A COMPLETE CLASSIFICATION OF FINITE MORSE INDEX SOLUTIONS TO ELLIPTIC SINE-GORDON EQUATION

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Abstract. The elliptic sine-Gordon equation in the plane has a family of explicit multiple-end solutions (soliton-like solutions). We show that all the finite Morse index solutions belong to this family. We also prove they are non-degenerate in the sense that the corresponding linearized operators have no nontrivial bounded kernel. We then show that solutions with 2n ends have Morse index n(n − 1)/2.

1. Introduction and statement of the main results

The classical sine-Gordon equation originally arises from the study of surfaces with constant negative curvature in the nineteenth century. It also appears in many physical contexts such as Josephson junction. It is an important partial differential equation and has been extensively studied partly due to the fact that it is an integrable system and one can use the technique of inverse scattering transform to analyze its solutions. There are vast literatures on this subject. We refer to the papers in the book [47] and the references cited there for more information about the background and detailed discussion for this equation.

In the space-time coordinate, the sine-Gordon equation has the form

\[ \partial_t^2 u - \partial_x^2 u = \sin u. \quad (1.1) \]

In this paper, we are interested in the elliptic version of this equation. More precisely, we shall consider the problem

\[ - \Delta u = \sin u \quad \text{in } \mathbb{R}^2, \quad |u| < \pi. \quad (1.2) \]

This is a special case of the Allen-Cahn type equations

\[ \Delta u = W'(u) \quad \text{in } \mathbb{R}^N, \quad (1.3) \]

where \( W \) are double well potentials. The equation (1.2) corresponds to \( W(u) = 1 + \cos u \). Note that if \( W(u) = \frac{1}{4} (u^2 - 1)^2 \), then the corresponding equation is the classical Allen-Cahn equation

\[ - \Delta u = u - u^3 \quad \text{in } \mathbb{R}^N, \quad |u| < 1. \quad (1.4) \]

Equation (1.2) has a one dimensional "heteroclinic" solution

\[ H(x) = 4 \arctan e^x - \pi. \]

It is monotone increasing. Translating and rotating it in the plane, we obtain a family of one dimensional solutions. The celebrated De Giorgi conjecture concerns the classification of monotone bounded solutions of the Allen-Cahn type equation (1.3). Many works have been done towards a complete resolution of this conjecture. We refer to [1, 10, 12–14, 20, 29, 36, 45] and the references therein for results in this direction. We will study in this paper non-monotone solutions of (1.2) in the plane.
Let us recall the following

**Definition 1.** ([8]) A solution $u$ of (1.2) is called a $2n$-ended solution, if outside a large ball, the nodal set of $u$ is asymptotic to $2n$ half straight lines.

These asymptotic lines are called ends of the solutions. One can show that actually along these half straight lines, the solution $u$ behaves like one dimensional solution $H$ in the transverse direction. (See [8].) The set of $2n$-ended solution is denoted as $\mathcal{M}_{2n}$. By a recent result of Wang-Wei [49], a solution to (1.2) is multiple-end if and only if it is finite Morse index. In [9], the infinite dimensional Lyapunov-Schmidt reduction method has been used to construct a family of $2n$-end solutions for the Allen-Cahn equation (1.4). The method can also be applied to general double well potentials, including the elliptic sine-Gordon equation (1.2). The nodal sets of these solutions are almost parallel, meaning that the angles between consecutive ends are close to 0 or $\pi$. Actually, the nodal curves are given approximately by a rescaled solutions of the Toda system. It is also known that locally around each $2n$-end solution, the moduli space of $2n$-end solutions has the structure of a real analytic variety. If the solution happens to be nondegenerate, then locally around this solution, the moduli space is indeed a $2n$-dimensional manifold [8]. For general nonlinearities, little is known for the structure of the moduli space of $2n$-end solutions, except in the $n = 2$ case. We now know [30–32] that the space of $4$-end solutions is diffeomorphic to the open interval $(0, 1)$, modulo translation and rotation (they give $3$-free parameters in the moduli space). Based on these four-end solutions, an end-to-end construction for $2n$-end solutions has been carried out in [33]. Roughly speaking, these solutions are in certain sense lying near the boundary of the moduli space.

The classification of $\mathcal{M}_{2n}$ is largely open for general nonlinearities. Important open question include: are solutions in $\mathcal{M}_{2n}$ nondegenerate? Is $\mathcal{M}_{2n}$ connected? What is Morse index in $\mathcal{M}_{2n}$? In a recent paper Mantoulidis [37], a lower bound $n - 1$ on Morse index of $\mathcal{M}_{2n}$ is given. In this paper we give a complete answer to the above question in the case of the special elliptic sine-Gordon equation (1.2).

It is well known that the classical sine-Gordon equation (1.1) is an integrable system. Methods from the theory of integrable systems can be used to find solutions of this system. In particular, it has soliton solutions. Note that (1.2) is elliptic, while (1.1) is hyperbolic in nature. We find in this paper that the Hirota direct method of integrable systems also gives us real nonsingular solutions of (1.2). Let $U_n$ be the functions defined by (3.2). Then $U_n - \pi$ are solutions to (1.2), they depends on $2n$ parameters, $p_j, \eta^0_j, j = 1, ..., n$. We are interested in the spectrum property of these solutions. In this paper, we shall use Bäcklund transformation to prove the following

**Theorem 2.** The $2n$-end solutions $U_n - \pi$ of elliptic sine-Gordon equation (1.2) are $L^\infty$-nondegenerated in the following sense: If $\phi$ is a bounded solution of the linearized equation

$$ -\Delta \phi + \phi \cos U_n = 0. $$

Then there exist constants $d_j, j = 1, ..., n$, such that

$$ \phi = \sum_{j=1}^{n} \left( d_j \partial_{\eta^0_j} U_n \right). $$
We remark that the nonlinear stability of 2-soliton solutions of the classical hyperbolic sine-Gordon equation \((1.1)\) has been proved recently by Munoz-Palacios [38], also using Bäcklund transformation. We refer to the references therein for more discussion on the dynamical properties of the hyperbolic sine-Gordon equation. For general background and applications of Bäcklund transformation, we refer to [43, 44].

The Morse index of \(U_n - \pi\) is by definition the number of negative eigenvalues of the operator \(-\Delta - \cos U_n\), in the space \(H^2(\mathbb{R}^2)\), counted with multiplicity. The Morse index can also be defined as the maximal dimension of the subspace of compactly supported smooth functions where the associated quadratic form of the energy functional is negative. Our next result is

**Theorem 3.** The Morse index of \(U_n - \pi\) is equal to \(\frac{n(n-1)}{2}\). Moreover, all the finite Morse index solutions of \((1.2)\) are of the form \(U_n - \pi\), with suitable choice of the parameters \(p_j, q_j, \eta_j^0\).

The classification result stated in this theorem follows from a direct application of the inverse scattering transform studied in [23]. Inverse scattering transform for elliptic sine-Gordon equation has also been used in [2, 3] to study solutions with periodic asymptotic behavior or vortex type singularities. Note that certain class of vortex type solutions were analyzed through Bäcklund transformation or direct method in [28, 35, 40, 46], and finite energy solutions with point-like singularities have been studied in [51]. It is also worth mentioning that more recently, some classes of quite involved boundary value problems of the elliptic sine-Gordon equation have been investigated via Fokas direct method in [15, 16, 41, 42].

Theorem 3 implies that in the special case \(n = 2\), the four-end solutions of the equation \((1.2)\) have Morse index one. In the family of four-end solutions, there is a special one, called saddle solution (see \((3.3)\)), explicitly given by

\[
4 \arctan \left( \frac{\cosh \left( \frac{\theta}{\sqrt{2}} \right)}{\cosh \left( \frac{\eta}{\sqrt{2}} \right)} \right) - \pi.
\]

The nodal set of this solution consists of two orthogonally intersected straight lines, hence the name saddle solution. Saddle-shaped solutions of Allen-Cahn type equation \(\Delta u = F'(u)\) in \(\mathbb{R}^k\) with \(k \geq 2\) has been studied by Cabre and Terra in a series of papers. In [5, 6] it is proved that in \(\mathbb{R}^4\) and \(\mathbb{R}^5\), the saddle-shaped solution is unstable, while in \(\mathbb{R}^{2k}\) with \(k \geq 7\), they are stable [4]. It is also conjectured in [4] that for \(k \geq 4\), the saddle-shaped solutions should be a global minimizer of the energy functional. However, for \(F(u) = 1 + \cos u\), the generalized elliptic sine-Gordon equation in even dimension higher than two is believed to be non-integrable, hence no explicit formulas are available for these saddle-shaped solutions. We expect that our nondegeneracy results in this paper will be useful in the construction of solutions of the generalized elliptic sine-Gordon equation in higher dimensions.

