PATH INDEPENDENCE OF THE ADDITIVE FUNCTIONALS FOR
STOCHASTIC VOLterra equations with singular kernels

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Abstract. In this paper, we characterise path-independence of additive functionals for
stochastic Volterra equations with singular kernels, extending the results of semimartingale
type stochastic differential equations to more general type stochastic differential
equations with singular kernels which includes stochastic differential equations driven by
fractional Brownian motions as a special case. This is done by linking the concerned
stochastic Volterra equations to mild formulation of parabolic type stochastic partial
differential equations and then utilising our previous result regarding path-independence
for stochastic evolution equations in [12].

1. Introduction

In recent years there has been increasing interest utilising stochastic differential equa-
tions (SDEs for short) to model the evolution of events and phenomena with randomness,
in particular in economics and finance, biology and medicine, as well as in ecology and
environmental data science, and so on. These diverse studies promote and also demand
further development of stochastic analysis with applied features. An important and re-
markable example is the path-independence of the Girsanov transformation, also named as
the transformation of drift (c.f., e.g., [6]), for stochastic differential equations. This prop-
erty was originated from the consideration of market efficiency in mathematical finance,
see [5] wherein the path-independent property of the Black-Scholes model (c.f. [16]) under
a risk neutral probability measure, in other words, the Girsanov transformation for linear
SDEs, was addressed in a naive manner. The mathematical formulation and justification
of the path-independence of the Girsanov transformation for general SDEs was initiated
in [17, 20], wherein a characterisation theorem with necessary and sufficient conditions
for the path-independence of the Girsanov transformation of (non-degenerated) SDEs is
established. Since then, there are a number of papers devoted to investigating this prop-
erty for various stochastic differential equations, see most recent works [7, 12, 13, 14, 15],
and the recent survey article [18] and references therein.

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On the other hand, SDEs driven by fractional Brownian motion and stochastic Volterra equations are interesting topics in stochastic analysis and applications, see e.g. [1, 2, 3, 4, 8, 11], just mention a few. As it is known that for SDEs driven by fractional Brownian motions and stochastic Volterra equations, the solutions are no longer semimartingales, so it is a natural question whether one can consider path-independent property for these two kinds of SDEs. We note the fact that by integral transformation of stochastic integrals with respect to fractional Brownian motions, one can link them to the stochastic integrals with respect to standard Brownian motions involving singular kernels, thus SDEs driven by fractional Brownian motions can be treated as stochastic Volterra equations involving singular kernels. This motives us in this paper to study the path-independence of certain additive functionals for stochastic Volterra equations with singular kernels.

The rest of our paper is organised as follows. In Section 2 we will set up the framework and explicate our characterisation theorem. Then two essential ingredients in derivation of the necessary and sufficient conditions for the characterisation theorem and a concise review for extensions of the characterisation theorem are presented in Section 3. In Section 4 we derive a new result on characterising the strong solutions of stochastic differential equations. Finally, we enclose the proofs of some used theorems in Section 5.

The following convention will be used throughout the paper: C with or without indices will denote different positive constants whose values may change from one place to another.

2. Preliminary

In the section, we introduce some notations and stochastic Volterra equations with singular kernels.

2.1. Notations. In the subsection, we introduce some notations used in the sequel.

Let $C(\mathbb{R})$ be the collection of continuous functions on $\mathbb{R}$ and $C^2(\mathbb{R})$ be the class of functions on $\mathbb{R}$ having continuous second derivatives. $C_c^\infty(\mathbb{R})$ denotes the collection of all real-valued $C^\infty$ functions of compact supports.

Set for $\lambda \in \mathbb{R}$,

$$
\|h\|_{\lambda, \infty} := \sup_{x \in \mathbb{R}} |h(x)| e^{-\lambda |x|},
$$

$$
C_{\text{tem}} := \{ h \in C(\mathbb{R}), \|h\|_{\lambda, \infty} < \infty \text{ for every } \lambda > 0 \},
$$

and we equip $C_{\text{tem}}$ with the topology induced by the norms $\| \cdot \|_{\lambda, \infty}$ for $\lambda > 0$. Similarly we define

$$
C_{\text{rap}} := \{ h \in C(\mathbb{R}), \|h\|_{\lambda, \infty} < \infty \text{ for every } \lambda < 0 \},
$$

and endow it with the topology induced by the norms $\| \cdot \|_{\lambda, \infty}$ for $\lambda < 0$. Let $C_{\text{tem}}^k$ (respectively $C_{\text{rap}}^k$) be the collection of functions in $C_{\text{tem}}$ (respectively in $C_{\text{rap}}$) which are $k$ times continuously differentiable with all the derivatives in $C_{\text{tem}}$ (respectively in $C_{\text{rap}}$).

Let $C(\mathbb{R}^+, C_{\text{tem}})$ be the space of all continuous functions on $\mathbb{R}^+$ taking values in $C_{\text{tem}}$. Let $\mathcal{M}$ be the space of finite measures on $\mathbb{R}$ endowed with the weak convergence topology.

2.2. Stochastic Volterra equations with singular kernels. In this subsection, we introduce stochastic Volterra equations with singular kernels and the notion of path-independence for their additive functionals.
Let \((Ω, \mathcal{F}, (\mathcal{F}_t)_{t≥0}, P)\) be a filtered probability space. Let \((B_t)_{t≥0}\) be a one-dimensional \((\mathcal{F}_t)_{t≥0}\)-Brownian motion. Consider the following stochastic Volterra equation on \(\mathbb{R}\):

\[
X_t = x_0 + \int_0^t (t-s)^{-α} b(s)ds + \int_0^t (t-s)^{-α} σ(s, X_s)dB_s, \quad t ≥ 0,
\]

where \(x_0 ∈ \mathbb{R}, 0 < α < \frac{1}{2}\), and the coefficients \(b : \mathbb{R}_+ → \mathbb{R}, σ : \mathbb{R}_+ × \mathbb{R} → \mathbb{R}\) are all Borel measurable. We assume:

\((H^1_b)\) \(b\) is bounded and continuous, and a constant \(L_b > 0\) denotes the bound of \(b\), i.e. \(|b(t)| ≤ L_b|\).

\((H^1_σ)\) There exists a constant \(L_σ ≥ 0\) such that for any \(t ≥ 0\) and \(x_1, x_2, x ∈ \mathbb{R}\)

\[
|σ(t, x_1) - σ(t, x_2)| ≤ L_σ|x_1 - x_2|, \\
|σ(t, x)| ≤ L_σ(1 + |x|).
\]

**Theorem 2.1.** Assume that \((H^1_b)\) and \((H^1_σ)\) hold. Then there exists a unique solution \((X_t)_{t≥0}\) to Eq.(1) fulfilling

\[
E|X_t|^p < ∞, \quad ∀p > \frac{2}{1 - 2α}.
\]

Theorem 2.1 was discussed in an early arXiv preprint [10] by Mytnik and Salisbury. For the completeness of our paper, we will present our verification of Theorem 2.1 in the appendix.

Next, we arbitrarily fix \(T > 0\) and introduce the following additive functional

\[
f_{s,t} := \int_s^t g_1(r, ω, X_r)dr + \int_s^t g_2(r, ω, X_r)dB_r, \quad 0 ≤ s < t ≤ T,
\]

where

\[
g_1 : [0, T] × Ω × \mathbb{R} → \mathbb{R}, \quad g_2 : [0, T] × Ω × \mathbb{R} → \mathbb{R}
\]

are progressively measurable, and moreover for almost all \(ω ∈ Ω\), both \(g_1(t, ω, x)\) and \(g_2(t, ω, x)\) are continuous in \((t, x)\), so that \(f_{s,t}\) is a well-defined semimartingale.

**Definition 2.2.** The additive functional \(f_{s,t}\) is called path-independent with respect to the solution \((X_t)_{t≤0}\) of the equation (1), if there exists a function \(v : [0, T] × \mathbb{R} → \mathbb{R}\) such that

\[
f_{s,t} = v(t, X_t) - v(s, X_s).
\]

3. **Main results**

In this section, we set up the connection of the equation (1) to a stochastic partial differential equation (SPDE for short), and then study the path-independence of the equation (1) by means of the SPDE.

