Quasisymmetrically minimal homogeneous perfect sets∗

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Abstract: In [6], the notion of homogenous perfect set as a generalization of Cantor type sets is introduced. Their Hausdorff, lower box-counting, upper box-counting and packing dimensions are studied in [6] and [8]. In this paper, we show that the homogenous perfect set be minimal for 1-dimensional quasisymmetric maps, which generalize the conclusion in [3] about the uniform Cantor set to the homogenous perfect set.

Key words: Homogenous perfect set; Quasisymmetric map; Quasisymmetrically minimal set

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1 Introduction

Given $M \geq 1$, a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is said to be $M$–quasisymmetric if and only if

$$M^{-1} \leq \frac{|f(I)|}{|f(J)|} \leq M$$

for all pairs of adjacent intervals $I, J$ of equal length, here and in sequel $|\cdot|$ stands for the 1-dimensional Lebesgue measure. A map is quasisymmetric if it is $M$–quasisymmetric for some $M \geq 1$. More generally a homeomorphism

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between metric spaces \((X, d_X)\) and \((Y, d_Y)\). If there is a homeomorphism \(\eta : [0, +\infty) \to [0, +\infty)\) such that
\[
\frac{d_X(a, x)}{d_X(b, x)} \leq t \Rightarrow \frac{d_Y(f(a), f(x))}{d_Y(f(b), f(x))} \leq \eta(t)
\]
for all triples \(a, b, x\) of distinct points in \(X\) and \(t \in [0, +\infty)\), then we call \(f\) is a quasisymmetric map. When \(X = Y = \mathbb{R}^n\), we also say that \(f\) is an \(n\)-dimensional quasisymmetric map.

Let \(QS(X)\) denote the collection of all quasisymmetric maps defined on \(X\). Conformal dimension of a metric space, a concept introduced by Pansu in [5], is the infimal Hausdorff dimension of quasisymmetric images of \(X\),
\[
\mathcal{C} \text{dim } X = \inf_{f \in QS(X)} \dim_H f(X).
\]
We say \(X\) is minimal for conformal dimension or just minimal if \(\mathcal{C} \text{dim } X = \dim_H X\). Euclidean spaces with standard metric are the simplest examples of minimal spaces. Basic analytic definitions and results about the conformal dimension and the quasisymmetric map are contained in [4].

Now, we introduce the notion of the homogeneous perfect set. The general references on the homogeneous perfect set are [6, 8]. In these paper, the authors obtained the Hausdorff, lower box-counting, upper box-counting and packing dimensions of the homogeneous perfect set.

**Homogeneous perfect sets.** Let \(J_0 = [0, 1] \subset \mathbb{R}\) be the fixed closed interval which we call the initial interval. Let \(\{n_k\}_{k=1}^{\infty}\) be a sequence of positive integers and \(\{c_k\}\) a sequence of positive real numbers such that for any \(k \geq 1, n_k \geq 2\) and \(0 < c_k < 1\). For any \(k \geq 1\), let \(D_k = \{(i_1, i_2, \ldots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}\), \(D = \bigcup_{k=0}^{\infty} D_k\), where \(D_0 = \{0\}\). We assume if \(\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) \in D_k, 1 \leq j \leq n_{k+1}\), then \(\sigma * j = (\sigma_1, \sigma_2, \ldots, \sigma_k, j) \in D_{k+1}\).

Suppose that \(J_0\) is the initial interval and \(\mathcal{J} = \{J_\sigma : \sigma \in D\}\) is a collection of closed subintervals of \(J_0\). We say that the collection \(\mathcal{J}\) fulfills the homogenous perfect structure provided:

1. For any \(k \geq 0, \sigma \in D_k, J_{\sigma * 1}, J_{\sigma * 2}, \ldots, J_{\sigma * n_{k+1}}\) are subintervals of \(J_\sigma\). Furthermore, \(\max\{x : x \in J_{\sigma * i}\} \leq \min\{x : x \in J_{\sigma * (i+1)}\}, 1 \leq i \leq n_{k+1} - 1,\) that is the interval \(J_{\sigma * i}\) is located at the left of \(J_{\sigma * (i+1)}\) and the interiors of the intervals \(J_{\sigma * i}\) and \(J_{\sigma * (i+1)}\) are disjoint.

