PROOF OF THE CHUDNOVSKY’S SERIES FOR $1/\pi$

JESÚS GUILLERA

ABSTRACT. We prove rational alternating Ramanujan-type series of level 1 discovered by the Chudnovsky’s brothers, by using a method of the author. We have carried out the computations with Maple (a symbolic software for mathematics).

1. INTRODUCTION

In his famous paper [11] of 1914 Ramanujan gave a list of 17 extraordinary formulas for the number $1/\pi$, which are of the following form

\[ \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)^3_n} (a + bn) \frac{z^n}{(a + bn)^n} = \frac{1}{\pi}, \quad (c)_n = 1, \quad (c)_n = \prod_{j=1}^{n}(c + j - 1), \]

where $s \in \{2, 3, 4, 6\}$, and $z, b, a$ are algebraic numbers. Instead of using $s$ to classify them, we will use the level $\ell$ of the family (the level of the modular forms that parametrize it). It is known that

\[ \ell = 4 \sin^2 \frac{\pi}{s}. \]

The only formulas of level $\ell = 1$ ($s = 6$) in the list recorded by Ramanujan are

\[ \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)^3_n} (11n + 1) \left(\frac{4}{125}\right)^n = \frac{5\sqrt{15}}{6\pi}, \]

and

\[ \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)^3_n} (133n + 8) \left(\frac{4}{85}\right)^{3n} = \frac{85\sqrt{255}}{54\pi}, \]

see [11, eq. 33 and 34]. However the most interesting series in this level are the alternating ones, which were discovered by the brothers David and Gregory Chudnovsky in 1987 [4]. The most impressive is

\[ \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)^3_n} (545140134n + 13591409) \left(-\frac{1}{53360}\right)^{3n} = \sqrt{640320} \frac{2}{12\pi}. \]

[4] and [11] are reprinted in [1]: A book collecting papers on the number $\pi$.

In this paper we will prove alternating Ramanujan-type series for $1/\pi$ of level 1 discovered by David and Gregory Chudnovsky, by using the formulas obtained by the author in [6]. The fastest of all rational series for $1/\pi$ (not only for level 1) is [4], which provides approximately $\log_{10}(53360^3) \approx 14.18$ correct digits of $\pi$ per term. In [7], we applied our method to prove the fastest series of level 3, an alternating one discovered by Chan, Liaw and Tan [2], and in [8] we proved the fastest series due to Ramanujan [11, eq. 44].

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2. Elliptic modular functions

According to the papers [6] and [7], we let

\[ F_\ell(x) = 2F_1\left(\frac{1}{s}, \frac{1}{\ell}; 1; x_\ell\right), \quad \ell = 4 \sin^2 \frac{\pi}{s}, \]

where \( s = 2, 3, 4, 6 \), and \( \ell = 4, 3, 2, 1 \) is the corresponding level. The following functions \( x_\ell(q) \) are modular functions of levels \( \ell = 4, 2, 3 \), respectively [5, p. 244 and p. 261], that parametrize \( x \) in such a way that \( F_\ell(x_\ell(q)) \) is a modular form of weight 2:

\[ x_4(q) = 16q \prod_{n=1}^{\infty} \left(1 + \frac{q^{2n}}{1 + q^{2n-1}}\right)^8, \quad x_2(q) = \frac{64q}{64q + \prod_{n=1}^{\infty} (1 + q^n)^{-24}}, \]

and

\[ x_3(q) = \frac{27q}{27q + \prod_{n=1}^{\infty} (1 + q^n + q^{2n})^{-12}}. \]

It is a well known theorem that all the elliptic modular functions are algebraically related. For example, one has [5, p. 274]:

\[ J = \frac{1728}{4x_1(1-x_1)} = \frac{64(1+3x_2)^3}{x_2(1-x_2)^2} = \frac{27(1+8x_3)^3}{x_3(1-x_3)^3} = \frac{16(1+14x_4+x_2^2)^3}{x_4(1-x_4)^4}, \]

where \( J \) is the modular invariant:

\[ J(q) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots. \]

If we let \( \beta = x_\ell(q) \), \( \alpha = x_\ell(q^d) \), and \( \Phi(u, v) \) is the Weber modular polynomial [10] of degree \( d \), then a modular equation of level \( \ell \) and degree \( 1/d \) (of \( \beta \) with respect to \( \alpha \)) or \( d \) (of \( \alpha \) with respect to \( \beta \)), is an algebraic relation \( A(\alpha, \beta) = 0 \). It is important to observe that in [6] we showed that all we need to know, in order to prove the Ramanujan-type series for \( 1/\pi \), are the modular equations satisfied by \( x_\ell(q) \).

