ON THE FINITE DIMENSIONALITY OF A K3 SURFACE

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Abstract: For a smooth projective surface \( X \) the finite dimensionality of the Chow motive \( h(X) \), as conjectured by S.I Kimura, has several geometric consequences. For a complex surface of general type with \( p_g = 0 \) it is equivalent to Bloch’s conjecture. The conjecture is still open for a K3 surface \( X \) which is not a Kummer surface. In this paper we prove some results on Kimura’s conjecture for complex K3 surfaces. If \( X \) has a large Picard number \( \rho = \rho(X) \), i.e \( \rho = 19, 20 \), then the motive of \( X \) is finite dimensional. If \( X \) has a non-symplectic group acting trivially on algebraic cycles then the motive of \( X \) is finite dimensional. If \( X \) has a symplectic involution \( i \), i.e a Nikulin involution, then the finite dimensionality of \( h(X) \) implies \( h(X) \simeq h(Y) \), where \( Y \) is a desingularization of the quotient surface \( X/ < i > \). We give several examples of K3 surfaces with a Nikulin involution such that the isomorphism \( h(X) \simeq h(Y) \) holds, so giving some evidence to Kimura’s conjecture in this case.

1. Introduction

For a smooth projective variety \( X \) over a field \( k \) we will denote by \( A^i(X) \) the Chow group of codimension \( i \) cycles with rational coefficients and by \( \mathcal{M}_{rat}(k) \) the (covariant ) category of Chow motives with rational coefficients over the field \( k \), which is is a \( \mathbb{Q} \)-linear, pseudoabelian, tensor category.

An object \( M \in \mathcal{M}_{rat}(k) \) is of the form \( M = (X, p, m) \), where \( X \) is a smooth projective variety over \( k \), \( p \) a correspondence in \( X \times X \) such that \( p^2 = p \) and \( m \in \mathbb{Z} \). We will denote by \( h(X) \) the motive \( (X, \Delta_X, 0) \), where \( \Delta_X \) is the diagonal in \( X \times X \). If \( X \) and \( Y \) are smooth (irreducible) projective varieties over \( k \) then

\[
\text{Hom}_{\mathcal{M}_{rat}(k)}(h(X), h(Y)) = A^{\dim X}(X \times Y)
\]

where \( A^*(X \times Y) = CH^*(X \times Y) \otimes \mathbb{Q} \).
We will consider a classical Weil cohomology theory $H^*$ with coefficients in a field $K$ of characteristic $0$ which induces a tensor functor $H^* : \mathcal{M}_{rat} \to \text{Vect}^g_K$ such that $H^i((X, p, m)) = p^* H^{i-2m}(X, K)$ (see [KMP 1.4]). If $\text{char } k = 0$ homological equivalence does not depend on the choice of $H^*$. By replacing rational equivalence with homological equivalence we get the category $\mathcal{M}_{hom}(k)$ of homological motives.

For an object $M \in \mathcal{M}_{rat}(k)$, one defines the exterior power $\wedge^n M \in \mathcal{M}_{rat}(k)$ (and similarly in $\mathcal{M}_{hom}$) and the symmetric power $S^n M$. (see [Ki]). A motive $M$ is finite dimensional if it can be decomposed as $M = M^+ \oplus M^-$ with $M^+$ evenly finite dimensional, i.e. such that $\wedge^n M = 0$ for some $n > 0$ and $M^-$ oddly finite dimensional, i.e. such that $S^n M = 0$ for $n > 0$.

S.I. Kimura and O’Sullivan (see [Ki]) have conjectured that all the motives in $\mathcal{M}_{rat}(k)$ are finite dimensional. The conjecture is known for curves, for abelian varieties and for some surfaces: rational surfaces, Godeaux surfaces, Kummer surfaces, surfaces with $p_g = 0$ which are not of general type, surfaces isomorphic to a quotient $(C \times D)/G$, where $C$ and $D$ are curves and $G$ is a finite group. It is also known for Fano 3-folds (see[G-G]). In all these known cases the motive $h(X)$ lies in the tensor subcategory of $\mathcal{M}_{rat}(k)$ generated by abelian varieties.

If $M = h(X)$ is the motive of a surface then the finite dimensionality of $M$ is equivalent to the vanishing of $\wedge^n t^2(X)$ for some $n > 0$, where $t^2(X)$ is the transcendental part of $h(X)$. This follows from the existence of a refined Chow-Künneth decomposition for the motive $h(X)$ of a surface

$$h(X) = 1 \oplus h_1(X) \oplus h^2_{ad}(X) \oplus t^2(X) \oplus h_3(X) \oplus L^2$$

where $1$ is the motive of a point and $L$ is the Lefschetz motive. (see [KMP]). In the above decomposition all the summands, but possibly $t^2(X)$, are finite dimensional because they lie in the subcategory of $\mathcal{M}_{rat}(k)$ generated by abelian varieties. Therefore the information necessary to study the above conjecture for a surface $X$ is concentrated in the transcendental part of the motive $t^2(X)$. More precisely, according to Murre’s Conjecture (see [Mu]), or equivalently to Bloch-Beilinson’s conjecture (see [J]) and to Kimura’s Conjecture the following results should hold for a surface $X$

(a) The motive $t^2(X)$ is evenly finite dimensional;

(b) $h(X)$ satisfies the Nilpotency conjecture, i.e. every homologically trivial endomorphism of $h(X)$ is nilpotent;

(c) Every homologically trivial correspondence in $CH^2(X \times X)_Q$ acts trivially on the Albanese kernel $T(X)$;
d) The endomorphism group of $t_2(X)$ (tensored with $\mathbb{Q}$) has finite rank (over a field of characteristic 0).

By a result of S.Kimura in [Ki], (a) implies (b).

If $X$ is a complex surface of general type with $p_g(X) = 0$, Bloch’s conjecture asserts that $A_0(X) \simeq \mathbb{Q}$. Then

$$(a) \iff A_0(X) = \mathbb{Q} \iff t_2(X) = 0$$

(see [G-P]).

A case where all the above conjectures are still unknown is that of a complex K3 surface which is not a Kummer surface. The aim of this paper is to prove some results about the finite dimensionality of $h(X)$ in the case $X$ is a K3 surface over $\mathbb{C}$.

Note that a result by Y.Andrè in [A 10.2.4.1] implies that the motive of a K3 surface is isomorphic to the motive of an abelian variety in a suitable category of motivated motives. Under the standard conjecture $B(X)$ this category coincides with $\mathcal{M}_{\text{hom}}$ (see [A p.100]. Therefore Andrè’s result suggests that the Chow motive of every K3 surface can be expressed in terms of the motives of abelian varieties.

In §2 we consider the case of a projective surface $X$ with an involution $\sigma$ and the desingularization $Y$ of the quotient surface $X/\langle \sigma \rangle$. Corollary 1 gives necessary and sufficient conditions on $\sigma$ for the existence of an isomorphism $t_2(X) \simeq t_2(Y)$ and for $t_2(Y) = 0$. In particular this result applies to a complex surface of general type $X$ with $p_g(X) = 0$ and an involution $\sigma$ for which $t_2(Y) = 0$.

