Vector calculus in two-dimensional space

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Vector calculus in three-dimensional space is ubiquitous in applications of mathematics in physics and engineering. Its two-dimensional version is, however, quite rare. Here we try to provide a pedagogical account of the subject. It is based on the logic of theory of differential forms. For readers not familiar with the latter, the results are presented in detail in the standard language and notations.

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1 Introduction

We live in three-dimensional space. Therefore also mathematics connected with the three-dimensional space turns out to be especially important as well as developed.

In particular, it turns out that vector calculus ideally suits for neat and clear mathematical description of a number of important phenomena around us (diffusion, heat flow, electricity, magnetism, gravitational field, ...; see e.g. [4], [7], [9], [10], etc.). Useful concepts are introduced in it (like flux of a vector field for a surface) and useful and non-trivial results (as, for example, Gauss’ theorem) are derived for them.

Two dimensional version of the vector calculus is, however, considerably less known. It should play the same role for 2D phenomena as the standard vector calculus does in 3D. Nevertheless, what and how exactly is to be changed in mathematical relations while switching from 3D to 2D may not always be quite evident.

In this text we therefore concentrate on this very question: How actually the analogy of the three dimensional vector calculus, applicable in two dimensions, looks like?

Several approaches might exist, see e.g. [5] or [8]. Here we try to be led by logic of the theory of differential forms. For reader unfamiliar with the theory it might look as an overkill. It is, however, well-known (see e.g. [1], [2], [11] or §8.5 in [3]), that this very point of view turns out to be extremely instructive and efficient in three-dimensional case: For example, all relevant differential operators do appear automatically as well as relations between them (differently looking curl, grad and div are just masked versions of universal exterior derivative of forms). This is also the case for all integral theorems in which the operators occur (as particular cases of each time the same Stokes theorem for forms).

In this text we can see that differential forms approach is definitely not a bad choice also here.

If forms in general (and so also vector calculus in the language of forms in particular) is virtually unknown to the reader, derivations might be technically unclear, but:

1. Results will be clear, since they will be also presented in elementary language, so that just to understand how it all falls out, forms are not needed.

2. In addition, rough idea perhaps still might be caught, and consequently the forms could captivate the reader, resulting in decision to learn more about them in the future.

2 How it works in three dimensions

In vector calculus one often hears phrases like “integrate vector field over a surface” (resulting in its flux for the surface). In integration theory, however, we learn (see e.g. [1], [2], [11] or Chapter 7 in [3]), that

- differential forms alone are integrated
- $p$-form is integrated over a $p$-dimensional domain.

So if something is integrated
- along a curve, it is necessarily a 1-form,
- over a surface, it is necessarily a 2-form,
- over a volume, it is necessarily a 3-form, respectively.

How should one understand the fact, then, that under surface integrals, where 2-form is to be standing, one sees expression of the structure $\mathbf{A} \cdot d\mathbf{S}$, i.e. not to see vector field $\mathbf{A}$ there is like even deny what everything someone does? Well, the answer is that what we actually see there is the whole $\mathbf{A} \cdot d\mathbf{S}$ and this in fact is a 2-form. It is, however, uniquely parametrized in terms of the vector field $\mathbf{A}$.

In total, parametrizations of all types of forms in $E^3$ (three dimensional Euclidean space) look as follows (see e.g. section 8.5 in [3] mentioned above):

There are 0-forms, 1-forms, 2-forms and 3-forms (this is just because of three dimensions of the space) and we have, at each of these degrees, operation of exterior derivative $d$, raising the degree in one unit and such that its square vanishes (this is so in any dimension). One can display the situation in terms of the diagram

$$
\begin{align*}
\Omega^0 \xrightarrow{d} \Omega^1 &\xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \\
\text{dd} = 0
\end{align*}
(1)
$$

(it is known as the de Rham complex; the space of $p$-forms in $E^3$ is denoted as $\Omega^p$).

Since there is also (usual Euclidean) metric tensor on $E^3$, we have in addition the Hodge operator on $p$-forms. It is, in general, a canonical linear isomorphism (denoted by star) of the linear space of $p$-forms and $(n-p)$-forms, which squares to plus or minus identity. (It is just plus on each degree on $E^3$. Here $n$ is dimension of the space under consideration.)

