

$L^\pm$ Operators and Quantum Enveloping Algebras *

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Abstract

The relations between $L^\pm$ operators and the generators in the quantum enveloping algebras are studied. The $L^\pm$ operators for $U_qA_N$ and $U_qG_2$ algebras are explicitly expressed by the generators as examples.

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I. Introduction

Recently, the theories of quantum enveloping algebras and quantum groups have drawn wide concerns, and made great progress \cite{1,2}. For a generic \( q \), there have been standard methods for calculating the highest weight representations and Clebsch-Gordan coefficients of quantum enveloping algebras \cite{3,4}. Based on them, a series of solutions \( R_q \) of the simple Yang-Baxter equation (without spectral parameter) and solutions \( R_q(x) \) of the Yang-Baxter equation \cite{5} were found. In terms of the quantum double, the general form of universal \( R \) matrix were obtained in principle \cite{3}. For \( q \) being a root of unity, the representation theory was studied, too \cite{6}. The representation of quantum affine algebras and their applications to the solvable lattice models has been made great advance \cite{7}. On the other hand, following the general idea of Connes on the non-commutative geometry \cite{8}, Woronowicz \cite{9} elaborated the framework of the non-commutative differential calculus. He introduced the bimodule over the quantum groups and presented varies theorems concerning the differential forms and exterior derivative. The \( q \)-deformed gauge theory was studied from several different viewpoints \cite{8−10}.

A quantum group is introduced as a Hopf algebra \( \mathcal{A} = \text{Fun}_q(G) \) which is both non-commutative and non-cocommutative. It is the continuous deformation of the Hopf algebra of the functions on a Lie group, and is freely generated by the non-commutative matrix elements \( T^a_b \):

\[
(R_q)^{ca} r_s T^r_b T^s_d = T^a_c (R_q)^{sr} b_d
\]  

(1.1)

where the matrix \( T \) takes value in the minimal representation, and \( R_q \) is the solution of the simple Yang-Baxter equation corresponding to the minimal representation.

Faddeev-Reshetikhin-Takhtajan \cite{11} introduced two linear functionals \( L^\pm \) which belong to the dual Hopf algebra \( \mathcal{A}' \). They are defined by their values on the elements \( T^a_b \):

\[
\left( L^+ \right)^a_b (T^c_d) = (R_q^{-1})^{ac} b_d, \quad \left( L^- \right)^a_b (T^c_d) = (R_q)^{ca} d_b
\]  

(1.2)

As is well known, the quantum enveloping algebra is also a Hopf algebra dual to \( \mathcal{A} \),
and the eq.(1.1) implies, in fact, this dual relation (see Sec.II), so it must tightly connect with the operators $L^\pm$. Ref.[11] gave their relations for $U_qA_1$, but their generalization is not trivial. Some authors tried to make clear the profound relations between quantum groups and quantum enveloping algebras [12]. In this paper, after a further discussion on those relations we carefully study the concrete expressions of the operators $L^\pm$ by the generators in a quantum enveloping algebra, and explicitly give the expressions in two typical examples: $U_qA_N$ and $U_qG_2$ algebras.

II. The dual relation

The generators $t_j$, $e_j$ and $f_j$ of a quantum enveloping algebra $U_qG$ satisfy the following algebraic relations:

$$
t_i t_j = t_j t_i, \quad t_j^{-1} t_j = t_j^{-1} = 1,
$$

$$
t_i e_j = q_i^{a_{ij}} e_j t_i, \quad t_i f_j = q_i^{-a_{ij}} f_j t_i, \quad q_i = q^{d_i},
$$

$$
[e_i, f_j] = \delta_{ij} \omega_j^{-1} (t_j - t_j^{-1}), \quad \omega_j = q_i - q_i^{-1}
$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right] q_i \begin{pmatrix} e^{1-a_{ij}-n} e_j e_i^n \end{pmatrix} = 0, \quad i \neq j
$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right] q_i \begin{pmatrix} f^{1-a_{ij}-n} f_j f_i^n \end{pmatrix} = 0, \quad i \neq j
$$

