A Stochastic Diffusion Model of Climate Change

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1. INTRODUCTION

There is widespread agreement that the obliquity and precession of the Earth’s orbit result in a significant portion of global temperature variance at frequencies near 40,000 and 20,000 years. Aside from this “external” forcing of the climate system, there is no consensus on the physical processes governing the remainder of the variance. It is generally agreed that the variance resulting from orbital forcing is superimposed on a continuous “background” spectrum with a variance that increases with decreasing frequency (Kerr 1978). Most attempts to model this background spectrum assume a nonlinear response of the climate system to the orbital forcing (Bhattacharya, Ghil, and Vulis 1982; Imbrie and Imbrie 1980; Le Treut and Ghil 1983; Nicolis 1993). Each of these studies has succeeded in reproducing only a limited portion of the spectrum of climatic variations.

Several studies have recognized that the low-frequency portion of the power spectrum, $S(f)$, of temperature variations inferred from ice and ocean cores has the form of a Lorentzian distribution: (Hasselmann 1971; Komintz and Pisias 1979; Lovejoy and Shertzer 1986)

$$S(f) \propto \frac{1}{f^2 + f_0^2}$$

(1)

This function is constant for very low frequencies and proportional to $f^{-2}$ for $f \gg f_0$. Hasselmann (1971) and Komintz and Pisias (1979) have suggested that a stochastic model may be the most appropriate model for this spectrum since a Lorentzian spectrum can result from dynamics in which a macroscopic variable is the sum of uncorrelated random pulses with a negative feedback mechanism limiting the variance at low frequencies.

At shorter time scales it is recognized that there is “persistence” or correlations in time in meteorological time series over a range of time scales. Persistence means that warm years (or weeks or months) are, more often than not, followed by warm years and cold years by cold years. Hurst, Black, and Simaika (1965) presented studies of these correlations using the rescaled-range technique (see Feder 1991 for an introduction). Hurst found that time series of annual mean temperature produced a power-law rescaled-range plot with an average exponent of 0.73. No persistence would yield an exponent of 0.5. Numerical studies have shown that series with a Hurst exponent of 0.73 have a power spectrum proportional to $f^{-2}$ (Higuchi 1990; Gomes da Silva and Turcotte 1994; Malamud and Turcotte 1995).

In this paper we present a model that provides a specific physical mechanism for the entire background spectrum from time scales of 1 day to 1 million years. We present spectral analyses of paleoclimatic proxy data and instrumental data that support the observation of a low-frequency Lorentzian spectrum and a higher-frequency spectrum proportional to $f^{-2}$. At very high frequencies we found a distinct difference between the power spectra of continental and maritime stations. Continental stations exhibit power spectra proportional to $f^{-2}$ at time scales less than one month while the spectra of maritime stations remain proportional to $f^{-2}$ down to time scales of one day. Our model is based upon an analytic approach to modeling the stochastic diffusion of heat in the atmosphere and ocean. The difference between continental and maritime stations arises because the air mass above maritime stations exchanges heat with both the atmosphere above and the ocean below while the air mass above continental stations exchanges heat with only the atmosphere above it.

“...turbulent transfer in a system is dominated by eddies much smaller than the system size, random convective action of turbulent eddies will be analogous to the molecular agitation responsible for molecular diffusion” (Moffatt 1983). In this approximation, turbulent transfer can be modeled as a stochastic diffusion process (Csanady 1980). This is equivalent to the Lagrangian theory of turbulence for times long compared to the Lagrangian time scale (time below which particle velocities are autocorrelated) (Tennekes and Lumley 1972; Gifford 1982). In models of climate change, it is common to model the turbulent transfer of heat as a deterministic diffusion process (Ghil 1983). The stochasticity of turbulent transfer results in temperature fluctuations from equilibrium not present in a deterministic model of turbulent heat transfer. We will present the power spectrum of temperature fluctuations from equilibrium resulting from stochastic heat transport in a two-layer geometry appropriate to the atmosphere and ocean.

