Exactly solvable models for 2+1D topological phases derived from crossed modules of semisimple Hopf algebras

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April 8, 2021

We define an exactly solvable model for 2+1D topological phases of matter on a triangulated surface derived from a crossed module of semisimple finite-dimensional Hopf algebras, the Hopf-algebraic higher Kitaev model. This model generalizes both the Kitaev quantum double model for a semisimple Hopf algebra and the full higher Kitaev model derived from a 2-group, and can hence be interpreted as a Hopf-algebraic discrete higher gauge theory.

We construct a family of crossed modules of semisimple Hopf algebras $\mathcal{F}_C(X) \otimes \mathbb{C}E \xrightarrow{\partial} \mathcal{F}_C(Y) \rtimes \mathbb{C}G \ltimes \mathcal{G}$ that depends on four finite groups $E, G, X$ and $Y$. We calculate the ground-state spaces of the resulting model when $G = E = \{1\}$ and when $Y = \{1\}$, both properly Hopf-algebraic constructions; prove that they are canonically independent of the triangulations; and find a 2+1D TQFT whose state spaces on surfaces give the ground-state spaces. These TQFTs are particular cases of Quinn’s finite total homotopy TQFT and hence the state spaces assigned to surfaces are free vector spaces on sets of homotopy classes of maps from a surface to homotopy finite spaces, in this case obtained as classifying spaces of finite groupoids and finite crossed modules of groupoids.

We leave it as an open problem whether the ground-state space of the Hopf-algebraic higher Kitaev model on a triangulated surface is independent of the triangulation for general crossed modules of semisimple Hopf algebras, whether a TQFT always exists whose state space on a surface gives the ground-state space of the model, and whether the ground-state space of the model obtained from $E, G, X, Y$ can always be given a homotopical explanation.

Acknowledgements: This paper was financed by the Leverhulme trust research project grant “RPG-2018-029: Emergent Physics From Lattice Models of Higher Gauge Theory”. JFM would like to express his gratitude to Tim Porter for discussions on Quinn-like TQFTs for the case of classifying spaces of groupoids and crossed modules / complexes of groupoids. VK would like to thank Ehud Meir for countless invaluable explanations about Hopf algebras, Christoph Schweigert for introducing him to the Kitaev model, and Catherine Meusburger and Thomas Voß for further helpful discussions.

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1 Introduction

In his seminal paper Kitaev [1] defined a lattice model, henceforth called the “Kitaev quantum double model”, for (2+1)-dimensional, (2+1D), topological phases of matter on an oriented surface $\Sigma$, with a triangulation $L$. Initially proposed in the context of quantum computing as a model for an error-correcting quantum code, the so-called toric code, it allows for fault-tolerant quantum gates by braiding (non-abelian) anyons [1, 2]. Due to its relation [3] to the Turaev-Viro / Barrett-Westbury constructions [4, 5] of quantum invariants of 3-manifolds from spherical fusion categories, the Kitaev model also provides a link between low-dimensional topology, Hopf algebras and tensor categories. In this paper we extend the Kitaev model to handle crossed modules of Hopf algebras more generally, hence extending it to Hopf-algebraic higher gauge theory, as we now elaborate.

The Kitaev model on $L$ has as input a finite group $G$, and can be seen as a lattice gauge theory model [6]. The total Hilbert space of the Kitaev model on $L$ is $\mathcal{H}_L^{\text{Kit}} := \mathbb{C}G^{L^1}$, where $L^1$ is the set of edges of $L$. This vector space is the free vector space on the set of discretised $G$-connections over $(\Sigma, L)$, called in [7, §2.1] gauge $G$-configurations over $(\Sigma, L)$. One can then define vertex operators $V_g : \mathcal{H}_L^{\text{Kit}} \rightarrow \mathcal{H}_L^{\text{Kit}}$ and plaquette operators $F_{P, a} : \mathcal{H}_L^{\text{Kit}} \rightarrow \mathcal{H}_L^{\text{Kit}}$, where $g, a \in G$, $v \in L^0$ is a vertex of $L$, and $P \in L^2$ is a plaquette of $L$, with a vertex $w \in \partial P$. The operator $V_g : \mathcal{H}_L^{\text{Kit}} \rightarrow \mathcal{H}_L^{\text{Kit}}$ performs a discrete gauge transformation supported on $v$, whereas the operator $F_{P, a} : \mathcal{H}_L^{\text{Kit}} \rightarrow \mathcal{H}_L^{\text{Kit}}$ ‘chooses’ the discrete gauge configurations whose holonomy around the plaquette $P$, with initial point $w$, is $a \in G$. These operators extend respectively
to representations of the Hopf algebras $\mathbb{C}G$ and $\mathcal{F}_C(G) = \text{span}_\mathbb{C}\{\delta_a \mid a \in G\}$, the algebra of functions on $G$, (and if $V^g_v$ and $F^g_{P,v}$, for the same $v$, are put together, of the quantum double $D(G) = \mathcal{F}_C(G) \rtimes \mathbb{C}G$). The local operator algebra of the Kitaev quantum double model is the algebra of operators $\mathcal{H}_L^{Kit} \rightarrow \mathcal{H}_L^{Kit}$ generated by all vertex and plaquette operators. The topological excitations of the Kitaev model are naturally those that are given by representations of the local operator algebra as these are the excitations that cannot be destroyed by local operators $[1, 5]$. The Hopf algebras $\mathbb{C}G$ and $\mathcal{F}_C(G)$ are both semisimple (and so is the quantum double $D(G) = \mathcal{F}_C(G) \rtimes \mathbb{C}G$). In particular $[9]$ they have unique Haar integrals, which take the form $\ell = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}G$ and $\lambda = \delta_1 \in \mathcal{F}_C(G)$. Define, given a vertex $v$, a projector $V_v := V^1_v = \frac{1}{|G|} \sum_{g \in G} V^g_v$, called a vertex projector, and given $P \in L^2$, the projector $F_P := F^\lambda_{P,v} = F^1_{P,v}$, called a plaquette projector (this is independent of the vertex $w$ in the boundary of $P$). The Hamiltonian of the Kitaev model is $H_0 : \mathcal{H}_L^{Kit} \rightarrow \mathcal{H}_L^{Kit}$ $[1, 5]$, where

$$H_0 = \sum_{v \in L^0} (1 - V_v) + \sum_{P \in L^2} (1 - F_P).$$

(Here $L^0$ and $L^2$ are the sets of vertices and plaquettes of $L$.) All projectors appearing in $H_0 : \mathcal{H}_L^{Kit} \rightarrow \mathcal{H}_L^{Kit}$ are mutually commuting. It is for this reason that the Kitaev model is called an exactly solvable model $[10]$. Given that the Hamiltonian is the sum of commuting projectors, it is diagonalisable, and it has a lowest-energy eigenspace, the ground-state space.

Even though the Kitaev model defined on $(\Sigma, L)$ explicitly depends on the triangulation (or cell decomposition) $L$ of $\Sigma$, its ground-state space is naturally a topological invariant, canonically given by the free vector space on the set of homotopy classes of maps $\Sigma \rightarrow B_G$, where $B_G$ is the classifying space of $G$. A 2+1D topological quantum field theory (TQFT) exists, sending a surface $\Sigma$ to the ground-state space of the Kitaev quantum double model on $\Sigma$ derived from $G$. Namely consider the Dijkgraaf-Witten TQFT $[11]$ with group $G$ and trivial cocycle. The latter TQFT is a special case of the Quinn’s finite total homotopy TQFT $\mathcal{Q}_B [12]$, which depends on the choice of a homotopy finite space $B$, and sends a surface $\Sigma$ to the free vector space on the set of homotopy classes of maps $\Sigma \rightarrow B$. (Dijkgraaf-Witten TQFT with group $G$ and trivial cocycle $\omega \in H^3(G, U(1))$ is just $\mathcal{Q}_{B_G}$.)

The Kitaev quantum double model has been extended in a number of ways. In $[13]$, see also $[3]$, a generalised 2+1D model, henceforth called the “Hopf-algebraic Kitaev model” was defined. The latter takes as input, instead of a finite group $G$, a finite-dimensional semisimple complex Hopf algebra $H$, with Haar integral $\ell \in H$, and can be formulated in the context of Hopf algebra gauge theory on a lattice $[3]$. The Hopf-algebraic Kitaev model is an exactly solvable model defined on the vector space $H_L^{Kit} := H \otimes L^1$. This is a Hilbert space, and the vertex and plaquette projectors are Hermitian, if one takes as input a finite-dimensional Hopf $C^*$-algebra $H$ $[13]$.

The local operator algebra of the Hopf-algebraic Kitaev model likewise contains vertex operators $V^h_{v,P} : H_L^{Kit} \rightarrow H_L^{Kit}$, where $v \in L^0$, $h \in H$, and $P \in L^2$ is an adjacent plaquette to $v$, which is used $[3$, Definition 2.2$]$ to define a total order on the set of edges adjacent to $v$ (a cyclic order is given by the orientation, and $P$ is used to specify an initial edge adjacent to $v$). This additional data to define vertex operators is essential when $H$ is non-cocommutative. We also have plaquette operators $F^w_{P,v} : H_L^{Kit} \rightarrow H_L^{Kit}$, where $v \in H^*$, $P \in L^2$ and $w$ a vertex in the boundary of $P$. Note that the dual vector space $H^*$ is also a finite-dimensional semisimple Hopf algebra $[9]$ and hence it has a unique Haar integral $\Lambda \in H^*$. As in the Kitaev model, we can define mutually commuting vertex projectors $V_v := V^\ell_{v,P} : H_L^{Kit} \rightarrow H_L^{Kit}$ (this is independent of the adjacent plaquette $[3$, §2.4$]$, because the Haar integral $\ell$ is cocommutative $[9]$ and hence any coproduct of it cyclically invariant) and plaquette projectors $F_P := F^\Lambda_{P,v} : H_L^{Kit} \rightarrow H_L^{Kit}$. (Likewise $F_{P,w}$ is independent of the vertex in the boundary of $P$.) The Hamiltonian of the
Hopf-algebraic Kitaev model is also given by \( (1) \), and hence reduces to the Kitaev quantum double model if \( H = \mathbb{C}G \).

The ground-state space of the Hopf-algebraic Kitaev model defined on a triangulated surface \((\Sigma, L)\) is, as for the Kitaev model, canonically independent of the triangulation \(L\) of \(\Sigma\). This follows from [3, Theorem 4.1] since the Turaev-Viro TQFT for the spherical fusion category \(H\)–mod of modules of a semisimple Hopf algebra \(H\) is such that the state space assigned to a surface \(\Sigma\) is canonically isomorphic to the ground-state space of the Hopf-algebraic Kitaev model for the Hopf algebra \(H\), on any triangulation of \(\Sigma\), see also [13]. However, the Kitaev model does not describe the state spaces of every Turaev-Viro TQFT; only those coming from spherical fusion categories that arise as representation categories of finite dimensional semisimple Hopf algebras. More generally, for any spherical fusion category the Levin-Wen string-net construction [15, 16] provides a commuting-projector Hamiltonian model whose ground-state spaces recover the state spaces assigned to surfaces by the corresponding Turaev-Viro TQFT. Those Hilbert spaces are smaller than the ones derived from the Hopf-algebraic Kitaev model in the cases when both models are defined.

The model constructed in the present paper generalizes the Kitaev model in a different direction, which we describe in the following.

A higher gauge theory ‘lifting’ of Kitaev’s quantum double model was constructed in [7, 17, 18]. This “higher Kitaev model” can be defined on manifolds of arbitrary dimension. In loc cit, instead of generalising to Hopf algebras, the Kitaev model was extended to 2-groups [19], from the point of view of discrete higher gauge theory [7, 20]. Let us give some details. Higher gauge theory is a categorified version of gauge theory over a manifold \(M\) where one has not only path-holonomy but also 2D holonomy along surfaces [21, 22] any time a 2-connection is defined on \(G\) is faithfully represented by a crossed module \(G\times L\times 1\) [23, §2.1 & 2.7]. Here \(G\) and \(E\) are groups, the boundary map \(\partial: E \rightarrow G\) is a homomorphism and \(\triangleright\) is a left action of \(G\) on \(E\) by automorphisms, satisfying appropriate compatibility relations (the Peiffer relations). Roughly speaking, path-holonomies take values in \(G\) and surface holonomies in \(E\), with a compatibility relation stating that the boundary of the 2D holonomy along a surface coincides with the path holonomy around its boundary. This compatibility relation was called fake-flatness in [7]. (This term originates from the closely related notion of fake-curvature of a 2-connection [24], where its vanishing is essential for surface holonomy to be defined, and implies the above compatibility relation between path and surface holonomy [21, 22].)

As for discrete connections [25], discrete gauge 2-group 2-connections on a manifold \(M\) can be discretised given a triangulation, or 2-lattice decomposition, \(L\) of \(M\) [7, 20] by specifying their local holonomies along edges (with a fixed orientation) and plaquettes \(P\), provided with a choice of (fixed) vertex \(v_P\) in the boundary of \(P\), its base-point. The underlying set of 2-gauge configurations in \((M, L)\) is hence \(G^{L_1} \times E^{L_2}\) [7, §3.2, 17]. Inside the set of 2-gauge configurations there is a subset of fake-flat configurations which satisfy the fake-flatness relations for all plaquettes [7]. These are the configurations that have well-defined 2-dimensional holonomy operators. The higher Kitaev model defined in [7, 17] is a model defined on the free vector space \(H^M\) on the set of all fake-flat 2-gauge configurations, and features vertex operators \(V^g_v\), where \(g \in G, v \in L^0\): they implement gauge transformations supported on a vertex \(v\); edge operators \(E^t_e\), where \(t \in L^1\) and \(e \in E\), which implement gauge transformations supported on an edge \(t\), and also blob operators \(B^b_a\), where \(a \in \ker(\partial)\) and \(b \in L\), which choose those fake-flat configurations that have 2D holonomy equal to \(a\) around a 3-cell (a blob) \(b \in L^3\). In order for the latter blob operators to be well-defined (and commute with all edge operators) it is essential to restrict to fake-flat gauge configurations [17]. That restriction is not necessary in the 2+1D case as there are no blob operators since a 2-lattice decomposition of a surface has
no 3-cells.

On that token, our starting point for the model constructed in this paper is a generalisation of the 2+1D version of the higher Kitaev model treated in [17, page 8], here called the “full higher Kitaev model”. This model and the 2+1D case of the higher Kitaev model have the same ground-state space, however the full higher Kitaev model has a larger Hilbert space $\mathcal{H}_{L}^{\text{full}} := \mathbb{C}(G^{L^{1}} \times E^{L^{2}})$, the free vector space on the set of all 2-gauge configurations, and a larger repertoire of operators, also containing plaquette operators $F_{P}^{h}$, where $g \in G$. Those plaquette operators choose the configurations whose fake curvature around the plaquette $P$ with initial point $v_{P}$ is $g$. Passing to the Haar integrals $\ell = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C} G$, $\Lambda = \frac{1}{|E|} \sum_{e \in E} e \in \mathbb{C} E$, and $\lambda = \delta_{\ell \Lambda} \in \mathbb{C} G^{*}$, we can define mutually commuting vertex projectors $V_{\psi} := V_{\psi}^{v}$, plaquette projectors $F_{P} := F_{P}^{h}$ and edge projectors $E_{\ell} := E_{\ell}^{A}$, on $\mathcal{H}_{L}^{\text{full}}$. The full higher Kitaev model is given by the following Hamiltonian on $\mathcal{H}_{L}^{\text{full}}$ (see [17, Equation (35)]):

$$
\sum_{v \in L^{0}} (1 - V_{\psi}) + \sum_{\ell \in L^{1}} (1 - E_{\ell}) + \sum_{P \in L^{2}} (1 - F_{P}).
$$

(2)

This model reduces to Kitaev’s quantum double model when considering crossed modules of the form $(\{1\} \xrightarrow{\partial} G)$; see [17].

In this paper, we define a Hopf-algebraic generalisation of the full higher Kitaev model in 2+1D, henceforth called the “Hopf-algebraic higher Kitaev model”. It takes as input a crossed module of semisimple Hopf algebras $(A \xrightarrow{\partial} H, \triangleright)$, as defined in [26], and also considered in [27] [28] [29]. Here $A$ and $H$ are Hopf algebras, $\triangleright$ is a left action of $H$ on $A$ making $A$ an $H$-module algebra and coalgebra, and $\partial : A \rightarrow H$ is a Hopf algebra map which together with $\triangleright$ satisfies two compatibility conditions (the Peiffer relations, for crossed modules of Hopf algebras). Crossed modules of Hopf algebras were characterized as a class of quantum 2-groups in [26]. Given an oriented surface $\Sigma$, with a triangulation $L = (L^{0}, L^{1}, L^{2})$, or a more generally a cell decomposition (see Definition 17), where each plaquette $P \in L^{2}$ comes with a choice of base-point, the total space of the Hopf-algebraic higher Kitaev model is:

$$
\mathcal{H}_{L} := H^{\otimes L^{1}} \otimes A^{\otimes L^{2}}.
$$

Combining the Hopf-algebraic setting in [13] [8] with the 2-group setting of [17] [7], in the model we can define:

- Vertex operators $V_{v,P}^{h} : \mathcal{H}_{L} \rightarrow \mathcal{H}_{L}$, where $v \in L^{0}$, $P$ is an adjacent plaquette, and $h \in H$. For each $(v, P)$ these operators define a representation of $H$ on $\mathcal{H}_{L}$. A vertex projector is defined as $V_{\psi} := V_{v,P}^{h}$, where $\ell$ is the Haar integral in $H$. This does not depend on the chosen adjacent plaquette $P$, for $\ell$ is cocommutative.

- Edge operators $E_{a,\ell}^{v} : \mathcal{H}_{L} \rightarrow \mathcal{H}_{L}$, where $t \in L^{1}$ and $a \in A$. For each edge $t$ we have a representation of $A$. The Haar integral $\Lambda \in A$ gives the vertex projector $E_{t} := E_{a,\ell}^{v}$.\n
- Plaquette operators $F_{P}^{\psi} : \mathcal{H}_{L} \rightarrow \mathcal{H}_{L}$, where $\psi \in H^{*}$. Again this defines for each plaquette $P$, which recall comes equipped with a base-point $v_{P}$, a representation of $H^{*}$. If $\lambda$ is the Haar integral of $H^{*}$, the plaquette projector $F_{P}^{\psi}$ is defined as $F_{P}^{\lambda}$.

We explicitly compute the commutation relations between these operators. It follows from the commutation relations, and the properties of Haar integrals for crossed modules of semisimple Hopf algebras, that the vertex, edge and plaquette projectors are mutually commuting. Therefore the Hamiltonian defined as in (2) defines an exactly solvable model. This is our proposal for the Hopf-algebraic higher Kitaev model.
The Hopf-algebraic higher Kitaev model reduces to the Hopf-algebraic Kitaev model when $A = \mathbb{C}$ in $(A \xrightarrow{\partial} H, \varepsilon)$. (Here $\partial : \mathbb{C} \rightarrow H$ is $z \in \mathbb{C} \mapsto z 1_H$, the unique Hopf algebra map. The action of $H$ is given by the counit map $\varepsilon : H \rightarrow \mathbb{C}$.) More precisely, the total spaces $\mathcal{H}_L$ and $\mathcal{H}_{\text{HKit}}^L$ of both models coincide, and so do the Hamiltonians and vertex and plaquette operators. The additional edge operators $E^e_z$, $e \in L^1$, $z \in \mathbb{C}$, that were not in the Hopf-algebraic Kitaev model, amount to multiplication by $z$.

If $(E \xrightarrow{\partial} G, \varepsilon)$ is a crossed module of groups, then, passing to the group algebras, we have a crossed module of semisimple Hopf algebras $(CE \xrightarrow{\partial} CG, \varepsilon)$. The resulting Hopf-algebraic higher Kitaev model coincides with the full higher Kitaev model in [17] for $(E \xrightarrow{\partial} G, \varepsilon)$. Given that the ground-state space of the full higher Kitaev model coincides with that of the higher Kitaev model on the same triangulated surface $(\Sigma, L)$, it hence follows from [7, §5.2], and also [17, 18], that the ground-state space of the full higher Kitaev model for $(E \xrightarrow{\partial} G, \varepsilon)$ on $(\Sigma, L)$ does not depend on the triangulation of $\Sigma$ and is canonically isomorphic to the free vector space on the set of homotopy classes of maps from $\Sigma$ to $B$, the classifying space of the crossed module $(E \xrightarrow{\partial} G, \varepsilon)$; see [30, 31, 32]. (This was interpreted in terms of the Yetter homotopy 2-type TQFT in [17].) Therefore Quinn’s finite total homotopy TQFT $\mathcal{Q}_B$ [12] again gives a TQFT whose state spaces give the ground-state spaces of the full higher Kitaev model.

The crossed module of Hopf algebras $(CE \xrightarrow{\partial} CG, \varepsilon)$ derived from a crossed module of groups can be generalised. Indeed if $f : Y \rightarrow X$ is a homomorphism of groups, and we have a left action of $G$ on $X$ and $Y$ by automorphisms, such that $f$ preserves the actions, and such that $\partial(E) \subseteq G$ acts trivially on $X$ and $Y$, we can define a crossed module of semisimple Hopf algebras $(\mathcal{F}_C(X) \otimes CE \xrightarrow{\partial} \mathcal{F}_C(Y) \rtimes CG, \varepsilon)$. Here $\partial$ means $f^* \otimes \partial$, and the action is the product of the obvious action of $CG$ on $\mathcal{F}_C(X) \otimes CE$ and the trivial action of $\mathcal{F}_C(Y)$.

We unpack the Hopf-algebraic Kitaev model derived from $(E, G, X, Y)$, and the remaining data, in some particular cases and compute its ground-state space, proving that in these particular cases the ground-state space is canonically independent of the triangulation $L$ of the surface. In all of these cases, the ground-state spaces can be derived from Quinn’s finite total homotopy TQFT [12] Lecture 4 $\mathcal{Q}_B$ for some homotopy finite space $B$. At this point we will not be self-contained, and use a deep result of Brown–Higgins [30, Theorem A] (see also [31] and the recent monograph by Brown–Higgins-Sivera [23, §11.4.iii]) describing the set of homotopy classes of maps $M \rightarrow \mathcal{B}_A$, where $A$ is a crossed complex and $M$ is CW-complex, in terms of homotopy classes of crossed complex maps $\Pi(M) \rightarrow A$, where $\Pi(M)$ denotes the fundamental crossed complex of a CW-complex $M$ [31]. (Below note that a crossed complex homotopy between groupoid functors [33] boils down to a natural transformation.) These techniques to prove triangulation independence have already been applied in [32, 7], and will be further developed in [34].

The cases of the $(E, G, X, Y)$-model that we will consider, on triangulated surfaces, $(\Sigma, L)$ are:

- The $(1, G, X, 1)$-case, where $E = Y = \{1\}$. (Our data boils down to a group $G$ acting on another group $X$ by automorphisms.) In this case we obtain a coupling between the Kitaev model for $G$ and the $|X|$-state Potts model [35] (cf. for example [36]). The ground-state space for $(\Sigma, L)$ is given by the free vector space on the set of equivalence classes of functors $\pi_1(\Sigma, \Sigma^0_\nu) \rightarrow X/\Gamma$, considered up to natural transformations, where $X/\Gamma$ is the action groupoid of the action of $G$ on $X$. Here $\pi_1(\Sigma, \Sigma^0_\nu)$ denotes the fundamental groupoid of $\Sigma$, with set of base-points being the set of vertices $\nu \in L^0$ in $\Sigma$.

By applying Brown–Higgins theorem, the latter space is canonically isomorphic to the free vector space on the set of homotopy classes of maps $\Sigma \rightarrow \mathcal{B}_{X/\Gamma}$. Here $\mathcal{B}_{X/\Gamma}$ is the classifying space of the action groupoid $X/\Gamma$. Hence the ground-state space does not
depend on $L$. Quinn’s finite total homotopy TQFT $\Omega_{Bx//G}$ gives a TQFT whose state space on a surface $\Sigma$ coincides with the ground-state space on $(\Sigma, L)$, for any triangulation of $L$ of $\Sigma$.

- More generally, the $(E, G, X, 1)$-case, where $Y = \{1\}$. This includes the full higher Kitaev model as a special case (when $X = \{1\}$). In this case, the ground-state space for a triangulated surface $(\Sigma, L)$ is canonically given by the free vector space on the set of homotopy classes of crossed module maps $\Pi_2(\Sigma, \Sigma^1_L, \Sigma^2_L) \to (X//E \to X//G, \triangleright)$, where $\Pi_2(\Sigma, \Sigma^1_L, \Sigma^2_L)$ is the fundamental crossed module of $(\Sigma, \Sigma^1_L, \Sigma^2_L)$, the underlying CW-complex the triangulation $L$, with its skeletal filtration; see [3] §3.3. Also $(X//E \to X//G, \triangleright)$ is a crossed module of groupoids, where $X//E$ and $X//G$ are action groupoids. (Here $E$ acts on $X$ trivially, and the action of the groupoid $X//G$ on the groupoid $E//G$ is derived from the action of $G$ on $E$).

By applying Brown–Higgins theorem, the ground-state space is similarly seen to be independent of the triangulation $L$ of $\Sigma$, and given by the free vector space on the set of homotopy classes of maps $\Sigma \to B$ where $B$ is the classifying space of $(X//E \to X//G, \triangleright)$. Likewise, Quinn’s finite total homotopy TQFT $\Omega_B$ gives a TQFT whose state space coincides with the ground-state space of the underlying Hopf-algebraic higher Kitaev model for each surface $\Sigma$.

- The $(1, 1, X, Y)$-case, when $E$ and $G$ are both the trivial group. In this case the ground-state space on $(\Sigma, L)$ is canonically triangulation independent, and given by $\Omega_{Bx//Y}(\Sigma)$. Here $B_{x//Y}$ is the classifying space of the action groupoid of the action $\triangleright$ of $Y$ on $X$, where $y \triangleright x = f(y)x$. In order to prove that the ground-state space is indeed canonically isomorphic, regardless of the chosen triangulation $L$ of $\Sigma$, to the free vector space on the set of homotopy classes of maps $\Sigma \to B_{x//Y}$, we use a slightly different trick as in the two previous cases. Namely, we consider the dual cell decomposition $(\Sigma, L^\ast)$ to $(\Sigma, L)$, and identify the ground-state space with the free vector space on the set of equivalence classes of groupoid functors $\pi_1(\Sigma, \Sigma^0_{L^\ast}) \to X//Y$, considered up to natural transformations. (Note that we now have a base-point of $\pi_1(\Sigma, \Sigma^0_{L^\ast})$ for each plaquette of $L$.)

Cf. [3] §2.3], indeed, the resulting $(1, 1, X, Y)$-model has a particularly simple expression in the dual cell decomposition $(\Sigma, L^\ast)$. Moreover the model reduces to the Kitaev model based on $Y$ on $(\Sigma, L^\ast)$ when $X = \{1\}$ (which essentially is [3] Lemma 2.6], to the $|X|$-state Potts model if $Y = \{1\}$, and to a groupoid version of Kitaev model (with groupoid $X//Y$), also featuring edge operators, in the general case. The $(1, 1, X, Y)$-model on the dual cell decomposition is also closely related to the construction in [37].

- Finally we have the $(1, G, 1, Y)$-case when $E$ and $X$ are the trivial group. This is not a proper crossed Hopf-algebraic higher Kitaev model, contrary to the previous three cases, in that the crossed module of Hopf algebras is $(\mathbb{C} \xrightarrow{\partial} \mathcal{F}_\mathbb{C}(Y) \rtimes \mathbb{C}G, \triangleright)$. The $(1, G, 1, Y)$-model is hence a special case of the Hopf-algebraic Kitaev model. In particular, the ground-state space is known to be canonically triangulation independent.

It is an open problem whether the Turaev-Viro TQFT derived from the spherical fusion category $(\mathcal{F}_\mathbb{C}(Y) \rtimes \mathbb{C}G)$–mod, whose state spaces give the ground-state spaces of the $(1, G, 1, Y)$-case, is also a particular case of Quinn’s finite total homotopy TQFT and, in particular, if it has a homotopical explanation.

We finish this introduction by presenting the following set of open problems which arise from the present paper:
1. Are the ground-state spaces of the full \((E, G, X, Y)\)-case triangulation independent and is there a TQFT giving these ground-state spaces, which is a particular case of Quinn’s finite total homotopy TQFT?

2. The following are likely the most important open problems resulting from our construction. Considering general crossed modules of semisimple Hopf algebras:
   - Is the ground-state space of the Hopf-algebraic higher Kitaev model in \((\Sigma, L)\) always canonically invariant of the triangulation \(L\) of \(\Sigma\)?
   - Is there an underpinning TQFT whose state spaces give the ground-state spaces of the Hopf-algebraic higher Kitaev model? In particular, what are the quantum gates that could be realized by the underlying mapping class group actions?

3. Majid defined in [26] a generalisation of crossed modules of Hopf algebras that he called “braided crossed modules of Hopf algebras” \((B \xrightarrow{\partial} H, \triangleright)\). Here \(B\) is a Hopf algebra in \(Z(H{-}\text{mod})\), the braided category of Yetter-Drinfeld modules over \(H\). It is an open problem whether our construction of the Hopf-algebraic higher Kitaev model generalises to this setting.

4. What local excitations does the Hopf-algebraic higher Kitaev model admit? Previously [38], defects of co-dimensions 2 and 1 (i.e. point-like and string-like excitations) in the Kitaev model have been studied in the general Hopf-algebraic setting. It is an interesting open problem to extend this to the Hopf-algebraic higher Kitaev model constructed in the present paper.