In §3 we apply the results in §2 to the case of a complex K3 surface $X$ with an involution $\sigma$. If $\sigma$ is symplectic, i.e $\sigma$ is a Nikulin involution, then the finite dimensionality of $h(X)$ implies the isomorphism $h(X) \simeq h(Y)$, see Theorem 3. If the rank of the Neron- Severi group of $X$ is 19 or 20, then $h(X)$ is finite dimensional (Theorem 2). If $\sigma$ is not symplectic then $t_2(Y) = 0$, with $Y = X/\langle \sigma \rangle$, hence $t_2(X) \neq t_2(Y)$, see Remark 3. If a K3 surface $X$ has a non-symplectic group acting trivially on algebraic cycles then the motive of $X$ is finite dimensional (Corollary 2). Note that, in all the cases where we can show that the motive $h(X)$ of a K3 surface is finite dimensional, $h(X)$ lies in the tensor subcategory of $\mathcal{M}_{\text{rat}}(k)$ generated by abelian varieties.

In §4, using the results in [VG-S], we describe several examples of K3 surfaces, with a Nikulin involution $i$ and Picard rank 9, such that $t_2(X) \simeq t_2(Y)$. We also show (see Theorem 7) that the same result holds if the K3 surface $X$ has an elliptic fibration $X \to \mathbb{P}^1$ with a section. This gives some evidence to Kimura’s conjecture for a K3 surface with a symplectic involution.
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2. Surfaces with an involution

In this section we prove some results on the transcendental part $t_2(X)$ of the motive of a surface $X$, with an involution $\sigma$.

We first note that, if $X$ is a smooth projective variety over a field $k$, and $G$ is a finite group acting on $X$, then the theory of correspondences can be extended to $Y = X/G$, if one uses rational coefficients in the Chow groups (see [Fu 16.1.13]). In particular this holds if $G = \langle \sigma \rangle$, where $\sigma$ is an involution.

Let $X$ be a smooth irreducible projective surface (over any field $k$) with a refined Chow-Künneth decomposition

$$\sum_{0 \leq i \leq 4} h_i(X)$$

where $h_2(X) = h_{alg}^2(X) + t_2(X)$ and $t_2(X) = (X, \pi^{tr}_2, 0)$, see [KMP 2.2]. Here

$$\pi_{alg}^2(X) = \sum_{1 \leq h \leq \rho} [D_h \times D_h]$$

where $\{D_h\}$ is an orthogonal basis of $NS(X) \otimes \mathbb{Q}$ and $\rho = \text{rank} \ NS(X)$. The map

(1) $\Psi_X : A^2(X \times X) \rightarrow \text{End}_{M_{rat}}(t_2(X))$

defined by $\Psi_X(\Gamma) = \pi_{alg}^2 \circ \Gamma \circ \pi_{alg}^2$ yields an isomorphism (see [KMP 4.3])

$$A^2(X \times X)/J(X) \simeq \text{End}_{M_{rat}}(t_2(X))$$

where $J(X)$ is the ideal of $A^2(X \times X)$ generated by the classes of correspondences which are not dominant over $X$ by either the first or the second projection. Let $k(X)$ be the field of rational functions and let $T(X_{k(X)})$ be the Albanese kernel of $X_{k(X)}$, i.e. the kernel of the Abel-Jacobi map $A_0(X_{k(X)}) \rightarrow Alb_X(k(X) \otimes \mathbb{Q})$. Let $\tau_X : A^2(X \times X) \rightarrow T(X_{k(X)})$ be the map

$$\tau_X(Z) = (\pi^{tr}_2 \circ Z \circ \pi^{tr}_2)(\xi)$$

with $\xi$ the generic point of $X$. Then $\tau_X$ induces an isomorphism (see [KMP 5.10])

$$\text{End}_{M_{rat}}(t_2(X)) \simeq \frac{T(X_{k(X)})}{H_{\leq 1} \cap T(X_{k(X)})}$$

Here $H_{\leq 1}$ is the subgroup of $A_0(X_{k(X)})$ generated by the subgroups $A_0(X_L)$, where $L$ runs over the subfields of $k(X)$ containing $k$ and which are of transcendence degree $\leq 1$ over $k$. If $q(X) = 0$ then $X$ has no odd
cohomology, $Alb_X(k) = 0$ and in the Chow-K"unneth decomposition we have $h_1(X) = h_3(X) = 0$. Therefore $A_0(X_{k(X)}) = T(X_{k(X)})$ and $T(X) = A_0(X)_0$, where $A_0(X)_0$ is the group of 0-cycles of degree 0. By [KMP 5.10] we have

$$H_{\leq 1} \cap T(X_{k(X)}) = T(X)$$

Hence, for a surface $X$ with $q(X) = 0$, the map $\tau_X$ yields an isomorphism

$$(2) \quad \text{End}_{\text{rat}}(t_2(X)) \simeq \frac{A_0(X_{k(X)})}{A_0(X)}$$

where the class $[\xi]$ in $\frac{A_0(X_{k(X)})}{A_0(X)}$ of the generic point $[\xi]$ of $X$ corresponds to the identity of the ring $\text{End}_{\text{rat}}(t_2(X))$

The definition of the map $\Psi_X$ in (1) can be extended to the case of two smooth projective surfaces $X$ and $X'$ as in [KMP 7.4]

$$\Psi_{X,X'} : A^2(X \times X') \to \text{Hom}_{\text{rat}}(t_2(X), t_2(X'))$$

and the following functorial relation holds

$$(3) \quad \Psi_{X,X''}(\Gamma') \circ \Gamma = \Psi_{X',X''}(\Gamma') \circ \Psi_{X,X'}(\Gamma)$$

where $X, X', X''$ are smooth projective surfaces, $\Gamma \in A^2(X \times X')$ and $\Gamma' \in A^2(X' \times X'')$. The proof of (3) immediately follows by taking refined Chow-K"unneth decompositions of the motives $h(X), h(X'), h(X'')$ and writing the elements in $\text{Hom}_{\text{rat}}(h(X), h(X'))$, and $\text{Hom}_{\text{rat}}(h(X'), h(X''))$ as lower triangular matrices defined by these decompositions, as in [KMP p.163]. Applying $\Psi$ corresponds to taking appropriate diagonal entries of such lower triangular matrices.

**Lemma 1.** Let $X$ and $Y$ be smooth projective surfaces and let $f : X \to Y$ be a finite morphism. Then $f$ induces homomorphisms $\bar{f}_* : \text{End}_{\text{rat}}(t_2(X)) \to \text{End}_{\text{rat}}(t_2(Y))$ and $\bar{f}^* : \text{End}_{\text{rat}}(t_2(Y)) \to \text{End}_{\text{rat}}(t_2(X))$.\n
**Proof.** The maps $\Psi_X : A^2(X \times X) \to \text{End}_{\text{rat}}(t_2(X))$ and $\Psi_Y : A^2(Y \times Y) \to \text{End}_{\text{rat}}(t_2(Y))$ give rise to the following commutative diagram

$$
\begin{array}{ccc}
0 & \to & \mathcal{J}(X) & \to & A^2(X \times X) & \xrightarrow{\Psi_X} & \text{End}_{\text{rat}}(t_2(X)) & \to & 0 \\
& & \downarrow{(f \times f)_*} & & \downarrow{\bar{f}_*} & & \downarrow{\bar{f}_*} & & \\
0 & \to & \mathcal{J}(Y) & \to & A^2(Y \times Y) & \xrightarrow{\Psi_Y} & \text{End}_{\text{rat}}(t_2(Y)) & \to & 0 \\
& & \downarrow{(f \times f)^*} & & \downarrow{\bar{f}^*} & & \downarrow{\bar{f}^*} & & \\
0 & \to & \mathcal{J}(X) & \to & A^2(X \times X) & \xrightarrow{\Psi_X} & \text{End}_{\text{rat}}(t_2(X)) & \to & 0
\end{array}
$$
where the map $(f \times f)_*$ sends a correspondence $Z \in A^2(X \times X)$ to
$\Gamma_f \circ Z \circ \Gamma_f$ and the map $f \times f)^*$ sends a correspondence $Z' \in A^2(Y \times Y)$
to $\Gamma_f' \circ Z' \circ \Gamma_f$. It is easy to see that these maps send the ideal $\mathcal{J}(X)$
to $\mathcal{J}(Y)$ and $\mathcal{J}(Y)$ to $\mathcal{J}(X)$ respectively, thus yielding the diagram above. 