The model we present was first solved by van Vliet, van der Ziel, and Schmidt (1980) to determine the power spectrum of variations due to the stochastic diffusion of heat in a metallic film in thermal equilibrium with a substrate. Temperature variations in the film and substrate occur as a result of fluctuations in the heat transport by electrons undergoing Brownian motion. The top of the film absorbs and emits blackbody radiation. In this paper we use van Vliet et al.’s model exactly as they presented it with the atmosphere as the metallic film and the ocean as the substrate. Turbulent eddies in the atmosphere and ocean are analogous to the electrons undergoing Brownian motion in a metallic film in contact with a substrate. The model studied by van Vliet et al. (1980), with phys-
ical constants appropriate to the atmosphere and ocean, yield a power spectrum in agreement with that of climatic variations recorded in instrumental records and inferred from ocean and ice cores from time scales of one day to one million years.

2. OBSERVATIONS OF CLIMATIC VARIATIONS

![Figure 1. Logarithm of the power spectrum of atmospheric temperature variance predicted by the model as a function of the logarithm of the frequency in years\(^{-1}\). The crossover frequencies observed in the climatological record are, from left to right, \(f_0 = \frac{1}{40,000\text{years}}\), \(f_1 = \frac{1}{2,000\text{years}}\), \(f_2 = \frac{1}{1\text{month}}\).](image1)

In Figure 1 we present the logarithm of the power spectrum of the temperature variations as a function of the logarithm of the frequency predicted by the model and supported by observational data. At low frequencies the power spectrum is constant. Above \(f \approx \frac{1}{40,000\text{years}}\), the power spectrum is proportional to \(f^{-2}\). Above \(f \approx \frac{1}{2,000\text{years}}\) the power spectrum is proportional to \(f^{-1.2}\). At very high frequencies (above \(f \approx \frac{1}{1\text{month}}\)) the spectrum varies as \(f^{-\frac{3}{2}}\) for continental stations and remains proportional to \(f^{-\frac{1}{2}}\) for maritime stations. The crossover frequencies quoted above are those observed in the climatological record. The model predicts crossover frequencies which are close to those observed in the climatological record (two out of three are within a factor of two of the observed frequencies). All of the power spectra that we present in this paper are plotted after taking the logarithm of the power spectrum and of the frequency against which the spectrum is plotted. All power-law functions appear as straight lines with a slope equal to the exponent of the power-law. The frequency unit of all observational data is \(\text{years}^{-1}\).

![Figure 2. Logarithm of the normalized Lomb periodogram of the temperature inferred from the Vostok ice core as a function of the logarithm of the frequency in years\(^{-1}\).](image2)

Figure 2 shows the logarithm of the normalized Lomb periodogram of the Vostok \(\delta D\) record extending back 220,000 years converted into degrees Celsius by the conversion factor 5.6\% per degree Celsius (Jouzel et al. 1987). We obtained the spectrum from M. Ghil and P. Yiou (1994). It is not possible to directly estimate the power spectrum of the Vostok record with the FFT since the data are unevenly sampled. Numerical Recipes suggests the use of the Lomb Periodogram for such data (Press et al. 1992). The periodogram shows a constant low-frequency region that changes to a region proportional to \(f^{-2}\) above \(f \approx \frac{1}{40,000\text{years}}\). Above \(f \approx \frac{1}{2,000\text{years}}\) the power spectrum of temperature variations is \(\propto f^{-1.2}\). To reduce the scatter of the periodogram at high frequencies, we averaged the periodogram in logarithmically-spaced bins of size \(\log f = 0.01\) above \(\log f = -4.0\).

