A PRESENTATION OF THE DEFORMED $W_{1+\infty}$ ALGEBRA

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Abstract. We provide a generators and relation description of the deformed $W_{1+\infty}$-algebra introduced in previous joint work of E. Vasserot and the second author. This gives a presentation of the (spherical) cohomological Hall algebra of the one-loop quiver, or alternatively of the spherical degenerate double affine Hecke algebra of $GL(\infty)$.

INTRODUCTION

In the course of their work on the cohomology of the moduli space of $U(r)$-instantons on $\mathbb{P}^2$ in relation to $W$-algebras and the AGT conjecture (see [SV]) E. Vasserot and the second author introduced a certain one-parameter deformation $SH_c$ of the enveloping algebra of the Lie algebra $W_{1+\infty}$ of algebraic differential operators on $\mathbb{C}^\ast$. The algebra $SH_c$—which is defined in terms of Cherednik’s double affine Hecke algebras—acts on the above mentioned cohomology spaces (with a central character depending on the rank $n$ of the instanton space). For the same value of the central character, $SH_c$ is also strongly related to the affine $W$ algebra of type $gl_n$, and has the same representation theory (of admissible modules) as the latter. The same algebra $SH_c$ arises again as the (spherical) cohomological Hall algebra of the quiver with one vertex and one loop, and as a degeneration of the (spherical) elliptic Hall algebra (see [SV, Sec. 4, 8]). It also independently appears in the work of Maulik and Okounkov on the AGT conjecture, see [MO].

The definition of $SH_c$ given in [SV] is in terms of a stable limit of spherical degenerate double affine Hecke algebras, and does not yield a presentation by generators and relations. In this note, we provide such a presentation, which bears some resemblance with Drinfeld’s new realization of quantum affine algebras and Yangians. Namely, we show that $SH_c$ is generated by families of elements in degrees $-1, 0, 1$, modulo some simple quadratic and cubic relations (see Theorems 3.1, 3.2).

The definition of $SH_c$ is recalled in Section 1. In the short Section 2 we briefly recall the links between $SH_c$ and Cherednik algebras, resp. $W$-algebras. The presentation of $SH_c$ is given in Section 3, and proved in Section 4. Although we have tried to make this note as self-contained as possible, there are multiple references to statements in [SV] and the reader is advised to consult that paper (especially Sections 1 and 8) for details.

1. Definition of $SH_c$

1.1. Symmetric functions and Sekiguchi operators. Let $\kappa$ be a formal parameter, and let us set $F = \mathbb{C}(\kappa)$. Let us denote by $\Lambda_F$ the ring of symmetric polynomials in infinitely many variables with coefficients in $F$, i.e.

$$\Lambda_F = F[X_1, X_2, \ldots]^{\otimes\infty} = F[p_1, p_2, \ldots].$$

For $\lambda$ a partition, we denote by $J_\lambda$ the integral form of the Jack polynomial associated to $\lambda$ and to the parameter $\alpha = 1/\kappa$. It is well-known that $\{J_\lambda\}$ forms a basis of $\Lambda_F$ (see e.g. [S], or [SV, Sec. 1.3, 1.6]).

The polynomials $J_\lambda$ arise as the joint spectrum of a family of commuting differential operators $\{D_{0,l}\}, l \geq 1$ called Sekiguchi operators. In the above normalization, these may be characterized through the following relations:
1.3. Grading and filtration

(see [SV, Prop. 1.18]). It may be characterized as follows, see [SV, Prop. 1.2]:

\[ \text{compatible with the rank grading, induced from the filtration by the order of differential operators.} \]

As operators on polynomials will be called the

\[ \text{generators} \]

\[ \text{the content of} \]

\[ \text{when} \]

\[ \text{according} \]

\[ \text{to the continental convention (see [SV, Sec. 0.1]).} \]

We denote by \( D_{l,0} \in \operatorname{End}(\Lambda_F) \) the operator of multiplication by the power-sum function \( p_l \).

1.2. The algebras \( \text{SH}^+ \) and \( \text{SH}^< \). Let \( \text{SH}^+ \) be the unital subalgebra of \( \operatorname{End}(\Lambda_F) \) generated by \( \{ D_{0,l}, D_{1,0} \mid l \geq 1 \} \). For \( l \geq 1 \) we set \( D_{1,l} = [D_{0,l+1}, D_{1,0}] \). This relation is still valid when \( l = 0 \), and we furthermore have

\[ \text{(1.2) } [D_{0,l}, D_{1,k}] = D_{1,k+l-1} \quad l \geq 1, k \geq 0. \]

We denote by \( \text{SH}^\circ \) the unital subalgebra of \( \text{SH}^+ \) generated by \( \{ D_{1,l} \mid l \geq 0 \} \), and by \( \text{SH}^0 \) the unital subalgebra of \( \text{SH}^+ \) generated by the Sekiguchi operators \( \{ D_{0,l} \mid l \geq 1 \} \). It is known (and easy to check from (1.1)) that the \( D_{0,l} \) are algebraically independent, i.e. \( \text{SH}^0 = F[D_{0,1}, D_{0,2}, \ldots] \).

