ON THE ASYMPTOTIC BEHAVIOR OF BERGMAN KERNELS FOR
POSITIVE LINE BUNDLES

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ABSTRACT. Let $L$ be a positive line bundle on a projective complex manifold. We study
the asymptotic behavior of Bergman kernels associated with the tensor powers $L^p$ of $L$
as $p$ tends to infinity. The emphasis is the dependence of the uniform estimates on
the positivity of the Chern form of the metric on $L$. This situation appears naturally when we
approximate a semi-positive singular metric by smooth positively curved metrics.

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1. INTRODUCTION

Let $L$ be an ample holomorphic line bundle over a projective manifold $X$ of dimension
$n$. Fix a (reference) smooth Hermitian metric $h_0$ on $L$ whose first Chern form $\omega_0$ is a
Kähler form. Recall that $\omega_0 = \frac{i}{2\pi} R^L_0$, where $R^L_0$ is the curvature of the Chern connection
on $(L, h_0)$.

Let $h^L$ be a semi-positive singular metric on $L$. For various applications, one needs to
understand the asymptotic behavior of the Bergman kernel associated with $L^p$ and $h^L$
when $p$ tends to infinity. A natural approach is to approximate the considered metric
by smooth positively curved metrics, and therefore, it is necessary to understand the
dependence of the Bergman kernels in terms of the positivity of the curvature of the
metric. See [4, 9, 10] for the regularization of metrics. This method was already used
in our previous work on the speed of convergence of Fekete points, see [1, 10]. In
[10, §2.3], inspired by [2], an $L^1$-estimate for Bergman kernels was obtained. Here, we
investigate the uniform estimate which can be useful for applications in geometry.

Fix a smooth Kähler form $\theta$ on $X$ (one can take $\theta = \omega_0$). Consider a metric $h = e^{-2\phi} h_0$
on $L$ with weight $\phi$ of class $C^{n+6}$ whose first Chern form $\omega := dd^c \phi + \omega_0$ (here $d^c := \sqrt{-1} \partial \ BAR \ \partial$) satisfies

\begin{equation}
\omega \geq \zeta \theta \quad \text{for some constant } 0 < \zeta \leq 1.
\end{equation}

Consider the natural metric on the space of smooth sections of $L^p$, induced by the metric
$h$ on $L$ and the volume form $\theta^n$ on $X$, which is defined by

\begin{equation}
\|s\|_{L^2(\theta^n)}^2 := \int_X |s(x)|_{p\phi}^2 \theta^n / n!.
\end{equation}

Here, $|s(x)|_{p\phi}$ stands for the norm of $s(x)$ with respect to the metric $h^{\otimes p}$ on $L^p$. Let
$\langle \cdot, \cdot \rangle_{p\phi}$ be the associated Hermitian product on $C^\infty(X, L^p)$, the space of smooth sections
of $L^p$. Let $P_p$ be the orthogonal projection from $(C^\infty(X, L^p), \langle \cdot, \cdot \rangle_{p\phi})$ onto the subspace
of holomorphic sections $H^0(X, L^p)$. The Bergman kernel associated with the above data
is the kernel associated with the last projection where we use the volume form $\theta^n / n!$
to integrate functions on $X$. This kernel is denoted by $P_p(x, x')$, with $x, x' \in X$. It is a section of the line bundle over $X \times X$ which is the tensor product of two line bundles: the first one is the pull-back to $X \times X$ of the line bundle $L^p$ over the first factor, and the second one is the pull-back of the dual line bundle $(L^*)^p$ of $L^p$ over the second factor. In particular, its restriction to the diagonal of $X \times X$, i.e., $P_p(x, x)$, can be identified to a positive-valued function on $X$. See [12] for details. In fact, if $\{s_j\}_j$ is an orthonormal basis of $(H^0(X, L^p), \langle \cdot, \cdot \rangle)$, then
\begin{equation}
(1.3) \quad P_p(x, x) = \sum_j |s_j(x)|^2_{p^0} = \sup \{ |s(x)|^2_{p^0}, \quad s \in H^0(X, L^p) \text{ with } ||s||_{L^2(p^0)} = 1 \}.
\end{equation}

Here is the main result in this paper which gives us a uniform estimate of the Bergman kernel in terms of $\phi, \omega, p$ and $\zeta$. This is a version of Tian’s theorem [14]. See [2, 5, 6, 7, 8, 11, 13, 15, 16] for various generalizations. We also refer to [12] for a comprehensive study of several analytic and geometric aspects of Bergman kernels. The last reference is inspired by the analytic localization technique in [3].

**Theorem 1.1.** Under the above assumptions, there exist $\delta > 0, c > 0$ satisfying the following condition: for any $l \in \mathbb{N}^+$, there is a constant $c_l > 0$ such that for $p \in \mathbb{N}^+$, $p\zeta > \delta$, and $x \in X$, we have
\begin{equation}
(1.4) \quad \left| p^{-n} P_p(x, x) - \frac{\omega(x)^n}{\theta(x)^n} \right| \leq c |d\phi|^{2n+8}_{n+5} |\omega|^{4n+20}_{n+5} |d\phi|^{2n+2}_{n+2} \zeta^{-2n-10} P^{-1} + c_l |\omega|^{2n+2}_{n} (|d\phi|_2)^{2\zeta-1}_{n} 6^{n+6+3l} P^{-1}.
\end{equation}

Note that $| \cdot |_k$ stands for $1 + \| \cdot \|_{C^k}$. As a direct consequence, we infer the following result by taking $l = 1$.

**Corollary 1.2.** There exist $\delta > 0, c > 0$ such that for any $0 < \zeta \leq 1$, any weight $\phi$ of class $C^{n+6}$ with $dd^c \phi + \omega_0 \geq \zeta \theta$, and any $p \in \mathbb{N}^+$ with $p\zeta > \delta$, we have
\begin{equation}
(1.5) \quad \left| p^{-n} P_p(x, x) - \frac{\omega(x)^n}{\theta(x)^n} \right| \leq c \zeta^{-6n-9} |d\phi|^{8n+30}_{n+5} P^{-1}.
\end{equation}

If $\phi \in C^{n+2k+6}$, we can adapt easily the proof of Theorem 1.1 to get the estimate for $C^k$-norm of the left hand side of (1.4). Cf. Remark 3.9.

The article is organized as follows. In Section 2, we reduce the problem to the local setting. In Section 3, we establish Theorem 1.1. We need an approach different from previous ones which use the normal coordinates and the extension of connections on $L$, see [8, §4.2] and [12, §4.1.3]. Note that throughout the paper, the constants $c, c', c_l, \ldots$ may be changed from line to line.

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### 2. Localization of the Problem

Recall that the complex structure on $X$ is given by a smooth section $J$ of the vector bundle $End(TX)$ such that $-J^2$ is the identity section. Here, $TX$ denotes the real tangent bundle of $X$. Denote also by $T^{(1,0)}X$ and $T^{(0,1)}X$ the holomorphic and anti-holomorphic
tangent bundles of $X$. They are complex vector sub-bundles of $TX \otimes_{\mathbb{R}} \mathbb{C}$. The Kähler form $\theta$ induces a Riemannian metric $g^{TX}$ on $X$ defined by $g^{TX} := \theta(\cdot, \cdot)$.

Let $\overline{\nabla}^{L^p}$ be the $\overline{\nabla}$-operator acting on $L^p$ and $\overline{\nabla}^{L^p,*}$ its dual operator with respect to the metric $h = e^{-2\phi}h_0$ on $X$. Consider the Dirac and Laplacian-type operators

\begin{equation}
D_p := \sqrt{2}(\overline{\nabla}^{L^p} + \overline{\nabla}^{L^p,*}) \quad \text{and} \quad \Box_p := \frac{1}{2}D^2_p = \overline{\nabla}^{L^p}\overline{\nabla}^{L^p,*} + \overline{\nabla}^{L^p,*}\overline{\nabla}^{L^p}.
\end{equation}

They act on $\Omega^{0,*}(X, L^p)$, the space of the forms of bi-degree $(0, \cdot)$ with values in $L^p$.

