Two Time Physics
with a Minimum Length

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Abstract

We study the possibility of introducing the classical analogue of Snyder’s Lorentz-covariant noncommutative space-time in two-time physics theory. In the free theory we find that this is possible because there is a broken local scale invariance of the action. When background gauge fields are present, they must satisfy certain conditions very similar to the ones first obtained by Dirac in 1936. These conditions preserve the local and global invariances of the action and leads to a Snyder space-time with background gauge fields.

1 Introduction

Two-Time Physics [1,2,3,4,5,6,7] is an approach that provides a new perspective for understanding ordinary one-time dynamics from a higher dimensional, more unified point of view including two time-like dimensions. This is achieved by introducing a new gauge symmetry that insure unitarity, causality and absence of ghosts. The new phenomenon in two-time physics is that the gauge symmetry of the free two-time physics action can be used, by imposing gauge conditions, to obtain various different actions describing different free and interacting dynamical systems in the usual one-time physics, thus uncovering a new layer of unification through higher dimensions.

An approach to the introduction of background gravitational and gauge fields in two-time physics was first presented in [7]. In [7], the linear realization of the $Sp(2,R)$ gauge algebra of two-time physics is required to be preserved when background gravitational and gauge fields come into play. To satisfy this requirement, the background gravitational field must satisfy a homothety condition [7], while in the absence of space-time gravitational fields the gauge field must satisfy the conditions [7]

\begin{align}
X.A(X) &= 0 \quad \text{(1.1a)} \\
\partial_{M}A^{M}(X) &= 0 \quad \text{(1.1b)}
\end{align}
which were first proposed by Dirac [8] in 1936. Dirac proposed these conditions as subsidiary conditions to describe the usual 4-dimensional Maxwell theory of electromagnetism as a theory in 6 dimensions which automatically displays \(SO(4,2)\) symmetry.

If we recall that in the transition to quantum mechanics \(X^M \to X^M\) and \(P_M \to i\frac{\partial}{\partial X^M}\), we can rewrite Dirac’s conditions (1.1) in the classical form

\[
(X \partial + 1)A_M(X) = 0 \quad (1.1c)
\]

\[
X.A(X) = 0 \quad (1.2a)
\]

\[
P.A(X) = 0 \quad (1.2b)
\]

\[
(-iX.P + 1)A_M(X) = 0 \quad (1.2c)
\]

In this paper we show how a set of subsidiary conditions very similar to (1.2) can be obtained in the classical Hamiltonian formalism for two-time physics. As in Dirac’s original paper on the \(SO(4,2)\)-invariant formulation of electromagnetism [8], the conditions we find in two-time physics are necessary for the \(SO(d,2)\) invariance of the interacting theory. A new result in this work is that we show that these conditions are also necessary for a perfect match between the number of physical degrees of freedom contained in the \((d + 2)\)-dimensional gauge field and the number of physical canonical pairs describing the dynamics in the reduced phase space.

The paper is divided as follows. In the next section we review the basic formalism of two-time physics and show how the \(SO(d,2)\) Lorentz generator for the free 2T action can be obtained from a local scale invariance of the Hamiltonian. Invariance under this local scale transformation of only the Hamiltonian reveals that two-time physics can also be consistently formulated in terms of another set of classical phase space brackets, which are the classical analogues of the Snyder commutators [9].

In 1947 Snyder proposed a quantized space-time model in a projective geometry approach to the de Sitter space of momenta with a scale \(\theta\) at the Planck scale. In this model, the energy and momentum of a particle are identified with the inhomogeneous projective coordinates. Then, the space-time coordinates become noncommutative operators \(\hat{x}^\mu\) given by the “translation” generators of the de Sitter (dS) algebra. Snyder’s space-time has attracted interest in the last few years in connection with generalizations of special relativity. In particular, it was pointed out [10] that there is a one-to-one correspondence between the Snyder space-time and a formulation of de Sitter-invariant special relativity [11] with two universal invariants, the speed of light \(c\) and the de Sitter radius of curvature \(R\).