Observe that by (1.2), the operators \( \text{ad}(D_{0,l}) \) preserve the subalgebra \( \text{SH}^\circ \). This allows us to view \( \text{SH}^+ \) as a semi-direct product of \( \text{SH}^0 \) and \( \text{SH}^\circ \). In fact, the multiplication map induces an isomorphism

\[ \text{SH}^\circ \otimes \text{SH}^0 \simeq \text{SH}^+ \]

(see [SV, Prop. 1.18]).

1.3. Grading and filtration. The algebra \( \text{SH}^+ \) carries an \( \mathbb{N} \)-grading, defined by setting \( D_{0,l}, D_{1,k} \) in degrees zero and one respectively. This grading, which corresponds to the degrees as operators on polynomials will be called the rank grading. It also carries an \( \mathbb{N} \)-filtration compatible with the rank grading, induced from the filtration by the order of differential operators. It may be characterized as follows, see [SV, Prop. 1.2]: \( \text{SH}^+[\leq d] \) is the space of elements \( u \in \text{SH}^\circ \) satisfying

\[ \text{ad}(z_1) \circ \cdots \circ \text{ad}(z_{d+1})(u) = 0 \]

for all \( z_1, \ldots, z_{d+1} \in F[D_{1,0}, D_{2,0}, \ldots] \). We have \( \text{SH}^+[\leq 0] = F[D_{1,0}, D_{2,0}, \ldots] \). The following is proved in [SV, Lemma 1.21]. Set \( D_{r,d} = [D_{0,d+1}, D_{r,0}] \) for \( r \geq 1, d \geq 0 \).

Proposition 1.1. (i) The associated graded algebra \( \text{grSH}^+ \) is equal to the free commutative polynomial algebra in the generators \( D_{r,d} \in \text{grSH}^+[r,d] \), for \( r \geq 0, d \geq 0, (r,d) \neq (0,0) \).

(ii) The associated graded algebra \( \text{grSH}^\circ \) is equal to the free commutative polynomial algebra in the generators \( D_{r,d} \in \text{grSH}^\circ[r,d] \), for \( r \geq 1, d \geq 0 \).

We will need the following slight variant of the above result, which can easily be deduced from [SV] Prop. 1.38. For \( r \geq 1 \), set \( D_{r,d} = \text{ad}(D_{0,2})^d(D_{r,0}) \). Then

\[ D_{r,d} \in r^{d-1}D_{r,d} \otimes \text{SH}^\circ[r, \leq d - 1]. \]

In particular, \( \text{grSH}^\circ \) is also freely generated by the elements \( D_{r,d} \in \text{grSH}^\circ[r,d] \).

1.4. The algebra \( \text{SH}^c \). Let \( \text{SH}^< \) be the opposite algebra of \( \text{SH}^\circ \). We denote the generator of \( \text{SH}^\circ \) corresponding to \( D_{1,l} \) by \( D_{-1,l} \). The algebra \( \text{SH}^c \) is generated by \( \text{SH}^+, \text{SH}^0, \text{SH}^< \) together with a family of central elements \( e = (c_0, c_1, \ldots) \) indexed by \( \mathbb{N} \), modulo a certain set of relations involving the commutators \( [D_{-1,k}, D_{1,l}] \) (see [SV, Sec. 1.8]). In order to write down these relations, we need a few notations. Set \( \xi = 1 - k \) and

\[ G_0(s) = -\log(s), \quad G_l(s) = (s^{-1} - 1)/l, \quad l \geq 1, \]

\[ \varphi_1(s) = \sum_{q=1,-\xi,-k} s^l(G_l(1 - qs) - G_l(1 + qs)), \quad l \geq 1, \]

\[ (1.1) \quad D_{0,l}(J_\lambda) = \sum_{s \in \lambda} c(s)^{l-1}J_\lambda \]

where \( s \) runs through the set of boxes in the partition \( \lambda \), and where \( c(s) = x(s) - \kappa y(s) \) is the content of \( s \). Here \( x(s), y(s) \) denote the \( x \) and \( y \)-coordinates of the box \( s \), when \( \lambda \) is drawn according to the continental convention (see [SV, Sec. 0.1]).
where the elements $E_l(1 + \xi)_{l \geq 0}$ with multiplication given by shuffle algebra of $g$.

