**$L_p$-improving convolution operators on finite quantum groups**

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**Abstract.** For $A$ being a finite dimensional C*-algebra equipped with a faithful tracial state $\tau$, and $T : A \to A$ being a unital trace preserving map on $A$, we prove that the $L_p$-improving property $\|T : L_p(A) \to L_2(A)\| = 1$ with some $1 < p < 2$ holds if and only if we have the "spectral gap": $\sup_{\alpha \in A \setminus \{0\}, \tau(\alpha) = 0} \|Tx\|_2/\|x\|_2 < 1$. As a result we characterize positive convolution operators on a finite quantum group $G$ which are $L_p$-improving. More precisely, it is proved that the convolution operator $T_\varphi : x \mapsto \varphi \ast x$ given by a state $\varphi$ on $C(G)$ satisfies

$$\exists ! 1 < p < 2, \quad \|T_\varphi : L_p(G) \to L_2(G)\| = 1$$

if and only if the Fourier series $\hat{\varphi}$ satisfies $\|\hat{\varphi}(\alpha)\| < 1$ for all nontrivial irreducible unitary representation $\alpha$, and only if the state $(\varphi \circ S) \ast \varphi$ is non-degenerate (where $S$ is the antipode).

We also prove that these $L_p$-improving properties are stable under taking free products, which gives a method to construct $L_p$-improving multipliers on infinite compact quantum groups.

**Introduction**

The convolution operators or multipliers constitute a central part of Fourier analysis. One among phenomena studied on the circle group $\mathbb{T}$ is the existence and behavior of positive Borel measures that convolve $L_p(\mathbb{T})$ into $L_q(\mathbb{T})$ with finite $q > p$ for a given $1 < p < \infty$, which are considered to be $L_p$-improving measures. An example due to Oberlin [Obe82] is the Cantor-Lebesgue measure supported by the usual middle-third Cantor set. Oberlin revealed that, after a careful analysis on the structure of this measure, this result can be reduced to proving that there exists $p < 2$ such that

$$\|\mu \ast f\|_2 \leq \|f\|_p, \quad f \in L_p(\mathbb{Z}/3\mathbb{Z})$$

where the $L_p$-norms are those taken with respect to the normalized counting measure on the cyclic group $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ with three elements and $\mu$ is the probability measure with mass $1/2$ at $0$ and $2$. Motivated by these results, Ritter showed in [Rit84] that, if $G$ is an arbitrary finite group and $T_\mu : f \mapsto \mu \ast f$ is the convolution operator associated to a probability measure $\mu$ on $G$, then

$$(\exists p < 2, \quad \|T_\mu : L_p(G) \to L_2(G)\| = 1) \iff G = \langle ij^{-1} : i, j \in \text{supp} \mu \rangle,$$

which provides a more general method to construct $L_p$-improving measures on groups.

In this paper we give an alternative approach related to these topics, in the context of quantum groups and noncommutative $L_p$-spaces. We show that, for a finite quantum group $G$ and a state $\varphi$ on $C(G)$, denoting by $\hat{\varphi}$ the Fourier series of $\varphi$ and writing $\psi = (\varphi \circ S) \ast \varphi$, $S$ being the antipode, the following assertions are equivalent (Theorem 3.7):

1. there exists $1 < p < 2$ such that,

   $$\forall x \in C(G), \quad \|\varphi \ast x\|_2 \leq \|x\|_p ;$$

2. $\|\hat{\varphi}(\alpha)\| < 1$ for all $\alpha \in \text{Irr}(G) \setminus \{1\}$ ;

3. For any nonzero $x \in C(G)_+$, there exists $n \geq 1$ such that $\psi^n(x) > 0$.

The last assertion, in a quantum opinion, should be interpreted as claiming that the “support” of $\varphi$ “generates” the quantum group $G$, which will be explained in the last section. We will illustrate by example in Remark 3.10 that the finiteness condition in the above conclusion is rather crucial and cannot be removed.

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In particular, the result characterizes the Fourier-Schur multipliers on finite groups which have an $L_p$-improving property. Let $\Gamma$ be a finite group and $\varphi$ be a positive definite function on $\Gamma$. Let $M_{\varphi}$ be the associated Fourier-Schur multiplier operator determined by $M_{\varphi}(\lambda(\gamma)) = \varphi(\gamma)\lambda(\gamma)$ for all $\gamma \in \Gamma$. Then

$$\exists 1 < p < 2, \quad \|M_{\varphi}x\|_2 \leq \|x\|_p, \quad x \in C^*(\Gamma)$$

if and only if $|\varphi(\gamma)| < 1$ for any $\gamma \in \Gamma \setminus \{e\}$.

We should emphasize that our argument relies essentially on new and interesting properties on the unital trace preserving operators on noncommutative $L_p$-spaces, based on the recent work of Ricard and Xu [RX14]. In fact, the following fact proved in Theorem 2.6 plays a key role in our argument. For a finite dimensional $C^*$-algebra $A$ equipped with a faithful tracial state $\tau$, and $T: A \to A$ a unital trace preserving map, the $L_p$-improving property

$$(0.1) \quad \exists 1 < p < 2, \quad \|T: L_p(A) \to L_2(A)\| = 1$$

holds if and only if we have the following “spectral gap”:

$$\sup_{x \in A \setminus \{0\}, \tau(x) = 0} \frac{\|Tx\|_2}{\|x\|_2} < 1.$$ 

We provide two proofs of this result, where one is based on very elementary arguments with an additional assumption of 2-positivity and another, which is rather short, on [RX14]. In Theorem 2.9 we also show that the $L_p$-improving property (0.1) remains stable under the free products.

We end this introduction with a brief description of the organization of the paper. In Section 1 we present some preliminaries and notation on noncommutative $L_p$-spaces, quantum groups and free products. We include in this section a short and explicit calculation of Fourier series for compact quantum groups, parallel to the case of classical compact groups, which does not exist in other literature. Section 2 deals with the characterization of unital trace preserving $L_p$-improving operators on finite dimensional $C^*$-algebras and their free products. The last Section 3 is devoted to the positive convolution operators on finite quantum groups and constructions of operators with similar properties on infinite compact quantum groups by free product. Some properties of non-degenerate states on a general compact quantum group are also obtained in this section.

1. Preliminaries

1.1. Noncommutative $L_p$-spaces. Here we recall some well-known basics of noncommutative $L_p$-spaces on finite von Neumann algebras. We refer to [Tak02] for the theory of von Neumann algebras and to [PX03] for more information on noncommutative $L_p$-spaces. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal faithful tracial state $\tau$. Let $1 \leq p < \infty$. For each $x \in \mathcal{M}$, we define

$$\|x\|_p = [\tau(|x|^p)]^{1/p}.$$ 

One can show that $\|\|_p$ is a norm on $\mathcal{M}$. The completion of $(\mathcal{M}, \|\|_p)$ is denoted by $L_p(\mathcal{M}, \tau)$ or simply by $L_p(\mathcal{M})$. The elements of $L_p(\mathcal{M})$ can be described by densely defined closed operators measurable with respect to $(\mathcal{M}, \tau)$, as in the commutative case. For convenience, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. Since $|\tau(x)| \leq \|x\|_1$ for all $x \in \mathcal{M}$, $\tau$ extends to a continuous functional on $L_1(\mathcal{M})$. Let $1 \leq p, q, r \leq \infty$ be such that $1/p + 1/q = 1/r$. If $x \in L_p(\mathcal{M})$ and $y \in L_q(\mathcal{M})$, then $xy \in L_r(\mathcal{M})$ and the following H"older inequality holds:

$$\|xy\|_r \leq \|x\|_p \|y\|_q.$$ 

In particular, if $r = 1$, $|\tau(xy)| \leq \|xy\|_1 \leq \|x\|_p \|y\|_q$ for arbitrary $x \in L_p(\mathcal{M})$ and $y \in L_q(\mathcal{M})$. This defines a natural duality between $L_p(\mathcal{M})$ and $L_q(\mathcal{M})$: $(x, y) = \tau(xy)$. For any $1 \leq p < \infty$ we have $L_p(\mathcal{M})^* = L_q(\mathcal{M})$ isometrically.

1.2. Compact quantum groups.
1.2.1. Basic properties. We recall some basic definitions and properties of compact quantum groups. All proofs of the facts mentioned below without references can be found in [Wor98] and [MVD98].

**Definition 1.1.** Consider a unital C*-algebra $A$ and a unital *-homomorphism $\Delta : A \to A \otimes A$ called **comultiplication** on $A$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and
\[
\{\Delta(a)(b \otimes 1) : a, b \in A\} \quad \text{and} \quad \{\Delta(a)(b \otimes 1) : a, b \in A\}
\]
are linearly dense in $A \otimes A$. Then $(A, \Delta)$ is called a **compact quantum group**. We denote $\mathbb{G} = (A, \Delta)$ and $A = C(\mathbb{G})$. We say that $\mathbb{G}$ is a **finite quantum group** if the space $A = C(\mathbb{G})$ is finite dimensional.

The following fact due to Woronowicz is fundamental in the quantum group theory.

**Proposition 1.2.** Let $\mathbb{G}$ be a compact quantum group. There exists a unique state $h$ on $C(\mathbb{G})$ (called the Haar state of $\mathbb{G}$) such that for all $x \in C(\mathbb{G})$,
\[
(h \otimes \iota) \Delta(x) = h(x)1 = (\iota \otimes h) \Delta(x).
\]

Let $\mathbb{G} = (A, \Delta)$ be a compact quantum group and consider an element $u \in A \otimes B(H)$ with $\dim H = n$. We identify $A \otimes B(H) = \mathbb{M}_n(A)$ and write $u = \{u_{ij}\}_{i,j=1}^n$. $u$ is called an $n$-dimensional representation of $\mathbb{G}$ if for all $j, k = 1, \ldots, n$ we have
\[
\Delta(u_{jk}) = \sum_{p=1}^n u_{jp} \otimes u_{pk}.
\]

A representation $u$ is said to be **non-degenerate** if $u$ is invertible, **unitary** if $u$ is unitary, and **irreducible** if the only matrices $T \in \mathbb{M}_n(\mathbb{C})$ with $Tu = uT$ are multiples of the identity matrix. Two representations $u, v \in \mathbb{M}_n(A)$ are called **equivalent** if there exists an invertible matrix $T \in \mathbb{M}_n(\mathbb{C})$ such that $Tu = vT$. Denote by $\mathrm{Irr}(\mathbb{G})$ the set of unitary equivalence classes of irreducible unitary representations of $\mathbb{G}$. For each $\alpha \in \mathrm{Irr}(\mathbb{G})$, let $u^\alpha \in \mathbb{C} \otimes B(H_\alpha)$ be a representative of the class $\alpha$ where $H_\alpha$ is the finite dimensional Hilbert space on which $u^\alpha$ acts.