However, a particle moving in a de Sitter or Anti-de Sitter space-time with signature \((d - 1, 1)\), where \(d\) is the number of spacelike dimensions, is only one of the many dual lower-dimensional systems that can be obtained by imposing
gauge conditions on the free two-time physics action (see, for instance, ref. [4]). Furthermore, a Snyder space-time with signature \((d - 1, 1)\) for a free massless relativistic particle has already been obtained from the \((d, 2)\) space-time of two-time physics by using the Dirac bracket technique, after imposing gauge conditions to reduce the gauge invariance of the free 2T action [12]. This shows that in the \((d - 1, 1)\) space-time there are inertial motions and inertial observers in the Snyder space-time, giving a principle of relativity for dS/AdS-invariant special relativity. The results of this work may be used to suggest that the other universal invariant of dS/AdS special relativity, the radius of curvature \(R\), can be interpreted as a very large integer multiple of the minimum spacelike length introduced by the Snyder commutators.

In the treatment of [12], the appearance of Snyder’s space-time in the reduced phase space of the Dirac brackets is a direct consequence of the fact that the gauge conditions break the conformal \(SO(1, 2) \sim Sp(2, R)\) gauge invariance of the 2T action, leaving only \(\tau\)-reparametrization invariance. Then a length scale induced by the Snyder commutators emerges in the resulting \((d - 1, 1)\) space-time, leaving the global scale and conformal invariances of the gauge-fixed action untouched. To preserve the powerful unifying properties of 2T physics, and retain the massless particle in a dS/AdS space-time as one of its gauge-fixed versions, it is then interesting to investigate the possibility of constructing a Snyder space-time with signature \((d, 2)\), in which the \(Sp(2, R)\) gauge invariance and consequently the full duality properties of the 2T action would be preserved. In this work we take this task and show that it can be done while also explicitly preserving the global Lorentz \(SO(d, 2)\) invariance of the action. These developments are the content of section two.

In section three we introduce interactions with a background gauge field by modifying the constraint structure of two-time physics according to the minimal coupling prescription to electrodynamic gauge fields. We show how a set of subsidiary conditions very similar to (1.2) emerge after requiring \(Sp(2, R)\) gauge invariance of the interacting theory and how these conditions lead to the same Snyder brackets we found in the free theory. Some concluding remarks appear in section four.

## 2 Two-time Physics

The central idea in two-time physics [1,2,3,4,5,6,7] is to introduce a new gauge invariance in phase space by gauging the duality of the quantum commutator \([X_M, P_N] = i\eta_{MN}\). This procedure leads to a symplectic \(Sp(2,R)\) gauge theory. To remove the distinction between position and momentum we set \(X_1^M = X^M\) and \(X_2^M = P^M\) and define the doublet \(X_1^M = (X_1^M, X_2^M)\). The local \(Sp(2, R)\) acts as

\[
\delta X^M_i(\tau) = \epsilon_{ik} \omega^{kl}(\tau) X^M_l(\tau)
\]

\(\omega^{ij}(\tau)\) is a symmetric matrix containing three local parameters and \(\epsilon_{ij}\) is the Levi-Civita symbol that serves to raise or lower indices. The \(Sp(2, R)\) gauge
field $A^{ij}$ is symmetric in $(i, j)$ and transforms as
\[
\delta A^{ij} = \partial_t \omega^{ij} + \omega^{ik} \epsilon_{kl} A^{lj} + \omega^{jk} \epsilon_{kl} A^{il}
\] (2.2)

The covariant derivative is
\[
D^\tau X_i^M = \partial_t X_i^M - \epsilon_{ik} A^{kl} X_l^M
\] (2.3)

An action invariant under the $Sp(2, R)$ gauge symmetry is
\[
S = \frac{1}{2} \int d\tau (D^\tau X_i^M) \epsilon^{ij} X_j^N \eta_{MN}
\] (2.4a)

After an integration by parts this action can be written as
\[
S = \int d\tau (\partial_t X_i^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N) \eta_{MN}
\]
\[
= \int d\tau [\dot{X}_i P - (\frac{1}{2} \lambda_1 P^2 + \lambda_2 X_i P + \frac{1}{2} \lambda_3 X^2)]
\] (2.4b)

where $A^{11} = \lambda_3$, $A^{12} = A^{21} = \lambda_2$, $A^{22} = \lambda_1$ and the canonical Hamiltonian is
\[
H = \frac{1}{2} \lambda_1 P^2 + \lambda_2 X_i P + \frac{1}{2} \lambda_3 X^2
\] (2.5)

The equations of motion for the $\lambda$’s give the primary [13] constraints
\[
\phi_1 = \frac{1}{2} P^2 \approx 0
\] (2.6)
\[
\phi_2 = X_i P \approx 0
\] (2.7)
\[
\phi_3 = \frac{1}{2} X^2 \approx 0
\] (2.8)

and therefore we can not solve for the $\lambda$’s from their equations of motion. The values of the $\lambda$’s in action (2.4b) are arbitrary. Constraints (2.6)-(2.8), as well as evidences of two-time physics, were independently obtained in [14].

