Abstract. We study the value distribution of holomorphic curves from a general open Riemann surface into a smooth logarithmic pair \((X, D)\). By stochastic calculus, we first obtain a version of tautological inequality (proposed by McQuillan) and a logarithmic derivative lemma. Then, one uses them to establish a Second Main Theorem of Nevanlinna theory for pair \((X, D)\) under certain conditions. Finally, we apply the Second Main Theorem to study the holomorphic curves from a general open Riemann surface into certain special moduli spaces of polarized Abelian varieties intersecting boundary divisors.

1. Introduction

1.1. Main results.

To begin with, we shall review a conjecture of Vojta in Nevanlinna theory. Let \((X, D)\) be a smooth logarithmic pair over \(\mathbb{C}\), i.e., \(X\) is a smooth complex projective variety and \(D\) is a normal crossing divisor on \(X\). Denote \(K_X(D) = K_X \otimes \mathcal{O}_X(D)\), where \(K_X\) is the canonical line bundle over \(X\) and \(\mathcal{O}_X(D)\) is the holomorphic line bundle defined by \(D\). Let us consider the finite ramified covering \(\pi : B \to \mathbb{C}\), where \(B\) is an open (connected) Riemann surface. Given an ample line bundle \(A\) over \(X\), Vojta conjectured the following Second Main Theorem (\cite{28}, Conjecture 27.5) that

**Conjecture 1.1** (Vojta, \cite{28}). For any holomorphic curve \(f : B \to X\) whose image is not contained in \(\text{Supp}D\), we have

\[
T_{f,K_X(D)}(r) \leq_{\text{exc}} N_f^{[1]}(r, D) + N_{\text{Ram}(\pi)}(r) + O\left(\log T_{f,A}(r) + \log r\right),
\]

where \(\leq_{\text{exc}}\) means that an inequality holds for \(r > 1\) outside a subset of finite Lebesgue measure.

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Recently, Sun [26] considered these pairs $(X, D)$ which can be interpreted as smooth compactification of the base space of a family, i.e., the pair $(X, D)$ whose complement $U = X \setminus D$ carries a family of smooth polarized varieties. Suppose that $(\psi : V \to U)$ is a smooth family of polarized smooth varieties with semi-ample canonical sheaves and fixed Hilbert polynomial $h$, such that the induced classifying mapping from $U$ to moduli scheme $M_h$ is quasi-finite.

In Section 5 ([26], Section 2), we will review that there exists an ample line bundle $A$ over the base space $X$, which is closely related to the direct image sheaf of the family $\psi$. Consider a holomorphic curve $f : B \to X$ whose image is not contained in $\text{Supp} D$, Sun showed the following Second Main Theorem

\[ T_{f,A}(r) \leq \text{exc} \left( \frac{d+1}{2} \left( N_f^{[1]}(r, D) + N_{\text{Ram}(\pi)}(r) \right) + O \left( \log T_{f,A}(r) + \log r \right), \right. \]

where $d$ is the fiber dimension of the family $\psi$. Furthermore, Sun applied his Second Main Theorem to study the holomorphic curves into certain modular varieties ([26], Theorem D). For more details about moduli spaces of smooth polarized varieties, we refer the reader to [20, 29].

In this paper, we will revisit Vojta’s conjecture and Sun’s results in a very different way. Instead of $B$, we study the value distribution of a holomorphic curve $f : S \to X$, from a geometric point of view, where $S$ is a general open Riemann surface. Instead of $N_{\text{Ram}(\pi)}(r)$, we wish to give a quantitative term depending only on geometric nature of $S$. As two most important cases, we also wish to include the classical results for $\mathbb{C}$ and the unit disc.

The main strategy used in the paper is the probabilistic approach, namely, the technique of Brownian motions. Applications of Brownian motion theory in Nevanlinna theory can be traced back to 1986, Carne [5] re-formulated the Nevanlinna’s functions of meromorphic functions on $\mathbb{C}$ in terms of Brownian motions. Via the stochastic calculus, Carne provided a probabilistic proof of Nevanlinna’s two fundamental theorems [17], i.e., First Main Theorem and Second Main Theorem. Later, Atsuji wrote a series of papers in developing this technique. One of the most important work of Atsuji on the Nevanlinna theory may be the establishment of a Second Main Theorem of meromorphic functions on a complete Kähler manifold of non-positive sectional curvature (see [1, 2]). Via the stochastic calculus of Brownian motions, Dong-He-Ru [7] also gave a probabilistic proof of H. Cartan’s theory for holomorphic curves into a complex projective space intersecting hyperplanes in general position. In this paper, we shall apply Brownian motion more technically to the study of value distribution of holomorphic curves in moduli spaces. Although the stochastic calculus of Brownian motions such as Coarea formula, Itô formula and Dynkin formula is similarly used as Carne and Atsuji, the technical route differs from what it used to be. Atsuji focused his attention on meromorphic functions on manifolds of higher dimension, however, what we are concerned with are holomorphic curves into a complex projective variety. They are two
different research branches in Nevanlinna theory and have their own research approaches. A quite different technical route in the paper is the Logarithmic Derivative Lemma (Theorem 3.1), that has not been crossed by Atsuji. This work, nevertheless, is also very benefited from the contributions of Atsuji to the estimation of Green functions for geodesic balls (Lemma 3.1).

Let us introduce the main theorems of this paper. Some notations will be provided later. By uniformization theorem, we can equip \( S \) with a complete Hermitian metric written as 
\[
ds^2 = 2gdzd\bar{z}
\]
in a local holomorphic coordinate \( z \), such that its Gauss curvature \( K_S \leq 0 \) associated to the metric \( g \), here \( K_S \) is defined by
\[
K_S = -\frac{1}{2} \Delta_S \log g = -\frac{1}{g} \frac{\partial^2 \log g}{\partial z \partial \bar{z}},
\]
where \( \Delta_S \) is the Laplace-Beltrami operator on \( S \) associated to the metric \( g \).

Evidently, \( S \) is a complete Kähler manifold with associated Kähler form
\[
\alpha = g \frac{\sqrt{-1}}{\pi} dz \wedge d\bar{z}
\]
in a local holomorphic coordinate \( z \). Now, fix \( o \in S \) as a reference point. We denote by \( D(r) \) the geodesic ball centered at \( o \) with radius \( r \), and by \( \partial D(r) \) the boundary of \( D(r) \). By Sard’s theorem, \( \partial D(r) \) is a submanifold of \( S \) for almost all \( r > 0 \). Set
\[
(1) \quad \kappa(t) = \min \{ K_S(x) : x \in \overline{D(t)} \}.
\]
Then \( \kappa \) is a non-positive, decreasing continuous function on \([0, \infty)\).

We establish a Second Main Theorem for pair \((X,D)\) as follows

**Theorem A** (=Theorem 5.2). Let \((X,D)\) be a smooth logarithmic pair over \( \mathbb{C} \) with \( U = X \setminus D \). Assume that there is a smooth family \((\psi : V \to U)\) of polarized smooth varieties with semi-ample canonical sheaves and a given Hilbert polynomial \( h \), such that the induced classifying mapping from \( U \) into moduli scheme \( \mathcal{M}_h \) is quasi-finite. Then for any holomorphic curve \( f : S \to X \) whose image is not contained in \( \text{Supp}D \), we have
\[
T_{f,A}(r) \leq \text{exc} \frac{d+1}{2} N_f^{[1]}(r,D) + O\left( \log T_{f,A}(r) - \kappa(r)r^2 + \log^+ \log r \right),
\]
where \( A \) is an ample line bundle over \( X \) close to \( \psi \) given as above, and \( d \) is the fiber dimension of the family \( \psi \). More precisely, if \( S \) is the Poincaré disc (take \( o \) as the center of disc), then
\[
T_{f,A}(r) \leq \text{exc} \frac{d+1}{2} N_f^{[1]}(r,D) + O\left( \log T_{f,A}(r) + r \right).
\]

We find in Theorem A that \( N_{\text{Ram}(\pi)}(r) \) is removed and a new term \(-\kappa(r)r^2\) is appeared, which depends on the curvature of \( S \), and is more intuitive than \( N_{\text{Ram}(\pi)}(r) \). In case \( S = \mathbb{C} \) (equipped with standard Euclidean metric), one
has \( \kappa(r) \equiv 0 \). By Remark 2.1, \( T_{f,A}(r) \) (characteristic function) and \( N_f^{[1]}(r, D) \) (truncated counting function) agree with the classical ones. The case \( S = \mathbb{D} \) (unit open disc equipped with Poincaré metric) will be discussed in Section 1.2 below.

**Remark 1.2.** In this paper, the Poincaré disc is the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) equipped with Poincaré metric

\[
ds^2 = \frac{4dzd\bar{z}}{(1 - |z|^2)^2},
\]

which is a complete hyperbolic metric of Gauss curvature \(-1\).

**Theorem B** (=Corollary 5.3). Assume the same conditions as in Theorem A. Let \( f : S \to X \) be a holomorphic curve which ramifies over \( D \) with order \( c > (d+1)/2 \), i.e., a constant \( c > (d+1)/2 \) such that \( f^*D \geq c \cdot \text{Supp} f^*D \). If \( f \) satisfies the growth condition

\[
\liminf_{r \to \infty} \frac{\kappa(r)r^2}{T_{f,A}(r)} = 0,
\]

then \( f(S) \) is contained in \( D \). More precisely, if \( S \) is the Poincaré disc (take \( o \) as the center of disc), then \( f(S) \) is contained in \( D \) provided that

\[
\limsup_{r \to \infty} \frac{r}{T_{f,A}(r)} = 0.
\]

Notice, if \( S = \mathbb{C} \), the first growth condition in Theorem B is automatically satisfied. To receive a degeneracy result, some growth condition is necessary. It is hard, however, to give a perfect estimate of Green functions in a general Riemannian manifold, hence the first growth condition in Theorem B is not optimal. However, we remark that the second growth condition in Theorem B is sharp (see comparisons with the classical results in Section 1.2 below).

**Theorem C** (=Corollary 5.4). Assume the same conditions as in Theorem A. Then for any holomorphic curve \( f : S \to X \) whose image is not contained in \( \text{Supp}D \), we have

\[
T_{f,K_X(D)}(r) \leq_{\text{exc}} \frac{k(d+1)}{2} N_f^{[1]}(r, D) + O\left( \log T_{f,A}(r) - \kappa(r)r^2 + \log r \right)
\]

for an integer \( k \) such that \( A^{\otimes k} \geq K_X(D) \). More precisely, if \( S \) is the Poincaré disc (take \( o \) as the center of disc), then

\[
T_{f,K_X(D)}(r) \leq_{\text{exc}} \frac{k(d+1)}{2} N_f^{[1]}(r, D) + O\left( \log T_{f,A}(r) + r \right)
\]

for an integer \( k \) such that \( A^{\otimes k} \geq K_X(D) \).

