Solving the Probabilistic Profitable Tour Problem on a tree

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Abstract

The profitable tour problem (PTP) is a well-known NP-hard routing problem searching for a tour visiting a subset of customers while maximizing profit as the difference between total revenue collected and traveling costs. PTP is known to be solvable in polynomial time when special structures of the underlying graph are considered. However, the computational complexity of the corresponding probabilistic generalizations is still an open issue in many cases. In this paper, we analyze the probabilistic PTP where customers are located on a tree and need, with a known probability, for a service provision at a predefined prize. The problem objective is to select a priori a subset of customers with whom to commit the service so to maximize the expected profit. We provide a polynomial time algorithm computing the optimal solution in \( O(n^2) \), where \( n \) is the number of nodes in the tree.

Keywords: traveling salesman problem with profits; probabilistic profitable tour problem; polynomial time complexity; special graph structure; tree.

1 Introduction

Routing problems with profits (prizes) represent some of the most challenging variants of vehicle routing problems. Several real-world routing problems jointly consider the optimization of revenues and transportation costs, the latter either bounded in the constraints or directly subtracted from the global revenue in the objective function (Feillet et al. (2005)).

In the literature, a gamut of contributions can be found on deterministic variants of single and multi vehicle routing problems combining prize collection to other real constraints (see, for instance, Yu et al. (2022) where arrival and service times affect the collection of profits, Hanafi et al. (2020) where multi-visits are allowed while complying with precedence constraints and Archetti et al. (2014) for a survey). Nevertheless, a limited number of papers deal with these routing problems under uncertainty.

Probabilistic problems, characterized by the inclusion of some probabilistic elements in the problem definition, can be seen as generalizations of the corresponding deterministic variants. Different features can be probabilistic in a routing problem: from traveling times, arc costs, service times, to customer requests and collected prizes. A large part of the literature has been focused on problems where the probabilistic feature in the graph is associated with nodes (see, for instance, Bellalouna et al. (1995) and Henchiri et al. (2014) and references therein). The introduction of probabilistic aspects typically allows the decision maker to face more realistic
problems in which uncertainty is a major concern. It can be used as a tool to make strategic planning, for instance to plan a service/product distribution by evaluating its expected convenience. Probabilistic problems are commonly tackled in literature by means of an a priori optimization strategy. It consists in finding an a priori solution and then applying a modification strategy to adapt this solution to the occurrence (Bertsimas et al. (1990)). It is evident how such a strategy can be expressed as a 2-stage decision scheme. For example, in Laporte et al. (1994), the a priori optimization is applied to the probabilistic traveling salesman problem. In a first stage, the computation of an a priori Hamiltonian tour is obtained and then, in a second stage, when information on the present nodes has been revealed, the a priori tour is followed by skipping the absent nodes (recourse action).

In the literature different works can be found on the probabilistic variants of both traveling salesman problem and vehicle routing problem (see Jaillet (1985), Jaillet and Odoni (1988), Bertsimas and Howell (1993) and Campbell and Thomas (2008)). In case minimization of waiting time instead of distance traveled is considered, the problem becomes a repairman problem whose probabilistic variant on a tree has been studied by Averbakh and Berman (1995). In Angelelli et al. (2017) the authors analyze the probabilistic orienteering problem providing a stochastic mixed integer linear program and deriving a deterministic equivalent formulation to solve by means of an exact branch-and-cut method.

In this paper, we consider a probabilistic routing problem where a company has to decide which potential customers to engage into a long-term business agreement (typically an annual contract) for service provision (e.g. assistance on demand) so to maximize the expected value of its daily profits (revenues minus traveling costs). More precisely, according to the agreement, a customer will receive the service in a given day as long as he/she has requested it the day before. The service is guaranteed and provided by the company against the payment of a prize contractually established (revenue for the company), which is paid by the customer for each service call. The company does not know how many times (if any) each potential customer will need the service and when during the year. However, it knows each customer, on every single day, will call for a service with a given probability. In practice, the company will serve a (possibly) different set of customers every day. The service will be carried out at the minimum operating cost measured as travel cost (distance) required to visit in a tour all the customers involved. The profit (revenue minus cost) earned daily by the company depends on the set of customers under contract and on those who, among them, issue a service request. Therefore, the profit is a random variable that depends on the set of customers under contract, and each one of its realizations (the daily profit) is related to the customers issuing a service request on each day. Since the global profit over the year is given by the sum of daily realizations, the goal of the company is to determine with which customers it will undertake a contract so to maximize the expected value of the daily profit. In terms of a 2-stage decision process, the company first selects the subset of potential customers entering the annual contract and for which the provided service will be guaranteed (a priori decision), then, every day, the service requests from customers occur (nature outcome), and as a recourse action the company serves the customers requiring the service at the minimum travel cost. This generalization of the deterministic profitable tour problem (PTP) is known as the Probabilistic Profitable Tour Problem (PPTP). In Zhang et al. (2017), the authors study this problem applied to the case of big e-tailing companies that need to determine, among the set of all potential customers, which customers to serve directly and which outsource.

In our problem, the potential customers are assumed to be geographically dispersed in a
mountain area characterized by a main valley surrounded by a number of lateral valleys. The road network consists of one main road that traverses the main valley down up and is characterized by the presence of several bifurcations, that in turns generate secondary roads, leading to the lateral valleys, with a similar structure to the main one. Each customer location is linked to the company’s depot from a unique path through the road network. Thus, the road network can be express as a tree rooted at the depot where the other nodes are customers or bifurcations traversed to reach farthest customers located on secondary roads. Edge weights represent the traveling cost to traverse them.

When considering specific network structure (line, cycle, tree, star), the computational complexity of the deterministic PTP and other deterministic variants of the traveling salesman problem with profits can be found in Angelelli et al. (2014). The specific case of deterministic PTP on trees was previously discussed in Klau et al. (2003) where a linear time algorithm is provided. Finally, in Angelelli et al. (2022), the probabilistic PTP is analyzed considering the special case of the underlying graph represented by a path. In that paper, the authors show that the problem can be solved in $O(n^2)$ time, where $n$ is the number of customers and provide an interesting characterization of the optimal solution space of the problem.

The present variant of the problem analyzed over a rooted tree, from now on called the Probabilistic Profitable Tour Problem on a Tree (PPTP-T), is the generalization of the PPTP on a path studied in Angelelli et al. (2022). We show that also PPTP-T can be optimally solved in polynomial time with complexity $O(n^2)$ on a general rooted tree with $n$ nodes.

We believe that our study can be relevant also for other application domains where the tree-like topology is commonly used (e.g. communication networks).

The paper is organized as follows. Section 2 provides main notation and introduces the formal problem definition. In Section 3, we discuss the main properties of the characteristic function of a generic sub-tree which we recursively compute on larger sub-trees up to the main one. Finally, in Section 4, we formalize the solution algorithm and discuss its computational complexity. Conclusions are drawn in Section 5.

2 Notation and problem definition

The central contribution of this paper is a polynomial time algorithm for the Probabilistic Profitable Tour Problem (PPTP) when restricted on trees (PPTP-T). The present section is devoted to introduce the main notation used in the paper while providing a formal definition of both PPTP and PPTP-T. In PPTP the topology is described by a general graph $\langle V, E \rangle$ where node set $V$ includes customers and the company’s depot, and every edge $e \in E$ has a non-negative length that represents its traveling cost.

For every customer node $v \in V$, the input also specifies the probability $\pi_v > 0$ that customer $v$ will actually issue a request and the prize $p_v > 0$ collected by the company when fulfilling this request. In fact, a Bernoulli random variable $r_v$ is associated with each customer node $v$ that takes value 1 with probability $\pi_v$ (node $v$ issues a request) and 0 with probability $1 - \pi_v$ (no request). Variables $r_v$ are assumed to be independent. Thus, the probability that a subset $\omega$ of the customers turns out to comprise precisely those customers willing to issue a request is $P(\omega) = \prod_{v \in \omega} \pi_v \cdot \prod_{v \notin \omega} (1 - \pi_v)$.

Given all these data as its input, the company has to take an offline decision: select the most convenient subset $S$ of the customers with which to commit, knowing that, after this decision is
taken, each customer $s$ in $S$ will issue its request (of value $p_s$) with probability $\pi_s$, independently one from the other (Bernoulli process). With the choice of $S$, the company has committed to fulfill each one of the issued requests in $S$, and is going to collect the relative prizes, but also to cover the whole costs of the cheapest tour that starts from the depot, visits each one of the customers in $S$ that have issued their request, and finally returns to the depot.

Given the a priori selection of $S$, we define and consider three random variables all depending on the outcome $\omega$: the revenue, the cost, and the profit ($\text{revenue} - \text{cost}$). The PPTP asks to select $S$ with the objective to maximize the expected value of the profit.

The expected value of the revenue is easy to compute. On the contrary, computing the cost of a given outcome $\omega$ is NP-hard on general graphs as it essentially amounts to the solution of a standard TSP on node set $S \cap \omega$. Thus, PPTP is an NP-hard problem. Observe that, when facing PPTP, the outcome $\omega$ is not given, and $S$ is actually the independent variable we should optimize on.

The PPTP-T is the restriction of PPTP on trees. Here, we assume the input graph to be an undirected tree rooted at the depot. Clearly, it is easy to identify a cheapest route to follow for a given $S$ and any outcome $\omega$. As a matter of fact, every solution to the resulting TSP, i.e., every optimal tour that starts from the root and ends in the root of the tree after having visited the node set $S \cap \omega$, is limited to those tours in which every edge is traversed at most once in each direction. In particular, an edge should be considered if and only if it lays on the unique path from a node in $S \cap \omega$ to the root of the tree. For sake of simplicity, from now on, we assume that the cost of an edge includes both traversing cost back and forth.

It might be the case that some of the nodes the tree is built upon are not real customers, but just bifurcation nodes of the underlying road/communication network. On one side, we are not willing to select bifurcation nodes in our solution; on the other hand, for sake of notation and analysis simplicity, we prefer not to formally distinguish between customer and bifurcation nodes. Thus, without any loss of generality, we assume that a prize equal to $-1$ and a probability equal to 1 are assigned to each node representing just a tree bifurcation. This is enough to guarantee that bifurcation nodes will be automatically excluded from any optimal node selection.

### 2.1 Notation

Let $T$ be a rooted weighted tree whose recursive structure is described on a generic sub-tree $A$ as follows:

$$A = \{a, (e^{(1)}, A^{(1)}), \ldots, (e^{(i)}, A^{(i)}), \ldots, (e^{(m)}, A^{(m)})\}$$

where $a$ is the root of $A$, $A^{(i)}$ is the $i$-th sub-tree of $A$, and $e^{(i)} \geq 0$ is the cost required to reach its root $a^{(i)}$ starting from the root $a$ of $A$. The cost required to reach the root $a$ from the root $t$ of the main tree $T$ is indicated with $d_a$, and we naturally have $d_t = 0$ for the root $t$ of the main tree $T$, and $d_{e^{(i)}} = d_a + e^{(i)}$ in general. If $m = 0$, the tree $A$ consists only of the root node (leaf node of $T$) and is indicated as $A = \{a, [\ ]\}$.

