Bulk and edge properties of the Chern-Simons Ginzburg-Landau theory for the fractional quantum Hall effect

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ABSTRACT

The Chern-Simons Ginzburg-Landau theory for the fractional Quantum Hall effect is studied in the presence of a confining potential. We review the bulk properties of the model and discuss how the plateau formation emerges without any impurity potential. The effect is related to changes, by accumulation of charge, at the edge when the chemical potential is changed. Fluctuations about the ground state are examined and an expression is found for the velocity of the massless edge mode in terms of the confining potential. The effect of including spin is examined for the case when the system is fully polarized in the bulk. In general a spin texture may appear at the edge, and we examine this effect in the case of a small spin down component. The low frequency edge modes are examined and a third order equation is found for velocities which indicates the presence of three different modes. The discussions are illustrated by numerical studies of the ground states, both for the one- and two-component cases.
1 Introduction

Since the experimental discovery of the quantum Hall effect (QHE) \cite{1,2}, which occurs in a two-dimensional electron gas subject to a strong perpendicular magnetic field, several approaches to a theoretical description of this system have been developed. As a useful supplement to existing microscopic descriptions \cite{3}, Zhang, Hansson and Kivelson \cite{4} in 1988 proposed a Chern-Simons Ginzburg-Landau (CSGL) model for the fractional quantum Hall effect (FQHE). This model is based on the concept of “statistical transmutation”, \textit{i.e.} the fact that in two dimensions, fermions can be described as (charged) bosons carrying an odd integer number of (statistical) flux quanta. At the Lagrangian level this is done by adding Chern-Simons fields coupled to the bosons. In this way, electrons in an external magnetic field are described as bosons in a combined external and statistical magnetic field. At special values of the filling fraction the statistical field cancels the external field (in the mean field sense) and the system is described as a gas of bosons feeling no net magnetic field. These bosons “condense” into a homogeneous ground state. In fact, this rather simple effective theory reproduces several of the key features of the FQHE \cite{4}, such as the quantization of the Hall conductance and the existence of (anti)vortex excitations. It is even possible to re-construct Laughlin’s electron wave functions from this model by considering quantum fluctuations about the (mean field) boson ground state \cite{5}.

An interesting aspect of the quantum Hall system is the existence of gapless edge modes \cite{6} which has important consequences for the transport properties of the system \cite{7}. Several authors have studied the CSGL model in the presence of an (infinitely) steep external confining potential and have shown the presence of gapless edge modes \cite{8,9,10} in this model. The description of these modes in the CSGL model has also been related \cite{11} to the chiral Luttinger liquid description of the edge excitations \cite{12,6}.

In this paper we study a number of aspects of the CSGL model, discussing both its ground state and excitations. We focus, in particular, the attention on edge effects and follow up the discussion of previous papers on this subject by considering the density profile and edge excitations for a smooth confining potential. We also examine the effects on the edge of including the electron spin in the description.

A short presentation of the model in its simplest form is given in section 2. It describes the quantum Hall system without spin at Laughlin fillings $\nu = 1/(2m + 1)$. In section 3, we review the ground state properties of the bulk and discuss how the existence of finite-energy vortex excitations leads to incompressibility of the ground state and the existence of quantum Hall plateaux, two key features of the QH system. The edge of the system in the presence of a smooth confining potential is then studied (section 4). In the ground state (with uniform bulk density) one finds a one-parameter family of solutions parametrized by the total charge at the edge \cite{13}. When this charge exceeds an upper critical value, the ground state with uniform bulk density becomes unstable due to formation of anti-vortices. Similarly, there is a lower critical value of the total charge beyond which the bulk tends to rearrange itself by formation of vortices. A numerical study of the ground state within these limits is presented. We further perform a systematic analysis of edge excitations and
show the presence of a massless mode with dispersion of the expected form \( \omega/q = < E > / B \), and with an explicit expression for the averaged electric field \(< E >\).

In section 5 we generalize the model to include spin by introducing a two-component description with a Zeeman interaction. The ground state is completely polarized in the bulk, however, a spin texture at the edge may be energetically favourable if the Zeeman energy is not too large. We study the ground state analytically and numerically; there is a two-parameter family of solutions parametrized by two conserved charges. The numerical analysis provides estimates of the critical Zeeman gap and it shows rotation of the spin vector along the edge for smaller values of the Zeeman energy. (The same effect has been demonstrated in a recent study by Karlhede, Kivelson, Lejnell and Sondhi [13] who used Hartree-Fock techniques and an effective spin model.) We further examine the edge excitations of the spin-polarized system and find an equation for the mode frequencies which indicates the presence of three massless modes at the edge.

Finally, in section 6 we summarize and discuss our results.

2 The model

The spinless system is described by the Lagrangian \([4]\)

\[
\mathcal{L} = \phi^*(i\partial_t - a_0 - eA_0 - V(x) + \mu)\phi - \kappa |(\nabla - i(a + eA))\phi|^2 - \lambda |\phi|^4 + \frac{1}{4\theta} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda. (1)
\]

The Bose field \(\phi\) is coupled both to an external field \(A = (0, Bx)\) with \(B < 0\), and to the statistical field \(a\). \(\mu\) is a chemical potential, \(e = -|e|\) denotes the electron charge, \(\theta = (2m + 1)\pi\) for the \(\nu = 1/(2m + 1)\) state and the term \(\lambda|\phi|^4\) models the Coulomb interaction which is present in the underlying microscopic theory. We choose \(A_0 = 0\). With the Bose field decomposed in terms of density and phase,

\[
\phi = \sqrt{\rho} e^{i\theta},
\]

the field equations are

\[
\kappa \nabla^2 \sqrt{\rho} = \left[ a_0 + V - \mu + \partial_t \theta + \kappa (A - \nabla \theta)^2 + 2\lambda \rho \right] \sqrt{\rho} = \partial_t \rho + \nabla \cdot j = 0
\]

and

\[
\nabla \times A = -2\theta (\rho - \bar{\rho}) \\
\partial_x a_0 + \partial_y a_x = -2\theta j_y \\
\partial_y a_0 + \partial_x a_y = 2\theta j_x,
\]

where \(A \equiv a + eA\), \(\bar{\rho} \equiv eB/2\theta\), and the current density is

\[
j = -2\kappa \rho (A - \nabla \theta).
\]
The Hamiltonian is
\[ H = \int d^2x \left[ \kappa |D\phi|^2 + V|\phi|^2 + \lambda |\phi|^4 - \mu |\phi|^2 \right], \] (6)
where \( D = \nabla - iA \) and the constraint (3) is implicit.

