Constructions stemming from nonseparating planar graphs and their Colin de Verdière invariant

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A planar graph $G$ is said to be nonseparating if there exists an embedding of $G$ in $\mathbb{R}^2$ such that, for any cycle $\mathcal{C} \subseteq G$, all vertices of $G \setminus \mathcal{C}$ are within the same connected component of $\mathbb{R}^2 \setminus \mathcal{C}$. Dehkordi and Farr classified the nonseparating planar graphs as either outerplanar graphs, subgraphs of wheel graphs, or subgraphs of elongated triangular prisms. We use maximal nonseparating planar graphs to construct examples of maximal linkless graphs and maximal knotless graphs. We show that, for a maximal nonseparating planar graph $G$ with $n \geq 7$ vertices, the complement $cG$ is $(n-7)$–apex. This implies that the Colin de Verdière invariant of the complement $cG$ satisfies $\mu(cG) \leq n - 4$. We show this to be an equality. As a consequence, the conjecture of Kotlov, Lovász and Vempala that, for a simple graph $G$, $\mu(G) + \mu(cG) \geq n - 2$ is true for 2–apex graphs $G$ for which $G - \{u, v\}$ is planar nonseparating. It also follows that complements of nonseparating planar graphs of order at least nine are intrinsically linked. We prove that the complements of nonseparating planar graphs $G$ of order at least ten are intrinsically knotted.

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1 Introduction

All graphs in this paper are finite and simple. A graph is intrinsically linked (IL) if every embedding of it in $\mathbb{R}^3$ (or $S^3$) contains a nontrivial 2–component link. A graph is linklessly embeddable if it is not intrinsically linked (nIL). A graph is intrinsically knotted (IK) if every embedding of it in $\mathbb{R}^3$ (or $S^3$) contains a nontrivial knot. The combined work of Conway and Gordon [1983], Sachs [1984] and Robertson, Seymour and Thomas [Robertson et al. 1993] fully characterize IL graphs: a graph is IL if and only if it contains a graph in the Petersen family as a minor. The Petersen family consists of seven graphs obtained from the complete graph $K_6$ by $\nabla Y$ moves and $Y \nabla$ moves, as described in Figure 1.

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The $\nabla Y$ move and the $Y \nabla$ move preserve the IL property. While $K_7$ and $K_{3,3,1,1}$ together with many other minor minimal IK graphs have been found [Goldberg et al. 2014; Conway and Gordon 1983; Foisy 2002], a characterization of IK graphs is not fully known. While the $\nabla Y$ move preserves the IK property [Motwani et al. 1988], the $Y \nabla$ move doesn’t preserve it [Flapan and Naimi 2008]. A graph is said to be $k$–apex if it can be made planar by removing $k$ vertices. If $G$ and $H$ denote two simple graphs with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, respectively, then the sum $G + H$ denotes the simple graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup L$, where $L$ denotes the set of all edges with one endpoint in $V(G)$ and the other in $V(H)$.

A planar graph $G$ is nonseparating if there exists an embedding of $G$ in $\mathbb{R}^2$ such that, for any cycle $\mathcal{C} \subset G$, all vertices of $G \setminus \mathcal{C}$ are within the same connected component of $\mathbb{R}^2 \setminus \mathcal{C}$. By [Dehkordi and Farr 2021], a nonseparating planar graph is one of three types:

1. an outerplanar graph,
2. a subgraph of a wheel,
3. a subgraph of an elongated triangular prism.

In Section 2, we consider sums between maximal nonseparating planar graphs and small empty graphs, complete graphs or paths to construct maximal linklessly embeddable graphs and maximal knotlessly embeddable graphs. A simple graph $G$ is called maximal linklessly embeddable (maxnIL) if it is not a proper subgraph of a nIL graph of the same order. A simple graph $G$ is called maximal knotlessly embeddable (maxnIK) if it is not a proper subgraph of a nIK graph of the same order. Constructions and properties of maxnIL graphs can also be found in [Aires 2021; Naimi et al. 2023], and for maxnIK graphs in [Eakins et al. 2023].