2. For any \(k \geq 1, \sigma \in D_{k-1}, 1 \leq j \leq n_k, \) we have
\[
\frac{|J_{\sigma * i}|}{|J_\sigma|} = c_k.
\]
3. There exists a sequence of nonnegative real numbers \( \{\eta_{k,j}, k \geq 1, 0 \leq j \leq n_k\} \) such that for any \( k \geq 0, \sigma \in D_k \), we have \( \min(J_{\sigma^*1}) - \min(J_{\sigma}) = \eta_{k+1,0}, \max(J_{\sigma}) - \max(J_{\sigma^*n_{k+1}}) = \eta_{k+1,n_{k+1}}, \) and \( \min(J_{\sigma^*(i+1)}) - \max(J_{\sigma^* i}) = \eta_{k+1,i} (1 \leq i \leq n_{k+1} - 1) \).

Suppose that the collection of intervals \( \mathcal{J} = \{J_{\sigma} : \sigma \in D\} \) satisfies the homogeneous perfect structure.

Let

\[
E_k = \bigcup_{\sigma \in D_k} J_{\sigma}
\]

for every \( k \geq 1 \). The set

\[
E := E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\}) = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_{\sigma} = \bigcup_{k \geq 0} E_k
\]

is called a homogeneous perfect set and the intervals \( J_{\sigma}, \sigma \in D_k \), the fundamental intervals of order \( k \).

For any \( k \geq 1 \), if \( \eta_{k,0} = \eta_{k,n_k} = 0 \) and \( \eta_{k,l} = c_k |J_{\sigma}| \) for all \( 1 \leq l \leq n_k - 1, \sigma \in D_{k-1} \). Then \( E \) is called a uniform Cantor set. This case has been considered by M.D. Hu and S.Y. Wen in [3]. They obtained

**Theorem 1** ([3]). Let \( E \) be a uniform Cantor set. If the sequence \( \{n_k\} \) is bounded and if \( \dim_H E = 1 \). Then \( \dim_H f(E) = 1 \) for all \( 1 \)-dimensional quasisymmetric maps \( f \).

In this paper, we generalize Theorem 1 to the homogeneous perfect set and show how the techniques of [3] can be applied to the homogeneous perfect set and obtain the following theorem.

**Theorem 2.** Let \( E := E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\}) \) be a homogeneous perfect set. If the sequence \( \{n_k\} \) is bounded and if \( \dim_H E = 1 \), then \( \dim_H f(E) = 1 \) for all \( 1 \)-dimensional quasisymmetric map \( f \).

This paper is organized as following. In section 2 we introduce the basic general definitions and results in fractal geometry. The proof of Theorem 2 appears in section 3.

## 2 Preliminary

In order to obtain our result, we need the following lemma from [9], the lemma can also be found in [2] or [3].
Lemma 1 (9). Let $f$ be an $M$-quasisymmetric map. Then

$$ (1 + M)^{-2} \left( \frac{|J|}{|I|} \right)^q \leq \left| \frac{f(J)}{f(I)} \right| \leq 4 \left( \frac{|J|}{|I|} \right)^p $$

for all pairs $J, I$ of intervals with $J \subset I$, where

$$ 0 < p = \log_2 (1 + M^{-1}) \leq 1 \leq q = \log_2 (1 + M). $$

**Hausdorff dimension.** In this subsection, we recall the definition of Hausdorff dimension. For more details we refer to [1, 7].

Let $K \subset \mathbb{R}^d$. For any $s \geq 0$, the $s$–dimensional Hausdorff measure of $K$ is given in the usual way by

$$ H^s(K) = \liminf_{\delta \to 0} \left\{ \sum_i |U_i|^s : K \subset \bigcup_i U_i, 0 < |U_i| < \delta \right\}. $$

This leads to the definition of the Hausdorff dimension of $K$:

$$ \dim_H K = \inf \{ s : H^s(K) < \infty \} = \sup \{ s : H^s(K) > 0 \}. $$

The Hausdorff dimension of the homogeneous perfect set $E$, which depends on $\{n_k\}, \{c_k\}$ and $\{\eta_{k,j}\}$ have been obtained in [6] as follows

