The static Kottler metric is the Schwarzschild vacuum metric extended to include a cosmological constant. Angular momentum is added to the Kottler metric by using Newman and Janis’ complexifying algorithm. The new metric is the $\Lambda$ generalization of the Kerr spacetime. It is stationary, axially symmetric, Petrov type $\text{II}$, and has Kerr-Schild form.

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I. INTRODUCTION

The cosmological constant, originally included in the field equations to create a static universe, is now included in models of the universe to drive acceleration. How the cosmological constant effects more local scenarios, both static and stationary, is also of interest. The 1918 Kottler solution, $g_{Kot}^{\alpha\beta}$, describes matter from cosmological constant $\Lambda$ around a spherical source of mass $m_0$, creating a $\Lambda$ generalization of the Schwarzschild vacuum solution. The Kottler metric is

$$g_{Kot}^{\alpha\beta} dx^\alpha dx^\beta = (1 - 2m_0/r - \Lambda r^2/3)du^2 + 2dudr - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2).$$

(1)

When Einstein’s field equations are extended to include $\Lambda g_{\alpha\beta}$, Birkhoff’s theorem extends to the Kottler solution as the unique consequence of spherical symmetry [1]. Kottler reduces to the vacuum Schwarzschild solution when $\Lambda = 0$. The Kottler spacetime is an Einstein space with Ricci tensor

$$R_{Kot}^{\alpha\beta} = \Lambda g_{Kot}^{\alpha\beta}.$$  

(2)

Chrusciel and Simon [2] have discussed the classification of static spherically symmetric solutions with $\Lambda$. For $m_0 = 0$ the spacetime is either de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$). Its conformal structure has been studied by Lake and Roeder [3]. The Kottler spacetime with $m_0 \neq 0$ is not cosmological since it is spherically symmetric rather than LRS. Examining geodesic orbits shows the $\Lambda$ term corresponds to a repulsive ($\Lambda > 0$) central force of magnitude $(\Lambda/3)r$. $\Lambda$ contributes to the periapsis precession of geodesic orbits but does not effect gravitational lensing [4]. This spacetime is Petrov type D, with $\Psi_2 = -m_0/r^3$, the only non-zero Weyl tensor component. Only the Schwarzschild mass contributes to the Weyl tensor, which explains why $\Lambda$ has no lensing effect. With $m_0 \neq 0$, one interpretation of $\Lambda$ in the field equations $G_{Kot}^{\alpha\beta} = -8\pi T_{Kot}^{\alpha\beta} = -\Lambda g_{Kot}^{\alpha\beta}$ is a vacuum energy-momentum. In the Newtonian limit $\Lambda$ is recognized as a uniform mass density

$$\rho_\Lambda = -\Lambda/4\pi.$$  

(3)

An upper limit [5] for $\Lambda \approx 2 \times 10^{-35}$ sec$^{-2}$ is equivalent to a vacuum density of $\rho_\Lambda \approx 2 \times 10^{-45}$ gm/cm$^3$. We will focus on $\Lambda$ as a background atmosphere.

With $m_0 = 0$ there is a trapped surface (horizon) at $R_h = \sqrt{3/\Lambda}$. With mass, the presence of the $\Lambda$- atmosphere modifies the horizon structure even in the static case, the
positions being given by the roots of the cubic function \(1 - 2m_0/r - \Lambda r^2/3\). While horizons in both Schwarzschild and Kottler have constant radial coordinates, the Kottler spacetime has two horizons for the mass range of the observed black holes. Using \(\Lambda \sim 10^{-51} m^{-2}\), these are black holes with \(m_0 < 10^{25} m\). One of the horizons occurs in the vicinity of the Schwarzschild horizon and the other at larger \(r\). As an example, consider the black hole in \(M87\) containing about a billion suns. For a black hole of this size, the Kottler horizons will occur at \(r \sim 2m_0\) and \(r \sim R_h \sim 5.5 \times 10^{26} m\). The radius of this horizon is bigger than a 13 billion year old universe expanding at the speed of light. As angular momentum is added, the horizon structure will change.

### A. Generalizations

The Newman-Janis (NJ) transformation has been used recently to obtain the AdS\(_3\) rotating black hole solution from the non-rotating black hole solution of Bañados, Teitelboim, and Zanelli. Drake and Turolla have used the NJ transformation to generate interior solutions for the Kerr spacetime, focusing on bounded interiors that cannot be written in Kerr-Schild form. Yazadjiev has applied the transformation to the rotating dilaton–axion black hole.

