FRÖLICH-NIJENHUIS BRACKET ON MANIFOLDS WITH SPECIAL HOLONOMY

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Abstract. In this article, we summarize our recent results on the study of manifolds with special holonomy via the Frölicher-Nijenhuis bracket. This bracket enables us to define the Frölicher-Nijenhuis cohomologies which are analogues of the $d^c$ and the Dolbeault cohomologies in Kähler geometry, and assigns an $L_\infty$-algebra to each associative submanifold. We provide several concrete computations of the Frölicher-Nijenhuis cohomology.

1. Introduction

The Frölicher-Nijenhuis bracket, which was introduced in [FN1956, FN1956b], defines a natural structure of a graded Lie algebra on the space of tangent bundle valued differential forms $\Omega^*(M, TM)$ on a smooth manifold $M$.

On a Riemannian manifold $(M, g)$, if there is a parallel differential form of even degree, we can define canonical cohomologies which are analogues of the $d^c$ and the Dolbeault cohomologies in Kähler geometry. See Section 3. We compute these cohomologies for $G_2$-manifolds in Theorem 4.1 and give a sketch of the proof in Section 4. A similar statement holds for $\text{Spin}(7)$ and Calabi-Yau manifolds. See [KLS2017b, KLS2018].

In the second part of our note, using the Frölicher-Nijenhuis bracket, we assign to each associative submanifold an $L_\infty$-algebra.

Notation: Let $(V, g)$ be an $n$-dimensional oriented real vector space with a scalar product $g$. Define the map $\partial$ by contraction of a form with the metric $g$, i.e.

\begin{equation}
\partial = \partial_g : \Lambda^k V^* \longrightarrow \Lambda^{k-1} V^* \otimes V, \quad \partial_g(\alpha^k) := (e_i \alpha^k) \otimes e^i,
\end{equation}

where $(e_i)$ is an orthonormal basis of $V$ with the dual basis $(e^i)$.

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2. Preliminaries

2.1. Graded Lie algebras and differentials. We briefly recall some basic notions and properties of graded (Lie) algebras. Let $V := (\bigoplus_{k \in \mathbb{Z}} V_k, \cdot)$ be a graded real vector space with a graded bilinear map $\cdot : V \times V \to V$, called a product on $V$. A graded derivation of $(V, \cdot)$ of degree $l$ is a linear map $D^l : V \to V$ of degree $l$ (i.e., $D^l(V_k) \subset V_{k+l}$) such that

$$
D^l(x \cdot y) = (D^l x) \cdot y + (-1)^{|x|} x \cdot (D^l y),
$$

(2.1)

where $|x|$ denotes the degree of an element, i.e., $|x| = k$ for $x \in V_k$. If we denote by $D^l(V)$ the graded derivations of $(V, \cdot)$ of degree $l$, then $D(V) := \bigoplus_{l \in \mathbb{Z}} D^l(V)$ is a graded Lie algebra with the Lie bracket

$$
[D_1, D_2] := D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1,
$$

(2.2)
i.e., the Lie bracket is graded anti-symmetric and satisfies the graded Jacobi identity,

$$
[x, y] = -(-1)^{|x||y|} [y, x],
$$

(2.3)

$$
(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|} [y, [x, z]] + (-1)^{|z||y|} [z, [x, y]] = 0.
$$

(2.4)

In general, if $L = (\bigoplus_{k \in \mathbb{Z}} L_k, [\cdot, \cdot])$ is a graded Lie algebra, then an action of $L$ on $V$ is a Lie algebra homomorphism $\pi : L \to D(V)$, which yields a graded bilinear map $L \times V \to V$, $(x, v) \mapsto \pi(x)(v)$ such that the map $\pi(x) : V \to V$ is a graded derivation of degree $|x|$ and such that

$$
[\pi(x), \pi(y)] = \pi[x, y].
$$

For instance, a graded Lie algebra acts on itself via the adjoint representation $ad : L \to D(L)$, where $ad_x(y) := [x, y]$.

For a graded Lie algebra $L$ we define the set of Maurer-Cartan elements of $L$ of degree $2k + 1$ as

$$
\mathcal{MC}^{2k+1}(L) := \{ \xi \in L_{2k+1} \mid [\xi, \xi] = 0 \}.
$$

If $\pi : L \to D(V)$ is an action of $L$ on $(V, \cdot)$, then for $\xi \in \mathcal{MC}^{2k+1}(L)$ we have $0 = [\pi(\xi), \pi(\xi)] = 2\pi(\xi)^2$, so that $\pi(\xi) : V \to V$ is a differential on $V$. We define the cohomology of $(V, \cdot)$ w.r.t. $\xi$ as

$$
H^i_{\xi}(V) := \frac{\ker(\pi(\xi) : V_i \to V_{i+2k+1})}{\operatorname{Im}(\pi(\xi) : V_{i-(2k+1)} \to V_i)}
$$

(2.5)

for $\xi \in \mathcal{MC}^{2k+1}(L)$.

Since $\pi(\xi)$ is a derivation, it follows that $\ker(\pi(\xi)) \cap \ker(\pi(\xi)) \subset \ker(\pi(\xi))$, whence there is an induced product on $H^*_{\xi}(V) := \bigoplus_{i \in \mathbb{Z}} H^i_{\xi}(V)$.

If $L = \bigoplus_{k \in \mathbb{Z}} L_k$ is a graded Lie algebra, then for $v \in L_0$ and $t \in \mathbb{R}$, we define the formal power series

$$
\exp(tv) : L \to L[[t]], \quad \exp(tv)(x) := \sum_{k=0}^{\infty} \frac{t^k}{k!} d^k_v(x).
$$

(2.6)

Observe that $ad_v(v) = 0$ for some $v \in L_0$ iff $ad_v(\xi) = 0$ iff $\exp(tv)(\xi) = \xi$ for all $t \in \mathbb{R}$. In this case, we call $v$ an infinitesimal stabilizer of $\xi$. 