We have introduced the **weak equality symbol** $\approx$. This is to emphasize that constraints (2.6)-(2.8) are numerically restricted to be zero on the submanifold of phase space defined by the constraint equations, but do not identically vanish throughout phase space [15]. This means, in particular, that they have nonzero Poisson brackets with the canonical variables. More generally, two functions $F$ and $G$ that coincide on the submanifold of phase space defined by constraints $\phi_i \approx 0$, $i = 1, 2, 3$ are said to be **weakly equal** [15] and one writes $F \approx G$. On the other hand, an equation that holds throughout phase space and not just on the submanifold $\phi_i \approx 0$, is called **strong**, and the usual equality symbol is used in that case. It can be demonstrated [15] that
\[
F \approx G \iff F - G = c_i(X, P) \phi_i
\] (2.9)
If we consider the Euclidean, or the Minkowski metric as the background space-time, we find that the surface defined by the constraint equations (2.6)-(2.8) is trivial. The only metric giving a non-trivial surface, preserving the unitarity of the theory, and avoiding the ghost problem is the flat metric with two time-like dimensions \([1,2,3,4,5,6,7]\). Following \([1,2,3,4,5,6,7]\) we introduce another space-like dimension and another time-like dimension and work in a Minkowski space-time with signature \((d,2)\).

We use the Poisson brackets

\[
\{P_M, P_N\} = 0 \quad (2.10a)
\]

\[
\{X_M, P_N\} = \eta_{MN} \quad (2.10b)
\]

\[
\{X_M, X_N\} = 0 \quad (2.10c)
\]

where \(M, N = 0, ..., d+1\), and verify that constraints (2.6)-(2.8) obey the algebra

\[
\{\phi_1, \phi_2\} = -2\phi_1 \quad (2.11a)
\]

\[
\{\phi_1, \phi_3\} = -\phi_2 \quad (2.11b)
\]

\[
\{\phi_2, \phi_3\} = -2\phi_3 \quad (2.11c)
\]

These equations show that all constraints \(\phi\) are first-class \([13]\). Equations (2.11) represent the symplectic \(Sp(2, R)\) gauge algebra of two-time physics. The 3-parameter local symmetry \(Sp(2, R)\) includes \(\tau\)-reparametrizations, generated by constraint \(\phi_1\), as one of its local transformations, and therefore the 2T action (2.4) is a generalization of gravity on the worldline. It corresponds to conformal \(SO(2,1)\) gravity on the worldling \([4,14]\). Since we have \(d + 2\) dimensions and 3 first-class constraints, only \(d + 2 - 3 = d - 1\) of the canonical pairs \((X_M, P_M)\) will correspond to true physical degrees of freedom.

Action (2.4) also has a global symmetry under Lorentz transformations \(SO(d,2)\) with generator \([1,2,3,4,5,6,7]\)

\[
L^{MN} = \epsilon^{ij} X_i^M X_j^N = X^M P^N - X^N P^M \quad (2.12)
\]

It satisfies the space-time algebra

\[
\{L_{MN}, L_{RS}\} = \delta_{MR} L_{NS} + \delta_{NS} L_{MR} - \delta_{MS} L_{NR} - \delta_{NR} L_{MS} \quad (2.13)
\]

and is gauge invariant because it has identically vanishing brackets with the first-class constraints (2.6)-(2.8), \(\{L_{MN}, \phi_i\} = 0\).

