ON THE INJECTIVITY OF THEhifted
FUNK-RADON TRANSFORM AND RELATED
HARMONIC ANALYSIS

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To the memory of Professor Lawrence Zalcman

Abstract. Necessary and sufficient conditions are obtained for injectivity of the shifted Funk-Radon transform associated with \( k \)-dimensional totally geodesic submanifolds of the unit sphere \( S^n \) in \( \mathbb{R}^{n+1} \). This result generalizes the well known statement for the spherical means on \( S^n \) and is formulated in terms of zeros of Jacobi polynomials. The relevant harmonic analysis is developed, including a new concept of induced Stiefel (or Grassmannian) harmonics, the Funk-Hecke type theorems, addition formula, and multipliers. Some perspectives and conjectures are discussed.

1. Introduction

Let \( X \) be an \( n \)-dimensional constant curvature space, \( \Xi \) be the set of all \( k \)-dimensional totally geodesic submanifolds of \( X \), \( 1 \leq k \leq n-1 \). Consider the Radon type transform

\[
(R_t f)(\xi) = \int_{d(x,\xi)=t} f(x) dm(x), \quad x \in X, \ \xi \in \Xi, \ t > 0,
\]

where \( d(\cdot, \cdot) \) stands for the geodesic distance on \( X \) and \( dm(x) \) is the relevant canonical measure.

**Question.** Suppose that \( t > 0 \) is fixed. How does the injectivity of \( R_t \) depend on the values of \( t \) and the class of functions \( f \) ?

For the spherical means on \( X \), formally corresponding to \( k = 0 \), this problem was studied by Berenstein and Zalcman [3, Section 6]. It falls into the scope of the wide class of Pompeiu’s problems. There is an extensive literature related to numerous aspects of the spherical

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means and the Pompeiu problem in general; see, e.g., [1, 33, 34, 35] and references therein.

The operator $R_t$ and its dual

$$
(R_t^* \varphi)(x) = \int \varphi(\xi) d\mu(\xi)
$$

are well known in integral geometry [11, 20]. Following Rouvière [16, p. 19], we call $R_t f$ and $R_t^* \varphi$ the \textit{shifted Radon transform} and the \textit{shifted dual Radon transform}, respectively. The terminology is motivated by the fact that the limiting case $t = 0$ yields the well known totally geodesic Radon transform and its dual [11].

In the present article we are focusing on the case when $X$ is the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ and call $R_t f$ the \textit{shifted Funk-Radon transform}, because this name is more precise. The functions on $\Xi$ can be thought of as the functions on the Grassmann manifold $G_{n+1,k+1}$ of $(k+1)$-dimensional linear subspaces of $\mathbb{R}^{n+1}$. Alternatively, they can be interpreted as right $\text{O}(n-k)$-invariant functions on the Stiefel manifold $V_{n+1,n-k}$ of orthonormal $(n-k)$-frames in $\mathbb{R}^{n+1}$.

\textbf{Main Results.} We invoke the Jacobi polynomials $P_{j/2}^{(\sigma,\rho)}$ [8] with $j$ even.

\textbf{Theorem 1.1.} Let $1 \leq k \leq n-1$, $\sigma = (n-k)/2 - 1$, $\rho = (k-1)/2$.

(i) The operator $R_t$ with fixed $t \in (0, \pi/2)$ is injective on $L^1_{\text{even}}(S^n)$ if and only if $P_{j/2}^{(\sigma,\rho)}(\cos 2t) \neq 0$ for all $j \in \{0, 2, 4, \ldots\}$.

(ii) More generally, given a positive integer $\ell$, let $f \in L^1_{\text{even}}(S^n)$ and suppose that $R_{t_i} f = 0$ a.e. for all $t_i \in (0, \pi/2)$; $i = 1, 2, \ldots, \ell$. If the equations

$$
P_{j/2}^{(\sigma,\rho)}(\cos 2t_i) = 0, \quad i = 1, 2, \ldots, \ell,
$$

have no common solution for $j \in \{0, 2, 4, \ldots\}$, then $f = 0$ a.e. on $S^n$. If these equations have a common solution, say, $j = j_0$, then $R_{t_i} Y_{j_0} = 0$ for all $i = 1, 2, \ldots, \ell$ and all spherical harmonics $Y_{j_0}$ of degree $j_0$.

This theorem agrees with the known case $k = n-1$ (cf. [3, Theorem 8]), when the Jacobi polynomial $P_{j/2}^{(\sigma,\rho)}$ can be written as the Gegenbauer polynomial $C_j^{(n-1)/2}$ with transformed argument; use, e.g., [8, formula 10.9 (21)].

\textbf{Corollary 1.2.} The set of all $t \in (0, \pi/2)$, for which $R_t$ is non-injective on $L^1_{\text{even}}(S^n)$, is everywhere dense in $(0, \pi/2)$ and so is the set for which it does.
This statement follows from the density property of zeros of ortho-
gonal polynomials (see, e.g., [30, Theorem 6.1.1]) and the fact that the set of all such zeros is countable. It mimics the celebrated Ungar’s freak theorem for spherical caps in $S^2$; see also Schneider [24, 25] and Beren-
stein and Zalcman [3] regarding the similar statement for hyperplane sections of $S^n$, $n \geq 2$.

Remark 1.3. Theorem 1.1 gives no answer about injectivity of $R_t$ for particular values of $t \in (0, \pi/2)$, say, $t = \pi/5$ or $t = \pi/6$. It only reformulates the problem in a different language. However, this reformulation is very important. For instance, it allows one to invoke the tools of number theory and asymptotic properties of Jacobi poly-
nomials for further investigation. Some results in this direction for $k = n - 1$ and associated Legendre functions can be found in [17], [21, Section 5.5].

To prove Theorem 1.1 we introduce a new concept of induced Stiefel harmonics on $V_{n+1,n-k}$. These harmonics are right $O(n-k)$-invariant, constitute an orthonormal system, and can be regarded as harmonics on the Grassmann manifold $G_{n+1,k+1}$. They are generated by the usual spherical harmonics on $S^n$. We prove the addition formula for such harmonics and establish new Funk-Hecke type theorems for $O(n+1)$-
itertwining operators, which connect functions on $S^n$ with functions on $V_{n+1,n-k}$ (or $G_{n+1,k+1}$). These theorems provide explicit formulas for the relevant Fourier-type multipliers. The Jacobi polynomials are the main ingredients of these formulas.

The developed harmonic analysis is applicable not only to the oper-
ators (1.1) and (1.2) but also to the Funk-Radon transforms (the case $t = 0$) and to the more general analytic families of generalized cosine transforms in integral geometry [18, 19]; see examples in Section 3.5.

Section 2 contains preliminaries. In Section 3 we prove the main results. More comments can be found in Conclusion, also containing some thoughts about possible developments in the future.

2. Preliminaries

2.1. Notation.
In the following, $\mathbb{R}^{n+1}$, $n \geq 2$, is the real $(n+1)$-dimensional Euclidean space with the coordinate unit vectors $e_1, \ldots, e_{n+1}$; $S^n \subset \mathbb{R}^{n+1}$ is the $n$-dimensional unit sphere with the area $\sigma_n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$. The points in $\mathbb{R}^{n+1}$ will be identified with the relevant column vectors.

For $x \in S^n$, we write $dx$ for the surface area measure on $S^n$ and set $d_x x = \sigma_n^{-1} dx$ for the corresponding normalized measure.
Given an integer $k$, $1 \leq k \leq n-1$, let $V_{n+1,n-k}$ be the Stiefel manifold of orthonormal $(n-k)$-frames in $\mathbb{R}^{n+1}$. Every element $v \in V_{n+1,n-k}$ is an $(n+1) \times (n-k)$ matrix satisfying $v^T v = I_{n-k}$, where $v^T$ is the transpose of $v$ and $I_{n-k}$ is the identity $(n-k) \times (n-k)$ matrix. We equip $V_{n+1,n-k}$ with the standard probability measure $d_v$, which is left $O(n+1)$-invariant and right $O(n-k)$-invariant. Given $v \in V_{n+1,n-k}$, we denote by $v^\bot$ the $(k+1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$ orthogonal to $v$. The Grassmann manifold of all such subspaces will be denoted by $G_{n+1,k+1}$. If $x \in S^n$, $v \in V_{n+1,n-k}$, and $\{v\}$ is the $(n-k)$-plane spanned by $v$, then $|x^T v|$ is the length of the orthogonal projection of $x$ onto $\{v\}$. The notation $x \cdot y = x_1 y_1 + \cdots + x_{n+1} y_{n+1}$ for the vectors $x, y \in \mathbb{R}^{n+1}$ is standard.

