KINETIC PROPERTIES OF FRACTAL STELLAR MEDIA

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ABSTRACT

Kinetic processes in fractal stellar media are analysed in terms of the approach developed in our earlier paper (Chumak & Rastorguev, 2015) involving a generalization of the nearest neighbour and random force distributions to fractal media. Diffusion is investigated in the approximation of scale-dependent conditional density based on an analysis of the solutions of the corresponding Langevin equations. It is shown that kinetic parameters (time scales, coefficients of dynamic friction, diffusion, etc.) for fractal stellar media can differ significantly both qualitatively and quantitatively from the corresponding parameters for a quasi-uniform random media with limited fluctuations. The most important difference is that in the fractal case kinetic parameters depend on spatial scale length and fractal dimension of the medium studied. A generalized kinetic equation for stellar media (fundamental equation of stellar dynamics) is derived in the Fokker-Planck approximation with the allowance for the fractal properties of the spatial stellar density distribution. Also derived are its limit forms that can be used to describe small departures of fractal gravitating medium from equilibrium.

Key words: stellar dynamics, relaxation processes in gravitating systems, diffusion, dynamic friction, fractal structure of stellar systems

1 INTRODUCTION

The earliest notions about hierarchical, fractal structure of the world of galaxies date back to Shapley (1934) and Carpenter (1938), who showed that spatial number density in clusters of galaxies obeys a power-law relation \(n(r) \sim r^{-\alpha}\) with the exponent of about \(\alpha \approx 1.5\), which is the same within the errors for clusters of different dimensions. Later, de Vaucouleurs (1970) concluded that the spatial distribution of galaxies obeys a universal power-law relation, albeit with a different exponent of \(\alpha \approx 1.7\). Based on his results, Mandelbrot (1977) proposed a hierarchical fractal model to describe the distribution of galaxies. It became gradually clear that such extragalactic structures as clusters and superclusters, walls, pancakes, etc., are not base elements but rather parts of a general stochastic hierarchy spanning a huge range of scales. In his fundamental studies Peebles (1980, 1993) finally proved that density distribution in all clusters of galaxies, irrespective of their sizes, exhibits properties that can be described in terms of two-point correlation functions. Such functions have the form of power-law relations with exponent \(\alpha \approx 1.7\), which is approximately the same for all scales, directions, and distances.

Davis and Peebles (1983) introduced for power-law density distribution the characteristic size \(r_0\) defined by relation

\[ n(r_0) = 1, \]

which later came to be called the "correlation length". This quantity is a matter of convention, because the actual correlation length for power-law density distributions is infinite, unlike the correlation length in molecular media or Debye radius in plasma. Because of the power-law nature of the density distribution, any definition of the size of the gravitating system is also only a matter of convention, because all gravitating systems actually correlate with each other and the nearest systems transit smoothly into each other, featuring the overall fractal pattern of the gravitating medium with a certain fractal dimension. However, in practise the "correlation length" \(r_0\) proposed by the above authors proved to be useful for various numerical estimates.

Subsequent studies showed that this self-similar stochastic hierarchy treated in terms of fractal concepts can describe quite well not only the observed features of the world of galaxies (Einasto, 1989; Klypin et al., 1989; Labini et al., 1996; Joyce et al., 1999), but also those of gas-and-dust nebulae where star clusters and associations form (Larson, 1981). The fractal model of the distribution of galaxies was also convincingly confirmed by direct numerical simulations (see Pietronero et al., 2002; Pietronero & Labini, 2005; Labini & Pietronero, 2007 and references therein). However,
researchers have noted the lack of spatial resolution in three-dimensional models of cosmological gravitational clustering, which complicates the study of their fractal properties down to smallest spatial scales. From the other hand, it is possible to derive exact solutions for one-dimensional cosmological clustering models. The 1D model was first formulated by Rouet et al. (1990). More recent studies (see, for example, Miller & Rouet, 2010; Joyce & Sicard, 2011; Benhaiem et al., 2013) have shown that power spectrum of initial perturbations in 1D models leads to fractal and multifractal structures in a wide range of scales. Detailed review of the problem can be found in Shiozawa & Miller (2016). One-dimensional cosmological models seem to give correct qualitative understanding of many events in the gravitational collapse of three-dimensional universe, but some ambiguity still remains as to how to connect the properties of 1D and 3D models (Joyce & Sicard, 2011).