In some parts of the paper it will be convenient to work with dimensionless quantities. Rescaling the fields in (1) such that length is measured in units of the magnetic length \( l_B = 1/\sqrt{eB} \), chemical potential in units of the cyclotron energy \( \omega_c = 2e\kappa B \), time in units of the inverse cyclotron frequency and density in units of its constant bulk value,

\[ \tilde{a}_0 = a_0/\omega_c, \quad \tilde{\mu} = \mu/\omega_c \]
\[ \tilde{\partial}_i = \partial_i/\omega_c \]
\[ \tilde{\phi} = \phi l_B, \quad \tilde{\phi}_i = l_B a_i \]
\[ \tilde{\rho} = \rho/\bar{\rho}, \]

one finds that the Lagrangian (rescaled by an overall factor \( 2\theta/2\kappa e^2 B^2 \)) becomes
\[ \mathcal{L} = \tilde{\phi}^* \left( i\tilde{\partial}_i - \tilde{a}_0 - \tilde{V} + \tilde{\mu} \right) \tilde{\phi} - \frac{1}{2} \left| (\tilde{\nabla} - i\tilde{A}) \tilde{\phi} \right|^2 - \tilde{\lambda} |\tilde{\phi}|^4 + \frac{1}{2} \epsilon^{\mu\nu\lambda} \tilde{a}_\mu \tilde{\partial}_\nu \tilde{a}_\lambda \] (7)
with
\[ \tilde{\lambda} \equiv \frac{\lambda}{4\kappa \theta}. \] (8)

Thus, the theory contains only one free, dimensionless parameter (in addition to the chemical potential) \( \theta \). In particular it is worth noting that the statistics parameter \( \theta \) is absorbed in the rescaled interaction parameter \( \tilde{\lambda} \).

3 Bulk properties

We start by reviewing the main features of the CSGL model in the absence of an external potential, \( V(x) = 0 \). It is known for the FQHE that at special fillings, \( \nu = 1/(2m+1) \), the ground state is characterized by a uniform electron density, \( \bar{\rho} = \nu eB/2\pi \). As is well known \( \mathbb{I} \), the CSGL model has this constant solution when the chemical potential takes a special value \( \mu_0 \),

\[ \rho = \bar{\rho} = \frac{\mu_0}{2\lambda} = \frac{eB}{2\theta} \]
(9)
\[ a = -eA \] (10)
\[ a_0 = 0. \] (11)

This solution simultaneously minimizes the kinetic term (\( D_i\phi = 0 \)) and the potential \( V(\phi) = \mu |\phi|^2 - \lambda |\phi|^4 \).
However, there exist other static solutions with the same constant density $\bar{\rho}$ away from the potential minimum. These solutions have $a_0 \neq 0$ and $\mu \neq \mu_0$ with $\mu - a_0 = \mu_0$. This is because the chemical potential enters the equations of motion only in the combination $\mu - a_0$; thus, a shift in the chemical potential can always be absorbed as a constant shift in $a_0$. In this sense, the solutions of the equations of motion are independent of the value of $\mu$. However, the energy does depend on $\mu$ as can be seen from Eq. (6). This observation is useful for understanding how the QH plateaux emerge in this model, as we will discuss below (see also Curnoe and Weiss [15]).

The low energy charged excitations of the QH system are the localized quasiparticles and quasiholes. These are believed to be formed when the density is changed relative to the special filling $\rho = \bar{\rho}$, e.g., by changing the strength of the magnetic field $B$. In the CSGL model these excitations are represented by vortex and anti-vortex solutions, whose existence has been shown in [4, 5] and their properties examined by several authors [16, 14, 15]. (We refer to the solutions with charge deficit as the vortex solutions and the solutions with excess charge as the anti-vortex solutions.) A general vortex solution is characterized by a topological quantum number $s$, which is $s = \pm 1$ for the elementary vortex/anti-vortex solutions. The corresponding charges are $\pm \nu e$, and the energies (relative to the ground state) we denote by $\tilde{e}^v$ for the vortex and $\tilde{e}^{av}$ for the anti-vortex.

For given $s$ and $\tilde{\mu}$, the dimensionless vortex/anti-vortex energies are functions of $\tilde{\lambda}$ only. These functions were studied numerically by Tafelmayer in Ref. [14], for $s = \pm 1, \pm 2$ and $\tilde{\mu} = \tilde{\mu}_0$. As demonstrated by these functions there are two separate regions of $\tilde{\lambda}$ where the vortex solutions show qualitatively different behaviour. The regions are separated by the special point $\tilde{\lambda} = 1/2$, referred to as the self-dual point. At this value of $\tilde{\lambda}$ the vortices are solutions of a linear (self-dual) differential equation, and the vortex energy is exactly proportional to the vorticity. Thus, the vortices are non-interacting. When $\tilde{\lambda} > 1/2$, the energy of $n$ vortices with topological charge 1 is lower than that of one vortex with vorticity $n$ (as demonstrated in Ref. [14] for $n = 2$). This means that a stable multi-vortex configuration with $s = 1$ can exist, since the vortices repel each other. For $\tilde{\lambda} < 1/2$ the situation is opposite, so vortices attract each other; $n$ vortices with $s = 1$ would tend to combine to one large vortex. We may then identify the interval $\tilde{\lambda} \geq 1/2$ as physically relevant for the FQH effect, since only in this interval a stable density of vortices will be formed when the filling fraction is decreased relative to the plateau value $1/(2m + 1)$. For anti-vortices there is no self-dual point. They have a repulsive interaction for all values of $\tilde{\lambda}$.

The existence of finite-energy vortex/anti-vortex solutions together with the freedom to vary the chemical potential for a given solution leads to the incompressibility of the ground state and the existence of QH plateaux within the CSGL model. We argued above that a given solution of the equations of motion is independent of the value of $\mu$, whereas the energy of the solution does depend on $\mu$. The energy (3) can be rewritten in a dimensionless form as

$$\tilde{E}(\tilde{\mu}) = \tilde{E}_0 - \tilde{\mu}\tilde{Q}, \quad (12)$$
where \( \tilde{Q} = 2\theta Q = 2\theta \int d^2 x \: \rho \), \( \tilde{\mu} = \mu/2e\kappa B \), \( \tilde{E} = 2\theta E/2e\kappa B \), and \( \tilde{E}_0 \) is independent of \( \tilde{\mu} \). (The occurrence of the additional (dimensionless) factor \( 2\theta \) in the rescaling of the total energy and charge is due to our choice to include this factor in the rescaling of the density.) For \( \tilde{\mu} = \tilde{\mu}_0 \) the energy is minimized by the constant solution \( \tilde{\rho} = 1 \), and the vortex- and anti-vortex solutions have positive energies, \( \tilde{\epsilon}^v(\tilde{\mu}_0) \) and \( \tilde{\epsilon}^{av}(\tilde{\mu}_0) \) respectively, relative to the ground state. When the chemical potential is increased by \( \delta \tilde{\mu} \), the energies of all the solutions will decrease by \( \delta \tilde{\mu} \) times their respective charge. Since the anti-vortex configuration has an excess charge as compared to \( \tilde{\rho} = 1 \), its energy will be lower than that of the uniform solution when \( \tilde{\mu} \) exceeds a critical value

\[
\tilde{\mu}^+ = \tilde{\mu}_0 + \tilde{\epsilon}^{av}(\tilde{\mu}_0)/2\pi
\]

with \( 2\pi \) as the (rescaled) charge of the anti-vortex. Thus, for \( \tilde{\mu} \gtrsim \tilde{\mu}^+ \) the uniform ground state becomes unstable with respect to anti-vortex formation. Similarly, a lower critical value exists,

\[
\tilde{\mu}^- = \tilde{\mu}_0 - \tilde{\epsilon}^v(\tilde{\mu}_0)/2\pi,
\]

and if this is exceeded the uniform ground state becomes unstable with respect to vortex formation. Thus, there is a total “window” in the (dimensionless) chemical potential,

\[
\Delta \tilde{\mu} = (\tilde{\epsilon}^v(\tilde{\mu}_0) + \tilde{\epsilon}^{av}(\tilde{\mu}_0))/2\pi
\]

within which the ground state is given by the uniform density solution.