5. Finally, is there a 3+1D version of the Hopf-algebraic higher Kitaev similar to the original 2-group higher Kitaev model [17, 7, 18]? This extension would seem quite tricky when the Hopf algebra \(H\) in \((A \xrightarrow{\partial} H, \triangleright)\) is not cocommutative. This is because there is not a natural way to define vertex operators, since there is no given cyclic order on the edges incident to a vertex. A model however exists for \((CE \xrightarrow{\partial} CG, \triangleright)\) where \((E \xrightarrow{\partial} G, \triangleright)\) is a crossed module of groups (as per the construction in [17, 7, 18]).

Some broader motivation.
A primary question for the field-theoretic approach to physical modelling of gauge phenomena is that of the relationship between the representation of space(-time), and the symmetry structure underlying the gauge fields. This may be cast as the choices of two categories (space and gauge, informally put) — yielding the category of functors between them (see e.g. [20, §2], [21, 7]) as (a basis for) the space of states. Such a categorical formulation can somewhat obscure the direct tie to the physics being modelled, but facilitates the development of candidate generalisations of the ‘classical’ suite of models. For example in [20, 7] one passes from the functor category of gauge configurations to a lift to higher gauge configurations — nominally 2-functors from a 2-lattice 2-category to a 2-group. In this setting space(-time) is modelled by a generalisation of a CW-complex — depending on the requirements of the Hamiltonian, and the gauge group becomes a kind of higher group (or indeed lower group! in the sense that in Potts models it can really be just a set, since the interaction is a delta-function, furthermore requiring only graph data for the lattice). In parallel to this approach it is natural to ask what other generalisations of group characterisations of symmetry can be supported in principle (the $64$ question is What is demanded by physical observation?, but this begs also the question of where and how to observe — see later, and cf. [15]). And what impact on the formalisation of space–time this might have. Hopf algebras, and in particular higher versions such as Hopf crossed modules [26, 27], offer a possible line of generalisation from gauge groups in this context.
because Hopf algebras (miraculously) bridge between the combinatorial and geometric worlds, at least in 2D [39]. However while the linear structure on the resultant Hilbert space in the group cases is essentially passive, it plays a crucial role in the Hopf case (confer e.g. [40, 4]). Here we take a first step to probe this challenge in a higher setting — a higher theory, formally, but staying on surfaces. The version of this analysis relevant for higher Kitaev models (and hence relatively straightforward use of Whitehead’s underlying free fundamental crossed module technology) is discussed for example in [7].

2 Crossed modules of semisimple Hopf algebras

Crossed module of Hopf algebras are defined for example in [26, 27, 28, 29]. Algebras here will by default be unital, associative and finite-dimensional over $\mathbb{C}$. Our discussion about semisimple Hopf algebras and Haar integrals follows that of [3, Section 1] closely.

A Hopf algebra $H$ is a unital associative algebra together with a compatible structure of a co-unital co-associative co-algebra given by an algebra map $\Delta : H \to H \otimes H$ and a co-unit $\varepsilon : H \to \mathbb{C}$. We denote its antipode as usual by $S : H \to H$, or $S_H : H \to H$ if we want to emphasize the Hopf algebra it belongs to.

We will make extensive use of the Sweedler notation for co-multiplication:

$$h(1) \otimes h(2) := \Delta(h) \quad \forall h \in H,$$

which is in general a sum of pure tensors even though we omit the summation symbol and variable. Due to co-associativity, any $n$-fold composition of the co-multiplication tensored with suitable identity maps has the same result and one may therefore write:

$$h(1) \otimes h(2) \otimes \cdots \otimes h(n) := \Delta^{(n-1)}(h) := (\Delta \otimes \text{id}_H^{\otimes (n-2)}(\cdots (\Delta(h)))) \quad \forall h \in H.$$

A fact we will frequently use when we construct our model is the following. Any multiple coproduct of a cocommutative element $h \in H$, i.e. one which satisfies $h(1) \otimes h(2) = h(2) \otimes h(1)$, is cyclically invariant, i.e.

$$h(1) \otimes h(2) \otimes \cdots \otimes h(n) = h(2) \otimes h(3) \otimes \cdots \otimes h(n-1) \otimes h(1) = \ldots = h(n) \otimes h(1) \otimes h(2) \cdots \otimes h(n-1). \quad (3)$$

This follows by combining $h(1) \otimes h(2) = h(2) \otimes h(1)$ with the coassociativity of $\Delta$. Any cocommutative element $\varphi : H \to \mathbb{C}$ of the dual Hopf algebra $H^*$ satisfies:

$$\varphi(xy) = \varphi(yx) \quad \text{for all } x, y \in H.$$

In this paper, we are assuming that the Hopf algebras $H$ are finite-dimensional over $\mathbb{C}$ and semisimple as algebras. In particular the algebra has a canonical 1-dimensional representation (since $\varepsilon : H \to \mathbb{C}$ is an algebra homomorphism) and unique corresponding primitive central idempotent (for example, the trivial representation and corresponding idempotent, respectively, of a finite group) [12 §9]. Two concrete consequences of the assumptions which are crucial for the construction of the commuting-projector Hamiltonian in this paper, just as in the Hopf-algebraic Kitaev model [13], are the following. The Hopf algebra is involutive, i.e. its antipode $S : H \to H$ is involutive, $S^2 = \text{id}_H$, and it possesses the above-mentioned idempotent, the Haar integral:

**Definition 1. / Proposition.** Let $H$ be a Hopf algebra. The Haar integral for $H$ (if it exists) is the unique idempotent $\ell \in H$ satisfying $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$. 

9
Proof. The idempotence implies \( \varepsilon(\ell) = 1 \). Let \( \ell' \in H \) be another Haar integral. Then we have 
\[
\ell' = \varepsilon(\ell)\ell' = \varepsilon(\ell') = \ell.
\]

**Proposition 2.** Let \( H \) be a complex finite-dimensional Hopf algebra. The Haar integral \( \ell \in H \) is cocommutative, \( \ell(1) \otimes \ell(2) = \ell(2) \otimes \ell(1) \), \( S(\ell) = \ell \), and the following are equivalent:

- The Haar integral \( \ell \in H \) exists.
- \( H \) is semisimple.
- The antipode is involutive, \( S^2 = \text{id}_H \).

**Proof.** See [9, Prop. 1.3] and also [3, Theorem 1.2]. In particular, the Haar integral can be expressed as 
\( \ell = \frac{1}{\dim(H)} \chi_H \ast \), where \( \chi_H \ast \in (H^*)^* \) is the regular character of the dual Hopf algebra \( H^* \) and where the canonical identification \( (H^*)^* \cong H \) is implicit. The cocommutativity of the Haar integral then is an immediate consequence of the cyclicity of the trace. \( \square \)

**Example 3.** Let \( G \) be a finite group. The group algebra \( \mathbb{C} G \) is semisimple by Maschke’s theorem. We will regard \( \mathbb{C} G \) as a Hopf algebra in the usual way with 
\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1},
\] for all \( g \in G \). The Haar integral is 
\[
\ell = \frac{1}{|G|} \sum_{g \in G} g.
\]

**Example 4.** Given a finite group \( G \), the algebra \( \mathcal{F}_C(G) \) of functions \( G \to \mathbb{C} \), with the pointwise product, can be turned into a Hopf algebra by putting 
\[
\varepsilon(f) = f(1_G), \quad \Delta(f) = f(xy) \quad \text{for all } x,y \in G,
\] noting the isomorphism \( \mathcal{F}_C(G) \otimes \mathcal{F}_C(G) \cong \mathcal{F}_C(G \times G) \). This algebra is also semisimple. The Haar integral is \( \delta_1 \), where \( \delta_1(x) = 1 \) if \( x = 1_G \), and otherwise 0.

**Definition 5** ([26, 27, 29]). A crossed module of Hopf algebras or Hopf crossed module \( (A \xrightarrow{\partial} H, \triangleright) \) consists of:

- Hopf algebras \( A \) and \( H \) with a Hopf algebra morphism \( \partial : A \to H \), called **boundary map**, and
- a left \( H \)-action \( \triangleright : H \otimes A \to A \), such that:
  - the action turns \( A \) into an \( H \)-module algebra, i.e.:
    \[
    h \triangleright (ab) = (h(1) \triangleright a) (h(2) \triangleright b), \quad h \triangleright 1_A = \varepsilon(h)1_A,
    \]
  and an \( H \)-module coalgebra, i.e.:
    \[
    \Delta(h \triangleright a) = (h(1) \triangleright a(1)) \otimes (h(2) \triangleright a(2)), \quad \varepsilon(h \triangleright a) = \varepsilon(h)\varepsilon(a),
    \]
  for all \( h \in H, a \in A \),
  - the **Yetter-Drinfeld** condition holds:
    \[
    h(1) \otimes (h(2) \triangleright a) = h(2) \otimes (h(1) \triangleright a) \quad \text{for all } h \in H, a \in A,
    \] (4)
satisfying the two **Peiffer** relations:

1. \( \partial(h \triangleright a) = h(1) \partial(a) Sh(2), \quad \text{for all } a \in A, h \in H; \)
2. \( \partial(a) \triangleright b = a(1) b Sa(2), \quad \text{for all } a, b \in A. \)
Remark 6. A crucial fact we will use about a Hopf crossed module \((A \xrightarrow{\partial} H, \triangleright)\) is that the antipode \(S_A : A \rightarrow A\) is \(H\)-linear, which follows from the module algebra and module coalgebra properties, see also [26, page 4]. We will explicitly check this property separately in the examples of Hopf crossed modules in Subsection 2.1.

Remark 7. Definition 5 can be equivalently rephrased as follows. \(H\) is a Hopf algebra, \((A, \triangleright)\) together with the trivial left \(H\)-co-action is a Hopf algebra in the braided monoidal category of \((\text{left-left})\) Yetter-Drinfeld-modules over \(H\), and \(\partial : A \rightarrow H\) is a Hopf algebra morphism satisfying the Peiffer relations. This follows straightforwardly by spelling out the definition of the braided monoidal category of Yetter-Drinfeld modules (see e.g. [43, §10.6]). Note that equation (4) is precisely the Yetter-Drinfeld condition in the case of a trivial co-action.

We remark also that from this point of view the antipode \(S : A \rightarrow A\) is by definition \(H\)-linear, since any morphism in the category of \(H\)-Yetter-Drinfeld modules is in particular an \(H\)-module morphism.

Remark 8. The Yetter-Drinfeld condition (4), combined with the coassociativity of \(\Delta\), implies the following property (which we note here because we will frequently make use of it). For any \(h \in H\) and \(a_1 \otimes \cdots \otimes a_k \in A^{\otimes k}\), the expression

\[
h(1) \otimes \cdots \otimes h(n) \otimes h(n+1) \triangleright a_1 \otimes \cdots \otimes h(n+k) \triangleright a_k \in H^{\otimes n} \otimes A^{\otimes k}
\]

is invariant under permutations of \((h(1), \ldots, h(n), h(n+1), \ldots, h(n+k))\) which preserve the order of the first \(n\) factors.

Example 9. A crossed module of groups \((E \xrightarrow{\partial} G, \triangleright)\) [32, 31, 30] is given by a group homomorphism \(\partial : E \rightarrow G\), together with a left action \(\triangleright\) of \(G\) on \(E\) by automorphisms, such that the Peiffer relations, in the group case, as below, are satisfied:

1. \(\partial(g \triangleright e) = g\partial(e)g^{-1}\) for all \(g \in G\) and \(e \in E\).
2. \(\partial(e) \triangleright f = ef e^{-1}\) for all \(e, f \in E\).

If we consider the induced map \(\partial : CE \rightarrow CG\) on group algebras and the induced ‘linearised’ action \(\triangleright\) of \(CG\) on \(CE\), then \((CE \xrightarrow{\partial} CG, \triangleright)\) is a crossed module of Hopf algebras; see [27]. This example will be generalised in Subsection 2.1.

The following structure of a crossed product (also known as smash product) is well known and arises naturally from a crossed module of Hopf algebras [26, 41].

Definition 10. / Proposition.

Let \(H\) be a Hopf algebra and let \(A\) be a left \(H\)-module algebra. Then the crossed product algebra \(A \rtimes H\) is the algebra with underlying vector space \(A \otimes H\) and multiplication:

\[
(a \otimes h) \cdot (b \otimes k) := a(h(1) \triangleright b) \otimes h(2)k,
\]

for \(a, b \in A\) and \(h, k \in H\).

If \(A\) is additionally an \(H\)-module coalgebra, such that the Yetter-Drinfeld condition (4), i.e. \(h(1) \otimes h(2) \triangleright a = h(2) \otimes h(1) \triangleright a\), holds, then the algebra \(A \rtimes H\) becomes a Hopf algebra with the usual tensor product coalgebra structure, \(\Delta(a \otimes h) := (a(1) \otimes h(1)) \otimes (a(1) \otimes h(2))\). In particular, the Yetter-Drinfeld condition clearly holds if \(H\) is cocommutative.

The following is a key lemma that we need in order to prove the commutativity between vertex and edge projectors in the commuting-projector Hamiltonian model defined in Section 3 below. Specifically, see Corollary 79.
Lemma 11. Let $H$ and $A$ be semisimple finite-dimensional Hopf algebras and let $\triangleright : H \otimes A \to A$ be a left action turning $A$ into an $H$-module coalgebra. Denote by $\Lambda \in A$ the Haar integral of $A$. Then $\Lambda$ is $H$-invariant, namely:

$$h \triangleright \Lambda = \varepsilon(h) \Lambda.$$  

As a consequence, $\Lambda \otimes 1 \in A \otimes H$ is central in the crossed product algebra $A \rtimes H$, i.e. we have

$$(\Lambda \otimes 1_H) \cdot (a \otimes h) = \Lambda a \otimes h = a(h_{(1)} \triangleright \Lambda) \otimes h_{(2)} = (a \otimes h) \cdot (\Lambda \otimes 1_H) \quad \forall a \in A, h \in H.$$  

Proof. The second equation follows from the first by noting that the Haar integral $\Lambda \in A$ is central in $A$, i.e. $\Lambda a = a \Lambda$ for all $a \in A$. Let us therefore show that $\Lambda$ is an $H$-invariant element, that is $h \triangleright \Lambda = \varepsilon(h) \Lambda$ for all $h \in H$. Due to Larson-Radford [9, Prop. 1.3], the Haar integral in $A$ can be expressed by the regular character $\chi_{A^*} \in (A^*)^*$ of the dual Hopf algebra $A^*$, i.e. $\Lambda = \frac{1}{\dim(A)} \chi_{A^*}$, where the canonical identification $(A^*)^* \cong A$ is implicit. The regular character of the dual Hopf algebra $A^*$ has the following expression as an element of $A$, in terms of the co-multiplication of $A$:

$$\chi_{A^*} = \sum_{i=1}^{\dim(A)} (e_i)_{(1)} e^i((e_i)_{(2)}),$$

where $(e_i)_{(1)}^{\dim(A)}$ and $(e^i)_{(1)}^{\dim(A)}$ are dual bases for $A$ and $A^*$, respectively. Hence, we compute, omitting the summation symbol and letting $h \in H$:

$$h \triangleright \Lambda = h \triangleright ((e_i)_{(1)} e^i((e_i)_{(2)}))$$

$$= (h \triangleright (e_i)_{(1)}) e^i((e_i)_{(2)})$$

$$= (h_{(1)} \triangleright (e_i)_{(1)}) e^i(S^{-1}(h_{(3)}) \triangleright h_{(2)} \triangleright (e_i)_{(2)})$$

$$= (h_{(1)} \triangleright (e_i)_{(1)}) e^i(h_{(2)} \triangleright (e_i)_{(2)})$$

$$A \text{-mod coalg.}$$

$$= (\varepsilon(h)(e_i)_{(1)} e^i((e_i)_{(2)}))$$

$$= (\varepsilon(h) \Lambda),$$

concluding the proof of the lemma. Here in the sixth step (*) we have used the elementary property of dual bases which states that $\sum_i f(e_i) \otimes e^i = \sum_i e_i \otimes (e^i \circ f)$ for any linear map $f : A \to A$ (which can be verified by evaluating both sides of the equation on the basis). \hfill \Box

Example 12. In the case of a crossed module of Hopf algebras ($CE \xrightarrow{\partial} CG, \triangleright)$ derived from a crossed module of finite groups ($E \xrightarrow{\partial} G, \triangleright)$, see Example 9, the compatibility relation between the Haar integral in $CE$ and the action of $CG$ simply means that:

$$g \triangleright \left( \frac{1}{|E|} \sum_{e \in E} e \right) = \frac{1}{|E|} \sum_{e \in E} e \quad \text{for all } g \in G.$$  

(5)

2.1 A class of examples of crossed modules of semisimple Hopf algebras from crossed products of groups algebras with dual group algebras

Proposition 13. Let $(E \xrightarrow{\partial} G, \triangleright)$ be a crossed module of groups. Let $X$ and $Y$ be finite groups on which $G$ acts by automorphisms, and let $f : Y \to X$ be a $G$-equivariant group morphism.
We require additionally that the restrictions of the actions of $G$ on $Y$ and $X$, respectively, to $\text{im}(\partial) \subseteq G$ are trivial.

Then this gives rise to a crossed product Hopf algebra $\mathcal{F}_C(Y) \rtimes CG$ (see Definition [10]) and we have a crossed module of Hopf algebras $(\mathcal{F}_C(X) \otimes CE \xrightarrow{\partial} \mathcal{F}_C(Y) \rtimes CG, \triangleright)$, where

$$\partial : \mathcal{F}_C(X) \otimes CE \longrightarrow \mathcal{F}_C(Y) \rtimes CG,$$

$$\xi \otimes e \longmapsto f^* \xi \otimes \partial(e),$$

(here $(f^* \xi)(y) = \xi(f(y))$, if $y \in Y$) and

$$(\varphi \otimes g) \triangleright (\xi \otimes e) := \varphi(1)(g \triangleright \xi) \otimes (g \triangleright e),$$

for $\varphi \otimes g \in \mathcal{F}_C(Y) \otimes CG$, $\xi \otimes e \in \mathcal{F}_C(X) \otimes CE$, where $(g \triangleright \xi)(x) := \xi(g^{-1} \triangleright x)$, for $x \in X$.

Proof.

$\mathcal{F}_C(Y)$ is a $CG$-module algebra and coalgebra.

For $g \in G$, $\varphi \in \mathcal{F}_C(Y)$ and $y \in Y$, we set $(g \triangleright \varphi)(y) := \varphi(g^{-1} \triangleright y)$.

We check the module algebra conditions, where $\varphi, \psi \in \mathcal{F}_C(Y)$:

$$(g \triangleright (\varphi \psi))(y) = (\varphi \psi)(g^{-1} \triangleright y)$$

$$= \varphi(g^{-1} \triangleright y) \psi(g^{-1} \triangleright y)$$

$$= (g \triangleright \varphi)(y) (g \triangleright \psi)(y)$$

$$= ((g \triangleright \varphi)(g \triangleright \psi))(y)$$

and $(g \triangleright 1_{\mathcal{F}_C(Y)})(y) = 1_{\mathcal{F}_C(Y)}(g^{-1} \triangleright y) = 1 = 1_{\mathcal{F}_C(Y)}(y)$.

We check the module coalgebra conditions, where $y, z \in Y$:

$$(\vartriangle(g \triangleright \varphi))(y, z) = (g \triangleright \varphi)(yz)$$

$$= \varphi(g^{-1} \triangleright (yz))$$

$$= \varphi((g^{-1} \triangleright y)(g^{-1} \triangleright z))$$

$$= \vartriangle(\varphi)(g^{-1} \triangleright y, g^{-1} \triangleright z)$$

$$= ((g \otimes g) \triangleright \vartriangle(\varphi))(y, z)$$

and $\varepsilon(g \triangleright \varphi) = (g \triangleright \varphi)(1) = \varphi(g^{-1} \triangleright 1) = \varphi(1) = \varepsilon(\varphi)$.

Let us further explicitly verify the $CG$-linearity of the antipode:

$$(g \triangleright S(\varphi))(y) = \varphi((g^{-1} \triangleright y)^{-1}) = \varphi(g^{-1} \triangleright y^{-1}) = S(g \triangleright \varphi)(y).$$

Since $CG$ is cocommutative (and hence the Yetter-Drinfeld condition automatically holds), we can thus form the crossed product $\mathcal{F}_C(Y) \rtimes CG$, see Definition [10], which is a Hopf algebra with underlying vector space $\mathcal{F}_C(Y) \otimes CG$ and the following multiplication and comultiplication:

$$(\varphi \otimes h) \cdot (\psi \otimes h) := \varphi(g \triangleright \psi) \otimes gh,$$

$$\vartriangle(\varphi \otimes g) := (\vartriangle(\varphi) \otimes g)(\varphi(2) \otimes g).$$

Next, consider the tensor product Hopf algebra $\mathcal{F}_C(X) \otimes CE$.

$\mathcal{F}_C(X) \otimes CE$ is a $(\mathcal{F}_C(Y) \rtimes CG)$-module algebra and coalgebra.

For $\varphi \otimes g \in \mathcal{F}_C(Y) \otimes CG$, $\xi \otimes e \in \mathcal{F}_C(X) \otimes CE$, we set

$$(\varphi \otimes g) \triangleright (\xi \otimes e) := \varphi(1)(g \triangleright \xi) \otimes (g \triangleright e),$$
where \( (g \triangleright x)(x) := \xi(g^{-1} \triangleright x) \) for \( x \in X \). This is indeed an \((\mathcal{F}_C(Y) \rtimes CG)\)-action:

\[
(\varphi \otimes g) \triangleright ((\psi \otimes h) \triangleright (\xi \otimes e)) = (\varphi \otimes g) \triangleright (\psi \otimes (h \triangleright (\psi \otimes (h \triangleright e)))
= \varphi(1)(\psi(1)(g \triangleright (h \triangleright \xi))) \otimes (g \triangleright (h \triangleright e))
= \varphi(1)(\psi(g^{-1} \triangleright 1)(g \triangleright (h \triangleright \xi))) \otimes (g \triangleright (h \triangleright e))
= (\varphi(g \triangleright \psi) \otimes g \triangleright h) \triangleright (\xi \otimes e)
= ((\varphi \otimes g) \triangleright (\psi \otimes h)) \triangleright (\xi \otimes e).
\]

Now we check the module algebra conditions, where \( \varphi \otimes g \in \mathcal{F}_C(Y) \otimes CG \) and \( \xi \otimes e, \zeta \otimes d \in \mathcal{F}_C(X) \otimes CE \):

\[
(\varphi \otimes g) \triangleright ((\xi \otimes e)(\zeta \otimes d)) = (\varphi \otimes g) \triangleright (\xi \otimes ed)
= \varphi(1)(g \triangleright (\xi \zeta)) \otimes g \triangleright (ed)
= \varphi(1)(g \triangleright \xi) \otimes (g \triangleright e)(g \triangleright d)
= \varphi(1)((g \triangleright \xi) \otimes (g \triangleright e))(\varphi(2)(g \triangleright \zeta) \otimes (g \triangleright d))
= ((\varphi(1) \otimes g) \triangleright (\xi \otimes e))((\varphi(2) \otimes g) \triangleright (\zeta \otimes d))
\]

and \((\varphi \otimes g) \triangleright (1 \triangleright 1) \otimes 1_E = \varphi(1)(1 \triangleright 1) \otimes (g \triangleright 1_E) = \varphi(1)(1 \triangleright 1) = \varepsilon(\varphi \otimes g) \otimes 1_E = 1 \otimes 1 \triangleright 1_E \).

Next we check the module coalgebra conditions, where \( \varphi \otimes g \in \mathcal{F}_C(Y) \otimes CG \) and \( \xi \otimes e \in \mathcal{F}_C(X) \otimes CE \):

\[
\Delta((\varphi \otimes g) \triangleright (\xi \otimes e)) = \varphi(1)(g \triangleright \xi) \otimes (g \triangleright e)
= \varphi(1)((g \triangleright \xi)(1) \otimes (g \triangleright e)) \otimes ((g \triangleright \xi)(2) \otimes (g \triangleright e))
= \varphi(1)((g \triangleright \xi(1)) \otimes (g \triangleright e)) \otimes ((g \triangleright \xi(2)) \otimes (g \triangleright e))
= \varphi(1)(1)((g \triangleright \xi(1)) \otimes (g \triangleright e)) \otimes \varphi(2)(1)((g \triangleright \xi(2)) \otimes (g \triangleright e))
= ((\varphi(1) \otimes g) \triangleright (\xi \otimes e)) \triangleleft \Delta(\xi \otimes e)
\]

and \(\varepsilon((\varphi \otimes g) \triangleright (\xi \otimes e)) = \varphi(1)\xi(g^{-1} \triangleright 1)\varepsilon(g \triangleright e) = \varphi(1)\xi(1) = \varepsilon(\varphi \otimes g)\varepsilon(\xi \otimes e) \).

Finally let us explicitly verify that the antipode of \( \mathcal{F}_C(X) \otimes CE \) is \((\mathcal{F}_C(Y) \rtimes CG)\)-linear:

\[
S((\varphi \otimes g) \triangleright (\xi \otimes e)) = S(\varphi(1)g \triangleright \xi \otimes g \triangleright e)
= \varphi(1)S(g \triangleright \xi) \otimes S(g \triangleright e)
= \varphi(1)(S(\xi) \otimes g \triangleright e^{-1})
= (\varphi \otimes g) \triangleright (S(\xi) \otimes e^{-1})
= (\varphi \otimes g) \triangleright S(\xi \otimes e).
\]

\( \mathcal{F}_C(X) \otimes CE \) satisfies the Yetter-Drinfeld condition (with trivial \( \mathcal{F}_C(Y) \rtimes CG \)-coaction).

For this we have to check that \((\varphi(1) \otimes g) \otimes (\varphi(2) \otimes g) \triangleright (\xi \otimes e) = (\varphi(2) \otimes g) \otimes (\varphi(1) \otimes g) \triangleright (\xi \otimes e) \)
holds for \( \varphi \otimes g \in \mathcal{F}_C(Y) \otimes CG \) and \( \xi \otimes e \in \mathcal{F}_C(X) \otimes CE \). Indeed, it is easy to see that both sides of the equation are equal to \((\varphi \otimes g) \otimes (g \triangleright \xi \otimes g \triangleright e) \).

There is a Hopf algebra morphism \( \partial : \mathcal{F}_C(X) \otimes CE \rightarrow \mathcal{F}_C(Y) \rtimes CG \).

We define

\[
\partial : \mathcal{F}_C(X) \otimes CE \rightarrow \mathcal{F}_C(Y) \rtimes CG,
\xi \otimes e \longmapsto f^*\xi \otimes \partial(e)
\]
and check that it is a morphism of algebras:
\[
\partial((\xi \otimes e)(\xi \otimes d)) = \partial(\xi \xi \otimes ed) \\
= f^*(\xi \xi) \otimes \partial(ed) \\
= (f^*\xi)(f^*\xi) \otimes \partial(e)\partial(d) \\
= (f^*\xi \otimes \partial(e))(f^*\xi \otimes \partial(d)) \\
= \partial(\xi \otimes e)\partial(\xi \otimes d)
\]
and \(\partial(1_{\mathcal{T}_C(X)} \otimes 1_E) = f^*(1_{\mathcal{T}_C(X)}) \otimes \partial(1_E) = 1_{\mathcal{T}_C(X)} \otimes 1_E\).

Next we check that \(\partial\) is a morphism of coalgebras:
\[
\Delta(\partial(\xi \otimes e)) = (((f^*\xi)(\xi) \otimes \partial(e)) \otimes ((f^*\xi)(\xi) \otimes \partial(e))) \\
= (((\xi \circ f)(\xi) \otimes \partial(e)) \otimes ((\xi \circ f)(\xi) \otimes \partial(e))) \\
= \partial(\xi(1)) \otimes e \otimes \partial(\xi(2)) \otimes e
\]
and \(\varepsilon(\partial(\xi \otimes e)) = \xi(f(1_Y))\varepsilon(\partial(e)) = \xi(1_X)\varepsilon(e) = \varepsilon(\xi \otimes e)\).

\(\partial : \mathcal{T}_C(X) \otimes \mathcal{CE} \rightarrow \mathcal{T}_C(Y) \otimes \mathcal{CG}\) satisfies the Peiffer conditions.

We check the first Peiffer condition: On the one hand we have, for \(y \in Y\),
\[
\partial((\varphi \otimes g) \triangleright (\xi \otimes e))(y) = \varphi(1)\partial(g \triangleright \xi) \otimes g \triangleright e \\
= \varphi(1)\xi(g^{-1} \triangleright f(y))\partial(g \triangleright e) \\
= \varphi(1)\xi(f(g^{-1} \triangleright y))\partial(g \triangleright e).
\]
On the other hand we have
\[
((\varphi(1) \otimes g)\partial(\xi \otimes e)S(\varphi(2) \otimes g))(y) = (((\varphi(1) \otimes g)(f^*\xi \otimes \partial(e))(1 \otimes g^{-1})(S(\varphi(2)) \otimes 1))(y) \\
= (((\varphi(1)(g \triangleright f^*\xi) \otimes g\partial(e))(1 \otimes g^{-1})(S(\varphi(2)) \otimes 1))(y) \\
= (((\varphi(1)(g \triangleright f^*\xi) \otimes g\partial(e))(1 \otimes g^{-1})(S(\varphi(2)) \otimes 1))(y) \\
= (((\varphi(1)(g \triangleright f^*\xi) \otimes g\partial(e))(1 \otimes g^{-1})(S(\varphi(2)) \otimes 1))(y) \\
= ((\varphi(1)(g \triangleright f^*\xi) \otimes \partial(g \triangleright e))(S(\varphi(2)) \otimes 1))(y) \\
= ((\varphi(1)(g \triangleright f^*\xi) \otimes \partial(g \triangleright e))(S(\varphi(2)) \otimes 1))(y) \\
\stackrel{(\ast)}{=} \varphi(1)(y)\xi(f(g^{-1} \triangleright y))\varphi(2)(y^{-1})\partial(g \triangleright e) \\
= \varphi(1)\xi(f(g^{-1} \triangleright y))\partial(g \triangleright e).
\]
where in \((\ast)\) we use the assumption that \(\text{im}(\partial)\) acts trivially on \(Y\). In fact, this is the only place where this condition is used.