![Figure 3. The average unnormalized power spectrum of 94 monthly temperature time series on a log-log plot as a function of frequency in years\(^{-1}\).](image3)
In Figure 3 we present the logarithm of the unnormalized power spectra of temperature time series as a function of the logarithm of the frequency taken instrumentally at higher frequencies. We plot the logarithm of the average power spectrum of time series of monthly mean temperature from 94 stations worldwide with the yearly trend removed. The power spectra were computed by taking the modulus squared of the complex Fourier coefficients obtained by computing the Fast Fourier Transform using the Numerical Recipes routine reallft (Press et al. 1992). We computed the power spectra of all complete temperature series of length greater than or equal to 1024 months from the climatological database compiled by Vose et al. (1992). The yearly trend was removed by subtracting from each monthly data point the average temperature for that month in the 86 year record for each station. All of the power spectra were then averaged at equal frequency values. The data yield a straight-line least-square fit with slope close to -0.5 indicating that \( S(f) \propto f^{-\frac{1}{2}} \) in this frequency region. This is consistent with the results of Hurst, Black, and Simaika (1965) who analyzed correlations at instrumental time scales with the rescaled-range method. Numerical studies have shown that a time series which yields a power-law rescaled-range plot with Hurst exponent \( H = 0.73 \) has a power spectrum \( S(f) \propto f^{-\frac{1}{2}} \) (Higuchi 1990; Gomes da Silva and Turcotte 1994; Malamud and Turcotte 1995). \( f^{-\frac{1}{2}} \) power spectra have also been observed in variations of atmospheric humidity on time scales of days to years (Vattay and Harnos 1994) and in tree-ring widths (a proxy for precipitation) up to several thousands of years (Pelletier 1995).

In Figure 4 we present the averaged unnormalized power spectrum of 50 continental and 50 maritime daily temperature time series on a log-log plot as a function of frequency in years \(^{-1}\). The crossover frequency for the spectral curve is \( f_2 = \frac{1}{\text{month}} \).

In Figure 3 we present the average unnormalized power spectrum of time series of daily mean temperature (estimated by taking the average of the maximum and minimum temperature of each day) from 50 continental and 50 maritime stations over 4096 days estimated by computing the Fast Fourier transform as before. Maritime stations were sites on small islands far from any large land masses. Continental stations, conversely, were well inland on large continents, far from any large bodies of water. We chose 50 stations at random from the complete records (those with greater than 4096 nearly consecutive days of data) provided by the Global Daily Summary database compiled by the National Climatic Data Center (1994). We found no records in the database that were without at least a few weeks worth of missing days over the 14 years of data covered by the longest of the station records. We filled in the missing days with the same value as the previous day. This will not introduce any discernible error since the erroneous datapoints make up at most a fraction of one percent of each series. The yearly periodicity in these data were removed by subtracting from each station’s temperature time series a converged least-squares fit (using Numerical Recipes (Press et al. 1992) routine mrqmin) to a function of the form

\[
T_i = T_{av} + A \cos \left( \frac{2\pi}{365} i + \phi \right)
\]

where \( i \) is the number of the day in the year and \( T_{av}, A, \) and \( \phi \) were the fitting parameters. This procedure is a standard one for subtracting the yearly periodicity in a meteorological time series (Janosi and Vattay 1992). Continental stations exhibit a \( f^{-\frac{1}{2}} \) high-frequency region. Maritime stations exhibit \( f^{-\frac{3}{2}} \) scaling up to the highest frequency.

3. TURBULENT TRANSPORT AS A STOCHASTIC DIFFUSION PROCESS

In most models of climate change, turbulent transfer of heat energy in the atmosphere and ocean is modeled by a diffusion process (Ghil 1983). This assumption is supported by the self-consistent results of experimental studies of tracer dispersion which parameterize transport as a diffusion process. The resulting measurements of eddy diffusivity are nearly constant as a function of depth in the ocean and height in the atmosphere. If heat transport was dominated by convection currents of the same scale as the height of the atmosphere or the depth of the ocean, the resulting diffusion coefficient from such large-scale tracer studies would be greatly inhomogeneous. Studies of the dispersion of Tritium vertically in the ocean have resulted in a depth-independent (except for the mixed layer extending down to about 100 m) eddy diffusivity of about 6x10\(^{-6}\) m\(^2\)/s (Garrett 1984). Vertical diffusivity in the atmosphere is considerably more uncertain, but a height-independent order of magnitude estimate of 1 m\(^2\)/s for stable air conditions has been quoted by a couple of studies (Pleume 1990; Seinfeld 1986). Radar studies of trace gases in the middle atmosphere obtained the same diffusivity to within an order of magnitude from 5-10 km (Fukao et al. 1994). Since the time scale of horizontal diffusion in the atmosphere and ocean is so much smaller
than the time scale of vertical diffusion, diffusion of heat into and out of a local air mass is one-dimensional. For this reason, we consider only the variations in local temperature resulting from heat exchange vertically in the atmosphere and ocean.

