WEAK APPROXIMATION FOR CERTAIN QUADRIC SURFACE BUNDLES

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ABSTRACT. We investigate weak approximation for a class of biquadratic fourfolds inside $\mathbb{P}^2 \times \mathbb{P}^3$, some of which appear in the recent work of Hassett–Pirutka–Tschinkel [13].

1. Introduction

Let $k$ be a number field of degree $d$ and let $\Omega$ denote the set of valuations on $k$. Given a smooth algebraic variety $X$ over $k$, we have the following embeddings

$$X(k) \hookrightarrow X(\mathbb{A}_k) \subset \prod_{\nu \in \Omega} X(k_{\nu}),$$

where the first map is the diagonal embedding of the rational points into the adèles. When the set of rational points on $X$ is non-empty, one would like to be able to discuss (either qualitatively or quantitatively) how the rational points on $X$ are distributed. We say that the variety $X$ satisfies weak approximation if the image of $X(k)$ is dense in $X(\mathbb{A}_k)$, under the product topology.

In his 1970 ICM talk, Manin observed that one can use the Brauer group, $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$, to define a set $X(\mathbb{A}_k)^{\text{Br}}$ with $X(k) \subset X(\mathbb{A}_k)^{\text{Br}} \subset X(\mathbb{A}_k)$, which can sometimes obstruct weak approximation. Colliot-Thélène [2] conjectured that this Brauer–Manin obstruction is the only obstruction to weak approximation for any smooth, projective, geometrically integral and rationally connected variety. Our first result is to confirm Colliot-Thélène’s conjecture for a particular class of fourfolds over $k$.

**Theorem 1.1.** Let $k$ be a number field and $X/k$ the biprojective variety in $\mathbb{P}^2 \times \mathbb{P}^3$ defined by the equation

$$xyt_1^2 + xzt_2^2 + yzt_3^2 + F(x, y, z)t_4^2 = 0,$$  \hspace{1cm} (1.1)

where $F$ is a non-degenerate ternary quadratic form over $k$. Then the Brauer–Manin obstruction is the only obstruction to weak approximation for any smooth projective model of $X$.

There exist two general methods by which to prove theorems of this nature. The first is the descent method, a generalisation of the classical descent theory...
of elliptic curves via the use of universal torsors (c.f. [22]). The second is the fibration method, in which one exploits the existence of a fibration $f : X \to Y$ where the base $Y$ and the fibres of $f$ satisfy the desired property. The varieties in Theorem 1.1 naturally admit the structure of quadric surface bundles over $\mathbb{P}^2$ by the projection map

$$X \to \mathbb{P}^2$$

$$(x : y : z ; t) \mapsto (x : y : z),$$

and so it is the latter method which will be relevant.

The earliest example of the use of a fibration to study local-global principles in a family of varieties is due to Hasse in his proof of the local-global principle for quadratic forms in 4 variables (see e.g. [18, Ch. IV, Thm. 8]). These ideas were then generalised by Colliot-Thélène and Sansuc [5] who replaced Hasse’s use of the prime number theorem in arithmetic progressions with an evocation of Schinzel’s Hypothesis, which allowed them to show conditionally that the Brauer–Manin obstruction is the only one for a large class of conic bundles over $\mathbb{P}^1$. This was extended by Colliot-Thélène and Swinnerton-Dyer [9](building on work of Serre [19], Swinnerton-Dyer [23] and Salberger [16]) to establishing the conjecture (conditionally on a variant of Schinzel’s Hypothesis) for pencils of generalised Severi–Brauer varieties (in the language of [9]), of which quadric surfaces are an example. Skorobogatov [21](and subsequently Colliot-Thélène–Skorobogatov [8]) was able to establish unconditionally that the Brauer–Manin obstruction is the only one for quadric surface bundles over $\mathbb{P}^1$ of rank $\leq 3$, by combining the fibration and descent methods. Finally, if all of the degenerate fibres of a quadric surface bundle over $\mathbb{P}^1$ are defined over $\mathbb{Q}$, then the Brauer–Manin obstruction is unconditionally known to be the only one thanks to the work of Browning–Mattheisen–Skorobogatov [1, Theorem 1.4].

Over higher dimensional bases, much less is known. Conditional on Schinzel’s Hypothesis, Wittenberg [26, Corollaire 3.6] has shown that a fibration over $\mathbb{P}^n$ into generalised Severi–Brauer varieties has the property that the Brauer–Manin obstruction is the only one. Theorem 1.1 gives a rare example of an unconditional proof that the Brauer–Manin obstruction is the only one, without a specific restriction on the number (or degree) of the degenerate fibres. We take this moment to point out that conic bundle fibrations have a vast and illustrious history in the literature and that fibrations over projective space into quadrics of dimension $\geq 3$ always satisfy weak approximation.

Our motivation for studying the particular class of varieties that we do comes from a recent breakthrough by Hassett–Pirutka–Tschinkel [13] on classical rationality problems in algebraic geometry. It was observed in that paper that the Brauer group of certain quadric surface bundles can contain non-constant
elements, hence it is natural to wonder if these Brauer group elements might obstruct weak approximation.