In one-time physics, a natural way to implement the notion of a minimum length \([16,17,18,19,20]\) in theories containing gravity is to formulate these models on a noncommutative space-time. By a minimum length it is understood that no experimental device subject to quantum mechanics, gravity and causality can
exclude the quantization of position on distances smaller than the Planck length [20]. It has been shown [21] that when measurement processes involve energies of the order of the Planck scale, the fundamental assumption of locality is no longer a good approximation in theories containing gravity. The measurements alter the space-time metric in a fundamental manner governed by the commutation relations \([x_\mu, p_\nu] = i\eta_{\mu\nu}\) and the classical field equations of gravitation [21]. This in-principle unavoidable change in the space-time metric destroys the commutativity (and hence locality) of position measurement operators. In the absence of gravitation locality is restored [21]. This effect of a minimum length can be modeled by introducing a nonvanishing commutation relation between the position operators [22].

Let us now consider how the classical analogue of Snyder’s noncommutative space-time can be made to emerge in two-time physics. To arrive at these classical Snyder brackets we use what can be considered as a broken local scale invariance of the free 2T action. This local scale invariance is a symmetry only of the 2T Hamiltonian. It is not a symmetry of the action because the kinetic term \(\dot{X}.P\) in the Legendre transformation, giving the Lagrangian from the Hamiltonian, is not invariant under this local scale transformation. This is why we can introduce Snyder brackets in two-time physics and still preserve the original invariances of the action.

Hamiltonian (2.5) is invariant under the local scale transformations

\[
X^M \rightarrow \tilde{X}^M = \exp\{\beta\}X^M \quad (2.14a)
\]

\[
P_M \rightarrow \tilde{P}_M = \exp\{-\beta\}P_M \quad (2.14b)
\]

\[
\lambda_1 \rightarrow \exp\{2\beta\}\lambda_1 \quad (2.14c)
\]

\[
\lambda_2 \rightarrow \lambda_2 \quad (2.14d)
\]

\[
\lambda_3 \rightarrow \exp\{-2\beta\}\lambda_3 \quad (2.14e)
\]

where \(\beta\) is an arbitrary function of \(X^M(\tau)\) and \(P_M(\tau)\). Keeping only the linear terms in \(\beta\) in transformation (2.14), we can write the brackets

\[
\{\tilde{P}_M, \tilde{P}_N\} = (\beta - 1)[\{P_M, \beta\}P_N + \{\beta, P_N\}P_M] + \{\beta, \beta\}P_MP_N \quad (2.15a)
\]

\[
\{\tilde{X}_M, \tilde{P}_N\} = (1 + \beta)[\eta_{MN}(1 - \beta) - \{X_M, \beta\}P_N] + (1 - \beta)X_M\{\beta, P_N\} - X_MX_N\{\beta, \beta\} \quad (2.15b)
\]

\[
\{\tilde{X}_M, \tilde{X}_N\} = (1 + \beta)[X_M\{\beta, X_N\} - X_N\{\beta, X_M\}] + X_MX_N\{\beta, \beta\} \quad (2.15c)
\]
for the transformed canonical variables. If we choose \( \beta = \phi_1 = \frac{1}{2}P^2 \approx 0 \) in equations (2.15) and compute the brackets on the right side using the Poisson brackets (2.10), we find the expressions

\[
\{ \tilde{P}_M, \tilde{P}_N \} = 0 \quad (2.16a)
\]

\[
\{ \tilde{X}_M, \tilde{P}_N \} = (1 + \frac{1}{2}P^2)[\eta_{MN}(1 - \frac{1}{2}P^2) - P_MP_N] \quad (2.16b)
\]

\[
\{ \tilde{X}_M, \tilde{X}_N \} = -(1 + \frac{1}{2}P^2)(X_MP_N - X_NP_M) \quad (2.16c)
\]

We see from the above equations that, on the constraint surface defined by constraints (2.6)-(2.8), brackets (2.16) reduce to

\[
\{ \tilde{P}_M, \tilde{P}_N \} = 0 \quad (2.17a)
\]

\[
\{ \tilde{X}_M, \tilde{P}_N \} = \eta_{MN} - P_MP_N \quad (2.17b)
\]

\[
\{ \tilde{X}_M, \tilde{X}_N \} = -(X_MP_N - X_NP_M) \quad (2.17c)
\]

To impose \( \phi_1 = \frac{1}{2}P^2 \approx 0 \) strongly at the end of the computation of brackets (2.16), the expressions for the corresponding Dirac brackets should be used in place of the Poisson brackets. However, for the special case \( \beta = \phi_1 = \frac{1}{2}P^2 \approx 0 \) we can use the property [15] of the Dirac brackets that, on the first-class constraint surface,

\[
\{ G, F \}_D \approx \{ G, F \} \quad (2.18)
\]

when \( G \) is a first-class constraint and \( F \) is an arbitrary function of the canonical variables. This justifies the use of Poisson brackets to arrive at (2.17).

