AN APPLICATION OF GUILLEMIN-ABREU THEORY
TO A NON-ABELIAN GROUP ACTION

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ABSTRACT. This note is a step towards demonstrating the benefits of a symplectic approach to studying equivariant Kähler geometry. We apply a local differential geometric framework from Kähler toric geometry due to Guillemin & Abreu to the case of the standard linear SU(n) action on \( \mathbb{C}^n \setminus \{0\} \). Using this framework we (re)construct a scalar-flat Kähler metric on the blow-up of \( \mathbb{C}^n \) at the origin from data on the moment polytope.

1. INTRODUCTION

A symplectic toric manifold \((M, \omega, \tau, \mu)\) is a compact symplectic manifold of dimension \(2n\) equipped with an effective hamiltonian action \(\tau\) of the real \(n\)-torus \(T^n = \mathbb{R}^n/2\pi \mathbb{Z}^n\) and moment map \(\mu : M \to (\mathbb{R}^n)^*\). By Delzant theory \((M, \omega, \tau, \mu)\) admits a canonical \(\omega\)-compatible \(T^n\)-invariant complex structure \(J\), [Del88]. Hence every symplectic toric manifold is canonically a Kähler toric manifold. Furthermore, symplectic toric manifolds are completely classified by Delzant polytopes \(\Delta \subset (\mathbb{R}^n)^*\) i.e. moment polytopes satisfying certain additional integrality conditions.

According to a differential geometric construction in toric geometry one can in fact encode all the \(T^n\)-invariant Kähler geometry of a symplectic toric manifold in terms of data on its Delzant moment polytope \(\Delta\). This construction relies on the interplay between complex and symplectic structures in Kähler geometry in the following sense. The data we refer to is a family of certain smooth functions on \(\Delta\) that determine every possible \(\omega\)-compatible \(T^n\)-invariant complex structure on \(M\). These functions are obtained via a Legendre coordinate transform from complex (holomorphic) coordinates on \(M\) to symplectic (action-angle) coordinates on \(\Delta\). This coordinate transform identifies Kähler potentials \(f\) over \(M\) (which determine \(T^n\)-invariant \(J\)-compatible symplectic structures on \(M\) within a fixed cohomology class) as Legendre duals to symplectic potentials \(g\) on \(\Delta\) (which determine \(T^n\)-invariant \(\omega\)-compatible complex structures on \(M\) within a fixed diffeomorphism class), [Gui94, Abr03]. We refer to this construction as Guillemin-Abreu theory.

In this paper we show how Guillemin-Abreu theory can be applied to the case of the standard linear SU(n)-action on \(\mathbb{C}^n \setminus \{0\}\). Using this construction we describe an interesting example. We (re)construct a \(U(n)\)-invariant, scalar-flat, Kähler metric on \(\mathbb{C}^n\), the blow-up of \(\mathbb{C}^n\) at the origin. This metric was originally identified by Simanca who generalized the well-known Burns metric on \(\mathbb{C}^2\), [Sim91]. The main purpose of this note is to illustrate that, in spirit of Guillemin & Abreu’s work, doing Kähler geometry in symplectic coordinates as opposed to the usual complex coordinates makes the formulae quite elegant and the calculations more manageable. Consequently the symplectic setting might be more appropriate for working with these metrics, as in [AP04] which uses the Burns-Simanca metric to construct constant scalar curvature Kähler metrics on blow-ups, for example.

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2. Guillemin-Abreu Theory

Let $(M, \omega, J, \tau, \mu)$ be a Kähler toric manifold and $\Delta = \mu(M)$ its Delzant polytope. We first describe the local structure of $M$. Consider the dense, open subset $M^o \subset M$ where the $T^a$-action is free. Let $T^a_n = C^o / 2\pi i \mathbb{Z}^n = \mathbb{R}^n \times iT^n = \{ w = a + ib : a \in \mathbb{R}^n, b \in T^n \}$. $(M^o, J) \cong T^a_n$ i.e. $(a, b)$ are complex (holomorphic) coordinates on $M^o$ (see Appendix A of [Abr03]). The $T^n$-action on $M^o$ is $(t, w) \mapsto w + it$. The Kähler form $\omega$ is given by $2i\partial\overline{\partial}f$ where $f \in C^{o}(M^o)$. Since $\omega$ is $T^n$-invariant $f = f(a) \in C^{o}(\mathbb{R}^n)$. Thus

$$\omega = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial a_i \partial a_k} da_i \wedge db_j.$$  