(2.1)

where $a_{ij}$ is the Cartan matrix element of a Lie algebra $G$, and $d_j$ is half-length of the simple root. For the longer simple root, $d_j = 1$. As a Hopf algebra, the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$ of the generators are defined as follows:

$$
\Delta(t_j) = t_j \otimes t_j, \quad \Delta(e_j) = e_j \otimes 1 + t_j \otimes e_j,
$$

$$
\Delta(f_j) = f_j \otimes t_j^{-1} + 1 \otimes f_j,
$$

$$\varepsilon(t_j) = 1, \quad \varepsilon(e_j) = \varepsilon(f_j) = 0,
$$

$$S(t_j) = t_j^{-1}, \quad S(e_j) = -t_j^{-1} e_j, \quad S(f_j) = -f_j t_j,
$$

(2.2)

In fact, the representation matrix elements $D_q^{[\lambda]}(\alpha)_{\mu\nu}$ of a quantum enveloping algebra give the dual relation between the generators $\alpha \in U_qG$ and the non-commutative
quantity $\left(T^{[\lambda]}\right)^{\mu}_{\nu}$ of the quantum group in the corresponding representation:

$$\langle \alpha, \left(T^{[\lambda]}\right)^{\mu}_{\nu} \rangle = D^{[\lambda]}(\alpha)_{\mu\nu}, \quad \alpha \in U_q G$$  \hfill (2.3)

When $[\lambda]$ is the minimal representation, $\left(T^{[\lambda]}\right)^{\mu}_{\nu}$ becomes $T^{q}_{\lambda b}$.

The universal $R$ matrix can be expressed as

$$R = \sum_{i} u^{i} \otimes v^{i} \in U_q G \otimes U_q G$$  \hfill (2.4)

where $u^{i}$ and $v^{i}$ are the dual bases of up- and down-Borel subalgebras, respectively.

Take the value of $R$ matrix in the minimal representation, we get the $R^{q}$ matrix:

$$\langle R, T^{c}_{d} \otimes T^{a}_{b} \rangle = (R^{q})^{ca}_{db}$$  \hfill (2.5)

The universal $R$ matrix satisfies [3,4]:

$$R \Delta(\alpha) = (P \circ \Delta(\alpha)) R, \quad \alpha \in U_q G$$  \hfill (2.6)

$$R^{-1} = (\text{id} \otimes S^{-1}) R$$  \hfill (2.7)

where $P$ is the space transposition operator. Substituting eq.(2.6) into eq.(2.5), we get

$$R^{ca}_{rs} \langle \Delta(\alpha), T^{c}_{b} \otimes T^{a}_{d} \rangle = \langle R \Delta(\alpha), T^{c}_{b} \otimes T^{a}_{d} \rangle$$

$$= \langle (P \circ \Delta(\alpha)) R, T^{c}_{b} \otimes T^{a}_{d} \rangle = \langle \Delta(\alpha), T^{a}_{s} \otimes T^{c}_{r} \rangle R^{rs}_{bd}$$  \hfill (2.8)

Due to arbitrariness of $\alpha$, one can immediately get eq.(1.1) from eq.(2.8). Similarly, it is easy to show that equation (1.1) holds for any other representation. So the consistent condition (1.1) is just the direct reflection of the dual relation between a quantum group and a quantum enveloping algebra.

Now we rewrite eq.(1.2) as the form of dual relation:

$$\langle (L^{-})^{a}_{b}, T^{c}_{d} \rangle = (R^{q})^{ca}_{db}$$

Comparing it with eq.(2.5), we have

$$\left(L^{-}\right)^{a}_{b} = \langle R, \text{id} \otimes T^{a}_{b} \rangle$$  \hfill (2.9)

Similarly,

$$\left(L^{+}\right)^{a}_{b} = \langle R^{-1}, T^{a}_{b} \otimes \text{id} \rangle$$
From eq.(2.7) we have:

\[(SL^+)^a_b = \langle R, T^a_b \otimes \text{id} \rangle\] (2.10)

In a different notation, the analogous relations were also given in Ref. [13]. By making use of the general form of the universal \( R \) matrix given by Jimbo \[3\], one can calculate the explicit relations between \((L^+)^a_b\) and the generators in a quantum enveloping algebra from eqs.(2.9) and (2.10).