Colin de Verdière [1990] introduced the graph invariant $\mu$, which is based on spectral properties of matrices associated with the graph $G$. He showed that $\mu$ is monotone under taking minors and that planarity is characterized by the inequality $\mu \leq 3$. By [Lovász and Schrijver 1998; Robertson et al. 1993], it is known that linkless embeddability is characterized by the inequality $\mu \leq 4$. By reformulating the definition of $\mu$ in terms of vector labelings, Kotlov, Lovász and Vempala [Kotlov et al. 1997] related the topological properties of a graph to the $\mu$ invariant of its complement: for $G$ a simple graph on $n$ vertices,

1. if $G$ is planar, then $\mu(cG) \geq n - 5$;
2. if $G$ is outerplanar, then $\mu(cG) \geq n - 4$;
3. if $G$ is a disjoint union of paths, then $\mu(cG) \geq n - 3$.

For $G$ a graph with $n$ vertices $v_1, v_2, \ldots v_n$, $cG$ denotes the complement of $G$ in the complete graph $K_n$. The graph $cG$ has the same set of vertices as $G$ and $E(cG) = \{v_i v_j \mid v_i v_j \notin E(G)\}$.

By [Battle et al. 1962], the complement of a planar graph with nine vertices is not planar. This is also implied by the inequality $\mu(cG) \geq n - 5$. Here we show a stronger inequality for maximal nonseparating planar graphs. In Section 3, we prove two theorems.
Theorem 1  If $G$ is a maximal nonseparating planar graph with $n \geq 7$ vertices, then $cG$ is $(n-7)$–apex.

Theorem 1 establishes the upper bound $\mu(cG) \leq n - 4$ for $G$ a maximal nonseparating planar graph, since $\mu \leq 3$ for planar graphs and adding one vertex increases the value of $\mu$ by at most one [van der Holst et al. 1999]. We prove this is an equality.

Theorem 2  For $G$ a maximal nonseparating planar graph with $n \geq 7$ vertices, $\mu(cG) = n - 4$.

Kotlov et al. [1997] conjectured that, for a simple graph $G$, $\mu(G) + \mu(cG) \geq n - 2$. We revisit results about $\mu$ to show the conjecture is true for planar graphs and 1–apex graphs. As a consequence of Theorem 2, the conjecture holds for 2–apex graphs $G$ for which $G - \{u, v\}$ is planar nonseparating. Theorem 2 also implies that, for $G$ a maximal nonseparating planar graph with nine vertices, $\mu(cG) = 5 > 4$, and thus $cG$ is intrinsically linked. While the relationship between the $\mu$ invariant and intrinsic linkness is well understood, the same is not true for intrinsic knottedness. The inequality $\mu(cG) \geq n - 5$ for planar graphs $G$ implies that complements of planar graphs with ten vertices are intrinsically linked. Theorem 2 establishes that, for $G$ a maximal nonseparating planar graph with ten vertices, $\mu(cG) = 6$, but this does not imply that $cG$ is intrinsically knotted. There are known IK graphs with $\mu = 5$ [Foisy 2003; Mattman et al. 2021], as well as nIK graphs with $\mu = 6$ [Flapan and Naimi 2008]. In Section 4, we do a case-by-case analysis to prove the following theorem:

Theorem 3  If $G$ is a nonseparating planar graph on ten vertices, then $cG$ is intrinsically knotted.

Since the complement of a nonseparating planar graph contains the complement of a maximal nonseparating planar graph of the same order as a subgraph, it suffices to prove Theorem 3 for maximal nonseparating planar graphs, namely

1. maximal outerplanar graphs,
2. the wheel graph,
3. elongated triangular prisms.

A similar approach to that presented in Section 4 works to prove that:

(a) If $G$ is a nonseparating planar graph on seven vertices, then $cG$ is not outerplanar.
(b) If $G$ is a nonseparating planar graph on eight vertices, then $cG$ is nonplanar.
(c) If $G$ is a nonseparating planar graph on nine vertices, then $cG$ is intrinsically linked.

For outerplanar graphs $G$ with at most nine vertices, these results can also be obtained using the graph invariant $\mu$, since, for such graphs $G$, $\mu(cG) \geq n - 4$ [Kotlov et al. 1997].

2  MaxnIL and maxnIK graphs

In this section, we use maximal nonseparating planar graphs to build examples of maxnIL and maxnIK graphs. Jørgensen [1989] and Dehkordi and Farr [2021] considered the class of graphs of the type $H + E_2$,
where $E_2$ denotes the graph with two vertices and no edges and $H$ is an elongated prism. Jørgensen proved that these graphs are maximal with no $K_6$ minors. Dehkordi and Farr proved that these graphs are maxnIL. Here we add to this type of example by taking the sum of maximal nonseparating planar graphs with small empty graphs, complete graphs and paths. Sachs [1984] proved that 1–apex graphs are nIL and 2–apex graphs are nIK. A theorem of Mader [1968] shows that a graph $G$ with $n$ vertices and $4n - 9$ edges, with $n \geq 6$, contains a $K_6$ minor, and a graph $G$ with $n$ vertices and $5n - 14$ edges, with $n \geq 7$, contains a $K_7$ minor. We combine these results into the following useful lemma:

**Lemma 4** A maximal 1–apex graph is maxnIL. A maximal 2–apex graph is maxnIK.

A vertex of a graph $H$ which is incident to all the other vertices of $H$ is a cone. We also say that $v$ cones over the subgraph induced by all the vertices of $H$ minus $v$. Let $W_n$ denote the wheel graph of order $n \geq 4$. Let $P_2$ be the graph with vertex set $V(P_2) = \{u, v, w\}$ and edge set $E(P_2) = \{\{u, w\}, \{v, w\}\}$. Let $K_3$ denote the complete graph on vertices $\{u, v, w\}$. Using Lemma 4, we derive the following result:

**Theorem 5**

1. The graph $G \simeq W_n + E_2$ is maxnIL.
2. If $H$ is a maximal outerplanar graph of order $n \geq 4$, then $G \simeq H + K_2$ is a maxnIL graph.
3. The graph $G \simeq W_n + P_2$ is maxnIK.
4. If $H$ is a maximal outerplanar graph of order $n \geq 4$, then $G \simeq H + K_3$ is a maxnIK graph.

**Proof** For the first two cases, the graph $G$ is maximal 1–apex, and thus maxnIL. For the last two cases, the graph $G$ is maximal 2–apex, and thus maxnIK. 

For the elongated prism case, we distinguish two cases, according to the number of nontriangular edges of the triangular prism which are subdivided.

**Theorem 6** Let $H$ denote an elongated prism of order $n \geq 6$ obtained by repeated subdivisions of at most two of three nontriangular edges of the prism graph. Then $G \simeq H + P_2$ is a maxnIK graph.

Figure 2: An elongated prism with only two edges subdivided (left) and a planar graph obtained by deleting the vertices $t$ and $w$ of $H + P_2$ (right).
Proof Assume that $H$ is isomorphic to the graph depicted in Figure 2, left, in which the edge $\{v_3, v_4\}$ is not subdivided. Perform a $\nabla Y$ move on the triangle induced by the vertices $\{v_3, v_4, u\}$ by deleting the edges $\{v_3, v_4\}, \{v_3, u\}$ and $\{v_4, u\}$ and adding a new vertex $t$ incident to all of $\{v_3, v_4, u\}$ to obtain a new graph $G'$. This graph is 2–apex, since deleting the vertices $t$ and $w$ gives the planar graph of Figure 2, right. Thus, $G'$ is nIK, and so must be $G$, as the $\nabla Y$ move preserves the IK property [Motwani et al. 1988].

To show that $G$ is maximal nIK, one notices that $G$ is isomorphic to a cone $w$ over $H + E_2$. Since $H + E_2$ is maxnIL by [Dehkordi and Farr 2021], adding any edge to $G$ produces a structure of a cone over an IL graph. This structure will contain a minor isomorphic to a graph in either the $K_7$ family or the $K_{3,3,1,1}$ family, and will therefore be IK.

Theorem 7 Let $H$ denote an elongated prism of order $n \geq 9$ obtained by repeated subdivisions of all three nontriangular edges of the prism graph. Then $G \simeq H + P_2$ is an IK graph.

Proof By repeated edge contractions applied to $G$, one obtains the minor $S \simeq P' + P_2$, where $P'$ is the graph depicted in Figure 3, left.

Foisy [2002] proved that, if a graph contains a doubly linked $D_4$ minor in every embedding, the graph must be IK. This result was also proved independently by Taniyama and Yasuhara [2001]. The graph $D_4$ is depicted in Figure 3, right. An embedding of the graph $D_4$ is doubly linked if the linking numbers $\text{lk}(C_1, C_3)$ and $\text{lk}(C_2, C_4)$ are both nonzero mod 2. We used a Mathematica program written by Naimi to show that $S$ has a doubly linked $D_4$ minor in every embedding.

3 The $\mu$ invariant

In this section we determine the value of the $\mu$ invariant for complements of maximal nonseparating planar graphs. By [van der Holst et al. 1999], if $G$ is planar with $n$ vertices, then $\mu(cG) \geq n - 5$. We first show the inequality $\mu(cG) \leq n - 4$ for graphs $G$ which are maximal nonseparating planar. In Theorem 2, we show this is in fact an equality.

