Three-Pronged Strings and 1/4 BPS States in N=4 Super-Yang-Mills Theory

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Abstract

We provide an explicit construction of 1/4 BPS states in four-dimensional N = 4 Super-Yang-Mills theory with a gauge group SU(3). These states correspond to three-pronged strings connecting three D3-branes. We also find curves of marginal stability in the moduli space of the theory, at which the above states can decay into two 1/2 BPS states.

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1 Introduction

$N = 4$ super-Yang-Mills theory is the simplest setting for the Olive-Montonen duality conjecture \cite{1}. This theory is in fact believed to possess an exact $SL(2, \mathbb{Z})$ symmetry \cite{2}, which includes an element corresponding to the original electric-magnetic duality of \cite{1}. This symmetry implies the existence of an infinite set of stable dyon states, obtained by acting with the duality group on the perturbative $W$-boson states. Stability of these states is guaranteed by the fact that they preserve $1/2$ of the underlying supersymmetry, and therefore transform in short multiplets of the superalgebra \cite{3, 4}. Existence is a more difficult question, and relies on identifying harmonic forms on the moduli space of monopoles. In the simplest case of $SU(2)$ with two monopoles this form was found by Sen \cite{2}. There are also indications that such forms exist as well for more than two monopoles \cite{5, 6}. For higher rank gauge groups the strategy is to embed the known $SU(2)$ solutions using the simple roots of the group \cite{7, 8}.

D-branes \cite{9} have allowed us to approach this problem from a new direction. The low-energy world-volume theory of $n$ parallel D$p$-branes is $p + 1$-dimensional super-Yang-Mills theory with sixteen supersymmetries, and a gauge group $U(n)$, which is spontaneously broken to $U(1)^n$ if none of the D-branes coincide. In particular, for D3-branes in type IIB string theory we get precisely four-dimensional $N = 4$ SYM \cite{10, 11}. In this picture the $W$-bosons appear as fundamental strings between different D3-branes. Monopoles and dyons correspond to D-strings and bound states of D-strings and fundamental strings, respectively, between different D3-branes. These bound states are required by $SL(2, \mathbb{Z})$ duality of type IIB string theory, and in fact shown to exist in \cite{12}.

Recently, a new kind of object has entered the game in type IIB string theory, the so-called “three-string junction” or “three-pronged string”. The basic configuration corresponds simply to a fundamental string ending on a D-string. The subtle observation made by Schwarz \cite{13} is that charge conservation requires that on one side the D-string turn into a $(1,1)$ string. Other three-pronged strings can be obtained from the basic one by acting with $SL(2, \mathbb{Z})$. This will ensure that the $(p, q)$ charges are always conserved. More generally, three-pronged strings are believed to exist for any three strings with relatively prime charges $(p_i, q_i)$, under the condition that the total $p$ charge and total $q$ charge vanish. Furthermore, these objects were argued to be BPS saturated if the angles between the three strings are adjusted properly \cite{14, 15, 16, 17, 18}. Multi-pronged strings have been used to explain the appearance of exceptional gauge symmetry enhancement in eight dimensions \cite{19}, and to construct certain BPS states in five-dimensional field theories \cite{20}.

Since D3-branes are self-dual, i.e. invariant under $SL(2, \mathbb{Z})$, any $(p, q)$ string can end on them. It is therefore fair to assume that they are allowed boundaries for three-pronged strings as well. The obvious question is what state such a configuration corresponds to in the world-volume theory. Since one needs at least three D3-branes to support a three-pronged
string, we only expect these states to arise for gauge groups $SU(3)$ and higher.

We will argue that these are the $1/4$ BPS states of $N = 4$ SYM, by showing that their supersymmetry transformation properties agree, as well as their gauge charges and masses. We will further argue that these states exist only in certain regions of moduli space, and derive the curves of marginal stability explicitly. On these curves the $1/4$ BPS states are unstable to decay into two states. This corresponds to a point where one of the prongs degenerates, and the two remaining prongs separate along the common D3-brane into two open strings.

