TWISTED KOECHER-MAASS SERIES OF THE IKEDA TYPE LIFT
FOR THE EXCEPTIONAL GROUP OF TYPE $E_{7,3}$

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Abstract. We compute the twisted Koecher-Maass series of the first and second kind of the Ikeda type lift for the exceptional group of type $E_{7,3}$. As an application, we obtain their rationality result.

1. Introduction

Let $J$ be the exceptional Jordan algebra consisting of $3 \times 3$ matrices with entries in the Cayley numbers, and $M'$ the group scheme over $\mathbb{Z}$ of type $E_{6,2}$. (See Section 2 for the definitions.) Let $\mathcal{T}$ be the exceptional domain in $\mathbb{C}^2$. Let $G$ be a connected reductive group of type $GE_{7,3}$. For a cusp form $F(Z) = \sum_{T \in J(\mathbb{Z}) > 0} a_F(T) \exp(2\sqrt{-1}(T,Z))$ of weight $k$ ($k$ even) with respect to $G(\mathbb{Z})$ and a Dirichlet character $\chi$, we define the twisted Koecher-Maass series $K^{(2)}(s,F,\chi)$ of the second kind as

$$K^{(2)}(s,F,\chi) = \sum_{T \in J(\mathbb{Z}) > 0} \frac{\chi(\det T) a_F(T)}{\epsilon(T) \det(T)^s},$$

where for $T \in J(\mathbb{Z}) > 0$, $\epsilon(T) = \#(U_T(\mathbb{Z}))$, and $U_T$ denotes the group scheme of type $F_4$. (See Section 5) For a positive integer $N$ let $\phi_N : M'(\mathbb{Z}) \to M'(\mathbb{Z}/N\mathbb{Z})$ be the homomorphism induced by the natural surjection $\pi_N : \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$, and put $M'(N;\mathbb{Z}) = \text{Ker}(\phi_N)$. We define the twisted Koecher-Maass series $K^{(1)}(s,F,\chi)$ of the first kind as

$$K^{(1)}(s,F,\chi) = \sum_{T \in J(\mathbb{Z}) > 0/M'(N;\mathbb{Z})} \frac{\chi(\text{Tr}(T)) a_F(T)}{\epsilon_N(T) \det(T)^s},$$

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where $\epsilon_N(T) = \#(\mathcal{M}'(N; \mathbb{Z}) \cap \mathbb{U}_T(\mathbb{Z}))$.

For a cuspidal Hecke eigenform $f \in S_{2k-8}(SL_2(\mathbb{Z}))$ ($2k \geq 20$), let $F_f$ be the Ikeda type lift which is a cuspidal Hecke eigenform of weight $2k$ with respect to $G(\mathbb{Z})$ constructed by Kim and Yamauchi. In this paper, we give explicit formulas of $K^{(1)}(s, F_f, \chi)$ and $K^{(2)}(s, F_f, \chi)$.

**Theorem 1.1.** Let $c = \frac{5!7!11!}{(2\pi)^{28}}$ and $L(s, \pi_f, \chi)$ be the $L$-function of the cuspidal automorphic representation $\pi_f$ attached to $f$ twisted by $\chi$. Then

$$K^{(2)}(s, F_f, \chi) = c\zeta(2)\zeta(6)\zeta(8)\zeta(12) \times \prod_{i=1}^{3} L(s - k - 9/2 + 4i - 3, \pi_f, \chi).$$

**Theorem 1.2.** Let $\chi$ be a primitive Dirichlet character mod $N$. Suppose that $\chi$ is not a quadratic character. Let $l = \gcd(3, \phi(N))$ with $\phi$ is the Euler phi-function, and let $u_0$ be a primitive $l$-th root of unity mod $N$.

(1) Suppose that $\chi(u_0) \neq 1$. Then $K^{(1)}(s, F_f, \chi) = 0$.

(2) Suppose that $\chi(u_0) = 1$ and fix a character $\tilde{\chi}$ such that $\chi = \tilde{\chi}^3$ (cf. Lemma 4.11). Then

$$K^{(1)}(s, F_f, \chi) = c\zeta(2)\zeta(6)\zeta(8)\zeta(12)d_N \sum_{\eta \in \mathbb{D}_N} J(\overline{\chi\eta}, \overline{\chi\eta}, \overline{\chi\eta}) \prod_{i=1}^{3} L(s - k - 9/2 + 4i - 3, \pi_f, \overline{\chi\eta}),$$

where $d_N = N^{64} \prod_{p|N}(1 - p^{-2})(1 - p^{-6})(1 - p^{-8})(1 - p^{-12})$,

$$\mathbb{D}_N = \{\eta \mid \eta \text{ is a Dirichlet character mod } N \text{ such that } \eta^l = 1\},$$

and $J(\xi, \xi, \xi)$ is the generalized Jacobi sum (cf. Section 4). In particular, if $N$ is odd, we have

$$K^{(1)}(s, F_f, \chi) = c\zeta(2)\zeta(6)\zeta(8)\zeta(12)d_N \sum_{\eta \in \mathbb{D}_N} \frac{W(\overline{\chi\eta})^3}{W(\overline{\chi})} \prod_{i=1}^{3} L(s - k - 9/2 + 4i - 3, \pi_f, \overline{\chi\eta}),$$

where $W(\xi)$ is the Gauss sum of the Dirichlet character $\xi$.

As an application, we obtain the rationality result for $K^{(1)}(m, F_f, \chi)$ and $K^{(2)}(m, F_f, \chi)$ for $9 \leq m \leq 2k - 9$ (Theorems 7.2 and 7.3).

In the case of Siegel modular forms, Choie-Kohnen studied the twisted Koecher-Maass series of the first kind and proved the analytic continuation and the functional equation, and studied their special values. On the other hand, the first named author studied the twisted Koecher-Maass series of both the first kind and the second kind, and gave explicit formulas for them associated with the Duke-Imamoglu-Ikeda lift.
Our method of proving the above two theorems is similar to that used in [1]. However, unlike the Siegel case, we cannot use explicit matrix decompositions in the exceptional group. It is one of the obstacles we need to overcome. This paper is organized as follows. In Section 2, we briefly review modular forms on the exceptional domain. Unlike in previous papers [1, 8], we need to consider the exceptional similitude group $GE_{7,3}$. We use the definition of $GE_{7,3}$ in [9].

In Section 3, we prove that the twisted Koecher-Maass series of the first kind for any cusp forms has analytic continuation and the functional equation. In Section 4, we obtain the relationship between the twisted Koecher-Maass series of the first kind and the second kind.

In Section 5, we briefly review the mass formula and local density $\beta_p(T)$ from [5], where we used them to compute the Rankin-Selberg series $R(s, F_f, F_f)$ for the Ikeda type lift $F_f$. In Section 6, we review the Ikeda type lift. By construction, the Fourier coefficients of $F_f$ are expressed in terms of a product of the local Siegel series. Therefore, using the mass formula in Section 5, we can express the twisted Koecher-Maass series of the second kind as an Euler product:

$$K^{(2)}(s, F_f, \chi) = c \prod_p H_p(\alpha_p, \chi(p))p^{-s+k+9/2},$$

where $c$ is in Theorem 1.1, $\alpha_p$ is the $p$-th Satake parameter for $f$, and $H_p(X, t)$ is a certain power series involving the Siegel series and the local density. It is remarkable that $K^{(2)}(s, F_f, \chi)$ has an Euler product since $K^{(2)}(s, F, \chi)$ does not have an Euler product for a general cusp form $F$. In Section 7, we prove Theorems 1.1 and 1.2 and obtain rationality result of the twisted Koecher-Maass series.

**Notation.** In addition to the standard symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, for a prime number $p$, let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ be the field of $p$-adic numbers and the ring of $p$-adic integers. For a commutative ring $R$ let $R^x$ denote the group of units in $R$.

Let $\sim$ be an equivalence relation on a set $S$. We denote by $S/\sim$ the set of equivalence classes of $S$ under $\sim$. We use the same symbol $S/\sim$ to denote a complete set of representatives. Let $G$ be a group acting on a set $S$. For two elements $a_1$ and $a_2$, we write $a_2 \sim_G a_1$ if $a_2 = g \cdot a_1$ with $g \in G$. The relation $\sim_G$ is an equivalence relation on $S$ and we write $S/G$ instead of $S/\sim_G$.

For an associate or non-associate algebra $R$ let $M_{mn}(R)$ denote the set of $m \times n$ matrix with entries in $R$. In particular we put $M_n(R) = M_{nn}(R)$. In particular if $R$ is a commutative ring, for an element $A \in M_n(R)$ let $\det A$ denote the determinant of $A$. We put $GL_n(R) = \{A \in M_n(R) \mid \det A \in R^x\}$. Moreover, for an $R$-module, $M$, let $GL(M)$ denote the group of $R$-linear
transformations on $M$. For square matrices $A_1, \ldots, A_r$, we write $A_1 \perp \cdots \perp A_r = \begin{pmatrix} A_1 & O & O \\ O & \ddots & O \\ O & O & A_r \end{pmatrix}$.

We sometimes write $\text{diag}(A_1, \ldots, A_r)$ instead of $A_1 \perp \cdots \perp A_r$. For $x \in \mathbb{R}$, let $e(x) = e^{2\pi \sqrt{-1} x}$.

For an element $a$ of $\mathbb{Z}/N\mathbb{Z}$, we use the same symbol $a$ to denote its representative mod $N$. Moreover, for a Dirichlet character $\chi \mod N$, we use the same symbol $\chi$ to denote the mapping $\mathbb{Z}/N\mathbb{Z} \ni a \mod N \mapsto \chi(a) \in \mathbb{C}$.

2. Modular forms on the exceptional domain

We will freely use the notations from [1, 8, 5]. Let $\mathfrak{C}_Q$ and $\mathfrak{o} \subset \mathfrak{C}_Q$ be the Cayley numbers and integral Cayley numbers, resp. The trace and the norm on $\mathfrak{C}_Q$ are defined by

$$\text{Tr}(x) = x + \bar{x} \quad \text{and} \quad N(x) = x\bar{x},$$

where $\bar{x}$ is the anti-involution in [5, Section 2]. Let $\mathfrak{J}_Q$ be the exceptional Jordan algebra consisting of matrices

$$X = (x_{ij})_{1 \leq i,j \leq 3} = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix}$$

with $a, b, c \in \mathbb{Q}$ and $x, y, z \in \mathfrak{C}_Q$. We define the determinant $\det X$ and the trace $\text{Tr}(X)$ by

$$\det X = abc - aN(z) - bN(y) - cN(x) + \text{Tr}((xz)\bar{y}), \quad \text{Tr}(x) = a + b + c.$$  

We define a lattice $\mathfrak{J}(\mathbb{Z})$ of $\mathfrak{J}_Q$ by

$$\mathfrak{J}(\mathbb{Z}) = \{ X = (x_{ij}) \in \mathfrak{J}_Q \mid x_{ii} \in \mathbb{Z} \text{ and } x_{ij} \in \mathfrak{o} \text{ for } i \neq j \}.$$  

For a commutative algebra $R$, we put $\mathfrak{J}(R) = \mathfrak{J}(\mathbb{Z}) \otimes \mathbb{Z} R$. Recall

$$\mathfrak{J}(R)^{\text{ns}} = \{ X \in \mathfrak{J}(R) \mid \det(X) \neq 0 \}, \quad R_3^+(R) = \{ X^2 \mid X \in \mathfrak{J}(R)^{\text{ns}} \}.$$  

We denote by $\overline{R}_3^+(\mathbb{R})$ the closure of $R_3^+(\mathbb{R})$ in $\mathfrak{J}(\mathbb{R}) \simeq \mathbb{R}^{27}$. For a subring $A$ of $\mathbb{R}$, set

$$\mathfrak{J}(A)_{>0} = \mathfrak{J}(A) \cap R_3^+(\mathbb{R}) \text{ and } \mathfrak{J}(A)_{\geq 0} = \mathfrak{J}(A) \cap \overline{R}_3^+(\mathbb{R}).$$
We also define $\mathcal{J}_{2, \mathbb{Q}}$ as the set of matrices of forms $X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix}$, $a, b \in \mathbb{Q}$, $x \in \mathbb{C}_\mathbb{Q}$, and its lattice $\mathcal{J}_2(\mathbb{Z})$ as

$$\mathcal{J}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \mid a, b \in \mathbb{Z}, x \in \mathfrak{o} \right\}.$$ 

We define the determinant $\det X$ and the trace $\operatorname{Tr}(X)$ of $X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathcal{J}_{2, \mathbb{Q}}$ by

$$\det X = ab - N(x), \quad \operatorname{Tr}(x) = a + b.$$ 

For a commutative algebra $R$, put $\mathcal{J}_2(R) = \mathcal{J}_2(\mathbb{Z}) \otimes \mathbb{Z} R$.

