GRAPHS AND $(\mathbb{Z}_2)^k$-ACTIONS

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Abstract. Let $\mathcal{A}_n^k$ denote all nonbounding effective smooth $(\mathbb{Z}_2)^k$-actions on $n$-dimensional smooth closed connected manifolds, each of which is cobordant to one with finite fixed set. Motivated by GKM theory, we can associate to each action of $\mathcal{A}_n^k$ a $(\mathbb{Z}_2)^k$-colored regular graph of valence $n$. Together with the combinatorics of colored graphs, equivariant cobordism and the tom Dieck-Kosniowski-Stong localization theorem,

- we give a lower bound for the number of fixed points of an action in $\mathcal{A}_n^k$, which can become the best possible in some cases;
- we determine the existence and the equivariant cobordism classification of all actions in $\mathcal{A}_n^k(h)$ with $h = 3, 4$, where $\mathcal{A}_n^k(h)$ is the subset of $\mathcal{A}_n^k$, each of which is equivariantly cobordant to an effective $(\mathbb{Z}_2)^k$-action fixing just $h$ isolated points (note: it is well-known that $\mathcal{A}_n^k(h)$ is empty if $h = 1, 2$);
- we characterize the explicit relationships among tangent representations at fixed points of each action in $\mathcal{A}_n^k(h)$ with $h = 3, 4$, which actually give the explicit solution of the Smith problem in such cases.

As an application, we also study the minimum number of fixed points of all actions in $\mathcal{A}_n^k$.

1. Introduction

1.1. Background. Using the work of Chang and Skjelbred [8], Goresky, Kottwitz and MacPherson in [17] established the GKM theory, saying that the equivariant cohomology of certain algebraic varieties with complex torus actions can explicitly be calculated in terms of their associated labeled regular graphs (also called GKM graphs). This indicates that there is an essential link between torus actions and the combinatorics of labeled regular graphs. Later on, Tolman and Weitsman [38] gave a simple proof of this result in the symplectic setting, and showed that such GKM graphs can be produced from a kind of $T^k$-manifolds (called the GKM manifolds). In subsequent works (see, e.g., [23]-[27]), Guillemin and Zara developed the GKM theory combinatorially. In [4], Biss, Guillemin and Holm studied the mod 2 version of the GKM theory, and showed that a mod 2 GKM manifold can still be associated to a unique labeled graph (called the mod 2 GKM graph), and its equivariant cohomology can be read out from this graph, where a mod 2 GKM manifold $X$ is the real locus of a GKM symplectic manifold $M$ with a Hamiltonian $T^k$-action, which is the fixed point set of an anti-symplectic involution (compatible with the

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$T^k$-action) on $M$. Note that $X$ naturally admits an action of $(\mathbb{Z}_2)^k$ such that $X^{(\mathbb{Z}_2)^k} = M^{T^k}$. In recent years, the GKM theory has been further developed in a variety of different directions (see, e.g., [10-11], [15-16], [19-22], [28, 33, 39]). For example, Guillemin and Holm in [19] established the GKM theory for torus actions with nonisolated fixed points, and Goldin and Holm in [16] generalized the GKM theory to the case in which the one-skeleton has dimension at most 4, so the mod 2 GKM theory may be generalized to the case in which the one-skeleton has dimension at most 2 (see [4]), where the one-skeleton of a $T^k$-manifold $M$ consists of the points $p \in M$ with its isotropy subgroup $G_p$ having dimension $\geq k - 1$.

1.2. Motivation and Problems. An important feature of the GKM theory is that equivariant topology can be associated with the combinatorics of labeled regular graphs. In this paper, we shall further extend this idea to general $(\mathbb{Z}_2)^k$-actions.

Throughout this paper assume that $G = (\mathbb{Z}_2)^k$ unless stated otherwise. Suppose that $(\Phi, M^n)$ is an effective smooth $G$-action on a smooth connected closed manifold $M^n$ with a finite fixed point set (i.e., $0 < |M^G| < +\infty$). Because the $G$-action is effective, the $G$-representation on the tangent space at a fixed point must be faithful and that implies $n \geq k$. Conner and Floyd showed in [12, Theorem 25.1] that if an involution on a closed manifold fixes a finite set, then the number of the fixed points must be even. Based upon this, we shall show in Section 2 that $(\Phi, M^n)$ can always be associated to an $n$-valent regular graph $\Gamma_{(\Phi,M)}$ with $M^G$ as its vertex set such that there is a natural map $\alpha$ from the set $E_{\Gamma_{(\Phi,M)}}$ of all edges of $\Gamma_{(\Phi,M)}$ to $\text{Hom}(G, \mathbb{Z}_2)$ with the following properties:

(P1) For each vertex $p$ of $\Gamma_{(\Phi,M)}$, the image set $\alpha(E_p)$ spans $\text{Hom}(G, \mathbb{Z}_2)$, where $E_p$ denotes the set of edges adjacent to $p$.

(P2) For each edge $e$ of $\Gamma_{(\Phi,M)}$,

$$\alpha(E_p) \equiv \alpha(E_q) \pmod{\alpha(e)}$$

where $p$ and $q$ are two endpoints of $e$.

Here we call the pair $(\Gamma_{(\Phi,M)}, \alpha)$ a $G$-colored graph of $(\Phi, M^n)$. Indeed, generally $\Gamma_{(\Phi,M)}$ is not determined uniquely, but the set $\{\alpha(E_p) \mid p \in M^G\}$ is independent of the choice of $\Gamma_{(\Phi,M)}$. The $G$-colored graph $(\Gamma_{(\Phi,M)}, \alpha)$ provides much important topological information of $(\Phi, M^n)$. Actually, since all irreducible real representations of $G$ can be identified with all elements of $\text{Hom}(G, \mathbb{Z}_2)$, the set $\{\alpha(E_p) \mid p \in M^G\}$ gives all $G$-representations on tangent spaces at fixed points. In particular, $\{\alpha(E_p) \mid p \in M^G\}$ also determines a complete equivariant cobordism invariant $\mathcal{P}_{(\Phi,M^n)}$ of $(\Phi, M^n)$ where $\mathcal{P}_{(\Phi,M^n)}$ is obtained by deleting all possible same pairs in $\{\alpha(E_p) \mid p \in M^G\}$ (see Theorem 3.2). This leads us to consider the following questions:

(Q1) What about the lower bound of the number $|M^G|$?

(Q2) How to determine the existence and the equivariant cobordism classification of such $G$-actions $(\Phi, M^n)$?

(Q3) How is the solution of the Smith problem for $(\Phi, M^n)$?

Remark 1. The original Smith problem [36] says that if a smooth closed manifold $X$ homotopic to a sphere admits an action of a finite group $H$ such that the fixed point set is formed by only two isolated points $u, v$, then are the tangent representations
at u and v isomorphic? The problem was solved by Atiyah-Bott [2] for H to be cyclic p-group (p a prime) and by Milnor [34] when X is a homology sphere with a semi-free H-action for H an arbitrary connected compact Lie group. For further development in this subject and counterexamples, see, e.g., [18], [35], [8]. More generally, if X is not restricted to be a homology sphere but a closed manifold and if the number |X|^H of fixed points is greater than two, as stated in [24], the question of how the tangent representations of H at distinct fixed points are related to each other is still open and is known as the Smith problem.

With respect to the above three questions, we shall pay more attention on the case in which G-actions are nonbounding. This is based upon the following result by Stong in [37], which implies that if a G-action (Φ, M^n) with 0 < |M|^G < +∞ is bounding, then the number |M|^G may be reduced to be zero from the viewpoint of cobordism.

**Theorem 1.1** (Stong). Suppose (Φ, M^n) is a smooth G-action on an n-dimensional smooth closed manifold M^n. Then (Φ, M^n) bounds equivariantly if and only if it is cobordant to a G-action (Ψ, N^n) with N^G empty.

**Remark 2.** Conner and Floyd [12] showed that when k = 1, any G-action (Φ, M^n) with |M|^G < +∞ always bounds equivariantly. This implies that k ≥ 2 if (Φ, M^n) with |M|^G < +∞ is nonbounding. Note that for n = 0, there is a trivial action on a single point which is nonbounding, and an action is nonbounding if and only if it has an odd number of fixed points. Implicitly we will have n > 0 and k > 1 throughout.

1.3. **Statements of main results.** Now let A^n_k denote the set of all nonbounding effective smooth G-actions on smooth connected closed n-manifolds such that each such G-action is cobordant to one fixing a finite set, where n ≥ k ≥ 2. Given a G-action (Φ, M^n) in A^n_k, let (Γ(Φ, M), α) be a G-colored graph of (Φ, M^n). With respect to the question (Q1), we shall use (Γ(Φ, M), α) to find a lower bound of |M|^G with a combinatorial argument, which is stated as follows.

**Theorem 1.2.** Let (Φ, M^n) be a G-action in A^n_k. Then the number |M|^G is at least 1 + \left\lceil \frac{n}{n-k+1} \right\rceil where \(\lceil r \rceil\) denotes the least integer greater than or equal to r.

**Remark 3.** First, we notice that the bound estimated in Theorem 1.2 is the best possible in some special cases. For instance, when n = k or 2^{k-1}, we shall see from Examples 2.3 in Section 5 that the bound is attainable, i.e., |M|^G = 1 + \left\lceil \frac{n}{n-k+1} \right\rceil. However, when k = 2 and n is not a power of 2, we can find from Lemma 5.2 that |M|^G is never equal to this bound. Second, since n ≥ k ≥ 2, we have that \(\left\lfloor \frac{n}{n-k+1} \right\rfloor ≥ 2\), so |M|^G ≥ 3.

**Corollary 1.3** (cf. [12] Theorem 31.3)). There is no manifold M equipped with a G-action fixing exactly a single point in any equivalence class of A^n_k.

**Proof.** If there is a G-action fixing exactly an isolated point, then this action must be nonbounding according to the classical Smith Theorem. □

With respect to the question (Q2), let A^n_k(h) be the subset of A^n_k, each of which is equivariantly cobordant to an effective G-action fixing exactly h isolated points. Then we know from Remark 3 that

- A^n_k(1) is empty.
• \(A_n^k(2)\) is also empty. Actually, any \(G\)-action fixing only two isolated points bounds equivariantly (see \[29, 30\]).

For the case \(h \geq 3\), as far as the author knows, the existence and equivariant cobordism classification of \(G\)-actions has not been completely determined yet except for the following cases:

• The case \(k = 2\). In \[12\] Theorem 31.2, Conner and Floyd determined the existence of all \((\mathbb{Z}2)^2\)-actions of \(A_2^k\) (i.e., \(A_2^k\) is nonempty if and only if \(n \geq 2\) is even), and they classified all possible \((\mathbb{Z}2)^2\)-actions of \(A_2^k\) up to equivariant cobordism.

• The case \(k = n = 3\). In \[32\], the author classified all \((\mathbb{Z}2)^3\)-actions of \(A_3^3\) up to equivariant cobordism.

Using colored graphs and the tom Dieck-Kosniowski-Stong localization theorem (see Theorem 33), we determine the existence of actions in \(A_n^k(h)\) with \(h = 3, 4\), and furthermore we completely classify up to equivariant cobordism all possible \(G\)-actions in \(A_n^k(h)\) with \(h = 3, 4\). Let \((\phi_i, \mathbb{R}P^i)\) be the standard linear \((\mathbb{Z}2)^i\)-action on \(\mathbb{R}P^i\) defined by

\[
\phi_i ((g_1, \ldots, g_i), [x_0, x_1, \ldots, x_i]) = [x_0, g_1 x_1, \ldots, g_i x_i]
\]

where \((g_1, \ldots, g_i) \in (\mathbb{Z}2)^i\), which fixes \(i + 1\) isolated points. In Definitions 1.1, 1.3 we will introduce two operations \(\Omega\) and \(\Delta\) on \(G\)-spaces, and apply \(\Omega\)-operation \(k - i\) times and \(\Delta\)-operation \(2^k\) times to \((\phi_i, \mathbb{R}P^i)\) to obtain \(\Delta^2 \Omega^{k-1}(\phi_i, \mathbb{R}P^i)\). We then show that this is typical example when there are three or four isolated fixed points, as follows.

**Theorem 1.4.** For \(h = 3\), we have that

(a) \(A_n^k(3)\) is nonempty if and only if \(k \geq 2\) and \(n = 2^k \geq 2k - 1\).

(b) Given an integer \(\ell \geq k - 1\) with \(n = 2^\ell\), each of \(A_n^k(3)\) is equivariantly cobordant to one of \(\{\sigma \Delta^2 \Omega^{k-1}(\phi_2, \mathbb{R}P^2) | \sigma \in \text{GL}(k, \mathbb{Z}_2)\}\), where \(\sigma \Delta^2 \Omega^{k-1}(\phi_1, \mathbb{R}P^i)\) denotes the action obtained by applying an automorphism \(\sigma \in \text{GL}(k, \mathbb{Z}_2)\) of \((\mathbb{Z}2)^k\) to \(\Delta^2 \Omega^{k-1}(\phi_i, \mathbb{R}P^i)\).

**Theorem 1.5.** For \(h = 4\), we have that

(a) \(A_n^k(4)\) is nonempty if and only if \(k \geq 3\) and \(n\) is in the range

\[
\bigcup_{\ell \geq k-3} [3 \cdot 2^\ell, 5 \cdot 2^\ell].
\]

(b) Given an integer \(\ell \geq k - 3\) such that \(3 \cdot 2^\ell \leq n \leq 5 \cdot 2^\ell\), each of \(A_n^k(4)\) is equivariantly cobordant to one of

\[
\{\sigma \Lambda^2 \Delta^2 \Omega^{k-3}(\phi_3, \mathbb{R}P^3) | v \in I^k(n - 3 \cdot 2^\ell), \sigma \in \text{GL}(k, \mathbb{Z}_2)\}
\]

where \(I^k(t)\) denotes the set of all lattices satisfying the equation \(x_1 + \cdots + x_{2k-3} = t\) with each \(0 \leq x_i \leq 2^k \cdot 4 + 4\) in \(\mathbb{R}^{2k-3}\), \(\Lambda^v\) is a special operation on \(\Delta^2 \Omega^{k-3}(\phi_3, \mathbb{R}P^3)\) (see Subsection 4.2), and \(\sigma \Lambda^v \Delta^2 \Omega^{k-3}(\phi_3, \mathbb{R}P^3)\) denotes the action obtained by applying an automorphism \(\sigma \in \text{GL}(k, \mathbb{Z}_2)\) to \(\Lambda^v \Delta^2 \Omega^{k-3}(\phi_3, \mathbb{R}P^3)\).

In theory, a colored graph \((\Gamma (\Phi, M), \alpha)\) of a \(G\)-action \((\Phi, M^n)\) fixing a finite set indicates a possible relationship among representations on tangent spaces at fixed
points, and the tom Dieck-Kosniowski-Stong localization theorem gives the algebraic relationships among them (see Theorem 3.3). Here combining two machineries leads us to obtain the explicit relationships for the cases $|M^G| = 3$ and $4$, giving the solution of Smith problem. (Note that the Smith problem for $|M^G| = 2$ can easily be solved by Theorem 3.3, i.e., the tangent representations at two fixed points are isomorphic.) In particular, this can also be associated with the geometrical realization of abstract colored graphs.

Let $\Gamma$ be a connected regular graph of valence $n \geq k$. If there is a map $\alpha : E_\Gamma \rightarrow \text{Hom}(G, \mathbb{Z}_2)$ satisfying (P1) and (P2) as stated before, then the pair $(\Gamma, \alpha)$ is called an abstract $1$-skeleton of type $(n, k)$, and the set $\{\alpha(E_p) | p \in V_\Gamma\}$ is called the vertex-coloring set of $(\Gamma, \alpha)$, where $V_\Gamma$ denotes the set of all vertices of $\Gamma$. We completely characterize the colored graphs of all actions in $\mathcal{A}_n^k(3)$ as follows.

**Theorem 1.6.** Let $(\Gamma, \alpha)$ be an abstract $1$-skeleton of type $(n, k)$ such that $\Gamma$ contains exactly three vertices $p, q, r$. Then $(\Gamma, \alpha)$ is realizable as a $G$-colored graph of some $G$-action in $\mathcal{A}_n^k(3)$ if and only if the following conditions are satisfied

(a) $k \geq 2$ and $n = 2^\ell$ with $\ell \geq k - 1$;

(b) there is a basis $\{\beta_1, ..., \beta_{k-1}, \gamma\}$ of $\text{Hom}(G, \mathbb{Z}_2)$ such that

$$\alpha(E_p) = \hat{\beta} \cup \hat{\gamma}, \quad \alpha(E_q) = \hat{\beta} \cup \hat{\delta}, \quad \alpha(E_r) = \hat{\delta} \cup \hat{\gamma}$$

where $\hat{\beta}$ is a multiset consisting of all $2^{k-2}$ different sums with same multiplicity $2^{k-1}$ formed by the odd number of elements of $\beta_1, ..., \beta_{k-1}$ (i.e., each sum appears exactly $2^{k-1}$ times in $\beta$ so $|\hat{\beta}| = 2^{\ell-1}$), $\hat{\gamma} = \{\gamma + \beta_1 + \beta | \beta \in \hat{\beta}\}$, and $\hat{\delta} = \{\gamma + \beta | \beta \in \hat{\beta}\}$.

We also characterize the tangent representation sets of all actions in $\mathcal{A}_n^k(4)$ as follows.

**Theorem 1.7.** Let $(\Gamma, \alpha)$ be an abstract $1$-skeleton of type $(n, k)$ such that $\Gamma$ contains exactly four vertices $p, q, r, s$. Then $\{\alpha(E_p), \alpha(E_q), \alpha(E_r), \alpha(E_s)\}$ is the fixed data of some $G$-action in $\mathcal{A}_n^k(4)$ if and only if the following conditions are satisfied

(a) $k \geq 3$ and $n$ is in the range $3 \cdot 2^\ell \leq n \leq 5 \cdot 2^\ell$ for some $\ell \geq k - 3$;

(b) there is a basis $\{\beta_1, ..., \beta_{k-2}, \gamma, \delta\}$ of $\text{Hom}(\mathbb{Z}_2^k, \mathbb{Z}_2)$ such that

$$\alpha(E_p) = \hat{\beta} \cup \hat{\gamma} \cup \hat{\delta} \cup \hat{\omega}, \quad \alpha(E_q) = \hat{\beta} \cup \hat{\eta} \cup \hat{\varepsilon} \cup \hat{\omega},$$

$$\alpha(E_r) = \hat{\gamma} \cup \hat{\varepsilon} \cup \hat{\lambda} \cup \hat{\omega}, \quad \alpha(E_s) = \hat{\delta} \cup \hat{\eta} \cup \hat{\lambda} \cup \hat{\omega}$$

where $\hat{\beta}$ is a multiset consisting of all $2^{k-3}$ different sums with the same multiplicity $2^{k-4}$ formed by the odd number of elements of $\beta_1, ..., \beta_{k-2}$, and

$$\begin{align*}
\hat{\gamma} &= \{\gamma + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
\hat{\delta} &= \{\delta + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
\hat{\varepsilon} &= \{\gamma + \beta | \beta \in \hat{\beta}\} \\
\hat{\eta} &= \{\delta + \beta | \beta \in \hat{\beta}\} \\
\hat{\lambda} &= \{\gamma + \delta + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
|\hat{\omega}| &= n - 3 \cdot 2^\ell
\end{align*}$$

and every element in $\hat{\omega}$ is chosen in the $2^{k-3}$ different elements of $\{\gamma + \delta + \beta | \beta \in \hat{\beta}\}$ and has multiplicity at most $2^{k-4}$. 


Remark 4. The reason why we only characterize the tangent representation sets of all actions in $A_k^n(4)$ is because generally each action $(\Phi, M^n)$ in $A_k^n(4)$ may not determine a unique regular graph $\Gamma(\Phi, M^n)$. But when $n = 3 \cdot 2^\ell$, each action in $A_{3 \cdot 2^\ell}^k(4)$ can determine a unique regular graph, so in this case, we can characterize the colored graphs of all actions in $A_{3 \cdot 2^\ell}^k(4)$ (see Theorem 8.23).

The paper is organized as follows. In Section 2 we review some basic facts for $G$-representations, and then show how to get colored regular graphs from $G$-actions. In Section 3 we reformulate Stong’s result (i.e., Theorem 1.1) into a complete equivariant cobordism invariant in terms of tangent representations at fixed points, and then review the tom Dieck-Kosniowski-Stong localization theorem. In Section 4 we introduce the $\Delta$-operation and the $\Omega$-operation. In Section 5 we prove Theorems 1.3 and 1.4. In Section 6 by using the colored graph of $(\phi_3, \mathbb{R}P^3)$, we construct two examples with four fixed points, which will play an essential role in the argument of the general case. Section 7 is the most subtle and delicate arguments in this paper. We spend much time on determining the essential structure of the tangent representations at fixed points of actions of $A_k^n(4)$ in Subsection 8.2. Then we determine the existence and the equivariant cobordism classification of all actions of $A_k^n(4)$ in Subsections 8.3-8.4, and completely characterize the relationships among tangent representations at fixed points of each action of $A_k^n(4)$ in Subsection 8.5. The proofs of Theorems 1.5 and 1.7 will be completed in Subsections 8.4 and 8.5, respectively. As an application, in Section 9 we also study the minimum number of fixed points of all actions in $A_k^n$.

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2. $G$-representations and graphs of actions

2.1. $G$-representations. Let $G = (\mathbb{Z}_2)^k$, and let $\text{Hom}(G, \mathbb{Z}_2)$ be the set of all homomorphisms $\rho : G \rightarrow \mathbb{Z}_2 = \{\pm 1\}$, which consists of $2^k$ distinct homomorphisms. Let $\rho_0$ denote the trivial element in $\text{Hom}(G, \mathbb{Z}_2)$, i.e., $\rho_0(g) = 1$ for all $g \in G$. Every irreducible real representation of $G$ is one-dimensional and has the form

$$\lambda_\rho : G \times \mathbb{R} \rightarrow \mathbb{R}$$

with $\lambda_\rho(t, r) = \rho(t) \cdot r$ for some $\rho \in \text{Hom}(G, \mathbb{Z}_2)$. It is well-known that there is a 1-1 correspondence between all irreducible real representations of $G$ and all elements of $\text{Hom}(G, \mathbb{Z}_2)$. $\text{Hom}(G, \mathbb{Z}_2)$ forms an abelian group with addition given by $(\rho + \sigma)(g) = \rho(g) \cdot \sigma(g)$, so it is also a $k$-dimensional vector space over $\mathbb{Z}_2$ with standard basis $\{\rho_1, ..., \rho_k\}$ where $\rho_i$ is defined by mapping $g = (g_1, ..., g_i, ..., g_k)$ to $g_i$.

Following [11]-[12], let $R_n(G)$ denote the set generated by the isomorphism classes of $G$-representations of dimension $n$, which naturally forms a vector space over $\mathbb{Z}_2$. Then

$$R_\ast(G) = \sum_{n \geq 0} R_n(G)$$

is a graded commutative algebra over $\mathbb{Z}_2$ with unit. The multiplication in $R_\ast(G)$ is given by $[V_1] \cdot [V_2] = [V_1 \oplus V_2]$. We can identify $R_\ast(G)$ with the graded polynomial
algebra over $\mathbb{Z}_2$ generated by $\text{Hom}(G, \mathbb{Z}_2)$. Namely, $R_*(G)$ is isomorphic to the graded polynomial algebra $\mathbb{Z}_2[\rho_1, ..., \rho_k]$. Based upon this, throughout this paper we use the convention that real representations of $G$ will be denoted by elements of $\mathbb{Z}_2[\rho_1, ..., \rho_k]$.

2.2. Graphs of actions. Now let $(\Phi, M^n)$ be an effective smooth $G$-action on a smooth closed connected manifold with $0 < |M^G| < +\infty$. Here $(\Phi, M^n)$ is not necessarily restricted to be nonbounding. Then we are going to construct a regular graph $\Gamma_{(\Phi, M)}$ associated with the action $(\Phi, M^n)$, such that the set of vertices of the graph is $M^G$ and the valence of the graph is exactly $n$.