In particular, Hassett–Pirutka–Tschinkel were able to show for the first time that rationality is not deformation invariant in families of complex fourfolds by exhibiting a family for which many of the fibres were rational but a ‘very general’ fibre was not even stably rational. This result was proven using the specialisation method pioneered by Voisin [25] and later refined by Colliot-Thélène–Pirutka [6] and Schreieder [17]. The upshot of this method is that if one can find a special fibre $V_b$ in the family such that

\begin{enumerate}
\item $H^2_{nr}(\mathbb{C}(V_b), \mathbb{Z}/2\mathbb{Z})$ is non-trivial,
\item $V_b$ admits a desingularisation which is universally $\text{CH}_0$-trivial,
\end{enumerate}

then a very general fibre is not stably rational. In [13], these conditions were shown to be satisfied by the fourfold $V_b$ defined by the equation

$$xyt_1^2 + xzt_2^2 + yzt_3^2 + F_{\text{HPT}}(x, y, z)t_4^2 = 0,$$

where

$$F_{\text{HPT}}(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz). \quad (1.2)$$

The unramified cohomology group (c.f. Section 3) used in the specialisation method satisfies

$$H^2_{nr}(\mathbb{C}(V_b), \mathbb{Z}/2\mathbb{Z}) = \text{Br}_\mathbb{C}(\tilde{V}_b)[2],$$

where $\tilde{V}_b$ is a desingularisation for $V_b$ (the group on the right hand side is independent of the choice of desingularisation). This means that Hassett–Pirutka–Tschinkel had to explicitly construct a non-trivial element, specifically the quaternion algebra $(x/z, y/z)_{\mathbb{C}(V_b)}$, in the ($\mathbb{C}$-)Brauer group of $V_b$. The major novelty of this paper is to use the techniques employed by Hassett–Pirutka–Tschinkel (and others in the area) over non algebraically closed fields in order to apply them, for the first time, to the study of problems of a Diophantine nature.

The main result of this paper is to completely understand when weak approximation does and does not hold for the varieties appearing in Theorem 1.1.

**Theorem 1.2.** Let $k$ be a number field and $X/k$ the biprojective variety in $\mathbb{P}^3 \times \mathbb{P}^2$ defined by the equation

$$xyt_1^2 + xzt_2^2 + yzt_3^2 + F(x, y, z)t_4^2 = 0,$$

where $F$ is a non-degenerate ternary quadratic form over $k$ such that

$$\begin{cases}
\text{The forms } F(0, y, z), F(x, 0, z), \text{ and } F(x, y, 0) \text{ are all squares}, \\
F(x, y, z) \notin k(x, y, z)^2.
\end{cases} \quad (1.3)$$
If $F$ takes only positive values in all real embeddings, then $X$ satisfies weak approximation. Otherwise, $X(k_{\nu})$ has two connected components for $\nu$ real and weak approximation holds in one but may fail in the other.

It is possible to construct examples where weak approximation fails in one of the components and examples where weak approximation is satisfied (c.f. the final remark of the paper). We note that this provides new families of examples of failure of weak approximation on rational varieties caused by a transcendental Brauer group element. This can never occur for the more familiar setting of pencils of conics but Harari has previously provided examples of conic bundles over higher dimensional bases [12].

**Remark.** The first condition is a little odd looking from a geometric point of view. Perhaps more natural would be the condition that the conic $F(x, yz) = 0$ is tangent to each of the coordinate axis. However, as we shall see in the course of the proof, this condition is not strong enough for fields $k$ which are not algebraically closed. Tangency would allow us to deduce that $F(0, y, z)$ is square in $k[y, z]$ modulo constants, however we really need for it to be a square.

**Remark.** The Hassett–Pirutka–Tschinkel example (1.2) certainly satisfies condition (1.3).

**Remark.** Our varieties are not smooth and hence we must clarify what we mean by weak approximation in this setting. Following Colliot-Thélène–Xiu [10, Section 8], we say that a singular variety $V$ satisfies weak approximation if for any finite set of places of $k$, there exists a resolution of singularities $\tilde{V} \xrightarrow{\varphi} V$ such that the smooth locus of $V$ is dense in the set $\prod_{\nu \in S} \varphi(\tilde{V}(k_{\nu}))$. Note that this definition is independent of the resolution of singularities chosen.

The second scenario in Theorem 1.2 where the rational points are restricted to a connected component occurs often when $\text{Br}(X)/\text{Br}(k)$ is finite, as is the case here, see for instance [24, §3] or [4, Prop 7.2].

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2. Proof of Theorem 1.1

A systematic study of the arithmetic of quadric surface bundles over $\mathbb{P}^1$ was conducted by Skorobogatov [21]. It was proved that if there are very few closed points on $\mathbb{P}^1$ above which the fibre is not geometrically integral then the Brauer–Manin obstruction is the only one for the total space. Our goal is to bootstrap this result into a result about our quadric surface bundles over $\mathbb{P}^2$, which themselves have very few bad fibres. We start by gathering together some salient facts about the geometry of quadric surface bundles.

