Abstract In this paper, we prove that there exists a unique solution to the Dirichlet boundary value problem for a general class of semilinear second order elliptic differential operators which do not necessarily have the maximum principle and are non-symmetric in general. Our method is probabilistic. It turns out that we need to solve a class of backward stochastic differential equations with singular coefficients, which is of independent interest itself. The theory of Dirichlet forms also plays an important role.

Keywords Dirichlet boundary value problem · Semilinear second order elliptic differential equations · Dirichlet forms · Backward stochastic differential equations

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1 Introduction

In this paper, we will use probabilistic methods to solve the Dirichlet boundary value problem for the semilinear second order elliptic partial differential equations (PDEs) of the following form:

\[
\begin{cases}
    A u(x) = - f(x, u(x)), & \forall x \in D, \\
    u(x)|_{\partial D} = \varphi, & \forall x \in \partial D,
\end{cases}
\]  

(1.1)
where $D$ is a bounded domain in $\mathbb{R}^d$, $f(\cdot, \cdot)$ is a nonlinear function and $\varphi \in C(\partial D)$. The operator $A$ is given by

$$Au = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} - \text{“div}(\hat{b}u)" + q(x)u,$$

where $a(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$ is a Borel measurable, (not necessarily symmetric) matrix-valued function on $\mathbb{R}^d$ satisfying

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \quad \forall \xi \in \mathbb{R}^d, x \in \mathbb{R}^d \quad (1.2)$$

and

$$a_{ij}(x) \leq \frac{1}{\lambda} \quad \forall x \in \mathbb{R}^d, 1 \leq i, j \leq d \quad (1.3)$$

for some constant $0 < \lambda \leq 1$; $b = (b_1, ..., b_d)^*$, $\hat{b} = (\hat{b}_1, ..., \hat{b}_d)^*$ and $q$ are Borel measurable functions and satisfy $|b|^2 I_D \in L^p(D; dx)$, $|\hat{b}|^2 I_D \in L^p(D; dx)$ and $q I_D \in L^p(D; dx)$ for some constant $p > \frac{d}{2}$. Here $*$ stands for the transpose of a vector or matrix.

There is a big literature regarding the probabilistic approaches of solving boundary value problems. The pioneering work is traced back to Kukutani [12] who used Brownian motion to represent the solution of the classical Dirichlet problem for Laplacian operators. In the linear case, i.e. $f(\cdot, \cdot) = 0$, if $\hat{b} = 0$, $q \leq 0$, the problem (1.1) is solved by the Feynman-Kac formula

$$u(x) = E_x \left[ e^{\int_0^{\tau_D} q(X(s)) ds} \varphi(X_{\tau_D}) \right],$$

where $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x, x \in \mathbb{R}^d)$ is the diffusion process associated with the generator $L_1$ given by

$$L_1 u = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i} \quad (1.4)$$

and $\tau_D$ is the first exit time of $X$ from $D$. We refer the readers to Chen and Zhao [8] for details. The first result on probabilistic interpretation of viscosity solutions of semilinear parabolic PDEs was obtained by Peng in [17] and [16], through the backward stochastic differential equations, which is different from the weak solution considered in this paper.

Since $\hat{b}$ is merely measurable, some of the coefficients of the operator $A$ are very singular (just distributions). When $\hat{b} \neq 0$, $f(\cdot, \cdot) \neq 0$ and the matrix $a(x)$ is symmetric, the boundary value problem (1.1) was considered by Zhang [18]. The time reversal of symmetric Markov processes and the backward stochastic differential equations (BSDEs) with random terminal time play an important role in [18]. When $a$ is non-symmetric and $f(\cdot, \cdot) = 0$ (i.e. the linear case), Chen, Sun and Zhang [5] obtained a probabilistic representation of the solution to problem (1.1) under the condition that

$$q - \text{div}\hat{b} \leq g \quad (1.5)$$

in the sense of distribution for a sufficient small non-negative function $g \in L^p(D)$.

In this paper, we consider the weak/Soblev solution (see the definition in the next section) of the semilinear problem (1.1). We will show that there exists a unique, continuous weak solution to the Dirichlet value problem (1.1) under appropriate conditions. We will use the $h$-transform method to tackle the singular term “div($\hat{b}$)”. To solve the semilinear boundary
value problem (1.1), we first produce a candidate to the solution by appealing to the theory of BSDEs. More precisely, we will solve a class of BSDEs with very singular coefficients and random terminal time. The BSDEs are driven by the martingale part of a diffusion process. The study of this class of BSDEs is of independent interest. It turns out that the classical $L^2$ setting of BSDEs is not suitable here. We have to work in the framework of $L^1$ and deal with class $D$ stochastic processes. We refer the readers to the nice article [4] for related results.

The rest of the paper is organized as follows. In Section 2, we set up the precise framework. Section 3 is devoted to establishing the existence and uniqueness of solutions of a backward stochastic differential equation with singular coefficients. The boundary value problem is solved in Section 4.

2 Framework

We assume that $d \geq 3$. Hereafter we fix $p > \frac{d}{2}$. We will use $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$ to denote respectively the inner product of the Euclidean space $\mathbb{R}^d$ and the inner product of the $L^2(D)$-space. $|| \cdot ||_\infty$ denotes the $L^\infty$ norm. Recall that $W^{1,2}(D) = (\int_D (\nabla u, \nabla u) dx + (u, u))^\frac{1}{2}$ and that $W^{1,2}_0(D)$ denotes the completion of $C_0^\infty(D)$ under the same norm $|| \cdot ||_{W^{1,2}(D)}$. The operator $A$ introduced in Section 1 is rigorously determined by the quadratic form $(Q, D(Q))$ with $D(Q) = W^{1,2}(D)$ and for $u, v \in W^{1,2}(D)$

\[
Q(u, v) = (-Au, v)_{L^2(\mathbb{R}^d)} = \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_{\mathbb{R}^d} b_i(x) \frac{\partial u}{\partial x_i} v(x) dx - \sum_{i=1}^d \int_{\mathbb{R}^d} \hat{b}_i(x) \frac{\partial v}{\partial x_i} u(x) dx - \int_D q(x) u(x) v(x) dx.
\]

Definition 2.1 $u \in W^{1,2}(D) \cap C(\overline{D})$ is called a weak solution of problem (1.1), if $u$ satisfies

\[
Q(u, \phi) = \int_D f(x, u(x)) \phi(x) dx \text{ for any } \phi \in C_0^\infty(D),
\]

and $u(x) = \varphi(x)$ when $x \in \partial D$.

Now we need to recall the notion of VMO functions in order to apply the $W^{1,p}$ estimates for the divergence operators. A locally integrable function $g$ is said to be in the VMO space, if

\[
\lim_{\epsilon \to 0} \sup_{0 \leq \theta \leq 1} \int_{B_r} |g(x) - g_{B_r}| \, dx = 0,
\]

where $B_r$ denotes a ball of radius $r$ and

\[
g_{B_r} = \frac{1}{|B_r|} \int_{B_r} g(x) \, dx.
\]

Assumption I

(I.1) $D$ is a bounded, connected $C^1$-domain.

(I.2) Each $a_{ij}, 1 \leq i, j \leq d$, is continuous and belongs to the VMO space.
Remark 2.1 Under the Assumption I, it follows from Theorem 1 in [3] that there exists a unique element \( v \in W_{0}^{1,2}(D) \) such that
\[
\int_{D} \langle a(x) \nabla v(x), \nabla \phi(x) \rangle dx = -2 \int_{D} \langle \hat{b}(x), \nabla \phi(x) \rangle dx, \quad \forall \phi \in W_{0}^{1,2}(D).
\]

Set \( L_{0}u = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_{i}} (a_{ij}(x) \frac{\partial u}{\partial x_{j}}) \). Let \((X_{t}, P_{0}^{0}, x \in \mathbb{R}^{d})\) be the diffusion process generated by the infinitesimal operator \( L_{0} \). It is well-known that \((X_{t}, P_{0}^{0})\) is a conservative Feller process on \( \mathbb{R}^{d} \) that has continuous transition density function which admits a two-sided Aronsons heat kernel estimate (see Aronson [1, 2]). Moreover, by Theorem IV.2.5 in [14], for quasi-everywhere (q.e.) \( x \in \mathbb{R}^{d} \) we have the Fukushima’s decomposition
\[
X(t) - X(0) = M_{t}^{0} + N_{t}^{0}, \quad P_{x}^{0} - a.s.,
\]
where \( M_{t}^{0} \) is a martingale additive functional (MAF) and \( N_{t}^{0} \) is a continuous additive functional (CAF) of locally zero energy. However the decomposition (2.1) can be strengthened to hold for every \( x \in \mathbb{R}^{d} \). This follows from the existence of the heat kernel, Theorem 3.5.4 in [15] and Theorem 2 in [11]. Let \( v \) be the function stated in Remark 2.1. By Sobolev embedding theorem, \( v \in C(\mathbb{R}^{d}) \) if we extend \( v = 0 \) on \( D^{c} \). Moreover, for any \( x \in \mathbb{R}^{d} \) the Fukushima’s decomposition holds
\[
v(X(t)) - v(X(0)) = M_{t}^{v} + N_{t}^{v}, \quad P_{x}^{0} - a.e.,
\]
where \( M_{t}^{v} \) is a MAF and \( N_{t}^{v} \) is a CAF of zero energy.

