COMBINATORIAL $\alpha$-CURVATURES AND $\alpha$-FLOWS ON POLYHEDRAL SURFACES, I

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ABSTRACT. We introduce combinatorial $\alpha$-curvature for piecewise linear metrics on polyhedral surfaces, which is a generalization of the classical combinatorial curvature on polyhedral surfaces. Then we prove the global rigidity of $\alpha$-curvature with respect to the discrete conformal factors. To study the corresponding Yamabe problem of $\alpha$-curvature, we introduce the combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow for piecewise linear metrics on surfaces. To handle the possible singularities along the flows, we do surgery on the flows by flipping. Then we prove that if $\alpha \chi(S) \leq 0$, there exists a piecewise linear metric with constant combinatorial $\alpha$-curvature on a polyhedral surface $S$, which is a parameterized generalization of the discrete uniformization theorem in [34]. We further prove that the combinatorial $\alpha$-Yamabe flow and $\alpha$-Calabi flow with surgery exists for all time and converges to a piecewise linear metric with constant combinatorial $\alpha$-curvature for any initial piecewise linear metric on a surface $S$ if $\alpha \chi(S) \leq 0$.

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1. INTRODUCTION

In this paper, we study the combinatorial $\alpha$-curvature of piecewise linear metrics on surfaces. Combinatorial $\alpha$-curvature was introduced by Ge and the author [24, 26] for Thurston’s circle packing metrics as a generalization of the classical combinatorial curvature $K$. The classical combinatorial curvature $K$, defined as angle deficit at conic points on polyhedral surfaces, is invariant under scaling of the circle packing metric. Furthermore, as the triangulation of the surface is finer and finer, the classical combinatorial curvature $K$ does not approximate the smooth Gaussian curvature on surfaces. These motivate us to define the combinatorial $\alpha$-curvature, which is

$$R_{\alpha,i} = \frac{K_i}{\gamma_i^\alpha}$$

at a vertex $v_i$ for Thurston’s Euclidean circle packing metrics [26]. In the case of $\alpha = 2$, the combinatorial curvature $R_2$ is an approximation of the smooth Gaussian curvature as the triangulation is finer and finer. Furthermore, if we take $g_i = \gamma_i^2$ as an analogue of the Riemannian metric, then the
combinatorial curvature $R_2$ transforms according to $R_{2,\lambda g} = \lambda^{-1} R_{2,g}$, parallel to the smooth case. The properties of combinatorial curvature $R_2$ along the combinatorial Ricci flow and combinatorial Calabi flow are also proved to be parallel to that of Gaussian curvature along the smooth surface Ricci flow and surface Calabi flow [26]. Combinatorial $\alpha$-curvatures can also be defined for sphere packing metrics on 3-dimensional manifolds, see [20, 23, 26]. There are lots of works on combinatorial curvatures and combinatorial curvature flows of circle packing metrics on surfaces and sphere packing metrics on 3-dimensional manifolds, see [7, 9–14, 16–26, 28–32, 35, 38, 40, 45–47, 49]. In this paper, we focus on the combinatorial $\alpha$-curvature of piecewise linear metrics on surfaces. The combinatorial $\alpha$-curvature of piecewise hyperbolic metrics on surfaces is studied in [48].

To study the conformal geometry of piecewise linear metrics on manifolds, Luo [37] and Röcek-Williams [44] independently introduced a discrete conformality for piecewise linear metrics (Euclidean polyhedral metrics), which is now called vertex scaling. Luo [37] further defined the combinatorial Yamabe flow for piecewise linear metrics on triangulated surfaces and obtained the combinatorial obstruction for the existence of constant combinatorial curvature piecewise linear metrics. Bobenko-Pinkall-Springborn [2] studied the vertex scaling introduced by Luo and Röcek-Williams and obtained the relationship between the vertex scaling and the geometry of ideal polyhedra in hyperbolic three space. They further introduced vertex scaling for piecewise hyperbolic metrics on triangulated surfaces. Based on Bobenko-Pinkall-Springborn's work [2] and Penner's work [41], Gu-Luo-Sun-Wu [34] recently proved a discrete uniformization theorem for piecewise linear metrics on surfaces via a variational principle established by Luo in [37]. Similar discrete uniformization theorem was established by Gu-Guo-Luo-Sun-Wu [33] for piecewise hyperbolic metrics on surfaces. Combinatorial Yamabe flow with surgery for polyhedral metrics were defined in [33, 34], where the long time existence and convergence of the combinatorial Yamabe flow with surgery are proved. Following Luo’s approach, Ge [11] introduced the combinatorial Calabi flow for piecewise linear metrics on surfaces. Recently, Zhu and the author [50] studied the combinatorial Calabi flow with surgery for piecewise linear and piecewise hyperbolic metrics on surfaces and proved the long-time existence and convergence of the flow. In this paper, we study the $\alpha$-curvature for piecewise linear metrics on polyhedral metrics. In [48], we study the $\alpha$-curvature for piecewise hyperbolic metrics on polyhedral metrics.

Suppose $S$ is a closed connected surface and $V$ is a finite subset of $S$, $(S, V)$ is called a marked surface. A piecewise linear metric (PL metric) on $(S, V)$ is a flat cone metric with cone points contained in $V$. Suppose $T = (V, E, F)$ is a triangulation of $(S, V)$, where $V, E, F$ represent the set of vertices, edges and faces respectively. We use $(S, V, T)$ to denote a triangulated surface. If a map $d : E \to (0, +\infty)$ satisfies that $d_{rs} < d_{rt} + d_{st}$ for $\{r, s, t\} = \{i, j, k\}$, where $d_{rs} = d(\{rs\})$ and $\{i, j, k\}$ is any triangle in $F$, then $d$ determines
a PL metric on \((S,V)\), which is still denoted by \(d\). Given \((S,V)\) with a triangulation \(\mathcal{T}\) and a map \(d : E \to (0, +\infty)\) determined by a PL metric \(d\) on \((S,V)\), the vertex scaling \([37, 44]\) of the PL metric \(d\) by a function \(w : V \to (0, +\infty)\) is defined to be the PL metric \(w \ast d\) on \((S,V)\) determined by the map \(w \ast d : E \to (0, +\infty)\) with
\[
(w \ast d)_{ij} := w_i w_j d_{ij}, \quad \forall \{ij\} \in E.
\]
The function \(w : V \to (0, +\infty)\) is called a conformal factor. For a triangulated surface \((S,V,\mathcal{T})\) with a PL metric \(d\), we denote the admissible space of conformal factors by \(\Omega^{\mathcal{T}}(d)\). Set \(u_i = \ln w_i, \ i = 1, \ldots, N\), and \(U^{\mathcal{T}}(d) = \ln \Omega^{\mathcal{T}}(d)\).

Suppose \((S,V)\) is a marked surface with a PL metric \(d\). The combinatorial curvature \(K_i\) of \(d\) at \(v_i \in V\) is \(2\pi\) less the cone angle at \(v_i\). If \(\mathcal{T}\) is a geometric triangulation of \((S,V)\) with a PL metric \(d\), we denote \(\theta_{ijk}\) as the inner angle at the vertex \(v_i\) of the triangle \(\triangle ijk\). Then the combinatorial curvature \(K_i = 2\pi - \sum_{\triangle ijk \in F} \theta_{ijk}\). Note that the combinatorial curvature \(K\) is independent of the geometric triangulations of \((S,V)\) with a PL metric \(d\).

**Definition 1.1.** Suppose \((S,V,\mathcal{T})\) is a triangulated surface with a PL metric \(d\) and \(w : V \to (0, +\infty)\) is a conformal factor of \(d\) on \((S,V,\mathcal{T})\). For any \(\alpha \in \mathbb{R}\), the combinatorial \(\alpha\)-curvature of \(w \ast d\) on \((S,V,\mathcal{T})\) is defined to be
\[
R^{\alpha,i} = \frac{K_i}{w_i^\alpha}.
\]

In the case that \(\alpha = 0\), the 0-curvature \(R_0\) is the classical combinatorial curvature \(K\). Furthermore, for any constant \(\lambda > 0\), we have \(R^{\alpha,i}(\lambda \ast l) = \lambda^{-\alpha} R^{\alpha,i}(l)\). Specially, for \(\alpha = 1\), we have
\[
R_{1,i}(\lambda \ast l) = \lambda^{-1} R_{1,i}(l),
\]
which is parallel to the smooth transformation of Gaussian curvature \(K_{\lambda g} = \lambda^{-1} K_g\).

For \(\alpha\)-curvature of PL metrics on triangulated surfaces, we have the following global rigidity.

**Theorem 1.1.** Suppose \((S,V,\mathcal{T})\) is a triangulated closed surface with a PL metric \(d\) and \(\alpha \in \mathbb{R}\) is a constant such that \(\alpha \chi(S) \leq 0\). \(\overline{R}\) is a given function defined on \(V\).