**Lemma 2.1** ([21, Corollary 4.1]). Let $Z/k$ be a quadric surface bundle over $\mathbb{P}^1_k$ such that the fibre above at most two closed geometric fibres is defined by a quadratic form of rank $\leq 2$. Then, for any smooth proper model of $Z$, the Brauer–Manin obstruction is the only obstruction to weak approximation and to the existence of rational points.

**Lemma 2.2** ([21, Lemma 3.1]). Let $Z/k$ be a quadric surface bundle over $\mathbb{P}^1_k$ and let $Q_\eta$ denote the quadratic form whose vanishing defines the generic fibre. If $\det(Q_\eta)$ is not a square in $k(\eta)$, then $\text{Pic}(Z_{k(\eta)})$ is isomorphic to $\mathbb{Z}$ as $\text{Gal}(k/k)$ modules.

As discussed in the introduction, we will use the fibration method to establish Theorem 1.1 specifically Harari's fibration method with a smooth section.

**Theorem 2.3** ([11, Théorème 4.3.1]). Let $V$ and $B$ be geometrically integral varieties over a field $k$ such that $B$ satisfies weak approximation and there exists a dominant morphism $V \twoheadrightarrow B$ which admits a section $s$. The Brauer–Manin obstruction is the only one for any smooth projective model of $V$ if the following are satisfied:

1. The generic fibre $V_\eta$ is a geometrically integral, geometrically rational variety over $k(B)$ and $s$ defines a smooth point in $V_\eta$.
2. For any smooth projective model $X$ of $V_\eta$, $\text{Br}(X) := \text{Br}(X \times_{k(B)} k(B))$ is trivial and $\text{Pic}(X)$ has no torsion.
3. There exists a non-empty open $U \subset B$ such that $\forall b \in U$ the Brauer–Manin obstruction is the only one for all smooth proper models of $V_b$.

**Proof of Theorem 1.1** We will apply Theorem 2.3 to establish that the Brauer–Manin obstruction is the only one for smooth proper models of $X$. We cannot apply this to the natural fibration $X \to \mathbb{P}^2$, however, as we cannot guarantee the existence of a smooth section. Instead we take an approach inspired by the proof of [1 Thm 1.5]. We will consider the quasi-projective variety obtained by dehomogenising the equation defining $X$. Setting $z = 1$, we have

$$xyt_1^2 + xt_2^2 + yt_3^2 + F(x, y, 1)t_4^2 = 0,$$
which defines a variety $V$ inside $\mathbb{A}^2 \times \mathbb{P}^3$. Again, the natural map to $\mathbb{A}^2$ does not necessarily have a smooth section. Instead, we use the following map

$$V \xrightarrow{\psi} \mathbb{A}^1$$

$$(x, y; t_1, \ldots, t_4) \mapsto x.$$ 

This map does admit a section given by

$$x \mapsto (x, 0; 0, 0, 1, 0).$$

Each fibre of $\psi$, say $V_x$, is a threefold which admits the structure of a quadric surface bundle over $\mathbb{A}^1$, with the map to $\mathbb{A}^1$ given by projecting on the $y$-variable. When $x \neq 0$, the fibre $V_x$ is a quadric surface bundle whose only essential singularity (i.e. fibre defined by a quadratic form of rank $\leq 2$) occurs above $y = 0$. It follows, by Lemma 2.1, that for any projective model of a fibre above the set $U = \{x \neq 0\}$ the Brauer–Manin obstruction is the only one. To handle the second condition, we first observe that by Lemma 2.2 the Picard group of the geometric generic fibre of $\psi$ is $\mathbb{Z}$. Hence, for any smooth projective model $X$ of $V_\eta$, Pic($X$) has no torsion. The Brauer group of $V_\eta$ is isomorphic to $\text{Br}(k(\mathbb{P}^2))$ (for instance, by Proposition 3.1). Hence, for any smooth projective model $X$ of $V_\eta$, we have $\text{Br}(X) \simeq \text{Br}(k(\mathbb{P}^2))$, which is trivial (because $k(\mathbb{P}^2)$ is a $C_1$ field). Finally, we note that the singular locus of the generic fibre of $\psi$ consists of those $(y; t)$ where precisely two of the $t_i$ are $0$, hence this section does indeed correspond to a smooth point on the generic fibre. Therefore, by Theorem 2.3, the Brauer–Manin obstruction is the only one for any smooth projective model of $V$. Since $V$ and $X$ are birational, any resolution of singularities for $X$ will be a smooth projective variety birational to $V$ and hence the conclusion applies. □

3. Computing the Brauer Group

Throughout this and all subsequent sections we fix a number field $k$, let $X/k$ be the variety in the statement of Theorem 1.2 and $\tilde{X}$ a fixed desingularisation. If we hope to exhibit failures of weak approximation, we need to find non-trivial elements in the Brauer group of $\tilde{X}$.

Let $X_\eta$ be the generic fibre of the map $X \to \mathbb{P}^2$. Specifically $X_\eta$ is a quadric surface over the function field $K = k(\mathbb{P}^2)$. We have the injection

$$\text{Br}(X_\eta) \hookrightarrow \text{Br}(k(X))$$

so we start by studying the Brauer group of the generic fibre $X_\eta$.