**Proposition 1.** Let $X$ be a smooth projective surface with an involution $\sigma$, such that the quotient surface $Y = X/\langle \sigma \rangle$ is smooth. Let $\xi$ denote the generic point of $X, \eta$ the generic point of $Y$ and let $[\xi] = \Psi_X(\Delta_X) \in \text{End}_{\mathcal{M}_{rat}}(t_2(X)), [\eta] = \Psi_Y(\Delta_Y) \in \text{End}_{\mathcal{M}_{rat}}(t_2(Y))$. Set $\alpha = \Psi_X(1 \times \sigma)\Delta_X = \Psi_X(\Gamma_\sigma) = \sigma([\xi])$. Then the map $f : X \to Y$ satisfies:

(i) $1/2(\Gamma_f \circ \Gamma_f^t) = \Delta_Y, \tilde{f}_*(\xi) = \tilde{f}_*(\xi) = 2[\eta]$ and $(\alpha)^2 = [\xi]$.
(ii) $\tilde{f}_*(\eta) = [\xi] + \alpha$ and $\tilde{f}^*(\tilde{f}_*(\xi)) = 2[\xi] + 2\alpha$.
(iii) Let $p = 1/2(\Gamma_f \circ \Gamma_f^t)$; then $p \circ p = p, \Psi_X(p) = 1/2([\xi] + \alpha)$ and 
$\Psi_X(\Delta_X - p) = 1/2([\xi] - \alpha)$. Hence $[\xi] = 1/2([\xi] + \alpha) + 1/2([\xi] - \alpha)$.

**Proof.** Regard the diagonals $\Delta_X$ and $\Delta_Y$ as cycles in $A^2(X \times X)$ and 
$A^2(Y \times Y)$. Then $\tilde{f}_*(\xi)$ is the image under $\Psi_Y$ of $\Gamma_f \circ \Gamma_f^t = 2\Delta_Y$.
Thus $\tilde{f}_*([\xi]) = 2\Psi_Y(\Delta_Y) = 2[\eta]$ and we also have

$$p \circ p = (1/4)\Gamma_f^t \circ (2\Delta_Y) \circ \Gamma_f = 1/2(\Gamma_f^t \circ \Gamma_f) = p$$

Since $\alpha$ is the image of $(1 \times \sigma)\Delta_X$, and $(1 \times \sigma)\Delta_X \cdot (1 \times \sigma)\Delta_X = \Delta_X$ we have $\alpha^2 = [\xi]$. Since $\Gamma_f \cdot (1 \times \sigma)\Delta_X = \Gamma_f$, the correspondence $(1 \times \sigma)\Delta_X$ also maps to $\Delta_Y$, so $f_*(\alpha) = 2[\eta]$. This establish (i) and (ii) follows immediately. Part (iii) follows from (ii) and $p \circ p = p$. 

Let $X$ be a smooth projective surface and let $\sigma$ be an involution on $X$. Let $k$ be the number of isolated fixed points of $\sigma$ and let $D$ the 1-dimensional part of the fixed-point locus. The divisor $D$ is smooth (possibly empty). Let $\tilde{X}$ be the blow-up of of the set of isolated fixed points . Then the involution $\sigma$ lifts to an involution on $\tilde{X}$ (which we will still denote by $\sigma$). The quotient $Y = \tilde{X}/\langle \sigma \rangle$ is a desingularization of $X/\langle \sigma \rangle$. $Y$ has $k$ disjoint nodal curves $C_1, \cdots, C_k$. The map $X \to X/\langle \sigma \rangle$ induces a commutative diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\beta} & X \\
\downarrow f & & \downarrow \\
Y & \longrightarrow & X/\langle \sigma \rangle
\end{array}$$
Since $t_2(-)$ is a birational invariant for smooth projective surfaces the maps $\beta : \tilde{X} \to X$ and $f : \tilde{X} \to Y$ induce a morphism
\[
\theta : t_2(\tilde{X}) = t_2(X) \to t_2(Y)
\]

**Corollary 1.** Let $X, \tilde{X}, Y$ be as in the diagram above. Then

(i) $\theta : t_2(X) \to t_2(Y)$ is the projection onto a direct summand.

(ii) $\theta$ is an isomorphism iff $\Psi_X(\Gamma_\sigma) = \text{id}_{t_2(X)}$, i.e iff $\bar{\sigma}([\xi]) = [\xi]$ in $\text{End}_{\text{rat}}(t_2(X))$.

(iii) If $q(X) = 0$ the conditions of (ii) are equivalent to $A_0(X)_0 = A_0(X)_0$.

(iv) $t_2(Y) = 0 \iff \Psi_X(\Gamma_\sigma) = -i_0 \iff \bar{\sigma}([\xi]) = -[\xi]$ in $\text{End}_{\text{rat}}(t_2(X))$.

Proof. Since $\Psi_{\tilde{X},X}(\Gamma_\beta)$ is an isomorphism and $\theta = \Psi_{\tilde{X},Y}(\Gamma_f) \circ \Psi_{\tilde{X},X}(\Gamma_\beta)^{-1}$ it is enough, after replacing $X$ by $\tilde{X}$, to prove the Corollary under the assumption $\tilde{X} = X$. Then $\theta = \Psi_{X,Y}(\Gamma_f)$. From Proposition 1 we get that $\Gamma_f$ has a right inverse $1/2(\Gamma_f^\prime)$ and $2p = \Gamma_f^\prime \circ \Gamma_f = \Delta_X + (1 \times \sigma)\Delta_X = \Delta_X + \Gamma_\sigma$. It follows from the functoriality of $\Psi$ in (3) that, if $t_2(X)^+$ and $t_2(X)^-$ are the direct summands of $t_2(X)$ on which the involution $\Psi_X(\Gamma_\sigma)$ acts respectively as $+1$ or $-1$, then the restriction of $\theta$ to $t_2(X)^-$ is 0 and to $t_2(X)^+$ is an isomorphism. This gives (i).

Also $\theta$ is an isomorphism iff $t_2(X)^- = 0$ which is equivalent to $\Psi_X(\Gamma_\sigma)$ being the identity in $\text{End}_{\text{rat}}(t_2(X))$. This gives (ii).

If $q(X) = 0$ then $A_0(X)_0 = T(X)$ and we have a canonical isomorphism
\[
\text{Hom}_{\text{rat}}(1, t_2(X)) \simeq A_0(X)_0
\]
which is compatible with the action of correspondences. Hence, by taking the action of $\Psi_X(\Gamma_\sigma)$ on $t_2(X)$ we get
\[
\text{Hom}_{\text{rat}}(1, t_2(X)^-) \simeq A_0(X)_0
\]
Therefore $\Psi_X(\Gamma_\sigma)$ acts as the identity on $t_2(X)$ iff $A_0(X)_0 = A_0(t_2(X)^-) = 0$. Since $A_i(t_2(X)) = 0$ for $i \neq 0$, we have $A_i(M) = 0$ for all $i$, where $M = t_2(X)^-$. It follows that $M = 0$ (see [C-G Lemma 1]. This proves (iii).

