Operators and their symbols in the optical probabilistic representation of quantum mechanics

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Abstract

Explicit expressions for most interesting quantum operators in optical tomography representation are found. General formalism of symbols of operators is presented in optical tomographic representation. The symbols of the operators are found explicitly for physical quantities.

Keywords: symbol of the operator, dual symbol, quantizer, dequantizer, optical tomogram of quantum state.

1 Introduction

The transition from quantum to classical mechanics has been an important research subject since the beginning of quantum mechanics (see [1, 2] for a review). A suitable setting for this problem is represented by the Wigner-Weyl-Moyal formalism where the operators corresponding to observables and the states, considered as linear functionals on the space of observables, are mapped onto functions on a suitable manifold. Such a representation for quantum mechanics has been later generalized yielding to the deformation quantization program [3]. There the operator noncommutativity is implemented by a noncommutative (star) product which is a generalization of the Moyal product [4, 5, 6]. Since then, most attention to the star-product quantization scheme has been devoted to the case where the functions (symbols of the operators) are defined on the classical phase space of the system [7, 8, 9, 10].

In a different setting it was recently established [11, 12] that the symplectic [13, 15, 16] and spin [17, 18] tomography, which furnish alternative formulations of quantum mechanics and quantum field theory [19], can be described as well within a star-product scheme. Moreover, in [12] different known star-product schemes were presented in a unified form. There, the symbols of the operators are defined in terms of a special family of operators using the trace formula (what we sometimes call the dequantization map because of its original meaning in the Wigner-Weyl formalism), while the reconstruction of operators in terms of their symbols (the quantization map) is determined using another family of operators. These two families determine completely the star-product scheme, including the kernel of the star-product.

The aim of our work is to find the explicit expressions of most physically interesting operators and their dual symbols in optical tomographic representation, necessary for practical calculations.

The paper is organized as follows. In Sec.2 we review optical tomography of quantum states. In Sec.3 the correspondence rules for physical operators in optical tomographic representation are found. In Sec.4 the general formalism of symbols of operators is presented in optical tomographic representation. The expressions for the dual symbols of the operators in terms of singular generalized functions and for the kernel of their star-product are presented. In Sec.5 the representation of dual symbols of operators in terms of regular generalized functions is given.
2 Optical tomographic representation of quantum states

In this section we give a short review of the tomographic representation of quantum mechanics by using so called optical tomogram [20, 21]. For the photon states this tomogram is measured experimentally [22, 23]. For simplicity of the formulas we consider a case of one degree of freedom with dimensionless variables, because the generalization of all our calculations and results to the case of any arbitrary number dimensional degrees of freedom is obvious.

If we have the density matrix of the quantum state \( \hat{\rho} \), the optical tomogram is defined as

\[
w(X, \theta) = \text{Tr}\{\hat{\rho}\delta(X - \hat{q}\cos\theta - \hat{p}\sin\theta)\} = \langle X, \theta|\hat{\rho}|X, \theta\rangle,
\]

where \( |X, \theta\rangle \) is an eigenvector of the hermitian operator \( \hat{q}\cos\theta + \hat{p}\sin\theta \) for the eigenvalue \( X \)

\[
\langle q|X, \theta\rangle = \frac{1}{\sqrt{2\pi|\sin\theta|}} \exp\left(\frac{iXq - \frac{q^2}{2}\cos\theta}{\sin\theta}\right).
\]

In terms of the Wigner function [24] the tomogram \( w(X, \theta) \) is expressed as

\[
w(X, \theta) = \frac{1}{2\pi^2} \int W(q, p)e^{i\eta(X - q\cos\theta - p\sin\theta)}d\eta dq dp.
\]

This relation can be reversed using the symmetry property of the optical tomogram

\[
w(X, \theta, t) = w((-1)^kX, \theta + \pi t), \quad k = 0, \pm 1, \pm 2, ...
\]

After calculations we can write

\[
W(q, p) = \frac{1}{2\pi} \int_0^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(X, \theta)|\eta|e^{i\eta(X - q\cos\theta - p\sin\theta)}d\eta dX.
\]

From (3) using the relations between the Wigner function and the density matrix \( \rho(q, q') \) in coordinate representation

\[
W(q, p) = \int \rho(q + u/2, q - u/2)e^{-ipu}du,
\]

\[
\rho(q, q') = \frac{1}{2\pi} \int W\left(\frac{q + q'}{2}, p\right)e^{ip(q - q')}dp,
\]

we can write the relations between the optical tomogram and the density matrix in coordinate representation as follows

\[
w(X, \theta) = \frac{1}{2\pi} \int \rho\left(q + \frac{u\sin\theta}{2}, q - \frac{u\sin\theta}{2}\right)e^{-iu(X - q\cos\theta)}du dq,
\]

\[
\rho(q, q') = \frac{1}{2\pi} \int_0^{\pi} \int_{-\infty}^{+\infty} w(X, \theta)|\eta|\exp\left\{i\eta\left(X - \frac{q + q'}{2}\cos\theta\right)\right\} \delta(q - q' - \eta\sin\theta) d\theta d\eta dX.
\]

Thus, the tomogram \( w(X, \theta) \) contains all information about the quantum state.

