Exact solutions for a type of electron pairing model with spin-orbit interactions and Zeeman coupling

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(Dated: March 28, 2011)

A type of electron pairing model with spin-orbit interactions or Zeeman coupling is solved exactly in the framework of Richardson ansatz. Based on the exact solutions obtained, we show rigorously that the pairing symmetry is of the $p$+$ip$-wave regardless of the strength of pairing interaction, as expected by the mean field theory. Intriguingly, how Majorana fermions can emerge in the system is also elaborated. Exact analytical results are illustrated for two simple systems respectively with spin-orbit interactions and Zeeman coupling.

PACS numbers: 71.70.Ej, 71.10.-w, 74.90.+n

Recently, significant research attentions have been paid to various physical systems with spin-orbit interactions, including Quantum Spin Hall Effects,[1,3], topological insulators[4], semiconductor heterostructures[5], and a number of artificial systems like ultra-cold atoms in optical lattices[6,7]. In particular, several important theoretical understandings have been obtained for pairing electrons in the presence of spin orbit interactions[4,8]. Nevertheless, all of these theoretical investigations on pairing systems have been conducted in the framework of mean field theory, which is known to be a good approximation merely for weak pairing interactions. Therefore, more rigorous theoretical understandings or even exact solutions for these electron pairing systems are highly appreciated, particularly for strong pairing cases, even though it is extremely challenging to find exact solutions of models for interacting many-electron systems. This is a central motivation of this work.

It is noted that Richardson obtained exact solutions of some pairing models in the 1960s[9]. As is known, Richardson’s exact solutions for pairing force models have played an important role in the research of interacting many-particle physics[10], including their connection with the well known BCS model[11].

In this Letter, we first consider a type of electron pairing model with spin-orbit interactions and solve it exactly in the framework of Richardson ansatz. As an illustration, an analytical result is derived for a very simple case. Based on the exact solutions obtained, we show rigorously that the pairing order parameter has always the $p$+$ip$-wave symmetry regardless of the strength of pairing interactions, which recovers an important conclusion deduced from the mean field theory. Then, we address the same model with the Zeeman coupling term[12]. Remarkably, we are also able to find an exact solution in the presence of a pure Zeeman term with the same scenario. Exact analytical results are presented for a special electron system. Moreover, we also elaborate how Majorana fermions can emerge in the system.

Let us consider a pairing electron Hamiltonian with spin-orbit interactions in a two dimensional lattice, which may be written as

$$H = H_0 + H_{int},$$

with

$$H_0 = \sum_k (c_{k\uparrow}^\dagger c_k^\dagger)(\varepsilon_k + \alpha k \cdot \sigma)(c_k^\uparrow, c_k^\downarrow)^T,$$

$$H_{int} = -\sum_{k,k'} V_0(k,k')c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{k'\uparrow},$$

where $\varepsilon_k$ is the spin-independent single electron energy, $c_{k\uparrow}^\dagger$ and $c_{k\downarrow}$ are the creation and annihilation operators of electrons, $k = (k_x, k_y)$ is the wave vector of the lattice[13], $\alpha$ is the effective strength of spin-orbit interaction, and $\sigma = (\sigma_x, \sigma_y)$ are the pauli matrices. We here consider an $s$-wave pairing interaction, namely $V_0(k,k') = V_0 > 0$. Although the above Hamiltonian has been studied recently under various mean field approximations, to the best of our knowledge, it has not been solved exactly. Motivated by this, we here want to find an exact solution in the framework of Richardson ansatz. We first diagonalize the single-particle Hamiltonian by making the following unitary transformation,

$$c_k^\uparrow = \frac{1}{\sqrt{2}}(a_{k,+} e^{-i\theta(k)} a_{k,-}),$$

$$c_k^\downarrow = \frac{1}{\sqrt{2}}(e^{i\theta(k)} a_{k,+} - a_{k,-}),$$

with $e^{i\theta(k)} = (k_x + ik_y)/|k|$[14] for $k \neq 0$ and $e^{i\theta(0)} = 1$ for $k = 0$. Physically, this unitary transformation corresponds to a local spin-basis rotation to align the spin-direction along the wave vector $k$, which actually introduces an effective local gauge field acting on electrons. The Hamiltonian is now rewritten as

$$H = \sum_{k,s} \varepsilon_{k,s} a_{k,s}^\dagger a_{k,s} - \frac{V_0}{4} \sum_{k,k',s'} e^{-is\theta(k)+is'\theta(k')} (A_{k,s}^\dagger - \delta_{k,0} A_{0,s}^\dagger)(A_{k',s'} - \delta_{k,0} A_{0,s'}).$$
where the dispersion $\varepsilon_{k,s} = \varepsilon_{k} + sak$ with $s = \pm 1$ denoting the two branches of the diagonalized single-particle spectrum in the new basis. Here, the pairing operators are defined by

$$A_{k,s}^\dagger \equiv a_{k,s}^\dagger a_{-k,-s}^\dagger (s = \pm), \quad A_{0,0}^\dagger = a_{0,+}^\dagger a_{0,-}^\dagger. \quad (4)$$

