COMPARING POWERS OF EDGE IDEALS

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ABSTRACT. Given a nontrivial homogeneous ideal $I \subseteq k[x_1, x_2, \ldots, x_d]$, a problem of great recent interest has been the comparison of the $r$th ordinary power of $I$ and the $m$th symbolic power $I^{(m)}$. This comparison has been undertaken directly via an exploration of which exponents $m$ and $r$ guarantee the subset containment $I^{(m)} \subseteq I^r$ and asymptotically via a computation of the resurgence $\rho(I)$, a number for which any $m/r > \rho(I)$ guarantees $I^{(m)} \subseteq I^r$. Recently, a third quantity, the symbolic defect, was introduced; as $I^t \subseteq I^{(t)}$, the symbolic defect is the minimal number of generators required to add to $I^t$ in order to get $I^{(t)}$.

We consider these various means of comparison when $I$ is the edge ideal of certain graphs by describing an ideal $J$ for which $I^{(t)} = I^t + J$. When $I$ is the edge ideal of an odd cycle, our description of the structure of $I^{(t)}$ yields solutions to both the direct and asymptotic containment questions, as well as a partial computation of the sequence of symbolic defects.

1. Introduction

Let $k$ be an algebraically closed field, and $I$ a nonzero proper homogeneous ideal in $R = k[x_0, x_1, x_2, \ldots, x_N]$. Recall that the $m$th symbolic power of $I$ is the ideal

$$I^{(m)} = R \cap \left( \bigcap_{P \in \text{Ass}(I)} I^m R_P \right).$$

Over the last 10–15 years, the structure of $I^{(m)}$ has been an object of ongoing study; see, e.g., the recent survey [5]. One avenue for this study has been the examination of the relationship between $I^{(m)}$ and the well-understood algebraic structure of $I^r$, the $r$th ordinary power of $I$. The naive context in which to examine this relationship is via subset containments, i.e., for which $m$ and $r$, $s$, and $t$ do we have $I^s \subseteq I^{(t)}$ and $I^{(m)} \subseteq I^r$? In fact, this line of inquiry has been extremely productive. It is straightforward to see that $I^s \subseteq I^{(t)}$ if and only if $s \geq t$, but determining which $r$ and $m$ give $I^{(m)} \subseteq I^r$ is more delicate.

Seminal results of Ein-Lazarsfeld-Smith and Hochster-Huneke [8, 11] established that for such ideals, $I^{(m)} \subseteq I^r$ if $m/r \geq N$. Additional information about the ideal under consideration generally leads to tighter results (see, e.g., [1, 6, 7]). This phenomenon led to Bocci and Harbourne’s introduction of a quantity known as the resurgence of $I$, denoted $\rho(I)$; it is the least upper bound of the set $T = \{ m/r \mid I^{(m)} \nsubseteq I^r \}$. Thus, if $m/r > \rho(I)$, we have $I^{(m)} \subseteq I^r$.

Recently, Galetto, Geramita, Shin, and Van Tuyl introduced a new measure of the difference between $I^{(m)}$ and $I^m$ known as the symbolic defect. Since $I^m \subseteq I^{(m)}$, the quotient
$I^{(m)}/I^m$ is a finite $R$-module; thus, we let $\text{sd}(I, m)$ denote the number of minimal generators of $I^{(m)}/I^m$ as an $R$-module. This is known as the symbolic defect, and the symbolic defect sequence is the sequence \{sdefect$(I, m)$\}$_{m \in \mathbb{N}}$. In [9], the authors study the symbolic defect sequences of star configurations in $\mathbb{P}^n_k$ and homogeneous ideals of points in $\mathbb{P}^2_k$.

Our work considers all these questions in the context of a class of edge ideals. Let $G = (V, E)$ be a (simple) graph on the vertex set $V = \{x_1, x_2, \ldots, x_n\}$ with edge set $E$. The edge ideal of $G$, introduced in [14], is the ideal $I(G) \subseteq R = k[x_1, x_2, \ldots, x_n]$ given by $I(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E\})$.

That is, $I(G)$ is generated by the products of pairs of those variables between which are edges in $G$.

In [12], the authors establish that, for an edge ideal $I = I(G)$, we have $I^{(m)} = I^m$ for all $m \geq 1$ if and only if $G$ is bipartite. A natural question, then, is to explore the relationship between $I(G)^{(m)}$ and $I(G)^r$ when $G$ is not bipartite, which is equivalent to $G$ containing an odd cycle. Thus, [15] sought to explore this relationship when $G = C_{2n+1}$ is a cycle on $2n + 1$ vertices.

We continue the problem of exploring the structure of the symbolic power $I(G)^{(t)}$ for certain classes of graphs $G$, with a focus on when $G$ is an odd cycle. The main results of this work are Theorem 4.4 and Corollary 4.5, which together describe a decomposition of the form $I^{(t)} = I^t + J$, where $J$ is a well-understood ideal. We are then able to use this decomposition to resolve a conjecture of [15], compute $\rho(I(C_{2n+1}))$ in Theorem 5.11, and establish a partial symbolic defect sequence in Theorem 5.12. We close by showing that our ideas in Theorem 4.4 apply for complete graphs and graphs which consist of an odd cycle plus an additional vertex and edge.

Remark. As preparation of this manuscript was concluding in summer 2017, Dao et. al posted the preprint [5]. In particular, their Theorem 4.13 bears a striking resemblance to our Corollary 5.3. While these similarities are worth noting, in part as evidence that interest in symbolic powers is high, it is also worth noting that the aims of these two works are distinct and complementary. The aim of the relevant sections of [5] is to investigate the packing property for edge ideals, while ours is to more directly describe the difference between the ordinary and symbolic powers by investigating the structure of a set of minimal generators for $I^{(t)}$. We then use information about these generators to compute invariants related to the containment $I^{(m)} \subseteq I^r$.

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2. BACKGROUND RESULTS

Edge ideals are an important class of examples of squarefree monomial ideals, i.e., an ideal generated by elements of the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $a_i \in \{0, 1\}$ for all $i$. When
I is squarefree monomial, it is well-known that the minimal primary decomposition is of the form

\[ I = P_1 \cap \cdots \cap P_r, \text{ with } P_j = (x_{j_1}, \ldots, x_{j_{s_j}}) \text{ for } j = 1, \ldots, r. \]

When \( I = I(G) \) is an edge ideal, the variables in the \( P_j \)'s are precisely the vertices in the minimal vertex covers of \( G \). Recall that, given a graph \( G = (V, E) \), a vertex cover of \( G \) is a subset \( V' \subseteq V \) such that for all \( e \in E \), \( e \cap V' \neq \emptyset \). A minimal vertex cover is a vertex cover minimal with respect to inclusion. The minimal vertex covers will be especially useful to us, as they describe the variables needed to decompose an edge ideal into its minimal primes (see, e.g., [13, Corollary 3.35]).

**Lemma 2.1.** Let \( G \) be a graph on the vertices \( \{x_1, x_2, \ldots, x_n\} \), \( I = I(G) \subseteq k[x_1, x_2, \ldots, x_n] \) be the edge ideal of \( G \) and \( V_1, V_2, \ldots, V_r \) the minimal vertex covers of \( G \). Let \( P_j \) be the monomial prime ideal generated by the variables in \( V_j \). Then

\[ I = P_1 \cap P_2 \cap \cdots \cap P_r \]

and

\[ I^{(m)} = P_1^m \cap P_2^m \cap \cdots \cap P_r^m. \]

Symbolic powers of squarefree monomial ideals (and, more specifically, edge ideals) have enjoyed a great deal of recent interest (see, e.g., [4, 3]). In [3], a linear programming approach is used to compute invariants related to the containment question. We adapt this technique in Lemma 5.8 for the edge ideals under consideration in this paper. One result of [3] which will be of use is the following, which reduces the problem of determining whether a given monomial is in \( I^{(m)} \) to a problem of checking certain (linear) constraints on the exponents of the variables.

**Lemma 2.2.** Let \( I \subseteq R \) be a squarefree monomial ideal with minimal primary decomposition \( I = P_1 \cap P_2 \cap \cdots \cap P_r \) with \( P_j = (x_{j_1}, \ldots, x_{j_{s_j}}) \) for \( j = 1, \ldots, r \). Then \( x_{a_1} \cdots x_{a_n} \in I^{(m)} \) if and only if \( a_{j_1} + \cdots + a_{j_{s_j}} \geq m \) for \( j = 1, \ldots, r \).

**Remark 2.3.** Throughout this work, we will be exploring questions related to ideals in \( R = k[x_1, x_2, \ldots, x_n] \) related to graphs on the vertex set \( \{x_1, x_2, \ldots, x_n\} \). We will use the \( x_i \)'s interchangeably to represent both vertices and variables. The specific use should be clear from context, and we see this as an opportunity to emphasize the close connection between the graph and the ideal.

