STABILITY OF HYPERSURFACES IN MINKOWSKI NORMED SPACES

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Abstract. We extend to Minkowski spaces the classical result of Barbosa and do Carmo [1] that characterizes the euclidean sphere as the unique compact stable CMC hypersurface of $\mathbb{R}^n$. More precisely, if $K$ is a smooth convex body in $\mathbb{R}^n$ with positive Gauss curvature, containing the origin in its interior and $M$ is an immersed hypersurface, there are well defined concepts of surface area measure, normal vector field and principal curvatures of $M$, with respect to $K$. Thus we introduce the concept of stability with respect to normal variations and compute the formula of second variation with respect to $K$. Finally we show that if $M$ is compact, has constant mean Minkowski curvature and is stable (with respect to $K$) then $M$ is homothetic to $\partial K$.

1. Introduction and main results

1.1. Euclidean variations. Let $M$ be an $n-1$ oriented differentiable manifold and $x : M \to \mathbb{R}^n$ a smooth immersion. The area of the immersion $x$ is defined as

$$A(x) = \int_M dS$$

where $dS$ is the area element induced by $x$. We can define the volume of $x$ as

$$V(x) = \frac{1}{n} \int_M \langle x, \xi \rangle dS$$

where $\xi$ denotes the unit normal vector field determined by the orientation of $M$. The definition of $V$ is justified by the fact that when $x$ is an embedding and $M$ is closed, $V$ represents the volume of the interior of $M$.

A variation of $M$ is a smooth function $F : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^n$ such that for every $t \in (-\varepsilon, \varepsilon)$, the function $F^t = F(t, \cdot)$ is also an immersion. For such a variation, the area and volume defined above give one-parameter functions $A(t) = A(F^t)$, $V(t) = V(F^t)$. We say that a variation has compact support if $F(t, p) = p$ for every $p$ outside a compact subset of $M$.

It is known that minimal surfaces, this is, surfaces with zero mean curvature, arise naturally as critical points of the area function, whereas CMC surfaces (surfaces with constant mean curvature) correspond to the critical points of the area with restricted volume. More precisely, we say that an immersion is minimal if for every variation with compact support we have $A'(0) = 0$. It is known that for any immersion $x$ and any variation $F$,

$$A'(0) = \int_M (n-1)H_x(p)\langle F_t(0, p), \xi(p) \rangle dS$$

where $H_x$ denotes the mean curvature and $F_t(0, p) = \frac{\partial}{\partial t} F(0, p)$, so minimal immersions are characterized as immersions with zero mean curvature. Minimal surfaces
arise naturally as solutions of the Plateau problem, that is finding a surface with prescribed boundary, minimizing the surface area measure. If we restrict to immersions having a fixed volume, we see from the well known formula
\[ V'(0) = \int_M \langle F_t(0, p), \xi(p) \rangle dS \]
that the immersions such that \( A'(0) = 0 \) for every volume-preserving variation of compact support, are precisely those with constant mean curvature.

Surfaces minimizing the area with restricted volume are CMC but the converse is not true, meaning that a CMC surface may be deformed locally into nearby surfaces with less area. This is the case for example, of cylindrical and plane surfaces.

We say that an immersion is \textit{stable} if it has constant mean curvature and if for every variation of compact support that preserves the volume, \( A''(0) \geq 0 \). For a variation of any minimal immersion, it is known that the formula of second variation holds
\[ A''(0) = \int_M f(\Delta f + B_e^2 f) dS \]
where \( f(p) = \langle F_t(0, p), \xi(p) \rangle \) is the normal component of the variation, \( \Delta \) is the Laplace-Beltrami operator and \( B_e \) is the norm of the second fundamental form of \( M \).

If \( M = \partial \Omega \) is a closed smooth embedded surface that minimizes the surface area measure among all “nearby” surfaces enclosing the same volume as \( M \), then \( M \) must be a round sphere. The same holds true if \( M \) is only closed and CMC. This fact is known since the classical work of Alexandrov [2] where he introduced the moving planes method. But for immersed surfaces this fact is false in general. In [14] Hopf showed that a CMC immersion \( S^2 \to \mathbb{R}^3 \) must be a round sphere. Latter Hsiang constructed a CMC immersed surface of genus one (the Wente torus) in \( \mathbb{R}^3 \) and Kapouleas [15] constructed similar surfaces in \( \mathbb{R}^3 \), of higher genus. In contrast, Barbosa and do Carmo in [1] showed that the surface must be a round sphere when the ingredient of stability is added. They proved the following.

\textbf{Theorem 1} (Theorem 1.3, [1]). \textit{Let} \( M^{n-1} \) \textit{be compact, orientable, and let} \( x : M \to \mathbb{R}^n \) \textit{be an immersion with non-zero constant mean curvature. Then} \( x \) \textit{is stable if and only if} \( x(M) \subset \mathbb{R}^n \) \textit{is a (round) sphere} \( S^{n-1} \subset \mathbb{R}^n \).

