THE TOTALLY NONNEGATIVE PART OF $G/P$ IS A CW COMPLEX.

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Dedicated to Professor Bertram Kostant on the occasion of his 80th birthday

Abstract. The totally nonnegative part of a partial flag variety $G/P$ has been shown in [18, 17] to be a union of semi-algebraic cells. Moreover the closure of a cell was shown in [19] to be a union of smaller cells. In this note we provide glueing maps for each of the cells to prove that $(G/P)_{\geq 0}$ is a CW complex. This generalizes a result of Postnikov, Speyer and the second author [15] for Grassmannians.

1. Introduction

In a reductive algebraic group over $\mathbb{C}$ split over $\mathbb{R}$ with a fixed choice of Chevalley generators in the Lie algebra, there is a well-defined notion of positive, or $\mathbb{R}_{>0}$-valued points due to Lusztig [9]. In the case of $GL_n$ with the standard choices, the resulting “$GL_n(\mathbb{R}_{>0})$” recovers the classical notion of totally positive matrices, that is, matrices such that all minors are positive. For general $G$ the set $G_{>0}$ is therefore called the totally positive part of $G$. The closure $G_{\geq 0}$ of $G_{>0}$ (in the real topology) is called the totally nonnegative part of $G$.

These notions have a natural extension to flag varieties $G/P$. That is, there is a notion of $(G/P)_{>0}$, and of $(G/P)_{\geq 0}$, the closure of $(G/P)_{>0}$, which Lusztig has described as a “remarkable polyhedral subspace” [9]. Lusztig has proved that $(G/P)_{\geq 0}$ is contractible [10] and the first author has shown that it is a union of semi-algebraic cells [18, 17]. Moreover, in [19] the first author showed that the closure of a totally nonnegative cell in $G/P$ is a union of totally nonnegative cells and described the closure relations in terms of the Weyl group. The combinatorics of these closure relations was then studied by the second author in [22], where it was shown that the partially ordered set (poset) of cells of $(G/P)_{\geq 0}$ is in fact the poset of cells of a regular CW complex. Recall that a CW complex is a union of cells with additional requirements on how cells are glued; a regular CW complex is one where the closure of each cell is homeomorphic to a closed ball and the closure minus the interior of each cell is homeomorphic to a sphere. The combinatorial results of [22] prompted the second author to conjecture that $(G/P)_{\geq 0}$ is a regular CW complex, which in particular would imply that $(G/P)_{\geq 0}$ is homeomorphic to a closed ball.

In [15], Postnikov, Speyer, and the second author proved that the non-negative part of the Grassmannian is a CW complex, by introducing an auxiliary toric
variety to each parameterization of a cell, and constructing a gluing map from
the non-negative part of that toric variety to the closure of the corresponding cell.
The construction of the toric variety and gluing map relied on explicit positivity
properties of the parameterizations of the cells, which had been described in terms
of certain graphs in [13].

In this paper we generalize the previous result and show that the non-negative
part of any flag variety \((G/P)_{\geq 0}\) is a CW complex. As in [15], we again construct
a toric variety for each parameterization of a cell. However, in our proof we use
the parameterizations of the cells due to Marsh and the first author [12], and use
Lusztig’s canonical basis [7] in order to prove that they have the desired positivity
properties. Once we have proved that \((G/P)_{\geq 0}\) is a CW complex, the combinatorics
from [22] implies that the closures of the individual cells have Euler characteristic
one.

The following result is our main theorem.

**Theorem 1.1.** \((G/P)_{\geq 0}\) is a CW complex.

In [22], the second author proved the following result.

**Theorem 1.2.** [22] The poset of cells of \((G/P)_{\geq 0}\) is the poset of cells of some
regular CW complex; therefore the poset of cells of \((G/P)_{\geq 0}\) is Eulerian.

In other words, the alternating sum of cells in the closure of a cell of \((G/P)_{\geq 0}\)
is 1. This result combined with Theorem 1.1 implies the following.

**Corollary 1.3.** The Euler characteristic of the closure of a cell of \((G/P)_{\geq 0}\) is 1.

The structure of this paper is as follows. In Section 2, we review basic results
on algebraic groups and flag varieties. In Sections 3 and 4 we introduce the notion
of total positivity for real reductive groups and flag varieties, and toric varieties,
respectively. In Section 5 we construct a toric variety associated to a parameteri-
zation of a cell. In Section 6 we prove a key proposition, and in Section 7 we prove
the main result.

2. Preliminaries

2.1. We recall some basic notation and results from algebraic groups, see e.g. [21].
Let \(G\) be a simply connected semisimple linear algebraic group over \(\mathbb{C}\) split over
\(\mathbb{R}\). We identify \(G\) and any related spaces with their \(\mathbb{R}\)-valued points in their
real topology (as real manifolds or subsets thereof). We write \(\mathbb{R}^*\) for \(\mathbb{R} \setminus \{0\}\).

Let \(T\) be a split torus and \(B^+\) and \(B^-\) opposite Borel subgroups containing \(T\).
We denote the character and the cocharacter groups of \(T\) by \(X^*(T)\) and \(X_*(T)\),
respectively. Let \(<,>\) denote the dual pairing between \(X^*(T)\) and \(X_*(T)\). The
unipotent radicals of \(B^+\) and \(B^-\) are denoted \(U^+\) and \(U^-\), respectively. Let \(\{\alpha_i \mid i \in I\} \subset X^*(T)\) be the set of simple roots associated to \(B^+\) and \(\{\alpha_i^\vee \mid i \in I\} \subset X_*(T)\)
the corresponding coroots. Then we have the simple root subgroups \(U_{\alpha_i}^+ \subseteq U^+\)
and \(U_{\alpha_i}^- \subseteq U^-\). Furthermore assume we are given homomorphisms

\[ \phi_i : SL_2(\mathbb{R}) \to G, \quad i \in I, \]

such that

\[ \phi_i \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = \alpha_i^\vee(t), \quad t \in \mathbb{R}^*, \]
and such that
\[ \phi_i \left( \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \right) := x_i(m), \quad \phi_i \left( \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \right) := y_i(m), \]
define isomorphisms \( x_i : \mathbb{R} \to U_{\alpha_i}^+ \) and \( y_i : \mathbb{R} \to U_{\alpha_i}^- \). Following [9], the datum \((T, B^+, B^-, x_i, y_i, i \in I)\) is called a pinning for \( G \).