Let $\nabla^L$ be the Chern connection on $(L, h = e^{-2\phi}h_0)$ and $R^L = (\nabla^L)^2$ its curvature which is related to the first Chern form $\omega$ by

\begin{equation}
\omega = \frac{\sqrt{-1}}{2\pi}R^L.
\end{equation}

Let $\nabla^{TX}$ be the Levi-Civita connection on $(TX, g^{TX})$. It preserves $T^{(1,0)}X$, $T^{(0,1)}X$, and its restriction to $T^{(1,0)}X$ is the Chern connection $\nabla^{T^{(1,0)}X}$. Let $\nabla^{\wedge^0,*}$ be the connection on $\Lambda(T^{*(0,1)}X)$ induced by $\nabla^{T^{(1,0)}X}$, and $\nabla^{\wedge^0,*}\otimes L^p$ the connection on $\Lambda(T^{*(0,1)}X) \otimes L^p$ induced by $\nabla^{\wedge^0,*}$ and $\nabla^L$. For $u \in T^{(1,0)}X$ and $v \in T^{(0,1)}X$, let $u^* \in T^{*(0,1)}X$ be the metric dual of $u$ with respect to $g^{TX}$, define the operator $c(\cdot)$ depending linearly on a vector in $T^{(1,0)}X \oplus T^{(0,1)}X$ by setting

\begin{equation}
c(u) := \sqrt{2}u^* \quad \text{and} \quad c(v) := \sqrt{2}v,
\end{equation}

where $i$ denotes, as usual, the contraction operator. Then by [12, p.31], for $\{e_j\}$ an orthonormal frame of $(TX, g^{TX})$, we have

\begin{equation}
D_p = \sum_j c(e_j) \nabla^{\wedge^0,*}_{e_j} \otimes L^p.
\end{equation}

Denote by $K_X^\wedge$ the anti-canonical bundle of $X$. The curvature of $K_X^\wedge$ with respect to the above Riemannian metric is denoted by $R^{K_X^\wedge}$. Then $\sqrt{-1}R^{K_X^\wedge}$ is the Ricci curvature of $(X, g^{TX})$. Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}X$ with dual frame $\{w_j^*\}_{j=1}^n$. Set

\begin{equation}
\omega_d := -\sum_{l,m} R^{L^1}(w_l, \overline{w}_m) \overline{w}^m \wedge i_{w_l}.
\end{equation}

Recall that $\sqrt{-1}R^L = \omega \geq \zeta \theta$. Then $\omega_d$ is a section of $\text{End}(\Lambda(T^{*(0,1)}X))$ and $R^L$ acts as the derivative $\omega_d$ on $\Lambda(T^{*(0,1)}X)$. By [12, (1.4.63)] and using that $\langle \Delta^{0,*}s, s \rangle_{\rho \phi} \geq 0$, we obtain for $s \in \Omega^{0,*}(X, L^p)$ that

\begin{equation}
\|D_p s\|^2_{L^2(\rho \phi)} = 2\langle \Box_p s, s \rangle_{\rho \phi} \geq -2p \langle \omega_d s, s \rangle_{\rho \phi} + 2 \sum_{l,m} \langle R^{K_X^\wedge}(w_l, \overline{w}_m) \overline{w}^m \wedge i_{w_l} s, s \rangle_{\rho \phi}.
\end{equation}

Now by (1.21), (2.2), (2.5) and some standard arguments, see the proof of [12, Theorem 1.5.5], there exists $\delta > 0$, depending only on the Ricci curvature $R^{K_X^\wedge}$, such that if $\zeta \rho > \delta$, then

\begin{equation}
\text{Spec}(D^2_p) \subset \{0\} \cup [2\pi \zeta \rho, +\infty[.
\end{equation}

Let $a_X$ denote the injectivity radius of $(X, \theta)$. For $0 < \epsilon_0 < a_X/4$, let $f_{\epsilon_0} : \mathbb{R} \to [0, 1]$ be a smooth even function such that

\begin{equation}
f_{\epsilon_0}(v) = \begin{cases} 
1 & \text{for } |v| \leq \epsilon_0/2, \\
0 & \text{for } |v| > \epsilon_0.
\end{cases}
\end{equation}
Set

\[
F_{\epsilon_0}(a) := \left( \int_{-\infty}^{+\infty} f_{\epsilon_0}(v)dv \right)^{-1} \int_{-\infty}^{+\infty} e^{i\epsilon_0 a} f_{\epsilon_0}(v)dv
\]

(2.9)

\[
= \left( \int_{-\infty}^{+\infty} f_{\epsilon_0}(\zeta^{-1}v)dv \right)^{-1} \int_{-\infty}^{+\infty} e^{i\epsilon_0} f_{\epsilon_0}(\zeta^{-1}v)dv.
\]

Then \( F_{\epsilon_0}(a) \) lies in Schwartz space \( \mathcal{S}(\mathbb{R}) \) and \( F_{\epsilon_0}(0) = 1 \).

**Proposition 2.1.** Let \( \delta > 0 \) verifying (2.7). Then, for all \( l \in \mathbb{N} \), \( 0 < \epsilon_0 < a \pi / 4 \) and \( F_{\epsilon_0} \) as above, there exists \( c > 0 \) such that for \( p \geq 1 \), \( \delta / p < \zeta \leq 1 \), \( x, x' \in X \)

\[
\| F_{\epsilon_0}(D_p)(x, x') - P_p(x, x') \|_{L^\infty(p\phi)} \leq c |\omega|^{2n+2}\zeta^{-6n-3l-6} p^{-l}.
\]

(2.10)

**Proof.** For \( a \in \mathbb{R} \), set

\[
\phi_p(a) := 1_{[\sqrt{\epsilon_0}+\epsilon, \infty)}(|a|) F_{\epsilon_0}(a).
\]

By (2.7) and (2.11), for \( \zeta p > \delta \), we get

\[
F_{\epsilon_0}(D_p) - P_p = \phi_p(D_p).
\]

(2.12)

By (2.9), for any \( m \in \mathbb{N} \) there exists \( c > 0 \) such that for all \( \zeta \in [0, 1] \),

\[
\sup_{a \in \mathbb{R}} |a|^m |F_{\epsilon_0}(a)| \leq c \zeta^{-m}.
\]

(2.13)

Thus, for any \( m \in \mathbb{N} \) and \( \zeta p > \delta \), we have

\[
\| (D_p)^m F_{\epsilon_0}(D_p) \|_{L^0(p\phi)} := \sup_{s \in \Omega^{R^*}(X,L^2)} \frac{\| (D_p)^m F_{\epsilon_0}(D_p) s \|_{L^2(p\phi)}}{\| s \|_{L^2(p\phi)}} \leq c \zeta^{-m}.
\]

(2.14)

As \( X \) is compact, there exists a finite set of points \( a_i, 1 \leq i \leq r \), such that the family of balls \( U_i := B^X(a_i, \epsilon_0) \) of center \( a_i \) and radius \( \epsilon_0 \) is a covering of \( X \). We identify the ball \( B^{T_{a_i}X}(0, \epsilon_0) \) in the tangent space of \( X \) at \( a_i \) with the ball \( B^X(a_i, \epsilon_0) \) using the exponential map. We then identify \( (TX)\gamma, \Lambda(T^{*(0,1)}X)\gamma, L^p_{\gamma} \) for \( Z \in B^{T_{a_i}X}(0, \epsilon_0) \) with \( T_{a_i}X, \Lambda(T^{*(0,1)}X)_{a_i}, L^p_{a_i} \) by parallel transport with respect to the connections \( \nabla^{TX}, \nabla^{A^L}, \nabla^{L^p} \) along the curve \( \gamma_Z : [0, 1] \ni t \mapsto \exp_{a_i}^X(tZ) \). Then \( (L, h)|_{U_i} \) is identified as the trivial bundle \( (L_{a_i}, h_{a_i}) \).