So in three-dimensional space it provides the following two canonical isomorphisms:

$$
\begin{align*}
\Omega^0 &\leftrightarrow \Omega^3 \\
\Omega^1 &\leftrightarrow \Omega^2 \\
\ast^{-1} &\ast = \ast
\end{align*}
(2)
$$

They enable us to “identify” 2-forms with 1-forms as well as 3-forms with 0-forms.

(If we need somewhere a 2-form $\beta$, we can parametrize it in terms of a 1-form $\alpha$, i.e. write it as $\beta = \ast \alpha$ where $\alpha$ is unique 1-form: $\alpha = \ast \beta$.) So “actually” we need just 0-forms (i.e. functions; we denote as $\mathcal{F}$ the linear space of functions) and 1-forms (the rest may be parametrized in terms of the two).

The metric tensor provides another two important canonical isomorphisms, namely the (mutually inverse) operations of the raising and lowering of indices (they are denoted as $\sharp$ and $\flat$, inspiration originating in musical notation). They turn 1-forms into vector fields ($\sharp$) and vice versa ($\flat$). If the linear space of vector fields on $E^3$ is denoted as $\mathfrak{X}$, we have

$$
\begin{align*}
\Omega^1 &\overset{\sharp}{\leftrightarrow} \mathfrak{X} \\
\sharp^{-1} = \flat \\
\flat^{-1} = \sharp
\end{align*}
(3)
$$
This is the reason we can also forget about 1-forms and manage with just functions and vector fields! So differential forms in $E^3$ become as if completely out of the game! We can concisely express the situation in terms of the commutative diagram

\[ \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \]

\[ \mathcal{F} \xrightarrow{a_0} \mathcal{X} \xrightarrow{a_1} \mathcal{X} \xrightarrow{a_2} \mathcal{F} \]

Vertical arrows denote the corresponding canonical isomorphisms of various degrees of forms (the upper line) onto scalar fields $\mathcal{F}$ and vector fields $\mathcal{X}$. When the arrows are reversed, we get equivalent diagram

\[ \Omega^0 \xleftarrow{d} \Omega^1 \xleftarrow{d} \Omega^2 \xleftarrow{d} \Omega^3 \]

\[ \mathcal{F} \xleftarrow{a_0} \mathcal{X} \xleftarrow{a_1} \mathcal{X} \xleftarrow{a_2} \mathcal{F} \]

When the (upward directed) arrows are applied onto scalar and vector fields (on some $f$ and $A$), we get exactly those standard expressions we used to see under integral sign (see section 8.5 in [3]):

\[ f \in \Omega^0 \quad A \cdot dr \in \Omega^1 \quad A \cdot dS \in \Omega^2 \quad f dV \in \Omega^3 \]

(6)

The arrows $a_0, a_1, a_2$ in the bottom line of both diagrams are effective operations (composition of corresponding three arrows - upward, to the right and downward) on objects sitting in the bottom line, which in a sense “substitute” operation of the exterior derivative (arrows $d$), which “really” acts on forms in the upper line. From the diagrams we can read off that

\[ a_0 = \sharp d \quad a_1 = \sharp * d \phi \quad a_2 = * d * \phi \]

(7)

All of them are clearly first order differential operators, since they contain one $d$. Computation in Cartesian coordinates in $E^3$ reveals that they are exactly notorious operators grad, curl and div so that the diagram (4) actually reads

\[ \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \]

\[ \mathcal{F} \xrightarrow{\text{grad}} \mathcal{X} \xrightarrow{\text{curl}} \mathcal{X} \xrightarrow{\text{div}} \mathcal{F} \]

(8)

\[ ^{1}\text{In this respect functions and vector fields resemble concept of essential amino-acids in nutrition science: Although there are roughly 20 amino-acids in total, it is actually enough to take care of the essential ones (roughly half of them), our body is then able to produce the remaining ones from them. In three dimensional Euclidean space our body is able to produce any differential form from functions and vector fields.} \]
From the fact that $dd = 0$ (see (1)) and commutativity of the diagram we immediately get well-known identities

$$\text{curl grad} = 0 \quad \text{div curl} = 0$$

(9)

(upstairs, composition of neighboring arrows results in zero operator, so the same must be true downstairs).

