Controllability for Sobolev type fractional integro-differential systems in a Banach space

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Abstract
In this paper, by using compact semigroups and the Schauder fixed-point theorem, we study the sufficient conditions for controllability of Sobolev type fractional integro-differential systems in a Banach space. An example is provided to illustrate the obtained results.

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1 Introduction
A Sobolev-type equation appears in a variety of physical problems such as flow of fluids through fissured rocks, thermodynamics and propagation of long waves of small amplitude (see [1–3]). Recently, there has been an increasing interest in studying the problem of controllability of Sobolev type integro-differential systems. Balachandran and Dauer [4] studied the controllability of Sobolev type integro-differential systems in Banach spaces. Balachandran and Sakthivel [5] studied the controllability of Sobolev type semilinear integro-differential systems in Banach spaces. Balachandran, Anandhi and Dauer [6] studied the boundary controllability of Sobolev type abstract nonlinear integro-differential systems.

In this paper, we study the controllability of Sobolev type fractional integro-differential systems in Banach spaces in the following form:

\[ \mathcal{D}^{\alpha}(E x(t)) + A x(t) = B u(t) + f(t, x(t)) + \int_0^t g(t, s, x(s), \int_0^s H(s, \tau, x(\tau)) \, d\tau) \, ds, \]

where \( E \) and \( A \) are linear operators with domain contained in a Banach space \( X \) and ranges contained in a Banach space \( Y \). The control function \( u(\cdot) \) is in \( L^2(J, U) \), a Banach space of admissible control functions, with \( U \) as a Banach space. \( B \) is a bounded linear operator from \( U \) into \( Y \). The nonlinear operators \( f \in C(J \times X, Y) \), \( H \in C(J \times J \times X, X) \) and \( g \in C(J \times J \times X, Y) \) are all uniformly bounded continuous operators. The fractional derivative \( \mathcal{D}^{\alpha} \), \( 0 < \alpha < 1 \) is understood in the Caputo sense.

2 Preliminaries
In this section, we introduce preliminary facts which are used throughout this paper.
**Definition 2.1** (see [7–9]) The fractional integral of order $\alpha > 0$ with the lower limit zero for a function $f$ can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma$ is the gamma function.

**Definition 2.2** (see [7–9]) The Caputo derivative of order $\alpha$ with the lower limit zero for a function $f$ can be written as

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{n-1-\alpha}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, 0 \leq n - 1 < \alpha < n.$$

If $f$ is an abstract function with values in $X$, then the integrals appearing in the above definitions are taken in Bochner’s sense.

The operators $A : D(A) \subset X \rightarrow Y$ and $E : D(E) \subset X \rightarrow Y$ satisfy the following hypotheses:

$(H_1)$ $A$ and $E$ are closed linear operators,

$(H_2)$ $D(E) \subset D(A)$ and $E$ is bijective,

$(H_3)$ $E^{-1} : Y \rightarrow D(E)$ is continuous.

The hypotheses $H_1$, $H_2$ and the closed graph theorem imply the boundedness of the linear operator $AE^{-1} : Y \rightarrow Y$.

$(H_4)$ For each $t \in [0, a]$ and for some $\lambda \in \rho(-AE^{-1})$, the resolvent set of $-AE^{-1}$, the resolvent $R(\lambda, -AE^{-1})$ is a compact operator.

**Lemma 2.1** [10] Let $S(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda; A)$ of $A$ is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.

From the above fact, $-AE^{-1}$ generates a compact semigroup $\{T(t), t \geq 0\}$ in $Y$, which means that there exists $M > 1$ such that

$$\max_{t \in J} \|T(t)\| \leq M. \quad (2.1)$$

**Definition 2.3** The system (1.1) is said to be controllable on the interval $J$ if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1.1) satisfies $x(a) = x_1$.

$(H_5)$ The linear operator $W$ from $U$ into $X$ defined by

$$Wu = \int_0^a E^{-1}(a-s)^{\alpha-1}T_\alpha(a-s)Bu(s) ds$$

has an inverse bounded operator $W^{-1}$ which takes values in $L^2(J, U)/\ker W$, where the kernel space of $W$ is defined by $\ker W = \{x \in L^2(J, U) : Wx = 0\}$, $B$ is a bounded linear operator and $T_\alpha(t)$ is defined later.