Assume that \( f(x,y) \) is a measurable function, which is continuous w.r.t. \( y \) and satisfies
\[
\begin{align*}
(y_{1} - y_{2})(f(x, y_{1}) - f(x, y_{2})) &\leq -J(x) |y_{1} - y_{2}|^{2}, \\
|f(x, y)| &\leq C(1 + J_{1}(x)|y|),
\end{align*}
\]
for some constant \( C \) and \( J, J_{1} \in L^{p}(D) \). Finally set
\[
\tilde{a} = \{\tilde{a}_{ij}(x)\}_{1 \leq i, j \leq d} = \left\{ \frac{a_{ij}(x) + a_{ji}(x)}{2} \right\}_{1 \leq i, j \leq d}.
\]

3 BSDEs with Singular Coefficients

In this section, we will obtain the existence and uniqueness of solutions of a class of BSDEs with singular coefficients and random terminal time. The results is of independent interest on its own right and will be used in the subsequent sections. Recall that \( X = (\Omega, \mathcal{F}, \mathcal{F}_{t}, X_{t}, P_{x}, x \in \mathbb{R}^{d}) \) denotes the diffusion process associated with the generator \( L_{1} \) given by
\[
L_{1}u = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_{i}} (a_{ij}(x) \frac{\partial u}{\partial x_{j}}) + \sum_{i=1}^{d} b_{i}(x) \frac{\partial u}{\partial x_{i}},
\]
where \( \mathcal{F}_{t} \) is the completed minimal admissible filtration generated by \( X_{t} \). Similar to (2.1) we have the following Fukushima’s decomposition: for any \( x \in \mathbb{R}^{d} \)
\[
X(t) - X(0) = M_{t} + N_{t}, \quad P_{x} - a.e.,
\]
where $M_t = (M^1_t, M^2_t, \cdots M^d_t)$ is a MAF of finite energy and $N_t$ is CAF of locally zero quadratic variation. Moreover,

$$\langle M^i, M^j \rangle_t = \int_0^t \tilde{a}_{ij}(X(s)) ds.$$  

Given a $h \in L^p(D)$. Let $F$ be a Borel measurable function on $D$ satisfying

$$|F(x)| \leq C(1 + |h(x)|),$$  

for some constant $C > 0$. Then similar to Theorem 3.18 and Theorem 4.6 in [9], we have the following result.

**Lemma 3.1** Assume there exists a $x_0 \in D$ such that

$$E_{x_0}[e^{-\int_0^{\tau_D} h(X(s)) ds}] < \infty,$$  

then for any $F$ satisfying (3.3) we have

$$\sup_{x \in D} E_x [\int_0^{\tau_D} e^{-\int_0^t h(X(s)) ds} F(X(t)) dt] < \infty.$$  

**Proof** Let $G_D(x, y)$ denote the Green function on the domain $D$ associated with the operator $L_1$. It is known from (2.14) in [5] that

$$\int_D G_D(x, y)|h(y)|dy = E_x[\int_0^{\tau_D} |h(X(s))| ds] \leq \sigma \|h\|_{L^p(D)},$$  

for some constant $\sigma$. Since $h$ is $L^p$ integrable, one can choose $\delta > 0$ such that for any $B \subset D$ with $m(B) < \delta$, we have $\|I_B h\|_{L^p(D)} < \frac{1}{\sigma}$. Hence

$$\sup_{B \subset D: m(B) < \delta} \sup_{x \in D} \int_B G_D(x, y)|h(y)|dy < 1.$$  

So $h$ belongs to the class of functions $K_1$ defined in [6]. It thus follows from Theorem 2.2 in [6] and (3.4) that

$$E_x[e^{-\int_0^{\tau_D} h(X(s)) ds}] \leq C_1, \quad \forall x \in D,$$  

where $C_1$ is a positive constant. Moreover by (3.6)

$$\inf_{x \in D} E_x[e^{-\int_0^{\tau_D} h(X(s)) ds}] \geq \inf_{x \in D} \exp(-E_x[\int_0^{\tau_D} h(X(s)) ds]) \geq \exp(-\sigma \|h\|_{L^p(D)}).$$  

On the other hand, it was shown in [1, 2] that the transition density function $p(t, x, y)$ of $(X_t, P_x, x \in \mathbb{R}^d)$ has a Gaussian upper bound estimate:

$$p(t, x, y) \leq \frac{\sigma_1}{t^{d/2}} e^{-\frac{\sigma_2|x-y|^2}{t}}, \quad \forall (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$$  

for some constants $\sigma_1, \sigma_2 > 0$. This bound together with $h \in L^p(D)$ implies that

$$\sup_{x \in D} E_x[\int_0^\delta |h(X(s))| ds] < 1 \text{ for some } \delta > 0.$$  

Hence similar to the proof of Theorem 4.6 in [9] we can show that

$$\int_0^{\tau_D} \sup_{x \in D} E_x[e^{-\int_0^t h(X(s)) ds}; t < \tau_D] dt < \infty.$$  

The remainder of the proof of (3.5) is the same as that of Theorem 3.18 in [9]. We omit the details.

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Fix a probability measure \( \mu \) on \( \mathbb{R}^d \) and let \( P = P_\mu \) denote the probability law of the diffusion process \( X \) starting with the initial distribution \( \mu \), namely, \( P(\cdot) = \int_{\mathbb{R}^d} P_x(\cdot) \mu(dx) \).

We now consider BSDEs driven by the martingale part \( M \) in (3.2) with random terminal time on the probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\). Denote by \( E \) the expectation under \( P \).

Let \( g(t, y, \omega) : [0, \infty) \times \mathbb{R} \times \Omega \to \mathbb{R} \) be a progressively measurable function. For notational convenience, we omit the random parameter \( \omega \). Given a finite stopping time \( \tau \) and a random variable \( \xi \in \mathcal{F}_\tau \). Let us recall the definition of a solution of the BSDEs with random terminal time.

**Definition 3.1** A pair \((Y, Z)\) is a solution to the following BSDE

\[
Y(t) = \xi + \int_t^\tau g(s, Y(s))ds - \int_t^\tau \langle Z(s), dM_s \rangle,
\]

if \( Y \) is a \( \mathbb{R} \)-valued progressively measurable process and \( Z \) is a \( \mathbb{R}^d \)-valued predictable process such that:

\[
|g(\cdot, Y(\cdot))| \in L^1([0, \tau], ds), \quad Z(\cdot) \in L^2([0, \tau], ds), \quad P\text{-a.s.}
\]

and (3.10) is valid \( P\text{-a.s.} \) for every \( t \geq 0 \).

To prove the existence and uniqueness of a solution to the BSDE (3.10), we need the following lemma which can be found in [4].

**Lemma 3.2** If \((Y, Z)\) is a solution to the BSDE (3.10), then for any \( 0 \leq t \leq u \leq \tau \),

\[
|Y(t)| \leq |Y(u)| + \int_t^u \frac{Y(s)}{|Y(s)|} I_{\{Y(s) \neq 0\}} g(s, Y(s))ds - \int_t^u \frac{Y(s)}{|Y(s)|} I_{\{Y(s) \neq 0\}} \langle Z(s), dM_s \rangle.
\]

Suppose that \( g \) is continuous with respect to the variable \( y \) and satisfies

\[
\begin{cases}
(y_1 - y_2)(g(t, y_1) - g(t, y_2)) \leq -K_1(t)|y_1 - y_2|^2,
|g(t, y) - g(t, 0)| \leq K_2(t) + K_3(t)|y|,
\end{cases}
\]

where \( K_1(t), K_2(t) \geq 0, K_3(t) \geq 0 \) are progressively measurable stochastic processes.

For any \( 0 < \beta < 1 \), denote by \( \varphi^\beta \) the set of the real-valued, adapted, continuous processes \( Y(t) \) such that

\[
||Y||_{\varphi^\beta} = E[\sup_{t \geq 0} |Y(t)|^\beta] < \infty.
\]

If we define \( \rho(Y_1, Y_2) = ||Y_1 - Y_2||_{\varphi^\beta} \) as a distance in \( \varphi^\beta \), then \( \varphi^\beta \) is complete under this distance. Denote by \( M^\beta \) the set of \( \mathbb{R}^d \)-valued predictable processes \( Z(\cdot) \) such that

\[
||Z||_{M^\beta} = E[(\int_0^\infty |Z(t)|^2 dt)^{\frac{\beta}{2}}] < \infty.
\]

Then \( M^\beta \) is also complete under the distance \( \rho'(Z_1, Z_2) = ||Z_1 - Z_2||_{M^\beta} \).

Let \( \Gamma \) denote the set of the stopping times. We say that a progressively measurable process \( Y \) is in Class (D) if the family \( \{Y(T)I_{\{T < \infty\}} : T \in \Gamma\} \) is uniformly integrable. We put

\[
||Y||_1 = \sup\{E[|Y(T)|I_{\{T < \infty\}}] : T \in \Gamma\}.
\]

Then it is known that the space of adapted continuous processes which belong to class (D) is complete under \( || \cdot ||_1 \) norm, see Chapter VI 21 in [10]. Here is the main result of this section.
Theorem 3.2 Assume
\[ E[\int_0^\tau e^{-\int_0^s K_1(s)ds} \xi] < \infty, \]
and
\[ E[\int_0^\tau (e^{-\int_0^s K_1(s)ds} (K_2(t) + |g(t,0)|) + |K_1(t)| + K_3(t))dt] < \infty. \]  

Then the BSDE (3.10) has a unique solution \((Y, Z)\) such that \(e^{-\int_0^\tau K_1(s)ds} Y(t)\) belongs to Class (D). Moreover, for any \(0 < \beta < 1\), \(e^{-\int_0^\tau K_1(s)ds} Y(t), e^{-\int_0^\tau K_1(s)ds} Z(t)\) \(\in \varphi^\beta \times M^\beta\).

