Mass dependence of the gravitationally-induced wave-function phase

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Abstract

The leading mass dependence of the wave function phase is calculated in the presence of gravitational interactions. The conditions under which this phase contains terms depending on both the square of the mass and the gravitational constant are determined. The observability of such terms is briefly discussed.

a. Introduction There has been a series of publications discussing the mass dependence of the gravitational effects on the phase of the wave function [1–3], and, due to a divergence in the results, a certain amount of controversy has resulted. Some results [1] indicate the presence of a gravitationally-induced phase proportional to the square of the masses, while other calculations show that such terms are in fact absent [2], and that the leading dependence in the gravitational contributions to the phase is proportional to the fourth power of the mass. In this short note I will attempt to resolve this (apparent) contradiction. I will first consider the case of neutrinos propagating in an arbitrary gravitational field and then generalize to other particles.
b. Neutrinos in a gravitational field Using a WKB approximation it is possible to derive the effective Hamiltonian for neutrinos propagating in a non-trivial metric. Following the procedure described in detail in [4] the relevant expression is

\[ H_{\text{eff}} = \frac{1}{4} p_a (\partial_v e_{b\mu}) e^\mu_{e\nu} \left[ \epsilon^{abcd} - 2 \eta^{ab} \epsilon^{cdef} \frac{p_e p_f}{p^2} \right] \gamma_5 + \frac{1}{2} m^2, \]  

where Latin indices refer to the local Lorentz frame, Greek indices to the global coordinate frame, and \( e^\mu_a \) denote the tetrads. \( p \) is the momentum of a null geodesic which the neutrino wave packets follow, within the WKB approximation, when \( m = 0 \). I will denote by \( \lambda \) the affine parameter along this geodesic

\[ p^\mu = \frac{dx^\mu}{d\lambda}. \]  

The vector \( \frac{\dot{r}}{\dot{p}} \) denotes the time component of \( p \) with respect to a comoving reference frame. The Hamiltonian \( H_{\text{eff}} \) generates translations in \( \lambda \).

Though the pure gravitational terms in (1) can generate interesting effects, here I will concentrate on the contributions generated by the term \( m^2/2 \). It is clear from the above expression that each mass eigenstate acquires a phase equal to

\[ \Phi^{(m)} = -\frac{1}{2} m^2 \lambda. \]  

There are subleading contributions to \( H_{\text{eff}} \) which depend on the mass [4], but these are of order \( m/R \) where \( R \) denotes the distance scale of the metric, and can be ignored for \( m > 10^{-11} \text{eV} \).

c. Other particles The same expression for the mass dependence of the phase can be obtained using the following argument. Within the WKB approximation the wave function of a general quantum system takes the form

1The Hamiltonian has units of mass\(^2\) due to the choice of evolution parameter, see below

2More precisely: denoting by \( \nu^\mu_A, A = 1, 2, 3 \) three independent (local) solutions to the geodesic deviation equation [3], \( \frac{\dot{r}}{\dot{p}} \) is the component of \( p \) orthogonal to the \( \nu^\mu_A \).
\[ \Psi = e^{iS} \chi, \]  \hspace{1cm} (4)

where \( S \) denotes the classical action and \( \chi \) a slowly-varying amplitude. When the particle moves in a non-trivial metric background \( S \) satisfies the following Hamilton-Jacobi equation \[ g^{\mu \nu} \partial_\mu S \partial_\nu S = m^2, \]  \hspace{1cm} (5)

(I ignored all other interactions and considered a single mass eigenstate). The action is generally covariant so that the corresponding phase is unambiguously determined (within the semiclassical approximation)