The evolution equation and energy level equation for optical tomograms were found explicitly in [25].
3 The correspondence rules for physical operators in optical tomographic representation

Using the relation (7) between the density matrix and the Wigner function one has the correspondence rules

\[
\frac{\partial \rho}{\partial t} \leftrightarrow \frac{\partial W}{\partial t}, \quad \frac{\partial \rho}{\partial x} \leftrightarrow \left(\frac{1}{2} \frac{\partial}{\partial q} + ip\right) W, \quad (10)
\]

\[
\frac{\partial \rho}{\partial x'} \leftrightarrow \left(\frac{1}{2} \frac{\partial}{\partial q} - ip\right) W;
\]

\[
x' \rho \leftrightarrow \left(q + \frac{i}{2} \frac{\partial}{\partial p}\right) W;
\]

\[
x \rho \leftrightarrow \left(q - \frac{i}{2} \frac{\partial}{\partial p}\right) W.
\]

Using the relation (11) between the Wigner function and the optical tomogram one can find the correspondence rules for the operators acting on the Wigner function and the optical tomogram:

\[
\cos \theta \frac{\partial}{\partial X} w(X, \theta) = \frac{1}{4\pi^2} \int ik \cos \theta e^{-ikq\cos \theta} W(q, p, t) e^{ik(X-p\sin \theta)} dk dq dp. \quad (11)
\]

In view of the equality

\[
ik \cos \theta e^{-ikq\cos \theta} = -\frac{\partial}{\partial q} e^{-ikq\cos \theta}
\]

and integrating (11) by parts with account of \(W(q, p) \rightarrow 0\) for \(q \rightarrow \pm \infty\) we arrive at

\[
\cos \theta \frac{\partial}{\partial X} w(X, \theta) = \frac{1}{4\pi^2} \int \frac{\partial W(q, p)}{\partial q} \delta(X - q \cos \theta - p \sin \theta) dq dp. \quad (12)
\]

It means that

\[
\frac{\partial}{\partial q} W(q, p) \leftrightarrow \cos \theta \frac{\partial}{\partial X} w(X, \theta). \quad (13)
\]

Analogously we have the correspondence rule

\[
\frac{\partial}{\partial p} W(q, p) \leftrightarrow \sin \theta \frac{\partial}{\partial X} w(X, \theta). \quad (14)
\]

Applying the operator \((\partial/\partial X)^{-1}\) which is defined by acting on plane wave as

\[
\left(\frac{\partial}{\partial X}\right)^{-1} e^{ikX} = \frac{1}{ik} e^{ikX}
\]

(15)
to (13) together with differentiation over the angle variable \(\theta\) and multiplication by \(\sin \theta\) we get the equality

\[
\sin \theta \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial X}\right)^{-1} w(X, \theta) + X \cos \theta \ w(X, \theta) = \frac{1}{4\pi^2} \int q \ W(q, p) e^{ik(X-q\cos \theta-p\sin \theta)} dk dq dp. \quad (16)
\]

We used identities

\[
(q \sin \theta - p \cos \theta) \sin \theta + X \cos \theta = q + (X - q \cos \theta - p \sin \theta) \cos \theta,
\]

\[
\delta(X - q \cos \theta - p \sin \theta) \ (X - q \cos \theta - p \sin \theta) = 0.
\]

The relation (16) gives the correspondence rule

\[
q \ W(q, p) \leftrightarrow \left(\sin \theta \left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \theta} + X \cos \theta\right) w(x, \theta). \quad (17)
\]
Analogously we get the correspondence rule

$$ p \ W(q, p) \leftrightarrow \left(-\cos \theta \left( \frac{\partial}{\partial x} \right)^{-1} \frac{\partial}{\partial \theta} + X \sin \theta \right) w(x, \theta). \tag{18} $$

If the product of the direct and the inverse Radon transform gives the unity, then the explicit form in the optical tomographic representation of the product of the operators is equal to the product of these operators in the optical tomographic representation.

More over, suppose we have a set of operators \( \{ \hat{A}_{ik} \} \), acting on the set of functions \( W(\vec{q}, \vec{p}) \in S^{2n} \) (we consider a multidimensional case), and let for any \( W(\vec{q}, \vec{p}) \in S^{2n} \), we have \( \hat{A}_{ik} W(\vec{q}, \vec{p}) \in S^{2n} \) for any \( \hat{A}_{ik} \in \{ \hat{A}_{ik} \} \), then we can write

$$ R \left[ \sum_i C_i \prod_k (\hat{A}_{ik})^{l_k} W(\vec{q}, \vec{p}) \right] (\vec{X}, \vec{\theta}) = \sum_i C_i \prod_k R \left[ \hat{A}_{ik}^{l_k} \right] (\vec{X}, \vec{\theta})^{l_k} R \left[ W(\vec{q}, \vec{p}) \right] (\vec{X}, \vec{\theta}), \tag{19} $$

where \( \sum_i \) and \( \prod_k \) – no more than countable.