Carter discovered a \(\Lambda\)– generalization of the vacuum Kerr solution which preserves the Einstein space property. Carter’s solution has been studied by Stuchlík et al. Demianski found a \(\Lambda\)– generalization of the NUT solution by considering a complex coordinate transformation

\[
r' = r + iF(\theta, \phi) \\
u' = u + iG(\theta, \phi)
\]

of metric

\[
ds^2 = M(r)du^2 + 2dudr - r^2d\Omega^2
\]

with functions \(F(\theta, \phi), G(\theta, \phi), M(r)\) determined from the field equations with only \(\Lambda g_{\alpha\beta}\) matter content. Vaidya discovered a vacuum generalization of the Kerr solution which he called "the associated Kerr metric". Vaidya’s solution is related to Demianski’s. The complex NJ transformation generates the vacuum Kerr spacetime from
the vacuum Schwarzschild solution. It is a special case of Demianski's transformation with $F = -G = \cos \theta$.

An NJ transformation of the curvature tensor was done by Quevedo [18], producing curvature tensors which satisfy the Einstein-Maxwell equations with $\Lambda$.

B. Our generalization

In this work we discuss the NJ spinning generalization (which we believe is a new solution) of the static Kottler spacetime, its matter content, and trapped surface position. The transformation can be used to investigate how $\Lambda$ will affect stationary generalizations of the Kottler spacetime with no assumptions about the resulting matter content. The matter content will follow directly from the field equations.

II. NJ APPLIED TO KOTTLER

A basis set of null vectors for $g^{\text{Kot}}_{\alpha\beta}$ is

$$l^\alpha \partial_\alpha = \partial_r,$$

$$n^\alpha \partial_\alpha = \partial_u - (1/2)(1 - 2m_0/r - \Lambda r^2/3)\partial_r,$$

$$m^\alpha \partial_\alpha = (1/\sqrt{2})(1/r)[\partial_\theta + (i/\sin \vartheta)\partial_\varphi],$$

The complex coordinate transformation of the NJ algorithm is

$$u' = u - ia \cos \vartheta,$$

$$r' = r + ia \cos \vartheta,$$

$$\vartheta' = \vartheta, \quad \varphi' = \varphi.$$

Note that there are no arbitrary functions in the transformation, only the parameter '$a$'. The transformed null tetrad is (with the primes omitted from the new coordinates)

$$L^\alpha \partial_\alpha = \partial_r,$$

$$N^\alpha \partial_\alpha = \partial_u - (1/2)(1 - 2m_0r/\Sigma - \Lambda \Sigma/3)\partial_r,$$

$$M^\alpha \partial_\alpha = \frac{1}{\sqrt{2}(r + ia \cos \vartheta)} \left[ \partial_\theta + \frac{i}{\sin \vartheta} \partial_\varphi + ia \sin \vartheta (\partial_u - \partial_r) \right],$$

$$\bar{M}^\alpha \partial_\alpha = \frac{1}{\sqrt{2}(r - ia \cos \vartheta)} \left[ \partial_\theta - \frac{i}{\sin \vartheta} \partial_\varphi - ia \sin \vartheta (\partial_u - \partial_r) \right].$$
Here Σ = \( r^2 + a^2 \cos^2 \vartheta \). This transformed tetrad yields a new metric

\[
g_{\alpha\beta}^{spK} = L^\alpha N^\beta + N^\alpha L^\beta - M^\alpha \bar{M}^\beta - \bar{M}^\alpha M^\beta
\]  

(7)

or, with \( \psi_\Lambda = 1 - 2m_0 r/\Sigma - (\Lambda/3)\Sigma \), and \( R^2_\Lambda = \Sigma + (2 - \psi_\Lambda)a^2 \sin^2 \vartheta \)

\[
g_{\alpha\beta}^{spK} dx^\alpha dx^\beta = \psi_\Lambda du^2 + 2du dr + (1 - \psi_\Lambda)a \sin^2 \vartheta 2du d\varphi
\]

\[- a \sin^2 \vartheta 2dr d\varphi - \Sigma d\vartheta^2 - R^2_\Lambda \sin^2 \vartheta d\varphi^2.
\]  

(8)

The new spacetime is stationary and axially symmetric with Killing vectors \( \partial_u \) and \( \partial_\varphi \).

