STRONGLY SEQUENTIALLY SEPARABLE FUNCTION SPACES,
VIA SELECTION PRINCIPLES

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Abstract. A separable space is strongly sequentially separable if, for each countable dense set, every point in the space is a limit of a sequence from the dense set. We consider this and related properties, for the spaces of continuous and Borel real-valued functions on Tychonoff spaces, with the topology of pointwise convergence. Our results solve a problem stated by Gartside, Lo, and Marsh.

1. Introduction

We apply methods of selection principles to a problem of Gartside, Lo, and Marsh [6, Problem 19].

By space we mean a Tychonoff topological space. A space is Fréchet–Urysohn if each point in the closure of a set is a limit of a sequence from the set. A separable space is strongly sequentially separable (SSS) [9] if, for each countable dense set, every point in the space is a limit of a sequence from the dense set. Every separable Fréchet–Urysohn space is strongly sequentially separable, but not conversely [1, Example 2.4].

For a space \( X \), let \( C(X) \) and \( B(X) \) be the spaces of continuous and Borel, respectively, real-valued functions on \( X \), with the topology of pointwise convergence. We are only concerned with uncountable spaces. In this case, the space \( B(X) \) is never Fréchet–Urysohn. Indeed, for an uncountable space, the constant function \( 1 \) is in the closure of the set of characteristic functions of finite subsets of the space and there is no sequence in the set converging to \( 1 \).

Strong sequential separability is hereditary for separable dense subspaces. Thus, if the space \( C(X) \) is separable, we have the following implications.

\[
\mathbb{R}^X \text{ is SSS} \rightarrow B(X) \text{ is SSS} \rightarrow C(X) \text{ is SSS} \rightarrow C(X) \text{ is Fréchet–Urysohn}
\]

It is consistent that the properties in this diagram hold only for countable spaces \( X \) and are, thus, equivalent [6, Corollary 17]. This motivates the following problem [6, Problem 19].

Problem 1 (Gartside–Lo–Marsh). Is there, consistently, a space \( X \) such that the space \( C(X) \) is strongly sequentially separable but not Fréchet–Urysohn, and the space \( \mathbb{R}^X \) is not strongly sequentially separable?

We solve this problem, and all other problems suggested by the above diagram. To this end, we extend Arhangel’skii’s local-to-global duality, dualize these problems to ones concerning covering properties, and apply the theory of selection principles.
2. Local-to-global Duality

A cover of a space is a family of proper subsets whose union is the entire space. For families $A$ and $B$ of covers of a space, the property that every cover in the family $A$ has a subcover in the family $B$ is denoted $(\frac{A}{B})$. An $\omega$-cover is a cover such that each finite subset of the space is contained in some set from the cover. A $\gamma$-cover is an infinite cover such that each point of the space belongs to all but finitely many sets from the cover.

An open cover is a cover by open sets. Similarly, we define Borel cover, clopen cover, etc. Given a space, let $\Omega$, $\Omega_{\mathrm{ctbl}}$, $\Omega_{\mathrm{coz}}$, $\Omega_{\mathrm{Bor}}$ and $\Gamma$, be the families of open $\omega$-covers, countable open $\omega$-covers, countable cozero $\omega$-covers, countable Borel $\omega$-covers, and $\gamma$-covers, respectively.

The property $(\Omega_{\Gamma})$ is the celebrated $\gamma$-property of Gerlits and Nagy, who proved that a space has this property if and only if the space $C(X)$ is Fréchet–Urysohn [7, Theorem 2].

