Abstract. In this paper, we initiate the study of lightlike hypersurfaces of an \((\varepsilon)\)-almost paracontact metric manifold which are tangent to the structure vector field. In particular, we give definitions of invariant lightlike hypersurfaces and screen semi-invariant lightlike hypersurfaces, and give some examples. Integrability conditions for the distributions involved in the screen semi-invariant lightlike hypersurface are investigated when the ambient manifold is an \((\varepsilon)\)-para Sasakian manifold.

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1 Introduction

The theory of submanifolds of semi-Riemannian manifolds is one of the most important topics in differential geometry. In case the induced metric on the submanifold of semi-Riemannian manifold is degenerate, the study becomes more difficult and is quite different from the study of nondegenerate submanifolds. The primary difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact that in the first case the normal vector bundle has non-trivial intersection with the tangent vector bundle, and moreover in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle. Lightlike submanifolds of semi-Riemannian manifolds were introduced by K.L. Duggal and A. Bejancu in [5].

In 1976, an almost paracontact structure \((\phi, \xi, \eta)\) satisfying \(\phi^2 = I - \eta \otimes \xi\) and \(\eta(\xi) = 1\) on a differentiable manifold, was introduced by I. Sato [13]. The structure is an analogue of the almost contact structure [12], [3] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, T. Takahashi [14] introduced almost contact manifolds equipped with an associated pseudo-Riemannian metric. In particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as \((\varepsilon)\)-almost contact metric manifolds and \((\varepsilon)\)-Sasakian manifolds, respectively (see [2, 4, 7]). Lightlike hypersurfaces and submanifolds of indefinite Sasakian manifolds were studied in 2003 and 2007 (see [9] and [7]). Also, in 1989, K. Matsumoto [10] replaced the structure vector field \(\xi\) by \(-\xi\) in an almost paracontact manifold and
associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold.

An $(\varepsilon)$-Sasakian manifold is always odd-dimensional. On the other hand, in a Lorentzian almost paracontact manifold given by K. Matsumoto, the semi-Riemannian metric has only index 1 and the structure vector field $\xi$ is always timelike. These circumstances motivated the authors of [15] to associate a semi-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and this indefinite almost paracontact metric structure is called an $(\varepsilon)$-almost paracontact structure, where the structure vector field $\xi$ is spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$ (see also [16]).

In the present paper, as a first step to study lightlike geometry of $(\varepsilon)$-almost paracontact metric manifolds we study lightlike hypersurfaces. The paper is organized as follows. In section 2, we give a brief account of lightlike hypersurfaces of a semi-Riemannian manifold, for later use. Section 3 is devoted to $(\varepsilon)$-almost paracontact metric manifolds. In section 4, we give investigate lightlike hypersurfaces of an $(\varepsilon)$-almost paracontact metric manifold. In section 5, we define invariant lightlike hypersurfaces and give an example. Screen semi invariant hypersurfaces are introduced in Section 6. Moreover, integrability conditions for the distributions involved in the screen semi-invariant lightlike hypersurface are investigated when the ambient manifold is an $(\varepsilon)$-para Sasakian manifold.

## 2 Lightlike Hypersurfaces

Let $(\widetilde{M}, \widetilde{g})$ be an $(n + 2)$-dimensional semi-Riemannian manifold of fixed index $q \in \{1, \ldots, n + 1\}$ and $\widetilde{M}$ a hypersurface of $\widetilde{M}$. Assume that the induced metric $g = \widetilde{g}|_M$ on the hypersurface is degenerate on $M$. Then there exist a vector field $E \neq 0$ on $M$ such that

$$g(E, X) = 0, \quad X \in \Gamma(TM).$$

The radical space [11] of $T_xM$, at each point $x \in M$, is defined by

$$Rad T_xM = \{ E \in T_xM : g(E, X) = 0, \quad X \in \Gamma(T_xM) \},$$

(2.1)

whose dimension is called the nullity degree of $g$ and $(M, g)$ is called a lightlike hypersurface of $(\widetilde{M}, \widetilde{g})$. Since $g$ is degenerate and any null vector is perpendicular to itself, $T_xM$ is also degenerate and

$$Rad T_xM = T_xM \cap T_xM^\perp.$$  

(2.2)

For a hypersurface $M$, $\dim T_xM = 1$ implies that

$$\dim Rad T_xM = 1, \quad Rad T_xM = T_xM^\perp.$$

We call $Rad TM$ the radical distribution and it is spanned by the null vector field $E$.

Consider a complementary vector bundle $S(TM)$ of $Rad TM$ in $TM$. This means that

$$TM = S(TM) \perp Rad TM,$$

(2.3)

