Holonomic quantum computation in the ultrastrong-coupling regime of circuit QED

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We present an experimentally feasible scheme to implement holonomic quantum computation in the ultrastrong-coupling regime of light-matter interaction. The large anharmonicity and the \( \mathbb{Z}_2 \) symmetry of the quantum Rabi model allow us to build an effective three-level \( \Lambda \)-structured artificial atom for quantum computation. The proposed physical implementation includes two gradiometric flux qubits and two microwave resonators where single-qubit gates are realized by a two-tone driving on one physical qubit, and a two-qubit gate is achieved with a time-dependent coupling between the field quadratures of both resonators. Our work paves the way for scalable holonomic quantum computation in ultrastrongly coupled systems.

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INTRODUCTION

The extensive progress in quantum information science has motivated continuous demand for implementing high-fidelity quantum operations. Holonomic quantum computation (HQC) represents a promising approach to achieve this goal because of its intrinsic noise-resilience features [1, 2]. The holonomic gates can be achieved by using either Abelian [3, 4] or non-Abelian [5, 6] geometric phases. The Abelian approach [7–9] utilizes quantum two-level systems (qubits) as elementary units, and the underlying idea is to choose a pair of orthogonal states that will evolve cyclically. In contrast, the non-Abelian approach [1, 10] embeds qubits in a proper subspace of the total Hilbert space. Recently, a scheme to build fast holonomic gates through the non-Abelian approach has been proposed in Refs. [2, 11]. The advent of this idea has triggered off a set of new proposals [12, 13]. Apart from the theoretical interest, high fidelity gates based on fast HQC schemes have been demonstrated in different systems such as transmon-based superconducting qubits [14], NMR systems [15], and diamond NV-centers [16, 17].

On the other hand, the light-matter interaction has been the focus of interest in recent years owing to the experimental realizations of the ultrastrong-coupling (USC) regime [18–24]. In this case, the light-matter coupling strength is comparable to the cavity and the qubit frequencies [25], and in the dipolar approximation, it is described by the quantum Rabi model (QRM) [26, 27]. Apart from the fundamental interest of the USC regime, it has been intensively studied for demonstrating novel quantum optics phenomena [28–32], implementing quantum information tasks [33, 34], as well as fast quantum computation [35–39] within circuit quantum electrodynamics (QED) [40, 41]. The latter provides a promising solid-state architecture for performing quantum computation due to the desirable properties of superconducting qubits, such as long coherence times, and most importantly, its controllability and scalability [42].

Meanwhile, many efforts have also been made to implement HQC in cavity QED system with natural atoms [43] and artificial atoms [44], as well as circuit QED system with superconducting qubits [45]. While the HQC scheme in Ref. [43] is performed adiabatically, the schemes in Refs. [44, 45] are designed in a non-adiabatic fashion in addition to their decoherence-free subspace (DFS) encoding, and thus integrate both the noise resilience of DFS and the operational robustness of holonomies. However, all of those schemes [43–45] are based on the strong coupling of light-matter interaction, which can be well described by the Jaynes-Cummings model. To the best of knowledge, so far no scheme has ever been proposed to construct holonomic gates in the ultrastrong-coupling regime with the quantum Rabi model. Ultrastrong coupling offers the possibility for ultrafast quantum gate operations even in the time scale of subnanosecond [39], therefore, the realization of HQC in USC is of particular interest yet challenging.

In this work, we propose an experimentally feasible scheme to implement universal non-adiabatic HQC in the USC regime of light-matter interaction. The large anharmonicity and the \( \mathbb{Z}_2 \) symmetry of the QRM allow us to construct an effective three-level \( \Lambda \)-type system for quantum computing. We show that non-commuting single-qubit holonomic gates can be obtained by means of a two-tone driving on one physical qubit, and nontrivial two-qubit holonomic gates can be achieved with a time-dependent coupling between the field quadratures of two bosonic modes. Moreover, we discuss the physical implementation by considering two gradiometric flux qubits each galvanically coupled to its transmission line resonator, which are then connected to each other through a superconducting quantum interference device. Compared to the existing proposals for implementing HQC in circuit QED, the strategy we pursue is different in the sense that we exploit the discrete \( \mathbb{Z}_2 \) symmetry of the quantum Rabi model instead of the continuous \( U(1) \) symmetry of the Jaynes-Cummings model. Therefore, our proposal works well in the ultrastrong-coupling regime, and it may find compelling applications for quantum information processing in ultrastrong coupling and
deep strong coupling regimes for various systems.