3.1. **Derivation of a SPDE relating to the equation (1).** Set \(θ := \frac{1}{α} - 2 > 0\) so that

\[
\frac{1}{2 + θ} = α.
\]

Define the operator \(∆_θ\) and its domain as

\[
∆_θ := \frac{2}{(2 + θ)^2} \frac{∂}{∂x}\left(|x|^{-θ} \frac{∂}{∂x}\right),
\]
\[ D(\Delta_\theta) := \{ \phi \in C^2_{rap} : \Delta_\theta \phi \in C_{rap} \}, \]
\[ D_{tem}(\Delta_\theta) := \{ \phi \in C^2_{tem} : \Delta_\theta \phi \in C_{tem} \}. \]

Next, we consider the following evolution equation on \( \mathbb{R}_+ \times \mathbb{R} \)
\[ \frac{\partial}{\partial t} u(t, x) = \Delta_\theta u(t, x), \]
\[ u(0, x) = \delta_0(x), \]
where \( \delta_0 \) stands for the Delta function, that is, \( \delta_0(y) = 0 \), for \( y \neq 0 \) and \( \delta_0(y) = 1 \), for \( y = 0 \). By direct calculation, one obtains that \( p^\theta_t(x) \) is the fundamental solution to the above evolution equation, where the normalised constant \( c_\theta \) is determined as follows
\[ c_\theta := \left( \int_{\mathbb{R}} e^{-\frac{|x|^{2+\theta}}{2t}} \, dx \right)^{-1} = \left( \int_{\mathbb{R}} e^{-\frac{1}{2} |x|^{2+\theta}} \, dx \right)^{-1}. \]

Note that \( p^\theta_t(\cdot) \) is a probability density on \( \mathbb{R} \) for each \( t \geq 0 \).
Let \( \{S_t^\theta, t \geq 0\} \) be the semigroup generated by \( \Delta_\theta \), namely
\[ (S_t^\theta \phi)(x) := \int_{\mathbb{R}} p^\theta_t(x, y) \phi(y) \, dy, \quad \phi \in C^2_{tem}, \]
where \( p^\theta_t(x, y) \) is the transition density for the process with generator \( \Delta_\theta \). The generator is ambiguous at \( x = 0 \), but we choose the semigroup to be symmetric, i.e. \( p_t(0, x) = p_t(0, -x) \). Because \( \Delta_\theta \) is in the divergence form, \( p^\theta_t(x, y) = p^\theta_t(y, x) \), one has \( p^\theta_t(x, 0) = p^\theta_t(0, x) = p^\theta_t(x) \).

Now let us consider the following stochastic partial differential equation
\[ dX_t = (\Delta_\theta X_t + E(t, \cdot)) \, dt + \frac{1}{c_\theta} \sigma(t, X_t(0)) \delta_0 \, dB_t \quad (5) \]
for a random field \( X_t = X_t(x) = X(t, x, \omega) : \mathbb{R}_+ \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \), where \( E : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R} \) is Borel measurable. Then the following expression
\[ X_t = S_t^\theta X_0 + \int_0^t S_{t-s}^\theta E(s, \cdot) \, ds + \int_0^t S_{t-s}^\theta \frac{1}{c_\theta} \sigma(s, X_s(0)) \delta_0 \, dB_s \quad (6) \]
gives an explicit mild solution to the equation (5).

**Theorem 3.1.** Assume that \( X_0 \in C_{tem} \) and \( E \in C(\mathbb{R}_+, C_{tem}) \cup C(\mathbb{R}_+, \mathcal{M}) \). Suppose further that \( \sigma \) fulfils \( (H_1^\gamma) \). Then, there exists a pathwise unique mild solution \( X \in C(\mathbb{R}_+, C_{tem}) \) to the equation (5).

Theorem 3.1 was also discussed in the early arXiv preprint [10] by Mytnik and Salisbury. For the completeness of our paper, we will give our verification of Theorem 3.1 in the appendix.
Next, let us set up a relation between weak solutions of the equation (11) and mild solutions of the equation (5). Take $X_0 = x_0$ and $E(t, \cdot) = \frac{1}{c_\theta} b(t) \delta_0$ in (6), and notice the following

$$
\left( S_t^\phi x_0 \right)(x) = \left( S_t^\phi \right)(x) = x_0,
$$

$$
\left( S_t^\phi E(s, \cdot) \right)(x) = \left( S_t^\phi \frac{1}{c_\theta} b(s) \delta_0 \right)(x) = \int_\mathbb{R} p_{t-s}^\phi (x, y) \frac{1}{c_\theta} b(s) \delta_0(y) dy = p_{t-s}^\phi (x, 0) \frac{1}{c_\theta} b(s),
$$

$$
\left( S_t^\phi \sigma(s, X_s(0)) \delta_0 \right)(x) = \int_\mathbb{R} p_{t-s}^\phi (x, y) \frac{1}{c_\theta} \sigma(s, X_s(0)) \delta_0(y) dy = p_{t-s}^\phi (x, 0) \frac{1}{c_\theta} \sigma(s, X_s(0)),
$$

and then, (6) becomes

$$
X_t(x) = x_0 + \int_0^t p_{t-s}^\phi (x, 0) \frac{1}{c_\theta} b(s) ds + \int_0^t p_{t-s}^\phi (x, 0) \frac{1}{c_\theta} \sigma(s, X_s(0)) dB_s.
$$

In particular for $x = 0$, we get

$$
X_t(0) = x_0 + \int_0^t p_{t-s}^\phi (0, 0) \frac{1}{c_\theta} b(s) ds + \int_0^t p_{t-s}^\phi (0, 0) \frac{1}{c_\theta} \sigma(s, X_s(0)) dB_s
$$

$$
= x_0 + \int_0^t (t-s)^{-\alpha} b(s) ds + \int_0^t (t-s)^{-\alpha} \sigma(s, X_s(0)) dB_s.
$$

That is to say that $X_t(0)$ is a weak solution of the equation (11). Thus, in order to obtain the path-independent property related to the equation (7), we are going to study the path-independence related to the equation (5).

### 3.2. An auxiliary SDE

In this subsection, based on the equation (5), we want to construct an auxiliary SDE and then study its path-independent property.

By Theorem 3.1, we conclude that $X_t$ belongs to the Banach space $C_{tem}$. Hence, we will consider a weak solution of the equation (5), i.e. for any $\varphi \in C(\mathbb{R}^+, D(\Delta_\theta))$ with $r \mapsto \frac{\partial \varphi(\cdot)}{\partial r} \in C(\mathbb{R}^+, \mathbb{R})$,

$$
\langle X_t, \varphi_t \rangle = \langle X_0, \varphi_0 \rangle + \int_0^t \left\langle X_r, \left( \Delta_\theta \varphi_r + \frac{\partial \varphi_r}{\partial r} \right) \right\rangle dr + \int_0^t \langle E(r, \cdot), \varphi_r \rangle dr
$$

$$
+ \int_0^t \frac{1}{c_\theta} \sigma(r, X_r(0)) \varphi_r(0) dB_r, \tag{7}
$$

where $\varphi_t(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ and $\langle X_t, \varphi_t \rangle := \int_\mathbb{R} X_t(y) \varphi_t(y) dy$. Based on the equivalence of weak solutions and mild solutions for the equation (5), there exists a pathwise unique solution $(X_t)_{t \geq 0}$ to the equation (7). On the other hand, for each $\varphi \in C(\mathbb{R}^+, D(\Delta_\theta))$, notice that the equation (7) is an Itô type SDE, thus by the Yamada-Watanabe theorem, the pair $\langle X_t, \varphi_t \rangle$ is a unique strong solution (in the Itô sense) of the equation (7). Now, we define the path-independent property related to the equation (7).