It is only left to check the second Peiffer condition: On the one hand we have, for \(x \in X\),
\[
((\xi \otimes e) \triangleright (\zeta \otimes d))(x) = ((f^*\xi \otimes \partial(e)) \triangleright (\zeta \otimes d))(x) \\
= \xi(f(1))((\partial(e) \triangleright \zeta) \otimes \partial(e) \triangleright d)(x) \\
= \xi(1)\zeta((\partial(e)^{-1} \triangleright x)\partial(e) \triangleright d) \\
\stackrel{(\ast)}{=} \xi(1)\zeta(x)\partial(e) \triangleright d,
\]
where in \((\ast)\) we use the assumption that \(\text{im}(\partial)\) acts trivially on \(X\). On the other hand we have
\[
((\xi(1) \otimes e)(\zeta \otimes d)S(\xi(2) \otimes e))(x) = ((\xi(1) \otimes e)(\zeta \otimes d)(1 \otimes e^{-1})(S(\xi(2)) \otimes 1))(x)
\]
\[= (\xi(1) \zeta S(\xi(2)))(x) e \Delta e^{-1} \]
\[= (\xi(1) \zeta S(\xi(2)))(x) \partial(e) \triangleright d \]
\[= \xi(x x^{-1}) \zeta(x) \partial(e) \triangleright d \]
\[= \xi(1) \zeta(x) \partial(e) \triangleright d. \]

\[\boxed{\text{Remark 14.} \ A \text{ variant of the construction, where one does not assume that } \text{im} \partial \subseteq G \text{ acts trivially on } X, \text{ would yield a pre-crossed module of Hopf algebras \cite{26, 27}. (This is defined like a crossed module, but omitting the second Peiffer condition.) For this, one has to replace the tensor product } \mathcal{F}_C(X) \otimes \mathbb{C}E \text{ by a crossed product } \mathcal{F}_C(X) \rtimes \mathbb{C}E, \text{ defined using the action of } E \text{ on } X \text{ via } \partial. \]

We furthermore note that the condition that \text{im} \partial \subseteq G \text{ acts trivially on } Y \text{ is only used for the first Peiffer condition.}

**Lemma 15.** The Hopf algebras \( \mathcal{F}_C(X) \otimes \mathbb{C}E \) and \( \mathcal{F}_C(Y) \rtimes \mathbb{C}G \) are semisimple with Haar integrals

\[\delta_{1_X} \otimes \left( \frac{1}{|E|} \sum_{e \in E} e \right) \in \mathcal{F}_C(X) \otimes \mathbb{C}E\]

and

\[\delta_{1_Y} \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right) \in \mathcal{F}_C(Y) \rtimes \mathbb{C}G.\]

**Proof.** We verify the defining properties of the Haar integral in Definition 1. For \( \mathcal{F}_C(X) \otimes \mathbb{C}E \) this is straightforward, since it is just a tensor product of two Hopf algebras and \( \delta_{1_X} \in \mathcal{F}_C(X) \) and \( \frac{1}{|E|} \sum_{e \in E} e \) are the Haar integrals of the two Hopf algebras.

To show the idempotence we calculate:

\[\left( \delta_{1_Y} \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right) \right) \cdot \left( \delta_{1_Y} \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right) \right) = \frac{1}{|G|^2} \sum_{g, h \in G} \delta_{1_Y}(g \triangleright \delta_{1_Y}) \otimes gh\]

\[= \delta_{1_Y} \delta_{1_Y} \otimes \frac{1}{|G|^2} \sum_{g, h \in G} gh\]

\[= \delta_{1_Y} \frac{1}{|G|^2} \sum_{g, h \in G} g\]

\[= \delta_{1_Y} \frac{1}{|G|} \sum_{g \in G} g.\]

Let us now show the remaining left and right invariance properties of the Haar integral. Let \( y \in Y \) and \( h \in G \). Then we have, on the one hand,

\[(\delta_y \otimes h) \cdot \left( \delta_{1_Y} \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right) \right) = \delta_y(h \triangleright \delta_{1_Y}) \otimes \left( \frac{1}{|G|} \sum_{g \in G} hg \right)\]

\[= \delta_y \delta_{1_Y} \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right)\]

\[= \delta_y(1_Y) \delta_{1_Y} \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right)\]

\[= \varepsilon(\delta_y \otimes h) \left( \delta_{1_Y} \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right) \right),\]
and on the other hand,

\[
\left( \delta_1 \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right) \right) \cdot \left( \delta_y \otimes h \right) = \frac{1}{|G|} \sum_{g \in G} \delta_1 \left( g \triangleright \delta_y \right) \otimes gh
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \delta_1 \delta_y (gh^{-1}g) \otimes g
\]

\[
= \delta_y (1_Y) \delta_1 \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right)
\]

\[
= \varepsilon (\delta_y \otimes h) \left( \delta_1 \otimes \left( \frac{1}{|G|} \sum_{g \in G} g \right) \right).
\]

In particular, by Proposition 2 the two Hopf algebras are semisimple. \qed

**Remark 16.** For the commutativity of vertex projectors and edge projectors in the commuting-projector Hamiltonian model defined in Section 3, the relation

\[
h \triangleright \lambda = \varepsilon (h) \lambda
\]

for any \( h \in H \) and the Haar integral \( \lambda \in A \), where \( (A \xrightarrow{\triangleright} H, \triangleright) \) is a crossed module of Hopf algebras, proven in Lemma 11 is crucial.

Here we provide an illustrative explicit calculation of the relation for the class of Hopf crossed modules defined in this section: Let \( y \in Y \) and \( g \in G \).

\[
(\delta_y \otimes g) \triangleright \left( \delta_1 \otimes \left( \frac{1}{|E|} \sum_{e \in E} e \right) \right) = \delta_y (1_Y) (g \triangleright \delta_1) \otimes \left( \frac{1}{|E|} \sum_{e \in E} g \triangleright e \right)
\]

\[
= \delta_y (1_Y) \delta_1 \otimes \left( \frac{1}{|E|} \sum_{e \in E} e \right)
\]

\[
= \varepsilon (\delta_y \otimes g) \left( \delta_1 \otimes \left( \frac{1}{|E|} \sum_{e \in E} e \right) \right).
\]

### 3 Hopf-algebraic higher Kitaev model:

**Commuting-projector Hamiltonian model for oriented surfaces with cell decomposition from a crossed module of semisimple Hopf algebras**

In [17] a 2-group commuting-projector Hamiltonian model for 2+1D and 3+1D topological phases was defined. The model, henceforth called higher Kitaev model, takes as input a 2-group, which can be represented by a crossed module \((E \xrightarrow{\delta} G, \triangleright)\) of groups Definition 9; see [19][23, §2.5 & 2.7]. Given a manifold \( M \), with a triangulation (or more generally a 2-lattice decomposition [7]) \( L \), the total Hilbert space of the higher Kitaev model is the free vector space on the set of all fake-flat 2-gauge configurations in \((M, L)\) [7, §3.2.1]. A variant of the higher Kitaev model defined in [17, Equation (35)] has as the total Hilbert space the free vector space on the set of all 2-gauge configurations in \((M, L)\). We will henceforth call the 2+1D version of the latter model the full higher Kitaev model. It was proven in [17, Page 8] that the full higher Kitaev model coincides with the Kitaev quantum double model [1] derived from \( G \) when \( E = \{1\} \).

Crossed modules of Hopf algebras were defined in [26], and there related to a special case of strict quantum 2-groups; see also [27, 29, 28]. In this section, we define a Hopf-algebraic generalization of the 2+1D full higher Kitaev model defined on a surface \( \Sigma \) with a triangulation \( L \), or something slightly more general which we define below, and simply call cell decomposition.
The other input datum is a crossed module \((A \xrightarrow{\partial} H, \triangleright)\) of finite-dimensional semisimple Hopf algebras \(A\) and \(H\), see Definition 3. To the model we construct in the section we will give the name: Hopf-algebraic higher Kitaev model.

3.1 Cell decompositions and the total state space of the model

Let \(\Sigma\) be a compact oriented surface.

Definition 17 (Cell decomposition of a surface). By a cell decomposition \(L\) of \(\Sigma\) we mean the following. We have an embedded graph \(L\) in \(\Sigma\) with no loops and no univalent vertices, with finite sets \(L^0\) and \(L^1\) of vertices and edges, such that its complement in \(\Sigma\) is a disjoint union of open 2-disks, forming the set \(L^2\) of plaquettes. Furthermore, each edge is oriented and each plaquette \(P\) has a distinguished vertex \(v_P\) in its boundary, called its base-point.

The choice of a base-point in each plaquette is essential to establish conventions for edge operators and plaquette operators, and was also considered in \([17, 7]\).

Remark 18. In particular, \(L\) is in this way a regular CW decomposition of \(\Sigma\), which additionally has that the attaching maps of the 2-cells are cellular, as the 2-lattices defined in \([7]\).

Remark 19. The cell decompositions considered here in particular include triangulations of surfaces, which in this paper we regard to be equipped with an orientation of each edge and a choice of a vertex (base-point) for each triangle, unless explicitly stated otherwise. Here triangulations as usual come with the restriction that two simplices can only intersect along a common face.

Let \((A \xrightarrow{\partial} H, \triangleright)\) be a crossed module of finite-dimensional semisimple Hopf algebras. We start by defining the total state space of the Hopf-algebraic higher Kitaev model:

\[
\mathcal{H}_L := H^\otimes L^1 \otimes A^\otimes L^2 := \bigotimes_{e \in L^1} H \otimes \bigotimes_{P \in L^2} A.
\] (6)

Thus for example if \((\Sigma, L)\) is the minimal cell decomposition of the sphere and the Hopf crossed module is \((\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2)\) (in \((E, G, X, Y)\) notation, cf. Prop \([13]\) then the Hilbert space is 11943936-dimensional.

On the vector space \(\mathcal{H}_L\) we will define three families of linear endomorphisms: vertex operators, edge operators and plaquette operators.

By the support of an endomorphism \(T\) of \(\mathcal{H}_L = H^\otimes L^1 \otimes A^\otimes L^2\) we mean the largest subset \(M = M^1 \cup M^2 \subseteq L^1 \sqcup L^2\) such that there exists an endomorphism \(T_S\) of \(\mathcal{H}_M := H^\otimes M^1 \otimes A^\otimes M^2\) such that \(T = T_M \otimes \text{id}_{\mathcal{H}_L \setminus \mathcal{H}_M}\).

We will occasionally write \(H_e\) or \(A_P\), where \(e \in L^1\) and \(P \in L^2\), for copies of the Hopf algebras \(H\) and \(A\) seen as the tensor factors in \(\mathcal{H}_L\) associated with the edge \(e\) or the plaquette \(P\), respectively.

Remark 20. As in \([13]\), one could consider \(C^*\)-Hopf algebras \(H\) and \(A\) and this would turn \(\mathcal{H}_L\) into a Hilbert space and make the projectors, and hence the Hamiltonian, Hermitian.

3.2 Vertex operators

Fix a pair \((\Sigma, L)\), as in Definition 17. Let \(v \in L^0\) be a vertex and let \(P \in L^2\) be an adjacent plaquette. Such a pair \((v, P)\) is also called a site \([1, 3]\) Definition 2.1 \([13]\) §3.1. Note that \(v\) is not necessarily the base-point of \(P\).
We define now a family of vertex operators $V_{v,P}^h, h \in H$ acting on $\mathcal{H}_L$, whose support is the set of the edges and plaquettes incident to the vertex $v$. Each adjacent plaquette $P$ gives as an auxiliary input information fixing the convention for these operators \cite[Definition 2.2]{7}. The definition of the vertex operators in our model follows that of the vertex operators in the Hopf-algebraic Kitaev model \cite{13}, and reduces to the latter when $A = \mathbb{C}$ (up to minor differences in conventions). At the same time, the vertex operators here recover the vertex gauge spikes in the full higher Kitaev model defined in \cite{7} for a Hopf crossed module $(A \xrightarrow{\partial} H, \triangleright) = (CE \xrightarrow{\partial} CG, \triangleright)$ induced by a crossed module of groups $(E \xrightarrow{\partial} G, \triangleright)$.

Consider the set of edges incident to (that is, ending or starting at) the vertex $v$. The orientation of the surface $\Sigma$ induces a cyclic order on this set, by going around the vertex in the counter-clockwise order with respect to the orientation of $\Sigma$. The choice of plaquette $P$ incident to $v$ furthermore lifts this cyclic order to a linear one by starting with the edge that comes right after $P$ in the counter-clockwise order around $v$. We write the edges in this specified linear order as $(e_1, \ldots, e_n)$. These edges might be oriented away from or towards $v$. Define $\theta_i := +1$ in the former case and $\theta_i := -1$ in the latter, for $i = 1, \ldots, n$. There are two canonical left actions of $H$ on itself: By left multiplication, $h \otimes x \mapsto hx$ for any $h, x \in H$, on the one hand, and by right multiplication pulled back along the antipode, $h \otimes x \mapsto xS(h)$, on the other hand; see \cite{13} §3.1. In order to treat both cases at once, we use the notation

$$h^{(\theta)} = \begin{cases} h^{(+1)} := h, & \text{if } \theta = +1, \\ h^{(-1)} := S(h), & \text{if } \theta = -1, \end{cases}$$

which is justified, since $S$, being involutive (Proposition 2), defines an action of the group $\{+1, -1\} \cong \mathbb{Z}_2$. In some cases this notation will be combined with the Sweedler notation, where it is understood that for instance $h^{(+1)} \otimes h^{(-1)}$ means $h^{(1)} \otimes S(h^{(2)})$, and $h^{(-1)} \otimes h^{(-1)}$ means $S(h^{(1)}) \otimes S(h^{(2)})$, so we always apply comultiplication before applying $S$.

Furthermore, consider the (possibly empty) set of all plaquettes $P \in L^2$ whose base-point is $v$ and choose any order on this set, say $(P_1, \ldots, P_k)$. The definition of the vertex operator will be independent of this choice. Then we define (note that our convention is opposite to the one in \cite{13}):

**Definition 21 (Vertex operator).** For any $h \in H$, the vertex operator $V_{v,P}^h$ on $\mathcal{H}_L$ based at the site $(v, P)$ is

$$V_{v,P}^h : \mathcal{H}_L \rightarrow \mathcal{H}_L, \quad v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k} \mapsto (h^{(1)} v_{e_1}^{(\theta_1)})^{(\theta_1)} \otimes \cdots \otimes (h^{(n)} v_{e_n}^{(\theta_n)})^{(\theta_n)} \otimes h^{(n+1)} \triangleright X_{P_1} \otimes \cdots \otimes h^{(n+k)} \triangleright X_{P_k},$$

where $v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k} \in H_{e_1} \otimes \cdots \otimes H_{e_n} \otimes A_{P_1} \otimes \cdots \otimes A_{P_k} = H^\otimes \otimes A^\otimes k$ are in the tensor factors of $\mathcal{H}_L$ associated with the edges $(e_1, \ldots, e_n)$ and plaquettes $(P_1, \ldots, P_k)$. It is implicitly understood that $V_{v,P}^h$ acts as the identity on all remaining tensor factors.

**Remark 22.** The definition of the vertex operator $V_{v,P}^h$ is independent of the order chosen on the set $\{P_1, \ldots, P_k\}$ because of the Yetter-Drinfeld condition \cite{4}, namely $h^{(1)} \otimes h^{(2)} \triangleright X = h^{(2)} \otimes h^{(1)} \triangleright X$ for any $h \in H$ and $X \in A$, satisfied by a Hopf crossed module $(A \xrightarrow{\partial} H, \triangleright)$, see Remark 8. The vertex operator does however in general depend on the order on the set $\{e_1, \ldots, e_n\}$, but this order is naturally given in terms of the orientation of the surface $\Sigma$ and the choice of plaquette $P$, as explained above.

**Lemma 23.** Given any vertex $v \in L^0$ and adjacent plaquette $P \in L^2$, the endomorphisms $(V_{v,P}^h : \mathcal{H}_L \rightarrow \mathcal{H}_L)_{h \in H}$ define a representation of $H$ on $\mathcal{H}_L$. 

19
Proof. This follows from the fact that $\Delta^{(n+k-1)} : H \to H^{\otimes (n+k)}$ is an algebra map. See Lemma \textbf{72} in the Appendix for a detailed proof. \qed

Owing to the fact that a semisimple Hopf algebra $H$ comes with a distinguished idempotent, the Haar integral $\ell \in H$, see Proposition \textbf{2} we can define the following projectors, that is idempotent endomorphisms, on $\mathcal{H}_L$, which will enter the definition of the Hamiltonian of the exactly solvable model. This is completely analogous to the construction of the Kitaev model for a semisimple Hopf algebra \textbf{13.4}.

(Cf. \textbf{[3] §2.4} for the similar case of vertex projectors on the Hopf-algebraic Kitaev model.) Note that the Haar integral $\ell \in H$ is cocommutative. By combining (\textbf{3}) with Remark \textbf{8} it follows that the operator $V^{\ell}_{v,P}$ depends only on the cyclic order of the set $(e_1,\ldots,e_n)$ of edges incident to the vertex $v$, and is hence independent of the choice of adjacent plaquette $P$.

**Definition 24** (Vertex projector). Let $\ell \in H$ be the Haar integral of $H$. Then we define the vertex projector $V_{v} : \mathcal{H}_L \longrightarrow \mathcal{H}_L$ based at the vertex $v$ as

$$V_v := V^{\ell}_{(v,P)}.$$  

(Note that a cyclic order on the set $(e_1,\ldots,e_n)$ is given by the orientation of the surface.)

### 3.3 Edge operators

Let $e \in L^1$ be an edge. With respect to the orientations of the edge $e$ and the surface $\Sigma$, we can define the plaquette $P \in L^2$ on the left side of the edge $e$ and the plaquette $Q \in L^2$ on the right side of $e$, as we go from the starting vertex to the target vertex of $e$.

We define now a family of edge operators acting on $\mathcal{H}_L$, whose support consists of the plaquettes $P$ and $Q$ and the set of the edges in the boundary of $P$ and $Q$. These operators do not appear in the Hopf-algebraic Kitaev model. For a Hopf crossed module $(A \overset{\partial}{\to} H, \triangleright) = (CE \overset{\partial}{\to} CG, \triangleright)$ induced by a crossed module of groups $(E \overset{\partial}{\to} G, \triangleright)$ they recover the edge gauge transformations in the full higher Kitaev model \textbf{17} and in its fake-flat subspace the gauge spikes in the higher Kitaev model defined in \textbf{7}.

Denote by $v_P,v_Q \in L^0$ the base-points of $P$ and $Q$, respectively, and let $v \in L^0$ be the starting vertex of the edge $e$, i.e. the vertex from which $e$ points away. Let $(e_1,\ldots,e_\ell)$ be the (possibly empty) ordered set of edges connecting $v_P$ with $v$ in the boundary of $P$, starting from $v_P$ and going in counter-clockwise direction (with respect to the orientation of $\Sigma$) around $P$. (Another way to see this sequence of edges is as the unique path from $v_P$ to $v$ that does not pass through $e$.) The set is empty precisely when $v_P = v$. For $i = 1,\ldots,\ell$, let $\theta_i := +1$ if $e_i$ is oriented in counter-clockwise direction around $P$, and let $\theta_i := -1$ otherwise.

Similarly, let $(d_1,\ldots,d_r)$ be the (possibly empty) ordered set of edges connecting $v_Q$ with $v$ in the boundary of $Q$, starting from $v_Q$ and going in clockwise direction (with respect to the orientation of $\Sigma$) around $Q$. The set is empty precisely when $v_Q = v$. For $j = 1,\ldots,r$, let $\sigma_j := +1$ if $d_j$ is oriented in clockwise direction around $Q$, and let $\sigma_j := -1$ otherwise.

**Definition 25** (Edge operator). For any $a \in A$, the edge operator $E^a_{e}$ on $\mathcal{H}_L$ based at the edge $e$ is

$$E^a_{e} : \mathcal{H}_L \longrightarrow \mathcal{H}_L,$$

$$v_e \otimes v_{e_1} \otimes \cdots \otimes v_{e_{\ell}} \otimes v_{d_1} \otimes \cdots \otimes v_{d_r} \otimes X_P \otimes X_Q \longrightarrow \partial a_{(3)} v_e$$

$$\otimes (v_{e_1})_{(1)} \otimes \cdots \otimes (v_{e_{\ell}})_{(1)}$$

$$\otimes (v_{d_1})_{(2)} \otimes \cdots \otimes (v_{d_r})_{(2)}$$

$$\otimes \left((v_{e_1})_{(2)} \cdots (v_{e_{\ell}})_{(2)} \triangleright a_{(1)}\right) X_P$$
where we use the notation $v^{(+1)} := v$ and $v^{(-1)} := S(v)$ for any $v \in H$, and where

$$
v_{e} \otimes v_{e_1} \otimes \cdots \otimes v_{e_\ell} \otimes v_{d_1} \otimes \cdots \otimes v_{d_r} \otimes X_P \otimes X_Q
\in H \otimes H_{e_1} \otimes \cdots \otimes H_{e_\ell} \otimes H_{d_1} \otimes \cdots \otimes H_{d_r} \otimes A_P \otimes A_Q
= H^{\otimes 1+\ell+r} \otimes A^{\otimes 2}
$$

are in the tensor factors of $H_L$ associated with the edges $e, (e_1, \ldots, e_\ell)$ and $(d_1, \ldots, d_r)$, and plaquettes $P$ and $Q$, respectively. Here for instance the term $v_{e_i} \otimes (v_{e_i})^{(\theta_i)}_{(2)}$ means $v_{e_i} \otimes (v_{e_i})_{(2)}$ if $\theta_i = 1$ and $v_{e_i} \otimes S((v_{e_i})_{(2)})$ if $\theta_i = -1$.

It is implicitly understood that the edge operator is defined to act as the identity on all remaining tensor factors in $H_L$.

We also note that:

1. Due to the Yetter Drinfeld condition (4), and using that $S$ is an anti-coalgebra-map, with $S^2 = id_H$, we can substitute any term $(v_{e_i})_{(1)} \otimes (v_{e_i})^{(\theta_i)}_{(2)}$ by $(v_{e_i})_{(2)} \otimes (v_{e_i})^{(\theta_i)}_{(1)}$ in the formula for the edge operator, and the same for the $v_{d_i}$. In general this follows since if $v \in H$ and $a \in A$ then:

$$v_{(1)} \otimes v_{(2)} \triangleright a = v_{(2)} \otimes v_{(1)} \triangleright a,$$

(straight from (4)) and

$$v_{(1)} \otimes S(v_{(2)}) \triangleright a = S(S(v_{(1)})) \otimes S(v_{(2)}) \triangleright a
= S(S(v_{(2)})) \otimes S(v_{(1)}) \triangleright a
\triangleleft D.
$$

Indeed this is trivial if $\theta_i = 1$ ($\sigma_i = 1$), and follows from the Yetter-Drinfeld condition (4) and $S^2 = id_H$ if $\theta_i = -1$ ($\sigma_i = -1$). So another formula for the edge operator is:

$$E_{\iota}^e : H_L \longrightarrow H_L,$n

$$v_{e} \otimes v_{e_1} \otimes \cdots \otimes v_{e_\ell} \otimes v_{d_1} \otimes \cdots \otimes v_{d_r} \otimes X_P \otimes X_Q \longrightarrow \partial a_{(3)} v_{e}
\otimes (v_{e_1})^{(\theta_1)}_{(1)} \otimes \cdots \otimes (v_{e_\ell})^{(\theta_\ell)}_{(1)}
\otimes (v_{d_1})^{(\sigma_1)}_{(2)} \otimes \cdots \otimes (v_{d_r})^{(\sigma_r)}_{(2)}
\otimes ((v_{e_1})^{(\theta_1)}_{(2)} \cdots (v_{e_\ell})^{(\theta_\ell)}_{(2)} \triangleright a_{(1)}) X_P
\otimes X_Q ((w_1^{(\sigma_1)})_{(1)} \cdots (v_{e_r})^{(\sigma_r)}_{(1)} \triangleright S a_{(2)}).
$$

2. We could also have used a similar notation to that when we defined edge operators, and will later use when defining plaquette operators, and substitute any of the $(v_{e_i})_{(1)} \otimes (v_{e_i})^{(\theta_i)}_{(2)}$ by $(v_{e_i})^{(\theta_i)}_{(1)} \otimes (v_{e_i})^{(\theta_i)}_{(2)},$ and $(v_{d_i})_{(2)} \otimes (v_{d_i})^{(\sigma_i)}_{(1)}$ by $(v_{d_i})^{(\sigma_i)}_{(2)} \otimes (v_{d_i})^{(\sigma_i)}_{(1)}$. Indeed this is trivial if $\theta_i = 1$ ($\sigma_i = 1$), and follows from the Yetter-Drinfeld condition (4) and $S^2 = id_H$ if $\theta_i = -1$ ($\sigma_i = -1$). So another formula for the edge operator is:

$$E_{\iota}^e : H_L \longrightarrow H_L,$n

$$v_{e} \otimes v_{e_1} \otimes \cdots \otimes v_{e_\ell} \otimes v_{d_1} \otimes \cdots \otimes v_{d_r} \otimes X_P \otimes X_Q \longrightarrow \partial a_{(3)} v_{e}
\otimes (v_{e_1})^{(\theta_1)}_{(1)} \otimes \cdots \otimes (v_{e_\ell})^{(\theta_\ell)}_{(1)}
\otimes (v_{d_1})^{(\sigma_1)}_{(2)} \otimes \cdots \otimes (v_{d_r})^{(\sigma_r)}_{(2)}
\otimes ((v_{e_1})^{(\theta_1)}_{(2)} \cdots (v_{e_\ell})^{(\theta_\ell)}_{(2)} \triangleright a_{(1)}) X_P
\otimes X_Q ((w_1^{(\sigma_1)})_{(1)} \cdots (v_{e_r})^{(\sigma_r)}_{(1)} \triangleright S a_{(2)}).
$$

3. Some of the edges in the list[1] in $(d_1, \ldots, d_r)$ might be the same as some of the edges in the list $(e_1, \ldots, e_\ell)$. If an edge $f$ is in both lists it is ‘inserted’ into the edge operator as $(v_f)_{(1)} \otimes (v_f)_{(2)} \otimes (v_f)_{(3)}$ or any other permutation of $(1, 2, 3)$. Which permutation it is does not matter, due to Yetter-Drinfeld condition (4).

[1]This issue does not arise when triangulations are considered.
As in the case of vertex operators we have:

**Lemma 26.** Given an edge \( e \in L^1 \), the edge operators \( E^e_a : H_L \to H_L \) form a representation of \( A \) on \( H_L \).

**Proof.** See Lemma 74 in the Appendix.

Analogously to the vertex projectors, we can now define the edge projectors by acting with the Haar integral of the Hopf algebra \( A \). This gives an idempotent endomorphism, because the Haar integral is idempotent and the edge operators are a representation of \( A \):

**Definition 27** (Edge projector). Let \( \Lambda \in A \) be the Haar integral of \( A \). Then we define the edge projector \( E^e : H_L \to H_L \) based at the edge \( e \) as

\[
E^e := E^\Lambda_e.
\]

### 3.4 Plaquette operators

Let \( P \in L^2 \) be a plaquette and let \( v_P \in L^0 \) be its base-point.

We define a family of plaquette operators acting on \( H_L \), whose support is \( P \) and set of the edges in the boundary of \( P \). The definition of the plaquette operators in our model follows that of the plaquette operators in the Hopf-algebraic Kitaev model [13], and reduces to the latter when \( A = C \) (up to minor differences in conventions).

Denote by \((e_1, \ldots, e_n)\) the ordered set of edges in the boundary of \( P \), starting at the base-point \( v_P \) and going around the boundary in a counter-clockwise way with respect to the orientation of \( \Sigma \). For \( i = 1, \ldots, n \) let \( \theta_i = +1 \) if the edge \( e_i \) is oriented counter-clockwise along the boundary of \( P \) and let \( \theta_i = -1 \) otherwise. Cf. the very similar case of plaquette operators in the Hopf-algebraic Kitaev model in [3, Definition 2.3] and [13, Definition 1], our conventions are opposite. In loc cit only the formula for the plaquette operators when all the edge orientations match the direction in which we go around the plaquette in is presented. It is however understood [13, §3.2] that if an edge does not have the same orientation we first apply \( S \) to the corresponding edge tensor factor.

**Definition 28.** For any \( \varphi \in H^* \), the plaquette operator \( F_P^\varphi \) on \( H_L \) based at the plaquette \( P \) is

\[
F_P^\varphi : H_L \to H_L, \quad v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \mapsto \left( v_{e_1}^{(\theta_1)} \right)^{(1)} \otimes \cdots \otimes \left( v_{e_n}^{(\theta_n)} \right)^{(1)} \otimes (X_P)^{(2)} \varphi \left( \left( v_{e_1}^{(\theta_1)} \right)^{(2)} \cdots \left( v_{e_n}^{(\theta_n)} \right)^{(2)} S \partial(X_P)^{(1)} \right),
\]

where we use the notation \( v^{(+1)} := v \) and \( v^{(-1)} := S(v) \) for any \( v \in H \), and where \( v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \in H_{e_1} \otimes \cdots \otimes H_{e_n} \otimes A_P = H^\otimes_n \otimes A \) are in the tensor factors of \( H_L \) associated with the edges \((e_1, \ldots, e_n)\) and the plaquette \( P \), respectively. It is implicitly understood that the plaquette operator acts as the identity on all remaining tensor factors of \( H_L \).

