Stit logic of justification announcements: a completeness result

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Abstract. We present a completeness result for a logical system which combines stit logic and justification logic in order to represent proving activity of the agents. This logic is interpreted over the semantics introduced in [7]. We define a Hilbert-style axiomatic system for this logic and show that this system is strongly complete relative to the intended semantics.

stit logic, justification logic, completeness, compactness

1 Introduction

Stit logic of justification announcements (JA-STIT) is a formalism for reasoning about proving activity of agents which combines expressive powers of stit logic (see e.g. [4]) with those of justification logic (see e.g. [2]). The two latter logics provide for the pure agency side and the pure proof ontology side of the proving activity, respectively, so that it is assumed that doing something is in effect seeing to it that something is the case, and that every actual proof can be understood as a realization of some proof polynomial from justification logic. The only missing element in this picture is then the link between the two components, i.e. how agents can see to it that a proof is realized. Such a realization may come in different forms, researchers may, for example, exchange emails or put the proofs they have found on a common whiteboard. In stit logic of justification announcements this rather common situation is idealized in that only public proving activity of agents is taken into account. In other words, taking up the whiteboard metaphor, the agents in question can only participate in proving activity by putting their proofs on the common whiteboard for everyone to see, and not by sending one another private messages or scribbling in their private notebooks. Therefore, the only type of communicative actions within this idealized community turns out to be a variant of public announcement of proof polynomials.

1Even though this type of public announcement actions plays a central role in our logic, finding any sort of meaningful connection to the well-known public announcement logic (PAL) looks like a
This idealization lends the medium of public announcement, i.e. the metaphorical community whiteboard, the status of the only interface between the agentive efforts of the community and the abstract realm of proofs. Proof polynomials may end up being presented on the whiteboard, and the agents may see to it that this or that particular proof is presented there. The whiteboard itself is also idealized in that we assume that there is always enough space on it to put up another proof, and that every proof, once on the whiteboard, remains there forever.

The language of stit logic of justification announcements then combines the full sets of justification and stit modalities with a new modality $Et$ which says that the proof polynomial $t$ is presented to the community (or, to continue with the whiteboard metaphor, that $t$ is put on the community whiteboard). In this way arises a non-trivial and expressively rich logic, and the main purpose of the present paper is to provide a strongly complete axiomatization for this logic.

The layout of the rest of the paper is as follows. In Section 2 we define the language and the semantics of the logic at hand. We also briefly characterize its relations with other formalisms combining the resources of justification logic and stit logic, studied in the earlier publications, namely, in [7] and [8]. We mention that the finite model property fails for the stit logic of justification announcements in a rather strong form, and show that the language of JA-STIT is expressive enough to distinguish between the full class of justification stit models and the class of justification stit models based on discrete time.

The system of axioms for JA-STIT is then presented in Section 3. We immediately show this system to be sound w.r.t. the semantics introduced in Section 2 and deduce some theorems in the system.

Section 4 then contains the bulk of technical work necessary for the proof of completeness of the presented axiomatization w.r.t. the class of normal stit models. It gives a stepwise construction and adequacy check for all the numerous components of the canonical model and ends with a proof of a truth lemma. Section 5 then gives a concise proof of the completeness result and draws some quick corollaries including the compactness property.

Then follows Section 6 giving some conclusions and drafting directions for future work.

In what follows we will be assuming, due to space limitations, a basic acquaintance with both stit logic and justification logic. We recommend to peruse [5, Ch. 2] for a quick introduction to the basics of stit logic, and [1] for the same w.r.t. justification logic.

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non-trivial matter. One obvious reason for this is the difference between the underlying action logics (stit logic in the case at hand vs. dynamic logic of PAL). Another reason is that we are studying public announcements of a special type of objects (i.e. proof polynomials) in a multi-agent setting, whereas in PAL sentence announcements are studied, and these sentence announcements are not tied to a particular agent. Moreover, in JA-STIT public announcements are not reducible to static formulas and are actually intended to be that way.
2 Basic definitions and notation

2.1 Preliminaries

We fix some preliminaries. First, we choose a finite set $A_g$ disjoint from all the other sets to be defined below. Individual agents from this set will be denoted by letters $i$ and $j$. Then we fix countably infinite sets $PVar$ of proof variables (denoted by $x, y, z, w, u$) and $PConst$ of proof constants (denoted by $a, b, c, d$). When needed, subscripts and superscripts will be used with the above notations or any other notations to be introduced in this paper. Set $Pol$ of proof polynomials is then defined by the following BNF:

$$t := x \mid c \mid s + t \mid s \times t \mid !t,$$

with $x \in PVar$, $c \in PConst$, and $s, t$ ranging over elements of $Pol$. In the above definition, $+$ stands for the sum of proofs, $\times$ denotes application of its left argument to the right one, and $!$ denotes the so-called proof-checker, so that $!t$ checks the correctness of proof $t$.

In order to define the set $Form$ of formulas, we fix a countably infinite set $Var$ of propositional variables to be denoted by letters $p, q, r, s$. Formulas themselves will be denoted by letters $A, B, C, D$, and the definition of $Form$ is supplied by the following BNF:

$$A := p \mid A \wedge B \mid \neg A \mid [j]A \mid \Box A \mid t:A \mid KA \mid Et,$$

with $p \in Var$, $j \in A_g$ and $t \in Pol$.

It is clear from the above definition of $Form$ that we are considering a version of modal propositional language. As for the informal interpretations of modalities, $[j]A$ is the so-called cstit action modality and $\Box$ is the historical necessity modality; both modalities are borrowed from stit logic. The next two modalities, $KA$ and $t:A$, come from justification logic and the latter is interpreted as “$t$ proves $A$”, whereas the former is the strong epistemic modality “$A$ is known”.

We assume $\Diamond$ as notation for the dual modality of $\Box$. As usual, $\omega$ will denote the set of natural numbers including 0, ordered in the natural way.

2.2 Semantics

For the language at hand, we assume the following semantics. A justification stit (or jstit, for short) model is a structure

$$\mathcal{M} = \langle Tree, \preceq, Choice, Act, R, R_e, \mathcal{E}, V \rangle$$

such that:

1. $Tree$ is a non-empty set. Elements of $Tree$ are called moments.

2. $\preceq$ is a partial order on $Tree$ for which a temporal interpretation is assumed. We will also freely use notations like $\succeq$, $\ll$, and $\gg$ to denote the inverse relation and the irreflexive companions.$^2$

$^2$A more common notation $\leq$ is not convenient for us since we also widely use $\leq$ in this paper to denote the natural order relation between elements of $\omega$. 
3. \( Hist(M) \) is a set of maximal \( \leq \)-chains in \( Tree \). Since \( Hist(M) \) is completely determined by \( Tree \) and \( \leq \), it is not included into the model structure as a separate component. Elements of \( Hist(M) \) are called histories. The set of histories containing a given moment \( m \) will be denoted \( H_m \). The following set:

\[
MH(M) = \{(m, h) \mid m \in Tree, h \in H_m\},
\]

called the set of moment-history pairs, will be used to evaluate the elements of \( Form \).

4. \( Choice \) is a function mapping \( Tree \times Agent \) into \( 2^{Hist} \) in such a way that for any given \( j \in Agent \) and \( m \in Tree \) we have as \( Choice(m, j) \) (to be denoted as \( Choice^m_j \) below) a partition of \( H_m \). For a given \( h \in H_m \) we will denote by \( Choice^m_j(h) \) the element of partition \( Choice^m_j \) containing \( h \).

5. \( Act \) is a function mapping \( MH(M) \) into \( 2^{Pol} \).

6. \( R \) and \( R_e \) are two pre-order on \( Tree \) giving two versions of epistemic accessibility relation. They are assumed to be connected by inclusion \( R \subseteq R_e \).

7. \( E \) is a function mapping \( Tree \times Pol \) into \( 2^{Form} \).

8. \( V \) is the evaluation function, mapping the set \( Var \) into \( 2^{MH(M)} \).

However, not all structures of the above described type are admitted as jstit models. A number of additional restrictions needs to be satisfied. More precisely, we assume satisfaction of the following constraints:

1. Historical connection:

\[(\forall m, m_1 \in Tree)(\exists m_2 \in Tree)(m_2 \leq m \land m_2 \leq m_1).\]

2. No backward branching:

\[(\forall m, m_1, m_2 \in Tree)((m_1 \leq m \land m_2 \leq m) \rightarrow (m_1 \leq m_2 \lor m_2 \leq m_1)).\]

3. No choice between undivided histories:

\[(\forall m, m' \in Tree)(\forall h, h' \in H_m)(m \triangleleft m' \land m' \in h \land h' \rightarrow Choice^m_j(h) = Choice^m_j(h'))\]

for every \( j \in Agent \).

4. Independence of agents:

\[(\forall m \in Tree)(\forall f : Ag \rightarrow 2^{H_m})(\forall j \in Ag)(f(j) \in Choice^m_j) \Rightarrow \bigcap_{j \in Ag} f(j) \neq \emptyset).\]

5. Monotonicity of evidence:

\[(\forall t \in Pol)(\forall m, m' \in Tree)(R_e(m, m') \Rightarrow E(m, t) \subseteq E(m', t)).\]
6. Evidence closure properties. For arbitrary $m \in \text{Tree}, s \in \text{Pol}$ and $A, B \in \text{Form}$ it is assumed that:
(a) $A \rightarrow B \in \mathcal{E}(m, s) \land A \in \mathcal{E}(m, t) \Rightarrow B \in \mathcal{E}(m, s \times t)$;
(b) $\mathcal{E}(m, s) \cup \mathcal{E}(m, t) \subseteq \mathcal{E}(m, s + t)$.
(c) $A \in \mathcal{E}(m, t) \Rightarrow t : A \in \mathcal{E}(m, !t)$.

7. Expansion of presented proofs:
$$(\forall m, m' \in \text{Tree})(m' \triangleleft m \Rightarrow \forall h \in H_m (\text{Act}(m', h) \subseteq \text{Act}(m, h))).$$

8. No new proofs guaranteed:
$$(\forall m \in \text{Tree})(\bigcap_{h \in H_m} (\text{Act}(m, h)) \subseteq \bigcup_{m' \triangleleft m, h \in H_m} (\text{Act}(m', h))).$$

9. Presenting a new proof makes histories divide:
$$(\forall m \in \text{Tree})(\forall h, h' \in H_m)((\exists m' \triangleright m)(m' \in h \cap h') \Rightarrow (\text{Act}(m, h) = \text{Act}(m, h'))).$$

10. Future always matters:
$$\triangleleft \subseteq R.$$

11. Presented proofs are epistemically transparent:
$$(\forall m, m' \in \text{Tree})(R_e(m, m') \Rightarrow (\bigcap_{h \in H_m} (\text{Act}(m, h)) \subseteq \bigcap_{h' \in H_{m'}} (\text{Act}(m', h')))).$$

We offer some intuitive explanation for the above defined notion of jstit model. Jstit models were introduced in [7] for the logics based on the combination of stit and justification modalities. Due to space limitations, we only explain the intuitions behind jstit models very briefly, and we urge the reader to consult [7, Section 3] for a more comprehensive explanations, whenever needed.

The components like Tree, $\triangleleft$, Choice and $V$ are inherited from stit logic, whereas $R$, $R_e$, and $\mathcal{E}$ come from justification logic. The only new component is the function $\text{Act}$, which gives out, to take up the whiteboard metaphor, the current state of this whiteboard at any given moment under any given history. When interpreting $\text{Act}$, we draw on the classical stit distinction between dynamic (agentive) and static (moment-determinate) entities, assuming that the presence of a given proof polynomial $t$ on the community whiteboard only becomes an accomplished fact at $m$ when $t$ is present in $\text{Act}(m, h)$ for every $h \in H_m$. On the other hand, if $t$ is in $\text{Act}(m, h)$ only for some $h \in H_m$ this means that $t$ is rather in a dynamic state of being presented, rather than being present, to the community.

The numbered list of semantical constraints above then just builds on these intuitions. Constraints 1–4 are borrowed from stit logic, constraints 5 and 6 are inherited from justification logic. Constraint 7 just says that nothing gets erased from the whiteboard, constraint 8 says a new proof cannot spring into existence as a static (i.e. moment-determinate) feature of the environment out of nothing, but rather has
to come as a result (or a by-product) of a previous activity. Constraint 9 is just a corollary to constraint 3 in the richer environment of \textit{jstit} models, constraint 10 says that the possible future of the given moment is always epistemically relevant in this moment, and constraint 11 says that the community immediately knows everything that has firmly made its way onto the whiteboard.

For the members of Form, we will assume the following inductively defined satisfaction relation. For every \textit{jstit} model \( M = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, E, V \rangle \) and for every \((m, h) \in MH(M)\) we stipulate that:

\[
\begin{align*}
M, m, h \models p & \Leftrightarrow (m, h) \in V(p); \\
M, m, h \models [j]A & \Leftrightarrow (\forall h' \in \text{Choice}_j^{m}(h))(M, m, h' \models A); \\
M, m, h \models \Box A & \Leftrightarrow (\forall h' \in H_m)(M, m, h' \models A); \\
M, m, h \models KA & \Leftrightarrow \forall m' \forall h'(R(m, m') \& h' \in H_{m'} \Rightarrow M, m', h' \models A); \\
M, m, h \models t:A & \Leftrightarrow A \in E(m, t) \& (\forall m' \in \text{Tree})(R_e(m, m') \& h' \in H_{m'} \Rightarrow M, m', h' \models A); \\
M, m, h \models Et & \Leftrightarrow t \in \text{Act}(m, h).
\end{align*}
\]

In the above clauses we assume that \( p \in \text{Var}; \) we also assume standard clauses for the Boolean connectives. We further assume standard definitions for satisfiability and validity of formulas and sets of formulas in the presented semantics.

One can in principle simplify the above semantics by introducing the additional constraint that \( R_e \subseteq R. \) This leads to a collapse of the two epistemic accessibility relation into one. Therefore, we will call \textit{jstit} models satisfying \( R_e \subseteq R \) \textit{unirelational} \textit{jstit} models. It is known that such a simplification in the context of pure justification logic does not affect the set of theorems (see, e.g., \cite{6} and \cite{2, Comment 6.5}), and we will show that this is also the case within the more expressive environment of JA-STIT.

In fact, the canonical model to be constructed in our completeness is unirelational, therefore, we offer some comments as to the simplifications of semantics available in the unirelational setting.