where $\perp$ denotes the orthogonal direct sum. The bundle $S(TM)$ is called the screen distribution on $M$. Since the screen distribution $S(TM)$ is non-degenerate, there exists a complementary orthogonal vector subbundle $S(TM)^\perp$ to $S(TM)$ in $T\widetilde{M}$ which is called the screen transversal bundle. The rank of the screen transversal bundle $S(TM)^\perp$ is 2.
Since $\text{Rad } TM$ is a lightlike vector subbundle of $S(TM)^ \perp$, therefore for any local section $E \in \Gamma(\text{Rad } TM)$ there exists a unique local section $N$ of $S(TM)^ \perp$ such that
\[
\tilde{g}(N, N) = 0, \quad \tilde{g}(E, N) = 1. \tag{2.4}
\]
Hence, $N$ is not tangent to $M$ and $\{E, N\}$ is a local frame field of $S(TM)^ \perp$. Moreover, we have a 1-dimensional vector subbundle $\text{ltr } TM$ of $\tilde{T}M$, namely lightlike transversal bundle, which is locally spanned by $N$. Then we set
\[
S(TM)^ \perp = \text{Rad } TM \oplus \text{ltr } TM,
\]
where the decomposition is not orthogonal. Thus we have the following decomposition of $\tilde{T}M$:
\[
\tilde{T}M = S(TM)^ \perp \text{Rad } TM \oplus \text{ltr } TM = TM \oplus \text{ltr } TM. \tag{2.5}
\]
From the above decomposition of a semi-Riemannian manifold $\tilde{M}$ along a lightlike hypersurface $M$, we may consider the following local quasi-orthonormal field of frames of $\tilde{M}$ along $M$:
\[
\{X_1, \ldots, X_n, E, N\},
\]
where $\{X_1, \ldots, X_n\}$ is an orthonormal basis of $\Gamma(S(TM))$. According to the decomposition given by (2.5), we have the following Gauss and Weingarten formulas, respectively:
\[
\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y) N, \tag{2.6}
\]
\[
\tilde{\nabla}_X N = - A_N X + \tau(X) N, \tag{2.7}
\]
where $B$ is a symmetric $(0, 2)$ tensor which is called the second fundamental form and $A$ is an endomorphism of $TM$ which is called the shape operator with respect to $N$ and $\tau$ is a 1-form on $M$ [5]. For each $X \in \Gamma(TM)$, we may write
\[
X = PX + \theta(X) E, \tag{2.8}
\]
where $P$ is the projection of $TM$ on $S(TM)$ and $\theta$ is a 1-form given by
\[
\theta(X) = \tilde{g}(X, N). \tag{2.9}
\]
From (2.7), for all $X, Y, Z \in \Gamma(TM)$, we get
\[
(\nabla_X g)(Y, Z) = B(X, Y) \theta(Z) + B(X, Z) \theta(Y), \tag{2.10}
\]
which implies that the induced connection $\nabla$ is a non-metric connection on $M$. From (2.3), we have
\[
\nabla_X W = \nabla^*_X W + C(X, W) E, \tag{2.11}
\]
\[
\nabla_X E = - A^*_E X - \tau(X) E \tag{2.12}
\]
for all $X \in \Gamma(TM)$, $W \in \Gamma(S(TM))$, where $C$, $A^*_E$ and $\nabla^*$ are the local second fundamental form, the local shape operator and the induced connection on $S(TM)$, respectively. Note that $\nabla^*_X W$ and $A^*_E X$ belong to $\Gamma(S(TM))$. Also, we have the following identities
\[
g(A^*_E X, W) = B(X, W), \quad g(A^*_E X, N) = 0, \quad B(X, E) = 0, \quad g(A_N X, N) = 0. \tag{2.13}
\]
Moreover, from the first and third equations of (2.13) we have
\[
A^*_E E = 0. \tag{2.14}
\]
For more details we refer to [5], [6] and [8].
3  \((\varepsilon)\)-almost paracontact metric manifolds

Let \(\widetilde{M}\) be an almost paracontact manifold [13] equipped with an almost paracontact structure \((\phi, \xi, \eta)\) consisting of a tensor field \(\phi\) of type \((1, 1)\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying

\[
\phi^2 = I - \eta \otimes \xi, \tag{3.1}
\]

\[
\eta(\xi) = 1, \tag{3.2}
\]

\[
\phi \xi = 0, \tag{3.3}
\]

\[
\eta \circ \phi = 0. \tag{3.4}
\]

Let \(\tilde{g}\) be a semi-Riemannian metric [11] such that

\[
\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \varepsilon \eta(X) \eta(Y), \quad X, Y \in \Gamma(T\widetilde{M}), \tag{3.5}
\]

where \(\varepsilon = \pm 1\). Then \(\widetilde{M}\) is called an \((\varepsilon)\)-almost paracontact metric manifold equipped with an \((\varepsilon)\)-almost paracontact metric structure \((\phi, \xi, \eta, \tilde{g}, \varepsilon)\) [15]. In particular, if \(\text{index}(g) = 1\), that is when the metric is a Lorentzian metric [1], then an \((\varepsilon)\)-almost paracontact metric manifold is called a Lorentzian almost paracontact manifold. From (3.5) we have

\[
\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) \tag{3.6}
\]

along with

\[
\tilde{g}(X, \xi) = \varepsilon \eta(X) \tag{3.7}
\]

for all \(X, Y \in \Gamma(T\widetilde{M})\). From (3.7) it follows that

\[
\tilde{g}(\xi, \xi) = \varepsilon \tag{3.8}
\]

that is, the structure vector field \(\xi\) is never lightlike.

Let \((\widetilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon)\) be an \((\varepsilon)\)-almost paracontact metric manifold (resp. a Lorentzian almost paracontact manifold). If \(\varepsilon = 1\), then \(\widetilde{M}\) will be said to be a spacelike \((\varepsilon)\)-almost paracontact metric manifold (resp. a spacelike Lorentzian almost paracontact manifold). Similarly, if \(\varepsilon = -1\), then \(\widetilde{M}\) will be said to be a timelike \((\varepsilon)\)-almost paracontact metric manifold (resp. a timelike Lorentzian almost paracontact manifold) [15].