**Remark 29.** Note that for concrete values of the edge orientations \( \theta_j \) the formula for the plaquette operators simplifies significantly, since the double appearance of the antipode in a term such as \( \left( v_{e_1}^{(\theta_1)} \right)^{(1)} \otimes \left( v_{e_1}^{(\theta_1)} \right)^{(2)} \) can always be cancelled out due to fact that the antipode is involutive, \( S^2 = \text{id}_H \), and an anti-coalgebra map. For instance the term \( \left( v_{e_1}^{(\theta_1)} \right)^{(1)} \otimes \left( v_{e_1}^{(\theta_1)} \right)^{(2)} \) is \( \left( v_{e_1} \right)^{(1)} \otimes \left( v_{e_1} \right)^{(2)} \), if \( \theta_1 = 1 \). Whereas if \( \theta_1 = -1 \) then, using the fact that \( S^2 = \text{id}_H \):

\[
\left( v_{e_1}^{(\theta_1)} \right)^{(1)} \otimes \left( v_{e_1}^{(\theta_1)} \right)^{(2)} = S \left( \left( S(v_{e_1}) \right)^{(1)} \otimes \left( S(v_{e_1}) \right)^{(2)} \right).
\]
\[
S(S((v_{e_1})_{(2)})) \otimes S((v_{e_1})_{(1)}) = (v_{e_1})_{(2)} \otimes S((v_{e_1})_{(1)}).
\]

The idea here is that on the tensor factors \(H_{e_1} \otimes \cdots \otimes H_{e_n}\) the plaquette operator acts via the left or right co-multiplication depending on the orientation of the edge \(e_j\) relative to the plaquette \(P\), and the formula in Definition 28 captures this in one closed form for all possible configurations of edge orientations.

**Remark 30.** A relation that we will use is, for all \(v \in H\) and \(\theta \in \{+1, -1\}\):

\[
(v^{(\theta)})_{(1)} \otimes (v^{(\theta)})_{(2)} = (S(v)^{(-\theta)})_{(1)} \otimes (S(v)^{(-\theta)})_{(2)}.
\]

Indeed if \(\theta = 1\) this follows from \(S^2 = 1\). If \(\theta = -1\) this follows since \(v^{(-1)} = S(v)\).

**Lemma 31.** Given a plaquette \(P \in L^2\), the plaquette operators \((F^x_P)_{\varphi \in H^*}\) form a representation of \(H^*\) on \(H_L\).

**Proof.** See Lemma 76 in the Appendix.

Since \(H\) is semisimple, then so is \(H^*\) [9, Proposition 1.3], as we are working only with finite dimensional Hopf algebras over \(\mathbb{C}\). Hence, we again obtain idempotent endomorphisms of \(H_L\) by acting with the Haar integral of \(H^*\) via the plaquette operators:

**Definition 32.** Let \(\lambda \in H^*\) be the Haar integral of \(H^*\). Then we define the plaquette projector \(F_P : H_L \rightarrow H_L\) based at the plaquette \(P\) as

\[
F_P := F^\lambda_P.
\]

For the case of a crossed module of Hopf algebras induced by a crossed module of groups \((E \xrightarrow{\partial} G, \triangleright)\), the plaquette projectors recover the plaquette projectors in [17, Equation (20)]. Those latter plaquette projectors choose the 2-gauge configurations that are fake flat around a plaquette. Plaquette projectors do not arise in the higher Kitaev model [7], due to the fake-flatness being imposed in the 2-gauge configurations.

### 3.5 Edge orientation reversal and plaquette base-point shift

The total state space \(H_L\) of the Hopf-algebraic higher Kitaev model defined in this section does not depend on the edge orientations or plaquette base-points of the cell decomposition \(L\), but the vertex, edge and plaquette operators defined on it do.

Now we define linear maps on \(H_L\) which represent the reversal of the orientation of an edge and the moving around of the base-point of a plaquette. By showing that these maps commute with vertex, edge and plaquette operators in a suitable way, our calculations in the Appendix in Subsection 5.2 showing the commutation relations of the latter will simplify significantly, because it will then suffice to carry out the computations for only the most convenient configurations of edge orientations and base-points.

The edge orientation reversals and plaquette base-point shifts can also be of independent interest, towards showing the independence of the ground-state space of the model of the cell decomposition, a problem which we leave for future work.

**Definition 33.** Consider the total state space \(H_L = H^{\otimes L^1} \otimes A^{\otimes L^2}\) of the Hopf-algebraic higher Kitaev model.

For any edge \(e \in L^1\), we define the edge orientation reversal

\[
R_e : H_L \rightarrow H_L
\]

to act as the antipode \(S\) of \(H\) on the tensor factor in \(H_L\) associated with the edge \(e\) and as the identity on all remaining tensor factors.
Due to the involutivity of the antipode, the edge orientation reversal is an isomorphism and, more specifically, $R^2_e = \text{id}_{H_L}$.

**Definition 34.** Consider the total state space $\mathcal{H}_L = H^{\otimes L^1} \otimes A^{\otimes L^2}$ of the Hopf-algebraic higher Kitaev model.

For any plaquette $P \in L^2$, we define the (positive) base-point shift

$$T^+_P : \mathcal{H}_L \to \mathcal{H}_L$$

as follows. Let $A$ be the tensor factor in $\mathcal{H}_L$ associated with the plaquette $P$ and let $H$ be the tensor factor in $\mathcal{H}_L$ associated with the edge in the boundary of $P$ incident to the base-point of $P$ in counter-clockwise direction around $P$. Then define $T^+_P$ to act as

$$H \otimes A \ni v \otimes X \mapsto v(1) \otimes v^{(-\theta)}(2) \triangleright X,$$

where $\theta \in \{+1, -1\}$ is $+1$ if the edge is oriented counter-clockwise around $P$ and otherwise $-1$, and to act as the identity on all remaining tensor factors in $\mathcal{H}_L$.

Analogously, we define the (negative) base-point shift

$$T^-_P : \mathcal{H}_L \to \mathcal{H}_L$$

as follows. Let $A$ be the tensor factor in $\mathcal{H}_L$ associated with the plaquette $P$ and let $H$ be the tensor factor in $\mathcal{H}_L$ associated with the edge in the boundary of $P$ incident to the base-point of $P$ in clockwise direction around $P$. Then define $T^-_P$ to act as

$$H \otimes A \ni v \otimes X \mapsto v(1) \otimes v^{(\sigma)}(2) \triangleright X,$$

where $\sigma \in \{+1, -1\}$ is $+1$ if the edge is oriented counter-clockwise around $P$ and otherwise $-1$, and to act as the identity on all remaining tensor factors in $\mathcal{H}_L$.

**Remark 35.** Observe that the base-point shifts are isomorphisms and that, more specifically, if $P'$ has the same underlying plaquette as $P$ but its base-point is shifted once in counter-clockwise direction around $P$, then $T^-_{P'} = (T^+_P)^{-1}$.

**Remark 36.** Note that moving the base-point of a plaquette $P$ all the way around its boundary, coming back to the initial base-point, does not in general induce the identity on $\mathcal{H}_L$. This operation however restricts to the identity on a certain subspace $\mathcal{H}^f_L$ (the fake-flat subspace) of $\mathcal{H}_L$, as shown in Lemma 71 in the Appendix.

In the Appendix in Subsection 5.1 we prove:

**Proposition 37.** Let $\Sigma$ be an oriented surface with cell decomposition $L$.

- Let $P \in L^2$ be a plaquette and let $e \in L^1$ be an edge. Denote by $\tilde{T}^+_P$ the base-point shift with respect to the cell decomposition $\tilde{L}$ obtained from $L$ by reversing the orientation of $e$. Then
  $$R_e \circ T^+_P = \tilde{T}^+_P \circ R_e.$$

- Let $e \in L^1$ be an edge and let $v \in L^0$ be a vertex with an adjacent plaquette $P \in L^2$. Denote by $\tilde{V}^h_{v,P}$ for any $h \in H$ the vertex operator at $v$ with respect to the cell decomposition $\tilde{L}$ obtained from $L$ by reversing the orientation of the edge $e$. Then:
  $$R_e \circ V^h_{v,P} = \tilde{V}^h_{v,P} \circ R_e \quad \text{for all } h \in H.$$
• Let $e$ and $f \in L^1$ be two distinct edges. Denote by $\tilde{E}_e^a$ for any $a \in A$ the edge operator at $e$ with respect to the cell decomposition $\tilde{L}$ obtained from the given one $L$ by reversing the orientation of the edge $f$. Then

$$R_f \circ E_e^a = \tilde{E}_e^a \circ R_f \quad \text{for all } a \in A.$$  

• Let $e \in L^1$ be an edge and let $P \in L^2$ be a plaquette. Denote by $\tilde{F}_P^\varphi$ for any $\varphi \in H^*$ the plaquette operator at $P$ with respect to the cell decomposition $\tilde{L}$ obtained from $L$ by reversing the orientation of the edge $e$. Then:

$$R_e \circ F_P^\varphi = \tilde{F}_P^\varphi \circ R_e \quad \text{for all } \varphi \in H^*.$$  

• Let $P \in L^2$ be any plaquette and let $v \in L^0$ be any vertex with adjacent plaquette $Q$. Denote by $(\tilde{V})^h_{v,Q}$ for any $h \in H$ the vertex operator for the vertex $v$ with respect to the cell decomposition $\tilde{L}$ obtained from the given one $L$ by shifting the base-point of $P$ once in counter-clockwise direction around $P$. Then:

$$(\tilde{V})^h_{v,Q} \circ T_P^+ = T_P^+ \circ (\tilde{V})^h_{v,Q} \quad \text{for all } h \in H.$$  

• Let $e \in L^1$ be an edge and let $P \in L^2$ be a plaquette adjacent to the edge $e$. We further require that the base-point of $P$ is not equal to the starting vertex of $e$ if $e$ is oriented counter-clockwise around $P$, and not equal to the target vertex of $e$ if $e$ is oriented clockwise around $P$. Then

$$T_P^+ \circ E_e^a = E_e^a \circ T_P^+ \quad \text{for all } a \in A,$$

where $E_e^a$ denotes the edge operator of the edge $e$ for the cell decomposition $L'$ obtained from the given one $L$ by shifting the base-point of $P$ once in counter-clockwise direction.

• Let $P$ and $Q$ be any two (not necessarily distinct) plaquettes. Denote by $(\tilde{F})^\varphi_P$ for any $\varphi \in H^*$ the plaquette operator for the plaquette $P$ with respect to the cell decomposition $L'$ obtained from $L$ by shifting the base-point of $Q$ once in counter-clockwise direction around $Q$. Then:

$$(\tilde{F})^\varphi_P \circ T_Q^+ = T_Q^+ \circ (\tilde{F})^\varphi_P \quad \text{for all cocommutative } \varphi \in H^*.$$  

In particular, since the Haar integral $\lambda \in H^*$ is cocommutative (Proposition 2) we have:

$$(\tilde{F})^\lambda_P \circ T_Q^+ = T_Q^+ \circ (\tilde{F})^\lambda_P.$$  

So base-point shifts commute with plaquette projectors.

### 3.6 Commutation relations between vertex, edge and plaquette operators

Having defined all operators, we can now state some of their commutation relations, which we prove in the Appendix. The following generalises [3, Theorem 2.4].

**Proposition 38.** Let $\Sigma$ be a compact oriented surface with cell decomposition $L$ and let $(A \xrightarrow{\partial} H)$ be a crossed module of finite-dimensional semisimple Hopf algebras.

Let $P \in L^2$ be a plaquette with base-point $v \in L^0$ and let $e \in L^1$ be an edge in the boundary of $P$, oriented counter-clockwise around $P$, with starting vertex $v$. Then the associated vertex, edge and plaquette operators enjoy the following relations:

$$E_e^{(1)a} \circ V^{(2)}_{e,P} = V^{(1)}_{e,P} \circ E_e^a, \quad \text{for all } h \in H, a \in A,$$

where $V^{(1)}_{e,P}$ denotes the vertex operator for the vertex $e$ with respect to the cell decomposition $L'$ obtained from the given one $L$ by shifting the base-point of $P$ once in counter-clockwise direction.

25
Let $\varphi(Sh_{(3)} \cdot h_{(1)}) : H \rightarrow \mathbb{C}$ be such that if $x \in H$ then $x \mapsto \varphi(Sh_{(3)} \cdot x \cdot h_{(1)})$.

**Proof.** See Lemmas 78, 80 and 81 in the Appendix.

NB: If in the proposition above, $e$ is oriented clockwise then the middle relation becomes:

$$[E^a_e, F^\varphi_p] = 0 \quad \text{for all } a \in A, \varphi \in H^*.$$  

As a consequence of the fact that the vertex, edge and plaquette operators define on $H_L$ representations of the algebras $H$, $A$ and $H^*$, respectively, and that they satisfy the commutation relations in Proposition 38, we obtain in total a representation of the algebra which has the underlying vector space of $H^* \otimes (A \rtimes H)$ with the multiplication defined by

$$(\varphi \otimes (a \otimes h)) \cdot (\psi \otimes (b \otimes k)) := \varphi \psi (S \delta a_{(3)} Sh_{(3)} \cdot h_{(1)} \partial a_{(1)}) \otimes (a_{(2)} \otimes h_{(2)})$$

$\varphi, \psi \in H^*, a, b \in A, h, k \in H$.

In the case that $A = \mathbb{C}$ is the trivial Hopf algebra, this algebra is the Drinfeld double $D(H)$ and we therefore recover the well-known fact that in the Kitaev model for a semisimple Hopf algebra $H$ the local operators form a representation of the Drinfeld double $D(H)$ [13, 3].

### 3.7 Commuting-projector Hamiltonian and its ground-state space

**Theorem 39.** Let $\Sigma$ be a compact oriented surface with cell decomposition $L$, and let $(A \xrightarrow{\partial} H)$ be a crossed module of finite-dimensional semisimple Hopf algebras. Then the following operators are (well-defined) pairwise commuting projectors:

$$V_v := V^\ell_{v,p}, \quad \text{for all } v \in L^0,$$
$$E_e := E^\Lambda_e, \quad \text{for all } e \in L^1,$$
$$F_P := F^\Lambda_P, \quad \text{for all } P \in L^2,$$

where $\ell \in H$, $\Lambda \in A$, $\lambda \in H^*$ are the Haar integrals of the respective Hopf algebras, all semisimple.

As a consequence, the following operator (the Hamiltonian) is a diagonalizable endomorphism of $H_L$:

$$h := \sum_{v \in L^0} (1 - V_v) + \sum_{e \in L^1} (1 - E_e) + \sum_{P \in L^2} (1 - F_P), \quad (8)$$

whose eigenspace for its lowest eigenvalue 0 (the ground-state space) is

$$\ker h = \text{im} \left( \prod_{v \in L^0} V_v \prod_{e \in L^1} E_e \prod_{P \in L^2} F_P \right) = \left\{ x \in H_L \mid \begin{array}{l}
V_v(x) = x \text{ for all } v \in L^0, \\
E_e(x) = x \text{ for all } e \in L^1, \\
F_P(x) = x \text{ for all } P \in L^2
\end{array} \right\}.$$  

**Proof.** The commutativity of the projectors is proven in the Appendix in Lemmas 78, 79, 81 and Corollary 79. The commutativity relations are proven more generally for $V^\ell_{v,p}, E^\Lambda_e$ and $F^\Lambda_P$, for all cocommutative elements $a \in H$, $b \in A$ and $\psi \in H^*$. In particular (by Proposition 2) the commutativity relations hold for $V^\ell_{v,p}, E^\Lambda_e$ and $F^\Lambda_P$.

Mutually commuting projectors are simultaneously diagonalizable and, hence, the map $h$ is diagonalizable with eigenvalues being all the non-negative integers up to $|L^0| + |L^1| + |L^2|$. The ground-state space is therefore the eigenspace for eigenvalue 0, i.e. the kernel, which is the space on which all the projectors occurring in the Hamiltonian act by 0. 

26
Proposition 40. The ground-state space \( \ker h \) can equivalently be described as follows. A vector
\[
x = \bigotimes_{v \in L^1} v_e \otimes \bigotimes_{P \in L^2} X_P \in H^{\otimes L^1} \otimes A^{\otimes L^2},
\]
which by abuse of notation we consider to be a sum of pure tensors while omitting the summation symbol, is a ground state if and only if the following conditions are satisfied:

- For every \( P \in L^2 \):
  \[
y \otimes \left( (v_{e_1}^{(\theta_1)})(1) \right)^{(\theta_1)} \otimes \cdots \otimes \left( (v_{e_n}^{(\theta_n)})(1) \right)^{(\theta_n)} \otimes (X_P)(2) \otimes \left( (v_{e_1}^{(\theta_1)})(2) \cdots (v_{e_n}^{(\theta_n)})(2) \right) S \partial (X_P)(1) \right) = y \otimes v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes 1_H,
\]
where \( (e_1, \ldots, e_n) \) are the edges in the boundary of \( P \) in counter-clockwise order starting at the base-point of \( P \), and where \( \theta_j = +1 \) if the edge \( e_j \) is oriented counter-clockwise around \( P \) and \( \theta_j = -1 \) otherwise, for \( j = 1, \ldots, n \).

- For every \( e \in L^1 \):
  \[
  E_e^a(x) = \varepsilon(a)x \quad \text{for all } a \in A.
  \]

- For every \( v \in L^0 \):
  \[
  V_v^h(x) = \varepsilon(h)x \quad \text{for all } h \in H \text{ (for any choice of } P \in L^2 \text{ adjacent to } v).\]

Proof. This follows from the representation-theoretic properties of the Haar integral; see e.g. [3, Corollary 1.4][6, Lemma B.9]. For any module \( M \) over a finite-dimensional semisimple Hopf algebra \( H' \), the Haar integral \( \ell' \in H' \) projects onto the subspace of \( H' \)-invariants
\[
\ell'.M = M^{H'} := \{ m \in M \mid h.m = \varepsilon(h)m \text{ for all } h \in H' \}.
\]
(This follows from the Haar integral \( \ell' \) is idempotent and \( x\ell' = \ell'x = \varepsilon(x)\ell' \).)

Now apply this respectively to \( H^* \), \( A \) and \( H \), and their actions on \( \mathcal{H}_L \) given respectively by plaquette operators, edge operators and vertex operators.

Note that the formula for the elements invariant under the plaquette operators \( F_P^\psi \) follows from the fact that an element \( x \in \mathcal{H}_L \) lies in the space of \( H^* \)-invariants under the representation \( F_P^\psi \) of \( H^* \) if and only if for any \( \psi \in H^* \):
\[
y \otimes \left( (v_{e_1}^{(\theta_1)})(1) \right)^{(\theta_1)} \otimes \cdots \otimes \left( (v_{e_n}^{(\theta_n)})(1) \right)^{(\theta_n)} \otimes (X_P)(2) \psi \left( (v_{e_1}^{(\theta_1)})(2) \cdots (v_{e_n}^{(\theta_n)})(2) \right) S \partial (X_P)(1) \right) = y \otimes v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \psi(1_H).
\]

4 Some particular cases of the Hopf-algebraic higher Kitaev model

4.1 The \((E, G, X, Y)\)-class of models

Let \((E \xrightarrow{\partial} G, \triangleright)\) be a crossed module of groups. Let \( X \) and \( Y \) be finite groups on which \( G \) acts by automorphisms, and let \( f : Y \to X \) be a \( G \)-equivariant group morphism, such that \( \text{im}(\partial) \subseteq G \) acts trivially on \( Y \) and \( X \). In [21] we constructed a crossed module of semisimple Hopf algebras \((\mathcal{F}_C(X) \otimes \mathcal{E} \xrightarrow{\partial} \mathcal{F}_C(Y) \rtimes \mathcal{E} \rtimes G, \triangleright)\). The resulting class of Hopf-algebraic higher Kitaev models as defined in Section [3] is here called the \((E, G, X, Y)\)-class of models.

In this section we investigate more closely three particular cases of the associated \((E, G, X, Y)\)-class of models. We will also determine their ground-state spaces (see Subsection [3.7]) on a
surface $\Sigma$, proving that they are canonically independent of the cell decomposition of $\Sigma$, at least for the case of triangulations.

In order to simplify our discussion, we will only discuss the typical case when $\Sigma$ is an oriented surface without boundary and $L$ is a triangulation of $\Sigma$, together with an orientation on each edge and a choice of base-point $v_P$ for each plaquette $P$, to be a vertex of $P$.

The particular cases considered for $(\mathcal{F}_C(X) \otimes \mathbb{C}E \xrightarrow{a} \mathcal{F}_C(Y) \rtimes \mathbb{C}G, \varphi)$ are:

- The $(1, G, X, 1)$-case, where the groups $E$ and $Y$ both are trivial. This turns out to give Potts model \[35\] when $G$ is trivial. The general case of the model is a coupling between Kitaev quantum-double model \[1\] and Potts model.

- A generalisation of $(1, G, X, 1)$-case, the $(E, G, X, 1)$ case, where also $E$ may be non-trivial will be briefly discussed.

- The $(1, 1, X, Y)$ case where the groups $G$ and $E$ each are trivial.

Note that the $(E, G, 1, 1)$-case, where the groups $X$ and $Y$ each are trivial, coincides with the $n = 2$ case of the 2-group Kitaev model in \[17\] Equation (35)], in the Introduction called the full higher Kitaev model. In loc cit, its ground-state space was proven to be related with Yetter homotopy 2-type TQFT \[45\], and in particular to be canonically independent of the triangulation of $\Sigma$. The $(E, G, 1, 1)$-case reduces to the Kitaev model \[1\] when $E$ is trivial; see \[17\] page 8.

In all the particular cases above, the ground-state space is identified with the $\mathbb{C}$-linear span of the set of homotopy classes of maps $\Sigma \to B$. Here $B$ is a certain space, actually a homotopy 1-type or 2-type, \[32\] \[46\], obtained as the classifying space of a certain groupoid, or crossed module of groupoids, \[30\] \[31\] derived from $(G, E, X, Y)$. (The way the groupoids and crossed modules of groupoids are constructed depends on the example, so we do not have a general treatment of the full $(E, G, X, Y)$-model.)

The latter identification of the ground-state spaces as free vector spaces on sets of homotopy classes of maps from $\Sigma$ into 1-types or 2-types, here homotopy finite spaces \[12\] \[34\], means in particular that Quinn’s finite total homotopy TQFT \[12\] \[34\] provides a TQFT whose state spaces are canonically isomorphic to the ground-state spaces of the corresponding Hopf-algebraic higher Kitaev model, in the particular cases we consider. A discussion is in Subsubsection 4.2.3.

The calculation of the ground-state space of the $(1, G, X, 1)$, $(G, E, X, 1)$ and $(1, 1, X, Y)$-models are all based on results of Brown and Higgins \[30\], expressing the set of homotopy classes of maps $M \to B_A$, where $A$ is a crossed complex of groupoids, a generalisation of crossed module, in terms of homotopy classes of crossed complex maps $\Pi(M) \to A$. Here $M$ is any space with a CW-complex structure and $\Pi(M)$ denotes its fundamental crossed complex \[31\]. It should however be noted that the tricks that permit the application of Brown-Higgins theorem are quite different in the $(E, G, X, 1)$- and $(1, 1, X, Y)$-models. It is an open problem whether the results can be generalised to the full $(E, G, X, Y)$-model.

In the $(1, G, 1, Y)$-case, the Hopf algebra $\mathcal{F}_C(X) \otimes \mathbb{C}E$ becomes $\mathbb{C}$, so this example is not a crossed module case. In particular our model reduces to the Hopf-algebraic Kitaev model (with Hopf algebra $\mathcal{F}_C(Y) \rtimes \mathbb{C}G$) \[13\]. Therefore its ground-state space is known to be triangulation-independent. However we do not know of a neat homotopy interpretation of the ground state of the $(1, G, 1, Y)$-model in terms of homotopy classes of maps to some classifying space.
4.2 The \((1, G, X, 1)\)-case and its relation to Quinn’s finite total homotopy TQFT

Let \(G\) be a finite group. Let \(X\) be a finite group on which \(G\) acts by automorphisms. The general construction in Subsection 2.1 for \(E = 1 = Y\) gives a crossed module of Hopf algebras:

\[
(\mathcal{F}_C(X) \xrightarrow{\partial} \mathbb{C}G, \triangleright)
\]

Here \(G\) acts on \(\mathcal{F}_C(X)\) as \((g \triangleright \psi)(x) := \psi(g^{-1} \triangleright x)\) for \(\psi \in \mathcal{F}_C(X)\) and \(x \in X\). The boundary map \(\partial\) is trivial: \(\partial(\psi) = \psi(1_X)1_G\). The Haar integral of \(\mathbb{C}G\) is \(\ell = \frac{1}{|G|} \sum_{g \in G} g\) and the Haar integral of \(\mathcal{F}_C(X)\) is \(\Lambda = \delta_{1_X}\), so clearly \(g \triangleright \Lambda = \Lambda\) for each \(g \in G\); cf. Lemma 11.

4.2.1 An explanation of the \((1, G, X, 1)\)-model

**Definition 41.** Let \(\Sigma\) be a surface with a triangulation \(L\), with an orientation in each edge, and a choice of a base-point \(v_P\) to each plaquette \(P\).

The total state space assigned to \((\Sigma, L)\) is

\[
\mathcal{H}_{L,X,G} = \mathbb{C}G^{|L|} \otimes \mathcal{F}_C(X)^{|L|^2}.
\]

The total state space assigned to \((\Sigma, L)\) is naturally described in terms of \((G, X)\)-colourings. We define:

**Definition 42.** For sets \(A\) and \(B\), an \((A, B)\)-colouring \(\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)\) of \(L\) is a labelling \(\mathcal{F}_1: L^1 \to A\) of the edges of \(L\) by elements of \(A\) and another (independent) labelling \(\mathcal{F}_2: L^2 \to B\) of the plaquettes of \(L\) by elements of \(B\).

Clearly \(\mathcal{H}_{L,X,G}\) is isomorphic to the free vector space on the set of \((G, X)\)-colourings \(\mathcal{F}\) of \(L\). The language of \((G, X)\)-colourings, similar to that of [11, 7, 32], is quite suitable for specifying the actions of the vertex, edge and plaquette operators of the \((1, G, X, 1)\)-model.

Let \(\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)\) be a \((G, X)\)-colouring of \((\Sigma, L)\).

**Vertex operators**

Let \(v \in L^0\) be a vertex and let \(P \in L^2\) be an adjacent plaquette. The vertex operators \(V^g_v, P\) depend here only on the underlying vertex \(v\), \(V^g_v, P = V^g_v\), due to cocommutativity of the group algebra \(\mathbb{C}G\). Given a vertex \(v \in L^0\) and an element \(g \in G\), we obtain the \((G, X)\)-colouring \(V^g_v(\mathcal{F}) = (V^g_v(\mathcal{F})_1, V^g_v(\mathcal{F})_2)\):

- given \(t \in L^1:\)

\[
(V^g_v(\mathcal{F}))(t) = \begin{cases} 
  g \mathcal{F}_1(t), & \text{if } v \text{ is the starting vertex of } t \\
  \mathcal{F}_1(t) g^{-1}, & \text{if } v \text{ is the target vertex of } t \\
  \mathcal{F}_1(t), & \text{if } v \text{ is not incident to } t
\end{cases}
\]

- given \(P \in L^2:\)

\[
(V^g_v(\mathcal{F}))(P) = \begin{cases} 
  g \triangleright \mathcal{F}_2(P), & \text{if } v \text{ is the base-point of } P \\
  \mathcal{F}_2(P), & \text{if } v \text{ is not the base-point of } P
\end{cases}
\]

Since the Haar integral of the group algebra \(\mathbb{C}G\) is \(\ell = \frac{1}{|G|} \sum_{g \in G} g\), the vertex projector is given by:

\[
V^\ell_v = V^\ell_v = \frac{1}{|G|} \sum_{g \in G} V^g_v.
\]
Remark 43. Cf. Subsubsection 5.2.1 in the Appendix. From the above formulas, it is easy to see that any two vertex projectors \( V_v \) and \( V_{v'} \) for vertices \( v, v' \in L^0 \) commute with each other. Moreover, the vertex operators \( (V^g_v)_{g \in G} \) form a representation of \( G \) and any two vertex operators \( V^g_v \) and \( V^{g'}_v \), for \( g, g' \in G \) and distinct vertices \( v \neq v' \) commute with each other.

**Edge operators**

In order to describe the edge operators, we need another bit of notation. Let \( \mathcal{F} \) be a \((G, X)\)-colouring of \((\Sigma, L)\). Let \( P \in L^2 \) be a plaquette. Let \( t \) be an edge in the boundary of \( P \), and \( w \) be the initial point of \( t \). Let as usual \( v_P \) denote the base-point of \( P \).

Let us define the following notation \( \text{hol}^P_{P,t} \) (where \( \text{hol} \) stands for holonomy):

- If \( w = v_P \), let \( \text{hol}^P_{P,t} = 1_G \)
- Otherwise consider the unique path from \( v_P \) to \( w \), that does not pass through \( t \). Then let \( \text{hol}^P_{P,t} \) be the product of each element of \( G \) assigned to the edges transcribed when going from \( v_P \) to \( w \) (or its inverse if the edge is transcribed in the opposite direction to its orientation, when going from \( v_P \) to \( w \)). Some examples are indicated in the diagram below (an edge going from a vertex \( a \) to a vertex \( b \) is denoted \( t_{(a,b)} \)):

\[
\begin{align*}
\begin{tikzpicture}[->,>=stealth, shorten >=1pt,auto,node distance=1.5cm, semithick]
  \node (v) at (0,0) {$v$};
  \node (wp) at (1,0) {$v_P$};
  \node (w) at (2,0) {$w$};
  \node (wpw) at (1,-2) {$w$};
  \node (wpwp) at (0,-2) {$v_P$};

  \draw (v) edge [bend right] node [auto] {$t_{(v,v_P)}$} (wp);
  \draw (wp) edge [bend right] node [auto] {$t_{(v,w)}$} (w);
  \draw (v) edge [bend right] node [auto] {$t_{(v,w)}^{-1}$} (wp);
  \draw (wp) edge [bend right] node [auto] {$t_{(w,v)}$} (w);

  \node (v) at (5,0) {$v$};
  \node (wp) at (6,0) {$v_P$};
  \node (w) at (7,0) {$w$};
  \node (wpw) at (6,-2) {$w$};
  \node (wpwp) at (5,-2) {$v_P$};

  \draw (v) edge [bend right] node [auto] {$t_{(v,v_P)}^{-1}$} (wp);
  \draw (wp) edge [bend right] node [auto] {$t_{(v,w)}^{-1}$} (w);
  \draw (v) edge [bend right] node [auto] {$t_{(v,w)}$} (wp);
  \draw (wp) edge [bend right] node [auto] {$t_{(w,v)}^{-1}$} (w);
\end{tikzpicture}
\end{align*}
\]

Note that the notation \( \text{hol}^P_{P,t} \) extends to more general cell decompositions of \( \Sigma \), including the 2-lattice decompositions in \([7]\).

