Monopulse beam synthesis using a sparse single-layer of weights

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Abstract—A conventional monopulse radar system uses three beams; sum beam, elevation difference beam and azimuth difference beam, which require different layers of weights to synthesize each beam independently. Since the multi-layer structure increases hardware complexity, many simplified structures based on a single layer of weights have been suggested. In this work, we introduce a new technique for finding disjoint and fully covering sets of weight vectors, each of which constitutes a sparse subarray, forming a single beam. Our algorithm decomposes the original non-convex optimization problem for finding disjoint weight vectors into a sequence of convex problems. We demonstrate the convergence of the algorithm and show that the interleaved array structure is able to meet difficult beam constraints.

Index Terms—Monopulse radar, sparse array, interleaved array, convex optimization, argumentative reselection algorithm, alternating projection method.

I. INTRODUCTION

A monopulse radar with an antenna array needs multiple beams; the sum beam and the delta beams, on a same antenna-array face. This, in turn, requires multiple layers of weights i.e., transmit-receive modules (TRMs) to shape each beam, independently and optimally. However, it is costly and structurally complicated to attach multiple TRMs on each antenna. Therefore, many researchers have engaged in the problem of subarraying and assigning a single weight on each antenna heuristically [1] and systematically [2]–[9].

The first approach is to use multiple and clustered subarrays, whose responses are combined to obtain multiple beams [4]–[6]. For example, Figure 1(a) shows two clustered non-overlapping subarrays, which form a single-layer of weights, that are used to obtain the sum beam \( F_1 \) and the delta beam \( F_2 \). Each subarray response is generated by combining the antenna responses in analog manner, and each beam response is obtained by combining the subarray responses in digital manner. Figure 1(a) shows non-overlapping subarrays [10], but partially overlapping [11], [12] structures have been studied as well.

The second approach is to use sparse and irregular subarrays. As an example, the fully interleaved thinned linear array (FITLA) structure [3] is shown in Figure 1(b). The larger aperture of each sparse sub-array, compared with that of a dense array with the same number of weights, gives sharper and narrower beam while the irregularity of the sparse array suppresses grating lobes.

Above mentioned beamforming structures can be synthesized by formulating a constrained minimization problem, which is in general, non-convex. Although this minimization can be carried out with an inefficient global optimization method, two ingeniously crafted methods, the alternating projection method and the hybrid method exist.

The alternating projection method [7], [13]–[17] iteratively finds an intersecting point of two sets \( \mathcal{M} \) and \( \mathcal{B} \), where \( \mathcal{M} \) specifies the excitation constraints and \( \mathcal{B} \), the beam pattern, by repeatedly projecting the point in a current set onto the other. A difficulty with this method is that one of the two sets, in general, is not convex, and therefore the starting point must be chosen carefully to ensure the convergence to the global minimum. We remark that the alternating projection method appears in a variety of algorithms, sometimes disguisedly, including the direction of arrival finding algorithm [18] and
the alternating direction implicit (ADI) method for solving partial differential equations [19].

The hybrid method [2, 3, 4, 9, 20] also decomposes the constrained minimization problem into two; one a convex problem and the other, usually a non-convex problem. This method has been successfully applied to the monopulse beam synthesis by iteratively finding the weights as well as subarray grouping for the structures in Figures 1(a) and 1(b) [2–9].

In general, however, both the alternating projection method and the hybrid method involve a non-convex optimization step, and therefore, require a global optimization algorithm [13] or a good choice of the starting point [9].

We propose a new algorithm which solves the original non-convex problem for finding the structure in Figure 1(b) by decomposing it into a sequence of l1-minimization problems, which are convex. In a sense, the proposed algorithm is a variant of the alternating projection method, where the non-convex constraint set is replaced with a more convenient convex set [21].