For $\xi \in MC^{2k+1}(L)$, we say that $x \in L_{2k+1}$ is an infinitesimal deformation of $\xi$ within $MC^{2k+1}(L)$ if $[\xi + tx, \xi + tx] = 0$ mod $t^2$. Evidently, this is equivalent to $[\xi, x] = 0$ or $x \in \ker ad_\xi$. Such an infinitesimal deformation is called trivial if $x = [\xi, v]$ for some $v \in L_0$, since in this case, $\xi + tx = \exp(-tv)(\xi)$ mod $t^2$, whence up to second order, it coincides with elements in the orbit of $\xi$ under the (formal) action of $\exp(tv)$. Thus, we have the following interpretation of some cohomology groups.

**Proposition 2.1.** Let $(L = \bigoplus_{i \in \mathbb{Z}} L_i, [\cdot, \cdot])$ be a real graded Lie algebra, acting on itself by the adjoint representation, and let $\xi \in MC^{2k+1}(L)$. Then the following holds.

1. If $L_{-(2k+1)} = 0$, then $H^0_\xi(L)$ is the Lie algebra of infinitesimal stabilizers of $\xi$.
2. $H^{2k+1}_\xi(L)$ is the space of infinitesimal deformations of $\xi$ within $MC^{2k+1}(L)$ modulo trivial deformations.

### 2.2. The Frölicher-Nijenhuis bracket.

We shall apply our discussion from the preceding section to the following example. Let $M$ be a manifold and $(\Omega^*(M), \wedge) = (\bigoplus_{k \geq 0} \Omega^k(M), \wedge)$ be the graded algebra of differential forms. Evidently, the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a derivation of $\Omega^*(M)$ of degree 1, whereas insertion $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ of a vector field $X \in \mathfrak{X}(M)$ is a derivation of degree $-1$.

More generally, for $K \in \Omega^k(M, TM)$ we define $\iota_K \alpha^l$ as the insertion of $K$ into $\alpha^l \in \Omega^l(M)$ pointwise by

$$\iota_{\kappa^k \wedge X} \alpha^l := \kappa^k \wedge (\iota_X \alpha^l) \in \Omega^{k+l-1}(M),$$

where $\kappa^k \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$, and this is a derivation of $\Omega^*(M)$ of degree $k - 1$. Thus, the Nijenhuis-Lie derivative along $K \in \Omega^k(M, TM)$ defined as

$$\mathcal{L}_K(\alpha^l) := [\iota_K, d](\alpha^l) = \iota_K(d\alpha^l) + (-1)^k d(\iota_K \alpha^l) \in \Omega^{k+l}(M)$$

is a derivation of $\Omega^*(M)$ of degree $k$.

Observe that for $k = 0$ in which case $K \in \Omega^0(M, TM)$ is a vector field, both $\iota_K$ and $\mathcal{L}_K$ coincide with the standard notion of insertion of and Lie derivative along a vector field.

In [FN1956, FN1956b], it was shown that $\Omega^*(M, TM)$ carries a unique structure of a graded Lie algebra, defined by the so-called Frölicher-Nijenhuis bracket,

$$[\cdot, \cdot]^{FN} : \Omega^k(M, TM) \times \Omega^l(M, TM) \to \Omega^{k+l}(M, TM)$$

such that $\mathcal{L}$ defines an action of $\Omega^*(M, TM)$ on $\Omega^*(M)$, that is,

$$\mathcal{L}_{[K_1, K_2]}^{FN} = [\mathcal{L}_{K_1}, \mathcal{L}_{K_2}] = \mathcal{L}_{K_1} \circ \mathcal{L}_{K_2} - (-1)^{|K_1||K_2|} \mathcal{L}_{K_2} \circ \mathcal{L}_{K_1}.$$
It is given by the following formula for \( \alpha_k \in \Omega^k(M) \), \( \beta^l \in \Omega^l(M) \), \( X_1, X_2 \in \mathfrak{X}(M) \) \[KMS1993\] Theorem 8.7 (6), p. 70:

\[
\begin{align*}
[\alpha_k \otimes X_1, \beta^l \otimes X_2]_{FN} &= \alpha_k \wedge \beta^l \otimes [X_1, X_2] \\
&+ \alpha_k \wedge (\mathcal{L}_{X_1} \beta^l) \otimes X_2 - (\mathcal{L}_{X_2} \alpha_k) \wedge \beta^l \otimes X_1 \\
&+ (-1)^k \left( d\alpha_k \wedge (\iota_{X_1} \beta^l) \otimes X_2 + (\iota_{X_2} \alpha_k) \wedge d\beta^l \otimes X_1 \right).
\end{align*}
\] (2.9)

In particular, for a vector field \( X \in \mathfrak{X}(M) \) and \( K \in \Omega^*(M, TM) \) we have \[KMS1993\] Theorem 8.16 (5), p. 75]

\[
\mathcal{L}_X(K) = [X, K]_{FN},
\] (2.10)

that is, the Frölicher-Nijenhuis bracket with a vector field coincides with the Lie derivative of the tensor field \( K \in \Omega^*(M, TM) \). This means that \( \exp(tX) : \Omega^*(M, TM) \to \Omega^*(M, TM) |t| \) is the action induced by (local) diffeomorphisms of \( M \). Thus, Proposition 2.1 now immediately implies the following result.

**Theorem 2.2.** Let \( M \) be a manifold and \( K \in \Omega^{2k+1}(M, TM) \) be such that \( [K, K]_{FN} = 0 \), and define the differential \( d_K(K') := [K, K']_{FN} \). Then

1. \( H^0_K(\Omega^*(M, TM)) \) is the Lie algebra of vector fields stabilizing \( K \).
2. \( H^{2k+1}_K(\Omega^*(M, TM)) \) is the space of infinitesimal deformations of \( K \) within the differentials of \( \Omega^*(M, TM) \) of the form \( \text{ad}_{\xi_{2k+1}} \), modulo \( \text{(local)} \) diffeomorphisms.

