Biquasile Colorings of Oriented Surface-Links

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Abstract

We introduce colorings of oriented surface-links by biquasiles using marked graph diagrams. We use these colorings to define counting invariants and Boltzmann enhancements of the biquasile counting invariants for oriented surface-links. We provide examples to show that the invariants can distinguish both closed surface-links and cobordisms and are sensitive to orientation.

Keywords: Biquasiles, counting invariants, surface-links, marked graph diagrams

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1 Introduction

In [9], a type of algebraic structure known as biquasiles was introduced and used to define invariants of oriented classical knots and links via counting colorings of graphs known as dual graph diagrams. Dual graph versions of a generating set of oriented Reidemeister moves were identified and used to motivate the biquasile axioms and to prove invariance of the set of biquasile colorings under these moves. Biquasiles can be understood as a special case of the ternary algebraic structures defined in [11].

In [8], planar graphs with extra information known as marked graph diagrams (also sometimes called marked vertex diagrams or ch-diagrams) were introduced as a way of encoding knotted and linked surfaces in \( \mathbb{R}^4 \), known as surface-links. In [13] a set of moves on marked graph diagrams encoding ambient isotopy of surfaces in \( \mathbb{R}^4 \) analogous to the Reidemeister moves was proposed. In [7], generating sets of these Yoshikawa moves which are necessary and sufficient for ambient isotopy of oriented surface-links were identified.

In this paper we define biquasile counting invariants for oriented surface-links and use them to define new nonnegative integer-valued invariants of oriented surface-links. We then enhance these invariants with Boltzmann weights taking values in a commutative ring \( R \) such that the multiset of Boltzmann weight values over the complete set of biquasile colorings of a marked graph diagram defines a stronger invariant of surface-links from which we can recover the biquasile counting invariant by taking the cardinality of the multiset. In particular, these invariants potentially provide obstructions to cobordism between classical knots and links. As this paper was nearing completion, the authors learned that similar results have been independently obtained and recently presented by Maciej Niebrzydowski [10].

The paper is organized as follows. In Section 2 we review the basics of marked graph diagrams and surface-links. In Section 3 we review biquasiles and define the biquasile counting invariant for surface-links. We provide examples to show that biquasile colorings can distinguish surface-links and can detect orientation reversals. In Section 4 we recall the biquasile Boltzmann weight enhancement and extend it to the case of oriented surface-links with some examples. As an application we show that these invariants can distinguish non-isotopic cobordisms between links. We end in Section 5 with some open questions for future research.

2 Marked Graph Diagrams

In this section, we review (oriented) marked graph diagrams representing surface-links.

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Definition 1. A **marked graph** is a 4-valent graph in \( \mathbb{R}^3 \) each of whose vertices has a marker that looks like

\[
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array}
\]

Two marked graphs are said to be **equivalent** if they are ambient isotopic in \( \mathbb{R}^3 \) with keeping the rectangular neighborhoods of markers. A marked graph in \( \mathbb{R}^3 \) can be described by a link diagram on \( \mathbb{R}^2 \) with some 4-valent vertices equipped with markers, called a **marked graph diagram**.

Definition 2. An **orientation** of a marked graph \( G \) in \( \mathbb{R}^3 \) is a choice of an orientation for each edge of \( G \). An orientation of a marked graph \( G \) is said to be **consistent** if every vertex in \( G \) looks like

\[
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array}
\quad \text{or} \quad 
\begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array}
\]

A marked graph \( G \) in \( \mathbb{R}^3 \) is said to be **orientable** if \( G \) admits a consistent orientation. Otherwise, it is said to be **non-orientable**.

Definition 3. By an **oriented marked graph**, we mean an orientable marked graph in \( \mathbb{R}^3 \) with a fixed consistent orientation. Two oriented marked graphs are said to be **equivalent** if they are ambient isotopic in \( \mathbb{R}^3 \) with keeping the rectangular neighborhood of the marker and consistent orientation.

For \( t \in \mathbb{R} \), we denote by \( \mathbb{R}^3_t \) the hyperplane of \( \mathbb{R}^4 \) whose fourth coordinate is equal to \( t \in \mathbb{R} \), i.e.,

\[
\mathbb{R}^3_t = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t \}.
\]

A surface-link \( \mathcal{L} \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \) can be described in terms of its **cross-sections** \( \mathcal{L}_t = \mathcal{L} \cap \mathbb{R}^3_t \), \( t \in \mathbb{R} \) (cf. [2]).