In parallel to CMC surfaces, the theory of surfaces immersed in general \( n \)-dimensional normed spaces (Minkowski spaces) was studied by Busemann [5], Petty [6]. The aspects of differential geometry were further developed more recently by Balexto et al. [7], [8] in a systematic way.

We briefly recall the basic definitions. Let \( B \) be a compact convex set with the origin in the interior (there is no reason to assume that \( B \) is symmetric), and suppose it has smooth boundary with positive Gauss curvature. The Gauss map \( \partial B \to S^{n-1} \) is a smooth diffeomorphism and we denote the inverse by \( u : S^{n-1} \to \partial B \).

Given an immersed surface \( M \) with Gauss map \( \xi : M \to S^{n-1} \) we define \( \eta = u \circ \xi \) and observe that \( \eta(p) \in \partial B \) is the unique point such that \( T_{\eta(p)} \partial B = T_p M \). We call \( \eta \) the Birkhoff-Gauss map of \( M \) with respect to \( B \).

The differential \( d_p \eta \) can thus be regarded as an endomorphism of \( T_p M \) and can be shown to be diagonalizable with real eigenvalues \( \lambda_1, \ldots, \lambda_{n-1} \). This is done
considering the following Riemannian metric in \( M \) called the \textit{Dupin metric} \((\cdot,\cdot)_b\). For \( v,w \in T_pM \) define \((v,w)_b = (d_p u^{-1}(v),w)\) and notice that \( d_p \eta \) is self-adjoint with respect to \((\cdot,\cdot)_b\). The Dupin metric is shown in \[8\], \[9\] to be a useful tool for translating properties that are valid in classical differential geometry, to the Minkowski case.

The Minkowskian mean curvature and Minkowskian Gauss curvature are defined by \( H_m = \frac{\lambda_1 + \cdots + \lambda_{n-1}}{n-1} \) and \( K_m = \lambda_1 \cdots \lambda_{n-1} \) respectively.

The support function of \( B \) is defined by
\[
h_B(v) = \max_{x \in B} \langle v, x \rangle
\]
and, for a unit vector \( v \in S^{n-1} \), it measures the distance from the origin to the supporting hyperplane of \( B \) perpendicular to \( v \). Notice that \( h_B(\xi(p)) = \langle \eta(p), \xi(p) \rangle \), since by definition, \( T_{\eta(p)} \partial B + \eta(p) \) is this supporting hyperplane.

The Minkowskian area measure \( \omega = \omega_B \) is defined by
\[
A_m(x) = \int_M d\omega = \int_M \langle \eta, \xi \rangle dS
\]
and if \( F(t,p) \) is a variation of \( M \) with compact support, the first variation formula for \( A_m \) is given by
\[
A_m'(0) = \int_M (n-1) H_m(p) N_p(F_t(0,p)) d\omega(p)
\]
where \( N_p \) is the projection to the second coordinate in the direct-sum decomposition \( \mathbb{R}^n = T_pM \oplus \langle \eta \rangle \). Formula \[8\] was proven in \[8\] for variations in a more restricted class than \[1\], but the formula extends easily to the general case. We prove this in Appendix B.

If \( M \) is the (smooth) boundary of a bounded open set \( \Omega \), this area measure already appears in the literature as the \textit{mixed volume} of \( \Omega \) and \( B \) (see \[10\]). The mixed volume of any two compact convex sets \( K \) and \( L \) with non-empty interior, is defined as
\[
V(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0} \frac{\text{vol}(K + \varepsilon L) - \text{vol}(K)}{\varepsilon}
\]
where \( K + L = \{x + y : x \in K, y \in L\} \) is the Minkowski sum of sets. The integral representation (see \[10\])
\[
V(K, L) = \frac{1}{n} \int_M h_L(\xi(p)) dS(p)
\]
shows that the Minkowskian area measure of \( M \) is precisely \( nV(\Omega, B) \). The mixed volume inequality (see Theorem 7.2.1, \[10\]) states that
\[
V(K, L) \geq \text{vol}(K)^{\frac{n-1}{n}} \text{vol}(L)^{\frac{1}{n}}
\]
with equality if and only if \( K \) and \( L \) are homothetic.

As a consequence of the mixed volume inequality we deduce that the least possible value of the Minkowskian area measure of an embedded surface \( M = \partial \Omega \) with fixed volume \( \text{vol}(\Omega) = \nu \) is \( A_m(M) = n\nu^{\frac{1}{n-1}} \text{vol}(B)^{\frac{1}{n}} \), and this value is attained if and only if \( \Omega \) is homothetic to \( B \) and thus \( M = \partial (x_0 + \lambda B) \) is a \textit{Minkowskian sphere}.

