Improved bounds for coloring locally sparse hypergraphs

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Abstract

We show that, for every $k \geq 2$, every $k$-uniform hypergraph of degree $\Delta$ and girth at least 5 is efficiently $(1 + o(1))(k - 1)(\Delta / \ln \Delta)^{1/(k-1)}$-list colorable. As an application (and to the best of our knowledge) we obtain the currently best algorithm for list-coloring random hypergraphs of bounded average degree.

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1 Introduction

In hypergraph coloring one is given a hypergraph $H(V,E)$ and the goal is to find an assignment of one of $q$ colors to each vertex $v \in V$ so that no hyperedge is monochromatic. In the more general list-coloring problem, a list of $q$ allowed colors is specified for each vertex. A graph is $q$-list-colorable if it has a list-coloring no matter how the lists are assigned to each vertex. The list chromatic number, $\chi_l(H)$, is the smallest $q$ for which $H$ is $q$-list colorable.

Hypergraph coloring is a fundamental constraint satisfaction problem with several applications in computer science and combinatorics, that has been studied for over 60 years. In this paper we consider the task of coloring locally sparse hypergraphs and its connection to coloring sparse random hypergraphs.

A hypergraph is $k$-uniform if every hyperedge contains exactly $k$ vertices. An $i$-cycle in a $k$-uniform hypergraph is a collection of $i$ distinct hyperedges spanned by at most $i(k-1)$ vertices. We say that a $k$-uniform hypergraph has girth at least $g$ if it contains no $i$-cycles for $2 \leq i < g$. Note that if a $k$-uniform hypergraph has girth at least 3 then every two of its hyperedges have at most one vertex in common.

The main contribution of this paper is to prove the following theorem.

**Theorem 1.1.** Let $H$ by any $k$-uniform hypergraph, $k \geq 2$, of maximum degree $\Delta$ and girth at least 5. For all $\epsilon > 0$, there exist a positive constant $\Delta_{\epsilon,k}$ such that if $\Delta \geq \Delta_{\epsilon,k}$, then

$$\chi_l(H) \leq (1 + \epsilon)(k - 1) \left( \frac{\Delta}{\ln \Delta} \right)^{\frac{1}{\epsilon + 1}}.$$  

Furthermore, if $H$ is a hypergraph on $n$ vertices then there exists an algorithm that constructs such a coloring in expected polynomial time in $n$.

**Remark 1.1.** If $\Delta$ is assumed to be constant, then the algorithm of Theorem 1.1 can be efficiently derandomized, i.e., there exists a deterministic algorithm that constructs the promised coloring in polynomial time in $n$.

Theorem 1.1 is interesting for a number of reasons. First, it generalizes a well-known result of Kim [21] for coloring graphs of degree $\Delta$ and girth 5, and it implies the classical theorem of Ajtai, Komlós, Pintz, Spencer and Szemerédi [4] regarding the independence number of $k$-uniform hypergraphs of degree $\Delta$ and girth 5. The latter is a seminal result in combinatorics, with applications in geometry and coding theory [22, 23, 24]. Second, Theorem 1.1 is tight up to a constant [8]. Note also that, without the girth assumption, the best possible bound [11] on the chromatic number of $k$-uniform hypergraphs is $O(\Delta^{1/(k-1)})$, i.e., it is asymptotically worse than the one of Theorem 1.1. For example, there exist graphs of degree $\Delta$ whose chromatic number is exactly $\Delta + 1$. Third, when it applies, Theorem 1.1 improves upon a result of Frieze and Mubayi [13] regarding the chromatic number of simple hypergraphs, who showed (1) with an unspecified large leading constant (of order at least $\Omega(k^4)$). Finally, Theorem 1.1 can be used to provide the currently best algorithm for list-coloring random $k$-uniform hypergraphs of bounded average degree (to the best of our knowledge). We discuss the connection between locally sparse hypergraphs and sparse random hypergraphs with respect to the task of coloring in the following section.

1.1 Application to coloring pseudo-random hypergraphs

The random $k$-uniform hypergraph $H(k,n,p)$ is obtained by choosing each of the $\binom{n}{k}$ $k$-element subsets of a vertex set $V$ ($|V| = n$) independently with probability $p$. The chosen subsets are the hyperedges of the hypergraph. Note that for $k = 2$ we have the usual definition of the random graph $G(n,p)$. We say that $H(k,n,p)$ has a certain property $A$ asymptotically almost surely or with high probability, if the probability that $H \in H(k,n,p)$ has $A$ tends to 1 as $n \to \infty$. 


In this paper we are interested in $H(k, n, d/\binom{n}{k-1})$, i.e., the family of random $k$-uniform hypergraphs of bounded average degree $d$. Specifically, we use Theorem 1.1 to prove the following theorem.

**Theorem 1.2.** For any constants $\delta \in (0, 1)$, $k \geq 2$, there exists $d_{\delta,k} > 0$ such that for every constant $d \geq d_{\delta,k}$, the random hypergraph $H(k, n, d/\binom{n}{k-1})$ can be $(1 + \delta)(k-1)(d/\ln d)^{1/(k-1)}$-list-colored by a deterministic algorithm whose running time is polynomial in $n$ asymptotically almost surely.

**Remark 1.2.** Note that, for $k, d$ constants, a very standard argument reveals that $H(k, n, d/\binom{n}{k-1})$ is essentially equivalent to $\mathbb{H}(k, n, dn/k)$, namely the uniform distribution over $k$-uniform hypergraphs with $n$ vertices and exactly $dn/k$ hyperedges. Thus, Theorem 1.2 extends to that model as well.

We note that previous approaches [3,31,13] for list-coloring random $k$-uniform hypergraphs of bounded average degree $d$ are either randomized, or require significantly larger lists of colors per vertex in order to succeed. Indeed, for $k = 2$ the approach of Achlioptas and Molloy [3] matches the bound of Theorem 1.2 but is randomized, while for $k \geq 3$, and to the best our knowledge, our algorithm uses less colors that any algorithm for (list-)coloring random hypergraphs of bounded average degree that has been rigorously analyzed in the literature. Moreover, it is believed that all efficient algorithms require lists of size at least $(1+o(1))(k-1)(d/\ln d)^{1/(k-1)}$, as this bound corresponds to the so-called shattering threshold [1,7,14] for coloring sparse random hypergraphs, which is also often referred to as the “algorithmic barrier” [1]. This threshold arises in a plethora of random constraint satisfaction problems, and it corresponds to a precise phase transition in the geometry set of solutions. In all of these problems, we are not aware of any efficient algorithm that works beyond the algorithmic barrier, despite the fact that solutions exist for constraint-densities larger than the one in which the shattering phenomenon appears. We refer the reader to [1,13] for further details.

In order to prove Theorem 1.2 we show that random $k$-uniform hypergraphs of bounded average degree $d$ can essentially be treated as hypergraphs of girth 5 and maximum degree $d$ for the purposes of list-coloring, and then apply Theorem 1.1. In particular, we identify a pseudo-random family of hypergraphs which we call *girth-reducible*, and show that almost all $k$-uniform hypergraphs of bounded average degree belong in this class. Then we show that girth-reducible hypergraphs can be colored efficiently using Theorem 1.1.

Formally, a $k$-uniform hypergraph $H$ is *$\kappa$-degenerate* if the induced subhypergraph of all subsets of its vertex set has a vertex of degree at most $\kappa$. The *degeneracy* of a hypergraph $H$ is the smallest value of $\kappa$ for which $H$ is $\kappa$-degenerate. Note that it is known that $\kappa$-degenerate hypergraphs are $(\kappa+1)$-list colorable and that the degeneracy of a hypergraph can be computed efficiently by an algorithm that repeatedly removes minimum degree vertices. Indeed, to list-color a $\kappa$-degenerate hypergraph we repeatedly find a vertex with (remaining) degree at most $\kappa$, assign to it a color that does not appear in any of its neighbors so far, and remove it from the hypergraph. Clearly, if the lists assigned to each vertex are of size at least $\kappa + 1$ this procedure always terminates successfully.

**Definition 1.3.** For $\delta \in (0, 1)$, we say that a $k$-uniform hypergraph $H(V, E)$ of average degree $d$ is $\delta$-girth-reducible if its vertex set can be partitioned in two sets, $U$ and $V \setminus U$, such that:

(a) $U$ contains all cycles of length at most 4, and all vertices of degree larger than $(1 + \delta)d$;

(b) subhypergraph $H[U]$ is $\left(\frac{d}{\ln d}\right)^{1/k-1}$-degenerate;

(c) every vertex in $V \setminus U$ has at most $\delta \left(\frac{d}{\ln d}\right)^{1/k-\delta}$ neighbors in $U$.

In words, a hypergraph is $\delta$-girth-reducible if its vertex set can be seen as the union of two parts: A “low-degeneracy” part, which contains all vertices of degree more than $(1 + \delta)d$ and all cycles of lengths at most 4, and a “high-girth” part, which induces a hypergraph of maximum degree at most $(1 + \delta)d$ and girth 5. Moreover, each vertex in the “high-girth” part has only a few neighbors in the “low-degeneracy” part.
Note that given a $\delta$-girth-reducible hypergraph we can efficiently find the promised partition $(U, V \setminus U)$ as follows. We start with $U := U_0$, where $U_0$ is the set of vertices that either have degree at least $(1 + \delta)d$, or they are contained in a cycle of length at most 4. Let $\partial U$ denote the vertices in $V \setminus U$ that violate property (c). While $\partial U \neq \emptyset$, update $U$ as $U := U \cup \partial U$. The correctness of the process lies in the fact that in each step we add to the current $U$ a set of vertices that must be in the low-degeneracy part of the hypergraph. Observe also that this process allows us to efficiently check whether a hypergraph is $\delta$-girth-reducible.

We prove the following theorem regarding the list-chromatic number of girth-reducible hypergraphs.

**Theorem 1.4.** For any constants $\delta \in (0, 1)$ and $k \geq 2$, there exists $d_{\delta, k} > 0$ such that if $H$ is a $\delta$-girth-reducible, $k$-uniform hypergraph of average degree $d \geq d_{\delta, k}$, then

$$\chi_l(H) \leq (1 + \epsilon)(k - 1) \left( \frac{d}{\ln d} \right)^{\frac{1}{k-1}},$$

where $\epsilon = 4\delta = O(\delta)$. Furthermore, if $H$ is a hypergraph on $n$ vertices then there exists a deterministic algorithm that constructs such a coloring in time polynomial in $n$.

**Proof of Theorem 1.4.** Let $\epsilon = 4\delta$. Given lists of colors of size $(1 + \epsilon)(k - 1) \left( \frac{d}{\ln d} \right)^{\frac{1}{k-1}}$ for each vertex of $H$, we first color the vertices of $U$ using the greedy algorithm which exploits the low degeneracy of $H[U]$. Now each vertex in $V - U$ has at most $\delta \left( \frac{d}{\ln d} \right)^{\frac{1}{k-1}}$ forbidden colors in its list as it has at most that many neighbors in $U$. We delete these colors from the list. Observe that if we manage to properly color the induced subgraph $H[V \setminus U]$ using colors from the updated lists, then we are done since every hyperedge with vertices both in $U$ and $V \setminus U$ will be automatically “satisfied”, i.e., it cannot be monochromatic. Notice now that the updated list of each vertex still contains at least $(1 + 3\delta)(k - 1) \left( \frac{d}{\ln d} \right)^{\frac{1}{k-1}}$ colors, for sufficiently large $d$. Since the induced subgraph $H[V \setminus U]$ is of girth at least 5 and of maximum degree at most $(1 + \delta)d$, it is efficiently $(1 + \delta)(k - 1) \left( \frac{(1+\delta)d}{\ln((1+\delta)d)} \right)^{\frac{1}{k-1}}$-list-colorable for sufficiently large $d$ per Theorem 1.4 and Remark 1.1. This concludes the proof since $(1 + \delta)(1 + \delta)^{\frac{1}{k-1}} < (1 + 3\delta)$.

Moreover, we show that girth-reducibility is a pseudo-random property which is admitted by almost all sparse $k$-uniform hypergraphs.

**Theorem 1.5.** For any constants $\delta \in (0, 1)$, $k \geq 2$, there exists $d_{\delta, k} > 0$ such that for every constant $d \geq d_{\delta, k}$, asymptotically almost surely, the random hypergraph $H(k, n, d/\binom{n}{k-1})$ is $\delta$-girth-reducible.

Theorem 1.5 follows by simple, although somewhat technical, considerations on properties of sparse random hypergraphs, which are mainly inspired by the results of Alon, Krivelevich and Sudakov [1, 6] and Łuczak [25]. Observe that combining Theorem 1.5 with Theorem 1.4 immediately implies Theorem 1.2.

Overall, the task of coloring locally sparse hypergraphs is inherently related to the average-case complexity of coloring. In particular, in this section we showed that Theorem 1.1 implies a robust algorithm for hypergraph coloring, namely a deterministic procedure that applies to worst-case $k$-uniform hypergraphs, while at the same using a number of colors that is only a $(k - 1)$-factor away from the algorithmic barrier for random instances (matching it for $k = 2$). We remark that this application is inspired by recent results that study the connection between local sparsity and efficient randomized algorithms for coloring sparse regular random graphs [26, 2, 10].
1.2 Technical overview

The intuition behind the proof of Theorem 1.1 comes from the following observation, which we explain in terms of graph coloring for simplicity. Let $G$ be a triangle-free graph of degree $\Delta$, and assume that each of its vertices is assigned an arbitrary list of $q$ colors. Fix a vertex $v$ of $G$, and consider the random experiment in which the neighborhood of $v$ is properly list-colored randomly. Since $G$ contains no triangles, this amounts to assigning to each neighbor of $v$ a color from its list randomly and independently. Assuming that $q \geq q^* := (1 + \epsilon)\Delta/\ln \Delta$, the expected number of available colors for $v$, i.e., the colors from the list of $v$ that do not appear in any of its neighbors, is at least $q(1 - 1/q)^\Delta = \omega(\Delta^{\epsilon/2})$. In fact, a simple concentration argument reveals that the number of available colors for $v$ in the end of this experiment is at least $\Delta^{\epsilon/2}$ with probability that goes to 1 as $\Delta$ grows. To put it differently, as long as $q \geq q^*$, the vast majority of valid ways to list-color the neighborhood of $v$ “leaves enough room” to color $v$ without creating any monochromatic edges.

A completely analogous observation regarding the ways to properly color the neighborhood of a vertex can be made for $k$-uniform hypergraphs. In order to exploit it we employ the so-called semi-random method, which is the main tool behind some of the strongest graph coloring results, e.g., [15, 16, 17, 18, 20, 27, 32], including the one of Kim [21]. (See also the very recent survey [19] of Kang et al. on the subject.) The idea is to gradually color the hypergraph in iterations until we reach a point where we can finish the coloring with a simple, e.g., greedy, algorithm. In its most basic form, each iteration consists of the following simple procedure (using graph vertex coloring as a canonical example): Assign to each vertex a color chosen uniformly at random; then uncolor any vertex that receives the same color as one of its neighbors. Using the Lovász Local Lemma [11] and concentration inequalities, one typically shows that, with positive probability, the resulting partial coloring has useful properties that allow for the continuation of the argument in the next iteration. (In fact, using the Moser-Tardos algorithm [29] this approach yields efficient, and often times deterministic [9], algorithms.) Specifically, one keeps track of certain parameters of the current partial coloring and makes sure that, in each iteration, these parameters evolve almost as if the coloring was totally random. For example, recalling the heuristic experiment of the previous paragraph, one of the parameters we would like to keep track of in our case is a lower bound on the number of available colors of each vertex in the hypergraph: If this parameter evolves “randomly” throughout the process, then the vertices that remain uncolored in the end are guaranteed to have a non-trivial number of available colors.

