Regular operators
between non-commutative \( L_p \)-spaces.

by

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Abstract
We introduce the notion of a regular mapping on a non-commutative \( L_p \)-space associated to a hyperfinite von Neumann algebra for \( 1 \leq p \leq \infty \). This is a non-commutative generalization of the notion of regular or order bounded map on a Banach lattice. This extension is based on our recent paper [P3], where we introduce and study a non-commutative version of vector valued \( L_p \)-spaces. In the extreme cases \( p = 1 \) and \( p = \infty \), our regular operators reduce to the completely bounded ones and the regular norm coincides with the \( cb \)-norm. We prove that a mapping is regular if and only if it is a linear combination of bounded, completely positive mappings. We prove an extension theorem for regular mappings defined on a subspace of a non-commutative \( L_p \)-space. Finally, let \( R_p \) be the space of all regular mappings on a given non-commutative \( L_p \)-space equipped with the regular norm. We prove the isometric identity \( R_p = (R_\infty, R_1)^\theta \) where \( \theta = 1/p \) and where \( (\cdot, \cdot, \cdot)^\theta \) is the dual variant of Calderón’s complex interpolation method.

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§0. Introduction.

Let $L, \Lambda$ be Banach lattices. An operator $u: L \to \Lambda$ is called regular (or sometimes “order bounded”) if there is a constant $C$ such that for all finite sequences $x_1, \ldots, x_n$ in $L$ we have

$$\|\sup_{i \leq n} |u(x_i)|\|_\Lambda \leq C \|\sup_{i \leq n} |x_i|\|_L.$$  

We denote by $\|u\|_r$ the smallest constant $C$ such that this holds. More generally, this definition makes sense for an operator $u: S \to \Lambda$ defined only on a subspace $S \subset L$ by considering (0.1) restricted to $n$-tuples of elements of $S$.

We will denote by $B_r(L, \Lambda)$ (resp. $B_r(S, \Lambda)$) the space of all such operators $u: L \to \Lambda$ (resp. $u: S \to \Lambda$) and we equip it with the norm $\|u\|_r$. Equivalently, let us denote $S[\ell^n_\infty]$ the space $S \otimes \ell^n_\infty$ equipped with the norm $\left\| \sum_{i=1}^n x_i \otimes e_i \right\| = \|\sup_{i \leq n} |x_i|\|_L$. Then an operator $u: S \to \Lambda$ is regular iff there is a constant $C$ such that for all $n$

$$\|u \otimes I_{\ell^n_\infty}\|_{S[\ell^n_\infty] \to \Lambda[\ell^n_\infty]} \leq C.$$  

Moreover $\|u\|_r$ is equal to the smallest constant $C$ for which this holds. It is well known (see [MN, p. 24]) that if $\Lambda$ is Dedekind complete (this is the terminology of [MN], the same notion is sometimes called order complete) then $u \in B_r(L, \Lambda)$ iff $u$ is a linear combination of bounded positive maps from $L$ into $\Lambda$. This is classical. More recently, we observed ([P2]) that the space $B_r(L_p, L_p)$ coincides with the complex interpolation space $(B(L_\infty, L_\infty), B(L_1, L_1))^\theta$ when $\theta = 1/p$. We also proved an extension theorem for $L_p$-valued regular maps defined on a subspace of $L_p$. In this paper, we wish to extend the notion of regularity and to present a non-commutative version of these results of [P2]. In particular, we will consider the Schatten $p$-class $S_p$, a subspace $S \subset S_p$ and an operator $u: S \to S_p$. It turns out that (0.1) and (0.2) can be formulated in this setting (although of course $S_p$ is not a Banach lattice in the usual sense).

In the non-commutative case, the space $S[\ell^n_\infty]$ is replaced by $S[M_n]$ where $M_n$ is the Banach space of all complex $n \times n$ matrices, equipped with the norm of $B(\ell^n_2)$. This extension is based on our recent paper [P3] (see also the announcement [P5]), where we introduce and study a non-commutative version of vector valued $L_p$-spaces. In that theory,
the space of values is an operator space, i.e. a closed subspace \( E \subset B(H) \) for some Hilbert space \( H \). In [P3] we defined the \( E \)-valued version of the Schatten \( p \)-class and we denoted it by \( S_p[E] \). If \( S \subset S_p \) is a closed subspace, we denote by \( S[E] \) the closure of \( S \otimes E \) in \( S_p[E] \). In particular this makes sense if \( E = M_n \) and it allows us to extend the definition of regular operators.

In particular, we will prove below

**Theorem 0.1.** Let \( 1 \leq p < \infty \). Let \( u: S_p \to S_p \) be a linear map. The following are equivalent.

(i) There is a constant \( C \) such that for all \( n \), \( u \otimes I_{M_n} \) is bounded on \( S_p[M_n] \) and

\[
\|u \otimes I_{M_n}\|_{S_p[M_n] \to S_p[M_n]} \leq C.
\]

(ii) The map \( u \) is a linear combination of bounded, completely positive maps on \( S_p \).

(iii) There are completely positive, bounded linear maps \( u_j: S_p \to S_p \) (\( j = 1, 2, 3, 4 \)) such that \( u = u_1 - u_2 + i(u_3 - u_4) \).

Here of course a linear map \( v: S_p \to S_p \) is called completely positive if for each \( n \geq 1 \) the map \( I_{M_n} \otimes v: S_p(\ell_2^n \otimes \ell_2) \to S_p(\ell_2^n \otimes \ell_2) \) is positive (i.e. positivity preserving). This extends the usual complete positivity in the case \( p = \infty \).

We will abbreviate completely positive by \( c.p. \). Note that in [P1] we proved the following result (see [P1, section 8]).

**Theorem 0.2.** Let \( 0 < \theta < 1 \) and \( p = 1/\theta \). Then the space

\[
(cb(S_\infty, B(\ell_2)), cb(S_1, S_1))^\theta
\]

coincides with the space of all operators \( u: S_p \to S_p \) which are linear combinations of bounded c.p. maps on \( S_p \) (or equivalently which satisfy (iii) in Theorem 0.1).

In section 1 below, we summarize the results from [P1, P3] that we will need. In section 2 we discuss completely positive maps and regular maps on \( S_p \). In section 3, we prove our main results including Theorem 0.1. Finally we will prove an extension theorem: any regular map \( u: S \to S_p \) defined on a subspace of \( S_p \) extends to a regular map on \( S_p \) with the same regular norm. This generalizes to \( S_p \) the well known extension property of c.b. maps into \( B(\ell_2) \).
We have restricted ourselves for simplicity to the simplest non-commutative $L_p$-space, i.e. $S_p$, but it is clearly possible to extend our results to the setting of $L_p$-spaces associated to a hyperfinite (or equivalently injective, by Connes’s well known results [Co]) von Neumann algebra $M$ equipped with a semi-finite trace as in [P3]. See [N, H2, Ko, Te1, Te2] and the references in [N], for the classical (i.e. scalar valued) theory of non-commutative $L_p$-spaces. The resulting statements can be viewed as an extension to a non-commutative $L_p$-space of Wittstock’s well known extension/decomposition theorem for $cb$ maps into an injective von Neumann algebra. Indeed, viewing a von Neumann algebra as a non-commutative $L_\infty$-space, Wittstock’s result corresponds to the case $p = \infty$. We state this extension without proof at the end of the paper. We leave the details to the reader.
\section{Background on operator spaces.}

In this section, we recall a number of notions and results that we will use.

