NOTES ON THE ISOTOPY FINITENESS

VINCENT COLIN, EMMANUEL GIROUX, AND KO HONDA

Abstract. This is the less official, English version of the proof of the fact that every closed
atoroidal 3-manifold carries finitely many isotopy classes of tight contact structures.

In this note we start where we left off in [CGH1] and present the proof of the following
theorem:

Theorem 0.1. Let $V$ be a closed, oriented, atoroidal 3-manifold. Then there exist only
finitely many isotopy classes of tight contact structures on $V$.

Unless indicated otherwise, $V$ is a closed, oriented, atoroidal 3-manifold, and contact
structures on $V$ are cooriented, i.e., oriented and positive. The space of tight contact 2-plane
fields on $V$ is denoted by $\text{Tight}(V)$, and the set of isotopy classes of tight contact structures
on $V$ by $\pi_0(\text{Tight}(V))$.

First Reduction: If $V$ is a connected sum $V_1 \# V_2$, then a gluing theorem of Colin [Col]
implies that $\pi_0(\text{Tight}(V_1)) \times \pi_0(\text{Tight}(V_2)) \cong \pi_0(\text{Tight}(V))$. Since tight contact structures
on $S^1 \times S^2$ are also understood (there is a unique one up to isotopy), we may assume that
$V$ is irreducible.

1. Weights on branched surfaces

In [CGH1], we constructed a finite number of pairs $(\mathcal{B}_i, \zeta_i)$, $i = 1, \ldots, k$, where:

(1) $\mathcal{B}_i$ is a branched surface, possibly with boundary,
(2) $N(\mathcal{B}_i) \subset V$ is a branched surface neighborhood of $\mathcal{B}_i$,
(3) $\zeta_i$ is a tight contact structure on $V \setminus N(\mathcal{B}_i)$,

such that every $\xi \in \text{Tight}(V)$, up to isotopy, is generated by some $(\mathcal{B}_i, \zeta_i)$, i.e.,

(1) the fibers of $\pi_i : N(\mathcal{B}_i) \to \mathcal{B}_i$ are Legendrian,
(2) $\zeta_i|_{V \setminus N(\mathcal{B}_i)} = \xi|_{V \setminus N(\mathcal{B}_i)}$.

Remark 1.1. Note that a “branched surface neighborhood” $N(\mathcal{B}_i)$ is usually not a neighbor-
hood (in the topological sense) of $\mathcal{B}_i$, if we think of $\mathcal{B}_i$ as embedded in $V$. Therefore, we will
only think of $N(\mathcal{B}_i)$ as embedded inside $V$; $\mathcal{B}_i$ will be an abstract branched surface which is
not embedded inside $V$, although there is a projection map $\pi_i : N(\mathcal{B}_i) \to \mathcal{B}_i$.

Fix some $(\mathcal{B}_i, \zeta_i)$ – for simplicity, we omit the index $i$. It suffices to prove the finiteness of
isotopy classes of tight contact structures generated by $(\mathcal{B}, \zeta)$. Fix a tight contact structure
Lemma 1.2. If \( L \) is the branch locus of \( \mathcal{B} \), then on each connected component \( B \) of \( \mathcal{B} \setminus L \) the difference in the number of twists is constant and is an integer.

Proof. Since \( \xi \) and \( \xi_0 \) both agree on \( \partial N(\mathcal{B}) \) and each fiber \( \pi^{-1}(pt) \) is Legendrian for both, the difference in the number of twists is an integer. Also, \( \pi^{-1}B = B \times I \) is fibered by Legendrian arcs \( \{pt\} \times I \) and \( B \times \{0, 1\} \) is fixed for both \( \xi_0 \) and \( \xi_1 \). Continuity then guarantees that the integer does not vary over \( B \).

Definition 1.3. A weight function \( w \) on \( B \) is a locally constant function \( w : B \setminus L \to \mathbb{Z} \) (hence constant on each sector) which satisfies the branch equations \( w(B_1) + w(B_2) = w(B_3) \), whenever \( B_1, B_2, B_3 \) are sectors adjacent along a component \( c \) of \( L \setminus Q \), and for \( B_1 \) and \( B_2 \) the branching direction is outward and for \( B_3 \) the branching direction is inward. We say \( w \) is positive (resp. nonnegative) if \( w(B \setminus L) \subset \mathbb{Z}^+ \) (resp. \( \subset \mathbb{N} \)).

Therefore, to each \( \xi \) generated by \((\mathcal{B}, \zeta)\), we assign a weight function \( w_\xi \) which on the connected component \( B \) is the difference \( t(\pi^{-1}(p), \xi_0) - t(\pi^{-1}(p), \xi) \in \mathbb{Z} \), where \( t(\delta, *) \) is the “twisting number” of the Legendrian arc \( \delta \) with respect to the contact structure \(*\), and \( p \in B \).

Lemma 1.4. Let \( w : B \setminus L \to \mathbb{Z} \) be a nonnegative weight function. Then \( w \) uniquely determines, up to isotopy rel \( V \setminus N(\mathcal{B}) \), a contact structure \( \xi \) generated by \((\mathcal{B}, \zeta)\).

Remark 1.5. Of course it is difficult to tell which nonnegative weight functions \( w \) correspond to tight contact structures.

Proof. The lemma is a consequence of the contractibility of \( \text{Diff}^+(I) \), the space of orientation-preserving diffeomorphisms of the unit interval.
Let \( \xi \) and \( \xi' \) be two contact structures on \( D^2 \times I \) with coordinates \(((x, y), t)\), which coincide on \( D^2 \times \{0, 1\} \), have Legendrian fibers \( \{pt\} \times I \), and have the same weight. Then they are given by \( \alpha = \cos f_0(x, y, t)dx - \sin f_0(x, y, t)dy \) and \( \alpha' = \cos f_1(x, y, t)dx - \sin f_1(x, y, t)dy \) satisfying \( \frac{\partial \alpha}{\partial t} > 0 \) and \( f_0(x, y, j) = f_1(x, y, j) \), where \( i, j = 0, 1 \). By the contractibility of \( Diff^+(I) \), there exists a 1-parameter family of functions \( f_s : D^2 \times I \to \mathbb{R} \), \( s \in [0, 1] \), which satisfy \( \frac{\partial f_s}{\partial t} > 0 \) and are independent of \( s \) on \( D^2 \times \{0, 1\} \). Therefore \( \xi \) and \( \xi' \) are isotopic relative to \( D^2 \times \{0, 1\} \) through contact structures which have \( \{pt\} \times I \) as Legendrian fibers.