Using the formulas given in this paragraph, it is possible to find explicit form of any interesting for the practice operators in optical probability representation.

Thus one can find the operators \( \hat{q}, \hat{p}, \hat{q}^2, \hat{p}^2, \hat{q}\hat{p} \) in the optical tomographic representation

\begin{align*}
\hat{q}_i &= \sin \theta_i \left( \frac{\partial}{\partial X_i} \right)^{-1} \frac{\partial}{\partial \theta_i} + X_i \cos \theta_i + \frac{i \hbar}{2 m_i \omega_{oi}} \sin \theta_i \frac{\partial}{\partial X_i}; \\
\hat{p}_i &= m_i \omega_{oi} \left( -\cos \theta_i \left( \frac{\partial}{\partial X_i} \right)^{-1} \frac{\partial}{\partial \theta_i} + X_i \sin \theta_i \right) - \frac{i \hbar}{2} \cos \theta_i \frac{\partial}{\partial X_i}; \\
\hat{q}_i^2 &= \sin^2 \theta_i \left( \frac{\partial}{\partial X_i} \right)^{-2} \left( \frac{\partial^2}{\partial \theta_i^2} + 1 \right) + X_i \left( \frac{\partial}{\partial x_i} \right)^{-1} \left( \sin 2 \theta_i \frac{\partial}{\partial \theta_i} - \sin^2 \theta_i \right) + X_i^2 \cos^2 \theta_i \\
&+ \frac{i \hbar}{m_i \omega_{oi}} \left( \sin^2 \theta_i \frac{\partial}{\partial \theta_i} + \frac{\sin 2 \theta_i}{2} \left( 1 + X_i \frac{\partial}{\partial x_i} \right) \right) \frac{\hbar^2}{4 m_i \omega_{oi}^2} \sin^2 \theta_i \frac{\partial^2}{\partial X^2}; \\
\hat{p}_i^2 &= m_i^2 \omega_{oi}^2 \left( \cos^2 \theta_i \left( \frac{\partial}{\partial x_i} \right)^{-2} \left( \frac{\partial^2}{\partial \theta_i^2} + 1 \right) - X_i \left( \frac{\partial}{\partial x_i} \right)^{-1} \left( \sin 2 \theta_i \frac{\partial}{\partial \theta_i} + \cos^2 \theta_i \right) + X_i^2 \sin^2 \theta_i \right) \\
&- \frac{i \hbar m_i \omega_{oi}}{2} \left( \cos^2 \theta_i \frac{\partial}{\partial \theta_i} \sin^2 \theta_i \frac{\partial^2}{\partial X^2} \right); \\
\hat{q}_i \hat{p}_i &= m_i^2 \omega_{oi}^2 \left( \frac{\sin 2 \theta_i}{2} \left( \frac{\partial}{\partial X_i} \right)^{-1} \left( \frac{\partial}{\partial \theta_i} \sin^2 \theta_i \frac{\partial^2}{\partial x_i} \right) + X_i^2 \sin^2 \theta_i \right) \\
&- \frac{i \hbar}{2} \frac{\sin 2 \theta_i}{2} \left( \frac{\partial}{\partial \theta_i} \sin^2 \theta_i \frac{\partial^2}{\partial x_i} \right) + \frac{\hbar^2}{8 m_i \omega_{oi}^2} \sin^2 \theta_i \frac{\partial^2}{\partial X^2}. \tag{20} 
\end{align*}

Let us find the momentum operator in the optical tomographic representation. As known in the density matrix representation \( \hat{l} = -i \hbar [\hat{q}, \nabla \hat{q}] \), i.e. \( \hat{l}_1 = \hat{q}_2 \hat{p}_3 - \hat{p}_2 \hat{q}_3 \), \( l_2 \) and \( l_3 \) are given from the relation for \( \hat{l}_1 \) by cyclic replacement of indices. In the Wigner representation

$$ \hat{l}_1 = -i \left\{ \frac{q_2}{2} \frac{\partial}{\partial q_3} + iq_2 p_3 + \frac{i}{4} \frac{\partial^2}{\partial q_3 \partial p_2} - \frac{p_3}{2} \frac{\partial}{\partial p_2} - \frac{q_3}{2} \frac{\partial}{\partial q_2} - iq_3 p_2 - \frac{i}{4} \frac{\partial^2}{\partial p_3 \partial q_2} + \frac{p_2}{2} \frac{\partial}{\partial p_3} \right\}, $$