Let $\Xi$ be the set of all $k$-dimensional totally geodesic submanifolds $\xi$ of $S^n$ ($k$-geodesics, for short) equipped with the canonical $O(n+1)$-invariant probability measure $d_\xi$. Every $\xi \in \Xi$ has the form $\xi = S^n \cap v^\bot$ for some $v \in V_{n+1,n-k}$.

The group $O(n+1)$ of orthogonal transformations of $\mathbb{R}^{n+1}$ and all subgroups of $O(n+1)$ will be equipped with the corresponding Haar measure of total mass one.

In the following, $v_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n+1,n-k}$ denotes the coordinate frame. If $v \in V_{n+1,n-k}$ and $r_v \in O(n+1)$ maps $v_0$ to $v$, we set $f_v(x) = f(r_v x)$. Similarly, if $x \in S^n$ and $r_x \in O(n+1)$ maps $e_{n+1}$ to $x$, we denote $\varphi_x(v) = \varphi(r_x v)$.

We say that an integral under consideration exists in the Lebesgue sense if it is finite when the integrand is replaced by its absolute value.

### 2.2. Bispherical means

In this section we give precise meaning to the shifted Funk-Radon transform $R_t$ on $S^n$ and its dual $R_t^\ast$. The operator $R_t$ will be realized as a certain bispherical mean associated with bispherical coordinates in $S^n$. We also recall some known facts about spherical harmonics and their representation in bispherical coordinates.

Let

$$\mathbb{R}^{n+1} = \mathbb{R}^{k+1} \times \mathbb{R}^{n-k}, \quad 1 \leq k \leq n-1.$$ (2.1)

Every point $x \in S^n$ can be represented as

$$x = \eta \sin \theta + \zeta \cos \theta,$$ (2.2)

where

$$\eta \in S^k \subset \mathbb{R}^{k+1}, \quad \zeta \in S^{n-k-1} \subset \mathbb{R}^{n-k}, \quad 0 \leq \theta \leq \pi/2,$$
\[ dx = \sin^k \theta \cos^{n-k-1} \theta \ d\eta \zeta d\theta, \]

\[ dx, d\eta, d\zeta \] being the corresponding non-normalized surface area measures; see, e.g., [32, pp. 12, 22]. We recall that the relevant normalized measures are denoted by \( dx_* \), \( d\eta_* \), \( d\zeta_* \).

The variables \((\eta, \zeta, \theta)\) are called the \textit{bispherical coordinates} of \( x \).

Let \( v_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n+1,n-k} \) be the coordinate frame, \( r_v \) be a rotation mapping \( v_0 \) to \( v \in V_{n+1,n-k} \), \( f_v(x) = f(r_v x) \). Consider the integral

\[ (M_{\cos \theta} f)(v) = \int_{S^{n-k-1}} d\zeta \int_{S^k} f_v(\eta \sin \theta + \zeta \cos \theta) d\eta_* \]

or (set \( \tau = \cos \theta \))

\[ (M_{\tau} f)(v) = \int_{S^{n-k-1}} d\zeta \int_{S^k} f_v(\eta \sqrt{1 - \tau^2} + \zeta \tau) d\eta_* . \]

We call \((M_{\tau} f)(v)\) the \textit{bispherical mean of} \( f \) \textit{in the direction of} \( v \in V_{n+1,n-k} \) \textit{at the level} \( \tau \). One can also write

\[ (M_{\tau} f)(v) = \int_{|x^Tv|=\tau} f(x) d_x x \]

(see Notation), where \( d_x x \) stands for the corresponding probability measure.

The integral (2.3) gives precise meaning to the shifted Funk-Radon transform (1.1). Specifically,

\[ (M_{\cos \theta} f)(v) = \int_{d(x,\xi)=t} f(x) \ dm(x) \equiv (R_{\xi} f)(\xi), \]

\[ t = \frac{\pi}{2} - \theta, \quad \xi = S^n \cap v^\perp \in \Xi . \]

By (2.2), for any \( v \in V_{n+1,n-k} \) we have

\[ \int_{S^n} f(x) d_x x = \frac{\sigma_{n-k-1} \sigma_k}{\sigma_n} \int_0^{\pi/2} (M_{\cos \theta} f)(v) \sin^k \theta \cos^{n-k-1} \theta \ d\theta . \]
To define the dual of \((M_\tau f)(v)\), we first write
\[
(M_\tau f)(v) = \int_{SO(n-k)} d\alpha \int_{SO(k+1)} f_v(\alpha e_{n+1} \cos \theta + e_{k+1} \sin \theta) d\alpha \]
where \(K' = SO(n-k) \times SO(k+1), \delta \in K = SO(n), \tau = \cos \theta, \) and \(g_{k+1,n+1}(\theta)\) is a rotation in the plane \((e_{k+1}, e_{n+1})\) with the matrix
\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}.
\]
Clearly, \(g_{k+1,n+1}(\theta)e_{n+1} = e_{n+1} \cos \theta + e_{k+1} \sin \theta.\) We define
\[
(M^*_\tau \varphi)(x) = \int_{K' \times K} \varphi_x(g^{-1} v_0) d\gamma d\delta,
\]
where \(g\) has the same meaning as in (2.9), \(\varphi_x(v) = \varphi(r_x v), r_x\) is a rotations which maps \(e_{n+1}\) to \(x.\)

If \(\varphi\) is a right \(O(n-k)\)-invariant function on \(V_{n+1,n-k}\), which is interpreted as a function of \(\xi \in \Xi,\) then, abusing notation, we can write (2.10) as
\[
(M^*_\cos \theta \varphi)(x) = \int_{d(x,\xi)=t} \varphi(\xi) d\mu(\xi) \equiv (R^*_t \varphi)(x), \quad t = \frac{\pi}{2} - \theta.
\]
This integral gives precise meaning to the shifted dual Funk-Radon transform (1.2).

We recall that \((R_t f)(\xi)\) and \((R^*_t \varphi)(x)\) with \(t = 0\) are the usual Funk-Radon transforms [11, 16, 18], so that
\[
(M_0 f)(v) = (R f)(S^n \cap v^\perp) = (R f)(\xi), \quad (M^*_0 \varphi)(x) = (R^* \varphi)(x),
\]
if we identify \(O(n-k)\)-invariant functions \(\varphi\) on \(V_{n+1,n-k}\) with functions on \(\Xi.\)

**Lemma 2.1.** For any \(\tau \in [0, 1],\)
\[
\int_{V_{n+1,n-k}} (M_\tau f)(v) \varphi(v) d_* v = \int_{S^n} f(x)(M^*_\tau \varphi)(x) d_* x,
\]
provided that either side of this equality exists in the Lebesgue sense.
Proof. This duality statement is well known in the general double fibration context \[11\]. For the sake of completeness, we present its proof in the Stiefel terms. Let \( G = SO(n+1), \tau = \cos \theta \). By (2.9),

\[
I = \int_{V_{n+1,n-k}} (M_\tau f)(v)\varphi(v)dv = \int_G (M_\tau f)(gv_0)\varphi(gv_0)dg
\]

\[
= \int_{K'} \int_G \varphi(gv_0) f(gg_{k+1,n+1}(\theta)\delta e_{n+1})dg.
\]

Now we change the notation \( gg_{k+1,n+1}(\theta)\delta \to g \), then integrate in \( \delta \in K \), and change the order of integration. This gives

\[
I = \int_G f(ge_{n+1})dg \int_{K'\times K} \varphi(g\delta^{-1}[g_{k+1,n+1}(\theta)]^{-1}\gamma^{-1}v_0)d\gamma d\delta
\]

\[
= \int_G f(ge_{n+1})(M_\tau^*\varphi)(ge_{n+1})dg,
\]

which implies (2.13). \( \square \)

2.3. Spherical harmonics in bispherical coordinates.

1. Let \( \mathcal{Y} = \{Y_{j,\lambda}^n(x)\} \) be an orthonormal basis of spherical harmonics in \( L^2(S^n) \). Here \( j \in \{0,1,\ldots\}, \lambda \in \{1,2,\ldots,d_n(j)\} \);

\[
d_n(j) = (n + 2j - 1) \frac{\Gamma(n + j - 1)}{\Gamma(j + 1) \Gamma(n)}
\]

is the dimension of the subspace of spherical harmonics of degree \( j \). Thus

\[
\int_{S^n} Y_{j,\lambda}^n(x) Y_{j',\lambda'}^n(x) d_s x = \begin{cases} 1 & \text{if } j = j' \text{ and } \lambda = \lambda', \\ 0 & \text{otherwise} \end{cases}
\]

(it is important to keep in mind that normalization of the spherical harmonics throughout the paper is understood with respect to the probability measure \( d_s x \), not with respect to the surface area measure \( dx \)).