Applications of the semi-random method tend to be technically intense and this is even more so in our case, where we have to deal with constraints of large arity. Large constraints introduce several difficulties, but the most important one is that our algorithm has to control many parameters that interact with each other. Roughly, in order to guarantee the properties that allow for the continuation of the argument in the next iteration, for each uncolored vertex $v$, each color $c$ in the list of $v$, and each integer $r \in [k - 1]$, we should keep track of a lower bound on the number of adjacent to $v$ hyperedges that have $r$ uncolored vertices and $k - 1 - r$ vertices colored $c$. Clearly, these parameters are not independent of each other throughout the process, and so the main challenge is to design and analyze a coloring procedure in which all of them, simultaneously, evolve essentially randomly.

1.3 Organization of the paper

The paper is organized as follows. In Section 2 we present the necessary background. In Section 3 we present the algorithm and state the key lemmas for the proof of Theorem 1.1 while in Section 4 we give the full details. Finally, in Section 5 we prove Theorem 1.5.
2 Background and preliminaries

In this section we give some background on the technical tools that we will use in our proofs.

2.1 The Lovász Local Lemma

As we have already mentioned, one of the key tools we will use in our proof is the Lovász Local Lemma [11].

**Theorem 2.1.** Consider a set \( B = \{B_1, B_2, \ldots, B_m\} \) of (bad) events. For each \( B \in B \), let \( D(B) \subseteq B \setminus \{B\} \) be such that \( \Pr[B \mid \bigcap_{C \in S} \overline{C}] = \Pr[B] \) for every \( S \subseteq B \setminus (D(B) \cup \{B\}) \). If there is a function \( x : B \rightarrow (0, 1) \) satisfying

\[
\Pr[B] \leq x(B) \prod_{C \in D(B)} (1 - x(C)) \quad \text{for all } B \in B,
\]

then the probability that none of the events in \( B \) occurs is at least \( \prod_{B \in B} (1 - x(B)) > 0 \).

In particular, we will need the following two corollaries of Theorem 2.1. For their proofs, the reader is referred to Chapter 19 in [28].

**Corollary 2.2.** Consider a set \( B = \{B_1, \ldots, B_m\} \) of (bad) events. For each \( B \in B \), let \( D(B) \subseteq B \setminus \{B\} \) be such that \( \Pr[B \mid \bigcap_{C \in S} \overline{C}] = \Pr[B] \) for every \( S \subseteq B \setminus (D(B) \cup \{B\}) \). If for every \( B \in B \):

(a) \( \Pr[B] \leq \frac{1}{4} \);

(b) \( \sum_{C \in D(B)} \Pr[C] \leq \frac{1}{4} \),

then the probability that none of the events in \( B \) occurs is strictly positive.

**Corollary 2.3.** Consider a set \( B = \{B_1, B_2, \ldots, B_m\} \) of (bad) events such that for each \( B \in B \):

(a) \( \Pr[B] \leq p < 1 \);

(b) \( B \) is mutually independent of a set of all but at most \( \Delta \) of the other events.

If \( 4p\Delta \leq 1 \) then with positive probability, none of the events in \( B \) occur.

2.2 Talagrand’s inequality

We will also need the following version of Talagrand’s inequality [30] whose proof can be found in Chapter 20 of [28].

**Theorem 2.4.** Let \( X \) be a non-negative random variable, not identically 0, which is determined by \( n \) independent trials \( T_1, \ldots, T_n \), and satisfying the following for some \( c, r > 0 \):

1. changing the outcome of any trial can affect \( X \) by at most \( c \), and

2. for any \( s \), if \( X \geq s \) then there is a set of at most \( ws \) trials whose outcomes certify that \( X \geq s \),

then for any \( 0 \leq t \leq \mathbb{E}[X] \),

\[
\Pr[|X - \mathbb{E}[X]| > t + 60c\sqrt{w\mathbb{E}[X]}] \leq 4e^{-\frac{t^2}{8c^2w\mathbb{E}[X]}}.
\]
3 List-coloring high-girth hypergraphs

In this section we describe the algorithm of Theorem 1.1. As we already explained, our approach is based on the semi-random method. For an excellent exposition both of the method and Kim’s result the reader is referred to [28].

We assume without loss of generality that $\epsilon < \frac{1}{10}$. Also, it will be convenient to define the parameter $\delta := (1 + \epsilon)(k - 1) - 1$, so that the list of each vertex initially has at least $(1 + \delta)(\frac{\Delta}{\ln \Delta})^{\frac{1}{k-1}}$ colors, and assume that $k \geq 3$. (The case $k = 2$ is Kim’s result.)

We analyze each iteration of our procedure using a probability distribution over the set of (possibly improper) colorings of the uncolored vertices of $H$ where, additionally, each vertex is either activated or deactivated. We call a pair of coloring and activation bits assignments for the uncolored vertices of hypergraph $H$ a state.

Let $V_i$ denote the set of uncolored vertices in the beginning of the $i$-th iteration. (Initially, all vertices are uncolored.) Also, for each $v \in V_i$ we denote by $L_v = L_v(i)$ the list of colors of $v$ in the beginning of the $i$-th iteration. Finally, we say that a color $c \in L_v$ is available for $v$ in a state $\sigma$ if assigning $c$ to $v$ does not cause any hyperedge whose initially uncolored vertices are all activated in $\sigma$.

For each vertex $v$, color $c \in L_v$ and iteration $i$, we define a few quantities of interest that our algorithm will attempt to control. Let $\ell_i(v)$ be the size of $L_v$. Further, for each $r \in [k]$, let $D_{i,r}(v,c)$ denote the set of hyperedges $h$ that contain $v$ and (i) exactly $r$ vertices $\{u_1, \ldots, u_r\} \subseteq h \setminus \{v\}$ are uncolored and $c \in L_{u_j}$ for every $j \in [r]$; (ii) the rest $k - 1 - r$ vertices of $h$ other than $v$ are colored $c$. We define $t_{i,r}(v,c) := |D_{i,r}(v,c)|$.

As it is common in the applications of the semi-random method, we will not attempt to keep track of the values of $\ell_i(v)$ and $t_{i,r}(v,c)$, $r \in [k - 1]$, for every vertex $v$ and color $c$, but rather we will focus on their extreme values. In particular, we will define appropriate $L_i, T_{i,r}$ such that for each $i$ the following property holds in the beginning of iteration $i$:

**Property P(i):** For each vertex $v \in V_i$, color $c \in L_v$ and $r \in [k - 1]$:

\[
\ell_i(v) \geq L_i; \quad t_{i,r}(v,c) \leq T_{i,r}.
\]

As a matter of fact, it would be helpful for our analysis (though not necessary) if the inequalities defined in $P(i)$ were actually tight. Given that $P(i)$ holds, we can always enforce this stronger property in a straightforward way as follows. First, for each vertex $v$ such that $\ell_i(v) > L_i$ we choose arbitrarily $\ell_i(v) - L_i$ colors from its list and remove them. Then, for each vertex $v$ and color $c \in L_i$ such that $t_{i,r}(v,c) < T_{i,r}$ we add to the hypergraph $T_{i,r} - t_{i,r}(v,c)$ new hyperedges of size $r + 1$ that contain $v$ and $r$ new “dummy” vertices. (As it will be evident from the proof, we can always assume that $L_i, T_{i,r}$ are integers, since our analysis is robust to replacing $L_i, T_{i,r}$ with $\lceil L_i \rceil$ and $\lceil T_{i,r} \rceil$.) We assign each dummy vertex a list of $L_i$ colors: $L_i - 1$ of them are new and do not appear in the list of any other vertex, and the last one is $c$.

**Remark 3.1.** Dummy vertices are only useful for the purposes of our analysis and can be removed at the end of the iteration. Indeed, one could use the technique of “equalizing coin flips” instead. For more details see e.g., [28].

Overall, without loss of generality, at each iteration $i$ our goal will be to guarantee that Property $P(i + 1)$ holds assuming Property $Q(i)$.

**Property Q(i):** For each vertex $v \in V_i$, color $c \in L_v$ and $r \in [k - 1]$:

\[
\ell_i(v) = L_i; \quad t_{i,r}(v,c) = T_{i,r}.
\]
An iteration. For the $i$-th iteration we apply the Local Lemma with respect to the probability distribution induced by assigning to each vertex $v \in V_i$ a color chosen uniformly at random from $L_v(i)$ and activating $v$ with probability $\alpha = \frac{K}{\ln \Delta}$, where $K = (100k^3)\Delta^{-1}$. That is, we apply the Moser-Tardos algorithm in the space induced by the $2|V_i|$ variables corresponding to the color and activation bit of each variable in $V_i$. (We will define the family of bad events for each iteration shortly.)

When the execution of the Moser-Tardos algorithm terminates, we uncolor some of the vertices in $V_i$ in order to get a new partial coloring. In particular, the partial coloring of the hypergraph, set $V_{i+1}$, and the lists of colors for each uncolored vertex in the beginning of iteration $i+1$ are induced as follows. Let $\sigma$ be the output state of the application of the Moser-Tardos algorithm in the $i$-th iteration. The list of each vertex $v$, $L_v(i+1)$, is induced from $L_v(i)$ by removing every non-available color $c \in L_v(i)$ for $v$ in $\sigma$. We obtain the partial coloring $\phi$ for the hypergraph and set $V_{i+1}$ for the beginning of iteration $i+1$ by removing the color from every vertex $v \in V_i$ which is either deactivated or is assigned a non-available for it color in $\sigma$.

Overall, the $i$-th iteration of our algorithm can be described at a high-level as follows.

1. Apply the Moser-Tardos algorithm to the probability space induced by assigning to each vertex $v \in V_i$ a color chosen uniformly at random from $L_v(i)$, and activating $v$ with probability $\alpha$.

2. Let $\sigma$ be the output state of the Moser-Tardos algorithm.

3. For each vertex $v \in V_i$, remove any non-available color $c \in L_v(i)$ in $\sigma$ to get a list $L_v(i+1)$.

4. Uncolor every vertex $v \in V_i$ that has either received a non-available color or is deactivated in $\sigma$, to get a new partial coloring $\phi$.

Controlling the parameters of interest. Next we describe the recursive definitions for $L_i$ and $T_{i,r}$ which, as we already explained, will determine the behavior of the parameters $\ell_i(v)$ and $t_{i,r}(v,c)$, respectively.

Initially, $L_1 = (1 + \delta)\left(\frac{\Delta}{\ln \Delta}\right)^{k^{-1}}$, $T_{1,k-1} = \Delta$ and $T_{1,r} = 0$ for every $r \in [k-2]$. Letting

$$\text{Keep}_i = \prod_{r=1}^{k-1} \left(1 - \left(\frac{\alpha}{L_i}\right)^r\right)^{T_{i,r}},$$

we define

$$L_{i+1} = L_i \cdot \text{Keep}_i - L_i^{2/3},$$

$$T_{i+1,r} = \sum_{j=r}^{k-1} \left(T_{i,j} \cdot \left(\frac{j}{r}\right) \left(\text{Keep}_i (1 - \alpha \text{Keep}_i)\right)^r \left(\frac{\alpha \text{Keep}_i}{L_i}\right)^{j-r}\right) + 4k^{2(k-r)}\alpha (\alpha^{-1} L_i)^r \ln \Delta \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}} \frac{1}{(\ln \Delta)^{2\ell}} + \left(\sum_{j=r}^{k-1} \left(\frac{j}{r}\right) \left(\alpha^{-r} \frac{T_{i,j}}{L_i^{2j-2r}}\right)^{2/3}\right).$$

To get some intuition for the recursive definitions (4), (5), observe that $\text{Keep}_i$ is the probability that a color $c \in L_v(i)$ is present in $L_v(i+1)$ as well. Note further that this implies that the expected value of $\ell_{i+1}(v,c)$ is $L_i \cdot \text{Keep}_i$, a fact which motivates (4). Calculations of similar flavor for $E[t_{i+1,r}(v,c)]$ motivate (5).
The key lemmas. We are almost ready to state the main lemmas which guarantee that our procedure eventually reaches a partial list-coloring of $H$ with favorable properties that allow us to extend it to a full list-coloring. Before doing so, we need to settle a subtle issue that has to do with the fact that $t_{i+1,r}(v, c)$ is not sufficiently concentrated around its expectation. To see this, notice for example that $t_{i+1,1}(v, c)$ drops to zero if $v$ is assigned $c$. (Similarly, for $r \in \{2, \ldots, k - 1\}$, if $v$ is assigned $c$ then $t_{i+1,r}(v, c)$ can be affected by a large amount.) To deal with this problem we will focus instead on variable $t'_{i+1,r}(v, c)$, i.e., the number of hyperedges $h$ that contain $v$ and (i) exactly $k - r - 1$ vertices of $h \setminus \{v\}$ are colored $c$ in the end of iteration $i$; (ii) the rest $r$ vertices of $h \setminus \{v\}$ did not retain their color during iteration $i$ and, further, $c$ would be available for them if we ignored the color assigned to $v$. Observe that if $c$ is not assigned to $v$ then $t_{i+1,r}(v, c) = t'_{i+1,r}(v, c)$ and $t'_{i+1,r}(v, c) \geq t_{i+1,r}(v, c)$ otherwise.

The first lemma that we prove estimates the expected value of the parameters at the end of the $i$-th iteration. Its proof can be found in Section 4.

Lemma 3.1. Let $S_i = \sum_{j=1}^{k-1} T_{i,j}^{(\ln \Delta)},$ and $Y_{i,r} = \sum_{j=r}^{k-1} T_{i,j}^{(\ln \Delta)}$. If $Q(i)$ holds and for all $1 < j, r \in [k-1]$, $L_j \geq (\ln \Delta)^{20(k-1)}$, $T_{i,r} \geq (\ln \Delta)^{20(k-1)}$, then, for every vertex $v \in V_{i+1}$ and color $c \in L_v$:

(a) $\mathbb{E}[\ell_{i+1}(v)] = \ell_i(v) \cdot \text{Keep}_i$;

(b) $\mathbb{E}[\ell'_{i+1,r}(v, c)] \leq \sum_{j=r}^{k-1} \left( T_{i,j}^{(j/r)} (\text{Keep}_i (1 - \text{Keep}_i))^{r} \left( \frac{\text{Keep}_i}{L_i} \right)^{j-r} \right) + 4k^{2(k-r)}(\alpha^{-1}L_i)^r S_i \ln \Delta + O(Y_{i,r}).$

The next step is to prove strong concentration around the mean for our random variables per the following lemma. Its proof can be found in Section 4.