De Vega et al. (1998) demonstrated the fundamental differences between the physics of gravitating and quasi-uniform media. The above authors studied the thermodynamics of self-gravitating systems using the theory of critical phenomena to show that gravity provides a dynamical mechanism to produce finite multiscaled structures. This inherent inhomogeneity of gravitating media, which has the form of multiscaled fractal and stochastic hierarchy, represents a fundamental difference between gravitating and quasi-uniform systems.

A specific feature of such stochastic fractals is the self-similarity of stochastic structures and the fact that the density \( n(r) \) of the distribution of structure elements does not tend to any particular finite value. This actually means that there is no such distance \( r_c \) that density fluctuations \( \Delta n(r) \) could be neglected at \( r > r_c \), i.e., \( \Delta n(r)/n(r) \) does not vanish at any \( r \).

This is due to the fact that voids, which contain practically no elements of the structure, are present at all observed scales, as is immediately apparent, e.g., in the Sloan Digital survey data (SDSS). This alternation property can be explained by the fact that the statistics of fractals is not Gaussian. Fractal statistics demonstrates other, non-Gaussian steady distributions (Mandelbrot, 1987). Because of these important differences, fractal statistics, in particular, has to be taken into account in the analysis of any gravitating media. The underlying tools in the study of the kinetics of such media include, in particular, Lévy’s steady distributions (Lévy, 1937), whose special cases are the power-law distributions like the Carpenter-de Vaucouleurs law in the world of galaxies. Kinetic processes in such media are therefore often referred to as Lévy processes, because they demonstrate the tendency of the medium to approach some Lévy distribution.

Note, however, that in contrast to purely mathematical models, where stochastic scale invariance can be infinite, all the real fractal distributions, without exception show finite depth of the hierarchy of scale invariance (scaling area). That is, these distributions have the minimum and maximum scale, which limit the applicability of fractal models. The key problem in the study of particular real fractal distributions is the determination of the limiting scale. This is not always an easy task, especially in astrophysics, since these scales may lie outside the range of available observational data. In the last two decades of the past century and in the first decade of this century an especially sharp and prolonged debate arose about determining the maximum size for the fractal model of the distribution of galaxies in the universe. The authors of different reputable studies found estimates for this upper scale ranging from 5 to 600 Mpc. The obvious reason for this variation was the lack of observational data of sufficient accuracy and scope to solve this problem. Only the latest data from the WiggleZ Dark Energy Survey team (Drinkwater et al., 2010) and the analysis of these data by Scrimgeour et al. (2012) made it possible to close the issue. Scrimgeour et al. (2012) used WiggleZ survey data in the red-shift interval \( 0.1 \leq z \leq 0.9 \) to compute the correlation dimension \( D_2 \) (third dimension of Renyi fractal dimensions spectrum). The above authors showed that the fractal dimension is within 1% of \( D_2 = 3 \) for scale lengths exceeding \( 80h - 1\)Mpc. In other words, the distribution of matter in the universe can be considered as practically uniform on the scale lengths greater than \( 80h - 1\)Mpc. Scrimgeour et al. (2012) also provide an impressive historical review of the problem, to which we refer the interested readers.

Chumak & Rastorguev (2015) analysed Geneva-Copenhagen Survey of local FG-type dwarfs (hereafter referred to as GCS) and showed that the space distribution of stars of this type, which represent a significant fraction of the local stellar population, differs significantly from uniform Poisson distribution. We also showed that the distribution of stars on GCS scales is consistent with fractal model.

As is well known, the fundamental equation of stellar dynamics is based on Poisson model (Binney & Tremaine, 2008). Hence the question arose whether it was the Poisson model is a proper tool to analyze kinetic processes in stellar media. Certain progress toward solving this problem was achieved by Vlad (1994), Chavanis (2009), Chumak & Rastorguev (2016), who generalized the Holtsmark distribution to fractal media to derive the distribution of random force acting on a test particle. The random force acting on a test star was shown to be fully determined by the nearest neighbour located at a certain effective distance \( r_m \). Distance \( r_m \) is determined from the generalized nearest-neighbour distribution. The average conditional density distribution was derived and the fractal dimension of the near-solar stellar medium was determined based on GCS data.

In this paper we consider the motion of a test star in stellar medium, and derive the fundamental equation of stellar dynamics that takes into account, to a first approximation, the fundamental properties of stellar gravitating media in terms of fractal model.