2.2. If \( G \) is not simply laced, then one can construct a simply laced group \( \hat{G} \) and an automorphism \( \tau \) of \( \hat{G} \) defined over \( \mathbb{R} \), such that there is an isomorphism, also defined over \( \mathbb{R} \), between \( G \) and the fixed point subset \( \hat{G}^\tau \) of \( \hat{G} \). Moreover the groups \( G \) and \( \hat{G} \) have compatible pinnings. Explicitly we have the following.

Let \( \hat{G} \) be simply connected and simply laced. We apply the same notations as above for \( G \), but with an added dot, to our simply laced group \( \hat{G} \). So we have a pinning \((\hat{T}, \hat{B}^+, \hat{B}^-, \hat{x}_i, \hat{y}_i, i \in \hat{I})\) of \( \hat{G} \), and \( \hat{I} \) may be identified with the vertex set of the Dynkin diagram of \( \hat{G} \).

Let \( \sigma \) be a permutation of \( \hat{I} \) preserving connected components of the Dynkin diagram, such that, if \( j \) and \( j' \) lie in the same orbit under \( \sigma \) then they are not connected by an edge. Then \( \sigma \) determines an automorphism \( \tau \) of \( \hat{G} \) such that
\[ \begin{align*}
(1) & \quad \tau(T) = \hat{T}, \\
(2) & \quad \tau(x_i(m)) = x_{\sigma(i)}(m) \quad \text{and} \quad \tau(y_i(m)) = y_{\sigma(i)}(m) \quad \text{for all} \ i \in \hat{I} \quad \text{and} \quad m \in \mathbb{R}.
\end{align*} \]
In particular \( \tau \) also preserves \( \hat{B}^+, \hat{B}^- \). Let \( \hat{I} \) denote the set of \( \sigma \)-orbits in \( \hat{I} \), and for \( i \in \hat{I} \), let
\[ \begin{align*}
x_i(m) & := \prod_{i \in \hat{I}} x_i(m), \\
y_i(m) & := \prod_{i \in \hat{I}} y_i(m).
\end{align*} \]
The fixed point group \( \hat{G}^\tau \) is a simply connected algebraic group with pinning \((T^\tau, B^{\tau^+}, B^{\tau^-}, x_i, y_i, i \in \hat{I})\). There exists, and we choose, \( \hat{G} \) and \( \tau \) such that \( \hat{G}^\tau \) is isomorphic to our group \( G \) via an isomorphism compatible with the pinnings.

2.3. Let \( W = N_G(T)/T \) be the Weyl group of \( G \). For \( i \in I \) the elements
\[ \hat{s}_i = x_i(-1)y_i(1)x_i(-1) \]
represent the simple reflections \( s_i \in W \). If \( w = s_{i_1} \ldots s_{i_m} \) is a reduced expression for \( w \) then we write \( \ell(w) = m \) for the length of \( w \). We note also that the representative
\[ \hat{w} = \hat{s}_{i_1} \ldots \hat{s}_{i_m} \]
of \( w \) is well-defined, independent of the reduced expression. Inside \( W \) there is a longest element which is denoted by \( w_0 \).

2.4. Let \( J \) be a subset of \( I \). The parabolic subgroup \( W_J \subseteq W \) is the subgroup generated by all of the \( s_j \) with \( j \in J \). Let \( w_J \) denote the longest element in \( W_J \). We also consider the set \( W_J^\ell \) of minimal-length coset representatives for \( W/W_J \), and the set \( W_{\ell_{\max}}^J = W_J^\ell w_J \) of maximal-length coset representatives.

The parabolic subgroup \( W_J \) of \( W \) corresponds to a parabolic subgroup \( P_J \) in \( G \) containing \( B^+ \). Namely, \( P_J \) is the subgroup of \( G \) generated by \( B^+ \) and the elements \( \hat{w} \) for \( w \in W_J \). Let \( \mathcal{P}_J \) be the set of parabolic subgroups \( P \) conjugate to \( P_J \). This
is a homogeneous space for the conjugation action of $G$ and can be identified with the partial flag variety $G/P_J$ via

$$G/P_J \sim \mathcal{P}^J : gP_J \mapsto gP_Jg^{-1}.$$ 

In the case $J = \emptyset$ we are identifying the full flag variety $G/B^+$ with the variety $B$ of Borel subgroups in $G$. We have the usual projection from the full flag variety to any partial flag variety which takes the form $\pi = \pi^J : \mathcal{B} \rightarrow \mathcal{P}^J$, where $\pi(B)$ is the unique parabolic subgroup of type $J$ containing $B$.

The conjugate of a parabolic subgroup $P$ by an element $g \in G$ will be denoted by $g \cdot P := gPg^{-1}$.

2.5. Recall the Bruhat decomposition for the full flag variety,

$$\mathcal{B} = \bigsqcup_{w \in W} B^+ \cdot B^+,$$

and the Bruhat order $\leq$ on $W$. The Bruhat cell $B^+ \cdot B^+$ is isomorphic to $\mathbb{R}^{\ell(w)}$. And the Bruhat order has the property

$$v \leq w \iff B^+ \cdot B^+ \subseteq B^+ \cdot B^+,$$

for $v, w \in W$.

It is a well-known consequence of Bruhat decomposition that $\mathcal{B} \times \mathcal{B}$ is the union of the $G$-orbits $O(w) = G \cdot (B^+, B^+)$, with $G$ acting diagonally. Therefore to any pair $(B_1, B_2)$ of Borel subgroups one can associate a unique $w \in W$ such that

$$(B_1, B_2) = (g \cdot B^+, B^+)$$

for some $g \in G$. We write $B_1^w \rightarrow B_2$ in this case and call $w$ the relative position of $B_1$ and $B_2$.