We observe that one can equivalently define a unirelational \textit{jstit} model as a structure \( M = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, E, V \rangle \) satisfying all the constraints for the \textit{jstit} models, except that in the numbered constraints one substitutes \( R \) for \( R_e. \) Also, in the context of unirelational \textit{jstit} models, it is possible to simplify the satisfaction clause for \( t:A \) as follows:

\[
M, m, h \models t:A \Leftrightarrow A \in E(m, t) \& M, m, h \models KA.
\]

Before we move on, we briefly clarify the relation of JA-STIT to other logics based on the combination of justification and stit modalities to be found in the existing literature. Firstly, JA-STIT is a fragment of the logic introduced in \cite{8} under the name ‘logic of E-notions’. The difference is that in the logic of E-notions an implicit version of \textit{Et}-modality is also present. This implicit version comes in the format \( EA, \) where \( A \in \text{Form} \) and has the meaning that \textit{some} proof of \( A \) is presented to the community.

The satisfaction clause for this additional modality looks as follows:

\[
M, m, h \models EA \Leftrightarrow (\exists t \in \text{Pol})(t \in \text{Act}(m, h) \& M, m, h \models t:A).
\]

It is pretty obvious that \( EA \) is not definable using expressive powers of JA-STIT, so that JA-STIT is a proper fragment of the logic of E-notions.
Another natural logic featuring the full set of justification and stit modalities is the basic jstit logic introduced in [7] and further explored in [8]. This logic is also interpreted over the class of jstit models which facilitates the comparison. In basic jstit logic justification and stit modalities are augmented with the following set of four modalities representing different modes of proving activity:

| Notation     | Informal interpretation   |
|--------------|----------------------------|
| Prove(j, A)  | Agent j proves A          |
| Prove(j, t, A) | Agent j proves A by t    |
| Proven(A)    | A has been proven         |
| Proven(t, A) | A has been proven by t    |

In the above table, \( A \in \text{Form} \), \( t \in \text{Pol} \) and \( j \in \text{Ag} \) are designating the type and arrangement of arguments for the listed modalities. It turns out that two of these four modalities, namely \( \text{Prove}(j, t, A) \) and \( \text{Proven}(t, A) \) are definable in JA-STIT. These modalities are interpreted by the following satisfaction clauses:

\[
\mathcal{M}, m, h \models \text{Prove}(j, t, A) \iff (\exists h' \in \text{Choice}^m_j(h))(t \in \text{Act}(m, h') & \mathcal{M}, m, h \models t:A) &
(\exists h'' \in H_m)(t \notin \text{Act}(m, h''));
\]

\[
\mathcal{M}, m, h \models \text{Proven}(t, A) \iff (\forall h' \in H_m)(t \in \text{Act}(m, h') & \mathcal{M}, m, h \models t:A)
\]

It is easy to see then that these modalities can be defined within JA-STIT as follows:

\[
\text{Prove}(j, t, A) =_{df} [j]Et \land \Diamond \neg Et \land t:A;
\]

\[
\text{Proven}(t, A) =_{df} \Box Et \land t:A.
\]

However, as for the other two modalities of the basic jstit logic, their indefinability within JA-STIT is rather obvious and can be easily shown. On the other hand, \( Et \)-modality itself does not seem to be definable within the basic jstit logic. Given all these facts, the relation between JA-STIT and the basic jstit logic can be described as follows. The fragment of basic jstit logic given by the two modalities \( \{\text{Prove}(j, t, A), \text{Proven}(t, A)\} \) plus the set of stit and justification modalities can be faithfully recovered within JA-STIT. This is a maximal fragment of basic jstit logic that can be recovered within JA-STIT, and JA-STIT itself is a proper extension of this fragment in terms of expressive power. In the other direction, \( Et \)-modality cannot be recovered within basic jstit logic, which means that only those fragments of JA-STIT can be recovered within basic jstit logic which are confined to combinations of modalities borrowed directly from justification and stit logics.

### 2.3 Concluding remarks

Before we start with the task of axiomatizing JA-STIT, we briefly mention some facts about its expressive powers which are relevant to our chosen format of completeness proof. Firstly, it is worth noting that under the presented semantics some satisfiable formulas cannot be satisfied over finite models, or even over infinite models where all histories are finite. The argument for this is the same as in implicit fragment of basic jstit logic, for which this claim was proved in [9] using \( K(\Diamond p \land \Diamond \neg p) \) as an example of a formula which is satisfiable over jstit models but not over jstit models.
with finite histories. This already rules out some methods of proving completeness like filtration method.

Secondly, it turns out that, even though JA-STIT is not, strictly speaking, a temporal logic, it can still tell something about the structure of histories generated in a given jstit model. Indeed, let us define that a jstit model $\mathcal{M}$ is based on discrete time iff every chain in $\text{Hist}(\mathcal{M})$ is isomorphic to an initial segment of $\omega$, the set of natural numbers. Then it can be shown that:

**Proposition 1.** The set of JA-STIT formulas valid over the class of (unirelational) jstit models is a proper subset of the set of JA-STIT formulas valid over the class of (unirelational) jstit models based on discrete time.

**Proof.** We clearly have the subset relation. As for the properness part, consider the formula $K(\neg \Box Ex \lor \Box Ey) \rightarrow (\neg Ex \lor Ey)$ with $x, y \in \text{PVar}$. We show that this formula is not valid over the class of all unirelational jstit models (hence not valid over the class of all jstit models either). Consider the following unirelational model $\mathcal{M} = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, E, V \rangle$ for the community of a single agent $j$:

- $\text{Tree} = \{a, b\} \cup \{r \in \mathbb{R} \mid 0 < r < 1\}$;
- $\preceq = \{(a, b)\} \cup \{(a, r) \mid r \in \mathbb{R} \cap \text{Tree}\} \cup \{(r, r') \mid r, r' \in \mathbb{R} \cap \text{Tree}, r \leq r'\}$;
- $\text{Choice}^m_j = H_m$ for all $m \in \text{Tree}$;
- $R = \preceq$;
- $E(m, t) = \text{Form}$, for all $m \in \text{Tree}$ and $t \in \text{Pol}$.
- $V(p) = \emptyset$ for all $p \in \text{Var}$.

It is straightforward to see that the above-defined components of $\mathcal{M}$ satisfy all the constraints imposed on normal jstit models except possibly those involving $\text{Act}$. Before we go on and define $\text{Act}$, let us pause a bit and reflect on the structure of histories in the model $\mathcal{M}$ that is being defined. We only have two histories in it, one is $h_1 = (a, b)$ and the other is $h_2 = \{a\} \cup \{r \in \mathbb{R} \mid 0 < r < 1\}$. So we define:

- $\text{Act}(a, h_2) = \{x\}$;
- $\text{Act}(a, h_1) = \text{Act}(b, h_1) = \emptyset$;
- $\text{Act}(r, h_2) = \{x, y\}$ for all $r \in \mathbb{R} \cap \text{Tree}$.

Again, most of the constraints on jstit models are now easily seen to be satisfied.

3Note that this model also satisfies any possible constant specification (as defined in Section 6) so that introducing any such specification cannot affect the counterexample at hand.
Now, consider $a \in Tree$. The set of $a$’s epistemic alternatives is $Tree$ itself. We have $M, a, h_1 \not\models Ex$, therefore $M, a, h_2 \models \neg \square Ex$, whence $M, a, h_2 \models \neg \square Ex \land \square Ey$. We also have, of course, that $M, a, h_1 \models \neg \square Ex$ and $M, a, h_1 \models \neg \square Ex \lor \square Ey$. In the same way, we see that $M, b, h_1 \models \neg \square Ex$ and $M, b, h_1 \models \neg \square Ex \lor \square Ey$

Moreover, if $r$ is a real number strictly between 0 and 1, then $M, r, h_2 \models Ex$, and, since $h_2$ is the only history passing through $r$, we get also $M, r, h_2 \models \square Ex$, and, further, $M, r, h_2 \models \neg \square Ex \lor \square Ey$. Thus the formula $\neg \square Ex \lor \square Ey$ holds at every epistemic alternative of $a$ for every history passing through this alternative. This means that $M, a, h_2 \models K(\neg \square Ex \lor \square Ey)$. Besides, we have that $M, a, h_2 \models Ex \land \neg Ey$, so that the pair $(a, h_2)$ falsifies the formula $K(\neg \square Ex \lor \square Ey) \rightarrow (\neg Ex \lor Ey)$ in $M$.

On the other hand, $K(\neg \square Ex \lor \square Ey) \rightarrow (\neg Ex \lor Ey)$ is valid in the class of jstit models based on discrete time (hence also over unirelational jstit models based on discrete time). In order to show this, we will assume its invalidity and obtain a contradiction. Indeed, let $M = \langle Tree, \preceq, Choice, Act, R, Re, E, V \rangle$ be a jstit model based on discrete time such that

$$M, m, h \not\models K(\neg \square Ex \lor \square Ey) \rightarrow (\neg Ex \lor Ey).$$

Then we will have both

$$M, m, h \models K(\neg \square Ex \lor \square Ey),$$

and

$$M, m, h \models Ex \land \neg Ey.$$  \hfill (1)

By (1), we know that:

$$M, m, h \models \neg \square Ex \lor \square Ey,$$  \hfill (2)

and, by (2), it follows that:

$$M, m, h \models \neg \square Ey.$$  \hfill (3)

Therefore, we know by (3) that $M, m, h \models \neg \square Ex$, so that there is an $h^\prime \in H_m$ such that $M, m, h^\prime \models \neg Ex$. In view of (2), we must have $h \not\models h^\prime$, so $H_m$ cannot be a singleton. Since histories are defined as maximal chains of moments, we know that $H_m$ is always a singleton when $m \in Tree$ is $\preceq$-maximal. Therefore $m$ cannot be $\preceq$-maximal and thus $m$ cannot be the $\preceq$-last moment along $h$. Since $M$ is based on discrete time, consider embedding $f$ of $h$ into an initial segment of $\omega$. Suppose that $f(m) = n$. Since $m$ is not the $\preceq$-last moment along $h$, there must be an $m^\prime \in h$ such that $f(m^\prime) = n + 1$. Since $f$ is an embedding, this means that $m < m^\prime$ and for no $m'' \in Tree$ it is true that $m < m'' < m^\prime$. By the future always matters constraint, we know that $R(m, m')$, therefore, by (1) we must have:

$$M, m', h \models \neg \square Ex \lor \square Ey.$$  \hfill (4)

On the other hand, let $g \in H_{m'}$ be arbitrary. Then, by the absence of backward branching, $g \in H_m$, and, moreover, $g$ is undivided from $h$ at $m$. Therefore, by the presenting a new proof makes histories divide constraint, we must have $Act(m, g) = Act(m, h)$. By (2) we know that $x \in Act(m, h)$, which means that also $x \in Act(m, g)$. Since $g \in H_{m'}$ was chosen arbitrarily, the latter means that $x \in \bigcap_{g \in H_{m'}} (Act(m, g))$, and, by the expansion of presented proofs constraint, $x \in \bigcap_{g \in H_{m'}} (Act(m', g))$. This, in turn, yields that:

$$M, m', h \models \square Ex.$$  \hfill (5)
whence, in view of (3), it follows that
\[ \mathcal{M}, m', h \models \Box Ey. \] (7)

The latter means that \( y \in \bigcap_{g \in H_m}(\text{Act}(m', g)) \), and by the no new proofs guaranteed constraint, it follows that for some \( g \in H_m \) and some \( m'' \in g \) such that \( m'' \triangleleft m' \), we must have \( y \in \text{Act}(m'', g) \). Now, if \( m'' \triangleleft m' \) it follows that \( m'' \preceq m \), since \( m' \) was chosen as the immediate \( \triangleleft \)-successor of \( m \) along \( h \). The latter means, by the expansion of presented proofs, that \( y \in \text{Act}(m, g) \). Since \( g \) is undivided from \( h \) at \( m \), this means, by the presenting a new proof makes histories divide constraint, that \( \text{Act}(m, g) = \text{Act}(m, h) \) and, further, that \( y \in \text{Act}(m, h) \). The latter is in obvious contradiction with (2).

Proposition 1 shows that if one wants to prove the completeness theorem for JA-STIT by constructing a canonical model, the histories in this model both have to be infinite and have to have a rather involved order structure. This shows that the canonical model used in the completeness proof that follows below, is not likely to allow for any major simplifications.

3 Axiomatic system and soundness

We consider the Hilbert-style axiomatic system \( \Sigma \) with the following set of axiomatic schemes:

- A full set of axioms for classical propositional logic (A0)
- S5 axioms for \( \Box \) and \([j]\) for every \( j \in Agent \) (A1)
- \( \Box A \rightarrow [j]A \) for every \( j \in Agent \) (A2)
- \( \bigcirc([j][A_1 \land \ldots \land [j][A_n]) \rightarrow \bigcirc([j][A_1 \land \ldots \land [j][A_n]) \) (A3)
- \( (s:(A \rightarrow B) \rightarrow (t:A \rightarrow (s \times t):B) \) (A4)
- \( t:A \rightarrow (lt:(t:A) \land KA) \) (A5)
- \( (s:A \lor t:A) \rightarrow (s + t):A \) (A6)
- S4 axioms for \( K \) (A7)
- \( KA \rightarrow \Box K\Box A \) (A8)
- \( \Box Et \rightarrow K\Box Et \) (A9)

The assumption is that in (A3) \( j_1, \ldots, j_n \) are pairwise different.

To this set of axiom schemes we add the following rules of inference:

- From \( A, A \rightarrow B \) infer \( B \); (R1)
- From \( A \) infer \( KA \); (R2)
- If \( A \) is an instance of (A0)–(A9) and \( c \in PConst \), then infer \( c:A \); (R3)
- From \( KA \rightarrow (\neg \Box Et_1 \lor \ldots \lor \neg \Box Et_n) \)
  infer \( KA \rightarrow (\neg Et_1 \lor \ldots \lor \neg Et_n) \). (R4)

Rule (R3) is obviously not satisfied over the general class of jstit models. However, we introduce it as an inheritance of justification logic with its constant specifications.
(R3) gives just one example of such constant specification, but it serves as a general case in our situation, since the form of our completeness proof allows for a straightforward adaptation to any other variant of constant specification allowed for in justification logic, including the empty constant specification which would correspond to omitting (R3) altogether. On the other hand, should we take the empty constant specification as our default example, it would not be clear how to adapt the proof to accommodate non-empty constant specification, since completeness proof for the empty specification allows for quite a bit of shortcuts, which are not available in the more general case. We postpone a more general discussion of constant specifications till Section 5, confining ourselves in the meantime to the particular case given by (R3).

In order to adapt the scope of our completeness result to the presence of (R3), we call a (unirelational) jstit model $M$ normal iff the following condition is satisfied:

$$(\forall c \in PConst)(\forall m \in \text{Tree})(\{A | A \text{ is a substitution case of a scheme in } (A1)–(A9)\} \subseteq \mathcal{E}(m, c)).$$

Our goal is now to obtain a strong completeness theorem for $\Sigma$ w.r.t. the class of normal models. Establishing soundness mostly reduces to a routine check that every axiom is valid and that rules preserve validity. We treat the less obvious cases in some detail:

**Theorem 1.** Every instance of $(A0)$–$(A9)$ is valid over the class of normal jstit models. Every application of rules $(R1)$–$(R4)$ to formulas which are valid over the class of normal jstit models yields a formula which is valid over the class of normal jstit models.

