The NLO $\mathcal{N}=4$ SUSY BFKL Green function in the adjoint representation

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1. Introduction

The Balitsky–Fadin–Kuraev–Lipatov (BFKL) formalism [2–4] has proven to be very useful to understand scattering amplitudes in the $\mathcal{N}=4$ supersymmetric Yang–Mills theory in multi-Regge (MRK) and quasi-multi-Regge (QMRK) kinematics [5,6]. In this context, corrections to the Bern–Dixon–Smirnov (BDS) iterative ansatz [7] were found in this limit for the six-point amplitude at leading order (MRK) in [10]. In the present Letter we improve on that study to address the much more complicated case of next-to-leading order (NLO) corrections, solving the corresponding equation presented by Fadin and Lipatov for QMRK in [1].

In Section 2 we provide an iterative representation for the NLO Green function in the adjoint representation directly in rapidity and transverse momentum space. As shown in [8,9], the infrared divergences can be factorized in a simple form, leaving an iterative infrared finite remainder that we numerically investigate, with Monte Carlo techniques [11] in Section 3. We write an iterative representation for the solution to the equation for arbitrary momentum transfer $q$. In the limit of large $q$ we obtain agreement with the studies in [8,9]. Besides investigating the collinear behaviour of the Green function, we find that the growth of energy of the different Fourier components in the azimuthal angle changes its structure when going from the large to the small momentum transfer limit. Finally, we present our conclusions and scope for future work.

2. The adjoint BFKL Green function at NLO in iterative form

In Ref. [12] the non-forward BFKL equation for QCD was presented for a general projection of the colour quantum numbers in the $t$-channel, at leading and next-to-leading order. The relevant representation for planar $\mathcal{N}=4$ SUSY maximally helicity violating amplitudes is the adjoint which, at NLO, was calculated in [1]. The main difference between the octet and the singlet representation is that the latter is infrared finite while the former is not. It is however possible to show that the extra infrared divergencies that appear in the non-singlet representations can be written as a simple overall factor in the gluon Green function [1,8,9]. The infrared finite remainder contains infrared divergencies in the real emission and gluon trajectory sectors which have to cancel against each other. Their corresponding divergencies can be regularized by a physical cut off in the transverse momentum, which we call $\lambda$. In this way we can then write the $\mathcal{N}=4$ SUSY BFKL equation in the adjoint representation at NLO in the form

$$
\omega + \frac{\bar{\alpha}}{2} \left(1 - \frac{\zeta_2}{2} \bar{\alpha}\right) \log \left(\frac{q^2_1 q^2_2}{q^2_1 q^2_2 + 3 / 4 \bar{\alpha}^2}\right) \mathcal{F}_\omega(q_1, q_2, q) = \delta^{(2)}(q_1 - q_2) + \int d^2 k \left(\frac{\bar{\alpha}}{4} \left(1 - \frac{\zeta_2}{2} \bar{\alpha}\right) \theta(k^2 - \lambda^2) \right).
$$
Reggeized gluons. We also have
\[ \Phi(k, q) = \frac{q_i^2}{(q_1 + k)^2} F_{\omega}(q_1 + k, q_2; q). \]

Let us note that we have used the notation \( q'_i = q_i - q \), where \( q \) is the momentum transfer and all two-dimensional vectors are represented in bold. \( q_{1,2} \) are the transverse momenta of the \( t \)-channel Reggeized gluons. We also have

\[ \Phi(q_1, q_1 + k) = \frac{\alpha^2}{32\pi q_i^2 (k + q_i')^2} \left\{ q^2 \ln \left( \frac{q_i^2}{q'^2} \right) \ln \left( \frac{q_i^2}{q^2} \right) + \ln \left( \frac{q_1 + k}{q} \right) \ln \left( \frac{q_i^2}{q^2} \right) + \frac{1}{2} \ln^2 \left( \frac{q_i^2}{(q_1 + k)^2} \right) + \frac{1}{2} \ln^2 \left( \frac{q_i^2}{q^2} \right) \right\} \]