Now, keeping the same order of approximation used to arrive at brackets (2.15), that is, retaining only the linear terms in \( \beta \), transformation equations (2.14a) and (2.14b) read

\[
\tilde{X}^M = \exp\{\beta\}X^M = (1 + \beta)X^M \quad (2.19a)
\]

\[
\tilde{P}_M = \exp\{-\beta\}P_M = (1 - \beta)P_M \quad (2.19b)
\]

Using again the same function \( \beta = \phi_1 = \frac{1}{2}P^2 \approx 0 \) in equations (2.19), we write them as

\[
\tilde{X}^M = X^M + \frac{1}{2}P^2X^M \quad (2.20a)
\]

\[
\tilde{P}_M = P_M - \frac{1}{2}P^2P_M \quad (2.20b)
\]

or, equivalently,

\[
\tilde{X}^M - X^M = C_i^M(X,P)\phi_i \quad (2.21a)
\]
\[ \tilde{P}_M - P_M = D^1_M(X, P)\phi_i \quad (2.21b) \]

with \( C^1_M = X^M \), \( C^2_M = C^3_M = 0 \) and \( D^1_M = -P_M, D^2_M = D^3_M = 0 \). Equations (2.21) are obviously in the form (2.9) and so we can write

\[ \tilde{X}^M \approx X^M \quad (2.22a) \]

\[ \tilde{P}_M \approx P_M \quad (2.22b) \]

Using these weak equalities in brackets (2.17) we rewrite them as

\[ \{P_M, P_N\} \approx 0 \quad (2.23a) \]

\[ \{X_M, P_N\} \approx \eta_{MN} - P_M P_N \quad (2.23b) \]

\[ \{X_M, X_N\} \approx -(X_M P_N - X_N P_M) \quad (2.23c) \]

to emphasize that these brackets are valid only on the constraint surface defined by constraints (2.6)-(2.8). But, as we saw above, the non-trivial surfaces corresponding to constraints (2.6)-(2.8) require a space-time with signature \((d, 2)\).

Brackets (2.23) are the classical 2T equivalent of the Lorentz-covariant Snyder commutators [9], which were proposed in 1947 as a way to solve the ultraviolet divergence problem in quantum field theory by introducing a minimum space-time length. In the canonical quantization procedure, where brackets are replaced by commutators according to the rule

\[ [\text{commutator}] = i\{\text{bracket}\} \]

the 2T brackets (2.23) will lead directly to a Lorentz-covariant noncommutative space-time for two-time physics, thus implementing the notion of a minimum length in the \((d + 2)\)-dimensional space-time for this theory.

The Snyder brackets (2.23) give an equally valid description of two-time physics at the classical level. If we compute the bracket \(\{L_{MN}, L_{RS}\}\) using the Snyder brackets we find that the same space-time algebra (2.13) is reproduced. This implies that the Snyder brackets (2.23) preserve the global \(SO(d, 2)\) Lorentz invariance of action (2.4). Since \(SO(d, 2)\) contains scale as well as conformal transformations we see that, although we may introduce a scale at the Planck length using the Snyder brackets (2.23), global scale and conformal invariances still exist. This is because to arrive at the Snyder brackets (2.23) we have used the local scale invariance (2.14) of the 2T Hamiltonian, which is a broken scale invariance from the Lagrangian point of view.

If we compute the brackets \(\{L_{MN}, \phi_i\}\) using (2.23) to verify the gauge invariance of \(L_{MN}\) in a phase space with Snyder brackets, we find that the \(\{L_{MN}, \phi_i\}\) identically vanish, proving that \(L_{MN}\) is gauge invariant in this phase space.
Computing the algebra of constraints (2.6)-(2.8) using (2.23) we arrive at the expressions

\[ \{\phi_1, \phi_2\} = -2\phi_1 + 4\phi_1^2 \tag{2.24a} \]

\[ \{\phi_1, \phi_3\} = -\phi_2 + 2\phi_1\phi_2 \tag{2.24b} \]

\[ \{\phi_2, \phi_3\} = -2\phi_3 + \phi_2^2 \tag{2.24c} \]

which show that the first-class property of constraints (2.6)-(2.8) is preserved by brackets (2.23). Equations (2.24) are the realization of the \(Sp(2,R)\) gauge algebra of two-time physics in a phase space with Snyder brackets. Equations (2.24) exactly reproduce the gauge algebra (2.11) if we take the linear approximation on the right side.