Although coherent air motions exist which lead to large-scale periodic motions of the atmosphere, these motions are strongest horizontally and are predominant above the troposphere where less than half of the heat capacity of the atmosphere resides (Dunkerton 1993). We will assume that such oscillatory air motions transport at most a small fraction of the total heat transported vertically in the atmosphere and ocean. If most of the heat is transported by eddies much smaller than the height of the atmosphere and the depth of the ocean, turbulent transport can be modeled as a diffusion process.

4. FLUCTUATION THEORY OF CLIMATE CHANGE

A stochastic diffusion process can be studied analytically by adding a noise term to the flux of a deterministic diffusion equation (van Kampen 1981):

$$
\rho c \frac{\partial \Delta T}{\partial t} = - \frac{\partial J}{\partial x} \quad (3)
$$

$$
J = -\sigma \frac{\partial \Delta T}{\partial x} + \eta(x, t) \quad (4)
$$

where $\Delta T$ is the fluctuation in temperature from equilibrium and the mean and variance of the noise is given by

$$
\langle \eta(x, t) \rangle = 0 \quad (5)
$$

$$
\langle \eta(x, t)\eta(x', t') \rangle \propto \sigma(x)(T(x))^2 \delta(x - x')\delta(t - t') \quad (6)
$$

Figure 5. Geometry of the diffusion calculation detailed in the text.

We will calculate the power spectrum of temperature fluctuations in a layer of width $2l$ of an infinite, one-dimensional, homogeneous space. The presentation we give is similar to that of Voss and Clarke (1976). The variations in total heat energy in the layer of width $2l$ is determined by the heat flow across the boundaries.

$$
\frac{dE(t)}{dt} = J(l, t) - J(-l, t) \quad (7)
$$

where $\alpha = \frac{\sigma}{\rho c}$ is the vertical thermal diffusivity. The flux of heat energy out of the layer at the boundary at $x = l$ (the other boundary is located at $x = -l$) is the rate of change of the total energy in the layer $E(t)$:

$$
\frac{dE(t)}{dt} = J(l, t) \quad (8)
$$

Therefore, the power spectrum of variations in $E(t)$, $S_E(\omega) = |\langle E(\omega) \rangle|^2$, is

$$
S_E(\omega) \propto \int_{-\infty}^{\infty} dk \frac{D}{4k^4 + \omega^2} \propto \omega^{-\frac{5}{2}} \quad (9)
$$

Since $\Delta T \propto \Delta E$, $ST(\omega) \propto \omega^{-\frac{3}{2}}$ also.

If we include the heat flux out of both boundaries, the rate of change of energy in the layer will be given by the difference in heat flux: $\frac{dE(t)}{dt} = J(l, t) - J(-l, t)$. The Fourier transform of $E(t)$ is now

$$
E(\omega) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} dk \sin(kl)J(k, \omega) \quad (10)
$$

Then,

$$
S_T(\omega) \propto S_E(\omega) \propto \int_{-\infty}^{\infty} \frac{dk}{D^2k^4 + \omega^2} \propto \omega^{-\frac{3}{2}}(1 + e^{-\theta(\sin \theta + \cos \theta)}) \quad (11)
$$

where $\theta = \frac{\omega}{\omega_o}$ and $\omega_o = D/2l^2$ is the frequency where the correlation length is equal to the width of the layer. When $\lambda << 2l$, the above expression reduces to $S_T(\omega) \propto \omega^{-\frac{3}{2}}$. When $\lambda >> 2l$, $S_T(\omega) \propto \omega^{-\frac{5}{2}}$ (Voss and Clarke 1976).