By an operator space, we mean a closed subspace $E \subset B(H)$ of the space $B(H)$ of all bounded operators on some Hilbert space $H$. We will use the theory of operator spaces, as developed recently in a series of papers [BP, ER2]. In this theory, an operator space structure is a sequence of norms on the spaces $M_n(E)$ of all $n \times n$ matrices with entries in $E$. If $E \subset B(H)$ is a subspace of $B(H)$, its natural operator space structure is by definition the one corresponding to the norms induced on $M_n(E)$ by $B(\ell_2^2(H))$, more precisely to the norms
\begin{equation}
\| (a_{ij}) \|_n = \| (a_{ij}) \|_{B(\ell_2^2(H))}.
\end{equation}

Given two embeddings $E \subset B(H), E \subset B(K)$ of the same space into $B(H)$ and $B(K)$ we will say that they define the same operator space structure if the associated norms (1.1) are the same for both embeddings.

By a fundamental result of Ruan [Ru] the sequences of norms $\| \|_n$ which arise in this way can be characterized by very simple axioms (see [Ru]). We will refer to the sequence (1.1) of norms $\| \|_n$ on $M_n(E)$ as “the operator space structure” on $E$. Note that the Banach space structure of $E$ is included via the case $n = 1$.

Thus, to give ourselves an operator space structure on a vector space $E$ is the same as to give ourselves an equivalence class of embeddings $E \rightarrow B(H)$ (two embeddings are equivalent if they give the same structure via (1.1)). Let $E, F$ be operator spaces, a map $u: E \rightarrow F$ is called completely bounded (in short c.b.) if $I_{M_n} \otimes u$ is bounded from $M_n(E)$ into $M_n(F)$ and $\sup_{n \geq 1} \| I_{M_n} \otimes u \| < \infty$. We define $\| u \|_{cb} = \sup_{n \geq 1} \| I_{M_n} \otimes u \|$. We denote by $cb(E,F)$ the space of all such maps equipped with this norm. We refer to [Pa] for more information. We will say that $u: E \rightarrow F$ is completely isomorphic (resp. completely isometric) if $u$ is an isomorphism and $u$ and $u^{-1}$ are c.b. (resp. if $I_{M_n} \otimes u: M_n(E) \rightarrow M_n(F)$ is isometric for all $n \geq 1$).

In [BP, ER2], it was proved that $cb(E,F)$ can be equipped with an operator space structure by giving to $M_n(cb(E,F))$ the norm of the space $cb(E, M_n(F))$. In particular, this defines an operator space structure on the dual $E^* = cb(E, \mathbb{C})$ so that we have
isometrically

\[(1.2) \quad M_n(E^*) = cb(E, M_n). \]

This duality for operator spaces has many of the nice properties of the usual Banach space duality. See [BP, ER2, B2, BS, ER4] for more details.

In [P1], we introduced complex interpolation for operator spaces. Let \((E_0, E_1)\) be a compatible couple of Banach spaces in the sense of interpolation theory. We refer to [BL, Ca] for the basic notions and the definitions of the complex interpolation methods \((E_0, E_1)_\theta\) and \((E_0, E_1)^\theta\).

Now assume \(E_0, E_1\) each equipped with an operator space structure (in the form of norms on \(M_n(E_0)\) and \(M_n(E_1)\) for all \(n\)). Let \(E_\theta = (E_0, E_1)_\theta\) and \(E^\theta = (E_0, E_1)^\theta\). Then, we can define an operator space structure on \(E_\theta\) (resp. \(E^\theta\)) by setting

\[(1.3) \quad M_n(E_\theta) = (M_n(E_0), M_n(E_1))_\theta \quad (\text{resp.} \quad M_n(E^\theta) = (M_n(E_0), M_n(E_1))^\theta). \]

In [P1] we observed that these norms verify Ruan’s axioms and hence they define an operator space structure on \(E_\theta\) (resp. \(E^\theta\)).

Let \(H, K\) be Hilbert spaces. Let \(1 \leq p < \infty\). Let us denote by \(S_p\) (resp. \(S_p(H)\), \(S_p(H, K)\)) the space of all operators \(T: \ell_2 \to \ell_2\) (resp. \(T: H \to H\), resp. \(T: H \to K\)) such that \(\text{tr}|T|^p < \infty\). We equip it with the norms

\[\|T\|_p = (\text{tr}|T|^p)^{1/p}.\]

For notational convenience, we denote by \(S_\infty\) (resp. \(S_\infty(H), S_\infty(H, K)\)) the space of all compact operators on \(\ell_2\) (resp. on \(H\), resp. from \(H\) into \(K\)), equipped with the operator norm.

Since \(S_\infty\) and \(B(H)\) are \(C^*\)-algebras they have an obvious operator space structure, which we will call the natural operator space structure (in short o.s.s.). More generally, the space \(B(H, K)\) has a natural o.s.s. corresponding to the norm induced by \(B(\ell_2^n(H), \ell_2^n(K))\) on \(M_n(B(H, K))\). Since \(S_1 = S_\infty\), we can equip \(S_1\) with the dual operator space structure, as defined above in (1.2). Similarly for any predual \(X\) of a von Neumann algebra, we obtain by (1.2) an o.s.s. on \(X\) which we will call the natural o.s.s. on \(X\).
In particular for any measure space \((\Omega, \mu)\), the spaces \(L_\infty(\mu)\) and \(L_1(\mu)\) can now be viewed as operator spaces with their \textit{natural} o.s.s. Let \(0 < \theta < 1\) and \(p = 1/\theta\). By interpolation, using

\[
L_p(\mu) = (L_\infty(\mu), L_1(\mu))_\theta
\]

and

\[
S_p = (S_\infty, S_1)_\theta
\]

we define using (1.3) an o.s.s. on \(L_p(\mu)\) and \(S_p\), which we will again call the \textit{natural} one.

We now turn to the vector valued case. Let \(E, F\) be operator spaces. We denote by \(E \otimes_{\text{min}} F\) the completion of \(E \otimes F\) for the minimal (= spatial) tensor product. If \(E \subset B(H)\), then \(L_\infty(\mu; E) \subset L_\infty(\mu; B(H))\) so that we can equip \(L_\infty(\mu; E)\) with the o.s.s. induced by \(L_\infty(\mu; B(H))\). Similarly, we define \(S_\infty[E] = S_\infty \otimes_{\text{min}} E\) and we equip it with the natural o.s.s. of the minimal (= spatial) tensor product. In the finite dimensional case, we define \(S_\infty^n[E] = M_n \otimes_{\text{min}} E\). This is the same as \(M_n(E)\), which we will sometimes also denote by \(M_n[E]\). This settles the simplest case \(p = \infty\).