1. If \(\alpha \overline{R} \equiv 0\), then there exists at most one conformal factor \(\overline{w} \in \Omega^{\mathcal{T}}(d)\) with \(\alpha\)-curvature \(\overline{R}\) up to scaling;
2. If \(\alpha \overline{R} \leq 0\) and \(\alpha \overline{R} \not\equiv 0\), then there exists at most one conformal factor \(\overline{w} \in \Omega^{\mathcal{T}}(d)\) with \(\alpha\)-curvature \(\overline{R}\).

If \(\alpha = 0\), there is no restriction on \(\overline{R}_0 = K\) and the global rigidity of \(R_0 = K\) is reduced to the rigidity proved in \([2, 37]\).

For \(\alpha\)-curvature, it is interesting to consider the corresponding Yamabe problem.
**Combinatorial \(\alpha\)-Yamabe Problem:** Suppose \((S, V, T)\) is a triangulated closed surface with a PL metric \(d\), does there exist any conformal factor \(w : V \to (0, +\infty)\) in \(\Omega^T(d)\) such that \(w \ast d\) has constant \(\alpha\)-curvature?

In the case of \(\alpha = 0\), Luo [37] obtained a combinatorial obstruction for the solvability of the combinatorial Yamabe problem, which implies that there exists no constant combinatorial curvature PL metric on some triangulated surfaces. Luo [37] further introduced the combinatorial Yamabe flow to search for the constant curvature metric and established a variational principle for the flow. Following Luo’s approach, Ge [11] introduced the combinatorial Calabi flow to find the constant curvature PL metric, which is a negative gradient flow of the combinatorial Calabi energy.

To study the combinatorial \(\alpha\)-Yamabe problem, we introduce the combinatorial \(\alpha\)-Yamabe flow and combinatorial \(\alpha\)-Calabi flow for PL metrics on surfaces. Chow and Luo [7] introduced the combinatorial Ricci flow for Thurston’s circle packing metrics and proved its convergence, which is the first work using combinatorial curvature flow to study circle packing metrics. Inspired by Chow and Luo’s work, Ge [11,12] introduced the combinatorial Calabi flow for Thurston’s Euclidean circle packing metrics and prove the convergence. Ge and the author [22] studied the combinatorial Calabi flow for Thurston’s hyperbolic circle packing metrics and obtained some partial results on the convergence of the flow. Recently, Ge and Hua [13] proved the convergence of the combinatorial Calabi flow for Thurston’s hyperbolic circle packing metrics under Thurston’s combinatorial condition. Ge and the author [24–26] and Ge and Jiang [16–18] studied the combinatorial \(\alpha\)-Ricci flow and \(\alpha\)-Calabi flow for circle packing metrics. There are also some works on combinatorial curvature flows on 3-dimensional manifolds, see [14,19,20,23,26,29] for example. We extend the definition of \(\alpha\)-flows for circle packing metrics to PL metrics on surfaces here. The definition of \(\alpha\)-flows for piecewise hyperbolic metrics on surfaces was given in [48].

**Definition 1.2.** Suppose \((S, V, T)\) is a triangulated closed surface with a PL metric \(d_0\) and \(\alpha \in \mathbb{R}\) is a constant. The normalized combinatorial \(\alpha\)-Yamabe flow is defined to be

\[
\begin{cases}
  \frac{dw_i}{dt} = (R_{\alpha,av} - R_{\alpha,i})w_i, \\
  w_i(0) = 1,
\end{cases}
\]  

where \(R_{\alpha,av} = \frac{2\pi\chi(M)}{\sum_{i=1}^n w_i^0}\).

When \(\alpha = 0\), this is the combinatorial Yamabe flow introduced by Luo [37]. The combinatorial \(\alpha\)-curvature \(R_{\alpha}\) evolves according to

\[
\frac{dR_{\alpha,i}}{dt} = (\Delta^T_{\alpha} R_{\alpha})_i + \alpha R_{\alpha,i}(R_{\alpha,i} - R_{\alpha,av})
\]
along the combinatorial $\alpha$-Yamabe flow \[1.2\], where the $\alpha$-Laplace operator $\Delta^{\alpha}_T$ on $(S, V, T)$ is defined to be

$$\Delta^{\alpha}_T f_i = \frac{1}{w_i^{\alpha}} \sum_{j \sim i} \left( \cot \theta^{ij}_k + \cot \theta^{ij}_l \right) (f_j - f_i)$$

for $f \in \mathbb{R}^V$. Here $\theta^{ij}_k$ and $\theta^{ij}_l$ are two inner angles facing the edge $\{ij\}$. \[1.3\] is similar to the evolution of Gaussian curvature along the normalised Ricci flow on surfaces \[6, 36\].

**Definition 1.3.** Suppose $(S, V, T)$ is a closed triangulated surface with a PL metric $d_0$ and $\alpha \in \mathbb{R}$ is a constant. The combinatorial $\alpha$-Calabi flow is defined to be

$$\begin{cases} \frac{dw_i}{dt} = (\Delta^{\alpha}_T R^\alpha)_i w_i, \\
   w_i(0) = 1. \end{cases} \tag{1.4}$$

When $\alpha = 0$, this is the combinatorial Calabi flow introduced by Ge \[11\]. The combinatorial $\alpha$-curvature $R^\alpha$ evolves according to

$$\frac{dR^\alpha_{,i}}{dt} = - (\Delta^\alpha_T)^2 R^\alpha_{,i} - \alpha R^\alpha_{,i} \Delta^\alpha_T R^\alpha_{,i}$$

along the combinatorial $\alpha$-Calabi flow \[1.4\], which is similar to the evolution of Gaussian curvature along the surface Calabi flow \[4, 5, 8\]. If we choose the parameters properly, the evolution equations of combinatorial $\alpha$-curvature along the combinatorial $\alpha$-Yamabe flow and $\alpha$-Calabi flow are formally parallel to the evolution equations of Gaussian curvature along the surface Ricci flow and surface Calabi flow respectively. See \[26\] for this.

The combinatorial $\alpha$-flows (combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow) may develop singularities along the corresponding $\alpha$-flows. To handle the possible singularities along the combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow, we do surgery on the flows, the idea of which comes from \[33, 34, 37\]. Note that the weight in the $\alpha$-Laplace operator is $\omega_{ij} = \cot \theta^{ij}_k \cot \theta^{ij}_l$. To ensure that the discrete $\alpha$-Laplace operator have good properties along the $\alpha$-flows, especially the discrete maximal principle could be applied on the combinatorial $\alpha$-Yamabe flow, we need the weight $\omega_{ij}$ to be nonnegative on every edge, which is equivalent to $\theta^{ij}_k + \theta^{ij}_l \leq \pi$ for every edge $\{ij\} \in E$. This is the Delaunay condition on the triangulation \[3\]. This condition is imposed on both the combinatorial $\alpha$-Yamabe flow and the combinatorial $\alpha$-Calabi flow. Note that every PL metric on $(S, V)$ admits at least one Delaunay triangulation \[1, 43\], so this additional condition is reasonable. Along the $\alpha$-flows on $(S, V)$ with a triangulation $T$, if $T$ is Delaunay in $w(t) \ast d_0$ for $t \in [0, T]$ and not Delaunay in $w(t) \ast d_0$ for $t \in (T, T + \epsilon)$, $\epsilon > 0$, there exists an edge $\{ij\} \in E$ such that $\theta^{ij}_k(t) + \theta^{ij}_l(t) \leq \pi$ for $t \in [0, T]$ and $\theta^{ij}_k(t) + \theta^{ij}_l(t) > \pi$ for $t \in (T, T + \epsilon)$. Then we replace the triangulation $T$ by a new triangulation $T'$ at time $t = T$ via replacing two triangles $\triangle ijk$ and $\triangle ijl$ adjacent to $\{ij\}$ by two new triangles $\triangle ikl$ and $\triangle jkl$. This is called a
surgery by flipping on the triangulation $T$, which is also an isometry of $(S,V)$ with PL metric $w(T) * d_0$. After the surgery at time $t = T$, we run the $\alpha$-flows on $(S,V,T')$ with initial metric coming from the corresponding $\alpha$-flow on $(S,V,T)$ at time $t = T$.

We prove the following result on combinatorial $\alpha$-Yamabe flow and $\alpha$-Calabi flow with surgery, which is similar to the combinatorial Ricci flow on surfaces [7, 24, 26] and the combinatorial Calabi flow [11–13, 22, 26] for Thurston’s circle packing metrics.

**Theorem 1.2.** Suppose $(S,V)$ is a closed connected marked surface with a PL metric $d_0$ and $\alpha \in \mathbb{R}$ is a constant with $\alpha \chi(S) \leq 0$. Then there exists a PL metric in the conformal class $D(d_0)$ with constant $\alpha$-curvature if and only if one of the following two conditions is satisfied:

1. The combinatorial $\alpha$-Yamabe flow with surgery exists for all time and converges exponentially fast to a PL metric $d^*$ with constant combinatorial curvature;
2. The combinatorial $\alpha$-Calabi flow with surgery exists for all time and converges exponentially fast to a PL metric $d^*$ with constant combinatorial curvature.

