Optimal pointwise estimates for derivatives of solutions to Laplace, Lamé and Stokes equations

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Abstract. Various optimal estimates for solutions of the Laplace, Lamé and Stokes equations in multidimensional domains, as well as new real-part theorems for analytic functions are obtained.

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1 Introduction

In the present paper we extend our study of the best constants in certain inequalities for solutions of the Laplace, Lamé and Stokes equations (see Kresin and Maz’ya [10]). We also deal with optimal estimates for analytic functions in the spirit of our recent article [9]. Let us formulate some results obtained in the sequel.

By $|\cdot|$ we denote the Euclidean length of a vector or absolute value of a scalar quantity. Let $\Omega$ be a domain in $\mathbb{R}^n$. By $d_x$ we mean the distance from a point $x \in \Omega$ to $\partial \Omega$ and by $\omega_n$ we denote the area of the $(n-1)$-dimensional unit sphere.

One of the results derived in Section 2 is the following pointwise estimate of the gradient of a bounded harmonic function in the complement $\Omega$ of a convex closed domain in $\mathbb{R}^n$:

$$|\nabla u(x)| \leq \frac{C_n}{d_x} \sup_{\Omega} |u|$$

for all $x \in \Omega$. Here

$$C_n = \frac{4(n-1)^{(n-1)/2} \omega_{n-1}}{n^{n/2} \omega_n}$$
is the best constant.

We also state here a limit estimate with the same best constant $C_n$, valid for arbitrary domains which is established in Section 2.

Let $\Omega$ be a domain in $\mathbb{R}^n$, and let $\mathcal{H}(\Omega)$ be the set of harmonic functions $u$ in $\Omega$ with $\sup_{\Omega} |u| \leq 1$. Suppose that a point $\xi \in \partial \Omega$ can be touched by an interior ball $B$. Then

$$\limsup_{x \to \xi} \sup_{u \in \mathcal{H}(\Omega)} |x - \xi| |\nabla u(x)| \leq C_n,$$

where $x$ is a point of the radius of $B$ directed from the center to $\xi$.

In Section 3 we obtain pointwise estimates for the directional derivative $(\ell, \nabla)u$, where $u(x) = (u_1(x), \ldots, u_m(x))$ is a vector field whose components are harmonic in $\Omega$. Assertions proved here are generalizations of the theorems given in Section 2. In Section 4 we present analogs of the theorems of Section 2 containing pointwise and limit estimates for $|\text{div } u(x)|$, $m = n$.

By $[C_b(\Omega)]^n$ we mean the space of vector-valued functions with $n$ components which are bounded and continuous on $\Omega$. This space is endowed with the norm $||u||_{[C_b(\Omega)]^n} = \sup\{|u(x)| : x \in \Omega\}$. By $[C^2(\Omega)]^n$ we denote the space of $n$-component vector-valued functions with continuous derivatives up to second order in $\Omega$.

Next, in Section 5 we find an optimal estimate for $|\text{div } u(x)|$, where $u$ is an elastic displacement vector in $\mathbb{R}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$. As a corollary, we obtain an optimal estimate for the pressure $p$ in a viscous incompressible fluid in $\mathbb{R}^n_+$. We formulate two statements following from these results.

Let $\Omega = \mathbb{R}^n \setminus G$, where $G$ is a convex domain in $\mathbb{R}^n$.

(i) Let $u \in [C^2(\Omega)]^n \cap [C_b(\Omega)]^n$ be a solution of the Lamé system

$$\Delta u + (1 - 2\sigma)^{-1} \text{grad div } u = 0$$

in $\Omega$, where $\sigma \in (-\infty, 1/2) \cup (1, +\infty)$ is the Poisson coefficient. Then for any point $x \in \Omega$ the inequality

$$|\text{div } u(x)| \leq \frac{(1 - 2\sigma)E_n}{(3 - 4\sigma)d_x} \sup_{\Omega} |u|$$

holds, where

$$E_n = \frac{4\omega_{n-1}}{\omega_n} \int_0^{\pi/2} [1 + n(n - 2) \cos^2 \vartheta]^{1/2} \sin^{n-2} \vartheta d\vartheta$$

is the best constant.

(ii) Let $u \in [C^2(\Omega)]^n \cap [C_b(\Omega)]^n$ be a vector component of the solution $\{u, p\}$ to the Stokes system

$$\Delta u - \text{grad } p = 0, \quad \text{div } u = 0 \quad \text{in } \Omega$$

and let $p(x)$ be the pressure vanishing as $d_x \to \infty$. Then for any point $x \in \Omega$ the inequality

$$|p(x)| \leq \frac{E_n}{d_x} \sup_{\Omega} |u|$$

holds with the same best constant $E_n$ as above.

The last Section 6 is dedicated to some new real-part theorems for analytic functions (see Kresin and Maz’ya [7] and the bibliography collected there). We derive the following results.
(i) Let $\Omega = \mathbb{C} \setminus G$, where $G$ is a convex domain in $\mathbb{C}$, and let $f$ be a holomorphic function in $\Omega$ with bounded real part. Then for any point $z \in \Omega$ the inequality

$$|f^{(s)}(z)| \leq \frac{K_s}{d_z} \sup_{\Omega} |\Re f|, \quad s = 1, 2, \ldots,$$

holds with $d_z = \text{dist}(z, \partial \Omega)$, where

$$K_s = \frac{s!}{\pi} \max_{\alpha} \int_{-\pi/2}^{\pi/2} \left| \cos \left( \alpha + (s + 1)\varphi \right) \right| \cos^{s-1} \varphi \, d\varphi \sup_{\Omega} |\Im f|,$$

is the best constant. In particular $K_{2l+1} = 2[(2l + 1)!]^2|\pi(2l + 1)|^{-1}$.

(ii) Let $\Omega$ be a domain in $\mathbb{C}$, and let $\mathcal{R}(\Omega)$ be the set of holomorphic functions $f$ in $\Omega$ with $\sup_{\Omega} |\Re f| \leq 1$. Assume that a point $\zeta \in \partial \Omega$ can be touched by an interior disk $D$. Then

$$\limsup_{z \to \zeta} \sup_{f \in \mathcal{R}(\Omega)} |z - \zeta|^s |f^{(s)}(z)| \leq K_s, \quad s = 1, 2, \ldots,$$

where $z$ is a point of the radius of $D$ directed from the center to $\zeta$. Here the constant $K_s$ is the same as above and cannot be diminished.

More details concerning the above formulations can be found in the statements of corresponding theorems, propositions and corollaries in what follows.

2 Estimates for the gradient of harmonic function

We introduce some notation used henceforth. Let $\mathbb{B} = \{ x \in \mathbb{R}^n : |x| < 1 \}$, $\mathbb{B}_R = \{ x \in \mathbb{R}^n : |x| < R \}$, and $\mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}$. By $h^\infty(\Omega)$ we denote the Hardy space of bounded harmonic functions on the domain $\Omega$ with the norm $||u||_{h^\infty(\Omega)} = \sup\{|u(x)| : x \in \Omega\}$.

Theorem 1. Let $\Omega = \mathbb{R}^n \setminus G$, where $G$ is a convex domain in $\mathbb{R}^n$, and let $u$ be a bounded harmonic function in $\Omega$. Then for any point $x \in \Omega$ the inequality

$$|\nabla u(x)| \leq \frac{C_n}{d_x} \sup_{\Omega} |u| \tag{2.1}$$

holds, where

$$C_n = \frac{4(n - 1)(n-1/2)}{n^{n/2} \omega_n} \omega_{n-1} \tag{2.2}$$

is the best constant in the inequality

$$|\nabla u(x)| \leq C_n x_n^{-1} \|u\|_{L^\infty(\partial \mathbb{R}^n_+)}$$

for a bounded harmonic function $u$ in the half-space $\mathbb{R}^n_+$. In particular,

$$C_2 = \frac{2}{\pi}, \quad C_3 = \frac{4}{3\sqrt{3}}.$$
Proof. Let $\xi \in \partial \Omega$ be a point at $\partial \Omega$ nearest to $x \in \Omega$ and let $T(\xi)$ be the hyperplane containing $\xi$ and orthogonal to the line joining $x$ and $\xi$. By $\mathbb{R}^n_\xi$ we denote the open half-space with boundary $T(\xi)$ such that $\mathbb{R}^n_\xi \subset \Omega$.

