Symmetry of Lie algebras associated with $(\varepsilon, \delta)$ Freudenthal-Kantor triple systems

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Abstract
Symmetry group of Lie algebras and superalgebras constructed from $(\varepsilon, \delta)$ Freudenthal-Kantor triple systems has been studied. Especially, for a special $(\varepsilon, \varepsilon)$ Freudenthal-Kantor triple, it is $SL(2)$ group. Also, relationship between two constructions of Lie algebras from structurable algebra has been investigated.

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1. Introduction and Summary of Main Results

Let $L$ be a Lie algebra over a field $F$. Suppose that it is endowed with a group homomorphism

$$G \to \text{Aut}_G(L)$$

for a group $G$. If $G$ is a finitely generated Abelian group, a grading of $L$ is given (see [Ko]) by the action of the automorphism (the group of characters) on the Lie algebras. Recently, we have studied actions of some small finite-dimensional non-Abelian groups by automorphism on a Lie algebra $L$. For instance, Lie algebras with symmetric group $S_4$ as its automorphism are coordinated ([E-O.1]) by some non-associative algebras. The unital algebras in this class turn out to be the structurable algebras of Allison ([A]). Moreover, Lie algebras graded by the non-reduced root system $BC_1$ of type $B_1$ ([B-S.]) are naturally among the Lie algebras with the $S_4$-symmetry. We have also studied the case of the invariant group $G$ being $S_3$ and more generally dicyclic group $Dic_3$ ([E-O.2,3]) for the characterization of $L$.

Since $(\varepsilon, \delta)$Freudenthal-Kantor triple systems (abbreviated hereafter as to FKTS) [Y-O] offer a simple method of constructing Lie algebras (for the case of $\delta = +1$) and Lie superalgebras (for the case of $\delta = -1$) with 5-graded structure, it may be of some interest to study its symmetry group in this note. In order to facilitate the discussion, let us briefly sketch its definition.

Let $(V, xyz)$ be a triple linear system, where $xyz$ for $x, y, z \in V$ is a tri-linear product in a vector space $V$ over a field $F$. We introduce two linear mappings $L$ and $K : V \otimes V \to \text{End} V$ by

$$L(x, y)z = xyz, \quad K(x, y)z = xzy - \delta yzx$$

for $\delta = +1$ or $-1$. If they satisfy

$$[L(u, v), L(x, y)] = L(L(u, v)x, y) + \varepsilon L(x, L(u, v)y),$$

$$K(K(u, v)x, y) = L(y, x)K(u, v) - \varepsilon K(u, v)L(x, y)$$

for $\delta = +1$ or $-1$. If they satisfy

$$K(u, v)x = L(u, v)x + \varepsilon L(u, v)x, \quad L(u, v)x = x(u, v) + \varepsilon u(u, v)x$$

for $\delta = +1$ or $-1$. If they satisfy

$$L(u, v)x + \varepsilon L(u, v)x = x(u, v) + \varepsilon u(u, v)x.$$
for any \( u, v, x, y \in V \) and \( \varepsilon = \pm 1 \), we call the triple system to be \((\varepsilon, \delta)\) FKTS.

One consequence of Eqs.(1.2) and (1.3) is the validity of the following important identity (see [Y-O.] Eqs.(2.9) and (2.10))

\[
K(u, v)K(x, y) = \varepsilon \delta L(K(u, v)x, y) - \varepsilon L(K(u, v)y, x)
\]

\(= L(v, K(x, y)u) - \delta L(u, K(x, y)v). \tag{1.4}\)

We can then construct a Lie algebra for \( \delta = +1 \) and a Lie superalgebra for \( \delta = -1 \) as follows:

Let \( W \) be a space of \( 2 \times 1 \) matrix over \( V \)

\[
W = \begin{pmatrix} V \\ V \end{pmatrix}
\]

and define a tri-linear product:

\[
W \otimes W \otimes W \to W \text{ by }
\]

\[
\left\[ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right\] = \left( \begin{array}{ccc}
L(x_1, y_2) - \delta L(x_2, y_1) & \delta K(x_1, x_2) \\
-\varepsilon K(y_1, y_2) & \varepsilon L(y_2, x_1) - \varepsilon \delta L(y_1, x_2)
\end{array} \right) \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}. \tag{1.6}\]

Then, it defines a Lie triple system for \( \delta = +1 \) and an anti-Lie triple system for \( \delta = -1 \). We then note

\[
\hat{L}0 = \text{span}\left\{ \begin{pmatrix} L(x, y) \\ -\varepsilon K(u, v) \end{pmatrix} : x, y, u, v \in V \right\} \tag{1.7}\]

is a Lie subalgebra of \( \text{Mat}_2(\text{End}(V))^- \), where \( B^- \) for an associative algebra \( B \) implies a Lie algebra with bracket; \([x, y] = xy - yx\). We note also then

\[
\hat{D} = \begin{pmatrix} L(x, y), & \delta K(z, w) \\
-\varepsilon K(u, v), & \varepsilon L(y, x) \end{pmatrix} \in L(W, W) \tag{1.8}\]

is a derivation of the triple system. Setting

\[
L_1 = \text{span}\{X = \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in V\} (= W), \tag{1.9}\]

then \( L \) defined by

\[
L = \hat{L}0 \oplus L_1 \tag{1.10}\]

gives a Lie algebra for \( \delta = +1 \), and a Lie superalgebra for \( \delta = -1 \), where

\[
L_0 = \{D | D \text{ is a derivation of } L\}, \tag{1.11}\]

i.e., \( D \) satisfies

\[
D[X_1, X_2, X_3] = [DX_1, X_2, X_3] + [X_1, DX_2, X_3] + [X_1, X_2, DX_3] \tag{1.12a}\]

and hence induces also

\[
[D, [X, Y]] = [DX, Y] + [X, DY], \tag{1.12b}\]
if we define the bracket by

\[ [D_1 \oplus X_1, D_2 \oplus X_2] = ([D_1, D_2] + L(X_1, X_2)) \oplus (D_1 X_2 - D_2 X_1). \] (1.13)

where

\[ [D_1, D_2] = D_1 D_2 - D_2 D_1 \]

and

\[ L(X_1, X_2) = [X_1, X_2] = \left( \begin{array}{cc} L(x_1, y_2) - \delta L(x_2, y_1) & \delta K(x_1, x_2) \\ -\varepsilon K(y_1, y_2) & \varepsilon L(y_2, x_1) - \varepsilon \delta L(y_1, x_2) \end{array} \right). \] (1.14)

Note that the endomorphism \( L(X, Y) \) is then an inner derivation of the triple system. Since \( L_0 \supset \hat{L}_0 \), we will mainly discuss a subsystem \( \hat{L} \) of \( L \), given by

\[ \hat{L} = \hat{L}_0 \oplus L_1 = L(W, W) \oplus W \] (1.15)

rather than the larger \( L \) except for in section 2. Then, \( \hat{L} \) is 5-graded

\[ \hat{L} = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \] (1.16)

where

\[ L_{-2} = \text{span}\{ \begin{pmatrix} 0 & 0 \\ -\varepsilon K(x, y) & 0 \end{pmatrix} \mid x, y \in V \} \] (1.17a)

\[ L_{-1} = \text{span}\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \mid x \in V \} \] (1.17b)

\[ L_0 = \text{span}\{ \begin{pmatrix} L(x, y) & 0 \\ 0 & \varepsilon L(y, x) \end{pmatrix} \mid x, y \in V \} \] (1.17c)

\[ L_1 = \text{span}\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in V \} \] (1.17d)

\[ L_2 = \text{span}\{ \begin{pmatrix} 0 & \delta K(x, y) \\ 0 & 0 \end{pmatrix} \mid x, y \in V \}. \] (1.17e)

Here, we utilized the following Proposition for some of its proof.

