WELL/ILL-POSEDNESS BIFURCATION FOR THE BOLTZMANN EQUATION WITH CONSTANT COLLISION KERNEL

XUWEN CHEN AND JUSTIN HOLMER

Abstract. We consider the 3D Boltzmann equation with the constant collision kernel. We investigate the well/ill-posedness problem using the methods from nonlinear dispersive PDEs. We construct a family of special solutions, which are neither near equilibrium nor self-similar, to the equation, and prove that the well/ill-posedness threshold in $H^s$ Sobolev space is exactly at regularity $s = \frac{1}{2}$, despite the fact that the equation is scale invariant at $s = \frac{1}{2}$.

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1. Introduction

The fundamental Boltzmann equation describes the time-evolution of the statistical behavior of a thermodynamic system away from a state of equilibrium in the mesoscopic regime, accounting for both dispersion under the free flow and dissipation as the result of collisions. So far, the well-posedness of the Boltzmann equation remains largely open even after the innovative work [44, 62], while we are not aware of any ill-posedness results. The goal of this paper is to investigate the fine properties of well-posedness versus ill-posedness of the Boltzmann equation via a scaling point of view, using the latest techniques from the field of nonlinear dispersive PDEs.

Trying out techniques which work for nonlinear dispersive PDEs on the Boltzmann equation is not without precursors. For years, there have been many nice developments [1, 2, 4, 5, 45, 46, 48, 61, 63, 69, 71, 73] which have hinted at or used space-time estimates like the Chemin-Lerner spaces or the harmonic analysis related to nonlinear dispersive PDEs, and many of them have reached global (strong and mild) solutions if the datum is close enough to the Maxwellians or satisfies some conditions. In the same period of time, the theory of nonlinear dispersive PDEs has matured into a stage on which the working function spaces have been cleanly unified, the well-posedness and ill-posedness (See, for example, the now well-known work [41, 42, 52, 54, 59, 60] and also the survey [72] and the references within.) away from the scale invariant spaces have been mostly

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settled, and global well-posedness for large solutions at critical scaling has made significant progress. Of course, the Boltzmann equation is certainly very different from the nonlinear dispersive PDEs and the nuts and bolts designed for one, so far, do not fit the other. Interestingly, the recent series of papers \cite{11-13} by T. Chen, Denlinger, and Pavlović suggests that a systematic study of the Boltzmann equation using tools built for the dispersive PDEs might indeed be possible.

It is well-known that, the Wigner transform turns the kinetic transport operator $\partial_t + v \cdot \nabla_x$ into the hyperbolic Schrödinger operator

\begin{equation}
  i\partial_t + \Delta_y - \Delta_{y'},
\end{equation}

and hence the Wigner transform turns the Boltzmann equation into a dispersive equation with \eqref{1.1} being the linear part. Technical problems remain (if they do not get worse) though, since the form of the collision operators does not improve and derivatives are complicated under the Wigner transform while the analysis of \eqref{1.1} was not very developed for a long time. In \cite{11-13}, incorporating the new developments \cite{14-34,36,39,40,47,49,50,55,58,66-68} on the quantum many-body hierarchy dynamics that rely on the analysis of \eqref{1.1}, with some highly nontrivial technical improvisions, T. Chen, Denlinger, and Pavlović provided an alternate dispersive PDE based route for proving the local well-posedness of the Boltzmann equation. The solutions provided in \cite{11-13} differ from previous work in the sense that they solve the Boltzmann equation a.e. instead of everywhere in time. But these solutions are by no means weak, as the datum-to-solution map is Lipschitz continuous and provides persistence of regularity.

We follow the lead of \cite{11-13}, but we study ill-posedness instead of well-posedness in this paper (and we are not using the Wigner transform). It is of mathematical and physical interest to prove well-posedness at the “optimal” regularity as it would mean all the nonlinear interactions of the complicated underlying nonlinear equation have been analyzed and accounted for. Thus knowing in advance where such “optimality” lies is instructive (if not important). We prove that, the 1st guess that the well/ill-posedness split at the critical scaling or the 1st expectation that self-similar solutions would provide the bad solutions, are in fact wrong, and the ill-posedness starts to happen in the scaling subcritical regime. Therefore ordinary perturbative methods in proving well-posedness fail here, and we have to address the full nonlinear equation with new ideas and we need our estimates to be as sharp as possible.

As we are using dispersive and harmonic analysis techniques, the constant collision kernel case would be the logical first-go-to context as it has a particularly clean form for the loss operator in Fourier variables and it would also clarify the role of the dispersive techniques for future references as there is not a (partial) convolution or angular integral in the loss term to generate smoothing. Denoting by $f(t, x, v)$ the phase-space density, we consider here the 3D Boltzmann equation with constant collision kernel without boundary:

\begin{equation}
  \partial_t f + v \cdot \nabla_x f = \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} \left[ f(u^*)f(v^*) - f(u)f(v) \right] du d\omega \text{ in } \mathbb{R}^{1+6}.
\end{equation}

The variables $u, v$ can be regarded as incoming velocities for a pair of particles, $\omega \in \mathbb{S}^2$ is a parameter for the deflection angle in the collision of these particles, and the outgoing velocities are $u^*, v^*$:

\[
  u^* = u + [\omega \cdot (v - u)]\omega \text{ and } v^* = v - [\omega \cdot (v - u)]\omega
\]

\footnote{Our $X_{s,b}$ method in fact does not require the Fourier transform of the collision kernel. See, for example, Appendix B.}
We adopt the usual gain term and loss term shorthands
\[
Q(f, g) = Q^{+}(f, g) - Q^{-}(f, g)
\]
\[
Q^{+}(f, g) = \int_{\mathbb{S}^{2}} \int_{\mathbb{R}^{3}} f(v^{*})g(u^{*}) \, du \, d\omega
\]
\[
Q^{-}(f, g) = \int_{\mathbb{S}^{2}} \int_{\mathbb{R}^{3}} f(v)g(u) \, du \, d\omega = 4\pi f(0) \int_{\mathbb{R}^{3}} g(u) \, du
\]
and equation (1.2) is invariant under the scaling
\[
f_{\lambda}(t, x, v) = \lambda^{\alpha+2\beta} f(\lambda^{\alpha}\xi, \lambda^{\alpha}x, \lambda^{\beta}v)
\]
for any \(\alpha, \beta \in \mathbb{R}\) and \(\lambda > 0\).

To draw a more direct connection between the analysis of (1.2) and the theory of nonlinear dispersive PDEs, we start with the linear part of (1.2), which, upon passing to the inverse Fourier variable \(v \mapsto \xi\), is the symmetric hyperbolic Schrödinger equation
\[
 i\partial_{t} \hat{f} + \nabla_{\xi} \cdot \nabla_{x} \hat{f} = 0,
\]
where \(\hat{f}(t, x, \xi)\) is the inverse Fourier transform in the third (velocity) variable only. Evolution of initial condition \(\phi(x, \xi)\) along (1.4) will be denoted as \(\hat{f} = e^{it\nabla_{\xi} \cdot \nabla_{x}} \phi\). Solutions to (1.4) automatically satisfy Strichartz estimates, as we shall review in Appendix A, that
\[
\|\hat{f}\|_{L_{t}^{r}L_{x,\xi}^{p}} \lesssim \|\hat{f}|_{t=0}\|_{L_{x,\xi}^{2}}, \text{ provided } \frac{2}{q} + \frac{6}{p} = 3, q \geq 2
\]
Thus one considers the Sobolev norms defined by
\[
\|\hat{f}\|_{H_{x,\xi}^{s}} = \|\langle \nabla_{x}\rangle^{s} \langle \nabla_{\xi}\rangle^{r} \hat{f}\|_{L_{x,\xi}^{2}} = \|\langle \nabla_{x}\rangle^{s} \langle \nabla_{\xi}\rangle^{r} f\|_{L_{x,v}^{2}} = \|f\|_{L_{v}^{2,s}H_{x}^{r}}.
\]
However, not only is it instinctive to require the same regularity of \(x\) and \(\xi\) due to the symmetry in \(x\) and \(\xi\) in (1.4), the scaling invariance (1.3) of (1.2) also suggests\(^2\) that it is natural to take \(s = r\) and seek to reach the scaling-invariant critical level\(^3\) of \(s = \frac{1}{4}\), even though the nonlinear part of (1.2) is not symmetric in \(x\) and \(\xi\). Thus we will use the Sobolev norm
\[
\|\hat{f}\|_{H_{x,\xi}^{s}} = \|\langle \nabla_{x}\rangle^{s} \langle \nabla_{\xi}\rangle^{s} \hat{f}\|_{L_{x,\xi}^{2}} = \|\langle \nabla_{x}\rangle^{s} \langle \nabla_{\xi}\rangle^{s} f\|_{L_{x,v}^{2}} = \|f\|_{L_{v}^{2,s}H_{x}^{s}}
\]
and we say\(^2\) (1.2) is \(H_{x,\xi}^{\frac{1}{4}}\)-invariant or \(L_{v}^{2,\frac{1}{4}}H_{x}^{-\frac{1}{2}}\)-invariant (in scale).

The focus of the paper is on estimates in the Fourier restriction norm spaces as in \([6,8,57,65]\), directly associated with the \(H_{x,\xi}^{s}\) space and the propagator \(e^{it\nabla_{\xi} \cdot \nabla_{x}}\), that we now define. We will denote by \(\eta\) the Fourier dual variable of \(x\) and by \(v\) the Fourier dual variable of \(\xi\). The function \(\hat{f}(\eta, v)\) denotes the Fourier transform of \(\hat{f}(x, \xi)\) in both \(x \mapsto \eta\) and \(\xi \mapsto v\), and is thus the Fourier transform of \(f(x, v)\) itself in only \(x \mapsto \eta\). The Fourier restriction norm spaces (or \(X\) spaces) associated with equation (1.4) are
\[
\|\hat{f}\|_{X_{x,\xi}^{s,b}} = \|\hat{f}(\tau, \eta, v) \langle \tau + \eta \cdot v \rangle^{b} \langle \eta \rangle^{s} \langle v \rangle^{s}\|_{L_{\tau,\eta,v}^{2}}
\]
and it is customary to define their finite time restrictions via
\[
\|\hat{f}\|_{X_{x,\xi}^{s,b}(T)} = \inf \left\{ \|F\|_{X_{x,\xi}^{s,b}} : F|_{[-T,T]} = \hat{f} \right\}.
\]
\(^2\)See a discussion in Appendix A.
\(^3\)Instead of based on scaling invariance of equation, some people define the critical regularity for the Boltzmann equation at \(s = \frac{3}{4}\), the continuity threshold.
The form of the gain and loss operators in the \((x, \xi)\) variables are:

\[
\begin{align*}
\tilde{Q}^+ \left( \tilde{f}, \tilde{g} \right)(\xi) &= \int_{\mathbb{R}^2} \tilde{f}(\xi^+)\tilde{g}(\xi^-)d\omega, \\
\tilde{Q}^- \left( \tilde{f}, \tilde{g} \right)(\xi) &= \tilde{f}(\xi)\tilde{g}(0)
\end{align*}
\]

where \(\xi^+ = \frac{1}{2}(\xi + |\xi|\omega)\) and \(\xi^- = \frac{1}{2}(\xi - |\xi|\omega)\), by the well-known Bobylev identity \(^7\).

As there are the \((x, v)\) and \((x, \xi)\) sides of (1.2) in this paper, to make things clear, we recall the definition of a strong solution and well-posedness.

**Definition 1.1.** We say \(f(t, x, v)\) is a strong \(L^2_v^s H^s_x\) or \(H^s_{x,\xi}\) solution to (1.2) on \([-T, T]\) if \(\tilde{f} \in X^T_{s, \frac{1}{2}+}\) (in particular, \(\tilde{f} \in C([-T, T], H^s_{x,\xi})\) and \(f \in C([-T, T], L^2_v H^s_x)\)) and satisfies

\[
(i \partial_t + \nabla_x \cdot \nabla_x) \tilde{f} = \tilde{Q} \left( \tilde{f}, \tilde{f} \right),
\]

in which both sides are well-defined as \(X^T_{s, -\frac{1}{2}+}\) functions (in particular, \(L^1_{[-T, T]} H^s_{x,\xi}\) or \(L^1_{[-T, T]} L^2_v H^s_x\) functions).

**Definition 1.2.** We say (1.2) is well-posed in \(H^s_{x,\xi}\) or \(L^2_v^s H^s_x\) if for each \(R > 0\), there exists a time \(T = T(R) > 0\), such that all of the following are satisfied.

(a) (Existence) For each \(f_0 \in L^2_v^s H^s_x\) with \(\|f_0\|_{L^2_v^s H^s_x} \leq R\). There is a \(f(t, x, v)\) such that \(f(t, x, v)\) is a strong \(L^2_v^s H^s_x\) or \(H^s_{x,\xi}\) solution to (1.2) on \([-T, T]\). Moreover, \(f(t, x, v) \equiv 0\) if \(f_0 \equiv 0\).

(b) (Conditional Uniqueness) Suppose \(f\) and \(g\) are two strong \(L^2_v^s H^s_x\) or \(H^s_{x,\xi}\) solutions to (1.2) on \([-T, T]\) with \(f|_{t=0} = g|_{t=0}\), then

\[
f - g = \tilde{f} - \tilde{g} = 0 \text{ on } [-T, T]
\]

as \(H^s_{x,\xi}\) or \(L^2_v^s H^s_x\) functions.

(c) (Uniform Continuity of the Solution Map) Suppose \(f\) and \(g\) are two strong \(L^2_v^s H^s_x\) or \(H^s_{x,\xi}\) solutions to (1.2) on \([-T, T]\), \(\forall \varepsilon > 0\), there exists \(\delta(\varepsilon)\) independent of \(f\) or \(g\) such that

\[
\|f(t) - g(t)\|_{C([-T, T], L^2_v^s H^s_x)} < \varepsilon \text{ provided that } \|f(0) - g(0)\|_{L^2_v^s H^s_x} < \delta(\varepsilon)
\]

In the context defined above, we have exactly found the separating index between well/ill-posedness for (1.2) to be \(s = 1\) though (1.2) is actually \(H^1_{x,\xi}\)-invariant or \(L^2_v^1 H^1_x\)-invariant (in scale).

**Theorem 1.3** (Main Theorem – Well-ill-posedness).

1. Equation (1.2) is locally well-posed in \(H^s_{x,\xi}\) or \(L^2_v^s H^s_x\) for \(s > 1\).
2. Equation (1.2) is ill-posed in \(H^s_{x,\xi}\) or \(L^2_v^s H^s_x\) for \(\frac{1}{2} < s < 1\) in the following senses:

   2a) The data-to-solution map is not uniformly continuous and hence (c) or (1.11) in Definition 1.2 is violated. In particular, given any \(t_0 \in \mathbb{R}\), for each \(M \gg 1\), there exists a time sequence \(\{t^M_0\}_M\) such that \(t^M_0 < t_0\) and two solutions \(f^M(t), g^M(t)\) to equation (1.2) in \([t^M_0, t_0]\) with \(\|f^M(t^M_0)\|_{L^2_v^s H^s_x}, \|g^M(t^M_0)\|_{L^2_v^s H^s_x} \sim 1\), such that \(f^M(t), g^M(t)\) are initially close at \(t = t^M_0\)

\[
\|f^M(t^M_0) - g^M(t^M_0)\|_{L^2_v^s H^s_x} \leq \frac{1}{\ln M} \ll 1,
\]

\(^4\)One could replace (c) with the Lipschitz continuity which is usually the case as well.
but become fully separated at $t = t_0$

$$\|f^M(t_0) - g^M(t_0)\|_{L^2_0 s H^s_x} \sim 1.$$ 

2b) Moreover, there exists a family of solutions having norm deflation forward in time (and hence norm inflation backward in time). In particular, given any $t_0 \in \mathbb{R}$, for each $M \gg 1$, there exists a solution $f_M$ to equation (1.2) and a $T_0 = T_0(M) < t_0$ such that

$$\|f_M(t_0)\|_{L^2_0 s H^s_x} \sim \frac{1}{\ln M} < 1 \text{ but } \|f_M(T_0)\|_{L^2_0 s H^s_x} \sim \frac{M^\delta}{\ln M} \gg 1.$$ 

The novelty of Theorem 1.3 is certainly the ill-posedness / part (b). Part (a) is essentially included in [11,12] already and is stated and proved here with different estimates, namely (1.13) and (1.15).