Since \( \tilde{\mu} = \mu/2e\kappa B \), a variation in \( \tilde{\mu} \) can be interpreted either as a variation in \( \mu \) at fixed \( B \),

\[
\delta \tilde{\mu} = \frac{1}{2e\kappa B} \delta \mu
\]

or as a variation in \( B \) at fixed \( \mu \),

\[
\delta \tilde{\mu} = -\frac{\mu}{2e\kappa B^2} \delta B,
\]

corresponding to the following interpretations (returning to dimensionful quantities):

1) The ground state density at fixed \( B \) remains unchanged within a finite interval

\[
\Delta \mu = (\epsilon^v(\mu_0) + \epsilon^{av}(\mu_0))/\nu
\]

which implies the incompressibility condition

\[
\left( \frac{\partial \rho}{\partial \mu} \right)_B = 0
\]
near \( \mu = \mu_0 \).

2) The density remains “locked” to the magnetic field through the condition \( b + eB = 0 \) when \( B \) is varied within a finite interval

\[
\Delta B = \left( \epsilon_v(\mu_0) + \epsilon_{av}(\mu_0) \right) \frac{\pi}{e\nu^2\lambda} = \left( \tilde{\epsilon}_v(\tilde{\mu}_0) + \tilde{\epsilon}_{av}(\tilde{\mu}_0) \right) \frac{B}{4\pi \tilde{\lambda}}. \tag{20}
\]

Since the Hall conductivity is proportional to \( \rho/B \) this implies that \( \sigma_H \) is constant within the interval \( \Delta B \), giving rise to the Hall plateaux.

So, at least qualitatively, the simple mean field theory reproduces the right picture as far as the plateaux are concerned. This is the case even without an impurity potential, since this is a model with a fixed chemical potential rather than fixed charge density. Thus, on a plateau the charge density changes with the magnetic field until it is energetically favoured to create vortices or anti-vortices in order to re-adjust the mean charge density. At this point the Hall conductivity leaves the plateau.

When the strength of \( \tilde{\lambda} \) is increased, the width of the plateaux will decrease. This follows from the expression (20) and the fact that the dimensionless vortex-and anti-vortex energies divided by \( \tilde{\lambda} \) decrease with \( \tilde{\lambda} \) (see Ref. [14]). This is in accordance with the expectation that an increase in the (Coulomb) interaction of the particles will tend to suppress variations in the charge density. If an impurity potential is added we expect a broadening of the plateaux beyond the point where vortices are formed, since the vortices will bind to the impurities and therefore not affect the Hall conductance. This gives the connection to the standard picture of the fractional quantum Hall effect.

4 The edge

In the presence of a confining potential \( V(x) \), corresponding to an external electric field \( eE = -\nabla V \), the size of the system becomes finite, with a density profile at the edge that depends on the form of \( V \). We choose here a geometry such that \( V \) is a function of \( x \) only and thus translationally invariant in the \( y \)-direction. This means that the edge is parallel to the \( y \)-axis. We further choose \( x = 0 \) to be an interior line in the bulk (far away from the edge), and assume the system to be symmetric about this line. Due to this symmetry it is sufficient to consider the system for \( x > 0 \). We first examine the ground state of the system and then, in Sec. 4.2, discuss the gapless edge modes obtained by perturbing the ground state. Static solutions have previously been obtained numerically by Orgad and Levit [11] in the case of an infinitely high step function potential. In this paper, we consider a smooth potential \( V(x) \) which is more relevant for real physical situations. [1]

\[1\] However, the translationally invariant form we assume for the ground state density at the edge may implicitly mean that the steepness of the potential should not be too small. A slowly varying potential may be considered as an \( x \)-dependent chemical potential, and from the discussion of the bulk properties, we in this case expect a broad edge with vortices trapped in the edge region. This corresponds in the FQHE to a composite edge with strips of compressible and incompressible Hall fluids, as discussed in Ref. [14].
4.1 Ground state

We consider ground states which are translationally invariant in the $y$-direction, i.e. they correspond to static solutions of the field equations where all the fields are functions of $x$ only. The field equations then reduce to the form,

\begin{align*}
\kappa \partial_x^2 \phi(x) &= \left(a_0(x) - \mu + V(x) + \kappa A_y(x)^2 + 2\lambda \rho\right) \phi(x) \\
\partial_x A_y(x) &= -2\theta \left(\rho(x) - \bar{\rho}\right) \\
\partial_x a_0(x) &= 4\kappa \theta A_y(x) \rho(x),
\end{align*}

where we have made the gauge choice $\phi = \sqrt{\rho}$, i.e. the wave functions are real. Again, $A_y = eA_y + a_y$ and due to the symmetry of the system (and our gauge choice) $a_x = 0$. As before, $A_x = 0$ and $A_y = Bx$. We further choose $a_y(0) = 0$, and as follows from the discussion in the previous section $a_0(0) = \mu - \mu_0$ in order for the density to take the constant value $\bar{\rho}$ in the bulk.

There is a conserved charge in this problem, $Q = \int dx \rho$, and consequently a one-parameter set of ground states parametrized by this charge \[9\]. This set of ground states can alternatively be parametrized by the chemical potential. Thus, the ground state energy, which has the $\mu$-dependence $E = E_0(Q) - \mu Q$, may initially be considered as depending on two independent parameters $Q$ and $\mu$. However, for the true ground state, which is found by minimizing the energy with respect to $Q$,

\begin{equation}
\frac{\partial E(\mu, Q)}{\partial Q} = 0,
\end{equation}

the chemical potential will be a function of $Q$,

\begin{equation}
\mu = \frac{dE_0}{dQ}.
\end{equation}

From the discussion of the previous section we know that there is no change in the bulk charge for small variations in $\mu$ around $\mu_0$. This means that when the chemical potential is changed within the limits determined by the vortex- and anti-vortex energies, all changes in the ground state take place at the edge. In particular, the additional charge which is introduced by a change in the chemical potential, is confined to the edge.

The energy is stationary with respect to variations in the wave function about the ground state. This is true, due to \[24\], also for variations where the total charge is changed. For an infinitesimal translation $\delta x$ of the edge, with $\delta Q = \bar{\rho} \delta x$ and $\delta \rho(x) = -\rho'(x) \delta x$, one finds

\begin{equation}
\delta E_0 = \delta x \left(\lambda \bar{\rho}^2 - \int dx V(x) \rho'(x)\right)
\end{equation}

which leads to

\begin{equation}
\mu = \frac{\mu_0}{2} + \frac{1}{\bar{\rho}} \int dx \ V'(x) \rho(x).
\end{equation}
The same expression, but with the additional term $a_0(\infty)$ on the right hand side can be derived from the equation of motion (21) by multiplying it with $\phi'(x)$ and integrating over $x$. This leads to the conclusion $a_0(\infty) = 0$. An interesting consequence of this is that the total current in the system is zero when $\mu = \mu_0$: Integrating Eq.(23) over $x$ gives

$$a_0(\infty) - a_0(0) = -2\theta J,$$

(28)

where $J$ is the total current (in the $y$-direction). Since $a_0(0) = \mu - \mu_0$ we have

$$\mu - \mu_0 = 2\theta J.$$

(29)

Thus, the integrated current, which in general will have two contributions of opposite sign at the edge, will have an overall sign which changes at $\mu = \mu_0$.