4
and corresponding Wigner symbol of this operator

\[ W_l^1(\vec{q}, \vec{p}) = q_2 p_3 - q_3 p_2. \]

In the optical distribution representation

\[
\hat{l}_1 = -i \left( \frac{1}{2} \left( \sin \theta_2 \left[ \frac{\partial}{\partial X_2} \right]^{-1} \frac{\partial}{\partial \theta_2} + X_2 \cos \theta_2 \right) \cos \theta_3 \frac{\partial}{\partial X_3} \right.
\]
\[
+ \left. i \left( \sin \theta_2 \left[ \frac{\partial}{\partial X_2} \right]^{-1} \frac{\partial}{\partial \theta_2} + X_2 \cos \theta_2 \right) \left( -\cos \theta_3 \left[ \frac{\partial}{\partial X_3} \right]^{-1} \frac{\partial}{\partial \theta_3} + X_3 \sin \theta_3 \right) \right)
\]
\[
+ \frac{i}{4} \sin \theta_2 \frac{\partial}{\partial X_2} \cos \theta_3 \frac{\partial}{\partial X_3} + \sin \frac{\theta_2}{2} \frac{\partial}{\partial X_2} \left( \cos \theta_3 \left[ \frac{\partial}{\partial X_3} \right]^{-1} \frac{\partial}{\partial \theta_3} - X_3 \sin \theta_3 \right) \right) + i \left\{ 2 \leftrightarrow 3 \right\}.
\]

Components \( \hat{l}_2 \) and \( \hat{l}_3 \) are given from (21) by cyclic replacement of indices.

The creation and annihilation operators acting on the density matrix in coordinate representation

\[
\hat{a} = \frac{1}{\sqrt{2}} \left( q + \frac{\partial}{\partial q} \right) ; \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( q - \frac{\partial}{\partial q} \right)
\]

In the optical probability representation

\[
\hat{a}_i = \frac{\exp(i \theta_i)}{\sqrt{2}} \left\{ \frac{1}{2} \frac{\partial}{\partial X_i} + X_i - i \left[ \frac{\partial}{\partial X_i} \right]^{-1} \frac{\partial}{\partial \theta_i} \right\},
\]
\[
\hat{a}_i^\dagger = \frac{\exp(-i \theta_i)}{\sqrt{2}} \left\{ \frac{1}{2} \frac{\partial}{\partial X_i} + X_i + i \left[ \frac{\partial}{\partial X_i} \right]^{-1} \frac{\partial}{\partial \theta_i} \right\}.
\]

For the number of quanta operator \( \hat{N}_i = \hat{a}_i^\dagger \hat{a}_i \) in \( i \)-th mode of \( n \)-dimensional oscillator we have

\[
\hat{N}_i \rho(\vec{q}, \vec{q}') = \hat{a}_i^\dagger a_i \rho(\vec{q}, \vec{q}') = \frac{1}{2} \left\{ q_i^2 - \frac{\partial^2}{\partial q_i^2} - 1 \right\} \rho(\vec{q}, \vec{q}'),
\]

in the Wigner representation

\[
(\hat{N}_i)_W \rho(\vec{q}, \vec{p}) = (\hat{a}_i^\dagger)_W \rho(\vec{q}, \vec{p})
\]
\[
= \frac{1}{2} \left\{ q_i^2 - \frac{1}{4} \left( \frac{\partial^2}{\partial p_i^2} + \frac{\partial^2}{\partial q_i^2} \right) + i q_i \frac{\partial}{\partial p_i} - i p_i \frac{\partial}{\partial q_i} + p_i^2 - 1 \right\} W(\vec{q}, \vec{p}).
\]

Using the formulas of Sec.3 of this work, or taking the product of two operators (23) we arrive at

\[
\hat{N}_i w(\vec{X}, \vec{\theta}) = \hat{a}_i^\dagger \hat{a}_i w(\vec{X}, \vec{\theta})
\]
\[
= \frac{1}{2} \left\{ \left[ \frac{\partial}{\partial X_i} \right]^{-2} \left( \frac{\partial^2}{\partial \theta_i^2} + 1 \right) + X_i^2 - X_i \left[ \frac{\partial}{\partial X_i} \right]^{-1} - \frac{1}{4} \frac{\partial^2}{\partial X_i^2} + i \frac{\partial}{\partial \theta_i} - 1 \right\} w(\vec{X}, \vec{\theta}).
\]
The operator $\hat{N}_i$ acts on the functions $w_n(\vec{X}, \vec{\theta})$ of harmonic oscillator according to the formula

$$\hat{N}_i w_n(\vec{X}, \vec{\theta}) = n_i w_{n_i}(\vec{X}, \vec{\theta}),$$

where $n_i$ is a number of quanta in the $i$–th mode. The result (24) also can be found as the product of creation and annihilation operators in the optical probability representation.