### 3. Factoring Monomials Along Odd Cycles

In this section, we introduce the main ideas of our approach to studying symbolic powers of edge ideals. We begin by defining a means of writing a monomial in a power of an edge ideal with respect to the minimal vertex covers of the graph. We then study this factorization and describe a situation in which it can be improved. In what follows, let \( R = k[x_1, x_2, \ldots, x_{2n+1}] \) and let \( I = I(C_{2n+1}) \) be the edge ideal of the odd cycle \( C_{2n+1} \), i.e.,

\[ I = (x_1 x_2, x_2 x_3, \ldots, x_{2n} x_{2n+1}, x_{2n+1} x_1). \]
**Definition 3.1.** Let \( m \in k[x_1, x_2, \ldots, x_{2n+1}] \) be a monomial. Let \( e_j \) denote the degree two monomial representing the \( j \)th edge in the cycle, i.e., \( e_j = x_jx_{j+1} \) for \( 1 \leq j \leq 2n \), and \( e_{2n+1} = x_{2n+1}x_1 \). We may then write

\[
m = x_1^{a_1} x_2^{a_2} \cdots x_{2n+1}^{a_{2n+1}} e_1^{b_1} e_2^{b_2} \cdots e_{2n+1}^{b_{2n+1}},
\]

where \( b(m) := \sum b_j \) is as large as possible (observe \( 0 \leq 2b(m) \leq \deg(m) \)) and \( a_i \geq 0 \). When \( m \) is written in this way, we will call this an **optimal factorization** of \( m \), or say that \( m \) is expressed in **optimal form**. In addition, each \( x_i^{a_i} \) with \( a_i > 0 \) in this form will be called an **ancillary**.

When \( m \) is written in optimal form, i.e., it can be rewritten as a product of strictly more edges. That is, we can re-express \( m \) and without loss of generality, let \( \sum b_i \geq x_i^{a_i} \) such that for all evenly indexed \( a_i > 0 \) and \( 0 \leq b_i \leq b_j \) for all \( i \) and \( j \).

**Lemma 3.2.** Let \( m = x_1^{a_1} x_2^{a_2} \cdots x_{2n+1}^{a_{2n+1}} e_1^{b_1} e_2^{b_2} \cdots e_{2n+1}^{b_{2n+1}} \) be an optimal factorization, where \( m \in I(C_{2n+1}) \). Then any \( m' = x_1^{a'_1} x_2^{a'_2} \cdots x_{2n+1}^{a'_{2n+1}} e_1^{b'_1} e_2^{b'_2} \cdots e_{2n+1}^{b'_{2n+1}} \) will also be an optimal factorization if \( 0 \leq a'_i \leq a_i \) and \( 0 \leq b'_i \leq b_i \) for all \( i \).

**Proof.** Let \( m' = x_1^{a'_1} x_2^{a'_2} \cdots x_{2n+1}^{a'_{2n+1}} e_1^{b'_1} e_2^{b'_2} \cdots e_{2n+1}^{b'_{2n+1}} \) such that for all \( i,j \), \( 0 \leq a'_i \leq a_i \) and \( 0 \leq b'_j \leq b_j \). Since each variable’s exponent from \( m' \) is less than or equal to the corresponding exponent from \( m \), we know that \( m' \) divides \( m \). Thus, there must exist some

\[
m'' = x_1^{(a_1-a'_1)} x_2^{(a_2-a'_2)} \cdots x_{2n+1}^{(a_{2n+1}-a'_{2n+1})} e_1^{(b_1-b'_1)} e_2^{(b_2-b'_2)} \cdots e_{2n+1}^{(b_{2n+1}-b'_{2n+1})}
\]

such that \( m = m' m'' \).

Suppose that \( m' \) is not in optimal form. Then there must exist some other way of expressing \( m' \) such that the sum of the exponents of edge factors will be greater in this new expression. That is, we can re-express \( m' \) as \( m' = x_1^{c_1} x_2^{c_2} \cdots x_{2n+1}^{c_{2n+1}} e_1^{d_1} e_2^{d_2} \cdots e_{2n+1}^{d_{2n+1}} \), such that

\[
\sum b'_i < \sum d_i. \\
\text{As } m = m'm'' = \prod x_i^{(a_i-a'_i+c_i)} (b_i-b'_i+d_i),
\]

\( m \) will have an edge exponent sum of \( \sum (b_i-b'_i+d_i) = \sum b_i - \sum b'_i + \sum d_i \). As \( \sum b'_i < \sum d_i \), it must be true that this edge exponent sum is greater than \( \sum b_i \). This contradicts the premise that \( m \) was expressed in optimal form, and thus \( m'' = x_1^{a'_1} x_2^{a'_2} \cdots x_{2n+1}^{a'_{2n+1}} e_1^{b'_1} e_2^{b'_2} \cdots e_{2n+1}^{b'_{2n+1}} \) is an optimal factorization of \( m' \). \( \square \)

The next lemma describes a process that will be critical in the proof of the main result. Intuitively, it says that if a monomial is factored as a product of an odd number of consecutive edges with ancillaries on both ends of this path of edges, the monomial is not written in optimal form, i.e., it can be rewritten as a product of strictly more edges.

**Lemma 3.3.** Let \( m = x_1^{a_j} x_2^{b_j+1} \cdots x_{2n+1}^{a_{j+2k}} e_1^{a_{j+2k+1}} \) where \( a_j, a_{j+2k+1} \geq 1 \). If it is the case that \( b_j+2h+1 \geq 1 \) for all \( h \in \{0, \ldots, k-1\} \), then \( m \) is not in optimal form.

**Proof.** Let \( m = x_1^{a_j} x_2^{b_j+1} \cdots x_{2n+1}^{a_{j+2k}} e_1^{a_{j+2k+1}} \) and notice that \( m \) is a string of adjacent edges with ancillaries on either end. Our goal is to rewrite \( m \) in a more optimal form. For clarity, and without loss of generality, let \( j = 1 \), and suppose that \( b_j \geq 1 \) for all evenly indexed edge exponents.
Let \( p = x_1 e_2 e_4 \cdots e_{2k} x_{2k+2} \) and note that by Lemma 3.2, \( p \) must be in optimal form as \( m \) is expressed optimally. However,

\[
p = x_1 e_2 e_4 \cdots e_{2k} x_{2k+2} = x_1 (x_2 x_3)(x_4 x_5) \cdots (x_{2k} x_{2k+1}) x_{2k+2} = (x_1 x_2)(x_3 x_4) (x_5 x_6) \cdots (x_{2k+1} x_{2k+2}) = e_1 e_3 e_5 \cdots e_{2k+1}
\]

Since \( p \) was not initially expressed in optimal form, we know that \( m \) could not have been an optimal factorization. \( \square \)

**Example 3.4.** Let \( G \) be a cycle with 111 vertices and consider \( m = x_1^3 x_2^4 x_3^2 x_4^3 x_5^3 x_6^3 x_7^2 x_8 \in I(G) \) with edge factorization:

\[
m = x_1 e_1^2 e_2 e_3 e_4 e_5^2 e_6 e_7 x_8
\]

where \( e_i = x_i x_{i+1} \). Note that in this factorization, there is an ancillary at \( x_1 \) and \( x_8 \). We will show that \( m \) is not in optimal form.

We can graphically represent \( m \) by drawing an edge between \( x_i \) and \( x_{i+1} \) for each \( e_i \) in \( m \) and creating a bold outline for each ancillary, as shown below:

Using the method outlined in Lemma 3.3, we will “break” each of the red (bolded) edges back into standard \( x_i \) notation so that we create new ancillaries at every vertex.

Note that if we define a new monomial \( p \) based on this graphical representation, where \( p = x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 e_1^2 e_2 e_4 e_5^2 e_7 \), it will still be true that \( m = p \) because we are merely changing the factorization of the monomial, not its value.

As one can see, there are now 8 consecutive ancillaries, which we can pair up in a new way, as shown below. New edges are highlighted in green (bolded in the second line).

Now we have a third possible representation \( q \) of this monomial. Note that \( q = e_1^3 e_2 e_3 e_4^2 e_5^3 e_7^2 \), and it is still true that \( q = p = m \). As you can see, this monomial representation has one more edge than our original representation, which means that \( m \) is not optimal.
Example 3.5. Let $G$ be a cycle with 111 vertices in it, and consider the following edge factorization of $m = x_1^4 x_2^4 x_3^2 x_4^3 x_5^3 x_6 x_7 x_8$:

$$m = x_1 e_1^3 e_2 e_3 e_4^2 e_5^3 e_7 x_8$$

Again, our goal is to determine whether or not $m$ is optimal.