For small \( m \) the action takes the form

\[ S = S_0 + m^2 S_1 + \cdots, \]  \hspace{1cm} (6)

which, when substituted into (5) yields

\[ g^{\mu \nu} p_\mu p_\nu = 0, \quad p^\mu \partial_\mu S_1 = -\frac{1}{2}, \]  \hspace{1cm} (7)

where

\[ p^\mu = -g^{\mu \nu} \partial_\nu S_0 = \frac{dx^\mu}{d\lambda}, \]  \hspace{1cm} (8)

is the tangent vector to the null geodesic. It then follows that \( p^\mu \partial_\mu S_1 = dS_1/d\lambda \), so that \( S_1 = -\lambda m^2/2 \) which gives the same contribution as (3). It follows that this result is general provided \( m \) is small compared to the particle’s momentum.

d. Observability of the gravitationally induced phase  I will consider a wave packet that can be decomposed into a coherent superposition of mass eigenstates. Using standard arguments the above expressions imply that, within the semiclassical approximation, the phase \( S_0 + \lambda m^2/2 \) may lead to oscillations between components of different mass. This will occur provided (a) the amplitude of the mass components of the state have a sufficient overlap at the observation point and, (b) the phase difference between them is sufficiently large (\( \gtrsim \pi \)).
I will first review briefly the conditions for under which there is significant amplitude overlap and then discuss the effects of the phase difference.

If the mass components are created in the same localized space-time region \( A \) then, for long distances and within the WKB approximation, they will overlap at the observation region \( B \) only if the corresponding amplitudes in momentum space are centered at different momenta. In this case the phase difference is simply \( \Delta m^2 (\lambda_B - \lambda_A)/2 \).

Alternatively we can assume that the initial mass amplitudes are localized in the same region of momentum space. Then, in order for them to arrive simultaneously at \( B \) (so that there is significant amplitude overlap), they must originate at different space-time locations \( A \) and \( A' \), which may be space or time-like separated (the creation region then contains \( A \) and \( A' \) and is sufficiently large to allow a small momentum uncertainty). In this case the phase difference is \( S_0(A) - S_0(A') + \Delta m^2 (\lambda_B - \lambda_A)/2 \), where \( S_0(A) - S_0(A') \) is also of order \( \Delta m^2 \). This is the scenario described in [2].

Leaving aside the issue of which of the two scenarios above is more natural, it is clear that the expressions for the phase difference will be different in each case because the assumed initial conditions are different [3].

Even if interference effects are present, an independent question is whether the phase difference between two mass eigenstates has a strong dependence on the gravitational field. I will consider for simplicity the first case described above, where the phase for each mass component is given by (3) and depends only on the geodesic length \( \lambda \). In order to determine the dependence on the gravitational force this expression should be re-written in terms of a set of physical observables, but such a set is no unique and different choices may exhibit varying dependences on the gravitational constant.

The expression for \( \Phi^{(m)} \) will depend on the geodesic parameters such as the energy and angular momentum. If, for example, the interference effects are measured by a freely-falling observer, these should be written in terms of the corresponding quantities measured by the observer (e.g. the energy-momentum vector equals \( e^\mu p_\mu \)). In addition, this phase will depend on the physical distance from the source to the observer. This distance can be defined as
half the time it takes for light to go from $A$ to $B$ and back to $A$ (in units where the speed of light is one); explicitly

$$\ell_{AB} = \int_A^B d\lambda \sqrt{(g_{ij} - g_{0i}g_{0j}/g_{00}) p^i p^j} = E \int_A^B \frac{d\lambda}{\sqrt{-g_{00}}}$$

(Latin indices denote the spatial components) the second expression is valid for null geodesics in a static space time (the integrals are evaluated along the geodesic). Note however, that $\ell_{AB}$ is not the only possible quantity that can be associated with the physical distance from the source to the observer. Other possibilities are in general available and, as will be illustrated below, the form of $\Phi^{(m)}$ are strongly dependent on the choice made.