Clearly $t_2(Y) = 0$ is equivalent to $\bar{\sigma}([\xi]) = -[\xi] \in \text{End}_{\text{rat}}(t_2(X))$.

Since the cycle class $[\xi]$ corresponds to the identity of $\text{End}_{\text{rat}}(t_2(X))$ under the isomorphism in (2), $\Psi_X(\Gamma_\sigma)$ act as $-1$ on $t_2(X)$. Let $q(X) = 0$; then, by the same argument as in the proof of (iii) we get $A_0(X)_0 = 0$. This gives (iv).

F. Severi in [Sev] has introduced the notions of valence and indices of a correspondence $T \in A^n(X \times X)$, where $X$ is a smooth projective
variety of dimension $n$. In the case when $X$ is a surface, Severi related these notions to the computation of the degree of the cycle $T \cdot \Delta_X$.

**Definition 1.** Let $X$ be a smooth projective variety of dimension $n$. A correspondence $T \in A^n(X \times X)$ has valence 0 if it belongs to the ideal of degenerate correspondences, i.e., the ideal generated by correspondences of the form $[V \times W]$, with $V, W$ proper subvarieties of $X$. A correspondence $\Gamma$ has valence $v$ if $T = \Gamma + v\Delta_X$ has valence 0. If $T = T_1 + T_2$ in $A^d(X \times X)$ and $T_1, T_2$ have valences $v_1, v_2$ then $T$ has valence $v_1 + v_2$. If the correspondences $T$ and $T'$ have valences $v$ and $v'$ then $v(T \circ T') = -v(T) \cdot v(T')$; see [Fu 16 1.5]. It follows that if $p$ is a projector in $A^2(X \times X)$ which has a valence, then $v(p)$ is either 0 or -1.

The indices of a correspondence $T$ are the numbers $\alpha(T) = \text{deg}(T \cdot [P \times X])$ and $\beta(T) = \text{deg}(T \cdot [X \times P])$, where $P$ is any rational point on $X$; see [Fu, 16 1.4]. The indices are additive in $T$ and $\beta(T) = \alpha(T')$.

**Theorem 1.** Let $X$ be a smooth projective surface with an involution $\sigma$ and let $Y$ be the desingularization of $X/\langle \sigma \rangle$. Assume $p_\sigma(X) > 0$ and let $\Gamma_\sigma = (1 \times \sigma)\Delta_X$. If the correspondence $\Gamma_\sigma$ has a valence, then

$$t_2(Y) = 0 \iff v(\Gamma_\sigma) = 1 \ ; \ \theta : t_2(X) \xrightarrow{\sim} t_2(Y) \iff v(\Gamma_\sigma) = -1$$

**Proof.** Let $[\xi] = 1/2([\xi] + \alpha) + 1/2([\xi] - \alpha)$ be the splitting in $\text{End}_{\text{rat}}(t_2(X))$ coming from Proposition 1, with $\alpha = \bar{\sigma}([\xi]) = \Psi_X(\Gamma_\sigma)$. If the correspondence $\Gamma_\sigma$ has a valence then also the projector $p = 1/2(\Delta_X + (1 \times \sigma)\Delta_X) = 1/2(\Delta_X + \Gamma_\sigma)$ has a valence and $v(p)$ is either 0 or -1. Since $v(\Delta_X) = -1$ we have

$$v(p) = 0 \iff v(\Gamma_\sigma) = 1 \ ; \ v(p) = -1 \iff v(\Gamma_\sigma) = -1$$

Suppose $v(\Gamma) = 1$; then $v(p) = 0$ i.e., $p$ belongs to the ideal of degenerate correspondences, which is contained in $K\text{er} \Psi_X$. From $\Psi_X(p) = 0$ we get $1/2([\xi] + \Psi_X(\Gamma_\sigma)) = 0$ hence $\Psi_X(\Gamma_\sigma) = -id_{t_2(X)}$. From Corollary (iv) we get $t_2(Y) = 0$. Conversely if $t_2(Y) = 0$, then $\Psi_X(\Gamma_\sigma) = -id_{t_2(X)}$ hence $\Psi_X(p) = 0$ and we get $v(p) = 0$.

If $\theta : t_2(X) \rightarrow t_2(Y)$ is an isomorphism then, by Corollary 1 (ii) $\Psi_X(\Gamma_\sigma) = id_{t_2(X)}$. By the same argument as before we get $v(\Gamma_\sigma) = -1$. \hfill \Box

**Remark 1.** The assumption $p_\sigma(X) > 0$ in Theorem 1 is necessary in order to have a uniquely defined valence for $\Gamma_\sigma$. If $p_\sigma(X) = 0$ and $X$ satisfies Bloch’s conjecture then, by the results in [B-S], $v(\Delta_X) = 0$, hence the correspondence $\Delta_X$ has 2 different valences: namely 0 and -1. Note that, for a surface $X$, a correspondence $\Gamma$ can have 2 different valences $v$ and $v'$ only if $p_\sigma(X) = 0$. This was first observed by Severi.
in [Sev p.761]). In fact then the multiple \((v - v')\) of the diagonal \(\Delta_X\) belongs to the ideal of degenerate correspondences and this implies that \(\Psi_X(\Delta_X) = 0\) in \(\text{End}_{M_{\text{rat}}}(t_2(X))\). Therefore the identity map is 0 in \(\text{End}_{M_{\text{rat}}}(t_2(X))\). Hence \(t_2(X) = 0\) and this may occur only if \(p_g(X) = 0\).

3. **Complex K3 surfaces**

A smooth (irreducible) projective K3 surface \(X\) over \(\mathbb{C}\) is a regular surface (i.e \(q(X) = 0\)), therefore it has a refined Chow-Künneth decomposition (see [KMP 2.2]) of the form \(h(X) = \sum_{0 \leq i \leq 4} h_i(X)\) with \(h_1(X) = h_3(X) = 0\). Moreover \(h_2(X) = h_2^{alg}(X) + t_2(X)\), where \(t_2(X) = (\pi_{2}^{tr}, 0)\) and \(h_2^{alg}(X) \cong \mathbb{L}^{\oplus \rho(X)}\). Here \(\rho(X)\) is the rank of the Néron-Severi group \(\text{NS}(X)\) so that \(1 \leq \rho \leq 20\). Moreover \(H^i(t_2(X)) = 0\) for \(i \neq 2\); \(H^2(t_2(X)) = \pi_{2}^{tr} H^2(X, \mathbb{Q}) = H_{tr}^2(X, \mathbb{Q})\),

\[A_i(t_2(X)) = \pi_{2}^{tr} A_i(X) = 0 \text{ for } i \neq 2; \quad A_0(t_2(X)) = T(X),\]

where \(T(X)\) is the Albanese Kernel. Since \(q(X) = 0\), we also have \(T(X) = A_0(X) = 0\) (0-cycles of degree 0) and

\[\dim H^2(X) = b_2(X) = 22; \quad \dim H_{tr}^2(X) = b_2(X) - \rho\]

A Nikulin involution \(i\) of a complex K3 surface \(X\) is a symplectic automorphism of order 2, i.e. such that \(i^* \omega = \omega\) for all \(\omega \in H^{2,0}(X)\). A K3 surface \(X\) with a Nikulin involution has rank \(\rho(X) \geq 9\). The Néron-Severi group \(\text{NS}(X)\) contains a primitive sublattice isomorphic to \(E_8(-2)\) where \(E_8\) is the unique even unimodular positive definite lattice of rank 8 (see [Mor p.106]). Here, if \(L\) is a lattice and \(m\) is an integer, \(L(m)\) denotes same free \(\mathbb{Z}\)-module \(L\) with a form which has been altered by multiplication by \(m\), that is \(b_{L(m)}(x, y) = m b_L(x, y)\), where \(b_L(x, y)\) is the \(\mathbb{Z}\)-valued symmetric bilinear form of \(L\).