II. DATA MODEL

Assuming omni-directional antennas, let us define the array response vector by

\[ a(\theta) = [1, e^{-jkdsin\theta}, \ldots, e^{-jk(N-1)sin\theta}]^T, \]

where \( j = \sqrt{-1} \), and \( k = 2\pi/\lambda \). The angle \( \theta \) denotes the bearing of a target and the constant \( d \), antenna spacing. Then sum beam response \( F_1 \) and delta beam response \( F_2 \) are respectively, expressed as

\[
\begin{align*}
F_1(\theta) &= a(\theta)^HM^Hw_1, \quad w_1 \in \mathbb{C}^{N \times 1}, \\
F_2(\theta) &= a(\theta)^HM^Hw_2, \quad w_2 \in \mathbb{C}^{N \times 1},
\end{align*}
\]

where \( M \) represents the mutual coupling matrix. The vectors \( w_1 \) and \( w_2 \) are disjoint and fully covering weight vectors. For example, for the array in Figure 1(b), we have

\[
\begin{align*}
w_1 &= [w_1, 0, 0, w_4, 0, w_6, w_7, w_8, 0, 0]^T, \\
w_2 &= [0, w_2, w_3, 0, w_5, 0, 0, 0, w_9, w_{10}]^T.
\end{align*}
\]

The role of weight vectors \( w_1 \) and \( w_2 \) is to compensate the mutual coupling as well as to shape the beams under the altered array response vector, \( \text{Ma}(\theta) \).

Let \( \Theta_1 \) and \( \Theta_2 \) be the sets of side-lobe angles of the sum beam and the delta beam, respectively, and \( \theta_0 \), the bore-sight angle. A monopulse radar functions properly, if \( F_1 \) and \( F_2 \) are synthesized to satisfy each beam constraints, i.e., the weight vectors \( w_1 \) and \( w_2 \) belong to the sets defined by

\[
\begin{align*}
C_1 &= \left\{ w_1 \mid F_1(\theta_0) = \mu_1, |F_1(\theta_{1,m_1})|^2 \leq \tau_1 \right\}, \\
C_2 &= \left\{ w_2 \mid F_2(\theta_0) = 0, |F_2(\theta_{2,m_2})|^2 \leq \tau_2, \frac{\partial F_2(\theta)}{\partial \theta} \bigg|_{\theta=\theta_0} = \mu s \right\},
\end{align*}
\]

where \( \theta_{1,m_1} \in \Theta_1 \) and \( \theta_{2,m_2} \in \Theta_2 \) indicate sampling points in the side-lobe regions. The constants \( M_1 \) and \( M_2 \) represent the number of samples in the side-lobe regions for each beam. Only one sample at \( \theta_0 \) is taken in the main-lobe regions. Here, \( \mu \) is defined as the array gain for the sum beam at the bore sight. The bounds \( \tau_1 \) and \( \tau_2 \) denote the maximum side-lobe levels (SLLs) of \( F_1 \) and \( F_2 \), respectively, and the constant \( s \) is the slope of \( F_2 \) at the bore sight. The sets \( C_1 \) and \( C_2 \) are convex sets, since \( F_1 \) and \( F_2 \) are linear functions of \( w_1 \) and \( w_2 \), respectively. See Equation (1) and (2).

III. ARGUMENTATIVE RESELECTION ALGORITHM

To find disjoint weight vectors such as \( w_1 \) and \( w_2 \) in Figure 1(b), we shall build an optimization problem and propose an algorithm to solve the problem. We call the process, argumentative reselection algorithm, since the process is comparable to the situation where many people argue for their individual benefit, but eventually reach a compromise with which all can accept.

A. The problem

Now let us define the two-variable cost function

\[
\hat{J}(w_1, w_2) \triangleq |w_1|^2 \cdot |w_2|,
\]

where \(|\cdot|\) takes the element-wise absolute value. Then the problem is to find \( \hat{w}_1 \) and \( \hat{w}_2 \) such that

\[
(\hat{w}_1, \hat{w}_2) = \arg\min_{w_1, w_2} \hat{J}(w_1, w_2)
\]

subject to \( w_1 \in C_1, w_2 \in C_2 \).