Due to the usual embedding $H^{s+} \hookrightarrow B^s_{2,1} \hookrightarrow H^s$, Theorem 1.3 also proves the well/ill-posedness separation at exactly $s = 1$ for the Besov spaces (like the ones in, for example, [50,69]). The “bad” solutions we found are not the usually expected self-similar solutions, they are actually implosion / cavity like solutions. See Figure 1 for an illustration.

Figure 1. Illustration at time $t < 0$ and time $t = 0$ of the $x$-support of $f_b(x, v, t)$ in blue and $f_l(x, v, t)$ in red. The function $f_b$ consists of $\sim (MN_2)^2$ tubes; here three typical tubes are depicted. Each tube moves in the direction of its long dimension. The ill-posedness part of Theorem 1.3 exploits solutions of the form is $f = f_b + f_l + f_c$, where $f_c$ is a small correction. As time evolves forward, the loss term between $f_b$ and $f_l$ drives the amplitude of $f_l$ downward exponentially fast.

Moreover, this family of “bad” solutions is uniformly bounded in mass, variance and kinetic energy. That is, one would still conclude ill-posedness in $L^2_0 s H^s_x$ via Theorem 1.3 even if one assume uniformly bounded mass, variance and kinetic energy. (See Remark 5.2.)

Per Theorem 1.3 these implosion / cavity like solutions are obstacles to the well-posedness for $s < 1$ while they do not pose any problems for $s > 1$. We wonder if this phenomenon is related to the fact that, though more difficult to solve, the Boltzmann equation is preferred over its various
There exist functions \( \theta(t) \) such that \( \theta(t) = 1 \) on \(-1 \leq t \leq 1\) and \( \text{supp} \theta \subset [-2, 2] \). For \( \tilde{f} \) supported in the dyadic regions \(|\eta_1| \sim M_1 \geq 1\), \(|v_1| \sim N \geq 1\), and \( \tilde{g} \) supported in the dyadic regions \(|\eta_2| \sim M_2 \geq 1\), \(|v_2| \sim N_2 \geq 1\),

\[
\|\theta(t)\hat{Q}^{-}(\tilde{f}, \tilde{g})\|_{X_{0,-\frac{1}{2}+}} \lesssim \min(M_1, M_2) N_2 B_{M_1,M_2} B_{N,N_2} \|\tilde{f}\|_{X_{0,\frac{1}{2}+}} \|\tilde{g}\|_{X_{0,\frac{1}{2}+}},
\]

where \( B_{M_1,M_2} \) and \( B_{N,N_2} \) are the one-sided bilinear gain factors

\[
B_{M_1,M_2} = \begin{cases} \left( \frac{M_1}{M_2} \right)^{1/2} & \text{if } M_1 \leq M_2, \\ 1 & \text{if } M_1 > M_2 \end{cases}, \quad B_{N,N_2} = \begin{cases} \left( \frac{N_2}{N} \right)^{1/2} & \text{if } N_2 \leq N, \\ 1 & \text{if } N_2 > N \end{cases}
\]

It follows that for any \( s > 1 \), for any \( f \) and \( g \) (without frequency support restrictions),

\[
\|\theta(t)\hat{Q}^{-}(\tilde{f}, \tilde{g})\|_{X_{s,-\frac{1}{2}+}} \lesssim \|\tilde{f}\|_{X_{s,\frac{1}{2}+}} \|\tilde{g}\|_{X_{s,\frac{1}{2}+}}
\]

There exist functions \( f \) and \( g \) with \( \tilde{f} \) supported in the dyadic regions \(|\eta_1| \sim M_1\), \(|v_1| \sim N\), and \( \tilde{g} \) supported in the dyadic regions \(|\eta_2| \sim M_2\), \(|v_2| \sim N_2\), such that

\[
\max(N,1) < N_2 \quad M_1 \geq 1, \quad M_2 \geq 1
\]

and

\[
\|\theta(t)\hat{Q}^{-}(\tilde{f}, \tilde{g})\|_{X_{0,-b'}} \gtrsim \min(M_1, M_2) N_2 B_{M_1,M_2} \|\tilde{f}\|_{X_{0,b_1}} \|\tilde{g}\|_{X_{0,b_2}}
\]

for all \( b', b_1, b_2 \in \mathbb{R} \).

We emphasize that the bilinear gain factors, typically present in these types of estimates, are both only one-sided in this case. This comes from the fact that the loss operator \( Q^{-} \) is highly nonsymmetric in \( f \) and \( g \), and not of convolution type in the variable \( v \). Estimate (1.13), which is (1.12) without the bilinear gain factors, can be proved rather quickly using the Strichartz estimates, but to see the “bad” solutions, one has to study (1.12) and how it becomes saturated.

In contrast to the loss term, the gain term has bilinear estimates which almost match the critical scaling \( s = \frac{1}{2} \).

**Theorem 1.5.** Let \( \theta(t) \) be a smooth function such that \( \theta(t) = 1 \) on \(-1 \leq t \leq 1\) and \( \text{supp} \theta \subset [-2, 2] \). For all \( s > \frac{1}{2} \), for any \( f \) and \( g \) (without frequency support restrictions),

\[
\|\theta(t)\hat{Q}^{+}(\tilde{f}, \tilde{g})\|_{X_{s,-\frac{1}{2}+}} \lesssim \|\tilde{f}\|_{X_{s,\frac{1}{2}+}} \|\tilde{g}\|_{X_{s,\frac{1}{2}+}}
\]

The \( s = \frac{1}{2} \) case of (1.15) might also be correct, but it cannot change the ill-posedness facts even if it is, due to the loss term. That is, in low regularity settings, the gain term is like an error, compared to the loss term. Or in other words, the long expected cancelation between the gain and loss terms does not happen. Such a disparity between the gain term and the loss term ultimately creates Theorem 1.3.
1.1. **Organization of the Paper.** The novelty of this paper lies in Theorem [1.3] which establishes the well/ill-posedness separation and proves the ill-posedness. Because the ill-posedness happens at scaling subcritical regime, ordinary scaling or perturbative methods for scaling critical or supercritical regimes fail here. We thus need new ideas addressing the full nonlinear equation. The 1st step would be getting sharp estimates.

In §2 we prove (1.13) in Theorem [1.4] by appealing to the Strichartz estimates (reviewed in Appendix [A]) and prove the sharpness of (1.12) from which the illposedness originates. In §3 we prove the gain term estimates in Theorem [1.5] by appealing to a Hölder type estimate in [3] together with a Littlewood-Paley decomposition. For completeness, we apply in the short section §4 the loss term estimate in Theorem [1.4] and the gain term estimate in Theorem [1.5] to prove the well-posedness part of Theorem [1.3].

We prove the illposedness part of Theorem [1.3] in §5. We work on the more usual \((x,v)\) side of (1.2) throughout §5 but it is still very much based on \(X_{a,b}\) theory. As the proof involves many delicate computations, we 1st provide a heuristic using the sharpness example of (1.12), suitably scaled, in §5.1 then we present the approximate solutions / ansatz \(f_a\) we are going to use in §5.2. We use a specialized perturbative argument, built around \(f_a\), to prove that there is an exact solution to (1.2) which is mostly \(f_a\). We put down the tools in §5.3 to §5.5 to conclude the perturbative argument in §5.6. During the estimate of the error terms, the proof involves many geometric techniques, like the ones in \(X_{a,b}\) theory (See, for example, [13,56]), on the fine nonlinear interactions. A good example would be §5.4.4. Such a connection between the analysis of (1.2) and the dispersive equations might be highly nontrivial and deserves further investigations. We then conclude the ill-posedness and the norm deflation in Corollaries [5.6] and [5.7]. Our method is general and can be pushed further: see [37] for a generalization to well/ill-posedness separation with general kernels, and then [38] for a sharp global well-posedness developed from this paper. Moreover, the derivation of the Boltzmann equation is also related to the well/ill-posedness threshold we found in the sense of [35].

After the proof of the main theorem in §5 we include in Appendix [A] a discussion of the scaling properties of (1.2) and review the Strichartz estimates, and in Appendix [B] a full proof of a more manageable version of (1.12) capturing its salient features. We have not used (1.12) in this paper but its sharpness has motivated the construction of the example generating the ill-posedness. Some additional comments are included in the arxiv.org version of this paper.

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2. **Loss term bilinear estimate and its sharpness**

In this section, we prove (1.13), the simpler version of the full optimal bilinear estimate (1.12), and establish the sharpness of (1.12). The proof of (1.12) is highly technical as it needs a lengthy

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5 We end up simplifying the well-posedness estimates in [11,12] a little bit and reducing the regularity requirement for the gain term, as co-products.

6 Both of the examples in §2.2 and §5.2 maximize (1.12). The one in §2.2 is more straight forward for the purpose of maximizing (1.12).
We can describe this as the cone with vertex $e$ parallel to the long side. Applying Hölder, Strichartz (1.5), we have

$$\eta$$

for $s > 1$. Recall (1.9),

$$\left\| \tilde{Q}^{-}(\tilde{f}, \tilde{g}) \right\|_{L^2_t H^s_{y\xi}} \approx \left\| \left( \langle \nabla_x \rangle^s \langle \nabla_x \rangle^s \tilde{f}(t, x, \xi) \right) \tilde{g}(t, x, 0) \right\|_{L^2_t L^2_{x\xi}}$$

Splitting into the two cases in which $\tilde{f}$ or $\tilde{g}$ has the dominating frequency, we have

$$\left\| \tilde{Q}^{-}(\tilde{f}, \tilde{g}) \right\|_{L^2_t H^s_{y\xi}} \approx \left\| \left( \langle \nabla_x \rangle^s \langle \nabla_x \rangle^s \tilde{f}(t, x, \xi) \right) \tilde{g}(t, x, 0) \right\|_{L^2_t L^2_{x\xi}}$$

Apply Hölder,

$$\left\| \tilde{Q}^{-}(\tilde{f}, \tilde{g}) \right\|_{L^2_t H^s_{y\xi}} \approx \left\| \langle \nabla_x \rangle^s \langle \nabla_x \rangle^s \tilde{f}(t, x, \xi) \right\|_{L^2_t L^2_{x\xi}} \left\| \tilde{g}(t, x, \xi) \right\|_{L^2_t L^2_{x\xi}}$$

Use $s > 1$ and apply Sobolev,

$$\left\| \tilde{Q}^{-}(\tilde{f}, \tilde{g}) \right\|_{L^2_t H^s_{y\xi}} \approx \left\| \langle \nabla_x \rangle^s \langle \nabla_x \rangle^s \tilde{f}(t, x, \xi) \right\|_{L^2_t L^2_{x\xi}} \left\| \langle \nabla_x \rangle^s \langle \nabla_x \rangle^s \tilde{g}(t, x, \xi) \right\|_{L^2_t L^2_{x\xi}}$$

Apply Strichartz (1.5), we have

$$\left\| \tilde{Q}^{-}(\tilde{f}, \tilde{g}) \right\|_{L^2_t H^s_{y\xi}} \approx \left\| \tilde{f} \right\|_{X_{s, \frac{1}{2}+}} \left\| \tilde{g} \right\|_{X_{s, \frac{1}{2}+}}$$

as claimed.

2.2. Proof of Sharpness. At this point, we turn to the sharpness example. We will consider $|\eta_2| \sim M_2$ and $|v_2| \sim N_2$, with $M_2 \gg 1$ and $N_2 \gg 1$ dyadic. On the unit sphere, lay down a grid of $J \sim M_2^2 N_2^2$ points $\{e_j\}_{j=1}^J$, where the points $e_j$ are roughly equally spaced and each have their own neighborhood of unit-sphere surface area $\sim M_2^{-1} N_2^{-1}$. Let $P_{e_j}$ denote the orthogonal projection onto the 1D subspace spanned by $e_j$. Let $P_{e_j}^\perp$ denote the orthogonal projection onto the 2D subspace span$\{e_j\}^\perp$.

For each $e_j$, let $B_j$ denote, in $v_2$ space, the conic neighborhood of $e_j$ obtained by taking all radial rays passing through the patch of surface area $M_2^{-1} N_2^{-1}$ on the unit sphere around the vector $e_j$. We can describe this as the cone with vertex $e_j$ and angular aperture $M_2^{-1} N_2^{-1}$. When this cone, in $v_2$-space, intersects the dyadic annulus at radius $N_2$, it creates a shape that is approximately a cube with long side $N_2$ and two shorter dimensions each of length $M_2^{-1}$. For convenience we think of it as a cube with shape $M_2^{-1} \times M_2^{-1} \times N_2$, with the line through $e_j$ passing through the axis and parallel to the long side.
For each \( e_j \), in \( \eta_2 \) space, let \( A_j \) consist of all vectors \( \eta_2 \) that are “nearly perpendicular” to \( e_j \), in the sense that the cosine of the angle between \( e_j \) and \( \eta_2/|\eta_2| \) is

\[
\cos(e_j, \frac{\eta_2}{|\eta_2|}) \leq \frac{1}{M_2 N_2}
\]

Since \( \sin(90^\circ - \theta) = \cos(\theta) \), this means that the angle between \( e_j \) and any vector \( \eta_2 \in A_j \) is within \( M_2^{-1}N_2^{-1} \) of 90 degrees. This property then clearly extends to replacing \( e_j \) with any vector in \( B_j \). We can then visualize \( B_j \), projected onto the unit sphere, as being obtained by taking the plane perpendicular to \( e_j \), calling that the “equator”, and then taking the band that is width \( M_2^{-1}N_2^{-1} \) around that equator. When \( A_j \) is intersected with the dyadic annulus at radius \( \sim M_2 \), this set looks like a thickened plane of width \( N_2^{-1} \), so it has approximate dimensions \( M_2 \times M_2 \times N_2^{-1} \), where the \( M_2 \times M_2 \) planar part is perpendicular to \( e_j \). \( A_j \) looks like the so-called bevelled washer.

The example proving (1.14) in Theorem 1.4 is produced as follows. Let \( \hat{\chi}(\eta) \) (or \( \hat{\chi}(v) \)) be a smooth nonnegative compactly supported function such that \( \hat{\chi}(0) = 1 \). We present the functions \( \hat{\phi} \) and \( \hat{\psi} \) as normalized in \( L^2_x L^2_\xi \).

\[
\hat{\phi}(\eta_1, v) = \frac{1}{M_1^{3/2}N^{3/2}} \hat{\chi}(\frac{\eta_1}{M_1}) \hat{\chi}(\frac{v}{N})
\]

\[
\hat{\psi}(\eta_2, v_2) = \frac{1}{M_2 N_2} \sum_{j=1}^J \hat{\chi}(\frac{P_{e_j} \eta_2}{M_2}) \hat{\chi}(N_2 P_{e_j} \eta_2) \hat{\chi}(M_2 P_{e_j} v_2) \hat{\chi}(\frac{P_{e_j} v_2}{N_2})
\]

where \( J \sim M_2^2 N_2^2 \). Since we are proving a lower bound, we can select any normalized dual function \( \hat{\zeta} \in L^2_x L^2_\xi \). We select

\[
\hat{\zeta}(\eta, v) = \frac{1}{\max(M_1, M_2)^{3/2}N^{3/2}} \hat{\chi}(\frac{\eta}{\max(M_1, M_2)}) \hat{\chi}(\frac{v}{N})
\]

Then let

\[
I = \int_{\eta} \int_{\eta_2} \int_v \int_{v_2} \hat{\theta}(\eta - \eta_2, v - v_2) \hat{\phi}(\eta - \eta_2, v) \hat{\psi}(\eta_2, v_2) \hat{\zeta}(\eta, v) \, dv \, dv \, d\eta_2 \, d\eta
\]

and we aim to prove

\[
I \geq \min(M_1, M_2) N_2 B_{M_1 M_2} \| \hat{\phi} \|_{L^2_{\eta_2 v_2}} \| \hat{\psi} \|_{L^2_{\eta_2 v_2}} \| \hat{\zeta} \|_{L^2_{\eta_2 v_2}}
\]

We are restricting to \( M_1 \geq 1, M_2 \geq 1 \) and \( N \leq M^{-1} \ll N_2 \) but otherwise do not assume any relationship between \( M_1 \) and \( M_2 \). The argument can be extended to weaken the restriction on \( N \) only \( N \ll N_2 \).