Note that $m$ is equal to the monomial $q$ from Example 3.4, except that there are now ancillaries at $x_1$ and $x_8$. Again, we will create a graphical representation of $m$, shown below:

![Graphical representation of m]

However, now it is impossible to remove the right combination of edges so that we create an ancillary at every vertex because no edge exists between $x_6$ and $x_7$.

Therefore, we cannot use Lemma 3.3 conclude that $m$ is not optimal. In fact, we have no conclusive way to determine whether $m$ is an optimal factorization at this point.

Despite that, this example has not been without value. Note the nonexistence of an edge between $x_6$ and $x_7$ and the fact we would have been able to prove that $m$ was not optimal if not for the nonexistence of at least one of the following: $e_2$, $e_4$, $e_6$. This will be useful for the latter stages of the proof of Theorem 4.4.

4. Powers of Edge Ideals and Their Structures

We will now turn to a decomposition of $I^{(t)}$ in terms of $I^t$ and another ideal $J$ so that $I^{(t)} = I^t + J$. Our approach has numerous strengths, including the ability to easily compute the symbolic defect of $I$ for certain powers, as well as determining which additional elements are needed to generate $I^{(t)}$ from $I^t$.

Although we will primarily focus on odd cycles in this section, we go on to show that the same underlying principles can be extended to edge ideals of other types of graphs; see Section 6 for more.

Definition 4.1. Let $V' \subseteq V(G) = \{x_1, x_2, \ldots, x_{2n+1}\}$ be a set of vertices. For a monomial $x^a \in k[x_1, x_2, \ldots, x_{2n+1}]$ with exponent vector $a = (a_1, a_2, \ldots, a_{2n+1})$, define the vertex weight $w_{V'}(x^a)$ to be

$$w_{V'}(x^a) := \sum_{x_i \in V'} a_i.$$

We will usually be interested in the case when $V'$ is a (minimal) vertex cover.

Using the language of vertex weights, the definition of the symbolic power of an edge ideal given in Lemma 2.2 becomes

$$I^{(t)} = (\{x^a | \text{for all minimal vertex covers } V', w_{V'}(x^a) \geq t\}).$$

Now define sets

$$L(t) = \{x^a | \deg(x^a) \geq 2t \text{ and for all minimal vertex covers } V', w_{V'}(x^a) \geq t\}$$

and

$$D(t) = \{x^a | \deg(x^a) < 2t \text{ and for all minimal vertex covers } V', w_{V'}(x^a) \geq t\},$$
and generate ideals \((L(t))\) and \((D(t))\), respectively. Note that \(I^{(t)} = (L(t)) + (D(t))\). The main work of this section is to show, for the edge ideal \(I\) of an odd cycle, that \(I^t = (L(t))\), which is the content of the Theorem 4.4.

**Lemma 4.2.** Let \(R = k[x_1, \ldots, x_r], G\) be a graph on \(\{x_1, \ldots, x_r\}\), \(I = I(G)\), and \(L(t)\) is as defined above. Then \(I^t \subseteq (L(t))\).

Proof. Suppose \(m \in I^t\). Write \(m\) in optimal form as \(m = x_1^{a_1} \cdots x_r^{a_r} \prod_{i < j} e_{i,j}^{b_{i,j}}\). We know that given an arbitrary minimal vertex cover \(V'\) and edge \(e_{i,j} = x_i x_j\) dividing \(m\), it must be true that \(x_i \in V'\) or \(x_j \in V'\) or both. Thus \(w_{V'}(m) \geq b(m)\). Further, since \(m \in I^t\), we know \(b(m) \geq t\) and \(\deg(m) \geq 2t\), which means that \(m \in (L(t))\). \(\square\)

**Lemma 4.3.** Let \(R = k[x_1, \ldots, x_r], G\) be a graph on \(\{x_1, \ldots, x_r\}\), \(I = I(G)\), and \(L(t)\) be as defined above. Then for all \(m \not\in I^t\), if \(m\) has no ancillaries or a single ancillary of degree 1 then \(m \not\in (L(t))\).

Proof. If there are no ancillaries in \(m\) then \(\deg(m) = 2b(m) < 2t\). Thus, \(m\) cannot be in \((L(t))\), which also means that it is not in \((L(t))\) as none of the divisors of \(m\) are in \((L(t))\) for a similar reason. Furthermore, we reach the same conclusion if there is only one ancillary in \(m\) and it has an exponent of 1, as \(\deg(m) = 2b(m) + 1 < 2t + 1\), and since \(2b(m) + 1\) and \(2t + 1\) are both odd, \(2b(m) + 1 < 2t\). \(\square\)

For the remainder of this section, let \(R = k[x_1, x_2, \ldots, x_{2n+1}], G\) be an odd cycle of size \(2n + 1\) with the vertices \(V(G) = \{x_1, x_2, \ldots, x_{2n+1}\}\), \(I\) be the edge ideal of \(G\), and \(V' \subseteq V(G)\) be a minimal vertex cover of \(G\). We make the following definition, which describes the sum of the exponents of a given monomial relative to a set of vertices.

**Theorem 4.4.** Given \(I\) and \((L(t))\) as defined above, \(I^t = (L(t))\).

Proof. By Lemma 4.2 we know that \(I^t \subseteq (L(t))\) so we must only show the reverse containment. Let \(m \not\in I^t\) (which implies that \(b(m) < t\)), then we will show that \(m \not\in (L(t))\).

Lemma 4.3 allows us to consider only cases where \(m\) either has multiple ancillaries or has a single ancillary of at least degree 2.

Given an arbitrary monomial \(m \not\in I^t\), let \(m = x_{\ell_1}^{a_{\ell_1}} x_{\ell_2}^{a_{\ell_2}} \cdots x_{\ell_r}^{a_{\ell_r}} e_1^{b_1} e_2^{b_2} \cdots e_{2n+1}^{b_{2n+1}}\) be an optimal factorization of \(m\) where \(x_{\ell_q}^{a_{\ell_q}}\) is an ancillary and \(1 \leq \ell_1 < \ell_2 < \cdots < \ell_r \leq 2n + 1\).

Our goal is to show that there exists some vertex cover with a weight equal to \(b(m)\), and as \(b(m) < t\), \(m\) cannot be in \((L(t))\). Since \((L(t))\) is the generating set of \((L(t))\), this will be sufficient to claim that \(m \not\in (L(t))\) because neither \(m\), nor any of its divisors whose vertex weights can only be less than that of \(m\), will be in the generating set.

We will construct a minimal vertex cover \(S\) of \(G\) out of a sequence of subsets \(S_1, S_2, \ldots, S_r\) of \(V\), where each \(S_q\) is a cover for the induced subgraph \(H_q\) of \(G\) on

\[ V_{H_q} = \{ x_{\ell_q}, x_{\ell_q+1}, \ldots, x_{\ell_{q+1}-1}, x_{\ell_{q+1}} \} \]

For the sake of simplicity, let \(x_i^{a_i}\) and \(x_j^{a_j}\) be a pair of consecutive ancillaries (or let \(x_i^{a_i} = x_{\ell_r}\) and \(x_j^{a_j} = x_{\ell_1}\) in the wraparound case, or let \(x_i^{a_i} = x_{\ell_1}\) and \(x_j^{a_j} = x_{\ell_1}^{a_{\ell_1}-1}\) in the case of
a single ancillary with degree greater than 1). In addition, let \( m_q = x_i^{a_i}e_i^{b_i}e_{i+1}^{b_{i+1}} \cdots e_{j-1}^{b_{j-1}}x_j^{a_j} \).

Note that by Lemma 3.2, \( m_q \) is in optimal form.

We will show for each subgraph \( H_q \), there exists some set of vertices \( S_q \subseteq V_{H_q} \) that covers \( H_q \) such that \( w_{S_q}(m_q) = b(m_q) \).

**Case 1:** Suppose that \( V_{H_q} \) has an odd number of elements. Consider

\[
S_q = \{ x_{i+1}, x_{i+3}, \ldots, x_{j-1} \}.
\]

We claim that \( w_{S_q}(m_q) = b(m_q) \). This can be shown as follows:

\[
m_q = x_i^{a_i}e_i^{b_i}e_{i+1}^{b_{i+1}} \cdots e_{j-2}^{b_{j-2}}e_{j-1}^{b_{j-1}}x_j^{a_j} \\
= x_i^{a_i}(x_{i+1}x_{i+1})^{b_i}(x_{i+1+1}x_{i+2})^{b_{i+1}} \cdots (x_{j-2}x_{j-1})^{b_{j-2}}(x_{j-1}x_j)^{b_{j-1}}x_j^{a_j} \\
= x_i^{a_i}x_{i+1}^{b_i+b_i}(x_{i+1+1}x_{i+2})^{b_{i+1}+b_{i+2}} \cdots (x_{j-2}+b_{j-2})x_{j-1}^{b_{j-1}+b_{j-1}}x_j^{b_{j-1}+a_j}.
\]

Intuitively, this is because we are selecting alternating vertices to be in \( S_q \), which would guarantee that no edge of \( m_q \) contributes to the weight twice because edges can only connect sequentially indexed vertices. Also, there are no ancillaries in \( m_q \) other than \( x_i^{a_i} \) and \( x_j^{a_j} \), which would increase the weight if they are included.