The paper is organized as follows. In section 2 we review the BPS states of $N = 4$ SYM and their representation in terms of strings between D3-branes. In section 3 we discuss three-pronged strings in general, and in the presence of D-branes. In section 4 we make the connection between three-pronged strings connecting D3-branes and $1/4$ BPS states of $N = 4$ SYM. We also derive the curves of marginal stability for these states, and explain the decay process in terms of the string picture. In section 5 we present a summary of the results, and offer some directions for future work.

2 BPS States in $N=4$ SYM

2.1 BPS states

Consider four-dimensional $N = 4$ super-Yang-Mills theory with a simple gauge group $G$. This theory can be obtained as the dimensional reduction of ten-dimensional $N = 1$ super-Yang-Mills theory on $T^6$. The ten-dimensional Lorentz group reduces to $SO(3,1) \times SO(6)$, where $SO(6)$ becomes a global R-symmetry. The vector supermultiplet consists of a gauge field $A_\mu (\mu = 0, \ldots , 3)$, six scalar fields $\phi^I (I = 4, \ldots , 9)$, transforming in the 6 of $SO(6)$, and four Weyl fermions transforming in the 4 of $Spin(6)$, all valued in the adjoint representation of $G$. The bosonic part of the action is given by

$$S = - \frac{1}{16\pi} \text{Im} \int \tau \text{Tr}(F \wedge F + iF \wedge *F) - \frac{1}{2e^2} \int \text{Tr}(|D\phi^I|^2 + \sum_{I<J} [\phi^I, \phi^J]^2),$$

where $\tau = i/e^2 + \theta_{YM}/2\pi$. We shall assume that $\theta_{YM} = 0$. The $N = 4$ supersymmetry algebra admits two central charges, given by $\mathbb{I}$ $\mathbb{Z}$

$$Z_\pm^2 = |Q^I_E|^2 + |Q^I_M|^2 \pm 2|Q^I_E||Q^I_M|\sin \gamma \quad (\gamma \geq 0),$$

where

$$Q^I_E = \int dS \cdot \text{Tr}(E\phi^I) \ , \quad Q^I_M = \int dS \cdot \text{Tr}(B\phi^I),$$

\[2.3\]
are the magnetic and electric charge vectors in $\mathbb{R}^6$, respectively, and $\gamma$ is the angle between them. Analysis of the Hamiltonian derived from (2.1) reveals a lower bound on the mass of charged states,

$$M^2 \geq Z^2_+.$$  \hspace{1cm} (2.4)

This is the Bogomol’nyi-Prasad-Sommerfeld (BPS) bound. There are two kinds of states (BPS states) that saturate this bound. For $\gamma = 0$, i.e. parallel electric and magnetic charge vectors $Q^I_M \propto Q^I_E$, the BPS states preserve 1/2 of the underlying supersymmetry, and therefore transform in a short (2$^4$) representation of the superalgebra, in which the highest spin is 1. Their mass is given by

$$M^2_{1/2\text{BPS}} = |Q^I_E|^2 + |Q^I_M|^2.$$ \hspace{1cm} (2.5)

For $\gamma > 0$, i.e. $Q^I_M \not\propto Q^I_E$, the BPS states preserve 1/4 of the supersymmetry, and therefore transform in a medium (2$^6$) representation of the superalgebra, in which the highest spin is 3/2. Their mass is given by

$$M^2_{1/4\text{BPS}} = |Q^I_E|^2 + |Q^I_M|^2 + 2|Q^I_E||Q^I_M|\sin \gamma.$$ \hspace{1cm} (2.6)

As we shall see, these states can only arise if the rank of the group is greater than 1. In both cases the masses are protected from quantum corrections, since both supermultiplets are shorter than the generic (2$^8$) representation [3].