**Proof.** First, note that if $M = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle$ is a normal jstit model, then $\langle \text{Tree}, \preceq, \text{Choice}, V \rangle$ is a model of stit logic. Therefore, axioms $(A0)$–$(A3)$, which were copy-pasted from the standard axiomatization of dstit logic, must be valid. Second, note that if $M = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle$ is a normal jstit model, then $\langle \text{Tree}, R, R_e, \mathcal{E}, V \rangle$ is what is called in [2, Section 6] a justification model with the form of constant specification defined by (R3). This means that also all of the $(A4)$–(A7) must be valid, whereas $(R1)$–(R3) must preserve validity, given that all these parts of our axiomatic system were borrowed from the standard axiomatization of justification logic. The validity of other parts of $\Sigma$ will be motivated below in some detail. In what follows, $M = \langle \text{Tree}, \preceq, \text{Choice}, \text{Act}, R, R_e, \mathcal{E}, V \rangle$ will always stand for an arbitrary normal jstit model, and $(m, h)$ for an arbitrary element of $MH(M)$.

As for $(A8)$, assume for reductio that $M, m, h \models KA \land K \neg A$. Then $M, m, h \models KA$ and also $M, m, h' \models K \land K \land \neg A$ for some $h' \in H_m$. By reflexivity of $R$, it follows that $\land \land \neg A$ will be satisfied at $(m, h)$ in $M$. The latter means that, for some $h'' \in H_m$, $A$ must fail at $(m, h'')$ and therefore, again by reflexivity of $R$, $KA$ must fail at $(m, h)$ in $M$, a contradiction.

We consider next $(A9)$. If $\Box Et$ is true at $(m, h)$ in $M$, then, by definition, $t \in \bigcap_{h' \in H_m} \text{Act}(m, h)$. Now, if $m' \in \text{Tree}$ is such that $R(m, m')$, then, by epistemic transparency of presented proofs constraint, we must have $t \in \bigcap_{h' \in H_m} \text{Act}(m', h')$ so that for every $g \in H_m$ we will have $M, m', g \models \Box Et$. Therefore, we must have $M, m, h \models K \diamond Et$ as well.

---

4See, e.g. [4, Ch. 17], although $\Sigma$ uses a simpler format closer to that given in [3, Section 2.3].

5The format for the variable assignment $V$ is slightly different, but this is of no consequence for the present setting.
It only remains to show that \([R4]\) preserves validity over normal \(jst\) models. Assume that \(KA \rightarrow (\neg \Box Et_1 \lor \ldots \lor \neg \Box Et_n)\) is valid over normal \(jst\) models, and assume also that we have:

\[
\mathcal{M}, m, h \models KA \land Et_1 \land \ldots \land Et_n.
\]  

(8)

Whence, by the assumed validity, we know that also:

\[
\mathcal{M}, m, h \models \neg \Box Et_1 \lor \ldots \lor \neg \Box Et_n,
\]

therefore, we can choose a natural \(k\) such that \(1 \leq k \leq n\) and \(\mathcal{M}, m, h \models \neg \Box Et_k\). The latter, in turn, means that for some \(h' \in H_m\) we have that:

\[
\mathcal{M}, m, h' \models \neg Et_k.
\]

(9)

Comparison between (8) and (9) shows that \(h \neq h'\). Therefore, we know that \(H_m\) is not a singleton, which means that \(m\) cannot be a \(\leq\)-maximal moment in \(Tree\) and we can choose an \(m' \in Tree\) such that \(h \in H_{m'}\) and \(m' \triangleright m\). By (8) we know that \(t_1, \ldots, t_n \in Act(m, h)\) and we know that every \(g \in H_{m'}\) is undivided from \(h\) at \(m\). Therefore, by the presenting a new proof makes histories divide constraint, we get that \(t_1, \ldots, t_n \in Act(m, g)\) for all \(g \in H_{m'}\), hence, by the expansion of presented proofs constraint, we get that \(t_1, \ldots, t_n \in \bigcap_{g \in H_{m'}} Act(m', g)\). This means that we have, on the one hand:

\[
\mathcal{M}, m', h \models \Box Et_1 \land \ldots \land \Box Et_n.
\]

(10)

And, on the other, hand, we know that by the future always matters constraint, we have \(R(m, m')\), which also means that, by (8) we get that:

\[
\mathcal{M}, m', h' \models KA.
\]

(11)

Taken together, (10) and (11) contradict the validity of \(KA \rightarrow (\neg \Box Et_1 \lor \ldots \lor \neg \Box Et_n)\).
1. There exists a $\Delta \subseteq \text{Form}$ such that $\Delta$ is maxiconsistent and $\Gamma \subseteq \Delta$.

2. If $\Gamma$ is maxiconsistent, then exactly one element of $\{A, \neg A\}$ is in $\Gamma$.

3. If $\Gamma$ is maxiconsistent, then $A \lor B \in \Gamma$ iff ($A \in \Gamma$ or $B \in \Gamma$).

4. If $\Gamma$ is maxiconsistent and $A, (A \rightarrow B) \in \Gamma$, then $B \in \Gamma$.

5. If $\Gamma$ is maxiconsistent, then $A \land B \in \Gamma$ iff ($A \in \Gamma$ and $B \in \Gamma$).

Proof. (Part 1) Just as in the standard case, we enumerate the elements of $\text{Form}$ as $A_1, \ldots, A_n, \ldots$ and form the sequence of sets $\Gamma_1, \ldots, \Gamma_n, \ldots$ such that $\Gamma_1 := \Gamma$ and for every natural $i \geq 1$:

$$\Gamma_{i+1} := \begin{cases} 
\Gamma_i, & \text{if } \Gamma_i \cup \{A_i\} \text{ is inconsistent}; \\
\Gamma_i \cup \{A_i\}, & \text{otherwise}.
\end{cases}$$

We now define $\Delta := \bigcup_{i \geq 1} \Gamma_i$. Of course, we have $\Gamma \subseteq \Delta$, and, moreover, $\Delta$ is maxiconsistent. To see this, note that for every $i \geq 1$ the set $\Gamma_i$ is consistent by construction. Now, if $\Delta$ is inconsistent, then there must be a valid implication from a finite conjunction of formulas in $\Delta$ to $\bot$. These formulas must be mentioned in our numeration of $\text{Form}$ so that the valid implication in question can be presented as $\vdash (A_{i_1} \land \ldots \land A_{i_n}) \rightarrow \bot$ for appropriate natural $i_1, \ldots, i_n$. Since all of $A_{i_1}, \ldots, A_{i_n}$ are in $\Delta$, we must have, by the construction of $\Gamma_1, \ldots, \Gamma_n, \ldots$, that $A_{i_1}, \ldots, A_{i_n} \in \Gamma_{\max(i_1, \ldots, i_n)}$. But then this latter set must be inconsistent which contradicts our construction.

Further, if some consistent $\Xi \subseteq \text{Form}$ is such that $\Delta \subseteq \Xi$, then let $A_n \in \Xi \setminus \Delta$. We must have then $\Gamma_n \cup \{A_n\}$ inconsistent, but we also have $\Gamma_n \cup \{A_n\} \subseteq \Xi$, which implies inconsistency of $\Xi$, in contradiction to our assumptions. Therefore, $\Delta$ is not only consistent, but also maxiconsistent.

(Part 2) We cannot have both $A$ and $\neg A$ in $\Gamma$, since we have, of course, $\vdash (A \land \neg A) \rightarrow \bot$. If, on the other hand, neither $A$, nor $\neg A$ are in $\Gamma$, then both $\Gamma \cup \{A\}$ and $\Gamma \cup \{\neg A\}$ must be inconsistent, so that for some $B_1, \ldots, B_n \in \Gamma$ we will have:

$$\vdash (B_1 \land \ldots \land B_n \land A) \rightarrow \bot,$$

whereas for some $C_1, \ldots, C_k \in \Gamma$ we will have:

$$\vdash (C_1 \land \ldots \land C_k \land \neg A) \rightarrow \bot,$$

whence we get, using (A0) and (K1):

$$\vdash (C_1 \land \ldots \land C_k) \rightarrow A,$$

and further:

$$\vdash (B_1 \land \ldots \land B_n \land C_1 \land \ldots \land C_k) \rightarrow \bot,$$

so that $\Gamma$ turns out to be inconsistent, contrary to our assumptions.

(Part 3) Assume $(A \lor B) \in \Gamma$. If neither $A$ nor $B$ are in $\Gamma$, then, by Part 2, both $\neg A$ and $\neg B$ are in $\Gamma$. Using (A0) and (K1) we get that:

$$\vdash ((A \lor B) \land \neg A \land \neg B) \rightarrow \bot,$$
showing that $\Gamma$ is inconsistent, contrary to our assumptions. In the other direction, if, say $A \in \Gamma$ and $(A \lor B) \notin \Gamma$, then, by Part 2, we must have $\neg(A \lor B) \in \Gamma$. Using $[A0]$ and $[R1]$ we get that:

$$
\vdash (\neg (A \lor B) \land A) \rightarrow \bot,
$$

showing, again, that $\Gamma$ is inconsistent, contrary to our assumptions. The case when $B \in \Gamma$ is similar.

Parts 4 and 5 are similar to Part 3.

We are now prepared to formulate our main result:

**Theorem 2.** Let $\Gamma \subseteq \text{Form}$. Then $\Gamma$ is consistent iff it is satisfiable in a normal (unirelational) jstit model.

The rest of the paper is mainly concerned with proving Theorem 2. One part of it we have, of course, right away, as a consequence of Theorem 1:

**Corollary 1.** If $\Gamma \subseteq \text{Form}$ is satisfiable in a normal (unirelational) jstit model, then $\Gamma$ is consistent.

**Proof.** Let $\Gamma \subseteq \text{Form}$ be satisfiable in a normal jstit model so that we have, say $\mathcal{M}, m, h \models \Gamma$ for some $(m, h) \in \text{MH}$. If $\Gamma$ were inconsistent this would mean that for some $A_1, \ldots, A_n \in \Gamma$ we would have $\vdash (A_1 \land \ldots \land A_n) \rightarrow \bot$. By Theorem 1 this would mean that:

$$
\mathcal{M}, m, h \models (A_1 \land \ldots \land A_n) \rightarrow \bot,
$$

whence clearly $\mathcal{M}, m, h \models \bot$, which is impossible. Therefore, $\Gamma$ must be consistent.

Further, if $\Gamma \subseteq \text{Form}$ is satisfiable in a normal unirelational jstit model, then $\Gamma$ must be satisfiable in a normal jstit model. Hence $\Gamma$ must be consistent by the above reasoning.

Before we move further, we mention some theorems in the above axiom system to be used later in the proof of the main result:

**Lemma 2.** The following holds for every $A \in \text{Form}$, $t \in \text{Pol}$, $x \in \text{PVar}$, and $j \in \text{Ag}$:

1. $\vdash t: A \rightarrow \Box t: A$;
2. $\vdash KA \rightarrow \Box KA$.

**Proof.** (Part 1) We have:

- $t: A \rightarrow \Box t: A$ (by $[A3]$)
- $\rightarrow Kt: A$ (by $[A3]$)
- $\rightarrow \Box K \Box t: A$ (by $[A8]$)
- $\rightarrow K \Box t: A$ (by $[A11]$)
- $\rightarrow \Box t: A$ (by $[A7]$)

Our theorem follows then by transitivity of implication.

(Part 2). By S5 properties of $\Box$ and S4 properties of $K$, we clearly have $\vdash \Box K \Box A \rightarrow \Box KA$. Part 2 follows then by $[A8]$ and transitivity of implication. □
4 The canonical model

The main aim of the present section is to prove the inverse of Corollary 1. The method used is a variant of the canonical model technique, but, due to the complexity of the case, we do not define our model in one full sweep. Rather, we proceed piecewise, defining elements of the model one by one, and checking the relevant constraints as soon, as we have got enough parts of the model in place. The last subsection proves the truth lemma for the defined model. As we have already indicated, the model to be built will be a normal unirelational jstit model, so that $R_e$ will be omitted, or, equivalently, assumed to coincide with $R$.

The ultimate building blocks of $\mathcal{M}$ we will call elements. Before going on with the definition of $\mathcal{M}$, we define what these elements are and explore some of their properties.

Definition 1. An element is a sequence of the form $(\Gamma_1, \ldots, \Gamma_n)$ for some $n \in \omega$ with $n \geq 1$ such that:

- For every $k \leq n$, $\Gamma_k$ is maxiconsistent;
- For every $k < n$, for all $A \in \text{Form}$, if $KA \in \Gamma_k$, then $KA \in \Gamma_{k+1}$;
- For every $k < n$, for all $t \in \text{Pol}$, if $Et \in \Gamma_k$, then $\Box Et \in \Gamma_{k+1}$.

We prove the following lemma:

Lemma 3. Whenever $(\Gamma_1, \ldots, \Gamma_n)$ is an element, there exists a $\Gamma_{n+1} \subseteq \text{Form}$ such that the sequence $(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1})$ is also an element.

Proof. Assume $(\Gamma_1, \ldots, \Gamma_n)$ is an element and consider the following set:

$$\Delta := \{KA \mid KA \in \Gamma_n\} \cup \{\Box Et \mid Et \in \Gamma_n\}.$$

We show that $\Delta$ is consistent. Of course, the set $\{KA \mid KA \in \Gamma_n\}$ is consistent since it is a subset of $\Gamma_n$ and the latter is assumed to be consistent. Further, if $\Delta$ is inconsistent, then, wlog, for some $KB_1, \ldots, KB_r, Et_1, \ldots, Et_u \in \Gamma_n$ we will have:

$$\vdash (KB_1 \land \ldots \land KB_r) \rightarrow (\neg \Box Et_1 \lor \ldots \lor \neg \Box Et_u),$$

whence, by $(A7)$:

$$\vdash K(B_1 \land \ldots \land B_r) \rightarrow (\neg Et_1 \lor \ldots \lor \neg Et_u),$$

and further, by $(R4)$:

$$\vdash K(B_1 \land \ldots \land B_r) \rightarrow (\neg Et_1 \lor \ldots \lor \neg Et_u).$$

The latter formula shows that $\Gamma_n$ is inconsistent which contradicts the assumption that $(\Gamma_1, \ldots, \Gamma_n)$ is an element.

Therefore, $\Delta$ must be consistent, and, by Lemma 1.1, it is also extendable to a maxiconsistent $\Gamma_{n+1}$. By the choice of $\Delta$, this means that $(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1})$ must be an element. \qed
The structure of elements will be important in what follows. If \( \xi = (\Gamma_1, \ldots, \Gamma_n) \) is an element and an element \( \tau \) is of the form \((\Gamma_1, \ldots, \Gamma_k)\) with \( k < n \), we say that \( \tau \) is a proper initial segment of \( \xi \). Moreover, if \( k = n - 1 \), then \( \tau \) is the greatest proper initial segment of \( \xi \). We define \( n \) to be the length of \( \xi \). Furthermore, we define that \( \Gamma_n \) is the end element of \( \xi \) and write \( \Gamma_n = \text{end}(\xi) \).

