Brane-Universes with Variable $G$ and $\Lambda_{(4)}$

J. Ponce de Leon*

Laboratory of Theoretical Physics, Department of Physics
University of Puerto Rico, P.O. Box 23343, San Juan,
PR 00931, USA

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Abstract

We investigate the cosmological consequences of a brane-world theory which incorporates time variations in the gravitational coupling $G$ and the cosmological term $\Lambda_{(4)}$. We analyze in detail the model where $\dot{G}/G \sim H$ and $\Lambda_{(4)} \sim H^2$, which seems to be favored by observations. We show that these conditions single out models with flat space sections. We determine the behavior of the expansion scale factor, as well as, the variation of $G$, $\Lambda_{(4)}$ and $H$ for different possible scenarios where the bulk cosmological constant, $\Lambda_{(5)}$, can be zero, positive or negative. We demonstrate that the universe must recollapse, if it is embedded in an Anti-de Sitter five-dimensional bulk, which is the usual case in brane models. We evaluate the cosmological parameters, using some observational data, and show that we are nowhere near the time of recollapse. We conclude that the models with zero and negative bulk cosmological constant agree with the observed accelerating universe, while fitting simultaneously the observational data for the density and deceleration parameters. The age of the universe, even in the recollapsing case, is much larger than in the FRW universe.

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*E-mail: jponce@upracd.upr.clu.edu
1 Introduction

In Einstein’s General Relativity (GR) there are two fundamental physical parameters, the gravitational coupling $G$ and the cosmological term $\Lambda(4)$, which are typically assumed to be constants. However, the data emerging from the experimental and/or observational verification of this assumption are not conclusive. As a matter of fact, recent measurements point toward the possibility that small variations of these parameters cannot be excluded a priori. \[1\]

The accuracy in the determination of $G$ is rather poor. Indeed, in the best case it does not exceed $10^{-4}$. Besides, some experiments give inconsistent values for $G$, which do not overlap within their range of accuracy. Taking into consideration the progress and increase of precision in measuring technology, this can be a sign of possible range variations of $G$ which might be induced by some “new physics” and/or non-Newtonian interactions. \[2\], \[3\].

In astronomical and geophysical experiments, measurements of possible variations of $G$ with cosmic time provide experimental bounds on $\dot{G}/G$, which are not very tight. They span from $10^{-10}$ to $10^{-12}$ $yr^{-1}$ depending on the experiment and/or observation. Therefore the time variation of $G$ in terms of the Hubble parameter $H$ can be written as

$$\frac{\dot{G}}{G} = gH,$$  \[1\]

where $g$ is a coefficient whose observational bound is $|g| \leq 0.1$. A comprehensive and updated discussion of the various experimental and observational constraints on the value of $g$ (as well as on the variation of other fundamental “constants” of nature) has recently been provided by Uzan. \[4\]. Therefore, here we will not review the cosmological constraints on $g$ imposed by the cosmic microwave background and nucleosynthesis. Instead, we will refer the interested reader to \[1\] and references therein. On the other hand, there are a number of theories, such as scalar-tensor theories of gravity \[5\] and multidimensional cosmological models \[6\], which lead to estimates for the variation of the gravitational coupling similar to \[1\].

The introduction of a cosmological term in GR is usually considered artificial. Therefore, commonly it is set equal to zero. However, the possibility of a small but non-zero cosmological term is suggested by some recent observational results from Type Ia supernovae in distant galaxies and the age problem. \[7\], \[8\]. Present data reveal that the energy density of the field (or quintessence) associated with $\Lambda(4)$ exceeds the density of ordinary matter. \[9\]. There is an extensive literature suggesting that the relation

$$\Lambda(4) \sim H^2,$$  \[2\]

plays a fundamental role in cosmology. This relation has been obtained in a number of empirical models \[10\], \[11\]. More recently, it was obtained from a model based on the quantum gravitational uncertainty principle and the discrete structure of spacetime at Plank length \[12\], \[13\]. The dependence $\Lambda(4) \sim H^2$ explains the current observations successfully and provides a much needed large age of the universe.

In a recent paper \[14\] we showed that the introduction of a time-varying $g_{44}$ in brane-world theory yields a number of cosmological models which have good physical properties but do not admit constant values for $G$ and $\Lambda(4)$. In particular we considered the five-dimensional metric

$$dS^2 = y^2 dx^2 - t^{2/\alpha} y^{2/(1-\alpha)}[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] - \alpha^2(1-\alpha)^{-2}t^2 dy^2,$$  \[3\]

where $\alpha$ is a constant. This is a solution of the Einstein field equations in five-dimensions. \[15\]. In four-dimensions (on the hypersurfaces $y = const.$) it corresponds to the 4D Friedmann-Robertson-Walker models with flat 3D sections. We used \[15\] as the generating 5D space for a $\mathbb{Z}_2$ symmetric brane-universe filled with perfect fluid, with energy density $\rho$ and pressure $p$ satisfying the equation of state $p = \gamma \rho$. From Israel’s boundary conditions we found that the effective $G$ and $\Lambda(4)$ vary as

$$\frac{\dot{G}}{G} = -\alpha H, \quad 8\pi G = \frac{k^2_{(5)}[3(\gamma + 1) - \alpha]}{3(\gamma + 1)} H,$$  \[4\]

and

$$\Lambda(4) = \frac{[3(\gamma + 1) - \alpha]^2}{(\gamma + 1)^2} H^2,$$  \[5\]

\[1\]The energy density $\rho_{eff}$ and pressure $p_{eff}$ of the effective 4D matter satisfy the equation of state $p_{eff} = n \rho_{eff}$, where $n = (2\alpha/3 - 1)$. Thus for $\alpha = 2$ we recover radiation, for $\alpha = 3/2$ we recover dust, etc.
respectively. Thus, the metric (3) leads an exact brane-model for a FRW universe where where (1) and (2) hold.

A crucial point here is that in brane-world theories the quantities $G$ and $\Lambda(4)$ are related to each other through the vacuum energy density of the 3-brane. They are not independent, as in Jordan–Brans–Dicke and other multidimensional theories. Thus, either both are truly constants or they vary simultaneously, viz.,

$$\dot{\Lambda}(4) \sim G\dot{G}.$$  \hspace{1cm} (6)

This places particular constrains upon the evolution of brane-world models. Namely, when we take into account the experimental bounds (1) and (2) we obtain an equation for the Hubble parameter which has to be solved together with the generalized Friedmann equation on the brane.

Another important feature here is that a cosmological term in 5D induces “natural” constant scales in time, say $\tau_s$, and length ($l_s = c\tau_s$) in the physics on the brane. Therefore, the evolution of the brane universe can be separated into three epochs. For $t << \tau_s$, the evolution of $H$ and $\Lambda(4)$ does not differ much from the familiar $H \sim t^{-1}$ and $\Lambda(4) \sim t^{-2}$, in agreement with the natural dimensions of these quantities in 4D. For $t \sim \tau_s$, the overall evolution of the expansion scale factor of the brane is dictated by the constant scale. For $t >\tau_s$, the evolution depends on whether the universe is embedded in a de Sitter or anti de Sitter five-dimensional bulk.

In this work we investigate the consequences of (1) and (2) in the framework of brane-world models. The natural question to ask is whether (6) could lead to a generic analysis of the time variations of $\Lambda(4)$ and $G$. Unfortunately, the answer to this question seems to be negative. We will see that in order to be able to integrate the equations we need to introduce some additional hypothesis or assumption.

Regarding $G$, here we consider the simplifying assumption that $g$ is a non-vanishing constant (otherwise $G$ and $\Lambda(4)$ are constants). Beforehand, we should note that this assumption is not equivalent to requiring $G \sim H$, in general. The physical meaning of this assumption is that the variation of $g$ is much “slower” than that of $H$ and $G$, namely, $|g/H| << |\dot{H}/H|, |\dot{g}/g| << |\dot{G}/G|$. Besides, the numerical value of $g$ should be small enough as to ensure that one obtains the correct abundances, in accordance with the discussion in (4), and not contradict nucleosynthesis.

Regarding $\Lambda(4)$, without any loss of generality we can write

$$\Lambda(4) = \xi(t)H^2,$$  \hspace{1cm} (7)

where $\xi$ is some function of time. This form of $\Lambda(4)$ is convenient for our purposes and is consistent with the fact that supernovae data give us $\Lambda(4) \sim H^2$ at present. In general $\xi$ is not constant, but a function which can be obtained from the equations of the model. We will see that, in models with non-vanishing bulk cosmological constant, $\Lambda(4)$ is not strictly proportional to $H^2$, i.e., $\Lambda(4) \neq \text{const} \times H^2$. These two quantities become exactly proportional to each other only asymptotically in time. This is an interesting feature specially in view of latest supernovae results [15] suggesting that $\Lambda(4)$ was less important is the past and therefore not proportional to $H^2$.