Given a (by definition, oriented) edge \( t \in L^1 \), let \( P \) and \( Q \) be the two plaquettes that have \( t \) in common, where \( P \) is on the left-hand side (noting that \( \Sigma \) is oriented). Then, unpacking Definition \( [25] \) for \( \xi \in \mathcal{F}_C(X) \):

\[
E^\xi_t(\mathcal{F}) = \xi \left( \left( (\text{hol}^P_{P,t})^{-1} \triangleright \mathcal{F}_2(P) \right) \left( (\text{hol}^Q_{Q,t})^{-1} \triangleright \mathcal{F}_2(Q) \right)^{-1} \right) \mathcal{F}.
\]

**Remark 44.** Cf. 5.2.2 From the above formula, it is clear that any two edge operators \( E^\xi_t \) and \( E^\xi'_t \) commute with each other, since they simply multiply a given \((G, X)\)-colouring \( \mathcal{F} \) by a scalar.

The Haar integral \( \Lambda \) in \( \mathcal{F}_C(X) \) is the delta function \( \delta_{1_X} \). So the edge projector \( E_t \) is such that:

\[
E_t(\mathcal{F}) = E^\Lambda_t(\mathcal{F}) = \begin{cases} 
\mathcal{F}, & \text{if } (\text{hol}^P_{P,t})^{-1} \triangleright \mathcal{F}_2(P) = (\text{hol}^Q_{Q,t})^{-1} \triangleright \mathcal{F}_2(Q) \\
0, & \text{otherwise.}
\end{cases}
\]

**Plaquette operators**

Let \( P \) be a plaquette, with base-point \( v_P \). Let \( \varphi \) be an element of the dual algebra of \( CG \), i.e. \( \varphi \) is canonically a function \( \varphi : G \to C \). Given a \((G, X)\)-colouring \( \mathcal{F} \), we need a new bit of notation, \( \text{hol}^P_{P,t} \) to be the product of the elements of \( G \) assigned to the edges of the boundary of \( P \), when we trace the boundary of \( P \) counterclockwise from \( v_P \) to \( v_P \). As in the definition
of $\text{hol}^F_{P,t}$ above, if an edge is traced in the opposite orientation, we put $\mathcal{F}_1(t)^{-1}$ instead. An example is below:

$$
\begin{align*}
\begin{array}{c}
v \quad \overset{t(v,v_P)}{\rightarrow} \quad v_P \\
\quad \overset{t(w,v_P)}{\leftarrow} \quad w \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{hol}^F_{P} \quad \overset{t(v,w)}{\rightarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\overset{t(w,v_P)}{\leftarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{hol}^F_{P} \quad \overset{t(v,w)}{\rightarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\overset{t(w,v_P)}{\leftarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{hol}^F_{P} \quad \overset{t(v,v_P)}{\rightarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\overset{t(w,v_P)}{\leftarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{hol}^F_{P} \quad \overset{t(v,w)}{\rightarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\overset{t(w,v_P)}{\leftarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\text{hol}^F_{P} \quad \overset{t(v,v_P)}{\rightarrow} \\
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\overset{t(w,v_P)}{\leftarrow} \\
\end{array}
\end{align*}
\end{align*}

Unpacking Definition 28, we have for the plaquette operator, given $\varphi \in (\mathbb{C}G)^{*}$:

$$
F^\varphi_P(\mathcal{F}) = \varphi(\text{hol}^F_P)\mathcal{F}.
$$

Remark 45. Cf. Subsubsection 5.2.3 in the Appendix. From the above formula, it is again clear that any two plaquette operators $F^\varphi_P$ and $F^{\varphi'}_{P'}$ commute with each other, since they simply multiply a given $(G,X)$-colouring $\mathcal{F}$ by a scalar.

The Haar integral in $(\mathbb{C}G)^*$ is $\delta_{1_G}$. The plaquette projector is hence given by $F_P(\mathcal{F}) = F^\delta_{1_G}_P(\mathcal{F})$. So clearly:

$$
F_P(\mathcal{F}) = \begin{cases}
\mathcal{F}, & \text{if } \text{hol}^F_P = 1_G \\
0, & \text{otherwise}.
\end{cases}
$$

Remark 46. Suppose $G$ is the trivial group. Then the $(1,G,X,1)$-model coincides with the Potts model on $(\Sigma,L)$ [35], considering the underlying set of $X$. The Hamiltonian does not depend on the group structure $X$, however the edge operators do.

Remark 47. Note that if $X$ has only one element then the $(1,G,X,1)$-model coincides with Kitaev’s quantum double model [1] for $G$. The full $(1,G,X,1)$-model can be seen as a coupling between the $|X|$-state Potts model and the Kitaev model for $G$.

Remark 48. We have already seen above, that all vertex projectors commute with each other, and likewise for the plaquette projectors and the edge projectors. Furthermore, it is clear from the above formulas that all edge operators commute with all plaquette operators since they act by scalars on the $(G,X)$-colourings. (The latter point is a special characteristic of the $(1,G,X,1)$-model.)

One can also see from the formulas that vertex operators commute with edge projectors and plaquette projectors. (Again the latter is a special characteristic of the $(1,G,X,1)$-model.) For vertices that lie in the interior of a path whose holonomy $\text{hol}^F$ enters the formula for the edge or plaquette projector, this follows from the fact such a vertex operator preserves the value of $\text{hol}^F$. For a starting vertex of one of the two paths entering an edge operator, which is also the base-point of one of the two adjacent plaquettes, the action of the corresponding vertex operator on the holonomy of the path and the action on the colouring of the plaquette cancel each other out in the formula for the edge operator. For the target vertex of the two paths entering the formula for the edge operator, the actions of the corresponding vertex operator on the holonomies of the two paths cancel each other out in the formula for the edge projector.

4.2.2 Fully flat $(G,X)$-colourings of $(\Sigma,L)$ and the ground-state space of the $(1,G,X,1)$-model

Definition 49. A $(G,X)$-colouring of $(\Sigma,L)$ is called fully flat if:

- (Flatness on plaquettes) Given any plaquette $P \in L^2$, $\text{hol}^F_{P} = 1_G$.

- (Flatness on edges) Given any edge $t \in L^1$,

$$
(\text{hol}^F_{P_1,t})^{-1} \triangleright \mathcal{F}_2(P_1) = (\text{hol}^F_{P_2,t})^{-1} \triangleright \mathcal{F}_2(P_2),
$$

where $P_1$ and $P_2$ are the two plaquettes that have $t$ as the common edge in their boundaries.
We let $\Phi_{(1,G,X,1)}(\Sigma, L)$ denote the set of fully flat $(G, X)$-colourings of $(\Sigma, L)$.

**Lemma 50.** The set $\Phi_{(1,G,X,1)}(\Sigma, L)$ is invariant under vertex operators $V^g_v$. Moreover, we have an action of $G^{L^0}$ on the set of fully flat $(G, X)$-colourings such that $(g_v)_{v \in L^0} \in G^{L^0}$ acts as: $\prod_{v \in L^0} V^g_v$.

Proof. An easy calculation proves that indeed $\Phi_{(1,G,X,1)}(\Sigma, L)$ is invariant under all vertex operators. (Cf. also Remark 48.) The second statement follows from the fact that vertex operators $V_v^g$ and $V^h_w$ commute for any two distinct vertices $v, w \in L^0$.

It is clear that the subspace of the total state space of the $(1, G, X, 1)$-model on $(\Sigma, L)$ of vectors that are invariant under all plaquette projectors and edge projectors is isomorphic to the free vector space on $\Phi_{(1,G,X,1)}(\Sigma, L)$. Moreover, the ground-state space of the $(1, G, X, 1)$-model is the subspace of $G^{L^0}$-invariant vectors in the free vector space on $\Phi_{G,X}(\Sigma, L)$. Therefore:

**Lemma 51.** Let $\Sigma$ be a surface with a triangulation $L$. The ground-state space of the $(1, G, X, 1)$-model on $(\Sigma, L)$ is canonically isomorphic to the free vector space on $\Phi_{(1,G,X,1)}(\Sigma, L)/G^{L^0}$.  

Since $G$ acts on the underlying set of $X$, we can define the action groupoid $X//G$. Objects of $X//G$ are elements $x \in G$. The set of morphisms $x \to y$ is given by the set of all pairs $(x, g) \in X \times G$ such that $g^{-1} \triangleright x = y$. The composition of the morphisms:

$$(x \xrightarrow{(x,g)} g^{-1} \triangleright x) \text{ and } (g^{-1} \triangleright x \xrightarrow{(g^{-1} \triangleright x, h)} (gh)^{-1} \triangleright x)$$

is

$$(x \xrightarrow{(x,gh)} (gh)^{-1} \triangleright x).$$

Given a triangulation $L$ of $\Sigma$, we put $\Sigma^0_L$ for the set of vertices of $L$, and $\Sigma^1_L$ for the subspace of $\Sigma$ made out of the union of 0- and 1-simplices of $L$. So $\Sigma^1_L$ is the $i$-skeleton of the CW-decomposition given by $L$; see e.g. [7] Definition 16).

**Lemma 52.** There is a one-to-one correspondence between fully flat $(G, X)$-colourings $\mathcal{F}$ of $(\Sigma, L)$ and groupoid functors $f_\mathcal{F} : \pi_1(\Sigma, \Sigma^0_L) \to X//G$.

This means that we have a one-to-one correspondence between fully flat $(G, X)$-colourings $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ of $(\Sigma, L)$ and pairs of maps $\mathcal{F}_0 : L^0 \to X$ and $\mathcal{F}_1 : L^1 \to G$ ($\mathcal{F}_1$ does not change), such that:

- **flatness around plaquettes**: given any plaquette $P \in L^2$, $\text{hol}_{\mathcal{F}}^P = 1_G$,
- **given any edge $t \in L^1$, with starting vertex $v$ and target vertex $w$**: $\mathcal{F}_0(w) = \mathcal{F}_1(t)^{-1} \triangleright \mathcal{F}_0(v)$.

Proof. Cf. [7] §2.5 & §5.1.6]. The fundamental groupoid $\pi_1(\Sigma, \Sigma^0_L)$ is the quotient of the free groupoid on the graph $(L^0, L^1)$ consisting of the vertices and edges of $L$, with a relation at each plaquette $P \in L^2$. That relation says that the composition of the elements of $\pi_1(\Sigma, \Sigma^0_L)$ assigned to the edges in the boundary $P$, when going around $P$ from the base-point $v_P$ to $v_{P'}$ again is the identity; see [31], as explained in [7] §2.5. Therefore $\mathcal{F}_1$ gives a functor $\pi_1(\Sigma, \Sigma^0_L) \to \ast//G$. To lift it to a functor $\pi_1(\Sigma, \Sigma^0_L) \to X//G$ we start by giving the value of the latter on vertices.

Let $(\mathcal{F}_1, \mathcal{F}_2)$ be a fully flat $(G, X)$-colouring. Then if $P$ is a plaquette, its base-point $v_P$ can be assigned the element $R(P, v_P) := \mathcal{F}_2(P) \in X$. We can extend this to the other vertices of $P$ using the rule $R(P, v') = \mathcal{F}_1(t)^{-1} \triangleright R(P, v)$, whenever $t$ is an edge in the boundary of $P$ oriented from $v$ to $v'$. Given the flatness condition of $\mathcal{F}_1$ around plaquettes, this gives a well-defined map $v' \mapsto R(P, v')$ from the set of vertices in the boundary of $P$ to $X$.  

32
Now let us see that $R(P,v)$ depends only on $v$ and not on the plaquette $P$ that $v$ belongs to. If $v$ is a vertex, and $v$ belongs to $P$ and $P'$, and there is an edge $t$ incident to $v$ separating $P$ and $P'$, then the flatness condition of $\mathcal{F}$ on the edge $t$ implies that $R(P,v) = R(P',v)$. This can easily be used to show that all plaquettes $Q$ containing $v$ give the same $R(Q,v)$.

So given $v \in L^0$ we can put $\mathcal{F}_0(v) := R(P,v)$, where $P$ is any plaquette containing $v$.

The rest of the statements follow straightforwardly. 

\textbf{Lemma 53.} Under the correspondence of Lemma 52, we have a one-to-one correspondence between orbits of $G^{L^0}$ in $\Phi_{G,X}(\Sigma, L)$ and equivalence classes of groupoid functors $f_\Sigma: \pi_1(\Sigma, \Sigma^0) \to X//G$, considered up to natural transformations.

\textbf{Proof.} Cf. [2] §4.3.1. These calculations only require checking compatibility between the languages of functors and natural transformations and that of vertex operators. 

Given a groupoid $\Gamma$, let $B_\Gamma$ be its classifying space. Classifying spaces of groupoids are particular cases of classifying spaces of crossed complexes, as defined in [30, 31, 32].

Cf. [7] §5.2. We will now use [30, Theorem A] (see also [31, Theorem 7.16] and [23, §11.4.iii]). This theorem gives a canonical identification between the set of groupoid functors $\pi_1(\Sigma, \Sigma^0) \to X//G$, considered up to natural transformation and homotopy classes of maps $\Sigma \to B_{X//G}$, for any triangulation $L$ of $\Sigma$. This permits us to see that the ground-state space of the $(1,G,X,1)$-model is canonically independent of the chosen triangulation $L$ of $\Sigma$.

\textbf{Theorem 54.} Consider a pair of finite groups $G$ and $X$, with $G$ acting on $X$ by automorphisms. Let $\Sigma$ be an oriented surface with a triangulation $L$. There is a canonical isomorphism between the ground-state space of the Hopf-algebraic higher Kitaev model for $(1,G,X,1)$ and the free vector space on the set of homotopy classes of maps $\Sigma \to B_{X//G}$, the classifying space of the groupoid $X//G$. In particular the ground-state space of the model is canonically independent of the triangulation $L$ of $\Sigma$.

\textbf{Proof.} This follows directly by applying Brown-Higgins classification theorem [30, Theorem A]. For explanation see [7] §5.2 [32].

\subsection*{4.2.3 Relation with Quinn’s finite total homotopy TQFT}

Let $m$ be a non-negative integer. A space $X$ is called an $m$-type [32, 46], if $\pi_i(X,x) = 0$ whenever $i > m$, for all possible choices of base-point $x \in X$. A space is called \textit{homotopy finite} [12, 34, 47] if furthermore it has only a finite number of path-components and all of their homotopy groups are finite. Classifying spaces of groupoids $\Gamma$ are homotopy 1-types, whose fundamental group at each point is isomorphic to a corresponding hom-group $\text{hom}_\Gamma(a,a)$ in $\Gamma$, where $a$ is an object of $\Gamma$. Consequently, classifying spaces of finite groupoids are homotopy finite spaces.

Quinn constructed in [12] what he called the \textit{finite total homotopy TQFT} $\mathcal{Q}_B$; see also [34]. It is a $(n+1)$-TQFT defined for all spatial dimensions $n$; in particular it restricts to a $(2+1)$-dimensional TQFT. Quinn’s TQFT $\mathcal{Q}_B$ depends on a (fixed) homotopy finite space $B$, called the base space. Explicitly, the $(n+1)$-TQFT $\mathcal{Q}_B$ sends a compact $n$-manifold $M$ to the free vector space on the set of homotopy classes of maps $M \to B$. The linear maps assigned to cobordisms $W: M \to N$ between manifolds are derived from the \textit{homotopy order}, called in [47, 48] \textit{homotopy cardinality}, of certain spaces of functions $W \to B$.

We hence have:

\footnote{In the case of functors between groupoids, crossed complex homotopies, the language of [30], boil down to natural transformations between groupoid functors.}
Theorem 55 (Relation with Quinn’s TQFT). There exists a $(2+1)$-dimensional TQFT whose state spaces are canonically isomorphic to the ground-state spaces of the Hopf-algebraic higher Kitaev model for $(1, G, X, 1)$. Namely consider Quinn’s finite total homotopy TQFT $\mathcal{Q}_B$ \cite{32}, \cite{34} with base space $\mathcal{B} = B_{X//G}$. This TQFT sends each surface $\Sigma$ to the free vector space on the set of homotopy classes of maps $\Sigma \to B_{X//G}$.

Proof. This follows directly from Theorem \cite{34} and the construction of Quinn’s finite total homotopy TQFT $\mathcal{Q}_B$. \hfill $\Box$

Note that the finite total homotopy TQFT $\mathcal{Q}_B$ is for $\mathcal{B} = B_{X//G}$ constructed as the Dijkgraaf-Witten TQFT, with trivial cocyle \cite{11}, except using a groupoid, in this case $X//G$, rather than a group. This follows from the discussion in \cite{32}. Details will appear in \cite{34}.

4.3 The ground-state space of the $(E, G, X, 1)$-model

The results in Subsubsection 4.2.2 extend to the $(E, G, X, 1)$-model. We will use the language of crossed modules of groupoids \cite{33} \cite{7 §2.1}, and in particular the fundamental crossed module $\Pi_2(\Sigma, \Sigma_L^1, \Sigma_L^0)$ of a surface – more generally a CW-complex $M$ – with a cell decomposition; see \cite{7}, and (for the theory) \cite{31}. Hence $\Sigma$ is given the CW-decomposition arising from the triangulation $L$.

Consider the trivial action $\triangleright$ of $E$ on $X$. Hence if $x \in X$ and $e \in E$ then: $e \triangleright x = x = \partial(e) \triangleright x$. In particular, the action groupoid $X//E$ is totally disconnected (i.e. there are no morphisms between different objects), and we have a groupoid functor $f: X//E \to X//G$, which is the identity on objects and sends $(x, e)$ to $(x, \partial(e))$. We have a groupoid action \cite{31} \cite{7 §2.1} of $X//G$ on $X//E$ such that

$$(x \xrightarrow{g} y) \triangleright (g^{-1} \triangleright x \xrightarrow{(x,e)} g^{-1} \triangleright x) = (x \xrightarrow{(x,g\circ e)} x).$$

This gives a crossed module of groupoids \cite{7} \cite{33} \cite{34} denoted by $\mathcal{G}(G, E, X)$. The same type of argument as in Subsubsection 4.2.2 combined with the construction in the last section in \cite{7} gives:

Theorem 56. Consider a crossed module of groups $(E \xrightarrow{\partial} G, \triangleright)$, and another group $X$ on which $G$ acts by automorphisms, such that $\text{im}(\partial) \subseteq G$ acts trivially on $X$. Consider the crossed module of Hopf algebras $(\mathcal{F}_C(X) \otimes \mathcal{C}E \xrightarrow{\partial} \mathcal{C}G, \triangleright)$ defined in Subsection 2.1. Let $(\Sigma, L)$ be an oriented surface with a triangulation.

There exists a canonical isomorphism between the ground-state space of the $(E, G, X, 1)$-model and the free vector space on the set of homotopy classes of crossed module maps $\Pi(\Sigma, \Sigma_L^1, \Sigma_L^0) \to \mathcal{G}(G, E, X)$, as defined in \cite{33} \cite{30}[7 §4.3.1]. In particular, there is a canonical isomorphism between the ground-state space of the $(E, G, X, 1)$-model and the free vector space on the set of homotopy classes of maps $\Sigma \to B_{\mathcal{G}(G,E,X)}$, the classifying space \cite{27} \cite{28} of $\mathcal{G}(G, E, X)$.

Proof. On the combinatorial side, the proof of the statements follows the same pattern as in the $(1, G, X, 1)$-case. The fact that $\partial(E) \subseteq G$ acts trivially in $X$ plays a key role. The final statement follows (directly) from \cite{30} Theorem A], as explained in \cite{7} §5.2]. \hfill $\Box$

Remark 57. Cf. Theorem \cite{55} including the preliminary discussion in \cite{4.2.3}. Classifying spaces of crossed modules of groupoids are homotopy 2-types \cite{30} \cite{31} \cite{32}. If the crossed modules are finite then their classifying spaces have only a finite number of path-components, each of which with finite fundamental and second homotopy groups for all choices of a based point. And hence they are homotopy finite spaces. In particular, Quinn’s finite total homotopy TQFT $\mathcal{Q}_B$ with base space $\mathcal{B} = B_{\mathcal{G}(G,E,X)}$ again gives a TQFT whose state spaces are the ground-state spaces of the $(E, G, X, 1)$-model.
4.4 The $(1, 1, X, Y)$-model and its relation to Quinn’s finite total homotopy TQFT

Let $f: Y \to X$ be a group homomorphism. We consider the crossed module defined in Subsection 2.1 for the case when $G$ and $E$ are both the trivial group. We thus obtain a crossed module of Hopf algebras $(\mathcal{F}_C(X) \xrightarrow{f^*} \mathcal{F}_C(Y), \triangleright)$, where the action of $\mathcal{F}_C(Y)$ on $\mathcal{F}_C(X)$ is the trivial one, namely:

$$\psi \triangleright \eta = \varepsilon(\psi) \eta = \psi(1_Y) \eta.$$

The Haar integrals in $\mathcal{F}_C(X)$ and $\mathcal{F}_C(Y)$ are respectively given by $\Lambda = \delta_{1_X}$ and $\ell = \delta_{1_Y}$. We will also need the Haar integral in $\mathcal{F}_C(Y)^* \cong \mathbb{C}Y$ which is $\lambda = \frac{1}{|X|} \sum_{x \in X} x$.

Let $(\Sigma, L)$ be an oriented surface (with no boundary) together with a triangulation, as usual so that each edge is oriented and each plaquette $P \in L^2$ is assigned a base-point $v_P \in L^0$.

First, as a vector space, the total state space $\mathcal{H}_L$ as defined in Section 3 is here:

$$\mathcal{H}_L = \mathcal{F}_C(Y)^{\otimes L^1} \otimes \mathcal{F}_C(X)^{\otimes L^2} \cong \text{span}_\mathbb{C}(B^{L^1} \times X^{L^2}).$$

As in Subsection 4.2 it is convenient to reformulate the total state space in terms of $(Y, X)$-colourings $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ of $(\Sigma, L)$, pairs of maps $\mathcal{R}_2: L^2 \to X$ and $\mathcal{R}_1: L^1 \to Y$, see Definition 42. Clearly $\mathcal{H}_L$ is isomorphic to the free vector space on the set of $(Y, X)$-colourings of $(\Sigma, L)$.

In order to simplify our notation and mainly reduce the number of different cases of possible orientation of the edges in the plaquettes, we will furthermore impose that we have a total order on the set of the vertices of the triangulation. This total order gives an orientation for each edge $(a, b)$, where $a < b$, going from $a$ to $b$. If $P$ is a plaquette, with vertices $a_P < b_P < c_P$, we put as base point $v_P = a_P$. Put $d_0(P) = (b_P, c_P)$, $d_1(P) = (a_P, c_P)$ and $d_2(P) = (a_P, b_P)$.

4.4.1 Plaquette operators

Let $z \in Y \subset \mathcal{F}_C(Y)^* = \mathbb{C}Y$. Let $P \in L^2$. We will now determine the plaquette operator $F_P^z$, in the $(1, 1, X, Y)$ case, according to the general definition of plaquette operators in Definition 28.

Suppose that the orientation on $P$ induced by the order of the vertices is opposite to the orientation induced by $\Sigma$. The plaquette operator $F_P^z$ is such that the colours on the plaquette $P$ and its edges transform as:

$$F_P^z(\mathcal{R})(a_P \ y_{d_2(P)} \ x \ y_{d_0(P)} \ b_P) = \sum_{p'p=y_{d_2(P)}} \sum_{q'q=y_{d_0(P)}} \sum_{r'=y_{d_1(P)}^{-1}} \sum_{s's=x} \delta(p, q) \delta(p, r'^{-1}) \delta(f(p), (s')^{-1}) \delta(z, p) \delta(f(z), (s')^{-1})$$

$$a_P \ x \ b_P$$

$$\delta(p, q) \delta(p, r'^{-1}) \delta(f(p), (s')^{-1}) \delta(z, p) \delta(f(z), (s')^{-1})$$

35
4.4.2 Edge operators

Let $t \in L^1$ be an oriented edge. Given $\psi \in \mathcal{F}_C(X) = \text{span}_C\{\delta_x \mid x \in X\}$, let us determine the edge operator $E^\psi_t$, as in Definition 25. Let $P$ and $Q$ be the two plaquettes that have $t$ as a common edge. Given that the action of $\mathcal{F}_C(Y)$ on $\mathcal{F}_C(X)$ is the trivial one, namely $\psi \triangleright \eta = \psi(1_Y)\eta$, the formula for the edge operators $E^\psi_t$ in Definition 25 simplifies enormously. In particular the edge operators $E^\psi_t$ are independent of the base-points of the plaquettes $P$ and $Q$, and only depend on the colouring of the edge $t$. Let us explain the general formula for $E^\delta_w_t$ if $w \in X$. In the figures below, $t$ is the middle edge coloured by $z \in Y$. The other edges in the figure can have arbitrary orientations. For $w \in X$, the edge operator $E^\delta_w_t$ is:

$$E^\delta_w_t = E^\delta_z X$$

The edge projector $E_t = E^\delta_z X$ is hence:

$$E_t = E^\delta_z X$$

4.4.3 Vertex operator

We now unpack how the vertex operators defined in the general case in Subsection 3.2 look like in the case of the $(1,1,X,Y)$-model. Let $v \in L^0$, and choose a plaquette $P$ that has $v$ as a vertex. Given $\psi \in \mathcal{F}_C(Y) = \text{span}_C\{\delta_z \mid z \in Y\}$, we determine the vertex operator $V^\psi_{v,P}$. It suffices to consider the case $V^\delta_z_{v,P}$, where $z \in Y$. 

Outside of $P$ the colourings on simplices do not change.

If the orientation on $P$ induced by the total order of the vertices of the triangle is the same as the orientation induced by that of $\Sigma$, then the plaquette operator $F^\delta_z_P$ is such that:

$$F^\delta_z_P(R) = b_P y_{d_2(P)} z^{-1} f(z)x y_{d_0(P)} z^{-1}. $$
Let \( t_1, \ldots, t_n \) be the edges of \((\Sigma, L)\) incident to \( v \), starting from the second (hence last) edge of \( P \) incident to \( v \), when going around \( v \) counterclockwise, and in counterclockwise order. We let \( \theta_{v,t_i} = -1 \) if the edge \( t_i \) is pointing towards \( v \) and \( \theta_{v,t_i} = 1 \) if \( t_i \) is pointing away from \( v \). Let \( P_1, \ldots, P_k \) be the plaquettes of \((\Sigma, L)\), possibly none, that have \( v \) as a base-point. Fix a \((Y, X)\)-colouring \( \mathcal{R} \). Then \( V_{v,P}^{\delta_{x,P}} \) can only change the colours of edges \( t_1, \ldots, t_n \) and the plaquettes \( P_1, \ldots, P_k \). Let \((y_1, \ldots, y_n; x_1, \ldots, x_k)\) be those colours. (It turns out that only the colours of the edges change under the vertex operators.) Put:

\[
\Delta^{(n+k)}(z) = \sum_{w_1, \ldots, w_{n+k} \in Y, w_1, \ldots, w_{n+k} = z} w_1 \otimes \cdots \otimes w_{n+k}.
\]

Noting that the action of \( \mathcal{F}_C(Y) \) on \( \mathcal{F}_C(X) \) is the trivial one \( \psi \circ \phi = \psi(1_Y)\phi \), it follows that:

\[
(y_1, \ldots, y_n; x_1, \ldots, x_k) \xrightarrow{V_{v,P}^{\delta_{x,P}}} \sum_{w_1, \ldots, w_{n+k} = z} \delta(w_1, y_1^{\theta_{v,t_1}}) \ldots \delta(w_n, y_n^{\theta_{v,t_n}}) \delta(w_{n+1}, 1_Y) \ldots \delta(w_{n+k}, 1_Y) (y_1, \ldots, y_n; x_1, \ldots, x_k)
\]

\[
= \sum_{w_1, \ldots, w_{n+k} = z} \delta(w_1, y_1^{\theta_{v,t_1}}) \ldots \delta(w_n, y_n^{\theta_{v,t_n}})(y_1, \ldots, y_n; x_1, \ldots, x_k)
\]

\[
= \delta(y_1^{\theta_{v,t_1}} \ldots y_n^{\theta_{v,t_n}}, z) (y_1, \ldots, y_n; x_1, \ldots, x_k).
\]

Hence given an \((Y, X)\)-colouring \( \mathcal{R} \) we have:

\[
V_{v,P}^{\delta_{x,P}}(\mathcal{R}) = \delta(\mathcal{R}_1(t_1)^{\theta_{v,t_1}} \ldots \mathcal{R}_1(t_n)^{\theta_{v,t_n}}, z) \mathcal{R}.
\]

The vertex projector \( V_v = V_{v, P}^{\delta_{x,Y}} \) takes the form:

\[
V_v(\mathcal{R}) = \delta(\mathcal{R}_1(t_1)^{\theta_{v,t_1}} \ldots \mathcal{R}_1(t_n)^{\theta_{v,t_n}}, 1_Y) \mathcal{R}.
\]

Note that the vertex projector \( V_v = V_{v,P}^{\delta_{x,Y}} \) does not depend the chosen plaquette \( P \) that \( v \) belongs to, as it should be: see before Definition 24.