We now define the canonical model using elements as our building blocks. We start by defining the following relation \( \equiv \):

\[
(\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}) \equiv (\Delta_1, \ldots, \Delta_n, \Delta_{n+1}) \iff (\Gamma_1 = \Delta_1 \& \ldots \& \Gamma_n = \Delta_n \& \\
(\forall A \in \text{Form})(\Box A \in \Gamma_{n+1} \Rightarrow A \in \Delta_{n+1}).
\]

It is routine to check that \( \equiv \) is an equivalence relation given that \( \Box \) is an S5 modality. The notation \( [(\Gamma_1, \ldots, \Gamma_n)]_{\equiv} \) will denote the \( \equiv \)-equivalence class generated by \((\Gamma_1, \ldots, \Gamma_n)\). Since all the elements inside a given \( \equiv \)-equivalence class are of the same length, we may extend the notion of length to these classes setting that the length of \([\Gamma_1, \ldots, \Gamma_n]_{\equiv}\) also equals \( n \).

We now proceed to definitions of components for the canonical model.

### 4.1 Tree, \( \preceq \), and Hist(\( \mathcal{M} \))

The first two elements of the canonical model \( \mathcal{M} \) are as follows:

- \( \textbf{Tree} = \{\dagger\} \cup \{([\xi]_{\equiv}, n) \mid n \in \omega, \xi \text{ is an element}\} \). Thus the elements of \( \textbf{Tree} \), with the exception of the special moment \( \dagger \), are \( \equiv \)-equivalence classes of elements coupled with natural numbers. Such moments we will call \textit{standard} moments, and the left projection of a standard moment \( m \) we will call its \textit{core} (and write \( \overline{m} \)), while the right projection of such moment we will call its \textit{height} (and write \( \vert m \vert \)). In this way, we get the equality \( m = (\overline{m}, \vert m \vert) \) for every standard \( m \in \textbf{Tree} \).

  

We further define that the length of a standard moment \( m \) is the length of its core. For the sake of completeness, we extend the above notions to \( \dagger \) setting both length and height of this moment to 0 and defining that \( \overline{\dagger} = \dagger \).

- We set that \( (\forall m \in \textbf{Tree} \setminus \{\dagger\})(\dagger \prec m \& \neg m \prec \dagger) \). We further set that for any two standard moments \( m \) and \( m' \), we have that \( m \prec m' \) if either (1) there exists a \( \xi \in \overline{m} \) such that for every \( \tau \in \overline{m'} \), \( \xi \) is a proper initial segment of \( \tau \), or (2) \( \overline{m} = \overline{m'} \) and \( |m'| < |m| \). The relation \( \preceq \) is then defined as the reflexive companion to \( \prec \).

Before we move on to the choice- and justifications-related components, let us pause to check that the restraints imposed by our semantics on \( \textbf{Tree} \) and \( \preceq \) are satisfied:

\textbf{Lemma 4.} The relation \( \preceq \), as defined above, is a partial order on \( \textbf{Tree} \), which satisfies both historical connection and no backward branching constraints.

\textbf{Proof.} Reflexivity of \( \preceq \) holds by definition. For \textbf{transitivity}, suppose that \( m, m', \) and \( m'' \) are in \( \textbf{Tree} \) and that we have \( m \preceq m' \) and \( m' \preceq m'' \). Then, if any two moments among \( m, m' \) and \( m'' \) coincide, or if one of those moments is \( \dagger \), we must clearly have \( m \preceq m'' \). So suppose that all of \( m, m' \) and \( m'' \) are standard and pairwise different so that we have \( m \prec m' \prec m'' \). We have then four cases to consider:
Case 1. There are \( \xi \in \overrightarrow{m} \) and \( \tau \in \overrightarrow{m} \) such that \( \xi \) is a proper initial segment of every element in \( \overrightarrow{m'} \) (and this clearly includes \( \tau \)), and \( \tau \) is a proper initial segment of every element in \( \overrightarrow{m'} \). It is immediate then that \( \xi \) is a proper initial segment of every element in \( \overrightarrow{m''} \), and \( m \prec m'' \) follows.

Case 2. We have \( |m| > |m'| > |m''| \) and also \( \overrightarrow{m} = \overrightarrow{m'} = \overrightarrow{m''} \). Then both \( |m| > |m''| \) and \( \overrightarrow{m} = \overrightarrow{m''} \) clearly follow so that we get \( m \prec m'' \).

Case 3. There is a \( \xi \in \overrightarrow{m} \) such that \( \xi \) is a proper initial segment of every element in \( \overrightarrow{m'} \). Additionally, we have both \( \overrightarrow{m'} = \overrightarrow{m''} \) and \( |m'| > |m''| \). Then clearly \( \xi \) must be a proper initial segment also of every element in \( \overrightarrow{m''} \) so that \( m \prec m'' \) holds.

Case 4. There is a \( \tau \in \overrightarrow{m} \) such that \( \tau \) is a proper initial segment of every element in \( \overrightarrow{m'} \). On the other hand, we have both \( \overrightarrow{m} = \overrightarrow{m'} \) and \( |m| > |m'| \). Then, of course, \( \tau \) is also in \( \overrightarrow{m} \) and again we get \( m \prec m'' \).

As for anti-symmetry, assume that we have both \( m \prec m' \) and \( m' \prec m \). Then both \( m \) and \( m' \) must be standard. Again we have to consider four cases, and we obtain a contradiction in each of them, showing that this situation never arises:

Case 1. There are \( \xi \in \overrightarrow{m} \) and \( (\tau) \in \overrightarrow{m'} \) such that \( \xi \) is a proper initial segment of every element in \( \overrightarrow{m'} \) and \( \tau \) is a proper initial segment of every element in \( \overrightarrow{m} \). It is clear then that both \( \xi \) is a proper initial segment of \( \tau \) and \( \tau \) a proper initial segment of \( \xi \), which gives us the contradiction.

Case 2. We have \( \overrightarrow{m} = \overrightarrow{m'} \) and also both \( |m| > |m'| \) and \( |m| < |m'| \). The contradiction is immediate.

Case 3. There is a \( \xi \in \overrightarrow{m} \) such that \( \xi \) is a proper initial segment of every element in \( \overrightarrow{m'} \). Besides, we have both \( \overrightarrow{m} = \overrightarrow{m'} \) and \( |m'| > |m| \). But then \( \xi \in \overrightarrow{m} \) and therefore must be its own proper initial segment, a contradiction.

Case 4. There is a \( \tau \in \overrightarrow{m} \) such that \( \tau \) is a proper initial segment of every element in \( \overrightarrow{m'} \), and also we have both \( \overrightarrow{m} = \overrightarrow{m'} \) and \( |m| > |m'| \). This case is similar to Case 3.

Historical connection is satisfied since \( \dagger \) is the \( \leq \)-least element of Tree.

Let us prove the absence of backward branching. Assume that we have both \( m \leq m'' \) and \( m' \leq m'' \) but neither \( m \leq m' \) nor \( m' \leq m \) holds. This means that all the three moments are pairwise different and none of them is \( \dagger \), otherwise our assumptions about them would be immediately falsified. Therefore, all the three moments are standard and we also have \( m \neq m' \), \( m \prec m'' \), and \( m' \prec m'' \). We will use the familiar fourfold partition of cases:

Case 1. There are \( \xi \in \overrightarrow{m} \) and \( \tau \in \overrightarrow{m} \) such that both \( \xi \) and \( \tau \) are proper initial segments of every element in \( \overrightarrow{m''} \). If \( \xi = \tau \), then we must have \( \overrightarrow{m} = \overrightarrow{m'} \) since moment cores are classes of equivalence. Hence we will have \( |m| \neq |m'| \), since \( m \neq m' \). But then, depending on whether we have \( |m| < |m'| \) or \( |m'| < |m| \), we get either \( m' \prec m \) or \( m \prec m' \). On the other hand, if \( \xi \) is different from \( \tau \), then either \( \xi \) must be a proper initial segment of \( \tau \) or vice versa. Assume, wlog, that \( \xi \) is a proper segment of \( \tau \). Then \( \xi \) is included in the greatest proper initial segment of \( \tau \) and since every element in \( \overrightarrow{m'} \) has the same greatest proper initial segment, this means that \( \xi \) is a proper initial segment of every element in \( \overrightarrow{m'} \) so that \( m \prec m' \).

Case 2. We have, on the one hand, \( \overrightarrow{m} = \overrightarrow{m'} \) and \( |m''| < |m| \), and, on the other
hand \( \overrightarrow{m} = \overrightarrow{m'} \) and \( |m''| < |m'| \). Then we immediately get that \( \overrightarrow{\bar{m}} = \overrightarrow{m'} \). Further, by \( m \neq m' \) we know that either \( |m| < |m'| \) or \( |m'| < |m| \). Whence we get, respectively, either \( m' < m \) or \( m < m' \).

Case 3. There is a \( \xi \in \overrightarrow{\bar{m}} \) such that \( \xi \) is a proper initial segment of every element in \( \overrightarrow{\bar{m}'} \), and, on the other hand, we have both \( \overrightarrow{m} = \overrightarrow{\bar{m'}} \) and \( |m''| < |m'| \). Then, of course \( \xi \) is also a proper initial segment of every element in \( m' \), and \( m < m' \) follows.

Case 4. There is a \( \tau \in \overrightarrow{\bar{m}} \) such that \( \tau \) is a proper initial segment of every element in \( \overrightarrow{\bar{m'}} \), and, on the other hand, we have both \( \overrightarrow{\bar{m}} = \overrightarrow{\bar{m'}} \) and \( |m''| < |m| \). This case is similar to Case 3.

Before we move on to the other components of the canonical model \( \mathcal{M} \) to be defined in this section, we look into the structure of \( \text{Hist}(\mathcal{M}) \) as induced by the above-defined \( \text{Tree} \) and \( \preceq \). We start by defining a basic sequence of elements. A basic sequence of elements is a set of elements of the form \( \{\xi_1, \ldots, \xi_n, \ldots\} \) such that for every \( n \geq 1 \):

- \( \xi_n \) is of length \( n \);
- \( \xi_n \) is the greatest proper initial segment of \( \xi_{n+1} \).

Basic sequences will be denoted by capital Latin letters \( S, T \), and \( U \) with subscripts and superscripts when needed. Every given basic sequence \( S \) induces the following \( [S] \subseteq \text{Tree} \):

\[
[S] = \{\top\} \cup \bigcup_{n \in \omega} \{([\xi_n]_{\equiv}, k) \mid k \in \omega\}.
\]

It is immediate that every basic sequence \( S \) induces a unique \( [S] \subseteq \text{Tree} \) in this way. It is, perhaps, less immediate that the mapping \( S \mapsto [S] \) is injective:

**Lemma 5.** Let \( S, T \) be basic sequences of elements. Then:

\[
[S] = [T] \Rightarrow S = T.
\]

**Proof.** Assume that \( S = \{\xi_1, \ldots, \xi_n, \ldots\} \) and that \( T = \{\tau_1, \ldots, \tau_n, \ldots\} \). We will show that \( \xi_n = \tau_n \) for arbitrary \( n \in \omega \). Indeed, note that it is immediate from the definition of \( S \mapsto [S] \), that both \( [S] \) and \( [T] \) contain exactly one moment of length \( n + 1 \) and height \( 0 \), and these moments are \( ([\xi_{n+1}]_{\equiv}, 0) \) and \( ([\tau_{n+1}]_{\equiv}, 0) \), respectively. Therefore, if \( [S] = [T] \), then we must have \( ([\xi_{n+1}]_{\equiv}, 0) = ([\tau_{n+1}]_{\equiv}, 0) \), whence, further, \( [\xi_{n+1}]_{\equiv} = [\tau_{n+1}]_{\equiv} \) and \( \xi_{n+1} = \tau_{n+1} \). Therefore, \( \xi_{n+1} \) and \( \tau_{n+1} \) must share the greatest proper initial segment which is \( \xi_n \) for \( \xi_{n+1} \) and \( \tau_n \) for \( \tau_{n+1} \). Since this segment is the same for \( \xi_{n+1} \) and \( \tau_{n+1} \), it follows that \( \xi_n = \tau_n \). \( \square \)

We now move on to a characterization of \( \text{Hist}(\mathcal{M}) \), first proving a number of technical lemmas:

**Lemma 6.** If \( h \in \text{Hist}(\mathcal{M}) \) and \( k \in \omega \), then \( h \) contains at least one moment of length exceeding \( k \).

**Proof.** Suppose otherwise, and let \( k \in \omega \) be such that every moment in \( h \) has length at most \( k \). We may assume that this is the least such \( k \) so that some elements of the length \( k \) are actually in \( h \). We have to consider two cases then:
Case 1. \( k = 0 \). Then \( h = \{ \dot{t} \} \). Take any maxconsistent \( \Gamma \subseteq \text{Form} \), it is immediate that \( \Gamma \) is an element. Then \( ((\Gamma)_\equiv, 0) \in \text{Tree} \) and, moreover \( \dot{t} \prec ((\Gamma)_\equiv, 0) \), so that \( \{ \dot{t}, ((\Gamma)_\equiv, 0) \} \) is a \( \preceq \)-chain properly extending \( h \), which contradicts the maximality of \( h \).

Case 2. \( k > 0 \). Then take an arbitrary moment \( m \) of the length \( k \) in \( h \), say \( m = ((\Gamma_1, \ldots, \Gamma_k)_\equiv, n) \). Then \( ((\Gamma_1, \ldots, \Gamma_k)_\equiv, 0) \) is an \( \preceq \)-upper bound for \( h \). Indeed, we clearly have \( m \preceq ((\Gamma_1, \ldots, \Gamma_k)_\equiv, 0) \). Now, if \( m' \in h \), then either \( m' \preceq m \), or \( m \preceq m' \). If \( m' \preceq m \), then, by transitivity, \( m' \preceq ((\Gamma_1, \ldots, \Gamma_k)_\equiv, 0) \) and we are done. If \( m \preceq m' \), then we cannot have any \( \xi \in m' \) such that \( \xi \) is a proper initial segment of every element in \( m' \) since every such \( \xi \) is of length \( k \) and this would mean that elements in \( m' \) must have a length greater than \( k \), which contradicts the choice of \( m' \). Therefore, we must have \( m' = m \) and also \( |m'| < |m| \). But then also \( m' \preceq ((\Gamma_1, \ldots, \Gamma_k)_\equiv, 0) \) clearly follows.

