Improved Uniqueness of Multi-breathers of the Modified Korteweg–de Vries Equation

Raphaël Côte · Alexander Semenov

Received: 20 September 2022 / Accepted: 28 December 2022 / Published online: 26 June 2023
© Vietnam Academy of Science and Technology (VAST) and Springer Nature Singapore Pte Ltd. 2023

Abstract
We consider multi-breathers of \((mKdV)\). In Semenov (2022), a smooth multi-breather was constructed, and proved to be unique in two cases: first, in the class of super-polynomial convergence to the profile (in the spirit of (Commun. Partial Differ. Equ. 46, 2325–2385, 2021)), and second, under the assumption that all speeds of the breathers involved are positive (without rate of convergence). The goal of this short note is to improve the second result: we show that uniqueness still holds if at most one velocity is negative or zero.

Keywords mKdV · Breathers · Solitons · Multi-breathers · Multi-solitons · Uniqueness

Mathematics Subject Classification (2010) Primary 35Q51 · 35Q53 · 35B40 · Secondary 37K10 · 37K40

1 Setting of the Problem

1.1 The Modified Korteweg–de Vries equation

We consider the modified Korteweg–de Vries equation on \(\mathbb{R}\):

\[
\begin{align*}
  u_t + (u_{xx} + u^3)_x &= 0, & (t, x) &\in \mathbb{R}^2, \\
  u(0) &= u_0, & u(t, x) &\in \mathbb{R}.
\end{align*}
\]

\((mKdV)\)

The \((mKdV)\) equation appears as a model of some physical problems as plasma physics, electrodynamics [10], fluid mechanics, ferromagnetic vortices, and more; we refer to [11] for further information about the physical applications. Let us recall that \((mKdV)\) is globally...
well-posed for any initial data in $H^2$ (see [6] for much stronger results), and for such data, three quantities are conserved in time:

- the $L^2$ mass $M[u](t) := \frac{1}{2} \int u^2(t, x)\, dx$,
- the energy $E[u](t) := \int \left( \frac{1}{2} u_x^2(t, x) - \frac{1}{4} u^4(t, x) \right)\, dx$,
- the second energy $F[u](t) := \int \left( \frac{1}{2} u_{xx}^2(t, x) - \frac{5}{2} u^2(t, x)u_x^2(t, x) + \frac{1}{4} u^6(t, x) \right)\, dx$.

Finally, (mKdV) is an integrable system, and there are (at least for smooth solutions) infinitely many conservation laws, but we point out that we will only use $H^2$ regularity and the above conservation laws.

### 1.2 Solitons and Breathers of (mKdV)

The special “basic” solutions of (mKdV) that we consider here are solitons and breathers.

**Definition 1** Let $c > 0$, $\kappa \in \{-1, 1\}$ and $x_0 \in \mathbb{R}$. A soliton $R_{c,\kappa}(x_0)$ of shape parameter (or velocity) $c$, of sign $\kappa$ and of translation parameter (or initial position) $x_0$ is a solution of (mKdV), given by the following formula:

$$\forall (t, x) \in \mathbb{R}^2, \quad R_{c,\kappa}(t, x; x_0) := \kappa Q_c(x - x_0 - ct),$$

where $Q_c$ is defined by the following formula:

$$\forall x \in \mathbb{R}, \quad Q_c(x) := \left( \frac{2c}{\cosh^2(c^{1/2} x)} \right)^{1/2}.$$

**Remark 2** There exists a constant $C > 0$ that depends only on $c$ such that

$$\forall (t, x) \in \mathbb{R}^2, \quad |R_{c,\kappa}(t, x; x_0)| \leq C \exp(-\sqrt{c}|x - x_0 - ct|).$$

Further properties of solitons (in particular, their $H^2$ variational structure) can be found in [11].