Kotlov et al. [1997] conjectured that, for a simple graph $G$, $\mu(G) + \mu(cG) \geq n - 2$. We review that the conjecture holds for planar graphs and 1–apex graphs. We show that, as a consequence of Theorem 2, the conjecture holds for 2–apex graphs $G$ for which $G - \{u, v\}$ is planar nonseparating.
Figure 4: A maximal outerplanar graph with seven vertices (left), the graph $G$, a wheel with $n$ vertices (center), and $cG \setminus \{v_7, v_8, \ldots, v_{n-1}\}$ (right).

**Theorem 1**  If $G$ is a maximal nonseparating planar graph with $n \geq 7$ vertices, then $cG$ is $(n-7)$–apex.

**Proof**  We treat the three types in turn:

**Outerplanar case** Any maximal outerplanar graph $H$ of order $n \geq 3$ can be represented by a triangulated $n$–cycle in the plane (with the unbounded face containing all vertices). The $n$–cycle contains at least one $2$–chord, an edge which forms a triangle with two adjacent edges along the cycle. We say that the $2$–chord isolates the vertex which is part of the triangle but is not incident to the $2$–chord. For example, in Figure 4, left, the $2$–chord $v_1v_6$ isolates the vertex $v_7$ and the $2$–chord $v_1v_5$ of $H \setminus \{v_7\}$ isolates $v_6$. The complement of the unique maximal outerplanar graph with five vertices is $P_3$, a path with three edges, together with an isolated vertex. It follows that the complement of any maximal outerplanar graph with seven vertices is planar, since the deletion of two vertices gives a path with three edges and an isolated vertex. For example, after the deleting the vertices $v_7$ and $v_6$, the complement of the graph in Figure 4, left, is the path $v_1v_3v_5v_2$ together with the isolated vertex $v_4$. Starting with a maximal outerplanar graph with $n \geq 7$ vertices, one can recursively delete $n-7$ isolated vertices and obtain a maximal outerplanar graph of order 7. The same sequence of $n-7$ vertex deletions gives a planar subgraph of $cG$. Thus, $cG$ is $(n-7)$–apex.

**Wheel case** Let $G$ be the wheel on $n$ vertices. Then $cG \simeq (K_{n-1} \setminus C_{n-1}) \cup K_1$. Let $\{v_1, v_2, \ldots, v_{n-1}\}$ be the vertices of $C_{n-1}$ in consecutive order, as in Figure 4, center. Then $cG \setminus \{v_7, v_8, \ldots, v_{n-1}\}$ is a planar graph (the triangular prism added one edge, together with an isolated vertex) and thus $cG$ is $(n-7)$–apex. See Figure 4, right.

**Elongated prism case** Let $G$ be an elongated prism with $n \geq 7$ vertices. Without loss of generality, let $v_1v_3v_5$ be one of two induced triangles of $G$. Let $a, b$ and $c$ denote their respective neighbors in $V(G) \setminus \{v_1, v_3, v_5\}$, as in Figure 5, left. Deleting all vertices but $\{v_1, v_3, v_5, a, b, c\}$ in $cG$ gives the subgraph of the outerplanar graph with six vertices in Figure 5, right. Deleting any $n-7$ vertices of $cG$ none of which is in the set $\{v_1, v_3, v_5, a, b, c\}$ yields a planar graph, and thus $cG$ is $(n-7)$–apex.

**Corollary 8**  For $G$ a maximal nonseparating planar graph with $n \geq 7$ vertices, $\mu(cG) \leq n-4$. 

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Proof By Theorem 1, \( cG \) is \((n-7)\)-apex. Let \( H \) be the planar subgraph of \( cG \) obtained by deleting \( n-7 \) vertices. Then \( \mu(H) \leq 3 \) and \( \mu(cG) \leq 3 + (n-7) = n-4 \), since adding one vertex to a graph increases the value of \( \mu \) by at most one (see [van der Holst et al. 1999, Theorem 2.7]).

Corollary 8 establishes an upper bound of \( n-4 \) for the values of \( \mu \) of complements of maximal nonseparating planar graphs on \( n \) vertices. We show that \( n-4 \) is the actual value of \( \mu \). We use [van der Holst et al. 1999, Theorem 5.5], which says that, for \( H \) a graph on \( n \) vertices and \( \nu(H) := n - \mu(cH) - 1 \), the inequality \( \nu(H) \leq 2 \) holds if and only if \( H \) does not contain as a subgraph any of the five graphs in Figure 6. We also use that, for a graph \( G \) with at least one edge, \( \mu(G + K_1) = \mu(G) + 1 \) by [van der Holst et al. 1999, Theorem 2.7].