Now, using Lemma \ref{lem-chain-maximal} we can choose a \( \Gamma_{k+1} \subseteq \text{Form} \) such that \( (\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}) \) is an element. Consider then \( m'' = ((\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1})_\equiv, 0) \in \text{Tree} \). We obviously have \( m'' \notin h \) since the length of \( m'' \) is \( k + 1 \). On the other hand, we have, by definition of \( \preceq \), that \( ((\Gamma_1, \ldots, \Gamma_k)_\equiv, 0) \prec m'' \). Hence \( h \cup \{ m'' \} \) is a \( \preceq \)-chain properly extending \( h \), which, again, contradicts the maximality of \( h \).

\begin{lemma}
If \( h \in \text{Hist}(\mathcal{M}) \) and \( k \in \omega \), then \( h \) contains at least one moment of the length \( k \).
\end{lemma}

\begin{proof}
Take an arbitrary \( k \in \omega \). If \( k = 0 \), then the lemma holds, since \( \dot{t} \), being the \( \preceq \)-least moment in \( \text{Tree} \), is of course in \( h \). Assume that \( k > 0 \). There are two cases to consider then.

Case 1. For every \( n+1 \in \omega \) it is true that whenever there is a moment of the length \( n+1 \) in \( h \), then there is also a moment of length \( n \) in \( h \). Then our lemma follows from Lemma \ref{lem-chain-maximal}.

Case 2. There is an \( n+1 \in \omega \) such that some \( m \in \text{Tree} \) of the length \( n+1 \) is in \( h \), but there are no moments of the length \( n \) in \( h \). Then consider \( m' \), say \( m = ((\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1})_\equiv, r) \). We show then that \( m' = ((\Gamma_1, \ldots, \Gamma_n)_\equiv, 0) \) must be in \( h \) as well, since \( h \cup \{ m' \} \) is a \( \preceq \)-chain and \( h \) is maximal. Indeed, we have \( m' \prec m \), since \( (\Gamma_1, \ldots, \Gamma_n) \) is a proper initial segment of every element in \( m' \). Now, if \( m'' \in h \), then either \( m \preceq m'' \), or \( m'' \prec m \). If \( m \preceq m'' \), then of course \( m'' \prec m'' \) by transitivity. If, on the other hand, \( m'' \prec m \), then, by the absence of backward branching, either \( m'' \preceq m' \) or \( m' \preceq m'' \).

Thus we have shown that \( m' \in h \), and since the length of \( m' \) equals \( n \), this gives us a contradiction with the hypothesis of Case 2.
\end{proof}

\begin{lemma}
Assume that \( h \in \text{Hist}(\mathcal{M}) \), that \( k \in \omega \), and that \( m, m' \in h \) are of the length \( k \). Then \( m = m' \).
\end{lemma}

\begin{proof}
We may assume that \( k > 0 \) since there is only one core of length 0. If \( m, m' \in h \) are standard moments, then either \( m \preceq m' \) or \( m' \preceq m \). Assume, wlog, that \( m \preceq m' \). Then there is no \( \xi \in m' \) such that \( \xi \) is a proper initial segment of every element in \( m' \), since the length of \( \xi \) is equal to the length of elements in \( m' \). Therefore, we must have \( m = m' \) by definition of \( \preceq \).
\end{proof}
We now offer the following characterization of $Hist(M)$:

**Lemma 9.** The following statements hold:

1. If $S = \{\xi_1, \ldots, \xi_n, \ldots\}$ is a basic sequence, then $[S] \in Hist(M)$, and the following presentation gives $[S]$ in the $\sqsubseteq$-ascending order:

   $\uparrow, \ldots, ([\xi_1]_{\sqsubseteq}, k), \ldots, ([\xi_1]_{\sqsubseteq}, 0), \ldots, ([\xi_n]_{\sqsubseteq}, k), \ldots, ([\xi_n]_{\sqsubseteq}, 0), \ldots,$

2. $Hist(M) = \{[S] \mid S$ is a basic sequence$\}$.

**Proof.** (Part 1). It is quite easy to see that for a given basic sequence $S = \{\xi_1, \ldots, \xi_n, \ldots\}$, $[S]$ is a $\sqsubseteq$-chain and that Part 1 of the Lemma represents this chain in the ascending order. We focus on maximality of $[S]$ as a $\sqsubseteq$-chain. Suppose $m \in Tree$ is such that $m \notin [S]$, but $[S] \cup \{m\}$ is still a $\sqsubseteq$-chain. Then $m$ must be standard, since $\uparrow$ is already in $[S]$. Suppose $m = ([\tau]_{\sqsubseteq}, k)$ for some element $\tau$ and $k \in \omega$, and suppose that the length of $m$ is $n \geq 1$. Consider then $([\xi_{n+1}]_{\sqsubseteq}, 0) \in [S]$. Since $[S] \cup \{m\}$ is a $\sqsubseteq$-chain we must have either $([\xi_{n+1}]_{\sqsubseteq}, 0) \leq m$ or $m \prec ([\xi_{n+1}]_{\sqsubseteq}, 0)$. But the length of $([\xi_{n+1}]_{\sqsubseteq}, 0)$ is greater than the length of $m$, therefore $[\xi_{n+1}]_{\sqsubseteq} \neq m$ and also no element in $[\xi_{n+1}]_{\sqsubseteq}$ can be a proper initial segment of any element in $m$. Therefore, we cannot have $([\xi_{n+1}]_{\sqsubseteq}, 0) \leq m$ and must then get $m \prec ([\xi_{n+1}]_{\sqsubseteq}, 0)$. Given that we have shown $[\xi_{n+1}]_{\sqsubseteq} \neq m$, $m \prec ([\xi_{n+1}]_{\sqsubseteq}, 0)$ must mean that some element $\tau \in m$ is a proper initial segment of every element in $[\xi_{n+1}]_{\sqsubseteq}$ including $\xi_{n+1}$. Since the length of $\xi_{n+1}$ is $n + 1$ and the length of $m$ is $n$, this means that $\tau'$ must be the greatest proper initial segment of $\xi_{n+1}$. But the greatest proper initial segment of $\xi_{n+1}$ is $\xi_n$, therefore $\tau' = \xi_n$ and, consequently, $m = ([\tau]_{\sqsubseteq}, k) = ([\xi_n]_{\sqsubseteq}, k) \in [S]$, which contradicts the choice of $m$. (Part 2). It follows from Part 1 that $Hist(M) \supseteq \{[S] \mid S$ is a basic sequence$\}$, so we only need to show the inverse inclusion. So, choose an arbitrary $h \in Hist(M)$. Consider the set

$$core(h) = \{m \mid m \in h\}.$$

It follows from Lemmas 7 and 8 that $core(h)$ contains exactly one moment core of the length $n$ for every $n \in \omega$. Therefore, $core(h)$ has the form $\{\uparrow, \alpha_1, \ldots, \alpha_n, \ldots\}$, where every $\alpha_k$ is an equivalence class of elements of length $k$. We now claim that if $k \geq 2$, then there is a $\xi_{k-1} \in \alpha_{k-1}$ such that $\xi_{k-1}$ is a proper initial segment of every element in $\alpha_k$. Indeed, we know that for some $r, r' \in \omega$ the moments $(\alpha_{k-1}, r), (\alpha_k, r')$ are in $h$. We cannot have $(\alpha_k, r') \leq (\alpha_{k-1}, r)$ since the length of $\alpha_{k-1}$ is strictly less than the length of $\alpha_k$. Therefore, since $h$ is a chain, we must have $(\alpha_{k-1}, r) \prec (\alpha_k, r')$, and, again by length considerations, there must be a $\xi_{k-1} \in \alpha_{k-1}$ such that $\xi_{k-1}$ is a proper initial segment of every element in $\alpha_k$.

So we choose such a $\xi_{k-1} \in \alpha_{k-1}$ for every $k \geq 2$. In this way we obtain the sequence $S = \{\xi_1, \ldots, \xi_n, \ldots\}$ with the following properties:

1. For all $k \geq 1$, $\xi_k \in \alpha_k$ (so that $\alpha_k = [\xi_k]_{\sqsubseteq}$ and $\xi_k$ itself is therefore of the length $k$);

2. For all $k \geq 1$, $\xi_k$ is a proper initial segment of every element in $\alpha_{k+1}$.

Now, for given $k \geq 1$, since $\xi_k$ is a proper initial segment of every element in $\alpha_{k+1}$, then $\xi_k$ is also a proper initial segment of $\xi_{k+1}$. And since the lengths of $\xi_k$ and $\xi_{k+1}$
are $k$ and $k + 1$, respectively, then $\xi_k$ is the greatest proper initial segment of $\xi_{k+1}$. This means that the sequence $S = \{\xi_1, \ldots, \xi_n, \ldots\}$ is in fact a basic sequence. We now show that $[S] \subseteq h$ and since, by Part 1, $[S]$ is itself a history, this will mean that $[S] = h$, and that, given that $h$ was chosen arbitrarily, we will be done.

Indeed, assume that $m \in [S]$. If $m = \dagger$, then of course $m \in h$ by maximality of $h$, since $\dagger$ is the $\preceq$-least element in $\text{Tree}$. Therefore, assume that $m$ is standard, say $m = ([\xi_n]_\equiv, k)$. Take an arbitrary $m' \in h$. We will show that we either have $m \preceq m'$ or $m' \preceq m$. In the case when $m' = \dagger$ we trivially get $m' \preceq m$ so we assume that $m'$ is standard so that for some appropriate $k', n' \in \omega$ we must have $m' = ([\xi_{n'}]_\equiv, k')$. We have then three cases to consider:

Case 1. $n' < n$. Then $\xi_{n'}$ must be a proper initial segment of every element in $[\xi_n]_\equiv$, and we immediately get $m' \preceq m$.

Case 2. $n < n'$. This case is an inversion of Case 1, giving us that $m \preceq m'$.

Case 3. $n = n'$. Then $\overrightarrow{m} = \overrightarrow{m'}$ and, depending on whether we have $k < k'$, $k' < k$, or $k = k'$ we obtain that $m' \preceq m$, $m \preceq m'$, or $m = m'$, respectively.

Thus we have shown that $h \cup \{m\}$ is an $\preceq$-chain, whence, by the maximality of $h$, it follows that $m \in h$. And since $m \in [S]$ was chosen arbitrarily, this means that $[S] \subseteq h$ and therefore $[S] = h$, as desired. \[\square\]

It follows from Lemmas 9 and 10 that not only every basic sequence generates a unique $h \in \text{Hist}(\mathcal{M})$, but also for every $h \in \text{Hist}(\mathcal{M})$ there exists a unique basic sequence $S$ such that $h = [S]$. We will denote this unique $S$ for a given $h$ by $[h]$. It is immediate from Lemmas 9 and 10 that for every $h \in \text{Hist}(\mathcal{M})$, $h = [(h)]$. Likewise, for every basic sequence $S$, we have $S = [([S])]$. As a further useful piece of notation, we introduce the notion of intersection of a standard moment $m$ with a history $h \in H_m$. Assume that $m$ is of the length $n$ and that $|h| = \{\xi_1, \ldots, \xi_n, \ldots\}$. Then $m$ must be of the form $([\xi_n]_\equiv, k)$ for some $k \in \omega$, and we will also have $\overrightarrow{m} \cap |h| = \{\xi_n\}$. We now define the only member of the latter singleton as the result $m \cap h$ of the intersection of $m$ and $h$, setting $m \cap h = \xi_n$. It can be shown that for any element $\xi$ in the core of a given standard moment $m$ there exists an $h \in H_m$ such that $\xi = m \cap h$.

**Lemma 10.** Let $(\Gamma_1, \ldots, \Gamma_k)$ be an element. Then, for every $n \in \omega$ there is at least one history $h \in H_{([\Gamma_1, \ldots, \Gamma_k])_\equiv, n}$ such that $(([\Gamma_1, \ldots, \Gamma_k])_\equiv, n) \cap h = (\Gamma_1, \ldots, \Gamma_k)$.

*Proof.* Using Lemma 3 and axiom of choice, we successively choose $\Gamma_{k+1}, \ldots, \Gamma_{k+l}, \ldots \subseteq \text{Form}$ such that all of the structures

$$(\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}), \ldots, (\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}, \ldots, \Gamma_{k+l}), \ldots,$$

are elements. But then, it is obvious that the set:

$S = \{(\Gamma_1), \ldots, (\Gamma_1, \ldots, \Gamma_k), (\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}), \ldots, (\Gamma_1, \ldots, \Gamma_k, \Gamma_{k+1}, \ldots, \Gamma_{k+l}), \ldots\}$

is a basic sequence and $([\Gamma_1, \ldots, \Gamma_k])_\equiv, n \in [S]$ so that $[S] \in H_{([\Gamma_1, \ldots, \Gamma_k])_\equiv, n}$. Further, it is clear that $([\Gamma_1, \ldots, \Gamma_k])_\equiv, n \cap [S] = (\Gamma_1, \ldots, \Gamma_k)$, as desired. \[\square\]

We offer some general remarks on what we have shown thus far. Lemma 9 shows that every history in the canonical model has a uniform order structure which can be otherwise described as follows. If $L$ and $L'$ are two linear orders then let $L \oplus L'$ be a
copy of $L$ with a copy of $L'$ appended at the end, let $L \otimes L'$ be the result of replacement of every element in $L'$ with a disjoint copy of $L$, and let $L^*$ be the inversion of $L$. Also, for any $n \in \omega$, let $(0, \ldots, n)$ be the first $n + 1$ natural numbers with their natural order. Then Lemma 9 tells us that every history in the canonical model is ordered in the type of $(0) \oplus (\omega^* \otimes \omega)$. Also, note that it follows from Lemma 9 that for every ordered couple of natural numbers $(k, n)$ with $k > 0$, every given history $h$ contains exactly one moment of length $k$ and height $n$. Another general observation is that histories in $M$ can only branch off at moments of height 0, so that at moments of other heights all the histories remain undivided. This last fact does not follow from the lemmas proved thus far and we end this subsection with its proof, also establishing a couple of technical facts to be used later:

**Lemma 11.** Let $m, m' \in \text{Tree}$, and let $h \in H_m$. If $\rightarrow m = \rightarrow m'$, then $h \in H_{m'}$ and also $m \cap h = m' \cap h$.

**Proof.** If $h \in H_m$, then there is an element $\xi \in \rightarrow m = \rightarrow m'$ such that $\xi \in |h|$. For this element we will also have $\xi \in m \cap h$. Since $\xi \in m'$, we further get that $m' = (\{\xi\}, m')$. It follows, by $\xi \in |h|$, that $m' \in [h] = h$ so that $h \in H_{m'}$. Now, consider $m' \cap |h|$. We know that this set must be a singleton with $m' \cap h$ as its only element, and we know also that $\{\xi\} = \rightarrow m \cap |h| = m' \cap |h|$. Therefore, $m' \cap h = \xi = m \cap h$ and thus we are done.