We now describe the interior $\Delta^o$ of $\Delta$. Suppose $\Delta$ consists of $d$ facets (codimension-1 faces). Then $\Delta = \{ x \in (\mathbb{R}^n)^* : (x, u_i) \geq \lambda_i, i = 1, \ldots, d \}$ where $u_i$ is the integral primitive inward pointing normal vector to the $i$th facet of $\Delta$ (see [Gui94]). Let $l_i$ be affine functions on $(\mathbb{R}^n)^*$ defined by $l_i : x \mapsto \langle x, u_i \rangle - \lambda_i$. Then $x \in \Delta^o$ if and only if $l_i(x) > 0$.

**Theorem 1 ([Gui94]).** $\mu$ factors into

$$\begin{array}{ccc}
\mathbb{R}^n \times iT^n & \xrightarrow{\mu} & (\mathbb{R}^n)^* \\
\downarrow p & & \downarrow \mu_f \\
\mathbb{R}^n & \xrightarrow{\mu_f} & (\mathbb{R}^n)^* \\
\end{array}$$

and this diagram commutes. Moreover, $\mu_f$ is the Legendre transform $a \mapsto df_a = x$ associated to $f$ and is a diffeomorphism onto $\Delta^o \subset (\mathbb{R}^n)^*$. Furthermore, there exists an inverse Legendre transform

$$\mu_f^{-1} : \Delta^o \to \mathbb{R}^n, \quad x \mapsto dg_x = a$$

where the function $g \in C^{o}(\Delta^o)$ is the Legendre dual to $f$ i.e.

$$f(a) + g(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial a_i} \frac{\partial g}{\partial x_i}.$$  

Guillemin’s set-up provides us with a means of encoding the Kähler data on $M$, originally given in terms of $f$ and the coordinates $(a, b)$, into symplectic (action-angle) coordinates $(x, y)$ on $\Delta^o \times T^n$ and $g$ through the map $(a, b) \mapsto (x, y)$ which is the Legendre transform $\mu_f$ on the first factor and the identity on the second factor. Guillemin introduces the function $g(x) = \frac{1}{2} \sum_{i=1}^{n} l_i(x) \log l_i(x)$. The particular form of this function guarantees that it is convex and smooth on $\Delta^o$ and has the appropriate singular behavior on the boundary $\partial \Delta$ of $\Delta$. It is the Legendre dual of the Kähler potential $f$ that defines the canonical $T^n$-invariant Kähler metric $\omega(\cdot, J)$ on $M$. Using Guillemin’s set-up one can construct any other $T^n$-invariant Kähler metric on $M$ in the class $[\omega]$ purely from the combinatorial data on $\Delta$ employing such functions $g \in C^{o}(\Delta^o)$. Those $g$ whose Legendre duals $f$ define $T^n$-invariant Kähler metrics can be regarded as ‘potentials’ for complex structures on $M$ in analogy to the Kähler potentials $f$ for symplectic structures on $M$.

Just as symplectic structures within a fixed cohomology class are parameterized in terms of Kähler potentials $f$ through the $\partial \bar{\partial}$-lemma, Abreu provides an analogous theorem for parameterizing complex structures within a fixed diffeomorphism class in terms of symplectic potentials $g$.

**Theorem 2 ([Abr03]).** Let $(M_\Delta, \omega_\Delta)$ be a symplectic toric 2n-manifold corresponding to the Delzant polytope $\Delta$ which has $d$ facets. Then a $T^n$-invariant, $\omega_\Delta$-compatible complex structure $J$ on $M_\Delta$ given at a point in the coordinates $(x, y)$ by

$$\begin{pmatrix}
0 & -G^{-1} \\
G & 0
\end{pmatrix}.$$
is determined by a smooth function

\[ g(x) = \frac{1}{2} \sum_{i=1}^{d} l_i(x) \log l_i(x) + h(x) \]

on \( \Delta^o \), where \( h(x) \in C^\infty(\Delta) \) (i.e. there is an open set \( U \subset (\mathbb{R}^n)^* \) containing \( \Delta \) and an \( \tilde{h} \in C^\infty(U) \) which restricts to \( h \) on \( \Delta \)), the hessian matrix \( G \) of (4) is positive definite on \( \Delta^o \) and

\[ \det G^{-1} = \delta(x) \prod_{i=1}^{d} l_i(x) \]

where \( \delta(x) \in C^\infty(\Delta) \) and is strictly positive on \( \Delta \). Conversely, every \( g \) of the form (4) determines a \( T^n \)-invariant, \( \omega_\Delta \)-compatible complex structure on \( (M_\Delta, \omega_\Delta) \) which in the \( (x,y) \) coordinates is of the form (3).