The purpose of this work is twofold. Firstly, to determine the general evolution of the expansion scale factor, as well as, the variation of $G$, $\Lambda(4)$ and $H$ in a brane-universe which is compatible with (1) and (2). Secondly, to examine in some detail the effects of a five-dimensional cosmological constant on the evolution of the brane universe, and perhaps on the signature of the extra dimension.

Our analysis is rather universal since we make no reference to any particular solution of the field equations in 5D. For generality, we do not impose the signature of the extra dimension either. We consider different possible scenarios where the bulk cosmological constant can be zero, positive or negative. For these scenarios, we show that models with spacelike extra dimension agree with the observed accelerating universe.

The extra dimension can be timelike only if the cosmological constant in the bulk is positive. The corresponding cosmological models are well behaved and exhibit interesting physical properties, but they do not seem to be of much observational significance.

This paper is organized as follows. In Section 2 we give a brief summary of the theory and of the generalized Friedmann equation on a spatially homogeneous and isotropic brane. In Section 3 we show how to incorporate a varying vacuum energy into the scheme. We analyze the compatibility of the generalized Friedmann equation with the observational requirements (1) and (2). We find that the coupled evolution of $G$ and $\Lambda(4)$ completely determines the behavior of the brane, except for some adjustable parameters which can be evaluated using some observational
data. In Sections 4, 5 and 6 we present a detailed study of the behavior of the brane-universes under consideration. In Section 7 we present a summary and discussion.

2 Field equations in 5D

In order to facilitate the presentation, and set the notation, we give a brief review of the equations in brane-world theory. We consider the metric

$$dS^2 = g_{\mu\nu}(x^\rho, y)dx^\mu dx^\nu + \epsilon \Phi^2(x^\rho, y)dy^2,$$

where \(\epsilon = -1\) or \(\epsilon = +1\) depending on whether the extra dimension is spacelike or timelike, respectively.

Everywhere we use signature \((+---\epsilon)\). The Einstein equations in five dimensions are

$$(5) G_{AB} = (5) R_{AB} - \frac{1}{2}g_{AB} (5) R = k_{(5)} (5) T_{AB},$$

where \(k_{(5)}\) is a constant introduced for dimensional considerations and \((5) T_{AB}\) is the five-dimensional energy-momentum tensor.

These equations contain the first and second derivatives of the metric with respect to the extra coordinate. These can be expressed in terms of geometrical tensors in 4D.

In absence of off-diagonal terms \((g_{4\mu} = 0)\) the dimensional reduction of the five-dimensional equations is particularly simple \[17\]. The usual assumption is that our spacetime is orthogonal to the extra dimension. Thus we introduce the normal unit \((n_A n^A = \epsilon)\) vector, orthogonal to hypersurfaces \(y = constant\),

$$n^A = \frac{\delta^A_4}{\Phi}, \quad n_A = (0, 0, 0, 0, \epsilon \Phi).$$

Then, the first partial derivatives can be written in terms of the extrinsic curvature

$$K_{\alpha\beta} = \frac{1}{2} \mathcal{L}_{\alpha g}\Phi = \frac{1}{2\Phi} \frac{\partial g_{\alpha\beta}}{\partial y}, \quad K_{A4} = 0,$$

The second derivatives, \((\partial^2 g_{\mu\nu}/\partial y^2)\), can be expressed in terms of the projection \(C_{\mu\nu4}\) of the bulk Weyl tensor in five-dimensions, viz.,

$$(5) C_{ABCD} = (5) R_{ABCD} - \frac{2}{3} (5) R_{A[C}g_{D]B} - (5) R_{B[C}g_{D]A} + \frac{1}{6} (5) R g_{A[C}g_{D]B}.$$

The field equations \[13\] can be split up into three parts. In terms of the above quantities, the effective field equations in 4D are,

$$(4) G_{\alpha\beta} = \frac{2}{3} k_{(5)}^2 \left[ (5) T_{\alpha\beta} + (5) T_4 - \frac{1}{4} (5) T g_{\alpha\beta} \right] - \epsilon \left( K_{\alpha\lambda} K^\lambda_{\beta} - K^\lambda_{\lambda} K_{\alpha}\right) + \frac{\epsilon}{2} g_{\alpha\beta} \left( K_{\lambda\rho} K^{\lambda\rho} - (K^\lambda_{\lambda})^2 \right) - \epsilon E_{\alpha\beta},$$

where

$$E_{\alpha\beta} = (5) C_{A|\alpha B|\beta} n^A n^B$$

and

$$E_{\mu\nu} = \frac{1}{\Phi} \frac{\partial K_{\alpha\beta}}{\partial y} + K_{\alpha\rho} K^\rho_{\beta} - \epsilon \frac{\Phi_{\alpha\beta}}{\Phi} - \epsilon \frac{k_{(5)}^2}{3} T_{\alpha,\beta} + \frac{1}{2} \left[(5) T_{\alpha,\beta} + (5) T_4 - \frac{1}{3} (5) T g_{\alpha\beta}\right].$$

Since \(E_{\mu\nu}\) is traceless, the requirement \(E^\mu_{\mu} = 0\) gives the inhomogeneous wave equation for \(\Phi\), viz.,

$$\Phi_{\mu,\mu} = -\epsilon \frac{\partial K_{\alpha\rho}}{\partial y} - \Phi (\epsilon K_{\lambda\rho} K^{\lambda\rho} + (5) R_4),$$
which is equivalent to \( G_{44} = k_2^2 (5) T_{44} \) from [10]. The remaining four equations are

\[
D_\mu \left( K_\alpha^\mu - \delta_\mu^\alpha K_\lambda^\lambda \right) = k_2^2 \frac{(5) T_{4\alpha}}{\Phi}.
\]  

(16)

In the above expressions, the covariant derivatives are calculated with respect to \( g_{a\beta} \), i.e., \( D g_{a\beta} = 0 \).

### 2.1 The brane-world paradigm

In the brane-world scenario our space-time is identified with a singular hypersurface (or 3-brane) embedded in an \( AdS_5 \) bulk [18], i.e., it is assumed that the five-dimensional energy-momentum tensor has the form

\[
(5) T_{AB} = \Lambda (5) \bar{g}_{AB},
\]

(17)

where \( \Lambda (5) < 0 \) is the cosmological constant in the bulk. For convenience, the coordinate \( y \) is chosen such that the hypersurface \( \Sigma : y = 0 \) coincides with the brane. Thus, the metric is continuous across \( \Sigma \), but the extrinsic curvature \( K_{\mu\nu} \) is discontinuous. Most brane-world models assume a \( Z_2 \) symmetry about our brane, namely,

\[
\begin{align*}
    ds^2 &= g_{\mu\nu}(x^\rho, +y) dx^\mu dx^\nu + \epsilon \Phi^2(x^\rho, +y) dy^2, & \text{for } y \geq 0 \\
    ds^2 &= g_{\mu\nu}(x^\rho, -y) dx^\mu dx^\nu + \epsilon \Phi^2(x^\rho, -y) dy^2, & \text{for } y \leq 0.
\end{align*}
\]

(18)

Thus

\[
K_{\mu\nu} \mid_{\Sigma^+} = -K_{\mu\nu} \mid_{\Sigma^-}.
\]

(19)

Therefore the field equations in the resulting \( Z_2 \)-symmetric brane universe can be written as

\[
(5) \bar{G}_{AB} = k_2^2 (5) \Lambda (5) \bar{g}_{AB} + (5) \bar{T}^{(brane)}_{AB},
\]

(20)

where \( \bar{g}_{AB} \) is taking as in [18] and \( (5) \bar{T}^{(brane)}_{AB} \), with \( (5) \bar{T}^{(brane)}_{AB} n^A = 0 \), is the energy-momentum tensor of the matter on the brane

\[
(5) \bar{T}^{(brane)}_{AB} = \delta_{AB}^{(5)} \delta(y) \frac{\Phi}{\Phi}.
\]

(21)

The delta function expresses the confinement of matter in the brane, hence

\[
\tau_{\mu\nu}(x^\rho, 0) = \lim_{\xi \to 0} \int_{-\xi/2}^{\xi/2} (5) \bar{T}^{(brane)}_{\mu\nu} \Phi dy.
\]