If \( X \) is a smooth complex projective curve, we can replace \( k(d+1)/2 \) by 1 in Theorem C, see Theorem 4.5 in Section 4. We obtain a direct consequence
of Theorem A: given any holomorphic curve \( f : S \to X \) whose image is not contained in \( \text{Supp} D \), there exists a positive constant \( c_\psi \), depending only on \( \psi \) and \( (X, D) \), such that

\[
T_{f, K_X(D)}(r) \leq \text{exc} c_\psi N_{f}^{[1]}(r, D) + O \left( \log T_{f, A}(r) - \kappa(r)r^2 + \log^+ \log r \right).
\]

More precisely, if \( S \) is the Poincaré disc, then

\[
T_{f, K_X(D)}(r) \leq \text{exc} c_\psi N_{f}^{[1]}(r, D) + O \left( \log T_{f, A}(r) + r \right).
\]

We apply Theorem A to Siegel modular varieties (see [25]), which will be introduced in Section 5. Let \( A_g^{[n]} (n \geq 3) \) be the moduli space of principally polarized Abelian varieties with level-\( n \) structure. Indeed, let \( \overline{A}_g^{[n]} \) be smooth compactification of \( A_g^{[n]} \) such that \( D = A_g^{[n]} \setminus \overline{A}_g^{[n]} \) is a normal crossing divisor. Then we obtain

**Theorem D** (=Theorem 5.5). For any holomorphic curve \( f : S \to \overline{A}_g^{[n]} \) whose image is not contained in \( \text{Supp} D \), we have

\[
T_{f, K_{\overline{A}_g^{[n]}}}(r) \leq \text{exc} \frac{(g + 1)^2}{2} N_{f}^{[1]}(r, D) + O \left( \log T_{f, A}(r) - \kappa(r)r^2 + \log^+ \log r \right).
\]

More precisely, if \( S \) is the Poincaré disc (take \( o \) as the center of disc), then

\[
T_{f, K_{\overline{A}_g^{[n]}}}(r) \leq \text{exc} \frac{(g + 1)^2}{2} N_{f}^{[1]}(r, D) + O \left( \log T_{f, A}(r) + r \right).
\]

### 1.2. Comparing Theorem A with the classical results.

We compare Theorem A obtained with the classical results in Nevanlinna theory. Without going into the details, we shall refer the reader to the recent excellent papers such as Păun-Sibony [19], Ru [22] and Ru-Sibony [23], etc., and refer the reader to some good books, for example, Noguchi-Winkelmann [18], Ru [21] and Vojta [28], etc.. The case \( B = \mathbb{C} \) is simple, we can see easily that Theorem A coincides with the classical results by Remark 2.1. In what follows, we compare Theorem A with the classical results when \( B = \mathbb{C} \).

Let \( \mathbb{D} \) denote the unit disc, we need to compare the Second Main Theorem of holomorphic curve \( f \) from \( \mathbb{D} \) into \( X \) under Poincaré metric and standard Euclidean metric, respectively. To avoid confusion, it is better to use \( r, \tilde{r} \) to stand for the geodesic radius under Poincaré metric and standard Euclidean metric, respectively. Clearly, note that \( 0 < r < \infty \) and \( 0 < \tilde{r} < 1 \). Combining the arguments of Ru-Sibony [23] with ones of Sun [26], we have the classical Second Main Theorem for \( \mathbb{D} \) (equipped with standard Euclidean metric)

\[
T_{f, A}(\tilde{r}) \leq \text{exc} \frac{d + 1}{2} N_{f}^{[1]}(\tilde{r}, D) + O \left( \log T_{f, A}(\tilde{r}) + \log \frac{1}{1 - \tilde{r}} \right).
\]
where $A, d$ are given in Theorem A. Note that the main error term $O(\log \frac{1}{1-\tilde{r}})$ is optimal in the classical Nevanlinna theory [18, 21]. Our result in Theorem A says that

$$T_{f,A}(r) \leq \text{exc} \frac{d+1}{2} N_f^{[1]}(r, D) + O \left( \log T_{f,A}(r) + r \right).$$

a) Comparing the main error terms

Taking $o$ as the center of $\mathbb{D}$. Let $r(x), \tilde{r}(x)$ denote the Riemannian distance function of a point $x$ from o under Poincaré metric and standard Euclidean metric, respectively. We compare the main error terms $O(r)$ and $O(\log \frac{1}{1-\tilde{r}})$. Let the radius $r$ correspond to the radius $\tilde{r}$.

By the relation ([21], Page 273)

$$r(x) = \log \frac{1 + \tilde{r}(x)}{1 - \tilde{r}(x)},$$

it follows that

$$r = \log \frac{1 + \tilde{r}}{1 - \tilde{r}} = \log \frac{1}{1 - \tilde{r}} + O(1)$$

due to $\tilde{r} < 1$. This implies that the main error terms $O(r), O(\log \frac{1}{1-\tilde{r}})$ (under the two metrics respectively) are actually equivalent.

b) Comparing the Nevanlinna’s functions

We only compare the characteristic functions $T_{f,A}(r)$ and $T_{f,A}(\tilde{r})$, and the other Nevanlinna’s functions (proximity functions and counting functions) can be compared similarly. Refer to the definition for Nevanlinna’s functions in our settings in Section 2.2, and the definition for the classical Nevanlinna’s functions in [18, 21]. In the following, we will prove that $T_{f,A}(r)$ and $T_{f,A}(\tilde{r})$ are a match (and so are the other Nevanlinna’s functions), if the radius $r$ is corresponding to the radius $\tilde{r}$ via the relation $r = \log((1 + \tilde{r})/(1 - \tilde{r}))$.

Let $\Delta, \tilde{\Delta}$ be the Laplace-Beltrami operators under Poincaré metric and standard Euclidean metric, respectively, and let $D(r), \tilde{D}(\tilde{r})$ be the geodesic balls with radius $r, \tilde{r}$ centered at $o$ under the two metrics, respectively. Moreover, we denote by $g_r(o, x), \tilde{g}_{\tilde{r}}(o, x)$ the Green functions (see [4]) of $\Delta/2, \tilde{\Delta}/2$ for $D(r), \tilde{D}(\tilde{r})$ with Dirichlet boundary condition and a pole $o$, respectively. Let the radius $r$ correspond to the radius $\tilde{r}$, then $D(r)$ corresponds to $\tilde{D}(\tilde{r})$.

Notice that

$$\tilde{g}_{\tilde{r}}(o, x) = \frac{1}{\pi} \log \frac{\tilde{r}}{\tilde{r}(x)}$$

which corresponds to the Green function

$$g_r(o, x) = \frac{1}{\pi} \log \frac{(e^r - 1)(e^{r(x)} + 1)}{(e^r + 1)(e^{r(x)} - 1)}$$

due to [2]. A direct computation shows that $\Delta udV = \tilde{\Delta} ud\tilde{V}$ for any smooth function $u$, where $dV, d\tilde{V}$ are the volume elements of $\mathbb{D}$ under Poincaré metric.
and standard Euclidean metric, respectively. Hence, we conclude that (see \(5\) for definition of characteristic function)

\[
T_{f,A}(r) = \pi \int_{D(r)} g_r(o,x)f^*c_1(A,h) \\
= -\frac{1}{4} \int_{D(r)} g_r(o,x)\Delta \log f(x) dV(x) \\
= -\frac{1}{4} \int_{D(\tilde{r})} \tilde{g}_r(o,x)\tilde{\Delta} \log f(x) d\tilde{V}(x) \\
= \pi \int_{D(\tilde{r})} \tilde{g}_r(o,x)f^*c_1(A,h) \\
= \int_0^{\tilde{r}} \frac{dt}{t} \int_{D(t)} f^*c_1(A,h) \\
= T_{f,A}(\tilde{r}),
\]

where \(h\) is a Hermitian metric on \(A\) such that the Chern form \(c_1(A,h) > 0\). This certifies that \(T_{f,A}(r), \tilde{T}_{f,A}(\tilde{r})\) are a match. It means that the two Second Main Theorems are equivalent.

**Remark 1.3.** Recently, Păun-Sibony [19] investigated the value distribution of a holomorphic mapping \(f: Y \rightarrow \mathbb{P}^1(\mathbb{C})\), where \(Y\) is a parabolic Riemann surface (i.e., an open Riemann surface with a parabolic exhaustion function \(\sigma\)). An earlier work concerning holomorphic curves from \(Y\) into \(\mathbb{P}^n(\mathbb{C})\) is due to Wu [32] (see also He-Ru [14] and Shabat [24]). Let us introduce the result of Păun-Sibony briefly. Given the parabolic ball \(B(r) := \{x \in Y : \sigma(x) < r\}\). The characteristic function is defined by

\[
T_f(r) = \int_1^r \frac{dt}{t} \int_{B(t)} f^*\omega_{FS},
\]

where \(\omega_{FS}\) is the Fubini-Study form on \(\mathbb{P}^1(\mathbb{C})\). The other Nevanlinna’s functions can be defined in this setting ([19], Page 13). Păun-Sibony proved that ([19], Theorem 3.2)

\[
(q - 2)T_f(r) \leq \text{exc} \sum_{j=1}^q N_f^{[1]}(r, a_j) + \mathcal{X}_\sigma(r) + O\left(\log T_f(r) + \log r\right)
\]

for distinct points \(a_1, \ldots, a_q\) in \(\mathbb{P}^1(\mathbb{C})\), where

\[
\mathcal{X}_\sigma(r) = \int_1^r |\chi_\sigma(t)| \frac{dt}{t}, \quad \chi_\sigma(t) = \chi(B(t)).
\]

It is expected to compare our result with Păun-Sibony’s result, but it seems difficult. In the setting of Păun-Sibony, \(\sigma\) may give more than one connected components of \(B(r)\), and each connected component may be multi-connected if the genus of \(Y\) is greater than 0. In our setting, however, the completeness
of metric implies that the geodesic ball \( D(r) \) is simply-connected (with genus 0). Therefore, one can hardly compare the Nevanlinna’s functions in the two settings. We also can hardly compare error terms \(-\kappa(r)r^2 \) (or \( r \)) and \( X_\sigma(r) \).

In my opinion, it gives two different ways in describing the value distribution of curve \( f \): one is to use the topology (Euler characteristic) of \( \mathcal{Y} \), the other is to use the geometry (Gauss curvautre) of \( \mathcal{Y} \).

1.3. Work of Carne and Atsuji on Nevanlinna theory.

In 1986, Carne \[5\] first noticed the relationship between Nevanlinna theory and Brownian motions, he formulated Nevanlinna’s functions of a meromorphic function on \( \mathbb{C} \) via the Brownian motion \( X_t \) in \( \mathbb{C} \), where \( X_t \) is generated by \( \Delta_C/2 \). By means of stochastic calculus, specially, Itô formula and Coarea formula, Carne re-obtained the Nevanlinna theory of meromorphic functions on \( \mathbb{C} \). We would mention that Carne’s method is also suitable to the case \( \mathbb{C}^n \), though he hadn’t pushed his work since then. A remarkable development of work of Carne is due to Atsuji \[1, 2\], who investigated the value distribution theory of meromorphic functions on Kähler manifolds, along a line of Carne, but developed techniques of Carne. The major work of Atsuji is to generalize Nevanlinna theory to the non-positively curved complete Kähler manifolds, and one of his main contributions is the estimation of lower bounds of Green functions. This work of estimation leads him to obtain the Calculus Lemma \([2\), Lemma 13\] which is a useful tool in the study of Nevanlinna theory. By combining another estimate \([2\), Lemma 12\], he established a Second Main Theorem (with an error term depending on the curvature of the manifolds) of meromorphic functions on such class of manifolds \([2\], Theorem 9\).

We compare Theorem A with the results of Atusji in case that \( X = \mathbb{P}^1(\mathbb{C}) \) and domain is the Riemann surface. For a general open Riemann surface \( S \), Theorem A agrees with Atsuji’s result. However, for \( S = \mathbb{D} \) (equipped with Poincaré metric), Theorem A gives a main error term \( O(r) \), but Atsuji gave \( O(r^2) \) according to his theorem \([2\], Theorem 9\). It implies that our results are better than Atsuji’s results.