Recall the exceptional domain:

$$\mathfrak{T} := \{ Z = X + Y \sqrt{-1} \in \mathcal{J}_C \mid X, Y \in \mathcal{J}_R, Y \in R_3^+(\mathbb{R}) \}$$

which is a complex analytic subspace of $\mathbb{C}^{27}$.

Define the group schemes $\mathcal{M}$ and $\mathcal{M}'$ over $\mathbb{Z}$ by

$$\mathcal{M}(R) = \{ g \in GL(\mathcal{J}(R)) \mid \det(g \cdot X) = \nu(g) \det X \text{ with } \nu(g) \in R^\times \},$$

$$\mathcal{M}'(R) = \{ g \in \mathcal{M}(R) \mid \nu(g) = 1 \}.$$ 

Put $\mathcal{M} = \mathcal{M} \otimes \mathbb{Z} \mathbb{Q}$ and $\mathcal{M}' = \mathcal{M}' \otimes \mathbb{Z} \mathbb{Q}$. Then $\mathcal{M}$ is an algebraic group over $\mathbb{Q}$ of type $GE_{6, 2}$ and $\mathcal{M}'$ is the derived group of $\mathcal{M}$, which is a simple group of type $E_{6, 2}$.

Let $X, X'$ be two $\mathbb{Q}$-vector spaces, each isomorphic to $\mathcal{J}$, and $\Xi, \Xi'$ be copies of $\mathbb{Q}$. Let $W = X \oplus \Xi \oplus X' \oplus \Xi'$, and for $w = (X, \xi, X', \xi') \in W$, define a quartic form $Q_w$ on $W$ by

$$Q_w = (X \times X, X' \times X') - \xi \det(X) - \xi' \det(X') - \frac{1}{4}((X, X') - \xi \xi')^2,$$

and a skew-symmetric bilinear form $\{ , \}$ by

$$\{ w_1, w_2 \} = (X_1, X_2') - (X_2, X_1') + \xi_1 \xi_2' - \xi_2 \xi_1'.$$

Recall the definition of the exceptional group of type $GE_{7, 3}$ [9]:

$$G(\mathbb{Q}) = \{ g \in GL(W) \mid Qg(w) = \mu(g)^2 Q(w), g\{ w_1, w_2 \} = \mu(g)\{ w_1, w_2 \} \},$$

for some $\mu(g) \in \mathbb{Q}^\times$. Then $\mu$ is a rational character of $G$. Let $G'(\mathbb{Q}) = \{ g \in G(\mathbb{Q}) \mid \mu(g) = 1 \}$.

Define the similitude factor $h_0(a)$ as

$$h_0(a)(X, \xi, X', \xi') = (aX, a^{-1} \xi, X', a^2 \xi').$$
Then $\mu(h_0(a)) = a$, and $G = \{h_0(a)\} \ltimes G'$. Here $G'$ is a simply connected algebraic $\mathbb{Q}$-group of type $E_{7,3}^\infty$ as in [I], and $G$ is a $\mathbb{Q}$-group of type $GE_{7,3}$. Let $I_W$ be the identity operator on $W$. Let $Z$ be the central torus of $GL(W)$, i.e.,

$$Z_q = \{\lambda I_W | \lambda \in \mathbb{Q}, \lambda \neq 0\}.$$

Then $Z$ is the central torus of $G$, and $G = Z \cdot G' = (Z \times G')/\mu_2$, where $\mu_2$ is embedded in both centers, and $G/Z$ is the adjoint exceptional group of type $E_{7,3}$. The real rank of $G'$ is 3, and it is split over $\mathbb{Q}_p$ for any prime $p$.

Let $\widetilde{M} = Z \cdot M$ (almost direct product). Then $P = \widetilde{M}N$ is the Siegel parabolic subgroup of $G$. We extend the character $\nu$ to $P$ by defining $\nu(zmn) = \nu(m)$ for $z \in Z$, $m \in M$, and $n \in N$.

Let $G(\mathbb{R})^+ = \{g \in G(\mathbb{R}) | \mu(g) > 0\}$. For a function $F : \mathfrak{T} \longrightarrow \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, and $g \in G(\mathbb{R})^+$, let $g = zg'$ with $z \in Z(\mathbb{R})$ and $z > 0$, and $g' \in G'(\mathbb{R})$. We define the “slash operator” by

$$F|_{k,g}(Z) := j(g',Z)^{-k}F(g'Z),$$

where $j(g',Z)$ is the canonical factor of automorphy in [7]. Hence the center acts trivially. We write $F|g$ instead of $F|_{k,g}$ when there is no confusion.

Let $\Gamma = G(\mathbb{Z})$. For a positive integer $N$, define $M(N; \mathbb{Z})$ to be the kernel of the map $M(\mathbb{Z}) \longrightarrow M(\mathbb{Z}/N\mathbb{Z})$.

Let $F \in S_k(\Gamma)$ be a cusp form of weight $k$ ($k$ even) with respect to $\Gamma$, and write

$$F(Z) = \sum_{T \in \mathfrak{I}(\mathbb{Z}) > 0} a_F(T)e((T,Z)).$$

For a primitive Dirichlet character $\chi \mod N$, let

$$F_{\chi}(Z) = \sum_{T \in \mathfrak{I}(\mathbb{Z}) > 0} \chi(tr(T))a_F(T)e((T,Z)).$$

Then for $m \in M(N; \mathbb{Z})$, $F_{\chi}(mZ) = F_{\chi}(Z)$.

Let $W(\bar{\chi}) = \sum_{a \, (\text{mod} \, N)} \bar{\chi}(a)e(a/N)$ be the Gauss sum associated to $\bar{\chi}$. Note that if $(a, N) > 1$, $\bar{\chi}(a) = 0$. Hence the sum is in fact over all $a \, (\text{mod} \, N)$ such that $(a, N) = 1$.

**Theorem 2.1.** Let $h_0(N) \in G(\mathbb{Q})$ be the similitude factor. Then $h_0(N)Z = N^{-1}Z$. Let $\iota_N = Nh_0(N)\iota h_0(N)^{-1} = h_0(N^2)$. Then

$$F_{\chi}|_{\iota_N} = W(\chi)^2N^{-1}F_{\chi}.$$
So by taking $Z = \sqrt{-1}Y$, we have

\begin{equation}
F_{\chi}(\sqrt{-1}(N^2Y)^{-1}) = (-1)^{\frac{N}{2}}W(\chi)^2N^{3k-1}\det(Y)^kF_{\chi}(\sqrt{-1}Y).
\end{equation}

**Proof.** For $(a, N) = 1$, let $\lambda, b \in \mathbb{Z}$ such that $\lambda N - ab = 1$. Then we claim that

\begin{equation}
p^{N\lambda}a_{1\ell}N = N\gamma p^{N\lambda}_B a_{1\ell}.
\end{equation}

for some $\gamma \in \Gamma$, where $p_B \in \mathbb{N}$. We prove it below. Since

\[
\sum_{a \equiv \chi \bmod N} \tilde{\chi}(a)e(\text{tr}(T)\frac{a}{N}) = \chi(\text{tr}(T))W(\tilde{\chi}),
\]

we have

\[
F_{\chi} = \frac{1}{W(\tilde{\chi})} \sum_{a \equiv \chi \bmod N} \tilde{\chi}(a)F|p^{N\lambda}_B a_{1\ell}.
\]

Then

\[
W(\tilde{\chi})F_{\chi}|_{tN} = \sum_{a \equiv \chi \bmod N \atop (a, N) = 1} \tilde{\chi}(a)F|p^{N\lambda}_B a_{1\ell}N = \sum_{a \equiv \chi \bmod N \atop (a, N) = 1} \tilde{\chi}(a)F|p^{N\lambda}_B a_{1\ell}
\]

\[
= \chi(-1) \sum_{b \equiv \chi \bmod N \atop (b, N) = 1} \chi(b)F|p^{N\lambda}_B a_{1\ell} = \chi(-1)W(\chi)F_{\chi}.
\]

Since $W(\chi)W(\tilde{\chi}) = \chi(-1)N$, we have the result. \hfill \square

**Proof of (2.3).** We show that $g = N^{-1}\tau^{-1}p^{-1}_B a_{1\ell}^- p^{-1}_A a^{-1} \in \Gamma$. Note that $\tau^{-1} = -\tau$ and $p^{-1}_A = p_A$. It is enough to show that $g(X, \xi, X', \xi') = W_\phi$ and $g^{-1}(X, \xi, X', \xi') \in W_\phi$, where $W_\phi = \{(X, \xi, X', \xi') \mid X, X' \in \mathfrak{F}(Z), \xi, \xi' \in \mathbb{Z}\}$. Recall $\nu(X, \xi, X', \xi') = (X', -\xi', X, \xi)$ and $h_0(N)^{-1} = h_0(N^{-1})$. Also $p_B' = \nu_B^{-1}$ is in the opposite unipotent subgroup. Hence $g = Np^{N\lambda}_B a_{1\ell}^- p^{-bN1\ell} h_0(N^{-2}).$

We can compute $g(X, \xi, X', \xi') = (X_1, \xi_1, X'_1, \xi'_1)$, where

\[
X_1 = \frac{X}{N} - \frac{b}{N^2}13 - \frac{2a}{N}13 \times (NX' - 2b13 \times X + \frac{b^2}{N}13) + \frac{a^2}{N^2}13(N^3\xi - bN^2(13, X') + b^2N(13, X) - b^3\xi'),
\]

\[
\xi_1 = N^3\xi - bN^2(13, X') + b^2N(13, X) - b^3\xi',
\]

\[
X'_1 = NX' - 2b13 \times X + \frac{b^2}{N}13 - \frac{a}{N}13(N^3\xi - bN^2(13, X') + b^2N(13, X) - b^3\xi'),
\]

\[
\xi'_1 = \frac{\xi}{N^3} - \frac{ab}{N^3}(13, X) + \frac{ab}{N^3}(13, 13) + \frac{a^2}{N^2}(13, NX' - 2b13 \times X + \frac{b^2}{N}13) - \frac{a^3}{N^3}(N^3\xi - bN^2(13, X') + b^2N(13, X) - b^3\xi').
\]
We use the fact that \((13, X) = \text{tr}(X)\), and \(13 \times X = \frac{1}{2} \text{tr}(X)13 - \frac{1}{2} X\). In particular, \((13, 13) = 3, 13 \times 13 = 13\). By using \(1 + ab = \lambda N\), we obtain our result. For example,

\[
\xi'_1 = -a^3 \xi + \frac{N^2}{N^3}(1 + ab)^3 - \frac{a}{N^2} \text{tr}(X)(1 + ab)^2 + \frac{a^2}{N} \text{tr}(X')(1 + ab) \in \mathbb{Z}.
\]

Since \(g^{-1} = NpN_{13}^{-1}p_{N_{13}}^{-1}g\), we can show in the same way that \(g^{-1}(X, \xi, X', \xi') \in W_o\). \(\Box\)

Remark 2.2. Consider \(\phi : G(\mathbb{Z}) \rightarrow G(\mathbb{Z}/N^2\mathbb{Z})\). Let \(I\) be the subgroup of \(G(\mathbb{Z}/N^2\mathbb{Z})\) generated by \(N(\mathbb{Z}/N^2\mathbb{Z})\) and \(M_0\), where \(M_0\) is the preimage of the scalar matrices under the map \(M(\mathbb{Z}/N^2\mathbb{Z}) \rightarrow M(\mathbb{Z}/NZ)\). Let \(\Gamma_0^s(N^2) = \phi^{-1}(I)\). Then \(M(N; Z) \subset \Gamma_0^s(N^2)\).