### 3. Frölicher-Nijenhuis Cohomology

Suppose that \( (M, g) \) is an \( n \)-dimensional Riemannian manifold with Levi-Civita connection \( \nabla \), and \( \Psi \in \Omega^{2k}(M) \) is a parallel form of even degree. We now make the following simple but crucial observation. The proof is given by a straightforward computation in geodesic normal coordinates.

**Proposition 3.1.** Let \( \hat{\Psi} := \partial_g \Psi \in \Omega^{2k-1}(M, TM) \) with the contraction map \( \partial_g \) from \( (1.1) \). Then \( \hat{\Psi} \) is a Maurer-Cartan element, i.e., \([\hat{\Psi}, \hat{\Psi}]_{FN} = 0\).

Thus, by the discussion in Section 2.1 the Lie derivative \( \mathcal{L}_{\hat{\Psi}} \) and the adjoint map \( \text{ad}_{\hat{\Psi}} \) are differentials on \( \Omega^*(M) \) and \( \Omega^*(M, TM) \), respectively, and for simplicity, we shall denote these by

\[
\mathcal{L}_{\Psi} : \Omega^*(M) \to \Omega^*(M), \quad \text{ad}_{\Psi} : \Omega^*(M, TM) \to \Omega^*(M, TM),
\]

or, if we wish to specify the degree,

\[
\mathcal{L}_{\Psi; l} : \Omega^{l-2k+1}(M) \to \Omega^l(M), \quad \text{ad}_{\Psi; l} : \Omega^{l-2k+1}(M, TM) \to \Omega^l(M, TM).
\]
The cohomology algebras we denote by \( H^*_\Psi(M) \) and \( H^*_\Psi(TM) \) instead of \( H^*_\Psi(\Omega^*(M)) \) and \( H^*_\Psi(\Omega^*(M,TM)) \), respectively. That is, the \( i \)-th cohomologies are defined as

\[
H^i_\Psi(M) := \frac{\ker \mathcal{L}_\Psi : \Omega^i(M) \to \Omega^{i+2k-1}(M)}{\text{Im} \mathcal{L}_\Psi : \Omega^{i-2k+1}(M) \to \Omega^i(M)},
\]

(3.1)

\[
H^i_\Psi(M,TM) := \frac{\ker \text{id} : \Omega^i(M,TM) \to \Omega^{i+2k-1}(M,TM)}{\text{Im} \text{id} : \Omega^{i-2k+1}(M,TM) \to \Omega^i(M,TM)}.
\]

Example 3.2. In the case of a Kähler manifold, using the Kähler form \( \Psi = \omega \), the differential \( L_\omega \) on \( \Omega^*(M) \) is the complex differential \( d^c := i(\bar{\partial} - \partial) \), whereas on \( \Omega^*(M,TM) \), \( \text{id} \) coincides with the Dolbeault differential \( \bar{\partial} : \Omega^{p,q}(M,TM) \to \Omega^{p,q+1}(M,TM) \). Thus, these differentials recover well known and natural cohomology theories. In particular, the cohomology algebras \( H^*_\omega(M) \) and \( H^*_\omega(M,TM) \) are finite dimensional if \( M \) is closed.

Now we give some general strategies to compute \( H^*_\Psi(M) \). First we summarize formulas of \( \mathcal{L}_\Psi \).

**Lemma 3.3** ([KLS2017b, Section 2.4]).

\[
\mathcal{L}_\Psi d\alpha = -d\mathcal{L}_\Psi \alpha, \quad \mathcal{L}_\Psi d^* \alpha = -d^* \mathcal{L}_\Psi \alpha \quad \text{and thus} \quad \mathcal{L}_\Psi \Delta \alpha = \Delta \mathcal{L}_\Psi \alpha.
\]

(3.2)

where \( \Delta \) is the Hodge Laplacian. As in the case of \( d^* \), the formal adjoint \( L^*_\Psi : \Omega^i(M) \to \Omega^{i-2k+1}(M) \) of \( L_\Psi : \Omega^{i-2k+1}(M) \to \Omega^i(M) \) is given by

\[
L^*_\Psi \alpha = (-1)^{n(n-1)} \ast \mathcal{L}_\Psi \ast \alpha.
\]

Recall that for a closed oriented Riemannian manifold \((M, g)\) there is the Hodge decomposition of differential forms

\[
\Omega^i(M) = \mathcal{H}(M) \oplus d\Omega^{i-1}(M) \oplus d^*\Omega^{i+1}(M),
\]

(3.3)

where \( \mathcal{H}(M) \subset \Omega^i(M) \) denotes the space of harmonic forms.

We define the space of \( \mathcal{L}_\Psi \)-harmonic forms as

\[
\mathcal{H}_\Psi^i(M) := \{ \alpha \in \Omega^i(M) \mid \mathcal{L}_\Psi \alpha = \mathcal{L}_\Psi \ast \alpha = 0 \}
\]

(3.4)

Evidently, the Hodge-\( \ast \) yields an isomorphism

\[
\ast : \mathcal{H}_\Psi^i(M) \longrightarrow \mathcal{H}_\Psi^{n-i}(M).
\]

(3.5)

Since \( \mathcal{H}_\Psi^i(M) \subset \ker \mathcal{L}_\Psi \) and \( \mathcal{H}_\Psi^i(M) \cap \text{Im} (\mathcal{L}_\Psi) = 0 \), there is a canonical injection

\[
\iota : \mathcal{H}_\Psi^i(M) \hookrightarrow H^i_\Psi(M).
\]

(3.6)

This is analogous to the inclusion of harmonic forms into the de Rham cohomology of a manifold, which for a closed manifold is an isomorphism.
due to the Hodge decomposition \(3.3\). Therefore, one may hope that the
maps \(\eta\) are isomorphisms as well. It is not clear if this is always true, but
we shall give conditions which assure this to be the case and show that in
the applications we have in mind, this condition is satisfied.