Notice that \(L_{MN}\) explicitly appears with a minus sign in the right hand side of the Snyder bracket (2.23c), establishing a connection between the global \(SO(d,2)\) Lorentz invariance of action (2.4) and the local scale invariance (2.14) of Hamiltonian (2.5).

The new result obtained in this section is that the classical and free two-time physics theory can also be consistently formulated in a phase space where the Snyder brackets (2.23) are valid. In the next section we will see that this remains true in the presence of a background gauge field \(A_M(X)\) when a set of subsidiary conditions very similar to (1.2) are satisfied.

### 3 2T Physics with Gauge Fields

To introduce a background gauge field \(A_M(X)\) we modify the free action (2.4b) according to the usual minimal coupling prescription to gauge fields, \(P_M \rightarrow P_M - A_M\). The interacting 2T action in this case is then

\[ S = \int d\tau \{ \dot{X}.P - \frac{1}{2}\lambda_1(P - A)^2 + \lambda_2X.(P - A) + \frac{1}{2}\lambda_3X^2 \} \tag{3.1} \]

where the Hamiltonian is

\[ H = \frac{1}{2}\lambda_1(P - A)^2 + \lambda_2X.(P - A) + \frac{1}{2}\lambda_3X^2 \tag{3.2} \]

The equations of motion for the multipliers now give the constraints

\[ \phi_1 = \frac{1}{2}(P - A)^2 \approx 0 \tag{3.3} \]

\[ \phi_2 = X.(P - A) \approx 0 \tag{3.4} \]

\[ \phi_3 = \frac{1}{2}X^2 \approx 0 \tag{3.5} \]
The Poisson brackets between the canonical variables and the gauge field are

\[ \{ X_M, A_N \} = 0 \]  \hspace{1cm} (3.6a)

\[ \{ P_M, A_N \} = - \frac{\partial A_N}{\partial X_M} \]  \hspace{1cm} (3.6b)

\[ \{ A_M, A_N \} = 0 \]  \hspace{1cm} (3.6c)

Computing the algebra of constraints (3.3)-(3.5) using the Poisson brackets (2.9) and (3.6) we obtain the equations

\[ \{ \phi_1, \phi_2 \} = -2\phi_1 + (P^M - A^M_N) \frac{\partial}{\partial X_M} (X.A) - (P - A).A \]  \hspace{1cm} (3.7a)

\[ X^M \frac{\partial}{\partial X^M} [(P - A).A] - X^M \frac{\partial}{\partial X^M} \left( \frac{1}{2} A^2 \right) \]  \hspace{1cm} (3.7b)

\[ \{ \phi_1, \phi_3 \} = -\phi_2 \]  \hspace{1cm} (3.7b)

\[ \{ \phi_2, \phi_3 \} = -2\phi_3 \]  \hspace{1cm} (3.7c)

Equations (3.7) exactly reproduce the \( Sp(2, R) \) gauge algebra (2.11) when the conditions

\[ X.A = 0 \]  \hspace{1cm} (3.8a)

\[ (P - A).A = 0 \]  \hspace{1cm} (3.8b)

\[ \frac{1}{2} A^2 = 0 \]  \hspace{1cm} (3.8c)

hold. Condition (3.8a) is the first of Dirac’s subsidiary conditions (1.2) on the gauge field. Conditions (3.8b) and (3.8c) are, however, different from (1.2b) and (1.2c). When conditions (3.8) hold, the \( Sp(2, R) \) gauge algebra (2.11) is reproduced by constraints (3.3)-(3.5). Thus, when (3.8) holds, the only possible space-time metric associated with constraints (3.3)-(3.5) giving a non-trivial surface and avoiding the ghost problem is the flat metric with two time-like dimensions.