It is known ([5, 8]) that any surface-link \( \mathcal{L} \) is equivalent to a surface-link \( \mathcal{L}' \) such that the projection \( p_{\mathcal{L}'} : \mathcal{L}' \to \mathbb{R} \) satisfies the following conditions:

1. A surface-link \( \mathcal{L}' \) has finitely many critical points and all critical points are non-degenerate.
2. All the index 0 critical points (minimal points) are in \( \mathbb{R}^3_{-1} \).
3. All the index 1 critical points (saddle points) are in \( \mathbb{R}^3_0 \).
4. All the index 2 critical points (maximal points) are in \( \mathbb{R}^3_1 \).

We call \( \mathcal{L}' \) a **normal form** of \( \mathcal{L} \).

Let \( \mathcal{L} \) be a surface-link in \( \mathbb{R}^4 \), and \( \mathcal{L}' \) a normal form of \( \mathcal{L} \). Then \( \mathcal{L}'_0 \) is a spatial 4-valent regular graph in \( \mathbb{R}^3_0 \). We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated.

\[
\begin{array}{c}
t = \epsilon \\
t = 0 \\
t = -\epsilon
\end{array}
\]

We choose an orientation for each edge of \( \mathcal{L}'_0 \) that coincides with the induced orientation on the boundary of \( \mathcal{L}' \cap \mathbb{R}^3 \times (-\infty, 0] \) from the orientation of \( \mathcal{L}' \). The resulting oriented marked graph \( G \) is called an **oriented marked graph** of \( \mathcal{L} \). As usual, \( G \) is described by a link diagram \( D \) with rigid marked vertices. Such a diagram \( D \) is called an **oriented marked graph diagram** or an **oriented ch-diagram** (cf. [12]) of \( \mathcal{L} \).
Let $D$ be an oriented marked graph diagram. We obtain two links $L_-(D)$ and $L_+(D)$ from $D$ by replacing each marked vertex in $D$ as shown.

\[L_-(D) \cup \{B_i\}\]

The links $L_-(D)$ and $L_+(D)$ are also called the negative resolution and the positive resolution of $D$, respectively. Conversely, we obtain an oriented surface-link by replacing a neighborhood of each marked vertex $v_i$ ($1 \leq i \leq n$) with an oriented band $B_i$ as illustrated.

Denote the disjoint union $B_1 \sqcup \cdots \sqcup B_n$ of bands by $\mathcal{B}(D)$. A marked graph diagram $D$ is said to be admissible if both resolutions $L_-(D)$ and $L_+(D)$ are trivial link diagrams.

Let us describe how to construct an oriented surface-link $\mathcal{L}$ in $\mathbb{R}^4$ from any given admissible marked graph diagram up to equivalence. Let $\Delta_1, \ldots, \Delta_a \subset \mathbb{R}^3$ be mutually disjoint 2-disks with $\partial(\cup_{j=1}^a \Delta_j) = L_+(D)$, and let $\Delta'_1, \ldots, \Delta'_b \subset \mathbb{R}^3$ be mutually disjoint 2-disks with $\partial(\cup_{k=1}^b \Delta'_k) = L_-(D)$. We
define a surface-link $S(D) \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ by

$$
(R_t^3, S(D) \cap \mathbb{R}^3_t) = \begin{cases} 
(R^3, \phi) & \text{for } t > 1, \\
(R^3, L_+(D) \cup (\cup_{j=1}^{p} \Delta_j)) & \text{for } t = 1, \\
(R^3, L_+(D)) & \text{for } 0 < t < 1, \\
(R^3, L_-(D) \cup (\cup_{i=1}^{n} B_i)) & \text{for } t = 0, \\
(R^3, L_-(D)) & \text{for } -1 < t < 0, \\
(R^3, L_-(D) \cup (\cup_{k=1}^{b} \Delta_k')) & \text{for } t = -1, \\
(R^3, \phi) & \text{for } t < -1.
\end{cases}
$$

It is proved in [5] that the isotopy type of $S(D)$ does not depend on choices of $\Delta_j$’s and $\Delta_k'$’s. We choose an orientation for the surface-link $S(D)$ so that the orientation of the cross-section $S(D)_0 = S(D) \cap \mathbb{R}_0^3$ induced from the chosen orientation of $S(D)$ is coherent to the orientation of $BL(D)$.