**Proposition 1.1 ([K-O.], [K-M-O.])**

Let \( (V, (xyz)) \) be a \((\varepsilon, \delta)\)-Freudenthal-Kantor triple system with an endomorphism \( P \) such that \( P^2 = -\varepsilon \delta \text{Id} \) and \( P(xyz) = (P_x P_y P_z) \). Then, \( (V, [xyz]) \) is a Lie triple system (for \( \delta = 1 \)) and anti-Lie triple system (for \( \delta = -1 \)) with respect to the product

\[ [xyz] = (xPyz) - \delta(yPzx) + \delta(xPzy) - (yPzx). \]

In passing, we note that the standard \( \hat{L} = \Sigma_{i=-2}^2 \oplus L_i \) is a result of Proposition 1.1 immediately with

\[ P = \begin{pmatrix} 0, \delta \\ -\varepsilon, 0 \end{pmatrix} \text{ and } x \rightarrow X \text{ etc.} \]

Next, we introduce \( \theta, \sigma(\lambda) \in \text{End}(\hat{L}) \) for any \( \lambda \in F(\lambda \neq 0) \), being non-zero constant by

\[ \theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\varepsilon y \\ \delta x \end{pmatrix} \] (1.18a)
in $W = L_1 = L_{-1} \oplus L_1$. We may easily verify that they are automorphism of $[W, W, W]$, i.e, we have for example

$$\theta([X, Y, Z]) = [\theta X, \theta Y, \theta Z]$$

for $X, Y, Z \in W$. We then extend their actions to the whole of $\hat{L}$ in a natural way to show that they will define automorphism of $\hat{L}$. They moreover satisfy

(i) $\sigma(1) = Id, \theta^4 = Id$

where $Id$ is the identity mapping

(ii) $\theta^2 = -\varepsilon\delta Id$ for $L_1$ but $\theta^2 = Id$ for $L_0$

(iii) $\sigma(\mu)\sigma(\nu) = \sigma(\mu\nu)$ for $\mu, \nu \in F, \mu\nu \neq 0$

(iv) $\sigma(\lambda)\theta\sigma(\lambda) = \theta$ for any $\lambda \in F, \lambda \neq 0$.

We call the group generated by $\sigma(\lambda)$ and $\theta$ satisfying these conditions simply as $D(\varepsilon, \delta)$ due to a lack of better terminology. If the field $F$ contains $\omega \in F$ satisfying $\omega^3 = 1$ but $\omega \neq 1$, then a finite sub-group of $D(\varepsilon, \delta)$ generated by $\theta$ and $\sigma(\omega)$ defines $Dic_3$ group for $\varepsilon = \delta$ but $S_3$ for $\varepsilon = -\delta$, as we have noted already ([E-O,4]).

Conversely any 5-graded Lie algebra (or Lie superalgebra) with such automorphism $\theta$ and $\sigma(\lambda)$ satisfying Eqs.(1.19) lead essentially to a $(\varepsilon, \delta)$FKTS in $L_1$ with a triple product defined by $[x, y, z] = [[x, \theta y], z]$ for $x, y, z \in L_1$ (see [E-K-O]).

We note that the corresponding local symmetry of $D(\varepsilon, \delta)$ yields a derivation of $\hat{L}$, given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfies

$$h[X, Y, Z] = [hX, Y, Z] + [X, hY, Z] + [X, Y, hZ]$$

as well as

$$[h, [X, Y]] = [hX, Y] + [X, hY]$$

for $X, Y, Z \in W$.

We can find a larger automorphism group of $\hat{L}$, if we impose some additional conditions. First suppose that $K(x, y)$ is now expresed as

$$K(x, y) = \varepsilon\delta L(y, x) - \varepsilon L(x, y)$$

for any $x, y \in V$. We call then the triple system to be a special $(\varepsilon, \delta)$FKTS ([E-O,4]). As we will see at the end of section 2, this triple system is intimately related to zero Nijenhuis tensor condition. Moreover for the case of special $(\varepsilon, \varepsilon)$FKTS (i.e. $\varepsilon = \delta$), the automorphism group of $\hat{L}$ turns out to be a larger $SL(2, F)(= Sp(2, F))$ group which contains $D(\varepsilon, \varepsilon)$ as its subgroup. In this case, the triple system $[W, W, W]$ becomes invariant under

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow U \begin{pmatrix} x \\ y \end{pmatrix}$$

for any $2 \times 2$ $SL(2, F)$ matrix $U$, i.e.

$$U = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix}, \text{Det } U = \alpha\nu - \beta\mu = 1,$$
as we will show in the next section (Section 2). Also, the associated Lie algebras or superalgebras are $BC_1$-graded algebra of type $C_1$.

Finally (Section 3), we consider a ternary system $(V, xy, xyz)$ where $xy$ and $xyz$ are binary and ternary products, respectively, in the vector space $V$. Suppose that they satisfy

(1) the triple system $(V, xyz)$ is a $(-1, 1)$ FKTS.
(2) The binary algebra $(V, xy)$ is unital and involution ($\bar{x} \bar{y} = \bar{y} \bar{x}$) with the involutive map $x \to \bar{x}$, $\bar{\bar{x}} = x$.
(3) The triple product $xyz$ is expressed in terms of the bi-linear products by

$$xyz = (z \bar{y})x - (z \bar{x})y + (x \bar{y})z.$$  \hspace{1cm} (1.24)

We may call the ternary system $(V, xy, xyz)$ to be Allison-ternary algebra or simply $A$-ternary algebra, since $A = (V, xy)$ is then the structurable algebra ($[A], [A-F]$).

This case is of great interest, first because structurable algebras exhibit a triality relation ($[A-F]$), and second because we can construct another type of Lie algebras independently of the standard construction of $(-1, 1)$ FKTS, which is $S_4$-invariant and of $BC_1$ graded Lie algebra of type $B_1$. The relationship between the Lie algebra constructed in the new way and that given as in Eq.(1.17) is by no means transparent. Note that the group $D(-1, 1)$ contains $S_3$ but not $S_4$ symmetry. In section 3, we will show that if the field $F$ contains the square root $\sqrt{-1}$ of $-1$, then Eqs.(1.17) can be prolonged to yield the Lie algebra for the structurable algebra.

2. Symmetry Group of Lie Algebras associated with $(\varepsilon, \delta)$FKTS

Although the invariance of the Lie algebra or superalgebra $\hat{L}$ under $\theta$ and $\sigma(\lambda)$ given by Eqs.(1.18) has been already noted in [E-O,4] let us recapitulate its proof briefly as follows: For $\theta$ given by Eq.(1.18a), it is easy to verify the validity of

$$\theta([X_1, X_2, X_3]) = [\theta X_1, \theta X_2, \theta X_3]$$

for $X_i = \begin{pmatrix} x_j \\ y_j \end{pmatrix} \in W$ ($j = 1, 2, 3$). Since $\hat{L}$ is 5-graded, it is also invariant under

$$\sigma_n(\lambda) : Z_n \to \lambda^n Z_n, \ (n = 0, \pm 1, \pm 2)$$

for any $Z_n \in L_n$ given in Eqs.(1.17). This implies the validity of Eq.(1.18b). Then, Eqs.(1.19) can be readily verified. Thus, a generic Lie algebra or superalgebra $\hat{L}$ associated with $(\varepsilon, \delta)$FKTS has the symmetry group $D(\varepsilon, \delta)$ generated by $\theta$ and $\sigma(\lambda)$ satisfying Eqs.(1.19). However, for some special $(\varepsilon, \delta)$FKTS, the invariance group for $\hat{L}$ can be larger as follows.

Let us consider the case of special $(\varepsilon, \varepsilon)$FKTS ([E-O,4]), where $K(x, y)$ is expressed as

$$K(x, y) = L(y, x) - \varepsilon L(x, y)$$  \hspace{1cm} (2.1)

in terms of $L(x, y)'s$. We can show that $\sigma \in \text{End} \hat{L}$ defined by

$$\sigma \begin{pmatrix} x \\ y \end{pmatrix} := U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \mu & \nu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ \mu x + \nu y \end{pmatrix}$$  \hspace{1cm} (2.2)
gives an automorphism of the Lie algebra or superalgebra $\hat{L}$, provided that we have

$$\text{Det} U = \alpha \nu - \beta \mu = 1.$$  \hfill (2.3)

We further define the action of $\sigma$ on $L(W, W)$ by

$$\sigma \left( \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \right) = [\sigma \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \sigma \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right)]] = \left[ \sigma \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \sigma \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \right] = U \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \right] U^{-1}. \hfill (2.4)$$

and will prove the following.