**Lemma 2.** Let $X$ be a space with a coarser second countable topology. The following assertions are equivalent:

1. The space $B(X)$ is strongly sequentially separable.
2. The space $X$ has the property $(\Omega_{\Gamma})$.

**Proof.** $(1) \Rightarrow (2):$ Since the space $X$ has a coarser second countable topology, there is a countable dense set $H$ in the space $C(X)$ [12, Theorem 1] and hence $H$ is dense in the space $B(X)$. Let $U \in \Omega_{\mathrm{Bor}}(X)$. For a Borel set $U \subseteq X$ and a function $h \in H$, let $f_{U,h} \in B(X)$ be the function such that $f_{U,h} | U := h | U$ and $f_{U,h} | (X \setminus U) := 1$. The set $D := \{ f_{U,h} : U \in \mathcal{U}, h \in H \}$ is a countable dense subset of $B(X)$. By (1), there is a sequence $\{ f_{U_n,h_n} : n \in \mathbb{N} \}$ in the set $D$, converging to the zero function 0. Let $F$ be a finite subset of $X$. The set $W := \{ f \in B(X) : f[F] \subseteq (-1,1) \}$ is a neighborhood of 0 in $B(X)$. For a natural number $n_i$, if $f_{U_n,h_n} \in W$, then $F \subseteq U_n$. Since all but finitely many elements of the sequence belong to the set $W$, we have $\{ U_n : n \in \mathbb{N} \} \in \Gamma(X)$. Thus, the space $X$ satisfies $(\Omega_{\Gamma})$.

$(2) \Rightarrow (1):$ The property $(\Omega_{\Gamma})$ implies that every point in the closure of a countable set in $B(X)$ is the limit of a sequence from that set [13, Lemma 2.8].

Let $\mathbb{N}$ be the set of natural numbers. For infinite sets $a,b \subseteq \mathbb{N}$ we write $a \subseteq^* b$ if the set $a \setminus b$ is finite. A pseudointersection of a family of infinite sets is an infinite set $a$ with $a \subseteq^* b$ for all sets $b$ in the family. A subfamily of $[\mathbb{N}]^\infty$ is centered if the finite intersections of its elements, are infinite. Let $p$ be the minimal cardinality of a family of infinite subsets of $\mathbb{N}$ that is centered and has no pseudointersection. The hypothesis $\aleph_1 < p$ and its negation ($\aleph_1 = p$) are both consistent [1, Theorem 5.1]. Information about the cardinal number $p$ is available, for example, in van Douwen’s survey [3].

Gartside, Lo and Marsh proved that a Tychonoff product $\mathbb{R}^X$ is strongly sequentially separable if and only if $|X| < p$ [6, Theorem 11]. They also proved that a function space $C(X)$ is strongly sequentially separable if and only if the space $X$ has a coarser second countable topology, and every coarser second countable topology for $X$ satisfies $(\Omega_{\Gamma})$ [6, Theorem 16]. The property $(\Omega_{\Gamma})$ implies $(\Omega_{\Gamma}^{\mathrm{coz}})$. Bonanzinga, Cammaroto and Matveev proved that a space $X$ has the property $(\Omega_{\Gamma}^{\mathrm{coz}})$ if and only if every coarser second countable topology for the space $X$ has the property $(\Omega_{\Gamma})$ [2, Theorem 54]. In summary, for spaces $X$ with a coarser second countable topology, the diagram from the previous section dualizes to the following
one.

\[ |X| < p \implies X \text{ satisfies } (\Omega_{\text{Bor}}_{\Gamma}) \implies X \text{ satisfies } (\Omega_{\text{coz}}_{\Gamma}) \\]

\[ X \text{ satisfies } (\Omega_{\Gamma}) \]

Problem 3 is thus reduced to the following problem.

**Problem 3.** Is there, consistently, a space \( X \) with a coarser second countable topology, that satisfies \((\Omega_{\text{coz}}_{\Gamma})\) but not \((\Omega_{\Gamma})\), with \(|X| \geq p\)?

We will solve this problem, as well as its variations.