With the notation above, the *-subalgebra $A$ spanned by $\{u^\alpha_{ij} : u^\alpha = \{u^\alpha_{ij}\}_{i,j=1}^n, \alpha \in \mathrm{Irr}(\mathbb{G})\}$, usually called the algebra of the polynomials on $\mathbb{G}$, is dense in $C(\mathbb{G})$, and the Haar state $h$ is faithful on this dense algebra. In the sequel we denote $A = \mathrm{Pol}(\mathbb{G})$. Consider the GNS representation $(\pi_h, H_h)$ of $C(\mathbb{G})$, then $\mathrm{Pol}(\mathbb{G})$ can be viewed as a subalgebra of $B(H_h)$. Define $C^*_r(\mathbb{G})$ (resp., $L^\infty(\mathbb{G})$) to be the C*-algebra (resp., the von Neumann algebra) generated by $\mathrm{Pol}(\mathbb{G})$ in $B(H_h)$. Then $h$ extends to a normal faithful state on $L^\infty(\mathbb{G})$.

It is known that there exists a linear antihomomorphism $S$ on $\mathrm{Pol}(\mathbb{G})$ such that
\[
S((a^{\ast})^\ast) = a, \quad a \in \mathrm{Pol}(\mathbb{G}),
\]
determined by
\[
S((u_{ij}^\alpha)^\ast) = (u_{ji}^\ast), \quad u^\alpha = \{u_{ij}^\alpha\}_{i,j=1}^n, \alpha \in \mathrm{Irr}(\mathbb{G}).
\]
$S$ is called the **antipode** of $\mathbb{G}$. For $a, b \in \mathrm{Pol}(\mathbb{G})$, we have
\[
S((i \otimes h)((\Delta(b)(1 \otimes a)))) = (i \otimes h)((1 \otimes b)\Delta(a)),
\]
\[
S((h \otimes i)((b \otimes 1)\Delta(a))) = (h \otimes i)(\Delta(b)(a \otimes 1)).
\]

We will use the **Sweedler notation** for the comultiplication of an element $a \in A$, i.e. omit the summation and the index in the formula $\Delta(a) = \sum a_{(1),i} \otimes a_{(2),i}$ and write simply $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

The Peter-Weyl theory for compact groups can be extended to the quantum case. In particular, it is known that for each $\alpha \in \mathrm{Irr}(\mathbb{G})$ there exists a positive invertible operator $Q_\alpha \in B(H_\alpha)$ such that $\text{Tr}(Q_\alpha) = \text{Tr}(Q_\alpha^{-1}) := d_\alpha$ and
\[
h(u_{ij}^\alpha)^\ast = \delta_{\alpha\beta} \delta_{ij} \frac{(Q_\alpha^\ast)_{mj}}{d_\alpha}, \quad h((u_{ij}^\alpha)^\ast u_{lm}^\beta) = \delta_{\alpha\beta} \delta_{jm} \frac{(Q_\alpha^{-1})_{il}}{d_\alpha}
\]
where $\beta \in \mathrm{Irr}(\mathbb{G})$, $1 \leq i, j \leq \dim H_\alpha$, $1 \leq l, m \leq \dim H_\beta$. 

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$L_p$-improving convolution operators on finite quantum groups
The dual quantum group $\hat{G}$ of $G$ is defined via its “algebra of functions”

$$\ell_\infty(\hat{G}) = \oplus_{\alpha \in \Irr(G)} B(H_\alpha)$$

where $\otimes_{\alpha} B(H_\alpha)$ refers to the direct sum of $B(H_\alpha)$, i.e. the bounded families $(x_\alpha)_\alpha$ with each $x_\alpha$ in $B(H_\alpha)$. We will not completely recall the quantum group structure on $\hat{G}$ as we do not need it in the following. We only remark that the (left) Haar weight $h$ on $\hat{G}$ can be explicitly given by (see e.g. [VD96, Section 5])

$$\hat{h} : \ell_\infty(\hat{G}) \ni x \mapsto \sum_{\alpha \in \Irr(G)} d_\alpha \Tr(Q_\alpha p_\alpha x),$$

where $p_\alpha$ is the projection onto $H_\alpha$ and $\Tr$ denotes the usual trace on $B(H_\alpha)$ for each $\alpha$.

Our main result will only concentrate on the case where $G$ is of Kac type, that is, its Haar state is tracial.

**Proposition 1.3 ([Wor98, Theorem 1.5]).** Let $G$ be a compact quantum group. The Haar state $h$ on $C(G)$ is tracial if and only if $Q_\alpha = Id_{H_\alpha}$ for all $\alpha \in \Irr(G)$ in the formula (1.4) and if and only if the antipode $S$ satisfies $S^2(x) = x$ for all $x \in C(G)$. In particular, if the above conditions are satisfied, then $S$ extends to a $*-\text{antihomomorphism}$ on $C_r(G)$ which is positive and bounded of norm one according to (1.2).

**Proposition 1.4 ([VD97]).** If $G$ is a finite quantum group, then the Haar state is tracial on $C(G)$.

For a compact quantum group $G$, we write $L_2(G)$ to be the Hilbert space associated to the GNS-construction with respect to the Haar state $h$. Similarly, denote by $\ell_2(\hat{G})$ the Hilbert space given by the GNS-construction for $\ell_\infty(\hat{G})$ with respect to the Haar weight $\hat{h}$. If $G$ is of Kac type, for $1 \leq p \leq \infty$, we denote additionally by $L_p(G)$ the $L_p$-space associated to the pair $(L_\infty(G), h)$, as defined in the previous subsection.

1.2.2. Fourier analysis. The Fourier transform for locally compact quantum groups has been defined in [Kah10], [Coo10] and [Cas13]. In the setting of compact quantum groups, we may give a more explicit description below. Let a compact quantum group $G$ be fixed. For a linear functional $\varphi$ on $\Pol(G)$, we define the Fourier transform $\hat{\varphi} = (\hat{\varphi}(\alpha))_{\alpha \in \Irr(G)} \in \otimes_{\alpha} B(H_\alpha)$ by

$$\hat{\varphi}(\alpha) = (\varphi \otimes i)((u^\alpha)^*) \in B(H_\alpha), \quad \alpha \in \Irr(G).$$

In particular, any $x \in L_\infty(G)$ (or $L_2(G)$) induces a continuous functional on $L_\infty(G)$ defined by $y \mapsto h(y x)$, and the Fourier transform $\hat{x} = (\hat{x}(\alpha))_{\alpha \in \Irr(G)}$ of $x$ is given by

$$\hat{x}(\alpha) = (h(\cdot \otimes i)((u^\alpha)^*)) \in B(H_\alpha), \quad \alpha \in \Irr(G).$$

The above definition is slightly different from that of [Cas13] or [Kah10]. Indeed, we replace the unitary $u^\alpha$ by $(u^\alpha)^*$ in the above formulas. This is just to be compatible with standard definitions in classical analysis on compact groups such as in [Fol95, Section 5.3], which will not cause essential difference. On the other hand, the notation $\hat{\varphi}$ has some slight conflict with the dual Haar weight $\hat{h}$ on $\hat{G}$ whereas one can distinguish them by the elements on which it acts, so we hope that this will not cause ambiguity for readers.

Denote by $F : x \mapsto \hat{x}$ the Fourier transform established above. It is easy to establish the Fourier inversion formula and the Plancherel theorem for $L_2(G)$. As we did not find them explicitly for compact quantum groups in the literature, we include the detailed calculation of the Fourier series in the following proposition.

**Proposition 1.5.** (a) For all $x \in L_2(G)$, we have

$$x = \sum_{\alpha \in \Irr(G)} d_\alpha (\iota \otimes \Tr)[(1 \otimes \hat{x}(\alpha)Q_\alpha)u^\alpha].$$

where the convergence of the series is in the $L_2$-sense.

(b) $F$ is a unitary from $L_2(G)$ onto $\ell_2(\hat{G})$. 

So each \( x(1.7) \times Pol(∥ \cdot ∥) \) is spanned by all the matrix units of the range of \( B \) matrix units of \( G \). For each \( α \) and \( \ell \)-dense in \( L_2(G) \), isometrically maps \( 2 \)-dense in \( L_2(G) \) and \( L_2(G) \) is a Hilbert direct sum of the orthogonal subspaces \( (E_α)_{α∈\text{Irr}(G)} \). Hence by Parseval’s identity,

\[
\text{Hence by Parseval’s identity,}
\]

Proof. (a) Denote by \( E_α \) the subspace spanned by the matrix units \( (u^α_{ij})_{i,j=1}^n \) for \( α ∈ \text{Irr}(G) \). Then \( Pol(G) \) is spanned by all the \( E_α, α ∈ \text{Irr}(G) \). It is easy to see from Hölder’s inequality that \( Pol(G) \) is \( ∥ \cdot ∥_2 \)-dense in \( L_∞(G) \), and also recall that \( L_2(G) \) is the \( ∥ \cdot ∥_2 \)-completion of \( L_∞(G) \), so \( Pol(G) \) is \( ∥ \cdot ∥_2 \)-dense in \( L_2(G) \) and \( L_2(G) \) is a Hilbert direct sum of the orthogonal subspaces \( (E_α)_{α∈\text{Irr}(G)} \). So each \( x ∈ L_2(G) \) can be written as

\[
(1.7) \quad x = \sum_{α∈\text{Irr}(G)} E_α x = \sum_{α∈\text{Irr}(G)} \sum_{i,j} x^α_{ij} u^α_{ij}, \quad (x^α_{ij} ∈ \mathbb{C})
\]

where \( E_α x := \sum_{i,j} x^α_{ij} u^α_{ij} \) is the orthogonal projection of \( x \) onto \( E_α \).

Now for \( α ∈ \text{Irr}(G) \), write \( u^α = \sum_{l,m} u^α_{lm} \otimes e^α_{lm} \) and \( X_α = [x^α_{ij}, i,j] \) where \( e^α_{lm} \) denote the canonical matrix units of \( B(H_α) \). Then

\[
\hat{x}(α) = (h(x) ⊗ i)((u^α)^*) = (h(E_α x) ⊗ i)((u^α)^*) \quad \text{by Parseval’s identity,}
\]

\[
hence \quad d_α \iota ⊗ Tr)[(1 ⊗ \hat{x}(α)Q_α)u^α] = \sum_{i,j,l,m} (\iota ⊗ Tr)[(1 ⊗ x^α_{ij} u^α_{lm})] = \sum_{i,j,l,m} (\hat{x}(α)Q_α)^{-1} = \sum_{i,j} x^α_{ij} u^α_{ji} = E_α x.
\]

Combining the last equality with (1.7) proves the desired (1.6).