We call the oriented surface-link $S(D)$ the oriented surface-link associated with $D$. It is easily seen that $D$ is a marked graph diagram associated with the oriented surface-link $S(D)$. In particular, an admissible diagram represents a closed surface-link while a non-admissible diagram represents a cobordism between the positive and negative resolutions.

**Example 1.** The oriented marked graph diagram below represents the pictured cobordism between the unknot and unlink of two components depicted by the broken surface diagram below.

Since the marked graph diagram is admissible, this cobordism can be capped off with disks to obtained a sphere in $\mathbb{R}^4$, in this case an unknotted sphere.

Two oriented marked graph diagrams represent ambient isotopic surface-links if and only if they are related by the following local moves, known as Yoshikawa moves, in addition to the usual oriented Reidemeister
3 Biquasile Colorings of Surface-Links

We begin this section with a definition from [9].

**Definition 4.** A biquasile is a set \( X \) with six binary operations \( *, \backslash, /, \cdot, \backslash, / \) satisfying the axioms

(i) For all \( x, y \in X \) we have

\[
\begin{align*}
y \backslash (y \star x) &= x = (x \star y) / y \\
y \star (y \cdot x) &= x = (x \cdot y) / y
\end{align*}
\]

and

(ii)

\[
\begin{align*}
a \star (x \cdot (y \star (a \cdot b))) &= (a \star (x \cdot y)) \star (x \cdot (y \star (a \star (x \cdot y) \cdot b))) \\
y \star ((a \star (x \cdot y) \cdot b)) &= (y \star (a \cdot b)) \star ((a \star (x \cdot (y \star (a \cdot b))) \cdot b))
\end{align*}
\]

Axiom (i) says that \( X \) forms a **quasigroup** under both the \( \star \) and \( \cdot \) operations (hence \( \text{“biquasile”} \)) with left and right inverse operations \( \backslash, / \) and \( \star, \cdot \) respectively (at least in the finite case; see remark 1 below) and axiom (ii) specifies the relationship between the operations, like a complicated form of distributivity.

**Remark 1.** The conditions in axiom (i) arising from the Riedemeister moves are really the requirements that for every \( y \in X \), the maps \( x \mapsto x \star y, y \star x, x \cdot y \) and \( y \cdot x \) are invertible with inverse maps given by \( x \mapsto x / y \) etc. If \( X \) is a finite set then the inverse maps commute with the original maps and we also have

\[
\begin{align*}
y \star (y \backslash x) &= x = (x / y) \star x \\
y \cdot (y \backslash x) &= x = (x \cdot y) \cdot y;
\end{align*}
\]

however, for infinite \( X \) these conditions are not imposed \textit{a priori} in the biquasile definition. In practice, we have not considered biquasiles which do not also satisfy these additional conditons; it may be of interest to do so in the future.
Example 2. Let $X$ be a module over $\mathbb{Z}[d^{\pm 1}, s^{\pm 1}, n^{\pm 1}]$. Then $X$ is a biquasile under the operations

$$x \ast y = -dsn^2 x + ny \quad \text{and} \quad x \cdot y = dx + sy.$$ 

We have left inverse operations given by

$$x \ast y = n^{-1} x + dsny \quad \text{and} \quad x \ast y = -ds^{-1} x + s^{-1} y$$

and right inverse operations

$$x \ast y = -d^{-1} s^{-1} n^{-2} x + d^{-1} s^{-1} n^{-1} y \quad \text{and} \quad x/y = -ds^{-1} x + s^{-1} y.$$ 

Biquasiles of this sort are called Alexander biquasiles.