Let \( \{ e_j \}_{j} \) be an orthonormal basis of \( T_{a_i}X \cong \mathbb{R}^{2n} \). Let \( \tilde{e}_j(Z) \) be the parallel transport of \( e_j \) with respect to \( \nabla^{TX} \) along the above curve. Let \( \Gamma^L, \Gamma^{A^L} \) be the corresponding connection forms of \( \nabla^L \) and \( \nabla^{A^L} \) with respect to any fixed frame for \( L \) and \( L^1(T^{*(0,1)}X) \) which is parallel along the curve \( \gamma_Z \) under the trivialization on \( U_i \). Denote by \( \nabla_v \) the ordinary differentiation operator on \( T_{a_i}X \) in the direction \( v \). As we are working in the Kähler case, by [12, Prop. 1.2.6, Th. 1.4.5, Rk. 1.4.8], we can write on \( U_i \)

\[
D_p = \sum_j c(\tilde{e}_j) \left( \nabla_{\tilde{e}_j} + p \Gamma^L(\tilde{e}_j) + \Gamma^{A^L}(\tilde{e}_j) \right).
\]

(2.15)

In fact, the last identity is a consequence of (2.4). Consider the radial vector field \( R = \sum_j Z_j e_j \). By [12, (1.2.32)], the Lie derivative \( L_{R} \Gamma^L \) is equal to \( i_R \Gamma^L \). Therefore, we get the identity

\[
\Gamma^L Z = \int_0^1 (i_R \Gamma^L)_{12} dt,
\]

(2.16)

which allows us to bound \( \Gamma^L \).
Let \{\varphi_i\} be a partition of unity subordinate to \{U_i\}. For \(m \in \mathbb{N}\), we define a Sobolev norm on the \(m\)-th Sobolev space \(H^m(X, \Lambda(T^*(0,1)X) \otimes L^0)\) by

\[
\|s\|_{H^m}^2 = \sum_{i=1}^r \sum_{k=0}^m \sum_{j_1, \ldots, j_k=1}^{2n} \|\nabla_{e_{j_1}} \cdots \nabla_{e_{j_k}} (\varphi_i s)\|_{L^2}^2.
\]

Note that here we trivialize the line bundle \(L\) using an unitary section; so the section \(s\) above is identified with a function. Therefore, we drop the subscript \(p\phi\) since this weight is already taken into account.

By (2.15), (2.16) and [12, (1.6.9)], for \(P\) a differential operator of order \(m \in \mathbb{N}\) with scalar principal symbol and with compact support in \(U_i\), we get

\[
\|Ps\|_{H^1} \leq c\left(\|D_pPs\|_{L^2} + p|\omega|_0\|Ps\|_{L^2}\right)
\]

\[
\leq c' \left(\|PD_pPs\|_{L^2} + p \sum_{k=0}^m |\omega|_k\|s\|_{H^{m-k}}\right),
\]

for some constants \(c, c' > 0\). From (2.18), we get by induction for (other) suitable constants \(c, c' > 0\)

\[
\|s\|_{H^{m+1}} \leq c \sum_{k=0}^{m+1} p^{m+1-k} \|D_p^k s\|_{L^2} \prod_{(k_r+1)=m-k+1} \|\omega|_{k_r} \leq c' \sum_{k=0}^{m+1} p^{m+1-k} \|D_p^k s\|_{L^2} |\omega|_{m-k+1}^{m-k+1}.
\]

Note that for \(k = m+1\) we set \(|\omega|_{m-k+1} = 1\).

Let \(Q\) be a differential operator of order \(m' \in \mathbb{N}\) with scalar principal symbol and with compact support in \(U_j\). Using the identity

\[
\langle D_p^k \phi_p(D_p)Qs, s' \rangle = \langle s, Q^* \phi_p(D_p)D_p^k s' \rangle,
\]

we deduce from (2.19) with suitable sections instead of \(s\) that

\[
\|P\phi_p(D_p)Qs\|_{L^2} \leq c \sum_{k=0}^m \sum_{k'=0}^{m'} p^{m+m'-k-k'} \|D_p^k \phi_p(D_p)D_p^k s\|_{L^2} |\omega|_{m-k-1}^{m-k'} |\omega|_{m'-k'-1}^{m'-k'} = c \sum_{k=0}^m \sum_{k'=0}^{m'} p^{m+m'-k-k'} \|D_p^{k+k'} \phi_p(D_p)s\|_{L^2} |\omega|_{m-k-1}^{m-k} |\omega|_{m'-k'-1}^{m'-k'}.
\]

Note that the operators, considered in the last two lines, commute. Thank to (2.7), (2.11), (2.12) and then (2.14), if \(0 < \zeta \leq 1\) and \(\zeta p \geq \delta\), for any \(q \in \mathbb{N}\), the main factor in the last line can be bounded using

\[
\|D_p^{k+k'} \phi_p(D_p)s\|_{L^2} \leq (\zeta p)^{-q/2} \|D_p^{k+k'+q} \phi_p(D_p)s\|_{L^2} \leq c(\zeta p)^{-q/2} \zeta^{-k-k'-q} \|s\|_{L^2}.
\]
Take any $l > 0$ and choose $q := 2(m + m' - k - k' + l)$. Then there exists $c_l > 0$ such that for $0 < \zeta \leq 1$, $\zeta p \geq \delta$, we have
\[
\|P\phi_p(D_p)Qs\|_{L^2} \leq c \sum_{k=0}^{m} \sum_{k'=0}^{m'} \rho^{m+m'-k-k'}(\zeta p)^{-q/2} \zeta^{-k-q-k'}|\omega|^{m-k}_{m-k-1}|\omega|^{m'-k'}_{m'-k'-1}\|s\|_{L^2} \leq c_l \zeta^{-3m-3m'-3l}|\rho|^{m}_{m-1}|\omega|^{m}_{m-1}\|s\|_{L^2}.
\]
(2.22)

Finally, on $U_1 \times U_j$, by using the standard Sobolev’s inequality and (2.12), we get (2.10). Proposition 2.1 follows.

Remark 2.2. By (2.9) and the finite propagation speed of solutions of hyperbolic equations [12, Theorem D.2.1], $F_{e_0}(D_p)(x, x')$ only depends on the restriction of $D_p$ to $B^X(x, \epsilon_0 \zeta)$, and
\[
F_{e_0}(D_p)(x, x') = 0 \quad \text{when} \quad \text{dist}(x, x') \geq \epsilon_0 \zeta.
\]
(2.23)

To get the uniform estimate of the Bergman kernels in terms of $\zeta, p$, we need an approach different from the use of the normal coordinates and the extension of connections on $L$ in [8, §4.2] and [12, §4.1.3]. Let $\psi : X \supset U \to V \subset \mathbb{C}^n$ be a holomorphic local chart such that $0 \in V$ and $V$ is convex (by abuse of notation, we sometimes identify $U$ with $V$ and $x$ with $\psi(x)$). Then, for any $x \in \frac{1}{2}V := \{y \in \mathbb{C}^n : 2y \in V\}$, we will use the holomorphic coordinates induced by $\psi$ and let $0 < \epsilon_0 \leq 1$ be such that $B(x, 4\epsilon_0) \subset V$ for any $x \in \frac{1}{2}V$. We choose $\epsilon_0$ smaller than $a_x/4$ in order to use the estimates given in the proof of Proposition 2.1. Consider the holomorphic family of holomorphic local coordinates $\psi_x : \psi^{-1}(B(x, 4\epsilon_0)) \to B(0, 4\epsilon_0)$ for $x \in \frac{1}{2}V$ given by $\psi_x(y) := \psi(y) - x$.

Let $\sigma$ be a holomorphic frame of $L$ on $U$ and define the function $\varphi(Z)$ on $U$ by $|\sigma|^2_\omega(Z) := e^{-2\varphi(Z)}$. Consider the holomorphic family of holomorphic trivializations of $L$ associated with the coordinates $\psi_x$ and the frame $\sigma$. These trivializations are given by $\Psi_x : L|_{\psi^{-1}(B(x, 4\epsilon_0))} \to B(0, 4\epsilon_0) \times \mathbb{C}$ with $\Psi_x(y, v) := (\psi_x(y), v/\sigma(y))$ for $v$ a vector in the fiber of $L$ over the point $y$.

Consider a point $x_0 \in \frac{1}{2}V$. Denote by $\varphi_{x_0} = \varphi \circ \psi_{x_0}^{-1}$ the function $\varphi$ in local coordinates $\psi_{x_0}$. Denote also by $\varphi_{x_0}^{[1]}$ and $\varphi_{x_0}^{[2]}$ the first and second order Taylor expansions of $\varphi_{x_0}$, i.e.,
\[
\varphi_{x_0}^{[1]}(Z) := \sum_{j=1}^{n} \left( \frac{\partial \varphi}{\partial z_j}(x_0) z_j + \frac{\partial \varphi}{\partial \bar{z}_j}(x_0) \bar{z}_j \right),
\]
(2.24)
\[
\varphi_{x_0}^{[2]}(Z) := \text{Re} \sum_{j,k=1}^{n} \left( \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(x_0) z_j z_k + \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(x_0) z_j \bar{z}_k \right),
\]
where we write $z = (z_1, \ldots, z_n)$ the complex coordinates of $Z$.