4.4.4 The \((1, 1, X, Y)\)-model on the dual cell decomposition.

The explicit forms of the plaquette, edge and vertex operators in the \((1, 1, X, Y)\)-model become very transparent in the dual cell decomposition to \((\Sigma, L)\). (This is essentially as in [3, Lemma 2.6], which corresponds to the \( X = \{1\} \) case.) So consider the dual cell decomposition \( L^* \rightarrow L \); see [4, Section 3]. We have a 0-cell \( P^* \) of \( L^* \) for each 2-cell \( L \) of \( P \) at its barycentre. We have an oriented 1-cell \( t^* \) of \( L^* \) for each oriented 1-cell \( t \) of \( L \), oriented from the vertex of \( L^* \) corresponding to the plaquette on the right-hand side of \( t \), towards the vertex corresponding to the plaquette on the left-hand side of \( t \). Finally we have a 2-cell \( v^* \) of \( L^* \) for each 0-cell \( v \) of \( L \). The local configuration is as in Figure 1.

The total Hilbert space of the \((1, 1, X, Y)\)-model is given by the free vector space on the set of all \((Y, X)\)-colourings of \((\Sigma, L)\). Dually, such a colouring \( \mathcal{R} = (\mathcal{R}_1 : L^1 \rightarrow Y, \mathcal{R}_2 : L^2 \rightarrow X) \) corresponds to labellings \( \mathcal{R}_1^* : (L^*)^1 \rightarrow Y \) and \( \mathcal{R}_2^* : (L^*)^0 \rightarrow X \). To \( \mathcal{R}^* = (\mathcal{R}_1^0, \mathcal{R}_2^1) \) we call the dual colouring to \( \mathcal{R} \). The total Hilbert space of the \((1, 1, X, Y)\)-model will from now on be seen as the free vector space on the set of those dual colourings \( \mathcal{R}^* \). We will graphically represent dual colourings as in Figure 2.
Figure 1: The local picture of a triangulation $L$ (in red) and its dual cell decomposition $L^*$ (in black).

Figure 2: A dual colouring $R^*$ to an $(Y,X)$-colouring $R$. The simplices of the original triangulation are in red. Cells of the dual cell decomposition are written in blue. The labellings given by the dual colouring $R^*$ are all in black. Hence $y_1, \ldots, y_5 \in Y$ and $x_1, \ldots, x_5 \in X$.

**Plaquette operators in the dual picture**

The plaquette operator $F^z_P : \mathcal{H}_L \rightarrow \mathcal{H}_L$, for $z \in Y$, is such that for an edge $t \in (L^*)^1$:

$$F^z_P(R^*)_1(t) = \begin{cases} \frac{1}{2} z (R^*)_1(t), & \text{if } t \in (L^*)^1 \text{ starts in } P^* \\ (R^*)_1(t) z^{-1}, & \text{if } t \in (L^*)^1 \text{ ends in } P^* \\ (R^*)_1(t), & \text{if } t \text{ is not adjacent to } P^* \end{cases}$$

And for a vertex $w \in (L^*)^0$:

$$F^z_P(R^*)_0(w) = \begin{cases} f(z) F^z_P(R^*)_0(w), & \text{if } w = P^* \\ F^z_P(R^*)_0(w), & \text{if } w \neq P^* \end{cases}$$
Figure 3: The plaquette operator $F_P^a$ in the dual colouring picture. The figure shows the action on the colouring in Figure 2.

The graphical picture for the action of $F_P^a$ on the dual colouring in Figure 2 is as in Figure 3. Then the plaquette projector is as always:

$$F_P^\lambda, \text{ where here } \lambda = \frac{1}{|Y|} \sum_{y \in Y} y \in CY.$$  

4.4.5 The edge operators in the dual picture

Let $t \in L^1$. Cf. the calculations in Subsubsection 4.4.2. The edge operator $E_t^{g_\times}$ for $x \in X$ is such that, if $Q$ is on the left-hand side of $t$ and $P$ is on the right-hand side of $t$, so we have an edge $P^* \xrightarrow{t^*} Q^*$ in the dual cell decomposition:

$$E_t^{g_\times}(R^*) = \begin{cases} R^*, & \text{if } f((R^*)_1(t^*)) x_1^{-1} (R^*)_0(Q^*) = (R^*)_0(P^*) \\ 0, & \text{otherwise.} \end{cases}$$

So the edge projector $E_t = E_t^{g_\times}$ is

$$E_t(R^*) = \begin{cases} R^*, & \text{if } f((R^*)_1(t^*)) (R^*)_0(Q^*) = (R^*)_0(P^*) \\ 0, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (9)

Looking at the dual colouring $R^*$ in Figure 2 the edge operators $E_{t_1}^{g_\times}(R)$ and $E_{t_2}^{g_\times'}(R)$ are such that:

$$E_{t_1}^{g_\times}(R^*) = \begin{cases} R^*, & \text{if } x f(y_1)^{-1} x_1 = x_2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad E_{t_2}^{g_\times'}(R^*) = \begin{cases} R^*, & \text{if } x' f(y_2)^{-1} x_3 = x_2 \\ 0, & \text{otherwise} \end{cases}$$

So the edge projectors $E_{t_1} = E_{t_1}^{g_\times}$ and $E_{t_2} = E_{t_2}^{g_\times'}$ are:

$$E_{t_1}(R^*) = \begin{cases} R^*, & \text{if } f(y_1)^{-1} x_1 = x_2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad E_{t_2}(R^*) = \begin{cases} R^*, & \text{if } f(y_2)^{-1} x_3 = x_2 \\ 0, & \text{otherwise.} \end{cases}$$
Remark 58. Later on in Subsubsection 4.4.6, we will consider the action groupoid \(X//Y\) of the action of \(Y\) on \(X\) such that \(y \triangleright x := f(y)x\). Note that the edge projector on a dual edge coloured as \(x \xrightarrow{y} x'\), where \(x, x' \in X\) and \(y \in Y\), hence chooses the configurations that make \(x \xrightarrow{(x,y)} x'\) an arrow in the action groupoid \(X//Y\). This will play a key role in determining the ground-state space of the \((1, 1, X, Y)\)-model.

**Vertex operators in the dual picture**

Let \(R^* = (R^*_0, R^*_1)\) be a dual colouring. Let \(v \in L^1\) and \(P\) be an adjacent plaquette. We resume the notation in Subsubsection 4.4.3. The choice of \(P\) equips the plaquette \(v^* \in (L^*)^2\) of the dual cell decomposition with a base-point \(P^* \in (L^*)^0\). Put:

\[
\text{hol}_{\partial v^*}^R = \mathcal{R}^*_1(t_1)^{\theta_{v,t}} \cdots \mathcal{R}^*_1(t_n)^{\theta_{v,t,n}}
\]

Let \(w \in Y\), the vertex operator \(V_{v,P}^{\delta_w}\) is such that:

\[
V_{v,P}^{\delta_w^w}(R^*) = \begin{cases} R^*, & \text{if } \text{hol}_{\partial v^*}^R = \mathcal{R}^*_1(t_1)^{\theta_{v,t}} \cdots \mathcal{R}^*_1(t_n)^{\theta_{v,t,n}} = w \\ 0, & \text{otherwise}. \end{cases}
\]

So the vertex projector is \(V_v = V_{v,P}^{\delta_Y^v}\):

\[
V_v(R^*) = \begin{cases} R^*, & \text{if } \text{hol}_{\partial v^*}^R = \mathcal{R}^*_1(t_1)^{\theta_{v,t}} \cdots \mathcal{R}^*_1(t_n)^{\theta_{v,t,n}} = 1_Y \\ 0, & \text{otherwise}. \end{cases}
\]

Cf. [3] §2.3] for the \(X = 1\) case. On a configuration \(R^*\) that locally looks like in Figure 2, the vertex operator \(V_{v,P}^{\delta_w}\), where \(w \in Y\), and the vertex projector \(V_v = V_{v,P}^{\delta_Y^v}\) take the form:

\[
V_{v,P}^{\delta_w}(R^*) = \begin{cases} R^*, & \text{if } y_1^{-1}y_2y_3^{-1}y_4y_5^{-1} = w \\ 0, & \text{otherwise}. \end{cases} \quad \text{and} \quad V_v(R^*) = \begin{cases} R^*, & \text{if } y_1^{-1}y_2y_3^{-1}y_4y_5^{-1} = 1 \\ 0, & \text{otherwise}. \end{cases}
\]

**Remark 59.** Cf. Proposition 38. We can easily see that the following commutation relations between plaquette, edge and vertex operators hold:

\[
[F_P^z, F_Q^w] = 0, \text{ if } P, Q \in L^2, \text{ with } P \neq Q, \text{ and } z, w \in Y;
\]

\[
F_P^z F_P^w = F_P^{zw}, \text{ if } P \in L^2 \text{ and } z, w \in Y;
\]

\[
[F_P^z, E_t^{\delta_x}] = 0, \text{ if } P \in L^2, t \in L^1, \text{ and } z \in Y, x \in X, \text{ if } t \text{ is not an edge in } P,
\]

\[
[E_t^{\delta_x}, E_{t'}^{\delta_{x'}}] = 0, \text{ if } t, t' \in L^1, \text{ and } w, w' \in X.
\]

Also if \(t\) is an edge in \(P\) then if \(z \in Y\) and \(x \in X\):

\[
[F_P^z, E_t^{\delta_x}] = 0, \text{ if } P \text{ is on the right of } t
\]

\[
F_P^z \circ E_t^{\delta_x} = E_t^{\delta_((z-x)\cdot f(x)^{-1})} \circ F_P^z, \text{ if } P \text{ is on the left of } t.
\]

In general, if \(P\) is a plaquette then for any vertex \(v \in L^0\) and edge \(t \in L^1\), and for all \(z \in Y\), we have:

\[
[F_P^z, E_t] = 0,
\]

\[
[F_P^z, V_v] = 0.
\]

If \(v\) is a vertex and \(P, Q\) are different plaquettes incident to \(v\) then given \(y \in Y\) and \(x \in X\) then:

\[
F_P^y \circ V_{v,P}^{\delta_x} = V_{v,P}^{\delta_y} \circ F_P^y,
\]

\[
[F_Q^y, V_{v,P}^{\delta_x}] = 0.
\]
4.4.6 The ground-state space of the $(1,1,X,Y)$-model and its relation with Quinn’s finite total homotopy TQFT

As for the $(1,G,X,1)$ model and, more generally the $(E,G,X,1)$ model, we can prove that the ground-state space of the $(1,1,X,Y)$-model on a surface $\Sigma$ with a triangulation $L$ is canonically isomorphic to the free vector space of homotopy classes of maps from the surface $\Sigma$ into a homotopy finite space $\mathcal{B} = B_{X//Y}$. From this it follows that the ground-state space is canonically independent of the triangulation $L$ of $\Sigma$.

We first consider the action groupoid of the action of $Y$ on $X$ defined as

$$y \triangleright x := f(y)x.$$  

So the set of objects of $X//Y$ is $X$ and the arrows look like:

$$x \xrightarrow{y} f(y)^{-1}x, \text{ where } x \in X, y \in Y.$$  

The composition of the arrows

$$x \xrightarrow{y} f(y)^{-1}x \text{ and } f(y)^{-1}x \xrightarrow{y'} f(y')^{-1}f(y)^{-1}x.$$  

is

$$x \xrightarrow{yy'} f(yy')^{-1}x.$$  

The following follows as for the $(1,G,X,1)$-model.

**Lemma 60.** There is a canonical isomorphism between the ground-state space of the $(1,1,X,Y)$-model and the free vector space on the set of equivalence classes of groupoid functors $\pi_1(\Sigma, \Sigma^0_L) \to X//Y$, considered up to natural transformation.

Here the notation $\Sigma^0_L$ means the set of vertices of $L^*$, i.e. $\Sigma^0_L$ is the 0-skeleton of the underlying CW-decomposition $L^*$ of $\Sigma$.

**Proof.** The result follows as in the proof of Lemma 52. First of all, the fundamental groupoid $\pi_1(\Sigma, \Sigma^0_L)$ is the free groupoid on the vertices and edges of the dual cell decomposition $L^*$, with one relation for each vertex $v \in L^0$ of $L$, i.e. plaquette $v^* \in (L^*)^2$ of the dual cell decomposition $L^*$; see [7, §2.5]. The latter relation imposes that the composition of the arrows around $v^*$ is an identity, as in the formula for the vertex projector in Equation (10).

Consider the subspace of the total Hilbert space $\mathcal{H}_L$ obtained as:

$$(\mathcal{H}_L)_0 := \left( \prod_{t \in L^1} E_t \prod_{v \in L^0} V_v \right)(\mathcal{H}_L)$$

Given the explicit formula of the vertex and edge projectors of the $(1,1,X,Y)$-model in the dual picture, it follows that $(\mathcal{H}_L)_0$ has a basis in one-to-one correspondence with the set of all groupoid functors $T: \pi_1(\Sigma, \Sigma^0_L) \to X//Y$. (Remark 58 is used here.)

Each plaquette operator $F_P^0: \mathcal{H}_L \to \mathcal{H}_L$ restricts to a map $F_P^0: (\mathcal{H}_L)_0 \to (\mathcal{H}_L)_0$. This can be seen directly or by applying Remark 59. Given that plaquette operators $F_P^0$ and $F_Q^h$ based at different plaquettes $P$ and $Q$ commute, we hence have a representation of $\prod_{P \in L^2} Y$ on $(\mathcal{H}_L)_0$ via the obvious product action by plaquette operators.

The space $(\mathcal{H}_L)_0$ is the free vector space on all functors $T: \pi_1(\Sigma, \Sigma^0_L) \to X//Y$. Each element of $\prod_{P \in L^2} Y$ acts on a functor $T: \pi_1(\Sigma, \Sigma^0_L) \to X//Y$ by inducing a natural transformation of functors. This correspondence between actions by elements of $Y^{L^2}$ and natural transformations is easily seen to be one-to-one. (Again this is just about checking compatibility between the languages of plaquette operators and that of natural transformations between functors.)

Hence the ground-state space of the $(1,1,X,Y)$-model, namely $(\prod_{P \in L^2} F_P)((\mathcal{H}_L)_0)$, is canonically isomorphic to the free vector space on the set of all groupoid functors $\pi_1(\Sigma, \Sigma^0_L) \to X//Y$, considered up to natural transformations.

\[ \square \]
As in Subsubsection 4.2.2, we will now use [30, Theorem A] (see also [31, Theorem 7.16] and [23, §11.4.iii]) to find a canonical identification between the set of isomorphism classes of groupoid functors $\pi_1(\Sigma, \Sigma^0_\ast) \to X//Y$, considered up to natural transformation, and homotopy classes of maps $\Sigma \to B_{X//Y}$. This permits us to remove the only apparent triangulation dependence of the ground state of the $(1, 1, X, Y)$-model on a triangulated surface $(\Sigma, L)$. More discussion is in [7, §5.2].

**Theorem 61.** Let $(\Sigma, L)$ be a surface with a triangulation with an orientation on each edge. Let $f : X \to Y$ be a map of finite groups. The ground-state space of the $(1, 1, X, Y)$-model is canonically isomorphic to the free vector space on the set of homotopy classes of maps $\Sigma \to B_{X//Y}$, where $B_{X//Y}$ is the classifying space of the groupoid $X//Y$ [30, 31]. In particular there is a (2+1)-dimensional TQFT, namely Quinn’s finite total homotopy TQFT $\Omega_{B_{X//Y}}$, whose state space on a surface $\Sigma$ is given by the ground-state space of the $(1, 1, X, Y)$-model on $(\Sigma, L)$.

**Proof.** Again the first statement follows from [30, Theorem A]. The second follows as in 4.2.3 from the construction of Quinn’s finite total homotopy TQFT [12, 34], considering the space $B_{X//Y}$. The latter is a 1-type, and moreover (as the groupoid $X//Y$ is finite) a homotopy finite space; see [30]. □

**Remark 62.** As we can see from Subsubsection 4.4.4, in the dual lattice decomposition to $(\Sigma, L)$, the $(1, 1, X, Y)$-model reduces to the Kitaev model based on $Y$ if $X = \{1\}$, [3, Lemma 2.6]. In the general case the $(1, 1, X, Y)$-model gives a groupoid version of the Kitaev model based on the groupoid $X//Y$. We have additional edge operators, that do not exist in the original Kitaev model, that impose an energy penalty on configurations outside $X//Y$; see Remark 58. The case when $Y = \{1\}$ is just Potts model on $|X|$; see [35].

## 5 Appendix (full calculations)

Let $\Sigma$ be a compact oriented surface with a cell decomposition $L$, as in Definition 17, with sets $L^0, L^1$ and $L^2$ of vertices, edges and plaquettes, respectively (for example, a triangulation). Recall that each edge is oriented and each plaquette has a distinguished vertex in its boundary as base-point.

We fix a crossed module of Hopf algebras $(A \xrightarrow{\partial} H, \triangleright)$. In the calculations in this Appendix we will indicate some of the key reasons for some of the equalities to hold by decorating the corresponding equal signs. The dictionary for what each less self-explanatory decoration means is as below:

1. $\overset{Y.-D.}{=} :$ using the Yetter-Drinfeld condition [4],
2. $\overset{A \text{ mod. alg. and } A \text{ H-mod.coalg.}}{=} :$ using the fact that $A$ is a module algebra or a module coalgebra over $H$,
3. $\overset{\text{Pf. 1 and Pf. 2}}{=} :$ using the Peiffer conditions 1. and 2. for crossed module of Hopf algebras in Definition 5,
4. $\overset{S_A \text{ H-lim.}}{=} :$ using the fact that the antipode $S_A$ in $A$ is $H$-linear, Remark 6.
5.1 Commutativity of edge orientation reversals and base-point shifts with vertex, edge and plaquette operators

5.1.1 Edge orientation reversals and base-point shifts commute with each other

Lemma 63. Let $P \in L^2$ be a plaquette and let $e \in L^1$ be an edge. Denote by $\tilde{T}_P^+$ the base-point shift with respect to the cell decomposition $\tilde{L}$ obtained from $L$ by reversing the orientation of $e$. Then

$$R_e \circ T_p^+ = \tilde{T}_P^+ \circ R_e.$$  

Proof. The supports of the edge orientation reversal $R_e$ and the base-point shift $T_p^+$ intersect non-trivially only if $e$ is the edge along which the base-point of $P$ is shifted, i.e. the edge lying next to the base-point in counter-clockwise direction around $P$. Let us therefore assume this is the case.

Then we indeed have for any $v_e \otimes X_P \in H_e \otimes A_P = H \otimes A$:

$$(\tilde{T}_P^+ R_e)(v_e \otimes X_P) = \tilde{T}_P^+ (S v_e \otimes X_P) = (S v_e)_1 \otimes (S v_e)_2 \otimes X_P \nonumber$$

and

$$(R_e T_p^+)(v_e \otimes X_P) = R_e((v_e)_1 \otimes (v_e)_2 \otimes X_P) = S(v_e)_1 \otimes (v_e)_2 \otimes X_P,$$

where by slight abuse of notation we omit the tensor factors in $\mathcal{H}_L$ on which all involved maps act as the identity. \hfill \Box

5.1.2 Edge orientation reversals commute with vertex operators

Lemma 64. Let $e \in L^1$ be an edge and let $v \in L^0$ be a vertex with an adjacent plaquette $P \in L^2$. Denote by $\tilde{V}_{v,P}^h$ for any $h \in H$ the vertex operator at $v$ with respect to the cell decomposition $\tilde{L}$ obtained from $L$ by reversing the orientation of the edge $e$. Then:

$$R_e \circ V_{v,P}^h = \tilde{V}_{v,P}^h \circ R_e \quad \text{for all } h \in H.$$

Proof. If $e$ is not adjacent to $v$, then $\tilde{V}_{v,P}^h = V_{v,P}^h$, and $V_{v,P}^h$ and $R_e$ have disjoint support in $\mathcal{H}_L$.

Hence, let $e$ be adjacent to $v$. Let $(e_1, \ldots, e_n)$ be the edges adjacent to $v$ in counter-clockwise order around $v$ starting and ending at the plaquette $P$ such that $e_j = e$ for $j \in \{1, \ldots, n\}$. Furthermore, let $(\theta_1, \ldots, \theta_n)$ be such that $\theta_i := -1$ if the edge $e_i$ is oriented away from $v$ and otherwise $\theta_i := 1$ for $i \in \{1, \ldots, n\}$. Let $(P_1, \ldots, P_k)$ be the plaquettes which have $v$ as their base-point. From Definition 21, we recall the definition of the vertex operator acting on $v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k} \in H_{e_1} \otimes \cdots \otimes H_{e_n} \otimes A_{P_1} \otimes \cdots \otimes A_{P_k} = H^{\otimes n} \otimes A^{\otimes k}$:

$$V_{(e,P)}^h(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k}) = (h(1)v_{e_1}^{(\theta_1)})^{(\theta_1)} \otimes \cdots \otimes (h(n)v_{e_n}^{(\theta_n)})^{(\theta_n)} \otimes h_{(n+1)} X_{P_1} \otimes \cdots \otimes h_{(n+k)} X_{P_k},$$

and

$$\tilde{V}_{(e,P)}^h(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k}) = (h(1)v_{e_1}^{(\theta_1)})^{(\theta_1)} \otimes \cdots \otimes (h(n)v_{e_n}^{(\theta_n)})^{(\theta_n)} \otimes (h_{(j+1)}v_{e_j}^{(\theta_j)})^{(\theta_j)} \otimes \cdots \otimes (h_{(j-1)}v_{e_{j-1}}^{(\theta_{j-1})})^{(\theta_{j-1})}.$$
\[ \otimes (h_{(j+1)} v_{e_{j+1}}^{(\theta_{j+1})}) \otimes \cdots \otimes (h_n v_{e_n}^{(\theta_n)}) \otimes h_{(n+1)} \triangleright X_{P_1} \otimes \cdots \otimes h_{(n+k)} \triangleright X_{P_k} \]

Now let us show the desired commutation relation:

\[ (R_{e_j}^h v_{e,P}^h)(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k}) = R_{e_j} \left( (h_{(1)} v_{e_1}^{(\theta_1)}) \otimes \cdots \otimes (h_n v_{e_n}^{(\theta_n)}) \otimes h_{(n+1)} \triangleright X_{P_1} \otimes \cdots \otimes h_{(n+k)} \triangleright X_{P_k} \right) \]
\[ = \left( h_{(1)} v_{e_1}^{(\theta_1)} \right) \otimes \cdots \otimes \left( h_{(j-1)} v_{e_{j-1}}^{(\theta_{j-1})} \right) \otimes \left( h_{(j)} v_{e_j}^{(\theta_j)} \right) \otimes \left( h_{(j+1)} v_{e_{j+1}}^{(\theta_{j+1})} \right) \otimes \cdots \otimes \left( h_n v_{e_n}^{(\theta_n)} \right) \otimes h_{(n+1)} \triangleright X_{P_1} \otimes \cdots \otimes h_{(n+k)} \triangleright X_{P_k} \]
\[ = \left( \tilde{V}_{e,P}^h R_{e_j} \right) (v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k}), \]

concluding the proof of the lemma.

5.1.3 Edge orientation reversals commute with edge operators (when the edges are different)

**Lemma 65.** Let \( e \) and \( f \) \( \in \mathbb{L}^1 \) be two distinct edges. Denote by \( \tilde{E}_e^a \) for any \( a \in A \) the edge operator at \( e \) with respect to the cell decomposition \( \tilde{L} \) obtained from the given one \( L \) by reversing the orientation of the edge \( f \). Then

\[ R_f \circ \tilde{E}_e^a = \tilde{E}_e^a \circ R_f \quad \text{for all } a \in A. \]

**Proof.** Denote by \( P \in \mathbb{L}^2 \) the plaquette on the left of \( e \) and by \( Q \in \mathbb{L}^2 \) the one on the right. Let \( (e_1, \ldots, e_k) \) be the edges in the boundary of \( P \) in counter-clockwise order starting at its base-point and ending at the starting vertex of the edge \( e \). Let \( \theta_j \), for \( j = 1, \ldots, k \) be \( +1 \) if \( e_j \) is oriented counter-clockwise around \( P \) and otherwise \(-1 \). Let \((d_1, \ldots, d_{\ell})\) be the edges in the boundary of \( Q \) in clockwise order starting at its base-point and ending at the starting vertex of the edge \( e \). Let \( \sigma_j \), for \( j = 1, \ldots, \ell \) be \( +1 \) if \( d_j \) is oriented counter-clockwise around \( Q \) and otherwise \(-1 \).

Recall with this notation the definition of the edge operator \( E_e^a \) for any \( a \in A \) acting on \( v_{e_1} \otimes \cdots \otimes v_{e_k} \otimes v_{d_1} \otimes \cdots \otimes v_{d_{\ell}} \otimes v_e \otimes X_P \otimes X_Q \in H_{e_1} \otimes \cdots \otimes H_{e_k} \otimes H_{d_1} \otimes \cdots \otimes H_{d_{\ell}} \otimes H_e \otimes A_P \otimes A_Q = H^{(k+\ell+1)} \otimes A^{(2)} \):

\[ E_e^a (v_{e_1} \otimes \cdots \otimes v_{e_k} \otimes v_{d_1} \otimes \cdots \otimes v_{d_{\ell}} \otimes v_e \otimes X_P \otimes X_Q) \]
\[ = (v_{e_1})_{(1)} \otimes \cdots \otimes (v_{e_k})_{(1)} \otimes (v_{d_1})_{(1)} \otimes \cdots \otimes (v_{d_{\ell}})_{(1)} \otimes \bar{a}_3 v_e \]
\[ \otimes \left( (v_{e_1})_{(2)} \cdots (v_{e_k})_{(2)} \right) a_1 X_P \otimes X_Q \left( (v_{d_1})_{(2)} \cdots (v_{d_{\ell}})_{(2)} \right) S a_{(2)}. \]

We assume that \( f \) is either \( e_j \) for \( j = 1, \ldots, k \) or \( d_i \) for \( i = 1, \ldots, \ell \), since otherwise the supports of the operators \( R_f \) and \( E_e^a = \tilde{E}_e^a \) are disjoint. We calculate for \( f = e_j \):

\[ (R_{e_j} E_e^a)(v_{e_1} \otimes \cdots \otimes v_{e_k} \otimes v_{d_1} \otimes \cdots \otimes v_{d_{\ell}} \otimes v_e \otimes X_P \otimes X_Q) \]
\[ = R_{e_j} \left( (v_{e_1})_{(1)} \otimes \cdots \otimes (v_{e_k})_{(1)} \right) \]
\[ \otimes \left( (v_{d_1})_{(1)} \otimes \cdots \otimes (v_{d_{\ell}})_{(1)} \otimes \bar{a}_3 v_e \right) \]
\[ \otimes \left( (v_{e_1})_{(2)} \cdots (v_{e_k})_{(2)} \right) a_1 X_P \]

44
The calculation for the case $f = d_i$ is completely analogous and, hence, we do not spell it out explicitly here.

5.1.4 Edge orientation reversals commute with plaquette operators

Lemma 66. Let $e \in L^1$ be an edge and let $P \in L^2$ be a plaquette. Denote by $\tilde{F}_P^\varphi$ for any $\varphi \in H^*$ the plaquette operator at $P$ with respect to the cell decomposition $\tilde{L}$ obtained from $L$ by reversing the orientation of the edge $e$. Then:

$$R_e \circ F_P^\varphi = \tilde{F}_P^\varphi \circ R_e \quad \text{for all } \varphi \in H^*.$$

Proof. If $e$ is not in the boundary of $P$, then any plaquette operator for $P$ and the edge orientation reversal at $e$ have disjoint support.