We now turn to the case \(p = 1\). We define

\[
L_1(\mu; E) = L_1(\mu) \otimes^\wedge E
\]

and

\[
S_1[E] = S_1 \otimes^\wedge E,
\]

where the symbol \(\otimes^\wedge\) refers to the operator space version of the projective tensor norm introduced in [BP, ER2] and developed in [ER6]. This defines operator space structures in \(L_1(\mu; E)\) and \(S_1[E]\). By interpolation, we define

\[
(1.4) \quad L_p(\mu; E) = (L_\infty(\mu; E), L_1(\mu; E))_\theta \text{ and } S_p[E] = (S_\infty[E], S_1[E])_\theta,
\]

and using (1.3) this gives us an operator space structure on \(L_p(\mu; E)\) and \(S_p[E]\). We will refer to all these structures in each case as the \textit{natural one}.

Clearly, a similar definition applies to \(S_p(H)\) and \(S_p(H, K)\). We will denote by \(S_p[H; E]\) and \(S_p[H, K; E]\) the resulting operator spaces. Moreover if \(H = \ell_2^n\) we will denote simply by \(S^n_p[E]\) the space \(S_p[\ell_2^n; E]\). Note that by definition \(S_p[E]\) is included into
the space $M_\infty(E)$, and $S_p^n[E]$ (which is clearly linearly isomorphic to $M_n(E)$) is completely isometric to the subspace of $S_p[E]$ formed by the matrices with entries in $E$ which have zero coefficients outside the “upper left corner” of size $n \times n$.

We will also need an explicit formula for the norm in the spaces $S_p[H;E]$ or $S_p[H,K;E]$ as described in [P3].

For any $x$ in $S_p[H,K;E]$ (viewed as an element of the larger space $S_\infty(H,K) \otimes_{\min} E$) we have

\[
\|x\|_{S_p[H,K;E]} = \inf \{ \|a\|_{S_{2p}(\ell_2,K)} \|y\|_{S_\infty(E)} \|b\|_{S_{2p}(H,\ell_2)} \}
\]

where the infimum runs over all possible decompositions of $x$ of the form

\[
x = (a \otimes I_E) \cdot (y) \cdot (b \otimes I_E),
\]

where the dot denotes the “matrix” product, and we identify $S_{2p}(\ell_2, K)$ and $S_{2p}(H, \ell_2)$ with the corresponding matrix spaces and $S_\infty[E]$ is viewed as a subset of the set of all matrices $(y_{ij})_{i,j \in \mathbb{N}}$ with entries in $E$. In the particular case when $H, K$ are finite dimensional, we have for all $x$ in $S_p^n[E]$

\[
\|x\|_{S_p^n[E]} = \inf \{ \|a\|_{S_{2p}^n} \|y\|_{M_n(E)} \|b\|_{S_{2p}^n} \}
\]

where the infimum runs over all possible decompositions

\[
x = (a \otimes I_E) \cdot y \cdot (b \otimes I_E)
\]

(matrix product). Indeed, if $H, K$ are $n$-dimensional in (1.5) we can replace $a$ and $b$ respectively by $aP$ and $Qb$ where $P$ (resp. $Q$) is the orthogonal projection onto the essential support of $a$ (resp. onto the range of $b$); if we also replace $y$ by $(P \otimes I_E) \cdot y \cdot (Q \otimes I_E)$ we easily obtain (1.5)′.

We will use the following version of Fubini’s theorem (cf. [P3, Theorem 1.9]):

Let $1 \leq p \leq \infty$. Let $H, K$ be arbitrary Hilbert spaces and let $E$ be an operator space. We have completely isometrically

\[
S_p[H;S_p[K;E]] \simeq S_p[H \otimes_2 K;E] \simeq S_p[K;S_p[H;E]].
\]
Let $m \geq 1$ be an arbitrary integer. For any matrix $a$ in $M_m$, let us denote by $tr_m(a)$ its trace in the usual sense. Let $z$ be an element of $S_1^m[M_m]$, of the form $z = (z_{ij})$ with $z_{ij} \in M_m$. Then we have linear identifications $S_1^m = M_m^*$ and $S_1^m[M_m] = S_1^m \otimes M_m$. These allow to define the trace functional $z \mapsto tr(z)$ associated to the pairing $M_m^* \times M_m \to \mathbb{C}$. It is easy to check that

$$tr(z) = \sum_{ij} \langle z_{ij}, e_{ij} \rangle = \sum_{ij} \langle e_i, z_{ij}(e_j) \rangle.$$ 

Moreover if $z = a \otimes b$ with $a \in S_1^m$ and $b \in M_m$ then we have

(1.7) 

$$tr(z) = tr_m(t^ab)$$

where $t^a$ is the transposed of $a$.

We will use the following elementary identity valid for any $\alpha, \beta$ in $M_m$

(1.8) 

$$tr((\alpha \otimes I) \cdot z \cdot (\beta \otimes I)) = tr((I \otimes t^\alpha) \cdot z \cdot (I \otimes t^\beta)).$$

This is easily checked as follows. It clearly suffices to prove this for $z = a \otimes b$. Then $(\alpha \otimes I) \cdot z \cdot (\beta \otimes I) = (aa\beta) \otimes b$ so that (1.8) reduces to

$$tr_m(t^1(aa\beta)b) = tr_m(t^1(a(t^1ab\beta)))$$

which follows from the well known identity satisfied by the trace of the product of two matrices.

Finally, we will use the inequality

(1.9) 

$$|tr(z)| \leq \|z\|_{S_1^m[M_m]}.$$ 

This is an immediate consequence of the construction of the projective tensor product $S_1^m \otimes^\vee M_m$ as developed in [BP, ER2]. For the convenience of the reader, we briefly sketch a direct deduction of (1.9) from (1.5)'.

Assume $\|z\|_{S_1^m[M_m]} < 1$. Then by homogeneity we can write $z = (a \otimes I_{M_m}) \cdot y \cdot (b \otimes I_{M_m})$ with $\|a\|_{S_2^m} < 1$, $\|y\|_{M_m(M_m)} < 1$, $\|b\|_{S_2^m} < 1$. Then $z = (z_{ij})$ with $z_{ij} \in M_m$ defined by

$$z_{ij} = \sum_{k\ell} a_{ik} y_{k\ell} b_{\ell j}.$$ 

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Hence we have

\begin{align*}
(1.10) \quad |\text{tr}(z)| &= \left| \sum_{ij} \langle e_i, z_{ij} e_j \rangle \right| \\
&= \left| \sum_{k\ell} \langle \alpha_k, y_{k\ell} \beta_\ell \rangle \right|
\end{align*}

where \( \alpha_k, \beta_\ell \in \ell^m_2 \) are defined by

\[ \beta_\ell = \sum_j b_{\ell j} e_j \quad \text{and} \quad \alpha_k = \sum_i a_{ik} e_i. \]

Now (1.10) implies

\[ |\text{tr}(z)| \leq \|y\|_{M_m(M_m)} \left( \sum_k \|\alpha_k\|^2_2 \right)^{1/2} \left( \sum_\ell \|\beta_\ell\|^2_2 \right)^{1/2} \leq \|y\|_{M_m(M_m)} \|a\|_{S^m_2} \|b\|_{S^m_2} < 1. \]

By homogeneity, this concludes the proof of (1.9).
§2. Completely positive maps and regular maps.