In derivation of the above Eq. (4), we have employed a useful relation $\theta(k) - \theta(-k) = \pm \pi$ for $k \neq 0$. It is obviously seen from Eq. (4) that $A_{k,s}^\dagger = 0$ for $k = 0$. The above operators satisfy the following commutation relations:

$$[A_{k,s}, A_{k',s'}^\dagger] = \delta_{k,k'} \delta_{s,s'} (1 - 2 A_{k,s}^\dagger A_{k,s}), \quad (5)$$

$$[A_{k,s}^\dagger A_{k',s'}, A_{k,s'}^\dagger] = \delta_{k,k'} \delta_{s,s'} A_{k,s}^\dagger$$

for $k$ or $k' \neq 0$, and

$$[A_{0,0}, A_{0,0}^\dagger] = 1 - 2 A_{0,0}^\dagger A_{0,0}, \quad (6)$$

for $k = k' = 0$. These relations play a crucial role in solving this model exactly. Although the pairing term in Eq. (3) is $k$-dependent, we still make an ansatz in the same framework of Richardson’s pioneering work on a pairing model [9]. In this framework, the eigenstates of Hamiltonian (3) should take the product form as

$$|n, S_+ , S_- \rangle = \prod_{k_1 \in S_+} \prod_{k_1 \in S_-} a_{k_1, s}^\dagger - \prod_{n=1} a_{k_1, s}^\dagger B_{n}^\dagger, \quad (7)$$

where

$$B_{n}^\dagger = \sum_{s, k_1 \in P_+, k \neq 0} \frac{e^{-i\theta(k)}}{2\varepsilon_{k,s} - E_{n}} A_{k,s}^\dagger + \frac{A_{0,0}^\dagger}{2\varepsilon_{0} - E_{n}}. \quad (8)$$

Remarkably, here we have demonstrated the pairing model of Eq. (3) to be an integrable problem [17], making such pairing model more promising and useful. The set of Eq. (4) is quite similar to Richardson’s one. In particular, when $\alpha = 0$, the two branches are degenerate and Eq. (9) recovers the usual Richardson’s equation [3, 13, 10]. It has been shown by Gaudin that Eq. (9) has a continuum limit form in the thermodynamic limit [13, 10].

Note that there are normally two kinds of spin-orbit interactions: one takes the form of $\mathbf{k} \cdot \mathbf{\sigma}$ as in Eq. (11) with the exact solution being given above, while the other has the form $(\mathbf{\sigma} \times \mathbf{k}) \cdot \hat{z}$ [22]. If the spin-orbit interaction in Eq. (11) is changed to the second form, one can replace $\theta(k)$ in Eqs. (2), (3) and (8) by $\theta(k) = \theta(k) - \pi/2$ and accordingly the pairing model with $(\mathbf{\sigma} \times \mathbf{k}) \cdot \hat{z}$-type spin-orbit interaction is exactly solvable as well.

Although it is still rather challenging to solve Eq. (9) even numerically, the computational loading is significantly reduced in comparison with the numerical exact-diagonalization. In terms of this exact solution for the system described by Hamiltonian (3), we are able to evaluate some quantities exactly and obtain relevant rigorous results, which are very helpful for validating or invalidating the related results based on the usual mean field framework.

As an important example, we use Eq. (7) to calculate exactly the following dimensionless order parameter,

$$\Delta_{k,s} = \frac{(0, 0, n - 1 | a_{-k,s}^\dagger a_{k,s} | (0, 0, 0))}{\sqrt{C_n C_{n-1}}} = e^{-i\theta(k)} \Delta_{k,s}^0 = \sqrt{n C_n C_{n-1}} \sum_{\nu=1}^{n} \sum_{\nu=1}^{n} \frac{g_{\nu}^{n}}{g_{\nu}^{n}} \pi^{-n} \nu \varepsilon_{k} - E_{n}. \quad (11)$$

where $C_n = (0, 0, n | n, 0, 0)$ and

$$\Delta_{k,s}^0 = \frac{1}{\sqrt{C_n C_{n-1}}} \sum_{\nu=1}^{n} \sum_{\nu=1}^{n} \frac{g_{\nu}^{n}}{g_{\nu}^{n}} \pi^{-n} \nu \varepsilon_{k} - E_{n}. \quad (12)$$

with

$$g_{\nu}^{n} = \sum_{\nu=1}^{n} \frac{1}{\pi \nu \varepsilon_{k} - E_{n}} \sum_{\nu=1}^{n} \frac{1}{\pi \nu \varepsilon_{k} - E_{n}}.$$ 