**Definition 3.2.** For the additive functional

$$
F_{s,t}^\phi := \int_s^t G_1(r, \omega, \langle X_r, \varphi_r \rangle) dr + \int_s^t G_2(r, \omega, \langle X_r, \varphi_r \rangle) dB_r, \quad 0 \leq s < t \leq T, \tag{8}
$$

where

$$
G_1 : [0, T] \times \Omega \times \mathbb{R} \mapsto \mathbb{R}, \quad G_2 : [0, T] \times \Omega \times \mathbb{R} \mapsto \mathbb{R}
$$
are progressively measurable, if there exists a function $V : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ such that $F^\varphi_{s,t}, V$ satisfy

$$F^\varphi_{s,t} = V(t, \langle X_t, \varphi_t \rangle) - V(s, \langle X_s, \varphi_s \rangle),$$  \hspace{1cm} (9)

we say that $F^\varphi_{s,t}$ is path-independent.

For the equation (7) and the additive functional $F^\varphi_{s,t}$, we have the following first main result.

**Theorem 3.3.** Assume that $\sigma(r, X_r(0))\varphi_r(0) \neq 0$ for any $r \in [0, T]$, and for $\omega \in \Omega$, $G_1(\cdot, \omega, \cdot) \in C([0, T] \times \mathbb{R} \mapsto \mathbb{R})$, $G_2(\cdot, \omega, \cdot) \in C([0, T] \times \mathbb{R} \mapsto \mathbb{R})$. Then, the additive functional $F^\varphi_{s,t}$ is path-independent with $V$ belonging to $C^1_\mathbb{R}([0, T] \times \mathbb{R})$ and all the derivatives of $V$ in $(t, z)$ being uniformly continuous, i.e., the equality (9) holds if and only if $V, G_1, G_2$ satisfy the following partial differential equations

$$\begin{cases}
\partial_t V(r, z) + \frac{1}{2} \partial^2_r V(r, z) \left( \frac{1}{c_\theta} \sigma(r, X_r(0))\varphi_r(0) \right)^2 \\
+ \partial_z V(r, z) \left( \langle X_r, (\Delta_r \varphi_r + \frac{\partial \varphi_r}{\partial r}) \rangle + \langle E(r, \cdot), \varphi_r \rangle \right) = G_1(r, z), \\
\partial_z V(r, z) \frac{1}{c_\theta} \sigma(r, X_r(0))\varphi_r(0) = G_2(r, z).
\end{cases} \hspace{1cm} (10)$$

**Proof. Necessity.** On one hand, applying the Itô formula to $V(t, \langle X_t, \varphi_t \rangle)$, we obtain that

$$V(t, \langle X_t, \varphi_t \rangle) = V(s, \langle X_s, \varphi_s \rangle) + \int_s^t \frac{\partial V(r, \langle X_r, \varphi_r \rangle)}{\partial r} dr$$

$$+ \int_s^t \frac{\partial V(r, \langle X_r, \varphi_r \rangle)}{\partial z} \left( \langle X_r, (\Delta_r \varphi_r + \frac{\partial \varphi_r}{\partial r}) \rangle + \langle E(r, \cdot), \varphi_r \rangle \right) dr$$

$$+ \int_s^t \frac{1}{c_\theta} \sigma(r, X_r(0))\varphi_r(0) dB_r$$

$$+ \frac{1}{2} \int_s^t \frac{\partial^2 V(r, \langle X_r, \varphi_r \rangle)}{\partial z^2} \left( \frac{1}{c_\theta} \sigma(r, X_r(0))\varphi_r(0) \right)^2 dr. \hspace{1cm} (11)$$

On the other side, since $F^\varphi_{s,t}, V$ satisfies (9), it holds that

$$V(t, \langle X_t, \varphi_t \rangle) - V(s, \langle X_s, \varphi_s \rangle) = \int_s^t G_1(r, \langle X_r, \varphi_r \rangle) dr + \int_s^t G_2(r, \langle X_r, \varphi_r \rangle) dB_r. \hspace{1cm} (12)$$

Comparing (11) and (12), by uniqueness of the decomposition of semimartingales, we get

$$\frac{\partial V(r, \langle X_r, \varphi_r \rangle)}{\partial r} + \frac{1}{2} \frac{\partial^2 V(r, \langle X_r, \varphi_r \rangle)}{\partial z^2} \left( \frac{1}{c_\theta} \sigma(r, X_r(0))\varphi_r(0) \right)^2$$

$$+ \frac{\partial V(r, \langle X_r, \varphi_r \rangle)}{\partial z} \left( \langle X_r, (\Delta_r \varphi_r + \frac{\partial \varphi_r}{\partial r}) \rangle + \langle E(r, \cdot), \varphi_r \rangle \right) = G_1(r, \langle X_r, \varphi_r \rangle),$$

$$\frac{\partial V(r, \langle X_r, \varphi_r \rangle)}{\partial z} \frac{1}{c_\theta} \sigma(r, X_r(0))\varphi_r(0) = G_2(r, \langle X_r, \varphi_r \rangle).$$

Furthermore, by the assumption that $\sigma(r, X_r(0))\varphi_r(0) \neq 0$ for any $r \in [0, T]$, one can get that the support of $\langle X_t, \varphi_t \rangle$ is $\mathbb{R}$. Thus, it holds that

$$\frac{\partial V(r, z)}{\partial r} + \frac{1}{2} \frac{\partial^2 V(r, z)}{\partial z^2} \left( \frac{1}{c_\theta} \sigma(r, X_r(0))\varphi_r(0) \right)^2$$
Proof. Set \( \int \phi \) replace \( g \) thus, it holds that \( F \).

That is, Theorem 3.4. section, we study the path-independence for the equation (1).

3.3. The path-independence for the stochastic Volterra equations. In this subsection, we study the path-independence for the equation (1).

Theorem 3.4. Suppose that \( \sigma \neq 0 \) and for \( \omega \in \Omega \), \( g_1(\cdot, \omega, \cdot) \in C([0, T] \times \mathbb{R} \mapsto \mathbb{R}) \), \( g_2(\cdot, \omega, \cdot) \in C([0, T] \times \mathbb{R} \mapsto \mathbb{R}) \) and \( g_1, g_2 \) are uniformly bounded. Then, under \( (H^1_1) \) \( (H^1_2) \), the additive functional \( f_{s,t} \) is path-independent with respect to \((X_t)_{t \in [0, T]} \) with \( v \) belonging to \( C^1([0, T] \times \mathbb{R}) \) and all the derivatives of \( v \) being uniformly continuous, if and only if

\[
\begin{align*}
\partial_r v(r, z) + \frac{1}{2} \partial^2_z v(r, z) \left( \frac{1}{c_\theta} \sigma(r, X_r(0)) \right)^2 + \partial_z v(r, z) \frac{1}{c_\theta} b(r) &= g_1(r, z), \\
\partial_z v(r, z) \frac{1}{c_\theta} \sigma(r, X_r(0)) &= g_2(r, z).
\end{align*}
\]

Proof. Set \( \psi^m(x) = \varphi^m(x) \), and then \( \psi^m \) weakly converges to \( \delta_0 \) as \( m \to \infty \). Thus, we replace \( \varphi_t \) by \( \psi^m \) in (1) and obtain that

\[
\int_s^t G_1(r, \omega, \langle X_r, \psi^m \rangle) \, dr + \int_s^t G_2(r, \omega, \langle X_r, \psi^m \rangle) \, dB_r = V(t, \langle X_t, \psi^m \rangle) - V(s, \langle X_s, \psi^m \rangle).
\]

Besides, by Theorem 3.3 it holds that the above equality is equivalent to the following conditions

\[
\begin{align*}
\partial_r V(r, z) + \frac{1}{2} \partial^2_z V(r, z) \left( \frac{1}{c_\theta} \sigma(r, X_r(0)) \psi^m(0) \right)^2 + \partial_z V(r, z) \frac{1}{c_\theta} \psi^m(0) &= G_1(r, z), \\
\partial_z V(r, z) \frac{1}{c_\theta} \sigma(r, X_r(0)) \psi^m(0) &= G_2(r, z).
\end{align*}
\]