**Proposition 2.1**

Under the assumption as in above, we have

$$\sigma \left( \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \right) = U \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \right] U^{-1}. \hfill (2.5)$$

**Proof**

First we note

$$\left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) = \left( \begin{array}{c} L(x_1, y_2) - \varepsilon L(x_2, y_1), -\varepsilon K(y_1, y_2) \\ -\varepsilon K(y_1, y_2), \varepsilon L(y_2, x_1) - L(y_1, x_2) \end{array} \right) \hfill (2.6)$$

since $\varepsilon = \delta$, so that the right side Eq.(2.4) is calculated to be

$$\left[ \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \right] = \left( \begin{array}{c} \alpha x_1 + \beta y_1 \\ \mu x_1 + \nu y_1 \end{array} \right), \left( \begin{array}{c} \alpha x_2 + \beta y_2 \\ \mu x_2 + \nu y_2 \end{array} \right) \right] = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \hfill (2.7)$$

where we have set

$$A = L(\alpha x_1 + \beta y_1, \mu x_2 + \nu y_2) - \varepsilon L(\alpha x_2 + \beta y_2, \mu x_1 + \nu y_1) = \alpha \mu L(x_1, x_2) - \varepsilon L(x_2, x_1) + \beta \nu L(y_1, y_2) + \alpha \nu L(x_1, y_2)$$

$$= \alpha \mu L(x_1, x_2) - \beta \nu K(y_1, y_2) + \{ \alpha \nu L(x_1, y_2) - \beta \mu L(y_1, x_2) \} + \{ \beta \nu L(y_1, x_2) - \varepsilon \mu L(x_1, y_2) \} \hfill (2.8a)$$

$$D = \varepsilon L(\mu x_2 + \nu y_2, \alpha x_1 + \beta y_1) - L(\mu x_1 + \nu y_1, \alpha x_2 + \beta y_2)$$

$$= \mu \alpha L(x_2, x_1) - L(x_1, x_2) + \nu \beta (L(y_2, y_1) - L(y_1, y_2))$$

$$= \mu \alpha L(x_2, x_1) + \nu \beta L(y_2, x_1) + \{ \mu \beta L(x_2, y_1) - \varepsilon L(y_1, x_2) \} \hfill (2.8b)$$

$$B = \varepsilon^{2} K(x_1, x_2) + \varepsilon^{2} K(y_1, y_2) = \varepsilon \alpha K(x_1, y_2) + \varepsilon \beta K(y_1, x_2)$$

$$C = -\varepsilon^{2} K(x_1, x_2) = \varepsilon^{2} K(y_1, y_2) = \varepsilon \mu K(x_1, y_2) - \varepsilon \nu K(y_1, x_2) \hfill (2.8d)$$

Here, we used Eq.(2.1) to simplify the last lines in Eqs.(2.8a) and (2.8b).

Then, we find further

$$\alpha A + \mu B = \alpha (L(x_1, y_2) - L(x_2, y_1)) - \beta \varepsilon K(y_1, y_2)$$

$$\beta A + \nu B = \beta (L(y_1, y_2) - L(y_1, x_2)) + \alpha \varepsilon K(x_1, x_2)$$

$$\alpha C + \mu D = \mu (L(x_1, y_2) - L(x_2, y_1)) - \varepsilon \nu K(y_1, y_2)$$

$$\beta C + \nu D = \nu (L(y_2, x_1) - L(y_1, x_2)) + \varepsilon \mu K(x_1, x_2)$$
in view of $\alpha \nu - \beta \mu = 1$, so that we have

$$[\sigma \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \sigma \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right)] U = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \left( \begin{array}{c} \alpha \\ \mu \end{array} \right) \left( \begin{array}{c} \beta \\ \nu \end{array} \right) = \left( \begin{array}{cc} \alpha A + \mu B & \beta A + \nu B \\ \alpha C + \mu D & \beta C + \nu D \end{array} \right)$$

which is rewritten further as

$$[\sigma \left( \begin{array}{c} x_1 \\ y_1 \end{array} \right), \sigma \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right)] U = \left( \begin{array}{cc} \alpha \beta \\ \mu \nu \end{array} \right) \left( \begin{array}{cc} L(x_1, y_2) - \varepsilon L(x_2, y_1) & \varepsilon K(x_1, x_2) \\ -\varepsilon K(y_1, y_2) & \varepsilon L(y_2, x_1) - L(y_1, x_1) \end{array} \right)$$

$$= U \left( \begin{array}{cc} x_1 \\ y_1 \end{array} \right), \left( \begin{array}{c} x_2 \\ y_2 \end{array} \right) \right].$$

This proves Eq.(2.5), and completes the proof.//

Then this yields also

$$\sigma([X_1, X_2, X_3]) = [\sigma X_1, \sigma X_2, \sigma X_3]$$

for $X_j = \left( \begin{array}{c} x_j \\ y_j \end{array} \right) \in W$ $(j = 1, 2, 3)$, since we calculate

$$\sigma[X_1, X_2, X_3] = U[X_1, X_2, X_3] \quad (since \ [X_1, X_2, X_3] \in W)$$

$$= U[X_1, X_2]X_3 \quad (by \ Eqs. \ (1.6) \ and \ (1.14))$$

$$= U[X_1, X_2]U^{-1}UX_3 \quad (since \ [X_1, X_2] \in L(W, W) \ is \ a \ 2 \times 2 \ matrix)$$

$$= [\sigma X_1, \sigma X_2]X_3 \quad (by \ Eqs. \ (2.2) \ and \ (2.5)).$$

In conclusion, the Lie algebra or superalgebra $\hat{L}$ constructed from any special $(\varepsilon, \varepsilon)$FKTS admits $SL(2)(=Sp(2))$ as its automorphism group. Note that $SL(2)$ contains the group $D(\varepsilon, \varepsilon)$ by

$$D(\varepsilon, \varepsilon) \to SL(2)$$

$$\theta \to \left( \begin{array}{cc} 0 & -\varepsilon \\ \varepsilon & 0 \end{array} \right)$$

$$\sigma(\lambda) \to \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right).$$

We then define an operator $\hat{\sigma}$ on $\hat{L}$ as follows:

$$\left( \begin{array}{cc} L(a, b) & \varepsilon K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{array} \right) \oplus \left( \begin{array}{c} x \\ y \end{array} \right) \to U \left( \begin{array}{cc} L(a, b) & \varepsilon K(c, d) \\ -\varepsilon K(e, f) & \varepsilon L(b, a) \end{array} \right) U^{-1} \oplus U \left( \begin{array}{c} x \\ y \end{array} \right)$$

where

$$\sigma := U = \left( \begin{array}{cc} \alpha & \beta \\ \mu & \nu \end{array} \right) \quad and \quad det \ U = 1.$$

Then $\hat{\sigma}$ is an automorphism of $\hat{L}$ induced from the triple system.

Moreover as the local version of the global $SL(2)$ symmetry, we can prove also the following.

**Proposition 2.2**
Under the assumption as in above, let

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]

(2.9)

which forms a \(sl(2)\) Lie algebra by the standard commutation relations. We also define their actions to \(X = \begin{pmatrix} x \\ y \end{pmatrix} \in W\) by

\[
[A, X] = A \begin{pmatrix} x \\ y \end{pmatrix}
\]

for \(A = h, f, g\). Noting \(W = h, f, g\) and similarly define \([A, M] = AM - MA\) for \(M \in L(W, W)\).

Then, \(h, f, g\) are derivations of \(\hat{L} = L(W, W) \oplus W\) for special \((\epsilon, \delta)\)FKTS and vice versa.

**Proof.**

First, \(h\) is actually a derivation of \(\hat{L}\) for any \((\epsilon, \delta)\)FKTS as we have already noted. Also, the fact that \(g\) is a derivation of \(\hat{L}\) for special \((\epsilon, \delta)\)FKTS has been proven in [K-O] (see Theorem 5.2). Here we will show similarly the validity of

\[
[f(X_1, X_2, X_3) = [f X_1, X_2, X_3] + [X_1, f X_2, X_3] + [X_1, X_2, f X_3]
\]

(2.10)

which prove these being derivations of \(W\). Noting

\[
f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix},
\]

we then calculate

\[
f\left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right)
= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L(x_1, y_2) - \epsilon L(x_2, y_1), \epsilon K(x_1, x_2) \\ -\epsilon K(y_1, y_2), \epsilon L(x_2, x_1) - L(y_1, x_2) \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}
\]

\[
= \begin{pmatrix} -\epsilon K(y_1, y_2), \epsilon L(y_2, x_1) - L(y_1, x_2) \\ 0, 0 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}
= \begin{pmatrix} -\epsilon K(y_1, y_2)x_3 + (\epsilon L(y_2, x_1) - L(y_1, x_2))y_3 \\ 0 \end{pmatrix}.
\]

But

\[
[f \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) = \left[ \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right]
= \begin{pmatrix} L(y_1, y_2) & \epsilon K(y_1, x_2) \\ 0, & \epsilon L(y_2, y_1) \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} L(y_1, y_2)x_3 + \epsilon K(y_1, x_2)y_3 \\ \epsilon L(y_2, y_1)y_3 \end{pmatrix},
\]

\[
\left[ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, f \left( \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right) = \left[ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \right]
= \begin{pmatrix} -\epsilon L(y_2, y_1), \epsilon K(x_1, y_2) \\ 0, & -L(y_1, y_2) \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} -\epsilon L(y_2, y_1)x_3 + \epsilon K(x_1, y_2)y_3 \\ -L(y_1, y_2)y_3 \end{pmatrix},
\]