**Theorem 3 (8).** Let $E = E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\})$ be a homogeneous perfect set. Suppose $n_k \leq D$ for all $k$, where $D$ is a constant, then

$$ \dim_H E = \liminf_{k \to \infty} \frac{\log(n_1 n_2 \cdots n_k)}{-\log(\sum_{l=1}^{n_k+1} \eta_{k+1,l} + n_k+1 c_1 c_2 \cdots c_{k+1})}. \quad (4) $$

Denote by $N_k$ the number of component intervals of $E_k$ and by $\delta_k$ their common length. Let $e_{k,l} = \eta_{k,l}/\delta_{k-1} \geq \eta_{k,l}$ for all $k \geq 1$ and $0 \leq l \leq n_k$. From the definition we obtain

$$ n_k c_k \leq 1, \quad N_k = n_k n_{k-1} \cdots n_1 \quad \text{and} \quad \delta_k = c_k c_{k-1} \cdots c_1 $$

for all $k \geq 1$. So we have the total length of $E_k$ is

$$ N_k \delta_k = \prod_{i=1}^{k} n_i c_i, $$

and

$$ \delta_k = \sum_{l=0}^{n_k+1} \eta_{k+1,l} + n_k+1 \delta_{k+1} = \sum_{l=0}^{n_k+1} e_{k+1,l} \delta_k + n_k+1 \delta_{k+1}. \quad (5) $$
Lemma 2. Let $E = E(J_0, \{n_k\}, \{c_k\}, \{\eta_{k,j}\})$ be a homogeneous perfect set. Suppose the sequence $\{n_k\}$ is bounded and $\dim_H E = 1$ then:

1. $\lim_{k \to \infty} (N_k \delta_k)^{1/k} = 1$.
2. $\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} e_i^p = 0$ for any $0 < p \leq 1$, where $e_i = \max_{0 \leq j \leq n_i} e_{i,j}$.
3. $\lim_{k \to \infty} \# \{i : 0 \leq i \leq k, e_i \geq \epsilon \}/k = 0$ for any $\epsilon \in (0, 1)$, where $\#$ denotes the cardinality.

Proof. (1) Since

$$N_k(\delta_k - \eta_{k,0} - \eta_{k,n_k+1}) \leq N_k \delta_k \leq 1,$$

Thus, we have

$$\frac{\log N_k}{- \log(\delta_k - \eta_{k,0} - \eta_{k,n_k+1})} \leq \frac{\log N_k}{- \log \delta_k} \leq 1.$$

As $\dim_H E = 1$, we get from Theorem 3

$$1 = \dim_H E = \liminf_{k \to \infty} \frac{\log N_k}{- \log(\delta_k - \eta_{k,0} - \eta_{k,n_k+1})} \leq \lim_{k \to \infty} \frac{\log N_k}{- \log \delta_k} \leq 1.$$ (6)

Thus we obtain

$$\lim_{k \to \infty} \frac{\log N_k}{- \log \delta_k} = \lim_{k \to \infty} \frac{\log N_k}{\log N_k - \log N_k \delta_k} = 1,$$

and

$$\lim_{k \to \infty} \frac{\log N_k \delta_k}{\log N_k} = 0.$$

Let $N = 1 + \sup_k n_k < \infty$. We obtain $N_k \leq N^k$, so

$$\lim_{k \to \infty} \frac{\log N_k \delta_k}{k \log N} = 0,$$

that gives the conclusion (1) of the lemma.

(2) Since

$$(N_k \delta_k)^{1/k} = (\prod_{i=1}^{k} n_i c_i)^{1/k} \leq \frac{1}{k} \sum_{i=1}^{k} n_i c_i \leq 1.$$
Thus, we have

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} n_i c_i = 1.$$  \hfill (7)

From the equation (5), we have

$$\delta_k = \sum_{l=0}^{n_{k+1}} e_{k+1,l} \delta_k + n_{k+1} c_{k+1} \delta_k.$$  \hfill (8)

Thus

$$e_{k+1} \leq 1 - n_{k+1} c_{k+1},$$

so

$$\frac{1}{k} \sum_{i}^{k} e_i \leq \frac{1}{k} \sum_{i}^{k} (1 - n_i c_i).$$

Since the equation (7), we obtain

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i}^{k} e_i = 0,$$

which together with Jensen’s inequality yields

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} e_i^p \leq \lim_{k \to \infty} \left( \frac{1}{k} \sum_{i=1}^{k} e_i \right)^p = 0$$

for any $0 < p \leq 1$. This proves the conclusion (2).