2.6. Finally, let us consider the two opposite Bruhat decompositions

$$\mathcal{B} = \bigsqcup_{w \in W} B^+ \cdot B^+ = \bigsqcup_{v \in W} B^+ \cdot B^+.$$

Note that $B^+ \cdot B^+ \cong \mathbb{R}^{\ell(w) - \ell(v)}$. The closure relations for these opposite Bruhat cells are given by $B^+ \cdot B^+ \subseteq B^+ \cdot B^+$ if and only if $v \leq v'$. We define

$$\mathcal{R}_{v, w} := B^+ \cdot B^+ \cap B^+ \cdot B^+,$$

the intersection of opposed Bruhat cells. This intersection is empty unless $v \leq w$, in which case it is smooth of dimension $\ell(w) - \ell(v)$, see [6, 11].

3. Total Positivity for $G$ and $\mathcal{B}$

Real projective space $\mathbb{P}^n$ has a natural open subset: the set of lines spanned by vectors with all coordinates positive. This subset is called the totally positive part of $\mathbb{P}^n$, and its closure, the set of lines spanned by vectors with all coordinates non-negative, is called the totally non-negative part of $\mathbb{P}^n$. These subsets can be defined more generally [9] for any split semisimple real algebraic group and any partial flag manifold of such a group.
3.1. **Total positivity in** $G$. The totally nonnegative part $G_{≥ 0}$ of $G$ is defined by Lusztig [9] to be the semigroup inside $G$ generated by the sets
\[
\{ x_i(t) \mid t \in \mathbb{R}_{>0}, i \in I \}, \\
\{ y_i(t) \mid t \in \mathbb{R}_{>0}, i \in I \}, \quad \text{and} \\
T_{>0} := \{ t \in T \mid \chi(t) > 0 \text{ all } \chi \in X^*(T) \}.
\]

When $G = SL_n(\mathbb{R})$ then by a theorem of A. Whitney this definition agrees with the classical notion of totally nonnegative matrices inside $SL_n(\mathbb{R})$, that is those matrices all of whose minors are nonnegative.

We recall some basic facts about total positivity for $G$ from [9]. Let $U^+_0 := G_{≥ 0} \cap U^+$ and $U^-_0 := G_{≥ 0} \cap U^-$. For $w \in W$ and $s_{i_1} \ldots s_{i_m} = w$ a reduced expression define
\[
U^+(w) := \{ x_{i_1}(t_1)x_{i_2}(t_2) \ldots x_{i_m}(t_m) \mid t_i \in \mathbb{R}_{>0} \}, \\
U^-(w) := \{ y_{i_1}(t_1)y_{i_2}(t_2) \ldots y_{i_m}(t_m) \mid t_i \in \mathbb{R}_{>0} \}.
\]

These sets are independent of the chosen reduced expression and give
\[
U^+(w) = U^+_{≥ 0} \cap B^- \hat{w} B^-, \\
U^-(w) = U^-_{≥ 0} \cap B^+ \hat{w} B^+.
\]

In particular $U^+_{≥ 0} = \bigsqcup_{w \in W} U^+(w)$ and $U^-_{≥ 0} = \bigsqcup_{w \in W} U^-(w)$. Moreover $U^+(w)$ and $U^-(w)$ are isomorphic to $\mathbb{R}^{k(w)}$ using the $t_i$ as coordinates. The totally positive parts for $U^+$ and $U^-$ are defined by
\[
U^+_{≥ 0} := U^+(w_0), \quad U^-_{≥ 0} := U^-(w_0).
\]

3.2. **Total positivity in** $B$. The totally positive and totally nonnegative parts of the flag variety $B$ are defined by
\[
B_{> 0} := \{ y \cdot B^+ \mid y \in U^-_{> 0} \}, \\
B_{≥ 0} := \overline{B_{> 0}}.
\]

The set $B_{≥ 0}$ has a decomposition into strata,
\[
R^+_{v,w} := R_{v,w} \cap B_{≥ 0},
\]
where $v \leq w$. These strata were defined and conjectured to be semi-algebraic cells by Lusztig [9], a result which was later proved in [18]. The conjecture was proved again in a different way in [12], this time with explicit parametrizations of the cells given.

We recall these parametrizations now.

Let $v \leq w$ and let $w = (i_1, \ldots, i_m)$ encode a reduced expression $s_{i_1} \ldots s_{i_m}$ for $w$. Then there exists a unique subexpression $s_{i_{j_1}} \ldots s_{i_{j_k}}$ for $v$ in $w$ with the property that, for $l = 1, \ldots, k$,
\[
s_{i_{j_1}} \ldots s_{i_{j_l}} \cdot s_{i_{j_{l+1}}} > s_{i_{j_1}} \ldots s_{i_{j_l}} \quad \text{whenever } j_l < r \leq j_{l+1},
\]
where $j_{k+1} := m$. This is loosely speaking the rightmost reduced subexpression for $v$ in $w$. It is called the ‘positive subexpression’ in [12], and we use the notation
\[
\mathbf{v}_+ := \{ j_1, \ldots, j_k \}, \\
\mathbf{v}_+^c := \{ 1, \ldots, m \} \setminus \{ j_1, j_2, \ldots, j_k \},
\]
when referring to this special subexpression for $v$ in $w$. 
Now we can define a map
\[
\phi_{\nu, w} : (\mathbb{C}^*)^{\nu^+} \to R_{v, w},
\]
\[
(t_r)_{r \in \nu^+} \mapsto g_1 \ldots g_m : B^+,
\]
where
\[
gr = \begin{cases} \hat{s}_i, & \text{if } r \in \nu, \\
y_{i_r}(t_r), & \text{if } r \in \nu^+.
\end{cases}
\]
It is shown in [12] that the map \( \phi_{\nu, w} \) is an embedding with image the open ‘Deodhar stratum’ of \( R_{v, w} \) associated to \( w \), see [4].