**SELECTION RULES IN THE QUANTUM RABI MODEL**

The model that we consider is schematically depicted in Fig. 1. It includes two ultrastrongly coupled qubit-cavity systems, which interact via a time-dependent coupling of strength $J(t)$. Each ultrastrongly coupled system, onwards called quantum Rabi system (QRS), is described by

$$H_p = \hbar \omega_a a^\dagger a + \hbar \omega_c \frac{1}{2} \sigma_z + \hbar g \sigma_x (a^\dagger + a),$$

where $\omega_a$, $\omega_c$, and $g$ stand for the qubit frequency, cavity frequency, and the qubit-resonator coupling strength, respectively. In addition, $a(a^\dagger)$ is the bosonic annihilation(creation) operator, and $\sigma_z$, $\sigma_x$ are the Pauli matrices of the qubit.

In the ultrastrong-coupling regime [46, 47], which is characterized by the ratio range $0.1 \leq g/\omega_c < 1$, the bosonic field and the qubit merge into dressed-state systems that feature the discrete $\mathbb{Z}_2$ symmetry, as shown in Fig. 2, where we plot the energy spectrum of the quantum Rabi model as a function of the coupling strength $g/\omega_c$. This symmetry is characterized by the parity operator $P = e^{\imath \pi a^\dagger a + \pi \sigma_z}$, such that $P|\psi_o\rangle = |\psi_o\rangle$, $P|\psi_e\rangle = -|\psi_e\rangle$. Note that in Fig. 2, even($|\psi_e\rangle$) and odd($|\psi_o\rangle$) eigenstates are represented by continuous-blue and dashed-red lines, respectively. The QRM can be rewritten as

$$H_p = \sum_{s=0} \hbar \omega_s |s\rangle\langle s|,$$

where we consider both even and odd parity states together and labeled them as eigenstates $|s\rangle$ of increasing energy $\hbar \omega_s$.

Formally, the parity in quantum mechanics is intimately related to the selection rules. For the QRM It can be shown that the matrix elements of an even operator are zero between states of different parity, $\langle \psi_e|\mathcal{A}_s|\psi_o\rangle = \langle \psi_o|\mathcal{A}_s|\psi_e\rangle = 0$, while the matrix elements of an odd operator are zero between states of equal parity $\langle \psi_e|\mathcal{A}_s|\psi_e\rangle = \langle \psi_o|\mathcal{A}_s|\psi_o\rangle = 0$. Also, from Fig. 2, we see that the spectrum is anharmonic enough such that the dressed states may be used as computational basis for quantum information processing. In particular, when

$$g/\omega_c \approx 0.3,$$

one can build an effective three-level system by choosing the lowest three levels, $|0\rangle \equiv |\psi_{o,0}\rangle$, $|1\rangle \equiv |\psi_{o,0}\rangle$ and $|2\rangle \equiv |\psi_{o,1}\rangle$ to implement holonomic quantum computation schemes.

**SINGLE-QUBIT GATE**

In this section we show how to construct an arbitrary single-qubit gate in the dressed-state basis of the quantum Rabi model with a non-adiabatic non-Abelian scheme [2]. We choose the two lower levels $|0\rangle$ and $|1\rangle$ to form the qubit subspace $S_1(0) \equiv \{0, 1\}$, leaving the upper level $|2\rangle$ as an auxiliary state.

In this encoding, the states $|0\rangle$ and $|2\rangle$ belong to different parity subspaces, such that the transitions between them can be induced by an odd parity operator, i.e. $\sigma_x$. Similarly, the states $|1\rangle$ and $|2\rangle$ have the same parity and the transition between them can be induced by an even parity operator such as $\sigma_z$. Therefore, a single-qubit holonomic quantum gate can be realized by making use of a two-tone driving scheme on the physical qubit. This can be modeled by the Hamiltonian

$$H_d = \Omega_1(t) \cos(\omega_1 t + \phi_1) \sigma_x + \Omega_2(t) \cos(\omega_2 t + \phi_2) \sigma_z.$$  

The qubit driving Hamiltonian Eq. (3) can be written in the dressed-state basis by using the completeness relation $I = \sum_s |s\rangle\langle s|$

$$H_d = \Omega_1(t) \cos(\omega_1 t + \phi_1) \sum_{s,t} \chi_{st} |s\rangle\langle t|$$

$$+ \Omega_2(t) \cos(\omega_2 t + \phi_2) \sum_{s,t} \varepsilon_{st} |s\rangle\langle t|.$$  

FIG. 1. (Color online) Schematic representation of our model. A system of a single qubit and a single cavity mode that interact in the ultrastrong-coupling regime constitutes the quantum Rabi system. The interaction between the two quantum Rabi systems is mediated by cavities through a time-dependent coupling of strength $J(t)$.  

