No-broadcasting theorem for non-signalling boxes and assemblages

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The no-broadcasting theorem is one of the most fundamental results in quantum information theory; it guarantees that the simplest attacks on any quantum protocol, based on eavesdropping and copying of quantum information, are impossible. Due to its fundamental importance, it is natural to ask whether it is an inherent quantum property or holds also for a broader class of non-classical theories. A relevant generalization is to consider non-signalling boxes. Subsequently Joshi, Grudka and Horodecki\textsuperscript{[5]} conjectured that one cannot locally broadcast nonlocal boxes. In this paper, we prove their conjecture based on fundamental properties of the relative entropy of boxes. Following a similar reasoning, we also obtain an analogous theorem for steerable assemblages.

I. INTRODUCTION

Among the many no-go theorems in quantum information theory, it is fair to assume that the most famous are the no-cloning and the no-broadcasting theorems. Broadly speaking, the no-cloning theorem states that there is no universal quantum cloning machine that can always create two exact copies from unknown states $\rho$. To a certain extent, the no-broadcasting result can be thought of as a generalization of the no-cloning theorem. In general lines, the impossibility of broadcasting quantum information can be seen as a fundamental limit, a statement showing that there is no universal quantum channel such that given an unknown quantum state $\rho$, it produces a bipartite quantum state whose either marginal coincides with the original state $\rho$. An even stronger formulation of the no-broadcasting theorem states that given two non-commuting quantum states $\rho_1$ and $\rho_2$, there is no quantum channel that could broadcast both of them.

No-cloning and no-broadcasting play a vital role in fields such as quantum cryptography, as they show that eavesdroppers are unable to perfectly copy unknown quantum information\textsuperscript{[1,2]} transmitted between two sites. Additionally, these no-go theorems together with the Heisenberg uncertainty principle can be used to obtain a limit to the amount of information an agent can learn from a single copy of a quantum state\textsuperscript{[7]}.

Following the earlier results, a generalized no-broadcasting theorem was demonstrated for any non-classical finite-dimensional probabilistic model satisfying a no-signalling criterion\textsuperscript{[8]}. Therefore, it can be said, that the impossibility put forward by the standard versions of the no-broadcasting theorem is not distinctive to quantum theory, as it is also present in broader non-classical theories as well.

Alternative and initially counterintuitive variations of the no-broadcasting theorem have also been discussed in the literature: local versions of the theorem are perhaps the best example. In these variations, the focus is on the potential broadcasting of a known bipartite quantum states using local operations. It turns out that only separable states can be broadcast using LOCC (local operations and classical communication)\textsuperscript{[9-11]}. If one considers only LO (local operations), then an even stronger restriction is imposed: nothing other than classical-classical bipartite states can be broadcast\textsuperscript{[12]}. Later it was shown that this kind of broadcasting is equivalent to the original one in the case of quantum theory\textsuperscript{[13-14]}.

In\textsuperscript{[11]}, the authors considered yet another variant of the broadcasting scenario. Their idea was to explore local broadcasting of bipartite non-signalling boxes with binary inputs and outputs—put another way, boxes in the usual $(2,2,2)$ Bell-scenario. They consider a class of operations that transform local boxes into local ones (ones that admit a LHV model) and prove that a nonlocal box cannot be broadcast in this scenario if we are restricted to these transformations. The unfortunate drawback of ref.\textsuperscript{[11]} is that their proof heavily relies on some properties of the $(2,2,2)$ scenario. Our work draws from that scholarship as we want to answer the question they raise: ’is there a no-broadcasting theorem in the general scenario?’, or in other words, ’is there a proof for general boxes?’.

A similar question also arises regarding the case of
assemblages [15][17]. We have discussed that the non-broadcasting theorem is not an exclusivity of quantum theories and that alternative local versions exist both for quantum states and for boxes. In a rough approximation, assemblages can be thought of as an object halfway between bipartite quantum states and boxes—assemblages combine probabilities and quantum states. Inspired by this crude approximation, a natural line of inquiry is the question of whether or not it is possible to broadcast with the appropriate transformations certain classes of assemblages. In this work, we prove a general local non-broadcasting theorem for boxes and also for assemblages using fundamental properties of the relative entropy of nonlocality and the relative entropy of steering.

II. NO-BROADCASTING FOR BOXES

In this section, we will state and prove a theorem about the impossibility of broadcasting nonlocal boxes (Thm. 1). Before we state our main result, we introduce what we mean by correlation scenarios, boxes, local boxes, and the like. For a more extensive introduction, we suggest ref. [18].

A. Correlation scenarios

Correlation scenarios are usually formulated in a device-independent language. Think of it as a collection of \( N \) boxes. Each box has \( m \) inputs, and for each input, there are \( o \) possible associated outputs (see Fig. 1). The framework is formulated to hide the inner physical mechanism of each box. As we do not have access to the physical details producing an outcome given that a certain button was pressed, the only description for this \( (N, m, o) \)-scenario is via the aggregated joint statistics called the behavior:

\[
P = \{ P(ab...c|xy...z) \}_{ab...c|xy...z} \in \mathbb{R}^{(om)^N}. \tag{1}
\]

Each \( P(ab...c|xy...z) \) means the joint probability of obtaining the outcome \( a \) out of the first box when the \( x \) button is pressed, and outcome \( b \) out of the second box when the \( y \) button is pressed, ..., and outcome \( c \) out of the \( N \)th-box when button \( z \) is pressed. Fig. 1 shows a pictorial representation of this situation.

B. Local Boxes

Definition 1. In a given \( (N, m, o) \)-scenario, we say that a behavior \( P = \{ P(ab...c|xy...z) \}_{ab...c|xy...z} \in \mathbb{R}^{(om)^N} \) is local whenever it satisfies

\[
P(ab...c|xy...z) = \sum_\lambda P(a|x, \lambda)P(b|y, \lambda)...P(c|z, \lambda)q(\lambda), \tag{2}
\]

for all measurements and for all outcomes, with \( q(\lambda) \) a probability distribution over an exogenous variable.

In sum, locality for boxes says that there is a hidden variable we do not have access to that fully explains the correlation across the boxes, or, more succinctly, that there is a classical explanation for such correlations. That there are more than classical and more than quantum correlations is not immediate, but they exist. The next sections address precisely these cases.

C. Quantum Boxes

Definition 2. In a given \( (N, m, o) \)-scenario, we say that a behavior \( P = \{ P(ab...c|xy...z) \}_{ab...c|xy...z} \in \mathbb{R}^{(om)^N} \) is quantum whenever it satisfies:

\[
P(ab...c|xy...z) = \text{tr}(\Pi_a^x \otimes \Pi_b^y \otimes ... \otimes \Pi_c^z \rho_{A_1A_2...A_N}), \tag{3}
\]

where \( \{ \Pi_a^x \}, \{ \Pi_b^y \}, ..., \{ \Pi_c^z \} \) are POVMs and \( \rho_{A_1A_2...A_N} \) is a density operator acting on \( \bigotimes_{i=1}^N \mathcal{H}_{A_i} \).

While the set of local correlations is a convex polytope, the set of quantum correlations has a far more complex geometry. It is a convex set that properly contains the local polytope, but its exact shape is only known in certain regions and in certain scenarios. For an in-depth analysis of the problem, we refer to [19]. Behaviors that can be obtained by post-quantum theories also exist, and although they have never been realized in the laboratory, their study still can enlighten the nature of physical theories and what they all have in common. The best example is given by non-signalling correlations that we go on to discuss now.

D. Non-Signalling Boxes

For sake of simplicity, we introduce the notion of non-signalling only for two boxes. All of the reasoning here can be easily extended for larger scenarios with \( N > 2 \).
boxes. We also refer to ref. [20], where non-signalling is motivated and rigorously defined for correlation scenarios with more than two boxes.

**Definition 3.** In a given $(2, m, o)$-scenario, we say that a behavior $P = \{P(ab|xy)\}_{ab,xy} \in \mathbb{R}^{(om)^2}$ is non-signalling whenever it satisfies

$$\sum_b P(ab|xy) = P(a|x) := \sum_b P(ab|xy'); \ \forall \ a, x, y, y',$$

(4)

$$\sum_a P(ab|xy) =: P(b|y) := \sum_a P(ab|x'y); \ \forall \ b, y, x, x'.$$

(5)

**Remark.** Simply saying, one may want to consider non-signalling behaviors as those behaviors for which all marginal probabilities are well-defined. For the $(2, m, o)$ case, it is equivalent of saying that $P(a|x, y) = P(a|x) = P(a|x, y')$ for all choices of measurements and outcomes.

### E. KL-divergence

We prove our main result for boxes using information measures. In general lines, our argument is the following: on the one hand, we will argue that these measures are monotones for a set of free transformation. On the other hand, we will show that broadcasting-like transformations increase the quantity of resource, so they are not allowed. To do so, we need, first and foremost, to decide upon an information measure. In this paper, we work with well-adapted extensions of the Kullback-Leibler divergence.

The Kullback-Leibler divergence (also called relative entropy) is an information-based measure of disparity among probability distributions [21, 22]. Formally, given two probability distributions $p(a)$ and $q(a)$, the Kullback-Leibler divergence (KL-divergence) of $p(a)$ from $q(a)$ is denoted by $S(p(a)||q(a))$ and defined as:

$$S(p(a)||q(a)) := \sum_a p(a) \log \frac{p(a)}{q(a)},$$

(6)

where the index $a$ runs over the alphabet that $p$ and $q$ are defined over. The KL-divergence is (i) non-negative; (ii) it is zero if, and only if, the distributions match exactly; (iii) it is not symmetric in $p(a)$ and $q(a)$; and can potentially be equal infinity. The KL-divergence is, in other words, a type of statistical distance: a measure of how one probability distribution $p(a)$ is different from a second, reference probability distribution $q(a)$. Especially, the KL-divergence measures the information lost when $q(a)$ is used to approximate $p(a)$ [23]. It plays a central role in the theory of statistical inference [24].