Next, we observe two integrals in the left side of (15). Note that as \( m \to \infty \),

\[
\langle X_r, \psi^m \rangle \to X_r(0) \quad \text{a.s.}
\]
Assume that $G_1, G_2$ are uniformly bounded. Thus, by the dominated convergence theorem, it holds that
\[
\int_s^t G_1(r, \omega, (X_r, \psi^m)) dr \xrightarrow{L^1} \int_s^t G_1(r, \omega, X_r(0)) dr,
\]
\[
\int_s^t G_2(r, \omega, (X_r, \psi^m)) dB_r \xrightarrow{L^2} \int_s^t G_2(r, \omega, X_r(0)) dB_r.
\]
Then we take the limit on two sides of (15) and attain that
\[
\int_s^t G_1(r, \omega, X_r(0)) dr + \int_s^t G_2(r, \omega, X_r(0)) dB_r = V(t, X_t(0)) - V(s, X_s(0)).
\] (18)

Besides, as $m \to \infty$, (16) and (17) become
\[
\partial_r V(r, z) + \frac{1}{2} \partial^2_z V(r, z) \left( \frac{1}{c_0} \sigma(r, X_r(0)) \right)^2 + \partial_z V(r, z) E(r, 0) = G_1(r, z),
\]
\[
\partial_z V(r, z) \frac{1}{c_0} \sigma(r, X_r(0)) = G_2(r, z).
\]
Set $X_0 = x_0$ and $E(t, \cdot) = \frac{1}{c_0} b(t) \delta_0$, and then by Subsection 3.1 it holds that $X_t(0) = X_t$ for $t \in [0, T]$ and
\[
\partial_r V(r, z) + \frac{1}{2} \partial^2_z V(r, z) \left( \frac{1}{c_0} \sigma(r, X_r) \right)^2 + \partial_z V(r, z) \frac{1}{c_0} b(r) = G_1(r, z),
\]
\[
\partial_z V(r, z) \frac{1}{c_0} \sigma(r, X_r) = G_2(r, z).
\]
Finally, taking $g_1 = G_1, g_2 = G_2$ and $v = V$, we conclude that the additive functional $f_{s,t}$ is path-independent with respect to $(X_t)_{t \in [0,T]}$ if and only if (13) and (14) hold. The proof is complete. \qed

4. Application

In this section, we apply Theorem 3.4 to a class of SDEs driven by fractional Brownian motions.

First of all, we recall some basics about fractional Brownian motions (c.f. [3]). Let $B^H$ be a fractional Brownian motion with the Hurst index $H \in (0, 1)$, which is a centered Gaussian process with the following covariance kernel
\[
R_H(s, t) = \frac{V_H}{2} (s^{2H} + t^{2H} - |t - s|^{2H}),
\]
where
\[
V_H = \frac{\Gamma(2 - 2H) \cos(\pi H)}{\pi H (1 - 2H)}.
\]
It is known that the fractional Brownian motion $B^H$ has the representation in law:
\[
B^H_t = \int_0^t K_H(t, s) dB_s,
\]
where $K_H(t,s)$ is the square root of the covariance operator, that is

$$R_H(s,t) = \int_0^1 K_H(s,r)K_H(t,r)dr.$$ 

More precisely,

$$K_H(t,r) = \frac{(t-r)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, 1 - \frac{t}{r}\right)I_{[0,t]}(r),$$

where $F$ is the Gauss hypergeometric function.

By [1], a SDE driven by a fractional Brownian motion means an equation of the form

$$X_t = x_0 + \int_0^t \tilde{K}_H(t,s)\tilde{b}(s)ds + \int_0^t \tilde{K}_H(t,s)\tilde{\sigma}(s,X_s)dB_s,$$

where $\tilde{K}_H(t,s) = C(t-s)^{H-\frac{1}{2}}I_{[0,1]}(s)$, $C > 0$ is a constant and $\tilde{b}, \tilde{\sigma}$ satisfy (H_{\tilde{b}}) (H_{\tilde{\sigma}}). Provided that $0 < H < \frac{1}{2}$ and $\alpha = \frac{1}{2} - H$, by Theorem 2.1, we can obtain that the equation (19) has a unique solution $(X_t)_{t \geq 0}$. Also assume that $\tilde{\sigma} \not\equiv 0, v \in C^1([0,T] \times \mathbb{R})$ and all the derivatives of $v$ are uniformly continuous, for $\omega \in \Omega$, $g_1(\cdot, \cdot, \cdot) \in C([0,T] \times \mathbb{R} \rightarrow \mathbb{R})$, $g_2(\cdot, \cdot, \cdot) \in C([0,T] \times \mathbb{R} \rightarrow \mathbb{R})$ and $g_1, g_2$ are uniformly bounded. Thus, by Theorem 3.4, we know that the additive functional $f_{s,t}$ is path-independent with respect to $(X_t)_{t \in [0,T]}$ if and only if

$$\partial_v(r,z) + \frac{1}{2} \partial^2_v(r,z) \left(\frac{1}{c_0}C\tilde{\sigma}(r,X_r)\right)^2 + \partial_z v(r,z) \frac{1}{c_0}C\tilde{b}(r) = g_1(r,z)$$

$$\partial_z v(r,z) \frac{1}{c_0}C\tilde{\sigma}(r,X_r) = g_2(r,z).$$

5. Appendix

In this section, we give proofs of Theorem 2.1 and Theorem 3.1. We first show Theorem 2.1. To this end, we begin with a key proposition.

**Proposition 5.1.** Assume that $g(t)$ is continuous and $\sigma$ satisfies (H_{\sigma}). Then the following equation

$$U_t = g(t) + \int_0^t (t-s)^{-\alpha}\sigma(s,U_s)dB_s$$

has a unique solution $(U_t)_{t \geq 0}$ with

$$\mathbb{E}|U_t|^p < \infty, \quad p > \frac{2}{1-2\alpha}.$$ (21)

**Proof.** Fix $T > 0$ and construct the Picard iterated sequence as follows: For $n \in \mathbb{N}$

$$\left\{\begin{array}{ll}
U_t^{(0)} & = g(t) \in \mathbb{R}, \\
U_t^{(n)} & = g(t) + \int_0^t (t-s)^{-\alpha}\sigma(s,U_s^{(n-1)})dB_s, \quad t \in [0,T].
\end{array}\right.$$ 

For $p > \frac{2}{1-2\alpha}$, it holds that

$$\mathbb{E}|U_t^{(n)}|^p \leq 2^{p-1}|g(t)|^p + 2^{p-1}\mathbb{E}\left|\int_0^t (t-s)^{-\alpha}\sigma(s,U_s^{(n-1)})dB_s\right|^p.$$
\[
\begin{align*}
&\leq 2^{p-1}|g(t)|^p + 2^{p-1}C_p \mathbb{E} \left| \int_0^t (t-s)^{-2\alpha|\sigma(s,U^{(n-1)}_s)|^2} ds \right|^{p/2} \\
&\leq 2^{p-1}|g(t)|^p + 2^{p/2} L_\sigma^p 2^{p-2} C_p \mathbb{E} \left| \int_0^t (t-s)^{-2\alpha(1 + |U^{(n-1)}_s|^2)} ds \right|^{p/2} \\
&\leq 2^{p-1}|g(t)|^p + L_\sigma^p 2^{p-2} C_p \left( \int_0^t (t-s)^{-2\alpha} ds \right)^{(p-2)/2} \int_0^t \mathbb{E}|U^{(n-1)}_s|^p ds \\
&\leq C_{p,T} + C_{p,T} \int_0^t \mathbb{E}|U^{(n-1)}_s|^p ds,
\end{align*}
\]
where we use the BDG inequality, \((H^1_\sigma)\), the Hölder inequality and the fact that for \(0 < \alpha < \frac{1}{2}\),
\[
\int_0^t (t-s)^{-2\alpha} ds \leq C_T, \quad \int_0^t (t-s)^{-2\alpha} ds \leq C_T.
\]
By iteration, we have that
\[
\mathbb{E}|U^{(n)}_t|^p \leq C_{p,T} e^{C_{p,T} T} + \sup_{t \in [0,T]} |g(t)|^p \left( \frac{(C_{p,T} t)^n}{n!} \right) \leq \left( C_{p,T} + \sup_{t \in [0,T]} |g(t)|^p \right) e^{C_{p,T} T},
\]
and furthermore
\[
\sup_n \sup_{t \in [0,T]} \mathbb{E}|U^{(n)}_t|^p \leq \left( C_{p,T} + \sup_{t \in [0,T]} |g(t)|^p \right) e^{C_{p,T} T}. \tag{22}
\]