### 3. The problems and their solutions

Consider the positions in the diagrams from the previous section. Write there “•” if the property holds, and “◦” if it does not. For example, sets \( X \subseteq \mathbb{R} \) of cardinality smaller than \( p \) realize the following setting.

\[ \bullet \implies \bullet \implies \bullet \]

that will be denoted \( \bullet \bullet \bullet \) for brevity. We consider the consistency of all settings that are not ruled out by the implications in the diagram. These are the following settings:

\[ \circ \circ \circ \quad \circ \circ \bullet \quad \circ \bullet \bullet \quad \bullet \circ \circ \quad \bullet \bullet \circ \quad \bullet \circ \bullet \quad \bullet \bullet \bullet \]

Problem 3 asks whether either of the the settings \( \circ \circ \bullet \) or \( \circ \circ \circ \) is consistent.

The following proposition is a variation of an earlier result [15, Example 4.7].

**Proposition 4 \( (\bullet \bullet \bullet) \).** The following assertions are equivalent:

1. There is a space \( X \) such that the space \( \mathbb{R}^X \) is strongly sequentially separable, but the space \( C(X) \) is not Fréchet–Urysohn.
2. \( \aleph_1 < p \).

**Proof.** Recall that the space \( \mathbb{R}^X \) is strongly sequentially separable if and only if \(|X| < p\).

(1) \( \implies \) (2): The given space \( X \) has \(|X| < p\). Had it been countable, the space \( C(X) \) would have been metrizable.

(2) \( \implies \) (1): A discrete space of cardinality \( \aleph_1 \) is not Lindelöf, and thus does not satisfy \((\Omega_{\Gamma})\). Apply duality.

The following folklore fact implies that discrete spaces of cardinality \( p \) or greater have none of the studied properties.

**Lemma 5 \( (\circ \circ \circ) \).** Let \( X \) be a set. Then \(|X| < p\) if and only if every countable \( \omega \)-cover consisting of subsets of \( X \) contains a \( \gamma \)-cover.
Proof. (⇒) Let \( \{ U_n : n \in \mathbb{N} \} \) be a countable \( \omega \)-cover consisting of subsets of \( X \). For each element \( x \in X \), let \( a_x := \{ n \in \mathbb{N} : x \in U_n \} \), an infinite subset of \( \mathbb{N} \). Since \( |X| < p \), the family \( \{ a_x : x \in X \} \) has a pseudointersection \( a \). Then \( \{ U_n : n \in a \} \) is a \( \gamma \)-cover of \( X \).

(⇐) Assume that \( |X| \geq p \). We may assume that \( X \subseteq [N]^\infty \) where \([N]^\infty\) is the family of infinite subsets of \( \mathbb{N} \). Then \( X \) is a family of infinite subsets of \( \mathbb{N} \) of cardinality \( p \), that is centered and has no pseudointersection. The family of sets \( \{ U_n : n \in \mathbb{N} \} \), defined by \( U_n := \{ x \in X : n \in x \} \) for natural numbers \( n \), is a countable \( \omega \)-cover of \( X \) and has no subfamily in \( \Gamma \). \( \square \)

A theorem of Galvin and Miller [5, Theorem 2] asserts that, if \( p = |\mathbb{R}| \), then there is a set \( X \subseteq \mathbb{R} \) of cardinality \( p \), satisfying (\( \Omega \)). The Galvin–Miller Theorem is refined by Theorem of Orenshtein and Tsaban [14, Theorem 3.6]. Since this result is central to the remainder of this paper, we include here a simpler proof, due to the third named author [20].

We identify the Cantor space \( \{0,1\}^\mathbb{N} \) with the family \( P(\mathbb{N}) \) of all subsets of the set \( \mathbb{N} \). Thus, we view the space \( P(\mathbb{N}) \) as a subset of the real line. The space \( P(\mathbb{N}) \) splits into two subspaces: the family of infinite subsets of \( \mathbb{N} \), denoted \([N]^\infty\), and the family of finite subsets of \( \mathbb{N} \), denoted \( \text{Fin} \). We identify every set \( a \in [N]^\infty \) with its increasing enumeration, an element of the Baire space \( \mathbb{N}^\mathbb{N} \). Thus, for a natural number \( n \), \( a(n) \) is the \( n \)-th element in the increasing enumeration of the set \( a \). This way, we have \([N]^\infty \subseteq \mathbb{N}^\mathbb{N} \), and the topology of the space \([N]^\infty \) (a subspace of the Cantor space \( P(\mathbb{N}) \)) coincides with the subspace topology induced by \( \mathbb{N}^\mathbb{N} \). When an element of \([N]^\infty \) is viewed as an element of \( \mathbb{N}^\mathbb{N} \), we refer to it as a function.