Note, that at a derivation of correspondence rules we actually used, that functions $W(\vec{q}, \vec{p})$ belong to the space $S^{2n}$ of well-behaved test functions, on which the space of the generalized functions of slow growth $S'^{2n}$ can be constructed.

4 General formalism of symbols of operators

The relation between the density matrix and the tomogram can be represented in the form

$$w(X, \theta) = \text{Tr}\{\hat{\rho}\hat{U}(X, \theta)\}, \quad \hat{\rho} = \int w(X, \theta)\hat{D}(X, \theta)dX \ d\theta,$$

where

$$\hat{U}(X, \theta) = \delta(X\hat{1} - \hat{\theta} \cos \theta - \hat{\theta} \sin \theta),$$

$$\hat{D}(X, \theta) = \frac{1}{2\pi} \int |\eta| e^{i\eta(X - \hat{\theta} \cos \theta - \hat{\theta} \sin \theta)d\eta$$

are the dequantizer and quantizer operator respectively. These operators satisfy to orthogonality and completeness conditions.

$$\text{Tr}\{\hat{U}(X, \theta)\hat{D}(X', \theta')\} = \delta(X \cos(\theta - \theta') - X')\delta(\sin(\theta - \theta')),$$

$$\int \hat{D}_{\hat{q}\hat{p}}(X, \theta)\hat{U}_{\hat{q}\hat{p}}(X, \theta)dX \ d\theta = \delta(\hat{q} - \hat{q}')\delta(\hat{p} - \hat{p}).$$

Let us associate the symbol $w_{\hat{A}}(X, \theta)$ to the arbitrary operator $\hat{A}$ by the definition

$$w_{\hat{A}}(X, \theta) = \text{Tr}\{\hat{A}\hat{U}(X, \theta)\}.$$ 

Taking into account the completeness condition (27) we can write the inverse relation

$$\hat{A} = \int w_{\hat{A}}(X, \theta)\hat{D}(X, \theta)dX \ d\theta.$$ 

The action of the operator $\hat{A}$ to the density matrix can be represented in tomographic representation as the integral operator

$$\text{Tr}\{\hat{A}\hat{\rho}\hat{U}(X, \theta)\} = \int w_{\hat{A}}(X', \theta')w(X'', \theta'')\text{Tr}\{\hat{D}(X', \theta')\hat{D}(X'', \theta'')\hat{U}(X, \theta)\}dX' \ d\theta' \ dX'' \ d\theta''.$$ 

The average value of the operator $\hat{A}$ equals

$$\text{Tr}\{\hat{A}\hat{\rho}\} = \int w(X, \theta)\text{Tr}\{\hat{A}\hat{D}(X, \theta)\}dX \ d\theta = \int w(X, \theta)w^{(d)}_{\hat{A}}(X, \theta)dX \ d\theta,$$

where we denote the designation for dual symbol of the operator $\hat{A}$

$$w^{(d)}_{\hat{A}}(X, \theta) = \text{Tr}\{\hat{A}\hat{D}(X, \theta)\}.$$ (28)
With the help of (27) the operator $\hat{A}$ can be found from its dual symbol

$$\hat{A} = \int w^{(d)}_{\hat{A}}(X, \theta)\hat{U}(X, \theta)dX \mathrm{d}\theta.$$  

Symbol $w_{\hat{A}}(X, \theta)$ and corresponding dual symbol $w^{(d)}_{\hat{A}}(X, \theta)$ are associated by the relations

$$w^{(d)}_{\hat{A}}(X, \theta) = \int w_{\hat{A}}(X', \theta')\text{Tr}\{\hat{D}(X', \theta')\hat{D}(X, \theta)\}dX' \mathrm{d}\theta',$$

$$w_{\hat{A}}(X, \theta) = \int w^{(d)}_{\hat{A}}(X', \theta')\text{Tr}\{\hat{U}(X', \theta')\hat{U}(X, \theta)\}dX' \mathrm{d}\theta'.$$

The dual symbol of the product of two operators $\hat{A}$ and $\hat{B}$ is equal to the star-product with the corresponding kernel

$$w^{(d)}_{\hat{A}\hat{B}}(X, \theta) = w^{(d)}_{\hat{A}}(X, \theta) * w^{(d)}_{\hat{B}}(X, \theta)$$

$$= \int K^{(d)}(X, \theta; X', \theta'; X'', \theta'')w^{(d)}_{\hat{A}}(X', \theta')w^{(d)}_{\hat{B}}(X'', \theta'')dX' \mathrm{d}\theta' \mathrm{d}X'' \mathrm{d}\theta'',$$

where

$$K^{(d)}(X, \theta; X', \theta'; X'', \theta'') = \text{Tr}\{\hat{U}(X', \theta')\hat{U}(X'', \theta'')\hat{D}(X, \theta)\}.$$  