For \(\gamma \in \Gamma_0^s(N^2)\), \(\gamma \equiv p \mod N\) for some \(p \in M(\mathbb{Z}/NZ)\). Define \(\chi(\gamma) = \chi(\nu(p))\). It is well-defined. If \(G\) is a holomorphic function on \(\mathfrak{T}\) which satisfies

\[
G|_{k\gamma}(Z) = \omega(\gamma)G(Z), \quad Z \in \mathfrak{T}, \gamma \in \Gamma_0^s(N^2),
\]

then \(G\) is called a modular form on \(\mathfrak{T}\) of weight \(k\) with respect to \(\Gamma_0^s(N^2)\) with the central character \(\omega\).

Let \(M_k(\Gamma_0^s(N^2), \omega)\) be the space of modular forms of weight \(k\) with respect to \(\Gamma_0^s(N^2)\) on \(\mathfrak{T}\) with the central character \(\omega\). We can also define the space of cusp forms \(S_k(\Gamma_0^s(N^2), \omega)\) using the Siegel \(\Phi\)-operator. Then we expect \(F_\chi \in S_k(\Gamma_0^s(N^2), \chi^2)\). We do not need this fact in this paper.

3. Twisted Koecher-Maass series of the first kind

From now on put \(e(x) = \exp(2\pi \sqrt{-1} x)\) for \(x \in \mathbb{C}\). Let \(F\) be a cusp form of weight \(k\) \((k\ even)\) on the exceptional domain \(\mathfrak{T}\) with respect to \(\Gamma\), and let \(F(Z) = \sum_{T \in \mathfrak{T}(Z) > 0} a_F(T)e((T, Z))\). Recall the twisted Koecher-Maass series \(K^{(1)}(s, F, \chi)\) of the first kind for a Dirichlet character \(\chi\);

\[
K^{(1)}(s, F, \chi) = \sum_{T \in \mathfrak{T}(Z) > 0/\mathcal{M}'(N; \mathbb{Z})} \frac{\chi(\text{Tr}(T))a_F(T)}{\epsilon_N(T) \det(T)^s}.
\]

In this section we prove the analytic continuation and the functional equation of \(K^{(1)}(s, F, \chi)\).

Recall that \(d^*Y = \det(Y)^{-9}dY\) is the invariant measure in \(R^+_3(\mathbb{R})\). Let \(\mathcal{R}\) be a fundamental domain for the action of \(\mathcal{M}(\mathbb{Z})\) on \(R^+_3(\mathbb{R})\). Let \(\mathcal{R}_N\) be a fundamental domain for the action of \(\mathcal{M}(N; \mathbb{Z})\) on \(R^+_3(\mathbb{R})\). We may take

\[
\mathcal{R}_N = \bigcup_{a=1}^r m_a \mathcal{R},
\]

where \(\{m_1, ..., m_r\}\) is a set of representatives for \(\mathcal{M}'(\mathbb{Z})/\mathcal{M}'(N; \mathbb{Z})\). We prove
Theorem 3.1. For $Re(s) > 9 + \frac{b}{2}$, $K^{(1)}(s, F, \chi)$ converges absolutely, and in this region we have the integral representation

$$\Lambda(s, F, \chi) = \pi^{12}(2\pi)^{-3s}N^{3s}\Gamma(s)\Gamma(s-4)\Gamma(s-8)K^{(1)}(s, F, \chi) = N^{3s}\int_{\mathcal{R}_N} F_X(\sqrt{-TY}) \det(Y)^s d^sY.$$ 

It has the analytic continuation to all of $\mathbb{C}$, and satisfies the functional equation

$$\Lambda(k-s, F, \chi) = (-1)^{\frac{b}{2}}\frac{W(\chi)^2}{N}\Lambda(s, F, \chi).$$

Proof. By Hecke’s bound, $a_F(T) \ll \det(T)^{\frac{b}{2}}$. Hence

$$K^{(1)}(s, F, \chi) \ll \sum_{T \in \mathfrak{I}(\mathbb{Z})_{>0}/\mathcal{M}(N;\mathbb{Z})} \frac{\det(T)^{\frac{b}{2}-Re(s)}}{\epsilon_N(T)}.$$ 

By integral test, the series is majorized by

$$\int_{{\mathcal{R}_N}^{+}(8)}^{\mathbb{R}_+} \det(Y)^{\frac{b}{2}-Re(s)+9} d^sY,$$

which converges for $Re(s) > 9 + \frac{b}{2}$.

We have $F_X(\sqrt{-TY}) = \sum_{T \in \mathfrak{I}(\mathbb{Z})_{>0}} \chi(\text{tr}(T))a_F(T)e^{-2\pi(T,Y)}$. Since $a_F(uT) = a_F(T)$ for $u \in \mathcal{U}_T(\mathbb{Z}) \cap \mathcal{M}(N;\mathbb{Z})$,

$$N^{3s}\int_{\mathcal{R}_N} F_X(\sqrt{-TY}) \det(Y)^s d^sY = N^{3s} \sum_{T \in \mathfrak{I}(\mathbb{Z})_{>0}} \chi(\text{tr}(T))a_F(T)\int_{\mathcal{R}_N} e^{-2\pi(T,Y)} \det(Y)^s d^sY$$

$$= N^{3s} \sum_{T \in \mathfrak{I}(\mathbb{Z})_{>0}/\mathcal{M}(N;\mathbb{Z})} \frac{\chi(\text{tr}(T))a_F(T)}{\epsilon_N(T)} \sum_{u \in \mathcal{U}_T(\mathbb{Z}) \cap \mathcal{M}(N;\mathbb{Z})} \int_{\mathcal{R}_N} e^{-2\pi(T,Y)} \det(Y)^s d^sY$$

$$= N^{3s} \pi^{12}(2\pi)^{-3s}\Gamma(s)\Gamma(s-4)\Gamma(s-8) \sum_{T \in \mathfrak{I}(\mathbb{Z})_{>0}/\mathcal{M}(N;\mathbb{Z})} \frac{\chi(\text{tr}(T))a_F(T)}{\epsilon_N(T)} \det(T)^{-s}.$$ 

Here we use the fact [11 page 538] that for $Re(s) > 8$,

$$\int_{\mathcal{R}_N^{+}(\mathbb{R})} e^{-2\pi(T,Y)} \det(Y)^s d^sY = \pi^{12}(2\pi)^{-3s}\Gamma(s)\Gamma(s-4)\Gamma(s-8) \det(T)^{-s}.$$ 

For the functional equation, we write

$$\Lambda(s, F) = N^{3s}\int_{\mathcal{R}_N \mathbb{R}_+} F_X(\sqrt{-TY}) \det(Y)^s d^sY + N^{3s} \int_{\mathcal{R}_N \mathbb{R}_+} F_X(\sqrt{-TY}) \det(Y)^s d^sY.$$
Notice that \( \det((N^2 Y)^{-1}) = N^{-6} \det(Y)^{-1} \). Use the change of variables \( Y \mapsto \frac{1}{N^2 Y} \), and the functional equation (2.2):

\[
N^3 s \int_{\det(Y) \leq N^{-3}} F_\chi(\sqrt{-1} Y) \det(Y)^s \, d^s Y = N^3 s \int_{\det(Y) \geq N^{-3}} F_\chi(\sqrt{-1} N^{-2} Y^{-1}) \det(Y)^{-s} \, d^s Y \\
= \int_{\det(Y) \geq N^{-3}} (-1)^{\frac{3k}{2}} N^{3k-3s} \frac{W(\chi)^2}{N} F_\chi(\sqrt{-1} Y) \det(Y)^{k-s} \, d^s Y.
\]

Hence

\[
\Lambda(s, F) = \int_{\det(Y) \geq N^{-3}} \left( F_\chi(\sqrt{-1} Y) (N^3 \det(Y))^s + (-1)^{\frac{3k}{2}} \frac{W(\chi)^2}{N} F_\chi(\sqrt{-1} Y) (N^3 \det(Y))^{k-s} \right) d^s Y.
\]

The analytic continuation and the functional equation follow from this. \( \square \)

**Remark 3.2.** Even though we do not need in this paper, one can ask: is it true that \( \epsilon_N(T) = 1 \) for \( N \geq 3 \)?

4. **Relationship between the twisted Koecher-Maass series of the first and the second kind**

In this section we express the twisted Koecher-Maass series of the first kind in terms of the twisted Koecher-Maass series of the second kind. Define \( h(A, \chi) \) as

\[
h(A, \chi) = \sum_{g \in M(\mathbb{Z})/M(N; \mathbb{Z})} \chi(\text{Tr}(g \cdot A)).
\]

**Proposition 4.1.** Let

\[
F(Z) = \sum_{T \in \mathbb{Z}^3 \setminus 0} a_F(T) \exp(2\pi \sqrt{-1}(T, Z))
\]

be a cusp form of weight \( k \) (\( k \) even) with respect to \( G(\mathbb{Z}) \) and \( \chi \) a Dirichlet character mod \( N \). Then we have

\[
K^{(1)}(s, F, \chi) = \sum_{T \in \mathbb{Z}^3 \setminus 0/M(\mathbb{Z})} \frac{h(T, \chi)a_F(T)}{\epsilon(T)(\det T)^s}.
\]

**Proof.** This can be proved in the same manner as in [6, Proposition 3.1]. But for readers' convenience, we give a proof. The assertion is trivial if \( N = 1 \). Suppose that \( N > 1 \). We have

\[
K^{(1)}(s, F, \chi) = \sum_{T \in \mathbb{Z}^3 \setminus 0/M(\mathbb{Z})} \frac{a_F(T)}{(\det T)^s} \sum_{T' \in \mathbb{Z}^3} \frac{\chi(\text{Tr}(T'))}{\epsilon_N(T')}.
\]
where
\[ J_T = \{ T' \in J(\mathbb{Z})_> \mid T' \sim_{M'} T \}. \]

We note that \( \epsilon_N(T') = \epsilon_N(T) \) for any \( T' \in J_T \), and
\[ \frac{\epsilon(T)}{\epsilon_N(T)} = \#(M'(N;\mathbb{Z})U_T(\mathbb{Z})/M'(N;\mathbb{Z})). \]