Only for sake of notation simplicity, when $m > 0$ we indicate as $A^{(0)}$ the root node intended as a sub-tree of a single node whose edge connection $e^{(0)}$ measures zero. This will allow us to describe tree $A$ as follows:

$$A = \{(e^{(0)} = 0, A^{(0)}), (e^{(1)}, A^{(1)}), \ldots, (e^{(m)}, A^{(m)})\} \quad \text{with} \quad A^{(0)} = \{a, [\ ]\}.$$
We indicate as \( A \) the set of nodes contained into the tree \( A \) and with \( A^{(i)} \) the node set associated with the sub-tree \( A^{(i)} \), so that \( A = \bigcup_{i=0}^{m} A^{(i)} \). Defined as \( N_A = \{0, 1, \ldots, m\} \) the index set of the sub-trees of \( A \), for each \( I \subseteq N_A \) we denote as \( A_I = \bigcup_{i \in I} A^{(i)} \) the set of nodes belonging to the group of sub-trees indexed in \( I \), so that \( A = A_{N_A} = \bigcup_{i \in N_A} A^{(i)} \).

In the following, we will use \( 1_Q(\omega) \) to represent the indicator function of node set \( Q \) taking value 1 if \( Q \cap \omega \neq \emptyset \), and value 0 otherwise. Accordingly, we have that the probability \( \pi_s \), that node \( s \in T \) issues a request, is \( \pi_s = \sum_{\omega \subseteq T} 1_{\{s\}}(\omega) \mathbb{P}(\omega) \) whereas the probability that at least one node in a set of nodes \( S \subseteq T \) issues a request is \( \Pi_S = \sum_{\omega \subseteq T} 1_S(\omega) \mathbb{P}(\omega) \) or equivalently \( \Pi_S = 1 - \prod_{s \in S} (1 - \pi_s) \). Naturally, if \( S = \emptyset \) then \( \Pi_S = 0 \). Table 1 summarizes the main notation used to define the problem.

| Notation | Description |
|----------|-------------|
| \( T \) | main tree on which the problem is defined |
| \( T \) | node set of the main tree \( T \) |
| \( t \) | root of the main tree \( T \) |
| \( A \) (\( A^{(i)} \)) | generic sub-tree of \( T \) (\( i \)-th sub-tree of \( A \)) |
| \( A \) (\( A^{(i)} \)) | set of nodes of the generic tree \( A \) (of \( i \)-th sub-tree of \( A \)) |
| \( a, \pi_a, p_a, d_a \) | root of \( A \), probability, prize and distance from the root of \( T \) |
| \( e^{(i)} \) | weight of the edge connecting root \( a \) of \( A \) with the root of \( i \)-th sub-tree \( A^{(i)} \) |
| \( N_A = \{0, 1, \ldots, m\} \) | index set of the sub-trees of \( A \) |
| \( A_I, I \subseteq N_A \) | set of nodes belonging to a subset \( I \) of sub-trees of \( A \) |
| \( \Pi_S = 1 - \prod_{s \in S} (1 - \pi_s) \) | probability that at least one node in the set \( S \) offers a non-null prize |

Table 1: Main notation

### 2.2 Problem definition

Let \( S \subseteq T \) be a set of nodes on which an agreement has been a priori established and let \( \omega \subseteq T \) be the set of nodes issuing a request of service (nature outcome). The nodes that have to be visited are then given by \( S \cap \omega \).

According to the a priori selection of \( S \), let us define three random variables all depending on the outcome \( \omega \): the \textit{revenue}, the \textit{cost}, and the \textit{profit} defined as difference between revenue and cost. The distributions, and in particular the expected value of these random variables, depend on the node set \( S \) selected a priori and their analytical expressions follow.

#### 2.2.1 Revenue

The revenue produced by a node set \( S \) is defined as

\[
R_{\omega}(S) = \sum_{s \in S} p_s 1_{\{s\}}(\omega),
\]
and if $S \subseteq A$ for a sub-tree $A$, then the revenue can be recursively computed as

$$R_\omega(S) = R_\omega(S \cap A) = \sum_{i=0}^{m} R_\omega(S \cap A^{(i)}).$$

The expected value of the revenue produced by node set $S$ is thus given by

$$R(S) \equiv \mathbb{E}[R_\omega(S)] = \sum_{\omega \subseteq T} \left( \sum_{s \in S} p_s 1_s(\omega) \right) P(\omega) = \sum_{s \in S} p_s \left( \sum_{\omega \subseteq T} 1_s(\omega) P(\omega) \right) = \sum_{s \in S} p_s \pi_s,$$

and by

$$R(S) = R(S \cap A) = \sum_{i=0}^{m} R(S \cap A^{(i)})$$

when $S \subseteq A$ for a sub-tree $A$.

### 2.2.2 Discounted cost

To describe the cost random variable, we introduce a bonus $x$ representing a discount applied to the cost of the common path starting from the root of the main tree $T$ and shared by all paths reaching nodes in the selected set $S$. This bonus, not strictly relevant to the problem definition, is used as a tool to define the optimal solutions and to prove problem complexity, as showed in the following.

Thus, let $S$ be an a priori node selection and $A$ any sub-tree of $T$ such that $S \subseteq A$. For all $x \in [0, d_a]$, the random discounted cost implied by $S$ with bonus $x$ is

$$C_\omega(S; x) = (d_a - x + C_\omega(S; d_a)) 1_S(\omega). \quad (1)$$

The random variable $C_\omega(S; x)$ indicates the minimum traveling cost of a tour visiting nodes in $S \cap \omega$ from root $t$ when we assume to have a bonus equal to $x$. This cost function is null if $1_S(\omega) = 0$ (i.e. $S \cap \omega = \emptyset$), otherwise it is computed as the cost to reach the root $a$ of $A$ (common to all the nodes in $S \cap \omega$) net of the bonus $x$ plus the cost to visit the nodes in $S \cap \omega$ computed starting from the root of $A$. Regarding the term $C_\omega(S; d_a)$, it can be exploded into the sum of costs incurred in the different branches of tree $A$:

$$C_\omega(S; d_a) = \sum_{i=0}^{m} C_\omega(S \cap A^{(i)}; d_a). \quad (2)$$

Indeed, nodes in $S \cap \omega$ jointly contribute to cover the common cost $d_a - x$ to reach the root of $A$, whereas they no longer have anything in common starting from the root $a$ and therefore no common cost to share. If we use $e^{(i)} = d_{a^{(i)}} - d_a$ and apply definition (1) recursively on (2) we get

$$C_\omega(S; d_a) = \sum_{i=0}^{m} \left( e^{(i)} + C_\omega(S \cap A^{(i)}; d_{a^{(i)}}) \right) 1_{S \cap A^{(i)}}(\omega)$$

from which we can observe that if $S \cap \omega = \emptyset$, it necessarily holds that $1_{S \cap A^{(i)}}(\omega) = 0$ for each $i \in N_A$ and thus $C_\omega(S; d_a)$ is null without requiring to be multiplied by $1_S(\omega) = 0$. As a consequence, we can rewrite expression (1) as follows:
\[ C_\omega(S; x) = (d_a - x)1_S(\omega) + C_\omega(S; d_a). \] \hspace{1cm} (3)

It is worth noticing that, as stated in the following proposition, computing \( C_\omega(S; x) \) returns always the same value for any tree \( A \) such that \( S \subseteq A \) and \( x \in [0, d_a] \):

**Proposition 1** Let \( A \) and \( B \) be two trees such that \( S \subseteq A \) and \( S \subseteq B \) and \( x \leq \min(d_a, d_b) \). Then, for each outcome \( \omega \), it holds that

\[ (d_a - x)1_S(\omega) + C_\omega(S; d_a) = (d_b - x)1_S(\omega) + C_\omega(S; d_b). \]

**Proof of Proposition** \( \Box \) If \( S = \emptyset \) the equality trivially holds with

\[ (d_a - x)1_S(\omega) + C_\omega(S; d_a) = (d_b - x)1_S(\omega) + C_\omega(S; d_b) = 0. \]

Otherwise, one tree contains the other as a sub-tree. Without any loss of generality, let us assume that \( B \) is contained in \( A \) and thus \( d_a \leq d_b \). Notice that as \( S \subseteq B \) and \( B \) is a subtree of \( A \), \( S \) can be contained in only one sub-tree \( A^{(h)} \) of \( A \) and thus \( S \cap A^{(h)} = S \) and \( S \cap A^{(i)} = \emptyset \) for all the remaining sub-trees \( i \neq h, i \in N_A \).

Starting from (3) and (2) and further developing them we get:

\[
C_\omega(S; x) = (d_a - x)1_S(\omega) + C_\omega(S; d_a) \\
= (d_a - x)1_S(\omega) + \sum_{i=0}^{m} C_\omega(S \cap A^{(i)}; d_a) \\
= (d_a - x)1_S(\omega) + C_\omega(S \cap A^{(h)}; d_a) \\
= (d_a - x)1_S(\omega) + (d_{a^{(h)}} - d_a)1_S(\omega) + C_\omega(S; d_{a^{(h)}}) \\
= (d_{a^{(h)}} - x)1_S(\omega) + C_\omega(S; d_{a^{(h)}}) \\
= C_\omega(S; x),
\]

observe that the first and last occurrences of \( C_\omega(S; x) \) correspond to computing, by formula (3), the realized cost from \( A \) and its sub-tree \( A^{(h)} \), respectively.

If \( B = A^{(h)} \) we have done, otherwise we should iterate down in the sub-tree \( A^{(h)} \) up to \( B \) keeping constant the value of \( C_\omega(S; x) \).