The moduli space of vacua is parameterized by the VEV’s of the adjoint Higgs fields $\phi^I$ in the Cartan subalgebra $H$,

$$\langle \phi^I \rangle = v^I \cdot H.$$ \hspace{1cm} (2.7)

Let us assume that the gauge group $G$ is maximally broken to $U(1)^r$, where $r$ is the rank of $G$. The electric and magnetic charge vectors are then given by

$$Q^I_E = e q \cdot v^I, \quad Q^I_M = \frac{4\pi}{e} g \cdot v^I,$$ \hspace{1cm} (2.8)

where $q$ and $g$ are vectors in the root lattice and co-root lattice of $G$, respectively. As such, they can be expanded in terms of the simple roots $\beta^{(a)}$ and simple co-roots $\beta^{(a)*}$,

$$q = \sum n^a_e \beta^{(a)}, \quad g = \sum n^a_m \beta^{(a)*},$$ \hspace{1cm} (2.9)

where $\beta^{(a)*} = \beta^{(a)}/\beta^{(a)}$, and $a = 1, \ldots, r$.

For $G = SU(2)$ the expressions for the charge vectors reduce to

$$Q^I_E = e n^a_e v^I, \quad Q^I_M = \frac{4\pi}{e} n^a_m v^I,$$ \hspace{1cm} (2.10)
so we see that $Q^I_M$ and $Q^I_E$ are always parallel, and therefore that all the BPS states preserve 1/2 of the supersymmetry. It also follows from (2.5) and from the triangular inequality that BPS states with $(n_e, n_m)$ relatively prime integers are stable for all $e$. The perturbative BPS spectrum consists of a neutral massless photon multiplet $(0, 0)$, and massive $W^\pm$-boson multiplets $(\pm 1, 0)$. S-duality maps these to BPS states $(n_e, n_m)$ with $n_e$ and $n_m$ relatively prime. For $n_m = 1$ this is just the 'tHooft-Polyakov $SU(2)$ monopole, and its dyonic generalizations. Finding the states with larger values of $n_m$ is harder, and is equivalent to the problem of finding normalizable harmonic forms on the moduli space of monopoles. This was done for $n_m = 2$ in [4], and substantial evidence was provided for the existence of such forms for $n_m > 2$ as well [3, 4].

The case $G = SU(3)$ is the first instance where both kinds of BPS states are allowed. For $n_m \propto n_e$, i.e. $g \propto q$ and $Q^I_M \propto Q^I_E$, the BPS states are in short multiplets (1/2 BPS). The massive perturbative BPS states are $W$-bosons with $n_e = \pm(1, 0), \pm(0, 1)$ and $n_e = \pm(1, 1)$, corresponding to the two simple roots $\beta^{(1)}, \beta^{(2)}$, and the non-simple positive root $\gamma = \beta^{(1)} + \beta^{(2)}$. S-duality maps these to states with $(n_e, n_m) = (k(1, 0), l(1, 0)), (k(0, 1), l(0, 1))$, and $(k(1, 1), l(1, 1))$ respectively, where $k$ and $l$ are relatively prime integers. The property of $n_m \propto n_e$ is preserved, so these are 1/2 BPS states as expected. In fact these are the only BPS states predicted by S-duality. It follows from the BPS mass formula (2.5) that the first two classes of states are absolutely stable, whereas the states obtained from the $n_e = (1, 1)$ $W$-boson are only marginally stable. The former are obtained by embedding $SU(2)$ monopoles with charge $l$ using the appropriate simple root. The latter class of states are in general more difficult to find. The special case of the $n_m = (1, 1)$ monopole was obtained by an $SU(2)$ embedding using the root $\gamma$ [7].

Unlike in the $SU(2)$ case, we now have the possibility of choosing electric and magnetic charge vectors which are not parallel, i.e. $n_m \not\propto n_e$. The corresponding BPS states would preserve only 1/4 of the supersymmetry, and would have a mass given by (2.4). The existence of such states is not predicted by S-duality, since they cannot be obtained from any perturbative states. In fact they lie on separate $SL(2, \mathbb{Z})$ orbits. Furthermore, there are no known solutions to the Bogomol'nyi equations with non-parallel electric and magnetic charge vectors. Nevertheless, we will show that they do indeed exist.

