FINITE $C^\infty$-ACTIONS ARE DESCRIBED BY ONE VECTOR FIELD

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Abstract. In this work one shows that given a connected $C^\infty$-manifold $M$ of dimension $\geq 2$ and a finite subgroup $G \subset \text{Diff}(M)$, there exists a complete vector field $X$ on $M$ such that its automorphism group equals $G \times \mathbb{R}$ where the factor $\mathbb{R}$ comes from the flow of $X$.

1. Introduction

This work fits within the framework of the so called Inverse Galois Problem: working in a category $\mathcal{C}$ and given a group $G$, decide whether or not there exists an object $X$ in $\mathcal{C}$ such that $\text{Aut}_\mathcal{C}(X) \cong G$.

This metaproblem has been addressed by researchers in a wide range of situations from Algebra [2] and Combinatorics [4], to Topology [3]. In the setting of Differential Geometry, Kojima shows that any finite group occurs as $\pi_0(\text{Diff}(M))$ for some closed 3-manifold $M$ [8, Corollary page 297], and more recently Belolipetsky and Lubotzky [1] have proven that for every $m \geq 2$, every finite group is realized as the full isometry group of some compact hyperbolic $m$-manifold, so extending previous results of Kojima [8] and Greenberg [5].

Here we consider automorphisms of vector fields. Although it is obvious that the automorphism group of a vector field is never finite, we show that a given finite group of diffeomorphisms can be determined by a vector field. More precisely:

Theorem. Consider a connected $C^\infty$ manifold $M$ of dimension $m \geq 2$ and a finite subgroup $G$ of diffeomorphisms of $M$. Then there exists a complete $G$-invariant vector field $X$ on $M$, such that the map

$$G \times \mathbb{R} \to \text{Aut}(X)$$

$$(g, t) \mapsto g \circ \Phi_t$$

is a group isomorphism, where $\Phi$ and $\text{Aut}(X)$ denote the flow and the group of automorphisms of $X$ respectively.

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Recall that, for any $m \geq 2$, every finite group $G$ is a quotient of the fundamental group of some compact, connected $C^\infty$-manifold $M'$ of dimension $m$. Therefore $G$ can be regarded as the group of desk transformations of a connected covering $\pi : M \to M'$ and $G \leq \text{Diff}(M)$. Consequently the result above solves the Galois Inverse Problem for vector fields. Thus:

**Corollary 1.** Let $G$ be a finite group and $m \geq 2$, then there exists a connected $C^\infty$-manifold $M$ of dimension $m$ and a vector field $X$ on $M$ such that $\pi_0(\text{Aut}(X)) \cong G$.

Our results fit into the $C^\infty$ setting, but it seems interesting to study the same problem for other kind of manifolds and, among them, the topological ones. Namely: given a finite group $\tilde{G}$ of homeomorphisms of a connected topological manifold $\tilde{M}$ prove, or disprove, the existence of a continuous action $\tilde{\Phi} : \mathbb{R} \times \tilde{M} \to \tilde{M}$ such that:

1. $\tilde{\Phi}_t \circ g = g \circ \tilde{\Phi}_t$ for any $g \in \tilde{G}$ and $t \in \mathbb{R}$.
2. If $f$ is a homeomorphism of $\tilde{M}$ and $\tilde{\Phi}_s \circ f = f \circ \tilde{\Phi}_s$ for every $s \in \mathbb{R}$, then $f = g \circ \tilde{\Phi}_t$ for some $g \in \tilde{G}$ and $t \in \mathbb{R}$ that are unique.

This work, reasonably self-contained, consists of five sections, the first one being the present Introduction. The others are organized as follows. In Section 2 some general definitions and classical results are given. Section 3 is devoted to the main result of this work (Theorem 1) and its proof. The extension of Theorem 1 to manifolds with non-empty boundary is addressed in Section 4. The manuscript ends with an Appendix where a technical result needed in Section 4 is proven.

For the general questions on Differential Geometry the reader is referred to [7] and for those on Differential Topology to [9].

## 2. Preliminary notions

Henceforth all structures and objects considered are real $C^\infty$ and manifolds without boundary, unless another thing is stated. Given a vector field $Z$ on a $m$-manifold $M$ the group of automorphisms of $Z$, namely $\text{Aut}(Z)$, is the subgroup of diffeomorphisms of $M$ that preserve $Z$, that is

$$\text{Aut}(Z) = \{ f \in \text{Diff}(M) : f_*(Z(p)) = Z(f(p)) \text{ for all } p \in M \}.$$ 

On the other hand, recall that a *regular trajectory* is the trace of a non-constant maximal integral curve. Thus any regular trajectory is oriented by the time in the obvious way and,
FINITE $C^\infty$-ACTIONS ARE DESCRIBED BY ONE VECTOR FIELD

if it is not periodic, its points are completely ordered. As usual, a singular trajectory is a singular point of $Z$.

If $Z(p) = 0$ and $Z'$ is another vector field defined around $p$ then $[Z', Z](p)$ only depends on $Z'(p)$; thus the formula $Z'(p) \to [Z', Z](p)$ defines an endomorphism of $T_pM$ called the linear part of $Z$ at $p$. For the purpose of this work, we will say that $p \in M$ is a source (respectively a sink) of $Z$ if $Z(p) = 0$ and its linear part at $p$ is the product of a positive (negative) real number by the identity on $T_pM$.