3. Weber modular equations

Instead of using modular equations in the R. Russel form [3] as we did in [7] and [8], we will use Weber modular equations for proving the Ramanujan-type series of level 1. If we let \( \beta = x_1(q) \) and \( \alpha = x_1(q^4) \), and \( \Phi(u, v) \) is the Weber modular polynomial [10] of degree \( d \), then

\[ \alpha(1-\alpha) = \frac{432u^{24}}{(u^{24}-16)^3}, \quad \beta(1-\beta) = \frac{432v^{24}}{(v^{24}-16)^3}, \quad \Phi(u, v) = 0, \]

is a modular equation of level 1 and degree \( d \). In this paper we will apply our method to prove the alternating series of level 1 and degrees 5, 7, 11, 17 and 41. Our proofs of 1A5, 1A11, 1A17 and 1A41 (Ad), where A means alternating, are completely analogues, and for proving them we will suitably modify the polynomial \( \Phi(u, v) \) into another polynomial \( P(u, v) \) in order to have a modular equation of the form

\[ \alpha(1-\alpha) = \frac{432u^{12}}{(u^{12}-16)^3}; \quad \beta(1-\beta) = \frac{432v^{12}}{(v^{12}-16)^3}; \quad P(u, v) = 0, \]

because we have observed that by doing it the computations are simpler. For proving 1A7 we do not modify the Weber polynomial.
4. THE FORMULAS OF OUR METHOD

From a modular equation of level \(\ell\) and degree \(d\) (\(\alpha\) respect to \(\beta\)), we can derive two real Ramanujan-type series for \(1/\pi\):

\[
\sum_{n=0}^{\infty} \frac{\frac{1}{2} \binom{1}{n} \frac{1}{2} \binom{1}{n} (1 - \frac{1}{2})^{n}}{(1)^{3}^{n}} (a + bn) z^{n} = \frac{1}{\pi}, \quad \ell = 4 \sin^{2} \frac{\pi}{8},
\]

one of positive terms \(z > 0\) and the other one being and alternating series \(z < 0\). In \([6]\) we proved that they correspond respectively to the following sets of formulas:

\[
q = e^{-\pi \sqrt{\frac{4d}{\ell}}}, \quad z = 4\alpha_{0}\beta_{0}, \quad b = (1 - 2\alpha_{0}) \sqrt{\frac{4d}{\ell}}, \quad a = -2\alpha_{0}\beta_{0}\frac{m'_{0}}{\alpha'_{0}} \frac{d}{\sqrt{\ell}},
\]

and

\[
q = -e^{-\pi \sqrt{\frac{4d}{\ell} - 1}}, \quad z = 4\alpha_{0}\beta_{0}, \quad b = (1 - 2\alpha_{0}) \sqrt{\frac{4d}{\ell} - 1}, \quad a = -2\alpha_{0}\beta_{0}\frac{m'_{0}}{\alpha'_{0}} \frac{d}{\sqrt{\ell}},
\]

where the multiplier \(m(\alpha, \beta)\) is given by the Ramanujan formula:

\[
m^{2} = \frac{1}{d} \frac{\beta(1 - \beta) \alpha'}{\alpha(1 - \alpha) \beta'},
\]

Taking logarithms in \((7)\) and differentiating, we get

\[
\frac{m'}{\alpha'} = \frac{m}{2\alpha'} \left( \frac{\beta'}{\beta} - \frac{\beta'}{1 - \beta} - \frac{\alpha'}{\alpha} + \frac{\alpha'}{1 - \alpha} + \frac{\alpha''}{\alpha'} - \frac{\beta''}{\beta'} \right).
\]

From \((7)\) and \((8)\), we obtain the following formulas:

\[
\frac{\beta'}{\alpha'} = \frac{1}{dm_{0}^{2}}, \quad \frac{m'}{\alpha'} = \frac{1}{2} \left( m_{0} + \frac{1}{dm_{0}} \right) \frac{\alpha_{0} - \beta_{0}}{\alpha_{0} \beta_{0}} + \frac{m_{0}}{2\alpha_{0}} \left( \frac{\alpha'}{\alpha_{0}} - \frac{\beta'}{\beta_{0}} \right).
\]

Hence for proving a Ramanujan-type series for \(1/\pi\) of degree \(d\) one only needs to know a modular equation of that degree. When we apply our method we begin making \(\beta = 1 - \alpha\) in the modular equation and choose a solution \(\alpha_{0}\). If with that solution we get \(|m_{0}| \neq 1/\sqrt{d}\) then it is not of degree \(d\) and we have to try another solution. A good test to select the correct solution \(\alpha_{0}\) was explained in \([6]\) and used in \([7]\) and \([8]\).