We also stress that \(W(u) = 1 + \cos u\) is essentially the only double well potential such that the corresponding equation is integrable [11]. It is also worth pointing out that the sine nonlinearity also appears in the Pierls-Nabarro equation whose solutions have been classified in [48]. In dimension two, a classification result like Theorem 3 for general double well potentials could be very difficult.
Finally we mention that recently there have been many interesting studies on the use of Allen-Cahn type equation in constructing minimal surfaces. We refer to [7], [17], [18], [21], [37] and the references therein.

Acknowledgement The research of J. Wei is partially supported by NSERC of Canada. Part of this work was finished while the first author was visiting the University of British Columbia in 2017. He thanks the institute for the financial support.

2. Soliton solutions of the hyperbolic sine-Gordon equation

In this section, we consider the classical sine-Gordon equation
\[
\partial^2_\phi - \partial^2_z \phi = \sin \phi. \tag{2.1}
\]
Hereafter, we shall call it hyperbolic sine-Gordon equation. It is well known that this equation has soliton solutions. Let us first recall the explicit \(n\)-soliton solutions of (2.1) in the form obtained in [24] using Hirota direct method. We also refer to [19, 25, 26, 50] for related results on soliton solutions.

Let \(P_j, Q_j\) be complex numbers with \(P_j^2 - Q_j^2 = 1\). Define
\[
\alpha(j, k) = \frac{(P_j - P_k)^2 - (Q_j - Q_k)^2}{(P_j + P_k)^2 - (Q_j + Q_k)^2}. \tag{2.2}
\]
Note that \(\alpha(j, k) = \alpha(k, j)\). Since
\[
P_j - Q_j = \frac{1}{P_j + Q_j},
\]
we can also rewrite \(\alpha\) in the form
\[
\alpha(j, k) = \frac{(P_j - P_k + Q_j - Q_k)\left(\frac{1}{P_j + Q_j} - \frac{1}{P_k + Q_k}\right)}{(P_j + P_k + Q_j + Q_k)\left(\frac{1}{P_j + Q_j} + \frac{1}{P_k + Q_k}\right)} = \frac{(P_j - P_k + Q_j - Q_k)^2}{(P_j + P_k + Q_j + Q_k)^2}.
\]
We also define \(a\) by
\[
a(i_1, i_2, ..., i_n) = 1, \text{ if } n = 0,1,
\]
\[
a(i_1, i_2, ..., i_n) = \prod_{k<i} \alpha(i_k, i_i), \text{ if } n \geq 2.
\]
Let us introduce the notation \(\eta_i = P_i x - Q_i z - \eta_i^0\), where \(\eta_i^0\) is a complex constant.

It has been proved in [24] that equation (2.1) has families of \(n\)-soliton solutions of the form
\[
\phi = 4 \arctan \frac{g}{f}, \tag{2.3}
\]
where the functions \(f, g\) are explicitly given by
\[
f = \sum_{k=0}^{[n/2]} \left( \sum_{\{i_1, ..., i_{2k}\}} [a(i_1, ..., i_{2k}) \exp (\eta_{i_1} + ... + \eta_{i_{2k}})] \right), \tag{2.4}
\]
\[
g = \sum_{k=0}^{[(n-1)/2]} \left( \sum_{\{i_1, ..., i_{2k+1}\}} [a(i_1, ..., i_{2k+1}) \exp (\eta_{i_1} + ... + \eta_{i_{2k+1}})] \right). \tag{2.5}
\]
Here the notation $\sum_{\{n,k\}}$ means taking sum over all possible $k$ different integers $i_1, ..., i_k$ from the set of integers $\{1, ..., n\}$. It is worth mentioning that these solutions can also be written in the Wronskian form (39). Here we choose to use the form (2.4), (2.5), because it is more convenient to check the positive condition of the function. This will be clear when we are dealing with the elliptic version of the sine-Gordon equation. Note that in the special case $n = 3$, we have

$$f = \sum_{n=0}^{1} \left( \sum_{\{3,2n\}} a (i_1, ..., i_{2n}) \exp (\eta_{i_1} + ... + \eta_{i_{2n}}) \right)$$

$$= 1 + a (1, 2) \exp (\eta_1 + \eta_2) + a (1, 3) \exp (\eta_1 + \eta_3) + a (2, 3) \exp (\eta_2 + \eta_3)$$

$$= 1 + \alpha (1, 2) \exp (\eta_1 + \eta_2) + \alpha (1, 3) \exp (\eta_1 + \eta_3) + \alpha (2, 3) \exp (\eta_2 + \eta_3).$$

$$g = \sum_{k=0}^{1} \left( \sum_{\{3,2k+1\}} a (j_1, ..., j_{2k+1}) \exp (\eta_{j_1} + ... + \eta_{j_{2k+1}}) \right)$$

$$= \exp (\eta_1) + \exp (\eta_2) + \exp (\eta_3) + a (1, 2, 3) \exp (\eta_1 + \eta_2 + \eta_3)$$

$$= \exp (\eta_1) + \exp (\eta_2) + \exp (\eta_3) + \alpha (1, 2) \alpha (1, 3) \alpha (2, 3) \exp (\eta_1 + \eta_2 + \eta_3).$$

2.1. Bäcklund transform and bilinear form of the hyperbolic sine-Gordon equation. Lamb [34] has established a superposition formula for the Bäcklund transformation of the hyperbolic sine-Gordon equation. In particular, this formula enables us to get multi-soliton solutions in an algebraic way. However, in this formulation, for $n$-soliton solutions with $n$ large, it will be quite tedious to write down the explicit expressions for the solutions. Nevertheless, it turns out that the soliton solutions (2.3) discussed above can be obtained through Bäcklund transformation. This will be discussed in more details in this section.

In the light-cone coordinate, the hyperbolic sine-Gordon equation has the form

$$u_{st} = \sin u, (s, t) \in \mathbb{R}^2.$$  \tag{2.6}

Let $\beta$ be a parameter. The Bäcklund transformation between two solutions $f$ and $g$ of (2.6) is given by (see for instance [44]):

$$\begin{align*}
\begin{cases}
\partial_s f &= \partial_s g + 2\beta \sin \frac{f+g}{2}, \\
\partial_t f &= -\partial_t g + 2\beta^{-1} \sin \frac{F^2}{2}.
\end{cases}
\end{align*}$$  \tag{2.7}

Next we recall the bilinear form of the hyperbolic sine-Gordon equation (27). Let $F = f + ig$ be a complex function, where $i$ is the complex unit and $f, g$ are real valued functions. The complex conjugate of $F$ will be denoted by $\bar{F}$. Now we write $u$ as

$$u = 2i \ln \frac{\bar{F}}{F} = 4 \arctan \frac{g}{f}.$$  \tag{2.6}

Note that

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i} = \frac{1}{2i} \left( F^2 - \bar{F}^2 \right).$$

We use $D$ to denote the bilinear operator (27). Then (2.6) has the bilinear form

$$D_s D_t F \cdot F = \frac{1}{2} \left( F^2 - \bar{F}^2 \right).$$
We see that the Bäcklund transformation can be written in the following bilinear form (see [27]):

\[
\begin{align*}
D_x G \cdot F &= \frac{h}{2} \tilde{G} \tilde{F}, \\
D_t G \cdot \tilde{F} &= \frac{1}{2\pi} GF.
\end{align*}
\]

If \( u = 2i \ln \frac{\tilde{P}}{P}, v = 2i \ln \frac{\tilde{Q}}{Q} \) satisfy (2.8), then they also satisfy (2.7).