Denote by \( Q_+ \) the set of all the non-negative integral combinations of simple roots, \( Q_+ = \left\{ \beta | \beta = \sum m_j r_j, m_j \in \mathbb{Z}_- \right\} \)

For a given \( \beta \in Q_+ \), we define two finite-dimensional spaces \( N^\pm_\beta \):

\[N^+_\beta : \left\{ e_{i_1} e_{i_2} \cdots e_{i_m} | \sum j r_{ij} = \beta \right\}
N^-_\beta : \left\{ f_{i_1} f_{i_2} \cdots f_{i_m} | \sum j r_{ij} = \beta \right\}\]

where \( r_{ij} \) is the simple root corresponding to the generators \( e_{ij} \) and \( f_{ij} \). Those two spaces both are Hopf algebras, and they are associated and dual to each other \[4\]. In these dual spaces we can adopt the following dual bases:

\[u^i \in N^+_\beta, \quad v_i \in N^-_\beta,\]

so that the general form of universal \( R \) matrix can be formally expressed as \[2, 4\]:

\[R = \left( \sum_{\beta \in Q_+} L_\beta \right) q^{-H}, \quad L_\beta = \sum_i u^i \otimes v_i.\] (2.11)

A representation is called integrable if:

i) \( V = \oplus \mu V_\mu \)

\[V_\mu = \left\{ v_\mu \in V | t_j v_\mu = q^{\mu j} v_\mu, \quad \mu = \sum \mu_j \lambda_j, \quad 1 \leq j \leq N \right\}\]

ii) For any \( v \in V \), there exists a common \( M \) such that:

\[e^M_j v = 0, \quad f^M_j v = 0.\]
And $q^{-H}$ is given by
\[ q^{-H} (v_\mu \otimes v_\nu) = q^{-\sum_{ij} d_i(a^{-1})_{ij} \mu_i \nu_j} (v_\mu \otimes v_\nu) \] (2.12)

where $(a^{-1})$ is the inverse Cartan matrix of the Lie algebra $G$.

When calculating $(L^-)^a_b$ by eq.(2.9), the action on the first subspace should keep its operator form, but for the second subspace, it should be taken value in the minimal representation, i.e., expressed as matrix form. By making use of this method, we are able to express $(L^\pm)^a_b$ operators by the generators in any quantum enveloping algebra in principle. In the rest of this paper we will compute the explicit expressions for two typical quantum enveloping algebras as examples.

III. $U_q A_N$ algebra

The reason for choosing $U_q A_N$ algebra as the first example is that it is the simplest and the most useful qutantum enveloping algebra. Jimbo [2] pointed out that for $U_q A_N$ any generator corresponding to a non-simple root can be expressed as a linear combination of the generators $e_j$ (or $f_j$) that correspond to the simple roots:

\[
E_{j(j+1)} = e_j, \quad E_{ij} = E_{ik} E_{kj} - q E_{kj} E_{ik}, \quad i < j < k
\]

Through calculation, we find that $(L^\pm)^a_b$ are directly related with these generators.

In the minimal representation of $U_q A_N$, the bases can be represented by only one index $a$, corresponding to the following weight:

\[
a \longrightarrow \lambda_a - \lambda_{a-1}, \quad \lambda_0 = \lambda_{N+1} = 0, \quad 1 \leq a \leq N + 1
\]

where $\lambda_a$ are the fundamental dominant weights of $U_q A_N$. The matrix forms of the generators in these bases are given as follows:

\[
t_{j} |a\rangle = q^{(S_{aj} - \delta_{a-1})_{jj}} |a\rangle \quad 1 \leq a \leq N + 1
\]
\[
e_{j} |a\rangle = |(a + 1)\rangle, \quad f_{j} |a\rangle = |(a - 1)\rangle
\]
From eq.(2.12), we have
\[ q^{-H} (|a\rangle \otimes |b\rangle) = \left\{ \prod_j t_j^{s_j} \otimes 1 \right\} (|a\rangle \otimes |b\rangle) \]
\[ \tau_{jb} = (a^{-1})_j(b^{-1})_j - (a^{-1})_jb = \begin{cases} \frac{j}{N+1} - 1, & j \geq b \\ \frac{j}{N+1}, & j < b \end{cases} \]
or briefly express it as:
\[ q^{-H} (|a\rangle \otimes |b\rangle) = \left\{ \prod_j t_j^{s_j/(N+1)} \right\} (\prod_{k=b}^N \frac{1}{k-1}) \otimes 1 \right\} (|a\rangle \otimes |b\rangle) \quad \text{(3.4)} \]

Now we use eq.(2.9) to calculate \((L^-)_b\). If \(a < b\), \((L^-)_b\) vanishes. If \(a = b\), \((L^-)_b\) only contains factor \(q^{-H}\) given in eq.(3.4). If \(a > b\), there is only one term in the summation of \(\mathcal{R}\) that has nonvanishing contribution to \((L^-)_b\), where \(v_i\) is:
\[ v_i = f_{a-1} f_{a-2} \cdots f_b, \quad a \geq b \quad \text{(3.5)} \]

The key to the problem is to find \(u^i\) dual to the above \(v_i\). In the calculation the following dual relations \([3,4]\) are used:
\[ \langle t_i^n, t_j^m \rangle = q_i^{-a_i} t_i^{-a_i}, \quad \langle e_i, f_j \rangle = -\delta_{ij} \omega^{-1}_i, \]
\[ \langle e_i, t_j^m \rangle = \langle t_i^n, f_j \rangle = 0 \quad \text{(3.6)} \]

When \((a-b)\) is small, the dual operator \(u^i\) is easy to calculate. We can drew the general form of \(u^i\) from these \(u^i\), then prove it by induction. Noting that for \(U_qA_N\) all \(\omega_j\) are equal to each other and can be denoted by one symbol \(\omega\). The result is:
\[ u^i = -\omega q^{a-b-1} \sum_P (-q)^{-n(P)} e_{p_1} e_{p_2} \cdots e_{p_{a-b}} \quad a > b \quad \text{(3.7)} \]
where \(P\) is a certain permutation of \((a-b)\) objects, that can be expressed by a product of some transpositions, and each transposition moves a smaller number from the right of a bigger one to the left.
\[ P = \begin{pmatrix} (a-1) & (a-2) & \cdots & b \\ p_1 & p_2 & \cdots & p_{a-b} \end{pmatrix} \]
In such a multiplication expression, each transposition appears at most once. We are only interested in the "neighboring transposition" that permutes the places of two neighboring numbers \( c \) and \((c - 1)\). Two \( P \) are called equivalent if their multiplication expressions contain same neighboring transpositions without concerning their order. The summation in eq.(3.6) runs over all inequivalent permutations include the unit element. \( n(P) \) is the number of neighboring transpositions contained in \( P \).

Substituting (3.4), (3.7) into (2.9) and (2.11), we finally get

\[
(L^-)_b^a = \begin{cases} 
0, & a < b \\
\left\{ \prod_{j=1}^{N} t_j^{j/(N+1)} \right\} \left\{ \prod_{k=b}^{a-1} t_k^{-1} \right\}, & a = b \\
(-1)^{a-b} \omega E_{ba} \left\{ \prod_{j=1}^{N} t_j^{j/(N+1)} \right\} \left\{ \prod_{k=b}^{a-1} t_k^{-1} \right\}, & a > b
\end{cases}
\]  

(3.8)

Similarly

\[
(L^+_b)_a = \begin{cases} 
0, & a > b \\
(-1)^{b-a+1} \omega \left\{ \prod_{j=1}^{N} t_j^{-j/(N+1)} \right\} \left\{ \prod_{k=a}^{b-1} t_k \right\} E_{ba}, & a < b \\
\left\{ \prod_{j=1}^{N} t_j^{-j/(N+1)} \right\} \left\{ \prod_{k=a}^{b-1} t_k \right\}, & a = b
\end{cases}
\]  

(3.9)

Note that it is only for \( U_q A_N \) where the generators corresponding to non-simple roots can be expressed as (3.1), so that the expressions of \((L^\pm)_b^a\) become so simple. For other algebras, the expressions become much more complicated. \( U_q G_2 \) algebra is an example that will be discussed in the next section.