For functions \( a, b \in [N]^\infty \), we write \( a \leq^* b \) if the set \( \{ n : b(n) < a(n) \} \) is finite. Let \( A \subseteq [N]^\infty \). For a function \( b \in [N]^\infty \), we write \( A \leq^* b \) if \( a \leq^* b \) for all functions \( a \in A \). The set \( A \) is unbounded if there is no function \( b \in [N]^\infty \) with \( A \leq^* b \). Let \( b \) be the minimal cardinality of an unbounded set in \([N]^\infty \). A set \( \{ x_\alpha : \alpha < b \} \subseteq [N]^\infty \) is an unbounded tower if it is unbounded and for all ordinal numbers \( \alpha, \beta < b \) with \( \alpha < \beta \), we have \( x_\alpha \nsubseteq x_\beta \). An unbounded tower of cardinality \( p \) exists if (and only if) \( p = b \) [14, Lemma 3.3].

**Theorem 6** (Orenshtein–Tsaban). For each unbounded tower \( T \subseteq [N]^\infty \) of cardinality \( p \), the set \( T \cup \text{Fin} \) of real numbers satisfies (\( \Omega \)).

In order to prove Theorem 6 we need the following notions and auxiliary results. Let \( n, m \) be natural numbers with \( n < m \). Define \( (n, m) := \{ i \in \mathbb{N} : n < i < m \} \). A set \( a \in [N]^\infty \) omits the interval \((n, m)\) if \( a \cap (n, m) = \emptyset \). For a space \( X \), let \( \Omega(X) \) be the family of all open \( \omega \)-covers of \( X \), and \( \Gamma(X) \) be the family of all open \( \gamma \)-covers of \( X \).

**Lemma 7** (Galvin–Miller [5, Lemma 1.2]). Let \( \mathcal{U} \) be a family of open sets in \( P(\mathbb{N}) \) such that \( \mathcal{U} \in \Omega(\text{Fin}) \). There are a function \( b \in [N]^\infty \) and distinct sets \( U_1, U_2, \ldots \in \mathcal{U} \) such that for each element \( x \in [N]^\infty \) and all natural numbers \( n \):

\[ \text{If } x \cap (b(n), b(n + 1)) = \emptyset, \text{ then } x \in U_n. \]

**Lemma 8** (Folklore [19, Lemma 2.13]). Let \( Y \) be a subset of \([N]^\infty\). The set \( Y \) is unbounded if and only if, for each function \( b \in [N]^\infty \), there is a set \( a \in Y \) that omits infinitely many intervals \((b(n), b(n + 1))\).

**Lemma 9**. Let \( X \subseteq P(\mathbb{N}) \) be a set such that \( \text{Fin} \subseteq X \) and \( |X| < p \). Let \( \mathcal{U} \) be a family of open sets in \( P(\mathbb{N}) \) such that \( \mathcal{U} \in \Omega(X) \), and \( Y \) be an unbounded set in \([N]^\infty\). There are a set \( a \in Y \), and sets \( U_1, U_2, \ldots \in \mathcal{U} \) such that \( \{ U_n : n \in \mathbb{N} \} \in \Gamma(X) \), and for each element
Let $x \in [\mathbb{N}]^\infty$ and all natural numbers $n$:

If $x \setminus \{1, \ldots, n\} \subseteq a$, then $x \in \bigcap_{k \geq n} U_k$.