By Definition 4.1, we know that the weight of a monomial with respect to a set of variables will be equal to the sum of the powers of those variables in the given monomial. In this case,

\[
w_{S_q}(m_q) = (b_i + b_{i+1}) + (b_{i+2} + b_{i+3}) + \cdots + (b_{j-2} + b_{j-1}) \\
= \sum_{h=i}^{j-1} b_h \\
= b(m_q)
\]

**Case 2:** Suppose now that \( V_{H_q} \) has an even number of elements. Note that it must contain more vertices than simply \( x_i \) and \( x_j \), because that would imply that there are no vertices between \( x_i \) and \( x_j \) and that the two ancillaries are adjacent and could thus be expressed as \( e_i \), which would contradict the statement that \( m \) is expressed in optimal form.

From Lemma 3.3, we know that for some \( h \) satisfying \( 1 \leq h \leq \frac{j-i-1}{2} \), the edge product \( e_{i+2h-1} \) does not appear in the current optimal form of \( m_q \), that is, \( b_{i+2h-1} = 0 \).

Consider \( S_q = \{ x_{i+1}, x_{i+3}, \ldots, x_{i+2h-1}, x_{i+2h}, x_{i+2h+2}, \ldots, x_{j-1} \} \). We claim that \( w_{S_q}(m_q) = b(m_q) \). We see
\[ m_q = a_i b_i c_i b_{i+1} \cdots e_j b_{j-1} a_j \]

\[ = x_i^{a_i} (x_i x_{i+1})^{b_i} (x_{i+1} x_{i+2})^{b_{i+1}} \cdots (x_{j-2} x_{j-1})^{b_{j-2}} (x_{j-1} x_j)^{b_j} a_j \]

\[ = x_i^{(a_i + b_i)} x_i^{(b_i + b_{i+1})} x_i^{(b_{i+1} + b_{i+2})} \cdots x_i^{(b_{j-2} + b_{j-1})} x_j^{(b_{j-1} + a_j)}. \]

Then:

\[ w_{S_q}(m_q) = (b_i + b_{i+1}) + (b_{i+2} + b_{i+3}) + \cdots + (b_{i+2h-2} + b_{i+2h-1}) + (b_{i+2h-1} + b_{i+2h}) \]

\[ + (b_{i+2h+1} + b_{i+2h+2}) + \cdots + (b_{j-2} + b_{j-1}) \]

\[ = b_{i+2h-1} + \sum_{h=i}^{j-1} b_h \]

\[ = b_{i+2h-1} + b(m_q) \]

\[ = 0 + b(m_q) \]

\[ = b(m_q). \]

Intuitively, this is because of the same reasons that were given when \( V_{H_q} \) had an odd number of elements, since alternating vertices are again chosen to be in \( S_q \) with the exception of \( x_{i+2h-1} \) and \( x_{i+2h} \). However, because the edge product \( e_{i+2h-1} \) does not appear in \( m_q \), we are not including any redundant powers in our weight, which means that \( w_{S_q}(m_q) = b(m_q) \). Hence, it does not matter whether \( V_{H_q} \) has an odd or even number of vertices because \( w_{S_q}(m_q) = b(m_q) \) regardless.

Now, since each \( S_q \) covers its respective set of vertices, the union of all of these disjoint subcovers \( S = \bigcup S_q \) is a vertex cover of \( G \). In addition, as each \( S_q \) is completely disjoint from any other subgraph’s cover, \( w_S(m) = \sum w_{S_q}(m_q) = \sum b(m_q) \). As each \( b(m_q) \) was the number of edges that existed in that induced subgraph representation, and no two subgraphs contained any of the same edges, \( \sum b(m_q) = b(m) \), the total number of edges in an optimal factorization of \( m \). That is, we have constructed a vertex cover \( S \) such that \( w_S(m) = b(m) < t \). Thus, \( m \notin (L(t)) \), and therefore \( I^t = (L(t)) \). \( \square \)

**Corollary 4.5.** Given \( I \) and \( (D(t)) \) as above, \( I^{(t)} = I^t + (D(t)) \)

**Proof.** Theorem 4.4 states that \( I^t = (L(t)) \). As we also know that \( I^{(t)} = (L(t)) + (D(t)) \), we can simply substitute \( (L(t)) \) with \( I^t \). Thus, \( I^{(t)} = I^t + (D(t)) \). \( \square \)

Now that we have proved that \( I^{(t)} = I^t + (D(t)) \), we will use this result to carry out various computations related to the interplay between ordinary and symbolic powers.

We close this section with a brief remark on the proof of Theorem 4.4. Specifically, it relies on the fact that \( G \) is a cycle, but not that \( G \) is an odd cycle. However, we focus on the odd cycle case as, when \( G \) is an even cycle, it is bipartite, and [12] showed in that case that \( I^t = I^{(t)} \) for all \( t \geq 1 \).
5. Applications to Ideal Containment Questions

Given the edge ideal $I$ of an odd cycle $C_{2n+1}$, Corollary 4.5 describes a structural relationship between $I^{(t)}$ and $I^t$ given any $t \geq 1$. In this section, we will exploit this relationship to establish the conjecture of [15]. We then will compute the resurgence of $I(C_{2n+1})$ and explore the symbolic defect of various powers of $I$.

Given $I = I(C_{2n+1}) \subseteq R = k[x_1, x_2, \ldots, x_{2n+1}]$, recall the definitions of $L(t)$ and $D(t)$, which generate ideals $(L(t)) = I^t$ and $(D(t))$, respectively.

$$L(t) = \{x^a | \deg(x^a) \geq 2t \text{ and for all minimal vertex covers } V', w_{V'}(x^a) \geq t\}$$

and

$$D(t) = \{x^a | \deg(x^a) < 2t \text{ and for all minimal vertex covers } V', w_{V'}(x^a) \geq t\}.$$  

We will begin by examining $D(t)$.

**Lemma 5.1.** For a given monomial $x^a$, if there exists some $i$ such that $a_i = 0$, then $x^a \notin (D(t))$.

**Proof.** Recall that $(D(t))$ is the ideal generated by $D(t)$.

Although each graph can have many different minimal vertex covers, there is a certain type of vertex cover that is guaranteed to exist for any odd cycle. This type of cover includes any two adjacent vertices and alternating vertices thereafter.

Without loss of generality, consider $x^a$ and suppose $a_1 = 0$. Two such minimal vertex covers that include $x_1$ are $\{x_1, x_2, x_4, x_6, \ldots, x_{2n}\}$ and $\{x_1, x_3, x_5, \ldots, x_{2n+1}\}$.

In order for $x^a$ to be in $D(t)$, it must be true that $w_{V'}(x^a) \geq t$. This means that $a_1 + a_2 + a_4 + \ldots + a_{2n} \geq t$ and $a_1 + a_3 + \ldots + a_{2n+1} \geq t$. Adding the inequalities yields $a_1 + (a_1 + a_2 + a_3 + \ldots + a_{2n+1}) \geq 2t$. As $a_1 = 0$, it follows that $\sum_{i=1}^{2n+1} a_i \geq 2t$, which contradicts the requirement that $\deg(x^a) < 2t$. Hence, any monomial $x^a$ with at least one exponent equal to 0 cannot be an element of $D(t)$ or, by extension, $(D(t))$. \qed

**Lemma 5.2.** For a given monomial $x^a$ in $D(t)$, if $\deg(x^a) = 2t - k$, then $x^a$ is divisible by $(x_1 x_2 \cdots x_{2n+1})^k$.

**Proof.** Let $x^a$ be an element of $D(t)$ such that $\deg(x^a) = 2t - k$, and suppose that $x^a$ is not divisible by $(x_1 x_2 \cdots x_{2n+1})^k$. This means that there exists an $i_0$ such that $a_{i_0} < k$. Moreover, since $x^a \in (D(t))$, we must have $a_j > 0$ for all $j$.