Here $k$ denotes $(k, s)$, the superscript $(n)$ corresponds to the $n$-pair state, and $P$ means the permutation of the corresponding terms. In the weak interaction limit ($V_0 \to 0$), $E_{n}$’s are all real, so that $\Delta_{k,s}^0$ is real as well. It is clearly seen that $\Delta_{k,s}^0$ has the $p_{\nu} + ip_{\nu}$ pairing symmetry. Notably, even when the solutions $E_{n}$ are complex numbers, we can show from Eq. (12) that $\Delta_{k,s}^0$ is still real, because the complex solutions of Eq. (10) appear in the form of conjugate pairs. This finding for the pairing symmetry justifies a result expected by the mean filed theory in the weak interaction limit [13, 14]. In addition, we can also evaluate the off-diagonal long range order

$$O(k', k) = \frac{e^{i\theta(k') - s\theta(k)}}{2\varepsilon_{0} - E_{n}} G(k', k), \quad (13)$$
with \( G(k',k) = C_n^{-1} \sum_{\mu,\nu=1}^{n} \sum_{j,\, j'\neq k}^{n} \sum_{i}^{n} g_{\mu \nu}^{(k')} g_{\mu \nu}^{(k)} \), which is always real and approaches \( \Delta_{k}^{0} \Delta_{k'}^{0} \) in the thermodynamic limit as expected.

For illustration of the complex solutions of Eq. (9), let us look into a toy model with four single-particle states \((k \uparrow, -k \downarrow, -k \uparrow, k \downarrow)\), which accommodate four electrons. In the presence of the spin-orbit interaction, the degenerate states are split into two groups with \( \varepsilon_{k,+} = \varepsilon_{-k,+} = 1 \) (as the energy unit) and \( \varepsilon_{k,-} = \varepsilon_{-k,-} = -1 \).

The set of Richardson equations for this model are two coupled equations, which are solved analytically to obtain,

\[
E_{1,2} = -V_0 \pm \sqrt{4 - V_0^2}. \tag{14}
\]

One can find readily that \( V_0 = 2 \) is a critical value for \( E_{1,2} \) to become complex, while the ground-state energy \( (E_1 + E_2) = -2V_0 \) is always real.

Next we turn to consider the Zeeman term [12] induced by an external magnetic field \( B = (B_x, B_y, B_z) \), which reads \( \hat{H}_Z = \sum_{k}(c_{k\uparrow}^\dagger c_{k\downarrow})B \cdot \sigma(c_{k\uparrow}, c_{k\downarrow})^T \) and is added to Hamiltonian [11]. We now make another transformation as

\[
\begin{align*}
&c_{k\uparrow} = \cos \varphi_{k} a_{k,+} + \sin \varphi_{k} e^{-i\theta_{k}} a_{k,-}, \\
&c_{k\downarrow} = \sin \varphi_{k} e^{i\theta_{k}} a_{k,+} - \cos \varphi_{k} a_{k,-},
\end{align*}
\tag{15}
\]

where

\[
\begin{align*}
&\tan(2\varphi_{k}) = \eta_{k}/B_z, \\
&e^{i\theta_{k}} = \left[ (B_x + \alpha_k x) + i(B_y + \alpha_k y) \right]/\eta_{k}, \\
&\eta_{k} = \sqrt{(B_x + \alpha_k x)^2 + (B_y + \alpha_k y)^2}.
\end{align*}
\tag{16}
\]

The single-particle spectrum still has two branches with \( \varepsilon_{k,s} = \varepsilon_{k} + s/2 \sqrt{\eta_{k}^2 + B_z^2} \). In addition to the operators in Eq. (11), we need new operators defined as

\[
A_{k,0}^\dagger \equiv a_{k,+}^\dagger a_{-k,+}, \quad A_{k,0} \equiv a_{-k,-} a_{k,+}.
\tag{17}
\]