$(H_6)$ The function $f$ satisfies the following two conditions:
(i) For each $t \in J$, the function $f(t, \cdot) : X \rightarrow Y$ is continuous, and for each $x \in X$, the function $f(\cdot, x) : J \rightarrow Y$ is strongly measurable.

(ii) For each positive number $k \in \mathbb{N}$, there is a positive function $g_k(\cdot) : [0, a] \rightarrow \mathbb{R}^+$ such that

$$\sup_{|t| \leq k} |f(t, x)| \leq g_k(t),$$

the function $s \rightarrow (t - s)^{-\alpha}g_k(s) \in L^1([0, t], \mathbb{R}^+)$, and there exists a $\beta > 0$ such that

$$\lim_{k \rightarrow \infty} \inf_{t \in [0, a]} \frac{\int_0^t (t - s)^{-\alpha}g_k(s) ds}{k} = \beta < \infty,$$

$(H_7)$ For each $(t, s) \in J \times J$, the function $H(t, s, \cdot) : X \rightarrow X$ is continuous, and for each $x \in X$, the function $H(\cdot, \cdot, x) : J \times J \rightarrow X$ is strongly measurable.

$(H_8)$ The function $g$ satisfies the following two conditions:

(i) For each $(t, s, x) \in J \times J \times X$, the function $g(t, s, \cdot, \cdot) : X \times X \rightarrow Y$ is continuous, and for each $x \in X$, the function $g(\cdot, s, \cdot, y) : J \times J \rightarrow Y$ is strongly measurable.

(ii) For each positive number $k \in \mathbb{N}$, there is a positive function $h_k(\cdot) : [0, a] \rightarrow \mathbb{R}^+$ such that

$$\sup_{|t| \leq k} \left| \int_0^t g(t, s, x, \int_0^s H(s, \tau, x) d\tau) ds \right| \leq h_k(t),$$

the function $s \rightarrow (t - s)^{-\alpha}h_k(s) \in L^1([0, t], \mathbb{R}^+)$, and there exists a $\gamma > 0$ such that

$$\lim_{k \rightarrow \infty} \inf_{t \in [0, a]} \frac{\int_0^t (t - s)^{-\alpha}h_k(s) ds}{k} = \gamma < \infty.$$ 

According to [11, 12], a solution of equation (1.1) can be represented by

$$x(t) = E^{-1}S_a(t)Ex_0 + \int_0^t (t - s)^{-\alpha - 1}T_a(t - s)E^{-1}f(s, x(s)) ds$$

$$+ \int_0^t (t - s)^{-\alpha - 1}T_a(t - s)Bu(s) ds$$

$$+ \int_0^t (t - s)^{-\alpha - 1}T_a(t - s) \left\{ \int_0^s g(s, \tau, x(\tau), R(\tau)) d\tau \right\} ds, \quad t \in J, \quad (2.2)$$

where

$$R(\tau) = \int_0^\tau H(\tau, \eta, x(\eta)) d\eta, \quad S_a(t)x = \int_0^\infty \xi_a(\theta)T(t^a \theta)x d\theta, \quad T_a(t)x = \alpha \int_0^\infty \theta \xi_a(\theta) T(t^a \theta)x d\theta$$

with $\xi_a$ being a probability density function defined on $(0, \infty)$, that is, $\xi_a(\theta) \geq 0, \theta \in (0, \infty)$ and $\int_0^\infty \xi_a(\theta) d\theta = 1$.

**Remark** $\int_0^\infty \theta \xi_a(\theta) d\theta = \frac{1}{t_0(t_{2a})}$.
**Definition 2.4** By a mild solution of the problem (1.1), we mean that the function \( x \in C(J, X) \) satisfies the integral equation (2.2).