If \( Y_j \) is a spherical harmonic of degree \( j \), \( \Omega(t)(1-t^2)^{n/2-1} \in L^1(-1,1) \), then, by the Funk-Hecke theorem,

\[
\int_{S^n} \Omega(x \cdot y) Y_j(x) d_s x = \omega_j Y_j(y),
\]
where
\[ \omega_j = \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} P_j(t) \Omega(t) (1 - t^2)^{n/2-1} dt, \]

(2.17)
\[ P_j(t) = \frac{j! (n-2)!}{(j+n-2)!} C_j^{(n-1)/2}(t), \]

\[ C_j^{(n-1)/2}(t) \] being the Gegenbauer polynomial. The polynomials (2.17) are called the spherical polynomials (other names are also known) and enjoy the following properties:

(2.18)
\[ P_j(1) = 1; \]

(2.19)
\[ \int_{-1}^{1} P_j(t) P_{j'}(t) (1 - t^2)^{n/2-1} dt = \begin{cases} 0, & \text{if } j \neq j', \\ \frac{\sigma_n}{d_n(j) \sigma_{n-1}}, & \text{if } j = j'; \end{cases} \]

(2.20)
\[ \sum_{\lambda=1}^{d_n(j)} Y_{j,\lambda}^n(x) Y_{j,\lambda}^n(y) = d_n(j) P_j(x \cdot y). \]

The reader is referred to [21, Section A.6], where these statements are proved in slightly different notation.

2. We will need representation of spherical harmonics in the bispherical coordinates
\[ x = \eta \sin \theta + \zeta \cos \theta, \]
\[ \eta \in S^k \subset \mathbb{R}^{k+1}, \quad \zeta \in S^{n-k-1} \subset \mathbb{R}^{n-k}, \quad 0 \leq \theta \leq \pi/2; \]
cf. (2.2). Let \( P_m^{(\rho, \sigma)}(t), m \in \{0, 1, 2, \ldots\}, \) be the Jacobi polynomials; \( \rho, \sigma > -1. \) The corresponding normalized polynomials are defined by

(2.21)
\[ R_m^{(\rho, \sigma)}(t) = \frac{P_m^{(\rho, \sigma)}(t)}{P_m^{(\rho, \sigma)}(1)}, \quad P_m^{(\rho, \sigma)}(1) = \frac{\Gamma(m + \rho + 1)}{m! \Gamma(\rho + 1)}. \]

We recall that

(2.22)
\[ \int_{-1}^{1} [R_m^{(\rho, \sigma)}(t)]^2 (1 - t)^\rho (1 + t)^\sigma dt \]
\[ = \frac{2^\rho+\sigma+1 m! \Gamma^2(\rho + 1)\Gamma(m + \sigma + 1)}{(2m + \rho + \sigma + 1)\Gamma(m + \rho + 1)\Gamma(m + \rho + \sigma + 1)} \]
and

\begin{equation}
\int_{-1}^{1} R_{\ell}^{(\rho,\sigma)}(t) R_{m}^{(\rho,\sigma)}(t)(1-t)^\rho(1+t)^\sigma dt = 0, \quad \ell \neq m;
\end{equation}

cf. [2] pp. 300, 301. Note also (see, e.g., [8] formula 10.9 (21)) that

\begin{equation}
R_{m}^{(n/2-1, -1/2)}(2t^2 - 1) = P_{2m}(t),
\end{equation}

$P_{2m}$ being the spherical polynomial \((2.17)\) of degree $j = 2m$.

Let $\{Y_{r,\mu}(\eta)\}$ and $\{Y_{s,\nu}^{n-k-1}(\zeta)\}$ be orthonormal bases of spherical harmonics in $L^2(S^k)$ and $L^2(S^{n-k-1})$, respectively. Here

$\mu = 1, \ldots, d_{k+1}(r); \quad \nu = 1, \ldots, d_{n-k}(s)$;

cf. \((2.14)\). We set

$\rho = r + (k - 1)/2, \quad \sigma = s + (n - k)/2 - 1,$

and consider the collection of functions

\begin{equation}
U_{M}^{j}(x) = \kappa_{M} Y_{r,\mu}^{k}(\eta) Y_{s,\nu}^{n-k-1}(\zeta) \sin^{r} \theta \cos^{s} \theta R_{m}^{(\rho,\sigma)}(\cos 2\theta),
\end{equation}

indexed by $M = (r, \mu; s, \nu; m)$ with $j = 2m + r + s$ and

\begin{equation}
\kappa_{M}^{2} = \frac{2\sigma_{n}(2m + \rho + \sigma + 1) \Gamma (m + \rho + 1) \Gamma (m + \rho + \sigma + 1)}{\sigma_{n-k-1}\sigma_{k} m! \Gamma (m + \sigma + 1) \Gamma^{2} (\rho + 1)}.
\end{equation}

Each $U_{M}^{j}(x)$ is a spherical harmonic of degree $j$. We denote by $\mathcal{U}$ the collection of all harmonics \((2.25)\). One can show [32] pp. 208 - 211 that $\mathcal{U}$ is an orthonormal basis in $L^2(S^n)$.

For convenience of the reader, let us check, for instance, that

\begin{equation}
||U_{M}^{j}||_{L^2(S^n)} = 1.
\end{equation}

Passing to bi-spherical coordinates \((2.2)\) and taking into account normalization, we have

\begin{align*}
||U_{M}^{j}||_{L^2(S^n)}^{2} &= \int_{S^n} |U_{M}^{j}(x)|^{2} d_s x \\
&= \kappa_{M}^{2} \frac{\sigma_{n-k-1}\sigma_{k}}{\sigma_{n}} \int_{S^k} |Y_{r,\mu}^{k}(\eta)|^{2} d_s \eta \int_{S^{n-k-1}} |Y_{s,\nu}^{n-k-1}(\zeta)|^{2} d_s \zeta \\
&\times \int_{0}^{\pi/2} \sin^{2r+k} \theta \cos^{2s+n-k-1} \theta |R_{m}^{(\rho,\sigma)}(\cos 2\theta)|^{2} d\theta \\
&= \kappa_{M}^{2} \frac{\sigma_{n-k-1}\sigma_{k}}{\sigma_{n}} I;
\end{align*}
\[
I = \int_0^{\pi/2} \sin^{2r+k} \theta \cos^{2s-n-k-1} \theta |R_m^{(\phi,\sigma)}(\cos 2\theta)|^2 d\theta.
\]

Changing variables and using (2.22), we obtain
\[
I = \frac{m! \Gamma^2(\rho + 1) \Gamma(m + \sigma + 1)}{2(2m + \rho + \sigma + 1) \Gamma(m + \rho + 1) \Gamma(m + \rho + \sigma + 1)}.
\]

Both orthonormal bases \( \mathcal{Y} = \{Y^n_{j,\lambda}(x)\} \) and \( \mathcal{U} = \{U^n_M(x)\} \) will be needed in the next sections.