The Guillemin-Abreu paradigm is that one can recover \( f \) from \( g \) so it is enough to work with \( g \) and data on \( \Delta \). One final result to recall from the Guillemin-Abreu framework is that Abreu applies the above ideas to formulate an elegant expression for the scalar curvature of the Kähler metric defined by (1). Furthermore, Abreu also provides ‘symplectic’ extremal Kähler condition. These are summed up in the following

**Theorem 3** ([Abr98]). The scalar curvature of (1) is

\[ S(g) = -\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 G^{ij}}{\partial x_i \partial x_j} \]

where \( G^{ij} \) is the \( (i,j) \)th entry of \( G^{-1} \). Furthermore, this Kähler metric is extremal if and only if

\[ \frac{\partial S}{\partial x_i} = \text{constant} \]

for \( i = 1, \ldots, n \) i.e. \( S \) is an affine function of \( x \).

### 3. A NON-ABELIAN GROUP ACTION

Let \( (M, \omega, J) \) be a Kähler \( n \)-fold. The scalar curvature \( S_J \) of the Kähler metric \( \omega(J, \cdot) \) is given by

\[ S_J \omega^n = n! \Theta_J \wedge \omega^{n-1} \]

where \( \Theta_J \) is the Ricci form with respect to the complex structure \( J \). Consider \( \mathbb{C}^2 \setminus \{0\} \) with standard complex coordinates \( z = (z_1, z_2) \) equipped with the standard linear SU(2)-action. Kähler potentials of SU(2)-invariant Kähler metrics are smooth functions on \( \mathbb{C}^2 \setminus \{0\} \) of the form \( f = f(s) \) where \( s = |z_1|^2 + |z_2|^2 \) is the square of the radius of the SU(2)-orbits. We are interested in studying SU(2)-invariant Kähler metrics on \( \mathbb{C}^2 \setminus \{0\} \). The standard approach is to fix the standard \( J_0 \) on \( \mathbb{C}^2 \setminus \{0\} \) and vary the symplectic structure using the Kähler potentials \( f(s) \) (through the ‘\( J \)-lemma’). The scalar curvature of the Kähler metric determined by an \( f(s) \) i.e.

\[ h_J = \left[ \frac{\partial^2 f}{\partial z_i \partial \overline{z_j}} \right]_{i,j=1}^{2} = \begin{pmatrix} f' + z_1 \overline{z_1} f'' & z_2 \overline{z_1} f'' \\ \overline{z_1} z_2 f'' & f' + z_2 \overline{z_2} f'' \end{pmatrix} \]

is then deduced from (8). Not all \( f(s) \) determine SU(2)-invariant Kähler metrics on \( \mathbb{C}^2 \setminus \{0\} \). \( h_J \) is positive definite if and only if \( f''(s) > 0, f'''(s) > -s^{-1} f'(s) \). The scalar curvature of (9) computes to be a complicated expression in \( f \), see [Cal82, Sim91].
3.1. Twisted Guillemin-Abreu theory. The key observation is that the SU(2)-invariant Kähler metrics on $\mathbb{C}^2 \setminus \{0\}$ are also invariant under U(2) and in particular under $T^2 \cong U(1) \times U(1) \subset U(2)$. We conclude from this observation that it is viable to employ Guillemin-Abreu theory for this standard linear SU(2)-action. Set $w_j = \log z_j$ where $w_j = a_j + ib_j$, $j = 1, 2$. Let $s = |z_1|^2 + |z_2|^2 = e^{a_1}e^{\bar{a}_1} + e^{a_2}e^{\bar{a}_2} = e^{2a_1} + e^{2a_2}$ parameterizes the SU(2) orbits. Now we invoke the Guillemin-Abreu theory introduced in the previous section. Applying the Legendre transform associated to a choice of $f(s)$ gives

$$x_i = \frac{\partial f}{\partial a_i} = f'(s) \frac{\partial s}{\partial a_i} = 2e^{2a_i}f'(s),$$