(22)

From Israel’s boundary conditions

\[
K_{\mu\nu} \mid_{\Sigma^+} = -K_{\mu\nu} \mid_{\Sigma^-} = -\epsilon k_2^2 (5) \left( \tau_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \right),
\]

(23)

and the \( Z_2 \) symmetry

\[
K_{\mu\nu} \mid_{\Sigma^+} = -K_{\mu\nu} \mid_{\Sigma^-} = -\epsilon k_2^2 (5) \left( \tau_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \right),
\]

(24)

we obtain

\[
\tau_{\mu\nu} = -\frac{2 \epsilon}{k_2^2 (5)} (K_{\mu\nu} - g_{\mu\nu} K).
\]

(25)

Then from [18] and [17] it follows that

\[
\tau_{\nu} = 0.
\]

(26)

Thus \( \tau_{\mu\nu} \) represents the total, vacuum plus matter, conserved energy-momentum tensor on the brane. It is usually separated in two parts,

\[
\tau_{\mu\nu} = \sigma g_{\mu\nu} + T_{\mu\nu},
\]

(27)
where \( \sigma \) is the tension of the brane in 5D, which is interpreted as the vacuum energy of the brane world, and \( T_{\mu\nu} \) represents the energy-momentum tensor of ordinary matter in 4D. From (24), (25) and (27) we finally get

\[
K_{\mu\nu} |_{\Sigma^+} = -\frac{\epsilon k^{(5)}_2}{2} \left( T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} (T + \sigma) \right).
\]

Substituting (28) and (17) into (13), we obtain the Einstein equations with an effective energy-momentum tensor in 4D as [19]-[23]

\[
\left( T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} (T + \sigma) \right) + \epsilon k^{(5)}_2 \Pi_{\mu\nu} - \epsilon E_{\mu\nu},
\]

where

\[
\Lambda(4) = \frac{1}{2} k^{(5)}_2 \left( \Lambda(5) - \epsilon \frac{k^{(5)}_4 \sigma^2}{6} \right),
\]

and

\[
\Pi_{\mu\nu} = \frac{1}{4} T_{\mu\alpha} T^\alpha_{\nu} - \frac{1}{12} T T_{\mu\nu} - \frac{1}{8} g_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} + 4 g_{\mu\nu} T^2.
\]

All these four-dimensional quantities have to be evaluated in the limit \( y \to 0^+ \). The above constitute the basic equations in brane-world models. They contain higher-dimensional modifications to general relativity. Namely, local quadratic energy-momentum corrections via the tensor \( \Pi_{\mu\nu} \), and the nonlocal effects from the free gravitational field in the bulk, transmitted by \( E_{\mu\nu} \). Another important novel feature is that they provide a working definition of the fundamental quantities \( \Lambda(4) \) and \( G \).

2.2 Cosmological settings

In cosmological applications the five-dimensional metric (8) is commonly taken in the form

\[
dS^2 = n^2(t, y) dt^2 - a^2(t, y) \left[ \frac{dr^2}{(1 - kr^2)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] + \epsilon \Phi^2(t, y) dy^2,
\]

where \( k = 0, +1, -1 \) and \( t, r, \theta \) and \( \phi \) are the usual coordinates for a spacetime with spherically symmetric spatial sections.

The metric coefficients are subjected to the conditions

\[
n(t, y)|_{brane} = 1, \quad a(t, y)|_{brane} = a_0(t).
\]

In this way the usual FLRW line element is recovered on the brane with \( a_0 \) as scale factor.

The corresponding field equations in the bulk can be written in a very compact form in terms of the function \( F \), which is a first integral of the field equations [24], namely,

\[
F(t, y) = ka^2 + \frac{(\dot{a} a)}{n^2} + \epsilon \frac{(a' a)}{\dot{\Phi}^2},
\]

where a prime denotes a derivative with respect to \( y \). The \( (\frac{3}{0}) \) and \( (\frac{4}{4}) \) components of (9) become

\[
F' = \frac{2 a' a^3}{3} k^{(5)}_2 T^0_0,
\]

and

\[
F = \frac{2 a^3}{3} k^{(5)}_2 T^4_4.
\]

---

2With this choice of signs, for perfect fluid \( \Pi_{\mu\nu} = (1/12) [\rho u_{\mu} u_{\nu} + \rho (\rho + 2p) h_{\mu\nu}] \) where \( h_{\mu\nu} = u_{\mu} u_{\nu} - g_{\mu\nu} \).
Now, the field equations \(1\) = \(2\) = \(3\) reduce to
\[
\left( \frac{\dot{a}}{na} \right)^2 = \frac{k_2^{(5)} \Lambda_{(5)}}{6} - \epsilon \left( \frac{a'}{a^2} \right)^2 - \frac{k}{a^2} + C a^4,
\]
where \(C\) is a constant of integration.

The ordinary matter on the brane is usually assumed to be a perfect fluid
\[
T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu},
\]
where the energy density \(\rho\) and pressure \(p\) satisfy the isothermal equation of state, viz.,
\[
p = \gamma \rho, \quad 0 \leq \gamma \leq 1.
\]

Thus, the boundary conditions \(23\), the \(Z_2\) symmetry \(24\), and \(28\) yield \(14\)
\[
\rho(t) = \frac{2\epsilon}{k_2^{(5)}(\gamma + 1)\Phi_{\text{brane}}} \left[ \frac{a'}{a} - n' \right]_{\text{brane}},
\]
and
\[
\sigma = \frac{2\epsilon}{k_2^{(5)}(\gamma + 1)\Phi_{\text{brane}}} \left[ (3\gamma + 2)\frac{a'}{a} + \frac{n'}{n} \right]_{\text{brane}}.
\]

These equations show that the tension of the brane and the energy density depend on the details of the model. They enable us to investigate the very intriguing possibility that \(\sigma\), and consequently \(G\) and \(\Lambda_{(4)}\), might vary with time.

Evaluating \(38\) at the brane we obtain the generalized Friedmann equation
\[
3 \left( \frac{\dot{a}}{a_0} \right)^2 = \Lambda_{(4)} + 8\pi G \rho - \frac{ck_2^{(5)}}{12} \rho^2 - \frac{3k}{a_0^2} + \frac{3C}{a_0^4},
\]
Except for the condition that \(n = 1\) at the brane, this equation is valid for arbitrary \(\Phi(t, y)\) and \(n(t, y)\) in the bulk. This equation allows us to examine the evolution of the brane without using any particular solution of the five-dimensional field equations. In what follows we will omit the subscript \(0\).

### 3 The equations for varying vacuum energy

In the above equation \(G\) and \(\Lambda_{(4)}\) are usually assumed to be “truly” constants. However, as we have already discussed there are several models, with reasonable physical properties, for which a variable \(\Phi\) induces a variation in the vacuum energy \(\sigma\) \(14\). In this section we show how \(13\) should be modified as to incorporate, or accommodate, the variation of these fundamental physical “constants”, matching observational predictions.

From \(20\) and \(27\) it follows that
\[
\sigma_{\nu} + T_{\nu\mu}^{\nu} = 0.
\]
For perfect fluid \(39\) this is equivalent to
\[
\dot{\rho} + (\rho + p)\Theta = -\dot{\sigma},
\]
\[
(\rho + p)a_{\nu} + p,\lambda h_{\nu}^\lambda = \sigma_{\lambda} h_{\nu}^\lambda,
\]
where \(\Theta = u_{\mu}^\mu\) is the expansion, \(a_{\nu} = u_{\nu,\lambda} u^\lambda\) is the acceleration and \(h_{\mu\nu} = u_{\mu} u_{\nu} - g_{\mu\nu}\) is the projector onto the spatial surfaces orthogonal to \(u_{\mu}\). In homogeneous cosmological models the second equation above is empty; only the first one is relevant. Namely, for the case of constant vacuum energy \(\sigma\), and the equation of state \(p = \gamma \rho\), it yields the familiar relation between the matter energy density and the expansion factor \(a\), viz.,
\[
\rho \sim \frac{1}{a^{3(\gamma + 1)}}.
\]
For the case where the vacuum energy is not constant we need some additional assumption. For example, if \(\sigma\) is a given function \(\sigma = \sigma(a)\), then we integrate the conservation equation \(15\), and substitute the resulting function \(\rho = \rho(a)\) into \(13\), and thus obtain the corresponding Friedmann equation. However, we have so far no theoretical/observational arguments for the evolution of \(\sigma\) in time.
The time variation of $G$ is usually written as $(\dot{G}/G) = qH$, where $g$ is a dimensionless parameter. As we have already mentioned in the Introduction, nucleosynthesis and the abundance of various elements are used to put constraints on $g$. The present observational upper bound is $|g| \leq 0.1$ [3]-[5]. In what follows we assume that $g$ is constant.