**Corollary 2.** If $h \in H_m$ and $m = (\overrightarrow{m}, k + 1)$, then for the $m' = (\overrightarrow{m}, k)$ it is true that $h \in H_{m'}$.

**Proof.** Immediate from Lemma 11.

**Corollary 3.** Let $m \in \text{Tree}$ be such that $|m| > 0$, and let $h, h' \in H_m$. Then $h$ and $h'$ are undivided at $m$.

**Proof.** Since $|m| > 0$, we know that $m = k + 1$ for some $k \in \omega$. Then, by Corollary 2, we must have $h, h' \in H_{m'}$ for $m' = (\overrightarrow{m}, k)$. It remains to notice that we clearly have $m \preceq m'$.

### 4.2 Choice

We now define the choice structures of our canonical model:

- $\text{Choice}^j_m(h) = \{h' \mid h' \in H_m (\forall A \in \text{Form})([j]A \in \text{end}(h \cap m) \Rightarrow A \in \text{end}(h' \cap m)\}$, if $m \neq \dagger$ and $|m| = 0$;

- $\text{Choice}^j_m = H_m$, otherwise.

Since for every $j \in Ag$, $[j]$ is an S5-modality, $\text{Choice}$ induces a partition on $H_m$ for every given $m \in \text{Tree}$. We check that the choice function verifies the relevant semantic constraints:

**Lemma 12.** The tuple $(\text{Tree}, \sqsubseteq, \text{Choice})$, as defined above, verifies both independence of agents and no choice between undivided histories constraints.
Proof. We first tackle no choice between undivided histories. Consider a moment \( m \) and two histories \( h, h' \in H_m \) such that \( h \) and \( h' \) are undivided at \( m \). Since the agents’ choices are only non-vacuous at moments represented by standard moments of height 0, we may safely assume that \( m \) is such a moment. Since \( h \) and \( h' \) are undivided at \( m \), this means that there is a moment \( m' \) such that \( m < m' \) and \( m' \) is shared by \( h \) and \( h' \). Hence we know that also \( m' \) is standard. Suppose the length of \( m \) is \( n \) and the length of \( m' \) is \( n' \). Then \( n < n' \) since \( m \) is of height 0 and therefore has no equivalence classes of elements of length \( n \) above itself. Therefore, \( h \cap m \) is the initial segment of length \( n \) of \( h \cap m' \), and similarly, \( h' \cap m \) is the initial segment of length \( n \) of \( h' \cap m' \). But both \( h \cap m' \) and \( h' \cap m' \) are, by definition, in \( m' \), therefore, they must share the greatest proper initial segment. Hence, their initial segments of length \( n \) must coincide as well, and we must have \( h \cap m = h' \cap m \), whence \( end(h \cap m) = end(h' \cap m) \). Now, if \( j \in Ag \) and \([j]A \in end(h \cap m)\), then, by (A1) and maxiconsistency of \( end(h \cap m) \), we will have also \( A \in end(h \cap m) = end(h' \cap m) \), and thus \( h' \in \text{Choice}^m_j(h) \), so that \( \text{Choice}^m_j(h) = \text{Choice}^m_j(h') \) since \( \text{Choice} \) is a partition of \( H_m \).

Consider, next, the independence of agents. Let \( m \in \text{Tree} \) and let \( f \) be a function on \( Ag \) such that \( \forall j \in Ag(f(j) \in \text{Choice}^m_j) \). We are going to show that in this case \( \bigcap_{j \in Ag} f(j) \neq \emptyset \). If \( m \) is not a standard moment of height 0, then this is obvious, since every agent will have a vacuous choice. We treat the case when \( m \) is a standard moment of height 0. Assume that \( m = ([(\Gamma_1, \ldots, \Gamma_{n+1})]_\Xi, 0) \). By (A1) we know that there is a set \( \Delta \) of formulas of the form \( \Box A \) which is shared by all sets of the form \( end(\xi) \) with \( \xi \in \overline{m} \) in the sense that if \( \xi \in \overline{m} \), then \( \Box A \in end(\xi) \) iff \( \Box A \in \Delta \). By the same axiom scheme and Lemma 10, we also know that for every \( j \in Ag \) there is set \( \Delta_j \) of formulas of the form \([j]A \) which is shared by all sets of the form \( end(\xi) \) such that \( \exists h(h \in f(j) \land \xi = m \cap h) \). More precisely:

\[
\xi \in \overline{m} \Rightarrow (\exists h(h \in f(j) \land \xi = m \cap h) \Leftrightarrow (\forall A \in \text{Form})([j]A \in end(\xi) \Leftrightarrow [j]A \in \Delta_j)).
\]

We now consider the set \( \Delta \cup \bigcup \{\Delta_j \mid j \in Ag\} \) and show its consistency. Indeed, if this set is inconsistent, then, wlog, we would have a provable formula of the following form:

\[
\vdash (\Box A \land \bigwedge_{j \in Ag} [j]A_j) \rightarrow \bot. \tag{12}
\]

But then, choose for every \( j \in Ag \) an element \( \xi_j \in \overline{m} \) such that

\[
(\forall A \in \text{Form})([j]A \in end(\xi_j) \Leftrightarrow [j]A \in \Delta_j)).
\]

This is possible, since we may simply choose an arbitrary \( h_j \in f(j) \) and set \( \xi_j := m \cap h_j \). Then we will have \([j]A_j \in \xi_j \) for every \( j \in Ag \). Next, consider \( \Gamma_{n+1} \). Since \( m = ([(\Gamma_1, \ldots, \Gamma_{n+1})]_\Xi, 0) \) and \( \Box \) is an S5-modality, we must have:

\[
\{\Box [j]A_j \in Ag\} \subseteq \Gamma_{n+1},
\]

whence, by Lemma 15:

\[
\bigwedge_{j \in Ag} \Box [j]A_j \in \Gamma_{n+1},
\]
and further, by \([A3]\) and Lemma 1.4:

$$\Diamond \bigwedge_{j \in Ag} [j]A_j \in \Gamma_{n+1}.\]$$

Also, by definition of \(\Delta\) and the fact that \((\Gamma_1, \ldots, \Gamma_{n+1}) \in \overrightarrow{m}\), we get successively:

$$\Box A \in \Gamma_{n+1},$$

then, by Lemma 1.5:

$$\Box A \land \Diamond \bigwedge_{j \in Ag} [j]A_j \in \Gamma_{n+1},$$

and finally, by the fact that \(\Box\) is an S5-modality:

$$\Diamond (\Box A \land \bigwedge_{j \in Ag} [j]A_j) \in \Gamma_{n+1}. \tag{13}$$

From (12), together with (13), it follows by S5 reasoning for \(\Box\) that \(\Box \bot \in \Gamma_{n+1}\), so that, again by S5 properties of \(\Box\) and Lemma 1.4, it follows that \(\bot \in \Gamma_{n+1}\), which is in contradiction with maxiconsistency of \(\Gamma_{n+1}\).

Hence \(\Delta \cup \bigcup \{\Delta_j \mid j \in Ag\}\) is consistent, and we can extend it to a maxiconsistent \(\Xi\). We now consider \((\Gamma_1, \ldots, \Gamma_n, \Xi)\) and show that it is an element. Indeed, if \(KA \in \Gamma_n\), then \(KA \in \Gamma_{n+1}\) by definition of an element. But then \(\Box KA \in \Gamma_{n+1}\) by Lemma 2.2 and maxiconsistency of \(\Gamma_{n+1}\), whence \(\Box KA \in \Delta\) and, therefore, \(\Box KA \in \Xi\).

By (A1) and maxiconsistency of \(\Xi\) we get then \(KA \in \Xi\). Similarly, if \(Et \in \Gamma_n\), then \(\Box Et \in \Gamma_n\) and, therefore, \(\Box Et \in \Xi\).

Therefore, \((\Gamma_1, \ldots, \Gamma_n, \Xi)\) is an element and since, moreover, \(\Delta \subseteq \Xi\), then also \((\Gamma_1, \ldots, \Gamma_n, \Xi) \in \overrightarrow{m}\) so that \(m = (\Gamma_1, \ldots, \Gamma_n, \Xi)\). Using Lemma 10 we can choose a \(g \in H_m\) such that \(g \cap m = (\Gamma_1, \ldots, \Gamma_n, \Xi)\). We also know that for every \(j \in Ag\), there is a history \(h_j \in f(j)\) such that \(h_j \cap m = \xi_j\) by the choice of \(\xi_j\). Therefore, for every \(j \in Ag\), \(Choice^m(h_j) = f(j)\). Also, if \([j]A \in end(\xi_j) = end(h_j \cap m)\), then \([j]A \in \Delta_j\), hence \([j]A \in \Xi = end(\overrightarrow{g \cap m})\), therefore, by (A1), \(A \in end(\overrightarrow{g \cap m})\). Thus we get that \(g \in \bigcap_{j \in Ag} Choice^m(h_j) = \bigcap_{j \in Ag} f(j)\) so that the independence of agents is verified.

4.3 \(R\) and \(E\)

We now define the justifications-related elements of our canonical model. We first define \(R\) as follows:

- \(R((\Gamma_1, \ldots, \Gamma_n, \Gamma)_\equiv, k), m') \iff \)
  \(\iff (m' \neq \top) \land (\forall \tau \in \overrightarrow{m'}(\forall A \in Form)(KA \in \Gamma \implies KA \in end(\tau)));

- \(R(\top, m)\), for all \(m \in Tree\).

Now, for the definition of \(E\):

- For all \(t \in Pol\): \(E(\top, t) = \{A \in Form \mid \vdash t : A\}\);
• For all $t \in \text{Pol}$ and $m \neq \dagger$: 

$$(\forall A \in \text{Form})(A \in \mathcal{E}(m, t) \Leftrightarrow (\forall \xi \in \overrightarrow{m})(t:A \in \text{end}(\xi))).$$

We start by mentioning a straightforward corollary to the above definition:

**Lemma 13.** For all $m \in \text{Tree}$ and $t \in \text{Pol}$ it is true that $\{A \in \text{Form} \mid \vdash t:A\} \subseteq \mathcal{E}(m, t)$.

**Proof.** This holds simply by the definition of $\mathcal{E}$ when $m = \dagger$. If $m \neq \dagger$, then, for every $\xi \in \overrightarrow{m}$, $\text{end}(\xi)$ is a maxiconsistent subset of $\text{Form}$ and must contain every provable formula. 

Note that since we know that for every instance $A$ of one of axiom schemes in the list $\{\text{A0}-\text{A9}\}$, it is true that $\vdash c:A$ for every $c \in PConst$ (by (R3)), it follows, among other things, that the above-defined function $\mathcal{E}$ satisfies the additional normality condition on jstit models.

**Lemma 14.** The relation $R$, as defined above, is a preorder on $\text{Tree}$, and, together with $\preceq$, verifies the future always matters constraint.

**Proof.** It is straightforward to check that $R$, as defined above, is a preorder on $\text{Tree}$, using (A7) and (A5). Let us look into why future always matters constraint is verified as well. Assume $m \in \text{Tree}$. If $m = \dagger$, then it is connected to all the elements in $\text{Tree}$ by both $\preceq$ and $R$, so this moment cannot falsify the constraint. Let us assume that $m \neq \dagger$, say $m = (\{(\Gamma_1, \ldots, \Gamma_n)\}|_{\Xi},k)$. If $m \preceq m'$, then $m'$ must be also standard. Now, if $\overrightarrow{m} = \overrightarrow{m'}$ and $KA \in \Gamma_n$, then, by maxiconsistency of $\Gamma_n$ and Lemma 2.2, we will have $\Box KA \in \Gamma_n$, which, by definition of $\Xi$, means that $KA \in \text{end}(\xi)$ for every $\xi \in \overrightarrow{m} = \overrightarrow{m'}$, and thus we get that $R(m,m')$. The other option is that $(\Gamma_1, \ldots, \Gamma_n)$ is a proper initial segment of every element in $m'$, so that we may assume, wlog, that $m' = (\{(\Gamma_1, \ldots, \Gamma_n')\}|_{\Xi},k')$ for some $n' > n$. But then take an arbitrary $A \in \text{Form}$. If $KA \in \Gamma_n$, then, since $(\Gamma_1, \ldots, \Gamma_n')$ is an element, $KA \in \Gamma_n'$. Moreover, by maxiconsistency of $\Gamma_n'$ and Lemma 2.2, we will have $\Box KA \in \Gamma_n'$. Now, by definition of $\Xi$, we get $KA \in \text{end}(\tau)$ for any given $\tau \in \overrightarrow{m'}$. It follows that, again, we have $R(m,m')$ as desired.

We further check that the semantical constraints for $\mathcal{E}$ are verified:

**Lemma 15.** The function $\mathcal{E}$, as defined above, satisfies both monotonicity of evidence and evidence closure properties.

**Proof.** We start with the monotonicity of evidence. Assume $R(m,m')$ and $t \in \text{Pol}$. If $m = \dagger$, then, by Lemma 13, $\mathcal{E}(m, t) = \{A \in \text{Form} \mid \vdash t:A\} \subseteq \mathcal{E}(m', t)$ for any $m' \in \text{Tree}$.

Assume, further, that $m$ is standard. Let $t \in \text{Pol}$ and $A \in \text{Form}$ be such that $A \in \mathcal{E}(m, t)$. Then, for every $\xi \in \overrightarrow{m}$, $t:A \in \text{end}(\xi)$, and, by Lemma 2.1, also $Kt:A \in \text{end}(\xi)$. Therefore, by $R(m,m')$, we get that, for every $\tau \in \overrightarrow{m'}$, $Kt:A \in \text{end}(\tau)$, so that, by (A7) and maxiconsistency of every $\text{end}(\tau)$, also $t:A \in \text{end}(\tau)$. Therefore, $A \in \mathcal{E}(m', t)$, as desired.
We turn now to the closure conditions. We verify the first two conditions, and the third one can be verified in a similar way, restricting attention to $t$ rather than considering both $s$ and $t$. Let $s, t \in \text{Pol}$. We need to consider two cases:

Case 1. $m = \dagger$. If $A \in \mathcal{E}(m, s)$, then $\vdash s: A$. Therefore, by (A6), we must also have $\vdash (s + t): A$ so that $A \in \mathcal{E}(m, s + t)$. Similarly, if $A \in \mathcal{E}(m, t)$, then also $A \in \mathcal{E}(m, s + t)$ and the closure constraint (b) is verified. If, on the other hand, it is true that for some $A, B \in \text{Form}$ we have both $A \rightarrow B \in \mathcal{E}(m, s)$ and $A \in \mathcal{E}(m, t)$, then, again, this means that both $\vdash s: A \rightarrow B$ and $\vdash t: A$. By (A4), it follows that $\vdash s \times t: B$ and, therefore, also $B \in \mathcal{E}(m, s \times t)$, so that the closure condition (a) is also verified.

