THE BALIAN-LOW TYPE THEOREMS ON $L^2(\mathbb{C})$

ANIRUDHA PORIA AND JITENDRIYA SWAIN

Abstract. In this paper it is shown that $\|Zg\|_2$ and $\|\bar{Z}g\|_2$ cannot both be simultaneously finite if the twisted Gabor frame generated by $g \in L^2(\mathbb{C})$ forms an orthonormal basis or an exact frame for $L^2(\mathbb{C})$. The operators $Z = \frac{d}{dz} + \frac{1}{2} \bar{z}$ and $\bar{Z} = \frac{d}{d\bar{z}} - \frac{1}{2} z$ are associated with the special Hermite operator $L = -\Delta_z + \frac{1}{4} |z|^2 - i \left( x \frac{d}{dy} - y \frac{d}{dx} \right)$ on $\mathbb{C}$, where $\Delta_z$ is the standard Laplacian on $\mathbb{C}$ and $z = x + iy$. Also the amalgam version of BLT is proved using Weyl transform and the distinction between BLT and amalgam BLT is illustrated by examples. The twisted Zak transform is introduced and using it several versions of the Balian-Low type theorems on $L^2(\mathbb{C})$ are established.

1. Introduction

The Balian-Low theorem (BLT) is one of the fundamental and interesting result in time-frequency analysis. It says that a function $g \in L^2(\mathbb{R})$ generating Gabor Riesz basis cannot be localized in both time and frequency domains. Precisely if $g \in L^2(\mathbb{R})$ and if a Gabor system $G(g,a,b) := \{ e^{2\pi imbt} g(t-na) \}_{m,n \in \mathbb{Z}}$ with $ab = 1$ forms an orthonormal basis for $L^2(\mathbb{R})$, then

$$\left( \int_{-\infty}^{\infty} |tg(t)|^2 dt \right) \left( \int_{-\infty}^{\infty} |\hat{g}(\gamma)|^2 d\gamma \right) = +\infty,$$

where $\hat{g}$ is the Fourier transform of $g$ formally defined by $\hat{g}(\gamma) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i \gamma t} dt$. This result was originally stated by Balian [3] and independently by Low in [20]. The proofs given by Balian and Low each contained a technical gap, which was filled by Coifman et al. [9] and extended the BLT to the case of Riesz bases. Battle [4] provided an elegant and entirely new proof based on the operator theory associated with the classical uncertainty principle. For general Balian-Low type results, historical comments and variations of BLT we refer to [7, 10].

The Balian-Low type results are proved for multi-window Gabor systems by Zibulski and Zeevi [30] and for superframes by Balan [2]. The BLT and its variations for symplectic lattices in higher dimensions (see [11, 15]), for the symplectic form on $\mathbb{R}^{2d}$ (see [5]) and on locally compact abelian groups (see [13]) are obtained in the literature. For further results on BLT we refer to [1, 6, 12, 16, 21, 22] and [29]. In this paper we establish the BLT and
some of its variations on $L^2(\mathbb{C})$ using the operators $Z$ and $\bar{Z}$ associated with the special Hermite operator $L$ on $\mathbb{C}$. The Weyl transform and the twisted convolution closely related to the Fourier transform on the Heisenberg group and play a significant role in proving our main results. Therefore we review the representation theory on the Heisenberg group to see various objects of interest arising from it.

One of the simple and natural example of non-abelian, non-compact groups is the famous Heisenberg group $H$, which plays an important role in several branches of mathematics. The Heisenberg group $H$ is a unimodular nilpotent Lie group whose underlying manifold is $\mathbb{C} \times \mathbb{R}$ and the group operation is defined by

$$(z, t) \cdot (w, s) = (z + w, t + s + \frac{1}{2} \text{Im}(z\bar{w})).$$

The Haar measure on $H$ is given by $dzdt$.

By Stone-von Neumann theorem, the only infinite dimensional unitary irreducible representations (up to unitary equivalence) are given by $\pi_\lambda$, $\lambda \in \mathbb{R} \setminus \{0\}$, where $\pi_\lambda$ is defined by

$$\pi_\lambda(z, t)\varphi(\xi) = e^{4\pi i \lambda t} e^{4\pi i \lambda (x\xi + \frac{1}{2}xy)} \varphi(\xi + y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R})$.

The group Fourier transform of $f \in L^1(H)$ is defined as

$$\hat{f}(\lambda) = \int_H f(z, t)\pi_\lambda(z, t)dzdt, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$ 

Note that for each $\lambda \in \mathbb{R} \setminus \{0\}$, $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R})$. Under the operation “group convolution” $L^1(H)$ turns out to be a non-commutative Banach algebra.

Let

$$f^\lambda(z) = \int_\mathbb{R} e^{4\pi i \lambda t} f(z, t)dt$$

denote the inverse Fourier transform of $f$ in the $t-$variable. Therefore $\hat{f}(\lambda) = \int_\mathbb{C} f^\lambda(z)\pi_\lambda(z, 0)dz$.

Thus we are led to consider operators of the form

$$(1.1) \quad W_\lambda(g) = \int_\mathbb{C} g(z)\pi_\lambda(z)dz,$$

where $\pi_\lambda(z, 0) = \pi_\lambda(z)$. For $\lambda = 1$ we call $[\square]$ as the Weyl transform of $g$. Thus for $g \in L^1(\mathbb{C})$ and writing $\pi(z)$ in place of $\pi_1(z)$ we have

$$(1.2) \quad W(g)\varphi(\xi) = \int_\mathbb{C} g(z)\pi(z)\varphi(\xi)dz, \quad \varphi \in L^2(\mathbb{C}).$$
For \( f, g \in L^1(\mathbb{C}) \), the twisted convolution is defined by

\[
    f \times g(z) = \int_{\mathbb{C}} f(z - w)g(w)e^{-2\pi i m(z \cdot \overline{w})}dw.
\]

Under twisted convolution \( L^1(\mathbb{C}) \) is a non-commutative Banach algebra. For \( f \in L^1 \cap L^2(\mathbb{C}) \) the Weyl transform of \( f \) can be explicitly written as

\[
    W(f) \varphi(\xi) = \int_{\mathbb{C}} f(z)e^{4\pi i (x \cdot \xi + \frac{1}{2} y \cdot y)}\varphi(\xi + y)dz, \quad \varphi \in L^2(\mathbb{R}), \quad z = x + iy,
\]

which maps \( L^1(\mathbb{C}) \) into the space of bounded operators on \( L^2(\mathbb{R}) \), denoted by \( \mathcal{B} \). The Weyl transform \( W(f) \) is an integral operator with kernel

\[
    K_f(\xi, \eta) = \int_{\mathbb{R}} f(x, \eta - \xi)e^{2\pi i x(\xi + \eta)}dx.
\]

If \( f \in L^2(\mathbb{C}) \), then \( W(f) \in \mathcal{B}_2 \), the space of all Hilbert-Schmidt operators on \( L^2(\mathbb{R}) \) and satisfies the Plancherel formula

\[
    \|W(f)\|_{\mathcal{B}_2} = \frac{1}{2} \|f\|_{L^2(\mathbb{C})}.
\]

In general, for \( f, g \in L^2(\mathbb{C}) \), we have

\[
    \langle W(f), W(g) \rangle_{\mathcal{B}_2} = \frac{1}{2} \langle f, g \rangle_{L^2(\mathbb{C})} = \langle K_f, K_g \rangle_{L^2(\mathbb{C})}.
\]

The inversion formula for Weyl transform is

\[
    f(z) = \text{tr}(\pi(z)^* W(f)),
\]

where \( \pi(z)^* \) is the adjoint of \( \pi(z) \) and \( \text{tr} \) is the usual trace on \( \mathcal{B} \). For detailed study on Weyl transform we refer to the text of Thangavelu [27, 28].

Let \( H_k \) denote the Hermite polynomial on \( \mathbb{R} \), defined by

\[
    H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}, \quad k = 0, 1, 2, \ldots,
\]

and \( h_k \) denote the normalized Hermite functions on \( \mathbb{R} \) defined by

\[
    h_k(x) = (2^k \sqrt{k!})^{-\frac{1}{2}} H_k(x)e^{-\frac{1}{2} x^2}, \quad k = 0, 1, 2, \ldots.
\]

Let \( A = -\frac{d}{dx} + x \) and \( A^* = \frac{d}{dx} + x \) denote the creation and annihilation operators in quantum mechanics respectively. The Hermite operator \( H \) is defined as

\[
    H = \frac{1}{2}(AA^* + A^*A) = -\frac{d^2}{dx^2} + x^2.
\]

The Hermite functions \( \{h_k\} \) are the eigenfunctions of the operator \( H \) with eigenvalues \( 2k + 1 \), \( k = 0, 1, 2 \cdots \). Using the Hermite functions, the special Hermite functions on \( \mathbb{C} \) are defined
as follows:
\[ \phi_{m,n}(z) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{i\xi x} h_m(\xi + \frac{1}{2}y) h_n(\xi - \frac{1}{2}y) d\xi, \]
where \( z = x + iy \in \mathbb{C} \) and \( m, n = 0, 1, 2, \cdots \). The functions \( \{\phi_{m,n} : m, n = 0, 1, 2, \cdots\} \) form an orthonormal basis for \( L^2(\mathbb{C}) \). The special Hermite functions are the eigenfunctions of a second order elliptic operator \( L \) on \( \mathbb{C} \). To define this operator \( L \) we need to define the operators \( Z \) and \( \bar{Z} \) as follows:
\[ Z = \frac{d}{dz} + \frac{1}{2} \bar{z}, \quad \bar{Z} = \frac{d}{d\bar{z}} - \frac{1}{2} z. \]
The functions \( \phi_{m,n} \) are eigenfunctions of the special Hermite operator
\[ L = -\Delta_z + \frac{1}{4} |z|^2 - i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -\frac{1}{2} (Z \bar{Z} + \bar{Z} Z), \]
(1.3) with eigenvalues \((2n + 1)\), where \( \Delta_z \) denotes the Laplacian on \( \mathbb{C} \). We list out some of the properties (see [27, 28]) of the operators \( Z \) and \( \bar{Z} \) in the following proposition, which will be useful at several places.

**Proposition 1.1.**

1. \( Z(\phi_{m,n}) = i\sqrt{2n} \phi_{m,n-1} \) and \( \bar{Z}(\phi_{m,n}) = i\sqrt{2n+2} \phi_{m,n+1} \).
2. \( W(Zf) = iW(f)A \) and \( W(\bar{Z}f) = iW(f)A^* \) for every Schwartz class function \( f \).
   (This expression is similar to the relation \( (\frac{d}{dx} f)(\gamma) = 2\pi i \gamma \hat{f}(\gamma) \)).
3. \( [Z, \bar{Z}] = -2I \), where \( [Z, \bar{Z}] = Z \bar{Z} - \bar{Z} Z \) is the commutator of \( Z \) and \( \bar{Z} \).
4. The adjoint \( Z^* \) of \( Z \) is \( -\bar{Z} \).