**Definition 3.4.** We say that the differential \(\mathcal{L}_\Psi\) is \(l\)-regular for \(l \in \mathbb{N}\) if
there is a direct sum decomposition

\[
\Omega^l(M) = \ker(\mathcal{L}_{\Psi,l}) \oplus \text{Im}(\mathcal{L}_{\Psi,l}).
\]

A standard result from elliptic theory states that \(\mathcal{L}_\Psi\) is \(l\)-regular if the
differential operator \(\mathcal{L}_{\Psi,l} : \Omega^{l-2k+1}(M) \to \Omega^l(M)\) is elliptic, overdetermined
elliptic or underdetermined elliptic, see e.g. [Besse1987, p.464, 32 Corollary].

The following theorem now relates the cohomology \(H^l(M)\) to the \(\mathcal{L}_\Psi\)-
harmonic forms \(H^l_{\Psi}(M)\).

**Theorem 3.5** ([KLS2017b, Theorem 2.7]).

1. If \(\mathcal{L}_\Psi\) is \(l\)-regular, then
   the map \(\eta\) from \((3.6)\) is an isomorphism.
2. There are direct sum decompositions
   \[
   \begin{align*}
   H^l_{\Psi}(M) &= \mathcal{H}^l(M) \oplus H^l_{\Psi}(M)_d \oplus H^l_{\Psi}(M)_{d^*}, \\
   \mathcal{H}^l_{\Psi}(M) &= \mathcal{H}^l(M) \oplus \mathcal{H}^l_{\Psi}(M)_d \oplus \mathcal{H}^l_{\Psi}(M)_{d^*},
   \end{align*}
   \]
   where \(\mathcal{H}^l(M)\) is the space of harmonic \(l\)-forms on \(M\), \(H^l_{\Psi}(M)_d\) and
   \(H^l_{\Psi}(M)_{d^*}\) are the cohomologies of \((d\Omega^*(M), \mathcal{L}_{\Psi})\) and \((d^*\Omega^*(M), \mathcal{L}_{\Psi})\),
   respectively, and where \(\mathcal{H}^l_{\Psi}(M)_d := \mathcal{H}^l_{\Psi}(M) \cap d\Omega^{l-1}(M), \mathcal{H}^l_{\Psi}(M)_{d^*} := \mathcal{H}^l_{\Psi}(M) \cap d^*\Omega^{l+1}(M)\). Moreover, the injective map \(\eta\) from \((3.6)\)
   preserves this decomposition, i.e.,

   \[
   \eta_1 : \mathcal{H}^l_{\Psi}(M)_d \hookrightarrow H^l_{\Psi}(M)_d \quad \text{and} \quad \eta_1 : \mathcal{H}^l_{\Psi}(M)_{d^*} \hookrightarrow H^l_{\Psi}(M)_{d^*}.
   \]
3. There are isomorphisms
   \[
   \begin{align*}
   d : H^l_{\Psi}(M)_{d^*} &\to H^{l+1}_{\Psi}(M)_d \quad \text{and} \quad d^* : H^l_{\Psi}(M)_d \to H^{l-1}_{\Psi}(M)_{d^*}, \\
   d : \mathcal{H}^l_{\Psi}(M)_{d^*} &\to \mathcal{H}^{l+1}_{\Psi}(M)_d \quad \text{and} \quad d^* : \mathcal{H}^l_{\Psi}(M)_d \to \mathcal{H}^{l-1}_{\Psi}(M)_{d^*}.
   \end{align*}
   \]
4. If \(\mathcal{L}_{\Psi}\) is \((l+1)\)-regular and \((l-1)\)-regular, then it is also \(l\)-regular.

Next, we consider another important case. We call a form \(\Psi \in \Omega^k(M)\)
multi-symplectic, if \(d\Psi = 0\) and for all \(v \in TM\)

\[
(3.12) \quad \eta_1 \Psi = 0 \iff v = 0.
\]

**Lemma 3.6.** If \(\Psi \in \Omega^{2k}(M)\) is multi-symplectic, then the differential operator
\(\mathcal{L}_{\Psi,l} : \Omega^{l-2k+1}(M) \to \Omega^l(M)\) is overdetermined elliptic for \(l = 2k - 1\)
and underdetermined elliptic for \(l = n\).

We also can make some statement for \(H^l_{\Psi}(M)\) for special values of \(l\).
Proposition 3.7. For a parallel form $\Psi \in \Omega^{2k}(M)$, we have

$$H^0_{\Psi}(\Omega^*(M)) \cong \mathcal{H}_{\Psi}^0(M) = \{ f \in C^\infty(M) \mid \iota_d^# \Psi = 0 \},$$

If $\Psi$ is multi-symplectic, then $\mathcal{H}_{\Psi}^0(M) = \mathcal{H}^0(M)$ and $H^0_{\Psi}(\Omega^*(M)) \cong H^0(M) = H^0(M)$.

Indeed, it can be shown that $H^0_{\Psi}(\Omega^*(M))$ is infinite dimensional if $\Psi$ is not multi-symplectic.

Proposition 3.8. Let $\Psi \in \Omega^{2k}(M)$ be a parallel multi-symplectic form. Then

$$H^{2k-1}_{\Psi}(\Omega^*(M)) = \mathcal{H}_{\Psi}^{2k-1}(M),$$

$$\ker(\mathcal{L}_{\Psi:2k}) = \{ \alpha \in \Omega^1(M) \mid \mathcal{L}_{\alpha#}(\ast \Psi) = 0 \text{ and } d\ast \alpha = 0 \}. $$

In particular, if $k \geq 2$ then $\ker(\mathcal{L}_{\Psi:2k}) = \mathcal{H}_{\Psi}^1(M) \cong H^1_{\Psi}(M)$ and

$$H^{n-1}_{\Psi}(M) = \{ \alpha \in \Omega^{n-1}(M) \mid \mathcal{L}_{(\ast \alpha)#}(\ast \Psi) = 0 \text{ and } d\alpha = 0 \}. $$

The first statement is an immediate consequence of Theorem 3.5 and Lemma 3.6. The second and the third statements follow from a direct computation and an integration by parts argument.