In order to prove the lemma, we relativize the above discussion. Let \( \xi \) and \( \xi' \) be two contact structures generated by \( \mathcal{B} \) and which have the same weight. If \( \mathcal{B} \) is a sector of \( \mathcal{B} \), then \( \partial \mathcal{B} \) is a polygon \( \delta_1 \cup \delta_2 \cup \cdots \cup \delta_m \), where \( \delta_i \) are edges and the consecutive edges meet along triple branch points of \( \mathcal{B} \). Since \( (B \times I) \cap \partial_v N(\mathcal{B}) \) consists of a union of \( \delta_i \cup [a_i, b_i] \) where \( [a_i, b_i] \subset [0, 1] \), we are considering \( \xi \) and \( \xi' \) which agree on \( \delta_i \times [a_i, b_i] \). We first line up \( \xi \) and \( \xi' \) along \( \delta_i \times ([0, 1] - [a_i, b_i]) \) using the contractibility of \( Diff^+(I) \), and then line up \( \xi \) and \( \xi' \) inside \( B \times I \), relative to \((\partial \mathcal{B}) \times I \). This proves the lemma.

Let \( \mathcal{T}(\mathcal{B}, \zeta) \) be the isotopy classes of tight contact structures which are generated by \((\mathcal{B}, \zeta)\). (Recall that \( \mathcal{B} \) may have nonempty boundary.) The following is a frequently used Amputation Lemma:

**Lemma 1.6** (Amputation). If there exists a point \( x \in \mathcal{B} \) such that the twisting number \( \min_{\zeta \in \mathcal{T}(\mathcal{B}, \zeta)} t(\pi^{-1}(x), \zeta) > -\infty \), then we may “amputate” all sectors of \( \mathcal{B} \) containing \( x \) from \( \mathcal{B} \) to obtain a new branched surface \( \mathcal{B}' \) (possibly with boundary) and tight contact structures \( \zeta_1, \ldots, \zeta_m \) such that \( \mathcal{T}(\mathcal{B}, \zeta) \) is generated by the \((\mathcal{B}', \zeta')\).

**Proof.** Let \( \mathcal{B} \) be a sector of \( \mathcal{B} \) which contains \( x \). For any \( y \in \mathcal{B} \) and \( \xi \in \mathcal{T}(\mathcal{B}, \zeta) \), \( t(\pi^{-1}(x), \xi) \) is also bounded below; hence there are only finitely many possibilities for \( w_c(\mathcal{B}) \). For each \( c \in \mathbb{Z}^+ \), there is a finite number of tight contact structures \( \zeta_i \) on \( V \setminus \pi^{-1}(B - B) \) which agree with \( \zeta \) on \( V \setminus N(\mathcal{B}) \) and which admit Legendrian fibrations on \( \pi^{-1}(B) \) with \( c = w_c(\mathcal{B}) \). Here, \( \zeta_i \) is not unique even if we fix \( c \) because of the following: If \( \delta \) is an edge of the polygon \( \partial \mathcal{B} \) (described in Lemma 1.4), then \( \partial_v N(\mathcal{B}) \cap (B \times I) \) may contain \( \delta \times [a, b] \), where \( [a, b] \subset [0, 1] \). Observe that there are finitely many ways of partitioning

\[
t(\{x\} \times [0, 1]) = t(\{x\} \times [0, a]) + t(\{x\} \times [a, b]) + t(\{x\} \times [b, 1]),
\]

where \( t(\{x\} \times [a, b]) \) is fixed. We can add an integer \( d \) to \( t(\{x\} \times [0, a]) \) (resp. add \(-d \) to \( t(\{x\} \times [b, 1]) \)), subject to the condition that \( t(\{x\} \times [0, a]) \) and \( t(\{x\} \times [b, 1]) \) remain negative.

By the following corollary, we may reduce to the case where \( \mathcal{B} \) has no boundary.

**Corollary 1.7.** Suppose \( \mathcal{T}(\mathcal{B}, \zeta) \) is generated by \((\mathcal{B}, \zeta)\). If the branched surface \( \mathcal{B} \) has boundary, then \( \mathcal{T}(\mathcal{B}, \zeta) \) is also generated by a finite collection \((\mathcal{B}_1, \zeta_1), \ldots, (\mathcal{B}_m, \zeta_m)\), where \( \mathcal{B}_i \) are branched surfaces without boundary.

**Proof.** Any point on \( \partial \mathcal{B} \) satisfies the conditions of the Amputation Lemma. Also, each amputation reduces the complexity of \( \mathcal{B} \), measured by the number of connected components of \( \mathcal{B} \setminus L \). Therefore, a finite number of applications of the Amputation Lemma yields the desired result.
We now have the following simplification:

**Proposition 1.8.** There exist finitely many pairs \((B_i, \zeta_i)\), \(i = 1, \ldots, k\), such that:

1. \(B_i\) is a branched surface without boundary.
2. \(\zeta_i\) is a tight contact structure on \(V \setminus N(B_i)\).
3. Every \(\xi_j \in T(B_i, \zeta_i)\), up to isotopy, is generated by \((B_i, \zeta_i)\). The corresponding weight function \(w_{\xi_j}\) is sufficiently positive, i.e., \(w_{\xi_j}(B) >> 0\) for all sectors \(B\) of \(B_i\).
4. \(\bigcup_{j=1}^{k} T(B_i, \zeta_i) = \pi_0(Tight(V))\).
5. \(\sup_j w_{\xi_j}(B) = +\infty\) for all sectors \(B\) of \(B_i\).

**Remark 1.9.** It is possible that \(B_i\) is empty, in which case \(T(B_i, \zeta_i)\) consists of just one element \(\zeta_i\).

**Proof.** This follows from the Amputation Lemma. Given a tight contact structure \(\xi\) generated by \((B, \zeta)\) for which there exists a sector \(B\) with insufficiently positive \(w_{\xi}(B)\), we amputate \(B\) to obtain \(B'\) (which we may assume has no boundary, after further amputations) and transfer \(\xi\) from \(T(B, \zeta)\) to \(T(B', \zeta')\). Moreover, if \(\sup_j w_{\xi_j}(B) < +\infty\) for some \(B\), then we can amputate \(B'\) from \(B\). \(\square\)

**2. Surfaces carried by the branched surface \(B\)**

Let \((B, \zeta)\) be one of the \((B_i, \zeta_i)\) from Proposition 1.8, assume \(\mathcal{B}\) is nonempty. Each positive weight function \(w_{\xi}\) corresponds to a closed surface \(\mathcal{T}\) which is fully carried by \(N(\mathcal{B})\). Since \(\mathcal{T}\) is transverse to the Legendrian fibration of \(N(\mathcal{B})\), it follows that \(\mathcal{T} \pitchfork \xi\). This implies that each component of \(\mathcal{T}\) is either a torus or a Klein bottle. The following proposition allows us to restrict attention to the case where there are no Klein bottles.