**Theorem 3.1.** [12, Theorem 11.3] The restriction of \( \phi_{\nu, w} \) to \( (\mathbb{R}_{>0})^{\nu^+} \) defines an isomorphism of semi-algebraic sets,
\[
\phi_{\nu, w}^0 : (\mathbb{R}_{>0})^{\nu^+} \to R_{v, w}^{>0}.
\]

### 3.3. Changes of coordinates under braid relations
In the simply laced case there is a simple change of coordinates [9] [10] which describes how two parameterizations of the same cell are related when considering two reduced expressions which differ by a commuting relation or a braid relation.

If \( s_i s_j = s_j s_i \) then \( y_i(a) y_j(b) = y_j(b) y_i(a) \) and \( y_i(a) \hat{s}_j = \hat{s}_j y_i(a) \).

If \( s_j s_i s_j = s_j s_i \) then
\[
(1) \quad y_i(a) y_j(b) y_i(c) = y_j(\frac{ab}{a+c}) y_i(a+c) y_j(\frac{ab}{a+c}).
\]
\[
(2) \quad y_i(a) \hat{s}_j y_i(b) = y_j(\frac{a}{b}) y_i(a) \hat{s}_j
\]
\[
(3) \quad \hat{s}_j \hat{s}_i y_j(a) = y_i(a) \hat{s}_j \hat{s}_i.
\]
The changes of coordinates have also been computed for more general braid relations and have been observed to be subtraction-free [1] [10].

### 3.4. Closure relations
We have the following closure relations.

**Theorem 3.2.** [19] Let \( v, w \in W \). Then
\[
\overline{R_{v,w}^{>0}} = \bigsqcup_{v \leq v' \leq w} R_{v',w}^{>0}.
\]

### 3.5. Total positivity and canonical bases, for simply laced \( G \)
We now assume that \( G \) is simply laced. Let \( U \) be the enveloping algebra of the Lie algebra of \( G \); this can be defined by generators \( e_i, h_i, f_i \) \( i \in I \) and the Serre relations. For any dominant character \( \lambda \) there is a finite-dimensional simple \( U \)-module \( V(\lambda) \) with a non-zero vector \( \eta \) such that \( e_i \eta = 0 \) and \( h_i \eta = \langle \lambda, \alpha_i^\vee \rangle \eta \) for all \( i \in I \). Moreover, the pair \( (V(\lambda), \eta) \) is determined up to unique isomorphism.

There is a unique \( G \)-module structure on \( V(\lambda) \) such that for any \( i \in I, a \in \mathbb{R} \) we have that \( x_i(a) \) acts by
\[
\exp(ae_i) : V(\lambda) \to V(\lambda),
\]
and \( y_i(a) \) acts by
\[
\exp(af_i) : V(\lambda) \to V(\lambda).
\]
Then \( x_i(a) \eta = \eta \) for all \( i \in I, a \in \mathbb{R} \), and \( t \eta = \lambda(t) \eta \) for all \( t \in T \). Let \( B(\lambda) \) be the canonical basis of \( V(\lambda) \) that contains \( \eta \). See [7] for details on the canonical basis.
In relation to $G_{\geq 0}$, the basis $B(\lambda)$ has the following positivity property [9] Prop. 3.2.

**Theorem 3.3.** [9] Let $g \in G_{\geq 0}$. Then the matrix entries of $g: V \rightarrow V$ with respect to $B(\lambda)$ are non-negative real numbers.

**Remark 3.4.** We remark that it is easy to see that a little more is true. The matrix entries of any $x_i(a)$ or $y_i(a) : V \rightarrow V$ with respect to $B(\lambda)$ are given by positive polynomials in $a$.

The following lemma comes from [11].

**Lemma 3.5.** [11 1.7(a)]. For any $w \in W$, the vector $w\eta$ is the unique element of $B(\lambda)$ which lies in the extremal weight space $V(\lambda)^{w(\lambda)}$. In particular, $w\eta \in B(\lambda)$.

## 4. Positivity for Toric Varieties

In this section we define projective toric varieties and recall some basic results.

We may define a (generalized) projective toric variety as follows [2 20]. Let $S = \{m_i \mid i = 1, \ldots, \ell\}$ be any finite subset of $\mathbb{Z}^n$, where $\mathbb{Z}^n$ can be thought of as the character group of the torus $(\mathbb{C}^*)^n$. Here $m_i = (m_{i1}, m_{i2}, \ldots, m_{in})$. Then consider the map $\phi : (\mathbb{C}^*)^n \rightarrow \mathbb{P}^{\ell-1}$ such that $x = (x_1, \ldots, x_n) \mapsto [x^{m_1}, \ldots, x^{m_\ell}]$. Here $x^{m_i}$ denotes $x_1^{m_{i1}}x_2^{m_{i2}} \ldots x_n^{m_{in}}$. We then define the toric variety $X_S$ to be the Zariski closure of the image of this map. The real part $X_S(\mathbb{R})$ of $X_S$ is defined to be the intersection of $X_S$ with $\mathbb{R}^{\ell-1}$; the positive part $X_S^{\geq 0}$ is defined to be the image of $(\mathbb{R}_{\geq 0})^n$ under $\phi$; and the non-negative part $X_S^{\geq 0}$ is defined to be the closure (in $X_S(\mathbb{R})$) of $X_S^{\geq 0}$.

Note that $X_S$ is not necessarily a toric variety in the sense of [5], as it may not be normal; however, its normalization is a toric variety in this sense. See [2] for more details.

Observe that if $S$ is the set of lattice points in the standard simplex, then $X_S^{\geq 0}$ and $X_S^{\geq 0}$ are the totally positive and totally non-negative parts of real projective space.

Let $P$ be the convex hull of $S$. The restriction of the moment map is a homeomorphism from $X_S^{\geq 0}$ to $P$ (see [5] Section 4.2, page 81) and [20] Theorem 8.4). In particular, $X_S^{\geq 0}$ is homeomorphic to a closed ball.

The following lemma, proved in [15], will be an important tool in the proof of our main result.