**Definition 3** Let $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$. A breather $B_{\alpha,\beta}(x_1, x_2)$ of shape parameters $\alpha, \beta$ and of translation parameters $x_1, x_2$, is a solution of (mKdV), given by the following formula:

$$\forall (t, x) \in \mathbb{R}^2, \quad B_{\alpha,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right],$$

where

$$y_1 := x + \delta t + x_1 \quad \text{and} \quad y_2 := x + \gamma t + x_2,$$

with $\delta := \alpha^2 - 3\beta^2$ and $\gamma := 3\alpha^2 - \beta^2$.

The velocity of $B_{\alpha,\beta}(x_1, x_2)$ is $-\gamma = \beta^2 - 3\alpha^2$ and its initial position is $-x_2$.

**Remark 4** There exists a constant $C > 0$ that depends only on $\alpha$ and $\beta$ such that

$$\forall (t, x) \in \mathbb{R}^2, \quad |B_{\alpha,\beta}(t, x; x_1, x_2)| \leq C \exp(-\beta|x + x_2 + \gamma t|).$$

Further properties of breathers can be found in [1] or [11].
1.3 Main Result

We consider multi-breathers made of \( K \) breathers and \( L \) solitons in the sense of [11], which we recall now. Let \( K, L \in \mathbb{N} \) and set \( J = K + L \). For \( 1 \leq k \leq K \), let \( \alpha_k, \beta_k > 0 \) and \( x_{1,k}^0, x_{2,k}^0 \in \mathbb{R} \), and define the breathers

\[
B_k := B_{\alpha_k, \beta_k}(x_{1,k}^0, x_{2,k}^0), \quad \text{with velocity } v_k^b := \beta_k^2 - 3\alpha_k^2. \tag{1}
\]

For \( 1 \leq l \leq L \), let \( c_l > 0, \kappa_l \in \{-1, 1\} \) and \( x_{0,l}^0 \in \mathbb{R} \) and consider the solitons

\[
R_l := R_{c_l, \kappa_l}(x_{0,l}^0), \quad \text{with velocity } v_l^s := c_l. \tag{2}
\]

An essential assumption in the present analysis is that \emph{all the velocities of the considered objects, solitons or breathers, must be distinct}:

\[
\forall k \neq k' \quad v_k^b \neq v_{k'}^b, \quad \forall l \neq l' \quad v_l^s \neq v_{l'}^s, \quad \forall k, l \quad v_k^b \neq v_l^s. \tag{3}
\]

We may therefore order the speeds and define an increasing function:

\[
\psi : \{1, \ldots, J\} \rightarrow \{v_k^b, 1 \leq k \leq K\} \cup \{v_l^s, 1 \leq l \leq L\}.
\]

The \( J \)-uple \((v_1, \ldots, v_J)\) is thus the ordered set of all possible velocities of our objects. This allows us to write in a convenient way the sum of breathers and solitons: for \( 1 \leq J \leq J \), we define \( P_j \) as the object (either the soliton \( R_l \) or the breather \( B_k \)) that corresponds to the velocity \( v_j \), so that \( P_1, \ldots, P_J \) are ordered by increasing velocity. We consider the sum of these breathers and solitons

\[
P := \sum_{j=1}^{J} P_j = \sum_{k=1}^{K} B_k + \sum_{l=1}^{L} R_l, \tag{4}
\]

and the associated multi-breather, that are solutions to (mKdV) which behave as \( P \) for large time as defined below:

**Definition 5** Given solitons and breathers (2), (1), whose sum is given by (4), a multi-breather associated to the sum \( P \) of solitons and breathers is a solution \( p \in C([T^*, +\infty), H^2(\mathbb{R})) \), for a constant \( T^* > 0 \), of (mKdV) such that

\[
\|p(t) - P(t)\|_{H^2} \to 0 \quad \text{as} \quad t \to +\infty. \tag{5}
\]

Let us recall from [11, Theorem 1.2] that as soon as the condition on the speeds (3) on \( P \) is satisfied, there exists a multi-breather \( p \) related to \( P \), such that \( p \) is smooth and the convergence \( p - P \to 0 \) occurs at an exponential rate in any \( H^s, s \in \mathbb{R} \).