**Lemma 2.2** (see [11]) The operators \( S_\alpha(t) \) and \( T_\alpha(t) \) have the following properties:

1. For any fixed \( x \in X \), \( \| S_\alpha(t)x \| \leq M \| x \| \), \( \| T_\alpha(t)x \| \leq \frac{aM}{T(\alpha+1)} \| x \| \);
2. \{\( S_\alpha(t), t \geq 0 \)\} and \{\( T_\alpha(t), t \geq 0 \)\} are strongly continuous;
3. For every \( t > 0 \), \( S_\alpha(t) \) and \( T_\alpha(t) \) are also compact operators if \( T(t), t > 0 \) is compact.

**3 Controllability result**

In this section, we present and prove our main result.

**Theorem 3.1** If the assumptions (H1)-(H6) are satisfied, then the system (1.1) is controllable on \( J \) provided that \( aM\Gamma(\alpha+1)(\beta + \gamma)\left[1 + \frac{aM\Gamma(\alpha+1)}{T(\alpha+1)}\| B \| \| W^{-1} \| \right] < 1 \).

**Proof** Using the assumption (H5), for an arbitrary function \( x(\cdot) \), define the control

\[
 u(t) = W^{-1}
 x_1 - E^{-1}S_\alpha(t)Ex_0 - \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)f(s,x(s))
 \]

\[
 - \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)\left\{\int_0^s g(s,\tau, x(\tau), R(\tau))
 \right\} d\tau
 \] \( (t) \).

It shall now be shown that when using this control, the operator \( Q \) defined by

\[
 (Qx)(t) = E^{-1}S_\alpha(t)Ex_0 + \int_0^t (t-s)^{\alpha-1}E^{-1}T_\alpha(t-s)f(s,x(s))
 \]

\[
 + \int_0^t (t-s)^{\alpha-1}E^{-1}T_\alpha(t-s)Bu(s)
 \]

\[
 + \int_0^t (t-s)^{\alpha-1}E^{-1}T_\alpha(t-s)\left\{\int_0^s g(s,\tau, x(\tau), R(\tau))
 \right\} d\tau
 \] \( (t) \)

from \( C(J, X) \) into itself for each \( x \in C = C(J, X) \) has a fixed point. This fixed point is then a solution of equation (2.2).

\[
 (Qx)(a) = E^{-1}S_\alpha(a)Ex_0 + \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)f(s,x(s))
 \]

\[
 + \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)BW^{-1}
 \]

\[
 \times \left[ x_1 - E^{-1}S_\alpha(a)Ex_0 - \int_0^a (a-\tau)^{\alpha-1}E^{-1}T_\alpha(a-\tau)f(\tau,x(\tau))
 \right] d\tau
 \]

\[
 - \int_0^a (a-\tau)^{\alpha-1}E^{-1}T_\alpha(a-\tau)\left\{\int_0^\tau g(\tau,\eta, x(\eta), R(\eta))
 \right\} d\tau
 \]

\[
 + \alpha \int_0^a (a-s)^{\alpha-1}E^{-1}T_\alpha(a-s)\left\{\int_0^s g(s,\tau, x(\tau), R(\tau))
 \right\} d\tau
 \] \( ds = x_1 \).

It can be easily verified that \( Q \) maps \( C \) into itself continuously.

For each positive number \( k > 0 \), let \( B_k = \{ x \in C : x(0) = x_0, \| x(t) \| \leq k, t \in J \} \). Obviously, \( B_k \) is clearly a bounded, closed, convex subset in \( C \). We claim that there exists a positive
number \( k \) such that \( QB_k \subset B_k \). If this is not true, then for each positive number \( k \), there exists a function \( x_k \in B_k \) with \( Qx_k \notin B_k \), that is, \( \|Qx_k\| \geq k \), then \( 1 \leq \frac{1}{k} \|Qx_k\| \), and so

\[
1 \leq \lim_{k \to \infty} k^{-1} \|Qx_k\|.
\]