A point $q \in M$ is called a rivet if

(a) $q$ is an isolated singularity of $Z$,
(b) around $q$ one has $Z = \psi \tilde{Z}$ where $\psi$ is a function and $\tilde{Z}$ a vector field with $\tilde{Z}(q) \neq 0$.

Note that by (b) a rivet is the $\omega$-limit of exactly one regular trajectory, the $\alpha$-limit of another one and an isolated singularity of index zero.

Consider a singularity $p$ of $Z$; let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of the linear part of $Z$ at $p$ and $\mu_1, \ldots, \mu_k$ the same eigenvalues but only taking each of them into account once regardless of its multiplicity. Assume that $\mu_1, \ldots, \mu_k$ are rationally independent; then $\lambda_j - \sum_{\ell=1}^m i_\ell \mu_\ell \neq 0$ for any $j = 1, \ldots, m$ and any non-negative integers $i_1, \ldots, i_m$ with $\sum_{\ell=1}^m i_\ell \geq 2$, and a theorem of linearization by Sternberg (see [10] and [9]) shows the existence of coordinates $(x_1, \ldots, x_m)$ such that $p \equiv 0$ and $Z = \sum_{j=1}^m \lambda_j x_j \partial/\partial x_j$. That is the case of sources ($\lambda_1 = \ldots = \lambda_m > 0$) and sinks ($\lambda_1 = \ldots = \lambda_m < 0$).

By definition, the outset (or unstable manifold) $R_p$ of a source $p$ will be the set of all points $q \in M$ such that the $\alpha$-limit of its $Z$-trajectory equals $p$. One has:

**Proposition 1.** Let $p$ be a source of a complete vector field $Z$. Then $R_p$ is open and there exists a diffeomorphism from $R_p$ to $\mathbb{R}^m$ that sends $p$ to the origin and $Z$ to a $\sum_{j=1}^m x_j \partial/\partial x_j$ for some $a \in \mathbb{R}^+$. In other words, there exist coordinates $(x_1, \ldots, x_m)$, whose domain $R_p$ is identified to $\mathbb{R}^m$, such that $p \equiv 0$ and $Z = a \sum_{j=1}^m x_j \partial/\partial x_j$, $a \in \mathbb{R}^+$.

Indeed, let $\Phi_t$ be the flow of $Z$; consider coordinates $(y_1, \ldots, y_m)$ such that $p \equiv 0$ and $Z = a \sum_{j=1}^m y_j \partial/\partial y_j$. Up to dilation and with the obvious identifications, one may suppose that $S^{m-1}$ is included in the domain of these coordinates. Then $R_p = \{\Phi_t(y) \mid t \in \mathbb{R}, y \in S^{m-1}\} \cup \{0\}$ and it suffices to send the origin to the origin and each $\Phi_t(y)$ to $e^{at}y$ for constructing the required diffeomorphism.
**Remark 1.** Observe that $R_p \cap R_q = \emptyset$ when $p$ and $q$ are different sources of $Z$.

Given a regular trajectory $\tau$ of $Z$ with $\alpha$-limit a source $p$, by the **linear $\alpha$-limit of $\tau$** one means the (open and starting at the origin) half-line in the vector space $T_pM$ that is the limit, when $q \in \tau$ tends to $p$, of the half-line in $T_qM$ spanned by $Z(q)$. From the local model around $p$ follows the existence of this limit; moreover if $Z$ is multiplied by a positive function the linear $\alpha$-limit does not change.

By definition, a **chain** of $Z$ is a finite and ordered sequence of two or more different regular trajectories, each of them called a **link**, such that:

(a) The $\alpha$-limit of the first link is a source.

(b) The $\omega$-limit of the last link is not a rivet.

(c) Between two consecutive links the $\omega$-limit of the first one equals the $\alpha$-limit of the second one. Moreover this set consists in a rivet.

The **order of a chain** is the number of its links and its $\alpha$-limit and linear $\alpha$-limit those of its first link.

For sake of simplicity, here countable includes the finite case as well. One says that a subset $Q$ of $M$ **does not exceed dimension** $\ell$, or it can be **enclosed in dimension** $\ell$, if there exists a countable collection $\{N_\lambda\}_{\lambda \in \Lambda}$ of submanifolds of $M$, all of them of dimension $\leq \ell$, such that $Q \subset \bigcup_{\lambda \in \Lambda} N_\lambda$. Note that the countable union of sets whose dimension do not exceed dimension $\ell$ does not exceed dimension $\ell$ too. On the other hand, if $\ell < m$ then $Q$ has measure zero so empty interior.

Given a $m$-dimensional real vector space $V$, a family $\mathcal{L} = \{L_1, \ldots, L_s\}$, $s \geq m$, of half-lines of $V$ is named in **general position** if any subfamily of $\mathcal{L}$ with $m$ elements spans $V$.

Now consider a finite group $H \subset GL(V)$ of order $k$. A family $\mathcal{L}$ of half-lines of $V$ is named a **control family with respect to** $H$ if:

(a) $h(L) \in \mathcal{L}$ for any $h \in H$ and $L \in \mathcal{L}$.