## 2 EQUATION OF MOTION OF TEST STAR

In the Lagrangian approach the equations of motion of test star \( i \) in the Galaxy in the local neighbourhood can be written in heliocentric Cartesian Galactic coordinates in the following form:

$$\frac{dv_{ik}}{dt} = -av_{ik} + F_{ik}^r + F_{ik}^g(t)$$  \( (1) \)

where \( k \) is used to indicate \( X, Y, Z \) – Cartesian coordinates with the \( X \)-axis pointing toward the Galactic center; \( a \) is the coefficient of dynamic friction; \( F_{ik}^r \), the regular force, and \( F_{ik}^g(t) \), the random (irregular) force produced by the
surrounding objects. For the sake of convenience we refer all forces to a unit-mass test star.

In the case of local circumsolar volume considered here regular force is time independent and reasonable estimates for the components of this force can be derived in terms of the Oort constants:

\[ F_{x}^{'9} \approx X_{i}A^{2}, \quad F_{y}^{'9} \approx Y_{i}B^{2}, \quad F_{z}^{'9} \approx 0. \]  

Equation set (1) can be rewritten in the form:

\[ \frac{dv_{ix}}{dt} = -av_{ix} + F_{ix}^{r}(t) \]
\[ \frac{dv_{iy}}{dt} = -av_{iy} + F_{iy}^{r}(t) \]
\[ \frac{dv_{iz}}{dt} = -av_{iz} + F_{iz}^{r}(t) \]

Hereafter we consider the change of the test star velocity produced by random force. In fractal gravitating media the random force acting on the test particle is practically fully determined by the gravitational influence of the nearest neighbour whose effective distance \( r_{n} \) from the test star is given by the following formula:

\[ r_{n} = \frac{3}{D} \left( \frac{3}{4\pi \hbar} \right)^{1/D} \Gamma \left( \frac{D + 1}{D} \right) \]  

(Chumak & Rastorguev, 2016). Here \( D \) is the fractal dimension of the medium; \( \Gamma (x) \), the Gamma function, and \( h \), the normalization parameter of the mean conditional density \( n(r) \) of the medium:

\[ n(r) = hr^{D-3}\]  

As is evident from equation (4), parameter \( h \) has fractional dimension equal to \( [h] = L^{-D} \), where \( L \) is the unit of length. Hereafter throughout this paper we restrict ourselves, as we do in equation (5), to considering dimension \( D \), i.e., the common Hausdorff dimension, exclusively. This dimension is a rough characteristic of the \( n(r) \) distribution. The distribution \( n(r) \) is actually multifractal, and \( D \) corresponds to \( D0 \) of multifractal spectrum of dimensions. This dimension describes the structural heterogeneity of the distribution \( n(r) \), which seems to be the most significant in the framework of the present work. For a more complete characterisation of the \( n(r) \) distribution one must, of course, calculate the entire spectrum of dimensions \( (D0, D1, D2, \ldots) \) of the multifractal, or at least the first three of these, which are the most important in practical applications (see, for instance, (Chumak, 2012)).

According to the above study, \( r_{n} \) for the solar neighbourhood is equal to a fraction of a parsec, and therefore when written in the coordinate system connected with the center of mass of interacting objects, equation set (3) contains small quantities on the order of \( r_{n} \) instead of \( X_{i}, Y_{i}, \) and \( Z_{i} \), and hence the effect of regular field on the act of encounter can be neglected in the first approximation. The corresponding terms in equation sets (1) and (3) can be dropped. Equation set (1) then becomes:

\[ \frac{dv_{ik}}{dt} = -av_{ik} + F_{ik}^{r}(t). \]  

Hereafter \( k = (x, y, z) \) are the coordinates of the star in the reference frame connected with the center of mass. The medium is locally isotropic on the scale length \( r_{n} \) of the encounter and therefore equations (6) are linearly independent. To simplify the notation we drop the ‘it’ superscript in the right-hand side and the equation (6) takes the form:

\[ \frac{dv_{ik}}{dt} = -av_{ik} + F_{k}(t), \]  

The right-hand side of equation (7) consists of two terms: the force of dynamical friction of the medium, which depends on the velocity of the star and isotropic fluctuating force \( F_{k}(t) \), which is independent of the velocity of the test star. The formal solution of equation (7) can be written as:

\[ v_{ik}(t) = e^{-at} \left[ v_{0ik} + \int_{0}^{t} e^{at} F(t') dt' \right], \]  

where \( v_{0ik} \) is the velocity of \( i \)-th star at time \( t = 0 \).

