A simplified formula for light-deflection in the quadrupole field of moving massive bodies has been obtained in [1–3], which will be applied for Gaia data reduction. So far, in Gaia data reduction it has been assumed that the positions of the giant planets should be computed at the retarded instant of time. The problem of light-deflection due to quadrupole field of moving planets has been re-considered in [4]. According to their solution, the position and velocity of the massive body have to be taken at retarded time. We show that the solution given in [4] coincides with our simplified quadrupole formula obtained in [1–3]. This coincidence implies that the positions of giant planets have, in fact, to be taken at retarded time.
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I. INTRODUCTION

An essential experiment of Gaia mission to test relativity concerns the new effect of light-deflection at giant planets due to their quadrupole field. Analytical solutions of light deflection by a quadrupole field of a massive body are well-known and have been investigated by many authors [5–12]. Because these formulas are rather complicated, they imply massive computations of quadrupole light deflection and are too time-consuming for Gaia data reduction. Therefore, we have derived a simplified quadrupole formula in [1–3], which is suitable for a time-efficient computation of quadrupole light-deflection on microarcsecond level of accuracy.

In reality, while the light-signal is being emitted at a position $x_0$ at time moment $t_0$ and received at position $x_1$ at a time moment $t_1$, the massive body changes the position from $x_A(t_0)$ to $x_A(t_1)$. Therefore, it is not obvious at which coordinate time $t_0 \leq t \leq t_1$ the position of the massive body $x_A(t)$ has to be chosen in the formula of quadrupole light-deflection. Up to now, in Gaia data reduction it is tacitly and implicitly assumed that the positions of multipoles (Jupiter and Saturn) should be computed at the retarded moments of time, given by the implicit light-cone equation:

$$s_A^1 = t_1 - \frac{|x_1 - x_A(s_A^1)|}{c}, \quad (1)$$

where $x_1$ is the position of observer at observation time $t_1$ and $x_A(s_A^1)$ is the coordinate of massive body at retarded time moment $s_A^1$. However, no theoretical proof of this assumption has been given. This problem has recently been solved by [4]. In our report we show that the solutions in [1–3] and in [4] agree with each other, implying that in the quadrupole formula the position of massive body has indeed to be taken at the retarded instant of time.

II. SIMPLIFIED QUADRUPOLE FORMULA

In [1–3] a simplified quadrupole formula has been derived which takes into account only those terms relevant for microarcsecond level of accuracy. In this Section we will give the main steps and results. Consider a gravitational field of $N$ massive bodies $A$, and the positions of these individual massive bodies are $x_A$. A light-ray is being emitted at a position $x_0$ at time moment $t_0$ and received at position $x_1$ at a time moment $t_1$. The unit coordinate direction of the light propagation at the moment of observation reads $n = \frac{\dot{x}(t_1)}{|\dot{x}(t_1)|}$ and the unit tangent vector of light path at infinitely past is $\sigma = \lim_{t \to -\infty} \frac{\dot{x}(t)}{c}$. Then, in post-Newtonian order, the transformation $\sigma$ to $n$ is given by [9, 10]

$$n = \sigma + \sum_i \delta \sigma_i + \mathcal{O}(c^{-4}), \quad (2)$$

where the sum runs over individual terms of various physical origin, that means monopole gravitational field, quadrupole field and higher multipole fields. The spherical symmetric part (monopole field) due to one massive body $A$ is given, for instance, by Eq. (102) in [13].

Here, we are only interested at the quadrupole light deflection term. For one massive body $A$, the quadrupole light-deflection is given as follows (see Eq. (40) in [2] or Eq. (9) in
\[ \delta \sigma_Q = -\frac{GM_A}{c^2} J_z A^2 \left( \frac{d^2}{dA^2} + 2 \cos \psi - \cos^3 \psi \right) \times \left[ \left( 1 - (\sigma \cdot e_3)^2 - 4 (n_A \cdot e_3)^2 \right) n_A + 2 (n_A \cdot e_3) e_3 - 2 (\sigma \cdot e_3) (n_A \cdot e_3) \sigma \right] . \]

Here, \( M_A \) is the mass of body \( A \), \( c \) is the speed of light, \( G \) is the gravitational constant, and the impact vector

\[ d_A = \sigma \times (r^A_1 \times \sigma) , \]

where \( r^A_1 = x_1 - x_A \), and the absolute value \( d_A = |d_A| \). Furthermore, the angle \( \psi = \delta (\sigma, r^A_1) \), the unit vector along the axis of symmetry (rotational axis of massive body) is denoted by \( e_3 \), \( P_A \) denotes the equatorial radius, and \( J_z^A \) is the coefficient of second zonal harmonic of the gravitational field of massive body \( A \). The unit vector \( n_A \) is defined by

\[ n_A = \frac{d_A}{d_A} . \]

Since the massive bodies move, the question arises at which instant of time the positions of the massive bodies have to be chosen. This problem has been solved in [4], and the main results of this work will be subject of the next Section.