**Lemma 4.1.** [15] Suppose we have a map $\Phi : (\mathbb{R}_{>0})^n \rightarrow \mathbb{P}^{N-1}$ given by

$$(t_1, \ldots, t_n) \mapsto [h_1(t_1, \ldots, t_n), \ldots, h_N(t_1, \ldots, t_n)],$$

where the $h_i$’s are Laurent polynomials with positive coefficients. Let $S$ be the set of all exponent vectors in $\mathbb{Z}^n$ which occur among the (Laurent) monomials of the $h_i$’s, and let $P$ be the convex hull of the points of $S$. Then the map $\Phi$ factors through the totally positive part $X_S^{\geq 0}$, giving a map $\tau_{>0} : X_S^{\geq 0} \rightarrow \mathbb{P}^{N-1}$. Moreover $\tau_{>0}$ extends continuously to the closure to give a well-defined map $\tau_{\geq 0} : X_S^{\geq 0} \rightarrow \overline{\tau_{>0}(X_S^{\geq 0})}$.

**Proof.** Let $S = \{m_1, \ldots, m_\ell\}$. Clearly the map $\Phi$ factors as the composite map $t = (t_1, \ldots, t_n) \mapsto [t^{m_1}, \ldots, t^{m_\ell}] \mapsto [h_1(t_1, \ldots, t_n), \ldots, h_N(t_1, \ldots, t_n)]$, and the image of $(\mathbb{R}_{>0})^n$ under the first map is precisely $X_S^{\geq 0}$. The second map, which we will call $\tau_{>0}$, takes a point $[x_1, \ldots, x_\ell]$ of $X_S^{\geq 0}$ to $[g_1(x_1, \ldots, x_\ell), \ldots, g_N(x_1, \ldots, x_\ell)]$, where
where the \( g_i \)'s are homogeneous polynomials of degree 1 with positive coefficients. By construction, each \( x_i \) occurs in at least one of the \( g_i \)'s.

Since \( X^0_S \) is the closure inside \( X_S \) of \( X^0_S \), any point \([x_1, \ldots, x_\ell]\) of \( X^0_S \) has all \( x_i \)'s non-negative; furthermore, not all of the \( x_i \)'s are equal to 0. And now since the \( g_i \)'s have positive coefficients and they involve all of the \( x_i \)'s, the image of any point \([x_1, \ldots, x_\ell]\) of \( X^0_S \) under \( \tau_{>0} \) is well-defined. Therefore \( \tau_{>0} \) extends continuously to the closure to give a well-defined map \( \tau_{\geq 0} : X^0_S \to \tau_{>0}(X^0_S) \).

\[
\square
\]

5. Construction of a toric variety associated to a parametrization of a cell

We begin by stating a key proposition.

**Proposition 5.1.** Given \( G \) we can construct a positivity preserving embedding \( i : G/B \to \mathbb{P}^N \), for some \( N \) with the following property. For any totally nonnegative cell \( \mathcal{R}^0_{v,w} \) and parameterization \( \phi^0_{v*,w} \) as in Theorem 4.1, the composition

\[
i \circ \phi^0_{v*,w} : (\mathbb{R}_{>0})^{v_*} \xrightarrow{\sim} \mathcal{R}^0_{v,w} \to \mathbb{P}^N
\]

takes the form

\[
i \circ \phi^0_{v*,w} : t = (t_r)_{r \in v_*} \mapsto [p_1(t), \ldots, p_{N+1}(t)],
\]

where the \( p_j \)'s are polynomials with nonnegative coefficients.

This proposition will be proved in the next section. Now assuming Proposition 5.1 is true, we can prove the following.

**Corollary 5.2.** There is a map \( \tau_{>0} : X^0_{v^*,w} \to \mathbb{P}^N \) which extends continuously to the closure to give a well-defined map

\[
\tau_{\geq 0} : X^0_{v^*,w} \to \tau_{>0}(X^0_{v^*,w}).
\]

Moreover we have

\[
\tau_{>0}(X^0_{v^*,w}) = i(\mathcal{R}^0_{v^*,w}) \xrightarrow{\sim} \mathcal{R}^0_{v^*,w}.
\]

The resulting map \( X^0_{v^*,w} \to \mathcal{R}^0_{v^*,w} \) is surjective, an isomorphism on the strictly positive parts, and takes the boundary of \( X^0_{v^*,w} \) to the boundary of \( \mathcal{R}^0_{v^*,w} \).

**Proof.** Let \( S_{v^*,w} \) be the set of all exponent vectors in \( \mathbb{Z}^{n+1} \) which occur among the monomials of the \( p_i \)'s, and let \( P_{v^*,w} \) be the convex hull of the points of \( S_{v^*,w} \). Let \( X_{v^*,w} \) be the toric variety associated with \( P_{v^*,w} \). By Lemma 4.1 the map \( i \circ \phi^0_{v^*,w} \) factors through \( X^0_{v^*,w} \),

\[
(\mathbb{R}_{>0})^{v_*} \xrightarrow{\sim} X^0_{v^*,w} \to \mathcal{R}^0_{v^*,w} \to \mathbb{P}^N,
\]

and we get a map \( \tau_{>0} \) from \( X^0_{v^*,w} \) to \( \mathbb{P}^N \). Moreover, this map extends continuously to a map \( \tau_{\geq 0} \) from \( X^0_{v^*,w} \) to \( \tau_{>0}(X^0_{v^*,w}) \).

Since the map \( i \circ \phi^0_{v^*,w} \) is a homeomorphism onto its image and it factors through \( X^0_{v^*,w} \) as above, the map \( \tau_{>0} \) restricts to a homeomorphism from \( X^0_{v^*,w} \) to \( i(\mathcal{R}^0_{v^*,w}) \). We now claim that that \( \tau_{\geq 0} \) takes the boundary of \( X^0_{v^*,w} \) to the boundary of \( i(\mathcal{R}^0_{v^*,w}) \).
To prove the claim, suppose that there is a point \( x \in \text{bd}(X_{v^+,w}^{\geq 0}) \) such that \( \tau_{\geq 0}(x) = y \) is in the interior of \( i(R_{v^+,w}^{\geq 0}) \). Since \( x \) is in the boundary of \( X_{v^+,w}^{\geq 0} \) we can find a sequence of points \( \{x_i\} \) in \( X_{v^+,w}^{\geq 0} \) which converge to \( x \). Let \( y_i = \tau_{\geq 0}(x_i) \). Since \( g_{\geq 0} \) is a homeomorphism on the interior, each \( y_i \) is in \( i(R_{v^+,w}^{\geq 0}) \), and since \( \tau_{\geq 0} \) is continuous, the sequence of points \( y_i \) converges to \( y \). Let \( g : i(R_{v^+,w}^{\geq 0}) \to X_{v^+,w}^{\geq 0} \) denote the inverse of the restriction of \( \tau_{\geq 0} \) to the interiors. We now have that \( g \) maps the convergent sequence \( \{y_i\} \) in \( i(R_{v^+,w}^{\geq 0}) \) to the divergent sequence \( \{x_i\} \) in \( X_{v^+,w}^{\geq 0} \), which contradicts the continuity of \( g \).