Since the Kerr metric has form \( [\psi = 1 - (2m_0 r)/\Sigma] \)

\[
g_{\alpha\beta}^{Kerr} dx^\alpha dx^\beta = \psi du^2 + 2du dr + (1 - \psi)a \sin^2 \vartheta 2du d\varphi
\]

\[- a \sin^2 \vartheta 2dr d\varphi - \Sigma d\vartheta^2 - [\Sigma + (2 - \psi)a^2 \sin^2 \vartheta] \sin^2 \vartheta d\varphi^2.
\]  

spinning Kottler is related to Kerr by

\[
g_{\alpha\beta}^{spK} = g_{\alpha\beta}^{Kerr} - (\Lambda/3)\Sigma L_\alpha L_\beta
\]  

(9)

III. MATTER CONTENT

A. Energy-momentum

The matter content of the spinning Kottler spacetime is abstracted from the transformed Einstein tensor with no assumptions. The Einstein tensor has components

\[
G_{\alpha\beta}^{spK} = -4\Phi_{11} (L_\alpha N_\beta + N_\alpha L_\beta) - 2\Phi_{22} L_\alpha L_\beta - \Lambda g_{\alpha\beta}^{spK}.
\]  

(10)

with

\[
\Phi_{11} = -(\Lambda/3)a^2 \cos^2 \vartheta/\Sigma,
\]  

(11)

\[
\Phi_{22} = -(\Lambda/3)a^2(1 - 3 \cos^2 \vartheta)/\Sigma.
\]

For \( G_{\mu\nu} = -8\pi T_{\mu\nu} \), the energy-momentum content may be interpreted as three fluids: the first fluid (with \( \Lambda g_{\alpha\beta}^{spK} \)) can be described as an isotropic cloud of strings, the second is a two-dimensional rotating string fluid with density \( \rho = 4\Phi_{11}/8\pi \) and pressure \( p = -4\Phi_{11}/8\pi \), \( \rho + p = 0 \). The third is a null fluid with magnitude \( 2\Phi_{22}/8\pi \). The matter content imposed by Demianski is only one of the three fluids that appear in \( g_{\alpha\beta}^{spK} \). This particular decomposition is, of course, not unique. Using the techniques of Coley and Tupper \[19\] one could describe the fluid in a number of formally equivalent ways.
B. Angular momentum

The Komar superpotential for metric $g^{spK}$ with axial Killing vector $k^\alpha \partial_\alpha = \partial_\phi$ is

$$U^{\alpha\beta} = (-g)^{1/2} k^{[\alpha} \partial^{\beta]}$$

(12)

where $(-g)^{1/2} = \Sigma \sin \vartheta$. Since $g^{spK}$ is stationary, and the rotation axis is a regular line in the spacetime, one can integrate the Komar superpotential on a two-surface of constant '$u'$ and '$r$', which bounds any $u = const$ three-volume ($3V$). The angular momentum is given by

$$J^{spK} = \oint_{\partial(3V)} U^{\alpha\beta}(\partial_\phi) u [u^r,\partial_\beta] d\Omega d\phi.$$  

(13)

We find

$$J^{spK} = -16\pi ma + 16\pi \left(\frac{\Lambda}{3}\right) \frac{4a^3 r}{15}.$$  

(14)

The standard Kerr angular momentum remains when $\Lambda \to 0$.

IV. GEOMETRIC STRUCTURE

Geometrically, the spinning Kottler spacetime is algebraically special, Petrov type II, and the rotation axis $[\theta : 0, 2\pi]$ is a regular line in the spacetime. The relation of the spK spacetime to Kerr is reflected in its generalized Kerr-Schild form (see Appendix C)

$$g^{spK}_{\mu\nu} = g^{Kerr}_{\mu\nu} - (\Lambda/3) L_\mu L_\nu$$

(15)

The Ricci tensor is

$$R^{spK}_{\alpha\beta} = -2\Phi_{11}(L_\alpha N_\beta + N_\alpha L_\beta + M_\alpha \bar{M}_\beta + \bar{M}_\alpha M_\beta) - 2\Phi_{22}L_\alpha L_\beta + (R/4)g^{spK}_{\alpha\beta}$$