2. Nevanlinna’s functions and First Main Theorem

2.1. Coarea formula and Dynkin formula.

Let \( (M, g) \) be a Riemannian manifold with Laplace-Beltrami operator \( \Delta_M \) associated to \( g \). Fix \( x \in M \), denote by \( B_x(r) \) the geodesic ball centered at \( x \) with radius \( r \), and by \( S_x(r) \) the geodesic sphere centered at \( x \) with radius \( r \). Apply Sard’s theorem, \( S_x(r) \) is a submanifold of \( M \) for almost all \( r > 0 \). A Brownian motion \( (X_t)_{t \geq 0} \) (written as \( X_t \) for short) in \( M \) is a heat diffusion process generated by \( \Delta_M/2 \) with transition density function \( p(t, x, y) \) which
is the minimal positive fundamental solution of the heat equation
\[
\frac{\partial}{\partial t} u(t, x) - \frac{1}{2} \Delta_M u(t, x) = 0.
\]
We denote by \( \mathbb{P}_x \) the law of \( X_t \) started at \( x \in M \) and by \( \mathbb{E}_x \) the corresponding expectation with respect to \( \mathbb{P}_x \). The reader may refer to [4, 7, 13, 15] to learn more about Brownian motions.

A. Coarea formula

Let \( D \) be a bounded domain with smooth boundary \( \partial D \) in \( M \). Fix \( x \in D \), we use \( d\pi^D_x \) to denote the harmonic measure on \( \partial D \) with respect to \( x \). This measure is a probability measure. Set
\[
\tau_D := \inf \{ t > 0 : X_t \notin D \}
\]
which is a stopping time. Let \( g_D(x, y) \) stand for the Green function of \( \Delta_M/2 \) for \( D \) with Dirichlet boundary condition and a pole \( x \), namely
\[
-\frac{1}{2} \Delta_M g_D(x, y) = \delta_x(y), \quad y \in D; \quad g_D(x, y) = 0, \quad y \in \partial D,
\]
where \( \delta_x \) is the Dirac function. For \( \phi \in C^b(D) \) (space of bounded continuous functions on \( D \)), the Coarea formula [4] says that
\[
\mathbb{E}_x \left[ \int_0^{\tau_D} \phi(X_t) dt \right] = \int_D g_D(x, y) \phi(y) dV(y),
\]
where \( dV(y) \) is the Riemannian volume element on \( M \). By Proposition 2.8 in [4], we have the following relation between harmonic measures and hitting times that
\[
\mathbb{E}_x \left[ \psi(X_{\tau_D}) \right] = \int_{\partial D} \psi(y) d\pi^D_x(y)
\]
for any \( \psi \in C^b(D) \). Coarea formula or [4] works if the set of singularities of \( \phi \) or \( \psi \) is pluripolar, since the Brownian motion \( X_t \) hits the pluripolar set in probability 0 (see [1, 2, 4, 9, 13, 15]).

B. Dynkin formula

Let \( u \in C^2(M) \) (space of bounded \( C^2 \)-class functions on \( M \)), we have the famous Itô formula (see [1] [2] [4] [9] [13] [15])
\[
u(X_t) - u(x) = B \left( \int_0^t \| \nabla_M u \|^2(X_s) ds \right) + \frac{1}{2} \int_0^t \Delta_M u(X_s) dt, \quad \mathbb{P}_x - a.s.
\]
where \( B_t \) is the standard Brownian motion in \( \mathbb{R} \) and \( \nabla_M \) is gradient operator on \( M \). Notice that \( B_t \) is a martingale, take expectation on both sides of the above formula, it follows Dynkin formula (see [1] [2] [4] [9] [13] [15])
\[
\mathbb{E}_x [u(X_{\tau_D})] - u(x) = \frac{1}{2} \mathbb{E}_x \left[ \int_0^{\tau_D} \Delta_M u(X_t) dt \right].
\]
Dynkin formula still works if the set of singularities of $u$ is pluripolar.

2.2. Nevanlinna’s functions and First Main Theorem.

Let $S$ be an open Riemann surface with Kähler form $\alpha$. Fixing $o \in S$ as a reference point. Denote by $D(r)$ the geodesic ball centered at $o$ with radius $r$, and by $\partial D(r)$ the boundary of $D(r)$. Moreover, one uses $g_r(o, x)$ to stand for the Green function of $\Delta S/2$ for $D(r)$ with Dirichlet boundary condition and a pole $o$, and $d\pi^r_o(x)$ to stand for the harmonic measure on $\partial D(r)$ with respect to $o$. Now, let $X_t$ be the Brownian motion in $S$ with generator $\Delta S/2$, started at $o$. Set the stopping time

$$
\tau_r = \inf\{ t > 0 : X_t \notin D(r) \}.
$$

Let $f : S \to X$ be a holomorphic curve into a compact complex manifold $X$. We introduce the generalized Nevanlinna’s functions over Riemann surface $S$. Let $L \to X$ be an ample holomorphic line bundle equipped with Hermitian metric $h$ such that the Chern form $c_1(L, h) > 0$. Let’s define the characteristic function of $f$ with respect to $L$ by

$$
T_{f,L}(r) = \pi \int_{D(r)} g_r(o, x) f^* c_1(L, h)
$$

$$
= -\frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log h \circ f(x) dV(x),
$$

where $dV(x)$ is the Riemannian volume measure of $S$. It can be easily checked that $T_{f,L}(r)$ is independent of the choices of metrics on $L$, up to a bounded term. Since a holomorphic line bundle can be represented as the difference of two ample holomorphic line bundles, the definition of $T_{f,L}(r)$ can extend to an arbitrary holomorphic line bundle. By Coarea formula, we obtain

$$
T_{f,L}(r) = -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \log h \circ f(X_t) dt \right].
$$

A divisor can be also expressed as the difference of two ample divisors. The Weil function of an effective divisor $D$ is well defined by

$$
\lambda_D(x) = -\log \| s_D(x) \|
$$

up to a bounded term, where $s_D$ is the canonical section of the line bundle $\mathcal{O}_X(D)$ over $X$, namely, $s_D$ is locally written as $s_D = \tilde{s}_D e$, where $e$ is a local holomorphic frame of $\mathcal{O}_X(D)$, and $\tilde{s}_D$ is a local defining function of $D$. Note that $\tilde{s}_D$ is a holomorphic function due to that $D$ is effective. We define the proximity function of $f$ with respect to $D$ by

$$
m_f(r, D) = \int_{\partial D(r)} \lambda_D \circ f(x) d\pi^r_o(x).
$$
By (4), one has

\[ m_f(r, D) = E_o[\lambda_D \circ f(X_{\tau r})]. \]

The *counting function* of \( f \) with respect to \( D \) is defined by

\[ N_f(r, D) = \pi \sum_{f' \cdot D \cap D(r)} g_{r}(o, x) \]
\[ = \pi \int_{D(r)} g_{r}(o, x) dd^c \left[ \log |\tilde{s}_D \circ f(x)|^2 \right] \]
\[ = \frac{1}{4} \int_{D(r)} g_{r}(o, x) \Delta_S \log |\tilde{s}_D \circ f(x)|^2 dV(x). \]

We can also define the *truncated counting function* (i.e., the simple counting function without counting multiplicities) of \( f \) with respect to \( D \) in a similar way, denote it by \( N_f^{[1]}(r, D) \).

**Remark 2.1.** The above definition of Nevanlinna’s functions is natural. In case \( S = \mathbb{C} \), the Green function for \( D(r) \) is computed as \((\log \frac{r}{|z|})/\pi\), and the harmonic measure on \( \partial D(r) \) is computed as \( d\theta/2\pi \). By integration by part, we can see that our Nevanlinna’s functions coincide with the classical ones.

Consider a holomorphic curve \( f : S \to X \) such that \( f(o) \notin \text{Supp} D \), where \( D \) is an effective divisor on \( X \). Apply Dynkin formula to \( \lambda_D \circ f(x) \), it follows that

\[ E_o[\lambda_D \circ f(X_{\tau r})] - \lambda_D \circ f(o) = \frac{1}{2} E_o \left[ \int_0^{\tau r} \Delta_S \lambda_D \circ f(X_t) dt \right]. \]

The first term on the left hand side of the above equality is equal to \( m_f(r, D) \), and the term on the right hand side equals

\[ \frac{1}{2} E_o \left[ \int_0^{\tau r} \Delta_S \lambda_D \circ f(X_t) dt \right] = \frac{1}{2} \int_{D(r)} g_{r}(o, x) \Delta_S \log \frac{1}{\|s_D \circ f(x)\|} dV(x) \]

due to Coarea formula. Since \( \|s_D\|^2 = \|\tilde{s}_D\|^2 \), where \( h \) is a Hermitian metric on \( \mathcal{O}_X(D) \) and \( \tilde{s}_D \) is a local defining function of \( D \) given as above, then

\[ \frac{1}{2} E_o \left[ \int_0^{\tau r} \Delta_S \lambda_D \circ f(X_t) dt \right] = -\frac{1}{4} \int_{D(r)} g_{r}(o, x) \Delta_S \log h \circ f(x) dV(x) \]
\[ -\frac{1}{4} \int_{D(r)} g_{r}(o, x) \Delta_S \log |\tilde{s}_D \circ f(x)|^2 dV(x) \]
\[ = T_{f, \mathcal{O}_X(D)}(r) - N_f(r, D). \]

To conclude, we obtain
**Theorem 2.2** (First Main Theorem). Let $X$ be a compact complex manifold. Let $f : S \to X$ be a holomorphic curve such that $f(o) \notin \text{Supp}D$, where $D$ is an effective divisor on $X$. Then

$$T_{f, o_X(D)}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

### 3. Logarithmic Derivative Lemma

Let $S$ be a complete open Riemann surface with Gauss curvature $K_S \leq 0$ associated to Hermitian metric $g$. In this section, we assume that $S$ is simply connected. Recall that

$$\tau_r = \inf \{ t > 0 : X_t \notin D(r) \}.$$

#### 3.1. Calculus Lemma.

Let $\kappa$ be given by (1). As noted before, $\kappa$ is a non-positive and decreasing continuous function on $[0, \infty)$. Consider the ordinary differential equation

$$(6) \quad G''(t) + \kappa(t)G(t) = 0; \quad G(0) = 0, \quad G'(0) = 1.$$  

Comparing (6) with the equation $y''(t) + \kappa(0)y(t) = 0$ under the same initial conditions, then $G$ can be easily estimated as

$$G(t) = t \quad \text{for } \kappa \equiv 0; \quad G(t) \geq t \quad \text{for } \kappa \neq 0.$$  

This implies that

$$(7) \quad G(r) \geq r \quad \text{for } r \geq 0; \quad \int_1^r \frac{dt}{G(t)} \leq \log r \quad \text{for } r \geq 1.$$  

On the other hand, rewrite (6) in the form

$$\log' G(t) \cdot \log' G'(t) = -\kappa(t),$$

where

$$\log' G(t) = \frac{G'(t)}{G(t)}, \quad \log' G''(t) = \frac{G''(t)}{G'(t)}.$$  

Since $G(t) \geq t$ is increasing, then the decrease and non-positivity of $\kappa$ imply that for each fixed $t$, $G$ must satisfy one of the following two inequalities

$$\log' G(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t > 0; \quad \log' G'(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t \geq 0.$$  

Since $G(t) \to 0$ as $t \to 0$, then by integration we obtain

$$(8) \quad G(r) \leq r \exp \left( r \sqrt{-\kappa(r)} \right) \quad \text{for } r \geq 0.$$  

**Lemma 3.1** (2). Let $\eta > 0$ be a constant. Then there is a constant $C > 0$ such that for $r > \eta$ and $x \in D(r) \setminus D(\eta)$

$$g_r(o, x) \int_\eta^r \frac{dt}{G(t)} \geq C \int_{r(x)}^r \frac{dt}{G(t)}.$$
Lemma 3.2 (Borel’s lemma). Let $u$ be a monotone increasing function on $[0, \infty)$ such that $u(r_0) > 1$ for some $r_0 \geq 0$. Then for any $\delta > 0$, there exists a set $E_\delta \subseteq [0, \infty)$ of finite Lebesgue measure such that
\[
u'(r) \leq \nu(r) \log^{1+\delta} \nu(r)
\]
holds for $r > 0$ outside $E_\delta$.