Proof. Since $|X| < p$, the set $X$ satisfies (\%) [16 Proposition 2]. Let $\mathcal{V} \in \Gamma(X)$ be a subfamily of $\mathcal{U}$. By Lemma 8, there is a function $b \in [\mathbb{N}]^\infty$, and distinct sets $V_1, V_2, \ldots \in \mathcal{V}$ such that for each element $x \in [\mathbb{N}]^\infty$, and all natural numbers $i$:

(1) If $x \cap (b(i), b(i + 1)) = \emptyset$, then $x \in V_i$.

By Lemma 8 there is a set $a \in Y$ such that the set

$$c := \{i \in \mathbb{N} : a \cap (b(i), b(i + 1)) = \emptyset\}$$

is infinite. Fix a natural number $n$. Let $k$ be a natural number with $n \leq k$, and $x \in [\mathbb{N}]^\infty$ be an element such that $x \setminus \{1, \ldots, n\} \subseteq a$. Then $n \leq c(k)$, and we have

$$x \cap (b(c(k)), b(c(k) + 1)) \subseteq a \cap (b(c(k)), b(c(k) + 1)) = \emptyset.$$

By (1), we have $x \in V_{c(k)}$. Thus, $x \in \bigcap_{k \geq n} V_{c(k)}$.

Since $\mathcal{V} \in \Gamma(X)$, we have $\{V_{c(i)} : i \in N\} \in \Gamma(X)$. \hfill \Box

Proof of Theorem 7. Let $\{x_\alpha : \alpha < b\} \subseteq [\mathbb{N}]^\infty$ be an unbounded tower. Let $X := \text{Fin} \cup \{x_\alpha : \alpha < b\}$, and for ordinal numbers $\gamma < b$, let $X_\gamma := \text{Fin} \cup \{x_\alpha : \alpha < \gamma\}$. Let $\mathcal{U} \in \Omega(X)$. Fix an ordinal number $\gamma_0 < b$. By induction, for a natural number $m > 0$, we proceed as follows. By Lemma 9 there are an ordinal number $\gamma_m < b$, and a subfamily $\{U^{(m)}_n : n \in \mathbb{N}\} \in \Gamma(X_{\gamma_m})$ of $\mathcal{U}$ such that, for each element $x \in [\mathbb{N}]^\infty$ and all natural numbers $n$:

(2) If $x \setminus \{1, \ldots, n\} \subseteq x_{\gamma_m}$, then $x \in \bigcap_{k \geq n} U^{(m)}_k$.

Let $\gamma := \sup_n \gamma_n$. There is a function $g \in [\mathbb{N}]^\infty$ such that $x_\gamma \setminus \{1, \ldots, g(n)\} \subseteq x_{\gamma_n}$ for all natural numbers $n$. Fix an ordinal number $\alpha$ with $\gamma \leq \alpha < b$. Since $x_\alpha \subseteq^* x_\gamma$, we have

$$x_\alpha \setminus \{1, \ldots, g(n)\} \subseteq x_\gamma \setminus \{1, \ldots, g(n)\} \subseteq x_{\gamma_n},$$

for all but finitely many natural numbers $n$. By (2), we have $x_\alpha \in \bigcap_{k \geq g(n)} U^{(n)}_k$ for all but finitely many natural numbers $n$. Thus, for any function $h \in [\mathbb{N}]^\infty$ with $g \leq^* h$, we have $\{U^{(n)}_h : n \in \mathbb{N}\} \in \Gamma(\{x_\alpha : \gamma \leq \alpha < b\})$.

For each element $x \in X_\gamma$, and each natural number $n$, define

$$f_x(n) := \min \left\{ m \in \mathbb{N} : x \in \bigcap_{k \geq m} U^{(n)}_k \right\}$$

if the set is nonempty, and $f_x(n) := 0$ otherwise. Since $|X_\gamma| < b$, there is a function $h \in [\mathbb{N}]^\infty$ such that $\{f_x : x \in X_\gamma\} \cup \{g\} \leq^* h$, and the sets $U^{(n)}_{h(n)}$ are distinct. Then $\{U^{(n)}_{h(n)} : n \in \mathbb{N}\} \in \Gamma(X_\gamma)$. Since $\{U^{(n)}_{h(n)} : n \in \mathbb{N}\} \in \Gamma(\{x_\alpha : \gamma \leq \alpha < b\})$ as well, we have $\{U^{(n)}_{h(n)} : n \in \mathbb{N}\} \in \Gamma(X)$. \hfill \Box
4. Subsets of the Real, Michael, and Sorgenfrey line