![Figure 1: Ground state edge profiles for $\tilde{\lambda} = 1$ and $\tilde{V}(x) = \frac{1}{2} \theta(x - 5) \cdot (x - 5)^2$. The curves correspond to (from above) the upper critical value $\mu^+$, $\mu = \mu_0$ and the lower critical value $\mu^-$.](image)

This change in sign of the current has an analogy in the microscopic picture of the real quantum Hall system: There are two contributions to the current. One is the drift current, caused by the drift of the cyclotron orbits in the external electric field. The other is the polarization current, due to the gradient of the density at the edge and with origin in the cyclotron motion of the electrons. These contributions will have opposite sign, and which
one will dominate is determined by the value of the chemical potential or, equivalently, by the charge collected at the edge.

A numerical study of charge and current densities in the ground state is presented in Figs. 1 and 2. In Fig. 1 the edge profile $\phi(x)$ is shown for three different values of the chemical potential in the case of a harmonic confining potential $\tilde{V}(x) = 0.5 \theta(x-5) \cdot (x-5)^2$ where $\theta(x)$ is the step function, and $\lambda = \lambda/(4\kappa \theta) = 1$. The upper curve corresponds to $\mu = 3.5 \omega_c$ which is the upper critical value as estimated from the numerical values in Ref. [14]. Note the excess charge accumulated at the edge. For $\mu$ larger than this value the ground state is expected to be an anti-vortex configuration in the bulk rather than a state with uniform density. The middle curve corresponds to $\mu = \mu_0 = 2 \omega_c$ whereas $\mu = 1.3 \omega_c$, the estimated lower critical value, for the lowest curve. For smaller values of the chemical potential the ground state is believed to reorganize into a density of vortices.

$$\theta(x) \text{ is the step function, and } \lambda = \lambda/(4\kappa \theta) = 1.$$  

Figure 2: Current densities corresponding to the profiles in Fig. 1. The total (static) current is negative at $\mu < \mu_0$, zero at $\mu = \mu_0$ and positive for $\mu > \mu_0$.

The current density profiles corresponding to these three solutions are shown in Fig. 2. Note the change in direction of the total static current: For $\mu = 1.3 \omega_c < \mu_0$ the current is negative; for $\mu = \mu_0$ the two contributions cancel (as anticipated in Eq. (28)), whereas the positive contribution dominates when $\mu > \mu_0$ as in the last plot.
4.2 Edge excitations

The linearized field equations for fluctuations of the fields about the ground state have the form

\[
\kappa \nabla^2 \sqrt{\rho} = \left[ \delta a_0 + \partial_t \delta \theta + 2\kappa A_y (\delta a_y - \partial_y \delta \theta) + 2\lambda \delta \rho \right] \sqrt{\rho} + \left[ a_0 + V - \mu + \kappa A^2 + 2\lambda \rho \right] \delta \sqrt{\rho} \tag{30}
\]

\[
\nabla \times \delta a = -2\theta \delta \rho \tag{31}
\]

\[
\partial_x \delta a_0 + \partial_t \delta a_x = 4\kappa \theta \left[ A_y \delta \rho + \rho (\delta a_y - \partial_y \delta \theta) \right] \tag{32}
\]

\[
\partial_y \delta a_0 + \partial_t \delta a_y = -4\kappa \theta \rho (\delta a_x - \partial_x \delta \theta) \tag{33}
\]

\[
\partial_t \delta \rho = 2\kappa \nabla \cdot \left[ A \delta \rho + \rho (\delta a - \nabla \delta \theta) \right]. \tag{34}
\]

For low-frequency fluctuations propagating along the edge we write, as a simple ansatz, all the fields as an \(x\)-dependent part modulated by a slowly varying \((y, t)\)-dependent harmonic part,

\[
\delta \rho(x, y, t) = \delta \rho(0)(x) e^{i(qy - \omega t)} \\
\delta \theta(x, y, t) = -i\eta(x) e^{i(qy - \omega t)} \\
\delta a_\mu(x, y, t) = \delta a_\mu(0)(x) e^{i(qy - \omega t)}. \tag{35}
\]

The field equations can then be reduced to purely \(x\)-dependent equations with the prescription \(\partial_y \rightarrow iq\) and \(\partial_t \rightarrow -i\omega\). We consider \(q\) as a small parameter and expand all the fields in powers of \(q\), e.g.

\[
\delta \rho(x) = \delta \rho(0)(x) + q \delta \rho(1)(x) + q^2 \delta \rho(2)(x) + \cdots . \tag{36}
\]

Since we are interested in the gapless modes we assume \(\omega\) to be of first order in \(q\). The field equations (30)–(34) are examined order by order in \(q\). As a starting point we assume that to lowest order only the phase fluctuation \(\eta\) contributes. This reduces the 0th order equations to

\[
4\kappa \theta (\rho \partial_x \eta) = \partial_x (\rho \partial_x \eta) = 0 \tag{37}
\]

so \(\eta(0)\) is a constant.

To first order in \(q\), in (32) and (33) only the terms involving \(\rho(\delta a_x - \partial_x \theta) \propto \delta j_x\) survive, simply giving

\[
\delta j_x^{(1)} = 0. \tag{38}
\]

The first three equations (31), (32), on the other hand, take the same form as those for a first order variation in the equations of motion (21)–(23) for the \(Q\)-dependent ground states. Thus, to \(O(q)\) we have the following solution:

\[
\delta \rho^{(1)}(x) = \delta \rho^{(s)}(x) \tag{39}
\]

\[
\delta a_0^{(1)}(x) = \delta a_0^{(s)}(x) - \delta \mu + \frac{\omega}{q} \eta(0) \tag{40}
\]

\[
\delta a_y^{(1)}(x) = \delta a_y^{(s)}(x) + \eta(0) \tag{41}
\]
where \((s)\) refers to variations between stationary fields obtained by a small change \(\delta Q\) in the ground state charge and a corresponding change \(\delta \mu\) in the chemical potential.

For the stationary ground state fields the current conservation is trivially satisfied. This is not the case for the modulated fields \((35)\), where the current conservation in fact determines the dispersion law for the edge waves. We note that to lowest order the current conservation involves the second order contribution to \(j_x\) and can be used to determine this from the first order fields. However, the integrated equation contains only first order fields, due to the boundary condition

\[ j_x(0) = j_x(\infty) = 0 \]  \hspace{1cm} (42)

and has the form

\[ \omega \delta Q = q \delta J. \]  \hspace{1cm} (43)

This equation is in fact correct to all orders in \(q\), with \(\delta Q\) as the fluctuation in the integrated charge and \(\delta J\) as the fluctuation of the total current in the \(y\)-direction. To first order in \(q\) this gives

\[ \frac{\omega}{q} = \frac{dJ}{dQ} = \frac{1}{2\theta} \frac{d\mu}{dQ} \]  \hspace{1cm} (44)

where the derivatives refer to the \(Q\)-dependent ground state. This expression for the dispersion law agrees with the expression found in Ref. \([9]\) in the case of an infinitely steep confining potential. In our case we can further express the dispersion in terms of the potential \(V(x)\). Inserting the expression \((27)\) for \(\mu\) derived in the previous section into \((44)\) gives

\[ \frac{\omega}{q} = \frac{1}{eB} \int_0^\infty dx \, V'(x) \frac{dp}{dQ}. \]  \hspace{1cm} (45)

For a confining potential which is approximately linear over the edge it has the form

\[ \frac{\omega}{q} = -\frac{E}{B}, \]  \hspace{1cm} (46)

with \(eE = -V'\), and more generally it can be read as an averaged form of this equation, with the \(E\)-field averaged over the edge. We note that the velocity of the edge waves is always in the same direction as the drift velocity of the electrons, and is not necessarily in the same direction as the (integrated) edge current in the ground state.