There is one final “experimental” issue concerning the observability of the phase difference. In many situations the distance to the source is only approximately known and the location of creation region is known only with limited accuracy. In such cases the gravitational effects will be distinguishable only if the phase difference due to this uncertainty can be neglected or otherwise taken into account.

e. Gravitational dependence of the quantum phase in a Kerr metric. The expression (3) contains a dependence on the gravitational interactions due to the non-trivial expression of $\lambda$ in terms of the coordinates. As an illustrative example I will consider the case of Kerr metric,

$$ds^2 = \left(1 - \frac{rr_g}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{rr_ga^2}{\rho^2} \sin^2 \theta\right) \sin^2 \theta d\phi^2 + \frac{2rr_ga}{\rho^2} \sin^2 \theta d\phi dt,$$

where

$$\Delta = r^2 - rr_g + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta.$$  

(11)

The gravitational radius $r_g$ equals $2MG$ where $M$ is the mass of the black hole and $G$ Newton’s constant. The black hole angular momentum equals $aM$.

In this case the Hamilton-Jacobi equation (3) with $m = 0$ is separable,

$$S_0 = -Et + W_r(r) + W_\theta(\theta) + L\phi,$$  

(12)
where $E$ denotes the energy and $L$ the (azimuthal) angular momentum for the particle (the $z$ axis is along the the black hole rotation axis). Both $E$ and $L$ are constant, the addition constant of the motion $K$ is related to the total angular momentum. Substituting, \[ \left( \frac{dW_\theta}{d\theta} \right)^2 = K - \left( aE \sin \theta - \frac{L}{\sin \theta} \right)^2, \quad \left( \frac{dW_r}{dr} \right)^2 = \frac{[(r^2 + a^2)E - aL]^2}{2} - K. \] Using (8) it follows that
\[
\lambda = \int^r \frac{\rho^2 \, dr}{\sqrt{[(r^2 + a^2)E - aL]^2 - K\Delta]},
\]
with $\rho$ and $\Delta$ defined in (11) where $\theta$ (buried in $\Delta$) is assumed to be expressed in terms of $r$ using
\[
\left( \frac{d\theta}{dr} \right)^2 = \frac{K - (aE \sin \theta - L/\sin \theta)^2}{[(r^2 + a^2)E - aL]^2 - K\Delta},
\]
which also follows from (8). The flat-space limit of (14) corresponds to $r_g = a = 0$ and $K = L^2$, in this case
\[
\lambda_{\text{flat space}} = \frac{1}{E} \sqrt{r^2 - (L/E)^2},
\]
which is a standard result. Note that $\lambda$ has units of mass$^{-2}$, so that $H_{\text{eff}}$ has units of mass$^2$ as noted earlier.

To lowest order in $r_g$, $a$, $K - L^2$ I find,
\[
\lambda = \frac{1}{E} \sqrt{r^2 - (L/E)^2} - \frac{(K - L^2) + 2ELa - E^2rr_g}{2E^3\sqrt{r^2 - (L/E)^2}} + \ldots.\]
In the limit of zero angular momentum $K = L^2 = 0, a = 0$ this expression retains a term proportional to $r_g$, but this term is actually an irrelevant constant, $r_g/(2E)$. It follows that for weak gravitational fields and small masses (3) becomes
\[
\Phi^{(m)} = -\frac{m^2}{2E} \sqrt{r^2 - (L/E)^2} + \frac{m^2[(K - L^2) + 2ELa]}{4E^3\sqrt{r^2 - (L/E)^2}} - \frac{m^2r_g(r - \sqrt{r^2 - (L/E)^2})}{4E\sqrt{r^2 - (L/E)^2}} + \ldots, \tag{18}
\]
3 $a$ is the black-hole angular momentum per unit mass
where I explicitly subtracted the above-mentioned constant. It follows that $\Phi^{(m)}$, when written in terms of $r$, does receive gravitational contributions, but these are proportional to an angular momentum (squared). This dependence on the angular momentum significantly suppresses the magnitude of these gravitational terms. In a region very distant from the black hole, $r \gg (L/E)$ the above expression reduces to (see also [3])

$$\Phi^{(m)} = -\frac{m^2 r}{2E} \left\{ 1 + \frac{(K - L^2) + 2ELa}{2E^2 r^2} + \cdots \right\}, \quad (r \gg L/E).$$ (19)