The Michael line \([10]\) is the set \(\mathbb{P}(\mathbb{N})\), with the topology where the points of the set \([\mathbb{N}]^\omega\) are isolated, and the neighborhoods of the points of the set \(\mathbb{F}\) are those induced by the Cantor space topology on \(\mathbb{P}(\mathbb{N})\). The Sorgenfrey line \([17]\) is the set \(\mathbb{R}\) with the topology generated by the half-open intervals \([a, b)\), for \(a, b \in \mathbb{R}\).

The forthcoming Theorem \([10]\)(2) solves the problem of Gartside–Lo–Marsh Problem (Problem \([1]\)). Recall that an unbounded tower in \([\mathbb{N}]^\omega\) of cardinality \(p\) exists if and only if \(p = b\). It is consistent that \(\aleph_1 < p = b\), e.g., it holds assuming the Martin Axiom with the negation of the Continuum Hypothesis.

**Theorem 10.** Let \(T \subseteq [\mathbb{N}]^\omega\) be an unbounded tower of cardinality \(p\).

1. (\(\circ \circ \ast\)) As a subset of \(\mathbb{R}\), the set \(T \cup F\) satisfies \((\Omega)\) but not \((\Omega_{\text{Bor}})\).
2. (\(\circ \circ \ast\)) Assume that \(\aleph_1 < p\). As a subset of the Michael line, the set \(T \cup F\) satisfies \((\Omega_{\text{ctbl}})\) but neither \((\Omega)\) nor \((\Omega_{\text{Bor}})\).

**Proof.** (1) By Theorem \([6]\) the set \(T \cup F\) satisfies \((\Omega)\). The set \(T\) is centered and has no pseudointersection. Thus, the set \(T\) does not satisfy \((\Omega_{\text{Bor}})\) \([18,\text{Lemma 22, Theorem 27}(1)]\). Since the set \(T\) is a Borel subset of \(T \cup F\), and the property \((\Omega_{\text{Bor}})\) is hereditary for Borel subsets \([18,\text{Theorem 48}]\), the set \(T \cup F\) does not satisfy \((\Omega_{\text{Bor}})\), too.

(2) For a set \(U \subseteq \mathbb{P}(\mathbb{N})\), let \(\text{Int}(U)\) be the interior of the set \(U\) in the Cantor space topology on \(\mathbb{P}(\mathbb{N})\). If \(U \subseteq \Omega(F)\) is a family of open sets in the Michael line, then \(\\{\text{Int}(U) : U \in \mathcal{U}\}\) is Borel subset \([18,\text{Lemma 24}]\). Since the set \(U\) is a countable family of open sets in the Michael line. Consequently, the proof of Theorem \([6]\) actually establishes that the set \(T \cup F\), as a subspace of the Michael line, satisfies \((\Omega_{\text{ctbl}})\).

Write \(T = \{x_\alpha : \alpha < b\}\) with \(x_\alpha \subseteq^* x_\beta\) for \(\beta < \alpha\). The set \(A := \{x \in T : x_\omega \subseteq^* x\}\) has cardinality \(\aleph_1\). The set \(A\) is \(F_{\sigma}\) in the Cantor space topology and, in particular, in the Michael line topology. Thus, the space \(T \cup F\) has an uncountable discrete \(F_{\sigma}\) subset. Since the Lindelöf property is hereditary for \(F_{\sigma}\) subsets, the space \(T \cup F\) is not Lindelöf. Every space with the property \((\Omega)\) is Lindelöf. Thus, the space \(T \cup F\) does not satisfy \((\Omega)\).