3. Intertwining Operators. Main results

Consider a dual pair of integral operators of the form
\[
(Af)(v) = \int_{S^n} a(|x^T v|) f(x) d_s x, \quad v \in V_{n+1,n-k},
\]
(3.1)
\[
(A^* \varphi)(x) = \int_{V_{n+1,n-k}} a(|x^T v|) \varphi(v) d_s v, \quad x \in S^n,
\]
(3.2)
which intertwine the action of the orthogonal group \( O(n+1) \) on \( S^n \) and \( V_{n+1,n-k} \). Here \( 1 \leq k \leq n-1 \) and \( a \) is a function on \([0,1]\).

Clearly, \((Af)(v\gamma) = (Af)(v)\) for all \( \gamma \in O(n-k) \), and therefore \((Af)(v)\) can be viewed as a function on the Grassmannians or on the space of \( k \)-geodesics. Specifically,
\[
(Af)(v) \equiv \begin{cases}
(A_1 f)(v^\perp), & v^\perp \in G_{n+1,k+1}, \\
(A_2 f)(\{v\}), & \{v\} \in G_{n+1,n-k}, \\
(A_3 f)(S^n \cap v^\perp), & S^n \cap v^\perp \in \Xi.
\end{cases}
\]
(3.3)

**Lemma 3.1.** Suppose that the integrals (3.1) and (3.2) are absolutely convergent. Then
\[
(Af)(v) = c_{n,k} \int_0^1 a(\tau)(M_\tau f)(v) \rho(\tau) d\tau,
\]
(3.4)
\[
(A^* \varphi)(x) = c_{n,k} \int_0^1 a(\tau)(M^*_\tau \varphi)(x) \rho(\tau) d\tau,
\]
(3.5)
where
\[
\rho(\tau) = (1 - \tau^2)^{(k-1)/2} \tau^{n-k-1}, \quad c_{n,k} = \frac{\sigma_{n-k-1} \sigma_k}{\sigma_n}.
\]
(3.6)
Proof. Since \((Af)(v) = \int_{S^n} a(|x^Tv_0|)f_v(x)dx,\) (3.4) follows from (2.7).
Furthermore,
\[
(A^*\varphi)(x) = \int_{V_{n+1,n-k}} a(|e_n^Tv|)\varphi_x(v)dv
\]
(3.7) \[= \int_G a(|e_n^Tv_0|)\varphi_x(gv_0)dg, \quad G = SO(n+1).\]

Replace \(g\) by \(\delta g^{-1}, \delta \in K = SO(n),\) and integrate in \(\delta.\) Then
\[
(A^*\varphi)(x) = \int_G a((ge_{n+1})^Tv_0)\omega(g)dg, \quad \omega(g) = \int_K \varphi_x(\delta g^{-1}v_0)d\delta.
\]

Since \(\omega(gK) = \omega(g),\) one can write \(\omega(g) = \Omega(ge_{n+1}),\) where \(\Omega\) is a function on \(S^n = G/K.\) Hence
\[
(A^*\varphi)(x) = \int_{S^n} a(|y^Tv_0|)\Omega(y)dy = (A\Omega)(v_0)
\]
\[= c_{n,k} \int_0^1 a(\tau)(M_\tau\Omega)(v_0)\rho(\tau) d\tau.
\]

Setting \(\tau = \cos \theta\) and using (2.9), we have
\[
(M_{\cos \theta}\Omega)(v_0) = \int_{K'} \Omega(\gamma g_{k+1,n+1}(\theta)e_{n+1})d\gamma
\]
\[= \int_{K'} \omega(\gamma g_{k+1,n+1}(\theta))d\gamma.
\]

The last integral coincides with (2.10), and the result follows. \(\square\)

3.1. Norm estimates. In the following, \(G = SO(n+1),\) \(\| \cdot \|_p\) and \(\| \cdot \|_p\) denote the \(L^p\)-norms of functions on \(V_{n+1,n-k}\) and \(S^n,\) respectively.

Lemma 3.2. For all \(1 \leq p \leq \infty\) and \(\tau = \cos \theta \in [0, 1],\)
\[
\|M_\tau f\|_p \leq \|f\|_p, \quad \|M_\tau^* \varphi\|_p \leq \|\varphi\|_p.
\]

Proof. Let \(F = M_\tau f.\) By (2.9),
\[
\|F\|_p = \|F(gv_0)\|_{L^p(G)} \leq \int_{K'} \left( \int_G |f(\gamma g_{k+1,n+1}(\theta)e_{n+1})|^p dg \right)^{1/p} d\gamma = \|f\|_p.
\]

The second inequality in (3.8) follows by duality (2.13). \(\square\)
Operators (3.1) and (3.2) are represented as convolutions on \( G \). Specifically, setting
\[
\tilde{f}(g) = f(ge_{n+1}), \quad \tilde{a}(g) = a(|e_{n+1}^T g v_0|), \quad g \in G,
\]
\[
\tilde{\varphi}(\gamma) = \varphi(\gamma v_0), \quad \tilde{a}^*(\gamma) = a(|(\gamma e_{n+1})^T v_0|), \quad \gamma \in G,
\]
we have
\[
(Af)(\gamma v_0) = \int_G \tilde{f}(\gamma g^{-1}) \tilde{a}(g) \, dg, \quad (A^* \varphi)(ge_{n+1}) = \int_G \tilde{\varphi}(g \gamma^{-1}) \tilde{a}^*(\gamma) \, d\gamma.
\]

**Lemma 3.3.** Let \( 1 \leq p \leq q \leq \infty, \quad 1 - p^{-1} + q^{-1} = r^{-1} \). Then
\[
\|Af\|_q \leq c_{n,k}^{1/r} \|f\|_p \|a\|_{r,\rho}, \quad \|A^* \varphi\|_q \leq c_{n,k}^{1/r} \|\varphi\|_p \|a\|_{r,\rho},
\]
where
\[
\|a\|_{r,\rho} = \left( \int_0^1 |a(\tau)|^r \rho(\tau) \, d\tau \right)^{1/r},
\]
\( \rho(\tau) \) and \( c_{n,k} \) being defined by (3.6).

**Proof.** The statement follows from Young’s inequality on \( G \) [12, Chapter 5, Theorem 20.18] if we notice that
\[
\|\tilde{f}\|_{L^p(G)} = \|f\|_p, \quad \|\tilde{\varphi}\|_{L^p(G)} = \|\varphi\|_p,
\]
\[
\|\tilde{a}\|_{L^r(G)} = \|\tilde{a}^*\|_{L^r(G)} = c_{n,k}^{1/r} \|a\|_{r,\rho}.
\]

### 3.2. The Funk-Hecke type theorems.

Below we introduce special orthonormal systems of functions on Stiefel and Grassmann manifolds. These systems are generated by spherical harmonics on \( S^n \). An analogue of the Funk-Hecke formula, leading to multiplier representation of the intertwining operators (3.1) and (3.2), is obtained.

To start with, we evaluate bispherical means of spherical harmonics. By (3.2), \((M_r Y_j)(v) = (M_r Y_j)_r(v_0)\). Let us decompose the function \((Y_j)_r(x) = Y_j(r v, x)\) in the orthonormal basis \( \mathcal{U} = \{U_M^j(x)\} \) according to (2.25). We obtain
\[
Y_j(r v, x) = \sum_M \alpha_M^j(v) U_M^j(x), \quad \alpha_M^j(v) = \int_{S^n} Y_j(r v, x) U_M^j(x) \, ds, \quad M = (r, \mu; s, \nu; m), \quad j = 2m + r + s.
\]