The proof is completed.
\begin{align*}
&\left(\begin{array}{c} x_1 \\ y_1 \\
\end{array}\right), \left(\begin{array}{c} x_2 \\ y_2 \\
\end{array}\right), f \left(\begin{array}{c} x_3 \\ y_3 \\
\end{array}\right) \right] = \left[\left(\begin{array}{c} x_1 \\ y_1 \\
\end{array}\right), \left(\begin{array}{c} x_2 \\ y_2 \\
\end{array}\right), \left(\begin{array}{c} y_3 \\
0 \\
\end{array}\right)\right] \\
&= \left(\begin{array}{c} L(x_1, y_2) - \varepsilon L(x_2, y_1), \\
\varepsilon K(x_1, x_2) \\
-\varepsilon K(y_1, y_2), \\
\varepsilon L(y_2, x_1) - L(y_1, x_2) \\
\end{array}\right) \left(\begin{array}{c} y_3 \\
0 \\
\end{array}\right) \\
&= \left(\begin{array}{c} (L(x_1, y_2) - \varepsilon L(x_2, y_1))y_3 \\
-\varepsilon K(y_1, y_2)y_3 \\
\end{array}\right).
\end{align*}

Hence, setting

\begin{align*}
&\left[\begin{array}{c} x_1 \\ y_1 \\
\end{array}\right], \left(\begin{array}{c} x_2 \\ y_2 \\
\end{array}\right), \left(\begin{array}{c} x_3 \\ y_3 \\
\end{array}\right) \right] + \left[\begin{array}{c} x_1 \\ y_1 \\
\end{array}\right], f \left(\begin{array}{c} x_2 \\ y_2 \\
\end{array}\right), \left(\begin{array}{c} x_3 \\ y_3 \\
\end{array}\right) \right] \\
&\left[\begin{array}{c} x_1 \\ y_1 \\
\end{array}\right], x_2, f \left(\begin{array}{c} x_3 \\ y_3 \\
\end{array}\right) \right] = \left(\begin{array}{c} z \\
w \\
\end{array}\right),
\end{align*}

we calculate

\begin{align*}
z &= L(y_1, y_2)x_3 + \varepsilon K(y_1, x_2)y_3 - \varepsilon L(y_2, y_1)x_3 + \varepsilon K(x_1, y_2)y_3 \\
&+ (L(x_1, y_2) - \varepsilon L(x_2, y_1))y_3 \\
&= \{L(y_1, y_2) - \varepsilon L(y_2, y_1)\}x_3 + \{\varepsilon K(y_1, x_2) + \varepsilon K(x_1, y_2) + L(x_1, y_2) - \varepsilon L(x_2, y_1)\}y_3 \\
&= K(y_2, y_1)x_3 + \{\varepsilon (L(x_2, y_1) - \varepsilon L(y_1, x_2))\}y_3 \\
&+ \{\varepsilon (L(y_2, x_1) - \varepsilon L(x_1, y_2)) + L(x_1, y_2) - \varepsilon L(x_2, y_1)\}y_3 \\
&= -\varepsilon K(y_1, y_2)x_3 + \{-L(y_1, x_2) + \varepsilon L(y_2, x_1)\}y_3
\end{align*}

and

\begin{align*}
w &= \varepsilon L(y_2, y_1)y_3 - L(y_1, y_2)y_3 - \varepsilon K(y_1, y_2)y_3 \\
&= \{\varepsilon L(y_2, y_1) - L(y_1, y_2) - \varepsilon K(y_1, y_2)\}y_3 = 0.
\end{align*}

Comparing these with the left hand of Eq.(2.10), we obtain

\[ f[X_1, X_2, X_3] = [fX_1, X_2, X_3] + [X_1, fX_2, X_3] + [X_1, X_2, fX_3], \]

so that \( f \) is also a derivation of the associated Lie algebra \( \hat{L} \). This completes the proof of Proposition 2.2."
Trivial modules consist of
\[
\begin{pmatrix}
L(y, x) + \varepsilon L(x, y) & 0 & L(y, x) + \varepsilon L(x, y) \\
L(y, x) & 0 & L(x, y) \\
0 & \varepsilon L(x, y) & 0
\end{pmatrix} + \varepsilon
\begin{pmatrix}
0 & \varepsilon L(y, x) \\
0 & 0 & \varepsilon L(y, x)
\end{pmatrix}
\in L(W, W)
\]
so that \(L\) is a \(BC_1\)-graded Lie algebra or superalgebra of type \(C_1\). This fact is in essential accord with results of Corollaries 3.8 and 4.6 of [E-O,4], which are based upon analysis of the \(J\)-ternary algebra [A-B-G].

**Remark 2.4**

A \((\varepsilon, \varepsilon)\) FKTS is called unitary (see [K-M-O]), if \(K(V, V)\) contains an identity map, i.e., there exist \(a_i, b_i \in V\) satisfying
\[
\sum_i K(a_i, b_i) = Id.
\]
Any unitary \((\varepsilon, \varepsilon)\) FKTS is special ([K-M-O],[E-O,4]). Moreover \((h, g, f)\) constructed above are now contained in \(L(W, W)\) by replacing \(K(x, y)\) in Remark 2.3 by \(\sum_i K(a_i, b_i) = Id\). Further a \((\varepsilon, \varepsilon)\) FKTS is said to be balanced if we have \(K(x, y) = \langle x|y \rangle Id\) for a non-zero bi-linear form \(\langle \cdot | \cdot \rangle\). Then, any balanced \((\varepsilon, \varepsilon)\) FKTS is unitary and hence special. If the field \(F\) is algebraically closed of zero characteristic, any simple Lie algebra except for \(sl(2)\) can be constructed standardly from some balanced \((1, 1)\) FKTS ([M],[Ka],[E-K-O]) so that any such simple classical Lie algebra is automatically \(BC_1\)-graded Lie algebra of type \(C_1\).

**Remark 2.5**

Let us set (for any \((\varepsilon, \delta)\)FKTS):
\[
J = f - g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
We can then verify the validity of
\[
J[X, Y]J^{-1} = \varepsilon \delta [JX, JY],
\]
as well as
\[
J[X, Y, Z] = \varepsilon \delta [JX, JY, JZ]
\]
for any \(X, Y, Z \in W\). We next introduce an analogue of Nijenhuis tensor in differential geometry ([K-N]) by
\[
N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y].
\]
Setting \(X = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\), and \(Y = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\), we then calculate
\[
N(X, Y) = \begin{pmatrix}
-\varepsilon(\Lambda(x_1, y_2) + \Lambda(y_1, x_2)), & \delta(\Lambda(y_1, y_2) - \Lambda(x_1, x_2)) \\
\varepsilon(\Lambda(y_1, y_2) - \Lambda(x_1, x_2)), & \delta(\Lambda(x_1, y_2) + \Lambda(y_1, x_2))
\end{pmatrix}.
\]
where
\[ \Lambda(x, y) = K(x, y) + \varepsilon L(x, y) - \varepsilon \delta L(x, y). \] (2.15)

Therefore, for any special \((\varepsilon, \delta)\) FKTS (which is defined by \(\Lambda(x, y) = 0\), see Eq.(1.22)), we get the zero Nijenhuis tensor condition of
\[ N(X, Y) = 0 \] (2.16)
for any \(X, Y \in W\) and vice versa. Moreover, \(J\) satisfies also the analogue of the almost complex structure condition of
\[ J^2 = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (2.17)

We remark that these facts are already noted in Proposition 5.3 of [K-O] for the special case of \(\varepsilon = \delta\). Further for \(\varepsilon = \delta\), \(J\) is a derivation (as well as an automorphism) of the Lie algebra or superalgebra \(\hat{L}\), and we may replace \(J\) by
\[ J \to \tilde{J} = UJU^{-1} \] (2.18)
for any \(2 \times 2\) matrix \(U\) satisfying \(\text{Det } U = 1\) by the following reason. First, we see that Eqs(2.13) with (2.16) is invariant under \(SL(2)\) transformation of
\[ J \to \tilde{J} = UJU^{-1}, \quad X \to \tilde{X} = UX, \quad Y \to \tilde{Y} = UY. \]

Moreover, since \(X\) and \(Y\) are arbitrary, we may replace \(X\) and \(Y\) by \(U^{-1}X\) and \(U^{-1}Y\), respectively, to obtain the desired result, i.e..
\[ \tilde{N}(X, Y) = [\tilde{J}X, \tilde{J}Y] - \tilde{J}[\tilde{J}X, Y] - \tilde{J}[X, \tilde{J}Y] + \tilde{J}^2[X, Y] = 0, \] (2.19)
with
\[ \tilde{J}^2 = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