Since \( X_{v^+,w}^{\geq 0} \) is homeomorphic to a closed ball the above corollary provides us with a glueing map for \( R_{v^+,w}^{\geq 0} \).

6. Proof of Proposition 5.1

In this section we will prove Proposition 5.1.

6.1. Simply laced case. We suppose \( G \) is simply laced. Consider the representation \( V = V(\rho) \) of \( G \) with a fixed highest weight vector \( \eta \) and corresponding canonical basis \( B(\rho) \). Let \( i : B \to \mathbb{P}(V) \) denote the embedding which takes \( g \cdot B_+ \in B \) to the line \( \langle g \cdot \eta \rangle \). This is the unique \( g \cdot B_+ \)-stable line in \( V \). We specify points in the projective space \( \mathbb{P}(V) \) using homogeneous coordinates corresponding to \( B(\rho) \).

Now let \( w_0 = (i_1, \ldots, i_N) \) be a fixed reduced expression of \( w_0 \).

**Lemma 6.1.** Let \( v \in W \), and let \( v_+ \) be the positive subexpression for \( v \) in \( w_0 \). The composition

\[
i \circ \phi_{v^+,w_0}^{>0} : (\mathbb{R}^{>0})^{v_+} \to R_{v^+,w_0}^{\geq 0} \to \mathbb{P}(V)
\]

is given by polynomials with positive coefficients.

**Proof.** Consider first a reduced expression of \( w_0 \) which ends in \( v \), i.e. consider a reduced expression \( s_{i_1} \ldots s_{i_k} s_{i_{k+1}} \ldots s_{i_m} \) of \( w_0 \), where \( s_{i_{k+1}} \ldots s_{i_m} \) is a reduced expression of \( v \) (which is clearly positive distinguished). Let \( \tilde{v} = s_{i_{k+1}} \ldots s_{i_m} \). Then \( i \circ \phi_{v^+,w_0}^{>0} \) maps \( (a_1, \ldots, a_k) \) to \( \langle y_{i_1}(a_1) \ldots y_{i_k}(a_k) \tilde{v} \cdot \eta \rangle \). By Lemma 3.3, \( \tilde{v} \cdot \eta \) is a canonical basis vector, and so by Remark 5.4, the coefficients which express \( \langle y_{i_1}(a_1) \ldots y_{i_k}(a_k) \tilde{v} \cdot \eta \rangle \) in terms of the canonical basis are positive polynomials in the \( a_i \)’s.

Now let the reduced expression for \( w_0 \) be arbitrary. This reduced expression can be obtained from our previous reduced expression \( s_{i_1} \ldots s_{i_k} s_{i_{k+1}} \ldots s_{i_m} \) using braid relations, so the results in Section 3.3 imply that the parameterization in question will change by a sequence of substitutions which involve only positive subtraction-free rational expressions. It follows that the image of the map \( i \circ \phi_{v^+,w_0}^{>0} \) is given by rational expressions with positive coefficients. Since we are in projective space, we can clear denominators to obtain polynomials with positive coefficients.

This proves Proposition 5.1 for simply laced \( G \) in the case where \( w = w_0 \). We now turn to the case of arbitrary \( w \).

**Proposition 6.2.** Proposition 5.1 holds when \( G \) is simply laced.
Proof. Choose \( v, w \in W \) with \( v \leq w \). Choose a reduced expression \( w_0 = (i_1, \ldots, i_N) \) for \( w_0 \) such that \( (i_1, \ldots, i_r) \) is a reduced expression for \( w_0 w^{-1} \). Then as in the proof of Lemma 4.3 in [19], we have
\[
\mathcal{R}^{>0}_{v, w_0} = \{ y_{s_1}(t_1) \cdots y_{s_r}(t_r) g_{r+1} \cdots g_N \cdot B^+ \mid g_{r+1} \cdots g_N \cdot B^+ \in \mathcal{R}^{>0}_{v, w_0} \}.
\]
Here each \( g_{r+j} \) is either \( \hat{s}_{i_{r+j}} \) or \( y_{i_{r+j}}(t_{r+j}) \) and the \( g \)'s give a parameterization of \( \mathcal{R}^{>0}_{v, w_0} \). It’s clear that our parameterization of \( \mathcal{R}^{>0}_{v, w_0} \) is obtained from the above parameterization of \( \mathcal{R}^{>0}_{v, w_0} \) by setting \( t_1, \ldots, t_r \) to 0. Since setting certain variables to zero in a positive polynomial results in another positive polynomial, Proposition 6.1 in the simply laced case now follows from Lemma 6.1. \( \square \)

6.2. General type case. Let \( \hat{G} \) be the simply laced group with automorphism corresponding to \( G \), as introduced in Section 2.2. Note that we have already proved Proposition 5.1 for \( \hat{G} \). Explicitly, we considered in Section 6.1 the projective space determined by the \( \rho \)-representation of \( \hat{G} \), that is, \( \mathbb{P}(V(\hat{\rho})) \), with homogeneous coordinates determined by the canonical basis \( B(\hat{\rho}) \). We have seen that for any totally nonnegative cell \( \mathcal{R}^{>0}_{v, w} \) with parameterization \( \hat{\phi}_{v, w} \), the composition \( i \circ \hat{\phi}_{v, w} \) of this parameterization with the usual embedding, \( i : \hat{G}/B^+ \rightarrow \mathbb{P}(V(\hat{\rho})) \), is given by polynomials with positive coefficients.