**Proposition 2.1.** Let \(V\) be a closed, oriented, irreducible, and atoroidal 3-manifold which contains a Klein bottle \(K\). Then \(\left|\pi_0(Tight(V))\right| < \infty\).

**Proof.** Let \(K\) be a Klein bottle, and let \(\rho : N(K) \to K\) be its tubular neighborhood. Then \(\partial N(K)\) is a torus, and since \(\partial N(K)\) is compressible, there exists a compressing disk \(D\) for \(\partial N(K)\) in \(V \setminus N(K)\). Now, irreducibility implies that either \(N(D) \cup N(K)\) is a 3-ball or \(V \setminus N(K)\) is a solid torus. The first option is not possible, since \(K\) does not separate \(B^3\), whereas every closed surface in \(B^3\) must separate. The latter implies that \(V = N(K) \cup (S^1 \times D^2)\). Now, \(N(K)\) admits a Seifert fibration over the disk with two singular fibers, where the regular fibers are isotopic to the boundary of a Möbius band \((\rho^{-1}(\delta)\) for an appropriate \(\delta \subset K\)). If the meridional curve of \(S^1 \times D^2\) is a regular fiber of \(N(K)\), then \(V\) is \(RP^3 \# RP^3\), which has a unique tight contact structure. Otherwise, \(V\) is a Seifert fibered space over \(S^2\) with three singular fibers and Seifert invariants \((\frac{1}{2}, \frac{1}{2}, \frac{2}{3})\). If \(\frac{2}{3}\) is an integer, \(V\) is a lens space and it is well-known that \(\left|\pi_0(Tight(V))\right| < \infty\). Otherwise, the finiteness of tight contact structures on these prism manifolds follows from Section 6. \(\square\)

From now on we assume that every connected surface carried by \(N(\mathcal{B})\) is a torus.

**Lemma 2.2.** There exists a surface \(\mathcal{T}\) fully carried by \(N(\mathcal{B})\) such that \(\mathcal{T} \supset \partial_h N(\mathcal{B})\), and there is an orientation on \(\mathcal{T}\) which agrees with the normal orientation on \(\partial N(\mathcal{B})\) along \(\mathcal{T} \cap \partial_h N(\mathcal{B})\).
Proof. By doubling \( T \) if necessary, we ensure that each fiber of \( N(B) \) intersects \( T \) at least twice. If \( \Sigma \subset \partial_h N(B) \) is a connected component, then we can flow \( \Sigma \) along the Legendrian fibers until we hit \( T \) for the first time. Reversing this process, we can pull that portion of \( \Sigma \) to the boundary component \( \Sigma T \), then we double \( T \) to obtain \( T_+ \) and \( T_- \), and place components of \( \partial_h N(B) \) of one orientation onto \( T_+ \) and components of the opposite orientation onto \( T_- \). \( \square \)

3. The Degree Lemma

Consider the pair \((B, \zeta)\). Let \( A \) be a connected component of \( \partial_v N(B) \) and \( c \) be a boundary component of \( A \). Define \( \deg(A) \), the degree of \( A \) to be the absolute value of the degree of the image of \( \zeta_x \) with respect to \( T_x A \) in the quotient \( T_x V/T_x(Legendrian \ fiber) \), as \( x \) ranges over \( c \). Here, we use the absolute value because there is not necessarily a coherent orientation on the fibers of \( \pi : N(B) \to B \).

Claim 3.1. We may assume that for each component \( A \) of \( \partial_v N(B) \) and each component \( c \) of \( \partial A \), there are exactly 2 \( \deg(A) \) points along \( c \) where \( T_x A = \zeta_x \).

Proof. Extend \( A \) along the Legendrian fibers to an annulus \( A' \), where \( A' - A \subset N(B) \), \( \xi|_{A'} \) agree for all \( \xi \in T(B, \zeta) \), and the condition of the claim holds for \( \partial A' \). (This is made possible by all \( \xi \in T(B, \zeta) \) having sufficiently large \( w_\xi \).)

The claim follows from "making a slit" along \( A - A' \). More precisely, consider a thickened annulus \( A' \times I \subset N(B) \), where \( A' \times \{1\} = A' \) and each \( \xi \) is \( I \)-invariant on the thickened annulus. (In particular, each \( A' \times \{t\} \) is fibered by Legendrian intervals.) Then the new \( N(B) \) is \( N(B) \setminus (A' \times I) \) with some corners rounded, so that \( \partial_v N(B) = A' \times \{0\} \). \( \square \)

From now on, all components of \( \partial_v N(B) \) are assumed to satisfy the above claim.

Lemma 3.2 (Degree Lemma). After possible amputations, we may assume that every connected component \( A \) of \( \partial_v N(B) \) with nonzero degree intersects \( T \) along homotopically trivial curves on \( T \).

Proof. Let \( T \) be a component of \( \Sigma \) which intersects a component \( A \) of \( \partial_v N(B) \) with \( \deg(A) \neq 0 \) along \( c \). Suppose \( c \) is homotopically essential on \( T \). For any \( \xi \in T(B, \zeta) \) with \( w_\xi \) sufficiently positive, there exists an embedding \( \phi : T \times [0, 1] \to N(B) \), where \( \phi(T, 0) = T \) and \( \phi^*\xi \) is given by \( \cos(f(x, y) + 2\pi t)dx - \sin(f(x, y) + 2\pi t)dy = 0 \). Here, the coordinates on \( T \times [0, 1] = \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1] \) are \((x, y, t)\), and \( f \) is a circle-valued function \( T \to \mathbb{R}/2\pi \mathbb{Z} \). We then have the following:

Claim 3.3. Inside \( \phi(T \times [0, 1]) \) there exists a torus \( T' \) isotopic to \( T \) and transverse to the Legendrian fibers, such that \( T' \) is convex and \( \# \Gamma_{T'} \leq 2 \deg(A) \).

