Toric degenerations of low-degree hypersurfaces

Nathan Ilten and Oscar Lautsch

Abstract. We show that a sufficiently general hypersurface of degree $d$ in $\mathbb{P}^n$ admits a toric Gröbner degeneration after linear change of coordinates if and only if $d \leq 2n - 1$.

1 Introduction

When does a projective variety $X$ admit a flat degeneration to a toric variety? Among other applications, such degenerations are used in the mirror-theoretic approach to the classification of Fano varieties [CCG+13], the construction of integrable systems [HK15], and in bounding Seshadri constants [Ito14]. The many applications of toric degenerations notwithstanding, there is as of yet no general method for determining if a given variety admits a toric degeneration.

In this note, we will consider the special case of toric degenerations of some $X \subset \mathbb{P}^n$ obtained as the flat limit of $X$ under a $\mathbb{G}_m$-action on $\mathbb{P}^n$. In the case that the $\mathbb{G}_m$-action arises as a one-parameter subgroup of the standard torus on $\mathbb{P}^n$, the situation may be well understood by studying the Gröbner fan and tropicalization of $X$ [MS15]. However, if we consider arbitrary $\mathbb{G}_m$-actions on $\mathbb{P}^n$, the situation becomes more complicated. As a test case, we investigate the existence of such toric degenerations when $X$ is a hypersurface.

In order to state our result, we introduce some notation. Throughout the paper, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. Let $\omega \in \mathbb{R}^{n+1}$. Consider any polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]$, where we write

\begin{equation}
 f = \sum_{u \in \mathbb{Z}_{\geq 0}^{n+1}} c_u x^u
\end{equation}

using multi-index notation. The initial term of $f$ with respect to the weight vector $\omega$ is

\[ \text{In}_\omega(f) = \sum_{u : \langle u, \omega \rangle = \lambda} c_u x^u, \]

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where \( \lambda \) is the maximum of \((u, \omega)\) as \( u \) ranges over all \( u \in \mathbb{Z}_{\geq 0} \) with \( c_u \neq 0 \). For an ideal \( J \subset \mathbb{K}[x_0, \ldots, x_n] \), its initial ideal with respect to the weight vector \( \omega \) is
\[
\text{In}_\omega(J) = \langle \text{In}_\omega(f) \mid f \in J \rangle.
\]
The weight of a monomial \( x^u \) with respect to \( \omega \) is the scalar product \( \langle u, \omega \rangle \in \mathbb{R} \).

**Definition 1.1** Let \( X \subset \mathbb{P}^n \) be a projective variety over \( \mathbb{K} \). We say that \( X \) **admits a toric Gröbner degeneration up to change of coordinates** if there exist a \( \text{PGL}(n+1) \) translate \( X' \) of \( X \) and a weight vector \( \omega \in \mathbb{R}^{n+1} \) such that the initial ideal
\[
\text{In}_\omega(I(X'))
\]
of the ideal \( I(X') \subset \mathbb{K}[x_0, \ldots, x_n] \) of \( X' \) is a prime binomial ideal.

We can now state our result.

**Theorem 1.2** Let \( d, n \in \mathbb{N} \). There is a non-empty Zariski open subset \( U \) of the linear system of degree \( d \) hypersurfaces in \( \mathbb{P}^n \) with the property that every hypersurface in \( U \) admits a toric Gröbner degeneration up to change of coordinates if and only if \( d \leq 2n - 1 \).

Before proving this theorem in the following section, we discuss connections to the existing literature.

A common source of toric degenerations of a projective variety \( X \subset \mathbb{P}^n \) arises by considering the Rees algebra associated with a full-rank homogeneous valuation \( v \) on the homogeneous coordinate ring of \( X \) [And13]. As long as the homogeneous coordinate ring of \( X \) contains a finite set \( S \) whose valuations generate the value semigroup of \( v \), one obtains a toric degeneration. Such a set \( S \) is called a **finite Khovanskii basis** for the coordinate ring of \( X \). This construction is in fact quite general: essentially any \( \mathbb{G}_m \)-equivariant degeneration of \( X \) over \( \mathbb{A}^1 \) arises by this construction (see [KMM23, Theorem 1.11] for a precise statement). There has been some work on algorithmically constructing valuations with finite Khovanskii bases (see, e.g., [BLMM17] for applications to degenerations of certain flag varieties), but as of yet, there is no general effective criterion for deciding when such a valuation exists.

Drawing on [KM19] which connects Khovanskii bases and tropical geometry, we may rephrase our results in the language of Khovanskii bases. It is straightforward to show that \( X \) admits a toric Gröbner degeneration up to change of coordinates if and only if there is some full-rank homogeneous valuation \( v \) for which the homogeneous coordinate ring has a finite Khovanskii basis consisting of degree one elements. Thus, our theorem shows the existence of finite Khovanskii bases for general hypersurfaces of degree at most \( 2n - 1 \), and shows that any finite Khovanskii basis for a general hypersurface of larger degree necessarily contains elements of degrees larger than one. In fact, we suspect that a general hypersurface of sufficiently large degree does not admit any finite Khovanskii basis at all.