The subject of Minkowskian differential geometry, although it was born more than half a century ago, remains vastly unexplored from the point of view of differential geometry. The above considerations imply that a Minkowskian version...
of the usual isoperimetric inequality holds for the area measure $A_m$. Also an immersed surface that minimizes the Minkowskian area with restricted volume must have constant Minkowskian mean curvature, as it is clear from the first variation formula (2). But to our knowledge, there is no treatment of the concept of Stability for the Minkowskian area measure.

Because of these facts, we propose to study the validity of Theorem 1 in the context of Minkowski geometry. Our contributions are the following:

**Theorem 2** (Second variation of the Minkowski area measure). Let $x : M \to \mathbb{R}^n$ be an immersed surface with Birkhoff-Gauss map $\eta$ and constant Minkowskian mean curvature, and let $F : (\varepsilon, \varepsilon) \times M \to \mathbb{R}^n$, be a volume-preserving variation of compact support given by

$$F(t, p) = F^t(p) = x(p) + g(t, p)\eta(p).$$

Denote $f(p) = \frac{\partial}{\partial t}y(t, p)\big|_{t=0}$ and $A_m(t) = A_m(F(t, \cdot))$ the area defined by (1). Then,

$$A_m''(0) = \int_M \left(-B^2_m f^2 + \langle \eta, \xi \rangle (\nabla_{b}f, \nabla_{b}f)_b\right) d\omega$$

$$= -\int_M f \left(B^2_m f + \langle \eta, \xi \rangle^{-1} \text{div}(\langle \eta, \xi \rangle^2 du(\nabla f))\right) d\omega$$

(4)

Here $\nabla_b f$ is the gradient of $f$ with respect to the Dupin metric and can be computed as $\nabla_b f = du(\nabla f)$ where $\nabla f$ is the gradient with respect to the usual metric. Also $B_m$ is the norm of the Minkowski second fundamental form, $B^2_m = \sum_{i=1}^{n-1} \lambda_i^2$.

The novelty in this formula is the term

$$\Delta_m(f) = \langle \eta, \xi \rangle^{-1} \text{div}(\langle \eta, \xi \rangle^2 du(\nabla f)),$$

that reduces to the usual Laplace-Beltrami operator when $B$ is the unit euclidean ball.

We also verify using formula (4), that the proof of Theorem 1 carries on to the Minkowski case. We say that an immersion is stable with respect to the Minkowskian structure if $A''_m(0) \geq 0$ for every variation of the form (3).

**Theorem 3.** Let $x : M \to \mathbb{R}^n$ be a compact immersed surface without boundary, with constant Minkowskian mean curvature and stable with respect to the Minkowskian structure. Then $x(M)$ is an embedded Minkowski sphere, this is, $x(M)$ is homothetic to $\partial B$.

The proof of Theorem 3 follows the same lines of Theorem 2 with some adaptations. The main difficulty here is to compute $\Delta_m(f)$ when $f$ is a suitable test function. This is done in Lemma 5.

The rest of the paper is organized as follows: In Section 2 we recall some basic definitions and lemmas. Then in Sections 3 and 4 we prove Theorems 2 and 3 respectively. The proof of Theorem 3 relies on a lengthy computation that we postpone to the Appendix A.

2. PRELIMINARIES

In this section we recall some observations that are important to the development of Section 3.
The following lemma is proved in [1] in the Euclidean context. Its extension to the Minkowskian case is immediate and we omit the proof.

**Lemma 1.** For any immersion with Minkowskian mean curvature $H_m$ and Minkowskian second fundamental form $II_m$ with $B_m = \|II_m\|$, we have $B_m^2 \geq (n-1)H_m^2$ with equality at a point $p$ and only if $p$ is an umbilic point (meaning that $\lambda_1 = \cdots = \lambda_{n-1}$).

The characterization of umbilical surfaces in Minkowskian geometry was done by Balestro, Martini, and Teixeira for dimension 3. The proof extends without modification to dimension $n$. We omit the proof.

**Lemma 2 ([7], Proposition 4.5).** A connected hypersurface immersed $\mathbb{R}^n$, all whose points are umbilic, is contained in a plane or in a Minkowski sphere.

The next lemma allows us to work with variations whose initial velocity has zero average, but are not necessarily volume-preserving. The proof is a standard application of the implicit function theorem.

**Lemma 3.** Let $x : M \to \mathbb{R}^n$ be an immersion with constant Minkowskian mean curvature and $F$ a variation of compact support of the form (3) and set $f(p) = \frac{\partial}{\partial t}g(t,p) \big|_{t=0}$.

Consider the functional $J_m(t) = A_m(t) - (n-1)H_mV(t)$, then we have

(a) If $\int_M fd\omega = 0$ then there exists a volume-preserving variation

$$\tilde{F}(t,p) = x(p) + \tilde{g}(t,p)\eta$$

such that $\frac{\partial}{\partial t}\tilde{g}(t,p) \big|_{t=0} = f(p)$.