Let us fix an \( n \in \mathbb{N} \). The \( n \)-soliton solutions discussed in the previous section are indeed Bäcklund transformation of certain \( n - 1 \) soliton type solutions. To see this, we write the solutions in another form. For \( j = 1, \ldots, n \), let \( k_j = P_j + Q_j \) and define \( \xi_j \) by

\[
c^{\xi_j} = \prod_{l<j} \frac{k_l + k_j}{k_l - k_j} \prod_{l>j} \frac{k_j + k_l}{k_j - k_l}.
\]

Since \( P_j^2 - Q_j^2 = 1 \), we know that \( k_j^{-1} = P_j - Q_j \). Let us now define \( \tilde{n}_j = n_j - \xi_j \), \( j = 1, \ldots, n \). It can be written as \( \tilde{n}_j = P_j x + Q_j z + \tilde{n}_{j,0} \), with \( \tilde{n}_{j,0} = n_{j,0} - \xi_j \).

With these notations, the function \( f_n \) can be rewritten as

\[
\sum_{k=0}^{[n/2]} \left( \sum_{i_1, \ldots, i_{2k}} a \left( i_1, \ldots, i_{2k} \right) \exp \left( \xi_{i_1} + \ldots + \xi_{i_{2k}} \right) \right) = \exp \left( \frac{1}{2} \left( \tilde{n}_1 + \ldots + \tilde{n}_n \right) \right) \tilde{f}_n \prod_{l<j \leq n} \frac{1}{k_l - k_j},
\]

where the function \( \tilde{f}_n \) is defined to be

\[
\sum_{\prod_{k=1}^{n} \varepsilon_k = (-1)^n} \left( \exp \left( \sum_{j=1}^{n} \frac{\varepsilon_j}{2} \left( \tilde{n}_j + \frac{\pi i}{2} \right) + \frac{n \pi i}{4} \right) \prod_{m<j \leq n} \left( k_m - \varepsilon_m \varepsilon_j k_j \right) \right),
\]

and \( \varepsilon_j \) are indices equal +1 or -1. Similarly, we can write

\[
g_n = \exp \left( \frac{1}{2} \left( \tilde{n}_1 + \ldots + \tilde{n}_n \right) \right) \tilde{g}_n \prod_{l<j \leq n} \frac{1}{k_l - k_j},
\]

with

\[
\tilde{g}_n = \sum_{\prod_{k=1}^{n} \varepsilon_k = (-1)^{n+1}} \left( \exp \left( \sum_{j=1}^{n} \frac{\varepsilon_j}{2} \left( \tilde{n}_j + \frac{\pi i}{2} \right) + \frac{(n-2) \pi i}{4} \right) \prod_{m<j \leq n} \left( k_m - \varepsilon_m \varepsilon_j k_j \right) \right).
\]

We see that the \( n \)-soliton solution (2.3) of the hyperbolic sine-Gordon equation also equals \( 4 \arctan \frac{\tilde{f}_n}{f_n} \).

We next would like to consider an \( n - 1 \)-soliton solutions closely related to \( 4 \arctan \frac{\tilde{g}_n}{f_n} \). More precisely, we define

\[
\gamma = \sum_{\prod_{k=1}^{n-1} \varepsilon_k = (-1)^{n-1}} \left( \exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{n}_j + \frac{\pi i}{2} \right) + \frac{(n-1) \pi i}{4} \right) \prod_{m<j \leq n-1} \left( k_m - \varepsilon_m \varepsilon_j k_j \right) \right),
\]

where the function \( \tilde{g}_n \) is defined to be
\[ \tau = \sum_{k=1}^{n-1} \left( \exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \left( \frac{n-3}{4} \pi i \right) \right) \prod_{m<j \leq n-1} (k_m - \varepsilon_m \varepsilon_j k_j) \right). \]

Let \( x = s + t, z = s - t \). We have following Bäcklund transformation.

**Lemma 4.** Let \( F = \gamma + i \tau, G = \tilde{f}_n + i \tilde{g}_n \). Suppose \( P_j, Q_j, \tilde{\eta}_j, 0, j = 1, \ldots, n, \) are real numbers. Then

\[
\begin{aligned}
D_s G \cdot F &= -\frac{1}{2k_n} \tilde{G} F, \\
D_s G \cdot \tilde{F} &= -\frac{1}{2k_n} \tilde{G} F.
\end{aligned}
\]

**Proof.** We sketch the proof of this fact for completeness. We only prove the first identity, the idea for the proof of the second one is similar.

We compute

\[
D_s G \cdot F = F \partial_s G - G \partial_s F
= \left( \partial_s \tilde{f}_n + i \partial_s \tilde{g}_n \right) (\gamma + \tau i) - \left( \tilde{f}_n + i \tilde{g}_n \right) (\partial_s \gamma + i \partial_s \tau)
= \gamma \partial_s \tilde{f}_n - \tau \partial_s \tilde{g}_n - \left( \tilde{f}_n \partial_s \gamma - \tilde{g}_n \partial_s \tau \right)
+ \left[ (\tau \partial_s \tilde{f}_n + \gamma \partial_s \tilde{g}_n) - (\tilde{f}_n \partial_s \tau + \tilde{g}_n \partial_s \gamma) \right] i.
\]

On the other hand,

\[
\tilde{G} F = \left( \tilde{f}_n - i \tilde{g}_n \right) (\gamma - \tau i)
= \tilde{f}_n \gamma - \tilde{g}_n \tau - i \left( \tilde{f}_n \tau + \tilde{g}_n \gamma \right).
\]

We proceed to show that the real part of \( D_s G \cdot F + \frac{1}{2k_n} GF \) is equal to zero, that is

\[
\gamma \partial_s \tilde{f}_n - \tau \partial_s \tilde{g}_n - \left( \tilde{f}_n \partial_s \gamma - \tilde{g}_n \partial_s \tau \right) + \frac{1}{2k_n} (\tilde{f}_n \gamma - \tilde{g}_n \tau) = 0.
\]

To see this, we first consider those terms involving \( \exp \left( \frac{1}{2} \varepsilon_n \tilde{\eta}_n \right) \) with \( \varepsilon_n = -1 \). Consider a typical sum \( I \) of terms in \( \gamma \partial_s \tilde{f}_n \), of the form

\[
\exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \left( \frac{n-3}{4} \pi i \right) \right) \prod_{m<j \leq n-1} (k_m - \varepsilon_m \varepsilon_j k_j)
\]

\[
\exp \left( \sum_{j=1}^{n} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \left( \frac{n}{4} \pi i \right) \right) \prod_{m<j \leq n} (k_m - \varepsilon_m \varepsilon_j k_j) \times \frac{1}{2} (\varepsilon_1 k_1^{-1} + \ldots + \varepsilon_n k_n^{-1})
\]

\[
+ \exp \left( \sum_{j=1}^{n} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \left( \frac{n-1}{4} \pi i \right) \right) \prod_{m<j \leq n-1} (k_m - \varepsilon_m \varepsilon_j k_j) \times \frac{1}{2} (\varepsilon_1 k_1^{-1} + \ldots + \varepsilon_n k_n^{-1})
\]

\[
+ \exp \left( \sum_{j=1}^{n} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \left( \frac{n}{4} \pi i \right) \right) \prod_{m<j \leq n} (k_m - \varepsilon_m \varepsilon_j k_j) \times \frac{1}{2} (\varepsilon_1 k_1^{-1} + \ldots + \varepsilon_n k_n^{-1})
\]
with \( \prod_{j=1}^{n-1} \varepsilon_j = -1 \), and \( \varepsilon_n = \hat{\varepsilon}_n = -1 \). The function \( I \) has a “dual” part \( I^* \) in \( \tilde{f}_n \partial s \gamma \), of the form

\[
\exp \left( \sum_{j=1}^{n} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{n \pi i}{4} \right) \prod_{m<j \leq n} \left( k_m - \varepsilon_m \varepsilon_j \right) \times \exp \left( \sum_{j=1}^{n-1} \frac{\hat{\varepsilon}_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{(n-1) \pi i}{4} \right) \prod_{m<j \leq n-1} \left( k_m - \hat{\varepsilon}_m \hat{\varepsilon}_j \right) \times \exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{(n-1) \pi i}{4} \right) \prod_{m<j \leq n-1} \left( k_m - \hat{\varepsilon}_m \hat{\varepsilon}_j \right) \times \varepsilon_n \frac{k_n^{-1}}{2} \left( \prod_{i<n} (k_i - \varepsilon_i \varepsilon_n k_n) + \prod_{i<n} (k_i - \hat{\varepsilon}_i \hat{\varepsilon}_n k_n) \right).
\]