In Section 2 we presented evidence that continental stations exhibit a $f^{-\frac{3}{2}}$ high-frequency region and maritime stations exhibit $f^{-\frac{5}{2}}$ scaling up to the highest frequency. This observation can be interpreted in terms of the diffusion model presented above. The power spectrum of temperature variations in an air mass exchanging heat by one-dimensional stochastic diffusion is $\propto f^{-\frac{1}{2}}$. 

Figure 5 illustrates the geometry of the layer exchanging thermal energy with diffusing regions above and below it. A diffusion process has a frequency-dependent correlation length $\lambda = (\frac{2D}{\omega})^{\frac{1}{2}}$ (Voss and Clarke 1976). Two different situations arise as a consequence of the length scale, $2l$, of the geometry. For very high frequencies, $\lambda << 2l$. In that case, the fluctuations in heat flow across the two boundaries are independent. For low frequencies, $\lambda >> 2l$ and the boundaries fluctuate coherently. First we consider high frequencies. Since the boundaries fluctuate independently, we can consider the flow across one boundary only. The flux of heat energy is given by eq. (5). Its Fourier transform is given by

$$
J(k, \omega) = \frac{i\omega \eta(k, \omega)}{\alpha k^2 - i\omega} \quad (7)
$$
if the air mass is bounded by two diffusing regions and is \( \propto f^{-\frac{3}{2}} \) if it interacts with only one. The maritime stations have a \( f^{-\frac{3}{2}} \) power spectrum up to the highest frequency because the air mass above a maritime station exchanges heat with both the atmosphere above and the ocean below. The fluctuation calculation appropriate for maritime stations is one in which the coherent fluctuations from two boundaries are considered as is in the calculation of the \( f^{-\frac{3}{2}} \) spectrum. The air mass above continental stations exchanges heat energy only with the atmosphere above it. The calculation appropriate for continental stations is the one-boundary model which predicts the observed \( f^{-\frac{3}{2}} \) spectrum. At low frequencies, horizontal diffusive heat exchange between continental and maritime air masses limits the variance of the continental stations. Low-frequency fluctuations of continental stations may correspond to fluctuations in the average temperature of the atmosphere above the oceans, giving a \( f^{-3} \) spectrum.

At lower frequencies, the atmosphere and ocean achieve thermal equilibrium. The variance in temperature of the atmosphere and ocean is then determined by the radiation boundary condition. The fluctuating temperature of the atmosphere adds and subtracts heat from the atmosphere and ocean through a linear radiation boundary condition; the heat flux out of the atmosphere is proportional to the temperature of the atmosphere. This results in temperature and irradiance variations with a random walk \( (f^{-2}) \) spectrum.

The fluctuating input and output of heat in the \( f^{-2} \) region will cause large variations from equilibrium. When the temperature of the atmosphere and ocean becomes larger than the equilibrium temperature, it will radiate, on average, more heat than at equilibrium. Conversely, when the temperature of the atmosphere and ocean wavers lower than the equilibrium temperature, less heat is radiated. This negative feedback limits the variance at low frequencies resulting in a constant power spectrum at very low frequencies.

To show this, we consider a coupled atmosphere-ocean model with an atmosphere of uniform density (equal to the density at sea level) in contact with an ocean of uniform density. The height of our model atmosphere is the scale height of the atmosphere (height at which the pressure falls by a factor of \( e \) from its value at sea level). Figure 6.1 illustrates the geometry and constants chosen (where \( \sigma \) is the vertical heat conductivity, \( \rho \) is the density, \( c \) is the specific heat per unit mass, \( \alpha \) is the vertical thermal diffusivity, and \( g \) is the thermal conduction of heat out of the Earth by emission of radiation). Primed constants denote values for the ocean. The physical constants which enter the model are the thermal conductance by emission of radiation and the density, specific heat, vertical thermal diffusivity, and depth of the ocean and atmosphere. The density and specific heat of air and water are well-known constants. We chose an ocean depth of 4 km and an atmospheric height equal to the scale height of 8 km as used by Hoffert, Callegari, and Hsieh (1980) in their climate modeling studies. The turbulent transfer of heat vertically in the ocean by diffusion and advection is commonly parametrized in studies of climate change as a diffusion process with an effective thermal diffusivity as we assume in this model (Ghil 1983). The diffusion coefficients of the atmosphere and ocean we employ were discussed in Section 3.