For \(n, m \in \mathbb{N}\), set \(Z^{n,m}_t := U^{(n)}_t - U^{(m)}_t\), and then by the BDG inequality, \((H^1_\sigma)\) and the Hölder inequality it holds that
\[
\begin{align*}
\mathbb{E}|Z^{n,m}_t|^p &= \mathbb{E} \left| \int_0^t (t-s)^{-\alpha} \sigma(s,U^{(n-1)}_s) dB_s - \int_0^t (t-s)^{-\alpha} \sigma(s,U^{(m-1)}_s) dB_s \right|^p \\
&\leq C_p \mathbb{E} \left( \int_0^t (t-s)^{-2\alpha} \sigma(s,U^{(n-1)}_s) - \sigma(s,U^{(m-1)}_s) \right|^2 ds)^{p/2} \\
&\leq C_p L_\sigma^p \left( \int_0^t (t-s)^{-2\alpha} ds \right)^{(p-2)/2} \int_0^t \mathbb{E}|U^{(n-1)}_s - U^{(m-1)}_s|^p ds \\
&\leq C_{p,T} \int_0^t \mathbb{E}|Z^{n-1,m-1}_s|^p ds.
\end{align*}
\]
Moreover, integrating two sides of the above inequality, we obtain that
\[
\int_0^t \mathbb{E}|Z^{n,m}_s|^p ds \leq C_{p,T} \int_0^t \left( \int_0^s \mathbb{E}|Z^{n-1,m-1}_r|^p dr \right) ds \leq C_{p,T} t \int_0^t \frac{1}{s} \left( \int_0^s \mathbb{E}|Z^{n-1,m-1}_r|^p dr \right) ds,
\]
and
\[
\frac{1}{t} \int_0^t \mathbb{E}|Z^{n,m}_s|^p ds \leq C_{p,T} \int_0^t \frac{1}{s} \left( \int_0^s \mathbb{E}|Z^{n-1,m-1}_r|^p dr \right) ds.
\]
Again set
\[ h^{n,m}_t := \frac{1}{t} \int_0^t \mathbb{E}|Z^{n,m}_s|^p ds, \]
and then it holds that
\[ h^{n,m}_t \leq C_{p,T} \int_0^t h^{n-1,m-1}_s ds. \]
Based on (22), it is easy to see that
\[ \sup_{n,m} \sup_{t \in [0,T]} h^{n,m}_t \leq C_{p,T}. \]
Thus, by the Fatou lemma, one can get that
\[ h_t \leq C_{p,T} \int_0^t h_s ds, \]
where \( h_t := \limsup_{n,m \to \infty} h^{n,m}_t \), which together with [19, Lemma 2.1] yields that \( h_t = 0 \). This means that \( \{U^{(n)}, n \in \mathbb{N}\} \) is a Cauchy sequence in \( \mathbb{L}^p := L^p([0,T] \times \Omega, \mathcal{P}_T, dt \times P; \mathbb{R}) \), where \( \mathcal{P}_T \) denotes the collection of the progressive measurable sets of \([0,T] \times \Omega\). Since \( \mathbb{L}^p \) is complete, there exists a process \( U \in \mathbb{L}^p \) satisfying
\[ \lim_{n \to \infty} \mathbb{E} \int_0^T |U_t^{(n)} - U_t|^p dt = 0. \]
Next, we prove that \( U \) is a solution of Eq.(20). Indeed, by the BDG inequality, (H\(_1\)) and the Hölder inequality it holds that
\[
\begin{align*}
\mathbb{E} \int_0^T \left| \int_0^t (t-s)^{-\alpha} \sigma(s,U_s^{(n)}) dB_s - \int_0^t (t-s)^{-\alpha} \sigma(s,U_s) dB_s \right|^p dt \\
\leq C_p \int_0^T \mathbb{E} \left( \int_0^t (t-s)^{-2\alpha} |\sigma(s,U_s^{(n)}) - \sigma(s,U_s)|^2 ds \right)^{p/2} dt \\
\leq L_p C_p \int_0^T \left( \int_0^t (t-s)^{-2\alpha p} ds \right)^{(p-2)/2} \left( \int_0^t \mathbb{E}|U_s^{(n)} - U_s|^p ds \right) dt \\
\leq C_{p,T} \mathbb{E} \int_0^T |U_t^{(n)} - U_t|^p dt,
\end{align*}
\]
which implies that
\[ \int_0^t (t-s)^{-\alpha} \sigma(s,U_s^{(n)}) dB_s \to \int_0^t (t-s)^{-\alpha} \sigma(s,U_s) dB_s \text{ in } L^p. \]
So, \( U \) is a solution of Eq.(20).

By the similar deduction, we also conclude that the solutions of Eq.(20) are pathwise unique.

**Proof of Theorem 2.1.**

By Proposition 5.1, we only prove that \( \int_0^t (t-s)^{-\alpha} b(s) ds \) is continuous in \( t \). In fact, for \( 0 \leq t < t + \theta \leq T \),
\[
\left| \int_0^{t+\theta} (t + \theta - s)^{-\alpha} b(s) ds - \int_0^t (t-s)^{-\alpha} b(s) ds \right|^p
\]
holds that

Thus, by the similar computation to that of \( I \)
The above deduction implies that

And we claim that

Since \( X \)

Next, we prove Theorem 3.1. And we show the existence and uniqueness of mild solutions to Eq. (23), respectively.

**Proof of the existence.**

First of all, we consider the following equation:

\[
U_t = g(t) + \int_0^t (t-s)^{-\alpha} \sigma(s, U_s) dB_s, \tag{23}
\]

where

\[
g(t) = \int_{\mathbb{R}} p_t^\alpha(0, y) X_0(y) dy + \int_0^t \int_{\mathbb{R}} p_{t-s}^\alpha(0, y) E(s, y) dy ds.
\]

And we claim that \( g(t) \) is continuous in \( t \). In fact, for any \( T > 0 \) and \( t_1, t_2 \in [0, T] \), it holds that

\[
\left| \int_{\mathbb{R}} p_{t_2}^\alpha(0, y) X_0(y) dy - \int_{\mathbb{R}} p_{t_1}^\alpha(0, y) X_0(y) dy \right| \\
= c_0 \left| \int_{\mathbb{R}} e^{-\frac{|y|^{2+\theta}}{2t_2}} X_0(y) dy - \int_{\mathbb{R}} e^{-\frac{|y|^{2+\theta}}{2t_1}} X_0(y) dy \right| \\
= c_0 \left| \int_{\mathbb{R}} e^{-\frac{|y|^{2+\theta}}{2t_2}} X_0(t_2\alpha u) du - \int_{\mathbb{R}} e^{-\frac{|y|^{2+\theta}}{2t_1}} X_0(t_1\alpha u) du \right| \\
\leq c_0 \int_{\mathbb{R}} e^{-\frac{|u|^{2+\theta}}{2}} \left| X_0(t_2\alpha u) - X_0(t_1\alpha u) \right| du.
\]