Subtracting \( I \) with \( I^* \), we obtain

\[
\exp \left( \sum_{j=1}^{n} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{n \pi i}{4} \right) \prod_{m<j \leq n} \left( k_m - \varepsilon_m \varepsilon_j \right) \times \exp \left( \sum_{j=1}^{n-1} \frac{\hat{\varepsilon}_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{(n-1) \pi i}{4} \right) \prod_{m<j \leq n-1} \left( k_m - \hat{\varepsilon}_m \hat{\varepsilon}_j \right) \times \varepsilon_n \frac{k_n^{-1}}{2} \left( \prod_{i<n} (k_i - \varepsilon_i \varepsilon_n k_n) + \prod_{i<n} (k_i - \hat{\varepsilon}_i \hat{\varepsilon}_n k_n) \right).
\]

This corresponds to the sum of two terms in \( -\frac{k_n^{-1}}{2} \tilde{f}_n \gamma \). Hence if one only considers those terms involving \( \exp \left( \frac{1}{2} \varepsilon_n \tilde{\eta}_n \right) \) with \( \varepsilon_n = -1 \), then \( \gamma \partial_s \tilde{f}_n - \tau \partial_s \tilde{g}_n = -\frac{k_n^{-1}}{2} \tilde{f}_n \gamma \), similarly for \( \tau \partial_s \tilde{g}_n - \tilde{g}_n \partial_s \tau + \frac{k_n^{-1}}{2} \tilde{g}_n \tau \).

For those terms involving \( \exp \left( \frac{1}{2} \varepsilon_n \tilde{\eta}_n \right) \) with \( \varepsilon_n = 1 \), there is a similar cancelation between \( \gamma \partial_s \tilde{f}_n - \tau \partial_s \tilde{g}_n \) and \(-\frac{k_n^{-1}}{2} \tilde{g}_n \tau \), also there is cancelation between \( \tau \partial_s \tilde{g}_n - \tilde{g}_n \partial_s \tau \) and \(-\frac{k_n^{-1}}{2} \tilde{f}_n \gamma \).

Summarizing, we get

\[
\gamma \partial_s \tilde{f}_n - \tau \partial_s \tilde{g}_n - \left( \tilde{f}_n \partial_s \gamma - \tilde{g}_n \partial_s \tau \right) + \frac{k_n^{-1}}{2} \left( \tilde{f}_n \gamma - \tilde{g}_n \tau \right) = 0.
\]

The proof is completed. \( \square \)

3. MULTIPLE-END SOLUTIONS AND BÄCKLUND TRANSFORMATION OF THE ELLIPTIC SINE-GORDON EQUATION

In this section, we consider the elliptic sine-Gordon equation in the form

\[
\partial_x^2 u + \partial_y^2 u = \sin u.
\]  
(3.1)

Note that \( u \) is a solution to (3.1) if and only if \( u - \pi \) is a solution to (1.2). The elliptic sine-Gordon equation has been studied by Leibrbrandt in [35], with an application to the Josephson effect. He uses the Bäcklund transformation method. However, the
solutions he found is singular at some points in the plane. Gutshabash-Lipovski˘ı [23] studied the boundary value problem of the elliptic sine-Gordon equation in the half plane using inverse scattering transform and obtained multi-soliton solutions in the determinant form. The boundary problems have also been studied in [15, 16, 41, 42] by the Fokas direct method.

Our observation in this paper is that in the hyperbolic sine-Gordon equation (2.1), if we introduce the changing of variable $z = yi$, where $i$ is the complex unit, then we get the elliptic sine-Gordon equation. Based on this, by choosing certain complex parameters in (2.4), (2.5) for the solutions of the hyperbolic sine-Gordon equation, we then get multiple-end solutions of the elliptic sine-Gordon equation. Let us describe this in more details.

Let $p_j, q_j$ be real numbers with $p_j^2 + q_j^2 = 1$. Similar as the hyperbolic sine-Gordon case, we define

$$\alpha(j, k) = \frac{(p_j - p_k)^2 + (q_j - q_k)^2}{(p_j + p_k)^2 + (q_j + q_k)^2},$$

We still use the notation

$$a(i_1, i_2, ..., i_n) = 1, \text{ if } n = 0, 1,$$
$$a(i_1, i_2, ..., i_n) = \prod_{k<l} \alpha(i_k, i_l), \text{ if } n \geq 2.$$

Define $\eta_i = p_i x - q_i y - \eta_i^0$. Then the elliptic sine-Gordon equation has the solution

$$U_n := 4 \arctan \frac{g}{f},$$

where

$$f = \sum_{k=0}^{[n/2]} \left( \sum_{\{n, 2k\}} a(i_1, ..., i_{2k}) \exp (\eta_{i_1} + ... + \eta_{i_{2k}}) \right),$$
$$g = \sum_{m=0}^{[(n-1)/2]} \left( \sum_{\{n, 2m+1\}} a(i_1, ..., i_{2m+1}) \exp (\eta_{i_1} + ... + \eta_{i_{2m+1}}) \right).$$

Note that $U_n - \pi$ is indeed a smooth $2n$-end solution of (1.2).

In the special case of $n = 2$, if we choose $p_1 = p_2 = p$ and $q_1 = -q_2 = q$, $\eta_1^0 = \eta_2^0 = \ln \frac{p}{q}$, then we get the solution

$$\varphi_{p, q}(x, y) := 4 \arctan \left( \frac{p \cosh(qy)}{q \cosh(px)} \right) - \pi.$$

This corresponds to a four-end solution of the elliptic sine-Gordon equation (1.2). Note that on the lines $px = \pm qy$, $\varphi_{p, q} = 4 \arctan \frac{p}{q} - \pi$. In the special case $p = q = \sqrt{2}$, the solution is

$$4 \arctan \left( \frac{\cosh \left( \frac{y}{\sqrt{2}} \right)}{\cosh \left( \frac{x}{\sqrt{2}} \right)} \right) - \pi.$$  

This is the classical saddle solution.

We remark that this family of 4-end solutions has analogous in the minimal surface theory. They are the so called Scherk second surface family, which are
embedded singly periodic minimal surfaces in \( \mathbb{R}^3 \). Explicitly, these surface can be described by
\[
\cos^2 \theta \cosh \frac{x}{\cos \theta} - \sin^2 \theta \sinh \frac{y}{\sin \theta} = \cos z.
\]
Here \( \theta \) is a parameter. Each of these surfaces has four wings, called ends of the surfaces. Geometrically, they are obtained by desingularized two intersected planes with intersection angle \( \theta \).

Next, we would like to investigate the Bäcklund transformation for the solutions of elliptic sine-Gordon equation. Let \( k_j = p_j + q_j i \). Define \( \xi_j \) by
\[
e^{\xi_j} = \prod_{l<j} \frac{k_l + k_j}{k_l - k_j} \prod_{j<l} \frac{k_j + k_l}{k_j - k_l}.
\]
Recall that for all \( j, p_j^2 + q_j^2 = 1 \). Hence the number \( \frac{k_j + k_j}{k_j - k_j} \) is purely imaginary and \( e^{\xi_j} \) is in general complex valued. We define \( \tilde{\eta}_j = \eta_j - \xi_j = p_j x + q_j y + q_j^0 - \xi_j, j = 1, \ldots, n \).