Given its extreme importance, it is natural to study extensions of KL-divergence to other objects. For correlation scenarios, a possible generalization was introduced in [25, 26]. The main idea is to associate a single probability distribution to a box, once this is done, we use the KL-divergence defined in eq. (6) to compare two boxes. More concretely, if we consider that the probability distribution of Alice’s and Bob’s choices of inputs is given by $\pi(x, y)$, then $P(a, b, x, y) := \pi(x, y)P(a, b|x, y)$ is the probability of the event where the input choices were $x, y$ and the outputs were $a, b$. Therefore, we can compare two boxes $P(a, b|x, y), Q(a, b|x, y)$ through the KL-divergence between $\pi(x, y)P(a, b|x, y)$ and $\pi(x, y)Q(a, b|x, y)$. More generally, if we want to measure how different two boxes are, it is natural to grant Alice and Bob the freedom to choose their inputs according to any probability distribution available to them, and then single out the one that optimizes this distance. That process can be used to define the KL-divergence for boxes between $P(a, b|x, y)$, and $Q(a, b|x, y)$ as:

$$S_b(P(a, b|x, y)||Q(a, b|x, y)) := \sup_{\pi(x, y)} S(\pi(x, y)P(a, b|x, y)||\pi(x, y)Q(a, b|x, y)), \quad (7)$$

where the supremum is taken over all possible discrete probability distributions over Alice and Bob’s inputs.

Remarkably, the expression in eq. (7) can be simplified. As shown in ref. [26], the optimal probability $\pi^*(x, y)$ is such that the KL-divergence for boxes can be expressed as:

$$S_b(P(a, b|x, y)||Q(a, b|x, y)) = \max_{x, y} S(P(a, b|x, y)||Q(a, b|x, y)). \quad (8)$$

### F. Broadcasting of a box

We start by recalling the standard definition of broadcasting of quantum states [2]. Given a quantum state $\rho \in \mathcal{L}(\mathcal{H})$, we say that $\rho_{XY} \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ is a broadcasting of $\rho$ when both reduced state of $\rho_{XY}$ is equal to $\rho$. Put another way, $\text{tr}_X(\rho_{XY}) = \rho = \text{tr}_Y(\rho_{XY})$. It is worth emphasizing that broadcasting is a generalization of cloning, where we now allow for the possibility of correlations between the two parts.

An analogous definition for boxes can be crafted in the following way: consider a scenario with four agents, labelled $A_0, A_1, B_0, B_1$, where $x_i \in X$ and $a_i \in A$ are the inputs and outputs for the agents $A_i$, respectively. Similarly, $y_i \in Y$ and $b_i \in B$ are the inputs and outputs for the agents $B_i$, respectively. The aggregated statistics for the experiment is described by the behavior $P(a_0, a_1, b_0, b_1|x_0, x_1, y_0, y_1)$. We say that the agents are realizing a broadcasting of a box $P^*(a, b|x, y)$ when the marginal statistics for the pair of agents $A_0, B_0$ is equal to the marginal statistics for the pair $A_1, B_1$ which must be, in turn, equal to $P(a, b|x, y)$. We summarize this discussion in the definition below.
Definition 4. We say that \( P(a_0, a_1, b_0, b_1 | x_0, x_1, y_0, y_1) \) is a broadcasting of \( P'(a, b | x, y) \) if:

\[
\sum_{a_1, b_1} P(a_0, a_1, b_0, b_1 | x_0, x_1, y_0, y_1) = P'(a_0, b_0 | x_0, y_0),
\]

and

\[
\sum_{a_0, b_0} P(a_0, a_1, b_0, b_1 | x_0, x_1, y_0, y_1) = P'(a_1, b_1 | x_1, y_1),
\]

where the equations above must be true for every choice of inputs and outputs.

Recalling the usual notion of non-signalling behaviors, we should notice that eq. (7) implies that \( P(a_0, a_1, b_0, b_1 | x_0, x_1, y_0, y_1) \) is non-signalling with respect to the partition \( A_0B_0|A_1B_1 \).

Remark: In order to not overload the notation, we will usually denote \((a_0, a_1)\) by \(a\) and similarly for \(b, x\) and \(y\). Nonetheless, we will alternate between those two notations when we feel it is convenient.

G. Nonlocality in the broadcasting scenario

Although our broadcasting scenario could be seen as a quadripartite situation, the concept of locality we will use is definitively bipartite, for it is related to the bipartition \( A_0A_1|B_0B_1 \). This partition is not artificial, as the resource is spatially divided between \( A_0, A_1 \) on one side and \( B_0, B_1 \) on the other. With that in mind, we define the set of Local-Realistic Non-Signalling boxes \([11]\). This set consists of boxes that are local on the partition \( A_0A_1|B_0B_1 \), and in addition, the local decomposition has non-signalling components between \( A_0 \) and \( A_1 \) and between \( B_0 \) and \( B_1 \). The second assumption is also natural since we are interested in the broadcasting boxes, so the marginals \( A_0B_0 \) and \( A_1B_1 \) must be well defined. More formally,

Definition 5. A behavior \( Q(a, b | x, y) \) is local realistic non-signalling if it can be decomposed as:

\[
Q(a_0, a_1, b_0, b_1 | x_0, x_1, y_0, y_1) = \sum_{\lambda} r(\lambda)Q_{\lambda}(a_0, a_1 | x_0, x_1)Q_{\lambda}(b_0, b_1 | y_0, y_1),
\]

where \( Q_{\lambda}(a_0, a_1 | x_0, x_1) \) and \( Q_{\lambda}(b_0, b_1 | y_0, y_1) \) are non-signalling for all \(\lambda\). The set of all boxes with such properties will be denoted by \( LR_{\text{ns}} \).

Now that we have introduced the set we will be interested in the most, we can ask whether or not a box belongs to this set. In the negative case, when a behavior does not belong to that set, it is fair to be still interested in how “distant” this box is from the elements of that set. To answer, we can use the KL-divergence for correlations scenarios, which, as discussed, compares how different two boxes are. Using the KL-divergence for boxes, we introduce the relative entropy of nonlocality in the following way \([26]\):

Definition 6. The relative entropy of nonlocality of a behavior \( P(a, b | x, y) \) is given by:

\[
E_{\text{LR}}(P(a, b | x, y)) := \inf_{Q \in LR_{\text{ns}}} S(P(a, b | x, y) || Q(a, b | x, y)).
\]

Remark: The KL-divergence for correlation scenarios is not a proper distance measure, as it is not symmetric. Nevertheless, it is fair to say that the relative entropy of nonlocality defined by eq. (11) measures how distinct \( P(a, b | x, y) \) is from the probability distributions that admit a local-realistic hidden model.

H. Transforming boxes into boxes

1. LOSR-Transformations

A significant part of our work is dedicated to the (in)possibility of broadcasting boxes—in the sense of def. \([4]\). To do so, we need to specify which set of box transformations we will be considering and, more importantly, why. Naturally, we will work with transformations that map a bipartite system \(AB\) into a quadripartite system \(A_0A_1B_0B_1\). Nonetheless, we still want to keep some sort of locality, so that we will deal with transformations that can be implemented without any communication between Alice and Bob. Finally, in our framework, any kind of classical correlation between the parts is allowed. This class of operations is called LOSR (Local Operations and Shared-Randomness) transformations \([27]\). A useful characterization of these transformations is given by the following definition \([26]\).

Definition 7. A map \(M\) from \(AB\) to \(A_0A_1B_0B_1\) is a LOSR transformation if there exist local boxes \(I(x, y|x, y)\) and \(O(a, b|x, x, a, y, y, b)\) such that:

\[
P(a, b | x, y) = M(P'(a, b | x, y)) = \sum_{x, y, a, b} I(x, y|x, y)P'(a, b | x, y)O(a, b|x, x, a, y, y, b).
\]

An illustration of LOSR transformations is depicted in Fig.\([2]\).

An important property of LOSR transformations is that it transforms local boxes into local boxes, i.e., if \(P(a, b|x, y)\) is local, then \(P'(a, b|x, y) = M(P(a, b|x, y))\) will be also local \([27]\). It should be highlighted that the set of transformations that preserves the local boxes is a larger set than that of LOSR transformations. The standard example is WPICC maps, these are transformations that preserve locality, but their implementation
I. No-broadcasting theorem for nonlocal boxes

This section deals with our main no-go theorem for boxes. We show that it is impossible to broadcast a nonlocal box using only LR\textsubscript{ns}-LOSR transformations. Our argument is by contradiction. In fact, we will establish that if it were possible to broadcast a nonlocal box $P'(a, b|x, y)$ by LR\textsubscript{ns}-LOSR transformations, then we would have $E_{LR}(P(a, b|x, y)) < E_{LR}(P'(a, b|x, y))$, which is a contradiction.

The proof will be divided into two parts, which we will state as propositions. First, we will show that the relative entropy of nonlocality is contractive over the set of LR\textsubscript{ns}-LOSR transformations.

**Proposition 1.** If $M$ is a LR\textsubscript{ns}-LOSR transformation, then

$$E_{LR}(M(P(a, b|x, y))) \leq E_{LR}(P(a, b|x, y)). \quad (14)$$

The proof of the Proposition above is mainly based on two special properties. One is the fact that LR\textsubscript{ns}-LOSR transformations preserve the set of LR\textsubscript{ns} boxes. The other is the contractivity of the boxes KL-divergence over LOSR transformations (see eq. (13)). Details of the proof are given in Appendix [A].