$D(d_0)$ is the discrete conformal class defined in the sense of Gu-Luo-Sun-Wu [34]. Please refer to Definition 3.1.

Applying the discrete maximal principle to the combinatorial $\alpha$-Yamabe flow with surgery, we further prove the existence of constant $\alpha$-curvature metric.

**Theorem 1.3.** Suppose $(S,V)$ is a closed connected marked surface with a PL metric $d_0$ and $\alpha \in \mathbb{R}$ is a constant such that $\alpha \chi(S) \leq 0$. Then there exists a PL metric in the conformal class $D(d_0)$ with constant $\alpha$-curvature. Theorem 1.3 is a parameterized generalization of Gu-Luo-Sun-Wu’s discrete uniformization theorem in [34]. Combining Theorem 1.2 and Theorem 1.3 we have the following result.

**Theorem 1.4.** Suppose $(S,V)$ is a closed connected marked surface with a PL metric $d_0$ and $\alpha \in \mathbb{R}$ is a constant with $\alpha \chi(S) \leq 0$. Then the combinatorial $\alpha$-Yamabe flow with surgery and the combinatorial $\alpha$-Calabi flow with surgery exist for all time and converge exponentially fast to a PL metric $d^*$ with constant combinatorial $\alpha$-curvature.

When $\alpha = 0$, the convergence of combinatorial Yamabe flow with surgery was proved in [34] and the convergence of combinatorial Calabi flow with surgery was proved in [50].

The paper is organized as follows. In Section 2 we prove Theorem 1.1 and study the stability of combinatorial $\alpha$-flows on triangulated surfaces. In Section 3 we prove Theorem 1.2 and Theorem 1.3 based on the discrete conformal theory established in [34].
2. α-CURVATURE AND α-FLOWS ON TRIANGULATED SURFACES

2.1. Rigidity of α-curvature on triangulated surfaces. Suppose \((S, V, \mathcal{T})\) is a triangulated surface with a PL metric \(d\) and \(w : V \to (0, +\infty)\) is a positive function defined on \(V\). Set \(h : \mathbb{R}^n_{>0} \to \mathbb{R}^n\) be the homeomorphism defined by \(u_i = h(w_i) = \ln w_i\). Then \(w\) is a conformal factor of \(d\) on \((S, V, \mathcal{T})\) if and only if \(u : V \to \mathbb{R}\) is in the following space

\[
\mathcal{U}_{ijk}^T(d) \triangleq \{(u_{i}, u_{j}, u_{k}) \in \mathbb{R}^3 | \frac{d_{rs}}{e^{w_r}} + \frac{d_{st}}{e^{w_s}} > \frac{d_{st}}{e^{w_{st}}}, \{r, s, t\} = \{i, j, k\}\}
\]

for every triangle \(\Delta_{ijk} \in F\). It is observed by Luo [37] that the non-convex simply connected space \(\mathcal{U}_{ijk}^T(d)\) is the image of the convex space \(\{(w_{i}, w_{j}, w_{k}) \in \mathbb{R}^3_{>0} | (\frac{d_{rs}}{w_r}, \frac{d_{st}}{w_s}, \frac{d_{st}}{w_{st}}) \in \Delta\}\) under the homeomorphism \(h\), where \(\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3_{>0} | x_i + x_j > x_k\}, \) where \(i, j, k\) are distinct}. Furthermore, Luo [37] proved the following lemma.

Lemma 2.1. The \(3 \times 3\) matrix \(\frac{\partial w}{\partial u_{jk}}\) is symmetric, negative semi-definite with null space \(\{(t, t, t) \in \mathbb{R}^3 | t \in \mathbb{R}\}\).

Lemma 2.2. Let \(l_1, l_2, l_3\) and \(\theta_1, \theta_2, \theta_3\) be the edge lengths and inner angles of a triangle \(\Delta\) in \(\mathbb{E}^2\), or \(\mathbb{H}^2\), or \(\mathbb{S}^2\) so that the \(l_i\)-th edge is opposite to the angle \(\theta_i\). Consider \(\theta_i = \theta_i(l)\) as a function of \(l = (l_1, l_2, l_3)\).

(1): If \(\Delta\) is Euclidean or hyperbolic, the angle function \(\theta_i\) defined on

\[
\Omega = \{(l_1, l_2, l_3) \in \mathbb{R}^3 | l_1 + l_2 > l_3, l_1 + l_3 > l_2, l_2 + l_3 > l_1 \}
\]

can be extended continuously by constant functions to a function \(\bar{\theta}_i\) on \(\mathbb{R}^3_{>0}\).

(2): If \(\Delta\) is spherical, the angle function \(\theta_i\) defined on

\[
\Omega = \{(l_1, l_2, l_3) \in \mathbb{R}^3 | l_1 + l_2 > l_3, l_1 + l_3 > l_2, l_2 + l_3 > l_1, l_1 + l_2 + l_3 < 2\pi \}
\]

can be extended continuously by constant functions to a function \(\bar{\theta}_i\) on \((0, \pi)^3\).

Before going on, we recall the following result of Luo in [38].
Definition 2.1. A differential 1-form \( w = \sum_{i=1}^{n} a_i(x) dx_i \) in an open set \( U \subset \mathbb{R}^n \) is said to be continuous if each \( a_i(x) \) is continuous on \( U \). A continuous differential 1-form \( w \) is called closed if \( \int_{\partial \tau} w = 0 \) for each triangle \( \tau \subset \mathbb{R}^n \).

Theorem 2.1 (Corollary 2.6). Suppose \( X \subset \mathbb{R}^n \) is an open convex set and \( A \subset X \) is an open subset of \( X \) bounded by a \( C^1 \) smooth codimension-1 submanifold in \( X \). If \( w = \sum_{i=1}^{n} a_i(x) dx_i \) is a continuous closed 1-form on \( A \) so that \( F(x) = \int_{a}^{x} w \) is locally convex on \( A \) and each \( a_i \) can be extended continuous to \( X \) by constant functions to a function \( \tilde{a}_i \) on \( X \), then \( \tilde{F}(x) = \int_{a}^{x} \sum_{i=1}^{n} \tilde{a}_i(x) dx_i \) is a \( C^1 \)-smooth convex function on \( X \) extending \( F \).

Using Lemma 2.2 and Theorem 2.1, we have

Lemma 2.3 (\cite{37}). The function \( F_{ijk}(u) \) in (2.2) could be extended by constant to a \( C^1 \)-smooth concave function \( \tilde{F}_{ijk}(u) = \int_{u_0}^{u} \tilde{\theta}_i du_i + \tilde{\theta}_j du_j + \tilde{\theta}_k du_k \) defined for \( u \in \mathbb{R}^3 \), where the extension \( \tilde{\theta}_i \) of \( \theta_i \) by constant is defined to be \( \tilde{\theta}_i = \pi \) when \( l_{jk} \geq l_{ik} + l_{ij} \) or \( l_{ij} \geq l_{jk} + l_{ik} \).

Proof of Theorem 1.1: The proof is parallel to that of Theorem 3.3 in \cite{25}. For completeness, we give the proof here. Suppose \( w_0 \in \Omega^{T}(d) \) is a conformal factor and \( u_0 = \ln w_0 \). Then we can define the following Ricci energy \( F(u) \) by

\[
F(u) = -\sum_{\Delta ijk \in \partial \tau} F_{ijk} + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \tilde{\theta}_i w_i^\alpha) du_i. \tag{2.4}
\]

Note that the function \( F_{ijk} \) is smooth on \( U^T(d) = h(\Omega^{T}(d)) \). By direct calculations, we have

\[
\text{Hess}_u F = L - \alpha \begin{pmatrix}
\tilde{\theta}_1 w_1^\alpha \\
\vdots \\
\tilde{\theta}_N w_N^\alpha
\end{pmatrix}, \tag{2.5}
\]

where

\[
L = (L_{ijk})_{N \times N} = \frac{\partial (K_1, \cdots, K_N)}{\partial (u_1, \cdots, u_N)} = \begin{pmatrix}
\frac{\partial K_1}{\partial u_1} & \cdots & \frac{\partial K_1}{\partial u_N} \\
\vdots & \ddots & \vdots \\
\frac{\partial K_N}{\partial u_1} & \cdots & \frac{\partial K_N}{\partial u_N}
\end{pmatrix}. \tag{2.6}
\]

The matrix \( L \) has the following property \cite{37}.