A simple example of a special \((\varepsilon, \delta)\) FKTS is given as follows. Let \(< \cdot | \cdot >\) be a bilinear form in a vector space \(V\), satisfying \( < x|y > = -\varepsilon < y|x >\), and define a tri-linear product \(xyz\) in \(V\) by
\[ xyz = < y|z > x. \]

Then \((V, xyz)\) is a special \((\varepsilon, \delta)\)FKTS. The fact that it gives a \((\varepsilon, \delta)\) FKTS has been already noted in ([K-O Proposition 2.8 (ii)]). In order to show it to be special, we calculate
\[ K(x, y)z = xyz - \delta yxz = < z|y > x - \delta < z|x > y = -\varepsilon < y|z > x + \varepsilon \delta < x|z > y = -\varepsilon xyz + \varepsilon \delta yxz \]
and hence
\[ K(x, y) = -\varepsilon L(x, y) + \varepsilon \delta L(y, x). \]

**Remark 2.6**
Let $\hat{L} = W \oplus L(W,W)$ be the Lie algebra derived from a Lie triple system $[W,W,W]$, and introduce an analogue of covariant derivative $\nabla : \hat{L} \to \text{End}\hat{L}$ by

$$\nabla_X Y = [X,Y], \quad \nabla_X [Y,Z] = [Y,Z,X],$$

$$\nabla_{[X,Y]} Z = -[X,Y,Z], \quad \nabla_{[X,Y]} [V,Z] = -[[X,Y],[V,Z]].$$

Then the Riemann curvature tensor defined by (see [K-N])

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

is identically zero, i.e., $R(X,Y) = 0$ in $\hat{L}$, as we demonstrate below. First we calculate

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)Z - \nabla_{[X,Y]}Z$$

$$= \nabla_X [Y,Z] - \nabla_Y [X,Z] + [X,Y,Z] = [Y,Z,X] - [X,Z,Y] + [X,Y,Z]$$

$$= [Y,Z,X] + [Z,X,Y] + [X,Y,Z] = 0.$$

Second,

$$R(X,Y)[V,Z] = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)[V,Z] - \nabla_{[X,Y]}[V,Z]$$

$$= [X,[V,Z,Y]] - [Y,[V,Z,X]] + [[X,Y],[V,Z]]$$

$$= [X,L(V,Z)Y] - [Y,L(V,Z)X] - L(V,Z)[X,Y] = 0.$$

where $L(W,W)$ is defined as before by

$$L(X,Y)Z = [X,Y,Z],$$

and we note that it is a derivation of the Lie triple system.

However the torsion tensor $T(X,Y)$ defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

is not zero, since it gives

$$T(X,Y) = [X,Y] - [Y,X] - [X,Y] = [X,Y].$$

In conclusion, we see that the Lie triple system associated with the $(\varepsilon,\delta)$ FKTS contains many interesting structures in it.

3. Structurable algebras and $S_4$- symmetry

Let $A = (V, xy)$ be a structurable algebra with the unit element $e$ and with involution map $x \to \bar{x}$. Let $l(x)$ and $r(x)$ be the left and right multiplication operators defined by

$$l(x)y = xy, \quad r(x)y = yx, \quad (3.1)$$

we introduce then $d_j : A \otimes A \to \text{End}A$ for $j = 1, 2, 3$ by

$$d_1(x,y) = l(\bar{y})l(x) - l(\bar{x})l(y) \quad (3.2a)$$

$$d_2(x,y) = r(\bar{y})r(x) - r(\bar{x})r(y) \quad (3.2b)$$

$$d_3(x,y) = r(\bar{y}x - \bar{y}x) + l(y)l(\bar{x}) - l(x)l(\bar{y})$$

$$= l(y\bar{x} - x\bar{y}) + r(y)r(\bar{x}) - r(x)r(\bar{y}), \quad (3.2c)$$
following [A-F]. Note that they satisfy

\[ d_j(x, y) = -d_j(y, x), \quad j = 1, 2, 3. \]  

\[ (3.3) \]

It is known ([A-F],[O]) then that they satisfy first the triality relation:

\[ \overline{d}_j(u, v)(xy) = (d_{j+1}(u, v)x)y + x(d_{j+2}(u, v)y) \]

\[ (3.4) \]

for any \( u, v, x, y \in A \) and \( j = 1, 2, 3 \). Here, \( d_j(u, v) \) is defined modulo 3 for the index \( j \), i.e..

\[ d_{j\pm 3}(u, v) = d_j(u, v) \]

and \( \overline{Q} \in \text{End} \ A \) for any \( Q \in \text{End} \ A \) is defined as usual by

\[ \overline{Q}x = \overline{Q}x. \]

Moreover, they satisfy

\[ (i) \quad d_3(x, y)z + d_3(y, z)x + d_3(z, x)y = 0 \]

\[ (3.5a) \]

\[ (ii) \quad d_1(\bar{x}, yz) + d_2(\bar{y}, zx) + d_3(\bar{z}, xy) = 0 \]

\[ (3.5b) \]

\[ (iii) \quad \overline{d}_j(x, y) = d_{3-j}(\bar{x}, \bar{y}) \]

\[ (3.5c) \]

\[ (iv) \quad [d_j(u, v), d_k(x, y)] = d_k(d_{j-k}(u, v)x, y) + d_k(x, d_{j-k}(u, v)y). \]

\[ (3.5d) \]

Conversely, the validity of Eqs.(3.4) and (3.5) imply that the algebra is structurable, if it is unital and involutive.

Although the structurable algebra is intimately related to \((-1,1)\)FKTS, we can construct another type of Lie algebra out of it, independently of the standard construction given in section 1 as follows: Let \( \rho_j(A) \) for \( j = 1, 2, 3 \) be 3 copies of \( A \), and we introduce symbols \( T_j(A, A) \) satisfying

\[ (i) \quad T_j(x, y) = -T_j(y, x) = T_{j\pm 3}(x, y) \]

\[ (3.6a) \]

\[ (ii) \quad T_1(\bar{x}, yz) + T_2(\bar{y}, zx) + T_3(\bar{z}, xy) = 0. \]

\[ (3.6b) \]

Let \( T(A, A) \) be a linear span of all \( T_j(x, y) \) for \( x, y \in A \) and consider

\[ L = \rho_1(A) \oplus \rho_2(A) \oplus \rho_3(A) \oplus T(A, A). \]

\[ (3.7) \]

Then, \( L \) is a Lie algebra (see [A-F], and [O]) with Lie brackets of

\[ (i) \quad [\rho_i(x), \rho_i(y)] = \gamma_j \gamma_k^{-1} T_{3-i}(x, y) \]

\[ (3.7a) \]

\[ (ii) \quad [\rho_i(x), \rho_j(y)] = -[\rho_j(y), \rho_i(x)] = -\gamma_j \gamma_i^{-1} \rho_k(xy) \]

\[ (3.7b) \]

\[ (iii) \quad [T_i(u, v), \rho_j(x)] = -[\rho_j(x), T_i(u, v)] = \rho_j(d_{j+i}(u, v)x) \]

\[ (3.7c) \]

\[ (iv) \quad [T_i(u, v), T_m(x, y)] = \]

\[ -[T_m(x, y), T_i(u, v)] = T_m(x, d_{l-m}(u, v)y) + T_m(d_{l-m}(u, v)x, y). \]

\[ (3.7d) \]

Here \((i, j, k)\) is any cyclic permutation of indices \((1, 2, 3)\) with \( \gamma_j's \) being any non-zero constants, while indices \( l \) and \( m \) are arbitrary integers and we assumed \( \rho_j(x) \) to be \( F \)-linear in \( x \). A economical choice for \( T_i(x, y) \) is to assume it to be a triple

\[ T_i(x, y) = T(d_i(x, y), d_{i+1}(x, y), d_{i+2}(x, y)) \]
as in [A-F] and [E], since the Eqs.(3.6) and (3.7d) are automatically satisfied by Eqs.(3.5b) and (3.5d). However, this choice is not suitable in what follows.