We now consider fixed sets of subgroups of $G$. Given a nontrivial irreducible representation $\rho$, then the kernel $\ker \rho$ is a subgroup of $G$ isomorphic to $(\mathbb{Z}_2)^{k-1}$. Let $C$ be a component of the fixed set of $\ker \rho$ and let $d = \dim C$. Then the fixed points of the group $G/\ker \rho \cong \mathbb{Z}_2$ acting on $C$ will be fixed points of $G$ acting on $M$. If $p$ is a fixed point of $G/\ker \rho$ acting on $C$, then $\rho$ occurs as a factor exactly $d$ times in the tangent $G$-representation (i.e., a monomial of degree $n$ in $\mathbb{Z}_2[\rho_1, ..., \rho_k]$) at $p$ in $M$. Assuming $d > 0$, Conner and Floyd have shown in [12] that the number of fixed points of $G/\ker \rho$-action on $C$ must be even. Thus we may always choose a connected regular graph $\Gamma_{\rho,C}$ with vertices the fixed points of $G/\ker \rho$-action on $C$ with $d$ edges meeting at each vertex. If $d = 1$, then $C$ is a circle with precisely two fixed points, and we may choose an edge joining these two fixed points. Clearly, in this case there is only one choice for $\Gamma_{\rho,C}$. If $d > 1$, let $p_1, p_2, ..., p_{2i-1}, p_{2i}$ be all fixed points of $G/\ker \rho$-action on $C$. In this case, there can be many choices for $\Gamma_{\rho,C}$ if $i > 1$. For example, when $d = i = 3$, one may gives five kinds of different choices as shown in Figure 1.

![Figure 1. 3-valent regular connected graphs with 6 vertices](image)

Our graph $\Gamma_{(\Phi, M)}$ is now the union of all of these subgraphs $\Gamma_{\rho,C}$ chosen for each $\rho$ and $C$ along fixed points of $M^G$. Because the tangent $G$-representation at a fixed point $p$ is a monomial of degree $n$ in $\mathbb{Z}_2[\rho_1, ..., \rho_k]$, exactly $n$ edges meet at $p$. So $\Gamma_{(\Phi, M)}$ is a regular graph of valence $n$ with $M^G$ as its vertex set.

Remark 5.

a) In general, there can be many choices for our graph $\Gamma_{(\Phi, M)}$.

b) Unlike GKM graphs, the orientation of associated regular graphs in our case will not be considered.

c) Generally, there also may be several edges having the same endpoints in $\Gamma_{(\Phi, M)}$, i.e., the number $|E_e|$ for some edge $e \in E_{\Gamma_{(\Phi, M)}}$ may not be 1, where $E_{\Gamma_{(\Phi, M)}}$ denotes the set of all edges of $\Gamma_{(\Phi, M)}$, and $E_e$ denotes the set of all edges joining two endpoints of $e$.

On the uniqueness of $\Gamma_{(\Phi, M)}$, it is easy to see that
Lemma 2.1. $\Gamma_{(\Phi,M)}$ is uniquely determined if for each $\rho$ and $C$, $G/\ker\rho$-action on $C$ fixes only two isolated points.

By the construction of $\Gamma_{(\Phi,M)}$, there is a natural map

$$\alpha : E_{\Gamma_{(\Phi,M)}} \rightarrow \text{Hom}(G, \mathbb{Z}_2)$$

such that each edge of $\Gamma_{(\Phi,M)}$ is colored (or labeled) by a nontrivial element in $\text{Hom}(G, \mathbb{Z}_2)$. Actually, given an edge $e$ in $E_{\Gamma_{(\Phi,M)}}$, there exists a nontrivial element $\rho$ in $\text{Hom}(G, \mathbb{Z}_2)$ and a component $C$ of $M^{\ker\rho}$ such that $e$ is an edge of $\Gamma_{\rho,C}$. Then $e$ is colored by $\rho$, namely $\alpha(e) = \rho$. The natural map $\alpha$ has the following two basic properties:

(P1) For each vertex $p$ of $\Gamma_{(\Phi,M)}$, the image set $\alpha(E_p)$ spans $\text{Hom}(G, \mathbb{Z}_2)$.

(P2) For each edge $e$ of $\Gamma_{(\Phi,M)}$,

$$\alpha(E_p) \equiv \alpha(E_q) \mod \alpha(e)$$

where $p$ and $q$ are two endpoints of $e$.

The property (P1) follows from the following argument. At a fixed point $p$ in $M^n$, its tangent $G$-representation is a monomial $\prod_{e \in E_p} \alpha(e)$ of degree $n$ in $\mathbb{Z}_2[\rho_1, ..., \rho_k]$ where each $\alpha(e) \in \text{Hom}(G, \mathbb{Z}_2)$ is non-trivial by the definition of $\alpha$. Then the condition that the action is effective means that $\alpha(e), e \in E_p$ span $\text{Hom}(G, \mathbb{Z}_2)$. The argument of (P2) is as follows: Since $p$ and $q$ are two endpoints of $e$, there must be a connected component $C$ of the fixed point set $M^{\ker \alpha(e)}$ such that $p, q \in C$. Thus, both $p$ and $q$ have the same ker $\alpha(e)$-representation. Further, the tangent $G$-representations at $p$ and $q$ in $M$ are the same when restricted to ker $\alpha(e)$, and in particular, as we have noted that $\alpha(e)$ occurs as a factor exactly $\dim C$ times in the tangent $G$-representations at both $p$ and $q$. If $\sigma$ is another nontrivial irreducible representation occurring at $\alpha(E_p)$ or $\alpha(E_q)$, then $\sigma$ and $\sigma + \alpha(e)$ become the same nontrivial representation when restricted to ker $\alpha(e)$. Thus, $\alpha(E_p) \equiv \alpha(E_q) \mod \alpha(e)$.

The pair $(\Gamma_{(\Phi,M^n)}, \alpha)$ is called a $G$-colored graph of $(\Phi,M^n)$. By $\mathcal{N}_{(\Phi,M^n)}$ we denote the set $\{\alpha(E_p)|p \in M^G\}$. Note that for $p \in M^G$, generally $\alpha(E_p)$ may be a multiset. Then

$$\left\{ \prod_{e \in E_p} \alpha(e) | p \in M^G \right\}$$

gives the tangent representations at all fixed points. Obviously, both $\alpha(E_p)$ and $\prod_{e \in E_p} \alpha(e)$ are determined by each other. Thus, throughout this paper $\mathcal{N}_{(\Phi,M^n)}$ will be understood as all tangent $G$-representations at fixed points.

Remark 6. Clearly $\mathcal{N}_{(\Phi,M^n)}$ is independent of the choice of $G$-colored graphs.

Example 1. Let $(\phi_n, \mathbb{R}P^n)$ with $n \geq 2$ be the standard linear $(\mathbb{Z}_2)^n$-action on $\mathbb{R}P^n$ defined by

$$\phi_i((g_1, ..., g_n), [x_0, x_1, ..., x_n]) = [x_0, g_1x_1, ..., g_nx_n], \ (g_1, ..., g_n) \in (\mathbb{Z}_2)^n,$$

which fixes $n+1$ isolated points $[0, ..., 0, 1, 0, ..., 0]$ with 1 in the $i$-th place for $i = 0, 1, ..., n$. It is not difficult to see that $(\phi_n, \mathbb{R}P^n)$ determines a unique regular graph $\Gamma_{(\phi_n, \mathbb{R}P^n)}$, which is just the 1-skeleton of an $n$-simplex, and the $\binom{n+1}{2}$ edges of $\Gamma_{(\phi_n, \mathbb{R}P^n)}$ are colored by $\rho_1, ..., \rho_n, \rho_1 + \rho_j, 1 \leq i < j \leq n$, respectively. Actually, $(\phi_n, \mathbb{R}P^n)$ is a small cover over an $n$-simplex $\Delta^n$, and all its possible orbit types
GRAPHS AND $(\mathbb{Z}_2)^k$-ACTIONS

can be read out from $\Delta^n$ (see [13]). When $n = 2, 3$, the colored graph $\Gamma_{(\phi_n, \mathbb{R}P^n)}$ is shown in Figure 2. Furthermore, the diagonal action on two copies of $(\phi_2, \mathbb{R}P^2)$ and $\rho_2$ $\rho_1 + \rho_2$

The case $n = 2$

$\rho_2 + \rho_3$

The case $n = 3$

![Figure 2. Colored graphs for the cases $n = 2, 3$.](image)

the twist involution on the product $\mathbb{R}P^2 \times \mathbb{R}P^2$ give a $(\mathbb{Z}_2)^3$-action on $\mathbb{R}P^2 \times \mathbb{R}P^2$ fixing three fixed points, whose colored graph is shown in Figure 3.

![Figure 3. The colored graph of the $(\mathbb{Z}_2)^3$-action on $\mathbb{R}P^2 \times \mathbb{R}P^2$.](image)

Remark 7. In mod 2 GKM theory, as noted in [4, Remark 5.9], when $(\Phi, M^n)$ satisfies the conditions that (1) $(\Phi, M^n)$ is equivariantly formal; (2) $M^G$ is finite; (3) the isotropy weights of the tangent representations at each fixed point are all distinct and non-zero (i.e., for each $\rho$ and $C$, $\dim C = 1$, so $|E_\rho| = 1$ for each edge $e \in \Gamma_{(\Phi,M)}$), its equivariant cohomology can be explicitly read out from $(\Gamma_{(\Phi,M)}, \alpha)$ as follows:

$$H^*_G(M^n; \mathbb{Z}_2) \cong \{ f : M^G \to \mathbb{Z}_2[\rho_1, ..., \rho_k] \mid f(p) \equiv f(q) \mod \alpha(e) \text{ for } e \in E_p \cap E_q \}.$$

In addition, as mentioned in Subsection 1.1 Biss, Guillemin and Holm in [4] also generalized the mod 2 GKM theory to the case where the one-skeleton has dimension at most 2 (i.e., for each $\rho$ and $C$, $\dim C \leq 2$).

A natural question is that if $(\Phi, M^n)$ with $M^G$ a finite set satisfies the conditions that (1) $(\Phi, M^n)$ is equivariantly formal; (2) for each $\rho \in \text{Hom}(G, \mathbb{Z}_2)$ and possible components $C$ with $\dim C > 2$ of $M^\ker \rho$, the action of $G/\ker \rho$ on $C$ fixes only two isolated points, whether can its equivariant cohomology be explicitly read out from its colored graph $(\Gamma_{(\Phi,M)}, \alpha)$? But this is beyond the scope of the current paper.

3. A COMPLETE EQUIVARIANT COBDISM IN Variant AND THE TOM Dieck-Kosinski-Stong LOCALIZATION THEOREM

In this section, we first reformulate Stong’s result (i.e., Theorem [13]) into a complete equivariant cobordism invariant in terms of tangent representations at
fixed points, and then review the tom Dieck-Kosniowski-Stong localization theorem \([14, 29]\).

Throughout the following, assume that \((\Phi, M^n)\) is an effective smooth \(G\)-action on a smooth closed connected manifold with \(0 < |M^G| < +\infty\), and \((\Gamma(\Phi, M^n), \alpha)\) is a colored graph of \((\Phi, M^n)\).

### 3.1. A complete equivariant cobordism invariant.

**Lemma 3.1.** \((\Phi, M^n)\) is equivariantly cobordant to a \(G\)-action \((\Psi, N)\) such that either \(\mathcal{N}(\Psi, N)\) is empty or it is non-empty but all elements of \(\mathcal{N}(\Psi, N)\) are different.

**Proof.** If \(p\) and \(q\) are two fixed points with \(\alpha(E_p) = \alpha(E_q)\), then one can cut out neighborhoods of \(p\) and \(q\), each of which looks like the disc in the associated representation space. One then glues the resulting boundaries together to obtain a new action, which is cobordant to the action \((\Phi, M^n)\) by \([37]\). This reduces the number of fixed points by two. One can carry out this procedure until one obtains the required action \((\Psi, N)\).

Obviously, \(\mathcal{N}(\Psi, N)\) is determined uniquely by \((\Phi, M^n)\). Set

\[
\mathcal{P}(\Phi, M^n) := \mathcal{N}(\Psi, N).
\]

Here one calls \(\mathcal{P}(\Phi, M^n)\) the prime tangent \(G\)-representation set of \((\Phi, M^n)\). By Theorem \([13]\) and Lemma \([3.1]\), one has

**Theorem 3.2.** \(\mathcal{P}(\Phi, M^n)\) is a complete equivariant cobordism invariant of \((\Phi, M^n)\).

### 3.2. The tom Dieck-Kosniowski-Stong localization theorem.

In \([14]\), tom Dieck showed that the equivariant cobordism class of \((\Phi, M^n)\) is completely determined by its equivariant Stiefel-Whitney characteristic numbers, and in particular, the existence of \((\Phi, M^n)\) can be determined in terms of its fixed data. Later on, Kosniowski and Stong \([29]\) gave a more precise formula for the characteristic numbers of \(M^n\) in terms of the fixed data. Combining their works, one has the following localization theorem.

**Theorem 3.3 (tom Dieck-Kosniowski-Stong).** Let \((\Gamma, \alpha)\) be an abstract 1-skeleton of type \((n, k)\). Then the necessary and sufficient condition that \(\{\alpha(E_p) | p \in V_\Gamma\}\) is the fixed data of some \((\mathbb{Z}_2)^k\)-action \((\Phi, M^n)\) is that for any symmetric polynomial function \(f(x_1, \ldots, x_n)\) over \(\mathbb{Z}_2\),

\[
\sum_{p \in V_\Gamma} \frac{f(\alpha(E_p))}{\prod_{e \in E_p} \alpha(e)} \in \mathbb{Z}_2[\rho_1, \ldots, \rho_k]
\]

where \(V_\Gamma\) denotes the set of all vertices of \(\Gamma\), and \(f(\alpha(E_p))\) means that \(x_1, \ldots, x_n\) in \(f(x_1, \ldots, x_n)\) are replaced by all elements in \(\alpha(E_p)\).

**Remark 8.** Originally, as stated in \([14]\) and \([29]\), the formula \([14]\) should be written as the following form

\[
\sum_{p \in V_\Gamma} \frac{f(\chi^G(p))}{\chi^G(p)} \in H^*\left(B(\mathbb{Z}_2)^k; \mathbb{Z}_2\right) = \mathbb{Z}_2[t_1, \ldots, t_k]
\]

with each \(t_i \in H^1\left(B(\mathbb{Z}_2)^k; \mathbb{Z}_2\right)\), where \(\chi^G(p)\) denotes the equivariant Euler class of the tangent representation at \(p\), which is a monomial of degree \(n\) in \(H^*\left(B(\mathbb{Z}_2)^k; \mathbb{Z}_2\right)\) and \(f(\chi^G(p))\) means that \(x_1, \ldots, x_n\) in \(f(x_1, \ldots, x_n)\) are replaced by \(n\) factors in
\( \chi^G(p) \). If \( \{ \alpha(E_p) | p \in V_1 \} \) is the fixed data of some \((\mathbb{Z}_2)^k\)-action \((\Phi, M^n)\), then the polynomial \( \sum_{p \in V_1} \frac{f(\chi^G(p))}{\chi^G(p)} \in H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2) \) actually means an equivariant Stiefel-Whitney number of the action. In other words, if we formally write the the equivariant total Stiefel-Whitney class of the tangent bundle \( \tau(M) \) as \( w^G(\tau(M)) = \prod_{i=1}^n (1 + x_i) \), then the equivariant Stiefel-Whitney number \( f(x_1, ..., x_n)[M] \) can be calculated by the following formula

\[
(3) \quad f(x_1, ..., x_n)[M] = \sum_{p \in V_1} \frac{f(\chi^G(p))}{\chi^G(p)}
\]

where \([M]\) denotes the fundamental homology class of \( M \). For more details, see [14] and [29]. The formula (3) is an analogue of the Atiyah-Bott-Berlin-Vergne formula for the case of torus actions (see [3] and [6]). Since \( H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2) \) is isomorphic to \( \mathbb{Z}_2[\rho_1, ..., \rho_k] \), each class \( \chi^G(p) \) uniquely corresponds to its tangent representation \( \alpha(E_p) \), so we can give another description of the formula (2) in terms of \( G \)-representations, as stated in [1].

4. Some operations on \( G \)-actions

Throughout the following, let \((\Phi, M^n)\) be a \((\mathbb{Z}_2)^k\)-action on a closed manifold \( M \) (note that here the fixed set of the action is not necessarily restricted to be finite).

4.1. Diagonal action.

**Definition 4.1.** Given an integer \( i \geq 1 \), the action \( \Phi^i \times \cdots \times \Phi^i \) on \( M^n \times \cdots \times M^n \)

defined by

\[
(g, (x_1, ..., x_i)) \mapsto (\Phi(g, x_1), ..., \Phi(g, x_i))
\]

is called an \( i \)-multi-diagonal action on \((\Phi, M^n)\), denoted by \( \Delta^i(\Phi, M^n) \).

For each \( i \), \( \Delta^i(\Phi, M^n) \) is still a \((\mathbb{Z}_2)^k\)-manifold and has dimension \( i \cdot n \) such that its fixed point set is \( \Delta^i(M(\mathbb{Z}_2)^k) = \prod_{0 \leq j \leq n} \xi^{n-j} \to F^j \) be the normal bundle of \( M(\mathbb{Z}_2)^k \) in \( M^n \), where \( F^j \) denotes the \( j \)-dimensional fixed part of \( M(\mathbb{Z}_2)^k \), which consists of a disjoint union of all \( j \)-dimensional connected components, and if \( M(\mathbb{Z}_2)^k \) contains no \( j \)-dimensional fixed part, then \( F^j \) is chosen to be empty. Now if \( i \) is a power of 2, say \( 2^s \), since \( \binom{n}{i} \equiv 0 \mod 2 \) for any \( 0 < h < 2^s \), then it is easy to see that \( \Delta^i(\Phi, M^n) \) is equivariantly cobordant to a \((\mathbb{Z}_2)^k\)-action \((\Psi, N)\) having fixed point set \( \bigsqcup_{0 \leq j \leq n} F^j \), where \( F^j = \left\{ F^j \right\} \times \cdots \times F^j \).

This gives the following result.

**Lemma 4.2.** If \( i \) is a power of 2, then \( \Delta^i(\Phi, M^n) \) is equivariantly cobordant to a \((\mathbb{Z}_2)^k\)-action \((\Psi, N)\) whose fixed point set is the diagonal copy of \( M(\mathbb{Z}_2)^k \) contained in \( \prod_{0 \leq j \leq n} F^j \)
Remark 9. If \( i \) is a power of 2 and \( M^{(\mathbb{Z}_2)^k} \) is a finite set, then \( (\Psi, N) \) has the same number of fixed points as \( (\Phi, M^n) \).

4.2. \( \Omega \)-operation.

Definition 4.3. An \( \Omega \)-operation on \( (\Phi, M^n) \) means that \( \Omega \) maps \( (\Phi, M^n) \) into a \((\mathbb{Z}_2)^{k+1}\)-action on \( M^n \times M^n \), denoted by \( \Omega(\Phi, M^n) \), which is given by the diagonal action \( \Phi \times \Phi \) together with the involution swapping the factors of \( M^n \times M^n \). The fixed set of \( \Omega(\Phi, M^n) \) is the copy of the fixed set of \( M^n \) in the diagonal copy of \( M^n \) contained in \( M^n \times M^n \).

\( \Omega(\Phi, M^n) \) has many same properties as \( (\Phi, M^n) \). For example, the fixed point set of \( \Omega(\Phi, M^n) \) has the same dimension as that of \( (\Phi, M^n) \). Also, as noted in [31], if the fixed set of \( (\Phi, M^n) \) possesses the linear independence, then this also is so for \( \Omega(\Phi, M^n) \).

Applying the \( \Omega \)-operation \( l \) times to \( (\Phi, M^n) \) gives a \((\mathbb{Z}_2)^{k+l}\)-action denoted by \( \Omega^l(\Phi, M^n) \) such that \( \dim \Omega(\Phi, M^n) = 2^l n \), and its fixed point set is

\[
\{ (p, \ldots, p) \mid p \in M^{(\mathbb{Z}_2)^k} \}.
\]

Clearly, if \( (\Phi, M^n) \) has exactly a finite fixed set, then for each \( l \geq 1 \), \( \Omega^l(\Phi, M^n) \) has a finite fixed set, and in particular, \( \Omega^l(\Phi, M^n) \) has the same number of fixed points as \( (\Phi, M^n) \).

Lemma 4.4. Suppose that \( (\Phi, M^n) \) is a nonbounding \((\mathbb{Z}_2)^k\)-action fixing a finite set. Then for each \( l \geq 1 \), \( \Omega^l(\Phi, M^n) \) is nonbounding.

Proof. It suffices to show that for \( l = 1 \), \( \Omega^1(\Phi, M^n) \) is nonbounding. Let \( (\Gamma_{(\Phi, M^n)}, \alpha) \) be a colored graph of \((\Phi, M^n)\), and let \( p \) be a fixed point of \( (\Phi, M^n) \). Then its tangent representation at \( p \) is \( \alpha(E_p) \). Let \( \Lambda \) and \( \bar{\Lambda} \) be two subsets of \( \text{Hom}((\mathbb{Z}_2)^{k+1}, \mathbb{Z}_2) \) such that both \( \Lambda \) and \( \bar{\Lambda} \) are isomorphic to \( \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \) as \( \mathbb{Z}_2 \) vector spaces, and each \( \delta_\rho \) in \( \Lambda \) is \( \rho \in \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \) on \( (\mathbb{Z}_2)^k \) and 1 on the new \( \mathbb{Z}_2 \) generator \( t_{k+1} \). Now it is easy to see that the tangent representation at the corresponding fixed point \( (p, p) \) in \( \Omega(\Phi, M^n) \) is \( \{ \delta_{\rho}, \rho \in \alpha(E_p) \} \cup \{ \delta_{\rho}, \rho \in \alpha(E_p) \} \). Since \( (\Phi, M^n) \) has a nonempty prime tangent representation set, it is also so for \( \Omega(\Phi, M^n) \). Then the lemma follows from Theorem [32].

4.3. Automorphisms of \((\mathbb{Z}_2)^k\). Given an automorphism \( \sigma \in \text{GL}(k, \mathbb{Z}_2) \) of \((\mathbb{Z}_2)^k\), the action \((\mathbb{Z}_2)^k \times M^n \to M^n\) defined by

\[
(g, x) \mapsto \Phi(\sigma(g), x)
\]

is called the \( \sigma \)-induced action of \((\Phi, M^n)\), denoted by \((\sigma \Phi, M^n)\) or \(\sigma(\Phi, M^n)\). Note that \((\sigma \Phi, M^n)\) is weakly equivariantly homeomorphic to \((\Phi, M^n)\), but it may not be equivariantly cobordant to \((\Phi, M^n)\) since generally \( \sigma \) will change the tangent representation set of \((\Phi, M^n)\).

Remark 10. If \((\Phi, M^n)\) has exactly a finite fixed set, let \((\Gamma_{(\Phi, M^n)}, \alpha)\) is a colored graph of \((\Phi, M^n)\), then it is easy to see that \((\Gamma_{(\Phi, M^n)}, \sigma \alpha)\) is a colored graph of \((\sigma \Phi, M^n)\), where \(\sigma \alpha : E_{\Gamma_{(\Phi, M^n)}} \to \text{Hom}(G, \mathbb{Z}_2)\) is defined by

\[
\sigma \alpha(e)(g) = \alpha(e)(\sigma(g)).
\]
for \( e \in E_{\Gamma_{(\Phi,M^n)}} \) and \( g \in G \).

5. THE LOWER BOUND OF \(|M^G|\) AND EXAMPLES

Let \((\Phi, M^n)\) be a \(G\)-action in \(A_n^k\), and let \((\Gamma_{(\Phi,M)}, \alpha)\) be a colored graph associated to \((\Phi, M^n)\). Since \((\Phi, M^n)\) is assumed to be nonbounding, by Theorem 3.2 there must be some edges in \(\Gamma_{(\Phi,M)}\), each of which has different colorings at its two endpoints.

**Lemma 5.1.** Let \( e \in E_{\Gamma_{(\Phi,M)}} \) be an edge with two endpoints \( p \) and \( q \) such that \( \alpha(E_p) \neq \alpha(E_q) \). Then the number \(|E_e|\) is at most \( n - k + 1 \), where \( E_e \) denotes the set of all edges joining two endpoints \( p \) and \( q \).