### 2.2 D3-brane construction

Four dimensional $N = 4$ SYM with a gauge group $U(n)$ (or $SU(n)$) can be realized as the low-energy effective theory on the world-volume of $n$ parallel D3-branes [10, 11]. As we are primarily interested in $SU(3)$, let us consider three parallel D3-branes, extended along the directions $(x^1, x^2, x^3)$. These correspond to three points $R_1, R_2, R_3$ in $\mathbb{R}^6$, and therefore define a plane. The generic gauge group is $U(1)^3$, and is enhanced to $U(3)$ when all three D3-branes coincide. Since the c.o.m. degrees of freedom are decoupled from the rest, we
shall focus on the $SU(3)$ subgroup of $U(3)$, corresponding to relative degrees of freedom. This allows us to fix the position of one of the three D3-branes, so we choose $R_3$ to be at the origin. For generic values of $R_1$ and $R_2$, the gauge group $SU(3)$ is broken to $U(1)^2$. We shall work in a basis in which the two $U(1)$’s are associated to the two D3-branes at $R_1$ and $R_2$, respectively. In this basis it is easy to identify all the $W$-bosons. The $\pm (1,0)$ bosons correspond to a fundamental string, of either orientation, between the D3-brane at $R_3$ and the one at $R_1$; the $\pm (0,1)$ bosons correspond to a fundamental string between $R_3$ and $R_2$, and the $\pm (1,1)$ bosons – to a fundamental string between $R_1$ and $R_2$ (Fig. 1).

![Figure 1: D3-brane representation of $N = 4$ SU(3) SYM.](image)

It is also straightforward to identify all the monopole and dyon states predicted by S-duality. These correspond to D-strings, and bound states of D-strings and fundamental strings, respectively. These bound states are predicted by S-duality in type IIB string theory, and were shown to exist in \[12\].

The D-brane separations are proportional to the Higgs VEV’s in the SYM theory. We can determine the precise numerical factors by comparing the masses of $W$-bosons and monopoles in the two approaches. The result can be written as\[1\]

$$
R_a n_e^a = \frac{2\pi \alpha'}{\sqrt{g_s}} Q_E^I , \quad R_a n_m^a = 2\pi \alpha' \sqrt{g_s} Q_M^I ,
$$

(2.11)

where $g_s$ is the type IIB string coupling, and with $Q_E^I$ and $Q_M^I$ given in (2.8).

3 Three-pronged strings and D-branes

Type IIB string theory contains, in addition to an $SL(2, \mathbb{Z})$ multiplet of $(p,q)$ strings, objects consisting of three strings meeting at a point (Fig. 2). These are known as “three-string

\[1\]Masses are measured in the Einstein metric.
junctions” [13], or “three-pronged strings” [19]. If the three strings are of type \((p_i, q_i)\) charge conservation requires

\[
\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} q_i = 0.
\]  

(3.1)

We shall therefore refer to these as “((\(p_1, p_2\), \((q_1, q_2)\)) strings”.

![Diagram](image)

Figure 2: Three-pronged string, (a) in the general case, and (b) for (1,0), (0,1) and (1,1) prongs. The angle between the (1,0) and (0,1) prongs is 90°, and the angle \(\theta\) is given by \(\tan \theta = 1/g_s\).

The angles between the strings are determined by the requirement that the net force on the junction point vanishes [13]. This is achieved if

\[
\sum_{i=1}^{3} T_{p_i,q_i} \hat{n}_i = 0,
\]

(3.2)

where \(\hat{n}_i\) is the direction of the \(i\)’th string and \(T_{p_i,q_i}\) is its tension. Recall that the tension of a \((p, q)\) string is given in the Einstein metric by

\[
T_{p,q} = \frac{\sqrt{g_s}}{2\pi\alpha'}|p + q\tau|,
\]

(3.3)

where \(\tau = i/g_s + \chi/2\pi\), and \(\chi\) is the expectation value of the type IIB R-R scalar field. We shall assume that \(\chi = 0\), which is consistent with our assumption that \(\theta_{YM} = 0\).

