Analogue Gravity and Trans-Atlantic Air Travel

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The problem of finding a minimal-time path for an aeroplane travelling in a wind flow has a simple formulation in terms of analogue gravity. This paper gives an elementary explanation with equations and some numerical solutions.

I. INTRODUCTION

Anyone flying across the Atlantic cannot fail to notice that the eastbound flight is shorter than the westbound flight. The more observant will also notice that the flightpath on the eastbound flight often travels further south than the westbound. The difference in times is due to the aeroplane taking advantage of the westerly jet stream. The inquisitive traveller may ponder what the optimal route may be. This problem is known as the Zermello navigation problem [1], and a solution is built into modern route planning algorithms [2–4]. The aim of the present paper is to explain the simple formulation of Zermello’s problem in terms of ideas from analogue gravity [5, 6]. (For a review of analogue gravity see [7]). The discussion is presented in elementary terms that should be accessible to non-experts in general relativity.

Aeroplanes are designed to fly efficiently at their optimal airspeed, which is typically around 500 kn\(^1\). In a wind with velocity \(v\), the speed along the ground \(u\) would have to satisfy the simple relation

\[(u - v)^2 = c^2,\]

(1)

where \(c\) is the airspeed. Later in the paper, this will be shown to be equivalent to finding paths in curved spacetime with metric

\[ds^2 = -(c^2 - v^2)dt^2 - 2v_i dx^i dt + g_{ij} dx^i dx^j,\]

(2)

This is the acoustic metric that forms the basis for a description of hydrodynamical waves and is the starting point for analogue models of gravity [8]. The aeroplane follows the same trajectory as the wave-front of a water wave in a flowing stream.

The object of the exercise is the minimise the time taken to get from point \(A\) to point \(B\) at some fixed altitude above the surface of the planet. For this, we appeal to Fermat’s principle: the path taken between two points by a ray of light is the path that can be traversed in the least time. In the analogue gravity context, this means the optimal route from \(A\) to \(B\) is defined by the analogue light rays, or null geodesics, in the acoustic metric.

There are other geometric reformulations of Zermello’s problem [9]. However, the analogue gravity approach with a good choice of affine parameter along the null geodesic helps simplify the equations, and in this respect the analogue gravity approach does seem to offer advantages over alternatives, including the engineering control theory methods [2–4]. Specifically, the equations obtained from the analogue gravity approach have a simple polynomial form.

II. METHODS

The first simplification is that motion will be restricted to a fixed altitude, and described by two coordinates \(x^i, i = 1, 2\). When explicit coordinates are needed, spherical polar angles \(x^i = (\phi, \theta)\) can be used. Basis vectors will not be normalised, so there is a distinction between vectors \(v^i\) and covectors \(v_i\). Tensor indices are lowered and raised using the metric tensor \(g_{ij}\) and its inverse \(g^{ij}\). For example, a flight at altitude \(h\) over a spherical earth of radius \(a\) would see local distances measured by the metric tensor,

\[ds^2 = g_{ij} dx^i dx^j = r^2 (d\theta^2 + \sin^2 \theta d\phi^2),\]

(3)

where \(r = a + h\).

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\(1\) Airplane speeds are measured in knots, abbreviated to kn, equal to 1.15078 mph or 0.51444 m s\(^{-1}\).
Figure 1. Illustration of Fermat’s principle. The null geodesics from the event O trace out the forward light cone. Any null line from another event P initially lies on the light cone from P, and cannot pass outside the light cone from O.

Analogue gravity arises from introducing an affine parameter $\tau$ along the trajectory, such that the velocity vector becomes

$$u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \frac{dt}{d\tau}.$$  \hfill (4)

The constant airspeed condition can be rewritten as

$$g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} - 2v^i \frac{dx^i}{d\tau} \frac{dt}{d\tau} - (c^2 - v^2) \left( \frac{dt}{d\tau} \right)^2 = 0.$$  \hfill (5)

In this form, the condition defines a path $(x^i(\tau), t(\tau))$ in curved spacetime, with zero length in the acoustic metric (2). The problem of finding the shortest-time path between two points on the surface of the Earth is solved by Fermat’s principle. A proof of Fermat’s principle for a time-independent metric is sketched out in Ref. [10], problem 40.3. If the metric depends on time, then the null geodesics either minimise the flight time, or they are extrema of the flight time if the null geodesics cross one another [11].