\[ \times \left\{ \ln \left( \frac{q_i^2}{(q_1 + k)^2} \right) \ln \left( \frac{q_i^2 (q_1 + k)^2}{q^2 k^2} \right) - \ln \left( \frac{q_i^2}{(q_1 + k)^2} \right) \ln \left( \frac{q_i^2 (q_1 + k)^2}{q^2 k^2} \right) \right\} \]

\[ - \frac{k^2}{2} \left\{ \ln^2 \left( \frac{q_i^2}{(q_1 + k)^2} \right) + \ln^2 \left( \frac{q_i^2}{(q_1 + k)^2} \right) \right\} \]

\[ + q^2 (k^2 - q_i^2 - (q_1 + k)^2 + 2q_i^2 (q_1 + k)^2 - q_i^2 (q_1 + k)^2 + \frac{q_i^2 (q_1 + k)^2 - q_i^2 (q_1 + k)^2}{k^2} (q_i^2 - (q_1 + k)^2) \]

\[ \times I(q_i^2, (q_1 + k)^2, k^2) \]

with \( k \) being the transverse momentum of the gluons in the \( s \)-channel and

\[ I(p^2, q_i^2, r^2) = \int_0^1 dx \frac{dx}{p^2(1 - x) + q_i^2 x - r^2 x(1 - x)} \]

\[ \times \ln \left( \frac{p^2(1 - x) + q_i^2 x}{r^2 x(1 - x)} \right) \]

The function \( \Phi \) was calculated by Fadin and Lipatov in [1].

After this regularization it is possible to iterate the equation and perform a Mellin transform back into energy (or rapidity) space. The final, iterative, representation in transverse momentum and rapidity space for the gluon Green function is

\[ \mathcal{F}(q_1, q_2; q, Y) = \left( \frac{q_i^2}{q_i^2} \right)^{\frac{1}{2}} e^{\frac{1}{2}(1 - \frac{q_i^2}{q^2})} \int_0^1 dx \ln \left( \frac{q_i^2}{q^2} \right) \delta(2) (q_1 - q_2) \]

\[ + \sum_{n=1}^{\infty} \prod_{i=1}^n \left[ \int d^2 k_i \frac{\theta(k_i^2 - \lambda^2)}{\pi k_i^2} \right] \left( 1 + \frac{1}{q_i^2 + \sum_{i=1}^{n-1} k_i^2} (q_1 + \sum_{i=1}^{n-1} k_i^2)^2 - q_i^2 k_i^2 \right) \]

\[ \times \Phi(q_1 + \sum_{i=1}^{n} k_i, q_1 + \sum_{i=1}^{n} k_i) \right\} \delta(2) (q_1 + \sum_{i=1}^{n} k_i - q_2) \]

\[ \times \int dy_i \left( q_{n+1} + \sum_{i=1}^{n} k_i \right) \frac{y_i}{2(1 - \frac{q_i^2}{q^2})} \]

\[ \times \left( \frac{q_i^2 + \sum_{i=1}^{n} k_i}{q_i^2 + \sum_{i=1}^{n} k_i} \right)^{\frac{y_i}{2(1 - \frac{q_i^2}{q^2})}} \right\}, \]

where \( y_0 = Y \) is the rapidity difference between the Reggeized gluons with transverse momenta \( q_{1,2} \).

It is very difficult to provide an analytic representation of this Green function. However, it is possible to solve it in the large \( q \) limit since there exists an asymptotic conformal invariance in that region [1,8,9]. In the present Letter we use advanced Monte Carlo integration techniques to study the gluon Green function. We have explicitly checked that our numerical solution to the exact equation agrees at large \( q \) with the results found in [1,8,9]. For this we have taken the large momentum transfer limit of the NLO BFKL kernel, which agrees with that in [1], and write the equation in the form

\[ \left\{ \omega + \frac{\tilde{\alpha}}{2} \left( 1 - \frac{2\zeta_2}{2} \right) \log \left( \frac{q_i^2}{\lambda^2} \right) - \frac{3}{4} \tilde{\alpha} \delta(2) \right\} \mathcal{F}(q_1, q_2; q) \]

\[ = \delta(2)(q_1 - q_2) \]