It follows from (3.1), (3.2), and (3.3) that a \((p, q)\) string which is part of a three-string junction, or more generally a string network, must be oriented along the vector \((p + q\tau)\) in the (complex) plane of the network [13]. This in turn implies that the unbroken supersymmetries of any three-pronged string, or string network, are given by \(\epsilon_L Q_L + \epsilon_R Q_R\), with [13]

\[
\begin{align*}
\epsilon_L &= \Gamma_1 \cdots \Gamma_8 \epsilon_L \\
\epsilon_R &= -\Gamma_1 \cdots \Gamma_8 \epsilon_R \\
\epsilon_L &= \Gamma_1 \cdots \Gamma_7 \Gamma_9 \epsilon_R,
\end{align*}
\]

(3.4)

\(^2\)Generalizations for \(\chi \neq 0\), and thus \(\theta_{YM} \neq 0\), will be discussed elsewhere.
where we have assumed that the network lies in the \((x^8, x^9)\)-plane. These leave 1/4 of the original type IIB supersymmetry unbroken. The first two conditions give the unbroken supersymmetries of a fundamental string extended along \(x^9\). The last condition can be turned into something more familiar using the ten-dimensional chirality operator. The original 16-component spinors have a definite ten-dimensional chirality,

\[ \Gamma^{11} \epsilon_L = +\epsilon_L, \quad \Gamma^{11} \epsilon_R = +\epsilon_R, \]  

(3.5)

where \(\Gamma^{11} = \Gamma_0 \cdots \Gamma_9\). This means that

\[ \Gamma_0 \Gamma_9 \epsilon_L = \Gamma_1 \cdots \Gamma_8 \epsilon_L = \epsilon_L, \]  

(3.6)

where the last equality is a result of the first condition in (3.4). Multiplying the third condition in (3.4) by \(\Gamma_0 \Gamma_9\) then gives

\[ \epsilon_L = \Gamma_0 \cdots \Gamma_7 \epsilon_R, \]  

(3.7)

which, by itself, gives the unbroken supersymmetries of a D7-brane extended along the directions \((x^1, \ldots, x^7)\). In summary, the supersymmetry preserved by a three-pronged string is the same as the supersymmetry preserved by a fundamental string oriented in the plane defined by the three prongs, together with a D7-brane transverse to that plane.

Since we know normal open strings can end on D-branes, whilst preserving a certain amount of supersymmetry, it is natural to ask whether the same is true for three-pronged strings. Since the supersymmetry conditions of the three-pronged strings already include effectively a D7-brane transverse to the plane of the three-pronged string, we conclude that adding D7-branes transverse to that plane does not break any more supersymmetry. On the other hand adding D3-branes or D(-1)-branes transverse to the plane further breaks the supersymmetry by half, leaving 1/8 of the original amount, and adding transverse D1-branes, D5-branes or D9-branes breaks all the supersymmetry.

Since the three prongs are mutually non-local, they cannot all end on D-branes in general. The exception is the D3-brane, on which any \((p, q)\) string can end. Three parallel D3-branes can therefore be connected by a three-pronged string, lying in a plane transverse to the D3-branes (Fig. 3). Furthermore, this configuration preserves 1/8 of the original supersymmetry. Of course since we do not yet know how to quantize three-pronged strings, our argument relies on the assumption that \((p, q)\) prongs are equivalent to \((p, q)\) strings as far as boundary conditions go.

The analogous situation with 7-branes is less clear, as those would have to be mutually non-local. It is not clear how much, if any, supersymmetry is left unbroken by a generic system of parallel, mutually non-local, 7-branes. Certain configurations of mutually non-local 7-branes, namely those that arise from F-theory compactification on K3, are known to preserve 1/2 of the supersymmetry. In those situations one could connect the 7-branes with
the appropriate three-pronged strings, leaving 1/4 of the supersymmetry unbroken. Such configurations were studied in [19], in the context of exceptional gauge enhancement in eight dimensions.