Let $n \geq 3$. According to Theorem 1 [3], the inequality
$$\left| \nabla u(x) \right| \leq \frac{C_n}{\partial x} \| u \|_{H^\infty(\mathbb{R}^n_\xi)}$$
(2.3)
holds, where $C_n$ is given by (2.2). Using (2.3) and the obvious inequality
$$\| u \|_{H^\infty(\mathbb{R}^n_\xi)} \leq \sup_{\Omega} \| u \|,$$
we arrive at (2.1).

The case $n = 2$ is considered analogously, the role of (2.3) being played by the estimate
$$|f'(z)| \leq \frac{2}{\pi |z|} \sup_{\mathbb{C}+} |f(z)|$$
(2.4)
(see [7], Sect. 3.7.3) by the change $f = u + iv$, $f'(z) = u'_x - iv'_y$, where $f$ is a holomorphic function in $\mathbb{C}_+ = \{ z \in \mathbb{C} : \Im z > 0 \}$ with bounded real part.

In what follows, we assume that the Cartesian coordinates with origin $O$ at the center of the ball are chosen in such a way that $x = |x|e_n$. By $\ell$ we denote an arbitrary unit vector in $\mathbb{R}^n$ and by $\nu_x$ we mean the unit vector of exterior normal to the sphere $|x| = r$ at a point $x$. Let $\ell_r$ be the orthogonal projection of $\ell$ on the tangent hyperplane to the sphere $|x| = r$ at $x$. If $\ell_r \neq 0$, we set $\tau_x = \ell_r / ||\ell_r||$, otherwise $\tau_x$ is an arbitrary unit vector tangent to the sphere $|x| = r$ at $x$. Hence
$$\ell = \ell_r \tau_x + \ell_x \nu_x,$$
(5.5)
where $\ell_r = ||\ell_r||$ and $\ell_x = (\ell, \nu_x)$.

We premise Lemmas 1 and 2 to Theorem 2. In Lemma 1 we derive a representation for the sharp coefficient $K_n(x)$ in the inequality
$$\left| \nabla u(x) \right| \leq K_n(x) \| u \|_{L^\infty(\partial \Omega)} ,$$
(6.6)
where $x \in \mathbb{B}$ and $u \in H^\infty(\mathbb{B})$. Here and elsewhere we say that a certain coefficient is sharp if it cannot be diminished for any point $x$ in the domain under consideration. The expression for $K_n(x)$, given below, contains two factors one of which is an explicitly given function increasing to infinity as $r \rightarrow 1$ and the second factor (the double integral) is a bounded function on the interval $0 \leq r \leq 1$.

Lemma 1. Let $u \in H^\infty(\mathbb{B})$, and let $x$ be an arbitrary point in $\mathbb{B}$. The sharp coefficient $K_n(x)$ in inequality (6.6) is given by
$$K_n(x) = \frac{2n^{-2}(n - 2)}{\pi(1 + r)^{n-1}(1 - r)} \sup_{\gamma \geq 0} \frac{1}{1 + \gamma^2} \int_0^\pi \sin^{-3} \varphi \, d\varphi \int_0^{\pi/2} G_n(\vartheta, \varphi; r, \gamma) \, d\vartheta ,$$
(7.7)
where
$$G_n(\vartheta, \varphi; r, \gamma) = \frac{n \cos 2\vartheta + n \gamma \sin 2\vartheta \cos \varphi + (n - 2)r}{1 + \left( \frac{1 + r}{1 - r} \right)^{2} \tan^2 \vartheta \left( n - 2 \right)} \sin^{n-2} \vartheta .$$
(8.8)

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Proof. 1. Representation for $K_n(x)$ by an integral over $\mathbb{S}^{n-1}$. Let $u$ stand for a harmonic function in $\mathbb{B}$ from the space $H^\infty(\mathbb{B})$. By Poisson formula we have

$$u(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{1-r^2}{|y-x|^n} u(y) d\sigma_y .$$

(2.9)

Fix a point $x \in \mathbb{B}$. By (2.9)

$$\frac{\partial u}{\partial x_i} = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \left[ \frac{-2x_i}{|y-x|^n} + \frac{n(1-r^2)(y_i-x_i)}{|y-x|^{n+2}} \right] u(y) d\sigma_y ,$$

that is

$$\nabla u(x) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{n(1-r^2)(y-x) - 2|y-x|^2x}{|y-x|^{n+2}} u(y) d\sigma_y .$$

Thus

$$(\nabla u(x), \ell) = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \frac{(n(1-r^2)(y-x) - 2|y-x|^2x, \ell)}{|y-x|^{n+2}} u(y) d\sigma_y ,$$

and therefore

$$K_n(x) = \frac{1}{\omega_n} \sup_{|\ell|=1} \int_{\mathbb{S}^{n-1}} \frac{|n(1-r^2)(y-x) - 2|y-x|^2x, \ell|}{|y-x|^{n+2}} d\sigma_y .$$

(2.10)

Using (2.5), we obtain

$$K_n(x) = \frac{1}{\omega_n} \sup_{|\ell|=1} \int_{\mathbb{S}^{n-1}} \frac{|n(1-r^2)((y, \nu_x) - r) - 2r|y-x|^2\ell_x + n(1-r^2)(y, \tau_x)\ell_r|}{|y-x|^{n+2}} d\sigma_y .$$

The last expression can be written as

$$K_n(x) = \frac{a_n(r)}{\omega_n} \sup_{|\ell|=1} \int_{\mathbb{S}^{n-1}} \frac{|b_n(r)(y, \tau_x)\ell_x + (y_n - c_n(r))\ell_r|}{(1 - 2ry_n + r^2)^{(n+2)/2}} d\sigma_y ,$$

(2.11)

where

$$a_n(r) = n(1-r^2) + 4r^2 , \quad b_n(r) = \frac{n(1-r^2)}{n(1-r^2) + 4r^2} , \quad c_n(r) = \frac{n(1-r^2) + 2(1+r^2)}{n(1-r^2) + 4r^2} r .$$

(2.12)

2. Representation for $K_n(x)$ by a double integral. Introducing the function

$$H_n(s,t; r, \ell) = \frac{|b_n(r)(y, \tau_x)\ell_x + (y_n - c_n(r))\ell_r|}{(1 - 2rt + r^2)^{(n+2)/2}} ,$$

(2.13)

we write the integral in (2.11) as the sum

$$\int_{\mathbb{S}^{n-1}} H_n((y, \tau_x), y_n; r, \ell) d\sigma_y + \int_{\mathbb{S}^{n-1}} H_n((y, \tau_x), y_n; r, \ell) d\sigma_y ,$$

(2.14)

where $S^{n-1}_+ = \{ y \in \mathbb{S}^{n-1} : (y, e_n) > 0 \}, S^{n-1}_- = \{ y \in \mathbb{S}^{n-1} : (y, e_n) < 0 \}$. 


Let \( y' = (y_1, \ldots, y_{n-1}) \in \mathbb{B}' = \{ y' \in \mathbb{R}^{n-1} : |y'| < 1 \} \). We put
\[
\tau'_x = \sum_{i=1}^{n-1} (\tau_x, e_i) e_i.
\]

Since \( y_n = \sqrt{1 - |y'|^2} \) for \( y \in S^{n-1}_+ \) and \( y_n = -\sqrt{1 - |y'|^2} \) for \( y \in S^{n-1}_- \) and since \( d\sigma_y = dy'/\sqrt{1-|y'|^2} \), it follows that each of integrals in (2.14) can be written in the form
\[
\int_{S^{n-1}_+} \mathcal{H}_n((y, \tau_x), y_n; r, \ell) d\sigma_y = \int_{\mathbb{B}'} \mathcal{H}_n\left((y', \tau_x'), \sqrt{1-|y'|^2}; r, \ell\right) \frac{dy'}{\sqrt{1-|y'|^2}},
\]
(2.15)
and
\[
\int_{S^{n-1}_-} \mathcal{H}_n((y, \tau_x), y_n; r, \ell) d\sigma_y = \int_{\mathbb{B}'} \mathcal{H}_n\left((y', \tau_x'), -\sqrt{1-|y'|^2}; r, \ell\right) \frac{dy'}{\sqrt{1-|y'|^2}}.
\]
(2.16)

