) = \sum_{i=0}^{m} a_i q^i \), by simple substitution \( q^{-1} \) instead of \( q \), and \( \sum_{i=0}^{m} a_i = 1 \) lead us to

\[
q(z) = \sum_{i=0}^{m} a_{m-i} q^i = \sum_{i=0}^{m} a_i q^i
\]

equating the coefficient of \( q^k \) to reach the statement.

**Example 7** simplest example of the previous corollary will be happened when \( \alpha(q, n) = \frac{2}{q} \). This case leads us to the following exponential function

\[
\mathcal{E}_{q,a} (z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q} \left( \frac{2}{q} \right) \text{ & } \mathcal{E}_{q^{-1},a(q^{-1})} (z) = \mathcal{E}_{q,a} (z)
\]

The another example will be occured if \( \alpha(q, n) = -q \frac{z^m}{2^n} \). By using \( q \)-binomial formula \( \alpha(q, n) = \frac{1}{2^n} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) q^{i-1-n} \). As we expect, where \( q \) tends 1 from the left side, \( \alpha(q, n) \) approach to 1.

This presentation is not in a form of previous corollary. However \( q(z) \frac{(1+z)(1+q^{-1})(1+q^{-2})\ldots(1+q^{-n})}{2^n} = \alpha(q, n) \). This parameter lead us to the improved \( q \)-exponential function

\[
\mathcal{E}_q (z) = \mathcal{E}_{q,a} (z) = \sum_{n=0}^{\infty} \frac{z^n (-1) q^n}{2^n} \text{ & } \mathcal{E}_{q^{-1}} (z) = \mathcal{E}_q (z)
\]

The properties of \( q \)-Bernoulli polynomials related to this improved \( q \)-exponential function was studied at [4].
Remark 8  It’s obvious that if we substitute \( q \) to \( q^{-1} \) in any kind of \( q \)-exponential function and achieve another \( q \)-anologue of exponential function, the parameter \( \alpha(q, n) \) will change to \( \beta(q, n) \), and \( q^{[z]} \alpha(q^{-1}, n) = \beta(q, n) \). The famous case is standard \( q \)-exponential function:

\[
e^{q^{-1}}(z) = E^{q^{-1}, \alpha(q^{-1})}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}
\]

\[
= \sum_{n=0}^{\infty} q^{[z]} \frac{z^n}{[n]_q!} = E_q(z) \quad \text{&} \quad q^{[z]} \alpha(q^{-1}, n) = q^{[z]} = \beta(q, n)
\]

\[\blacksquare\]

Proposition 9  The new \( q \)-exponential function \( E_{q, \alpha}(z) \) satisfy \( E_{q, \alpha}(-z) = (E_{q, \alpha}(z))^{-1} \), if and only if

\[
\alpha(q, 0) = 1 \& 2 \sum_{k=0}^{p-1} \binom{n}{k} q^{k} (1-1)^{k} \alpha_{k} \alpha_{n-k} = \binom{n}{p} q^{p+1} \alpha_{p}^{2} \text{ where } n = 2p \text{ and } p = 1, 2, ...
\]

Proof. Since \( E_{q, \alpha}(-z).E_{q, \alpha}(z) = 1 \) has to be hold, we write the expansion for this equation.

\[
E_{q, \alpha}(-z).E_{q, \alpha}(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \alpha_{k} \alpha_{n-k} \right) \frac{z^n}{[n]_q!} = 1
\]

Let call the expression on a bracket as \( \beta_{k, q} \). If \( n \) is an odd number, then

\[
\beta_{n-k, q} = \binom{n-k}{k} (-1)^{n-k} \alpha(q, k) \alpha(q, n-k) = - \binom{n}{k} (-1)^{k} \alpha(q, k) \alpha(q, n-k) = -\beta_{k, q} \text{ where } k = 0, 1, ..., n
\]

Therefore for \( n \) as an odd number, we have the trivial equation. since \( \binom{n-k}{k} = \binom{n}{k} \), The same discussion for even \( n \) and equating \( z^n \)-coefficient togheter lead as to the proof.