$i = 1, 2$ and $x_1 + x_2 = 2(e^{2a_1} + e^{2a_2})f' = 2xf' = \gamma(s)$. Let $h$ be the inverse function to $\gamma$, then $s = h(x_1 + x_2)$. Substituting $e^{2a_1} = e^{2a_2}x_1x_2^{-1}$ into $s = e^{2a_1} + e^{2a_2}$ gives $s = e^{2a_1} + e^{2a_2} = e^{2a_2}(x_1 + x_2)x_1^{-1}$, i.e. $e^{2a_1} = sx_1(x_1 + x_2)^{-1}$. Similarly $e^{2a_2} = sx_2(x_1 + x_2)^{-1}$. The Legendre dual of $f$ is given by (2) in Theorem 1. We have established above that $s = h(x_1 + x_2)$ hence $f = f(s) = f(h(x_1 + x_2)) = f(x_1 + x_2)$. A brief calculation shows that

$$g(x) = \frac{1}{2}(x_1 \log x_1 + x_2 \log x_2 + F(x_1 + x_2))$$

with

$$F(x_1 + x_2) = F(t) = t \log(h(t)t^{-1}) - 2f(s(t))$$

where we have set $t = x_1 + x_2$. The Guillemin-Abreu approach is to fix the standard $\omega_0$ on $\mathbb{C}^2 \setminus \{0\}$ and vary the complex structure using the symplectic potentials $g(x)$ (through Theorem 2). The standard $\omega_0$ is given by $f(s) = s/2$. Hence the moment polytope $\Delta_{\mathbb{C}^2}$ for the standard $T^2$-action is the positive orthant $\mathbb{R}^2_{\geq 0} \subset \mathbb{R}^2$ with standard symplectic (action) coordinates $(x_1, x_2) = (|z_1|^2, |z_2|^2)$. $g(x)$ is a smooth function on $\Delta_{\mathbb{C}^2}$ while $F(t)$ is a smooth function on $(0, \infty)$. We call $F(t)$ the t-potential of $g$ since it is the $t$-part of the symplectic potential $g$. The hessian matrix of $g$ is

$$G = (G_{ij}) = \frac{1}{2} \begin{pmatrix} 1 + F''(t) & F''(t) \\ F''(t) & 1 + F''(t) \end{pmatrix}$$

and $\det G = 4^{-1}(x_1x_2)^{-1} + t(x_1x_2)^{-1}F''$. It follows from the discussion in the previous section that $G$ must be positive definite. This in turn implies that $F''(t) > -t^{-1}$ and, furthermore, this condition is also sufficient i.e. only functions of the form $F(t)$ that satisfy this property determine SU(2)-invariant Kähler metrics on $\mathbb{C}^2 \setminus \{0\}$. To summarize

**Proposition 4.** The Kähler metric (9) has a symplectic potential given by (10). Conversely, any such smooth function on $\mathbb{R}^2_{\geq 0}$ with $F$ satisfying $F''(t) > -t^{-1}$ determines such a Kähler metric.

The inverted hessian of $g$ is

$$G^{-1} = (G^{'ij}) = \frac{2x_1x_2}{1 + tF''} \begin{pmatrix} 1 + F''(t) & -F''(t) \\ -F''(t) & 1 + F''(t) \end{pmatrix}.$$  

Applying Abreu’s scalar curvature formula (6) to (12) shows (see [Raz04]) that

**Proposition 5.** The scalar curvature of the Kähler metric (9) is given by

$$S(g) = r^{-1}(r^3 F''(1 + tF'')^{-1}).$$

Propositions 4 and 5 hold true for arbitrary dimension. Let $z = (z_1, \ldots, z_n)$ be standard complex coordinates on $\mathbb{C}^n \setminus \{0\}$. Let $s = \sum_{i=1}^n |z_i|^2$. An SU($n$)-invariant Kähler metric on $\mathbb{C}^n \setminus \{0\}$ is given by

$$h^n = \left[ \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right]^{n}_{i,j=1} = \left[ f' \delta_{ij} + z_i \bar{z}_j f'' \right]^{n}_{i,j=1}.$$
Repeating the construction we carried out for the case \( n = 2 \) we obtain symplectic coordinates \( x = (x_1, \ldots, x_n) \) on the positive orthant \( \mathbb{R}_{>0}^n = \Delta_{\mathbb{C}^n} \) i.e. the moment polytope for the standard \( T^n \)-action on \( \mathbb{C}^n \) (with its standard symplectic structure). One simply has to work through the algebra and extend the identities already derived earlier to the \( n \) case. Thus