Lemma 3.2. If $Q(i)$ holds and $L_i, T_{i,r} \geq (\ln \Delta)^{20(k-1)}$, $r \in [k-1]$, then for every vertex $v \in V_{i+1}$ and color $c \in L_v$,

(a) $\Pr \left[ |\ell_{i+1}(v) - \mathbb{E}[\ell_{i+1}(v)]| < L_i^{2/3} \right] < \Delta^{-\ln \Delta};$

(b) $\Pr \left[ |\ell'_{i+1,r}(v, c) - \mathbb{E}[\ell'_{i+1,r}(v, c)]| > \frac{1}{2} \left( \sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} T_{i,j}^{(j/r)} \right)^{2/3} \right] < \Delta^{-\ln \Delta}.$

Armed with Lemmas 3.1, 3.2 a straightforward application of the symmetric Local Lemma, i.e., Corollary 2.3 reveals the following.

Lemma 3.3. With positive probability, $P(i)$ holds for every $i$ such that for all $1 < j < i : L_j, T_{j,r} \geq (\ln \Delta)^{20(k-1)}$ and $T_{j,k-1} \geq \frac{1}{10k^2} T_{j-1}^{k-1}$.

The proof of Lemma 3.3 can be found in Section 4.

In analyzing the recursive equations (4), (5), it would be helpful if we could ignore the “error terms”. The next lemma shows that this is indeed possible. Its proof can be found in Section 4.
Lemma 3.4. Define $L'_i = (1 + \delta)\left(\frac{\Delta}{\ln \Delta}\right)^{\frac{1}{k-1}}, T'_{i,k-1} = \Delta, T'_{1,r} = 0$ for $r \in [k-2]$, and recursively define

$$L'_{i+1} = L'_i \cdot \text{Keep}_i,$$

$$T'_{i+1,r} = \sum_{j=r}^{k-1} T'_{i,j} \cdot \left(\text{Keep}_i \cdot (1 - \alpha \text{Keep}_i)^r \left(\frac{\alpha \text{Keep}_i}{L'_i}\right)^j \right)^r$$

$$+ 4k^{2(k-r)} \alpha (\alpha^{-1} L'_i)^r \ln \Delta \sum_{\ell=1}^{k-1} \frac{T'_{i,\ell}}{\ell^2 (\ln \Delta)^{2\ell}}.$$

If for all $1 < j < i$, $L_j \geq (\ln \Delta)^{20(k-1)}$, $T_{j,r} \geq (\ln \Delta)^{20(k-1)}$ for every $r \in [k-1]$, and $T_{j,k-1} \geq \frac{L_{k-1}}{100^2}$, then

(a) $|L_i - L'_i| \leq (L'_i)^{\frac{5}{2}}$;

(b) $|T_{i,r} - T'_{i,r}| \leq (T'_{i,r})^{\frac{100r}{100r+1}}$.

Remark 3.2. Note that Keep\textsubscript{i} in Lemma 3.4 is still defined in terms of $L_i, T_{i,r}$ and not $L'_i, T'_{i,r}$. Note also that in the definition of $T'_{i+1,r}$, the second summand is a function of $T_{i,\ell}, L_i, \ell \in [r-1]$, and not $T'_{i,\ell}, L'_i$.

Using Lemma 3.4 we are able to prove the following in Section 4

Lemma 3.5. There exists $i^* = O(\ln \Delta \ln \ln \Delta)$ such that

(a) For all $1 < i \leq i^*, T_{i,r} \geq (\ln \Delta)^{20(k-1)}, L_i \geq \Delta^{\frac{\epsilon^3}{(k-1)(1+\epsilon/2)}},$ and $T_{i,k-1} \geq \frac{1}{100^2} L_i^{k-1}$;

(b) $T_{i,r+1} \leq \frac{1}{100^2} L_{i+1,r},$ for every $r \in [k-1]$ and $L_{i^*+1} \geq \Delta^{\frac{\epsilon^3}{(k-1)(1+\epsilon/2)}}$.

Remark 3.3. Notice that Lemmas 3.3, 3.5 imply that with positive probability, after $i^*$ iterations the resulting proper coloring $\phi$ satisfies property $P(i^* + 1)$, and therefore condition (b) of Lemma 3.5.

Lemmas 3.3, 3.5 and 3.6 imply Theorem 1.1

Lemma 3.6. Let $i^*$ be the integer promised by Lemma 3.5 and assume property $P(i^* + 1)$ holds for the partial coloring $\phi$ in the beginning of the $(i^* + 1)$-th iteration. Given $\phi$, we can find a full list-coloring of $H$ in expected polynomial time in the number of vertices of $H$. Also, if $\Delta$ is assumed to be constant, then such a coloring can be constructed deterministically in polynomial time.

Proof of Theorem 1.1 We carry out $i^*$ iterations of our procedure. If $P(i)$ fails to hold for any iteration $i$, then we halt. By Lemmas 3.3 and 3.5 $P(i)$ (and, therefore, $Q(i)$) holds with positive probability for each iteration and so it is possible to perform $i^*$ iterations. Further, since the application of the LLL in the proof of Lemma 3.3 is within the scope of the so-called variable setting, i.e., the setting considered by [29], the Moser-Tardos algorithm applies and terminates in expected polynomial time. (If $\Delta$ is assumed to be constant the deterministic version of the Moser-Tardos algorithm also applies and terminates in polynomial time.) Thus, we can perform $i^*$ (successful) iterations in polynomial time.

After $i^*$ iterations we apply the algorithm of Lemma 3.6 and complete the list-coloring of the input hypergraph.
3.1 Proof of Lemma 3.6

Let $U_\phi$ denote the set of uncolored vertices in $\phi$, and $U_\phi(h)$ the subset of $U_\phi$ that belongs to a hyperedge $h$. Our goal is to color the vertices in $U_\phi$ to get a proper list-coloring.

Towards that end, let $L_v = L_v(\phi)$ denote the list of colors for $v$ in $\phi$, and $D_r(v, c) := D_{i^*+1,r}(v, c)$ the set of hyperedges (of size $l_{i^*+1,r}(v, c)$) with $r$ uncolored vertices in $\phi$ whose vertices “compete” for $c$ with $v$, and recall the conclusion of Lemma 3.5. Let $\mu$ be the probability distribution induced by giving each vertex $v \in U_\phi$ a color from $L_v$ uniformly at random. For every hyperedge $h$ and color $c$ such that (i) $c \in \bigcap_{v \in U_\phi(h)} L_v$, and (ii) $\phi(v) = c$ for every vertex in $h \setminus U_\phi(h)$, we define $A_{h,c}$ to be the event that all vertices of $h$ are colored $c$. Let $A$ be the family of these (bad) events, and observe that any elementary event (list-coloring) that does not belong in their union is a proper. In other words, if we avoid these bad events we have found a proper list-coloring of the hypergraph. Moreover, for every $A_{h,c} \in A$:

$$
\mu(A_{h,c}) \leq \frac{1}{\prod_{v \in U_\phi(h)} |L_v(\phi)|} < \frac{1}{4},
$$

for large enough $\Delta$, since $L_{i^*+1} = L_{i^*+1}(\Delta) \xrightarrow{\Delta \to +\infty} +\infty$.

Define

$$
D(A_{h,c}) := \bigcup_{v \in U_\phi(h)} \bigcup_{c' \in L_v} \bigcup_{r=1}^{k-1} D_r(v, c')
$$

and observe that $A_{h,c}$ is mutually independent of the events in $A \setminus D(A_{h,c})$. The existential claim of Lemma 3.6 follows from Corollary 2.2 as, for every $A_{h,c} \in A$:

$$
\sum_{A \in D(A_{h,c})} \mu(A) \leq \sum_{v \in U_\phi(h)} \sum_{c' \in L_v} \sum_{r=1}^{k-1} \sum_{h' \in D_r(v, c')} \mu(A_{h',c'})
= \sum_{v \in U_\phi(h)} \sum_{c' \in L_v} \sum_{i=1}^{k-1} \sum_{h' \in D_r(v, c')} \frac{1}{\prod_{u \in U_\phi(h')} |L_u|}
\leq \max_{v \in U_\phi(h)} \frac{k}{|L_v|} \sum_{c' \in L_v} \sum_{r=1}^{k-1} \frac{|D_r(v, c')|}{L_{i^*+1}^r}
\leq \frac{k}{10k^2} \max_{v \in U_\phi(h)} \frac{L_{i^*+1}^r |L_v|}{|L_v| \cdot L_{i^*+1}^r}
\leq \frac{1}{10} < \frac{1}{4},
$$

concluding the proof. Note that in (8) we used the facts that every hyperedge has at most $k$ vertices and $L_{i^*+1} \geq \Delta^{(k-1)/(1+r/2)}$, and in (9) we used the fact that $|D_r(v, c')| \leq T_{i^*+1}^r \leq \frac{1}{10k^2} L_{i^*+1}^r$.

As for the algorithmic claim of the lemma, since we apply the LLL in the variable setting the Moser-Tardos algorithm applies and it terminates in expected polynomial time. Further, if $\Delta$ is assumed to be constant then its deterministic version also applies and terminates in polynomial time, concluding the proof of the lemma.

4 Hypergraph list-coloring proofs

In this section we prove Lemmas 3.1, 3.2, 3.3, 3.4, 3.5.
We start by showing a couple of important technical lemmas that will be helpful for these proofs. It will be convenient to define \( R_{i,r} = \frac{T_{i,r}}{L_i} \), \( R'_{i,r} = \frac{T'_{i,r}}{(L_i)^r} \) for every \( r \in [k-1] \).

**Lemma 4.1.** If for all \( 1 < j < i, r \in [k-1] \), \( L_j, T_{j,r} \geq (\ln \Delta)^{20(k-1)} \), then
\[
R_{i,r} \leq k^{2(k-1-r)} \ln \Delta.
\]

The proof of Lemma 4.1 can be found in Appendix A. A straightforward corollary of Lemma 4.1 is the following.

**Corollary 4.2.** If \( L_i, T_{i,r} \geq (\ln \Delta)^{20(k-1)} \) and \( R_{i,k-1} \geq \frac{1}{10k^2} \), then
\[
C := \exp \left( -\frac{Kk^2(2-k)}{1 - \frac{\delta}{10k}} \right) \leq \text{Keep}_i \leq 1 - \frac{K^{k-1}}{12k^2(\ln \Delta)^{k-1}},
\]

where for the second equality we used our assumption that \( R_{i,k-1} \geq \frac{1}{10k^2} \) which implies that
\[
\text{Keep}_i \leq e^{-\sum_{r=1}^{k-1} \alpha^r R_{i,r}} \leq e^{-\alpha^{k-1} R_{i,k-1}} \leq e^{-\frac{K^{k-1}}{10k^2(\ln \Delta)^{k-1}}} < 1 - \frac{K^{k-1}}{12k^2(\ln \Delta)^{k-1}},
\]
for sufficiently large \( \Delta \).

The proof of the following lemma can also be found in Appendix A.

**Lemma 4.3.** If \( L_j, T_{j,r} \geq (\ln \Delta)^{20(k-1)} \) for all \( 1 < j < i \), then for every \( r \in [k-1] \):
\[
R'_{i,r} \leq (1 - \alpha C)^{r(i-1)} \ln \Delta \cdot \frac{(1 + \frac{\delta}{100})k^{k-1-r}}{(1 + \frac{\delta}{100})k^{k-1-r}C^{k-1-r} \prod_{p=r}^{k-2} (p+1)}.
\]

We are now ready to prove Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5.

### 4.1 Proof of Lemma 3.1

**Proof of part (a).** For every color \( c \in L_v(i) \),
\[
\Pr[c \in L_v(i+1)] = \prod_{r=1}^{k-1} \prod_{h \in D_{i,r}(v,c)} \left( 1 - \prod_{u \in (h \setminus \{v\}) \cap V_i} \frac{\alpha}{\ell_i(u)} \right) = \prod_{r=1}^{k-1} \left( 1 - \left( \frac{\alpha}{L_i} \right)^r \right) = \text{Keep}_i,
\]
where for the second equality we used our assumption that \( Q(i) \) holds. Therefore, the proof of the first part of the lemma follows from the linearity of expectation.

**Proof of part (b).** Recall the definition of \( t'_{i+1,r}(v,c) \) and note that only hyperedges in \( \bigcup_{j=r}^{k-1} D_{i,j}(v,c) \) can be potentially counted by \( t'_{i+1,r}(v,c) \). In particular, unless every uncolored vertex of an edge \( h \in D_{i,j}(v,c) \), \( j \geq r \), is assigned the same color with \( v \) in iteration \( i \), then if \( h \) is counted by \( t'_{i+1,r}(c) \), it is also counted by \( t_{i+1,r}(c) \). Therefore,
\[
\mathbb{E}[t'_{i+1,r}(v,c)] \leq \mathbb{E}[t_{i+1,r}(v,c)] + O \left( \sum_{j=r}^{k-1} \frac{T_{i,j}}{L_i^j} \right), \tag{12}
\]
and so we focus on bounding \( \mathbb{E}[t_{i+1,r}(v,c)] \).