Proof of Claim 3.3. After a \( C^\infty \)-small perturbation of \( T \), we may assume \( T \) is convex. Since \( T \cap \xi \) for any \( \xi \in T(B, \zeta) \), the characteristic foliation \( \xi T \) is nonsingular, and hence \( \# \Gamma_T \) is the same as the number of closed orbits \( \gamma_i \) of \( \xi T \). \( T \setminus \cup \gamma_i \) are annular components which are either Reeb (no transverse arc with endpoints on the boundary which intersects every
leaf) or taut (there exists such a transverse arc). We may assume $c$ is transverse to $∪_i γ_i$. By inspecting the connected components of $c \setminus ∪_i γ_i$, every (separating or nonseparating) arc inside a Reeb component contributes at least one tangency, whereas arcs inside taut components do not necessarily contribute. Therefore, the number of Reeb components is bounded above by $2 \deg(A)$ (= the number of tangencies of $c$), if the $γ_i$ have nontrivial geometric intersection with $c$. To see that the $γ_i$ have nontrivial geometric intersection with $c$, observe that $2 \deg(A)$, the signed count of tangencies of $c$ and $ξT$, is invariant under isotopy. If the geometric intersection number is zero, then the degree must be zero. Finally, all the taut components can be removed by isotoping $T$ a bounded distance within $φ(T × [0, 1])$. □

The key feature of the convex torus $T$ modified as in the above lemma is that $#Γ_T$ is bounded independently of the choice of $ξ ∈ T(B, ξ)$. Suppose that $ξ ∈ T(B, ξ)$ satisfies $w_ξ >> nw_T$, where $n = \deg(A) = \frac{1}{2} #Γ_T$. Then there exists an embedding $ψ : T × [0, n] → N(B)$, where $T × \{0\} = T'$ and $ψ^*ξ$ is given by $\cos(g(x, y) + 2πt)dx − \sin(g(x, y) + 2πt)dy = 0$. If we could remove $ψ(T × [0, n])$ and reglue $ψ(T × \{0\})$ with $ψ(T × \{n\})$ via the natural identification given by the Legendrian fibration, we obtain a contact structure $ξ'$ corresponding to the weight $w_ξ − nw_T$. Now, $ξ$ and $ξ'$ are isomorphic, since they differ by Dehn twists along tori. They are isotopic, since either (i) $T$ bounds a solid torus or (ii) $T$ bounds a knot complement inside $B^3$, and we can use the fact that a diffeomorphism of $B^3$ relative to the boundary is isotopic to the identity rel boundary. Therefore, we may inductively reduce $w_ξ → w_ξ − nw_T$ until some sector $B$ has small weight. Such a sector $B$ can be amputated. □

With the Degree Lemma at hand, we prove the following useful proposition:

**Proposition 3.4.** After possible amputations, we may assume that, given a connected component $A$ of $∂_c N(B)$,

1. $\deg(A) = 0$ if and only if both components of $∂A$ are essential on $Τ$.
2. $\deg(A) = 1$ if and only if both components of $∂A$ bound disks in $Τ$.

**Proof.** By the Degree Lemma, if a component of $∂A$ is essential, then $\deg(A) = 0$. On the other hand, if a component $c$ of $∂A$ bounds a disk $D$ in $Τ$, then, by Claim 3.1 and the nonsingularity of the characteristic foliation on $D$, there can only be two points along $c$ where $T_x A = ζ_x$. Hence $\deg(A) = 1$. Therefore, either both components of $∂A$ are essential, or both bound disks. The proposition follows. □

**Remark 3.5.** Observe that if both components of $∂A$ bound disks, then the disks must both be on the same side of $A$.

4. Elimination of disks of contact

In this section, we simplify the branched surface neighborhood $N(B)$ by eliminating disks of contact. A disk of contact is a properly embedded disk $D ⊂ N(B)$ transverse to the fibers of $N(B)$, whose boundary is on $∂_c N(B)$.

**Lemma 4.1.** Let $A$ be a component of $∂_c N(B)$. If there exists a disk of contact $D$ with boundary on $A$, then the boundary components $c_1, c_2$ of $A$ bound disks $D_1, D_2 ⊂ Τ$ so that $D_i$ is in the interior of $N(B)$ near $∂D_i$. 


Proof. Since \( A \) admits a disk of contact \( D \) and the characteristic foliation on \( D \) is nonsingular, \( \deg(A) \) must equal one. By the Degree Lemma, \( c_i \) must bound a disk \( D_i \) in \( T \). Note that \( D_i \) cannot be in the “opposite direction” from \( D \), namely \( D_i \) cannot contain the component of \( \partial_h N(B) \) adjacent to \( c_i \). Otherwise \( D \cup D_i \) (together with some pieces of \( A \), and after some rounding) will form an immersed 2-sphere transverse to the Legendrian fibration, a contradiction.

Remark 4.2. It is conceivable that \( D_1 \subset D_2 \) or vice versa.

Proposition 4.3. Let \( V \) be a closed, atoroidal, irreducible manifold. There exists a finite number of pairs \((N_i, \zeta_i), i = 1, \ldots, k\), satisfying the following:

1. \( N_i \subset V \) is a finite union of thickened tori \( T^2 \times [0,1] \) and annuli \( A \times [0,1] \), where each \( A \times \{j\}, j = 0,1, \) is glued essentially onto some \( \partial(T^2 \times [0,1]) \), and some boundary components of the \( T^2 \times I \) may be identified.
2. \( \zeta_i \) is a tight contact structure on \( V \setminus N_i \).
3. If \( T(N_i, \zeta_i) \) is the set of isotopy classes of tight contact structures \( \xi \) on \( V \) which agree with \( \zeta_i \) on \( V \setminus N_i \), then \( \cup_i T(N_i, \zeta_i) = \pi_0(\text{Tight}(V)) \).

Proof. We first eliminate all disks of contact from \( N(B) \), while preserving the condition that \( N(B) \) fully carry a union of tori \( T \). (See Remark 4.2 below.) If there is a disk of contact for \( A \), then using Lemma 4.1, we may replace it with disks \( D_1 \) and \( D_2 \) of contact (also for \( A \)) in \( T \). Without loss of generality, assume \( D_1 \) is an innermost disk of contact for \( T \). Then either \( D_1 \) and \( D_2 \) are disjoint, or \( D_1 \subset D_2 \). Since \( D_1 \) may contain some disks of \( \partial_h N(B) \), we make a small isotopy of \( D_1 \) along the fibers, to push \( D_1 \) away from \( \partial_h N(B) \cap \text{int}(D_1) \). Call the isotoped disk \( D'_1 \). Then modify \( N(B) \to N(B) \setminus D'_1 \), \( T \to (T \setminus D_1) \cup D'_1 \), and \( D_2 \to (D_2 \setminus D_1) \cup D'_1 \) if \( D_1 \subset D_2 \) (rename them \( N(B), T \), and \( D_2 \)).

