On the interpolation of sparse-grid InSAR data without need of phase unwrapping

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Abstract
Sparse phase measurements often need to be interpolated on regular grids, to extend the information to unsampled locations. Typical cases involve the removal of atmospheric phase screen information from Interferometric Synthetic Aperture Radar (InSAR) stacks, or the retrieval of displacement information over extended areas in Persistent Scatterers Interferometry (PSI) applications, when sufficient point densities are available. This operation is usually done after a phase unwrapping (PU) of the sparse measurements to remove the sharp phase discontinuities due to the wrap operation. PU is a difficult and error-prone operation, especially for sparse data. In this work, we investigate from the empirical point of view an alternative procedure, which involves an interpolation of the complex field derived from the sparse phase measurements. Unlike traditional approaches, our method allows to bypass the PU step and obtain a regular-grid complex field from which a wrapped phase field can be extracted. Under general conditions, this can be shown to be a good approximation of the original phase without noise. Moreover, the interpolated, wrapped phase field can be fed to state-of-the-art, regular-grid PU algorithms, to obtain an improved absolute phase field, compared to the canonical method consisting of first unwrapping the sparse-grid data. We evaluate the performance of the method in simulation, comparing it to the classical methodology described above, as well as to an alternative procedure, recently proposed, to reduce a sparse PU problem to a regular-grid one, through a nearest-neighbor interpolation step. Results confirm the increased robustness of the proposed method with respect to the effects of noise and undersampling.

Keywords: interpolation, SAR Interferometry, phase unwrapping, COSMO/SkyMed.

Introduction
Applications such as SAR interferometry [Hanssen, 2001] are increasingly used in “sparse” contexts, in which information about some geophysical parameters (e.g. millimetric terrain deformations) are only available over some of the imaged pixels [Ferretti et al., 2001]. Often, however, sparse point maps are not sufficient to retrieve reliable information about certain
physical phenomena. In such cases, it is useful to interpolate such sparse measurements on the regular grid where the original dataset is defined. Typical cases include the retrieval of regular-grid deformation maps over large areas starting from PSI measurements, or the retrieval of atmospheric contributions over unsampled pixels in an InSAR stack. The former case can be very critical when dealing with spatially-variable deformation phenomena on terrain with medium-low density of point scatterers, since, in PSI applications, the focus is only on point-wise scatterers. In the latter case, the above-mentioned limits are somewhat relaxed thanks to the generally smooth nature of atmospheric InSAR phase contributions. Removing the interpolated phase fields from the regular-grid interferograms allows to highlight the phase stability of more image pixels, which then add to the available data to infer useful information about terrain displacements or other phenomena of interest.

Phase samples are, as all phase measurements, obtained through an arctangent operation, i.e. they are given as principal values, limited to the interval \([-\pi,\pi]\). Before their interpolation, it is usually necessary to perform the operation known as phase unwrapping (PU), consisting of obtaining absolute phase values (i.e. defined over the whole real interval) from the corresponding principal values. Two-dimensional PU is a straightforward operation only if the phase signal is well-sampled, i.e. it satisfies the Nyquist condition, which for phase data translates in the condition that any difference between two neighboring phase samples be confined in the interval \([-\pi,\pi]\) [Itoh, 1982]. Otherwise, PU becomes an ill-posed mathematical problem, meaning that multiple solutions are possible for a given initial condition. In these cases, PU cannot be solved univocally without additional, external information [Tikhonov and Arsenin, 1977]. When passing from regular-grid to sparse data, phase sampling conditions get usually worse, so that PU becomes in these cases a critical step, often hindering all subsequent processing.

Research on sparse PU has been getting oriented towards either adaptation of existing algorithms for regular-grid data [Costantini and Rosen, 1999], or, more recently, towards more innovative procedures exploiting redundancy in the connection of the sparse samples, to better estimate phase gradients and improve performances [Shanker and Zebker, 2010; Costantini et al., 2012]. Such solutions allow to retrieve reasonably good absolute phase fields in sparse contexts, in a fair number of sampling conditions. However, application of such algorithms is not always straightforward, essentially because they require the solution of linear programming problems that are time-consuming, and also because they require some effort to devise the best set of processing parameters to obtain acceptable results.