Proof We set \(g(s, y) = K_1(s) = K_2(s) = K_3(s) = 0\) for \(s \geq \tau\) for convenience. Set \(\bar{Y}(t) = e^{-\int_0^t K_1(s)ds} Y(t), \bar{Z}(t) = e^{-\int_0^t K_1(s)ds} Z(t)\), \(\bar{g}(t, y) = e^{-\int_0^t K_1(s)ds} g(t, e^{\int_0^t K_1(s)ds} y) + K_1(t)y\) and \(\bar{\xi} = e^{-\int_0^\tau K_1(s)ds} \xi\). Then it is easy to see that \((\bar{Y}, \bar{Z})\) is a solution to the BSDE (3.10) if and only if \((\bar{Y}, \bar{Z})\) satisfies the following BSDE:

\[ \bar{Y}(t) = \bar{\xi} + \int_t^\tau \bar{g}(s, \bar{Y}(s))ds - \int_t^\tau \langle \bar{Z}(s), dM_s \rangle. \]

Therefore, we only need to establish the existence and uniqueness of the BSDE (3.16). Note that \(\bar{g}(t, y)\) is also continuous with respect to \(y\) and

\[ \{ (y_1 - y_2)(\bar{g}(t, y_1) - \bar{g}(t, y_2)) \leq 0, \]  
\[ E[\int_0^\tau |\bar{g}(t, 0)|dt] < \infty. \]

Moreover for any \(n \in \mathbb{N}\),

\[ E[\int_0^n e^{-\int_0^s K_1(s)ds} K_2(t)dt] = E[\int_0^{T+n} e^{-\int_0^s K_1(s)ds} K_2(t)dt] \leq E[\int_0^\tau e^{-\int_0^s K_1(s)ds} K_2(t)dt] < \infty, \]

where the first inequality is due to the fact \(K_1(s) = K_2(s) = 0\) when \(s \geq \tau\).

By (3.11) and (3.15), for any \(r > 0\), we have

\[ \sup_{|y| \leq r} |\bar{g}(t, y) - \bar{g}(t, 0)| \leq \sup_{|y| \leq r} e^{-\int_0^t K_1(s)ds} |g(t, e^{\int_0^t K_1(s)ds} y) - g(t, 0)| + |K_1(t)|r \]

\[ \leq e^{-\int_0^t K_1(s)ds} K_2(t) + (|K_1(t)| + K_3(t))r \in L^1([0, n] \times \Omega), \quad \forall n \in \mathbb{N}. \]

Set \(\bar{\xi}^N = (-N) \vee \bar{\xi} \wedge N\). Note that the martingale representation theorem with respect to the martingale part \(M\) is valid according to Theorem 2.1 in [18]. Thus, by (3.17)–(3.18) and Proposition 6.4 in [4] for any \(n \in \mathbb{N}\), there exists a solution pair \((\bar{Y}_n^N, \bar{Z}_n^N)\) to the following BSDE:

\[ \bar{Y}_n^N(t) = E[\bar{\xi}^N | \mathcal{F}_n] + \int_t^\tau I_{s \leq \tau} \bar{g}(s, \bar{Y}_n^N(s))ds - \int_t^\tau \langle \bar{Z}_n^N(s), dM_s \rangle, \quad 0 \leq t \leq n, \]

and moreover \(\bar{Y}_n^N \in \text{Class (D)}, (\bar{Y}_n^N, \bar{Z}_n^N) \in \bigcap_{0 < \beta < 1} \varphi^\beta \times M^\beta\), where we set \(\bar{Y}_n^N(t) = E[\bar{\xi}^N | \mathcal{F}_n], \bar{Z}_n^N(t) = 0\) for \(t > n\).

Next we are going to show that \(\{\bar{Y}_n^N\}_{n \geq 1}\) is a Cauchy sequence under norm \(\| \cdot \|_1\).

For \(n, i \in \mathbb{N}\), set \(\delta \bar{Y}_{n+i}^N(s) = \bar{Y}_{n+i}^N(s) - \bar{Y}_n^N(s), \delta \bar{Z}_{n+i}^N(s) = \bar{Z}_{n+i}^N(s) - \bar{Z}_n^N(s)\) and \(\delta \bar{g}(s, y) = I_{s \leq \tau} (\bar{g}(s, \bar{Y}_n^N(s) + y) - \bar{g}(s, \bar{Y}_n^N(s)) + \bar{g}(s, \bar{Y}_n^N(s)) I_{s \geq \tau}\). Then \((y_1 - y_2)(\delta \bar{g}(t, y_1) - \delta \bar{g}(t, y_2)) \leq 0\). By (3.19) we have for any \(t \geq 0\)

\[ \delta \bar{Y}_{n+i}^N(t) = E[\bar{\xi}^N | \mathcal{F}_n] - E[\bar{\xi}^N | \mathcal{F}_n] + \int_{t \vee (n+i)}^{n+i} (\delta \bar{g}(s, \delta \bar{Y}_{n+i}^N(s))ds \]

\[ - \int_{t \vee (n+i)}^{n+i} \langle \delta \bar{Z}_{n+i}^N(s), dM_s \rangle. \]  

(20)
Choose a sequence of stopping times \( \{ \tau_k \}_{k \geq 1} \) that increases to \( \infty \) and such that
\[
\int_0^{T \wedge \tau_k} \frac{\delta Y_{n+i}(s)}{|\delta Y_{n+i}(s)|} \, dZ_{n+i}(s), \, dM_s \]
is a martingale. Then by (3.20) we have
\[
\delta \tilde{Y}_{n+i}(t \wedge \tau_k) = \delta \tilde{Y}_{n+i}((n+i) \wedge \tau_k) + \int_{I \cap \tau_k} \delta \tilde{g}_n(s, \delta \tilde{Y}_{n+i}(s)) \, ds
- \int_{I \cap \tau_k} \langle \delta \tilde{Z}_{n+i}(s), dM_s \rangle.
\]

It follows from Lemma 3.2 that
\[
|\delta \tilde{Y}_{n+i}(t \wedge \tau_k)| \leq |\delta \tilde{Y}_{n+i}((n+i) \wedge \tau_k)| + \int_{I \cap \tau_k} \frac{\delta \tilde{g}_n(s, \delta \tilde{Y}_{n+i}(s))}{|\delta \tilde{g}_n(s, \delta \tilde{Y}_{n+i}(s))|} \, ds
- \int_{I \cap \tau_k} \langle \delta \tilde{Z}_{n+i}(s), dM_s \rangle
\leq |\delta \tilde{Y}_{n+i}((n+i) \wedge \tau_k)| + \int_{I \cap \tau_k} \delta \tilde{g}_n(s, 0) \, ds
- \int_{I \cap \tau_k} \langle \delta \tilde{Z}_{n+i}(s), dM_s \rangle.
\]

Take the conditional expectation with respect to \( F_t \) to obtain
\[
|\delta \tilde{Y}_{n+i}(t \wedge \tau_k)| \leq E[|\delta \tilde{Y}_{n+i}((n+i) \wedge \tau_k)||F_t] + E[\int_{I \cap \tau_k} |\delta \tilde{g}_n(s, 0)| \, ds|F_t]
= E[|\delta \tilde{Y}_{n+i}((n+i) \wedge \tau_k)||F_t] + E[\int_{I \cap \tau_k} |\delta \tilde{g}_n(s, 0)| \, ds|F_t],
\]
here we have used the fact that \( \int_0^{(n+i) \wedge \tau_k} \frac{\delta \tilde{Y}_{n+i}(s)}{|\delta \tilde{Y}_{n+i}(s)|} \, dZ_{n+i}(s), dM_s \), \( t \geq 0 \) is also a martingale with respect to the filtration \( F_t \), \( t \geq 0 \).

Since \( \delta \tilde{Y}_{n+i} = \tilde{Y}_{n+i} - Y_{n+i} \) is also in Class (D) and \( \delta \tilde{Y}_{n+i}(t) \) is continuous, we get
\[
\lim_{k \to \infty} \delta \tilde{Y}_{n+i}((n+i) \wedge \tau_k) = \delta \tilde{Y}_{n+i}(n+i) \in L^1(P; \Omega).
\]

Since
\[
\int_{I \cap \tau_k} |\delta \tilde{g}_n(s, 0)| \, ds
\leq \int_{I \cap \tau_k} |\tilde{g}(s, Y_{n+i}(n)) - \tilde{g}(s, 0)| + |\tilde{g}(s, 0)| \, ds
\leq \int_{I \cap \tau_k} |K_2(s)e^{-\int_0^s K_1(r) \, dr} + (|K_1(s)| + K_3(s))|Y_{n+i}(n)| \, ds
+ \int_n^{(n+i) \wedge \tau} |\tilde{g}(s, 0)| \, ds
\leq \int_{I \cap \tau_k} |K_2(s)e^{-\int_0^s K_1(r) \, dr} + N \int_n^{(n+i) \wedge \tau} (|K_1(s)| + K_3(s)) \, ds
+ \int_n^{(n+i) \wedge \tau} |\tilde{g}(s, 0)| \, ds \in L^1(P; \Omega),
\]
by the dominated convergence theorem we have
\[
\lim_{k \to \infty} \int_{I \cap \tau_k} |\delta \tilde{g}_n(s, 0)| \, ds = \int_{I \cap \tau} |\delta \tilde{g}_n(s, 0)| \, ds \in L^1(P; \Omega).
\]

Letting \( k \to \infty \) in (3.21) and noticing \( \delta \tilde{g}_n(s, 0) = 0 \) for \( s > \tau \), it follows from (3.22) and (3.24) that
\[
|\delta \tilde{Y}_{n+i}(t)| \leq E[|\delta \tilde{Y}_{n+i}(n+i)||F_T] + E[\int_n^{(n+i) \wedge \tau} |\delta \tilde{g}_n(s, 0)| \, ds|F_T].
\]