Fix \( h \in D_{i,j}(v,c) \), where \( j \geq r \). Our goal will be to show that

\[
\Pr[h \in D_{i+1,r}(v,c)] \leq \left( \frac{j}{r} \right) (\text{Keep}_i(1 - \alpha \text{Keep}_i))^r \left( \frac{\alpha \text{Keep}_i}{L_i} \right)^{j-r} + 4r \left( \frac{j}{r} \right) \frac{\text{Keep}_i^{j-1} \alpha^{j-r+1} S_i}{L_i^{j-r}} + O \left( \frac{1}{L_i^j} \right),
\]

since combining (13) with (12) implies the lemma. To see this, observe that

\[
T_{i,j} \cdot 4r \left( \frac{j}{r} \right) \frac{\text{Keep}_i^{j-1} \alpha^{j-r+1} S_i}{L_i^{j-r}} = 4\alpha r \left( \frac{j}{r} \right) \cdot \alpha^r \frac{T_{i,j}}{L_i} \frac{\text{Keep}_i^{j-1} \cdot (\alpha^{-1} L_i)^r S_i}{L_i^{j-r}} \leq \begin{cases} 
4T_{i,1} S_{i}, & \text{if } j = r = 1, \\
4\alpha r \left( \frac{j}{r} \right) \cdot (\alpha^{-1} L_i)^r S_i, & \text{otherwise}.
\end{cases}
\]

Note that in deriving the second part of the inequality in (14) we first used that \( 1 - x \leq \exp(-x) \) for every \( x \geq 0 \) in order to bound \( \text{Keep}_i \) by \( \exp(-T_{i,j}/L_i^r) \), and then that \( \max_x x e^{-\ell x} \leq \frac{1}{\ell e} \) for every \( \ell \). Therefore,

\[
\mathbb{E}[t_{i+1,r}(v,c)] \leq \sum_{j=r}^{k-1} T_{i,j} \max_{h \in D_{i,j}(v,c)} \Pr[h \in D_{i+1,r}(v,c)]
\]

\[
< \sum_{j=r}^{k-1} \left( T_{i,j} \cdot \left( \frac{j}{r} \right) (\text{Keep}_i(1 - \alpha \text{Keep}_i))^r \left( \frac{\alpha \text{Keep}_i}{L_i} \right)^{j-r} \right)
\]

\[
+4k^{2(k-r)} \alpha(\alpha^{-1} L_i)^r S_i \ln \Delta + O \left( \sum_{j=r}^{k-1} \frac{T_{i,j}}{L_i^j} \right).
\]

for sufficiently large \( \Delta \). In deriving (16) we used (14) and the facts that:

\[
T_{i,1} = \frac{T_{i,1}}{L_i} \cdot L_i \leq L_i \cdot k^{2(k-1-r)} \ln \Delta, \quad \text{according to Lemma 4.1}
\]

\[
\sum_{j=r}^{k-1} \frac{r(j)}{e^{(j-1)}} < \ln \Delta \cdot k^{2(k-1-r)} \text{ for sufficiently large } \Delta \text{ and } j > 1.
\]

Towards proving (13), for any vertex \( u \in h \setminus \{v\} \), consider the events

\[
E_{u,1} = \text{“} u \text{ does not retain its color and } c \in L_u(i+1), \text{”}
\]

\[
E_{u,2} = \text{“} u \text{ is assigned } c \text{ and retains its color} \text{”}.
\]

Let also \( B_c \) be the event that \( v \) and \( j - 1 \) other uncolored vertices of \( h \) receive color \( c \). Since we have assumed that our hypergraph is of girth at least 5, for any neighbor \( u \) of \( v \) and \( f \in \{1,2\} \) the event \( E_{u,f} \) is mutually independent of all events \( E_{u',f}, \ell \in \{1,2\} \), \( u \neq u' \), conditional on \( B_c \) not occurring. Thus, if \( \Pr[E_{u,\ell} | B_c] \leq p_\ell, \ell \in \{1,2\}, \) for every vertex \( u \in h \setminus \{v\} \), we obtain

\[
\Pr[h \in D_{i+1,r}(v,c)] \leq \left( \frac{j}{r} \right) \frac{p_1^r p_2^{j-r}}{L_i^r} + \Pr[B_c] \leq \left( \frac{j}{r} \right) \frac{p_1^r p_2^{j-r}}{L_i^r} + \frac{2k}{L_i^r},
\]

(17)
since \( \Pr[B_c] \leq 2^k L_i^{-\delta} \).

Now we claim that for any \( u \in h \setminus \{v\} \), and sufficiently large \( \Delta \),

\[
\Pr[E_{u,2} | B_c] \leq \frac{\alpha \text{Keep}_i}{L_i} + \frac{2}{(L_i \ln \Delta)^j} =: q_2 + \delta_2.
\]  

(18)

To see this, notice that conditional on \( B_c \), the probability that \( u \) is activated is \( \alpha \); it is assigned \( c \) with probability at most \( 1/L_i \), and it retains \( c \) with probability at most

\[
\prod_{r \in [k-1] \setminus \{j\}} \left(1 - \frac{\alpha^r}{L_i^r}\right)^{T_{i,r}} \cdot \left(1 - \frac{\alpha^j}{L_i^j}\right)^{T_{i,j}-1} = \frac{\text{Keep}_i}{1 - \frac{\alpha^j}{L_i^j}}.
\]

(19)

Thus,

\[
\Pr[E_{u,2} | B_c] = \frac{\alpha \text{Keep}_i}{L_i(1 - \frac{\alpha^j}{L_i^j})} \leq \alpha \cdot \text{Keep}_i \cdot \frac{1 + \frac{2\alpha^j}{L_i^j}}{L_i} \leq \frac{\alpha \text{Keep}_i}{L_i} + \frac{2}{(L_i \ln \Delta)^j}
\]

for sufficiently large \( \Delta \), concluding the proof of (18).

Further, we claim that

\[
\Pr[E_{u,1} | B_c] \leq \text{Keep}_i(1 - \alpha \text{Keep}_i) + (2\alpha S_i + (L_i \ln \Delta)^{-j}(3 + 4\alpha S_i)) =: q_1 + \delta_1.
\]

(20)

To show (20) we consider three cases. The first case is that \( u \) is not activated and \( c \in L_u(i + 1) \) (notice that these are two independent events). In this case \( u \) will not retain its color, and observe that

\[
\Pr[c \in L_u(i + 1) | B_c] \leq \text{Keep}_i + 2(L_i \ln \Delta)^{-j},
\]

(21)

using essentially the same calculations as in showing (18). Thus,

\[
\Pr[u \text{ is not activated and } c \in L_u(i + 1) | B_c] \leq (1 - \alpha) \left( \text{Keep}_i + 2(L_i \ln \Delta)^{-j} \right).
\]

(22)

In the second case we consider the scenario where \( u \) is activated and is assigned \( c \). Clearly then, the probability that \( c \in L_u(i + 1) \) and \( u \) does not retain \( c \) is zero. Finally, suppose that \( u \) is activated and is assigned a color \( \gamma \neq c \). Our goal is to compute \( \Pr[(u \text{ is activated and assigned } \gamma) \land E_{u,1} | B_c] \) for each \( \gamma \) so that we can sum up these probabilities over all possible \( \gamma \neq c \) along with (22).

For a vertex \( w \) let \( F_w^\gamma \) denote the event that \( w \) is activated and assigned \( \gamma \). Using this notation we have:

\[
\Pr[F_u^\gamma \land E_{u,1} | B_c] = \Pr[F_u^\gamma | B_c] \cdot \Pr[(\gamma \notin L_u(i + 1)) \land (c \in L_u(i + 1)) | F_u^\gamma, B_c]
\]

\[
= \frac{\alpha}{L_i} \cdot \Pr[(\gamma \notin L_u(i + 1)) \land (c \in L_u(i + 1)) | F_u^\gamma, B_c]
\]

\[
= \frac{\alpha}{L_i} \Pr[c \in L_u(i + 1) | F_u^\gamma, B_c] \cdot \Pr[\gamma \notin L_u(i + 1) | c \in L_u(i + 1), F_u^\gamma, B_c]
\]

\[
= \frac{\alpha}{L_i} \Pr[c \in L_u(i + 1) | B_c] \cdot \Pr[\gamma \notin L_u(i + 1) | c \in L_u(i + 1), F_u^\gamma, B_c]
\]

\[
\leq \frac{\alpha}{L_i} \cdot (\text{Keep}_i + 2(L_i \ln \Delta)^{-j}) \cdot \Pr[\gamma \notin L_u(i + 1) | c \in L_u(i + 1), F_u^\gamma, B_c]
\]

(23)

and so below we focus on bounding for each \( \gamma \in L_u(i) \setminus \{c\} \) the probability that \( \gamma \notin L_u(i + 1) \) and \( c \in L_u(i + 1) \) conditional on that \( u \) is activated and assigned \( \gamma \) and \( B_c \) did not occur. Note that in deriving (23) we used (21).
We have:

\[
\Pr[\gamma \notin L_u(i+1) \mid c \in L_u(i+1), F_u^\gamma, \overline{B_c}] = 1 - \Pr[\gamma \in L_u(i+1) \mid c \in L_u(i+1), F_u^\gamma, \overline{B_c}]
\]

\[
= 1 - \prod_{k=1}^{k-1} \prod_{\ell=1}^{\ell} \prod_{g \in D_{i,\ell}(u,\gamma)} (1 - \Pr[\cap_{w \in (g \setminus \{u\}) \cap V_i} F_u^w \mid c \in L_u(i+1), F_u^\gamma, \overline{B_c}])
\]

\[
= 1 - \prod_{k=1}^{k-1} \prod_{\ell=1}^{\ell} \prod_{g \in D_{i,\ell}(u,\gamma)} (1 - \Pr[\cap_{w \in (g \setminus \{u\}) \cap V_i} F_u^w \mid c \in L_u(i+1), \overline{B_c}])
\]

(24)

Note that in deriving (24) we use the fact the girth of the hypergraph is at least 5 which, in particular, implies that any two vertices that contain \(u\) do not have any other vertex in common.

To further bound (25), we consider the probability that every vertex in \((g \setminus \{u\}) \cap V_i\) is activated and assigned \(\gamma\), conditional on that \(c \in L_u(i+1)\) and \(\overline{B_c}\), for any fixed \(\ell \in [k-1]\) and \(g \in D_{i,\ell}(u,\gamma)\). We consider two cases depending on whether \(g = h\) or not.

We start with the case where \(g \neq h\). Let \(A_g\) be the event that not every vertex in \((g \setminus \{u\}) \cap V_i\) is activated and assigned \(c\). Since our hypergraph has girth at least 5 and the color activations and color assignments are independent over different vertices, we have:

\[
\Pr[\cap_{w \in (g \setminus \{u\}) \cap V_i} F_u^w \mid c \in L_u(i+1), \overline{B_c}] = \Pr[\cap_{w \in (g \setminus \{u\}) \cap V_i} F_u^w \mid A_g]
\]

\[
= \frac{\Pr[\cap_{w \in (g \setminus \{u\}) \cap V_i} F_u^w \land A_g]}{\Pr[A_g]}
\]

\[
= \frac{\Pr[\cap_{w \in (g \setminus \{u\}) \cap V_i} F_u^w]}{\Pr[A_g]}
\]

\[
= \frac{\alpha^{k-1}L_i^{-(k-1)}}{1 - \alpha^{k-1}L_i^{-(k-1)}} \leq \left(\frac{\alpha}{L_i}\right)^{k-1} + \frac{1}{L_i^{2\ell}(\ln \Delta)^{2\ell}},
\]

(26)

for sufficiently large \(\Delta\), since \(K < 1\).

Next we consider the case \(g = h\). The difference here is that the event \(\cap_{w \in (h \setminus \{u\}) \cap V_i} F_u^w\) is not independent of \(\overline{B_c}\) as before. However, notice that since \(g \in D_{i,\ell}(u,\gamma), h \in D_{i,j}(v, c)\) and \(\gamma \neq c\), we can only have \(g = h\) when \(j = \ell = k - 1\). This means that the occurrence of event the event \(\overline{B_c}\) prohibits the occurrence of event event \(A_h\). Therefore, the event \(\cap_{w \in (h \setminus \{u\}) \cap V_i} F_u^w\) is independent of the event \(c \in L_u(i+1)\) and, thus, we have:

\[
\Pr[\cap_{w \in (h \setminus \{u\}) \cap V_i} F_u^w \mid c \in L_u(i+1), \overline{B_c}] = \Pr[\cap_{w \in (h \setminus \{u\}) \cap V_i} F_u^w \mid \overline{B_c}]
\]

\[
= \frac{\Pr[\cap_{w \in (h \setminus \{u\}) \cap V_i} F_u^w]}{\Pr[\overline{B_c}]} = \frac{\alpha^{k-1}L_i^{-(k-1)}}{1 - \alpha^{k-1}L_i^{-(k-1)}} \leq 2\left(\frac{\alpha}{L_i}\right)^{k-1},
\]

(27)

for sufficiently large \(\Delta\).

Combining (25), (26) and (27), we are able to show the following proposition.

**Proposition 4.4.** For every color \(\gamma \neq c\):

\[
\Pr[\gamma \notin L_u(i+1) \mid c \in L_u(i+1), F_u^\gamma, \overline{B_c}] \leq 1 - \text{Keep}_i + 2\sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln \Delta)^{2\ell}}.
\]

(28)
Proof: Towards proving (28), a helpful observation is the following.

\[
\prod_{\ell=1}^{k-1} \left( 1 - \left( \frac{\alpha}{L_i} \right)^\ell - \frac{1}{L_i^{2\ell} (\ln \Delta)^{2\ell}} \right) T_{i,\ell}(u,\gamma) = \prod_{\ell=1}^{k-1} \left( 1 - \left( \frac{\alpha}{L_i} \right)^\ell \right) T_{i,\ell}(u,\gamma) \\
\times \left( 1 - \frac{1}{L_i^{2\ell} (\ln \Delta)^{2\ell} \cdot (1 - (\frac{\alpha}{L_i})^\ell)} \right) T_{i,\ell}(u,\gamma)
\]

\[
= \text{Keep}_i \cdot \exp \left( -\sum_{\ell=1}^{k-1} \frac{T_{i,\ell}(u,\gamma)}{L_i^{2\ell} (\ln \Delta)^{2\ell} \cdot (1 - (\frac{\alpha}{L_i})^\ell) - 1} \right)
\]

\[
\geq \text{Keep}_i \left( 1 - \frac{3}{2} \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell} - (\frac{\alpha}{L_i})^\ell} \right)
\]

\[
\geq \text{Keep}_i \left( 1 - \frac{3}{2} \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell} - L_i^{\ell} (\ln \Delta)^{\ell}} \right)
\]

\[
\geq \text{Keep}_i - \frac{19}{10} \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell}},
\]

for sufficiently large \(\Delta\). Note that in (29) we used the fact that \(1 - x \geq e^{-x}\) for any \(x \geq 2\).

Using (26), (27) and (30), we have:

\[
\prod_{\ell=1}^{k-1} \prod_{g \in D_{i,\ell}(u,\gamma)} \left( 1 - \Pr[\cap_{u \in (g \setminus \{u\}) \cap V_i, \kappa_{g_u}^\gamma | c \in L_u(i + 1) \cup \bar{c}] \right) \geq
\]

\[
\geq \prod_{\ell=1}^{k-1} \left( 1 - \left( \frac{\alpha}{L_i} \right)^\ell - \frac{1}{L_i^{2\ell} (\ln \Delta)^{2\ell}} \right) T_{i,\ell}(u,\gamma) \left( 1 - 2 \left( \frac{\alpha}{L_i} \right)^{k-1} \right)
\]

\[
\geq \left( \text{Keep}_i - \frac{19}{10} \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell}} \right) \left( 1 - 2 \left( \frac{\alpha}{L_i} \right)^{k-1} \right)
\]

\[
\geq \text{Keep}_i - \frac{19}{10} \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell}} - 2 \cdot \text{Keep}_i \left( \frac{\alpha}{L_i} \right)^{k-1}
\]

\[
\geq \text{Keep}_i - 2 \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell}},
\]

for sufficiently large \(\Delta\). Note that in deriving (31) we used our previous observation that (27) applies only when \(h = g\), and this can potentially happen only when \(\ell = j = k - 1\).

Combining (25) with (33) concludes the proof.

Overall, combining (22), (23) and Proposition 4.4 we see that \(\Pr[E_{u,1} \cup \bar{c}]\) is at most

\[
(1 - \alpha) \text{Keep}_i + 2(L_i \ln \Delta)^{-j} + \alpha \frac{L_i - 1}{L_i} (\text{Keep}_i + 2(L_i \ln \Delta)^{-j}) \left( 1 - \text{Keep}_i + 2 \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell} (\ln \Delta)^{2\ell}} \right),
\]

\[
\leq \text{Keep}_i (1 - \alpha \text{Keep}_i) + (2\alpha S_i + (L_i \ln \Delta)^{-j} (3 + 4\alpha S_i)),
\]

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Lemma 4.5. Let $\ell(v)$ be the binomial random variable that counts the number of successes in $n$ Bernoulli trials, where each trial succeeds with probability $p$. We will find the following lemma useful (see, e.g., Exercise 2.12 in [28]):

**Lemma 4.5.** For any $c, k, n$ we have

$$\Pr \left[ \text{Bin} \left( n, \frac{c}{n} \right) \geq k \right] \leq \frac{e^k}{k!}.$$

**Proof of Part of Lemma 4.5.** We will use Theorem 2.4 to show that the number of colors, $\overline{\ell}_v$, which are removed from $L_v$ during iteration $i$ is highly concentrated.