In this work, we explore empirically the possibility of avoiding the error-prone sparse PU operation, by using an alternative interpolation step applied directly to the complex phasor derived from the wrapped phase. This gives as a result a regular-grid complex dataset, whose phase is an approximation of the wrapped value of the ideal regular-grid phase. This interpolated field can then be used in subsequent InSAR processing steps, thus avoiding the problems connected with error propagation, which often result from erroneous PU results. Moreover, the interpolated, wrapped phase field can be fed to regular-grid PU algorithms, giving as a result an unwrapped, regular-grid phase. Under proper conditions, this field has more desirable properties compared to those obtained as a result of the “standard” procedure.

Attempts at bypassing the PU step in InSAR processing are not new, both in single-[Massonnet et al., 1996], and in multi-interferogram frameworks [Sandwell and Price, 1998]. Recently, a method to reduce the PU problem of a sparse-grid field to one corresponding to a regular grid has been proposed [Shanker and Zebker, 2009], based on a preliminary
nearest-neighbor interpolation step. The solution to the sparse problem is shown to be mathematically equivalent to that of a corresponding regular grid problem, properly derived from the former. The approach allows, as in our case, to employ existing, well-established algorithms for regular-grid PU [Chen and Zebker, 2001]. However, the regular-grid phase field unwrapped through the approach just mentioned can hardly be used in any subsequent processing step, due to the staircase-like appearance of any field interpolated through the nearest neighbor method. In the case in which the original, continuous phase field, from which the sparse samples are extracted, can be assumed to satisfy general conditions of smoothness and homogeneity [Refice et al., 2011], our approach allows to exploit at best such characteristics, and to have finally an absolute phase regular matrix more representative of the real data, and then more effective to use in subsequent processing steps [Bovenga et al., 2004].

In this paper, the proposed interpolation approach is investigated in detail from the empirical point of view. We use Kriging for the interpolation step, which, under proper smoothness conditions, is the best linear unbiased estimator (BLUE) for a 2-D field at unsampled locations [Cressie, 1989], and can be shown to be equivalent to a number of other interpolation approaches. Performances are evaluated in simulation, as a function of both the regularity conditions of the original sampled surface, and the sampling density. The paper is organized as follows. In the next section, we introduce the problem in mathematical form; then, in the following section, we describe the data simulation and the analysis methodologies. Then, the results are reported. Finally, we comment and discuss these results, and draw some conclusions.

**Problem definition**

We cast the problem in a mathematical form as follows: let us suppose to have a phase field $\phi^s$, defined over a (sparse) set of points $s = 1, \ldots, S$ over a 2-D plane, placed at coordinates $(x_s, y_s)$. Without loss of generality, we can assume that this phase field is actually the sum of a smooth phase signal and a noise term:

$$\phi^s = \phi^{s,0} + n; \quad s = 1, \ldots, S, \quad [1]$$

where $n$ is a white-spectrum random variable. This position is valid if $\phi^{s,0}$ comes from a smooth physical process such as the atmospheric phase screen (APS) contribution of an InSAR interferogram. The absolute phase in [1] is not accessible, since only its principal phase is available, which can be expressed as the result of a “wrapping” operation:

$$\psi^s = W(\phi^s) = \phi^s + 2\pi k^s; \quad s = 1, \ldots, S; \quad [2]$$

where the $k^s$ are defined as the set of integers such that $\psi^s \in [-\pi, \pi], \forall s = 1, \ldots, S$. Wrapping is, as known, a nonlinear operation, and brings with it the chance of generating a phase aliasing condition. The PU operation consists in finding the best set of $k^s$ integers which restore the absolute phase samples $\phi^s$ as faithfully as possible.