5 Two-component system
5.1 The model

The CSGL Lagrangian \( \mathcal{L} \) can be extended to include spin [18]. This is done by replacing \( \phi \) by a two-component complex scalar field and introducing a Zeeman term

\[
\mathcal{L}_{\text{Zeem}} = -\frac{g\mu_B B}{2} \phi^\dagger \sigma_z \phi. \tag{47}
\]

The model then takes the form

\[
\mathcal{L} = \sum_{\alpha=1}^{2} \phi^\dagger_\alpha \left( i\partial_t - a_0 + \mu_\alpha - V \right) \phi_\alpha - \kappa \sum_{\alpha=1}^{2} \left| (\nabla - i\mathbf{A}) \phi_\alpha \right|^2 \\
- \lambda (\rho_1 + \rho_2)^2 + \frac{1}{4\theta} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda, \tag{48}
\]

where the index \( \alpha \) denotes the two spin states (up and down), so that \( \rho = \rho_1 + \rho_2 \) is the total (charge) density, while \( \rho_S = \rho_1 - \rho_2 \) is the spin density. A spin unit vector may be introduced by

\[
\mathbf{n} = \frac{1}{\rho} \phi^\dagger \sigma \phi \tag{49}
\]

with components in the plane as well as in the direction of the \( B \)-field (\( z \)-direction). In [18] the Zeeman term is represented by the difference between the chemical potentials of the two components.

With the densities and phases of the two components of \( \phi_\alpha \) as variables, the field equations are,

\[
\kappa \nabla^2 \sqrt{\rho_\alpha} = \left[ a_0 + V - (\mu_\alpha - \partial_t \theta_\alpha) + \kappa (\mathbf{A} - \nabla \theta_\alpha)^2 + 2\lambda (\rho_1 + \rho_2) \right] \sqrt{\rho_\alpha} \\
\nabla \times \mathbf{a} = -2\theta (\rho_1 + \rho_2) \\
\partial_x a_0 + \partial_y a_x = 4k\theta \sum_\alpha \rho_\alpha (A_y - \partial_y \theta_\alpha) \\
\partial_y a_0 + \partial_x a_y = -4k\theta \sum_\alpha \rho_\alpha (a_x - \partial_x \theta_\alpha) \\
\partial_t \rho_\alpha + \nabla \cdot \mathbf{j}_\alpha = 0, \tag{50}
\]

where \( \mathbf{A}_i = a_i + eA_i \) as before, and the current densities are given by

\[
\mathbf{j}_\alpha = -2\kappa \rho_\alpha (\mathbf{A} - \nabla \theta_\alpha). \tag{51}
\]

This system has two conserved charges \( Q_\alpha \equiv \int d^2x \rho_\alpha \).

5.2 Ground state properties

The form of the Hamiltonian

\[
\mathcal{H} = \kappa \sum_{\alpha=1}^{2} \left| (\nabla - i\mathbf{A}) \phi_\alpha \right|^2 + V(\rho_1 + \rho_2) + \lambda (\rho_1 + \rho_2)^2 - \mu_1 \rho_1 - \mu_2 \rho_2 \tag{52}
\]
implies that the bulk ground state \((V = 0)\) is fully polarized and has a uniform density,

\[
\begin{align*}
\rho_1 &= \bar{\rho} = \frac{eB}{2\theta} \equiv \frac{\mu_0}{2\lambda} \\
\rho_2 &= 0,
\end{align*}
\]  

(53)  

(54)

where we have assumed \(\mu_1 > \mu_2\). The chemical potential of the upper component is related to \(\mu_0\) by

\[
\mu_1 - a_0(0) = \mu_0.
\]  

(55)

Even though the ground state in the bulk is fully polarized, this may not be the case at the edge of the system. Tilting the spins (i.e. \(\rho_2 \neq 0\)) at the edge may lower the energy if the Zeeman gap is not too large. This problem was recently studied numerically in Ref. [13]; their calculations were based on Hartree-Fock and an effective action approach [19] in the low-energy limit of the CSGL model. Here, we shall examine some of the ground state properties at the edge both analytically and numerically, within the framework of CSGL theory.

As before, we introduce a confining potential \(V(x)\), which gives rise to a straight edge parallel to the \(y\)-axis. We consider stationary solutions, with densities \(\rho_1\) and \(\rho_2\) that are translationally invariant in the \(y\)-direction. With the gauge choice \(\theta_1 = 0\), the solutions take the general form

\[
\begin{align*}
\phi_1 &= \sqrt{\rho_1(x)} \\
\phi_2 &= \sqrt{\rho_2(x)}e^{i(ky - \beta t)}
\end{align*}
\]  

(56)  

(57)

with \(\rho_1 = \bar{\rho}, \rho_2 = 0\) in the bulk. The gauge choice implies \(A_x = a_x = 0\) with \(a_0\) and \(a_y\) as functions of \(x\) only. The parameters \(k\) and \(\beta\) are undetermined at this stage. However, we see that if \(k \neq 0\) this means that the spins will rotate around the \(B\)-field when moving along the edge. With the ansatz (56) and (57) the equations of motion reduce to

\[
\begin{align*}
\kappa \partial_x^2 \sqrt{\rho_1} &= \left[ a_0 + V - \mu_1 + \kappa A_y^2 + 2\lambda(\rho_1 + \rho_2) \right] \sqrt{\rho_1} \\
\kappa \partial_x^2 \sqrt{\rho_2} &= \left[ a_0 + V - \mu_2 - \beta + \kappa(A_y - k)^2 + 2\lambda(\rho_1 + \rho_2) \right] \sqrt{\rho_2} \\
\partial_x A &= -2\theta (\rho_1 + \rho_2 - \bar{\rho}) \\
\partial_x a_0 &= 4\kappa \theta [\rho_1 A_y + \rho_2(A_y - k)]
\end{align*}
\]  

(58)  

(59)  

(60)  

(61)

The corresponding energy (per unit length in the \(y\)-direction) is

\[
\begin{align*}
E &= \int dx \left[ \kappa \left(\partial_x \sqrt{\rho_1}\right)^2 + \kappa \left(\partial_x \sqrt{\rho_2}\right)^2 + \kappa A_y^2 \rho_1 + \kappa (A_y - k)^2 \rho_2 + V(\rho_1 + \rho_2) \\
&\quad + \lambda(\rho_1 + \rho_2)^2 \right] - \mu_1 Q_1 - \mu_2 Q_2,
\end{align*}
\]  

(62)
where $Q_{\alpha}$ now denotes charge per unit length, $Q_{\alpha} = \int dx \, \rho_{\alpha}(x)$.