However,

\[
\lim_{k \to \infty} k^{-1} \|Qx_k\|
\leq \lim_{k \to \infty} k^{-1} \left\{ M \|E^{-1}\| \|E\| \|x_0\| + \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha + 1)} \int_0^\infty (a-s)^{\alpha-1} g_k(s) \, ds \\
+ \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha + 1)} \int_0^\infty (a-s)^{\alpha-1} \left[ \|x_1\| + M \|E^{-1}\| \|E\| \|x_0\| \right] \, ds \\
+ \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha + 1)} \int_0^\infty (a-s)^{\alpha-1} h_k(s) \, ds \right\}
\]

\[
\leq \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha + 1)} \beta + \frac{\alpha^2 M^2 (\|E^{-1}\|^2)}{(\Gamma(\alpha + 1)^2)} \|B\| \|W^{-1}\| (\beta + \gamma) + \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha + 1)} \gamma
\]

\[
= \frac{\alpha M \|E^{-1}\|}{\Gamma(\alpha + 1)} (\beta + \gamma) \left[ 1 + \frac{\alpha^2 M \|E^{-1}\|}{\Gamma(\alpha + 1)} \|B\| \|W^{-1}\| \right] < 1,
\]

a contradiction. Hence, \( QB_k \subset B_k \) for some positive number \( k \). In fact, the operator \( Q \) maps \( B_k \) into a compact subset of \( B_k \). To prove this, we first show that the set \( V_k(t) = \{(Qx)(t) : x \in B_k\} \) is a precompact in \( X \); for every \( t \in J \); This is trivial for \( t = 0 \), since \( V_k(0) = \{x_0\} \). Let \( t, 0 < t \leq a; \) be fixed. For \( 0 < \epsilon < t \) and arbitrary \( \delta > 0 \); take

\[
(Q^{\alpha}x)(t) = \int_0^t t^{\alpha} \xi_0(\sigma) E^{-1} (t^\alpha \sigma) E x_0 \, d\sigma
\]

\[
+ \alpha \int_0^{t-\epsilon} \int_0^\infty \theta (t-s)^{\alpha-1} \xi_0(\sigma) E^{-1} (t^\alpha \sigma) f(s,x(s)) \, d\sigma \, ds
\]

\[
+ \alpha \int_0^{t-\epsilon} \int_0^\infty \theta (t-s)^{\alpha-1} \xi_0(t) E^{-1} (t^\alpha \sigma) \, d\sigma \, ds \times B W^{-1} \left[ \xi_0(\sigma) E^{-1} (t^\alpha \sigma) E x_0 \, d\sigma \right]
\]

\[
- \alpha \int_0^{t-\epsilon} \int_0^\infty \theta (a-\tau)^{\alpha-1} \xi_0(\sigma) E^{-1} (a^\alpha \tau) f(\tau,x(\tau)) \, d\tau \, d\theta
\]

\[
- \alpha \int_0^{t-\epsilon} \int_0^\infty \theta (a-\tau)^{\alpha-1} \xi_0(\sigma) E^{-1} (a^\alpha \tau) \, d\tau \, d\sigma \times \left\{ \int_0^\tau g(\tau,\eta,x(\eta),R(\eta)) \, d\eta \right\} \, d\sigma \, d\tau
\]

\[
+ \alpha \int_0^{t-\epsilon} \int_0^\infty \theta (t-s)^{\alpha-1} \xi_0(\sigma) E^{-1} (t^\alpha \sigma) \, d\sigma \, ds \times \left\{ \int_0^\tau g(s,\tau,x(\tau),R(\tau)) \, d\tau \right\} \, d\sigma \, d\tau
\]
\[ T(e^{\alpha \delta}) \int_{\delta}^{\infty} \xi_\alpha(x) E^{-1} T(t^\alpha \theta - e^{\alpha \delta}) E x_0 \, d\theta \]

\[ + T(e^{\alpha \delta}) \alpha \int_{0}^{t^\alpha - e^{-\alpha \delta}} \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((t - s)^\alpha \theta - e^{\alpha \delta}) f(s, x(s)) \, d\theta \, ds \]

\[ + T(e^{\alpha \delta}) \alpha \int_{0}^{t^\alpha - e^{-\alpha \delta}} \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((t - s)^\alpha \theta - e^{\alpha \delta}) \]

\[ \times BW^{-1} \left[ x_1 - \int_{0}^{\infty} \xi_\alpha(x) E^{-1} T(a^\alpha \theta) E x_0 \, d\theta \right] \]

