Ambiguities in the derivation of retrodictive probability

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The derivation of the quantum retrodictive probability formula involves an error, an ambiguity. The end result is correct because this error appears twice, in such a way as to cancel itself. In addition, however, the usual expression for the probability itself contains the same ambiguity; this may lead to errors in its application. A generally applicable method is given to avoid such ambiguities altogether.

I. INTRODUCTION

In Ref. 1, Aharonov, Bergmann, and Lebowitz (ABL) examined, in quantum mechanics, the following problem:

After a system is prepared, the system variable \( P \) is observed, followed by the observation of the variable \( Q \). It is desired to determine ("retrodict") \( \Pr(p_j \mid q_k) \), the probability of the earlier event \( P = p_j \) given the later occurrent fact \( Q = q_k \).

They obtained, for a system initially prepared in the state \( \rho \), the retrodiction formula

\[
\Pr(p_j \mid q_k) = \frac{| \langle q_k \mid p_j \rangle |^2 \langle p_j \mid \rho \rangle p_j}{\sum_s | \langle q_k \mid p_s \rangle |^2 \langle p_s \mid \rho \rangle p_s}.
\] (1)

Their derivation of this (correct) expression contains an error which appears twice, canceling itself. Further, the probability expression on the left side itself is ambiguous, which may lead to confusion in application.

II. THE DERIVATION

The following derivation is implicit in Ref. 1; it appears somewhat more explicitly in Ref. 2 and in Ref. 3. Ordinarily one would solve this problem quite directly, using Bayes’s Formula:

\[
\Pr(p_j \mid q_k) = \frac{\Pr(q_k \mid p_j) \Pr(p_j)}{\Pr(q_k)}.
\] (2)

But this results in

\[
\Pr(p_j \mid q_k) = \frac{| \langle q_k \mid p_j \rangle |^2 \langle p_j \mid \rho \rangle p_j}{| \langle q_k \mid \rho \rangle q_k |^2},
\] (3)

which “doesn’t work” in quantum mechanics! (Consider a spin-1/2 system prepared as \( \rho = |z+\rangle \langle z+| \), with \( |q_k \rangle = |z-\rangle \) and \( |p_{1,2} \rangle = |y+,-\rangle \).)

Of course, ABL recognized this, and assumed that the event \( q_k \) arises following the events \( \{ p_j \} \); for the denominator of Eq. (2), they used the marginal-probability formula,

\[
\Pr(q_k) = \sum_s \Pr(p_s \wedge q_k) = \sum_s \Pr(q_k \mid p_s) \Pr(p_s).
\] (4)

In quantum-mechanical terms, this is

\[
\Pr(q_k) = \sum_s | \langle q_k \mid p_s \rangle |^2 \langle p_s \mid \rho \rangle p_s, \tag{5}
\]

which, in Eq. (2), results in the ABL formula Eq. (1). Further, Eq. (4) leads to the classical retrodiction expression

\[
\Pr(p_j \mid q_k) = \frac{\Pr(q_k \mid p_j) \Pr(p_j)}{\sum_s \Pr(q_k \mid p_s) \Pr(p_s)}, \tag{6}
\]

which directly implies the quantum ABL Formula Eq. (1).)

However, as Margenau[4] noted, there is something wrong with Eq. (5): for the pure state \( \rho = | \Psi \rangle \langle \Psi | \), Eq. (5) is

\[
| \langle q_k | \Psi \rangle |^2 = \sum_s | \langle q_k | p_s \rangle |^2 | \langle p_s | \Psi \rangle |^2,
\] (7)

which, for \( | \Psi \rangle \notin \{ | p_j \rangle \} \), is impossible.

Thus Eq. (2) and Eq. (4), which seem to be correct in ordinary probability theory, both fail in quantum mechanics. It has been suggested[4] that this implies a special, different “quantum probability.” However, the reality is more pedestrian: Quantum mechanics is a probability theory of sequences of events in systems of several variables. In such sequences issues arise which are unfamiliar in, but not foreign to, ordinary probability theory.[5] In the above derivation these issues have not been dealt with, with the result that neither Eq. (2) nor Eq. (4) is correct.