(3) Fixed $\epsilon \in (0, 1)$, we obtain from the conclusion (2)

$$\frac{1}{k} \sum_{i=1}^{k} \{i : 0 \leq i \leq k, e_i \geq \epsilon\} = \frac{1}{k} \sum_{i=1}^{k} 1 \leq \frac{1}{k \epsilon} \sum_{i=1}^{k} e_i \to 0$$

as $k$ tends to $\infty$. This proves the conclusion (3).

3 The proof of Theorem 2

In order to obtain our result, we need the following mass distribution principle to estate the lower bound.
Lemma 3 \([1]\). Let \(\mu\) be a mass distribution supported on \(E\). Suppose that for some \(t\) there are numbers \(c > 0\) and \(\eta > 0\) such that for all sets \(U\) with \(|U| \leq \eta\) we have \(\mu(U) \leq c|U|^t\). Then \(\dim H E \geq t\).

The proof of Theorem 2: Let \(E = \bigcap_{k=0}^{\infty} E_k\) be a homogeneous perfect set satisfying the conditions of Theorem 2. Let \(f : \mathbb{R} \to \mathbb{R}\) be an \(M\)-quasisymmetric map and \(q\) is the number defined as in \([3]\). Without loss of generality assume that \(f([0,1]) = [0,1]\). Then \(f(E) = \bigcap_{k=1}^{\infty} f(E_k)\). The images of component intervals of \(E_k\) are component intervals of \(f(E_k)\).

We define a mass distribution \(\mu\) on \(f(E)\) as follows: Let \(\mu([0,1]) = 1\). For every \(k \geq 1\) and for every component interval \(J\) of \(f(E_{k-1})\), let \(J_{k1}, J_{k2}, \ldots, J_{k_n}\) denote the \(n\) component intervals of \(f(E_k)\) lying in \(J\). Define

\[
\mu(J_{ki}) = \frac{|J_{ki}|^d}{||J||_d} \mu(J), \quad i = 1, 2, \ldots, n_k,
\]

where

\[
||J||_d = \sum_{i=1}^{n_k} |J_{ki}|^d
\]

and

\[
d \in \begin{cases} 
(0, 1) & \text{when } q = 1, \\
(1/q, 1) & \text{when } q > 1.
\end{cases}
\]

We are going to prove that the measure \(\mu\) satisfy

\[
\mu(J) \leq C|J|^d
\]

for any interval \(J \subset [0,1]\), where \(C\) is a positive constant independent of \(J\). We do this as following two steps.

Step 1. Suppose that \(J\) is a component interval of \(f(E_k)\). For every \(i, 0 \leq i \leq k\), let \(J_i\) be the component interval of \(f(E_i)\) such that

\[
J = J_k \subset J_{k-1} \subset \cdots J_1 \subset J_0 = [0,1]
\]

By the definition of \(\mu\), we have

\[
\frac{\mu(J)}{|J|^d} = \frac{1}{||J||_d} \frac{|J_{k-1}|^d}{||J_{k-1}||_d} \cdots \frac{|J_1|^d}{||J_0||_d} = \frac{|J_{k-1}|^d}{||J_{k-1}||_d} \cdots \frac{|J_1|^d}{||J_1||_d} \frac{|J_0|^d}{||J_0||_d}.
\]

Let

\[
r_i = \frac{||J_i||_d}{|J_i|^d}, \quad i = 0, 1, 2, \ldots, k - 1.
\]
So the above equality can be rewritten as
\[
\frac{\mu(J)}{|J|^d} = \left( \prod_{i=1}^{k} r_{i-1} \right)^{-1}.
\] (13)

In order to prove (10), it suffices to show
\[
\lim_{k \to \infty} k \prod_{i=1}^{k} r_{i-1} = \infty.
\] (14)