Thus,
\[
(M_r Y_j)(v) \equiv (M_r Y_j)_r(v_0) = \sum_M (M_r U_M^j)(v_0) \alpha_M^j(v),
\]
\[
(3.9) \quad Y_j(r v, x) = \sum_M \alpha_M^j(v) U_M^j(x), \quad \alpha_M^j(v) = \int_{S^n} Y_j(r v, x) U_M^j(x) \, ds, \quad M = (r, \mu; s, \nu; m), \quad j = 2m + r + s.
\]

\[
(3.10) \quad (M_r Y_j)(v) \equiv (M_r Y_j)_r(v_0) = \sum_M (M_r U_M^j)(v_0) \alpha_M^j(v),
\]
where, by (2.3) and (2.25),
\[
(M_{\tau}U_{M}^{j})(v_{0}) = \kappa_{M}(1 - \tau^{2})^{r/2} \tau^{s} R_{m}^{(\rho,\sigma)}(2\tau^{2} - 1)
\times \int_{S^{n-k-1}} Y_{n-k-1}^{r,s}(\zeta) d_{s} \zeta \int_{S^{k}} Y_{r,s}(\eta) d_{s} \eta.
\]
The last two integrals are zero, unless \( s = 0 \) and \( r = 0 \). If \( r = 0 \) and \( s = 0 \), we have
\[
\mu = 1, \quad \nu = 1, \quad M = (0, 1; 0, 1; m),
\]
and \( \kappa_{M} = \kappa_{j} \), where
\[
(3.11) \quad \rho = (k - 1)/2, \quad \sigma = (n - k)/2 - 1, \quad j = 2m,
\]
and \( \kappa_{j} = \sqrt{d_{n}(j)} \). Hence, for \( \tau = \cos \theta \),
\[
(M_{\tau}Y_{j})(v_{0}) = \kappa_{j} R_{j/2}^{(\rho,\sigma)}(\cos 2\theta) = \kappa_{j} R_{j/2}^{(\rho,\sigma)}(2\tau^{2} - 1)
\]
and
\[
(3.12) \quad \kappa_{j} = \sqrt{\frac{\sigma_{n}(2j + n - 1) \Gamma\left(\frac{j + k + 1}{2}\right) \Gamma\left(\frac{j + n - 1}{2}\right)}{\sigma_{n-k-1} \sigma_{k} \Gamma^{2}\left(\frac{k + 1}{2}\right) \Gamma\left(\frac{j}{2} + 1\right) \Gamma\left(\frac{j + n - k}{2}\right)}}^{1/2}
\]
(it is worth noting that if \( k = n - 1 \), then (3.12) gives \( \kappa_{j}^{2} = d_{n}(j) \); cf. (2.14)). Hence, for \( \tau = \cos \theta \),
\[
(M_{\tau}Y_{j})(v_{0}) = \kappa_{j} R_{j/2}^{(\rho,\sigma)}(\cos 2\theta) = \kappa_{j} R_{j/2}^{(\rho,\sigma)}(2\tau^{2} - 1)
\]
and
\[
(3.13) \quad \hat{Y}_{j}(v) = \kappa_{j} \int_{S^{n}} Y_{j}(r_{v}, x) R_{j/2}^{(\rho,\sigma)}(2|x^{T}v|^{2} - 1) d_{s} x
\]
\[
= \kappa_{j} \int_{S^{n}} Y_{j}(x) R_{j/2}^{(\rho,\sigma)}(2|x^{T}v|^{2} - 1) d_{s} x.
\]
Taking into account that the sum in (3.10) has only one term, we obtain
\[
(M_{\tau}Y_{j})(v) = \kappa_{j}^{2} R_{j/2}^{(\rho,\sigma)}(2\tau^{2} - 1) \int_{S^{n}} Y_{j}(x) R_{j/2}^{(\rho,\sigma)}(2|x^{T}v|^{2} - 1) d_{s} x.
\]
For technical reasons, it is convenient to set
\[
(3.14) \quad \hat{Y}_{j}(v) = \alpha_{j} \int_{S^{n}} Y_{j}(x) R_{j/2}^{(\rho,\sigma)}(2|x^{T}v|^{2} - 1) d_{s} x, \quad \alpha_{j} = \kappa_{j} \sqrt{d_{n}(j)}
\]
(the role of the coefficient \( \alpha_{j} \) will be clarified later) and
\[
(3.15) \quad \hat{m}_{r}(j) = \frac{\kappa_{j}}{\sqrt{d_{n}(j)}} R_{j/2}^{(\rho,\sigma)}(2\tau^{2} - 1).
\]
This gives the following statement.

**Theorem 3.4.** Let \( v \in V_{n+1,n-k}, 1 \leq k \leq n-1 \). For any spherical harmonic \( Y_j(x) \) of degree \( j \) on \( S^n \) and any \( \tau \in [0,1] \),

\[
(3.15) \quad (M_\tau Y_j)(v) = \begin{cases} 0 & \text{if } j \text{ is odd}, \\ \hat{m}_\tau(j) \hat{Y}_j(v) & \text{if } j \text{ is even}. \end{cases}
\]

An analogue of (3.15) for the operator (3.1) follows from (3.4).

**Theorem 3.5.** Let \( x \in S^n, v \in V_{n+1,n-k}, 1 \leq k \leq n-1 \). If

\[
(3.16) \quad \int_0^1 |a(\tau)| \rho(\tau) d\tau < \infty, \quad \rho(\tau) = (1 - \tau^2)^{(k-1)/2} \tau^{n-k-1},
\]

then for any spherical harmonic \( Y_j(x) \) of degree \( j \),

\[
(3.17) \quad (AY_j)(v) \equiv \int_{S^n} a(|x^T v|) Y_j(x) d_\ast x = \begin{cases} 0 & \text{if } j \text{ is odd}, \\ \hat{a}(j) \hat{Y}_j(v) & \text{if } j \text{ is even}, \end{cases}
\]

where

\[
(3.18) \quad \hat{a}(j) = c_{n,k,j} \int_0^1 R_j^{(\rho,\sigma)} (2\tau^2 - 1) a(\tau) \rho(\tau) d\tau, \quad c_{n,k,j} = \frac{\kappa_j \sigma_{n-k-1} \sigma_k}{\sigma_n \sqrt{d_n(j)}}.
\]

**Definition 3.6.** The function \( \hat{Y}_j(v) \), generated by the spherical harmonic \( Y_j(x) \) according to (3.13), will be called the induced Stiefel harmonic. Because \( \hat{Y}_j(v) \) is right \( O(n-k) \)-invariant, it can be regarded as a harmonic on the space \( \Xi \) of \( k \)-geodesics or on the Grassmannian \( G_{n+1,k+1} \). Following standard terminology in harmonic analysis, we call \( \hat{a}(j) \) and \( \hat{m}_\tau(j) \) the multipliers of the respective operators \( A \) and \( M_\tau \).

As we shall see below, the induced Stiefel harmonics have a number of remarkable properties, similar to those of usual spherical harmonics.

A routine calculation shows that in the case \( k = n-1 \), Theorems 3.4 and 3.5 agree with the corresponding statements for spherical harmonics in [21, Appendix]. In particular, Theorem 3.5 resembles the classical Funk-Hecke theorem; cf. [21, Theorem A.34].

### 3.3. Properties of the induced Stiefel harmonics.

Let \( \mathcal{Y} = \{ Y_{j,\lambda}(x) \} \) be an orthonormal basis of spherical harmonics in \( L^2(S^n) \). For \( j \) even, we denote

\[
(3.19) \quad \hat{Y}_{j,\lambda}(v) = \alpha_j \int_{S^n} Y_{j,\lambda}(x) R_j^{(\rho,\sigma)} (2|x^T v|^2 - 1) d_\ast x, \quad \alpha_j = \kappa_j \sqrt{d_n(j)},
\]

where, as in (3.13), \( \rho = (k-1)/2, \sigma = (n-k)/2 - 1 \).
**Lemma 3.7.** Let \( j \in \{0, 2, 4, \ldots \} \). The following addition formula holds:

\[
R^{(\rho, \sigma)}_{j/2}(2|x^T v|^2 - 1) = \alpha_j^{-1} \sum_{\lambda=1}^{d_n(j)} Y_{j,\lambda}(x) \hat{Y}_{j,\lambda}(v).
\]

**Proof.** Let us write (3.20) as \( I(x) = J(x) \), assuming \( v \) fixed. The statement will be proved if we establish the coincidence of the Fourier-Laplace coefficients of \( I(x) \) and \( J(x) \), that is,

\[
\int_{S^n} Y_{i,\mu}^n(x) I(x) dx = \int_{S^n} Y_{i,\mu}^n(x) J(x) dx
\]

for any harmonic \( Y_{i,\mu}^n \in \mathcal{Y} \), not necessarily even. Let us show that

\[
\text{l.h.s.} \equiv \int_{S^n} Y_{i,\mu}^n(x) I(x) dx = \begin{cases} \alpha_j^{-1} \hat{Y}_{j,\mu}(v) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
\]