To carry out the computation, we need to introduce \( C_k \), subsets of \( \eta_2 \) space, which are cone-like with angular aperture \( M_2^{-1}N_2^{-1} \) with vertex vector \( e_k \). (That is, \( C_k \) for \( k = 1, \ldots, M_2^2 N_2^2 \) is the same type of decomposition of \( \eta_2 \) space as the decomposition \( B_j \), \( j = 1, \ldots, M_2^2 N_2^2 \) of \( v_2 \)-space, except that in the case of \( \{C_k\} \), the cones intersect the \( M_2 \) dyad, and in the case \( \{B_j\} \), the cones intersect the \( N_2 \) dyad.). The cone \( C_k \) intersects the \( M_2 \) dyad in the \( \eta_2 \) space in a tube of geometry \( N_2^{-1} \times N_2^{-1} \times M_2 \), whereas the cone \( B_j \) intersects the \( N_2 \) dyad in \( v_2 \) space in a tube of geometry \( M_2^{-1} \times M_2^{-1} \times N_2 \). These cone-like decompositions are widely used in the analysis of dispersive equations. See, for example, \([43, 56, 70] \).
For each \( j \) and corresponding direction \( e_j \), there are \( \sim M_2 N_2 \) directions \( e_k \) that are perpendicular to \( e_j \) - let us call this set \( K(j) \). Then we have that for each \( j \), the thickened planes/beveled washers \( A_j \) are:

\[
A_j = \bigcup_{k \in K(j)} C_k
\]

We can now look at this in the other direction as well – fix \( k \) and corresponding direction \( e_k \). There are \( \sim M_2 N_2 \) directions \( e_j \) that are perpendicular to \( e_k \) – call this set of indices \( J(k) \). The union of these becomes a thickened plane/beveled washer in \( v_2 \) space.

Now we carry out the integral in (2.1) by first noting that since \( N \ll N_2 \), \( v_2 - v \) can effectively be replaced by \( v_2 \) in \( \hat{\theta}(\eta_2 \cdot (v_2 - v)) \). This allows us to simply carry out the \( v \) and \( v_2 \) integration after enforcing the orthogonality of \( v_2 \) and \( \eta_2 \) through restriction to appropriate sets \( B_j \) and \( C_k \), respectively.

(2.2) 
\[
I = \int_{\eta} \int_{\eta_2} \left( \int_{v} \hat{\phi}(\eta - \eta_2, v) \hat{\zeta}(\eta, v) \, dv \sum_{k} 1_{C_k}(\eta_2) \sum_{j \in J(k)} \int_{v_2} 1_{B_j}(v_2) \hat{\psi}(\eta_2, v_2) \, dv_2 \right) \, d\eta_2 \, d\eta
\]

The components inside (2.2) are

\[
\int_{v} \hat{\phi}(\eta - \eta_2, v) \hat{\zeta}(\eta, v) \, dv = \frac{1}{M_1^{3/2} \max(M_1, M_2)^{3/2}} \hat{\chi}(\eta - \eta_2) \hat{\chi}(\frac{\eta}{\max(M_1, M_2)})
\]

and

(2.3) 
\[
\sum_{k} \sum_{j \in J(k)} 1_{C_k}(\eta_2) \int_{v_2} 1_{B_j}(v_2) \hat{\psi}(\eta_2, v_2) \, dv_2 = \frac{1}{M_2^3} \sum_{k} \sum_{j \in J(k)} 1_{C_k}(\eta_2) \hat{\chi} \left( \frac{P_{e_j} \eta_2}{M_2} \right) \hat{\chi}(N_2 P_{e_j} \eta_2)
\]

For (2.3), consider that \( \hat{\chi} \left( \frac{P_{e_j} \eta_2}{M_2} \right) \hat{\chi}(N_2 P_{e_j} \eta_2) \) is basically \( 1_{A_j}(\eta_2) \), but since \( e_k \) and \( e_j \) are perpendicular, it follows that \( C_k \subset A_j \) and \( 1_{C_k}(\eta_2) 1_{A_j}(\eta_2) = 1_{C_k}(\eta_2) \). Thus we continue the evaluation as

\[
\sum_{k} \sum_{j \in J(k)} 1_{C_k}(\eta_2) \int_{v_2} 1_{B_j}(v_2) \hat{\psi}(\eta_2, v_2) \, dv_2 = \frac{1}{M_2^3} \sum_{k} \sum_{j \in J(k)} 1_{C_k}(\eta_2)
\]

\[
= \frac{1}{M_2^3} \sum_{k} \text{card } J(k) 1_{C_k}(\eta_2) = \frac{N_2}{M_2^3} \sum_{k} 1_{C_k}(\eta_2) = \frac{N_2}{M_2^3} \hat{\chi}(\frac{\eta_2}{M_2})
\]

where \( \text{card}(A) \) of a set \( A \) means the cardinality of set \( A \). That is, since \( \{ C_k \} \) just partitions the whole \( M_2 \) dyad, we end up with simply the projection onto the whole dyad. Plugging these results back into (2.2),

\[
I = \int_{\eta} \int_{\eta_2} \frac{1}{M_1^{3/2} \max(M_1, M_2)^{3/2}} \hat{\chi}(\eta - \eta_2) \hat{\chi}(\frac{\eta}{\max(M_1, M_2)}) \frac{N_2}{M_2^3} \hat{\chi}(\frac{\eta_2}{M_2}) \, d\eta_2 \, d\eta
\]

Now we revert back to the spatial side via Plancherel:

\[
I = M_1^{3/2} \max(M_1, M_2)^{3/2} M_2 N_2 \int \chi(M_1 x) \chi(\max(M_1, M_2) x) \chi(M_2 x) \, dx
\]

The function \( \chi(\max(M_1, M_2) x) \) is redundant, and the \( x \)-integral evaluates to \( \max(M_1, M_2)^{-3} \). Thus

\[
I = \frac{M_1^{3/2} M_2 N_2}{\max(M_1, M_2)^{3/2}} = \min(M_1, M_2) N_2 B_{M_1 M_2}
\]

which is precisely the lower bound in the statement of Theorem 1.4.
3. Gain term bilinear estimate

We prove Theorem 1.5 in this section. We will need the following lemma.

Lemma 3.1 (Hölder Inequality for $\tilde{Q}^+$). Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ where $p, q > \frac{3}{2}$ and let $b(\frac{\xi}{|\xi|}, \omega) \in C (S^2 \times S^2)$, we then have the estimate that

$$\left\| \int_{S^2} \tilde{f}(\xi^+) \tilde{g}(\xi^-) b\left(\frac{\xi}{|\xi|}, \omega\right) d\omega \right\|_{L^q_x} \leq \left\| \tilde{f} \right\|_{L^p_x} \left\| \tilde{g} \right\|_{L^q_x} \left\| b \right\|_{L^1_{|\xi|} \times L^1_{|\xi|}}$$

Proof. This is actually a small modification of a special case of [3] Theorem 1. Their proof actually works for all uniformly bounded $b$ with little modification. We omit the details here. $lacksquare$

Denote the Littlewood-Paley projector of $x/\xi$ at frequency $N/M$ by $P_N^x/P_M^x$. We will use Lemma 3.1 in the following format.

Lemma 3.2. Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ where $p, q > \frac{3}{2}$, we then have

$$\left\| P_N^x P_M^x \tilde{Q}^+ \left( P_{N_1}^x P_{M_1}^x \tilde{f}, P_{N_2}^x P_{M_2}^x \tilde{g} \right) \right\|_{L^q_x} \leq \min\left(1, \frac{\max(N_1, N_2)}{N}\right) \min\left(1, \frac{\max(M_1, M_2)}{M}\right)$$

$$\times \left\| P_{N_1}^x P_{M_1}^x \tilde{f} \right\|_{L^p_x} \left\| P_{N_2}^x P_{M_2}^x \tilde{g} \right\|_{L^q_x}$$

Proof. The estimate can be obtained by interpolating the four inequalities:

$$\left\| P_N^x P_M^x \tilde{Q}^+ \left( P_{N_1}^x P_{M_1}^x \tilde{f}, P_{N_2}^x P_{M_2}^x \tilde{g} \right) \right\|_{L^q_x} \leq \left\| P_{N_1}^x P_{M_1}^x \tilde{f} \right\|_{L^p_x} \left\| P_{N_2}^x P_{M_2}^x \tilde{g} \right\|_{L^q_x}$$

$$\left\| P_N^x P_M^x \tilde{Q}^+ \left( P_{N_1}^x P_{M_1}^x \tilde{f}, P_{N_2}^x P_{M_2}^x \tilde{g} \right) \right\|_{L^q_x} \leq \frac{\max(M_1, M_2)}{M} \left\| P_{N_1}^x P_{M_1}^x \tilde{f} \right\|_{L^p_x} \left\| P_{N_2}^x P_{M_2}^x \tilde{g} \right\|_{L^q_x}$$

$$\left\| P_N^x P_M^x \tilde{Q}^+ \left( P_{N_1}^x P_{M_1}^x \tilde{f}, P_{N_2}^x P_{M_2}^x \tilde{g} \right) \right\|_{L^q_x} \leq \frac{\max(N_1, N_2)}{N} \left\| P_{N_1}^x P_{M_1}^x \tilde{f} \right\|_{L^p_x} \left\| P_{N_2}^x P_{M_2}^x \tilde{g} \right\|_{L^q_x}$$

$$\left\| P_N^x P_M^x \tilde{Q}^+ \left( P_{N_1}^x P_{M_1}^x \tilde{f}, P_{N_2}^x P_{M_2}^x \tilde{g} \right) \right\|_{L^q_x} \leq \frac{\max(N_1, N_2) \max(M_1, M_2)}{M}$$

$$\times \left\| P_{N_1}^x P_{M_1}^x \tilde{f} \right\|_{L^p_x} \left\| P_{N_2}^x P_{M_2}^x \tilde{g} \right\|_{L^q_x}$$

The main concern is definitely the $\xi$-part as $\tilde{Q}^+$ commutes with $P_N^x$ and $\nabla_x$, it suffices to prove

$$\left\| P_M^x \tilde{Q}^+ \left( P_{M_1}^x \tilde{f}, P_{M_2}^x \tilde{g} \right) \right\|_{L^q_x} \leq \left\| P_{M_1}^x \tilde{f} \right\|_{L^p_x} \left\| P_{M_2}^x \tilde{g} \right\|_{L^q_x},$$

and

$$\left\| P_M^x \tilde{Q}^+ \left( P_{M_1}^x \tilde{f}, P_{M_2}^x \tilde{g} \right) \right\|_{L^q_x} \leq \frac{\max(M_1, M_2)}{M} \left\| P_{M_1}^x \tilde{f} \right\|_{L^p_x} \left\| P_{M_2}^x \tilde{g} \right\|_{L^q_x}.$$

While (3.1) is directly from Lemma 3.1 we prove only (3.2). Notice that

$$\partial_{\xi^-} \left( \tilde{f}(\xi^+) \tilde{g}(\xi^-) \right) = \left( \partial_{\xi^+} \tilde{f} \right)(\xi^+) \tilde{g}(\xi^-) \partial_{\xi^-} \xi^+ + \left( \partial_{\xi^-} \tilde{g} \right)(\xi^-) \partial_{\xi^+} \xi^-$$
and $|\partial_\xi \xi^\pm| \leq 1$, we can thus apply Lemma 3.1 with $b = \partial_\xi \xi^\pm$. That is,

$$\left\| P_M^\xi \tilde{Q}^+ \left( P_M^\xi \tilde{f}, P_M^\xi \tilde{g} \right) \right\|_{L^p_\xi} = M^{-1} \left\| P_M^\xi \nabla_\xi \tilde{Q}^+ \left( P_M^\xi \tilde{f}, P_M^\xi \tilde{g} \right) \right\|_{L^p_\xi} \leq M^{-1} \left( \left\| P_M^\xi \nabla_\xi \tilde{f} \right\|_{L^p_\xi} \left\| P_M^\xi \tilde{g} \right\|_{L^p_\xi} + \left\| P_M^\xi \tilde{f} \right\|_{L^p_\xi} \left\| P_M^\xi \nabla_\xi \tilde{g} \right\|_{L^p_\xi} \right)$$

which, by Bernstein, is

$$\left\| P_M^\xi \tilde{Q}^+ \left( P_M^\xi \tilde{f}, P_M^\xi \tilde{g} \right) \right\|_{L^p_\xi} \leq \frac{\max(M_1, M_2)}{M} \left\| P_M^\xi \tilde{f} \right\|_{L^p_\xi} \left\| P_M^\xi \tilde{g} \right\|_{L^p_\xi}$$

as claimed in (3.2). \hfill \Box

3.1. Proof of Theorem 1.5 It suffice to prove

$$\left\| \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^s \tilde{Q}^+ (\tilde{f}, \tilde{g}) \right\|_{L^1_\xi L^2_{2\xi}} \leq \left\| \tilde{f} \right\|_{X_{s,1}^+} \left\| \tilde{g} \right\|_{X_{s,1}^+}.$$

We start by decomposing $\tilde{Q}^+ (f, g)$ into Littlewood-Paley pieces

$$\left\| \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^s \tilde{Q}^+ (\tilde{f}, \tilde{g}) \right\|_{L^1_\xi L^2_{2\xi}} \leq \sum_{M,M_1, M_2 \text{ N}N_1, N_2} M^s N^s \left\| P_N^\xi P_M^\xi \tilde{Q}^+ \left( P_N^\xi P_{M_1}^\xi \tilde{f}, P_N^\xi P_{M_2}^\xi \tilde{g} \right) \right\|_{L^1_\xi L^2_{2\xi}} = A + B + C + D$$

where we have seperated the sum into 4 cases in which case A is $M_1 \geq M_2, N_1 \geq N_2$, case B is $M_1 \leq M_2, N_1 \geq N_2$, case C is $M_1 \leq M_2, N_1 \leq N_2$, and case D is $M_1 \geq M_2, N_1 \leq N_2$. We handle only cases A and B as the other two cases follow similarly.