Let us turn now to the other processing step considered in the paper. Interpolation can be generally
defined as the operation of inferring the value of a function, defined on a certain set of points, on another set of unsampled locations. In our case, such unsampled locations define a regular grid, \((x_j, y_k); j = 1, \ldots, N, k = 1, \ldots, M,\) such that \(\forall s = 1, \ldots, S; x_1 \leq x_s \leq x_N, y_1 \leq y_s \leq y_M.\)

Interpolation necessarily relies on some assumptions on the smoothness of the sought continuous field, which cannot be satisfied by the wrapped phase, due to the mentioned nonlinear nature of the wrapping operator. The “canonical” procedure in this case is to apply a PU step to the wrapped phase, previous to the interpolation, i.e. retrieve \(\tilde{\phi} = \psi^s + 2\pi \tilde{k}^s; s = 1, \ldots, S.\) PU of the sparse phase field is typically performed through integration of estimated gradients [Ghiglia and Pritt, 1998]. This operation is not an easy task, and gets critical in aliasing conditions. In fact, strong caveats should be taken into account when dealing with spatial estimators of phase gradients from wrapped phase fields [Loffeld and Arndt, 1997].

We can, instead, consider a complex field derived from the phase information. In the case of InSAR, a natural choice is to use the complex value of the interferogram from which the phase is derived, which can be written as \(I e^{i\psi},\) where the dependence on the position has been omitted for brevity, and where \(i = \sqrt{-1}.\) In this case the modulus \(I\) is the interferogram amplitude, which carries information about the terrain average backscatter. Since we are interested only in the phase, however, we ignore the amplitude information, and thus consider:

\[
e^{i\psi} = \cos\psi + i \sin\psi \quad [3]
\]

i.e. we consider a signal which has unit magnitude, \(\left|e^{i\psi}\right| = 1\) This choice limits the width of the signal spectrum to its minimum value. Enforcing this constraint, or, equivalently, any other constraint on the signal magnitude, during all the interpolation operations would be actually equivalent to considering only the phase as independent variable, which leads to the mentioned difficulties with the PU step. We can instead treat the real and imaginary parts of the above complex signal as two independent, real variables: \(A^s = \cos\psi^s; B^s = \sin\psi^s,\) and interpolate them on the same regular grid defined earlier:

\[
A^s, s = 1, \ldots, S \rightarrow \tilde{A}(x_j, y_k), j = 1, \ldots, N, k = 1, \ldots, M, \quad [4]
\]

\[
B^s, s = 1, \ldots, S \rightarrow \tilde{B}(x_j, y_k), j = 1, \ldots, k = 1, \ldots, M. \quad [5]
\]

After this step, we extract the regular-grid phase: \(\tilde{\psi}(x_j, y_k) = \angle \left[\tilde{A}(x_j, y_k) + \tilde{B}(x_j, y_k)\right].\)

The real variables \(A^s\) and \(B^s\) do not exhibit the sharp jumps which are typical of the phase fringes, due to the nonlinearity of the wrapping operator, so their interpolated versions lead to a phase \(\tilde{\psi}\) which has some characteristics of smoothness, and can thus be considered a good approximation of the original, wrapped, regular-grid phase \(\psi.\) Most importantly, this field has been obtained without use of any PU step, and so does not suffer from error propagation
effects. The complex field $\tilde{A} + i\tilde{B}$ can be used in subsequent processing steps, for example in case it represents spurious phase contributions such as the so-called atmospheric phase screen (APS) in PSI: the contribution can be subtracted from a stack of SAR images through conjugate complex multiplication. Optionally, the interpolated, wrapped field $\hat{\psi}$ can be fed to a regular-grid PU algorithm, to obtain a final, unwrapped regular-grid phase $\phi$.

**Data and Processing**

We work on simulated phase data. Random fields of prescribed spatial correlation can be generated through spectral filtering techniques, as shown in previous work [Refice et al., 2011]. In this case, we generated a $5000 \times 5000$ samples, regular-grid phase field satisfying a Gaussian autocorrelation function, corresponding to a variogram with parameters $N = 0$ (nugget), $S = 3$ rad (sill), $L = 1000$ samples (range). These parameters are considered realistic for atmospheric phase delay InSAR contributions. For a metric resolution SAR image, the simulated range corresponds to an order of magnitude of about a km, which is the correct one for tropospheric wet delay InSAR contributions [Hanssen, 2001].