Theorem 2 For \( G \) a maximal nonseparating planar graph with \( n \geq 7 \) vertices, \( \mu(cG) = n-4 \).

Proof Corollary 8 established the inequality \( \mu(cG) \leq n-4 \). Here we show that \( \mu(cG) \geq n-4 \). If \( G \) is outerplanar, then \( \mu(cG) \geq n-4 \) [Kotlov et al. 1997]. If \( G \) is the wheel graph on \( n \) vertices, \( cG = cC_{n-1} \cup K_1 \). By [van der Holst et al. 1999, Theorem 5.5], \( \nu(C_{n-1}) \leq 2 \) and we have
\[
\mu(cG) = \mu(cC_{n-1}) = n - 1 - \nu(C_{n-1}) - 1 \geq n-4.
\]

For elongated prisms, we distinguish two cases, according to the number of nontriangular edges of the prism which are being subdivided:

Case 1 Consider \( G \) the elongated prism in Figure 7, left, with exactly one nontriangular edge of the prism graph subdivided, \( v_1v_2 \). If at least two vertices are added along \( v_1v_2 \), as in Figure 7, left, consider...
the graph $H = G - \{v_1, v_2\}$. Then $\mu(cH) = (n - 2) - v(H) - 1 \geq n - 5$, by [van der Holst et al. 1999, Theorem 5.5]. Since in $cG$ the set of adjacent vertices $\{v_1, v_2\}$ cones over $cH$, $\mu(cG) \geq n - 4$ by [van der Holst et al. 1999, Theorem 2.7]. If only one vertex is added along the one edge, as in Figure 7, center, the set of adjacent vertices $\{v_1, v_2\}$ no longer cones over $cH$. However, in this case, $cG$ contains a $K_4$ minor, and thus $\mu(cG) \geq 3$. See Figure 7, right.

Case 2 Assume $G$ is obtained from the triangular prism by subdividing edges $v_1v_2$ and $v_5v_6$ along the way, as in Figure 8, left. The graph $H = G - \{v_1, v_6\}$ is a path with $n-2$ vertices, so $\mu(cH) \geq n - 5$ [Kotlov et al. 1997]. In $cG$, the set of adjacent vertices $\{v_1, v_6\}$ cones over $cH$, yielding $\mu(cG) \geq \mu(cH) + 1 \geq n - 4$ by [van der Holst et al. 1999, Theorem 2.7].

We briefly discuss the state of a conjecture of [Kotlov et al. 1997], that, for a simple graph $G$ on $n$ vertices, $\mu(G) + \mu(cG) \geq n - 2$. By [Kotlov et al. 1997; Colin de Verdière 1990; van der Holst et al. 1999], the conjecture holds if either one of $G$ or $cG$ is planar. We note that the conjecture holds if $\mu(G) \geq n - 6$ or $\mu(cG) \geq n - 6$. Assume $\mu(G) \geq n - 6$. If $\mu(cG) \geq 4$, then $\mu(G) + \mu(cG) \geq n - 2$; if $\mu(cG) < 4$, $\mu(G)$ is planar, and the conjecture holds.

**Proposition 9** The conjecture holds for 1–apex graphs.

**Proof** Let $G$ be a 1–apex graphs with $n$ vertices and $H = G - \{v\}$ planar. Then $\mu(cH) \geq (n-1) - 5 = n - 6$ [Kotlov et al. 1997]. We have that $cH$, the complement of $H$ in $K_{n-1}$, is a subgraph of $cG$, the complement of $G$ in $K_n$, since $cG$ may have additional edges incident to $v$, and so $n - 6 \leq \mu(cH) \leq \mu(cG)$. Thus, the conjecture holds for $G$. 

![Figure 7: An elongated prism with one nontriangular edge subdivided by more than one vertex (left), an elongated prism with one nontriangular edge subdivided by exactly one vertex (center) and the complement of that graph (right).](image1)

![Figure 8: An elongated prism $G$ with two subdivided edges (left) and $H = G - \{v_1, v_6\}$ (right).](image2)
Corollary 10  Let $G$ be a 2–apex graph with $n$ vertices with $H = G - \{u, v\}$ planar nonseparating. Then $\mu(G) + \mu(cG) \geq n - 2$.