Lemma 2.4 (\cite{37}). For a triangulated surface \((S, V, T)\) with a PL metric \( d \), the matrix \( L \) is symmetric and positive semi-definite on \( U^T(d) \) with kernel \{ \( t1 \mid t \in \mathbb{R} \) \}, where \( 1 = (1, \cdots, 1) \).
If $\alpha R \equiv 0$, then we have $\text{Hess}_u F$ is positive semi-definite with kernel $\{(t, \cdots, t) | t \in \mathbb{R} \}$ and $F$ is locally convex. If $\alpha R \leq 0$ and $\alpha R \neq 0$, then $\text{Hess}_u F$ is positive definite and $F$ is locally strictly convex.

By Lemma 2.3, $F_{ijk}$ defined on $U^T(d)$ could be extended to $\tilde{F}_{ijk}$ defined by (2.3) on $\mathbb{R}^3 \hookrightarrow \mathbb{R}^N$. And the second term $\int_{u_0}^u \sum_{i=1}^N (2\pi - \bar{R}_i w_i^\alpha) du_i$ in (2.4) can be naturally defined on $\mathbb{R}^N$, then we have the following extension $\tilde{F}(u)$ defined on $\mathbb{R}^N$ of the Ricci energy function $F(u)$

$$\tilde{F}(u) = - \sum_{\Delta_{ijk} \in F} \tilde{F}_{ijk} + \int_{u_0}^u \sum_{i=1}^N (2\pi - \bar{R}_i w_i^\alpha) du_i.$$ 

As $\tilde{F}_{ijk}$ is $C^1$-smooth concave by Lemma 2.3 and $\int_{u_0}^u \sum_{i=1}^N (2\pi - \bar{R}_i w_i^\alpha) du_i$ is a well-defined convex function on $\mathbb{R}^N$ for $\bar{R} \leq 0$, we have $\tilde{F}(u)$ is a $C^1$-smooth convex function on $\mathbb{R}^N$. Furthermore,

$$\nabla_{u_i} \tilde{F} = - \sum_{\Delta_{ijk} \in F} \tilde{\theta}_i + 2\pi - \bar{R}_i w_i^\alpha = \bar{K}_i - \bar{R}_i w_i^\alpha,$$

where $\bar{K}_i = 2\pi - \sum_{\Delta_{ijk} \in F} \tilde{\theta}_i$. Then we have $\tilde{F}(u)$ is convex on $\mathbb{R}^N$ and locally strictly convex on $U^T(d) \cap \{\sum_{i=1}^N u_i = 0\}$ for $\alpha R \equiv 0$. Similarly, $\tilde{F}(u)$ is convex on $\mathbb{R}^N$ and locally strictly convex on $U^T(d)$ for $\alpha R \leq 0$ and $\alpha R \neq 0$.

If there are two different conformal factors $\bar{w}_A, \bar{w}_B$ with the same combinatorial $\alpha$-curvature $\bar{R}$, then $\bar{w}_A = \ln \bar{w}_A \in U^T(d), \bar{w}_B = \ln \bar{w}_B \in U^T(d)$ are both critical points of the extended Ricci potential $\tilde{F}(u)$. It follows that

$$\nabla \tilde{F}(\bar{w}_A) = \nabla \tilde{F}(\bar{w}_B) = 0.$$

Set

$$f(t) = \tilde{F}((1 - t)\bar{w}_A + t\bar{w}_B)$$

$$= \sum_{\Delta_{ijk} \in F} f_{ijk}(t) + \int_{u_0}^u (1 - t)\bar{w}_A + t\bar{w}_B \sum_{i=1}^N (2\pi - \bar{R}_i w_i^\alpha) du_i,$$

where

$$f_{ijk}(t) = - \tilde{F}_{ijk}((1 - t)\bar{w}_A + t\bar{w}_B).$$

Then $f(t)$ is a $C^1$-smooth convex function on $[0, 1]$ and $f'(0) = f'(1) = 0$, which implies that $f'(t) \equiv 0$ for $t \in [0, 1]$. Note that $\bar{w}_A$ is in the open set $U^T(d)$, there exists $\epsilon > 0$ such that $(1 - t)\bar{w}_A + t\bar{w}_B \in U^T(d)$ for $t \in [0, \epsilon]$. Then $f(t)$ is smooth on $[0, \epsilon]$.

In the case of $\alpha R \leq 0$ and $\alpha R \neq 0$, the strict convexity of $\tilde{F}(u)$ on $U^T(d)$ implies that $f(t)$ is strictly convex on $[0, \epsilon]$ and $f'(t)$ is a strictly increasing function on $[0, \epsilon]$. Then $f'(0) = 0$ implies $f'(\epsilon) > 0$, which contradicts $f'(t) \equiv 0$ on $[0, 1]$. So there exists at most one conformal factor with combinatorial $\alpha$-curvature $\bar{R}$. 
For the case of \( \alpha R \equiv 0 \), we have \( f(t) \) is \( C^1 \) convex on \([0, 1]\) and smooth on \([0, \varepsilon]\). \( f'(t) \equiv 0 \) on \([0, 1]\) implies that \( f''(t) \equiv 0 \) on \([0, \varepsilon]\). Note that, for \( t \in [0, \varepsilon] \),

\[
f''(t) = (\underline{w}_A - \underline{w}_B)L(\underline{w}_A - \underline{w}_B)^T.
\]

By Lemma 2.4, we have \( \underline{w}_A - \underline{w}_B = c I \) for some constant \( c \in \mathbb{R} \), which implies that \( \underline{w}_A = e^t \underline{w}_B \). So there exists at most one conformal factor with combinatorial \( \alpha \)-curvature \( R \) up to scaling. Q.E.D.

Theorem 1.1 has a direct corollary.

**Corollary 2.1.** Suppose \((S, V, T)\) is a triangulated closed surface with a PL metric \( d \) and \( \alpha \in \mathbb{R} \) is a constant such that \( \alpha \chi(S) \leq 0 \). Then there exists at most one \( u^* \in \mathcal{U}^T(d) \) such that the PL metric \( e^{u^*}d \) has constant combinatorial \( \alpha \)-curvature (up to scaling for \( \alpha \chi(S) = 0 \)).

### 2.2. Combinatorial Yamabe flow of \( \alpha \)-curvature on triangulated surfaces

By direct calculations, we have the following properties of combinatorial \( \alpha \)-Yamabe flow.

**Lemma 2.5.** If \( \alpha = 0 \), \( \sum_{i=1}^{N} u_i \) is invariant along the normalized combinatorial \( \alpha \)-Yamabe flow (1.2). If \( \alpha \neq 0 \), \( \|w\|_{\alpha} = \sum_{i=1}^{N} w_i^\alpha \) is invariant along the normalized combinatorial \( \alpha \)-Yamabe flow (1.2).

**Theorem 2.2.** Suppose \( d_0 \) is a PL metric on a triangulated surface \((S, V, T)\) and \( \alpha \in \mathbb{R} \). If the solution of normalized combinatorial \( \alpha \)-Yamabe flow (1.2) on \((S, V, T)\) converges, then the limit metric is a constant combinatorial \( \alpha \)-curvature PL metric. Furthermore, suppose there exists a constant combinatorial \( \alpha \)-curvature PL metric \( d^* = e^{u^*}d_0 \) on a triangulated surface \((S, V, T)\) with \( \alpha \chi(S) \leq 0 \), then there exists a constant \( \delta > 0 \) such that if \( \|R_\alpha(u(0)) - R_\alpha(u^*)\| < \delta \), then the combinatorial \( \alpha \)-Yamabe flow (1.2) on \((S, V, T)\) exists for all time and converges exponentially fast to \( u^* \).

**Proof.** Suppose \( u(t) \) is a solution of the normalized combinatorial \( \alpha \)-Yamabe flow (1.2). If \( u(\infty) = \lim_{t \to +\infty} u(t) \) exists in \( \mathcal{U}^T(d) \), then we have \( R_\alpha(u(\infty)) = \lim_{t \to +\infty} R_\alpha(u(t)) \) exists. Furthermore, there exists \( \xi_n \in (n, n+1) \) such that

\[
u_i(n+1) - u_i(n) = u_i(\xi_n) = R_{\alpha, av} - R_{\alpha, i}(\xi_n) \to 0,
\]

which implies that \( R_\alpha(u(\infty)) = R_{av} \) and \( u(\infty) * d_0 \) is a constant \( \alpha \)-curvature PL metric.