Fix an element \( T \) of \( J(\mathbb{Z})_> \). Let \( g_1, g_2 \in M'(\mathbb{Z}) \). Then \( g_1 \cdot T \sim_{M'(N;\mathbb{Z})} g_2 \cdot T \) if and only if \( g_2 \in g_1 M'(N;\mathbb{Z})U_T(\mathbb{Z}) \). Therefore, the set \( \{ g \cdot T \mid g \in M'(\mathbb{Z})/M'(N;\mathbb{Z})U_T(\mathbb{Z}) \} \) is a complete set of representatives of \( J_T/M'(N;\mathbb{Z}) \). Moreover, we note that \( \text{Tr}(g \cdot T) \mod N \) depends only on \( g \in M'(\mathbb{Z})/M'(N;\mathbb{Z})U_T(\mathbb{Z}) \). Hence we have
\[
\epsilon(T) \sum_{T' \in J_T/M'(\mathbb{Z})} \frac{\chi(\text{Tr}(T'))}{\epsilon_N(T')} = \sum_{g \in M'(\mathbb{Z})/M'(N;\mathbb{Z})U_T(\mathbb{Z})} \chi(\text{Tr}(g \cdot T)) \frac{\epsilon(T)}{\epsilon_N(T)}
\]
\[ = \sum_{g \in M'(\mathbb{Z})/M'(N;\mathbb{Z})U_T(\mathbb{Z})} \chi(\text{Tr}(g \cdot T)) \#(M'(N;\mathbb{Z})U_T(\mathbb{Z})/M'(N;\mathbb{Z})) \]
\[ = \sum_{g \in M'(\mathbb{Z})/M'(N;\mathbb{Z})} \chi(\text{Tr}(g \cdot T)). \]

This proves the assertion. \( \square \)

**Remark 4.2.**

1. We need not assume that \( \epsilon_N(T) = 1 \) for \( N > 1 \) unlike [6, Proposition 3.1].
2. There is a typo in the proof of [6, Proposition 3.1]. The equality ‘\( \text{tr}(A[U_1]) = \text{tr}(A[U_2])' \) on page 463. line 7 should be ‘\( \text{tr}(A[U_1]) \equiv \text{tr}(A[U_2]) \mod N. \)

Let \( \chi \) be a Dirichlet character mod \( N \). Fix a prime factor \( p \) of \( N \). For an integer \( n \) prime to \( p \), take an integer \( m \) such that
\[ m \equiv \begin{cases} 
 n \mod p^\ell \\
 1 \mod N/p^\ell 
\end{cases}. \]

We then put
\[ \chi^{(p)}(n) = \begin{cases} 
 \chi(m) & \text{if } (n, p) = 1 \\
 0 & \text{if } (n, p) \neq 1 
\end{cases}. \]

Then it is independent of the choice of \( m \), and \( \chi^{(p)} \) is a character mod \( p^\ell \), and we have \( \chi = \prod_{p|N} \chi^{(p)} \).
For Dirichlet characters $\chi_1, \ldots, \chi_r \mod N$, we define the generalized Jacobi sum $J(\chi_1, \ldots, \chi_r)$ by

$$J(\chi_1, \ldots, \chi_r) = \sum_{a_1, \ldots, a_r \in \mathbb{Z}/N\mathbb{Z}} \chi_1(a_1) \cdots \chi_r(a_r).$$

We note that $J(\chi_1, \chi_2)$ is the usual Jacobi sum. Moreover, we define the Jacobi sum $J_2(\chi_1, \chi_2)$ on $J_2$ by

$$J_2(\chi_1, \chi_2) = \sum_{B \in J_2(\mathbb{Z}/N\mathbb{Z})} \chi_1(\det B) \chi_2(1 - \text{Tr}(B)).$$

**Proposition 4.3.** (1) Let $A \in J(\mathbb{Z})$ and $\chi$ a Dirichlet character mod $N$. Then we have

$$h(A, \chi) = \prod_{p | N} h(A, \chi^{(p)}).$$

(2) Let $\chi_1, \ldots, \chi_r$ be Dirichlet characters mod $N$.

(2.1) Suppose that $r \geq 3$ and that $\chi_1 \cdots \chi_{r-1}$ is primitive. Then,

$$J(\chi_1, \ldots, \chi_r) = J(\chi_1 \cdots \chi_{r-1}, \chi_r)J(\chi_1, \ldots, \chi_{r-1}).$$

(2.2) Suppose that $\chi_1 \chi_2$ is primitive. Then,

$$J(\chi_1, \chi_2) = \frac{W(\chi_1)W(\chi_2)}{W(\chi_1 \chi_2)}.$$

**Proof.** The assertion (1) can be proved by the Chinese remainder theorem. To prove (2.1) and (2.2), again by the Chinese remainder theorem, we may assume that $N = p^m$ with $p$ a prime number. We have

$$J(\chi_1, \ldots, \chi_r) = \sum_{a_1, \ldots, a_r \in \mathbb{Z}/p^m\mathbb{Z}} \chi_1(a_1) \cdots \chi_{r-2}(a_{r-2}) \chi_{r-1}(1 - a_1 - \cdots - a_{r-2} - a_r) \chi_r(a_r)$$

$$= \sum_{a_1, \ldots, a_{r-2} \in \mathbb{Z}/p^m\mathbb{Z}, v \in \mathbb{Z}/p^{m-1}\mathbb{Z}} \chi_1(a_1) \cdots \chi_{r-2}(a_{r-2}) \chi_{r-1}(-pv - a_1 - \cdots - a_{r-2}) \chi_r(pv - 1)$$

$$+ \sum_{a_1, \ldots, a_{r-2} \in \mathbb{Z}/p^m\mathbb{Z}, a_r \not\equiv 1 \mod p} \chi_1(a_1) \cdots \chi_{r-2}(a_{r-2}) \chi_{r-1}(1 - a_1 - \cdots - a_{r-2} - a_r) \chi_r(a_r).$$

Since $\chi_1 \cdots \chi_{r-1}$ is primitive, the first term of the right-hand side of the above equation is zero. Hence, putting $a_1 = (1 - a_r)b_1, \ldots, a_{r-2} = (1 - a_r)b_{r-2}$ in the second term of the right-hand side
of the above equation, we have

$$J(\chi_1, \cdots, \chi_r) = \sum_{b_1, \ldots, b_{r-2}, a_r \in \mathbb{Z}/p^m\mathbb{Z}} \chi_1((1 - a_r)b_1) \cdots \chi_{r-2}((1 - a_r)b_{r-2})$$

$$\times \chi_{r-1}(1 - a_r - (1 - a_r)(b_1 + \cdots + b_{r-2}))\chi_r(a_r)$$

$$= \sum_{b_1, \ldots, b_{r-2} \in \mathbb{Z}/p^m\mathbb{Z}} \chi_1(b_1) \cdots \chi_{r-2}(b_{r-2})\chi_{r-1}(1 - b_1 - \cdots - b_{r-2})$$

$$\times \sum_{a_r \in \mathbb{Z}/p^m\mathbb{Z}} (\chi_1 \cdots \chi_{r-1})(1 - a_r)\chi_r(a_r).$$

This proves the assertion (2.1). The second assertion (2.2) is well known in the case $m = 1$ (e.g. [3, Ch.8, Theorem 1]), and the general case can also be proved in the same manner.

Recall from the introduction, $\mathcal{D}_N = \{\eta \mid \eta \text{ is a Dirichlet character mod } N \text{ such that } \eta^l = 1\}$, where $l = \text{GCD}(3, \phi(N))$.

**Corollary 4.4.** Let $\bar{\chi}$ be a Dirichlet character mod $N$ and put $\chi = \bar{\chi}^3$. Suppose that $N$ is odd and $\chi$ is primitive. Then, for any $\eta \in \mathcal{D}_N$, we have

$$J(\bar{\chi}\eta, \bar{\chi}\eta, \bar{\chi}\eta) = \frac{W(\bar{\chi}\eta)^3}{W(\bar{\chi})}.$$  

**Proof.** By the assumption, $\chi^2$ is also primitive, and so is $\bar{\chi}^2\eta^2$ for any $\eta \in \mathcal{D}_N$. Hence, by Lemma 4.9 and Proposition 4.3 we have

$$J_{3_2}(\bar{\chi}\eta, \bar{\chi}\eta) = N^4 J(\bar{\chi}\eta, \bar{\chi}\eta, \bar{\chi}\eta) = N^4 J(\bar{\chi}\eta, (\bar{\chi}\eta)^2)J(\bar{\chi}\eta, \bar{\chi}\eta)$$

$$= N^4 \frac{W(\bar{\chi}\eta)W((\bar{\chi}\eta)^2)}{W((\bar{\chi}\eta)^3)} \frac{W(\bar{\chi}\eta)W(\bar{\chi}\eta)}{W((\bar{\chi}\eta)^2)} = N^4 \frac{W(\bar{\chi}\eta)^3}{W(\chi)^3}. \tag*{\blacksquare}$$

**Remark 4.5.** For a primitive character $\bar{\chi}$ mod $2^m$, $\bar{\chi}^2$ is not primitive.

We will show in Lemma 4.9 that $J_{3_2}(\chi_1, \chi_2) = N^4 J(\chi_1, \chi_1, \chi_2)$.

For a commutative ring $R$, let $S_l(R)$ denote the set of symmetric matrices of size $l$ with entries in $R$. For $S \in S_l(R)$ and $X \in M_{lm}(R)$, put $S[X] = ^tXSX$. Then, $GL_l(R)$ acts on $S_l(R)$ in the following way:

$$GL_l(R) \times S_l(R) \ni (g, S) \mapsto S[^t g].$$

Let $A_1$ and $A_2$ be elements of $S_l(R)$. Then, by definition, we have $A_1 \sim_{GL_l(R)} A_2$ if there exists an element $g \in GL_l(R)$ such that $A_1[^t g] = A_2$. (This is equivalent to saying that there exists
an element \( g \in GL_l(R) \) such that \( A_1[g] = A_2 \). Let \( R \) be an integral domain and \( K \) its quotient field. Suppose that the characteristic of \( K \) is different from 2. We denote by \( \mathcal{H}_l(R) \) the set of half-integral symmetric matrices of size \( l \) over \( R \), that is, the set of symmetric matrices \( A = (a_{ij}) \) in \( S_l(K) \) such that \( 2a_{ij}, a_{ii} \in R \).

Let \( S \in \mathcal{H}_l(\mathbb{Z}_p) \) with \( l \) even and \( \det S \neq 0 \). Then we put

\[
\chi(S) = \chi_p(S) = \begin{cases} 
1 & \text{if } S \sim_{GL_l(\mathbb{Z}_p)} H \perp \cdots \perp H \\
-1 & \text{otherwise}
\end{cases},
\]

where \( H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \). For \( w \in \mathbb{C}_Q \), let \( N(w) \in \mathbb{Q} \) be the norm. Then \( N \) defines a quadratic form over \( \mathbb{Q} \) in 8 variables, and in particular, \( N|_o \) defines an integral quadratic form. Let \( S_N \) be the Gram matrix of \( N|_o \) with respect to the standard basis. By construction, \( S_N \in \mathcal{H}_8(\mathbb{Z}) \cap \frac{1}{2}GL_8(\mathbb{Z}) \) and positive definite. Hence for any prime number \( p \), we have

\[
S_N \sim_{GL_8(\mathbb{Z}_p)} H \perp H \perp H \perp H.
\]

Hence \( \chi_p(S_N) = 1 \). For \( Z_1 = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \in \mathcal{J}_2(\mathbb{Q}) \) and \( w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in M_{2,1}(\mathbb{C}_Q) \), \( Z_1[w] := \bar{w}Z_1w \) is well-defined as the usual matrix multiplication. Then

\[
Z_1[w] = aN(w_1) + dN(w_2) + \text{Tr}(\bar{w}_1bw_2),
\]

and

\[
Q : M_{2,1}(\mathbb{C}_Q) \rightarrow \mathbb{Q}, \quad w \mapsto Z_1[w],
\]
defines a quadratic form over \( \mathbb{Q} \) in 16 variables. In particular if \( Z_1 \in \mathcal{J}_2(\mathbb{Z}) \), then \( Q|_{M_{2,1}(o)} \) defines an integral quadratic form.

For \( A \in \mathcal{H}_l(\mathbb{Z}) \) and \( c \in \mathbb{Z} \), let

\[
A_{p^m}(A, c) = \{ x \in M_{l1}(\mathbb{Z}/p^m\mathbb{Z}) \mid A[x] \equiv c \mod p^m \},
\]

and

\[
A_{p^m}^{\text{prim}}(A, c) = \{ x \in A_{p^m}(A, c) \mid x \not\equiv 0 \mod p \},
\]

and put \( A_{p^m}(A, c) = \#A_{p^m}(A, c) \) and \( A_{p^m}^{\text{prim}}(A, c) = \#A_{p^m}^{\text{prim}}(A, c) \).