We may now define $\text{SH}^c$ as the algebra generated by $\text{SH}^>, \text{SH}^<, \text{SH}^0$ and $F[c_0,c_1,\ldots]$ modulo the following relations:

\begin{align}
[D_{0,l}, D_{1,k}] &= D_{1,k+l-1}, \quad [D_{-1,k}, D_{0,l}] = D_{-1,k+l-1}, \\
[D_{-1,k}, D_{1,l}] &= E_{k+l}, \quad l, k \geq 0,
\end{align}

where the elements $E_k$ are determined through the formulas

\begin{align}
1 + \xi \sum_{l \geq 0} E_l s^{l+1} &= \exp \left( \sum_{l \geq 0} (-1)^{l+1} c_l \phi_l(s) \right) \exp \left( \sum_{l \geq 0} D_{0,l+1} \psi_l(s) \right).
\end{align}

Set $\text{SH}^{0,c} = \text{SH}^0 \otimes F[c_0,c_1,\ldots]$. One can show that the multiplication map provides an isomorphism of $F$-vector spaces

$$
\text{SH}^> \otimes \text{SH}^{0,c} \otimes \text{SH}^< \simeq \text{SH}^c.
$$

Putting the generators $D_{k+1}$ in degree $\pm 1$ and the generators $D_{0,l}, c_l$ in degree zero induces an $\mathbb{Z}$-grading on $\text{SH}^c$. One can show that the order filtration on $\text{SH}^>$, $\text{SH}^<$ can be extended to a filtration on the whole $\text{SH}^c$, but we won’t need this last fact.

2. Link to W-algebras, Cherednik algebras and shuffle algebras

2.1. Relation the Cherednik algebras. Let $\omega$ be a new formal parameter and let $\text{SH}^{\omega}$ be the specialization of $\text{SH}$ at $c_0 = 0, c_1 = -\kappa'/\omega'$. Let $H_n$ be Cherednik’s degenerate (or trigonometric) double affine Hecke algebra with parameter $\kappa$ (see [C]). Let $\text{SH}_n \subset H_n$ be its spherical subalgebra. The following result shows that $\text{SH}^\omega$ may be thought of as the stable limit of $\text{SH}_n$ as $n$ goes to infinity (see [SV, Sec. 1.7]):

**Theorem.** For any $n$ there exists a surjective algebra homomorphism $\Phi_n : \text{SH}^\omega \rightarrow \text{SH}_n$ such that $\Phi_n(\omega) = n$. Moreover $\bigcap_n \text{Ker} \Phi_n = \{0\}.$

2.2. Realization as a shuffle algebra. Consider the rational function

$$
g(z) = \frac{h(z)}{z}, \quad h(z) = (z + 1 - \kappa)(z - 1)(z + \kappa).
$$

Following [FQ], we may associate to $g(z)$ an $\mathbb{N}$-graded associative $F$-algebra $A_{g(z)}$, the *symmetric shuffle algebra* of $g(z)$ as follows. As a vector space,

$$
A_{g(z)} = \bigoplus_{n \geq 0} F[z_1,\ldots,z_n]_{\Sigma_n}
$$

with multiplication given by

$$
P(z_1,\ldots,z_r) \star Q(z_1,\ldots,z_s) = \sum_{\sigma \in \text{Sh}_{r,s}} \sigma \cdot \left( \prod_{1 \leq i \leq r} g(z_i - z_j) \cdot P(z_1,\ldots,z_r) Q(z_{r+1},\ldots,z_{r+s}) \right)
$$

where $\text{Sh}_{r,s} \subset \Sigma_{r+s}$ is the set of $(r,s)$ shuffles inside the symmetric group $\Sigma_{r+s}$. Let $S_{g(z)} \subseteq A_{g(z)}$ denote the subalgebra generated by $A_{g(z)}[1] = F[z]$. The following is proved in [SV] Cor. 6.4:

**Theorem.** The assignment $S_{g(z)}[1] \ni z^l \mapsto D_{l,1}, \ l \geq 0$ induces an isomorphism of $F$-algebras

$$
S_{g(z)} \simeq \text{SH}^>.
$$
2.3. Relation to $W$-algebras. Let $W_{1+\infty}$ be the universal central extension of the Lie algebra of all differential operators on $\mathbb{C}^*$ (see e.g. [FKRW]). This is a Z-graded and $\mathbb{N}$-filtered Lie algebra. The following result shows that $\mathbf{SH}$ may be thought of as a deformation of the universal enveloping algebra $U(W_{1+\infty})$ of $W_{1+\infty}$ (see [SV] App. F) :

**Theorem.** The specialization of $\mathbf{SH}^c$ at $\kappa = 1$ and $c_i = 0$ for $i \geq 1$ is isomorphic to $U(W_{1+\infty})$.

More interesting is the fact that, for certain good choices of the parameters $c_0, c_1, \ldots$, a suitable completion of $\mathbf{SH}^c$ is isomorphic to the current algebra of the (affine) $W$-algebra $W(\mathfrak{gl}_r)$ (see e.g. [A] Sec. 3.11]). Fix an integer $r \geq 1$, $k \in \mathbb{C}$ and let $(\varepsilon_1, \ldots, \varepsilon_r)$ be new formal parameters. Let $\mathfrak{U}(W_k(\mathfrak{gl}_r))^\prime$ be the formal current algebra of $W(\mathfrak{gl}_r)$ at level $k$, defined over the field $F(\varepsilon_1, \ldots, \varepsilon_r)$ (see [SV] Sec. 8.4 for details). Let $\mathbf{SH}^{(r)}$ be the specialization of $\mathbf{SH}^c$ to $\kappa = k + r$, $c_i = \varepsilon_1^i + \cdots + \varepsilon_r^i$ for $i \geq 1$. The following is proved in [SV] Cor. 8.24, to which we refer for details.