Since $G \sim \sigma$ and $H = \dot{a}/a$ we have $\sigma(a) = f_0 a^\gamma$, where $f_0$ is a constant of integration. Thus,

$$\dot{\rho} + 3\rho(\gamma + 1) \frac{\dot{a}}{a} = -f_0 ga(\gamma - 1) \dot{a}. \quad (47)$$

First, we consider the case where $\rho$ can be expressed in a way similar to (46), i.e., as a power function of $a$. Therefore, we find $g = \frac{-3D(\gamma + 1)}{f_0 + D}$ and $\rho = D a^{-3(\gamma + 1)/(f_0 + D)}$, where $D$ is a positive constant. In order to simplify the notation we set $f_0 = F_0 D$ and $(\gamma + 1)(1 + F_0) = \beta + 1$, (49)

$$\rho = \frac{D}{a^{(\beta+1)}}, \quad \sigma = \frac{D(\gamma - \beta)}{(\beta + 1)a^{(\beta+1)}}. \quad (50)$$

Notice that $F_0 \neq 0$ ($\beta \neq \gamma$), otherwise $G = 0$. We also have,

$$\frac{\dot{G}}{G} = -3(\beta + 1)H. \quad (51)$$

Before going on, we should become aware of the observational bounds of $\beta$. The lower bound comes from the obvious requirement $d\rho/da < 0$, while the upper one comes from the observation that $|g| \leq 0.1$ Thus,

$$-1 < \beta \leq -0.966, \quad \beta = -0.983 \pm 0.016. \quad (52)$$

We will see that $\beta$ is related to $q$ and $\Omega_\rho$, the deceleration and density parameters, respectively$^3$.

In order to obtain the evolution equation for $a$ we substitute (50) into (43)

$$3 \left( \frac{\dot{a}}{a} \right)^2 = \frac{k_5^2}{2}\Lambda(5) - \frac{e k_5^4}{12} \left( \frac{\gamma + 1}{\beta + 1} \right)^2 \frac{D^2}{a^{(\beta+1)}} - \frac{3k}{a^2} + \frac{3C}{a^4}. \quad (53)$$

Let us now introduce the quantities

$$x = a^{(\beta+1)}, \quad A = -\frac{ek_5^4}{4} (\gamma + 1)^2 D^2, \quad C = \frac{3k_5^2}{2} (\beta + 1)^2 \Lambda(5), \quad (54)$$

in terms of which (53) becomes

$$\left( \frac{dx}{dt} \right)^2 = A + Cx^2 - 9(\beta + 1)^2 \left[ k_x(x^{(6\beta+4)/(3\beta+3)}) - C_x(x^{(6\beta+2)/(3\beta+3)}) \right]. \quad (55)$$

$^3$Because of the difficulty of reliably determining $g$, the limit set by (52) is just exploratory. We use this number here, not to make specific observational suggestions, but in order to be able to discriminate between what is possible or not.
This equation admits exact integration, in terms of elementary functions, for a wide variety of parameters $A$, $C$, $\beta$ and $C$. However, we are interested in the physical models for which $\Lambda \sim H^2$. This introduces some additional features which we discuss below.

Let us now notice that (48) is not the general solution of (47), which is

$$\rho = \frac{D}{a^{3(\gamma + 1)}} - \frac{f_0 g}{(3\gamma + 3 + g)} a^g,$$

where $D$ is the integration constant. Indeed, setting $D = 0$ and denoting $g = -3(\beta + 1)$ and $D = f_0(\beta + 1)/(\gamma - \beta)$, we recover the solution (50). We will see that, for the models under consideration, the observational requirement (52) demands $D = 0$.

### 3.2 $\Lambda(4) \sim H^2$

Without loss of generality we now set

$$\Lambda(4) = \xi H^2,$$

where $\xi$ is an unknown function of time which we will determine below. Then using (50) and (50) we get

$$\Lambda(4) = \frac{1}{2} k^2(5) \Lambda(5) - \epsilon \frac{k^2(5) D^2(\gamma - \beta)^2}{12(\beta + 1)^2 a^{6(\beta + 1)}} = \xi \left( \frac{\dot{a}}{a} \right)^2.$$

In terms of the notation (54), this equation becomes

$$\left( \frac{dx}{dt} \right)^2 = 3 \xi \left[ A \frac{(\gamma - \beta)^2}{(\gamma + 1)^2} + C x^2 \right].$$

Thus, equating (55) and (55) we get

$$A \left[ \frac{3(\gamma - \beta)^2}{\xi(\gamma + 1)^2} - 1 \right] + C \left( \frac{3}{\xi} - 1 \right) x^2 + 9(\beta + 1)^2 \left[ k x^{(6\beta + 4)/(3\beta + 3)} - C x^{(6\beta + 2)/(3\beta + 3)} \right] = 0.$$

This equation should take place for all $t$ and $x$. Therefore, it is important to notice that for the allowed values of $\beta$, given by (52), the powers of $x$ in front of $k$ and $C$ are far from being zeroth or second order. Consequently, from (60) we get two compatibility conditions. The first one is a condition of physical nature. Namely,

$$k = 0, \quad C = 0.$$

Thus, the resulting cosmological models have flat space sections. This is compatible with astrophysical data\textsuperscript{4} from BOOMERANG\textsuperscript{25} and WMAP\textsuperscript{26}. Besides, the constant $C$, which is related to the bulk Weyl tensor and corresponds to an effective radiation term, is constrained to be small enough at the time of nucleosynthesis and it should be negligible today\textsuperscript{27}.

The second condition consists of three different options of mathematical nature:

1. If $A \neq 0$, $C \neq 0$, then (60) requires $\xi = 3$ and $(\gamma - \beta)^2 = (\gamma + 1)^2$;

2. If $C = 0$, then

$$\xi = \frac{3(\gamma - \beta)^2}{(\gamma + 1)^2}.$$

3. $\xi$ is a function of $x$.

\textsuperscript{4}In order to avoid misunderstanding, here do not use cosmological parameters emerging from BOOMERANG or WMAP. The point is that the conditions in (54), for a model which does not obey the usual Einstein equations, do not contradict well known results obtained under the assumption that the (usual) Einstein equations are valid.
Certainly, the first option cannot take place by virtue of (52); only the second and the third are possible ones. We will see them at work in Sections 4 and 5, respectively.

If we apply the same arguments used above to the general solution (56) and \( \sigma = f_0 a^2 \), instead of (50), we get two compatibility conditions. The first one for \( D \neq 0 \) is very restrictive and requires \( g = -3(\gamma + 1) \), which we disregard on observational grounds since \( |g| \leq 0.1 \). The second condition for \( D = 0 \) yields the case discussed above.

The conclusion from this section is that the cosmological models where \( \dot{G}/G \sim H \) and \( \Lambda(4) \sim H^2 \) have flat space sections and zero (or negligible) Weyl radiation from the bulk.

Their time evolution follows from

\[
\int \frac{dx}{\sqrt{A + C x^2}} = (t - t_0),
\]

where \( t_0 \) is a constant of integration.

The solution of this integral depends on the constants \( A \) and \( C \), which contain \( \epsilon \) and \( \Lambda(5) \). Thus, it allows us to study some important questions such as: How does the bulk cosmological constant affect the behavior of the brane universe? Does it influence the choice of the signature of the extra dimension?

In the next Sections we analyze the solutions of (63) for \( \Lambda(5) = 0, \Lambda(5) > 0 \) and \( \Lambda(5) < 0 \).

4 Bulk with \( \Lambda(5) = 0 \)

We consider first the behavior of the model with vanishing bulk cosmological constant. This case is important because, as we will see later, it is the limit of the models with \( \Lambda(5) \neq 0 \) and spacelike signature.