Hence,
\[
|||\delta \tilde{Y}_{n+i}|||_1 = \sup_{T : T \in \Gamma} E[|\delta \tilde{Y}_{n+i}(T)|I_{[T < \infty]}]
\leq \sup_{T : T \in \Gamma} E[E[|\delta \tilde{Y}_{n+i}(n+i)||F_T] + E[\int_n^{(n+i) \wedge \tau} |\delta \tilde{g}_n(s, 0)| \, ds|F_T]]
= E[|\delta \tilde{Y}_{n+i}(n+i)|] + E[\int_n^{(n+i) \wedge \tau} |\tilde{g}(s, Y_{n+i}(n))| \, ds].
\]

Since \( \tilde{g}^N \) is bounded, it holds that
\[
\delta \tilde{Y}_{n+i}(n+i) = E[\tilde{g}^N |F_{n+i}] - E[\tilde{g}^N |F_n] \to 0 \quad \text{in} \ L^1(\Omega)
\]
as \( n \to \infty \). By (3.11), we have

\[
|\tilde{g}(s, Y^n_n(n))| \\
\leq e^{-\int_0^s K_1(r)dr} |g(s, e^{\int_0^s K_1(r)dr} Y^n_n(n)) - g(s, 0))| \\
+ e^{-\int_0^s K_1(r)dr} |g(s, 0)| + N|K_1(s)| \\
\leq e^{-\int_0^s K_1(r)dr} K_2(s) + N K_3(s) \\
+ e^{-\int_0^s K_1(r)dr} |g(s, 0)| + N|K_1(s)|. 
\]

(3.28)

Thus the condition (3.15) implies that

\[
E[\int_0^\tau |\tilde{g}(s, Y^n_n(n))| ds] < \infty. 
\]

(3.29)

Combining (3.26), (3.27) and (3.29) we see that \( \{Y^n_n\}_{n \geq 1} \) is a Cauchy sequence with respect to the \( || \cdot ||_1 \) norm. Thus there exists a \( Y^N \in \text{Class (D)} \) such that \( \lim_{n \to \infty} ||Y^n_n - Y^N||_1 = 0 \). It follows from [10] that there exists a subsequence of \( \{Y^n_n\}_{n \geq 1} \) that converges almost surely to \( Y^N \) uniformly on \([0, \infty)\). Hence \( Y^N \) is a continuous process.

Next we will show that \( \tilde{Y}^N \) also converges to \( \tilde{Y} \) in \( \varphi^\beta \) for any \( 0 < \beta < 1 \).

Let \( \eta_n = |\delta \tilde{Y}^{n+i} (n+i) + f^{(n+i)} I \hat{g}_n(s, 0))| ds, M_n(t) = E[\eta_n F_\tau] \). Then by Lemma 6.1 in [4] we have \( E[\sup_{t \geq 0} |M_n(t)|^\beta] \leq \frac{1}{1-\beta} E[|\eta_n|^\beta] \). Together with (3.25) we get

\[
E[\sup_{t \geq 0} |\delta \tilde{Y}^{n+i}(t)|^\beta] \leq E[\sup_{t \geq 0} |M_n(t)|^\beta] \leq \frac{1}{1-\beta} E[|\eta_n|^\beta] \to 0
\]
as \( n \to 0 \). Therefor \( \tilde{Y}^N \in \varphi^\beta \) and

\[
\lim_{n \to \infty} E[\sup_{t \geq 0} |Y^n_n(t) - \tilde{Y}(t)|^\beta] = 0.
\]

(3.30)

Next we will establish the convergence of \( \{\hat{Z}^n_n\}_{n \geq 1} \).

By (3.20) and Ito’s formula,

\[
|\delta \hat{Z}^{n+i}_n(s)|^2 ds \leq |\delta \tilde{Y}^{n+i}_n(n+i)|^2 + 2 \int_0^{n+i} \langle \delta \hat{Z}^{n+i}_n(s), \delta \tilde{Z}^{n+i}_n(s) \rangle ds \\
- 2 \int_0^{n+i} \langle \delta \hat{Z}^{n+i}_n(s) |\delta \tilde{Z}^{n+i}_n(s), dM_s \rangle.
\]

Using (1.2) and (3.17), we deduce that

\[
\lambda \int_0^{n+i} |\delta \hat{Z}^{n+i}_n(s)|^2 ds \leq |\delta \tilde{Y}^{n+i}_n(n+i)|^2 + 2 \int_0^{n+i} \langle \delta \hat{Z}^{n+i}_n(s), \delta \tilde{g}_n(s, 0) ds \\
- 2 \int_0^{n+i} \langle \delta \hat{Z}^{n+i}_n(s) |\delta \tilde{Z}^{n+i}_n(s), dM_s \rangle.
\]

It follows now from the Burkholder’s inequality that

\[
E[\int_0^\infty |\delta \hat{Z}^{n+i}_n(s)|^2 ds]^{\frac{\beta}{2}} = E[\int_0^\infty |\delta \tilde{Z}^{n+i}_n(s)|^2 ds]^{\frac{\beta}{2}} \\
\leq C_\beta E[|\delta \tilde{Y}^{n+i}_n(n+i)|^\beta] + 2 E[\sup_{t \geq 0} |\delta \tilde{Y}^{n+i}_n(t)|^\beta |(\int_0^{n+i} \delta \tilde{g}_n(s, 0) ds)\beta] \\
+ 2C_\beta E[\sup_{t \geq 0} |\delta \tilde{Z}^{n+i}_n(s)|^2 ds]^{\frac{\beta}{2}} \\
\leq C_\beta [E[|\delta \tilde{Y}^{n+i}_n(n+i)|^\beta] + E[\sup_{t \geq 0} |\delta \tilde{Y}^{n+i}_n(t)|^\beta] + E[(\int_0^{n+i} \delta \tilde{g}_n(s, 0) ds)^\beta] \\
+ 2C_\beta E[\sup_{t \geq 0} |\delta \tilde{Z}^{n+i}_n(t)|^2 ds]^{\frac{\beta}{2}} \\
\leq C_\beta [2 \sup_{t \geq 0} E[|\delta \tilde{Y}^{n+i}_n(t)|^\beta] + E[(\int_0^{n+i} |\delta \tilde{g}_n(s, 0) ds)|^\beta] \\
+ 2C_\beta E[\sup_{t \geq 0} |\delta \tilde{Y}^{n+i}_n(t)|^\beta] + \frac{1}{2} E[(\int_0^{n+i} |\delta \tilde{Z}^{n+i}_n(s)|^2 ds)^{\frac{\beta}{2}}].
\]
Thus

$$E[\int_0^\infty |\tilde{Z}^{n+1}_n(s)|^2ds]^\beta \leq (4C_\beta + 4C_\beta^2)E[\sup_{t \geq 0} |\delta \tilde{Y}^{n+1}_n(t)|^\beta] + 2C_\beta [E(\int_0^{n+1} \delta \tilde{g}_n(s,0)ds)^\beta \to 0$$

as $n \to \infty$, where (3.17) was used. So there exists $\tilde{Z}^N \in M^\beta$ such that

$$\lim_{n \to \infty} E[\int_0^\infty |\tilde{Z}^{n+1}_n(s) - \tilde{Z}^N(s)|^2ds]^\beta] = 0.$$

Now we will show that $(\tilde{Y}^N, \tilde{Z}^N)$ is a solution of the equation

$$\tilde{Y}^N(t) = \tilde{\xi}^N + \int_{\tau \land T} \tilde{g}(s, \tilde{Y}^N(s))ds - \int_{\tau \land T} \langle \tilde{Z}^N(s), dM_s \rangle. \quad (3.31)$$

By (3.30) we can find a subsequence $\{n_k\}_{k \geq 1}$ such that

$$\lim_{n_k \to \infty} \sup_{s \geq 0} |\tilde{Y}^N_{n_k}(s) - \tilde{Y}^N(s)| = 0, \quad P\text{-a.e.}$$

By (17) and (18) for $P$-a.e. $\omega$,

$$|\tilde{g}(s, \tilde{Y}^N_{n_k}(s))| \leq |\tilde{g}(s, 0)| + K_2(s)e^{-\int_0^s K_1(r)dr} + |K_1(s)\tilde{Y}^N_{n_k}(s)| + |K_3(s)\tilde{Y}^N_{n_k}(s)|$$

$$\leq |\tilde{g}(s, 0)| + K_2(s)e^{-\int_0^s K_1(r)dr} + C(\omega)(|K_1(s)| + K_3(s)) \in L^1([0, \tau(\omega)], ds),$$

where $C(\omega) = \sup_{s \geq 0} \sup_{n_k \geq 1} |\tilde{Y}^N_{n_k}(s, \omega)| + |\tilde{Y}^N(s, \omega)|$. Hence using the dominate convergence theorem we get

$$\lim_{n_k \to \infty} \int_{\tau \land T} \tilde{g}(s, \tilde{Y}^N_{n_k}(s))ds = \int_{\tau \land T} \tilde{g}(s, \tilde{Y}^N(s))ds, \quad P\text{-a.s.}.$$

On the other hand, since $\tilde{Z}^N_n$ converges to $\tilde{Z}^N$ in $M^\beta$, $\int_{\tau \land T} \langle Z^N_n(s), dM_s \rangle$ converges to $\int_{\tau \land T} \langle \tilde{Z}^N(s), dM_s \rangle$ in probability as $n \to \infty$. Note also that $E[\tilde{\xi}^N |F_{t_\land T}]$ converges to $\tilde{\xi}^N$ in probability as $k \to \infty$. So replacing $n$ by $n_k$ in (3.19) and letting $k \to \infty$ we see that $(\tilde{Y}^N, \tilde{Z}^N)$ satisfies (3.31).