By \(T_X\) we will denote the transcendental lattice of \(X\), i.e \(T_X = \text{NS}(X)^\perp \subset H^2(X, \mathbb{Z})\).

For any K3 surface with a Nikulin involution \(i\) there is an isomorphism

\[H^2(X, \mathbb{Z}) \cong U^3 \oplus E_8(-1) \oplus E_8(-1)\]

where \(U\) is the hyperbolic plane, such that \(i^*\) acts as follows

\[i^*(u, x, y) = (u, y, x)\]

The invariant sublattice is \(H^2(X, \mathbb{Z})^i \cong U^3 \oplus E_8(-2)\) and \((H^2(X, \mathbb{Z})^i)^\perp \cong E_8(-2)\). Since \(i^* \omega = \omega\) for all \(\omega \in H^{2,0}(X)\) we also have \((H^2(X, \mathbb{Z})^i)^\perp \subset \text{NS}(X)\) (see [VG-S 2.1]). Therefore the involution \(i\) acts as the identity on \(H_{tr}^2(X, \mathbb{Q})\).

Let \(X \rightarrow X/ \langle i \rangle\) be the quotient map. The surface \(X/ \langle i \rangle\) has 8 ordinary double points \(Q_1, \ldots, Q_8\) corresponding to the 8 fixed
points $P_1, \ldots, P_8$ of the involution $i$ on $X$. The minimal model $Y$ of $X/ < i >$ is a K3 surface, hence $p_g(Y) > 0$. In the following we will always consider the standard diagram for a K3 surface with a Nikulin involution $i$ (see [Mor sec. 3])

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\beta} & X \\
f & \downarrow & \downarrow \\
Y & \longrightarrow & X/ < i > \\
\end{array}
$$

$	ilde{X}$ is the blow up of $X$ at the points $P_1, \ldots, P_8$ with exceptional divisors $\beta^{-1}(P_j) = E_j$. The Nikulin involution extends to an involution $i$ on $\tilde{X}$ and $Y = \tilde{X}/ < i >$. $f$ is a double cover branched on the divisor $\sum_{1 \leq j \leq 8} C_j$ where $C_j = f(E_j)$ are disjoint smooth irreducible rational curves corresponding to the points $Q_1, \ldots, Q_8$. Therefore $1/2(\sum_j C_j) \in NS(Y)$. The map $f_* \circ \beta^*$ induces an isomorphism of rational Hodge structures

$$
T_{\tilde{X}} \otimes \mathbb{Q} \simeq T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}
$$

where $T_X$ and $T_Y$ are the transcendental lattices. In particular the vector spaces $H^2_{tr}(X, \mathbb{Q})$ and $H^2_{tr}(Y, \mathbb{Q})$ have the same dimension, so that $22 - \rho(X) = 22 - \rho(Y)$.

Suppose conversely that a K3 surface $Y$ admits an even set of $k$ disjoint rational curves $C_1, \ldots, C_k$: this means that there exists a $\delta \in \text{Pic}Y$ such that

$$
C_1 + \cdots + C_k \sim 2\delta
$$

This is equivalent to the existence of a double cover $X$ of $Y$ branched on $C_1 + \cdots + C_k$. Then, by [N 1], $k = 0, 8, 16$. If $k = 16$ then $X$ is birational to an abelian surface $A$ and $Y$ is the Kummer surface of $A$. Therefore the motives $h(X)$ and $h(Y)$ are finite dimensional and $t_2(A) \simeq t_2(X) \simeq t_2(Y)$ (see [KMP 6.13]). If $k = 8$ then $X$ is a K3 surface, $Y$ is the desingularization of the quotient of $X$ by a Nikulin involution $i$. Hence $Y$ is a K3 surface.

**Theorem 2.** Let $X$ be a smooth projective K3 surface over $\mathbb{C}$ with $\rho(X) = 19, 20$. Then the motive $h(X) \in M_{\text{rat}}(\mathbb{C})$ is finite dimensional and lies in the subcategory of $M_{\text{rat}}(\mathbb{C})$ generated by the motives of abelian varieties.

**Proof.** By [Mor 6.4] $X$ admits a Shioda-Inose structure, i.e there is a Nikulin involution $i$ on $X$ such that the desingularization $Y$ of the quotient surface $X/ < i >$ is a Kummer surface, associated to an abelian surface $A$. Hence $h(Y)$ is finite dimensional. The rational map
$f : X \to Y$ induces a splitting $t_2(X) \simeq t_2(Y) \oplus N$. Since $t_2(Y)$ is finite dimensional we are left to show that $N = 0$. From Corollary 1 the vanishing of $N$ is equivalent to $A_0(X)^i_0 = A_0(X)_0$. By [Mor 6.3 (iv)] the Neron-Severi group of $X$ contains the sublattice $E_8(-1)^2$. Hence by the results in [Huy 6.3, 6.4], the symplectic automorphism $i$ acts as the identity on $A_0(X)$. From Corollary 1 we get $t_2(X) = t_2(Y)$. By [KMP 6.13] $t_2(Y) = t_2(A)$; therefore $h(X)$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(\mathbb{C})$ generated by the motives of abelian varieties. □

Remark 2. Note that by [Mo 2.10 (i), 4.4(i)] there exist K3 surfaces with $\rho(X) = 19, 20$ which are not Kummer surfaces.

Next we show that for every K3 surface with a Nikulin involution the finite dimensionality of $h(X)$ implies $h(X) \simeq h(Y)$.

Lemma 2. Let $X$ be a K3 surface over $\mathbb{C}$ with a Nikulin involution $i$ and let $Y$ be a desingularization of the quotient surface $X/\langle i \rangle$. Let $e(-)$ be the topological Euler characteristic. Then we have

$$e(X) + t + 2 + 2k = 2e(Y)$$

where $t$ is the trace of the involution $i$ on $H^2(X, \mathbb{C})$ and $k = 8$ is the number of the isolated fixed points of $i$. Therefore $\rho(X) = \rho(Y)$ and $t = 6$

Proof. : We use the same argument as in [D-ML-P 4.2]. Since $i$ has only isolated fixed points from the topological fixed point formula we get

$$e(X) + t + 2 = 2e(Y) - 2k$$

Since $X$ and $Y$ are both K3 surfaces we have $e(X) = e(Y) = 24$. Therefore, we get $t = 6$. Since $\dim H^2_{tr}(X) = \dim H^2_{tr}(Y)$ and $b_2(X) = b_2(Y) = 22$, we have $\rho(X) = \rho(Y)$.

Theorem 3. Let $X$ be a K3 surface with a Nikulin involution $i$. If $h(X)$ is finite dimensional then $h(X) \simeq h(Y)$.