A special choice of $\gamma_1 = \gamma_2 = \gamma_3 = 1$ is of particular interest, since the Lie algebra $L$ is then invariant under $S_4$-symmetry as follows: First, $L$ is invariant under the cyclic permutation group $Z_3$ generated by the permutation $(1, 2, 3)$, i.e., $1 \to 2 \to 3 \to 1$ by

$$\rho_j(x) \to \rho_{j+1}(x), \quad T_j(x, y) \to T_{j-1}(x, y).$$

(3.8)

The action of $\tau = (1, 2)$ of the $S_3$-group is given by

$$\rho_1(x) \leftrightarrow -\rho_2(x), \quad \rho_3(x) \to -\rho_3(x)$$

(3.9a)

$$T_1(x, y) \leftrightarrow T_2(x, y), \quad T_3(x, y) \to T_3(x, y).$$

(3.9b)

Then, $S_3$-group generated by $(1, 2, 3)$ and $\tau = (1, 2)$ can be shown to be automorphism of the Lie algebra $L$. Next, we consider the Klein’s 4-group $K_4$ corresponding to permutations

$$\tau_1 = (2, 3)(1, 4), \quad \tau_2 = (1, 3)(2, 4), \quad \tau_3 = (1, 2)(3, 4)$$

(3.10)

which satisfy

$$\tau_i \tau_j = \tau_j \tau_i, \quad \tau_i \tau_i = 1, \quad \tau_1 \tau_2 \tau_3 = 1$$

(3.11)

for $i, j = 1, 2, 3$. The actions of $\tau_1$ for example for $L$ is then realized by

$$\tau_1 : \rho_1(x) \to \rho_1(x), \quad \rho_2(x) \to -\rho_2(x), \quad \rho_3(x) \to -\rho_3(x)$$

(3.12a)

$$T_1(x, y) \to T_1(x, y)$$

and similarly for $\tau_2$ and $\tau_3$. Then, again we see that $L$ is invariant under this Klein’s 4-group. Since $S_4$ can be generated by $S_3$ and $K_4$, this shows that $L$ is invariant under $S_4$ as has been noted in [E-O,1].

Now, a question arises about relations between this construction of Lie algebra and that based upon the standard construction from $(-1, 1)$FKTS as in section 1. Note that the symmetry group $D(-1, 1)$ admits $S_3$-symmetry but not $S_4$. The purpose of this section is to show that the Lie algebra $L$ given in this section can be obtained from that of $\hat{L}$ constructed in section 1 by prolonging $\hat{L}$ when we take note of $A$ to be structurable, provided that the underlying field $F$ contains the square root $\sqrt{-1}$ of $-1$. More precisely, we will prove the following theorem.

**Theorem 3.1**

Let $A = (V, xy)$ be a structurable algebra. Let $\hat{L} = L(W,W) \oplus W$ be the Lie algebra constructed as in section 1 from the associated $(-1, 1)$FKTS. First, for any non-zero constants $\alpha, \beta, k \in F$, we introduce the ratio \( \frac{\gamma_2}{\gamma_3} \) and \( \frac{\gamma_3}{\gamma_2} \) for some $\gamma_j \in F$ by

(i) \( \frac{\gamma_2}{\gamma_3} = -2\alpha\beta \)

(ii) \( \frac{(\gamma_3)^2}{\gamma_1\gamma_2} = -k^2 \)

(3.13a)  

(3.13b)
and second define $\rho_j(x)$ and $T_j(x,y)$ in $\hat{L} = L(W,W) \oplus W$ by

\begin{align}
1) \quad \rho_1(x) &= \begin{pmatrix} \alpha x \\ \beta x \end{pmatrix} \\
2) \quad \rho_2(x) &= \begin{pmatrix} k\alpha x \\ -k\beta x \end{pmatrix} \\
3) \quad \rho_3(x) &= k\left(\begin{pmatrix} \alpha \beta (x + \bar{x}), & \alpha^2l(x - \bar{x}) \\ -\beta^2l(x - \bar{x}), & -\alpha \beta l(x + \bar{x}) \end{pmatrix} \right) \\
4) \quad T_1(x,y) &= \frac{\gamma_1}{\gamma_2} \left( \begin{pmatrix} \alpha \beta (L(x,y) - L(y,x)) \\ -\beta^2 K(x,y) \end{pmatrix} \right) \\
5) \quad T_2(x,y) &= \frac{\gamma_1}{\gamma_2} \left( \begin{pmatrix} \alpha \beta (L(x,y) - L(y,x)) \\ -\alpha \beta^2 K(x,y) \end{pmatrix} \right) \\
6) \quad T_3(x,y) &= \frac{\gamma_1}{\gamma_2}[\rho_3(x), \rho_3(y)].
\end{align}

Then, $\rho_3(x)$ and $T_3(x,y)$ are elements of $L(W,W) \oplus W$ and satisfy the Lie algebra relation of Eqs. (3.7) and (3.6).

**Remark 3.2**

It is not self-evident that $\rho_3(x)$ and $T_3(x,y)$ are elements of $L(W,W) \oplus W$. First, as we will show soon, we have

\begin{align}
1) \quad l(x + \bar{x}) &= L(e, x) + L(x, e) \\
2) \quad l(x - \bar{x}) &= K(x, e) = -K(e, x)
\end{align}

which prove $\rho_3(x) \in L(W,W)$, where $e$ is the unit element of $A$. Then this also implies $T_3(x,y)$ to be an element of $L(W,W)$ since $\hat{L} = L(W,W) \oplus W$ is a $\mathbb{Z}_2$-graded Lie algebra as we noted in section 1.

Before going into a proof of Theorem 3.1, we make a comment on the $S_4$-symmetry of the Lie algebra constructed here. The special choice of $\gamma_1 = \gamma_2 = \gamma_3 = 1$ requires $k^2 = -1$ and $2\alpha \beta = -1$ by Eqs. (3.13) so that we must assume $F$ to contain the element $\sqrt{-1}$.

We now proceed for a proof of Theorem 3.1. First, we show

**Lemma 3.3**

Under the assumption as in above, we have

\begin{align}
1) \quad L(x,y) + L(y,x) &= l(x\bar{y} + y\bar{x}) \\
2) \quad L(x,y) - L(y,x) &= -d_2(\bar{x}, \bar{y}) - d_0(x,y) \\
3) \quad K(x,y) &= d_2(\bar{x}, \bar{y}) - d_0(x,y) = l(x\bar{y} - y\bar{x}).
\end{align}

**Proof**

Since $xyz = (x\bar{y})z - (z\bar{x})y + (z\bar{y})x$, this is rewritten as

\[ L(x,y) = r(x)r(\bar{y}) - r(y)r(\bar{x}) + l(x\bar{y}). \]

On the other side, we note

\[ K(x,y)z = xz - yz = (y\bar{z})x - (y\bar{z})x + (z\bar{x})y - (z\bar{x})y + (x\bar{y})z - (y\bar{z})x \]
\[ = (x\bar{y} - y\bar{x})z = l(x\bar{y} - y\bar{x})z = \{d_2(\bar{x}, \bar{y}) - d_0(x,y)\}z. \]
Then, Eqs. (3.17) follow readily from these relations. Setting $y = e$ in Eq. (3.17a) gives Eqs. (3.16a), while Eq. (3.16b) is a simple consequence of Eq. (3.17c) for $y = e$. This completes the proof. //

Following ([KMO]), we note that

$D_{x,y} := L(x, y) - L(y, x)$ is a derivation of $A$ with respect to (abbreviation hereafter as to w.r.t) the triple product $(xyz)$,

$A_{x,y} := L(x, y) + L(y, x)$ is an anti-derivation of $A$ w.r.t the triple product $(xyz)$ i.e.,

$$[D_{x,y}, L(a, b)] = L(D_{x,y}a, b) + L(a, D_{x,y}b)$$

$$[A_{x,y}, L(a, b)] = L(A_{x,y}a, b) - L(a, A_{x,y}b),$$

and furthermore, note

$[K(x, y), K(a, b)]$ is a derivation of $A$ w.r.t. the triple product,

$[A_{x,y}, A_{a,b}]$ is a derivation of $A$ w.r.t. the triple product, and

$[A_{x,y}, D_{a,b}]$ is an anti-derivation of $A$ w.r.t. the triple system.