From (54) it follows that the extra dimension has to be spacelike (\( \epsilon = -1 \)). The expansion factor is given by

\[
a(t) = \left[ \frac{k_2^2(5)}{2} (\gamma + 1) D t \right]^{1/(3(\beta + 1))},
\]

where we have set \( t_0 = 0 \), in such a way that the big bang occurs at \( t = 0 \). Here \( \beta \) is related to the deceleration parameter \( q = -\ddot{a}/a^2 \) as

\[
q = 2 + 3\beta.
\]

Therefore, the ratio (51) becomes \( \dot{G}/G = -(q + 1) H \), which is the same as in separable models (14). For the vacuum energy we find

\[
\sigma = \frac{2(\gamma - \beta)}{(\beta + 1)(\gamma + 1) k_2^2(5)t},
\]

Consequently, the 4D effective cosmological term \( \Lambda(4) \) and the gravitational coupling \( G \) vary as

\[
\Lambda(4) = \frac{3(\gamma - \beta)^2}{(\gamma + 1)^2} H^2, \quad 8\pi G = \frac{k_2^2(5)(\gamma - \beta)}{\gamma + 1} H,
\]

where \( H = \dot{a}/a \) is the Hubble parameter, viz.,

\[
H = \frac{1}{3(\beta + 1)} \frac{1}{t}.
\]

Thus, here the second option (62) is taking place. There is a remarkable connection between \( \beta, \gamma \) and the density parameter \( \Omega_\rho \). Specifically,

\[
\frac{8\pi G \rho}{3H^2} = \Omega_\rho = \frac{2(\gamma - \beta)(\beta + 1)}{(\gamma + 1)^2}.
\]

Thus, giving \( \gamma \) and \( \Omega_\rho \) we obtain \( \beta \), viz.,

\[
(\beta + 1) = \frac{1}{2} (\gamma + 1) \left( 1 \pm \sqrt{1 - 2\Omega_\rho} \right),
\]
which gives an upper bound for $\Omega_\rho$. Namely,

$$\Omega_\rho \leq \frac{1}{2},$$  \hspace{1cm} (71)

regardless of the specific value of $\gamma$ and $q$.

From the above equations, it follows that the density parameter $\Omega_\rho$ can, in principle, be determined by measurements of the deceleration parameter $q$

$$\Omega_\rho = \frac{2(3\gamma - q + 2)(q + 1)}{9(1 + \gamma)^2}. \hspace{1cm} (72)$$

This is a useful feature which results from the assumption that $\Lambda$ vanishes. A similar situation occurs in the familiar FRW models. We also note the relationship between the density parameters, viz.,

$$\frac{\Lambda_{(4)}}{3H^2} = \Omega_\Lambda = \frac{9\Omega_\rho^2(1 + \gamma)^2}{4(1 + q)^2}. \hspace{1cm} (73)$$

4.1 Behavior of the model for different $\Omega_\rho$

Let us study the behavior of the model for different values of the density parameter $\Omega_\rho$, within the range of values allowed in (71). From (64) and (70) it follows that

$$a(t) \sim t^{2/3(\gamma + 1)[1 \pm \sqrt{1 - 2\Omega_\rho}]}. \hspace{1cm} (74)$$

Thus, in the upper limit, for $\Omega_\rho \to 1/2$, the evolution of the scale factor of the universe matches the one in familiar FRW models, viz., $a(t) \approx t^{2/3(\gamma + 1)}$. Accordingly, the deceleration parameter (65) reduces to $q \approx (1 + 3\gamma)/2$ which is the usual one in FRW cosmologies. An interesting feature of the model is that, in this limit neither the gravitational coupling is constant, nor the cosmological term becomes zero. Instead we have

$$8\pi G = \frac{k_0^2}{2} H, \quad \Lambda_{(4)} = \frac{3}{4} H^2, \quad H = \frac{2}{3(\gamma + 1)} \frac{1}{t},$$

where $H$ is the usual Hubble parameter in FRW models, as expected.

Let us now study the behavior of our model for other values of $\Omega_\rho$.

It is clear that the deviation from the FRW models increases as $\Omega_\rho$ moves away from 1/2. But for every value of $\Omega_\rho$ there are two possible models; one for each sign in front of the root in (70). They both reach the FRW model for $\Omega_\rho = 1/2$. The question now is whether they satisfy physical conditions.

Since a reliable and definitive determination of $\Omega_\rho$ has thus far eluded cosmologists, in our discussion we consider several values of $\Omega_\rho$, although $\Omega_\rho \approx 0.1 - 0.3$ seem to be the most probably options.

4.1.1 Accelerated expansion

We consider the behavior of dust-filled universes ($\gamma = 0$). In Table 1 we illustrate the evolution of such universes corresponding to the negative sign in (70). It is interesting to note that there are no arbitrary parameters or constants in the solution. Specifying $\Omega_\rho$ we find $\beta$, $q$ and $\Omega_\Lambda \equiv \Lambda_{(4)}/3H^2$. The age of the universe is calculated assuming $H \approx 0.7 \times 10^{-10} \text{yr}^{-1}$.

| $\Omega_\rho$ | $\Omega_\Lambda$ | $\beta_{(-)}$ | $q_{(-)}$ | $(t_{(-)}/10^{10})$ yr |
|-------------|----------------|--------------|-----------|-------------------------|
| 0.5         | 0.25           | -0.500       | 0.500     | 0.95                    |
| 0.4         | 0.522          | -0.723       | -0.170    | 1.71                    |
| 0.3         | 0.664          | -0.816       | -0.448    | 2.58                    |
| 0.2         | 0.783          | -0.887       | -0.661    | 4.21                    |
| 0.1         | 0.889          | -0.947       | -0.841    | 8.98                    |
| 0.08        | 0.907          | -0.958       | -0.874    | 11.33                   |
| 0.04        | 0.936          | -0.979       | -0.938    | 22.67                   |
This case is interesting because it shows an accelerated expansion of the universe, in agreement with modern observations. The acceleration is driven by the repulsive effect of the “dark energy” associated with $\Lambda$, which clearly dominates the evolution here. We notice that $\Omega_\rho + \Omega_\Lambda \neq 1$. There is a contribution, $\Omega_\rho^2$, from the quadratic correction in the generalized FRW equation (43), so that $\Omega_\rho + \Omega_\Lambda + \Omega_\rho^2 = 1$. This contribution decreases as the universe gets older. For $\Omega_\rho < 0.4$, the universe is quite old and $\Omega_\rho^2$ is “negligible”.

We have already mentioned that for $\Omega_\rho = 0.5$, the expansion scale factor is the same as in the de Einstein-de Sitter solution, although the matter content is totally different. We now see that $\Omega_\rho^2 = \Omega_\Lambda = 0.25$ for this solution. This is another example of the well known fact that in GR the same geometry can be attributed to different matter distributions.

Dynamical mass measurements from WMAP Mission reveal that the matter content of the universe is about 27% of the critical density (4% Atoms, 23% Cold Dark Matter). The rest 73% is usually declared either as “missing”, or Dark energy. Also, according to recent measurements the acceleration parameter is, roughly, $-0.5 \pm 0.2$. We note that the entries in the third and fourth rows are consistent with this picture of the universe. Another remarkable feature here is that the age of the universe is much larger than in the usual FRW model (first row).

### 4.1.2 Decelerated expansion

We now consider the solution with positive sign in front of the root in (70). For the purpose of comparison, we again consider the evolution of dust universes under the same set of values of $\Omega_\rho$ as in the model with negative sign. The relevant parameters are presented in Table 2.

| $\Omega_\rho$ | $\Omega_\Lambda$ | $\beta_{(+)}$ | $q_{(+)}$ | $(\ddot{T}_{(+)}/10^{10})\text{ yr}$ |
|-------------|-----------------|----------------|---------|-----------------------------|
| 0.5         | 0.25            | -0.500         | 0.500   | 0.95                        |
| 0.4         | 0.076           | -0.276         | 1.170   | 0.65                        |
| 0.3         | $3.37 \times 10^{-2}$ | -0.184       | 1.448   | 0.58                        |
| 0.2         | $1.27 \times 10^{-2}$ | -0.113       | 1.661   | 0.53                        |
| 0.1         | $0.37 \times 10^{-2}$ | -0.053       | 1.841   | 0.50                        |
| 0.08        | $0.17 \times 10^{-2}$ | -0.042       | 1.874   | 0.49                        |
| 0.04        | $0.41 \times 10^{-3}$ | -0.020       | 1.938   | 0.48                        |

In the present case the repulsion associated with the cosmological term is negligible. Here the evolution is dominated by the quadratic correction term. As an illustration consider the entries on the third row, for which $\Omega_\rho^2 \approx 0.7$. The huge gravitational attraction produced by this term, for a spacelike extra dimension, explains the large deceleration parameter.

This solution presents a number of interesting features. For example, a decrease in $\Omega_\rho$ entails a decrease in the age of the universe. This is the opposite of what we see in Table 1. Also, $\ddot{T}_{(+)}$ is almost the same for $\Omega_\rho \approx 0.1 - 0.3$, while in Table 1 the age of the universe changes almost four times in the same range.