This solution is of no practical interest because the coefficient of dynamic friction and random forces appearing to the right of the integral sign are not written explicitly.

3 Dynamic Friction

Dynamic friction is determined by random two-point encounters, and this fact allows the phenomenon to be analysed in terms of Markov type processes. However, one has to take into account the dependence of local density on scale length (5). As a result, the formula for dynamic friction coefficient in equation (7) can be rewritten as follows, by dropping small terms on the order of \( 1/\lambda \) and \( p_{\min}/p_{\max} \) (see, e.g., Chapter 7 in the monograph of Binney and Tremaine (2008)):

\[ a \approx \frac{8\pi G^{2} m^{2} \lambda v_{0}^{2}}{v_{0}^{2}} \]  

where \( \lambda = \ln(p_{\min}/p_{\max}) \) is the Coulomb logarithm; \( G \), the gravitational constant; \( m_{i} \), and \( m \), the mass of the test and field star, respectively; \( n \), the number density of field stars; \( p_{\min} = 2Gm/v_{i}^{2} \), the impact parameter of the close encounter in which the test star is deflected by \( \pi/2 \) angle, and \( p_{\max} \), the cutoff parameter for distant interactions (in our case \( p_{\max} = 2r_{m} \)).

In this case the dynamical friction coefficient \( a \) is computed by analysing two-point encounters, because the average conditional density \( n \) (5) can be considered to be uniform and isotropic on scale lengths on the order of \( r_{n} \). We set \( m_{i} = m \) to derive the following formula for \( a \):

\[ a \approx \frac{4\pi G^{2} m^{2} \lambda v_{0}^{2}}{v_{0}^{2}} r^{(D-3)}. \]  

The dynamic friction coefficient \( a \) of fractal and uniform media differ fundamentally in that the former explicitly on scale length \( r \), because the corresponding formula contains conditional density instead of uniform density. Correspondingly, there is no unique dynamic friction coefficient for a fractal medium. The average friction coefficient obviously depends on the volume considered, i.e., for fractal media the definition of dynamic friction is also a matter of convention. This is rather unusual; however, this is due to specific of the kinetics in real fractal gravitating media: instead of fixed average diffusion coefficients we have to deal with a continuous spectrum of coefficient values. It is immediately apparent
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from equation (10) that, as expected, it acquires the classical form in the special case of uniform distribution: $D \rightarrow 3$; $h \rightarrow n_q$.

Dynamic friction coefficient can approximately estimated using the "correlation length" $\tau_0$ proposed by Davis & Peebles (1983). Thus

$$a_0 = a(\tau_0) \approx \frac{8\pi G^2 m^2 \lambda}{v_0^2} \tau_0^{-3}$$  \hspace{1cm} (10)

where $r$ can be determined from equality $n(\tau_0) = 1$ and, in accordance with equation (5), it is equal to:

$$r_0 = h - \frac{\tau_0}{\pi}$$  \hspace{1cm} (10)

4 RANDOM FORCE

According to Vlad (1994) and Chumak & Rastorguev (2016), the asymptotic distribution $W(\mid \vec{F}'; D)$ of the random force $\mid \vec{F}$’ intensity for large forces in a medium with fractal dimension can be written in the form:

$$W(\mid \vec{F}; D)d\mid \vec{F} =$$

$$= 4\pi h(Gm)^{D/2}e^{\frac{4\pi h}{3GM^{1/2}}(\mid \vec{F}^2)}e^{-\frac{4\pi h}{3GM^{1/2}}(\mid \vec{F}^2)}$$  \hspace{1cm} (11)

Let us rewrite formula (11) using a more compact notation:

$$W(F_m; D)dF_m = 3b \cdot e^{(-b \cdot F_m^{-D/2})}dF_m,$$  \hspace{1cm} (12)

where, to simplify the formula, we adopted $\mid \vec{F} = F_m$,

$$b = \frac{4\pi h}{3}(Gm)^{D/2}.$$  \hspace{1cm} (13)