The ground state conditions are

$$\frac{\partial E}{\partial Q_1} = \frac{\partial E}{\partial Q_2} = 0$$

(63)

and

$$\frac{\partial E}{\partial k} = 0.$$  

(64)

These conditions determine the chemical potentials as functions of the charges, $\mu_{\alpha} = \mu_{\alpha}(Q_1, Q_2)$. So here we have a two-parameter set of ground state solutions parametrized by the charges $Q_1$ and $Q_2$.

From the condition (63), that the ground state energy is invariant under any first order variation of the two charges, we can derive expressions for $\mu_1$ and $\beta$. We first consider a perturbation in $Q_1$ caused by an infinitesimal translation of both components in the $x$-direction (thus keeping $Q_2$ fixed). This gives

$$\mu_1 = \lambda \bar{\rho} + \frac{1}{\bar{\rho}} \int dx \, V'(x) \rho(x)$$

(65)

which corresponds to Eq.(27) in the one-component case. Again, the same expression but with an additional term $a_0(\infty)$ can be derived directly from the equations of motion, giving $a_0(\infty) = 0$ in the ground state, and consequently

$$\mu_1 - \mu_0 = 2\theta J,$$

(66)

where $J = J_1 + J_2$ is the total (integrated) current along the edge.

Similarly, we can slightly perturb the “spin charge” $Q_S = Q_1 - Q_2$ keeping the total charge fixed. This is done by an infinitesimal $SU(2)$ rotation of $\phi$,

$$\phi_1 \rightarrow \phi_1 + \epsilon \phi_2$$

(67)

$$\phi_2 \rightarrow \phi_2 - \epsilon^* \phi_1$$

(68)

with

$$\epsilon = \epsilon(y) = \epsilon_0 e^{-iky}$$

(69)

and $\epsilon_0$ as an infinitesimally small real constant. This introduces the following variations in the charges,

$$\delta Q_S = 4\epsilon_0 \int dx \sqrt{\rho_1(x)} \sqrt{\rho_2(x)}, \quad \delta Q = 0,$$

(70)

and the condition of stationary energy, $\partial E/\partial Q_S = 0$, gives

$$\mu_1 - \mu_2 = -\kappa k^2 + 2\kappa k \int dx \frac{A_y(x) \sqrt{\rho_1(x)} \sqrt{\rho_2(x)}}{\int dx \sqrt{\rho_1(x)} \sqrt{\rho_2(x)}}.$$  

(71)
The same expression, but with an additional factor $\beta$ on the right hand side, can be deduced from the equations of motion by multiplying (58) and (59) by $\sqrt{\rho_2}$ and $\sqrt{\rho_1}$ respectively, subtracting the two equations and integrating over $x$. This leads us to the conclusion that the ground state condition $\partial E/\partial Q_S = 0$ implies $\beta = 0$. Thus, there is no $t$-dependence in the phase of $\phi_2$. Note that the relations (66) and (71) were derived only demanding that the energy be minimized with respect to the charges; they are therefore valid even for values of $k$ that do not minimize the energy. Finally, also demanding $\partial E/\partial k = 0$ one finds

$$J_2 = -2\kappa \int dx \, \rho_2(x) (A_y(x) - k) = 0$$

which gives an expression for the ground state value of $k$,

$$k \equiv k_0 = \frac{1}{Q_2} \int dx \, \rho_2(x) A_y(x).$$

As already stated, we expect a critical value of the Zeeman gap $\Delta \mu = \mu_1 - \mu_2$ to exist, so that if this critical value is exceeded the system will be fully polarized also at the edge. If the gap is close to, but smaller than this value we expect the second component of the wave function to be small. We will make an analysis of this case and illustrate the case by numerical evaluations of the ground state wave function.

For small $Q_2$ we can expand $\mu_2$ in powers of this charge (for given $Q_1$ and $k$)

$$\mu_2 = \mu_2^{(0)} + \mu_2^{(1)} Q_2 + \cdots$$

and similarly for the density $\rho_2$,

$$\hat{\rho}_2 \equiv \rho_2/Q_2 = \rho_2^{(0)} + \rho_2^{(1)} Q_2 + \cdots.$$ 

The lowest order terms correspond to the limit $Q_2 \to 0$. This means that $\mu_2^{(0)}$ is the value of $\mu_2$ which corresponds to the critical Zeeman gap for given $\mu_1$. Similarly, $\rho_2^{(0)}$ represents the profile of $\rho_2$ in the same limit. In this limit $\rho_2$ is negligible compared to $\rho_1$, and the problem is simplified since the equations for the two densities (58)–(61) then decouple: Eqns. (58), (60) and (61) get reduced to the single-component equations (21)–(23) discussed in Sec. 4.1, and the equation (59) for $\sqrt{\rho_2}$ simply becomes a one-dimensional potential problem,

$$\kappa \partial_x^2 \sqrt{\rho_2(x)} = [U(x, k) - \mu_2] \sqrt{\rho_2(x)}$$

where the potential is

$$U(x, k) = a(x) + V(x) + \kappa(A_y(x) - k)^2 + 2\lambda \rho_1(x).$$

The ground state is found by minimizing the energy with respect to $k$, and the chemical potential $\mu_2$ is determined as the (lowest) eigenvalue of the potential problem. This determines the critical value of the Zeeman gap for the chosen value of $\mu_1$. 

Figure 3: The (dimensionless) potential \( U(x, \tilde{k}_0) \) determining the density \( \hat{\rho}_2(x) \) of the lower component in the limit \( \rho_2 \ll \rho_1 \). Here, the confining potential is \( V(x) = 0.05 \theta(x - 5) \cdot (x - 5)^2 \), \( \tilde{\mu}_1 \approx 1.5 \) and \( \tilde{k}_0 = k_0 \cdot l_B = 0.49 \).

The expansion in \( Q_2 \) may be used to correct iteratively the lowest order results. Thus, the lowest order contribution to \( \hat{\rho}_2 \) gives first order corrections to the equations for \( \rho_1, a_0, A_y \). This in turn gives first order corrections to the potential \( U(x, k) \) and to \( \mu_2, \hat{\rho}_2 \) and \( k_0 \), etc. This gives a systematic way to determine the ground state density profiles for small \( Q_2 \).

We have solved this problem numerically only to lowest order in \( Q_2 \). Harmonic potentials of varying strength have been used, and several values for \( \mu_1 \) have been chosen in the allowed interval (as determined in the one-component case).

Fig. 3 shows the potential \( U(x, k_0) \) for one specific value of \( \mu_1 \). The minimum of \( U \) is located at the edge as expected. The wave number \( k_0 \) takes a non-vanishing value. Thus, the spin vector is rotating when moving along the edge.

The corresponding density profiles are shown in Fig. 4. The figure shows that the lower component \( (\varphi_2) \) of ground state density is relatively weakly bound in the potential \( U(x, k) \) for this particular choice of \( V(x) \) and \( \mu_1 \). The localization gets sharper for values of \( \mu_1 \) corresponding to larger \( k_0 \). A similar effect is seen in Fig. 5 when \( k \) is changed relative to the ground state value \( k_0 \).

Table 1 shows numerically estimated values of \( k_0 \) and the critical Zeeman gap for dif-
Figure 4: Edge profiles $\sqrt{\tilde{\rho}_1(x)} \equiv \varphi_1(x)$ and $\sqrt{\tilde{\rho}_2(x)} \equiv \varphi_2(x)$ corresponding to the potential in Fig. 3. The corresponding (critical) Zeeman gap is $\tilde{\mu}_1 - \tilde{\mu}_2 \approx 0.002$.