| ATMOSPHERE |
| --- |
| \( \alpha = 1.0 \text{ m/s} \) |
| \( \sigma = 1000 \text{ W/m K} \) |
| \( \rho = 1 \text{ kg/m}^3 \) |
| \( c = 1000 \text{ J/kg K} \) |
| \( w = 8000 \text{ km} \) |

| OCEAN |
| --- |
| \( \alpha' = 6 \times 10^4 \text{ m/s} \) |
| \( \sigma' = 25 \text{ W/m K} \) |
| \( \rho' = 1000 \text{ kg/m}^3 \) |
| \( c' = 4200 \text{ J/kg K} \) |
| \( w' = 4000 \text{ km} \) |

Figure 6. Geometry and constants of the model described in the text.

The equation for temperature fluctuations in space and time in the model is eq. (12):

\[
\frac{\partial \Delta T(x, t)}{\partial t} - \alpha(x) \frac{\partial^2 \Delta T(x, t)}{\partial x^2} = - \frac{\partial \eta(x, t)}{\partial x} \tag{12}
\]

with

\[
\langle \eta(x, t) \rangle = 0 \tag{13}
\]

\[
\langle \eta(x, t) \eta(x', t') \rangle \propto \sigma(x) \langle T(x) \rangle^2 \delta(x - x') \delta(t - t') \tag{14}
\]

The boundary conditions are that there be no heat flow out of bottom of the ocean and continuity of temperature and heat flux at the atmosphere-ocean boundary:

\[
\sigma' \frac{\partial T}{\partial x} \bigg|_{x=w_2} = 0 \tag{15}
\]

\[
\Delta T(x = w_2^+ ) = \Delta T(x = w_2^-) \tag{16}
\]

\[
\sigma \frac{\partial \Delta T}{\partial x} \bigg|_{x=w_1^-} = \sigma' \frac{\partial \Delta T}{\partial x} \bigg|_{x=w_1^+} \tag{17}
\]

At the top of the atmosphere we will impose a blackbody radiation boundary condition. Most (65\%) of the energy incident on the Earth is emitted as long-wavelength blackbody radiation from the H\(_2\)O and CO\(_2\) in the atmosphere (Peixoto and Oort 1992). This heat
emitted from the atmosphere is dependent on the temperature of the atmosphere at the point of emission according to the Stefan-Boltzmann law. It is common practice to assume that temperature variations from equilibrium are small (the global mean temperature has fluctuated by only about eight degrees Celsius during the last glaciation). Within a linear approximation, the emitted temperature will be proportional to the temperature difference from equilibrium (Ghil 1983). The boundary condition at the scale height of the atmosphere (which we take to be representative of the average elevation where radiation is emitted from the atmosphere) is then

$$
\left. \alpha \frac{\partial \Delta T}{\partial x} \right|_{x=0} = g \Delta T(x = 0)
$$

(18)

We will use the value $g = 1.7 \text{ W/m}^2\text{K}$ as used by Ghil (1983) and Harvey and Schneider (1985).

van Vliet et al. (1980) used Green’s functions to solve this model. The Green’s function of the Laplace-transformed diffusion equation is defined by

$$
i \omega G(x, x', \omega) - \alpha(x) \frac{\partial^2 G(x, x', \omega)}{\partial x^2} = \delta(x - x')
$$

(19)

where $G$ is governed by the same boundary conditions as $\Delta T$. This equation can be solved by separating $G$ into two parts: $G_a$ and $G_b$ with $x < x'$ and $x > x'$, respectively, where $G_a$ and $G_b$ satisfy the homogeneous (no forcing) diffusion equation with a jump condition relating $G_a$ and $G_b$:

$$
\left. \frac{\partial G_a}{\partial x} \right|_{x=x'} - \left. \frac{\partial G_b}{\partial x} \right|_{x=x'} = \frac{1}{\alpha(x')}
$$