Note that changing the assignment to any neighboring vertex of $v$ can change $\overline{\ell}_v$ by at most 1, and changing the assignment to any other vertex cannot affect $\overline{\ell}_v$ at all.

If $\overline{\ell}_v \geq s$, there are at most $s$ groups of at most $k - 1$ neighbors of $v$, so that each vertex in each group received the same color, and each group corresponds to a different color from $L_v$. Thus, the color assignments and activation choices of these vertices certify that $\overline{\ell}_v \geq s$.

Since, according to Corollary 4.2, $\text{Keep}_i = \Omega(1)$, applying Theorem 2.4 with $t = L_i^{1/3}$, $w = 2k$, $c = 1$, we obtain

$$\Pr \left[ |\overline{\ell}_v - \mathbb{E}[\overline{\ell}_v]| > L_i^{2/3} \right] < \Delta^{-\ln \Delta},$$

for sufficiently large $\Delta$.

Finally, $\mathbb{E}[\ell_{i+1}(v)] = \ell_i(v) - \mathbb{E}[\ell_v]$ implies that

$$\Pr \left[ |\ell_{i+1}(v) - \mathbb{E}[\ell_{i+1}(v)]| > L_i^{2/3} \right] = \Pr \left[ |\overline{\ell}_v - \mathbb{E}[\overline{\ell}_v]| > L_i^{2/3} \right] < \Delta^{-\ln \Delta}.$$
Proof of Part (b). Recall the definition of $D_{i,r}(v, c)$ and let $Z_{i,r}(v, c) = \bigcup_{j=r}^{k-1} D_{i,j}(v, c)$. Let $X_{i+1,r}(v, c)$ denote the number of hyperedges in $Z_{i,r}(v, c)$ which (i) contain exactly $r$ uncolored vertices other than $v$; and (ii) the rest of their vertices are assigned $c$ in the end of the $i$-th iteration. Let also $Y_{i+1,r}(v, c)$ be the number of these hyperedges which they contain an uncolored vertex $u \neq v$ such that (i) $c \not\in L_u(i+1)$; and (ii) $c$ would still not be in $L_u(i+1)$ even if we ignored the color of $v$.

By the linearity of expectation, it suffices to show that $X_{i+1,r}(v, c)$ and $Y_{i+1,r}(v, c)$ are both sufficiently concentrated. This is because

$$
\Pr \left[ t_{i+1,r}(v, c) - \mathbb{E}[t'_{i+1}(v, c)] > \frac{1}{2} \left( \sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_{i+r}^{j-r}} \right)^{2/3} \right],
$$

$$
= \Pr \left[ X_{i+1,r}(v, c) - \mathbb{E}[X_{i+1}(v, c)] - (Y_{i+1,r}(v, c) - \mathbb{E}[Y_{i+1,r}(v, c)]) > \frac{1}{2} \left( \sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_{i+r}^{j-r}} \right)^{2/3} \right],
$$

and, therefore, it is sufficient to prove that

$$
\Pr \left[ X_{i+1,r}(v, c) - \mathbb{E}[X_{i+1,r}(v, c)] > \frac{1}{4} \left( \sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_{i+r}^{j-r}} \right)^{2/3} \right] \leq \frac{1}{2} \Delta^{-\ln \Delta}, \quad (37)
$$

$$
\Pr \left[ Y_{i+1,r}(v, c) - \mathbb{E}[Y_{i+1,r}(v, c)] < -\frac{1}{4} \left( \sum_{j=r}^{k-1} \binom{j}{r} \alpha^{j-r} \frac{T_{i,j}}{L_{i+r}^{j-r}} \right)^{2/3} \right] \leq \frac{1}{2} \Delta^{-\ln \Delta}. \quad (38)
$$

We first focus on $X_{i+1,r}(v, c)$. Let $X'_{i+1,r}(v, c)$ denote the number of hyperedges in $Z_{i,r}(v, c)$ which (i) contain exactly $r$ uncolored vertices other than $v$; and (ii) the rest of their vertices were activated and assigned $c$ (but did not retain their color necessarily). Clearly, $X'_{i+1,r}(v, c) \geq X_{i+1,r}(v, c)$. Further, let $W_{i+1,r}(v, c)$ denote the random variable that counts all the hyperedges counted by $X'_{i+1,r}(v, c)$, except for those whose $r$ uncolored vertices (other than $v$) were uncolored because they were activated and received the same color as $v$. Finally, let $W_{i+1,r}(v, c)$ be the number of hyperedges which (i) contain exactly $r$ vertices that are activated and received the same color as $v$; and (ii) the rest of their $k-1-r$ vertices were activated and assigned $c$.

Observe that $X_{i+1,r}(v, c) \leq W_{i+1,r}(v, c) + W_{i+1,r}(v, c)$. The idea here is that we cannot directly apply Talagrand’s inequality to $X_{i+1,r}(v, c)$ and so we consider $W^1_{i+1,r}(v, c), W^2_{i+1,r}(v, c)$ instead.

First, we consider $W^1_{i+1,r}(v, c)$. Since our hypergraph is of girth at least 5, changing a choice for some vertex of a hyperedge $h \in Z_{i,r}(v, c)$ can only affect whether or not the vertices of $h$ remain uncolored, and thus affect $W^1_{i+1,r}(v, c)$ by at most 1. Furthermore, changing a choice for a vertex outside the ones that correspond to the hyperedges in $Z_{i,r}(v, c)$ can affect at most one vertex of at most one hyperedge in $Z_{i,r}(v, c)$ and, therefore, can affect $W^1_{i+1,r}(v, c)$ by at most 1.

We claim now that if $W^1_{i+1,r}(v, c) \geq s$, then there exist at most $2k^2s$ random choices that certify this event. To see this, notice that if a hyperedge $h$ is counted by $W^1_{i+1,r}(v, c)$, then for every $u \in h \setminus \{v\}$ that remained uncolored, it must be that either $u$ was deactivated, or $u$ is contained in a hyperedge $h' \neq h$ such that all the vertices in $(h' \setminus \{u\}) \cap V_i$ were activated and received the same color as $u$. Moreover, the event that a variable $u \in h \setminus \{v\}$ was activated and received $c$ can be verified by the outcome of two random choices. So, overall, we can certify that $h$ was counted by $W^1_{i+1,r}(v, c)$ by using the outcome of at most $2k^2$ random choices.

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Finally, observe that $\mathbb{E}[W_{i+1,r}^1(v,c)] \leq \sum_{j=r}^{k-1} \binom{j}{r} \alpha^j - r \frac{T_{i,j}}{L_i}$ and, thus, applying Theorem 2.4 with $c = 1$, $w = 2k^2$ and $t = \left(\sum_{j=r}^{k-1} \binom{j}{r} \frac{T_{i,j}}{L_i}\right)^{1.9/3}$ we obtain

$$
\Pr \left[ |W_{i+1,r}^1(v,c) - \mathbb{E}[W_{i+1,r}^1(v,c)]| > \frac{1}{8} \left( \sum_{j=r}^{k-1} \binom{j}{r} \alpha^j - r \frac{T_{i,j}}{L_i} \right)^{2/3} \right] \leq \frac{1}{4} \Delta^{-\ln \Delta},
$$

(39)

for sufficiently large $\Delta$, since we have assumed that $T_{i,j} \geq (\ln \Delta)^{20(k-1)}$.

As far as $W_{i+1,r}^2(v,c)$ is concerned, note that it is stochastically dominated by $\sum_{j=r}^{k-1} \text{Bin}(T_{i,j}, \binom{j}{r} \alpha^j / L_i)$ and recall Lemma 4.1. Since $T_{i,j} \geq (\ln \Delta)^{20(k-1)}$ for every $j \in [k-1]$, Lemma 4.5 implies that

$$
\Pr \left[ W_{i+1,r}^2(v,c) > \mathbb{E}[W_{i+1,r}^2(v,c)] + \frac{1}{8} \left( \sum_{j=r}^{k-1} \binom{j}{r} \alpha^j - r \frac{T_{i,j}}{L_i} \right)^{2/3} \right] \leq \frac{1}{4} \Delta^{-\ln \Delta}.
$$

(40)

Combining (39) and (40) implies (37).

We follow the same approach for $Y_{i+1,r}^r(v,c)$. Let $Y_{i+1,r}^r(v,c)$ be the number of hyperedges counted by $X_{i+1,r}^r(v,c)$ and which also contain an uncolored vertex $u \neq v$ such that (i) $c \notin L_u(i + 1)$; and (ii) $c$ would still not be in $L_u(i + 1)$ even if we ignored the color of $v$. Further, let $Y_{i+1,r}^r(v,c)$ be the random variable that counts all the hyperedges counted by $Y_{i+1,r}^r(v,c)$, except for those whose $r$ uncolored vertices were uncolored because they were activated and received the same color as $v$, and observe that $Y_{i+1,r}^r(v,c) \leq Y_{i+1,r}^r(v,c) \leq N_{i+1,r}(v,c) + W_{i+1,r}^2(v,c)$.

Moreover, if $Y_{i+1,r}^r(v,c) \geq s$, then there exist at most $(2k^2 + 2k)s$ random choices that certify this event. To see this, observe that for each hyperedge $h$ counted by $Y_{i+1,r}^r(v,c)$, we need the output of at most $2k^2$ choices to certify that it is counted by $X_{i+1,r}^r(v,c)$, and the output of at most $2k$ extra random choices to certify that there is a vertex $u \in h \setminus \{v\}$ for which $c \notin L_u(i + 1)$, and $c$ would still not be in $L_u(i + 1)$ even if we ignored the color of $v$.

Finally, $\mathbb{E}[Y_{i+1,r}^r(v,c)] \leq \sum_{j=r}^{k-1} \binom{j}{r} \alpha^j - r \frac{T_{i,j}}{L_i}$ and, therefore, an almost identical argument to the case for $X_{i+1,r}(v,c)$ implies (38). 

\[ \square \]

### 4.3 Proof of Lemma 3.3

We use induction on $i$. Property $P(1)$ clearly holds, so we assume that property $P(i)$ holds and we prove that with property $P(i + 1)$ holds with positive probability. Recall our discussion in the previous section in which we argued that we can assume without loss of generality that property $Q(i)$ holds.

For every $v$ and $c \in L_v$ let $A_v$ be the event that $\ell_{i+1}(v) < L_{i+1}$ and $B_{i,c}$ to be the event that $t_{i+1,r}(v,c) > T_{i+1,r}$. Clearly, if these bad events are avoided, then $P(i + 1)$ holds.

Since property $Q(i)$ holds, we have that $\ell_{i}(v) = L_i$. Therefore, by (4) and Lemmas 5.1 5.2 we have:

$$
\Pr[A_v] = \Pr \left[ \ell_{i+1}(v) < L_i \cdot \text{Keep - } L_i^{2/3} \right] = \Pr \left[ \ell_{i+1}(v) < \mathbb{E}[\ell_{i+1}(v)] - L_i^{2/3} \right] < \Delta^{-\ln \Delta}.
$$

(41)
Similarly, by (5) and Lemmas 3.1, 3.2 we have:

\[
\Pr[B'_{v,c}] \leq \Pr[t'_{i+1,r}(v, c) > T_{i+1,r}]
\]

\[
= \Pr \left[ t'_{i+1,r}(v, c) - \mathbb{E}[t'_{i+1,r}(v, c)] > \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{j-r} \frac{T_{i,j}}{L_i^j} \right)^{2/3} - C' \cdot \sum_{j=r}^{k-1} \frac{T_{i,j}}{L_i^j} \right]
\]

\[
\leq \Pr \left[ t'_{i+1,r}(v, c) - \mathbb{E}[t'_{i+1,r}(v, c)] > \frac{1}{2} \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{j-r} \frac{T_{i,j}}{L_i^j} \right)^{2/3} \right]
\]

\[
\leq \Delta^{-\ln \Delta},
\]

where \( C' > 0 \) is the hidden constant that multiplies \( Y_i \) in the statement of Lemma 3.1 Part (b). Note that in deriving (42) we used Lemma 4.1 — which implies that \( \sum_{j=r}^{k-1} \frac{T_{i,j}}{L_i^j} = O(\ln \Delta) \) — and our assumptions that \( T_{i,1} \geq (\ln \Delta)^{20(k-1)} \) — which implies that \( \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{j-r} \frac{T_{i,j}}{L_i^j} \right)^{2/3} = o(\ln \Delta) \).

Notice now that each bad event \( f_v \in \{ A_v, B'_{v,c} \} \) event is determined by the colors assigned to vertices of distance at most 3 from \( v \). Therefore, \( f_v \) is mutually independent of all but at most \( (k\Delta)^3(1+\delta) \left( \frac{\Delta}{\ln \Delta} \right)^{1/5} < \Delta^5 \) other bad events. For \( \Delta \) sufficiently large, \( \Delta^{-\ln \Delta} \Delta^5 < \frac{1}{4} \) and so the proof is concluded by applying Corollary 2.3 using (41), (43).

### 4.4 Proof of Lemma 3.4

Since \( L_i < L'_i \), for the first part of the lemma it suffices to prove that \( L'_i \leq L_i + (L'_i)^{5/6} \). Towards that end, at first we observe that for sufficiently large \( \Delta \), Corollary 4.2 and the fact that \( K = \frac{1}{100k^{3/2}} \) imply:

\[
\text{Keep}_{5/6} - \text{Keep}_i \geq \left( 1 - \frac{K^{k-1}}{12k^2(\ln \Delta)^{k-1}} \right)^{5/6} - \left( 1 - \frac{K^{k-1}}{12k^2(\ln \Delta)^{k-1}} \right)
\]

\[
\geq \left( 1 - \frac{5}{6} \cdot \frac{K^{k-1}}{12k^2(\ln \Delta)^{k-1}} \right) - \left( 1 - \frac{K^{k-1}}{12k^2(\ln \Delta)^{k-1}} \right) = \frac{K^{k-1}}{72k^2(\ln \Delta)^{k-1}} .
\]

Note that in deriving the first inequality we used the fact that the function \( x^{5/6} - x \) is decreasing on the interval \( [C, 1] \) since \( K \) is sufficiently small. For the second one, we used the Taylor Series for \( (1 - y)^{5/6} \) around \( y = 0 \).