**Lemma 4.6.** Let \( l \) be a positive even integer. Let \( S \in \mathcal{H}_l(\mathbb{Z}) \cap \frac{1}{2}GL_l(\mathbb{Z}_p) \) and \( c \in \mathbb{Z} \).
(1) We have
\[ A_p^{\text{prim}}(S, c) = \begin{cases} p^{l-1} (1 - p^{-l/2} \chi(S)), & \text{if } c \not\equiv 0 \mod p, \\ p^{(l-1)} (1 - p^{-l/2} \chi(S))(1 + p^{-l/2+1} \chi(S)), & \text{if } c \in p\mathbb{Z}. \end{cases} \]

(2) We have
\[ A_p(S, c) = \begin{cases} p^{l-1} (1 - p^{-l/2} \chi(S)), & \text{if } c \not\equiv 0 \mod p, \\ p^{(l-1)} (1 - p^{-l/2} \chi(S))(1 + p^{-l/2+1} \chi(S)) + 1, & \text{if } c \in p\mathbb{Z}. \end{cases} \]

(3) Suppose that \( m \geq 2 \). Then
\[ A_{p^m}(S, c) = A_p^{\text{prim}}(S, c) + p^l A_{p^{m-2}}(S, p^{-2}c). \]
Here we put the convention that \( A_{p^{m-2}}(S, p^{-2}c) = 0 \) if \( \text{ord}_p(c) \leq 1 \), and \( A_{p^{m-2}}(S, p^{-2}c) = 1 \) if \( m = 2 \) and \( \text{ord}_p(c) \geq 2 \).

Proof. The assertion (2) follows from [10, Theorem 1.3.2] and [10, Lemma 5.5.9]. The assertion (1) follows from [10, Theorem 1.3.2] and [10, Lemma 1.3.1]. Suppose that \( m \geq 2 \). Then
\[ A_{p^m}(S, c) = A_p^{\text{prim}}(S, c) \cup A'_{p^m}(S, c), \]
where
\[ A'_{p^m}(S, c) = \{ x \in A_{p^m}(S, c) \mid x \equiv 0 \mod p \}. \]

Put
\[ A''_{p^m}(S, c) = \{ x \in M_1(\mathbb{Z}_p/p^{m-1}\mathbb{Z}_p) \mid A[x] \equiv p^{-2}c \mod p^{m-2} \}. \]
Then clearly we have
\[ \# A''_{p^{m-1}}(S, p^{-2}c) = p^l \# A_{p^{m-2}}(A, p^{-2}c). \]
For \( x = py \in A'_{p^m}(A, c) \), the element \( y \mod p^{m-1} \) belongs to \( A''_{p^{m-1}}(S, p^{-2}c) \), and the mapping \( x \mapsto y \mod p^{m-1} \) gives a bijection from \( A'_{p^m}(A, c) \) to \( A''_{p^{m-1}}(S, p^{-2}c) \). This proves the assertion (3). \( \square \)

Corollary 4.7. Fix \( S \in \mathcal{H}_l(\mathbb{Z}) \cap \frac{1}{2} GL_1(\mathbb{Z}_p) \). Then, \( A_{p^m}(S, c) \) depends only on \( \text{ord}_p(c) \).

Lemma 4.8. Let \( l, S \) and \( c \) be as in Lemma 4.6. Let \( \eta \) be a primitive character \( \mod p^m \). Put
\[ I_{\eta, S, c} = \sum_{w \in (\mathbb{Z}/p^m\mathbb{Z})^l} \eta(S^l[w] + c). \]
Then
\[
I_{\eta,S,c} = p^{m/2} \eta(c) \times \begin{cases} 
\chi(S) & \text{if } m \text{ is odd,} \\
1 & \text{if } m \text{ is even.}
\end{cases}
\]

Proof. Suppose that \( m \geq 2 \). Then, by Lemma 4.6, we have
\[
I_{\eta,S,c} = \sum_{u \in \mathbb{Z}/p^m \mathbb{Z}} \eta(u) A_{p^m}(S, u - c)
\]

\[
= p^{l-1}(1 - p^{-l/2}\chi(S)) \sum_{u \in \mathbb{Z}/p^m \mathbb{Z}} \eta(u)
\]

\[
+ \sum_{u \in \mathbb{Z}/p^m \mathbb{Z}} \left( p^{l-1}(1 - p^{-l/2}\chi(S))(1 + p^{-l/2+1}\chi(S)) + p^l A_{p^m-2}(S, p^{-2}(u - c)) \right) \eta(u).
\]

Since \( \eta \) is primitive, we have
\[
I_{\eta,S,c} = p^l \sum_{u \in \mathbb{Z}/p^m \mathbb{Z}} A_{p^m-2}(S, p^{-2}(u - c)) \eta(u) = p^l \sum_{v \in \mathbb{Z}/p^{m-2} \mathbb{Z}} A_{p^m-2}(S, v) \eta(c + p^2 v).
\]

Suppose that \( c \in p\mathbb{Z} \). Then \( \eta(c + p^2 v) = 0 \) for any \( v \), and therefore \( I_{\eta,S,c} = 0 \). Suppose that \( c \notin p\mathbb{Z} \). For \( r \leq m/2 \), put
\[
L(r) = \sum_{v \in \mathbb{Z}/p^{m-2r} \mathbb{Z}} A_{p^{m-2r}}(S, v) \eta(c + p^{2r} v).
\]

Then, \( I_{\eta,S,c} = p^l L(1) \). Using the same argument as above and by Corollary 4.7, we have
\[
L(r) = p^l L(r + 1)
\]

if \( 2r + 2 \leq m \). Hence we have
\[
I_{\eta,S,c} = \begin{cases} 
 p^{ml/2} L(m/2) & \text{if } m \text{ is even,} \\
 p^{(m-1)l/2} L((m-1)/2) & \text{if } m \text{ is odd.}
\end{cases}
\]

By definition, we have \( L(m/2) = \eta(c) \). The relation \( L((m-1)/2) = p^{l/2}\chi(S)) \eta(c) \) follows from [4 Lemma 5.3] in the case \( p \) is odd, and it can also be proved in the case \( p = 2 \) in the same way. This proves the assertion.

\[ \square \]

Lemma 4.9. Let \( \chi \) and \( \eta \) be Dirichlet characters mod \( N \). Suppose that \( \chi \) is primitive. Then
\[
J_{\mathbb{Z}}(\chi, \eta) = N^4 J(\chi, \chi, \eta).
\]
Proof. By Chinese remainder theorem, we may assume that $N = p^m$ with $p$ a prime number. By definition,

$$J_{32}(\chi, \eta) = \sum_{(z_{11}, z_{22}) \in \mathbb{Z}/p^m \mathbb{Z}^2} \chi(-N(z_{12}) + z_{11}z_{22})\eta(1 - z_{11} - z_{22}).$$

Here, the norm $N(z_{12})$ can be regarded as a quadratic form over $\mathbb{Z}/p^m \mathbb{Z}$ of rank 8 with determinant 1. Hence, by Lemma 4.8, we have

$$J_{32}(\chi, \eta) = \sum_{(z_{11}, z_{22}) \in \mathbb{Z}/p^m \mathbb{Z}^2} p^{5m} \chi(z_{11}z_{22})\eta(1 - z_{11} - z_{22}).$$

This proves the assertion. □

For a positive integer $N$, let $d_N = N^{64} \prod_{p \mid N} (1 - p^{-2})(1 - p^{-6})(1 - p^{-8})(1 - p^{-12})$ as in Theorem 1.2. For $T \in J(\mathbb{Z}/N \mathbb{Z})$ and a positive integer $N$, let $\phi_{T,N}: M'(\mathbb{Z}/N \mathbb{Z}) \ni g \mapsto g \cdot T \in J(\mathbb{Z}/N \mathbb{Z})$.

Lemma 4.10. Let $p$ be a prime number and $m$ a positive integer. Let $T, S \in J(\mathbb{Z}/p^m \mathbb{Z})$ such that $\det T \in (\mathbb{Z}/p^m \mathbb{Z})^\times$. Then $\phi_{T,p^m}^{-1}(S) \neq \emptyset$ if and only if $\det S = \det T$. Moreover

$$\#(\phi_{T,p^m}^{-1}(S)) = p^{52m} (1 - p^{-2})(1 - p^{-6})(1 - p^{-8})(1 - p^{-12}).$$

Proof. The proof will be given after Proposition 5.1. □

The following lemma is easy to prove.

Lemma 4.11. Let $l = \gcd(3, \phi(N))$ and let $u_0$ be a primitive $l$-th root of unity mod $N$. Let $\chi$ be a Dirichlet character mod $N$ and suppose that $\chi(u_0) = 1$. Then there exists a Dirichlet character $\tilde{\chi}$ such that $\tilde{\chi}^3 = \chi$.

Theorem 4.12. Let $\chi$ be a primitive character mod $p^m$. Suppose that $\chi$ is not a quadratic character. Let $l = \gcd(3, \phi(p^m))$ and let $u_0$ be a primitive $l$-th root of unity mod $p^m$.

1. Suppose that $\chi(u_0) \neq 1$. Then $h(A, \chi) = 0$.

2. Suppose that $\chi(u_0) = 1$. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^3 = \chi$. Then

$$h(A, \chi) = d_{p^m} \sum_{\eta \in D_{p^m}} (\tilde{\chi}\eta)(\det A)J(\tilde{\chi}\eta, \tilde{\chi}\eta, \tilde{\chi}\eta).$$

In particular if $p$ is odd, then

$$h(A, \chi) = d_{p^m} \sum_{\eta \in D_{p^m}} (\tilde{\chi}\eta)(\det A) \frac{W(\tilde{\chi}\eta)^3}{W(\tilde{\chi})}. $$
Proof. The case $\chi = 1$ is trivial. Suppose that $\chi \neq 1$.

First, suppose that $\det A \in p\mathbb{Z}$. We may assume that $A = A_0 \perp pA_1$ with $A_0 \in \mathfrak{J}_2(\mathbb{Z}/p^m\mathbb{Z})$ and $A_1 \in \mathcal{C}(\mathbb{Z}/p^m\mathbb{Z})$. First let $p$ be odd. Then there exists an element $\xi_0 \in \mathbb{Z}/p^m\mathbb{Z}$ such that $\xi_0 \equiv 1 \mod p^{m-1}$ and $\chi(\xi_0)^2 \neq 1$. Indeed, if $m = 1$, this follows just from the assumption. If $m \geq 2$, since $\chi$ is primitive mod $p^m$, there exists an element $\xi_0$ such that $\xi_0 \equiv 1 \mod p^{m-1}$ and $\chi(\xi_0) \neq 1$. Then we have $\xi_0^p \equiv 1 \mod p^m$, and hence we can easily see that $\chi(\xi_0)^2 \neq 1$. We have $\theta(1, 1, \xi_0^{-3}) \cdot A = A$ and $\nu(\theta(1, 1, \xi_0^3) = \xi_0^{-6}$. Define $g_0 \in \mathcal{M}(\mathbb{Z}/p^m\mathbb{Z})$ as

$$g_0 : \mathfrak{J}(\mathbb{Z}/p^m\mathbb{Z}) \ni (t_{ij}) \mapsto (\xi_0^2 t_{ij}) \in \mathfrak{J}(\mathbb{Z}/p^m\mathbb{Z}).$$

Then, $\nu(g_0) = \xi_0^6$, and hence $g_0 \theta(1, 1, \xi_0^{-3}) = 1$. Hence

$$h(A, \chi) = \sum_{g \in \mathcal{M}(\mathbb{Z}/p^m\mathbb{Z})} \chi(\text{Tr}((g_0 \theta(1, 1, \xi_0^{-3}) \cdot A) = \sum_{g \in \mathcal{M}(\mathbb{Z}/p^m\mathbb{Z})} \chi(\text{Tr}(g \cdot (g \cdot A)) = \chi(\xi_0^2) h(A, \chi).$$

Hence we have $h(A, \chi) = 0$. Next, let $p = 2$. Then, $m \geq 4$. Put $\xi_0 = 1 + 2^{m-1} \mod 2^m$ and $\zeta_0 = 1 + 2^{m-2} \mod 2^m$. Then, $\xi_0^2 = \xi_0$ and $\zeta_0^6 = \zeta_0 = \xi_0$. We have $\theta(1, 1, \zeta_0^{-1}) \cdot A = A$ and $\nu(\theta(1, 1, \zeta_0^{-1})) = \xi_0^{-1}$. Then, similarly as above we have

$$h(A, \chi) = \sum_{g \in \mathcal{M}(\mathbb{Z}/p^m\mathbb{Z})} \chi(\text{Tr}((g_0 \theta(1, 1, \zeta_0^{-1}) \cdot A) = \sum_{g \in \mathcal{M}(\mathbb{Z}/p^m\mathbb{Z})} \chi(\text{Tr}(g \cdot (g \cdot A)) = \chi(\xi_0) h(A, \chi).$$

Hence we have $h(A, \chi) = 0$.

