On Two-Dimensional Fuzzy Random Data as Vague Perceptions of Random Phenomena *

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Abstract

In this paper, the author investigates numerically a class of two-dimensional fuzzy random sets (abbreviated as FRSs), which is one case of those proposed by the author[1] as models of vague capricious perceptions of random phenomena.

First, the basic results concerned with two-dimensional FRSs as vague capricious perceptions of random phenomena are reviewed. Expectations of two-dimensional FRSs, their estimators, and distances between expectations and their estimators are also reviewed briefly.

Finally, estimates for expectations of two-dimensional FRSs and their estimation errors are numerically examined by simulation studies, when their level sets are given by disk-form ones at each levels.

1 Introduction

Fuzzy random sets or fuzzy random variables have been intensively investigated for a long time by many researchers with various definitions motivated by the importance for treating the data exhibiting both vagueness and randomness. For instance, the concept of fuzzy random variables obtained as vague linguistic observations of crisp random data was firstly presented by Kwakernaak[2], and investigated by e.g., Kruse[3]. On the other hand, Puri and Ralescu[4] defined firstly fuzzy random variables as the generalized random sets and discussed their statistical properties by many researchers, e.g.,[5, 6].

The purpose of this paper is to investigate numerically a class of two-dimensional fuzzy random sets (abbreviated as two-dimensional FRSs, or more shortly FRSs), which is one of those proposed by the author[1] as models of vague capricious perceptions of crisp random phenomena.

Section 2 devotes to review the basic results concerned with two-dimensional FRSs as vague capricious perceptions of crisp random phenomena. The expectation of a FRS and its estimator, and the distance between an expectation and its estimator are also reviewed briefly in this section.

2 Fuzzy Random Sets(FRSs)

2.1 Two-Dimensional FRSs

In order to consider the vague perceptions of random phenomena, two types of randomness should be considered, one of which is the randomness due to the capricious person’s feelings and another of which is the randomness of the phenomena themselves.

Let $(\Omega_1, A_1, P_1)$ be an elementary probability space describing the randomness of capricious persons’ minds, and let $(\Omega_2, A_2, P_2)$ be a probability space, on which a random point $u_0 \in \mathbb{R}^2$ as the model of a random phenomenon is defined. Then, the FRS as a vague perception of the random vector $u_0$ (hereafter called ‘original random point’) is defined on $(\Omega, A, P) = (\Omega_1 \times \Omega_2, A_1 \otimes A_2, P_1 \times P_2)$.

Let $p$ be a point in $[1, +\infty)$, $\mathbb{F}^p(\mathbb{R}^2)$ be the family of fuzzy sets given in [7], and let $\bar{U}_M(\omega) \in \mathbb{F}^p(\mathbb{R}^2)$ be a FRS given by

$$\bar{U}_M(\omega) = \sum_{i,j=0,\pm1,\pm2,\cdots,\pm M} I_{\bar{u}_{ij}}(\omega) \cdot \bar{U}_{ij},$$

(1)

where

$$I_{\bar{u}_{ij}}(\omega) = \begin{cases} 1 & \text{if } \omega \in (\bar{u}_{ij}^{(1)}) \times \Omega_2 \\ 0 & \text{otherwise.} \end{cases}$$

(2)

$\{\bar{U}_{ij}; i, j = 0, \pm1, \pm2, \cdots, \pm M\}$ in (1) is a collection of fuzzy sets representing the realized values of $\bar{U}_M(\omega)$, and is given by

$$\bar{U}_{ij} = (\mathbb{R}^2, [\bar{U}_{ij}], s_{\bar{U}_{ij}}) \in \mathbb{F}^p(\mathbb{R}^2)$$

(3)

with

$$[\bar{U}_{ij}] = \{[\bar{U}_{ij}]_\alpha \mid \alpha \in \varepsilon\}.$$

(4)