Finally we show that there exists a solution to BSDE (3.16). For $N, N_1 \in \mathbb{N}$, let

$$\delta \tilde{Y}^N(t) = \tilde{Y}^{N+N_1}(t) - \tilde{Y}^N(t), \delta \tilde{Z}^N(t) = \tilde{Z}^{N+N_1}(t) - \tilde{Z}^N(t) \text{ and } \delta \tilde{g}(t, y) = \tilde{g}(t, y + \tilde{Y}^N(t)) - \tilde{g}(t, \tilde{Y}^N(t)).$$

Then by (3.17)

$$y \cdot \delta \tilde{g}(t, y) = y(\tilde{g}(t, y + \tilde{Y}^N(t)) - \tilde{g}(t, \tilde{Y}^N(t))) \leq 0. \quad (3.32)$$

From the equation (3.31) we have

$$\delta \tilde{Y}^N(t) = \tilde{\xi}^{N+N_1} - \tilde{\xi}^N + \int_{\tau \land T} \delta \tilde{g}(s, \delta \tilde{Y}^N(s))ds - \int_{\tau \land T} \langle \delta \tilde{Z}^N(s), dM_s \rangle. \quad (3.33)$$

Since $\delta \tilde{Y}^N(t) = \tilde{Y}^{N+N_1}(t) - \tilde{Y}^N(t)$ is also in Class (D), using Lemma 2.2 and the similar arguments as in the proof of (2.25), we obtain

$$|\delta \tilde{Y}^N(t)| \leq E[(\tilde{\xi}^{N+N_1} - \tilde{\xi}^N |F_{t_\land T}]. \quad (3.34)$$

This together with (3.14) imply

$$||\delta \tilde{Y}^N||_1 = \sup_{T \in \Gamma} E[||\delta \tilde{Y}^N(T)|_{T<T_\infty}|] \leq E[(\tilde{\xi}^{N+N_1} - \tilde{\xi}^N ||_1 \to 0$$

as $N \to \infty$. Hence there exists a continuous process $\tilde{Y} \in \text{Class (D)}$ such that

$$\lim_{N \to \infty} ||\tilde{Y}^N - \tilde{Y}||_1 = 0.$$
Applying Ito’s formula to $\tilde{Y}^N$ in (3.33) and using (3.32), we obtain

$$\langle \delta \tilde{Y}^N(t) \rangle \leq \tilde{\xi}_N + \tilde{\xi}_N - \tilde{\xi}_N^2 + 2 \int_{t}^{\tau} \delta \tilde{Y}^N(s) \delta \tilde{Z}^N(s) \, ds$$

(3.37)

By (1.2), (3.37) and Burkholder’s inequality, we have

$$E[f(x, y)] \leq C(1 + |J_1(x)| + |J_2(x)| |y|),$$

\(0 < \epsilon < 1\).

Letting \(N \to \infty\) as \(N \to \infty\).

By Lemma 3.2, we have

$$\lim_{N \to \infty} E[|\tilde{Z}^N(s) - \tilde{Z}(s)|^2 \, ds] = 0.$$
where \( J_1(x), J_2(x) \in L^p(D) \) and \( C \) is a constant. We have the following important corollary:

**Corollary 3.3** Assume there exists a \( x_0 \in D \) such that
\[
E_{x_0}[e^{-\int_0^T J_1(X(s))ds}] < \infty. \tag{3.40}
\]
Then for any \( F_{T_D} \)-measurable bounded r.v. \( \xi \) and \( x \in D \), the following two statements hold:

(i) There exists a unique pair \( (Y_x, Z_x) \) satisfying the following BSDE:
\[
Y_x(t) = \xi + \int_{t \wedge T_D}^{T_D} f(X(s), Y_x(s))ds - \int_{t \wedge T_D}^{T_D} \langle Z_x(s), dM_s \rangle, \; P_x\text{-a.e.}, \tag{3.41}
\]
such that \( e^{-\int_0^T J_1(X(s))ds}Y_x(t) \in \text{Class} (D) \). Furthermore,
\[
\sup_{x \in D} Y_x(0) < \infty.
\]

(ii) There exists at most one solution \( (Y_x, Z_x) \) satisfying the BSDE (3.41) such that \( Y_x \) is a bounded process.

**Remark 3.1** Please be ware of the subtle difference of (i) and (ii). We do not assume \( e^{-\int_0^T J_1(X(s))ds}Y_x(t) \in \text{Class} (D) \) in (ii). The statement (ii) is needed in the next section.

**Proof** (i) Since \( J_1, J_2 \in L^p(D) \), we know from (3.6) that \( E_x[\int_0^T (|J_1(X(s))| + J_2(X(s)))ds] < \infty \). By Lemma 3.1 and (3.40) we see that all the conditions in Theorem 3.2 are satisfied. Hence for any \( x \in D \), there exists a unique solution \( (Y_x, Z_x) \) to (3.41) such that \( e^{-\int_0^T J_1(X(s))ds}Y_x(t) \in \text{Class} (D) \).

Next we prove the last statement of the Corollary (i).

Set \( \tilde{Y}_x(t) = e^{-\int_0^T J_1(X(s))ds}Y_x(t), \tilde{Z}_x(t) = e^{-\int_0^T J_1(X(s))ds}Z_x(t) \) and \( \tilde{f}(t, y) = J_1(X(t))y + e^{-\int_0^T J_1(X(s))ds}f(X(t), e^{-\int_0^T J_1(X(s))ds}y), \tilde{\xi} = e^{-\int_0^T J_1(X(s))ds}\xi \) as in Theorem 2.1. It follows from (3.41) that
\[
\tilde{Y}_x(t) = \tilde{\xi} + \int_{t \wedge T_D}^{T_D} \tilde{f}(s, \tilde{Y}_x(s))ds - \int_{t \wedge T_D}^{T_D} \langle \tilde{Z}_x(s), dM_s \rangle. \tag{3.42}
\]

Choose a sequence of stopping times \( \{t_n\} \) that increases to \( \infty \) and such that
\[
\int_{T_D \wedge t_n}^{T_D} \frac{\tilde{Y}_x(s)}{\tilde{Y}_x(s)} ds \text{ is a uniformly integrable martingale under } P_x. \] By Lemma 3.2, it follows from (3.42) that
\[
|\tilde{Y}_x(t \wedge t_n)| = |\tilde{Y}_x(t_D \wedge t_n)| + \int_{t_D \wedge t_n}^{T_D \wedge t_n} \frac{\tilde{Y}_x(s)}{\tilde{Y}_x(s)} ds - \int_{t_D \wedge t_n}^{T_D \wedge t_n} \langle \tilde{Z}_x(s), dM_s \rangle \leq |\tilde{Y}_x(t_D \wedge t_n)| - \int_{t_D \wedge t_n}^{T_D \wedge t_n} \langle \tilde{Z}_x(s), dM_s \rangle.
\]

Taking the conditional expectation with respect to \( F_t \) on both sides of the above inequality, we have
\[
|\tilde{Y}_x(t \wedge t_n)| \leq E_x[|\tilde{Y}_x(t \wedge t_n)||F_t]. \tag{3.43}
\]
Since \( \tilde{Y}_x(t) \in \text{Class} (D) \), letting \( n \to \infty \) in (3.43) we get
\[
|\tilde{Y}_x(t)| \leq E_x[|\tilde{\xi}||F_t]. \tag{3.44}
\]
In particular, we have
\[
\sup_{x \in D} |Y_x(0)| = \sup_{x \in D} |\tilde{Y}_x(0)| \leq \sup_{x \in D} E_x[|\tilde{\xi}|] \\
\leq ||\tilde{\xi}||_\infty \sup_{x \in D} E_x[e^{-\int_0^T J_1(X(s))ds}] < \infty.
\]

(ii) Suppose that \((Y_x, Z_x)\) and \((Y'_x, Z'_x)\) are both bounded processes. Since \(J_1 \in L^p(D)\), there exists constant \(\alpha > 0\), \(\gamma > 1\) such that that \(||\gamma (-J_1 - \alpha^+)||_{L^p(D)} < \frac{1}{\sigma}\) where \(\sigma\) is defined in (3.6). Then by (3.6) and Kahamiskii’s inequality, we have
\[
E_x[e^{0^\gamma (Y(-J_1 - \alpha^+)(X(s))ds}] \leq E_x[e^{0^\gamma (Y(-J_1 - \alpha^+)(X(s))ds}] < \infty.\]
Moreover by (3.7) and (3.8)
\[
\gamma_1 < E_x[e^{0^\gamma (Y(-J_1 - \alpha^+)(X(s))ds}] < \gamma_2, \quad \forall x \in \mathcal{D}, \tag{3.45}
\]
for some positive constants \(\gamma_1, \gamma_2 > 0\).