4. The Frölicher-Nijenhuis cohomology of manifolds with special holonomy

On a $G_2$-manifold, there is a canonical parallel 4-form $\ast \varphi$, the Hodge dual of the $G_2$-structure $\varphi$. We may consider the differentials $\mathcal{L}_{\ast \varphi}$ and $\text{ad}_{\ast \varphi}$.

On closed manifolds, we obtain the following results on their cohomology groups.

Theorem 4.1. Let $(M^7, \varphi)$ be a closed $G_2$-manifold. Then for the cohomologies $H^i_{\ast \varphi}(M^7)$ and $H^i_{\ast \varphi}(M^7, TM^7)$ defined above, the following hold.

1. There is a Hodge decomposition

$$H^i_{\ast \varphi}(M^7) = \mathcal{H}^i(M^7) \oplus (H^i_{\ast \varphi}(M^7) \cap d\Omega^{i-1}(M^7))$$

$$\oplus (H^i_{\ast \varphi}(M^7) \cap d\Omega^{i+1}(M^7)),$$

where $\mathcal{H}^i(M^7)$ denotes the spaces of harmonic forms.

2. The Hodge-$\ast$ induces an isomorphism $\ast : H^i_{\ast \varphi}(M^7) \to H^{7-i}_{\ast \varphi}(M^7)$.

3. $H^i_{\ast \varphi}(M^7) = \mathcal{H}^i(M^7)$ for $i = 0, 1, 6, 7$. For $i = 2, 3, 4, 5$, $H^i_{\ast \varphi}(M^7)$ is infinite dimensional.

4. $\dim H^0_{\ast \varphi}(M^7, TM^7) = b^1(M^7)$; in particular, $H^0_{\ast \varphi}(M^7) = 0$ if $M^7$ has full holonomy $G_2$.

5. $\dim H^2_{\ast \varphi}(M^7, TM^7) \geq b^3(M^7) > 0$, as it contains all torsion free deformations of the $G_2$-structure modulo deformations by diffeomorphisms.

In [KLS2017], Theorem 3.5, we give a precise description of the cohomology ring $H^*_{\ast \varphi}(M^7)$.
Remark 4.2. On a Spin(7)-manifold, there is also a canonical parallel 4-form and we obtain the similar results. For more details, see [KLS2017b, Theorem 4.2].

Recently, we also computed $H^l_{\phi}(M)$ for the real part of a holomorphic volume form $\Psi$ in 4$n$-dimensional Calabi-Yau manifolds in [KLS2018]. When $n = 1$, it is isomorphic to the de Rham cohomology. When $n \geq 2$, as in the $G_2$ and Spin(7)-case, it is regular again, and all summands involved other than the harmonic forms are infinite dimensional.

Outline of the proof of Theorem 4.1. We begin by showing the $l$-regularity of $L_{\psi,\phi}$. For $l < 3$ and $l > 7$, this is obvious as then $L_{\psi,\phi} = 0$. By Lemma 3.6, $L_{\psi,\phi}$ is overdetermined elliptic for $l = 3$ and underdetermined elliptic for $l = 7$, whence $L_{\psi,\phi}$ is also 3- and 7-regular.

By a simple calculation, it follows that $L_{\psi,\phi}$ is overdetermined elliptic for $l = 4$ and underdetermined elliptic for $l = 6$, whence $L_{\psi,\phi}$ is 4-regular and 6-regular. Thus it is also 5-regular by Theorem 3.5(3). Therefore, the $l$-regularity of $L_{\psi,\phi}$ for all $l$ is established, whence by Theorem 3.5(1), $H^l_{\psi}(M) = \mathcal{H}^l_{\psi}(M)$.

For $l = 0, 7$, $H^l_{\psi}(M) \cong \mathcal{H}^l(M)$ by Proposition 3.7.

For $l = 1$, $H^1_{\psi}(M) = \ker L_{\psi,\phi}|_{\Omega^1(M)}$. Thus, by Proposition 3.8, $\alpha \in H^1_{\psi}(M)$ implies that $L_{\alpha,\phi}(\psi) = 0$, which in turn implies that $\alpha^\#$ is a Killing vector field. Since a $G_2$-manifold is Ricci flat, it follows by Bochner’s theorem that $\alpha^\#$ is parallel, whence so is $\alpha$. In particular, $\alpha$ is harmonic, showing that $H^1_{\psi}(M) = \mathcal{H}^1(M)$. For $l = 6$, we have $H^6_{\psi}(M) = *H^1_{\psi}(M) = *\mathcal{H}^1(M) = \mathcal{H}^6(M)$. This shows that $H^l_{\psi}(M) \cong \mathcal{H}^l(M)$ for $l = 1, 6$.

Next, for $l = 2$, we have $H^2_{\psi}(M)_d = 0$ by (3.10). Thus, we need to determine $\mathcal{H}^2_{\psi}(M)_d = \{ \alpha^2 \in d^\ast \Omega^2(M) | d^\ast (\alpha^2 \wedge * \psi) = 0 \}$.

We can investigate this space in detail by the irreducible decomposition of $\Omega^2(M)$ under the $G_2$-action and the Hodge decomposition. Then we can prove that $\mathcal{H}^2_{\psi}(M)_d$ is isomorphic to an infinite dimensional function space. We can prove the case of $l = 3$ similarly.

Again, since $* : \mathcal{H}^2_{\psi}(M) \rightarrow \mathcal{H}^{7-l}_\psi(M)$ is an isomorphism, the assertions for $l = 4, 5$ follow.