Proof. Since $u$ is monotone increasing, then $u'(r)$ exists for almost all $r \geq 0$. Set
\[S = \{r \geq 0 : u'(r) > u(r) \log^{1+\delta} u(r)\}.
\]
We have
\[
\int_S dr \leq \int_0^{r_0} dr + \int_{S \setminus [0, r_0]} dr \leq r_0 + \int_{r_0}^{\infty} \frac{u'(r)}{u(r) \log^{1+\delta} u(r)} dr < \infty.
\]

Theorem 3.3 (Calculus Lemma). Let $k \geq 0$ be a locally integrable function on $S$ such that it is locally bounded at $o \in S$. Then for any $\delta > 0$, there is a constant $C > 0$ independent of $k, \delta$, and a subset $E_\delta \subseteq (1, \infty)$ of finite Lebesgue measure such that
\[
\mathbb{E}_o [k(X_{r_\nu})] \leq \frac{F(\hat{k}, \kappa, \delta) e^r \sqrt{-\kappa(r)}}{2\pi C} \log r \mathbb{E}_o \left[ \int_{r_\nu}^{r} k(X_t) dt \right]
\]
holds for $r > 1$ outside $E_\delta$, where $\kappa$ is defined by (1) and $F$ is defined by
\[
F(\hat{k}, \kappa, \delta) = \left( \log^+ \hat{k}(r) \cdot \log^+ \left( r e^r \sqrt{-\kappa(r)} \hat{k}(r) \left( \log^+ \hat{k}(r) \right)^{1+\delta} \right) \right)^{1+\delta}
\]
with
\[
\hat{k}(r) = \frac{\log r}{C} \mathbb{E}_o \left[ \int_{0}^{r} k(X_t) dt \right].
\]
Moreover, we have the estimate
\[
\log^+ F(\hat{k}, \kappa, \delta) \leq O \left( \log^+ \log^+ \mathbb{E}_o \left[ \int_{0}^{r_\nu} k(X_t) dt \right] + \log^+ r \sqrt{-\kappa(r)} + \log^+ \log r \right).
\]

Proof. The argument refers to Atsuji [2]. From (6), the simple-connectedness and non-positivity of sectional curvature of $S$ imply $2\pi r d\sigma_r(x) \leq d\sigma_r(x)$, where $d\sigma_r(x)$ is the volume element on $\partial D(r)$ which is induced by the volume.
By Lemma 3.1 and (7), we have
\[
E_o \left[ \int_0^r k(X_t)dt \right] = \int_{D(r)} g_r(o,x)k(x)dV(x)
\]
\[
= \int_0^r dt \int_{\partial D(t)} g_r(o,x)k(x)d\sigma_t(x)
\]
\[
\geq C \int_0^r \int_t^r \frac{G^{-1}(s)}{G(s)} ds dt \int_{\partial D(t)} k(x)d\sigma_t(x)
\]
\[
\geq \frac{C}{\log r} \int_0^r dt \int_t^r \frac{ds}{G(s)} \int_{\partial D(t)} k(x)d\sigma_t(x),
\]
\[
E_o [k(X_\tau_r)] = \int_{\partial D(r)} k(x)d\pi_\tau_r(x) \leq \frac{1}{2\pi r} \int_{\partial D(r)} k(x)d\sigma_r(x).
\]
Thus,
\[
E_o \left[ \int_0^r k(X_t)dt \right] \geq \frac{C}{\log r} \int_0^r dt \int_t^r \frac{ds}{G(s)} \int_{\partial D(t)} k(x)d\sigma_t(x),
\]
(9)
\[
E_o [k(X_\tau_r)] \leq \frac{1}{2\pi r} \int_{\partial D(r)} k(x)d\sigma_r(x).
\]
Set
\[
\Lambda(r) = \int_0^r dt \int_t^r \frac{ds}{G(s)} \int_{\partial D(t)} k(x)d\sigma_t(x).
\]
Then
\[
\Lambda(r) \leq \frac{\log r}{C} E_o \left[ \int_0^r k(X_t)dt \right] = \hat{k}(r).
\]
Since
\[
\Lambda'(r) = \frac{1}{G(r)} \int_0^r dt \int_{\partial D(t)} k(x)d\sigma_t(x),
\]
it yields from (9) that
\[
E_o [k(X_\tau_r)] \leq \frac{1}{2\pi r} \frac{d}{dr} \left( \Lambda'(r)G(r) \right).
\]
By Lemma 3.2 twice and (8), we see that for any \( \delta > 0 \)
\[
\frac{d}{dr} \left( \Lambda'(r)G(r) \right)
\]
\[
\leq G(r) \left[ \log^+ \Lambda(r) \cdot \log^+ \left( G(r)\Lambda(r) \{ \log^+ \Lambda(r) \}^{1+\delta} \right) \right]^{1+\delta} \Lambda(r)
\]
\[
\leq re^{\sqrt{-\kappa(r)}} \left[ \log^+ \hat{k}(r) \cdot \log^+ \left( re^{\sqrt{-\kappa(r)}}\hat{k}(r) \{ \log^+ \hat{k}(r) \}^{1+\delta} \right) \right]^{1+\delta} \hat{k}(r)
\]
\[
= \frac{F(\hat{k},\kappa,\delta)re^{\sqrt{-\kappa(r)}}log r}{C} E_o \left[ \int_0^r k(X_t)dt \right]
\]
holds outside a subset $E_\delta \subseteq (1, \infty)$ of finite Lebesgue measure. Thus,

$$E_o[k(X_r)] \leq \frac{F(\hat{k}, \kappa, \delta) e^{r\sqrt{-\kappa(r)}} \log r}{2\pi C} \left[ \int_0^r k(X_t) dt \right].$$

Therefore, we get the desired inequality. Indeed, for $r > 1$ we compute that

$$\log^+ F(\hat{k}, \kappa, \delta) \leq O\left( \log^+ \log^+ \hat{k}(r) + \log^+ r \sqrt{-\kappa(r)} + \log^+ \log r \right)$$

and

$$\log^+ \hat{k}(r) \leq \log^+ E_o\left[ \int_{\tau} \kappa(X_t) dt \right] + \log^+ \log r.$$

We have the desired estimate. The proof is completed. \(\square\)

3.2. Logarithmic Derivative Lemma.

Note from [8] that each open Riemann surface admits a nowhere-vanishing holomorphic global vector field. Fix a nowhere-vanishing global holomorphic vector field $X$ over $S$. Let $\psi$ be a meromorphic function on $(S, g)$. The norm of the gradient of $\psi$ is defined by

$$\|\nabla_S \psi\|^2 = \frac{2}{g} \left| \frac{\partial \psi}{\partial z} \right|^2$$

in a local holomorphic coordinate $z$. Locally, we may write $\psi = \psi_1/\psi_0$, where $\psi_0, \psi_1$ are local holomorphic functions without common zeros. Regard $\psi$ as a holomorphic mapping into $\mathbb{P}^1(\mathbb{C})$ by $x \mapsto [\psi_0(x) : \psi_1(x)]$. Define

$$T_\psi(r) = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log \left( |\psi_0(x)|^2 + |\psi_1(x)|^2 \right) dV(x),$$

i.e., $T_\psi(r) := T_{f, \psi^3(1)}(r)$. Also, define $T(r, \psi) := m(r, \psi) + N(r, \psi)$, where

$$m(r, \psi) = \int_{\partial D(r)} \log^+ |\psi(x)| d\pi_o^r(x),$$

$$N(r, \psi) = \pi \sum_{x \in (\psi)^{\infty} \cap D(r)} g_r(o, x).$$

Let $i : \mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ be an inclusion defined by $z \mapsto [1 : z]$. Via the pull-back by $i$, we have a $(1,1)$-form $i^* \omega_{FS} = dd^c \log(1 + |\zeta|^2)$ on $\mathbb{C}$, where $\zeta := w_1/w_0$ and $[w_0 : w_1]$ is the homogeneous coordinate system of $\mathbb{P}^1(\mathbb{C})$. The characteristic function of $\psi$ with respect to $i^* \omega_{FS}$ is defined by

$$\hat{T}_\psi(r) = \frac{1}{4} \int_{D(r)} g_r(o, x) \Delta_S \log(1 + |\psi(x)|^2) dV(x).$$
Clearly, $\hat{T}_\psi(r) \leq T_\psi(r)$. We adopt the spherical distance $\|\cdot, \cdot\|$ on $\mathbb{P}^1(\mathbb{C})$, the proximity function of $\psi$ with respect to $a \in \mathbb{P}^1(\mathbb{C})$ is defined by

$$m_\psi(r, a) = \int_{\partial D(r)} \log \frac{1}{\|\psi(x), a\|} d\pi^r_o(x).$$

It is easy to check that $m(r, \psi) = \hat{m}_\psi(r, \infty) + O(1)$ which follows that

$$T(r, \psi) = \hat{T}_\psi(r) + O(1).$$

Similarly as Theorem 2.2, by Dykin formula and Coarea formula, it is trivial to show the First Main Theorem

$$T\left(r, \frac{1}{\psi - a}\right) = T(r, \psi) + O(1).$$

Therefore, we arrive at

$$(11) \quad T(r, \psi) + O(1) = \hat{T}_\psi(r) \leq T_\psi(r) + O(1).$$

**Theorem 3.4 (LDL).** Let $\psi$ be a nonconstant meromorphic function on $S$. Then

$$m\left(r, \frac{X^k(\psi)}{\psi}\right) \leq \text{exc} \frac{5k}{4} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r\right),$$

where $X^j = X \circ X^{j-1}$ with $X^0 = \text{Id}$, and $\kappa$ is defined by (1). More precisely, if $S$ is the Poincaré disc (take $o$ as the center of disc), then

$$m\left(r, \frac{X^k(\psi)}{\psi}\right) \leq \text{exc} \frac{5k}{4} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) + r\right).$$

Take a singular form

$$\Phi = \frac{1}{|\zeta|^2(1 + \log^2|\zeta|)} \frac{\sqrt{-1}}{4\pi^2} d\zeta \wedge d\bar{\zeta}$$

on $\mathbb{P}^1(\mathbb{C})$. A direct computation gives that

$$\int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 2\pi \psi^* \Phi = \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2|\psi|)} \alpha,$$

where $\alpha = (g\sqrt{-1}/\pi)dz \wedge d\bar{z}$ is the Kähler form of $S$. Set

$$T_\psi(r, \Phi) = \frac{1}{2\pi} \int_{D(r)} g_r(o, x) \frac{\|\nabla S\psi\|^2}{|\psi|^2(1 + \log^2|\psi|)}(x) dV(x).$$
By Fubini’s theorem
\[ T_\psi(r, \Phi) = \int_{D(r)} g_r(o, x) \frac{\psi^* \Phi}{\alpha} dV(x) \]
\[ = \pi \int_{\zeta \in \mathbb{P}^1(\mathbb{C})} \Phi \sum_{(\psi - \zeta) \cap D(r)} g_r(o, x) \]
\[ = \int_{\zeta \in \mathbb{P}^1(\mathbb{C})} N_\psi(r, \zeta) \Phi \leq T(r, \psi) + O(1), \]
which follows that
(12) \[ T_\psi(r, \Phi) \leq T(r, \psi) + O(1). \]

**Lemma 3.5.** Assume that \( \psi(x) \neq 0 \). Then
\[ \frac{1}{2} \mathbb{E}_o \left[ \log^+ \frac{\| \nabla S \psi \|^2}{\| \psi \|^2 (1 + \log^2 |\psi|)} (X_{r_\tau}) \right] \leq \text{exc} \frac{1}{2} \log T(r, \psi) + O \left( \log^+ \log T(r, \psi) + r \sqrt{-\kappa(r)} + \log^+ \log r \right), \]
where \( \kappa \) is defined by (11).