We will now explain how to realize \( D'_1 \) as a convex surface (with Legendrian boundary), so that the tight contact structure \( \zeta \) on \( N(B) \) extends uniquely to a contact structure on \( N(B) \cup N(D'_1) \). After rounding the corners of \( \partial N(B) \) and perturbing, \( \partial N(B) \) becomes convex. The fact that \( \deg(A) = 1 \) translates into \( tb(\partial D'_1) = -1 \), when we realize \( \partial D'_1 \) as a Legendrian curve on \( \partial N(B) \). Now, if \( D'_1 \) is perturbed into a convex surface with Legendrian boundary, there is only one possibility (up to isotopy) for \( \Gamma_{D'_1} \). Hence, after applying the Flexibility Theorem, we may assume that for any \( \xi \in T(N(B), \zeta) \), \( D'_1 \) can be taken to have the same characteristic foliation.

Since (the new) \( T \) is not fully carried by (the new) \( N(B) \), we modify \( T \) as follows: Let \( T \) be the connected component of \( T \) containing \( D_2 \), and let \( T' \) be a parallel push-off. Surger \( T \to (T \setminus D_2) \cup A \cup D'_1 \) and round. Doubling the surgered torus, we obtain a fully carried \( T \) containing \( \partial_h N(B) \). Since there are only finitely many components of \( \partial_h N(B) \), we can eliminate all the disks of contact. Observe that components of \( \partial_h N(B) \cap T \) which were homotopically essential (resp. homotopically trivial) remain essential (resp. trivial) after the surgery and doubling operations.

Having eliminated all disks of contact, we now examine the connected components of \( \partial_h N(B) \). Indeed, there are only three possibilities: (i) disks, (ii) annuli which are essential on \( T \), and (iii) tori. All the disk components of \( \partial_h N(B) \) can be eliminated as follows: Let \( D \) be a disk component of \( \partial_h N(B) \) and \( A \) an annulus of \( \partial_v N(B) \) which shares a boundary
component with $D$. Then $\deg(A)$ must be nonzero, and if $S$ is a component of $\partial_h N(\mathcal{B})$ which intersects the other boundary component of $A$, then, by the Degree Lemma, $S$ cannot be a homotopically essential annulus. Hence, $S$ is also a disk. Now, $D \cup A \cup S$ is a sphere which bounds a 3-ball $B^3$ on one side or another. In one case, we take $N(\mathcal{B}) \cup B^3$, and in the other case we take $N(\mathcal{B}) \setminus B^3$. Eventually, the horizontal disk components are removed. This implies that all the components of $N(\mathcal{B}) \setminus T$ are thickened tori or thickened annuli which are glued essentially onto the boundary of the thickened tori. \hfill \Box

Remark 4.4. In eliminating disks of contact, we lose control over the Legendrian fibration, although the topological fibration still exists. Therefore, instead of isotopy classes of tight contact structures which are generated by a pair $(N(\mathcal{B}), \zeta)$, we must consider isotopy classes of tight contact structures on $V$ which simply agree with $\zeta$ on $V \setminus N$. Due to this loss of information, we must repeat the finiteness study for simpler $N$ and $V$, namely when $V$ is a small Seifert space. This study will be conducted in the next two sections.

5. Reduction to the small Seifert case

Let $(N, \zeta) = (N_i, \zeta_i)$ be a pair as in Proposition 4.3.

Lemma 5.1. $N$ is a graph manifold with nonempty boundary.

Remark 5.2. Our graph manifolds may be disconnected, and the components may be Seifert fibered spaces.

Proof. Suppose $N = V$. Then $N$ consists only of $T^2 \times I$ components, glued successively to give a torus bundle over $S^1$. Therefore $V$ is toroidal, a contradiction. Now, whenever two $T^2 \times I$ components share a common boundary (they cannot share both boundary components), they can be merged into a single $T^2 \times I$. Next, if we cut $N$ along the union of $T^2 \times \{\frac{1}{2}\}$, then the connected components are diffeomorphic to $S^1$ times a compact surface with boundary. Therefore $N$ is a graph manifold. \hfill \Box

Proposition 5.3. If $\mathcal{T}(N, \zeta)$ is not finite, then $V$ is a Seifert fibered space over $S^2$ with 3 singular fibers.

Proof. Let $T$ be a boundary component of $N$, which is a graph manifold. Since $T$ is compressible, there is a compressing disk $D$ for $T$. By using an innermost argument and switching $T$ if necessary, we may assume that $D \subset N$ or $D \subset V \setminus N$.

Suppose $D \subset N$. Since $N$ is a graph manifold, it is irreducible. Hence $T$ bounds a solid torus $W$ which contains components of $N$. By the finiteness of tight contact structures on $W$, all components of $N$ inside $W$ can be removed from $N$ without affecting the infiniteness of $\mathcal{T}(N, \zeta)$.

Now suppose $D \subset V \setminus N$. By the irreducibility of $V$, either $T$ bounds a solid torus $W$ in $V \setminus N$, or $T \cup D$ is contained in a 3-ball whose boundary lies outside $N$. In the latter case, we may throw away all the components of $N$ inside the 3-ball (including the component of $N$ which is bounded by $T$). In the former case, we consider $N \cup W$. Let $M$ be the maximal Seifert fibered component of $N$ with $M \cap W \neq \emptyset$, as given by the canonical torus (Jaco-Shalen-Johannson) decomposition. Also let $\pi : M \to S$ be the projection onto $S$, a compact surface with boundary. If $S$ is a disk with at most one singular point, then $M$ is a solid torus
and \( V \) is a lens space. If \( S \) is an annulus without any singular points, then \( M = T^2 \times I \) is a connected component of \( N \), and \( M \cup W \) is a solid torus with a finite number of tight contact structures, hence can be excised from \( N \).

For any other \( S \), if the meridian of \( W \) does not bound a regular fiber in \( M \), then \( M \cup W \) is a maximal Seifert fibered component of \( N \cup W \). In this case, \( N \cup W \) is a graph manifold – we will rename this \( N \). If the meridian of \( W \) does bound, then let \( c \) be a boundary component of \( S \) which corresponds to \( T \). There exists an arc \( d \subset S - \{ \text{ singular points} \} \) with endpoints on \( c \), which is not \( \partial \)-parallel in \( S - \{ \text{ singular points} \} \). Now, the union of \( \pi^{-1}(d) \) and two meridional disks of \( W \) is a 2-sphere \( K \), which must bound a 3-ball \( B^3 \) on one side, by irreducibility. This implies, first of all, that \( S \) is a planar surface; otherwise there exists an arc \( d \) and a closed curve \( \delta \) on \( S \) which intersect precisely once, contradicting the fact that \( K \) separates. Next, consider \( M' = M \cup W \cup B^3 \). One of the components of \( S \setminus d \) is contained in \( B^3 \), and the other is a planar surface \( S' \) with fewer boundary components. Since \( K \) bounds a 3-ball, \( S' \) bounds a solid torus in \( M' \). Renaming \( N \cup W \cup B^3 \) and \( M \cup W \cup B^3 \) to be the new \( N \) and \( M \), and continuing in this manner, we see that \( M \cup W \) is a solid torus (or \( V \) is a lens space).