We can summarize our results as follows: For a sufficiently soft confining potential and for Zeeman energies smaller than some critical value, the spins at the edge are tilted in the ground state ($n_z \neq 1$) and rotate around the $z$-axis with a wave number $k_0$ as one moves along the edge. When $\mu_1$ is increased from its lowest allowed value $\mu^-$ at fixed confining potential, both $k_0$ and the critical Zeeman gap $\Delta \mu_{cr}$ decrease until $\Delta \mu_{cr}$ goes to zero. Similarly, one finds solutions with $\Delta \mu_{cr} > 0$ near the upper critical value $\mu^+$. These solutions are characterized by negative wave numbers whose absolute value decreases along with the critical Zeeman energy as $\mu_1$ is lowered.

We can summarize our results as follows: For a sufficiently soft confining potential and for Zeeman energies smaller than some critical value, the spins at the edge are tilted in the ground state ($n_z \neq 1$) and rotate around the $z$-axis with a wave number $k_0$ as one moves along the edge. When $\mu_1$ is increased from its lowest allowed value $\mu^-$ at fixed confining potential, both $k_0$ and the critical Zeeman gap $\Delta \mu_{cr}$ decrease until $\Delta \mu_{cr}$ goes to zero. Near the highest allowed value $\mu^+$ the edge is again spin textured, with negative $k_0$ and a critical Zeeman gap which approaches zero as $\mu_1$ is decreased. The ranges in $\mu_1$ where the edge is spin textured, decrease as the confining potential gets steeper.
As mentioned in Sec. 4, the edge is expected to become unstable with respect to a redistribution of the charge at soft confining potentials; for $\mu_1$ away from $\mu_0$ this reconstruction is expected to take place at steeper $V(x)$ than is the case for $\mu_1 = \mu_0$. This means that for fixed $\mu_1$ texturing may occur only in some intermediate region where the confining potential is neither too steep nor too soft.

Since the “window” (Eq. (18)) of allowed $\mu_1$ depends on the interaction strength $\tilde{\lambda}$, the results discussed above may change quantitatively for other values of $\tilde{\lambda}$; here we have only examined the case $\tilde{\lambda} = 1$. For this value of $\tilde{\lambda}$ we have found values for the critical Zeeman gap (see Table I) which are of the same order of magnitude as the Zeeman gap reported in real experimental situations; typically, $\Delta\mu/\omega_c \sim 1/60$ in QH experiments in GaAs heterojunctions [20].

5.3 Edge excitations

When the edge is spin textured one may expect new massless edge modes to appear; as argued in Ref. [13], translational symmetry along the edge is spontaneously broken by the phase of $\phi_2$, and one may expect a corresponding Goldstone mode to exist (at least at the

Figure 5: Shape of $\varphi_2(x) = \sqrt{\rho_2(x)}$ for values of $k$ away from $k_0$. The confining potential is $\tilde{V}(x) = 0.05 \theta(x - 5) \cdot (x - 5)^2$ and $\tilde{\mu}_1 = 1.5$. The curves correspond to (from below) $\tilde{k} = 0.49 (= \tilde{k}_0)$, 0.55, 0.60.
Table 1: Wave number and critical Zeeman gap $\Delta \mu_{cr} = \mu_1 - \mu_2$ for some values of the chemical potential $\mu_1$ at $\lambda = 1$. The confining potential is $V(x) = 0.05 \cdot \theta(x - 5) \cdot (x - 5)^2$. Chemical potentials are measured in units of the cyclotron energy, whereas the dimensionless wave number is $k = k \cdot l_B$.

| $\mu_1$ | $k_0$ | $\Delta \mu_{cr}$ |
|---------|-------|------------------|
| 1.3     | 1.0   | 0.06             |
| 1.4     | 0.6   | 0.01             |
| 1.5     | 0.5   | 0.002            |
| 1.6     | 0.3   | -0.007           |

mean field level). We have examined the edge modes by considering quadratic fluctuations in the fields around the ground state configuration in the same way as was done for the spinless case.

Again, we assume the fluctuations in the fields to be separable in the $x$- and $y$-dependence,

$$\delta \rho_\alpha(x, y, t) = \delta \rho_\alpha(x) e^{i q y - \omega t}$$  \hspace{1cm} (78)

$$\delta \theta_\alpha(x, y, t) = -i \eta_\alpha(x) e^{i q y - \omega t}$$  \hspace{1cm} (79)

$$\delta a_\mu(x, y, t) = \delta a_\mu(x) e^{i q y - \omega t}$$  \hspace{1cm} (80)

and thus replacing $\partial_y \rightarrow iq$ and $\partial_t \rightarrow -i \omega$, one finds for the linearized field equations

$$\kappa \left( \partial_x^2 - q^2 \right) \delta \sqrt{\rho_\alpha} = \left[ \delta a_0 - \omega \eta_\alpha + 2 \kappa (A_y - k_\alpha) \delta a_y - q \eta_\alpha \right] + 2 \lambda \delta \rho \sqrt{\rho_\alpha}$$  \hspace{1cm} (81)

$$\partial_x \delta a_y - iq \delta a_x = -2 \theta \delta \rho$$  \hspace{1cm} (82)

$$\partial_x \delta a_0 - i \omega \delta a_x = 4 \kappa \theta \sum_\alpha [\delta \rho_\alpha (A_y - k_\alpha) + \rho_\alpha (\delta a_y - q \eta_\alpha)]$$  \hspace{1cm} (83)

$$iq \delta a_0 - i \omega \delta a_y = -4 \kappa \theta \sum_\alpha [\rho_\alpha \delta a_x + i \rho_\alpha \partial_x \eta_\alpha]$$  \hspace{1cm} (84)

$$-i \omega \delta \rho_\alpha = 2 \kappa [iq \delta \rho_\alpha (A_y - k_\alpha) + i q \rho_\alpha (\delta a_y - q \eta_\alpha)$$

$$\quad + \partial_x (\rho_\alpha (\delta a_x + i \partial_x \eta_\alpha))]$$  \hspace{1cm} (85)

with $k_1 = 0$ and $k_2 = k$. As in Eq.(36) we expand all the $x$-dependent functions in powers of $q$ and assume that the phases $\eta_1$ and $\eta_2$ dominate to $0^{th}$ order. In analogy with Eq.(37), one finds

$$\sum_\alpha \rho_\alpha \partial_x \eta_\alpha^{(0)} = \partial_x \left( \rho_\alpha \partial_x \eta_\alpha^{(0)} \right) = 0$$  \hspace{1cm} (86)

which implies that $\eta_1^{(0)}$ and $\eta_2^{(0)}$ are constants.
To \( \mathcal{O}(q) \), the last two equations (84) and (85) give

\[
\delta j^{(1)}_{x,\text{tot}} = \partial_x \delta j^{(1)}_{x,\alpha} = 0,
\]
and therefore \( \delta j^{(1)}_{x,\alpha} = 0 \), as follows from the boundary conditions on the current deep in the bulk and outside the edge.