(20)

The power spectrum of the average temperature in the atmosphere in terms of $G$ is given by van Vliet et al. (1980) as:

$$
S_{\Delta T_{av}}(f) \propto \text{Re}(\int_0^{w_1} \int_0^{w_1} G_1(x, x', \omega) dx dx')
$$

(21)

where $G_1$ stands for the solution to the differential equation for $G$ where the source point is located in the atmosphere. $\text{Re}$ denotes the real part of the complex expression. Two forms of $G_{1a}$ and $G_{1b}$ are necessary for $x$ located above and below $x'$, respectively, due to the discontinuity in the derivative of $G_1$ created by the delta function (the jump condition). The solution of $G_1$ which satisfies the above differential equation and boundary conditions is

$$
G_{1a} = \frac{L}{\alpha K} \frac{\sigma' L}{\sigma L'} \sinh\left(\frac{w_1-x'}{L}\right) \sinh\left(\frac{w_2}{L}\right) + \cosh\left(\frac{w_1-x'}{L}\right) \cosh\left(\frac{w_2}{L}\right) (\sinh\left(\frac{x}{L}\right) + \frac{\sigma}{Lg} \cosh\left(\frac{x}{L}\right))
$$

(23)

and

$$
G_{1b} = G_{1a} + \frac{L}{\alpha} \sinh\left(\frac{x'-x}{L}\right)
$$

(24)

where

$$
K = \left(\sinh\left(\frac{w_1}{L}\right) + \frac{\sigma}{Lg} \cosh\left(\frac{w_1}{L}\right)\right) \frac{\sigma' L}{\sigma L'} \sinh\left(\frac{w_2}{L}\right)
$$

$$
+ \left(\cosh\left(\frac{w_1}{L}\right) + \frac{\sigma}{Lg} \cosh\left(\frac{w_1}{L}\right)\right) \cosh\left(\frac{w_2}{L}\right)
$$

(25)

$$
\text{and } L = \left(\frac{\alpha}{\omega}\right)^{\frac{1}{2}} \text{ and } L' = \left(\frac{\alpha'}{\omega'}\right)^{\frac{1}{2}}.
$$

Performing the integration van Vliet et al. obtained

$$
S_{\Delta T_{av}}(f) \propto \text{Re}(L^2 \frac{\sigma' L}{\sigma L'} \tanh\left(\frac{w_2}{L}\right) ((\frac{\omega_1}{\sigma} - 1)
$$

$$
\tanh\left(\frac{w_1}{L}\right) - 2gL \cosh(w_1/L) - \frac{1}{\sigma} \cosh(w_1/L) + \frac{w_2}{L})
$$

$$
+ (\frac{\omega_1}{\sigma} + (\frac{w_1}{L} - \frac{g}{\sigma} \tanh(w_1/L)) ((\tanh(w_1/L) + \frac{\sigma L}{g}) \frac{\sigma' L}{\sigma L'} \tanh\left(\frac{w_2}{L}\right) + (1 + \frac{\sigma}{Lg} \tanh(w_1/L))^{-1}
$$

(26)

For very low frequencies,

$$
\tanh\left(\frac{w_1}{L}\right) \approx \frac{w_1}{L}, \tanh\left(\frac{w_2}{L}\right) \approx \frac{w_2}{L}
$$

(27)

$$
\frac{\cosh(w_1/L) - 1}{\cosh(w_1/L)} \approx \frac{1}{2} \frac{w_1^2}{L^2}
$$

(28)

Reducing eq. (28),

$$
S_{\Delta T_{av}}(f) \propto \frac{1}{1 + \frac{\omega_1^2}{\omega^2}} \propto \frac{1}{f^2 + \frac{\omega_0^2}{f^2}}
$$

(29)

which is the low-frequency Lorentzian spectrum observed in the Vostok data. The crossover frequency as a function of the constants chosen for the model is

$$
f_0 = \frac{g}{2\pi(w_1 c \rho + w_2 c' \rho'(1 + \frac{\omega_1}{\omega_2}))} \approx \frac{\sigma}{2\pi w_1 w_2 c' \rho'}
$$

$$
\approx \frac{1}{25,000 \text{ years}}
$$

(30)

which is within an order of magnitude of the observed crossover frequency of the Vostok data, $f_0 = 40,000 \text{ years}$.