Let $K, H$ be Hilbert spaces. Consider (closed) subspaces $S \subset S_p(K)$ and $T \subset S_p(H)$, with $1 \leq p \leq \infty$. Let $E$ be an operator space. We will denote simply by $S[M_n]$ (resp. $S[E]$) the space $S \otimes M_n$ equipped with the norm induced by $S_p[K; M_n]$ (resp. the closure of $S \otimes E$ in $S_p[K; E]$). We equip $S[M_n]$ and $S[E]$ with the operator space structures induced by the natural ones on the spaces $S_p[K; M_n]$ and $S_p[K; E]$. We will again refer to these o.s.s. on $S[M_n]$ and $S[E]$ as the natural ones.

Definition 2.1. In the preceding situation, we will say that a linear map $u: S \to T$ is regular if

$$\sup_{n \geq 1} \|u \otimes I_{M_n}\|_{S[M_n] \to T[M_n]} < \infty,$$

and we denote

$$\|u\|_r = \sup_{n \geq 1} \|u \otimes I_{M_n}\|_{S[M_n] \to T[M_n]}.$$

We will denote by $B_r(S, T)$ the space of all regular maps from $S$ into $T$. Equipped with the norm $\| \cdot \|_r$, this clearly is a Banach space. (Actually, it can be equipped with the operator space structure of the direct sum $\bigoplus_{n \geq 1} cb(S[M_n], T[M_n])$.)

Remark. In the case $p = \infty$ (resp. $p = 1$), the spaces $S[M_n]$ and $T[M_n]$ (resp. $S_1[K, M_n]$ and $S_1[H, M_n]$) are tensor products associated with the minimal tensor product (resp. with the operator space version of the projective tensor product). Therefore, in these two extreme cases every $cb$ map $u$ is regular and satisfies $\|u\|_{cb} = \|u\|_r$. Hence, in the case $p = \infty$ we have an isometric identity $B_r(S, T) = cb(S, T)$, in particular

$$B_r(S_\infty(K), S_\infty(H)) = cb(S_\infty(K), S_\infty(H)),$$

and in the case $p = 1$, we have isometrically

$$B_r(S_1(K), S_1(H)) = cb(S_1(K), S_1(H)).$$

Let $u: S \to T$ be a regular linear map. Then for any operator space $E$, $u \otimes I_E$ extends to a bounded map from $S[E]$ into $T[E]$ with

$$\|u \otimes I_E\|_{S[E] \to T[E]} \leq \|u\|_r. \quad (2.1)$$

This is rather easy to check using the following identity.
For any $a$ in $S[E]$

\[(2.2) \quad \|a\|_{S[E]} = \sup \| (I_S \otimes w)(a) \|_{S[M_m]} \]

where the supremum runs over all $m \geq 1$ and all maps $w: E \to M_m$ with $\|w\|_{cb} \leq 1$. To prove (2.2), we can assume that $E = B(H)$ for some $H$ and that $S = S_p(K)$. The proof can then be completed using Lemma 1.12 and Corollary 1.8 in [P3]. We leave the details to the reader.

**Proposition 2.2.** Let $S, T$ be as above. Every regular map $u: S \to T$ is c.b. and satisfies

\[(2.3) \quad \|u\|_{cb} \leq \|u\|_r.\]

**Proof.** This follows from Remark 2.4 in [P3]. There it is proved that the c.b. norm of $u$ can be written equivalently as follows

\[(2.4) \quad \|u\|_{cb} = \sup_{n \geq 1} \|I_{S^n_p} \otimes u\|_{S^n_p[S] \to S^n_p[T]}.\]

Actually, (2.4) is valid for a map $u: S \to T$ between arbitrary operator spaces. But now if $S \subset S_p(K)$ and $T \subset S_p(H)$ we have by (1.6) isometric identities

\[S^n_p[S] = S[S^n_p] \quad \text{and} \quad S^n_p[T] = T[S^n_p].\]

Hence applying (2.1) with $E = S^n_p$, we obtain (2.3). \hfill \square

**Lemma 2.3.** Consider a linear map $u: S_p \to S_p$. If $u$ is completely positive and bounded, then $u$ is regular and $\|u\|_r \leq \|u\|$. Moreover, $u: S_p \to S_p$ is regular iff its adjoint $u^*: S_p' \to S_p'$ is regular and $\|u\|_r = \|u^*\|_r$.

**Proof.** By a simple density argument, it suffices to prove this for $u: S^n_p \to S^n_p$ with $N \geq 1$ arbitrary. Then, if $u$ is c.p., by Stinespring’s theorem (cf. [Pa, p. 53]) there is a finite set $y_1, \ldots, y_m$ in $M_N$ such that $u$ is of the form

\[(2.5) \quad u(x) = \sum_{i=1}^m y_i x y_i^*, \quad \forall x \in S^n_p.\]
Assume \( \|u\| = 1 \). Consider \( a = (a_{ij})_{ij \leq N} \) with \( a_{ij} \in M_n \) and \( \|a\|_{S_p^N[M_n]} < 1 \). Then, by (1.5)', we can write \( a = \alpha \cdot b \cdot \beta \) with \( \alpha, \beta \in B_{S_p^N} \) and with \( b = (b_{ij}) \) in the unit ball of \( M_N(M_n) \). Then we have

\[
(u \otimes I_{M_n})(a) = \alpha_1 \cdot \begin{pmatrix}
  b & & \\
  & \ddots & \\
  & & b
\end{pmatrix} \cdot \beta_1^*
\]

where \( \alpha_1 = (y_1 \alpha, \ldots, y_m \alpha) \) and \( \beta_1 = (y_1 \beta^*, \ldots, y_m \beta^*) \). By (1.5) this implies

\[
\|(u \otimes I_{M_n})(a)\|_{S_p^N[M_n]} \leq \|\alpha_1\|_2 \|\beta_1^*\|_{2p},
\]

whence

\[
\|(u \otimes I_{M_n})(a)\|_{S_p^N[M_n]} \leq (\text{tr}(\alpha_1 \alpha_1^*)^p)^{1/p}(\text{tr}(\beta_1 \beta_1^*)^p)^{1/2p}
\]

\[
\leq \left\| \sum y_i \alpha \alpha^* y_i^* \right\|_p^{1/2} \left\| \sum y_i \beta \beta^* y_i^* \right\|_p^{1/2}
\]

\[
\leq \|u(\alpha\alpha^*)\|_p^{1/2} \|u(\beta\beta^*)\|_p^{1/2}
\]

\[
\leq \|u\| \leq 1.
\]

This prove the first assertion.

We turn to the second part. Using the fact that \( S_p[M_n^*]^* = S_{p'}[M_n] \) completely isometrically (cf. [P3]) and using (2.1) with \( E = M_n^* \) we find

\[
\|u^*\|_r \leq \|u\|_r.
\]

Since \( u = (u^*)^* \), the converse is obvious. This completes the proof.