Also, from [11, Theorem 1.4], under the extra assumption that all speeds are positive, that is \( v_1 > 0 \), this multi-breather is unique in the class of (5).

In this note, we revisit the proof given in [11], in order to improve this last uniqueness result: it actually holds if at most one of the velocities is non positive, that is under the assumption \( v_2 > 0 \). Here is the precise statement.

**Theorem 6** Given breathers (1) and solitons (2), whose velocities satisfy (3), let \( P \) be the sum of the considered solitons and breathers given in (4).

Assume that \( v_2 > 0 \), so that all velocities, except possibly one, are positive. Then the multi-breather of [11] \( p \in C([T^*, +\infty), H^2) \) associated to \( P \) is the unique solution of (mKdV) such that (5) holds.
Multi-solitons have been constructed for many dispersive models (see for example [3, 4, 7, 9] for (NLS), Klein-Gordon, or water-waves), but the question of uniqueness (or classification) is generally open. The examples that we are aware of, where such uniqueness is known, are the generalized Korteweg–de Vries equation [8], the generalized Benjamin–Bona–Mahony equation [5] and the Zakharov–Kuznetsov equation [12]. The underlying difficulty is the interaction of the nonlinear object and linear dispersion: for the two models, solitons move to the right, and dispersion to the left, which allows a very nice decoupling, that one can express via a monotonicity property. This feature is however absent in other dispersive models, which explains why uniqueness of multi-solitons remains an open problem in general.

In this note, we consider the more complex multi-breather of (mKdV). The point is that breathers may travel to the left, in the Airy dispersion zone: but if at most one moves there, our result shows that uniqueness still holds.

2 Proof

In order to prove Theorem 6, we consider $P$ as in (4), we assume that $v_2 > 0$, and we consider a multi-breather $u \in C([T, +\infty), H^2)$ such that

$$\|u(t) - P(t)\|_{H^2} \to 0 \quad \text{as} \quad t \to +\infty.$$ 

Our goal is to prove that $u \equiv p$, the multi-breather associated to $P$ constructed in [11, Theorem 1.2]. For this, the main step is to prove that $u(t)$ converges actually exponentially fast to its profile $P(t)$.

**Proposition 7** There exists $\varpi > 0$, $T_0 \geq T$ and $C > 0$ such that

$$\forall t \geq T_0, \quad \|u(t) - P(t)\|_{H^2} \leq Ce^{-\varpi t}.$$ 

This corresponds to Proposition 4.10 in [11] (where of course the assumption on $P$ is different: there one supposes that $v_1 > 0$).

2.1 Proof of Proposition 7

The proof follows mainly the lines of Section 4.2 of [11], with several changes that we will detail here. An important ingredient is (almost) monotonicity properties of localized quantities.

We do our best to treat breathers and solitons together. To this end, we define the shape parameters as follows: for $j = 1, \ldots, J$,

- if $P_j = B_k$ is a breather, then
  $$ (a_j, b_j) = (\alpha_k, \beta_k), $$ \hspace{1cm} (6)

- if $P_j = R_l$ is a soliton, then
  $$ (a_j, b_j) = (0, \sqrt{c_l}). $$ \hspace{1cm} (7)

The shape of the cut-off function that we will use is given by $\Psi$:

$$ \Psi(x) := \frac{2}{\pi} \arctan\left(\exp(-\sqrt{\sigma} x/2)\right), $$

where $\sigma > 0$ is small enough (and precise conditions will be given in the proof).
We consider a cut-off function $\Phi_j$ given for $j = 1, \ldots, J - 1$ by
\[
\Phi_j(t, x) := \Psi(x - m_j t),
\]
and $\Phi_J \equiv 1$. For $j \in \{1, \ldots, J - 1\}$, $\Psi_j$ tend to 1 at $-\infty$, to 0 at $+\infty$, with an exponentially localized transition between the centers of $P_j$ and $P_{j+1}$. This requires that for all such $j$, $v_j < m_j < v_{j+1}$.