An \((\varepsilon)\)-almost contact metric structure is called an \((\varepsilon)\)-para Sasakian structure if

\[
(\tilde{\nabla}_X \phi)Y = -\tilde{g}(\phi X, \phi Y)\xi - \varepsilon \eta(Y) \phi^2 X, \quad X, Y \in \Gamma(T\widetilde{M}), \tag{3.9}
\]

where \(\tilde{\nabla}\) is the Levi-Civita connection with respect to \(\tilde{g}\). A manifold endowed with an \((\varepsilon)\)-para Sasakian structure is called an \((\varepsilon)\)-para Sasakian manifold [15]. In an \((\varepsilon)\)-para Sasakian manifold, we have

\[
\tilde{\nabla} \xi = \varepsilon \phi \tag{3.10}
\]

and

\[
\Phi(X, Y) = \tilde{g}(\phi X, \phi Y) = \varepsilon \tilde{g}(\tilde{\nabla}_X \xi, Y) = (\tilde{\nabla}_X \eta)Y, \quad X, Y \in \Gamma(T\widetilde{M}) \tag{3.11}
\]

where

\[
\Phi(X, Y) = \tilde{g}(X, \phi Y). \tag{3.12}
\]
From (3.12) we have
\[ \Phi (X, \xi) = 0. \]  
(3.13)

In an \((\varepsilon)-\)para Sasakian manifold the following equations hold for any \(X, Y, Z \in \Gamma(T\tilde{M})\) [15]:
\[
\tilde{R} (X, Y) \xi = \eta (X) Y - \eta (Y) X, \quad (3.14)
\]
\[
\tilde{R} (X, Y, Z, \xi) = - \varepsilon \eta (X) \tilde{g} (Y, Z) + \varepsilon \eta (Y) \tilde{g} (X, Z), \quad (3.15)
\]
\[
\eta (\tilde{R} (X, Y) Z) = - \varepsilon \eta (X) \tilde{g} (Y, Z) + \varepsilon \eta (Y) \tilde{g} (X, Z), \quad (3.16)
\]
\[
\tilde{R} (\xi, X) Y = - \varepsilon \tilde{g} (X, Y) \xi + \eta (Y) X, \quad (3.17)
\]
\[
\tilde{S} (Y, \xi) = - (n - 1) \eta (Y), \quad (3.18)
\]
where \(\tilde{R}\) is the Riemannian curvature tensor and \(S\) is the Ricci tensor of \(\tilde{M}\).

4 Lightlike hypersurfaces of \((\varepsilon)-\)para Sasakian manifolds

Let \((\tilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon)\) be an \((n + 2)\)-dimensional \((\varepsilon)\)-para Sasakian manifold and \(M\) be a lightlike hypersurface of \(\tilde{M}\), such that the structure vector field \(\xi\) is tangent to \(M\). Since \(\xi\) is a non-null vector field, it belongs to the screen distribution \(S(TM)\). If \(\text{index} (\tilde{g}) = 1\), in order that \(M\) is a lightlike hypersurface, it is necessary that the structure vector field \(\xi\) must be a spacelike vector field, that is, \(\tilde{M}\) must be a spacelike para Sasakian manifold. If \(\text{index} (\tilde{g}) > 1\), then \(\tilde{M}\) may also be a timelike para Sasakian manifold.

For local sections \(E\) and \(N\) of \(\text{Rad} TM\) and \(\text{ltr} TM\), respectively, in view of (3.7), we have
\[
\eta (E) = 0, \quad \eta (N) = 0. \nonumber
\]
From (3.5), it is easy to see that \(\phi E\) and \(\phi N\) are lightlike vector fields and
\[
\phi^2 E = E, \quad \phi^2 N = N. \nonumber
\]
Now, for \(X \in \Gamma(TM)\), we write
\[
\phi X = \varphi X + u (X) N, \quad \text{(4.1)}
\]
where \(\varphi X \in \Gamma(TM)\) and
\[
u (X) = \tilde{g} (\phi X, E) = \tilde{g} (X, \phi E). \quad \text{(4.2)}
\]

**Proposition 4.1** Let \((\tilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon)\) be an \((n + 2)\)-dimensional \((\varepsilon)\)-para Sasakian manifold and \(M\) be a lightlike hypersurface of \(\tilde{M}\), such that the structure vector field \(\xi\) is tangent to \(M\). Then we have
\[
\tilde{g} (\phi E, E) = 0, \quad \text{(4.3)}
\]
\[
\tilde{g} (\phi E, N) = \varepsilon g (A_N E, \xi), \quad \text{(4.4)}
\]
where \(E\) is a local section of \(\text{Rad} TM\) and \(N\) is a local section of \(\text{ltr} TM\).
Proof. From (3.10) and (2.14), we get (4.3). By using (3.10), (2.4) and (2.7) we obtain
\[ \varepsilon \tilde{g}(\phi E, N) = \tilde{g}(\tilde{\nabla}_E \xi, N) = -\tilde{g}(\xi, \tilde{\nabla}_EN) = g(A_N E, \xi), \]
which implies (4.4). ■

Remark 4.2 From (4.3) we see that there is no component of \( \phi E \) in \( ltr \ TM \), thus \( \phi E \in \Gamma(TM) \). Moreover, (4.4) implies that there may be a component of \( \phi E \) in \( Rad TM \). Thus, in view of (2.8) and (4.3), we observe that
\[ \phi E = \varphi E = P \phi E + \theta(\phi E)E. \] (4.5)