Let us hence assume that $e$ is in the boundary of $P$. Denote by $(e_1, \ldots, e_n)$ the edges in the boundary of $P$ in counter-clockwise order starting and ending at the base-point of $P$ such that $e_j = e$ for some $j \in \{1, \ldots, n\}$. Let $(\theta_1, \ldots, \theta_n)$ such that $\theta_i := +1$ if the edge $e_i$ is oriented away from $v$ and otherwise $\theta_i := -1$ for $i \in \{1, \ldots, n\}$. With Definition 28 we recall the plaquette operator acting on $v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \in H_{e_1} \otimes \cdots \otimes H_{e_n} \otimes A_P = H^\otimes n \otimes A$:

$$F_P^\varphi(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P) = (v_{e_1}^{(\theta_1)})(\theta_1) \otimes \cdots \otimes (v_{e_n}^{(\theta_n)})(\theta_n) \otimes (X_P)(2) \varphi \left( (v_{e_1}^{(\theta_1)})^{(1)}(2) \cdots (v_{e_n}^{(\theta_n)})^{(2)}(2) \right) S\partial(X_P)(1)$$

and

$$\tilde{F}_P^\varphi(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P) = (v_{e_1}^{(\theta_1)})(\theta_1) \otimes \cdots \otimes (v_{e_{j-1}}^{(\theta_{j-1})})(\theta_{j-1}) \otimes (v_{e_j}^{(-\theta_j)})(\theta_j) \otimes (v_{e_{j+1}}^{(\theta_{j+1})})(\theta_{j+1}) \otimes \cdots \otimes (v_{e_n}^{(\theta_n)})(\theta_n) \otimes (X_P)(2) \varphi \left( (v_{e_1}^{(\theta_1)})^{(2)}(2) \cdots (v_{e_{j-1}}^{(\theta_{j-1})})^{(2)}(2) (v_{e_j}^{(-\theta_j)})^{(2)}(2) \cdots (v_{e_n}^{(\theta_n)})^{(2)}(2) \right) S\partial(X_P)(1).$$
Let us now show the claim by calculating:

\[(R_{e_j}F_P^v)(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P)\]

\[= R_{e_j}((v_{e_1}^{(\theta_1)})^{(1)}(1), (v_{e_2}^{(\theta_2)})^{(1)}(2) \otimes (X_P)(2) \varphi((v_{e_1}^{(\theta_1)})(2) \cdots (v_{e_n}^{(\theta_n)})(2)S\partial(X_P)(1)))\]

\[= (v_{e_1}^{(\theta_1)})^{(1)}(1) \cdots \otimes (v_{e_{j-1}}^{(\theta_{j-1})})^{(-\theta_j)}(1) \otimes (v_{e_j}^{(\theta_j)})^{(-\theta_j)}(1) \otimes (v_{e_{j+1}}^{(\theta_{j+1})})^{(\theta_j)}(1) \cdots \otimes (v_{e_n}^{(\theta_n)})^{(\theta_n)}(1) \otimes (X_P)(2)\]

\[= (v_{e_1}^{(\theta_1)})^{(1)}(1) \cdots \otimes (v_{e_{j-1}}^{(\theta_{j-1})})^{(-\theta_j)}(1) \otimes ((Sv_{e_j})(\theta_j))^{(-\theta_j)}(1) \otimes (X_P)(2)\]

\[\varphi((v_{e_1}^{(\theta_1)})(2) \cdots (v_{e_{j-1}}^{(\theta_{j-1})})(2) \otimes (Sv_{e_j})(\theta_j)(2) \cdots (v_{e_n}^{(\theta_n)})(2)S\partial(X_P)(1))\]

\[= (\tilde{F}_P^vR_{e_j})(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P),\]

concluding the proof of the lemma. In the penultimate step we used remark 30. \(\square\)

### 5.1.5 Base-point shifts commute with vertex operators

**Lemma 67.** Let \(\Sigma\) be an oriented surface with cell decomposition \(L\). Let \(P \in L^2\) be any plaquette and let \(v \in L^0\) be any vertex with adjacent plaquette \(Q\). Denote by \((V^\prime)^h_{v,Q}\) for any \(h \in H\) the vertex operator for the vertex \(v\) with respect to the cell decomposition \(L^\prime\) obtained from the given one \(L\) by shifting the base-point of \(P\) once in counter-clockwise direction around \(P\). Then:

\[(V^\prime)^h_{v,Q} \circ T^+_P = T^+_P \circ V^h_{v,Q}\]

for all \(h \in H\).

**Proof.** First note that this clearly holds when \(v\) is not in the boundary of \(P\), because then \((V^\prime)^h_{v,Q} = V^h_{v,Q}\), and \(V^h_{v,Q}\) and \(T^+_P\) have disjoint support.

Let hence \(v\) be in the boundary of \(P\). Unless the base-point of \(P\) is equal to \(v\) or comes right before \(v\) in the counter-clockwise order, we again have that \((V^\prime)^h_{v,Q} = V^h_{v,Q}\), and \(V^h_{v,Q}\) and \(T^+_P\) have disjoint support.

Let us consider the case that the base-point of \(P\) is equal to \(v\). Let \((e_1, \ldots, e_n)\) be the edges incident to \(v\) in counter-clockwise order around \(v\), such that \(e_n\) is the edge right after \(v\) in the boundary of \(P\) in the counter-clockwise direction, i.e. \(e_n\) is the edge along which the base-point shift \(T^+_P\) happens. By applying edge orientation reversals where necessary, which commute with the vertex operator as well as the base-point shift, we may assume that all edges \((e_1, \ldots, e_n)\) are oriented away from \(v\). Let \((P_1, \ldots, P_k)\) be the plaquettes with \(v\) as their base-point such that \(P_1 = P\). For the sake of being explicit we assume in the following calculation that \(Q = P\), but it is easy to see that any other choice of \(Q\) would allow for a completely analogous calculation. Then for any \(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k} \in H_{e_1} \otimes \cdots \otimes H_{e_n} \otimes A_{P_1} \otimes \cdots \otimes A_{P_k} = H^\otimes n \otimes A^\otimes k\) we calculate:

\[(T^+_P)^h_{v,Q}(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k})\]

\[= T^+_P(v_{e_1} \otimes \cdots \otimes h_{(n)}v_{e_n} \otimes h_{(n+1)} \triangleright X_{P_1} \otimes \cdots \otimes h_{(n+k)} \triangleright X_{P_k})\]

46
concluding the proof of the lemma.
5.1.6 Base-point shifts commute with edge operators (when the base-point shift is not parallel to the edge)

We show now that base-point shifts commute with edge projectors if the edge is not the one along which we shift the base-point.

Lemma 68. Let $\Sigma$ be an oriented surface with cell decomposition $L$. Let $e \in L^1$ be an edge and let $P \in L^2$ be a plaquette adjacent to the edge $e$. We further require that the base-point of $P$ is not equal to the starting vertex of $e$ if $e$ is oriented counter-clockwise around $P$, and not equal to the target vertex of $e$ if $e$ is oriented clockwise around $P$. Then

$$T_P^+ \circ E^a_e = E^a_e \circ T_P^+ \quad \text{for all } a \in A,$$

where $E^a_e$ denotes the edge operator of the edge $e$ for the cell decomposition $L'$ obtained from the given one $L$ by shifting the base-point of $P$ once in counter-clockwise direction.

Proof. We assume for the proof that $P$ is the plaquette on the left of the edge $e$ (with respect to the orientations of $e$ and of the underlying surface $\Sigma$), so that $e$ is oriented in counter-clockwise direction around $P$. The calculations for the case that $e$ is oriented clockwise with respect to $P$ can be done analogously.

Let us consider the edge operator $E^a_e$, $a \in A$, for the edge $e$ and the base-point shift operator $T_P^+$ for the plaquette $P$. The condition that the base-point of $P$ and the starting vertex of $e$ differ is just to say that we shift the base-point along an edge different from the edge $e$.

Denote by $(e_1, \ldots, e_\ell)$ the (non-empty) set of edges in the boundary of $P$ connecting the base-point with the starting vertex of $e$ in counter-clockwise order. By applying edge orientation reversals where necessary, since these commute with the plaquette operator and base-point shifts, we may assume that the edges $(e_1, \ldots, e_\ell)$ are oriented in counter-clockwise direction around $P$. Similarly, let $Q \in L^2$ be the plaquette on the right of the edge $e$ with respect to its orientation and let $(d_1, \ldots, d_r)$ be as in Definition 25. Using edge orientation reversals, we may assume that any edge in $(d_1, \ldots, d_r)$ is oriented clockwise around $Q$.

Using Definition 25 for the edge operator we calculate:

$$(T_P^+ \circ E^a_e)(v_{e_1} \otimes \cdots \otimes v_{e_\ell} \otimes v_e \otimes v_{d_1} \otimes \cdots \otimes v_{d_r} \otimes X_P \otimes X_Q)$$

$$= T_P^+ \left( (v_{e_1}(1)) \otimes \cdots \otimes (v_{e_\ell}(1)) \otimes \partial a(3)v_e \otimes (v_{d_1}(2)) \otimes \cdots \otimes (v_{d_r}(2)) \right.$$

$$\otimes \left( (v_{e_1}(2)) \cdots (v_{e_\ell}(2)) \triangleright a(1) \right) X_P$$

$$\otimes X_Q \left( (v_{d_1}(2)) \cdots (v_{d_r}(2)) \triangleright S a(2) \right) \big)$$

$$= \left( (v_{e_1}(1)) \otimes (v_{e_1}(2)) \cdots (v_{e_\ell}(1)) \otimes \partial a(3)v_e \otimes (v_{d_1}(2)) \otimes \cdots \otimes (v_{d_r}(2)) \right.$$

$$\otimes S((v_{e_1}(1)) \triangleright (v_{e_1}(2)) \cdots (v_{e_\ell}(2)) \triangleright a(1)) X_P$$

$$\otimes X_Q \left( (v_{d_1}(1)) \cdots (v_{d_r}(1)) \triangleright S a(2) \right) \big)$$

$$\overset{\text{coassoc.}}{=} \left( (v_{e_1}(1)) \otimes \cdots \otimes (v_{e_\ell}(1)) \otimes \partial a(3)v_e \otimes (v_{d_1}(2)) \otimes \cdots \otimes (v_{d_r}(2)) \right.$$

$$\otimes S(v_{e_1}(2)) \triangleright (v_{e_1}(3)) (v_{e_2}(2)) \cdots (v_{e_\ell}(2)) \triangleright a(1)) X_P$$

$$\otimes X_Q \left( (v_{d_1}(1)) \cdots (v_{d_r}(1)) \triangleright S a(2) \right) \big)$$

$$\overset{A \mod \text{ alg.}}{=} \left( (v_{e_1}(1)) \otimes \cdots \otimes (v_{e_\ell}(1)) \otimes \partial a(3)v_e \otimes (v_{d_1}(2)) \otimes \cdots \otimes (v_{d_r}(2)) \right.$$

$$\otimes \left( (S(v_{e_1}(2))(1))(v_{e_1}(3))(v_{e_2}(2)) \cdots (v_{e_\ell}(2)) \triangleright a(1)) \right) \left( (S(v_{e_1}(2))(2) \triangleright X_P \right)$$

48
\[ \otimes X_Q((v_{d_1})(1)\cdots(v_{d_r})(1) \triangleright S_{a(2)}) \]

\[ = (v_{e_1})(1) \otimes\cdots\otimes (v_{e_\ell})(1) \otimes \partial a(3)^v_e \otimes (v_{d_1})(2) \otimes\cdots\otimes (v_{d_r})(2) \]

\[ \otimes \left( (S(v_{e_2})(3)(v_{e_3})(4)(v_{e_4})(2)\cdots(v_{e_\ell})(2) \triangleright a(1)) \right) \left( S(v_{e_1})(2) \triangleright X_P \right) \]

\[ \otimes X_Q((v_{d_1})(1)\cdots(v_{d_r})(1) \triangleright S_{a(2)}) \]

\[ \overset{\text{antip. prop.}}{=} (v_{e_1})(1) \otimes\cdots\otimes (v_{e_\ell})(1) \otimes \partial a(3)^v_e \otimes (v_{d_1})(2) \otimes\cdots\otimes (v_{d_r})(2) \]

\[ \otimes \left( (v_{e_2})(2)\cdots(v_{e_\ell})(2) \triangleright a(1)) \right) \left( S(v_{e_1})(2) \triangleright X_P \right) \]

\[ \otimes X_Q((v_{d_1})(1)\cdots(v_{d_r})(1) \triangleright S_{a(2)}) \]

\[ = E^\alpha_e((v_{e_1})(1) \otimes v_{e_2} \otimes\cdots\otimes v_{e_\ell} \otimes v_e \otimes v_{d_1} \otimes\cdots\otimes v_{d_r} \otimes S(v_{e_1})(2) \triangleright X_P \otimes X_Q) \]

\[ = (E^\alpha_e \circ T_P^+) (v_{e_1} \otimes\cdots\otimes v_{e_\ell} \otimes v_e \otimes v_{d_1} \otimes\cdots\otimes v_{d_r} \otimes X_P \otimes X_Q) . \]

The calculation proving the commutativity for the case where \( P \) lies to the right of \( e \) is analogous.

\[ \square \]

5.1.7 Base-point shifts commute with plaquette projectors

Lemma 69. Let \( \Sigma \) be an oriented surface with cell decomposition \( L \). Let \( P \) and \( Q \) be any two (not necessarily distinct) plaquettes. Denote by \( (F^\alpha)^Q_P \) for any \( \varphi \in H^* \) the plaquette operator for the plaquette \( P \) with respect to the cell decomposition \( L' \) obtained from \( L \) by shifting the base-point of \( Q \) once in counter-clockwise direction around \( Q \). Then:

\[ (F^\alpha)^Q_P \circ T_Q^+ = T_Q^+ \circ F_P^\varphi \quad \text{for all cocommutative} \; \varphi \in H^*. \]

In particular, putting \( \varphi = \lambda \), the Haar integral of \( H^* \), the equation above holds for the respective plaquette projectors at \( P \).

Proof. If \( P \) and \( Q \) are distinct and not adjacent, then the two maps have disjoint support in \( \mathcal{H}_L \) and hence commute with each other. Let \( P \) and \( Q \) be distinct but adjacent to each other and let \( e \in L_1 \) be the edge shared by the boundaries of both plaquettes and assume that the base-point shift \( T_Q^+ \) shifts the base-point of \( Q \) along \( e \). Then the base-point shift \( T_Q^+ \) and any plaquette operator \( F_P^\varphi \), \( \varphi \in H^* \), for the adjacent plaquette \( P \) commute with each other. Indeed, the two operators affect the tensor factor \( H \) in \( \mathcal{H}_L \) associated with the edge \( e \) only by applying co-multiplication and then acting with the additional resulting tensor factor of \( H \) on a copy of \( A \) associated with the two different plaquettes \( P \) and \( Q \). Due to the Yetter-Drinfeld condition these two operators thus commute with each other.

Now assume that \( P = Q \) and denote by \( (e_1,\ldots,e_n) \) the edges in the boundary of the plaquette \( P \) in counter-clockwise order starting at the base-point of \( P \). Then for the transformed cell decomposition \( L' \) the edges in the boundary of the plaquette \( P \) in counter-clockwise order starting at the base-point of \( P \) are \( (e_2,\ldots,e_n,e_1) \). By applying edge orientation reversals where necessary, we may assume that the edges \( (e_1,\ldots,e_n) \) are all oriented in counter-clockwise direction around \( P \).

Let \( v_{e_1} \otimes\cdots\otimes v_{e_n} \in H^\otimes n \) and \( X \in A \). We calculate, on the one hand:

\[ (T_P^+ F_P^\varphi)(v_{e_1} \otimes\cdots\otimes v_{e_n} \otimes X) \]
\[
\begin{align*}
= T_P^+ \left( (v_e)_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes X_{(2)} \right) \varphi \left( (v_e)_{(2)} \cdots (v_{e_n})_{(2)} S \partial X_{(1)} \right) \\
= \left( ((v_e)_{(1)})_{(1)} \otimes (v_{e_2})_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes S((v_e)_{(1)})_{(2)} \triangleright X_{(2)} \right) \\
\quad \varphi \left( (v_e)_{(2)} \cdots (v_{e_n})_{(2)} S \partial X_{(1)} \right)
\end{align*}
\]

co-assoc. \[
= (v_e)_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes (S(v_e)_{(2)} \triangleright X_{(2)}) \varphi \left( (v_e)_{(3)} \cdots (v_{e_n})_{(2)} S \partial X_{(1)} \right)
\]

On the other hand:
\[
((F')_P^* T_P^+) (v_e \otimes \cdots \otimes v_{e_n} \otimes X)
\]

\[
= F_P^* \left( (v_e)_{(1)} \otimes \cdots \otimes v_{e_n} \otimes S((v_e)_{(1)})_{(2)} \triangleright X \right)
\]

\[
= \left( ((v_e)_{(1)})_{(1)} \otimes (v_{e_2})_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes (S(v_e)_{(1)})_{(2)} \triangleright X_{(2)} \right) \\
\quad \varphi \left( (v_e)_{(2)} \cdots (v_{e_n})_{(2)} ((v_e)_{(1)})_{(2)} S \partial (S(v_e)_{(2)})_{(1)} \triangleright X_{(1)} \right)
\]

A mod. coalg. \[
= \left( (v_e)_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes (S(v_e)_{(2)} \triangleright X_{(2)}) \right) \\
\quad \varphi \left( (v_e)_{(2)} \cdots (v_{e_n})_{(2)} (v_e)_{(4)} S \partial (S(v_e)_{(2)})_{(1)} \triangleright X_{(1)} \right)
\]

co-assoc. \[
\text{Y} \in \mathcal{D}.
\]
\[
= (v_e)_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes S(v_e)_{(2)} \triangleright X_{(2)} \\
\quad \varphi \left( (v_e)_{(2)} \cdots (v_{e_n})_{(2)} (v_e)_{(4)} S \partial (S(v_e)_{(2)})_{(1)} \triangleright X_{(1)} \right)
\]

antipod. \[
= \text{prop.}
\]
\[
= (v_e)_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes S(v_e)_{(2)} \triangleright X_{(2)} \\
\quad \varphi \left( (v_e)_{(2)} \cdots (v_{e_n})_{(2)} S \partial X_{(1)} (v_e)_{(3)} \right)
\]
\[
= \varphi \text{ cocom.}
\]
\[
= (v_e)_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes (S(v_e)_{(2)} \triangleright X_{(2)}) \varphi \left( (v_e)_{(3)} (v_e)_{(2)} \cdots (v_{e_n})_{(2)} S \partial X_{(1)} \right),
\]

showing that indeed \((F')_P^* \circ T_P^+ = T_P^+ \circ F_P^*\) for any cocommutative \(\varphi \in H^*\). \qed

Remark 70. Now we know that base-point shifts commute with plaquette operators for cocommutative elements of \(H^*\), in particular for the Haar integral \(\lambda\). Therefore base-point shifts preserve the subspace:
\[
H_L^{ff} := \text{im} \left( \prod_{P \in L^2} F_P \right) \subseteq H_L.
\]

We call \(H_L^{ff}\) the \textit{fake-flat subspace}. In the full higher Kitaev model, this subspace coincides with the subspace of the total Hilbert space spanned by the fake-flat configurations \([7]\).

Let \(P\) be a plaquette. As mentioned before, moving the base-point \(v_P\) of \(P\) around the boundary of the plaquette \(P\), to return to \(v_P\) again, by using base-point shifts, does not yield the identity in \(H_L\). However this operation restricts to the identity over the subspace \(H_L^{ff}\).
Lemma 71. Let $P \in L^2$ be a plaquette. Let $T_P^\omega : \mathcal{H}_L \rightarrow \mathcal{H}_L$ be the composition of successive base-point shifts, moving the base-point counter-clockwise once around the entire plaquette. Then:

$$T_P^\omega |_{\mathcal{H}_L^\omega} = \text{id}_{\mathcal{H}_L^\omega}.$$ 

Proof. Let $(e_1,\ldots,e_n)$ be the edges in the boundary of $P$ in counter-clockwise order starting and ending at its base-point, and assume that they are each oriented in counter-clockwise direction, which we may by applying edge orientation reversals where necessary. For $v_{e_1}\otimes \cdots \otimes v_{e_n} \otimes X_P \in H^\otimes n \otimes A$ we then have

$$T_P^\omega(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P) = (v_{e_1})_1 \otimes \cdots \otimes (v_{e_n})_1 \otimes (S((v_{e_n})_2) \cdots S((v_{e_1})_2) \triangleright X_P).$$

Now assume that $(\cdots \otimes v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes \cdots) \in (\cdots \otimes H^\otimes n \otimes A \otimes \cdots) = \mathcal{H}_L$ is a fake-flat state, in particular (cf. the proof of Proposition 10):

$$(v_{e_1})_1 \otimes \cdots \otimes (v_{e_n})_1 \otimes (X_P)_2 \otimes ((v_{e_1})_2 \cdots (v_{e_n})_2)S\partial(X_P)_n) = v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes 1_H.$$ 

Then we see that for such a state we have:

$$T_P^\omega(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P) = (v_{e_1})_1 \otimes \cdots \otimes (v_{e_n})_1 \otimes (S((v_{e_n})_2) \cdots S((v_{e_1})_2) \triangleright X_P) = (v_{e_1})_1 \otimes \cdots \otimes (v_{e_n})_1 \otimes (S((v_{e_1})_2) \cdots (v_{e_n})_2) \triangleright X_P) = (v_{e_1})_1 \otimes \cdots \otimes (v_{e_n})_1 \otimes (S\partial(X_P)_n) \triangleright (X_P)_2) \approx v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P,$$

which proves the claim. \hfill \qed

5.2 Commutation relations between vertex, edge and plaquette operators

5.2.1 Commutation relations among vertex operators

Lemma 72. Let $v \in L^0$ be a vertex and $P \in L^2$ an adjacent plaquette. The vertex operators $(V^h_{v,P})_{h \in H}$ define a representation of $H$ on $\mathcal{H}_L$, that is:

$$V^h_{v,P} \circ V^{h'}_{v,P} = V^{hh'}_{v,P} \quad \text{for all } h, h' \in H.$$ 

Proof. By applying edge orientation reversals where necessary, we may assume that all edges incident to $v$ are oriented away from $v$. Denote these edges by $(e_1,\ldots,e_n)$ in counter-clockwise order starting and ending at $P$. Denote further the plaquettes which have $v$ as their base-point by $(P_1,\ldots,P_k)$. Then we have for any $v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k} \in H^\otimes n \otimes A^\otimes k$:

$$V^h_{v,P} \circ V^{h'}_{v,P} (v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k}) = V^h_{v,P} (h'_1 v_{e_1} \otimes \cdots \otimes h'_n v_{e_n} \otimes h'_{n+1} \triangleright X_{P_1} \otimes \cdots \otimes h'_{n+k} \triangleright X_{P_k}) = h'(1) h'_1 v_{e_1} \otimes \cdots \otimes h'(n) h'_n v_{e_n} \otimes h'(n+1) \triangleright h'_{n+1} \triangleright X_{P_1} \otimes \cdots \otimes h'_{(n+k)} \triangleright X_{P_k} = (V^{hh'}_{v,P}) (v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_{P_1} \otimes \cdots \otimes X_{P_k}).$$

\hfill \qed
Lemma 73. Let $v_1$ and $v_2 \in L^0$ be two distinct vertices. Then the corresponding vertex operators commute with each other, that is:

$$V_{v_1,P_1}^{h_1} \circ V_{v_2,P_2}^{h_2} = V_{v_2,P_2}^{h_2} \circ V_{v_1,P_1}^{h_1} \quad \text{for all } h_1, h_2 \in H,$$

where $P_1$ and $P_2 \in L^2$ are any plaquettes adjacent to $v_1$ and $v_2$, respectively.

Proof. The vertex operators only have intersecting supports if $v_1$ and $v_2$ are the end-points of a common edge $e \in L^1$. In this case, the intersection of their support is only the copy of $H$ that is associated with $e$. On this tensor factor the two vertex operators in question act via left or right multiplication, depending on whether the edge $e$ is directed away from or towards a given vertex. Since this differs for the two vertices, one of the operators acts via left multiplication and the other via right multiplication, and hence they commute with each other. \qed

In particular, putting $h_1 = h_2 = \ell \in H$, and using this last lemma and the previous one, vertex projectors commute: $[V_{v_1}, V_{v_2}] = 0$, for each $v_1, v_2 \in L^0$.

5.2.2 Commutation relations among edge operators (for cocommutative elements)

Lemma 74. Let $e \in L^1$ be an edge. The edge operators $(E_e^a)_{a \in A}$ define a representation of $A$ on $\mathcal{H}_L$, that is:

$$E_e^a \circ E_e^{a'} = E_e^{aa'} \quad \text{for all } a, a' \in A.$$

Proof. Let $P$ and $Q \in L^2$ be the two plaquettes adjacent to $e$, such that $P$ is on the left side of $e$ with respect to the orientations of $e$ and $\Sigma$, and $Q$ is on its right. By applying base-point shifts where necessary (and then undoing the shifts in the reverse way, applying Remark 35), we may assume that the base-points of $P$ and $Q$ are both equal to the starting vertex of the edge $e$. Note that none of the base-point shifts is parallel to $e$, so we can apply Lemma 68. Then the support of the edge operators in $\mathcal{H}_L$ is $H_e \otimes A_P \otimes A_Q = H \otimes A \otimes A$ and we have for any $v_e \otimes X_P \otimes X_Q \in H \otimes A \otimes A$:

$$(E_e^a \circ E_e^{a'})(v_e \otimes X_P \otimes X_Q)
= E_e^{a'}(\partial a'(3)v_e \otimes a(1)X_P \otimes X_Q Sa(2))
= \partial a'(3)v_e \otimes a(1)a'(1)X_P \otimes X_Q Sa(2)Sa(2)
= \partial(aa')(3)v_e \otimes (aa')(1)X_P \otimes X_Q Sa(2)
= (E_e^{aa'})(v_e \otimes X_P \otimes X_Q).$$

\qed

Lemma 75. Let $e_1$ and $e_2 \in L^1$ be two distinct edges. Then

$$E_{e_1}^{a_1} \circ E_{e_2}^{a_2} = E_{e_2}^{a_2} \circ E_{e_1}^{a_1} \quad \text{for any two cocommutative elements } a_1, a_2 \in A. \quad (11)$$

Proof. The edge operators $E_{e_1}^{a_1}$ and $E_{e_2}^{a_2}$, where $a_1, a_2 \in A$, have intersecting supports only if they lie in the boundary of a common plaquette, or if not but there are edges that lie simultaneously in the boundary of a plaquette adjacent to $e_1$ and in the boundary of a plaquette adjacent to $e_2$.

In the latter case the operators’ supports only intersect in edges different from $e_1$ and $e_2$. Since the edge operators for $e_1$ and $e_2$ act on such edges only by applying the co-multiplication of $H$ and acting with the additional resulting tensor factor on a copy of $A$ associated with one of the plaquettes, we can see that due to the Yetter-Drinfeld condition such operators $E_{e_1}^{a_1}$ and $E_{e_2}^{a_2}$ commute with each other.
We may hence assume that \(e_1\) and \(e_2\) lie in the boundary of a common plaquette \(P \in L^2\). We show in what follows that then \(E_{e_1}^{a_1} \circ E_{e_2}^{a_2} = E_{e_2}^{a_2} \circ E_{e_1}^{a_1}\) for any cocommutative \(a_1, a_2 \in A\). Denote by \(P_1\) the other plaquette adjacent to \(e_1\) and denote by \(P_2\) the other plaquette adjacent to \(e_2\). By applying base-point shifts that do not affect \(e_1\) and \(e_2\) (Lemma \([63]\)) and exchanging \(e_1\) and \(e_2\) if necessary, we may assume that the base-point of \(P\) is equal to the starting vertex of \(e_1\). Furthermore, we may assume that the base-point of \(P_1\) is also equal to the starting vertex of \(e_1\), and, for the case that \(P_1 \neq P_2\), that the base-point of \(P_2\) is equal to the starting vertex of \(e_2\).

For the explicit calculation of the commutation relations of edge operators it is convenient to distinguish between the cases \(P_1 = P_2\) and \(P_1 \neq P_2\) and within each of these to further distinguish between the following cases:

(i) Both edges \(e_1\) and \(e_2\) are oriented counter-clockwise with respect to \(P\).

(ii) Both edges \(e_1\) and \(e_2\) are oriented clockwise with respect to \(P\).

(iii) \(e_1\) and \(e_2\) are in opposite orientation to each other with respect to \(P\).