However, in order to the monotonicity argument to work, we need to choose cut-off functions that move to the right, i.e. have positive velocities $m_j > 0$. For $j = 2, \ldots, J - 1$, we set
\[
m_j := \frac{v_j + v_{j+1}}{2}.
\]
(Then we indeed have $m_j > 0$ because $v_j \geq v_2 > 0$.) On the other hand, $m_1$ needs better tuning: we define $m_1$ in the following way.

1. We first set $0 < v_1 < 1$ such that
\[
(b_1^2 - a_1^2) + v_1 (a_1^2 + b_1^2) > 0,
\]
2. then we choose $m_1$ such that
\[
\max(0, v_1) < m_1 < v_2, \quad \text{and}
\]
\[
m_1 (b_1^2 - a_1^2) > \frac{1}{2} (v_1 - 1) (a_1^2 + b_1^2)^2.
\]
Condition (8) corresponds to a choice of $v_1$ sufficiently near 1. Condition (9) can be satisfied: indeed, if $b_1^2 - a_1^2 \leq 0$ then $v_1 < 0$ and it suffices to choose $m_1 > 0$ sufficiently small; if $b_1^2 - a_1^2 > 0$, it is always satisfied as the righthand side is negative.

These conditions will be used in order to derive suitable monotonicity properties, see Step 3.

We set $\tau_0 > 0$ the minimal distance between $\{v_1, \ldots, v_J\}$ and $\{m_1, \ldots, m_{J-1}\}$.

As mentioned, the scheme of the proof is here roughly the same as in [11]; however, some specificities change.

First, we modulate the breathers $P_j$ into $\tilde{P}_j$ (by translation only), so that the default $w = u - \tilde{P}$ (defined in Lemma 8) enjoys orthogonality properties. Then, we prove that $\|w(t)\|_{H^2} \leq C e^{-\sigma t}$ by induction, where $\sigma > 0$ is a constant depending on the data of the problem.

For $j = 1, \ldots, J$, proposition $\mathcal{P}_j$ reads
\[
\forall t \geq T', \quad \int (w^2 + \tilde{w}_x^2 + \tilde{w}_xx^2) \Phi_j + \sum_{i=1}^j \int \tilde{P}_i w \leq C_j e^{-2\sigma t},
\]
where $T' \geq T$ is to be defined in the proof. $\mathcal{P}_0$ is the assertion “True”.

Given $j \in \{1, \ldots, J\}$, we assume $\mathcal{P}_{j-1}$, and our goal is to prove $\mathcal{P}_j$. We finally infer $\|w(t) - P(t)\|_{H^2} \leq C e^{-\sigma t}$, in the concluding step of the proof.

One difference with [11] is that here we make our proof by induction on the modulated difference $w$ and not on $u - P$. This is not crucial, but we find it nicer to perform a modulation for all the objects at once. One key difference compared to [11], though, is that we need and prove monotonicity for a functional which is slightly weaker than the natural Lyapunov functional required for the proof. When $v_1 < 0$, the proof requires a careful interpolation between positive terms in order to balance negative terms.
Step 1: Modulation

This step is devoted to the proof of the following modulation lemma.

**Lemma 8** There exists $C > 0$, $T_2 \geq T$, such that there exist unique $C^1$ functions $y_{1,k}, y_{2,k}, y_{0,l} : [T_2, +\infty) \to \mathbb{R}$ such that if we set:

$$w(t, x) := u - \tilde{P},$$

where

$$\tilde{P}(t, x) := \sum_{k=1}^{K} \tilde{B}_k(t, x) + \sum_{l=1}^{L} \tilde{R}_l(t, x)$$