Next, we consider $H^l_{\psi}(M, TM^7)$. First, note the following.

Lemma 4.3. Let $V$ be an oriented 7-dimensional vector space, and let $\Lambda^3 G_2^* \subset \Lambda^3 V^*$ be the set of $G_2$-structures on $V$. By definition, the group $GL^+(V)$ of orientation preserving automorphisms of $V$ act transitively on $\Lambda^3 G_2^*$ so that $\Lambda^3 G_2^* = GL^+(V)/G_2$. Then the map

$$
\mathcal{C} : \Lambda^3 G_2^* \rightarrow \Lambda^3 V^* \otimes V, \quad \varphi \mapsto \partial_{g_\varphi}(*_{g_\varphi} \varphi)
$$

is a $GL^+(V)$-equivariant injective immersion. Here, $\partial_g$ is the map from (1.4), and $g_\varphi$ denotes the metric induced by $\varphi$.
Proposition 4.4.

\( H^0_{\varphi}(M^7, TM^7) = \{ X \in \mathfrak{X}(M^7) \mid \mathcal{L}_X \varphi = 0 \} = \{ X \in \mathfrak{X}(M^7) \mid \nabla X = 0 \}. \)

This proposition implies the 4th part of Theorem 4.1.

Proof. Let \( X \in \mathfrak{X}(M^7) \) be a vector field, \( p \in M^7 \) and denote by \( F^t_X \) the local flow along \( X \), defined in a neighborhood of \( p \). Then because of the pointwise equivariance of \( \mathcal{C} \) we have

\[
(F^t_X)^* \left( \partial_g * \varphi \right)_{F^t_X(p)} = (F^t_X)^* \left( \mathcal{C}(\varphi)_{F^t_X(p)} \right) = \mathcal{C} \left( (F^t_X)^* (\varphi_{F^t_X(p)}) \right)
\]

and taking the derivative at \( t = 0 \) yields

\[
(4.1) \quad \mathcal{L}_X (\partial_g * \varphi)_p = \mathcal{L}_X (\mathcal{C}(\varphi))_p = \mathcal{C}_* (\mathcal{L}_X \varphi)_p.
\]

Now \( \mathcal{L}_X (\partial_g * \varphi) = [X, \partial_g * \varphi]^{FN} \), and since \( \mathcal{C} \) is an immersion by Lemma 4.3, it follows that \( X \in H^0_{\varphi}(M^7, TM^7) = \ker ad_{\partial_g * \varphi} \) iff \( \mathcal{L}_X \varphi = 0 \).

Since \( \varphi \) uniquely determines the Riemannian metric \( g_p \) on \( M^7 \), any vector field satisfying \( \mathcal{L}_X \varphi = 0 \) must be a Killing vector field. Since \( M^7 \) is closed, the Ricci flatness of \( G_2 \)-manifolds and Bochner’s theorem imply that \( X \) must be parallel, showing that in this case, \( \dim H^0_{\varphi}(M^7, TM^7) = b^1(M^7) \). \( \square \)

It was shown in [KLS2017, Theorem 1.1] that a \( G_2 \)-structure \( \varphi' \) is torsion-free if and only if \( [\chi_{\varphi'}, \chi_{\varphi'}]^{FN} = 0 \), where \( \chi_{\varphi'} := \mathcal{C}(\varphi') = \partial_{g_{\varphi'}} * g_{\varphi'}, \varphi' \in \Omega^3(M^7, TM^7) \). Therefore, for a family of torsion-free \( G_2 \)-structures \( \{ \varphi_t \} \) with \( \varphi_0 = \varphi \), we have

\[
0 = \frac{d}{dt} \bigg|_{t=0} [\chi_{\varphi_t}, \chi_{\varphi_t}] = 2 \left[ \chi_{\varphi_0}, \frac{d}{dt} \bigg|_{t=0} \chi_{\varphi_t} \right] = 2 \left[ \chi_{\varphi_0}, \mathcal{C}_* (\varphi_0) \right],
\]

so that \( \varphi_0 \in \Omega^3(M^7) \) is a torsion free infinitesimal deformation of \( \varphi_0 \) iff \( \mathcal{C}_* (\varphi_0) \in \ker (ad_{\chi_{\varphi_0}} : \Omega^3(M^7, TM^7) \to \Omega^0(M^7, TM^7)) \). Since \( \mathcal{C} \) is an immersion and hence \( \mathcal{C}_* \) injective by Lemma 4.3, we have an isomorphism

\[
\{ \text{torsion free infinitesimal deformations of } \varphi_0 \} \cong \ker \left( ad_{\chi_{\varphi_0}} : \Omega^3(M^7, TM^7) \to \Omega^0(M^7, TM^7) \right) \cap \text{Im } (\mathcal{C}_*).
\]

Observe that by (4.1)

\[
\mathcal{C}_* (\mathcal{L}_X \varphi_0) = \mathcal{L}_X (\mathcal{C}(\varphi_0)) = [X, \chi_{\varphi_0}]^{FN} = -ad_{\chi_{\varphi_0}}(X),
\]

whence there is an induced inclusion

\[
\{ \text{torsion free infinitesimal deformations of } \varphi_0 \} \quad \mathcal{C}_* \to H^0_{\varphi_0}(M^7, TM^7). \]

This implies the 5th part of Theorem 4.1. \( \square \)
5. Strongly homotopy Lie algebra associated with associative submanifolds

In this section we assign to each associative submanifold in a $G_2$-manifold an $L_\infty$-algebra, using the Frölicher-Nijenhuis bracket and Voronov’s derived bracket construction of $L_\infty$-algebras. The main purpose of this section is to explain the motivation that led us to study the Frölicher-Nijenhuis bracket on $G_2$-manifolds. We refer the reader to [FLSV2018] for detailed and general treatment of the theory discussed here.