**Lemma 3.4**

*Under the assumption as in above, we write*

$$[X,Y] = XY - YX, \text{ and } \{X,Y\}_+ = XY + YX \text{ for } X,Y \in \text{ End } V.$$

**Proof**

We rewrite the triality relation Eq. (3.4) as

$$d_{3-j}(\bar{x}, \bar{y})(zw) = (d_{j+1}(x, y)z)w + z(d_{j+2}(x, y)z)$$

by Eq. (3.5c), which gives

$$d_{3-j}(\bar{x}, \bar{y})l(z) = l(d_{j+1}(x, y)z) + l(z)d_{j+2}(x, y).$$

Letting $x \to \bar{x}$ and $y \to \bar{y}$ with $j \to 1 - j$, we have also

$$d_{j+2}(x, y)l(z) = l(d_{2-j}(\bar{x}, \bar{y})z) + l(z)d_{3-j}(\bar{x}, \bar{y}).$$

Adding and for subtracting both relations, we obtain

$$[d_{3-j}(\bar{x}, \bar{y}) + d_{j+2}(x, y), l(z)] = l((d_{j+1}(x, y) + d_{2-j}(\bar{x}, \bar{y}))z)$$

$$\{d_{3-j}(\bar{x}, \bar{y}) - d_{j+2}(x, y), l(z)\}_+ = l((d_{j+1}(x, y) - d_{2-j}(\bar{x}, \bar{y}))z).$$

Setting $j = 2$ and $3$, these give Eqs. (3.18). This completes the proof. //</ref>
where we have set
\[ j \]
from Eqs.(3.2). Also, we note
\[ T \]
Eq.(3.17). In order to prove it for \( z \)
Since by putting \( d \)
Also, it is still not a trivial matter to see that
Under the assumptions as in above, we have
\[
T_j(x, y) = -\frac{\gamma_3}{\gamma_2} \left( \alpha\beta(d_{j+1}(x, y) + d_{1-j}(\bar{x}, \bar{y})), \alpha^2(d_{j+1}(x, y) - d_{1-j}(\bar{x}, \bar{y})), \alpha\beta(d_{j+1}(x, y) + d_{1-j}(\bar{x}, \bar{y})) \right) \tag{3.19}
\]
for \( j = 1, 2, 3 \).

**Proof**

The case of \( j = 1 \) and \( 2 \) are simple rewriting of Eqs.(3.15d) and (3.15e) together with Eq.(3.17). In order to prove it for \( j = 3 \), we calculate
\[
T_3(x, y) = \frac{\gamma_3}{\gamma_1} \{ \rho_3(x)\rho_3(y) - \rho_3(y)\rho_3(x) \}
= (\frac{\gamma_3}{\gamma_1})^2k^2 \left( \begin{array}{cc}
\alpha\beta l(x + \bar{x}), & \alpha^2 l(x - \bar{x}) \\
-\beta^2 l(x - \bar{x}), & -\alpha\beta l(x + \bar{x})
\end{array} \right) \cdot \left( \begin{array}{cc}
\alpha\beta l(y + \bar{y}), & \alpha^2 l(y - \bar{y}) \\
-\beta^2 l(y - \bar{y}), & -\alpha\beta l(y + \bar{y})
\end{array} \right)
= k^2\frac{\gamma_1}{\gamma_2} \alpha\beta \left( \begin{array}{cc}
\alpha\beta A(x, y) & \alpha^2 B(x, y) \\
\beta^2 B(x, y) & \alpha\beta A(x, y)
\end{array} \right)
\]
where we have set
\[
A(x, y) = [l(x + \bar{x}), l(y + \bar{y})] - [l(x - \bar{x}), l(y - \bar{y})]
= 2\{[l(x), l(y)] + [l(\bar{x}), l(\bar{y})]\} = -2\{d_1(x, y) + d_1(\bar{x}, \bar{y})\}
\]
and
\[
B(x, y) = \{l(x + \bar{x}), l(y + \bar{y})\} - \{l(x - \bar{x}), l(y + \bar{y})\} + \\
= 2\{\{l(x), l(y)\} + \{l(\bar{x}), l(\bar{y})\}\} = 2(d_1(x, y) - d_1(x, y))
\]
from Eqs.(3.2). Also, we note
\[
k^2\frac{\gamma_1}{\gamma_2} \alpha\beta = (\frac{\gamma_3}{\gamma_1})^2(\frac{\gamma_1}{\gamma_2})\frac{\gamma_2}{\gamma_3} = \frac{\gamma_3}{2\gamma_2}
\]
so that these yield
\[
T_3(x, y) = -\frac{\gamma_3}{\gamma_2} \left( \begin{array}{cc}
\alpha\beta(d_1(x, y) + d_1(\bar{x}, \bar{y})), & \alpha^2(d_1(x, y) - d_1(\bar{x}, \bar{y})) \\
\beta^2(d_1(x, y) - d_1(\bar{x}, \bar{y})), & \alpha\beta(d_1(x, y) + d_1(\bar{x}, \bar{y}))
\end{array} \right)
\]
This completes the proof of Lemma 3.5.//

**Remark 3.6**

Eq.(3.19) together with Eqs.(3.5) prove the validity of Eq.(3.6b), i.e.,
\[
T_1(x, yz) + T_2(\bar{y}, zx) + T_3(\bar{z}, xy) = 0.
\]
Also, it is still not a trivial matter to see that \( T_3(x, y) \in L(W, W) \). To show it, we note
\[
d_1(x, y) - d_1(\bar{x}, \bar{y}) = -K(\bar{x}, \bar{y}) + K(x, y) \tag{3.20a}
d_1(x, y) + d_1(\bar{x}, \bar{y}) = L(y, x) - L(x, y) + L(e, \bar{x}y) - L(\bar{x}y, e). \tag{3.20b}
\]
Since by putting \( z = e \) and \( y = e \) in Eq.(3.5b) and changing notations suitably, we obtain
\[
d_1(x, y) = -d_3(e, \bar{x}y) + d_2(\bar{x}, \bar{y}) = d_3(\bar{x}, \bar{y}) - d_2(e, y\bar{x})
\]
and hence
\[ d_1(x, y) - d_1(\bar{x}, \bar{y}) = d_2(x, y) - d_3(x, y) - d_3(e, \bar{x}y) + d_2(e, \bar{y}x) = K(x, y) + K(e, \bar{x}y). \]

However,
\[ K(e, \bar{x}y) = l(e(\bar{x}y) - (\bar{x}y)e) = l(\bar{y}x - \bar{x}y) = -K(\bar{x}, \bar{y}), \]
and also
\[ d_1(x, y) + d_1(\bar{x}, \bar{y}) = d_2(x, y) + d_3(x, y) - d_3(e, \bar{x}y) - d_2(e, \bar{y}x) = -\{L(x, y) - L(y, x) - L(e, \bar{x}y) + L(\bar{x}y, e)\}. \]

We next show

**Proof of Eq.(3.7a)**

The case of \( j = 3 \) is trivial in view of Eq.(3.15f)

(i) For \( j = 1 \), we calculate

\[
[r_1(x), r_1(y)] = \left[ \begin{array}{c} \alpha x \\ \beta x \\ L(\alpha x, \beta y) - L(\alpha y, \beta x) \\ K(\alpha x, \alpha y) \\ K(\beta x, \beta y) \\ L(\beta x, \alpha y) - L(\beta y, \alpha x) \\ \alpha \beta (L(x, y) - L(y, x)) \\ \beta^2 K(x, y) \\ \alpha \beta (L(x, y) - L(y, x)) \end{array} \right] = \frac{\gamma_2}{\gamma_3} T_2(x, y).
\]

(ii) Similarly, we compute for \( j = 2 \)

\[
[r_2(x), r_2(y)] = k^2 \left[ \begin{array}{c} \alpha \bar{x} \\ -\beta \bar{x} \\ -\beta \bar{y} \\ K(\alpha \bar{x}, \alpha \bar{y}) \\ K(\beta \bar{x}, \beta \bar{y}) \\ L(\beta \bar{x}, \alpha \bar{y}) - L(\beta \bar{y}, \alpha \bar{x}) \end{array} \right] = -k^2 \left[ \frac{\gamma_2}{\gamma_3} T_1(x, y) \right] = \frac{\gamma_2}{\gamma_3} T_1(x, y).
\]

This completes the proof of (3.7a).