Suppose \( u^* \) corresponds to a constant \( \alpha \)-curvature metric. Set \( \Gamma_i(u) = R_{\alpha, av} - R_{\alpha, i} \). By direct calculations, we have

\[
\frac{\partial \Gamma_i}{\partial u_j}|_{u=u^*} = -\frac{1}{w_i^\alpha} \frac{\partial K_i}{\partial u_j} + \alpha R_{\alpha, av}(\delta_{ij} - \frac{w_j^\alpha}{\|w\|_{\alpha}}) = \alpha R_{\alpha, av}(\delta_{ij} - \frac{w_j^\alpha}{\|w\|_{\alpha}}) + \frac{1}{w_i^\alpha} \frac{\partial K_i}{\partial u_j} \frac{w_i^\alpha w_j^\alpha}{\|w\|_{\alpha}^2}.
\]
Set $w^\alpha = (w_1^\alpha, \cdots, w_N^\alpha)^T$ and $\Sigma = \text{diag}\{w_1, \cdots, w_N\}$, then

$$
\left.D\Gamma\right|_{u=u^*} = \alpha R_{\alpha,av} I - \Sigma^{-\alpha}(L + \alpha R_{\alpha,av} \frac{w^\alpha \cdot (w^\alpha)^T}{\|w\|_\alpha^\alpha})
$$

$$
\left.\right. = - \Sigma^{-\alpha/2} \left( \Lambda_\alpha - \alpha R_{\alpha,av} [I - \frac{w^\alpha/2 \cdot (w^\alpha/2)^T}{\|w\|_\alpha^\alpha}] \right) \Sigma^{\alpha/2},
$$

where $\Lambda_\alpha = \Sigma^{-\alpha/2} L \Sigma^{-\alpha/2}$. Note that the matrix $I - \frac{w^\alpha/2 \cdot (w^\alpha/2)^T}{\|w\|_\alpha^\alpha}$ has eigenvalues $1$ ($N - 1$ times) and $0$ (1 time) and kernel $\{cw^\alpha | c \in \mathbb{R}\}$ and $\Lambda_\alpha$ is positive semi-definite with 1-dimensional kernel $\{cw^\alpha | c \in \mathbb{R}\}$. Then if the first nonzero eigenvalue $\lambda_1(\Lambda_\alpha)$ of $\Lambda_\alpha$ satisfies

$$
\lambda_1(\Lambda_\alpha) > \alpha R_{\alpha,av},
$$

we have $\left.D\Gamma\right|_{u=u^*}$ has $N - 1$ negative eigenvalues and a zero eigenvalue with eigenspace $\{cw^\alpha | c \in \mathbb{R}\}$, which is orthogonal to the space $\{w \in \mathbb{R}^N | \sum_{i=1}^N w_i^\alpha = N\}$. Specially, if $\alpha R_{\alpha,av} \leq 0$, we have $u^*$ is a local attractor of the normalized combinatorial $\alpha$-Yamabe flow (1.2). Then the conclusion follows from the Lyapunov Stability Theorem (12), Chapter 5). Q.E.D.

2.3. Combinatorial Calabi flow of $\alpha$-curvature on triangulated surfaces. Similar to the combinatorial $\alpha$-Yamabe flow, we have the following properties of combinatorial $\alpha$-Calabi flow.

Lemma 2.6. If $\alpha = 0$, $\sum_{i=1}^N u_i$ is invariant along the combinatorial $\alpha$-Calabi flow (1.4). If $\alpha \neq 0$, $\|w\|_\alpha^\alpha = \sum_{i=1}^N w_i^\alpha$ is invariant along the combinatorial $\alpha$-Calabi flow (1.4).

Theorem 2.3. Suppose $d_0$ is a PL metric on a triangulated surface $(S,V,T)$ and $\alpha \in \mathbb{R}$. If the solution of combinatorial $\alpha$-Calabi flow on $(S,V,T)$ converges, then the limit metric is a constant combinatorial $\alpha$-curvature PL metric. Furthermore, suppose there exists a constant combinatorial $\alpha$-curvature PL metric $d^* = e^{u^*} d_0$ on $(S,V,T)$ with $\alpha \chi(S) \leq 0$, there exists a constant $\delta > 0$ such that if $\|R_\alpha(u(0)) - R_\alpha(u^*)\| < \delta$, then the combinatorial $\alpha$-Calabi flow (1.4) on $(S,V,T)$ exists for all time and converges exponentially fast to $u^*$.

Proof. The proof of Theorem 2.3 is similar to that of Theorem 2.2, we just give some key calculations. Set $\Gamma_i(u) = (\Delta^T_i R_\alpha)_i$, then

$$
\left.\frac{\partial \Gamma_i}{\partial u_j}\right|_{u=u^*} = - \frac{1}{w_i^\alpha} \sum_{k=1}^N L_{ik} \frac{1}{w_k^\alpha} L_{kj} + \alpha R_{\alpha,av} \frac{1}{w_i^\alpha} L_{ij}.
$$

In matrix form, we have

$$
\left.D\Gamma\right|_{u=u^*} = - \Sigma^{-\alpha} L \Sigma^{-\alpha} L - \alpha R_{\alpha,av} \Sigma^{-\alpha} L
$$

$$
= - \Sigma^{-\alpha/2} \left( \Sigma^{-\alpha/2} L \Sigma^{-\alpha/2} L - \alpha R_{\alpha,av} \Sigma^{-\alpha/2} L \Sigma^{-\alpha/2} / 2 \right) \Sigma^{\alpha/2}.
$$
If $\alpha \chi(S) \leq 0$, then we have $D\Gamma|_{u=u^*}$ has $N-1$ negative eigenvalue and a zero eigenvalue with 1-dimensional kernel orthogonal to the space $\{w \in \mathbb{R}^N | \sum_{i=1}^N w^\alpha = N\}$, which implies that $u^*$ is a local attractor of the combinatorial $\alpha$-Calabi flow (1.4). Then the conclusion follows from the Lyapunov Stability Theorem ([12], Chapter 5). Q.E.D.

3. $\alpha$-curvature and $\alpha$-flows on discrete Riemann surfaces

Theorem 2.2 and Theorem 2.3 gives the long time existence and convergence of the combinatorial $\alpha$-Yamabe flow (1.2) and combinatorial $\alpha$-Calabi flow (1.4) for initial PL metrics with small initial energy. However, for general initial PL metrics, the combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow may develop singularities, including the conformal factor tends to infinity and some triangle degenerates along the combinatorial $\alpha$-Yamabe flow and combinatorial $\alpha$-Calabi flow. To handle the possible singularities along the $\alpha$-flows, we do surgery on the flows by flipping as described in Section 1.

To analysis the behavior of the $\alpha$-flows with surgery, we need to use the discrete conformal theory established by Gu-Luo-Sun-Wu [34] for PL metrics. In the following, we briefly recall some results in [34]. For details of the theory, please refer to Gu-Luo-Sun-Wu’s important work [34].

3.1. Gu-Luo-Sun-Wu’s work on discrete uniformization theorem.

Definition 3.1 ([34] Definition 1.1). Two PL metrics $d,d'$ on $(S,V)$ are discrete conformal if there exist sequences of PL metrics $d_1=d,\cdots,d_m=d'$ on $(S,V)$ and triangulations $T_1,\cdots,T_m$ of $(S,V)$ satisfying

(a): (Delaunay condition) each $T_i$ is Delaunay in $d_i$,

(b): (Vertex scaling condition) if $T_i = T_{i+1}$, there exists a function $u: V \to \mathbb{R}$ so that if $e$ is an edge in $T_i$ with end points $v$ and $v'$, then the lengths $l_{d_{i+1}}(e)$ and $l_{d_i}(e)$ of $e$ in $d_{i+1}$ and $d_i$ are related by

$$l_{d_{i+1}}(e) = l_{d_i}(e)e^{u(v)+u(v')}$$

(c): if $T_i \neq T_{i+1}$, then $(S,d_i)$ is isometric to $(S,d_{i+1})$ by an isometry homotopic to identity in $(S,V)$.

The discrete conformal class of a PL metric is called a discrete Riemann surface.

The space of PL metrics on $(S,V)$ discrete conformal to $d$ is called the conformal class of $d$ and denoted by $\mathcal{D}(d)$.

Recall the following result for Delaunay triangulations.

Lemma 3.1 ([1,3]). If $T$ and $T'$ are Delaunay triangulations of $d$, then there exists a sequence of Delaunay triangulations $T_1,T_2,\cdots,T_k = T'$ so that $T_{i+1}$ is obtained from $T_i$ by a diagonal switch.
The diagonal switch in Lemma 3.1 is the surgery by flipping described in Section 1.

The following discrete uniformization theorem was established in [34].

**Theorem 3.1** ([34] Theorem 1.2). Suppose \((S, V)\) is a closed connected marked surface and \(d\) is a PL metric on \((S, V)\). Then for any \(K^* : V \to (-\infty, 2\pi)\) with \(\sum_{v \in V} K^*(v) = 2\pi \chi(S)\), there exists a PL metric \(d'\), unique up to scaling and isometry homotopic to the identity on \((S, V)\), such that \(d'\) is discrete conformal to \(d\) and the discrete curvature of \(d'\) is \(K^*\).

Denote the Teichmüller space of all PL metrics on \((S, V)\) by \(T_{PL}(S, V)\) and decorated Teichmüller space of all equivalence class of decorated hyperbolic metrics on \(S - V\) by \(T_D(S - V)\). In the proof of Theorem 3.1, Gu-Luo-Sun-Wu proved the following result.