(b) Let

\[
x = \sum_{α∈\text{Irr}(G)} E_α x = \sum_{α∈\text{Irr}(G)} \sum_{i,j} x^α_{ij} u^α_{ij} ∈ L_2(G).
\]

For each \( α ∈ \text{Irr}(G) \),

\[
∥E_α x∥_2^2 = h(E_α x, E_α x) = \sum_{i,j,l,m} x^α_{ij} x^α_{lm} h ((u^α_{lj})^* u^α_{im}) = d_α^{-1} \sum_{i,j,l,m} x^α_{ij} x^α_{lj} (Q_α^{-1})_{li} \]

and also by (1.8)

\[
d_α^2 \text{Tr}(Q_α \hat{x}(α)^* \hat{x}(α)) = \text{Tr}(X_α^* X_α Q_α^{-1}) = \sum_{i,j,l} x^α_{ij} x^α_{lj} (Q_α^{-1})_{li}.
\]

Hence by Parseval’s identity,

\[
∥x∥_2^2 = \sum_{α} ∥E_α x∥_2^2 = \sum_{α} d_α^{-1} \sum_{i,j,l} x^α_{ij} x^α_{lj} (Q_α^{-1})_{li} \]

\[
= \sum_{α} d_α^{-1} d_α^2 \text{Tr}(Q_α \hat{x}(α)^* \hat{x}(α)) = ∥\hat{x}∥_2^2.
\]

Thus \( F \) maps isometrically \( L_2(G) \) into \( ℓ_2(\hat{G}) \). From (1.6) and the isometric relation we see that the range of \( F \) contains the subset of all finitely supported families \( (a_α) ∈ ∆_α B(H_α) \), which is dense in \( ℓ_2(\hat{G}) \). Therefore \( F \) is surjective and hence unitary. □

**Example 1.6.** (1) Let \( G \) be a compact group and define

\[
\Delta(f)(s,t) = f(st), \quad f ∈ C(G), \quad s,t ∈ G.
\]
Then $G = (C(G), \Delta)$ is a compact quantum group. The elements in $\text{Irr}(G) := \text{Irr}(G)$ coincide with the usual strongly continuous irreducible unitary representations of $G$. Any continuous functional $\varphi$ on $C(G)$ corresponds to a complex Radon measure $\mu$ on $G$ by the Riesz representation theorem. By definition (1.5), the Fourier series of $\mu$ is given by
\[
\hat{\mu}(\pi) = (\varphi \otimes 1)(\pi(\cdot)^*) = \int_G \pi(g)^* \, d\mu(g), \quad \pi \in \text{Irr}(G).
\]
In particular for $f \in L_2(G)$, we have
\[
\hat{f}(\pi) = \int_G \pi(g)^* \, f(g) \, dg, \quad \pi \in \text{Irr}(G)
\]
and we have the Fourier expansion and the Plancherel formula
\[
f = \sum_{\pi \in \text{Irr}(G)} d_\pi \text{Tr} (f(\pi) \pi), \quad \|f\|_2^2 = \sum_{\pi \in \text{Irr}(G)} d_\pi \|\hat{f}(\pi)\|_{\text{HS}}^2
\]
where $d_\pi$ is the dimension of the Hilbert space on which the representation $\pi$ acts and $\|\cdot\|_{\text{HS}}$ denotes the usual Hilbert-Schmidt norm. We refer to [Pol95, Section 5.3] and [HR70, pp.77-87] for more information.

(2) Let $\Gamma$ be a discrete group with its neutral element $e$ and $C^*_r(\Gamma)$ be the associated reduced group $C^*$-algebra generated by $\lambda(\Gamma) \subset B(\ell^2(\Gamma))$, where $\lambda$ denotes the left regular representation. The “dual” $\hat{G} = \hat{\Gamma}$ of $\Gamma$ is a compact quantum group such that $C(\hat{G})$ is the group $C^*$-algebra $C^*_r(\Gamma)$ equipped with the comultiplication $\Delta : C^*_r(\Gamma) \to C^*_r(\Gamma) \otimes C^*_r(\Gamma)$ defined by
\[
\Delta(\lambda(\gamma)) = \lambda(\gamma) \otimes \lambda(\gamma), \quad \gamma \in \Gamma.
\]
The Haar state of $G$ is the unique trace $\tau$ on $C^*_r(\Gamma)$ such that $\tau(1) = 1$ and $\tau(\lambda(\gamma)) = 0$ for $\gamma \in \Gamma \setminus \{e\}$. The elements of $\lambda(\Gamma)$ give all irreducible unitary representations of $G$, which are all of dimension 1. It is easy to check from definition that for any $f \in C^*_r(\Gamma)$,
\[
\hat{f}(\gamma) = \tau(f \lambda(\gamma)^*), \quad \gamma \in \Gamma.
\]
And any $f \in L_2(\hat{G})$ has an expansion such that
\[
f = \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \lambda(\gamma), \quad \|f\|_2^2 = \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|^2.
\]

Return back to a general compact quantum group $G$. For $a = (a_\alpha)_\alpha \in \oplus_\alpha B(H_\alpha)$, we define the left multiplier $m_a : \text{Pol}(\hat{G}) \to \text{Pol}(\hat{G})$ associated to $a$ by
\[
m_a x = \sum_{\alpha \in \text{Irr}(G)} d_\alpha (\textup{Tr} \circ \hat{x}(\alpha) a_\alpha Q_\alpha u^\alpha), \quad x \in \text{Pol}(\hat{G}),
\]
which means that
\[
(m_a x)^*(\alpha) = \hat{x}(\alpha) a_\alpha, \quad \alpha \in \text{Irr}(G).
\]
In the same way we may define the right multiplier $m'_a : \text{Pol}(\hat{G}) \to \text{Pol}(\hat{G})$ such that
\[
m'_a x = \sum_{\alpha \in \text{Irr}(G)} d_\alpha (\textup{Tr} \circ \hat{x}(\alpha) a_\alpha Q_\alpha u^\alpha), \quad x \in \text{Pol}(\hat{G}).
\]
Observe that the multiplier $m_a$ (or $m'_a$ resp.) is unital, i.e. $m_a(1) = 1$ ($m'_a(1) = 1$ resp.) if and only if $a_1 = 1$.

**Remark 1.7.** In case $G$ is of Kac type, that is, $Q_\alpha = \text{Id}_{H_\alpha}$ for all $\alpha \in \text{Irr}(G)$ by Proposition 1.3, the multipliers $m_a$ and $m'_a$ can be equivalently defined by
\[
(m_a \otimes 1)(u^\alpha) = (1 \otimes a_\alpha)(u^\alpha), \quad (m'_a \otimes 1)(u^\alpha) = (a^\alpha)(1 \otimes a_\alpha), \quad \alpha \in \text{Irr}(G),
\]
which corresponds to the standard definition of left and right multipliers on locally compact quantum groups in [JNR09] and [Daw12]. If $G$ is not of Kac type, the above formula (1.11) gives a similar equality corresponding to (1.10), that is, $(m_a x)^*(\alpha) = \hat{x}(\alpha) Q_\alpha a_\alpha Q_\alpha^{-1}, \alpha \in \text{Irr}(G)$.
We will use the standard definition of convolution products given by Woronowicz. Let \( x \in C(G) \) and \( \varphi, \varphi' \) be linear functionals on \( C(G) \). We define
\[
\varphi \ast \varphi' = (\varphi \otimes \varphi') \circ \Delta,
\]
\[
x \ast x = (\varphi \otimes \varphi)(x),
\]
\[
\varphi \ast x = (\varphi \otimes \varphi)(x).
\]
Observing the embedding \( x \mapsto h(\cdot x) \) from \( C(G) \) into \( C(G)^* \), the convolution products defined above are related as follows according to (1.3) (see also [VD07, Proposition 2.2]): on \( \operatorname{Pol}(G) \) we have
\[
(1.12) \quad h(\cdot x) \ast h = h([\varphi \circ S \ast x]), \quad \varphi \ast h(\cdot x) = h(\cdot [x \ast (\varphi \circ S^{-1})]).
\]
We note that for \( \alpha \in \operatorname{Irr}(G) \) and \( u^\alpha = [u^\alpha_i]_{i,j \leq n} \),
\[
\left[ \varphi((u^\alpha_i)) \right]_{i,j} = (\varphi \circ \iota)((u^\alpha)^*) = \varphi(\alpha).
\]
Then by (1.1), a straightforward calculation shows that
\[
(1.13) \quad (\varphi \ast \varphi')(\alpha) = \varphi'(\alpha)\varphi(\alpha).
\]
Hence together with (1.12),
\[
(1.14) \quad (x \ast \varphi')(\alpha) = x(\alpha)(\varphi \circ S)^*(\alpha), \quad (\varphi \ast x)(\alpha) = (\varphi \circ S^{-1})^*(\alpha)x(\alpha).
\]

1.3. Free products. We firstly recall some constructions of free product of \( C^* \)-algebras, for which we refer to [VDN92] and [NS06] for details. Consider a family of unital \( C^* \)-algebras \( (A_i, \phi_i)_{i \in I} \) with distinguished faithful states \( \phi_i \) and associated GNS constructions \( (\pi_i, H_i) \). Set \( A_i = \ker \phi_i \) and \( a_i = a_i - \phi_i(a_i)I \) for each \( i \) and \( a_i \in A_i \). Construct a vector space
\[
(1.15) \quad A = \mathbb{C}1 \oplus \bigoplus_{n \geq 1} \left( \bigoplus_{i_1 \neq i_2 \cdots \neq i_n} \hat{A}_{i_1} \otimes \hat{A}_{i_2} \otimes \cdots \otimes \hat{A}_{i_n} \right).
\]

We equip \( A \) with an algebra structure such that \( 1 \) is the identity and the multiplication of a letter \( a \in A_i \) with an elementary tensor \( a_1 \otimes a_2 \otimes \cdots \otimes a_n \) in \( \hat{A}_{i_1} \otimes \hat{A}_{i_2} \otimes \cdots \otimes \hat{A}_{i_n} \) is defined as
\[
a \ast (a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \begin{cases} a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n, & \text{if } i_1 \neq i, \\
+ \phi_i(aa_1) a_2 \otimes \cdots \otimes a_n, & \text{if } i_1 = i.
\end{cases}
\]
Moreover, we give an involution on \( A \) by
\[
(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^* = a_1^* \otimes a_2^* \otimes \cdots \otimes a_n^*.
\]
In this sense \( A \) becomes a \( \ast \)-algebra, and each \( A_i \) can be viewed as a \( \ast \)-subalgebra in \( A \) by identifying \( A_i \) with \( \mathbb{C}1 \oplus \hat{A}_i \) in the big direct sum. We call \( A \) the algebraic free product of \( (A_i)_{i \in I} \).