Now that we have seen that action (3.1) has a local \( Sp(2, R) \) gauge invariance when conditions (3.8) hold, we may consider the question of which is the \( SO(d, 2) \) Lorentz generator for action (3.1). A possible answer is obtained if we use the minimal coupling prescription \( P_M \rightarrow P_M - A_M \) in the expression for \( L_{MN} \) in the free theory, thus obtaining in the interacting theory

\[ L^I_{MN} = X_M (P_N - A_N) - X_N (P_M - A_M) \]

\[ = (X_M P_N - X_N P_M) - (X_M A_N - X_N A_M) \]
\[ L_{MN} - S_{MN} \]  

We can even use the first-class gauge function \( \beta = \phi_1 = \frac{1}{2}(P - A)^2 \approx 0 \) and formally construct a set of Snyder brackets for the interacting theory in which \( L_{MN}^I \) appears on the right side of the bracket \( \{X_M, X_N\} \), exactly in the same way as \( L_{MN} \) appears on the right side of (2.23c) in the free theory. However, it can be verified (using Poisson brackets) that the global transformations generated by \( S_{MN} = X_M A_N - X_N A_M \) identically vanish,

\[
\delta X_R = \frac{1}{2} \epsilon_{MN} \{S_{MN}, X_R\} = 0 \quad (3.10a)
\]

\[
\delta P_R = \frac{1}{2} \epsilon_{MN} \{S_{MN}, P_R\} = 0 \quad (3.10b)
\]

\[
\delta A_R = \frac{1}{2} \epsilon_{MN} \{S_{MN}, A_R\} = 0 \quad (3.10c)
\]

The Lorentz generator for the interacting theory is then effectively identical to \( L_{MN} \) in the free theory. This agrees with Dirac’s interpretation of the conformal \( SO(4,2) \) symmetry of Maxwell’s theory as being the Lorentz symmetry in 6 dimensions. This was also pointed out, but in a rather unclear way, in reference [7] (see section four of [7]).

The above conclusion implies that \( L_{MN} \) must be invariant under the gauge transformations generated by constraints (3.3)-(3.5). Using the Poisson brackets (2.9) and (3.6) we find the equations

\[
\{L_{MN}, \phi_1\} = X_M \frac{\partial}{\partial X_N}[(P - A)A] + X_M \frac{\partial}{\partial X_N} \left( \frac{1}{2} A^2 \right)
\]

\[
- X_N \frac{\partial}{\partial X_M}[(P - A)A] - X_N \frac{\partial}{\partial X_M} \left( \frac{1}{2} A^2 \right) + P_M \frac{\partial}{\partial P_N}[(P - A)A]
\]

\[
- P_N \frac{\partial}{\partial P_M}[(P - A)A] \quad (3.11a)
\]

\[
\{L_{MN}, \phi_2\} = X_M \frac{\partial}{\partial X_N}(X.A) - X_N \frac{\partial}{\partial X_M}(X.A) \quad (3.11b)
\]

\[
\{L_{MN}, \phi_3\} = 0 \quad (3.11c)
\]

We see from the above equations that \( L_{MN} \) is gauge invariant, \( \{L_{MN}, \phi_i\} = 0 \), when conditions (3.8) are valid. Action (3.1), complemented with the subsidiary conditions (3.8), gives therefore a consistent classical Hamiltonian description of two-time physics with background gauge fields in a phase space with the Poisson brackets (2.9) and (3.6). But, as we saw in section two, there is another Hamiltonian description of two-time physics based in a phase space with the Snyder brackets (2.23). Let us then consider this Hamiltonian formulation in the case when background gauge fields are present.
Since the Lorentz generator \( L_{MN} \) in the interacting theory is identical to the one in the free theory, the form (2.23) of the Snyder brackets must also be preserved in the interacting theory because \( L_{MN} \) explicitly appears in the right side of (2.23c). This creates a mathematical difficulty because the gauge function \( \beta = \frac{1}{2}P^2 \) we used to arrive at (2.23) in the free theory is no longer a first-class function on the constraint surface defined by (3.3)-(3.5). Consequently, equations (2.9) and (2.18) cannot be used. To solve this difficulty we incorporate conditions (3.8) as new constraints for the interacting theory.