**Proof.** By Jensen’s inequality, it is clear that
\[ \mathbb{E}_o \left[ \log^+ \frac{\| \nabla S \psi \|^2}{\| \psi \|^2 (1 + \log^2 |\psi|)} (X_{r_\tau}) \right] \leq \mathbb{E}_o \left[ \log \left( 1 + \frac{\| \nabla S \psi \|^2}{\| \psi \|^2 (1 + \log^2 |\psi|)} (X_{r_\tau}) \right) \right] \]
\[ \leq \log^+ \mathbb{E}_o \left[ \frac{\| \nabla S \psi \|^2}{\| \psi \|^2 (1 + \log^2 |\psi|)} (X_{r_\tau}) \right] + O(1). \]
By Theorem 3.3 with Coarea formula and (12)
\[ \log^+ \mathbb{E}_o \left[ \frac{\| \nabla S \psi \|^2}{\| \psi \|^2 (1 + \log^2 |\psi|)} (X_{r_\tau}) \right] \leq \text{exc} \log^+ \mathbb{E}_o \left[ \int_0^{r_\tau} \frac{\| \nabla S \psi \|^2}{\| \psi \|^2 (1 + \log^2 |\psi|)} (X_t) dt \right] + \log \left( F(\hat{k}, \kappa, \delta)e^{\sqrt{-\kappa(r)} \log r} \right) \]
\[ \leq \log T_\psi(r, \Phi) + \log F(\hat{k}, \kappa, \delta) + r \sqrt{-\kappa(r)} + \log^+ \log r + O(1) \]
\[ \leq \log T(r, \psi) + O \left( \log^+ \log^+ \hat{k}(r) + r \sqrt{-\kappa(r)} + \log^+ \log r \right), \]
where
\[ \hat{k}(r) = \frac{\log r}{C} \mathbb{E}_o \left[ \int_0^{r_\tau} \frac{\| \nabla S \psi \|^2}{\| \psi \|^2 (1 + \log^2 |\psi|)} (X_t) dt \right]. \]
Indeed, we note that
\[ \hat{k}(r) = \frac{2\pi \log r}{C} T_\psi(r, \Phi) \leq \frac{2\pi \log r}{C} T(r, \psi). \]
Hence, we have the desired inequality. \( \square \)
Lemma 3.6. Let $\psi$ be a nonconstant meromorphic function on $S$. Then

$$m\left(r, \frac{X(\psi)}{\psi}\right) \leq \text{exc} \frac{5}{4} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) - \kappa(r) r^2 + \log^+ \log r\right),$$

where $\kappa$ is defined by (1). More precisely, if $S$ is the Poincaré disc (take $o$ as the center of disc), then

$$m\left(r, \frac{X(\psi)}{\psi}\right) \leq \text{exc} \frac{5}{4} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) + r\right).$$

Proof. In terms of a local holomorphic coordinate $z$, one can write $X = a \frac{\partial}{\partial z}$ satisfying $\|X\|^2 = g|a|^2$, where $a$ is a local holomorphic function and $g$ is the Hermitian metric on $S$. Then

$$m\left(r, \frac{X(\psi)}{\psi}\right) = \int_{\partial D(r)} \log^+ \left|\frac{X(\psi)}{\psi}\right|(x) d\pi_0^r(x)$$

$$\leq \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{|X(\psi)|^2}{\|X\|^2 |\psi|^2 (1 + \log^2 |\psi|)}(x) d\pi_0^r(x)$$

$$+ \frac{1}{2} \int_{\partial D(r)} \log(1 + \log^2 |\psi(x)|) d\pi_0^r(x)$$

$$+ \frac{1}{2} \int_{\partial D(r)} \log^+ \|X_x\|^2 d\pi_0^r(x)$$

$$:= A + B + C.$$

We handle $A, B, C$ respectively. For $A$, it follows from (10) and Lemma 3.5 that

$$A = \frac{1}{2} \int_{\partial D(r)} \log^+ \frac{|a|^2 |\frac{\partial \psi}{\partial z}|^2}{g|a|^2 |\psi|^2 (1 + \log^2 |\psi|)}(x) d\pi_0^r(x)$$

$$= \frac{1}{4} \int_{\partial D(r)} \log^+ \frac{\|\nabla S\psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)}(x) d\pi_0^r(x)$$

$$\leq \text{exc} \frac{1}{4} \log T(r, \psi) + O\left(\log^+ \log T(r, \psi) + r \sqrt{-\kappa(r)} + \log^+ \log r\right).$$

For $B$, the Jensen’s inequality and First Main Theorem implies that

$$B \leq \int_{\partial D(r)} \log \left(1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|}\right) d\pi_0^r(x)$$

$$\leq \log \int_{\partial D(r)} \left(1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|}\right) d\pi_0^r(x)$$

$$= \log \left(m(r, \psi) + m\left(r, \frac{1}{\psi}\right) + 1\right)$$

$$\leq \log T(r, \psi) + O(1).$$
Finally, we estimate $C$. Since $X$ never vanishes, we obtain $\|X\| > 0$. Since $S$ is non-positively curved and $a$ is holomorphic, then $\log \|X\|$ is subharmonic, i.e., $\Delta_S \log \|X\| \geq 0$. It is easy to check that
\begin{equation}
\Delta_S \log^+ \|X\| \leq \Delta_S \log \|X\|
\end{equation}
for $x \in S$ satisfying $\|X_x\| \neq 1$, and
\begin{equation}
\log^+ \|X_x\| = 0
\end{equation}
for $x \in S$ satisfying $\|X_x\| \leq 1$. Note that Dynkin formula cannot be directly applied to $\log^+ \|X_x\|$, but by virtue of (13) and (14), it is not hard to verify
\[ C = \frac{1}{2} \mathbb{E}_o [\log^+ \|X(X_{\tau_r})\|^2] \]
\[ \leq \frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \log \|X(X_t)\|^2 \, dt \right] + O(1) \]
\[ = \frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \log g(X_t) \, dt \right] + \frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_S \log |a(X_t)|^2 \, dt \right] + O(1) \]
\[ = -\frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r} K_S(X_t) \, dt \right] + O(1) \]
\[ \leq -2\kappa(r) \mathbb{E}_o [\tau_r] + O(1), \]
where we utilize the fact $K_S = -(\Delta_S \log g)/2$. Therefore, we deduce the first conclusion of the lemma by using $\mathbb{E}_o [\tau_r] \leq 2r^2$, due to Lemma 3.7 below.

If $S$ is the Poincaré disc, then we have $g = 2/(1 - |z|^2)^2$ and $\kappa(r) \equiv -1$. Take $X = \partial/\partial z$, then $C$ can be estimated as follows
\[ C = \frac{1}{2} \mathbb{E}_o [\log^+ \|X(X_{\tau_r})\|^2] \]
\[ = \frac{1}{2} \int_{\partial D(r)} \log \left( \frac{2}{1 - (e^r - 1)^2/(e^r + 1)^2} \right) \, d\theta \]
\[ = \log \frac{\sqrt{2}}{1 - (e^r - 1)^2/(e^r + 1)^2} \]
\[ \leq r + O(1). \]
This proves the second conclusion of the lemma. \qed

**Lemma 3.7.** Let $\tau_r$ be defined as above. Then
\[ \mathbb{E}_o [\tau_r] \leq 2r^2. \]

**Proof.** The argument follows essentially from Atsuji [2], but here we provide a simpler proof though a rougher estimate. Let $X_t$ be the Brownian motion

\[ \mathbb{E}_o [\tau_r] \leq 2r^2. \]
in $S$ started at $o \neq o_1$, here $o_1 \in D(r)$. Let $r_1(x)$ be the Riemannian distance function of $x$ from $o_1$. Apply Itô formula to $r_1(x)$

\begin{equation}
  r_1(X_t) - r_1(X_0) = B_t - L_t + \frac{1}{2} \int_0^t \Delta_{Sr_1}(X_s) ds,
\end{equation}

here $B_t$ is the standard Brownian motion in $\mathbb{R}$, and $L_t$ is a local time on cut locus of $o$, an increasing process which increases only at cut loci of $o$. Since $S$ is simply connected and non-positively curved, then

$$
\Delta_{Sr_1}(x) \geq \frac{1}{r_1(x)}, \quad L_t \equiv 0.
$$

By (15), we arrive at

$$
  r_1(X_t) \geq B_t + \frac{1}{2} \int_0^t \frac{ds}{r_1(X_s)}.
$$

Let $t = \tau_r$ and take expectation on both sides of the above inequality, then it yields that

$$
  \max_{x \in \partial D(r)} r_1(x) \geq \frac{E \tau_r}{2 \max_{x \in \partial D(r)} r_1(x)}.
$$

Let $o' \to o$, we are led to the desired inequality. $\square$

**Proof of Theorem 3.4**

The assertion can be confirmed by

$$
  m(r, \frac{X^k(\psi)}{\psi}) \leq \sum_{j=1}^k m(r, \frac{X^j(\psi)}{X^{j-1}(\psi)})
$$

together with the following Lemma 3.8.

**Lemma 3.8.** We have

$$
  m(r, \frac{X^{k+1}(\psi)}{X^k(\psi)}) \leq \text{exc} \left( 2^{k+1} \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) - \kappa(r) r^2 + \log^+ \log r \right) \right),
$$

where $\kappa$ is defined by (11). More precisely, if $S$ is the Poincaré disc (take $o$ as the center of disc), then we have

$$
  m(r, \frac{X^{k+1}(\psi)}{X^k(\psi)}) \leq \text{exc} \left( 2^k \log T(r, \psi) + O\left( \log^+ \log T(r, \psi) + r \right) \right).
$$

**Proof.** We claim that

\begin{equation}
  T(r, X^k(\psi)) \leq 2^k T(r, \psi) + O\left( \log T(r, \psi) - \kappa(r) r^2 + \log^+ \log r \right).
\end{equation}


By virtue of Lemma 3.6 when \( k = 1 \)
\[
T(r, \mathcal{X}(\psi)) = m(r, \mathcal{X}(\psi)) + N(r, \mathcal{X}(\psi)) 
\leq m(r, \psi) + 2N(r, \psi) + m \left( r, \frac{\mathcal{X}(\psi)}{\psi} \right) 
\leq 2T(r, \psi) + m \left( r, \frac{\mathcal{X}(\psi)}{\psi} \right) 
\leq 2T(r, \psi) + O \left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right)
\]
holds for \( r > 1 \) outside a set of finite Lebesgue measure. Assuming now that
the claim holds for \( k \leq n - 1 \). By induction, we only need to prove the claim
in the case when \( k = n \).