**Theorem 3.2** ([34]). There is a \(C^1\)-diffeomorphism \(A : T_{PL}(S, V) \to T_D(S, V)\) between \(T_{PL}(S, V)\) and \(T_D(S - V)\). Furthermore, the space \(D(d) \subset T_{PL}(S, V)\) of all equivalence classes of PL metrics discrete conformal to \(d\) is \(C^1\)-diffeomorphic to \(\{p\} \times \mathbb{R}_V^> 0\) under the diffeomorphism \(A\), where \(p\) is the unique hyperbolic metric on \(S - V\) determined by the PL metric \(d\) on \((S, V)\).

Set \(u_i = \ln w_i\) for \(w = (w_1, w_2, \cdots, w_n) \in \mathbb{R}_V^> 0\). Using the map \(A\), Gu-Luo-Sun-Wu defined the curvature map

\[
F : \mathbb{R}^n \to (-\infty, 2\pi)^n \\
F(u) = K_{A^{-1}(p,w(u))}
\]

and proved the following property of \(F\).

**Proposition 3.1** ([34]).

1. For any \(k \in \mathbb{R}\), \(F(v + k(1, 1, \cdots, 1)) = F(v)\).
2. There exists a \(C^2\)-smooth convex function \(W : \mathbb{R}^n \to \mathbb{R}\) so that its gradient \(\nabla W\) is \(F\) and the restriction \(W : \{u \in \mathbb{R}^n | \sum_{i=1}^n u_i = 0\} \to \mathbb{R}\) is strictly convex.

Theorem 3.2 implies that the union of the admissible spaces \(\Omega_T^T(d')\) of conformal factors such that \(T\) is Delaunay for \(d' \in D(d)\) is \(\mathbb{R}_V^> 0\). Furthermore, \(F\) defined on \(\mathbb{R}_V^> 0\) is a \(C^1\)-extension of the curvature \(K\) defined on the space of conformal factors \(\Omega_T^T(d')\) for \(d' \in D(d)\). Then we can extend the Euclidean discrete \(\alpha\)-Laplace operator to be defined on \(\mathbb{R}_V^> 0\), which is the space of the conformal factors for the discrete conformal class \(D(d)\).

**Definition 3.2.** Suppose \((S, V)\) is a marked surface with a PL metric \(d_0\), For a function \(f : V \to \mathbb{R}\) on the vertices, the discrete conformal \(\alpha\)-Laplace operator of \(d \in D(d_0)\) on \((S, V)\) is defined to be the map

\[
\Delta_\alpha : \mathbb{R}^V \to \mathbb{R}^V \\
f \mapsto \Delta_\alpha f
\]
where the value of $\Delta_\alpha f$ at $v_i$ is
\[
\Delta_\alpha f_i = \frac{1}{w_i^\alpha} \sum_{j \sim i} (\frac{\partial F_i}{\partial u_j})(f_j - f_i) = -\frac{1}{w_i^\alpha}(\tilde{L}f)_i,
\] (3.2)
where $(p, w) = A(d)$ and $\tilde{L}_{ij} = \frac{\partial F_i}{\partial u_j}$ is an extension of $L_{ij} = \frac{\partial K_i}{\partial u_j}$ for $u = \ln w \in U_T^D(d') = \ln \Omega_T^D(d'), d' \in D(d)$.

Remark 3.1. Note that $F$ is $C^1$-smooth in $u \in \mathbb{R}^n$ and $\Delta^T$ is independent of the Delaunay triangulations of a PL metric, so the operator $\Delta_\alpha$ is well-defined on $\mathbb{R}^n$. Furthermore, $\Delta_\alpha$ is continuous and piecewise smooth on $\mathbb{R}^n$ as a matrix-valued function of $u$ (\cite{34}, Lemma 5.1).

3.2. Rigidity of $\alpha$-curvature on discrete Riemannian surfaces. Following Gu-Luo-Sun-Wu’s approach, we can define the $\alpha$-curvature on discrete Riemannian surfaces as follows.

Definition 3.3. Suppose $(S, V)$ is a marked closed surface with a PL metric $d$, $\alpha \in \mathbb{R}$ is a constant and $F$ is the curvature map in (3.1). The $\alpha$-curvature on the discrete Riemannian surface $D(d)$ is defined to be
\[
F_{\alpha,i} = \frac{F_i}{w_i^\alpha}.
\] (3.3)

Remark 3.2. Note that $\alpha$-curvature on a discrete Riemann surface is well-defined and in some sense an extension of the combinatorial $\alpha$-curvature on a triangulated surface. If $T$ is a Delaunay triangulation of the marked surface $(S, V)$, $\Omega_T^D(d')$ is the space of conformal factors such that $T$ is Delaunay for $d' \in D(d)$, then $F_{\alpha,|U_T^D(d')} = R_{\alpha}$. Denote the space of conformal factors by $\Omega(d)$ and set $U(d) = \ln \Omega(d)$. Similar to Theorem \[1.1\] for $\alpha$-curvature on triangulated surfaces, we have the following global rigidity for $\alpha$-curvature on discrete Riemann surfaces.

Theorem 3.3. Suppose $(S, V)$ is a marked surface with a PL metric $d$ and $\alpha \in \mathbb{R}$ is a constant with $\alpha \chi(S) \leq 0$. $\overline{F}$ is a function defined on the vertices.

1: If $\alpha \overline{F} \equiv 0$, then there exists at most one conformal factor $u^* \in U(d)$ up to scaling such that $A^{-1}(p, w(u^*)) \in D(d)$ has combinatorial $\alpha$-curvature $\overline{F}$.

2: If $\alpha \overline{F} \leq 0$ and $\alpha \overline{F} \not\equiv 0$, then there exists at most one conformal factor $u^* \in U(d)$ such that $A^{-1}(p, w(u^*)) \in D(d)$ has combinatorial $\alpha$-curvature $\overline{F}$.

Proof. Define the energy function
\[
W_\alpha(u) = W(u) - \int_0^u \sum_{i=1}^N \overline{F}_i u_i^\alpha du_i.
\] (3.4)
By Proposition \[3.1\] $W_\alpha$ is a well-defined $C^2$-smooth function defined on $\mathbb{R}^n$. Furthermore, we have
\[
\nabla_{u_i} W_\alpha = F_i - \overline{F}_i u_i^\alpha.
\]
A^{-1}(p, w(u^*)) \in D(d) for u^* \in U(d) has combinatorial \(\alpha\)-curvature \(\bar{F}\) if and only if \(\nabla W_\alpha(u^*) = 0\). By direct calculations, we have

\[
\text{Hess } W_\alpha = L - \alpha \begin{pmatrix}
F_1 w_1^\alpha \\
\vdots \\
F_N w_N^\alpha
\end{pmatrix}.
\]

If \(\alpha \bar{F} \equiv 0\), Hess \(W_\alpha\) is positive semi-definite with kernel \(t_1 = (t, \ldots, t)\) and \(W_\alpha|_{\Sigma_0}\) is a strictly convex function on \(\Sigma_0 = \{u_1 + \cdots + u_N = 0\}\). If \(\alpha \bar{F} \leq 0\) and \(\alpha \bar{F} \not\equiv 0\), then Hess \(W_\alpha\) is positive definite and \(W_\alpha\) is strictly convex on \(\mathbb{R}^n\).

Recall the following well known fact from analysis.

**Lemma 3.2.** If \(W : \Omega \to \mathbb{R}\) is a \(C^1\)-smooth strictly convex function on an open convex set \(\Omega \subset \mathbb{R}^m\), then its gradient \(\nabla W : \Omega \to \mathbb{R}^m\) is an embedding.

Then the rigidity follows from Lemma 3.2. Q.E.D.

**Corollary 3.1.** Suppose \((S, V)\) is a marked surface with a PL metric \(d\) and \(\alpha \in \mathbb{R}\) is a constant with \(\alpha \chi(S) \leq 0\). Then the constant combinatorial \(\alpha\)-curvature PL metric in \(D(d)\) is unique (up to scaling if \(\alpha \chi(S) = 0\)).

### 3.3. Combinatorial \(\alpha\)-Yamabe flow with surgery

By Gu-Luo-Sun-Wu’s discrete conformal theory \[34\], the normalized combinatorial \(\alpha\)-Yamabe flow with surgery takes the following form.

**Definition 3.4.** Suppose \((S, V)\) is a marked surface with a PL metric \(d_0\). The combinatorial \(\alpha\)-Yamabe flow with surgery is defined to be

\[
\begin{align*}
\frac{du_i}{dt} &= F_{\alpha,av} - F_{\alpha,i} \\
u_i(0) &= 0,
\end{align*}
\]

where \(F_{\alpha,av} = \frac{2\pi \chi(S)}{\sum_{i=1}^N u_i^\alpha}\).