3.1.1. Case A: $M_1 \geq M_2, N_1 \geq N_2$.

$$A \leq \sum_{M,M_1, M_2 \text{ N}N_1, N_2} \frac{M^s N^s}{M^1_{1} N^1_{1}} \left\| P_N^\xi P_M^\xi \tilde{Q}^+ \left( P_N^\xi P_{M_1}^\xi \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^s \tilde{f}, P_N^\xi P_{M_2}^\xi \tilde{g} \right) \right\|_{L^1_\xi L^2_{2\xi}}$$

Use Lemma 3.2 and Hölder,

$$A \leq \sum_{M,M_1, M_2 \text{ N}N_1, N_2} \frac{M^s N^s}{M^1_{1} N^1_{1}} \min(1, N_1) \min(1, M_1) \times \left\| P_N^\xi P_M^\xi \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^s \tilde{f} \right\|_{L^2_{2\xi} L^2_{2\xi}} \left\| P_N^\xi P_M^\xi \tilde{g} \right\|_{L^2_{2\xi} L^2_{2\xi}}.$$
By Sobolev and Bernstein,
\[
A \lesssim \sum_{M_1, M_2} \frac{M^s N^s}{M_1^s N_1^s} \min\left(1, \frac{N_1}{N} \right) \min\left(1, \frac{M_1}{M} \right) \left\| P_{N_1} P_{M_1} \left\langle \nabla_x \right\rangle^s \left\langle \nabla_\xi \right\rangle^s f \right\|_{L_t^2 L_x^{3s}}^2 \left\| P_{N_2} P_{M_2} \left\langle \nabla_x \right\rangle^{\frac{3}{2}} \left\langle \nabla_\xi \right\rangle^{\frac{3}{2}} \tilde{g} \right\|_{L_t^2 L_x^{3s}}^2
\]

\[
\lesssim \sum_{M_1, M_2} \frac{M^s N^s}{M_1^s N_1^s} \min\left(1, \frac{N_1}{N} \right) \min\left(1, \frac{M_1}{M} \right) \frac{1}{N_2^{s-\frac{3}{2}}} \frac{1}{M_2^{s-\frac{3}{2}}}
\]

\[
\times \left\| P_{N_1} P_{M_1} \left\langle \nabla_x \right\rangle^s \left\langle \nabla_\xi \right\rangle^s f \right\|_{L_t^2 L_x^{3s}} \left\| P_{N_2} P_{M_2} \left\langle \nabla_x \right\rangle^{s} \left\langle \nabla_\xi \right\rangle^{s} \tilde{g} \right\|_{L_t^2 L_x^{3s}}
\]

By Strichartz,
\[
A \lesssim \sum_{M_1, M_2} \frac{M^s N^s}{M_1^s N_1^s} \min\left(1, \frac{N_1}{N} \right) \min\left(1, \frac{M_1}{M} \right) \frac{1}{N_2^{s-\frac{3}{2}}} \frac{1}{M_2^{s-\frac{3}{2}}}
\]

\[
\times \left\| P_{N_1} P_{M_1} \tilde{f} \right\|_{X^{s, \frac{1}{2}}} + \left\| P_{N_2} P_{M_2} \tilde{g} \right\|_{X^{s, \frac{1}{2}}}
\]

Cauchy-Schwarz in $N_2, M_2$ to carry out the $N_2, M_2$ sum, we have
\[
A \lesssim \left\| \tilde{g} \right\|_{X^{s, \frac{1}{2}}} \sum_{M_1, M_2} \frac{M^s N^s}{M_1^s N_1^s} \min\left(1, \frac{N_1}{N} \right) \min\left(1, \frac{M_1}{M} \right) \left\| P_{N_1} P_{M_1} f \right\|_{X^{s, \frac{1}{2}}}
\]

Separate the sum into four cases $M_1 \leq M, M_1 \geq M, N_1 \leq N,$ and $N_1 \geq N$, we are done by Schur’s test or Cauchy-Schwarz.

3.1.2. Case B: $M_1 \leq M_2, N_1 \geq N_2$.

\[
B \lesssim \sum_{M_1, M_2} \frac{M^s N^s}{M_1^s N_1^s} \left\| P_{N_1} P_{M_1} \left\langle \nabla_x \right\rangle^s f \right\|_{L_t^2 L_x^{3s}} \left\| P_{N_2} P_{M_2} \left\langle \nabla_\xi \right\rangle^s \tilde{g} \right\|_{L_t^2 L_x^{3s}}
\]

Use Lemma 3.2 and Hölder,
\[
B \lesssim \sum_{M_1, M_2} \frac{M^s N^s}{M_1^s N_1^s} \min\left(1, \frac{N_1}{N} \right) \min\left(1, \frac{M_1}{M} \right)
\]

\[
\times \left\| P_{N_1} P_{M_1} \left\langle \nabla_x \right\rangle^s f \right\|_{L_t^2 L_x^{3s} L_\xi^{3s}} \left\| P_{N_2} P_{M_2} \left\langle \nabla_\xi \right\rangle^s \tilde{g} \right\|_{L_t^2 L_\xi L_\xi^{3s}}
\]
By Sobolev and Bernstein,
\[ B \leq \sum_{M,M_1,M_2} \frac{M^s N^s}{M_2^s N_1^s} \min(1, \frac{N_1}{N}) \min(1, \frac{M_2}{M}) \]
\[ \times \| P_{N_1} P_{M_1} \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^{\frac{1}{2}} \tilde{f} \|_{L_t^2 L_{x\xi}^{3}} \| P_{N_2} P_{M_2} \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^{\frac{1}{2}} \tilde{g} \|_{L_t^2 L_{x\xi}^{3}} \]
\[ \leq \sum_{M,M_1,M_2} \frac{M^s N^s}{M_2^s N_1^s} \min(1, \frac{N_1}{N}) \min(1, \frac{M_2}{M}) \frac{1}{M_{1}^s} \frac{1}{M_{2}^s} \frac{1}{N_{1}^s} \frac{1}{N_{2}^s} \frac{1}{2} \frac{1}{2} \]
\[ \times \| P_{N_1} P_{M_1} \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^{\frac{1}{2}} \tilde{f} \|_{L_t^2 L_{x\xi}^{3}} \| P_{N_2} P_{M_2} \langle \nabla_x \rangle^s \langle \nabla_\xi \rangle^{\frac{1}{2}} \tilde{g} \|_{L_t^2 L_{x\xi}^{3}} \]
One can then proceed in the exact same manner as in Case A. We omit the details.

4. Well-posedness

For completeness, we provide a quick (and not so complete) treatment of the well-posedness statement in Theorem 1.3 using (1.13) and (1.15). As mentioned before, this part is already proved in [11,12]. We prove only the existence, uniqueness, and the Lipschitz continuity of the solution map here. The nonnegativity of \( f \) follows from the persistence of regularity [12].

Write \( R = 2 \| \tilde{f}_0 \|_{H_x^s} \) and let \( \theta(t) \) be a smooth function such that \( \theta(t) = 1 \) for all \( |t| \leq 1 \) and \( \theta(t) = 0 \) for \( |t| \geq 2 \). Put \( T = \delta(R)^{-2} \), where \( \delta > 0 \) is an absolute constant (selected depending upon constants appearing in the estimates such as (4.1) that we apply). The estimates (1.13) and (1.15) admit the following minor adjustments that account for restriction in time:

\[ \| \theta \left( \frac{t}{T} \right) \tilde{Q}(\tilde{f}, \tilde{g}) \|_{X_{s,-\frac{1}{2}+}} \leq T^{1/2} \| \tilde{f} \|_{X_{s,-\frac{1}{2}+}} \| \tilde{g} \|_{X_{s,-\frac{1}{2}+}} \quad \text{for } s > 1. \]

Denote \( B = \{ \tilde{f} : \| \tilde{f} \|_{X_{s,-\frac{1}{2}+}} \leq R \} \) the ball of radius \( R \) in \( X_{s,-\frac{1}{2}+} \), and introduce the nonlinear mapping \( \Lambda : B \to \mathcal{S}' \) given by

\[ \Lambda(\tilde{f}) = \theta(t)e^{it \nabla_x \nabla_\xi} \tilde{f}_0 + D(\tilde{f}, \tilde{f}), \quad D(\tilde{f}, \tilde{f}) = \theta(t) \int_0^t e^{i(t-t') \nabla_x \nabla_\xi} \theta(t'/T) \tilde{Q}(\tilde{f}(t'), \tilde{f}(t')) \, dt'. \]

A fixed point \( \tilde{f} = \Lambda(\tilde{f}) \) will satisfy (1.10) for \( -T \leq t \leq T \). Using (4.1) one can show that

\[ \| \Lambda(\tilde{f}) - \Lambda(\tilde{g}) \|_{X_{s,-\frac{1}{2}+}} \leq \frac{1}{2} \| \tilde{f} - \tilde{g} \|_{X_{s,-\frac{1}{2}+}} \]

and thus \( \Lambda \) is a contraction. From this it follows that there is a fixed point \( \tilde{f} \) in \( X_{s,-\frac{1}{2}+} \), and this fixed point is unique in \( X_{s,-\frac{1}{2}+} \).

Lipschitz continuity of the data to solution map is also straightforward. Given two initial conditions \( \tilde{f}_0 \) and \( \tilde{g}_0 \), let \( R = 2 \max(\| \tilde{f}_0 \|_{H^s}, \| \tilde{g}_0 \|_{H^s}) \) and \( \tilde{f}, \tilde{g} \) be the corresponding fixed points of \( \Lambda \) constructed above. Then

\[ \tilde{f} - \tilde{g} = \theta(t)e^{it \nabla_x \nabla_\xi}(\tilde{f}_0 - \tilde{g}_0) + D(\tilde{f} - \tilde{g}, \tilde{f} - \tilde{g}) + D(\tilde{f} - \tilde{g}, \tilde{g}) + D(\tilde{g}, \tilde{f} - \tilde{g}) \]

The embedding \( X_{s,-\frac{1}{2}+} \hookrightarrow C([-T,T], H^s_x) \) and estimate (4.1) applied to the above equation give the Lipschitz continuity of the data-to-solution map on the time \([-T,T]\).
5.ILL-POSEDNESS

In this section, we prove the ill-posedness portion of Theorem 1.3. We will take

\[ 0 < N \ll \max(M_1, M_2)^{-1} \ll 1, \quad M_1 \geq 1, \quad M_2 \geq 1 \]

5.1. Imprecise motivating calculations. We start by taking the two linear solutions that generated the “bad” interaction in the loss term from \( \delta \) now scaled so that \( \| \hat{\phi} \|_{H^s_x} \sim 1 \) and \( \| \hat{\psi} \|_{H^s_x} \sim 1 \). The bad solution will have \( O(1) \) \( H^s_x \)-norm initially but more obvious ill-behavior in \( H^s_{x\xi} \)-norm with \( s_0 < s \). The functions are:

\[ \hat{\phi}(\eta, v) = \frac{1}{M_1^{2+s} N^{3/2}} \hat{\chi}(\frac{\eta}{M_1}) \hat{\chi}(\frac{v}{N}) \]

\[ \hat{\psi}(\eta_2, v_2) = \frac{1}{M_2^{1+s} N_2^{1+s}} \sum_{j=1}^{J} \hat{\chi}(\frac{P_{e_j}^{+} \eta_2}{M_2}) \hat{\chi}(\frac{M_2 P_{e_j}^{+} v_2}{N_2}) \hat{\chi}(\frac{P_{e_j} v_2}{N_2}) \]

where \( J \sim M_2^2 N_2^2 \). Let \( \tilde{f} = e^{i\nabla_x \cdot \nabla \cdot \eta} \hat{\phi} \) and \( \tilde{g} = e^{i\nabla_x \cdot \nabla \cdot \eta} \hat{\psi} \). To deduce the form of the \((x,v)\) wave packets, we need to pass from \( \eta \) frequency support to \( x \) spatial support. This is done using the scaling principles of the Fourier transform. Since we have chosen \( 0 < N \ll \max(M_1, M_2)^{-1} \ll 1 \), the \( x \)-support of \( \tilde{f} \) hardly moves on an \( O(1) \) time scale:

\[ f(x,v,t) \approx \frac{M_1^{3-s}}{N^{3/2}} \chi(M_1 x) \hat{\chi}(\frac{v}{N}) \]

On the other hand, the \( g \) wave packet tubes move, each in their velocity confined direction parallel to \( e_j \) with speed \( \sim N_2 \). On short time scales, we regard them as stationary:

\[ g(x,v_2,t) \approx \frac{M_2^{1-s}}{N_2^{1+s}} \sum_{j=1}^{J} \chi(M_2 P_{e_j}^{+} x) \hat{\chi}(\frac{P_{e_j} x}{N_2}) \hat{\chi}(\frac{P_{e_j} v_2}{N_2}) \]

For the loss term, we need

\[ \int_{\mathbb{R}^3} g(x,v_2,t) dv_2 \approx \frac{1}{M_2^{1+s} N_2^{1+s}} \sum_{j=1}^{J} \chi(M_2 P_{e_j}^{+} x) \hat{\chi}(\frac{P_{e_j} x}{N_2}) \approx M_2^{1-s} N_2^{1-s} \chi(M_2 x) \]

We get

\[ Q^- (f,g) \approx \frac{M_1^{3-s}}{N^{3/2}} M_2^{1-s} N_2^{1-s} \chi(M_2 x) \chi(M_1 x) \hat{\chi}(\frac{v}{N}) \]

Formal Duhamel iteration of \( f + g \) suggests a prototype approximate solution of the form

\[ f_{a,0}(x,v,t) \approx \frac{M_1^{3-s}}{N^{3/2}} \exp[-M_2^{1-s} N_2^{1-s} \chi(M_2 x) t] \chi(M_1 x) \hat{\chi}(\frac{v}{N}) \]

\[ + \frac{M_2^{1-s}}{N_2^{1+s}} \sum_{j=1}^{J} \chi(M_2 P_{e_j}^{+} x) \hat{\chi}(\frac{P_{e_j} x}{N_2}) \hat{\chi}(\frac{P_{e_j} v}{N_2}) \]

The prototype approximate solution \( f_{a,0} \) is just \( f + g \) above with \( f \) preceded by an exponentially decaying factor in time (which is suggested by formal Duhamel iteration). Notice that when \( s < 1 \), the exponential term decays substantially on the short time scale \( \sim M_2^{s-1} N_2^{s-1} \ll 1 \). Thus we seek to show that this is an approximate solution on a time interval \( |t| \leq M_2^{s-1} N_2^{s-1} \ln M_2 \), which is long enough for the size of the first term with exponential coefficient to change substantially.

Let us take
We calculate, for any $0 < s \leq 1$, $s_0 \geq 0$,
\begin{equation}
\|f_{a,0}\|_{L^2_{t}H^s_x} \sim M^{s_0-s} \exp[-(MN_2)^{1-s}t]((MN_2)^{1-s}t)^{s_0} + (MN_2)^{s_0-s}
\end{equation}
In the second term involving the sum in $j$ over $J \sim (MN_2)^2$ terms, the velocity supports are almost disjoint and the square of the sum is approximately the sum of the squares. Thus, if we take
\begin{equation}
0 < s < 1, \quad s_0 = s - \frac{\ln M}{\ln N}, \quad T_* = -\delta(MN_2)^{s-1}\ln M \leq t \leq 0
\end{equation}
then at the ends of this interval:
\[\|f_{a,0}(0)\|_{L^2_{t}H^s_x} \lesssim \frac{1}{\ln M} \ll 1, \quad \|f_{a,0}(T_*)\|_{L^2_{t}H^s_x} \sim M^\delta \gg 1\]
Note that, as $M \not\to \infty$, $s_0 \not\to s$, and this approximate solution, in $H^s_x$, starts very small in $H^s_x$ at time 0, and rapidly inflates at time $T_* < 0$ to large size in $H^s_x$ backwards in time. By considering the same approximate solution starting at $T_* < 0$ and evolving forward to time 0, we have an approximate solution that starts large and deflates to a small size in a very short period of time.