We can conclude that the random variable \( C_\omega(S; x) \) is well defined for each value of \( x \) provided that there exists a tree \( A \) such that \( S \subseteq A \) and \( x \leq d_a \). The expected value of the cost function can thus be computed as follows:

\[
C(S; x) \equiv \mathbb{E}[C_\omega(S; x)] = (d_a - x)\Pi_S + C(S; d_a) \\
= (d_a - x)\Pi_S + \sum_{i=0}^{m} C(S \cap A^{(i)}; d_a).
\hspace{1cm} (4)
\]
Expression (4) immediately follows from:

$$
E [C_\omega(S; x)] = \sum_{\omega \subseteq T} \left[ (d_a - x) I_S(\omega) + C_\omega(S; d_a) \right] \mathbb{P}(\omega)
$$

$$
= (d_a - x) \sum_{\omega \subseteq T} I_S(\omega) \mathbb{P}(\omega) + \sum_{\omega \subseteq T} C_\omega(S; d_a) \mathbb{P}(\omega)
$$

$$
= (d_a - x) \Pi_S + C(S; d_a),
$$

whereas (5) can be proved as follows:

$$
E [C_\omega(S; d_a)] = \sum_{\omega \subseteq T} \left[ \sum_{i=0}^{m} C_\omega(S \cap A^{(i)}; d_a) \right] \mathbb{P}(\omega)
$$

$$
= \sum_{i=0}^{m} \left[ \sum_{\omega \subseteq T} C_\omega(S \cap A^{(i)}; d_a) \mathbb{P}(\omega) \right]
$$

$$
= \sum_{i=0}^{m} C(S \cap A^{(i)}; d_a).\tag{6}
$$

### 2.2.3 Profit

Finally, for any node set $S$ and $x \in [0, d_a]$ for some sub-tree $A$ such that $S \subseteq A$, the random variable profit implied by $S$ is

$$
G_\omega(S; x) = R_\omega(S) - C_\omega(S; x)
$$

to represent the profit gained to serve nodes $S \cap \omega$ with a cost discount $x$. Hence, the expected value of $G_\omega(S; x)$ can be computed as follows:

$$
G(S; x) \equiv E [G_\omega(S; x)] = R(S) - C(S; x)
$$

$$
= G(S; d_a) - (d_a - x) \Pi_S
$$

$$
= \sum_{i=0}^{m} G(S \cap A^{(i)}; d_a) - (d_a - x) \Pi_S.\tag{7}
$$

PPTP-T looks for a set $S$ that maximizes the expected value of the profit random variable $G_\omega(S; x)$ when the bonus $x$ is null:

$$
z^* = \max_{S \subseteq T} E [G_\omega(S; 0)].\tag{8}
$$

Solving problem (8) may appear a difficult task as the number of potential options is exponentially large in the size of the node set $T$. To solve it, we study the properties and generation of a function $f^{(A; I)}(x)$ defined as follows:

$$
f^{(A; I)}(x) = \max_{S \subseteq A, I} G(S; x).\tag{9}
$$

We call this function the sub-trees characteristic function with respect to the group of subtrees $I$, whereas a solution of the corresponding optimization problem is called optimal set.
Analogously, we indicate as \( f^{(A;NA)}(x) \) the tree characteristic function of \( \mathcal{A} \) which, for sake of simplicity, will be indicated as 
\[ f^{(A)}(x) \tag{10} \]
It follows that:
\[ z^* = f^{(T)}(0). \]

In the following sections, we show that problem (8) can be optimally solved in time \( O(|T|^2) \).

3 Properties

In this section, we discuss the properties of the sub-trees characteristic function for a generic tree \( \mathcal{A} \) and any group of sub-trees \( I \subseteq N_A \).

First of all, we show that \( f^{(A;I)}(x) \) is a monotone non-decreasing piece-wise linear function with a number of linear traits not larger than \( |A_I| + 1 \) over the domain \([0,d_a]\) of the bonus \( x \).

In Section 3.1 we show that each linear segment is characterized by a unique optimal set, each corner point is characterized by a unique minimal optimal set (left segment) and a unique maximal optimal set (right segment), and that the sequence of optimal sets defines a sort of matryoshka which grows while the bonus moves from left to right in the domain of \( f^{(A;I)}(x) \).

On this basis, in Section 3.2 we finally show the properties of sub-trees characteristic functions \( f^{(A;I)}(x) \) that can be exploited to efficiently compute the tree characteristic function \( f^{(A)}(x) \).

Let us start by observing that, given a tree \( \mathcal{A} \) and a group of sub-trees \( I \subseteq N_A \), the expected profit of a set of nodes \( S \subseteq A_I \subseteq A \) can be expressed, for all \( x \in [0,d_a] \), by means of formula (6) as
\[ G(S; x) = (G(S; d_a) - \Pi_S \cdot d_a) + \Pi_S \cdot x \tag{11} \]
to emphasize its dependency on the bonus \( x \). We call (11) the set profit line of node set \( S \).

Note that the set profit line of a node set \( S \) can be graphically represented as a line with non negative slope equal to \( \Pi_S \), that is the probability that at least one node in \( S \) issues a request. In particular, the slope is positive for each set \( S \neq \emptyset \), and null for \( S = \emptyset \) only; in such a case, \( G(\emptyset; x) = 0 \) for all \( x \in [0,d_a] \).

Thus, from the definition of function \( f^{(A;I)}(x) \) as the superior envelope of a finite number of linear segments with non negative slopes, we can claim the following fact:

**Fact 2** For any tree \( \mathcal{A} \) and group of sub-trees \( I \subseteq N_A \), the function \( f^{(A;I)}(x) \) is well defined for \( x \in [0,d_a] \), and is a continuous, convex, non-decreasing, piece-wise linear function. Each linear piece is the set profit line of an optimal set within the corresponding range.

3.1 Properties of the sub-trees characteristic function

**Maximal and minimal optimal sets.** Given a tree \( \mathcal{A} \) and a group \( I \subseteq N_A \) of sub-trees, for each value of the bonus \( x \in [0,d_a] \), there might be several optimal sets defining the sub-trees characteristic function \( f^{(A;I)}(x) \). In Proposition 4 we show that there is only one minimal optimal set and only one maximal optimal set. Before discussing Proposition 4 we need the following technical result whose proof is provided in Appendix.
Proposition 3 Let \( S_1 \) and \( S_2 \) be two node sets and \( \mathcal{A} \) a tree containing both \( S_1 \) and \( S_2 \) (i.e. \( S_1, S_2 \subseteq \mathcal{A} \)). Then for any \( x \in [0, d_a] \) we have

\[
G(S_1; x) + G(S_2; x) \leq G(S_1 \cup S_2; x) + G(S_1 \cap S_2; x). \tag{12}
\]

Proposition 4 Given a tree \( \mathcal{A} \), a group of sub-trees \( I \subseteq N_\mathcal{A} \) and a bonus \( x \in [0, d_a] \), there exists a unique minimal optimal set and a unique maximal optimal set corresponding to \( f(\mathcal{A}; I)(x) \).

Proof of Proposition 4 First of all, we observe that given two distinct optimal sets \( S_1 \) and \( S_2 \) for the same bonus \( x \) (i.e. \( G(S_1; x) = G(S_2; x) = f(\mathcal{A}; I)(x) \)), then also \( S_1 \cup S_2 \) and \( S_1 \cap S_2 \) are optimal sets.

Indeed, we have \( G(S_1; x) = G(S_2; x) \leq \max(G(S_1 \cup S_2; x), G(S_1 \cap S_2; x)) \), otherwise we get \( G(S_1; x) + G(S_2; x) > G(S_1 \cup S_2; x) + G(S_1 \cap S_2; x) \) in contrast with (12). For the optimality of \( S_1 \) and \( S_2 \) we conclude that at least one between \( S_1 \cap S_2 \) and \( S_1 \cup S_2 \) is optimal. Actually, they are both optimal because from inequality (12) we know that

\[
G(S_1; x) + G(S_2; x) \leq G(S_1 \cup S_2; x) + G(S_1 \cap S_2; x)),
\]

and subtracting on both side the optimal value (assume w.l.o.g. \( G(S_1; x) = G(S_1 \cup S_2; x) \)) we remain with

\[
G(S_2; x) \leq G(S_1 \cap S_2; x),
\]

which, from the optimality of \( S_2 \), also proves the optimality of \( S_1 \cap S_2 \).

Now we proceed by contradiction, let \( S_1 \) and \( S_2 \) be two distinct maximal optimal sets (i.e. \( S_1 \nsubseteq S_2 \) and \( S_2 \nsubseteq S_1 \)), we know that their proper superset \( S_1 \cup S_2 \) and their proper subset \( S_1 \cap S_2 \) are optimal, thus neither \( S_1 \) nor \( S_2 \) can be a maximal optimal set or a minimal optimal set.

Matryoshka property. In Propositions 5, we show that two distinct optimal sets for \( f(\mathcal{A}; I)(x) \) may exist only if \( x \) is a corner point. Moreover the unique minimal optimal set (guaranteed by Proposition 4) is the unique optimal set for the linear trait on the left of the corner point, and, analogously, the unique maximal optimal set is the unique optimal set on the linear trait on the right of the corner point. Finally, in Proposition 6, we show that the optimal sets in the domain [0, \( d_a \)] form a matryoshka implying that function \( f(\mathcal{A}; I)(x) \) has at most \( |\mathcal{A}_I| \) corner points and \( |\mathcal{A}_I| + 1 \) linear traits (possibly including the empty set).

Proposition 5 Let us consider a tree \( \mathcal{A} \) and a group of sub-trees \( I \subseteq N_\mathcal{A} \). If, for some \( \bar{x} \in (0, d_a) \), there exist two distinct optimal sets, then there is only one maximal optimal set which is optimal in \( [\bar{x}, \bar{x} + \varepsilon) \) and only one minimal optimal set which is optimal in \( (\bar{x} - \varepsilon, \bar{x}] \) for some \( \varepsilon > 0 \). In particular, \( \bar{x} \) is a corner point of \( f(\mathcal{A}; I)(x) \).

Proof of Proposition 5 From Proposition 4 we know that, for each given \( x \in [0, d_a] \), a unique minimal and a unique maximal optimal set exist, the first subset of the second one. If, for a given \( \bar{x} \in (0, d_a) \) we have two distinct optimal sets, then the minimal and maximal ones are distinct, and the minimal optimal set is a proper subset of the maximal optimal set. Let us call them \( Y \) and \( Z \), respectively. Clearly, any other optimal set \( W \) is a proper superset
of $Y$ and a proper subset of $Z$. It is evident from formula (11) that the slopes of their set profit lines are $\Pi_Y < \Pi_W < \Pi_Z$ (larger node sets manifest with higher probability). Thus, from

$$G(Y; \bar{x}) = G(W; \bar{x}) = G(G; \bar{x}),$$

we immediately derive $G(Y; x) > G(W; x) > G(Z; x)$ for $x < \bar{x}$ and vice-versa $G(Y; x) < G(W; x) < G(Z; x)$ for $x > \bar{x}$. For any other non optimal set $S$ in $\bar{x}$, we know that $G(S; \bar{x}) < G(Y; \bar{x})$, and the inequality holds in a neighborhood of $\bar{x}$ for the continuity of the set profit lines. Finally, if for a fixed $\varepsilon > 0$ there were some distinct optimal sets in interval $(\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ other than $Y$ and $Z$, then we can repeatedly halve the value of $\varepsilon$; the process must come to an end as we have only a finite number of potential optimal sets. Point $\bar{x}$ is thus a corner point since function $f^{(\mathcal{A}; I)}(x)$ takes different slopes around $\bar{x}$. □

**Proposition 6**  Given a tree $\mathcal{A}$ and a group of sub-trees $I \subseteq \mathcal{N}_\mathcal{A}$, the domain $[0, d_a]$ of function $f^{(\mathcal{A}; I)}(x)$ contains at most $|\mathcal{A}_I|$ corner points and $|\mathcal{A}_I| + 1$ linear pieces. Furthermore, the optimal sets corresponding to the sequence of linear traits form a chain with respect to inclusion (matryoshka).