By repeating the above argument, we inductively reduce the number of connected components of \( \partial N \), while ensuring that \( N \) is either a graph manifold or the empty set (which is disallowed by hypothesis). Since \( V \) is atoroidal, \( V = N \) would then be a small Seifert fibered space or a lens space (which is also disallowed).  

\[ \square \]

6. The small Seifert case

Let \( V \) be a small Seifert space, i.e., a Seifert fibered space with 3 singular fibers over \( S^2 \). The tubular neighborhoods of the singular fibers \( F_i, i = 1, 2, 3 \), are denoted by \( V_i \).

6.1. Case 1. We restrict attention to the set of tight contact structures for which there exists a Legendrian regular fiber with twisting number = 0, where the twisting is measured using the projection.

Remark 6.1. (Well-definition of twisting number) Take a neighborhood of a regular fiber, and on the boundary we look at curves which bound on either side. If they are different, the framing is well-defined; if they are the same, then that means that there is a horizontal surface (after making the surface incompressible). This works in our case. You can also use the uniqueness of the Seifert fibration.

Claim 6.2. Given a tight contact structure \( \xi \) with a zero-twisting Legendrian regular fiber, there exists an isotopy of \( \xi \) for which \( \partial V_i \) are convex, \( \Gamma_{\partial V_i} \) are vertical, and \( \# \Gamma_{\partial V_i} = 2 \).

Let \( W = V \setminus \bigcup_{i=1}^3 V_i \). Let \( \pi : W \to S \) be the projection induced by the fibration, where \( S \) is the 3-punctured sphere.

Claim 6.3. Let \( \hat{S} \) be a convex surface which is the image of a section \( s : S \to W \). Then \( \Gamma_{\hat{S}} \) consists of three nonseparating arcs, each of which connects distinct boundary components of \( \hat{S} \). 

Proof. Suppose there is a separating arc \( c \) on \( \hat{S} \). Then it is \( \partial \)-parallel. Let \( V'_i \) be the solid torus to which the bypass corresponding to \( c \) is attached. Then the thickening \( V'_i \) of \( V_i \) has
Remark 6.5. Let $\Gamma_0V_i'$ parallel to $\partial\hat{S} \cap \partial V_i$. Now $V_i'$ can be thickened again to $V_i''$ so as to contain Legendrian fibers with twisting number zero (taken, for instance, by pushing off a zero twisting curve on $V_j$, $j \neq i$). Some intermediate convex torus between $\partial V_i'$ and $\partial V_i''$ will then have dividing curves parallel to the meridian of $V_i''$, and hence the contact structure on $V_i''$ is overtwisted.

Now, if $c$ is a closed component of $\Gamma_0\hat{S}$, it is parallel to some boundary component $\partial V_i \cap \hat{S}$, and the same argument as above shows that the contact structure is overtwisted. \hfill $\square$

Write $W = S \times S^1$. If $\partial S = c_1 \sqcup c_2 \sqcup c_3$, then $\partial W = \bigsqcup_{i=1}^{3} c_i \times S^1$.

**Proposition 6.4.** There is a 1-1 correspondence between isotopy classes of tight contact structures on $W = S \times S^1$ with fixed convex boundary where $\Gamma_{c_i \times S^1}$, $i = 1, 2, 3$, consists of $2k_i$ parallel curves isotopic to the regular fiber, and isotopy classes of multicurves on $S$ which have no homotopically trivial components and which have $2k_i$ fixed endpoints on $c_i$.

The proposition can be found in [Gi5, Ho2].

Observe that there are infinitely many isotopy classes of possible dividing sets on $S$ relative to the boundary; however, non-relative isotopy classes are finite in number (in fact there are two). Moreover, if $\Gamma$ and $\Gamma'$ are two allowable dividing sets on $S$ which are isotopic but not isotopic relative to the boundary, they differ by Dehn twists parallel to the boundary components. In other words, we may assume that $\Gamma = \Gamma'$ when restricted to $\hat{S}' = \hat{S} - \bigcup_i A_i$, where $A_i$ is a collared neighborhood of $\hat{S} \cap \partial V_i$. Let $W' = \pi^{-1}(\pi(\hat{S}'))$, and $V_i''$ be the component of $V \setminus W'$ containing $V_i$.

Given a tight contact structure $\xi$ on $V$ with a zero-twisting Legendrian regular fiber, there exists an isotopy of $\xi$ so that $\xi|W'$ is one of two types. Now since there are only finitely many isotopy classes of tight contact structures on solid tori with a fixed boundary condition, we conclude that there are finitely many isotopy classes of tight contact structures on $V$ with a Legendrian regular fiber with zero twisting.

6.2. Case 2. In this case we only consider the set of tight contact structures on $V$ which do not have Legendrian regular fibers with zero twisting number.

**Remark 6.5.** The Seifert fibered space with 3 singular fibers over base $S^2$ is denoted $(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3})$. $\frac{\beta_i}{\alpha_i}$ is then the slope of the meridional disk of $V_i$, seen from $W = S \times S^1$. Here, $\partial S \cap V_i$ has slope zero and the regular fibers have slope $\infty$.

We define $\frac{\beta_i'}{\alpha_i'}$ with $\text{GCD}(\beta_i', \alpha_i') = 1$ to be the greatest rational number such that $\beta_i' \alpha_i - \beta_i \alpha_i' = 1$. (Viewed on the Farey tessellation, $\frac{\beta_i'}{\alpha_i'}$ is the point closest to $+\infty$ on $(\frac{\beta_i}{\alpha_i}, +\infty)$ with an edge to $(\frac{\beta_i}{\alpha_i})$)

**Claim 6.6.** Suppose $\xi$ is a contact structure so that $V_i$ is the standard neighborhood of a Legendrian singular fiber and $\Gamma_{\partial V_i}$ has slope in $(\frac{\beta_i}{\alpha_i}, \frac{\beta_i'}{\alpha_i'})$. If there exists a bypass along a Legendrian regular fiber on $\partial V_i$, then $\xi$ is isotopic to a contact structure $\xi'$ for which $V_i$ is the standard neighborhood of a Legendrian singular fiber with one higher twisting number.