The second proposition tells us that the amount of nonlocality that a broadcasting from a nonlocal box has is greater than the initial box. In fact,

**Proposition 2.** If $P(a, b|x, y)$ is a broadcasting of a nonlocal box $P'(a, b|x, y)$, then

$$E_{LR}(P(a, b|x, y)) > E_{LR}(P'(a, b|x, y)). \quad (15)$$

The proof of this proposition is more elaborate and the details are given in Appendix [B]. To summarize, we adapt the well-known chain rule of probability distributions for the box case. The chain rule is an identity between the KL-divergence of a joint probability distribution and the KL-divergence of one of the marginals of this distribution plus the mean of the KL-divergence of the conditional distributions (see eq. (12)). From this identity, we could recover the relative entropy of the marginal distribution as the first term of the Chain Rules of the expression. On the other hand, it was possible to prove if $P'(a, b|x, y)$ is nonlocal, then the second term is strictly positive. Concluding therefore that $E_{LR}(P(a, b|x, y)) > E_{LR}(P'(a, b|x, y))$.

We can now establish one of the main results of this work.

**Theorem 1** (No-broadcasting for boxes). It is impossible to broadcast nonlocal boxes using LR\textsubscript{ns}-LOSR transformations.

**Proof.** Let’s suppose, by contradiction, that there is a LR\textsubscript{ns}-LOSR transformation $M$ that broadcasts a nonlocal box $P'(a, b|x, y)$. Then, by **Proposition 1** we have:

$$E_{LR}(M(P'(a, b|x, y))) \leq E_{LR}(P'(a, b|x, y)) \quad (16)$$

requires communication between the parties [25, 26]. Despite being a subset of the transformations that preserve locality, as discussed in [27], the set of LOSR transformations is the most natural set of “free” transformations in the context of nonlocality. This comes from the fact that its implementation only uses resources that preserve the initial causal structure of the scenario, namely, classical correlations and local operations.

2. **LOSR Transformations and KL-Divergence**

An interesting property of the KL-divergence is that it is contractible over the set of LOSR transformations [28], i.e., if $P(a, b|x, y), Q(a, b|x, y)$ are two non-signalling boxes and $M$ is a LOSR transformation, then

$$S_b(M(P(a, b|x, y)))|\mathcal{M}(Q(a, b|x, y))) \leq S_b(P(a, b|x, y))|Q(a, b|x, y)). \quad (13)$$

As discussed in the previous section, for the problem of broadcasting, we are interested in the set of local boxes whose local decomposition is given by non-signalling components. With this in mind, the transformations that we will consider must also preserve the set of local realistic boxes in addition to being LOSR. Informally, we can summarize this discussion in the definition below.

**Definition 8.** A box transformation is said to be an LR\textsubscript{ns}-LOSR transformation when (i) it belongs to the set of LOSR transformations, and in addition, (ii) it transforms boxes from LR\textsubscript{ns} into boxes in LR\textsubscript{ns}.

![Fig. 2. Illustration of a LOSR transformation of a box. The box $P'$ is in green in the middle, with two wavy lines between the two parts representing possibly non-classical correlations, its inputs are $x$ and $y$, while the outputs are $a$ and $b$. The red boxes represent the input box $I$, while the blue ones the output box $O$. They have single wavy lines between them to represent classical correlations. The final box has inputs $x_0, x_1, y_0, y_1$ and outputs $a_0, a_1, b_0, b_1$ (colors online).](image)
On the other hand, as \(\mathcal{M}(P'(a, b|x, y))\) is a broadcasting of \(P'(a, b|x, y)\), by Proposition 2

\[
E_{LR}(P'(a, b|x, y)) < E_{LR}(\mathcal{M}(P'(a, b|x, y))). \tag{17}
\]

Therefore, by equations (16) and (17):

\[
E_{LR}(P'(a, b|x, y)) < E_{LR}(P'(a, b|x, y)), \tag{18}
\]

which is a contradiction. Thus, it is impossible to broadcast nonlocal boxes by LR-LOS transformations.

III. NO-BROADCASTING FOR STEERABLE ASSEMBLAGES

In this section, we show that steerable assemblages cannot be broadcast by local operations. To prove this claim, we adopt a strategy essentially analogous to the one we implemented for boxes. However, several adaptations are needed. In a bipartite steering scenario, the situation is described by a collection of ensembles of quantum states on Bob’s side together with a conditional probability distribution (boxes) on Alice’s. There is, then, right at the start, a combination of states and probabilities. This combination is reflected in our proof, for we are putting together the previous section’s strategies and the strategies from the no-local-broadcasting theorem for quantum states [28].

We start by recalling the basic ideas behind quantum steering—more extensive discussions abound, and we refer to [13-17]. Then, again based on the KL-divergence, we elaborate on a definition of relative entropy appropriate for the steering scenario. Following that, we address a set of physically motivated transformations that will become the basis of our analysis for the no-go theorem we conclude this section with.

A. Unsteerable assemblages in the Broadcasting scenario

Similarly to the correlation scenarios, steering scenarios are usually determined by the number of agents involved in the process, how those agents are grouped, and also by what resources those agents have access to. In a situation with multiple agents, one can think of granting a fraction of those agents tomographically complete measurements, whereas another fraction could be granted only boxes. The former group can talk about quantum states and measurements, and the latter can only provide aggregated statistics. Similar to Bell scenarios, the central purpose of steering theory is to describe correlations between the parties involved.

For concreteness, consider the bipartite scenario \(A\)B, involving two parts (two agents), namely \(A\) and \(B\). The characterization of the part \(B\) is given by a family of ensembles of quantum states, whereas the description of the part \(A\) is given by conditional probabilities \(p(a|x)\). Each input of part \(A\) is associated with an ensemble of the part \(B\), and each output is associated with a member state of such an ensemble. In fact, if part \(A\) has inputs \(x \in [r]\) and outputs \(a \in [s]\), the physical description of the system on \(B\)’s side for the pair \(a|x\) is given by a quantum state \(\rho_{a|x} \in \mathcal{L}(\mathcal{H}_B)\). We can join both descriptions using the idea of assemblage. An assemblage is a collection of unnormalized density matrices \(\{\varrho_{a|x}\}_{a,x} \subset \mathcal{L}(\mathcal{H}_B)\), where \(\varrho_{a|x} = p(a|x)\rho_{a|x}\).

We will restrict ourselves to non-signalling assemblages, i.e., those assemblages where the reduced state of the part \(B\) does not depend on the input choice of \(A\). More formally, an assemblage \(\{\varrho_{a|x}\}\) is non-signalling if \(\sum_a \varrho_{a|x} = \rho_B, \forall x [17]\).

As we mentioned, analogously to Bell scenarios, the central purpose of the steering framework is to describe correlations between the parties involved. Due to the nature of the framework, this analysis is performed on the assemblages, rather than merely on the probability distributions arising from boxes—the behaviors. An assemblage is said unsteerable if the correlations between \(A\) and \(B\) can be generated from a classical latent variable. In this case, we say that the assemblage admits a local hidden state (LHS) model. More formally, an assemblage \(\{\varrho_{a|x}\}\) is unsteerable if each element can be written as

\[
\varrho_{a|x} = \sum_\lambda r(\lambda)p_{\lambda}(a|x)\rho_\lambda, \tag{19}
\]

where \(\rho_\lambda\) are density matrices in \(\mathcal{L}(\mathcal{H}_B)\) and \(r(\lambda), p_{\lambda}(a|x)\) are probability distributions. Assemblages that are not unsteerable are said to be steerable [16]. We should note the similarity between the definition of local boxes, eq. [2], and unsteerable assemblages, eq. [19]. Both definitions follow the same idea, that correlations between parts can be generated from, and mediated by, a classical latent variable \(\lambda\).

An alternative representation of an assemblage is given by a set of quantum states \(\{\rho_{AB}(x)\}_{x} \subset \mathcal{D}(\mathcal{H}_E \otimes \mathcal{H}_B)\), given by:

\[
\rho_{AB}(x) = \sum_{a=1}^{s} |a\rangle\langle a| \otimes \varrho_{a|x} = \sum_{a=1}^{s} p(a|x) |a\rangle\langle a| \otimes \rho_{a|x}. \tag{20}
\]

Furthermore, if we take into account the probability distribution of the choice of inputs of the part \(A\) we can even associate a unique quantum state to an assemblage. Assuming that \(\pi(x)\) is such a probability distribution, we define \(\rho_{XAB} \in \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_E \otimes \mathcal{H}_B)\) to be the quantum state associated with an assemblage \(\varrho_{a|x}\) and a probability distribution \(\pi(x)\) as:

\[
\rho_{XAB} = \sum_{x=1}^{r} \pi(x) |x\rangle\langle x| \otimes \rho_{AB}(x)
= \sum_{x=1}^{r} \sum_{a=1}^{s} \pi(x)p(a|x) |x\rangle\langle x| \otimes |a\rangle\langle a| \otimes \rho_{a|x}. \tag{21}
\]
The sets of vectors \( \{ |x\rangle \}; \{ |a\rangle \} \) are orthonormal basis of the auxiliary Hilbert spaces \( \mathcal{H}_V, \mathcal{H}_E \). The states \( |x\rangle, |a\rangle \) are merely abstract flag states to express the inputs and outputs of the part \( A \), respectively. They do not describe the system inside the box on \( A \)’s side \([22, 30]\).

Now we have the ingredients to introduce the central notion of broadcasting for assemblages. Let’s start with an assemblage \( \{ \rho'_a|_x \} \subset \mathcal{L}(\mathcal{H}_B) \), where \( x \in [r] \) and \( a \in [s] \) labels the inputs and outputs on \( A \)’s side respectively. Let \( \{ \rho_{a_0a_1}|_{x_0x_1} \} \subset \mathcal{L}(\mathcal{H}_{B_0} \otimes \mathcal{H}_{B_1}) \) be another assemblage, where \( x_0, x_1 \in [r] \), \( a_0, a_1 \in [s] \) and \( \mathcal{H}_{B_0} = \mathcal{H}_B = \mathcal{H}_{B_1} \). We say that \( \{ \rho_{a_0a_1}|_{x_0x_1} \} \) is a broadcast of \( \{ \rho'_a|_x \} \) if the marginals \( A_0B_0 \) and \( A_1B_1 \) of \( \{ \rho_{a_0a_1}|_{x_0x_1} \} \) are equal to \( \{ \rho'_a|_x \} \). More formally:

**Definition 9** (Broadcasting for Assemblages). We say that \( \{ \rho_{a_0a_1}|_{x_0x_1} \} \) is a broadcast of \( \{ \rho'_a|_x \} \) if:

\[
\sum_{a_1} \text{tr}_{B_1}(\rho_{a_0a_1}|_{x_0x_1}) = \rho'_a|_{x_0}, \; \forall a_0, x_0, x_1 \tag{22a}
\]

and

\[
\sum_{a_0} \text{tr}_{B_0}(\rho_{a_0a_1}|_{x_0x_1}) = \rho'_a|_{x_1}, \; \forall a_1, x_0, x_1. \tag{22b}
\]

**Remark:** Notice that while the broadcasting for behaviors and for states are defined via marginalizing and tracing-out larger objects onto smaller ones, our definition uses both marginalization and partial trace. This combination should be thought of as reflecting the fact that assemblages are something in-between behaviors and states.