Our goal in this paper is to obtain the Balian-Low type Theorem (BLT) and its variations on \( L^2(\mathbb{C}) \). The motivation to prove the BLT on \( L^2(\mathbb{C}) \) arises from the classical Heisenberg’s uncertainty principle on \( L^2(\mathbb{R}) \). Let \( P \) and \( M \) be the position and the momentum operators defined by
\[ Pf(t) = tf(t) \quad \text{and} \quad Mf(t) = \frac{d}{dt} f(t), \]
on a suitable domain.

**Theorem 1.2. (Classical Heisenberg’s uncertainty principle on \( L^2(\mathbb{R}) \))** Let \( f \in L^2(\mathbb{R}) \). Then
\[ \| Pf \|_2 \| Mf \|_2 \geq \frac{1}{4\pi}\|f\|_2^2. \]
Observe that the Laplacian \( L_0 \) on \( \mathbb{R} \) can be written as
\[ L_0 = \frac{d^2}{dx^2} = \frac{1}{4}(A^* - A)(A^* - A) = \frac{1}{4}(A^*B + AB^*). \]
and satisfies
\[ [A, B] = [A, A^*] = -2I, \]
where \( B = A^* - A \). The expression for special Hermite operator \( L \) is similar to the Laplacian \( L_0 \) on \( \mathbb{R} \) (see (1.3) and (1.4)) with \( [Z, \bar{Z}] = -2I \). The classical uncertainty principle requires the operators \( P \) and \( M \) to be self-adjoint and uses the fact that \( [P, M] = -I \), whereas the operators \( Z \) and \( \bar{Z} \) are not self-adjoint. However we obtain the following variation of Heisenberg’s uncertainty inequality for \( L^2(\mathbb{C}) \).

**Theorem 1.3.** Let \( f \in L^2(\mathbb{C}) \). Then
\[
\int_{\mathbb{C}} |Zf(z)|^2 \, dz + \int_{\mathbb{C}} |\bar{Z}f(z)|^2 \, dz \geq 2\|f\|_2^2.
\]

In view of the above facts we obtain the following BLT for exact frames on \( L^2(\mathbb{C}) \).

**Theorem 1.4.** Let \( g \in L^2(\mathbb{C}) \). If the twisted Gabor system \( \mathcal{G}^t(g, 1, 1) = \{ T_{(m,n)}^t g : m, n \in \mathbb{Z} \} \) forms an exact frame for \( L^2(\mathbb{C}) \), then \( \|Zg\|_2 \|\bar{Z}g\|_2 = +\infty \).

Using Plancherel formula for Weyl transform we have
\[
\|Zg\|_2\|W(g)A^*\|_{L^2} = \frac{1}{2}\|Zg\|_2 \|\bar{Z}g\|_2 = +\infty.
\]

This expression is analogous to the conclusion of the classical BLT. Observe that unlike the Fourier transform of functions in \( L^1(\mathbb{R}) \), the Weyl transform of functions in \( L^1(\mathbb{C}) \) are bounded operators on \( L^2(\mathbb{R}) \). Therefore it is bit technical to deal with the Weyl transform and estimate the oscillations of the twisted Zak transform on \( L^2(\mathbb{C}) \) in terms of \( \|Zf\|_2 \) and \( \|\bar{Z}f\|_2 \).

The paper is organized as follows. In section 2, we provide necessary background for proving BLT and discuss basic properties of frames. In section 3, we define twisted Gabor frames, twisted Zak transform and deduce some of its properties. Also we prove the amalgam BLT and provide examples illustrating the distinction between the BLT and the amalgam BLT. Further, using the operators \( Z, \bar{Z} \) and the continuity of twisted Zak transform we obtain a version of amalgam BLT. In section 4, we prove the BLT for exact frames on \( L^2(\mathbb{C}) \) using the operators \( Z, \bar{Z} \) and the calculations are non-distributional. In section 5, we obtain a variation of Heisenberg uncertainty principle (Theorem 1.3), weaker version of BLT (Theorem 5.2) and show the equivalence of weak BLT and BLT (Theorem 5.5). Finally, we
discuss several consequences of BLT and weak BLT in terms of the operators $L, Z$ and $\bar{Z}$ in Remark 5.6.

2. Notations and Background

2.1. Frame and Riesz basis.

**Definition 2.1.** A sequence $\{f_k : k \in \mathbb{Z}\}$ is a frame for a separable Hilbert space $H$ if there exist constants $A, B > 0$ such that for all $f \in H$,

$$A\|f\|^2 \leq \sum_k |\langle f, f_k \rangle|^2 \leq B\|f\|^2.$$ 

A frame $\{f_k\}$ is exact if it ceases to be a frame when any single element $f_n$ is deleted, that is, $\{f_k\}_{k \neq n}$ is not a frame for any $n$. For any frame $\{f_k\}$ there will exist a dual frame $\{\tilde{f}_k\}$, so that for all $f \in H$, have a series representation given by

$$f = \sum_k \langle f, f_k \rangle \tilde{f}_k = \sum_k \langle f, \tilde{f}_k \rangle f_k.$$

**Definition 2.2.** A sequence $\{f_k : k \in \mathbb{Z}\}$ is called a Riesz basis for a Hilbert space $H$ if there exists a continuous, invertible, linear mapping $T$ on $H$ such that $\{Tf_k\}$ forms an orthonormal basis for $H$.

The concept of a Riesz basis and an exact frame for a frame sequence on a separable Hilbert space coincides.

2.2. Gabor frames and density. For $a, b > 0$, $g \in L^2(\mathbb{R}^d)$ and $n, k \in \mathbb{Z}^d$ define $M_{bn}g(x) := e^{2\pi ibnx}g(x)$ and $T_{ak}g(x) := g(x - ak)$. The collection of functions $G(g, a, b) = \{M_{bn}T_{ak}g : k, n \in \mathbb{Z}^d\}$ in $L^2(\mathbb{R}^d)$, is called a Gabor frame or a Weyl-Heisenberg frame if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k, n \in \mathbb{Z}^d} |\langle f, M_{bn}T_{ak}g \rangle|^2 \leq B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}^d).$$

The associated frame operator called the Gabor frame operator has the form

$$S_G f := \sum_{k, n \in \mathbb{Z}^d} \langle f, M_{bn}T_{ak}g \rangle M_{bn}T_{ak}g, \quad f \in L^2(\mathbb{R}^d).$$

If $g \in L^2(\mathbb{R}^d)$ generates a Gabor frame $G(g, a, b)$ then there exists a dual window (canonical dual window) $\tilde{g} = S_G^{-1}(g) \in L^2(\mathbb{R}^d)$ such that $G(\tilde{g}, a, b) = \{M_{bn}T_{ak}\tilde{g} : k, n \in \mathbb{Z}^d\}$ is also a
frame for $L^2(\mathbb{R}^d)$ called the dual Gabor frame. Consequently every $f \in L^2(\mathbb{R}^d)$ possess the expansion

$$f = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{bn} T_{ak} g \rangle M_{bn} T_{ak} \hat{g} = \sum_{k,n \in \mathbb{Z}^d} \langle f, M_{bn} T_{ak} \hat{g} \rangle M_{bn} T_{ak} g$$

with unconditional convergence in $L^2(\mathbb{R}^d)$.

One of the important and interesting concept in frame theory is to obtain the necessary condition on the lattice parameters $a, b$ so that the Gabor system $\mathcal{G}(g, a, b)$ constitute a frame. The algebraic structure of the lattice $\Lambda = \{(ak, bn) : k, n \in \mathbb{Z}^d\}$ has been exploited to derive the necessary condition for a Gabor system $\mathcal{G}(g, a, b)$ to be complete, a frame or an exact frame in terms of the product $ab$. The following results are known for Gabor frames in one dimension case ($d = 1$) with a rectangular lattice $\Lambda = a \mathbb{Z} \times b \mathbb{Z}$. In [26], Rieffel proved that the Gabor system $\mathcal{G}(g, a, b)$ is incomplete for any $g$ if $ab > 1$. Daubechies [9] proved Rieffel’s result for the case when $ab$ is rational and exceeds one. Assuming further decay on $g$ and $\hat{g}$, Landau [19] proved that $\mathcal{G}(g, a, b)$ cannot be a frame for $L^2(\mathbb{R})$ if $ab > 1$.

For $a, b \in \mathbb{R}^d, g \in L^2(\mathbb{R}^d)$ and the lattice $\Lambda \subset \mathbb{R}^{2d}$, Ramanathan and Steger [25] proved the incompleteness of Gabor systems that are uniformly discrete (i.e. there is a minimum distance $\delta$ between elements of $\Lambda$) in terms of the Beurling density defined as follows:

Let $\Lambda \subset \mathbb{R}^d$ be a uniformly discrete. Let $B$ be the ball of volume one in $\mathbb{R}^d$ centered at origin. For each $r > 0$, $\nu^+(r)$ and $\nu^-(r)$ denote the maximum and minimum number of points of $\Lambda$ that lie in any translate of $rB$ i.e. $\nu^+(r) = \max \{\lambda \in \Lambda : \lambda \in \Lambda \cap (x + rB)\}$ and $\nu^-(r) = \min \{\lambda \in \Lambda : \lambda \in \Lambda \cap (x + rB)\}$. Since $\Lambda$ is uniformly discrete, both $\nu^+(r)$ and $\nu^-(r)$ are finite for every $r > 0$. The upper and lower densities are defined by

$$D^+(\Lambda) = \limsup_{r \to \infty} \frac{\nu^+(r)}{r^d} \quad \text{and} \quad D^-(\Lambda) = \liminf_{r \to \infty} \frac{\nu^-(r)}{r^d}.$$ 

In [18], Landau shown that these quantities are independent of the particular choice of the set $B$ with measure 1. If $D^+(\Lambda) = D^-(\Lambda)$, then the set $\Lambda$ is said to have uniform Beurling density $D(\Lambda) = D^+(\Lambda) = D^-(\Lambda)$. Ramanathan and Steger [25] proved the following result.

**Theorem 2.3. (Density theorem)**

Let $g \in L^2(\mathbb{R}^d)$, and $\Lambda \subset \mathbb{R}^{2d}$ be a uniformly discrete set.

(a) If $D^+(\Lambda) < 1$, then $\{\rho(p, q)g : (p, q) \in \Lambda\}$ is not a frame for $L^2(\mathbb{R}^d)$ where $\rho(p, q)g(x) = e^{2\pi i q \cdot x}g(x - p)$.
If $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ is a rectangular lattice with uniform Beurling density $D(\Lambda) < 1$ then $\{\rho(p,q)g : (p,q) \in \Lambda\}$ is incomplete in $L^2(\mathbb{R}^d)$.