Since \( u(s) \) is bounded and \( T(e^{\alpha \delta}) \), \( e^{\alpha \delta} > 0 \) is a compact operator, then the set \( V_{t^\alpha}(t) = \{(Q^{\alpha \delta}) x(t) : x \in B_k \} \) is a precompact set in \( X \) for every \( \alpha, 0 < \alpha < t \), and for all \( \delta > 0 \). Also, for \( x \in B_k \), using the defined control \( u(t) \) yields

\[ \| (Q)x(t) - (Q^{\alpha \delta}) x(t) \| \]

\[ \leq \left\| \int_{0}^{\delta} \xi_\alpha(x) E^{-1} T(t^\alpha \theta) E x_0 \, d\theta \right\| \]

\[ + \alpha \left\| \int_{t^\alpha - e^{-\alpha \delta}}^{t^\alpha} \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((t - s)^\alpha \theta) f(s, x(s)) \, d\theta \, ds \right\| \]

\[ + \alpha \left\| \int_{0}^{t^\alpha - e^{-\alpha \delta}} \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((t - s)^\alpha \theta) \right\| \]

\[ \times BW^{-1} \left[ x_1 - \int_{0}^{\infty} \xi_\alpha(x) E^{-1} T(a^\alpha \theta) E x_0 \, d\theta \right] \]

\[ - \alpha \left\| \int_{0}^{a} \int_{\delta}^{\infty} \theta(\alpha - \tau)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((\alpha - \tau)^\alpha \theta) f(\tau, x(\tau)) \, d\theta \, d\tau \right\| \]

\[ - \alpha \left\| \int_{0}^{a} \int_{\delta}^{\infty} \theta(\alpha - \tau)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((\alpha - \tau)^\alpha \theta) \right\| \]

\[ \times \left\{ \left\| \int_{0}^{\tau} g(\tau, x(\tau), R(\tau)) \, d\eta \right\| \, d\tau \, ds \right\} \]

\[ + \alpha \left\| \int_{t^\alpha - e^{-\alpha \delta}}^{t^\alpha} \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((t - s)^\alpha \theta) \left\{ \int_{0}^{\tau} g(s, \tau, x(\tau), R(\tau)) \, d\tau \right\} \, d\theta \, ds \right\| \]

\[ + \alpha \left\| \int_{0}^{t^\alpha - e^{-\alpha \delta}} \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((t - s)^\alpha \theta) f(s, x(s)) \, d\theta \, ds \right\| \]