From the three neighboring squares of the diagram (8) and the parametrization (6) one can assemble the following three useful differential relations:

$$df = \text{grad } f \cdot d\mathbf{r} \quad d(\mathbf{A} \cdot d\mathbf{r}) = (\text{curl } \mathbf{A}) \cdot d\mathbf{S} \quad d(\mathbf{A} \cdot d\mathbf{S}) = (\text{div } \mathbf{A})dV$$

(10)

In order to obtain corresponding integral statements, we can make reference to general Stokes theorem (see [1], [2], [11] or Section 7.5 in [3]) from integration theory of differential forms: If $\alpha$ is a $p$-form, $D$ is a $(p + 1)$-dimensional domain and $\partial D$ its $p$-dimensional boundary, then

$$\int_D d\alpha = \int_{\partial D} \alpha$$

(11)

For the three particular expressions $d\alpha$ from (10) we get the three essential integral theorems of the vector calculus:

**gradient theorem**

$$\int_C \text{grad } f \cdot d\mathbf{r} := f(B) - f(A)$$

(12)

**Stokes theorem**

$$\int_S (\text{curl } \mathbf{A}) \cdot d\mathbf{S} := \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}$$

(13)

**Gauss’s theorem**

$$\int_V (\text{div } \mathbf{A})dV := \oint_{\partial V} \mathbf{A} \cdot d\mathbf{S}$$

(14)

### 3 How it works in two dimensions

In two-dimensional case, the de Rham complex, i.e. the analogue of (1), just simplifies to

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \quad dd = 0$$

(15)

Key difference is in action of the Hodge star:

$$\Omega^0 \leftrightarrow \Omega^2 \quad \Omega^1 \leftrightarrow \Omega^1$$

(16)

The left expression is a natural analogue of the left expression in (2), it identifies the edge degrees of forms; explicitly

$$* f = f dS \quad *(f dS) = f \quad dS = dx \wedge dy = \text{area 2-form}$$

(17)

This enables us to forget about 2-forms and express them in terms of 0-forms, i.e. in terms of functions (scalar fields).
The right expression is, however, completely different: While in (\ref{eq:iso}) it is an isomorphism of two different spaces, in (\ref{eq:iso3}) it is an isomorphism of a single space (the space of 1-forms) on itself (differing from the identity). Explicitly, in usual Cartesian coordinates \((x, y)\) in the plane,

\[ * dx = dy \quad * dy = -dx \quad (18) \]

This does not mean that we cannot forget about 1-forms. Rather, it means that in 2D there are as many as two canonical ways how 1-forms may be replaced by vector fields! One of them is the same like it was in 3D, i.e. via the operation \(\sharp\) (raising of index). It reads

\[ \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \]

\[ \id \downarrow \quad \sharp \downarrow \quad * \downarrow \quad \id \downarrow \quad \sharp \downarrow \quad * \downarrow \quad (19) \]

When the arrows (directed upwards) are applied on scalar and vector fields (on \(f\) and \(A\)) we again get standard expressions we used to see under integral sign:

\[ f \in \Omega^0 \quad A \cdot dx \equiv A_x dx + A_y dy \in \Omega^1 \quad f dS \in \Omega^2 \quad (20) \]

Another one is, however, available, now - we first apply the star operator (after which we still remain within 1-forms) and only then we raise the index. So we have two different analogues of the diagram (\ref{eq:iso1}):

\[ \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \]

\[ \id \downarrow \quad \sharp \downarrow \quad * \downarrow \quad \id \downarrow \quad \sharp \downarrow \quad * \downarrow \quad (21) \]

We see that also in 2D-vector calculus we make do with vector and scalar fields (bottom lines). From the diagrams one can express explicitly the bottom (effective) arrows and get

\[ a_0 = \sharp d \quad a_1 = * db \quad b_0 = \sharp * d \quad b_1 = * d *^{-1} b \quad (22) \]

When the (differential) operators are computed in (Cartesian) coordinates \((x, y)\), we learn that two of them are natural analogues of the situation in 3D (gradient and divergence), the remaining two are “new”, specific for 2D:

\[ \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \quad \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \]

\[ \id \downarrow \quad \sharp \downarrow \quad * \downarrow \quad \id \downarrow \quad \sharp \downarrow \quad * \downarrow \quad (23) \]
(Notation $b_0 = \text{Ham}$ is justified in section 3.1 and for $a_1 = \text{curl}_3$ in section 3.2.)