Then the solution \( U_n \) can be written as 4 arctan \( \frac{p_j}{q_j} \), where
\[
\tilde{f}_n = \sum_{\prod_{j=1}^n \varepsilon_j = (-1)^n} \left( \exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{n\pi i}{4} \right) \prod_{m<j \leq n} (k_m - \varepsilon_m \varepsilon_j k_j) \right). 
\]
\[
\tilde{g}_n = \sum_{\prod_{j=1}^n \varepsilon_j = (-1)^{n+1}} \left( \exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{(n-2)\pi i}{4} \right) \prod_{m<j \leq n} (k_m - \varepsilon_m \varepsilon_j k_j) \right). 
\]
Furthermore, we define
\[
\gamma = \sum_{\prod_{j=1}^{n-1} \varepsilon_j = (-1)^{n-1}} \left( \exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{(n-1)\pi i}{4} \right) \prod_{m<j \leq n-1} (k_m - \varepsilon_m \varepsilon_j k_j) \right), 
\]
\[
\tau = \sum_{\prod_{j=1}^{n-1} \varepsilon_j = (-1)^n} \left( \exp \left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2} \left( \tilde{\eta}_j + \frac{\pi i}{2} \right) + \frac{(n-3)\pi i}{4} \right) \prod_{m<j \leq n-1} (k_m - \varepsilon_m \varepsilon_j k_j) \right). 
\]

Let \( x = s + t, y = -i(s-t) \) and \( u = U_n, v = 4 \arctan \frac{\tau}{\gamma} \). A direct consequence of Lemma 4 is the following

**Lemma 5.** The functions \( u \) and \( v \) are connected through the following Bäcklund transformation:
\[
\left\{ \begin{array}{l}
\partial_x u = -i \partial_y v + \tilde{k}_n \sin \frac{\alpha u}{2} + k_n \sin \frac{\alpha v}{2}, \\
i \partial_y u = -\partial_x v - k_n \sin \frac{\alpha u}{2} + \tilde{k}_n \sin \frac{\alpha v}{2}.
\end{array} \right.
\]

We remark that \( \frac{\gamma}{\tau} \) is purely imaginary. The function \( \sin \frac{\alpha u}{2} \) is understood to be
\[
\sin \left( 2 \arctan \frac{\tau}{\gamma} \right) = \frac{2\gamma \tau}{\gamma^2 + \tau^2},
\]
\[
\cos \left( 2 \arctan \frac{\tau}{\gamma} \right) = \frac{\gamma^2 - \tau^2}{\gamma^2 + \tau^2}.
\]
and $\partial_x v = 4\frac{\partial_x \tau \partial_x \gamma}{\gamma + \tau^2}$.

4. **Linearized Bäcklund transformation and nondegeneracy of the 2n-end solution of the elliptic sine-Gordon equation**

Linearizing the Bäcklund transformation (3.4) at $(u, v)$, we get the linearized system

$$\begin{cases}
\partial_x \phi = -i \partial_y \eta + \bar{k}_n \cos \frac{u+v}{2} \left( \frac{\phi+n}{2} \right) + k_n \cos \frac{u-v}{2} \left( \frac{\phi-n}{2} \right), \\
i \partial_y \phi = -\partial_x \eta - \bar{k}_n \cos \frac{u+v}{2} \left( \frac{\phi+n}{2} \right) + k_n \cos \frac{u-v}{2} \left( \frac{\phi-n}{2} \right).
\end{cases}$$

It can be written as

$$\begin{cases}
L \phi = M \eta, \\
T \phi = N \eta,
\end{cases} \tag{4.1}$$

where

$L \phi = \partial_x \phi - \left( \bar{k}_n \cos \frac{u+v}{2} + k_n \cos \frac{u-v}{2} \right) \phi$, \\
$T \phi = i \partial_y \phi + \left( \bar{k}_n \cos \frac{u+v}{2} - k_n \cos \frac{u-v}{2} \right) \phi$, \\
$M \eta = -i \partial_y \eta + \left( \bar{k}_n \cos \frac{u+v}{2} - k_n \cos \frac{u-v}{2} \right) \eta$, \\
$N \eta = -\partial_x \eta - \left( \bar{k}_n \cos \frac{u+v}{2} + k_n \cos \frac{u-v}{2} \right) \eta$.

To simplify the notation, we write $\tilde{f}_n$ as $f$, and $\tilde{g}_n$ as $g$. Explicitly, using the formulas of $u$ and $v$, we find that $L \phi$ is equal to

$$\partial_x \phi - \left( \bar{k}_n \left( \frac{(f \gamma - g \tau)^2}{(f^2 + g^2)(\gamma^2 + \tau^2)} - 1 \right) + k_n \left( \frac{(f \gamma + g \tau)^2}{(f^2 + g^2)(\gamma^2 + \tau^2)} - 1 \right) \right) \phi.$$ 

The analysis of this operator is complicated by the fact that the function $\frac{\tau}{\gamma}$ is purely imaginary, hence $\gamma^2 + \tau^2$ will be equal to zero at some points of $\mathbb{R}^2$. We define this singular set as

$$S := \{(x, y) : \gamma^2 + \tau^2 = 0\}.$$ 

Note that the asymptotic behavior of $\gamma$ and $\tau$ are determined by some explicit exponential functions. It follows that for each fixed $y$, there are only finitely many points in $S$. Now we denote

$$\Gamma(x, y) := k \left( \frac{2(f \gamma - g \tau)^2}{(f^2 + g^2)(\gamma^2 + \tau^2)} - 1 \right).$$

Then $\Gamma$ is singular at $S$ and

$L \phi = \partial_x \phi - \text{Re} \left( \Gamma - \bar{k}_n \right) \phi$, \\
$T \phi = i \partial_y \phi + i \text{Im} \left( \Gamma - \bar{k}_n \right) \phi$.

Rotating the axis if necessary, we can assume $p_j \neq 0$, for any $j$, and $p_n > 0$.

**Lemma 6.** For any fixed $y \in \mathbb{R}$, 

$$\Gamma(x, y) \to 0 \text{ as } x \to \pm \infty.$$
Proof. This follows directly from analyzing the main order of \( f, g \) and \( \gamma, \tau \), as \(|x| \to +\infty\). Indeed, \( \Gamma \) decays exponentially fast as infinity.

We define the function
\[
\xi(x, y) := \exp \left( -x \Re \bar{k}_n + y \Im \bar{k}_n + \int_{-\infty}^{x} \Re(\Gamma(s, y)) \, ds \right).
\]

Then formally \( L\xi = 0 \), with \( \xi(x, y) \to e^{-x \Re \bar{k}_n + y \Im \bar{k}_n} \), as \( x \to -\infty \). However, since \( \Gamma \) has singularities, it is not clear at this moment whether \( \xi \) is well defined. Nevertheless, we will show that \( \xi \) is continuous.

We would like to analyze the singular set of \( \Gamma \) away from the origin.

Lemma 7. Let \((x_j, y_j)\) be a sequence of points in \( S \) such that \( x_j^2 + y_j^2 \to +\infty \), as \( j \to +\infty \). Then up to a subsequence, there is an index \( j_0 \) and sequence \( A_j \in \mathbb{R} \), such that
\[
\Gamma(x_j, y_j) k_{j_0} (p_{j_0}, x_j + q_{j_0} y_j + A_j) \to 1, \text{ as } j \to +\infty.
\]

Proof. It will be convenient to multiply both \( \gamma \) and \( \tau \) by \( \exp \left( \frac{1}{2} \sum (\tilde{\eta}_n + \cdots + \tilde{\eta}_{n-1}) \right) \).

Using the fact that \( |z/7| = 1 \) in \( S \), we first infer that there exists an index \( j_0 \) and a universal constant \( C \) such that \( |\eta_{j_0}| \leq C \) for a subsequence of \( \{(x_j, y_j)\}_{j=1}^{+\infty} \). (Otherwise, \( |z/7| \) will be tending to \( +\infty \) or 0, depending on the parity of \( n \).

We still denote this subsequence by \((x_j, y_j)\). Without loss of generality, we may assume that as \( j \to +\infty \),
\[
\eta_m \to -\infty, \text{ for } m = 1, \ldots, j_0 - 1, \\
\eta_m \to +\infty, \text{ for } m = j_0 + 1, \ldots, n.
\]

We only consider the case that \( n - j_0 \) is odd. The other case is similar.