\[ + \alpha \left\| \int_{0}^{t^\alpha - e^{-\alpha \delta}} \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \xi_\alpha(x) E^{-1} T((t - s)^\alpha \theta) \right\| \]
\[
\times BW^{-1} \left[ x_1 - \int_0^\infty \xi_\alpha(\theta) E^{-1} (a^\alpha \theta) E x_0 \, d\theta \right] \\
- \alpha \int_0^a \int_0^\infty \theta (a-\tau)^{\alpha-1} \xi_\alpha(\theta) E^{-1} ((a-\tau)^\alpha \theta) f(\tau, x(\tau)) \, d\theta \, d\tau \\
- \alpha \int_0^a \int_0^\infty \theta (a-\tau)^{\alpha-1} \xi_\alpha(\theta) E^{-1} ((a-\tau)^\alpha \theta) \\
\times \left\{ \int_0^\delta g(\tau, \eta, x(\eta), R(\eta)) \, d\eta \right\} (s) \, d\theta \, ds \\
+ \alpha \left. \int_0^t \int_0^\delta \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) E^{-1} ((t-s)^\alpha \theta) \left\{ \int_0^\delta g(s, \tau, x(\tau), R(\tau)) \, d\tau \right\} \, d\theta \, ds \right\} \\
\leq M \| E^{-1} \| \| E \| \| x_0 \| \int_0^\delta \xi_\alpha(\theta) \, d\theta \\
+ \alpha M \| E^{-1} \| \left( \int_{t-\epsilon}^t (t-s)^{\alpha-1} g_s(s) \, ds \right) \left( \int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right) \\
+ \alpha M \| E^{-1} \| \| B \| \| W^{-1} \| \int_{t-\epsilon}^t (t-s)^{\alpha-1} \left\| x_1 \right\| + M \| E^{-1} \| \| x_0 \| \\
+ \frac{\alpha}{\Gamma(\alpha+1)} M \| E^{-1} \| \int_0^a (a-\tau)^{\alpha-1} h_k(\tau) \, d\tau \\
+ \frac{\alpha}{\Gamma(\alpha+1)} M \| E^{-1} \| \int_0^a (a-\tau)^{\alpha-1} h_k(\tau) \, d\tau \left( \delta \right) \left( \int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right) \\
+ \alpha M \| E^{-1} \| \left( \int_{t-\epsilon}^t (t-s)^{\alpha-1} h_k(s) \, ds \right) \left( \int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right) \\
+ \alpha M \| E^{-1} \| \left( \int_{t-\epsilon}^t (t-s)^{\alpha-1} g_s(s) \, ds \right) \left( \int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right) \\
+ \alpha M \| E^{-1} \| \| B \| \| W^{-1} \| \int_{t-\epsilon}^t (t-s)^{\alpha-1} \left\| x_1 \right\| + M \| E^{-1} \| \| x_0 \| \\
+ \frac{\alpha}{\Gamma(\alpha+1)} M \| E^{-1} \| \int_0^a (a-\tau)^{\alpha-1} g_k(\tau) \, d\tau \\
+ \frac{\alpha}{\Gamma(\alpha+1)} M \| E^{-1} \| \int_0^a (a-\tau)^{\alpha-1} h_k(\tau) \, d\tau \left( \delta \right) \left( \int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right) \\
+ \alpha M \| E^{-1} \| \left( \int_{t-\epsilon}^t (t-s)^{\alpha-1} h_k(s) \, ds \right) \left( \int_0^\infty \theta \xi_\alpha(\theta) \, d\theta \right) \\
.\]

Therefore, as \( \epsilon \to 0^+ \) and \( \delta \to 0^+ \), there are precompact sets arbitrary close to the set \( V_k(t) \) and so \( V_k(t) \) is precompact in \( X \).

Next, we show that \( QB_k = \{ Qx : x \in B_k \} \) is an equicontinuous family of functions.

Let \( x \in B_k \) and \( t, \tau \in I \) such that \( 0 < t < \tau \), then

\[
\| (Qx)(t) - (Qx)(\tau) \| \\
\leq \| T(t^\alpha \theta) - T(\tau^\alpha \theta) \| \| E^{-1} \| \| E \| \| x_0 \| \\
+ \frac{\alpha}{\Gamma(\alpha+1)} \int_0^t \| (t-s)^{1-\alpha} T((t-s)^\alpha \theta) - (\tau-s)^{1-\alpha} T((\tau-s)^\alpha \theta) \| g_k(s) \, ds \\
+ \frac{\alpha M}{\Gamma(\alpha+1)} \int_t^\delta (\tau-s)^{1-\alpha} g_k(s) \, ds
\]
Now, $T(t)$ is continuous in the uniform operator topology for $t > 0$ since $T(t)$ is compact, and the right-hand side of the above inequality tends to zero as $t \to \tau$. Thus, $QB_k$ is both equicontinuous and bounded. By the Arzela-Ascoli theorem, $QB_k$ is precompact in $C(J, X)$. Hence, $Q$ is a completely continuous operator on $C(J, X)$.

From the Schauder fixed-point theorem, $Q$ has a fixed point in $B_k$. Any fixed point of $Q$ is a mild solution of (1.1) on $J$ satisfying $(Qx)(t) = x(t) \in X$. Thus, the system (1.1) is controllable on $J$.

\section{Example}

In this section, we present an example to our abstract results.