5.2. Precise reformulation. The calculations in Section 5.1 were imprecise (with $\approx$ symbols) although they served only to motivate the form of the actual approximate solution $f_a$ we are going to use. From now on, the error estimates will be made more precise. The following is a better way to write the form of $f_a$ leading to a more accurate approximate solution. Start with
\begin{equation}
f_b(t, x, v) = \frac{M^2}{N^{2+s}} \sum_{j=1}^{J} \chi(M_2 P_{e_j}^2(x - vt)) \chi(P_{e_j}(x - vt)) \chi(M_2 P_{e_j}^1 v) \chi(10 P_{e_j}(v - N_2 e_j))
\end{equation}
We note that $f_b$ is a linear solution:
\[\partial_t f_b + v \cdot \nabla_x f_b = 0\]
Another feature is that if $|i - j| \geq 4$, then the $i$th and $j$th terms have disjoint $v$-support. Now set
\begin{equation}
f_t(t, x, v) = \frac{M^{2-s}}{N^{3/2}} \exp \left[ -\int_{s=0}^{t} \int_{v} f_b(s, x, v) dv ds \right] \chi(M_1 x) \chi(v N), \quad N \leq M^{-1}
\end{equation}
Note that $f_t$ is a solution to a drift-free linearized Boltzmann equation:
\[\partial_t f_t + v \cdot \nabla_x f_t = -f_t(t, x, v) \int_{v} f_b(t, x, v) dv = -Q^-(f_t, f_b)\]
The approximate solution of interest is then
\[f_a = f_t + f_b\]
Let
\[F_{\text{err}} = \partial_t f_a + v \cdot \nabla_x f_a + Q^- (f_a, f_a) - Q^+(f_a, f_a)\]
\begin{equation}
= v \cdot \nabla_x f_t + Q^- (f_t, f_t) + Q^- (f_t, f_t)
+ Q^- (f_b, f_b) - Q^+ (f_a, f_a)
\end{equation}
Note that this is missing the “bad terms” $v \cdot \nabla_x f_b$ and $Q^-(f_t, f_b)$ since $f_a$ already incorporates the evolutionary impact of these interactions. Indeed $Q^-(f_t, f_b)$ is the cause of the backwards-in-time growth of $f_t$. 
For convenience, let us take
\[ M = M_1 = M_2 \gg 1, \quad N_2^{1-s} \geq M^\delta, \quad N \leq M^{-1} \]

We will show in Prop. 5.5 that there exist \( f_c \) (for “correction”) so that \( f_{\text{ex}} \) (for “exact”) given by
\[ f_{\text{ex}} = f_a + f_c \]

exactly solves
\[ \partial_t f_{\text{ex}} + v \cdot \nabla_x f_{\text{ex}} = -Q^-(f_{\text{ex}}, f_{\text{ex}}) + Q^+(f_{\text{ex}}, f_{\text{ex}}) \]
where
\[ T_\ast = -\delta (MN_2)^{s-1} \ln M \leq t \leq 0 \]
and
\[ (5.6) \quad \| f_c(t) \|_{L_v^{2,1} H_x^1} \lesssim M^{-\delta/4} \]
where the exponential term comes from the iteration of the local theory approximation argument below over time steps of length \( \delta(MN)^{s-1} \). The equation for \( f_c \) is
\[ \partial_t f_c + v \cdot \nabla_x f_c = G 
\]
\[ G = \pm Q^\pm(f_c, f_a) \pm Q^\pm(f_a, f_c) \pm Q^\pm(f_c, f_c) - F_{\text{err}} \]

To prove Proposition 5.5 we will do estimates in local time steps of size \( \delta(MN_2)^{s-1} \ll 1 \) by working with the norm:
\[ \| f_c(t) \|_Z = M \| f_c(t) \|_{L_v^{2,1} L_x^2} + \| \nabla_x f_c(t) \|_{L_v^{2,1} L_x^2} + \| f_c(t) \|_{L_v^1 L_x^\infty} + M^{-1} \| \nabla_x f_c(t) \|_{L_v^1 L_x^\infty} \]
to complete a perturbation argument to prove that \( f_c \) (and thus \( f_{\text{ex}} \)) exists. As we are trying to construct a fairly smooth solution with specific properties (some of which cannot be efficiently treated using Strichartz and, in any case, we do not have (1.13) for \( s = 1 \)) to the Boltzmann equation linearized near a special solution, we use the \( Z \)-norm instead of only the \( L_v^{2,1} H_x^1 \) norm to better serve our purpose here. Inside the \( Z \)-norm, the \( L_v^{2,1} L_x^\infty \) norm is preserved by the linear flow and scales the same way as the norm \( L_v^{2,1} H_x^1 \), and thus lies at regularity scale \textit{above} \( L_v^{2,1} H_x^1 \). Nevertheless, due to the specific structure of \( f_a(t) \), we have (see Lemma 5.1 below)
\[ \| f_a(t) \|_Z \approx \| f_a(t) \|_{L_v^{2,1} H_x^1} \]
and thus it is effective for our purposes. The extra terms \( M \| f_c(t) \|_{L_v^{2,1} L_x^2} \) and \( M^{-1} \| \nabla_x f_c(t) \|_{L_v^1 L_x^\infty} \)
are added to the definition of the norm to account for the case in which \( \nabla_x \) lands on the “wrong” term in the \( Z \) bilinear estimate (see Lemma 5.4 below).

In the next three subsections, Lemmas 5.1, 5.3, and 5.4 are proved and will provide the key technical tools to carry out the proof of Proposition 5.5.

5.3. Size of the approximate solution \( f_a \). Before proceeding, we will give a useful pointwise bound on \( f_b \). For \(-\frac{1}{4} \leq t \leq \frac{1}{4} \), given the constraints on \( v \),
\[ M^{1-s} N_2^{2-s} \sum_{j=1}^{J} \chi(10MP_{e_j}^j x) \chi(10P_{e_j} x/N_2) \chi(MP_{e_j}^j v) \chi(10P_{e_j} (v - N_2 e_j)) \]
\[ \leq f_b(t, x, v) \leq M^{1-s} N_2^{2-s} \sum_{j=1}^{J} \chi(10MP_{e_j}^j x) \chi(P_{e_j} x/10N_2) \chi(MP_{e_j}^j v) \chi(P_{e_j} (v - N_2 e_j)/10N_2) \]
In addition, we can pointwise estimate this sum by
\[ M^{-1-s} N_2^{-1-s} \int f_0(t, x, v) \, dv \leq M^{-1-s} N_2^{-1-s} \sum_{j=1}^{J} \chi(10MP_{e_j}x) \chi(\frac{10P_{e_j}x}{N_2}) \]
\[
\leq \int f_0(t, x, v) \, dv \leq M^{-1-s} N_2^{-1-s} \sum_{j=1}^{J} \chi(\frac{MP_{e_j}x}{10}) \chi(\frac{P_{e_j}x}{10N_2}) \]

We can pointwise estimate this sum by
\[
(5.8) \quad \int f_0(t, x, v) \, dv \sim M^{-1-s} N_2^{-1-s} \left( \frac{N_2}{|x| + M^{-1}} \right)^2 \chi\left( \frac{x}{N_2} \right). \]
To see (5.8), we split the size of $|x|$ into 3 cases. For $|x| \lesssim M^{-1}$, all the tubes are overlapping and hence, the $J$ sum has $(MN_2)^2$ summands. For $|x| \sim N_2$, the tubes are effectively disjoint (up to tubes of distance comparable to $M^{-1}$), that is, the $J$ sum has $O(1)$ summands. For the intermediate case, the overlap count would be the number of points with adjacent distance $\frac{|x|}{MN_2}$ in a $M^{-1} \times M^{-1}$ box, and is thus $M^{-2}/(\frac{|x|}{MN_2})^2$. Hence, we conclude (5.8).
In particular, we have
\[
(5.9) \quad \int f_0(t, x, v) \, dv \sim M^{1-s} N_2^{1-s} \text{ for } |x| \lesssim M^{-1}. \]
The inequalities (5.8) and (5.9) remain true with the extra factor $M^k$ if $\nabla^k_z$ is applied, for $k \geq 0$ and $k \in \mathbb{Z}$. Let
\[
\beta(t, x) = \int_0^t \int f_0(t_0, x, v) \, dv \, dt_0
\]
Then (5.8) yields the pointwise upper bounds
\[
|\nabla^k_z \beta(t, x)| \lesssim |t| M^{-1+k-s} N_2^{-1-s} \left( \frac{N_2}{|x| + M^{-1}} \right)^2 \chi\left( \frac{x}{N_2} \right)
\]
which implies
\[
(5.10) \quad \|\beta(t, x)\|_{L^2_x} \lesssim |t| M^{1+k-s} N_2^{1-s}
\]

**Lemma 5.1** (bounds on $f_a$). For all $q \geq 0$, we have for $t \leq 0$,
\[
(5.11) \quad \|f_a(t)\|_{L^2_q H^s_x} \sim M^{q-s} \exp[|t|(MN_2)^{1-s}] \langle |t|(MN_2)^{1-s} \rangle + (MN_2)^{q-s}
\]
In addition,
\[
\|f_a(t)\|_Z \lesssim M^{1-s} \exp[|t|(MN_2)^{1-s}] \langle |t|(MN_2)^{1-s} \rangle + (MN_2)^1
\]
Since we further assume that $N_2^{-1-s} \geq M^\delta$, this bound simplifies to
\[
\|f_a(t)\|_Z \lesssim (MN_2)^{1-s}
\]
Before proceeding to the proof, note that if we take $q = s_0$ in (5.11), where
\[
(5.12) \quad s_0 = s - \frac{\ln \ln M}{\ln M}
\]
then
\[
(5.13) \quad \|f_a(t)\|_{L^2_{s_0} H^{s_0}_x} \sim \frac{1}{\ln M} \exp[|t|(MN_2)^{1-s}] \langle |t|(MN_2)^{1-s} \rangle
\]
In particular,
\[
\|f_a(0)\|_{L^2_{s_0} H^{s_0}_x} \sim \frac{1}{\ln M} \ll 1, \quad \|f_a(T_*)\|_{L^2_{s_0} H^{s_0}_x} \sim \delta M^\delta \gg 1
\]
Remark

As we can take the only exception is the treatment of the bound on \( L^2 \) and recalling the estimate (5.10) and that \(|\cdot|\) bounds on \( \parallel \cdot \parallel \) similar bound is obtained for \( t \).

Analysis of the calculation here.

Remark 5.2. One can compute directly from (5.3) and (5.4) that

\[
\int \int (1 + |x|^2 + |v|^2) f_a(t, x, v) dxdv \lesssim M^{-1-s} N_2^4 + M^{-\frac{1}{2}} - s + \delta N^\frac{3}{2}
\]

As we can take \( N_2 \sim M^\delta \) and \( N \leq M^{-1} \), the 2nd moments of \( f_a \) is uniformly bounded in \( M \). That is, an uniform bound of the 2nd moment will not turn the ill-posedness into well-posedness.

5.4. Bounding of forcing terms \( F_{err} \).

Lemma 5.3 (bounds on \( F_{err} \)). For \( s > 1/2 \) and for \( T_* \leq t \leq 0 \),

\[
\| \int_{t_0}^t e^{-(t-t_0)\nabla_x} F_{err}(t_0) dt_0 \| Z \lesssim M^{-\delta}
\]

Proof. We address all of the terms in (5.5) separately below. The terms \( Q^\pm(f_a, f_b) \) need the most attention and require \( s > \frac{1}{2} \), thus we put them last. For many terms, the estimate is achieved by moving the \( t_0 \) integration to the outside:

\[
\| \int_{t_0}^t e^{-(t-t_0)\nabla_x} F_{err}(t_0) dt_0 \| Z \lesssim (MN_2)^{s-1} \ln M \| F_{err} \| L^\infty \cdot Z
\]

The only exception is the treatment of the bound on \( L_v^1 \cdot L^\infty_x \) of \( Q^\pm(f_a, f_b) \), where a substantial gain is captured by carrying out the \( t_0 \) integration first.

5.4.1. Analysis of \( v \cdot \nabla_x f_r \). Starting with

\[
f_t(t, x, v) = \frac{M^{\frac{3}{2}}}{N^{3/2}} \exp \left[ -\beta(x, t) \right] \chi(Mx) \chi \left( \frac{v}{N} \right)
\]

and recalling the estimate (5.10) and that \( t \| \lesssim \delta (MN_2)^{s-1} \ln M \)

\[
\| v \cdot \nabla_x f_t \|_{L^2_{v} \cdot H^1_x} \lesssim NM^{\frac{3}{2} - s} \left( M \| \exp[-\beta(t, x)] \nabla \chi(Mx) \|_{L^2_x} \right.
\]

\[
\left. + \| \nabla_x \beta(t, x) \exp[-\beta(t, x)] \chi(Mx) \|_{L^2_x} \right)
\]

\[
\lesssim NM^{\frac{3}{2} - s} \left( M^{-1/2} \| \exp[-\beta(t, x)] \|_{L^2_x} \right.
\]

\[
\left. + M^{-3/2} \| \nabla_x \beta(t, x) \|_{L^2_x} \| \exp[-\beta(t, x)] \|_{L^2_x} \right)
\]

\[
\lesssim NM^{1-s} \exp[\delta \ln M] \ln M = NM^{1+\delta-s} \langle \ln M \rangle \lesssim M^{\delta-s} \langle \ln M \rangle
\]

When multiplied by \( |T^*| \), this produces a bound \( M^{2\delta-1} N_2^{s-1} \), which suffices provided \( \delta < \frac{1}{3} \). A similar bound is obtained for \( M^{-1} \| \nabla_x [v \cdot \nabla_x f_t] \|_{L^2_{v} \cdot H^1_x} \).
Thus if

\[ \text{Thus, with} \]

When

\[ \text{since} \]

Upon multiplying by \(|T_\alpha|\), we obtain the bound \( M^{2\delta - 1} N_2^{s - 1} \), which suffices provided \( \delta < \frac{1}{3} \).

### 5.4.2. Analysis of \( Q^+ (f_r, f_b) \)

By [3, Theorem 2] with \( \lambda = 0, r = 1, p = 1, q = 1 \),

\[ \| Q^+ (f_r, f_b) \| L_t^1 L_x^\infty \lesssim \| f_r \| L_t^1 L_x^\infty \| f_b \| L_t^1 L_x^\infty \]

Since

\[ \| f_r \| L_t^1 L_x^\infty \sim M^{\frac{3}{2} - s} N_2^{3/2} \exp\left[ |t|(MN_2)^{1-s}\right] \lesssim M^{\delta - s}, \quad \| f_b \| L_t^1 L_x^\infty \lesssim (MN_2)^{1-s} \]

when \( N \leq M^{-1} \), we obtain

\[ (5.15) \quad \| Q^+ (f_r, f_b) \| L_t^1 L_x^\infty \lesssim M^{\delta - s} (MN_2)^{s - 1} \]

Upon multiplying by \(|T_\ast|\), this yields a bound \( M^{\delta - s} \langle \delta \ln M \rangle \), which suffices provided \( s > \frac{1}{2} \) and \( \delta < \frac{1}{4} \).

On the other hand,

\[ \| \nabla_x Q^+ (f_r, f_b) \| L_x^2 \lesssim Q^+ (\| \nabla_x f_r \|_{L_t^1 L_x^2}, \| f_b \| L_x^\infty) + Q^+ (\| f_r \|_{L_t^1 L_x^\infty}, \| \nabla_x f_b \|_{L_t^1 L_x^2}) \]

A weight of \(|v|\) is transferred to \( f_b (u^*) \), since \( f_r (v^*) \) forces \(|v^*| \lesssim N \) which in turn implies \(|P_{u^*} v| \lesssim N \ll 1 \) via

\[ v^* = P_{u^*} v + P_{\omega} \]

Thus if \(|v| \gg N \) (in particular if \(|v| \geq 1 \)), then \(|P_{u^*} v| \sim |v| \). Since

\[ u^* = P_{\omega} u + P_{\omega} v \]

it follows that \(|u^*| \gg |P_{\omega} v| \sim |v| \). Now by [3, Theorem 2], with \( \lambda = 0, r = 2, p = 1, q = 2 \),

\[ \| \nabla_x Q^+ (f_r, f_b) \|_{L_t^1 L_x^2} \lesssim \| \nabla_x f_r \|_{L_t^1 L_x^2} \| f_b \|_{L_t^1 L_x^\infty} + \| f_r \|_{L_t^1 L_x^\infty} \| \nabla_x f_b \|_{L_t^1 L_x^2} \]

(As an expository note, such a bound is not possible for \( Q^- \). In that case, the \( L_t^1 \) norm must go on \( f_b \).)

\[ \| \nabla_x f_r \|_{L_t^1 L_x^2} \sim M^{1-s} N_2^{3/2} \exp[(MN_2)^{1-s}|t|], \quad \| f_b \|_{L_t^1 L_x^\infty} \sim M^{1-s} N_2^{\frac{3}{2} - s} \]

\[ \| f_r \|_{L_t^1 L_x^\infty} \sim M^{\frac{3}{2} - s} N_2^{3/2} \exp[(MN_2)^{1-s}|t|], \quad \| \nabla_x f_b \|_{L_t^1 L_x^2} \sim (MN_2)^{1-s} \]

Thus, with \( N \lesssim M^{-1} \),

\[ (5.16) \quad \| \nabla_x Q^+ (f_r, f_b) \|_{L_t^1 L_x^2} \lesssim (MN_2)^{1-s} M^{-s} \exp[(MN_2)^{1-s}|t|] \]

Upon multiplying by \(|T_\ast|\), we obtain a bound of \( M^{\delta - s} \), which suffices provided \( \delta < \frac{1}{3} \) and \( s > \frac{1}{2} \).