Next, suppose that $\det A$ is a $p$-unit. For $c \in \mathbb{Z}/p^m\mathbb{Z}$, put

$$\mathcal{R}_p^m(A, c) = \{g \in \mathcal{M}(\mathbb{Z}/p^m\mathbb{Z}) \mid \text{Tr}(g \cdot A) = c\}.$$ 

Then we have

$$h(A, \chi) = \sum_{c \in \mathbb{Z}/p^m\mathbb{Z}} \chi(c) \#(\mathcal{R}_p^m(A, c)).$$

Let

$$\mathcal{S}_p^m(A, c) = \{B \in \mathfrak{J}(\mathbb{Z}/p^m\mathbb{Z}) \mid \det B = \det A \text{ and } \text{Tr}(B) = c\}.$$ 

Then, by Lemma 4.10,

$$\#(\mathcal{R}_p^m(A, c)) \cdot a_p^m \#(\mathcal{S}_p^m(A, c)),$$
where \( a_{p^m} = p^{52m}(1 - p^{-2})(1 - p^{-6})(1 - p^{-8})(1 - p^{-12}) \). Let

\[
\tilde{S}_{p^m}(A, c) = \{(Z_1, w) \in J_2(\mathbb{Z}/p^m\mathbb{Z}) \times M_{2,1}(\mathcal{C}(\mathbb{Z}/p^m\mathbb{Z})) \mid \det \begin{pmatrix} Z_1 & w \\ t \bar{w} & 1 - \text{Tr}(Z_1) \end{pmatrix} c^3 = \det A \}.
\]

Then we have

\[
\#(\tilde{S}_{p^m}(A, c)) = \#(S_{p^m}(A, c)).
\]

First, suppose that \( \chi(u_0) \neq 1 \). Then we have

\[
\tilde{S}(A, cu_0) = \tilde{S}(A, c),
\]

for any \( c \in (\mathbb{Z}/p^m\mathbb{Z})^\times \), and hence

\[
\sum_{c \in \mathbb{Z}/p^m\mathbb{Z}} \chi(c)\#\tilde{S}(A, c) = \sum_{c \in \mathbb{Z}/p^m\mathbb{Z}} \chi(u_0c)\#\tilde{S}(A, c) = \chi(u_0) \sum_{c \in \mathbb{Z}/p^m\mathbb{Z}} \chi(c)\#\tilde{S}(A, c).
\]

Therefore, \( h(A, \chi) = 0 \).

Suppose that \( \chi(u_0) = 1 \). Then we have

\[
h(A, \chi) = a_{p^m} \sum_{(Z_1, w_1) \in J_2(\mathbb{Z}/p^m\mathbb{Z}) \times M_{2,1}(\mathcal{C}(\mathbb{Z}/p^m\mathbb{Z}))} \tilde{\chi}(\det \begin{pmatrix} Z_1 & w \\ t \bar{w} & 1 - \text{Tr}(Z_1) \end{pmatrix}),
\]

where \((Z, w)\) runs over all elements of \( J_2(\mathbb{Z}/p^m\mathbb{Z}) \times M_{2,1}(\mathcal{C}(\mathbb{Z}/p^m\mathbb{Z}))\) such that

\[
(4.1) \quad \det \begin{pmatrix} Z_1 & w \\ t \bar{w} & 1 - \text{Tr}(Z_1) \end{pmatrix} \equiv u^3 \mod p^m,
\]

with some \( u \in \mathbb{Z} \cap \mathbb{Z}_p^\times \). For such a matrix \( \begin{pmatrix} Z_1 & w \\ t \bar{w} & 1 - \text{Tr}(Z_1) \end{pmatrix} \), there exist exactly \( l \) elements satisfying \( 4.1 \). We have

\[
\sum_{\eta \in \mathcal{D}_{p^m}} \tilde{\chi}\eta(v) = l\tilde{\chi}(v) \text{ or } 0,
\]

according as \( v \equiv u^m \mod p^m \) with some \( u \in \mathbb{Z} \cap \mathbb{Z}_p^\times \) or not. Hence we have

\[
h(A, \chi) = a_{p^m} \sum_{\eta \in \mathcal{D}_{p^m}} \sum_{(Z_1, w) \in J_2(\mathbb{Z}/p^m\mathbb{Z}) \times M_{2,1}(\mathcal{C}(\mathbb{Z}/p^m\mathbb{Z}))} (\tilde{\chi}\eta)(\det A)(\bar{\tilde{\chi}}\eta) \left( \det \begin{pmatrix} Z_1 & w \\ t \bar{w} & 1 - \text{Tr}(Z_1) \end{pmatrix} \right).
\]

For \( Z_1 \), put

\[
I(Z_1) = \sum_{w \in M_{2,1}(\mathcal{C}(\mathbb{Z}/p^m\mathbb{Z}))} (\tilde{\chi}\eta)(\det A)(\bar{\tilde{\chi}}\eta) \left( \det \begin{pmatrix} Z_1 & w \\ t \bar{w} & 1 - \text{Tr}(Z_1) \end{pmatrix} \right).
\]
Then
\[ I(Z_1) = \sum_{\eta, \chi \in \mathcal{E}(\mathbb{Z}/p^m\mathbb{Z})} (\chi \eta) \left( -\operatorname{Ad}(Z_1)[w] + (\det Z_1)(1 - \operatorname{Tr}(Z_1)) \right), \]
where \( \operatorname{Ad}(Z_1) = \begin{pmatrix} z_{22} & -z_{12} \\ -z_{12} & z_{11} \end{pmatrix} \) for \( Z_1 = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \). Then we may suppose that \( Z_1 = O \) or \( Z_1 = \xi_1 \perp p\xi_2 \) with \( \xi_1, \xi_2 \in \mathbb{Z} \) such that \( \xi_1 \notin p\mathbb{Z} \). In the former case, clearly we have \( I(Z_1) = 0 \). In the latter case, \( I(Z_1) \) can be expressed as
\[ I(Z_1) = \sum_{w_1} \sum_{w_2} (\chi \eta) \left( -N(w_1) + (\det Z_1)(1 - \operatorname{Tr}(Z_1)) - p\xi_1 N(w_2) \right). \]
Then the Gram matrix of the quadratic form \(-N(w)\) is \( -S_N \) and \( \chi(-S_N) = 1 \). We note that \((\det Z_1)(1 - \operatorname{tr}(Z_1)) - p\xi_1 N(w_2) \in p\mathbb{Z}_p \). Thus by Lemma 4.8, we have \( I(Z_1) = 0 \). Suppose that \( \det Z_1 \in \mathbb{Z}_p^\times \). Then we may suppose that \( Z_1 = \xi_1 \perp \xi_2 \) with \( \xi_1, \xi_2 \in \mathbb{Z}_p^\times \) and
\[ I(Z_1) = \sum_{w_2} \sum_{w_1} (\chi \eta) \left( -\xi_1 N(w_1) - \xi_2 N(w_2) + (\det Z_1)(1 - \operatorname{Tr}(Z_1)) \right). \]
Then the Gram matrix of the quadratic form \(-\xi_1 N(w_1) - \xi_2 N(w_2)\) is \( -\xi_1 S_N \perp \xi_2 S_N \), and we have \( \chi(-\xi_1 S_N \perp \xi_2 S_N) = 1 \). Thus, by Lemma 4.8,
\[ I(Z_1) = p^m \left( \chi \eta \right) ((\det Z_1)(1 - \operatorname{Tr}(Z_1))). \]
Hence we have
\[ h(A, \chi) = a_{p^m} p^m \sum_{\eta \in \mathcal{D}_{p^m}} (\chi \eta)(\det A)J_2(\chi \eta, \chi \eta). \]
Thus the assertion follows from Lemma 4.9 and Corollary 4.4. \( \square \)

**Corollary 4.13.** Let \( \chi \) be a primitive character mod \( N \). Suppose that \( \chi \) is not a quadratic character. Let \( l = \operatorname{GCD}(3, \phi(N)) \) and let \( u_0 \) be a primitive \( l \)-th root of unity mod \( N \).

1. Suppose that \( \chi(u_0) \neq 1 \). Then \( h(A, \chi) = 0 \).
2. Suppose that \( \chi(u_0) = 1 \). Fix a character \( \tilde{\chi} \) such that \( \tilde{\chi}^3 = \chi \). Then
\[ h(A, \chi) = d_N \sum_{\eta \in \mathcal{D}_N} (\chi \eta)(\det A)J(\chi \eta, \chi \eta, \chi \eta). \]
In particular, if \( N \) is odd, then
\[ h(A, \chi) = d_N \sum_{\eta \in \mathcal{D}_N} (\chi \eta)(\det A) \frac{W(\chi \eta)^3}{W(\chi)}. \]
Proof. Let $N = p_1^{e_1}\cdots p_r^{e_r}$ with $p_1, \ldots, p_r$ distinct primes such that $e_1, \ldots, e_r$ are positive integers. Then the mapping

$$
\mathcal{D}_N \ni \chi \mapsto (\chi(p_1), \ldots, \chi(p_r)) \in \mathcal{D}_{p_1^{e_1}} \times \cdots \times \mathcal{D}_{p_r^{e_r}},
$$

is a bijection. Hence the assertion follows from Proposition 4.13 and Theorem 4.12.

Theorem 4.14. Let $\chi$ be a primitive character mod $N$. Suppose that $\chi$ is not a quadratic character. Let $l = \gcd(3, \phi(N))$ and let $u_0$ be a primitive $l$-th root of unity mod $N$.

1) Suppose that $\chi(u_0) \neq 1$. Then

$$K^{(1)}(s, F, \chi) = 0.$$ 

2) Suppose that $\chi(u_0) = 1$. Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^3 = \chi$. Then

$$K^{(1)}(s, F, \chi) = d_N \sum_{\eta \in \mathcal{D}_N} K^{(2)}(s, F, \tilde{\chi}\eta) J(\tilde{\chi}\eta, \tilde{\chi}\eta, \tilde{\chi}\eta).$$

In particular if $N$ is odd, then

$$K^{(1)}(s, F, \chi) = d_N \sum_{\eta \in \mathcal{D}_N} K^{(2)}(s, F, \tilde{\chi}\eta) \frac{W(\tilde{\chi}\eta)^3}{W(\tilde{\chi})}.$$ 

5. Mass formula for the exceptional group of type $F_4$

In this section, we recall the mass formula and local density from [5] for the sake of completeness. For $T \in \mathfrak{J}_R$, we define a group scheme $U_T$ over $R$ by

$$U_T(S) = \{ g \in M(S) \mid g \cdot T = T \}$$

for any commutative $R$-algebra $S$. By definition, $U_T(S) \subset M'$. In particular, for $T \in \mathfrak{J}(\mathbb{Z})_{>0}$, put $U_T = U_T \otimes_{\mathbb{Z}} \mathbb{Q}$. It is easy to see that $U_T$ is a connected regular algebraic group over $\mathbb{Q}$ by (geometric) fiberwise argument. Further, $U_T$ is an exceptional group of type $F_4$ [11, p.108], and therefore we call $U_T$ the group scheme of type $F_4$.