Putting
\[
\mathcal{M}_n(s, t; r, \ell) = \mathcal{H}_n(s, t; r, \ell) + \mathcal{H}_n(s, -t; r, \ell),
\]
(2.17)
and using (2.13)-(2.16), we rewrite (2.11) as
\[
\kappa_n(x) = \frac{a_n(r)}{\omega_n} \sup_{|s|=1} \int_{\mathbb{B}'} \mathcal{M}_n\left((y', \tau_x'), \sqrt{1-|y'|^2}; r, \ell\right) \frac{dy'}{\sqrt{1-|y'|^2}}.
\]
(2.18)
By the identity
\[
\int_{\mathbb{B}^n} g((y, \xi), |y|) d\nu = \omega_{n-1} \int_0^1 \rho^{n-1} d\rho \int_0^\pi g(|\xi| \rho \cos \varphi, \rho) \sin^{n-2} \varphi d\varphi
\]
(see, e.g., [13, 3.3.2(3)]), we transform the integral in (2.18):
\[
\int_{\mathbb{B}'} \mathcal{M}_n\left((y', \tau_x'), \sqrt{1-|y'|^2}; r, \ell\right) \frac{dy'}{\sqrt{1-|y'|^2}}
\]
(2.19)
\[
= \omega_{n-2} \int_0^1 \rho^{n-2} d\rho \int_0^\pi \mathcal{M}_n\left(\rho \cos \varphi, \sqrt{1-\rho^2}; r, \ell\right) \sin^{n-3} \varphi d\varphi.
\]
The change \( \rho = \sin \theta \) in (2.19) gives
\[
\int_{\mathbb{B}'} \mathcal{M}_n\left((y', \tau_x'), \sqrt{1-|y'|^2}; r, \ell\right) \frac{dy'}{\sqrt{1-|y'|^2}}
\]
(2.20)
\[
= \omega_{n-2} \int_0^{\pi/2} \sin^{n/2} \theta d\theta \int_0^\pi \mathcal{M}_n\left(\sin \theta \cos \varphi, \cos \theta; r, \ell\right) \sin^{n-3} \varphi d\varphi.
\]
Applying (2.13), (2.17) and introducing the notation
\[
\mathcal{F}_n(\theta, \varphi; r, \ell) = \mathcal{H}_n\left(\sin \theta \cos \varphi, \cos \theta; r, \ell\right)
\]
\[
= \frac{|b_n(r)\ell_r \sin \theta \cos \varphi + (\cos \theta - c_n(r)) \ell_{\nu}|}{(1 - 2r \cos \theta + r^2)^{(n+2)/2}},
\]

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we write (2.20) as follows
\[\int_{\mathbb{B}} M_n^{(y,\tau)}(\sqrt{1-|y|^2};r,\ell) \frac{dy'}{\sqrt{1-|y'|^2}} = \omega_{n-2} \int_{\pi/2}^{\pi} \sin^{n-2} \theta d\theta \int_{0}^{\pi} \left( F_n(\theta,\varphi;r,\ell) + F_n(\pi-\theta,\varphi;r,\ell) \right) \sin^{n-3} \varphi \, d\varphi. \] (2.21)

Changing the variable \(\psi = \pi - \theta\), we obtain
\[\int_{\pi/2}^{\pi} \sin^{n-2} \theta d\theta \int_{0}^{\pi} F_n(\pi-\theta,\varphi;r,\ell) \sin^{n-3} \varphi \, d\varphi = \int_{\pi/2}^{\pi} \sin^{n-2} \psi d\psi \int_{0}^{\pi} F_n(\psi,\varphi;r,\ell) \sin^{n-3} \varphi \, d\varphi,\]
which together with (2.21) leads to the representation of (2.18):

\[K_n(x) = a_n(r) \omega_{n-2} \sup_{|\ell|=1} \int_{0}^{\pi} \sin^{n-2} \theta d\theta \int_{0}^{\pi} F_n(\theta,\varphi;r,\ell) \sin^{n-3} \varphi \, d\varphi. \] (2.22)

3. Transformation of representation for \(K_n(x)\). We make the change of variable
\[\theta = 2 \arctan \left( \frac{1-r}{1+r} \tan \vartheta \right)\]
in (2.22). Then
\[\sin \theta = \frac{2 \left( \frac{1-r}{1+r} \right) \tan \vartheta}{1 + \left( \frac{1-r}{1+r} \right)^2 \tan^2 \vartheta}, \] (2.23)
\[d\theta = \frac{2(1-r)}{(1+r) \cos^2 \vartheta \left( 1 + \left( \frac{1-r}{1+r} \right)^2 \tan^2 \vartheta \right)} \, d\vartheta, \] (2.24)
\[1 - 2r \cos \theta + r^2 = \frac{(1-r)^2}{\cos^2 \vartheta \left( 1 + \left( \frac{1-r}{1+r} \right)^2 \tan^2 \vartheta \right)}, \] (2.25)
\[b_n(r) \ell_r \sin \theta \cos \varphi + \left( \cos \theta - c_n(r) \right) \ell_\varphi = \frac{1-r)^2 [n \ell_r \sin 2\vartheta \cos \varphi + (n \cos 2\vartheta + (n-2)r) \ell_\varphi]}{[n(1-r^2) + 4r^2] \cos^2 \vartheta \left( 1 + \left( \frac{1-r}{1+r} \right)^2 \tan^2 \vartheta \right)}. \] (2.26)

Substituting (2.23)-(2.26) in (2.22), we arrive at
\[K_n(x) = \frac{2^{n-2}(n-2)}{\pi(1+r)^{n-1}(1-r) \sup_{|\ell|=1} \int_{0}^{\pi} \sin^{n-3} \varphi \, d\varphi \int_{0}^{\pi/2} G_n(\theta,\varphi;r,\ell) \, d\theta}, \] (2.27)
where
\[ G_n(\vartheta, \varphi; r, \ell) = \frac{n \ell \sin 2\vartheta \cos \varphi + (n \cos 2\vartheta + (n - 2)r) \ell \cos \varphi}{1 + \left(\frac{1 - r}{1 + r}\right)^2 \tan^2 \vartheta} \sin^{n-2} \vartheta. \]

Since the integrand in (2.10) does not change when the unit vector \( \ell \) is replaced by \(-\ell\), we may assume that \( \ell = (\ell, \nu_x) > 0 \) in (2.27). Introducing the parameter \( \gamma = \ell / \ell \) in (2.27) and using the equality \( \ell_\vartheta^2 + \ell_\nu^2 = 1 \), we arrive at (2.7) with \( G_n(\vartheta, \varphi; r, \gamma) \) given by (2.8).

By dilation, we obtain the following result, equivalent to Lemma 1 and involving the ball \( B_R \) with an arbitrary \( R \).

**Lemma 2.** Let \( u \in H^\infty(B_R) \), and let \( x \) be an arbitrary point in \( B_R \). The sharp coefficient \( K_{n,R}(x) \) in the inequality
\[ |\nabla u(x)| \leq K_{n,R}(x) ||u||_{L^\infty(\partial B_R)} \]
is given by
\[ K_{n,R}(x) = \frac{2 \pi (n - 2) R^{n-1}}{\pi R + |x| |R - |x||} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1 + \gamma^2}} \int_0^\pi \sin^{n-3} \varphi \ d\varphi \int_0^{\pi/2} G_n(\vartheta, \varphi; \frac{|x|}{R}, \gamma) \ d\vartheta, \]
where
\[ G_n(\vartheta, \varphi; r, \gamma) = \frac{n \gamma \sin 2\vartheta \cos \varphi + n \cos 2\vartheta + (n - 2)r}{1 + \left(\frac{1 - r}{1 + r}\right)^2 \tan^2 \vartheta} \sin^{n-2} \vartheta. \]

Now, we prove a limit estimate for the gradient of a bounded harmonic function.

**Theorem 2.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \), and let \( \mathcal{H}(\Omega) \) be the set of harmonic functions \( u \) in \( \Omega \) with \( \sup_{\Omega} |u| \leq 1 \). Assume that a point \( \xi \in \partial \Omega \) can be touched by an interior ball \( B \). Then
\[ \limsup_{x \to \xi} \sup_{u \in \mathcal{H}(\Omega)} |x - \xi| |\nabla u(x)| \leq C_n, \]
where \( x \) is a point at the radius of \( B \) directed from the center to \( \xi \). Here the constant \( C_n \) is the same as in Theorem 1.