Analogously to the box broadcasting scenario, Section [11, 30] the broadcasting scenario configures a multipartite scenario, however, the concept of unsteerability that we will use is related to the bipartition \( A_0A_1|B_0B_1 \).

Again, this is the natural partition, as the resource is spatially divided with \( A_0A_1 \) on one side and \( B_0B_1 \) on the other. With that in mind, and in parallel with the previous case for boxes, we define the set of *Unsteerable-Realistic Non-Signalling assemblages*. This set consists of assemblages that are unsteerable on the partition \( A_0A_1|B_0B_1 \), and in addition, the decomposition has non-signalling components between \( A_0 \) and \( A_1 \). The second assumption is also natural since we are interested in broadcasting assemblages, so the marginals \( A_0B_0 \) and \( A_1B_1 \) ought to be well defined (see eq. (22)). More formally,

**Definition 10.** An assemblage \( \varsigma_{a_0a_1}|_{x_0x_1} \) is Unsteerable-Realistic, if it can be decomposed as:

\[
\varsigma_{a_0a_1}|_{x_0x_1} = \sum_{\lambda} r(\lambda) q(\lambda, a_0, a_1|x_0, x_1) \sigma_\lambda. \tag{23}
\]

where \( q(\lambda, a_0, a_1|x_0, x_1) \) are non-signalling for all \( \lambda \). The set of all Unsteerable-Realistic assemblages will be denoted by \( \mathcal{U} \mathcal{R} \).

### B. Relative Entropy of Steering

#### 1. Quantum KL-Divergence

We proved the no-broadcasting theorem for behaviors in an informational manner. We defined a monotone information measure for some physically motivated transformations, which signalled the increase of the same resource if we sent it through a broadcasting channel. As we will see, the strategy for the steering case follows the same lines. In this sense, we need to decide upon an appropriate monotone. For obvious reasons we will use the quantum version of the Kullback-Leibler divergence.

The quantum Kullback-Leibler divergence, also known as quantum relative entropy, of two density matrices \( \rho, \sigma \in \mathcal{D}(\mathcal{H}) \) is given by \([31]\):

\[
S_Q(\rho||\sigma) = \text{tr}[\rho(\log(\rho) - \log(\sigma))]. \tag{24}
\]

The quantum KL-divergence was first studied by Umegaki \([32]\) as the noncommutative extension of KL-divergence \(-\varepsilon_{ij} \). It also has the same statistical interpretation as its classical analogue: it tells us how hard it is to differentiate the state \( \rho \) from the state \( \sigma \) \([33, 41]\).

Analogously to the classical KL-divergence, quantum KL-divergence is (i) non-negative; (ii) it is zero if, and only if, the quantum states are equal; (iii) it is not symmetric in \( \rho \) and \( \sigma \); and (iv) it can potentially be equal infinity. Another interesting property of quantum KL-divergence is the contractivity over quantum channel actions \([34]\).

Indeed, for any completely positive trace-preserving map \( \mathcal{E} \), we have that \( S_Q(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq S_Q(\rho||\sigma) \). This property preserves this property that we will use in the following paragraphs.

#### 2. KL-Divergence for Assemblages

We have seen that for an assemblage \( \{ \theta_a|_x \} \) and a probability distribution \( \pi(x) \) we can associate to them a quantum state \( \rho_{XAB} \) — eq. (21). With this association, we can make emerge in the steering scenario concepts developed for quantum states \([36]\). For instance, we can use the KL-divergence of quantum states along with the classical-quantum state associated to an assemblage to define the KL-divergence for assemblages \([30, 37]\).

**Definition 11.** Given \( \{ \theta_a|_x \}, \{ \varsigma_a|_x \} \subset \mathcal{L}(\mathcal{H}_B) \), the KL-divergence for assemblages between \( \{ \theta_a|_x \} \) and \( \{ \varsigma_a|_x \} \) is given by:

\[
S_A(\{ \theta_a|_x \}||\{ \varsigma_a|_x \}) = \sup_{\pi(x)} S_Q(\rho_{XAB}||\sigma_{XAB}), \tag{25}
\]

where \( \rho_{XAB} \) and \( \sigma_{XAB} \) are classical-quantum states associated to \( \theta_a|_x \) and \( \varsigma_a|_x \), respectively, over the same probability distribution \( \pi(x) \).
Since we already have a set of assemblages of interest — the set $UR_{ns}$ — and a divergence, we can introduce a relative entropy in the steering scenario. Such a definition was initially introduced in [30]. However, in [30] the set of free operations used is 1W-LOCC, while in our work we focus on LOSR transformations. This enables us to introduce a simplified version of the relative entropy introduced in [30] — we refer to [37] for more details.

**Definition 12.** The Relative Entropy of Steering is defined as:

$$E_A\left(\{\varrho_{a|0x_0}\}\right) = \inf_{\varrho_{0x_0} \in UR_{ns}} S_A(\{\varrho_{a|0x_0}\}||\{\varrho_{a|0x_0}\}),$$  

where the infimum is taken over all possible assemblages that accept an unsteerable-realistic decomposition—see def. 10 and the discussion around it.

**C. LOSR transformation and the no-local-broadcasting theorem for assemblages**

Analogously to the correlation scenario, we are interested in transformations that do not require communication between parts $A_0A_1$ and $B_0B_1$. Nevertheless, any kind of local operations and classical correlations are, in principle, allowed to them. In this way, we enter into the LOSR paradigm again, where the difference from the case of boxes is that Bob now acts on a quantum state [38]. As we are interested in the (im)possibility of performing broadcasting by this type of transformation, we will introduce the special class of maps LOSR between an arbitrary steering scenario $AB$ and its expanded broadcasting scenario $A_0A_1B_0B_1$.

**Definition 13.** A map $M$ from $\mathcal{L}(\mathcal{H}_B)$ to $\mathcal{L}(\mathcal{H}_{B_0} \otimes \mathcal{H}_{B_1})$ is a LOSR transformation between assemblages if it can be decomposed as

$$\varrho_{a|0x_0} = M(\varrho_{a|x}) = \sum_{\lambda,c} \sum_a r(\lambda) I(c,x|0x_1,\lambda) p(a|x) O(a_0,a_1|a,c) E_{\lambda}(\varrho_{a|x}),$$

where $r(\lambda), I(c,x|0x_1,\lambda), O(a_0,a_1|a,c)$ are probability distributions on appropriate finite alphabets, and $E_{\lambda}$ are CPTP maps. See Fig.3 for an illustration.

**D. A no-broadcasting theorem for assemblages**

Now we are ready to formulate our main no-go theorem for the case of assemblages.

**Theorem 2.** [No-broadcasting for assemblages] It is impossible to broadcast steerable assemblages using $UR_{ns}$-LOSR transformations.

This will be proven using the relative entropy of steering. A fundamental property of Relative Entropy of Steering is that it is contractive over LOSR. This result was demonstrated in [30] and [37]. For the sake of completeness, we re-state it below:
Proposition 3. If $M$ is a LOSR transformation, then:
\[
E_A(M(\rho_{a|x})) \leq E_A(\rho_{a|x}).
\] (28)

This is the first property that is required to have a contradiction proof just as in Theorem 1

The other desired property is the following. If $\{\rho_{a|x_1}^0\}$ is a broadcasting of $\rho_{a|x}$, then the relative entropy steering of the former must be strictly greater than that of latter. The proof of this result follows in a similar way to Proposition 2. The main difference is that in place of the chain rule, we will use Proposition 3.

Proposition 4. If $\{\rho_{a|x_1}^0\}$ is a broadcasting of a steerable assemblage $\{\rho_{a|x}\}$, then:
\[
E_A(\rho_{a|x_1}^0) > E_A(\rho_{a|x}).
\] (29)

The proof of the proposition can be found in Appendix C.

At this point we see that we have similar propositions as in the box case and we only need to exchange the respective propositions to arrive at showing Theorem 2; see Appendix X for more details.

IV. DISCUSSION

In this work, we provided negative answers for the possibility of local broadcasting for either boxes or assemblages by using fundamental properties of the relative entropy of nonlocality and the relative entropy of steering. Answering, therefore, the long-standing conjecture raised Joshi, Grudka and Horodecki.

Similarly to the quantum case, we concluded that the local copying of information from assemblages and boxes is impossible, provided that the object in question possesses a non-classical resource such as steerability or non-locality. Since non-steerable assemblages, and local boxes are locally broadcastable, we may say that failure of local broadcasting indicates the non-classicality of the correlations. As we mentioned in the introduction, the fact we were able to prove alternative forms of no-broadcasting theorems for other resources underlines the importance of the standard version of the no-broadcasting theorem—the impossibility of copying unknown information is not distinctive of the formalism of quantum theory, but of a broader class of probabilistic models satisfying no-signalling criteria.

Generalized versions of the local no-broadcasting theorem might also exist for any non-classical finite-dimensional probabilistic model satisfying a no-signalling criterion. It is also possible that such a generalized local no-broadcasting theorem is equivalent to the original generalized no-broadcasting theorem of Barnum, Barrett, Leifer and Wilce. We leave those points as open questions to be investigated in the future.