So, abstractly we have

$$\begin{align*}
\text{grad} &= \sharp d \\
\text{curl}_3 &= *db \\
\text{Ham} &= \sharp * d \\
\text{div} &= -*d *^{-1} b
\end{align*} \quad (24)$$

and their actions, in Cartesian coordinates $(x, y)$, read:

$$\begin{align*}
\text{grad} : & \quad f \mapsto (\partial_x f, \partial_y f) \\
\text{curl}_3 : & \quad (A_x, A_y) \mapsto (\partial_x A_y - \partial_y A_x) \\
\text{Ham} : & \quad f \mapsto (-\partial_y f, \partial_x f) \\
\text{div} : & \quad (A_x, A_y) \mapsto (\partial_x A_x + \partial_y A_y)
\end{align*} \quad (25-28)$$

One can easily check vanishing of composition of the two arrows in bottom lines (in both diagrams):

$$\text{curl}_3 \circ \text{grad} = 0 \quad \text{div} \circ \text{Ham} = 0 \quad (29)$$

Indeed,

$$\begin{align*}
\text{curl}_3 \circ \text{grad} : & \quad f \mapsto (\partial_x f, \partial_y f) \mapsto (\partial_y \partial_x f - \partial_y \partial_x f) = 0 \\
\text{div} \circ \text{Ham} : & \quad f \mapsto (-\partial_y f, \partial_x f) \mapsto (-\partial_x \partial_y f, \partial_y \partial_x f) = 0
\end{align*} \quad (30-31)$$

This is the counterpart of (9). Notice that there are also two identities, here. However, not because we put to use $dd = 0$ on different degrees in a single diagram, but rather on a single degree in two different diagrams.

It is also worth noticing that when two arrows from different diagrams are composed, we get (for both possible cases) another well-known operator, the Laplace operator $\triangle$:

$$\begin{align*}
\text{div} \circ \text{grad} &= \triangle \\
\text{curl}_3 \circ \text{Ham} &= \triangle
\end{align*} \quad (32)$$

Indeed,

$$\begin{align*}
\text{div} \circ \text{grad} : & \quad f \mapsto (\partial_x f, \partial_y f) \mapsto (\partial_x^2 + \partial_y^2)f \equiv \triangle f \\
\text{curl}_3 \circ \text{Ham} : & \quad f \mapsto (-\partial_y f, \partial_x f) \mapsto (\partial_x^2 + \partial_y^2)f \equiv \triangle f
\end{align*} \quad (33-34)$$

### 3.1 Hamiltonian fields

Notation $\text{Ham} f$ in (23) and (27) is abbreviation for Hamiltonian field generated by function $f$. It is key concept in theory of Hamiltonian systems (for more details see, e.g., [2], [11] or Chapter 14 in [3]). How could it occur here?

Well, it turns out that our surface form $\omega \equiv dS$ mentioned in (17) happens to be, at the same time, symplectic form (it satisfies corresponding definition; in general it is to be “closed and non-degenerate 2-form”). And whenever symplectic form is available, a rule for construction of Hamiltonian (vector) fields is available as well:

$$f \mapsto \text{Ham} f \quad i_{\text{Ham} f} \omega := -df$$ \quad (35)
Now, when we compute how (35) works here, what we get is exactly (27).

This also sheds a new light onto the second result in (29). The fact, that divergence of Hamiltonian field (generated by any function) vanishes is equivalent to well-known Liouville theorem from classical mechanics (see 14.3.7 in [3]). It says that, for Hamiltonian systems, volume (here area) in phase space (here our 2D space) is conserved w.r.t. time evolution.

### 3.2 Auxiliary third dimension

One can easily check that:

1. If, from a function $f(x, y)$ in 2D plane, one constructs the following auxiliary vector field in 3D space

   $$
   u = (0, 0, -f(x, y))
   $$

   (try to think how it looks like), then its curl reads

   $$
   \text{curl} \, u = (-\partial_y f, \partial_x f, 0) \equiv (\text{Ham} \, f, 0)
   $$

2. If, from a vector field $A$ in 2D plane, one constructs the following auxiliary vector field in 3D space

   $$
   \mathcal{A} = (A, 0) \equiv (A_x(x, y), A_y(x, y), 0)
   $$

   (try to think how it looks like), then its curl reads

   $$
   \text{curl} \, \mathcal{A} = (0, 0, \partial_x A_y - \partial_y A_x) \equiv (0, 0, \text{curl} \, 3 \mathcal{A})
   $$

What these two simple computations do reveal?