From the claim for \( k = 1 \) showed above and Lemma 3.6 repeatedly, we conclude that
\[
T(r, \mathcal{X}^n(\psi)) \leq 2^n T(r, \mathcal{X}^{n-1}(\psi)) + O \left( \log T(r, \mathcal{X}^{n-1}(\psi)) - \kappa(r)r^2 + \log^+ \log r \right) 
\]
\[
\leq 2^n T(r, \psi) + O \left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right) 
+ O \left( \log T(r, \mathcal{X}^{n-1}(\psi)) - \kappa(r)r^2 + \log^+ \log r \right) 
\leq 2^n T(r, \psi) + O \left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right) 
+ O \left( \log T(r, \mathcal{X}^{n-1}(\psi)) \right) 
\]
\[
\cdots 
\]
\[
\leq 2^n T(r, \psi) + O \left( \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right).
\]

Thus, claim (16) is confirmed. Using Lemma 3.6 and (16) to get
\[
m \left( r, \frac{\mathcal{X}^{k+1}(\psi)}{\mathcal{X}^k(\psi)} \right) 
\leq \frac{5}{4} \log T(r, \mathcal{X}^k(\psi)) + O \left( \log^+ \log T(r, \mathcal{X}^k(\psi)) - \kappa(r)r^2 + \log^+ \log r \right) 
\leq \frac{5}{4} \log T(r, \psi) + O \left( \log^+ \log T(r, \psi) - \kappa(r)r^2 + \log^+ \log r \right).
\]

This proves the first conclusion of the lemma. The second conclusion of the
lemma can be proved similarly by replacing \( \kappa(r)r^2 \) by \( -r \), due to the second
conclusion of Lemma 3.6.

\[ \square \]

4. Tautological inequality

In this section, we provide a version of tautological inequality for a general open
Riemann surface \( S \) from a geometric point of view by using stochastic
calculus. This inequality plays an essential role in proving the main theorem.
Let \((X, D)\) be a smooth logarithmic pair over \(\mathbb{C}\). Denote by \(\Omega^1_X(\log D)\) the logarithmic cotangent sheaf over \(X\) which is the sheaf of germs of logarithmic 1-forms with poles at most on \(D\), namely

\[
\Omega^1_X(\log D) = \sum_{j=1}^{s} \mathcal{O}_X \frac{d\sigma_j}{\sigma_j} + \Omega^1_X,
\]

where \(\sigma_1, \ldots, \sigma_s\) are irreducible and \(\sigma_1 \cdots \sigma_s = 0\) is a local defining equation of \(D\). Note that \(\Omega^1_X(\log D)\) is locally free. For a sheaf \(E\), we have the familiar symbols (see [28]):

\[
P(E) = \text{Proj} \bigoplus_{d \geq 0} S^d E, \quad \text{V}(E) = \text{Spec} \bigoplus_{d \geq 0} S^d E, \quad \text{Sym}^* = \bigoplus_{d \geq 0} S^d.
\]

Associate a nonconstant holomorphic curve \(f : S \to X\) whose image \(f(S)\) is not contained in \(\text{Supp} D\). Then the curve \(f\) induces a lifted curve

\[
f' : S \to P(\Omega^1_X(\log D))
\]

which is holomorphic on \(S\). Let

\[
p : B \to P(\Omega^1_X(\log D) \oplus \mathcal{O}_X)
\]

be the blow-up of \(P(\Omega^1_X(\log D) \oplus \mathcal{O}_X)\) along the zero section of \(\text{V}(\Omega^1_X(\log D))\), namely, the section corresponding to the projection \(\Omega^1_X(\log D) \oplus \mathcal{O}_X \to \mathcal{O}_X\). That is to say, \(B\) is the closure of the graph of the induced rational mapping

\[
P(\Omega^1_X(\log D) \oplus \mathcal{O}_X) \to P(\Omega^1_X(\log D)).
\]

Let \([0]\) be the exceptional divisor on \(B\). There is the natural lifted curve

\[
\partial f : S \to P(\Omega^1_X(\log D) \oplus \mathcal{O}_X)
\]

of the curve \(f\).

**Definition 4.1.** Let \(\phi : S \to B\) be the lift of \(\partial f\). The D-modified ramification counting function of a holomorphic curve \(f : S \to X\) is the counting function for \(\phi^*[0] : N_{f,\text{Ram}(D)}(r) := N_\phi(r, [0])\).

**Theorem 4.2** (Tautological inequality). Let \(\mathcal{A}\) be an ample line sheaf over a smooth logarithmic pair \((X, D)\). Let \(\mathcal{O}(1)\) be the tautological line sheaf over \(P(\Omega^1_X(\log D))\). Then

\[
T_{f', \mathcal{O}(1)}(r) + N_{f,\text{Ram}(D)}(r)
\]

\[# \leq \text{exc} \ N_{f}^{[1]}(r, D) + O \left( \log^+ T_{f,\mathcal{A}}(r) - \kappa(r)r^2 + \log^+ \log r \right),
\]

where \(\kappa\) is defined by (1). More precisely, if \(S\) is the Poincaré disc (take \(o\) as the center of disc), then

\[
T_{f', \mathcal{O}(1)}(r) + N_{f,\text{Ram}(D)}(r) \leq \text{exc} \ N_{f}^{[1]}(r, D) + O \left( \log^+ T_{f,\mathcal{A}}(r) + r \right).
\]
We have a natural embedding 
\[ V(\Omega^1_X(\log D)) \hookrightarrow \mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X) \]
that realizes \( \mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X) \) as the projective closure on fibers of \( V(\Omega^1_X(\log D)) \). Let \( [\infty] \) the (reduced) divisor \( \mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X) \setminus V(\Omega^1_X(\log D)) \).

In order to prove Theorem 4.2, we need the following lemma

**Lemma 4.3.** We have
\[
m_{\partial f}(r, [\infty]) \leq \text{exc} O \left( \log T_{f, \mathcal{O}}(r) - \kappa(r)r^2 + \log^+ \log r \right),
\]
where \( \kappa \) is defined by (1). More precisely, if \( S \) is the Poincaré disc (take \( o \) as the center of disc), then
\[
m_{\partial f}(r, [\infty]) \leq \text{exc} O \left( \log T_{f, \mathcal{O}}(r) + r \right).
\]

**Proof.** Without loss of generality, we may assume that \( S \) is simply connected (see Remark 4.4). Notice that there exists a finite set \( \mathcal{H} \) of rational functions on \( X \) with properties: for each point \( y \in X \), there exists a subset \( \mathcal{H}_y \subseteq \mathcal{H} \) such that

(i) \( \mathcal{H}_y \) generates \( \Omega^1_X(\log D) \);

(ii) \( dh/h \) is a regular section of \( \Omega^1_X(\log D) \) at \( y \) for all \( h \in \mathcal{H}_y \).

By the definition of \( [\infty] \) and the compactness of \( X \), it follows that
\[
\lambda_{[\infty]} \circ \partial f \leq \log^+ \max_{h \in \mathcal{H}} \left| \frac{X(h \circ f)}{h \circ f} \right| + O(1).
\]
Therefore, by the first conclusion of Theorem 3.4 we arrive at
\[
m_{\partial f}(r, [\infty]) \leq C \cdot \max_{h \in \mathcal{H}} m \left( r, \frac{X(h \circ f)}{h \circ f} \right) + O(1)
\leq O \left( \max_{h \in \mathcal{H}} \log T(r; h \circ f) - \kappa(r)r^2 + \log^+ \log r \right)
\leq O \left( \log T_{f, \mathcal{O}}(r) - \kappa(r)r^2 + \log^+ \log r \right).
\]

The second conclusion of the lemma can be also proved by using the second conclusion of Theorem 3.4.

**Remark 4.4.** Let \( \pi: \tilde{S} \to S \) be the (analytic) universal covering. Equip \( \tilde{S} \) with the metric induced from the metric of \( S \). Then, \( \tilde{S} \) is a simply-connected and complete open Riemann surface of non-positive Gauss curvature. Take a diffusion process \( \tilde{X}_t \) in \( \tilde{S} \) satisfying that \( X_t = \pi(\tilde{X}_t) \), then \( \tilde{X}_t \) is a Brownian motion with generator \( \Delta_{\tilde{S}}/2 \) induced from the pull-back metric. Now let \( \tilde{X}_t \) start at \( \tilde{o} \in \tilde{S} \) with \( o = \pi(\tilde{o}) \), we have
\[
\mathbb{E}_o[\phi(X_t)] = \mathbb{E}_{\tilde{o}}[\phi \circ \pi(\tilde{X}_t)].
\]
for $\phi \in \mathcal{C}_b(S)$. Set
\[ \tilde{\tau}_r = \inf \{ t > 0 : \tilde{X}_t \notin \tilde{D}(r) \}, \]
where $\tilde{D}(r)$ is a geodesic ball centered at $\tilde{o}$ with radius $r$ in $\tilde{S}$. If necessary, one can extend the filtration in probability space where $(X_t, \mathbb{P}_o)$ are defined so that $\tilde{\tau}_r$ is a stopping time with respect to a filtration where the stochastic calculus of $X_t$ works. With the above arguments, we can suppose $S$ is simply connected, without loss of generality, by lifting $f$ to the universal covering.

**Proof of Theorem 4.2**

The proof essentially follows McQuillan [16]. Recall the blow-up $p : B \to \mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X)$.

The $B$ admits a morphism $q : B \to \mathbb{P}(\Omega^1_X(\log D))$ which extends the rational mapping $\mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X) \to \mathbb{P}(\Omega^1_X(\log D))$ associated to the canonical mapping $\Omega^1_X(\log D) \hookrightarrow \Omega^1_X(\log D) \oplus \mathcal{O}_X$. Using the symbol $\mathcal{O}(1)$ to denote the tautological line sheaf over $\mathbb{P}(\Omega^1_X(\log D))$ and $\mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X)$.

Take a nonzero rational section $s$ of $\Omega^1_X(\log D)$ over $X$, then $s$ determines a rational section $s_1$ of $\mathcal{O}(1)$ over $\mathbb{P}(\Omega^1_X(\log D))$. The divisor $(s_1)$ is the sum of a generic hyperplane section (on fibers over $X$) and the pull-back of a divisor (on $X$). Notice that $(s, 0)$ is also a nonzero rational section of $\Omega^1_X(\log D) \oplus \mathcal{O}_X$ over $X$, hence it determines a rational section $s_2$ of $\mathcal{O}(1)$ over $\mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X)$. Note $(s_2)$ is again the sum of a generic hyperplane section and the pull-back of a divisor. Comparing $q^* (s_1)$ with $p^*(s_2)$, we find that they coincide except that $p^*(s_2)$ contains $[0]$ with multiplicity 1. Therefore, we obtain
\[ q^* \mathcal{O}(1) \cong p^* \mathcal{O}(1) \otimes \mathcal{O}([-0]). \]

Observing the following commutative diagram

\[ \begin{array}{ccc}
S & \xrightarrow{f} & \mathbb{P}(\Omega^1_X(\log D)) \\
\partial f & \quad & \\
\mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X) & \xrightarrow{p} & B,
\end{array} \]

we see that there is a unique holomorphic lift $\phi : S \to B$ satisfying $f' = q \circ \phi$ and $\partial f = p \circ \phi$. Combining (17) with the above diagram, we obtain
\[ T_{f', \mathcal{O}(1)}(r) = T_{\phi, q^* \mathcal{O}(1)}(r) + O(1) \]
\[ = T_{\phi, p^* \mathcal{O}(1)}(r) - T_{\phi, \mathcal{O}([-0])}(r) + O(1) \]
\[ = T_{\partial f, \mathcal{O}(1)}(r) - T_{\phi, \mathcal{O}([-0])}(r) + O(1). \]
Since the global section \((0, 1)\) of \(\Omega^1_X(\log D) \oplus \mathcal{O}_X\) over \(X\) corresponds to the divisor \([\infty]\) on \(\mathbb{P}(\Omega^1_X(\log D) \oplus \mathcal{O}_X)\), then \(\mathcal{O}(\infty) \cong \mathcal{O}(1)\). This implies that
\[
T_{\phi, 1}(r) = m_\phi(r, [\infty]) + N_\phi(r, [\infty]) + O(1).
\]
Indeed, we notice that \(\partial f\) meets \([\infty]\) only over \(D\), with multiplicity at most 1, then \(N_\phi(r, [\infty]) \leq N_f^{[1]}(r, D)\). Thus, it follows from (18) that
\[
T_{f, \phi(1)}(r) \leq m_\phi(r, [\infty]) + N_f^{[1]}(r, D) - m_\phi(r, [0]) - N_\phi(r, [0]) + O(1).
\]
In the above inequality, \(m_\phi(r, [0])\) is bounded from below; \(N_\phi(r, [0])\) is equal to \(N_{f, \text{Ram}}(r)\); and \(m_\phi(r, [\infty])\) is bounded from below by \(O(\log T_{f, \phi(r)}(r) - \kappa(r)r^2 + \log^+ \log r)\), more precisely, bounded from above by \(O(\log T_{f, \phi(r)}(r) + r)\) if \(S\) is the Poincaré disc, due to Lemma 4.3. This proves the theorem.