Proposition 4.3 Let \( (\widetilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon) \) be an \((n+2)\)-dimensional \((\varepsilon)\)-almost paracontact metric manifold and \( M \) be a lightlike hypersurface of \( \widetilde{M} \) such that the structure vector field \( \xi \) is tangent to \( M \). Then we have
\[ g(X, \varphi Y) = g(\varphi X, Y) + (u \wedge \theta)(X, Y), \] (4.6)
\[ g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y) - u(X)\theta(\varphi Y) - u(Y)\theta(\varphi X) \] (4.7)
for any \( X, Y \in \Gamma(TM) \)

Proof. From (4.1) and (4.2), we get
\[ \tilde{g}(\phi X, Y) = g(\varphi X, Y) + u(X)\theta(Y). \]
Hence in view of (3.6) we get (4.6). Using (4.1) we have
\[ \tilde{g}(\phi X, \phi Y) = g(\varphi X, \varphi Y) + u(X)\theta(\varphi Y) + u(Y)\theta(\varphi X). \] (4.8)
Thus by using (4.8) and (3.5) we complete the proof. ■

Corollary 4.4 Let \( (\widetilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon) \) be an \((n+2)\)-dimensional \((\varepsilon)\)-almost paracontact metric manifold and \( M \) be a lightlike hypersurface of \( \widetilde{M} \), such that the structure vector field \( \xi \) is tangent to \( M \). Then we have
\[ g(\xi, \varphi X) = 0, \quad X \in \Gamma(TM). \]

Proposition 4.5 Let \( (\widetilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon) \) be an \((n+2)\)-dimensional \((\varepsilon)\)-para Sasakian manifold and \( M \) be a lightlike hypersurface of \( \widetilde{M} \), such that the structure vector field \( \xi \) is tangent to \( M \). Then, for any \( X \in \Gamma(TM) \) we have
\[ \varphi^2 X = X - \eta(X)\xi - u(\varphi X)N - u(X)\phi N, \] (4.9)
\[ \varphi X = \varepsilon \nabla_X \xi; \] (4.10)
\[ B(X, \xi) = \varepsilon u(X). \] (4.11)

Proof. From (4.1) and (3.1), we get (4.9). Next, from (3.10), (2.6) and (4.1) we have
\[ \varepsilon \nabla_X \xi + \varepsilon B(X, \xi)N = \varphi X + u(X)N. \]
Then by equating the tangential and the transversal parts in the previous equation we get (4.10) and (4.11), respectively. ■
5 Invariant lightlike hypersurfaces

We begin with the following

**Definition 5.1** Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon)$ be a $(n+2)$-dimensional an $(\varepsilon)$-almost paracontact metric manifold and $M$ be a lightlike hypersurface of $\tilde{M}$. If $\phi(S(TM)) = S(TM)$, then $M$ will be called an invariant lightlike hypersurface of $\tilde{M}$.

**Example 5.2** Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system $(x, y, z, t, s)$. Defining

$$\eta = ds - ydx - t dz, \quad \xi = \frac{\partial}{\partial s},$$

$$\phi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \quad \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y},$$

$$\phi\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z} + t\frac{\partial}{\partial s}, \quad \phi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \quad \phi\left(\frac{\partial}{\partial s}\right) = 0,$$

$$\tilde{g} = -(dx)^2 - (dy)^2 + (dz)^2 + (dt)^2 + (ds)^2 - t (dz \otimes ds + ds \otimes dz) - y (dx \otimes ds + ds \otimes dx),$$

the set $(\phi, \xi, \eta, g)$ is a spacelike $(\varepsilon)$-almost paracontact structure with index$(g) = 3$ on $\mathbb{R}^5$. Consider a hypersurface $M$ of $\mathbb{R}^5$ given by $y = t$. It is easy to check that $M$ is a lightlike hypersurface whose radical distribution $Rad TM$ is spanned by

$$E = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}.$$

Then the lightlike transversal vector bundle $ltr TM$ is spanned by

$$N = \frac{1}{2}\left(-\frac{\partial}{\partial y} + \frac{\partial}{\partial t}\right),$$

and the screen bundle $S(TM)$ is spanned by

$$\{U_1, U_2, \xi\},$$

where $U_1 = \frac{\partial}{\partial x}$ and $U_2 = \frac{\partial}{\partial z}$. We easily check that

$$\phi E = -E, \quad \phi N = -N.$$

Thus $M$ is a invariant lightlike hypersurface of $\mathbb{R}^5$.

In the following we give a characterization of an invariant lightlike hypersurface.

**Theorem 5.3** Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon)$ be an $(\varepsilon)$-almost paracontact metric manifold. Then $M$ is an invariant lightlike hypersurface of $\tilde{M}$ if and only if

$$\phi Rad TM = Rad TM \quad \text{and} \quad \phi ltr TM = ltr TM.$$
Proof. Let \( M \) be an invariant lightlike hypersurface of \( \tilde{M} \). From (4.5), for any \( X \in \Gamma(TM) \), we get \( g(P\phi E, PX) = 0 \), that is, there is no component of \( \phi E \) in \( S(TM) \) and \( \phi Rad TM = Rad TM \). For any local section \( N \) of \( ltr TM \), we can write

\[
\phi N = P\phi N + \tilde{g}(\phi N, N) E + \tilde{g}(\phi N, E) N. \tag{5.1}
\]

From (5.1), for any \( X \in \Gamma(TM) \), we get \( g(P\phi N, PX) = 0 \), that is, there is no component of \( \phi N \) in \( S(TM) \). If we apply \( \phi \) to (5.1), then we get

\[
2\tilde{g}(\phi N, N) \tilde{g}(\phi N, E) = 0.
\]

Since \( \ker \phi = \text{Span} \{\xi\} \), we obtain \( \tilde{g}(\phi N, N) = 0 \). Thus we get \( \phi N = \tilde{g}(\phi N, E) N \), that is \( \phi ltr TM = ltr TM \).