**Theorem 6** ([Raz04]). The Kähler metric (13) has a symplectic potential given by

\[
g(x) = \frac{1}{2} \left[ \sum_{i=1}^n x_i \log x_i + F(t) \right]
\]

with

\[
F(t) = t \log (s(t)t^{-1}) - 2f(s(t))
\]

where \( t = \sum_{i=1}^n x_i = 2sf'(s) \) and its scalar curvature is given by

\[
S(g) = t^{1-n} \left( t^{n+1}F''(1+tF''^{-1}) \right)''.
\]

Conversely, any function of the form (14) on \( \Delta_{\mathbb{C}^n} \) with (15) satisfying \( F''(t) > -t^{-1} \) determines such a Kähler metric.

4. Applications of Theorem 6

As a first application of Theorem 6 we verify the well-known result that the scalar curvature of the Fubini-Study metric on \( \mathbb{CP}^n \) is constant. The \( t \)-potential of the Fubini-Study metric on \( \mathbb{CP}^n \) is \( F_{\mathbb{CP}^n}(t) = (1-t) \log(1-t) \). Substituting \( F_{\mathbb{CP}^n}(t) \) into (16) shows that

**Corollary 7.** The Fubini-Study metric on \( \mathbb{CP}^n \) has scalar curvature \( n(n+1) \).

Let \( \mathbb{C}^n \) denote the blow-up of \( \mathbb{C}^n \) at the origin. Recall that the Burns metric on \( \mathbb{C}^2 \) is the restriction of the standard product metric on the ambient space \( \mathbb{C}^2 \times \mathbb{CP}^1 \) when \( \mathbb{C}^2 \) is considered as a hypersurface in \( \mathbb{C}^2 \times \mathbb{CP}^1 \) [Sim91]. We refer to the restriction of the standard product metric on \( \mathbb{C}^n \times \mathbb{CP}^{n-1} \) to \( \mathbb{C}^n \) as the generalized Burns metric. In light of this we are led to ask whether the generalized Burns metric on \( \mathbb{C}^n \) is also scalar-flat. Unsurprisingly this is not the case. The \( t \)-potential of the generalized Burns metric is \( F_{\mathbb{C}^n}(t) = (t-1) \log(t-1) - t \log t - t + 1 \). Substituting this into (16) shows that the scalar curvature of the generalized Burns metric on \( \mathbb{C}^n \) is \( S(g_{\mathbb{C}^n}) = (n^2 - 3n + 2)t^{-2} \). So when do \( t \)-potentials of this form give rise to \( SU(n) \)-invariant scalar-flat Kähler metrics of this form? Setting \( S(g_{\mathbb{C}^n}) \) to zero gives the quadratic \( n^2 - 3n + 2 = 0 \) whose solutions are \( n = 1 \) and \( n = 2 \). Thus

**Corollary 8.** The generalized Burns metric on \( \mathbb{C}^n \) is scalar-flat in and only in dimension 1 and 2 i.e. regarding \( \mathbb{C}^n \) as a hypersurface in \( \mathbb{C}^n \times \mathbb{CP}^{n-1} \), the restriction of the standard ambient product metric on this space to \( \mathbb{C}^n \) is scalar-flat in and only in dimension 1 and 2.