5. PROOFS OF THE CHUDNOVSKY’S SERIES FOR 1/\(\pi\)

First we will prove the alternating series of degree 17. The proofs of the alternating series of degrees \(d = 5, 11, 41\) are completely similar. Finally we will prove the series 1A17. Other proofs of the Chudnovsky’s series are given in \([12]\) and \([9]\).

5.1. Proof of the formula 1A17. We see in the tables of \([6]\) that the alternating Ramanujan-type series for \(1/\pi\) of level 1 and degree 17 is

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n} \left(\frac{1}{2}\right)_{n} \left(\frac{3}{2}\right)_{n}}{(1)^{3}^{n}} (261702n + 10177) \left( -\frac{1}{440} \right)^{3n} = \frac{3 \cdot 440^{2}}{\sqrt{330} \pi}.
\]

With other methods one needs to use a modular equation of degree \(4d - 1 = 67\) to prove it. Here we will show how to prove it from the Weber modular equation of degree 17:

\[
\alpha(1 - \alpha) = \frac{432 u^{24}}{(u^{24} - 16)^{3}}, \quad \beta(1 - \beta) = \frac{432 v^{24}}{(v^{24} - 16)^{3}}, \quad \Phi_{17}(u, v) = 0,
\]
see $\Phi_{17}(u, v)$ at [10] second web-page]. However, we prefer to transform it in the following way: First write $\Phi_{17}(u, v) = 0$ as

\[ Q(u, v) = uvR(u, v), \]

where

\[
Q(u, v) = (u^{18} + v^{18}) + 17(u^{16}v^{10} + u^{10}v^{16}) + 119(u^{12}v^6 + u^6v^{12}) + 272(u^8v^2 + u^2v^8), \\
R(u, v) = -u^{16}v^{16} - 34(u^{14}v^2 + u^2v^{14}) + 34u^{12}v^{12} + 340u^8v^8 + 544u^4v^4 - 256.
\]

Squaring we have $Q^2(u, v) = u^2v^2R^2(u, v)$. Finally, replacing $u$ with $\sqrt{u}$ and $v$ with $\sqrt{v}$ we obtain $Q^2(\sqrt{u}, \sqrt{v}) - uvR^2(\sqrt{u}, \sqrt{v}) = 0$. It is clear that the left hand side is a polynomial $P(u, v)$, and we obtain the following modular equation:

\[ \alpha(1 - \alpha) = \frac{432u^{12}}{(u^{12} - 16)^3}, \quad \beta(1 - \beta) = \frac{432v^{12}}{(v^{12} - 16)^3}, \quad P(u, v) = 0. \]

We will use (12) instead of (11) because the calculations are simpler, and we will use Maple (a symbolic software for mathematics) to make those computations.

Our method begins taking $\beta = 1 - \alpha$. Hence, we have to find a solution of the system

\[ \frac{u^{12}}{(u^{12} - 16)^3} = \frac{v^{12}}{(v^{12} - 16)^3}, \quad P(u, v) = 0. \]

We choose the following solution (check!):

\[
u_0 = \left( \frac{91}{1200} - \frac{3\sqrt{201}}{400} \right) H^2 + \frac{1}{3} H - \frac{2}{3}, \\
v_0 = \left\{ \left( \frac{3\sqrt{201}}{800} - \frac{91}{2400} \right) H^2 - \frac{1}{6} H - \frac{2}{3} \right\} + \left\{ \left( \frac{-9\sqrt{67}}{800} + \frac{91\sqrt{3}}{2400} \right) H^2 - \frac{\sqrt{3}}{6} H \right\} i,
\]

where

\[ H = \left( \frac{91 + 9\sqrt{201}}{3} \right)^\frac{1}{3}. \]

Substituting in (12), we get

\[ \alpha_0 = \frac{1}{2} - \frac{651}{193600} \sqrt{22110}, \quad \beta_0 = \frac{1}{2} + \frac{651}{193600} \sqrt{22110}. \]

Hence

\[ z_0 = 4\alpha_0\beta_0 = \frac{-1}{440^3}. \]

Then, from the formula for $b$ in (6), we get

\[ b = \frac{43617}{96800} \sqrt{330}. \]