5.1. Voronov’s construction of $L_\infty$-algebras. Strongly homotopy Lie algebras, also called $L_\infty$-algebras, were defined by Lada and Stasheff in [LS1993], see also [Voronov2005] for a historical account. In [Voronov2005] Voronov suggested a relatively simple method to construct an $L_\infty$-algebra based on algebraic data, now called V-data. A V-data is a quadruple $(L, P, a, \triangle)$, where

(1) $L$ is a $\mathbb{Z}_2$-graded Lie algebra $L = L_0 \oplus L_1$ (we denote its bracket by $[\cdot, \cdot]$),
(2) $a$ - an abelian Lie subalgebra of $L$,
(3) $P : L \to a$ is a projection whose kernel is a Lie subalgebra of $L$,
(4) $\triangle \in (\ker P) \cap L_1$ is an element such that $[\triangle, \triangle] = 0$.

When $\triangle$ is an arbitrary element of $L_1$ instead of $\ker(P) \cap L_1$, we refer to $(L, a, P, \triangle)$ as a curved V-data.

Recall that a $(k, l)$-shuffle is a permutation of indices $1, 2, \cdots, k + l$ such that $\sigma(1) \prec \cdots \prec \sigma(k)$ and $\sigma(k + 1) \prec \cdots \prec \sigma(k + l)$.

Definition 5.1 ([Voronov2005, Definition 1]). A vector space $V = V_0 \oplus V_1$ endowed with a sequence of odd $n$-linear operations $m_n$, $n = 0, 1, 2, 3, \cdots$, is a strongly homotopy Lie algebra or $L_\infty$-algebra if: (a) all operations are symmetric in the $\mathbb{Z}_2$-graded sense:

$$m_n(a_1, \cdots, a_i, a_{i+1}, \cdots, a_n) = (-1)^{\bar{a}_i \bar{a}_{i+1}} m_n(a_1, \cdots, a_{i+1}, a_i, \cdots, a_n),$$

and (b) the “generalized Jacobi identities”

$$\sum_{k+l=n \ (k, l)-shuffles} (-1)^{\bar{a}} m_{k+1}(m_k(a_{\sigma(1)}, \cdots, a_{\sigma(k)}), a_{\sigma(k+1)}, \cdots, a_{\sigma(k+l)}) = 0$$

hold for all $n = 0, 1, 2, \cdots$. Here $\bar{a}$ is the degree of $a \in V$ and $(-1)^{\alpha}$ is the sign prescribed by the sign rule for a permutation of homogeneous elements $a_1, \cdots, a_n \in V$.

Henceforth symmetric will mean $\mathbb{Z}_2$-graded symmetric.

A 0-ary bracket is just a distinguished element $\Phi$ in $V$. We call the $L_\infty$-algebras with $\Phi = 0$ strict. In this case $m_2^2 = 0$ and we also write $d$ instead of $m_1$. For strict $L_\infty$-algebras, the first three “generalized Jacobi identities” simplify to

$$d^2 a = 0,$$
$$dm_2(a, b) + m_2(da, b) + (-1)^{\bar{a} \bar{b}} m_2(db, a) = 0,$$
Lemma 5.4. The image of the map \( \text{Fr"olicher-Nijenhuis bracket} \)
graded Lie algebra \( \Omega_{\ast} \Omega_{\ast} T_{NL} \) via the decomposition
We define a linear embedding
Usually in the literature a strict
Remark 5.3. Usually in the literature a strict \( L_{\infty} \)-algebra is called an \( L_{\infty} \)-
algebra, which we also adopt in this paper.

5.2. \( L_{\infty} \)-algebra associated with an associative submanifold \( L \subset M^7 \). Let \( L \) be a closed submanifold in a Riemannian \( M \). There exists an open neighborhood \( N_{\epsilon} L \) of the zero section \( L \) in the normal bundle \( NL \) such that the exponential mapping \( \text{Exp} : N_{\epsilon} L \to M \) is a local diffeomorphism.

Given such an open neighborhood \( N_{\epsilon} L \), we consider the pullback operator \( \text{Exp}^* : \Omega^* (M, TM) \to \Omega^* (N_{\epsilon} L, TNL) \)
\[
\text{Exp}^* (K)_x (X_1, \ldots, X_k) = (\text{Exp}_x)^{-1} ((K_{\text{Exp}(x)}) ((\text{Exp}_x)_x(X_1), \ldots, (\text{Exp}_x)_x(X_k)))
\]
for \( x \in N_{\epsilon} L, X_i \in T_x N_{\epsilon} L \) and \( K \in \Omega^k (M, TM) \) given by [KMS1993] 8.16, p. 74. It is known that [KMS1993] Theorem 8.16 (2), p. 74
\[
(5.1) \quad [\text{Exp}^* (K), \text{Exp}^* (L)]_{FN} = \text{Exp}^* ([K, L]_{FN}).
\]
Let \( \pi \) denote the projection \( NL \to L \). For each section \( X \in \Gamma(NL) \), define the vector field \( \hat{X} \) on \( N_{\epsilon} L \subset NL \) by the restriction of the vertical lift of \( X \) to \( N_{\epsilon} L \). That is,
\[
\hat{X} (y) = \frac{d}{dt} (y + tX(\pi(y))) \bigg|_{t=0} \quad \text{for } y \in N_{\epsilon} L.
\]
Let \( \Omega^* (N_{\epsilon} L, TNL) \) be the space of all smooth \( TNL \)-valued forms on \( N_{\epsilon} L \). We define a linear embedding \( I : \Omega^* (L, NL) \to \Omega^* (N_{\epsilon} L, TNL) \) by
\[
I(\phi \otimes X) := \pi^* (\phi) \otimes \hat{X}
\]
and extend it linearly on the whole \( \Omega^* (L, NL) \).
Let \( P \) denote the composition of the restriction \( r : \Omega^* (N_{\epsilon} L, TNL) \to \Omega^* (L, TNL) \) and the projection \( P r^N : \Omega^* (L, TNL) \to \Omega^* (L, NL) \) defined via the decomposition \( TNL|_L = NL \oplus TL \). Set
\[
\tilde{P} := I \circ P : \Omega^* (N_{\epsilon} L, TNL) \to \Omega^* (N_{\epsilon} L, TNL).
\]
Lemma 5.4. The image of the map \( \tilde{P} \) is an abelian subalgebra of the \( \mathbb{Z}_2 \)-graded Lie algebra \( \Omega^* (N_{\epsilon} L, TNL), [\cdot, \cdot]_{FN} \). The space \( \ker \tilde{P} \) is closed under the Fr"olicher-Nijenhuis bracket.
Proof. Note that the image of $\tilde{P}$ is equal to the image of $I$. To prove the first assertion of Lemma 5.4 it suffices to prove that