If $i_0$ is odd, consider minimal vertex covers

$$V_1 = \{x_1, x_3, \ldots, x_{i_0}, x_{i_0+1}, x_{i_0+3}, \ldots, x_{2n}\}$$

and

$$V_2 = \{x_2, x_4, \ldots, x_{i_0-1}, x_{i_0}, x_{i_0+2}, x_{i_0+4}, \ldots, x_{2n+1}\}$$

(if $i_0$ is even, use

$$V_1 = \{x_1, x_3, \ldots, x_{i_0-1}, x_{i_0}, x_{i_0+2}, \ldots, x_{2n}\}$$

and

$$V_2 = \{x_2, x_4, \ldots, x_{i_0}, x_{i_0+1}, x_{i_0+3}, \ldots, x_{2n+1}\}).$$

In order for $x^a$ to be in $D(t)$, it must be true that $w_{V_j}(x^a) \geq t$ for $j = 1, 2$. When $i_0$ is odd, this means that $a_1 + a_3 + \ldots + a_{i_0} + a_{i_0+1} + \ldots + a_{2n} \geq t$ and $a_2 + a_4 + \ldots + a_{2n+1} \geq t$. If $i_0$ is even, this results in $a_2 + a_4 + \ldots + a_{2n+1} \geq t$. Hence, we can conclude that $x^a$ is divisible by $(x_1 x_2 \cdots x_{2n+1})^k$. \qed
\[a_{i_0} - 1 + a_{i_0} + a_{i_0} + 2 + \cdots + a_{2n+1} \geq t\] (and similarly if \(i_0\) is even). Combining these, we see \(2t \leq a_{i_0} + (a_1 + a_2 + a_3 + \cdots + a_{2n+1}) = a_{i_0} + \left(\sum_{s=1}^{2n+1} a_s\right) = 2t - k + a_{i_0} < 2t,\) a contradiction.

The following corollary partially answers [15, Conjecture 16] in the affirmative. Note that this is a restatement of [5, Theorem 4.13].

**Corollary 5.3.** Let \(G\) be an odd cycle of size \(2n + 1\) and \(I\) be its edge ideal. Then \(I^{(t)} = I^t\) for \(1 \leq t \leq n\).

**Proof.** Suppose that \(1 \leq t \leq n\), and recall that \(I^{(t)} = I^t + (D(t))\), and any element of the generating set \(D(t)\) of \((D(t))\) must have degree less than \(2t\). However, since there are \(2n + 1 > 2t\) variables, at least two of them would need to have an exponent of 0 in order to be an element of \(D(t)\). But from Lemma 5.1, we know that none of the variables in a monomial in \(D(t)\) can have an exponent of 0.

Therefore, there are no monomials that satisfy all of the conditions for being in \(D(t)\), which means that it is empty, and thus \(I^{(t)} = I^t\) when \(t \leq n\).

A recent paper of Galetto, Geramita, Shin, and Van Tuyl [9] introduced the notion of symbolic defect, denoted \(\text{sdefect}(I, t)\), to measure the difference between the symbolic power \(I^{(t)}\) and ordinary power \(I^t\); it is the number of minimal generators of \(I^{(t)}/I^t\) as an \(R\)-module. Corollary 5.3 thus implies that \(\text{sdefect}(I, t) = 0\) for all \(t\) satisfying \(1 \leq t \leq n\).

**Corollary 5.4.** Let \(G\) be an odd cycle of size \(2n + 1\) and \(I\) be its edge ideal. Then \(\text{sdefect}(I, n + 1) = 1\). In particular, \(I^{(n + 1)} = I^{n + 1} + (x_1 x_2 \cdots x_{2n+1})\).

**Proof.** If we let \(t = n + 1\), Proposition 4.5 states that \(I^{(n + 1)} = I^{n + 1} + (D(n + 1))\). Again, recall that \((D(n + 1))\) is the ideal generated by \(D(n + 1) = \{x^k\mid \deg(x^k) < 2(n + 1)\}\) and for all minimal vertex covers \(V', w_{V'}(x^k) \geq n + 1\}.

From this, we know that the degree of any monomial in \(D(n + 1)\) must be strictly less than \(2n + 2\), and from Lemma 5.1, we also know that all variables have an exponent of at least 1.

As there are \(2n + 1\) variables, we can see that if any of the variables has an exponent of at least 2, the total degree of the monomial becomes at least \(2n + 2\), which is not valid. Thus every monomial that is not \(x_1 x_2 \cdots x_{2n+1}\) is not in \(D(n + 1)\). It is straightforward to check that \(x_1 x_2 \cdots x_{2n+1} \in (D(n + 1))\) and therefore that \((D(n + 1)) = (x_1 x_2 \cdots x_{2n+1})\). Thus, \(I^{(n + 1)} = I^{n + 1} + (x_1 x_2 \cdots x_{2n+1})\).

Recall that, if \(0 \neq I \subsetneq R = k[x_1, x_2, \ldots, x_n]\) is a homogenous ideal, the minimal degree of \(I\), denoted \(\alpha(I)\), is the least degree of a nonzero polynomial in \(I\). In particular, if \(I\) is an edge ideal, \(\alpha(I) = 2\), and \(\alpha(I^r) = 2r\) for any \(r \geq 1\). In general, if \(\alpha(I^{(t)}) < \alpha(I^s)\), we may conclude that \(I^{(t)} \subseteq I^s\), but the converse need not hold. When \(I = I(C_{2n+1})\), however, it does, as the next lemma demonstrates.

**Lemma 5.5.** Let \(I\) be the edge ideal of an odd cycle. Then \(\alpha(I^{(t)}) < \alpha(I^s)\) if and only if \(I^{(t)} \subsetneq I^s\).
Proof. The forward direction is clear.

For the converse, suppose that \( \alpha(I^{(t)}) \geq \alpha(I^s) \). From our definition of symbolic powers, we know

\[
I^{(t)} = \{ m \mid \text{for all minimal vertex covers } V', w_{V'}(m) \geq t \}.
\]

As \( I^t \subseteq I^{(t)} \), we note that \( 2t = \alpha(I^{(t)}) \geq \alpha(I^s) = 2s \). Thus, if \( m \in I^{(t)} \), \( w_{V'}(m) \geq t \geq s \) and \( \deg(m) \geq \alpha(I^{(t)}) \geq \alpha(I^s) = 2s \), and we observe

\[
I^{(t)} \subseteq \{ m \mid \text{deg}(m) \geq 2s \text{ and for all minimal vertex covers } V', w_{V'}(m) \geq s \}
\]

\[
= (L(s))
\]

\[
= I^s,
\]

which completes the proof. \( \square \)

Despite providing a condition which guarantees containments of the form \( I^{(t)} \subseteq I^s \), Lemma 5.5 does not actually compute \( \alpha(I^{(t)}) \), which is more delicate than computing \( \alpha(I^s) \). We next adapt Lemma 2.2 and the linear programming approach of [3] to compute it. In order to do so, we make the following definition.

**Definition 5.6.** Fix a list of minimal vertex covers \( V_1, V_2, \ldots, V_r \) for \( C_{2n+1} \) such that \( |V_i| \leq |V_{i+1}| \). We define the minimal vertex cover matrix \( A = (a_{ij}) \) to be the matrix of 0’s and 1’s defined by:

\[
a_{ij} = \begin{cases} 
0 & \text{if } x_j \notin V_i \\
1 & \text{if } x_j \in V_i.
\end{cases}
\]

**Remark 5.7.** Note the minimum cardinality for a minimal vertex cover of \( C_{2n+1} \) is \( n + 1 \); in fact, there are \( 2n + 1 \) minimal vertex covers of size \( n + 1 \). As we have seen, there do exist minimal vertex covers of size greater than \( n + 1 \). These covers will be accounted for in rows \( 2n + 2 \) and higher of the minimal vertex cover matrix \( A \).

We first seek a lower bound of \( \alpha(I^{(t)}) \) using linear programming. Let

\[
t = s(n + 1) + d, \text{ where } 0 \leq d \leq n.
\]

Consider the following linear program (*), where \( A \) is the minimal vertex cover matrix,

\[
b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \text{ and } c = \begin{pmatrix} t \\ \vdots \\ t \end{pmatrix}.
\]

minimize \( b^T y \)

subject to \( Ay \geq c \) and \( y \geq 0 \).

We refer to (*), the *alpha program*, and observe that if \( y^* \) is the value which realizes (*), we have \( \alpha(I^{(t)}) \geq b^T y^* \).

Consider the following partition of \( A \): let \( A' \) be the submatrix of \( A \) consisting of the first \( 2n + 1 \) rows (and thus corresponding to the \( 2n + 1 \) minimal vertex covers which contain exactly \( n + 1 \) vertices) and \( B \) the matrix consisting of the remaining rows of \( A \). We thus create the following sub-program of (*),
minimize \( b^T y \)
subject to \( A'y \geq c \) and \( y \geq 0 \).

\[ (\dagger) \]

Lemma 5.8. The value of \((\dagger)\) is \( \frac{(2n+1)t}{n+1} \).