We now proceed by using induction. The base case is trivial, so assume that the statement is true for \( i \), and consider \( i + 1 \). Since by our assumption \( L_i \geq (\ln \Delta)^{20(k-1)} \) we obtain

\[
L'_{i+1} = \text{Keep}_i L'_i
\]

\[
\leq \text{Keep}_i \left( L_i + (L'_i)^{5/6} \right)
\]

\[
= L_{i+1} + L_i^{2/3} + \text{Keep}_i(L'_i)^{5/6}
\]

\[
\leq L_{i+1} + L_i^{2/3} + \text{Keep}_i(L'_i)^{5/6} \left( L'_i \right)^{5/6} - \frac{K^{k-1}}{72k^2(\ln \Delta)^{k-1}} (L'_i)^{5/6}
\]

\[
\leq L_{i+1} + (L'_{i+1})^{5/6} + L_i^{2/3} - \frac{K^{k-1}}{72k^2(\ln \Delta)^{k-1}} (L'_i)^{5/6}
\]

\[
< L_{i+1} + (L'_{i+1})^{5/6} ,
\]
for sufficiently large $\Delta$. Note that in deriving (45) we used the inductive hypothesis; for (46) we used (44); and for (47) the fact that $L_i \geq (\ln \Delta)^{20(k-1)}$ and the inductive hypothesis.

The proof of the second part of the lemma is conceptually similar, but more technical, so we present its proof in Appendix A.

### 4.5 Proof of Lemma 3.5

We proceed by induction. Let $\eta := \frac{c^{\frac{3}{2}}}{(k-1)(1+\epsilon/2)}$. We will assume that $L_j \geq \Delta^q, T_{j,r} \geq (\ln \Delta)^{20(k-1)}$ for all $2 \leq j \leq i < i^*$, and prove that $L_{i+1} \geq \Delta^q, T_{i+1,r} \geq (\ln \Delta)^{20(k-1)}$. Towards that end, it will be useful to focus on the family of ratios $R_{i,r}, r \in [k-1]$. Note that, according to Lemma 3.4, this family is well-approximated by the family $R'_{i,r}, r \in [k-1]$. In particular, recalling Lemma 3.4 and applying Lemma 3.5 we obtain:

$$R_{i,r} \leq R'_{i,r} \cdot \frac{1 + (T'_{i,r})^{-\frac{1}{20(k-1)}}}{(1 - (L_i)^{-1/6})} \leq (1 - \alpha C)^{r(i-1)} \ln \Delta \cdot \frac{\prod_{p=r}^{k-2} (p + 1)}{(1 + \delta - \frac{1}{k^9})^{k-1} C^{k-1-r}},$$

(48)

for sufficiently large $\Delta$, since $L_i, T_{i,r} \geq (\ln \Delta)^{20(k-1)}$.

Using (48) and the fact that $1 - \frac{1}{x} > e^{-\frac{1}{x}}$ for $x \geq 2$ we can get an improved lower bound for $\text{Keep}_i$ as follows.

$$\text{Keep}_i \geq \exp \left( -\frac{1}{(1 - \frac{\delta}{k^{100}})} \sum_{r=1}^{k-1} \alpha^r R_{i,r} \right) \geq \exp \left( -\frac{1}{(1 + \delta - \frac{1}{2^6})^{k-1}} \sum_{r=1}^{k-1} (1 - \alpha C)^{r(i-1)} \frac{K^r \prod_{p=r}^{k-2} (p + 1)}{(\ln \Delta)^{r-1} C^{k-1-r}}, \right)$$

(49)

for sufficiently large $\Delta$.

Recall that $\delta = (1 + \epsilon)(k - 1) - 1$. Using (49) we get

$$\prod_{j=1}^{i-1} \text{Keep}_j \geq \exp \left( -\frac{1}{(1 + \delta - \frac{1}{2^6})^{k-1}} \sum_{r=1}^{k-1} \left( K^r \prod_{p=r}^{k-2} (p + 1) \frac{1}{(\ln \Delta)^{r-1} C^{k-1-r}} \sum_{j=1}^{i-1} (1 - \alpha C)^{r(j-1)} \right) \right) \geq \exp \left( -\frac{1}{(1 + \delta - \frac{1}{2^6})^{k-1}} \sum_{r=1}^{k-1} \left( C^{-(k-2)} \frac{1}{(1 + \delta - \frac{1}{k^{100}})(k - 1)! \ln \Delta} \right) \right) \geq \exp \left( -\frac{\ln \Delta}{(1 + \frac{\delta}{2})(k - 1)} \right),$$

(50)

for sufficiently large $\Delta$.

Using (50) we can now bound $L'_i$ as follows.

$$L'_i = L'_i \prod_{j=1}^{i-1} \text{Keep}_j \geq (1 + \delta) \left( \frac{\Delta}{\ln \Delta} \right)^{\frac{1}{k-1}} \Delta^{-\frac{1}{(1 + \frac{\delta}{2})(k - 1)}} \geq \Delta^\eta,$$

(51)
for sufficiently large $\Delta$. Thus, $L'_i$ never gets too small for the purposes of our analysis. Lemma 3.4 implies that neither does $L_i$.

The proof is concluded by observing that (48) implies that $R_{i,r}, r \in [k-1]$, becomes smaller than $\frac{1}{10k^2}$ for $i = O(\ln \Delta \ln \ln \Delta)$.

5 A sufficient pseudo-random property for coloring

In this section we present the proof of Theorem 1.5. To do so, we build on ideas of Alon, Krivelevich and Sudakov [6] and show that the random hypergraph $H(k, n, d/(k-1))$ asymptotically almost surely admits a few useful features.

The first lemma we prove states that all subgraphs of $H(k, n, d/(k-1))$ with not too many vertices are sparse and, therefore, of small degeneracy.

Lemma 5.1. For every constant $k \geq 2$, there exists $d_k > 0$ such that for any constant $d \geq d_k$, the random hypergraph $H(k, n, d/(k-1))$ has the following property asymptotically almost surely: Every $s \leq nd^{-\frac{1}{k-1}}$ vertices of $H$ span fewer than $s \left(\frac{d}{(\ln d)^2}\right)^{\frac{k-1}{k}}$ hyperedges. Therefore, any subhypergraph of $H$ induced by a subset $V_0 \subset V$ of size $|V_0| \leq nd^{-\frac{1}{k-1}}$, is $k \left(\frac{d}{(\ln d)^2}\right)^{\frac{k-1}{k}}$-degenerate.

Proof: Given the statement about the sparsity of any subhypergraph of $H$ induced by a set of $s \leq nd^{-\frac{1}{k-1}}$ vertices, the claim about its degeneracy follows from the fact that its average (and, therefore its minimum) degree is at most

$$k \cdot \frac{s}{\left(\frac{d}{(\ln d)^2}\right)^{\frac{k-1}{k}}} = \left(\frac{d}{(\ln d)^2}\right)^{-\frac{1}{k}}.$$

So we are left with proving the statement regarding sparsity.

Letting $r = \left(\frac{d}{(\ln d)^2}\right)^{\frac{k-1}{k}}$, we see that the probability that there exists a subset $V_0 \subset V$ which violates the statement of the lemma is at most

$$\sum_{i=r}^{nd^{-\frac{1}{k-1}}} \left(\frac{n}{i}\right) \left(\frac{i}{k}\right)^r \left(\frac{d}{(n-1)}\right)^r \leq \sum_{i=r}^{nd^{-\frac{1}{k-1}}} \left[\frac{en}{r} \left(\frac{ek^{-1}}{d}\right)^r \left(\frac{d}{(k-1)}\right)^r \right]^i \leq \sum_{i=r}^{nd^{-\frac{1}{k-1}}} \left[e^{1+\frac{1}{k-1}(k-1)} \frac{d}{(r)^{k-1}} \left(\frac{ek^{-1}d}{r(k-1)}\right)^{r\frac{1}{k-1}} \right]^i = o(1),$$

for sufficiently large $d$. Note that in the lefthand side of (52) we used the fact that any subset of vertices of size $s < r^{-\frac{1}{k-1}}$ cannot violate the assertion of the lemma, since it can span at most $\binom{n}{k} \leq s^k = s^{k-1} \cdot s < r \cdot s$ hyperedges. In deriving (54) we used the fact that, for sufficiently large $d, n$, every summand in (53) is at most $n^{-(k-1)(d/(\ln d)^2)^{1/(k-1)/2}}$, and there exist at most $n$ summands. Throughout our derivation we used that for any pair of positive integers $\alpha, \beta$, we have $\left(\frac{\alpha}{\beta}\right)^{\delta} \leq \left(\frac{\alpha}{\beta}\right) < \left(\frac{\alpha}{\beta}\right)^{\delta}$.

$\square$
Next we show that, that for any constant $c$, the number of vertices of $H(k, n, d / (n_{k-1}))$ that have degree $c$ essentially behaves as a Poisson random variable with mean $d$.

**Lemma 5.2.** For constants $c \geq 1$, $k \geq 2$ and $d$ sufficiently large, let $X_c$ denote the number of vertices of degree $c$ in $H(k, n, d / (n_{k-1}))$. Then, asymptotically almost surely,

$$X_c \leq d^c e^{-d} c! \frac{n}{1 + O \left( \frac{\log n}{\sqrt{n}} \right)}.$$  

**Proof.** The lemma follows from standard ideas for estimation of the degree distribution of random graphs (see for example the proof of Theorem 3.3 in [12] for the case $k = 2$). In particular, assume that the vertices of $H(k, n, d / (n_{k-1}))$ are labeled $1, 2, \ldots, n$. Then,

$$\mathbb{E}[X_c] = n \Pr[\deg(1) = c]$$

$$= n \binom{n-1}{k-1} \left( \frac{d}{n} \right)^c \left( 1 - \frac{d}{n} \right)^{n_{k-1} - c} \leq n \binom{n-1}{k-1} \frac{d}{n} \left( 1 - \left( \frac{n-1}{k-1} - c \right) \frac{d}{n_{k-1}} \right) \leq n \frac{d^c e^{-d}}{c!} \left( 1 + O \left( \frac{1}{n^{k-1}} \right) \right).$$

Note that in the first inequality above we used the fact that for every any pair of positive integers $\alpha, \beta$, we have $\binom{\alpha}{\beta} < \frac{\alpha^\beta}{\beta!}$.

To show concentration of $X_c$ around its expectation, we will use Chebyshev’s inequality. In order to do so, we need to estimate the second moment of $X_c$ and, in particular, $\Pr[\deg(1) = \deg(2) = c]$. Letting $p = \frac{d}{C_{k-1}}$ and $C_{c,d}$ be a sufficiently large constant, we see that $\Pr[\deg(1) = \deg(2) = c]$ equals

$$\sum_{\ell=0}^{c} \binom{n-2}{k-2} p^\ell (1-p)^{(n-2)-(c-\ell)} \left( \binom{n-1}{k-1} - \binom{n-2}{k-2} \right) p^{c-\ell} (1-p)^{n_{k-1}-(n_{k-2})-(c-\ell)}$$

$$\leq \sum_{\ell=0}^{c} \binom{n-2}{k-2} p^\ell (1-p)^{(n-2)-(c-\ell)} \left( C_{c,d} \frac{n-1}{k-1} \right) p^{c-\ell} (1-p)^{(n_{k-1}-(n_{k-2})-(c-\ell)}$$

$$= \left( \binom{n-1}{k-1} p^c (1-p)^{(n-1)-(c-\ell)} \right)^2 \sum_{\ell=0}^{c} C_{c,d}^2 \binom{n-2}{k-2} p^\ell (1-p)^{(n_{k-2})+(c-\ell)}$$

$$= \Pr[\deg(1) = c] \cdot \Pr[\deg(2) = c] \left( 1 + O \left( \frac{1}{n} \right) \right).$$

Note that in deriving (57) we used that for any $\ell \in \{1, \ldots, c-1\}$:

$$\left( \frac{n-1}{k-1} - \binom{n-2}{k-2} \right) \cdot p^{-\ell} \leq \left( \binom{n-1}{k-1} - \binom{n-2}{k-2} \right) \cdot \left( \frac{n_{k-1}}{d} \right)^\ell$$

$$\leq \frac{(n_{k-1})^c}{(c-\ell)d^c} e^{-\ell} \cdot \left( \frac{n_{k-1}}{d} \right)^\ell$$

$$= \frac{(n_{k-1})^c}{(c-\ell)d^c} e^{-\ell} \leq C_{c,d} \cdot \left( \frac{n_{k-1}}{c} \right),$$
for large enough $C_{c,d}$. Moreover, in deriving (58) we use that for every $\ell \in \{0, \ldots, c\}$

$$C_{c,d}^{2} \left( \frac{(n-2)^{\ell}}{\ell!} \right) p^{\ell} = O \left( n^{\ell(k-2)} \cdot \frac{1}{n^{\ell(k-1)}} \right) \begin{cases} O(1) & \text{if } \ell = 0, \\ O(1/n) & \text{otherwise,} \end{cases}$$

and that:

$$(1 - p)^{-\binom{n-2}{k-2} + \ell} \leq \exp \left( \frac{d(n-2)}{\binom{n-2}{k-2}} \right) = 1 + O \left( \frac{1}{n} \right).$$

Therefore, letting $I_{j}$ denote the indicator random variable which equals 1 if vertex $j$ has degree $c$ and 0 otherwise,

$$\Var[X_{c}] = \mathbb{E} \left[ X_{c}^{2} \right] - (\mathbb{E}[X_{c}])^{2}$$

$$= \mathbb{E} \left( \sum_{j=1}^{n} I_{j} \right)^{2} - (\mathbb{E}[X_{c}])^{2}$$

$$= \mathbb{E} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} I_{i} I_{j} \right) - (\mathbb{E}[X_{c}])^{2}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (\Pr[\deg(i) = c, \deg(j) = c] - \Pr[\deg(1) = c] \Pr[\deg(2) = c])$$

$$\leq \sum_{i=1}^{n} \sum_{j \in [n] \setminus \{i\}} (\Pr[\deg(i) = c, \deg(j) = c] - \Pr[\deg(1) = c] \Pr[\deg(2) = c])$$

$$+ \sum_{i=1}^{n} \Pr[\deg(i) = c]$$

$$\leq A n$$

for some constant $A = A(c, d)$. Note that in deriving (62) we used (58) and that $\sum_{i=1}^{n} \Pr[\deg(i) = c] = \mathbb{E}[X_{c}] = O(n)$ according to (56).

Finally, applying the Chebyshev’s inequality, we obtain that, for any $t > 0$,

$$\Pr \left[ |X_{c} - \mathbb{E}[X_{c}]| \geq t \sqrt{n} \right] \leq \frac{A}{t^{2}},$$

and, thus, the proof is concluded by choosing $t = \log n$. \hfill \qed

Lemma 5.2 implies the following useful corollary.

**Corollary 5.3.** For any constants $\delta \in (0, 1), k \geq 2, d > 0$, let $X = X(\delta, k, d)$ denote the random variable equal to the number of vertices in $H(k, n, d/\binom{n}{k-1})$ whose degree is in $[(1 + \delta)d, 3(k - 1)^{k-1}d]$. There exists a constant $d_{\delta} > 0$ such that if $d \geq d_{\delta}$ then, asymptotically almost surely, $X \leq \frac{A}{d^{2}}$.