**Proof.** Without loss of generality, one may assume that
\[
\alpha(E_p) = \{\beta_1, \beta_2, \ldots, \beta_t, \gamma_1, \ldots, \gamma_{n-t}\}
\]
and
\[
\alpha(E_q) = \{\beta_1, \beta_2, \ldots, \beta_t, \gamma'_1, \gamma'_2, \ldots, \gamma'_{n-t}\}
\]
where \(\beta_1, \beta_2, \ldots, \beta_t\) are the representations for the \(t\) edges joining \( p \) to other fixed points and at most \( n - 1 - t \) edges at \( G \), where \( s \leq n - t \leq k - 2 \). Since \(\alpha(E_p)\) spans \(\text{Hom}(G, \mathbb{Z}_2)\) by (P1) of Section 2 one can choose at least two of the \(\beta\)'s, say \(\beta_1\) and \(\beta_2\), so that the set \(\{\beta_1, \beta_2, \gamma_1, \ldots, \gamma_s\}\) is linearly independent.

Since \(\alpha(E_p) \neq \alpha(E_q)\), there must be some nonzero element \(\delta\) in \(\text{Hom}(G, \mathbb{Z}_2)\) which occurs more times in \(\alpha(E_p)\) than in \(\alpha(E_q)\). Since each \(\beta_i\) occurs the same number of times in both \(\alpha(E_p)\) and \(\alpha(E_q)\), one has that \(\delta \not\in \{\gamma_1, \ldots, \gamma_{n-t}\}\). Since the number of times \(\delta\) and \(\delta + \beta_1\) occurring in \(\alpha(E_p)\) is the same as the number of times \(\delta\) and \(\delta + \beta_1\) occurring in \(\alpha(E_q)\) by (P2) of Section 2 \(\delta + \beta_1\) must occur more times in \(\alpha(E_p)\) than in \(\alpha(E_q)\). Then \(\delta + \beta_1 + \beta_2\) must occur more times in \(\alpha(E_p)\) than in \(\alpha(E_q)\). Thus one has \(\delta + \beta_1 + \beta_2 \in \{\gamma_1, \ldots, \gamma_{n-t}\}\). Then \(\beta_1 + \beta_2\) is in the span of \(\{\gamma_1, \gamma_2, \ldots, \gamma_s\}\) contradicting the linear independence of \(\{\beta_1, \beta_2, \gamma_1, \ldots, \gamma_s\}\). \(\square\)

Now let us give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** By Lemma 3.1 we can replace \((\Phi,M)\) by \((\Psi,N)\) with \(N(\Psi,N) = \mathcal{P}_{(\Phi,M^n)}\) prime tangent representation set. This decreases the number of fixed points. Let \( p \) be one fixed point of \( N^G \). Then \( n \) edges at \( p \) of the graph \(\Gamma_{(\Psi,N)}\) would connect \( p \) to other fixed points and at most \( n - k + 1 \) edges can connect \( p \) to the same point by Lemma 5.1. Thus there must be at least \( \frac{n}{n-k+1} \) fixed points not equal to \( p \), and so the number of fixed points is at least \( 1 + \left[ \frac{n}{n-k+1} \right] \). \(\square\)

Next let us give two examples to show that the bound established in Theorem 1.2 is attainable in some special cases.

**Example 2.** For \( n = k \), one has that \( \frac{n}{n-k+1} = n = k \), so this says that the action has at least \( k+1 \) fixed points. Consider the standard linear \((\mathbb{Z}_2)^k\)-action \((\phi_k, \mathbb{R}P^k)\). This action fixes precisely \( k+1 \) fixed points.

**Example 3.** For \( k \geq 2 \), applying the 2-operation \( k - 2 \) times to \((\phi_2, \mathbb{R}P^2)\) gives a \((\mathbb{Z}_2)^k\)-action \(\Omega^{k-2}(\phi_2, \mathbb{R}P^2)\), which has 3 fixed points and dimension \( 2^{k-1} \). Then \( 1 + \left[ \frac{2^{k-1}}{2^{k-1}-k+1} \right] = 3 \) for all \( k \geq 2 \).
The following example illustrates that the bound established in Theorem 1.2 can be much smaller than the actual number of fixed points.

For $k = 2$, Conner and Floyd [12] found the equivariant cobordism classes of $(\mathbb{Z}_2)^2$-actions with finite fixed set. Begin with the standard linear $(\mathbb{Z}_2)^2$-action $(\varphi_2, \mathbb{RP}^2)$ having three fixed points, Conner and Floyd wrote its fixed data as a polynomial $\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1$ in $\mathbb{Z}_2[\rho_1, \rho_2, \rho_3]$. They then proved that every nonbounding $(\mathbb{Z}_2)^2$-action $(\Phi, M^n)$ having a finite fixed set is equivariantly cobordant to the $m$-multi-diagonal $(\mathbb{Z}_2)^2$-action $\Delta^m(\varphi_2, \mathbb{RP}^2)$ where $n = 2m$. Up to equivariant cobordism, the fixed data of this action can be written as $(\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1)^m$. The minimum number of fixed points is then the number of monomials $\rho_1^i \rho_2^j \rho_3^k$ in $(\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1)^m$ which have nonzero coefficient modulo 2. To find this number, let $m = 2^p_1 + \cdots + 2^p_r$ be the 2-adic expansion of $m$. Then

$$(\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1)^m \equiv (\rho_1^{2^p_1} \rho_2^{2^p_1} + \rho_2^{2^p_1} \rho_3^{2^p_1} + \rho_3^{2^p_1} \rho_1^{2^p_1}) \cdots (\rho_1^{2^p_r} \rho_2^{2^p_r} + \rho_2^{2^p_r} \rho_3^{2^p_r} + \rho_3^{2^p_r} \rho_1^{2^p_r}) \mod 2.$$

This product has $3^r$ monomials and the monomials are all distinct—for $p_i^1 p_j^1 p_k^1$ the 2-adic expansions of $i, j, k$ determine which factors were taken. This gives

**Lemma 5.2.** Let $(\Phi, M^n)$ be a nonbounding $(\mathbb{Z}_2)^2$-action in $\mathcal{A}_n^2$. Then $n$ is even and $(\Phi, M^n)$ has at least $3^r$ fixed points where $r$ is the number of terms in the 2-adic expansion of $n$.

Of course, $1 + \left\lceil \frac{n}{2} \right\rceil = 3$, so the bound in Theorem 1.2 is now precise exactly for $n = 2^s$.

**Remark 11.** The reason why the bound established in Theorem 1.2 can be much smaller than the actual number of fixed points is because the proof of Theorem 1.2 is only a local analysis for the colored graph.

### 6. Actions with three fixed points

#### 6.1. The existence and the equivariant cobordism classification of actions in $\mathcal{A}_n^k(3)$.

Suppose that $\mathcal{A}_n^k(3)$ is nonempty. Take a $G$-action $(\Phi, M^n)$ in $\mathcal{A}_n^k(3)$, without the loss of generality, assume that $(\Phi, M^n)$ has exactly three fixed points $p, q, r$. Let $(\Gamma_{(\Phi, M^n)}, \alpha)$ be a colored graph of $(\Phi, M^n)$.

**Lemma 6.1.** $\Gamma_{(\Phi, M^n)}$ is uniquely determined, and $\mathcal{N}_{(\Phi, M^n)} = \{\alpha(E_p), \alpha(E_q), \alpha(E_r)\}$ is prime.

**Proof.** Let $p, q$ be connected by $a$ edges; $p, r$ by $b$ edges; and $q, r$ by $c$ edges. Then

\[
\begin{align*}
    a + b &= n \\
    b + c &= n \\
    a + c &= n
\end{align*}
\]

so $n$ must be even and $a = b = c = \frac{n}{2}$. This means that $(\Phi, M^n)$ determines a unique graph $\Gamma_{(\Phi, M^n)}$.

It is obvious that $\mathcal{N}_{(\Phi, M^n)} = \{\alpha(E_p), \alpha(E_q), \alpha(E_r)\}$ is prime since any $(\mathbb{Z}_2)^k$-action can not fix a single isolated point.  \[\square\]
By Lemma 6.1, one can write
\[ \alpha(E_\rho) = \hat{\beta} \cup \hat{\gamma}, \quad \alpha(E_\delta) = \hat{\delta} \cup \hat{\delta}, \quad \alpha(E_v) = \hat{\delta} \cup \hat{\gamma} \]
where \( \hat{\beta}, \hat{\gamma}, \hat{\delta} \) are three multisets formed by elements of \( \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \) with \( |\hat{\beta}| = |\hat{\gamma}| = |\hat{\delta}| = n/2 \).

**Lemma 6.2.** For any \( \beta \in \hat{\beta}, \gamma \in \hat{\gamma}, \text{and} \ \delta \in \hat{\delta}, \)
\[ \beta + \gamma \in \hat{\delta}, \quad \gamma + \delta \in \hat{\beta}, \quad \beta + \delta \in \hat{\gamma}. \]

**Proof.** Consider an irreducible nontrivial representation \( \rho \) and a fixed component \( C \) of \( \ker \rho \) acting on \( M^n \). Then \((\mathbb{Z}_2)^k/\ker \rho \cong \mathbb{Z}_2\) fixes an even number of points of \( C \). If \( C \) has a fixed point, then it has exactly two fixed points, so \( C \) exactly determines a unique \( \dim C \)-valent regular graph \( \Gamma_{p,C} \) with those two fixed points as its vertices. This means that \( \hat{\beta}, \hat{\gamma}, \hat{\delta} \) are all disjoint. Further, the lemma follows from the property (P2) in Section 2.

**Lemma 6.3.** There exists an integer \( m \geq 1 \) and a basis \( \{\beta_1, \ldots, \beta_{k-1}, \gamma\} \) of \( \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \) such that
\[
\begin{align*}
(i) & \quad n = m \cdot 2^{k-1}; \\
(ii) & \quad \hat{\beta} \text{ is a multiset consisting of all odd sums with multiplicity } m \text{ formed by } \beta_1, \ldots, \beta_{k-1}; \\
(iii) & \quad \hat{\gamma} = \{\gamma + \beta_1 + \beta | \beta \in \hat{\beta}\}; \\
(iv) & \quad \hat{\delta} = \{\gamma + \beta | \beta \in \hat{\beta}\}
\end{align*}
\]
where an odd sum formed by \( \beta_1, \ldots, \beta_{k-1} \) means the sum of an odd number of elements in \( \beta_1, \ldots, \beta_{k-1} \).

**Proof.** Choose one \( \gamma \in \hat{\gamma} \). Then \( \hat{\delta} = \{\gamma + \beta | \beta \in \hat{\beta}\} \) by Lemma 6.2. Choosing any \( \beta' \in \hat{\beta} \), one has \( \hat{\gamma} = \{\gamma + \beta + \beta' | \beta \in \hat{\beta}\} \) by Lemma 6.2 again. Similarly, \( \hat{\delta} = \{\delta + \beta + \beta' | \beta \in \hat{\beta}\} \) for any \( \delta \in \hat{\delta} \).

is spanned by elements \( \beta \) of \( \hat{\beta} \) and \( \gamma = \gamma + \beta + \beta' \). Since \( \hat{\beta} \cup \hat{\gamma} \) spans \( \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \) by (P1) in Section 2, one has that at least \( k-1 \) elements of \( \hat{\beta} \) must be linearly independent, so \( n/2 \geq k-1 \).

**Claim I.** Every sum of an odd number of elements of \( \hat{\beta} \) is again an element of \( \hat{\beta} \) and no sum of an even number of elements of \( \hat{\beta} \) is again in \( \hat{\beta} \). The elements \( \gamma + \text{ even sum of } \beta \)'s all belong to \( \hat{\gamma} \) and the elements \( \gamma + \text{ odd sum of } \beta \)'s all belong to \( \hat{\delta} \).

In fact, one has from the above argument that \( \hat{\beta} = \{\gamma + \beta | \beta \in \hat{\beta}\} \) and \( \hat{\delta} = \{\delta + \beta + \beta' | \beta \in \hat{\beta}\} \) for any \( \delta \in \hat{\delta} \) and any \( \beta' \in \hat{\beta} \). Thus, for any \( \beta'' \in \hat{\beta} \), taking \( \delta = \gamma + \beta'' \) gives
\[ \hat{\beta} = \{\beta | \beta \in \hat{\beta}\} = \{\beta + \beta' + \beta'' | \beta \in \hat{\beta}\}. \]
Replacing \( \beta \) by \( \beta + \beta' + \beta'' \) and repeating this procedure, one has that every sum of an odd number of elements of \( \hat{\beta} \) is again an element of \( \hat{\beta} \). Furthermore, by Lemma 6.2 one has that the elements \( \gamma + \text{ odd sum of } \beta \)'s all belong to \( \hat{\delta} \), the elements \( \gamma + \text{ even sum of } \beta \)'s all belong to \( \hat{\gamma} \), and any sum of an even number of elements of \( \hat{\beta} \) is not in \( \hat{\beta} \).

**Claim II.** There are exactly \( k-1 \) elements of the \( \beta \)'s which are linearly independent.
Actually, one knows from Claim I that an odd sum of the $\beta$'s is again in $\hat{\beta}$ and if $\gamma \in \hat{\gamma}$ is an even sum of the $\beta$'s, then any $\delta$ in $\hat{\delta}$ is an odd sum of the $\beta$'s, so one has that $\delta \cap \hat{\beta} \neq \emptyset$. But this is impossible. Thus, $\gamma \in \hat{\gamma}$ cannot be a linear combination of elements of $\hat{\beta}$, so exactly $k-1$ elements of the $\beta$'s are linearly independent, say $\beta_1, \ldots, \beta_{k-1}$. In particular, $\beta_1, \ldots, \beta_{k-1}, \gamma$ form a basis of $\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$.

Now let $V$ be the vector space spanned by the elements of $\hat{\beta}$. Then, by Claims I and II, $V$ is a $(k-1)$-dimensional vector space over $\mathbb{Z}_2$ and $\beta_1, \ldots, \beta_{k-1}$ form a basis of $V$. The set of elements of $V$ belonging to $\hat{\beta}$ must then be the set of the sums of an odd number of the elements $\beta_1, \ldots, \beta_{k-1}$ by Claim I. Thus one has

**Claim III.** $\hat{\beta}$ contains $2^{k-2}$ different elements.

Since any two odd sums of the elements of $\hat{\beta}$ differ by an even sum, and adding an even sum permutes the elements of $\hat{\beta}$, the elements of $\hat{\beta}$ must occur with the same multiplicity. Since $\hat{\beta}$ contains $2^{k-2}$ different elements, one has that $n/2 = m \cdot 2^{k-2}$ so $n = m \cdot 2^{k-1}$, where $m$ is the common multiplicity. This gives

**Claim IV.** All elements of $\hat{\beta}$ occur with the same multiplicity $m$ so $n = m \cdot 2^{k-1}$.

Combining the above arguments completes the proof of the lemma. $\square$

We see from Lemma 6.3 that the tangent representation set $N_{(\Phi,M^n)} = \{\alpha(E_p), \alpha(E_q), \alpha(E_r)\}$ is uniquely determined by some basis of $\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$. On the other hand, any basis of $\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ can be translated into a given basis by an automorphism of $(\mathbb{Z}_2)^k$. Thus, by Theorem 3.2 and Remark 11 it follows that

**Proposition 6.4.** Let $(\Psi,N^n)$ be another $(\mathbb{Z}_2)^k$-action in $A^k_n(3)$. Then there is one $\sigma \in \text{GL}(k, \mathbb{Z}_2)$ such that $(\Phi,M^n)$ is equivariantly cobordant to $(\sigma \Psi, N^n)$.

Next, let us look at the case $m = 1$.

When $m = 1$, one has that $n = 2^{k-1}$. In this case, Example 3 in Section 5 provides an example $\Omega^{k-2}(\phi_2, \mathbb{R}P^2)$ which fixes exactly three isolated points, so by Proposition 6.4 any $(\mathbb{Z}_2)^k$-action in $A^k_{2k-1}(3)$ is equivariantly cobordant to the $(\mathbb{Z}_2)^k$-action obtained by applying an automorphism of $(\mathbb{Z}_2)^k$ to $\Omega^{k-2}(\phi_2, \mathbb{R}P^2)$ to switch representations around. This gives

**Corollary 6.5.** $A^k_{2k-1}(3)$ is nonempty. Furthermore, each of $A^k_{2k-1}(3)$ is equivariantly cobordant to one of $\sigma \Omega^{k-2}(\phi_2, \mathbb{R}P^2)$, $\sigma \in \text{GL}(k, \mathbb{Z}_2)$.

**Remark 12.** For $k = 2$, it is easy to see that up to equivariant cobordism, there is a unique $(\mathbb{Z}_2)^2$-action in $A^2_2(3)$, which is exactly $\Omega^0(\phi_2, \mathbb{R}P^2) = (\phi_2, \mathbb{R}P^2)$. However, for $k > 2$, up to equivariant cobordism there may be more $(\mathbb{Z}_2)^k$-actions in $A^k_{2k-1}(3)$.
For example, for \( k = 3 \), up to equivariant cobordism there are seven different \((\mathbb{Z}_2)^3\)-actions in \(A_3^2(3)\), whose tangent representation sets are listed as follows:

| \( i \) | tangent representation set \( N_i \) |
|---|---|
| 1 | \{\{\rho_1, \rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}, \{\rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}\} |
| 2 | \{\{\rho_1, \rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}, \{\rho_1, \rho_2, \rho_1 + \rho_3, \rho_2 + \rho_3\}\} |
| 3 | \{\{\rho_1, \rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}, \{\rho_1, \rho_2, \rho_1 + \rho_3, \rho_2 + \rho_3\}\} |
| 4 | \{\{\rho_1, \rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}, \{\rho_1, \rho_2, \rho_1 + \rho_3, \rho_2 + \rho_3\}\} |
| 5 | \{\{\rho_1, \rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}, \{\rho_1, \rho_3, \rho_1 + \rho_2 + \rho_3\}\} |
| 6 | \{\{\rho_1, \rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}, \{\rho_1, \rho_3, \rho_1 + \rho_2 + \rho_3\}\} |
| 7 | \{\{\rho_1, \rho_2, \rho_3, \rho_1 + \rho_2 + \rho_3\}, \{\rho_1, \rho_3, \rho_1 + \rho_2 + \rho_3\}\} |

where \{\rho_1, \rho_2, \rho_3\} is the standard basis of \(\text{Hom}(\mathbb{Z}_2^3, \mathbb{Z}_2)\) as stated in Section 2.

Now let us look at the general case \( m \geq 1 \).

**Lemma 6.6.** Let \( m \geq 1 \). Then \( m \) is a power of \( 2 \).

**Proof.** By Lemma 6.3, \( \tilde{\beta} \) contains \( 2^{k-2} \) different elements (say \( \tilde{\beta}_1, ..., \tilde{\beta}_{2^{k-2}} \)) with same multiplicity \( m \), which consist of all odd sums formed by \( \beta_1, ..., \beta_{2^{k-1}} \). Without loss of generality, assume that \( \tilde{\beta}_1 = \beta_1 \). Applying Theorem 5.3 we use the symmetric function \( f = 1 \) to deduce that

\[
\frac{1}{(\prod_{i=1}^{2^{k-2}} \tilde{\beta}_i)^m (\prod_{i=1}^{2^{k-2}} (\gamma + \tilde{\beta}_i))^m} + \frac{1}{(\prod_{i=1}^{2^{k-2}} (\gamma + \tilde{\beta}_i))^m (\prod_{i=1}^{2^{k-2}} (\gamma + \beta_1)^i)^m} + \frac{1}{(\prod_{i=1}^{2^{k-2}} (\gamma + \beta_1)^i)^m (\prod_{i=1}^{2^{k-2}} (\gamma + \tilde{\beta}_i))^m}
\]

must belong to \( \mathbb{Z}_2[\beta_1, ..., \beta_k] \). Taking the common denominator, one has that

\[
(\prod_{i=1}^{2^{k-2}} \tilde{\beta}_i)^m + (\prod_{i=1}^{2^{k-2}} (\gamma + \tilde{\beta}_i))^m + (\prod_{i=1}^{2^{k-2}} (\gamma + \beta_1 + \tilde{\beta}_i))^m
\]

must belong to \( \mathbb{Z}_2[\beta_1, ..., \beta_k] \). Since the numerator has smaller degree than the denominator, this is only possible if the numerator is zero. Thus, one has

\[
(\prod_{i=1}^{2^{k-2}} \tilde{\beta}_i)^m + (\prod_{i=1}^{2^{k-2}} (\gamma + \tilde{\beta}_i))^m + (\prod_{i=1}^{2^{k-2}} (\gamma + \beta_1 + \tilde{\beta}_i))^m = 0.
\]

By Lemma 6.3 and Corollary 5.3 when \( m = 1 \), the above expression still holds, i.e.,

\[
\prod_{i=1}^{2^{k-2}} \tilde{\beta}_i + \prod_{i=1}^{2^{k-2}} (\gamma + \tilde{\beta}_i) + \prod_{i=1}^{2^{k-2}} (\gamma + \beta_1 + \tilde{\beta}_i) = 0.
\]
Furthermore, one has
\[
\prod_{i=1}^{2^{k-2}} (\gamma + \bar{\beta}_i) = \prod_{i=1}^{2^{k-2}} (\gamma + \beta_1 + \bar{\beta}_i) + \prod_{i=1}^{2^{k-2}} \bar{\beta}_i = \gamma \prod_{i \neq 1} (\gamma + \beta_1 + \bar{\beta}_i) + \prod_{i=1}^{2^{k-2}} \bar{\beta}_i
\]
and then
\[
\begin{align*}
(\prod_{i=1}^{2^{k-2}} \bar{\beta}_i)^m + (\prod_{i=1}^{2^{k-2}} (\gamma + \beta_1 + \bar{\beta}_i))^m + (\prod_{i=1}^{2^{k-2}} (\gamma + \beta_1 + \bar{\beta}_i))^m \\
= (\prod_{i=1}^{2^{k-2}} \bar{\beta}_i)^m + \left\{ \gamma \prod_{i \neq 1} (\gamma + \beta_1 + \bar{\beta}_i) + \prod_{i=1}^{2^{k-2}} \bar{\beta}_i \right\}^m + \left\{ \gamma \prod_{i \neq 1} (\gamma + \beta_1 + \bar{\beta}_i) \right\}^m \\
= \sum_{0 < j < m} \binom{m}{j} \left( \gamma \prod_{i \neq 1} (\gamma + \beta_1 + \bar{\beta}_i) \right)^j \left( \prod_{i=1}^{2^{k-2}} \bar{\beta}_i \right)^{m-j}.
\end{align*}
\]
If \(m\) is not a power of 2, then there is a largest \(j\), \(0 < j < m\), with \(\binom{m}{j} \neq 0\) and this sum has a nonzero coefficient for
\[
(\gamma^{2^{k-2}})^j \left( \prod_{i=1}^{2^{k-2}} \bar{\beta}_i \right)^{m-j}.
\]
This is a contradiction. Thus \(m\) must be a power of 2. \(\Box\)

Now, let us complete the proof of Theorem 1.4.

**Proof of Theorem 1.4** If \(A_n^k(3)\) is nonempty, then we may choose a \(G\)-action \((\Phi, M^n)\) in \(A_n^k(3)\). Without loss of generality, assume that \((\Phi, M^n)\) exactly fixes three isolated points. By Lemmas 6.3 and 6.6 one has that \(k \geq 2\) and \(n = 2^\ell\) for some \(\ell \geq k-1\). Conversely, let \(n = 2^\ell\) with \(\ell \geq k-1 \geq 1\). Then the diagonal action on \(2^{\ell-k+1}\) copies of \(\Omega^{k-2}(\phi_2, \mathbb{R}P^2)\) gives a \((\mathbb{Z}_2)^k\)-action \(\Delta^{2^{\ell-k+1}} \Omega^{k-2}(\phi_2, \mathbb{R}P^2)\), which has dimension \(2^\ell\) and fixes \(3^{2^{\ell-k+1}}\) isolated points. However, by Lemma 4.2, this action is actually cobordant to the action with exactly three fixed points. Thus, \(A_n^k(3)\) is nonempty. This completes the proof of Theorem 1.4(a).

Now if \(A_n^k(3)\) is nonempty, then \(n = 2^\ell\) with \(\ell \geq k-1 \geq 1\). Furthermore, by Proposition 6.4, each action of \(A_n^k(3)\) is equivariantly cobordant to one of
\[
\sigma \Delta^{2^{\ell-k+1}} \Omega^{k-2}(\phi_2, \mathbb{R}P^2), \quad \sigma \in \text{GL}(k, \mathbb{Z}_2).
\]
\(\Box\)

6.2. The characterization of the colored graphs–Proof of Theorem 1.6

Let \((\Gamma, \alpha)\) be an abstract 1-skeleton of type \((n, k)\) with exactly three vertices \(p, q, r\).