Let $G$ be a group (resp. a group scheme) acting on a set (resp. a scheme) $S$. Then, for $s \in S$, we denote by $O_G(s)$ the orbit (resp. the orbit scheme) of $s$ under $G$, that is

$$O_G(s) = \{ g \cdot s \mid g \in G \}.$$ 

For $T \in \mathcal{M}(\mathbb{Z}_p)^{ns}$ we note that we have $\int_{O_{\mathcal{M}(\mathbb{Z}_p)}(T)} |d\sigma(x)|_p \neq 0$.

Recall the definition of the local density $\beta_p(T)$ of $T$ for $T \in \mathcal{M}(\mathbb{Z}_p)^{ns}$ [5]:

$$\beta_p(T) = \frac{(1 - p^{-1})\delta_p}{\int_{O_{\mathcal{M}(\mathbb{Z}_p)}(T)} |d\sigma(x)|_p},$$
where \( \delta_p = (1 - p^{-2})(1 - p^{-5})(1 - p^{-6})(1 - p^{-8})(1 - p^{-9})(1 - p^{-12}) \).

**Proposition 5.1.** For \( T \in \mathfrak{J}(\mathbb{Z}_p)_{\text{ns}} \) and an integer \( n \geq \text{ord}_p(\det T) + 1 \), we have

\[
\beta_p(T) = p^{-52n} \# \mathcal{U}_T(\mathbb{Z}_p/p^n\mathbb{Z}_p).
\]

**Proof.** The assertion follows from [5, Lemmas 3.3 and 6.8, Theorem 3.7]. □

**Proof of Lemma 4.10.** The first assertion follows from [5, Lemma 3.1 (2.2)]. Suppose that \( \phi_{T,p^m}(S) \neq \emptyset \). Then, clearly we have

\[
\# \phi_{T,p^m}^{-1}(S) = \# \phi_{T,p^m}^{-1}(T) = \# \mathcal{U}_T(\mathbb{Z}_p/p^n\mathbb{Z}_p),
\]

and the second assertion follows from [5, Corollary 6.2, (1)]. □

Let \( T \) be an element of \( \mathfrak{J}(\mathbb{Z})_{>0} \). For \( T' \sim_{\mathfrak{M}'(\mathbb{Z})} T \), we say that \( T' \) belongs to the same \( \mathfrak{M}_A \)-genus as \( T \) and write \( T' \approx T \) if \( T' \sim_{\mathfrak{M}'(\mathbb{Z}_p)} T \) for any prime number \( p \). For \( T \in \mathfrak{J}(\mathbb{Z})_{>0} \), let

\[
\mathcal{G}(T) = \{ T' \in \mathfrak{J}(\mathbb{Z})_{>0} \mid T' \approx T \}.
\]

Put

\[
\text{Mass}(T) = \sum_{T' \in \mathcal{G}(T)/\mathfrak{M}'(\mathbb{Z})} \frac{1}{\epsilon(T')},
\]

where \( \epsilon(T') = \# \mathcal{U}_{T'}(\mathbb{Z}). \)

Then we have the mass formula for \( T \) (cf. [5, Theorem 3.8]).

**Theorem 5.2.** (Mass-formula) Let \( T \) be an element of \( \mathfrak{J}(\mathbb{Z})_{>0} \). Then we have

\[
\text{Mass}(T) = c \frac{(\det T)^9}{\prod_{p<\infty} \beta_p(T)}, \quad c = \frac{5!7!11!}{(2\pi)^{28}}.
\]

For \( p \leq \infty \), let \( \iota_p : \mathfrak{J}(\mathbb{Q}) \rightarrow \mathfrak{J}(\mathbb{Q}_p) \) be the natural embedding, and let \( \varphi : \mathfrak{J}(\mathbb{Q}) \rightarrow \prod_{p \leq \infty} \mathfrak{J}(\mathbb{Q}_p) \) be the diagonal embedding. Let

\[
\mathbb{J} = \prod_p (\mathfrak{J}(\mathbb{Z}_p)/\mathfrak{M}'(\mathbb{Z}_p)).
\]

Then \( \varphi \) induces a mapping from \( \mathfrak{J}(\mathbb{Z})_{>0}/\prod_p \mathfrak{M}'(\mathbb{Z}_p) \) to \( \mathbb{J} \), which will be denoted also by \( \varphi \). For \( d \in \mathbb{Z}_p \setminus \{0\} \), put

\[
\mathfrak{J}(d,\mathbb{Z}_p) = \{ T \in \mathfrak{J}(\mathbb{Z}_p) \mid \det T = d \}.
\]

Moreover, for a positive integer \( d \), put

\[
\mathfrak{J}(d,\mathbb{Z}) = \{ T \in \mathfrak{J}(\mathbb{Z}) \mid \det T = d \},
\]
and
\[ \mathfrak{J}(d) = \prod_p (\mathfrak{J}(d, \mathbb{Z}_p)/\mathcal{M}'(\mathbb{Z}_p)). \]

Now we have the following local-global principle (cf. [5, Proposition 3.10]).

**Proposition 5.3.** The mapping \( \varphi \) induces a bijection from \( \mathfrak{J}(d, \mathbb{Z})_0/\prod_p \mathcal{M}'(\mathbb{Z}_p) \) to \( \mathfrak{J}(d) \).

6. **Twisted Koecher-Maass series of the second kind of the Ikeda type lift for the exceptional group of type \( E_{7,3} \)**

We review the Ikeda type lift of a cuspidal Hecke eigenform in [8], and consider its twisted Koecher-Maass series of the second kind. Let \( k \geq 10 \) be a positive integer, and let
\[ f(\tau) = \sum_{m=1}^{\infty} a_f(m) \exp(2\pi \sqrt{-1} m \tau) \]
be in \( S_{2k-8}(SL_2(\mathbb{Z})) \). For a prime number \( p \), let \( \alpha_p \) be a complex number such that
\[ a_f(p) = p^{(2k-9)/2} (\alpha_p + \alpha_p^{-1}). \]
By Deligne’s theorem we have \( |\alpha_p| = 1 \). For a Dirichlet character \( \chi \) we define the automorphic \( L \)-function \( L(s, \pi_f, \chi) \) of the cuspidal representation \( \pi_f \) attached to \( f \) as
\[ L(s, \pi_f, \chi) = \prod_p \left\{ (1 - p^{-s} \alpha_p \chi(p))(1 - p^{-s} \alpha_p^{-1} \chi(p)) \right\}^{-1}. \]
If \( \chi \) is the principal character, we simply write \( L(s, \pi_f, \chi) \) as \( L(s, \pi_f) \).

Let \( p \) be a prime number. For \( T \in \mathfrak{J}(\mathbb{Q}_p) \), let \( T \sim_{\mathcal{M}(\mathbb{Z}_p/p^n \mathbb{Z}_p)} e_{1} p^{a_1} \perp e_{2} p^{a_2} \perp e_{3} p^{a_3} \) with \( a_1, a_2, a_3 \in \mathbb{Z} \cup \{\infty\} \), \( a_1 \leq a_2 \leq a_3 \), and \( e_i \in \mathbb{Z}_{p}^\times \). Define \( \kappa_p(T) \) by \( \kappa_p(T) = \prod_{1 \leq i \leq 3} p^{a_i} \). Here we make the convention that \( \kappa_p(T) = 1 \) if \( T = O \). We note that \( \kappa_p(T) \) is uniquely determined by \( T \) mod \( \mathfrak{J}(\mathbb{Z}_p) \).

Moreover, for \( x \in \mathbb{Q}_p \), put \( e_p(x) = \exp(2\pi \sqrt{-1} \mathrm{Frac}(x)) \), where \( \mathrm{Frac}(x) \) is the fractional part of \( x \). For \( T \in \mathfrak{J}(\mathbb{Z}_p)^{ns} \), let \( S_p(T) \) be the local Siegel series defined by
\[ S_p(s, T) = \sum_{T' \in \mathfrak{J}(\mathbb{Q}_p)/\mathfrak{J}(\mathbb{Z}_p)} e_p((T, T')) \kappa_p(T')^{-s}. \]
Then, there is a polynomial \( f_T^p(X) \) in \( X \) such that
\[ S_p(s, T) = (1 - p^{-s})(1 - p^{4-s})(1 - p^{8-s}) f_T^p(p^{9-s}). \]
Put
\[ \tilde{f}_T^p(X) = X^{\mathrm{ord}_p(\det T)} f_T^p(X^{2^{-s}}). \]
Then it satisfies the functional equation

\[ \tilde{f}_T^p(X^{-1}) = \tilde{f}_T^p(X). \] (6.1)

For \( T \in \mathfrak{A}(\mathbb{Z})_> \) put \( a_{F_f}(T) = \det(T)^{2k-9} \prod_{p|\det(T)} \tilde{f}_T^p(\alpha_p) \), and define the Fourier series \( F_f(Z) \) on \( \mathfrak{A} \) by

\[ F_f(Z) = \sum_{T \in \mathfrak{A}(\mathbb{Z})_> \cap \mathbb{M}(\mathbb{Z})_e} a_{F_f}(T) e((T, Z)) \quad (Z \in \mathfrak{A}). \]

Then, Kim and Yamauchi \[8\] showed that \( F_f \) is a cuspidal Hecke eigenform of weight \( 2k \) for \( G(\mathbb{Z}) \).