However, the solution with positive sign in (70) seems to have little in common with present observations. Firstly, the universe is quite young. Secondly, it does not fit the observational requirements on the ratio $\dot{G}/G$, i.e. on $\beta$, neither on the deceleration parameter.

We would like to finish this Section by stressing the fact that the parameter $\beta$ is related to (i) the ratio $\dot{G}/G$, (ii) the deceleration parameter, and (iii) the density parameter $\Omega_\rho$ through the equations (52), (53) and (70), respectively.

5 We would like to emphasize that here we are only using the measured values of $q$, the rest of the parameters are obtained from the model.

6 This is way the quadratic corrections $\Omega_\rho^2$ dominate.
5 de Sitter bulk, $\Lambda_{(5)} > 0$

Although in brane-world theory our universe is embedded in a higher-dimensional space with negative cosmological constant, the solutions to the evolution equation \( (5) \) depend analytically on $\Lambda_{(5)}$, allowing us to continue to the range of positive bulk cosmological constant. In this Section we explore the case with $\Lambda_{(5)} > 0$. An attractive feature of this case is that the extra dimension can be either spacelike or timelike. We will discuss these two cases separately.

5.1 Spacelike extra dimension $\epsilon = -1$

The integration of (63) yields

\[
a^{3(\beta+1)} = \frac{k_{(5)}(\gamma + 1)D}{(\beta + 1)\sqrt{6\Lambda_{(5)}}}\sinh \sqrt{C}t, \tag{76}
\]

where once more we have set $t_0 = 0$ and

\[
\sqrt{C} = k_{(5)}(\beta + 1)\sqrt{\frac{3\Lambda_{(5)}}{2}}. \tag{77}
\]

The deceleration parameter here is

\[
q = 2 + 3\beta - 3(\beta + 1)\tanh^2 \sqrt{C}t. \tag{78}
\]

The matter and vacuum density vary as

\[
\rho = \frac{2\sqrt{C}}{k_{(5)}^2(\gamma + 1)\sinh \sqrt{C}t}, \quad \sigma = \frac{2(\gamma - \beta)}{k_{(5)}^2(\gamma + 1)(\beta + 1)\sinh \sqrt{C}t}. \tag{79}
\]

As in the previous solution we have $\dot{G}/G = -3(\beta + 1)H$ and

\[
\Lambda_{(4)} = \frac{3(\gamma - \beta)^2}{(\gamma + 1)^2}H^2 + \frac{k_{(5)}^2\Lambda_{(5)}}{2}\left[\frac{(\gamma + 1)^2 - (\gamma - \beta)^2}{(\gamma + 1)^2}\right], \tag{80}
\]

\[
8\pi G = \frac{k_{(5)}^2(\gamma - \beta)}{(\gamma + 1)H\sqrt{1 - \frac{k_{(5)}^2\Lambda_{(5)}}{6H^2}}}, \tag{81}
\]

where

\[
H = \frac{\sqrt{C}}{3(\beta + 1)\tanh \sqrt{C}t}. \tag{82}
\]

We note that $\Lambda_{(4)}$ is always positive and $H$ is a monotonically decreasing function of time bounded below by $H = k_{(5)}\sqrt{\Lambda_{(5)}/6}$. This assures the positivity of $G$, as long as $\gamma > \beta$.

The general behavior of the solution is as follows

1. For $\Lambda_{(5)} = 0$ we recover the previous solution.
2. For small values of $t$, near the big bang, this model behaves exactly as the previous one (61)-(65).
3. At large times the universe is expanding with positive acceleration, $q \approx -1$. The expansion is exponential and $\Lambda_{(4)}$ tends to $\tilde{\Lambda} \equiv k_{(5)}^2\Lambda_{(5)}/2$.

Now, if we combine (68) and (82) we obtain

\[
C = 3H^2(2 + 3\beta - q)(1 + \beta). \tag{83}
\]
Using this expression into $8\pi G \rho$ from (78) we get

$$\beta = \gamma - \frac{3\Omega_\rho(\gamma + 1)^2}{2(1 + q)}.$$  \hfill (84)

The condition $(\beta + 1) > 0$ sets an upper limit on the density parameter, viz.,

$$\Omega_\rho < \frac{2(1 + q)}{3(1 + \gamma)}. \hfill (85)$$

We also obtain an interesting expressions for $\Lambda(4)$. Namely,

$$\Lambda(4) = \left[ \frac{(\gamma - \beta)^2(1 + q) + (1 + \gamma)^2(2 + 3\beta - q)}{(1 + \beta)(1 + \gamma)^2} \right] H^2. \hfill (86)$$

Coming back to (77), here the factor $(\xi)$ in front of $H^2$ is not constant, because $q$ is a function of time. Thus, this solution corresponds to the third option mentioned after (62).

For the gravitational “constant” we get

$$8\pi G = \frac{k^2_{(5)}}{\sqrt{3(\gamma + 1)\sqrt{1 + \beta}}} H \hfill (87)$$

The above equations (84), (86) relate the observational quantities $\gamma$, $\Omega$, $q$ and $\beta$. The age of the universe is

$$T = \frac{1}{C} \tanh^{-1} \left( \frac{2 + 3\beta - q}{3(\beta + 1)} \right)^{1/2}, \hfill (88)$$

where $C$ is obtained from (83). Finally, the five-dimensional quantities $k_{(5)}$ and $\Lambda_{(5)}$ can be evaluated from (87) and (77). We note that there are no free parameters left in the solution.

### 5.1.1 Characteristic time

It is important to notice that a non-vanishing cosmological constant in the bulk induces a natural time scale, in $4D$. We define it as $\tau_s = \sqrt{3/\Lambda}$, where $\Lambda \equiv k^2_{(5)}\Lambda_{(5)}/2$. Thus, $\tau_s = (\beta + 1)/\sqrt{C}$ or, in terms of observational quantities

$$\tau_s = \frac{1}{H} \left[ \frac{4(1 + q)^2 - 9\Omega_\rho^2(1 + \gamma)^2}{4(1 + q)^2\Omega_{\Lambda} - 9\Omega_\rho^2(1 + \gamma)^2} \right]^{1/2}. \hfill (89)$$

We remark that although the quantities on the r.h.s. are functions of time, $\tau_s$ is a “universal” constant fixed by the five-dimensional embedding bulk. This equation shows how we can evaluate this constant from measurements performed in $4D$. The denominator vanishes and $\tau_s \rightarrow \infty$ when the density parameters are related as in (73). Thus, the “upper” bound for $\tau_s$ takes place in the models discussed in the previous Section.

In order to get an expression for the lower bound, we rewrite (89) as

$$\frac{4(1 + q)^2}{9\Omega_\rho^2(1 + \gamma)^2} = \frac{H^2\tau_s^2 - 1}{\Omega_{\Lambda}H^2\tau_s^2 - 1}. \hfill (90)$$

From (84) it follows that the numerator in (87) is always positive. Then, from $\Omega_{\Lambda} < 1$ we obtain $(H^2\tau_s^2 - 1) > 0$. Consequently from (90) we get $(\Omega_{\Lambda}H^2\tau_s^2 - 1) > 0$. Thus

$$\frac{1}{H\sqrt{\Omega_{\Lambda}}} < \tau_s < \infty \hfill (91)$$

For a universe with $\Omega_{\Lambda} \approx 0.7$ and $H \approx 0.7 \times 10^{-10}$ yr$^{-1}$, the lower bound for the characteristic scale is $\approx 1.7 \times 10^{10}$ years or 17 billion years, which is more that the age of the universe according to the data from WMAP Mission.
5.1.2 Observational constraints

Let us now use the observational bound (92). From (84) we obtain

\[ \frac{3\Omega_\rho(1 + \gamma)}{2} - 1 < q < \frac{3\Omega_\rho(1 + \gamma)^2}{2(\gamma + 0.966)} - 1, \]

where we have assumed \((\gamma + 0.966) > 0\). This equation implies, \(q < 0\), an accelerated expansion of the universe in the range of values allowed for \(\Omega_\rho\). However, not all \(q\) and \(\Omega_\rho\) generate adequate physical models. We have to take into account that here \(\Lambda(5) > 0\). Therefore, from (54) it follows that \(C > 0\). Consequently, (83) requires \((2 + 3\beta - q) > 0\).

If we apply the observational bounds (92), as well as (85), we get

\[ -1 < q < -0.88, \quad \Omega_\rho \leq \frac{0.08}{(1 + \gamma)}, \]

which are even more restrictive than (92). According to the present estimates for \(q\) and \(\Omega_\rho\) we are nowhere near these values. Thus, although the present model is attractive from a theoretical point of view, it seems to have limited practical application.