If \((\Gamma, \alpha)\) is a colored graph of some action \((\Phi, M^n)\) in \(A_n^k(3)\), then by Lemmas 6.3 and 6.6 the necessity of Theorem 1.6 holds.

Conversely, suppose that \((\Gamma, \alpha)\) satisfies the conditions (a) and (b) of Theorem 1.6. Then it is easy to see that \(\Gamma\) is uniquely determined. Moreover, to prove that \((\Gamma, \alpha)\) is a colored graph of some action \((\Phi, M^n)\) in \(A_n^k(3)\), it suffices to show that \(\{\alpha(E_p), \alpha(E_q), \alpha(E_r)\}\) is the fixed data of some action \((\Phi, M^n)\) in \(A_n^k(3)\). By the construction of \(\hat{\beta}, \hat{\gamma}, \hat{\delta}\) in Theorem 1.6(b), we see that \(\hat{\beta}\) exactly contains \(2^{k-2}\) different elements (which consist of all odd sums formed by \(\beta_1, \ldots, \beta_{k-1}\), and so
are \( \hat{\gamma} \) and \( \hat{\delta} \). By \( \hat{\beta}' \) (resp. \( \hat{\gamma}', \hat{\delta}' \)) we denote the set formed by those \( 2^k-2 \) different elements in \( \hat{\beta} \) (resp. \( \hat{\gamma}, \hat{\delta} \)). Then we have that \( \hat{\gamma}' = \{ \gamma + \beta_1 + \beta | \beta \in \hat{\beta}' \} \) and \( \hat{\delta}' = \{ \gamma + \beta | \beta \in \hat{\beta}' \} \). Obviously, \( \hat{\beta}' \), \( \hat{\gamma}' \), \( \hat{\delta}' \) are uniquely determined by the basis \( \{ \beta_1, ..., \beta_{k-1}, \gamma \} \).

**Claim.** \( \{ \hat{\beta}' \cup \hat{\gamma}', \hat{\beta}' \cup \hat{\delta}', \hat{\delta}' \cup \hat{\gamma}' \} \) is the fixed data of some action in \( \mathcal{A}_{k-1}^l(3) \).

We proceed by induction on \( k \). When \( k = 2 \), we have that \( \hat{\beta}' = \{ \beta_1, \gamma \} \), \( \hat{\gamma}' = \{ \gamma \} \), and \( \hat{\delta}' = \{ \gamma + \beta_1 \} \). Since \( \text{Hom}(\mathbb{Z}_2^2, \mathbb{Z}_2) \) contains only three nontrivial elements, without the loss of generality one may assume that \( \beta_1 = \rho_1 \) and \( \gamma = \rho_2 \). Then \( \hat{\beta}' \cup \hat{\gamma}' = \{ \rho_1, \rho_2 \} \), \( \hat{\beta}' \cup \hat{\delta}' = \{ \rho_1, \rho_1 + \rho_2 \} \), and \( \hat{\delta}' \cup \hat{\gamma}' = \{ \rho_1 + \rho_2, \rho_2 \} \) are exactly the fixed data of the standard linear \( (\mathbb{Z}_2)^2 \)-action on \( (\phi_2, \mathbb{R}P^2) \). When \( k = l \geq 2 \), suppose inductively that \( \hat{\beta}' \cup \hat{\gamma}', \hat{\beta}' \cup \hat{\delta}', \hat{\delta}' \cup \hat{\gamma}' \) are the fixed data of some \( (\mathbb{Z}_2)^l \)-action in \( \mathcal{A}_{k-1}^l(3) \).

When \( k = l + 1 \), let \( \hat{\beta}'_1 \) denote the set of all odd sums formed by \( \beta_1, ..., \beta_{l-1} \). Then \( \hat{\beta}'_1 \subset \hat{\beta}' \) and \( \hat{\beta}'_1 \) contains \( 2^l-2 \) different elements. Let \( \hat{\beta}'_2 \) denote the set formed by all elements \( \beta_1 + \beta, \beta, \beta \in \hat{\beta}'_1 \). Then \( \hat{\beta}'_2 \subset \hat{\beta}' \) and \( \hat{\beta}'_2 \) contains \( 2^{l-2} \) different elements, too. Since \( \beta_1, ..., \beta_l \) are linearly independent, one has that the intersection of \( \hat{\beta}'_1 \) and \( \hat{\beta}'_2 \) is empty and \( \hat{\beta}' = \hat{\beta}'_1 \cup \hat{\beta}'_2 \). Then one has

\[
\hat{\gamma}' = \hat{\gamma}'_1 \cup \hat{\gamma}'_2 \quad \text{with} \quad \hat{\gamma}'_1 \cap \hat{\gamma}'_2 = \emptyset
\]

where \( \hat{\gamma}'_1 = \{ \gamma + \beta_1 + \beta | \beta \in \hat{\beta}'_1 \} \) and \( \hat{\gamma}'_2 = \{ \gamma + \beta_1 + \beta | \beta \in \hat{\beta}'_2 \} \), and

\[
\hat{\delta}' = \hat{\delta}'_1 \cup \hat{\delta}'_2 \quad \text{with} \quad \hat{\delta}'_1 \cap \hat{\delta}'_2 = \emptyset
\]

where \( \hat{\delta}'_1 = \{ \gamma + \beta | \beta \in \hat{\beta}'_1 \} \) and \( \hat{\delta}'_2 = \{ \gamma + \beta | \beta \in \hat{\beta}'_2 \} \). Now let us look at \( \hat{\beta}'_1, \hat{\gamma}'_1, \) and \( \hat{\delta}'_1 \). Clearly \( \hat{\beta}'_1, \hat{\gamma}'_1, \) and \( \hat{\delta}'_1 \) are exactly formed by \( \beta_1, ..., \beta_{l-1}, \gamma \). One sees that \( \beta_1, ..., \beta_{l-1}, \gamma \) span a \( l \)-dimensional subspace of \( \text{Hom}(\mathbb{Z}_2^{l+1}, \mathbb{Z}_2) \) which is isomorphic to \( \text{Hom}(\mathbb{Z}_2^l, \mathbb{Z}_2) \). Now, regarding \( \{ \beta_1, ..., \beta_{l-1}, \gamma \} \) as a basis of \( \text{Hom}(\mathbb{Z}_2^l, \mathbb{Z}_2) \), one has by induction that

\[
\begin{align*}
\hat{\beta}'_1 \cup \hat{\gamma}'_1, \quad \hat{\beta}'_1 \cup \hat{\delta}'_1, \quad \hat{\delta}'_1 \cup \hat{\gamma}'_1
\end{align*}
\]

are the fixed data of some \( (\mathbb{Z}_2)^l \)-action, denoted by \( (\Psi, N^{2^{-1}}) \). Then by applying \( \Omega \)-operation to \( (\Psi, N^{2^{-1}}) \), as in the proof of Lemma 1.4 (see also [31], Lemma 4.1), the fixed data of \( \Omega(\Psi, N^{2^{-1}}) \) exactly consists of

\[
\begin{align*}
\hat{\beta}' \cup \hat{\gamma}' &= \hat{\beta}'_1 \cup \hat{\beta}'_2 \cup \hat{\gamma}'_1 \cup \hat{\gamma}'_2 \\
\hat{\beta}' \cup \hat{\delta}' &= \hat{\beta}'_1 \cup \hat{\beta}'_2 \cup \hat{\delta}'_1 \cup \hat{\delta}'_2 \\
\hat{\delta}' \cup \hat{\gamma}' &= \hat{\delta}'_1 \cup \hat{\delta}'_2 \cup \hat{\gamma}'_1 \cup \hat{\gamma}'_2.
\end{align*}
\]

This completes the induction and the proof of the claim.

Now by Corollary 6.5, there is an automorphism \( \sigma \in \text{GL}(k, \mathbb{Z}_2) \) such that \( \{ \hat{\beta}' \cup \hat{\gamma}', \hat{\beta}' \cup \hat{\delta}', \hat{\delta}' \cup \hat{\gamma}' \} \) is the fixed data of \( \sigma \Omega^{k-2}(\phi_2, \mathbb{R}P^2) \). Moreover, applying \( \Delta \)-operation \( 2^{l-k+1} \) times to \( \sigma \Omega^{k-2}(\phi_2, \mathbb{R}P^2) \) gives an action \( \Delta^{2^{l-k+1}} \sigma \Omega^{k-2}(\phi_2, \mathbb{R}P^2) \) in \( \mathcal{A}_k^l(3) \). By Lemma 1.2, \( \Delta^{2^{l-k+1}} \sigma \Omega^{k-2}(\phi_2, \mathbb{R}P^2) \) is equivariantly cobordant to a \( (\mathbb{Z}_2)^k \)-action such that its fixed data is exactly \( \{ \alpha(E_p), \alpha(E_q), \alpha(E_r) \} \). This completes the proof of Theorem 1.6.
7. Examples of actions with four fixed points

This section is to show how to obtain new \((\mathbb{Z}_2)^3\)-actions from \((\mathbb{Z}_2)^3\)-action \((\phi_3, \mathbb{R}P^3)\), which will play an important role in the study of the general case with four fixed points.

Begin with the standard linear \((\mathbb{Z}_2)^3\)-action \((\phi_3, \mathbb{R}P^3)\) with four fixed points \(p = [1, 0, 0, 0], q = [0, 1, 0, 0], r = [0, 0, 1, 0],\) and \(s = [0, 0, 0, 1]\). One can easily read off the tangent representations at four fixed points, and then its colored graph \((\Gamma_{(\phi_3, \mathbb{R}P^3)}, \alpha)\) can explicitly be shown in Figure 4:

\[
\begin{array}{c}
p \quad \rho_1 \\
\rho_2 \\
\rho_3 \\
\rho_1 + \rho_2 \\
s \quad \rho_1 + \rho_2 \\
\rho_2 + \rho_3 \\
\rho_1 + \rho_3 \\
\end{array}
\]

Figure 4. The colored graph \((\Gamma_{(\phi_3, \mathbb{R}P^3)}, \alpha)\) of \((\phi_3, \mathbb{R}P^3)\)

Our approach to obtain new \((\mathbb{Z}_2)^3\)-actions is to modify \((\Gamma_{(\phi_3, \mathbb{R}P^3)}, \alpha)\) into abstract 1-skeleta by adding colored edges on \((\Gamma_{(\phi_3, \mathbb{R}P^3)}, \alpha)\), and then to use the abstract 1-skeleta to prove the existence of required new \((\mathbb{Z}_2)^3\)-actions. We shall see that only 4- and 5-dimensional new \((\mathbb{Z}_2)^3\)-actions can be obtained in such a way.

7.1. 4-dimensional case. Clearly, by adding two edges to \(\Gamma_{(\phi_3, \mathbb{R}P^3)}\) we can produce a unique connected regular graph of valence 4 (up to combinatorial equivalence), denoted by \(\Gamma\), shown in the following figure.

The connected regular graph \(\Gamma\) of valence 4

Now by adding two colored edges with same coloring \(\rho_1 + \rho_2 + \rho_3\) into \((\Gamma_{(\phi_3, \mathbb{R}P^3)}, \alpha)\), one can obtain three abstract 1-skeleta \((\Gamma, \alpha_1), (\Gamma, \alpha_2), (\Gamma, \alpha_3)\) of type \((4, 3)\), as shown in Figure 5. Obviously, these three abstract 1-skeleta \((\Gamma, \alpha_1), (\Gamma, \alpha_2), (\Gamma, \alpha_3)\) have the same vertex-coloring set, and in particular, they exactly give all possible abstract 1-skeleta with such a vertex-coloring set.

Lemma 7.1. There exists a \((\mathbb{Z}_2)^3\)-action \((\Upsilon, M^4)\) on a 4-dimensional closed manifold \(M^4\) with four fixed points such that its colored graph \((\Gamma_{(\Upsilon, M^4)}, \alpha)\) is one of three abstract 1-skeleta \((\Gamma, \alpha_1), (\Gamma, \alpha_2), (\Gamma, \alpha_3)\).

Proof. If there is a \((\mathbb{Z}_2)^3\)-action on a 4-dimensional closed manifold such that its colored graph has the same vertex-coloring set as those three abstract 1-skeleta \((\Gamma, \alpha_1), (\Gamma, \alpha_2), (\Gamma, \alpha_3)\), then it must be one of \((\Gamma, \alpha_1), (\Gamma, \alpha_2), (\Gamma, \alpha_3)\). Thus, to complete the proof, it suffices to show that the vertex-coloring set

\[\{\alpha_1(E_p), \alpha_1(E_q), \alpha_1(E_r), \alpha_1(E_s)\}\]
of \((\Gamma, \alpha_1)\) is the fixed data of some \((\mathbb{Z}_2)^3\)-action on a closed 4-manifold. By Theorem 3.3, this is equivalent to showing that for any symmetric function \(f(x_1, x_2, x_3, x_4)\) of four variables over \(\mathbb{Z}_2\)

\[
\hat{f} = \frac{f(\alpha_1(E_p))}{(p_1 + p_2 + p_3) p_1 p_2 p_3} + \frac{f(\alpha_1(E_q))}{(p_1 + p_2 + p_3) p_1 (p_1 + p_2) p_3}
\]

\[
+ \frac{f(\alpha_1(E_r))}{(p_1 + p_2 + p_3) p_2 p_1 (p_2 + p_3)}
\]

belongs to \(\mathbb{Z}_2[p_1, p_2, p_3]\). One may write \(f(x_1, x_2, x_3, x_4) = \sum_{i=0}^j x_i f_i(x_2, x_3, x_4)\) to obtain

\[
\hat{f} = \frac{1}{(p_1 + p_2 + p_3)} \left\{ \frac{f_0(p_1, p_2, p_3)}{p_1 p_2 p_3} + \frac{f_0(p_1, p_1 + p_2, p_1 + p_3)}{p_1 (p_1 + p_2) (p_1 + p_3)} \right. \\
+ \left. \frac{f_0(p_2, p_1 + p_2, p_2 + p_3)}{p_2 (p_1 + p_2) (p_2 + p_3)} \right. \\
+ \sum_{i=1}^j (p_1 + p_2 + p_3)^{i-1} \left\{ \frac{f_i(p_1, p_2, p_3)}{p_1 p_2 p_3} + \frac{f_i(p_1, p_1 + p_2, p_1 + p_3)}{p_1 (p_1 + p_2) (p_1 + p_3)} \right. \\
+ \left. \frac{f_i(p_2, p_1 + p_2, p_2 + p_3)}{p_2 (p_1 + p_2) (p_2 + p_3)} \right. \\
+ \sum_{i=1}^j (p_1 + p_2 + p_3)^{i-1} \left\{ \frac{f_i(p_1, p_2, p_3)}{p_1 p_2 p_3} + \frac{f_i(p_1, p_1 + p_2, p_1 + p_3)}{p_1 (p_1 + p_2) (p_1 + p_3)} \right. \\
+ \left. \frac{f_i(p_2, p_1 + p_2, p_2 + p_3)}{p_2 (p_1 + p_2) (p_2 + p_3)} \right\}
\]

Note that for \(0 \leq i \leq j\), \(f_i\) is symmetric. Then, from the colored graph \((\Gamma_{(\phi_3, \mathbb{R}P^3)}, \alpha)\) of the action \((\phi_3, \mathbb{R}P^3)\), one sees that for \(0 \leq i \leq j\), each term

\[
\hat{f}_i = \frac{f_i(p_1, p_2, p_3)}{p_1 p_2 p_3} + \frac{f_i(p_1, p_1 + p_2, p_1 + p_3)}{p_1 (p_1 + p_2) (p_1 + p_3)} + \frac{f_i(p_2, p_1 + p_2, p_2 + p_3)}{p_2 (p_1 + p_2) (p_2 + p_3)}
\]

belongs to \(\mathbb{Z}_2[p_1, p_2, p_3]\) by Theorem 3.3.

Clearly, \(\hat{f}\) belongs to \(\mathbb{Z}_2[p_1, p_2, p_3]\) if and only if \(\hat{f}_0\) is divisible by \(p_1 + p_2 + p_3\). So, to complete the proof, it remains to show that \(\hat{f}_0\) is divisible by \(p_1 + p_2 + p_3\).
To check that this is true, we put $\rho_1 + \rho_2 + \rho_3 = 0$ and then

$$
\hat{f}_0 = \frac{f_0(\rho_1, \rho_2, \rho_3)}{\rho_1 \rho_2 \rho_3} + \frac{f_0(\rho_1, \rho_1 + \rho_2, \rho_1 + \rho_3)}{\rho_1 (\rho_1 + \rho_2) (\rho_1 + \rho_3)} + \frac{f_0(\rho_1, \rho_1 + \rho_2, \rho_1 + \rho_3)}{\rho_2 (\rho_1 + \rho_2) (\rho_2 + \rho_3)}
$$

which is zero since $f_0$ is symmetric. This means that $\hat{f}_0$ is divisible by $\rho_1 + \rho_2 + \rho_3$.

\[\square\]

Remark 13. It is easy to see that any two of $(\Gamma, \alpha_1), (\Gamma, \alpha_2), (\Gamma, \alpha_3)$ can be translated to each other by automorphisms of $\text{Hom}((\mathbb{Z}_2)^3, \mathbb{Z}_2)$, i.e., for any $\alpha_i$ and $\alpha_j$, there is an automorphism $\theta$ such that the following diagram commutes.

Thus, each of $(\Gamma, \alpha_1), (\Gamma, \alpha_2), (\Gamma, \alpha_3)$ can become a colored graph of some $(\mathbb{Z}_2)^3$-action.

7.2. 5-dimensional case. An easy observation shows that up to combinatorial equivalence, by adding four edges to $\Gamma_{(\phi_3, \mathcal{R}P^3)}$ we can merely produce two regular graphs $\Gamma_1, \Gamma_2$ of valence 5, shown in Figure 6. Furthermore, by adding four colored edges with same coloring $\rho_1 + \rho_2 + \rho_3$ on $(\Gamma_{(\phi_3, \mathcal{R}P^3)}, \alpha)$, we can obtain six abstract 1-skeleta of type $(5, 3)$, which are shown in Figure 7. These six abstract 1-skeleta have the same vertex-coloring set, and they give all possible abstract 1-skeleta with such a vertex-coloring set. Thus, if there is a $(\mathbb{Z}_2)^3$-action on a 5-dimensional closed manifold such that its colored graph has the same vertex-coloring set as those six abstract 1-skeleta, then its colored graph must be one of six abstract 1-skeleta.
Lemma 7.2. There exists a \((\mathbb{Z}_2)^3\)-action \((\Lambda, M^3)\) on a 5-dimensional closed manifold with four fixed points such that its colored graph \((\Gamma_{(\Lambda, M^3)}, \alpha)\) is one of six abstract 1-skeleta in Figure 6.

Proof. In a similar way to Lemma 7.1, it suffices to show that the vertex-coloring set

\[
\{\alpha_1(E_p), \alpha_1(E_q), \alpha_1(E_r), \alpha_1(E_s)\}
\]

of \((\Gamma_1, \alpha_1)\) is the fixed data of some \((\mathbb{Z}_2)^3\)-action on a closed 5-manifold. Let \(f(x_1, x_2, x_3, x_4, x_5)\) be a symmetric polynomial function in 5 variables over \(\mathbb{Z}_2\) and

\[
\hat{f} = \frac{f(\alpha_1(E_p))}{(\rho_1 + \rho_2 + \rho_3)^2 \rho_1 \rho_2 \rho_3} + \frac{f(\alpha_1(E_q))}{(\rho_1 + \rho_2 + \rho_3)^2 \rho_1 (\rho_1 + \rho_2)(\rho_1 + \rho_3)} + \frac{f(\alpha_1(E_r))}{(\rho_1 + \rho_2 + \rho_3)^2 \rho_2 (\rho_1 + \rho_2)(\rho_2 + \rho_3)} + \frac{f(\alpha_1(E_s))}{(\rho_1 + \rho_2 + \rho_3)^2 \rho_3 (\rho_1 + \rho_3)(\rho_2 + \rho_3)}.
\]

By Theorem 3.3 it needs to show that \(\hat{f} \in \mathbb{Z}_2[\rho_1, \rho_2, \rho_3]\). For this, write

\[
f(x_1, x_2, x_3, x_4, x_5) = \sum_{i=0}^{j} x_i x_2^{i-1} f_{i,1}(x_3, x_4, x_5) = \sum_{i=0}^{j} x_i x_2^{i-1} f_{i,1}(x_3, x_4, x_5)
\]
with each \( f_{i,l} \) a symmetric function in 3 variables \( x_3, x_4, x_5 \) and
\[
\hat{f}_{i,l} = \frac{f_{i,l}(\rho_1, \rho_2, \rho_3)}{\rho_1 \rho_2 \rho_3} + \frac{f_{i,l}(\rho_1, \rho_1 + \rho_2, \rho_1 + \rho_3)}{\rho_1 (\rho_1 + \rho_2) (\rho_1 + \rho_3)} + \frac{f_{i,l}(\rho_2, \rho_1 + \rho_2, \rho_2 + \rho_3)}{\rho_2 (\rho_1 + \rho_2) (\rho_2 + \rho_3)} + \frac{f_{i,l}(\rho_3, \rho_1 + \rho_3, \rho_2 + \rho_3)}{\rho_3 (\rho_1 + \rho_3) (\rho_2 + \rho_3)}.
\]

Then by direct calculations, one has
\[
\hat{f} = \frac{\hat{f}_{0,0}}{(\rho_1 + \rho_2 + \rho_3)^2} + \frac{\hat{f}_{0,1} + \hat{f}_{1,0}}{(\rho_1 + \rho_2 + \rho_3)} + \sum_{i + l \geq 2} (\rho_1 + \rho_2 + \rho_3)^{i + l - 2} \hat{f}_{i,l}.
\]

One knows from the proof of Lemma 7.1 that for any \( i \) and \( l \), \( \hat{f}_{i,l} \) belongs to \( \mathbb{Z}_2[\rho_1, \rho_2, \rho_3] \) and is divisible by \( \sigma = \rho_1 + \rho_2 + \rho_3 \). Thus \( \hat{f} \in \mathbb{Z}_2[\rho_1, \rho_2, \rho_3] \) if and only if \( \hat{f}_{0,0} \) is divisible by \( \sigma^2 \). So it remains to prove that \( \hat{f}_{0,0} \equiv 0 \mod \sigma^2 \).

To do this, one will write the symmetric function \( f_{0,0}(x_3, x_4, x_5) \) of 3 variables \( x_3, x_4, x_5 \) in terms of elementary symmetric functions \( \sigma_1(x_3, x_4, x_5), \sigma_2(x_3, x_4, x_5) \) and \( \sigma_3(x_3, x_4, x_5) \).

Now on the four fixed points, one has that \( \sigma_1(\rho_1, \rho_2, \rho_3) = \sigma_1(\rho_1, \rho_1 + \rho_2, \rho_1 + \rho_3) = \sigma_1(\rho_2, \rho_1 + \rho_2, \rho_2 + \rho_3) = \sigma_1(\rho_3, \rho_1 + \rho_3, \rho_2 + \rho_3) = \sigma_1(\rho_1 + \rho_2 + \rho_3, \rho_1 + \rho_2 + \rho_3, \rho_1 + \rho_2 + \rho_3) = \sigma \). Thus, if \( f_{0,0}(x_3, x_4, x_5) = \sigma_1(f_3, x_4, x_5) f'(x_3, x_4, x_5) \) then \( \hat{f}_{0,0} = \sigma f' \), and since \( f' \) is divisible by \( \sigma \) by Lemma 7.1 \( \hat{f}_{0,0} \) is divisible by \( \sigma^2 \).

So it suffices to only consider \( f_{0,0}(x_3, x_4, x_5) = \sigma_1^w(x_3, x_4, x_5) \sigma_3^w(x_3, x_4, x_5) \).

Write \( \rho_2 = \rho_1 + \rho_2 + \sigma \) and consider everything in \( \mathbb{Z}_2[\rho_1, \rho_2, \sigma] \cong \mathbb{Z}_2[\rho_1, \rho_2, \rho_3] \).