\[(5.2) \quad [\pi^*(\alpha_1) \otimes \hat{X}_1, \pi^*(\alpha_2) \otimes \hat{X}_2]^{FN} = 0\]

for any $X, Y \in \Gamma(NL)$ and any $\alpha_1, \alpha_2 \in \Omega^*(L)$.

Using (2.9) and taking into account the following identities

\[\hat{X}_1 \pi^*(\alpha_2) = 0 = \hat{X}_2 \pi^*(\alpha_1),\]
\[d\pi^*(\alpha_1) = \pi^*(d\alpha_1),\]

for any $\alpha_1, \alpha_2 \in \Omega^*(L)$ and any $X_1, X_2 \in \mathfrak{X}(L)$ we obtain (5.2) immediately.

Now let us prove the second assertion of Lemma 5.4. Since $I$ is injective, we have $\ker \tilde{P} = \ker P$. Moreover, $\ker P$ is generated by the $TNL$-valued differential forms $\alpha \otimes X$ such that $X(x) \in T_x L$ for all $x \in L$. Assume that $\alpha_1 \otimes X_1, \alpha_2 \otimes X_2 \in \ker P$. Using (2.9) and the fact that if $X_1, X_2 \in \mathfrak{X}(N_L L)$ and $(X_1)|_L \in \mathfrak{X}(L), (X_2)|_L \in \mathfrak{X}(L)$ then

\[[X_1, X_2]|_L \in \mathfrak{X}(L),\]

we obtain immediately

\[[\alpha_1 \otimes X_1, \alpha_2 \otimes X_2] \in \ker P = \ker \tilde{P}.\]

This completes the proof of Lemma 5.4. □

**Theorem 5.5.** Assume that $L$ is an associative submanifold of a $G_2$-manifold $(M^7, \varphi)$. There is an $L_\infty$-algebra structure on the space $\Omega^*(L, NL)$ given by the following family of graded multi-linear maps

\[m_k : \Omega^*(L, NL)^k \to \Omega^*(L, NL)\]

\[m_k(\omega_1, \ldots, \omega_k) = P[\cdots[Exp^*(\chi), I(\omega_1)]^{FN}, I(\omega_2)]^{FN} \cdots, I(\omega_k)]^{FN},\]

where $\chi = \partial_\varphi \ast \varphi \in \Omega^3(M, TM)$ with $\partial_\varphi$ from (1.1).

**Proof.** By Proposition 3.1 and using (5.1), we have

\[(5.3) \quad [Exp^*(\chi), Exp^*(\chi)]^{FN} = 0.\]

**Lemma 5.6.** A submanifold $L$ in a $G_2$-manifold $(M^7, \varphi)$ is associative iff

\[Exp^*(\chi) \in \ker \tilde{P}.\]

**Proof.** It is known that a 3-submanifold $L$ in a $G_2$-manifold is associative iff $\chi|_L = 0$ [HIL1982]. Since $\chi(x \wedge y \wedge z)$ is orthogonal to each $x, y, z$, it follows that $L$ is associative, iff $Pr^N(\chi|_L) = 0 \in \Omega^*(L, NL)$, where $Pr^N$ is the orthogonal projection from $TM|_L$ to the normal bundle of $L$. This implies Lemma 5.6 taking into account the injectivity of $I$ and $Exp^*$. □

Lemmas 5.4, 5.6 and the equation (5.3) imply that $(\Omega^*(N_L TNL), I(\Omega^*(L, NL)), \tilde{P}, Exp^*(\chi))$ is a $V$-data. This and Proposition 5.2 yield Theorem 5.5 immediately. □
Lemma 5.7. Let $V_1, \ldots, V_k \in \Gamma(NL) = \Omega^0(L, NL)$. Then
\[ m_k(V_1, \ldots, V_k) = P(\mathcal{L}_{I(-V_1)} \cdots \mathcal{L}_{I(-V_k)}(\text{Exp}^* (\chi))). \]

Proof. Let $V_i \in \Gamma(NL)$. Then $\{I(V_i)\}$ are mutually commuting vector fields on $N_{\epsilon}L$. Using (2.10) and noting that
\[ [\text{Exp}^* (\chi), I(V_i)]^F N = [I(-V_i), \text{Exp}^* (\chi)]^F N \]
we obtain Lemma 5.7 immediately. \qed

We shall denote the map $m_1 : \Omega^*(L, NL) \to \Omega^{*+3}(L, NL)$ by $d_L$. Since $\dim L = 3$, $d_L$ is non-trivial only on $\Omega^0(L, NL)$.

Remark 5.8. Using the formal deformation theories developed in [LO2016] and [LS2014] it is not hard to see that the $L_\infty$-algebra associated to a closed associative submanifold $L$ encodes the formal and smooth associative deformations of $L$. This and further generalization have been considered in [FLSV2018]. A search for an $L_\infty$-algebra associated to each associative submanifold in $G_2$-manifolds led us to discover the role of the Fr"{o}licher-Nijenhuis bracket in geometry of $G_2$-manifolds.

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