**Proof of Proposition 6.** According to Proposition 5 at each corner point the optimal set gains additional nodes, and since we have $|\mathcal{A}_I|$ nodes, we can have $|\mathcal{A}_I|$ corner points at most. Starting from the empty set the number of optimal sets cannot be greater than $|\mathcal{A}_I| + 1$. Finally, moving from left to right in $[0, d_a]$, at each corner point we add one or more new nodes to the previous optimal set so that the new optimal set is a superset of the previous one. □

### 3.2 Computing $f^{(\mathcal{A})}(x)$

In this section, we propose a data structure $\mathcal{D}_{(\mathcal{A}; I)}$ to describe $f^{(\mathcal{A}; I)}(x)$ as a list of at most $|\mathcal{A}_I| + 1$ records, and then illustrate the properties that allow us to efficiently compute $f^{(\mathcal{A})}(x)$ for any tree $\mathcal{A}$. This goal is achieved by a bottom-up approach starting from the tree characteristic function $f^{(\mathcal{A})}(x)$ of trees without descendants (leaf nodes of $\mathcal{T}$) and by aggregating (or merging) ever larger groups of sub-trees until we have $f^{(\mathcal{A}; I)}(x)$ for $I = \mathcal{N}_\mathcal{A}$. Discussion will go through the following steps:

- building the description of $f^{(\mathcal{A})}(x)$ starting from leaf sub-trees;
- the descriptions $\mathcal{D}_{(\mathcal{A}_I)}$ and $\mathcal{D}_{(\mathcal{A}_I''''')}$ of the sub-trees characteristic function of two disjoint groups of sub-trees $I'$ and $I''$ can be “merged” into the description $\mathcal{D}_{(\mathcal{A}_I \cup I''')}$ of the union group $I' \cup I''$ by exploring the Cartesian product of the two descriptions instead of the power set of $\mathcal{A}_{I' \cup I''}$;
- the computation of the description $\mathcal{D}_{(\mathcal{A}_I'} \cup I''')$ of $f^{(\mathcal{A}; I' \cup I''')}(x)$ can be even more efficient with an appropriate exploration strategy of the descriptions of $f^{(\mathcal{A}; I')}(x)$ and $f^{(\mathcal{A}; I'')}(x)$.

#### 3.2.1 Characteristic function representation

Given a tree $\mathcal{A}$ and a group of its sub-trees $I \subseteq \mathcal{N}_\mathcal{A}$, we refer to

$$\mathcal{R}^{(i)}_{(\mathcal{A}; I)} = \langle S, x_{\min}^i, x_{\max}^i, \pi, q \rangle$$

as the tuple of the fields fully describing the $i$-th linear piece of the sub-trees characteristic function $f^{(\mathcal{A}; I)}(x)$ on the domain $[0, d_a]$.
In particular, $S$ indicates the maximal optimal set on the interval $[x^{\min}, x^{\max})$, whereas $\pi$ and $q$ represents the slope of the set profit line associated with $S$ (i.e. $\Pi_S$) and the value of the set profit line when $x = d_a$ (i.e. $G(S; d_a)$). The complete description of the linear pieces of $f^{(A; I)}(x)$ is thus given by a vector of tuples

$$D_{(A; I)} = [\mathcal{R}_{(A; I)}^{(i)}]_{i=0}^{n_A}$$

where the $n_A + 1 \leq |A_I| + 1$ records are sorted in such a way that $\mathcal{R}_{(A; I)}^{(i)} \cdot x^{\min} < \mathcal{R}_{(A; I)}^{(i+1)} \cdot x^{\min}$ for each $i = 0, 1, \ldots, n_A - 1$. Then, it follows that

$$\mathcal{R}_{(A; I)}^{(i)} \cdot x^{\max} = \mathcal{R}_{(A; I)}^{(i+1)} \cdot x^{\min}$$
$$\mathcal{R}_{(A; I)}^{(i)} \cdot S \subset \mathcal{R}_{(A; I)}^{(i+1)} \cdot S$$
$$\mathcal{R}_{(A; I)}^{(i)} \cdot \pi < \mathcal{R}_{(A; I)}^{(i+1)} \cdot \pi$$

and, in particular, $\mathcal{R}_{(A; I)}^{(0)} \cdot x^{\min} = 0$, $\mathcal{R}_{(A; I)}^{(n_A)} \cdot x^{\max} = d_a$. Observe that, accordingly to the previous discussion, $\mathcal{R}_{(A; I)}^{(i)} \cdot S$ is the only maximal optimal set for $x = \mathcal{R}_{(A; I)}^{(i)} \cdot x^{\min}$, the only optimal set for $x \in (\mathcal{R}_{(A; I)}^{(i)} \cdot x^{\min}, \mathcal{R}_{(A; I)}^{(i)} \cdot x^{\max})$, the only minimal optimal set for $x = \mathcal{R}_{(A; I)}^{(i)} \cdot x^{\max}$. Also note that, while $\mathcal{R}_{(A; I)}^{(i)} \cdot x^{\min} < \mathcal{R}_{(A; I)}^{(i)} \cdot x^{\max}$ for all $i < n_A$, it could be given the case that range $[\mathcal{R}_{(A; I)}^{(n_A)} \cdot x^{\min}, \mathcal{R}_{(A; I)}^{(n_A)} \cdot x^{\max}]$ reduces to a single point if a new node enters the maximal optimal set exactly when $x = d_a$. Finally, if $d_a = 0$ we have $n_A = 0$ and only one record $\mathcal{R}_{(A; I)}^{(0)}$ to describe $f^{(A)}(x)$ in the domain $[0, 0]$. For presentation convenience, we indicate with

$$\mathcal{M}_{(A; I)} = [\mathcal{R}_{(A; I)}^{(0)} \cdot S \subset \mathcal{R}_{(A; I)}^{(1)} \cdot S \subset \mathcal{R}_{(A; I)}^{(2)} \cdot S \subset \ldots \subset \mathcal{R}_{(A; I)}^{(n_A)} \cdot S]$$

the nested sequence of the optimal sets (matryoshka).

Note that all the information in $D_{(A; I)}$ is directly implied by the matryoshka $\mathcal{M}_{(A; I)}$ from which all other information can be computed. This is the reason why we focus only on the calculation of the matryoshka $\mathcal{M}_{(A; I)}$. Nevertheless, for computational efficiency reasons, it is convenient to explicitly store all the elements of the tuple $\mathcal{R}_{(A; I)}^{(i)}$, how illustrated in Section 4 where the algorithm implementation is commented.

For sake of simplicity, we remove index $I$ from notation when the group of sub-trees is complete (i.e. $I = N_A$) and use $D_A$ and $\mathcal{M}_A$ instead of $D_{(A; N_A)}$ and $\mathcal{M}_{(A; N_A)}$, in the same way as $f^{(A)}(x)$ is an alias for $f^{(A; N_A)}(x)$.

### 3.2.2 The basic case (A is a leaf)

To compute $f^{(A)}(x)$ for a tree $A$ without descendants, the decision is only whether or not to select the root of the tree. The choice is determined by the following expression:

$$f^{(A)}(x) = \max(G(\emptyset; x), G(\{a\}; x))$$
$$= \max(0, (p_a - (d_a - x))\pi_a)$$
$$= \left\{ \begin{array}{ll} (x - (d_a - p_a))\pi_a & \text{if } x \geq d_a - p_a \\ 0 & \text{otherwise.} \end{array} \right.$$
Thus, the description of \( D_A \) is given by

\[
D_A = \begin{cases} 
[ R^{(0)}_A ] = \langle \emptyset, 0, d_a, 0, 0 \rangle & \text{if } p_a < 0 \\
[ R^{(0)}_A ] = \langle \emptyset, 0, d_a - p_a, 0, 0 \rangle, \; \; \; R^{(1)}_A = \langle \{ a \}, d_a - p_a, d_a, \pi_a, p_a \pi_A \rangle & \text{if } p_a \in (0, d_a) \\
[ R^{(0)}_A ] = \langle \{ a \}, 0, d_a, \pi_a, p_a \pi_A \rangle & \text{if } p_a \geq d_a 
\end{cases}
\]

We only highlight two special cases. The case \( d_a = 0 \) reduces the domain of \( f(A)(x) \) to a single point interval \([0, 0]\). On the other hand, having \( p_a < 0 \) leads to only one optimal set (the empty one) on the whole domain, excluding a priori node \( a \) from all the possible optimal sets independently of its probability. As mentioned at the beginning of Section 2 this is particularly useful for modeling nodes representing just a bifurcation in the network and not a customer.

### 3.2.3 The general case

Unfortunately, we have no guarantee that for a certain \( x \in [0, d_a] \) the optimal set for \( f(A)(x) \) is the union of the optimal sets corresponding to the same \( x \) for \( f(A^{(i)})(x) \) of the sub-trees \( A^{(i)} \) with \( i = 0, 1, \ldots, m \). According to \( 12 \) we know that the union of two optimal sets \( S_i \) and \( S_j \) for the sub-trees characteristic functions \( f(A^{(i)})(x) \) and \( f(A^{(j)})(x) \) of distinct sub-trees in \( A \) may lead to an expected profit larger than the sum of two profits alone, and the optimal profit may be even larger. In a formula

\[
f(A^{(i)})(x) + f(A^{(j)})(x) = G(S_i; x) + G(S_j; x) \leq G(S_i \cup S_j; x) \leq f(A^{[i,j]})(x) \leq f(A)(x).
\]

Based on monotony of the tree characteristic function and the matryoshka property of its maximal optimal sets, intuition tells us that the selection of the optimal set for \( f(A^{(i)})(x) \) encourages the selection of a node set in \( A^{(j)} \) even larger of the optimal set for \( f(A^{(j)})(x) \) because the cost to reach nodes in \( A^{(j)} \) is partially paid by the engagement on optimal set in sub-tree \( A^{(i)} \). Proposition 7 confirms, quantifies and generalizes the intuition providing a technical basis for the proof of the main statement in Proposition 8 stating that the maximal optimal sets for \( f(A)(x) \) have to be taken within the Cartesian product of the sub-tree matryoshkas.

**Proposition 7** Given a tree \( A \), let \( I', I'' \subseteq N_A \) be two disjoint groups of sub-trees (i.e. \( I' \cap I'' = \emptyset \)) and let \( \Psi \subseteq A_{I'} \) and \( \Phi \subseteq A_{I''} \) two subsets of nodes. Then for any \( x \in [0, d_a] \) we have

\[
G(\Psi \cup \Phi; x) = G(\Psi; x) + G(\Phi; d_a \Pi_\Psi + x(1 - \Pi_\Psi))
\]

where \( \Pi_\Psi = 0 \) in case \( \Psi = \emptyset \).