Since \( X_0 \in C_{tem} \), we obtain that

\[
\lim_{t_2 \to t_1} \left| X_0(t_2\alpha u) - X_0(t_1\alpha u) \right| = 0,
\]

\[
\leq 2^{p-1} \left| \int_t^{t+\theta} (t + \theta - s)^{-\alpha} b(s) ds \right|^p \\
+ 2^{p-1} \left| \int_0^t ((t + \theta - s)^{-\alpha} - (t - s)^{-\alpha}) b(s) ds \right|^p \\
= I_1 + I_2.
\]

For \( I_1 \), by the simple calculation we get that

\[
I_1 = 2^{p-1} \left| \int_0^\theta (\theta - r)^{-\alpha} b(t + r) dr \right|^p \leq 2^{p-1} L_b \theta^{(1-\alpha)p} C_p.
\]

For \( I_2 \), note that

\[
0 \leq \int_0^t ((t - s)^{-\alpha} - (t + \theta - s)^{-\alpha}) ds \leq \frac{\theta^{1-\alpha}}{1-\alpha}.
\]

Thus, by the similar computation to that of \( I_1 \), one can have that

\[
I_2 \leq 2^{p-1} L_b \frac{\theta^{(1-\alpha)p}}{(1-\alpha)^p}.
\]

The above deduction implies that \( \int_0^t (t-s)^{-\alpha} b(s) ds \) is continuous in \( t \). The proof is over.
and
\[ e^{-\frac{|u|^{2+\theta}}{2} + \lambda T^\alpha |u|} |X_0(t_2^u u) - X_0(t_1^u u)| \]
\[ \leq e^{-\frac{|u|^{2+\theta}}{2} + \lambda T^\alpha |u|} \left( \sup_{t_1^u u \in \mathbb{R}} |X_0(t_2^u u)|e^{-\lambda |t_2^u u|} + \sup_{t_1^u u \in \mathbb{R}} |X_0(t_1^u u)|e^{-\lambda |t_1^u u|} \right) \]
\[ \leq 2e^{-\frac{|u|^{2+\theta}}{2} + \lambda T^\alpha |u|} \|X_0\|_{\lambda, \infty}. \]

Noting that for suitable \( \lambda \), \( \int e^{-\frac{|u|^{2+\theta}}{2} + \lambda T^\alpha |u|} du < \infty \). Therefore, the dominated convergence theorem implies that \( \int_F p_t^\theta (0, y) X_0(y) dy \) is continuous in \( t \). By the same deduction to that for \( \int_F p_t^\theta (0, y) X_0(y) dy \), one can have that \( \int_0^t \int_F p_t^\theta (0, y) E(s, y) dy ds \) is continuous in \( t \). Thus, by Proposition 5.1, we know that Eq. (23) has a unique solution denoted as \( U \).

Set
\[ X_t(x) = \int_R p_t^\theta (x, y) X_0(y) dy + \int_0^t \int_R p_t^\theta (x, y) E(s, y) dy ds \]
\[ + \int_0^t \int_R p_t^\theta (x, y) \frac{\sigma(s, U_x)}{c_\theta} \delta_0(y) dy dB_s \]
\[ = (S_t^\theta X_0)(x) + \int_0^t (S_t^\theta E(s, \cdot))(x) ds + \int_0^t \left( S_t^\theta \frac{\sigma(s, X_0(0))}{c_\theta} \right)(x) dB_s, \]
and then it is easy to check that \( X \) is a mild solution to Eq. (5) with \( X_t(0) = U_t, t \geq 0 \) and \( X \in C(R_+, C_{tem}) \). Moreover, by the simple calculation and (21), it holds that for any \( T, \lambda > 0 \)
\[ \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} E(\|X_t(x)|^p e^{-\lambda |x|}) < \infty, \quad p > \frac{2}{1-2\alpha}. \tag{24} \]

**Proof of the uniqueness.**
The method below is taken from [9].

**Step 1.** Let \( X^1, X^2 \) be two weak solutions of Eq. (5) with \( X_0^1 = X_0^2 = X_0 \). Set \( Z_t := X_t^1 - X_t^2 \), and then for \( \Phi^m_x(y) = p_{m-1/\alpha}^\theta (x, y) \), it holds that
\[ \langle Z_t, \Phi^m_x \rangle = \int_0^t \langle Z_r, \Delta_\theta \Phi^m_x \rangle dr + \int_0^t \frac{1}{c_\theta} \left( \sigma(r, X_t^1(0)) - \sigma(r, X_t^2(0)) \right) \Phi^m_x(0) dB_r. \]

Next, we define a sequence of functions \( \phi_n \) as follows. First of all, let \( \{a_n\} \) be a strictly decreasing sequence such that \( a_0 = 1 \) and
\[ \int_{a_n}^{a_{n-1}} \frac{1}{x} dx = n, \quad n \geq 1, \quad n \in \mathbb{N}. \]
Then let \( \{\psi_n\} \) be functions in \( C_{c}^\infty (\mathbb{R}) \) such that \( \text{supp}(\psi_n) \subset (a_n, a_{n-1}) \) and
\[ 0 \leq \psi_n(x) \leq \frac{2}{n|x|}, \quad \text{for } \forall x \in \mathbb{R}, \quad \text{and } \int_{a_n}^{a_{n-1}} \psi_n(x) dx = 1. \tag{25} \]
Finally, set
\[ \phi_n(x) = \int_0^{|x|} \int_{a_n}^{y} \psi_n(z) dz dy, \]
and then we know that \( \phi_n(x) \uparrow |x| \) and
\[
\phi'_n(x) = \text{sgn}(x) \int_0^{|x|} \psi_n(y) dy,
\]
\[
\phi''_n(x) = \psi_n(|x|).
\]
Moreover, \(|\phi'_n(x)| \leq 1\) and \(\int \phi''_n(x) h(x) dx \to h(0)\) for any function \(h\) which is continuous at \(0\).

In the following, applying the Itô formula to \(\phi_n((Z_t, \Phi^m_x))\), we obtain that
\[
\phi_n((Z_t, \Phi^m_x)) = \int_0^t \phi'_n((Z_r, \Phi^m_x)) (Z_r, \Delta \Phi^m_x) \, dr
\]
\[
+ \int_0^t \phi''_n((Z_r, \Phi^m_x)) \frac{1}{C^{\phi}} \left( \sigma(r, X^1_r(0)) - \sigma(r, X^2_r(0)) \right) \Phi^m_x(0) dB_r
\]
\[
+ \frac{1}{2} \int_0^t \phi''_n((Z_r, \Phi^m_x)) \frac{1}{C^{\phi}} \left( \sigma(r, X^1_r(0)) - \sigma(r, X^2_r(0)) \right)^2 \Phi^m_x(0)^2 dr. \tag{26}
\]
Take a nonnegative test function \(\Psi \in C([0, \infty), \mathcal{D}(\Delta \phi))\) such that
\[
\Psi_s(0) > 0, \forall s \geq 0 \quad \text{and} \quad \sup_{0 \leq s \leq t} \left| \int_{0^+} |x|^{-\theta} \left( \frac{\partial \Psi_s(x)}{\partial x} \right)^2 \left( \Psi_s(x) \right)^{-1} dx \right| < \infty, \quad \forall t > 0, \tag{27}
\]
and \(s \mapsto \frac{\partial \Psi_s(.)}{\partial s} \in C(\mathbb{R}^+, C_{\text{rop}})\). Also assume that \(\Gamma(t) := \{ x : \exists s \leq t, \Psi_s(x) > 0 \} \subset B(0, J(t))\) for some \(J(t) > 0\). So, applying the Itô formula to \(\phi_n((Z_t, \Phi^m_x)) \Psi_t(x)\), integrating two sides with respect to \(x\) and taking the expectation on two sides, we have that
\[
\mathbb{E} \langle \phi_n((Z_t, \Phi^m_x)), \Psi_t \rangle
\]
\[
= \mathbb{E} \int_0^t \langle \phi'_n((Z_r, \Phi^m_x)), (Z_r, \Delta \Phi^m_x), \Psi_r \rangle \, dr
\]
\[
+ \mathbb{E} \int_0^t \langle \phi''_n((Z_r, \Phi^m_x)) \Phi^m(0), \Psi_r \rangle \frac{1}{C^{\phi}} \left( \sigma(r, X^1_r(0)) - \sigma(r, X^2_r(0)) \right) dB_r
\]
\[
+ \frac{1}{2} \mathbb{E} \int_0^t \langle \phi''_n((Z_r, \Phi^m_x)) \Phi^m(0)^2, \Psi_r \rangle \frac{1}{C^{\phi}} \left( \sigma(r, X^1_r(0)) - \sigma(r, X^2_r(0)) \right)^2 dr
\]
\[
+ \mathbb{E} \int_0^t \langle \phi_n((Z_r, \Phi^m_x)), \frac{\partial \Psi_s(.)}{\partial r} \rangle \, dr. \tag{28}
\]
We claim that
\[
\lim_{n,m \to \infty} \mathbb{E} \langle \phi_n((Z_t, \Phi^m_x)), \Psi_t \rangle = \mathbb{E} \int_0^t \int_\mathbb{R} |Z_r(x)| \left( \Delta \Psi_r(x) + \frac{\partial \Psi_r(x)}{\partial r} \right) dx dr. \tag{29}
\]
Based on the claim and the Fatou lemma, it holds that
\[
\int_\mathbb{R} \mathbb{E} |Z_t(x)| \Psi_t(x) dx \leq \int_0^t \int_\mathbb{R} |Z_r(x)| \left( \Delta \Psi_r(x) + \frac{\partial \Psi_r(x)}{\partial r} \right) dx dr. \tag{30}
\]
Next, let \(\{g_N, N \in \mathbb{N}\}\) be a sequence of functions in \(C^\infty_c(\mathbb{R})\) such that \(g_N : \mathbb{R} \mapsto [0, 1]\), \(g_N(x) = 1\) for \(|x| \leq N\), \(g_N(x) = 0\) for \(|x| > N + 1\) and
\[
\sup_N \left( \| |x|^{-\theta} g_N \|_\infty + \| \Delta \phi g_N \|_\infty \right) < \infty.
\]
Again set for \((r, x) \in [0, t] \times \mathbb{R}\)