FIG. 2. Energy levels of the quantum Rabi model as a function of the dimensionless parameter $g/\omega_c$ with $\omega_c/\omega_o = 0.8$. Energies are rescaled in order to set the ground level to zero. The parity of the corresponding eigenstates is identified, continuous-blue line for even states and dashed-red lines for odd states.
where the transition elements are given by $x_{st} = \langle s|\sigma_z|t\rangle$ and $z_{st} = \langle s|\sigma_z|t\rangle$. Notice that according to the selection rule for even and odd operators, $x_{st} = 0$ if $|s\rangle$ and $|t\rangle$ belong to the same parity subspace, and $z_{st} = 0$ if $|s\rangle$ and $|t\rangle$ belong to a different parity subspace. Furthermore, we can interpret the projector $|s\rangle\langle t|$ as a flip operator between dressed states of either equal or different parity depending on the nature of the system operator (in our case, either $\sigma_x$ or $\sigma_z$). Therefore, such a Hamiltonian Eq. (4) induces coherent excitation transfer between all the accessible dressed states.

As we have shown in Fig. 2, for $g/\omega_c \leq 1$, the energy spectrum of the quantum Rabi system has a large anharmonicity such that one can access one particular transition frequency $\omega_{st} = \omega_i - \omega_s$. Let us consider the total Hamiltonian $H = H_p + H_d$, and we move to the interaction picture with respect to the Rabi Hamiltonian in Eq. (2). If the condition $|\Omega_{3}(t)| \ll |\omega_{st}|, \bar{\omega}_d$ is satisfied, $H$ can be approximated by a RWA and neglect fast oscillating terms. Moreover, when bringing the frequency of the driving close to resonance with the transitions in which we are interested, i.e., $\bar{\omega}_1 = \omega_20$ and $\bar{\omega}_2 = \omega_{21}$, the interaction Hamiltonian reads

$$H_I = \frac{\Omega_{1}(t)}{2} e^{-i\varphi_1} x_{20} |0\rangle \langle 0| + \frac{\Omega_{2}(t)}{2} e^{-i\varphi_2} z_{21} |1\rangle \langle 1| + \text{H.c.},$$

with $x_{20} = \langle 2|\sigma_x|0\rangle$ and $z_{21} = \langle 2|\sigma_z|1\rangle$. Therefore, by engineering the driving amplitudes and frequencies, Rabi oscillations between two specific dressed states can be performed.

In Fig. 3, we show Rabi oscillations for the lowest three dressed states in the quantum Rabi model. This simulation has been performed by making use of the full Hamiltonian $H = H_p + H_d$. As shown in Fig. 3 (a), by driving the qubit in the $\sigma_x$ direction on resonance with the transition $\bar{\omega}_1 = \omega_{20}$, we observe Rabi oscillations between the two different parity states $|2\rangle$ and $|0\rangle$. Similarly, by driving the qubit in the $\sigma_z$ direction on resonance with the transition $\bar{\omega}_2 = \omega_{21}$, the complete population transfer between the two same parity states $|2\rangle$ and $|1\rangle$ is shown in Fig. 3 (b). Moreover, by tuning the driving frequency and amplitude to make both $\sigma_x$ and $\sigma_z$ rotations, we may have full control of any structured three-level system built from the dressed states of the quantum Rabi model, as shown in Fig. 3 (c). In this manner, we are able to implement the effective Hamiltonian in Eq. (5).

Now we show how to construct an arbitrary holonomic single-qubit gate with the effective Hamiltonian $H_I$. By setting $\Omega_{2}(t) = \Omega_{1}(t) = \Omega_{2}(t) = \Omega_{2}(t) |0\rangle \langle 0| + \omega_{21} |1\rangle \langle 1| / 2$, $\varphi = \varphi_2 - \varphi_1$, and $\theta = -\arctan(|\Omega_{1}(t)x_{20}|/|\Omega_{2}(t)x_{21}|)$, we can rewrite Eq. (5) as follows

$$H_I = \Omega(t) \left(e^{i\varphi} \sin \frac{\theta}{2} |0\rangle \langle 0| - \cos \frac{\theta}{2} |1\rangle \langle 1| + \text{H.c.} \right).$$

In this case, we construct a $\Lambda$-system Hamiltonian in the dressed-state basis, from which an arbitrary single-qubit holonomic gate can be obtained. The effective Hamiltonian $H_I$ in Eq. (6) can be recast as the auxiliary state $|2\rangle$ coupled to the bright state $|b\rangle = e^{-i\varphi} \sin \frac{\theta}{2} |0\rangle - \cos \frac{\theta}{2} |1\rangle$ and decoupled from the dark state $|d\rangle = e^{i\varphi} \sin \frac{\theta}{2} |1\rangle + \cos \frac{\theta}{2} |0\rangle$. Initially, quantum information is stored in the qubit states of subspace $S_1(0)$. When $H_I$ is applied, the subspace $S_1$ is driven out of $S_1(0)$, and we obtain the Rabi oscillation between states $|b\rangle$ and $|2\rangle$ with Rabi frequency of $\Omega(t)$. When the condition $\int_0^t \Omega(t) dt = \pi$ is satisfied, the system states return to the original subspace $S_1(0)$ after a cyclic evolution. The corresponding unitary operator $U_I(\tau)$ acting on $S_1(0)$ reads $U_I(\tau) = \mathbf{n} \cdot \sigma$, where $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ being Pauli operators [2]. It is clear that two non-commuting single-qubit holonomic quantum gates can be achieved based on $U_I(\tau)$. Moreover, there is no dynamical contribution to $U_I(\tau)$ since $\langle m|H_I|n\rangle = 0$ $(m, n \in \{0, 1\})$ and hence $\langle m|H_I|n\rangle = 0$. The result shows the pure geometric nature of the obtained gate. Therefore, the desired single-qubit gates for universal quantum computation can be implemented in our system based on the non-adiabatic non-Abelian scheme [2, 11].