**Remark.** Let \( S \) be a subspace of \( \ell_p \). Since \( \ell_p \) can be viewed as embedded in \( S_p \) via the diagonal matrices, there are two notions of regularity for an operator \( u: S \to \ell_p \). But it is easy to check that \( u: S \to \ell_p \) is regular in the Banach lattice sense (see the introduction) iff it is regular in our sense (viewing \( S \) and \( \ell_p \) as embedded into \( S_p \)).

It can also be easily checked that \( u: S \to \ell_p \) is completely positive iff it is positive in the usual Banach lattice sense. This remark clearly extends to more general \( \ell_p \)-spaces and explains why we allowed ourselves to use the same word “regular” as in the Banach lattice case (while “completely regular” would have been tempting!).

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§3. **Main result and applications.**

To prove our main result on regular operators, we will need to describe the predual of the space of regular operators on $S_p$. It will be simpler to work in the finite dimensional case, so let $n$ and $m$ be fixed integers.

Consider $a$ in $S_p^n \otimes S_p^m$ and assume that there are $\alpha, \beta$ in $S_{2p'}^m$ and $y$ in $S_p^n[M_m]$ such that

\begin{equation}
(3.1) \quad a = (I \otimes \alpha) \cdot y \cdot (I \otimes \beta).
\end{equation}

We define

\begin{equation}
(3.2) \quad \rho_p(a) = \inf \{ \| \alpha \|_{S_{2p'}^m} \| y \|_{S_p^n[M_m]} \| \beta \|_{S_{2p'}^m} \},
\end{equation}

where the infimum runs over all possible representations of the form (3.1).

A more symmetric equivalent definition is as follows. Consider all representations of $a$ of the form

\begin{equation}
(3.1)' \quad a = (\gamma \otimes \alpha) \cdot g \cdot (\delta \otimes \beta),
\end{equation}

with $\alpha, \beta$ as above, with $g$ in $M_n(M_m)$ and with $\gamma$ and $\delta$ in $S_{2p'}^n$. Then by (1.5)' we clearly have

\begin{equation}
(3.2)' \quad \rho_p(a) = \inf \{ \| \gamma \|_{S_{2p'}^n} \| \alpha \|_{S_{2p'}^m} \| g \|_{M_n(M_m)} \| \beta \|_{S_{2p'}^m} \| \delta \|_{S_{2p'}^n} \},
\end{equation}

We will prove below that $\rho_p$ is a norm. Surprisingly this is not so obvious. The argument will be based on interpolation (as in [P2] for the commutative case). Let $X_p^{nm}$ be the space $S_p^n \otimes S_p^m$ equipped with the norm $\rho_p$. We will prove below that

\begin{equation}
(X_p^{nm})^* = B_r(S_p^n, S_p^m)
\end{equation}

isometrically and also that if $\theta = 1/p$ we have isometrically

\begin{equation}
X_p^{nm} = (S_1^n[M_n], S_1^m[M_m])_{\theta}.
\end{equation}

Note that in (3.1) we have denoted by $I$ the identity matrix in $M_n$ and the product appearing in (3.1) is the product in $M_n \otimes M_m$. However, we will frequently need in the
sequel to identify $M_n \otimes M_m$ and $M_m \otimes M_n$ in the usual manner ($x \otimes y \to y \otimes x$) so that $y$ in (3.1) can be viewed alternatively as an element of $M_m \otimes M_n$ so that (3.1) becomes

$$(3.1)' \quad a = (\alpha \otimes I) \cdot y \cdot (\beta \otimes I).$$

We warn the reader that we will use both ways to write (3.1) in the sequel. The context will always make clear whether we work in $M_m \otimes M_n$ or $M_n \otimes M_m$. This identification is also used to give a meaning to the various interpolation theorems we consider in this section. For instance it is used to view the couple appearing in (3.3) as a compatible interpolation couple.

The proof is based on the matricial version of Szegő’s classical theorem due to Masani-Wiener-Helson-Lowdenslager (see [He]) which can be stated as follows (this form suffices for our purposes).

Let $w: \mathbb{T} \to M_m$ be a measurable matrix valued Lebesgue integrable function. Assume that for some $\epsilon > 0$ we have

$$\forall t \in \mathbb{T} \quad w(t) \geq \epsilon I.$$  

Then there is an analytic function $F: D \to M_m$ with entries in $H^2$ such that the boundary values (= radial limits) satisfy

$$F^*(z)F(z) = w(z) \quad \text{a.e. on } \partial \Omega$$

and moreover such that $F(z)$ is invertible for all $z$ in $D$ and $z \to F(z)^{-1}$ is bounded and analytic on $D$.

**Theorem 3.1.** Consider $a$ in $S_p^n \otimes S_{p'}^m$. Let $\theta = 1/p$. The following are equivalent.

(i) $\rho_p(a) < 1$,

(ii) $a$ is in the open unit ball of the space $(S_m^m[M_n], S^n_m[M_m]))_\theta$,

Therefore $\rho_p$ is a norm and we have the isometric identity (3.3).

**Proof.** Assume first $\rho_p(a) < 1$. Then $a$ admits a representation (3.1)’ with $\alpha, \beta$ (resp. $\gamma, \delta$) in the open unit ball of $S^m_{2p'}$ (resp. $S^m_{2p}$) and $g$ in the open unit ball of $M_n(M_m)$. Since

$$(3.4) \quad S^m_{2p'} = (S^m_2, S^m_\infty)_\theta \quad \text{and} \quad S^m_{2p} = (S^m_\infty, S^m_2)_\theta,$$
we may apply the classical multilinear interpolation theorem (cf. [BL, p. 96] to the multilinear map

$$(\alpha, \beta, \gamma, \delta) \longrightarrow (\gamma \otimes \alpha) \cdot g \cdot (\delta \otimes \beta).$$

We claim that this map is a contraction from

$$S^m_{2p'} \times S^m_{2p'} \times S^m_{2p} \times S^m_{2p} \text{ into } (S^m_1[M_n], S^n_1[M_m])_{\theta}.$$ 

Indeed this is true if $\theta = 0$ and if $\theta = 1$ (with $\theta = 1/p$), hence by interpolation this also holds for any intermediate $0 < \theta < 1$. Hence we obtain that (i) implies (ii).