(c) If $\Lambda$ has uniform Beurling density $D(\Lambda)$ such that $\{\rho(p,q)g : (p,q) \in \Lambda\}$ is a Riesz basis then $D(\Lambda) = 1$.

By the density theorem, there is a clear separation between overcomplete frames and undercomplete Riesz sequences with Riesz bases corresponding to the critical density lattices that satisfy $D(\Lambda) = 1$. The classical BLT [7] on $L^2(\mathbb{R})$ says that the window $g$ of any Gabor Riesz basis $G(g,a,b)$ must either not be smooth or must decay poorly at infinity.

3. Twisted Zak transform and Amalgam BLT

**Definition 3.1.** Let $f \in L^2(\mathbb{C})$ and $a, b > 0$. For $(m,n) \in \mathbb{Z}^2$ we define twisted translation of $f$, denoted by $T^t_{(am, bn)}f$, as

$$T^t_{(am, bn)}f(z) = e^{2\pi i(bnx - amy)}f(x - am, y - bn), \quad z = x + iy \in \mathbb{C}. \quad (3.1)$$

For $a = b = 1$ the properties of twisted translation are listed below (see [23]).

1. The adjoint $(T^t_{(m,n)})^*$ of $T^t_{(m,n)}$ is $T^t_{(-m, -n)}$.
2. $T^t_{(m_1, n_1)}T^t_{(m_2, n_2)} = T^t_{(m_1 + m_2, n_1 + n_2)}$.
3. $T^t_{(m,n)}$ is a unitary operator on $L^2(\mathbb{C})$ for all $(m,n) \in \mathbb{Z}^2$.
4. The Weyl transform of $T^t_{(m,n)}f$ is given by $W(T^t_{(m,n)}f) = \pi(m,n)W(f)$.

**Definition 3.2.** For $a, b > 0$, $g \in L^2(\mathbb{C})$ the collection of functions $G^t(g,a,b) = \{T^t_{(am, bn)}g : m, n \in \mathbb{Z}\}$ in $L^2(\mathbb{C})$, is called a twisted Gabor frame or a twisted Weyl-Heisenberg frame if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k,n \in \mathbb{Z}} |\langle f, T^t_{(am, bn)}g \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{C}). \quad (3.2)$$

Then one can define the twisted Gabor tight frames, Riesz basis and the frame operator analogously. It is natural to ask about the density result as in Theorem 2.3 for twisted Gabor frames. For $a, b > 0$ and $g \in L^2(\mathbb{C})$, the sequence $\{T^t_{(am, bn)}g : m, n \in \mathbb{Z}\}$ is complete in $L^2(\mathbb{C})$ if and only if the system $\{\rho(p,q)g : (p,q) \in \Lambda \subset \mathbb{R}^4\}$ is complete in $L^2(\mathbb{R}^2)$, where $p = (am, bn), q = (bn, -am)$. In this case uniform Beurling density is $D(\Lambda) = \frac{1}{(ab)^2}$. So by Theorem 2.3 if $ab > 1$ then the twisted Gabor system $G^t(g,a,b) = \{T^t_{(am, bn)}g : m, n \in \mathbb{Z}\}$
is incomplete in $L^2(\mathbb{C})$. Therefore without loss of generality we consider the case when $a = b = 1$ throughout the paper. Now we define the twisted Zak transform which will be an important tool to prove our main results.

3.1. Twisted Zak transform. The Zak transform $\mathcal{Z}f$ on $L^2(\mathbb{R})$ is formally defined by $\mathcal{Z}f(x, t) = \sum_{k \in \mathbb{Z}} T_k f(x) \cdot M_k(1)(t)$, $x, t \in \mathbb{R}$, where $1$ is the constant function $1$. Since we are interested to obtain the BLT for twisted Gabor frames we define the twisted Zak transform with a slight modification in the following way.

**Definition 3.3.** Let $f \in L^2(\mathbb{C})$. The twisted Zak transform $Z^t f$ of $f$ is the function on $\mathbb{C}^2$ defined by

$$(Z^t f)(z, w) = \sum_{k \in \mathbb{Z}^2} T_k f(z) \cdot T_k^1(-w) = \sum_{k \in \mathbb{Z}^2} f(z - k) e^{2\pi i M(wk)}, \quad z, w \in \mathbb{C},$$

where $\bar{k}$ is the complex conjugate of $k$ and $\text{Im}(wk)$ is the imaginary part of $wk$.

Clearly $Z^t f$ is well-defined for continuous functions with compact support and converges in $L^2-$sense for $f \in L^2(\mathbb{C})$. In fact $Z^t$ is a unitary map of $L^2(\mathbb{C})$ onto $L^2(Q \times Q)$, where $Q := [0, 1) \times [0, 1)$. The idea of the proof is similar to the Zak transform on $L^2(\mathbb{R})$ as in [8]. The unitary nature of twisted Zak transform allows to transfer certain conditions on frames for $L^2(\mathbb{C})$ into conditions on $L^2(Q \times Q)$. More precisely, $\{f_k\}$ is complete or a frame or an exact frame or an orthonormal basis for $L^2(\mathbb{C})$ if and only if the same is true for $\{Z^t f_k\}$ in $L^2(Q \times Q)$.

As in case of Zak transform on $L^2(\mathbb{R})$ we obtain the similar properties of twisted Zak transform on $L^2(\mathbb{C})$ in the following lemma. However, our main results are still valid if Zak transform on $L^2(\mathbb{R}^2)$ is applied in place of twisted Zak transform on $L^2(\mathbb{C})$.

**Lemma 3.4.** Let $f \in L^2(\mathbb{C})$. Let $z = x + iy$, $w = r + is$ and $Q := [0, 1) \times [0, 1)$. Then the following holds:

(i) $Z^t f(z + 1, w) = e^{2\pi i s} Z^t f(z, w)$, $Z^t f(z + i, w) = e^{-2\pi i r} Z^t f(z, w)$

and $Z^t f(z, w + 1) = Z^t f(z, w + i) = Z^t f(z, w)$.

(ii) $Z^t (T_{(m,n)}^f)(z, w) = e^{2\pi i (x.n - y.m)} e^{2\pi i (r.n - s.m)} Z^t f(z, w)$.

(iii) $\{T_{(m,n)}^f\}$ is complete in $L^2(\mathbb{C})$ if and only if $Z^t f \neq 0$ a.e.

(iv) $\{T_{(m,n)}^f\}$ is minimal and complete in $L^2(\mathbb{C})$ if and only if $1/(Z^t f) \in L^2(Q \times Q)$. 
10 ANIRUDHA PORIA AND JITENDRIYA SWAIN

\( \{ T_{(m,n)} f \} \) is a frame for \( L^2(\mathbb{C}) \) with frame bounds \( A, B \) if and only if \( 0 < A^{1/2} \leq |Z^t f| \leq B^{1/2} < \infty \) a.e. In this case, \( \{ T^t_{(m,n)} f \} \) is an exact frame for \( L^2(\mathbb{C}) \).

(vi) \( \{ T^t_{(m,n)} f \} \) is an orthonormal basis for \( L^2(\mathbb{C}) \) if and only if \( |Z^t f|^2 = 1 \), a.e.

(vii) \( \{ T^t_{(m,n)} f \} \) is a Riesz basis for \( L^2(\mathbb{C}) \) with bounds \( A, B \) if and only if \( 0 < A^{1/2} \leq |Z^t f| \leq B^{1/2} < \infty \) a.e.

(viii) If \( Z^t f \) is continuous on \( \mathbb{C}^2 \) then \( Z^t f \) has a zero in \( Q \times Q \).

Proof. The proof of the lemma follows similarly as in the Zak transform for \( L^2(\mathbb{R}) \) (see [7, 8, 14] or [17]). We only prove part (viii). Assume that \( Z^t f(z,w) \neq 0 \) for all \( (z,w) \in \mathbb{C}^2 \).

Since \( Z^t f \) is continuous on a simply connected domain \( \mathbb{C}^2 \), there is a continuous function \( \varphi(z,w) \) such that

\[ Z^t f(z,w) = |Z^t f(z,w)|e^{2\pi i \varphi(z,w)} \quad \text{for} \quad (z,w) \in [0,1]^2 \times [0,1]^2. \]

By part (i), we have \( Z^t f(z+i,w) = e^{-2\pi i r} Z^t f(z,w) \) and \( Z^t f(z,w+1) = Z^t f(z,w+i) \). Therefore for each \( z \) and \( w \) there are integers \( l_z \) and \( k_w \) such that \( \varphi(z,1) = \varphi(z,i) + 2\pi l_z \) and \( \varphi(i,w) = \varphi(0,w) + 2\pi k_w - 2\pi r \). Since \( \varphi(z,1) - \varphi(z,i) \) and \( \varphi(i,w) - \varphi(0,w) + 2\pi r \) are continuous functions of \( z \) and \( w \) respectively, so \( l_z = l \) (say) and \( k_w = k \) (say), for all \( z, w \in \mathbb{C} \). Therefore,

\[
0 = \varphi(0,1) - \varphi(0,i) + \varphi(0,i) - \varphi(i,i) + \varphi(i,i) - \varphi(i,1) + \varphi(i,1) - \varphi(0,1) \\
= 2\pi l - 2\pi k - 2\pi l + 2\pi k - 2\pi \\
= -2\pi,
\]

contradicting our assumption. \( \square \)

3.2. The Amalgam BLT. In this section we prove a variation of the BLT called the amalgam BLT in terms of Wiener amalgam space using certain properties of twisted Zak transform.

Definition 3.5. The Wiener amalgam space \( W(L^p, \ell^q) \) is the Banach space of all complex-valued measurable functions \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) for which the norm

\[
\|f\| := \left( \sum_{k \in \mathbb{Z}^d} \|f : T_k \chi_{[0,1]^d}\|_p^q \right)^{1/q} < \infty,
\]

with the obvious modification for \( q = \infty \).
For $p \geq 1$, consider the amalgam space defined by $W(C_0, \ell^p) = \{ f \in W(L^\infty, \ell^p) : f \text{ is continuous} \}$. Clearly $W(C_0, \ell^1) \subseteq L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. The amalgam BLT in terms of $W(C_0, \ell^1)$ and a subspace of $B_2$ is obtained in the following theorem.

**Theorem 3.6. (Amalgam BLT)** Let $g \in L^2(\mathbb{C})$. If the twisted Gabor system $\mathcal{G}'(g, 1, 1)$ is an exact frame for $L^2(\mathbb{C})$ then

$$g \notin W(C_0, \ell^1) \quad \text{and} \quad W(g) \notin \mathcal{W},$$

where $\mathcal{W} = \{ T \in B_2 : h(z) = \text{tr}(\pi(z)^*T) \text{ and } h \in W(C_0, \ell^1) \}$.