In view of the main order terms of \( \tau \) and \( \gamma \), we get
\[
\frac{\tau}{\gamma} \to \exp(\tilde{\eta}_{j_0}) \prod_{j=1}^{j_0-1} \frac{k_j + k_{j_0}}{k_j - k_{j_0}} \prod_{j=j_0+1}^{n-1} \frac{k_{j_0} - k_j}{k_{j_0} + k_j}. \tag{4.2}
\]

On the other hand, along this sequence \((x_j, y_j)\),
\[
\frac{g}{f} \to \exp(-\tilde{\eta}_{j_0}) \prod_{j=1}^{j_0-1} \frac{k_j - k_{j_0}}{k_j + k_{j_0}} \prod_{j=j_0+1}^{n} \frac{k_{j_0} + k_j}{k_{j_0} - k_j}.
\]

Recall that \( \gamma^2 + \tau^2 = 1 \) at \((x_j, y_j)\). Hence
\[
\frac{g^2}{f^2} \to -\left( \frac{k_{j_0} + k_n}{k_{j_0} - k_n} \right)^2. \tag{4.3}
\]

Now we compute
\[
\Gamma = \bar{k}_n \frac{2(f\gamma - g\tau)^2}{(f^2 + g^2)(\gamma^2 + \tau^2)} \\
= 2\bar{k}_n \frac{(1 - \bar{\tau}\gamma)^2}{(1 + \bar{\tau}^2)(1 + \bar{\gamma}^2)}.
\]
Then by (4.2) and (4.3), as \( j \to +\infty, \)
\[
\Gamma(x_j, y_j) \left(1 + \frac{x_j^2}{\gamma^2}\right) \to 2k_n \left(1 - \frac{k_{j0} + k_n}{k_{j0} - k_n}\right)^2 = -2\bar{k}_{j0}.
\]
This then leads to the assertion of the lemma.

By (4.2), away from the origin, the singular set \( S \) consists of finitely many components, each of them is asymptotic to a straight line.

**Lemma 8.** Let \( T_1 := L\phi - M\eta, T_2 := T\phi - N\eta. \) Suppose that \( \Delta \eta = \eta \cos u \) and \( T_1 = 0. \) Then
\[
\partial_x T_2 = \left(\frac{k_n}{2} \cos \frac{u + v}{2} + \frac{k_n}{2} \cos \frac{u - v}{2}\right) T_2.
\]

**Proof.** Let \( \beta = \bar{k}_n. \) Consider the system
\[
\begin{cases}
- \partial_x u - i\partial_y v + \beta \sin \frac{u + v}{2} + \beta^{-1} \sin \frac{u - v}{2} = 0 \\
- i\partial_y u - \partial_x v - \beta \sin \frac{u + v}{2} + \beta^{-1} \sin \frac{u - v}{2} = 0
\end{cases}
\]
Denoting the right hand side of the first equation by \( A_1, \) and that of the second equation by \( A_2, \) we have
\[
\partial_x A_2 - i\partial_y A_1 = -\Delta v - \beta \cos \frac{u + v}{2} \left(\frac{\partial_x u + \partial_x v}{2}\right) + \beta^{-1} \cos \frac{u - v}{2} \left(\partial_x u - \partial_x v\right)
- \beta i \cos \frac{u + v}{2} \left(\frac{\partial_y u + \partial_y v}{2}\right) - \beta^{-1} i \cos \frac{u - v}{2} \left(\partial_y u - \partial_y v\right)
= -\Delta v - \beta \cos \frac{u + v}{2} \left(\partial_x u + \partial_x v + i \left(\partial_y u + \partial_y v\right)\right)
+ \frac{\beta^{-1}}{2} \cos \frac{u - v}{2} \left(\partial_x u - \partial_x v - i \left(\partial_y u - \partial_y v\right)\right)
= -\Delta v - \beta \cos \frac{u + v}{2} \left(2\beta^{-1} \sin \frac{u - v}{2} - A_1 - A_2\right)
+ \frac{\beta^{-1}}{2} \cos \frac{u - v}{2} \left(2\beta \sin \frac{u + v}{2} - A_1 + A_2\right)
= -\Delta v + \sin v + A_1 \left(\frac{\beta}{2} \cos \frac{u + v}{2} - \frac{\beta^{-1}}{2} \cos \frac{u - v}{2}\right)
+ A_2 \left(\frac{\beta}{2} \cos \frac{u + v}{2} + \frac{\beta^{-1}}{2} \cos \frac{u - v}{2}\right).
\]
Differentiating this equation in \( u, v, \) we get the desired (4.4).

**Proposition 9.** \( \xi \) is well defined in \( \mathbb{R}^2. \) Near each point \( (x_0, y_0) \in S, \) \( \xi(x, y) = O \left|x - x_0\right|. \) Moreover, \( T\xi = 0 \) in \( \mathbb{R}^2. \)

**Proof.** Let \( (x_0, y_0) \in S. \) First we consider the case that \( |y_0| \) is large. From Lemma 7, we infer that near \( x_0, \) \( \xi(x, y_0) = O \left|x - x_0\right|^\alpha, \) where \( \alpha \) is close to 1. Hence \( \xi \) is well defined for \( |y| \) large, say \( |y| > C_0. \)

We wish to show that in the region \( \Omega_1 := \{(x, y) : y > C_0\}, T\xi = 0. \) Let \( y_1 \in [C_0, +\infty). \) Suppose \( S \cap \{(x, y_1) : x \in \mathbb{R}\} = \{s_1, ..., s_k\}, \) where \( s_j < s_{j+1} \) and they
depends on $y_1$. Let $x_1 \in (-\infty, s_1)$ and $\rho$ be a function to be determined. Consider the problem

$$
\begin{cases}
T(\rho \xi) = 0, & \text{for } x = x_1, \\
\rho(y_1) = 1.
\end{cases}
$$

(4.5)

Note that

$$
T(\rho \xi) = \rho' \xi + (\partial_y \xi - (\operatorname{Im} \Gamma - \operatorname{Im} \tilde{k}_n) \xi) \rho.
$$

Therefore the problem (4.5) is an ODE for $\rho$ and has a unique solution $\rho$ in a small interval $(y_1 - \delta, y_1 + \delta)$. Using Lemma 8, we know that $T(\rho \xi) = 0$ in the strip

$$
\Omega_2 := (-\infty, x_1 + \delta) \times (y_1 - \delta, y_1 + \delta).
$$

Hence in this region,

$$
\rho' \xi + (\partial_y \xi - (\operatorname{Im} \Gamma - \operatorname{Im} \tilde{k}_n) \xi) \rho = 0.
$$

Dividing both sides by $\xi$ and letting $x \to -\infty$, we find that $\rho'(y) = 0$, thus $\rho(y) = 1$. This implies that the function $\xi$ solves $T(\xi) = 0$ in $\Omega_2$.

Next we proceed to analyze the asymptotic behavior of $\xi$ near the left most singularity $s_1$, when $\partial_x \Gamma^{-1}$ is nonzero at $s_1$ (This holds when $y$ is large). Assume $\xi$ has the form $\beta(y)e^{y \operatorname{Im} \tilde{k}_n}(s_1(y) - x)^{\alpha(y)}$, $\alpha, \beta$ are unknown functions, and $\beta \neq 0$, $\alpha$ is close to 1. We call $\alpha$ the vanishing order of $\xi$. Then

$$
T(\xi)e^{-y \operatorname{Im} \tilde{k}_n} = \partial_y \xi - (\operatorname{Im} \Gamma - \operatorname{Im} \tilde{k}_n) \xi
$$

$$
= \beta'(y)(s_1 - x)^{\alpha(y)} + \beta(y)\alpha(y)(s_1 - x)^{\alpha(y) - 1}s_1'
$$

$$
+ \beta(y)(s_1 - x)^{\alpha(y)} \ln(s_1 - x)\alpha'(y)
$$

$$
- \beta(y)(s_1 - x)^{\alpha(y)} \operatorname{Im} \Gamma
$$

$$
= 0.
$$

(4.6)

Here $s_1$ is evaluated at $y$. In the last identity, dividing both sides with $(s_1 - x)^{\alpha(y) - 1}$ and letting $x \to s_1$, we obtain

$$
\alpha(y)s_1' - [(s_1 - x)\operatorname{Im} \Gamma]|_{x = s_1} = 0.
$$

(4.7)

Using the real analyticity of $\Gamma^{-1}$, we can expand $\Gamma$ around $x = s_1$. Dividing (4.6) by $(s_1 - y)^{\alpha(y)}$ and using (4.7), we find that $\alpha'(y) = 0$. Hence $\alpha$ is a constant.