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Appendix A: Proof of Proposition 1

Proposition 1. If $\mathcal{M}$ is a $LR_{\text{ns}}$-LOSR transformation, then

$$E_{LR}(\mathcal{M}(P(a,b|x,y))) \leq E_{LR}(P(a,b|x,y)). \quad (14)$$

Proof. By definition,

$$E_{LR}(\mathcal{M}(P(a,b|x,y))) = \inf_{Q \in LR_{\text{ns}}} S_b(\mathcal{M}(P(a,b|x,y))||Q(a,b|x,y)) \quad (A1)$$

As $\mathcal{M}$ is $LR_{\text{ns}}$ preserving, then the image of $\mathcal{M}$ over the set of local boxes is a subset of $LR_{\text{ns}}$. Therefore, if we take...
the infimum of the equation above over the set $\text{Im}(\mathcal{M})$, we will get a upper bound for $E_{LR}(\mathcal{M}(P(a,b|x,y)))$. Indeed,

$$E_{LR}(\mathcal{M}(P(a,b|x,y))) \leq \inf_{Q \in \text{Im}(\mathcal{M})} S_b(\mathcal{M}(P(a,b|x,y))) |Q(a,b|x,y))$$

$$= \inf_{Q \in \text{LR}_{\text{ns}}} S_b(\mathcal{M}(P(a,b|x,y))) |\mathcal{M}(Q'(a,b|x,y))).$$

(A2)

Now, by the contractivity of $S_b$ over LOSR transformation — eq. [13], we have:

$$E_{LR}(\mathcal{M}(P(a,b|x,y))) \leq \inf_{Q \in \text{LR}_{\text{ns}}} S_b(\mathcal{M}(P(a,b|x,y))) |\mathcal{M}(Q'(a,b|x,y))) = E_{LR}(P(a,b|x,y)).$$

(A3)

\[ \square \]

Appendix B: Proof of Proposition 2

First, let’s introduce some concepts involving probability distributions and KL-divergence. Given two joint probability distributions $p(a,b), q(a,b)$, let $p(b|a), q(b|a)$ be the conditional probabilities distributions. An interesting identity relating the KL-divergence of the joint probability distribution with the KL-divergence of the marginal and the average KL-divergence of the conditional is given by:

$$S(p(a,b)||q(a,b)) = S(p(a)||q(a)) + \sum_a p(a)S(p(b|a)||q(b|a)).$$

(B1)

This identity is called by Chain Rule.

Let $P(a,b|x,y)$ and $Q(a,b|x,y)$ be non-signalling boxes. If we fix the inputs $x, y$, then we can apply the Chain Rule for $S(P(a,b|x,y)||Q(a,b|x,y))$, resulting in:

$$S(P(a,b|x,y)||Q(a,b|x,y)) = S(P(a_0,b_0|x_0,y_0)||Q(a_0,b_0|x_0,y_0))$$

$$+ \sum_{a_0,b_0} P(a_0,b_0|x_0,y_0)S(P(a_1,b_1|x,y,a_0,b_0)||Q(a_1,b_1|x,y,a_0,b_0)),$$

(B2)

where, In eqs. [B2] we have used the non-signalling feature for $P(a,b|x,y)$ and $Q(a,b|x,y)$.

In order to prove Proposition 2 we will need to prove some preliminaries results.

**Lemma 1.** Let $P(a,b|x,y)$ be a NS box s.t the marginal $P(a_1,b_1|x_1,y_1)$ is nonlocal. Then, for every $x_0,y_0$, there exist $a_0,b_0$ such that $P(a_1,b_1|x,y,a_0,b_0)$ is nonlocal and $P(a_0,b_0|x_0,y_0) \neq 0$.

**Proof.** By the definition of a conditional probability distribution:

$$P(a_1,b_1|a_0,b_0,x,y) = \frac{P(a_0,a_1,b_0,b_1|x,y)}{P(a_0,b_0|x_0,y_0)}$$

(B3)

Thus,

$$P(a_0,a_1,b_0,b_1|x,y) = P(a_0,b_0|x_0,y_0)P(a_1,b_1|a_0,b_0,x,y).$$

(B4)

In this way, summing the equation above over $a_0,b_0$, we have:

$$P(a_1,b_1|x_1,y_1) = \sum_{a_0,b_0} P(a_0,a_1,b_0,b_1|x,y) = \sum_{a_0,b_0} P(a_0,b_0|x_0,y_0)P(a_1,b_1|a_0,b_0,x,y).$$

(B5)

Given $x_0,y_0$, let’s suppose, by contradiction, that for every $a_0,b_0$ with $P(a_0,b_0|x_0,y_0) \neq 0$ we have that $P(a_1,b_1|a_0,b_0,x,y)$ is local. Thus, for each $a_0,b_0$ with $P(a_0,b_0|x_0,y_0) \neq 0$ there exist $r_{x_0,y_0,a_0,b_0}(\lambda), P_{x_0,y_0,a_0,b_0,\lambda}(a_1|x_1)$ and $P_{x_0,y_0,a_0,b_0,\lambda}(b_1|y_1)$ such that:

$$P(a_1,b_1|a_0,b_0,x,y) = \sum_{\lambda} r_{x_0,y_0,a_0,b_0}(\lambda)P_{x_0,y_0,a_0,b_0,\lambda}(a_1|x_1)P_{x_0,y_0,a_0,b_0,\lambda}(b_1|y_1).$$

(B6)
Thus,

\[ P(a_1, b_1|x_1, y_1) = \sum_{a_0, b_0} P(a_0, b_0|x_0, y_0)P(a_1, b_1|a_0, b_0, x, y) \]

\[ = \sum_{\lambda} \sum_{a_0, b_0} P(a_0, b_0|x_0, y_0)r_{x_0, y_0, a_0, b_0}(\lambda)P_{x_0, y_0, a_0, b_0, \lambda}(a_1|x_1)P_{x_0, y_0, a_0, b_0, \lambda}(b_1|y_1). \quad (B7) \]

Let \( \lambda' = (a_0, b_0, \lambda) \) and \( r_{x_0, y_0}(\lambda') = P(a_0, b_0|x_0, y_0)r_{x_0, y_0, a_0, b_0}(\lambda) \). Then it's easily seen that \( r_{x_0, y_0}(\lambda') \) is a probability distribution on the variable \( \lambda' \), i.e., \( r_{x_0, y_0}(\lambda') \geq 0 \) and \( \sum_{\lambda'} r_{x_0, y_0}(\lambda') = 1 \). We also have:

\[ P(a_1, b_1|x_1, y_1) = \sum_{\lambda'} r_{x_0, y_0}(\lambda')P_{x_0, y_0, \lambda'}(a_1|x_1)P_{x_0, y_0, \lambda'}(b_1|y_1). \quad (B8) \]

Therefore, \( P(a_1, b_1|x_1, y_1) \) is local, which is a contradiction with the hypothesis.

**Lemma 2.** If \( Q(a, b|x, y) \) be a LR\(_{ns} \) box, then \( Q(a_1, b_1|x, y, a_0, a_1) \) will be local for every choice of \( a_0, b_0, x_0, y_0 \).

**Proof.** Let’s start fixing \( a_0, b_0, x_0, y_0 \). By definition,

\[ Q(a_1, b_1|a_0, b_0, x, y) = \frac{Q(a_0, a_1, b_0, b_1|x, y)}{Q(a_0, b_0|x_0, y_0)}. \quad (B9) \]

As \( Q(a, b|x, y) \) is a LR\(_{ns} \) box, we have:

\[ Q(a, b|x, y) = \sum_{\lambda} r(\lambda)Q_\lambda(a_0, a_1|x_0, x_1)Q_\lambda(b_0, b_1|y_0, y_1), \quad (B10) \]

where \( Q_\lambda(a_0, a_1|x_0, x_1) \) and \( Q_\lambda(b_0, b_1|y_0, y_1) \) are non-signalling. So, we have:

\[ Q_\lambda(a_0, a_1|x_0, x_1) = Q_\lambda(a_1|x_0, x_1, a_0)Q_\lambda(a_0|x_0, x_1) = Q_\lambda(a_1|x_0, x_1, a_0)Q_\lambda(a_0|x_0). \quad (B11) \]

and

\[ Q_\lambda(b_0, b_1|y_0, y_1) = Q_\lambda(b_1|y_0, y_1, b_0)Q_\lambda(b_0|y_0, y_1) = Q_\lambda(b_1|y_0, y_1, b_0)Q_\lambda(b_0|y_0). \quad (B12) \]

Therefore,

\[ Q(a, b|x, y) = \sum_{\lambda} r(\lambda)Q_\lambda(a_0|x_0, x_1, a_0)Q_\lambda(b_0|y_0, y_1)Q_\lambda(b_1|y_0, y_1, b_0) \quad (B13) \]

\[ = \sum_{\lambda} r(\lambda)Q_\lambda(a_0|x_0)Q_\lambda(b_0|y_0)Q_\lambda(a_1|x_0, x_1, a_0)Q_\lambda(b_1|y_0, y_1, b_0) \quad (B14) \]

Then,

\[ Q(a_1, b_1|a_0, b_0, x, y) = \sum_{\lambda} \frac{r(\lambda)Q_\lambda(a_0|x_0, x_1, a_0)Q_\lambda(b_0|y_0)}{Q(a_0, b_0|x_0, y_0)}Q_\lambda(a_1|x_0, x_1, a_0)Q_\lambda(b_1|y_0, y_1, b_0) \quad (B15) \]

Let

\[ r_{a_0, b_0, x_0, y_0}(\lambda) = \frac{r(\lambda)Q_\lambda(a_0|x_0)Q_\lambda(b_0|y_0)}{Q(a_0, b_0|x_0, y_0)}. \quad (B16) \]