**IV. \( U_q G_2 \) algebra**

\( U_q G_2 \) is another typical example where the lengths of two simple roots are different. Let

\[
q_1 = q, \quad q_2 = q^{1/3}, \quad [m] = \frac{q_2^m - q_2^{-m}}{q_2 - q_2^{-1}}, \quad \omega_2 = q_2 - q_2^{-1}, \quad \omega_1 = q_1 - q_1^{-1} = [3] \omega_2
\]  

(4.1)
The minimal representation of $U_qG_2$ algebra is given as follows:

\[ D_q(t_1) = \text{diag} \{ 1, q, q^{-1}, 1, q, q^{-1}, 1 \}, \]
\[ D_q(t_2) = \text{diag} \{ q_2, q_2^{-1}, q_2^2, q_2, q_2^{-1} \}, \]
\[ D_q(e_{12}) = D_q(f_1)_{32} = D_q(e_{56}) = D_q(f_1)_{65} = 1, \quad (4.2) \]
\[ D_q(e_{23}) = D_q(f_2)_{21} = D_q(e_{67}) = D_q(f_2)_{76} = 1, \]
\[ D_q(e_{34}) = D_q(f_2)_{43} = D_q(e_{45}) = D_q(f_2)_{54} = [2]^{1/2} \]

The rest of elements are vanishing. Through tedious calculation, we obtain:

\[ (L^-)^1_1 = t_1^{-1} t_2^{-2}, \quad (L^-)^2_2 = t_1^{-1} t_2^{-1}, \]
\[ (L^-)^3_3 = t_2^{-1}, \quad (L^-)^4_4 = 1, \]
\[ (L^-)^5_5 = t_2, \quad (L^-)^6_6 = t_1 t_2, \]
\[ (L^-)^7_7 = t_1 t_2, \]
\[ (L^-)^2_1 = -\omega_2 e_2 t_1^{-1} t_2^{-2}, \quad (L^-)^3_2 = -\omega_1 e_1 t_1^{-1} t_2^{-1}, \]
\[ (L^-)^4_3 = -\omega_2 [2]^{1/2} e_2 t_2^{-1}, \quad (L^-)^5_4 = -\omega_2 [2]^{1/2} e_2, \]
\[ (L^-)^6_5 = -\omega_1 e_1 t_2, \quad (L^-)^7_6 = -\omega_2 e_2 t_1 t_2, \]
\[ (L^-)^3_1 = \omega_2 (q e_1 e_2 - e_2 e_1) t_1^{-1} t_2^{-2}, \]
\[ (L^-)^4_2 = -\omega_2 [2]^{1/2} (e_1 e_2 - q e_2 e_1) t_1^{-1} t_2^{-1}, \]
\[ (L^-)^5_3 = \omega_2 q_2^{-1} e_2 t_2^{-1}, \]
\[ (L^-)^6_4 = \omega_2 [2]^{1/2} (q e_1 e_2 - e_2 e_1), \]
\[ (L^-)^7_5 = -\omega_2 (e_1 e_2 - q e_2 e_1) t_2, \]
\[ (L^-)^4_1 = \omega_2 q_2 [2]^{-1/2} \{ e_1 e_2^2 - ([6]/[3]) e_2 e_1 e_2 + e_2 e_1 \} t_1^{-1} t_2^{-2}, \]
\[ (L^-)^5_2 = -\omega_2 [2]^{-1} \{ e_1 e_2^2 - q_2^2 [2] e_2 e_1 e_2 + q_2^4 e_2 e_1 \} t_1^{-1} t_2^{-1}, \]
\[ (L^-)^6_3 = -\omega_2 [2]^{-1} \{ q_2^4 e_1 e_2^2 - q_2^2 [2] e_2 e_1 e_2 + e_2 e_1 \} t_2^{-1}, \]
\[ (L^-)^7_4 = \omega_2 q_2^2 [2]^{-1/2} \{ e_1 e_2^2 - ([6]/[3]) e_2 e_1 e_2 + e_2 e_1 \}, \]
\[ (L^-)^5_1 = \omega_2 q_2 [2]^{-1} \{ e_1 e_2^3 - ([4] - q_2) e_2 e_1 e_2 + q_2 ([4] - q_2) e_2 e_1 e_2 - q_2 e_2 e_1 \} t_1^{-1} t_2^{-2}, \]
\[ (L^-)^6_2 = -\omega_1 \omega_2 q_2^3 [6]^{-1} \{ e_1 e_2^3 + [3] e_2 e_1 e_2 - ([6][5]/[3][2]) e_2 e_1 e_2 + e_2 e_1 \} t_1^{-1} t_2^{-1}, \]
\[ (L^-)^7_3 = -\omega_2 q_2 [2]^{-1} \{ q_2 e_1 e_2^3 - q_2 ([4] - q_2) e_2 e_1 e_2 + ([4] - q_2) e_2 e_1 e_2 - e_2 e_1 \} t_2^{-1}, \]
\[(L^-)^6_a = \omega_1 q([6][2])^{-1} \left\{ q_2^2 e_1^2 e_2^3 + q_2 e_2^3 e_2 + q_2([4] - q_2^{-1}) e_2 e_1 e_2^2 \\
+ q_2^{-1}([4] - q_2) e_2^3 e_2 - q_2^2([4] - q_2)([6]/[3]) e_2 e_1 e_2 \\
- q_2^2([4] - q_2^{-1})([6]/[3]) e_2 e_1 e_2 + ([6][4]/[3][2]) e_1 e_2 e_1 \right\} t_1^{-1} t_2^{-2},
\]
\[(L^-)^7_a = \omega_1 q([6][2])^{-1} \left\{ q_2^2 e_1^2 e_2^3 + q_2 e_2^3 e_1^2 + q_2([4] - q_2) e_2 e_1 e_2^2 \\
+ q_2([4] - q_2^{-1}) e_2^3 e_1^2 - q_2^2([4] - q_2^{-1})([6]/[3]) e_2 e_1 e_2 \\
- q_2^2([4] - q_2)([6]/[3]) e_2 e_1 e_2 + ([6][4]/[3][2]) e_1 e_2 e_1 \right\} t_1^{-1} t_2^{-1},
\]
\[(L^-)^7_b = \omega_1 q([6][2])^{-1} \left\{ ([6][4]^2/[3][2]) e_2 e_1 e_2 e_1 - ([6]/[3][2]) e_1 e_2 e_1 \\
-(4[2] - 1) e_2^3 e_2^2 + (e_2^2 e_1^2 + e_1^2 e_2^2) - ([6][5]/[3][2]) (e_1 e_2^3 e_1 e_2^2 + e_2^3 e_1 e_2) \right\} t_1^{-1} t_2^{-2},
\]
\[(4.8)\]

\[(L^+)^a_b \text{ can be obtained from } (L^-)^b_a \text{ by following transformations: replace } q_j \text{ by } q_j^{-1} \text{ (so } \omega_j \text{ becomes } -\omega_j), \ e_j \text{ by } f_j, \ t_i \text{ by } t_i^{-1}, \text{ and reverse the product order of operators in each term. For example,}
\]
\[
\left( L^+ \right)_4^1 = -\omega_2 q_2^{-2}[2]^{-1/2} t_1 t_2 \left\{ f_2^2 f_1 - ([6]/[3]) f_2 f_1 f_2 + f_1 f_2^2 \right\}
\]

It seems to us that this relation between \((L^+)^a_b\) and \((L^-)^b_a\) holds for all quantum enveloping algebras.

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