(b) There exists a family $\mathcal{L}'$ of $\mathcal{L}$ with $km + 1$ elements, which is in general position, such that $H \cdot \mathcal{L}' = \{h(L) \mid h \in H, L \in \mathcal{L}'\}$ equals $\mathcal{L}$.

**Lemma 1.** Let $\mathcal{L}$ be a control family with respect to $H$ and $\varphi$ an element of $GL(V)$. If $\varphi$ sends each orbit of the action of $H$ on $\mathcal{L}$ into itself, then $\varphi = ah$ for some $a \in \mathbb{R}^+$ and $h \in H$. 
Indeed, as for every \( L \in \mathcal{L}' \) there is \( h' \in H \) such that \( \varphi(L) = h'(L) \), there exist a subfamily \( \mathcal{L}'' = \{L_1, \ldots, L_{m+1}\} \) of \( \mathcal{L}' \) and a \( h \in H \) such that \( \varphi(L_j) = h(L_j), j = 1, \ldots, m+1. \) Therefore \( h^{-1} \circ \varphi \) sends \( L_j \) into \( L_j, j = 1, \ldots, m+1, \) and because \( \mathcal{L}'' \) is in general position \( h^{-1} \circ \varphi \) has to be a multiple of the identity. Since every \( L_j \) is a half-line this multiple is positive.

3. The main result

This section is devoted to prove the following result on finite groups of diffeomorphisms of a connected manifold.

**Theorem 1.** Consider a connected manifold \( M \) of dimension \( m \geq 2 \) and a finite group \( G \subset \text{Diff}(M) \). Then there exists a complete vector field \( X \) on \( M \), which is \( G \)-invariant, such that the map

\[
(g, t) \in G \times \mathbb{R} \rightarrow g \circ \Phi_t \in \text{Aut}(X)
\]

is a group isomorphism, where \( \Phi \) denotes the flow of \( X \).

Consider a Morse function \( \mu: M \rightarrow \mathbb{R} \) that is \( G \)-invariant, proper and non-negative, whose existence is assured by a result of Wasserman (see the remark of page 150 and the proof of Corollary 4.10 of [11]). Denote by \( C \) the set of its critical points, which is closed, discrete (that is without accumulation points in \( M \)) so countable. As \( M \) is paracompact, there exists a locally finite family \( \{A_p\}_{p \in C} \) of disjoint open set such that \( p \in A_p \) for every \( p \in C \).

**Lemma 2.** There exists a \( G \)-invariant Riemannian metric \( \tilde{g} \) on \( M \) such that if \( J(p): T_pM \rightarrow T_pM, p \in C, \) is defined by \( H(\mu)(p)(v, w) = \tilde{g}(p)(J(p)v, w) \), where \( H(\mu)(p) \) is the hessian of \( \mu \) at \( p \), then:

1. If \( p \) is a maximum or a minimum then \( J(p) \) is a multiple of the identity.
2. If \( p \) is a saddle, that is \( H(\mu)(p) \) is not definite, then the eigenvalues of \( J(p) \) avoiding repetitions due to the multiplicity are rationally independent.

**Proof.** We start constructing a 'good' scalar product on each \( T_pM, p \in C \). If \( p \) is a minimum [respectively maximum] one takes \( H(\mu)(p) \) [respectively \( -H(\mu)(p) \)]. When \( p \) is a saddle consider a scalar product \( \langle \cdot, \cdot \rangle \) on \( T_pM \) invariant by the linear action of the isotropy group \( G_p \) of \( G \) at \( p \). In this case as \( J(p) \) is \( G_p \)-invariant (of course here \( J(p) \) is defined with respect to \( \langle \cdot, \cdot \rangle \)), \( T_pM = \bigoplus_{j=1}^{k} E_j \) and \( J(p)|_{E_j} = a_j Id_{E_j} \) where each \( E_j \) is \( G_p \)-invariant, \( a_j \neq 0, \langle E_j, E_\ell \rangle = 0 \) and \( a_j \neq a_\ell \) if \( j \neq \ell \).
Besides one may suppose $a_1,\ldots,a_k$ rationally independent by taking, if necessary, a new scalar product $\langle \cdot, \cdot \rangle'$ such that $\langle E_j, E_\ell \rangle' = 0$ when $j \neq \ell$ and $\langle \cdot, \cdot \rangle'_{|E_j} = b_j \langle \cdot, \cdot \rangle_{|E_j}$ for suitable scalars $b_1,\ldots,b_k$.

In turns this family of scalar products on $\{T_pM\}_{p \in C}$ can be construct $G$-invariant. Indeed, this is obvious for maxima and minima since $\mu$ is $G$-invariant. On the other hand, if $C' \subset C$ is a $G$-orbit consisting of saddles take a point $p$ in $C'$, endow $T_pM$ with a 'good' scalar product and extend to $C'$ by means of the action of $G$.

It is easily seen, through the family $\{A_p\}_{p \in C}$, that of all these scalar products on $\{T_pM\}_{p \in C}$ extend to a Riemannian metric $\tilde{g}$ on $M$. Finally, if $\tilde{g}$ is not $G$-invariant consider $\sum_{g \in G} g^*(\tilde{g})$.

Let $Y$ be the gradient vector field of $\mu$ with respect to some Riemannian metric $\tilde{g}$ as in Lemma 2. We will assume that $Y$ is complete by multiplying, if necessary, $\tilde{g}$ by a suitable $G$-invariant positive function (more exactly by $e^{(Y \cdot \rho)^2}$ where $\rho$ is a $G$-invariant proper function).