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and \( s_{\tilde{U}_{ij}} \) is the predicate associated with the statement such as
\[
s_{\tilde{U}_{ij}}(u) = \begin{cases} 
    u \in \tilde{U}_{ij} \text{ coincides with the original random point } u_0 
  \end{cases},
\]
(5)
Assume that there exists a FRS \( \tilde{U}(\omega) \in \mathbb{F}_{\infty}(\mathbb{R}^2) \) such that
\[
\lim_{M \to \infty} \rho_p \left( \tilde{U}(\omega), \tilde{U}_M(\omega) \right) = 0 \quad \text{a.s.}
\]
(6)
or equivalently
\[
\tilde{U}(\omega) = \lim_{M \to \infty} \tilde{U}_M(\omega) \quad \text{a.s.},
\]
(7)
where \( \rho_p \) is the metric between \( \tilde{U}(\omega) \) and \( \tilde{U}_M(\omega) \) given by
\[
\rho_p \left( \tilde{U}(\omega), \tilde{U}_M(\omega) \right) = \left( \int \left| \tilde{p}_U(\omega, \alpha, x) \right|^p \, d\mu(\alpha) \otimes \mu_{\tilde{U}}(\alpha) \right)^{\frac{1}{p}}.
\]
(8)
In (8), \( \mathbb{S}^1 \) is the unit circle, and \( \tilde{p}_U(\cdot, \cdot, \cdot) \) is the fuzzy support function given by
\[
\tilde{p}(\tilde{A}, \alpha, x) = \begin{cases} 
    \sup \{ (x, \alpha) | a \in [\tilde{A}]_\alpha \} & \text{for } \alpha \in (0, 1] \\
    0 & \text{at } \alpha = 0
  \end{cases},
\]
(9)
for any fuzzy set \( \tilde{A} \in \mathbb{F}_{\infty}(\mathbb{R}^2) \), where
\[
\tilde{p}(x, [\tilde{A}]_\alpha) = \sup \left\{ (x, a) | a \in [\tilde{A}]_\alpha \right\}
\]
(10)
(see e.g.,[7]). Then, the following definition is obtained:

**Definition 2.1.1.** A FRS \( \tilde{U}(\omega) \) on \((\Omega, \mathcal{A}, P)\) obtained as the capricious vague perception of an original random point \( u_0(\omega) \) on \((\Omega_2, \mathcal{A}_2, P_2)\) is defined by
\[
\tilde{U}(\omega) = \left( \mathbb{R}^2, \{ \tilde{U}(\omega) \}, s_{\tilde{U}} \right) \in \mathbb{F}_{\infty}(\mathbb{R}^2)
\]
(11)
with
\[
\tilde{U}(\omega) = \left\{ [\tilde{U}(\omega) \omega] | \alpha \in I \right\},
\]
(12)
where \( s_{\tilde{U}} \) is the predicate associated with the proposition such as
\[
s_{\tilde{U}}(u) = \left\{ u \text{ in } \tilde{U} \text{ coincides with the original random point } u_0 \right\}.
\]
(13)
Then, using (1) and (7), we can rewrite \( \tilde{U}(\omega) \) in (11) by
\[
\tilde{U}(\omega) = \sum_{i,j=0,1, \pm 2, \cdots} 1_{\tilde{U}_{ij}}(\omega) \cdot \tilde{U}_{ij},
\]
(14)
where \( [\tilde{U}_{ij}]_{i,j=0,1, \pm 2, \cdots} \) is a collection of fuzzy sets given by (3), and \( 1_{\tilde{U}_{ij}}(\omega) \) is a characteristic function given by (2).