It follows that \( r_{a_0, b_0, x_0, y_0}(\lambda) \geq 0 \). On the other hand,

\[ Q(a_0, b_0|x_0, y_0) = \sum_{a_1, b_1} Q(a_0, a_1, b_0, b_1|x, y) = \sum_{a_1, b_1} \sum_{\lambda} r(\lambda)Q_\lambda(a_0|x_0)Q_\lambda(b_0|y_0)Q_\lambda(a_1|x_0, x_1, a_0)Q_\lambda(b_1|y_0, y_1, b_0) \]

\[ = \sum_{\lambda} r(\lambda)Q_\lambda(a_0|x_0)Q_\lambda(b_0|y_0) \left( \sum_{a_1} Q_\lambda(a_1|x_0, x_1, a_0) \right) \left( \sum_{b_1} Q_\lambda(b_1|y_0, y_1, b_0) \right) \]

\[ = \sum_{\lambda} r(\lambda)Q_\lambda(a_0|x_0)Q_\lambda(b_0|y_0). \quad (B17) \]
Therefore,
\[ \sum_{\lambda} r_{a_0,b_0,x_0,y_0}(\lambda) = \sum_{\lambda} r(\lambda)Q_{\lambda}(a_0|x_0)Q_{\lambda}(b_0|y_0) = 1. \]  
(B18)

So, \( r_{a_0,b_0,x_0,y_0}(\lambda) \) is a probability distribution in \( \lambda \). In this way:
\[ Q(a_1,b_1|a_0,b_0,x,y) = \sum_{\lambda} r_{a_0,b_0,x_0,y_0}(\lambda)Q_{\lambda}(a_1|x_0,x_1,a_0)Q_{\lambda}(b_1|y_0,y_1,b_0). \]  
(B19)

Thus, \( Q(a_1,b_1|a_0,b_0,x,y) \) is local. \( \square \)

**Lemma 3.** If \( Q(a,b|x,y) \) is a LR\(_{ns} \) box and \( P(a,b|x,y) \) is a NS box with \( P(a_1,b_1|x_1,y_1) \) nonlocal, then
\[ \max_{x_1,y_1} \sum_{a_0,b_0} P(a_0,b_0|x_0,y_0)S(P(a_1,b_1|x,y,a_0,b_0)||Q(a_1,b_1|x,y,a_0,b_0)) > 0, \]
for every \( x_0,y_0 \).

**Proof.** Given \( x_0,y_0 \), by Lemma 1 there exist \( a_0',b_0' \) such that \( P(a_1,b_1|x,y,a_0',b_0') \) is nonlocal and \( P(a_0',b_0'|x_0,y_0) \neq 0 \). On the other hand, by Lemma 2 \( Q(a_1,b_1|x,y,a_0',b_0') \) is local. Therefore:
\[ \max_{x_1,y_1} \sum_{a_0,b_0} P(a_0,b_0|x_0,y_0)S(P(a_1,b_1|x,y,a_0,b_0)||Q(a_1,b_1|x,y,a_0,b_0)) \geq \max_{x_1,y_1} P(a_0',b_0'|x_0,y_0)S(P(a_1,b_1|x,y,a_0',b_0')||Q(a_1,b_1|x,y,a_0',b_0')) \]
(B21)

But, as boxes, \( P(a_1,b_1|x,y,a_0',b_0') \) and \( Q(a_1,b_1|x,y,a_0',b_0') \) are different, since the first is nonlocal while the second is local. Therefore,
\[ S(P(a_1,b_1|x,y,a_0',b_0')||Q(a_1,b_1|x,y,a_0',b_0')) > 0. \]
Thus,
\[ \max_{x_1,y_1} \sum_{a_0,b_0} P(a_0,b_0|x_0,y_0)S(P(a_1,b_1|x,y,a_0,b_0)||Q(a_1,b_1|x,y,a_0,b_0)) > 0. \]  
(B22)

**Lemma 4.** There is always \( \tilde{Q}(a,b|x,y) \in LR_{ns} \) such that \( E_{LR}(P(a,b|x,y)) = S_0(P(a,b|x,y)||\tilde{Q}(a,b|x,y)) \). In other words, the infimum on the definition of the relative entropy of nonlocality is always achieved.

**Proof.** It’s well known that the set of LR\(_{ns} \) boxes is a polytope, therefore is a compact set. On the other hand, fixing a box \( P(a,b|x,y) \), let’s define \( f(Q(a,b|x,y)) := S_0(P(a,b|x,y)||Q(a,b|x,y)) \) is lower semi-continuous in on the set of LR\(_{ns} \) boxes. Therefore, by the extended Bolzano–Weierstrass theorem, the function \( f \) has a minimum in LR\(_{ns} \). \( \square \)

Given these results, let’s now move on to the main goal of this section, prove **Proposition 2**

**Proposition 2.** If \( P(a,b|x,y) \) is a broadcasting of a nonlocal box \( P'(a,b|x,y) \), then
\[ E_{LR}(P(a,b|x,y)) > E_{LR}(P'(a,b|x,y)). \]  
(15)

**Proof.** By the definitions,
\[ E_{LR}(P(a,b|x,y)) = \inf_{Q \in LR_{ns}} S_0(P(a,b|x,y)||Q(a,b|x,y)) \]
\[ = S_0(P(a,b|x,y)||\tilde{Q}(a,b|x,y)) \]
(B23)

where we have used Lemma 4 in the last equation. Then,
\[ E_{LR}(P(a,b|x,y)) = S_0(P(a,b|x,y)||\tilde{Q}(a,b|x,y)) \]
\[ = \max_{x,y} S(P(a,b|x,y)||\tilde{Q}(a,b|x,y)). \]  
(B24)
Using now (B2), we have:

\[ E_{LR}(P(a, b | x, y)) \]

\[ = \max_{x, y} \left[ S(P(a_0, b_0 | x_0, y_0) || \bar{Q}(a_0, b_0 | x_0, y_0)) + \sum_{a_0, b_0} P(a_0, b_0 | x_0, y_0) S(P(a_1, b_1 | x, y, a_0, b_0) || \bar{Q}(a_1, b_1 | x, y, a_0, b_0)) \right] \]

\[ = \max_{x_0, y_0} \max_{x_1, y_1} \left[ S(P(a_0, b_0 | x_0, y_0) || \bar{Q}(a_0, b_0 | x_0, y_0)) + \sum_{a_0, b_0} P(a_0, b_0 | x_0, y_0) S(P(a_1, b_1 | x, y, a_0, b_0) || \bar{Q}(a_1, b_1 | x, y, a_0, b_0)) \right] \]

\[ = \max_{x_0, y_0} \left[ S(P(a_0, b_0 | x_0, y_0) || \bar{Q}(a_0, b_0 | x_0, y_0)) + \max_{x_1, y_1} \sum_{a_0, b_0} P(a_0, b_0 | x_0, y_0) S(P(a_1, b_1 | x, y, a_0, b_0) || \bar{Q}(a_1, b_1 | x, y, a_0, b_0)) \right]. \quad (B25) \]

Applying Lemma 3

\[ E_{LR}(P(a, b | x, y)) > \max_{x_0, y_0} S(P(a_0, b_0 | x_0, y_0) || \bar{Q}(a_0, b_0 | x_0, y_0)) \]

\[ = S_0(P(a_0, b_0 | x_0, y_0)). \quad (B26) \]

It’s trivial to see that \( \bar{Q}(a_0, b_0 | x_0, y_0) \) is a local box, therefore:

\[ E_{LR}(P(a, b | x, y)) > \inf_{Q \in LR_{as}} S_0(P(a_0, b_0 | x_0, y_0) || Q(a_0, b_0 | x_0, y_0)) \]

\[ = E_{LR}(P'(a, b | x, y)), \quad (B27) \]

where in the last equation we have used that as \( P(a, b | x, y) \) is a broadcast of \( P'(a, b | x, y) \), then \( P(a_0, b_0 | x_0, y_0) = P'(a_0, b_0 | x_0, y_0) \).

\[ \textbf{Appendix C: Proof of Proposition 4} \]

Let’s first introduce the concept of a measurement map

**Definition 15.** Given a POVM \( \{ F_i \}_{i=1}^n \subset \mathcal{L}(\mathcal{H}) \) a POVM, the measurement map associate it is define as \( F : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathbb{C}^n) \), with

\[ F(X) = \sum_{i=1}^n \text{tr}(F_i X) |i\rangle\langle i|, \quad (C1) \]

where \( \{ |i\rangle \} \) is an orthogonal basis.

It’s easy to see that every measurement map is a CPTP. In this way, as the quantum relative entropy is contractible over CPTPs maps, then

\[ S_Q(F(\rho) || F(\sigma)) \leq S_Q(\rho || \sigma) \quad \forall \rho, \sigma \in \mathcal{L}(\mathcal{H}). \quad (C2) \]

On the other hand, let \( \rho_{XY} \) and \( \sigma_{XY} \) be bipartite quantum states and \( F : \mathcal{L}(\mathcal{H}_X) \rightarrow \mathcal{L}(\mathbb{C}^n) \) a measurement map. In [28], Piani showed the following inequality:

\[ S_Q(\rho_{XY} || \sigma_{XY}) \geq S_Q(F(\rho_X) || F(\sigma_X)) + S_Q\left( \rho_Y || \sum_k \alpha_k \sigma_Y^k \right), \quad (C3) \]

where \( \alpha_k = \text{tr}(F_k \rho_X) \) and \( \sigma_Y^k = \frac{\text{tr}_2((F_k \otimes I) \sigma_{XY})}{\text{tr}_2((F_k \otimes I) \sigma_{XY})} \).