Since $\mu$ is non-negative and proper, the $\alpha$-limit of any regular trajectory of $Y$ is a local minimum or a saddle of $\mu$, whereas its $\omega$-limit is empty, a local maximum or a saddle of $\mu$.

Now $Y^{-1}(0) = C$ and, by the Sternberg’s Theorem, around each $p \in C$ (note that the linear part of $Y$ at $p$ equals $J(p): T_pM \to T_pM$ defined in Lemma 2) there exist a natural $1 \leq k \leq m-1$ and coordinates $(x_1,\ldots,x_m)$ such that $p \equiv 0$ and $Y = \sum_{j=1}^{m} \lambda_j x_j \partial/\partial x_j$ where $\lambda_1,\ldots,\lambda_k > 0$ and $\lambda_{k+1},\ldots,\lambda_m < 0$, or $Y = a \sum_{j=1}^{m} x_j \partial/\partial x_j$ where $a > 0$ if $p$ is a source (that is a minimum of $\mu$) and $a < 0$ if $p$ is a sink (a maximum of $\mu$).

Let $I$ be the set of local minima of $\mu$, that is the set of sources of $Y$, and $S_i, i \in I$, the outset of $i$ relative to $Y$. Obviously $G$ acts on the set $I$.

**Lemma 3.** In $M$ the family $\{S_i\}_{i \in I}$ is locally finite and the set $\bigcup_{i \in I} S_i$ dense.

**Proof.** First notice that $\mu(S_i)$ is low bounded by $\mu(i)$. But $I$ is a discrete set and $\mu$ a non-negative proper Morse function, so in every compact set $\mu^{-1}((\infty,a])$ there are only a finite number of elements of $I$. Therefore $\mu^{-1}((\infty,a])$ and of course $\mu^{-1}(\infty,a)$ only intersect a finite number of $S_i$. Finally, observe that $M = \bigcup_{a \in \mathbb{R}} \mu^{-1}(\infty,a)$.

If the $\alpha$-limit of the $Y$-trajectory of $q$ is a saddle $s$, with the local model given above there exists $t \in \mathbb{Q}$ such that $\Phi_t(q)$ is close to $s$ and $x_{k+1}(\Phi_t(q)) = \ldots = x_{m}(\Phi_t(q)) = 0$. Since the submanifold given by the equations $x_{k+1} = \ldots = x_{m} = 0$ has dimension $\leq m-1$ and $\mathbb{Q}$ and
the set of saddles are countable, it follows that the set of points coming from a saddle may be enclosed in dimension \( m - 1 \) and its complementary, that is \( \bigcup_{i \in I} S_i \), has to be dense. \( \square \)

The vector field \( Y \) has no rivets since all its singularities are isolated with index \( \pm 1 \), therefore it has no chain; moreover the regular trajectories are not periodic.

For each \( i \in I \), let \( \mathcal{L}_i \) be a control family on \( T_i M \) with respect to the action of the isotropy group \( G_i \) of \( G \) at \( i \), such that if \( g(i) = i' \) then \( g \) transforms \( \mathcal{L}_i \) in \( \mathcal{L}_{i'} \). These families can be constructed as follows: for every orbit of the action of \( G \) on \( I \) choose a point \( i \) and \( k_i m + 1 \) different half-lines in general position, where \( k_i \) is the order of \( G_i \); now \( G_i \)-saturate this first family for giving rise to \( \mathcal{L}_i \). For other points \( i' \) in the same orbit choose \( g \in G \) such that \( g(i) = i' \) and move \( \mathcal{L}_i \) to \( i' \) by means of \( g \).

Let \( \mathcal{L} \) be the set of all elements of \( \mathcal{L}_i \), \( i \in I \). By Proposition [1] each element of \( \mathcal{L} \) is the linear \( \alpha \)-limit of just one trajectory of \( Y \); let \( \mathcal{T} \) be the set of such trajectories. Clearly \( G \) acts on \( \mathcal{T} \), since \( Y \) and \( \mathcal{L} \) are \( G \)-invariant, and the set of orbits of this action is countable. Therefore this last one can be regarded as a family \( \{ P_n \}_{n \in \mathbb{N}'} \) where \( \mathbb{N}' \subset \mathbb{N} - \{0, 1\} \), each \( P_n \) is a \( G \)-orbit and \( P_n \neq P_{n'} \) if \( n \neq n' \).

In turns, in each \( T \in P_n \) one may choose \( n - 1 \) different points in such a way that if \( T' = g(T) \) then \( g \) sends the points considered in \( T \) to those of \( T' \). Denoted by \( W_n \) the set of all points chosen in the trajectories of \( P_n \).

Since \( \{ S_i \}_{i \in I} \) is locally finite (Lemma [3]), the set \( W = \bigcup_{n \in \mathbb{N}} W_n \) is discrete, countable, closed and \( G \)-invariant. Therefore there exists a \( G \)-invariant function \( \psi : M \to \mathbb{R} \), which is non negative and bounded, such that \( \psi^{-1}(0) = W \). Set \( Y = \varphi Z \). One has:

(a) \( G \) is a subgroup of \( \text{Aut}(X) \).