4.1. A scalar-flat Kähler metric on \( \mathbb{C}^n \). Consider now \( U(n) \)-invariant Kähler metrics of zero scalar curvature on \( \mathbb{C}^n \). Then Theorem 6 leads to the ODE (16) = 0 which solves (for \( F'' \)) to give

\[
F''(t) = \frac{At+B}{t(t^n-(At+B))}
\]

where \( A \) and \( B \) are constants. Simanca proved the existence of a \( U(n) \)-invariant scalar-flat complete Kähler metric on the total space of the bundle \( \mathcal{O}(-1) \to \mathbb{CP}^{n-1} \) i.e. the blow-up \( \mathbb{C}^n \) of \( \mathbb{C}^n \) at the origin, [Sim91]. We shall now describe this metric in the symplectic coordinate setting. Since the Kähler metric we seek is also \( T^n \)-invariant we employ Guillemin-Abreu theory, in particular Theorem 2, to find our boundary condition. The Delzant moment polytope \( \Delta_{\mathbb{C}^n} \) corresponding to \( \mathbb{C}^n \) is the positive orthant \( \mathbb{R}_{>0}^n \) with the
vertex \( p = (0, \ldots, 0) \) replaced by the \( n \) vertices \( p + x_i, i = 1, \ldots, n \). We refer to \( \Delta_{\mathbb{C}^n} \) as the moment polytope of the 1-symplectic blow-up of \( \mathbb{C}^n \) at the origin (see [Raz04]). As a result \( \Delta_{\mathbb{C}^n} \) has \( (n + 1) \) facets. Let \( l_1(x), l_2(x), \ldots, l_{n+1}(x) \) be the affine functions corresponding to these facets i.e. \( l_i(x) = x_i \) and \( l_{n+1}(x) = \sum_{i=1}^{n} x_i - 1 = t - 1 \). That is, each affine function determines a hyperplane in \( \mathbb{R}^n \) and these hyperplanes together trace out the boundary of \( \Delta_{\mathbb{C}^n} \). The interior of \( \Delta_{\mathbb{C}^n} \) is \( \Delta_{\mathbb{C}^n}^0 = \{ x \in \mathbb{R}^n : l_i(x) > 0, i = 1, \ldots, n + 1 \} \). Let \( g_{\mathbb{C}^n}(x) \) be the symplectic potential of the metric we seek. By Theorem 6

\[
(18) \quad g_{\mathbb{C}^n}(x) = \frac{1}{2} \left[ \sum_{i=1}^{n} x_i \log x_i + F_{\mathbb{C}^n}(t) \right]
\]

where \( F_{\mathbb{C}^n}(t) \) is the \( t \)-potential of \( g_{\mathbb{C}^n}(x) \). Furthermore, for \( g_{\mathbb{C}^n}(x) \) to determine a scalar-flat Kähler metric \( F_{\mathbb{C}^n}(t) \) must be of the form (17). The determinant of the hessian \( G_{\mathbb{C}^n} \) of \( g_{\mathbb{C}^n}(x) \) is \( \det G_{\mathbb{C}^n} = 2^{-n} \left( 1 + tF_{\mathbb{C}^n}'(t) \right) \prod_{i=1}^{n} x_i^{-1} \) and so

\[
(19) \quad \det G_{\mathbb{C}^n}^{-1} = \prod_{i=1}^{n} x_i \left( 1 + tF_{\mathbb{C}^n}'(t) \right)^{-1} = 2^n \prod_{i=1}^{n} x_i t^n - (At + B) t^{-n}.
\]

By Theorem 2 \( \det G_{\mathbb{C}^n}^{-1} \) should be of the form (5) with \( l_i \) as given above and \( \delta(x) \in C^\infty(\Delta_{\mathbb{C}^n}) \) (in the sense described in Theorem 2) and positive. Writing (19) as \( \det G_{\mathbb{C}^n}^{-1} = \delta_{\mathbb{C}^n}(x) \prod_{i=1}^{n+1} l_i(x) \) such that

\[
(20) \quad \delta_{\mathbb{C}^n}(x) = \frac{2^n (t^n - (At + B))}{t^n (t - 1)}
\]