**Proof of Proposition 7** Result can be proved directly by the two following identities:

\[
R(\Psi \cup \Phi) = R(\Psi) + R(\Phi)
\]

\[
C(\Psi \cup \Phi; x) = C(\Psi; x) + C(\Phi; d_a \Pi_\Psi + x(1 - \Pi_\Psi))
\]

The first identity can be easily verified since \( \Psi \cap \Phi = \emptyset \). Regarding the second identity, we first observe that \( d_a \Pi_\Psi + x(1 - \Pi_\Psi) \leq d_a \), thus \( C(\Phi; d_a \Pi_\Psi + x(1 - \Pi_\Psi)) \) is well defined being \( \Phi \subseteq A \).
Since \( \Psi \cup \Phi \subseteq A \), the identity can be proved by decomposing \( C(\Psi \cup \Phi; x) \) as shown in (1) and (5).

\[
C(\Psi \cup \Phi; x) = (d_a - x)\Pi_{\Psi \cup \Phi} + \sum_{i=0}^{m} C((\Psi \cup \Phi) \cap A^{(i)}; d_a) \\
= (d_a - x)\Pi_{\Psi \cup \Phi} + \sum_{i \in I'} C(\Psi \cap A^{(i)}; d_a) + \sum_{i \in I''} C(\Phi \cap A^{(i)}; d_a) + \sum_{i \in N_A \setminus (I' \cup I'')} C(\emptyset \cap A^{(i)}; d_a)
\]

\[= (d_a - x)\Pi_{S} + C(\Psi; d_a) + C(\Phi; d_a)
\]

\[= (d_a - x)\Pi_{S} + [(d_a - x)\Pi_{\Psi} + C(\Psi; d_a)] - (d_a - x)\Pi_{\Psi} + C(\Phi; d_a)
\]

\[= C(\Psi; x) + (d_a - x)\Pi_{S} - (d_a - x)\Pi_{\Psi} + C(\Phi; d_a)
\]

\[= C(\Psi; x) + (d_a - x)(\Pi_{S} - \Pi_{\Psi}) + C(\Phi; d_a),
\]

and then, according to (4),

\[C(\Phi; d_a\Pi_{\Psi} + x(1 - \Pi_{\Psi})) = (d_a - (d_a\Pi_{\Psi} + x(1 - \Pi_{\Psi})))\Pi_{\Psi} + C(\Phi; d_a)
\]

\[= (d_a - x)(1 - \Pi_{\Psi})\Pi_{\Phi} + C(\Phi; d_a).
\]

Now, it is enough to show that the following equality holds

\[(d_a - x)(\Pi_{S} - \Pi_{\Psi}) = (d_a - x)(1 - \Pi_{\Psi})\Pi_{\Phi}
\]

that, through algebraic computations, corresponds to the identity

\[\Pi_{S} = \Pi_{\Psi \cup \Phi} = \Pi_{\Phi} + \Pi_{\Psi} - \Pi_{\Psi}\Pi_{\Phi}.
\]

\[\square\]

**Proposition 8** Let \( I \subseteq N_A \) be a group of sub-trees and let \( I = I' \cup I'' \) with \( I' \cap I'' = \emptyset \) an arbitrary partition of \( I \). For each set of nodes \( S \in \mathcal{M}(A; I) \) let \( S' = S \cap A_{I'} \) and \( S'' = S \cap A_{I''} \) be its components in the two groups of sub-trees (i.e. \( S = S' \cup S'' \) and \( S' \cap S'' = \emptyset \)).

Then the following condition holds:

\[(S', S'') \in \mathcal{M}(A; I') \times \mathcal{M}(A; I'')
\]

**Proof of proposition 8** Let \( \hat{x} \in [0, d_a] \) be a value of the bonus for which \( S \) is a maximal optimal set of the sub-trees characteristic function \( f^{(A; I)}(x) \).

We first prove that \( S' \in \mathcal{M}(A; I') \) by contradiction. From Proposition 7 we know that

\[G(S; \hat{x}) = G(S''; \hat{x}) + G(S'; d_a\Pi_{S''} + \hat{x}(1 - \Pi_{S''}))
\]

if \( S' \notin \mathcal{M}(A; I') \) then \( S' \) cannot be a maximal optimal set for any \( x \in [0, d_a] \). If \( S' \) is not optimal for any value of the bonus, then it is not even for \( x = d_a\Pi_{S''} + \hat{x}(1 - \Pi_{S''}) \), and thus \( G(S; \hat{x}) \) can be strictly increased which contradicts the optimality of \( S \). If \( S' \) is optimal for \( x = d_a\Pi_{S''} + \hat{x}(1 - \Pi_{S''}) \), but not maximal, then also \( S \) can not be a maximal optimal set. Thus, \( S' \) is necessarily a maximal optimal set for \( f^{(A; I')}(x) \) for at least one value in \( [0, d_a] \) and thus it belongs to \( \mathcal{M}(A; I') \) by definition.

A similar discussion proves that \( S'' \in \mathcal{M}(A; I'') \). \[\square\]
It is worth noticing that the Cartesian product $\mathcal{M}_{(A; I')} \times \mathcal{M}_{(A; I'')} \times \mathcal{M}_{(A; I'')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I'''')} \times \ldots \times \mathcal{M}_{(A; I'(m-1)) \times \mathcal{M}_{(A; I'(m)}}$

with any order of product execution; hence, once known the matryoshka $\mathcal{M}_{A(i)}$, each one with $O(|A(i)|)$ optimal sets, the computation of $\mathcal{M}_{A}$ does not require the evaluation of $O(2^{|A|})$ set profit lines anymore, but in the worst case of $O(\prod_{i=0}^{m} |A(i)|) = O(|A|^{m+1})$.

It is now appropriate to specify that, while matryoshka $\mathcal{M}_{A}$, which we are trying to build, describes $f(A)(x)$ in the interval $[0, d_{a}]$, matryoshkas $\mathcal{M}_{A(i)}$ describe functions $f(A(i))(x)$ in larger intervals $[0, d_{a(i)}]$ and might contain more elements than necessary. Proposition 9 will help us to adequately limit matryoshka extension for each sub-tree from which $\mathcal{M}_{A}$ will be obtained. Their size remain, however, in the order of $O(|A(i)|)$.

**Proposition 9** Let us consider a tree $A$. The last maximal optimal set of the matryoshka $\mathcal{M}_{A}$ is the set union of maximal optimal sets for $f(A(i))(d_{a})$.

**Proof of Proposition 9.**

From expression (7) we know that for any node set $S \subseteq A$ and $x = d_{a}$ we have

$$G(S; d_{a}) = \sum_{i=0}^{m} G(S \cap A(i); d_{a}).$$

Thus, maximization of $G(S; d_{a})$ with respect to $S \subseteq A$ is equivalent to solve $m + 1$ independent problems on the $m + 1$ sub-trees of $A$. ■

### 3.2.4 Order of magnitude of matryoshka’s Cartesian products

Computing the whole Cartesian Product of all matryoshkas associated to sub-trees may be not computationally convenient. In fact, if $m = O(|A|)$ also the Cartesian product contains $O(2^{|A|})$ elements. Consider the case of a simple tree with $m$ nodes distributed in $m$ sub-trees of one node each (including the root sub-tree $A(0)$); the matryoshka of each sub-tree potentially contains 2 optimal sets and their Cartesian product has cardinality $O(2^m)$.

The situation changes radically if, to obtain the matryoshka $\mathcal{M}_{A}$, we first calculate the matryoshkas of two complementary non-empty sub-trees groups $I' \subseteq N_A$ and $I'' = N_A \setminus I'$. In this case, the two matryoshkas $\mathcal{M}_{(A; I')} \times \mathcal{M}_{(A; I'')} \times \mathcal{M}_{(A; I'')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I'''')} \times \ldots \times \mathcal{M}_{(A; I'(m-1)) \times \mathcal{M}_{(A; I'(m)}}$ of node sets involved, we can easily observe that

$$\mathcal{M}_{(A; I' \cup I'')} \subset \mathcal{M}_{(A; I')} \times \mathcal{M}_{(A; I'')} \times \mathcal{M}_{(A; I'')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I''')} \times \mathcal{M}_{(A; I'''')} \times \ldots \times \mathcal{M}_{(A; I'(m-1)) \times \mathcal{M}_{(A; I'(m)}}.$$
a Cartesian product of dimension $O(|A|^2)$. At this point we have a strong clue that the computation of $D_A$ for $f^{(A)}(x)$ can be done in polynomial time. However, before drawing conclusions, let’s make a few more observations that will allow us to further lower the degree of the polynomial.

3.2.5 Avoiding the Cartesian product of matryoshkas

We have shown that the matryoshka $M_A$ is contained in the Cartesian product of two matryoshkas each of $O(|A|)$ elements. Therefore, the product has cardinality $O(|A|^2)$. In this section we show that it is not necessary to evaluate all the $O(|A|^2)$ elements of the Cartesian product and that we can instead select in $O(|A|)$ time a matryoshka $M_K$ which certainly contains $M_A$. As a final step we will see that we can get $M_K$ in $O(|A|)$ time. This will be the decisive step that allows us to define a quadratic algorithm to compute $f^{(A)}(x)$ and solve the problem.

Thus, we will discuss how, the matryoshkas

$$M_{(A,B)} = [B_0 \subset B_1 \subset \ldots \subset B_{n_B}]$$

and

$$M_{(A,C)} = [C_0 \subset C_1 \subset \ldots \subset C_{n_C}]$$

do not contain $M_{K_n}$ of two disjoint groups $B \subset N_A$ and $C \subset N_A$ of sub-trees can be processed to give rise to the matryoshka

$$M_{(A,D)} = [D_0 \subset D_1 \subset \ldots \subset D_{n_D}]$$

do the union group $D = B \cup C$. The whole operation can be done in time $O(n_B + n_C)$. On the basis of the Proposition we assume that $M_{(A,B)}$ and $M_{(A,C)}$ cover only the domain $[0, d_n]$.

First we will identify a matryoshka $M_K \subseteq M_{(A,B)} \times M_{(A,C)}$ such that $M_{(A,D)} \subseteq M_K$. Then we will see how to extract the matryoshka $M_{(A,D)}$.

The idea is to start with a matryoshka $M^{(0)}_K = [K_0 = B_0 \cup C_0]$ whose only element is clearly a subset of $D_0 \in M_{(A,D)} \subseteq M_{(A,B)} \times M_{(A,C)}$.

Then, for $h = 1, 2, \ldots, n_B + n_C$, a set $K_h \in M_{(A,B)} \times M_{(A,C)}$ is iteratively added to $M^{(h-1)}_K$ so that a new matryoshka $M^{(h)}_K$ compliant with the following definition of coherence with $M_{(A,D)}$ is obtained at each step.