The measurability of \( \tilde{U} \) is given through
\[
\tilde{U}^{-1}(B) \in \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \quad \text{for any } B \in \mathcal{B},
\]
(15)
where \( \mathcal{B} \) is a \( \sigma \)-algebra generated by the subsets of \( \tilde{U} = \{ \tilde{U}_{ij} \}_{i,j=0,1, \pm 2, \cdots} \), and the admissible class of possible original random points \( \mathfrak{A}_e \) is assumed to be given by
\[
\mathfrak{A}_e = \left\{ u \, | \, u(\omega) = \sum_{i,j=0,1, \pm 2, \cdots} 1_{\tilde{U}_{ij}}(\omega) \cdot \xi(\omega^{2i}) \right\}
\]
(16)
\( \xi(\omega) \) is the integrable random variable on \((\Omega_2, \mathcal{A}_2, P_2)\).

**2.2 Expectations of FRSS**

We can show that the expectation of a FRS \( \tilde{U} \) may be given as follows:

**Proposition 2.2.1.** Let \( \tilde{U} = (\mathbb{R}^2, [\tilde{U}], s_{\tilde{U}}) \) be a FRS given by (11). Then, the expectation of \( \tilde{U} \) is given by
\[
E[\tilde{U}] = \left( \mathbb{R}^2, [E[\tilde{U}]], s_{E[\tilde{U}]} \right)
\]
(17)
with
\[
E[\tilde{U}] = \left\{ E([\tilde{U}]_{\omega}) | \alpha \in I \right\},
\]
(18)
where \( s_{E[\tilde{U}]}(x) = \left\{ x \text{ coincides with the expectation of } u_0 \right\} \).

and \([E[\tilde{U}]]\) is the set representation of \( E[\tilde{U}] \) given through
\[
E([\tilde{U}]_\omega) = \sum_{i,j=0,1, \pm 2, \cdots} [\tilde{U}_{ij}]_\omega \cdot P(\omega_{ij}^{(1)}),
\]
(19)
where \( P(\omega_{ij}^{(1)}) \) is the marginal probability, i.e., \( P(\omega_{ij}^{(1)}) = P([\omega_{ij}]_\omega, \Omega_2) \).

Let here \( \mathcal{S} \) be the sub \( \sigma \)-algebra of \( \mathcal{A} \) consisting all cylinder sets of the form \( \mathcal{A} = \Omega_1 \times A^{(2)} \) with \( A^{(2)} \in \mathcal{A}_2 \). Then, the conditional expectation of \( \tilde{U} \) concerned with \( \mathcal{S} \) should be given as follows:
\[
E[\tilde{U} | \mathcal{S}] = \left( \mathbb{R}^2, [E[\tilde{U} | \mathcal{S}]], s_{E[\tilde{U} | \mathcal{S}]} \right)
\]
(20)
with
\[
E[\tilde{U} | \mathcal{S}] = \left\{ E([\tilde{U}]_{\omega}) | \alpha \in I \right\}.
\]
(21)

**Proposition 2.2.2.** Let \( \tilde{U} = (\mathbb{R}^2, [\tilde{U}], s_{\tilde{U}}) \) be a FRS given by (11). Then, it follows
\[
E[\tilde{U}] = E[E[\tilde{U} | \mathcal{S}]]
\]
(22)
where \( E[\tilde{U} | \mathcal{S}] \) defined by (20) is given by
\[
E[\tilde{U} | \mathcal{S}] = \sum_{i,j=0,1, \pm 2, \cdots} \tilde{U}_{ij} \cdot P(\omega_{ij}^{(1)}) | \mathcal{S}.
\]
(23)
2.3 Consistent Estimator of $E[\widetilde{\mathcal{U}}]$  

The distribution of a FRS $\widetilde{U}(\omega) \in \tilde{\mathcal{U}}$ is a probability measure on $\tilde{\mathcal{U}}$ defined by

$$P_{\tilde{\mathcal{U}}}(B) = P(\widetilde{U}^{-1}(B))$$

(24)

for any $B \in \mathcal{B}$. Let $\mathcal{A}_{\tilde{\mathcal{U}}}$ be the $\sigma$-algebra generated by $\widetilde{U}^{-1}(B)$, i.e.,

$$\mathcal{A}_{\tilde{\mathcal{U}}} = \sigma\left(\widetilde{U}^{-1}(B) \in \mathcal{A}_1 \otimes \mathcal{A}_2; B \in \mathcal{B}\right).$$