To treat the general case, we now identify \( G \) with \( \hat{G}^\tau \) and use all of the notation from Section 2.2. For any \( i \in \mathcal{I} \) there is a simple reflection \( s_i \) in \( W \), which is represented in \( \hat{G} \) by
\[
s_i := \prod_{j \in \mathcal{I}} \hat{s}_i.
\]
In this way any reduced expression \( w = (\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_m) \) in \( W \) gives rise to a reduced expression \( \hat{w} \) in \( \hat{W} \) of length \( \sum_{k=1}^{m} |\tilde{i}_k| \), which is determined uniquely up to commuting elements [13, Prop. 3.3]. To a subexpression \( v \) of \( w \) we can then associate a unique subexpression \( \hat{v} \) of \( \hat{w} \) in the obvious way.

Lemma 6.3. Let \( w \) be a reduced expression for \( w \) in \( W \) and \( v \leq w \). If \( v_+ \) is the positive subexpression for \( v \) in \( w \), then \( \hat{v}_+ \) is the positive subexpression for \( v \) (now viewed as element of \( \hat{W} \)) in \( \hat{w} \).

Proof. Let \( w = (\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_m) \) be a reduced expression for \( w \) in \( W \) and let \( \hat{v}_{(j)} \) denote the product (in order) of all simple generators of \( v_+ \), which come from \( (\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_j) \). The fact that \( v_+ \) is positive in \( W \) means that \( v_{(j-1)} < \hat{v}_{(j-1)} \hat{s}_{i_j} \) for all \( j = 1, \ldots, n \). In particular, \( \ell(\hat{v}_{(j-1)} \hat{s}_{i_j}) = \ell(\hat{v}_{(j-1)}) + 1 \) in \( \hat{W} \). For the remainder of this proof let \( \hat{v}_{(j-1)} \) denote the element \( \hat{v}_{(j-1)} \) viewed as an element of \( \hat{W} \). Then by the relationship of lengths in \( W \) versus \( \hat{W} \), \( \ell(\hat{v}_{(j-1)} \prod_{i_j \in \mathcal{I}_j} \hat{s}_{i_j}) = \ell(\hat{v}_{(j-1)}) + |\tilde{i}_j| \) in \( \hat{W} \). This fact together with the fact that the \( \hat{s}_{i_j} \)’s for \( i_j \in \mathcal{I}_j \) commute with each other implies that for any \( i_j \in \mathcal{I}_j \), \( \ell(\hat{v}_{(j-1)} \hat{s}_{i_j}) = \ell(\hat{v}_{(j-1)}) + 1 \) in \( \hat{W} \). Therefore for any \( i_j \in \mathcal{I}_j \), \( \hat{v}_{(j-1)} \hat{s}_{i_j} > \hat{v}_{(j-1)} \). Letting \( i_j = \{i_{j_1}, \ldots, i_{j_r}\} \), this now shows that \( \hat{v}_{(j-1)} \hat{s}_{i_{j_1}} \cdots \hat{s}_{i_{j_k}} > \hat{v}_{(j-1)} \hat{s}_{i_{j_1}} \cdots \hat{s}_{i_{j_k}} \) for all \( 0 \leq k \leq r - 1 \) (note that we are again using the commutativity of the simple generators coming from \( \tilde{i}_j \)). A little thought now shows that \( \hat{v}_+ \) is positive in \( \hat{W} \). \( \square \)

Lemma 6.4. Let \( v, w \in W \) with \( v \leq w \).
THE TOTALLY NONNEGATIVE PART OF $G/P$ IS A CW COMPLEX

(1) We have

$$R_{v,w}^\geq = \hat{R}_{v,w}^\geq \cap \mathcal{B}_\tau.$$ 

In particular the composition $i': R_{v,w} \hookrightarrow \hat{R}_{v,w} \rightarrow \mathbb{P}(V(\hat{\rho}))$ is positivity preserving.

(2) Suppose $w = (\bar{i}_1, \ldots, \bar{i}_m)$ is a reduced expression for $w$ in $W$, and $v_+ = (j_1, \ldots, j_k)$ is the positive subexpression for $v$. Then we have a commutative diagram,

$$\begin{array}{ccc}
R_{v,w}^\geq & \overset{i}{\longrightarrow} & \hat{R}_{v,w}^\geq \\
\phi^\geq_{v_+} & \uparrow & \phi^\geq_{v_+} \\
R_{v_+}^\geq & \overset{i}{\longrightarrow} & \hat{R}_{v_+}^\geq
\end{array}$$

where the top arrow is the usual inclusion, the vertical arrows are both isomorphisms, and the map $i$ has the form

$$(t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_k),$$

where each $t_i$ is repeated $|\bar{i}_{j_i}|$ times on the right hand side.

Proof. (1) We have $B_{\geq 0} = \hat{B}_{\geq 0} \cap \hat{\mathcal{B}}_\tau$ by [9]. Clearly $R_{v,w}^\geq \subset \hat{R}_{v,w}^\geq$. However since $B_{\geq 0} = \bigsqcup_{v,w \in W} R_{v,w}^\geq = \hat{B}_{\geq 0} \cap \hat{\mathcal{B}}_\tau$ it follows that $R_{v,w}^\geq = \hat{R}_{v,w}^\geq \cap \hat{\mathcal{B}}_\tau$.

(2) This is a consequence of Lemma 6.3.

We can now combine the two parts of Lemma 6.4 with Proposition 6.2 to prove Proposition 5.1 for general type $G$.