**Proof.** Let \( n \geq 3 \). By Lemma 2, the relations
\[ \limsup_{|x| \to R} \sup_{|u| \in \mathcal{H}(\Omega)} \left\{ (R - |x|)|\nabla u(x)| : ||u||_{L^\infty(\mathbb{R}^n)} \leq 1 \right\} \leq \lim_{|x| \to R} (R - |x|) K_{n,R}(x) = C_n \]
hold, where
\[ C_n = \frac{n - 2}{2\pi} \sup_{\gamma \geq 0} \frac{1}{\sqrt{1 + \gamma^2}} \int_0^\pi \sin^{n-3} \varphi \ d\varphi \int_0^{\pi/2} |P_n(\vartheta, \varphi; \gamma)| \sin^{n-2} \vartheta \ d\vartheta, \]
with
\[ P_n(\vartheta, \varphi; \gamma) = n \gamma \sin 2\vartheta \cos \varphi + n \cos 2\vartheta + (n - 2) \]
\[ = 2 \left[ n \gamma \cos \vartheta \sin \vartheta \cos \varphi + (n \cos^2 \vartheta - 1) \right]. \]
According to Proposition 1 in [8], the sharp coefficient \( C_n(x) \) in the inequality
\[
|\nabla u(x)| \leq C_n(x) ||u||_{h^\infty(\mathbb{R}^n_+)} ,
\] (2.31)
where \( u \) is a bounded harmonic function in the half-space \( \mathbb{R}^n_+ \), is equal to \( C_n(x) = C_n/x_n \) with the best constant \( C_n \) given by (2.30). By Theorem 1 in [8], the value of \( C_n \) is given by the formula
\[
C_n = \frac{4(n-1)(n-1/2) \omega_{n-1}}{n^{n/2} \omega_n} .
\] (2.32)

Let \( R \) denote the radius of the ball \( B \subset \Omega \) tangent to \( \partial \Omega \) at the point \( \xi \). We put the origin \( O \) at the center of \( B \). Let the point \( x \) belong to the interval joining \( O \) and \( \xi \). Then \( R - |x| = |x - \xi| \).

By (2.29) with \( C_n \) from (2.32) on the right-hand side we conclude the proof in the case \( n \geq 3 \) by reference to the inequality
\[
||u||_{h^\infty(B)} \leq \sup_\Omega |u| .
\] (2.33)

The proof of Theorem 2 in the case \( n = 2 \) is analogous, estimate (2.29) follows from D. Khavinson’s [6] inequality
\[
|f'(z)| \leq \frac{4R}{\pi(R^2 - |z|^2)} \sup_{|\zeta| < R} |\Re f(\zeta)|
\] (2.34)
by the change \( f = u + iv, \ f'(z) = u'_x - iv'_y, \) where \( f \) is holomorphic in \( D_R = \{ z \in \mathbb{C} : |z| < R \} \).

The estimate (2.31) results from (2.4) by the change \( f = u + iv, \ f'(z) = u'_x - iv'_y, \) where \( f \) is holomorphic in \( \mathbb{C}_+ \).

**Remark 1.** The following inequality for the modulus of the gradient of a harmonic function is known (see [12], Ch. 2, Sect. 13)
\[
|\nabla u(x)| \leq \frac{A_n}{d_x} \osc_{\Omega}(u) ,
\]
where
\[
A_n = \frac{n\omega_{n-1}}{(n-1)\omega_n} .
\]
It is equivalent to the estimate
\[
|\nabla u(x)| \leq \frac{2A_n}{d_x} \sup_\Omega |u| ,
\] (2.35)
where \( u \) is a bounded harmonic function in \( \Omega \subset \mathbb{R}^n, \ n \geq 2, \) and \( \osc_{\Omega}(u) \) is the oscillation of \( u \) on \( \Omega \).

The coefficient on the right-hand side of (2.35) is less than that in the well known gradient estimate (see, e.g., [2], Sect. 2.7)
\[
|\nabla u(x)| \leq \frac{n}{d_x} \sup_\Omega |u| .
\]

By
\[
\frac{C_n}{2A_n} = \frac{2}{\sqrt{n}} \left( 1 - \frac{1}{n} \right)^{(n+1)/2} < 1 ,
\]
inequality (2.1) with \( C_n \) from (2.2) improves (2.35) for domains complementary to convex closed domains.

Sharp estimates of derivatives of harmonic functions can be found in the books [7], [10]. We also mention the articles [1], [3], [5] dealing with estimates of harmonic functions.
3 Estimates for the maximum value of the modulus of directional derivative of a vector field with harmonic components

Let in the domain $\Omega \subset \mathbb{R}^n$, there is a $m$-component vector field $\mathbf{a}(x) = (a_1(x), \ldots, a_m(x))$, $m \geq 1$. Let, further $\ell = (\ell_1, \ldots, \ell_n)$ be a unit $n$-dimensional vector. The derivative of the field $\mathbf{a}(x)$ in the direction $\ell$ is defined by

$$\frac{\partial \mathbf{a}}{\partial \ell} = \lim_{t \to 0} \frac{\mathbf{a}(x + t\ell) - \mathbf{a}(x)}{t},$$

that is

$$\frac{\partial \mathbf{a}}{\partial \ell} = (\ell, \nabla) \mathbf{a}. \quad (3.1)$$

Let us introduce some notation used in the sequel. By $||u||_{L^\infty(\partial \Omega)} = \sup\{||u(x)|| : x \in \partial \Omega\}$ we denote the norm in the space $[L^\infty(\partial \Omega)]^m$ of vector-valued functions $u$ on $\partial \Omega$ with $m$ components from $L^\infty(\partial \Omega)$. By $[h^\infty(\Omega)]^m$ we mean the Hardy space of vector-valued functions $u(x) = (u_1(x), \ldots, u_m(x))$ with bounded harmonic components on $\Omega$ endowed with the norm $||u||_{h^\infty(\Omega)} = \sup\{||u(x)|| : x \in \Omega\}$.

It is known that any element of $[h^\infty(\mathbb{R}^n_+)]^m$ can be represented by the Poisson integral

$$u(x) = \frac{2}{\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{x_n}{|y - x|^n} u(y)dy', \quad (3.2)$$

with boundary values in $[L^\infty(\partial \mathbb{R}^n_+)]^m$, where $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$.

Now, we find a representation for the sharp coefficient $C_{m,n}(x)$ in the inequality

$$\begin{align*}
\max_{|\ell| = 1} |(\ell, \nabla)u(x)| &\leq C_{m,n}(x)||u||_{L^\infty(\partial \mathbb{R}^n_+)}^m, \\
&= \mathcal{C}_{m,n}(x) ||u||_{L^\infty(\partial \mathbb{R}^n_+)}^m, \quad (3.3)
\end{align*}$$

where $u \in [h^\infty(\mathbb{R}^n_+)]^m$ and $x \in \mathbb{R}^n_+$.

**Lemma 3.** Let $u \in [h^\infty(\mathbb{R}^n_+)]^m$, and let $x$ be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $C_{m,n}(x)$ in (3.3) is given by

$$C_{m,n}(x) = C_{m,n} x_n^{-1}, \quad (3.4)$$

where

$$C_{m,n} = \frac{1}{\omega_n} \max_{|\ell| = 1} \int_{\mathbb{S}^{n-1}} |(e_n - n(e_\sigma, e_n) e_\sigma, \ell)| d\sigma, \quad (3.5)$$

and $e_\sigma$ stands for the $n$-dimensional unit vector joining the origin to a point $\sigma$ on the sphere $\mathbb{S}^{n-1}$.

**Proof.** Let $x = (x', x_n)$ be a fixed point in $\mathbb{R}^n_+$. The representation (3.2) implies

$$\frac{\partial \mathbf{u}}{\partial x_j} = \frac{2}{\omega_n} \int_{\partial \mathbb{R}^n_+} \left[ \frac{\delta_{nj}}{|y - x|^n} + \frac{nx_n(y_j - x_j)}{|y - x|^{n+2}} \right] u(y)dy',$$

that is, by (3.1),

$$\frac{\partial \mathbf{u}}{\partial \ell} = \frac{2}{\omega_n} \sum_{j=1}^n \ell_j \int_{\partial \mathbb{R}^n_+} \left[ \frac{\delta_{nj}}{|y - x|^n} + \frac{nx_n(y_j - x_j)}{|y - x|^{n+2}} \right] u(y)dy',$$

$$= \frac{2}{\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{\langle e_n - n(e_{xy}, e_n) e_{xy}, \ell \rangle}{|y - x|^n} u(y)dy,$$
Proposition 2. Let \( C \) holds, where the best constant we arrive at (3.7).

The case \( C \) where \( C \) is given by (2.2). Then, using the inequality
\[
\left| e - n(e_{xy}, e_{xy}, \ell) \right| \leq C \left| \frac{n}{y - x} \right|^n (u(y), z) dy.
\]

Hence,
\[
C_{m,n}(x) = \frac{2}{\omega_n} \max_{\ell \in \mathbb{R}^n} \int_{\partial \mathbb{R}^n} \left| \frac{n}{y - x} \right|^n dy

= \frac{1}{\omega_n} \max_{\ell \in \mathbb{R}^n} \int_{\mathbb{R}^n_+} \left| \frac{n}{y - x} \right|^n d\sigma.
\]
The last equality proves (3.4) and (3.5). \( \square \)

By Lemma 3, the sharp coefficient \( C_{m,n}(x) \) in inequality (3.3) does not depend on \( m \). Thus, \( C_{m,n}(x) = C_{1,n}(x) = C_n(x) \), where \( C_n(x) = C_n(x_n^{-1}) \) is the sharp coefficient in (2.31). Thus, we arrive at the following generalization of Theorem 1 in our paper [8], where the case \( m = 1 \) is treated.