In what follows we shall focus on the D3-brane configuration of section 2, in which the three D3-branes are located at the points 0, $R_1$, and $R_2$ in the $(x^8, x^9)$ plane.

4 1/4 BPS States and Marginal Stability

4.1 1/4 BPS states

Since the configuration of three parallel D3-branes connected by a three-pronged string preserves 1/8 of the space-time supersymmetry, namely 4 supersymmetries, it also preserves 1/4 of the D3-brane world-volume supersymmetry. This strongly suggests that we identify the corresponding world-volume states with the 1/4 BPS states of $N = 4$ SYM discussed in section 2. In fact, since the three prongs, and in particular the two prongs ending on the D3-branes at $R_1$ and $R_2$, are mutually non-local, the corresponding electric and magnetic charge vectors $n_e, n_m$ will be non-parallel, as required for the 1/4 BPS states. Our conjecture is then that the ground state of the $((p_1, p_2), (q_1, q_2))$ string corresponds in the world-volume theory to the 1/4 BPS state with $(n_e, n_m) = ((p_1, p_2), (q_1, q_2))$.

As further evidence for this conjecture, let us compute the mass of a specific 1/4 BPS state, and compare it with the mass of the corresponding three-pronged string. With the aid of (2.11) we can rewrite the mass formula for 1/4 BPS states (2.6) in terms of the geometric variables $R_a$ as

$$M_{1/4BPS}^2 = \left(\frac{1}{2\pi \alpha'} \right)^2 \left( g_s |R_a n_e^a|^2 + \frac{1}{g_s} |R_a n_m^a|^2 + 2 |R_a n_e^a||R_a n_m^a| \sin \gamma \right). \quad (4.1)$$
Let us now consider a specific state, with
\[(n_e, n_m) = ((1, 0), (0, 1)).\] (4.2)
This state is electrically charged under the first \(U(1)\), and magnetically charged under the second \(U(1)\). Its mass is given by
\[M^2_{((1,0),(0,1))} = \left(\frac{1}{2\pi \alpha'}\right)^2 \left(\frac{g_s}{\sqrt{|R_1|^2 + \frac{1}{g_s}|R_2|^2 + 2|R_2| |R_2| \sin \gamma}\right).\] (4.3)
The corresponding three-pronged string has charges
\[(p_1, q_1) = (1, 0), \quad (p_2, q_2) = (0, 1), \quad (p_3, q_3) = (-1, -1).\] (4.4)
The orientations of the prongs are shown in Fig. 2b and Fig. 4. The \((1, 0)\) and \((0, 1)\) prongs meet at a right angle, and the \((1, 1)\) prong comes in at an angle \(\theta\) given by
\[\tan \theta = 1/g_s.\] (4.5)
Let us assume that all the angles of the triangle formed by the three D3-branes \(\alpha, \beta, \gamma\) are less than 90°. We shall discuss what happens when one of these angles grows beyond 90° in the next subsection.

\[\begin{align*}
\text{Figure 4: A } & (\text{1,0,0,1)} \text{ three-pronged string ending on three D3-branes at } R_1^I, R_2^I \text{ and } 0. \\
& \text{The angles of the triangle defined by these points are } \alpha, \beta \text{ and } \gamma, \text{ respectively, and the lengths of the corresponding prongs are } A_1, A_2 \text{ and } A_3. \text{ The angle } \theta \text{ is given by } \tan \theta = 1/g_s. \\
& \text{The mass of the three-pronged string connecting the three D3-branes is simply given by the sum of the masses of the individual prongs:}
\end{align*}\]
\[
M^2_{\text{3-string}} = \left( T_{(1,0)} A_1 + T_{(0,1)} A_2 + T_{(-1,-1)} A_3 \right)^2
\]
\[
= \frac{g_s}{(2\pi \alpha')^2} \left( A_1 + \frac{1}{g_s} A_2 + \sqrt{1 + \frac{1}{g_s^2} A_3} \right)^2.
\] (4.6)
From the geometry of Fig. 4 it follows that the lengths of the prongs $A_1, A_2, A_3$ are related to the lengths $R_1, R_2$ and the angle $\gamma$ by