III. QUADRUPOLE FORMULA BY KOPEIKIN & MAKAROV

A. Description of the approach

The light-deflection at moving monopoles and quadrupoles has been re-investigated by in [4]. In this Subsection we describe the basic steps of this approach. The gravitational field is described by \( g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \) where \( h_{\alpha\beta} \) is the metric perturbation in post-Minkowski approximation. Using harmonic gauge, the linearized Einstein equations for the perturbation \( h_{\alpha\beta} \) are homogeneous wave equations,

\[ \left( -\frac{\partial^2}{\partial t^2} + \nabla^2 \right) h_{\alpha\beta} = 0 . \]

A general solution of (6) is given in terms of multipole expansion [14, 15], where the terms in the perturbation \( h_{\alpha\beta} \) are taken at the retarded instant of time \( s^A_1 \). The retardation follows directly from the retarded (causal) solution of the homogeneous wave equations (6).

In the approach [4] the monopole, dipole and quadrupole terms of the general multipole expansion of \( h_{\alpha\beta} \), given in Eqs. (12) - (14) in [4], are taken into account. The center of coordinate system is shifted from the mass center of the massive body by a spatial distance. Then, by means of parallel axis theorem (Huygens-Steiner theorem), they apply general expressions for the quadrupole moment and their time derivative, see Eq. (17) and Eq. (20) in [4].

Then, in [4] the geodesic equation is rewritten into a considerably simpler form, given by Eq. (19) in [16]. The integration leads to the expressions (28) and (29) in [4]. Using their developed integration method, presented in part by Eqs. (30) and (31) in [4], they succeed to integrate analytically the differential equation (29) in [4]. Furthermore, in [4] all those terms are neglected which they proof to contribute less than 1 microarcsecond. The result of their approach is finally given by Eqs. (39) - (41) in [4].
B. The analytical solution for quadrupole light-deflection

In Eq. (44) in [4], the following form for the light-deflection of sources at infinite distances due to one moving quadrupole $A$ has been given:

$$\delta\sigma_A^Q = 4 \left( 1 - \frac{\sigma \cdot v_A}{c} \right) \frac{GM_A}{c^2} \frac{P_A^2}{d_A^3} \times \left[ \left( e_3 \cdot n_A \right)^2 - \left( e_3 \cdot m_A \right)^2 \right] n_A - 2 \left( e_3 \cdot n_A \right) \left( e_3 \cdot m_A \right) m_A .$$

(7)

It is essential to noticed, that all time-dependent quantities in Eq. (7), that is $x_A$, $v_A$ and $e_3$, and therefore also $n_A$ and $m_A$, are computed at retarded instant of time $s_1^A$ given by Eq. (1). It should be noticed that in Eq. (7) we have omitted a term proportional to the displacement of the planetary center from the origin of the coordinate system which is an artificial term of the approach and not relevant for Gaia data reduction. Furthermore, the unit vector $m_A$ is defined by

$$m_A = \sigma \times n_A .$$

(8)

In Eq. (7) we have used our notational conventions in [2, 3]. As mentioned in the introductory Section, we have replaced the vector $k$ by $\sigma$, since $\sigma = k + O \left( \frac{m}{c} \right)$, that means such a replacement would cause effects of higher order beyond the post-Newtonian approximation.

C. Comparison of the formula of quadrupole light-deflection

The quadrupole light-deflection (7) can be estimated by

$$\left| \delta\sigma_A^Q \right| \leq 4 \left( 1 - \frac{\sigma \cdot v_A}{c} \right) \frac{GM_A}{c^2} \left| J_2^A \right| \frac{P_A^2}{d_A^3} .$$

(9)

Accordingly, for Jupiter and Saturn we have

$$\left| \sigma_{Q, \text{Jupiter}} \right| \leq \left( 1 - \frac{\sigma \cdot v_{\text{Jupiter}}}{c} \right) 240 \text{ } \mu\text{as} ,$$

(10)

$$\left| \sigma_{Q, \text{Saturn}} \right| \leq \left( 1 - \frac{\sigma \cdot v_{\text{Saturn}}}{c} \right) 95 \text{ } \mu\text{as} ,$$