Proof. We claim that

\[ y^* = \begin{pmatrix} \frac{t}{n+1} \\ \frac{n}{n+1} \\ \vdots \\ \frac{1}{n+1} \end{pmatrix}, \]

a \((2n+1) \times 1\) column vector, is a feasible solution to \((\dagger)\). Indeed, \( A'y^* \) is a column vector whose entries are all \( t = s(n+1) + d \), satisfying the constraint of the LP. In this case, \( b^T y^* = \frac{(2n+1)t}{n+1} \).

To show that this is the value of \((\dagger)\), we make use of the fundamental theorem of linear programming by showing the existence of an \( x^* \) which produces the same value for the dual linear program:

maximize \( c^T x \)
subject to \( (A')^T x \leq b \) and \( x \geq 0 \)

\[ (\ast) \]

Specifically, let

\[ x^* = \begin{pmatrix} \frac{1}{n+1} \\ \frac{n}{n+1} \\ \vdots \\ \frac{n}{n+1} \end{pmatrix}. \]

As the rows of \((A')^T\) again have exactly \( n+1 \) 1’s, we see \((A')^T x^* \leq b\) is satisfied, and it is straightforward to check that \( c^T x^* = b^T y^* = \frac{(2n+1)t}{n+1} \). \(\Box\)

Lemma 5.9. The value of \((\dagger)\) is bounded below by \( \frac{(2n+1)t}{n+1} \).

Proof. Observe that \((\dagger)\) is obtained from \((\dagger)\) by (possibly) introducing additional constraints. Thus, the value of \((\dagger)\) is at least the value of \((\dagger)\), which is \( \frac{(2n+1)t}{n+1} \). \(\Box\)

Proposition 5.10. Let \( t = s(n+1) + d \), where \( 0 \leq d \leq n \). Then \( \alpha(I^{(t)}) = 2t - \lfloor \frac{t}{n+1} \rfloor \).

Proof. By Lemma 5.9, we see that \( \alpha(I^{(t)}) \) is bounded below by the value of \((\dagger)\), i.e.,

\[ \alpha(I^{(t)}) \geq \frac{(2n+1)t}{n+1} = \frac{(2n+1)(s(n+1)+d)}{n+1} = (2n+1)s + 2d - \frac{d}{n+1}. \]

As \( 0 \leq \frac{d}{n+1} < 1 \), it’s enough to find an element of degree \((2n+1)s + 2d\) in \( I^{(t)} \). We claim that

\[ m = x_1^{s+d} x_2^{s+d} x_3^s \cdots x_{2n+1}^s \]

is such an element. Note that any minimal vertex cover \( V' \) (and hence minimal prime of \( I \)) will contain one of \( x_1 \) and \( x_2 \), and at least \( n-1 \) (if it contains both \( x_1 \) and \( x_2 \)) or \( n \) (if it contains only one of \( x_1 \) and \( x_2 \)) other vertices.

In the former case, \( w_{V'}(m) \geq 2(s+d) + s(n-1) = s(n+1) + 2d \geq t \), and so \( m \in I^{(t)} \). In the latter case, \( w_{V'}(m) \geq (s+d) + sn = s(n+1) + d = t \), and again we see \( m \in I^{(t)} \).
Thus, \( \alpha(I^{(t)}) \) is an integer satisfying \((2n + 1)s + 2d - \left\lfloor \frac{d}{n+1} \right\rfloor \leq \alpha(I^{(t)}) \leq (2n + 1)s + 2d\), whence \( \alpha(I^{(t)}) = (2n + 1)s + 2d - s = 2t - \left\lfloor \frac{d}{n+1} \right\rfloor = 2t - \left\lfloor \frac{t}{n+1} \right\rfloor \).

Recall that, given a nontrivial homogeneous ideal \( I \subseteq k[x_1, x_2, \ldots, x_{2n+1}] \), the resurgence of \( I \), introduced in [2] and denoted \( \rho(I) \), is the number \( \rho(I) = \sup \left\{ \frac{m}{r} \mid I^{(m)} \not\subseteq I^r \right\} \).

**Theorem 5.11.** If \( G \) is an odd cycle of size \( 2n + 1 \) and \( I \) its edge ideal, then \( \rho(I) = \frac{2n+2}{2n+1} \).

**Proof.** Let \( T = \left\{ \frac{m}{r} \mid I^{(m)} \not\subseteq I^r \right\} \), and suppose that \( I^{(m)} \not\subseteq I^r \) so that \( \frac{m}{r} \in T \). In order for \( I^{(m)} \) to not be a subset of \( I^r \), it must be true that \( \alpha(I^{(m)}) < \alpha(I^r) \) by Lemma 5.5. Since we know \( \alpha(I^r) = 2r \) and \( \alpha(I^{(m)}) = 2m - \left\lfloor \frac{m}{n+1} \right\rfloor \) by 5.10, it follows that \( 2m - \left\lfloor \frac{m}{n+1} \right\rfloor < 2r \), and that \( 2m - \frac{m}{n+1} \leq 2m - \left\lfloor \frac{m}{n+1} \right\rfloor < 2r \). Thus \( 2m - \frac{m}{n+1} < 2r \), and we conclude that \( \frac{m}{r} < \frac{2n+2}{2n+1} \).

Our next goal is to prove that \( \frac{2n+2}{2n+1} \) is the smallest upper bound of \( T \), and we will do this by finding a sequence \( a_k = \frac{m_k}{r_k} \in T \) with \( \lim_{k \to \infty} a_k = \frac{2n+2}{2n+1} \).

We first make the following claim.

**Claim:** If \( m/r \in T \), then \((m + 2n + 2)/(r + 2n + 1) \in T \).

**Proof of Claim:** By Lemma 5.5, because \( m/r \in T \) it follows that \( \alpha(I^{(m)}) < \alpha(I^r) \), and it is then enough to show that \( \alpha(I^{(m+2n+2)}) < \alpha(I^r + 2n + 1) \) to conclude that \((m + 2n + 2)/(r + 2n + 1) \in T \). By Proposition 5.10, we have

\[
\alpha(I^{(m+2n+2)}) = 2(m + 2n + 2) - \left\lfloor \frac{m + 2n + 2}{n + 1} \right\rfloor \\
= 2m + 4n + 4 - \left\lfloor \frac{m}{n + 1} + \frac{2n + 2}{n + 1} \right\rfloor \\
= 4n + 2 + 2m - \left\lfloor \frac{m}{n + 1} \right\rfloor \\
= 4n + 2 + \alpha(I^{(m)}) \\
< 2(n + 1) + \alpha(I^r) \\
= \alpha(I^{(r+2n+1)}).
\]

Recall that for any odd cycle of size \( 2n + 1 \), \( I^{(n+1)} \not\subseteq I^{n+1} \), so let \( m_0 = r_0 = n + 1 \) and \( a_0 = \frac{m_0}{r_0} \). Then recursively define \( a_k = \frac{m_k}{r_k} \) where \( m_k = m_{k-1} + 2n + 2 \) and \( r_k = r_{k-1} + 2n + 1 \). By the claim above, we have \( a_k = \frac{m_k}{r_k} \in T \). Note that this definition of the sequence \( a_k \) is equivalent to the explicit formula \( a_k = \frac{n+1+k(2n+2)}{n+1+k(2n+1)} \). Moreover, \( \lim_{k \to \infty} a_k = \frac{2n+2}{2n+1} \), which finally implies that \( \rho(I) = \frac{2n+2}{2n+1} \).

In [9], a new measure of the failure of \( I^m \) to contain \( I^{(m)} \) was introduced. This measure is known as the symbolic defect, and, for a given \( m \), is the number \( \mu(m) \) of minimal generators \( F_1, F_2, \ldots, F_{\mu(m)} \) such that \( I^{(m)} = I^m + (F_1, F_2, \ldots, F_{\mu(m)}) \). Recall that Corollaries
and imply, for \( I = I(C_{2n+1}) \), that
\[
s\text{defect}(I, t) = \begin{cases} 
0 & \text{if } t \leq n \\
1 & \text{if } t = n + 1.
\end{cases}
\]

Next, we explore additional terms in the symbolic defect sequence. Our general approach is to rely on the decomposition described in Corollary 4.5. In the parlance of our work, the symbolic defect is the size of a minimal generating set for the ideal \( D(t) \). Observe that in general this is not the same as computing the cardinality of the set \( D(t) \), as there may be monomials in \( D(t) \) which are divisible by other monomials in the set. Thus, our goal is to determine the cardinality of the subset \( D'(t) \) of \( D(t) \) which forms a minimal generating set of \( D(t) \).