(16)

with $\Phi_{11}$ and $\Phi_{22}$ given in Eq.(11). The Ricci scalar $\mathcal{R}$ (not to be confused with $R = r - ia \cos \vartheta$) is

$$\mathcal{R}/4 = (\Lambda/3)(3r^2 + a^2 \cos^2 \vartheta)/\Sigma.$$  

The non-zero Weyl tensor components are

$$\Psi_2 = (\Lambda/3) \frac{2a^2 \cos^2 \vartheta}{\Sigma} - \frac{m_0}{R^3},$$  

(17)

$$\Psi_3 = \frac{a \sin \vartheta}{\sqrt{2R}} \left[ (\Lambda/3) \frac{2a \cos \vartheta}{R} + i \frac{3m_0}{R^3} \right],$$  

(18)

$$\Psi_4 = \frac{a^2 \sin^2 \vartheta}{R^2} \left[ -(\Lambda/3) \frac{\Sigma}{R^2} + \frac{3m_0}{R^3} \right].$$  

(19)
The Kretschmann invariant is
\[
R_{\alpha\beta\mu\nu}^{\text{spK}} R_{\alpha\beta\mu\nu}^{\text{spK}} = \frac{8}{\Sigma_6} \left[ (\Lambda/3)^2 \Sigma^4 (3\Sigma^2 + a^4 \cos^4 \vartheta - 4r^2 a^2 \cos^2 \vartheta) 
+ 8(\Lambda/3) m_0 r \Sigma^2 a^2 \cos^2 \vartheta (3a^2 \cos^2 \vartheta - r^2) 
+ 6m_0^2 (\Sigma - 2a^2 \cos^2 \vartheta)(\Sigma^2 - 16r^2 a^2 \cos^2 \vartheta) \right].
\] (20)

The limit \( \Lambda \to 0 \) yields the Kretschmann scalar for the vacuum Kerr solution. The quadratic Ricci scalar is
\[
R_{\alpha\beta}^{\text{spK}} R_{\alpha\beta}^{\text{spK}} = 4\Lambda^2 \left[ 1 - \frac{4a^2 \cos^2 \vartheta}{9\Sigma^2} (3r^2 + a^2 \cos^2 \vartheta) \right].
\] (21)

This scalar has directionality, unlike Carter’s in Eq.(A2).

The ergosphere and horizon structure of the spinning Kottler and Kerr spacetimes are quite interesting to compare. To locate an horizon we search for trapped surfaces. Using the method of Senovilla [20] (see Appendix B) to identify the location of the marginally trapped surface, whose time history is the apparent horizon, one finds
\[
(\Lambda/3)(r^2 + a^2 \cos^2 \vartheta)^2 - \Delta = 0
\] (22)

where \( \Delta = r^2 - 2m_0 r + a^2 \). The solutions to this equation provide an interesting contrast to the Kerr horizon. For Kerr \( (\Lambda = 0) \) this equation has solutions for \( a \leq m_0 \). When \( a = m_0 \) there is a single horizon position, \( r = m_0 \). With the \( \Lambda \) field added, the extreme angular momentum value changes and the value of ‘\( a \)’ can increase. For Kerr, the horizon position are the points were \( \Delta \) cuts the \( r \) axis. For spinning Kottler, the positions are the multiple intersections of \( \Delta \) with the first term in equation (22). The presence of the cosmological constant changes the structure of the extreme rotating solution. The ergosurfaces bound the trapped surfaces and are found at
\[
(\Lambda/3)(r^2 + a^2 \cos^2 \vartheta)^2 - r^2 + 2m_0 r - a^2 \cos^2 \vartheta = 0.
\] (23)

The outer trapped surface touches the outer ergosurface at \( \vartheta = 0 \). At \( \vartheta = \pi/2 \), the spK ergosurface has the same equation and roots as the Kottler trapped surface
\[
g^K_{\text{ot}}(u) = 1 - 2m_0/r - (\Lambda/3)r^2 = 0.
\]

In the vacuum Kerr solution outside a black hole, specific angular momentum \((J/m_0)_{\text{ker}} \) is restricted by \( m_0 \geq a \). This restriction preserves the existence of a trapped surface inside
the ergosphere. For values of \( a \) beyond the extreme black hole limit, a naked singularity is present in the Kerr solution. This limit appears to change when the cosmological constant is included, and may possess implications for microscopic black holes.