**Theorem 4.5.** Let \(X\) be a smooth complex projective curve, and let \(D\) be an effective (reduced) divisor on \(X\). Then for any holomorphic curve \(f : S \to X\) which is nonconstant, we have
\[
T_{f, \mathcal{X}_D}(r) \leq \text{exc } N_f^{[1]}(r, D) + O\left(\log T_{f, \phi} - \kappa(r)r^2 + \log^+ \log r\right),
\]
where \(\kappa\) is defined by (11). More precisely, if \(S\) is the Poincaré disc (take \(o\) as the center of disc), then
\[
T_{f, \mathcal{X}_D}(r) \leq \text{exc } N_f^{[1]}(r, D) + O\left(\log T_{f, \phi(r)} + r\right).
\]

**Proof.** Since \(X\) is a curve, then \(\Omega^1_X(\log D)\) is isomorphic to \(\mathcal{X}_X(D)\) and thus the canonical projection \(\pi : \mathbb{P}(\Omega^1_X(\log D)) \to X\) is an isomorphism. It yields that \(\mathcal{O}(1) \cong \pi^*\mathcal{X}_X(D)\) and \(f' = \pi^{-1} o f\). Therefore, we have
\[
T_{f', \phi(1)}(r) = T_{f, \mathcal{X}_D}(r) + O(1).
\]
By Theorem 4.2 we have the theorem holds.

**Corollary 4.6.** Let \(X\) be a smooth complex projective curve with \(\mathcal{X}_X\) ample. Then there exist positive constants \(a, b\) such that
\[
T_{f, \mathcal{X}_X}(r) \leq \text{exc } a - \kappa(r)r^2,
\]
where \(\kappa\) is defined by (11). More precisely, if \(S\) is the Poincaré disc (take \(o\) as the center of disc), then
\[
T_{f, \mathcal{X}_X}(r) \leq \text{exc } a + br.
\]

**Proof.** According to the first conclusion of Theorem 4.5, we have
\[
T_{f, \mathcal{X}_X}(r) \leq \text{exc } O\left(\log T_{f, \phi}(r) - \kappa(r)r^2 + \log^+ \log r\right).
\]
If \(\kappa(r) \equiv 0\), then the universal covering of \(S\) is \(\mathbb{C}\). By lifting \(f\) to the universal covering, one can assume that \(S = \mathbb{C}\). Since \(\mathcal{X}_X > 0\), then we note from (18) that \(T_{f, \mathcal{X}_X}(r) \geq \text{exc } O(\log r)\) if \(f\) is nonconstant. Using the above inequality,
has to be a constant. If \( \kappa(r) \neq 0 \), we see that \( \log^+ \log r \leq \text{exc} - \kappa(r)r^2 \) since \( -\kappa(r) \geq 0 \) is increasing. Again, by \( \log T_{\mathcal{X}}(r) = o(T_{\mathcal{X}}(r)) \), one obtains \( T_{\mathcal{X}}(r) \leq \text{exc} O(-\kappa(r)r^2) \). So, we show the first conclusion of the corollary. The second conclusion of the corollary can be proved similarly by using the second conclusion of Theorem 4.5. \( \square \)

5. Second Main Theorem

5.1. Curvature current inequality.

In order to prove the main theorem (Theorem 5.2), it remains to introduce a curvature current inequality obtained by Sun [26, Proposition 2.4]. Since this inequality was demonstrated, then we are not going to give details again. However, in order to make it more readable for the reader to understand the hypotheses of Sun’s curvature current inequality, we shall provide a concise explanation, but the details please refer to [26, Section 2], [30, Section 4] and [31, Section 6].

5.1.1. Higgs bundles and Hodge bundles.

Higgs bundles were introduced by Hitchin [10] as solutions of the so-called Hitchin equations, the 2-dimensional reduction of the Yang-Mills self-duality equations, given by

\[
F_A + [\tau, \tau^*] = 0, \quad \bar{\partial}_A \tau = 0,
\]

where \( F_A \) is the curvature of a unitary connection \( \nabla_A = \partial_A + \bar{\partial}_A \) associated to a Dolbeault operator \( \bar{\partial}_A \) on a holomorphic principal \( G_C \)-bundle \( F \). Exactly, a Higgs bundle over a smooth projective variety \( X \) is a pair \((F, \tau)\) such that

a) \( F \) is a holomorphic vector bundle over \( X \);

b) \( \tau \) is a holomorphic 1-form with values in the bundle of endomorphisms of \( F \), satisfying \( \tau \wedge \tau = 0 \). We call \( \tau \) the Higgs field.

To learn Viehweg-Zuo’s construction of certain Higgs bundles, we refer the reader to [30, Section 4] and [31, Section 4]. Viehweg-Zuo’s construction gives a method in constructing (pseudo) Finsler metrics. By which, Sun [26] obtained a curvature current inequality (Lemma 5.1 below).

Let \( \mathcal{M}_g \) be the moduli space of algebraic curves of genus \( g \) over a scheme. A Hodge bundle over \( \mathcal{M}_g \) is a vector bundle \( E \) whose fiber at a point \( C \in \mathcal{M}_g \) is the space of holomorphic differentials on the curve \( C \). To be precise, see [11], let \( \pi : \mathcal{C}_g \to \mathcal{M}_g \) be the universal algebraic curve of genus \( g \), and let \( \mathcal{E} \) be its relative dualizing sheaf, the Hodge bundle \( E \) is the pushforward of \( \mathcal{E} \), i.e.,

\[
E = \pi_* \mathcal{E}.
\]
5.1.2. Curvature current inequality.

Now, we introduce an inequality of curvature currents proved by Sun [26], who showed his inequality via Viehweg-Zuo’s construction [30][31]. Without going into the details about moduli spaces, we refer the reader to [3, 20].

Let \((X, D)\) be a smooth logarithmic pair over \(\mathbb{C}\). Put \(\hat{S} = S \setminus f^*D\), where \(f: S \to X\) is a holomorphic curve whose image is not contained in \(\text{Supp}D\). Let \(\gamma: \hat{S} \to X \setminus D\) be the restriction of \(f\) to \(\hat{S}\), which induces a natural lift \(\gamma': \hat{S} \to \mathbb{P}(T_X(-\log D))\),

where

\[
\mathbb{P}(T_X(-\log D)) := \text{Proj Sym}^* \Omega_X^1(\log D)
\]

is the projective logarithmic tangent bundle. Suppose \((\psi: V \to U := X \setminus D)\) is a smooth family of polarized smooth varieties with semi-ample canonical sheaves and a given Hilbert polynomial \(h\), such that the induced classifying mapping from \(U\) to the moduli scheme \(M_h\) is quasi-finite. Follow the theory of Viehweg-Zuo ([30], Section 4; [31], Section 6), we shall have the following geometric objects over \(X\) ([26], Section 2): an ample line bundle \(A\) whose restriction on smooth locus \(X \setminus D\) is isomorphic to some Viehweg line bundle \(\det(\psi_*\omega_{V/U}^\mu)\) ([31], Corollary 2.4 (ix) and Section 4); a deformation Higgs bundle \((F, \tau)\) associated to \(\psi\); a logarithmic Hodge bundle \((E, \theta)\) with poles along \(D+T\), where \(T\) is a normal crossing divisor; and a comparison mapping \(\rho\) which fits into the following commutative diagram ([30], Lemma 4.4; [31], Lemma 6.3)

\[
\begin{array}{ccc}
F^{p,q} & \xrightarrow{\tau^{p,q}} & F^{p-1,q+1} \otimes \Omega_X^1(\log D) \\
\rho^{p,q} & & \downarrow \rho^{p-1,q+1} \otimes \iota \\
A^{-1} \otimes E^{p,q} & \xrightarrow{\text{Id} \otimes \theta^{p,q}} & A^{-1} \otimes E^{p-1,q-1} \otimes \Omega_X^1(\log (D + T))
\end{array}
\]

where \(\mathcal{O}_X\) is a subsheaf of \(F_{d,0}\) and that ([31], Lemma 6.5)

\[
(E, \theta) = \bigoplus_{p,q} (E^{p,q}, \theta^{p,q}), \quad (F, \tau) = \bigoplus_{p,q} (F^{p,q}, \tau^{p,q}).
\]

To avoid excessive complexity and verbosity of the statements (since that is not the point of the paper), the reader may refer to [31], Pages 20-21 for the definitions of \(F^{p,q}, E^{p,q}, \tau^{p,q}, \theta^{p,q}\), and [31], Section 4 for the concept of determinant bundle \(\det(\psi_*\omega_{V/U}^\mu)\). Let \(d\) be the fiber dimension of family \(\psi\). By iterating the Higgs mappings \(\tau^{p,q}\), it follows that ([30], Lemma 4.4 (vi))

\[
\tau^{d-q+1,q-1} \circ \cdots \circ \tau^{d,0}: F^{d,0} \to F^{d-q,0} \otimes \bigotimes_{q} \Omega_X^1(\log D).
\]
Since $\tau \wedge \tau = 0$ (Section 5.1, Part A), then the composition factors through \(\mathbf{[30]}, \text{Lemma 4.4 (vi)}\)

$$\tau^q : F^{d,0} \longrightarrow F^{d-q,q} \otimes \mathrm{Sym}^q \Omega^1_X(\log D).$$

The pull-back of the Higgs bundle \((F, \tau)\) over \(X\) by \(\gamma\) induces a Higgs bundle \((F_\gamma, \tau_\gamma)\) over \(\hat{S}\) in the following manner

$$F_\gamma := \gamma^* F, \quad \tau_\gamma : F_\gamma \longrightarrow \gamma^* \tau \longrightarrow F_\gamma \otimes \gamma^* \Omega^1_X(\log D) \longrightarrow F_\gamma \otimes \Omega^1_{\hat{S}}.$$

Similarly as above, we define the holomorphic Hodge bundle \((E_\gamma, \theta_\gamma)\), where \(\theta_\gamma\) has logarithmic pole along \(\gamma^* T\). Over \(\hat{S}\), we have the commutative diagram \(\mathbf{[30]}, \text{Lemma 4.4 (i)}\)

$$\begin{array}{ccc}
F^{p,q}_{\gamma} & \longrightarrow & F^{p-1,q+1}_{\gamma} \otimes \Omega^1_{\hat{S}} \\
\tau^{p,q}_{\gamma} \downarrow & & \downarrow \rho^{p-1,q+1}_{\gamma} \\
A^{-1}_{\gamma} \otimes E^{p,q}_{\gamma} \longrightarrow & & A^{-1}_{\gamma} \otimes E^{p-1,q-1}_{\gamma} \otimes \Omega^1_{\hat{S}}(\log \gamma^* T),
\end{array}$$

where \(A_\gamma := \gamma^* A\). Similarly, we can define the iterations of Higgs mappings

$$\tau^{\otimes q}_\gamma : T^{\otimes q}_{\hat{S}} \longrightarrow F^{d-q,q}_{\gamma}.$$