**Proof.** Suppose that $g \in W(C_0, \ell^1)$. Then by the definition of twisted Zak transform, $Z'g$ is continuous. By Lemma 3.4(viii), $Z'g$ must have a zero. Therefore $|Z'g|^{-1}$ is unbounded and by Lemma 3.4(v), $\mathcal{G}'(g, 1, 1)$ cannot be a frame. Again assume that $\mathcal{G}'(g, 1, 1)$ is an exact frame and $W(g) \in \mathcal{W}$. So by the inversion formula for Weyl transform $g(z) = \text{tr}(\pi(z)^*W(g))$ and $g \in W(C_0, \ell^1)$, leads to a contradiction. \qed

The BLT and amalgam BLT are two distinct results. There exists a function $g \in L^2(\mathbb{C})$ satisfying BLT but not amalgam BLT and vice-versa. The following examples illustrate the difference between the BLT and amalgam BLT.

**Example 3.7.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} e^{-\frac{1}{(x^2+y^2)^p}}, & (x, y) \in (0, 1) \times (0, 1), \\ 0, & \text{otherwise}. \end{cases}$$

Let $z = x + iy$, and define $g : \mathbb{C} \to \mathbb{R}$ by

$$g(z) = \sum_{(k_1, k_2) \in \mathbb{N}^2} \frac{1}{k_1^2 + k_2^2} f(x - k_1, y - k_2).$$

Then clearly $g \in W(C_0, \ell^1)$. Further,

$$W(g) = \sum_{(k_1, k_2) \in \mathbb{N}^2} \frac{1}{k_1^2 + k_2^2} Wf(x - k_1, y - k_2).$$

Clearly $W(g) \in B_2$. From the inversion formula for Weyl transform it follows that $W(g) \in \mathcal{W}$. Next we show that $\|Zg\|_2 = \infty$. Consider
We shall construct a function $f$. Therefore

\[ \|Zg\|_2^2 = \int_C |Zg(z)|^2 dz \]

\[ \geq \sum_{m,n \in \mathbb{N}} \int_{[m,m+1] \times [n,n+1]} Zg(z) \cdot \overline{Zg(z)} \, dz \]

\[ = \sum_{m,n \in \mathbb{N}} \int_{[m,m+1] \times [n,n+1]} \left| \frac{dg(z)}{dz} \right|^2 + \frac{1}{4}|zg(z)|^2 - \text{Re} \left( \bar{z}g(z) \frac{dg(z)}{dz} \right) \, dz. \]

Note that for each $m, n \in \mathbb{N}$ and $(x, y) \in (m, m+1) \times (n, n+1)$, the integrand

\[ \left| \frac{dg(z)}{dz} \right|^2 - \text{Re} \left( \bar{z}g(z) \frac{dg(z)}{dz} \right) \]

\[ = \frac{1}{4m^2n^2} e^{-\left( \frac{m^2 + n^2}{m^2n^2} \right)} \left[ \frac{(2x - 1)^2}{x^2(1-x)^2} + \frac{(2y - 1)^2}{y^2(1-y)^2} + \frac{2(2x - 1)}{x(1-x)^2} + \frac{2(2y - 1)}{y(1-y)^2} \right] \geq 0. \]

Therefore

\[ \|Zg\|_2^2 \geq \frac{1}{4} \sum_{m,n \in \mathbb{N}} \int_{[m,m+1] \times [n,n+1]} |zg(z)|^2 \, dz \]

\[ \geq \frac{1}{4} \sum_{m,n \in \mathbb{N}} \int_{[m,m+1] \times [n,n+1]} \frac{m^2 + n^2}{m^2n^2} |f(x - m, y - n)|^2 \, dxdy \]

\[ \geq \frac{1}{4} \|f\|_2^2 \sum_{m \in \mathbb{N}} \frac{1}{m} = \infty. \]

**Example 3.8.** We shall construct a function $f$ such that $Zf$ and $\tilde{Z}f \in L^2(C)$ but $f \not\in W(C_0, \ell^1)$ and $W(f) \not\subset W$. For sufficiently large $k$ (say $k > N$) choose $a_k \neq b_k$ such that $[a_k - \frac{1}{k}, b_k + \frac{1}{k}] \subset [k, k+1]$ and $b_k^2 - a_k^2 < k$. Define the continuous function $g_k$ by

\[ g_k(x) = \begin{cases} 
\frac{1}{\log k} (x - a_k + \frac{1}{k}), & x \in [a_k - \frac{1}{k}, a_k], \\
\frac{1}{k \log k} (b_k + \frac{1}{k} - x), & x \in [b_k, b_k + \frac{1}{k}], \\
0, & x \not\in [a_k - \frac{1}{k}, b_k + \frac{1}{k}].
\end{cases} \]

Clearly the function $g = \sum_{k=N}^{\infty} g_k$ is continuous on $\mathbb{R}$. Also $\|g\|_2 \leq 2 \sum_{k=N}^{\infty} \frac{1}{k(\log k)} < \infty$, $\|xg\|_2 \leq 3 \sum_{k=N}^{\infty} \frac{1}{k(\log k)^2} < \infty$, and $\|g'\|_2 \leq 2 \sum_{k=N}^{\infty} \frac{1}{k(\log k)} < \infty$, where $g'$ is the classical derivative of $g$, defined except at countably many points.

Define $f(z) = f(x, y) = g(x)g(y)$. Since $Zf = \frac{1}{2}(f_x - if_y + xf + iyf)$ we have

\[ \|Zf\|_2 \leq \frac{1}{2} (\|f_x\|_2 + \|f_y\|_2 + \|xf\|_2 + \|yf\|_2) \]

\[ = \frac{1}{2} (\|g'\|_2 \|g\|_2 + \|g'\|_2 \|g\|_2 + \|xg\|_2 \|g\|_2 + \|yg\|_2 \|g\|_2) < \infty. \]
Similarly $\|\hat{Z}f\|_2 < \infty$. Further,

$$\|f\|_{W(L^{\infty}, \ell^1)} = \sum_{k \in \mathbb{Z}^2} \|f \cdot T_k \chi_{(0,1)}^2\|_\infty = \sum_{k_1, k_2 = N}^{\infty} \frac{1}{k_1 \log k_1} \frac{1}{k_2 \log k_2} = \infty.$$ 

Again if $W(f) \in W$ then the inversion formula for Weyl transform gives $f \in W(C_0, \ell^1)$, which is a contradiction.

Now we investigate the relationships between the operators $Z, \hat{Z}$ and the continuity of twisted Zak transform. A version of BLT assuming the Wiener amalgam condition is obtained in the following theorem:

**Theorem 3.9.** If $g \in L^2(\mathbb{C})$ and

$$\hat{Z}g, \hat{Z}g \in W(C_0, \ell^2),$$

then $\{T_{(m,n)}^i g\}$ cannot be a twisted Gabor frame for $L^2(\mathbb{C})$.

**Proof.** Given that $g$ is continuous and hence Fundamental theorem of calculus for complex variables and ML-inequality can be applied. Now we claim that $g \in W(C_0, \ell^2)$. To prove the claim it is sufficient to show

$$\sum_{k} |g(z_k + k)|^2 < \infty$$

for every sequence $\{z_k\} \in [0,1] \times [0,1]$. Since $g \in L^2(\mathbb{C})$ we have

$$\sum_{k} |g(z + k)|^2 < \infty, \text{ a.e. on } [0,1] \times [0,1].$$

For fixed $z_0 \in [0,1] \times [0,1]$ and any sequence $\{z_k\} \in [0,1] \times [0,1]$ together with (3.6) gives

$$\left(\sum_{k} |g(z_k + k)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{k} |g(z_k + k) - g(z_0 + k)|^2\right)^{\frac{1}{2}} + \left(\sum_{k} |g(z_0 + k)|^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_{k} \left| \int_{z_0}^{z_k} \partial g(z + k) \, dz \right|^2 \right)^{\frac{1}{2}} + \left(\sum_{k} |g(z_0 + k)|^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{2} \left(\sum_{k} (M_k + m_k)^2\right)^{\frac{1}{2}} + \left(\sum_{k} |g(z_0 + k)|^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{2} \left[\left(\sum_{k} M_k^2\right)^{\frac{1}{2}} + \left(\sum_{k} m_k^2\right)^{\frac{1}{2}} + \left(\sum_{k} |g(z_0 + k)|^2\right)^{\frac{1}{2}}\right] < \infty,$$
where \( \gamma_k \) is the straight line joining the points \( z_0 \) and \( z_k \), with
\[
M_k = \text{ess sup}_{z \in \gamma_k} |Zg(z)|, \quad \text{and} \quad 2m_k = \text{ess sup}_{z \in \gamma_k} |zg(z)|.
\]
Observe that \( \sum_k M_k^2 \) and \( \sum_k m_k^2 \) are finite, since \( g \) satisfies (3.4). Without loss of generality we choose the curve \( \gamma_k \), because Fundamental theorem of calculus assures that the complex line integral is independent of path. Therefore \( g \in W(C_0, \ell^2) \). Using this fact and the definition of twisted Zak transform yields \( Z^t g \) is continuous on \( \mathbb{C} \). Thus \( \{T^t_{(m,n)} g\} \) cannot be a twisted Gabor frame for \( L^2(\mathbb{C}) \) (see Lemma 3.3 (v) and (viii)).

4. Non-distributional Calculations and the BLT

In this section we prove Theorem 1.4 using non-distributional calculations. Unlike the Fourier transform of functions in \( L^1(\mathbb{R}) \), the Weyl transform of functions in \( L^1(\mathbb{C}) \) are bounded operators on \( L^2(\mathbb{R}) \). Therefore it is difficult to estimate the suitable upper bound for the oscillation of the twisted Zak transform of \( f \in L^2(\mathbb{C}) \) in terms of \( \|Zf\|_2 \) and \( \|\tilde{Z} f\|_2 \). We make use of Weyl transform to estimate the variation of twisted Zak transform of \( f \in L^2(\mathbb{C}) \) over small cubes of length \( r < 1 \). We start with the following lemma.