Distribution (11) is stationary. Let us assume that there are no temporal correlations in the random process of the motion of a test star and all directions of random force are equally represented. Subscript ‘k’ can then be dropped in the force terms in equations (7) and (8) and $F_k(t) = F(t)$ can be written as the product of:

$$F(t; t_e) = F_m \varphi_i(t; t_e),$$  \hspace{1cm} (14)

where $t_e$ is the time scale over which the test star interacts (collides) with a field object. Let us assume that the collision (closest encounter) of $i$-th star occurs at time $t$. At that time the force is maximal. Suppose, for simplicity, that the force acting on the star is symmetric with respect to time $t = t'$ and decreases with time in accordance with the following law:

$$F(t; t_e) = \frac{F_m}{t_e \sqrt{\pi}} e^{-(t-t_e)^2}.$$  \hspace{1cm} (15)

If $t_e << l/|v|$, where $l$ is the characteristic free path, we let $t_e$ to zero to derive the following formula instead of equation (15):

$$F(t) = F_m \delta(t - t').$$  \hspace{1cm} (16)

Function $\varphi_i(t)$ then describes a delta-correlated random process and, correspondingly, random forces $F(t)$ in equation (14) satisfy the following conditions:

$$\langle F(t) \rangle = 0,$$

$$\langle F(t)F(t') \rangle \approx [W(F_m; D)F_m]^2 \delta(t - t'),$$  \hspace{1cm} (17)

where angular braces denote averaging over the ensemble consisting of $n$ independent realizations. For example,

$$\langle \varphi(t) \rangle = \frac{1}{n} \sum_{i=1}^{n} \varphi_i(t).$$  \hspace{1cm} (18)

Each realization represents a random element of stellar medium containing one test star.

Formulae (17) have simple physical meaning: interaction is practically instantaneous and all interactions are mutually uncorrelated. Equation (7) in this case is the classical Langevin equation. Below we apply the appropriate procedures and methods described, e.g., in monographs of Résumé and Leener (1982) and van Kampen (1990) and other studies, to solve this equation and perform its subsequent analysis.

5 DIFFUSION COEFFICIENTS

Given the first condition (17), we conclude from formal solution (7) that

$$\langle v_k(t) \rangle = v_0 e^{-at},$$  \hspace{1cm} (19)

where $v_0$ is the initial velocity component of the test star at initial time $t = 0$.

We then multiply equation (8) by a similar expression, and, given that $\langle v_k(t) \cdot v_i(t) \rangle = 0$ for $k \neq l$, and $\langle v_k(t) \cdot v_l(t) \rangle = \langle v_k^2(t) \rangle$, we obtain:

$$\langle v_k^2(t) \rangle = v_0^2 e^{-2at} \int_0^t dt' \int_0^{t'} dt'' e^{a(t' + t'' - t)} \langle F(t')F(t'') \rangle.$$  \hspace{1cm} (20)

Hereafter subscript ‘i’ is not due to averaging over the ensemble of realizations (see. (18) as an example). Given the second condition (17) we obtain, after performing the necessary operations:

$$\langle v_k^2(t) \rangle = v_0^2 e^{-2at} + \frac{1}{2a^2} [W(F_m; D)F_m]^{-2} (1 - e^{-2at}).$$  \hspace{1cm} (21)

The expression $W(F_m; D)F_m dF_m$ is equal to the probability of finding in the ensemble of two-point interactions considered here, a random field intensity in the interval $F_m + dF_m$ subject to the asymptotic form of generalized Holtsmark distribution (12). For numerical estimates the expression in squared brackets should be replaced by some value averaged over the ensemble of physically realizable $F_m$:

$$\langle F_m \rangle = \int_{F_{max}}^{F_{min}} W(F_m; D)F_m dF_m,$$  \hspace{1cm} (22)

where $F_{max}$ and $F_{min}$ are the boundaries of the interval of such physically realizable $F_m$ values. We now rewrite formula (11) in terms of dimensionless variable $x$:

$$x = \frac{4\pi h}{3} \frac{Gm}{\mid \vec{F} \mid} = \frac{4\pi h}{3} \frac{Gm}{F_m}^{D/2}.$$  \hspace{1cm} (23)

Formula (22) then becomes

$$\langle F_m \rangle = \gamma D F_0, \hspace{0.5cm} \gamma D = \int_{x_{min}}^{x_{max}} e^{-2x} x^{2/D} dx,$$  \hspace{1cm} (24)

where

$$F_0 = \frac{2Gm}{3} \frac{(4\pi h)^{2/D}}{3}.$$
and
\[ x_{\text{min}} = \frac{4\pi h}{3} p_{\text{min}}, x_{\text{max}} = \frac{4\pi h}{3} p_{\text{max}}. \] (25)