Conversely assume (ii). By definition of the complex interpolation space, we can write $a = f(\theta)$ where $f$ is an analytic function on the strip $\Omega = \{0 < \Re z < 1\}$, with values in $M_m \otimes M_n$, bounded and continuous on $\overline{\Omega} = \{0 \leq \Re z \leq 1\}$ and such that

$$\sup_{t \in \mathbb{R}} \|f(it)\|_{S^m_1[M_n]} < 1, \quad \sup_{t \in \mathbb{R}} \|f(1 + it)\|_{S^n_1[M_m]} < 1.$$

Let $\partial_0 = \{z \in \mathbb{C} \mid \Re z = 0\}$ and $\partial_1 = \{z \in \mathbb{C} \mid \Re z = 1\}$. By a classical continuous selection argument, this implies that we can find bounded continuous functions $\alpha: \partial\Omega \rightarrow M_m$ and $\beta: \partial\Omega \rightarrow M_m$ and $y: \partial\Omega \rightarrow M_n \otimes M_m$ such that

$$(3.5) \quad \forall z \in \partial\Omega \quad f(z) = (I \otimes \alpha(z)) \cdot y(z) \cdot (I \otimes \beta(z)).$$

$$(3.6)^1 \quad \sup_{z \in \partial_0} \|\alpha(z)\|_{S^m_2} < 1, \quad \sup_{z \in \partial_0} \|\beta(z)\|_{S^m_2} < 1 \quad \text{and} \quad \sup_{z \in \partial_0} \|y(z)\|_{M_m(M_n)} < 1.$$ 

$$(3.6)^2 \quad \sup_{z \in \partial_1} \|\alpha(z)\|_{S^n_{\infty}} < 1, \quad \sup_{z \in \partial_1} \|\beta(z)\|_{S^n_{\infty}} < 1 \quad \text{and} \quad \sup_{z \in \partial_1} \|y(z)\|_{S^n_1[M_m]} < 1.$$ 

Furthermore, we can write

$$(3.5)' \quad \forall z \in \partial\Omega \quad y(z) = (\gamma(z) \otimes I) \cdot g(z) \cdot (\delta(z) \otimes I)$$

with $\gamma(z), \delta(z) \in M_n$ and $g(z) \in M_n(M_m)$ such that

$$(3.6)^3 \quad \sup_{z \in \partial_0} \|\gamma(z)\|_{S^n_{\infty}} < 1, \quad \sup_{z \in \partial_0} \|\delta(z)\|_{S^n_{\infty}} < 1, \quad \text{and} \quad \sup_{z \in \partial_0} \|g(z)\|_{M_n(M_m)} < 1.$$ 

$$(3.6)^4 \quad \sup_{z \in \partial_1} \|\gamma(z)\|_{S^m_2} < 1, \quad \sup_{z \in \partial_1} \|\delta(z)\|_{S^m_2} < 1, \quad \text{and} \quad \sup_{z \in \partial_1} \|g(z)\|_{M_n(M_m)} < 1.$$ 

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Note that for \( z \in \partial_1 \) (resp. \( z \in \partial_0 \)) we can take \( \alpha(z) \) and \( \beta(z) \) (resp. \( \gamma(z) \) and \( \delta(z) \)) equal to a multiple of the identity matrix. This yields

\[(3.5)'' \quad \forall z \in \partial \Omega \quad f(z) = (\gamma(z) \otimes \alpha(z)) \cdot g(z) \cdot (\delta(z) \otimes \beta(z)).\]

Of course the functions \( \alpha, \beta, \gamma, \delta \) and \( g \) are a priori no longer (boundary values of) analytic functions, but we will correct this using the matricial Szegő theorem. We will now choose \( \varepsilon > 0 \) (small enough to be specified later) and we let for all \( z \) in \( \partial \Omega \)

\[
w_1(z) = \varepsilon I + \alpha(z)\alpha(z)^* \quad v_1(z) = \varepsilon I + \gamma(z)\gamma(z)^*
\]

and

\[
w_2(z) = \varepsilon I + \beta(z)^*\beta(z) \quad v_2(z) = \varepsilon I + \delta(z)^*\delta(z).
\]

Then by the matricial version of Szegő's theorem (and by a conformal mapping argument), there are \( M_n\)-valued (resp. \( M_n\)-valued) analytic functions \( z \to \tilde{\alpha}(z) \) and \( z \to \tilde{\beta}(z) \) (resp. \( z \to \tilde{\gamma}(z) \) and \( z \to \tilde{\delta}(z) \)) such that \( z \to \tilde{\alpha}(z)^{-1} \) and \( z \to \tilde{\beta}(z)^{-1} \) (resp. \( z \to \tilde{\gamma}(z)^{-1} \) and \( z \to \tilde{\delta}(z)^{-1} \)) are well defined, bounded and analytic in \( \Omega \) and such that for almost all \( z \) in \( \partial \Omega \)

\[(3.7) \quad w_1(z) = \tilde{\alpha}(z)\tilde{\alpha}(z)^*, \quad v_1(z) = \tilde{\gamma}(z)\tilde{\gamma}(z)^*, \quad w_2(z) = \tilde{\beta}(z)^*\tilde{\beta}(z), \quad v_2(z) = \tilde{\delta}(z)^*\tilde{\delta}(z).
\]

We can then write for all \( z \) in \( \Omega \)

\[
f(z) = (\tilde{\gamma}(z) \otimes \tilde{\alpha}(z)) \cdot \tilde{g}(z) \cdot (\tilde{\delta}(z) \otimes \tilde{\beta}(z))
\]

where we have set

\[(3.8) \quad \tilde{g}(z) = (\tilde{\gamma}(z)^{-1} \otimes \tilde{\alpha}(z)^{-1}) \cdot f(z) \cdot (\tilde{\delta}(z)^{-1} \otimes \tilde{\beta}(z)^{-1}).\]

Note that by (3.8) \( \tilde{g} \) is bounded and analytic in \( \Omega \) and its boundary values satisfy (by (3.5)"") a.e. on \( \partial \Omega \)

\[
\tilde{g}(z) = u(z) \cdot g(z) \cdot v(z),
\]

where \( u(z) = \tilde{\gamma}(z)^{-1}\gamma(z) \otimes \tilde{\alpha}(z)^{-1}\alpha(z) \) and \( v(z) = \delta(z)\tilde{\delta}(z)^{-1} \otimes \beta(z)\tilde{\beta}(z)^{-1} \). But by (3.7) for almost all \( z \) in \( \partial \Omega \) we have

\[
\tilde{\alpha}(z)\tilde{\alpha}(z)^* \geq \alpha(z)\alpha(z)^* \quad \text{and} \quad \tilde{\gamma}(z)\tilde{\gamma}(z)^* \geq \gamma(z)\gamma(z)^*
\]
hence \(\|u(z)\|_{M_n \otimes_{\min} M_m} = \|u(z)\|_{M_n(M_m)} \leq 1\). Similarly, \(\|v(z)\|_{M_n(M_m)} \leq 1\). Therefore, the boundary values of \(\tilde{g}\) satisfy the same bounds \((3.6)^3\) and \((3.6)^4\) as \(g\) on \(\partial \Omega = \partial_0 \cup \partial_1\).