(25)

Then, the FRSs $\{\widetilde{U}(\omega); n = 1, 2, \cdots\}$ are said to be independent if $\{\mathcal{A}_{\tilde{\mathcal{U}}_n}; n = 1, 2, \cdots\}$ are independent, and identically distributed if all $\{P_{\tilde{\mathcal{U}}_n}; n = 1, 2, \cdots\}$ are identical, and independent identically distributed (i.i.d.), if they are independent and identically distributed [5].

Proposition 2.3.1. Let $\{\widetilde{U}(\omega); n = 1, 2, \cdots\}$ be a series of i.i.d. FRSs, and the estimator of $E[\widetilde{\mathcal{U}}]$ be heuristically given by

$$\widehat{\mathcal{U}}_N = \frac{1}{N} \sum_{n=1}^{N} \widetilde{U}(\omega).$$

Then, if

$$E(\|\widehat{\mathcal{U}}_N\|_1) = E(\rho_p(\widehat{\mathcal{U}}, \{0\})) < +\infty,$$

(27)

it can be shown that

$$\rho_p(\widehat{\mathcal{U}}_N, E[\widetilde{\mathcal{U}}]) \rightarrow 0 \text{ a.s. as } N \rightarrow \infty.$$  

(28)

3 Step-wise Approximations of Set Representations

In this paper, for convenience of numerical feasibility, the set representation of $\widetilde{U}(\omega)$ is approximated by the collection of step-wise membership levels, i.e.,

$$[\widetilde{U}(\omega)] = \left\{[\widetilde{U}(\omega)]_{\alpha_k} \bigg| k = 0, 1, 2, \cdots, L \right\},$$

(29)

where $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_L = 1$ and $[\widetilde{U}(\omega)]_{\alpha_k}$ is the level set at the level $\alpha_k$ given by

$$[\widetilde{U}(\omega)]_{\alpha_k} = L_{\pi_k} \widetilde{U}(\omega)$$

(30)

for $k = 1, 2, \cdots, L$, and

$$[\widetilde{U}(\omega)]_{\alpha_0} = [\widetilde{U}(\omega)]_0 = cl(L_0 \widetilde{U}(\omega))$$

(31)

satisfying

$$[\widetilde{U}(\omega)]_1 \subseteq [\widetilde{U}(\omega)]_{\alpha_1} \subseteq [\widetilde{U}(\omega)]_{\alpha_2} \subseteq \cdots \subseteq [\widetilde{U}(\omega)]_{\alpha_L} = [\widetilde{U}(\omega)]_0,$$

where $L_{\pi_k} \widetilde{U}(\omega)$ and $L_0 \widetilde{U}(\omega)$ are the level set and the strong cut of $\widetilde{U}(\omega)$ at the level $\alpha_k$ and 0, respectively (see e.g., [8]).

Hence, the set representations of $\{[\widetilde{U}_{i,j}]_{\alpha_k}; i, j = 0, 1, 2, \cdots\}$ in (14) are also approximated by the step-wise membership levels, i.e.,

$$[\widetilde{U}_{i,j}] = \left\{[\widetilde{U}_{i,j}]_{\alpha_k} \bigg| k = 0, 1, 2, \cdots, L \right\},$$

(32)

and $[\widetilde{U}_{i,j}]_{\alpha_k}$ is given by

$$[\widetilde{U}_{i,j}]_{\alpha_k} = L_{\pi_k} \widetilde{U}_{i,j}$$

(33)

for $k = 1, 2, \cdots, L$, and

$$[\widetilde{U}_{i,j}]_{\alpha_0} = [\widetilde{U}_{i,j}]_0 \in cl \left( L_0 \widetilde{U}_{i,j} \right)$$