We choose $u$ as the independent variable. Differentiating $P(u, v) = 0$ with respect to $u$ at $u = u_0$ we obtain $v_0'$, and differentiating twice $P(u, v)$ with respect to $u$ at $u = u_0$, we get $v_0''$. Differentiating

\[ \alpha(1 - \alpha) = \frac{432u^{12}}{(u^{12} - 16)^3}, \quad \beta(1 - \beta) = \frac{432v^{12}}{(v^{12} - 16)^3}, \]

with respect to $u$ at $u = u_0$, we obtain $\alpha'_0$ and $\beta'_0$, and we get

\[ m_0 = \sqrt{\frac{1}{d} \beta'_0} = \frac{\sqrt{67}}{34} + \frac{1}{34} i, \quad |m_0| = \frac{1}{\sqrt{17}}. \]
Then, differentiating (14) twice, we obtain $\alpha''_0$ and $\beta''_0$. Finally, from the formula for $a$ in (6) and the formulas (7) and (9), we obtain

$$a = \frac{10177}{580800} \sqrt{330}.$$  

5.2. **Proof of the formulas** 1A5, 1A11, 1A41 and 1A7. The proofs of the formulas 1A5, 1A11, 1A41 are completely similar to the proof of 1A7: Modify the Weber polynomial $\Phi_d(u, v)$ in the same way that we have done in the case of degree $d = 17$. For proving 1A7 do not modified the Weber polynomial. Then, continue choosing the values of $u_0$ and $v_0$ that we indicate below.

5.2.1. **Proof of the formula** 1A5. Choose

$$u_0 = \left( \frac{-1}{192} + \frac{\sqrt{57}}{64} \right) H^2 - \frac{1}{3} H + \frac{2}{3},$$

$$v_0 = \left\{ \left( \frac{-\sqrt{57}}{128} + \frac{1}{384} \right) H^2 + \frac{1}{6} H + \frac{2}{3} \right\} - \left\{ \left( \frac{\sqrt{3}}{384} - \frac{\sqrt{171}}{128} \right) H^2 - \frac{\sqrt{3}}{6} H \right\} i,$$

where

$$H = \left( 1 + 3\sqrt{57} \right)^{\frac{1}{3}}.$$  

Then follow the steps of (5.1).

5.2.2. **Proof of the formula** 1A11. Choose

$$u_0 = \left( \frac{-35}{48} + \frac{\sqrt{129}}{16} \right) H^2 - \frac{1}{3} H + \frac{4}{3},$$

$$v_0 = \left\{ \left( \frac{35}{96} - \frac{\sqrt{129}}{32} \right) H^2 + \frac{1}{6} H + \frac{4}{3} \right\} + \left\{ \left( \frac{-35\sqrt{3}}{96} + \frac{3\sqrt{43}}{32} \right) H^2 + \frac{\sqrt{3}}{6} H \right\} i,$$

where

$$H = \left( 35 + 3\sqrt{129} \right)^{\frac{1}{3}}.$$  

Then follow the steps of (5.1).

5.2.3. **Proof of the formula** 1A41. Choose

$$u_0 = \left( \frac{-467}{13872} + \frac{11\sqrt{489}}{4624} \right) H^2 - \frac{1}{3} H + \frac{4}{3},$$

$$v_0 = \left\{ \left( \frac{467}{27744} - \frac{11\sqrt{489}}{9248} \right) H^2 + \frac{1}{6} H + \frac{4}{3} \right\} + \left\{ \left( \frac{467\sqrt{3}}{27744} - \frac{33\sqrt{163}}{9248} \right) H^2 - \frac{\sqrt{3}}{6} H \right\} i,$$

where

$$H = \left( 467 + 33\sqrt{489} \right)^{\frac{1}{3}}.$$  

Then follow the steps of (5.1).
5.2.4. **Proof of the formula 1A7.** Take $P(u, v) = \Phi_7(u, v)$ (that is, do not modify the Weber polynomial), and choose

$$u_0 = \sqrt[3]{2} \left(\sqrt[3]{2} - 1\right)^{\frac{1}{3}},$$

$$v_0 = \sqrt[3]{2} \left(\sqrt[3]{2} - 1\right)^{\frac{1}{3}} \left(\frac{1}{2} (\sqrt[4]{4} + 1) - \frac{\sqrt{3}}{6} (1 + \sqrt[4]{4} + 2 \sqrt{2}) i\right).$$

Then follow the steps of (5.1).

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**University of Zaragoza, Department of mathematics, 50009 Zaragoza (Spain)**

**E-mail address:** jguillera@gmail.com