Example 3. For finite biquasiles we can specify the biquasile structure with a block matrix encoding the operation tables of $\cdot$ and $\ast$. For example, the Alexander biquasile structure on $X = \mathbb{Z}_4$ with $d = s = 1$ and $n = 3$ has operations

$$x \ast y = -dsn^2 x + ny = -(1)(1)(3^2)x + 3y = 3x + 3y \quad \text{and} \quad x \cdot y = dx + sy = x + y$$

so we have operation tables (using 4 for the class of 0 mod 4 since we number our rows and columns starting with 1):

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 1 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 3 & 2 \\
4 & 3 & 2 & 1 \\
\end{array} \quad \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
\end{array}$$

or in matrix form

$$\begin{bmatrix}
2 & 1 & 4 & 3 \\
1 & 4 & 3 & 2 \\
4 & 3 & 2 & 1 \\
3 & 2 & 1 & 4 \\
\end{bmatrix} \begin{bmatrix}
2 & 3 & 4 & 1 \\
3 & 4 & 1 & 2 \\
4 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
\end{bmatrix}.$$

Definition 5. Let $X$ be a biquasile. Then a biquasile coloring of an oriented marked graph diagram $D$ is an assignment of elements of $X$ to the regions in the planar complement of $D$ such that at every classical crossing or marked vertex, we have the following:

Theorem 1. If an oriented marked graph diagram $D$ has a biquasile coloring by a biquasile $X$ before a Yoshikawa move taking $D$ to $D'$, there is a unique biquasile coloring of $D'$ which agrees with the given coloring of $D$ outside the neighborhood of the move.

Proof. We verify for each of the moves $\Gamma_4$, $\Gamma'_4$, $\Gamma_5$, $\Gamma_6$, $\Gamma'_6$, $\Gamma_7$ and $\Gamma_8$; see [9] for the case of the classical
Reidemeister moves.

\[ a = c \cdot (d \cdot b) \]

\[ c = a \cdot (d \cdot b) \]
Corollary 2. For any finite biquasile $X$, the number $\Phi^X_Z(L)$ of biquasile colorings of a marked graph diagram $D$ representing an oriented surface-link $L$ is a surface-link invariant. We call this invariant the biquasile counting invariant.

Example 4. Let $X$ be the Alexander biquasile structure on $\mathbb{Z}_7$ with $d = 2$, $s = 3$ and $n = 4$, so we have

\[
\begin{align*}
x \ast y &= 5x + 4y \\
x \cdot y &= 2x + 3y.
\end{align*}
\]

Consider the marked graph diagram below which represents the surface-link $6_1$.

\[
\begin{pmatrix}
5 & 5 & 0 & 6 & 1 \\
6 & 5 & 0 & 5 & 1 \\
5 & 5 & 6 & 0 & 1 \\
6 & 5 & 5 & 0 & 1
\end{pmatrix}
\]

Assigning variables $x_1, \ldots, x_6$ to the regions, we obtain system of coloring equations

\[
\begin{align*}
x_1 \ast (x_5 \cdot x_2) &= x_4 & 5x_1 + 4(2x_5 + 3x_2) &= x_4 & 5x_1 + 5x_2 + 6x_4 + x_5 &= 0 \\
x_4 \ast (x_5 \cdot x_2) &= x_1 & 5x_4 + 4(2x_5 + 3x_2) &= x_1 & 6x_1 + 5x_2 + 5x_4 + x_5 &= 0 \\
x_1 \ast (x_5 \cdot x_2) &= x_3 & 5x_1 + 4(2x_5 + 3x_2) &= x_3 & 5x_1 + 5x_2 + 6x_3 + x_5 &= 0 \\
x_3 \ast (x_5 \cdot x_2) &= x_1 & 5x_3 + 4(2x_5 + 3x_2) &= x_1 & 6x_1 + 5x_2 + 5x_3 + x_5 &= 0
\end{align*}
\]

Then we have coefficient matrix

\[
\begin{pmatrix}
5 & 5 & 0 & 6 & 1 \\
6 & 5 & 0 & 5 & 1 \\
5 & 5 & 6 & 0 & 1 \\
6 & 5 & 5 & 0 & 1
\end{pmatrix}
\text{ row moves over } \mathbb{Z}_7
\begin{pmatrix}
1 & 2 & 0 & 2 & 6 \\
0 & 1 & 3 & 2 & 3 \\
0 & 0 & 1 & 6 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

so the kernel has dimension 2 and the space of colorings has $7^2 = 49$ $X$-colorings, i.e. $\Phi^X_Z(6_1) = 49$. Since
the corresponding unlinked object (a standard unlinked torus and sphere)

\[
\begin{array}{c}
\xymatrix{
& x_2 & x_2 \\
& x_1 & x_1 \\
x_3 & & \\
}\end{array}
\]

has \(7^3 = 343\) colorings by \(X\), the biquasile counting invariant with respect to this biquasile \(X\) detects the nontriviality of the surface-link \(6_1\).