(b) For such a variation $\tilde{F}$ we have $A''_m(0) = J''_m(0)$.

As mentioned in the introduction, the Minkowskian isoperimetric inequality is proven in [13] (with different notation) by extending the concept of mixed volume, and the mixed volume inequality to general domains.

**Lemma 4 ([13], Lemma 3.2).** Let $K \subseteq \mathbb{R}^n$ be a compact domain with smooth boundary and $B$ a compact convex set with the origin in the interior. Then

$$\frac{1}{n} \int_{\partial K} d\omega_B \geq \text{vol}(K)^{\frac{n-1}{n}} \text{vol}(B)^{\frac{1}{n}}$$

with equality if and only if $K$ and $B$ are homothetic.

3. **Second variation formula**

In this section we prove Theorem 2. In view of Lemma 3b it suffices to compute $J''_m$.

Let $p \in M$, there is a neighborhood of $p$ where the restriction of $x$ is an embedding. Without loss of generality we will assume for the local computations, that $M \subseteq \mathbb{R}^n$ is a submanifold and $x$ is the identity. Take an orthonormal basis $e_1, \ldots, e_{n-1}$, of $T_pM$ consisting of euclidean principal directions. For $t \in (-\varepsilon, \varepsilon)$ the vectors $e^t_i = d_pF^t(e_i)$ span the tangent space of $M^t = F^t(M)$ at $F(t,p)$. We denote respectively by $\eta^t_i, g^t_{ij}$ and $H^t_m$, the Birkhoff-Gauss map, the coefficients of the metric and the Minkowskian mean curvature, of $M^t$. Again without loss of generality the functions $\eta^t_i, g^t_{ij}, H^t_m$ can be regarded as functions defined on (neighborhoods of) $M$ or $M^t$ via composition with $F^t$. Denote by $\nabla$ the usual connection in $\mathbb{R}^n$. 
Proof of Theorem 3. Take coefficients $a_{k,i}^t$ such that

$$d\eta^t(e^t_i) = \nabla e^t_i(\eta^t \circ F^t) = \sum_k a_{k,i}^t e^t_k$$

and note that

$$(e^t_j, \nabla e^t_i \eta^t) = \sum_k a_{k,i}^t g_{j,k}^t$$

$$(n-1)H_m^t = \sum_i a_{i,i}^t = \sum_{i,j} g^{i,j,t} (e^t_j, \nabla e^t_i \eta^t)$$

where $g^{i,j,t}$ are the coefficients of the inverse matrix of $g_{i,j}^t$. 
Now we compute the derivative

\[ \frac{\partial}{\partial t}(n-1)H_m \bigg|_{t=0} = \sum_{i,j} \frac{\partial}{\partial t}g_{j,i} \bigg|_{t=0} \langle \varepsilon^1_j, \nabla_{\varepsilon^1_i} \eta \rangle + \delta_{i,j} \frac{\partial}{\partial t} \langle \varepsilon^1_j, \nabla_{\varepsilon^1_i} \eta \rangle \bigg|_{t=0} \]

\[ = \sum_{i,j} (e_i(f)\eta + f\nabla_{e_i} \eta, e_j) + (e_j(f)\eta + f\nabla_{e_j} \eta, e_i)\langle e_j, \nabla_{e_i} \eta \rangle \]

\[ + \sum_i (e_i(f)\eta + f\nabla_{e_i} \eta, \nabla_{e_i} \eta) + \left\langle e_i, \frac{\partial}{\partial t} \nabla_{\varepsilon^1_i} \eta \right\rangle \bigg|_{t=0} \]

\[ = -f \left( \sum_i \|\nabla_{e_i} \eta\|^2 + \sum_{i,j} \langle e_i, \nabla_{e_j} \eta \rangle \langle e_j, \nabla_{e_i} \eta \rangle \right) + f \sum_i \|\nabla_{e_i} \eta\|^2 \]

\[ - \sum_{i,j} (e_i(f)\eta, e_j) + (e_j(f)\eta, e_i)\langle e_j, \nabla_{e_i} \eta \rangle + \sum_i (e_i(f)\eta, \nabla_{e_i} \eta) + \left\langle e_i, \frac{\partial}{\partial t} \nabla_{\varepsilon^1_i} \eta \right\rangle \bigg|_{t=0} \]

\[ = -f B^2_m - \sum_i e_i(f) \left( \sum_j \langle \eta, e_j \rangle e_j, \nabla_{e_i} \eta \right) - \sum_{i,j} e_j(f) \langle \eta, e_i \rangle \langle e_j, \nabla_{e_i} \eta \rangle \]