By direct calculations one has that
\[
\begin{align*}
\sigma_2(\rho_1, \rho_2, \rho_3) &= \tau + (\rho_1 + \rho_2)\sigma \\
\sigma_2(\rho_1, \rho_1 + \rho_2, \rho_1 + \rho_3) &= \tau + \rho_2\sigma \\
\sigma_2(\rho_2, \rho_1 + \rho_2, \rho_2 + \rho_3) &= \tau + \rho_1\sigma \\
\sigma_2(\rho_3, \rho_1 + \rho_3, \rho_2 + \rho_3) &= \tau + \sigma^2 \equiv \tau \mod \sigma^2 \\
\end{align*}
\]

where \( \tau = \rho_1^2 + \rho_1 \rho_2 + \rho_2^2 \), and
\[
\begin{align*}
\sigma_3(\rho_1, \rho_2, \rho_3) &= \rho_1 \rho_2 \rho_3 = \varphi + \rho_1 \rho_2 \sigma \\
\sigma_3(\rho_1, \rho_1 + \rho_2, \rho_1 + \rho_3) &= \rho_1 (\rho_1 + \rho_2) (\rho_1 + \rho_3) = \varphi + (\rho_1^2 + \rho_1 \rho_2) \sigma \\
\sigma_3(\rho_2, \rho_1 + \rho_2, \rho_2 + \rho_3) &= \rho_2 (\rho_1 + \rho_2) (\rho_2 + \rho_3) = \varphi + (\rho_2^2 + \rho_1 \rho_2) \sigma \\
\sigma_3(\rho_3, \rho_1 + \rho_3, \rho_2 + \rho_3) &= \rho_3 (\rho_1 + \rho_3) (\rho_2 + \rho_3) \\
&\equiv \varphi + (\rho_1^2 + \rho_1 \rho_2 + \rho_2^2) \sigma \mod \sigma^2 \\
\end{align*}
\]

where \( \varphi = \rho_1 \rho_2 (\rho_1 + \rho_2) \). Beginning with the easiest case, take \( f_{0,0}(x_3, x_4, x_5) = \sigma_1^w(x_3, x_4, x_5) \sigma_3^w(x_3, x_4, x_5) \) with \( w \geq 1 \). This cancels the denominators in the expression of \( \hat{f}_{0,0} \) and
\[
\begin{align*}
\dot{f}_{0,0} &= [\tau + (\rho_1 + \rho_2) \sigma]^w [\varphi + \rho_1 \rho_2 \sigma]^w [\tau + \rho_2 \sigma]^w [\varphi + (\rho_1^2 + \rho_1 \rho_2) \sigma]^w \\
+ [\tau + \rho_1 \sigma]^w [\varphi + (\rho_2^2 + \rho_1 \rho_2) \sigma]^w [\tau + \rho_1 \sigma]^w [\varphi + (\rho_2^2 + \rho_1 \rho_2) \sigma]^w \\
&\equiv 4 \tau^w \varphi^{w-1} + 4 \tau^w \varphi^{w-1} [\rho_1^2 + \rho_1 \rho_2] \sigma + 4 \tau^w \varphi^{w-1} [\rho_2^2 + \rho_1 \rho_2] \sigma + 4 \tau^w \varphi^{w-2} [\rho_1^2 + \rho_1 \rho_2 + \rho_2^2] \sigma \\
&\equiv 0 \mod \sigma^2 \\
\end{align*}
\]
For the case $f_{0,0}(x_3, x_4, x_5) = \sigma_2^\ast(x_3, x_4, x_5)$, it is convenient to take the common denominator in $f_{0,0}$ to be the product of the four choices for $\sigma_3(x_3, x_4, x_5)$; i.e.,

$$(\varphi + \rho_1 \rho_2 \sigma)[\varphi + (\rho_1^2 + \rho_1 \rho_2)\sigma][\varphi + (\rho_2^2 + \rho_1 \rho_2)\sigma][\varphi + (\rho_1^2 + \rho_1 \rho_2 + \rho_2^2)\sigma]$$

and then the numerator in $f_{0,0}$ becomes (modulo $\sigma^2$)

$$[\tau + (\rho_1 + \rho_2)\sigma][\varphi^3 + \varphi^2((\rho_1^2 + \rho_1 \rho_2)\sigma + (\rho_2^2 + \rho_1 \rho_2)\sigma + (\rho_1^2 + \rho_1 \rho_2 + \rho_2^2)\sigma)] + [\tau + \rho_2 \sigma][\varphi^3 + \varphi^2(\rho_1 \rho_2 \sigma + (\rho_2^2 + \rho_1 \rho_2)\sigma + (\alpha_1^2 + \alpha_1 \alpha_2 + \alpha_2^2)\sigma)]$$

$$+ [\tau + \rho_1 \sigma][\varphi^3 + \varphi^2(\rho_1 \rho_2 \sigma + (\rho_2^2 + \rho_1 \rho_2)\sigma + (\rho_1^2 + \rho_1 \rho_2 + \rho_2^2)\sigma)]$$

$$+ \tau^v[\varphi^3 + \varphi^2(\rho_1 \rho_2 \sigma + (\rho_2^2 + \rho_1 \rho_2)\sigma + (\rho_1^2 + \rho_1 \rho_2 + \rho_2^2)\sigma)]$$

$$\equiv 4\tau^v \varphi^3 + \rho_1 \rho_2 \sigma + 6\rho_2^2 \sigma + \rho_1 \sigma + 0$$

$$+ \tau^v \varphi^2[12\rho_1 \rho_2 \sigma + 6\rho_1^2 \sigma + 6\rho_2^2 \sigma]$$

$$\equiv 0 \mod \sigma^2.$$

Thus $f_{0,0}$ is always divisible by $\sigma^2$. \hfill \Box

**Remark 14.** We see easily that any two of $(\Gamma_1, \alpha_1)$, $(\Gamma_1, \alpha_2)$, $(\Gamma_1, \alpha_3)$ can be translated to each other by automorphisms of $\text{Hom}((\mathbb{Z}^d, \mathbb{Z}^d)$). Thus, if one of $(\Gamma_1, \alpha_1)$, $(\Gamma_1, \alpha_2)$, $(\Gamma_1, \alpha_3)$ can become a colored graph of some $(\mathbb{Z}^d)^3$-action, then so are all three abstract 1-skeleta $(\Gamma_1, \alpha_1), (\Gamma_1, \alpha_2), (\Gamma_1, \alpha_3)$. This is also true for $(\Gamma_2, \alpha_4)$, $(\Gamma_2, \alpha_5)$, $(\Gamma_2, \alpha_6)$. However, we don’t know whether two kinds of abstract 1-skeleta all can become the colored graphs of $(\mathbb{Z}^d)^3$-actions, and in addition, we cannot determine which of six abstract 1-skeleta is the colored graph of $(\Lambda, M^5)$.

**Remark 15.** In the proof of Lemma 7.2 when

$$f_{0,0}(x_3, x_4, x_5) = \sigma_2(x_3, x_4, x_5)\sigma_3(x_3, x_4, x_5)$$

we have that

$$\hat{f}_{0,0} = [\tau + (\rho_1 + \rho_2)\sigma] + (\tau + \rho_2 \sigma) + (\tau + \rho_1 \sigma) + (\tau + \sigma^2) = \sigma^2.$$

Thus, we cannot improve the divisibility of $\hat{f}_{0,0}$. This also implies that by the above method, we cannot further modify $(\phi_3, \mathbb{R}P^3)$ to obtain a 6-dimensional example with four fixed points.

Finally, applying the $\Delta$-operation and $\Omega$-operation to $(\phi_3, \mathbb{R}P^3)$, $(\Upsilon, M^4)$ and $(\Lambda, M^5)$ gives the following result.

**Corollary 7.3.** When $n = 3 \cdot 2^\ell, 4 \cdot 2^\ell, 5 \cdot 2^\ell$ with $\ell \geq k - 3 \geq 0$, $\mathcal{A}_n^k(4)$ is nonempty.

8. Actions with Four Fixed Points

Suppose that $\mathcal{A}_n^k(4)$ is nonempty. Then $k \geq 3$ since any nonbounding $(\mathbb{Z}^d)^2$-action cannot exactly fix four isolated points by the work of Conner and Floyd [12, Theorem 31.2] (also see Lemma 5.2 in Section 5). Given a $G$-action $(\Phi, M^n)$ in $\mathcal{A}_n^k(4)$, without the loss of generality assume that $(\Phi, M^n)$ has exactly 4 fixed points $p, q, r, s$. Let $(\Gamma(\Phi, M^n), \alpha)$ be a colored graph of $(\Phi, M^n)$. Then the tangent representation set of $(\Phi, M^n)$ is $N(\Phi, M^n) = \{\alpha(E_p), \alpha(E_q), \alpha(E_r), \alpha(E_s)\}$. 
8.1. The simple description of $N_{(\Phi, M^n)}$. Consider a nontrivial irreducible representation $\rho$ in $\text{Hom}(G, \mathbb{Z}_2)$ and a component $C$ of fixed set of $\ker \rho$. Then $C$ must contain an even number of fixed points of $(\Phi, M^n)$, so $C$ contains either all 4 fixed points or 2 fixed points of $M^G = \{p, q, r, s\}$.

If $C$ contains all 4 fixed points, then $\rho$ appears the same number of times in the tangent representation at each fixed point. Actually, $\rho$ appears exactly $\text{dim} C$ times in the tangent representation at each fixed point. Obviously, $\text{dim} C$ must be more than 1, so generally the pair $(\rho, C)$ doesn’t determine a unique connected $\text{dim} C$-valent regular graph $\Gamma_{\rho, C}$ except for $\text{dim} C = 2$. Let $\hat{\omega}$ be the multiset of representations written in $\text{Hom}(G, \mathbb{Z}_2)$ occurring for all such components $C$ (with multiplicities $\text{dim} C$’s).

If $C$ contains 2 fixed points, then $\rho$ appears the same number of times in the tangent representations at these two fixed points. Each pair $(\rho, C)$ determines a unique $\text{dim} C$-valent regular graph $\Gamma_{\rho, C}$ with two vertices. Denote the representations occurring for all such components $C$ with $\Gamma_{\rho, C}$ containing $p$ and $q$ by $\hat{\beta}$, $p$ and $r$ by $\hat{\gamma}$, $p$ and $s$ by $\hat{\delta}$, $q$ and $r$ by $\hat{\epsilon}$, $q$ and $s$ by $\hat{\eta}$, $r$ and $s$ by $\hat{\lambda}$, respectively, as shown in Figure 8. Then we can write

\[ \alpha(E_p) = \hat{\omega} \cup \hat{\beta} \cup \hat{\gamma} \cup \hat{\delta}, \quad \alpha(E_q) = \hat{\omega} \cup \hat{\beta} \cup \hat{\eta} \cup \hat{\lambda}, \quad \alpha(E_r) = \hat{\omega} \cup \hat{\epsilon} \cup \hat{\gamma} \cup \hat{\lambda}, \quad \alpha(E_s) = \hat{\omega} \cup \hat{\delta} \cup \hat{\eta} \cup \hat{\lambda}. \]

Obviously, each of the above four expressions consists of the union of four disjoint sets. The only possible nonempty intersections are $\hat{\beta} \cap \hat{\lambda}$, $\hat{\gamma} \cap \hat{\eta}$, and $\hat{\delta} \cap \hat{\epsilon}$.

Furthermore, we have that

\[
\begin{align*}
 n &= |\hat{\omega}| + |\hat{\beta}| + |\hat{\gamma}| + |\hat{\delta}| \\
 n &= |\hat{\omega}| + |\hat{\beta}| + |\hat{\epsilon}| + |\hat{\eta}| \\
 n &= |\hat{\omega}| + |\hat{\delta}| + |\hat{\gamma}| + |\hat{\lambda}| \\
 n &= |\hat{\omega}| + |\hat{\delta}| + |\hat{\eta}| + |\hat{\lambda}|
\end{align*}
\]

and thus

(4) \[ |\hat{\beta}| = |\hat{\lambda}|, \quad |\hat{\gamma}| = |\hat{\eta}|, \quad |\hat{\delta}| = |\hat{\epsilon}|. \]

Now let $\hat{\beta}_0 = \hat{\lambda}_0$ be the common part of $\hat{\beta}$ and $\hat{\lambda}$ and let $\hat{\beta}_1 = \hat{\beta} \setminus \hat{\beta}_0$ and $\hat{\lambda}_1 = \hat{\lambda} \setminus \hat{\lambda}_0$. By (4) we then have

\[ |\hat{\beta}_1| = |\hat{\beta}| - |\hat{\beta}_0| = |\hat{\lambda}| - |\hat{\lambda}_0| = |\hat{\lambda}_1|. \]
Similarly we form $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\eta}_0, \tilde{\eta}_1$ with $\gamma_0 = \tilde{\eta}_0$ and $|\gamma_1| = |\tilde{\eta}_1|$, and $\tilde{\delta}_0, \tilde{\delta}_1, \tilde{\varepsilon}_0, \tilde{\varepsilon}_1$ with $\delta_0 = \tilde{\varepsilon}_0$ and $|\tilde{\delta}_1| = |\tilde{\varepsilon}_1|$. Then four tangent representations of $\mathcal{N}_{(\Phi, M^n)}$ become

\[
\alpha(E_p) = \tilde{\omega} \cup \tilde{\delta}_0 \cup \tilde{\gamma}_0 \cup \tilde{\delta}_1 \cup \tilde{\gamma}_1 \cup \tilde{\delta}_1,
\]
\[
\alpha(E_q) = \tilde{\omega} \cup \tilde{\delta}_0 \cup \tilde{\gamma}_0 \cup \tilde{\delta}_1 \cup \tilde{\varepsilon}_1 \cup \tilde{\eta}_1,
\]
\[
\alpha(E_r) = \tilde{\omega} \cup \tilde{\delta}_0 \cup \tilde{\gamma}_0 \cup \tilde{\delta}_1 \cup \tilde{\varepsilon}_1 \cup \tilde{\gamma}_1 \cup \tilde{\lambda}_1,
\]
\[
\alpha(E_s) = \tilde{\omega} \cup \tilde{\delta}_0 \cup \tilde{\gamma}_0 \cup \tilde{\delta}_1 \cup \tilde{\eta}_1 \cup \tilde{\lambda}_1
\]

with $|\tilde{\delta}_1| = |\tilde{\lambda}_1|$, $|\tilde{\gamma}_1| = |\tilde{\eta}_1|$ and $|\tilde{\delta}_1| = |\tilde{\varepsilon}_1|$.

Note that if $\tilde{\omega}$ is empty, then $\Gamma_{(\Phi, M^n)}$ is uniquely determined.

8.2. **The essential structure of $\mathcal{N}_{(\Phi, M^n)}$**. Following the notions of the above subsection, now let us study the essential structure of $\mathcal{N}_{(\Phi, M^n)}$.

**Lemma 8.1.**

(a) $\tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$ are all nonempty.

(b) The sets $\tilde{\omega}, \tilde{\delta}_0 = \lambda_0, \tilde{\gamma}_0 = \tilde{\eta}_0, \tilde{\delta}_1 = \varepsilon_0, \tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$ are all disjoint.

**Proof.** Since no two of the fixed points have the same representation, one of the sets $\tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$ must be nonempty. Say $\tilde{\beta}_1 \neq \emptyset$ (then also $\tilde{\lambda}_1 \neq \emptyset$). Then by comparing $\alpha(E_p)$ and $\alpha(E_q)$ one of $\tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1$ must be nonempty. Say $\tilde{\gamma}_1 \neq \emptyset$ (then also $\tilde{\eta}_1 \neq \emptyset$).

Let $\beta \in \tilde{\beta}$ and $\gamma \in \tilde{\gamma}_1$, and consider $\beta + \gamma$. The representation $\gamma$ occurs more times in $\alpha(E_p)$ than in $\alpha(E_q)$, but the number of times that $\gamma$ and $\beta + \gamma$ occur in $\alpha(E_p)$ is the same as the number of times that $\gamma$ and $\beta + \gamma$ occur in $\alpha(E_q)$ by the property (P2) in Section 2 so $\beta + \gamma$ must occur more times in $\alpha(E_q)$ than in $\alpha(E_p)$. Thus $\beta + \gamma$ belongs to $\tilde{\eta}_1 \cup \tilde{\varepsilon}_1$. If $\beta \in \tilde{\beta}_0$, then $\beta \in \tilde{\lambda}$ and $\gamma$ occurs more times in $\alpha(E_r)$ than in $\alpha(E_q)$, so $\beta + \gamma$ occurs more times in $\alpha(E_q)$ than in $\alpha(E_r)$, and then $\beta + \gamma$ belongs to $\tilde{\delta}_1 \cup \tilde{\eta}_1$. But $\beta + \gamma$ is in both $\tilde{\eta}_1 \cup \tilde{\varepsilon}_1$ and $\tilde{\delta}_1 \cup \tilde{\eta}_1$, so $\beta + \gamma$ belongs to $\tilde{\eta}_1$. If $\beta \in \tilde{\beta}_1$, then $\beta$ occurs more times in $\alpha(E_p)$ than in $\alpha(E_r)$ so $\beta + \gamma$ must occur more times in $\alpha(E_r)$ than in $\alpha(E_p)$, and $\beta + \gamma$ belongs to $\tilde{\lambda}_1 \cup \tilde{\varepsilon}_1$. But $\beta + \gamma$ is in both $\tilde{\eta}_1 \cup \tilde{\varepsilon}_1$ and $\tilde{\lambda}_1 \cup \tilde{\varepsilon}_1$, so $\beta + \gamma$ belongs to $\tilde{\varepsilon}_1$. Since we had supposed that $\tilde{\beta}_1$ and $\tilde{\gamma}_1$ are nonempty, we have that $\tilde{\varepsilon}_1$ is nonempty so all of the sets $\tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$ are nonempty.

Further, if $\beta \in \tilde{\beta}_0$ and $\beta \in \tilde{\beta}_1$, then $\beta + \gamma$ belongs to both $\tilde{\eta}_1$ and $\tilde{\varepsilon}_1$, which is impossible. Thus $\tilde{\beta}_0$ and $\tilde{\beta}_1$ must be disjoint. Similarly, $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are disjoint, and so on. Thus, the sets

$\omega, \tilde{\beta}_0 = \lambda_0, \tilde{\gamma}_0 = \tilde{\eta}_0, \tilde{\delta}_0 = \varepsilon_0, \tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$

are all disjoint. \qed

Now let us analyze the structures of $\tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$.

**Lemma 8.2.**

(a) $\tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$ have the same number of elements, and all elements of $\tilde{\beta}_1, \tilde{\gamma}_1, \tilde{\delta}_1, \tilde{\varepsilon}_1, \tilde{\eta}_1, \tilde{\lambda}_1$ occur with the same multiplicity.
(b) All odd sums of elements of $\hat{\beta}_1$ are again in $\hat{\beta}_1$ and choose $\gamma \in \hat{\gamma}_1$ and $\delta \in \hat{\delta}_1$, one has that
\begin{align}
\begin{cases}
\hat{\gamma}_1 = \{ \gamma + \beta + \beta' | \beta, \beta' \in \hat{\beta}_1 \} \\
\hat{\delta}_1 = \{ \delta + \beta + \beta' | \beta, \beta' \in \hat{\beta}_1 \} \\
\hat{\varepsilon}_1 = \{ \gamma + \beta | \beta \in \hat{\beta}_1 \} \\
\hat{\eta}_1 = \{ \delta + \beta | \beta \in \hat{\beta}_1 \} \\
\hat{\lambda}_1 = \{ \gamma + \beta + \beta' | \beta, \beta' \in \hat{\beta}_1 \}
\end{cases}
\end{align}
\begin{align}
\text{and } \gamma, \delta \text{ are linearly independent of the span of } \hat{\beta}_1.
\end{align}

Proof. First, by Figure 5 Lemma 5.1 and the property (P2) in Section 2 we easily obtain that for any $\beta \in \hat{\beta}_1, \gamma \in \hat{\gamma}_1, \delta \in \hat{\delta}_1, \varepsilon \in \hat{\varepsilon}_1, \eta \in \hat{\eta}_1, \lambda \in \hat{\lambda}_1$,
\begin{align}
\begin{cases}
\beta + \gamma + \delta \in \hat{\delta}_1, \beta + \varepsilon \in \hat{\varepsilon}_1, \beta + \eta \in \hat{\eta}_1, \lambda \in \hat{\lambda}_1, \\
\beta + \gamma + \delta \in \hat{\delta}_1, \beta + \varepsilon \in \hat{\varepsilon}_1, \beta + \eta \in \hat{\eta}_1, \lambda \in \hat{\lambda}_1, \\
\beta + \gamma + \delta \in \hat{\delta}_1, \beta + \varepsilon \in \hat{\varepsilon}_1, \beta + \eta \in \hat{\eta}_1, \lambda \in \hat{\lambda}_1,
\end{cases}
\end{align}

Now for $\gamma \in \hat{\gamma}_1$ and $\beta \in \hat{\beta}_1$, the number of times that $\gamma$ occurs in $\alpha(E_\beta)$ is the same as the number of times that $\beta + \gamma$ occurs in $\alpha(E_\beta)$, so the number of times that $\gamma$ occurs in $\hat{\gamma}_1$ is the same as the number of times that $\beta + \gamma$ occurs in $\hat{\varepsilon}_1$. Also, by (6) if $\varepsilon \in \hat{\varepsilon}_1$, then $\beta + \varepsilon \in \hat{\gamma}_1$. Thus if $\hat{\gamma}_1 = \{ \gamma_1, \ldots, \gamma_t \}$ then $\hat{\varepsilon}_1 = \{ \gamma_1 + \beta, \ldots, \gamma_t + \beta \}$. Thus $|\hat{\beta}_1| = |\hat{\varepsilon}_1|$ and similarly, $|\hat{\beta}_1| = |\hat{\eta}_1|$, so all of the sets $\hat{\beta}_1, \hat{\gamma}_1, \hat{\delta}_1, \hat{\varepsilon}_1, \hat{\eta}_1, \hat{\lambda}_1$ have the same number of elements.

Moreover, the number of times that $\beta + \gamma$ occurs in $\hat{\varepsilon}_1$ is not only the same as the number of times that $\gamma$ occurs in $\hat{\gamma}_1$, and but it also is the same as the number of times that $\beta$ occurs in $\hat{\beta}_1$. Thus, the elements of $\hat{\beta}_1, \hat{\gamma}_1, \hat{\varepsilon}_1$ all occur with the same multiplicity. Similarly, this also happens for $\hat{\delta}_1, \hat{\gamma}_1, \hat{\lambda}_1$ (resp. $\hat{\beta}_1, \hat{\delta}_1, \hat{\eta}_1$). Therefore, the elements of $\hat{\beta}_1, \hat{\gamma}_1, \hat{\delta}_1, \hat{\varepsilon}_1, \hat{\eta}_1, \hat{\lambda}_1$ all occur with the same multiplicity.

By (5) and the property (P2) of Section 2 one has that $\hat{\eta}_1 = \{ \delta + \beta | \beta \in \hat{\beta}_1 \}$, and for any $\eta \in \hat{\eta}_1$ and $\beta' \in \hat{\beta}_1, \eta + \beta' \in \hat{\delta}_1$ and $\hat{\eta}_1 = \{ \eta + \beta + \beta' | \beta \in \hat{\beta}_1 \}$. Thus, for any $\beta'' \in \hat{\beta}_1$, choosing $\eta = \delta + \beta''$ gives
\begin{align}
\hat{\beta}_1 = \{ \beta | \beta \in \hat{\beta}_1 \} = \{ \beta + \beta' + \beta'' | \beta \in \hat{\beta}_1 \}.
\end{align}
Replacing $\beta$ by $\beta + \beta' + \beta''$ and repeating the above procedure, we have that every sum of an odd number of elements in $\hat{\beta}_1$ is again in $\hat{\beta}_1$. Then (4) follows from this and (1).