**Example 5.** We selected three biquasiles of order 3

\[
X_1 = \begin{bmatrix}
1 & 2 & 3 & 1 & 2 & 3 \\
2 & 3 & 1 & 3 & 1 & 2 \\
3 & 1 & 2 & 2 & 3 & 1
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
1 & 2 & 3 & 1 & 2 & 3 \\
3 & 1 & 2 & 2 & 3 & 1 \\
2 & 3 & 1 & 3 & 1 & 2
\end{bmatrix}, \quad X_3 = \begin{bmatrix}
2 & 1 & 3 & 1 & 3 & 2 \\
3 & 1 & 2 & 3 & 2 & 1 \\
3 & 2 & 1 & 2 & 1 & 3
\end{bmatrix}.
\]

and computed the counting invariant for all of the orientable surface-links with \(ch\)-index up to 10 with the orientations as shown
Example 6. Biquasile counting invariants are sensitive to orientation-reversals, More precisely, reversing the orientation of the sphere component in the surface-link \( L = 9_{1,1} \) to obtain \( L' \) as depicted results in different numbers of \( X \)-colorings for all three of the biquasiles in example [5]

\[
\begin{array}{|c|ccc|ccc|ccc|ccc|}
\hline
L & 2_1 & 6_{1,1} & 8_{1,1} & 9_{1,1} & 10_1L & 10_2 & 10_3 & 10_{1,1} & 10_{2,1} & 10_{1,1} \\
\hline
\Phi X_1 & 9 & 27 & 9 & 27 & 9 & 9 & 9 & 27 & 3 & 9 & 27 \\
\Phi X_2 & 9 & 27 & 9 & 27 & 3 & 27 & 9 & 9 & 9 & 27 & 0 & 27 & 81 \\
\Phi X_3 & 9 & 27 & 9 & 27 & 9 & 9 & 9 & 27 & 0 & 27 & 0 & 27 & 81 \\
\hline
\end{array}
\]

4 Boltzmann Weight Enhancement

In [7], biquasile colorings of oriented link diagrams are enhanced with Boltzmann weights. In this section we extend this construction to the case of oriented surface-links.

Definition 6. Let \( X \) be a biquasile and \( A \) an abelian group. Then an \( A \)-linear map \( \phi : A[X^3] \to A \) from \( A \)-linear combinations of ordered triples of elements of \( X \) to \( A \) is a Boltzmann weight if for all \( x, y, a, b \in X \) we have

(i) \( \phi(x, a, a \backslash (x \setminus x)) = \phi(x, (x \setminus x)/b, b) = 0 \)

and

(ii) \[
\phi(x, a, b) + \phi(b, x \ast (a \cdot b), y) + \phi(x \ast (a \cdot b), a, b \ast ([x \ast (a \cdot b)] \cdot y)) \\
= \phi(b, x, y) + \phi(x, a, b \ast (x \cdot y)) + \phi(b \ast (x \cdot y), a \cdot [b \ast (x \cdot y) \cdot y]).
\]

The Boltzmann weight definition collects the conditions required to make the sum of \( \phi(x, a, b) \) values at all crossings in a biquasile-labeled oriented link diagram using the rule

unchanged by the oriented Reidemeister moves. Then for a classical oriented link \( L \) the multiset

\[
\Phi^{\phi,M}_X(L) = \left\{ \sum_{\text{crossing in } L_f} \pm \phi(x, a, b) \mid L_f X \text{ coloring of } L \right\}
\]
is an invariant of links whose cardinality is the biquasile counting invariant. The “$M$” here stands for “multiset” (as opposed to the polynomial version of the invariant also defined [1]) and $L_f$ is a biquasile coloring of the surface link diagram $L$ with coloring map $f$ assigning elements $f(r_j) \in X$ to each region $r_j$ in $L$. See [1] for more.