**Proposition 3.1** ([7 Prop. 6.2.3]). Let $K$ be a field with $\text{char}(K) \neq 2$ and $Q/K$ a smooth projective quadric of dimension 1 or 2. Then the map $\text{Br}(K) \to \text{Br}(Q)$ is surjective. Moreover,
(1) Suppose $Q$ is a conic. If $Q(K) \neq \emptyset$ then the map is an isomorphism. If $Q(K) = \emptyset$, then the kernel of the map is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(2) Suppose $Q$ is a quadric surface. If the discriminant of $Q$ is a square in $K$ and $Q(K) \neq \emptyset$ then the map is an isomorphism. Otherwise, the kernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Since $F(x, y, z)$ is non-square in $k(x, y, z)$, the map $\text{Br}(k(\mathbb{P}^2)) \to \text{Br}(X_\eta)$ is an isomorphism and we have the injection
\[
\text{Br}(k(\mathbb{P}^2)) \xrightarrow{\sim} \text{Br}(X_\eta) \hookrightarrow \text{Br}(k(X)).
\]

This means that non-trivial elements of $\text{Br}(k(\mathbb{P}^2))$ are sent to non-trivial elements of $\text{Br}(k(X))$, which is isomorphic to $\text{Br}(k(\tilde{X}))$ by birationality. Henceforth, we write $\alpha$ as $(-x, -y)$ using the affine coordinates on this chart. The quaternion algebra $(-x, -y)$ has nontrivial residues (e.g. along the axis $\{x = 0\}$) hence it is nontrivial in $\text{Br}(k(\mathbb{P}^2))$. Therefore, we conclude that its image under the map $\text{Br}(k(\mathbb{P}^2)) \hookrightarrow \text{Br}(k(\tilde{X}))$ is nontrivial.

We claim that in fact this image $\beta$ lies in the subgroup $\text{Br}(\tilde{X}) \subset \text{Br}(k(X))$. The tool which we will use to establish this is unramified cohomology. A large part of the appeal of this approach is that one never needs to explicitly compute a smooth model of the singular variety in order to study the Brauer group.

### 3.1. Unramified Cohomology.

For the basic definitions of unramified cohomology groups, we refer the reader to [14]. For smooth projective varieties, by the purity theorem, we can compute the unramified cohomology by studying residues at codimension one points. Indeed
\[
H^i_{nr}(k(V)/k, \mu_n^{\otimes j}) = \bigcap_{x \in V^{(1)}} \text{Ker} \left( H^i(k(V), \mu_n^{\otimes j}) \xrightarrow{\partial_x^i} H^{i-1}(k(x), \mu_n^{\otimes j-1}) \right),
\]
with the intersection here running over DVR’s associated to codimension one points $x$ where $\partial_x^i$ is the associated residue map and $k(x)$ the residue field. One important property of these groups is their connection to torsion in the Brauer group.

**Proposition 3.2** ([14, Prop 3.7]). If $V$ is a smooth projective variety over $k$ then
\[
H^2_{nr}(k(V)/k, \mathbb{Z}/2\mathbb{Z}) \simeq \text{Br}(V)[2].
\]

Since quaternion algebras are 2 torsion elements in the Brauer group, it suffices to prove that our Brauer class $\beta$ lies in $H^2_{nr}(k(\tilde{X})/k, \mathbb{Z}/2\mathbb{Z})$. Note that
we can compute the residues of quaternion algebras explicitly. Suppose that 
\(a, b \in k(V)^*\) and \(\nu\) a rank one valuation on \(k(V)\) with residue field \(\kappa(\nu)\). Then, 
the residue in \(\text{Br}(k(V))[2]\) of the cup product \((a, b) := a \cup b \in H^2(k(V), \mathbb{Z}/2\mathbb{Z})\) 
is given by 
\[
\partial^2_{\nu}(a, b) = (-1)^{\nu(a)\nu(b)} \frac{a^{\nu(b)}}{b^{\nu(a)}},
\]
where \(\frac{a^{\nu(b)}}{b^{\nu(a)}}\) is the image of \(a^{\nu(b)}\) in \(\kappa(\nu)^*/(\kappa(\nu))^2\). We will use this formula 
to compute the residue of \(\alpha\) along codimension one points in \(\mathbb{P}^2\). This can be 
related to the residue of \(\beta\) along codimension one points in \(\tilde{X}\) by the following 
compatibility result.

**Proposition 3.3** ([14, Prop 3.4]). Let \(A \subset B\) be discrete valuation rings with 
fields of fractions \(M \subset L\) respectively. Assume that char\((M)\) and char\((L)\) are 
both not 2. Let \(\pi_A\) (resp. \(\pi_B\)) be a uniformising parameter of \(A\) (resp. \(B\)) and 
\(\kappa(A)\) (resp. \(\kappa(B)\)) the residue field of \(A\) (resp. \(B\)). Let \(e\) be the valuation of 
\(\pi_A\) in \(B\). Then the following diagram commutes 
\[
\begin{array}{ccc}
H^i(L, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial^2_{\nu}} & H^{i-1}(\kappa(B), \mathbb{Z}/2\mathbb{Z}) \\
\uparrow{\text{Res}_{M/L}} & & \uparrow{e\text{Res}_{\kappa(A)/\kappa(B)}} \\
H^i(M, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial^2_{\mu}} & H^{i-1}(\kappa(A), \mathbb{Z}/2\mathbb{Z})
\end{array}
\]
where \(\text{Res}_{K/L}\) and \(e\text{Res}_{\kappa(A)/\kappa(B)}\) are the restriction maps in Galois cohomology.