**Proof.** Let $X_{r}$ denote the number of vertices of degree $r$ in $H(k, n, d/\binom{n}{k-1})$. Since $k, d$ are constants, using Lemma 5.2 and Stirling’s approximation we see that, asymptotically almost surely,

$$\sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d} X_{r} \leq n \left( 1 + O \left( \frac{\log n}{\sqrt{n}} \right) \right) \sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d} \frac{d^{r} e^{-d}}{r!} \leq n(1 + \delta) \sum_{r=(1+\delta)d}^{3(k-1)^{k-1}d} \frac{d^{r} e^{-d}}{\sqrt{2\pi r \left( \frac{r}{e} \right)^{r}}} \leq \frac{n}{d^{2}},$$

for sufficiently large $d$ and $n$. \hfill \qed
Using Lemma 5.1 and Corollary 5.3 we show that, asymptotically almost surely, only a small fraction of vertices of $H(k, n, d/(\binom{k}{k-1}))$ have degree that significantly exceeds its average degree.

**Lemma 5.4.** For every constants $k \geq 2$ and $\delta \in (0, 1)$, there exists $d_{k, \delta} > 0$ such that for any constant $d \geq d_{k, \delta}$, all but at most $\frac{2n}{d^2}$ vertices of the random hypergraph $H(k, n, d/(\binom{n}{k-1}))$ have degree at most $(1 + \delta)d$, asymptotically almost surely.

**Proof.** Corollary 5.3 implies that the number of vertices with degree in the interval $[(1 + \delta)d, 3(k - 1)^{k-1}d]$ is at most $\frac{2n}{d^2}$, for sufficiently large $d$.

Suppose now there are more than $\frac{n}{d^2}$ vertices with degree at least $3(k - 1)^{k-1}d$. Denote by $S$ a set containing exactly $\frac{n}{d^2}$ such vertices. According to Lemma 5.1 asymptotically almost surely, the induced subhypergraph $H[S]$ has at most

$$e(H[S]) \leq \left(\frac{d}{(\ln d)^2}\right)^{\frac{n}{d^2}} |S| = \frac{n}{d^2(\ln d)^{\frac{n}{d^2}}}$$

hyperedges. Therefore, the number of hyperedges between the sets of vertices $S$ and $V \setminus S$ is at least

$$3(k - 1)^{k-1}d|S| - ke(H[S]) \geq \frac{2.9(k - 1)^{k-1}n}{d} = N.$$

for sufficiently large $d$. However, the probability that $H(k, n, d/(\binom{n}{k-1}))$ contains such a subhypergraph is at most

$$\left(\frac{n}{d^2}\right)\binom{n}{k-1} \left(\frac{d}{n}\right)^k \left(\frac{d}{n}\right)^{k-1} \leq (ed^2)^{\frac{n}{d^2}} \left(\frac{n^k e}{d^2 N} \cdot \frac{d}{n^{k-1}}\right)^N = o(1),$$

for sufficiently large $d$. Note that in deriving the final equality we used that for any pair of integers $\alpha, \beta$, we have that $\binom{\alpha}{\beta} \geq \binom{\alpha}{\beta}$. Therefore, asymptotically almost surely there are at most $\frac{n}{d^2}$ vertices in $G$ with degree greater than $3(k - 1)^{k-1}d$, concluding the proof. \qed

Finally, we show that the neighborhood of a typical vertex of $H(k, n, d/(\binom{n}{k-1}))$ is locally tree-like.

**Lemma 5.5.** For every constants $k \geq 2, \delta \in (0, 1)$, asymptotically almost surely, the random hypergraph $H(k, n, d/(\binom{n}{k-1}))$ has a subset $U \subseteq V(H)$ of size at most $n^{1-\delta}$ such that the induced hypergraph $H[V \setminus U]$ is of girth at least 5.

**Proof.** Let $Y_2, Y_3, Y_4$, denote the number of 2-, 3- and 4-cycles in $H(n, k, d/(\binom{n}{k-1}))$, respectively. A straightforward calculation reveals that for $i \in \{2, 3, 4\}$:

$$\mathbb{E}[Y_i] \leq \sum_{s=1}^{i(k-1)} \binom{n}{s} \binom{s}{k-1} \binom{d}{n}^i \leq i(k-1) \left(\frac{n \cdot e}{i(k-1)}\right)^{i(k-1)} \left(\frac{i(k-1)}{i}\right)^i \left(\frac{d}{n^{k-1}}\right)^i \leq i(k-1) \left(\frac{n \cdot e}{i(k-1)}\right)^{i(k-1)} \left(\frac{i \cdot e^{k-1}}{i}\right)^i \left(\frac{d(k-1)^{k-1}}{n^{k-1}}\right)^i = i(k-1) \left(\frac{e^{2(k-1)d}}{i}\right)^i = O(1).$$

By Markov’s inequality this implies that $Y_2 + Y_3 + Y_4 \leq n^{1-\sqrt{3}}$ asymptotically almost surely. Denote by $U$ the union of all 2-, 3- and 4-cycles in $H$. Then the induced subhypergraph $H[V \setminus U]$ has girth at least 5 and, asymptotically almost surely, $|U| \leq n^{1-\delta}$. \qed
We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Our goal will be to find a subset \( U \subset V \) of size \(|U| \leq nd^{-k^{-1}}\) that (i) contains all cycles of length at most 4 and every vertex of degree more than \((1 + \delta)d\); and (ii) such that, every vertex \( v \) in \( V \setminus U \) has at most \( 9k^2 \left( \frac{d}{(\ln d)^2} \right)^{\frac{1}{k-1}} \) neighbors in \( U \). Note that in this case, according to Lemma 5.1, \( H[U] \) is \( k \left( \frac{d}{(\ln d)^2} \right)^{\frac{1}{k-1}} \)-degenerate, concluding the proof assuming \( d \) is sufficiently large. A similar idea has been used in [5, 6, 25].

Towards that end, let \( U_1 \) be the set of vertices of degree more than \((1 + \delta)d\), and \( U_2 \) the set of vertices that are contained in a 2-, 3- or a 4-cycle. Notice that \( U_1, U_2 \), can be found in polynomial time and, according to Lemmas 5.4 and 5.5, the size of \( U_0 := |U_1 \cup U_2| \) is at most \( 3n d^2 \) for sufficiently large \( n \) and \( d \).

We now start with \( U := U_0 \) and as long as there exists a vertex \( v \in V \setminus U \) having at least \( 9k^2 \left( \frac{d}{(\ln d)^2} \right)^{\frac{1}{k-1}} \) neighbors in \( U \) we do the following. Let \( S_v = \{u_1, u_2, \ldots, u_N\} \) be the neighbors of \( v \) in \( U \). We choose an arbitrary hyperedge \( h \) that contains \( v \) and \( u_1 \) and update \( U \) and \( S_v \) by defining \( U := U \cup h \) and \( S_v := S_v \setminus h \). We keep repeating this operation until \( S_v \) is empty.

This process terminates with \( |U| < nd^{-k^{-1}} \) because, otherwise, we would get a subset \( U \subset V \) of size \(|U| = nd^{-k^{-1}}\) spanning more than

\[
\frac{1}{k} \left( \frac{n}{d^{k-1}} - |U_0| \right) \times 9k^2 \left( \frac{d}{(\ln d)^2} \right)^{\frac{1}{k-1}} \times \frac{1}{k} > \frac{n}{d^{k-1}} \times \left( \frac{d}{(\ln d)^2} \right)^{\frac{1}{k-1}}
\]

hyperedges, for sufficiently large \( d \). According to Lemma 5.1 however, \( H \) does not contain any such set asymptotically almost surely.

\[\square\]

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A Proofs omitted from Section 4

In this section we prove Lemmas 4.1, 4.3 and Lemma 3.4 Part (b).

A.1 Proof of Lemma 4.1

We proceed by induction. The case $i = 1$ is straightforward to verify since $R_{1,r} = 0$ for every $r \in [k - 2]$, while $R_{1,k-1} = \frac{\ln \Delta}{(1+\delta)^k}$. Therefore, we inductively assume the claim for $i$, and consider the case $i+1$. Note that the inductive hypothesis implies that $\text{Keep}_i = \Omega(1)$ since $1 - \frac{1}{x} \geq e^{-\frac{1}{x+1}}$ for every $x \geq 2$ and, thus,

$$\text{Keep}_i \geq \exp \left( -\sum_{r=1}^{k-1} \frac{T_{i,r}}{\alpha^{-1} L_i} r - 1 \right) \geq \exp \left( -\frac{Kk^{2(k-2)}}{1 - \frac{3}{100k}} \right),$$

(59)
In deriving (61) we used the inductive hypothesis, and the that

\[ R_{i+1,r} = \sum_{j=r}^{k-1} \left( \frac{T_{i,j}}{L_{i+1}^r} \cdot (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left( \frac{j}{r} \right) \left( \frac{\text{Keep}_i}{L_i} \right)^{j-r} \right) \]

\[ + \frac{1}{L_{i+1}^r} \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{-r} \frac{T_{i,j}}{L_i^{2-r}} \right)^{2/3} + 4k^2(k-r) \alpha^{-r+1} \left( \frac{L_i}{L_{i+1}} \right)^{r} \ln \Delta \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^\ell (\ln \Delta)^{2\ell}} \]

\[ = \sum_{j=r}^{k-1} \left( \frac{T_{i,j}}{L_i^r \left( \text{Keep}_i - L_i^{-1/3} \right)^r} \cdot (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left( \frac{j}{r} \right) \left( \frac{\text{Keep}_i}{L_i} \right)^{j-r} \right) + O \left( \frac{1}{\ln \Delta} \right) \]

\[ \leq \sum_{j=r}^{k-1} \left( 1 - \frac{\alpha \text{Keep}_i}{2} \right)^r \left( R_{i,j} + \sum_{j=r+1}^{k-1} \left( \frac{j}{r} \right) R_{i,j} \alpha^{j-r} \right) + O \left( \frac{1}{\ln \Delta} \right) \]

\[ \leq \left( 1 - \frac{\alpha \text{Keep}_i}{2} \right) \left( k^2(k-1-r) \ln \Delta + \sum_{j=r+1}^{k-1} \left( \frac{j}{r} \right) k^2(k-1-j) K^{j-r} \right) + O \left( \frac{1}{\ln \Delta} \right) \]

\[ \leq \left( 1 - \frac{\alpha \text{Keep}_i}{2} \right) \left( k^2(k-1-r) \ln \Delta + k^2(k-1-(r+1)) (r+1) K \right) + O \left( \frac{1}{\ln \Delta} \right) \]

\[ \leq k^2(k-1-r) \ln \Delta - K \left( \text{Keep}_i k^2(k-1-r) \right)^{2} - k^2(k-1-(r+1)) (r+1) \right) + O \left( \frac{1}{\ln \Delta} \right) \]

\[ \leq k^2(k-1-r) \ln \Delta, \] (62)

for sufficiently large \( \Delta \), concluding the proof. Note that in deriving (60) we used the inductive hypothesis and that \( L_i \geq (\ln \Delta)^{20(k-1)} \) to obtain:

\[ \frac{1}{L_{i+1}^r} \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{j-r} \frac{T_{i,j}}{L_i^{2-r}} \right)^{2/3} = \left( \frac{1}{L_{i+1}^r} \right)^{1/3} \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{j-r} \frac{T_{i,j}}{L_i^{2-r}} \right)^{2/3} = O \left( \frac{1}{\ln \Delta} \right), \]

\[ 4k^2(k-r) \alpha^{-r+1} \left( \frac{L_i}{L_{i+1}} \right)^{r} \ln \Delta \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^\ell (\ln \Delta)^{2\ell}} = 4k^2(k-r) \ln \Delta \cdot \alpha^{-r+1} \left( \frac{L_i}{L_{i+1}} \right)^{r} \sum_{\ell=1}^{k-1} \frac{R_{i,\ell}}{L_i^\ell (\ln \Delta)^{2\ell}} \]

\[ = O \left( \frac{1}{\ln \Delta} \right). \]

In deriving (61) we used the facts that \( \text{Keep}_i = \Omega(1) \), \( L_i, T_{i,r} \geq (\ln \Delta)^{20(k-1)} \). In deriving (63) we used the fact that \( K = (100)^{3k} \cdot k^{-1} \) and, therefore, the second term in (62) is a negative constant, the inductive hypothesis, and the that \( L_i \geq (\ln \Delta)^{20(k-1)}, \text{Keep}_i = \Omega(1) \) and \( k \geq 3 \).
A.2 Proof of Lemma 4.3

We proceed by induction. The base cases are easy to verify since $R'_{i,r} = 0$ for every $r \in [k - 2]$ and $R'_{i,k-1} = \frac{\ln \Delta}{(1 + \delta)^{k-1}}$.

We first focus on the case $r = k - 1$. We assume that the claim is true for $i - 1$ and consider $i$. Note that the inductive hypothesis, the facts that $L_j \geq (\ln \Delta)^{20(k-1)}$ for every $1 < j < i$ and $\text{Keep}_j \geq C$, imply:

$$4k^2(k-r)\alpha^{-r+1} \left( \frac{L_i}{L_{i+1}} \right)^r \ln \Delta \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i^{2\ell}(\ln \Delta)^{2\ell}} \leq \frac{1}{(\ln \Delta)^{10(k-1)}},$$

for sufficiently large $\Delta$, for every $r \in [k-1]$ and $1 < j \leq i$. Therefore, recalling (6), (7) and using (64), we have:

$$R'_{i,k-1} \leq R'_{i-1,k-1}(1 - \alpha \text{Keep}_{i-1})^{k-1} + \frac{1}{(\ln \Delta)^{10(k-1)}}$$

$$\leq R'_{i-2,k-1}(1 - \alpha \text{Keep}_{i-2})^{k-1}(1 - \alpha \text{Keep}_{i-1})^{k-1} + \frac{1}{(\ln \Delta)^{10(k-1)}} + \frac{1}{(\ln \Delta)^{10(k-1)}}$$

$$\leq R'_{i-2,k-1}(1 - \alpha C)^{2(k-1)} + \frac{1}{(\ln \Delta)^{10(k-1)}} + \frac{1}{(\ln \Delta)^{10(k-1)}}$$

$$\leq \ldots$$

$$\leq (1 - \alpha C)^{(i-1)(k-1)} R_{1,k-1} + \frac{1}{(\ln \Delta)^{10(k-1)}} \sum_{\ell=0}^{i-1} (1 - \alpha C)^{(k-1)\ell}$$

$$\leq (1 - \alpha C)^{(i-1)(k-1)} \frac{\ln \Delta}{(1 + \delta)^{k-1}} + \frac{1}{(\ln \Delta)^{5(k-1)}}$$

$$\leq (1 - \alpha C)^{(i-1)(k-1)} \frac{\ln \Delta}{(1 + \delta - \frac{\delta}{k^{10}})^{k-1}},$$

for sufficiently large $\Delta$, concluding the proof for the case $r = k - 1$.