(b) \( X^{-1}(0) = Y^{-1}(0) \cup W \), the rivets of \( X \) are just the points of \( W \) and \( X \) has no periodic regular trajectories.

(c) \( X \) and \( Y \) have the same sources, sinks and saddles. Moreover if \( R_i \), \( i \in I \), is the \( X \)-outset of \( i \) , then \( R_i \subset S_i \) and \( \bigcup_{i \in I} (S_i - R_i) \subset \bigcup_{T \in P_n, n \in \mathbb{N}'} T \), so \( \{ R_i \}_{i \in I} \) is locally finite and \( \bigcup_{i \in I} R_i \) is dense.

(d) Let \( C_T, T \in P_n, n \in \mathbb{N}' \), be the family of \( X \)-trajectories of \( T - W \) endowed with the order induced by that of \( T \) as \( Y \)-trajectory. Then \( C_T \) is a chain of \( X \) of order \( n \) whose rivets are the points of \( T \cap W \) and whose \( \alpha \)-limit and linear \( \alpha \)-limit are those of \( T \). Besides \( C_T, T \in P_n \), are the only chain of \( X \) of order \( n \).
As each \( P_n \) is a \( G \)-orbit in \( \mathcal{T} \), the group \( G \) acts on the set of chains of \( X \) and every \( \{C_T \mid T \in P_n\} \) is an orbit. Thus \( G \) acts transitively on the set of \( \alpha \)-limit and on that of linear \( \alpha \)-limit of the chains \( C_T, T \in P_n \). Recall that:

**Lemma 4.** Any map \( \varphi : \mathbb{R}^k \to \mathbb{R}^s \) such that \( \varphi(ay) = a\varphi(y) \), for all \( (a, y) \in \mathbb{R}^+ \times \mathbb{R}^k \), is linear.

**Remark 2.** As it is well known, the foregoing lemma does not hold for continuous maps (in this work maps are \( C^\infty \) unless another thing is stated).

**Proposition 2.** Given \( f \in \text{Aut}(X) \) and \( i \in I \) there exists \( (g, t) \in G \times \mathbb{R} \) such that \( f = g \circ \Phi_t \) on \( R_i \).

*Proof.* Consider \( n \in \mathbb{N} \) such that \( i \) is the \( \alpha \)-limit of some chain of order \( n \). Then \( f(i) \) is the \( \alpha \)-limit of some chain of order \( n \) and there exists \( g \in G \) such that \( g(i) = f(i) \); therefore \( (g^{-1} \circ f)(i) = i \), which reduces the problem, up to change of notation, to consider the case where \( f(i) = i \).

Note that every \( L \in \mathcal{L}_i \) is the linear \( \alpha \)-limit of some \( T \in \mathcal{T} \), so the linear \( \alpha \)-limit of \( C_T \); moreover \( \mathcal{L}_i \) is the family of linear \( \alpha \)-limit of all chains starting at \( i \). As \( f \) sends chains starting at \( i \) into chains starting at \( i \) because \( f \) is an automorphism of \( X \), follows that \( f_\ast(i) \) sends \( \mathcal{L}_i \) into itself.

On the other hand, since for any \( T \in P_n \) one has \( f(C_T) = C_{T'} \) where \( T' \) belongs to \( P_n \) as well, it has to exists \( h \in G \) that sends the linear \( \alpha \)-limit of \( C_T \) to the linear \( \alpha \)-limit of \( C_{T'} \). But both chains start at \( i \) so \( h \in G_i \), which implies that \( f_\ast(i) \) preserves each orbit of the action of \( G_i \) on \( \mathcal{L}_i \). From Lemma [1] follows that \( f_\ast(i) = c h_\ast(i) \) with \( c > 0 \) and \( h \in G_i \). Therefore considering \( h^{-1} \circ f \) we may suppose, up to a new change of notation, that \( f_\ast(i) = c \text{Id}, c > 0 \).

Now Proposition [1] allows us to regard \( f \) on \( R_i \) as a map \( \varphi : \mathbb{R}^m \to \mathbb{R}^m \) that preserves the vector field \( X = a \sum_{j=1}^m x_j \partial/\partial x_j, a \in \mathbb{R}^+ \). But this last property implies that \( \varphi(bx) = b \varphi(x) \) for any \( b \in \mathbb{R}^+ \) and \( x \in \mathbb{R}^m \); therefore \( \varphi \) is linear (Lemma [4]). Since \( f_\ast(i) = c \text{Id} \) one has \( \varphi = c \text{Id}, c > 0 \); that is to say \( \varphi \) and \( f_{|R_i} \) equal \( \Phi_t \) for some \( t \in \mathbb{R} \). \( \square \)

Given \( f \in \text{Aut}(X) \), consider a family \( \{(g_t, t_i)\}_{i \in I} \) of elements of \( G \times \mathbb{R} \) such that \( f = g_t \circ \Phi_{t_i} \) on each \( R_i \). We will show that \( f = g \circ \Phi_t \) for some \( g \in G, t \in \mathbb{R} \).

**Lemma 5.** If all \( g_t \) are equal then all \( t_i \) are equal too.
Proof. The proof reduces to the case where all \( g_i = e_G \) (neutral element of \( G \)) by composing \( f \) on the left with a suitable element of \( G \). Obviously \( f = \Phi_{t_i} \) on \( R_i \).