This allows us to compute the residue at a codimension 1 point of \(\tilde{X}\) with 
codimension 1 image in \(\mathbb{P}^2\) by studying the residue along its image. However, 
codimension 1 points in \(\tilde{X}\) could also map down to the generic point of \(\mathbb{P}^2\) or 
a closed point. To handle some of these cases we will have to make use of the 
following.

**Proposition 3.4** ([14, Cor 3.12]). Let \(A\) be a local ring with field of fractions \(F\), 
with char\((F)\) \(\neq 2\) and residue field \(\kappa\). Let \(Q/F\) be a quadric corresponding 
to the vanishing of the quadratic form \(q \simeq (1, -a, -b, abd)\). Let \(\nu\) be a discrete 
valuation on \(F(Q)\) with valuation ring \(B\). Assume that there is an injection 
\(A \rightarrow B\) of local rings and that upto multiplication by a square in \(F\), the element 
\(d\) is a unit in \(A\) and a square in \(\kappa\). Let \(\alpha = (a, b) \in H^2(F, \mathbb{Z}/2\mathbb{Z})\) and \(\alpha'\) its 
image in \(H^2(F(Q), \mathbb{Z}/2\mathbb{Z})\). Then \(\partial^2_{\nu}(\alpha') = 0\).

Let \(\nu : k(X)^* \rightarrow \mathbb{Z}\) be a valuation, \(S\) the associated valuation ring and \(\kappa_{\nu}\) 
the residue field. If \(k(\mathbb{P}^2) \subset S\) then \(\nu(x) = \nu(y) = 0\) hence \((-x, -y)_{k(\mathbb{P}^2)}\) has 
trivial residue. Otherwise, \(S \cap k(\mathbb{P}^2) = R\) is a discrete valuation ring with 
field of fraction \(k(\mathbb{P}^2)\). The image of \(\text{Spec}(R)\) in \(\mathbb{P}^2\) is either a codimension one 
point or a closed point.
**Codimension one.** If the codimension one point \( P \) is not the generic point of a component of \( \{ xy = 0 \} \) then \((-x, -y)\) is unramified at \( P \). The diagram in Proposition 3.3 in this case is

\[
\begin{array}{ccc}
H^2(k(X), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial^2_\nu} & H^1(\kappa_\nu, \mathbb{Z}/2\mathbb{Z}) \\
\uparrow & & \uparrow \\
H^2(k(\mathbb{P}^2), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial^2_P} & H^1(\kappa_R, \mathbb{Z}/2\mathbb{Z})
\end{array}
\]

where \( \kappa_R \) denotes the residue field of the DVR \( R = S \cap k(\mathbb{P}^2) \). Since \( \alpha \) is unramified at \( P \), we have \( \partial^2_\nu(\alpha) = 0 \) and therefore, by the commutativity of the diagram, we must have \( \partial^2_P(\beta) = 0 \).

Otherwise, suppose that \( P \) is the generic point of the line \( \{ x = 0 \} \). In this case, the discriminant \( d \) is given by \( F(0, y, z) \) in the residue field, which is square by (1.3). Furthermore, since the valuation of \( F \) with respect to \( x \) is 0, it is indeed a unit in the associated local ring. Hence, by Proposition 3.4, we conclude that the residue of \( \beta \) along \( v \) in \( K(Q) \) is trivial. The same reasoning holds if \( P \) is the generic point of \( \{ y = 0 \} \).

**Codimension two.** If \( P \) is a closed point of \( \mathbb{P}^2_k \), then there is a local homomorphism of local rings \( \mathcal{O}_{\mathbb{P}^2,P} \to S \) which induces an embedding \( k \to \kappa_\nu \). If \( P \) does not lie in \( \{ xy = 0 \} \) but not \( \{ y = 0 \} \) then \( \mathcal{O}_{\mathbb{P}^2,P} \) and hence in \( S \). Therefore, \( \partial^2_\nu(\beta) = 0 \). If \( P \) lies on \( \{ x = 0 \} \) but not \( \{ y = 0 \} \) then in the residue field \( \kappa(P) \), the image of \(-y\) in \( \kappa(P) \) lies in \((k^*)^2 \). Hence, it is a square in \( S \). Therefore, we have \( \beta = 0 \) in \( H^2(K(Q)_\nu, \mathbb{Z}/2\mathbb{Z}) \) and thus \( \partial_\nu(\beta) = 0 \). Finally, if \( P \) is on both lines then we know that \( F(0, 0, 1) \) is a nonzero square in the residue field and thus by Hensel’s Lemma, it is a square in the completion \( k(X)_\nu \). We can apply Proposition 3.4 as above to conclude that \( \partial^2_\nu(\beta) = 0 \).