Define $\mathcal{G}S$ to be the isotopy classes of tight contact structures for which there exists a representative $\xi$ satisfying the following conditions:
(1) $V_1$ and $V_2$ are standard neighborhoods of singular Legendrian fibers.
(2) The annulus $A$ connecting $\partial V_1$ to $\partial V_2$ is convex, contains no $\partial$-parallel dividing curves, and is fibered by Legendrian regular fibers with maximal twisting number.

**Claim 6.7.** There are only finitely many isotopy classes of tight contact structures which are not in $\mathcal{GCS}$.

**Proof.** Let $F$ be a Legendrian regular fiber with maximal twisting number, $V_i, i = 1, 2$, be standard neighborhoods of Legendrian singular fibers with boundary slopes in $(\frac{\beta}{\alpha_1}, \frac{\beta}{\alpha_2})$, and $A$ be a convex annulus from $\partial V_1$ to $\partial V_2$ which contains $F$. If $A$ has no $\partial$-parallel arc, then the contact structure $\xi$ is in $\mathcal{GCS}$. Otherwise, we attach the corresponding bypass to thicken $V_i$ by using Claim 6.6. Thus, we are reduced to considering the case when the boundary slopes of $V_i$ are $\frac{\beta}{\alpha_i}$.

Consider $\gamma_i$, the shortest increasing path in the Farey tessellation from $\frac{\beta}{\alpha_i}$ to $+\infty$. If the slope of $\partial V_i$ is in (the vertices of) $\gamma_i$, then attaching a bypass corresponding to a $\partial$-parallel component of $\Gamma_A$ produces a solid torus with boundary slope which is the next term in the path $\gamma_i$. Hence, repeating this operation, we obtain a contact structure for which $A$ has no $\partial$-parallel dividing curves and the boundary slopes of $V_1$ and $V_2$ are in the sequences $\gamma_1, \gamma_2$.

Since the vertices of $\gamma_i$ are finite in number, and $\Gamma_{\partial V_i}$ is determined by the above data, the proof follows from using the finiteness of tight contact structures on solid tori with fixed boundary slopes and a fixed number of dividing curves. \hfill \Box

We are now left to consider $\mathcal{GCS}$.

**Proposition 6.8.** If $\sum_{i=1}^{3} \frac{\beta}{\alpha_i} \neq 0$, then $\mathcal{GCS}$ is finite.

**Proof.** We argue by contradiction. Suppose there exists an infinite sequence $\xi_k$ of tight contact structures in $\mathcal{GCS}$. Then the boundary slopes on $V_1$ and $V_2$ for $\xi_k$ converge to meridional slopes $\frac{\beta_1}{\alpha_1}$ and $\frac{\beta_2}{\alpha_2}$. (In fact, if the two boundary slopes remain bounded away from the meridional slopes, finiteness follows from the finiteness of tight contact structures on solid tori, and if one boundary slope tends to its meridional slope, the other must also tend to its meridional slope because of the connecting annulus $A$.)

Letting $V_3$ be the torus obtained from $V \setminus (V_1 \cup V_2 \cup A)$ by rounding edges, the boundary slopes $s_k$ of $\xi_k$ on $V_3$ tend to $s = -\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}$. Let $\gamma$ be the interval $(\frac{s}{\alpha_3}, s)$, where if $s < \frac{s}{\alpha_3}$ we understand it to be $(\frac{s}{\alpha_3}, +\infty) \cup [-\infty, s)$. Now, if $s < \frac{s}{\alpha_3}$, then for $k$ large enough there exists a convex torus $T$ in $V_3$ parallel to $\partial V_3$ with slope $\infty$. This is ruled out by the assumption of Case 2. Therefore we assume $s > \frac{s}{\alpha_3}$.

Now consider $\xi_k$ where $s_k > s$. Then there exists a convex torus $T$ in $V_3$ parallel to $\partial V_3$ with slope $s$. For $k$ large enough, $\Gamma_T$ intersects the regular fiber fewer times than $\Gamma_{\partial V_3}$; this contradicts the maximality of the twisting number of the Legendrian regular fiber in $A$.

Finally, consider $\xi_k$ where $s_k < s$. On the interval $\gamma$, let $s'$ be the vertex of the Farey tessellation closest to $\frac{s}{\alpha_3}$ with an edge to $s$. For $k$ large enough, $s_k$ is in the interval $(s', s)$, and hence there exists a convex torus $T$ in $V_3$ with slope $s'$. Its dividing curves intersect the regular fiber in fewer points than those of $\partial V_3$, which is again a contradiction. \hfill \Box

Suppose now that $\sum_{i=1}^{3} \frac{\beta}{\alpha_i} = 0$. 

Convention: We will normalize the Seifert invariants so that $0 < \frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2} < 1$ and $-2 < \frac{\beta_3}{\alpha_3} < 0$, and we take the $\alpha_i$ to be positive integers.

**Theorem 6.9.** If $GCS$ is infinite, then $V$ is an elliptic torus bundle over the circle, and hence is toroidal.

**Remark 6.10.** Seifert fibered spaces over $S^2$ with three singular fibers which are torus bundles over the circle are classified – they have Seifert invariants $\pm\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{7}\right), \pm\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{1}\right)$, and $\pm\left(-\frac{2}{3}, \frac{4}{7}, \frac{2}{5}\right)$. They satisfy the property that $\sum \frac{1}{\alpha_i} = 1$.

Suppose $GCS$ is infinite. Then there exists an infinite number of positive pairs $(k_1, k_2)$ so that $k_1\alpha_1 + \alpha'_1 = k_2\alpha_2 + \alpha'_2$, and tight contact structures whose corresponding boundary slopes of $V_i$ are equal to $\frac{k_1\beta_1 + \beta'_1}{k_1\alpha_1 + \alpha'_1}$. Indeed, only finitely many contact structures induce the given boundary slopes on $V_1$ and $V_2$. So the existence of infinitely many contact structures in $GCS$ implies the existence of infinitely many $(k_1, k_2)$.

The solutions of the equation $k_1\alpha_1 + \alpha'_1 = k_2\alpha_2 + \alpha'_2$ are parametrized by a number $k \in \frac{1}{GCS(\alpha_1, \alpha_2)}N$ in the following way: Given a particular solution $(r_1, r_2)$, other solutions are parametrized by $k_1 = k\alpha_2 + r_1$ and $k_2 = k\alpha_1 + r_2$. Observe that there exists a subsequence $k \to +\infty$ and tight contact structures $\xi_k$ in $GCS$.