Finally, let us show that $\gamma, \delta$ are linearly independent of the span of $\hat{\beta}_1$. This is equivalent to proving that $\gamma + \delta$ cannot be a sum of elements of $\hat{\beta}_1$. If $\gamma + \delta$ is an odd sum of elements of $\hat{\beta}_1$, then $\gamma + \delta \in \hat{\lambda}_1 \cap \hat{\beta}_1$ so $\hat{\lambda}_1 \cap \hat{\beta}_1$ is nonempty. This is impossible by Lemma 5.1(b). If $\gamma + \delta$ is an even sum of elements of $\hat{\beta}_1$, then $\gamma \in \hat{\delta}_1 \cap \hat{\gamma}_1$, but this is also impossible. \hfill \Box

Remark 16. Let $m$ be the multiplicity of each element of $\hat{\beta}_1$. In a similar way to the proof of Lemma 5.3 it is easy to check that all elements of $\hat{\beta}_1$ span a $(k - 2)$-dimensional subspace in $\text{Hom}(\mathbb{Z}_2^k, \mathbb{Z}_2)$, so $|\hat{\beta}_1| = m \cdot 2^{k-3}$. Thus, $|\hat{\gamma}_1| = |\hat{\delta}_1| = |\hat{\varepsilon}_1| = |\hat{\eta}_1| = |\hat{\lambda}_1| = m \cdot 2^{k-3}$. 

Definition 8.3. Let 
\[ \hat{\omega} := \omega \cup \hat{\beta}_0 \cup \hat{\gamma}_0 \cup \hat{\delta}_0 \]
which is called the changeable part of \( N_{(\Phi, M^*n)} \).

Indeed, \((\phi_3, \mathbb{R}P^3)\) is an example with \(\hat{\omega}\) being empty, while Lemmas \[7\] and \[2\] in Section \[7\] provide examples with \(\hat{\omega}\) nonempty.

Lemma 8.4. When \(|\hat{\omega}| > 0\), every element in \(\hat{\omega}\) has the form \(\gamma + \delta + \beta, \beta \in \hat{\beta}_1\), where \(\gamma \in \hat{\gamma}_1\) and \(\delta \in \hat{\delta}_1\).

Proof. Let \(\xi \in \hat{\omega}\), \(\gamma \in \hat{\gamma}_1\), and \(\delta \in \hat{\delta}_1\). If \(\xi \in \hat{\omega} \cup \hat{\beta}_0 \cup \hat{\gamma}_0\) but \(\xi \not\in \hat{\delta}_0\), since \(\gamma \in \alpha(E_p)\) and \(\gamma \not\in \alpha(E_q)\), one then has that \(\gamma + \xi \in \alpha(E_q)\) so \(\gamma + \xi \in \hat{\gamma}_1 \cup \hat{\eta}_1\) by Lemma \[8.1\].

On the other hand, \(\gamma \in \alpha(E_p)\) but \(\gamma \not\in \alpha(E_q)\), so \(\gamma + \xi \in \alpha(E_q)\) and \(\gamma + \xi \in \hat{\lambda}_1 \cup \hat{\eta}_1\). Thus \(\gamma + \xi \in \hat{\eta}_1\). Further, by Lemma \[8.2\] there is an element \(\beta \in \hat{\beta}_1\) such that \(\gamma + \xi = \delta + \beta\), so \(\xi = \gamma + \delta + \beta^\prime\) as desired. If \(\xi \in \hat{\gamma}_0\), since \(\delta \in \alpha(E_p)\) but \(\delta \not\in \alpha(E_q)\) and \(\delta \not\in \alpha(E_q)\), similarly to the above argument, one has that \(\delta + \xi \in \hat{\eta}_1 \cup \hat{\varepsilon}_1\) and \(\delta + \xi \in \hat{\lambda}_1 \cup \hat{\varepsilon}_1\), so \(\delta + \xi \in \hat{\varepsilon}_1\) and then by Lemma \[8.2\] there is an element \(\beta^\prime \in \hat{\beta}_1\) such that \(\delta + \xi = \gamma + \beta^\prime\). Thus \(\xi = \gamma + \delta + \beta^\prime\) as desired. \(\square\)

Remark 17. The sum of two elements in \(\hat{\omega}\) is an even sum of elements in \(\hat{\beta}_1\) and cannot occur as an element in \(\alpha(E_p), \alpha(E_q), \alpha(E_r),\) and \(\alpha(E_s)\).

Together with Lemmas \[8.2, 8.4\] and Remark \[10\] one has

Proposition 8.5. There is a basis \(\{\beta_1, \ldots, \beta_{k-2}, \gamma, \delta\}\) of \(\text{Hom}(\mathbb{Z}_2)^k, \mathbb{Z}_2\) and an integer \(m\) such that
\begin{enumerate}
\item \(n = 3m \cdot 2^{k-3} + |\hat{\omega}|\);
\item \(\hat{\beta}_1\) is a multiset consisting of all odd sums with same multiplicity \(m\) formed by \(\beta_1, \ldots, \beta_{k-2}\), and
\begin{align*}
\hat{\gamma}_1 &= \{\gamma + \beta_1 + \beta \mid \beta \in \hat{\beta}_1\} \\
\hat{\delta}_1 &= \{\delta + \beta_1 + \beta \mid \beta \in \hat{\beta}_1\} \\
\hat{\varepsilon}_1 &= \{\gamma + \beta \mid \beta \in \hat{\beta}_1\} \\
\hat{\eta}_1 &= \{\delta + \beta \mid \beta \in \hat{\beta}_1\} \\
\hat{\lambda}_1 &= \{\gamma + \delta + \beta_1 + \beta \mid \beta \in \hat{\beta}_1\}
\end{align*}
\item Every element in \(\hat{\omega}\) has the form \(\gamma + \delta + \beta, \beta \in \hat{\beta}_1\) if \(\hat{\omega}\) is nonempty.
\end{enumerate}

Definition 8.6. If \(\hat{\omega} = \emptyset\), then \((\Phi, M^n)\) is said to be \(\hat{\omega}\)-empty; otherwise, it is said to be \(\hat{\omega}\)-nonempty.

Remark 18. If \(\hat{\omega} = \emptyset\), then we see from Proposition \[8.5\] that the tangent representation set \(N_{(\Phi, M^n)} = \{\alpha(E_p), \alpha(E_q), \alpha(E_r), \alpha(E_s)\}\) is completely determined by the basis \(\{\beta_1, \ldots, \beta_{k-2}, \gamma, \delta\}\) of \(\text{Hom}(\mathbb{Z}_2)^k, \mathbb{Z}_2\). Thus, for any two \(\hat{\omega}\)-empty actions \((\Phi_1, M^n_1)\) and \((\Phi_2, M^n_2)\) in \(A^n_k(4)\), there is an automorphism \(\sigma \in \text{GL}(k, \mathbb{Z}_2)\) such that \((\Phi_1, M^n_1)\) is equivariantly cobordant to \((\sigma \Phi_2, M^n_2)\).

To keep the notation manageable, \(\prod \hat{\beta}_1\) means \(\prod_{\beta \in \hat{\beta}_1} \beta\), and similarly for \(\prod \hat{\gamma}_1, \prod \hat{\delta}_1, \prod \hat{\varepsilon}_1, \prod \hat{\eta}_1, \prod \hat{\lambda}_1, \prod \hat{\omega}\). Let \(\hat{\beta}_{11}\) (resp. \(\hat{\gamma}_{11}, \hat{\delta}_{11}, \hat{\varepsilon}_{11}, \hat{\eta}_{11}, \hat{\lambda}_{11}\)) denote the set of consisting of all \(2^{k-3}\) different elements in \(\hat{\beta}_1\) (resp. \(\hat{\gamma}_1, \hat{\delta}_1, \hat{\varepsilon}_1, \hat{\eta}_1, \hat{\lambda}_1\)). Then, by Proposition \[8.5\] one has that \(\prod \hat{\beta}_1 = (\prod \hat{\beta}_{11})^m\) where \(\prod \hat{\beta}_{11}\) means \(\prod_{\beta \in \hat{\beta}_{11}} \beta\), and...
Corollary 8.7. \(\bar{\beta}_{11} \cup \bar{\gamma}_{11} \cup \bar{\delta}_{11}, \bar{\beta}_{11} \cup \bar{\eta}_{11} \cup \bar{\varepsilon}_{11}, \bar{\varepsilon}_{11} \cup \bar{\gamma}_{11} \cup \bar{\lambda}_{11}, \) and \(\bar{\lambda}_{11} \cup \bar{\eta}_{11} \cup \bar{\delta}_{11}\) are the fixed data of some action in \(A_{3}^{5-2k-3}(4)\).

Proof. We proceed by induction on \(k\). When \(k = 3\), we have that \(\bar{\beta}_{11} = \{\beta_{1}\}, \bar{\gamma}_{11} = \{\gamma\}, \bar{\delta}_{11} = \{\delta\}, \bar{\varepsilon}_{11} = \{\gamma + \beta_{1}\}, \bar{\eta}_{11} = \{\delta + \beta_{1}\}, \bar{\lambda}_{11} = \{\gamma + \delta\}\). Then it is easy to see that \(\bar{\beta}_{11} \cup \bar{\gamma}_{11} \cup \bar{\delta}_{11} = \{\beta_{1}, \gamma, \delta\}, \bar{\beta}_{11} \cup \bar{\eta}_{11} \cup \bar{\varepsilon}_{11} = \{\beta_{1}, \delta + \beta_{1}, \gamma + \delta\}, \bar{\varepsilon}_{11} \cup \bar{\gamma}_{11} \cup \bar{\lambda}_{11} = \{\gamma + \beta_{1}, \gamma, \gamma + \delta\}, \) and \(\bar{\lambda}_{11} \cup \bar{\eta}_{11} \cup \bar{\delta}_{11} = \{\gamma + \delta, \delta + \beta_{1}, \delta\}\) form the fixed data of \((\sigma \phi_{3}, \mathbb{R}P^{3})\) for some \(\sigma \in \text{GL}(3, \mathbb{Z}_{2})\). When \(k = l \geq 3\), suppose inductively that \(\bar{\beta}_{11} \cup \bar{\gamma}_{11} \cup \bar{\delta}_{11}, \bar{\beta}_{11} \cup \bar{\eta}_{11} \cup \bar{\varepsilon}_{11}, \bar{\varepsilon}_{11} \cup \bar{\gamma}_{11} \cup \bar{\lambda}_{11}, \) and \(\bar{\lambda}_{11} \cup \bar{\eta}_{11} \cup \bar{\delta}_{11}\) are the fixed data of some action in \(A_{3}^{5-2k-3}(4)\).

When \(k = l + 1\), in a similar way to the proof of Claim in Section 6.2 we let \(\bar{\beta}^{*}_{11}\) denote the set of all odd sums formed by \(\beta_{1}, \ldots, \beta_{l-2}\). Then \(\bar{\beta}^{*}_{11} \subset \bar{\beta}_{11}\) contains \(2^{l-3}\) different elements. Let \(\bar{\beta}^{u}_{11}\) denote the set formed by all elements \(\beta_{1} + \beta_{l-1} + \beta, \beta \in \bar{\beta}^{*}_{11}\). Then, \(\bar{\beta}^{u}_{11} \subset \bar{\beta}_{11}\) contains \(2^{l-3}\) different elements, too. Moreover, one has that \(\bar{\beta}_{11} = \bar{\beta}^{*}_{11} \cup \bar{\beta}^{u}_{11}\) and \(\bar{\beta}^{*}_{11} \cap \bar{\beta}^{u}_{11} = \emptyset\). Now let

\[
\begin{align*}
\bar{\gamma}^{*}_{11} &= \{\gamma + \beta_{1} + \beta | \beta \in \bar{\beta}^{*}_{11}\}, \\
\bar{\delta}^{*}_{11} &= \{\delta + \beta_{1} + \beta | \beta \in \bar{\beta}^{*}_{11}\}, \\
\bar{\varepsilon}^{*}_{11} &= \{\gamma + \beta | \beta \in \bar{\beta}^{*}_{11}\}, \\
\bar{\eta}^{*}_{11} &= \{\delta + \beta | \beta \in \bar{\beta}^{*}_{11}\}, \\
\bar{\lambda}^{*}_{11} &= \{\gamma + \delta + \beta_{1} + \beta | \beta \in \bar{\beta}^{*}_{11}\}
\end{align*}
\]

Then

\[
\begin{align*}
\bar{\gamma}_{11} &= \bar{\gamma}^{*}_{11} \cup \bar{\gamma}^{u}_{11} \text{ with } \bar{\gamma}^{*}_{11} \cap \bar{\gamma}^{u}_{11} = \emptyset, \\
\bar{\delta}_{11} &= \bar{\delta}^{*}_{11} \cup \bar{\delta}^{u}_{11} \text{ with } \bar{\delta}^{*}_{11} \cap \bar{\delta}^{u}_{11} = \emptyset, \\
\bar{\varepsilon}_{11} &= \bar{\varepsilon}^{*}_{11} \cup \bar{\varepsilon}^{u}_{11} \text{ with } \bar{\varepsilon}^{*}_{11} \cap \bar{\varepsilon}^{u}_{11} = \emptyset, \\
\bar{\eta}_{11} &= \bar{\eta}^{*}_{11} \cup \bar{\eta}^{u}_{11} \text{ with } \bar{\eta}^{*}_{11} \cap \bar{\eta}^{u}_{11} = \emptyset, \\
\bar{\lambda}_{11} &= \bar{\lambda}^{*}_{11} \cup \bar{\lambda}^{u}_{11} \text{ with } \bar{\lambda}^{*}_{11} \cap \bar{\lambda}^{u}_{11} = \emptyset.
\end{align*}
\]

We see that \(\bar{\beta}^{*}_{11}, \bar{\gamma}^{*}_{11}, \bar{\delta}^{*}_{11}, \bar{\varepsilon}^{*}_{11}, \bar{\eta}^{*}_{11}, \bar{\lambda}^{*}_{11}\) are exactly formed by \(\beta_{1}, \ldots, \beta_{l-2}, \gamma, \delta\). Now, regarding \(\{\beta_{1}, \ldots, \beta_{l-2}, \gamma, \delta\}\) as a basis of \(\text{Hom}(\mathbb{Z}_{2}^{l}, \mathbb{Z}_{2})\), one has by induction that \(\bar{\beta}^{*}_{11} \cup \bar{\gamma}^{*}_{11} \cup \bar{\delta}^{*}_{11}, \bar{\beta}^{*}_{11} \cup \bar{\eta}^{*}_{11} \cup \bar{\varepsilon}^{*}_{11}, \bar{\varepsilon}^{*}_{11} \cup \bar{\gamma}^{*}_{11} \cup \bar{\lambda}^{*}_{11}, \) and \(\bar{\lambda}^{*}_{11} \cup \bar{\eta}^{*}_{11} \cup \bar{\delta}^{*}_{11}\) are the fixed data of some \((\mathbb{Z}_{2})^{k-1}\)-action, denoted by \((\Psi, N)\). Then by applying \(\Omega\)-operation to \((\Psi, N)\), as in the proof of Lemma 4.4, the fixed data of \(\Omega(\Psi, N)\) exactly consists of

\[
\bar{\beta}_{11} \cup \bar{\gamma}_{11} \cup \bar{\delta}_{11}, \quad \bar{\beta}_{11} \cup \bar{\eta}_{11} \cup \bar{\varepsilon}_{11}, \quad \bar{\varepsilon}_{11} \cup \bar{\gamma}_{11} \cup \bar{\lambda}_{11}, \quad \bar{\lambda}_{11} \cup \bar{\eta}_{11} \cup \bar{\delta}_{11}.
\]

This completes the induction and the proof. \(\square\)

Corollary 8.8. \(m\) is a power of 2.
Proof. Using the formula of Theorem 3.3 we consider 1 as a symmetric polynomial in $n = 3m \cdot 2^k - 3 + |\hat{\omega}|$ variables. Then we have

\[
\frac{1}{(\prod \hat{\omega})(\prod \beta_{11})^m(\prod \gamma_{11})^m(\prod \delta_{11})^m} + \frac{1}{(\prod \hat{\omega})(\prod \beta_{11})^m(\prod \eta_{11})^m(\prod \hat{\xi}_{11})^m} + \frac{1}{(\prod \hat{\omega})(\prod \beta_{11})^m(\prod \hat{\eta}_{11})^m(\prod \hat{\lambda}_{11})^m}
\]

\[\times \left\{ (\prod \hat{\xi}_{11})^m(\prod \hat{\eta}_{11})^m(\prod \hat{\lambda}_{11})^m + (\prod \hat{\gamma}_{11})^m(\prod \hat{\delta}_{11})^m(\prod \hat{\lambda}_{11})^m + (\prod \hat{\beta}_{11})^m(\prod \hat{\delta}_{11})^m(\prod \hat{\eta}_{11})^m + (\prod \hat{\beta}_{11})^m(\prod \hat{\gamma}_{11})^m(\prod \hat{\xi}_{11})^m \right\}
\]

must belong to the polynomial algebra $\mathbb{Z}_2[\rho_1, \ldots, \rho_k]$. Because the degree of the numerator is smaller than the degree of the denominator, this means the numerator must be zero, i.e.,

\[
(\prod \hat{\xi}_{11})^m(\prod \hat{\eta}_{11})^m(\prod \hat{\lambda}_{11})^m + (\prod \hat{\gamma}_{11})^m(\prod \hat{\delta}_{11})^m(\prod \hat{\lambda}_{11})^m + (\prod \hat{\beta}_{11})^m(\prod \hat{\delta}_{11})^m(\prod \hat{\eta}_{11})^m + (\prod \hat{\beta}_{11})^m(\prod \hat{\gamma}_{11})^m(\prod \hat{\xi}_{11})^m = 0
\]

By Corollary 8.7 the above equation still holds for $m = 1$, so one has

\[
(\prod \hat{\xi}_{11})(\prod \hat{\eta}_{11})(\prod \hat{\lambda}_{11}) + (\prod \hat{\gamma}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\lambda}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\eta}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\gamma}_{11})(\prod \hat{\xi}_{11}) = 0
\]

and then

\[
(*) \quad [(\prod \hat{\gamma}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\lambda}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\eta}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\gamma}_{11})(\prod \hat{\xi}_{11})]^m + (\prod \hat{\beta}_{11})^m(\prod \hat{\gamma}_{11})^m(\prod \hat{\lambda}_{11})^m + (\prod \hat{\beta}_{11})^m(\prod \hat{\delta}_{11})^m(\prod \hat{\eta}_{11})^m + (\prod \hat{\beta}_{11})^m(\prod \hat{\gamma}_{11})^m(\prod \hat{\xi}_{11})^m = 0.
\]

Next, we are going to show that if $m$ is not a power of 2, then $(*)$ does not hold. Let $m = 2^{p_r} + 2^{p_r-1} + \cdots + 2^{p_1} = j + 2^{p_1}$ where $p_r > p_{r-1} > \cdots > p_1$. Using the 2-adic expansion of $m$ and write

\[
[(\prod \hat{\gamma}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\lambda}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\eta}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\gamma}_{11})(\prod \hat{\xi}_{11})]^m
\]

\[= [(\prod \hat{\gamma}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\lambda}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\eta}_{11}) + (\prod \hat{\beta}_{11})(\prod \hat{\gamma}_{11})(\prod \hat{\xi}_{11})]^j \times \left\{ [(\prod \hat{\gamma}_{11})(\prod \hat{\delta}_{11})(\prod \hat{\lambda}_{11})]^{2^{p_1}} + [(\prod \hat{\beta}_{11})(\prod \hat{\gamma}_{11})(\prod \hat{\xi}_{11})]^{2^{p_1}} \right\}.
\]
In the above equation we seek the terms of largest degree in \( \gamma \) and \( \delta \). For this we have
\[
\begin{align*}
\prod \hat{\gamma}_{11} &= \gamma^{2^{k-3}} + \text{terms of lower degree} \\
\prod \hat{\delta}_{11} &= \delta^{2^{k-3}} + \text{terms of lower degree} \\
\prod \hat{\varepsilon}_{1} &= \gamma^{2^{k-3}} + \text{terms of lower degree} \\
\prod \hat{\eta}_{1} &= \delta^{2^{k-3}} + \text{terms of lower degree} \\
\prod \hat{\lambda}_{1} &= (\gamma + \delta)^{2^{k-3}} + \text{terms of lower degree}
\end{align*}
\]
and \( \prod \hat{\beta}_{11} \) has no \( \gamma \)'s and \( \delta \)'s. In the \( j \)-th power, the term of largest degree in \( \gamma \) and \( \delta \) is \([\gamma \delta (\gamma + \delta)]^{j^{2^{k-3}}} \) occurring in the monomial \([\prod \hat{\gamma}_{11} (\prod \hat{\delta}_{11} (\prod \hat{\lambda}_{11})]^{j} \). In (\( * \)) this monomial is multiplied by
\[
\left( \prod \hat{\beta}_{11} \right) \left( \prod \hat{\delta}_{11} \right) \left( \prod \hat{\eta}_{1} \right) \left( \prod \hat{\varepsilon}_{1} \right) \left( \prod \hat{\lambda}_{1} \right) = \left( \prod \hat{\beta}_{11} \right)^{2^{p_{1}}} \left( \gamma^{2^{k-3}} + \delta^{2^{k-3}} + \text{terms of lower degree} \right).
\]
Thus, in (\( * \)) the term with largest degree in \( \gamma \) and \( \delta \) is
\[
\left( \prod \hat{\beta}_{11} \right)^{2^{p_{1}}} \gamma^{j^{2^{k-3}}} \delta^{j^{2^{k-3}}} (\gamma + \delta)^{j^{2^{k-3}}} + \text{terms of lower degree}
\]
which is nonzero. But this is impossible, so \( m \) must be a power of 2.

Throughout the following, assume that \( m \) is always a power of 2.

By Corollary 8.7 let \( \hat{\beta}_{11} \cup \hat{\gamma}_{11} \cup \hat{\delta}_{11}, \hat{\beta}_{11} \cup \hat{\eta}_{11} \cup \hat{\varepsilon}_{11}, \hat{\beta}_{1} \cup \hat{\gamma}_{1} \cup \hat{\delta}_{1}, \hat{\beta}_{1} \cup \hat{\eta}_{1} \cup \hat{\varepsilon}_{1}, \hat{\beta}_{1} \cup \hat{\gamma}_{1} \cup \hat{\lambda}_{1}, \) and \( \hat{\lambda}_{1} \cup \hat{\eta}_{1} \cup \hat{\delta}_{1} \) be the fixed data of some action \((\Psi, N)\). Then, applying \( \Delta \)-operation \( m \) times to \((\Psi, N)\) gives an action \( \Delta^{m}(\Psi, N) \). Since \( m \) is a power of 2, by Lemma 4.2 \( \Delta^{m}(\Psi, N) \) is equivariantly cobordant to an action whose fixed data exactly consists of \( \hat{\beta}_{1} \cup \hat{\gamma}_{1} \cup \hat{\delta}_{1}, \hat{\beta}_{1} \cup \hat{\eta}_{1} \cup \hat{\varepsilon}_{1}, \hat{\beta}_{1} \cup \hat{\gamma}_{1} \cup \hat{\lambda}_{1}, \) and \( \hat{\lambda}_{1} \cup \hat{\eta}_{1} \cup \hat{\delta}_{1} \). This gives

**Corollary 8.9.** \( \hat{\beta}_{1} \cup \hat{\gamma}_{1} \cup \hat{\delta}_{1}, \hat{\beta}_{1} \cup \hat{\eta}_{1} \cup \hat{\varepsilon}_{1}, \hat{\beta}_{1} \cup \hat{\gamma}_{1} \cup \hat{\lambda}_{1}, \) and \( \hat{\lambda}_{1} \cup \hat{\eta}_{1} \cup \hat{\delta}_{1} \) are the fixed data of some action in \( A^{3m \cdot 2^{k-3} - 4}(4) \).

Now let us further analyze the structure of the changeable part \( \hat{\omega} \).

Let \( f(x_{1}, ..., x_{3m \cdot 2^{k-3}}) \) be a symmetric function in \( 3m \cdot 2^{k-3} \) variables over \( \mathbb{Z}_{2} \) where \( m \) is a power of 2. Write
\[
\hat{f} = \frac{f(\hat{\beta}_{1}, \hat{\gamma}_{1}, \hat{\delta}_{1})}{\prod \hat{\beta}_{1} \prod \hat{\gamma}_{1} \prod \hat{\delta}_{1}} + \frac{f(\hat{\beta}_{1}, \hat{\varepsilon}_{1}, \hat{\eta}_{1})}{\prod \hat{\beta}_{1} \prod \hat{\varepsilon}_{1} \prod \hat{\eta}_{1}} + \frac{f(\hat{\varepsilon}_{1}, \hat{\gamma}_{1}, \hat{\lambda}_{1})}{\prod \hat{\varepsilon}_{1} \prod \hat{\gamma}_{1} \prod \hat{\lambda}_{1}} + \frac{f(\hat{\lambda}_{1}, \hat{\eta}_{1}, \hat{\delta}_{1})}{\prod \hat{\lambda}_{1} \prod \hat{\eta}_{1} \prod \hat{\delta}_{1}}
\]
Then \( \hat{f} \) has degree \( \deg f - 3m \cdot 2^{k-3} \). By Corollary 8.9 and Theorem 3.3 it follows that

**Lemma 8.10.** \( \hat{f} \in \mathbb{Z}_{2}[\rho_{1}, ..., \rho_{k}] \) is a polynomial. In particular, if \( \deg f < 3m \cdot 2^{k-3} \), then \( \hat{f} = 0 \).