Proof. $Y$ is a K3 surface and we have $t_2(\tilde{X}) = t_2(X)$ because $t_2(-)$ is a birational invariant for surfaces. Also

$$H^2_{tr}(X) \simeq H^2_{tr}(\tilde{X}) \simeq H^2_{tr}(Y)$$

because the Nikulin involution acts trivially on $H^2_{tr}(X)$. Let $t$ be the trace of the involution $\sigma$ on the vector space $H^2(X, \mathbb{C})$. From Lemma 2 we get $t = 6$. The involution $i$ acts trivially on $H^2_{tr}(X)$ and $H^2_{tr}(X)$ is a subvector space of $H^2(X, \mathbb{C})$ of dimension $22 - \rho$. Therefore the trace of the action of $i$ on $NS(X) \otimes \mathbb{C}$ equals $\rho - 16$. Since the only eigenvalues
of an involution are $+1$ and $-1$ we can find an orthogonal basis for $NS(X) \otimes \mathbb{C}$ of the form $H_1, \ldots, H_r; D_1, \ldots, D_8$, with $r = \rho - 8 \geq 1$ such that $i_*(H_j) = H_j$ and $i_*(D_l) = -D_l$. Then $NS(X) \otimes \mathbb{C}$ has a basis of the form $E_1, \ldots, E_8; H_1, \ldots, H_r; D_1, \ldots, D_8$, where $E_h$, for $1 \leq h \leq 8$ are the exceptional divisors of the blow up $\tilde{X} \rightarrow X$. The set of $r + 8 = \rho$ divisors $f_*(E_h) = C_k$, for $1 \leq h \leq 8$ and $f_*(H_j) \simeq H_j$, for $1 \leq j \leq r$ gives an orthogonal basis for $NS(Y) \otimes \mathbb{Q}$. Since $q(X) = q(Y) = q(\tilde{X}) = 0$ we can find Chow-K"unneth decompositions for $h(X)$, $h(\tilde{X})$ and $h(\tilde{Y})$ of the form

$$h(X) = 1 \oplus h^2_{alg}(X) \oplus t_2(X) \oplus L^2 \cong 1 \oplus L^{\otimes \rho} \oplus t_2(X) \oplus L^2$$

$$h(\tilde{X}) = 1 \oplus h^2_{alg}(\tilde{X}) \oplus t_2(X) \oplus L^2 \cong h(X) \oplus L^{\otimes 8}$$

$$h(Y) = 1 \oplus h^2_{alg}(Y) \oplus t_2(Y) \oplus L^2 \cong 1 \oplus L^{\otimes \rho} \oplus t_2(Y) \oplus L^2$$

where $\rho(X) = \rho(Y) = \rho$. By Corollary 1 we have $t_2(X) \simeq t_2(Y) \oplus N$ for some $N \in \mathcal{M}_{rat}$. Since $h(X)$ is finite dimensional also $N$ is finite dimensional. Since $H(X)$ and $H(Y)$ are isomorphic as graded vector spaces $H(N) = 0$. By [Ki 7.3] $N = 0$. Therefore $h(X) \simeq h(Y)$. \hfill $\square$

**Theorem 4.** Let $X$ be a K3 surface with a Nikulin involution $i$. Then the following conditions are equivalent:

(i) the correspondence $\Gamma_i = (1 \times i)\Delta_X$ has a valence.

(ii) $\theta : t_2(X) \rightarrow t_2(Y)$.

(iii) $\bar{i}([\xi]) = [\xi]$ in $End_{\mathcal{M}_{rat}}(t_2(X))$.

(iv) $i$ acts as the identity on $A_0(X) \otimes \mathbb{Q}$

**Proof.** Let

$$\Delta_X = 1/2(\Delta_X + (1 \times i)\Delta_X) + 1/2(\Delta_X - (1 \times i)\Delta_X)$$

as in Proposition 1 and let $\Gamma_i = (1 \times i)\Delta_X$. If $\Gamma_i$ has a valence then also the projector $q = 1/2(\Delta_X - (1 \times i)\Delta_X)$ has a valence and $v(q)$ is either 0 or -1. Suppose that $v(1/2(\Delta_X - (1 \times i)\Delta_X)) = -1$; then the correspondence $(1 \times i)\Delta_X$ has valence 1. From Theorem 1 we get $t_2(Y) = 0$ hence a contradiction because $Y$ is a K3 surface. Therefore $v(1/2(\Delta_X - (1 \times i)\Delta_X)) = 0$, so that $v(1 \times i)\Delta_X = v(\Delta_X) = -1$. By Theorem 1 $\theta : t_2(X) \rightarrow t_2(Y)$ is an isomorphism. Therefore (i) $\Rightarrow$ (ii).

Conversely if $\theta : t_2(X) \rightarrow t_2(Y)$ then by Corollary 1 (ii) $\Psi_X(\Delta_X - \Gamma_i) = 0$, hence $\Delta_X - \Gamma_i \in Ker \Psi_X$. Since $q(X) = 0$ Ker $\Psi_X$ is coincides with the ideal of degenerate correspondences. This proves (i).
The equivalences $(ii) \Leftrightarrow (iii)$ and $(iii) \Leftrightarrow (iv)$ come from Corollary 1.

\[\square\]

**Remark 3.** Let $\sigma$ is an involution on a K3 surface $X$ which is not symplectic, i.e $\sigma^*(\omega) = -\omega$, where $\omega$ is a generator of the vector space $H^{2,0}(X)$. By [Zh 1.2 ] if $X^\sigma = \emptyset$ the quotient surface $Y = X/ \langle \sigma \rangle$ is an Enriques surface, while $Y$ is a rational surface if $X^\sigma \neq \emptyset$. In any case the motive $h(Y)$ has no transcendental part. Therefore $t_2(Y) = 0$ and $t_2(X) \neq t_2(Y)$, because $t_2(X) \neq 0$ for a K3 surface. From the identity in $\text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X))$

\[\bar{\sigma}([\xi]) = -[\xi] \text{ and } [\xi] \neq -[\xi] \text{ in } \text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X))\]

we get

\[\bar{\sigma}([\xi]) = -[\xi] \text{ and } [\xi] \neq -[\xi] \text{ in } \text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X))\]

because otherwise we would get $t_2(X) = 0$. Hence Theorem 4 does not hold true.

Following the example in Remark 3 we now consider the case of a complex K3 surface $X$ with a non-symplectic group $G$ acting trivially on the algebraic cycles. Any automorphism $g$ of $X$ preserves the 1-dimensional vector space $H^{2,0} = H^2(X, \Omega^2_X) \simeq \mathbb{C}\omega$. Hence $g$ is non-symplectic iff there exists a complex number $\alpha(g) \neq 1$ such that $g^*(\omega) = \alpha(g)\omega$. Let $NS(X)$ and $T_X$ be the lattices of algebraic and transcendental cycles on $X$. $X$ is said to be unimodular if $\det T_X = \pm 1$. Let $H_X$ be the finite cyclic group defined as the kernel of the map $\text{Aut}(X) \to O(\text{NS}(X))$, where $O(\text{NS}(X))$ denotes the group of isometries of $\text{NS}(X)$. Then there are only finitely many values for $m = \vert H_X \vert$.

By [LSY Th. 5] one has the following result.

**Theorem 5.** Let $X$ be a complex K3 surface $X$ with a non-symplectic group $G$ acting trivially on the algebraic cycles. Let $m = \vert H_X \vert \neq 3$ : then there exists a surjective morphism $F_n \to X$, where $F_n \subset \mathbb{P}^3$ is the Fermat surface, of degree $n \geq 4$

\[F_n : X_0^n + X_1^n + X_2^n + X_3^n = 0\]

Here $n = m$ if $X$ is unimodular and $n = 2m$, if $X$ is not unimodular.