Beside \( (5.15), (5.16) \), the other two norms \( M \| \cdot \|_{L_t^2 L_x^2} \) and \( M^{-1} \| \nabla_x \cdot \|_{L_t^1 L_x^\infty} \) comprising \( Z \) are similarly bounded since the \( x \)-frequency of both terms in \( Q^+ (f_r, f_b) \) is \( \sim M \).
5.4.3. Analysis of $Q^\pm(f_b, f_r)$ and $Q^\pm(f_r, f_r)$. These terms will be treated simultaneously with the two estimates

\begin{align}
(5.17) & \quad \| \nabla_x Q^\pm(f_1, f_2) \|_{L^{2,1}_0 L^2_x} + M \| Q^\pm(f_1, f_2) \|_{L^{2,1}_0 L^2_x} \lesssim M \| f_1 \|_{L^{2,1}_0 L^2_x} \| f_2 \|_{L^{1,1}_0 L^\infty_x} \\
(5.18) & \quad \| Q^\pm(f_1, f_2) \|_{L^1_0 L^\infty_x} + M^{-1} \| \nabla_x Q^\pm(f_1, f_2) \|_{L^1_0 L^\infty_x} \lesssim \| f_1 \|_{L^1_0 L^\infty_x} \| f_2 \|_{L^1_0 L^\infty_x}
\end{align}

for

\[(1, 2) \in \{(b, r), (r, r)\}\]

We have

\[
\| f_r \|_{L^{2,1}_0 L^2_x} \sim M^{-s} \exp[(MN^2)^{1-s/2}|t|] \\
\| f_r \|_{L^1_0 L^\infty_x} \sim M^{2-s} N^{3/2} \exp[(MN^2)^{1-s/2}|t|] \lesssim M^{-s} \exp[(MN^2)^{1-s/2}|t|] \\
\| b_0 \|_{L^{2,1}_0 L^2_x} \sim M^{-s} N^{1-s} \\
\| b_0 \|_{L^1_0 L^\infty_x} \sim (MN^2)^{1-s}
\]

Substituting $(f_1, f_2) = (f_b, f_r)$ and $(f_1, f_2) = (f_r, f_r)$ into the two bounds (5.17), (5.18), we obtain

\[
\| Q^\pm(f_b, f_r) \|_{L^1_0 L^\infty_x} \lesssim (MN^2)^{1-s} M^{-s} \exp[(MN^2)^{1-s/2}|t|] \\
\| Q^\pm(f_r, f_r) \|_{L^1_0 L^\infty_x} \lesssim M^{1-2s} \exp[(MN^2)^{1-s/2}|t|]
\]

which, upon multiplying by the time factor $T_*$, yield bounds of $M^\delta - s$ and $M^\delta - s N_{2s}^s$, respectively, which suffice provided $s > \frac{1}{2}$ and $\delta < \frac{1}{4}$. 

5.4.4. Analysis of $Q^\pm(f_b, f_b)$. It suffices to estimate $\| Q^\pm(f_b, f_b) \|_{L^{2,1}_0 L^2_x}$ and

\[
\left\| \int_0^t e^{-(t-t_0)\nabla_x Q^\pm(f_b, f_b)} \right\|_{L^1_0 L^\infty_x}
\]

without the derivatives due to the specific and clear structure of $f_b$. These estimates rely on the geometric gain of some integrals based on the fine structure of the nonlinear interactions. We start with the $L^{2,1}_0 L^2_x$ estimates on which the smallness comes from the $L^2_x$ integral. The gain inside the $L^1_0 L^\infty_x$ estimate comes from the time integration in the Duhamel integral and uses extensively the $X_{s,b}$ techniques.

The $L^{2,1}_0 H^1_x$ estimates. For either the gain or loss term, the weight on $v$ and the $x$-derivative produce factors of $N_2$ and $M$ respectively, so we can reduce to estimating the $L^2_x L^2_x$ norm. The loss term, the factor $\int_v f_b(x, v, t) dv$ effectively restricts to $|x| \lesssim M^{-1}$, as in (5.9), and this effectively truncates the $x$-tubes in the other $f_b(x, v, t)$ factor.

\[
Q^-(f_b, f_b) \sim M^{2-2s} N_2^{1-2s} \chi(Mx) \sum_j \chi(MP_{e_j}^1 v) \chi(10N_2^{-1} P_{e_j}(v - N_2 e_j))
\]

Upon applying the $L^2_x$ norm, we use the disjointness of $v$-supports between terms in the sum to obtain

\[
\| Q^-(f_b, f_b) \|_{L^2_x L^2_x} \lesssim M^{2-2s} N_2^{1-2s} M^{-3/2} (MN^2) M^{-1} N_{2s}^{1/2} = M^{1-2s} N_2^{1/2 - 2s}
\]

Thus

\[
\| Q^-(f_b, f_b) \|_{L^{2,1}_0 H^1_x} + M \| Q^-(f_b, f_b) \|_{L^{2,1}_0 L^2_x} \lesssim M^{3-2s} N_2^{3/2 - 2s}
\]

Upon multiplying by the time factor $|T_*|$, this yields a bound of $M^{3/2 - s} N_2^{1/2 - s}$, which suffices for $s > \frac{1}{2}$ and $\delta \leq 2(s - \frac{1}{2})$. 

The same gain in the effective $x$-width is achieved in the gain term $Q^+(f_b, f_b)$, although it is a little more subtle. Recall (5.3) and write

\begin{equation}
    f_b(t, x, v) = \frac{M^{1-s}}{N_2^{2+s}} \sum_{j=1}^J I_j(t, x, v),
\end{equation}

then

\[ Q^+(f_b, f_b) = M^{2-2s}N_2^{4-2s} \sum_{j,k} Q^+(I_j, I_k) \]

and therefore

\[ \|Q^+(f_b, f_b)\|_{L_x^2} \lesssim M^{2-2s}N_2^{4-2s} \left( \sum_{j_1,j_2,k_1,k_2} \int x Q^+(I_{j_1}, I_{k_1})Q^+(I_{j_2}, I_{k_2}) \, dx \right)^{1/2} \]

The $x$ supports of $I_{j_1}, I_{k_1}, I_{j_2}, I_{k_2}$ are parallel to $e_{j_1}, e_{k_1}, e_{j_2}, e_{k_2}$, respectively. Typically, at least two of these directions are transverse which means that the two $x$-tubes intersect to a set with diameter $\sim M^{-1}$. Thus, upon carrying out the $x$ integration, we obtain a factor $M^{-3}$:

\[ \|Q^+(f_b, f_b)\|_{L_x^2} \lesssim M^{1-2s}N_2^{4-2s} \left( \sum_{j_1,j_2,k_1,k_2} Q^+(\hat{I}_{j_1}, \hat{I}_{k_1})Q^+(\hat{I}_{j_2}, \hat{I}_{k_2}) \right)^{1/2} \]

where $\hat{I}$ is only the $v$-part.

Thus

\[ \|Q^+(f_b, f_b)\|_{L_x^2} \lesssim M^{1-2s}N_2^{4-2s}\|Q^+(\hat{I}, \hat{I})\|_{L_\hat{v}^2} \]

where $\hat{I}$ is the $x$-independent function

\[ \hat{I}(v) = \sum_{j=1}^J \chi(M_2P_{e_j}v)\chi(10\frac{P_{e_j}(v-N_2e_j)}{N_2}) \sim 1_{\frac{1}{10}N_2\leq|v|\leq\frac{1}{10}N_2}(v) \]

Thus, using [3] Theorem 2] with $\lambda = 0, r = 2, p = 1, q = 2$ like before,

\[ \|Q^+(\hat{I}, \hat{I})\|_{L_\hat{v}^2} \lesssim N_2^{9/2}, \]

from which it follows that

\[ \|Q^+(f_b, f_b)\|_{L_x^2} \lesssim M^{1-2s}N_2^{1-2s} \]

as in the loss case.

The $L_v^1L_x^\infty$ estimates. We 1st deal with $Q^-(f_b, f_b)$ as it is shorter and prepares for the gain term which is more difficult. We estimate the Duhamel term as that is how it is going to be used

\[ D^- = \int_\tau^t e^{-(t-t_0)v\cdot\nabla_x} Q^-(f_b, f_b)(t_0) \, dt_0 \]

Plugging in (5.3) which is a linear solution and (5.8), we have

\[ D^- \sim \frac{M^{1-s}}{N_2^{2+s}} \sum_{j=1}^J \chi(MP_{e_j}(x-vt))\chi(10\frac{P_{e_j}(x-vt)}{N_2})\chi(MP_{e_j}v)\chi(10\frac{P_{e_j}(v-N_2e_j)}{N_2}) \]

\[ \times \int_\tau^t M^{-1-s} \left( \frac{N_2}{|x-v(t-t_0)|+M^{-1}} \right)^2 \chi(\frac{x-v(t-t_0)}{N_2}) \, dt_0 \]

\[ \text{The rare nearly parallel cases can be handled by a dyadic angular decomposition and corresponding reduction in the summation count, similar to that given below in the treatment of the $L_v^1L_x^\infty$ estimate for the gain term.} \]
The point is we pick up a $\frac{1}{|v|}$ in the $dt_0$ integral

$$
\int_{t}^{1} \frac{1}{|x - v(t - t_0)| + M^{-1}} \chi(\frac{x - v(t - t_0)}{N_2})dt_0
$$

which is like $N_2^{-1}$ due to the cutoff $\chi(10\frac{P_{e_j}(v-N_2e_j)}{N_2})$, and a $M$ from actually carrying out the 1D $dt_0$ integral (with some leftovers having no effects under $L^\infty_x$). So we have

$$
\|D^-\|_{L^1_tL^\infty_x} \lesssim \frac{M^{-s}}{N_2^{2+s}} (M N_2^{-1}) (M^{-2} N_2) (M N_2)^{2} \lesssim M^{-2-s} N_2^{-1-2s}
$$

where the factors $M^{-2} N_2$ and $(M N_2)^{2}$ come from $L^1_v$ and the number of summands in $J$ respectively. It is small if $s > \frac{1}{2}$.

We now turn to $Q^+(f_b, f_b)$. Just like the loss term, we would like to exploit the time integration in the Duhamel

$$
D^+ = \int_{t}^{1} e^{-(t-t_0)v \cdot \nabla_x} Q^+(f_b, f_b)(t_0)dt_0.
$$

Use the short hand (5.19)

$$
D^+ = \frac{M^{-2-2s}}{N_2^{4+2s}} \sum_{k} \sum_{j} \sum_{j} du \int_{S^2} d\omega \int_{t}^{1} e^{-(t-t_0)v \cdot \nabla_x} I_j(t_0, x, u^*) I_k(t_0, x, v^*) dt_0
$$

: $= \frac{M^{-2-2s}}{N_2^{4+2s}} \sum_{k} \sum_{j} \sum_{j} du \int_{S^2} d\omega S_{j,k}(t, x, u^*, v^*)$.

We estimate by

$$
\|D^+\|_{L^1_tL^\infty_x} \leq \frac{M^{-2-2s}}{N_2^{4+2s}} \left( \sum_{k} \sum_{j} \sum_{j} \int_{S^2} d\omega \|S_{j,k}(t, x, u^*, v^*)\|_{L^\infty_x} \right)
$$

\leq \frac{M^{-2-2s}}{N_2^{4+2s}} \sum_{k} \sum_{j} \sum_{j} \int_{S^2} d\omega \int_{t}^{1} \|S_{j,k}(t, x, u^*, v^*)\|_{L^\infty_x} dt_0
$$

\leq \frac{M^{-2-2s}}{N_2^{4+2s}} \sum_{k} \sum_{j} \sum_{j} \int_{S^2} d\omega \int_{t}^{1} \int_{S^2} d\omega \|S_{j,k}(t, x, u^*, v^*)\|_{L^\infty_x} \int_{S^2} d\omega
$$

\leq \frac{M^{-2-2s}}{N_2^{4+2s}} \sum_{k} \sum_{j} \sum_{j} \int_{S^2} d\omega \int_{t}^{1} \int_{S^2} d\omega \|S_{j,k}(t, x, u^*, v^*)\|_{L^\infty_x} \int_{S^2} d\omega
$$

We turn to

$$
\int_{S^2} d\omega \|S_{j,k}(t, x, u^*, v^*)\|_{L^\infty_x}.
$$

Notice that, $I_j(t_0, x, u^*)$ or $I_k(t_0, x, v^*)$ are not linear solutions in $(t, x, v)$ but in $(t, x, v^*)$ or $(t, x, u^*)$, doing $dt_0$ 1st does not net us a $1/|v|$ directly like in the loss term. Recall

$$
v^* = P_{\omega}^{\perp} u + P_{\omega}^{\perp} v, \quad u^* = P_{\omega}^{\perp} v + P_{\omega}^{\perp} u,
$$

$$
v = P_{\omega}^{\perp} v + P_{\omega}^{\perp} u^*, \quad u = P_{\omega}^{\perp} v^* + P_{\omega}^{\perp} u^*,
$$

we have

$$
v - u^* = P_{\omega}^{\perp}(v^* - u^*), \quad v - v^* = -P_{\omega}(v^* - u^*),$$
and hence

\[ x - v(t - t_0) - u^*t_0 = x - vt + P^\perp_\omega(v^* - u^*)t_0, \]

\[ x - v(t - t_0) - v^*t_0 = x - vt - P^\perp_\omega(v^* - u^*)t_0. \]

Hence, fixing \( u^*, v^* \), we have

\[ S_{j,k}(t, x, u^*, v^*) \]

\[ \lesssim \int_0^t \chi \left(MP_{e_j}^\perp (x - vt + P^\perp_\omega(v^* - u^*)t_0)\right) \chi \left(\frac{P_{e_j}(x - vt + P^\perp_\omega(v^* - u^*)t_0)}{N_2}\right) \chi \left(MP_{e_j}^\perp(v^*)\right) \chi \left(\frac{P_{e_k}(x - vt - P^\perp_\omega(v^* - u^*)t_0)}{N_2}\right) \chi \left(MP_{e_k}^\perp(v^*)\right) \chi \left(\frac{P_{e_k}(u^* - N_2e_k)}{N_2}\right) dt_0 \]

Substitute \( t_0 \) by \( P^\parallel_\omega(v^* - u^*) \cdot \omega \) \( t_0 \) and \( x \) by \( x - vt \), we pick up a factor \( \left| P^\parallel_\omega(v^* - u^*) \right|^{-1} \). That is

\[ \int_{S^2} d\omega \left\| S_{j,k}(t, x, u^*, v^*) \right\|_{L^\infty} \]

\[ \lesssim \chi(MP_{e_k}^\perp u^*) \chi(10 \frac{P_{e_k}(u^* - N_2e_k)}{N_2}) \chi(MP_{e_j}^\perp v^*) \chi(10 \frac{P_{e_j}(v^* - N_2e_j)}{N_2}) \]

\[ \int_{S^2} d\omega \left| P^\parallel_\omega(v^* - u^*) \right|^{-1} \int_0^\infty \chi(MP_{e_k}^\perp(x - t_0\omega)) \chi(\frac{P_{e_k}(x - t_0\omega)}{N_2}) dt_0 \]

with

\[ \int_0^\infty \chi(MP_{e_k}^\perp(x - t_0\omega)) \chi(\frac{P_{e_k}(x - t_0\omega)}{N_2}) dt_0 \]

\[ \lesssim \left| P^\parallel_\omega \left\{ x : \left( P^\perp_{e_k} x < \frac{1}{M} \right) \& \& (P_{e_k} x \lesssim N_2) \right\} \right| = \left| P^\parallel_\omega A \right| \]

where \( \left| P^\parallel_\omega A \right| \) of a set \( A \) means the length of the set \( A \) projected onto the direction \( \omega \). That is,

\[ \int_{S^2} d\omega \left\| S_{j,k}(t, x, u^*, v^*) \right\|_{L^\infty} \]

\[ \lesssim \chi(MP_{e_k}^\perp u^*) \chi(10 \frac{P_{e_k}(u^* - N_2e_k)}{N_2}) \chi(MP_{e_j}^\perp v^*) \chi(10 \frac{P_{e_j}(v^* - N_2e_j)}{N_2}) \]

\[ \int_{S^2} d\omega \left| P^\parallel_\omega(v^* - u^*) \right|^{-1} \left| P^\parallel_\omega A \right|. \]

We split into 2 cases in which Case I is when \( e_j \) and \( e_k \) are not near parallel and Case II is when \( e_j \) and \( e_k \) are near parallel. Due to the \( u^* \) and \( v^* \) cutoffs, \( u^* \) is nearly parallel to \( e_j \) and \( u^* \) is nearly parallel to \( e_k \).