Combining conditions (3.8) with constraints (3.3)-(3.5), we get the irreducible [15] set of constraints

\[
\phi_1 = \frac{1}{2}P^2 \approx 0 \tag{3.12}
\]
\[
\phi_2 = X.P \approx 0 \tag{3.13}
\]
\[
\phi_3 = \frac{1}{2}X^2 \approx 0 \tag{3.14}
\]
\[
\phi_4 = X.A \approx 0 \tag{3.15}
\]
\[
\phi_5 = P.A \approx 0 \tag{3.16}
\]
\[
\phi_6 = \frac{1}{2}A^2 \approx 0 \tag{3.17}
\]

Note that Dirac’s equations (1.2a) and (1.2b) are now reproduced by constraints \( \phi_4 \) and \( \phi_5 \) above. But now there is a clear meaning for the third condition: the gauge field must remain massless. Constraints (3.12)-(3.17) obey the \( Sp(2,R) \) gauge algebra (2.11) together with the equations

\[
\{\phi_1, \phi_4\} = -P_M \frac{\partial}{\partial X_M}(X.A) \approx 0 \tag{3.18a}
\]
\[
\{\phi_1, \phi_5\} = -P_M \frac{\partial}{\partial X_M}(P.A) \approx 0 \tag{3.18b}
\]
\[
\{\phi_1, \phi_6\} = -P_M \frac{\partial}{\partial X_M}\left(\frac{1}{2}A^2\right) \approx 0 \tag{3.18c}
\]
\[
\{\phi_2, \phi_4\} = -X_M \frac{\partial}{\partial X_M}(X.A) \approx 0 \tag{3.18d}
\]
\[
\{\phi_2, \phi_5\} = P.A - X_M \frac{\partial}{\partial X_M}(P.A) \approx 0 \tag{3.18e}
\]
\[
\{\phi_2, \phi_6\} = -X_M \frac{\partial}{\partial X_M}\left(\frac{1}{2}A^2\right) \approx 0 \tag{3.18f}
\]
Equations (2.11) together with equations (3.18) show that all constraints (3.12)-(3.17) are first-class constraints.

Hamiltonian (3.2) will be invariant under the local scale transformations (2.14) when the gauge field effectively transforms as

$$A_M \rightarrow \tilde{A}_M = \exp\{-\beta\}A_M$$  \hspace{1cm} (3.19)

Using this local scale invariance we can again construct the same brackets (2.15). On the constraint surface defined by equations (3.12)-(3.17) we can use again equation (2.9) and the property (2.18) of the Dirac bracket and choose $\beta = \phi_1 = \frac{1}{2}P^2 \approx 0$ to arrive, by the same steps described in the previous section, at the same Snyder brackets (2.23).

Finally, let us consider the role of conditions (3.8). As in the free theory, the first-class constraints (3.12)-(3.14) reduce the number of physical canonical pairs $(X, P)$ to be $d - 1$. We introduced a gauge field $A_M(X)$ with $d + 2$ components, but as a consequence of (3.8) now there are 3 first-class constraints (3.15)-(3.17) acting on these $d + 2$ components. These constraints can be used to reduce the number of independent components of the gauge field to be $d + 2 - 3 = d - 1$, creating a perfect match of the number of independent components of the gauge field with the number of physical canonical pairs. It is this perfect match that preserves the local $Sp(2, R)$ invariance in the presence of the gauge field.

### 4 Concluding remarks

In this investigation we considered the possibility of introducing a minimum length in the classical two-time physics theory by constructing its Hamiltonian formulation in a phase space with Snyder brackets. It makes sense to try to introduce this minimum length in two time physics because the action is a generalization of gravity on the world-line and gravity introduces additional uncertainties in the quantum position measurement process.

We saw that it is possible to introduce a minimum length in the free theory and in the presence of background gauge fields, while at the same time preserving the usual symmetries of two-time physics, because of the existence of a broken local scale invariance which is a symmetry only of the Hamiltonian. We clarified a previous observation of the fact that the global $SO(d, 2)$ Lorentz generator in the presence of background gauge fields is identical to the one in the free theory and exposed the connection of this Lorentz generator with the concept of a minimum length. We also revealed the mechanism for the preservation of
the local $Sp(2, R)$ invariance of the action, which consists in a perfect match, in the Hamiltonian formalism, between the number of physical canonical pairs describing the dynamics and the number of physical components in the gauge field.

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