Here, like in formula (9), \( p_{\text{min}} = 2Gm/v_{\text{f}}^2, p_{\text{max}} \approx 2r_{\text{a}}. \)

The integral in formula (24) for \( x \to \infty \) can be written in terms of Whittaker function \( W_{\delta,\lambda}(u) \):
\[ \langle F_m \rangle = F_0 e^{-\frac{\delta}{2} e^{-\frac{2u_{\text{min}}}{\delta}}} W_{\frac{1}{2}, \frac{1}{2}} \left(\frac{u_{\text{min}}}{\delta}\right) \] (26)

In the particular cases \( D = 1 \) and \( D = 2 \), integral (24) can be expressed in terms of hypergeometric function \( Ei(x) \). For arbitrary \( x_{\text{min}}, x_{\text{max}}, \) and \( D \) integral (24) has to be computed numerically. We finally rewrite formula (21) in the form:
\[ \langle v_k^2(t) \rangle = v_0^2 e^{-2at} + \frac{1 - e^{-2at}}{2a^2} \langle F_m \rangle^2 \] (27)

We then use equation (19) to compute the time scale \( \tau_{\text{ff}} \) of the deceleration of a star due to dynamic friction:
\[ \frac{dv(t)}{dt} \sim \frac{\langle v(t) \rangle}{\tau_{\text{ff}}} \Rightarrow \tau_{\text{ff}} = \frac{\langle v(t) \rangle}{\langle v(t) \rangle / dt} = a^{-1}. \] (28)

It is evident from formula (21) that
\[ \sigma_{cc}^2 = \lim_{t \to \infty} \langle v^2(t) \rangle \rightarrow \langle F_m \rangle^2 / 2a^2. \] (29)

Physically this means that whereas in this limit dynamic friction (19) reduces the initial velocity of test stars to zero, diffusion due to random force occurs independently of initial velocity and leads to the equilibrium dispersion \( \sigma^2 \) determined by formula (29). As a consequence of these two processes the test ensemble of stars completely "forgets" its initial conditions, and the final dispersion of the ensemble becomes equal to the equilibrium velocity dispersion as defined by equation (28). Let us denote the distribution function of the equilibrium state of fractal stellar environment as \( f_0 \). Consider a small deviation of the distribution function \( f \) from the equilibrium state \( f_0 \). In this case, by analogy with (27) and taking into account equation (28), we obtain the following formula for the characteristic diffusion time \( \tau_{\text{df}} \):
\[ \tau_{\text{df}} = \frac{\sigma_{cc}^2 - \langle v_k^2(t) \rangle}{d\langle v_k^2(t) \rangle / dt} = (2a)^{-1}. \] (30)

In view of the fluctuation-dissipation theorem in its simplest form, we conclude that \( \sigma^2 = \sigma_{cc}^2 \) is the average local velocity dispersion determined in the observable volume of size \( r_{\text{obs}} \), whereas parameter \( a \) is, according to equation (10), a function of \( r \). In view of formulae (10) and (24), after simple manipulations we derive from equation (27) the following formula:
\[ \sigma_{cc}^2 = 2\left(\frac{4\pi}{3}\right) \frac{GmG}{D\sqrt{\lambda}} \frac{v_0}{\gamma_D} \frac{r^{D-3}}{r^{D-3}} \] (31)

where
\[ a_1 = \frac{8\pi G^2 m^2 \lambda}{v_0^3}, \]
and the value of integral \( \gamma_D \) and other constants were determined above.

It follows from formula (29) that the cause of the existence of equilibrium velocity dispersion in a stellar medium is the non-Poissonian (nonuniform) structure of this medium. We pass to Poisson limit in equation (29) \( D \to 3 \) to derive a formula for equilibrium velocity dispersion in a uniform Poisson medium:
\[ \sigma_{cc,3}^2 = 2\left(\frac{4\pi}{3}\right) \frac{GmG}{3\sqrt{\lambda}} \frac{r^{4}}{r^{4}} \] (32)

where \( v_0 \) is the acceleration produced by random force. In the Fokker-Planck approximation the usual practise is to expand flow \( \vec{J}_{\text{ir}} \) into a series and leave only its first two terms:
\[ J_{\text{ir}} = c_k f + c_{kl} \frac{\partial f}{\partial v_k} + c_{klm} \frac{\partial^2 f}{\partial v_k \partial v_l} + ... \approx v_k \frac{\partial f}{\partial v_k} + v_l \frac{\partial f}{\partial v_l}. \] (33)