Under this transformation, the total Hamiltonian with Zeeman term can be rewritten as

\[
H = \sum_{k,s=\pm} \varepsilon_{k,s} a_{k,s}^\dagger a_{k,s} - V_0 \sum_{k,s,k',s'} e^{-i\sigma_{s-k}+i\sigma_{s'}-i\sigma_{s-k'}} \lambda_{s}(k) \lambda_{s'}(k') A_{k,s}^\dagger A_{k',s'},
\tag{18}
\]

where

\[
\begin{align*}
&\lambda_{s}(k) \equiv s \cos \varphi_{k} \sin \varphi_{s(-k)} \\
&\lambda_{0}(k) \equiv - \left( \cos \varphi_{k} \cos \varphi_{-k} + \sin \varphi_{k} \sin \varphi_{-k} e^{i\theta_{k}} - e^{-i\theta_{k}} \right)
\end{align*}
\]

and \( s, s' = 0, \pm 1 \) in the second summation of Eq. (18).

Generally, because \( [A_{k,0}^\dagger, A_{k',0}] \neq 0 \) for \( k \neq k' \) and \( \lambda_{s}(k) \) is \( k \)-dependent, it is hard to find an exact solution of Hamiltonian [13] by adopting a similar ansatz used above. Nevertheless, Hamiltonian [13] can still be solved exactly for some special but relevant cases. When the external magnetic field \( B = 0 \), we have \( \varphi_{k} = \varphi_{-k} = \sqrt{2}/2 \) and \( \theta_{k} = \theta(k) \), so that \( \lambda_{s}(k) = s/2 \) (\( s = \pm \)), \( \lambda_{0}(k) = 0 \) (\( k \neq 0 \)) and \( \lambda_{0}(0) = -1 \). In this case, Hamiltonian [13] reduces to Eq. (3).

On the other hand, when \( \alpha = 0 \), \( B_x = B_y = 0 \), and \( B_z \neq 0 \), only the Zeeman term is present. For this case, \( \lambda_{s}(k) = 0 \) (\( s = \pm \)) and \( \lambda_{0}(k) = -1 \). Since we have

\[
\begin{align*}
&\sum_{s=\pm} \varepsilon_{k,s} a_{k,s}^\dagger a_{k,s}, A_{k,0}^\dagger \equiv (\varepsilon_{k,+} + \varepsilon_{k,-}) A_{k,0}^\dagger, \tag{19}
\end{align*}
\]

we can take another ansatz

\[
C_{\nu}^\dagger = \sum_{k} \frac{A_{k,0}^\dagger}{\varepsilon_{k,+} + \varepsilon_{k,-} - E_{\nu}} \tag{20}
\]

to replace \( B_{\nu}^\dagger \) in Eq. (7). Solving the Schrödinger equation with Hamiltonian [13] and the corresponding eigenvector, we obtain the equations that the parameters \( E_{\nu} \)'s satisfy,

\[
1 - \sum_{k} \frac{V_0}{\varepsilon_{k,+} + \varepsilon_{k,-} - E_{\nu}} + \sum_{\mu \neq \nu} \frac{2V_0}{E_{\mu} - E_{\nu}} = 0, \tag{21}
\]

where \( \nu = 1, 2, \ldots, n \). The expression of the eigenenergy of the whole system is the same as Eq. (10). Eq. (21) implies that even when the single-particle energies of electrons are spin-dependent, the Hamiltonian is still exactly solvable. With the solutions of Eq. (21), we can similarly evaluate the dimensionless order parameter for this system with Eq. (11). Now the order parameter \( \Delta_{k,0} = \Delta_{k,0} \), which has just the usual s-wave symmetry with the same reason being mentioned before regardless of the strength of pairing interactions.

As an interesting example, we also address a special case, where all \( N_e \) electrons are on the Fermi surface \( k = k_F \) in Eq. (21), which is an approximation for the considered system as many physical phenomena are only closely related to the electrons near the Fermi surface. In this case, we are able to obtain the total energy of the system without having all \( E_{\nu} \) to be solved from Eq. (21). Supposing the degeneracy of Fermi level to be \( \Omega \) and considering that \( 2\varepsilon_{k_F} = \varepsilon_{k,+} + \varepsilon_{k,-} \), we multiply \( 2\varepsilon_{k_F} - E_{\nu} \) on Eq. (21) and take the sum of all \( n \) equations to obtain one equation. From this equation, we get the total pairing energy \( E_P \) of the pairing state \( \ket{n,0,0} \) as

\[
E_P = 2n\varepsilon_{k_F} - V_0\Omega n + 2V_0 \sum_{\nu} \frac{n}{\mu \neq \nu} \sum_{\nu} \frac{2\varepsilon_{k_F} - E_{\nu}}{E_{\mu} - E_{\nu}}. \tag{22}
\]