The holonomic single-qubit gate performance under loss mechanisms in the ultrastrongly coupled system can be studied by means of the time-convolutionless projection operator method [48]. In this approach the master equation reads

$$\dot{\rho} = \frac{1}{i\hbar} [H_I, \rho] + \sum_{n \in \{0, 1\}} \sum_{\nu \in \{0, 1\}} \sum_{\alpha \in \{\sigma_x, \sigma_y, \sigma_z\}} \left( U_{\nu \rho} S_n + S_n^\dagger U_{\nu \rho}^\dagger - S_n U_{\nu \rho}^\dagger - S_n^\dagger U_{\nu \rho} \right) S_n,$$

where $S_n$ are Hermitian system operators, and the operators...
are defined as
\[
U_n = \int_0^\infty dt \nu_n(t)e^{-iℏH_t^\tau} S_n e^{iℏH_t^\tau}
\]
\[
\nu_n(t) = \frac{γ_n(ω)}{2π} [\tilde{N}_n(ω) e^{iωt} + (\tilde{N}_n(ω) + 1) e^{-iωt}].
\] (8)

Here, we consider independent thermal baths for each loss mechanism acting on the system described by the Hamiltonian \( H_t(t) = Ω(t) (|T_0⟩⟨0| + |T_1⟩⟨1| + \text{H.c.}) \) [cf. Eq. (6)]. In our simulation we include loss mechanisms acting on the dressed-state system via transversal noise \((γ_c)\), longitudinal noise \((γ_z)\), and noise acting on the field quadrature \((γ_e)\), through operators \(S_x = σ_x, S_z = σ_z\), and \(S_e = a + a^\dagger\), respectively. In our treatment, each loss mechanism is described by independent thermal baths with bare loss rates \(γ_j\). This leads to \(γ_j(ω) = (γ_j/ω_j)ωθ(ω)\), where \(θ(ω)\) is the Heaviside step function.

Following Ref. [2] we have studied the performance of the Hadamard gate under loss mechanisms through the gate fidelity \(F = ⟨χ|U′(C)ρ_{\text{out}} U(C)|χ⟩\), where \(U(C) = (σ_x + σ_z)/\sqrt{2}\), and \(ρ_{\text{out}}\) is the density matrix of the output state obtained from the master equation (7). The gate fidelity is computed numerically for 4000 input states \(|χ⟩\), uniformly distributed over the Bloch sphere. In \(H_t(t)\), we choose \(Ω(t) (|T_0⟩⟨0| + |T_1⟩⟨1| + \text{H.c.}) \) for parameters \(g = 0.3Ω, Ω_a = 0.8Ω_r\), and \(γ_c = γ_x = γ_e = 10^2ω_c\). Note that loss mechanisms in the dressed-state basis, including even and odd operators in the parity Hilbert space, will induce a complete decay to the fundamental state \(|0⟩\) at a scale time of \(\sim 100/ω_c\). Despite of this, if the pulse is sufficiently short compared with the decay time, i.e. \(β/γ_x \gg 1\), the fidelity of the nonadiabatic gate approaches to unity.

**TWO-QUBIT GATE**

In what follows, we will demonstrate a nontrivial two-qubit gate by using a non-adiabatic Abelian scheme [9] in a four-dimensional space spanned by the encoded logic qubit states \(S_2 \equiv \{|0dx⟩, |0dx⟩, |0r⟩, |0r⟩, |1l⟩, |1r⟩\}. This can be proven by considering two ultrastrongly coupled systems that interact via a time-dependent coupling strength \(J(t)\), as depicted in Fig. 1. The Hamiltonian describing the whole system is composed of the sum of two quantum Rabi models and a coupling between the field quadratures [49, 50]

\[
H_{\text{tot}} = H_{p,l} + H_{p,r} + H_{\text{int}},
\]

\[
H_{\text{int}} = hJ(t)(a_l^\dagger + a_l)(a_r^\dagger + a_r),
\] (9)

with \(H_{p,j}\) \((j = l, r)\) being the Hamiltonian for the left and right quantum Rabi system.