V. DISCUSSION

In this paper we have described the NJ-\( \Lambda \) generalization of Kerr spacetime. The matter content is spinning and reduces to the Kerr vacuum in the \( \Lambda = 0 \) limit. The spinning Kottler (spK) metric is similar to the Kerr-Newman (KN) metric, and differs from KN only in the functional form of \( \psi \)

\[
\text{KN: } \psi_Q = 1 - 2m_0 r / \Sigma + Q^2 / \Sigma \\
\text{spK: } \psi_\Lambda = 1 - 2m_0 r / \Sigma - (\Lambda / 3) \Sigma.
\]

The spK metric differs from Demianski’s NUT generalization in several ways. Demianski’s solution (and the NUT solution) has a wire singularity on the rotation axis. The axis is not singular in the spK solution. The spK metric reduces to Kerr in the \( \Lambda = 0 \) limit while Demianski’s solution goes to NUT in this limit. The NJ spin up of Kottler is also different from Carter’s \( \Lambda \) generalization of Kerr. Carter’s spacetime contains only a ”\( \Lambda \) fluid”, \( \Lambda g^{\text{Cart}} \), while \( g^{\text{spK}} \) has more complex fluid content. That the two spacetimes are different is most clearly seen by comparing the quadratic Ricci invariants, Eq. (21) and Eq. (A2).

Some interesting questions remain to be examined. Future work could include a stability analysis and an examination of geodesic orbits for this metric. Spinning Kottler presents a new equilibrium spacetime for future study.

APPENDIX A: CARTER’S SOLUTION

In Boyer-Lindquist coordinates, Carter’s metric \[10\] is

\[
g^{\text{Cart}}_{\alpha \beta} dx^\alpha dx^\beta = \frac{1}{I^2} [1 - \frac{\Lambda}{3} (r^2 + a^2 \sin^2 \vartheta) - \frac{2mr}{\Sigma} dt^2 - \frac{\Sigma}{\Delta} dr^2 \\
+ \frac{\sin^2 \vartheta A}{I^2} \frac{\Lambda}{3} (r^2 + a^2) + \frac{2mr}{\Sigma} ] dt d\varphi + \frac{\Sigma}{\Delta \vartheta} d\vartheta^2 \\
- \frac{\sin^2 \vartheta}{I^2} [\Sigma + \frac{\Lambda}{3} a^2 (r^2 + a^2) + a^2 \sin^2 \vartheta (1 + 2mr / \Sigma)] d\varphi_{B-L}.
\]
\[ \Delta_r = (1 - \frac{\Lambda}{3} r^2)(r^2 + a^2) - 2mr, \quad \Delta_\vartheta = (1 + \frac{\Lambda}{3} a^2 \cos^2 \vartheta), \]
\[ I = 1 + \frac{\Lambda}{3} a^2, \quad \Sigma = r^2 + a^2 \cos^2 \vartheta \]

(In the Black Holes article, Carter has \(-\Lambda\) of \(\Lambda\) here). The limit \(\Lambda \to 0\) gives the Kerr solution in Boyer-Lindquist coordinates. The limit \(a \to 0\) gives the Kottler solution. The Ricci tensor is
\[ R^\text{Cart}_{\alpha\beta} = \Lambda g^\text{Cart}_{\alpha\beta} \] (A1)
with invariant square
\[ R^\text{Cart}_{\alpha\beta} R^\text{Cart}_{\alpha\beta} = 4\Lambda^2 \] (A2)
Carter’s metric can be mapped to \((u, r, \vartheta, \varphi)\) coordinates by
\[ dt = du + \frac{I(r^2 + a^2)}{\Delta_r} dr, \quad d\varphi = Ia \frac{1}{\Delta_r} dr \]
\[ g^\text{Cart}_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{I^2} \left[ 1 - \frac{\Lambda}{3} (r^2 + a^2 \sin^2 \vartheta) - \frac{2mr}{\Sigma} \right] du^2 + \frac{1}{I^2} 2 du dr 
+ \frac{2 \sin^2 \vartheta}{I^2} \left[ \frac{\Lambda}{3} (r^2 + a^2) + \frac{2mr}{\Sigma} \right] d\vartheta 
- \frac{2 \sin^2 \vartheta}{I \Delta_r} \left[ \frac{\Lambda}{3} (r^2 + a^2)(1 - r^2) - 2mr \right] d\varphi - \frac{\Sigma}{\Delta_\vartheta} d\vartheta^2 
- \frac{\sin^2 \vartheta}{I} \left[ \Sigma + \frac{\Lambda}{3} a^2 (r^2 + a^2) + a^2 \sin^2 \vartheta (1 + 2mr/\Sigma) \right] d\varphi^2 \]

**APPENDIX B: TRAPPED SURFACE**

We follow the notion of a trapped surface given in Hawking and Ellis \[21\].