gives us the appropriate form of \( \det G_{\mathbb{C}^n}^{-1} \). We now have to find the correct \( A, B \) in (20) so that it satisfies the condition of Theorem 2. In its current form \( \delta_{\mathbb{C}^n}(x) \) becomes singular at the boundary i.e. when \( t \to 1 \). Also \( 2^n t^{-n} > 0 \) and smooth. Therefore \( A, B \) must be such that \( (t^n - At - B) \approx (t - 1) \) for \( t = 1 + \epsilon \) for small \( \epsilon > 0 \). Then \( (1 + \epsilon)^n - A(1 + \epsilon) - B = 1 + n\epsilon + O(\epsilon^2) - A(1 + \epsilon) - 1 = (1 + \epsilon) - 1 \) which gives \((1 - (A + B)) + (n - A) \epsilon = \epsilon \). Hence \( n - A = 1 \) and \( 1 - (A + B) = 0 \) and we get \( A = n - 1, B = 2 - n \). It follows that \( (t^n - (n - 1) t - (n - 2))^2 = \sum_{i=1}^{n} t^i - (n - 2) \) which is clearly smooth and positive on the whole of \( \Delta_{\mathbb{C}^n} \). Hence \( \delta_{\mathbb{C}^n}(t) = 2^n t^{-n} (\sum_{i=1}^{n} t^i - (n - 2)) \). As a result

\[
(21) \quad F_{\mathbb{C}^n}'(t) = \frac{(n - 1) t + 2 - n}{t^n - (n - 1) t - 2 + n}
\]

such that \( n > 1 \). We need to make sure that \( F_{\mathbb{C}^n}'(t) \) is non-singular for \( t > 1 \) i.e. \( t^n - (n - 1) t - 2 + n \neq 0 \) if \( t > 1 \). We have \( t^n - (n - 1) t - (2 - n) = (t - 1) (\sum_{i=1}^{n-1} t^i - (n - 2)) \) and this is zero if \( (t - 1) = 0 \) or if \( \sum_{i=1}^{n-1} t^i - (n - 2) = 0 \). It is clear that for \( t \geq 1 \) and \( n > 1, \sum_{i=1}^{n-1} t^i > n - 2 \). It follows that (21) is non-singular for all \( t > 1 \) and hence we have

**Corollary 9.** There exists a U(n)-invariant, scalar-flat, Kähler metric on \( \mathbb{C}^n \) determined by the symplectic potential (18) on \( \Delta_{\mathbb{C}^n} \) where \( F_{\mathbb{C}^n}(t) - (t - 1) \log(t - 1) \) is a smooth function on \([1, \infty)\) such that \( F_{\mathbb{C}^n}(t) \) satisfies (21).

We refer to the Kähler metric determined by the symplectic potential (18) as the Burns-Simanca metric. The case \( n = 2 \) is special since in that case the metric on \( \mathbb{C}^2 \) determined by (18) is just the restriction of the product metric on \( \mathbb{C}^2 \times \mathbb{C}P^1 \). The crucial point is that for \( n > 2 \) the Burns-Simanca metric on \( \mathbb{C}^n \) is not the restriction of the standard product metric on the ambient space \( \mathbb{C}^n \times \mathbb{C}P^{n-1} \).
4.1.1. **Behaviour of the Burns-Simanca metric away from the blow-up.** Let \( d = (\delta_{ij}) \) be the standard flat euclidean metric on \( \mathbb{C}^n \). Then a Kähler metric \( h \) on a non-compact Kähler \( n \)-fold \( M \) is called asymptotically euclidean (AE) with rate of decay \( r^{-n} \), where \( r \) is the radius function on \( \mathbb{C}^n \), if \( h \) approximates \( d \) under an appropriate biholomorphism between \( M \) (minus a compact subset) and \( \mathbb{C}^n \).

**Proposition 10.** The Burns-Simanca metric on \( \hat{\mathbb{C}}^n \) is AE with rate of decay \( r^{2-2n} \).

*Proof.* Let \( (y_1, \ldots, y_n) \) be the toric (angle) coordinates. The flat metric on \( \mathbb{C}^n \setminus \{0\} \) is given in the symplectic coordinates \((x,y)\) by

\[
d_{(x,y)} = \sum_{i=1}^{n} \frac{1}{2x_i} dx_i \otimes dx_i + 2x_i dy_i \otimes dy_i.
\]