First, that our two “new” operations in 2D vector calculus, $a_1 \equiv \text{curl} \, 3$ and $b_0 \equiv \text{Ham}$, may also be discovered as hidden at special places of outputs of specially chosen inputs of standard 3D vector calculus. The approach treated in this paper finds their form and properties in “intrinsic” way, with no reference to auxiliary additional dimensions.

And, second, we see motivation for notation of the operation $\text{curl} \, 3$. Well, mainly for the index 3 (in particular, that $\text{curl} \, 3 \mathcal{A} = (\text{curl} \, \mathcal{A})_3$). Motivation for the notation $\text{curl}$ itself (which should be related to kind of rotation of something) is discussed already in standard 3D vector calculus (it stems for example from vortex-like motion of fluid flow; see a few more words on this in Section 4.2).

### 3.3 Integral identities and Green’s theorem

Integral theorems for 2D-vector calculus may be again derived from universal Stokes theorem [11] for forms. Corresponding technical procedure just mimics the one used in the 3D case at the end of Section 2.

Since there are altogether as many as four basic squares (with upper arrow $d$) in two diagrams in [23], four integral theorems are expected.

8
First, we assemble the corresponding four differential identities (analogues of expressions (10)). From the individual squares we get

\[
\begin{align*}
    df &= \text{grad} f \cdot dr \\
    d(A \cdot dr) &= (\text{curl}_3 A) dS \\
    df &= - \ast (\text{Ham} f \cdot dr) \\
    d \ast (A \cdot dr) &= (\text{div} A) dS
\end{align*}
\]  

(40)

Their integration and application of Stokes theorem (11) leads to

\[
\begin{align*}
    \int_C \text{grad} f \cdot dr &= f(B) - f(A) \\
    \int_C \ast (\text{Ham} f \cdot dr) &= f(A) - f(B) \\
    \int_S (\text{curl}_3 A) dS &= \oint_{\partial S} A \cdot dr \\
    \int_S (\text{div} A) dS &= \oint_{\partial S} \ast (A \cdot dr)
\end{align*}
\]  

(42)

Now, when we explicitly write down expressions under integral signs (with the help of (18) and (25) - (28)), we discover that both identities in (42) say the same thing and, similarly, the same is true for both identities in (43). So we get just two mutually different integral identities. When appropriate notations are used, they read as follows:

**gradient theorem**

\[
\int_C (\partial_x f) dx + (\partial_y f) dy := f(B) - f(A)
\]  

(44)

**Green’s theorem**

\[
\int_S (\partial_x g - \partial_y f) dxdy := \oint_{\partial S} f dx + gdy
\]  

(45)

The first one is a 2D version of the gradient theorem. This is well known and holds in any dimension. It relates an integral along a curve \(c\) to an “integral” over its boundary \(\partial c\) (which reduces, here, to just two points, \(A\), its beginning and \(B\), its end).

The second one is Green’s theorem. This is well-known, again, and is specific for two dimensional space. It relates an integral over a 2D-domain \(S\) to an integral over its boundary (being a closed circuit \(\partial S\)).

### 3.4 Perpendicular vector field

As it was already mentioned in Section 3, in two dimensions we encounter a specific situation that the Hodge star maps 1-forms on 1-forms again (see (16)). The latter are, however, in bijection with vector fields (via \(\sharp\) and \(\flat\)). So, effectively, the Hodge star induces a canonical isomorphism of vector fields on themselves

\[
\sharp \ast \flat : \mathcal{X} \rightarrow \mathcal{X}
\]  

(46)

A computation gives

\[
\sharp \ast \flat : (A_x, A_y) \mapsto (-A_y, A_x) \quad \text{i.e.} \quad A \mapsto A_{\pi/2}
\]  

(47)

We denote the result as \(A_{\pi/2}\), since it is evident from components that the new vector is given, at each point, by rotation of the vector of the original vector field
by $\pi/2$ in positive sense, i.e. in counter-clock-wise direction. (This is compatible with the fact that, on 1-forms, the square of the star is the minus identity, see (18)).

The mapping (47) is equivalent to a handy formula

$$\star \left( A \cdot dr \right) =: A_{\pi/2} \cdot dr$$

(48)

Then, when combined with (40), we get the following useful relation of the gradient field and the Hamiltonian one:

$$\text{Ham} f = (\text{grad} f)_{\pi/2}$$

(49)

This is also evident from their expressions in (25) and (27). If $f(x, y)$ describes height of a hill somewhere in a landscape, we know that the vector field grad $f$ shows, at any point, the direction of the steepest ascent (and $-\text{grad} f$ the direction of the steepest descent). Then the vector field Ham $f$ shows, at any point, the direction of no ascent, i.e. the direction along contour lines.