**FIG. 4.** Performance of the Hadamard gate under loss mechanisms acting upon the ultrastroongly coupled system. The fidelity is averaged on an ensemble of 4000 input states uniformly distributed over the Bloch sphere. In this simulation we have considered parameters \(g = 0.3ω_c, ω_a = 0.8ω_c, γ_x = γ_e = 10^2ω_c\).

By using the completeness relation, the system Hamiltonian Eq. (9) can be rewritten as

\[
H_{\text{tot}} = \sum_{j=0}^{n} (hω_{x,r}|s⟩⟨s| + hω_{x,l}|s⟩⟨s|) + hJ(t)\times \left[ \sum_{s,i,j,l} (f_{s,l}|s⟩⟨i| + \text{H.c.}) \otimes \sum_{u_r, u_r > u_r} (f_{u_r, u_r}|v_r⟩⟨v_r| + \text{H.c.}) \right],
\] (10)

where \(f_{s,lj} = ⟨s|⟨a_j + a_j^\dagger|t⟩\rangle, j = l, r\), is the transition matrix elements for the left \((l)\) and right \((r)\) system. Here, we have used the fact that the transition matrix elements are zero between states of the same parity, i.e., \(f_{s,lj} = 0\). Similar to the single-qubit case, the operator \(|s⟩⟨t|\) is the raising operator for the left or the right system. Let us consider the interaction picture with respect to Hamiltonian \(H_{p,l} + H_{p,r}\). In this case, the interaction Hamiltonian reads

\[
H_{\text{int}}^f = hJ(t)\sum_{s,i,j,l} (f_{s,l}|s⟩⟨i|e^{-iω_{l,r}t} + \text{H.c.}) \times \sum_{u_r, u_r > u_r} (f_{u_r, u_r}|v_r⟩⟨v_r|e^{-iω_{l,r}t} + \text{H.c.})
\] (11)

where \(ω_{x, j} = ω_{x,j} - ω_{x,j} > 0\). In particular, the cavity-cavity coupling parameter can be a time-dependent function \(J(t) = J_0(t) \cos(ω_αt + \varphi_d)\). In this case, if one chooses the resonance condition for two specific dressed states, i.e., \(ω_d = ω_{x,l} - ω_{x,r}\) and the cavity-cavity coupling strength satisfies the condition \(|J_0(t)| \ll ω_{x,l} + ω_{x,r}\), we can apply the rotating-wave approximation and the interaction Hamiltonian effectively reduces to

\[
H_{\text{int}} = \frac{hJ_0(t)}{2} f_{x, l} f_{x, r} e^{-iϕ_d}|s⟩⟨t| ⊗ |v_r⟩⟨v_r| + \text{H.c.}
\] (12)
It is clear that such a Hamiltonian produces entanglement and induces coherent excitation transfer between specific dressed states $|s⟩$ and $|l⟩$ of the left and the right systems [50]. It is worthy noting that the coupling operator $(a_0^+ + a_l^+)(a_0^+ + a_l^+)$ is an odd operator for the left and right quantum Rabi system individually, so it only induces transitions between states with opposite parity. Specifically, if we choose $\omega_d = ω_{1,0} - ω_{1,0}$, the system can be effectively described

$$H_{\text{int}} = \hbar \frac{J_0(t)}{2} f_{01} f_{01}^* e^{-i\omega_d t} |01, r⟩⟨10, r| + \text{H.c.}, \quad (13)$$

which gives the interaction between two specific states $|01, r⟩$ and $|10, r⟩$. Meanwhile, Eq. (13) is our target Hamiltonian for the two-qubit HQC with the lowest three states in the dressed state basis. The interaction Hamiltonian (13) also indicates that, out of the nine possible tensor states in the Hilbert space of the total system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, there are only two states that are correlated. This restricts us to a two-dimensional subspace which is spanned by vectors $[[10, r], |01, r⟩]$, and allows us to demonstrate a two-qubit gate on Abelian geometric phases. To illustrate our scheme, we encode the states $|01, r⟩$ and $|10, r⟩$ into logical single-qubit states $|1⟩_L$ and $|0⟩_L$, respectively. In the logical representation, Eq. (13) reads

$$H_{\text{int}} = \hbar \frac{J_0(t)}{2} f_{01} f_{01}^* (\cos \varphi_d S_x + \sin \varphi_d S_y), \quad (14)$$

where $S_x$ and $S_y$ are Pauli operators on the logical basis.