It then can be shown that the algebra \( A \) admits a faithful \( \ast \)-representation \( (\pi, H, \xi) \) such that \( \pi|A_i = \pi_i \) for each \( i \in I \) and \( \phi(\cdot) := \langle \pi(\cdot)\xi, \xi \rangle \) restricted on \( A_i \) coincides with \( \phi_i \). Moreover the state \( \phi \) is faithful on \( A \). Then the reduced \( C^* \)-algebraic free product of \( (A_i)_{i \in I} \) is the \( C^* \)-algebra generated by \( \pi(A) \) in \( B(H) \), i.e., the norm closure of \( \pi(A) \) in \( B(H) \), denoted by \( *_{i \in I} A_i \); and the state extends to \( *_{i \in I} \phi_i \), called the free product state of \( (\phi_i)_{i \in I} \) and denoted by \( *_{i \in I} \phi_i \). If moreover each \( A_i = M_i \) is a von Neumann algebra and each \( \phi_i \) is normal, then the weak closure of \( \pi(A) \) in \( B(H) \), is defined to be the von Neumann algebraic free product of \( (M_i)_{i \in I} \), denoted by \( *_{i \in I} M_i \), and the free product state \( \phi = *_{i \in I} \phi_i \) is also normal. Also, we remark that if each \( \phi_i \) is a tracial state, then \( \phi = *_{i \in I} \phi_i \) is also tracial.

Let \( A_i \) and \( B_i \) be unital \( C^* \)-algebras with distinguished faithful states \( \phi_i \) and \( \psi_i \) \( (i \in I) \) respectively, and let \( T_i : A_i \to B_i \) be a unital state preserving map for each \( i \in I \). Set \( (A, \phi) = *_{i \in I} (A_i, \phi_i) \) and \( (B, \psi) = *_{i \in I} (B_i, \psi_i) \). Then it is obvious that
\[
T(a_1 a_2 \cdots a_n) = T_1(a_1) \cdots T_n(a_n) \quad (a_k \in \hat{A}_{i_k}, \forall k, i_1 \neq i_2 \neq \cdots \neq i_n)
\]
defines a unital state preserving map from the algebraic free products \( (A, \phi) \) to \( (B, \psi) \). We denote by \( T = *_{i \in I} T_i \), and call it the free product map of the \( T_i \)'s.
Now we turn to the dual free product of compact quantum groups. The following construction is given by [Wan95].

**Proposition 1.8.** Let \( \mathcal{G}_1 = (A, \Delta_A) \) and \( \mathcal{G}_2 = (B, \Delta_B) \) be two compact quantum groups with Haar states \( h_A, h_B \) respectively. There exists a unique comultiplication \( \Delta \) on \( A *_{c_0} B \) such that the pair \( (A *_{c_0} B, \Delta) \) forms a compact quantum group, denoted by \( \mathcal{G} = \mathcal{G}_1 * \mathcal{G}_2 \) and we have

\[
\Delta|_A = (i_A \otimes i_A) \circ \Delta_A, \quad \Delta|_B = (i_B \otimes i_B) \circ \Delta_B,
\]

where \( i_A \) and \( i_B \) are the natural embedding of \( A \) and \( B \) into \( A *_{c_0} B \) respectively. Moreover the Haar state on \( \mathcal{G} \) is the free product state \( h_A * h_B \).

2. \( L_p \)-improvement and spectral gaps

Let \( A \) be a finite dimensional C*-algebra equipped with a faithful tracial state \( \tau \). The associated noncommutative \( L_p \)-spaces will be denoted by \( L_p(A) \). For a subset \( E \subset A \), we denote by \( E_{+} \) the positive part of \( E \).

Recall that \( A \) can be identified with a direct sum of matrix algebras, that is, there exist some finite dimensional Hilbert spaces \( H_1, \ldots, H_m \) such that the following *-isomorphism holds

\[
A \simeq B(H_1) \oplus \cdots \oplus B(H_m).
\]

We will not distinguish the above two C*-algebras in the sequel. For each \( k \) finite dimensional Hilbert spaces \( H_1, \ldots, H_m \) such that the following *-isomorphism holds

\[
A \simeq B(H_1) \oplus \cdots \oplus B(H_m).
\]

We will prove in this section the result below.

**Theorem 2.1.** Let \( A \) be a finite dimensional C*-algebra equipped with a faithful tracial state \( \tau \), and \( T: A \to A \) be a unital 2-positive trace preserving map on \( A \). Then

\[
\exists 1 \leq p < 2, \quad \|Tx\|_2 \leq \|x\|_p, \; x \in A
\]

if and only if

\[
\sup_{x \in A \setminus \{0\}, \tau(x) = 0} \frac{\|Tx\|_2}{\|x\|_2} < 1. \tag{2.2}
\]

**Remark 2.2.** Equivalently we can rewrite the above condition (2.2) as

\[
\sup_{x \in A \setminus \{0\}, \tau(x) = 0} \frac{\|Tx\|_2}{\|x\|_2} < 1,
\]

which means exactly that the whole eigenspace of \( |T| \) for the eigenvalue 1 is just \( C_1 \). In this sense we refer to the above inequality as a spectral gap phenomenon of \( T \).

Recall that the \( L^2 \)-norms assert some differential properties. The following lemma is elementary.

**Lemma 2.3.** Let \( A \) be a C*-algebra with a state \( \varphi \) and \( T: A \to A \) be a positive map on \( A \). Let \( O \subset A_{h} \) be an open set in the space \( A_{h} \) of all selfadjoint elements in \( A \). The function \( f: O \ni x \mapsto \varphi(Tx^2) \) is infinitely (Fréchet) differentiable in \( O \) and for \( x \in O \), \( f'(x) = \varphi(TxT^*:x) + \varphi(T:T^*) \), \( f'' \equiv 2\varphi(TT^*) \), \( f^{(n)} \equiv 0 \), \( n \geq 3 \).

In general a norm estimate can be reduced to the argument on positive cones.
Lemma 2.4 ([RX14, Remark 9]). Let \( M \) be a von Neumann algebra and \( T : L_p(M) \to L_q(M) \) be a bounded linear map for \( 1 \leq p, q \leq \infty \). Assume that \( T \) is 2-positive in the sense that \( \text{Id}_{\mathbb{M}_2} \otimes T \) maps the positive cone of \( L_p(\mathbb{M}_2 \otimes M) \) to that of \( L_q(\mathbb{M}_2 \otimes M) \). Then
\[
\|Tx\|_q \leq \|T(|x|)\|_q^{1/2}\|T(|x^*|)\|_q^{1/2}, \quad x \in L_p(M).
\]
Consequently,
\[
\|T\| = \sup\{\|Tx\|_q : x \in L_p(M)_+, \|x\|_p \leq 1\}.
\]

Now we give the proof of the theorem.

Proof of the theorem. Assume firstly \( 1 \leq p < 2 \) and \( \|Tx\|_2 \leq \|x\|_p \) for all \( x \in A \). Suppose by contradiction that there exists nonzero \( x_0 \in A \) with \( \tau(x_0) = 0 \) such that \( \|Tx_0\|_2 = \|x_0\|_2 \). Then we have
\[
\|x_0\|_2 = \|Tx_0\|_2 \leq \|x_0\|_p.
\]
Recall that the Hölder inequality \( \|x_0\|_2 \leq \|x_0\|_p \|1/p' + 1/p = 1/2\) becomes an equality if and only if \( x_0 = \lambda 1 \) with some \( \lambda \in \mathbb{C} \), which contradicts the choice of \( x_0 \).

Now we suppose (2.2) holds. Set \( A = \{ x \in A \mid \tau(x) = 0 \} \) and take \( \sigma = \{ x \in A_+ \mid \tau(x) = 1 \} = (1 + \hat{A})_+ \), which is exactly the positive elements in the unit sphere of \( L_1(A) \). We first show that there exists \( 1 \leq p < 2 \) and a neighborhood \( U \) of 1 such that
\[
(2.3) \quad \forall x \in U \cap \sigma, \|Tx\|_2 \leq \|x\|_p.
\]
To begin with, we consider
\[
F(x) = \|Tx\|_2 - \|x\|_2, \quad x \in A_+.
\]
Using the previous lemma we see that \( F \) is infinitely differentiable at any \( x \in A_+ \setminus \{0\} \) and
\[
F'(x)(y) = \|Tx\|_2^{-1} \tau((Tx)(Ty)) - \|x\|_2^{-1} \tau(xy), \quad y \in A
\]
\[
F''(x)(y_1, y_2) = -\|Tx\|_2^{-3} \tau((Tx)(Ty_1)) \tau((Tx)(Ty_2)) + \|Tx\|_2^{-1} \tau((Ty_1)(Ty_2)) + \|x\|_2^{-3} \tau(xy_1) \tau(xy_2) - \|x\|_2^{-1} \tau(y_1y_2), \quad y_1, y_2 \in A.
\]
Since \( T \) is unital and preserves the trace, it follows that for \( y \in \hat{A} \),
\[
F'(1)(y) = 0, \quad F''(1)(y, y) = \|Ty\|_2^2 - \|y\|_2^2.
\]
Then consider the second order Taylor expansion of \( F \) at 1. We can find a \( \delta_1 > 0 \) such that for all \( \|y\|_2 \leq \delta_1 \), \( y \in \hat{A} \), we have \( 1 + y \in A_+ \) and
\[
F(1 + y) = F(1) + F'(1)(y) + \frac{1}{2} F''(1)(y, y) + R_1(y)
\]
\[
= \frac{1}{2}(\|Ty\|_2^2 - \|y\|_2^2) + R_1(y), \quad R_1(y) = o(\|y\|_2^2).
\]
Recall that by (2.2), \( \|Ty\|_2^2 - \|y\|_2^2 \leq 0 \) for \( y \in \hat{A} \). Thus by continuity,
\[
c := \sup\{\|Ty\|_2^2 - \|y\|_2^2 : y \in \hat{A}, \|y\|_2 = 1\} < 0.
\]
Since the function \( y \mapsto \|Ty\|_2^2 - \|y\|_2^2 \) is 2-homogeneous, we get
\[
\forall y \in \hat{A}, \|Ty\|_2^2 - \|y\|_2^2 \leq c \|y\|_2^2.
\]
Take \( \delta_0 \in (0, \delta_1) \) such that
\[
\forall y \in \hat{A}, \|y\|_2 \leq \delta_0, \quad \frac{|R_1(y)|}{\|y\|_2^2} < \frac{|c|}{4}.
\]
Then for \( y \in \hat{A}, \|y\|_2 \leq \delta_0 \),
\[
(*) \quad F(1 + y) = \frac{1}{2}(\|Ty\|_2^2 - \|y\|_2^2) + R_1(y) \leq \frac{c}{4} \|y\|_2^2.
\]
On the other hand, consider
\[
G(x) = \|x\|_2 - \|x\|_p, \quad x = 1 + y, y = y^* \in A, \|y\|_2 < \delta_0.
\]
Let $y = y^* \in A$ with $\|y\|_2 < \delta_0$, then by (2.1) we may take some $K \in \mathbb{N}$ and $\beta_1, \ldots, \beta_K \in [0, 1]$ such that the $L^p$-norm of $x = 1 + y$ for $1 \leq p < \infty$ is exactly

$$(**)
\|1 + y\|_p = \left( \sum_{i=1}^{K} \beta_i(1 + \lambda_i)^p \right)^{\frac{1}{p}}$$

where $(\lambda_i) \subset \mathbb{R}$ is the list of eigenvalues of $y$. So in order to estimate $G$, we consider the function $g$ on $\mathbb{R}^K$ defined as

$$g(\xi) = \left( \sum_{i=1}^{K} \beta_i(1 + \xi_i)^2 \right)^{\frac{1}{2}} - \left( \sum_{i=1}^{K} \beta_i(1 + \xi_i)^p \right)^{\frac{1}{p}}, \quad \xi = (\xi_1, \ldots, \xi_K) \in \mathbb{R}^K.$$ 