(34)

satisfying

$$[\widetilde{U}_{i,j}]_1 \subseteq [\widetilde{U}_{i,j}]_{\alpha_1} \subseteq [\widetilde{U}_{i,j}]_{\alpha_2} \subseteq \cdots \subseteq [\widetilde{U}_{i,j}]_{\alpha_L} = [\widetilde{U}_{i,j}]_0.$$  

Then, the set representation $E[\widetilde{\mathcal{U}}]$ given by (17) is approximated by

$$[E[\widetilde{\mathcal{U}}]] = \left\{ [E[\widetilde{\mathcal{U}}]]_k \bigg| k = 0, 1, 2, \cdots, L \right\}$$

(35)

and

$$E[\widetilde{\mathcal{U}}]_{\alpha_k} = E[[\widetilde{\mathcal{U}}]]_{\alpha_k} = \sum_{i,j,0=0,1,\cdots} [\widetilde{U}_{i,j}]_{\alpha_k} \cdot P(\omega_{i,j}^{(1)}).$$

(36)

where $[\widetilde{U}_{i,j}]_{\alpha_k}$ are given by (33) and (34).

4 Numerical Studies

4.1 Estimates of $E[\widetilde{\mathcal{U}}]$

In this subsection, each element of (32) is assumed to be the disk on the $\alpha_k$-level plane, i.e.,

$$[\widetilde{U}_{i,j}]_{\alpha_k} = \left\{ \text{Disk on the } \alpha_k\text{-level plane} \right\},$$

(37)

where the radius $r_{i,j}^{(i,k)}$ is given by

$$r_{i,j}^{(i,k)} = \frac{\rho_{i,j}^{(i,k)}}{2 \cos^{-1}(2\alpha_k - 1)}$$

(38)

for $0 \leq \cos^{-1}(2\alpha_k - 1) \leq \pi$, where $r_{i,j}^{(i)}$ is the radius of the disk $[\widetilde{U}_{i,j}]_{\alpha_k}$.

The original random point $u_0(\omega^{(2)})$ is assumed to be a Gaussian random vector with the mean $m_u$ and the variance $\Sigma_u$ such that

$$m_u = (m_{u_1}, m_{u_2})'$$

(39)

$$\Sigma_u = \begin{pmatrix} \sigma_{u_1}^2 & \rho_u \sigma_{u_1} \sigma_{u_2} \\ \rho_u \sigma_{u_1} \sigma_{u_2} & \sigma_{u_2}^2 \end{pmatrix}$$

(40)

and its realized value is recognized to be in some rectangle

$$I_{p, q} = \left[ 2p - \frac{1}{4} \sigma_{u_1}, \frac{2p + 1}{4} \sigma_{u_1} \right] \times \left[ 2q - \frac{1}{4} \sigma_{u_2}, \frac{2q + 1}{4} \sigma_{u_2} \right].$$

(41)
and the center $c^{(ij,k)}$ in (37) is assumed by
\[ c^{(ij,k)} = \left( \frac{1}{4} \sigma_{\omega_1}, \frac{1}{4} \sigma_{\omega_2}, \alpha_k \right). \]  
(42)

Then, the subset $A_{p,q}$ is given by
\[ A_{p,q} = \Omega_1 \times A_{p,q}^{(2)} \]  
(43)
and
\[ A_{p,q}^{(2)} = \{ \omega^{(2)} \in \Omega_2 \mid u_0(\omega^{(2)}) \in I_{p,q} \}. \]  
(44)

Let the sub-$\sigma$-algebra $\mathcal{S} \subset \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ be generated by the cylinder sets of the form $\{A_{p,q}; p, q = 0, \pm 1, \pm 2, \ldots \}$. Then, the conditional probability $P(B|\mathcal{S})$ is given by
\[ P(B|\mathcal{S}) = \sum_{p,q=0,1,2,\ldots} \mathbf{1}_{A_{p,q}}(\omega) \cdot \frac{P(B \cap A_{p,q})}{P(A_{p,q})} \]  
(45)
for all $B \in \mathcal{A}$. Hence, if $\omega \in A_{p,q}$, it follows
\[ P(B|\mathcal{S}) = \frac{P(B \cap A_{p,q})}{P(A_{p,q})}. \]  
(46)