**Remark.** In the above proof we have made pivotal use of the conditions (1.3). One intuitive justification for the above theorem is given by Abhyankar’s Lemma (as observed by Colliot-Thélène [3, §3]). Namely, since the ramification locus of the quaternion algebra \((-xy, -yz)_{k(\mathbb{P}^2)}\) is contained within the ramification locus of \( X \to \mathbb{P}^2 \), the ramification cancels in \( \tilde{X} \) (see e.g. [20] p. 116 for an example of this phenomenon). Concretely, in this case \(-yz\) is not a square modulo \( x \) in \( k(\mathbb{P}^2) \) but is a square modulo \( x \) in \( k(X) \). This explains why the quaternion algebra \( \alpha \) has nontrivial residues but these residues vanish when you move from \( \mathbb{P}^2 \) to \( X \).

### 3.2. The full Brauer group

The purpose of this section is to prove the following result.

**Proposition 3.5.** Let \( X \) the variety defined in Theorem 1.2. Then,

\[
\operatorname{Br}(X)/\operatorname{Br}(k) \cong \mathbb{Z}/2\mathbb{Z},
\]

where \( \operatorname{Br}(X) \) is the Brauer group of \( X \) and \( \operatorname{Br}(k) \) is the Brauer group of \( k \).
and is generated by $\beta$.

Proposition 3.1 lets us compare the Brauer group of the generic fibre with the Brauer group of the function field $k(\mathbb{P}^2)$. We shall also need the following result in degree 1 (see e.g. [13, Thm 3.10], although the result is originally due to Arason).

**Lemma 3.6.** Let $Q$ be a quadric defined by the vanishing of a non-degenerate quadratic form $q$ over $K$.

1. If $q$ has dimension at least 3, then the natural map
   \[ H^1(K, \mathbb{Z}/2\mathbb{Z}) \to H^{nr}_1(K(Q)/K, \mathbb{Z}/2\mathbb{Z}) \]
   is injective.
2. If $q$ has dimension 2, then the map
   \[ H^1(K, \mathbb{Z}/2\mathbb{Z}) \to H^{nr}_1(K(Q)/K, \mathbb{Z}/2\mathbb{Z}) \]
   has kernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by the class of the discriminant of $q$.

Our first step in the proof of Proposition 3.5 is to show that we can restrict our attention to the 2-torsion part of the Brauer group.

**Lemma 3.7.** The quotient $\text{Br}(X)/\text{Br}(k)$ is 2-torsion.

**Proof.** We will prove this by exploiting the fibration $f : X \to \mathbb{P}^2$. We make use of the following commutative diagram (c.f. [7, Eqn (10.1)]) coming from the purity sequences for the two varieties

\[
\begin{array}{cccccc}
0 & \to & \text{Br}(X) & \to & \text{Br}(X_\eta) & \to & \bigoplus_P H^1(k(X_P), \mathbb{Q}/\mathbb{Z}) \\
\to & \text{Br}(k) & \to & \text{Br}(K) & \to & \bigoplus_{P \in (\mathbb{P}^2)^{(1)}} H^1(k(P), \mathbb{Q}/\mathbb{Z}).
\end{array}
\]

Let $\gamma \in \text{Br}(X)$. If $P$ is a codimension one point on $\mathbb{P}^2$ which is not an irreducible component of the ramification divisor $\{xyz = 0\}$ then the residue $\partial_P(\gamma) = 0$ (see for instance [7, Prop 10.1.4]). Hence, the only points in the direct sum which we need consider are $P \subset \{xyz = 0\}$. If $P$ is such a point, by the commutativity of the diagram, we have

\[ \partial_P(\gamma) \in \text{Ker}[\text{res}_{k(X_P)/k(P)}H^1(k(P), \mathbb{Q}/\mathbb{Z}) \to H^1(k(X_P), \mathbb{Q}/\mathbb{Z})]. \]

Note that by the nature of the fibration, the non-split fibres are given by the vanishing of quadratic forms of rank two. Therefore, there exists a quadratic
extension $F_P$ over which the non-split fibres can be written as a union of two transversal planes. In particular, this means
\[
\partial_P(\gamma) \in \ker\{\text{res}_{F_P/k(P)}H^1(k(P), \mathbb{Z}/2\mathbb{Z}) \to H^1(F_P, \mathbb{Z}/2\mathbb{Z})\}.
\]
Since there are exactly 3 codimension 1 points on $\mathbb{P}^2$ over which the fibre is non-split, we deduce the existence of a sequence $\text{Br}(k) \to \text{Br}(X) \to (\mathbb{Z}/2\mathbb{Z})^3$, which gives the claim.

It remains to determine the group $\text{Br}(X)[2]$. Our proof is based on [14, Lemma 3.10]. We know that every non-trivial element of $\text{Br}(X)[2]$ comes from a class in $H^2(K, \mathbb{Z}/2\mathbb{Z})$ and we'll aim to show that if a class maps to something non-trivial in $\text{Br}(X)[2]$ then its residue along any codimension one point in $\mathbb{P}^2$ must be in the subgroup generated by the residue of $\alpha$ along that codimension one point. From this, we will then deduce that this element can differ from $\alpha$ at most by a constant class.