**Lemma 3.** Let $L$ be a marked graph diagram with a coloring by a biquasile $X$ and let $\phi : A[X^3] \to A$ be a Boltzmann weight for an abelian group $A$. Then the sum

$$\phi(L_f) = \sum_{\text{crossing in } L_f} \pm \phi(x, a, b)$$

of contributions at the crossings and marked vertices according to the rule

is unchanged by Yoshikawa moves.

**Proof.** The case of classical Reidemeister moves is covered in [1]. We observe that for each of the additional moves $\Gamma_4, \Gamma'_4, \Gamma_5, \Gamma_6, \Gamma'_6, \Gamma_7$ and $\Gamma_8$ the contributions on each side of each move are equal. \hfill \Box

**Corollary 4.** Let $L$ be a marked graph diagram, $X$ a biquasile, $A$ an abelian group and $\phi : A[X^3] \to A$ a Boltzmann weight. Then the multiset

$$\Phi_{\phi,M}^X(L) = \{ \phi(L_f) \mid L_f \text{ X-coloring of } L \}$$

where $\phi(L_f)$ is as defined in Lemma 3, and the polynomial

$$\Phi_X^\phi(L) = \sum_{L_f, \text{X-coloring of } L} u^{\phi(L_f)}$$

are invariants of surface-links known as biquasile Boltzmann enhancements of $\Phi_X^\phi$.

**Proposition 5.** If $L$ is a marked graph diagram representing a cobordism between oriented classical links $L_1$ and $L_2$ then we have inclusions

$$\Phi_{X}^{\phi,M}(L) \subset \Phi_{X}^{\phi,M}(L_1) \quad \text{and} \quad \Phi_{X}^{\phi,M}(L) \subset \Phi_{X}^{\phi,M}(L_2)$$

for every Boltzmann weight $\phi$.

**Proof.** Every $X$-coloring of a marked graph diagram extends to an $X$-coloring of both of its resolutions. Since the Boltzmann weight is determined by its values at classical crossings, the Boltzmann weight at the level $t = 0$ is the same as that for $t > 0$ and for $t < 0$. \hfill \Box

**Corollary 6.** If $L$ is an marked graph diagram representing a closed surface-link, i.e. a cobordism between unlinks, then $\Phi_{X}^{\phi}(L) = |\Phi_{X}^{\phi}(L)|$ for every Boltzmann weight $\phi$, i.e., the Boltzmann enhancement is trivial for closed surface-links.

We also have the following easy observation:
Proposition 7. The biquasile counting invariant and its Boltzmann enhancement do not detect the genus of the surface-link or cobordism.

Proof. We simply note that stabilization moves do not change either the number of colorings or the Boltzmann weight of a coloring:

\[ \begin{array}{c}
\text{a} & \text{b} \\
\end{array} \leftrightarrow \begin{array}{c}
\text{a} & \text{b} \\
\end{array} \]

Our final example shows that biquasile Boltzmann enhancements can detect knottedness of cobordisms. More precisely, a classical link diagram can be considered as a marked graph diagram without marked vertices, which corresponds to the trivial cobordism \( L \times [0,1] \) between two copies of \( L \).

Example 7. Let \( L2a1 \) be the trivial cobordism between two Hopf links and let \( L \) be the cobordisms between two Hopf links determined by the marked graph diagram below:

Let \( X \) be the biquasile with matrix

\[ \begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 1 & 1 & 2 \\
\end{bmatrix}, \]

let \( R = \mathbb{Z}_5 \) and let \( \phi = \chi(2,1,2) + \chi(2,2,1) \). Then we have

\[ \Phi_X^\phi(L) = 4u + 4 \neq 4 = \Phi_X^\phi(L2a1) \]

and the biquasile Boltzmann enhancement detects the difference between \( L \) and \( L2a1 \).

5 Questions

We end with some questions for future research.

A marked diagram which is not admissible, i.e., such that \( \mathcal{L}_+, \mathcal{L}_- \) or both are not unlinks, defines a cobordism between \( \mathcal{L}_+ \) and \( \mathcal{L}_- \). In particular, if a knot \( K \) is slice then a cobordism exists between \( K \) and an unknotted. Can biquasile invariants be used to detect when a knot is not slice?

Currently biquasile invariants have only been defined for classical knots and links of dimensions 1 and 2. What about biquasile invariants for virtual knots and links and virtual surface-links?
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