**Theorem.** There is an embedding $\mathbf{SH}^{(r)} \to \mathfrak{U}(W_k(\mathfrak{gl}_r))^\prime$ with a dense image, which induces an equivalence between the category of admissible $\mathbf{SH}^{(r)}$-modules and the category of admissible $\mathfrak{U}(W_k(\mathfrak{gl}_r))^\prime$-modules.

## 3. Presentation of $\mathbf{SH}^+$ and $\mathbf{SH}^c$

### 3.1. Generators and relations for $\mathbf{SH}^+$

Consider the $F$-algebra $\widetilde{\mathbf{SH}}^+$ generated by elements $\{\widetilde{D}_{0,l} \mid l \geq 1\}$ and $\{\widetilde{D}_{1,k} \mid k \geq 0\}$ subject to the following set of relations :

(3.1) $[\widetilde{D}_{0,l}, \widetilde{D}_{0,k}] = 0$, \hspace{1cm} $\forall \ l, k \geq 1$,

(3.2) $[\widetilde{D}_{0,l}, \widetilde{D}_{1,k}] = \widetilde{D}_{1,l+k-1}$, \hspace{1cm} $\forall \ l \geq 1, k \geq 0$,

(3.3) $(3[\widetilde{D}_{1,2}, \widetilde{D}_{1,1}] - [\widetilde{D}_{1,3}, \widetilde{D}_{1,0}] + [\widetilde{D}_{1,1}, \widetilde{D}_{1,0}]) + \kappa(\kappa - 1)(\widetilde{D}_{1,0}^2 + [\widetilde{D}_{1,1}, \widetilde{D}_{1,0}]) = 0$

(3.4) $[\widetilde{D}_{1,0}, [\widetilde{D}_{1,0}, \widetilde{D}_{1,1}]] = 0$.

Let $\widetilde{\mathbf{SH}}^0 = F[\widetilde{D}_{0,1}, \widetilde{D}_{0,2}, \ldots]$ denote the subalgebra of $\widetilde{\mathbf{SH}}^+$ generated by $\widetilde{D}_{0,l}, l \geq 1$, and let $\widetilde{\mathbf{SH}}^\geq$ be the subalgebra generated by $\widetilde{D}_{1,k}, k \geq 0$. The algebras $\mathbf{SH}^+, \mathbf{SH}^0, \mathbf{SH}^\geq$ are all $\mathbb{N}$-graded, where $\widetilde{D}_{0,l}$ and $\widetilde{D}_{1,k}$ are placed in degrees zero and one respectively. According to the terminology used for $\mathbf{SH}^+$, we call this grading the rank grading.

**Theorem 3.1.** The assignment $\widetilde{D}_{0,l} \mapsto D_{0,l}, \widetilde{D}_{1,k} \mapsto D_{1,k}$ for $l \geq 1, k \geq 0$ induces an isomorphism of graded $F$-algebras

$$\phi : \widetilde{\mathbf{SH}}^+ \xrightarrow{\sim} \mathbf{SH}^+.$$  

Obviously, the map $\phi$ restricts to isomorphisms $\widetilde{\mathbf{SH}}^0 \simeq \mathbf{SH}^0, \mathbf{SH}^\geq \simeq \mathbf{SH}^\geq$. Note however that $\mathbf{SH}^\geq$ is not generated by the elements $D_{1,k}$ with the sole relations (3.3, 3.4). Theorem 3.1 is proved in Section 4.
3.2. Generators and relations for $\text{SH}^\circ$. For the reader’s convenience, we write down the presentation of $\text{SH}^\circ$, an immediate corollary of Theorem 3.1 above. Let $\text{SH}^\circ$ be the algebra generated by elements $\{\tilde{D}_{0,l} \mid l \geq 1\}$, $\{\tilde{D}_{\pm 1,k} \mid k \geq 0\}$ and $\{c_i \mid i \geq 0\}$ subject to the following set of relations:

\begin{align}
(3.5) & \quad [\tilde{D}_{0,l}, \tilde{D}_{0,k}] = 0, \quad \forall \ l, k \geq 1, \\
(3.6) & \quad [\tilde{D}_{0,l}, \tilde{D}_{1,k}] = \tilde{D}_{1,l+k-1}, \quad [\tilde{D}_{-1,k}, \tilde{D}_{0,l}] = \tilde{D}_{-1,l+k-1} \quad \forall \ l \geq 1, k \geq 0, \\
(3.7) & \quad (3[\tilde{D}_{1,2}, \tilde{D}_{1,1}] - [\tilde{D}_{1,3}, \tilde{D}_{1,0}] + [\tilde{D}_{1,1}, \tilde{D}_{1,0}]) + \kappa(\kappa - 1)((\tilde{D}_{1,0}^2 + [\tilde{D}_{1,1}, \tilde{D}_{1,0}]) = 0 \\
(3.8) & \quad (3[\tilde{D}_{-1,2}, \tilde{D}_{-1,1}] - [\tilde{D}_{-1,3}, \tilde{D}_{-1,0}] + [\tilde{D}_{-1,1}, \tilde{D}_{-1,0}]) + \kappa(\kappa - 1)((\tilde{D}_{-1,0}^2 + [\tilde{D}_{-1,1}, \tilde{D}_{-1,0}]) = 0 \\
(3.9) & \quad [\tilde{D}_{1,0}, [\tilde{D}_{1,0}, \tilde{D}_{1,1}]] = 0, \quad [\tilde{D}_{-1,0}, [\tilde{D}_{-1,0}, \tilde{D}_{-1,1}]] = 0, \\
(3.10) & \quad [\tilde{D}_{-1,k}, \tilde{D}_{1,l}] = \tilde{E}_{k+l}, \quad l, k \geq 0,
\end{align}

where the $\tilde{E}_l$ are defined by the formula \((3.7)\).

**Theorem 3.2.** The assignment $\tilde{D}_{0,l} \mapsto D_{0,l}, \tilde{D}_{\pm 1,k} \mapsto D_{\pm 1,k}$ for $l \geq 1, k \geq 0$ and $c_i \mapsto c_i$ for $i \geq 0$ induces an isomorphism of $F$-algebras

$$\phi : \text{SH}^\circ \overset{\sim}{\rightarrow} \text{SH}^\circ.$$

Coupled with the Theorems in Section 2.3., this provides a potential ‘generators and relations’ approach to the study of the category of admissible modules over the $W$-algebras $W_k(\mathfrak{gl}_n)$.

4. Proof of Theorem 3.1

4.1. Let us first observe that $\phi$ is a well-defined algebra map, i.e. that relations \((3.1)\) \((3.3)\) hold in $\text{SH}^\circ$. For \((3.1)\) \((3.2)\) this follows from the definition of $\text{SH}^\circ$ and \((1.38)\). Equation \((3.3)\) may be checked directly, e.g. from the Pieri rules (see \((1.26)\)), or from the shuffle realization of $\text{SH}^\circ$ (see \((2.3)\), below). As for equation \((3.1)\), we have by \((1.35)\), \([[D_{1,1}, D_{1,0}], D_{1,0}] = [D_{2,0}, D_{1,0}] = 0\). The map $\phi$ is surjective by construction; in the rest of the proof, we show that it is injective as well.

4.2. Using relation \((3.2)\) it is easy to see that any monomial in the generators $\tilde{D}_{0,l}, \tilde{D}_{1,k}$ may be expressed as a linear combination of similar monomials, in which all $\tilde{D}_{0,l}$ appear on the right of all $\tilde{D}_{1,k}$. Hence the multiplication map $\text{SH}^\circ \otimes \text{SH}^0 \rightarrow \text{SH}^\circ$ is surjective. Since $\phi$ clearly restricts to an isomorphism $\text{SH}^\circ \simeq \text{SH}^0$ we only have to show, by \((1.3)\), that $\phi$ restricts to an isomorphism $\text{SH}^\circ \simeq \text{SH}^\circ$. Our strategy will be to construct a suitable filtration on $\text{SH}^\circ$ mimicking the order filtration of $\text{SH}^\circ$ and to pass to the associated graded algebras.

4.3. We begin by proving directly, using the shuffle realization of $\text{SH}^\circ$, that $\phi$ is an isomorphism in ranks one and two. This is obvious in rank one since $\phi$ is a graded map and the only relation in rank one is \((3.2)\).

Suppose $\sum \alpha_i D_{1,k_i} D_{1,l_i} = 0$ is a relation in rank two. The shuffle realization then implies $\sum \alpha_i z_i^{k_i} z_i^{l_i} = 0$ so that

$$h(z_1 - z_2)(\sum \alpha_i z_i^{k_i} z_i^{l_i}) = h(z_2 - z_1)(\sum \alpha_i z_i^{l_i} z_i^{k_i}).$$
Therefore \( \sum \alpha_i z_i^k z_j^l = h(z_2 - z_1)P(z_1, z_2) \) where \( P(z_1, z_2) \) is some symmetric polynomial in \( z_1, z_2 \). Hence \( \sum \alpha_i z_i^k z_j^l \) is a linear combination of polynomials of the form \( h(z_2 - z_1)(z_i^k z_j^l + z_i^l z_j^k) \) so that \( \sum \alpha_i D_{1,k} D_{1,l} \) is a linear combination of expressions of the form

\[
3[D_{1,t+2}, D_{1,k+l+1}] - 3[D_{1,t+1}, D_{1,k+l+2}] - [D_{1,t+3}, D_{1,k}] + [D_{1,t}, D_{1,k+3}] + [D_{1,t+1}, D_{1,k}] - [D_{1,t}, D_{1,k+1}]
\]

(4.1)

If \( I \) denotes the image of \((4.3)\) under the action of \( F[adD_{0,2}, adD_{0,3}, \ldots] \) then using \((4.2)\) we see that each such expression lies in \( \phi(I) \) so that \( \phi \) is indeed an isomorphism in rank two.