It is straightway to check that \(\sum_{i=1}^N w_i^\alpha \left(\sum_{i=1}^N u_i\right)\) for \(\alpha = 0\) is invariant along the combinatorial \(\alpha\)-Yamabe flow with surgery \[3.5\].

Similar to the results in \[7, 12, 26\], we have the following result for combinatorial \(\alpha\)-Yamabe flow with surgery.

**Theorem 3.4.** Suppose \((S, V)\) is a closed connected marked surface with a PL metric \(d_0\). \(\alpha \in \mathbb{R}\) is a constant such that \(\alpha \chi(S) \leq 0\). Then there exists a constant \(\alpha\)-curvature PL metric in \(D(d_0)\) if and only if the combinatorial \(\alpha\)-Yamabe flow with surgery \[3.5\] exists for all time and converges to some \(u^* \in U(d_0)\).

**Proof.** When \(\alpha = 0\), the combinatorial \(\alpha\)-Yamabe flow with surgery \[3.5\] is the Yamabe flow with surgery studied in \[34, 37\], where the conclusion has been proved. We only prove the case \(\alpha \neq 0\) here.
If the solution $u(t)$ of combinatorial $\alpha$-Yamabe flow with surgery (3.5) converges to $u^* \in \mathcal{U}(d_0)$, then we have $F_\alpha(u^*) = \lim_{t \to +\infty} F_\alpha(u(t))$ by the $C^1$ smoothness of $F$. For any $n \in \mathbb{N}$, there exists $\xi_n \in (n, n + 1)$ such that

$$u_i(n + 1) - u_i(n) = u_i(\xi_n) = F_{\alpha, av} - F_{\alpha, i}(u(\xi_n)).$$

Set $n \to +\infty$, then we have

$$F_{\alpha, i}(u^*) = \lim_{n \to +\infty} F_{\alpha, i}(u(\xi_n)) = F_{\alpha, av},$$

which implies that $u^*$ is a conformal factor in $\mathcal{U}(d_0)$ with constant combinatorial $\alpha$-curvature.

Conversely, suppose $u^*$ is a conformal factor in $\mathcal{U}(d_0)$ with constant combinatorial $\alpha$-curvature. Then the constant curvature must be the constant $F_{\alpha, av} =\frac{2\pi \chi(S)}{\sum_{i=1}^{N} w_i}$. Set

$$W_\alpha(u) = W(u) - F_{\alpha, av} \int_0^u \sum_{i=1}^{N} w_i^\alpha du_i.$$

Then $W_\alpha$ is a well-defined $C^2$-smooth convex function defined on $\mathbb{R}^n$ and $W_\alpha(u + k1) = W_\alpha(u)$. Note that $\nabla W_\alpha(u^*) = 0$, we have $\lim_{u \to +\infty} W_\alpha(u)|_P = +\infty$, where $P = \{u \in \mathbb{R}^n| \sum_{i=1}^{N} w_i^\alpha = N\}$. This implies that $W_\alpha(u)|_P$ is a proper function on $P$.

Note that

$$\frac{dW_\alpha(u(t))}{dt} = \sum_{i=1}^{N} \frac{\partial W_\alpha}{\partial u_i} \cdot \frac{du_i}{dt} = \sum_{i=1}^{N} (F_i - F_{\alpha, av}w_i^\alpha)(F_{\alpha, av} - F_{\alpha, i})$$

$$= -\sum_{i=1}^{N} (F_{\alpha, av} - F_{\alpha, i})^2 w_i^\alpha \leq 0.$$

So we have $0 \leq W_\alpha(u(t)) \leq W_\alpha(u(0))$. Note that $\sum_{i=1}^{N} w_i^\alpha$ is invariant along the combinatorial $\alpha$-Yamabe flow with surgery (3.5), we have the solution $u(t)$ of the combinatorial $\alpha$-Yamabe flow with surgery lies in a compact subset of $P$ by the properness of $W_\alpha$ on $P$. Then the solution of the combinatorial $\alpha$-Yamabe flow with surgery (3.5) exists for all time and $\lim_{t \to +\infty} W_\alpha(u(t))$ exists. Furthermore,

$$0 = \lim_{n \to +\infty} (W_\alpha(u(n + 1) - W_\alpha(u(n)))) = \lim_{n \to +\infty} \frac{dW_\alpha(u(t))}{dt} |_{t=\xi_n}$$

$$= -\lim_{n \to +\infty} \sum_{i=1}^{N} (F_{\alpha, av} - F_{\alpha, i})^2 w_i^\alpha |_{t=\xi_n}.$$

Then we have $\lim_{n \to +\infty} F_\alpha(u(\xi_n)) = F_{\alpha, av} = F_\alpha(u^*)$, which implies that $\lim_{n \to +\infty} u(\xi_n) = u^*$ by Theorem 3.3.

Set $\Gamma_i(u) = F_{\alpha, av} - F_{\alpha, i}$. Similar to the proof of Theorem 2.2 we can check that $D\Gamma|_{u^*}$ has $N - 1$ negative eigenvalue and a zero eigenvalue. The kernel of $D\Gamma|_{u^*}$ is orthogonal to $P$. Then the convergence of the solution of
combinatorial $\alpha$-Yamabe flow with surgery to $u^*$ follows from the Lyapunov Stability Theorem (\cite{42}, Chapter 5). Q.E.D.

Remark 3.3. The proof of Theorem 3.4 suggests a generalization of the combinatorial $\alpha$-Yamabe flow with surgery. Suppose $F$ is a function defined on the vertices, then $F$ is the combinatorial $\alpha$-curvature of a PL metric in $D(d_0)$ if and only if the solution of combinatorial $\alpha$-Yamabe flow with surgery for $F$ (defined similarly) exists for all time and converges to a conformal factor $u^* \in \mathcal{U}(d_0)$.

In the case of $\alpha \chi(S) \leq 0$, we can prove the existence of constant combinatorial $\alpha$-curvature metric and then obtain a generalization of the discrete uniformization theorem obtained in \cite{34}.

By direct calculations, the curvature $F_{\alpha,i}$ evolves according to the following type equation

$$\frac{dF_{\alpha,i}}{dt} = \Delta F_{\alpha,i} + \alpha F_{\alpha,i}(F_{\alpha,i} - F_{\alpha,av}).$$

(3.6)

along the combinatorial $\alpha$-Ricci flow.

Note that the surgery ensures that the weight

$$\omega_{ij} = \frac{1}{w_i^\alpha} \frac{\partial}{\partial u_j} = \frac{\cot \theta_k^j + \cot \theta_l^j}{w_i^\alpha} \geq 0$$

along the combinatorial $\alpha$-Yamabe flow with surgery (3.5). This motives us to use the following discrete maximal principle. The readers can refer to \cite{26} for a proof.

Theorem 3.5. (Maximum Principle) Let $f : V \times [0,T) \to \mathbb{R}$ be a $C^1$ function such that

$$\frac{\partial f_i}{\partial t} \geq \Delta f_i + \Phi_i(f_i), \quad \forall (i,t) \in V \times [0,T)$$

where the Laplacian operator is defined as

$$\Delta f_i = \sum_{j \sim i} a_{ij}(t)(f_j - f_i)$$

with $a_{ij} \geq 0$ and $\Phi_i : \mathbb{R} \to \mathbb{R}$ is a local Lipschitz function. Suppose there exists $C_1 \in \mathbb{R}$ such that $f_i(0) \geq C_1$ for all $i \in V$. Let $\varphi$ be the solution to the associated ODE

$$\begin{cases}
\frac{d\varphi}{dt} = \Phi_i(\varphi) \\
\varphi(0) = C_1,
\end{cases}$$

then

$$f_i(t) \geq \varphi(t)$$

for all $(i,t) \in V \times [0,T)$ such that $\varphi(t)$ exists.

Similarly, suppose $f : V \times [0,T) \to \mathbb{R}$ be a $C^1$ function such that

$$\frac{\partial f_i}{\partial t} \leq \Delta f_i + \Phi_i(f_i), \quad \forall (i,t) \in V \times [0,T).$$
Suppose there exists \( C_2 \in \mathbb{R} \) such that \( f_i(0) \leq C_2 \) for all \( i \in V \). Let \( \psi \) be the solution to the associated ODE
\[
\begin{cases}
\frac{d\psi}{dt} = \Phi_i(\psi) \\
\psi(0) = C_2,
\end{cases}
\]
then
\[ f_i(t) \leq \psi(t) \]
for all \((i, t) \in V \times [0, T]\) such that \( \psi(t) \) exists.

Applying the discrete maximal principle, we have the following result.

**Theorem 3.6.** Suppose \((S, V)\) is a marked surface with a PL metric \( d_0 \). \( \alpha \in \mathbb{R} \) is a constant such that \( \alpha F_{\alpha,i}(u(0)) < 0 \) for all \( i \in V \), then the normalized \( \alpha \)-Yamabe flow with surgery exists for all time and converges exponentially fast to a constant \( \alpha \)-curvature PL metric.