In order to see the source of these errors, let us rewrite Eq. (2), using a more-careful notation which denotes the ordinal position of each event by bracketed superscripts:

\[
\Pr(p_{j[1]} \mid q_{k[2]}) = \frac{\Pr(q_{k[2]} \mid p_{j[1]}) \Pr(p_{j[1]})}{\Pr(q_{k[2]})}. \tag{2'}
\]

The denominator of Eq. (2') is obviously ambiguous: what event\(^{[1]} \) precedes \( q_{k[2]}^{} \)? I have shown, in Ref. 5, that this ambiguity is not innocent, even in ordinary non-quantal probability—the expression \( \Pr(q_{k[2]}^{}) \) may be undefined (and is undefinable in quantum mechanics). The same ambiguity appears on the left side of Eq. (4), which is thus also incorrect.

III. A DERIVATION WITHOUT AMBIGUITIES

Let us derive an expression for Bayes’s Formula which avoids this ambiguity; this will allow us to show that Eq. (6)
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Section 2

Given the complete, disjoint set of values \( \{ p_j \} \), we introduce the notation \( M_P \equiv \bigvee_s p_s \); then \( p_j \equiv p_j \land M_P \), and

\[
\Pr(p_j^{[1]} \land q_k^{[2]}) = \Pr(p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}).
\] (8)

The derivation of Bayes’s Formula involves applying the definition of conditional probability to the conjunction \( \Pr(p_j^{[1]} \land q_k^{[2]} \) in either order: First, using Eq. (8),

\[
\Pr(p_j^{[1]} \land q_k^{[2]}) = \Pr(q_k^{[2]} \land p_j^{[1]} \land M_P^{[1]}) \Pr(p_j^{[1]} \land M_P^{[1]}),
\] (9)

which simplifies to (the expected)

\[
\Pr(p_j^{[1]} \land q_k^{[2]}) = \Pr(q_k^{[2]} \land p_j^{[1]}) \Pr(p_j^{[1]}).
\] (10)

Second, again using Eq. (8),

\[
\Pr(p_j^{[1]} \land q_k^{[2]}) = \Pr(p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}) \Pr(M_P^{[1]} \land q_k^{[2]});
\] (11)

since \( \Pr(M_P) = 1 \),

\[
\Pr(p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}) = \Pr(M_P^{[1]} \land p_j^{[1]} \land q_k^{[2]}) \Pr(p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]});
\] (12)

so

\[
\Pr(p_j^{[1]} \land q_k^{[2]}) = \Pr(p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}) \Pr(M_P^{[1]} \land q_k^{[2]}). \] (13)

Combining Eqs. (9) and (13), we obtain, in place of Eq. (2), the correct expression of Bayes’s Formula,

\[
\Pr(p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}) = \frac{\Pr(q_k^{[2]} \land p_j^{[1]}) \Pr(p_j^{[1]})}{\Pr(M_P^{[1]} \land q_k^{[2]})}.
\] (14)

The no-longer ambiguous denominator expands to

\[
\Pr(M_P^{[1]} \land q_k^{[2]}) = \sum_s \Pr(p_s^{[1]} \land q_k^{[2]})
\] (15)

replacing this in Eq. (14), we obtain

\[
\Pr(p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}) = \frac{\Pr(q_k^{[2]} \land p_j^{[1]}) \Pr(p_j^{[1]})}{\Pr(M_P^{[1]} \land q_k^{[2]})}.
\] (16)

IV. COMMENTS

We see from Eq. (14) that the error in Eq. (2) is indeed the ambiguity in the denominator \( \Pr(q_k^{[2]} \land p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}) \); the missing event \( \Pr(q_k^{[2]} \land p_j^{[1]} \land M_P^{[1]} \land q_k^{[2]}) \), the ignored observation of \( P \); the denominator may be written simply \( \Pr(q_k^{[2]} \land M_P^{[1]} \land q_k^{[2]} \), “the probability of \( q_k \) following the ignored complete observation of \( P \)”.