**Lemma 4.1.** Let \( f, Zf \) and \( \tilde{Z}f \in L^2(\mathbb{C}) \). Fix \( \epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{C} \). If \( \tilde{f}(z) = f(z)e^{2\pi i (y\epsilon_1 - x\epsilon_2)}, \tau_{\epsilon} f(z) = f(z - \epsilon) \) and \( f_{\epsilon}(z) = f(z - \epsilon)e^{2\pi i (z\epsilon_2 - y\epsilon_1)} \), then there exists an \( N_\epsilon \in \mathbb{N} \) such that
\[
\begin{align*}
(i) & \quad \|\tilde{f} - f\|_2 \leq 2\pi|\epsilon|(1 + |\epsilon|)^N\|f\|_2, \\
(ii) & \quad \|\tau_{\epsilon} f - f\|_2 \leq \frac{4\pi}{\epsilon} |\epsilon| \left( \|Z\tilde{f}\|_2 + \|\tilde{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1 + |\epsilon|)^N\|f\|_2 \right), \\
(iii) & \quad \|f_{\epsilon} - f\|_2 \leq \frac{4\pi}{\epsilon} |\epsilon| \left( \|Z\tilde{f}\|_2 + \|\tilde{Z}\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1 + |\epsilon|)^N\|f\|_2 \right).
\end{align*}
\]

**Proof.** (i) Choose a smallest positive integer \( N_\epsilon \) such that \( \frac{1}{1 + |\epsilon|} < |\epsilon| \). Therefore
\[
\|\tilde{f} - f\|_2 \leq 2\|f\|_2 \leq 2\pi|\epsilon|(1 + |\epsilon|)^N\|f\|_2.
\]
(ii) Notice that \( Zf \in L^2(\mathbb{C}) \Leftrightarrow \tilde{Z}f \in L^2(\mathbb{C}) \) and \( \tilde{Z}f \in L^2(\mathbb{C}) \Leftrightarrow \tilde{Z}f \in L^2(\mathbb{C}) \). Since \( f \in L^2(\mathbb{C}) \) we have \( \|\tau_{\epsilon} f - f\|_2 = \|W(\tau_{\epsilon} f) - W(f)\|_{B_2} \). Then
\[
W(\tau_{\epsilon} f)\phi(\xi) = \int_{\mathbb{C}} f(z - \epsilon) e^{4\pi i (z\xi + \frac{x}{2}\epsilon_2)} \phi(\xi + y) \, dx dy
\]
\[
= \int_{\mathbb{C}} f(z) e^{4\pi i (z\xi + \frac{1}{2}(x+\epsilon_1)(y+\epsilon_2))} \phi(\xi + y + \epsilon_2) \, dx dy
\]
\[
= e^{4\pi i (\epsilon_1 \xi + \frac{1}{2}(\epsilon_1 \epsilon_2))} W(\tilde{f})\phi(\xi + \epsilon_2), \forall \phi \in L^2(\mathbb{R}).
\]
Applying mean value theorem on the Schwartz class function $\phi$ on $\mathbb{R}$ we have

$$||W(\tau,f) - W(\tilde{f})\phi(\xi)||$$

$$= |e^{i\pi (\epsilon_1 + \epsilon_2)}W(\tilde{f})\phi(\xi + \epsilon_2) - W(\tilde{f})\phi(\xi)|$$

$$\leq |e^{i\pi (\epsilon_1 + \epsilon_2)} - 1|W(\tilde{f})\phi(\xi + \epsilon_2) + |W(\tilde{f})(\phi(\xi + \epsilon_2) - \phi(\xi))|$$

$$\leq 4\pi |\epsilon||W(\tilde{f})(\phi(\xi + \epsilon_2) + \pi|\epsilon||W(\tilde{f})\phi(\xi + \epsilon_2) + |\epsilon||W(\tilde{f})\phi'(\xi + \theta\epsilon_2)|,$$

for some $\theta \in (0, 1)$. Writing $2\epsilon\phi(\xi) = (A + A^*)\phi(\xi)$ and $2\epsilon'(\xi) = (A^* - A)\phi(\xi)$ we get

$$||W(\tau,f) - W(\tilde{f})\phi||_2 \leq 4\pi |\epsilon||W(\tilde{f})\phi||_2 + \pi|\epsilon||W(\tilde{f})\phi'||_2$$

$$\leq \frac{5}{2}\pi|\epsilon|(|W(\tilde{f})A\phi||_2 + ||W(\tilde{f})A^*\phi||_2 + ||W(\tilde{f})\phi||_2).$$

But $||W(\tau,f) - W(f)||_2 \leq ||W(\tau,f) - W(\tilde{f})||_2 + ||W(\tilde{f}) - W(f)||_2$. Thus for any orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ for $L^2(\mathbb{R})$ we have

$$||W(\tau,f) - W(\tilde{f})||^2_2$$

$$= \sum_{j=1}^{\infty} ||W(\tau,f) - W(\tilde{f})\phi_j||^2_2$$

$$\leq \frac{25}{4}\pi^2|\epsilon|^2 \sum_{j=1}^{\infty} (||W(\tilde{f})A\phi_j||_2 + ||W(\tilde{f})A^*\phi_j||_2 + ||W(\tilde{f})\phi_j||_2)^2$$

$$\leq \frac{75}{4}\pi^2|\epsilon|^2 \sum_{j=1}^{\infty} (||W(\tilde{f})A\phi_j||^2_2 + ||W(\tilde{f})A^*\phi_j||^2_2 + ||W(\tilde{f})\phi_j||^2_2)$$

$$= \frac{75}{4}\pi^2|\epsilon|^2 (||W(\tilde{f})A||^2_2 + ||W(\tilde{f})A^*||^2_2 + ||W(\tilde{f})||^2_2)$$

$$= \frac{75}{4}\pi^2|\epsilon|^2 \left(||ZF||^2_2 + ||ZF||^2_2 + ||F||^2_2\right).$$

Therefore by (i) we get

$$||W(\tau,f) - W(f)||_2 \leq \frac{15}{2}\pi|\epsilon| \left(||ZF||_2 + ||ZF||_2 + ||F||_2\right) + 2\pi|\epsilon|(1 + |\epsilon|)^N ||f||_2$$

$$\leq \frac{15}{2}\pi|\epsilon| \left(||ZF||_2 + ||ZF||_2 + ||F||_2 + (1 + |\epsilon|)^N ||f||_2\right).$$

(iii) From (i) and (ii) we get

$$||f_\epsilon - f||_2 \leq ||f_\epsilon - \tau_\epsilon f||_2 + ||\tau_\epsilon f - f||_2$$

$$\leq \frac{15}{2}\pi|\epsilon| \left(||ZF||_2 + ||ZF||_2 + ||F||_2\right) + 4\pi|\epsilon|(1 + |\epsilon|)^N ||f||_2$$

$$\leq \frac{15}{2}\pi|\epsilon| \left(||ZF||_2 + ||ZF||_2 + ||F||_2 + (1 + |\epsilon|)^N ||f||_2\right).$$
Theorem 4.2. Let \( x = (t, w) \in \mathbb{R}^2 \) and \( r > 0 \). Then \( Q(x; r) \) is the square centered at \( x \) with radius \( r \), i.e.
\[
Q(x; r) = \left[ t - \frac{r}{2}, t + \frac{r}{2} \right] \times \left[ w - \frac{r}{2}, w + \frac{r}{2} \right]
\]
\[
= \left\{ (u, v) : u \in \left[ t - \frac{r}{2}, t + \frac{r}{2} \right], v \in \left[ w - \frac{r}{2}, w + \frac{r}{2} \right] \right\}.
\]
Thus the square \( Q = [0, 1] \times [0, 1) \) can be represented as \( Q(0, 1/2; 1) \).

Theorem 4.2. Let \( f, Zf, \tilde{Z}f \in L^2(\mathbb{C}), \ G = Z^t f, \alpha_0 = (z_0, w_0) \in Q(z_0, 1) \times Q(w_0, 1) := Q[\alpha_0, 1], z_0 \in \left[ -\frac{3}{2}, \frac{3}{2} \right] \times \left[ -\frac{3}{2}, \frac{3}{2} \right] \) and \( w_0, \epsilon \in \mathbb{C} \) be given. Let \( f, \tilde{f} \) be as in Lemma \( 4.1 \). Then there exists an \( N_\epsilon \in \mathbb{N} \) such that
\[
\|T_{\tau_j}^t G - G\|_{L^2(Q[\alpha_0, 1])} \leq 8\pi|\epsilon|\|f\|_2 + \frac{15}{2}\pi|\epsilon| \left( \|Z\tilde{f}\|_2 + \|Z\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1 + |\epsilon|)^N\|f\|_2 \right),
\]
and
\[
\|T_{\tau_j}^t G - G\|_{L^2(Q[\alpha_0, 1])} \leq 2\pi|\epsilon|(1 + |\epsilon|)^N\|f\|_2 + 8\pi|\epsilon|\|f\|_2,
\]
where \( T^t_{\tau_j} G(z, w) \) is the twisted translation of \( G \) in the \( j \)th variable for \( j = 1, 2 \).

Proof. Notice that \( T_{\tau_j}^t G(z, w) = e^{2\pi i m(z\tau_j)} Z^t(\tau_j f)(z, w) \). Then by using the fact that the twisted Zak transform \( Z^t \) is an unitary operator of \( L^2(\mathbb{C}) \) onto \( L^2(Q[\alpha_0, 1]) \) we get,
\[
\|T_{\tau_j}^t G - G\|_{L^2(Q[\alpha_0, 1])} \leq \|T_{\tau_j}^t G - Z^t(\tau_j f)\|_{L^2(Q[\alpha_0, 1])} + \|Z^t(\tau_j f) - G\|_{L^2(Q[\alpha_0, 1])}
\]
\[
= \left( e^{2\pi i m(z\tau_j)} - 1 \right) Z^t(\tau_j f)\|L^2(Q[\alpha_0, 1]) + \|Z^t(\tau_j f) - Z^t f\|_{L^2(Q[\alpha_0, 1])}
\]
\[
\leq 8\pi|\epsilon|\|f\|_2 + \|\tau_j f - f\|_2
\]
\[
\leq 8\pi|\epsilon|\|f\|_2 + \frac{15}{2}\pi|\epsilon| \left( \|Z\tilde{f}\|_2 + \|Z\tilde{f}\|_2 + \|\tilde{f}\|_2 + (1 + |\epsilon|)^N\|f\|_2 \right),
\]
by Lemma \( 4.1 \) (ii). Observe that
\[
T_{\tau_j}^t G(z, w) = G(z, w - \epsilon)e^{2\pi i m(z\epsilon)} = \sum_{k \in \mathbb{Z}^2} f(z - k)e^{2\pi i m(w-\epsilon)k}e^{2\pi i m(\epsilon)}
\]
\[
= e^{-2\pi i m(z\epsilon)} \sum_{k \in \mathbb{Z}^2} f(z - k)e^{2\pi i m((z-k)\epsilon)}e^{2\pi i m(\epsilon)}
\]
\[
= e^{2\pi i m(w-z)\epsilon} \sum_{k \in \mathbb{Z}^2} h(z - k)e^{2\pi i m(\epsilon)}
\]
\[
= e^{2\pi i m(w-z)\epsilon} Z^t h(z, w),
\]
where \( h(z) = f(z)e^{2\pi i f(z)} \). Therefore
\[
\|T_{\epsilon,2}^{f}G - G\|_{L^2(Q(a_0,1))} \leq \|(2\pi i f(\overline{w-z})\epsilon - 1)Z^{f}h\|_{L^2(Q(a_0,1))} + \|Z^{f}h - Z^{f}f\|_{L^2(Q(a_0,1))}
\]
Choose same \( N_\epsilon \) as in Lemma 4.1 (i) to get \(|(2\pi i f(\overline{w-z})\epsilon - 1)| \leq 2 \leq 2\pi |\epsilon|(1 + |\epsilon|)^N_\epsilon \). Then
\[
\|(2\pi i f(\overline{w-z})\epsilon - 1)Z^{f}h\|_{L^2(Q(a_0,1))} \leq 2\pi |\epsilon|(1 + |\epsilon|)^N_\epsilon \|f\|_2.
\]
Thus
\[
\|T_{\epsilon,2}^{f}G - G\|_{L^2(Q(a_0,1))} \leq 2\pi |\epsilon|(1 + |\epsilon|)^N_\epsilon \|f\|_2 + 8\pi |\epsilon|\|f\|_2.
\]