By (1), since every Borel set in the Cantor space is also Borel in the Michael line, the space \(T \cup F\) does not satisfy \((\Omega_{\text{Bor}})\). \(\Box\)

**Corollary 11.** Let \(T \subseteq [\mathbb{N}]^\omega\) be an unbounded tower of cardinality \(p\).

1. (\(\circ \circ \ast\)) For the real line topology, the space \(C(T \cup F)\) is Fréchet-Urysohn but the space \(B(T \cup F)\) is not strongly sequentially separable.
2. Assume that \(\aleph_1 < p\). For the Michael line topology, the space \(C(T \cup F)\) is strongly sequentially separable and not Fréchet–Urysohn, and the space \(B(T \cup F)\) is not strongly sequentially separable. \(\Box\)

Assuming the Continuum Hypothesis, there is an uncountable set of real numbers satisfying \((\Omega_{\text{Bor}})\) \([3,\text{Theorem 4.1}, [11,\text{Theorem 5}])\). If \(\aleph_1 < p\), then any subset of real numbers of cardinality \(\aleph_1\) satisfies \((\Omega_{\text{Bor}})\) \([18,\text{Lemma 22, Theorem 27}(1)]\).

**Theorem 12.** Let \(X \subseteq \mathbb{R}\) be an uncountable set satisfying \((\Omega_{\text{Bor}})\).

1. (\(\circ \circ \ast\)) As a subset of \(\mathbb{R}\), the set \(X\) satisfies \((\Omega)\).
2. As a subset of the Sorgenfrey line, the set \(-X \cup X\) satisfies \((\Omega_{\text{Bor}})\) but not \((\Omega)\).
In particular, if the Continuum Hypothesis holds, we obtain the setting \( \Omega \) from (1), and the setting \( \Omega \) from (2). If \( \aleph_1 < p \), we obtain the settings \( \Omega \) and \( \Omega \).

**Proof.** (1) For subsets of \( \mathbb{R} \), the property \( (\Omega_{Bor}^\Gamma) \) implies \( (\Omega) \).

(2) Let \( Y \subseteq \mathbb{R} \) be an uncountable set satisfying \( (\Omega_{Bor}^\Gamma) \). The disjoint union \( Y \sqcup Y \) satisfies \( (\Omega_{Bor}^\Gamma) \) as well: Let \( U \) be a countable Borel \( \omega \)-cover of \( Y \sqcup Y \). The family

\[
\mathcal{V} := \{ U \cap V : U \cup V \in \mathcal{U}, U \subseteq Y \cup \emptyset, V \subseteq \emptyset \cup Y \}
\]

is a countable Borel \( \omega \)-cover of \( Y \). Let \( W \subseteq \mathcal{V} \) be a \( \gamma \)-cover of \( Y \). Then the family

\[
\{ U \cup V \in \mathcal{U} : U \cap V \in \mathcal{W}, U \subseteq Y \cup \emptyset, V \subseteq \emptyset \cup Y \}
\]

is a \( \gamma \)-cover of \( Y \sqcup Y \).

The set \( X := Y \cup \{ -y : y \in Y \} \), a continuous image of the space \( Y \sqcup Y \), satisfies \( (\Omega_{Bor}^\Gamma) \), too. Consider this set as a subspace of the Sorgenfrey line. Since the Borel sets in the real line and the Sorgenfrey line are the same, the space \( X \) satisfies \( (\Omega_{Bor}^\Gamma) \).

The product space \( X \times X \) contains the uncountable closed discrete set \( \{ (x, -x) : x \in X \} \), and thus does not satisfy \( (\Omega) \). The property \( (\Omega) \) is preserved by finite powers [9, Theorem 3.6]. Thus, the space \( X \) does not satisfy \( (\Omega) \).