### 3.5 Poincaré lemma

Recall, first, what this useful lemma states.

The fact $dd = 0$ says that if a form $\alpha$ is exact (i.e. if $\alpha = d\beta$), then it is necessarily closed ($d\alpha = 0$). It turns out that the opposite implication does not, in general, hold. Poincaré lemma, however, guarantees that on a domain which is contractible to a point (for example on a coordinate patch) the opposite implication still does hold.

In 3D vector calculus this leads frequently used and useful statements (conversions of implications from (9))

$$\begin{align*}
curl a &= 0 \implies a = \text{grad} f \\
div a &= 0 \implies a = \text{curl} b
\end{align*}$$

(50)

When expressed in terms of the diagram (8), it says that if some arrow in the bottom line gives zero, the input is necessarily an output of the previous arrow.

And what exactly the deduction gives in 2D vector calculus, i.e. for diagrams (23)? This:

$$\begin{align*}
curl 3a &= 0 \implies a = \text{grad} f \\
div a &= 0 \implies a = \text{Ham} f
\end{align*}$$

(51)

We can see an application of these facts in Section 4.

### 4 2D vector calculus in 2D hydrodynamics

2D vector calculus may be useful, as an example, in description of 2D flows in hydrodynamics. Such flows are described in terms of velocity field $\mathbf{v} = (v_x, v_y)$. It results from continuity equation (which encodes conservation of mass under the flow) that if the mass density $\rho$ is constant, the velocity field is divergence-free:

$$\text{div} \mathbf{v} = 0 \quad \text{incompressible fluid}$$

(52)
4.1 Incompressible fluid and stream function

Compare the second implications in (50) and (51). We see that if a vector field in 2D happens to be divergence-free, it is not the curl of an arbitrary vector field, as we are accustomed to in 3D, but rather it is *Hamiltonian* field generated by an arbitrary function \( f(x, y) \).

\[
\text{div} \mathbf{v} = 0 \Rightarrow \mathbf{v} = \text{Ham} f \quad \text{i.e.} \quad (v_x, v_y) = (-\partial_y f, \partial_x f) \tag{53}
\]

One can say that the velocity field is *potential* in the sense that components of the field may be computed in terms of derivatives of some quantity, namely of a function (potential) \( f \), here. The field is, however, not computed as the usual gradient of the potential (as it is the case, for example, for electric field in electrostatics), but rather other combinations of partial derivatives are needed. In particular what we found is that one is to construct Hamiltonian field from \( f \).

Now, what is the physical meaning of the function \( f \)?

First, one easily checks that directional derivative of the function \( f \) along streamlines of the flow vanishes:

\[
\dot{f} \equiv \frac{d}{dt} f(r(t)) = \dot{x}(\partial_x f) + \dot{y}(\partial_y f) = \dot{x} v_y - \dot{y} v_x = 0 \tag{54}
\]

since \( \mathbf{v} = (\dot{x}, \dot{y}) \) on streamline \( r(t) \equiv (x(t), y(t)) \).

So the function \( f \) is *constant* on streamlines. If we were able to find \( f \), we could determine the *shape of streamlines*, i.e. the pattern of our 2D flow, from the equation

\[
f(x, y) = \text{const.} \tag{55}
\]

That’s why \( f \) is known as the *stream function*.

Second, we should realize the *value* itself of the function \( f \) at a point \( A \) cannot carry any direct physical meaning since it is evident that there is a freedom in additive constant for \( f \). What can carry a real physical meaning, however, is the *difference* of values at two points \( A \) and \( B \), i.e. \( f(B) - f(A) \). In electrostatics, where the (electric) field is (minus) the gradient of the potential, we get *voltage* between the two points in this way. What we get here, where the velocity field turns out to be Hamiltonian field corresponding to this “potential”?

It turns out that it gives net *area flux* of the fluid per unit time through a path connecting the two points. So if we draw two streamlines which pass through the two points, it represents the net area flux under a bridge over the “river”, which runs between the two streamlines.