**Corollary 4.3.** Let \( f, zf, \bar{z}f \in L^2(\mathbb{C}), G = Z^{f}f \). For \( 0 < r < 1 \), fix \( a_0 = (z_0, w_0) \in Q(z_0, r) \times Q(w_0, r) := Q[a_0, r], z_0 \in [-\frac{3}{2}, \frac{3}{2}] \times [-\frac{3}{2}, \frac{3}{2}] \) and \( w_0, \epsilon \in \mathbb{C} \). Then
\[
\int_{Q(a_0, r)} |T_{\epsilon,1}^{f}G(z, w) - G(z, w)|dzdw \leq r^2 |\epsilon|C_1 f(r)
\]
and
\[
\int_{Q(a_0, r)} |T_{\epsilon,2}^{f}G(z, w) - G(z, w)|dzdw \leq r^2 |\epsilon|C_2 f(r),
\]
where \( T_{\epsilon,j}^{f}G(z, w) \) is the twisted translation of \( G \) in the \( j \)th variable with \( r \rightarrow 0 \) \( C_{\epsilon,j} f(r) = 0 \) for \( j = 1, 2 \).

**Proof.** As in the proof of Theorem 4.2 we have
\[
\|T_{\epsilon,1}^{f}G - G\|_{L^2(Q(a_0, r))} \leq \|\left(2\pi i f(\overline{w-z})\epsilon - 1\right)Z^{f}(\tau_{r}f)\|_{L^2(Q(a_0, r))} + \|Z^{f}(\tau_{r}f) - Z^{f}f\|_{L^2(Q(a_0, r))}
\]
\[
= \|\left(2\pi i f(\overline{w-z})\epsilon - 1\right)Z^{f}(\tau_{r}f) \cdot \chi_{Q(z_0, r)}\|_{L^2(Q(a_0,1))} + \|(Z^{f}(\tau_{r}f) - Z^{f}f) \cdot \chi_{Q(z_0, r)}\|_{L^2(Q(a_0,1))}
\]
\[
\leq 8\pi |\epsilon|\|f \cdot \chi_{Q(z_0, r)}\|_2 + \frac{15}{2} \pi |\epsilon| \left(\|Z\bar{\phi} \cdot \chi_{Q(z_0, r)}\|_2 + \|\bar{Z}\bar{\phi} \cdot \chi_{Q(z_0, r)}\|_2\right).
\]
Again
\[
\|T_{\epsilon,2}^{f}G - G\|_{L^2(Q(a_0, r))} \leq \|\left(2\pi i f(\overline{w-z})\epsilon - 1\right)Z^{f}h\|_{L^2(Q(a_0, r))} + \|Z^{f}h - Z^{f}f\|_{L^2(Q(a_0, r))}
\]
\[
= \|\left(2\pi i f(\overline{w-z})\epsilon - 1\right)Z^{f}h \cdot \chi_{Q(z_0, r)}\|_{L^2(Q(a_0,1))} + \|(Z^{f}h - Z^{f}f) \cdot \chi_{Q(z_0, r)}\|_{L^2(Q(a_0,1))}
\]
\[
\leq 8\pi |\epsilon|(1 + |\epsilon|)^N_\epsilon \|f \cdot \chi_{Q(z_0, r)}\|_2 + 8\pi |\epsilon|\|f \cdot \chi_{Q(z_0, r)}\|_2.
\]
Applying Cauchy-Schwartz inequality in the left hand side of (4.1) and (4.2) the proof follows immediately, where

\[ C_{\epsilon,1,f}(r) = 8\pi \| f \cdot \chi_{Q(z_0,r)} \|_2 + \frac{15}{2} \pi \left( \| \tilde{Zf} \cdot \chi_{Q(z_0,r)} \|_2 + \| \bar{\tilde{Zf}} \cdot \chi_{Q(z_0,r)} \|_2 \right) \]

and

\[ C_{\epsilon,2,f}(r) = 8\pi (1 + |\epsilon|) N \| f \cdot \chi_{Q(z_0,r)} \|_2 + 8\pi \| f \cdot \chi_{Q(z_0,r)} \|_2. \]

Further, using the fact that \( \| f \cdot \chi_{Q(z_0,r)} \|_2 \to 0 \) as \( r \to 0 \), we have \( \lim_{r \to 0} C_{\epsilon,j,f}(r) = 0 \) for \( j = 1, 2 \).

Now we are in a position to prove Theorem 1.4. Motivated by the proof of BLT on \( L^2(\mathbb{R}) \) (Coifman and Semmes [9], Benedetto et al. [7] etc.), we obtain the following proof of BLT on \( L^2(\mathbb{C}) \).

**Proof of Theorem 1.4:** Assume that \( \{ T_{(m,n)}^{t}g \} \) is an exact frame for \( L^2(\mathbb{C}) \). Then by Lemma 3.4 (v) we have

\[ 0 < A^{1/2} \leq |Z^t g| \leq B^{1/2} < \infty \text{ a.e.} \]

Assume that both \( Zg \) and \( \tilde{Z}g \in L^2(\mathbb{C}) \). We will show our assumption together with (4.3) leads to a contradiction in the following three steps.

**Step 1:** (Construction of an continuous averaged function \( G_r(z,w) \) that approximating \( G(z,w) = Z^t g(z,w) \).) Let \( \rho(z,w) = \chi_{[0,1]^4}(z,w) \) and for \( r > 0 \), let \( \rho_r(z,w) = \frac{1}{r^4} \rho \left( \frac{z}{r}, \frac{w}{r} \right) \).

Define

\[ G_r(z,w) = G \times \rho_r(z,w) = \int_{[0,1]^4} G(z-z',w-w') \rho_r(z',w') e^{-2\pi i m(z\bar{z'} + w\bar{w'})} \, dz' dw'. \]

Then \( G_r \) satisfies the following properties:

(a) \( |G_r(z_1,w_1) - G_r(z_2,w_2)| \leq 2 \left( \pi (r + \max\{|z_1|,|w_1|\}) + \frac{1}{r} \right) B^{1/2} (|z_1 - z_2| + |w_1 - w_2|) \).
Similarly we can obtain the other identities with 
\[ |G_r(z_1, w_1) - G_r(z_2, w_2)| \]
\[ = \frac{1}{4} \left| \int_{Q[z_1^*, w_1^*]} G(u, v)e^{2\pi i m(z_1 u + w_1 v)} \, du \right| \]
\[ \leq \frac{1}{4} \left| \int_{Q[z_2^*, w_2^*]} G(u, v)e^{2\pi i m(z_2 u + w_2 v)} \, du \right| \]
\[ + \frac{1}{4} \int_{Q[z_1^*, w_1^*]} G(u, v)e^{2\pi i m(z_2 u + w_2 v)} \, du \]
\[ \leq \frac{2\pi}{r} B^{\frac{1}{2}} \left( |z_1 - z_2| + |w_1 - w_2| \right) \Delta Q[z_1^*, w_1^*; r] \]
\[ = 2 \left( \pi + \max \{|z_1|, |w_1|\} \right) \Delta Q[z_1^*, w_1^*; r] \]
\[ \leq 2 \left( \pi + \max \{|z_1|, |w_1|\} \right) \Delta Q[z_1^*, w_1^*; r] \]

where \( \Delta \) is the symmetric difference operator and \( Q[z_j^*, w_j^*; r] = Q[z_j - \frac{r}{2}(1+i), w_j - \frac{r}{2}(1+i); r] \), \( j=1,2 \).

(b) (i) \( G_r(z, w + 1) = G_r(z, w) + \psi_1(r, z, w) \) and \( G_r(z, w + i) = G_r(z, w) + \psi_2(r, z, w) \),

(ii) \( G_r(z + 1, w) = e^{2\pi i m(w)} G_r(z, w) + \psi_3(r, z, w) \) and \( G_r(z + i, w) = e^{-2\pi i m(iw)} G_r(z, w) + \psi_4(r, z, w) \), where \( |\psi_{j,r}(z, w)| \leq 2\pi B^{1/2}r, \ j = 1, 2, 3, 4. \)

\[ G_r(z, w + 1) = \int_{[0, 1]^4} G(z - z', w + 1 - w') \rho_r(z', w') e^{-2\pi i m(zz' + w1')} \, dz' \, dw' \]
\[ = \int_{[0, 1]^4} G(z - z', w - w') \rho_r(z', w') e^{-2\pi i m(zz' + w1')} \, dz' \, dw' + \psi_1(r, z, w) \]
\[ = G_r(z, w) + \psi_1(r, z, w). \]

where \( \psi_1(r, z, w) = \int_{[0, 1]^4} \left( e^{-2\pi i m(w')} - 1 \right) G(z - z', w - w') \rho_r(z', w') e^{-2\pi i m(zz' + w1')} \, dz' \, dw'. \)

Further
\[ |\psi_1(r, z, w)| \]
\[ \leq B^{1/2} \int_{[0, 1]^4} \left| 2\pi i m(w') \rho_r(z', w') \right| \, dz' \, dw' \]
\[ \leq 2\pi B^{1/2}r. \]

Similarly we can obtain the other identities with \( |\psi_{j,r}(z, w)| \leq 2\pi B^{1/2}r, \ j = 2, 3, 4. \)
(c) Fix \((z, w), (z', w') \in \mathbb{C}^2\) and using (a) one has
\[
|G(z, w) - G_r(z, w)| \geq |G(z, w)| - |G_r(z, w) - G_r(z', w')| - |G_r(z', w')|
\]
\[
\geq A^\frac{1}{2} - 2B^\frac{1}{2} \left( \pi (r + \max \{|z|, |w|\}) + \frac{1}{r} \right) (|z - z'| + |w - w'|) - |G_r(z', w')|.
\]
In particular for fixed \((z, w) \in [0,1]^4, c < 1\) and \((z, w) \in Q[z', w'; cr]\) we have
\[
|G(z, w) - G_r(z, w)| \geq A^\frac{1}{2} - 2cr \left( \pi (cr + \max \{|z|, |w|\}) + \frac{1}{cr} \right) B^\frac{1}{2} - |G_r(z', w')|.
\]