Since \(\tilde{g}\) is bounded and analytic, this implies by the maximum principle

\[
\forall \ z \in \Omega \quad \|\tilde{g}(z)\|_{M_n(M_m)} \leq 1.
\]

On the other hand, if we choose \(\varepsilon\) small enough we can by \((3.7)\) guarantee that \(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}\) still satisfy the estimates \((3.6)^1\), \((3.6)^2\), \((3.6)^3\) and \((3.6)^4\). We then obtain

\[
\|\tilde{\alpha}(\theta)\|_{(S^m_2, S^m_\infty)} < 1, \quad \|\tilde{\gamma}(\theta)\|_{(S^m_u, S^m_2)} < 1, \quad \|\tilde{\beta}(\theta)\|_{(S^m_2, S^m_u)} < 1, \quad \|\tilde{\delta}(\theta)\|_{(S^m_u, S^m_2)} < 1.
\]

Then using \((3.4)\) and recalling \((3.2)'\) we conclude that \(\rho_p(a) < 1\) follows from the identity

\[
a = f(\theta) = (\tilde{\gamma}(\theta) \otimes \tilde{\alpha}(\theta)) \cdot \tilde{g}(\theta) \cdot (\tilde{\delta}(\theta) \otimes \tilde{\beta}(\theta)).
\]

This concludes the proof \((ii) \Rightarrow (i)\).

\[\]

**Theorem 3.2.** In the situation of Theorem 3.1, we have isometrically

\[
(X^p_{nm})^* = B_r(S^m_p, S^m_p).
\]

**Proof.** By Theorem 3.1 we have

\[
(X^p_{nm})^* = (S^m_1[M_n]^*, S^m_1[M_m]^*).\]

Observe the isometric identities

\[
S^m_1[M_n]^* \simeq cb(M_n, M_m) = B_r(M_n, M_m)
\]

\[
S^m_1[M_m]^* \simeq cb(S^m_1, S^m_1) = B_r(S^m_1, S^m_1).
\]

Using \((1.4)\) with \(E = M_k\) with \(k\) arbitrary, it is easy to show that (if \(p = 1/\theta\)) we have a norm one inclusion

\[
(B_r(M_n, M_m), \ B_r(S^m_1, S^m_1))_{\theta} \subset B_r(S^m_p, S^m_p).
\]

This shows that we have a norm one inclusion

\[
(X^p_{nm})^* \subset B_r(S^m_p, S^m_p).
\]
To prove the converse it suffices to prove the following claim: for any \( u \) in \( B(B(S^n_S, S^m_S)) \) and any \( a \) in \( X_{nm}^{nm} \) we have

\[
|\langle u, a \rangle| \leq \rho_p(a)\|u\|_r.
\]

Let us verify this. By homogeneity we may assume \( \rho_p(a) < 1 \). Then we can assume that (3.1) holds with

\[
\|\alpha\|_{s_{2p}^m} < 1, \quad \|y\|_{s_{p}^m[M_m]} < 1, \quad \|\beta\|_{s_{2p}^m} < 1.
\]

Let \( z = (u \otimes I_{M_m})(y) \). Note \( z \in S_{p}^m[M_m] \). We have by definition of \( \|u\|_r \)

\[
\|z\|_{s_{p}^m[M_m]} \leq \|u\|_r\|y\|_{s_{p}^m[M_m]} < \|u\|_r.
\]

Let \( \tilde{z} = (\alpha \otimes I_{M_m}) \cdot z \cdot (\beta \otimes I_{M_m}) \). We have by (1.5)'

\[
\|\tilde{z}\|_{s_{1}^m[M_m]} \leq \|\alpha\|_{2p'}\|z\|_{s_{p}^m[M_m]}\|\beta\|_{2p'} < \|u\|_r.
\]

On the other hand by (1.8)

\[
\langle u, a \rangle = tr((I_{M_m} \otimes \alpha)z(I_{M_m} \otimes \beta)) = tr(\tilde{z})
\]

hence by (1.9)

\[
|\langle u, a \rangle| \leq \|\tilde{z}\|_{s_{1}^m[M_m]} < \|u\|_r.
\]

This concludes the proof of our claim.

**Corollary 3.3.** Let \( n, m \) be arbitrary integers, \( 0 \leq \theta \leq 1 \), \( p = 1/\theta \). Then we have isometric identities

\[
(3.9)' \quad (cb(S^n_S, S^n_S), \quad cb(S^n_S, S^n_S)_{\theta} = B_r(S^n_S, S^n_S),
\]

\[
(3.9)'' \quad (cb(S_{\infty}, B(\ell_2)), \quad cb(S_I, S_I)_{\theta}^\theta = B_r(S^p_S, S^p_S).
\]

**Proof.** Let \( X_{\infty}^{nm} = S_{1}^m[M_n] \) and \( X_{1}^{nm} = S_{1}^n[M_m] \). Then by Theorem 3.1 we have isometrically

\[
X_{p}^{nm} = (X_{\infty}^{nm}, X_{1}^{nm})_{\theta}.
\]

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On the other hand, by our definitions and known estimates (cf. [BP, ER2, ER6])

\[ X_1^{nm*} = (S_1^n)^* \otimes_{\min} (M_m)^* \simeq cb(S_1^n, S_1^m) \]

and

\[ X_\infty^{nm*} = (S_1^m)^* \otimes_{\min} (M_n)^* \simeq cb(M_n, M_m). \]

Hence (3.9)' follows from (3.10) by duality, using Theorem 3.2. By an entirely elementary approximation argument (left to the reader) we can obtain (3.9)''.

**Proof of Theorem 0.1.** We simply combine (3.9)'' with our earlier result Theorem 0.2.

We may clearly extend the definition (3.2) to the infinite dimensional case. For any \( a \) in \( S_p(H) \otimes S_{p'}(K) \) we consider all possible factorizations of \( a \) of the form

\[ a = (I \otimes \alpha) \cdot y \cdot (I \otimes \beta) \]

with \( \alpha, \beta \in S_{2p'}(K) \) and \( y \in S_p[H; S_\infty(K)] \). We then let

\[ \rho_p(a) = \inf \{ \| \alpha \|_{2p'} \| y \|_{S_p[H; S_\infty(K)]} \| \beta \|_{2p'} \} \]

and we define \( X_{p}^{H,K} \) as the completion of \( S_p(H) \otimes S_{p'}(K) \) for this norm.

Actually we will work only with \( H = K = \ell_2 \) and in that case we set \( X_p = X_{p}^{H,K} \).

Alternatively, we could use the norms defined in (3.2), form the inductive limit \( \bigcup_{n,m} S_p^n \otimes S_{p'}^m \) and define \( \rho_p \) as the resulting completion. In any case, we have clearly by (3.3) and Theorem 3.1 the following result

**Corollary 3.4.** The following are isometric identities (\( \theta = 1/p, \ 1 < \theta < 1 \))

\[ X_{p}^{H,K} = (S_1[K; S_\infty(H)], \ S_1[H; S_\infty(K)])_\theta \]

and

\[ (X_{p}^{H,K})^* = B_r(S_p(H), S_p(K)). \]

We will now prove an extension theorem, which in the commutative case with \( p = 1 \) goes back to M. Lévy [Lé]. See [P2] for the commutative case with \( 1 \leq p < \infty \) arbitrary.
Theorem 3.5. Let \( 1 \leq p < \infty \). Let \( S \subset S_p \) be a closed subspace. Then any regular operator \( u: S \to S_p \) admits a regular extension \( \tilde{u}: S_p \to S_p \) with \( \|\tilde{u}\|_r = \|u\|_r \).