and

$$\tilde{R}_l(t, x) := \kappa_l Q_{cl}(x - x_{0,l}^0 + y_{0,l}(t) - c_l t),$$

$$\tilde{B}_k(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k} + y_{1,k}(t), x_{2,k} + y_{2,k}(t)).$$

then, $w(t)$ satisfies, for any $t \in [T_2, +\infty)$,

$$\forall l = 1, \ldots, L, \int (\tilde{R}_l)_x(t) w(t) = 0,$$

$$\forall k = 1, \ldots, K \int (\tilde{B}_k)_1(t) w(t) = \int (\tilde{B}_k)_2(t) w(t) = 0,$$

where we denote:

$$(\tilde{B}_k)_1(t, x) := \partial_{x_1} \tilde{B}_k, \quad (\tilde{B}_k)_2(t, x) := \partial_{x_2} \tilde{B}_k.$$

Moreover, for any $t \in [T_2, +\infty)$,

$$\|w(t)\|_{H^2} + |y_{1,k}(t)| + |y_{2,k}(t)| + |y_{0,l}(t)| \leq C \|v(t)\|_{H^2}, \quad (11)$$

and, if $\sigma$ is small enough,

$$|y'_{1,k}(t)| + |y'_{2,k}(t)| + |y'_{0,l}(t)| \leq C \left( \int w(t)^2 \Phi_j \right)^{1/2} + C e^{-\sigma t}. \quad (12)$$

**Proof** The proof of this lemma can be performed in the same manner as in [11, Lemma 2.8]. □

As above for (4), we denote $\tilde{P}_j = \tilde{B}_k$ if $P_j = B_k$ is a breather, and $\tilde{P}_j = \tilde{R}_l$ if $P_j = R_l$ is a soliton, so that

$$\tilde{P} = \sum_{j=1}^{J} \tilde{P}_j.$$

The difference with [11] is that

- the modulation that we perform here does not modify any shape parameter (that is why there is only one modulation direction for each soliton here),
- we perform the modulation once and not on each step of the induction.
Step 2: Approximation of the Lyapunov functional

This step is devoted to define a localized Lyapunov functional. Let $j \in \{1, \ldots, J\}$. First, we define the localized quantities related to conservation laws as follows:

$$M_j(t) := \int u^2(t)\Phi_j(t),$$
$$E_j(t) := \int \left[ \frac{1}{2} u^2_x - \frac{1}{4} u^4 \right] \Phi_j(t),$$
$$F_j(t) := \int \left[ \frac{1}{2} u^2_{xx} - \frac{5}{2} u^2 u^2_x + \frac{1}{4} u^6 \right] \Phi_j(t).$$

Then, the localized Lyapunov functional is

$$\mathcal{H}_j(t) := F_j(t) + 2 \left( b_j^2 - a_j^2 \right) E_j(t) + \left( a_j^2 + b_j^2 \right)^2 M_j(t),$$

where $a_j, b_j$ stand for generalized shape parameters defined in (6)–(7).

This Lyapunov functional was already introduced in [11]. We prove the following lemma that somehow quantifies how far is a modulated sum of solitons and breathers from being a critical point for $\mathcal{H}_j$.

First, there hold the following Taylor expansions.

**Lemma 9** There exists $C > 0, T_1 \geq T$ such that the following holds for any $t \geq T_1$:

$$\left| M_j(t) - \sum_{i=1}^{j} M[\tilde{P}_i] - \sum_{i=1}^{j} \int \tilde{P}_i w - \frac{1}{2} \int w^2 \Phi_j \right| \leq C e^{-2\sigma t}, \quad (13)$$

$$\left| E_j(t) - \sum_{i=1}^{j} E[\tilde{P}_i] - \sum_{i=1}^{j} \int \left[ (\tilde{P}_i)_x w - \tilde{P}_i^3 w \right] - \int \left[ \frac{1}{2} w^2_x - \frac{3}{2} \tilde{P}^2 w^2 \right] \Phi_j \right| \leq C e^{-2\sigma t} + o \left( \int w^2 \Phi_j \right), \quad (14)$$