**Proof of Proposition 3.3.** Let $\gamma \in H^2(K, \mathbb{Z}/2\mathbb{Z})$ and $D \in (\mathbb{P}^2)^{(1)}$ such that $\delta_D(\gamma) \neq 0$. Let $q$ be the quadratic form over $K$ which defines the quadric $Q$. The divisor associated to the closed subscheme $q = 0$ in $\mathbb{P}^2_D$ is either integral or a union of two planes. Let $\nu$ be the valuation attached either to the integral divisor or one of the two planes. Then Proposition 3.3 gives the following diagram
\[
\begin{array}{ccc}
H^2(K(Q), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial_D} & H^1(\kappa_D, \mathbb{Z}/2\mathbb{Z}) \\
\uparrow & & \uparrow \text{Res}_{D/\nu} \\
H^2(K, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\partial_{\nu}} & H^1(\kappa_{\nu}, \mathbb{Z}/2\mathbb{Z}).
\end{array}
\]

The field $\kappa_{\nu}$ is the function field of a quadric over $\kappa_D$, which we denote $\overline{q}$. If at most one of the coefficients of $q$ has odd valuation with respect to $D$ then $\overline{q} = 0$ is a smooth conic or quadric surface. Hence, by Lemma 3.6 (1), the map $\text{Res}_{D/\nu}$ is injective. Since $\gamma$ lands in $H^2(K(Q), \mathbb{Z}/2\mathbb{Z})$ the map along the top arrow is trivial. Therefore, $\partial_D(\gamma) = 0$.

This means at least 2 of the coefficients of $q$ must have odd valuation. The only way this can happen is if $D$ is one of the components of \{xyz = 0\}. Suppose $D$ corresponds to the line $\{x = 0\}$. Then, $\kappa(\nu)$ is the function field of the quadric over $\kappa_D$ defined by the equation
\[
\overline{y}zU^2 + F(x, y, z)V^2 = 0.
\]
By Lemma 3.6 (2), $\ker\text{Res}_{D/\nu}$ is generated by the class of the discriminant of $\overline{q}$ which is $-yzF(x, y, z) \in (\kappa(D)^*)^2$. However, $F(x, y, z)$ is in $(\kappa(D)^*)^2$, by conditions (1.3). Thus the class is given by $-\overline{y}z$, which is precisely the
residue of \((−x/z, −y/z\)) along \(D\) (by applying the formula (3.1)). The exact same calculation holds for the other two lines.

Hence, if \(γ\) is a non-trivial class in \(H^2(K, \mathbb{Z}/2\mathbb{Z})\) giving rise to a non-constant element of \(\text{Br}(X)\) and \(D\) is one of the three coordinate axes, then \(∂_D(γ) \in \{0, ∂_D(α)\}\). It remains to rule out the possibility that \(γ\) is just ramified along one or two of the coordinate axes (not all three like \(α\)). However, \(γ\) cannot just be ramified along one axis because \(\text{Br}(\mathbb{A}^2)\) is trivial. For the same reason, \(γ − α\) cannot be ramified along just one axis which means that \(γ\) cannot be ramified along exactly two axes. Thus, \(γ\) must be ramified along all three axes and thus must satisfy \(∂_D(γ) = ∂_D(α)\) for all codimension one points \(D\).

Hence, it follows that the induced element in \(\text{Br}(X)\) can differ from \(β\) by at most constant classes. □

4. The Brauer–Manin Obstruction set

In this final section, we explicitly compute the Brauer–Manin obstruction, which combined with Theorem 1.1 finishes the proof of Theorem 1.2. Recall the definition of the Brauer–Manin obstruction set (e.g. [15, Defn 8.2.5])

\[
X(\mathbb{A}_k)^{\text{Br}} = \{(P_ν)_ν \in X(\mathbb{A}_k) : \sum_ν \text{inv}_ν\text{ev}_γ(P) = 0 \in \mathbb{Q}/\mathbb{Z} \forall γ \in \text{Br}(X)\}.
\]

Theorem 1.2 follows immediately from explicitly determining the image of the evaluation maps.

**Proposition 4.1.** Let \(β \in \text{Br}(X)\) be the non-trivial class from the previous section. For every non-archimedean place \(ν\), \(\text{inv}_ν\text{ev}_β\) is identically zero on all \(k_ν\) points.

Since the Brauer–Manin obstruction is the only obstruction to weak approximation, and \(β\) the only non-trivial Brauer class, the above shows that the failure of weak approximation is completely determined by the archimedean places.