**Theorem 5.12.** Let \( I = I(C_{2n+1}) \). Then, for \( t \) satisfying \( n + 2 \leq t \leq 2n + 1 \), we have
\[
s\text{defect}(I, t) = \sum_{\ell=1}^{2n+1} \binom{2n+1}{\ell} \left( t - (n+1) - \ell \right).
\]

**Proof.** As stated above, we wish to count the number of minimal generators in \( D'(t) \). Recall that \( \alpha(I^{(t)}) = 2t - \left\lfloor \frac{t}{n+1} \right\rfloor = 2t - 1 \); by definition, as everything in \( D(t) \) has degree less than \( 2t \), we see that \( D(t) \) consists only of monomials of degree \( 2t - 1 \). The collection of all distinct monomials of degree \( 2t - 1 \) is itself linearly independent, and thus \( D(t) \) is a minimal generating set for \( (D(t)) \), i.e., \( D(t) = D'(t) \).

Consider an arbitrary \( m \in D(t) \), and note that \( \deg(m) = 2t - 1 = 2(t - 1) + 1 \). Since the edge monomials \( e_i = x_i x_{i+1} \) (where again \( e_{2n+1} = x_{2n+1} x_1 \)) have degree 2, we see that \( m \) is divisible by the product of at most \( t - 1 \) edge monomials. Further, as \( \alpha(I^{(t)}) = 2t - 1 \geq 2(t - 1) = \alpha(I^{t-1}) \) by Proposition 5.10, Lemma 5.5 gives that \( m \in I^{t-1} \), and thus \( m \) is divisible by at least \( t - 1 \) edge monomials. Thus, \( m \) must be divisible by exactly \( t - 1 \) edge monomials, and an optimal factorization of \( m \) is
\[
m = x_i^{b_1} e_1^{b_2} \cdots e_{2n+1}^{b_{2n+1}}, \text{ where } \sum b_i = t - 1.
\]
That is, \( m \) has a single ancillary with exponent 1.

By Lemma 5.2, \( m \) must be divisible by \( x_1 x_2 \cdots x_{2n+1} \). Write \( x_1 x_2 \cdots x_{2n+1} \) in optimal form as
\[
x_1 x_2 \cdots x_{2n+1} = x_i e_1 e_3 \cdots e_{i_0} e_{i_0+1} \cdots e_{2n}
\]
if \( i_0 \) is odd, and as
\[
x_1 x_2 \cdots x_{2n+1} = x_i e_2 e_4 \cdots e_{i_0} e_{i_0+1} \cdots e_{2n+1}
\]
if \( i_0 \) is even. Observe that, in either case, \( x_1 x_2 \cdots x_{2n+1} \) is the product of a single variable and \( n \) edge monomials. Thus, the monomial \( p = m/x_1 x_2 \cdots x_{2n+1} \) is the product of exactly \( (t - 1) - n \) edge monomials.

We have thus factored any \( m \in D(t) \) as \( m = x_1 x_2 \cdots x_{2n+1} p \), where \( p \) is the product of exactly \( t - n - 1 \) edge monomials. Observe that, if \( m' = x_1 x_2 \cdots x_{2n+1} p' \), where \( p' \) is the product of exactly \( t - n - 1 \) edge monomials, \( \deg(m') = 2t - 1 \) and, if \( V \) is any minimal vertex cover of \( C_{2n+1} \), \( w_V(m') = w_V(x_1 x_2 \cdots x_{2n+1}) + w_V(p') \geq n + 1 + t - n - 1 = t \), where \( w_V(p') \) follows from the fact that \( p' \in I^{t-1} \) by definition; thus, \( m' \in D(t) \).
Therefore, to count the monomials in $D(t)$, it suffices to count all monomials $p$ that are products of $t - n - 1$ edge monomials.

We can visualize this problem by counting the number of ways to place these $t - n - 1$ ‘edges’ around the cycle, assuming that we can place multiple edges between the same two vertices. To that end, let $\ell$ be the number of pairs of vertices between which we place at least one edge. Then there are $\binom{2n+1}{\ell} \binom{(t-(n+1)) - \ell}{\ell}$ ways to place the $t - (n + 1)$ edges: first we choose from among the $\binom{2n+1}{\ell}$ choices for pairs of vertices between which to place the edges, and then we choose from the $\binom{(t-(n+1)) - \ell}{\ell}$ ways to arrange the edges. Thus,

$$s\text{defect}(I,t) = \sum_{\ell=1}^{2n+1} \binom{2n+1}{\ell} \binom{2n+1}{\ell-\ell} \binom{(t-(n+1)) - \ell}{\ell}.$$

In particular,

$$s\text{defect}(I(C_{2n+1}), n + 2) = 2n + 1.$$

The computation of $s\text{defect}(I,t)$ becomes much more complicated as $t \gg 2n + 1$.

6. An additional containment question

Our proof that $I^{(t)} = I^t + (D(t))$ does not hold for any graph other than a cycle, as it relies on the fact that each path between ancillaries is disjoint from every other path. This is not true for general non-cycles. This leads naturally to the following question.

**Question 6.1.** Let $G$ be a graph on the vertices $V = \{x_1, x_2, \ldots, x_d\}$ containing an odd cycle. Suppose $I = I(G)$ is the edge ideal of $G$ in $R = k[x_1, x_2, \ldots, x_d]$, and let $L(t)$ and $D(t)$ retain their usual definitions with respect to $G$. Then $I^{(t)} = I^t + (D(t))$ for all $t \geq 1$.

The following example answers Question 6.1 in the negative.

**Example 6.2.** Consider the graph $G$ defined by $V(G) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $E(G) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1, x_1x_6, x_6x_7\}$ (where we write the edges as products of vertices), and let $m = x_1^2x_2^2x_3^2x_4^2x_5^2x_7^2$. Observe that $m \notin I^6$, but as every minimal vertex cover $V$ of $G$ contains three of $x_1, x_2, x_3, x_4, x_5$, we have $w_V(m) \geq 2 \cdot 3 = 6$. Thus, $I^6 \neq (L(6))$.

However, we observe in the following two theorems that $I^t = (L(t))$ for certain classes of graphs.

One case in which Question 6.1 holds is the case in which $G$ is an odd cycle with one additional vertex connected to exactly one vertex of the cycle (see Figure 1 for an example of such a graph constructed from $C_9$).

**Theorem 6.3.** Let $G$ be a graph consisting of $2n + 2$ vertices and $2n + 2$ edges such that $2n + 1$ of them form a cycle and the remaining edge connects the remaining vertex to any existing vertex of the cycle. Further, let $I$ be the edge ideal of $G$ and let $L(t)$ and $D(t)$ retain their usual definitions with respect to $G$. Then $I^{(t)} = I^t + (D(t))$. 
Proof. Without loss of generality, consider the cycle formed by \( x_1, \ldots, x_{2n+1} \) with \( e_{2n+2} = x_1x_{2n+2} \) being the newly added edge. Recall that \( e_i = x_ix_{i+1} \) when \( i \leq 2n \) and \( e_{2n+1} = x_{2n+1}x_1 \).

Let \( m \) be a monomial expressed in optimal form \( m = \sum_{a_1} x_1^{a_1} x_2^{a_2} \cdots x_{2n+2}^{a_{2n+2}} b_1^{e_{b_1}} \cdots b_{2n+2}^{e_{b_{2n+2}}} \) and recall that \( b(m) = \sum b_i \). As with the cycle, if \( I^t = (L(t)) \), it will follow that \( I^{(i)} = I^t + (D(t)) \).

By Lemma 4.2 we know that \( I^t \subseteq (L(t)) \) so we must only show the reverse containment. Let \( m \not\in I^t \) (which implies that \( b(m) < t \)), then we will show that \( m \not\in (L(t)) \). Lemma 4.3 allows us to consider only cases where \( m \) either has multiple ancillaries or has a single ancillary of at least degree 2. Note that \( \deg(m) \geq 2t \), else \( m \not\in (L(t)) \) by definition. We will construct a minimal vertex cover \( V' \) of \( G \) such that \( w_{V'}(m) = b(m) < t \).

First, assume that \( x_{2n+2} \) is the only ancillary of \( m \), and observe that \( a_{2n+2} \geq 2 \). We may write \( m = x_{2n+2} e_1^{b_1} e_2^{b_2} \cdots e_{2n+2}^{a_{2n+2}} - 1 \). It cannot be true that \( b_i \geq 1 \) for all \( i \in \{1, 3, 5, \ldots, 2n+1\} \), because it would then be possible to divide \( m \) by some monomial \( p = x_{2n+2} e_1^{b_1} e_2^{b_2} \cdots e_{2n+2}^{a_{2n+2}} - 1 \) which must be in optimal form by Lemma 3.2; however, in this case, \( p = e_{2n+2} e_2^{b_4} \cdots e_{2n} e_{2n+2} \), contradicting that \( p \) was in optimal form. Thus, at least one \( b_{2j+1} \) is 0. Then construct \( V' \) as follows:

1. If \( b_1 = 0 \), let \( V' = \{x_1, x_2, x_4, \ldots, x_{2n}\} \). Then \( w_{V'}(m) = (b_{2n+1} + b_{2n+2} + b_1) + (b_1 + b_2) + (b_3 + b_4) + \cdots + (b_{2n-1} + b_{2n}) = b_1 + \sum_{i=1}^{2n+2} b_i = 0 + \sum_{i=1}^{2n+2} b_i = b(m) < t \).