By using the iterations of Higgs mappings, we shall define a Higgs subbundle of \((E_\gamma, \theta_\gamma)\). For each integer \(q \geq 0\), we define \(G^{d-q,q}\) as the saturation of the image of

$$A \otimes T^{\otimes q}_{\hat{S}} \longrightarrow A \otimes F^{d-q,q}_{\gamma} \longrightarrow E^{d-q,q}_{\gamma}$$

in \(E^{d-q,q}_{\gamma}\). We have \(\theta^{d-q,q}_{\gamma}(G^{d-q,q}) \subseteq G^{d-q-1,q+1} \otimes \Omega^1_{\hat{S}}\) \(\mathbf{[26]}, \text{Lemma 2.1}\) and \(c_1(\det G, h) \leq 0\) \(\mathbf{[25]}, \text{Proposition 2.2}\), where \(h\) is the Hermitian metric on the determinant bundle \(\det G\) induced by the Hodge metric on \(E\). Let \(\mathcal{O}(-1)\) be the tautological line bundle over \(\mathbb{P}(T_X(\log D))\). Since the iterations of Higgs mappings \(\tau^{\otimes q}_\gamma\) also factors through

$$T^{\otimes q}_{\hat{S}} \longrightarrow \gamma^* \mathcal{O}(-q) \longrightarrow F^{d-q,q}_{\gamma},$$

in which \(\tau^q : \mathcal{O}(-q) \to \pi^* F^{d-q,q}\) is the lift of \(\tau^q\) and \(\pi : \mathbb{P}(T_X(\log D)) \to X\) is the nature projection, then \(G^{d-q,q}\) is also the saturation of the image of

$$A_\gamma \otimes \gamma^* \mathcal{O}(-q) \longrightarrow A_\gamma \otimes F^{d-q,q}_{\gamma} \longrightarrow E^{d-q,q}_{\gamma},$$

where \(\pi : \mathbb{P}(T_X(\log D)) \to X\) is the nature projection. Then this gives the mappings \(\mathbf{[26]}, (2.4)\)

$$\zeta^q : \gamma^* \mathcal{O}(-q) \longrightarrow A^{-1}_\gamma \otimes G^{d-q,q}, \quad q = 0, \cdots, d.$$
Note that $\rho^d_1 \circ \tau^1_1$ is nonzero, then it implies that $\zeta^1$ is nonzero. So, there exists a positive integer $m$ such that $\zeta^m \neq 0$ and $\zeta^{m+1} = 0$, here $m$ is called the maximal length of iteration. We have $m \leq d$ and $\det G = \bigotimes_{q=0}^{m} G^{d-q,q}$ ($G^{d-q,q}=0$ for $q > m$). For every $q$, one can construct a (pseudo) metric $F_q$ on $\mathcal{O}(-1)$ via the following composition mapping

$$
\mathcal{O}(-q) \xrightarrow{\tilde{\tau}^q} \pi^* F^{d-q,q} \xrightarrow{} \pi^*(A^{-1} \otimes E^{d-q,q}) .
$$

Here, we note that $F_q$ is a bounded pseudo metric with possible degeneration on $\mathbb{P}(T_X(-\log D))$. So, we can write $F_q = \phi_q F$, where $F$ is a smooth metric on $\mathcal{O}(-1)$, and $\phi_q$ is a bounded function with at most a discrete set of zeros. Then we have

$$
c_1(\mathcal{O}(-1), F) = c_1(\mathcal{O}(-1), F_q) + dd^c \log \phi_q.
$$

This will derive a curvature current inequality as follows

**Lemma 5.1** ([20], Proposition 2.4). Suppose that $m$ is the maximal length of iteration, i.e., the largest integer such that $\zeta^m \neq 0$. Then

$$
\frac{m+1}{2} f^* c_1(\mathcal{O}(-1)) \leq - f^* c_1(A) + \frac{1}{m} \sum_{q=1}^{m} q f^* dd^c \log \phi_q,
$$

where $f, A, \phi_q$ are given as above.

**5.2. Second Main Theorem.**

We show the following Second Main Theorem

**Theorem 5.2** (Second Main Theorem). Let $(X, D)$ be a smooth logarithmic pair over $\mathbb{C}$ with $U = X \setminus D$. Assume that there is a smooth family $(\psi : V \rightarrow U)$ of polarized smooth varieties with semi-ample canonical sheaves and a given Hilbert polynomial $h$, such that the induced classifying mapping from $U$ into moduli scheme $\mathcal{M}_h$ is quasi-finite. Then for any holomorphic curve $f : S \rightarrow X$ whose image is not contained in $\text{Supp} D$, we have

$$
T_{f, A}(r) \leq_{\text{exc}} \frac{d+1}{2} N^{[1]}_f (r, D) + O\left(\log T_{f, A}(r) - \kappa(r) r^2 + \log^+ \log r\right),
$$

where $A$ is an ample line bundle over $X$ given in Lemma 5.1, $d$ is the fiber dimension of the family $\psi$, and $\kappa$ is defined by [1]. More precisely, if $S$ is the Poincaré disc (take $o$ as the center of disc), then

$$
T_{f, A}(r) \leq_{\text{exc}} \frac{d+1}{2} N^{[1]}_f (r, D) + O\left(\log T_{f, A}(r) + r\right).
$$
Proof. By lifting $f$ to the (analytic) universal covering, we may assume that $S$ is simply connected. By Lemma 5.1 it follows immediately that

\[-\frac{d+1}{2} T_{f',\mathcal{O}(1)}(r) \leq -\frac{m+1}{2} T_{f',\mathcal{O}(1)}(r) \leq -T_{f,A}(r) + \frac{\pi}{m} \sum_{q=1}^{m} q \int_{D(r)} g_{r}(o,x)d\mathcal{C} \log \phi_{q} \circ f'.\]

Since $\phi_{1}, \cdots, \phi_{q}$ are bounded, then it yields from Coarea formula and Dynkin formula that

\[\pi \int_{D(r)} g_{r}(o,x)d\mathcal{C} \log \phi_{q} \circ f' = \frac{1}{4} \mathbb{E}_{o} \left[ \int_{0}^{\tau_{r}} \Delta_{S} \log \phi_{q} \circ f'(X_{t}) dt \right] = \frac{1}{2} \mathbb{E}_{o} \left[ \log \phi_{q} \circ f'(X_{\tau_{r}}) \right] + O(1) \leq O(1).\]

Combining the above with Theorem 4.2, we can prove the theorem. \qed

Corollary 5.3. Assume the same conditions as in Theorem 5.2 Let $f: S \to X$ be a holomorphic curve ramifying over $D$ with multiplicity $c > (d+1)/2$. If $f$ satisfies the growth condition

\[\lim_{r \to \infty} \frac{\kappa(r)r^{2}}{T_{f,A}(r)} = 0,\]

where $\kappa$ is defined by (11), then $f(S)$ is contained in $D$. More precisely, if $S$ is the Poincaré disc (take $o$ as the center of disc), then $f(S)$ is contained in $D$ provided that

\[\lim_{r \to \infty} \frac{r}{T_{f,A}(r)} = 0.\]

Proof. Set $b = 2c/(d+1)$, then $b > 1$. By contradiction, we assume that $f(S)$ is not contained in $D$. By the first conclusion of Theorem 5.2 and condition $f^{*}D \geq c \cdot \text{Supp} f^{*}D$, we obtain

\[bT_{f,A}(r) \leq \text{exc} \ cN_{f}^{[1]}(r,D) + O \left( \log T_{f,A}(r) - \kappa(r)r^{2} + \log^{+} \log r \right) \leq N_{f}(r,D) + O \left( \log T_{f,A}(r) - \kappa(r)r^{2} + \log^{+} \log r \right).\]

If $\kappa(r) \neq 0$, by First Main Theorem, then the above inequality implies that $b \leq 1$, it is a contradiction. If $\kappa(r) \equiv 0$, we can regard $S$ as $\mathbb{C}$ by lifting $f$ to the universal covering, then we obtain $T_{f,A}(r) \geq \text{exc} O(\log r)$ (see [18]). But, it still contradicts with $b > 1$. Therefore, the first conclusion of the corollary holds. The second conclusion of the corollary is similarly proved. \qed
Corollary 5.4. Assume the same conditions as in Theorem 5.2. Then for any holomorphic curve \( f : S \rightarrow X \) whose image is not contained in \( \text{Supp}D \), we have
\[
T_{f,K_X(D)}(r) \leq \text{exc} \frac{k(d+1)}{2} N_f^{[1]}(r,D) + O \left( \log T_{f,A}(r) - \kappa(r)r^2 + \log^+ \log r \right)
\]
for an integer \( k \) such that \( A^\otimes k \geq K_X(D) \), where \( \kappa \) is defined by (1). More precisely, if \( S \) is the Poincaré disc (take \( o \) as the center of disc), then
\[
T_{f,K_X(D)}(r) \leq \text{exc} \frac{k(d+1)}{2} N_f^{[1]}(r,D) + O \left( \log T_{f,A}(r) + r \right)
\]
for an integer \( k \) such that \( A^\otimes k \geq K_X(D) \).

Proof. Since \( A^\otimes k \geq K_X(D) \), then we see that \( T_{f,K_X(D)}(r) \leq kT_{f,A}(r)+O(1) \). Therefore, the corollary follows from Theorem 5.2. \( \square \)

Let us consider Siegel modular varieties [25]. A Siegel modular variety is a moduli space of principally polarized Abelian varieties of a fixed dimension. Exactly speaking, the Siegel modular variety \( A_g \) parametrizes the principally polarized Abelian varieties of dimension \( g \), which can be constructed as the complex analytic spaces (constructed as the quotient of the Siegel upper half-space of degree \( g \) by the action of a symplectic group). Refer to [12, 27], \( A_g \) has dimension \( g(g+1)/2 \), and is of general type for \( g \geq 7 \). A Siegel modular variety \( A_g^{[n]} \), which parametrizes the principally polarized Abelian varieties of dimension \( g \) with level-\( n \) structure, arises as the quotient of Siegel upper half-space by the action of the principal congruence subgroup of level-\( n \) of a symplectic group.

Theorem 5.5. Let \( A_g^{[n]} (n \geq 3) \) be the moduli space of principally polarized Abelian varieties with level-\( n \) structure. Let \( \overline{A}_g^{[n]} \) be the smooth compactification of \( A_g^{[n]} \) such that \( D = \overline{A}_g^{[n]} \setminus A_g^{[n]} \) is a normal crossing (boundary) divisor. For any holomorphic curve \( f : S \rightarrow \overline{A}_g^{[n]} \) whose image is not contained in \( \text{Supp}D \), we have
\[
T_{f,K_{\overline{A}_g^{[n]}}(D)}(r) \leq \text{exc} \frac{(g+1)^2}{2} N_f^{[1]}(r,D) + O \left( \log T_{f,A}(r) - \kappa(r)r^2 + \log^+ \log r \right),
\]
where \( \kappa \) is defined by (1). More precisely, if \( S \) is the Poincaré disc (take \( o \) as the center of disc), then
\[
T_{f,K_{\overline{A}_g^{[n]}}(D)}(r) \leq \text{exc} \frac{(g+1)^2}{2} N_f^{[1]}(r,D) + O \left( \log T_{f,A}(r) + r \right).
\]

Proof. We just need to carry the arguments of Sun ([26], Corollary 4.3) and use Corollary 5.4 then the theorem can be proved. \( \square \)
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