Given an \(i\), \(1 \leq i \leq k\), we are going to estimate \(r_{i-1}\). Let \(J_{i-1}\) be the component interval of \(f(E_{i-1})\) in the sequence (11). Let \(J_{i1}, J_{i2}, \ldots, J_{in_i}\) be the \(n_i\) component intervals of \(f(E_i)\) lying in \(J_{i-1}\). Recall that \(J_i \subset J_{i-1}\) is a component interval of \(f(E_i)\). So there must exist \(1 \leq i_0 \leq n_i\) such that \(J_i = J_{i_{i0}}\). Let \(G_{i0}, G_{i1}, \ldots, G_{in_i}\) be the \(n_i + 1\) gaps in the \(J_{i-1}\). Put
\[
I_{i-1} = f^{-1}(J_{i-1}), \quad I_i = f^{-1}(J_i) = f^{-1}(J_{i_{i0}}) \quad \text{and} \quad I_{ij} = f^{-1}(J_{ij}),
\]
for \(j = 1, 2, \ldots, n_i\). Then \(I_{i1}, \ldots, I_{in_i}\) are component intervals of \(E_i\) lying in the component interval \(I_{i-1}\) of \(E_{i-1}\). Since \(f\) is \(M\)-quasisymmetric, it follows Lemma 1 and the construction of \(E\) that
\[
\left| \frac{G_{ij}}{|J_{i-1}|} \right| \leq 4 \left( \frac{|f^{-1}(G_{ij})|}{|f^{-1}(J_{i-1})|} \right)^p \leq 4e_i^p, \quad j = 0, 1, 2, \ldots, n_i,
\] (15)
where \(e_i = \max_{0 \leq l \leq n_i} e_{i,l}\) and that
\[
\frac{|J_{ij}|}{|J_{i-1}|} \geq (1 + M)^{-2} \left( \frac{|I_{ij}|}{|I_{i-1}|} \right)^q = (1 + M)^{-2} e_i^q.
\] (16)

Here \(p, q\) are numbers defined in Lemma 1. The inequality (15) yields
\[
\frac{|J_{i1}| + \cdots + |J_{in_i}|}{|J_{i-1}|} = \frac{|J_{i-1}| - |G_{i0}| - \cdots - |G_{in_i}|}{|J_{i-1}|} \geq 1 - 4(n_i + 1)e_i^p.
\] (17)

From inequality (16), we have
\[
r_{i-1} = \frac{|J_{i1}|^d + \cdots + |J_{in_i}|^d}{|J_{i-1}|^d} \geq n_i \left( \frac{|J_{ij}|}{|J_{i-1}|} \right)^d \geq \frac{n_i}{(1 + M)^{2d} e_i^d}.
\] (18)
Let
\[ S(k, p) = \{ i : 1 \leq i \leq k, e_i^p \leq \min\{ a, |I_i|^p \} \} \]
where \( a = 1 - \frac{4N^4}{4N+4} \), where \( N = 1 + \sup \{ n_i \} \). Since \( \eta_{j,l} \leq e_{i,t} \). Thus, If \( i \in S(k, p) \) we have
\[ c_i = \frac{|I_{ij}|}{|I_{i-1}|} = \frac{|I_{ij}|}{n_i |I_{ij}| + \sum_{t=0}^{n_i} \eta_{t,l}} \geq \frac{1}{n_i |I_{ij}| + (n_i + 1)\eta_i} \]
\[ \geq \frac{1}{2n_i + 1} \]
\[ \geq \frac{1}{2N} \]
for \( j = 1, \ldots, n_i \), where \( \eta_i = \max_{0 \leq t \leq n_i} \eta_{i,t} \).

From the conclusion (3) of Lemma 2, we obtain
\[ \lim_{k \to \infty} \frac{\#S(k, p)}{k} = 1. \] (20)

Then follows from the left hand inequality of (2) that
\[ 1 \geq \frac{|J_{ij}|}{|J_i|} = \frac{|f(I_{ij})|}{|f(I_i)|} \geq (1 + M)^{-2} \left( \frac{|I_{ij}|}{|I_{i-1}|} \right)^q \geq A \]
for \( j = 1, 2, \ldots, n_i \), where \( A = \left( \frac{(1+M)^{-2}}{(2N)^q} \right) \). Therefore,
\[ \frac{|J_i|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d}{(|J_i| + |J_{i1}| + \cdots + |J_{in_i}|)^d} = \frac{1 + x_1^d + \cdots + x_{n_i}^d}{(1 + x_1 + \cdots + x_{n_i})^d} \]
\[ \geq (1 + A)^{1-d} \]
(21)
where \( x_j = \frac{|J_{ij}|}{|J_i|} \in [A, 1] \).