After partitioning the summation term into two parts with the sum indices \( \nu, \mu \) in one of them being interchanged, the total energy of pairing system is given by

\[
E_P = 2n\varepsilon_{k_F} - V_0\Omega n(\Omega - n + 1). \tag{23}
\]
Note that $\varepsilon_{k-} = (\varepsilon_{k_F} - B_z) < \varepsilon_{k+} = (\varepsilon_{k_F} + B_z)$, single electrons prefer to occupy the $S_-$ set. Therefore, the total energy of $|n, 0, S_-\rangle$ is found to be

$$E = (N_e - 2n)(\varepsilon_{k_F} - B_z) + 2n\varepsilon_{k_F} - V_0(n\Omega' - n + 1),$$

where $\Omega' = \Omega - (N_e - 2n)$ in consideration of that $(N_e - 2n)$ levels are blocked by single electrons. Thus the condensation energy $\Delta E = E - E_0$ with $E_0$ as the no-pair state energy is given by

$$\Delta E = -V_0n^2 + n(2B_z - V_0(\Omega - N_e + 1)).$$  \(\tag{24}\)

Combining the condition $\Delta E \leq 0$ with the requirement of minimum $E$, we can readily find a critical value $B_c = V_0(\Omega - N_e/2 + 1)/2$. When $B_z < B_c$, the system is in the pairing state, otherwise the ferromagnetic state.

We now attempt to elaborate how Majorana fermions (MF) can emerge in the system described by Hamiltonian \(1\) based on our exact solution. To capture the essential physics but without loss of generality, we consider that 2$n$ electrons occupy the states in a narrow ribbon around $\varepsilon_{k_F}$ and confined in an annular region $r_0 < r < R_0$. In the continuum limit and from $H_{0n}$, in addition to the bulk states $\varepsilon_{k_F,n}$ one can also find the inner and outer edge states with the energies $E_{in} = \varepsilon_{k_F} + \alpha L_z/r_0$ and $E_{out} = \varepsilon_{k_F} - \alpha L_z/R_0$, where $L_z$ is the angular momentum. In the presence of the pairing interaction, the zero modes with $L_z = 0$ survive due to the topological protection \(\text{[12]}\) and they could be occupied by pairs of MFs: $a_{MF}(m\varepsilon_{k_F}) = (\gamma_{1m} - i\gamma_{2m})/2$ \((m = \pm)\) with $\gamma_{im} = \gamma^\dagger_{im}$ the MF-operators, while the occupied bulk states are in the pairing states described by Eq. \(\text{[7]}\) with a lower energy. If an occupied pairing state in the branch $\varepsilon_{k_+}$ is lowered by the pairing energy to touch the edge state energy level $\varepsilon_{k_F} = E_{MF}$ with $E_{MF}$ the occupation energy for one pair of $m$-MFs, i.e., $E(n, 0, 0) = E(n - 1, 2, 0) = E(n - 1, 0, 0) + 2E_{MF}$ in terms of Eqs. \(\text{[7]}\) and \(\text{[10]}\), the MFs may emerge as gapless excitations. This condition also shows the degeneracy of occupation and vacuum of MF states, which the nonabelian statistics originates from. Note that the probability amplitude for the emergence of a pair of $m$-MFs is proportional to $\langle 0, 1_{k_F} + 1_{-k_F}, n - 1|\gamma_{1m}^\dagger \gamma_{2m}^\dagger|1_{k_F} + 1_{-k_F}, n \rangle \neq 0$ for the present system. The above analysis asserts some important results of the mean field theory \(\text{[21, 22]}\).

In summary, by making a spin-rotation unitary transformation and in the framework of Richardson ansatz, we have found for the first time that a class of electron pairing model with two kinds of spin-orbit interactions is exactly solvable, which is clearly relevant to recent research hot spots on topological superconductors and Dirac fermions. More importantly, based on the exact solution, we have rigorously shown that the pairing symmetry is of the $p+i p$-wave regardless of the strength of pairing interactions. Intriguingly, we have also elaborated how Majorana fermions can emerge in the system. Moreover, we have addressed a system with the Zeeman term included and presented an exact solution as well. Exact analytical results have been illustrated for two simple examples. Finally, we wish to pinpoint that the present exact solutions for the mentioned pairing systems may shed light on profound understandings of topological superfluids.

We would like to thank L. A. Wu, Y. C. He, Y. Chen and Y. Li for helpful discussions. This work was supported by the RGC of Hong Kong (Nos. HKU7044/08P and HKU7055/09P) and a CRF of Hong Kong.

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