**Corollary 13.** Let \( X \subseteq \mathbb{R} \) be an uncountable set satisfying \( (\Omega_{Bor}^\Gamma) \). As a subset of the Sorgenfrey line, the space \( C(-X \cup X) \) is not Fréchet–Urysohn, but the space \( B(-X \cup X) \) is strongly sequentially separable. \( \square \)

5. Additional results

The space from Theorem 10(2) has the property \( (\Omega_{ctbl}^\Gamma) \), that is formally stronger than \( (\Omega_{ctbl}^\Gamma) \). In the forthcoming Proposition 14, we show that the properties \( (\Omega_{ctbl}^\Gamma) \) and \( (\Omega_{ctbl}^\Gamma) \) are different.

A family of sets is **almost disjoint** if the intersection of any pair of sets of this family is finite. For an almost disjoint family \( A \) in \( [\mathbb{N}]^\infty \), the Mrówka–Isbell space \( \Psi(A) \) is the set \( A \cup \mathbb{N} \), with the points of \( \mathbb{N} \) isolated, and with the sets \( \{ a \} \cup a \setminus b \) (for \( b \in \text{Fin} \)) as neighborhoods of the points \( a \in A \).

**Proposition 14.** There is a maximal almost disjoint family \( A \) in \( [\mathbb{N}]^\infty \) such that the Mrówka–Isbell space \( \Psi(A) \) satisfies \( (\Omega_{ctbl}^\Gamma) \) but not \( (\Omega_{ctbl}^\Gamma) \).

**Proof.** There is a maximal almost disjoint family \( A \) in \( [\mathbb{N}]^\infty \), of cardinality \( |\mathbb{R}| \), such that the space \( \Psi(A) \) satisfies \( (\Omega_{ctbl}^\Gamma) \) [2, Example 61 and Theorem 54]. Let \( A = \{ a_r : r \in \mathbb{R} \} \). Since \( \mathbb{R} \) does not satisfy \( (\Omega_{ctbl}^\Gamma) \), there is a family \( U \in \Omega_{ctbl}(\mathbb{R}) \) with no subfamily in \( \Gamma(\mathbb{R}) \). For each set \( U \in \mathcal{U} \), let \( U' := \{ a_r : r \in U \} \cup \mathbb{N} \). The family \( \{ U' : U \in \mathcal{U} \} \) is in \( \Omega_{ctbl}(\Psi(A)) \) and has no subfamily in \( \Gamma(\Psi(A)) \). Thus, the space \( \Psi(A) \) does not satisfy \( (\Omega_{ctbl}^\Gamma) \). \( \square \)

A space is **projectively** \( (\Omega) \) if each continuous second countable image of this space satisfies \( (\Omega) \) [2].

**Proposition 15.** For a space \( X \), the following assertions are equivalent:

1. The space \( C(X) \) is strongly sequentially separable.
2. The space \( X \) has a coarser second countable topology, and it is projectively \( (\Omega) \).
Proof. (1) $\Rightarrow$ (2): By a result of Noble [12, Theorem 1], the space $X$ has a coarser second countable topology. In order to prove that the space $X$ is projectively $\Omega(\Gamma)$, we show that it satisfies the equivalent property $(\Omega_{\text{coz}} \Gamma)$ [2, Theorem 54]. Let $F \subseteq C(X)$ be a countable set such that the family $U = \left\{ f^{-1}[\mathbb{R} \setminus \{0\}] : f \in F \right\}$ is an $\omega$-cover of $X$. Let $B$ be a countable basis of $\mathbb{R}$, and $B'$ be a countable basis of a coarser topology on $X$. Let $Y$ be the set $X$ with the topology generated by the family $\left\{ f^{-1}[B] : B \in B \right\} \cup B'$. The space $Y$ is second countable. By a result of Gartside, Lo, and Marsh [6, Theorem 16], the space $Y$ satisfies $(\Omega \Gamma)$. Since $U \in \Omega(Y)$, the family $U$ contains a cover $V \in \Gamma(Y)$. Thus, $V \in \Gamma(X)$.

(2) $\Rightarrow$ (1): By (2), every coarser second countable topology for the space $X$ satisfies $(\Omega \Gamma)$. By a result of Gartside, Lo and Marsh [6, Theorem 16], the space $C(X)$ is strongly sequentially separable. □

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