2. If \( b_{2j+1} = 0 \) for some \( j \geq 0 \), let \( V' = \{x_1, x_3, x_5, \ldots, x_{2j+1}, x_{2j+2}, x_{2j+4}, x_{2j+6}, \ldots, x_{2n}\} \). Then \( w_{V'}(m) = (b_{2n+1} + b_{2n+2} + b_1) + (b_2 + b_3) + \cdots + (x_{2j} + x_{2j+1}) + (x_{2j+1} + x_{2j+2}) + \cdots + (b_{2n-1} + b_{2n}) = b_{2j+1} + \sum_{i=1}^{2n+2} b_i = 0 + \sum_{i=1}^{2n+2} b_i = b(m) < t \).

Now suppose that all ancillaries of \( m \) are in the cycle, i.e., the ancillaries come from the set \( \{x_1, x_2, \ldots, x_{2n+1}\} \). By adapting the argument from Theorem 4.4, we may assume that there is either one ancillary with exponent at least 2, or that there are multiple ancillaries. Use the construction in the proof of Theorem 4.4 to decompose the subgraph \( C_{2n+1} \) of \( G \) as \( H_1, H_2, \ldots, H_r \). Define \( m_{C_{2n+1}} = x_1^{a_1} \cdots x_{2n+1}^{a_{2n+1}} b_1^{e_1} \cdots b_{2n+1}^{e_{2n+1}} \), i.e., \( m_{C_{2n+1}} = m/(x_{2n+2} e_{2n+2}) \). The proof of Theorem 4.4 provides minimal subcovers \( S_1, S_2, \ldots, S_r \) such that \( S = \bigcup S_q \) and \( w_S(m_{C_{2n+1}}) = \sum_{i=1}^{2n+1} b_i \).
If \( x_1 \in S \), then \( S \) covers \( G \) and \( w_S(m) = w_S(m_{c_{2n+1}}) + b_{2n+2} = \Sigma_{i=1}^{2n+2} b_i = b(m) < t \). In this case, we may let \( V' = S \).

On the other hand, if \( x_1 \notin S \), let \( V' = S \cup \{x_{2n+2}\} \). Then \( w_{V'}(m) = w_S(m) + b_{2n+2} = \Sigma_{i=1}^{2n+2} b_i = b(m) < t \).

Next, assume that the ancillaries of \( m \) are \( x_{2n+2} \) and at least one \( x_j \) in the cycle (where \( j \neq 1 \); if \( j = 1 \), we may write \( x_{2n+2} x_1 = e_{2n+2} \), contradicting the assumption that \( m \) is in optimal form). Use the construction of Theorem 4.4 to decompose the cycle into subgraphs \( H_1, H_2, \ldots, H_r \) and note that \( x_1 \) is a vertex in \( H_r \). Observe that since \( x_1 \) is not ancillary, \( x_1 \notin H_i \) for any \( i \neq r \). Let the vertices of \( H_i \) be represented by \( \{x_{\ell_1}, x_{\ell_1+1}, \ldots, x_{\ell_r}\} \), where \( x_{\ell_1}, \ldots, x_{\ell_r} \) are ancillaries, and we wrap around with \( x_{\ell_{r+1}} \) representing \( x_{\ell_1} \). For all \( i \neq r \), the proof of Theorem 4.4 gives a construction of a minimal vertex subcover \( S_i \) with the required properties. Now construct a subgraph \( H'_r \) of \( G \) as follows: \( V(H'_r) = V(H_r) \cup \{x_{2n+2}\} \) and \( E(H'_r) = E_{H_r} \cup \{x_1, x_{2n+2}\} \). Decompose \( H_r \) as two induced subgraphs \( H'_r \) and \( H''_r \) of \( G \) on the vertices \( \{x_{\ell_1}, \ldots, x_1, x_{2n+2}\} \) and \( \{x_{2n+2}, x_1, \ldots, x_{\ell_1}\} \). We observe that we may now use the construction in the proof of Theorem 4.4 to build minimal covers of \( H'_r \) and \( H''_r \) containing \( x_1 \) (and not \( x_{2n+2} \)) whose union gives a cover \( S_r \) of \( H'_r \). Given \( m_r = x_{\ell_1}^{d_1} x_{\ell_1+1}^{d_2} x_{2n+2}^{b_{2n+2}} e_{\ell_1}^{b_{\ell_1+1}} \ldots e_{2n+2}^{b_{2n+2}} e_1^{b_1} \ldots e_{\ell_1-1}^{b_{\ell_1-1}} \) note that \( w_{S_r}(m_r) = b(m_r) \). Then the union \( V' = \cup_{i=1}^r S_i \) has the required property that \( w_{V'}(m) = b(m) \).

In all cases, \( m \notin L(t) \), and, by extension, \( m \notin (L(t)) \).

We also verify that the answer to Question 6.1 is positive when \( G \) is a complete graph. Thus, additional study is needed to identify the precise graph-theoretic property for which Question 6.1 has an affirmative answer.

**Theorem 6.4.** Let \( R = k[x_1, \ldots, x_n] \) and let \( K_n \) denote the complete graph on \( \{x_1, \ldots, x_n\} \). Further, let \( I = I(K_n) \) and \( L(t) \) and \( D(t) \) maintain their definitions as above. Then \( I^{(t)} = I^t + (D(t)) \).

**Proof.** Let \( e_{i,j} \) denote the edge between \( x_i \) and \( x_j \) such that \( i < j \). We will show that \( I^t = (L(t)) \). By Lemma 4.2, we must only show the reverse containment. Let \( m \notin I^t \) (which implies that \( b(m) < t \)), and recall that Lemma 4.3 allows us to consider only cases where \( m \) either has multiple ancillaries or has a single ancillary of at least degree 2. Let \( m = x_1^{a_1} \ldots x_n^{a_n} e_{1,1}^{b_{1,1}} \ldots e_{n,n}^{b_{n,n}} \) be a monomial in optimal form. Then \( m \) has at most 1 ancillary because if \( x_i^{a_i} \) and \( x_j^{a_j} \) were both ancillaries, then \( m \) could be expressed in a more optimal form as

\[
m = x_1^{a_1} \ldots x_{i-1}^{a_{i-1}} x_i^{a_i-1} \ldots x_j^{a_j-1} \ldots x_n^{a_n} e_{1,1}^{b_{1,1}} e_{1,2}^{b_{1,2}} e_{1,3}^{b_{1,3}} \ldots e_{i,j}^{b_{i,j}+1} \ldots e_{n,n}^{b_{n,n}}
\]

for some \( s \) because there is guaranteed to be an edge between \( x_i \) and \( x_j \) as \( K_n \) is complete. Thus \( m \) has exactly 1 ancillary and it must have a degree of at least 2.

Without loss of generality, let \( x_1^{a_1} \) be the ancillary of \( K_n \). Note that \( b_{i,j} = 0 \) if \( i, j \neq 1 \). If this was not the case, \( m \) could be expressed in a more optimal form as

\[
m = x_1^{a_1-2} \ldots x_n^{a_n} e_{1,1}^{b_{1,1}} e_{1,2}^{b_{1,2}} e_{1,3}^{b_{1,3}} \ldots e_{1,i}^{b_{1,i}+1} \ldots e_{i,j}^{b_{i,j}+1} \ldots e_{n,n}^{b_{n,n}}
\]
Let $V' = \{x_2 \cdots x_n\}$. Observe that $V'$ covers $K_n$ and
\[
w_{V'}(m) = w_{V'}(x_2^{b_{1,2}+b_{2,3}+b_{2,4}+\cdots +b_{2,n}} + x_3^{b_{1,3}+b_{2,3}+b_{3,4}+\cdots +b_3,n} + \cdots + x_n^{b_{1,n}+b_{2,n}+b_{3,n}+\cdots +b_{n-1,n}})
= w_{V'}(x_2^{b_{1,2}+0+\cdots+0} + x_3^{b_{1,3}+0+\cdots+0} + \cdots + x_n^{b_{1,n}+0+\cdots+0})
= \sum_{j=2}^{n} b_{1,j}
= \sum_{j=2}^{n} b_{1,j} + \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} b_{1,j}
= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{1,j}
= b(m)
\]
Thus $w_{V'}(m) = b(m) < t$, so $m \not\in L(t)$ and by the same argument, no divisor of $m$ is in $L(t)$, which means $m \not\in (L(t))$.

Therefore, $I^t = (L(t))$. Because $I^{(t)} = (L(t)) + (D(t))$ by Corollary 4.5, this leads to the desired result that $I^{(t)} = I^t + (D(t))$. □

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