(11)

while for other massive bodies of the solar system we find considerably smaller values. Furthermore, the orbital speed of these planets in respect to the barycenter is of the order $v_A \sim 10^{-4} c$. Hence, the contribution of the velocity-term in the quadrupole light-deflection is by far less than about $0.1 \mu\text{as}$ and can be neglected for Gaia astrometric accuracy. Thus, the quadrupole light-deflection formula (7) can be simplified as follows:

$$\delta\sigma_A^Q = 4 \frac{GM_A}{c^2} \frac{J_2^A P_A^2}{d_A^3} \times \left[ \left( e_3 \cdot n_A \right)^2 - \left( e_3 \cdot m_A \right)^2 \right] n_A - 2 \left( e_3 \cdot n_A \right) \left( e_3 \cdot m_A \right) m_A .$$

(12)

Eq. (12) determines the quadrupole light-deflection for moving massive bodies, while the rotational axis $e_3$ and the unit vectors $n_A$ and $m_A$ have to be computed at retarded instant of time $s_1^A$ given by Eq. (1).
The expression given by Eq. (12) in [4] coincides with our expression given in Eq. (3), see [1–3]. This can be shown by means of the relation \( \cos^2 x = 1 - \sin^2 x \), that is \((e_3 \cdot m_A)^2 = 1 - (e_3 \times m_A)^2\). Then, by taking into account \(e_3 \times m_A = e_3 \times (\sigma \times n_A)\) we obtain the relation

\[
(e_3 \cdot m_A)^2 = 1 - (e_3 \cdot n_A)^2 - (e_3 \cdot \sigma)^2.
\]

(13)

Inserting (13) into (12) we obtain

\[
\delta \sigma_Q = -4 \frac{G M_A}{c^2} J_2^A \frac{P_A^2}{d_A^3}
\times \left[ (1 - 2 (e_3 \cdot n_A)^2 - (e_3 \cdot \sigma)^2) n_A + 2 (e_3 \cdot n_A) (e_3 \cdot m_A) m_A \right].
\]

(14)

Furthermore, for the vector \(e_3\) we have to use the linear combination in terms of the unit vectors \(d_A, \sigma\) and \(m_A\), that means \(e_3 = (e_3 \cdot \sigma) \sigma + (e_3 \cdot n_A) n_A + (e_3 \cdot m_A) m_A\), from which we conclude

\[
(e_3 \cdot m_A) m_A = e_3 - (e_3 \cdot \sigma) \sigma - (e_3 \cdot n_A) n_A.
\]

(15)

Inserting (15) into (14), we obtain

\[
\delta \sigma_Q^A = -4 \frac{G M_A}{c^2} J_2^A \frac{P_A^2}{d_A^3}
\times \left[ (1 - (\sigma \cdot e_3)^2 - 4 (n_A \cdot e_3)^2) n_A + 2 (n_A \cdot e_3) e_3 - 2 (\sigma \cdot e_3) (n_A \cdot e_3) \sigma \right].
\]

(16)

This expression coincides with our quadrupole light-deflection for a massive body, given by Eq. (3) if we approximate \(2 + 3 \cos \psi - \cos^3 \psi\) by a factor 4. This approximation follows from \(\sin \psi = \frac{d_A}{r_A^1}\), that means \(2 + 3 \cos \psi - \cos^3 \psi = 4 + \mathcal{O} \left( \frac{d_A^4}{r_A^4} \right)\). Thus, we have shown the coincidence of our quadrupole formula given by Eq. (3) (that means Eq. (40) in [2] or Eq. (9) in [3]) with the quadrupole formula in [4] given by Eq. (12). This coincidence implies that for Gaia data reduction the position of giant planets, in fact, to be taken at the retarded instant of time, given by the implicit light-cone equation (1).

**IV. SUMMARY**

A simplified quadrupole formula for massive bodies has been derived in [1–3], which will be used for a time-efficient computation of quadrupole light-deflection in Gaia data reduction. However, since the massive bodies move during the light-signal travelling, it is not obvious at which coordinate time the position of the massive body \(x_A(t)\) has to be taken in this formula of quadrupole light-deflection. So far, in Gaia data reduction it has been tacitly assumed that the positions of the giant planets should be computed at the retarded instant of time \(x_A(s_A^1)\). However, no theoretical proof of this assumption has been performed.

Therefore, in this report, we have re-considered the results of [4] where the light-deflection at moving quadrupoles has been determined. We have shown that the quadrupole formula obtained in [4] given by Eq. (12) coincides with our simplified quadrupole formula obtained in [1–3] and given by Eq. (3). This coincidence implies that the position of massive body has, in fact, to be taken at the retarded instant of time.
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