![Figure 1](image-url) - (Top-left): regular-grid, $5000 \times 5000$ samples, simulated data used in the experiments. The three representative, $400 \times 500$ samples sub-areas are indicated by the black rectangles labeled 1, 2, and 3. The other three panels show the sparse point patterns for the three sub-areas, with point colors encoding the phase values.
Samples were taken from this field. In order to make the experiment more meaningful, we used spatial point patterns taken from real data, namely the PS distribution from a PSI dataset obtained by COSMO/SkyMed data. We show results from three representative sub-areas of size 400x500 samples, extracted from the above-mentioned test site, representing, respectively, dense, intermediate, and sparse sampling distributions (indicated as Area 1, 2, and 3, respectively, in Fig. 1). The sample average spatial densities range from about 7.1% to about 0.9% of the original regular-grid pixels. We note that PS densities of more than 5-6% can be obtained only on high-resolution SAR data (COSMO/SkyMed or TerraSAR-X), while legacy, medium resolution sensors such as ENVISAT lead typically to much lower densities, especially on areas far from urban centers. Our sampling rates therefore cover a relatively large spectrum of situations. For the validation, a Gaussian white noise is added to the sampled phase data, with different standard deviations of the related statistical distribution, ranging from 0.2 to 1.5 radians.

![Figure 2](image-url)  
**Figure 2** - Sample sparse residue maps for the three sub-areas, obtained by adding Gaussian noise with increasing standard deviation to the real PSI maps shown at the bottom of Fig. 1. White and black triangles correspond to positive and negative residues, respectively.

An image of the different sample densities arising from different additive noise levels can be seen in Figure 2, where the distribution of sparse residues is also shown, as black and white triangles in the corresponding Delaunay triangulation. It can be noticed that the absolute number...
of residues increases as the noise levels increase. As sampling density decreases (going from left to right in the maps on each line of Fig. 2), the triangles mean size gets larger, so that an error in PU potentially impacts a larger area of the interpolated field. Therefore, bypassing the sparse PU step would seem to potentially reduce the error budget on the final surface retrieval. On the other hand, interpolating from sparser point patterns also involves a higher degree of reconstruction error. In other words, different noise levels and sampling densities influence in a complicated way the two steps of PU and interpolation, which are performed in different orders in the classical and the proposed procedures. It is therefore interesting to track how the performances of the two methods respond to these two variables. The rest of the paper deals with this analysis, which is possible as we work on simulated data.

In Figure 3 we represent in schematic form the methodologies we consider and compare, together with the adopted nomenclature for the symbols used to indicate the various quantities used for the tests. All the methods are tested on sparse phase data, sampled from simulated regular-grid fields generated as described above.

![Diagram](image)

Figure 3 - Scheme of the experimental procedures, showing the symbols used in the text to represent the various quantities.

In the “canonical” procedure (indicated as “a” in Fig. 3), we use an implementation of the weighted MCF algorithm on a sparse grid (WMCFS) [Ahuja et al., 1993] to unwrap the...
sparse samples and obtain the $\phi^{a,s}$ set. The subsequent interpolation is performed through the Kriging approach, by estimating first the variogram parameters (for which we use the known Gaussian analytic expression), then using these to infer phase values at unsampled locations [Cressie, 1989].

In our proposed procedure (procedure "b" in Fig. 3), we perform the interpolation of the real and imaginary parts $A^s = \cos \psi^s$ and $B^s = \sin \psi^s$ of the complex signal obtained from the phase, through the same Kriging methodology used in the canonical method, to give the two regular-grid fields $A^b$ and $B^b$.

The choice of the Kriging methodology is to make the results as general as possible, since Kriging has been shown to be a generalization of many other interpolation approaches [Stein, 1999]. Experiments with different interpolation methods, not shown here for brevity, support the same conclusions.