Set \(\overline{Y}_x(s) = e^{0^\gamma J_1(X(r))dr} Y_x(s), \overline{Z}_x(s) = e^{0^\gamma J_1(X(r))dr} Z_x(s)\) and \(\overline{Y}'_x(s) = e^{0^\gamma J_1(X(r))dr} Y'_x(s), \overline{Z}'_x(s) = e^{0^\gamma J_1(X(r))dr} Z'_x(s)\). Then
\[
\delta \bar{Y}_x(s) = \bar{Y}_x(s) - \bar{Y}'_x(s), \delta \bar{Z}_x(s) = \bar{Z}_x(s) - \bar{Z}'_x(s), \bar{f}(s, y) = e^{0^\gamma J_1(X(r))dr} f(X(s), e^{0^\gamma J_1(X(r))dr} y)(J_1 + \alpha)(X(s))y.\]

\[
(y_1 - y_2)(\bar{f}(s, y_1) - \bar{f}(s, y_2)) \leq \alpha(y_1 - y_2)^2. \tag{3.46}
\]

Choose a sequence of stopping times \(\tau_n\) that increases to \(\infty\) and such that
\[
\int_0^{\tau_n \wedge \tau_D} (\delta \bar{Y}_x(s))ds \text{ is a uniformly integrable martingale. By (3.45)}
\]
\[
\sup_n \gamma_1^{-1} E_x\left[\int_0^{\tau_n \wedge \tau_D} \delta \bar{Y}_x(s)ds\right] \leq \sup_n \gamma_1^{-1} E_x\left[\int_0^{\tau_n \wedge \tau_D} \gamma(-J_1 - \alpha^+)(X(s))ds\right] < \infty,
\]
which implies that \(\{e^{0^\gamma J_1(X(r))dr} (-J_1 - \alpha^+)(X(s))ds\}_{n \geq 1}\) is \(\text{P}_x\)-uniformly integrable. Hence \(\{\delta \bar{Y}_x(t \wedge \tau_D)\}_{n \geq 1}\) is \(\text{P}_x\)-uniformly integrable because \(Y_x - Y'_x\) is bounded. Similarly, we can show that \(\{\delta \bar{Y}_x(t \wedge \tau_n \wedge \tau_D)\}_{n \geq 1}\) is also \(\text{P}_x\)-uniformly integrable.

Note that
\[
\delta \bar{Y}_x(t) = \int_0^{\tau_D} (\bar{f}(s, \bar{Y}_x(s)) - \bar{f}(s, \bar{Y}'_x(s)))ds - \int_0^{\tau_D} \langle \delta \bar{Z}(s), dM_x \rangle.
\]

By Lemma 3.2 and (3.46), it holds that
\[
|\delta \bar{Y}_x(t \wedge \tau_n \wedge \tau_D)| \leq |\delta \bar{Y}_x(t \wedge \tau_D)| + \int_0^{\tau_n \wedge \tau_D} \left(\frac{\delta \bar{Y}_x(s)}{|\delta \bar{Y}_x(s)|}\right) \langle \bar{f}(s, \bar{Y}_x(s)) - \bar{f}(s, \bar{Y}'_x(s))ds\rangle
\]
\[
- \int_0^{\tau_n \wedge \tau_D} \left(\frac{\delta \bar{Y}_x(s)}{|\delta \bar{Y}_x(s)|}\right) \langle \delta \bar{Z}(s), dM_x \rangle
\]
\[
\leq |\delta \bar{Y}_x(t \wedge \tau_D)| + \alpha \int_0^{\tau_n \wedge \tau_D} |\delta \bar{Y}_x(s)|ds - \int_0^{\tau_n \wedge \tau_D} \langle \delta \bar{Z}(s), dM_x \rangle, \quad \text{P}_x - a.e.\tag{3.47}
\]

Taking the expectation on the both sides of (3.47) we get
\[
E_x[|\delta \bar{Y}_x(t \wedge \tau_n \wedge \tau_D)|] \leq E_x[|\delta \bar{Y}_x(t \wedge \tau_D)|] + \alpha E_x[\int_0^{\tau_n \wedge \tau_D} |\delta \bar{Y}_x(s)|ds]. \tag{3.48}
\]
Notice that \( \lim_{n \to \infty} |\delta \overline{Y}_x (\tau'_n \wedge \tau_D)| = e^{|\int_0^\tau_D (-J_1 - \alpha)(X(s))ds|} |\xi - \xi| = 0, P_x-\text{a.e.} \) Using the uniformly integrability of \( \{\delta \overline{Y}_x (t \wedge \tau'_n \wedge \tau_D)\}_{n \geq 1} \) and \( \{\delta \overline{Y}_x (\tau'_n \wedge \tau_D)\}_{n \geq 1} \), letting \( n \to \infty \) in (3.48) we find

\[
E_x[|\delta \overline{Y}_x (t)|] \leq \alpha \int_t^\infty E_x[|\delta \overline{Y}_x (s)|] ds.
\]

By Gronwall’s inequality we conclude \( E_x[|\delta \overline{Y}_x (t)|] = 0. \) Hence \( Y_x = Y'_x. \) Furthermore we deduce that \( \int_0^\tau (\delta \overline{Z}_x (s), dM_s) = 0 \) and hence \( Z_x = Z'_x. \) The uniqueness is proved. \( \square \)

### 4 Dirichlet Boundary Value Problem

#### 4.1 Linear Case

Consider the following second order elliptic operator

\[
L_2 u = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + q(x).
\]

The quadratic form generated by \( L_2 \) is

\[
\begin{align*}
\mathcal{E}^q(u, v) &= (-L_2 u, v) \\
&= \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \sum_{i=1}^d \int_D b_i(x) \frac{\partial u}{\partial x_i} v(x) dx - \int_D q(x) u(x) v(x) dx, \\
D(\mathcal{E}^q) &= W_0^{1,2}(D).
\end{align*}
\]

Since \( |b|^2, q \in L^p \) for \( p > \frac{d}{2} \), it is known that \( (\mathcal{E}^q, D(\mathcal{E}^q)) \) is a coercive closed form and there exist a \( \alpha_0 > 0 \) such that

\[
\lambda_{\alpha_0} \mathcal{E}(u, u) \leq \mathcal{E}^q_0(u, u) \leq \Lambda_{\alpha_0} \mathcal{E}(u, u)
\]

for some constant \( \lambda_{\alpha_0}, \Lambda_{\alpha_0} \), where \( \mathcal{E}(u, u) = \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \) and \( \mathcal{E}^q_0(u, u) = \mathcal{E}^q(u, u) + \alpha_0 (u, u) \). By Theorem 3.2 in [13] or Lemma 4.1 in [7], for sufficient large \( \alpha > 0 \), the \( \alpha \)-resolvent operator of \( (\mathcal{E}^q, D(\mathcal{E}^q)) \) is given by

\[
G^q_\alpha f(x) = E_x[\int_0^{\tau_D} e^{\int_0^t q(X(s))ds - \alpha t} f(X(t)) dt], \quad \forall f \in L^2(D),
\]

where \( X = (X_t, P_x, x \in D) \) is the diffusion associated with the operator \( L_1 \) in (1.4).

Let \( F \) be a measurable function satisfying

\[
|F(x)| \leq C + C|q(x)|,
\]

for some constant \( C \). Take \( \varphi \in C(\partial D) \) and consider the boundary value problem

\[
\begin{align*}
L_2 u(x) &= -F(x), \quad \forall x \in D, \\
u(x)|_{\partial D} &= \varphi, \quad \forall x \in \partial D.
\end{align*}
\]

We have the following result:

**Theorem 4.1** Assume there exists a \( x_0 \in D \) such that

\[
E_{x_0}[\exp(\int_0^{\tau_D} q(X(s)) ds)] < \infty.
\]
Then there exists a unique continuous weak solution to problem (4.4) which is given by
\[ u(x) = E_x \left[ e^{\int_{\tau_D}^T q(X(s))ds} \varphi(X(\tau_D)) + \int_0^{\tau_D} e^{\int_t^{\tau_D} q(X(s))ds} F(X(t))dt \right]. \] (4.5)

**Proof** It is known from Lemma 2.1 in [5] that \( u_1(x) = E_x[\varphi(X(\tau_D))] \) is the unique continuous weak solution of the problem
\[ \begin{aligned} L_1u_1(x) &= 0, \quad \forall x \in D, \\ u_1(x)|_{\partial D} &= \varphi, \quad \forall x \in \partial D. \end{aligned} \] (4.6)

Let \( F_1(x) = F(x) + q(x)u_1(x) \) and \( u_2(x) = E_x[\int_{\tau_D}^T e^{\int_t^{\tau_D} q(X(s))ds} F_1(X(t))dt]. \) By Lemma 3.1 and (4.3) \( u_2 \) is well defined. Let \( u(x) \) be defined as (4.5). Then \( u = u_1 + u_2. \) Hence, to prove that \( u \) is a continuous weak solution to (4.4) we only need show that \( u_2 \) is a continuous weak solution of the following problem:
\[ \begin{aligned} L_2u_2(x) &= -F_1, \quad \forall x \in D, \\ u_2(x)|_{\partial D} &= 0, \quad \forall x \in \partial D. \end{aligned} \] (4.7)

Since \( u_1 \) is bounded, by (4.3) we have \( |F_1(x)| \leq c_1(1 + |q(x)|) \) for some \( c_1 > 0. \) By Lemma 5.7 in [8] and Lemma 2.1 in [5], we know that the semigroup generated by \( L_2 \) is strong Feller. Hence, using the similar proof as that of Theorem 3.18 in [9] we obtain that \( u_2 \in C(\overline{D}). \)

Next we show that \( u_2 \in W^{1,2}_0(D). \)

Since \( q \in L^p \) for \( p > \frac{d}{2} \) and since the transition density function of \((X_t, P^0_x, x \in \mathbb{R}^d)\) has the Gaussian upper bound estimate by (3.9), we can easily deduce that \( \alpha G^q_\theta \) is strongly continuous on \( L^p. \)

By the Markov property, it is easy to see that
\[ \alpha(u_2 - \alpha G^q_\theta u_2) = \alpha G^q_\theta F_1(x). \] (4.8)

Hence we get
\[ \lim_{\alpha \to \infty} \alpha(u_2 - \alpha G^q_\theta u_2, u_2) \leq \lim_{\alpha \to \infty} ||\alpha G^q_\theta F_1(x)||_{L^p} ||u_2||_{L^{p'}} < \infty, \]
where \( \frac{1}{p} + \frac{1}{p'} = 1. \) By Theorem 1.1.4 in [15] we deduce that \( u_2 \in W^{1,2}_0(D). \)

Finally we show that \( u_2 \) is a continuous weak solution of (4.7). For any \( \phi \in C^\infty_0(D), \) by (4.8) we have
\[ \mathcal{E}^q(u_2, \phi) - (F_1, \phi) = \lim_{\alpha \to \infty}(\alpha(u_2 - \alpha G^q_\theta u_2, F_1) - (F_1, \phi) = \lim_{\alpha \to \infty}(\alpha G^q_\theta F_1, \phi) - (F_1, \phi) = 0. \]

Hence \( u_2 \) is a continuous weak solution of (4.7).