\[ + \sum_i (e_i(f)\eta, \nabla_{e_i} \eta) + \left\langle e_i, \frac{\partial}{\partial t} \nabla_{\varepsilon^1_i} \eta \right\rangle \bigg|_{t=0} \]

\[ = -f B^2_m - \langle \eta^T, \nabla_{\eta^T} \eta \rangle - \sum_i \langle \eta, e_i \rangle \langle \nabla f, \nabla_{e_i} \eta \rangle + \langle \eta^T, \nabla_{\eta^T} \eta \rangle + \left\langle e_i, \frac{\partial}{\partial t} \nabla_{\varepsilon^1_i} \eta \right\rangle \bigg|_{t=0} \]

\[ = -f B^2_m - \langle \nabla f, \nabla_{\eta^T} \eta \rangle + \sum_i \left\langle e_i, \frac{\partial}{\partial t} \nabla_{\varepsilon^1_i} \eta \right\rangle \bigg|_{t=0} \]

\[ = -f B^2_m - \langle \nabla f, du(\nabla_{\eta^T} \xi) \rangle + \sum_i \left\langle e_i, \nabla_{e_i} \left( \frac{\partial}{\partial t} \eta \right) \bigg|_{t=0} \right\rangle \]

\[ = -f B^2_m - \langle du(\nabla f), \nabla(\eta, \xi) \rangle + \text{div} \left( \frac{\partial}{\partial t} \eta \bigg|_{t=0} \right) \]

\[ = -f B^2_m - \langle du(\nabla f), \nabla(\eta, \xi) \rangle - \text{div}(\langle \eta, \xi \rangle du(\nabla f)) \]

\[ = -f B^2_m - \langle \eta, \xi \rangle^{-1} \text{div}(\langle \eta, \xi \rangle^2 du(\nabla f)) \]
Finally we observe that
\[ J'(t) = A'(t) - (n - 1)H_m^0 \nu'(t) \]
\[ = (n - 1) \int_M g_t(H_m^t - H_m^0) d\omega \]
\[ J''(t) = (n - 1) \int_M f \left( \frac{\partial}{\partial t} H_m \bigg|_{t=0} \right) d\omega \]
and the result follows. \(\square\)

4. Stability

We start by showing that the Minkowski sphere is stable.

**Theorem 4.** Let \( B = x_0 + \lambda(\mathbb{R}) \) and \( F_t \) a volume-preserving variation of \( \partial B \) of the form \( \mathbf{4} \), then \( A'_m(0) = 0 \) and \( A''_m(0) \geq 0 \).

**Proof.** Since \( \eta \) is a transversal vector field we have that for small values of \( t \), \( F(t, \partial B) \) is the (smooth) boundary for some compact domain \( B_t \). By Lemma \( \mathbf{4} \)
\[ A_m(t) \geq n \text{vol}(B_t)^\frac{n-1}{n} \text{vol}(B)^\frac{1}{n} = n \text{vol}(B) = A_m(0) \]
and the result follows. \(\square\)

The following lemma is the key component of our stability theorem. The proof is a lengthy computation and will be presented in Appendix A, to improve readability.

**Lemma 5.** Let \( x : M \to \mathbb{R}^n \) have constant Minkowskian mean curvature and let \( \rho(x) = \langle \eta, \xi \rangle^{-1}(x, \xi) \). Then
\[ \langle \eta, \xi \rangle^{-1} \text{div}(\langle \eta, \xi \rangle^2 du(\nabla \rho)) = (n - 1)H_m - \rho B_m^2 \]

We shall verify the Minkowski identity for the function \( \rho \) in \( \mathbf{5} \) (see equation \( (5.2), [11] \)). To this end, consider the variation \( x(t, p) = (t + 1)p \) and notice that
\[ A_m(t) = (t + 1)^{n-1} A(0) \]
\[ A'_m(0) = (n - 1) \int_M d\omega. \]
On the other hand, by formula \( \mathbf{2} \) we have
\[ A'_m(0) = (n - 1) \int_M H_m(p) \rho(p) d\omega \]
then we obtain
\[ \int_M \rho H_m d\omega = \int_M d\omega. \]
Thus taking \( f(p) = 1 - \rho(p)H_m \) we have
\[ (6) \int_M f d\omega = 0. \]

Now we are in conditions to prove our main theorem:

**Proof of Theorem \( \mathbf{3} \).** Let \( \rho \) be as in Lemma \( \mathbf{5} \) By \( \mathbf{5} \) and by Lemma \( \mathbf{3} \) there is a volume-preserving variation
\[ F(t, p) = p + g(t, p) \eta \]
with \( g_t(0, p) = f \).
The second variation formula reads

\[ 0 \leq A''(0) = \int_M f (-B_m^2 f - \langle \eta, \xi \rangle^{-1} \text{div}((\eta, \xi)^2 du(\nabla f)))d\omega. \]