Let $\rho : \mathbb{R} \to [0, 1]$ be a smooth even function such that
\[
\rho(t) = 1 \text{ if } |t| < 2; \quad \rho(t) = 0 \text{ if } |t| > 4.
\]
(2.25)

We denote in the sequel $X_0 = \mathbb{R}^{2n} \simeq T_{x_0}X$ and equip $X_0$ with the metric $g^{TX_0}(Z) := g^X(\rho(\epsilon_0^{-1}|Z|) |Z|)$. Now let $0 < \epsilon < \epsilon_0$ and define
\[
\varphi_{x}(Z) := \rho(\epsilon^{-1}|Z|)\varphi_{x_0}(Z) + (1 - \rho(\epsilon^{-1}|Z|))\left(\varphi(x_0) + \varphi_{x_0}^{[1]}(Z) + \varphi_{x_0}^{[2]}(Z)\right).
\]
(2.26)
Let $h_{\epsilon}^{L_0}$ be the metric on $L_0 = X_0 \times \mathbb{C}$ defined by
\begin{equation}
|1|_{\epsilon\phi}^2(Z) := e^{-2\varphi_{\epsilon}(Z)}.
\end{equation}
Here, as above, the subscript $\varphi_{\epsilon}$ informs the use of the weight $\varphi_{\epsilon}$. Let $\nabla_{\epsilon}^{L_0}$ be the Chern connection on $(L_0, h_{\epsilon}^{L_0})$ and $R_{\epsilon}^{L_0}$ be the curvature of $\nabla_{\epsilon}^{L_0}$.

Then there exists a constant $A$ with $c|d\phi|_2^{-1} < A < 1$ for $c > 0$ such that when $\epsilon \leq A\zeta$, the following estimate holds for every $x_0 \in U$,
\begin{equation}
(2.28) \quad \inf \left\{ \sqrt{-1}R_{\epsilon,z}^{L_0}(u, Ju)/|u|_{g_{TX_0}}^2 : u \in T_Z X_0, Z \in X_0 \right\} \geq \frac{4}{5} \zeta,
\end{equation}
because there exists $C > 0$ such that for $|Z| \leq 4\epsilon$, $0 \leq j \leq 2$, we have
\begin{equation}
(2.29) \quad |\varphi_{x_0}(Z) - (\varphi(x_0) + \varphi_{x_0}^{[1]}(Z) + \varphi_{x_0}^{[2]}(Z))|_{\epsilon} \leq C|d\phi|_2|Z|^{3-j}.
\end{equation}

From now on, we take
\begin{equation}
(2.30) \quad \epsilon := \epsilon_0 A\zeta.
\end{equation}
Let $S_{x_0}$ be the unitary section of $(L_0, h_{\epsilon}^{L_0})$ that can be written as $S_{x_0} = e^{-\tau}1$ with $\tau(x_0) = \varphi(x_0)$. So we have
\begin{equation}
(2.31) \quad \nabla_{\epsilon}^{L_0} S_{x_0} = i_Z (-d\tau - 2\partial\varphi) S_{x_0} = 0
\end{equation}
and hence the function $\tau$ is given by
\begin{equation}
(2.32) \quad \tau(Z) = \varphi(x_0) - 2 \int_0^1 (i_Z \partial\varphi)_{1z} dt.
\end{equation}
Let
\begin{equation}
(2.33) \quad D_p^{X_0} = \sqrt{2} (\overline{\partial}_{\epsilon}^{L_0} + \overline{\partial}_{\epsilon \phi}^{L_0})
\end{equation}
be the Dolbeault operator on $X_0$ associated with the above data, i.e., $\overline{\partial}_{\epsilon \phi}^{L_0}$ is the adjoint of $\overline{\partial}_{\epsilon}^{L_0}$ with respect to the metrics $g^{TX_0}$ and $h_{\epsilon}^{L_0}$. Over the ball $B(x_0, 2\epsilon)$, $D_p$ is just the restriction of $D_p^{X_0}$. Now by [12] Th. 1.4.7, and observe that the tensors associated with $g^{TX_0}$ do not depend on $\zeta$ and $\epsilon$, as in (2.7), we get from (2.28), the existence of a constant $\delta > 0$ such that for $\zeta p > \delta$,
\begin{equation}
(2.34) \quad \text{Spec} (D_p^{X_0})^2 \subset \{0\} \cup [\zeta p, +\infty[.
\end{equation}

Using $S_{x_0}$, we get an isometry $L_0^p \simeq \mathbb{C}$. Let $P^0_p$ be the orthogonal projection from $\mathcal{C}^\infty(X_0, L_0^p) \simeq \mathcal{C}^\infty(X_0, \mathbb{C})$ on $\text{Ker} D_p^{X_0}$. Let $P^0_p(x, x')$ be the smooth kernel of $P^0_p$ with respect to the volume form $dv_{X_0}(x')$ induced by the metric $g^{TX_0}$. We have the following result.

**Proposition 2.3.** For all $l \in \mathbb{N}$, there exists $c > 0$ such that for $\zeta p > \delta$, $x, x' \in B(x_0, \epsilon)$,
\begin{equation}
(2.35) \quad \left\| (P^0_p - P_p)(x, x') \right\|_{g_0} \leq c \left( |d\phi|_2^{-1} \zeta \right)^{-6n-3l-6p-l} |\omega|_l^{2n+2}.
\end{equation}

**Proof.** First, we replace $f_{\epsilon_0}(v)$ in (2.8) by $f_{\epsilon_0}(v/A)$. By Remark 2.2 and (2.30), for $x, x' \in B(x_0, \epsilon)$, we have $F_\epsilon(D_p)(x, x') = F_\epsilon(D_p^0)(x, x')$. Now we have a version of Proposition 2.1 for $P^0_p$ with $A\zeta$ instead of $\zeta$. Estimate (2.35) follows. \qed
3. Uniform estimate of the Bergman kernels

We continue to use the notations introduced at the end of the last section. By Proposition [2.3] in order to study the kernel $P_p$, it suffices to study the kernel $P_0$. For this purpose, we will rescale the operator $(D_p^2, X_0)^2$. Let $dV_X$ be the Riemannian volume form of $(T_{x_0}, X, g_{T_{x_0}})$, and let $\kappa$ be the smooth positive function defined by the equation

\begin{equation}
\kappa(Z) = \kappa(Z)dV_X(Z),
\end{equation}

with $\kappa(0) = 1$.

Let $\{e_j\}_{j=1}^{2n}$ be an oriented orthonormal basis of $T_{x_0}$, and let $\{e_j\}_{j=1}^{2n}$ be its dual basis. They allow us to identify $X_0 = \mathbb{C}^n$ with $\mathbb{R}^{2n}$ and we write $Z = (Z_1, \ldots, Z_{2n})$. If $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a multi-index, set $Z^\alpha := Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}$. Denote by $\nabla_U$ the ordinary differentiation operator on $T_{x_0}$ in the direction $U$, and set $t_j := \nabla_{e_j}$. Then we denote by $t := p^{-1/2}$. For $s \in C^\infty(\mathbb{R}^{2n}, \mathbb{C})$ and $Z \in \mathbb{R}^{2n}$, define

\begin{equation}
(S_t(s))(Z) = s(Z/t), \quad \nabla_t := tS_t^{-1} \kappa^{-1/2} \nabla L_0^t \kappa^{-1/2} S_t,
\end{equation}

\begin{equation}
L_t := \kappa^{-1/2} (D^X) \kappa^{-1/2} S_t.
\end{equation}

Once we did the trivialization of $L_0$ on $X_0$, (3.2) is well-defined for any $p \in \mathbb{R}$, $p \geq 1$

The notations $(\cdot, \cdot)_0$ and $\| \cdot \|$ mean respectively the inner product and the $L^2$-norm on $C^\infty(X_0, \mathbb{C})$ induced by $g^X_0$. For $s \in C^\infty_0(X_0, \mathbb{C})$, set

\begin{equation}
\|s\|_{t,0}^2 := \int_{\mathbb{R}^{2n}} |s(Z)|^2 dV_X(Z),
\end{equation}

\begin{equation}
\|s\|_{t,m}^2 := \sum_{t=0}^m \sum_{j_1, \ldots, j_t=1}^{2n} \|\nabla_{t,e_{j_1}} \cdots \nabla_{t,e_{j_t}} s\|_{t,0}^2.
\end{equation}

We then, for convenience, denote by $(s, s')_{t,0}$ the inner product on $C^\infty(X_0, L_0^\infty)$ corresponding to the norm $\| \cdot \|_{t,0}$. Let $H^m_t$ be the Sobolev space of order $m$ with norm $\| \cdot \|_{t,m}$. Let $H^{-1}_t$ be the Sobolev space of order $-1$ and let $\| \cdot \|_{t,-1}$ be the norm on $H^{-1}_t$ defined by $\|s\|_{t,-1} := \sup_{0 \neq \phi \in H^{-1}_t} \frac{\|\phi s\|_{t,0}}{\|\phi\|_{t,0}}$. If $B : H^m_t \rightarrow H^{m'}_t$ is a bounded linear operator for $m, m' \in \mathbb{Z}$, denote by $\|B\|^{m,m'}_{t}$ the norm of $B$ with respect to the norms $\| \cdot \|_{t,m}$ and $\| \cdot \|_{t,m'}$.