Proof Since $H$ is planar nonseparating, by Theorem 2, $\mu(cH) \geq (n - 2) - 4 = n - 6$, with equality if $H$ is maximal. We have that $cH$, the complement of $H$ in $K_{n-2}$, is a subgraph of $cG$, the complement of $G$ in $K_n$, since $cG$ may have additional edges incident to $u$ and $v$, and so $\mu(cG) \geq \mu(cH) \geq n - 6$. Thus, the conjecture holds for $G$. \hfill \square

4 Graphs of order ten

The relationship between the $\mu$ invariant and the property of being intrinsic knotted is not well understood. While Theorem 2 establishes that, for $G$ a maximal nonseparating planar graph with ten vertices, $\mu(cG) = 6$, this information has no bearing on whether $cG$ is intrinsically knotted. Flapan and Naimi [2008] prove that the IK property is not preserved by the $Y \nabla$ move by showing a graph in the $K_7$ family which is not intrinsically knotted. Since $\mu(K_7) = 6$ and both the $\nabla Y$ move and the $Y \nabla$ move preserve $\mu$ for $\mu \geq 4$ [van der Holst et al. 1999], this nIK graph has $\mu = 6$. On the other hand, Foisy [2003] and Mattman et al. [2021] provide examples of IK graphs with $\mu = 5$. In this section, we do a case-by-case analysis to prove that, for $G$ a maximal nonseparating planar graph with ten vertices, $cG$ is intrinsically knotted. We recall that the $\nabla Y$ move preserves the IK property. In some cases, graphs are shown to be IK because they are obtained through one or more $\nabla Y$ moves from IK graphs such as $K_7$ or $K_{3,3,1,1}$. In other cases, graphs $G$ are shown to be IK because the graphs obtained from $G$ by one or more $Y \nabla$ moves contain $K_7$ or $K_{3,3,1,1}$ minors.

Lemma 11  If $G$ is a maximal outerplanar graph with ten vertices, then $cG$ is intrinsically knotted.

Proof We label the vertices of $G$ by $v_1, v_2, \ldots, v_9, v_{10}$ in clockwise order around the cycle $\mathcal{C}$ bordering the outer face of a planar embedding. See Figure 9. We organize the proof according to the longest chord of $\mathcal{C}$. The length of a chord is defined as the length of the shortest path in $\mathcal{C}$ between the endpoints of the chord. In each case we show the complement $cG$ contains an intrinsically knotted graph as a minor. We remark that, within any triangulation of the disk bounded by $\mathcal{C}$, out of a total of seven chords, at most six have length 2 or 3. Thus there exist chords of length 4 or 5.

Case (a) If the cycle $\mathcal{C}$ has a chord of length 5, we may assume without loss of generality that $v_1v_6 \in E(G)$. Consider the cycles $\mathcal{C}_1 := v_1v_6v_7v_8v_9v_{10}$ and $\mathcal{C}_2 := v_1v_2v_3v_4v_5v_6$. We note that $\mathcal{C}$ necessarily contains a 3–chord or a 4–chord with one endpoint at $v_1$ or $v_6$ and the other endpoint among the vertices of $\mathcal{C}_i$ for $i = 1, 2$. We distinguish six cases, according to whether there are any 4–chords at all and whether these chords share one of their ends:

(a1) Assume there exists a 4–chord incident to $v_1$ or $v_6$, say $v_1v_5 \in E(G)$.

(i) If $v_1v_7 \in E(G)$ (see Figure 9, far left), then the complement $cG$ contains as a subgraph the graph obtained through two $\nabla Y$ moves from $K_7$ with vertex set $\{v_2, v_3, v_4, v_8, v_9, v_{10}, v_6\}$: one $\nabla Y$
move over the triangle $v_2v_3v_4$ with new vertex $v_7$ and one $\nabla Y$ move over the triangle $v_8v_9v_{10}$ with new vertex $v_5$.

(ii) If $v_1v_7 \notin E(G)$ and $v_1v_8 \in E(G)$ (see Figure 9, center left), then, in $cG$, delete any edges incident to $v_5$ except $v_5v_8$, $v_5v_9$ and $v_5v_{10}$, then perform a $Y \nabla$ move at $v_5$ to create a graph containing the triangle $v_8v_9v_{10}$. This graph contains a $K_{3,3,1,1}$ minor with partition $\{v_2, v_3, v_4\}, \{v_6, v_7, v_8\}, \{v_9\}, \{v_{10}\}$.