Remark 10  The previous proposition can be rewritten as a system of nonlinear equations. The following system shows a condition for \( \alpha_k \). we mention that \( \alpha_k \rightarrow 1 \) where \( q \rightarrow 1^- \) and \( \alpha_0 = 1 \).

\[
\begin{aligned}
2 \alpha_2 \alpha_1 - 2(\frac{2}{1})_q \alpha_0 \alpha_0 &= 0 \\
2 \alpha_2 \alpha_1 - 2(\frac{4}{1})_q \alpha_0 \alpha_2 + 2(\frac{5}{2})_q \alpha_2 \alpha_2 &= 0 \\
2 \alpha_6 \alpha_1 - 2(\frac{6}{1})_q \alpha_5 \alpha_2 + 2(\frac{6}{2})_q \alpha_4 \alpha_3 - 2(\frac{6}{3})_q \alpha_3 \alpha_3 &= 0 \\
&\vdots \\
2 \alpha_{n} \alpha_1 - 2(\frac{n}{1})_q \alpha_{n-1} \alpha_2 + 2(\frac{n}{2})_q \alpha_{n-2} \alpha_3 - ... + (-1)^{\frac{n}{2}}(\frac{n}{\frac{n}{2}})_q \alpha_\frac{n}{2} \alpha_\frac{n}{2} &= 0
\end{aligned}
\]

(3)

For even \( n \), we have \( \frac{n}{2} \) equations and \( n \) unknown variables. In this case we can find \( \alpha_k \) respect to \( \frac{n}{2} \) parameters by the recurrence formula. For example, some few terms can be found as follow

Corollary 11  Remark 12

\[
\begin{aligned}
\alpha_0 &= 1 \\
\alpha_2 &= \frac{1+q}{2} \alpha_1 \\
\alpha_4 &= \frac{4}{\alpha_2} \left( \frac{2}{1}_q \alpha_3 - \frac{3}{1}_q \right) \\
\alpha_6 &= \frac{6}{1}_q + \frac{6}{1}_q - \frac{1}{2} \left( \frac{6}{2}_q \frac{1+q}{2} \alpha_7 \left( \frac{4}{1}_q \frac{4}{2}_q \left( \frac{2}{1}_q \alpha_3 - \frac{3}{1}_q \right) \right) \right)
\end{aligned}
\]
The familiar solution of this system is $\alpha(q, k) = \frac{(-1)^k}{2^k}$. This $\alpha(q, k)$ leads us to the improved exponential function. On the other hand, we can assume that all $\alpha_k$ for odd $k$ are 1. Then by solving the system for these parameters, we reach the another exponential function that satisfies $E_{q, \alpha}(-z) = \left(\mathcal{E}_{q, \alpha}(z)\right)^{-1}$.

**Lemma 13** If $\frac{\alpha(q, n+1)}{\alpha(q, n)}$ can be demonstrated as a polynomial of $q$, that means $\frac{\alpha(q, n+1)}{\alpha(q, n)} = \sum_{k=0}^{m} a_k q^k$, then

$$D_q(\mathcal{E}_{q, \alpha}(z)) = \sum_{k=0}^{m} a_k \mathcal{E}_{q, \alpha}\left(z q^k\right).$$

**Proof.** The proof is based on the following identity

$$D_q(\mathcal{E}_{q, \alpha}(z)) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n-1} q \alpha(q, n) = \sum_{n=0}^{\infty} \frac{z^n}{n!} q \left(\alpha(q, n) \sum_{k=0}^{m} a_k q^k \right) = \sum_{k=0}^{m} a_k \left(\frac{z q^k}{n!} q\right)^n \alpha(q, n) = \sum_{k=0}^{m} a_k \mathcal{E}_{q, \alpha}(z q^k).$$

**Example 14** For $\alpha(q, n) = 1$, $q^n$ and $\frac{(-1)^q}{2^n}$, the ratio of $\frac{\alpha(q, n+1)}{\alpha(q, n)}$ becomes 1, $q^n$ and $\frac{1+q^n}{2}$ respectively. Therefore the following derivatives hold true

$$D_q(\mathcal{E}_q(z)) = e_q(z) \quad \& \quad D_q(E_q(z)) = E_q(qz) \quad \& \quad D_q(\mathcal{E}_q(z)) = \frac{\mathcal{E}_q(z) + E_q(zq)}{2}.$$

3 Related $q$-Bernoulli polynomial

In this section, we will study the related $q$–Bernoulli polynomials, $q$–Euler polynomials and $q$–Gennocchi polynomials. The discussion of properties of general $q$-exponential at the previous section, give us the proper tools to reach to the general properties of these polynomials related to $\alpha(q, n)$. First, we give the general form of addition theorem.