\[
\Psi_r^{(N)}(x) := (S_t^\theta h)(x)g_N(x), \quad h \in C_c^\infty(\mathbb{R}),
\]

and then one can verify that \(\Psi_r^{(N)} \in C_c^\infty(\mathbb{R})\) and for \(\lambda > 0\), there exists a constant \(C > 0\) such that for all \(N\)

\[
|\Delta^\theta \Psi_r^{(N)}(x) + \frac{\partial \Psi_r^{(N)}(x)}{\partial r}| = 4\alpha^2|x|^{-\theta} \frac{\partial (S_t^\theta h)(x)}{\partial x} \frac{\partial g_N(x)}{\partial x} + (S_t^\theta h)(x) \Delta^\theta g_N(x) 
\leq C e^{-\lambda|x|} I_{|x| > N}.
\]

Replacing \(\Psi_t\) by \(\Psi_t^{(N)}\) in (30), by the above inequality, we conclude that

\[
\int_\mathbb{R} \mathbb{E}|Z_t(x)| h(x) dx \leq C \int_0^t \int_\mathbb{R} \mathbb{E}|Z_r(x)| e^{-\lambda|x|} I_{|x| > N} dx dr.
\]

Note that by (24), the right side of the above inequality tends to zero as \(N \to \infty\). Therefore, for \(t > 0, x \in \mathbb{R}\)

\[
\mathbb{E}|Z_t(x)| = 0,
\]

and furthermore

\[
X_t^1(x) = X_t^1(x).
\]

Finally, the continuity of \(X_t^1(x), X_t^1(x)\) in \(t, x\) gives the required result.

**Step 2.** We prove the claim (29). To do this, by (28) we divide \(\mathbb{E}\langle \phi_n(\langle Z_t, \Phi_m^\gamma \rangle), \Psi_t \rangle\) into \(I_1, I_2, I_3, I_4\), where

\[
I_1 := \mathbb{E} \int_0^t \langle \phi'_n(\langle Z_r, \Phi_m^\gamma \rangle) \langle Z_r, \Delta^\theta \Phi_m^\gamma \rangle, \Psi_r \rangle dr,
\]

\[
I_2 := \mathbb{E} \int_0^t \langle \phi'_n(\langle Z_r, \Phi_m^\gamma \rangle) \Phi_m^\gamma(0), \Psi_r \rangle \frac{1}{c^\theta} \left( \sigma(r, X_r^1(0)) - \sigma(r, X_r^2(0)) \right) dB_r,
\]

\[
I_3 := \frac{1}{2} \mathbb{E} \int_0^t \langle \phi''_n(\langle Z_r, \Phi_m^\gamma \rangle) \Phi_m^\gamma(0)^2, \Psi_r \rangle \frac{1}{c^\theta} \left( \sigma(r, X_r^1(0)) - \sigma(r, X_r^2(0)) \right)^2 dr,
\]

\[
I_4 := \mathbb{E} \int_0^t \langle \phi_n(\langle Z_r, \Phi_m^\gamma \rangle), \frac{\partial \Psi_r(\cdot)}{\partial r} \rangle dr.
\]

**Estimation for** \(I_1\). Note that

\[
\int_\mathbb{R} Z_r(y) \Delta_{y, \theta} \Phi_m^\gamma(y) dy = \int_\mathbb{R} Z_r(y) \Delta_{x, \theta} \Phi_m^\gamma(y) dy = \Delta_{x, \theta} \int_\mathbb{R} Z_r(y) \Phi_m^\gamma(y) dy,
\]

where the fact \(\Delta_{y, \theta} \Phi_m^\gamma(y) = \Delta_{x, \theta} \Phi_m^\gamma(y)\) is used. Thus, by integration by parts, it holds that

\[
\int_0^t \langle \phi'_n(\langle Z_r, \Phi_m^\gamma \rangle) \langle Z_r, \Delta^\theta \Phi_m^\gamma \rangle, \Psi_r \rangle dr = \int_0^t \int_\mathbb{R} \phi'_n(\langle Z_r, \Phi_m^\gamma \rangle) \Delta_{x, \theta}(\langle Z_r, \Phi_m^\gamma \rangle) \Psi_r(x) dx dr
\]

\[
= -2\alpha^2 \int_0^t \int_\mathbb{R} \frac{\partial}{\partial x} \left( \phi'_n(\langle Z_r, \Phi_m^\gamma \rangle) \right) |x|^{-\theta} \frac{\partial}{\partial x} (\langle Z_r, \Phi_m^\gamma \rangle) \Psi_r(x) dx dr
\]
For any $r \in [0,t]$, set

$$A^+: = \left\{ x : \left( \frac{\partial}{\partial x} \left( \langle Z_r, \Phi^m_x \rangle \right) \right)^2 \psi_r(x) \leq \frac{\partial}{\partial x} \left( \langle Z_r, \Phi^m_x \rangle \right) \langle Z_r, \Phi^m_x \rangle \frac{\partial}{\partial x} \psi_r(x) \right\} \cap \{ x : \psi_r(x) > 0 \}$$

$$A^-: = A^+ \cap \{ x : \frac{\partial}{\partial x} \left( \langle Z_r, \Phi^m_x \rangle \right) > 0 \},$$

$$A^0: = A^+ \cap \{ x : \frac{\partial}{\partial x} \left( \langle Z_r, \Phi^m_x \rangle \right) = 0 \}.$$