Assume that the set of these \( t_i \) has more than one element. Fixed one of them, say \( t \), set \( D_1 \) the union of all \( R_i \) such that \( t_i = t \) and \( D_2 \) the union of all \( R_i \) such that \( t_i \neq t \). Since \( \{ R_i \}_{i \in I} \) is locally finite and \( \bigcup_{i \in I} R_i \) dense, the family \( \{ R_i \}_{i \in I} \) is locally finite too and \( \bigcup_{i \in I} R_i = M \).

Thus \( D_1 \) and \( D_2 \) are closed and \( M = D_1 \cup D_2 \). On the other hand if \( p \in D_1 \cap D_2 \) then \( \Phi_t(p) = \Phi_{t_i}(p) \) for some \( t \neq t_i \), so \( \Phi_{t-t_i}(p) = p \) and \( X(p) = 0 \) since \( X \) has no periodic regular trajectories, which implies that \( D_1 \cap D_2 \) is countable. Consequently \( M - D_1 \cap D_2 \) is connected. But \( M - D_1 \cap D_2 = (D_1 - D_1 \cap D_2) \cup (D_2 - D_1 \cap D_2) \) where the terms of this union are non-empty, disjoint and closed in \( M - D_1 \cap D_2 \), contradiction.

Choose an \( i_0 \in I \). Composing \( f \) on the left with a suitable element of \( G \) we may assume \( g_{i_0} = e_G \). On the other hand, \( f \) sends each orbit of the actions of \( G \) on \( I \) into itself because the points of every orbit are just the starting points of the chains of order \( n \) for some \( n \in \mathbb{N}' \). Thus \( f \) equals a permutation on each orbit of \( G \) in \( I \) and there exists \( \ell > 0 \) such that \( f^\ell \) is the identity on these orbits; for instance \( \ell = r! \) where \( r \) is the order of \( G \).

Now suppose that \( f^\ell = h_i \circ \Phi_{s_i} \) on \( R_i \), \( i \in I \). Then \( h_i \in G_i \). Since the order of \( G_i \) divides that of \( G \) one has \( f^{r \ell} = \Phi_{r \ell s_i} \) on \( R_i \). In short, there exists a natural number \( k > 0 \) such that \( f^k = \Phi_{u_i} \) on \( R_i \), and by Lemma 5 one has \( f^k = \Phi_u \) on every \( R_i \) for some \( u \in \mathbb{R} \).

In turns, composing \( f \) with \( \Phi_{-u/k} \) we may assume, without lost of generality, that \( f^k = Id \) on \( M \).

On \( R_{i_0} \) one has \( f^k = \Phi_{kt_{i_0}} \), so \( t_{i_0} = 0 \) and \( f = Id \). But \( f \) spans a finite group of diffeomorphisms of \( M \), which assure us that \( f \) is an isometry of some Riemannian metric \( \hat{g} \) on \( M \). Recall that isometries on connected manifolds are determined by the 1-jet at any point. Therefore from \( f = Id \) on \( R_{i_0} \) follows \( f = Id \) on \( M \).

In other words the map \( (g, t) \in G \times \mathbb{R} \to g \circ \Phi_t \in \text{Aut}(X) \) is an epimorphism. Now the proof of Theorem \( \text{[I]} \) will be finished showing that it is an injection.

Assume that \( g \circ \Phi_t = Id \) on \( M \). As \( g' = e_G \) follows \( \Phi_{rt} = Id \) whence \( t = 0 \) because \( X \) has no periodic regular trajectories. Thus \( g = e_G \).

Remark 3. From the proof of Theorem \( \text{[I]} \) above, follows that this theorem holds for \( X' = \rho X \) where \( \rho \colon M \to \mathbb{R} \) is any \( G \)-invariant positive bounded function. Indeed, reason as before with \( (\rho \psi)Y \) instead of \( \psi Y \).
4. Actions on manifolds with boundary

Let $P$ be an $m$-manifold with non-empty boundary $\partial P$. First recall that there always exist a manifold $\tilde{P}$ without boundary and a function $\tilde{\varphi} : \tilde{P} \to \mathbb{R}$ such that zero is a regular value of $\tilde{\varphi}$ and $P$ diffeomorphic to $\tilde{\varphi}^{-1}((-\infty, 0])$; so let us identify $P$ and $\tilde{\varphi}^{-1}((-\infty, 0])$.

Now assume that $G$ is a finite subgroup of $\text{Diff}(P)$, $P$ is connected and $m \geq 2$. Then $G$ sends $\partial P$ to $\partial P$ and $M$ to $M$; thus by restriction $G$ becomes a finite subgroup of $\text{Diff}(M)$.

Let $X'$ be a vector field as in the proof of Theorem 1 with respect to $M$ and $G \subset \text{Diff}(M)$. By Proposition 3 in the Appendix (Section 5) applied to $M$ and $X'$, there exists a bounded function $\varphi : \tilde{P} \to \mathbb{R}$, which is positive on $M$ and vanishes elsewhere, such that the vector field $\varphi X'$ on $M$ prolongs by zero to a (differentiable) vector field on $\tilde{P}$.

**Lemma 6.** For every $g \in G$ the vector field $X_g$ equal to $(\varphi \circ g)X'$ on $M$ and vanishing elsewhere is differentiable.