**Definition 10** A matryoshka $M^{(h)}_K = [K_0 \subset K_1 \subset \ldots \subset K_h]$ is said to be coherent with a matryoshka $M_{(A,D)}$ if for any $D \in M_{(A,D)}$, it holds that

$$D \in M^{(h)}_K \quad \forall \quad K_h \subseteq D.$$
$\mathcal{D}_{n_B} \subseteq K_h$ and there is no need to extend matryoshka $\mathcal{M}^{(h)}_K$. The smallest step we can do is to take either $K_{h+1} = B_{i+1} \cup C_j$ or $K_{h+1} = B_i \cup C_{j+1}$. Both choices may be correct, but in some cases only one preserves the property of coherence with $\mathcal{M}_{(A,D)}$. For example, consider matryoshka $\mathcal{M}^{(0)}_K = [K_0]$ with $K_0 = B_0 \cup C_0$ and its candidate extensions $Y = B_1 \cup C_0$ and $Z = B_0 \cup C_1$. In the case $D_0 = B_2 \cup C_3$, then both $Y$ and $Z$ are valid extensions that preserve coherence with $\mathcal{M}_{(A,D)}$, but in the case $D_0 = B_0 \cup C_3$, only $Z$ is a correct choice as $Y$ would irremediably broke the coherence of the new matryoshka with $\mathcal{M}_{(A,D)}$.

Thus, we need a rule that prevent us from taking a wrong choice. Proposition 13 gives us the rule we need. The idea exploits equation (13) in Proposition 7. Let us consider a bonus $x < d_A$, two sub-tree groups $B$ and $C$ and the corresponding matryoshkas $\mathcal{M}_{(A,B)}$ and $\mathcal{M}_{(A,C)}$. A commitment to serve node set $B_i \in \mathcal{M}_{(A,B)}$ with a bonus $x$ offers a larger bonus $d_A \pi_{B_i} + x(1 - \pi_{B_i})$ to visit nodes in the sub-tree group $C$. Which set in $\mathcal{M}_{(A,C)}$ is better to select (given the commitment to $B_i$) depends on the bonus $x$. Thus, we define the entry threshold of set $C_j$ given the commitment to $B_i$ as the minimum bonus value such that the choice of $C_j$ is better than any other (given the commitment to $B_i$). Of course the same considerations can be made on the other side, by considering a commitment to some $C_j \in \mathcal{M}_{(A,C)}$ and subsequent choice of an optimal set in $\mathcal{M}_{(A,B)}$. The following definition helps to formalize the concept of entry threshold and Proposition 12 states the main properties of these indices that will be used in Proposition 13.

**Definition 11** Let $B$ and $C$ be two disjoint groups of sub-trees in tree $A$ (i.e. $B, C \subseteq N_A$, $B \cap C = \emptyset$), and let $B_i \in \mathcal{M}_{(A,B)}$ and $C_j \in \mathcal{M}_{(A,C)}$.

We define entry threshold of the 'entering' node set $C_j$ given the committed node set $B_i$ the value

$$S(C_j; B_i) = \arg \min_x \left\{ G(B_i \cup C_j; x) = \max_h G(B_i \cup C_h; x) \right\}$$

Symmetrically, we define entry threshold of the entering node set $B_i$ given the committed node set $C_j$ the value

$$S(B_i; C_j) = \arg \min_x \left\{ G(B_i \cup C_j; x) = \max_h G(B_h \cup C_j; x) \right\}$$

**Proposition 12** Let $B$ and $C$ be two disjoint groups of sub-trees in tree $A$, and let $\mathcal{M}_{(A,B)} = [B_0, \ldots, B_{n_B}]$ and $\mathcal{M}_{(A,C)} = [C_0, \ldots, C_{n_C}]$ be the respective matryoshkas of maximal optimal sets. For each $i \leq n_B$, $j \leq n_C$ we have:

$$S(C_j; B_i) = \frac{R_{(A,C)}^{(j)} x_{\text{min}} - d_a \pi_{B_i}}{1 - \pi_{B_i}} \leq R_{(A,C)}^{(j)} x_{\text{min}} \quad \text{entry advance}$$

(14)

$$S(C_1; B_i) < S(C_2; B_i) < \ldots < S(C_{n_C}; B_i) \quad \text{monotonicity with respect to the entering set}$$

(15)

$$S(C_j; B_0) > S(C_j; B_1) > \ldots > S(C_j; B_{n_B}) \quad \text{monotonicity with respect to the committed set}$$

(16)
Obviously the symmetrical ones also apply

\[ S(B_i; C_j) = \frac{\mathcal{R}^{(i)}_{(A:B)} - d_i \Pi_{C_j}}{1 - \Pi_{C_j}} \leq \mathcal{R}^{(i)}_{(A:B)} \cdot x^{\min} \] (17)

\[ S(B_1; C_j) < S(B_2; C_j) < \ldots < S(B_{n_B}; C_j) \] (18)

\[ S(B_i; C_0) > S(B_i; C_1) > \ldots > S(B_i; C_{n_C}) \] (19)

The proof of Proposition 13 is provided in Appendix. Here, we observe that inequality (14) shows that node set \( C_j \in M_{(A:C)} \) starts to dominate smaller sets in its matryoshka for smaller values of the bonus when also set \( B_i \in M_{(A:B)} \) is committed. Moreover, the inequality chain (15) attests that the relative order of set selection in \( M_{(A:C)} \) with respect to the bonus remains unchanged when set \( B_i \in M_{(A:B)} \) is committed. Finally, the inequality chain (16) shows that the entry threshold of \( C_j \) becomes smaller and smaller as the set committed in \( M_{(A:B)} \) becomes larger. Proposition 13 will use these facts to define a rule to coherently extend a matryoshka.

The proof is provided in the Appendix.

**Proposition 13** Let \( B \) and \( C \) be two disjoint groups of sub-trees in tree \( A \), let the matryoshka \( M^{(h)}_K = [K_0 \subset K_1 \subset \ldots \subset K_h] \subseteq M_{(A:B)} \times M_{(A:C)} \) be coherent with \( M_{(A:B;C)} \), and let \( K_h = B_i \cup C_j \) denote its last element with \( i + j < n_B + n_C \). Then, by choosing \( K_{h+1} \) with the following rule

- If \( (i < n_B, \ j = n_C) \Rightarrow K_{h+1} = B_{i+1} \cup C_j \)
- Otherwise if \( (i = n_B, \ j < n_C) \Rightarrow K_{h+1} = B_i \cup C_{j+1} \)
- Otherwise if \( S(B_i+1; C_j) \leq S(C_{j+1}; B_i) \Rightarrow K_{h+1} = B_{i+1} \cup C_j \)
- Otherwise \( S(B_i+1; C_j) > S(C_{j+1}; B_i) \Rightarrow K_{h+1} = B_i \cup C_{j+1} \)

the extension of \( M^{(h)}_K \) with \( K_{h+1} \) produces a matryoshka \( M^{(h+1)}_K \) coherent with \( M_{(A:B;C)} \).

In conclusion of this Section we observe that so far we have seen how to build the matryoshka of a leaf node and commented that the matryoshkas of two (groups of) sub-trees can be “merged” to form ever larger groups of sub-trees up to form the matryoshka of the father tree. We also showed that a matryoshka containing the matryoshka of the union of two (groups of) sub-trees can be computed in a number of steps which is linear with respect to the number on nodes involved. How to assemble all this stuff and obtain an efficient algorithm is described in Section 4 where we provide an algorithm definition and discuss its time complexity.

4 An optimal algorithm

In this section, we formalize the algorithm and discuss its computational complexity. As a first step we discuss the fundamental step of creating the description of the sub-trees characteristic function of a group of sub-trees which is the union of two disjointed groups of sub-trees of which we have the descriptions (Algorithm Merge). Then, we provide the definition of a recursive procedure that cumulatively merge the descriptions of all sub-trees in a given tree (Algorithm SolveTree), and finally the main function providing the required problem solution (Algorithm PPTP–Tree).
Proposition 14 Let $A$ be a tree and $B \in N_A$, $C \in N_A$ be two disjointed groups of sub-trees. The description $\mathcal{D}_{(A,B,C)}$ of the sub-trees characteristic function of the union group $B \cup C$ can be computed in time $O(|\mathcal{A}_{(B,C)}|)$ starting from $\mathcal{D}_{(A,B)}$ and $\mathcal{D}_{(A,C)}$.

Algorithm 1: Merge

```
input : $\mathcal{D}_{(A,B)}$, $\mathcal{D}_{(A,C)}$ descriptions of $f^{(A,B)}(x)$ and $f^{(A,C)}(x)$
output: $\mathcal{D}_{(A,B\cup C)}$ description of $f^{(A,B\cup C)}(x)$
1 - $d_a = \mathcal{D}_{(A,B)}(x)^{\text{max}}$
2 - $E$ = empty stack
3 - $K.S = \mathcal{D}_{(A,B)}^{(0)} x \cup \mathcal{D}_{(A,C)}^{(0)} x$
4 - $K.\pi = 1 - (1 - \mathcal{D}_{(A,B)}^{(0)} \cdot \pi(1 - \mathcal{D}_{(A,C)}^{(0)} \cdot \pi)$
5 - $K.q = \mathcal{D}_{(A,B)}^{(0)} q + \mathcal{D}_{(A,C)}^{(0)} q // \text{value of } G(K.S;d_a)$
6 - $K.x^{\text{min}} = 0, K.x^{\text{max}} = d_a$
7 - $E.push(K)$
8 - $i = 0, j = 0$
9 while $(i < n_B \lor j < n_C)$ do
  10 if $(i == n_B)$ then $j = j + 1$
  11 else if $(j == n_C)$ then $i = i + 1$
  12 else
    13 - $S_B = \mathcal{D}_{(A,B)}^{(i+1)} x \cup \mathcal{D}_{(A,C)}^{(j)} x$
    14 - $S_C = \mathcal{D}_{(A,B)}^{(i)} x \cup \mathcal{D}_{(A,C)}^{(j+1)} x$
    15 if $(S(S_B;K.S) < S(S_C;K.S))$ then $i = i + 1$
    16 else $j = j + 1$
    17 - $K'.S = \mathcal{D}_{(A,B)}^{(i)} x \cup \mathcal{D}_{(A,C)}^{(j)} x$
    18 - $K'.\pi = 1 - (1 - \mathcal{D}_{(A,B)}^{(i)} \cdot \pi(1 - \mathcal{D}_{(A,C)}^{(j)} \cdot \pi)$
    19 - $K'.q = \mathcal{D}_{(A,B)}^{(i)} q + \mathcal{D}_{(A,C)}^{(j)} q // \text{value of } G(K'.S;d_a)$
    20 - $K'.x^{\text{min}} = 0, K'.x^{\text{max}} = d_a$
    21 - $x = d_a - \frac{K'.q-K.q}{K'.\pi-K.\pi} // \text{Solution of equation } G(K'.S;x) = G(K.S;x)$
    22 while $(x \leq K.x^{\text{min}})$ do
      23 - $E.pop()$
      24 - $K = E.peek()$
      25 - $x = d_a - \frac{K'.q-K.q}{K'.\pi-K.\pi}$
    26 - $K.x^{\text{max}} = x$
    27 - $K'.x^{\text{min}} = x$
    28 - $E.push(K')$
    29 - $K = K'$
  30 - $\mathcal{D}_{(A,B\cup C)} = \text{revert } E$
```

Proof of Proposition 14 We refer to Algorithm 1 (Merge) for the pseudo-code description of the procedure that calculates the result.
Correctness. The procedure is based on Proposition \[13\] and starts by building the smallest matryoshka \(M_K^{(0)}\) coherent with \(M_{(A;B;C)}\) using the smallest maximal optimal sets of the matryoshkas in the descriptions \(D_{(A;B)}\) and \(D_{(A;C)}\) of the sub-trees characteristic functions of the groups \(B\) and \(C\)\[3.6\]. The corresponding record descriptor \(K\) is then added to the stack \(E\) which represents the under construction description \(D_{(A;B;C)}\) of \(f^{(B;C)}(x)\). In particular, the calculation of the field \(q\) comes from the formula \(7\), and being the only set currently considered, its domain is over the whole interval \([0, d_a]\).