Note that on $A^+$, it holds that

$$0 < \frac{\partial}{\partial x} \left( \langle Z_r, \Phi^m_x \rangle \right) \psi_r(x) \leq \langle Z_r, \Phi^m_x \rangle \frac{\partial}{\partial x} \psi_r(x),$$

which implies that

$$\int_t^0 \int_{A^+} \psi_n(\langle Z_r, \Phi^m_x \rangle)|x|^{-\theta} \frac{\partial}{\partial x} \left( \langle Z_r, \Phi^m_x \rangle \right) \langle Z_r, \Phi^m_x \rangle \frac{\partial}{\partial x} \psi_r(x) dx dr \leq \int_t^0 \int_{A^+} \psi_n(\langle Z_r, \Phi^m_x \rangle)|x|^{-\theta} \langle Z_r, \Phi^m_x \rangle^2 \frac{\partial^2}{\partial x^2} \psi_r(x) dx dr \leq \int_t^0 \int_{A^+} \frac{2}{n} I_{\{a_n < \langle Z_r, \Phi^m_x \rangle \leq a_{n-1}\}} |x|^{-\theta} \langle Z_r, \Phi^m_x \rangle \left( \frac{\partial}{\partial x} \psi_r(x) \right)^2 dx dr \leq \frac{2a_{n-1}}{n} \int_0^t \int \frac{\partial^2}{\partial x^2} \psi_r(x) dx dr.$$
\[
\begin{align*}
\text{where \([9]\) Lemma 2.1] is used in the last second inequality and the constant } \varepsilon > 0 \text{ is small enough such that } \\
B(0, \varepsilon) \subset \Gamma(t), \text{ and } \inf_{r \in t, x \in B(0, \varepsilon)} \Psi_r(x) > 0.
\end{align*}
\]

On \(A^{-r}\), we know that
\[
0 > \frac{\partial}{\partial x} \left( (Z_r, F^m_x) \right) \Psi_r(x) \geq \langle Z_r, F^m_x \rangle \frac{\partial}{\partial x} \Psi_r(x),
\]
\[
\frac{\partial}{\partial x} \left( (Z_r, F^m_x) \right) \langle Z_r, F^m_x \rangle \frac{\partial}{\partial x} \Psi_r(x) \leq \langle Z_r, F^m_x \rangle^2 \left( \frac{\partial}{\partial x} \Psi_r(x) \right)^2.
\]

Therefore, by similar deduction to the above, one can get that
\[
\int_0^t \int_{A^{-r}} \psi_n(|(Z_r, F^m_x)|) |x|^{-\theta} \frac{\partial}{\partial x} \left( (Z_r, F^m_x) \right) \langle Z_r, F^m_x \rangle \frac{\partial}{\partial x} \Psi_r(x) dx dr \leq \frac{2a_{n-1}}{n} C_{\Psi,t}.
\]

Finally, it is easy to see that
\[
\int_0^t \int_{A^{0, r}} \psi_n(|(Z_r, F^m_x)|) |x|^{-\theta} \frac{\partial}{\partial x} \left( (Z_r, F^m_x) \right) \langle Z_r, F^m_x \rangle \frac{\partial}{\partial x} \Psi_r(x) dx dr = 0.
\]

By the above deduction, we obtain that
\[
\mathbb{E} J_{11} \leq 4a^2 C_{\Psi,t} \frac{a_{n-1}}{n}.
\]

Now we observe \(J_{12}\). Note that \(\phi_n(x) \uparrow |x|\) uniformly in \(x\) as \(n \to \infty\), and \(\langle Z_r, F^m_x \rangle\) tends to \(Z_r(x)\) a.s. as \(m \to \infty\) for all \(r, x\). Hence, \(\phi_n(|(Z_r, F^m_x)|) \langle Z_r, F^m_x \rangle \to |Z_r(x)|\) a.s. as \(n, m \to \infty\). Besides, the Jensen inequality and \([24]\) imply that \(\langle Z_r, F^m_x \rangle\) is \(L^p\) bounded on \([0, t] \times B(0, J(t)) \times \Omega, dr \times dx \times \mathbb{P}\) uniformly in \(m\). This gives uniform integrability of \(\langle Z_r, F^m_x \rangle\) in \(m\) on \([0, t] \times B(0, J(t)) \times \Omega\). Moreover, we know that
\[
|\phi_n((Z_r, F^m_x))| \leq |(Z_r, F^m_x)| \leq \langle Z_r, F^m_x \rangle.
\]

So, \(\phi_n((Z_r, F^m_x))\) is uniformly integrable. Since \(\Psi_r = 0\) outside \(B(0, J(t))\), this yields that
\[
\lim_{n, m \to \infty} \mathbb{E} J_{12} = \mathbb{E} \int_0^t \int_{\mathbb{R}} |Z_r(x)| \Delta_{\theta} \Psi_r(x) dx dr.
\]

Collecting the pieces, we obtain that
\[
\lim_{n, m \to \infty} I_1 = \mathbb{E} \int_0^t \int_{\mathbb{R}} |Z_r(x)| \Delta_{\theta} \Psi_r(x) dx dr.
\]

**Estimation for \(I_2\).** Note that the quadratic variation of
\[
\int_0^t \left\langle \phi_n((Z_r, F^m_x)) \Phi^m(0), \Psi_r \right\rangle \frac{1}{c_\theta} \left( \sigma(r, X^1_r(0)) - \sigma(r, X^2_r(0)) \right) dB_r,
\]

\[
\leq 2a_{n-1} C_{\Psi,t}.
\]
satisfies
\[
\int_0^t \langle \phi_n'((Z_r, \Phi^m)) \rangle \Phi^m(0), \Psi_r \rangle^2 \frac{1}{c^2} \sigma(r, X^1_r(0)) - \sigma(r, X^2_r(0)) \rangle^2 dr \\
\leq 2L^2 \int_0^t \langle \phi_n'((Z_r, \Phi^m)) \rangle \Phi^m(0), \Psi_r \rangle^2 \frac{1}{c^2} \langle X^1_r(0) \rangle^2 + \langle X^2_r(0) \rangle^2 dr \\
\leq 2L^2 ||\Psi||_\infty \frac{1}{c^2} \int_0^t \langle X^1_r(0) \rangle^2 + \langle X^2_r(0) \rangle^2 dr.
\]
Thus, for any \( t \in [0, T] \), we have that
\[
I_2 = 0. \tag{32}
\]

**Estimation for \( I_3 \).** By the Fatou lemma, \((H^1_a)\) and \((25)\), it holds that
\[
\lim_{m \to \infty} I_3 \leq \frac{1}{2c^2} \mathbb{E} \int_0^t \psi_n(|Z_r(0)|) \Psi_r(0) \left( \sigma(r, X^1_r(0)) - \sigma(r, X^2_r(0)) \right)^2 dr \\
\leq \frac{L^2}{nc^2} \mathbb{E} \int_0^t \Psi_r(0)|Z_r(0)| dr \\
\leq \frac{C}{n},
\]
where \((2)\) is used in the last inequality. Thus, we obtain that
\[
\lim_{n,m \to \infty} I_3 = 0. \tag{33}
\]

**Estimation for \( I_4 \).** Note that
\[
\phi_n((Z_r, \Phi^m)) \leq \langle Z_r, \Phi^m \rangle \leq \langle Z_r, \Phi^m \rangle,
\]
which together with uniform integrability of \( \langle Z_r, \Phi^m \rangle \) in \( m \) on \([0, t] \times B(0, J(t)) \times \Omega \) implies that \( \phi_n((Z_r, \Phi^m)) : n, m \) is uniformly integrable on \([0, t] \times B(0, J(t)) \times \Omega \). Moreover, we know that \( \phi_n((Z_r, \Phi^m)) \to |Z_r(x)| \) as \( n, m \to \infty \) a.s. for all \( x \) and all \( r \leq t \). Thus, it follows that
\[
\lim_{n,m \to \infty} I_4 = \mathbb{E} \int_0^t \int_\mathbb{R} |Z_r(x)| \frac{\partial \Psi_r(x)}{\partial r} dx dr, \tag{34}
\]
where we use the fact that
\[
\left| \frac{\partial \Psi_r(\cdot)}{\partial r} \right| \leq CI_{|x| \leq J(t)}.
\]
Finally, from \((31)\) \((32)\) \((33)\) \((34)\) and \((28)\), the claim follows.

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