**Proof of Eq.(3.7b)**

(i) We calculate

\[
[r_1(x), r_2(y)] = \left[ \begin{array}{c} \alpha x \\ \beta x \\ k\alpha \bar{x} \\ -k\beta \bar{y} \end{array} \right] = k \left[ \begin{array}{c} L(\alpha x, -\beta \bar{y}) - L(\alpha \bar{y}, -\beta x) \\ K(\beta x, -\beta \bar{y}) \\ L(\beta x, \alpha \bar{y}) - L(-\beta \bar{y}, \alpha x) \end{array} \right] = -k \left[ \begin{array}{c} \alpha \beta (L(x, \bar{y}) + L(\bar{y}, x)) \\ \beta^2 K(x, \bar{y}) \\ -\alpha \beta (L(x, \bar{y}) + L(\bar{y}, x)) \end{array} \right] = -\frac{\gamma_2}{\gamma_1} r_3(\bar{y}x) = -\frac{\gamma_2}{\gamma_1} r_3(x\bar{y}),
\]

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(ii) Similarly we have
\[
[r_2(x), r_3(y)] = - [r_3(y), r_2(x)]
\]
\[
= -k^2 \left( \frac{\alpha \beta l(y + \bar{y}), \quad \alpha^2 l(y - \bar{y})}{-\beta^2 l(y - \bar{y}), \quad -\alpha \beta l(y + \bar{y})} \right) \left( \begin{array}{c} k\alpha \bar{x} \\ -k\beta \bar{x} \end{array} \right)
\]
\[
= -k^2 \left( \frac{(\alpha^2 \beta(y + \bar{y}) - \alpha^2 \beta(y - \bar{y}))\bar{x}}{(-\beta^2 \alpha(y - \bar{y}) + \beta^2 \alpha(y + \bar{y}))\bar{x}} \right)
\]
\[
= -k^2 \left( \frac{2\alpha^2 \beta \bar{y} \bar{x}}{2\beta^2 \alpha \bar{y} \bar{x}} \right)
\]
\[
= -2k^2 \left( \frac{\alpha \beta \bar{y} \bar{x}}{\beta \bar{y} \bar{x}} \right) = -\frac{2\alpha}{\gamma^2} \rho_1(\bar{x} \bar{y})
\]

(iii) Similarly we have
\[
[r_3(x), r_1(y)] = k \left( \frac{\alpha \beta l(x + \bar{x}), \quad \alpha^2 l(x - \bar{x})}{-\beta^2 l(x - \bar{x}), \quad -\alpha \beta l(x + \bar{x})} \right) \left( \begin{array}{c} \alpha y \\ \beta y \end{array} \right)
\]
\[
= k \left( \frac{(\alpha^2 \beta(x + \bar{x})y + \alpha^2 \beta(x - \bar{x})y)}{(-\beta^2 \alpha(x - \bar{x})y - \alpha \beta^2(x + \bar{x})y)} \right)
\]
\[
= k \left( \frac{2\alpha^2 \beta xy}{-2\beta^2 \alpha xy} \right)
\]
\[
= 2 \left( \frac{\alpha \beta \rho_2(\bar{x} \bar{y})}{\beta \rho_2(\bar{x} \bar{y})} \right) = -\frac{2\alpha}{\gamma^2} \rho_1(\bar{x} \bar{y})
\]

This completes the proof of Eq. (3.7b).

**Proof of Eq. (3.7c)**

1. The case of \( j = 1 \), we have

\[
[T_1(x, y), \rho_1(z)]
\]
\[
= -\frac{\alpha \beta (d_{i+1}(x, y) + d_{i-1}(\bar{x}, \bar{y}))}{\beta^2 (d_{i+1}(x, y) - d_{i-1}(\bar{x}, \bar{y}))} \left( \begin{array}{c} \alpha z \\ \beta z \end{array} \right)
\]
\[
= -\frac{\alpha \beta (d_{i+1}(x, y) + d_{i-1}(\bar{x}, \bar{y}))}{\beta^2 (d_{i+1}(x, y) - d_{i-1}(\bar{x}, \bar{y}))} \left( \begin{array}{c} \alpha z \\ \beta z \end{array} \right)
\]
\[
= -\frac{2\alpha \beta}{\gamma^2} \alpha (d_{i+1}(x, y)z) = \rho_1(d_{i+1}(x, y)z).
\]

2. Case of \( j = 2 \), we have

\[
[T_1(x, y), \rho_2(z)]
\]
\[
= -\frac{\alpha \beta (d_{i+1}(x, y) + d_{i-1}(\bar{x}, \bar{y}))}{\beta^2 (d_{i+1}(x, y) - d_{i-1}(\bar{x}, \bar{y}))} \left( \begin{array}{c} \alpha z \\ \beta z \end{array} \right)
\]
\[
= -\frac{2\alpha \beta}{\gamma^2} \alpha (d_{i+1}(x, y)z) = \rho_2(d_{i+1}(x, y)z).
\]

3. Case of \( j = 3 \).
For simplicity, we write
\[ T_l(x, y) = -\frac{\gamma_3}{\gamma_2} \left( \alpha \beta C_l(x, y), \alpha^2 D_l(x, y) \right) \]
with
\[
C_l(x, y) = d_{l+1}(x, y) + d_{1-l}(\bar{x}, \bar{y})
\]
\[
D_l(x, y) = d_{l+1}(x, y) - d_{1-l}(\bar{x}, \bar{y})
\]
and calculate
\[
[T_l(x, y), \rho_3(z)]
\]
\[
= -\frac{\gamma_3}{\gamma_2} \left( \frac{k \gamma_1}{\gamma_2} \right) \left( \begin{array}{c}
\alpha \beta C_l(x, y), \alpha^2 D_l(x, y) \\
\beta^2 D_l(x, y), \alpha \beta C_l(x, y)
\end{array} \right)
\]
\[
= -\frac{\gamma_3}{\gamma_2} \left( \frac{k \gamma_1}{\gamma_2} \right) \left( \begin{array}{c}
\beta^3 \alpha M_{11}, \alpha^2 \beta M_{12} \\
\alpha \beta \alpha M_{21}, \alpha^2 \beta^2 M_{22}
\end{array} \right)
\]
where we have
\[
M_{11} = -M_{22} = [C_l(x, y), l(z + \bar{z})] - \{D_l(x, y), l(z - \bar{z})\} +,
\]
\[
M_{12} = -M_{21} = [C_l(x, y), l(z - \bar{z})] - \{D_l(x, y), l(z + \bar{z})\} +
\]
and hence
\[
M_{11} = [d_{l+1}(x, y) - d_{1-l}(\bar{x}, \bar{y}), l(z + \bar{z})] - \{d_{l+1}(x, y) - d_{1-l}(\bar{x}, \bar{y}), l(z - \bar{z})\} +
\]
\[
= l((d_l(x, y) + d_{3-l}(\bar{x}, \bar{y}))(z + \bar{z})) + l((d_l(x, y) - d_{3-l}(\bar{x}, \bar{y}))(z - \bar{z}))
\]
\[
= 2l(d_l(x, y)z + d_{3-l}(\bar{x}, \bar{y})\bar{z}) = 2l(d_l(x, y)z + \bar{d_l}(x, y)\bar{z})
\]
and similarly
\[
M_{12} = 2l(d_l(x, y)z - d_{3-l}(\bar{x}, \bar{y})\bar{z}) = 2l(d_l(x, y)z - \bar{d_l}(x, y)\bar{z}).
\]
Thus, we find
\[
[T_l(x, y), \rho_3(z)]
\]
\[
= -\frac{\gamma_3}{\gamma_2} \left( \frac{k \gamma_1}{\gamma_2} \right) 2k \alpha \beta 
\]
\[
\left( \begin{array}{c}
\alpha \beta l(d_l(x, y)z + \bar{d_l}(x, y)\bar{z}), \alpha^2 l(d_l(x, y)z - \bar{d_l}(x, y)\bar{z}) \\
-\beta^2 l(d_l(x, y)z - \bar{d_l}(x, y)\bar{z}), -\alpha \beta l(d_l(x, y)z + \bar{d_l}(x, y)\bar{z})
\end{array} \right)
\]
\[
= \rho_3(d_l(x, y)z).
\]
This completes the proof of Eq.(3.7c).

**Proof of Eq.(3.7d)**

This follows from Eqs.(3.7a-c) and the fact that the Lie algebra associated with structurable algebra is contained into \( \hat{L} = L(W, W) \oplus W \), when we replace \( T_j(x, y) \) by
\[
T_j(x, y) = \frac{\gamma_{2-j}}{\gamma_{1-j}} [\rho_{3-j}(x), \rho_{3-j}(y)].
\]

These completes the proof of Theorem 3.1.//

**Concluding Remark**

Theorem 3.1 clarifies the following fact. The Lie algebra constructed in Eqs.(3.7) on the basis of the structurable algebra is known to be a \( B C_1 \)-graded Lie algebra of type \( B_1 \),
provided that the ground field $F$ contain the square root $\sqrt{-1}$ of $-1$ ([E-O,2]). On the other side, for any left unital $(-1,1)$FKTS (i.e., we have $eex = x$ for any $x \in A$ where $e$ is a left unital element), and hence for any $A$-ternary algebra, the associated standard Lie algebra constructed as in Eqs.(1.13) and (1.14) is also a $BC_1$-graded Lie algebra of type $B_1$ without assuming $\sqrt{-1} \in F$, (see [E-K-O]). Also, if $F$ is an algebraically closed field of characteristic zero, then any simple Lie algebra is known to be $S_3$-invariant and can be constructed by some structurable algebra, so that any such Lie algebra is also a $BC_1$-graded Lie algebra of type $B_1$, (as well as of type $C_1$). Of course, the underlying $sl(2)$ symmetry is different for both $B_1$ and $C_1$ cases.

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