A simple geometrical argument can be used to illustrate the general idea. Figure 1 shows the light rays emanating from an event $O$ at $x_0$ and time $t = 0$ in a spacetime diagram. These define the forward light cone for the event $O$. The cone intersects the world-line of the destination $x_1$ for the first time at time $t$. Any null curve from $O$ lies on or inside the forward light cone of $O$, and the curve will always intersect the world-line of the point $x_1$ at a later time than $t$. Where this simple picture breaks down, is ignoring the possibility that strong winds can refocus some null geodesics from $O$ to another spacetime point, complicating the simple geometry of the light cone. When this happens, the optimal route is always one that avoids the refocussing point [11].

A standard way to obtain the null geodesics is to use an action principle, starting from the Lagrangian,

$$L = -\frac{1}{2} (c^2 - v^2) \dot{t}^2 - v_i \dot{x}^i - \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j.$$  \hfill (6)

After introducing the momenta $p_i = \partial L/\partial \dot{x}^i$ and $p_t = \partial L/\partial \dot{t}$, the Hamiltonian constructed form the action is

$$H = -\frac{1}{2} p_t^2 - p_i \frac{v_i}{c} p_i + \frac{1}{2} \left( g^{ij} - \frac{v_i v^j}{c^2} \right) p_i p_j.$$  \hfill (7)

Hamilton’s equations are then

$$\dot{t} = -\frac{p_t}{c} - \frac{v_i}{c^2} p_i,$$  \hfill (8)

$$\dot{x}^i = g^{ij} p_j - \frac{v^i v^j}{c^2} p_j - p_t \frac{v^i}{c},$$  \hfill (9)

$$\dot{p}_t = \frac{p_t}{c^2} \frac{\partial v_i}{\partial t} + \frac{v_j}{c} \frac{p_j}{c} \frac{\partial v^i}{\partial t},$$  \hfill (10)

$$\dot{p}_i = -\frac{1}{2} \frac{\partial g^{jk}}{\partial x^i} p_j p_k + p_i \frac{v_j}{c} \frac{\partial v^j}{\partial x^i} + p_k v^k \frac{\partial v^i}{\partial x^j}.$$  \hfill (11)
These equations are valid for any analogue spacetime geodesic, but for null geodesics an additional constraint $H = 0$ is imposed. This constraint is the hamiltonian form of the original airspeed condition (1). The solutions to the equations and the constraint solve the Zermello navigation problem problem of finding the minimum-time paths between the endpoints $x(0) = x_0$ and $x(\tau) = x_1$.

A simplified discussion is where the wind velocity is taken to be independent of time. The time-momentum becomes constant, and up to rescaling of the affine parameter it is possible to set $p_t = -c$. The remaining equations are as follows,

\begin{align}
\dot{t} & = \gamma, \\
\dot{x}^i & = g^{ij}p_j + \gamma v^i, \\
\dot{p}_i & = -\frac{1}{2} \frac{\partial g^{jk}}{\partial x^i} p_j p_k - \gamma p_j \frac{\partial v^j}{\partial x^i},
\end{align}

where the analogue time dilation factor $\gamma$ between the time and the affine parameter is

\begin{equation}
\gamma = 1 - \frac{p_i v^i}{c^2}.
\end{equation}

Unlike in relativistic time dilation, the dilation factor can be larger or less than unity. The constraint $H = 0$ becomes,

\begin{equation}
p^2 = \gamma^2 c^2.
\end{equation}

In order to find an optimal path between the endpoints $x_0$ and $x_1$ numerically, the equations are integrated from $x_0$ with the initial momentum along an arbitrary unit vector $n_i(\alpha)$. When substituted into the constraint, this gives a formula for the initial momentum $p_i(0)$,

\begin{equation}
p_i(0) = \frac{c n_i(\alpha)}{1 + n_i(\alpha)v^i/c}.
\end{equation}

The distance of closest approach of the geodesic to the final point $x_1$ defines a distance function $d(\alpha)$. The zeros of the distance function can be found by Newton’s method. Each of these zeros represents a viable null geodesic. The smallest local time at the point of closest approach is the optimal journey time.