We consider the fractional integro-partial differential equation in the form

$$\frac{\partial^\alpha}{\partial \tau^\alpha} (z(t, x) - z_{xx}(t, x)) - z_{xx}(t, x) = Bu + \mu_1(t, z_{xx}(t, x))$$

$$+ \int_0^t \mu_2(s, z_{xx}(s, x)) \int_0^s \mu_3(t, s, z_{xx}(s, x)) \, ds \, dt, \quad 0 \leq x \leq \pi, t \in J, \quad (4.1)$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \in J,$$

$$z(0, x) = z_0(x), \quad x \in [0, \pi],$$

where $\partial^\alpha/\partial \tau^\alpha$ is the Caputo fractional partial derivative of order $0 < \alpha < 1$.

Take $X = Y = L^2[0, \pi]$ and define the operators $A : D(A) \subset X \rightarrow Y$ and $E : D(E) \subset X \rightarrow Y$ by $Az = -z_{xx}$ and $Ez = z - z_{xx}$, where each domain $D(A)$ and $D(E)$ is given by $\{z \in X : z, z_x \text{ are absolutely continuous, } z_{xx} \in X, z(0) = z(\pi) = 0\}$.

Then $A$ and $E$ can be written respectively as [13]

$$Az = \sum_{n=1}^{\infty} n^2 (z, z_n)z_n, \quad z \in D(A),$$
where \( z_n(x) = \sqrt{2/n} \sin n x, \) \( n = 1, 2, \ldots, \) is the orthonormal set of eigenvectors of \( A \) and \( (z, z_n) \) is the \( L^2 \) inner product. Moreover, for \( z \in X \), we get

\[
Ez = \sum_{n=1}^{\infty} (1 + n^2)(z, z_n)z_n, \quad z \in D(E),
\]

We assume that

(i) For each \( t \in J \), the nonlinear operator \( \mu_1 : J \times X \to Y \) satisfies the following three conditions:

\( f(t, z)(x) = \int_0^t \mu_1(t, s)z(s(x)) \) ds

(A1): The operator \( B : U \to Y \), with \( U \subset J \), is a bounded linear operator.

(A2): The linear operator \( W : U \to X \) defined by

\[
Wu = \int_0^s E^{-1}(a - s)^{\nu - 1} T_u(a - s)Bu(\tau)\,d\tau
\]

has an inverse bounded operator \( W^{-1} \) which takes values in \( L^2(J, U)/\ker W \), where the kernel space of \( W \) is defined by \( \ker W = \{ x \in L^2(J, U) : Wx = 0 \} \), \( B \) is a bounded linear operator.

(A3): The nonlinear operator \( \mu_2 : J \times J \times X \to X \) satisfies the following two conditions:

\( f(t, z)(x) = \int_0^t \mu_2(t, s)z(s(x)) \) ds

(A4): The nonlinear operator \( \mu_3 : J \times J \times X \to X \) satisfies the following three conditions:

\( f(t, z)(x) = \int_0^t \mu_3(t, s)z(s(x)) \) ds

Define an operator \( f : J \times X \to Y \) by

\[
f(t, z)(x) = \mu_1(t, z_{\nu x}(x))
\]
and let

\[
H(t,s,z)(x) = \mu_2(t,s,z_{xx}(x)), \quad (t,s,z) \in J \times J \times X,
\]

\[
g\left(t,s,z, \int_0^s H(s,\tau,z) d\tau\right)(x) = \mu_3\left(t,s,z_{xx}, \int_0^s \mu_2(s,\tau,z_{xx}(x)) d\tau\right), \quad x \in [0,\pi].
\]

Then the problem (4.1) can be formulated abstractly as:

\[
\mathcal{D}^{\alpha} (Ez(t)) + Az(t) = Bu(t) + f(t,z(t)) + \int_0^t g\left(t,s,z, \int_0^s H(s,\tau,z(\tau)) d\tau\right) ds, \quad t \in J, z(0) = z_0.
\]

It is easy to see that \(-AE^{-1}\) generates a uniformly continuous semigroup \(\{S(t)\}_{t \geq 0}\) on \(Y\) which is compact, and (2.1) is satisfied. Also, the operator \(f\) satisfies condition \((H_6)\) and the operator \(H\) and \(g\) satisfy \((H_7)\) and \((H_8)\). Also all the conditions of Theorem 3.1 are satisfied. Hence, the equation (4.1) is controllable on \(J\).

Competing interests
The author declare that he has no competing interests.

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