\[ + \int d^2 k_i \left( \frac{\tilde{\alpha} \delta_2}{4} \left( 1 - \frac{2\zeta_2}{2} \right) \theta(k_i^2 - \lambda^2) \right) \left( 1 + \frac{(q_1 + k)^2 - k_i^2}{q_i^2} \right) \]

\[ \times \Phi(q_1, q_1 + k) \right\} \frac{q_i^2}{(q_1 + k)^2} \mathcal{F}_{\omega}(q_1 + k, q_2; q). \]

with a much simpler function

\[ \Phi(q_1, q_1 + k) = \frac{\alpha^2}{32\pi q_i^2} \left( \frac{1}{2} - \frac{(q_i^2 - (q_1 + k)^2)}{k^2} \right) \ln^2 \left( \frac{q_i^2}{(q_1 + k)^2} \right) \]

\[ - \frac{1}{2} \left( \frac{q_i^2 - (q_1 + k)^2}{k^2} \right) \ln \left( \frac{q_i^2}{(q_1 + k)^2} \right) \ln \left( \frac{q_i^2 (q_1 + k)^2}{k^2} \right) \]

\[ + \left( k^2 - 2q_i^2 - 2(q_1 + k)^2 + \frac{q_i^2 - (q_1 + k)^2}{k^2} \right) \]

\[ \times I(q_i^2, (q_1 + k)^2, k^2) \].

We present the results obtained from the numerical implementation of these representations in the following section.
We first present our solution for the gluon Green function in the collinear/anti-collinear regions where we fix \( q_2 = 20 \text{ GeV} \) and vary \( q_1 \) from 10 to 40 GeV. This is shown, for \( Y = 4, 8 \), in Fig. 1. The discontinuity seen at \( q_1 = q_2 \) corresponds to the delta function initial condition chosen for the Green function. Its width is smaller at larger rapidities since a larger number of iterations of the kernel are needed to reach a convergent result, rapidly screening the information stemming from the initial condition. Hence, the weight in the full solution of the initial term is reduced.

We have checked that all of our results are \( \lambda \) independent for small \( \lambda \), in agreement with the infrared finiteness of \( \mathcal{F} \). The qualitative collinear behaviour of the solution at large \( q \) (\( q = 50 \text{ GeV} \) in Fig. 1 (bottom)) is not very different to the one we previously found at LO in Ref. [10]. In the present work we have also investigated the low \( q \) region and found a flatter Green function for large \( q_1 \) than in the case of large \( q \). It would be interesting to investigate how this is related to the lack of \( SL(2, C) \) invariance at low \( q \) and how it might affect the analytic calculation of the anomalous dimensions.

In studies of the BFKL gluon Green function it is always interesting to investigate its expansion in Fourier components in the azimuthal angle between the two transverse momenta \( \mathbf{q}_1 \) and \( \mathbf{q}_2 \). We have performed this analysis and briefly present some of the results in Fig. 2. For large \( q \) (bottom of the figure) we find qualitatively the same behaviour as at LO, with the only rising with energy component being the \( n = 1 \) one. This is a common feature of the adjoint solution at large momentum transfer. We find an interesting change in this trend when \( q \) is small. In this case, as it can be seen in Fig. 2 (top) for \( q = 5 \text{ GeV} \) the dominant Fourier component is that with \( n = 0 \), with all the other components decreasing at high energies. Again, this should indicate the departure from conformal invariance.

This is a brief presentation of our numerical studies. It will be very interesting to numerically integrate this solution with the corresponding impact factors and extract information for the MHV amplitudes, to complete the already available studies in the literature.

4. Conclusions and scope

We have presented the exact solution of the BFKL equation in the adjoint representation for the \( \mathcal{N} = 4 \) SUSY theory at NLO accuracy. We have found agreement with the approximations to this solution in the case of large momentum transfer discussed in [1,8,9]. The NLO non-forward BFKL gluon Green function plays a fundamental role in the construction of the “finite remainder function” of MHV and planar amplitudes in QMRK [1,8,9]. It has been investigated in terms of energy growth, collinear limits and azimuthal angle behaviour. We have shown that the factorization of infrared divergencies is complete, generating an infrared finite
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