**Corollary 2.** Let $X$ be a complex K3 surface with a non-symplectic group $G$ acting trivially on the algebraic cycles. Let $m = \vert H_X \vert \neq 3$. Then the motive of $X$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{\text{rat}}(\mathbb{C})$ generated by the motives of abelian varieties. K3 surfaces satisfying these conditions have $\rho(X) = 2, 4, 6, 10, 12, 16, 18, 20$. 
Proof. From Theorem 5 there is surjective morphism $F_n \to X$, with $F_n$ a Fermat surface. By [SK] the motive $h(F_n)$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(C)$ generated by the motives of abelian varieties. By [Ki 6.6 and 6.8] if $f : Z \to X$ is a surjective morphism of smooth projective varieties, then $h(X)$ is a direct summand of $h(Z)$. Therefore $h(X)$ is finite dimensional and lies in the subcategory of $\mathcal{M}_{rat}(C)$ generated by the motives of abelian varieties. The computation of the rank $\rho(X)$ appears in [LSY Th. 1 and Th. 2]. □

4. Examples

In this section we describe some examples of K3 surfaces with a Nikulin involution $i$, such that $t_2(X) \simeq t_2(Y)$. Hence $h(X) \simeq h(Y)$. We will use the classification given by Van Geemen and Sarti in [VG-S] and by Garbagnati and Sarti in [G-S]. Their results are based on the following Theorem.

Theorem 6. ([VG-S 2.2]) Let $X$ be K3 surface with $\rho(X) = 9$ and a Nikulin involution $i$. Let $L$ be a generator of $E_8(-2) \subset NS(X)$ with $L^2 = 2d > 0$ which we may assume to be ample. Let

$$\Lambda_{2d} = ZL \oplus E_8(-2)$$

Then, if $L^2 \equiv 2 \pmod 4$, we have $\Lambda_{2d} \simeq NS(X)$. If $L^2 \equiv 0 \pmod 4$ we have either $NS(X) \simeq \Lambda_{2d}$ or $NS(X) \simeq \Lambda_{2d}$. Here $\Lambda_{2d}$ is the unique even lattice containing $\Lambda_{2d}$ with $\Lambda_{2d}/\Lambda_{2d} \simeq Z/2Z$ and such that $E_8(-2)$ is a primitive sublattice of $\Lambda_{2d}$. For every $\Gamma = \Lambda_{2d}$ with $d > 0$ or $\Gamma = \Lambda_{2d}$ with $d = 2m > 0$, there exists a K3 surface with a Nikulin involution $i$ such that $NS(X) = \Gamma$ and $(H^2X, Z)i \perp \simeq E_6(-2)$.

Let’s consider the following cases described in [VG-S]:

(i) $X$ is a double cover of $P^2$ branched over a sextic curve and $Y$ a double cover of a quadric cone in $P^3$;

(ii) $X$ is a double cover of a quadric in $P^3$ and $Y$ is a double cover of $P^2$ branched over a reducible sextic;

(iii) the image of $X$ under the map $\Phi_L$ is the intersection of 3 quadrics in $P^5$ and $Y$ is a quartic surface in $P^3$.

First we I show that in the cases (i),(ii) and (iii) the map $f : X \to Y$ induces an isomorphism

$$t_2(X) \simeq t_2(Y)$$

Then, in Theorem 7 we prove that the same result holds if $g : X \to P^1$ is a general elliptic fibration with a section and also $Y$ is an elliptic fibration.
In the case (i) $NS(X) \simeq \mathbb{Z}L \oplus E_8(-2)$, with $L^2 = 2$ and $i^* L \simeq L$ (see [VG-S 3.2]). The map $\Phi_L : X \to \mathbb{P}^2$ is a double cover branched over a sextic curve $C$ and $X/ < i >$ is a double cover of a quadric cone in $\mathbb{P}^3$. Let $\sigma$ denote the covering involution on $X$. Then $\sigma \neq i$. The quotient surface $Y = X/ < \sigma >$ is isomorphic to $\mathbb{P}^2$. Let $j = \sigma \circ i = i \circ \sigma$ and let $G = \langle 1, \sigma, i, j \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The quotient surfaces $\mathbb{P}^2 = X/ < \sigma >$ and $S = X/ < j >$ are both rational, because $S$ is a Del Pezzo surface of degree 1 (see [VG-S 3.2]). The motives $h(\mathbb{P}^2)$ and $h(S)$ have no transcendental part. Therefore $t_2(X) \neq t_2(Y)$ and $t_2(X) \neq t_2(S)$, because $t_2(X) \neq 0$ for a K3 surface. From the identities in $End_{M_{rat}}(t_2(X))$

$$[\xi] = 1/2([\xi] + \sigma([\xi]) + 1/2([\xi] - \sigma([\xi])$$

$$[\xi] = 1/2([\xi] + \bar{j}([\xi]) + 1/2([\xi] - \bar{j}([\xi])$$

we get, by Corollary 1 (iv), $[\xi] + \sigma([\xi]) = [\xi] + \bar{j}([\xi]) = 0$ in $End_{M_{rat}}(t_2(X))$.

We have

$$\Psi(\Gamma_i) = \Psi_X((1 \times i)\Delta_X) = \Psi_X((1 \times \sigma \circ j)\Delta_X) = \Psi(\Gamma_{\sigma \circ j})$$

Therefore the class of $\bar{i}([\xi])$ in $End_{M_{rat}}(t_2(X))$ equals $\bar{\sigma}([\xi]) \circ \bar{j}([\xi]) = (-[\xi]) \circ (-[\xi]) = ([\xi])^2 = [\xi]$ because $[\xi]$ is the identity of $End_{M_{rat}}(t_2(X))$. Hence

$$\bar{i}([\xi]) - [\xi] = 0 \text{ in } End_{M_{rat}}(t_2(X))$$

From Corollary 1 (ii) we get $\theta : t_2(X) \xrightarrow{\sim} t_2(Y)$.

The proof for (ii) is similar to the previous one. In this case the lattice $\mathbb{Z}L \oplus E_8(-2)$ has index 2 in $NS(X)$ and we may assume that $NS(X)$ is generated by $L$, $E_8(-2)$ and $E_1 = (L + v)/2$, with $v \in E_8(-2)$, such that $v^2 = -4$. Then $E_1$ and $E_2$, where $E_2 = (L - v)/2$, are the classes of 2 elliptic fibrations. The map

$$\Phi_L : X \to \mathbb{P}^3$$

is a 2:1 map to a quadric $Q$ in $\mathbb{P}^3$ and it is ramified on a curve $C$ of bidegree $(4, 4)$ ([VG-S 3.5]). The quadric $Q$ is smooth, hence it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The covering involution $\sigma : X \to X$ of $X \to Q$ and the Nikulin involution $i$ commute, the elliptic pencils $E_1$ and $E_2$ are permuted by $i$ because $i^* L = L$ and $i^* v = -v$. $i$ induces an involution $i_Q$ on $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ which acts sending a point $\{(s, t), (u, v)\}$ to $\{(u, v), (s, t)\}$. The quotient $Q/ < i_Q >$ is isomorphic to $\mathbb{P}^2$. Let $j = i \circ \sigma = \sigma \circ i$ in $Aut(X)$ and let $G = \{1, \sigma, i, j\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. $S = X/ < j >$ is a Del Pezzo surface of degree 2, by [VG-S 3.5]. The
motives $h(Q)$ and $h(S)$ have no transcendental part, hence from the same argument as in (i), we get an isomorphism $t_2(X) \simeq t_2(Y)$.