When $y \to +\infty$, we know from Lemma 7 that $\alpha(y) \to 1$. It follows that $\alpha$ is identically equal to one along each unbounded connected component of $S$ containing $s_1$.

In principle, $S$ could have bounded connected components (We don’t know whether this can actually happen). Assume now that $s_1$ is belonging to a bounded component $B_1$. Using the previous argument, one can first prove that the vanishing order $\alpha$ of $\xi$ in $B_1$ is constant. We now show that $\alpha$ is actually positive. Indeed, observe that the functions $f, g, \gamma, \tau$ contain parameters $k_1, \ldots, k_n$. We can deform these parameters to the situation that all $k_j$ are close to $k_n$. For a generic deformation, the vanishing order of the corresponding functions $\xi$ (also depends on $k_j$) will not change sign. (Note that we don’t know whether the vanish order will change along this deformation). But in the case that $k_j$ are all close to $k_n$, bounded components of singular set will not appear and thus the vanishing order are equal to one, thus positive. This tells us that $\alpha > 0$.

Now we have proved that $\xi$ solves $T\xi = 0$ for $x < s_1(y)$. To prove that $T\xi = 0$ for any $x_1 \in (s_1(y), s_2(y))$, we still consider the function $\phi := \rho(y)\xi(x, y)$, with $\rho(y) = 1$. One can solve the problem $T\phi = 0$ for $x = x_1$. Due to the asymptotic
behavior of $\phi$ at $x \to s_1(y)$, $\rho' = 0$ and hence $\rho = 1$. Arguing in this way, we finally prove that $T \xi = 0$ in $\mathbb{R}^2$. The proof is thus completed. 

With the vanishing order of $\xi$ being understood, we proceed to solve the system (4.1), with $\eta$ being a bounded kernel of the linearized elliptic sine-Gordon equation

$$\Delta \eta + \eta \cos u = 0. \quad (4.8)$$

For each fixed $y$, the first inhomogeneous equation in (4.1) has a solution of the form

$$\phi(x, y) = \xi(x, y) \int_{-\infty}^{x} \xi^{-1}(s, y) M \eta ds. \quad (4.9)$$

**Lemma 10.** Let $\eta$ be a bounded solution of (4.8). The function $\phi$ defined by (4.9) satisfies system (4.1). As a consequence, $\phi$ is a kernel of the linearized elliptic sine-Gordon equation at $v$, that is,

$$\Delta \phi + \phi \cos v = 0. \quad (4.10)$$

**Proof.** By the definition of $\xi$, it is always nonnegative. By multiplying $\xi$ by $+1$ or $-1$ in different connected components of $\mathbb{R}^2 \setminus S$, we get a $C^1$ function $\xi^*$ solving $L \xi^* = T \xi^* = 0$. We wish to show that $\phi$ solves $T \phi = N \eta$. Let $(x_1, y_1) \in \mathbb{R}^2 \setminus S$. Consider the function

$$\Phi(x, y) := \phi(x, y) + \rho(y) \xi^*(x, y),$$

where $\rho$ satisfies

$$\begin{cases}
\rho' \xi^*(x, y) = -T \phi + N \eta, & \text{for } x = x_1, y \in (y_1 - \delta, y_1 + \delta) \\
\rho(y_1) = 0.
\end{cases}$$

Then $T \Phi = 0$ for $x = x_1, y \in (y_1 - \delta, y_1 + \delta)$. Using Lemma 8, for $y \in (y_1 - \delta, y_1 + \delta)$, $\Phi$ satisfies the system

$$\begin{cases}
L \Phi = M \eta, \\
T \Phi = N \eta.
\end{cases}$$

Hence

$$\rho' \xi^* (x, y) = -T \phi + N \eta, \text{ for } y \in (y_1 - \delta, y_1 + \delta).$$

For each fixed $y$, sending $x$ to $-\infty$ in the above equation, we get $\rho'(y) = 0$. Hence $\rho = 0$ and $\Phi$ satisfies system (4.1). It then follows from the linearization of the Bäcklund transformation that $\phi$ satisfies (4.10). The proof is completed.

Now we are ready to prove the nondegeneracy theorem.

**Proof of Theorem 2.** Let us fixed a $2n$-end solution $u = U_n$ of (3.1). Suppose $\eta$ is nontrivial bounded kernel of the linearized operator. Note that in the definition of $U_n$, there are $2n$ real parameters $\text{Re} \ k_j, \eta_{j0}^n, j = 1, \ldots, n$. Differentiating with respect to these parameters in the elliptic sine-Gordon equation, we obtain $2n$ linearly independent solutions of the equation (4.8), denoting them by $\zeta_1, \ldots, \zeta_{2n}$. By adding suitable linear combinations of $\zeta_j, j = 1, \ldots, 2n$, if necessary, we can assume that $\eta(x, y)$ decays to zero exponentially fast, as $x \to -\infty$. Applying Lemma 10, we get a corresponding kernel $\phi_{n-1}$ of the linearized operator at the function $4 \arctan \frac{y}{x}$, which can be regarded as a $n - 1$-soliton type solution of elliptic sinh-Gordon equation having singularities. Moreover, $\phi_{n-1}$ is bounded and decays to zero as $x \to -\infty$. 

ELLIPTIC SINE-GORDON EQUATION 15
Now similarly as before, 4 arctan $\frac{x}{y}$ is the Bäcklund transformation of an $n - 2$-soliton type solution, which will be denoted by 4 arctan $\frac{x_{n-2}}{y}$. Repeating this procedure, we may consider the Bäcklund transformation between 4 arctan $\frac{x}{y}$ and 4 arctan $\frac{x_{n-1}}{y}$, where 4 arctan $\frac{x}{y}$ is a $j$-soliton, and 4 arctan $\frac{x_{n}}{y} = 0$. Linearizing these Bäcklund transformation and solving them similarly as in Lemma 10 (One also need to be careful about the point singularities in these systems), we finally get a bounded kernel $\varphi_0$ of the operator
\[
\Delta \varphi_0 - \varphi_0 = 0.
\]
Moreover, we may assume that $\varphi_0$ is decaying to zero as $x \to -\infty$. Hence $\varphi_0 = 0$. This together with an analysis of the reverse Bäcklund transformation ultimately tell us that $\eta = 0$. This finishes the proof. $\square$

5. INVERSE SCATTERING TRANSFORM AND THE CLASSIFICATION OF MULTIPLE-END SOLUTIONS

The rest of the paper will be devoted to the proof of Theorem 3. We consider the elliptic sine-Gordon equation in the form
\[
\partial_x^2 u + \partial_y^2 u = \sin u, \quad 0 < u < 2\pi.
\] (5.1)

Multiple-end solutions of (1.2) are corresponding to those solutions of (5.1) whose $\pi$ level sets are asymptotic to finitely many half straight lines at infinity. Along these half lines, the solutions resemble the one dimensional heteroclinic solution $\exp(\lambda)$ in the transverse direction. In this section, we will classify these solutions using the inverse scattering transform framework developed in [23].

Let $\sigma_i, i = 1, 2, 3$ be the Pauli spin matrices, that is,
\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Let $\lambda$ be a complex spectral parameter. The equation (5.1) has a Lax pair
\[
\Phi_x = \frac{1}{2} \left( \begin{bmatrix} \frac{i\lambda}{2} + \frac{\cos u}{2i\lambda} \\ \frac{\cos u}{2} \end{bmatrix} \sigma_3 - \frac{i}{2} (u_x + iu_y) \sigma_2 - \frac{i\sin u}{2\lambda} \sigma_1 \right) \Phi, \quad (5.2)
\]
\[
\Phi_y = \frac{1}{2} \left( -\frac{\lambda}{2} + \frac{\cos u}{2\lambda} \right) \sigma_3 + \frac{1}{2} (u_x - iu_y) \sigma_2 - \frac{\sin u}{2\lambda} \sigma_1 \Phi. \quad (5.3)
\]

Let $k(\lambda) = \lambda - \frac{1}{\lambda}$. Note that due to the asymptotic behavior of $u$, as $x \to \pm\infty$, the coefficient matrix of the righthand side of (5.2) tends to the constant matrix $\frac{i}{4}k\sigma_3$.