We now focus on $r \in [k - 2]$. We first observe that

$$R'_{2,r} \leq \sum_{j=r}^{k-1} \left( \frac{T_{i,j}'}{L_i'} \right)^r (\text{Keep}_1 (1 - \alpha \text{Keep}_1))^r \left( \frac{j}{r} \right) \left( \frac{\alpha \text{Keep}_1}{L_i'} \right)^{j-r} + \frac{1}{(\ln \Delta)^{10(k-1)}}$$

$$= R'_{i,k-1} \cdot (\text{Keep}_1 (1 - \alpha \text{Keep}_1))^r \left( \frac{k-1}{r} \right) (\alpha \text{Keep}_1)^{k-1-r} + \frac{1}{(\ln \Delta)^{10(k-1)}}$$

$$\leq \frac{(\ln \Delta)^{-(k-2-r)}}{(1 + \delta)^{k-1}} K^{k-1-r} \left( \frac{k-1}{r} \right) + \frac{1}{(\ln \Delta)^{10(k-1)}}$$

$$\leq \frac{(1 + \delta)^{k-1-r} (\ln \Delta)^{-(k-2-r)}}{(1 + \delta - \frac{\delta}{k^{10}})^{k-1}} K^{k-1-r} \prod_{p=r}^{k-2} (p+1),$$

concluding the proof of the base cases.

Assume that the claim holds for all pairs $(r', i')$, where $r' \in \{r, \ldots, k-1\}$ and $i' \leq i - 1$. It suffices to
prove that it also holds for any pair \((r, i)\), where \(i > 2\) and \(r \in [k - 2]\). To see this, observe that

\[
R'_{i, r} \leq \sum_{j=r}^{k-1} \left( \frac{T'_{i-1,j}}{(L'_{i-1})^r} \cdot \left( \text{Keep}_{i-1} \left( 1 - \alpha \text{Keep}_{i-1} \right) \right)^j \right) \left( \frac{\alpha \text{Keep}_{i-1}}{L'_{i-1}} \right)^{j-r} + \frac{1}{(\ln \Delta)^{10(k-1)}}
\]

\[
= \sum_{j=r}^{k-1} \left( \frac{T'_{i-1,j}}{(L'_{i-1})^r} \cdot \text{Keep}_{i-1} \left( 1 - \alpha \text{Keep}_{i-1} \right)^j \alpha^{j-r} \right) + \frac{1}{(\ln \Delta)^{10(k-1)}}
\]

\[
= (1 - \alpha \text{Keep}_{i-1})^r \sum_{j=r}^{k-1} \left( R'_{i-1,j} \left( \frac{j}{r} \right) \alpha^{j-r} \right) + \frac{1}{(\ln \Delta)^{10(k-1)}}
\]

\[
\leq (1 - \alpha C)^r \sum_{j=r}^{k-1} \left( R'_{i-1,j} \left( \frac{j}{r} \right) \alpha^{j-r} \right) + \frac{1}{(\ln \Delta)^{10(k-1)}}
\]

\[
\leq (1 - \alpha C)^r R'_{i-2, r} + \sum_{j=r+1}^{k-1} \left( \frac{j}{r} \right) K^{j-r} (\ln \Delta)^{1-(j-r)} \left( 1 - \alpha C \right)^{j(2)+r} \frac{(1 + \frac{\delta}{k^p r})^{k-1-j}}{1 + \frac{\delta}{k^p r} k^{r-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1)
\]

\[
+ \frac{1 + (1 - \alpha C)^r}{(\ln \Delta)^{10(k-1)}} \qquad (65)
\]

\[
\leq (1 - \alpha C)^3 r R'_{i-3, r} + \sum_{j=r+1}^{k-1} \left( \frac{j}{r} \right) K^{j-r} (\ln \Delta)^{1-(j-r)} \left( 1 - \alpha C \right)^{j(3)+2r} \frac{(1 + \frac{\delta}{k^p r})^{k-1-j}}{1 + \frac{\delta}{k^p r} k^{r-1} C^{k-1-j}} \prod_{p=j}^{k-2} (p+1)
\]

\[
+ \frac{1 + (1 - \alpha C)^r}{(\ln \Delta)^{10(k-1)}} \qquad (66)
\]
\[ \leq \ldots \]
\[ \leq (1 - \alpha C)^{(i-1)r} R_{1,r}' \]
\[ + \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r}(\ln \Delta)^{1-(j-r)} \prod_{p=j}^{k-2} (p+1) \frac{(1 + \delta)_{k-1-j}}{(1 + \delta - \frac{\alpha}{k})_{k-1-j}} \sum_{\ell=0}^{i-1} (1 - \alpha C)^{j(\ell-1)+\ell r} \]
\[ + O \left( \frac{1}{(\ln \Delta)^{5(k-1)}} \right) \]
\[ \leq \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r}(\ln \Delta)^{1-(j-r)} \prod_{p=j}^{k-2} (p+1) \frac{(1 + \delta)_{k-1-j}}{(1 + \delta - \frac{\alpha}{k})_{k-1-j}} \sum_{\ell=0}^{i-1} (1 - \alpha C)^{(i-1)r+(\ell-1)(j-r)} \]
\[ + O \left( \frac{1}{(\ln \Delta)^{5(k-1)}} \right) \]
\[ \leq \frac{(1 - \alpha C)^{(i-1)r}}{(1 + \delta - \frac{\alpha}{k})_{k-1}} \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r}(\ln \Delta)^{1-(j-r)} \prod_{p=j}^{k-2} (p+1) \frac{(1 + \delta)_{k-1-j}}{C^{k-1-j}} \sum_{\ell=0}^{i-1} (1 - \alpha C)^{(i-1)r+(\ell-1)(j-r)} \]
\[ = \left( \frac{1 - \alpha C)^{(i-1)r}}{(1 + \delta - \frac{\alpha}{k})_{k-1}} \right) \sum_{j=r+1}^{k-1} \binom{j}{r} K^{j-r}(\ln \Delta)^{1-(j-r)} \prod_{p=j}^{k-2} (p+1) \frac{(1 + \delta)_{k-1-j}}{C^{k-1-j}} \sum_{\ell=0}^{i-1} (1 - \alpha C)^{(i-1)r+(\ell-1)(j-r)} \]
\[ \leq (1 - \alpha C)^{(i-1)r} \ln \Delta \cdot \frac{(1 + \delta)_{k-1-r}}{(1 + \delta - \frac{\alpha}{k})_{k-1}} \sum_{p=r}^{k-2} (p+1) \]
for sufficiently large \( \Delta \), concluding the proof. Note that in order to get (66) we upper bound \( R_{i-1,r}' \) in the same way we upper bounded \( R_{i-2,r}' \). We keep using the same steps to bound \( R_{i-2,r}', R_{i-3,r}', \ldots \) until we get (67). In deriving (68) we used that \( R_{1,r} = 0 \) for every \( r \in [k-2] \). In going from (69) to (70) we start the summation in the last term from \( \ell = 0 \) instead from \( \ell = 1 \) in order to subsume the \( O(1/(\ln \Delta)^{5(k-1)}) \) error term. Finally, in going from (71) to (72) we multiply the term that corresponds to \( j = r + 1 \) in the summation by \( (1 + \frac{\delta}{k})_{k-1-r} \) in order to subsume the terms of the summation that correspond to \( j > r + 1 \).

**A.3 Proof of Lemma 3.4, Part(b)**

We observe that it suffices to show that \( T_{i,r}' \geq T_{i,r} - \frac{100r}{100r+1} (T_{i,r}')^{100r+1} \) (since \( T_{i,r} \geq T_{i,r}' \) for every \( i \) by definition) and proceed by using induction. Again, the base case is trivial, so we assume the statement is true for \( i \), and
consider $i + 1$. We obtain the following.

\[
T_{i+1,r} = \sum_{j=r}^{k-1} \left( T_{i,j} \cdot (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left( \frac{j}{r} \right) \left( \frac{\alpha \text{Keep}_i}{L_i'} \right)^{j-r} \right)
\]

\[
+ 4k^{2(k-r)} \alpha (\alpha^{-1} L_i)^r \ln \Delta \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i'^{2\ell} (\ln \Delta)^{2\ell}}
\]

\[
\geq \sum_{j=r}^{k-1} \left( T_{i,j} - (T_{i,j})^{100r+1} \right) (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left( \frac{j}{r} \right) \left( \frac{\alpha \text{Keep}_i}{L_i'} \right)^{j-r}
\]

\[
+ 4k^{2(k-r)} \alpha (\alpha^{-1} L_i)^r \ln \Delta \sum_{\ell=1}^{k-1} \frac{T_{i,\ell}}{L_i'^{2\ell} (\ln \Delta)^{2\ell}}
\]

\[
\geq T_{i+1,r} - \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{j-r} \frac{T_{i,j}}{L_i'^{j-r}} \right)^{2/3}
\]

\[
- \sum_{j=r}^{k-1} T_{i,j} (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r (\alpha \text{Keep}_i)^{j-r} \left( \frac{1}{L_i'^{j-r}} - \frac{1}{(L_i')^{j-r}} \right)
\]

\[
- \sum_{j=r}^{k-1} (T_{i,j})^{100r+1} (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left( \frac{j}{r} \right) \left( \frac{\alpha \text{Keep}_i}{L_i'} \right)^{j-r}
\]

\[
\geq T_{i+1,r} - \left( \sum_{j=r}^{k-1} \left( \frac{j}{r} \right) \alpha^{j-r} \frac{T_{i,j}}{L_i'^{j-r}} \right)^{2/3}
\]

\[
- \sum_{j=r+1}^{k-1} (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r (\alpha \text{Keep}_i)^{j-r} \frac{T_{i,j}}{L_i'^{j-r}} O \left( \frac{L_i^{-1/6}}{L_i} \right)
\]

\[
- (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \frac{r}{100r+1} (T_{i+1,r})^{100r+1}
\]

\[
(73)
\]

\[
\geq T_{i+1,r} - \left( 1 - \frac{K^{k-1}}{1300 \alpha^{k} (\ln \Delta)^{k-1}} \right) (T_{i+1,r})^{100r}
\]

\[
- O \left( \sum_{j=r+1}^{k-1} \alpha^{j-r} R_{i,j} L_i'^{r-1/6} + T_{i,r}^{2/3} + \left( \sum_{j=r+1}^{k-1} \alpha^{j-r} R_{i,j} L_i'^{r} \right)^{2r/3} \right)
\]

\[
(74)
\]

\[
\geq T_{i+1,r} - (T_{i+1,r})^{100r+1},
\]

\[
\]

(75)

for sufficiently large $\Delta$, concluding the proof of the lemma. Note that in deriving (74) we used the first part of Lemma 3.3, i.e., the fact that $L_i' - L_i \leq (L_i')^{5/6}$ to obtain

\[
1 - \frac{L_i}{L_i'} \leq \left( \frac{L_i'}{L_i} \right)^{5/6} = \left( \frac{1}{L_i} \right)^{1/6}
\]
and the fact that
\[
\sum_{j=r}^{k-1} (T'_{i,j})_{\frac{100r}{100r+1}} (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left(\frac{j}{r}\right) \left(\frac{\alpha \text{Keep}_i}{L_i'}\right)^{j-r}
\]
\[
= (\text{Keep}_i (1 - \alpha \text{Keep}_i))_{\frac{100r}{100r+1}} \sum_{j=r}^{k-1} (T'_{i,j})_{\frac{100r}{100r+1}} (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left(\frac{j}{r}\right) \left(\frac{\alpha \text{Keep}_i}{L_i'}\right)^{j-r}
\]
\[
\leq (\text{Keep}_i (1 - \alpha \text{Keep}_i))_{\frac{100r}{100r+1}} \sum_{j=r}^{k-1} (T'_{i,j})_{\frac{100r}{100r+1}} (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left(\frac{j}{r}\right) \left(\frac{\alpha \text{Keep}_i}{L_i'}\right)^{j-r}
\]
\[
\leq (\text{Keep}_i (1 - \alpha \text{Keep}_i))_{\frac{100r}{100r+1}} \left( \sum_{j=r}^{k-1} (T'_{i,j}) (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left(\frac{j}{r}\right) \left(\frac{\alpha \text{Keep}_i}{L_i'}\right)^{j-r} \right)
\]
\[
= (\text{Keep}_i (1 - \alpha \text{Keep}_i))_{\frac{100r}{100r+1}} (T'_{i,r+1} - 4k^{2(k-r)}\alpha(\alpha^{-1}L_i)^r \ln \Delta \sum_{\ell=1}^{k-1} L_i^{2\ell}(\ln \Delta)^2\ell)_{\frac{100r}{100r+1}}
\]
\[
< (\text{Keep}_i (1 - \alpha \text{Keep}_i))_{\frac{100r}{100r+1}} (T'_{i,r+1})_{\frac{100r}{100r+1}}
\]
for sufficiently large \(\Delta\). In deriving (75) we used Lemma 4.1 to obtain that
\[
\left(\sum_{j=r}^{k-1} \left(\frac{j}{r}\right) \alpha^{j-r} \frac{T_{i,j}}{L_i'^{-1}}\right)^{2/3} = \left( T_{i,r} + \sum_{j=r+1}^{k-1} \left(\frac{j}{r}\right) \alpha^{j-r} R_{i,j} L_i^r \right)^{2/3}
\]
\[
= O \left( T_{i,r}^{2/3} + \left( \sum_{j=r+1}^{k-1} \alpha^{j-r} R_{i,j} L_i^r \right)^{2r/3} \right)
\]
and, similarly, that
\[
\sum_{j=r+1}^{k-1} (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left(\frac{j}{r}\right) \left(\frac{\alpha \text{Keep}_i}{L_i'}\right)^{j-r} O \left( L_i^{-\frac{1}{2}}\right) = O \left( \sum_{j=r+1}^{k-1} \alpha^{j-r} R_{i,j} L_i^{r-1/6} \right).
\]
We also used the fact that, using Corollary 3.2, we obtain
\[
(\text{Keep}_i (1 - \alpha \text{Keep}_i))_{\frac{100r}{100r+1}} \leq \left( 1 - \frac{K^{k-1}}{12k^2(\ln \Delta)^{k-1}} \right)_{\frac{100r}{100r+1}} \leq \left( 1 - \frac{K^{k-1}}{1300k^2(\ln \Delta)^{k-1}} \right)
\]
for sufficiently large \(\Delta\). Finally, to derive (76) we observe that:
\[
\left( T_{i+1,r} \right)^{\frac{100r}{100r+1}} \left( \frac{1}{1300k^2(\ln \Delta)^{k-1}} \sum_{j=r}^{k-1} \left( T'_{i,j} \left(\frac{j}{j}\right) (\text{Keep}_i (1 - \alpha \text{Keep}_i))^r \left(\frac{\alpha \text{Keep}_i}{L_i'}\right)^{j-r} \right) \right)^{\frac{100r}{100r+1}}
\]
\[
= \Omega \left( \frac{1}{1300k^2(\ln \Delta)^{k-1}} \left( T_{i,r} + \sum_{j=r+1}^{k-1} \alpha^{j-r} R_{i,j} L_i^r \right)^{\frac{100r}{100r+1}} \right)
\]
\[
= \omega \left( \sum_{j=r+1}^{k-1} \alpha^{j-r} R_{i,j} L_i^{r-1/6} + T_{i,r}^{2/3} + \left( \sum_{j=r+1}^{k-1} \alpha^{j-r} R_{i,j} L_i^r \right)^{2r/3} \right),
\]
where in deriving (77) we used our assumption that $L_i, T_{i,r} \geq (\ln \Delta)^{20(k-1)}$ and the inductive hypothesis according to which $R'_{i,j}, L'_i$ are within a constant factor (arbitrarily close to 1) of $R_{i,j}, L_i$. 