Let $B_{i,j}$ be given by
\[ \Omega = \bigcup_{i,j} B_{i,j} \quad \text{and} \quad B_{i,j} = [c_{\omega_{i,j}}^{(1)}] \times \Omega_2, \]  
(47)
then, for $\omega \in A_{p,q}$, we have
\[ E[\widetilde{U}|\mathcal{S}](\omega) = \mathcal{E} \left( \sum_{i,j=0,1,2,\ldots} \mathbf{1}_{U_{i,j}}(\omega) \cdot \widetilde{U}_{i,j} \right) |\mathcal{S}(\omega) \]  
(48)
where
\[ P(B_{i,j}|A_{p,q}) = \frac{P(B_{i,j} \cap A_{p,q})}{P(A_{p,q})}. \]  
(49)

In order to describe the capricious mind of each individual, which may fluctuates slightly but randomly, the conditional probability $P(B_{i,j}|A_{p,q})$ defined in (49) is assumed to be given by
\[
\begin{align*}
P(B_{2p,2q}|A_{p,q}) &= P_{0,0}, \quad P(B_{2p+1,2q+1}|A_{p,q}) = P_{1,1}, \\
P(B_{2p+1,2q}|A_{p,q}) = P_{1,0}, \quad P(B_{2p,2q+1}|A_{p,q}) = P_{0,1}, \\
P(B_{2p-1,2q}|A_{p,q}) = P_{0,1}, \quad P(B_{2p+1,2q-1}|A_{p,q}) = P_{1,0}, \\
P(B_{2p,2q-1}|A_{p,q}) = P_{0,-1}, \quad P(B_{2p+1,2q+1}|A_{p,q}) = P_{1,-1} \quad (50)
\end{align*}
\]
and otherwise, $P(B_{i,j}|A_{p,q}) = 0$. Then, using (48) and (50), the conditional expectation of $\widetilde{U}$ at $\omega \in A_{p,q}$ is given by
\[ E[\widetilde{U}|\mathcal{S}](\omega) = \sum_{i,j=0,1} P_{i,j} \cdot \widetilde{U}_{2p+1,2q+j}. \]  
(51)
with its set representation given by
\[ \{E[\widetilde{U}|\mathcal{S}](\omega)\}_{k=0,1,2,\ldots,L} \]  
(52)
The shapes of the level sets of \( \bar{U}_{i,j} \) and \( E[\bar{U}] \) are given in Fig. 2, where those of \( E[\bar{U}] \) is also depicted by the red-colored circles. The shapes of level sets for the estimated expectations are depicted in Fig. 3, where the blue-colored circles denote the estimated ones at \( N = 2, 5, 10, 20, 30, 50, 70, 80 \) and 100 whereas the red-colored circles denote the true ones depicted only for the convenience of the comparisons between the estimated and true ones.

### 4.2 Evaluation of Estimation Errors

Let \( \tilde{D}_{i,j}(i, j = 0, \pm 1, \pm 2, \cdots) \) be the series of fuzzy sets as follows:

\[
\tilde{D}_{i,j} = (\mathbb{R}^2, [\tilde{D}_{i,j}], s_{\tilde{D}_{i,j}}) \in F_{\tilde{c}}(\mathbb{R}^2),
\]

where the step-wise approximated set representation of \( \tilde{D}_{i,j} \) is given by

\[
[\tilde{D}_{i,j}]_{\alpha_k} = \left\{ [\tilde{D}_{i,j}]_{\alpha_k} | k = 0, 1, 2, \cdots, L \right\}
\]

and each element \( [\tilde{D}_{i,j}]_{\alpha_k} \) is the disk on the \( \alpha_k \)-level plane with the center \((0, 0, \alpha_k)\) and the radius \( r^{(i,j,k)} \) given by (38), i.e.,

\[
[\tilde{D}_{i,j}]_{\alpha_k} = \left( \text{Disk on the } \alpha_k \text{-level plane with the center } (0, 0, \alpha_k) \text{ and the radius } r^{(i,j,k)} \right)
\]

\[
= \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq (r^{(i,j,k)})^2, x_3 = \alpha_k \right\}.
\]