Conversely, let \( \phi Rad TM = Rad TM \) and \( \phi ltr TM = ltr TM \). For any \( X \in \Gamma(S(TM)) \) we have

\[
\tilde{g}(\phi X, E) = \tilde{g}(X, \phi E) = 0;
\]

thus there is no component of \( \phi X \) in \( ltr TM \). Similarly, we get

\[
\tilde{g}(\phi X, N) = \tilde{g}(X, \phi N) = 0,
\]

which implies that there is no component of \( \phi X \) in \( Rad TM \). This completes the proof. \( \blacksquare \)

**Theorem 5.4** Let \( (\tilde{M}, \phi, \xi, \eta, \tilde{g}, \epsilon) \) be an \((\epsilon)\)-almost paracontact metric manifold. Let \( M \) be an invariant lightlike hypersurface of \( \tilde{M} \). Then \((M, \varphi, \xi, \eta, g, \epsilon)\) is an \((\epsilon)\)-almost paracontact metric manifold.

**Proof.** Let \( M \) be an invariant lightlike hypersurface of \( \tilde{M} \). Let us assume that \( X, Y \in \Gamma(TM) \). From (4.1), we get

\[
\phi X = \varphi X. \tag{5.2}
\]

Using (3.1) and (5.2), we have

\[
\varphi^2 X = X - \eta(X)\xi. \tag{5.3}
\]

Also from (5.2), it follows that

\[
\varphi \xi = 0. \tag{5.4}
\]

Next, in view of (5.3) and (5.4) one can easily see that

\[
\eta \circ \varphi = 0,
\]

\[
\eta(\xi) = 1.
\]

Moreover, from (4.7) we have

\[
g(\varphi X, \varphi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y).
\]

This completes the proof. \( \blacksquare \)

**Proposition 5.5** Let \( M \) be an invariant lightlike hypersurface of an \((\epsilon)\)-para Sasakian manifold \((\tilde{M}, \phi, \xi, \eta, \tilde{g}, \epsilon)\). Then we have

\[
g(A_N PX, \xi) = 0, \quad X \in \Gamma(TM).
\]
Proof. Since $g(\xi, N) = 0$, using (3.10), we get

$$\tilde{g}\left(\tilde{\nabla}_X N, \xi\right) = -\varepsilon \tilde{g}(N, \phi X).$$

From (2.7), we have the assertion of the proposition. ■

**Theorem 5.6** An invariant lightlike hypersurface of an $(\varepsilon)$-para Sasakian manifold is always $(\varepsilon)$-para Sasakian. Moreover,

$$B(X, \varphi Y)N - B(X, Y)\phi N = 0,$$

(5.5)

$$\varphi(A_N X) = A_{\phi N} X - \theta(X)\xi$$

(5.6)

for all $X, Y \in \Gamma(TM)$.

Proof. We have

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y = \phi(\tilde{\nabla}_X Y)
= \tilde{\nabla}_X \phi Y - \phi(\nabla_X Y + B(X, Y)N)
= \nabla_X \varphi Y + B(X, \varphi Y)N - \varphi \nabla_X Y - B(X, Y)\phi N
= (\nabla_X \varphi)Y + B(X, \varphi Y)N - B(X, Y)\phi N,$$

which in view of (3.9) gives

$$-g(\varphi X, \varphi Y)\xi - \varepsilon \eta(Y)\varphi^2 X = (\nabla_X \varphi)Y + B(X, \varphi Y)N - B(X, Y)\phi N.$$  \hspace{1cm} (5.7)

Equating tangential parts in (5.7) provides

$$(\nabla_X \varphi)Y = -g(\varphi X, \varphi Y)\xi - \varepsilon \eta(Y)\varphi^2 X,$$ \hspace{1cm} (5.8)

In view of (5.8) and Theorem 5.4 we see that $M$ is $(\varepsilon)$-para Sasakian. Equating transversal parts in (5.7) yields (5.5).

Next, using (3.9) and (2.7) we have

$$-\theta(X)\xi = (\tilde{\nabla}_X \phi)N = \tilde{\nabla}_X \phi N - \phi(\tilde{\nabla}_X N)
= -A_{\phi N} X + \tilde{g}\left(\tilde{\nabla}_X \phi N, E\right)N + \phi(A_N X) - \tau(X)\phi N.$$

In the last equation, if we equate the tangential parts, we get

$$-\theta(X)\xi = -A_{\phi N} X + \varphi(A_N X).$$

This completes the proof. ■

**Remark 5.7** It is well-known that, if there exists a lightlike hypersurface in an $(\varepsilon)$-Sasakian manifold, then the dimension of the Sasakian manifold must be greater than or equal to 5. But in the case of an $(\varepsilon)$-paracontact metric manifold there is no such restriction on the dimension of the ambient manifold for the existence of lightlike hypersurfaces.
6 Screen semi-invariant lightlike hypersurfaces

We begin with the following:

**Definition 6.1** Let \((\widetilde{M}, \phi, \xi, \eta, \widetilde{g}, \varepsilon)\) be a \((n + 2)\)-dimensional \((\varepsilon)\)-almost paracontact metric manifold and \(M\) be a lightlike hypersurface of \(\widetilde{M}\). If \(\phi \text{Rad } TM \subset S(TM)\) and \(\phi \text{ltr } TM \subset S(TM)\), then \(M\) will be called a screen semi-invariant lightlike hypersurface of \(\widetilde{M}\).