The selection rule described in Proposition \[13\] (lines \[10.16\]) is applied to each iteration of the main cycle to extend the matryoshka \(M_{K}^{(h)}\) to \(M_{K}^{(h+1)}\) and generate a new record \(K'\) to add to the stack \(E\) (lines \[17.20\]).

Note that the matryoshka coherent with \(M_{(A;B;C)}\), although calculated element by element in the variable \(K'\), is not explicitly stored as the relative records are immediately used to build the envelope of \(f^{(B;C)}(x)\) in the stack \(E\). Indeed, node set \(K'.S\) certainly has the right to be included in \(E\) since, by virtue of its being a superset of all the sets inserted up to now in \(E\), it turns out to have an expected profit not smaller than all these at least for \(x = d_a\) (see equation \(7\)). Thus, before inserting \(K'\) into \(E\), the records referred to sets dominated, over their entire range, by \(K'.S\) are removed from \(E\) (lines \[21.25\]). In general, the first set that is not fully dominated will yield part of its interval to the new set \(K'.S\), so an adjustment of the two ranges follows at the intersection of their respective set profit lines (lines \[26.27\]). Finally the stack \(E\) is reverted to sort records of \(D_{(A;B;C)}\) in the right order (line \[30\]).

Complexity. The main loop is executed no more than \(n_B + n_C \leq |A_{B;C}|\) times because at each iteration the index \(i\) or the index \(j\) is incremented by 1. Observe that, despite their unsettling definition, the entry thresholds computed at line \[15\] can be computed in \(O(1)\) time according to formulas \(14\) and \(17\) in Proposition \[12\]. Thus, all operations in the loop have a cost of \(O(1)\) except for the inner loop which removes a certain number of records from the stack \(E\) before inserting the record for the new \(K\). Each iteration of the inner loop has a cost of \(O(1)\), and, since at each iteration a record is removed from the stack \(E\), the number of iterations is bounded by the number of records in the stack. The number of records remaining at loop termination determines the bound for the maximum number of iterations on the next pass. Basically, in the first pass we have \(|E| = 1\), and \(h_1 \leq 1\) iterations are performed, for \(2 - h_1\) records in \(E\) at the exit from the loop. In the second pass, \(h_2 \leq 2 - h_1\) iterations are performed for \(3 - h_1 - h_2\) records in \(E\) when exiting the loop. In the third pass, \(h_3 \leq 3 - h_1 - h_2\) iteration are performed for \(4 - h_1 - h_2 - h_3\) records in \(E\) when leaving the loop. In the \(k\)-th pass, \(h_k \leq k - \sum_{j<k} h_j\) iterations are performed for \(k - \sum_{j\leq k} h_j\) records in \(E\) upon exiting the loop. After \(n_B + n_C\) steps, we get \(\sum_{j\leq n_B + n_C} h_j \leq n_B + n_C\) which shows that, overall, the internal loop to extract records from the stack \(E\) has a cost \(O(n_B + n_C)\). This ends the proof.

We can now state the complexity to find the optimal solution for the PPTP-T:

**Theorem 15** An optimal set for problem \(8\) can be computed in \(O(|T|^2)\) time.
Algorithm 2: PPTP-Tree

\textbf{input} : \mathcal{T} \text{ tree} \\
\textbf{output}: S \text{ optimal node set of } \mathcal{T}; G \text{ optimal value}

1 - D_{\mathcal{T}} = \text{SolveTree}(\mathcal{T}, N_{\mathcal{T}}) // description of tree characteristic function \\
\text{ }\text{ }\text{ }\text{ }\text{ }\text{ }f^{(\mathcal{T})}(x) \\
2 - S = R^{(0)}_{\mathcal{T}} \cdot S \\
3 - G = R^{(0)}_{\mathcal{T}} \cdot q - d_{t} \cdot R^{(0)}_{\mathcal{T}} \cdot \pi

Algorithm 3: SolveTree

\textbf{input} : \mathcal{A} \text{ a tree} \\
\textbf{output}: D_{\mathcal{A}} \text{ description of tree characteristic function } f^{(\mathcal{A})}(x)

1 - compute \( D^{(0)}_{\mathcal{A}} \) // See base case in Section 3.2 \\
2 - D_{\mathcal{A}} = \text{Truncate}(D^{(0)}_{\mathcal{A}}, d_{a}) // truncate description at root of tree \mathcal{A} \\
3 \text{ for } (i = 1 \text{ to } |N_{\mathcal{A}}| - 1) \text{ do} \\
4 \text{ }\text{ }\text{ }\text{ }\text{ }\text{ }D^{(i)}_{\mathcal{A}} = \text{SolveTree}(\mathcal{A}^{(i)}) \\
5 \text{ }\text{ }\text{ }\text{ }\text{ }\text{ }D^{(i)}_{\mathcal{A}} = \text{Truncate}(D^{(i)}_{\mathcal{A}}, d_{a}) \\
6 \text{ }\text{ }\text{ }\text{ }\text{ }\text{ }D_{\mathcal{A}} = \text{Merge}(D_{\mathcal{A}}, D^{(i)}_{\mathcal{A}})

\textbf{Proof of Theorem 15.} We refer to Algorithm 2 (PPTP-Tree) which builds \( D_{\mathcal{T}} \) of the tree characteristic function \( f^{(\mathcal{T})}(x) \) in one step and then extracts from the first record the optimal set and the expected profit corresponding to a null bonus \( x = 0 \). Obviously, the complexity lies in the function \text{SolveTree} \text{ called on Line 1 (we refer to Algorithm 3 for a pseudo-code description of this function). Then, it will be sufficient to show that Algorithm 3 (SolveTree) computes the tree characteristic function \( f^{(\mathcal{A})}(x) \) of a tree \( \mathcal{A} \) in time \( O(|\mathcal{A}|^{2}) \).

Building the description \( D_{\mathcal{A}} \) of \( f^{(\mathcal{A})}(x) \) starts with the description \( D^{(0)}_{\mathcal{A}} \) of its root sub-tree \( \mathcal{A}^{(0)} \) (see Section 3.2.2) and iteratively cumulates the descriptions \( D^{(i)}_{\mathcal{A}} \) of all other sub-trees \( \mathcal{A}^{(i)} \) for \( i = 1, \ldots, m \). As commented in Proposition 9 all descriptions are truncated appropriately at \( d_{a} \).

\textbf{Correctness.} The correctness of Algorithm 3 is based on Proposition 8 which establishes that the matryoshka of a group of trees can be obtained from the matryoshkas relating to a partition of the same group, and on Proposition 14 that guarantees this task is correctly performed by Algorithm \text{Merge}.

\textbf{Complexity.} To prove complexity we proceed by induction on the depth of the tree.

If the tree \( \mathcal{A} \) has zero depth (i.e. a leaf tree with no descendants), then \( N_{\mathcal{A}} = \{0\} \) and the description \( D_{\mathcal{A}} \) of its tree characteristic function \( f^{(\mathcal{A})}(x) \) is computed in \( O(1) \) time as described in Section 3.2.2 from a vector of up to 2 records.

Now, let us assume that the thesis holds for trees of any depth less than or equal to \( h \) and show that it holds for trees of depth \( h + 1 \).

At iteration \( i \geq 1 \), three tasks are performed:

(a) description \( D^{(i)}_{\mathcal{A}} \) of \( f^{(\mathcal{A}^{(i)})}(x) \) is computed, by the induction hypothesis, in \( O(|\mathcal{A}^{(i)}|^{2}) \) time; 
(b) description truncation up to \( d_{a} \) is computed in \( O(|\mathcal{A}^{(i)}|) \) time;
(c) cumulated description $\mathcal{D}(\mathbb{A}; I)$ with $I = \{0, \ldots, i\} \subseteq \mathbb{N}_\mathbb{A}$ is computed from $\mathcal{D}(\mathbb{A}; I')$ with $I' = \{0, \ldots, i - 1\}$ and $\mathcal{D}(\mathbb{A}; \{i\})$ in $O(|\mathbb{A}|)$ time. Since $|\mathbb{A}| = \sum_{i \in \mathbb{N}_\mathbb{A}} |\mathbb{A}^{(i)}|$, then total time spent over all iterations is $O(|\mathbb{A}|^2)$ for task (a) and $O(|\mathbb{A}|)$ for task (b). Finally overall time spent in task (c) is $O(\sum_{i \in \mathbb{N}_\mathbb{A}} |\mathbb{A}_{\{0, \ldots, i\}}|) = O(\sum_{i \in \mathbb{N}_\mathbb{A}} \sum_{k=0}^{i} |\mathbb{A}^{(k)}|) = O(|\mathbb{A}|^2)$. The total time is thus $O(|\mathbb{A}|^2)$. \ \ \ \ □

5 Conclusions

In this paper, we analyze the probabilistic profitable tour problem on a tree and prove that it can be efficiently solved in $O(n^2)$ time where $n$ is the number of nodes. The problem finds application in service provision contexts where customers are located on special road network typical of mountain areas.

As future developments, one can consider other specific topological networks or study variants of the problem where budget constraints on time or costs are taken into account. Finally, the case of multi-valued prizes paid by the customers can be investigated.

References

Angelelli, E., Archetti, C., Filippi, C., and Vindigni, M. (2017). The probabilistic orienteering problem. Computers & Operations Research, 81:269–281.

Angelelli, E., Bazgan, C., Speranza, M. G., and Tuza, Z. (2014). Complexity and approximation for traveling salesman problems with profits. Theoretical Computer Science, 531:54–65.

Angelelli, E., Mansini, R., and Rizzi, R. (2022). The probabilistic profitable tour problem under a specific graph structure. submitted.

Archetti, C., Speranza, M. G., and Vigo, D. (2014). Vehicle Routing Problems with Profits, chapter 10, pages 273–297. SIAM.

Averbakh, I. and Berman, O. (1995). Probabilistic sales-delivery man and sales-delivery facility location problems on a tree. Transportation Science, 29(2):184–197.

Bellalouna, M., Murat, C., and Paschos, V. T. (1995). Probabilistic combinatorial optimization problems on graphs: A new domain in operational research. European journal of operational research, 87(3):693–706.

Bertsimas, D. and Howell, L. H. (1993). Further results on the probabilistic traveling salesman problem. European Journal of Operational Research, 65(1):68–95.

Bertsimas, D., Jaillet, P., and Odoni, A. R. (1990). A priori optimization. Operations Research, 38(6):1019–1033.