Case 2. $m \neq \dagger$. If $A \in \text{Form}$ and $A \in \mathcal{E}(m, s)$, then, for every $\xi \in \mathcal{m}$, $s: A \in \text{end}(\xi)$, and, by (A6) and maxiconsistency of every $\text{end}(\xi)$, we get that $s + t: A \in \text{end}(\xi)$. Therefore, $A \in \mathcal{E}(m, s + t)$. Similarly, if $A \in \mathcal{E}(m, t)$, then $A \in \mathcal{E}(m, s + t)$ as well, and closure condition (b) is verified.

On the other hand, if $A, B \in \text{Form}$ and we have both $A \rightarrow B \in \mathcal{E}(m, s)$ and $A \in \mathcal{E}(m, t)$, then, for every $\xi \in \mathcal{m}$, we have $t: A, s: (A \rightarrow B) \in \text{end}(\xi)$. By (A4) and maxiconsistency of every $\text{end}(\xi)$, we get that $s \times t: B \in \text{end}(\xi)$, thus $B \in \mathcal{E}(m, s \times t)$, and closure condition (a) is verified.

4.4 Act and $V$

It only remains to define $\text{Act}$ and $V$ for our canonical model, and we define them as follows:

- $(m, h) \in V(p) \Leftrightarrow p \in \text{end}(m \cap h)$, for all $p \in \text{Var}$;
- $\text{Act}(\dagger, h) = \emptyset$ for all $h \in \text{Hist}(\mathcal{M})$;
- $\text{Act}(m, h) = \{t \in \text{Pol} \mid Et \in \text{end}(m \cap h)\}$, if $m \neq \dagger$, $|m| = 0$ and $h \in H_m$;
- $\text{Act}(m, h) = \{t \in \text{Pol} \mid Et \in \bigcap_{g \in H_m} \text{end}(m \cap g)\}$, if $m \neq \dagger$, $|m| > 0$ and $h \in H_m$

We first draw some of the immediate consequences of the above definitions:

**Lemma 16.** Assume that $m \in \text{Tree} \setminus \{\dagger\}$ and $t \in \text{Pol}$. Then the following statements are true:

1. $Et \in \bigcap_{h \in H_m} (\text{end}(m \cap h)) \Leftrightarrow t \in \bigcap_{h \in H_m} (\text{Act}(m, h))$;
2. If $|m| > 0$ and $h, h' \in H_m$, then $\text{Act}(m, h) = \text{Act}(m, h')$;
3. If $h, h' \in H_m$ and $m \cap h = m \cap h'$, then $\text{Act}(m, h) = \text{Act}(m, h')$.

**Proof.** (Part 1). Let $g \in H_m$ be arbitrary. If $Et \in \bigcap_{h \in H_m} (\text{end}(m \cap h))$, then $t \in \text{Act}(m, g)$ whatever the height of $m$ is. Since $g$ was chosen arbitrarily, this means that $t \in \bigcap_{h \in H_m} (\text{Act}(m, h))$. In the other direction, assume that $t \in \text{Act}(m, g)$. Then, again irrespectively of the height, $Et \in \text{end}(m \cap g)$. Therefore, if $t \in \bigcap_{h \in H_m} (\text{Act}(m, h))$, then $Et \in \bigcap_{h \in H_m} (\text{end}(m \cap h))$.

(Part 2). In the assumptions of this part, we get that:

$$t \in \text{Act}(m, h) \Leftrightarrow Et \in \bigcap_{g \in H_m} (\text{end}(m \cap g)) \Leftrightarrow t \in \text{Act}(m, h'),$$
for an arbitrary \( t \in \text{Pol} \).

(Part 3). We have to distinguish between two cases. If \( |m| = 0 \), then, for an arbitrary \( t \in \text{Pol} \), we get that:

\[
 t \in \text{Act}(m, h) \iff \text{Et} \in \text{end}(m \cap h) \iff \text{Et} \in \text{end}(m \cap h') \iff t \in \text{Act}(m, h').
\]

On the other hand, if \( |m| > 0 \), then we are done by Part 2.

We now check the remaining semantic constraints on normal jstit models:

**Lemma 17.** The canonical model, as defined above, satisfies the constraints as to the expression of presented proofs, no new proofs guaranteed, presenting a new proof makes histories divide, and epistemic transparency of presented proofs.

**Proof.** We consider the **expansion of presented proofs** first. Let \( m' < m \) and let \( h \in H_m \). If \( m' = \hat{1} \), then we have \( \text{Act}(\hat{1}, h) = \emptyset \), so that the expansion of presented proofs holds. If \( m' \neq \hat{1} \), then \( m \) is also standard. Consider then \( m' \cap h \) and \( m \cap h \). Both these elements must be in the basic sequence \( |h| \), therefore, one of them must be an initial segment of another. By \( m' < m \) we know that \( m' \cap h \) must be a proper initial segment of \( m \cap h \). So we may assume that \( m' \cap h = (\Gamma_1, \ldots, \Gamma_k) \) and \( m \cap h = (\Gamma_1, \ldots, \Gamma_n) \) for some appropriate \( \Gamma_1, \ldots, \Gamma_n \subseteq \text{Form} \) and \( n > k \). Now, if \( t \in \text{Act}(m', h) \), then \( \text{Et} \in \text{end}(m' \cap h) = \Gamma_k \). Then, since \( (\Gamma_1, \ldots, \Gamma_n) \) is an element, we must have \( \Box \text{Et} \in \Gamma_n \). By definition of \( \Box \), it follows that for every \( \xi \in m' \) we must have that \( \text{Et} \in \text{end}(\xi) \). Now, if \( g \in H_m \), then of course \( m \cap g \in m' \). Therefore, we get that \( \text{Et} \in \bigcap_{g \in H_m} \text{end}(m \cap g) \), whence, by Lemma 16.1, \( t \in \text{Act}(m, h) \) immediately follows.

We consider next the **no new proofs guaranteed** constraint. Let \( m \in \text{Tree} \). If \( m = \hat{1} \), then \( \bigcap_{h \in H_m} (\text{Act}(m, h)) = \bigcup_{m' \lesssim m, h \in H_m} (\text{Act}(m', h)) = \emptyset \) and the constraint is trivially satisfied. Assume that \( m \neq \hat{1} \). Then \( m \) must be of the form \( ([\Gamma_1, \ldots, \Gamma_n])_\equiv, k \) for appropriate \( \Gamma_1, \ldots, \Gamma_n \subseteq \text{Form} \) and \( k \in \omega \). Assume that \( t \in \bigcap_{h \in H_m} (\text{Act}(m, h)) \). By Lemma 16.1, we get then that \( \text{Et} \in \bigcap_{h \in H_m} \text{end}(m \cap h) \). Now, consider \( m' = ([\Gamma_1, \ldots, \Gamma_n])_\equiv, k + 1 \). We clearly have \( m' < m \), therefore, if \( g \in H_m \), then also \( g \in H_{m'} \). In the other direction, if \( g \in H_m \), then, by Corollary 2 we get \( g \in H_m \), so that the fans of histories passing through \( m \) and \( m' \) are identical. Further, we have \( m = m' \), hence it follows from Lemma 16.1 that \( \text{end}(g \cap m) = \text{end}(g \cap m') \) for every \( g \in H_m = H_m \), and further, \( \bigcap_{h \in H_m} \text{end}(m \cap h) = \bigcap_{h \in H_{m'}} \text{end}(m' \cap h) \). Therefore, \( \text{Et} \in \bigcap_{h \in H_m} \text{end}(m' \cap h) \) and it follows, by Lemma 16.1, that \( t \in \text{Act}(m', h) \subseteq \bigcup_{m' \lesssim m, h \in H_m} (\text{Act}(m', h)) \).

We turn next to the **presenting a new proof makes histories divide** constraint. Consider an \( m, m' \in \text{Tree} \) such that \( m < m' \) and arbitrary \( h, h' \in H_{m'} \). We immediately get then that \( h, h' \in H_m \). If \( m = \hat{1} \), then the constraint is verified trivially. If \( m \neq \hat{1} \), then we have two cases to consider:

**Case 1.** \( m = m' \) and \( |m| > |m'| \). Then we must have \( |m| > 0 \), and by Lemma 16.2 it follows that in this case for all \( h, h' \in H_m \) we will have \( \text{Act}(m, h) = \text{Act}(m, h') \) so that the constraint is verified.

**Case 2.** There is a \( \xi \in m' \) such that \( \xi \) is a proper initial segment of every \( m \in m' \). Consider then \( m' \cap h \) and \( m' \cap h' \). These are elements in \( m' \), and hence \( \xi \) is a proper initial segment of both \( m' \cap h \) and \( m' \cap h' \). It follows that \( m' \cap h = m \cap h' = \xi \) whence, by Lemma 16.3, we immediately get \( \text{Act}(m, h) = \text{Act}(m, h') \).
That we have, of course \( m \in \text{Act}(m, h) \). If we have \( m = \dagger \), then, by definition, we must have \( \forall h \in H_m \text{(Act}(m, h)) = \emptyset \), and the constraint is verified in a trivial way. If, on the other hand, \( m \neq \dagger \), then, by \( R(m, m') \), we must also have \( m' \neq \dagger \). Assume that \( t \in \bigcap_{h \in H_m} \text{Act}(m, h) \). Then, by Lemma 16, we also have \( Et \in \bigcap_{h \in H_m} \text{Act}(m, h) \). Let \( h \in H_m \) be arbitrary. We claim that under these assumptions, we must have \( \Box Et \in \text{end}(m \cap h) \). Indeed, if \( \Box Et \notin \text{end}(m \cap h) \), then consider the following set \( \Xi \) of formulas:

\[
\Xi = \{ \Box B \mid \Box B \in \text{end}(m \cap h) \} \cup \{ \neg Et \}.
\]

We claim that \( \Xi \) is consistent. Otherwise we would have

\[
\vdash (\Box B_1 \land \ldots \land \Box B_n) \rightarrow Et
\]

for some \( \Box B_1, \ldots, \Box B_n \in \text{end}(m \cap h) \), and the latter, by S5 reasoning for \( \Box \), would mean that

\[
\vdash (\Box B_1 \land \ldots \land \Box B_n) \rightarrow \Box Et,
\]

whence, by Lemma 14 and maxiconsistency of \( \text{end}(m \cap h) \), \( \Box Et \notin \text{end}(m \cap h) \) would follow, contrary to our hypothesis. But then we can extend \( \Xi \) to a maxiconsistent \( \Delta \). Assume that \( m \cap h = (\Gamma_1, \ldots, \Gamma_k, \Gamma) \), so that \( \Gamma = \text{end}(m \cap h) \). We show that \( (\Gamma_1, \ldots, \Gamma_k, \Delta) \in \overline{m} \). We start by showing that \( (\Gamma_1, \ldots, \Gamma_k, \Gamma) \) is an element. If \( KB \in \Gamma_k \), then, since \( (\Gamma_1, \ldots, \Gamma_k, \Gamma) \) is an element, it follows that \( KB \in \Gamma \). By Lemma 2 and maxiconsistency of \( \Gamma \), we further get that \( \Box KB \in \Gamma \), whence \( \Box KB \in \Delta \), and, by \( \{ A_1 \} \) and maxiconsistency of \( \Delta \), \( KB \in \Delta \). Similarly, if \( Es \in \Gamma_k \), then \( \Box Es \in \Gamma \) and further, \( \Box Es \in \Delta \). Once \( (\Gamma_1, \ldots, \Gamma_k, \Delta) \) is thus shown to be an element, \( (\Gamma_1, \ldots, \Gamma_k, \Gamma) \equiv (\Gamma_1, \ldots, \Gamma_k, \Delta) \) follows immediately just by the choice of \( \Xi \) and the fact that \( \Delta \) extends \( \Xi \). Therefore, \( (\Gamma_1, \ldots, \Gamma_k, \Delta) \in \overline{m} \). By Lemma 10 there is a \( g \in H_m \) such that \( m \cap g = (\Gamma_1, \ldots, \Gamma_k, \Delta) \). Then \( \Delta = \text{end}(m \cap g) \), but we also know that \( \neg Et \in \Delta \). Therefore, by maxiconsistency, \( Et \notin \Delta = \text{end}(m \cap g) \). But this is in contradiction with our assumption that \( Et \in \bigcap_{h \in H_m} (m \cap h) \).

The obtained contradiction shows that \( \Box Et \notin \text{end}(m \cap h) \), and by maxiconsistency of \( \text{end}(m \cap h) \) and \( \{ A_9 \} \), this means that also \( K \Box Et \notin \text{end}(m \cap h) \). It remains to note that we have, of course \( m = ([m \cap h]_{\Xi}, |m|) \), whence by \( R(m, m') \) we get that \( K \Box Et \in \tau \) for every \( \tau \in \overline{m} \). This means, by maxiconsistency of every such \( \tau \), \( \{ A_1 \} \), and \( \{ A_7 \} \), that \( Et \in \tau \) for every \( \tau \in \overline{m} \). Note, further, that if \( g \in H_m \), then \( m' \cap g \in \overline{m} \), so that we have shown that \( Et \in \bigcap_{g \in H_m} (m' \cap g) \), and hence, by Lemma 16, also \( t \in \bigcap_{g \in H_m} (\text{Act}(m', g)) \), as desired.

4.5 The truth lemma

It follows from Lemmas 14, 17 that our above-defined canonical model is in fact a normal unirelational jst model. Now we need to supply a truth lemma:

**Lemma 18.** Let \( A \in \text{Form}, m \in \text{Tree} \setminus \{ \dagger \} \) be such that \( |m| = 0 \), and let \( h \in H_m \).

Then:

\[ M, m, h \models A \iff A \in \text{end}(m \cap h). \]
Proof. As is usual, we prove the lemma by induction on the construction of $A$. The basis of induction with $A = \varphi \in Var$ we have by definition of $V$, whereas Boolean cases for the induction step are trivial. We treat the modal cases:

Case 1. $A = \Box B$. If $\Box B \in end(m \cap h)$, then note that for every $h' \in H_m$ we must have $m \cap h' \in m$ so that $m \cap h' \equiv m \cap h$. By definition of $\equiv$ and the fact that $m \in Tree \setminus \{↑\}$, we must have then $B \in end(m \cap h')$ for all $h' \in H_m$ and thus, by induction hypothesis, we obtain that $M, m, h \models \Box B$. If, on the other hand, $\Box B \notin end(m \cap h)$, then let $m \cap h = (\Gamma_1, \ldots, \Gamma_k, \Gamma)$ so that $end(m \cap h) = \Gamma$. Then the set

$$\Xi = \{\Box C \mid \Box C \in \Gamma\} \cup \{-B\}$$

must be consistent, since otherwise we would have

$$\vdash (\Box C_1 \land \ldots \land \Box C_n) \rightarrow B$$

for some $\Box C_1, \ldots, \Box C_n \in \Gamma$, whence, since $\Box$ is an S5-modality, we would get

$$\vdash (\Box C_1 \land \ldots \land \Box C_n) \rightarrow \Box B,$$

which would mean that $\Box B \in \Gamma$, contrary to our assumption. Therefore, $\Xi$ is consistent and we can extend $\Xi$ to a maxiconsistent $\Delta \subseteq Form$. Of course, in this case $B \notin \Delta$.