Theorems [3.1], [3.2], [3.4] and Proposition [3.3] below are the analogues of [12] Th. 4.1.9-4.1.14 (cf. also [8] Th. 4.7-4.10). The emphasis here is the precise dependence of the involved constants on the curvature form $\omega$.

**Theorem 3.1.** There exist $c_1, c_2, c_3 > 0$ such that for $t \in [0, 1]$, $\zeta \in [0, 1]$, and $s, s' \in C^\infty_0(\mathbb{R}^{2n}, \mathbb{C})$,

\begin{equation}
(s, s')_{t,0} \geq c_1 \|s\|_{t,1}^2 - c_2 |\omega|_0 \|s\|_{t,0}^2,
\end{equation}

\begin{equation}
|\langle L_t s, s' \rangle_{t,0}| \leq c_3 |\omega|_0 \|s\|_{t,1} \|s'\|_{t,1}.
\end{equation}

**Proof.** By using the Lichnerowicz formula [12] (4.1.33)], the same arguments as in [12] (4.1.38)-(4.1.39) give the result. \[\square\]

Let $\delta_\zeta$ be the counter-clockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\zeta/2$. 


Theorem 3.2. There exists $\delta > 0$ such that the resolvent $(\lambda - \mathcal{L}_t)^{-1}$ exists for all $\lambda \in \delta_c$ and $t \in [0, \sqrt{\zeta/\delta}]$. There exists $c > 0$ such that for all $t \in [0, \sqrt{\zeta/\delta}]$, $\lambda \in \delta_c$, we have
\[
(3.5) \quad \|(\lambda - \mathcal{L}_t)^{-1}\|_t^0 \leq 2\zeta^{-1}, \quad \|(\lambda - \mathcal{L}_t)^{-1}\|_t^{-1,1} \leq c \sqrt{\zeta^{-1}}.
\]
Proof. By (2.34) and (3.2), we have
\[
(3.6) \quad \text{Spec}(\mathcal{L}_t) \subset \{0\} \cup [\zeta, +\infty].
\]
Thus, the resolvent $(\lambda - \mathcal{L}_t)^{-1}$ exists for $\lambda \in \delta_c$ and $t \in [0, \sqrt{\zeta/\delta}]$, and we get the first inequality of (3.5).

By (3.4), $(\lambda_0 - \mathcal{L}_t)^{-1}$ exists for $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leq -2c_2 |\omega|_0$. Moreover, as $c_1 \|s\|_{t,1}^2 \leq -(\mathcal{L}_0 - \mathcal{L}_t) s, s\|_t,0 \leq \|(\lambda_0 - \mathcal{L}_t) s\|_{t,1} \|s\|_{t,1}$, we have
\[
(3.7) \quad \|(\lambda_0 - \mathcal{L}_t)^{-1}\|_t^{-1,1} \leq \frac{1}{c_1}.
\]
On the other hand, we have
\[
(3.8) \quad (\lambda - \mathcal{L}_t)^{-1} = (\lambda_0 - \mathcal{L}_t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathcal{L}_t)^{-1}(\lambda_0 - \mathcal{L}_t)^{-1}.
\]
Therefore, for $\lambda \in \delta_c$, from the first estimate in (3.5) and (3.8), we get
\[
(3.9) \quad \|(\lambda - \mathcal{L}_t)^{-1}\|_t^{-1,0} \leq \frac{1}{c_1} (1 + 2|\lambda - \lambda_0|\zeta^{-1}).
\]
In (3.8), we can interchange the last two factors. Then, applying (3.7) and (3.9) gives
\[
(3.10) \quad \|(\lambda - \mathcal{L}_t)^{-1}\|_t^{-1,1} \leq \frac{1}{c_1} + \frac{|\lambda - \lambda_0|}{c_1^2} (1 + 2|\lambda - \lambda_0|\zeta^{-1}) \leq c \sqrt{\zeta^{-1}}.
\]

The theorem follows.

Proposition 3.3. Take $m \in \mathbb{N}^*$. There exists $c > 0$ such that for $t \in [0, 1]$, $Q_1, \ldots, Q_m \in \{\nabla_{t,e_i}, Z_j\}_{j=1}^{2n}$ and $s, s' \in \mathcal{C}_\infty^0(X_0, \mathbb{C})$,
\[
(3.11) \quad |\langle [Q_1, [Q_2, \ldots [Q_m, \mathcal{L}_t] \ldots] s, s'\rangle |_t,0 \| \leq c|d\phi|_{m+1}^{\min(2,m)} \|s\|_{t,1} \|s'\|_{t,1}.
\]

Proof. By (12.1(6.31)) and as in the proof of (12 Proposition 1.6.9), we know that $[Q_1, [Q_2, \ldots [Q_m, \mathcal{L}_t] \ldots]$ has the same structure as $\mathcal{L}_t$ for $t \in [0, 1]$. More precisely, it has the form
\[
(3.12) \quad \sum_{i,j} a_{ij}(t, tZ) \nabla_{t,e_i} \nabla_{t,e_j} + \sum_j d_j(t, tZ) \nabla_{t,e_j} + c(t, tZ),
\]
where $a_{ij}(t, Z)$ and its derivatives in $Z$ are uniformly bounded, $d_j(t, Z), c(t, Z)$ and their first derivatives in $Z$ are bounded by $c|d\phi|_{m+1}^{\min(2,m)}$ for $Z \in \mathbb{R}^{2n}$ and $t \in [0, 1]$ and a constant $c > 0$. We then get estimate (3.11).

Theorem 3.4. For $Q_1, \ldots, Q_m \in \{\nabla_{t,e_i}, Z_j\}_{j=1}^{2n}$, there exists $c > 0$ such that for $t \in [0, \sqrt{\zeta/\delta}]$, $\lambda \in \delta_c$ and $s \in \mathcal{C}_\infty^0(X_0, \mathbb{C})$,
\[
(3.13) \quad \|Q_1 \cdots Q_m (\lambda - \mathcal{L}_t)^{-1} s\|_{t,1} \leq c \sum_{k=0}^m \sum_{1 \leq j_1 < \cdots < j_k \leq m} |d\phi|_{m-k}^{\min(2,m)} (|\omega|_0^2)^{\min(2,m)} \|Q_{j_1} \cdots Q_{j_k} s\|_{t,0}.
\]
Proof. For $Q_1, \ldots, Q_m \in \{\nabla_{t,e_i}, Z_i\}_{i=1}^{2n}$, we can express $Q_1 \cdots Q_m (\lambda - \mathcal{L}_t)^{-1}$ as the sum of $(\lambda - \mathcal{L}_t)^{-1} Q_1 \cdots Q_m$ with a linear combination of operators of the type

$$[Q_{j_1}, [Q_{j_2}, \ldots [Q_{j_m}, (\lambda - \mathcal{L}_t)^{-1}] \ldots]] Q_{j_{m+1}} \cdots Q_{j_m},$$

with $j_1 < j_2 < \cdots < j_m, j_{m+1} < \cdots < j_m$. The coefficients of this combination are bounded when $m$ is bounded. Let $\mathcal{S}_t$ be the family of operators

$$[Q_{j_1}, [Q_{j_2}, \ldots [Q_{j_m}, \lambda - \mathcal{L}_t] \ldots]].$$

Note that

$$[Q, (\lambda - \mathcal{L}_t)^{-1}] = -(\lambda - \mathcal{L}_t)^{-1}[Q, \lambda - \mathcal{L}_t](\lambda - \mathcal{L}_t)^{-1} = (\lambda - \mathcal{L}_t)^{-1}[Q, \mathcal{L}_t](\lambda - \mathcal{L}_t)^{-1},$$

thus by the recurrence on $m_1$ we know that every commutator $[Q_{j_1}, [Q_{j_2}, \ldots [Q_{j_m}, \lambda - \mathcal{L}_t] \ldots]]$ is a linear combination of operators of the form

$$(\lambda - \mathcal{L}_t)^{-1} S_1(\lambda - \mathcal{L}_t)^{-1} S_2 \cdots S_{m_2} (\lambda - \mathcal{L}_t)^{-1}$$

with $S_1, \ldots, S_{m_2} \in \mathcal{S}_t$ and $m_2 \leq m_1$. The coefficients of this combination are bounded when $m_1$ is bounded.