In what follows, we demonstrate a nontrivial two-qubit gate based on the Hamiltonian Eq. (14) according to the non-adiabatic Abelian scheme presented in Ref. [9]. A geometric phase shift gate $U_β$ acting on the eigenstates of $S_z$, namely $|\pm⟩_L = (|0⟩_L \pm |1⟩_L)/\sqrt{2}$, can be achieved by letting the system evolve along a cyclic path based on properly designed four-step evolution up to a global phase. Step-1.— By setting $\varphi_d = \pi/2$, we apply a rotation $e^{i\pi S_z/4}$ to the basis, changing the states from $|+⟩_L$ and $|−⟩_L$ to $|0⟩_L$ and $|1⟩_L$, respectively. Step-2.— By setting $\varphi_d = 0$, the states $|0⟩_L$ and $|1⟩_L$ are swapped with each other by a rotation $e^{i\pi S_y/2}$. Step-3.— With a proper choice of the parameter $\varphi_d = \beta \neq m\pi (m \in \mathbb{N})$, we evolve the states from $|0⟩_L$ and $|1⟩_L$ to $e^{i\beta} |1⟩_L$ and $e^{−i\beta} |0⟩_L$, respectively, by a rotation $e^{i\pi \cos \beta S_x + \sin \beta S_y}/2$. Step-4.— By choosing $\varphi_d = \pi/2$ again, the resulting states $|0⟩_L$ and $−|1⟩_L$ can be changed back to $|+⟩_L$ and $|−⟩_L$ by using the rotation $e^{−i\pi S_y/4}$, eliminating the minus sign obtained in the first step. Therefore, the system undergoes a cyclic evolution

$$|+⟩_L \rightarrow |0⟩_L \rightarrow |1⟩_L \rightarrow e^{−i\beta} |0⟩_L \rightarrow e^{−i\beta} |+⟩_L, \quad (15)$$

$$|−⟩_L \rightarrow |−⟩_L \rightarrow |0⟩_L \rightarrow −e^{i\beta} |1⟩_L \rightarrow e^{i\beta} |−⟩_L, \quad (16)$$

and the obtained geometric phase shift gate $U_β$ written in the states $|±⟩_L$ is of the following form,

$$U_β = \begin{pmatrix}
        e^{−i\beta} & 0 \\
        0 & e^{i\beta}
      \end{pmatrix}. \quad (17)$$

There is no dynamical phase accompanied during the cyclic evolution since the evolution is along geodesic lines. Eq. (17) is nothing but a non-trivial two-qubit gate in the basis $\{|0⟩_r, |1⟩_r, |0⟩_l, |1⟩_l⟩\}$ with

$$U_2 = \begin{pmatrix}
        1 & 0 & 0 & 0 \\
        0 & \cos \beta & −i \sin \beta & 0 \\
        0 & −i \sin \beta & \cos \beta & 0 \\
        0 & 0 & 0 & 1
      \end{pmatrix}. \quad (18)$$

It is apparent that $U_2$ is nontrivial when $\beta \neq m\pi$ with $m \in \mathbb{N}$. Together with the non-commuting single-qubit gates, we have demonstrated a universal set of holonomic quantum gates for ultrastrongly coupled system.

**PHYSICAL IMPLEMENTATION**

Here, we propose the use of a gradiometric flux qubit with a tunable gap galvanically connected to a cavity, see Fig. 5, to implement HQC. This circuit QED architecture in the strong-coupling regime has been implemented in [51]. Also, the ultrastrong-coupling regime may be achieved by implementing a longer and thinner shared line between the gradiometric flux qubit and the microwave resonator [52].

We stress that the gradiometric configuration is unaffected by homogeneous magnetic fields, so it has the advantage to overcome flux crosstalk [51]. Also, an inhomogeneous magnetic field in the outer-loop of the flux qubit enables the coupling to a microwave resonator. The additional $\alpha$-loop in the gradiometric configuration allows for a tunable qubit gap. This mechanism is completely independent of the flux line that controls the frustration parameter in the outer loop [51, 52]. Therefore, our two-tone driving scheme for the single-qubit gate may be implemented in the gradiometric

![FIG. 5. Schematic of circuit QED design for the holonomic quantum computation. Two transmission line resonators (cavities) are grounded through a SQUID. Each cavity is galvanically coupled to a gradiometric tunable-gap flux qubit, that is constituted by four Josephson junctions. The time-dependent interaction between two resonators can be realized by modulating the external magnetic flux $\Phi_t(t)$ through the SQUID.](image-url)
qubit by applying two independent magnetic fluxes of different frequencies to the outer loop and the α-loop [37].

The time-dependent coupling $J(t)$ between the two cavities can be implemented by means of a superconducting quantum interference device (SQUID) [49], threaded by an external flux $\Phi_{x}(t)$, as shown in Fig. 5. Although the effective cavity length is oscillating with small deviations from its average value, we can still consider the system as a single-mode resonator, see Ref. [49] for a detailed discussion. In particular, the specific form of cavity-cavity coupling strength $J(t) = J_{0}(t)$ achieved by choosing the time-dependent external magnetic flux $\Phi_{x}(t)$ to be composed of the sum of a small amplitude-modulated signal oscillating at the driving frequency $\omega_{d}$ and a constant offset $\Phi$, namely, $\Phi_{x}(t) = \Phi + \Delta \Phi(t) \cos(\omega_{d} t + \varphi_{d})$. By controlling the driving frequency $\omega_{d}$, it will allow us to selectively activate the interaction between two specific energy states of the system and to obtain the effective Hamiltonian Eq. (13) for the two-qubit gate, see the Appendix for a detailed discussion.