The \((i,j)\)th entries of the hessian matrix of \((18)\) and its inverse are

\[
(G_{\hat{\mathbb{C}}^n})_{ij} = \frac{1}{2} \left\{ \begin{array}{ll}
    x_i^{-1} + F^n_{\hat{\mathbb{C}}^n}, & i = j \\
    F''_{\hat{\mathbb{C}}^n}, & i \neq j
  \end{array} \right.
\]

and

\[
(G_{\hat{\mathbb{C}}^n})^{ij} = \frac{2}{1 + tF''_{\hat{\mathbb{C}}^n}} \left\{ \begin{array}{ll}
    x_i (1 + (t-x_i)tF''_{\hat{\mathbb{C}}^n}(t))x_i, & i = j \\
    -F''_{\hat{\mathbb{C}}^n} x_i x_j, & i \neq j
  \end{array} \right.
\]

respectively. See [Raz04]. By the Guillemin-Abreu construction the Burns-Simanca metric is given by the \( 2n \times 2n \) matrix \( h_{BS} = \text{diag}(G_{\hat{\mathbb{C}}^n}, G_{\hat{\mathbb{C}}^n}^{-1}) \). (23) can be written as the sum

\[
G_{\hat{\mathbb{C}}^n} = A + B \quad \text{where} \quad A = \text{diag}((2x_1)^{-1}, \ldots, (2x_n)^{-1}) \quad \text{and} \quad B = ((n-1)t^2 + 2n - (n-1)t + 2n)P \quad \text{where} \quad P \text{ is an } n \times n \text{ matrix of } 1's.
\]

This gives us the upper left block of \( h_{BS} \). For \((24)\) the coefficient term is \( 2r^{-n}(t^n - (n-1)t + n - 2) \) hence the off-diagonal terms of \( G_{\hat{\mathbb{C}}^n}^{-1} \) are \((G_{\hat{\mathbb{C}}^n}^{-1})^{ij} = -2(n-1)t + 2n \). Note that in \((24)\) the term \( (1 + F''_{\hat{\mathbb{C}}^n}(t)\sum_{k=1,k \neq i}^{n} k) x_i = (1 + (t-x_i)tF''_{\hat{\mathbb{C}}^n}(t))x_i \). Hence the diagonal terms in \( G_{\hat{\mathbb{C}}^n}^{-1} \) are \((G_{\hat{\mathbb{C}}^n}^{-1})^{ii} = (2(1 + x_i)(1 + tF''_{\hat{\mathbb{C}}^n}(t))(1 + tF''_{\hat{\mathbb{C}}^n}(t))^{-1} \) which using the coefficient term gives \((G_{\hat{\mathbb{C}}^n}^{-1})^{ii} = 2x_i (t^n - (n-1)t + n - 2) \). Thus we have that the inverted hessian matrix of \((18)\) splits into \( G_{\hat{\mathbb{C}}^n}^{-1} = C + D \) where \( C = t^{-n}(t^n - (n-1)t + n - 2) \), \( D = 2r^{-n}(t^n - (n-1)t + n - 2) \), and \( Q = \text{diag}(tx_1, \ldots, tx_n) - [x_i x_j]_{i<j} \). This gives us the lower right block of \( h_{BS} \). Therefore we have \( h_{BS} = \text{diag}(A,C) + \text{diag}(B,D) = E + F \) so that \( E = \sum_{i=1}^{n} \frac{1}{2x_i} dx_i \otimes dx_i + 2x_i t^{-n} dt \). Now we use the flat metric coordinates \((\lambda, \mu)\). Set

\[
\lambda_i = \sqrt{2x_i} \cos y_i, \quad \mu_i = \sqrt{2x_i} \sin y_i,
\]

\( i = 1, \ldots, n \). Then the flat metric \((22)\) becomes of the standard form \( d(\lambda, \mu) = \sum_{i=1}^{n} d\lambda_i \otimes d\lambda_i + d\mu_i \otimes d\mu_i \). The component \( E \) of \( h_{BS} \) in these \((\lambda, \mu)\) coordinates is given by \( E = \sum_{i=1}^{n} \lambda_i^2 + \mu_i^2 + O(u^{-n+1}) \) where \( u = 2^{-1} \). Then the variable \( r \) in the \((\lambda, \mu)\) coordinates. For the component \( F \) of \( h_{BS} \) its clear that for \( u \to \infty \) the coefficient terms are \( O(u^{-n}) \). It follows that as \( u \to \infty \), \( h_{BS} = d(\lambda, \mu) + O(u^{-n+1}) \). Since by construction the coordinate \( u \) represents the square of the radius function \( r \) on \( \mathbb{C}^n \) we deduce that \( u^{-n} \equiv r^{2(1-n)} \).

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