From Proposition 3.3 we deduce that the $\| \cdot \|_{t,1}^{-1}$ norms of the operators $[Q_{j_1}, [Q_{j_2}, \ldots [Q_{j_m}, \mathcal{L}_t] \ldots]]$ are uniformly bounded from above by a constant times $|d\phi|_{t+1}^2$. Hence, by Theorem 3.2, the $\| \cdot \|_{t,1}^1$ norm of the operator (3.15) is bounded by a constant times

$$\zeta^{-m_2-1} |\omega|_0^{2m_2+2} \sum_{l_1 + \cdots + l_{m_2} = m_1} \prod_{j=1}^{m_2} |d\phi|_{l_j+1}^\min(2,l_j).$$

The theorem follows. \square

Let $\mathcal{P}_t : (\mathcal{C}^\infty(X_0, \mathbb{C}), \| \cdot \|_t) \to \text{Ker}(\mathcal{L}_t)$ be the orthogonal projection corresponding to the norm $\| \cdot \|_{t,0}$ given in (3.3). Let $\mathcal{P}_t(Z, Z')$, with $Z, Z' \in X_0$ be the smooth kernel of $\mathcal{P}_t$ with respect to $d\nu_{TX}(Z')$. Note that $\mathcal{L}_t$ is a family of differential operators on $T_{x_0}X$ with coefficients in $\mathbb{C}$. Let $\pi : TX \times_X TX \to X$ be the natural projection from the fiberwise product of $TX$ with itself on $X$. We can view $\mathcal{P}_t(Z, Z')$ as smooth functions over $TX \times_X TX$ by identifying a section $F \in \mathcal{C}^\infty(TX \times_X TX, \mathbb{C})$ with the family $(F_{x_0})_{x_0 \in X}$, where $F_{x_0} := F|_{\pi^{-1}(x_0)}$. In the following result we adapt [12, Theorem 4.1.24] to the present situation.

**Theorem 3.5.** For any $r \in \mathbb{N}$, $\sigma > 0$, there exists $c > 0$, such that for $t \in ]0, \sqrt{\zeta/\delta}]$ and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| \leq \sigma,$

$$\left\| \frac{\partial^r}{\partial t^r} \mathcal{P}_t(Z, Z') \right\|_{\mathcal{C}^\infty(X)} \leq c\zeta^{-2n-4r-2} |d\phi|_{2r+n+1}^4 |d\phi|_{n+2}^{2n+2} |\omega|_0^{8r+4n+4}.$$

**Proof.** By (3.6), for every $k \in \mathbb{N}^*$,

$$\mathcal{P}_t = \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{R}^n} x^{k-1} (\lambda - \mathcal{L}_t)^{-k} d\lambda.$$

For $m \in \mathbb{N}$, let $\mathcal{Q}^m$ be the set of operators $\nabla_{t,e_i} \cdots \nabla_{t, e_j}$ with $j \leq m$. We apply Theorem 3.4 to $m-1$ operators $Q_2, \ldots, Q_m$ instead of $m$ operators. We deduce that for $l, m \in \mathbb{N}^*$
with \( l \geq m \), and \( Q = (Q_1, \ldots, Q_m) \in \mathbb{Q}^m \), there are \( c, c' > 0 \) such that for \( t \in [0, \sqrt{\zeta/\delta}] \), \( \zeta \in [0, 1] \), \( s \in C^0(X_0, \mathbb{C}) \), and \( \lambda \in \delta_\zeta \\
\|Q_1 \cdots Q_m(\lambda - \mathcal{L}_t)^{-l}s\|_{l,0} \leq c\|Q_2 \cdots Q_m(\lambda - \mathcal{L}_t)^{-l}s\|_{l,1} \\
(3.18) \leq c' \sum_{k=0}^{m-1} \sum_{1 \leq i_1 < \cdots < i_k \leq m} |d\phi|^{m-k-1}_{m-k}(|\omega|^{2}_{0} \zeta^{-1})^{m-k}\|Q_{i_1} \cdots Q_{i_k}(\lambda - \mathcal{L}_t)^{-l+1}s\|_{l,0}.
\)

Then, by induction and using (3.3), we get

\( (3.19) \|Q_1 \cdots Q_m(\lambda - \mathcal{L}_t)^{-l}s\|_{l,0} \leq c\zeta^{-m-l+1}|d\phi|_{m-1}^{m-1}|\omega|^{2m}_{0}\|s\|_{l,0}. \)

As \( \mathcal{L}_t \) is symmetric, we can consider the adjoint of the operator in (3.19) and get for \( Q' = (Q'_1, \ldots, Q'_{m'}) \in \mathbb{Q}^{m'} \\
(3.20) \|Q(\lambda - \mathcal{L}_t)^{-l}Q_1 \cdots Q_{m'}s\|_{l,0} \leq c\zeta^{-m'-l+1}|d\phi|^{m'-1}_{m'}|\omega|^{2m'}_{0}\|s\|_{l,0}. \)

Note that for \( m = 0 \) and \( l \in \mathbb{N} \) we also have \( \|Q(\lambda - \mathcal{L}_t)^{-l}s\|_{l,0} \leq c\zeta^{-l}|s|_{l,0}. \) Thus, for \( Q \in \mathbb{Q}^m, Q' \in \mathbb{Q}^{m'} \) with \( m, m' > 0 \), by taking \( k = m + m' \), we get

\( (3.21) \|QP_tQ'\|_{l,0}^{0,0} \leq \frac{1}{2\pi} \int_{\delta_\zeta} |\lambda|^{m+m'-1}Q(\lambda - \mathcal{L}_t)^{-m-m'}Q'\|_{l,0}^{0,0}|d\lambda| \\
\leq c|d\phi|^{m-1}_{m}|\omega|^{2m}_{0}|d\phi|^{m-1}_{m'}|\omega|^{2m'}_{0}\zeta^{-2m-2m'+2}\zeta^{m+m'} \\
= c|d\phi|^{m-1}_{m}|\omega|^{2m}_{0}|d\phi|^{m-1}_{m'}|\omega|^{2m'}_{0}\zeta^{-m-m'+2}.
\)

By [12, Lemma 1.2.4], (2.31), (2.32) and (3.2), on \( \mathbb{B}^{T_{x_0}X}(0, \varepsilon/t) \),

\( (3.22) \nabla_{t, e_i} |Z = \nabla_{e_i} + \frac{1}{2} R_{x_0}^L(Z, e_i) + O(t|Z|^2) \),

Let \( | \cdot |_{(\sigma), m} \) denote the usual Sobolev norm on \( C^0(\mathbb{B}^{T_{x_0}X}(0, \sigma + 1), \mathbb{C}) \) induced by the volume form \( du_{T_X}(Z) \) as in (3.3). Observe that by (3.3), (3.22), for \( m > 0 \), there exists \( c > 0 \) such that for \( s \in C^\infty(X_0, \mathbb{C}) \) with supp \( (s) \subset B(0, \sigma + 1) \),

\( (3.23) \frac{1}{c|d\phi|^{m}_{m}}\|s\|_{l,m} \leq |s|_{(\sigma), m} \leq c|d\phi|^{m}_{m+1}\|s\|_{l,m}. \)

Now, we want to estimate \( Q_2Q'_Z P_l(Z, Z') \) using the standard Sobolev’s inequality for \( Q \in \mathbb{Q}^m \) and \( Q' \in \mathbb{Q}^{m'} \). If we define \( S := QP_tQ' \) then we have for \( |Z|, |Z'| \leq \sigma \)

\( (3.24) |Q_2Q'_Z P_l(Z, Z')| \leq c\sup\left\{ \frac{|\partial^\sigma|}{\partial Z^\sigma} \frac{\partial^\alpha |s|}{\partial Z^\alpha} \right\}|_{(\sigma), n+1}, \\
\|s\|_{L^2} = 1, \text{supp } (s) \subset B(0, \sigma + 1), |\alpha|, |\alpha'| \leq n + 1 \}
\)