It may be also seen on a picture (see Fig.1) and then computed “on fingers”. Indeed, the second identity in (40) gives

\[
2\text{The direction of the flow as well as the magnitude of the velocity of the flow cannot be detected in this way. The picture shows, at the same time, contour lines of the hill with height given by } f(x, y), \text{ mentioned at the end of Section 3.4. If the stream function were used in the role of the height of a fictitious hill, the fluid would flow, on the corresponding tourist map, along contour lines.}
\]
Figure 1: Streamline a passes through point A, streamline b passes through point B. Streamlines in between them form a “river”. A bridge over the troubled water connects the point A with the point B. The area of the rhomboid is the area flux (per unit time) of the water underneath the small piece \((dx, dy)\) of the bridge.

\[
df = - \star (\mathbf{v} \cdot d\mathbf{r}) = - \star (v_x dx + v_y dy) = v_y dx - v_x dy
\]

where we used (18). So if we step forth (along the path connecting A and B) a small vector \((dx, dy)\), the value of the function \(f\) changes by \(df = v_y dx - v_x dy\). Now the expression on the r.h.s. is nothing but the area of the rhomboid spanned by edges \((dx, dy)\) and \((v_x, v_y)\). And this is exactly the area flux of a fluid with velocity \((v_x, v_y)\) through a small piece \((dx, dy)\) of the connecting line, i.e. the area, which flows through the piece per unit time. (In 3D situation it would be volume flux, i.e. the volume of the fluid which flows through the small surface \(dS\) and the corresponding expression were \(\mathbf{v} \cdot d\mathbf{S}\).) And the net change of the value of \(f\) going from A to B is then corresponding integral along the connecting path which gives the net area flux for the entire path.

### 4.2 Vorticity-free flow and its potential

Now let us turn our attention to the first implication in (51). We see that if a vector field \(\mathbf{a}\) in 2D has vanishing curl, it is necessarily a gradient of some function \(g(x,y)\). So for the velocity field \(\mathbf{v}\) it holds

\[
curl \mathbf{v} = 0 \Rightarrow \mathbf{v} = \text{grad } g \quad \text{i.e.} \quad (v_x, v_y) = (\partial_x g, \partial_y g)
\]
But what is the physical meaning of the fact \( \text{curl}_3 \mathbf{v} = 0 \)?

In 3D case, vector field \( \text{curl} \mathbf{v} \) is known as the vorticity field. It turns out that, in a given point, it represents twice the vector of the angular velocity, by which the “droplet” centered in the point rotates.

The scalar function \( \text{curl}_3 \mathbf{v} \) is the 2D version of the vorticity of the flow. (It is called vorticity as well.) If we placed a small body made of core somewhere on the surface of the 2D flow, it would rotate with angular velocity \( 2 \text{curl}_3 \mathbf{v} \).

So the condition \( \text{curl}_3 \mathbf{v} = 0 \) describes a vorticity-free flow. As we see from (57), such flow is also “potential” one, moreover in the standard sense, now (i.e. that the vector field is computed as the gradient of the potential).

### 4.3 Where from complex analysis arises

Consider a 2D flow which happens to be “incompressible” and at the same time vorticity-free. Then, according to (53) and (57) it holds

\[
\mathbf{v} = \text{Ham} f = \text{grad} g \quad \text{i.e.} \quad (v_x, v_y) = (-\partial_y f, \partial_x f) = (\partial_x g, \partial_y g) \quad (58)
\]

The last equality sign in (58) expresses, however, exactly the Cauchy-Riemann equations; they say that (complex) function of complex variable

\[
h(z) = f + ig, \quad z = x + iy
\]

is analytic (holomorphic). We can easily check that velocity field is hidden in its derivative

\[
h'(z) = v_y + iv_x \quad (60)
\]

Indeed,

\[
2\partial_z h(z) = (\partial_x - i\partial_y)(f + ig) = 2(v_y + iv_x) \quad (61)
\]

One can learn much more on this in standard textbooks on hydrodynamics, e.g. see [6].

### 5 Conclusion

In this paper the 2D analogue of the standard 3D vector calculus is discussed. The approach is based on using of differential forms. The standard logic, which helps so much for understanding 3D vector calculus, is repeated. The approach is completely “intrinsic”, it does not use any embedding of the 2D space into an auxiliary ambient 3D space. We show what differential operators occur here (analogues of grad, curl and div from 3D), how they are related and in which integral identities they may be found. All results are also presented in standard language (i.e. with no forms whatsoever).
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