**Example 6.2** Let \(\mathbb{R}^5\) be the 5-dimensional real number space with a coordinate system \((x, y, z, t, s)\). We define

\[
\eta = \frac{1}{2} (z dx + t dy + ds), \quad \xi = 2 \frac{\partial}{\partial s},
\]

\[
\phi X = -X_3 \frac{\partial}{\partial x} - X_4 \frac{\partial}{\partial y} - X_1 \frac{\partial}{\partial z} - X_2 \frac{\partial}{\partial t} + (X_3 z + X_4 t) \frac{\partial}{\partial s},
\]

\[
\widetilde{g} = -\frac{1}{4} (dx \otimes dx + dz \otimes dz) + \frac{1}{4} (dy \otimes dy + dt \otimes dt) + \eta \otimes \eta.
\]

Here \(X\) is a vector field given by

\[
X = X_1 \frac{\partial}{\partial x} + X_2 \frac{\partial}{\partial y} + X_3 \frac{\partial}{\partial z} + X_4 \frac{\partial}{\partial t} + X_5 \frac{\partial}{\partial s}.
\]

Then \((\phi, \xi, \eta, \widetilde{g})\) is a spacelike almost \((\varepsilon)\)-paracontact structure on \(\mathbb{R}^5\). We note that \(\text{index}(\widetilde{g}) = 2\).

Now consider a hypersurface \(M\) given by

\[
t = z.
\]

Then the tangent bundle \(TM\) of \(M\) is spanned by

\[
\left\{ U_1 = \frac{\partial}{\partial x}, \ U_2 = \frac{\partial}{\partial y}, \ U_3 = \frac{\partial}{\partial z} + \frac{\partial}{\partial t}, \ U_4 = \frac{\partial}{\partial s} \right\},
\]

and \(\text{Rad } TM\) is spanned by \(E = U_3\). Also the lightlike transversal vector bundle is

\[
\text{ltr } TM = \text{Span} \left\{ N = 2 \left( -\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right) \right\}.
\]

Furthermore

\[
\phi E = - \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + (z + t) \frac{\partial}{\partial s} \in \Gamma (S(TM)),
\]

\[
\phi N = 2 \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + (t - z) \frac{\partial}{\partial s} \right) \in \Gamma (S(TM)).
\]

Thus \(M\) is a screen semi-invariant lightlike hypersurface of \(\mathbb{R}^5\).
Let $M$ be a screen semi-invariant lightlike hypersurface of a $(n + 2)$-dimensional $(\varepsilon)$-almost paracontact metric manifold $\tilde{M}$. We set

$$V = \phi E \quad \text{and} \quad U = \phi N.$$  \hfill (6.1)

Then, from the second equation of (2.4) and (3.5), we obtain

$$g(V, U) = 1.$$  \hfill (6.2)

Therefore $\langle V \rangle \oplus \langle U \rangle$ is a non-degenerate vector subbundle of $S(TM)$ of rank 2. Since $\xi$ belongs to $S(TM)$ and

$$g(V, \xi) = g(U, \xi) = 0,$$

therefore there exists a non-degenerate distribution $D_0$ of rank $n - 3$ on $M$ such that

$$S(TM) = D_0 \perp \{\langle V \rangle \oplus \langle U \rangle\} \perp \langle \xi \rangle.$$  \hfill (6.3)

We note that $D_0$ is an invariant distribution with respect to $\phi$, that is, $\phi D_0 = D_0$. Denoting

$$D = D_0 \perp \text{Rad TM} \perp \langle V \rangle \quad \text{and} \quad D' = \langle U \rangle,$$

we have

$$TM = D \oplus D' \perp \langle \xi \rangle.$$  \hfill (6.4)

Thus, every $X \in \Gamma(TM)$ can be expressed as

$$X = RX + QX + \eta(X) \xi,$$

where $R$ and $Q$ are the projections of $TM$ into $D$ and $D'$, respectively. Hence, we may write

$$\phi X = \phi RX, \quad X \in \Gamma(TM).$$

From (3.2), (4.1) and (4.2), we obtain

$$\phi^2 X = \phi^2 X + u(X) U + u(\phi X) N.$$  \hfill (6.5)

By comparing the tangential and transversal parts in (6.5) we get

$$\phi^2 = I - \eta \otimes \xi - u \otimes U,$$  \hfill (6.6)

$$u \circ \phi = 0,$$  \hfill (6.7)

respectively. Next, from (3.3) one can easily see that

$$\phi \xi = 0 \quad \text{and} \quad u(\xi) = 0.$$  \hfill (6.8)

Since $\phi^2 N = N$, by using (4.1) we also have

$$\phi U = 0 \quad \text{and} \quad u(U) = 1.$$  \hfill (6.9)

Furthermore, from (3.4) we have

$$\eta(U) = 0.$$  \hfill (6.10)

Finally, we get

$$(\eta \circ \phi) X = \eta(\phi X - u(X) N),$$

which gives

$$\eta \circ \phi = 0.$$  \hfill (6.11)