4.4. We now turn to the definition of the analog, on \( \overline{\text{SH}} \), of the order filtration on \( \text{SH}^\geq \). We will proceed by induction on the rank \( r \). For \( r = 1, d \geq 0 \), we set

\[
\text{SH}^\geq [1, \leq d] = \bigoplus_{k \leq d} F \overline{D}_{1,k}.
\]

Assuming that \( \overline{\text{SH}}^\geq [r', \leq d'] \) has been defined for all \( r' < r \) we let \( \overline{\text{SH}}^\geq [r, \leq d] \) be the subspace spanned by all products

\[
\text{SH}^\geq [r', \leq d'] \cdot \overline{\text{SH}}^\geq [r'', \leq d''], \quad r' + r'' = r, d' + d'' = d
\]

and by the spaces

\[
ad(\overline{D}_{1,l})(\overline{\text{SH}}^\geq [r - 1, \leq d - l + 1]), \quad l = 0, \ldots, d + 1.
\]

From the above definition, it is clear that \( \overline{\text{SH}}^\geq \) is a \( \mathbb{Z} \)-filtered algebra. Note that it is not obvious at the moment that \( \text{SH}^\geq [r, \leq d] = \{0\} \) for \( d < 0 \). Because the associated graded \( gr \text{SH}^\geq \) is commutative, it follows by induction on the rank \( r \) that \( \phi : \overline{\text{SH}}^\geq \to \text{SH}^\geq \) is a morphism of filtered algebras. We denote by \( gr \overline{\text{SH}}^\geq \) the associated graded of \( \overline{\text{SH}}^\geq \) and we let \( \overline{\phi} : gr \overline{\text{SH}}^\geq \to gr \text{SH}^\geq \) be the induced map. The map \( \overline{\phi} \) is graded with respect to both rank and order. Moreover \( \overline{\phi} \) is an isomorphism in ranks 1 and 2 (indeed, that the filtration as defined above coincides with the order filtration in rank 2 can be seen directly from \( [SV] \ (1.84)) \). The rest of the proof of Theorem 4.1 consists in checking that \( \overline{\phi} \) is an isomorphism. Once more, we will argue by induction. So in the remainder of the proof, we fix an integer \( r \geq 3 \) and assume that \( \overline{\phi} \) is an isomorphism in ranks \( r' < r \).

4.5. By our assumption above, the algebra \( gr \overline{\text{SH}}^\geq \) is commutative in ranks less than \( r \), that is \( ab = ba \) whenever \( \text{rank}(a) + \text{rank}(b) < r \). Our first task is to extend this property to the rank \( r \).

**Lemma 4.1.** The algebra \( gr \overline{\text{SH}}^\geq \) is commutative in rank \( r \).

**Proof.** We have to show that for \( a \in \text{SH}^\geq [r_1, \leq d_1], b \in \text{SH}^\geq [r_2, \leq d_2] \) and \( r_1 + r_2 = r \) we have

\[
[a, b] \in \text{SH}^\geq [r, \leq d_1 + d_2 - 1].
\]

(4.3)

We argue by induction on \( r_1 \). If \( r_1 = 1 \) then \((4.3)\) holds by definition of the filtration. Now let \( r_1 > 1 \) and let us further assume that \((4.3)\) is valid for all \( r_1', r_2' \) with \( r_1' + r_2' = r \) and \( r_1' < r_1 \). We will now prove \((4.3)\) for \( r_1, r_2 \), thereby completing the induction step. According to the definition of the filtration, there are two cases to consider:
Case 1) We have $a = a_1a_2$ with $a_1 \in \mathcal{SH}^> [s', \leq d'], a_2 \in \mathcal{SH}^> [s'', \leq d'']$ such that $s' + s'' = r_1d' + d'' = d_1$. Then $[a, b] = a_1[a_2, b] + [a_1, b]a_2$. By our induction hypothesis on $r$, $[a_2, b] \in \mathcal{SH}^> [s'' + r_2, \leq d'' + d_2 - 1]$ hence $a_1[a_2, b] \in \mathcal{SH}^> [r, \leq d_1 + d_2 - 1]$. The term $[a_1, b]a_2$ is dealt with in a similar fashion.