We compute

\[ \langle \eta, \xi \rangle^{-1} \text{div}((\eta, \xi)^2 du(\nabla f)) = -(n - 1)H^2_m + \rho B^2_m H_m \]

\[ -B_m^2 f^2 - \langle \eta, \xi \rangle^{-1} \text{div}((\eta, \xi)^2 du(\nabla f))f = -B_m^2 f^2 + (n - 1)H^2_m f - \rho H_m B^2_m f \]

\[ = -B_m^2 f^2 + (n - 1)H^2_m f - (1 - f)B_m^2 f \]

\[ = ((n - 1)H^2_m - B_m^2) f. \]

And using that \( \int_M f d\omega = 0 \) we obtain

\[ A''(0) = -\int_M B_m^2 f d\omega = -\int_M B_m^2 (1 - \rho H_m) d\omega. \]

Using (5) and the fact that \( H_m \) is constant we obtain

\[ 0 = \int_M \text{div}((\eta, \xi)^2 du(\nabla \rho))dS = \int_M \langle \eta, \xi \rangle^{-1} \text{div}((\eta, \xi)^2 du(\nabla \rho))d\omega = \int_M (n - 1)H_m - \rho B_m^2 d\omega \]

\[ \int_M (n - 1)H_m^2 d\omega = \int_M B_m^2 \rho H_m d\omega. \]

Substituting in (7) we get

\[ A''(0) = -\int_M B_m^2 - (n - 1)H_m^2 d\omega \leq 0 \]

where the integrand is non-negative by Lemma 1 implying that \( B_m^2 = (n - 1)H_m^2 \) for every \( p \in M \), hence all points of \( M \) are umbilic. Since \( M \) is compact, Lemma 2 implies that \( x(M) \subset \mathbb{R}^n \) is a Minkowski sphere.

\[ \square \]

5. Appendix A: Proof of Lemma 5

The proof is divided in several lemmas. Let \( p \in M \) and \( e_1, \ldots, e_{n-1} \) an orthonormal basis of \( T_pM \) consisting of euclidean principal directions.

Recall that \( u : S^{n-1} \to \partial \mathbb{B} \) is the inverse of the euclidean Gauss map of \( \partial \mathbb{B} \), \( \eta = u \circ \xi \) and that \( \rho = \langle \eta, \xi \rangle^{-1} \langle x, \xi \rangle \)

**Lemma 6.** Concerning \( \rho \) we have the following properties:

a) \( e_i(\rho) = \langle \eta, \xi \rangle^{-1} \langle x - \rho \eta, \nabla e_i, \xi \rangle \)

b) \( \sum_j \langle \nabla e_j, du(e_i), e_j \rangle \nabla e_i e_i = -\sum_j \langle du(e_j), \nabla e_j, \xi \rangle e_i \)

c) \( \sum_{i,j} e_i(\rho) \langle \eta, \xi \rangle \langle \nabla e_j, du(e_i) \rangle e_j = -\sum_{i,j} \langle x - \rho \eta, \nabla e_j, \xi \rangle \langle du(e_i), e_j \rangle \)

**Proof.** The first assertion is obvious, and the third one follows directly from the first two. For the second assertion, translate the basis \( \{e_i\} \), by parallel transport along geodesics issuing from \( p \), to all points in a geodesic neighborhood in \( M \). Extend the vector fields \( \{e_i\} \) to a neighbourhood of \( p \) in \( \mathbb{R}^n \) and notice that \( \nabla e_i e_j(p) = 0 \), \( [e_i, e_j](p) = 0 \) and thus \( \nabla e_i \nabla e_j X(p) = -\nabla e_i \nabla e_j X(p) = \nabla_{[e_i, e_j]} X(p) = 0 \) for every
vector field $X$. Now using that $\{e_i\}$ are eigenvalues of $d\xi$, that $H_m$ is constant and $du$ is self-adjoint,

$$
\langle \nabla e_j du(e_j), e_j \rangle = e_j (\langle du(e_j), e_j \rangle) \nabla e_j \xi \\
= e_j (\langle du(e_j), e_j \rangle) \nabla e_j \xi \\
= \langle \nabla e_j du(e_j), \nabla e_j \xi \rangle e_i \\
= (-\langle du(e_j), \nabla e_j \xi \rangle + e_j (\langle du(e_j), \nabla e_j \xi \rangle)) e_i
$$

$$
\sum_j e_j (\langle du(e_j), \nabla e_j \xi \rangle) = \sum_j e_j (\langle e_j, \nabla e_j \eta \rangle) \\
= \sum_j \langle e_j, \nabla e_j \nabla e_j \eta \rangle \\
= \sum_j \langle e_j, \nabla e_j \nabla e_j \eta \rangle \\
= e_i (\sum_j \langle e_j, \nabla e_j \eta \rangle) \\
= e_i ((n-1)H_m) = 0
$$