Proposition 1. Let \( u \in [h^\infty(\mathbb{R}^n_+)]^m \) and let \( x \) be an arbitrary point in \( \mathbb{R}^n_+ \). The inequality
\[
\max_{|\ell| = 1} |(\ell, \nabla)u(x)| \leq C_n x_n^{-1} ||u||_{L^\infty(\partial \mathbb{R}^n_+)}^m
\]
holds, where the best constant \( C_n \) is the same as in Theorem 1.

The assertion below is an extension of Theorem 1.

Proposition 2. Let \( \Omega = \mathbb{R}^n \setminus G \), where \( G \) is a convex subdomain of \( \mathbb{R}^n \), and let \( u \) be a vector-valued function with \( m \) bounded harmonic components in \( \Omega \). Then for any point \( x \in \Omega \) the inequality
\[
\max_{|\ell| = 1} |(\ell, \nabla)u(x)| \leq C_n \frac{d_x}{d_x} \sup_{\Omega} |u|
\]
holds, where the constant \( C_n \) is the same as in Theorem 1.

Proof. Let \( \xi \in \partial \Omega \) be the point at \( \partial \Omega \) nearest to \( x \in \Omega \). Let the notation \( \mathbb{R}^n_\xi \) be the same as in the proof of Theorem 1. By Proposition 1
\[
\max_{|\ell| = 1} |(\ell, \nabla)u(x)| \leq C_n \frac{d_x}{d_x} ||u||_{[h^\infty(\mathbb{R}^n_\xi)]^m},
\]
where \( C_n \) is given by (2.2). Then, using the inequality
\[
||u||_{[h^\infty(\mathbb{R}^n_\xi)]^m} \leq \sup_{\Omega} |u|,
\]
we arrive at (3.7). \( \square \)

Any element of \([h^\infty(\mathbb{B})]^m \) can be represented as the Poisson integral
\[
u(x) = \frac{1}{\omega_n} \int_{\mathbb{R}^n_+} \frac{1 - r^2}{y - x} u(y) d\sigma_y
\]
with boundary values in \([L^\infty(\partial \mathbb{B})]^m \).

In the next assertion we find a representation for the sharp coefficient \( K_{m,n}(x) \) in the inequality
\[
\max_{|\ell| = 1} |(\ell, \nabla)u(x)| \leq K_{m,n}(x) ||u||_{[L^\infty(\partial \mathbb{B})]^m}.
\]
Lemma 4. Let \( u \in [h^{\infty}(\mathbb{B})]^m \), and let \( x \) be an arbitrary point in \( \mathbb{B} \). The sharp coefficient \( K_{m,n}(x) \) in (3.10) is given by

\[
K_{m,n}(x) = \frac{1}{\omega_n} \sup_{\ell \in \mathbb{S}^{n-1}} \int_{S^{n-1}} \frac{\left| (n \left( 1 - r^2 \right) (y - x) - 2|y - x|^2 x, \ell \right) \right|}{|y - x|^{n+2}} d\sigma_y. \tag{3.11}
\]

Proof. Fix a point \( x \in \mathbb{B} \). By (3.9)

\[
\frac{\partial u}{\partial x_j} = \frac{1}{\omega_n} \int_{S^{n-1}} \left[ \frac{-2x_j}{|y - x|^n} + \frac{n \left( 1 - r^2 \right) (y_j - x_j)}{|y - x|^{n+2}} \right] u(y) d\sigma_y,
\]

that is

\[
\frac{\partial u}{\partial \ell} = (\ell, \nabla)u(x) = \frac{1}{\omega_n} \sum_{j=1}^n \ell_j \int_{S^{n-1}} \frac{n \left( 1 - r^2 \right) (y_j - x_j) - 2|y - x|^2 x_j}{|y - x|^{n+2}} u(y) d\sigma_y.
\]

For any \( z \in S^{n-1} \) we have

\[
( (\ell, \nabla)u(x), z ) = \frac{1}{\omega_n} \int_{S^{n-1}} \frac{n \left( 1 - r^2 \right) (y - x) - 2|y - x|^2 x, \ell \right) \right|}{|y - x|^{n+2}} (u(y), z) d\sigma_y,
\]

which implies (3.11). \( \square \)

The next assertion is a generalization of Theorem 2.

Proposition 3. Let \( \Omega \) be a domain in \( \mathbb{R}^n \). Let \( \mathfrak{U}(\Omega) \) be the set of \( m \)-component vector-valued functions \( u \) whose components are harmonic in \( \Omega \), with \( \sup_{\Omega} |u| \leq 1 \). Assume that a point \( \xi \in \partial \Omega \) can be touched by an interior ball \( B \). Then

\[
\limsup_{x \to \xi} \sup_{\mathfrak{U}(\Omega)} \max_{|\ell| = 1} |x - \xi|| (\ell, \nabla)u(x) | \leq C_n, \tag{3.12}
\]

where \( x \) is a point of the radius of \( B \) directed from the center to \( \xi \). Here the constant \( C_n \) is the same as in Theorem 1.

Proof. By Lemma 4 \( K_{m,n}(x) \) does not depend on \( m \) and therefore \( K_{m,n}(x) = K_{1,n}(x) = K_n(x) \), where \( K_n(x) \) is the sharp coefficient in (2.4). Hence (3.10) can be written in the form

\[
\max_{|\ell| = 1} | (\ell, \nabla)u(x) | \leq K_n(x)||u||_{L^\infty(\partial \mathbb{B})^m}.
\]

By dilation in the last inequality we obtain the analogue of Lemma 2

\[
\max_{|\ell| = 1} | (\ell, \nabla)u(x) | \leq K_{n,R}(x)||u||_{L^\infty(\partial \mathbb{B}_{R})^m}, \tag{3.13}
\]

where \( x \in \mathbb{B}_R \) and \( u \in [h^{\infty}(\mathbb{B}_R)]^m \). Now, (3.13) along with the representation of \( K_{n,R}(x) \) from Lemma 2 leads to the inequality

\[
\limsup_{|x| \to R} \sup_{|\ell| = 1} \left| (R - |x|)(\ell, \nabla)u(x) \right| : |\ell| = 1, ||u||_{h^{\infty}(\mathbb{B}_R)} \leq 1 \leq \lim_{|x| \to R} (R - |x|)K_{n,R}(x) = C_n,
\]

where \( C_n \) is given by (2.30). The proof is completed in the same way as that of Theorem 2, with the only difference that (2.33) is replaced by the inequality

\[
||u||_{h^{\infty}(\mathbb{B})} \leq \sup_{\Omega} |u|, \tag{3.14}
\]

\( \square \)
4 Estimates for the divergence of a vector field with harmonic components

Let $u(x) = (u_1(x), \ldots, u_n(x))$ be a vector field with $n$ bounded harmonic components in $\Omega \subset \mathbb{R}^n$.