**CONCLUSION**

In conclusion, we have presented a proposal to implement a holonomic quantum computation scheme in the ultrastrong-coupling regime of circuit QED. The effective three-level Λ artificial atom to carry out the quantum gate operations is built from the eigenstates of the quantum Rabi model in the dressed-state basis, which is based on its large anharmonic-

**APPENDIX**

Quantization of the circuit model and its quantum dynamics

A detailed analysis of circuit quantization of Fig. 5 can be found in Ref. [49]. The full system Hamiltonian that includes the two quantum Rabi models and the resonator-resonator coupling reads

$$H = H_{p,l} + H_{p,r} + \hbar \sum_{j=l,r} [\hat{J}_{j} + J_{j}(t) \cos(\omega_{d} t + \varphi_{d})](a_{j} + a_{j}^{\dagger})^{2} + \hbar [\hat{J}_{0} + J_{0}(t) \cos(\omega_{d} t + \varphi_{d})](a_{l} + a_{r}^{\dagger})(a_{r} + a_{l}^{\dagger}),$$

with

$$\hat{J}_{j} = \frac{\phi_{0}}{4 \Phi \cos \Phi Z_{j}^{2} C_{j}} \frac{\omega_{j}}{J_{0}} + \hbar \sum_{j=l,r} [\hat{J}_{j} + J_{j}(t) \cos(\omega_{d} t + \varphi_{d})](a_{j} + a_{j}^{\dagger})^{2} + \hbar [\hat{J}_{0} + J_{0}(t) \cos(\omega_{d} t + \varphi_{d})](a_{l} + a_{r}^{\dagger})(a_{r} + a_{l}^{\dagger}),$$

$$J_{j}(t) = \frac{\phi_{0}}{4 \Phi \cos \Phi Z_{j}^{2} C_{j}} \Delta \Phi \Omega(t), \quad J_{0}(t) = 2 \sqrt{J_{j} J_{r}},$$

The above circuit Hamiltonian is obtained by considering a weak harmonic magnetic flux with frequency $\omega_{d}$ applied to the SQUID, that is,

$$\Phi_{x}(t) = \Phi + \Delta \Phi \Omega(t) \cos(\omega_{d} t + \varphi_{d}).$$

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with $\Delta \Phi \ll \Phi$ and $\Omega(t)$ being the normalized temporal envelope of the applied flux. It implies $\bar{\phi} = \pi \Phi_0/\Phi$ and $\Delta \phi = \pi \Delta \Phi/\Phi_0$.

Notice that the above Hamiltonian Eq. (A-1) is different from the Hamiltonian discussed in the main text (9). In particular, except the time-dependent resonator-resonator coupling terms, the Hamiltonian (A-1) also includes single-mode squeezing terms and time-independent coupling terms. Nonetheless, one can demonstrate that Eq. (A-1) reduces to Eq. (11) if we consider realistic system parameters. In order to see this correspondence, let us assume identical resonators, $\omega_r = \omega_c = \omega_r$, with impedances $Z_j = 80 \Omega$ and capacitances $C_j = 200 \text{ fF}$. For the critical current of the SQUID we consider $I_c = 180 \mu A$. Notice that recent experiments in circuit QED have considered critical currents in the range of $1-2 \mu A$ [53], however, one can increase the critical current by to a larger value or even a few orders of magnitude into the mA regime by making the junctions bigger, this would involve a trilayer fabrication process [54, 55]. Moreover, we consider a flat ($\Omega(t) = 1$) magnetic flux pulse applied to the SQUID with parameters $\bar{\phi} = \pi/4$ and $\Delta \phi = 0.1 \bar{\phi}$. The reduced flux quantum is $\phi_0 = \hbar/(2e) = 3.2911 \times 10^{-16} \text{ Wb}$. These parameters lead to $J_j \approx 5 \times 10^{-4} \omega_c$, $J_0 \approx 10^{-3} \omega_c$, $J_j \approx 4 \times 10^{-5} \omega_c$, and $J_0 \approx 8 \times 10^{-5} \omega_c$.