(iii) If $v_6v_{10} \in E(G)$ (see Figure 9, center right), then, in $cG$, delete any edges incident to $v_1$ except $v_1v_7$, $v_1v_8$ and $v_1v_9$, then perform a $Y \nabla$ move at $v_1$ to create a graph containing the triangle $v_7v_8v_{10}$. Further, delete any edges incident to $v_6$ except $v_2v_6$, $v_3v_6$ and $v_4v_6$, then perform a $Y \nabla$ move at $v_6$ to create a graph containing the triangle $v_2v_3v_4$. Within this new graph, contract $v_5v_{10}$ to a new vertex $t$ to obtain a $K_7$ minor with vertices $\{v_2, v_3, v_4, v_7, v_8, v_9, t\}$.

(iv) If $v_6v_{10} \notin E(G)$ and $v_6v_9 \in E(G)$ (see Figure 9, far right), then, in $cG$, delete any edges incident to $v_6$ except $v_6v_2$, $v_6v_3$ and $v_6v_4$, then perform a $Y \nabla$ move at $v_6$ to create a graph containing the triangle $v_2v_3v_4$. Within this new graph, contract the edge $v_5v_9$ to a vertex $t$, and contract the edge $v_1v_7$ to a vertex $t_7$ to obtain a $K_7$ minor with vertices $\{v_2, v_3, v_4, t_7, v_8, v_{10}, t\}$.

(a2) Assume there is no 4–chord of $\mathcal{C}$ incident to $v_1$ or $v_6$. There are two 3–chords of $\mathcal{C}$ incident to $v_1$ or $v_6$ and endpoints in each $\mathcal{C}_1$ and $\mathcal{C}_2$. Assume $v_1v_4 \in E(G)$.

(i) If $v_1v_8 \in E(G)$ (see Figure 10, far left), for any choice of edges which triangulate the quadrilaterals $v_1v_2v_3v_4$ and $v_8v_9v_{10}v_1$, the complement $cG$ contains as a subgraph the graph Cousin 12 of

![Figure 9: Outerplanar graphs with ten vertices.](image)

![Figure 10: Outerplanar graphs with ten vertices.](image)
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Figure 11: A wheel graph with ten vertices (left) and the complement of \( E_9 + e \) in \( K_{10} \) (right).

\( K_{3,3,1,1} \) described in [Goldberg et al. 2014]. This is a minor minimal IK graph with nine vertices obtained from \( K_{3,3,1,1} \) by two \( \nabla Y \) moves followed by a \( Y \nabla \) move.

(ii) If \( v_6v_9 \in E(G) \) (see Figure 10, center left), obtain a \( K_7 \) minor of \( cG \) by contracting the edges \( v_1v_8, v_2v_6 \) and \( v_4v_9 \).

Case (b) Assume the cycle \( \mathcal{C} \) has no chord of length 5. Then it has at least a chord of length 4. Assume \( v_1v_7 \in E(G) \). Up to symmetry, we recognize two cases.

(b1) If \( v_1v_5 \in E(G) \) (see Figure 10, center right), then the complement \( cG \) contains the graph obtained through two \( \nabla Y \) moves from \( K_7 \) with vertex set \( \{v_2, v_3, v_4, v_6, v_8, v_9, v_{10}\} \): one \( \nabla Y \) move over the triangle \( v_2v_3v_4 \) with new vertex \( v_7 \) and one \( \nabla Y \) move over the triangle \( v_8v_9v_{10} \) with new vertex \( v_5 \).

(b2) If \( v_1v_4, v_4v_7 \in E(G) \) (see Figure 10, far right), then, in \( cG \), delete any edge incident to \( v_4 \) except \( v_4v_8, v_4v_9 \) and \( v_4v_{10} \), then perform a \( Y \nabla \) move at \( v_4 \) to create a graph containing the triangle \( v_8v_9v_{10} \). Within this graph, contract the edges \( v_1v_5 \) to \( t_5 \) and \( v_2v_7 \) to \( t_2 \) obtain a \( K_7 \) with vertex set \( \{t_2, v_3, t_5, v_6, v_8, v_9, v_{10}\} \).

Lemma 12 If \( G \) is a wheel with ten vertices, then \( cG \) is intrinsically knotted.

Proof The graph \( E_9 + e \) is a minor minimal intrinsically knotted graph with nine vertices described in [Goldberg et al. 2014]. The complement of \( E_9 + e \) in \( K_{10} \) contains the 10–wheel as a subgraph. See Figure 11. Thus, the complement \( cG \) contains \( E_9 + e \) as a subgraph and therefore it is intrinsically knotted.