**Proposition 4.** Let $u \in [h^\infty(\mathbb{R}^n_+)]^n$, and let $x$ be an arbitrary point in $\mathbb{R}^n_+$. The sharp coefficient $D_n(x)$ in the inequality

$$|\text{div} \ u(x)| \leq D_n(x)||u||_{L^\infty(\partial \mathbb{R}^n_+)}$$

is given by

$$D_n(x) = D_n x_n^{-1},$$

where

$$D_n = \frac{2\omega_{n-1}}{\omega_n} \int_0^{\pi/2} [1 + n(n-2) \cos^2 \vartheta]^{1/2} \sin^{n-2} \vartheta d\vartheta.$$

In particular,

$$D_2 = 1, \quad D_3 = 1 + \frac{\sqrt{3}}{6} \ln (2 + \sqrt{3}).$$

**Proof.** By (3.2),

$$\text{div} \ u = \frac{2}{\omega_n} \sum_{j=1}^n \int_{\partial \mathbb{R}^n_+} u_j(y) \frac{\partial}{\partial x_j} \left( \frac{x_n}{|y-x|^n} \right) dy' =$$

$$\frac{2}{\omega_n} \sum_{j=1}^n \int_{\partial \mathbb{R}^n_+} \left( \frac{\delta_{jn}}{|y-x|^n} + \frac{n x_n (y_j - x_j)}{|y-x|^{n+2}} \right) u_j(y) dy' =$$

$$\frac{2}{\omega_n} \sum_{j=1}^n \int_{\partial \mathbb{R}^n_+} \left( \delta_{jn} - n(e_{xy}, e_n) e_{xy}, u_j(y) \right) dy'.$$

which implies

$$\text{div} \ u = \frac{2}{\omega_n} \int_{\partial \mathbb{R}^n_+} \left( e_n - n(e_{xy}, e_n) e_{xy}, u(y) \right) \frac{y}{|y-x|^n} dy'.$$

This equality shows that the sharp coefficient $D_n(x)$ in (4.1) is represented in the form

$$D_n(x) = \frac{2}{\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{|e_n - n(e_{xy}, e_n) e_{xy}|}{|y-x|^n} dy'.$$

Then

$$D_n(x) = \frac{2}{\omega_n x_n} \int_{\partial \mathbb{R}^n_+} \frac{|e_n - n(e_{xy}, e_n) e_{xy}|}{|y-x|^n} dy' = \frac{D_n}{x_n},$$

where

$$D_n = \frac{2}{\omega_n} \int_{S^{n-1}} |e_n - n(e_\sigma, e_n) e_\sigma| d\sigma = \frac{1}{\omega_n} \int_{S^{n-1}} |e_n - n(e_\sigma, e_n) e_\sigma| d\sigma.$$

The identity

$$|e_n - n(e_\sigma, e_n) e_\sigma|^2 = 1 + n(n-2)(e_\sigma, e_n)^2,$$
along with (4.7) leads to the formula

\[ D_n = \frac{1}{\omega_n} \int_{S^{n-1}} \left( 1 + n(n-2)(\mathbf{e}_n, \mathbf{e}_n)^2 \right)^{1/2} d\sigma. \]  

(4.9)

Using

\[ \int_{S^{n-1}} f((\xi, y)) d\sigma = \omega_{n-1} \int_{-1}^{1} f(|\xi|) (1 - t^2)^{(n-3)/2} dt \]  

(see, e.g., [14, 4.3.2(2)] and the change of variable \( t = \cos \vartheta \)), we obtain

\[ \int_{S^{n-1}} \left( 1 + n(n-2)(\mathbf{e}_n, \mathbf{e}_n)^2 \right)^{1/2} d\sigma = 2\omega_{n-1} \int_{0}^{\pi/2} \left[ 1 + n(n-2) \cos^2 \vartheta \right]^{1/2} \sin^{n-2} \vartheta \sin d\vartheta. \]  

(4.11)

By (4.6), (4.9) and (4.11), we arrive at (4.2) and (4.3). \( \square \)

The next assertion is analogous to Proposition 2. Here the divergence replaces the directional derivative.

**Proposition 5.** Let \( \Omega = \mathbb{R}^n \setminus \overline{G} \), where \( G \) be a convex subdomain of \( \mathbb{R}^n \), and let \( u \) be a \( n \)-component vector-valued function with bounded harmonic components in \( \Omega \). Then for any point \( x \in \Omega \) the inequality

\[ |\text{div} \ u(x)| \leq \frac{D_n}{d_x} \sup_{\Omega} |u| \]  

(4.12)

holds, where the constant \( D_n \) is the same as in Proposition 4.

**Proof.** Let \( \xi \in \partial \Omega \) be a point at \( \partial \Omega \) nearest to \( x \in \Omega \). Let the notation \( \mathbb{R}^n_\xi \) be the same as in the proof of Theorem 1. By Proposition 4

\[ |\text{div} \ u(x)| \leq \frac{D_n}{d_x} \left( \left| u \right|_{L^\infty(\mathbb{R}^n_\xi)} \right)^n, \]

where \( D_n \) is defined by (4.3). Then by (3.3) with \( m = n \), we arrive at (4.12). \( \square \)

**Lemma 5.** Let \( u \in [L^\infty(\mathbb{B})]^n \), and let \( x \) be an arbitrary point in \( \mathbb{B} \). The sharp coefficient \( T_n(x) \) in the inequality

\[ |\text{div} \ u| \leq T_n(x) \left| u \right|_{L^\infty(\partial \mathbb{B})} \]  

(4.13)

is given by

\[ T_n(x) = \frac{2^{n-1} \omega_{n-1}}{\omega_n (1+r)^{n-1}} \int_0^{\pi/2} \left[ \left( n - (n-2)r \right)^2 + 4n(n-2)r \cos^2 \vartheta \right]^{1/2} \frac{\sin^{n-2} \vartheta \sin d\vartheta}{1 + \left( \frac{r-x}{1+r} \right)^2 \tan^2 \vartheta}. \]  

(4.14)

In particular,

\[ T_2(x) = \frac{2}{1-r^2}, \quad T_3(x) = \frac{1}{1-r^2} \left( 2 + \frac{3-r^2}{2\sqrt{3}} \ln \frac{\sqrt{3}+r}{\sqrt{3}-r} \right). \]
Finally, setting which leads to the formula

\[
div \mathbf{u} = \frac{1}{\omega_n} \int_{S^{n-1}} \left( \frac{-2x_j}{|y-x|^n} + \frac{n(1-r^2)(y_j-x_j)}{|y-x|^{n+2}} \right) u_j(y) d\sigma_y.
\]

Therefore,

\[
div \mathbf{u} = \frac{1}{\omega_n} \int_{S^{n-1}} \left( \frac{-2x}{|y-x|^n} + \frac{n(1-r^2)(y-x)}{|y-x|^{n+2}} , \mathbf{u}(y) \right) d\sigma_y.
\]

This implies that the sharp coefficient \( T_n(x) \) in (4.13) has the form

\[
T_n(x) = \frac{1}{\omega_n} \int_{S^{n-1}} \frac{|-2x| y - x|^2 + n(1-r^2)(y-x)|}{|y-x|^{n+2}} d\sigma_y,
\]

which leads to the formula

\[
T_n(x) = \frac{1}{\omega_n} \int_{S^{n-1}} \frac{(4r^2 + a^2(r) - 4a(r)(x,y))^{1/2}}{(1 - 2(x,y) + r^2)^{(n+1)/2}} d\sigma_y, \tag{4.15}
\]

where \( a(r) = 2r^2 + n(1-r^2) \). Transforming the integral in (4.15) with help of (4.10), we obtain

\[
T_n(x) = \frac{\omega_{n-1}}{\omega_n} \int_{-1}^{1} \frac{(4r^2 + a^2(r) - 4a(r)t)^{1/2}}{(1 - 2rt + r^2)^{(n+1)/2}} (1 - t^2)^{(n-3)/2} dt.
\]

Changing the variable \( t = \cos \theta \), we derive

\[
T_n(x) = \frac{\omega_{n-1}}{\omega_n} \int_{0}^{\pi} \frac{(4r^2 + a^2(r) - 4a(r) \cos \theta)^{1/2}}{(1 - 2r \cos \theta + r^2)^{(n+1)/2}} \sin^{n-2} \theta d\theta. \tag{4.16}
\]

Finally, setting

\[
\theta = 2 \arctan \left( \frac{1-r}{1+r} \tan \vartheta \right)
\]

in (4.16) and using (2.23) - (2.25), we arrive at (4.14). \( \square \)

By dilation in Lemma 5, we obtain

Lemma 6. Let \( \mathbf{u} \in [h^\infty(\mathbb{B}_R)^n] \), and let \( x \) be an arbitrary point in \( \mathbb{B}_R \). The sharp coefficient \( T_{n,R}(x) \) in the inequality

\[
|div \mathbf{u}(x)| \leq \mathcal{T}_{n,R}(x)||\mathbf{u}||_{L^\infty(\partial\mathbb{B}_R)^n}
\]

is given by

\[
T_{n,R}(x) = \frac{2^{n-1} \omega_{n-1} R^{n-1}}{\omega_n (R + |x|)^{n-1}(R - |x|)} \int_{0}^{\pi/2} Q_n \left( \vartheta; \frac{|x|}{R} \right) \sin^{n-2} \vartheta d\vartheta,
\]

where

\[
Q_n(\vartheta; r) = \left[ \frac{(n - (n-2)r)^2 + 4n(n-2)r \cos^2 \vartheta}{1 + (\frac{1-r}{1+r})^2 \tan^2 \vartheta} \right]^{(n-2)/2}.
\]

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Proposition 6. Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $\mathfrak{U}(\Omega)$ be the set of $n$-component vector-valued functions $\mathbf{u}$ whose components are harmonic in $\Omega$, and $\sup_{\Omega} |\mathbf{u}| \leq 1$. Suppose that a point $\xi \in \partial \Omega$ can be touched by an interior ball $B$. Then
\[
\limsup_{x \to \xi} \sup_{\mathbf{u} \in \mathfrak{U}(\Omega)} |x - \xi| |\text{div} \mathbf{u}(x)| \leq D_n ,
\]
where $x$ is a point of the radius of $B$ directed from the center to $\xi$. Here the constant $D_n$ is the same as in Proposition 4.