We note in passing that a general hypersurface of arbitrary degree will admit a toric degeneration in a weaker sense. Indeed, the universal hypersurface over the linear system of degree \( d \) hypersurfaces is a flat family, and for any degree \( d \), there is a toric hypersurface of degree \( d \). However, such a degeneration is not \( \mathbb{G}_m \)-equivariant.
An interesting comparison of our result can be made with [KMM21], which states that after a \textit{generic} change of coordinates, any arithmetically Cohen Macaulay variety \( X \subset \mathbb{P}^n \) has a Gröbner degeneration to a (potentially non-normal) variety equipped with an effective action of a codimension-one torus. Such varieties, called complexity-one \( T \)-varieties, are in a sense one step away from being toric. The hypersurfaces we consider in our main result (Theorem 1.2) are \textit{arithmetic} arithmetically Cohen Macaulay, so they admit Gröbner degenerations to complexity-one \( T \)-varieties. Our result characterizes when we can go one step further and Gröbner degenerate to something toric. When \( d \leq 2n - 1 \) and we are in the range for which this is possible for a generic hypersurface, the change of coordinates required is a special one as opposed to the generic change of coordinates of [KMM21].

2 Proof of the theorem

2.1 Setup

Throughout, we will assume that \( d, n > 1 \) since the theorem is clearly true if \( d = 1 \) or \( n = 1 \). We will view the coefficients \( c_u \) of \( f \) in (1.1) as coordinates on affine space \( \mathbb{A}^{\binom{d+n}{n-1}} \).

To indicate the dependence of \( f \) on the choice of coefficients \( c \), we will often write \( f = f_c \). Let \( K \) be the subset of all \( u \in \mathbb{Z}^{n+1}_{\geq 0} \) such that \( u_0 + u_1 = d \), \( u_i = 0 \) for \( i > 1 \), and \( u_1 < d \). We then set

\[
W = V(\langle c_u \rangle_{u \in K}) \subset \mathbb{A}^{\binom{d+n}{n-1}}.
\]

The family of polynomials parameterized by \( W \) consists of all degree \( d \) forms such that the only monomial involving only \( x_0 \) and \( x_1 \) is \( x_1^d \).

We will be considering the map

\[
\phi : \text{GL}(n+1) \times W \to \mathbb{K}[x_0, \ldots, x_n]^d
\]

\[
(A, c) \mapsto A.f_c,
\]

where \( A.f_c \) denotes the action of \( A \in \text{GL}(n+1) \) on a polynomial \( f_c = \sum c_u x^u \) via linear change of coordinates. We will be especially interested in the differential of \( \phi \) at \((e, c)\), where \( e \in \text{GL}(n+1) \) is the identity. A straightforward computation shows that the image of the differential at \((e, c)\) is generated by

\[
(2.1) \quad \frac{\partial f_c}{\partial x_i} \cdot x_j \quad 0 \leq i, j \leq n.
\]

The following lemma is the key to our proof.

**Lemma 2.1** The differential \( \phi \) is surjective at \((e, c)\) for general \( c \in W \) if and only if \( d \leq 2n - 1 \).

**Proof** Consider the image of the differential of \( \phi \) at \((e, c)\). From (2.1), we obtain the span of all monomials of \( \mathbb{K}[x_0, \ldots, x_n]^d \) with the exceptions of the \( d \) monomials \( x_0^d, x_0^{d-1}x_1, \ldots, x_0x_1^{d-1} \). From (2.2) with \( i = 1 \) and \( j = 0 \), modulo (2.1), we additionally
obtain the monomial $x_0 x_1^{d-1}$. We do not obtain anything new from (2.2) when $i = 1$ and $j = 1$, when $i = 0$, or when $j > 1$.

It remains to consider the contributions to the image from (2.2) with $i > 1$ and $j = 0, 1$. For $2 \leq i \leq n$ and $1 \leq m \leq d - 1$, let $u(i, m) \in \mathbb{Z}^{n+1}$ be the exponent vector with $u_i = 1, u_0 = m, u_1 = d - m - 1$. Modulo the span of (2.1) and $x_0 x_1^{d-1}$, from (2.2), we obtain

$$\frac{\partial f_c}{\partial x_i} \cdot x_0 \equiv c_u(i,d-1)x_0^d + c_u(i,d-2)x_0^{d-1}x_1 + \cdots + c_u(i,1)x_0^2x_1^{d-2},$$

$$\frac{\partial f_c}{\partial x_i} \cdot x_1 \equiv c_u(i,d-1)x_1^d + c_u(i,2)x_0^2x_1^{d-2}.$$

Varying $i$ from 2 to $n$, we obtain $2n - 2$ polynomials of degree $d$. The $(2n - 2) \times (d - 1)$ matrix of their coefficients has the form

$$
\begin{pmatrix}
0 & c_u(2,d-2) & \cdots & c_u(2,1) \\
0 & c_u(3,d-2) & \cdots & c_u(3,1) \\
\vdots & \vdots & & \vdots \\
0 & c_u(n,d-2) & \cdots & c_u(n,1) \\
0 & c_u(n,d-1) & \cdots & c_u(n,2)
\end{pmatrix}
$$

Since $c \in W$ is general, this matrix has full rank, that is, its rank is $\min\{d - 1, 2n - 2\}$. Hence, the image of the differential of $\phi$ has codimension

$$d - 1 - \min\{d - 1, 2n - 2\},$$

so the differential is surjective if and only if $d \leq 2n - 1$.