**Proof.** Obviously $X_g$ is smooth on $\tilde{P} - \partial P$. Now consider any $p \in \partial P$. As $g : P \to P$ is a diffeomorphism, there exist an open neighborhood $A$ of $p$ on $\tilde{P}$ and a map $\hat{g} : A \to \tilde{P}$ such that $\hat{g} = g$ on $A \cap P$. Shrinking $A$ allows to assume that $B = \hat{g}(A)$ is open, $\hat{g} : A \to B$ is a diffeomorphism and $A - \partial P$ has two connected components $A_1, A_2$ with $A_1 \subset M$ and $A_2 \subset \tilde{P} - P$; note that $\hat{g}(A_1) \subset M$, $\hat{g}(A_2) \subset \tilde{P} - P$ and $\hat{g}(A \cap \partial P) \subset \partial P$.

Thus $(X_g)|_A = \hat{g}^{-1}(X_{\varphi})|_B$ since $X'$ is $G$-invariant. □

On $P$ set $X = \sum_{g \in G} X_g$. Then $X|_{\partial P} = 0$ and $X|_M = \rho X'$ where $\rho = \sum_{g \in G} (\varphi|_M) \circ g$. Clearly $\rho : M \to \mathbb{R}$ is positive bounded and $G$-invariant, so by Remark 3 Theorem 1 also holds for $X|_M$. Moreover $X$ is complete on $P$.

If $f : P \to P$ belongs to $\text{Aut}(X)$ then $f|_M$ belongs to $\text{Aut}(X|_M)$ and $f = g \circ \Phi_t$ on $M$ and by continuity on $P$. In other words, Theorem 1 also holds for any connected manifold $P$, of dimension $\geq 2$, with non-empty boundary.

5. Appendix

In this appendix we prove Proposition 3 that was needed in the foregoing section. First consider a family of compact sets $\{K_r\}_{r \in \mathbb{N}}$ in an open set $A \subset \mathbb{R}^n$, such that $K_r \subset \tilde{K}_{r+1}$, $r \in \mathbb{N}$, and $\bigcup_{r \in \mathbb{N}} K_r = A$. 
Lemma 7. Given a family of positive continuous functions $\{f_r : A \rightarrow \mathbb{R}\}_{r \in \mathbb{N}}$ there exists a function $f : A \rightarrow \mathbb{R}$ vanishing on $\mathbb{R}^n - A$ and positive on $A$ such that, whenever $r \in \mathbb{N}$, one has $f \leq f_j$, $0 \leq j \leq r$, on $A - K_r$.

Proof. One may assume $f_0 \geq f_1 \geq \ldots \geq f_r \geq \ldots$ by taking $\min\{f_0, \ldots, f_r\}$ instead of $f_r$ if necessary. Consider functions $\varphi_r : \mathbb{R}^n \rightarrow [0, 1] \subset \mathbb{R}$, $r \in \mathbb{N}$, such that each $\varphi_r^{-1}(0) = K_{r-1} \cup (\mathbb{R}^n - K_{r+1})$ [as usual $K_j = \emptyset$ if $j \leq -1$].

Let $D$ be a partial derivative operator. Multiplying each $f_r$ by some $\varepsilon_r > 0$ small enough allows to suppose, without loss of generality, $\varphi_r \leq f_r/2$ on $A$ and $|D\varphi_r| \leq 2^{-r}$ on $\mathbb{R}^n$ for any $D$ of order $\leq r$.

Set $f = \sum_{r \in \mathbb{N}} \varphi_r$. By the second condition on functions $\varphi_r$, whenever $\tilde{D}$ is a partial derivative operator the series $\sum_{r \in \mathbb{N}} \tilde{D}\varphi_r$ uniformly converges on $\mathbb{R}^n$, which implies that $f$ is differentiable. On the other hand it is easily checked that $f(\mathbb{R}^n - A) = 0$, $f > 0$ on $A$ and $f \leq f_r \leq \ldots \leq f_0$ on $A - K_r$.

One will say that a function defined around a point $p$ of a manifold is flat at $p$ if its $\infty$-jet at this point vanishes. Note that given a function $\psi$ on a manifold and a function $\tau : \mathbb{R} \rightarrow [0, 1] \subset \mathbb{R}$ flat at the origin and positive on $\mathbb{R} - \{0\}$ (for instance $\tau(t) = e^{-1/t^2}$ if $t \neq 0$ and $\tau(0) = 0$), then $\tau \circ \psi$ is flat at every point of $(\tau \circ \psi)^{-1}(0) = \psi^{-1}(0)$ and $\text{Im}(\tau \circ \psi) \subset [0, 1]$.

Lemma 8. Consider an open set $A$ of a manifold $M$ and a function $f : A \rightarrow \mathbb{R}$. Then there exists a function $\varphi : M \rightarrow \mathbb{R}$ vanishing on $M - A$ and positive on $A$, such that the function $\hat{f} : M \rightarrow \mathbb{R}$ given by $\hat{f} = \varphi f$ on $A$ and $\hat{f} = 0$ on $M - A$ is differentiable.