Thus we have the following
Proposition 6.3 Let $M$ be a screen semi-invariant lightlike hypersurface of an $(\varepsilon)$-almost paracontact metric manifold $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon)$. Then $M$ possesses a para $(\varphi, \xi, \eta, U, u)$-structure, that is,

\[
\varphi^2 = I - \eta \otimes \xi - u \otimes U, \quad \varphi \xi = 0, \quad \varphi U = 0, \quad \eta \circ \varphi = 0,
\]

\[
u \circ \varphi = 0, \quad \eta(\xi) = 1, \quad u(U) = 1, \quad \eta(U) = 0, \quad u(\xi) = 0.
\]

Next, we have the following:

Theorem 6.4 Let $M$ be a screen semi-invariant lightlike hypersurface of an $(\varepsilon)$-para Sasakian manifold $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \varepsilon)$. Then we have

\[
(\nabla_X \varphi) Y = u(Y) A_N X + B(X, Y) U
- g(X, Y) \xi + 2\varepsilon \eta(X) \eta(Y) \xi - \varepsilon \eta(Y) X,
\]

(6.12)

\[
(\nabla_X \varphi) Y = u(Y) A_N X + B(X, Y) U
- (g(\varphi X, \varphi Y) + u(X)\theta(\varphi Y) + u(Y)\theta(\varphi X)) \xi
- \varepsilon \eta(Y) (\varphi^2 X + u(X) U),
\]

(6.13)

\[
(\nabla_X u) Y = -B(X, \varphi Y) - u(Y) \tau(X),
\]

(6.14)

\[
\nabla_X U = -\varphi(A_N X) + \tau(X) U,
\]

(6.15)

\[
B(X, U) = -u(A_N X)
\]

(6.16)

for all $X, Y \in \Gamma(TM)$.

Proof. We have

\[
(\tilde{\nabla}_X \phi) Y = (\nabla_X \varphi) Y - u(Y) A_N X - B(X, Y) U
+ \{ (\nabla_X u) Y + u(Y) \tau(X) + B(X, \varphi Y) \} N,
\]

(6.17)

where (4.1), (2.6), (2.7) and (6.1) are used. Next, from (3.9) we have

\[
(\tilde{\nabla}_X \phi) Y = -g(X, Y) \xi + 2\varepsilon \eta(X) \eta(Y) \xi - \varepsilon \eta(Y) X, \quad X, Y \in \Gamma(TM).
\]

(6.18)

Using (4.8), (6.5) and (6.7) in (3.9), for all $X, Y \in \Gamma(TM)$, we also have

\[
(\tilde{\nabla}_X \phi) Y = - (g(\varphi X, \varphi Y) + u(X)\theta(\varphi Y) + u(Y)\theta(\varphi X)) \xi
- \varepsilon \eta(Y) (\varphi^2 X + u(X) U).
\]

(6.19)

From (6.17) and (6.18) we get (6.12). Similarly, from (6.17) and (6.19) we get (6.13). Next, taking transversal part in (6.17) to be zero, we get (6.14).

Using (2.6), (2.7) and (4.1) we get

\[
(\tilde{\nabla}_X \phi) N = \nabla_X U + \varphi(A_N X) - \tau(X) U
+ (B(X, U) + u(A_N X)) N.
\]

(6.20)

Since, from (3.9) we have

\[
(\tilde{\nabla}_X \phi) N = 0,
\]

(6.21)
then using (6.20) and 6.21 we have
\[ 0 = \nabla_X U + \varphi(A_N X) - \tau(X) U + (B(X, U) + u(A_N X))N. \]

By equating the tangential and transversal parts in the previous equation we get (6.15) and (6.16), respectively. ■

**Proposition 6.5** Let \( M \) be a screen semi-invariant lightlike hypersurface of an \((\varepsilon)\)-para Sasakian metric manifold \((\tilde{M}, \phi, \xi, \eta, \tilde{\eta}, \varepsilon)\). Then for any \( X, Y \in \Gamma(TM) \), the Lie derivative of \( g \) with respect to the vector field \( V \) is given by
\[ (\mathcal{L}_V g) = X(u(Y)) + Y(u(X)) + u([X, Y]) - 2u(\nabla_X Y). \]  

**Proof.** We have
\[ (\mathcal{L}_Z \tilde{g})(X, Y) = (\mathcal{L}_Z g)(X, Y), \quad X, Y, Z \in \Gamma(TM). \]  

Hence, using \( V = \tilde{\varphi}E \) and \( \tilde{g}((\nabla_X \tilde{\varphi})E, Y) = 0 \) we have
\[ (\mathcal{L}_V g)(X, Y) = (\mathcal{L}_V \tilde{g})(X, Y) = \tilde{g}(\nabla_X E, \tilde{\varphi}Y) + \tilde{g}(\tilde{\varphi}X, \nabla_Y E). \]  

Next,
\[ \tilde{g}(\nabla_X E, \phi Y) = \tilde{g}(\nabla_X E, \varphi Y + u(Y)N) \]
\[ = \tilde{g}(\nabla_X E, \varphi Y) + u(Y)\tilde{g}(\nabla_X E, N) \]
\[ = -\tilde{g}(E, \nabla_X \varphi Y) - u(Y)\tilde{g}(E, \nabla_X N) \]
\[ = -B(X, \varphi Y) - u(Y)\tau(X) \]
\[ = (\nabla_X u)Y, \]

which gives
\[ \tilde{g}(\nabla_X E, \phi Y) = X(u(Y)) - u(\nabla_X Y). \]  