Lemma 13 If \( G \) is an elongated triangular prism with ten vertices, then \( cG \) is intrinsically knotted.

Proof An elongated prism with ten vertices is obtained by subdividing the three nontriangular edges of the prism with four vertices. These four vertices can be added in four different ways:

(a) on three different edges,
(b) on two edges with a 2-2 partition,
Figure 12: Elongated prisms with ten vertices. Dashed edges are edges of the complement graph.

(c) on two edges with a 3-1 partition,
(d) all on one edge.

See Figure 12. In each case, we show that $cG$ contains a $K_{3,3,1,1}$ minor.

**Case (a)** The four vertices are added on three different edges of the elongated prism, as in Figure 12, far left. Within $cG$, contract the edge $ac$ to the vertex $t$ and $bd$ to $u$ to obtain a $K_{3,3,1,1}$ minor of $cG$ given by the partition $\{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}, \{t\}, \{u\}$.

**Case (b)** The four vertices are added to two edges of the elongated prism with a 2-2 partition, as in Figure 12, center left. Within $cG$, contract $dv_5$ to $t_5$ and $av_6$ to $t_6$ to obtain a $K_{3,3,1,1}$ minor of $cG$ given by the partition $\{v_1, v_3, c\}, \{v_2, v_4, b\}, \{t_5\}, \{t_6\}$.

**Case (c)** The four vertices are added to two edges of the elongated prism with a 3-1 partition, as in Figure 12, center right. Within $cG$, contract $av_5$ to $t_5$ and $cd$ to $t$ to obtain a $K_{3,3,1,1}$ minor of $cG$ given by the partition $\{v_1, v_3, t_5\}, \{v_2, v_4, v_6\}, \{b\}, \{t\}$.

**Case (d)** The four vertices are added all on one edge of the elongated prism, as in Figure 12, far right. Within $cG$, contract $bv_5$ to $t_5$ and $cv_4$ to $t_4$ to obtain a $K_{3,3,1,1}$ minor of $cG$ given by the partition $\{v_1, v_3, a\}, \{v_2, v_6, d\}, \{t_4\}, \{t_5\}$.

Since $cG \subseteq cH$ for $H$ a subgraph of $G$ of the same order, Lemmas 11, 12 and 13 give the following theorem:

**Theorem 3** If $G$ is a nonseparating planar graph on ten vertices, then $cG$ is intrinsically knotted.

**Corollary 14** For $n \geq 10$, the complement of a nonseparating planar graph on $n$ vertices is IK.

**Remark 15** The bound $n \geq 10$ in Corollary 14 is the best possible. If $G$ is the 9–wheel, then $cG \setminus v = K_8 \setminus C_8$. Here $v$ is the isolated point within the complement of the wheel. Since it has 20 edges, $K_8 \setminus C_8$ is 2–apex and it is therefore knotlessly embeddable [Mattman 2011]. As $cG$ is isomorphic to $K_8 \setminus C_8$ with the isolated vertex $v$ added, it is also 2–apex, and thus nIK.

Note that, in the proof of Theorem 3, we’ve showed that the complements of nonseparating planar graphs of order 10 all have minor minimal intrinsically knotted minors of smaller order. From this it follows that there are no minor minimal intrinsically knotted (MMIK) graphs of order ten or more with nonseparating planar complements. On the other hand, by the combined work of [Blain et al. 2007; Conway and Gordon...
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1983; Campbell et al. 2008; Foisy 2002; Goldberg et al. 2014; Kohara and Suzuki 1992; Mattman et al. 2017], the eleven MMIK graphs of order at most 9 are known. Considering their complements, there are just four MMIK graphs with nonseparating planar complements.

Corollary 16 There are exactly four minor minimal intrinsically knotted graphs whose complements are nonseparating planar: $K_7$, $K_{3,3,1,1}$, $K_7^\n$ (the graph obtained by performing a single $\n$–move on $K_7$) and $G_9,28$ (the complement of a 7–cycle and an independent edge inside $K_9$).

Proof By inspection, the complements of the four graphs are nonseparating. The complements of the remaining seven order 9 graphs are planar, but none of them are nonseparating as:

- They cannot be subgraphs of an elongated prism of order 9 (size 12), since their size (14–15) is too big.
- They all have at least two vertices of degree bigger than 3; thus, they cannot be subgraphs of a wheel graph.
- They all have a $K_4$ minor; thus, they cannot be outerplanar.

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