If \( i = j \), this equality holds by definition (3.19) (set \( \mu = \lambda \)). If \( i \neq j \) we write out (3.17) with \( j \) replaced by \( i \), that is,

\[
(AY_i)(v) \equiv \int_{S^n} a(|x^T v|) Y_i(x) dx = \begin{cases} 0 & \text{if } i \text{ is odd}, \\ \hat{a}(i) \hat{Y}_i(v) & \text{if } i \text{ is even}, \end{cases}
\]

and then set \( a(|x^T v|) = R^{(\rho, \sigma)}_{i/2}(2|x^T v|^2 - 1), Y_i = Y_{i,\mu}^n \). This gives

\[
\text{l.h.s.} \equiv \int_{S^n} Y_{i,\mu}^n(x) I(x) dx = \hat{a}(i) Y_{i,\mu}^n(v) = 0
\]

because, by (3.18),

\[
\hat{a}(i) = c_{n,k,i} \int_0^1 (1 - \tau^2)^{(k-1)/2} \tau^{n-k-1} R^{(\rho, \sigma)}_{i/2}(2\tau^2 - 1) \, R^{(\rho, \sigma)}_{j/2}(2\tau^2 - 1) \, d\tau
\]

\[
= c_{n,k,i} \int_{-1}^1 (1 - s)^{\rho}(1 + s)^{\sigma} R^{(\rho, \sigma)}_{i/2}(s) R^{(\rho, \sigma)}_{j/2}(s) ds = 0
\]

due to orthogonality of Jacobi polynomials; cf. [22,23].

For the right-hand side we have

\[
\text{r.h.s.} \equiv \int_{S^n} Y_{i,\mu}^n(x) J(x) dx = \alpha_j^{-1} \sum_{\nu=1}^{d_n(j)} \hat{Y}_{j,\nu}(v) \int_{S^n} Y_{i,\mu}^n(x) Y_{j,\nu}(x) dx = 0.
\]

This completes the proof. \( \square \)
Consider the set of all induced Stiefel harmonics (3.19) and recall the notation \( \mathcal{Y} = \{ \hat{Y}_{j,\lambda} : j = 0, 2, \ldots; \lambda = 1, 2, \ldots d_n(j) \} \), \( d_n(j) \) being defined by (2.14).

**Lemma 3.8.** The set \( \mathcal{Y} \) is orthonormal, that is,

\[
\int_{V_{n+1,n-k}} \hat{Y}_{j,\lambda}(v) \hat{Y}_{j',\lambda'}(v) dv = \begin{cases} 1 & \text{if } j = j' \text{ and } \lambda = \lambda', \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** We denote by \( I \) the left-hand side of (3.23) and make use of (3.19). Changing the order of integration, we obtain

\[
I = \int_{S^n} Y_{j,\lambda}(x) \int_{S^n} Y_{j',\lambda'}(y) \Omega_{j,j'}(x, y) dy dx,
\]

\[
\Omega_{j,j'}(x, y) = \alpha_j \alpha_{j'} \int_{V_{n+1,n-k}} R_{j/2}^{(\rho,\sigma)} (2|x^T v|^2 - 1) R_{j'/2}^{(\rho,\sigma)} (2|y^T v|^2 - 1) dv.
\]

By the rotation invariance, \( \Omega_{j,j'}(x, y) \) is a single-variable function of \( x \cdot y \). Abusing notation, we set \( \Omega_{j,j'}(x, y) \equiv \Omega_{j,j'}(x \cdot y) \). Then the Funk-Hecke formula (2.16) yields

\[
I = \omega_{j,j'} \int_{S^n} Y_{j,\lambda}(x) Y_{j',\lambda'}(x) dx = \omega_{j,j'} \delta_{jj'} \delta_{\lambda\lambda'},
\]

\[
\omega_{j,j'} = \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^1 P_j(t) \Omega_{j,j'}(t) (1 - t^2)^{n/2-1} dt.
\]

To complete the proof, we need to justify the equality

\[
\omega_{j,j} = 1.
\]

First, let us show that

\[
\Omega_{j,j}(t) = \frac{\alpha^2_j}{x_j^2} P_j(t).
\]

By (3.25) and (3.20),

\[
\Omega_{j,j}(x \cdot y) = \alpha_j \sum_{\lambda=1}^{d_n(j)} Y_{j,\lambda}(x) \int_{V_{n+1,n-k}} \hat{Y}_{j,\lambda}(v) R_{j/2}^{(\rho,\sigma)} (2|y^T v|^2 - 1) dv.
\]

It follows that \( \Omega_{j,j}(x \cdot y) \) is a spherical harmonic of degree \( j \) in the \( x \)-variable for each \( y \in S^n \), and therefore, \( \Omega_{j,j}(t) \) is a constant multiple
of the spherical polynomial, i.e., $\Omega_{j,j}(t) = A_j P_j(t) A_j = \text{const}$; see, e.g., [21, Lemma A.26].

To find $A_j$, we set $x = y = e_{n+1}$ and $i = j$ in (3.25). This gives

$$\Omega_{j,j}(1) = A_j P_j(1) = \alpha_j^2 \int_{V_{n+1,n-k}} [R_{j/2}^{(\rho,\sigma)}(2|x^Tv|^2 - 1)]^2 d_*v.$$ 

Taking into account that $P_j(1) = 1$ and using (3.5) with $\phi = 1$ and $x = e_{n+1}$, we obtain

$$A_j = \alpha_j^2 c_{n,k} \int_0^1 (1 - \tau^2)^{(k-1)/2} \tau^{n-k-1} [R_{j/2}^{(\rho,\sigma)}(2\tau^2 - 1)]^2 (M^*_1)(e_{n+1}) d\tau,$$

where $(M^*_1)(e_{n+1}) = 1$, $c_{n,k} = \sigma_{n-k-1}\sigma_k/\sigma_n$. Furthermore, from (2.22) and (3.12) we have

$$\int_0^1 (1 - \tau^2)^{(k-1)/2} \tau^{n-k-1} [R_{j/2}^{(\rho,\sigma)}(2\tau^2 - 1)]^2 d\tau$$

$$= \frac{\Gamma^2 \left( \frac{k + 1}{2} \right) \Gamma \left( \frac{j}{2} + 1 \right) \Gamma \left( \frac{j + n - k}{2} \right)}{(2j + n - 1) \Gamma \left( \frac{j + k + 1}{2} \right) \Gamma \left( \frac{j + n - 1}{2} \right)} = \frac{1}{x_j^2 c_{n,k}}.$$

Hence $A_j = \alpha_j^2/x_j^2$, and (3.28) follows.