For zero orbital angular momentum there is still a contribution to (3) proportional to $r_g$, but it is also proportional to the square of the black-hole angular momentum. For example, for a geodesic in the $\theta = \pi/2$ plane

$$\Phi^{(m)} = -\frac{m^2 r}{2E} \left( 1 + \frac{a^2}{r^2} + \cdots \right); \quad (L = 0 \theta = \pi/2).$$ (20)

The expressions (19,20) contain no explicit gravitational contribution when the geodesics and the black hole have zero angular momentum. However when expressed in terms of the invariant length (9) the phase (18) becomes

$$\Phi^{(m)} = -\frac{m^2}{2E} \left[ \ell - \frac{r_g}{2} \arcsinh(E\ell/L) + \cdots \right]$$ (21)

which does contain a term $\sim r_g m^2 \ln \ell$ which persists even in the $L \to 0$ limit (note that in this limit there is a term proportional to $\ln L$, but it is constant and will cancel when considering phase differences). Yet, despite appearances, (18) and (21) are equivalent.

The calculation presented in [2] was done for radial geodesics in a Schwarzschild metric and corresponds to the result (18) when $K = L = a = 0$, in which case there is no explicit gravitational terms present in the phase. The same expression corresponds to the results in [1] provided the quantity $\int dL/r$ used in these references (where $L$ represents a distance and should not be confused with the angular momentum as used in this paper) is identified with $2 \ln \ell$. It is worth emphasizing that both $\ell$ and $r$ have reasonable physical interpretations as measures of the distance to the source [1] and both reduce t0 the usual flat-space distance

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4For example, in the case $a = 0$, $r$ can be defined in terms of the circumference of circles around
in the limit of zero gravitational interactions.

In order to extract the gravitational term from these expressions one must subtract the non-gravitational contribution. This, however, is an ambiguous quantity. For example, an observer defining the flat space contribution as \( \Phi_{\text{flat}}^{(m)} = -m^2 r/(2E) \) will find a small gravitational contribution of order \((\text{ang. momentum})^2/(rE)^2\) to (3). A second observer might alternatively define \( \Phi_{\text{flat}}^{(m)} = -m^2 \ell/(2E) \) and find a much more significant gravitational contribution \( \sim r_g \ln \ell \) to (3). Equivalently, an experiment done in flat space over a distance \( d \) will generate a phase \(-m^2 d/(2E)\), in comparing this to a similar measurement in curved space, one must determine what quantity \((r, \ell, \text{etc.})\) is to be used in lieu of \( d \), and the expression for \( \Phi_{\text{flat}}^{(m)} \) depends significantly on the choice made. In this sense the results presented in [1] and [2] are, in fact, consistent.

As mentioned in paragraph f the the observability of these effects is a separate issue, which I discuss briefly in paragraph g below. Here I merely note that the phase (in terms of \( r \)) is of order \((m^2/E)(r + r_g^2/r)\) when \( L \sim r_g E \); the second term is absent for the case of zero angular momentum. In terms of \( \ell \) and the phase contains a term \( \sim (m^2 r_g/E) \ln \ell \), even when the angular momenta vanish. It is possible, however, that these simple results are consequences of the high degree of symmetry of the metric (10). This possibility is investigated below.

f. General spherically symmetric metric The metric for this configuration is [6,5], using polar coordinates,

\[
ds^2 = e^\nu dt^2 - e^\gamma dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \tag{22}\]

where \( \nu \) and \( \gamma \) depend on \( r \) and \( t \) only. The geodesics of this metric lie on a plane which I take as the \( \theta = \pi/2 \); then (3) with \( m = 0 \) is again separable and its solution takes the form \( S_0 = -Et + L\phi + W_r(r) \) where \( W_r'' = e^\gamma (E^2 e^{-\nu} - L^2/r^2) \) so that