For the spacetime pair of manifold \(\mathcal{M}\) and metric \(g\), let \(S\) be a two-dimensional surface with intrinsic coordinates \(\{x^A : A, B = 2, 3\}\) imbedded in \(\mathcal{M}\) by parametric equations \(x^\alpha = F^\alpha(x^A)\). Tangent vectors of \(S\) are
\[ \vec{e}_A := \left[ \frac{\partial F^\alpha}{\partial x^A} \frac{\partial}{\partial x^\alpha} \right]_S \]
with first fundamental form
\[ \gamma_{AB} = \left[ \frac{\partial F^\mu}{\partial x^A} \frac{\partial F^\nu}{\partial x^B} g_{\mu\nu} \right]_S . \]
Assume $\gamma_{AB}$ is negative definite so that $S$ is spacelike. There exist two independent null one-forms $\omega_\mu^{(\pm)}$ on $S$ such that

$$\omega_\mu^{(\pm)} e_A^\mu = 0, \quad \omega_\mu^+ \omega_-^\mu = 0, \quad \omega_\mu^- \omega_-^\mu = 0, \quad \omega_\mu^+ \omega_-^\mu = 1,$$

with scale freedom

$$\omega_\mu^+ \rightarrow \tilde{\omega}_\mu^+ = f^2 \omega_\mu^+, \quad \omega_-^\mu \rightarrow \tilde{\omega}_- ^\mu = f^{-2} \omega_- ^\mu$$

for positive function $f^2$. The two second fundamental forms on $S$ are

$$K_{AB}^{(\pm)} = -\omega_\mu^{(\pm)} e_A^\nu \nabla_\nu e_B^\mu$$

with traces $K^{(\pm)} := \gamma^{AB} K_{AB}^{(\pm)}$ where $\gamma^{AC} \gamma_{CB} = \delta^A_B$. The traces allow a "trapping scalar" $\kappa$ to be defined.

$$\kappa := 2K^{(+)} K^{(-)}.$$  (B1)

The traces are the expansions of the two null geodesic congruences emerging orthogonally from $S$. Thus, $S$ is trapped if both congruences converge.

S is

- \begin{cases} 
  \text{trapped} & \kappa > 0 \\
  \text{marginally trapped} & \kappa = 0 \\
  \text{non-trapped} & \kappa < 0 
\end{cases}

1. **Kottler Trapped Surface Location**

First consider the static Kottler trapped surface:

$$\partial_u \cdot \partial_u = 1 - 2m_0/r - (\Lambda/3)r^2 = 0.$$  (B2)

For $\Lambda < 0$ there is one real root $r_0$ such that $0 < r_0 < 2m_0$. With $\Lambda > 0$ there are three cases.

1. For $3m_0 > \Lambda^{-1/2}$ there is one real root $< 0$.
2. For $3m_0 = \Lambda^{-1/2}$ there are three real roots

$$r_1 = r_2 = \Lambda^{-1/2} > 0, \quad r_3 = -2\Lambda^{-1/2}.$$  

3. For $3m_0 < \Lambda^{-1/2}$ there are three distinct roots ($r_4, r_5, r_6$)

$$0 < 2m_0 < r_4 < 3m_0 < r_5 < r_6 < 0.$$
2. spK TRAPPED SURFACE LOCATION

Since the roots of a quartic locate the spK trapped surfaces, we use *Descartes’ Rule of Signs* \[22\] to enumerate the roots: The number of positive roots of equation \( f(x) = 0 \) with real coefficients is equal to the number of sign changes in the polynomial \( f(x) \) or is less than that number by a positive integer. The number of negative roots is equal to the number of sign changes in \( f(-x) \) or is less than that number by a positive integer. A root of multiplicity \( n \) is counted as \( n \) roots.