$$
R_1^2 = A_1^2 + A_3^2 + 2A_1A_3 \cos \theta
$$

$$
R_2^2 = A_2^2 + A_3^2 + 2A_2A_3 \sin \theta
$$

$$
A_1^2 + A_2^2 = R_1^2 + R_2^2 - 2|R_1||R_2| \cos \gamma.
$$

These can be combined to get a fourth relation,

$$
|R_1||R_2| \sin \gamma = A_1A_2 + A_1A_3 \sin \theta + A_2A_3 \cos \theta.
$$

Using (4.5) together with the above relations in the mass formula (4.6), we find

$$
M_{3-string}^2 = \left(\frac{1}{2\pi \alpha'}\right)^2 \left( g_s |R_1|^2 + \frac{1}{g_s} |R_2|^2 + 2|R_1||R_2| \sin \gamma \right).
$$

This agrees with the mass of the $((1,0),(0,1))$ BPS state (4.3). We therefore conclude that the ground state of the $((1,0),(0,1))$ three-pronged string does indeed correspond to the $((1,0),(0,1))$ 1/4 BPS state in $N = 4$ SYM. Other 1/4 BPS states can be obtained from this one by the action of the field-theoretic duality group $SL(2, \mathbb{Z})$. On the other hand other three-pronged strings can be obtained from the basic one (4.4) by the action of the type IIB duality group $SL(2, \mathbb{Z})$. Since in both cases the states transform as doublets under $SL(2, \mathbb{Z})$, we conclude that the correspondence between three-pronged strings and 1/4 BPS states extends to all charges that can be obtained from the ones above by S-duality.

When the three D3-branes coincide, i.e. when $R_1 = R_2 = 0$, these states become massless. The medium multiplets then decompose into short multiplets, some of which contain particles with spin $> 1$. The existence of massless particles with spin $> 1$ seems surprising at first sight, but is actually expected since the theory is conformal.

### 4.2 marginal stability

It follows from (4.4) that

$$
M_{((1,0),(0,1))} \leq M_{((1,0),(0,0))} + M_{((0,0),(0,1))},
$$

where the inequality is saturated for $\gamma = 90^\circ$. This suggests that when $\gamma = 90^\circ$, the above 1/4 BPS state is only marginally stable, and may decay into the purely electric state $((1,0),(0,0))$ and the purely magnetic state $((0,0),(0,1))$. It does not however mean that the state does indeed decay. In particular the inequality still holds for $\gamma > 90^\circ$, so it appears that if the above BPS state still exists it should be stable.
On the other hand it is clear from the three-pronged string representation of this state that it does not exist if $\gamma > 90^\circ$, since the $(1,0)$ and $(0,1)$ prongs would have to meet outside of the triangle formed by the three D3-branes (Fig. 5c). This representation also provides the mechanism by which the $((1,0),(0,1))$ state decays. For $\gamma < 90^\circ$ (Fig. 5a) there exists a three-pronged string, with the required charges, connecting the D3-branes. It follows from (4.9) that this state is lighter than the state consisting of separate $(1,0)$ and $(0,1)$ strings. As $\gamma$ is increased, the length of the $(1,1)$ prong decreases, until it vanishes when $\gamma = 90^\circ$ (Fig. 5b). At this point we just have a $(1,0)$ string and $(0,1)$ string that meet at a point on the common D3-brane. There is now a flat direction associated to separating the two strings along this D3-brane. This corresponds in the world-volume theory to the decay of the $1/4$ BPS state to a purely electric $1/2$ BPS state and a purely magnetic $1/2$ BPS state. For $\gamma > 90^\circ$ (Fig. 5c) the two string state is the only one carrying the above charges, so we conclude that $\gamma = 90^\circ$ defines a curve of marginal stability $C_3$ on the plane of the three-pronged string, given by the smallest circle passing through $R_1$ and $R_2$.