This formula can be transformed to a form suitable for practical use as follows

$$K^{(d)}(X, \theta; X', \theta'; X'', \theta'') = \frac{1}{(2\pi)^2} \int \delta(X' - q \cos \theta' - p \sin \theta')\delta(X'' - q \cos \theta'' - p \sin \theta'')[\eta]$$

$$\times \exp\{i\eta(X - q \cos \theta - p \sin \theta)\} \exp\left\{i\eta \frac{2\sin(\theta - \theta')\sin(\theta - \theta'')}{\sin(\theta' - \theta'')}\right\} d\eta \mathrm{d}q \mathrm{d}p.$$  

(31)

From the definition of dual symbol (28) for the operators $\hat{1}, \hat{q}$ and $\hat{p}$ after calculations we arrive at the formulas

$$w^{(d)}_{\hat{1}}(X, \theta) = \delta(\sin(\theta - \theta_o)), \quad \theta_o \in [0, \pi];$$

$$w^{(d)}_{\hat{q}}(X, \theta) = X \cos \theta \delta(\sin \theta);$$

$$w^{(d)}_{\hat{p}}(X, \theta) = X \delta(\theta - \pi/2);$$

$$w^{(d)}_{\hat{q}\hat{p}}(X, \theta) = X^2 \delta(\theta - \pi/4) - \frac{1}{2}X^2 \delta(\sin \theta) - \frac{1}{2}X^2 \delta(\theta - \pi/2) + \frac{i}{2\pi}.$$  

Note, that for symplectic tomography the quantizer and dequantizer operators are given by the formulas

$$\hat{U}(X, \mu, \nu) = \delta(X \hat{1} - \hat{q}\mu - \hat{p}\nu),$$

$$\hat{D}(X, \mu, \nu) = \frac{1}{2\pi} e^{i(X - \hat{q}\mu - \hat{p}\nu)},$$

and for the corresponding symbol and dual symbol for the operator $\hat{A}$ we have

$$w_{\hat{A}}(X, \mu, \nu) = \text{Tr}\{\hat{A}\hat{U}(X, \mu, \nu)\}, \quad \hat{A} = \int w_{\hat{A}}(X, \mu, \nu)\hat{D}(X, \mu, \nu)dX \mathrm{d}\mu \mathrm{d}\nu,$$

$$w^{(d)}_{\hat{A}}(X, \mu, \nu) = \text{Tr}\{\hat{A}\hat{D}(X, \mu, \nu)\}, \quad \hat{A} = \int w^{(d)}_{\hat{A}}(X, \mu, \nu)\hat{U}(X, \mu, \nu)dX \mathrm{d}\mu \mathrm{d}\nu.$$
5 Representation of symbols of operators in terms of regular generalised functions

The dual symbols of the operators for the optical probability allow the representation in terms of regular generalised functions.

The dual symbol \( w^{(d)}_A(X, \theta) \) of some operator \( \hat{A} \) defines the linear continuous functional on the set of optical distributions \( w(X, \theta) \), belonging to space \( S^{2n} \) of well-behaved test functions. Thus, the set of \( w^{(d)}_A(X, \theta) \) actually defines the set of generalised functions on \( S^{2n} \). Obviously, that the equality of two symbols of one operator have to define as the functional equality or equality of two generalised functions, i.e. two symbols are equal each other when for any distribution \( w(X, \theta) \in S^{2n} \) we have the equality of values of corresponding functionals, denoted by these symbols. Thus there are set of symbols for any operator \( \hat{A} \) which are equal each other in the meaning of generalized functions.

In the previous paragraph we have found the general expression for the dual symbol of arbitrary operator and presented the singular forms of some operators. Singular forms of operators are convenient for analitical calculations, but for the numerical calculations and for the processing of experimental data the representation of symbols in the form of regular generalised functions can be more preferable.

If \( \hat{A} \) is an arbitrary operator in Wigner representation with existing average value, then the integral

\[
\int R[\hat{A}\ W(q,p)](\vec{X}, \vec{\theta})d^nX = \langle \hat{A} \rangle
\]

does not depend on \( \vec{\theta} \). Taking into account the definition of dual symbol of operator we can write

\[
\langle \hat{A} \rangle = \int w^{(d)}_A(\vec{X}, \vec{\theta})w(\vec{X}, \vec{\theta})d^nXd^n\theta
\]

\[
= \frac{1}{\pi^n} \int R[\hat{A}W(q,p)](\vec{X}, \vec{\theta})d^nXd^n\theta = \frac{1}{\pi^n} \int R[\hat{A}]R[W(q,p)](\vec{X}, \vec{\theta})d^nXd^n\theta. \tag{32}
\]

where \( R[\hat{A}] \) is an explicit form of the operator \( \hat{A} \) in the optical tomographic representation which we found in Sec. 3. In turn, the continuous linear functionals (generalized functions) of the form

\[
\frac{1}{\pi^n} \int d^nXd^n\theta R[\hat{K}]w(X, \theta)
\]