Hence, by (3.23), applied twice to \( n + 1 \) instead of \( m \), and also (3.21), applied to \( m + n + 1, m' + n + 1 \) instead of \( m, m' \), we get

\( (3.25) \sup_{|Z|, |Z'| \leq \sigma} |Q_2Q'_Z P_l(Z, Z')| \\
\leq c'|d\phi|^{m+n}_{m+n+1}|\omega|^{2m+2n+2}_{0}|d\phi|^{m'+n}_{m'+n+1}|\omega|^{2m'+2n+2}_{0}|d\phi|^{n+2}_{n+2} \zeta^{-m-m'-2n}.
\)

By (3.22) and (3.25) for \( m = m' = 0 \), estimate (3.16) holds for \( r = 0 \).
Consider now \( r \geq 1 \). Set
\[
I_{k,r} := \left\{ (k, r) = \{ (k_i, r_i) \}_{i=0}^j : \sum_{i=0}^j k_i = k + j \text{, } \sum_{i=1}^j r_i = r \text{, } k_i, r_i \in \mathbb{N}^r \right\}.
\]
Then there exist \( a_k^r \in \mathbb{R} \) such that
\[
A_k^r(\lambda, t) = \left( \lambda - \mathcal{L}_t \right)^{-k_0} \frac{\partial^{k_1} \mathcal{L}_t}{\partial t_1} \left( \lambda - \mathcal{L}_t \right)^{-k_1} \cdots \frac{\partial^{k_j} \mathcal{L}_t}{\partial t_j} \left( \lambda - \mathcal{L}_t \right)^{-k_j},
\]
(3.27)
\[
\frac{\partial^r}{\partial t^r} \left( \lambda - \mathcal{L}_t \right)^{-k} = \sum_{(k,r) \in I_{k,r}} a_k^r A_k^r(\lambda, t).
\]
Set \( g_{ij}(Z) := \langle \frac{\partial}{\partial Z_i}, \frac{\partial}{\partial Z_j} \rangle_Z \) and \( (g^{ij}) \) the inverse matrix of \( (g_{ij}) \). Note that \( \frac{\partial^n}{\partial t^n}(g^{ij}(tZ)) \), \( \frac{\partial^n}{\partial t^n}(\nabla_{t,e_i} - \frac{1}{2} \Gamma^L(tZ)) \) are functions which do not depend on \( \zeta \), and \( \frac{\partial^n}{\partial t^n} R^L(tZ), \frac{\partial^n}{\partial t^n} \Gamma(tZ) \) are functions of type \( d^2(tZ)Z^\beta \) and \( \nabla_{e_i} \cdots \nabla_{e_j} d^2(tZ) \) is uniformly controlled by \( |d\phi|_{l+u+1} \).

We handle now the operator \( A_k^r(\lambda, t)Q' \). We will move first all the terms \( Z^\beta \) in \( d^2(tZ)Z^\beta \) (defined above) to the right hand side of this operator. To do so, we always use the commutator trick as in the proof of [12] Theorem 1.6.10, i.e., each time, we perform only the commutation with \( Z_t \), not directly with \( Z^\beta \) with \(|\beta| > 1\). Then \( A_k^r(\lambda, t)Q' \) is as the form \( \sum_{|\beta| \leq 2r} \mathcal{L}_{\beta,t} \mathcal{Q}_{\beta} Z^\beta \), and \( \mathcal{Q}_{\beta} \) is obtained from \( Q' \) and its commutation with \( Z^\beta \). Observe that \( [Z_t, \mathcal{L}_t] \) is a first order differential operator and \([Z_{ij}, [Z_{j2}, \mathcal{L}_t]] = g^{j2}(tZ) \) is a bounded function. Therefore, \( \mathcal{L}_{\beta,t} \) is a linear combination of operators of the form
\[
(\lambda - \mathcal{L}_t)^{-k_1} S_1(\lambda - \mathcal{L}_t)^{-k_1} S_2 \cdots S_{r}(\lambda - \mathcal{L}_t)^{-k_r},
\]
with \( S_i \in \{ a(tZ) \nabla_{e_{i1}} \nabla_{e_{i2}}, d_{j1}(tZ) \nabla_{e_{j1}}, d_{j2}(tZ) \} \) and the number of \( \nabla_{e_{i1}} \) in all \( \{ S_i \} \), is less than \( \sum_i r_i + 2j = r + 2j \). As \( k > 2(r + 1) + m + m' \), we can split the above operator into two parts as in [12] (4.1.51) and use the fact that the term \( \nabla_{e_{i1}}(\lambda - \mathcal{L}_t)^{-k_1} \) will contribute \( \zeta^{-1} \). Similarly to (3.18), we get that \( A_k^r(\lambda, t) \) is well defined and for \( m, m' \in \mathbb{N} \), \( k > 2(r + 1) + m + m' \), \( Q \in Q^m, Q' \in Q^{m'} \), there exists \( c > 0 \) such that for \( \lambda \in \delta_\zeta \) and \( t \in [0, \sqrt{\zeta/\delta}] \),
\[
\| Q A_k^r(\lambda, t)Q's \|_{t,0} \leq c |d\phi|_{m+2r}\| \omega_0^{2m+4r} |d\phi|_{m+2r}^{m'-2r-1} |\omega_0^{2m'+4r} \zeta - \sum_{l=0}^{l=0} k_-m_-m_-3r \sum_{|\beta| \leq 2r} \| Z^\beta \|_{t,0} \]
\]
(3.29)
\[
\leq c |d\phi|_{m+2r}\| \omega_0^{2m+4r} |d\phi|_{m+2r}^{m'-2r-1} |\omega_0^{2m'+4r} \zeta - k_-m_-m_-4r \sum_{|\beta| \leq 2r} \| Z^\beta \|_{t,0} \]
\]

By (3.17), (3.27) and (3.29), as in (3.21), for \( m, r \in \mathbb{N}, Q \in Q^m \) and \( Q' \in Q^{m'} \), there exists \( c > 0 \) such that for \( t \in [0, \sqrt{\zeta/\delta}] \) and \( s \in C^0_0(X_0, C) \),
\[
\| \frac{\partial^r}{\partial t^r} P_t Q's \|_{t,0} \leq c |d\phi|_{m+2r}|d\phi|_{m+2r}^{m'+2r-1} |\omega_0^{2m+2m'+8r} \zeta - m_-m_-4r \sum_{|\beta| \leq 2r} \| Z^\beta \|_{t,0} \]
\]
(3.30)

Finally, (3.23) and (3.30) together with Sobolev’s inequalities imply for \( |Z|, |Z'| \leq \sigma \),
\[
\sup_{|Z|, |Z'| \leq \sigma} \left| \frac{\partial^r}{\partial t^r} P_t(Z, Z') \right| \leq c |d\phi|_{2(n+4r)}(2|2n+2+4r)|d\phi|_{2n+2} \zeta - 2n-4r-2 .
\]

This ends the proof of the theorem.
For $k$ big enough, set
\begin{equation}
F_r := \frac{1}{2\pi r!} \int_{\mathbb{C}} \chi^{k-1} \sum_{(r,k) \leq f_i} a^k_{i}(\lambda, 0) d\lambda.
\end{equation}
Let $F_r(Z, Z') \in \mathcal{C}^\infty(TX \times TX, \mathbb{C})$ be the smooth kernel of $F_r$ with respect to $dv_{TX}(Z')$.

**Theorem 3.6.** For all $j \in \mathbb{N}, \sigma > 0$, there exists $c > 0$ such that for $t \in [0, \sqrt{\zeta/\delta}]$ and $Z, Z' \in T_{x_0}X, |Z|, |Z'| \leq \sigma$, we have
\begin{equation}
\| (P_t - \sum_{r=0}^j F_r t^r) (Z, Z') \|_{\mathcal{C}^0(X)} \leq c |d\phi|_{2j+n+3}^2 |\omega|_{2j+2n+6}^2 |d\phi|_{n+2}^{2n+2} s^{-4j-2n-6+2j+1}.
\end{equation}

**Proof.** By [12, (4.1.69)], we have
\begin{equation}
\frac{1}{r!} \frac{\partial^r}{\partial t^r} P_t \big|_{t=0} = F_r.
\end{equation}
Recall that the Taylor expansion with integral rest of a function $G \in \mathcal{C}^{j+1}([0, 1])$ is
\begin{equation}
G(t) - \sum_{r=0}^j \frac{1}{r!} \frac{\partial^r G}{\partial t^r}(0) t^r = \frac{1}{j!} \int_0^t (t-t_0) \frac{\partial^{j+1} G}{\partial t^{j+1}}(t_0) dt_0, \quad t \in [0, 1].
\end{equation}

Theorem 3.5 and (3.34) show that the estimate (3.16) holds if we replace $\frac{1}{t!} \frac{\partial^r}{\partial t^r} P_t$ with $F_r$. Using this new estimate together with (3.35) and (3.16), we obtain (3.33).