Then, since each level set of \( \tilde{U}_{i,j} \) is given by (37), it can be rewritten by

\[
[\tilde{U}_{i,j}]_{\alpha_k} = [\tilde{D}_{i,j}]_{\alpha_k} + \{ \epsilon^{(i,j,k)} \}.
\]

Furthermore, let \( r^{(k)} \) and \( c^{(k)} \) be the random variables on \((\Omega, \mathcal{A}, P)\) such that

\[
c^{(k)}(\omega) = \sum_{i,j=0,1,2,\cdots} 1_{\omega_{i,j}}(\omega) \cdot c^{(i,j,k)},
\]

\[
r^{(k)}(\omega) = \sum_{i,j=0,1,2,\cdots} 1_{\omega_{i,j}}(\omega) \cdot r^{(i,j,k)}
\]

and \( \tilde{D}(\omega) \) be also a FRS on \((\Omega, \mathcal{A}, P)\) defined by

\[
\tilde{D}(\omega) = \sum_{i,j=0,1,2,\cdots} 1_{\omega_{i,j}}(\omega) \cdot [\tilde{D}_{i,j}]_{\alpha_k} \in F_{\tilde{c}}(\mathbb{R}^2),
\]

where \( c^{(i,j,k)} \) and \( r^{(i,j,k)} \) are defined respectively in (42) and (38). Them, using (14) and (61) to (67), it follows

\[
\tilde{U}(\omega) = \tilde{D}(\omega) + \{ E(c^{(k)}) \}
\]

and we have

\[
E[\tilde{U}] = E[\tilde{D}] + \{ E(c^{(k)}) \}.
\]

From (9), we have

\[
sp(E[\tilde{U}], \alpha_k, x) = sp(x, E[\tilde{U}]_{\alpha_k}) = \begin{cases} sp(x, [E(\tilde{U})]_{\alpha_k}) & \text{at } k = 1, 2, \cdots, L \\ 0 & \text{at } k = 0, \end{cases}
\]

where (17) has been used. Then, using (10) and (19), it follows that

\[
sp(E[\tilde{U}], \alpha_k, x) = sp(x, E[\tilde{U}]_{\alpha_k}) = \sum_{i,j=0,1,2,\cdots} sp(x, [\tilde{D}_{i,j}]_{\alpha_k}) \cdot P(\omega^{(i,j)}_{\alpha_k})
\]

at \( k = 1, 2, \cdots, L \), and using (10) and (64), we have

\[
sp(x, [\tilde{D}_{i,j}]_{\alpha_k}) = sp \left( x, [\tilde{D}_{i,j}]_{\alpha_k} \right) + sp \left( x, [c^{(i,j,k)}] \right).
\]