acting on the set of marginal distributions of \( w(X, \theta) \), found by the above rules easily can be represented in the form of regular generalised functions. For example, let us regularize the functional \( \int [\partial/\partial X_i]^{-2}w(X, \theta)d^nX \). After a double integration by parts we have:

\[
\int \left[ \frac{\partial}{\partial X_i} \right]^{-2}w(\vec{X}, \vec{\theta})d^nX = \int d^{n-1}X_i \left\{ X_i \left[ \frac{\partial}{\partial X_i} \right]^{-2}w(\vec{X}, \vec{\theta}) \right\}^{+\infty}_{-\infty} - \frac{X_i^2}{2} \left[ \frac{\partial}{\partial X_i} \right]^{-1}w(\vec{X}, \vec{\theta})^{+\infty}_{-\infty} + \frac{X_i^2}{2} w(\vec{X}, \vec{\theta})dX_i. \tag{33}
\]

Clarify the behavior of \( x_i[\partial/\partial X_i]^{-2}w(\vec{X}, \vec{\theta}) \) when \( x_i \to \pm \infty \)

\[
X_i \left[ \frac{\partial}{\partial X_i} \right]^{-2}w(\vec{X}, \vec{\theta}) = X_i \int W(\vec{X}, \vec{p}) \exp(-ik_\sigma(X_\sigma - q_\sigma \cos \theta_\sigma - p_\sigma \sin \theta_\sigma))/(-ik_\sigma)^2 d^nkd^nqd^np. \tag{34}
\]

We can see that the expression under the integral oscillates strongly at \( k_i \) when \( x_i \to \pm \infty \) and final \( q_i, p_i \), but when \( q_i, p_i \to \pm \infty \) we have \( W(q,p) \to 0 \) exponentially. Thus whole the integral \( (34) \to 0 \) when
$x_i \to \pm \infty$. Similarly, we can find that the second term on the right side of the integrand (33) is zero. Finally, we obtain
\[
\int \left[ \frac{\partial}{\partial x_i} \right]^{-2} w(\vec{X}, \vec{\theta}) d^n X = \int \frac{X_i^2}{2} w(\vec{X}, \vec{\theta}) d^n X, \quad \text{or}
\]
\[
\left[ \frac{\partial}{\partial x_i} \right]^{-2} \to \frac{x_i^2}{2}.
\]
We write down the regular representations of the operators most frequently used in intermediate calculations:
\[
\frac{\partial}{\partial X_i} \to 0; \quad X_i \frac{\partial}{\partial X_i} \to -1; \quad \frac{\partial}{\partial x_i} \to -X_i; \quad X_i \left[ \frac{\partial}{\partial X_i} \right]^{-1} \to - \frac{X_i^2}{2}; \quad \frac{\partial^2}{\partial \theta_i^2} \to -\delta'(\theta_i - \pi) + \delta'(\theta_i) = 0;
\]
\[
\sin \theta_i \frac{\partial}{\partial \theta_i} \to - \cos \theta_i; \quad \sin^2 \theta_i \frac{\partial^2}{\partial \theta_i^2} \to - \sin 2\theta_i;
\]
\[
\sin 2\theta_i \frac{\partial}{\partial \theta_i} \to - \cos \theta_i; \quad \sin^2 \theta_i \frac{\partial}{\partial \theta_i} \to 2 \cos 2\theta_i;
\]
\[
\sin 2\theta_i \frac{\partial^2}{\partial \theta_i^2} \to - (\delta(\theta_i - \pi) - \delta(\theta_i)) - 2 \sin 2\theta_i = -2 \sin 2\theta_i;
\]
\[
\cos \theta_i \frac{\partial}{\partial \theta_i} \to - (\delta(\theta_i - \pi) + \delta(\theta_i)) + \sin i
\]
The presence of $\delta-$functions and their derivatives in these formulas is a removable singularity due to the fact that the final regularization occurs on the set of functions $w(\vec{X}, \vec{\theta})$, with the additional symmetry property
\[
\frac{\partial^k}{\partial \theta_i^k} w(X_i, \theta_i = 0) = \frac{\partial^k}{\partial \theta_i^k} w(-X_i, \theta_i = \pi), \quad \text{where} \quad k = 0, 1, 2, 3 \ldots,
\]
In addition, we give one more useful formula for the dual symbol of the product of two operators. Let the operators $\mathcal{A}$ and $\mathcal{B}$, act on the set of such $W(\vec{q}, \vec{p}) \in S^{2n}$, that for every $W(\vec{q}, \vec{p}) \in S^{2n}$ the function $\mathcal{A}W(\vec{q}, \vec{p})$ and $\mathcal{B}W(\vec{q}, \vec{p})$ also belong to $S^{2n}$, then from (19) for the product of two operators and from the formula (32) we have equality:
\[
\langle \mathcal{A}\mathcal{B} \rangle = \int w^{(d)}_{\mathcal{A}\mathcal{B}}(\vec{X}, \vec{\theta}) w(\vec{X}, \vec{\theta}) d^n X d^n \theta = \int w^{(d)}_{\mathcal{A}}(\vec{X}, \vec{\theta}) R[\mathcal{B}] R[W(\vec{q}, \vec{p})](\vec{X}, \vec{\theta}) d^n X d^n \theta,
\]
or
\[
w^{(d)}_{\mathcal{A}}(\vec{X}, \vec{\theta}) R[\mathcal{B}](\vec{X}, \vec{\theta}) \to w^{(d)}_{\mathcal{A}\mathcal{B}}(\vec{X}, \vec{\theta}).
\]
Let us find a regular symbol of the operator $q_i$. From (20), (32), (35) and (36) we have the chain of equalities:
\[
\langle \hat{q}_i \rangle = \int f_{\hat{q}_i} w d^n X d^n \theta = \frac{1}{\pi^n} \int R[\hat{q}_i W] d^n X d^n \theta
\]
\[
= \frac{1}{\pi^n} \int \left\{ \sin \theta_i \left[ \frac{\partial}{\partial X_i} \right]^{-1} \frac{\partial}{\partial \theta_i} + X_i \cos \theta_i \right\} w d^n X d^n \theta = \frac{2}{\pi^n} \int X_i \cos \theta_i w d^n X d^n \theta.
\]
Thus we can write
\[
w^{(d)}_{\hat{q}_i}(\vec{X}, \vec{\theta}) = \frac{2}{\pi^n} X_i \cos \theta_i.
\]
Similarly, we find