Proof. By Lemma 6, the relations
\[
\limsup_{|x| \to R} \sup_{\mathbf{u}} \{ (R - |x|) |\text{div} \mathbf{u}(x)| : ||\mathbf{u}||_{\mathcal{H}(\mathcal{S}_R^n)} \leq 1 \} \leq \lim_{|x| \to R} (R - |x|) \mathcal{T}_{n,R}(x) = D_n \quad (4.17)
\]
hold, where $D_n$ is the same as in Proposition 4.

Using the notation introduced in Theorem 2, by (4.17) and (3.14) with $m = n$ the result follows. \qed

5 Estimates for the divergence of an elastic displacement field and the pressure in a fluid

Let $[C_b(\partial \mathbb{R}^n_+)]^n$ be the space of vector-valued functions with $n$ components which are bounded and continuous on $\partial \mathbb{R}^n_+$. This space is endowed with the norm $||\mathbf{u}||_{[C_b(\partial \mathbb{R}^n_+)]^n} = \sup \{ |\mathbf{u}(x)| : x \in \partial \mathbb{R}^n_+ \}$.

In the half-space $\mathbb{R}^n_+$, $n \geq 2$, consider the Lamé system
\[
\Delta \mathbf{u} + (1 - 2\sigma)^{-1} \text{grad} \text{div} \mathbf{u} = 0 , \quad (5.1)
\]
and the Stokes system
\[
\Delta \mathbf{u} - \text{grad} p = 0 , \quad \text{div} \mathbf{u} = 0 , \quad (5.2)
\]
with the boundary condition
\[
\mathbf{u}|_{x_n = 0} = \mathbf{f} , \quad (5.3)
\]
where $\sigma$ is the Poisson coefficient, $\mathbf{f} \in [C_b(\partial \mathbb{R}^n_+)]^n$, $\mathbf{u} = (u_1, \ldots, u_n)$ is the displacement vector of an elastic medium or the velocity vector of a fluid, and $p(x)$ is the pressure in the fluid vanishing as $x_n \to \infty$.

We assume that $\sigma \in (-\infty, 1/2) \cup (1, +\infty)$ which means the strong ellipticity of system (5.1). By $\lambda$ and $\mu$ we denote the Lamé constants. Since $\sigma = \lambda/2(\lambda + \mu)$ the strong ellipticity is equivalent to the inequalities $\mu > 0, \lambda + \mu > 0$ and $-\mu < \lambda + \mu < 0$.

A unique solution $\mathbf{u} \in [C^2(\mathbb{R}^n_+)]^n \cap [C_b(\overline{\mathbb{R}^n_+})]^n$ of problem (5.1), (5.3) and the vector component $\mathbf{u} \in [C^2(\mathbb{R}^n_+)]^n \cap [C_b(\overline{\mathbb{R}^n_+})]^n$ of a solution $\{\mathbf{u}, p\}$ to problem (5.2), (5.3) admit the representation (see, e.g., [10], pp. 64-65)
\[
\mathbf{u}(x) = \int_{\partial \mathbb{R}^n_+} \mathcal{H} \left( \frac{y - x}{|y - x|} \right) \frac{x_n}{|y - x|^n} \mathbf{f}(y') dy' , \quad (5.4)
\]
where $x \in \mathbb{R}^n_+$, $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$. Here $\mathcal{H}$ is the $(n \times n)$-matrix-valued function on $\mathbb{S}^{n-1}$ with elements
\[
2 \omega_n \left( (1 - \kappa)\delta_{jk} + n\kappa \frac{(y_j - x_j)(y_k - x_k)}{|y - x|^2} \right) , \quad (5.5)
\]
where $\kappa = 1$ for the Stokes system and $\kappa = (3 - 4\sigma)^{-1}$ for the Lamé system.

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Proposition 7. (i) Let \( u \in [C^2(\mathbb{R}_+^n)]^n \cap [C_b(\mathbb{R}_+^n)]^n \) be a solution of the Lamé system in \( \mathbb{R}_+^n \). The sharp coefficient \( E_n(x) \) in the inequality

\[
|\text{div } u(x)| \leq E_n(x)||u||_{[C_b(\partial \mathbb{R}_+^n)]^n}
\]

is given by

\[
E_n(x) = \frac{1 - 2\sigma}{3 - 4\sigma} E_n x_n^{-1},
\]

where

\[
E_n = \frac{4\omega_{n-1}}{\omega_n} \int_0^{\pi/2} [1 + n(n - 2) \cos^2 \vartheta]^{1/2} \sin^{n-2} \vartheta \, d\vartheta.
\]

In particular,

\[
E_2 = 2, \quad E_3 = 2 \left(1 + \frac{\sqrt{3}}{6} \ln (2 + \sqrt{3})\right).
\]

(ii) Let \( u \in [C^2(\mathbb{R}_+^n)]^n \cap [C_b(\mathbb{R}_+^n)]^n \) be the vector component of a solution \( \{u, p\} \) of the Stokes system (1.2) in \( \mathbb{R}_+^n \) and \( p(x) \) be the pressure vanishing as \( x_n \to \infty \). The sharp coefficient \( S_n(x) \) in the inequality

\[
|p(x)| \leq S_n(x)||u||_{[C_b(\partial \mathbb{R}_+^n)]^n}
\]

is given by

\[
S_n(x) = E_n x_n^{-1},
\]

where the constant \( E_n \) is defined by (5.8).

Proof. (i) Proof of inequality (5.6). By (5.4) and (5.5),

\[
u_j(x) = \frac{2}{\omega_n} \int_{\partial \mathbb{R}_+^n} \left(1 - \kappa\right) e_j + n\kappa \frac{(y_j - x_j)(y - x)}{|y - x|^2}, f(y') \right) \frac{x_n}{|y - x|^n} \, dy'.
\]

Noting that \( y_n = 0 \) in (5.11), we find

\[
\sum_{j=1}^n \frac{\partial}{\partial x_j} \left(\frac{(y_j - x_j)(y - x, f(y')) x_n}{|y - x|^{n+2}}\right) = \sum_{j=1}^n \frac{(n + 2)(y_j - x_j)^2(y - x, f(y')) x_n}{|y - x|^{n+4}} + \\
\sum_{j=1}^n \frac{-(y - x, f(y')) x_n + (y_j - x_j)f(y') x_n + (y_j - x_j)(y - x, f(y')) \delta_{nj}}{|y - x|^{n+2}} = \\
\frac{-n(y - x, f(y')) x_n - (y - x, f(y')) x_n + (y_n - x_n)(y - x, f(y')) + (n + 2)(y - x, f(y'))}{|y - x|^{n+2}} = 0.
\]

This together with (5.11) gives

\[
\text{div } u(x) = \frac{2}{\omega_n} (1 - \kappa) \sum_{j=1}^n \int_{\partial \mathbb{R}_+^n} f_j(y') \frac{\partial}{\partial x_j} \left(\frac{x_n}{|y - x|^n}\right) \, dy'.
\]

Hence using (4.4), (4.5) and \( \kappa = (3 - 4\sigma)^{-1} \), we have

\[
\text{div } u(x) = \frac{4(1 - 2\sigma)}{\omega_n (3 - 4\sigma)} \int_{\partial \mathbb{R}_+^n} \left(e_n - n(e_{xy}, e_n)e_{xy}, f(y')\right) \frac{x_n}{|y - x|^n} \, dy'.
\]
Therefore the sharp coefficient $\mathcal{E}_n(x)$ in (5.6) is represented in the form
\[
\mathcal{E}_n(x) = \frac{4(1 - 2\sigma)}{\omega_n(3 - 4\sigma)} \int_{\partial \mathbb{R}^n_+} \frac{|e_n - n(e_{xy}, e_n) e_{xy}| x_n}{|y - x|^n} dy' .
\]

Thus,
\[
\mathcal{E}_n(x) = \frac{4(1 - 2\sigma)}{\omega_n(3 - 4\sigma)x_n} \int_{\partial \mathbb{R}^n_+} |e_n - n(e_{xy}, e_n) e_{xy}| \frac{x_n}{|y - x|^n} dy' = \frac{(1 - 2\sigma) E_n}{(3 - 4\sigma)x_n} ,
\]

where
\[
E_n = \frac{4}{\omega_n} \int_{S^{n-1}} |e_n - n(e_{\sigma}, e_n) e_{\sigma}| d\sigma = \frac{2}{\omega_n} \int_{S^{n-1}} |e_n - n(e_{\sigma}, e_n) e_{\sigma}| d\sigma .
\]

Using (1.8), we write the last equality as
\[
E_n = \frac{2}{\omega_n} \int_{S^{n-1}} \left( 1 + n(n - 2)(e_{\sigma}, e_n)^2 \right)^{1/2} d\sigma .
\]

By (5.13), (5.14) and (4.11), we arrive at (5.7) and (5.8).