Proof of Proposition 5.1. Firstly, Proposition 6.2 gives the map

$$i \circ \phi^\geq_{\hat{v}_+} : \mathbb{R}_{\geq 0}^\hat{v}_+ \rightarrow \hat{R}_{v,w}^\geq \overset{i}{\longrightarrow} \mathbb{P}(V(\hat{\rho})).$$

Secondly, for non simply laced $G$ we will use the inclusion to projective space given by $i' : R_{v,w} \hookrightarrow \mathbb{P}(V(\hat{\rho}))$, which is positivity preserving by Lemma 6.4 (1). And thirdly, by Lemma 6.4 (2), we have that

$$i' \circ \phi^\geq_{\hat{v}_+} : \mathbb{R}_{\geq 0}^\hat{v}_+ \rightarrow \hat{R}_{v,w}^\geq \overset{i'}{\longrightarrow} \mathbb{P}(V(\hat{\rho}))$$

can be rewritten as

$$i' \circ \phi^\geq_{\hat{v}_+} = (i \circ \phi^\geq_{\hat{v}_+}) \circ \bar{i}.$$ 

By Proposition 6.2 for $\hat{G}$ and the description of $\bar{i}$ in Lemma 6.4 (2) we see that $i' \circ \phi^\geq_{\hat{v}_+}$ is given by positive polynomials. This proves Proposition 5.1 for $G$.

7. Generalization to partial flag varieties

In this section we generalize the previous results to partial flag varieties.
It is not hard to see that $B_L$ and $B_R$ are uniquely determined as the Borel subgroups in $P$ closest to $B^+$ respectively $B^-$ with regard to their relative position, and the projection maps $R_{x,u,w} \to P_{x,u,w}^J$ and $R_{xu^{-1},w} \to P_{x,u,w}^J$ are isomorphisms. In particular $P_{x,u,w}^J$ is nonempty if and only if $x \leq wu$, in which case it is smooth of dimension $\ell(w) + \ell(u) - \ell(x)$.

Let us denote the indexing set for this decomposition of $P^J$ by $Q^J$. So

$$Q^J := \{(x, u, w) \in W_{\max}^J \times W_J \times W_J^J \mid x \leq wu\}.$$  

### 7.2. Totally nonnegative cells in $P^J$.

The totally positive and nonnegative parts of $P^J$ are defined in [10] by

$$P^J_{\geq 0} = \pi^J(B_{\geq 0}),$$

$$P^J_{\geq 0} = \pi^J(B_{\geq 0}).$$

Since $\pi^J$ is closed it follows that $P^J_{\geq 0} = P^J_{\geq 0}$.

We decompose $P^J_{\geq 0}$ by intersecting it with the strata $P^J_{x,u,w}$ from Section [11].

From the definitions and the fact that reduction preserves total positivity it follows that (17 Lemma 3.2)

$$P^J_{x,u,w;\geq 0} := P^J_{x,u,w} \cap P^J_{\geq 0} = \pi^J(R_{x,u,w}^{\geq 0}) = \pi^J(R_{xu^{-1},w}^{\geq 0}).$$

Keeping in mind that $\pi^J : R_{x,u,w} \to P^J_{x,u,w}$, say, is an isomorphism, we see that

$$P^J_{x,u,w;\geq 0} \cong R_{x,u,w}^{\geq 0} \cong R_{\geq 0}^{\ell(w) + \ell(u) - \ell(x)},$$

for any triple $(x, u, w) \in Q^J$.

**Theorem 7.1.** [11] Let $(x, u, w) \in Q^J$. Then $P^J_{x,u,w;\geq 0}$ is the disjoint union of $P^J_{x,u,w;\geq 0}$ and some lower-dimensional cells.

### 7.3. Proof of Theorem 7.1.

Finally we are ready to prove Theorem 7.1. Let us recall the notion of a CW complex.

In a CW complex $X$, a cell is attached by glueing a closed $i$-dimensional ball $D^i$ to the $(i-1)$-skeleton $X_{i-1}$, i.e. the union of all lower dimensional cells. The gluing is specified by a continuous function $f$ from $\partial D^i = S^{i-1}$ to $X_{i-1}$. CW complexes are defined inductively as follows. Given $X_0$ a discrete space (a discrete union of 0-cells), we inductively construct $X_i$ from $X_{i-1}$ by attaching some collection of $i$-cells. The resulting colimit space $X$ is called a CW complex provided it is given the weak topology and the closure-finite condition is satisfied for its closed cells. Recall that the closure-finite condition requires that every closed cell is covered by a finite union of open cells.
Proof. The cell decomposition of $P_{\geq 0}$ has only finitely many cells; therefore the closure-finite condition in the definition of a CW complex is automatically satisfied. What we need to do is define the attaching maps for the cells.

Recall from Corollary 5.2 that for each parameterization of a cell $R_{v,w,0}$ of $B_{\geq 0}$ we have a toric variety $X_{v,w}$ and a map $\tau_{\geq 0} : X_{v,w}^0 \to R_{v,w,0}$ which is surjective, an isomorphism on the strictly positive parts, and which takes the boundary of $X_{v,w}^0$ to the boundary of $R_{v,w,0}$. Since $X_{v,w}^0$ is homeomorphic to a closed ball, this provides a glueing map for the cell $R_{v,w,0}$ of $B_{\geq 0}$.

To construct the glueing map for the cell $P_{x,u,w;\geq 0}$ of $P_{\geq 0}$, we compose the map $\pi^J$ from $B_{\geq 0}$ to $P_{\geq 0}$ with $\tau_{\geq 0}$. Since $\pi^J : R_{x,u,w} \to P_{x,u,w;\geq 0}$ is a homeomorphism, we obtain a map $\pi^J \circ \tau_{\geq 0}$ from $X_{x,u,w,u}^0 \to P_{x,u,w;\geq 0}$ which again is surjective and an isomorphism on the strictly positive parts.

Now as in the proof of Corollary 5.2, it must take the boundary of $X_{x,u,w,u}^0$ to the boundary of $P_{x,u,w;\geq 0}$. Therefore the composition $\tau_{\geq 0} \circ \pi^J$ provides a glueing map for cells of $P_{\geq 0}$.

The only thing that remains to check is that the boundary of each $i$-cell is mapped to the $i-1$-skeleton. But this follows from Theorem 7.1.

As explained in the Introduction, Theorem 1.1 together with Theorem 1.2 from [22] implies the following result.

**Corollary 7.2.** The Euler characteristic of the closure of a cell of $(G/P)_{\geq 0}$ is 1.

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