Now (3.26) and (2.19) yield

$$\omega_{j,j} = \frac{\sigma_{n-1}}{\sigma_n \alpha_j^2} \int_{-1}^1 [P_j(t)]^2 (1 - t^2)^{n/2 - 1} dt = \frac{\alpha_j^2}{x_j^2 d_n(j)} = 1.$$ 

This completes the proof. $\square$

**Corollary 3.9.** If the Stiefel harmonic $\tilde{Y}_j$ is induced by the spherical harmonic $Y_j$ of even degree $j$, then

$$Y_j(x) = \alpha_j \int_{V_{n+1,n-k}} \tilde{Y}_j(v) R_{j/2}^{(\rho,\sigma)}(2|x^Tv|^2 - 1)d_*v, \quad \alpha_j = x_j \sqrt{d_n(j)}.$$ 

**Proof.** By the addition formula (3.20), the right-hand side of (3.30) is a spherical harmonic of degree $j$. Hence it suffices to show that

$$Y_{j,\lambda}^\alpha(x) = \alpha_j \int_{V_{n+1,n-k}} \tilde{Y}_j,\lambda(v) R_{j/2}^{(\rho,\sigma)}(2|x^Tv|^2 - 1)d_*v$$ 

(3.31)
for all basic harmonics \( Y_{n,j}^\lambda(x), \lambda \in \{1, 2, \ldots, d_n(j)\} \). To this end, we represent \( R_{j/2}^{(\rho,\sigma)}(2|x^Tv|^2-1) \) by \( 3.20 \) and make use of the orthogonality \( 3.23 \). This gives
\[
\text{r.h.s.} = \sum_{\lambda' = 1}^{d_n(j)} Y_{n,j}^\lambda(x) \int_{V_{n+1,n-k}} \hat{Y}_{j,\lambda}(v) \hat{Y}_{j,\lambda'}(v) d_s v = Y_{n,j,\lambda}(x),
\]
as desired. \( \square \)

3.4. The Dual Statements.
The next Funk-Hecke type statement is dual to Theorem \( 3.5 \).

**Theorem 3.10.** If the Stiefel harmonic \( \hat{Y}_j \) is induced by the spherical harmonic \( Y_j \) of even degree \( j \) and \( a(\tau) \) satisfies \( 3.16 \), then
\[
(3.32) \quad \int_{V_{n+1,n-k}} a(|x^Tv|) \hat{Y}_j(v) d_s v = \hat{a}(j) Y_j(x),
\]
\( \hat{a}(j) \) being the multiplier \( 3.18 \).

**Proof.** As above, it suffices to show that
\[
(3.33) \quad \int_{V_{n+1,n-k}} a(|x^Tv|) \hat{Y}_{j,\lambda}(v) d_s v = \hat{a}(j) Y_{n,j,\lambda}(x),
\]
for all basic harmonics \( Y_{n,j,\lambda}(x), \lambda \in \{1, 2, \ldots, d_n(j)\} \). To prove \( 3.33 \), we evaluate the Fourier-Laplace coefficients of both sides. Let \( Y_{i,\mu}^n \) be an arbitrary spherical harmonic belonging to the orthonormal basis \( \mathcal{Y} \) of \( L^2(S^n) \). Changing the order of integration, owing to \( 3.17 \) and \( 3.23 \), we have
\[
\int_{S^n} Y_{i,\mu}^n(x) d_s x \int_{V_{n+1,n-k}} a(|x^Tv|) \hat{Y}_{j,\lambda}(v) d_s v = \hat{a}(i) \delta_{ij} \delta_{\mu\lambda}.
\]
Since the right-hand side has the same Fourier-Laplace coefficients, the result follows. \( \square \)

The following theorem is dual to Theorem \( 3.4 \).

**Theorem 3.11.** If the Stiefel harmonic \( \hat{Y}_j(v) \) is induced by the spherical harmonic \( Y_j(x) \) of even degree \( j \), then
\[
(3.34) \quad (M^*_\tau \hat{Y}_j)(x) = \hat{m}(j) Y_j(x),
\]
where \( \hat{m}(j) \) is the multiplier \( 3.14 \).
Proof. It suffices to prove (3.34) on basic harmonics, that is,

\[(3.35) \quad (M_r^* \hat{Y}_{j,\lambda})(x) = \hat{m}_r(j) Y_{j,\lambda}(x), \quad Y_{j,\lambda}^n \in \mathcal{Y}.\]

For any spherical harmonic \(Y_{j',\lambda'}^n \in \mathcal{Y}\), owing to (2.13), (3.15) and (3.8), we have

\[
\int_{S^n} Y_{j',\lambda'}^n(x)(M_r^* \hat{Y}_{j,\lambda})(x) \, d_s x = \int_{V_{n+1,n-k}} \hat{Y}_{j,\lambda}(v)(M_r Y_{j',\lambda'})(v) \, d_s v
\]

\[= \hat{m}_r(j) \int_{V_{n+1,n-k}} \hat{Y}_{j,\lambda}(v) \hat{Y}_{j',\lambda'}(v) \, d_s v = \hat{m}_r(j) \delta_{jj'} \delta_{\lambda\lambda'}.\]

Similarly, for the right-hand side of (3.35) we have

\[
\hat{m}_r(j) \int_{S^n} Y_{j',\lambda'}^n(x) Y_{j,\lambda}^n(x) \, d_s x = \hat{m}_r(j) \delta_{jj'} \delta_{\lambda\lambda'}.\]

Thus all the Fourier-Laplace coefficients of the both sides of (3.35) coincide, and the result follows. \(\square\)

3.5. Examples.

Example 3.12. Consider the Funk-Radon transforms

\[(R f)(S^n \cap v^\perp) = (M_0 f)(v), \quad (R^* \varphi)(x) = (M_0^* \varphi)(x),\]

where \(v \in V_{n+1,n-k}, x \in S^n\); cf. (2.12). Setting \(\tau = 0\) in (3.15) and (3.34), we compute the multiplier \(\hat{m}_j = \hat{m}_0(j)\) of these operators. Specifically, by (3.14),

\[
\hat{m}_j = \frac{x_j}{d_n(j)} R_{j/2}^{(\rho,\sigma)}(-1).
\]

Here, by (2.21),

\[
R_{j/2}^{(\rho,\sigma)}(-1) = \frac{P_{j/2}^{(\rho,\sigma)}(-1)}{P_{j/2}^{(\rho,\sigma)}(1)} = \frac{(-1)^{j/2} P_{j/2}^{(\sigma,\rho)}(1)}{P_{j/2}^{(\rho,\sigma)}(1)} = (-1)^{j/2} \frac{\Gamma \left( \frac{k + 1}{2} \right) \Gamma \left( \frac{j + n - k}{2} \right)}{\Gamma \left( \frac{n - k}{2} \right) \Gamma \left( \frac{j + k + 1}{2} \right)}.\]
Hence, by (3.12) and (2.14), a simple calculation yields
\[
\hat{m}_j = (-1)^j \delta_{n,k} \left[ \frac{\Gamma\left(\frac{j + n - k}{2}\right)}{\Gamma\left(\frac{j + n}{2}\right)} \frac{\Gamma\left(\frac{j + k + 1}{2}\right)}{\Gamma\left(\frac{j + k}{2}\right)} \right]^{1/2},
\]
\[
\delta_{n,k} = \left[ \frac{\Gamma\left(\frac{k + 1}{2}\right)}{\Gamma\left(\frac{n - k}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{1/2}} \right]^{1/2}.
\]
In particular, if \(k = n - 1\), then
\[
\hat{m}_j = (-1)^{j/2} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{j + 1}{2}\right)}{\pi^{1/2} \Gamma\left(\frac{j + n}{2}\right)}.
\]
This expression agrees with the known Fourier-Laplace multiplier of the Funk transform on \(S^{n-1}\); cf. [21, formula (5.1.3)].

**Example 3.13.** The intertwining operator (3.1) with the kernel \(a(t) = t^{\alpha-n+k}, \, \text{Re} \, \alpha > 0\), is a constant multiple of the generalized cosine transform in integral geometry [18, 19]. Let
\[
(A_{\alpha} f)(v) = \int_{S^n} |x^T v|^{\alpha-n+k} f(x) \, d_\ast x, \quad v \in V_{n+1,n-k},
\]
\[
(A_{\alpha}^* \varphi)(x) = \int_{V_{n+1,n-k}} |x^T v|^{\alpha-n+k} \varphi(v) \, d_\ast v, \quad x \in S^n.
\]
By (3.18), the multiplier of these operators is
\[
\hat{a}_\alpha(j) = c_{n,k,j} \int_0^1 (1 - \tau^2)^{(k-1)/2} \tau^{\alpha-1} R_{j/2}^{(\rho,\sigma)}(2\tau^2 - 1) \, d\tau.
\]
This integral can be evaluated using [13 2.22.2(9)] and the properties of Jacobi polynomials (we skip the routine calculations). The result is
\[
\hat{a}_\alpha(j) = (-1)^{j/2} \delta_{\alpha,n,k} \hat{m}_\alpha(j),
\]
where
\[
\delta_{\alpha,n,k} = \left[ \frac{\sigma_{n-k-1} \sigma_k \Gamma(n)}{2^n \sigma_n} \right]^{1/2} \frac{\Gamma(\alpha/2)}{\Gamma((n - k - \alpha)/2)}.
\]
\[ \dot{m}_\alpha(j) = \left[ \frac{\Gamma \left( \frac{j + k + 1}{2} \right)}{\Gamma \left( \frac{j + n - k - \alpha}{2} \right) \Gamma \left( \frac{j + 1}{2} \right)} \right]^{1/2} \frac{\Gamma \left( \frac{j + 1}{2} \right)}{\Gamma \left( \frac{j + n - k + \alpha + 1}{2} \right)} \]

In the case \( k = n - 1 \), this expression agrees (up to notation) with the known Fourier-Laplace multiplier of the \( \alpha \)-cosine transform on \( S^{n-1} \); cf. [21, formula (5.1.9)]. If \( j \to \infty \), then \( \dot{a}_\alpha(j) = O(j^{-k/2-\alpha}) \).