Now we prove the uniqueness. If \( \tilde{u} \) is another continuous weak solution of (4.4), then \( \tilde{u} - u_2 \) is a continuous weak solution of (4.6). By the uniqueness of (4.6), we get \( \tilde{u} = u. \)

### 4.2 Semilinear Case

Recall that \( M^0 \) is the martingale part of the diffusion process \((X_t, P^0_x)\) generated by \( L_0. \) Define the exponential martingale
\[ U(t) = \exp\left( \int_0^t \langle \tilde{a}^{-1}b(X(s)), dM^0_x \rangle - \frac{1}{2} \int_0^t \langle \tilde{a}^{-1}b(X(s)), b(X(s)) \rangle ds \right). \]
Then we have
\[ \frac{dP_x}{dP^0_x}|_{\mathcal{F}_t} = U(t). \]
Let \( g(x, y) : D \times \mathbb{R} \to \mathbb{R} \) be a measurable function such that \( g \) is continuous w.r.t. \( y \) and satisfies
\[
\begin{cases}
(y_1 - y_2)(g(x, y_1) - g(x, y_2)) \leq -K(x)|y_1 - y_2|^2, \\
g(x, y) \leq C(1 + |q(x) - K(x)| + |K_1(x)y|),
\end{cases}
\]
for some constant \( C \) and \( K, K_1 \in L^p(D) \).

Consider the following semilinear boundary value problem:
\[
\begin{aligned}
& L_2 u(x) = -g(x, u(x)), \quad \forall x \in D, \\
& u(x)|_{\partial D} = \varphi, \quad \forall x \in \partial D.
\end{aligned}
\] (4.9)

**Lemma 4.1** Assume there exists a \( x_0 \in D \) such that
\[
E_{x_0}^0 \left[ \exp \left( \int_0^{\tau_D} \langle \tilde{a} - 1 b(X(s)), dM_s^0 \rangle - \frac{1}{2} \int_0^{\tau_D} (b^* \tilde{a} - 1 b)(X(s))ds \right) \right] < \infty.
\] (4.10)

Then there exists a unique continuous weak solution to the problem (4.9).

**Proof** Set \( g_1(x, y) = g(x, y) + q(x)y \). By (4.10) and Corollary 3.3 for any \( x \in D \), there exists a unique solution \((Y_x, Z_x)\) to the following BSDE:
\[
Y_x(t) = \varphi(X(\tau_D)) + \int_{t \wedge \tau_D}^{\tau_D} g_1(X(s), Y_x(s))ds - \int_{t \wedge \tau_D}^{\tau_D} \langle Z_x(s), dM_s \rangle, \quad P_x-a.e.. \quad (4.11)
\]

Define \( u_0(x) := Y_x(0), \quad v_0(x) := Z_x(0) \). By the strong Markov property of \( X \) and the uniqueness of the BSDE (4.11), it is easy to see that \( P_x-a.e. \)
\[
u_0(X(t)) = Y_x(t), \quad v_0(X(t)) = Z_x(t), \quad 0 \leq t \leq \tau_D.
\] (4.12)

Consider the following linear problem:
\[
\begin{aligned}
& L_1 u(x) = -g_1(x, u_0(x)), \quad \forall x \in D, \\
& u(x)|_{\partial D} = \varphi, \quad \forall x \in \partial D.
\end{aligned}
\]

Since \( u_0 \) is bounded by Corollary 3.3, by Theorem 4.1 the above equation admits a unique continuous weak solution \( u \in W^{1, 2}(D) \). On the other hand, by the Fukushima’s decomposition we have for q.e. \( x \in D, \quad P_x-a.e. \)
\[
u(X(t)) - u(x) = \int_0^t \langle \nabla u(X(s)), dM_s \rangle + \int_0^t g_1(X(s), u_0(X(s)))ds \\
= \int_0^t \langle \nabla u(X(s)), dM_s \rangle + \int_0^t g_1(X(s), Y_x(s))ds, \quad \forall t < \tau_D.
\] (4.13)

Choose a sequence of stopping times \( \{T_n\}_{n \geq 1} \) that increases to \( \infty \) and such that \( \int_0^{\tau_n \wedge \tau_D} \langle Z_x(s), dM_s \rangle \) and \( \int_0^{\tau_n \wedge \tau_D} \langle \nabla u(X(s)), dM_s \rangle \) are \( P_x \)-uniformly integrable martingales. It follows from (4.11) and (4.13) that
\[
Y_x(t \wedge \tau_n \wedge \tau_D) - Y_x(\tau_n \wedge \tau_D) = \int_{t \wedge \tau_n \wedge \tau_D}^{\tau_n \wedge \tau_D} g_1(X(s), Y_x(s))ds - \int_{t \wedge \tau_n \wedge \tau_D}^{\tau_n \wedge \tau_D} \langle Z_x(s), dM_s \rangle,
\]
\[
u(X(t \wedge \tau_n \wedge \tau_D)) - u(X(\tau_n \wedge \tau_D)) = \int_{t \wedge \tau_n \wedge \tau_D}^{\tau_n \wedge \tau_D} g_1(X(s), Y_x(s))ds - \int_{t \wedge \tau_n \wedge \tau_D}^{\tau_n \wedge \tau_D} \langle \nabla u(X(s)), dM_s \rangle.
\]
Taking the conditional expectations respect to $\mathcal{F}_t$ on both sides of the above equations, we get
\[ u(X(t \wedge T_n \wedge \tau_D)) - E_x[u(X(T_n \wedge \tau_D))|\mathcal{F}_t] = Y_x(t \wedge T_n \wedge \tau_D) - E_x[Y_x(T_n \wedge \tau_D)|\mathcal{F}_t]. \] (4.14)
Since $u \in C(\overline{D})$ and since $Y_x(t)$ is bounded (see (4.12)), let $T_n \to \infty$ in (4.14) to obtain
\[ u(X(t \wedge \tau_D)) - E_x[u((X(\tau_D))|\mathcal{F}_t] = Y_x(t \wedge \tau_D) - E_x[\varphi(\tau_D)|\mathcal{F}_t]. \]
Since $u(x)|_{\partial D} = \varphi$, letting $t = 0$ we deduce that $u(x) = Y_x(0) = u_0(x)$ for q.e. $x \in D$. Hence $u$ is a weak solution to (4.9).

Now we proceed to prove the uniqueness of the boundary value problem (4.9). Suppose $u_1$ is another continuous weak solution of (4.9). Then by Fukushima’s decomposition $(u_1(X(t)), \nu_{u_1}(X(t)))$ is also a solution of BSDE (4.11) for q.e. $x \in D$. Since $u(X(t))$ and $u_1(X(t))$ are bounded, by Corollary 3.3(ii) we have $u_1(X(t)) = u(X(t))$. In particular, $|u_1(x) - u(x)| = E_x[|u_1(X(0)) - u(X(0))|] = 0$ for q.e. $x \in D$. By the continuity of $u_1$ and $u$, it holds for every $x \in D$. The uniqueness is proved.

Let $f(x, y) : D \times \mathbb{R} \to \mathbb{R}$ be a measurable function that is continuous w.r.t. $y$ and satisfies
\[
\begin{cases}
(y_1 - y_2)(f(x, y_1) - f(x, y_2)) \leq -J(x)|y_1 - y_2|^2, \\
|f(x, y)| \leq C(1 + J_1(x)|y|),
\end{cases}
\] (4.15)
for some constant $C$ and $J, J_1 \in L^p(D)$. Consider the following semilinear elliptic PDEs:
\[
\begin{align*}
&\mathcal{A}u(x) = -f(x, u(x)), \quad \forall x \in D, \\
&u(x)|_{\partial D} = \varphi, \quad \forall x \in \partial D.
\end{align*}
\] (4.16)

Now we can state the main result in this section.