Case 2) We have $a = [\tilde{D}_{1, l}, a']$ with $a' \in \mathcal{SH}^> [r_1 - 1, \leq d_1 - l + 1]$. Then $[a, b] = [[\tilde{D}_{1, l}, a'], b] = [\tilde{D}_{1, l}, [a', b]] - [a', [\tilde{D}_{1, l}, b]]$. By our induction hypothesis on $r$, $[a', b] \in \mathcal{SH}^> [r_1 + r_2 - 1, \leq d_1 + d_2 - l]$ hence $[\tilde{D}_{1, l}, [a', b]] \in \mathcal{SH}^> [r, \leq d_1 + d_2 - 1]$. Similarly, $[\tilde{D}_{1, l}, b] \in \mathcal{SH}^> [r_2 + 1, \leq d_2 + l - 1]$. The inclusion $[a', [\tilde{D}_{1, l}, b]] \in \mathcal{SH}^> [r, \leq d_1 + d_2 - 1]$ now follows from the induction hypothesis on $r_1$.

We are done. \hfill \Box

4.6. We now focus on the filtered piece of order $\leq 0$ of $\mathcal{SH}^>$. We inductively define elements $\tilde{D}_{l, 0}$ for $l \geq 2$ by

$$\tilde{D}_{l, 0} = \frac{1}{l-1}[\tilde{D}_{l, 1}, \tilde{D}_{l-1, 0}].$$

From [SV (1.35)] we have $\phi(\tilde{D}_{l, 0}) = D_{l, 0}$. Since we assume are assuming that $\overline{\phi}$ is an isomorphism in ranks less than $r$, we have $[\tilde{D}_{l, 0}, \tilde{D}_{l', 0}] = 0$ whenever $l + l' < r$.

Lemma 4.2. We have $[\tilde{D}_{l, 0}, \tilde{D}_{l', 0}] = 0$ for $l + l' = r$.

Proof. If $r = 3$ this reduces to the cubic relation (3.4). For $r = 4$ we have to consider

$$[\tilde{D}_{3, 0}, \tilde{D}_{1, 0}] = \frac{1}{2}[[\tilde{D}_{1, 1}, \tilde{D}_{2, 0}], \tilde{D}_{1, 0}]$$

$$= \frac{1}{2}[\tilde{D}_{1, 1}, [\tilde{D}_{2, 0}, \tilde{D}_{1, 0}]] - \frac{1}{2}[\tilde{D}_{2, 0}, [\tilde{D}_{1, 1}, \tilde{D}_{1, 0}]]$$

$$= -\frac{1}{2}[\tilde{D}_{2, 0}, \tilde{D}_{2, 0}] = 0.$$

Now let us fix $l, l'$ with $l + l' = r$. We have

$$[\tilde{D}_{l, 0}, \tilde{D}_{l', 0}] = \frac{1}{l-1}[\tilde{D}_{l, 1}, \tilde{D}_{l-1, 0}, \tilde{D}_{l', 0}]$$

$$= \frac{1}{l-1}[\tilde{D}_{l, 1}, [\tilde{D}_{l-1, 0}, \tilde{D}_{l', 0}]] - \frac{1}{l-1}[\tilde{D}_{l-1, 0}, [\tilde{D}_{l, 1}, \tilde{D}_{l', 0}]]$$

$$= -\frac{l'}{l-1}[\tilde{D}_{l-1, 0}, \tilde{D}_{l'+1, 0}].$$

If $r = 2k$ is even then by repeated use of (1.4) we get

$$[\tilde{D}_{l, 0}, \tilde{D}_{l', 0}] = c[\tilde{D}_{l}, \tilde{D}_{k}] = 0$$

for some constant $c$. Next, suppose that $r = 2k + 1$ is odd, with $k \geq 2$. Applying $ad(\tilde{D}_{1, 1})$ to $[\tilde{D}_{k+1, 0}, \tilde{D}_{k-1, 0}] = 0$ yields the relation

$$(4.5) \quad (k + 1)[\tilde{D}_{k+2, 0}, \tilde{D}_{k-1, 0}] + (k - 1)[\tilde{D}_{k+1, 0}, \tilde{D}_{k, 0}] = 0.$$

Similarly, applying $ad(\tilde{D}_{2, 1})$ to $[\tilde{D}_{k, 0}, \tilde{D}_{k-1, 0}] = 0$ and using the relation $[\tilde{D}_{k, 1}, \tilde{D}_{l, 0}] = kt[\tilde{D}_{l+1, 0}, \tilde{D}_{k, 0}]$ in $\mathcal{SH}^>$ (see [SV (1.91), (8.47)]) we obtain the relation

$$(4.6) \quad k[\tilde{D}_{k+2, 0}, \tilde{D}_{k-1, 0}] + (k - 1)[\tilde{D}_{k, 0}, \tilde{D}_{k+1, 0}] = 0.$$
4.7. Recall that \( gr\mathop{SH}\) is a free polynomial algebra in generators in the generators \( D'_{s,d} \) for \( s \geq 1, d \geq 0 \). In order to prove that \( \widetilde{\phi} \) is an isomorphism in rank \( r \), it suffices, in virtue of Lemma 4.1, to show that the factor space

\[
U_{r,d} = gr\mathop{SH} [r, d] / \left\{ \sum_{r'+r''=r \atop d'+d''=d} gr\mathop{SH} [r', d'] \cdot gr\mathop{SH} [r'', d'']\right\}
\]

is one dimensional for any \( d \geq 0 \). Let us set, for any \( s \geq 1, d \geq 0 \)

\[
\hat{D}'_{s,d} = ad(\hat{D}_{0,2})^d(\hat{D}_{s,0}) \in \mathop{SH} [s, \leq d].
\]