![Figure 5](image-url)

Figure 5: Moving a D3-brane through the curve of marginal stability. In (a) $\gamma < 90^\circ$, and the D3-branes can be connected by the $((1,0),(0,1))$ three-pronged string. In (b) $\gamma = 90^\circ$, and the three-pronged string becomes degenerate with a $(1,0)$ string plus a $(0,1)$ string. In (c) $\gamma < 90^\circ$, and only the two open string state exists.

More generally we can see from the geometry of Fig. 4 that the conditions on the angles for a $((1,0),(0,1))$ three-pronged string to exist are

$$\alpha < \theta + 90^\circ, \quad \beta < 180^\circ - \theta, \quad \gamma < 90^\circ,$$

where $\theta$ is determined by the value of the string coupling constant (4.5). We have argued that when $\gamma$ exceeds its bound the three-pronged string decays into a $(1,0)$ string and a $(0,1)$ string. On the other hand, if $\alpha$ exceeds its bound the three-pronged string will decay into a $(0,1)$ string and a $(1,1)$ string. In the world-volume theory these correspond to a monopole of charge $(n_e,n_m) = ((0,0),(-1,1))$, and a dyon of charge $(n_e,n_m) = ((1,0),(1,0))$, respectively. Similarly if $\beta$ exceeds its bound the three-pronged string will decay into a $(1,0)$ string...
and a $(1,1)$ string, which correspond to a $W$-boson of charge $(n_e, n_m) = ((1, -1), (0, 0))$ and a dyon of charge $(n_e, n_m) = ((0, 1), (0, 1))$.

The conditions on $\alpha$ and $\beta$ also define curves of marginal stability $C_1, C_2$ passing through the D3-branes at $0, R'_2$ and $0, R'_1$, respectively. The curve $C_1$, for example, is given by the set of points $\{Q\}$ in the $(0, R'_1, R'_2)$ plane for which the angle between the segments $Q0$ and $QR'_2$ is $\theta + 90^\circ$. The curve $C_2$ is defined similarly.

5 Summary and Outlook

We have shown that a configuration of three parallel D3-branes connected by a three-pronged string preserves $1/8$ of original space-time supersymmetry, and therefore $1/4$ of the D3-brane world-volume supersymmetry. This, together with the fact that the world-volume electric and magnetic charge vectors were not parallel, was used to argue that the ground state of such a string corresponds to a $1/4$ BPS state in the world-volume $N = 4$ SYM theory. As further evidence for this conjecture we showed that the masses agree.

The D3-brane approach has allowed us to identify curves of marginal stability in the moduli space of $N = 4$ SYM. On these curves the $1/4$ BPS states are marginally stable to decay into two $1/2$ BPS states. In the D3-brane picture this is seen as the dissociation of the three-pronged string into two open strings when one of the prongs degenerates.

Strictly speaking, our result applies to the $1/4$ BPS states which can be obtained from the $(n_e, n_m) = ((1, 0), (0, 1))$ state by $SL(2, \mathbb{Z})$, and for $\theta_{YM} = 0$. It would be interesting to generalize it for $\theta_{YM} \neq 0$. But more importantly we would like to generalize the result to other $SL(2, \mathbb{Z})$ conjugacy classes. In particular, the representation of such states as three-pronged strings may simplify the problem of counting them, and this may be relevant to black-hole entropy.

Three-pronged strings may also play a role in four-dimensional $N = 2$ superconformal field theories, which were studied in [22, 23]. From the point of view of type IIB string theory these correspond to the world-volume theories of a D3-brane, when it coincides with two mutually non-local 7-branes.

There are still many unanswered questions, the most important of which is how to actually quantize three-pronged strings, and thus get a better handle on their space-time properties.

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