Going back to our goal, let \( \{ \theta_{a_0|x_0, x_1} \} \) and \( \{ \varsigma_{a_0|x_0, x_1} \} \) be two non-signalling assemblages in the a broadcasting scenario. Let \( \pi(x_0, x_1) = \pi_0(x_0) \pi_1(x_1) \) be a independent probability distribution for \( A_0A_1 \) choices. In order not to
overload the notation, we will denote $Z = X_0A_0B_0$ and $W = X_1A_1B_1$. The CQ-states associated to the assemblage \( \{a_{0|a_1|0}, a_{1|a_1|0}\} \) are, respectively:

\[
\rho_{ZW} = \sum_{a_0,a_1,x_0,x_1} \pi_0(x_0)\pi_1(x_1)p(a_0,a_1|x_0,x_1)|x_0x_1\rangle\langle x_0x_1| \otimes |a_0a_1\rangle\langle a_0a_1| \otimes \rho_{a_0a_1|x_0x_1},
\]

and

\[
\sigma_{ZW} = \sum_{a_0,a_1,x_0,x_1} \pi_0(x_0)\pi_1(x_1)q(a_0,a_1|x_0,x_1)|x_0x_1\rangle\langle x_0x_1| \otimes |a_0a_1\rangle\langle a_0a_1| \otimes \sigma_{a_0a_1|x_0x_1},
\]

where \( q_\lambda(a_0,a_1|x_0,x_1) \) is non-signalling for all \( \lambda \).

Now, let \( \{E_i\}_{i=1}^n \subset \mathcal{L}(\mathcal{H}_{B_0}) \) be an arbitrary Informational Complete POVM (IC-POVM). In this way, we define \( \{F_k\} \subset \mathcal{L}(C^* \otimes C^* \otimes \mathcal{H}_{B_0}) \) as \( F_k = F_{x_0,a_0,i} = |x_0\rangle\langle x_0| \otimes |a_0\rangle\langle a_0| \otimes E_i \) (we are using the abbreviation \( k = (x_0,a_0,I) \)). It is easy to see that \( \{F_{x_0,a_0,i}\} \) is a POVM. Let \( \mathcal{F} : \mathcal{L}(C^* \otimes C^* \otimes \mathcal{H}_{B_0}) \to \mathcal{L}(C^* \otimes C^* \otimes C^\alpha) \) the measurement map associated to \( F_k \), i.e.,

\[
\mathcal{F}(X) = \sum_{x_0,a_0,i} \text{tr}[|x_0\rangle\langle x_0| \otimes |a_0\rangle\langle a_0| \otimes E_i]X|x_0\rangle\langle x_0| \otimes |a_0\rangle\langle a_0| \otimes |i\rangle\langle i|. \]

Then, apply eq. (C3) for the states \( \rho_{ZW}, \sigma_{ZW} \) and the measurement map \( \mathcal{F} \), we have:

\[
S_Q(\rho_{ZW}|\sigma_{ZW}) \geq S_Q(\mathcal{F}(\rho_{ZW})|\mathcal{F}(\sigma_{ZW})) + S_Q\left(\rho_W||\sum_k \alpha_k \sigma_W^k\right).
\]

We will now demonstrate some auxiliary statements that will help us reach the desired conclusion.

**Lemma 5.** Let \( \{\tau_{a_0|a_1|0}\} \) a non-signalling assemblage and \( \pi(x_0,x_1) \) a probability distribution. If \( \tau_{ZW} \) be the CQ-state associated to \( \{\tau_{a_0|a_1|0}, \pi(x_0,x_1)\} \), then \( \tau_Z = \text{tr}_W(\tau_{ZW}) \) is the CQ-state associated to \( \{\tau_{a_0|0}, \pi(x_0)\} \).

**Proof.** By definition of CQ-state,

\[
\tau_{ZW} = \sum_{a_0,a_1,x_0,x_1} \pi(x_0,x_1)p(a_0,a_1|x_0,x_1)|x_0x_1\rangle\langle x_0x_1| \otimes |a_0a_1\rangle\langle a_0a_1| \otimes \tau_{a_0a_1|x_0x_1}.
\]

Thus,

\[
\tau_Z = \text{tr}_W(\tau_{ZW}) = \text{tr}_{X_1A_1}\left[ \sum_{a_0,a_1,x_0,x_1} \pi(x_0,x_1)|x_0x_1\rangle\langle x_0x_1| \otimes |a_0a_1\rangle\langle a_0a_1| \otimes \tau_{a_0a_1|x_0x_1} \right]
\]

\[
= \sum_{a_0,a_1,x_0,x_1} \pi(x_0,x_1)|x_0\rangle\langle x_0| \otimes |a_0\rangle\langle a_0| \otimes \text{tr}_{B_1}(\tau_{a_0a_1|x_0x_1})
\]

\[
= \sum_{a_0,x_0,x_1} \pi(x_0,x_1)|x_0\rangle\langle x_0| \otimes |a_0\rangle\langle a_0| \otimes \sum_{a_1} \text{tr}_{B_1}(\tau_{a_0a_1|x_0x_1})
\]

As \( \{\tau_{a_0|a_1|0}\} \) is a non-signalling assemblage, \( \sum_{a_1} \text{tr}_{B_1}(\tau_{a_0a_1|x_0x_1}) = \tau_{a_0|0}. \) Consequently,

\[
\tau_Z = \sum_{a_0,x_0,x_1} \left( \sum_{x_1} \pi(x_0,x_1)\right)|x_0\rangle\langle x_0| \otimes |a_0\rangle\langle a_0| \otimes \tau_{a_0|0}
\]

\[
= \sum_{a_0,x_0,x_1} \pi(x_0)|x_0\rangle\langle x_0| \otimes |a_0\rangle\langle a_0| \otimes \tau_{a_0|0}.
\]

Therefore, \( \tau_Z \) is the CQ-state associated to \( \{\tau_{a_0|0}, \pi(x_0)\} \). \( \square \)
Lemma 6. The states \( \{ \sigma_W^k \} \) are CQ-states associated to unsteerable assemblages and the probability distribution \( \pi(x_1) \).

Proof. Given a fixed \( k = (x'_0, a'_0, i) \), by the definition of \( \sigma_W^k \),

\[
\sigma_W^k = \frac{\text{tr}_Z((F_k \otimes \text{Id})\sigma_{ZW})}{\text{tr}((F_k \otimes \text{Id})\sigma_{ZW})}.
\]

So,

\[
(F_k \otimes \text{Id})\sigma_{ZW} = (F_{x'_0, a'_0, i} \otimes \text{Id})\sigma_{ZW}
\]

\[
= \sum_{\lambda} \sum_{a_0, a_1, x_0, x_1} \pi_0(x_0)\pi_1(x_1) r(\lambda) q_\lambda (a_0, a_1 | x_0, x_1) (|x'_0\rangle|x'_0\rangle |x_0\rangle |x_0\rangle) \otimes |x_1\rangle|x_1\rangle \otimes (|a'_0\rangle|a'_0\rangle) \otimes |a_1\rangle|a_1\rangle \otimes F_i \sigma_{B_0 B_1}^\lambda
\]

Therefore,

\[
\text{tr}_Z \left[(F_{x'_0, a'_0, i} \otimes \text{Id})\sigma_{ZW}\right] = \sum_{\lambda} \sum_{a_1, x_1} \pi_0(x'_0)\pi_1(x_1) r(\lambda) q_\lambda (a_1 | x_1, a_0) q_\lambda (a_0 | x_0, x_1) = q_\lambda (a_1 | x_0, x_1, a_0) q_\lambda (a_0 | x_0, x_1).
\]

Thus,

\[
\sigma_W^{x'_0, a'_0, i} = \frac{\text{tr}_Z((F_{x'_0, a'_0, i} \otimes \text{Id})\sigma_{ZW})}{\text{tr}((F_{x'_0, a'_0, i} \otimes \text{Id})\sigma_{ZW})}
\]

\[
= \frac{\sum_{\lambda} \sum_{a_1, x_1} \pi_0(x'_0)\pi_1(x_1) r(\lambda) q_\lambda (a_1 | x_1, a_0) q_\lambda (a_0 | x_0, x_1) |x_1\rangle|x_1\rangle \otimes |a_1\rangle|a_1\rangle \otimes \sigma_{B_1}^\lambda}{\sum_{\lambda} \sum_{a_1, x_1} \pi_0(x'_0)\pi_1(x_1) r(\lambda) q_\lambda (a'_0 | x'_0) \otimes |a_1\rangle|x_1\rangle \otimes \sigma_{B_1}^\lambda}
\]

(18)
Let
\[ r_k(\lambda) = \frac{r(\lambda)q_\lambda(a_0'|x_0') \text{tr}(\sigma^{\lambda,i}_{B_1})}{\sum_{\lambda} r(\lambda)q_\lambda(a_0'|x_0') \text{tr}(\sigma^{\lambda,i}_{B_1})}. \]

It follows that \( r_k(\lambda) \geq 0 \) and that \( \sum_{\lambda} r_k(\lambda) = 1 \). Let’s also define \( q^k_\lambda(a_1|x_1) = q_\lambda(a_1|x_0',x_1,a_0') \) and
\[ \sigma^{\lambda,i}_{B_1} = \frac{\sigma^{\lambda,i}_{B_1}}{\text{tr}(\sigma^{\lambda,i}_{B_1})}. \]

Then,
\[ \sigma^k_W = \sum_{\lambda} \sum_{a_1,x_1} \pi(x_1) r_k(\lambda) q^k_\lambda(a_1|x_1) |x_1|x_1| \otimes |a_1a_1| \otimes \sigma^{\lambda,i}_{B_1}. \tag{C19} \]

Thus, \( \sigma^k_W \) are CQ-states associated to unsteerable assemblages by \( \pi(x_1) \).

\[ \square \]

**Lemma 7.** Given \( \{\sigma^k_W\}_k \) a set of CQ-states associated to unsteerable assemblages by the same probability distribution \( \pi(x_1) \), and let \( \{\alpha_k\} \subset \mathbb{R}^+ \) with \( \sum_k \alpha_k = 1 \). Then, \( \sum_k \alpha_k \sigma^k_W \) is also a CQ-state associated to an unsteerable assemblage by the probability distribution \( \pi(x_1) \).