The disjoint requirement of \( w_1 \) and \( w_2 \) is built-into the cost function because if we are able to minimize the cost of \( \hat{J} \) down to zero, then we shall obtain a disjoint pair \( w_1 \in C_1 \) and \( w_2 \in C_2 \). Otherwise, there are no disjoint \( w_1 \in C_1 \) and \( w_2 \in C_2 \), and we need to relax the specifications in \( C_1 \) and \( C_2 \). This trial and error approach of the parameter selection is quite common for beam synthesis problems [7].

B. The algorithm

A difficulty in the above optimization problem is that the two-variable cost function is not convex [22]. However, if
one of the weight vectors \( w_1 \) and \( w_2 \) is constant, then the resulting one-variable cost function \( J(\cdot) \) becomes convex. This leads us to build a new algorithm for Problem (7), which utilizes penaltying vectors \( p_1 \) and \( p_2 \):

Choose random \( w_1 \) and \( w_2 \);

repeat

(i) Set \( p_1 := |w_2| \) and solve

\[
\arg\min_{w_1} J(w_1, p_1) \text{ s.t. } w_1 \in C_1. \]

(ii) Set \( p_2 := |w_1| \) and solve

\[
\arg\min_{w_2} J(w_2, p_2) \text{ s.t. } w_2 \in C_2; \]

until converge

1) Intuitive analysis of the algorithm: Let us consider the statement (ii) in the above algorithm, where the penalizing vector \( p_2 \triangleq [p_1, p_2, \ldots, p_N]^T \) is the result \( w_1 \) after (i) in the current iteration, and \( w_2 \triangleq [w_1, w_2, \ldots, w_N]^T \) is the result after (ii) in the previous iteration. From

\[
J(w_2, p_2) = |w_1| \cdot p_1 + |w_2| \cdot p_2 + \cdots + |w_N| \cdot p_N, \quad (8)
\]

the new weight \( w_2 \) is to be found by minimizing \( J(w_2, p_2) \), or in other words, by taking a smaller \( w_i \) for a larger \( p_i \). Therefore, the vectors \( w_1 \) and \( w_2 \) tend to become disjoint as the iteration continues.

Figure 2 shows an example of antenna distributions in the course of the argumentative reselection process. The initial values for the elements of \( w_1 \) and \( w_2 \) are all 1s. After the first iteration (See Figure 2(a)), there are 18 shared antennas by the sum beam and the delta beam. However, as the algorithm proceeds, the shared antennas are removed gradually (See Figure 2(b)), and then completely (See Figure 2(c)).

2) Argumentative reselection algorithm for multiple beams:

In the case of \( K \) beams, i.e., \( K \) weight vectors in the sets \( \{C_k\}_{k=1}^K \), the \( K \)-variable cost function of the optimization problem is defined as:

\[
\hat{J}(w_1, w_2, \cdots, w_K) = \frac{1}{2} \sum_{i,j=1}^{K} |w_i| \cdot |w_j|. \quad (9)
\]

If we define the penalizing vectors and the one-variable cost function, respectively as:

\[
p_k := \sum_{j=1, j \neq k}^{K} \omega_j, \quad J(w_k, p_k) = |w_k|^T \cdot p_k, \quad (10)
\]

then we have Algorithm 1 below.

The above stated algorithm with two weight vectors is a special case of Algorithm 1, when \( K = 2 \). Now the convergence of Algorithm 1 is proven below.

Theorem: The sequence \( J(l) \) in Algorithm 1 is monotonically decreasing and bounded below by zero, and thus convergent.