\[
\frac{dr}{d\lambda} = e^{-(\nu+\gamma)/2} \sqrt{E^2 - \frac{L^2 e^{\nu}}{r^2}} \quad \frac{dt}{d\lambda} = E e^{-\nu}, \tag{23}\]

the origin while the interpretation of \( \ell \) was provided in paragraph h.
which can in principle be solved for $t(\lambda), r(\lambda)$ and inverted to obtain $\lambda = \lambda(r)$. For the interesting case of time-independent metric this gives

$$\Phi^{(m)} = -\frac{m^2}{2E} \int dr \frac{e^{(\nu+\gamma)/2}}{\sqrt{1 - \frac{L^2}{E^2r^2}}}.$$  \hspace{1cm} (24)

It appears possible for this to retain a gravitational contribution even when $L = 0$ (radial geodesics). To determine whether this is in fact the case I will consider the case of an ideal fluid of energy density $\rho$ and pressure $p$ both of which vanish for $r > R$. The solution to Einstein’s equations gives

$$\gamma = \ln[1 - 2m(r)G/r], \quad m(r) = \int_0^r 4\pi u^2 \rho(u) du$$

$$\nu = -2G \int_r^\infty du \frac{m(u) + 4\pi u^3 p(u)}{u[u - 2m(u)G]},$$  \hspace{1cm} (25)

where $G$ denotes Newton’s constant; for $r > R \nu$ and $\gamma$ reduce to their Schwarzschild expressions. To first order in $G$ I find

$$e^{(\nu+\gamma)/2} = 1 + 4\pi G \int_r^\infty du \frac{\rho(u) + p(u)}{u}$$  \hspace{1cm} (26)

which, when substituted into (24) with $L = 0$ gives the phase for radial motion. If the initial point of the geodesic $r_i$ satisfies $r_i < R$ and final point lies beyond $R$ then a simple estimate gives $\Phi^{(m)} \sim (m^2/2E)(r + cr_g)$ where $r_g$ denotes the gravitational radius of the matter distribution, and $c$ is a numerical constant $\lesssim O(1)$ that depends on the detailed form of $\rho$ and $p$. Note that, just as in (19,20), the leading gravitational contribution to the phase depends on the initial point and the details of the metric, but not on $r$; if expressed in terms of $\ell$, however, the phase again acquires an $m^2 \ln \ell$ contribution.

\textit{g. Weak gravitational interactions} \quad For the case of a general weak metric it is possible to obtain a rather simple expression for the phase $\Phi^{(m)}$. Expanding $S_0$ (cf. (6)) in powers of the metric perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \quad S_0 = S_0^{(0)} + S_0^{(1)} + \cdots; \quad S_0^{(1)} = O(h),$$  \hspace{1cm} (27)

(where $\eta$ denotes the flat-space metric) and defining
\[ p_\mu = -\partial_\mu S_0^{(0)}. \]  

(28)

I find \( p^2 = 0 \) and

\[ p^\mu \partial_\mu S_0^{(0)} = -\frac{1}{2} h_{\mu\nu} p^\mu p^\nu, \]  

(29)

where indices are raised and lowered using the flat metric \( \eta \).

It proves convenient to choose coordinates such that

\[ p^\mu = E(1, 1, 0, 0), \]  

(30)

then the equation for \( S_0^{(1)} \) becomes

\[ \frac{\partial S_0^{(1)}}{\partial x_+} = -\frac{1}{2}(h_{00} + 2h_{01} + h_{11}), \quad x_+ = \frac{x^0 + x^1}{2}, \]  

(31)

so that

\[ S_0 = -p \cdot x - \frac{1}{2} E \int dx_+(h_{00} + 2h_{01} + h_{11}) + \cdots. \]  

(32)

From this and the Hamilton-Jacobi equation I find

\[ \frac{dx^\mu}{d\lambda} = p^\mu + \frac{1}{2} E \partial_\mu \int dx_+(h_{00} + 2h_{01} + h_{11}) - h^{\mu\nu} p_\nu + \cdots, \]  