\[

w^{(d)}_{\hat{p}_i}(\vec{X}, \vec{\theta}) = \frac{2}{\pi^n} X_i \sin \theta_i; \quad w^{(d)}_{\hat{q}_i}(\vec{X}, \vec{\theta}) = \frac{X_i^2}{\pi^n} (1 + 2 \cos 2\theta_i); \\
w^{(d)}_{\hat{p}_i^2}(\vec{X}, \vec{\theta}) = \frac{X_i^2}{\pi^n} (1 - 2 \cos 2\theta_i); \quad w^{(d)}_{\hat{q}_i \hat{p}_i}(\vec{X}, \vec{\theta}) = \frac{2}{\pi^n} X_i^2 \sin 2\theta_i + \frac{i}{2\pi^n}.
\]

From (21) we can find symbols of the components of the angular momentum of the particle

\[

w^{(d)}_{\hat{l}_1}(X_1, X_2, X_3; \theta_1, \theta_2, \theta_3) = \frac{4}{\pi^{2n}} X_2 X_3 \sin(\theta_3 - \theta_2); \\
w^{(d)}_{\hat{l}_2}(X_1, X_2, X_3; \theta_1, \theta_2, \theta_3) = \frac{4}{\pi^{2n}} X_1 X_3 \sin(\theta_1 - \theta_3); \\
w^{(d)}_{\hat{l}_3}(X_1, X_2, X_3; \theta_1, \theta_2, \theta_3) = \frac{4}{\pi^{2n}} X_1 X_2 \sin(\theta_2 - \theta_1);
\]

From formulas (23), (24), (35) and (36) and we can find

\[

w^{(d)}_{\hat{a}_i}(\vec{X}, \vec{\theta}) = \sqrt{\frac{2}{\pi^n}} X_i (\cos \theta_i + i \sin \theta_i); \quad w^{(d)}_{\hat{a}_i^*}(\vec{X}, \vec{\theta}) = \sqrt{\frac{2}{\pi^n}} X_i (\cos \theta_i - i \sin \theta_i); \\
w^{(d)}_{\hat{a}_i^* \hat{a}_i}(\vec{X}, \vec{\theta}) = w^{(d)}_{\hat{N}_i}(\vec{X}, \vec{\theta}) = \frac{1}{\pi^n} (X_i^2 - 1/2).
\]

Having different forms of dual symbols of the same operators, we can write test expressions for the experimentally measured tomograms. Thus, for the \( \hat{q} \) quadrature we have

\[

\langle \hat{q} \rangle = \int X \cos \theta \delta(\sin \theta) w(X, \theta) dX \ d\theta = \frac{2}{\pi} \int X \cos \theta w(X, \theta) dX \ d\theta. \quad (37)
\]

Similar test expressions can be written for the other operators.

6 Conclusion

To summarize, we point out the main results of this work.

We obtained the correspondence rules and explicit expressions for operators of physical quantities in optical tomographic representation. We presented the general formalism for symbols of operators in this representation. We found an explicit expressions for dual symbols of physical quantities in terms of regular generalised functions. The expressions for operators found in the work provide the possibility of direct calculations of physical quantities from the optical tomogram without recalculation of it to the Wigner function or to the density matrix.

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