Let $\Phi_{\pm}$ be the solution of (5.2) such that $\Phi_{\pm}(x, y) \sim \exp(\pm \frac{i}{4}k\sigma_3 x), \quad x \to \pm\infty$. Note that $\Phi_{+}$ and $\Phi_{-}$ are solutions of the same ODE system. For $\lambda \in \mathbb{R}$, they are related by
\[
\Phi_{+}(x, y, \lambda) = \Phi_{-}(x, y, \lambda) \begin{bmatrix} a(\lambda) & b(\lambda) \\ -b(-\lambda) & a(-\lambda) \end{bmatrix}.
\]

The functions $a(\lambda, y), b(\lambda, y)$ are called the scattering data, which is a priori depending on $y$ and the spectral parameter $\lambda$. In equation (5.3), sending $x \to -\infty$, we know that they obey the following evolution laws along the $y$ direction:
\[
a(\lambda, y) = a(\lambda, 0), \quad b(\lambda, y) = b(\lambda, 0) \exp \left( -\frac{1}{4} (\lambda + \lambda^{-1}) y \right).
\]
Since \( u \) is a smooth bounded solution which looks like the gluing of finitely many one dimensional heteroclinic solution as \( |y| \to +\infty \), we must have \( b(\lambda, y) = 0 \) for nonzero \( \lambda \in \mathbb{R} \) (otherwise, it blows up exponentially fast).

Since \( u - \pi \) is a multiple-end solution of (1.2), there exists a choice of parameters \( p_j, q_j, \eta_j^0 \) such that the zero level set of the corresponding solution \( U_n - \pi \) has the same asymptotic lines as \( u - \pi \), as \( y \to +\infty \). We denote the \( a \) part of the scattering data of \( U_n \) by \( a_{U_n}(\lambda, y) \), and that of \( u \) by \( a_u(\lambda, y) \). Then since \( U_n \) and \( u \) have the same asymptotic behavior as \( y \to +\infty \), we must have

\[
a_{U_n}(\lambda, y) = a_u(\lambda, y).
\]

The potentials \( U_n \) and \( u \) in the Lax pair can be recovered by the inverse scattering procedure (See equations (14), (15) in [23]). It follows that \( U_n \) and \( u \) are two reflection-less potential having the same scattering data. Therefore \( u = U_n \).

6. Morse index of the multiple-end solutions

In this section, we shall compute the Morse index of the multiple-end solutions through a deformation argument. We have proved that the multiple-end solutions \( U_n - \pi \) are the only 2\( n \)-end solutions. Therefore, the space \( M_n \) of 2\( n \)-end solutions endowed with the natural topology defined in [8] has exactly one connected component. We now know that they are \( L^\infty \) nondegenerate. Hence for fixed \( n \), the Morse index of all the solutions in \( M_n \) are same.

**Proposition 11.** The Morse index of \( U_n - \pi \) is equal to \( n(n - 1)/2 \).

**Proof.** First of all, we observe that by the result of [22], when \( n = 2 \), the Morse index of \( U_n - \pi \) is equal to 1. We have developed in [33] an end-to-end construction scheme for multiple-end solutions of the Allen-Cahn equation. Roughly speaking, for each \( n \geq 2 \), we can glue \( n(n - 1)/2 \) four-end solutions together by matching their ends. Geometrically, the centers of these four-end solutions are far away from each other. The zero level set of the solution looks like a desingularization of the intersection of \( n \) lines, where the intersection points are far away from each other.

It will be suffice for us to show that the Morse index of the solutions \( u \) obtained from the end-to-end construction have Morse index \( n(n - 1)/2 \). We use \( z_1(u), \ldots, z_{n(n-1)/2}(u) \) to denote the centers of the corresponding four-end solutions \( g_1(u), \ldots, g_{n(n-1)/2}(u) \), and use \( \eta_j(u) \) with \( \|\eta_j\|_{L^\infty} = 1 \) to denote a choice of the negative eigenfunctions of the operator \( -\Delta + \cos g_j \). Since \( z_j \) are far away from each other and \( \eta_j \) decays exponentially fast at infinity, we can show that the Morse index of \( u \) is at least \( n(n - 1)/2 \), and each \( \eta_j \) can be perturbed into a true eigenfunction \( \eta^*_j \) with negative eigenvalue.

We now show that the Morse index of \( u \) is at most \( n(n - 1)/2 \), if the distances between any two centers for the four-end solutions are large enough. We will argue by contradiction and assume to the contrary that there exists a sequence of solutions \( u_k \) and a sequence of corresponding negative eigenfunction \( \phi_k \) of the operator \( -\Delta - \cos u_k \), with eigenvalue \( \lambda_k \), such that \( \phi_k \) is orthogonal to each \( \eta^*_j(u_k) \), \( j = 1, \ldots, n(n - 1)/2 \). We normalize it such that \( \|\phi_k\|_{L^\infty} = 1 \). We consider two cases.

Case 1. There is a sequence of points \( Z_k \) such that \( |\phi_k(Z_k)| > \frac{1}{2} \), and as \( k \to +\infty \), \( \min_j \text{dist}(Z_k, z_j(u_k)) \to +\infty \).
Note that in this case, the distance of \( Z_k \) to the zero level set of \( u_k \) has to be uniformly bounded, otherwise, \( \phi_k \) will converge around \( Z_k \), to a nontrivial bounded solution \( \Phi \) of the equation
\[
-\Delta \Phi + \Phi = 0 \text{ in } \mathbb{R}^2,
\]
which is impossible. Since away from the centers \( z_j(u_k) \), \( u_k \) looks like the one dimensional heteroclinic solution, we can show that \( \lambda_k \to 0 \) as \( k \to +\infty \). Recall that for each \( u_k \), the operator \(-\Delta - \cos u_k \) has \( 2n \) linearly independent kernels \( \zeta_{k,1}, \ldots, \zeta_{k,2n} \), which grow at most linearly at infinity. Analyzing the asymptotic behavior of \( \phi_k \) more closely, we know that actually \( \phi_k \) is close to certain linear combination of \( \zeta_{k,j}, j = 1, \ldots, 2n \). But this contradicts with the fact that \( \phi_k \) is orthogonal to \( \zeta_{k,j} \).

Case 2. \( \phi_k(z) \to 0 \) as \( |z - z_j(u_k)| \to +\infty \), for each \( j \), uniformly in \( k \).

In this case, we still choose \( Z_k \) such that \( \phi_k(Z_k) = \frac{1}{2} \). Then \( \text{dist}(Z_k, z_{jk}(u_k)) \leq C \) for some index \( j_k \). Consider the function \( \varphi_k(z) := \phi(z - Z_k) \). Then \( \varphi_k \) converges to a decaying eigenfunction \( \varphi_\infty \) of a four-end solution. The corresponding eigenvalue has to be negative, since the linearized operator of the four-end solution has no decaying kernel. This contradicts with the assumption that \( \phi_k \) is orthogonal to \( \eta_j^\ast(u_k), j = 1, \ldots, 2n \).

In conclusion, the Morse index of \( u \) has to be \( n(n - 1)/2 \) if the distance between those \( z_j(u) \) are large. \( \square \)

We remark that in [9], multiple-end solutions with almost parallel ends have been constructed. The zero level set of these solutions are close to solutions of the \( n \)-component Toda system
\[
\begin{align*}
q''_1 &= -e^{q_1 - q_2}, \\
q''_j &= e^{q_j - q_{j+1}} - e^{q_{j-1} - q_j}, j = 2, \ldots, n - 1, \\
q''_n &= e^{q_{n-1} - q_n}.
\end{align*}
\tag{6.1}
\]
The Morse index of these solutions is equal to the Morse index of the Toda system. Since (6.1) is a system of ODE, its solutions are automatically \( L^\infty \) nondegenerate. A corollary of Proposition 11 is that each solution of (6.1) has Morse index \( \frac{2(n^2 - 1)}{n} \).

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