We now show that $(\Gamma_1, \ldots, \Gamma_k, \Delta)$ is an element. Indeed, if for any $C \in Form$ we have that $KC \in \Gamma_k$, then, since $(\Gamma_1, \ldots, \Gamma_k, \Gamma)$ is an element, we will have $KC \in \Gamma$, whence, by maxiconsistency of $\Gamma$ and (A8), $\Box KC \in \Gamma$, and since every boxed formula from $\Gamma$ is also in $\Delta$, we get that $\Box KC \in \Delta$, whence $KC \in \Delta$ by maxiconsistency of $\Delta$ and S5 reasoning for $\Box$. Further, if we have $Et \in \Gamma_k$, for some $t \in Pol$, then, since $(\Gamma_1, \ldots, \Gamma_k, \Gamma)$ is an element, we will have $\Box Et \in \Gamma$, and since every boxed formula from $\Gamma$ is also in $\Delta$, we get that $\Box Et \in \Delta$.

Once we know that $(\Gamma_1, \ldots, \Gamma_k, \Delta)$ is an element, it follows by the choice of $\Xi$ and $\Delta$ that $(\Gamma_1, \ldots, \Gamma_k, \Gamma) \equiv (\Gamma_1, \ldots, \Gamma_k, \Delta)$. By Lemma 10 for some $h' \in H_m$ we will have $(\Gamma_1, \ldots, \Gamma_k, \Delta) = m \cap h'$ and, therefore, $\Delta = end(m \cap h')$. Since $B \notin \Delta$, it follows, by induction hypothesis, that $M, m, h' \not\models B$, hence $M, m, h \not\models \Box B$ as desired.

Case 2. $A = [j]B$ for some $j \in A_q$. Then, if $[j]B \in end(m \cap h)$, by definition of Choice and the fact that both $m \neq ↑$ and $|m| = 0$ we must have:

$$Choice^m_j(h) = \{h' \in H_m \mid (\forall C \in Form)\{[j]C \in end(h \cap m) \Rightarrow C \in end(h' \cap m)\}\}.$$

Therefore, if $h' \in Choice^m_j(h)$, then we must have that $B \in end(h' \cap m)$, and further, by induction hypothesis, that $M, m, h' \models B$, so that we get $M, m, h \models [j]B$. On the other hand, if $[j]B \notin end(m \cap h)$, we again assume that $m \cap h = (\Gamma_1, \ldots, \Gamma_k, \Gamma)$ so that $end(m \cap h) = \Gamma$. Then the set

$$\Xi = \{[j]C \mid [j]C \in \Gamma\} \cup \{-B\}$$

must be consistent, since otherwise we would have

$$\vdash ([j]C_1 \land \ldots \land [j]C_n) \rightarrow B$$

for some $[j]C_1, \ldots, [j]C_n \in \Gamma$, whence, since $[j]$ is an S5-modality, we would get

$$\vdash ([j]C_1 \land \ldots \land [j]C_n) \rightarrow [j]B,$$
which would mean that \([j]B \in \Gamma\), contrary to our assumption. Therefore, \(\Xi\) is consistent and we can extend \(\Xi\) to a maxiconsistent \(\Delta \subseteq Form\). Of course, in this case \(B \notin \Delta\). Arguing as in Case 1, we can show that \((\Gamma_1, \ldots, \Gamma_k, \Delta)\) is an element.

Now, if \(D \in Form\) is such that \(\Box D \in \Gamma\), then, by \([A2]\) and maxiconsistency of \(\Gamma\), we know that \([j]D \in \Gamma\), so that also \([j]D \in \Delta\), and hence, by \([A1]\) and maxiconsistency of \(\Delta\), \(D \in \Delta\). We have thus shown that:

\[
(\forall D \in Form)(\Box D \in \Gamma \Rightarrow D \in \Delta),
\]

and it follows that \((\Gamma_1, \ldots, \Gamma_k, \Gamma) \equiv (\Gamma_1, \ldots, \Gamma_k, \Delta)\) by definition of \(\equiv\). By Lemma[10] for some \(h' \in H_m\) we will have \((\Gamma_1, \ldots, \Gamma_k, \Delta) = m \cap h'\) and, therefore, \(\Delta = \text{end}(m \cap h')\).

Also, since \(\Delta\) contains all the \([j]\)-modalized formulas from \(\Gamma\), we know that for any such \(h'\) we will have \(h' \in \text{Choice}^m(h)\). Since \(B \notin \Delta\), it follows, by induction hypothesis, that \(\mathcal{M}, m, h' \not\models B\), hence \(\mathcal{M}, m, h \not\models [j]B\) as desired.

**Case 3.** \(A = KB\). Assume that \(KB \in \text{end}(m \cap h)\). We clearly have then \(m = (\{(m \cap h)\}^m, 0)\). Hence, by definition of \(R\) and the fact that \(m \neq \dagger\) we must have for every \(m' \in \text{Tree}\):

\[
R(m, m') \Rightarrow (\forall \tau \in m')((\forall C \in Form)(KC \in end(m \cap h) \Rightarrow KC \in end(\tau))).
\]

Therefore, if \(R(m, m')\) and \(h' \in H_{m'}\) is arbitrary, then, of course, \((h' \cap m') \in m'\) so that \(KB \in \text{end}(h' \cap m')\), and, further, \(B \in \text{end}(h' \cap m')\) by S4 reasoning for \(K\). Therefore, by induction hypothesis, we get that \(\mathcal{M}, m', h' \models B\), whence \(\mathcal{M}, m, h \models KB\). On the other hand, if \(KB \notin \text{end}(m \cap h)\), then consider the set

\[
\Xi = \{KC | KC \in end(m \cap h)\} \cup \{\neg \Box B\}.
\]

This set must be consistent, since otherwise we would have

\[
\vdash (KC_1 \wedge \ldots \wedge KC_n) \rightarrow \Box B
\]

for some \(KC_1, \ldots, KC_n \in \Gamma\), whence, since \(K\) is an S4-modality, we would get

\[
\vdash (KC_1 \wedge \ldots \wedge KC_n) \rightarrow \Box \Box B,
\]

which would mean that \(K \Box \Box B \in \text{end}(m \cap h)\), hence, by \([A1]\), \([A7]\) and maxiconsistency of \(\text{end}(m \cap h)\), that \(KB \in \text{end}(m \cap h)\), contrary to our assumption. Therefore, \(\Xi\) is consistent and we can extend \(\Xi\) to a maxiconsistent \(\Delta \subseteq Form\). Of course, in this case \(\Box B \notin \Delta\). We will have then that \((\Delta)\) is an element. So we set \(m' = (\{(\Delta)\}^m, 0)\). Assume that \((\Delta') \equiv (\Delta)\). Then every boxed formula from \(\Delta\) will be in \(\Delta'\). In particular, whenever \(KC \in \Delta\), also \(\Box KC \in \Delta\) and thus \(KC \in \Delta'\), by \([A1]\), \([A3]\) and maxiconsistency of \(\Delta\). Therefore, whenever \(KC \in \text{end}(m \cap h)\) and \(\tau \in m' = (\{(\Delta)\}^m)\), we have that \(KC \in \text{end}(\tau)\) so that we must have \(R(m, m')\). On the other hand, since \(\Box B \notin \Delta\), then, by Case 1, there must be a \(\tau \in m'\) such that \(B \notin \text{end}(\tau)\). But then, by Lemma[10] we can choose an \(h' \in H_{m'}\) in such a way that \(\tau = m' \cap h'\), and we get that \(B \notin \text{end}(m' \cap h')\). Therefore, by induction hypothesis, we get \(\mathcal{M}, m', h' \not\models B\). In view of the fact that also \(R(m, m')\), this means that \(\mathcal{M}, m, h \not\models KB\) as desired.

**Case 4.** \(A = t: B\) for some \(t \in Pol\). If \(t: B \in \text{end}(m \cap h)\), then, by maxiconsistency of \(\text{end}(m \cap h)\) and Lemma[21], we must have \(\Box t:B \in \text{end}(m \cap h)\). Now, if \(\xi \in m', \xi \models
then we must have, of course $\xi \equiv m \cap h$, whence \( t:B \in \text{end}(\xi) \). Therefore, we must have $B \in \mathcal{E}(m, t)$. Also, by maxiconsistency of $\text{end}(m \cap h)$ and (A5), we will have $KB \in \text{end}(m \cap h)$. Therefore, by Case 3, we will have that $\mathcal{M}, m, h \models KB$ and further, by $B \in \mathcal{E}(m, t)$, that $\mathcal{M}, m, h \models t:B$. On the other hand, if $t:B \not\in \text{end}(m \cap h)$, then, since clearly $m \cap h \in \neg \neg m$, we must have $B \not\in \text{end}(m, t)$, whence $\mathcal{M}, m, h \not\models t:B$.

Case 5. $A = Et$ for some $t \in \text{Pol}$. Then, given that $m \not\models \dagger$ and $|m| = 0$, we have, simply by definition of $\text{Act}$, that:

\[
Et \in \text{end}(m \cap h) \iff t \in \text{Act}(m, h) \iff M, m, h \models Et.
\]

This finishes the list of the modal induction cases at hand, and thus the proof of our truth lemma is complete.}

\section{The main result}

We are now in a position to prove Theorem 2. The proof proceeds as follows. One direction of the theorem was proved as Corollary 1. In the other direction, assume that $\Gamma \subseteq \text{Form}$ is consistent. Then, by Lemma 1, $\Gamma$ can be extended to a maxiconsistent $\Delta$. But then consider $\mathcal{M} = \langle \text{Tree}, \leq, \text{Choice}, \text{Act}, R, \mathcal{E}, V \rangle$, the canonical model defined in Section 4. It is clear that $\Delta$ is an element, therefore $m = \langle \langle \Delta \rangle \equiv \varepsilon, 0 \rangle \in \text{Tree}$. By Lemma 10 there is a history $h \in H_m$ such that $\Delta = \langle \langle \Delta \rangle \equiv \varepsilon, 0 \rangle \cap h$. For this $h$, we will also have $\Delta = \text{end}((\langle \Delta \rangle \equiv \varepsilon, 0) \cap h)$. By Lemma 18, we therefore get that:

\[
M, m, h \models \Delta \models \Gamma,
\]

and thus $\Gamma$ is shown to be satisfiable in a normal jstit unirelational model, hence in a normal jstit model.

\textbf{Remark}. Note that the canonical model used in this proof is universal in the sense that it satisfies every subset of $\text{Form}$ which is consistent relative to $\Sigma$.

As an obvious corollary of Theorem 2, we get the compactness property:

\textbf{Corollary 4}. An arbitrary $\Gamma \subseteq \text{Form}$ is satisfiable in a normal (unirelational) jstit model iff every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable in a normal (unirelational) jstit model.

The construction of the canonical model defined in Section 4 allows for a generalization. Let us call a \textit{constant specification} any set $\mathcal{CS}$ such that:

- $\mathcal{CS} \subseteq \{c_n: \ldots c_1:A \mid c_1, \ldots , c_n \in PConst, A \text{ an instance of } \{A_0 \ldots A_9\}\}$;

- Whenever $c_{n+1}:c_n: \ldots c_1:A \in \mathcal{CS}$, then also $c_n: \ldots c_1:A \in \mathcal{CS}$.

For a given constant specification, we can define the corresponding inference rule $R_{\mathcal{CS}}$ as follows:

\[
\text{If } c_n: \ldots c_1:A \in \mathcal{CS}, \text{ infer } c_n: \ldots c_1:A. \quad (R_{\mathcal{CS}})
\]

It is easy to see that the least constant specification will be just $\emptyset$ and that $R3$ is in fact $R_{\emptyset}$ where $\mathcal{CS}$ is the following constant specification:

\[
\{c:A \mid c \in PConst, A \text{ an instance of } \{A_0 \ldots A_9\}\}.
\]

We note that Theorem 2 is accordingly but a particular instance, obtained by setting $\mathcal{CS} := \mathcal{CS}$, of the following more general theorem:
Theorem 3. Let $CS$ be a constant specification. Then an arbitrary $\Gamma \subseteq \text{Form}$ is consistent relative to the axiomatic system $\Sigma_{CS} = \{ (A0) - (A9), (R1), (R2), (R4), (R_{CS}) \}$ iff $\Gamma$ is satisfiable in an (unirelational) jstit model satisfying the following additional condition:

$$(\forall c \in P\text{Const})(\forall m \in \text{Tree})(\{ A | c:A \in CS \} \subseteq \mathcal{E}(m, c)).$$

We further note that the proof of this more general theorem can be obtained from the proof of Theorem 2 above simply by replacing every reference to $\Sigma = \Sigma_{CS}$ by a reference to $\Sigma_{CS}$. We end this section by the observation that it follows from Theorem 3 that the axiomatization of the validities over the whole unrestricted class of (unirelational) jstit models is given by $\Sigma_{0} = \{ (A0) - (A9), (R1), (R2), (R4) \}$.

6 Conclusion

Building up on an earlier work on jstit formalisms, we have defined stit logic of justification announcements (JA-STIT) — a natural logic which combines justification logic with stit logic to provide a natural environment for representing proving activity of agents within a (somewhat idealized) finite community of researchers. For this logic, we have defined the semantics originally presented in [7]. The main import of this paper is that JA-STIT admits of a strongly complete axiomatization w.r.t. this semantics and that this axiomatization can be straightforwardly accommodated to a wide range of possible constant specifications.

The main result of the present paper also leads to a number of natural questions which we hope to be able to answer in our future publications. One problem is posed by the fact, established in Proposition 1 that JA-STIT is expressive enough to distinguish between the class of all jstit models and the class of all models based on discrete time. This fact implies that our axiomatization will no longer be complete once the time is assumed to be discrete. However, jstit models based on discrete time form a very natural subclass within the class of jstit models, and it would be nice to find out how to axiomatize our logic over this particular subclass.

Another problem for future research is finding a separate axiomatization for the explicit fragment of basic jstit logic. It was mentioned above that even though in JA-STIT one can retrieve explicit proving modalities of this logic, the inverse reduction does not seem to go through, so that in terms of expressive power JA-STIT appears to be a proper extension of the explicit fragment of basic jstit logic. A natural further move would be then to find a separate axiomatization for the explicit fragment of basic jstit logic and compare it to the axiomatization presented in this paper. Yet another natural, although by no means trivial, further move would be to take on board also the implicit version $EA$ of $Et$-modality and axiomatize the full logic of $E$-notions.

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To be inserted.
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