Proof. This follows from the Hahn-Banach theorem and a special property of the norm of the predual of \( B_r(S_p, S_p) \). It clearly suffices to consider the case of a map \( u: S \to S_p^m \).

Assume \( \|u\|_r \leq 1 \). We first claim that it suffices to prove the following: for all \( v \in S \otimes S_p^m \) the element \( \tilde{v} \in S_p \otimes S_p^m \) associated to \( v \) by the inclusion \( S \to S_p \) satisfies

\[
\langle u, v \rangle \leq \|v\|_{X_p}.
\] (3.11)

Here for simplicity we again denote by \( X_p \) the space \( X_{HK}^p \) associated to \( H = \ell_2, K = \ell_2^m \). Indeed, if (3.11) holds the Hahn-Banach theorem provides an element \( \tilde{u} \in X_p^* \) with \( \|\tilde{u}\|_{X_p^*} \leq 1 \) such that \( \langle \tilde{u}, v \rangle = \langle u, v \rangle \) for all \( v \in S \otimes S_p^m \). Since this last condition means that \( \tilde{u} \) extends \( u \) and since \( X_p^* = B_r(S_p, S_p^m) \) we obtain the desired conclusion.

Hence it suffices to show (3.11). Assume \( \|\tilde{v}\|_{X_p} < 1 \). Then there is a factorization \( \tilde{v} = (I \otimes \alpha)y(I \otimes \beta) \) with \( \|\alpha\|_{S^m_{2p'}} < 1, \|\beta\|_{S^m_{2p'}} < 1 \) and \( \|y\|_{S_p[M_m]} < 1 \). By a perturbation argument, we may clearly assume that \( \alpha \) and \( \beta \) are invertible. Then we have

\[
y = (I \otimes \alpha^{-1})\tilde{v}(I \otimes \beta^{-1})
\]

which shows that \( y \in S \otimes M_m \) since \( \tilde{v} \) comes from \( v \in S \otimes S_p^m \). Hence we can write \( v = (I \otimes \alpha)y(I \otimes \beta) \).

The rest of the proof is a variant of the proof of Theorem 3.2. Let \( z = (u \otimes I_{M_m})(y) \), note \( z \in S^m_p[M_m] \), and let \( \tilde{z} = (t^\alpha \otimes I)z(t^\beta \otimes I) \). Then by (1.5)' we have

\[
\|\tilde{z}\|_{S^m_t[M_m]} < \|z\|_{S^m_p[M_m]} < \|u\|_r.
\] (3.12)

Hence we conclude using (1.8) and (1.9)

\[
|\langle u, v \rangle| = |tr((I \otimes \alpha)z(I \otimes \beta))| = |tr(\tilde{z})| \leq \|\tilde{z}\|_{S^m_t[M_m]}.
\]

This shows that (3.12) implies (3.11).

To illustrate the preceding results, we consider the subspace \( T_p \subset S_p \) of all the upper triangular matrices. This space is often regarded as a noncommutative analogue of \( H^p \).
Using the results of [P4], it is not hard to show that we have isomorphically if \(1 < p < \infty\) and \(\theta = 1/p\)

\[T_p[S_\infty] = (T_\infty[S_\infty], T_1[S_\infty])_\theta.\]

Hence by repeating mutatis mutandis the proof of [P2] in the case of \(H^p\), we find

**Corollary 3.6.** We have (if \(1 < p < \infty\) and \(\theta = 1/p\)) an isomorphic identity

\[B_r(T_p, S_p) = (B_r(T_\infty, S_\infty), B_r(T_1, S_1))^\theta.\]

Following the framework of [P3], it is easy to extend the preceding results to the case when \(S_p\) or \(S_p^m\) is replaced by a non-commutative \(L_p\)-space associated to a *hyperfinite* von Neumann algebra \(M\). Recall that a von Neumann algebra \(M\) is called hyperfinite if it is the \(\sigma(M, M_*)\)-closure of the union of an increasing net of finite dimensional subalgebras.

The extension of the definition of a regular operator is immediate:

Let \(M\) (resp. \(N\)) be a hyperfinite von Neumann algebra equipped with a faithful normal semi-finite trace \(\varphi\) (resp. \(\psi\)). We will denote by \(L_p(\varphi)\) (resp. \(L_p(\psi)\)) the associated non-commutative \(L_p\)-space for \(1 \leq p < \infty\). (Note that it is natural to identify \(L_\infty(\varphi)\) with \(M\) and \(L_\infty(\psi)\) with \(N\).) These spaces are equipped with their *natural o.s.s.* as explained in section 1. Similarly, if \(E\) in an operator space, the space \(L_p(\varphi; E)\) is defined by interpolation as in (1.4) and it is equipped with the natural o.s.s. defined by (1.4).

Consider then a closed subspace

\[S \subset L_p(\varphi) \quad \text{(resp. } T \subset L_p(\psi))\].

We will denote by \(S[E]\) the closure of \(S \otimes E\) in the space \(L_p(\varphi; E)\). We can then define a regular map \(u: S \to T\) exactly as in definition 2.1.

We again denote by \(B_r(S, T)\) the space of all regular maps from \(S\) into \(T\) equipped with the norm \(\| \cdot \|_r\). It is then very easy to adapt the preceding proofs to obtain the following two statements. We leave the details to the reader.

**Theorem 3.7.** Let \(M\) (resp. \(N\)) be a hyperfinite von Neumann algebra equipped with a semi-finite faithful normal trace \(\varphi\) (resp. \(\psi\)). Let \(1 < p < \infty\) and \(\theta = 1/p\). We have an isometric identity

\[(cb(M, N), cb(L_1(\varphi), L_1(\psi)))^\theta = B_r(L_p(\varphi), L_p(\psi)).\]
Moreover the space \( B_r(L_p(\varphi), L_p(\psi)) \) coincides with the set of all maps \( u: L_p(\varphi) \rightarrow L_p(\psi) \) which are linear combinations of completely positive, bounded maps from \( L_p(\varphi) \) to \( L_p(\psi) \).

**Corollary 3.8.** Let \( 1 \leq p < \infty \). Let \((M, \varphi)\) and \((N, \psi)\) be as in Theorem 3.7. Let \( S \subset L_p(\varphi) \) be a closed subspace. Then any regular operator \( u: S \rightarrow L_p(\psi) \) admits an extension \( \tilde{u}: L_p(\varphi) \rightarrow L_p(\psi) \) with \( \|\tilde{u}\|_r = \|u\|_r \).

**Remark.** For the case \( p = 1 \) in corollary 3.8, it is useful to remind the reader that (by e.g. [Ta, p. 126-127]) there is a norm one, completely positive and completely contractive projection from \( N^* = L_1(\psi)^{**} \) onto \( L_1(\psi) \).

**Remark.** Note that in the case \( p = \infty \), the preceding two statements reduce to the well known decomposition and extension properties of a completely bounded map with values into a hyperfinite (=injective, by Connes’s well known results [Co]) von Neumann algebra, which follow from Wittstock’s theorem (cf. [Pa, p.100-107]). Note that these properties are only true in the injective case (cf. [H1]).
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