**Theorem 4.2** Under the Assumption I, if there exists a $x_0 \in D$ such that
\[
E_{x_0}^{0,1} \exp\left(\int_0^T \left(\begin{array}{c}
\hat{a}^{-1}(b - \hat{\beta})(X(s)) \, dM_s^0 + \frac{1}{2} \int_0^T (b - \hat{\beta})^*\hat{a}^{-1}(b - \hat{\beta})(X(s)) \, ds \\
N_{\tau_D}^0 + \int_0^T (q(X(s)) - J(X(s))) \, ds
\end{array}\right)\right) < \infty,
\] (4.17)
where $E_{x_0}^{0,1}$ stands for the expectation under $P_{x_0}^{0}$, $N_{\tau_D}^0$ is the zero-energy part of the Fukushima’s decomposition defined in (2.2). Then there exists a unique continuous weak solution to problem (4.16). Moreover, if $f(x, y) = F(x)$, the solution is given by
\[ u(x) = E_x^{0,1} \left[ \exp\left(\int_0^{T_D} \left(\begin{array}{c}
\hat{a}^{-1}(b - \hat{\beta})(X(s)) \, dM_s^0 + \frac{1}{2} \int_0^T (b - \hat{\beta})^*\hat{a}^{-1}(b - \hat{\beta})(X(s)) \, ds \\
N_{\tau_D}^0 + \int_0^T q(X(s)) \, ds
\end{array}\right)\right) \right] F(X(t_D))dt.
\]

**Proof** Introduce the following operator on $L^2(D)$:
\[
\hat{A} = \frac{1}{2} \sum_{i,j=1}^d \frac{\hat{a}}{\alpha_x} (a_{ij}(x) \frac{\hat{a}}{\alpha_x}) + \sum_{i=1}^d \left[ b_i - \hat{b}_i - (\hat{a} \nabla v)_i \right] \frac{\hat{a}}{\alpha_x} + \frac{1}{2} (\hat{a} \nabla v)^* \hat{a} \nabla v - (\hat{b} - \hat{\beta})^* \nabla v + q.
\] (4.18)
Set $h = e^v$ and consider the following semi-linear elliptic PDE:
\[
\begin{align*}
&\hat{A}\hat{u}(x) = -h(x)f(x, h^{-1}(x)\hat{u}(x)), \quad \forall x \in D, \\
&\hat{u}(x)|_{\partial D} = h(x)\varphi(x), \quad \forall x \in \partial D.
\end{align*}
\] (4.19)
Note that:

\[
\begin{align*}
&\int_0^{t_D} \left( a^{-1}(b - \hat{b} - \tilde{a} \triangledown v)(X(s)), dM^0_s \right) - \frac{1}{2} \int_0^{t_D} \left( (b - \hat{b} - \tilde{a} \triangledown v)^* a^{-1}(b - \hat{b} - \tilde{a} \triangledown v)(X(s)) ds \\
+ \int_0^{t_D} \left( \frac{1}{2} \triangledown v \right)^* a \triangledown v + q - J(X(s)) \right) ds \\
&= \int_0^{t_D} \left( a^{-1}(b - \hat{b})(X(s)), dM^0_s \right) - \frac{1}{2} \int_0^{t_D} \left( (b - \hat{b})^* a^{-1}(b - \hat{b})(X(s)) ds \\
+ \frac{1}{2} \int_0^{t_D} \left( (b - \hat{b})^* \triangledown v \right)(X(s)) ds + \frac{1}{2} \int_0^{t_D} \left( \triangledown v \right)^* (b - \hat{b} - \tilde{a} \triangledown v)(X(s)) ds \\
&\quad + \int_0^{t_D} \left( \frac{1}{2} \triangledown v \right)^* a \triangledown v + q - J(X(s)) \right) ds \\
(4.20)
\end{align*}
\]

Thus condition (4.17) is equivalent to

\[
E_x[\exp(\int_0^{t_D} \left( a^{-1}(b - \hat{b} - \tilde{a} \triangledown v)(X(s)), dM^0_s \right) \\
- \frac{1}{2} \int_0^{t_D} \left( (b - \hat{b} - \tilde{a} \triangledown v)^* a^{-1}(b - \hat{b} - \tilde{a} \triangledown v)(X(s)) ds \\
+ \int_0^{t_D} \left( \frac{1}{2} \triangledown v \right)^* a \triangledown v + q - J(X(s)) ds) < \infty.
(4.21)
\]

Since \( h(x) > 0 \), (4.15) implies

\[
\begin{cases}
(y_1 - y_2)(h(x)f(x, h^{-1}(x)y_1) - h(x)f(x, h^{-1}(x)y_2)) \leq -J(x)|y_1 - y_2|^2, \\
|h(x)f(x, h^{-1}(x)y) - C(\sup_{x \in D} h(x) + |J_1(x)y|). \\
\end{cases}
(4.22)
\]

By (4.21), (4.22) and Lemma 4.1, we conclude that there exists a unique continuous weak solution \( \hat{u} \) to problem (4.19). Let \( u(x) = h^{-1}(x)\hat{u}(x) \). Now following the same arguments as in the proof of Theorem 5.1 in [18], we can show that \( u \) is the continuous weak solution of problem (4.16). We omit the details here.

Let us now consider the special case, \( f(x, y) = F(x) \). Define \( \hat{U}(t) = \exp(\int_0^t \left( a^{-1}(b - \hat{b} - \tilde{a} \triangledown v)(X(s)), dM^0_s \right) - \frac{1}{2} \int_0^t \left( (b - \hat{b} - \tilde{a} \triangledown v)^* a^{-1}(b - \hat{b} - \tilde{a} \triangledown v)(X(s)) ds \\
+ \int_0^t \left( \frac{1}{2} \triangledown v \right)^* a \triangledown v + q - J(X(s)) ds) h(X(t))F(X(t))dt \)

Since \( b - \hat{b} - \tilde{a} \triangledown v \in L^2(D), \) by Novikov’s criterion \( \hat{U}(t) \) is a \( P^0_x \) exponential martingale. Define

\[
\frac{d\hat{P}_x}{dP^0_x}|_{\mathcal{F}_t} = \hat{U}(t).
(4.23)
\]

Then it follows from Theorem 4.1 that the solution to problem (4.19) admits the representation:

\[
\hat{u}(x) = \hat{E}_x[\exp(\int_0^{t_D} \left( \frac{1}{2} \triangledown v \right)^* a \triangledown v + q)(X(s)) ds \varphi(X(\tau_D))] \\
+ \hat{E}_x[\int_0^{t_D} \exp(\int_0^t \left( \frac{1}{2} \triangledown v \right)^* a \triangledown v + q)(X(s)) ds h(X(t))F(X(t))dt]
:= \hat{u}_1(x) + \hat{u}_2(x),
\]

where \( \hat{E}_x \) is the expectation under \( \hat{P}_x \). By the calculations in (4.20), we have

\[
\hat{u}_1(x) = E^0_x[\hat{U}_{\tau_D} e^{\int_0^{\tau_D} \left( \frac{1}{2} \triangledown v \right)^* a \triangledown v - (b - \hat{b})^* \triangledown v + q)(X(s)) ds \varphi(X(\tau_D))] \\
= E^0_x[\exp(\int_0^{\tau_D} \left( a^{-1}(b - \hat{b})(X(s)), dM^0_s \right) - \frac{1}{2} \int_0^{\tau_D} \left( (b - \hat{b})^* a^{-1}(b - \hat{b})(X(s)) ds \\
+ \int_0^{\tau_D} q(X(s)) ds \varphi(X(\tau_D)))] e^{\varphi(x)},
\]
Then by the diagonal method there exists a subsequence 

Then condition (4.17) in Theorem 4.2 is satisfied.

Hence the solution of (4.16) in this case is given by

Example 4.3 If \( q - J - \text{div} \hat{b} \leq g \) for some \( g \in L^1(D) \) in the sense of distribution and there exists a \( x_0 \in D \) such that

Then condition (4.17) in Theorem 4.2 is satisfied.

Proof First we show that \( P_{x_0}^0 -a.e., N_{x_0}^\gamma + \int_0^t (q - J)(X(s))ds \leq \int_0^t g(X(s))ds, \quad \forall t > 0 \).

Take an even, nonnegative function \( \psi \in C_\infty^0(\mathbb{R}^d) \) such that \( \int_{\mathbb{R}^d} \psi(x)dx = 1 \). Set \( \psi_k(x) := k\psi(kx) \) and \( \hat{b}_k(x) := \int_{\mathbb{R}^d} \hat{b}_1(x)|\psi_k(x - y)|dy \). Then \( \hat{b}_k \in C_\infty^0(\mathbb{R}^d) \). Set \( (g_k - q_k + J_k)(x) := \int_{\mathbb{R}^d} (g - q + J)(y)|\psi_k(x - y)|dy \). Then it’s easy to see for any \( x \in D \)

Set \( f_k(x) := (g - q + J)(x) - (g_k - q_k + J_k)(x) \), \( P_{x_0}^0 f_k(x) := E_{x_0}^0 [f_k(X(x))] \). Let \( \hat{P}_k \) be the dual operator of \( P_k \) in \( L^2(D) \). Since \( g_k - q_k + J_k \) converges to \( g - q + J \) in \( L^1(D) \),

as \( k \to \infty \). Hence \( \int_0^t |f_k(X(s))|ds \) converges to 0 in \( P_{x_0}^0 \) where \( P_{x_0}^0(\cdot) = \int_D P_k(\cdot)dx \). Then by the diagonal method there is a subsequence \( \{f_{k_j}\} \) such that \( \int_0^t |f_{k_j}(X(s))|ds \) converges to 0 uniformly in \( t \) on any compact set of \( [0, \infty) \) \( P_{x_0}^0 -a.e. \). For any \( \delta \geq 0 \), set

It’s easy to show that \( \Lambda_{\delta} = \{\omega : \text{ for any } n \in \mathbb{N}, \lim_{k_i \to \infty} \int_\delta^n |f_{k_i}(X(\omega, s))|ds = 0\} \).
Hence (4.17) is satisfied.

By (4.25) and (4.28), taking $k_l \to \infty$ and $n \to \infty$ in (4.28) we have $P_x^0$-a.e.

Thus we have

$$E_{x_0}^0\left[\exp\left(\int_0^{T_D} \widetilde{a}^{-1}(b - \hat{b})(X(s)) \, dM^0_x + \frac{1}{2} \int_0^{T_D} (q - J)(X(s)) \, ds\right)\right]$$

$$\leq E_{x_0}^0\left[\exp\left(\int_0^{T_D} \widetilde{a}^{-1}(b - \hat{b})(X(s)) \, dM^0_x + \frac{1}{2} \int_0^{T_D} g(X(s)) \, ds\right)\right] < \infty.$$
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