To compare the two approaches, we obtain an unwrapped phase field estimate $\phi^b$ by feeding the interpolated principal phase, obtained as $\psi^b = \angle(A^b + iB^b)$, to the SNAPHU algorithm [Chen and Zebker, 2001].

We include also a comparison with the methodology proposed in [Agram and Zebker, 2009], which is referred to as procedure "c" in Figure 3. This procedure involves a nearest-neighbor interpolation step, which gives a regular-grid phase $\psi^c$. This is then also fed to the SNAPHU regular-grid PU algorithm to retrieve a regular-grid absolute phase $\phi^c$. Of course, procedure c produces as output an unwrapped phase which has no meaning for points different from the sampled ones, so we use it here only as a reference benchmark.

In the other two cases, we compare both the unwrapped, interpolated phase data with the original, simulated absolute phase, and the wrapped, interpolated phase with the original principal phase, in order to highlight various aspects of the proposed methodology.

Results

In this section we illustrate the results obtained from the processing of the simulated data through the three methods illustrated in Figure 3. Figure 4 shows examples of regular-grid fields reconstructed through the three methods described, for area 3 (see Fig. 1). Similar results, not shown here for brevity, are obtained for the other two areas. It can be seen that, for low noise, procedures $a$ and $b$ give similar results, while method $c$ gives in practice an approximated version of the original field, but with values which remain constant within the Voronoi polygons related to each sample. This is expected, due to the discrete nature of the nearest neighbor interpolator. It can be also noted that, as noise levels get higher, procedure $a$ has, progressively, slightly worse performance than procedure $b$. This is due to the failure of the sparse PU for higher noise levels.

Now we turn to a statistical analysis of the reconstruction performances of the three methods. The evaluation is performed at three levels. First, we evaluate the interpolated, wrapped phase fields $\psi^a = W(\phi^a), \psi^b$, and $\psi^c$, in terms of phase residues. Then, we analyze the reconstruction capabilities of the three methods by comparing the interpolated phase fields with the original phase, both in its wrapped and unwrapped form.
Residues

As a first step, we analyze the number and spatial density of residues of the interpolated fields. We report in Table 1, and plot in Figure 5, the absolute and relative numbers of residues measured on the various sparse fields corresponding to the three subareas, as a function of the additive noise level to the original phase. We also report, for reference, the numbers and percentages of residues in the original, simulated regular-grid phase fields. Residue relative numbers, i.e. spatial densities, are obtained by normalizing the total number of residues to the number of grid elements, for regular-grid data, and to the number of triangles in the Delaunay triangulation, for the sparse data. The plots in Figure 5 are shown in logarithmic scale to better highlight the trends, while reducing the dynamics of the variations. As expected, the numbers of residues in the regular grids, interpolated through the Kriging smooth interpolant, are lower than those in the original, noisy phase field. Also, they are lower than those in the sparse fields, as well as in the fields interpolated through the nearest neighbor method. Of course, the fact that interpolated grids have less residues...
does not represent automatically a quality index on the reconstruction of interpolated field in the general case. However, since the simulated fields are smooth and have no residues, the presence of noise-induced residues in the sampled fields is likely to cause problems in the subsequent PU phase. An interesting observation is that the number of residues on the regular-grid field obtained by nearest-neighbor interpolation is nearly identical to that of the sparse data. This confirms empirically the fact, mathematically proven in [Agram and Zebker, 2009], that nearest-neighbor interpolation preserves the residues of the sparse phase field.