We compute the boundary slope of $V_3$ to be:

$$s_k = \frac{1 - ((k\alpha_2 + r_1)\beta_1 + \beta'_1) - ((k\alpha_1 + r_2)\beta_2 + \beta'_2)}{(k\alpha_2 + r_1)\alpha_1 + \alpha'_1} = \frac{k(-\alpha_2\beta_1 - \alpha_1\beta_2 + (1 - r_1\beta_1 - \beta'_1 - r_2\beta_2 - \beta'_2)}{k(\alpha_1\alpha_2) + r_1\alpha_1 + \alpha'_1},$$

where $(k\alpha_2 + r_1)\alpha_1 + \alpha'_1 = (k\alpha_1 + r_2)\alpha_2 + \alpha'_2$.

**Lemma 6.11.** $s_k > \frac{\beta_3}{\alpha_3}$ and, for sufficiently large $k$, there is an edge of the Farey tessellation from $\frac{\beta_3}{\alpha_3}$ to $s_k$.

**Proof.** If $s_k < \frac{\beta_3}{\alpha_3}$, then there is an intermediate convex torus in $V_3$ parallel to $\partial V_3$ with infinite slope, which contradicts the assumptions of Case 2. Now, if there is no edge from $\frac{\beta_3}{\alpha_3}$ to $s_k$, then let $s$ be the greatest rational number in $(\frac{\beta_3}{\alpha_3}, s_k)$ with an edge to $\frac{\beta_3}{\alpha_3}$, and let $s'$ be the rational number $> s_k$ with an edge to $\frac{\beta_3}{\alpha_3}$ and $s$. (Such an $s'$ exists provided $k$ is sufficiently large.) Observe that the denominator of $s$ is strictly smaller than the denominator of $s_k$ (in absolute value). Now, there exists a convex torus in $V_3$ with slope $s$, contradicting the maximality of the twisting number of a Legendrian fiber in $A$. □

**Claim 6.12.** The numerator $k(-\alpha_2\beta_1 - \alpha_1\beta_2 + (1 - r_1\beta_1 - \beta'_1 - r_2\beta_2 - \beta'_2)$ and the denominator $k(\alpha_1\alpha_2) + r_1\alpha_1 + \alpha'_1$ of $s_k$ are relatively prime.

**Proof.** If not, $\partial V_3$ has more than two dividing curves, and the twisting number of the Legendrian regular fiber on $A$ is not maximal. □

**Proof of Theorem 6.9.** According to Lemma 6.11 and Claim 6.12, we know that the determinant of $(\alpha_3, \beta_3)$ and $(k(\alpha_1\alpha_2) + r_1\alpha_1 + \alpha'_1, k(-\alpha_2\beta_1 - \alpha_1\beta_2) + (1 - r_1\beta_1 - \beta'_1 - r_2\beta_2 - \beta'_2))$ is
equal to 1. In other words, the determinant of \((\alpha_1\alpha_2, -\alpha_2\beta_1 - \alpha_1\beta_2)\) and \((r_1\alpha_1 + \alpha'_1, 1-r_1\beta_1 - \beta'_1 - r_2\beta_2 - \beta'_2)\) is equal to \(\frac{\alpha_1\alpha_2}{\alpha_3}\). A straightforward computation gives \(\alpha_1\alpha_2 - \alpha_1 - \alpha_2 = \frac{\alpha_1\alpha_2}{\alpha_3}\), that is, \(\sum_{i=1}^{3} \frac{1}{\alpha_i} = 1\). This is precisely the condition for \(V\) to be a torus bundle over \(S^1\). \(\square\)

Acknowledgements. We thank Francis Bonahon for topological help.

REFERENCES

[Be] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque 107-108 (1983), 87–161.
[Co1] V. Colin, *Chirurgies d’indice un et isotopies de sphères dans les variétés de contact tendues*, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), 659–663.
[Co2] V. Colin, *Sur la torsion des structures de contact tendues*, Ann. Sci. École Norm. Sup. (4), 34 (2001), 267–286.
[Co3] V. Colin, *Une infinité de structures de contact tendues sur les variétés toroïdales*, Comment. Math. Helv. 76 (2001), 353–372.
[CGH1] V. Colin, E. Giroux, and K. Honda, *On the coarse classification of tight contact structures*, to appear in the proceedings of the 2002 Georgia International Topology Conference.
[CGH2] V. Colin, E. Giroux, and K. Honda, *Finitude homotopique et isotopique des structures de contact tendues*, in preparation.
[El1] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. 98 (1989), 623–637.
[El2] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet’s work*, Ann. Inst. Fourier (Grenoble) 42 (1992), 165–192.
[ET] Y. Eliashberg and W. Thurston, *Confoliations*, University Lecture Series 13, Amer. Math. Soc., Providence (1998).
[Et] J. Etnyre, *Tight contact structures on lens spaces*, Commun. Contemp. Math. 2 (2000), 559–577.
[EH] J. Etnyre and K. Honda, *Tight contact structures with no symplectic fillings*, Invent. Math. 148 (2002), 609–626.
[Ga] D. Gabai, *Essential laminations and Kneser normal form*, J. Differential Geom. 53 (1999), 517–574.
[Gi1] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991), 637–677.
[Gi2] E. Giroux, *Une structure de contact, même tendue, est plus ou moins tordue*, Ann. Sci. École Norm. Sup. (4) 27 (1994), 697–705.
[Gi3] E. Giroux, *Une infinité de structures de contact tendues sur une infinité de variétés*, Invent. Math. 135 (1999), 789–802.
[Gi4] E. Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. 141 (2000), 615–689.
[Gi5] E. Giroux, *Structures de contact sur les variétés fibrées au-dessus d’une surface*, Comment. Math. Helv. 76 (2001), 218–262.
[Ha] W. Haken, *Theorie der Normalflächen. Ein Isotopiekriterium für den Kreisknoten*, Acta Math. 105 (1961), 245–375.
[Ho1] K. Honda, *On the classification of tight contact structures I*, Geom. Topol. 4 (2000), 309–368.
[Ho2] K. Honda, *On the classification of tight contact structures II*, J. Differential Geom. 55 (2000), 83–143.
[Ho3] K. Honda, *Gluing tight contact structures*, Duke Math. J. 115 (2002), 435–478.
[HKM] K. Honda, W. Kazez and G. Matić, *Convex decomposition theory*, Internat. Math. Res. Notices 2002, 55–88.
[Kn] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jahres. der Deut. Math.-Verein. 38 (1929), 248–260.
[KM] P. Kronheimer and T. Mrowka, *Monopoles and contact structures*, Invent. Math. 130 (1997), 209–255.
Université de Nantes
E-mail address: Vincent.Colin@math.univ-nantes.fr

École Normale Supérieure de Lyon
E-mail address: Emmanuel.GIROUX@umpa.ens-lyon.fr

University of Southern California
E-mail address: khonda@math.usc.edu
URL: http://math.usc.edu/~khonda