For Case I, in which \( e_j \) and \( e_k \) are not near parallel, if we write \( \theta \) to be the angle between \( \omega \) and \( v^* - u^* \), then

\[ \left| P^\parallel_\omega(v^* - u^*) \right| \sim N_2 \cos \theta = N_2 \sin \left(\frac{\pi}{2} - \theta \right) \sim N_2 \left(\frac{\pi}{2} - \theta \right) \]

and we partition \( \theta \) so that \( (\frac{\pi}{2} - \theta) \) is a dyadic number going down to the scale of \((MN_2)^{-1}\). On the other hand, let \( \alpha \) be the angle between \( \omega \) and \( e_k \), then

\[ \left| P^\parallel_\omega A \right| \lesssim M^{-1} (\sin \alpha)^{-1} \sim M^{-1} \alpha^{-1} \]
due to the size of the tubes and we partition \( \alpha \) into dyadics going down to the scale of \((MN_2)^{-1}\).
For a given \((\frac{\pi}{2} - \theta)\) and \(\alpha\), the measure of the corresponding \(\omega\)-set is then

\[
\alpha \min \left( \frac{\pi}{2} - \theta, \alpha \right) \leq \alpha \left( \frac{\pi}{2} - \theta \right)
\]

Hence,

\[
\int_{\mathbb{S}^2} d\omega \left| P^\dagger_\omega (v^* - u^*) \right|^{-1} \left| P^\dagger_\omega A \right| \leq \sum_{\alpha,(\frac{\pi}{2} - \theta)} (MN_2)^{-1} \left( N_2^\alpha (\frac{\pi}{2} - \theta) \right)^{-1} \frac{M^{-1}}{\alpha} \alpha \left( \frac{\pi}{2} - \theta \right)
\]

\[
\leq (MN_2)^{-1} \sum_{\alpha,(\frac{\pi}{2} - \theta)} 1
\]

\[
\leq (MN_2)^{-1} (\ln MN_2)^2
\]

That is, in Case I, we have

\[
\| D^{7}_1 \|_{L^1_t L^3_x} \leq \frac{M^{2-2s}}{N_{2}^{4+2s}} \sum_k \sum_j \int du^* \int dv^* \int_{\mathbb{S}^2} d\omega \| S_{j,k}(t, x, u^*, v^*) \|_{L^\infty_x}
\]

\[
\leq \frac{M^{2-2s}}{N_{2}^{4+2s}} (MN_2)^4 (M^{-2}N_2)^2 (MN_2)^{-1} (\ln MN_2)^2
\]

\[
\leq (MN_2)^{1-2s} (\ln MN_2)^2
\]

where \((M^{-2}N_2)^2\) and \((MN_2)^4\) come from the \(L^1_{u,*} L^1_{u,*}\) integrals and the number of summands in \(\sum_k \sum_j\).

For Case II, in which \(e_j\) and \(e_k\) are near parallel, we reuse the computation in Case I but with \(N_2\) replaced by \(N_2\beta\) where \(\beta\) is the angle between \(e_j\) and \(e_k\). For some \(a, b \in \left[ \frac{9}{10}N_2, \frac{11}{10}N_2 \right]\), we can estimate

\[
|v^* - u^*|^2 = |ae_j - be_k|^2 = (a - b)^2 + 2ab(1 - \cos \beta) \gtrsim N_2^2 (1 - \cos \beta) \sim N_2^2 \beta^2
\]

The above also gives a reduction of the double sum \(\sum_k \sum_j\): Fix a \(e_j\), the summands in the \(k\)-sum got reduced to \((MN_2)^2 \beta^2\). Then the computation in Case I results in

\[
\| D^{7}_1 \|_{L^1_t L^3_x} \leq (MN_2)^{1-2s}
\]

That is

\[
\| D^+ \|_{L^1_t L^3_x} \leq (MN_2)^{1-2s} (\ln MN_2)^2
\]

which is good enough as long as \(s > \frac{1}{2}\) and \(\delta \leq 2(s - \frac{1}{2})\).

5.5. Bilinear \(Z\) norm estimates.

**Lemma 5.4** (Bilinear \(Z\) norm estimates for loss/gain operator \(Q^\pm\)). For any \(f_1, f_2\) and any fixed \(t \in \mathbb{R}\),

\[
\| Q^\pm(f_1, f_2) \|_Z \leq \| f_1 \|_Z \| f_2 \|_Z
\]

**Proof.** First we carry out the \(Q^-\) estimates. For the \(L^2_{v,x}\) norms,

\[
M \| \langle v \rangle Q^-(f_1, f_2) \|_{L^2_{v,x}} \leq M \| \langle v \rangle f_1 \|_{L^2_{v,x}} \| f_2 \|_{L^1_t L^3_x} \leq \| f_1 \|_Z \| f_2 \|_Z
\]

Also,

\[
\| \langle v \rangle Q^- (\nabla_x f_1, f_2) \|_{L^2_{v,x}} \leq \| \langle v \rangle \nabla_x f_1 \|_{L^2_{v,x}} \| f_2 \|_{L^1_t L^3_x} \leq \| f_1 \|_Z \| f_2 \|_Z
\]
(5.20) \[ \langle v \rangle Q^-(f_1, \nabla_x f_2) \|_{L^2_{t,x}} \leq M \| \langle v \rangle f_1 \|_{L^2_{t,x}} M^{-1} \| \nabla_x f_2 \|_{L^1_t L^\infty_x} \leq \| f_1 \| \| f_2 \|_Z \]

Next, for the \( L^1_t L^\infty_x \) norms,
\[ \| Q^- (f_1, f_2) \|_{L^1_t L^\infty_x} \leq \| f_1 \|_{L^1_t L^\infty_x} \| f_2 \|_{L^1_t L^\infty_x} \leq \| f_1 \| \| f_2 \|_Z \]

Also,
\[ M^{-1} \| Q^- (\nabla_x f_1, f_2) \|_{L^1_t L^\infty_x} \leq \| \nabla_x f_1 \|_{L^1_t L^\infty_x} \| f_2 \|_{L^1_t L^\infty_x} \leq \| f_1 \| \| f_2 \|_Z \]
\[ M^{-1} \| Q^- (f_1, \nabla_x f_2) \|_{L^1_t L^\infty_x} \leq \| f_1 \|_{L^1_t L^\infty_x} M^{-1} \| \nabla_x f_2 \|_{L^1_t L^\infty_x} \leq \| f_1 \| \| f_2 \|_Z \]

The estimate (5.20) shows the need to include the terms \( M \| \langle v \rangle f \|_{L^2_{t,x}} \) and \( M^{-1} \| \nabla_x f \|_{L^1_t L^\infty_x} \) in the definition of the \( Z \) norm.

The proofs for \( Q^+ \) follow similarly but instead use the estimates in [3, Theorem 2] as was done in §5.4.2.

5.6. Perturbation argument. We are looking to prove (5.6) for \( f_c \) solving (5.7) on
\[ T_* = -\delta(MN_2)^{s-1} \ln M \leq t \leq 0 \]

To do so, we will in fact prove bound slightly stronger than (5.6) in the \( Z \) norm, making use of the bilinear estimates for \( Q^\pm \) in Lemma 5.4, the bounds on \( f_a \) in Lemma 5.1 and the bounds on \( F_{err} \) in Lemma 5.3.

**Proposition 5.5.** Given \( s > \frac{1}{2} \), suppose that \( f_c \) solves (5.7) with \( f_c(0) = 0 \). Then for all \( t \) such that
\[ T_* = -\delta(MN_2)^{s-1} \ln M \leq t \leq 0 \]
we have the bound
\[ \| f_c(t) \|_Z \leq M^{-\delta/4} \]

**Proof.** Let the time interval \( T_* \leq t \leq 0 \) be partitioned as
\[ T_* = T_n < T_{n-1} < T_{n-2} < \cdots < T_2 < T_1 < T_0 = 0 \]
where \( T_j = -\delta j(MN_2)^{s-1} \) and \( n = \delta \ln M \). Thus, the length of each time interval \( I_j = [T_{j+1}, T_j] \) is
\[ |I_j| = \delta(MN_2)^{s-1} \]

We have
\[ \partial_t f_c(t, x - tv, v) = G(t, x - tv, v) \]
from which it follows that, for \( t \in I_j = [T_{j+1}, T_j] \),
\[ f_c(t, x - tv, v) = f_c(T_j, x - T_j v, v) + \int_{T_j}^t G(t_0, x - t_0 v, v) dt_0 \]
where \( f_c(T_0) = 0 \). Replacing \( x \) by \( x + tv \),
\[ f_c(t, x, v) = f_c(T_j, x + (t - T_j)v, v) + \int_{T_j}^t G(t_0, x + (t - t_0)v, v) dt_0 \]
Applying the $Z$-norm
\[
\|f_c\|_{L^\infty_T Z} \leq \|f_c(T_j)\|_Z + \left\| \int_{T_j}^t G(t_0, x + (t - t_0)v, v) \, dt_0 \right\|_{L^\infty_T Z}
\]
\[
\leq |I_j| \|Q^\pm(f_c, f_a)\|_{L^\infty_T Z} + |I_j| \|Q^\pm(f_a, f_c)\|_{L^\infty_T Z} + \|I_j\| \|Q^\pm(f_c, f_c)\|_{L^\infty_T Z}
\]
\[
+ \left\| \int_{T_j}^t F_{\text{err}}(t_0, x + (t - t_0)v, v) \, dt_0 \right\|_{L^\infty_T Z}
\]
For the terms on the first line, we apply the bilinear estimate in Lemma 5.4, and then the estimate on $\|f_a\|_{L^\infty_T Z}$ from Lemma 5.1. For the term on the last line, we apply the estimate from Lemma 5.3. This yields the following bound
\[
\|f_c\|_{L^\infty_T Z} \leq \|f_c(T_j)\|_Z + 2\delta C \|f_c\|_{L^\infty_T Z} + \delta C(MN_2)^{s-1} \|f_c\|_{L^\infty_T Z}^2 + M^{-\delta}
\]
where $C$ is some absolute constant independent of $\delta$. For $\delta$ sufficiently small in terms of $C$, the terms with $\|f_c\|_{L^\infty_T Z}$ on the right can be absorbed on the left yielding
\[
\|f_c\|_{L^\infty_T Z} \leq 2\|f_c(T_j)\|_Z + CM^{-\delta}
\]
Applying this successively for $j = 0, 1, \ldots$, we obtain
\[
\|f_c\|_{L^\infty_T Z} \leq (2^{j+1} - 1)CM^{-\delta}
\]
Evaluating this with $j = n = \delta \ln M$, we obtain the desired conclusion that
\[
\|f_c(T_n)\|_Z \leq 2M^{-(1-\ln 2)^{\delta}} \leq M^{-\delta/4} \ll 1
\]

**Corollary 5.6** (norm deflation). Given $s > 1/2$ and let $s_0$ be given by (5.12). There exists an exact solution $f_{\text{ex}}$ that satisfies the same estimates as (5.13), specifically
\[
\|f_{\text{ex}}(t)\|_{L^{2, s_0}_v H^s_x} \approx \frac{1}{\ln M} \exp[t(MN_2)^{1-s}]
\]
In particular,
\[
\|f_{\text{ex}}(0)\|_{L^{2, s_0}_v H^s_x} \approx \frac{1}{\ln M} \ll 1, \quad \|f_{\text{ex}}(T_n)\|_{L^{2, s_0}_v H^s_x} \approx \frac{M^\delta}{\ln M} \gg 1
\]

**Proof.** In Proposition 5.5, we have obtained the bound (5.21) for $f_c$ solving (5.7). However (5.7) is equivalent to the statement that $f_{\text{ex}} = f_a + f_c$ is an exact solution of Boltzmann. The corollary just follows from the fact that the norm $L^{2, s_0}_v H^s_x$ is controlled by the $Z$ norm, so that $f_c$ is much smaller than $f_a$ on the whole time interval $T^* \leq t \leq 0$ in $L^{2, s_0}_v H^s_x$, and the size of $f_{\text{ex}}$ in this norm matches the size of $f_a$ in this norm. \]

**Corollary 5.7** (failure of uniform continuity of the data-to-solution map). Given $s > 1/2$ and let $s_0$ be given by (5.12). For each $M \gg 1$, there exists a sequence of times $t_0^M < 0$ such that $t_0^M \not\rightarrow 0$ and two exact solutions $f_{\text{ex}}^M(t), g_{\text{ex}}^M(t)$ to Boltzmann on $t_0^M \leq t \leq 0$ such that
\[
\|f_{\text{ex}}^M(t_0^M)\|_{L^{2, s_0}_v H^s_x} \approx 1, \quad \|g_{\text{ex}}^M(t_0^M)\|_{L^{2, s_0}_v H^s_x} \approx 1
\]
with initial closeness at $t = t_0^M$
\[
\|f_{\text{ex}}^M(t_0^M) - g_{\text{ex}}^M(t_0^M)\|_{L^{2, s_0}_v H^s_x} \leq \frac{1}{\ln M} \ll 1
\]
and full separation at \( t = 0 \)
\[
\| f_{\text{ex}}^M(0) - g_{\text{ex}}^M(0) \|_{L_v^{2,s_0} H_x^{s_0}} \sim 1
\]

**Proof.** For any \( M \gg 1 \), let \( f_{\text{ex}}(t) \) be the solution given in Corollary 5.6. Let \( t_0 \) be the time \( T_0 = t_0 \leq 0 \) at which \( \| f_t(t_0) \|_{L_v^{2,s_0} H_x^{s_0}} = 1 \). Let \( g_{\text{ex}} \) be the exact solution to Boltzmann with \( g_{\text{ex}}(t_0) = f_t(t_0) \). Applying the same methods to approximate \( g_{\text{ex}}(t) \), we obtain that for all \( t_0 \leq t \leq 0 \),
\[
g_{\text{ex}}(t) = f_t(t_0) + g_c(t)
\]
where
\[
\| g_c \|_{L_v^{2,s_0} H_x^{s_0}} \leq M^{-\delta}
\]
(Conceptually, this just results from the fact that on this short time interval, \( f_t \) is nearly stationary and moveover has small self-interaction through the gain and loss terms, and thus the constant function \( f_t(t_0) \) is a good approximation to an exact solution to Boltzmann.)