and the Lemma follows. \hfill \Box

**Proof of Lemma 5** Using Lemma 6-a compute

$$
e_j e_i(\rho) = -\langle \eta, \xi \rangle^{-2} \langle \eta, \nabla e_j \xi \rangle \langle x - \rho \eta, \nabla e_i \xi \rangle \\
+ \langle \eta, \xi \rangle^{-1} (e_j - \rho \nabla e_j \eta - e_j(\rho) \eta, \nabla e_i \xi) \\
+ \langle \eta, \xi \rangle^{-1} (x - \rho \eta, \nabla e_j \nabla e_i \xi) \\
= \langle \eta, \xi \rangle^{-1} (-e_i(\rho) \langle \eta, \nabla e_j \xi \rangle + \langle e_j - \rho \nabla e_j \eta - e_j(\rho) \eta, \nabla e_i \xi \rangle + \langle x - \rho \eta, \nabla e_j \nabla e_i \xi \rangle).
$$

Since $\{e_i\}$ is orthonormal we compute the divergence as

$$
\text{div}(X) = \sum_i \langle \nabla e_i, X, e_i \rangle.
$$

In the following we will omit the summation sings

$$
\text{div}(\langle \eta, \xi \rangle^2 du(\nabla \rho)) \\
= 2 \langle \langle \eta, \xi \rangle \nabla \langle \eta, \xi \rangle, du(\nabla \rho) \rangle + \langle \eta, \xi \rangle^2 \langle \nabla e_i (du(\nabla \rho)), e_i \rangle \\
= 2 \langle \langle \eta, \xi \rangle (\langle \eta, \nabla e_i \xi \rangle e_i + e_j(\rho) du(e_j)) + \langle \eta, \xi \rangle^2 \langle \nabla e_i (e_j(\rho) du(e_j)), e_i \rangle, e_i \rangle \\
= 2 \langle \langle \eta, \xi \rangle (\langle \eta, \nabla e_i \xi \rangle e_j(\rho) (e_i, du(e_j)) + \langle \eta, \xi \rangle^2 \langle e_i e_j(\rho)(du(e_j), e_i) \\
+ \langle \eta, \xi \rangle^2 e_j(\rho)(\nabla e_i (du(e_j)), e_i) \rangle.\n$$
Applying Lemma 6-c, using that $du$ is self-adjoint, and recalling that $e_i$ are eigenvectors of $d\xi$, 

\[
\langle \eta, \xi \rangle^{-1} \text{div}(\langle \eta, \xi \rangle^2 du(\nabla \rho)) = 2 \langle \eta, \nabla e_i \rangle e_j(\rho) \langle e_i, du(e_j) \rangle + \langle \eta, \xi \rangle e_j(\rho) \langle \nabla e_i, du(e_j) \rangle e_i
\]

\[
+ \langle \eta, \xi \rangle e_j(\rho) \langle \nabla e_i \rangle^2(du(e_j)), e_i \rangle = 2 \langle \eta, \nabla e_i \rangle e_j(\rho) \langle e_i, du(e_j) \rangle
\]

\[
+ \langle \eta, \xi \rangle e_j(\rho) \langle \nabla e_i, du(e_j) \rangle e_i.\]

Since $d\eta$ is diagonalizable, $[d\eta] = C.D.C^{-1}$ where $[d\eta]$ is the matrix of $d\eta$ in the basis $\{e_i\}$, $D$ is diagonal and $C$ is invertible. Then

\[
\sum_{i,j} \langle d\eta(e_j), e_i \rangle \langle e_j, d\eta(e_i) \rangle = \text{tr}([d\eta]^2) = \text{tr}(C.D^2.C^{-1}) = \sum_i \lambda_i^2
\]

and the result follows.

6. Appendix B: First variation formula for the area

In this final section we prove the first variation formula (2) for general variations.

**Theorem 5.** Assume $M$ is a closed manifold, $x : M \to \mathbb{R}^n$ a smooth immersion and $F : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^n$ smooth with compact support, with $F(0, p) = x(p)$.

\[
A'_m(0) = \int_M (n - 1)H_m(p)N_\eta \left( \frac{\partial F}{\partial t}(0, p) \right) d\omega(p)
\]

where $N_\eta$ is the projection to the second coordinate in the direct-sum decomposition $\mathbb{R}^n = T_p M \oplus \langle \eta \rangle$.

First we need some technical lemmas.