We will denote by the same symbol \( \hat{D}'_{s,d} \) the corresponding element of \( gr\mathop{SH} [s, d] \). Note that \( \hat{D}'_{s,0} = D_{s,0} \) We claim that in fact \( U_{r,d} = F\hat{D}'_{r,d} \). Observe that \( \phi(\hat{D}'_{s,d}) = D'_{s,d} \) for any \( s, d \), hence \( \hat{D}'_{s,d} \in U_{s,d} \) for any \( s \leq r, d \geq 0 \). Moreover, by our general induction hypothesis on \( r \) we have \( U_{s,d} = F\hat{D}'_{s,d} \) for any \( s < r \) and \( d \geq 0 \).

We will prove that \( U_{r,d} = F\hat{D}'_{r,d} \) by induction on \( d \). For \( d = 0 \), this comes from Lemma 4.2.

So fix \( d > 0 \) and let us assume that \( U_{r,l} = F\hat{D}'_{r,l} \) for all \( l < d \). By definition of the filtration on \( \mathop{SH} \), \( U_{r,d} \) is linearly spanned by the classes of the elements

\[
[\hat{D}_{1,0}, \hat{D}'_{r-1,d+1}], [\hat{D}_{1,1}, \hat{D}'_{r-1,d}], \ldots, [\hat{D}_{1,d+1}, \hat{D}'_{r-1,0}].
\]

By our induction hypothesis on \( d \), the elements

\[
[\hat{D}_{1,0}, \hat{D}'_{r-1,d}], [\hat{D}_{1,1}, \hat{D}'_{r-1,d-1}], \ldots, [\hat{D}_{1,d}, \hat{D}'_{r-1,0}]
\]

all belong to \( F\hat{D}'_{r,d-1} \otimes \mathop{SH} [r, \leq d-2] \). Applying \( ad(\hat{D}_{0,2}) \), we see that

\[
[\hat{D}_{1,0}, \hat{D}'_{r-1,d+1}] + [\hat{D}_{1,1}, \hat{D}'_{r-1,d-1}] + \cdots + [\hat{D}_{1,d}, \hat{D}'_{r-1,1}] + [\hat{D}_{1,d+1}, \hat{D}'_{r-1,0}]
\]

all belong to \( F\hat{D}'_{r,d} \otimes \mathop{SH} [r, \leq d-1] \). Next, applying \( ad(\hat{D}_{0,2}) \) to the equality \[ \hat{D}_{1,0}, \hat{D}'_{r-1,0} \] yields

\[
[\hat{D}_{1,0}, \hat{D}'_{r-1,d}] + [\hat{D}_{1,d+1}, \hat{D}'_{r-1,0}] = 0
\]

which implies, by (4.4), that

\[
[\hat{D}_{1,0}, \hat{D}'_{r-1,d+1}] + r^d[\hat{D}_{1,d+1}, \hat{D}'_{r-1,0}] \in [\hat{D}_{1,0}, \mathop{SH} [r-1, \leq d]] \subseteq \mathop{SH} [r, \leq d-1].
\]

The collection of inclusions (4.7), (4.8) may be considered as a system of linear equations in \( U_{r,d} \) modulo \( F\hat{D}'_{r,d} \) in the variables \[ \hat{D}_{1,0}, \hat{D}'_{r-1,d+1}, \ldots, [\hat{D}_{1,d+1}, \hat{D}'_{r-1,0}] \] whose associated matrix is

\[
M = \begin{pmatrix}
1 & 0 & \cdots & 0 & 1 \\
1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 1 & -r^d
\end{pmatrix}
\]

is invertible. We deduce that \( [\hat{D}_{1,0}, \hat{D}'_{r-1,d+1}], \ldots, [\hat{D}_{1,d+1}, \hat{D}'_{r-1,0}] \) all belong to the space \( F\hat{D}'_{r,d} \otimes \mathop{SH} [r, \geq d-1] \) as wanted. This closes the induction step on \( d \). We have therefore proved that \( U_{r,d} = F\hat{D}'_{r,d} \) for all \( d \geq 0 \), and hence that \( \widetilde{\phi} \) and \( \phi \) is an isomorphism in rank \( r \). This closes the induction step on \( r \). Theorem 4.1 is proved.

\[ \Box \]

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A PRESENTATION OF THE DEFORMED $W_{1+\infty}$ ALGEBRA

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