We now move on to prove the theorem.

### 2.2 Existence

We will first show that if $d \leq 2n - 1$, a general degree $d$ hypersurface admits a toric Gröbner degeneration up to change of coordinates. As noted above, the family of polynomials parameterized by $W$ consists of all degree $d$ forms such that the only monomial involving only $x_0$ and $x_1$ is $x_1^d$. Consider any $\omega \in \mathbb{R}^{n+1}$ such that

$$\omega_0 > \omega_1 > \omega_2 > \cdots > \omega_n \quad (d - 1) \omega_0 + \omega_2 = d \omega_1.$$

For general $c \in W$, the initial term of $f_c$ is

$$a x_1^d + b x_0^{d-1} x_2$$

for some $a, b \neq 0$; this is a prime binomial. Thus, we will be done with our first claim if we can show that the image of $\phi$ contains a non-empty open subset of $\mathbb{K}[x_0, \ldots, x_n]_d$.

To this end, we consider the image of the differential at $(e, c)$ for general $c \in W$. By Lemma 2.1, we conclude that $\phi$ has surjective differential at $(e, c)$ for general $c \in W$; it follows that $\phi$ has surjective differential at a general point of $\text{GL}(n + 1) \times W$. 
Thus, the dimension of the image of $\phi$ is the dimension of $\mathbb{K}[x_0, \ldots, x_n]_d$, and the image of $\phi$ contains a non-empty open subset of $\mathbb{K}[x_0, \ldots, x_n]_d$.

### 2.3 Nonexistence

Assume now that $d > 2n - 1$. We first give an overview of the proof strategy. There are only finitely many prime binomials $g$ of degree $d$. Likewise, there are only finitely many linear orderings $<$ of the variable indices $0, \ldots, n$. We say that a weight vector $\omega$ is compatible with $<$ and $g$ if whenever $i < j$ in the linear ordering, then $\omega_i \geq \omega_j$, and the two monomials of $g$ have the same weight with respect to $\omega$.

For fixed $g$ and linear ordering on the variables, we may consider the set $S$ of all polynomials $f$ in $\mathbb{K}[x_0, \ldots, x_n]_d$ for which there exists a compatible weight vector $\omega \in \mathbb{R}^{n+1}$ such that initial term of $f$ with respect to $\omega$ is $g$. We will show that up to permutation of the coordinates, this set $S$ can be identified as a subfamily of $W$. By Lemma 2.1, the map $\phi$ has nowhere surjective differential. Thus, by generic smoothness, the dimension of the image of $\phi$ must be strictly less than the dimension of $\mathbb{K}[x_0, \ldots, x_n]_d$. It follows that there cannot be a Zariski-open subset of $\mathbb{K}[x_0, \ldots, x_n]_d$ such that every hypersurface in this subset admits a toric Gröbner degeneration up to change of coordinates.

To complete the proof, we will fix a prime binomial $g = g' + g''$ of degree $d$ and a linear ordering of the variables. Here, $g'$ and $g''$ are the two terms of $g$. After permuting the variables and appropriately adapting $g$, we may assume without loss of generality that the indices are ordered as $0 < 1 < 2 < \cdots < n$. The irreducibility of $g$ implies that $g$ involves at least three distinct variables, and no variable appears in both $g'$ and $g''$. Let $p$ be the smallest index such that $x_p$ appears in $g$; we denote the corresponding term by $g'$. Let $q$ be the smallest index such that $x_q$ appears in the term $g''$. If $g'$ only involves variables $x_i$ with indices $i < q$, then any compatible term order $\omega$ must satisfy $\omega_p = \omega_q = \omega_j$ for all $p \leq j \leq q$. Indeed, if not, the term $g''$ would necessarily have smaller weight. Without loss of generality, we may thus permute indices without changing the set of compatible weight vectors to also assume that $g'$ involves some $x_i$ with $i > q$. For this, we are using that the irreducibility of $g$ guarantees that at least one of $g'$ and $g''$ is not a $d$th power.

Consider the set $S$ of polynomials $f_c$ such that there is a compatible weight $\omega$ for which $f_c$ has $g$ as its initial term. We claim that $S$ is a subset of the family parameterized by $W$. Indeed, since $q > 0$, $g''$ has weight at most equal to the weight of $x_1^d$. The monomials $x_0^d, x_0^{d-1}x_1, \ldots, x_0x_1^{d-1}$ all have weight at least as big as the weight of $x_1^d$, and are not scalar multiples of $g'$ or $g''$. Hence, none of these monomials can appear in any element of $S$, and the claim follows.

The proof of the theorem now follows from the argument given above.

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Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, BC V5A 1S6, Canada
e-mail: nilten@sfu.ca  oscar_lautsch@sfu.ca