**Step 2:** For any \((z_0, w_0) \in [0,1]^4, c < 1\) and \(r < 1\) we have
\[
c^4r^4(A^\frac{1}{2} - 2crz_{z,w}B^\frac{1}{2} - |G_r(z', w')|) \leq \int_{Q[z, w; cr]} |G(z, w) - G_r(z, w)| dzdw \leq c^2r^4C(r),
\]
where \(c_{z,w} = (\pi (cr + \max \{|z|, |w|\}) + \frac{1}{cr})\) and \(C(r)\) is independent on the point \((z, w)\) and
\[
\lim_{r \to 0} C(r) = 0.
\]
Using Corollary \ref{cor:4.3} and the fact that \(|z'| < 1, |w'| < 1\), we have \(|C_{1,g}^2(r)| < C_{1,g}(r)\) and \(|C_{2,g}^2(r)| < C_{2,g}(r)\), where
\[
C_{1,g}(r) = 8\pi \|g \cdot \chi_{Q(z_0, r)}\|_2 + \frac{15}{2} \pi \left( \|Z\tilde{g} \cdot \chi_{Q(z_0, r)}\|_2 + \|\bar{Z}\tilde{g} \cdot \chi_{Q(z_0, r)}\|_2 \right)
+ \|\tilde{g} \cdot \chi_{Q(z_0, r)}\|_2 + 2\|g \cdot \chi_{Q(z_0, r)}\|_2
\]
and
\[
C_{2,g}(r) = 16\pi \|g \cdot \chi_{Q(z_0, r)}\|_2 + 8\pi \|g \cdot \chi_{Q(z_0, r)}\|_2.
\]
Putting \(C(r) = C_{1,g}(r) + C_{2,g}(r)\) we get \ref{eq:4.3}. Then the inequality \ref{eq:4.5} can be obtained by \ref{eq:4.4} and applying Cauchy-Schwartz inequality in the last term of the above calculation.

**Step 3:** Claim: \(\inf_{(z, w) \in [0,1]^4} |G(z, w)| = 0\).
From (1.5) we get \( |G_r(z', w')| \geq A^+ - 2cr_c^\varepsilon z_u B^+ - \frac{C(r)}{c^2} \). Choose \( c < 1 \) such that \( A^+ - 2cr_c^\varepsilon z_u B^+ > \frac{4c^2}{\pi} \) and letting \( r \to 0 \) we get \( |G_r(z, w)| \geq \frac{4c^2}{\pi} \). Since \( G_r(z, w) \) is continuous real valued function on \([0, 1]^4\) (see [21], pp. 377-385), there exists a continuous real valued function \( \theta_r \) such that \( G_r(z, w) = |G_r(z, w)|e^{i\theta_r(z, w)} \). Define
\[
\delta_{1, r}(z, w) = 1 + \frac{\psi_{1, r}(z, w)}{G_r(z, w)}, \\
\delta_{2, r}(z, w) = 1 + \frac{\psi_{2, r}(z, w)}{G_r(z, w)}, \\
\delta_{3, r}(z, w) = 1 + \frac{\psi_{3, r}(z, w)}{e^{2\pi i \text{Im}(w)}G_r(z, w)}, \\
\delta_{4, r}(z, w) = 1 + \frac{\psi_{4, r}(z, w)}{e^{-2\pi i \text{Im}(w)}G_r(z, w)}.
\]
Clearly \( \delta_{j, r} \) is continuous and non-vanishing on \([0, 1]^4\) for each \( r > 0 \) and every \( j = 1, 2, 3, 4 \). Therefore there exists a continuous real valued function \( \theta_{j, r} \) such that \( \delta_{j, r}(z, w) = |\delta_{j, r}(z, w)|e^{i\theta_{j, r}(z, w)} \) for \( j = 1, 2, 3, 4 \). Since
\[
G_r(z, w + 1) = G_r(z, w)\delta_{1, r}(z, w), \\
G_r(z, w + i) = G_r(z, w)\delta_{2, r}(z, w), \\
G_r(z + 1, w) = e^{2\pi i \text{Im}(w)}G_r(z, w)\delta_{3, r}(z, w), \\
G_r(z + i, w) = e^{-2\pi i \text{Im}(w)}G_r(z, w)\delta_{4, r}(z, w),
\]
for each \( r > 0 \) and for all \( z, w \in [0, 1] \times [0, 1] \), there are integers \( I_r, J_r, K_r \) and \( L_r \) such that
\[
\theta_r(z, 1) = \theta_r(z, 0) + \theta_{1, r}(z, 0) + 2\pi I_r, \\
\theta_r(z, i) = \theta_r(z, 0) + \theta_{2, r}(z, 0) + 2\pi J_r, \\
\theta_r(1, w) = 2\pi I \text{m}(w) + \theta_r(0, w) + \theta_{3, r}(0, w) + 2\pi K_r, \\
\theta_r(i, w) = -2\pi I \text{m}(iw) + \theta_r(0, w) + \theta_{4, r}(0, w) + 2\pi L_r.
\]
Now
\[
0 = |\theta_r(0, 1) - \theta_r(0, i)| + |\theta_r(0, i) - \theta_r(i, i)| + |\theta_r(i, i) - \theta_r(i, 1)| + |\theta_r(i, 1) - \theta_r(0, 1)| \\
= |\theta_{1, r}(0, 0) - \theta_{2, r}(0, 0) + 2\pi (I_r - J_r)| + [-\theta_{4, r}(0, i) - 2\pi L_r] \\
+ |\theta_{2, r}(i, 0) - \theta_{1, r}(i, 0) + 2\pi (J_r - I_r)| + [-2\pi + \theta_{4, r}(0, 1) + 2\pi L_r] \\
= |\theta_{1, r}(0, 0) - \theta_{2, r}(0, 0) - \theta_{4, r}(0, i) + \theta_{2, r}(i, 0) - \theta_{1, r}(i, 0) - 2\pi + \theta_{4, r}(0, 1)|.
\]
Letting \( r \to 0 \) we get \( 0 = -2\pi \), a contradiction. \( \square \)
5. Uncertainty Principle approach to BLT

Motivated by the proofs of BLT for orthonormal basis and Riesz basis (see [4, 10]), we prove the analogue of BLT on $L^2(\mathbb{C})$. We start with the proof of Theorem 1.3, which is a variation of Heisenberg uncertainty inequalities for $L^2(\mathbb{C})$.

Proof of Theorem 1.3: Let $f \in L^2(\mathbb{C})$. Then $f(z) = \sum_{m,n=0}^{\infty} \langle f, \phi_{m,n} \rangle \phi_{m,n}(z)$. Using the properties of the operators $Z$ and $\bar{Z}$ we get

$$\|Zf\|^2_2 + \|\bar{Z}f\|^2_2 = \sum_{m,n=0}^{\infty} (4n + 2) |\langle f, \phi_{m,n} \rangle|^2 \geq 2 \sum_{m,n=0}^{\infty} |\langle f, \phi_{m,n} \rangle|^2 = 2 \|f\|^2_2.$$

Using the fact that $Z\phi_{m,0} = 0$ for $m = 0, 1, 2 \ldots$ we conclude that equality holds in the above inequality if and only if $n = 0$ i.e. $f = \sum_{m=0}^{\infty} c_m \phi_{m,0}$.

5.1. The Weak BLT. Now we prove the weaker version of BLT. We start with the following lemma.

Lemma 5.1. The operator $T^t_{(m,n)}$ commutes with $Z$ and $\bar{Z}$ i.e. $T^t_{(m,n)}Z = ZT^t_{(m,n)}$ and $T^t_{(m,n)}\bar{Z} = \bar{Z}T^t_{(m,n)}$.

Proof. Enough to show that the commutators $[Z, T^t_{(m,n)}] = [\bar{Z}, T^t_{(m,n)}] = 0$. For a Schwartz class function $f$ on $\mathbb{C}$ we have

$$\langle [\bar{Z}, T^t_{(m,n)}] f, f \rangle = \langle \bar{Z}T^t_{(m,n)} f - T^t_{(m,n)} \bar{Z} f, f \rangle$$

$$= \langle \bar{Z}T^t_{(m,n)} f, f \rangle - \langle T^t_{(m,n)} \bar{Z} f, f \rangle$$

$$= -\langle f, T^t_{(-m,-n)} Z f \rangle + \langle f, ZT^t_{(-m,-n)} f \rangle$$

$$= -\langle W(f), i\pi(-m, -n)W(f)A \rangle + \langle W(f), i\pi(-m, -n)W(f)A \rangle$$

$$= 0.$$

Similarly we can show that $[Z, T^t_{(m,n)}] = 0$.

Theorem 5.2. Assume $g \in L^2(\mathbb{C})$ is such that $\{g_{m,n}\}$ is an exact twisted Gabor frame for $L^2(\mathbb{C})$ and $\bar{g}$ be the dual function. Then we cannot have all of $Zg, Z\bar{g}, \bar{Z}g, \bar{Z}\bar{g} \in L^2(\mathbb{C})$, i.e., we must have

$$\|Zg\|_2 \|Z\bar{g}\|_2 \|\bar{Z}g\|_2 \|\bar{Z}\bar{g}\|_2 = +\infty.$$
Proof. Assume that $Zg, Z\tilde{g}, \tilde{Z}g, \tilde{Z}\tilde{g} \in L^2(\mathbb{C})$. Since $\{g_{m,n}\}$ is a twisted Gabor frame for $L^2(\mathbb{C})$, any $f \in L^2(\mathbb{C})$ can be expressed as $f = \sum_{m,n} \langle f, g_{m,n} \rangle \tilde{g}_{m,n} = \sum_{m,n} \langle f, \tilde{g}_{m,n} \rangle g_{m,n}$. Using Lemma 3.4 we get

$$\langle Zg, Z\tilde{g} \rangle = \sum_{m,n} \langle Zg, \tilde{g}_{m,n} \rangle \langle g_{m,n}, Z\tilde{g} \rangle$$

$$= \sum_{m,n} \langle g_{m,-n}, Z\tilde{g} \rangle \langle Zg, \tilde{g}_{m,-n} \rangle$$

$$= \sum_{m,n} \langle Z\tilde{g}, \tilde{g}_{m,n} \rangle \langle g_{m,n}, \tilde{Z}g \rangle$$

$$= \langle Zg, Z\tilde{g} \rangle.$$

(5.1)

Therefore, bi-orthogonality relation (Proposition 5.4.8 of [8], pp. 101) and the above equality gives

$$1 = \langle g, \tilde{g} \rangle = -\frac{1}{2} \langle (g, [Z, Z]\tilde{g}) \rangle = -\frac{1}{2} \langle (Zg, Z\tilde{g}) - \langle Zg, Z\tilde{g} \rangle \rangle = 0,$$

a contradiction. \qed

Remark 5.3. If the twisted Gabor frame $\{g_{m,n}\}$ forms an orthonormal basis then $g = \tilde{g}$ and the above theorem is precisely analogue of Battle’s proof of BLT in [4]. The BLT will follow from the weak BLT if $Zg \in L^2(\mathbb{C}) \iff Z\tilde{g} \in L^2(\mathbb{C})$ and $Zg \in L^2(\mathbb{C}) \iff Z\tilde{g} \in L^2(\mathbb{C})$. However we show that the BLT and the weak BLT are actually equivalent.