5.2 Timelike extra dimension \(\epsilon = +1\)

For completeness we now consider the case where the fifth dimension is timelike, although these models are controversial in many respects, including the problems of causality and quantization. We are not going to discuss these problems here, instead we refer the interested reader to [28] and references therein.

In this case the scale factor is given by

\[ a^{3(\beta+1)} = \frac{k(5)(\gamma + 1)D}{(\beta + 1) \sqrt{6\Lambda_{(5)}}} \cosh \sqrt{C}t, \]

where once more we have set \(t_0 = 0\). The expressions for the physical quantities \(\rho, \sigma, G, \Lambda(4)\) and \(H\) are formally obtained from the solution discussed in Section 5.1 by changing \(\sinh \sqrt{C}t \to \cosh \sqrt{C}t\) and \(\tanh \sqrt{C}t \to 1/\tanh \sqrt{C}t\). However, there is a qualitative difference between the solutions.

1. This is a “bouncing” solution where the universe never collapses to a singularity. The universe is always expanding with positive acceleration.

2. The Hubble parameter is an increasing function of time and is bounded above by \(k(5)\sqrt{\Lambda_{(5)}}/6\).

3. The vacuum energy must be negative in order to ensure the positiveness of \(G\). Namely,

\[ G = \frac{k^2(5)(\beta - \gamma)\sqrt{C}}{3(\gamma + 1)(\beta + 1) \cosh \sqrt{C}t}, \quad \beta > \gamma \]

We conclude that the model with a timelike extra dimension seems to be of no observational significance because for all \(t\) we have \(q < -1\), which is the opposite to what we expect \((q > -1)\), based on recent measurements.

5.3 Asymptotic behavior and classical inflation

The late time behavior of solutions with \(\Lambda_{(5)} > 0\) is similar for both signatures, \(\epsilon = -1\) or \(\epsilon = +1\), although they drastically differ at the origin of the universe. In both cases the expansion factor becomes identical to the one in the de Sitter solution,

\[ a(t) \sim \exp \sqrt{(\Lambda/3)t} = \exp(t/\tau_s), \]
as \( t \gg \tau_s \). It is clear that we are far from this asymptotic regime. It should be emphasized that this exponential behavior does not arise from a “false-vacuum” equation of state \( p = -\rho \) as in inflation, because \( \gamma \neq -1 \) in these solutions. The reason for this is that the assumption \( \dot{G}/G = g H \) is equivalent to the requirement that \( \rho \) and \( \sigma \) form a combined fluid with energy density \( \bar{\rho} = \rho + \sigma \) and pressure \( \bar{p} = \beta \bar{\rho} \). Then, the observational constraint \(-1 < \beta < -0.966 \) implies that the combined fluid behaves nearly like a cosmological constant, which dominates at late times and thus producing inflation.

5.3.1 Vacuum equation of state

For consistency, we should now show that this model is compatible with classical inflation for the equation of state of false-vacuum, \( \gamma = -1 \). Indeed, if \( \gamma = -1 \), then from (54) it follows that \( A = 0 \). Thus, the evolution equation (63) requires \( C > 0 \), which entails \( \Lambda(5) > 0 \). Integrating (63) we get \( x = \exp \sqrt{|C|}(t - t_0) \), which in terms of the original notation is identical to the de Sitter expression (96).

We conclude this section with the following comments.

(i) From (95) it follows that for a every given value of \( \Omega_\rho \), there is a range of possible values for \( \Omega_\rho \). Similarly, for a given \( \Omega_\rho \), there is a range of allowed values for \( q \), which is given by (96). This is different from the case with \( \Lambda(5) = 0 \) where they are related by (72). Therefore, for \( \Lambda(5) \neq 0 \) the sole specification of one of these parameters is not enough to determine the characteristics of the model. Here we have to specify both, independently. This additional degree of freedom is a consequence of the introduction of a non-vanishing cosmological constant in the bulk.

(ii) Measurements in 4D allow to “predict” the value of the five-dimensional constants \( k(5) \) and \( \Lambda(5) \). Namely, if we measure \( (\Omega_\rho, q) \), then \( \beta \) follows from (84). The value of \( \Omega_\Lambda \) is then obtained from (86). Finally, from (87) and (89) we evaluate the 5D constants \( k(5)^2 \) and \( \tau_s \) (and/or \( \Lambda(5) \)) in terms of \( H^{-1} \). If we have \( (\Omega_\rho, \Omega_\Lambda) \), then \( \beta \) and \( q \) are given by the solutions of the system of equations (84) and (86), then following the same steps as above we get the rest of the parameters.

6 Anti-de Sitter bulk, \( \Lambda(5) < 0 \)

This case is important, it corresponds to the brane world scenario where our universe is identified with a singular hypersurface (or a three-brane) embedded in an AdS_5 bulk.

For \( \Lambda(5) < 0 \) the extra dimension has to be spacelike and the evolution of the scale factor is given by

\[
a^{3(\beta+1)} = \frac{k(5)(\gamma + 1)D}{(\beta + 1)\sqrt{6|\Lambda(5)|}} \sin \sqrt{|C|}t, \quad (97)
\]

where once more we have set \( t_0 = 0 \) and

\[
\sqrt{|C|} = k(5)(\beta + 1)\sqrt{3|\Lambda(5)|/2}. \quad (98)
\]

In the present case the deceleration parameter is

\[
q = 2 + 3\beta + 3(\beta + 1)\tan^2 \sqrt{|C|}t, \quad (99)
\]

where

\[
|C| = 3H^2(q - 2 - 3\beta)(1 + \beta). \quad (100)
\]

This model is formally obtained from the solution in Section 5.1 by making the change \( \sqrt{\Lambda(5)} \to i\sqrt{|\Lambda(5)|} \). Thus \( \sqrt{C} \to i\sqrt{|C|} \). Consequently, \( (\sinh \sqrt{C}t)/i \to \sin \sqrt{|C|}t \). However, they are drastically different from each other.
The present solution \( (\Lambda(5) < 0) \) represents a spatially flat but recollapsing universe. The recollapse time \( T_{\text{rec}} \) is given by \( \sin \sqrt{|C|}T_{\text{rec}} = 0 \), which in terms of the characteristic time \( \tau_s \) defined in Section 5.1.1 becomes\(^7\)

\[
T_{\text{rec}} = \frac{\pi \tau_s}{6(\beta + 1)}.
\]

We note that for the physical values \( |C| = 2 \), \( T_{\text{rec}} > 15\tau_s \). The age of the universe can be written as

\[
T = \frac{2T_{\text{rec}}}{\pi} \tan^{-1}\left( \frac{q - 2 - 3\beta}{3(\beta + 1)} \right)^{1/2}.
\]

In Table 3 we use this solution to estimate the parameters \( \Omega_\Lambda, q \), in dust-filled universes, with various values of \( \Omega_\rho \). We also evaluate, in units of \( H^{-1} \), the characteristic and recollapse time, \( \tau_s \) and \( T_{\text{rec}} \), as well as the age of the universe \( T \).

| \( \Omega_\rho \) | \( \Omega_\Lambda(\Lambda) \) | \( q \) | \( H\tau_s \) | \( H\frac{T_{\text{rec}}}{T} \) |
|---|---|---|---|---|
| 0.5 | 0.495 | \(-0.236 \pm 0.013\) | 0.278 | 7.943 | 6.569 |
| 0.4 | 0.596 | \(-0.389 \pm 0.011\) | 0.311 | 9.044 | 7.650 |
| 0.3 | 0.697 | \(-0.542 \pm 0.008\) | 0.359 | 10.760 | 8.396 |
| 0.2 | 0.798 | \(-0.694 \pm 0.006\) | 0.487 | 13.004 | 9.246 |
| 0.1 | 0.898 | \(-0.846 \pm 0.003\) | 1.005 | 20.254 | 10.048 |
| 0.08 | 0.919 | \(-0.877 \pm 0.002\) | 1.211 | 25.996 | 11.417 |

For a given \( \Omega_\rho \) the values of \( q \), as obtained from \( (12) \), are spread over a “small” range. In other words, if we know \( \Omega_\rho \), we get \( q \) with a great accuracy, which is up to 1\% - 5\%. We should emphasize that this precision comes from the experimental bounds on \( \dot{G}/\dot{G} \). Then, we use the mean value of \( q \) to obtain \( \beta \) from \( (34) \), which we substitute into \( (30) \) to get \( \Omega_\Lambda \).