Further, the left side of Eq. (1) should be written \( \Pr(p_j^{[1]} \land q_k^{[2]} \land M_P^{[1]} \land q_k^{[2]} \); the ignored complete observation of \( P \) must be explicitly accounted for. Why is this important? First, rotate \( \{ |p_s\rangle \} \) about \( |p_1\rangle \) to get \( \{ |p_{s'}\rangle |p_{s'}\rangle = |p_1\rangle \}, \) the eigenstates of a variable \( P' \) which has its \( j = 1 \)-vector in common with \( P \); ignoring this complete observation yields a different value: \( \Pr(p_{s'} | q_k \rangle \neq \Pr(p_1 | q_k \rangle \). (This is the “something very curious” which arises in Ref. 6.) Second, the observation needn’t be complete: for example, merely observe \( p_j \) or not \( p_j \); in this case the denominator of Eq. (1)

\[
\langle q_k | p_j \rangle^2 \langle p_j | p_j \rangle + \sum_{s \neq j} \langle p_j | q_k \rangle^2 \langle q_k | p_s \rangle^2 \langle p_s | p_j \rangle^2.
\]

This leads to the “Three-Box Paradox”[2], which is surprising partly because of a failure to explicitly note the difference between the condition \( (p_1 \lor p_2 \lor p_3)^{[1]} \land q_k^{[2]} \) and the condition \( (p_1 \lor p_1)^{[1]} \land q_k^{[2]} \).

The derivation presented by Aharonov, Bergmann, and Lebowitz[1] is implicit: they start with Eq. (2) (which appears as the first part of their Eq. (2.4)). The result Eq. (1) is equivalent to the quantum expressions in their Eqs. (2.4) and (2.5). The transition from Eq. (2) to Eq. (1) is done in a single step, without comment, using the quantum equivalent of Eq. (4). Thus the ambiguous \( \Pr(q_k^{[2]} \) appears at the beginning of their derivation, and then simply vanishes. The cancelation of these errors is more good fortune (and good intuition) than good physics; Margenau was not so lucky.

Eq. (15) is the marginal probability identity appropriate to summing over the earlier event. It is interesting that, deriving a form of the marginal-probability formula, Ballentine[7] used a technique very similar to the above, but the \( M_P \), after being introduced, was dropped, resulting in the incorrect Eq. (4).

Quantum mechanics involves the classical probability of sequences of events involving more than one variable. Few treatments of probability deal with such sequences; the resulting unfamiliarity has lead to numerous errors in the understanding of the quantum-mechanical probability formulas. For the purpose of extending our fundamental understanding, the various formal approaches to measurement (involving POVMs, POVMs, effects and operations) are inadequate: based entirely on the the Hilbert-space formalism, their connection with probability theory is loose and ill-understood.
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[1] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, “Time symmetry in the quantum process of measurement,” Phys. Rev. 134B, 1410–1416 (1964).
[2] Y. Aharonov and L. Vaidman, “Complete description of a quantum system at a given time,” J. Phys. A 24, 2315–2328 (1991).
[3] S. M. Barnett, D. T. Pegg, and J. Jeffers, “Bayes’ theorem and quantum retrodiction,” J. Mod. Opt. 47(11), 1779–1789 (2000).
[4] H. Margenau, “Measurements in quantum mechanics,” Ann. Phys. 23, 469–485 (1963).
[5] K. A. Kirkpatrick, “Quantal behavior in classical probability,” quant-ph/0106072 (2001).
[6] D. Z. Albert, Y. Aharonov, and S. D’Amato, “Curious new statistical prediction of quantum mechanics,” Phys. Rev. Lett. 54(1), 5–7 (1985).
[7] L. E. Ballentine, “Probability theory in quantum mechanics,” Am. J. Phys. 54(10), 883–889 (1986).