**Remark 19.** \( \hat{f} \) will play an important role in determining the structure of \( \hat{\omega} \).

**Lemma 8.11.** Let \( \hat{\omega} \) be nonempty. Then \( \hat{f} \) is divisible by \( \prod \hat{\omega} \).

**Proof.** Since \( \hat{\omega} \) is nonempty, we have that \( n = 3m \cdot 2^{k-3} + |\hat{\omega}| \). Take the following polynomial function over \( \mathbb{Z}_{2} \) which is symmetric in variables \( x_{1}, ..., x_{n} \)
\[
g(x_{1}, ..., x_{n}; \hat{\beta}_{11}, \hat{\gamma}_{11}, \hat{\delta}_{11}, \hat{\varepsilon}_{11}, \hat{\eta}_{11}, \hat{\lambda}_{11}) = \sum_{i_{1}, ..., i_{n}} [\hat{h}(x_{i_{1}}, ..., x_{i_{n}}; \hat{\beta}_{11}, \hat{\gamma}_{11}, \hat{\delta}_{11}, \hat{\varepsilon}_{11}, \hat{\eta}_{11}, \hat{\lambda}_{11}) f(x_{i_{1}i_{1}+1}, ..., x_{i_{n}i_{n}+3m \cdot 2^{k-3}})]
\]
where each \( \{i_1, ..., i_n\} \) is a permutation of \( \{1, ..., n\} \) and
\[
h(x_{i_1}, ..., x_{i_n}; \hat{\beta}_{11}, \gamma_{11}, \delta_{11}, \epsilon_{11}, \eta_{11}, \lambda_{11}) = \prod_{j=1}^{[\mathfrak{w}]} \left[ \prod \left( x_{i_j} + \hat{\beta}_{11} \right) \prod \left( x_{i_j} + \hat{\gamma}_{11} \right) \prod \left( x_{i_j} + \hat{\delta}_{11} \right) \prod \left( x_{i_j} + \hat{\epsilon}_{11} \right) \prod \left( x_{i_j} + \hat{\eta}_{11} \right) \prod \left( x_{i_j} + \hat{\lambda}_{11} \right) \right]
\]
and \( x_{i_j} + \hat{\beta}_{11} = \{x_{i_j} + \beta | \beta \in \hat{\beta}_{11}\} \) (similarly for \( x_{i_j} + \hat{\gamma}_{11}, x_{i_j} + \hat{\delta}_{11}, x_{i_j} + \hat{\epsilon}_{11}, x_{i_j} + \hat{\eta}_{11}, x_{i_j} + \hat{\lambda}_{11}\)). Using the formula of Theorem 8.3, we then have that
\[
\hat{g} = \frac{g(\alpha(E_p); \hat{\beta}_{11}, \gamma_{11}, \delta_{11}, \epsilon_{11}, \eta_{11}, \lambda_{11})}{\prod \mathfrak{w} \prod \beta_1 \prod \gamma_1 \prod \delta_1} + \frac{g(\alpha(E_q); \hat{\beta}_{11}, \gamma_{11}, \delta_{11}, \epsilon_{11}, \eta_{11}, \lambda_{11})}{\prod \mathfrak{w} \prod \beta_1 \prod \eta_1 \prod \lambda_1}
\]
which belongs to \( \mathbb{Z}_2[p_1, ..., p_k] \). We know from Proposition 8.7 that each element of \( \mathfrak{w} \) has the form \( \beta + \gamma + \delta, \beta \in \hat{\beta}_1 \). An easy argument shows that for any \( \beta \in \hat{\beta}_1 \)
\[
\beta + \gamma + \delta + \hat{\beta}_{11} = \hat{\lambda}_{11}, \quad \beta + \gamma + \delta + \hat{\gamma}_{11} = \hat{\eta}_{11}, \quad \beta + \gamma + \delta + \hat{\delta}_{11} = \hat{\epsilon}_{11},
\]
\[
\beta + \gamma + \delta + \hat{\epsilon}_{11} = \hat{\delta}_{11}, \quad \beta + \gamma + \delta + \hat{\eta}_{11} = \hat{\gamma}_{11}, \quad \beta + \gamma + \delta + \hat{\lambda}_{11} = \hat{\beta}_{11}
\]
so
\[
h(\mathfrak{w}; \hat{\beta}_{11}, \gamma_{11}, \delta_{11}, \epsilon_{11}, \eta_{11}, \lambda_{11}) = \frac{1}{\prod \mathfrak{w}} \left[ \left( \prod \hat{\lambda}_{11} \right) \left( \prod \hat{\gamma}_{11} \right) \left( \prod \hat{\delta}_{11} \right) \left( \prod \hat{\epsilon}_{11} \right) \right]^{[\mathfrak{w}]}. \]
Obviously, \( h(\mathfrak{w}; \hat{\beta}_{11}, \gamma_{11}, \delta_{11}, \epsilon_{11}, \eta_{11}, \lambda_{11}) \) is not divisible by any \( \beta + \gamma + \delta, \beta \in \hat{\beta}_1 \), so it is not divisible by \( \prod \mathfrak{w} \). Thus we must have that \( \hat{f} \) is divisible by \( \prod \mathfrak{w} \).

**Lemma 8.12.** \( \hat{f} \) is divisible by \( \left[ \prod (\gamma + \delta + \hat{\beta}_1) \right]^2 = \prod_{\beta \in \hat{\beta}_{11}} (\beta + \gamma + \delta)^{2n} \).

**Proof.** Applying \( \Omega \)-operation \( k-3 \) times to the \( (\mathbb{Z}_2)^3 \)-action \( \Lambda, M^5 \) of Lemma 7.2 in Section 7 gives a \( (\mathbb{Z}_2)^{k-3} \)-action \( \Omega^{k-3}(\Lambda, M^5) \). It is easy to see that \( \Omega^{k-3}(\Lambda, M^5) \) is a \( \mathfrak{w} \)-nonempty action in \( A^k_{\mathbb{Z}_2} \), which has the property that \( |\mathfrak{w}| = 2^{k-2} \) and each element of \( \mathfrak{w} \) occurs exactly two times. In particular, by applying an automorphism
\( \sigma \in \text{GL}(k, \mathbb{Z}_2) \) of \((\mathbb{Z}_2)^k \) to \( \Omega^{k-3}(\Lambda, M^5) \) (if necessary), we can choose the same basis as that in Proposition 8.3 so that \( \hat{\omega} \) is exactly the disjoint union

\[
\{ \beta + \gamma + \delta \mid \beta \in \hat{\beta}_{11} \} \cup \{ \beta + \gamma + \delta \mid \beta \in \hat{\beta}_1 \},
\]

Now by applying \( \Delta \)-operation \( m \) times to \( \sigma \Omega^{k-3}(\Lambda, M^5) \), one may obtain a \( \hat{\omega} \)-nonempty \((\mathbb{Z}_2)^k\)-action \( \Delta^m[\sigma \Omega^{k-3}(\Lambda, M^5)] \) in \( A_{5m,2^{k-3}}^k(4) \) such that its changeable part

\[
\hat{\omega} = \{ \beta + \gamma + \delta \mid \beta \in \hat{\beta}_1 \} \cup \{ \beta + \gamma + \delta \mid \beta \in \hat{\beta}_{11} \},
\]

i.e., each element of \( \{ \beta + \gamma + \delta \mid \beta \in \hat{\beta}_{11} \} \) has multiplicity \( 2m \) times in \( \hat{\omega} \).

Similarly to the proof of Lemma 8.11 taking the following polynomial function over \( \mathbb{Z}_2 \) which is symmetric in variables \( x_1, \ldots, x_{5m,2^{k-3}} \)

\[
g(x_1, \ldots, x_{5m,2^{k-3}}; \hat{\beta}_{11}, \hat{\gamma}_{11}, \hat{\delta}_{11}, \hat{\epsilon}_{11}, \hat{\eta}_{11}, \hat{\lambda}_{11})
= \sum_{i_1, \ldots, i_n} \left[ h(x_{i_1}, \ldots, x_{i_{m,2^{k-2}}}; \hat{\beta}_{11}, \hat{\gamma}_{11}, \hat{\delta}_{11}, \hat{\epsilon}_{11}, \hat{\eta}_{11}, \hat{\lambda}_{11}) f(x_{i_{m,2^{k-2}}+1}, \ldots, x_{i_{5m,2^{k-3}}}) \right]
\]

where each \( \{i_1, \ldots, i_n\} = \{1, \ldots, n\} \) and

\[
h(x_{i_1}, \ldots, x_{i_{m,2^{k-2}}}; \hat{\beta}_{11}, \hat{\gamma}_{11}, \hat{\delta}_{11}, \hat{\epsilon}_{11}, \hat{\eta}_{11}, \hat{\lambda}_{11})
= \prod_{j=1}^{m,2^{k-2}} \left[ \prod(x_{ij} + \hat{\beta}_{11}) \prod(x_{ij} + \hat{\gamma}_{11}) \prod(x_{ij} + \hat{\delta}_{11}) \prod(x_{ij} + \hat{\epsilon}_{11}) \prod(x_{ij} + \hat{\eta}_{11}) \times \prod(x_{ij} + \hat{\lambda}_{11}) \right].
\]

By direct calculations one has that

\[
\hat{g} = \frac{h(\hat{\omega}, \hat{\beta}_{11}, \hat{\gamma}_{11}, \hat{\delta}_{11}, \hat{\epsilon}_{11}, \hat{\eta}_{11}, \hat{\lambda}_{11}) f}{\prod_{\beta \in \hat{\beta}_{11}} (\beta + \gamma + \delta)^{2m}}
\]

which belongs to \( \mathbb{Z}_2[\rho_1, \ldots, \rho_k] \) by Theorem 8.3 and so \( \hat{f} \) is divisible by \( \prod_{\beta \in \hat{\beta}_{11}} (\beta + \gamma + \delta)^{2m} \). \( \square \)

**Lemma 8.13.** Let \( f(x_1, \ldots, x_{3m,2^{k-3}}) \) be the product

\[
\sigma_{m,2^{k-2}}(x_1, \ldots, x_{3m,2^{k-3}}) \cdot \sigma_{3m,2^{k-3}}(x_1, \ldots, x_{3m,2^{k-3}})
\]

of the \((m,2^{k-2})\)-th elementary symmetric function and the \((3m,2^{k-3})\)-th elementary symmetric function in \(3m \cdot 2^{k-3}\) variables. Then

\[
\hat{f} = \prod_{\beta \in \hat{\beta}_{11}} (\beta + \gamma + \delta)^{2m}.
\]

**Proof.** Since \( \deg f = 5m \cdot 2^{k-3} \), one has that \( \deg \hat{f} = 2m \cdot 2^{k-3} = m \cdot 2^{k-2} \). Thus, in order to prove that \( \hat{f} = \prod_{\beta \in \hat{\beta}_{11}} (\beta + \gamma + \delta)^{2m} \), by Lemma 8.12 it suffices to show that \( \hat{f} \) is nonzero. Since \( \sigma_{m,2^{k-2}}(x_1, \ldots, x_{3m,2^{k-3}}) \) and \( \sigma_{3m,2^{k-3}}(x_1, \ldots, x_{3m,2^{k-3}}) \) are elementary symmetric functions and \( m \) is a power of 2, by direct calculations one
has that

\[
\hat{f} = \frac{f(\hat{\beta}_1, \hat{\gamma}_1, \hat{\delta}_1) + f(\hat{\beta}_1, \hat{\eta}_1, \hat{\varepsilon}_1) + f(\hat{\varepsilon}_1, \hat{\gamma}_1, \hat{\lambda}_1) + f(\hat{\lambda}_1, \hat{\eta}_1, \hat{\delta}_1)}{\prod \hat{\beta}_1 \prod \hat{\gamma}_1 \prod \hat{\delta}_1} = \sigma_{m, 2^{k-2}}(\hat{\beta}_1, \hat{\gamma}_1, \hat{\delta}_1) + \sigma_{m, 2^{k-2}}(\hat{\beta}_1, \hat{\eta}_1, \hat{\varepsilon}_1) + \sigma_{m, 2^{k-2}}(\hat{\varepsilon}_1, \hat{\gamma}_1, \hat{\lambda}_1) + \sigma_{m, 2^{k-2}}(\hat{\lambda}_1, \hat{\eta}_1, \hat{\delta}_1)
\]

so \(\hat{f} = (\hat{\sigma}_{2^{k-2}})^m\). Thus, this may be reduced to considering the case \(m = 1\), i.e., it only needs to show that \(\hat{\sigma}_{2^{k-2}} \neq 0\) for \(k \geq 3\).

We proceed by induction on \(k\). When \(k = 3\), by Remark 15 in Section 7 we know that \(\hat{\sigma}_2\) is nonzero. When \(k = l\), suppose inductively that \(\hat{\sigma}_{2^{l-2}} \neq 0\). Now consider the case in which \(k = l + 1\).

Let \(\hat{\beta}'_{11}\) denote the set of all odd sums formed by \(\beta_1, ..., \beta_{l-2}\) (note that \(l - 2 = k - 3\)), so \(\hat{\beta}'_{11}\) exactly contains the half of all elements of \(\hat{\beta}_{11}\) (i.e., \(\hat{\beta}'_{11}\) has just \(2^{k-4}\) different elements). Write \(\hat{\beta}'_{11} = \{\beta' + \beta_1 + \beta_{l-1} \mid \beta' \in \hat{\beta}'_{11}\}\). Then \(\hat{\beta}'_{11} \cap \hat{\beta}''_{11} = \emptyset\) and \(\hat{\beta}'_{11} \cup \hat{\beta}''_{11} = \hat{\beta}_{11}\). Similarly, let

\[
\begin{align*}
\hat{\gamma}'_{11} &= \{\gamma + \beta_1 + \beta' \mid \beta' \in \hat{\beta}'_{11}\} \\
\hat{\delta}'_{11} &= \{\delta + \beta_1 + \beta' \mid \beta' \in \hat{\beta}'_{11}\} \\
\hat{\varepsilon}'_{11} &= \{\gamma + \beta' \mid \beta' \in \hat{\beta}'_{11}\} \\
\hat{\eta}'_{11} &= \{\delta + \beta' \mid \beta' \in \hat{\beta}'_{11}\} \\
\hat{\lambda}'_{11} &= \{\gamma + \delta + \beta_1 + \beta' \mid \beta' \in \hat{\beta}'_{11}\}
\end{align*}
\]

so \(\hat{\gamma}'_{11}\) (resp. \(\hat{\delta}'_{11}, \hat{\varepsilon}'_{11}, \hat{\eta}'_{11},\) and \(\hat{\lambda}'_{11}\)) also contains exactly the half of all elements of \(\hat{\gamma}_{11}\) (resp. \(\hat{\delta}_{11}, \hat{\varepsilon}_{11}, \hat{\eta}_{11},\) and \(\hat{\lambda}_{11}\)). Furthermore, one has that

\[
\begin{align*}
\hat{\gamma}''_{11} &= \{\gamma + \beta' + \beta_{l-1} \mid \beta' \in \hat{\beta}'_{11}\} \text{ with } \hat{\gamma}'_{11} \cap \hat{\gamma}''_{11} = \emptyset \text{ and } \hat{\gamma}'_{11} \cup \hat{\gamma}''_{11} = \hat{\gamma}_{11} \\
\hat{\delta}''_{11} &= \{\delta + \beta' + \beta_{l-1} \mid \beta' \in \hat{\beta}'_{11}\} \text{ with } \hat{\delta}'_{11} \cap \hat{\delta}''_{11} = \emptyset \text{ and } \hat{\delta}'_{11} \cup \hat{\delta}''_{11} = \hat{\delta}_{11} \\
\hat{\varepsilon}''_{11} &= \{\gamma + \beta' + \beta_1 + \beta_{l-1} \mid \beta' \in \hat{\beta}'_{11}\} \text{ with } \hat{\varepsilon}'_{11} \cap \hat{\varepsilon}''_{11} = \emptyset \text{ and } \hat{\varepsilon}'_{11} \cup \hat{\varepsilon}''_{11} = \hat{\varepsilon}_{11} \\
\hat{\eta}''_{11} &= \{\delta + \beta' + \beta_1 + \beta_{l-1} \mid \beta' \in \hat{\beta}'_{11}\} \text{ with } \hat{\eta}'_{11} \cap \hat{\eta}''_{11} = \emptyset \text{ and } \hat{\eta}'_{11} \cup \hat{\eta}''_{11} = \hat{\eta}_{11} \\
\hat{\lambda}''_{11} &= \{\gamma + \delta + \beta' + \beta_{l-1} \mid \beta' \in \hat{\beta}'_{11}\} \text{ with } \hat{\lambda}'_{11} \cap \hat{\lambda}''_{11} = \emptyset \text{ and } \hat{\lambda}'_{11} \cup \hat{\lambda}''_{11} = \hat{\lambda}_{11}.
\end{align*}
\]
Obviously, after reducing modulo $\beta_1 + \beta_{-1}$, we see that $\hat{\beta}'_{11}$ (resp. $\hat{\gamma}'_{11}$, $\hat{\delta}'_{11}$, $\hat{\varepsilon}'_{11}$, $\hat{\eta}'_{11}$, and $\hat{\lambda}'_{11}$) becomes $\hat{\beta}'_{11}$ (resp. $\hat{\gamma}'_{11}$, $\hat{\delta}'_{11}$, $\hat{\varepsilon}'_{11}$, $\hat{\eta}'_{11}$, and $\hat{\lambda}'_{11}$). Thus
\[ \sigma_{2l-1} = \sigma_{2l-1}(\hat{\beta}'_{11}) + \sigma_{2l-1}(\hat{\gamma}'_{11}, \hat{\delta}'_{11}) + \sigma_{2l-1}(\hat{\varepsilon}'_{11}, \hat{\eta}'_{11}, \hat{\lambda}'_{11}) \]
\[ = \sigma_{2l-2}(\hat{\beta}'_{11}, \hat{\gamma}'_{11}, \hat{\delta}'_{11}) + \sigma_{2l-2}(\hat{\varepsilon}'_{11}, \hat{\eta}'_{11}, \hat{\lambda}'_{11}) \]
\[ = \left( \sigma_{2l-2}(\hat{\beta}'_{11}, \hat{\gamma}'_{11}, \hat{\delta}'_{11}) + \sigma_{2l-2}(\hat{\varepsilon}'_{11}, \hat{\eta}'_{11}, \hat{\lambda}'_{11}) \right)^2 \]
\[ = (\hat{\sigma}_{2l-2})^2 \mod (\beta_1 + \beta_{-1}) \]
\[ \neq 0 \text{ by induction} \]
so $\hat{\sigma}_{2l-1}$ must be nonzero. This completes the induction, and thus $\hat{f} \neq 0$. \[ \square \]

Combining Lemmas 8.11, 8.13 and Proposition 8.5 it easily follows that

**Corollary 8.14.** Each element of $\{\beta + \gamma + \delta \mid \beta \in \hat{\beta}_{11}\}$ occurs at most $2m$ times in $\hat{\omega}$ and so $|\hat{\omega}| \leq m \cdot 2^{k-2}$. Furthermore, $n$ is in the range $3m \cdot 2^{k-3} \leq n \leq 5m \cdot 2^{k-3}$.

Together with all arguments above, now we can give a description of the essential structure of $\mathcal{N}_{(\Phi, \mathcal{M}^n)}$, which is stated as follows.

**Theorem 8.15.** Let $(\Phi, \mathcal{M}^n) \in A^k_n(4)$ be an action with four fixed points $p, q, r, s$ and let $(\Gamma_{(\Phi, \mathcal{M}^n)}, \alpha)$ be a colored graph of $(\Phi, \mathcal{M}^n)$. Then
(i) $k \geq 3$ and $n$ is in the range $3 \cdot 2^\ell \leq n \leq 5 \cdot 2^\ell$ for some $\ell \geq k-3$;
(ii) there is a basis $\{\beta_1, \ldots, \beta_{k-2}, \gamma, \delta\}$ of $\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ such that
\[
\alpha(E_p) = \hat{\beta} \cup \hat{\gamma} \cup \hat{\delta} \cup \hat{\omega}, \quad \alpha(E_q) = \hat{\beta} \cup \hat{\eta} \cup \hat{\varepsilon} \cup \hat{\omega},
\]
\[
\alpha(E_r) = \hat{\gamma} \cup \hat{\varepsilon} \cup \hat{\lambda} \cup \hat{\omega}, \quad \alpha(E_s) = \hat{\delta} \cup \hat{\eta} \cup \hat{\lambda} \cup \hat{\omega}
\]
where $\hat{\beta}$ is a multiset consisting of all sums with same multiplicity $2^{\ell-k+3}$ formed by the odd number of elements of $\beta_1, \ldots, \beta_{k-2}$, and
\[
\gamma = \{\gamma + \beta_1 + \beta \mid \beta \in \hat{\beta}\}, \quad \delta = \{\delta + \beta_1 + \beta \mid \beta \in \hat{\beta}\},
\]
\[
\varepsilon = \{\gamma + \beta \mid \beta \in \hat{\beta}\}, \quad \eta = \{\delta + \beta \mid \beta \in \hat{\beta}\},
\]
\[
\lambda = \{\gamma + \delta + \beta_1 + \beta \mid \beta \in \hat{\beta}\}, \quad |\hat{\omega}| = n - 3 \cdot 2^\ell
\]
and every element in $\hat{\omega}$ has the form $\gamma + \delta + \beta$ and occurs at most $2^{\ell-k+4}$ times if $\hat{\omega}$ is non-empty, where $\beta \in \hat{\beta}$.

8.3. The existence of actions in $A^k_n(4)$.

**Definition 8.16.** Given a basis $B = \{\beta_1, \ldots, \beta_{k-2}, \gamma, \delta\}$ of $\text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ and an integer $\ell \geq k-3$ where $k \geq 3$. We say that $S_{B}^{k, \ell} = \{\hat{\beta}, \hat{\gamma}, \hat{\delta}, \hat{\varepsilon}, \hat{\eta}, \hat{\lambda}\}$ is a $\hat{\omega}$-empty
If \( \ell \) is fixed, then any two \( \hat{\omega} \)-empty \( 2^{-k+3} \)-multi-structures \( S_B^{k, \ell} \) and \( S_B^{k, \ell} \) can be translated to each other by automorphisms of \( \text{Hom}(\mathbb{Z}_2^k, \mathbb{Z}_2) \).

By Proposition 8.18, Corollary 8.19 and Theorem 3.2, one has that

\[
\hat{f} = \frac{f(\hat{\beta}, \hat{\gamma}, \hat{\delta})}{\prod \beta \prod \gamma \prod \delta} + \frac{f(\hat{\beta}, \hat{\epsilon}, \hat{\eta})}{\prod \beta \prod \epsilon \prod \eta} + \frac{f(\hat{\epsilon}, \hat{\gamma}, \hat{\lambda})}{\prod \epsilon \prod \gamma \prod \lambda} + \frac{f(\hat{\lambda}, \hat{\eta}, \hat{\delta})}{\prod \lambda \prod \eta \prod \delta},
\]

belongs to \( \mathbb{Z}_2[\rho_1, ..., \rho_k] \). Further, by Lemma 8.12, \( \hat{f} \) is divisible by \( [\prod (\gamma + \delta + \beta)]^2 \).