Proposition 5.4. If $g \in L^2(\mathbb{C})$ and $\{g_{m,n}\}$ is an exact twisted Gabor frame for $L^2(\mathbb{C})$, then there is a unique $\tilde{g} \in L^2(\mathbb{C})$ such that $Z^t\tilde{g} = 1/Z\tilde{g}$.

Proof. Let $h = Z^t \left( \frac{1}{Zg} \right)$. By Lemma 3.4 (v), $h$ is well defined and $h \in L^2(\mathbb{C})$. Let $z = x+iy$ and $w = r+is \in \mathbb{C}$. Using Lemma 3.4 (ii) and bi-orthogonality relation (Proposition 5.4.8 of [8], pp. 101) we have

$$\langle h, g_{m,n} \rangle = \langle Z^t h, Z^t g_{m,n} \rangle = \int_{Q \times Q} \frac{1}{Z^t g(z,w)} e^{-2\pi i(z_1 x - y_1 m)} e^{-2\pi i(z_2 s - t_2 m)} Z^t g(z,w) \, dz \, dw$$

$$= \delta_{m,0} \delta_{n,0} = \langle \tilde{g}, g_{m,n} \rangle, \quad \forall \ m, n \in \mathbb{Z}.$$

Since $\{g_{m,n}\}$ is complete in $L^2(\mathbb{C})$ and $h, \tilde{g} \in L^2(\mathbb{C})$, it follows that $h = \tilde{g}$. \qed

Theorem 5.5. If $g \in L^2(\mathbb{C})$ and $\{g_{m,n}\}$ is an exact twisted Gabor frame for $L^2(\mathbb{C})$, then

$\tilde{Z}g \in L^2(\mathbb{C}) \iff Z\tilde{g} \in L^2(\mathbb{C})$ and $Zg \in L^2(\mathbb{C}) \iff Z\tilde{g} \in L^2(\mathbb{C})$. 

Proof. Assume that \( Zg \in L^2(\mathbb{C}) \). Then
\[
Z^t(Zg)(z, w) = \sum_k Zg(z - k)e^{2\pi i \text{Im}(w,k)}
\]
\[
= \sum_k \left( \frac{d}{dz} + \frac{1}{2z} \right) g(z - k)e^{2\pi i \text{Im}(w,k)}
\]
(5.2)
\[
= \partial_z(Z^t g)(z, w) + \frac{1}{2} \overline{z}(Z^t g)(z, w) - \frac{1}{2\pi} \partial_w(Z^t g)(z, w).
\]
Similarly,
(5.3)
\[
Z^t(\overline{Z}g)(z, w) = \partial_z(Z^t g)(z, w) - \frac{1}{2} \overline{z}(Z^t g)(z, w) - \frac{1}{2\pi} \partial_w(Z^t g)(z, w).
\]
Now using Proposition 5.4, we compute
\[
\overline{Z^t(Z\overline{g})}(z, w) = \frac{\partial_z(Z^t \overline{g})(z, w) + \frac{1}{2} \overline{z}(Z^t \overline{g})(z, w) - \frac{1}{2\pi} \partial_w(Z^t \overline{g})(z, w)}{\overline{(Z^t g)^2}(z, w)}
\]
(5.4)
Thus it follows that \( \overline{Zg} \in L^2(\mathbb{C}) \Leftrightarrow Z\overline{g} \in L^2(\mathbb{C}) \) provided all the calculations are justified in distribution point of view. Similarly the other equivalent relation can be obtained. \( \square \)

Remark 5.6. (1) The functions \( L^\frac{1}{2}g \) and \( L^\frac{1}{2}\overline{g} \) cannot both be in \( L^2(\mathbb{C}) \): If \( L^\frac{1}{2}g \in L^2(\mathbb{C}) \), then
\[
\|L^\frac{1}{2}g\|^2 = \langle L^\frac{1}{2}g, L^\frac{1}{2}g \rangle = \langle g, Lg \rangle = \frac{1}{2}(\|Zg\|^2 + \|Z\overline{g}\|^2).
\]
Therefore \( L^\frac{1}{2}g \in L^2(\mathbb{C}) \Leftrightarrow Zg, \overline{Zg} \in L^2(\mathbb{C}) \). Now if \( L^\frac{1}{2}g \) and \( L^\frac{1}{2}\overline{g} \) \( L^2(\mathbb{C}) \) then \( Zg, \overline{Zg}, \overline{Zg} \in L^2(\mathbb{C}) \), contradicting the Theorem 5.2.

(2) The functions \( Z\overline{Z}g \) and \( \overline{Z}Zg \) cannot both be in \( L^2(\mathbb{C}) \): If \( Z\overline{Z}g, \overline{Z}Zg \in L^2(\mathbb{C}) \), then
\( Lg \in L^2(\mathbb{C}) \) and \( L^\frac{1}{2}g = L^{-\frac{1}{2}}(Lg) \in L^2(\mathbb{C}) \). This would imply \( Zg, \overline{Zg} \in L^2(\mathbb{C}) \), a contradiction to Theorem 1.4.

(3) The functions \( Lg \) and \( g \) cannot both be in \( L^2(\mathbb{C}) \).

(4) Consider the operators \( R \) and \( \overline{R} \) (Riesz transforms) defined by \( Rg = ZL^{-\frac{1}{2}}g \) and \( \overline{R}g = \overline{Z}L^{-\frac{1}{2}}g \). Then the functions \( \overline{R}g \) and \( Z\overline{R}g \) cannot both be in \( L^2(\mathbb{C}) \): If \( \overline{R}g, Z\overline{R}g \in L^2(\mathbb{C}) \), then
\( L^\frac{1}{2}g = -(\frac{1}{4}(Z\overline{Z} + Z\overline{Z})L^{-\frac{1}{2}}g = -(\frac{1}{4}(\overline{Z}R + Z\overline{R})g \in L^2(\mathbb{C}) \), leading to a contradiction.
ACKNOWLEDGMENTS

The first author wishes to thank the Ministry of Human Resource Development, India for the research fellowship and Indian Institute of Technology Guwahati, India for the support provided during the period of this work.

REFERENCES

1. G. Ascensi, H.G. Feichtinger, and N. Kaiblinger, Dilation of the Weyl symbol and Balian–Low theorem, Trans. Amer. Math. Soc. 366 (2014), no. 7, 3865–3880.
2. R. Balan, Extensions of no-go theorems to many signal systems, in: A. Aldroubi, E.B. Lin (Eds.), Wavelets, Multitawavelets and Their Applications, in: Contemp. Math. 216, American Mathematical Society, Providence, RI, 1998, pp. 3–14.
3. R. Balan, Un principe d’incertitude fort en théorie du signal ou en mécanique quantique, C.R. Acad. Sci. Paris 292 (1981), no. 2, 1357–1362.
4. G. Battle, Heisenberg proof of the Balian–Low theorem, Lett. Math. Phys. 15 (1988), no. 2, 175–177.
5. J.J. Benedetto, W. Czaja, and A. Ya. Maltsev, The Balian–Low theorem for the symplectic form on \( \mathbb{R}^{2d} \), J. Math. Phys. 44 (2003), no. 4, 1735–1750.
6. J.J. Benedetto, W. Czaja, A.M. Powell, and J. Sterbenz, An endpoint \((1, \infty)\) Balian-Low theorem, Math. Res. Lett. 13 (2006), no. 3, 467–474.
7. J.J. Benedetto, C. Heil, and D. F. Walnut, Differentiation and the Balian-Low theorem, J. Fourier Anal. Appl. 1 (1995), no. 4, 355–402.
8. O. Christensen, An introduction to frames and Riesz bases, Birkhäuser, Boston, 2003.
9. I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, IEEE Trans. Inform. Theory 36 (1990), no. 5, 961–1005.
10. I. Daubechies and A. J. E. M. Janssen, Two theorems on lattice expansions, IEEE Trans. Inform. Theory 39 (1993), no. 1, 3–6.
11. H.G. Feichtinger and K. Gröchenig, Gabor frames and time-frequency analysis of distributions, J. Funct. Anal. 146 (1997), no. 2, 464–495.
12. S.Z. Gautam, A critical-exponent Balian–Low theorem, Math. Res. Lett. 15 (2008), no. 3, 471–483.
13. K. Gröchenig, Aspects of Gabor analysis on locally compact abelian groups, in: H.G. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms: Theory and Applications, Birkhäuser, Boston, 1998, pp. 211–231.
14. K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.
15. K. Gröchenig, D. Han, C. Heil, and G. Kutyniok, The Balian–Low theorem for symplectic lattices in higher dimensions, Appl. Comput. Harmon. Anal. 13 (2002), no. 2, 169–176.
16. C. Heil and A.M. Powell, Gabor Schauder bases and the Balian-Low theorem, J. Math. Phys. 47 (2006), no. 11, 113506.
17. A. J. E. M. Janssen, The Zak transform: A signal transform for sampled time-continuous signals, Philips J. Res. 43 (1988), no. 1, 23–69.
18. H. J. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, Acta. Math. 117 (1967), 37–52.
19. H. J. Landau, On density of phase space expansions, IEEE Trans. Inform. Theory 39 (1993), 1152–1156.
20. F. Low, Complete sets of wave packets, A Passion for Physics–Essays in Honor of Geoofrey Chew (C. DeTar et al., eds.), World Scientific (1985), 17–22.
21. S. Nitzan and J.F. Olsen, Frames and Riesz bases twisted shift-invariant spaces in \( L^2(\mathbb{R}^{2n}) \), J. Math. Anal. Appl. 434 (2016), no. 2, 1442–1461.
22. T. Rado and P. Reichelderfer, Continuous Transformations in Analysis (1955), Springer-Verlag, New York.
23. J. Ramanathan and T. Steger, Incompleteness of sparse coherent states, Appl. Comp. Harm. Anal. 2 (1995), 148–153.
24. M. Rieffel, Von Neumann algebras associated with pair of lattice in Lie groups, Math. Ann. 257 (1981), 403–418.
25. S. Thangavelu, Lectures on Hermite and Laguerre expansions, vol. 42, Princeton University Press, 1993.
26. S. Thangavelu, Harmonic Analysis on the Heisenberg Group, Birkhäuser, Boston, 1997.
29. R. Tinaztepe and C. Heil, *Modulation spaces, BMO, and the Balian–Low theorem*, Sampl. Theory Signal Image Process. **11** (2012), no. 1, 25–41.
30. M. Zibulski and Y.Y. Zeevi, *Analysis of multiwindow Gabor-type schemes by frame methods*, Appl. Comput. Harmon. Anal. **4** (1997), no. 2, 188–221.

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781039, India.

E-mail address: anirudhamath@gmail.com, jitumath@iitg.ernet.in