3.6. **Proof of Theorem 1.1.**

(i) Let \( \tau = \sin t \). By (2.6), it suffices to prove injectivity of the mapping

\[ L^1_{\text{even}}(S^n) \ni f \longrightarrow M_\tau f \in L^1(V_{n+1,n-k}). \]

Denote \( \mathcal{P}_j(\tau) = R_{j/2}^{(\rho,\sigma)}(2\pi^2 - 1) \) and suppose that \( M_\tau \) is injective, i.e., \( M_\tau f = 0 \) implies \( f = 0 \) a.e. for every \( f \in L^1_{\text{even}}(S^n) \). Assuming the contrary, that is, \( \mathcal{P}_j(\tau) = 0 \) for some \( j = j_0 \in \{0, 2, 4, \ldots\} \), we obtain that \( M_\tau Y_{j_0} = c \mathcal{P}_j(\tau) Y_{j_0} \equiv 0 \) for every spherical harmonic \( Y_{j_0} \) on \( S^n \). Hence, by the injectivity assumption, \( Y_{j_0} \equiv 0 \), which gives a contradiction.

Conversely, suppose that \( \mathcal{P}_j(\tau) \neq 0 \) for all \( j \in \{0, 2, 4, \ldots\} \), and let \( M_\tau f = 0 \) for some \( f \in L^1_{\text{even}}(S^n) \). Then for every spherical harmonic \( Y_j \) on \( S^n \) with \( j \) even and the corresponding Stiefel harmonic \( \hat{Y}_j \), owing to (2.13) and (3.34), we obtain

\[
0 = \int_{V_{n+1,n-k}} (M_\tau f)(v) \hat{Y}_j(v) d_\nu v = \int_{S^n} f(x)(M_\tau^* \hat{Y}_j)(x) d_\nu x
\]

(3.40) \[ = c \mathcal{P}_j(\tau) \int_{S^n} f(x) Y_j(x) d_\nu x. \]

Because \( \mathcal{P}_j(\tau) \neq 0 \), it follows that all the Fourier-Laplace coefficients of \( f \) are zero. Hence \( f(x) = 0 \) for almost all \( x \in S^n \) (use, e.g., [21, Proposition A.18]). Now the statement of Theorem 1.1 follows if we set

\[
R_{j/2}^{(\rho,\sigma)}(2\pi^2 - 1) = R_{j/2}^{(\rho,\sigma)}(2\sin^2 t - 1) = (-1)^{j/2} R_{j/2}^{(\rho,\sigma)}(\cos 2t).
\]

(ii) More generally, let \( \tau_i = \sin t_i \). Then \( R_{t_i} f = M_{\tau_i} f \) and the system (1.3) is equivalent to

\[
\mathcal{P}_j(\tau_i) = 0, \quad i = 1, 2, \ldots, \ell.
\]

If these equations have no common solution in the \( j \)-variable, then for any \( j \in \{0, 2, 4, \ldots\} \) there exists at least one \( i = i(j) \) such that
$\mathcal{P}_j(\tau_{i(j)}) \neq 0$. If $M_{\tau_i} f = 0$ for all $i = 1, 2, \ldots, \ell$, then, in particular, $M_{\tau_{i(j)}} f = 0$ and, as in (3.40),

$$
\mathcal{P}_j(\tau_{i(j)}) \int_{S^n} f(x) Y_j(x) d_s x = 0.
$$

This implies $\int_{S^n} f(x) Y_j(x) d_s x = 0$. Because $j \in \{0, 2, 4, \ldots\}$ is arbitrary, it follows that $f = 0$ a.e. on $S^n$.

Conversely, if the equations (3.42) have a common solution, say, $j = j_0$, then, as above, for any spherical harmonic $Y_{j_0}$ and all $i = 1, 2, \ldots, \ell$, we have

$$
M_{\tau_i} Y_{j_0} = c \mathcal{P}_{j_0}(\tau_i) \hat{Y}_{j_0} \equiv 0.
$$

This completes the proof if we take into account (3.41). □.

4. Conclusion

Some comments are in order.

1. The purpose of the paper was two-fold. On the one hand, it would be interesting to investigate injectivity of the shifted Radon transforms on an arbitrary constant curvature space $X$. This setting of the problem extends the well known consideration of spherical means with center at a point to the case of a 'multidimensional center'. To start with, we restricted to the case $X = S^n$ and obtained necessary and sufficient conditions of injectivity of the shifted Funk-Radon transform on $L^1(S^n)$. The cases, when $X$ is the Euclidean or hyperbolic space, are left for the future.

On the other hand, our study needs a suitable harmonic analysis, which makes a bridge between functions on the sphere and functions on the Stiefel (or Grassmann) manifolds. This analysis is of independent interest and has many aspects. We considered only some of them, which are related to the induced orthonormal systems, the corresponding Funk-Hecke type theorems, the addition formula, and multipliers.

It is natural to conjecture that our consideration paves the way to further investigations. The corresponding theory for translation invariant linear operators in $\mathbb{R}^n$ is well known; see, e.g., Hörmander [14], Stein and Weiss [26], Grafakos [9]. Operators on the unit sphere commuting with rotations were studied by Coifman and Weiss [4], Dunkl [7], Rubin [21], Sections A.10 - A.13, Samko [23], to mention a few.

2. Our formula (3.36) for the multiplier of the Funk-Radon transform differs from that suggested by Strichartz [28, 29]. Nevertheless, multipliers in (3.36) and in [28] have the same order $O(j^{-k/2})$ as $j \to \infty$. 
Strichartz’s approach relies on his previous group-theoretic considerations in [27]; cf. formulas (4.2) and (4.4) in [28], where the multipliers of the Funk-Radon transform and its dual have different analytic expression. Our approach is essentially different, self-contained, and invokes Jacobi polynomials. It is applicable to more general intertwining operators, yields the relevant Funk-Hecke type theorems, and the addition formula for the corresponding harmonics. Moreover, unlike [28], our multipliers for the intertwining operators and the dual ones are the same.

3. According to Strichartz [29, Theorem 4.2], the asymptotics $O(j^{-k/2})$ of the multiplier corresponding to the Funk-Radon transform $R$ and combined with the oscillating factor $(-1)^{j/2}$, yields $L^p$-$L^q$ estimates of $R$. However, there is an intimate connection between $R$ and the $k$-plane transform on $\mathbb{R}^n$ (see, e.g., [22, Section 3]), which allows one to convert boundedness results for one class of operators to the similar results for another. This conversion is performed with preservation of the operator norms and yields important geometric inequalities for sections of convex bodies in integral geometry.

It might be of interest to convert Strichartz’s estimates from [29, Theorem 4.2] to those for the $k$-plane transforms and compare the obtained statements with known results by Christ [5] and Drury [6]. We conjecture that this approach is applicable to more general analytic family of operators (3.37) with the oscillatory multiplier (3.39), having the order $O(j^{-k/2-\alpha})$.

4. Natural higher-rank generalizations of the spherical means and the corresponding shifted Radon transforms arise in integral geometry on Grassmann manifolds and matrix spaces [10] [15]. In this setting, an analogue of the shift $t$ is matrix-valued and represented by a positive definite matrix. It might be of interest to study injectivity of such higher-rank mean value operators and develop the relevant harmonic analysis.

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