Now, for any integer \( t \leq 2^\ell+1 \), one can always choose a multisets \( \Theta \) formed by \( \hat{\beta}_1, ..., \hat{\beta}_{k-2}, \hat{\gamma}, \hat{\delta} \) such that \( |\Theta| = t \), and each element of \( \Theta \) is chosen in the \( 2^{k-3} \) different elements of \( \{\beta + \gamma + \delta | \beta \in \hat{\beta}\} \) and has multiplicity at most \( 2^{\ell-k+4} \). Note that if \( t = 0 \) then \( \hat{\theta} \) is empty. Since \( |\hat{\beta}| = 2^\ell \) and \( \ell \geq k-3 \), one has that \( [\prod (\gamma + \delta + \beta)]^2 \) is divisible by \( \prod \Theta \), so \( \hat{f} \) is also divisible by \( \prod \Theta \). Consider any symmetric function \( g(x_1, ..., x_n) \) in \( n \) variables. One can write \( g(x_1, ..., x_n) \) as a sum

\[
\sum_{h,f} h(x_1, ..., x_t)f(x_{t+1}, ..., x_n)
\]

of products \( h(x_1, ..., x_t)f(x_{t+1}, ..., x_n) \) such that each \( h \) is a function in \( t \) variables \( x_1, ..., x_t \) and each \( f \) is always a symmetric function in \( n-t \) variables \( x_{t+1}, ..., x_n \).
(note that for the cases \( n = 4, 5 \), see the proofs of Lemmas \( \text{7.1, 7.2} \). Then one has that
\[
\hat{g} = \frac{g(\Theta, \hat{\beta}, \hat{\gamma}, \hat{\delta})}{\prod \Theta} \prod \beta \prod \hat{\gamma} \prod \hat{\delta} + \frac{g(\Theta, \hat{\beta}, \hat{\eta}, \hat{\varepsilon})}{\prod \Theta} \prod \beta \prod \hat{\eta} \prod \hat{\varepsilon} + \frac{g(\Theta, \hat{\varepsilon}, \hat{\gamma}, \hat{\lambda})}{\prod \Theta} \prod \hat{\varepsilon} \prod \hat{\gamma} \prod \hat{\lambda}
\]
\[+ \frac{g(\Theta, \hat{\lambda}, \hat{\eta}, \hat{\delta})}{\prod \Theta} \prod \hat{\lambda} \prod \hat{\eta} \prod \hat{\delta}
\]

\[
= \sum \frac{h(\Theta)}{\prod \Theta} \left\{ \frac{f(\hat{\beta}, \hat{\gamma}, \hat{\delta})}{\prod \beta} \prod \hat{\gamma} \prod \hat{\delta} + \frac{f(\hat{\beta}, \hat{\eta}, \hat{\varepsilon})}{\prod \beta} \prod \hat{\eta} \prod \hat{\varepsilon} + \frac{f(\hat{\varepsilon}, \hat{\gamma}, \hat{\lambda})}{\prod \hat{\varepsilon}} \prod \hat{\gamma} \prod \hat{\lambda} + \frac{f(\hat{\lambda}, \hat{\eta}, \hat{\delta})}{\prod \hat{\lambda}} \prod \hat{\eta} \prod \hat{\delta} \right\}
\]
\[= \sum \frac{h(\Theta)\hat{f}}{\prod \Theta}
\]

so \( \hat{g} \) belongs to \( \mathbb{Z}_2[\rho_1, ..., \rho_k] \). Thus by Theorem \( 8.3 \) and Proposition \( 8.5 \) there is a \( \mathbb{Z}_2 \)-action such that its changeable part \( \hat{\omega} \) is just \( \Theta \). \( \Box \)

Together with Corollaries \( 8.8, 8.14 \) and Proposition \( 8.18 \) we have completely determined the existence of all actions in \( \mathcal{A}_n^k(4) \). The result is stated as follows.

**Theorem 8.19.** \( \mathcal{A}_n^k(4) \) is nonempty if and only if \( k \geq 3 \) and \( n \) is in the range \( \bigcup_{\ell \geq k-3} [3 \cdot 2^\ell, 5 \cdot 2^\ell] \).

We see from the proof of Proposition \( 8.18 \) that \( \hat{\omega} \) can always happen as long as \( 0 \leq |\hat{\omega}| \leq 2^{\ell+1} \). However, when we fix an integer \( t \) with \( 0 \leq t \leq 2^{\ell+1} \), generally there may be different choices of \( \hat{\omega} \) with \( |\hat{\omega}| = t \) if \( t > 0 \). This can be seen from the following example.

**Example 4.** Consider the \( \mathbb{Z}_2^4 \)-manifold \( \Omega(\phi_3, \mathbb{R}P^3) \). This is a \( \hat{\omega} \)-empty \( \mathbb{Z}_2^4 \)-action, whose colored graph is shown in Figure 9. It is easy to see that there are two

**Figure 9.** The colored graph of \( \Omega(\phi_3, \mathbb{R}P^3) \)

different choices for \( \hat{\omega} \) with \( |\hat{\omega}| = 1 \), which are \( \{\rho_1 + \rho_2 + \rho_3\} \) and \( \{\rho_1 + \rho_2 + \rho_3 + \rho_4\} \) respectively, and there are three different choices for \( \hat{\omega} \) with \( |\hat{\omega}| = 2 \), which are \( \{\rho_1 + \rho_2 + \rho_3, \rho_1 + \rho_2 + \rho_3, \rho_1 + \rho_2 + \rho_3 + \rho_4\} \), \( \{\rho_1 + \rho_2 + \rho_3 + \rho_4, \rho_1 + \rho_2 + \rho_3 + \rho_4, \rho_1 + \rho_2 + \rho_3 + \rho_4\} \) and \( \{\rho_1 + \rho_2 + \rho_3 + \rho_4, \rho_1 + \rho_2 + \rho_3 + \rho_4, \rho_1 + \rho_2 + \rho_3 + \rho_4\} \) respectively. Also, these different choices of \( \hat{\omega} \) with \( |\hat{\omega}| = 1 \) or \( 2 \) determine different actions up to equivariant cobordism.

Now fix an \( \hat{\omega} \)-empty \( 2^{\ell-k+3} \)-multi-structure \( \mathcal{S}_B^k \) over \( B = \{\beta_1, ..., \beta_{k-2}, \gamma, \delta\} \) and then let us look at how many there are choices of \( \hat{\omega} \) with \( |\hat{\omega}| = t \).
First, write $2^{k-3}$ different elements in $\{\beta + \gamma + \delta | \beta \in \hat{\beta}\}$ as $w_1, \ldots, w_{2^{k-3}}$. Next, choose an integer vector $v = (v_1, \ldots, v_{2^{k-3}})$ in $\mathbb{Z}^{2^{k-3}}_{\geq 0}$ such that
\[ |v| = v_1 + \cdots + v_{2^{k-3}} = t \]
and each $v_i \leq 2^{t-k+4}$. This vector $v$ determines a unique choice of $\hat{\omega}$ with $|\hat{\omega}| = t$, denoted by $\hat{\omega}^Y_B$, in such a way that each $w_i$ has multiplicity just $v_i$ in $\hat{\omega}$. Thus, it is not difficult to see that the number of different choices for $\hat{\omega}$ with $|\hat{\omega}| = t$ is exactly the number of those lattices satisfying the equation $x_1 + \cdots + x_{2^{k-3}} = t$ with $0 \leq x_i \leq 2^{t-k+4}$ in $\mathbb{R}^{2^{k-3}}$, i.e., the number of all integer vectors in the set
\[ \mathcal{I}^k(t) := \left\{ (v_1, \ldots, v_{2^{k-3}}) \in \mathbb{Z}^{2^{k-3}}_{\geq 0} \mid \sum_{i=1}^{2^{k-3}} v_i = t \text{ with each } 0 \leq v_i \leq 2^{t-k+4} \right\}. \]

Obviously, if $k = 3$, then $\hat{\omega}$ with $|\hat{\omega}| = t$ has a unique choice.

We see from the proof of Proposition 8.18 that each $\hat{\omega}^Y_B$ determines a $(\mathbb{Z}_2)^k$-action with the tangent representation set
\[ \{\hat{\omega}^Y_B \cup \hat{\beta} \cup \hat{\gamma} \cup \hat{\delta}, \hat{\omega}^Y_B \cup \hat{\beta} \cup \hat{\eta} \cup \hat{\epsilon}, \hat{\omega}^Y_B \cup \hat{\epsilon} \cup \hat{\gamma} \lor \hat{\lambda}, \hat{\omega}^Y_B \lor \hat{\lambda} \lor \hat{\eta} \lor \hat{\delta}\}, \]
which is called the $(\mathbb{Z}_2)^k$-action of type $\hat{\omega}^Y_B$ in $A^k_{3^{\ell}t+4}(4)$.

**Lemma 8.20.** Let $k > 3$ and let $v_1$ and $v_2$ be two different vectors in $\mathcal{I}^k(t)$. Then the $(\mathbb{Z}_2)^k$-action of type $\hat{\omega}^Y_B_v$ is not equivariantly cobordant to the $(\mathbb{Z}_2)^k$-action of $\hat{\omega}^Y_B_w$.

**Proof.** Obviously, both $v_1$ and $v_2$ with $v_1 \neq v_2$ give two different $\hat{\omega}^Y_B_v$ and $\hat{\omega}^Y_B_w$. Then two $(\mathbb{Z}_2)^k$-actions of types $\hat{\omega}^Y_B_v$ and $\hat{\omega}^Y_B_w$ determine two different tangent representation sets. Furthermore, the lemma follows from this by Theorem 8.22 \(\square\)

**Remark 22.** We see by Lemma 8.20 that up to equivariant cobordism, the number of all $(\mathbb{Z}_2)^k$-actions of types $\hat{\omega}^Y_B$ with $|v| = t$ in $A^k_{3^{\ell}t+4}(4)$ is the same as that of elements in $\mathcal{I}^k(t)$.

**8.4. The equivariant cobordism classification of all actions in $A^k_{n}(4)$.** Suppose that $A^k_{n}(4)$ is nonempty. Then by Theorem 8.19 there are integers $\ell$ and $t$ with $\ell \geq k - 3$ and $t \leq 2^{\ell+1}$ such that $n = 3 \cdot 2^\ell + t$.

Now let us consider the equivariant cobordism classification of all actions in $A^k_{3^{\ell}t+4}(4)$.

For $t = 0$, one knows from Remark 18 that for any two actions $(\Phi_1, M_1)$ and $(\Phi_2, M_2)$ in $A^k_{3^{\ell}2}(4)$, there is an automorphism $\sigma \in \text{GL}(k, \mathbb{Z}_2)$ such that $(\Phi_1, M_1)$ is equivariantly cobordant to $(\sigma \Phi_2, M_2)$. On the other hand, applying $\Omega$-operation $k - 3$ times and $\Delta$-operation $2^\ell-k+3$ times to $(\phi_3, \Omega P^3)$ gives a $(\mathbb{Z}_2)^k$-action $\Delta^2^{k-3}\Omega^3(\phi_3, \Omega P^3)$ in $A^k_{3^{\ell}2}(4)$. Thus one has

**Proposition 8.21.** Each of $A^k_{3^{\ell}2}(4)$ with $\ell \geq k - 3$ is equivariantly cobordant to one of
\[ \sigma \Delta^2^{k-3}\Omega^{k-3}(\phi_3, \Omega P^3), \quad \sigma \in \text{GL}(k, \mathbb{Z}_2). \]

For $t > 0$, as shown in Subsection 8.3 generally the changeable part $\hat{\omega}$ with $|\hat{\omega}| = t$ may have many possible choices, but we can give a description for those
possible choices. In particular, this description also works for the case \( t = 0 \). Thus, throughout the following assume that \( 0 \leq t \leq 2^{\ell+1} \).

Beginning with the \((\mathbb{Z}_2)^k\)-action \( \Delta^{2^{\ell+3}} \Omega^{k-3}(\phi_3, \mathbb{R}P^3) \) in \( A_{3,2^{\ell+1}}^k(4) \), which is equivariantly cobordant to an action \((\Psi, M)\) fixing exactly four isolated points. Then, by Lemma 8.17 we can obtain a unique \( \tilde{\omega}\)-empty \( 2^{\ell+k+3}\)-multi-structure \( S_B^k \) for \( B = \{ \beta_1, \ldots, \beta_{k-2}, \gamma, \delta \} \) such that the tangent representation set of \((\Phi, M)\) consists of

\[
\tilde{\beta} \cup \tilde{\gamma} \cup \tilde{\delta}, \; \tilde{\beta} \cup \tilde{\gamma} \cup \tilde{\delta} \cup \tilde{\epsilon} \cup \tilde{\lambda}, \; \tilde{\gamma} \cup \tilde{\delta} \cup \tilde{\lambda}.
\]

Now, given an integer vector \( v \in I^k(t) \) with \( |v| = t \), in the way as in Subsection 8.19 one can obtain a changeable part \( \tilde{\omega}_B^n \). Furthermore, by Lemma 8.20 up to equivariant cobordism there is a unique action denoted by \( \Lambda^*(\Delta^{2^{\ell+k+3}} \Omega^{k-3}(\phi_3, \mathbb{R}P^3)) \) such that its tangent representation set is

\[
\{ \tilde{\omega}_B^n \cup \tilde{\beta} \cup \tilde{\gamma} \cup \tilde{\delta}, \tilde{\omega}_B^n \cup \tilde{\beta} \cup \tilde{\gamma} \cup \tilde{\delta} \cup \tilde{\epsilon} \cup \tilde{\lambda}, \tilde{\gamma} \cup \tilde{\delta} \cup \tilde{\lambda} \}.
\]

Let \( \sigma \Lambda^*(\Delta^{2^{\ell+k+3}} \Omega^{k-3}(\phi_3, \mathbb{R}P^3)) \) denote the action produced by applying an automorphism \( \sigma \in \text{GL}(k, \mathbb{Z}_2) \) to \( \Lambda^*(\Delta^{2^{\ell+k+3}} \Omega^{k-3}(\phi_3, \mathbb{R}P^3)) \).

**Theorem 8.22.** Up to equivariant cobordism,

\[
\{ \sigma \Lambda^*(\Delta^{2^{\ell+k+3}} \Omega^{k-3}(\phi_3, \mathbb{R}P^3)) \mid v \in I^k(t), \sigma \in \text{GL}(k, \mathbb{Z}_2) \}
\]

gives all possible nonbounding actions in \( A_{3,2^{\ell+1}}^k(4) \) where \( \ell \geq k - 3 \geq 0 \) and \( 0 \leq t \leq 2^{\ell+1} \).

**Proof.** Let \((\Psi, N)\) be an action in \( A_{3,2^{\ell+1}}^k(4) \). Without loss of generality assume that \((\Psi, N)\) has exactly four fixed points. It is not difficult to see from Proposition 8.19 that \((\Psi, N)\) determines a unique \( \tilde{\omega}\)-empty \( 2^{\ell+k+3}\)-multi-structure over some basis \( B' \) of \( \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \). In addition, \((\Psi, N)\) also determines a unique changeable part \( \tilde{\omega}_B^n \). Thus, the tangent representation set \( N_{(\Psi, N)} \) is uniquely constructed by \( B' \). Since any two bases in \( \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \) can always be translated to each other by automorphisms of \( \text{Hom}((\mathbb{Z}_2)^k, \mathbb{Z}_2) \), there must be one \( \sigma \in \text{GL}(k, \mathbb{Z}_2) \) such that \((\Psi, N)\) and \( \sigma \Lambda^*(\Delta^{2^{\ell+k+3}} \Omega^{k-3}(\phi_3, \mathbb{R}P^3)) \) have the same tangent representation set. The theorem then follows from Theorem 8.22. \( \square \)

**Remark 23.** It should be pointed out that \( \Lambda^*(\Delta^{2^{\ell+k+3}} \Omega^{k-3}(\phi_3, \mathbb{R}P^3)) \) is not a concrete action, but its tangent representation set is concrete and can be constructed. We try to construct a concrete action, but fail.

Now, with Theorems 8.19 and 8.22 together, we complete the proof of Theorem 1.5.

### 8.5. The characterization of the colored graphs of actions in \( A_n^k(4) \)

Suppose that \( A_n^k(4) \) is nonempty. Then one knows from Theorem 8.19 there are integers \( \ell \) and \( t \) with \( \ell \geq k - 3 \) and \( t \leq 2^{\ell+1} \) such that \( n = 3 \cdot 2^\ell + t \).

Now let \((\Gamma, \alpha)\) be an abstract 1-skeleton of type \((3 \cdot 2^\ell + t, k)\) such that \( \Gamma \) contains exactly four vertices \( p, q, r, s \). Then we consider the following question.

**P** When can \((\Gamma, \alpha)\) become a colored graph of some action in \( A_{3,2^{\ell+1}}^k(4) \)?

If \( t = 0 \), we can characterize the colored graphs of actions in \( A_{3,2^{\ell+1}}^k(4) \), and our result is stated as follows.
Theorem 8.23. If $t = 0$, then $(\Gamma, \alpha)$ is realizable as a colored graph of some action in $A^k_{3,2^t}(4)$ if and only if there is a basis $\{\beta_1, \ldots, \beta_{k-2}, \gamma, \delta\}$ of $\text{Hom}(\mathbb{Z}_2^k, \mathbb{Z}_2)$ such that

$$
\alpha(E_p) = \hat{\beta} \cup \hat{\gamma} \cup \hat{\delta}, \quad \alpha(E_q) = \hat{\beta} \cup \hat{\eta} \cup \hat{\varepsilon}, \\
\alpha(E_r) = \hat{\varepsilon} \cup \hat{\gamma} \cup \hat{\lambda}, \quad \alpha(E_s) = \hat{\lambda} \cup \hat{\eta} \cup \hat{\delta}
$$

where $\hat{\beta}$ is a multiset consisting of all odd sums with same multiplicity $2^{\ell-k+3}$ formed by $\beta_1, \ldots, \beta_{k-2}$, and

$$
\begin{cases}
\hat{\gamma} = \{\gamma + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
\hat{\delta} = \{\delta + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
\hat{\varepsilon} = \{\varepsilon + \beta | \beta \in \hat{\beta}\} \\
\hat{\eta} = \{\delta + \beta | \beta \in \hat{\beta}\} \\
\hat{\lambda} = \{\gamma + \delta + \beta_1 + \beta | \beta \in \hat{\beta}\}
\end{cases}
$$

Proof. Clearly, all actions of $A^k_{3,2^t}(4)$ determine a unique connected regular graph with four vertices. Then Theorem 8.23 immediately follows from Proposition 8.5 and Corollary 8.9.

If $t > 0$, then generally $\Gamma_{(\Phi, M)}$ of an action $(\Phi, M^n)$ in $A^k_{3,2^t+t}(4)$ may not be uniquely determined, so this leads to the difficulty of determining an abstract 1-skeleton to be a colored graph of some action. The problem seems to be quite complicated. Actually, even if $n = 5$, as seen in the 5-dimensional example $(\Lambda, M^5)$ of Lemma 7.2, we still cannot determine which of six possible abstract 1-skeletons in Figure 7 is the colored graphs of $(\Lambda, M^5)$. However, we can characterize the tangent representation set of each action in $A^k_{3,2^t+t}(4)$. By Theorem 8.15 and Proposition 8.18 one has that

Theorem 8.24. If $t > 0$, then the vertex-coloring set $\{\alpha(E_p), \alpha(E_q), \alpha(E_r), \alpha(E_s)\}$ of $(\Gamma, \alpha)$ is the tangent representation set of some $G$-action $(\Phi, M)$ in $A^k_{3,2^t+t}(4)$ if and only if there is a basis $\{\beta_1, \ldots, \beta_{k-2}, \gamma, \delta\}$ of $\text{Hom}(\mathbb{Z}_2^k, \mathbb{Z}_2)$ such that

$$
\alpha(E_p) = \hat{\beta} \cup \hat{\gamma} \cup \hat{\delta} \cup \hat{\omega}, \quad \alpha(E_q) = \hat{\beta} \cup \hat{\eta} \cup \hat{\varepsilon} \cup \hat{\omega}, \\
\alpha(E_r) = \hat{\varepsilon} \cup \hat{\gamma} \cup \hat{\lambda} \cup \hat{\omega}, \quad \alpha(E_s) = \hat{\delta} \cup \hat{\eta} \cup \hat{\lambda} \cup \hat{\omega}
$$

where $\hat{\beta}$ is a multiset consisting of all sums with same multiplicity $2^{\ell-k+3}$ formed by the odd number of elements of $\beta_1, \ldots, \beta_{k-2}$, and

$$
\begin{cases}
\hat{\gamma} = \{\gamma + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
\hat{\delta} = \{\delta + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
\hat{\varepsilon} = \{\varepsilon + \beta | \beta \in \hat{\beta}\} \\
\hat{\eta} = \{\delta + \beta | \beta \in \hat{\beta}\} \\
\hat{\lambda} = \{\gamma + \delta + \beta_1 + \beta | \beta \in \hat{\beta}\} \\
|\hat{\omega}| = t
\end{cases}
$$

and every element in $\hat{\omega}$ is chosen in the $2^{k-3}$ different elements of $\{\gamma + \delta + \beta | \beta \in \hat{\beta}\}$ and has multiplicity at most $2^{\ell-k+4}$. 

Finally, combining Theorems 8.15, 8.23-8.24 and Proposition 8.18, we complete the proof of Theorem 1.7.

9. AN OBSERVATION ON THE MINIMUM NUMBER OF FIXED POINTS OF ACTIONS

Define

\[ m(n, k) := \min \{|M^G| | (\Phi, M^n) \in A_n^k \} \].

Theorem 1.2 has given an estimation of the lower bound for the number of fixed points of each action in \( A_n^k \), so one has that

\[ m(n, k) \geq 1 + \left\lceil \frac{n}{n - k + 1} \right\rceil \].

Then it easily follows that

**Lemma 9.1.** If there is a \( G \)-action \( (\Phi, M^n) \) in \( A_n^k \) such that \( |M^G| = 1 + \left\lceil \frac{n}{n - k + 1} \right\rceil \), then

\[ m(n, k) = 1 + \left\lceil \frac{n}{n - k + 1} \right\rceil \].

By Lemma 9.1, we know from Examples 2-3 that

\[ m(n, k) = \begin{cases} n + 1 & \text{if } n = k \geq 2 \\ 3 & \text{if } n = 2^\ell \geq 2^{k-1} \geq 2. \end{cases} \]

On the other hand, Theorems 1.4(a) tells us that

\[ m(n, k) = 3 \iff n = 2^\ell \geq 2^{k-1} \geq 2. \]

This gives all possible values of \( n \) and \( k \) for \( m(n, k) = 3 \). Set

\[ \mathcal{X}_1 = \left\{ (n, k) \in \mathbb{N}^2 | n = 2^\ell \geq 2^{k-1} \geq 2 \right\} \]

and

\[ \mathcal{X}_2 = \left\{ (n, k) \in \mathbb{N}^2 | k \geq 3, n \in \bigcup_{\ell \geq k-3} [3 \cdot 2^\ell, 5 \cdot 2^\ell] \right\} \].

Since 3 is the minimum possible value of \( m(n, k) \), by Theorem 1.5(a) we have that

\[ m(n, k) = 4 \iff (n, k) \in \mathcal{X}_2 \setminus \mathcal{X}_1. \]

Combining the above arguments, we have that

**Proposition 9.2.**

(a) \( m(n, k) = 3 \) if and only if \( (n, k) \in \mathcal{X}_1 \).

(b) \( m(n, k) = 4 \) if and only if \( (n, k) \in \mathcal{X}_2 \setminus \mathcal{X}_1 \).

(c) If \( n = k \geq 2 \), then \( m(n, k) = n + 1 \).

By applying the \( \Omega \)-operation and the \( \Delta \)-operation to \((\phi_4, \mathbb{R}P^4)\), one can obtain the \((\mathbb{Z}_2)^k\)-action \( (\Phi, M^n) \) fixing five isolated points with

\[ (n, k) \in \mathcal{X}_3 = \left\{ (n, k) \in \mathbb{N}^2 | n = 2^\ell \geq 2^{k-2} \geq 4 \right\}. \]

It is easy to see that \( \mathcal{X}_3 \setminus (\mathcal{X}_1 \cup \mathcal{X}_2) \) is nonempty. For example, \((4, 4) \in \mathcal{X}_3 \) but \((4, 4) \notin \mathcal{X}_1 \cup \mathcal{X}_2 \). By Proposition 9.2 one has

**Corollary 9.3.** If \( (n, k) \in \mathcal{X}_3 \setminus (\mathcal{X}_1 \cup \mathcal{X}_2) \), then \( m(n, k) = 5 \).

By Lemma 5.2 it easily follows that

**Corollary 9.4.** Let \( n \geq 2 \) be even and let \( n = 2^{p_1} + \cdots + 2^{p_r} \) with \( 0 < p_1 < \cdots < p_r \) be the \( 2 \)-adic expansion of \( n \). Then \( m(n, 2) = 3^r \).
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