We note that the entries on rows two, three and four are similar to those in Table 1. Here the deceleration parameter \( q \) is a monotonically increasing function of time and changes its sign for \( \beta \) less than \( \approx -0.66 \). Therefore, for the values allowed by \( (52) \), the universe initially expands with acceleration \( (q < 0) \), then changes sign becoming positive and \( q \to +\infty \) for \( t \to T_{\text{rec}} \).

This behavior is a consequence of the effective cosmological term \( \Lambda_{(4)} \) which is positive near the big bang and, as \( t \to T_{\text{rec}} \), becomes

\[
\Lambda_{(4)} \to \frac{k^2_{(5)}\Lambda(5)}{2} \left[ 1 - \frac{(\gamma - \beta)^2}{(\gamma + 1)^2} \right],
\]

which is negative for \( \Lambda(5) < 0 \).

Finally, we would like to note some similarities between solutions with a spacelike extra dimension \( \epsilon = -1 \). Firstly, they have an analogous behavior near the big bang, regardless of the value of the bulk cosmological constant. Secondly, the gravitational coupling of geometry to matter, \( G \), is divergent for \( t \to 0 \). This last feature is of particular interest, it suggests that the gravitational interaction was much stronger in the past than at the present time.

### 7 Summary and conclusions

In summary, we have studied the consequences of the conditions \( \dot{G}/\dot{G} = gH \), with \( |g| \leq 0.1 \), and \( \Lambda_{(4)} \sim H^2 \) on cosmological models based on the brane-world scenario. These two conditions lead to the requirement \( k = 0, C = 0 \), in \( (11) \). The same physics is obtained if we assume \( \dot{G}/\dot{G} = gH \) and \( k = 0, C = 0 \). In this case \( \Lambda_{(4)} \sim H^2 \) is not an extra condition but a consequence of these assumptions. However, if we choose to postulate \( \Lambda_{(4)} \sim H^2 \), and \( k = 0 \),\(^7\)In this case we use the magnitude of \( \Lambda_{(5)} \), namely, \( \Lambda = k^2_{(5)}\Lambda(5)|1/2 \). Therefore, for the evaluation of \( \tau_s \) in \( (30) \) we have to take the magnitude of the quantity in parenthesis, otherwise \( \tau_s \) would be a complex number.
\( C = 0 \), then we do not obtain \( \dot{G}/G = gH \), except for the case where \( \Lambda_{(5)} = 0 \). In other words, we recover only the models discussed in Section 4.

The study of these cosmological models is attractive for several reasons. Firstly, conditions (11) and (12) are motivated by a number of theoretical and experimental observations. Secondly, brane theory provides a framework within which to consider the simultaneous variation of \( G \) and \( \Lambda_{(4)} \), because they are related to the vacuum energy, which in principle could vary with time. Thirdly, the theory has no arbitrary parameters.

Although we introduce two “new” parameters, besides the observed \( q \) (or \( \beta \)), viz., \( \Lambda_{(5)} \) and \( k_{(5)} \), they can be evaluated by means of quantities measured in 4D only. Specifically, for \( \Lambda_{(5)} \) we have

\[
\Lambda_{(5)} = \frac{(2 + 3\beta - q)(\gamma - \beta)\sqrt{1 + q}}{\sqrt{3(\gamma + 1)(1 + \beta)^{3/2}} \left( \frac{H^3}{4\pi G} \right)}.
\]

As an illustration we consider the entries on the third row in Table 3, for which we get \( \Lambda_{(5)} \approx -8.8 \times 10^{-46} \text{ gr cm}^{-3} \text{ s}^{-1} \). The value of \( k_{(5)}^2 \) can be obtained from (57) as \( k_{(5)}^2 \approx 26.23 \times 10^{10} \text{ gr}^{-1} \text{cm}^3 \text{ s}^{-1} \). Thus, the r.h.s. of the field equations in (9) is a very “small” number, viz., \( k_{(5)}^2 \Lambda_{(5)} \approx 2.3 \times 10^{-34} \text{ s}^{-2} \). In terms of the Plank units of time \( t_{\text{Plank}} = 10^{-43} \text{ s} \), we get \( k_{(5)}^2 \Lambda_{(5)}/2 \approx |\Lambda_{(4)}| \approx 10^{-120} t_{\text{Plank}}^{-2} \), as one expected. Although this number is small, it has a crucial influence on the global behavior of the universe.

The model requires the universe to have flat \((k = 0)\) space sections, which is consistent with observations. Also, it is interesting that all possible scenarios, except those in Table 2, agree with the observed accelerated expansion of the universe and are dominated by the Dark energy associated with \( \Lambda_{(4)} \).

In addition, the observed value of \( q = -0.5 \pm 0.2 \) fits the ones predicted by the models with \( \Lambda_{(5)} \leq 0 \), as shown in Tables 1 and 3 for \( \Omega_\rho \approx 0.1 - 0.3 \). This is different from the models with positive bulk cosmological constant. These present good physical properties but do not appear to be compatible with current observations, regardless of the signature of the extra dimension.

In all cases the cosmological models with spacelike extra dimension have similar behavior near the big-bang. If \( \Lambda_{(5)} = 0 \), then the induced cosmological term in 4D is positive and the brane universe expands continuously in a power-law time-dependence. If \( \Lambda_{(5)} > 0 \), then the universe becomes dominated by a positive cosmological term \( \Lambda_{(4)} \), which tends to a constant value. The effect is an asymptotic de Sitter expansion, which occurs without a false-vacuum equation of state and regardless of the signature of the extra dimension. If \( \Lambda_{(5)} < 0 \), then \( \Lambda_{(4)} \) is initially positive, but later changes its sign. The universe becomes dominated by a negative cosmological term that causes the universe to recollapse.

Consequently, an inevitable conclusion of our work is that the universe must recollapse at some time in the future if it is embedded in an Anti-de Sitter five-dimensional bulk, which is the usual case in brane models. Fortunately, according to Table 3, we are nowhere near the time of recollapse. We emphasize that this behavior is independent on the (small) size of the (negative) bulk cosmological constant.

However, we note that the entries in Tables 1 and 3 are not very different from each other for \( \Omega_\rho \approx 0.1 - 0.3 \). This is explained by the “negligible” difference between the numbers in the r.h.s. of the field equations (9) in both cases. The entries in Table 1 are calculated for \( k_{(5)}^2 \Lambda_{(5)} = 0 \), while the ones in Table 3 for \( k_{(5)}^2 \Lambda_{(5)} \approx 2.3 \times 10^{-120} t_{\text{Plank}}^{-2} \). Such a small difference does not significantly alter the evolution of the universe.

Therefore, the solution with accelerated expansion of Section 4.1.1 gives a good description, to a first order approximation, of the evolution of a 4D universe embedded in a five-dimensional spacetime with non-vanishing \( \Lambda_{(5)} \) and spacelike extra dimension. We would like to emphasize that the whole analysis in this paper is independent of any particular solution used in the five-dimensional bulk. This is a great virtue of brane-world models as noted at the end of Section 2.

However, one could still ask whether the brane metrics described here can be embedded in a five-dimensional bulk. The answer to this question is positive. Clearly the model with \( \Lambda_{(5)} = 0 \), which generalizes the familiar power-law solution characteristic of flat FRW universes, is embedded in the cosmological metric with separation of variables (3) which is discussed in [11]. Regarding the models with \( \Lambda_{(5)} \neq 0 \), they can be embedded in five-dimensional “wave-like” cosmologies of the type discussed in [23]. If in equation (38) of [23] we take variable \( \sigma \) as here in (50), then the scale factor \( a \) for such wave-like models is governed by an equation which is identical to (53) in this paper.
We would like to finish with the following comments. The issue of constraining brane universes with observational data like supernova, CMB and quasars has been done in detail for brane cosmology in [29] and [30]. Our aim in this work has been to study the simultaneous variation of $G$ and $\Lambda_{(4)}$ and constraining the model with observations. Such study has not been done before. Besides we have very little indication of the evolution of $G$ and $\Lambda_{(4)}$ in time. Our working hypothesis, that $\dot{G}/G = gH$, is an extrapolation of the present limit of $|\dot{G}/G|$. It is not clear whether it remains valid as early as at nucleosynthesis. Many other important questions remain open. Among them the behavior of perturbations and structure formation. Also the issues of obtaining the 4-dimensional Newton law as well as the quantization in models with positive bulk cosmological constant and models with timelike extra dimension, respectively. These questions are beyond the scope of the present work.

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