**Proof.** Proposition 4.1 will be proved by an explicit computation of the invariant maps. Throughout, \((a, b)_ν\) will refer to the Hilbert symbol associated to the local field \(k_ν\). The evaluation of the class \(β\) at a point \(P \in X(\mathbb{A}_k)\) is equal to the evaluation of the pre-image \(α \in \text{Br}(k(\mathbb{P}^2))\) at the point \(B \in \mathbb{P}^2(\mathbb{A}_k)\) below \(P\). Fix a point \(P = (x_ν : y_ν : z_ν : t_ν)_ν \in X(\mathbb{A}_k)\), then

\[
\text{inv}_ν\text{ev}_β(x_ν : y_ν : z_ν : t_ν) = \frac{1}{2} \iff (−x_νy_ν, −y_νz_ν)_ν = −1.
\]

Let \(p\) a prime ideal in \(\mathcal{O}_k\) and consider the conic

\[
x_pz_pU^2 + y_pz_pV^2 + W^2 = 0
\]
in \( k_p \). By the definition of the Hilbert symbol, \( \text{inv}_p \text{ev}_\beta(x_p : y_p : z_p ; t_p) = 0 \) if and only if the conic has a \( k_p \) solution. The vector \((t_{1,p}, t_{2,p}, t_{3,p}, t_{4,p})\) is a non-trivial solution to the quadratic form in four variables defined by \((\ref{eq:11})\). The discriminant of this quadratic form is \( F(x_p, y_p, z_p) \), which is a square modulo \( p \) and hence a square in \( k_p \) by Hensel’s lemma. Therefore, by the criteria for local solubility of quadratic forms \([18, \text{Ch 6, Thm 6}]\), we must have

\[
\prod_{i<j} (a_i, a_j)_p = (1, 1)_p,
\]

where \( a_i \) are the coefficients of the quadratic form. In our case, after simplifying the corresponding expression, we have

\[
(x_p y_p, y_p z_p)_p (x_p y_p, x_p z_p)_p (y_p z_p, x_p z_p)_p = (1, 1)_p.
\]

Using the multiplicativity of the Hilbert symbol, we write

\[
(-x_p z_p, -y_p z_p)_p = (x_p z_p, y_p z_p)_p (-1, y_p z_p)_p (x_p z_p, -1)_p (-1, -1)_p.
\]

Combining these expressions, we see that

\[
(-x_p z_p, -y_p z_p)_p = (x_p z_p, x_p y_p)(x_p y_p, y_p z_p)(-1, y_p z_p)(x_p z_p, -1) = (x_p y_p, -x_p y_p)_p.
\]

This Hilbert symbol is certainly +1 because the associated conic \( x_p y_p U^2 - x_p y_p V^2 = W^2 \) always has the non-trivial rational point \((1, 1, 0)\).

\[\square\]

**Remark.** Informally speaking, the reason this result is true is similar to the reason why \( \beta \) lives in \( \text{Br}(X) \). Namely, if a prime divides \( x \), then the nature of the equation forces \(-yz\) to be a square, which kills the invariant map.

Finally, we will study the invariant map at archimedean places of \( k \). If \( \nu \) is a complex place then the conic \( x_\nu z_\nu U^2 + y_\nu z_\nu V^2 + W^2 = 0 \) always has points and thus the invariant map is identically zero. The value of the invariant map at real places is determined by the signs of \( x, y \) and \( z \).

**Proposition 4.2.** Let \( \nu \) be a real place of \( k \) and let \( P_\nu = (x_\nu : y_\nu : z_\nu : t_\nu) \in X(k_\nu) \). If \( F(x_\nu, y_\nu, z_\nu) > 0 \) then \( \text{inv}_\nu \text{ev}_\beta(P_\nu) = +1 \). Otherwise,

\[
\text{inv}_\nu \text{ev}_\beta(P_\nu) = \begin{cases} 
\frac{1}{2} & \text{if } x, y, z \text{ all have the same sign in } k_\nu, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** If \( F(x_\nu, y_\nu, z_\nu) > 0 \) then it is a square in \( \mathbb{R} \) and we may appeal to the local solubility criterion as in the proof of Proposition 4.1. The same Hilbert symbol computation shows that \( (-x_\nu z_\nu, -y_\nu z_\nu)_\nu = +1 \). Otherwise, by the explicit description of the real Hilbert symbol

\[
(-x_\nu z_\nu, -y_\nu z_\nu)_\nu = \begin{cases} 
-1 & \text{if } -x_\nu z_\nu < 0 \text{ and } -y_\nu z_\nu < 0, \\
+1 & \text{otherwise}.
\end{cases}
\]

This condition is equivalent to the one in the statement of the proposition. \[\square\]
Together Propositions 4.1 and 4.2 completely describe the obstruction set and hence the statement of Theorem 1.2 follows.

**Remark.** Consider the Hassett–Pirutka–Tschinkel example over $\mathbb{Q}$ where $F(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$. The point $P = (x, y, z; t) \in X(\mathbb{Q})$ given by $P = (1, 1, 1; 1, 1, 1, 1)$ has the property that $F(x, y, z) < 0$ and $x, y$ and $z$ share the same sign. Therefore weak approximation fails. However, if one takes instead $F(x, y, z) = x^2 + y^2 + z^2 + 2(xy + xz + yz)$, then the conditions (1.3) are met, and $F$ always takes positive values. Hence, weak approximation holds for the resulting fourfold.

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