Figure 5 - (Top) numbers of residues and (bottom) residue densities as a function of noise st.dev. for the three sub-areas, calculated on the original regular-grid simulated field, on the sparse samples, on the nearest-neighbor (NN) and on the Kriging (K) interpolated fields, respectively. Residue densities are obtained by normalizing absolute values to the number of minimal sample loops, which is equal to the number of samples on the regular grids, and to the number of Delaunay triangles in the sparse sample case.
Table 1 - Residue numbers and densities.

| Std. noise | Regular-grid residues | Sampled residues | Interp. (kriging) residues | Interp. (NN) residues | Regular-grid res. density | Sampled res. density | Interp. (kriging) res. density | Interp. (NN) res. density |
|------------|-----------------------|------------------|---------------------------|----------------------|-------------------------|---------------------|-----------------------------|--------------------------|
| 0.2        | 0                     | 0                | 0                         | 0                    | 0.018                   | 0.013               | 4×10⁻⁴                     | 0.002                    |
| 0.4        | 3689                  | 373              | 90                        | 341                  | 0.042                   | 0.029               | 0.001                      | 0.004                    |
| 0.9        | 8385                  | 845              | 226                       | 753                  | 0.250                   | 0.191               | 0.010                      | 0.024                    |
| 1.5        | 50057                 | 5450             | 2118                      | 4894                 |                         |                     |                             |                          |

| AREA 2 - # samples = 4675, # Delaunay triangles = 9301 |
|---------------------------------------------------------|
| 0.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.4 | 3832 | 126 | 12 | 116 | 0.019 | 0.014 | 6×10⁻⁵ | 6×10⁻⁴ |
| 0.9 | 8471 | 300 | 50 | 279 | 0.042 | 0.032 | 2×10⁻⁴ | 0.001 |
| 1.5 | 49693 | 1766 | 512 | 1654 | 0.248 | 0.190 | 0.0026 | 0.008 |

| AREA 3 - # samples = 1830, # Delaunay triangles = 3640 |
|---------------------------------------------------------|
| 0.2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.4 | 3663 | 64 | 16 | 64 | 0.018 | 0.017 | 8×10⁻⁵ | 3×10⁻⁴ |
| 0.9 | 8070 | 123 | 26 | 112 | 0.040 | 0.033 | 1×10⁻⁴ | 6×10⁻⁴ |
| 1.5 | 49122 | 660 | 184 | 630 | 0.245 | 0.181 | 9×10⁻⁴ | 0.003 |

This first analysis leads to the expectation that the proposed, “interpolation-first” method gives a smoother wrapped phase, which then poses less problems in a possible, subsequent PU step. This is confirmed in the following sections.

The residue densities plot at the bottom of Figure 5 shows that the residue spatial density of the sparse samples remains very similar to that of the original regular grids, with very low dependence on the spatial density of the sparse sampling (the three red curves are almost completely overlapped). Residue density is instead again somewhat lower in the interpolated fields, and its values get lower as the original point density gets lower (passing from Area 1, with maximum, to Area 3 with minimum density). The trends appear mostly similar in all cases.

Moreover, as mentioned previously, when the samples are sparse, the associated Delaunay triangles result very extended in size (see Fig. 2), so that phase inconsistencies associated to a wrong removal of such residues will have an appreciable impact on the final regular-grid phase in this case. These two facts exacerbate the influence of the sparse PU step on the whole chain of the “canonical” methodology.

**Interpolated fields**

In this section, we analyze the root-mean squared error (RMSE) of the interpolated phase fields, with respect to the original one. First, we compare the interpolated, wrapped phase with the original, wrapped simulated phase. We define, as mentioned above, \( \psi^a = W(\phi^a) \) as the wrapped version of the regular-grid phase obtained through the “canonical” approach. We also define a “wrapped phase without noise”, as \( \psi^0 = W(\phi^0) \), i.e. the wrapped version of the original smooth phase, although rigorously the nonlinear nature of the wrapping operator breaks the additive noise model, so this position is valid only for low noise levels. Then, we can pose:

\[
R_w^e = \sqrt{\frac{\sum_i w(\psi^e_i - \psi^0_i)^2}{N}} \quad [6]
\]
with $\ell = a, b, c$ indicating the three methods as in Figure 3. The quantities defined in [6] give a measure of the quality of the interpolation step, neglecting for the moment the PU step. In fact, both the used PU algorithms, namely the WMCFS and SNAPHU, are congruent algorithms, in the sense introduced in [Ghiglia and Pritt, 1998], i.e. they only add integer numbers of phase cycles to the wrapped phase. Therefore, wrapping their results gives exactly the same initial wrapped phase. 