We examine the quartic \( \Delta_\Lambda = 0 \) at \( \vartheta = 0 \).

\[
\left(\frac{\Lambda}{3}\right)r_H^4 + (2\frac{\Lambda}{3}a^2 - 1)r_H^2 + 2m_0r_H + (\frac{\Lambda}{3}a^2 - 1)a^2 = 0 \tag{B3}
\]

(\( \Lambda > 0 \))

(i) \( \Lambda > 3/a^2 \), polynomial signs \([+ + + +]\). No sign changes \( \Rightarrow \) no roots.

(ii) \( \Lambda > 3/(2a^2) \), polynomial signs \([+ + + −]\). One sign change \( \Rightarrow \) one real root.

For \( \Lambda = 3/a^2 \) the one real root is

\[
r_H = (m_0a^2)^{1/3}b_0^{1/3} - (a^2/3)(m_0a^2)^{-1/3}b_0^{-1/3}, \quad b_0 = \sqrt{1 + a^2/(3m_0^2) - 1} \tag{B4}
\]

(\( \Lambda < 0 \)) polynomial signs \([− − + −]\). Two sign changes \( \Rightarrow \) two real roots or no roots.

The quartic \( \Delta_\Lambda = 0 \) at \( \vartheta = \pi/2 \) is

\[
(\Lambda/3)r_H^4 - r_H^2 + 2m_0r_H - a^2 = 0 \tag{B5}
\]

(\( \Lambda > 0 \))

(i) \( f(r_H) \), polynomial signs \([+ − + −]\). Three sign changes \( \Rightarrow \) three real roots or one root

(ii) \( f(−r_H) \), polynomial signs \([+ − − −]\). One sign change \( \Rightarrow \) one real root.

For \( \Lambda = 1/(4a^2) \) the four roots are

\[
r_{H1,H2} = m_0(h_2 \pm h_3), \quad \tag{B6}
\]

\[
r_{H3,H4} = m_0(−h_2 \pm h_3)
\]
with

\[ h_1 = \left( \frac{1}{3}(a/m_0)^4 - \frac{8}{27}(a/m_0)^6 \right)^{1/3}, \]

\[ h_2 = \left[ 3h_1 + 2(a/m_0)^2 \right]^{1/2}, \]

\[ h_3 = \left[ 4(a/m_0)^2 - 3h_1 - 6(a/m_0)^2/h_2 \right]^{1/2}. \]

\( (\Lambda < 0) \) polynomial signs \([- - + -]\). Two sign changes \(\Rightarrow\) two real roots or no roots.

The differences are clearly seen in the graph below, a plot of \((x-1)^2\) and \((\Lambda m_0^2/3)x^4\) versus \(x\) \((x = r/m_0)\), \(\Lambda m_0^2/3 = 0.1\). With no cosmological constant, the horizon positions would be the intersection of the decreasing graph with the \(x\) axis at \(x = 1\), \(r = m_0\). With a cosmological constant, the trapped surface positions are the multiple intersections of the two curves. For the parameter chosen, the four intersections occur at \(x = -6.34, 0.86, 1.32\) and 4.16. Only two of the intersections are shown in the graph. As \(\Lambda\) decreases, the outermost trapped surface moves to larger values. For \(\Lambda m_0^2/3 = .001\), the positive intersections are \(x = .98, 1.02, 53.8\).

![Graph showing trapped surface positions](image)

**Figure 1.** Trapped surface positions

### APPENDIX C: KERR-SCHILD FORM

Consider \( \hat{g}_{\alpha\beta} = g_{\alpha\beta} + H(x^\mu)l_\alpha l_\beta \), where \( l^\alpha \) is null with respect to both metrics, \( l^\alpha l^\beta g_{\alpha\beta} = l^\alpha l^\beta \hat{g}_{\alpha\beta} = 0 \). Xanthopoulos’ generalized superposition theorem states:

If \( G_{\mu\nu}(g) = 0 \) and \( D_\gamma G_{\mu\nu}(H l l) = 0 \), then \( G_{\mu\nu}(\hat{g}) = 0 \). \hfill (C1)
Here $D_g G_{\mu\nu}$ represents the first functional derivative of the Einstein tensor, i.e. Einstein’s equations linearized about $g$. The generalized Kerr-Schild form of $g^{spK}$ is vacuum to non-vacuum. See Sopuerta and references therein.

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