Let $\mathcal{P}$ be the orthogonal projection from $L^2(X_0, \mathbb{C})$ onto $\text{Ker}(\mathcal{L}_0)$, and let $\mathcal{P}(Z, Z')$ be the smooth kernel of $\mathcal{P}$ with respect to $dv_{TX}(Z')$. Then $\mathcal{P}(Z, Z')$ is the Bergman kernel of $\mathcal{L}_0$. By [12, (4.1.84)], if we choose $\{w_j\}$ to be an orthonormal basis of $T_{x_0}^{(1,0)} X$ such that $\hat{R}_{x_0}^L = \text{diag}(a_1, \ldots, a_n) \in \text{End}(T_{x_0}^{(1,0)} X)$ with $\langle \hat{R}_{x_0}^L W, W \rangle = R^L(W, \overline{Y})$ for $W, Y \in T_{x_0}^{(1,0)} X$, then
\begin{equation}
\mathcal{P}(Z, Z') = \prod_{i=1}^n \frac{a_i}{2\pi} \exp \left( -\frac{1}{4} \sum_{i} a_i \left( |z_i|^2 + |z_i'|^2 - 2z_i \bar{z}_i' \right) \right).
\end{equation}
The following result was established in [12, Theorem 4.1.21].

**Theorem 3.7.** There exist polynomials $J_r(Z, Z')$ in $Z, Z'$ with the same parity as $r$ and $\deg J_r(Z, Z') \leq 3r$, whose coefficients are polynomials in $R^{TX}$ (resp. $R^L$) and their derivatives of order $\leq r - 2$ (resp. $\leq r$), and reciprocals of linear combinations of eigenvalues of $R^k$ at $x_0$, such that
\begin{equation}
F_r(Z, Z') = J_r(Z, Z') \mathcal{P}(Z, Z').
\end{equation}
Moreover, we have
\begin{equation}
J_0 = 1 \quad \text{and} \quad F_0 = \mathcal{P}.
\end{equation}

Owing to (3.1), (3.2), as in [12, (4.1.96)], we have
\begin{equation}
P^0_p(Z, Z') = t^{-2n} \kappa^{-1/2}(Z) \mathcal{P}_t(Z/t, Z'/t) \kappa^{-1/2}(Z'), \quad \text{for all } Z, Z' \in \mathbb{R}^{2n}.
\end{equation}

From Theorems 3.6 and 3.7 and (3.39), we get the following near-diagonal expansion of the Bergman kernels. Recall that we are working with $t = p^{-1/2}$.
**Theorem 3.8.** For every \( j \in \mathbb{N} \), there exists \( c > 0 \) such that the estimate

\[
(3.40) \quad \left| \frac{1}{p^n} P^n_p(Z, Z') - \sum_{r=0}^{j} F_r(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-1/2}(Z) \kappa^{-1/2}(Z') p^{-r/2} \right| \leq c |d\phi|^2_{2j+n+2} \omega_0^{2(2n+4j+6)} |d\phi|^2_{n+2} p^{-(j+1)/2} \zeta^{-2n-4j-6}
\]

holds for all \( 0 < \zeta \leq 1 \), \( \zeta p > \delta \), and all \( Z, Z' \in T_{x_0} X \) with \( |Z|, |Z'| \leq \min\{ \sigma/\sqrt{\delta}, \epsilon \} \).

**End of the proof of Theorem 1.1.** We apply Theorem 3.8 to \( Z = Z' = 0 \) and \( j = 1 \). Note that \( F_1(0, 0) = 0 \) because the function \( F_1(0, 0) \) is odd. By (3.36), \( P(0, 0) = \omega(x_0)^{n}/\theta(x_0)^{n} \). So from (3.40), we get

\[
(3.41) \quad \left\| \frac{1}{p^n} P^n_p(0, 0) - \frac{\omega(x_0)^{n}}{\theta(x_0)^{n}} \right\|_{\mathcal{E}^n(X)} \leq c |d\phi|^{2n+8} |\omega|_{n+5}^4 |d\phi|^{n+2} \zeta^{-2n+10} p^{-1}.
\]

We then deduce the result from Propositions 2.1, 2.3 and (3.41).

**Remark 3.9.** Assume now \( \phi \in \mathcal{E}^{n+2k+6} \). Then by the usual \( \mathcal{C}^k \)-norm on each \( U_j \) and Sobolev embedding theorem, from (2.22), we get

\[
(3.42) \quad \| F_{\zeta_0}(D_p)(x, x') - P_p(x, x') \|_{\mathcal{E}^k} \leq c \| \omega_{n+k}^{2n+2+2k} \zeta^{-6n-3l-6k} p^{-l} \|^{-1}.
\]

Note that \( \nabla^{L'} = d + p\Gamma \), cf. (2.15), thus if we use the \( \mathcal{E}^k \)-norm induced by \( \nabla^{L'} \), then we get

\[
(3.43) \quad \| F_{\zeta_0}(D_p)(x, x') - P_p(x, x') \|_{\mathcal{E}^k(X \times X)}
\]

\[
\leq c \sum_{r=0}^{k} \| \omega_{n+m}^{2n+2+2r} \zeta^{-6n-3l-6k} p^{-l} \|^{-1} \| \omega_{n+k}^{r} p^{-r} \|^{-1}
\]

\[
\leq c \| \omega_{n+k}^{2n+2+2k} \zeta^{-6n-9-6k} p^{-1} \|^{-1}.
\]

In the same way as (2.35) and above, we get

\[
(3.44) \quad \left\| \frac{P^n_p}{\theta^n} - P_p(x, x') \right\|_{\mathcal{E}^k(X \times X)} \leq c \{ |d\phi|_2^{-1} \zeta^{-6n-6k} p^{-1} \} |\omega|_{n+k}^{2n+2+2k}.
\]

Combining (12) (4.1.64) and the argument for (3.16), we get

\[
(3.45) \quad \left\| \frac{\partial^r}{\partial t^r} P_t(Z, Z') \right\|_{\mathcal{E}^m} \leq c \zeta^{-2n-4l-4m} |d\phi|_{2(r+m'+1)}^{4(r+m')-2} |\omega|_{n+m'+1}^{2(r+m')+4} |d\phi|_{n+2}^{2n+2}.
\]

Here \( \mathcal{E}^{m'} \) is the usual \( \mathcal{E}^{m'} \)-norm for the parameter \( x_0 \).

Thus we get an extension of (1.4),

\[
(3.46) \quad \left\| P^n_p(x, x) - \frac{\omega(x)^n}{\theta(x)^n} \right\|_{\mathcal{E}^k} \leq c \| d\phi \|_{n+2k+5} |\omega|_{n+2k+5}^{4n+8k+20} |d\phi|_{n+2}^{2n+2} \zeta^{-2n-4k-10} p^{-1}
\]

\[
+ c \| \omega |_{n+k}^{2n+2k+2} |d\phi| \zeta^{-6n+9+6k} p^{-1}.
\]

**Remark 3.10.** Let \( \phi \) be a function of class \( \mathcal{E}^\alpha \), \( 0 < \alpha \leq 1 \), which is \( \omega_0 \)-p.s.h., i.e., \( dd^c \phi + \omega_0 \geq 0 \). For each \( 0 < \zeta \leq 1 \), we can find a smooth \( \omega_0 \)-p.s.h. function \( \phi_\zeta \) such that

\[
\| \phi_\zeta \|_{\mathcal{E}^k} \leq c \zeta^{-k+\alpha} \quad \text{and} \quad dd^c \phi_\zeta + \omega_0 \geq \zeta \omega_0,
\]

see [10]. As mentioned in Introduction, we can study \( \phi \) by applying our results to \( \phi_\zeta \). Some steps in the proof of our estimates can be strengthened using \( \| \phi_\zeta \|_{\mathcal{E}^k} \leq c \zeta^{-k+\alpha} \) for each \( 0 \leq k \leq n + 6 \) instead of using only the \( \mathcal{E}^{n+6} \)-norm.
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