Note that the equality (17) and (21), for any \( i \in S(k, p) \) we obtain
\[ r_{i-1} = \frac{|J_i|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d}{|J_{i-1}|^d} \]
\[ = \frac{|J_i|^d + |J_{i1}|^d + \cdots + |J_{in_i}|^d (|J_i| + |J_{i1}| + \cdots + |J_{in_i}|)^d}{(|J_i| + |J_{i1}| + \cdots + |J_{in_i}|)^d |J_{i-1}|^d} \]
\[ \geq \alpha_2 (1 - 4(n_i + 1)\epsilon_i^p)^d, \] (22)
where $\alpha_2 = (1 + A)^{1-d} > 1$.

Since

$$1 - m x \geq (1 - x)^{m+1}$$

for all $x \in (0, 1 - \sqrt{\frac{m}{m+1}})$, so we have

$$1 - 4m x \geq (1 - x)^{4m+1}$$

for all $x \in (0, a)$ where $a = 1 - \frac{4N + 4}{4N + 5}$ and all positive integers $m \leq N$.

Note that $n_i < N$ and $e_i^p \in (0, a)$ for all $i \in S(k, p)$, thus we obtain

$$r_{i-1} \geq \alpha_2 (1 - e_i^p)^{4n_i + 4}$$

Using the estimate (18) and (23), we obtain

$$\prod_{i=1}^{k} r_{i-1} \geq \prod_{i \in S(k, p)} \frac{n_i c_i^{dq}}{(1 + M)^{2d}} \prod_{i \in S(k, p)} \alpha_2 (1 - 4(n_i + 1)e_i^p)^d$$

$$\geq \prod_{i \notin S(k, p)} \frac{n_i c_i^{dq}}{(1 + M)^{2d}} \prod_{i \in S(k, p)} \alpha_2 (1 - e_i^p)^{4n_i + 4}d$$

$$= \alpha_2^{\#S(k, p)} [1 + M)^{2d} \prod_{i \notin S(k, p)} n_i c_i^{dq} \prod_{i \in S(k, p)} (1 - e_i^p)^{4n_i + 4}d.$$ (24)

If $q = 1$, since $n_i c_i \leq 1$ then we have

$$\prod_{i \notin S(k, p)} n_i c_i^{dq} \prod_{i \in S(k, p)} n_i c_i^d \geq \prod_{i \notin S(k, p)} n_i c_i \geq \prod_{i=1}^{k} n_i c_i = N_k \delta_k.$$ (25)

If $q > 1$, we have

$$\prod_{i \notin S(k, p)} n_i c_i^{dq} = \prod_{i \notin S(k, p)} (n_i c_i)^{dq} n_i^{1-dq} \geq \prod_{i=1}^{k} (n_i c_i)^{dq} \prod_{i \notin S(k, p)} n_i^{1-dq}$$

$$= \prod_{i=1}^{k} (n_i c_i)^{dq} \prod_{i \notin S(k, p)} n_i^{1-dq} \geq (N_k \delta_k)^{dq} \prod_{i \notin S(k, p)} N_i^{1-dq}$$

$$= (N_k \delta_k)^{dq} (N_i^{1-dq})_{k-\#S(k, p)}$$
for \( d \in (1/q, 1) \).

Let

\[
\xi_k = \alpha_2^{g^{S(k,p)}} ((1 + M)^{-2d})^{k - \frac{d}{q}S(k,p)} (N_k \delta_k)^{dq} (N^{1-dq})^{k - \frac{d}{q}S(k,p)}
\]

and

\[
\zeta_k = \prod_{i \in S(k,p)} (1 - e_i^p)^{(4n_i + 4)d}.
\]

Thus, we have

\[
\prod_{i=1}^{k} r_{i-1} \geq \xi_k \zeta_k.
\]