Proof. The manifold $M$ can be seen as a closed imbedded submanifold of some $\mathbb{R}^n$. Let $\pi : E \rightarrow M$ be a tubular neighborhood of $M$. If the result is true for $\pi^{-1}(A)$ and $f \circ \pi : \pi^{-1}(A) \rightarrow \mathbb{R}$, by restriction it is true for $A$ and $f$. In other words, it suffices to consider the case of an open set $A$ of $\mathbb{R}^n$.

We will say that a function $\psi : A \rightarrow \mathbb{R}$ is neatly bounded if, for each point $p$ of the topological boundary of $A$ and any partial derivative operator $D$, there exists an open neighborhood $B$ of $p$ such that $|D\psi|$ is bounded on $A \cap B$. First assume that $f$ is neatly bounded. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function that is positive on $A$ and flat at every point of $\mathbb{R}^n - A$; then $\varphi$ satisfies Lemma \[ \square \]
Indeed, only the points $p \in (\bar{A} - A)$ need to be examined. Consider an natural $1 \leq j \leq n$; since $\int_0^\infty \varphi = 0$ near $p$ one has $\varphi(x) = \sum_{i=1}^n (x_i - p_i) \hat{\varphi}_i(x)$ and from the definition of partial derivative follows that $(\partial \hat{f}/\partial x_j)(p) = 0$. Thus $\partial \hat{f}/\partial x_j = (\partial \varphi/\partial x_j)f + \varphi \partial f/\partial x_j$ on $A$ and $\partial \hat{f}/\partial x_j = 0$ on $\mathbb{R}^n - A$, which shows that $f$ is $C^1$.

Since obviously the function $\partial f/\partial x_j$ is neatly bounded and $\partial \varphi/\partial x_j$ is flat on $\mathbb{R}^n - A$, the same argument as before applied to $(\partial \varphi/\partial x_j)f$ and $\varphi \partial f/\partial x_j$ shows that $f$ is $C^2$ and, by induction, the differentiability of $f$.

Let us see the general case. On $A$ the continuous functions $|Df| + 1$, where $D$ is any partial derivative operator, rise to a countable family of continuous positive functions $g_0, 0, g_1, 0, \ldots$. Let $\{K_r\}_{r \in \mathbb{N}}$ be a collection of compact sets such that $K_r \subset K_{r+1}$, $r \in \mathbb{N}$, and $\bigcup_{r \in \mathbb{N}} K_r = A$. By Lemma [\ref{lem:countable}] there exists a function $\rho : \mathbb{R}^n \to \mathbb{R}$ vanishing on $\mathbb{R}^n - A$ and positive on $A$ such that $\rho \leq g_j^{-1}, 0 \leq j \leq r$, on $A - K_r, r \in \mathbb{N}$.

For every $k \in \mathbb{N}$ let $\lambda_k : \mathbb{R} \to \mathbb{R}$ be the function defined by $\lambda_k(t) = t^{-k}e^{-1/t}$ if $t > 0$ and $\lambda_k(t) = 0$ elsewhere. Then the function $\tilde{f} = \lambda_0(\rho/2)f$ is neatly bounded on $A$. Indeed, consider any $p \in (\bar{A} - A)$ and any partial derivative operator $D$. Then $D\tilde{f}$ equals a linear combination, with constant coefficients, of products of some partial derivatives of $\rho$, a function $\rho^{-k}e^{-2/\rho} = \lambda_k(\rho)e^{-1/\rho}$ and some partial derivative $D'f$. On the other hand, there always exists a natural $\ell$ such that $\rho \leq g_{\ell} \leq |D'f| + 1$. But near $p$ one has $e^{-1/\rho} |D'f| \leq \rho |D'f| \leq \rho \rho \leq 1$; therefore $D\tilde{f}$ is bounded close to $p$.

Finally, take a function $\hat{\varphi} : \mathbb{R}^n \to \mathbb{R}$ positive on $A$ and flat at every point of $\mathbb{R}^n - A$ and set $\varphi = \hat{\varphi}\lambda_0(\rho/2)$. \hfill \Box

**Proposition 3.** Consider a vector field $X$ on an open set $A$ of a manifold $M$. Then there exists a bounded function $\varphi : M \to \mathbb{R}$, which is positive on $A$ and vanishes on $M - A$, such that the vector field $\hat{X}$ on $M$ defined by $\hat{X} = \varphi X$ on $A$ and $\hat{X} = 0$ on $M - A$ is differentiable.

**Proof.** Regard $M$ as a closed imbedded submanifold of some $\mathbb{R}^n$; let $\pi : E \to M$ be a tubular neighborhood of $M$. Then there exists a vector field $X'$ on $\pi^{-1}(A)$ such that $X' = X$ on $A$ and, by restriction of the function, it suffices to show our result for $X'$ and $\pi^{-1}(A)$. That is to say, we may suppose, without loss of generality, that $A$ is an open set of $\mathbb{R}^n$.

In this case on $A$ one has $X = \sum_{j=1}^n f_j \partial/\partial x_j$. Applying Lemma [\ref{lem:sum}] to every function $f_j$ yields a family of functions $\varphi_1, \ldots, \varphi_n$. Now it is enough setting $\varphi = \varphi_1 \cdots \varphi_n$.

Finally, if $\varphi$ is not bounded take $\varphi/(\varphi + 1)$ instead of $\varphi$. \hfill \Box
FINITE C∞-ACTIONS ARE DESCRIBED BY ONE VECTOR FIELD

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