**Proposition 15** (Addition Theorems) For all $x, y \in \mathbb{C}$ we have

$$\mathcal{B}_{n, q, \alpha}(x, y) = \sum_{k=0}^{n} \binom{n}{k} q \mathcal{B}_{k, q, \alpha}(x \oplus y)^{n-k}, \quad \mathcal{E}_{n, q, \alpha}(x, y)$$

$$= \sum_{k=0}^{n} \binom{n}{k} q \mathcal{E}_{k, q, \alpha}(x \oplus y)^{n-k}, \quad \mathcal{G}_{n, q, \alpha}(x, y) = \sum_{k=0}^{n} \binom{n}{k} q \mathcal{G}_{k, q, \alpha}(x \oplus y)^{n-k},$$

$$\mathcal{B}_{n, q, \alpha}(x, y) = \sum_{k=0}^{n} \binom{n}{k} q \alpha(q, n-k) \mathcal{B}_{k, q, \alpha}(x)^{y^{n-k}}, \quad \mathcal{E}_{n, q, \alpha}(x, y) = \sum_{k=0}^{n} \binom{n}{k} q \alpha(q, n-k) \mathcal{E}_{k, q, \alpha}(x)^{y^{n-k}}.$$
Setting $y = 1$ in (5), we get

$$B_n(x + 1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x), \quad E_n(x + 1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x), \quad G_n(x + 1) = \sum_{k=0}^{n} \binom{n}{k} G_k(x),$$

respectively. We mention that, from the definition of $E_{q,\alpha}(t)$, by using the Cauchy product, we reach to

$$\mathcal{E}_{q,\alpha}(tx)\mathcal{E}_{q,\alpha}(ty) = \sum_{n=0}^{\infty} t^n (xq, y)_n,$$

putting this equality in (2) and writing the product of single sums as a double sum, at the end equating coefficient of $t^n$ we lead to the proof of lemma.

**Lemma 16** The condition $\mathcal{E}_{q,\alpha}(-z) = (\mathcal{E}_{q,\alpha}(z))^{-1}$ and $\alpha(q, 1) = 1$ together provides that the odd coefficient of related $q$-Bernoulli numbers except the first one becomes zero. That means $B_{n,q,\alpha} = 0$ where $n = 2r + 1, (r \in \mathbb{N})$.

**Proof.** It follows from the fact that the function

$$f(t) = \sum_{n=0}^{\infty} \frac{B_{n,q,\alpha} t^n}{[n]_q!} - \frac{t}{\mathcal{E}_{q,\alpha}(t) - 1} + \frac{t}{2} \left( \frac{\mathcal{E}_{q,\alpha}(t) + 1}{\mathcal{E}_{q,\alpha}(t) - 1} \right)$$

is even. We recall that, if $\mathcal{E}_{q,\alpha}(-z) = (\mathcal{E}_{q,\alpha}(z))^{-1}$, then (3) is hold and $B_{1,q,\alpha} = -\frac{\alpha(q, 2)}{\alpha(q, 1)[2]_q}$. Since $\alpha(q, 1) = 1$, $B_{1,q,\alpha} = -\frac{1}{2}$.

**Lemma 17** If $\alpha(q, n)$ as a parameter of $\mathcal{E}_{q,\alpha}(z)$, satisfy $\frac{\alpha(q, n + 1)}{\alpha(q, n)} = \sum_{k=0}^{m} a_k q^k$, then we have

$$D_{q, x} B_{n,q,\alpha}(x) = [n]_q \sum_{k=0}^{m} a_k B_{n-1,q,\alpha} \left( x q^k \right), \quad D_{q, x} E_{n,q,\alpha}(x) = [n]_q \sum_{k=0}^{m} a_k E_{n-1,q,\alpha} \left( x q^k \right),$$

$$D_{q, x} \mathcal{E}_{n,q,\alpha}(x) = [n]_q \sum_{k=0}^{m} a_k \mathcal{E}_{n-1,q,\alpha} \left( x q^k \right).$$

**Example 18**

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