Proof. Writing the intermediate results explicitly, let \( w_k^{(l)} \) be the optimal vector obtained after the \( k \)th inner-iteration of the \( l \)th outer-iteration, and we define

\[
J_k^{(l)} \triangleq \hat{J}(w_1^{(l)}, w_2^{(l)}, \cdots, w_k^{(l)}, w_{k+1}^{(l-1)}, \cdots, w_K^{(l-1)}). \quad (11)
\]

**Algorithm 1** Algorithm for the argumentative reselection process

initialize \( w_k := \text{randn}(N,1) + j \text{randn}(N,1) \) for \( k \in \{1, \ldots, K\} \), and set \( J^{(0)} := \inf \)

for \( l = 1, 2, \ldots \)

for \( k = 1 : K \)

\[
p_k := \sum_{j=1, j \neq k}^{K} \omega_j; \quad w_k := \arg\min_{w_k} J(w_k, p_k), \text{ subject to } w_k \in C_k;
\]

end

\[
J(l) := \hat{J}(w_1, w_2, \cdots, w_K);
\]

if \( J(l-1) - J(l) < \epsilon \) exit

end

return \( \hat{w}_k := w_k \) for \( k \in \{1, \ldots, K\} \);

Then when \( k = 1 \),

\[
J_1^{(l+1)} = \arg\min_{w_1} J(w_1, p_1) \quad (12)
\]

\[
J_1^{(l+1)} = \arg\min_{w_1} |w_1|^T \cdot \left( \sum_{j=2}^{K} \left| w_j^{(l)} \right| \right) \quad (13)
\]

Therefore,

\[
J_1^{(l+1)} = |w_1^{(l)}|^T \cdot \left( \sum_{j=2}^{K} \left| w_j^{(l)} \right| \right) + \frac{1}{2} \sum_{i,j=2, i \neq j}^{K} \left| w_i^{(l)} \right|^T \cdot \left| w_j^{(l)} \right| \quad (14)
\]

\[
\leq |w_1^{(l)}|^T \cdot \left( \sum_{j=2}^{K} \left| w_j^{(l)} \right| \right) + \frac{1}{2} \sum_{i,j=2, i \neq j}^{K} \left| w_i^{(l)} \right|^T \cdot \left| w_j^{(l)} \right| \quad (15)
\]

\[
= J_1^{(l)}, \quad (16)
\]

so that \( J_1^{(l)} \geq J_1^{(l+1)} \), Similarly \( J_k^{(l+1)} \geq J_k^{(l+1)} \), and therefore \( J_K^{(l+1)} \geq J_K^{(l+2)} \geq \cdots \). Now the proof is complete because \( J_k^{(l)} \geq J(l), \forall l \), and \( J(l) \) are bounded below by zero from the definition of \( J \) in Equation (9).

IV. NUMERICAL RESULTS

We use the MOSEK solver which uses an interior point method \[22\] to solve the convex optimization problem. The simulation is performed with MATLAB under the hardware condition of i7-4790-3.6GHz (CPU) and 16GB RAM.

To demonstrate the applicability of the proposed algorithm under mutual coupling, we shall consider an idealized coupling matrix \[23\]:

\[
M = \begin{bmatrix}
1 & \rho & \rho^2 & \cdots & \rho^{N-1} \\
\rho & 1 & \rho & \cdots & \rho \\
\rho^2 & \rho & 1 & \cdots & \rho^2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\rho^{N-1} & \rho^2 & \cdots & \rho & 1 \\
\end{bmatrix} \quad (17)
\]
Namely, we approximate $\mathbf{M}$ with the covariance matrix of an autoregressive process of order 1. According to our simulation study, the proposed algorithm appears to be robust under different coefficient values of $\rho$. However, a proper coupling matrix must be determined experimentally for each particular antenna array before the algorithm is applied.

First, let us consider a uniform linear array of $N = 120$ antennas with $d = \lambda/2$, and therefore, of an aperture size, $59.5\lambda$. We assume the mutual coupling constant $\rho$ to be 0.1. The sidelobe regions for the sum and delta beams are respectively $\Theta_1 = [1^\circ, 90^\circ] \cup [-90^\circ, -1^\circ]$ and $\Theta_2 = [1.2^\circ, 90^\circ] \cup [-90^\circ, -1.2^\circ]$. The maximum SLLs are assumed to be $\tau_1 = \tau_2 = -16.7$dB, and the slope, $s = -100\,\text{deg}^{-1}$. Letting $\mu = 1$, we calculate the exact value of maximum SLLs. However, the value of the parameter $\mu$ is not important, since the maximum SLLs are relative value (in dB) of $\mu$. We have taken 1001 samples distributed evenly in $[-90^\circ, 90^\circ]$.