We consider the twisted Koecher-Maass series of \( F_f \) of the second kind. For a Dirichlet character \( \chi \mod N \), recall

\[ K^{(2)}(s, F_f, \chi) = \sum_{T \in \mathfrak{A}(\mathbb{Z})_> \cap \mathbb{M}^e(\mathbb{Z})_e} \frac{\chi(\det T) a_{F_f}(T)}{\epsilon(T) \det(T)^s}. \]

Even though \( K^{(2)}(s, F, \chi) \) does not have an Euler product for a general \( F \), we show that \( K^{(2)}(s, F_f, \chi) \) has an Euler product, which enables us to reduce its computation to each \( p \)-adic place. For \( d \in \mathbb{Z}_p \setminus \{0\} \), put

\[ \lambda_p(d, X) = \sum_{T \in \mathfrak{A}(\mathbb{Z}_p) / \mathbb{M}^e(\mathbb{Z}_p)} \frac{\tilde{f}_T^p(X)}{\beta_p(T)}, \]

and for a positive integer \( d \), put

\[ C(d; f) = \prod_{p<\infty} \lambda_p(d, \alpha_p). \]

**Theorem 6.1.** We have

\[ K^{(2)}(s, F_f, \chi) = c \sum_{d=1}^{\infty} C(d, f) \chi(d) d^{-s+k+\frac{9}{2}}, \]

where \( c \) is as in Theorem \[5.2\].
Proof. Let $G = \mathfrak{g}(\mathbb{Z})_{>0} / \approx$ be the set of all genera of $\mathfrak{g}(\mathbb{Z})_{>0}$. We note the Fourier coefficient $a_{F,p}(T)$ is uniquely determined by $G(T)$. Hence, by the Mass formula $[5$, Theorem 3.8$]$, we have

$$K^{(2)}(s, F_p, \chi) = \sum_{T \in G} \sum_{T' \in G(T)/M(Z)} (\det T)^{-s} a_{F,p}(T') \chi(\det T') \frac{\epsilon(T')}{\epsilon(T)} = \sum_{T \in G} \text{Mass}(T) a_{F,p}(T) \chi(\det T^s)$$

Thus the assertion follows from $[5$, Proposition 3.10$]$. \hfill \square

For $d \in \mathbb{Z}_{\geq 0}$, define a formal power series $H_p(d; X, t)$ as

$$H_p(d; X, t) = \sum_{m=0}^{\infty} \lambda_p(p^m d, X) t^m.$$ 

As in $[5$, Lemma 5.2$]$, we can show that $\lambda_p(d; X)$ is determined by $\text{ord}_p(d)$, and hence $H_p(d; X, t)$ does not depend on the choice of $d \in \mathbb{Z}_{\geq 0}$, and we write it as $H_p(X, t)$. Therefore,

**Theorem 6.2.** We have

$$K^{(2)}(s, F, \chi) = c \prod_p H_p(\alpha_p, \chi(p) p^{-s+9/2+k}).$$

Recall the formula for $\overline{f}_T(X)$ from $[5$, Corollary 7.2$]$: For $T = \perp p^{m_1+m_2} p^{m_3}$ with $0 \leq m_1, 0 \leq m_2 \leq m_3$,

$$\overline{f}_T(X) = \frac{X^{-m_2-m_3-3m_1}}{(1-X^2)(1-p^{4}X^2)(1-p^{6}X^2)} + \frac{X^{m_2+m_3+3m_1}}{(1-X^{-2})(1-p^{4}X^{-2})(1-p^{8}X^{-2})}$$

$$\frac{p^{8m_1+8} X^{-m_1-m_2-m_3+2}}{(1-X^2)(1-p^{4}X^2)(1-p^{8}X^2)} \frac{p^{8m_1+4(m_2+1)} X^{-m_3+m_2-m_1+2}}{(1-X^{-2})(1-p^{4}X^{-2})(1-p^{8}X^{-2})}$$

Now we have

**Theorem 6.3.**

$$H_p(X, t) = \{(1-p^{-2})(1-p^{-6})(1-p^{-8})(1-p^{-12})\}^{-1} \frac{1}{\prod_{i=1}^{3} (1-p^{-4i+3}X^{-1}t)(1-p^{-4i+3}Xt)}.$$
Then, as in [5, Theorem 7.4], we have

\[ Y_i = 1 \]

From now on, put

\[ P(X, t) = \frac{1}{1 - p^{-6}X^{-1}t(1 - p^{-5}X^{-1}t)(1 - p^{-4}X^{-1}t)} \]

Moreover, we have

\[ P_3(X, t) = \frac{1 + (p^{-5} + p^{-9})tX^{-1} + (p^{-10} + p^{-14})t^2 + p^{-19}t^3X^{-1}}{(1 - p^{-19}X^{-1}t^3)(1 - p^{-9}t^2)(1 - p^{-1}X^{-1}t)} \]
This proves the above equality. Hence, again by a simple computation, we have

\[
P_4(X,t) = \frac{1 + (p^{-5} + p^{-9})tX + (p^{-10} + p^{-14})t^2 + p^{-19}t^3X}{(1 - p^{-19}X^{-1}t^3)(1 - p^{-6}t^2)(1 - p^{-1}Xt)}.
\]

We prove the following equality.

\[
A_3(X)P_3(X,t) + A_4(X)P_4(X) = -X^2\{(1 - X^2)(1 - p^{-4}X^2)(1 - p^{4}X^2)\}^{-1} \\
\times \frac{(1 + p^4)(1 + p^{-14}t^2) + (X^{-1} + X)p^{-5}t}{(1 - p^{-19}X^{-1}t^3)(1 - p^{-6}t^2)(1 - p^{-1}Xt)}
\]

To prove this, put

\[
Q_{2,3}(X,t) = -X^{-2}(1 - X^2)^2(1 - p^4X^2)(1 - p^{-4}X^2) \\
\times (1 - p^{-19}X^{-1}t^3)(1 - p^{-6}t^2)(1 - p^{-1}Xt)(1 - p^{-1}X^{-1}t) \\
\times (A_3(X)P_3(X,t) + A_4(X)P_4(X,t)).
\]

Then, by a simple computation, we have

\[
Q_{2,3}(X,t) = (p^4 - X^2)(1 - p^{-1}Xt)(1 + (p^{-5} + p^{-9})tX^{-1} + (p^{-10} + p^{-14})t^2 + p^{-19}t^3X^{-1}) \\
+ (1 - p^4X^2)(1 - p^{-1}X^{-1}t)(1 + (p^{-5} + p^{-9})tX + (p^{-10} + p^{-14})t^2 + p^{-19}t^3X) \\
= (p^4 - X^2)\left((1 + ((p^{-5} + p^{-9})X^{-1} - p^{-1}X) + (p^{-10} + p^{-6})t^2 \\
+ (p^{-19}X^{-1} - (p^{-11} + p^{-15})X)t^3 - p^{-20}t^4) \\
+ (1 - p^4X^2)\left((1 + ((p^{-5} + p^{-9})X - p^{-1}X^{-1})t + (p^{-10} - p^{-4}t^2 \\
+ (p^{-19}X - (p^{-11} + p^{-15})X^{-1})t^3 - p^{-20}t^4) \\
= (1 - X^2)\left((1 + p^4)\left((1 + (p^{-10} + p^{-6})t^2 - p^{-20}t^4 \right) + (X^{-1} + X)p^{-5}t - p^{-11}t^3\right) \right) \\
= (1 - X^2)(1 - p^{-6}t^2)\left((1 + p^4)(1 + p^{-14}t^2) + (X^{-1} + X)p^{-5}t\right) \right).
\]

This proves the above equality. Hence, again by a simple computation, we have

\[
(6.2) \quad A_2(X)P_2(X,t) + A_3(X)P_3(X,t) + A_4(X)P_4(X) \\
= -X^2\{(1 - X^2)(1 - p^{-4}X^2)(1 - p^{8}X^2)\}^{-1} \frac{(1 + p^4 + p^8)}{(1 - p^{-5}X^{-1}t)(1 - p^{-1}X^{-1}t)(1 - p^{-1}Xt)}.
\]

Similarly we have

\[
P_5(X,t) = \frac{1}{(1 - p^{-9}Xt)(1 - p^{-5}Xt)(1 - p^{-1}Xt)}.
\]
Theorem 7.1. Recall the following rationality result of Shimura [12]:

\[ s \text{ the form} \]

Put

\[ L \]

Then, by (6.1), (6.2), (6.3) and (6.4), we have

\[ \begin{align*}
&\frac{1 + p^4 + p^8}{(1 - p^{-5}Xt)(1 - p^{-1}X^{-1}t)(1 - p^{-1}Xt)}. \\
&\end{align*} \]

We note that

\[ K \]

\[ \text{Therefore,} \]

\[ K' \]

\[ \text{Then, by (6.1), (6.2), (6.3) and (6.4), we have} \]

\[ K_p(X, t) = (1 - p^{-2})(1 - p^{-6})(1 - p^{-8})(1 - p^{-12})H_p(X, t) \prod_{i=1}^{3}(1 - p^{-4i+3}X^{-1}t)(1 - p^{-4i+3}Xt). \]

\[ \text{We note that} \]

\[ K_p(X, t) \text{ is a polynomial in} t \text{ of degree at most 3, and} \]

\[ K_p(X, p^9X^\pm 1) = K_p(X, p^5X^\pm 1) = 1. \]

\[ \text{Therefore,} \]

\[ K_p(X, t) = 1 \text{ as a polynomial in} t. \]

This proves the assertion. □

7. Proof of Theorems 1.1 and 1.2 and some rationality result

Theorem 1.1 and 1.2 are immediate consequences of Theorems 4.10, 6.2, and 6.3.

By the functional equation of \( L(s, \pi_f, \chi) \), we have the functional equation of \( K^{(1)}(s, F_f, \chi) \) of the form \( s \mapsto 2k - s \), which is compatible with that of the general case (Theorem 3.1).

From the mass formula (Theorem 5.2), \( c = \frac{5711}{(2\pi)^2} \). So \( c\zeta(2)\zeta(6)\zeta(8)\zeta(12) = \frac{691}{215.36.37.13} \in \mathbb{Q}. \)

Let \( L(s, f, \chi) \) be the unnormalized \( L \)-function, and \( K_f, K_\chi \) be Hecke fields. Then since \( L(s, \pi_f, \chi) = L(s + k - \frac{9}{2}, f, \chi) \), from Theorem 1.1,

\[ K^{(2)}(s, F_f, \chi) = c\zeta(2)\zeta(6)\zeta(8)\zeta(12) \times \prod_{i=1}^{3} L(s + 4i - 12, f, \chi). \]

Recall the following rationality result of Shimura [12]:

**Theorem 7.1.** For a Dirichlet character \( \chi \), let \( A(m, f, \chi) = (2\pi \sqrt{-1})^{-m}W(\chi)^{-1}L(m, f, \chi) \), and \( u^+ = A(2k - 9, f, \phi), u^- = A(2k - 9, f, \phi') \), where \( \phi, \phi' \) are any fixed real odd (even, resp.)
characters. Then
\[ A(m, f, \chi) \in \begin{cases} u^+ K_f K_\chi, & \text{if } \chi(-1) = (-1)^m \\ u^- K_f K_\chi, & \text{if } \chi(-1) = (-1)^{m-1} \end{cases}, \]
for every positive integer \( m < 2k - 8 \), and \( \pi W(\chi)(f, f) \sqrt{-1} \in u^+ u^- K_f. \)

Therefore, we have

**Theorem 7.2.** Let \( \chi \) be a Dirichlet character. For every integer \( m \), \( 9 \leq m \leq 2k - 9 \),
\[
(2\pi \sqrt{-1})^{-3m+12} W(\chi)^{-3} K(2)(m, F_f, \chi) \in \begin{cases} (u^+)^3 K_f K_\chi, & \text{if } \chi(-1) = (-1)^m \\ (u^-)^3 K_f K_\chi, & \text{if } \chi(-1) = (-1)^{m-1} \end{cases}.
\]

For \( K^{(1)}(s, F_f, \chi) \), we assume that \( \chi \) is a primitive character mod \( N \), and \( \chi = \tilde{\chi}^3 \) for some \( \tilde{\chi} \in (\mathbb{Z}/N\mathbb{Z})^\times \). Notice that if \( \eta^3 = 1, \eta(-1) = 1 \). Hence

**Theorem 7.3.** Let \( \chi \) be a primitive character mod \( N \). Then, for every integer \( m \), \( 9 \leq m \leq 2k - 9 \),
\[
(2\pi \sqrt{-1})^{-3m+12} W(\tilde{\chi}) K^{(1)}(m, F_f, \chi) \in \begin{cases} (u^+)^3 K_f K_{\tilde{\chi}} Q(\sqrt{-3}), & \text{if } \tilde{\chi}(-1) = (-1)^m \\ (u^-)^3 K_f K_{\tilde{\chi}} Q(\sqrt{-3}), & \text{if } \tilde{\chi}(-1) = (-1)^{m-1} \end{cases}.
\]
In particular if \( \phi(N) \) is not divisible by 3, then
\[
(2\pi \sqrt{-1})^{-3m+12} W(\tilde{\chi}) K^{(1)}(m, F_f, \chi) \in \begin{cases} (u^+)^3 K_f K_{\tilde{\chi}}, & \text{if } \tilde{\chi}(-1) = (-1)^m \\ (u^-)^3 K_f K_{\tilde{\chi}}, & \text{if } \tilde{\chi}(-1) = (-1)^{m-1} \end{cases}.
\]

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