(ii) Proof of inequality (5.9). We write (5.1) as
\[
\Delta u - \text{grad } p = 0 , \quad p = - \frac{1}{1 - 2\sigma} \text{ div } u .
\]

It follows from (5.12) that $\text{div } u(x) \to 0$ for every $x \in \mathbb{R}^n_+$ as $\sigma \to 1/2$. We also see that
\[
p(x) = - \frac{1}{1 - 2\sigma} \text{ div } u(x) = - \frac{4}{\omega_n(3 - 4\sigma)} \int_{\partial \mathbb{R}^n_+} \left( e_n - n(e_{xy}, e_n) e_{xy}, f(y') \right) \frac{d\sigma}{|y - x|^n}
\]
tends to
\[
- \frac{4}{\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{(e_n - n(e_{xy}, e_n) e_{xy}, f(y'))}{|y - x|^n} dy' .
\]
as $\sigma \to 1/2$. Hence
\[
p(x) = - \frac{4}{\omega_n} \int_{\partial \mathbb{R}^n_+} \frac{(e_n - n(e_{xy}, e_n) e_{xy}, f(y'))}{|y - x|^n} dy' .
\]
Replacing $\text{div } u(x)$ by $(2\sigma - 1)p(x)$ in (5.6), and taking the limit as $\sigma \to 1/2$, we arrive at (5.9) with the sharp coefficient (5.10). \hfill \Box

By Proposition 7 with the same argument as in Proposition 5, we derive

**Corollary 1.** Let $\Omega = \mathbb{R}^n \setminus \overline{G}$, where $G$ is a convex domain in $\mathbb{R}^n$. Let $u, \psi \in [C^2(\Omega)]^n \cap [C_b(\Omega)]^n$ be a solution of the Lamé system in $\Omega$. Then for any point $x \in \Omega$ the inequality
\[
|\text{div } u(x)| \leq \frac{(1 - 2\sigma) E_n}{(3 - 4\sigma)d_x} \sup_{\Omega} |u|
\]
holds, where the constant $E_n$ is the same as in Proposition 7.
Corollary 2. Let $\Omega = \mathbb{R}^n \setminus \overline{G}$, where $G$ is a convex domain in $\mathbb{R}^n$. Let $\mathbf{u} \in [C^2(\Omega)]^n \cap [C_b(\overline{\Omega})]^n$ be the vector component of a solution $\{\mathbf{u}, p\}$ of the Stokes system (5.2) in $\Omega$ and let $p(x)$ be the pressure vanishing as $d_x \to \infty$. Then for any point $x \in \Omega$ the inequality

$$|p(x)| \leq \frac{E_n}{d_x} \sup_\Omega |\mathbf{u}|$$

holds, where the constant $E_n$ is the same as above.

6 Real-part estimates for derivatives of analytic functions

Theorem 3. Let $\Omega = \mathbb{C} \setminus \overline{G}$, where $G$ is a convex domain in $\mathbb{C}$, and let $f$ be a holomorphic function in $\Omega$ with bounded real part. Then for any point $z \in \Omega$ the inequality

$$|f^{(s)}(z)| \leq \frac{K_s}{d_z^s} \sup_\Omega |\Re f|, \quad s = 1, 2, \ldots ,$$

holds with $d_z = \text{dist} (z, \partial \Omega)$, where

$$K_s = \frac{s!}{\pi} \max_\alpha \int_{-\pi/2}^{\pi/2} \left| \cos (\alpha + (s + 1)\varphi) \right| \cos^{s-1} \varphi \, d \varphi$$

(6.2)

is the best constant in the inequality

$$|f^{(s)}(z)| \leq \frac{K_s}{(3z)^s} \|\Re f\|_{L^\infty(\partial \mathbb{C}_+)}$$

(6.3)

for holomorphic functions $f$ in the half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$ with bounded real part.

In particular,

$$K_{2l+1} = \frac{2((2l + 1)!!)^2}{\pi (2l + 1)},$$

(6.4)

and

$$K_2 = \frac{3\sqrt{3}}{2\pi},$$

(6.5)

$$K_4 = \frac{3(16 + 5\sqrt{5})}{4\pi}.$$  

(6.6)

Proof. Inequality (6.3) with the best constant (6.2) can be found in [9]. Let $\zeta \in \partial \Omega$ be the point nearest to $z \in \Omega$ and let $T(\zeta)$ be the line containing $\zeta$ and orthogonal to the line passing through $z$ and $\zeta$. By $\mathbb{C}_\zeta$ we denote the half-plane with the boundary $T(\zeta)$ which is contained in $\Omega$. Then by (5.3),

$$|f^{(s)}(z)| \leq \frac{K_s}{d_z^s} \|\Re f\|_{h^\infty(\mathbb{C}_\zeta)},$$

(6.7)

where $K_s$ is given by (6.2). Using

$$\|\Re f\|_{h^\infty(\mathbb{C}_\zeta)} \leq \sup_\Omega |\Re f|,$$

we obtain (6.1).
Theorem 4. Let $\Omega$ be a domain in $\mathbb{C}$, and let $\mathcal{R}(\Omega)$ be the set of holomorphic functions $f$ in $\Omega$ with $\sup_{\Omega} |\Re f| \leq 1$. Assume that a point $\zeta \in \partial \Omega$ can be touched by an interior disk $D$. Then

$$ \limsup_{z \to \zeta} \sup_{f \in \mathcal{R}(\Omega)} |z - \zeta|^s |f^{(s)}(z)| \leq K_s, \quad s = 1, 2, \ldots ,$$

where $z$ is a point of the radius of $D$ directed from the center to $\zeta$. Here the constant $K_s$ is the same as in Theorem 3 and cannot be diminished.

Proof. In Theorem 7.1 of paper [9] (see also Corollary 1 in [11]) the limit relation was proved:

$$ \lim_{r \to R} (R - r)^s \mathcal{H}_s(z) = K_s, \quad (6.8) $$

where $r = |z|$, $K_s$ is the best constant in inequality (6.3), and $\mathcal{H}_s(z)$ is the sharp coefficient in the inequality

$$ |f^{(s)}(z)| \leq \mathcal{H}_s(z)||\Re f||_{L^\infty(\partial D_R)}. \quad (6.9) $$

Here $f$ is an analytic function with bounded real part in the disk $D_R = \{ z \in \mathbb{C} : |z| < R \}$.

Therefore, by (6.8) and (6.9), the relations

$$ \limsup_{r \to R} \sup \{ (R - r)^s |f^{(s)}(z)| : ||\Re f||_{L^\infty(\partial D_R)} \leq 1 \} \leq \lim_{|z| \to R} (R - r)^s \mathcal{H}_s(z) = K_s \quad (6.10) $$

hold.

Let $R$ be the radius of the interior disk $D$ tangent to $\partial \Omega$ at a point $\zeta$. We place the origin $O$ at the center of $D$. Let $z$ belong to the interval connecting $O$ and $\zeta$. Then $R - r = |z - \zeta|$. By (6.10) and the inequality

$$ ||\Re f||_{L^\infty(\partial D_R)} \leq \sup_{\Omega} ||\Re f||, $$

the result follows. \qed

Remark 2. We note that the estimate

$$ |f^{(s)}(z)| \leq \frac{4s!}{\pi d_z^s} \sup_{\Omega} ||\Re f||, \quad s = 1, 2, \ldots , $$

with a rougher constant than in (6.1), holds for an arbitrary domain $\Omega \subset \mathbb{C}$. The estimate follows from the sharp inequality

$$ |f^{(s)}(0)| \leq \frac{4s!}{\pi R^s} \sup_{|\zeta| < R} ||\Re f(\zeta)|| $$

obtained in [7], Section 5.3. Certain estimates for $|f^{(s)}(z)|$ in an arbitrary complex domain are obtained in [4].

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