At higher frequencies

$$
\tanh\left(\frac{w_1}{L}\right) \approx \frac{w_1}{L}, \tanh\left(\frac{w_2}{L}\right) \approx 1
$$

(31)

$$
\frac{\cosh(w_1/L) - 1}{\cosh(w_1/L)} \approx \frac{1}{2} \frac{w_1^2}{L^2}
$$

(32)

then

$$
S_{\Delta T}(f) \propto \frac{1}{2} \left(\frac{2\omega_1}{\sigma}\right)^{\frac{1}{2}} \left(\frac{c \rho}{c' \rho'}\right)^{\frac{1}{2}} \left(\frac{g}{w_1 c \rho f}\right)^{\frac{1}{2}} \propto f^{-\frac{1}{2}}
$$

(33)
as observed. The high and low-frequency spectra meet at

\[ f_1 = \frac{g}{w_1 \rho c} \left( \frac{\sigma}{2gw_1} \right)^{\frac{1}{2}} \left( \frac{c' \rho' \sigma'}{cp\sigma} \right)^{\frac{1}{2}} \left( \frac{cpw_2}{c' \rho' w_2} \right)^{\frac{1}{2}} \]  

\[ \approx \frac{1}{10,000 \text{ years}} \]  

(35)

which also agrees well with that observed in the Vostok data \( (f_1 \approx \frac{1}{20,000 \text{ years}}) \).

The time scales of \( f_1 \) and \( f_0 \) correspond to thermal and radiative equilibration of the coupled climate system, respectively. At time scales of 2000 years, the entire atmosphere and ocean are in thermal equilibrium. Besides the Vostok data, observational support for this thermal equilibration time is provided by comparisons of Antarctic and Greenland ice cores. Bender et al. (1994) found that temperature variations in Antarctica and Greenland are correlated above time scales of 2000 years and uncorrelated below it. This suggests that 2000 years represents the time scale of global thermal equilibrium.

Our model makes a specific prediction of the ability of the oceans to absorb an increase in atmospheric temperature resulting from the greenhouse effect. If the rate of increase of atmospheric temperature produced by the greenhouse effect is less than the vertical heat conductivity of the oceans, all of the heat produced by the greenhouse effect can be stored in the oceans for time scales up to 2000 years, the thermal equilibration time of the atmosphere and ocean. A 1°C per century warming is seven orders of magnitude smaller than the vertical heat conductivity, indicating that the ocean can easily absorb such an increase. A time scale of 2000 years is much larger than other values quoted for the delay in atmospheric warming due to the absorptive capacity of the oceans. Most studies have suggested time scales on the order of decades (Schneider and Thompson 1981).

We have applied the same model presented in this paper to variations in the solar luminosity from time scales of minutes to months (Pelletier 1995). A stochastic diffusion model of the turbulent heat transfer between the granulation layer of the sun (modeled as a homogeneous thin layer with a radiative boundary condition) and the rest of the convection zone (modeled as a homogeneous thick layer with thermal and diffusion constants appropriate to the lower convection zone) predicts the same spectral form observed in solar irradiance data recorded by the ACRIM project and observed in the climate spectra reported here. The time scales of thermal and radiative equilibration of the solar convection zone based upon thermal and diffusion constants estimated from mixing-length theory match those observed in the ACRIM data.

5. CONCLUSIONS

We have presented evidence that the power spectrum of atmospheric temperatures exhibits four scaling regions. We presented a model originally due to van Vliet et al. (1980) proposed to study temperature fluctuations in a metallic film (atmosphere) supported by a substrate (ocean) that matches the observed frequency dependence of the power spectrum of temperature fluctuations well. The difference between the high-frequency spectra of continental and maritime stations may be interpreted in terms of the fact that air masses above maritime stations interact with the atmosphere above and the ocean below while air masses above continental stations interact with only the atmosphere above.

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