**Proof.** Note that the combinatorial \( \alpha \)-curvature \( F_{\alpha,i} \) evolves according to (3.6) along the normalized \( \alpha \)-Yamabe flow with surgery. By the maximum principle, if \( \alpha > 0 \) and \( \alpha F_{\alpha,i}(u(0)) < 0 \) for all \( i \in V \), we have
\[
(F_{\alpha,\min}(0) - F_{\alpha,av}) e^{\alpha F_{\alpha,av} t} \leq F_{\alpha,i} - F_{\alpha,av} \leq (F_{\alpha,\max}(0) - F_{\alpha,av}) e^{\alpha F_{\alpha,av} t}.
\]
If \( \alpha < 0 \) and \( \alpha F_{\alpha,i}(u(0)) > 0 \) for all \( i \in V \), we have
\[
\frac{F_{\alpha,av}}{F_{\alpha,\min}(0)} (F_{\alpha,\min}(0) - F_{\alpha,av}) e^{\alpha F_{\alpha,av} t} \leq F_{\alpha,i} - F_{\alpha,av} \leq (F_{\alpha,\max}(0) - F_{\alpha,av}) e^{\alpha F_{\alpha,av} t}.
\]
In summary, if \( \alpha F_{\alpha,i}(u(0)) < 0 \) for all \( i \in V \), there exists constants \( C_1 \) and \( C_2 \) such that
\[
C_1 e^{\alpha F_{\alpha,av} t} \leq F_{\alpha,i}(u(t)) - F_{\alpha,av} \leq C_2 e^{\alpha F_{\alpha,av} t},
\]
which implies the long-time existence and exponential convergence of the normalized \( \alpha \)-Yamabe flow with surgery (3.5). Q.E.D.

**Proof of Theorem 1.3:** In the case \( \alpha \chi(S) = 0 \), we have \( \alpha = 0 \) or \( \chi(S) = 0 \). For \( \alpha = 0 \), \( \alpha \)-curvature \( F_{\alpha} \) is the classical discrete curvature \( F \). The existence of constant curvature PL metric is ensured by Theorem 3.1. If \( \chi(S) = 0 \), the constant \( \alpha \)-curvature metric is a zero \( \alpha \)-curvature metric for all \( \alpha \in \mathbb{R} \). Specially, it is a PL metric with zero \( F \) curvature, the existence of which is ensured by Theorem 3.1.

In the case of \( \alpha \chi(S) < 0 \), by Theorem 3.1 there is a PL metric \( d' \in D(d_0) \) with constant \( F \) curvature \( \frac{2\alpha \chi(S)}{N} \), which implies that the combinatorial \( \alpha \)-curvature of \( d' \) satisfies \( \alpha F_{\alpha,i} < 0 \) for all \( i \in V \). Applying Theorem 3.6 with initial metric \( d' \) gives the conclusion. Q.E.D.

**Remark 3.4.** There is another way to extend the combinatorial Yamabe flow initiated by Ge-Jiang [15]. Ge-Jiang’s extension comes from [2,38] and is designed for surfaces with fixed triangulations. For a fixed triangulated
surface, Ge-Jiang’s extension of $\alpha$-Yamabe flow ensures the long-time existence of the extended flow, while the extended $\alpha$-Yamabe flow may converge to a virtual constant $\alpha$-curvature PL metric. Gu-Luo-Sun-Wu’s extension we use here ensures the combinatorial $\alpha$-Yamabe flow with surgery converges to a real constant $\alpha$-curvature metric. Furthermore, Gu-Luo-Sun-Wu’s extension could be applied to extend the combinatorial $\alpha$-Calabi flow, while Ge-Jiang’s extension is not valid for this case. The readers can refer to Subsection 3.4 for the $\alpha$-Calabi flow with surgery.

3.4. Combinatorial $\alpha$-Calabi flow with surgery. We can also define the combinatorial $\alpha$-Calabi flow with surgery.

**Definition 3.5.** Suppose $d_0$ is a PL metric on a marked surface $(S,V)$ and $\alpha \in \mathbb{R}$. The combinatorial $\alpha$-Calabi flow with surgery on $(S,V)$ is defined as

$$\frac{du_i}{dt} = (\Delta_\alpha F_\alpha)_i, \quad u_i(0) = 0,$$

where $\Delta_\alpha$ is the discrete $\alpha$-Laplace operator of $A^{-1}(p,w(u(t))) \in \mathcal{D}(d_0)$ on $(S,V)$ defined by (3.2).

Similar to the combinatorial $\alpha$-Yamabe flow on discrete Riemann surface, $\sum_{i=1}^N w_i^\alpha (\sum_{i=1}^N u_i$ for $\alpha = 0$) is invariant along the combinatorial $\alpha$-Calabi flow with surgery (3.7). It is straightforward to check that if the combinatorial $\alpha$-Calabi flow with surgery (3.7) converges, the limit metric is a constant $\alpha$-curvature PL metric.

We have the following result for combinatorial $\alpha$-Calabi flow with surgery (3.7).

**Theorem 3.7.** Suppose $(S,V)$ is a closed connected marked surface with a PL metric $d_0$ and $\alpha \in \mathbb{R}$ is a constant such that $\alpha \chi(S) \leq 0$. Then the combinatorial $\alpha$-Calabi flow with surgery (3.7) exists for all time and converges exponentially fast to a constant $\alpha$-curvature metric in $\mathcal{D}(d_0)$.

**Proof.** By Theorem 1.3, there exists a unique PL metric $d = A^{-1}(p,e^{u^*}) \in \mathcal{D}(d_0)$ with $\sum_{i=1}^N e^{\alpha u_i^*} = N$ such that $d$ has constant combinatorial $\alpha$-curvature $F_\alpha$.

Similar to the proof of Theorem 3.4, we can define

$$W_\alpha(u) = W(u) - F_{\alpha,av} \int_{u^*}^u \sum_{i=1}^N w_i^\alpha \, du_i.$$  

Then $W_\alpha$ is a well-defined $C^2$-smooth convex function defined on $\mathbb{R}^n$ under the condition $\alpha \chi(S) \leq 0$. Furthermore, $W_\alpha(u) = W_\alpha(u + k1), k \in \mathbb{R}$. Note that $\nabla W_\alpha(u^*) = 0$, we have $\lim_{u \to \infty} W_\alpha(u)|_P = +\infty$, where $P = \{u \in \mathbb{R}^n | \sum_{i=1}^N u_i^\alpha = N\}$. This implies that $W_\alpha(u)|_P$ is a proper function on $P$. 

By direct calculations, we have
\[
\frac{dW_\alpha(u(t))}{dt} = \sum_{i=1}^{N} \frac{\partial W_\alpha}{\partial u_i} \frac{du_i}{dt} = \sum_{i=1}^{N}(F_{\alpha,i} - F_{\alpha,av}w_i^\alpha)(\Delta_\alpha F_{\alpha,i})
\]
\[
= -(F_\alpha - F_{\alpha,av})^T \cdot L \cdot (F_\alpha - F_{\alpha,av}) \leq 0.
\]

Then \(W_\alpha(u(t))\) is bounded along the combinatorial \(\alpha\)-Calabi flow with surgery (3.7). By the properness of \(W_\alpha\), \(u(t)\) is bounded along the combinatorial \(\alpha\)-Calabi flow with surgery (3.7), which implies the long-time existence of combinatorial \(\alpha\)-Calabi flow with surgery.

As \(W_\alpha(u(t))\) is bounded along the combinatorial \(\alpha\)-Calabi flow with surgery and \(\frac{dW_\alpha(u(t))}{dt} \leq 0\), we have \(\lim_{t \to +\infty} W_\alpha(u(t))\) exists.

Note that
\[
0 = \lim_{n \to +\infty} (W_\alpha(u(n+1)) - W_\alpha(u(n)))
\]
\[
= - \lim_{n \to +\infty} (F_\alpha - F_{\alpha,av})^T \cdot L \cdot (F_\alpha - F_{\alpha,av})|_{t=\xi_n},
\]
there is a subsequence \(\xi_{n_k}\) of \(\xi_n \in (n, n+1)\) such that \(F_\alpha(u(\xi_{n_k})) \to F_{\alpha,av}\), which implies that \(u(\xi_{n_k}) \to u^* \in P\). So we have \(\lim_{t \to +\infty} W_\alpha(u(t)) = W_\alpha(u^*)\). By the strictly convexity of \(W_\alpha\) on \(P\), we have \(\lim_{t \to +\infty} u(t) = u^*\).

Q.E.D.

**Remark 3.5.** In the case of \(\alpha = 0\), the combinatorial Calabi flow with surgery is studied in [50], where the long-time existence and convergence of the combinatorial Calabi flow with surgery is proved.

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