Using (6.25) in (6.24) we complete the proof. ■

### 6.1 Integrability of \( D \perp \langle \xi \rangle \)

We note that \( X \in \Gamma(D \perp \langle \xi \rangle) \) if and only if \( u(X) = 0 \). Now from (6.14), we have for all \( X, Y \in \Gamma(TM) \)
\[ u(\nabla_X Y) = \nabla_X u(Y) + B(X, \varphi Y) + u(Y)\tau(X), \]
from which we get
\[ u(X, Y] = B(X, \varphi Y) - B(\varphi X, Y) + \nabla_X u(Y) - \nabla_Y u(X) + u(Y)\tau(X) - u(X)\tau(Y). \]

Now, let \( X, Y \in \Gamma(D \perp \langle \xi \rangle) \). Then \( u(X) = 0 = u(Y) \), and from the previous equation we get
\[ u(X, Y] = B(X, \varphi Y) - B(\varphi X, Y) \]
for all \( X, Y \in \Gamma(D \perp \langle \xi \rangle) \). Thus we get a necessary and sufficient condition for the integrability of the distribution \( D \perp \langle \xi \rangle \) in the following:

**Theorem 6.6** Let \( M \) be a screen semi-invariant lightlike hypersurface of an \((\varepsilon)\)-para Sasakian manifold \((\tilde{M}, \phi, \xi, \eta, \tilde{\eta}, \varepsilon)\). Then the distribution \( D \perp \langle \xi \rangle \) is integrable if and only if
\[ B(X, \varphi Y) = B(\varphi X, Y), \quad X, Y \in \Gamma(D). \]
6.2 Integrability of $D' \perp \langle \xi \rangle$

Here we find a necessary and sufficient condition for the distribution $D' \perp \langle \xi \rangle$ to be integrable.

**Theorem 6.7** Let $M$ be a screen semi-invariant lightlike hypersurface of an $(\varepsilon)$-para Sasakian manifold $(\widetilde{M}, \phi, \xi, \eta, \mathcal{F}, \varepsilon)$. Then the distribution $D' \perp \langle \xi \rangle$ is integrable if and only if

$$A_N \xi + \varepsilon U = 0.$$  \hspace{1cm} (6.26)

**Proof.** Note that $X \in \Gamma(D' \perp \langle \xi \rangle)$ if and only if $\varphi X = 0$. Now for all $X, Y \in \Gamma(TM)$, in view of (6.12), we have

$$\varphi (\nabla_X Y) = \nabla_X \varphi Y - u (Y) A_N X - B(X, Y) U + g(X, Y) \xi - 2 \varepsilon \eta (X) \eta (Y) \xi + \varepsilon \eta (Y) X.$$

From the above equation we get

$$\varphi [X, Y] = \nabla_X \varphi Y - \nabla_Y \varphi X + u (X) A_N Y - u (Y) A_N X + \varepsilon \eta (Y) X - \varepsilon \eta (X) Y.$$

In particular, for $X, Y \in \Gamma(D' \perp \langle \xi \rangle)$ we get

$$\varphi [X, Y] = u (X) A_N Y - u (Y) A_N X + \varepsilon \eta (Y) X - \varepsilon \eta (X) Y. \hspace{1cm} (6.27)$$

But $D'$ and $\langle \xi \rangle$ are integrable, hence $D' \perp \langle \xi \rangle$ is integrable if and only if

$$\varphi [U, \xi] = 0,$$

which, in view of (6.27), is equivalent to (6.26). □

**References**

[1] J.K. Beem and P.E. Ehrlich, *Global Lorentzian geometry*, Marcel Dekker, New York, 1981.

[2] A. Bejancu and K.L. Duggal, *Real hypersurfaces of indefinite Kaehler manifolds*, Internat. J. Math. Math. Sci. 16 (1993), no. 3, 545-556.

[3] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhauser Boston, Inc., Boston, MA, 2002.

[4] K.L. Duggal, *Space time manifolds and contact structures*, Internat. J. Math. Math. Sci. 13 (1990), no. 3, 545–553.

[5] K.L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and its applications*, Kluwer, Dordrecht 1996.

[6] K.L. Duggal and D.H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific Publishing Co. Pvt. Ltd., 2007.
[7] K.L. Duggal and B. Sahin, *Lightlike submanifolds of indefinite Sasakian manifolds*, Int. J. Math. Math. Sci. 2007, Art. ID 57585, 21 pp.

[8] K.L. Duggal and B. Sahin, *Differential geometry of lightlike submanifolds*, Birkhäuser, 2010.

[9] T.H. Kang, S.D. Jung, B.H. Kim, H.K. Pak and J.S. Pak, *Lightlike hypersurfaces of indefinite Sasakian manifolds*, Indian J. Pure Appl. Math. 34 (2003), no. 9, 1369-1380.

[10] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. 12 (1989), no. 2, 151-156.

[11] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, 1983.

[12] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure I*, Tôhoku Math. J. 12 (1960), 459-476.

[13] I. Satô, *On a structure similar to the almost contact structure*, Tensor (N.S.) 30 (1976), no. 3, 219-224.

[14] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Tôhoku Math. J. 21 (1969), 644-653.

[15] M.M. Tripathi, E. Kılıç, S. Yüksel Perktaş and S. Keleş, *Indefinite almost paracontact metric manifolds*, Int. J. Math. Math. Sci. 2010 (2010), Art. Id. 846195, pp. 19.

[16] S. Yüksel Perktaş, E. Kılıç, M.M. Tripathi and S. Keleş, *On $(\varepsilon)$-para Sasakian 3-manifolds*, Int. J. Pure Appl. Math. 77 (2012), no.4, 485-499.