A straightforward calculation gives

$$\frac{\partial g}{\partial \xi_i}(0) = 0, \quad \frac{\partial^2 g}{\partial \xi_i \partial \xi_j}(0) = (p - 2)\beta_i \beta_j, \quad \frac{\partial^2 g}{\partial \xi_i \partial \xi_k}(0) = (2 - p)(\beta_i - \beta_k^2), \quad 1 \leq i \neq j \leq K.$$ 

So by the Taylor formula

$$g(\xi) = \frac{1}{2} \sum_{i=1}^{K} (2 - p)(\beta_i - \beta_i^2) \xi_i^2 + \frac{1}{2} \sum_{j \neq k} (p - 2)\beta_j \beta_k \xi_j \xi_k + R(\xi), \quad R(\xi) = O(\|\xi\|^2).$$

If $2 - \frac{|c|}{2} \leq p \leq 2$ and $0 < \delta < \delta_0$ is such that $|R(\xi)| \leq \frac{|c|}{4} \sum_{i=1}^{K} \beta_i \lambda_i^2$ whenever $\sum_{i=1}^{K} \beta_i \lambda_i^2 \leq \delta^2$, then for any $\xi \in \mathbb{R}^K$ with $\sum_{i=1}^{K} \beta_i \lambda_i^2 \leq \delta^2$,

$$|g(\xi)| \leq \frac{1}{2} (2 - p) \sum_{i=1}^{K} \beta_i - \beta_i^2 \lambda_i^2 + \frac{1}{2} (2 - p) \sum_{i=1}^{K} \beta_i \lambda_i^2 + \frac{|c|}{8} \sum_{i=1}^{K} \beta_i \lambda_i^2 < \frac{|c|}{4} \sum_{i=1}^{K} \beta_i \lambda_i^2.$$ 

This, together with (**), implies that, putting $\lambda = (\lambda_1, \ldots, \lambda_K)$,

$$G(1 + y) = g(\lambda) \leq \frac{|c|}{4} \sum_{i=1}^{K} \beta_i \lambda_i^2 = \frac{|c|}{4} \|y\|_2, \quad \|y\|_2 \leq \delta.$$ 

Combined with (*) we deduce

$$\|Tx\|_2 - \|x\|_p = F(1 + y) + G(1 + y) \leq 0, \quad x = 1 + y, \quad y \in \tilde{A}, \quad \|y\|_2 \leq \delta,$$ 

for all $p \geq 2 - \frac{|c|}{2} := p_1$. So $U = \{y \mid y = y^* \in A, \|y\|_2 < \delta\}$ is the desired neighborhood in (2.3).

Now we can derive the inequality for all $x \in \sigma$. For $x \in \sigma \setminus U \subset (1 + \tilde{A})_+ \setminus \{1\}$, we write $x = 1 + y$ with $y \in \tilde{A}_+ \setminus \{0\}$ and then by (2.2) and the trace preserving property we have $\|Tx\|_2^2 = 1 + \|Ty\|_2^2 < 1 + \|y\|_2^2 = \|x\|_2^2$. Note also that $\sigma$ is compact, so we can find $M < 1$ such that $\|Tx\|_2/\|x\|_2 < M$ for all $x \in \sigma \setminus U$. Given $p < 2$, let $C_p$ be the optimal constant for the inequality $\|x\|_2 \leq C_p\|x\|_p$ for $x \in A$, then $C_p \to 1$ when $p \to 2$. Take $p_0 \geq p_1$ such that $C_{p_0} \leq M^{-1}$. We get then

$$\forall x \in \sigma \setminus U, \quad \frac{\|Tx\|_2}{\|x\|_p} \leq MC_p \leq 1, \quad p_0 \leq p \leq 2.$$ 

As a result, for all $p \in [p_0, 2]$, it holds that

$$\|Tx\|_2 \leq \|x\|_p, \quad x \in \sigma.$$ 

Since the norm is homogeneous and $T$ is 2-positive, the above inequality holds for all $x \in A$ as well.

Apart from the above elementary proof, we would like to give an alternative simpler approach which yields a little bit stronger conclusion. The argument, however, depends heavily on the following recent and deep result on the convexity of $L^p$-spaces:
Theorem 2.5 ([RX14, Theorem 1]). Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful semifinite normal trace $\phi$. Let $\mathcal{N}$ be a von Neumann subalgebra such that the restriction of $\phi$ to $\mathcal{N}$ is semifinite. Denote by $E$ the unique $\phi$-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. For $1 < p \leq 2$, we have
\[
\|x\|_p^p \geq \|Ex\|_p^p + (p-1)\|x-Ex\|_p^2, \quad x \in L_p(\mathcal{M}).
\]
For $2 < p < \infty$, the inequality is reversed.

Immediately we get:

Theorem 2.6. Let $A$ be a finite dimensional C*-algebra equipped with a faithful tracial state $\tau$, and $T : A \to A$ be a unital trace preserving map on $A$. Then
\[
\exists 1 < p < 2, \forall x \in A, \|Tx\|_2 \leq \|x\|_p
\]
if and only if
\[
\lambda := \sup_{x \in A \setminus \{0\}, \tau(x) = 0} \frac{\|Tx\|_2}{\|x\|_2} < 1.
\]
Moreover, if the above assertions are satisfied, then
\[
\lambda \leq c_p^{-1} \sqrt{p-1}, \quad \text{where } c_p = \sup_{x \in A \setminus \{0\}, \tau(x) = 0} \frac{\|x\|_2}{\|x\|_p}.
\]

Proof. It is easy to see (2.4) implies $\lambda < 1$ since $\|x\|_{p_1} < \|x\|_{p_2}$ for any nonzero $x$ with trace 0 if $1 \leq p_1 < p_2 < \infty$.

Now assume $\lambda < 1$. Let $x \in A$ and $y = x - \tau(x)1$. Write $a = \tau(x)$. Since $T$ is trace preserving, $\tau(Ty) = \tau(y) = 0$. For $p \leq 2$ we denote by $c_p$ the best constant with $\| \cdot \|_2 \leq c_p \| \cdot \|_p$. Then $(p-1)/c_p^2 \to 1$ when $p \to 2$. Take $p < 2$ such that $(p-1)/c_p^2 > \lambda^2$, then we have
\[
\|Tx\|_2^2 = \|a + Ty\|_2^2 = \|a\|^2 + \|Ty\|_2^2 \leq \|a\|^2 + \lambda^2 y_2^2 \\
\leq \|a\|^2 + \lambda^2 x_p^2 \|y\|_p^2 \leq \|a\|^2 + (p-1)\|y\|_p^2 \leq \|x\|_p^2,
\]
whence (2.4). \hfill \Box

Remark 2.7. Let $A$ be a finite dimensional C*-algebra equipped with a faithful tracial state $\tau$, and $T : A \to A$ be a unital trace preserving map on $A$. Consider the restriction of $T$ on the subspace $\{x \in A : \tau(x) = 0\}$ of $A$ and its adjoint, then we see that
\[
\sup_{x \in A \setminus \{0\}, \tau(x) = 0} \frac{\|Tx\|_2}{\|x\|_2} = \sup_{x \in A \setminus \{0\}, \tau(x) = 0} \frac{\|T^*x\|_2}{\|x\|_2}.
\]
Then the above theorem also implies that if there exists $1 < p < 2$ such that
\[
\forall x \in A, \|Tx\|_2 \leq \|x\|_p,
\]
then
\[
\forall x \in A, \|T^*x\|_2 \leq \|x\|_p,
\]
and equivalently for $2 < q < \infty$ with $1/p + 1/q = 1$,
\[
\forall x \in A, \|Tx\|_q \leq \|x\|_2.
\]

It is easy to see that the free product of unital trace preserving completely positive maps can be extended to the $L_p$-spaces on using the interpolation between $L_1$ and $L_\infty$. But in general it is a delicate problem for the extension of algebraic free product of unital trace preserving maps onto the associated $L_p$-spaces. Here we provide a method to construct unital trace preserving $L_p$-improving operators on the free product of finite-dimensional C*-algebras. To see this we need the following trivial claim.
Claim 2.8. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a faithful tracial state $\tau$. If the vectors $e_1, \ldots, e_m \in \mathcal{M}$ are orthonormal in $L_2(\mathcal{M}, \tau)$ and denote $c = \max_{1 \leq k \leq m} \|e_k\|_{\infty}^2$, then for $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ and $2 \leq q \leq \infty$,
\[
\| \sum_{k=1}^{m} \alpha_k e_k \|_q \leq (cm)^{\frac{1}{q} - \frac{1}{2}} \| \sum_{k=1}^{m} \alpha_k e_k \|_2.
\]

Proof. Note that
\[
\| \sum_{k=1}^{m} \alpha_k e_k \|_\infty \leq c^{1/2} \| \sum_{k=1}^{m} |\alpha_k| \|_{2} \leq c^{1/2} m^{1/2} \left( \sum_{k=1}^{m} |\alpha_k|^{2} \right)^{1/2},
\]
which gives the claim for $q = \infty$. The inequality for $2 \leq q \leq \infty$ then follows from the Hölder inequality. \qed

Theorem 2.9. Let $(A_i, \tau_i), 1 \leq i \leq n$ be a finite family of finite dimensional $C^*$-algebras and set $(A, \tau) = \ast_{1 \leq i \leq n}(A_i, \tau_i)$ to be the von Neumann algebraic free product. For each $1 \leq i \leq n$, $T_i$ is a unital trace preserving map such that
\[
\| T_i : L_p(A_i) \rightarrow L_2(A_i) \| = 1
\]
for some $1 < p < 2$. Then the (algebraic) free product map $T = \ast_{1 \leq i \leq n}T_i$ on $\ast_{1 \leq i \leq n}A_i$ extends to a map such that
\[
\| T : L_{p'}(A) \rightarrow L_2(A) \| = 1
\]
for some $1 < p' < 2$.