It is obvious that

\[
\lim_{k \to \infty} \xi_k^{1/k} = \alpha_2 > 1.
\]

due to the conclusion (1) of Lemma 2 and the equality (20). On the other hand, since \( \log(1 - x) \geq -2x \) when \( 0 < x < 1 \), the conclusion (2) of Lemma 2 we obtain

\[
\frac{1}{k} \log \xi_k = \frac{1}{k} \log \prod_{i \in S(k,p)} (1 - e_i^p)^{(4n_i + 4)d}
\]

\[
= \frac{1}{k} \sum_{i \in S(k,p)} \log(1 - e_i^p)^{(4n_i + 4)d}
\]

\[
= \frac{1}{k} \sum_{i \in S(k,p)} (4n_i + 4)d \log(1 - e_i^p)
\]

\[
\geq \frac{(4N + 4)d}{k} \sum_{i \in S(k,p)} \log(1 - e_i^p)
\]

\[
\geq -2 \frac{(4N + 4)d}{k} \sum_{i \in S(k,p)} e_i^p
\]

\[
\geq -2 \frac{(4N + 4)d}{k} \sum_{i=1}^{k} e_i^p \to 0.
\]

as \( k \to \infty \). This show that

\[
\lim_{k \to \infty} \xi_k^{1/k} = 1.
\]
From (27), (28), (30), we obtain
\[
\liminf_{k \to \infty} \left( \prod_{i=1}^{k} r_{i-1} \right)^{1/k} \geq \alpha_2 > 1.
\]
This implies
\[
\lim_{k \to \infty} \left( \prod_{i=1}^{k} r_{i-1} \right) = \infty.
\]

**Step 2.** Let \( J \subset [0, 1] \) be any interval. For such \( J \), let \( k \) be the unique positive inter such that
\[
\delta_k \leq |f^{-1}(J)| \leq \delta_{k-1},
\]
where \( \delta_k \) denotes the length of component intervals of \( E_k \). Then the set \( f^{-1}(J) \) meets at most two component intervals of \( E_{k-1} \) and hence at most \( 2n_{k+1} \) component intervals of \( E_k \). Thus, the set \( J \) meets at most \( 2n_{k+1} \) component intervals of \( f(E_k) \).

Let \( J_1, J_2, \ldots, J_l, l \leq 2n_{k+1} \), be those component intervals of \( f(E_k) \) meeting \( J \). Using the conclusion of step 1. we obtain
\[
\mu(J) \leq \sum_{i=1}^{l} \mu(J_i) \leq C \sum_{i=1}^{l} |J_i|^d.
\] (31)
Since \( \delta_k \leq |f^{-1}(J)| \), we obtain
\[
f^{-1}(J_i) \subset 3f^{-1}(J), \quad i = 1, 2, 3 \ldots l,
\]
where \( 3f^{-1}(J) \) denote the interval of length \( 3|f^{-1}(J)| \) concentric with \( f^{-1}(J) \). Thus we obtain
\[
|J_i| \leq f(3f^{-1}(J)) \leq K|J|, \quad i = 1, 2, 3 \ldots l,
\]
where \( K \) is a positive constant depending on \( M \) only. This together with gives
\[
\mu(J) \leq C lK^d |J|^d \leq 2lCK^d |J|^d.
\]
This show that (10).

By Lemma (3), it follows from that \( \dim_H f(E) \geq d \) for \( d \). As \( d \) could be chosen as closed to 1 as one would. Hence \( \dim_H f(E) = 1 \). This completes the proof of Theorem 2.

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References

[1] K.J. Faconer, Fractal Geometry: Mathematical Foundations and Applications, John Wile Sons (1990).

[2] H.Hakobyan, Cantor sets minimal for quasisymmetric maps, J.Contemp. Math.Anal. 41(2), 2006, 5-13.

[3] M.D.Hu, S.Y.Wen, Quasisymmetrically minimal uniform cantor sets, Topology and its Applications, 155, 2008, no.6,515-521.

[4] J.Heinonen, Lectures on Analysis in Metric Spaces. Universitext. Springer-Verlag, New York, 2001.

[5] P.Pansu, Dimension conforme et sphère à l’infini des variétés à courbure négative, Ann.Acad.Sci.Fenn. 14(214 177-212.

[6] Z.Y.Wen, J.Wu, Hausdorff dimension of homogeneous perfect sets, Acta Math.Hungar.,107,2005,35-44.

[7] Z.Y.Wen, Mathematical Foundations of Fractal geometry, Shanghai Scientific and Technological Education Publishing House, 2000.

[8] X.Y.Wang, J.Wu, Packing dimensions of homogeneous perfect sets, Acta Math.Hungar., 118(1-2),2008,29-39.

[9] J.M.Wu, Null sets for doubling and dyadic doubling measures, Ann.Acad.Sci.Fenn.Math.18, 1993,77-91.