**Lemma 7.** Let $X^\top$ denote the orthogonal projection of $X$ in $T_p M$.

a) $\frac{dF}{dt} \bigg|_0 = -\nabla \langle F_t, \xi \rangle + \nabla_{F_t^\top} \xi$

b) $\nabla_{\eta^\top} \xi = \nabla \langle \eta, \xi \rangle$
Proof. For the first part let \( \{ e_i \} \) be an orthonormal basis for \( T_p M \) and \( e_i^t = dF_p^t(e_i) \).
\[
\left\langle \frac{\partial}{\partial t} \xi \bigg|_{t=0}, e_i \right\rangle = \frac{\partial}{\partial t} \left\langle \xi^t, e_i^t \right\rangle \bigg|_{t=0} - \left\langle \xi, \frac{\partial}{\partial t} e_i^t \bigg|_{t=0} \right\rangle
\]
As \( e_i^t \) is orthogonal to \( \xi^t \)
\[
\left\langle \frac{\partial}{\partial t} \xi \bigg|_{t=0}, e_i \right\rangle = -\langle \xi, \nabla e_i F_t \rangle
\]
Using that \([F_i, e_i^t] = 0\) and \( d\xi_p \) is a self-adjoint operator
\[
\left\langle \frac{\partial}{\partial t} \xi \bigg|_{t=0}, e_i \right\rangle = -\langle \xi, \nabla e_i F_t \rangle
\]
\[
= -(e_i \langle \xi, F_t \rangle - \langle \nabla e_i \xi, F_t \rangle)
\]
\[
= -(e_i \langle \xi, F_t \rangle - \langle e_i, \nabla F_t^\top \xi \rangle)
\]
\[
= -e_i \langle \xi, F_t \rangle + \langle e_i, \nabla F_t^\top \xi \rangle
\]
For the second part compute for any \( v \in T_p M \), \( \langle \nabla_{\gamma^\top} \xi, v \rangle = \langle \gamma^\top, \nabla \xi \rangle = v \langle \eta, \xi \rangle = \langle \gamma \langle \eta, \xi \rangle, v \rangle \).
\]
Proof of Proposition 3
\[
\frac{\partial}{\partial t} \langle \eta(t), \xi(t) \rangle \bigg|_{t=0} = \left\langle \eta, \frac{\partial}{\partial t} \xi \bigg|_{t=0} \right\rangle,
\]
since \( \frac{\partial}{\partial t} \eta \big|_{t=0} \in T_p M \). Thus (see eq. 1.44, [12])
\[
A'_m(0) = \int M \left\{ \left\langle \eta, \frac{\partial}{\partial t} \xi \bigg|_{t=0} \right\rangle + \langle \eta, \xi \rangle \text{div } F_t \right\} dS.
\]
Writing \( F_t = F_t^\top + \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \eta \), we have
\[
\text{div } F_t = \text{div } F_t^\top + \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \eta.
\]
On the other hand,
\[
\text{div } \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \eta = \left\langle \nabla \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \eta, \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \right\rangle + \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \text{div } \eta
\]
\[
= \left\langle \nabla \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle}, \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \right\rangle + \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} (n-1)H_m.
\]
Then
\[
A'(0) = \int M \left\{ \left\langle \eta, \frac{\partial}{\partial t} \xi \bigg|_{t=0} \right\rangle + \langle \eta, \xi \rangle \text{div } F_t^\top + \langle \eta, \xi \rangle \left\langle \nabla \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle}, \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \right\rangle + \langle \eta, \xi \rangle \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} (n-1)H_m \right\} dS.
\]
Using again the decomposition \( F_t = F_t^\top + \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \eta \), we obtain
\[
\frac{\partial}{\partial t} \xi \big|_{t=0} = -\nabla \langle F_t, \xi \rangle + \nabla F_t^\top \xi + \nabla \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \eta \gamma^\top \xi
\]
\[
= -\nabla \langle F_t, \xi \rangle + \nabla F_t^\top \xi + \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \nabla \eta \gamma^\top \xi.
\]
Furthermore,
\[
\nabla \langle F_t, \xi \rangle = \nabla \left( \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \right) = \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \nabla \langle \eta, \xi \rangle + \langle \eta, \xi \rangle \nabla \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle}.
\]
Using (8) and the Lemma 6, the Lemma 7 yields
\[ \frac{\partial}{\partial t} \xi \bigg|_{t=0} = -\langle \eta, \xi \rangle \nabla \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} + \nabla F_t^\top \xi. \]
Replacing in \( A'(0) \), we have:
\[ A'(0) = \int_M (n-1)H_m \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \, d\omega + \int_M \left\{ \langle \nabla F_t^\top \xi, \eta \rangle + \langle \eta, \xi \rangle \operatorname{div} F_t^\top \right\} \, dS, \]
as
\[ \operatorname{div} \langle \eta, \xi \rangle F_t^\top = F_t^\top \langle \eta, \xi \rangle + \langle \eta, \xi \rangle \operatorname{div} F_t^\top = \langle \nabla F_t^\top \xi, \eta \rangle + \langle \eta, \xi \rangle \operatorname{div} F_t^\top, \]
we have
\[ A'(0) = \int_M (n-1)H_m \frac{\langle F_t, \xi \rangle}{\langle \eta, \xi \rangle} \, d\omega. \]