Proof. By the previous theorem and remark,
\[
(2.5) \quad \lambda = \max_{1 \leq i \leq n} \sup_{x \in A_i} \| T_i x \|_2 = \max_{1 \leq i \leq n} \sup_{x \in A_i} \| T_i^* x \|_2 < 1.
\]
Consider $R = T^*$ and $R_i = T_i^*$ for all $1 \leq i \leq n$, then $R = R_1 \cdots R_n$. By density, consider $x \in \ast_{1 \leq i \leq n}(A_i, \tau_i)$ in the algebraic free product and we will show that
\[
\| Rx \|_q \leq \| x \|_2
\]
for some $q > 2$ independent of the choice of $x$. Now fix some $r \geq 1$. For each $i$, choose a family $(e_{k,i}^{(r)})_{k=1}^{n_{ij}}$ of eigenvectors of $|R_i|$ which forms an orthonormal basis of $A_{i}$ under $\tau_{i}$, then $E_r = \{ e_k^{(1)} \cdots e_k^{(r)} : 1 \leq k \leq n_j, 1 \leq j \leq \tau, i_1 \neq \cdots \neq i_r \}$ forms an orthonormal basis of $\oplus_{i_1 \neq \cdots \neq i_r} \hat{A}_{i_1} \otimes \cdots \otimes \hat{A}_{i_r}$ which are also eigenvectors of $|R|$. Note that $|E_r| \leq n^r m^r$ for $m = \max j n_j$. Write additionally $c = \max_{k,i} \| e_{k,i}^{(r)} \|_{\infty}^2$. Then for any $y_r \in \oplus_{i_1 \neq \cdots \neq i_r} \hat{A}_{i_1} \otimes \cdots \otimes \hat{A}_{i_r}$, the above claim yields
\[
(2.6) \quad \| y_r \|_q \leq (cnm)^{r(\frac{1}{q} - \frac{1}{2})} \| y_r \|_2.
\]
Write $x = \tau(x)1 + \sum_{r \geq 1} x_r$ where $x_r \in \oplus_{i_1 \neq \cdots \neq i_r} \hat{A}_{i_1} \otimes \cdots \otimes \hat{A}_{i_r}$. Note that $\| Rx_r \|_2 \leq \lambda^r \| x_r \|_2$ according to (2.5) and the choice of $E_r$. Together with Theorem 2.5 and (2.6),
\[
\| Rx \|_q^2 \leq |\tau(x)|^2 + (q - 1) \sum_{r \geq 1} \| Rx_r \|_q^2 \leq |\tau(x)|^2 + (q - 1) \left( \sum_{r \geq 1} \| Rx_r \|_q^2 \right)^2
\]
\[
\leq |\tau(x)|^2 + (q - 1) \left( \sum_{r \geq 1} (cnm)^{r(\frac{1}{q} - \frac{1}{2})} \| Rx_r \|_2^2 \right)^2
\]
\[
\leq |\tau(x)|^2 + (q - 1) \left( \sum_{r \geq 1} (cnm)^{r(\frac{1}{q} - \frac{1}{2})} \lambda^r \| x_r \|_2^2 \right)^2.
\]
Observe that \((q - 1)(\text{cmn})^{q - \frac{1}{q}}\) tends to 1 whenever \(q \to 2\) and that \(\lambda < 1\), so we may choose \(2 < q < \infty\) such that \(\lambda(\text{cmn})^{q - \frac{1}{q}} \leq (q - 1)^{-1}\). For such a \(q\) we then have

\[
\|Rx\|_2^2 \leq |\tau(x)|^2 + (q - 1) \left( \sum_{r \geq 1} (q - 1)^{-r} \|x_r\|_2^2 \right)^2
\]

\[
\leq |\tau(x)|^2 + (q - 1) \sum_{k \geq 1} (q - 1)^{-2k} \sum_{r \geq 1} \|x_r\|_2^2
\]

\[
< |\tau(x)|^2 + \sum_{r \geq 1} \|x_r\|_2^2 = \|x\|_2^2.
\]

Take \(1 < p' < 2\) such that \(1/p' + 1/q = 1\). Then we get \(\|T : L_p(A) \to L_2(A)\| = 1\).

\[
\square
\]

3. \(L_p\)-improving convolution operators for quantum groups

In this section we aim to give several characterizations of \(L_p\)-improving convolutions given by states on finite quantum groups, and also give the constructions for the free product of finite quantum groups. We will start with some discussions on multipliers on compact quantum groups. As for a compact quantum group \(G\) we are only interested in the elements in the \(L_p\)-spaces \(L_p(G)\) or that in the reduced algebra \(C_r(G)\) by density, we always assume in this section that the Haar state \(h\) on \(G\) is faithful on \(C(G)\) and write \(C(G) = C_r(G)\). In this section we keep the notation of multipliers \(m_a, m'_a\) and convolutions \(\varphi_1 \ast \varphi_2\) given in Section 1.2.2.

**Lemma 3.1.** Let \(G\) be a compact quantum group of Kac type. Suppose \(a \in \ell_\infty(G)\) such that \(m_a\) (resp. \(m'_a\)) extends to a unital left (resp., right) multiplier on \(C(G)\) and \(b = aa^*\). Then \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n m_k^ax = h(x)1\) for all \(x \in C(G)\) if and only if \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n m_k^ax = h(x)1\) for all \(x \in C(G)\) if and only if \(\|a_\alpha\| < 1\) for all \(\alpha \in \text{Irr}(G) \setminus \{1\}\).

**Proof.** Without loss of generality we only discuss the left multiplier \(m_a\). Assume that \(\|a_\alpha\| < 1\) for all \(\alpha \in \text{Irr}(G) \setminus \{1\}\). By the Plancherel theorem 1.5 and the formula (1.9) we note that \(m_k^a\) extends to a bounded map of norm one on \(L_2(G)\). We first consider the case \(x \in \text{Pol}(G)\), so that \(x\) is finitely supported. Let \(\alpha \in \text{Irr}(G) \setminus \{1\}\) and \(\|a_\alpha\| < 1\). Then

\[
\left\|\left(\frac{1}{n} \sum_{k=1}^n m_k^ax\right)\right\|_2 = \left\|\frac{1}{n} \sum_{k=1}^n \hat{x}(\alpha)(a_\alpha a_\alpha^*)^k\right\|_2 \leq \frac{1}{n} \sum_{k=1}^n \|a_\alpha\|^{2k} \left\|\hat{x}(\alpha)\right\|_2 \to 0
\]

whenever \(n \to \infty\). And for \(\alpha = 1\),

\[
(\frac{1}{n} \sum_{k=1}^n m_k^ax)^{(1)} = \hat{x}(1) = h(x).
\]

Thus by the Plancherel theorem

\[
\left\|\frac{1}{n} \sum_{k=1}^n m_k^ax - h(x)1\right\|_2^2 = \sum_{\alpha \neq 1} d_\alpha \left\|\frac{1}{n} \sum_{k=1}^n m_k^ax\right\|_2^2 \to 0
\]

when \(n \to \infty\). Since \(\frac{1}{n} \sum_{k=1}^n m_k^ax\) is a contraction on \(L_2(G)\) and \(\text{Pol}(G)\) is \(\|\cdot\|_2\)-dense in \(C(G)\), we get the convergence \(\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n m_k^ax = h(x)1\) for all \(x \in C(G)\).

Conversely, if \(\exists \alpha_0 \in \text{Irr}(G) \setminus \{1\}\), \(\|a_{\alpha_0}\| = 1\), then viewing \(b_{\alpha_0}\) as a matrix in \(M_{n_{\alpha_0}}\), we observe that \(1 \in \sigma(b_{\alpha_0})\) and there exists a nonzero \(x_{\alpha_0} \in M_{n_{\alpha_0}}\) such that \(x_{\alpha_0}b_{\alpha_0} = x_{\alpha_0}\). Take \(x \in C(G)\) such that \(\hat{x}(1) = 1\), \(\hat{x}(\alpha_0) = x_{\alpha_0}\), \(\hat{x}(\alpha) = 0\) for \(\alpha \in \text{Irr}(G) \setminus \{1, \alpha_0\}\). Then \(m_{\alpha_0}x = x\) and hence \(\frac{1}{n} \sum_{k=1}^n m_k^ax = x\) does not converge to \(h(x)1\).

\[
\square
\]

**Remark 3.2.** In case the compact quantum group \(G\) is not of Kac type, the above argument still remains true for right multipliers.

The following first main result is now in reach. We will consider the case where \(G\) is a finite quantum group.
Theorem 3.3. Let $G$ be a finite quantum group. Suppose $a \in \ell_\infty(\hat{G})$ is such that $m_a$ (resp., $m'_a$) is a unital left (resp., right) multiplier on $C(G)$. Then the following assertions are equivalent:

1. there exists $1 \leq p < 2$ such that,
$$\forall x \in C(G), \|m_ax\|_2 \leq \|x\|_p ;$$
2. there exists $1 \leq p < 2$ such that,
$$\forall x \in C(G), \|m'_ax\|_2 \leq \|x\|_p ;$$
3. $\|a_\alpha\| < 1$ for all $\alpha \in \text{Irr}(G) \setminus \{1\}$ ;
4. $\lim_n \frac{1}{n} \sum_{k=1}^n m^k_b x = h(x)1$ for all $x \in C(G)$ when $b = aa^*$;
5. $\lim_n \frac{1}{n} \sum_{k=1}^n m'^k_b x = h(x)1$ for all $x \in C(G)$ when $b = aa^*$.

Proof. Without loss of generality we only discuss the left multiplier $m_a$ and prove the equivalence (1) $\iff$ (3) $\iff$ (4).