\(P_1 = P_2\): \([i]\)

Let \((d_1, \ldots, d_k)\) be the edges between \(e_1\) and \(e_2\) (possibly none) in counter-clockwise order around \(P\). By applying edge orientation reversals where necessary, we may assume that the edges \((d_1, \ldots, d_k)\) are all oriented in counter-clockwise direction around \(P\). Let \(v_{e_1} \otimes v_{d_1} \otimes \cdots \otimes v_{d_k} \in H_{d_1} \otimes H_{d_2} \otimes \cdots \otimes H_{d_k} \otimes H_{e_2} = H^{\otimes (k+2)}\) and let \(X_P \otimes X_{P_1} \in A_P \otimes A_{P_1}\). For the edge operators \(E_{e_1}^a\) and \(E_{e_2}^{a'}\), if \(a \in A\) is cocommutative, we then have:

\[
(E_{e_2}^{a'}E_{e_1}^a)(v_{e_1} \otimes v_{d_1} \otimes \cdots \otimes v_{d_k} \otimes v_{e_2} \otimes X_P \otimes X_{P_1})
\]

\[= E_{e_2}^{a'}(\partial(a(3))v_{e_1} \otimes v_{d_1} \otimes \cdots \otimes v_{d_k} \otimes v_{e_2} \otimes a(1)X_P \otimes X_{P_1}S(a(2)))\]

\[= (\partial(a(3))v_{e_1})_1 \otimes (v_{d_1})_1 \otimes \cdots \otimes (v_{d_k})_1 \otimes \partial(a'(3))v_{e_2}
\]

\[\otimes \left((\partial(a(3))v_{e_1})_3 \otimes (v_{d_1})_3 \otimes \cdots \otimes (v_{d_k})_3 \otimes a'(1)_1\right) X_P
\]

\[\otimes X_{P_1}S(a(2))\left((\partial(a(3))v_{e_1})_2 \otimes (v_{d_1})_2 \otimes \cdots \otimes (v_{d_k})_2 \otimes S(a'(2))\right)\]

\[= \partial(a(3))v_{e_1})_1 \otimes (v_{d_1})_1 \otimes \cdots \otimes (v_{d_k})_1 \otimes \partial(a'(3))v_{e_2}
\]

\[\otimes \left((\partial(a(3))v_{e_1})_3 \otimes (v_{d_1})_3 \otimes \cdots \otimes (v_{d_k})_3 \otimes a'(1)_1\right) X_P
\]

\[\otimes X_{P_1}S(a(2))\left((\partial(a(4))v_{e_1})_2 \otimes (v_{d_1})_2 \otimes \cdots \otimes (v_{d_k})_2 \otimes S(a'(2))\right)\]

\[\overset{P_1 = 2}{=} \partial(a(3))v_{e_1})_1 \otimes (v_{d_1})_1 \otimes \cdots \otimes (v_{d_k})_1 \otimes \partial(a'(3))v_{e_2}
\]

\[\otimes \left((\partial(a(3))v_{e_1})_3 \otimes (v_{d_1})_3 \otimes \cdots \otimes (v_{d_k})_3 \otimes a'(1)_1\right) S(a(6))a(1)_1X_P
\]

\[\otimes X_{P_1}S(a(2))\left((\partial(a(4))v_{e_1})_2 \otimes (v_{d_1})_2 \otimes \cdots \otimes (v_{d_k})_2 \otimes S(a'(2))\right)\]
\[ a \text{ cocomm.} \implies \partial(a_{(2)})(v_{e_1})(1) \otimes (v_{d_1})(1) \otimes \cdots \otimes (v_{d_k})(1) \otimes \partial(a'_{(3)})v_{e_2} \]
\[ \otimes a(4)((v_{e_1})(3)(v_{d_1})(3) \cdots (v_{d_k})(3) \triangleright a'_{(3)})S(a_{(5)})a_{(6)}X_P \]
\[ \otimes X_{P_1}S(a_{(1)})(\partial(a_{(3)})(v_{e_1})(2)(v_{d_1})(2) \cdots (v_{d_k})(2) \triangleright S(a'_{2})) \]

\[ \text{antip. prop.} \implies \partial(a_{(2)})(v_{e_1})(1) \otimes (v_{d_1})(1) \otimes \cdots \otimes (v_{d_k})(1) \otimes \partial(a'_{(3)})v_{e_2} \]
\[ \otimes a(4)((v_{e_1})(3)(v_{d_1})(3) \cdots (v_{d_k})(3) \triangleright a'_{(3)})X_P \]
\[ \otimes X_{P_1}S(a_{(1)})(\partial(a_{(3)})(v_{e_1})(2)(v_{d_1})(2) \cdots (v_{d_k})(2) \triangleright S(a'_{2})) \]

\[ \partial \text{ coalg. map} \implies \partial(a_{(3)})(v_{e_1})(1) \otimes (v_{d_1})(1) \otimes \cdots \otimes (v_{d_k})(1) \otimes \partial(a'_{(3)})v_{e_2} \]
\[ \otimes a(4)((v_{e_1})(3)(v_{d_1})(3) \cdots (v_{d_k})(3) \triangleright a'_{(3)})X_P \]
\[ \otimes X_{P_1}S(a_{(1)})(\partial(a_{(2)})(v_{e_1})(2)(v_{d_1})(2) \cdots (v_{d_k})(2) \triangleright S(a'_{2})) \]

\[ \text{Proof.} \quad 2 \implies \partial(a_{(4)})(v_{e_1})(1) \otimes (v_{d_1})(1) \otimes \cdots \otimes (v_{d_k})(1) \otimes \partial(a'_{(3)})v_{e_2} \]
\[ \otimes a(5)((v_{e_1})(3)(v_{d_1})(3) \cdots (v_{d_k})(3) \triangleright a'_{(3)})X_P \]
\[ \otimes X_{P_1}S(a_{(1)})a_{(2)}((v_{e_1})(2)(v_{d_1})(2) \cdots (v_{d_k})(2) \triangleright S(a'_{2}))S(a_{(3)}) \]

\[ \text{antip. prop.} \implies \partial(a_{(2)})(v_{e_1})(1) \otimes (v_{d_1})(1) \otimes \cdots \otimes (v_{d_k})(1) \otimes \partial(a'_{(3)})v_{e_2} \]
\[ \otimes a(3)((v_{e_1})(3)(v_{d_1})(3) \cdots (v_{d_k})(3) \triangleright a'_{(3)})X_P \]
\[ \otimes X_{P_1}((v_{e_1})(2)(v_{d_1})(2) \cdots (v_{d_k})(2) \triangleright S(a'_{2}))S(a_{(1)}) \]

\[ a \text{ cocomm.} \implies \partial(a_{(3)})(v_{e_1})(1) \otimes (v_{d_1})(1) \otimes \cdots \otimes (v_{d_k})(1) \otimes \partial(a'_{(3)})v_{e_2} \]
\[ \otimes a(1)((v_{e_1})(3)(v_{d_1})(3) \cdots (v_{d_k})(3) \triangleright a'_{(3)})X_P \]
\[ \otimes X_{P_1}((v_{e_1})(2)(v_{d_1})(2) \cdots (v_{d_k})(2) \triangleright S(a'_{2}))S(a_{(2)}) \]

\[ = (E_{e_{1}}^{a}E_{e_2}^{a'})v_{e_1} \otimes v_{e_2} \otimes X_{P} \otimes X_{P_1}. \]

\[ (ii) \]
Here the calculation is as for \[ (i) \] but since now the edges \( e_1 \) and \( e_2 \) are counter-clockwise oriented with respect to \( P_1 = P_2 \), the roles of \( P \) and \( P_1 \) are reversed and \( (d_1, \ldots, d_k) \) are now the edges in counter-clockwise order around \( P_1 \) from \( e_1 \) to \( e_2 \), each oriented in counter-clockwise direction around \( P_1 \).

\[ (iii) \]
In this case, \( E_{e_1}^{a} \) and \( E_{e_2}^{a'} \) for any \( a, a' \in A \) act on the tensor factors \( A_P \) and \( A_{P_1} \) by multiplication from opposite sides and therefore it is clear that they commute with each other.
$P_1 \neq P_2$.

The calculations for the case that $P_1 \neq P_2$ are analogous to the ones for $P_1 = P_2$, but simpler, since now instead of the previous tensor factor $A_P$, there are two tensor factors $A_{P_1}$ and $A_{P_2}$, which are each only in the support of one of the two edge operators.

In particular, putting $a_1 = a_2 = \Lambda \in A$, and using this last lemma and the previous one, edge projectors commute: $[E_{e_1}, E_{e_2}] = 0$, for each $e_1, e_2 \in L^1$.

### 5.2.3 Commutation relations among plaquette operators

**Lemma 76.** Let $P \in L^2$ be a plaquette. Then the plaquette operators $(F_P^\varphi)_{\varphi \in H^*}$ define a representation of $H^*$ on $H_L$, that is:

$$F_P^\varphi \circ F_P^{\varphi'} = F_P^{\varphi \cdot \varphi'} \text{ for all } \varphi, \varphi' \in H^*.$$

**Proof.** By applying edge orientation reversals where necessary we may assume that the edges $(e_1, \ldots, e_n)$, in counter-clockwise order around $P$, are each oriented in counter-clockwise direction around $P$. Then we have for any $v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \in H^{\otimes n} \otimes A$:

$$
(F_P^\varphi \circ F_P^{\varphi'})(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P)
= F_P^\varphi ((v_{e_1}(1) \otimes \cdots \otimes (v_{e_n}(1) \otimes (X_P)(2))\varphi'((v_{e_1}(2) \cdots (v_{e_n}(2)S \partial(X_P)(1))
= ((v_{e_1}(1))_1 \otimes \cdots \otimes ((v_{e_n}(1))(1) \otimes ((X_P)(2))_2\varphi(((v_{e_1}(2) \cdots (v_{e_n}(2))S \partial((X_P)(2)(1))
\varphi'((v_{e_1}(2) \cdots (v_{e_n}(2))S \partial(X_P)(1))
= \text{anti-coalg.map}
= (v_{e_1}(1) \otimes \cdots \otimes (v_{e_n}(1) \otimes (X_P)(2))\varphi(((v_{e_1}(2) \cdots (v_{e_n}(2))S \partial(X_P)(1))
\varphi'((v_{e_1}(2) \cdots (v_{e_n}(2))S \partial(X_P)(1))
= (F_P^\varphi \varphi')(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P),
$$

concluding the proof of the lemma.

**Lemma 77.** Let $P_1 \in L^2$ and $P_2 \in L^2$ be any two distinct plaquettes. Then

$$(F_{P_1}^{\varphi_1} \circ F_{P_2}^{\varphi_2} = F_{P_2}^{\varphi_2} \circ F_{P_1}^{\varphi_1} \text{ for all } \varphi_1, \varphi_2 \in H^*.$$\n
**Proof.** If $P_1$ and $P_2$ are not adjacent to each other, then the corresponding plaquette operators $F_{P_1}^{\varphi_1}$ and $F_{P_2}^{\varphi_2}$ for any $\varphi_1, \varphi_2 \in H^*$ have disjoint support in $H_L$ and therefore they commute.

Let us assume that $P_1$ and $P_2$ are adjacent to each other, i.e. there exist edges which lie in both the boundary of $P_1$ and the boundary of $P_2$. These edges form the intersection of the supports of the plaquette operators $F_{P_1}^{\varphi_1}$ and $F_{P_2}^{\varphi_2}$. For the question of commutativity of the operators it is thus sufficient to restrict our attention to these edges. Since any such edge will have opposite orientations relative to $P_1$ as opposed to $P_2$, the corresponding operators will act via opposite-sided co-multiplication (that is, one of them left and the other right) of the Hopf algebra $H$ assigned to the edges. Therefore, due to coassociativity the operators commute with each other.

In particular, putting $\varphi_1 = \varphi_2 = \lambda \in H^*$, and using this last lemma together with the previous one, plaquette projectors commute: $[F_{P_1}, F_{P_2}] = 0$, for each $P_1, P_2 \in L^2$.  

55
5.2.4 Commutation relations between vertex and edge operators

Lemma 78. Let $v \in L^0$ be a vertex, with adjacent plaquette $P \in L^2$, and let $e \in L^1$ be an edge. Then for any $h \in H$ and $a \in A$ the following hold:

(i) $E_e^{h(1)\partial a} \circ V_{v,P}^{h(2)} = V_{v,P}^h \circ E_e^a$, if $v$ is the starting vertex of $e$.

(ii) $E_e^a \circ V_{v,P}^h = V_{v,P}^h \circ E_e^a$, if $v$ is not the starting vertex of $e$.

Proof. Let $P_1 \in L^2$ be the plaquette on the left of the edge $e$ with respect to the orientations of $\Sigma$ and $e$ and let $P_2 \in L^2$ be the plaquette on the right.

The supports of the corresponding vertex operator $V_{v,P}^h$, where $h \in H$, and edge operator $E_e^a$, where $a \in A$, intersect only if $v$ lies in the boundary of one of the two (or both) plaquettes $P_1$ and $P_2$ adjacent to $e$.

By using base-point shifts that do not affect the edge $e$ we may assume that both adjacent plaquettes $P_1$ and $P_2$ have the starting vertex of $e$ as their base-points. By using edge orientation reversals that do not affect the edge $e$, we may further assume that all edges in the boundary of the plaquette $P_1$ are oriented in counter-clockwise direction around $P_1$, and that all edges in the boundary of $P_2$ except $e$ are oriented in counter-clockwise direction around $P_2$.

Let $(e_1, e_2, \ldots, e_n)$ be the edges in the boundary of $P_1$ in counter-clockwise order starting at the base-point of $P_1$, and let $(d_1, \ldots, d_m = e)$ be the edges in the boundary of $P_2$ in counter-clockwise order starting at the base-point of $P_2$.

We distinguish the following cases:

(i) $v$ is the starting vertex of $e$ (and also the base-point of both $P_1$ and $P_2$).

(ii) $v$ is the target vertex of $e$.

(iii) $v$ is not adjacent to $e$ (but in the boundary of $P_1$ or $P_2$).

Assume $v$ is the starting vertex of $e$. Furthermore, assume that the adjacent plaquette $P$ needed to define the vertex operator $V_{v,P}^h$ is $P_1$, even though the following relation and its calculation hold for any choice of adjacent plaquette. Denote by $(f_1, \ldots, f_k)$ the edges incident to $v$ in counter-clockwise order around $v$, starting with $f_1 = e_n$. Then $f_{k-1} = d_1$ and $f_k = e$.

Again, by applying edge orientation reversals where necessary, we may assume without loss of generality that the edges $(f_2, \ldots, f_{k-2})$ are all oriented away from the vertex $v$. Also we may assume, by applying base-point shifts on plaquettes other than $P_1$ and $P_2$, that $v$ is not the base-point of any other plaquettes than $P_1$ and $P_2$.

Then we show that for all $h \in H$ and $a \in A$

\[
E_e^{h(1)\partial a} \circ V_{v,P}^{h(2)} = V_{v,P}^h \circ E_e^a.
\]

\[
(E_e^{h(1)\partial a}V_{v,P}^{h(2)})(v_e \otimes v_{e_2} \otimes \cdots \otimes v_{e_n} \otimes v_{f_2} \otimes \cdots \otimes v_{f_{k-2}} \otimes v_{d_1} \otimes \cdots \otimes v_{d_{m-1}} \otimes X_{P_1} \otimes X_{P_2})
\]

\[
= E_e^{h(1)\partial a}(h_{k+1})v_e \otimes v_{e_2} \otimes \cdots \otimes v_{e_{n-1}} \otimes v_{e_n}Sh_{(2)}(h_{3})v_{f_2} \otimes \cdots \otimes h_{(k-1)}v_{f_{k-2}} \times h_{(k)}v_{d_1} \otimes v_{d_2} \otimes \cdots \otimes v_{d_{m-1}} \otimes h_{(k+2)}^{-1}X_{P_1} \otimes h_{(k+3)}^{-1}X_{P_2})
\]

\[
= \partial(h_{(1)} \triangleright a)(h_{(k+1)})v_e \otimes v_{e_2} \otimes \cdots \otimes v_{e_{n-1}} \otimes v_{e_n}Sh_{(2)}(h_{3})v_{f_2} \otimes \cdots \otimes h_{(k-1)}v_{f_{k-2}} \times h_{(k)}v_{d_1} \otimes v_{d_2} \otimes \cdots \otimes v_{d_{m-1}} \otimes (h_{(1)} \triangleright a)(h_{(k+1)} \triangleright X_{P_1}) \otimes (h_{(k+2)} \triangleright X_{P_2})S(h_{(1)} \triangleright a)(h_{(k+3)} \triangleright X_{P_2})\]
Assume $v$ is the target vertex of $e$. Furthermore, assume that the adjacent plaquette $P$ needed to define the vertex operator $V_{e,P}^h$ is $P_1$, even though the following relation and its calculation hold for any choice of adjacent plaquette. Denote by $(f_1, \ldots, f_k)$ the edges incident to $v$ in counter-clockwise order around $v$, starting with $f_1 = e$. Then $f_2 = d_{m-1}$ and $f_k = e_2$. Again, by applying edge orientation reversals where necessary, we may assume without loss of generality that the edges $(f_3, \ldots, f_{k-1})$ are all oriented away from the vertex $v$. Also we may assume, by applying base-point shifts on plaquettes other than $P_1$ and $P_2$, that $v$ is not the base-point of any plaquette.

Then we show that for all $h \in H$ and $a \in A$

$$E_c^a \circ V_{v,P}^h = V_{v,P}^h \circ E_c^a$$

$$(E_c^a V_{v,P}^h)(v_e \otimes v_{f_2} \otimes v_{f_3} \otimes \cdots \otimes v_{f_{k-1}} \otimes v_{e_2} \otimes \cdots \otimes v_{e_n} \otimes v_{d_1} \otimes \cdots \otimes v_{d_{m-2}} \otimes X_{P_1} \otimes X_{P_2})$$

$$= E_c^a(v_e S(h_{(1)}) \otimes v_{f_2} S(h_{(2)}) \otimes h_{(3)} v_{f_3} \otimes \cdots \otimes h_{(k-1)} v_{f_{k-1}} \otimes h_{(k)} v_{e_2})$$

$\textbf{(ii)}$
\[
\otimes v_{e3} \otimes \cdots \otimes v_{en} \otimes v_{d1} \otimes \cdots \otimes v_{dm-2} \otimes X_{P1} \otimes X_{P2}
\]

\[
= (\partial(a_{(3)})v_{e}S(h_{(1)}) \otimes v_{f2}S(h_{(2)}) \otimes h_{(3)}v_{f3} \otimes \cdots \otimes h_{(k-1)}v_{f_{k-1}} \otimes h_{(k)}v_{e2}
\otimes v_{e3} \cdots \otimes v_{en} \otimes v_{d1} \cdots \otimes v_{dm-2} \otimes a_{(1)}X_{P1} \otimes X_{P2}S_{a(2)})
\]

\[
= (V^{h}_{e,P}E^{a}_{e})(v_{e} \otimes v_{f2} \otimes v_{f3} \otimes \cdots \otimes v_{f_{k-1}} \otimes v_{e2} \otimes \cdots \otimes v_{en} \otimes v_{d1} \otimes \cdots \otimes v_{dm-2} \otimes X_{P1} \otimes X_{P2}).
\]

(iii) Since \(v\) is not the base-point of \(P_1\) or \(P_2\), which we may without loss of generality assume to be the starting vertex of the edge \(e\) (see above), and since \(v\) is not adjacent to \(e\), the corresponding vertex operator and edge operator have disjoint support and therefore clearly commute with each other.

In particular, due to Lemma 11, the previous lemma implies that:

**Corollary 79.** The vertex projector \(V_v = V_{v,P}^{\ell}\) and edge projector \(E_e = E_e^{\Lambda}\) commute with each other, where \(\ell \in H\) and \(\Lambda \in A\) are the Haar integrals of the respective Hopf algebras, if \(v\) is the starting vertex of the edge \(e\).

The previous two results imply that if \(e \in E^1\) and \(v \in L^0\) then \([E_e, V_v] = 0\).

### 5.2.5 Commutation relations between edge and plaquette operators

**Lemma 80.** Let \(P \in L^2\) be a plaquette and let \(e \in L^1\) be an edge. If \(e\) is in the boundary of \(P\), then

\[
F_{e}^{\varphi} \circ E_{e}^{a} = E_{e}^{a} \circ F_{e}^{\varphi} \quad \text{for all cocommutative } a \in A, \text{ and cocommutative } \varphi \in H^*.
\]

Moreover, if additionally the starting vertex of \(e\) is the base-point of \(P\), then

\[
F_{e}^{\varphi(S\partial a_{(3)}; \partial a_{(1)})} \circ E_{e}^{a(2)} = E_{e}^{a} \circ F_{e}^{\varphi} \quad \text{for all } a \in A, \varphi \in H^*,
\]

if \(e\) is oriented counter-clockwise around \(P\) (note that the two previous formulae coincide if \(a\) and \(\varphi\) are both cocommutative), and

\[
F_{e}^{\varphi} \circ E_{e}^{a} = E_{e}^{a} \circ F_{e}^{\varphi} \quad \text{for all } a \in A, \varphi \in H^*,
\]

if \(e\) is oriented clockwise around \(P\).

If \(e\) is not in the boundary of \(P\), then

\[
F_{e}^{\varphi} \circ E_{e}^{a} = E_{e}^{a} \circ F_{e}^{\varphi} \quad \text{for all } a \in A, \varphi \in H^*.
\]

**Proof.** There are two types of situations in which the plaquette operators \(F_{e}^{\varphi}\) and the edge operators \(E_{e}^{a}\) can have a non-trivially intersecting support. Either the edge \(e\) is in the boundary of the plaquette \(P\), or it is not but some of the edges in the path connecting the base-point of a plaquette, say \(Q\), which is adjacent to the edge \(e\), with the starting vertex of \(e\) lie in the boundary of \(P\). In the latter case, any of the plaquette operators \(F_{e}^{\varphi}\) commutes with any of the edge operators \(E_{e}^{a}\) because each edge in the common support is oppositely oriented relative to the plaquettes \(P\) and \(Q\) and, hence, the operators are defined in terms of different-sided (that is, one of them left and the other right) co-multiplication of the Hopf algebra \(H\) associated to such an edge. Therefore they commute with each other for any \(a \in A\) and \(\varphi \in H^*\).

We may thus for the rest of the proof assume that \(e\) lies in the boundary of \(P\).
Let us first assume that \(e\) is oriented in counter-clockwise direction around \(P\). For now assume further that the starting vertex of \(e\) is the base-point of \(P\). By applying base-point shifts where necessary, we may assume that starting vertex of \(e\) is also the base-point of the other plaquette \(Q\). Denote by \((e_1, \ldots, e_n)\) the edges in the boundary of \(P\) in counter-clockwise order starting and ending at the base-point of \(P\). Recall that \(e = e_1\) is oriented in counter-clockwise direction around \(P\). By applying edge orientation reversals where necessary, we may assume that also the edges \((e_2, \ldots, e_n)\) are oriented in counter-clockwise direction around \(P\). Then for \(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes X_Q \in H_{e_1} \otimes \cdots \otimes H_{e_n} \otimes A_P \otimes A_Q = H^\otimes n \otimes A \otimes A\), the plaquette operator \(F^\varphi_P\) for any \(\varphi \in H^*\) acts, according to Definition \ref{def:plaquette}, as

\[
F^\varphi_P(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes X_Q) = (v_{e_1})_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes (X_P)_{(2)} \otimes X_Q \varphi((v_{e_1})_{(2)} \cdots (v_{e_n})_{(2)} S\partial(X_P)_{(1)})
\]

and the edge operator \(E^a_e\) for any \(a \in A\) acts, according to Definition \ref{def:edge}, as

\[
E^a_e(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes X_Q) = \partial(a_{(3)})v_{e_1} \otimes v_{(e_2)} \otimes \cdots \otimes v_{e_n} \otimes a_{(1)}X_P \otimes X_Q S(a_{(2)}).
\]

We calculate:

\[
(F^\varphi_P(S\partial a_{(3)} : \partial a_{(1)}) E^{a_{(2)})}(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes X_Q)
\]

\[
= F^\varphi_P(S\partial a_{(5)} : \partial a_{(1)}) (\partial(a_{(4)})v_{e_1} \otimes v_{e_2} \otimes \cdots \otimes v_{e_n} \otimes a_{(2)}X_P \otimes X_Q S(a_{(3)})
\]

\[
= (\partial(a_{(4)})v_{e_1})_{(1)} \otimes (v_{e_2})_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes (a_{(2)}X_P)_{(2)} \otimes X_Q S(a_{(3)}) \varphi(S\partial a_{(5)}(\partial(a_{(4)})v_{e_1}_{(2)}(v_{e_2})_{(2)} \cdots (v_{e_n})_{(2)}S\partial(a_{(2)}X_P)_{(1)}\partial a_{(1)})
\]

\[
= \partial(a_{(3)})(v_{e_1})_{(1)} \otimes (v_{e_2})_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes a_{(3)}(X_P)_{(2)} \otimes X_Q S(a_{(4)}) \varphi(S\partial a_{(7)}(\partial(a_{(6)})v_{e_1}_{(2)}(v_{e_2})_{(2)} \cdots (v_{e_n})_{(2)}S\partial(X_P)_{(1)}S\partial a_{(2)}\partial a_{(1)})
\]

antipode prop.

\[
\partial(a_{(3)})(v_{e_1})_{(1)} \otimes (v_{e_2})_{(1)} \otimes \cdots \otimes (v_{e_n})_{(1)} \otimes a_{(3)}(X_P)_{(2)} \otimes X_Q S(a_{(2)}) \varphi((v_{e_1})_{(2)}(v_{e_2})_{(2)} \cdots (v_{e_n})_{(2)}S\partial(X_P)_{(1)})
\]

\[
= (E^a_e F^\varphi_P)(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes X_Q).
\]

Now for cocommutative elements \(a \in A\) and \(\varphi \in H^*\) the relation just shown implies that the operators commute:

\[
E^a_e \circ F^\varphi_P = F^\varphi_P(S\partial a_{(3)} : \partial a_{(1)}) \circ E^{a_{(2)}_e} = F^\varphi_P(S\partial a_{(1)} S\partial a_{(3)}) \circ E^{a_{(2)}_e} = F^\varphi_P \circ E^a_e.
\]

Next assume that \(e\) lies in the boundary of \(P\), but the starting vertex of \(e\) is not equal to the base-point of \(P\). Then, since all edge operators at \(e\) commute with base-point shifts along edges other than \(e\), and the plaquette operators for cocommutative elements of \(H^*\) commute
with base-point shifts, see Lemma 69, we may reduce the situation to the above situation where the starting vertex of $e$ and the base-point of $P$ were the same, and we obtain:

$$F^\varphi_P(\partial a(3)\partial a(1)) \circ E^a_e = E^a_e \circ F^\varphi_P$$

for all $a \in A$, and cocommutative $\varphi \in H^*$,

and, hence, for $a \in A$ and $\varphi \in H^*$ both cocommutative we obtain again: $F^\varphi_P \circ E^a_e = E^a_e \circ F^\varphi_P$.

Finally, assume now that the edge $e$ in the boundary of $P$ is oriented clockwise around $P$. We again first consider the case where the starting vertex of $e$ coincides with the base-point of $P$. Again by applying base-point shifts where necessary, we may assume that the base-point of the other plaquette $Q$ adjacent to $e$ is also equal to the starting vertex of $e$. Denoting by $(e_1, \ldots, e_n = e)$ the edges in the boundary of $P$ in counter-clockwise order, we may assume by applying edge orientation reversals where necessary, that the edges $(e_1, \ldots, e_{n-1})$ are oriented counter-clockwise around $P$, whereas $e_n = e$ is by assumption oriented clockwise around $P$. Now a very similar calculation as the first one in this proof above shows that

$$F^\varphi_P \circ E^a_e = E^a_e \circ F^\varphi_P$$

for all $a \in A$, $\varphi \in H^*$.

We calculate:

$$(F^\varphi_P E^a_e)(v_{e_1} \otimes \cdots \otimes v_{e_n} \otimes X_P \otimes X_Q)$$

$$= F^\varphi_P(v_{e_1} \otimes \cdots \otimes v_{e_{n-1}} \otimes \partial a(3)v_{e_n} \otimes X_PSa(2) \otimes a(1)X_Q)$$

$$= (v_{e_1})(1) \otimes \cdots \otimes (v_{e_{n-1}})(1) \otimes \partial a(3)(v_{e_n})(2) \otimes (X_PSa(2))(2) \otimes a(1)X_Q$$

$$\varphi((v_{e_1})(2) \cdots (v_{e_{n-1}})(2)S(\partial a(3)v_{e_n})(1)S\partial(X_PSa(2))(1))$$

By applying the antipodal property and using the antipodal orientation of $H$, we obtain:

$$\Delta \text{ alg. map}$$

$$\text{S anti-adj. map}$$

$$\text{antipod. prop.}$$

Finally, for the case that the starting vertex of $e$ and the base-point of $P$ do not coincide, as above it follows using base-point shifts that $F^\varphi_P \circ E^a_e = E^a_e \circ F^\varphi_P$ still holds for all cocommutative $\varphi \in H^*$ and all $a \in A$.

Again by passing to edge projectors $E_e = E^\lambda_e$ and plaquette projectors $F_P = F^\lambda_P$, and noting that the Haar integrals $\Lambda \in A$ and $\lambda \in H^*$ are cocommutative, it follows that $[E_e, F_P] = 0$, for all $P \in L^2$ and $e \in L^1$.

5.2.6 Commutation relations between vertex and plaquette operators

Lemma 81. Let $P \in L^2$ be a plaquette and let $v \in L^0$ be a vertex. If $v$ is in the boundary of $P$, then

$$F^\varphi_P \circ V_{v,P}^h = V_{v,P}^h \circ F^\varphi_P$$

for all cocommutative $h \in H$, and cocommutative $\varphi \in H^*$.}

60
where \( P' \in L^2 \) may be any plaquette with base-point \( v \).

If additionally \( v \) is the base-point of \( P \), then

\[
F_P^{\varphi(S_h)} \cdot V_{v,P}^{h(2)} = V_{v,P}^h \cdot F_P^\varphi \quad \text{for all } h \in H, \varphi \in H^*.
\]

(Note that the two previous formulae coincide if \( h \) and \( \varphi \) are both cocommutative.)

If \( v \) is not in the boundary of \( P \), then

\[
F_P^\varphi \cdot V_{v,P'}^h = V_{v,P'}^h \cdot F_P^\varphi \quad \text{for all } h \in H, \varphi \in H^*.
\]

Proof. If \( v \) is not in the boundary of \( P \), then the operators have disjoint support and therefore clearly commute with each other as claimed. Let hence \( v \) be in the boundary of \( P \).

Let us first assume that \( v \) is the base-point of \( P \). Denote by \((e_1, \ldots, e_n)\) the edges in the boundary of \( P \) in counter-clockwise order starting at the base-point of \( P \). By applying edge orientation reversals where necessary we may assume that these edges are all oriented in counter-clockwise direction around \( P \).

Let \( (f_1, \ldots, f_k) \) be the edges incident to the vertex \( v \) in counter-clockwise order starting and ending at \( P \). Note that \( e_1 = f_k \) and \( f_1 = e_n \). By applying edge orientation reversals where necessary we may assume that the edges \( f_2, \ldots, f_{k-1} \) are oriented away from \( v \). Let \((Q_1, \ldots, Q_\ell)\) be the plaquettes which have \( v \) as their base-points, such that \( Q_1 = P \). Let us calculate for \((v_{e_2} \otimes \cdots \otimes v_{e_n} \otimes v_{f_2} \otimes \cdots \otimes v_{f_k} \otimes X_{P} \otimes X_{Q_2} \otimes \cdots \otimes X_{Q_\ell}) \in H^\otimes n+k-2 \otimes A^\otimes \ell :

\[
(F_P^{\varphi(S_{h_{3-7}})} \cdot V_{v,P}^{h(2)})(v_{e_2} \otimes \cdots \otimes v_{e_n} \otimes v_{f_2} \otimes \cdots \otimes v_{f_k} \otimes X_{P} \otimes X_{Q_2} \otimes \cdots \otimes X_{Q_\ell})
\]

\[
= F_P^{\varphi(S_{h_{3-7}})} \cdot (v_{e_2} \otimes \cdots \otimes v_{e_n} \otimes v_{f_2} \otimes \cdots \otimes v_{f_k} \otimes X_{P} \otimes X_{Q_2} \otimes \cdots \otimes X_{Q_\ell})
\]

\[
= (v_{e_2})_1 \otimes \cdots \otimes (v_{e_n})_1 \otimes (v_{f_2})_1 \otimes (v_{f_k})_1 \otimes (X_{P})_2 \otimes (X_{Q_2})_2 \otimes \cdots \otimes (X_{Q_\ell})_2
\]

\[
= (v_{e_2})_1 \otimes \cdots \otimes (v_{e_n})_1 \otimes (v_{f_2})_1 \otimes (v_{f_k})_1 \otimes (X_{P})_2 \otimes (X_{Q_2})_2 \otimes \cdots \otimes (X_{Q_\ell})_2
\]
shifts where necessary, in order to reduce it to the previous case where the vertex and plaquette operators commute: proving the second relation in the statement of the lemma.

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