Campbell, A. M. and Thomas, B. W. (2008). Probabilistic traveling salesman problem with deadlines. Transportation Science, 42(1):1–21.

Feillet, D., Dejax, P., and Gendreau, M. (2005). Traveling salesman problems with profits. Transportation science, 39(2):188–205.
Hanafi, S., Mansini, R., and Zanotti, R. (2020). The multi-visit team orienteering problem with precedence constraints. European journal of operational research, 282(2):515–529.

Henchiri, A., Bellalouna, M., and Khaznaji, W. (2014). A probabilistic traveling salesman problem: a survey. In FedCSIS (Position Papers), pages 55–60. Citeseer.

Jaillet, P. (1985). Probabilistic traveling salesman problems. PhD thesis, Massachusetts Institute of Technology.

Jaillet, P. and Odoni, A. R. (1988). The probabilistic vehicle routing problem. In: Golden B. L., Assad A. A. (eds): Vehicle routing: methods and studies, pages 293–318. North-Holland, Amsterdam.

Klau, G. W., Ljubić, I., Mutzel, P., Pferschy, U., and Weiskircher, R. (2003). The fractional prize-collecting steiner tree problem on trees. In Di Battista, G. and Zwick, U., editors, Algorithms - ESA 2003, pages 691–702, Berlin, Heidelberg. Springer Berlin Heidelberg.

Laporte, G., Louveaux, F. V., and Mercure, H. (1994). A priori optimization of the probabilistic traveling salesman problem. Operations research, 42(3):543–549.

Yu, Q., Adulyasak, Y., Rousseau, L.-M., Zhu, N., and Ma, S. (2022). Team orienteering with time-varying profit. INFORMS Journal on Computing, 34(1):262–280.

Zhang, M., Wang, J., and Liu, H. (2017). The probabilistic profitable tour problem. International Journal of Enterprise Information Systems (IJEIS), 13(3):51–64.

Appendix

Proof of Proposition 3. Once we have shown that the following expression holds

\[
\begin{align*}
R(S_1) + R(S_2) &= R(S_1 \cup S_2) + R(S_1 \cap S_2) \\
C(S_1; x) + C(S_2; x) &\geq C(S_1 \cup S_2; x) + C(S_1 \cap S_2; x)
\end{align*}
\]

it can be easily proved that:

\[
\begin{align*}
G(S_1 \cup S_2; x) &= R(S_1 \cup S_2) - C(S_1 \cup S_2; x) \\
&\geq [R(S_1) + R(S_2) - R(S_1 \cap S_2)] - [C(S_1; x) + C(S_2; x) - C(S_1 \cap S_2; x)] \\
&\geq [R(S_1) - C(S_1; x)] + [R(S_2) - C(S_2; x)] - [R(S_1 \cap S_2) - C(S_1 \cap S_2; x)] \\
&\geq G(S_1; x) + G(S_2; x) - G(S_1 \cap S_2; x).
\end{align*}
\]

The first identity can be verified immediately. We demonstrate the second one by induction on the depth of the tree \( \mathcal{A} \) such that \( S_1 \cup S_2 \subseteq \mathcal{A} \). The inequality is certainly true if \( \mathcal{A} \) is a leaf; in this case \( \mathcal{A} \) consists of only one node and each term of the inequality is then calculated on an empty set and, is zero, or it is calculated on exactly the same node and takes the same value. It is easy to verify that the inequality holds in all cases where the leaf belongs to \( S_1 \) and/or \( S_2 \).
Now, let us assume that the inequality holds for any tree with depth not larger than \( n \) for a fixed \( n \geq 0 \) and show that it also holds for any tree with depth \( n + 1 \). Let \( \mathcal{A} \) be such a tree. Developing the terms of inequality through (4) and (5), we get:

\[
C(S_1 \cup S_2; x) = (d_a - x)\Pi_{(S_1 \cup S_2)} + \sum_{i=0}^{m} C((S_1 \cup S_2) \cap A^{(i)}; d_a)
\]
\[
C(S_1 \cap S_2; x) = (d_a - x)\Pi_{(S_1 \cap S_2)} + \sum_{i=0}^{m} C(S_1 \cap S_2 \cap A^{(i)}; d_a)
\]
\[
C(S_1; x) = (d_a - x)\Pi_{S_1} + \sum_{i=0}^{m} C(S_1 \cap A^{(i)}; d_a)
\]
\[
C(S_2; x) = (d_a - x)\Pi_{S_2} + \sum_{i=0}^{m} C(S_2 \cap A^{(i)}; d_a).
\]

Thanks to induction’s assumption, it is immediate to verify that:

\[
\sum_{i=0}^{m} C((S_1 \cup S_2) \cap A^{(i)}; d_a) + \sum_{i=0}^{m} C(S_1 \cap S_2 \cap A^{(i)}; d_a) \leq \sum_{i=0}^{m} C(S_1 \cap A^{(i)}; d_a) + \sum_{i=0}^{m} C(S_2 \cap A^{(i)}; d_a).
\]

Then, it is sufficient to show that:

\[
(d_a - x)\Pi_{(S_1 \cup S_2)} + (d_a - x)\Pi_{(S_1 \cap S_2)} \leq (d_a - x)\Pi_{S_1} + (d_a - x)\Pi_{S_2}
\]

or equivalently

\[
\Pi_{(S_1 \cup S_2)} - \Pi_{S_1} \leq \Pi_{(S_1 \cap S_2)} - \Pi_{S_2}
\]

recalling that \( \Pi_{S} = 1 - \prod_{s \in S} (1 - \pi_s) \), we develop the two members of the inequality and we collect a common factor obtaining

\[
\prod_{s \in S_1} (1 - \pi_s) \cdot \left[ 1 - \prod_{s \in (S_1 \cap S_2)} (1 - \pi_s) \right] \leq \prod_{s \in (S_1 \cup S_2)} (1 - \pi_s) \cdot \left[ 1 - \prod_{s \in (S_2 \setminus S_1)} (1 - \pi_s) \right]
\]

which can be further simplified by removing the factors common to the two members, first in

\[
\prod_{s \in S_1} (1 - \pi_s) \leq \prod_{s \in (S_1 \cap S_2)} (1 - \pi_s)
\]

and finally in

\[
\prod_{s \in (S_1 \setminus S_2)} (1 - \pi_s) \leq 1
\]

which concludes the proof.

**Proof of Proposition 12.** We prove only the first group of equations (14), (15), (16). The second group derives in a completely analogous way for symmetry.
1. Formula \([13]\). Using formula \([13]\) of Proposition \([7]\) we get
\[
G(B_i \cup C_i; x) = G(B_i; x) + G(C_i; d_a \Pi_{B_i} + x(1 - \Pi_{B_i}))
\]
Therefore
\[
G(B_i \cup C_i; x) = \max_h G(B_i \cup C_i; x)
\]
if and only if
\[
G(C_i; d_a \Pi_{B_i} + x(1 - \Pi_{B_i})) = \max_h G(C_i; d_a \Pi_{B_i} + x(1 - \Pi_{B_i}))
\]
which happens, by construction, if and only if
\[
\mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{min}} \leq d_a \Pi_{B_i} + x(1 - \Pi_{B_i}) \leq \mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{max}}
\]
that is for
\[
x \in \left[ \frac{\mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{min}} - d_a \Pi_{B_i}}{1 - \Pi_{B_i}}, \frac{\mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{max}} - d_a \Pi_{B_i}}{1 - \Pi_{B_i}} \right]
\]
and thus
\[
S(C_j; B_i) = \frac{\mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{min}} - d_a \Pi_{B_i}}{1 - \Pi_{B_i}}.
\]
Observe that, being \(\Pi_{B_i} > 0\) and \(\mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{min}} \leq d_a\), it holds that \(S(C_j; B_i) \leq \mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{min}}\).

2. Chain \([15]\).
   It comes directly from \([14]\) and increasing monotony of the points \(\mathbb{R}_{(A,C)}^{(j)} \cdot x_{\text{min}}\) with respect to index \(j\).

3. Chain \([16]\).
   It comes directly from \([14]\) and increasing monotony of the probabilities \(\mathbb{R}_{(A,B)}^{(i)} \cdot \pi\) with respect to index \(i\).

\[\blacksquare\]

**Proof of Proposition \([13]\)**. The first two cases are trivial. If \(i < n_B\) and \(j = n_C\), then all elements of \(\mathcal{M}_A\) not yet contained in \(\mathcal{M}_K^{(h)}\) are of the type \(B_{i,k} \cup C_{n_C}\). Since \(B_{i+1} \cup C_{n_C}\) is the smallest of these sets, setting \(K_{h+1} = B_{i+1} \cup C_{n_C}\) allows us to increase the matryoshka without compromising coherence. Case \(i = n_B, j < n_C\), is discussed symmetrically.

Excluding the first two cases, of the remaining two we treat only the first; the last one derives in a completely analogous way for symmetry.

We therefore assume that \(i < n_B, j < n_C\) and \(S(B_{i+1}; C_j) \leq S(C_{j+1}; B_i)\). We show that no set of the type \(B_{i,k} \cup C_{j+k}\) with \(k \geq 1\) can be an optimal set. Consequently, adding \(K_{h+1} = B_{i+1} \cup C_{j+1}\) to \(\mathcal{M}_K^{(h)}\), not only retains the matryoshka properties for \(\mathcal{M}_K^{(h+1)}\), but also retains coherence with \(\mathcal{M}_A\) because the choice made excludes only sets proved not to belong to \(\mathcal{M}_A\) and \(K_{h+1}\) remains the smallest set of \(\mathcal{M}_{(A,B)} \times \mathcal{M}_{(A,C)}\) containing \(K_h\).

We proceed by showing first that sets of the type \(B_i \cup C_{j+k}\) with \(k \geq 1\) cannot be optimal sets if \(x < S(B_{i+1}; C_j)\). Indeed, \(x < S(C_{j+1}; B_i)\) also holds and therefore \(x < S(C_{j+k}; B_i)\) for any
$k \geq 1$ due to the monotony (15) with respect to the entering set. It follows, for the definition of entry threshold, that $B_i \cup C_{j+k}$ is strictly dominated when $x < S(B_{i+1}; C_j)$.

We now proceed to show that sets of the type $B_i \cup C_{j+k}$ with $k \geq 1$ cannot be optimal sets even if $x \geq S(B_{i+1}; C_j)$.

For the monotony (19) with respect to the committed set we have $S(B_{i+1}; C_{j+k}) < S(B_{i+1}; C_j)$ for all $k \geq 1$ so $x > S(B_{i+1}; C_j)$ implies $x > S(B_{i+1}; C_{j+k})$ for each $k \geq 1$. Now let $i' \geq i + 1$ be the maximum index for which $x > S(B_{i'}; C_{j+k})$ and, from the definition of entry threshold

$G(B_i \cup C_{j+k}; x) < G(B_{i'} \cup C_{j+k}; x)$,

which proves the non-optimality of sets $B_i \cup C_{j+k}$. \blacksquare