It is easy to see from Plancherel’s theorem that (3) is just (2.2) for $T = m_a$. In fact note that for $x \in C(G)$, $h(x) = 0$ if and only if $\hat{x}(1) = 0$, so (3) implies (2.2) via Plancherel’s theorem. On the other hand, suppose by contradiction that there exists $\alpha \in \text{Irr}(G) \setminus \{1\}$ such that $\|a_\alpha\| = 1$. By Proposition 1.5 we may take a nonzero $x \in C(G)$ such that $\hat{x}(1) = 0$, $\hat{x}(\alpha) = 0$ when $\|a_\alpha\| < 1$, and $\|\hat{x}(\alpha)a_\alpha\|_2 = \|\hat{x}(\alpha)\|_2$ when $\|a_\alpha\| = 1$. Then $\|m_ax\|_2 = \|x\|_2$ with $h(x) = 0$. As a consequence the equivalence (1) $\iff$ (3) follows from Theorem 2.6.

The equivalence between (3) and (4) was proved in the previous lemma. Therefore the theorem is established.

Now we turn to the corresponding convolution problems. We need the following lemma adapted from [Wor98, Lemma 2.1].

Lemma 3.4. Let $G$ be a compact quantum group and $A = C(G)$. Suppose that $(\rho_i)_{i \in I}$ is a family of states on $A$, i.e., $\forall x \in A_+ \setminus \{0\}$, $\exists i \in I$, $\rho_i(x) > 0$. If $\rho$ is a state on $A$ such that
$$\forall i \in I, \quad \rho \ast \rho_i = \rho_i \ast \rho = \rho,$$
then $\rho$ is the Haar state $h$ of $G$.

Proof. Set
$$I = \{ q \in A \otimes A : \forall i \in I, (\rho_i \otimes \rho)(q^*q) = 0 \}.$$

Then $I$ is a closed left ideal of $A \otimes A$. Define
$$\Psi_L(x) = (\rho \otimes \iota)\Delta(x) - \rho(x)1, \quad x \in A.$$

Since $\Psi_L$ is a difference of two unital completely positive maps, we see that $\Psi_L$ is a completely bounded map with norm at most 2. We will prove that
$$(\Psi_L \otimes \iota)\Delta(A) \subset I.$$

In fact, given $x \in A$, by the coassociativity of $\Delta$ we have
$$q := (\Psi_L \otimes \iota)\Delta(x) = (\rho \otimes \iota \otimes \iota)(\iota \otimes \Delta)\Delta(x) - 1 \otimes [(\rho \otimes \iota)\Delta(x)]$$
$$= \Delta((\rho \otimes \iota)\Delta(x)) - 1 \otimes [(\rho \otimes \iota)\Delta(x)].$$

Thus
$$q^*q = \Delta([(\rho \otimes \iota)\Delta(x)]^*[(\rho \otimes \iota)\Delta(x)]) - \Delta((\rho \otimes \iota)\Delta(x))^*[1 \otimes (\rho \otimes \iota)\Delta(x)]$$
$$- [1 \otimes ((\rho \otimes \iota)\Delta(x))^*] \Delta((\rho \otimes \iota)\Delta(x)) + 1 \otimes ([(\rho \otimes \iota)\Delta(x)]^*[\rho \otimes \iota)\Delta(x)])$$
and hence for any $i \in I$ we may write
$$(\rho_i \otimes \rho)(q^*q) = q_1 - q_2 - q_3 + q_4$$
where by the convolution invariance assumption and the coassociativity of \( \Delta \) we have
\[
q_1 = (\rho \otimes \rho) \Delta \left( \left( \rho \otimes \rho \Delta(x) \right)^* \rho \otimes \rho \Delta(x) \right) = \rho \left( \left( \rho \otimes \rho \Delta(x) \right)^* \rho \otimes \rho \Delta(x) \right), \\
q_2 = q_3, \\
q_3 = (\rho \otimes \rho) \left( \left( 1 \otimes \left( (\rho \otimes \rho) \Delta(x) \right)^* \Delta \left( (\rho \otimes \rho) \Delta(x) \right) \right) \right) \\
= \rho \left( \left( \rho \otimes \rho \Delta(x) \right)^* \rho \otimes \rho \Delta(x) \right), \\
q_4 = (\rho \otimes \rho) \left( 1 + \left( (\rho \otimes \rho) \Delta(x) \right)^* \rho \otimes \rho \Delta(x) \right).
\]

Note that \( q_1 = q_2 = q_3 = q_4 \). So \( (\rho \otimes \rho)(q^*q) = 0 \) and \( (\Psi_L \otimes \iota)\Delta(A) \subset \mathcal{I} \) is proved.

Now by the density of \((1 \otimes A)\Delta(A)\) in \( A \otimes A \) and the complete boundedness of \( \Psi_L \), it follows that \( (1 \otimes A)\Delta(A) = \Delta(A) \subset \mathcal{I} \) is also contained in the closed left ideal \( \mathcal{I} \), which means that for any \( i \in I \) and \( x \in A \),
\[
\rho_i(\Psi_L(x)^*\Psi_L(x)) = \rho_i \otimes \rho(\Psi_L(x)^*\Psi_L(x) \otimes 1) = 0.
\]

Recall that \( (\rho_i)_{i \in I} \) separates the points of \( A_i \), so we have \( \Psi_L(x) = 0 \) and \( (\rho \otimes \rho)\Delta(x) = \rho(x)1 \) for all \( x \in A \).

A similar argument applies as well to the map \( \Psi_R(x) = (\iota \otimes \rho)\Delta(x) - \rho(x)1, x \in A \). So \( \rho = h \) is the Haar state.

**Remark 3.5.** We remark that in case \( G \) is a finite quantum group, we can provide a simpler proof of the above lemma. Indeed, since \( C(G) \) is finite-dimensional, its dual space \( C(G)^* \) is also finite-dimensional, and we can take a maximal linear independent family \( \{\rho_1, \ldots, \rho_s\} \subset (\rho_i)_{i \in I} \) which form a basis of the subspace spanned by \( \{\rho_i\}_{i \in I} \) in \( C(G)^* \). Given a nonzero \( x \in A_i \), there is an \( i \in I \) such that \( \rho_i(x) > 0 \). Write \( \rho_i = \sum_{k=1}^{n} a_k \rho_{ik} \), then we see clearly that there exists at least one \( k \in \{1, \ldots, s\} \) such that \( \rho_{ik}(x) \neq 0 \) in order that \( \rho_i(x) > 0 \). Then \( \rho' = \frac{1}{n} \sum_{k=1}^{n} \rho_{ik} \) is faithful on \( C(G) \) and \( \rho \ast \rho' = \rho' \ast \rho = \rho \), thus \( \rho \) is the Haar state by [Wor98, Lemma 2.1].

We immediately obtain the following fact.

**Lemma 3.6.** Let \( G \) be a compact quantum group and \( \varphi \) be a state on \( C(G) \). Then
\[
w^* - \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^k h = h
\]
if and only if \( \varphi \) is non-degenerate on \( C(G) \) in the sense that for all nonzero \( x \in C(G)^+ \) there exists \( k \geq 0 \) such that \( \varphi^k(x) > 0 \). (Recall that \( h \) has been assumed to be faithful on \( C(G) \) in the beginning of the section.)

**Proof.** If the above limit holds, then clearly \( \varphi \) is non-degenerate since if there existed a nonzero \( x \geq 0 \) such that \( \varphi^n(x) = 0 \) for all \( n \), then we would have \( \lim_n \frac{1}{n} \sum_{k=1}^{n} \varphi^k(x) = 0 \), which contradicts the faithfulness of \( h \). On the other hand, if \( \varphi \) is non-degenerate, the family of states \( \{\frac{1}{n} \sum_{k=1}^{n} \varphi^k : n \geq 1\} \) separates the points of \( C(G)^+ \), so any accumulation point of this family becomes the unique Haar state by our previous lemma.

Now let \( \varphi \in C(G)^* \) for a compact quantum group \( G \). Recall the formula (1.13), and then we have
\[
[\varphi^* \left( (u^\varphi_n)^* \right)] = (\varphi^n)^* = \hat{\varphi}(\alpha)^n.
\]
Note that the convergence \( \frac{1}{n} \sum_{k=1}^{n} m_{\varphi^n}(x) \to h(x)1 \) for all \( x \in \text{Pol}(G) \), by (1.10) can be reformulated in terms of Fourier coefficients as
\[
\left(\frac{1}{n} \sum_{k=1}^{n} m_{\varphi^n}(x) \right)^* \rho(1) = h(x)\hat{\varphi}(1)^* = h(x)1,
\]
\[
\lim_n \left(\frac{1}{n} \sum_{k=1}^{n} m_{\varphi^n}(x) \right)^* = \lim_n \frac{1}{n} \sum_{k=1}^{n} (\hat{\varphi}(\alpha)^k \hat{\varphi}(\alpha) = 0, \quad \alpha \in \text{Irr}(G) \setminus \{1\}.
\]
This is to say,
\[ \hat{\varphi}(1) = 1, \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}(\alpha) = 0, \quad \alpha \in \text{Irr}(G) \setminus \{1\}, \]
which, according to (3.1), is equivalent to \( \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}(u_{ij}^\alpha) \to h(u_{ij}^\alpha) \) for all \( \alpha \in \text{Irr}(G) \) and \( 1 \leq i, j \leq n, \) or in other words,
\[ \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}(x) \to h(x), \quad n \to \infty, \quad x \in \text{Pol}(G). \]

Any state \( \varphi \) on \( C(G) \) induces two convolution operators on \( C(G) \)
\[ T_\varphi : C(G) \ni x \mapsto x \ast \varphi = (\varphi \otimes \iota)(\Delta(x)), \quad T'_\varphi : C(G) \ni x \mapsto \varphi \ast x = (\iota \otimes \varphi)(\Delta(x)). \]
If additionally \( G \) is of Kac type, then by Proposition 1.3 the antipode \( S \) extends to a positive linear operator on \( C(G) \) and \( S = S^{-1} \), and hence \( \varphi \circ S = \varphi \circ S^{-1} \) is also a state. In this case we have
\[ \varphi(u_{ij}^\alpha)_{i,j} = \hat{\varphi}(\alpha), \quad (\varphi \circ S)(\alpha) = [\varphi(u_{ij}^\alpha)]_{i,j} = \hat{\varphi}(\alpha)^* \]
and by (1.14) we have \( (x \ast \varphi)(\alpha) = \hat{x}(\alpha)\hat{\varphi}(\alpha)^* \) and \( (\varphi \ast x)(\alpha) = \hat{\varphi}(\alpha)^*\hat{x}(\alpha) \) for all \( \alpha \in \text{Irr}(G) \), \( x \in C(G) \). So \( T_\varphi = m_{\hat{\varphi}} \) and \( T'_\varphi = m'_{\hat{\varphi}} \), are unital completely positive left and right multipliers, respectively.