We now consider the description given in [VG-S 3.7] of (iii). Let $Y$ be the desingularization of the quotient surface $X/ < i >$. In this case there is a line bundle $M \in \text{NS}(Y)$ such that $\beta^*L \simeq f^*M$ and

$$H^0(X,L) \simeq f^*(H^0(Y,M)) \oplus f^*(H^0(Y,M-C))$$

where $\beta : \tilde{X} \to X$ is the blow-up at the 8 fixed points $P_1, P_2, \ldots, P_8$ of $i$, $f : \tilde{X} \to Y$ and $C = (\sum_{1 \leq i \leq 8} C_i)/2 \in \text{NS}(Y)$, with $C_i$ the rational curves on $Y$ corresponding to the 8 singular points $Q_1, \ldots, Q_8$ of $X/ < i >$. The above decomposition is the decomposition of $H^0(X,L)$ into the $i^*$ eigenspaces. We have $L^2 = 8, M^2 = 4, h^0(M) = 4, h^0(M-C) = 2$ so that

$$\Phi_L : X \to \mathbb{P}^5 ; \Phi_M : Y \to \mathbb{P}^3 ; \Phi_{M-C} : Y \to \mathbb{P}^1$$

The image of $X$ under $\Phi_L$ is the intersection of 3 quadrics in $\mathbb{P}^5$ and the involution $i$ is induced by the involution

$$C^6 : (x_0, x_1, x_2, x_3, y_0, y_1) \to (x_0, x_1, x_2, x_3, -y_0, -y_1)$$

The fixed points $(P_1, P_2, \ldots, P_8)$ lie in $X \cap \{y_0 = y_1 = 0\}$. The quadrics in the ideal of $X$ are of the form

$$y_0^2 = Q_1(x), y_0y_1 = Q_2(x), y_1^2 = Q_3(x)$$

where $x = (x_0, x_1, x_2, x_3)$. The line

$$l : x_0 = x_1 = x_2 = x_3 = 0$$

in $\mathbb{P}^5$ is fixed under $i$ and $l \cap X = \emptyset$. The image of $Y$ by $\Phi_M$ is the projection of $X$ from the invariant line to the invariant $\mathbb{P}^3$ which is defined by $y_0 = y_1 = 0$. The image is the quartic surface in $\mathbb{P}^3$ defined by

$$Q_1(x)Q_3(x) - Q_2^2(x) = 0$$

which can be identified with $Y$.

We now use a result in [Vois 1.18] : if $X$ is the K3 surface obtained as the intersection of 3 quadrics in $\mathbb{P}^5$ which are invariant under the involution

$$i : (x_0, x_1, x_2, x_3, y_0, y_1) \to (x_0, x_1, x_2, x_3, -y_0, -y_1)$$

then $i^*(\omega) = \omega$ for $\omega \in H^{2,0}(X)$ and $i$ acts trivially on $A_0(X)$. Therefore, by Corollary 1 (iii) we get an isomorphism $\theta : t_2(X) \simrightarrow t_2(Y)$.

Next we consider the case of a K3 surface $X$ which has an elliptic fibration $g : X \to \mathbb{P}^1$ with a global section $\sigma : \mathbb{P}^1 \to X$. The set of sections of $g$ is the Mordell-Weil group $\text{MW}_g$ with identity element $\sigma$. 
MW_0 is the subgroup of Aut X consisting of all automorphisms acting on a general fiber as translations and these translations preserve the holomorphic two form on X. Therefore, if there is an element \( \tau \) of order 2 in MW_0 then the translation by \( \tau \) defines a Nikulin involution \( i \) on X.

**Theorem 7.** Let \( X \) a general elliptic fibration \( g : X \to \mathbb{P}^1 \) with sections \( \sigma, \tau \) as above. Let \( i \) be the corresponding Nikulin involution on \( X \) and let \( Y \) be the desingularization of \( X/ < i > \). Then the map \( f : X \to X/ < i > \) induces an isomorphism

\[
\theta : t_2(X) \cong t_2(Y)
\]

**Proof.** In [VG-S 4.2] it is shown that for a general elliptic fibration \( g : X \to \mathbb{P}^1 \) there is an isomorphism \( MW_g = \{ \sigma, \tau \} \cong \mathbb{Z}/2\mathbb{Z} \) where \( \sigma : \mathbb{P}^1 \to X \). Hence the translation by \( \tau \) defines a Nikulin involution \( i \) on \( X \). The Weierstrass equation of \( X \) can be put in the form

\[
X : y^2 = x(x^2 + a(t)x + b(t))
\]

where the degree of \( a(t) \) and \( b(t) \) are 4 and 8 respectively. There are 8 singular fibers of type \( I_1 \), which are rational curves with a node, corresponding to the zeroes \( \{a_1, \ldots, a_8\} \) of \( a^2(t) - 4b(t) \) and 8 singular fibers of type \( I_2 \), which are union of two \( \mathbb{P}^1 \) meeting in 2 points, corresponding to the zeroes \( \{b_1, \ldots, b_8\} \) of \( b(t) \). The fixed points of the translation by \( \tau \) are the 8 nodes in the \( I_1 \)-fibers. \( \tau \) acts on the generic fiber \( E_t \) as the translation by a point of order 2, i.e. \( 2\tau(x) = 2x \). The desingularization \( Y \) of the quotient surface \( X/ < \tau > \) is an elliptic fibration with Weierstrass equation

\[
Y : y^2 = x(x^2 - 2a(t)x + 9a(t)^2 - 4b(t)).
\]

The generic fiber \( F_t \) of \( Y \) is the elliptic curve \( E_t/ < P > \), where \( E_t \) is the generic fiber on \( X \). Let

\[
V = \bigcup_{t \in A} g^{-1}(t) = \bigcup E_t
\]

where \( A = \mathbb{P}^1 - \{a_1, \ldots, a_8, b_1, \ldots, b_8\} \). Then \( V \) is open in \( X \) and, for every point \( x \in V \), the involution \( \tau \) acts as translation by a point of order 2 on \( E_t \), so that \( 2\tau(x) = 2x \). Therefore \( 2(1 \times \tau)(x, x) = (2x, 2x) \), for all \( x \in V \) i.e. \( (1 \times \tau)\Delta_V = \Delta_V \) with \( \Delta_V = \Delta_X \cap (V \times X) \). We get \( (1 \times \tau)\Delta_X = \Delta_X \) on \( V \times V \), hence \( (1 \times \tau)\Delta_X - \Delta_X \in \mathcal{J}(X) \), with \( \mathcal{J}(X) = Ker \Psi_X \) and \( \Psi_X : A^2(X \times X) \to End_{\mathcal{M}_{rat}}(t_2(X)) \). Therefore \( \Psi_X(1 \times \tau)\Delta_X = \Psi_X(\Delta_X) \) and

\[
\bar{\tau}([\xi]) = [\xi]
\]
in $\text{End}_{\mathcal{M}_{\text{rat}}}(t_2(X))$, where $\xi$ is the generic point of $X$. From Corollary 1 (ii) we get
\[
\theta : t_2(X) \sim t_2(Y)
\]

**Remark 4.** By [VG-S 4.1], if $X$ is as in theorem 7, then the Neron-Severi group $NS(X)$ has rank $\rho(X) = 10$, and $\text{dim } T_{X,\mathbb{Q}} = 12$ is even. In this case the isomorphism of Hodge structures $\phi_i : T_{X,\mathbb{Q}} \simeq T_{Y,\mathbb{Q}}$, induced by the involution $i$, is an isometry. On the contrary, in the cases described in (i),(ii), (iii), where $\rho(X) = 9$, $\phi_i$ is not an isometry. This follows from [VG-S 2.5] because $\text{dim } T_{X,\mathbb{Q}}$ is odd.

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