Thus we have two solutions with the decompositions
\[
f_{\text{ex}}(t) = f_t(t) + f_b(t) + f_c(t)
\]
\[
g_{\text{ex}}(t) = f_t(t_0) + g_c(t)
\]
which gives
\[
f_{\text{ex}}(t) - g_{\text{ex}}(t) = (f_t(t) - f_t(t_0)) + f_b(t) + f_c(t) - g_c(t)
\]
For all \( t \),
\[
\| f_b(t) \|_{L_v^{2,s_0} H_x^{s_0}} \sim (MN_2)^{\alpha - \delta} \sim \frac{1}{(\ln M)^{1+\mu}}
\]
where \( N_2 = M^\mu \) (recall we required \( \mu \geq \delta \)). Moreover,
\[
\| f_c(t) \|_{L_v^{2,s_0} H_x^{s_0}} \leq \| f_c(t) \|_{Z} \leq M^{-\delta} \exp[|t|((MN_2)^{\delta - 1})] \leq M^{-\delta(1-\ln 2)}
\]
just like in the end of the proof of Proposition 5.5 and
\[
\| g_c(t) \|_{L_v^{2,s_0} H_x^{s_0}} \leq M^{-\delta}
\]
Thus
\[
\| f_{\text{ex}}(t_0) - g_{\text{ex}}(t_0) \|_{L_v^{2,s_0} H_x^{s_0}} \sim \frac{1}{(\ln M)^{1+\mu}}
\]
and
\[
\| f_{\text{ex}}(0) - g_{\text{ex}}(0) \|_{L_v^{2,s_0} H_x^{s_0}} \sim \| f_t(0) - f_t(t_0) \|_{L_v^{2,s_0} H_x^{s_0}} \sim 1
\]

**Appendix A. Remarks on scaling and Strichartz estimates**

In this appendix, we give some remarks on the scaling of (1.2) and its effects. Let \( f_\lambda \) be as in (1.3), then
\[
\| |\nabla| \gamma \| f_\lambda \|_{L_v^{2_\gamma}} = \| |\nabla| \gamma \| f \|_{L_v^{2_\gamma}}
\]
if and only if
\[
\alpha s - \beta r = \frac{1}{2}(\alpha - \beta).
\]
At criticality, (A.1) must hold for all \( \alpha, \beta \in \mathbb{R} \). Taking \( \alpha = \beta = 1 \), (A.1) implies \( s = r \). Setting \( s = r \) in (A.1), we obtain \( s(\alpha - \beta) = \frac{1}{2}(\alpha - \beta) \), from which we conclude that \( s = r = \frac{1}{2} \). Hence, the norm
is defined as in (1.6) and we call (1.2) $H_{\frac{s}{2}}^v$-invariant. Moreover, the case $s = r$ allows a well-defined notion of subcriticality: since

\begin{equation}
(A.2) \quad \|\nabla_x |s| v^s f_X\|_{L^2_v} = \lambda^{(s-\frac{1}{2})(\alpha-\beta)}\|\nabla_x |s| v^s f\|_{L^2_v}
\end{equation}

a large data problem over a short time can be converted into a small data problem over a long time when $s = r > \frac{1}{2}$.

Such a scaling property also affects the Strichartz estimates (1.5). Suppose $\tilde{f}$ solves (1.4). Then an estimate of the type

\begin{equation}
(A.3) \quad \|\tilde{f}\|_{L^p_{t\in I}L^p_\xi} \leq \|\tilde{f}\|_{t=0}\|L^2_\xi
\end{equation}

can be a Strichartz estimate, where either $I = (-\infty, +\infty)$, or $I$ is some fixed subinterval of time. A necessary condition for such an estimate (regardless of $I$) is that $r = p$. This follows from the fact that if $f$ solves (1.4), then

\[ \tilde{g}(x, \xi, t) = \tilde{f}(\lambda x, \lambda^{-1}\xi, t) \]

solves (1.4) on the same time interval. If $p \neq r$, then we obtain a contradiction to (A.3) by either sending $\lambda \to 0$ or $\lambda \to \infty$.

The estimate (A.3) is valid for $p = r$ when the scaling condition in (1.5) is met. The case $q = \infty$, $p = 2$ is simply the fact that the equation preserves the $L^2_\xi$ norm. The case $q = 2$, $p = 3$ follows from the endpoint argument of Keel & Tao [51], since the corresponding linear propagator $e^{it\nabla_x \cdot \nabla_\xi}$ satisfies the dispersive estimate

\[ \|e^{it\nabla_x \cdot \nabla_\xi} \phi\|_{L^\infty_{t\in I}L^\infty_{\xi}} \lesssim t^{-3/2}\|\phi\|_{L^1_{t\xi}}. \]

We note that the Strichartz estimates (1.5) are not the same as those labeled in some literature as “Strichartz estimate for the kinetic transport equation” – see [9, 10, 64] for examples.

**Appendix B. Proof of the asymmetric bilinear estimate for the loss term**

In order to give a shorter proof of (1.12) in Theorem 1.4, we shall prove instead an estimate for the $X_{0,0}$ norm (which is actually larger than $X_{0,-\frac{1}{2}+}$), although the bilinear gain factor is not fully realized in every case. Specifically, we obtain

\begin{equation}
(B.1) \quad \|\theta(t)\tilde{Q}^-(\tilde{f}, \tilde{g})\|_{X_{0,0}} \leq \min(M_1, M_2)N_2B_{M_1, M_2, N, N_2}\|\tilde{f}\|_{X_{0,\frac{1}{2}+}}\|\tilde{g}\|_{X_{0,\frac{1}{2}+}}
\end{equation}

with the bilinear gain factor

\[ B_{M_1, M_2, N, N_2} = \begin{cases} 
1 & \text{if } M_2 \leq M_1, \ N \leq N_2 \\
(M_1/M_2)^{1/2} & \text{if } M_1 \ll M_2 \sim M, \ N \ll N_2 \\
(M_2/N)^{1/2} & \text{if } M_2 \ll M_1 \sim M, \ N_2 \ll N \\
(M_1)^{1/4}(N_2/N)^{1/4} & \text{if } M_1 \ll M_2, \ N_2 \ll N
\end{cases} \]

Note that in the statement of Theorem 1.4 the estimate claimed for the $X_{0,-\frac{1}{2}+}$ norm in (1.12) for the case $M_1 \ll M_2, N_2 \ll N$ has the stronger bilinear gain factor:

\[ B_{M_1, M_2}B_{N, N_2} = \left(\frac{M_1}{M_2}\right)^{1/2}\left(\frac{N_2}{N}\right)^{1/2} \]

Otherwise, the estimate (B.1) matches (1.12) for the $X_{0,-\frac{1}{2}+}$ norm. The main point here is the fact that the bilinear gain factor is asymmetric in $f$ and $g$. One could bring in the cone-washer decomposition like (2.2) to reach the stronger bilinear gain factor, but that could be too much for one estimate not used in this paper.
To prove (B.1), it suffices, by a standard reduction, to assume that \( \hat{f} = e^{it\nabla_x\cdot\nabla_\xi \tilde{\phi}} \) and \( \hat{g} = e^{it\nabla_x\cdot\nabla_\psi \tilde{\psi}} \). Then
\[
\| \mathcal{Q}^-(\hat{f}, \hat{g}) \|_{X_{0,0}} = \| e^{-it\nabla_x\cdot\nabla_\xi \hat{\phi}} (e^{it\nabla_x\cdot\nabla_\xi \hat{\phi}} - e^{it\nabla_x\cdot\nabla_\xi \hat{\psi}}) \|_{L^2_{x,t}}.
\]

Passing to the Fourier side \((x, \xi) \mapsto (\eta, v)\), and using a dual pairing with a function \( \hat{\zeta}(\eta, v, \tau) \in L^2_{\eta,v,\tau} \), it suffices to bound\(^9\)
\[
\int_{\eta, \eta_2, v, v_2} \hat{\phi}(\eta - \eta_2, v) \hat{\psi}(\eta_2, v_2) \hat{\zeta}(\eta, v, \eta_2 \cdot (v_2 - v)) d\eta d\eta_2 dv dv_2
\]
\[
\lesssim \min(M_1, M_2) N_2 B_{M_1, M_2 B_{N_2}} \| \hat{\phi} \|_{L^2} \| \hat{\psi} \|_{L^2} \| \hat{\zeta} \|_{L^2}
\]
where we can assume each factor \( \hat{\phi} \geq 0, \hat{\psi} \geq 0, \hat{\zeta} \geq 0 \). Replacing \( w_2 = v_2 - v \),
\[
= \int_{\eta, \eta_2, v, w_2} \hat{\phi}(\eta - \eta_2, v) \hat{\psi}(\eta_2, w_2 + v) \hat{\zeta}(\eta, v, \eta_2 \cdot w_2) d\eta d\eta_2 dv dw_2
\]
We use superscripts to denote components, for example \( \eta_2 = (\eta^1_2, \eta^2_2, \eta^3_2) \).

**Case 1.** \( M_2 \leq M_1 \). Assume that
\[
|\eta^2_2| = \max(|\eta^1_2|, |\eta^2_2|, |\eta^3_2|) \sim |\eta_2| \sim M_2
\]
(the other two cases are similar). Move the \( \eta, \eta_2, w_1^2, w_2^2 \) integration to the outside and the \( v, w_3^2 \) integration on the inside. Cauchy-Schwarz in \( \eta, w_3^2 \) to obtain
\[
\lesssim \int_{v, \eta_2, w_1^2, w_2^2} \| \hat{\phi}(\eta - \eta_2, v) \hat{\psi}(\eta_2, w_2 + v) \|_{L^2_{\eta, w_2^2}} \| \hat{\zeta}(\eta, v, \eta_2 \cdot w_2) \|_{L^2_{\eta, w_2^2}} d\eta_2 dw_1^2 dw_2^2 dv
\]
For the \( \hat{\zeta} \) term, change variable from \( w_2^3 \) to \( \tau = \eta_2 \cdot w_2 \) (here, \( \eta_2 \) and \( w_1^3, w_2^3 \) can be regarded as fixed, since they are in the outside integration). The change of differential is \( d\tau = |\eta^2_2| dw_3^2 \), and since \( |\eta^2_3| \sim M_2 \),
\[
\lesssim M_2^{-1/2} \int_v \left( \int_{\eta_2, w_1^2, w_2^2} \| \hat{\psi}(\eta_2, w_2 + v) \|_{L^2_{w_2^2}} d\eta_2 dw_1^2 dw_2^2 \right) \| \hat{\phi}(\eta_1, v) \|_{L^2_{\eta_1}} \| \hat{\zeta}(\eta_1, v, \tau) \|_{L^2_{\eta_1}} dv
\]
For fixed \( v \), the \( w_1^2, w_2^2 \) integrations are confined to sets of width \( N_2 \) (even if \( N \gg N_2 \)), due to the fact that \( w_2 = v + v_2 \). In the inner integral, Cauchy-Schwarz in \( \eta_2, w_1^2, w_2^2 \) with the whole integrand in one factor and 1 in the other to obtain the factor \( M_2^{3/2} N_2 \) coming from the support,
\[
\lesssim M_2^{-1/2} M_2^{3/2} N_2 \| \hat{\psi} \|_{L^2_{w_2^2}} \int_v \| \hat{\phi}(\eta_1, v) \|_{L^2_{\eta_1}} \| \hat{\zeta}(\eta_1, v, \tau) \|_{L^2_{\eta_1}} dv
\]
Cauchy-Schwarz in \( v \) to obtain
\[
\lesssim M_2 N_2 \| \hat{\phi} \|_{L^2} \| \hat{\psi} \|_{L^2} \| \hat{\zeta} \|_{L^2}
\]

**Case 2.** \( M_1 \ll M_2 \sim M \). Divide the \( \eta \) and \( \eta_2 \) space into cubes of size \( M_1 \). Once one of these cubes in \( \eta_2 \) space is selected, the inner \( \eta \) integral is confined to \( O(1) \) cubes. We can therefore carry the Case 1 argument above out and it will yield instead the factor \( M_2^{-1/2} M_1^{3/2} N_2 \). We finish with Cauchy-Schwarz over the cube partition (which incurs no loss),
\[
\lesssim M_2^{-1/2} M_1^{3/2} N_2 \| \hat{\phi} \|_{L^2} \| \hat{\psi} \|_{L^2} \| \hat{\zeta} \|_{L^2}
\]
\(^9\)Notice that, such a process does not require the Fourier transform of the collision kernel as we estimate in the \( v \)-side.
Case 3. $M_2 \leq M_1$ and $N \gg N_2$. Without loss we may assume that
$$|w^3_2| = \max(|w^1_2|, |w^2_2|, |w^3_2|) \sim N$$
(the other two cases are similar). Move the $v, w_2, \eta_1, \eta_2^3$ integration to the outside, bring the $\eta, \eta_2^3$ integration to the inside, and Cauchy-Schwarz in $\eta, \eta_2^3$ to obtain
$$\leq \int_{v, w_2, \eta_1, \eta_2^3} \|\phi(\eta - \eta_2, v)v\|_{L^2} \|\hat{\psi}(\eta_2, w_2 + v)\|_{L^2} \|\hat{\zeta}(\eta, v, \eta_2 \cdot w_2)\|_{L^2} \, dv \, dw_2 \, d\eta \, d\eta_2^3$$
For the $\hat{\zeta}$ term, change variable from $\eta^3_2$ to $\tau = \eta_2 \cdot w_2$, which has differential conversion $d\tau = |w^3_2| d\eta^3_2$ ($w_2$ is fixed, since the term is inside the $w_2$ integration). Since $|w^3_2| \sim N$,
$$\leq N^{-1/2} \int_{v, w_2, \eta_1, \eta_2^3} \|\phi(\eta_1, v)\|_{L^2} \|\hat{\psi}(\eta_2, w_2 + v)\|_{L^2} \|\hat{\zeta}(\eta, v, \tau)\|_{L^2} \, dv \, dw_2 \, d\eta \, d\eta_2^3$$
Converting back from $w_2$ to $v = w_2 + v$, the remaining integrals split:
$$= N^{-1/2} \int_v \|\phi(\eta_1, v)\|_{L^2} \|\hat{\zeta}(\eta, v, \tau)\|_{L^2} \, dv \int_{v_2, \eta_1, \eta_2^3} \|\hat{\psi}(\eta_2, v_2)\|_{L^2} \, dv_2 \, d\eta \, d\eta_2^3$$
In the first integral, Cauchy-Schwarz in $v$, and in the second integral, Cauchy-Schwarz with the whole integrand in one part and 1 in the other to pick up the support factor $N^{3/2}_2 M_2$.
$$\leq N^{-1/2} N^{3/2}_2 M_2 \|\hat{\phi}\|_{L^2} \|\hat{\psi}\|_{L^2} \|\hat{\zeta}\|_{L^2}$$
Case 4. $M_1 \ll M_2$ and $N \gg N_2$. Divide the $\eta$ and $\eta_2$ space into cubes of size $M_1$. Once one of these cubes in $\eta_2$ space is selected, the inner $\eta$ integral is confined to $O(1)$ cubes. We can therefore carry the Case 3 argument above out and it will yield instead the factor $N^{-1/2} N^{3/2}_2 M_1$. We finish with Cauchy-Schwarz over the cube partition (which incurs no loss).
$$\leq N^{-1/2} N^{3/2}_2 M_1 \|\hat{\phi}\|_{L^2} \|\hat{\psi}\|_{L^2} \|\hat{\zeta}\|_{L^2}$$
Alternatively, we can appeal to Case 2 to obtain
$$\leq M_2^{-1/2} M_1^{3/2} N_2 \|\hat{\phi}\|_{L^2} \|\hat{\psi}\|_{L^2} \|\hat{\zeta}\|_{L^2}$$
Taking the average gives
$$\leq \left( \frac{M_1}{M_2} \right)^{1/4} \left( \frac{N_2}{N} \right)^{1/4} M_1 N_2 \|\hat{\phi}\|_{L^2} \|\hat{\psi}\|_{L^2} \|\hat{\zeta}\|_{L^2}$$

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Department of Mathematics, University of Rochester, Rochester, NY 14627
Email address: xuwenmath@gmail.com
URL: http://www.math.rochester.edu/people/faculty/xchen84/

Department of Mathematics, Brown University, 151 Thayer Street, Providence, RI 02912
Email address: justin_holmer@brown.edu
URL: http://www.math.brown.edu/~holmer/