Now with these remarks and Lemma 3.6 in hand, we may reformulate Theorem 3.3 in terms of convolution operators using the above arguments.

**Theorem 3.7.** Let \( G \) be a finite quantum group and \( \varphi \) be a state on \( C(G) \). Denote \( \psi = (\varphi \circ S) \ast \varphi \). The following assertions are equivalent:

1. there exists \( 1 \leq p < 2 \) such that,
\[ \forall x \in C(G), \quad \|x \ast \varphi\|_p \leq \|x\|_p ; \]
2. there exists \( 1 \leq p < 2 \) such that,
\[ \forall x \in C(G), \quad \|x \ast \varphi\|_p \leq \|x\|_p ; \]
3. \( \|\hat{\varphi}(\alpha)\| < 1 \) for all \( \alpha \in \text{Irr}(G) \setminus \{1\} ; \)
4. \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \psi^{*k} = h; \)
5. For any nonzero \( x \in C(G)_+ \), there exists \( n \geq 1 \) such that \( \psi^{*n}(x) > 0 \).

Note that if \( C(G) \) is commutative, i.e. \( C(G) = C(G) \) where \( G \) is a finite group, then \( \varphi \in C(G)^* \) corresponds to a Radon measure \( \mu \) via the Riesz representation theorem. The above condition (5) in the theorem just asserts that \( G \) is the union of \( D_n := \text{supp}(\nu^n) \), \( n \geq 1 \), where \( \nu \) denotes the Radon measure corresponding to \( \psi \). It is easy to see that \( D_1 = \{ i^{-1}j : i,j \in \text{supp}(\mu) \} \) and \( D_n = D_n^\# \). So the above corollary covers Ritter’s result [Rit84].

**Corollary 3.8.** Let \( G \) be a finite group and \( \mu \) be a probability measure on \( G \). Then there is a \( 1 \leq p < 2 \) such that
\[ \|x \ast \mu\|_p \leq \|x\|_p, \quad x \in L_p(G) \]
if and only if \( G \) is equal to the subgroup generated by \( \{ i^{-1}j : i,j \in \text{supp}(\mu) \} \).

On the other hand, let \( \Gamma \) be a finite group with neutral element \( e \) and \( C^*(\Gamma) \) be the associated group \( C^*-\)algebra generated by \( \lambda(\Gamma) \subset B(l^2(\Gamma)) \), where \( \lambda \) denotes the left regular representation. Recall that if \( G = \hat{\Gamma} \), then \( C(G) \) is the group \( C^*-\)algebra \( C^*(\Gamma) \) equipped with the comultiplication \( \Delta : C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma) \) defined by
\[ \Delta(\lambda(\gamma)) = \lambda(\gamma) \otimes \lambda(\gamma), \quad \gamma \in \Gamma. \]
Note that any state \( \Phi \) on \( C(G) \) corresponds to a positive definite function \( \varphi \) on \( \Gamma \) with \( \varphi(e) = 1 \) via the relation \( \Phi(\lambda(\gamma)) = \varphi(\gamma) \) for all \( \gamma \in \Gamma \). Therefore we have
\[ \Phi \ast \lambda(\gamma) = \lambda(\gamma) \ast \Phi = (\Phi \otimes \iota)\Delta(\lambda(\gamma)) = (\Phi \otimes \iota)(\lambda(\gamma) \otimes \lambda(\gamma)) = \varphi(\gamma)\lambda(\gamma), \]
so the convolution operators associated to \( \Phi \) are just the Fourier-Schur multiplier on \( \Gamma \) associated to \( \varphi \). Our precedent argument in particular yields the following result extending [Rit84, Theorem 2(a)].

**Corollary 3.9.** Let \( \Gamma \) be a finite group and \( \varphi \) be a positive definite function on \( \Gamma \) with \( \varphi(e) = 1 \). Let \( M_\varphi \) be the associated Fourier-Schur multiplier operator determined by \( M_\varphi(\lambda(\gamma)) = \varphi(\gamma)\lambda(\gamma) \) for all \( \gamma \in \Gamma \). Then there exists \( 1 \leq p < 2 \) such that

\[
\|M_\varphi x\|_2 \leq \|x\|_p, \quad x \in C^*(\Gamma)
\]

if and only if \( |\varphi(\gamma)| < 1 \) for any \( \gamma \in \Gamma \setminus \{e\} \).

**Remark 3.10.** We remark that the finite-dimensional condition cannot be removed in any of our results, including Theorem 2.1, Theorem 3.3, Corollary 3.7-3.9. Here we give a counterexample illustrating this. Let \( \mathbb{T} \) be the unit circle in the complex plane, then \( \mathbb{T} \) gives an infinite compact quantum group. Define an operator \( T : C(\mathbb{T}) \to C(\mathbb{T}) \) by

\[
T(f) = (1 - \lambda)\tau(f) + \lambda f, \quad f \in C(\mathbb{T}),
\]

where \( 0 < \lambda < 1 \) and \( \tau \) denote the usual integral against the normalized Lebesgue measure on \( \mathbb{T} \). Then \( T \) is obviously unital completely positive since so are \( \tau \) and the identity map. It is a left multiplier satisfying \( T(f^*)(0) = f(0) \) and \( T(f^*)(n) = \lambda f(n) \) for \( n \neq 0 \). Also we may view \( T \) as a convolution operator associated to the state \( f \mapsto (1 - \lambda)\tau(f) + \lambda f(1) \) on \( C(\mathbb{T}) \), which is faithful since \( \tau \) is faithful. Note that \( T \) admits the spectral gap inequality (2.2) as well, and in fact, \( \|Tf\|_2 = \lambda\|f\|_2 < \|f\|_2 \) for any \( f \in C(\mathbb{T}) \) with \( \tau(f) = 0 \). However, there doesn’t exist any \( p < 2 \) such that \( \|Tf\|_2 \leq \|f\|_p \) for all \( f \in C(\mathbb{T}) \). Indeed if such a \( p \) existed, then for any \( f \in C(\mathbb{T}) \), we would have

\[
\|f\|_2^2 \geq \|f\|_p^2 \geq \|Tf\|_2^2 = \tau(f)^2 + \lambda^2\|f - \tau(f)\|_2^2 \geq \lambda^2(\tau(f)^2 + \|f - \tau(f)\|_2^2) = \lambda^2\|f\|_2^2,
\]

which yields an impossible equivalence between the norms \( \| \cdot \|_2 \) and \( \| \cdot \|_p \).

In spite of the above general remark, Theorem 2.9 still gives constructions of \( L_p \)-improving positive convolution operators on infinite compact quantum groups. Let \( \mathbb{G}_1, \ldots, \mathbb{G}_n \) be finite quantum groups and let each \( \varphi_i \) be a state on \( C(\mathbb{G}_i) \), \( i \in \{1, \ldots, n\} \). Denote \( \mathbb{G} = \mathbb{G}_1 \ast \cdots \ast \mathbb{G}_n \) with the Haar state \( h \) and consider the convolution operators \( T_i : x \mapsto x \ast \varphi_i, x \in C(\mathbb{G}_i) \). Note that the free product map \( T = \ast_{1 \leq i \leq n} T_i \) on \( C(\mathbb{G}) \) is just the convolution operator given by the free product state \( \varphi = \ast_{1 \leq i \leq n} \varphi_i \), i.e.,

\[
T(x) = (\varphi \circ \iota)\Delta(x), \quad x \in C(\mathbb{G}).
\]

In fact, we note that if \( h(a) = 0 \), then \( h(a(1)) = h(a(2)) = 0 \) by (1.1), where \( \Delta(a) := \sum a(1) \otimes a(2) \) denotes the Sweedler notation. Now for a reduced word \( x = x^1 \cdots x^m \) with \( x^k \in C(\mathbb{G}_{i_k}) \) such that

\[
h(x^k) = 0, \; i_1 \neq \cdots \neq i_m, \; i_k \in \{1, \ldots, n\} \quad \text{for each} \; k = 1, \ldots, m,
\]

we have

\[
T(x) = T_{i_1}(x^1) \cdots T_{i_m}(x^m) = \sum \varphi_{i_1}(x^1_{(1)})x^1_{(2)} \cdots \sum \varphi_{i_m}(x^m_{(1)})x^m_{(2)} = \sum \varphi(x^1_{(1)})x^2_{(2)} = (\varphi \circ \iota)\Delta(x)
\]

where we have used the fact that the comultiplication \( \Delta \) is an homomorphism. Then the equality (3.2) follows from a standard density argument. Now taking in Theorem 2.9 each \( T_i \) to be a convolution operator on a finite quantum group, we get the following corollary:

**Corollary 3.11.** Let \( \mathbb{G}_1, \ldots, \mathbb{G}_n \) be finite quantum groups and let each \( \varphi_i \) be a state on \( C(\mathbb{G}_i) \), \( i \in \{1, \ldots, n\} \). Denote \( \mathbb{G} = \mathbb{G}_1 \ast \cdots \ast \mathbb{G}_n \) and \( \varphi = \varphi_1 \ast \cdots \ast \varphi_n \). If each \( \varphi_i \) satisfies any one of the conditions (1)-(5) in Corollary 3.7, then the free product convolution operator given by \( T : x \mapsto x \ast \varphi, x \in C(\mathbb{G}) \) is a unital left multiplier on \( \mathbb{G} \) satisfies

\[
\|T : L_p(\mathbb{G}) \to L_2(\mathbb{G})\| = 1
\]

for a certain \( 1 < p < 2 \).
Example 3.12. Now we give a simple method to create nontrivial $L_p$-improving positive convolutions (i.e., the associated state is different from the Haar state) on finite and infinite compact quantum groups. Let $G$ be a finite quantum group and $h$ the Haar state on it. Given any state $\varphi$ on $C(G)$ and any $0 < \lambda < 1$, we can associate a state $\rho$ on $C(G)$ given by

$$\rho = \lambda \varphi + (1 - \lambda) h.$$ 

It is easy to see that $\|\hat{\rho}(\alpha)\| < 1$ for all $\alpha \in \text{Irr}(G) \setminus \{1\}$ and hence the convolution operator $T_\rho: x \mapsto x \ast \rho, x \in C(G)$ satisfies

$$\|T_\rho: L_p(G) \to L_2(G)\| = 1$$

for a certain $1 < p < 2$ according to Theorem 3.7. Moreover by Corollary 3.11, the convolution operator $T_{\rho'}: x \mapsto x \ast \rho', x \in C(G+G)$ given by the free product state $\rho' = \rho \ast \rho$ satisfies

$$\|T_{\rho'}: L_p(G+G) \to L_2(G+G)\| = 1$$

for some $1 < p' < 2$.

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