Now we consider the completeness relation for both quantum Rabi systems so that (A-1) can be expressed as

$$H = \sum_{s=0} (\hbar \omega_s) |s_i⟩⟨s_i| + \sum_{j=1} [J_j + J_j(t) \cos(\omega_d t + \varphi_d)] \left\{ \sum_{s_j, s_j > s_j} (X_{s_j}^a |s_j⟩⟨t_j| + X_{s_j}^a |s_j⟩⟨t_j|) + \sum_{s_j} X_{s_j^r} |s_j⟩⟨s_j| \right\} + [J_0 + J_0(t) \cos(\omega_d t + \varphi_d)] \left\{ \sum_{s_0} (f_{s_0} |s_0⟩⟨t_0| + \text{H.c.}) \otimes \sum_{u_r, v_r > u_r} (f_{s_r}^* |u_r⟩⟨v_r| + \text{H.c.}) \right\}, \quad (A-4)$$

where $X_{s_j} = (s_j |a_j + a_j^a|^2 |t_j⟩$ and $f_{s_j} = (s_j |a_j^a|^2 |t_j⟩$. Notice that the single mode squeezing operator $(a_j + a_j^a)^2$ is an even operator according with the parity symmetry of the system. It means that will connect states within the same parity subspace. Now, if we go to an interaction picture with respect to $H_{p_x} + H_{p_y}$, the Hamiltonian (A-4) reads

$$H(t) = \sum_{j=1} [J_j + J_j(t) \cos(\omega_d t + \varphi_d)] \left\{ \sum_{s_j, s_j > s_j} (e^{-i\omega_d t} X_{s_j}^r |s_j⟩⟨t_j| + e^{i\omega_d t} X_{s_j}^r |s_j⟩⟨t_j|) + \sum_{s_j} X_{s_j^r} |s_j⟩⟨s_j| \right\} + [J_0 + J_0(t) \cos(\omega_d t + \varphi_d)] \left\{ \sum_{s_0} (e^{-i\omega_d t} f_{s_0} |s_0⟩⟨t_0| + \text{H.c.}) \otimes \sum_{u_r, v_r > u_r} (e^{-i\omega_d t} f_{s_r}^* |u_r⟩⟨v_r| + \text{H.c.}) \right\}. \quad (A-5)$$

FIG. A1. Population inversion between states $|1, 0⟩$ and $|0, 1⟩$ for non identical quantum Rabi systems with parameters $\omega_0 = 1$, $\omega_0 = 0.8 \omega_0$, $\gamma_l = 3 \omega_0$, $\gamma_r = 0.3 \omega_0$ for the left quantum Rabi system, and $\omega_0 = 1$, $\omega_0 = 0.8 \omega_0$, $\gamma_r = 0.9 \omega_0$ for the right quantum Rabi system. These values lead to an effective cavity-cavity coupling strength $J_{eff} = 5.5 \times 10^{-4} \omega_c$. This simulation has been performed with the full Hamiltonian (A-1) through the Runge-Kutta algorithm.
If we restrict the three lowest energy levels for each quantum Rabi system, the above Hamiltonian reads

\[ H_1(t) = \sum_{j=1}^{J} \left[ J_j + J_1 \cos(\omega_1 t + \varphi_d) \right] \left( e^{i\omega_1 t} |1_j,1| \langle 2_j,1 \rangle + e^{-i\omega_1 t} |1_j,1| \langle 1_j,1 \rangle + \sum_{j'=0}^{J} X_{j,j'} \bar{s}_j (\bar{s}_{j'}) \right] \\
+ \left[ J_0 + J_0 \cos(\omega_0 t + \varphi_d) \right] \left( e^{i\omega_0 t} |0_j,0| \langle 1_j,0 \rangle + e^{-i\omega_0 t} |0_j,0| \langle 0_j,0 \rangle + \text{H.c.} \right) \otimes \left( e^{i\omega_0 t} |0_r,0| \langle 1_r,0 \rangle + e^{-i\omega_0 t} |0_r,0| \langle 0_r,0 \rangle + \text{H.c.} \right). \]

(A-6)

For non identical quantum Rabi systems, the above Hamiltonian can produce single excitations transfer if the resonance condition for the driving frequency is \( \omega_0 = J_0,1,1 + \bar{J}_0 |X_{1,1,1},0,0| - |\omega_0,0,0,0| \). Notice that single-mode squeezing terms \( J_j X_{j,j'} \) produce energy shifts for each dressed state. Furthermore, the rotating-wave approximation holds under conditions \( \bar{J}_0 |X_{1,1,1},0,0| < |\omega_2,1,1 \pm \omega_0| \), \( J_j |X_{j,j'}| \ll \omega_0 \), and \( \bar{J}_0 f_{0,1,0,1} \ll |\omega_0,0 - \omega_0,1| \). The effective coupling strength between the two specific dressed states of the two quantum Rabi models is given by \( J_{\text{eff}} = \bar{J}_0 f_{0,1,0,1} \).

We have performed \textit{ab initio} numerics by considering the Hamiltonian (A-1). Figure A1 shows the population inversion between states \(|1,0,r\rangle \) and \(|0,1,r\rangle \) for non identical quantum Rabi systems, see the caption to check parameters. For simplicity we have taken \( \varphi_d = 0 \). It is quite clear the correspondence with the Hamiltonian (13) discussed in the main text.
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