The first three equations, (81)-(83), again take the form of those for variations in the \( t \)- and \( y \)-independent (ground state) fields \( (\delta \rho_{\alpha}^{(s)}, \delta a_{\mu}^{(s)}, \delta \mu_{\alpha}, \delta k) \) (see Eqs. (58)-(61)), provided we make the following identifications

\[
\begin{align*}
\delta a_{y}^{(1)} - \eta^{(0)}_1 &= \delta a_{y}^{(s)} \\
\delta a_{y}^{(1)} - \eta^{(0)}_2 &= \delta a_{y}^{(s)} - \delta k \\
\delta a_{0}^{(1)} - \frac{\omega}{q} \eta^{(0)}_1 &= \delta a_{0}^{(s)} - \delta \mu_1 \\
\delta a_{0}^{(1)} - \frac{\omega}{q} \eta^{(0)}_2 &= \delta a_{0}^{(s)} - \delta \mu_2.
\end{align*}
\]

One should note that in these expressions \( \delta k \) is not restricted by the condition (64), only the ground state conditions (63) are satisfied. Thus the variations in the fields depend on three parameters, which we may choose to be \( Q_1, Q_2 \) and \( k \). However, the identifications (88) implicitly give an additional constraint,

\[
\omega \delta k = q(\delta \mu_1 - \delta \mu_2) = -q \delta \Delta \mu
\]
and with this condition included, the fields depend only on two parameters, \( Q_1 \) and \( Q_2 \).

The continuity equations for the currents,

\[
\omega \delta Q_{\alpha} = q \delta J_{\alpha}
\]

give two linear, homogenous equations in the two variables which determine the (low frequency) normal modes. The equations have the form

\[
M_{\alpha\beta} \delta Q_{\beta} = 0
\]
with

\[
M_{\alpha\beta} = \left( \frac{\omega}{q} + \Delta \mu_k \right) \left( \frac{\omega}{q} \delta_{\alpha\beta} - J_{\alpha\beta} \right) + J_{\alpha k} \Delta \mu_{\beta},
\]
where we have introduced the notations

\[
\begin{align*}
\Delta \mu_k &= \frac{\partial \Delta \mu}{\partial k} \\
\Delta \mu_{\alpha} &= \frac{\partial \Delta \mu}{\partial Q_{\alpha}} \\
J_{\alpha k} &= \frac{\partial J_{\alpha}}{\partial k} \quad J_{\alpha \beta} &= \frac{\partial J_{\alpha}}{\partial Q_{\beta}}.
\end{align*}
\]
Eq. (91) corresponds to Eq. (44) for the one-component case, which there gave $\omega/q$ in terms of derivatives of $J$ and $\mu$ with respect to $Q$. Here the matrix equation gives a secular equation which is third order in $\omega/q$,

$$\left(\frac{\omega}{q}\right)^3 + A \left(\frac{\omega}{q}\right)^2 + B \frac{\omega}{q} + C = 0 \tag{94}$$

with

\begin{align*}
A &= \Delta \mu_k - J_{11} - J_{22} \\
B &= J_{11}J_{22} - J_{12}J_{21} - \Delta \mu_k(J_{11} + J_{22}) + \Delta \mu_1J_{1k} + \Delta \mu_2J_{2k} \\
C &= \Delta \mu_k(J_{11}J_{22} - J_{12}J_{21}) - \Delta \mu_1(J_{1k}J_{22} - J_{2k}J_{12}) - \Delta \mu_2(J_{2k}J_{11} - J_{1k}J_{21}). \tag{95}
\end{align*}

The parameters of this equation are determined by the solutions of the (ground state) equations (61). We do not have any explicit expressions for the parameters, but one should note that they are not all independent, due to relations like (66).

The fact that (94) is third order in $\omega/q$ indicates that with spin effects included there are three massless chiral edge modes (rather than two). We cannot give the velocities of these modes in explicit form, but we note that the expressions (65) and (66), which relate the total current to the confining potential, imply that the velocities can be written as

$$\frac{\omega}{q} = \frac{1}{eB} \int_0^\infty dx \frac{V'(x)}{\delta Q} \frac{\delta \rho}{\delta Q}. \tag{96}$$

where $\delta \rho$ and $\delta Q$ are the variations in total charge density and integrated charge, respectively, for each of the normal modes (provided $\delta Q \neq 0$). When the variation in the charge density is located at the edge this shows that the velocities of all the modes can be written in the general form $- < E > /B$, with $< E >$ as an averaged value of $E = -V'/e$ at the edge. However, the $E$-field will be averaged with functions which in general are different for each of the three modes.

6 Discussion

The Chern-Simons Ginzburg-Landau theory gives a mean field description of the fractional quantum Hall effect which reproduces correctly several of its main features. Here we have examined, in particular, edge effects in this framework, both by analytical and numerical methods. The starting point has been the observation that if the chemical potential is varied within a certain interval there is no change in the uniform bulk density, but a change in the edge profile is introduced. Thus, there is a set of ground states parametrized by the edge charge. The edge current also varies with the chemical potential, and we have found that it changes sign for the value of the chemical which minimizes the potential energy in bulk. This value can be considered as corresponding to the central point of the
Hall plateau, the point where the uniform charge density of the electrons neutralizes the (constant) background charge.

The low-frequency edge modes have been examined in terms of quadratic fluctuations in the fields and with an expansion in the wave number \( q \) for fluctuations propagating along the edge. As expected (and previously shown for an infinitely steep confining potential) there exists in the spinless case one gapless chiral edge mode. To first order in \( q \) the density profile of this mode is the same as for an (infinitesimal) variation in the ground state profile introduced by a variation in the charge (and the chemical potential). An expression for the velocity of propagation has been found in terms of the confining potential; it has the form of an averaged drift velocity \( E/B \), averaged over the edge with the density profile of the gapless mode.

The edge effects of a fully polarized system with spin have been examined in a similar way. In this case there will be a spin texturing of the edge if the confining potential is sufficiently soft and if the Zeeman energy is not too large. The ground state then depends on two edge charges, one for each spin component, and we have examined the case were the second charge (corresponding to the spin down component) is small. The critical Zeeman gap has been examined numerically as a function of the chemical potential (for a particular value of the interaction strength). It is different from zero in intervals of \( \mu_1 \) near the upper and lower bounds \( \mu^+ \) and \( \mu^- \) and decreases when \( \mu_1 \) approaches \( \mu_0 \) from either side. The width of these intervals increases with softening of the confining potential.

Also in this case there are gapless edge modes which can be related to variations in the ground state fields introduced by variations in the charges. However, the variations involve changes in the wave number \( k \) of the spin down component which are not restricted by the ground state condition \( \partial E/\partial k = 0 \).

The continuity equations for the charges give a third order equation for the mode frequencies. This indicates the presence of three chiral modes. However, the equation has been given in terms of a set of undetermined parameters. It will be of interest to examine further these parameters to see if there are more relations or constraints than have been established in this paper. Also a numerical study of these modes is an interesting subject for further research.

The edge modes have here been studied as quadratic fluctuations about the mean field ground state. A question which deserves further study is whether quantum effects will qualitatively change this picture. The reason that this cannot be ruled out is the quasi-one dimensionality of the system and the suggested connection to spontaneous breaking of translational symmetry of the ground state \[21\]. As is well known, spontaneous symmetry breaking is not supported in one-dimensional systems due to (quantum) fluctuations in the symmetry breaking variable \[22\].

Acknowledgements

We wish to thank Jan Myrheim, Serguei Isakov, Hans Hansson and Dmitri Khveshchenko for useful discussions and comments.
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