Algorithm 1 with the constant $\epsilon = 10^{-5}$ successfully finds a pair of disjoint weight vectors; 53 weights for $F_1$ and 67 weights for $F_2$ (Figure 3(a)), using the MOSEK solver. The computation time is 6.12 second. The corresponding beam patterns (Figure 3(c)) meet the specifications we set above. Here, the 3dB beam width is 0.99$^\circ$.

On the other hand, the non-overlapping common weight approaches [10] could not find a feasible solution satisfying the specifications, (even when the mutual coupling matrix is the identity). If we relax the SLL requirements until the remaining beam requirements as well as the initial settings are met, then a solution could be found as shown in Figure 3(b). The corresponding beam patterns (Figure 3(d)) meet the specifications we set above. Here, the 3dB beam width is 0.99$^\circ$.

To verify the reliability of the argumentative reselection algorithm, we examine the success rate of finding a feasible solution through Monte Carlo simulation. We count successful runs of 500 trials with random initial penalizing vectors $p_1$, for each SLL from $-16.69$dB to $-16.76$dB. As shown in Figure 3(e), the proposed algorithm converges to zero with a 96.2% rate when the SLL is above $-16.76$dB.

Next, we consider a 2D planar array of 756 antennas with $d = \lambda/2$, which forms three beams; the sum beam $F_1$, the azimuth difference beam $F_2$ and the elevation difference beam $F_3$. We ignore the mutual coupling effect for simple exposition. Our beam specifications for $F_2$ and $F_3$ will be the same, and therefore, we shall describe $F_1$ and $F_2$ only. The side-lobe regions for $F_1$ and $F_2$ are $\Theta_1 = [5^\circ, 20^\circ] \cup (-20^\circ, -5^\circ)$ and $\Theta_2 = [20^\circ, 90^\circ] \cup [-90^\circ, -20^\circ]$, respectively. The maximum SLLs are assumed to be $\tau_1 = \tau_2 = -25$dB, and the slope $s = -22\,\text{deg}^{-1}$.

To speed up the computation of Algorithm 1, we divide the side-lobe regions into two, region $a$ and region $b$:

\begin{align}
\Theta_{1a} &= [5^\circ, 20^\circ] \cup (-20^\circ, -5^\circ), \\
\Theta_{1b} &= [20^\circ, 90^\circ] \cup [-90^\circ, -20^\circ], \\
\Theta_{2a} &= [8^\circ, 20^\circ] \cup (-20^\circ, -8^\circ), \\
\Theta_{2b} &= [20^\circ, 90^\circ] \cup [-90^\circ, -20^\circ].
\end{align}

Then we use $10 \times 2$ evenly spaced samples each, from $\Theta_{1a}$ and $\Theta_{2a}$, and $35 \times 2$ evenly spaced samples each, from $\Theta_{1b}$ and $\Theta_{2b}$. Again Algorithm 1 with $\epsilon = 10^{-5}$ finds three disjoint weight vectors: 301 weights for $F_1$, 233 weights for $F_2$ and 222 weights for $F_3$. See Figure 4(a). The computation time is 33.89 minutes. The corresponding beam patterns, Figure 4(b), (c), and (d) meet the given specifications, and the convergence is shown in Figure 4(e). The 3dB beamwidth of the sum beam is 4.2$^\circ$ for both azimuth and elevation angles.

**V. Concluding Remarks**

We have presented the argumentative reselection algorithm which partitions an antenna array into sparse sets, such that the sets of weights are disjoint and give independent desired beam patterns. Sparse subarrays with disjoint weights have a more degree of freedom, and therefore have better control on the beamsshapes compared to common (i.e., shared) weight
structure. As future work, it may be considered to include the crossing counts of the beamforming network into the objective function to reduce the complexity of the feeder structure.

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