Some further simplifying assumptions will be made about the wind velocity field for an illustration of the general ideas with specific calculations. First of all, the atmosphere will be approximated by an incompressible gas which is stratified into layers at fixed altitude (or more properly air pressure). Such flows are described as quasi-geostrophic, i.e. Coriolis and pressure gradients are the dominant forces [12]. The fluid velocity field is given by a two-dimensional stream function $\psi$, according to

\begin{equation}
v^i = \epsilon^{ij} \partial_j \psi,
\end{equation}

where the $\epsilon^{ij}$ is the alternating tensor, with $\epsilon^{12} = (\det g_{ij})^{-1/2}$. Small oscillations on the stationary flow patterns are known as Rossby waves. Waves with small wavelengths are advected with the eastward directed flow, but westward movement of the long wavelength Rossby waves partially compensates for the underlying drift to leave a relatively slowly varying wind pattern [12].

An exact solution is always useful for testing numerical codes. If the Earth is assumed to be a sphere, then the stream function $\psi = -v \cos \theta$ gives a soluble set of equations,

\begin{align}
\dot{\theta} & = \frac{p_\theta}{r^2} \\
\dot{\phi} & = \frac{p_\phi}{r^2 \sin^2 \theta} + \gamma \frac{v}{r}
\end{align}

along with $\dot{p}_\phi = 0$. The constraint closes the system,

\begin{equation}
p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} = \gamma^2 r^2 c^2.
\end{equation}

The velocity can be absorbed by introducing a new angular variable $\tilde{\phi} = \phi - vt/r$, and defining a renormalised speed $c' = \gamma c$. The resulting equations describe geodesics on a sphere, which are the the great circles in the $\theta$ and $\tilde{\phi}$ coordinates. The fightpaths are great circles shifted to the east at the speed $v$. 

For a slightly more realistic model of the jet stream, consider the following stream function

\[ \psi = C \arctan \left( \frac{\theta - \Theta(\phi)}{w} \right), \quad \Theta(\phi) = \theta_0 (1 + b \sin \theta \cos (m \phi - \phi_0)). \]  

(22)

The constant \( C \) is fixed by setting the maximum speed of the jet stream. The latitude and width of the stream are set by \( \theta_0 \) and \( w \), whilst \( b \) and \( m \) represent the amplitude and the wavelength of the Rosby waves. An example of the windspeed obtained from this stream function is shown in Fig. 2.

Figure 3 shows some of the optimal flightpaths for an aeroplane trajectory in the wind pattern shown in Fig. 2 on a spherical Earth. As expected, the eastbound flightpaths follow a more southerly route than the geodesic route. The westbound flightpaths are not as useful because then pilot has an option to fly at lower altitudes where the jet stream is not as strong. Nevertheless, an interesting phenomenon arises for wind speed above 100 kn, when a second route opens up over eastern North America. This second route is a local minimum of the flight-time. For strong wind speeds above 140 kn, the eastern route becomes the shortest route to LA.

The detailed flightpaths depend, naturally, on the particular parameters used to describe the wind pattern. Figure 4 gives an idea of the changes as a result of the Rossby wave phase \( \phi_0 \) and the latitude \( \theta_0 \).

III. DISCUSSION

The aim of this paper has been to present the reader with an example of analogue gravity in a ‘real-world’ setting. Many mathematical idealisations have been used to simplify the analysis and the results are not meant to be taken seriously as practical solutions to designing aeroplane flight paths. On the other hand, most of the simplifications used here can easily be improved upon. Replacing the spherical geometry with the Earth’s spheroidal shape is a trivial extension of the results. More realistic wind velocity profiles can be included by taking data from existing meteorological sources, either in grid or spherical harmonic form. Extending the results for time-dependent wind patterns is also perfectly possible. A harder problem is to take into account the different wind velocity fields in different strata for the atmosphere. The analogue metric can be extended to apply in three dimensions of space, but then other factors such as variable airspeed have to be taken into account. Whether this is would give any improvements on current navigational technology is highly questionable, but hopefully this account has been informative.

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Figure 3. Eastbound (green) and westbound (red) optimal paths for a journey between Los Angeles and London with windspeed 100 kn (left) and 180 kn (right). The dashed line is a local minimum of the travel time, the fastest route being the one over the Canadian Rockies. The sensible pilot will travel below the powerful jet stream on the westbound route making the westbound paths less relevant.

Figure 4. Variation of the Eastbound paths for changes in wind patterns with wind speed 100 kn. On the left, the jet stream latitude $\theta_0$ is 50N, and on the right 55N. Values of the longitude $\phi_0$ differ by 90 degrees.

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