The accelerations due to random forces and appearing
in the right-hand side of equation (32) can be determined from formulas (19) and (21). Let is consider the variations of our solutions (19) and (21) with time over small time scales on the order of $\Delta t \ll a^{-1}$. We restrict the expansion of exponential function in formula (19) to the first two terms to obtain:

$$\dot{v} \approx \frac{\Delta v}{\Delta t} v(t) - v_0 = -av_0 + \varepsilon(\Delta t)^2.$$

We similarly derive from equation (21) :

$$\dot{v}_{kl} = \frac{\Delta v^2}{\Delta t} = \frac{(v^2(t) - v^2_0)}{\Delta t} = -2av^2_0 + a^{-1}(F_m)^2 + \varepsilon(\Delta t)^2.$$

Here the problem reduces to one dimension and therefore the symmetric second-rank tensor $\dot{v}_{ij}$ is represented by its single trace term.

We now drop the second-order terms in $\Delta t$ in formulae (35) and (36) and take into account formulae (33) and (34) to write the term in the right-hand side of equation (32) in the following form:

$$\left(\frac{\partial f}{\partial t}\right)_{r} = av_{0k} \frac{\partial f}{\partial v_k} + 2av^2_0 - a^{-1}(F_m)^2 \frac{\partial^2 f}{\partial v^2_k} = av_{0k} \frac{\partial f}{\partial v_k} + 2\sigma_{eq} \frac{\partial^2 f}{\partial v^2_k}$$

(36)

Here we omit the '0' subscript at the test-star velocity.

Equation (31) with the right-hand side of the form (36) can be viewed as the fundamental equation of stellar dynamics in the Fokker-Planck approximation. Its last form in the rightmost-hand side in formula (37) can be used in the case of very small deviations of fractal gravitating medium from equilibrium. One of the main difficulties in solving kinetic equation (31) is due to the complex structure of collision integral (30). Approximation (37) also does not eliminate these difficulties. It is a common practice when dealing with equations of the form (31) to use the so-called model collision integrals - simpler operators preserving, without the fine details, the basic meaning and the main qualitative properties of the original exact collision integral. Equations of the form (31) with model collision integrals are usually referred to as model kinetic equations. One of the most famous equations of this kind is the BGK equation (Bhatnagar et al., 1954). If the deviation from the equilibrium distribution $f_0$ is small, then in the approximation of BGK model equation we have:

$$\left(\frac{\partial f}{\partial t}\right)_{r} \approx \frac{f - f_0}{\tau}$$

where, given equations (27'), (28), we have

$$\tau = \tau_{df} = \frac{\tau_{df}}{2a} \frac{\sigma_{eq}^2}{(F_m)^2}.$$

(39)

The approach represented by formulae (34) and (35) yields for $\tau_{fr}$ the estimate identical to that given by formula (27). For $\tau_{fr}$ we get another expression:

$$\tau_{df} = \frac{(v^2 - v^2_0)}{\tau_{df}} = 2a_{eq} \frac{\sigma_{eq}^2 - v^2_0}{(F_m)^2}.$$

(40)

It is evident from equation (40) that in the case $v^2_0 \rightarrow \sigma_{eq}^2$ and $\tau_{fr}$, we obtain, in view of equations (21) and (40),

$$\tau_{df}, \ v^2_0 \approx \frac{1}{2}(1 - e^{-2at}) \rightarrow 0$$

for small $t$. We pointed out a similar behaviour of characteristic time scales and velocity dispersion in our earlier paper (Chumak & Rastorguev 2014), where we derived a direct solution of Landau equation for non-equilibrium stellar media. On the other hand, if $\frac{(v^2 - v^2_0)}{\tau_{fr}}$, we obtain the same result as presented in equation (27) and, respectively, obtain the time-scale ratio $\tau_{fr}/\tau_{fr} = 2$. Since we use approximation (36) and BGK approximation (38), it is preferable to use the $\tau_{fr}$ value from formula (39).

Strictly speaking, the BGK approximation was proposed as a tool to derive qualitative estimates for solving the Boltzmann equation. However, it appears that the application of model collision integral (38) is also justified in this case, because here we consider the transition of the medium to stable equilibrium distribution, which is the main condition for the application of the model BGK approximation (Cercignani, 1975). Equation (31) with approximate collision integral (38) allows one to derive integral equations for the hydrodynamic variables in any problem considered. Many interesting problems require solving these equation. Note, however, that this approach can give only a qualitative estimate. Whenever possible, such solutions should be verified by direct numerical simulation, or by solving equation (31) with collision integral (36).