It can be also shown that

\[
sp(x, [c^{(i,j,k)}]) = (x, c^{(i,j,k)}).
\]
\( \mu^{(i,j,k)} \) given by (38). Then, we have
\[
\text{sp} \left( \{ \bar{D}_{i,j} \} ; \alpha \right) = \sup \left\{ \{ x, a \} \bigg| a = (a_1, a_2, a_3)^\prime \quad \in [ \bar{D}_{i,j} ] ; x = (x_1, x_2)^\prime \in \mathbb{S}^1 \right\}
\leq (a_1^2 + a_2^2)^\frac{1}{2} (x_1^2 + x_2^2)^\frac{1}{2}
= \rho^{(i,j,k)},
\]  
when \( x = (x_1, x_2)^\prime \in \mathbb{S}^1 \). Then, coupling these result, it follows
\[
\text{sp}(x, \{ \bar{U}_{i,j} \} ; \alpha) = \rho^{(i,j,k)} + (x, c^{(i,j,k)}).
\]  
Then, using (71) and (75), it follows
\[
\tilde{\text{sp}}(E[\bar{U}], \alpha_k, x) = E(\rho^{(k)}) + (x, E(c^{(k)})),
\]  
which means
\[
\tilde{\text{sp}}(E[\bar{U}], \alpha_k, x) = \begin{cases}  
E(\rho^{(k)}) + (x, E(c^{(k)})) & \text{at } k = 1, 2, \cdots, L \\ 0 & \text{at } k = 0.
\end{cases}
\]  

Since the fuzzy random data \( \{ \bar{U}^{(n)} ; n = 1, 2, \cdots, N \} \) is the correction of the realizations of the i.i.d. FRS \( \bar{U}^{(\omega)} \) defined by (14), \( \bar{U}^{(n)} \) should be one of \( \bar{U}_{i,j}, (i, j = 0, \pm 1, \pm 2, \cdots) \). Therefore, \( \{ \bar{U}^{(n)} \} ; k = 0, 1, 2, \cdots, L \) should be a disk with the center \( c^{(k)} \) and the radius \( r^{(k)} \) on the \( \alpha_k \)-level plane, where \( c^{(k)} \) and \( r^{(k)} \) are some ones of \( \rho^{(i,j,k)} \) and \( c^{(i,j,k)} \) with same \( i \) and \( j \), which means
\[
\tilde{\text{sp}}(\bar{U}^{(n)}, \alpha_k, x) = \begin{cases}  
\rho^{(k)} + (x, c^{(k)}) & \text{at } k = 1, 2, \cdots, L \\ 0 & \text{at } k = 0.
\end{cases}
\]  

Then, we have
\[
\tilde{\text{sp}} \left( \frac{1}{N} \sum_{n=1}^{N} \bar{U}^{(n)}, \alpha_k, x \right) = \frac{1}{N} \sum_{n=1}^{N} \tilde{\text{sp}}(\bar{U}^{(n)}, \alpha_k, x)
\leq \rho^{(k)} + (x, c^{(k)}),
\]  
where
\[
\rho^{(k)} = \frac{1}{N} \sum_{n=1}^{N} \rho^{(n)}, 
\quad c^{(k)} = \frac{1}{N} \sum_{n=1}^{N} c^{(n)}.
\]  

Finally, summarizing (70) and (79), the estimation error between \( E[\bar{U}] \) and its estimate (26) with \( p = 1 \) is evaluated as follows:
\[
\rho_1 \left( \bar{U}_N, E[\bar{U}] \right) \leq \frac{1}{L} \sum_{k=1}^{L} \left( \rho^{(k)} - E(\rho^{(k)}) \right)
\leq \left( \rho^{(k)} - E(\rho^{(k)}) \right).
\]  
As shown in (81), although the strict value of the estimation error \( \rho_1 \left( \bar{U}_N, E[\bar{U}] \right) \) is hard to obtain, its upper bound can be evaluated. The simulation result of upper bounds of estimation errors (100 sample paths) up to the number of data \( N = 100 \) is depicted in Fig. 4.

5 Conclusion

In this paper, the author has investigated numerically two-dimensional FRSSs, which are one case of those proposed by the author as models of vague capricious perceptions of crisp random phenomena, especially the asymptotic properties of estimators for expected FRSSs have been investigated numerically.

By adopting the step-wise approximated set representations of fuzzy sets and disk-form level sets, the numerical operations between two-dimensional FRSSs have easily been performed. Then, the estimates of expected FRSSs have been calculated and their estimation errors have been evaluated numerically.

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