**Proof.** By hypotheses,
\[ \sigma^k_W = \sum_{\lambda} \sum_{a_1,x_1} \pi(x_1) r_k(\lambda) q^k_\lambda(a_1|x_1) |x_1|x_1| \otimes |a_1a_1| \otimes \sigma^{\lambda,i}_{B_1}. \tag{C20} \]

Then,
\[ \sum_k \alpha_k \sigma^k_W = \sum_{\lambda} \sum_{k,a_1,x_1} \pi(x_1) \alpha_k r_k(\lambda) q^k_\lambda(a_1|x_1) |x_1|x_1| \otimes |a_1a_1| \otimes \sigma^{\lambda,i}_{B_1}. \tag{C21} \]

Let \( \lambda' = (\lambda,k) \), \( r(\lambda') = r(\lambda,k) = \alpha_k r_k(\lambda) \), \( q^k_{\lambda'}(a_1|x_1) = q^k_\lambda(a_1|x_1) \) and \( \sigma^{\lambda',i}_{B_1} = \sigma^{\lambda,i}_{B_1} \). We can see that, \( r(\lambda') \geq 0 \) and \( \sum_{\lambda'} r(\lambda') = \sum_k \alpha_k (\sum_{\lambda} r_k(\lambda)) = \sum_k \alpha_k = 1 \). Thus,
\[ \sum_k \alpha_k \sigma^k_W = \sum_{\lambda'} \sum_{a_1,x_1} \pi(x_1) r(\lambda') q_{\lambda'}(a_1|x_1) |x_1|x_1| \otimes |a_1a_1| \otimes \sigma^{\lambda',i}_{B_1}. \tag{C22} \]

Therefore, \( \sum_k \alpha_k \sigma^k_W \) is a CQ-state associated to an unsteerable assemblage by \( \pi(x_1) \).

\[ \square \]

**Lemma 8.** Given \( \pi(x) \) a probability distribution s.t \( \pi(x) > 0 \) \( \forall x \) and \( \{\rho_{a|x}\}, \{\sigma_{a|x}\} \) assemblages. Let \( \{E_i\} \) an IC-POVM and \( F_{x,a,i} = |x|a|a| \otimes E_i \). Then, \( \mathcal{F}(\rho_{XAB}) = \mathcal{F}(\sigma_{XAB}) \iff \{\rho_{a|x}\} = \{\sigma_{a|x}\} \).

**Proof.**
\[ \mathcal{F}(\rho_{XAB}) = \sum_{x,a,i} \text{tr}(F_{x,a,i}\rho_{XAB}) |x|a\langle a| \otimes \langle i|i|. \tag{C23} \]

Expanding the trace:
\[ \text{tr}(F_{x,a,i}\rho_{XAB}) = \sum_{x',a'} \pi(x)p(a|x) \text{tr} (|x|x'|a'a'|a'\langle a| \otimes E_i \rho_{a|x}) \]
\[ = \pi(x)p(a|x) \text{tr}(E_i \rho_{a|x}). \tag{C24} \]

Thus,
\[ \mathcal{F}(\rho_{XAB}) = \sum_{x,a,i} \pi(x)p(a|x) \text{tr}(E_i \rho_{a|x}) |x|x\langle a| \otimes \langle i|i|. \tag{C25} \]

Analogously,
\[ \mathcal{F}(\sigma_{XAB}) = \sum_{x,a,i} \pi(x)q(a|x) \text{tr}(E_i \rho_{a|x}) |x|x\langle a| \otimes \langle i|i|. \tag{C26} \]
As \( \{|x\}, \{|a\} \) and \( \{|i\} \) are orthonormal basis, \( F(\rho_{XAB}) = F(\sigma_{XAB}) \) implies in:

\[
\pi(x)p(a|x) \text{tr}(E_i \rho_{a|x}) = \pi(x)q(a|x) \text{tr}(E_i \sigma_{a|x}) \quad \forall x, a, i.
\] (C27)

Dividing both sides by \( \pi(x) \) and summing over \( i \):

\[
\begin{align*}
p(a|x) \text{tr}(\sum_i E_i \rho_{a|x}) &= q(a|x) \text{tr}(\sum_i E_i \sigma_{a|x}) \quad \forall x, a, i. \\
p(a|x) &= q(a|x) \quad \forall x, a. \\
p(a|x) &= q(a|x) \quad \forall x, a. \\
\end{align*}
\] (C28)

Thus, by eqs. (C27) and (C28), we have \( \text{tr}(E_i \rho_{a|x}) = \text{tr}(E_i \sigma_{a|x}) \quad \forall x, a, i. \)

As \( \{E_i\} \) is a IC-POVM, this implies in \( \rho_{a|x} = \sigma_{a|x} \quad \forall x, a. \) Thus,

\[
\theta_{a|x} = p(a|x)\rho_{a|x} = q(a|x)\sigma_{a|x} = \varsigma_{a|x} \quad \forall x, a.
\]

Therefore, \( \{\theta_{a|x}\} = \{\varsigma_{a|x}\}. \)

We are now able to show the main result of this section:

**Proposition 4.** If \( \{\theta_{a|a_1|x_0x_1}\} \) is a broadcasting of a steerable assemblage \( \{\varrho_{a|x}\} \), then:

\[
E_A(\theta_{a|a_1|x_0x_1}) > E_A(\varrho_{a|x}).
\] (29)

**Proof.** By definition,

\[
E_A(\{\theta_{a|a_1|x_0x_1}\}) = \sup_{\pi(x_0,x_1)} \inf_{\sigma_{x_0} \sigma_{x_1}} S_Q(\rho_{ZW}||\sigma_{ZW})
\]

where in the second line we are taking the supremum over the smaller set of product probability distributions, i.e, \( \pi(x_0, x_1) = \pi_0(x_0)\pi_1(x_1) \). Using eq. (C3), we have:

\[
E_A(\{\theta_{a|a_1|x_0x_1}\}) \geq \sup_{\pi_0(x_0)\pi_1(x_1)} \inf_{\sigma_{x_0} \sigma_{x_1}} S_Q(\mathcal{F}(\rho_Z)||\mathcal{F}(\sigma_Z)) + S_Q\left(\rho_W\|\sum_k \alpha_k \sigma_W^k\right) \tag{29}
\]

For each probability distribution \( \pi_0(x_0)\pi_1(x_1) \), let \( \varsigma_{a|a_1|x_0x_1} \) be the assemblage that achieves the infimum of the equation above (The existence of this state that reaches the minimum is guaranteed by arguments totally analogous to those of Lemma 4). Then,

\[
E_A(\{\theta_{a|a_1|x_0x_1}\}) \geq \sup_{\pi_0(x_0)\pi_1(x_1)} \left[ S_Q(\mathcal{F}(\rho_Z)||\mathcal{F}(\sigma_Z)) + S_Q\left(\rho_W\|\sum_k \alpha_k \sigma_W^k\right) \right]
\]

Now, by Lemma 5 \( \rho_Z \) and \( \sigma_Z \) are the CQ-states associated to \( \{\theta_{a|x}\}, \pi_1(x) \) and \( \{\varsigma_{a|x}\}, \pi_1(x) \), respectively. On the other hand, as \( \{\theta_{a|x}\} \) is steerable, and \( \{\varsigma_{a|x}\} \) is unsteerable, it follows that \( \{\theta_{a|x}\} \neq \{\varsigma_{a|x}\} \). Therefore, by Lemma 8, if \( \pi_0(x) > 0 \quad \forall x \), then \( \mathcal{F}(\rho_Z) \neq \mathcal{F}(\sigma_Z) \). But, we also have that \( S_Q(\rho||\sigma) = 0 \iff \rho = \sigma \). Therefore, for positive probability distributions, we have \( S_Q(\mathcal{F}(\rho_Z)||\mathcal{F}(\sigma_Z)) > 0 \). In this way,

\[
\sup_{\pi_0(x_0)} S_Q(\mathcal{F}(\rho_Z)||\mathcal{F}(\sigma_Z)) > 0.
\] (C32)

Applying this fact to eq. (C31), we have:

\[
E_A(\{\theta_{a|a_1|x_0x_1}\}) > \sup_{\pi_1(x_1)} S_Q\left(\rho_W\|\sum_k \alpha_k \sigma_W^k\right) \tag{33}
\]
However, by Corollaries 6 and 7, $\sum_k \alpha_k \bar{\sigma}^k_W$ is a CQ-state associated to an unsteerable assemblage. Thus,

$$E_A(\{\varrho_{a_0|x_0}\}) > \sup_{\pi_1(x_1)} S_Q(\rho_W || \sum_k \alpha_k \bar{\sigma}^k_W)$$

$$\geq \sup_{\pi_1(x_1) \in UR_{ns}} \inf_{\varsigma \in UR_{ns}} S_Q(\rho_W || \sigma_W)$$

$$= E_A(\{\varrho_{a|x}\}) = E_A(\{\varrho'_{a|x}\}).$$

(C34)

(C35)

Appendix D: Proof of Theorem 2

**Theorem 2.** [No-broadcasting for assemblages] It is impossible to broadcast steerable assemblages using UR_{ns}-LOSR transformations.

**Proof.** Let’s suppose, by contradiction, that there is a UR_{ns}-LOSR transformation $\mathcal{M}$ that broadcasts a steerable assemblage $\{\varrho'_{a|x}\}$. So, by Proposition 3 we have:

$$E_A(\mathcal{M}(\{\varrho'_{a|x}\})) \leq E_A(\{\varrho'_{a|x}\}).$$

(D1)

On the other hand, as $\mathcal{M}(\{\varrho'_{a|x}\})$ is a broadcasting of $\{\varrho'_{a|x}\}$, by Proposition 4

$$E_A(\{\varrho'_{a|x}\}) < E_A(\mathcal{M}(\{\varrho'_{a|x}\})).$$

(D2)

Therefore, by eqs. (D1) and (D2):

$$E_A(\{\varrho'_{a|x}\}) < E_A(\{\varrho'_{a|x}\}),$$

(D3)

which is a contradiction. Thus, it is impossible to broadcast a steerable assemblage by UR_{ns}-LOSR transformations.

\[\square\]