7 DISCUSSION

To demonstrate significant differences of the fractal model of gravitating medium from the Poisson model generally adopted in these models numerically estimated for the local solar neighbourhood based on the results of a fractal analysis of the GCS. We adopt $D \approx 1.23$ and $h \approx 1.644$ for the distribution of conditional density in the fractal medium from our earlier paper (Chumak & Rastorguev, 2015).

Formula (3) allows us to estimate the average effective distance to the nearest neighbour. We obtain $r_f \approx 0.48$ pc for the fractal model. The corresponding estimate for the Poisson model with $D \approx 3$ and $h \approx n \approx 0.10$ pc$^{-3}$ (we adopt local stellar density from Binney & Tremaine, 2008) is $r \approx 1$ pc, i.e., almost twice greater than for the fractal case.

Formula $r_0 = h^{-1/2}$ yields the "correlation length" of $r_0 \approx 2.41$ pc for the fractal model, which is about five times the effective interparticle distance. In the Poisson model such a parameter does not exist by definition.
We derive from equation (10) the following formula for the time scale of test star deceleration due to dynamic friction of fractal medium:

$$\tau_f = \frac{v_0^3}{8\pi G^2 m^2 \lambda^3} r^{D-3}. \quad (41)$$

In the Poisson limit ($D \to 3; h \to n_p$) we obtain the well-known formula for the relaxation time

$$\tau_p = \frac{v_0^3}{8\pi G^2 m^2 n^3 \lambda}. \quad (42)$$

Hence

$$\tau_f = n h^{-1} (D-3) \tau_p. \quad (43)$$

We substitute into this formula the parameter values for the solar neighbourhood mentioned above and adopt $r = r_0$ to obtain $\tau_f \approx 0.032 \tau_p$. In other words, on this scale the characteristic time of dynamic deceleration in terms of fractal model is about a factor of 30 shorter than in the Poisson limit. As is evident from formula (42), this characteristic time is even shorter on scale lengths $r < r_0$. It becomes equal to the Poisson-limit time scale only on scale lengths of $\sim 300$ pc. At greater distances $\tau_f > \tau_p$.

In this framework of this approach there exist small deviation from the current dispersion from its equilibrium value, $(\sigma_{0h}(t)) \to \sigma_{eq}$, and we obtain the relation $\tau_{eq} = 2 \tau_f r$. However, it should be noted that for larger deviations from the equilibrium formula (39), the ratio between the characteristic times will differ from that, and, accordingly, an approximate model (38) is not applicable. In these cases, the approximation (34), (35) is no more applicable, and we should use more realistic approximation of the collision integral.

And one more point. The kinetics of gravitating media differs fundamentally from that of uniform media (on scale lengths that are long compared to interparticle distance) because in the former case the diffusion coefficients appearing in kinetic equations depend on spatial scale length. This difference is emphasized quite clearly in the approach to kinetics developed by a number of authors and in this paper, involving averaged conditional density and average fractal dimension on the scale length of the fragment of the gravitating medium studied. However, this is by no means the only difference. Because of its alternating pattern the structure of fractal medium with all its multiscale concentrations and voids is much more complex. To establish a consistent kinetic theory of such a medium, a special mathematical apparatus has to be developed that should free of such strong assumptions as the validity of using average conditional density and average fractal dimension on the scale length of the fragment of the gravitating medium studied. However, this is by no means the only difference. Because of its alternating pattern the structure of fractal medium with all its multiscale concentrations and voids is much more complex. To establish a consistent kinetic theory of such a medium, a special mathematical apparatus has to be developed that should free of such strong assumptions as the validity of using average conditional density and average fractal dimension. The deviations of the actual values of these quantities from their means may be quite significant. Therefore our results should be viewed only as a first approximation, which appears to be closer to reality than the classical approximation based on the model of uniform Poisson space distribution. An alternative approach to the development of kinetics of fractal media is based on the derivation of fractional kinetic equations for each particular problem (see, e.g., Saichev & Zaslavsky, 1997). See also the monograph by Uchaikin (2008) for a review of publications in this field, which includes more than one hundred references. Fractional generalization of kinetic equations is based on the use of fractional derivatives with respect to time and coordinates. This technique will possibly produce more accurate approximations for the problem considered in this paper.

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