(33)

and, in particular,

\[ \frac{dx^1}{d\lambda} = E - \frac{1}{2} E \partial_1 \int dx_+(h_{00} + 2h_{01} + h_{11}) + E(h_{11} + h_{10}) + \cdots. \]  

(34)

For a time-independent metric the above expression simplifies to

\[ \frac{dx^1}{d\lambda} = E - \frac{1}{2} E(h_{00} - h_{11}), \]  

(35)

whence \( \lambda E = x + \frac{1}{2} \int dx(h_{00} - h_{11}) \) and

\[ \Phi^{(m)} = -\frac{m^2}{2E} \left[ x + \frac{1}{2} \int dx(h_{00} - h_{11}) + \cdots \right], \]  

(36)

where the phase difference must be obtained by evaluating the quantity in brackets at two different points on a geodesic with the line integral being along the same geodesic.
The gravitational terms vanish if the geodesic has zero angular momentum for the case of a distant (localized) matter distribution since in this case $h_{00} = -r_g/r$ and $h_{ij} = -r_g x^i x^j / r^3$ (in Cartesian coordinates). Moreover, since $\Phi^{(m)}$ is independent of $h_{0i}$, the dependence on the angular momenta will be quadratic. This result is not accidental: linear terms are absent since the phase is a scalar and the angular momentum is a pseudovector. In terms of the invariant length

$$\Phi^{(m)} = -\frac{m^2}{2E} \left[ \ell + \frac{1}{2} \int d\ell \, h_{00} + \cdots \right]$$

(37)

which again has a linear+logarithmic dependence on $\ell$.

h. Discussion  The leading mass dependence of the phase of a general wave function is, within the WKB approximation, proportional to the affine parameter along the geodesic followed by the wave packets and may contain explicit gravitationally-induced contributions. The expression for the gravitationally-induced phase depends on the initial conditions assumed for the system and on the choice of physical variables. This last point is particularly delicate since the separation of $\Phi^{(m)}$ into gravitational and flat-space contributions is sensitive to the quantity used as physical distance. The results presented in [1–3] are consistent with each other provided these points are taken into account.

In all interesting cases, however, the uncertainty in the distances to regions of strong gravitational field are too large for these effects to be unambiguously isolated experimentally. In terms of the radial coordinate $r$ (defined at large distances using the Schwarzschild metric) and for the case of non-zero angular momentum, $\Phi^{(m)} \sim -(m^2/2E)(r + r_0^2/r_i)$ where $r_i$ denotes the initial point of the geodesic and $r_0$ is a characteristic length of the problem (such as the geodesic impact parameter or the gravitational angular momentum per unit mass) such that the angular momenta are of order $r_0 E$. In all cases considered $r_0 < r_g < r_i$ so that $\Phi^{(m)} \sim -(m^2/2E)(r + r_g)$. These gravitational effects (for the case where the flat-space phase is defined as $-m^2 r / 2E$) can be isolated experimentally only when $r$ is known to an accuracy better than $r_g$, and this is not usually attainable.

In terms of the invariant length $\ell$ the phase takes the form $\Phi^{(m)} \simeq (m^2/2E)[\ell +
\[(r_g/2) \ln(\ell/r_g)\] with a more significant dependence on \(r_g\). Nonetheless the precision in \(\ell\) required to extract the gravitational contribution (where the flat-space phase is now defined as \(m^2\ell/2E\)) is still not available. For example for a 10\(M_\odot\) supernova 10\(^5\) light years away, \(\ell\) should be known better to a precision better than 2,000 km or \(10^{-13}\)%; for a 10\(^9\)\(M_\odot\) supermassive quasar 10\(^8\) light years away, \(\ell\) should be known to a precision better than 10\(^{11}\) km or \(10^{-10}\)%.

Despite the logarithmic enhancement the current experimental sensitivity precludes an accurate discrimination of the gravitational contributions for realistic situations.

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