In Figure 6 we plot the results. As these appear generally weakly dependent on the spatial pattern of the sampled points (see previous section), we average the RMSE values obtained on the three areas. We also average on the slightly different point patterns obtained from the real PSI data by adopting three different coherence thresholds (0.8, 0.85, and 0.9) for each area, so that each point in the plots are obtained averaging 9 independent experiments. The error bars correspond to the dispersions of these measurements.

![Figure 6. RMSE of the difference between wrapped, interpolated phase values, and the original, wrapped phase field without noise, $\psi^0$, for procedures a, b, and c, averaged on the various average sampling densities. Error bars represent the dispersion of the distribution.]

The plots all show a common trend, with errors steadily increasing with the noise variance. In all cases, as expected, RMSE values obtained by using the proposed interpolation methodology (procedure b in Fig. 3) are lower than those obtained by the Shanker and Zebker method (procedure c). They are at least equal, and sometimes slightly better than those obtained through the “canonical” procedure a. The relatively large dispersion of the data from procedure a at high noise levels is due to the mentioned, occasional failure of the sparse PU algorithm in critical noise conditions.

**Unwrapped fields**

In this section we analyze the reconstructed unwrapped phases, by defining:
\[ R_\ell = \sqrt{\frac{\sum (\phi^\ell - \phi^0)^2}{N}} \quad [7] \]

with, again, \( \omega = a, b, c \) for the three methods. Figure 7 shows the computed values of \( R_\ell \), as a function of noise level, for the three methods, with the same averaging methodology adopted in the previous section. The RMSE measure defined in [7] accounts for the effects of both the interpolation and the unwrapping operations. We remark again that unwrapping of the interpolated field \( \psi^b \) obtained with our method is not strictly necessary for most of the applications which require the regular grid field to “flatten” other information layers of interest (such as in the prototypical case of PSI applications). Nevertheless, we evaluate here the performance of the method by adding the final regular grid PU step, in order to facilitate comparisons with the classical method.

![Figure 7. RMSE of the difference between absolute, interpolated phase values, and the original, absolute phase field without noise, \( \psi^0 \). Colors and symbols are the same as in Figure 6.](image)

It can be seen that procedure \( b \) leads again to performances generally analogous or better than the “canonical” method \( a \) involving the sparse PU step.

To explain, at least in part, this result, it has to be considered again that many of the considered, simulated sparse phase fields are quite hard to unwrap properly, especially when sampling density gets very low, which means that points are located at high average distances from each other. In such cases, the proposed method, which is able to interpolate the complex phasor throughout these large triangles, giving as result an interpolated phase which is more similar to the original, provides a clear advantage from the empirical point of view.
Conclusions
We have proposed an empirical evaluation of a method to obtain regular-grid, principal phase matrices from sparse measurements, which consists of an interpolation of the real and imaginary parts of the complex field obtained from the wrapped phase. The method has been tested in simulation, adopting a synthetic regular-grid phase field with prescribed smoothness characteristics, and a series of sample patterns taken from real high-resolution SAR data, with different sparsity levels. The results show an advantage in using the proposed interpolation method, both in terms of error with respect to the original wrapped phase, and with respect to the reconstructed phase when PU is taken into account. In particular, interpolating the complex vector reduces the detected residues with respect to the original, sparse dataset. This helps in further PU operations. Moreover, the interpolated dataset has superior characteristics in terms of noise reduction, as long as the additive noise model holds for the original phase. The main advantage resides in the fact that the proposed approach bypasses the sparse PU operation, which is known to be a critical step in several instances.

Regarding the proposed procedure as an alternative PU methodology, i.e. performing a regular-grid PU on the interpolated data, rather than interpolating a sparse, unwrapped phase, results show that this also gives encouraging results, since the proposed methodology gives absolute phase fields which are more similar to the original, simulated ones. It can be concluded that the proposed procedure is useful for estimating efficiently regular-grid layers such as APS or velocity fields in PSI processing chains. The method appears then promising for inclusion in an operational approach.

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