$$\left| F_j(t) - \sum_{i=1}^{j} F[\tilde{P}_i] - \sum_{i=1}^{j} \int \left[ (\tilde{P}_i)_{xx} w_{xx} - 5 \tilde{P}_i (\tilde{P}_i)_x^2 w - 5 \tilde{P}_i^2 (\tilde{P}_i)_x w_x + \frac{3}{2} \tilde{P}_i^5 w \right] \right.$$

$$\left. - \int \left[ \frac{1}{2} w^2_{xx} - \frac{5}{2} w^2 \tilde{P}_x^2 - 10 \tilde{P} w \tilde{P}_x w_x - \frac{5}{2} \tilde{P}^2 w^2_x + \frac{15}{4} \tilde{P}^4 w^2 \right] \Phi_j(t) \right| \leq C e^{-2\sigma t} + o \left( \int \left( w^2 + w^2_x \right) \Phi_j \right). \quad (15)$$

**Proof** See for example [11, Proposition 2.12]. We emphasize that we do not use here the induction assumption. \qed

The Lyapunov functional is constructed so as to make the linear terms in $w$ cancel, as seen below.
Lemma 10  There exists $T_2 \geq T_1$ such that the following holds for $t \geq T_2$:

$$\mathcal{H}_j(t) = \sum_{i=1}^{j} F[\tilde{P}_i] + 2 \left( b_j^2 - a_j^2 \right) \sum_{i=1}^{j} E[\tilde{P}_i] + \left( a_j^2 + b_j^2 \right) \sum_{i=1}^{j} M[\tilde{P}_i] + H_j(t) + O(e^{-2\sigma t}) + o \left( \int (w^2 + w_x^2) \Phi_j \right),$$

where

$$H_j(t) := \int \left[ \frac{1}{2} w_{xx}^2 - \frac{5}{2} w_x^2 \tilde{P}_j^2 + \frac{5}{2} w^2 (\tilde{P}_j)_x^2 + 5 w^2 \tilde{P}_j (\bar{P}_j)_{xx} + \frac{15}{4} w^2 \tilde{P}_j^4 \right] \Phi_j(t) + \left( b_j^2 - a_j^2 \right) \int \left[ w_x^2 - 3 w^2 \tilde{P}_j^2 \right] \Phi_j(t) + \frac{1}{2} \left( a_j^2 + b_j^2 \right)^2 \int w^2 \Phi_j(t).$$

Proof  The proof of the lemma above can be performed as in [11, Proposition 2.12]: it uses Lemma 9, the elliptic equation satisfied by $P_j$ or $\tilde{P}_j$ (we do not need to make a distinction whether it is a soliton or a breather here), and the induction assumption for the contributions in the region $x \leq m_{j-1}t$. \qed

Step 3: Monotonicity

Let us first recall some monotonicity properties related to the localized conservations laws.

Lemma 11  Let $\omega > 0$ as small as desired. There exists $T_3 = T_3(\omega) \geq T_2$ and $C > 0$ such that for $t \geq T_3$,

$$\sum_{i=1}^{j} M[P_i] - M_j(t) \geq -Ce^{-2\sigma t},$$

$$\sum_{i=1}^{j} (E[P_i] + \omega M[P_i]) - (E_j(t) + \omega M_j(t)) \geq -Ce^{-2\sigma t},$$

$$\sum_{i=1}^{j} (F[P_i] + \omega M[P_i]) - (F_j(t) + \omega M_j(t)) \geq -Ce^{-2\sigma t}.$$

Proof  The lemma above may be proved in the same manner as in [11, Lemma 4.11]. \qed

We emphasize that some extra $L^2$ mass is needed in order to gain monotonicity for $E_j$ or $F_j$: this fact was already noted in [8], and is related to a lack of control of non linear terms far away from the breathers/solitons.

We now turn to the main monotonicity result that we will use.

Let $0 < \nu < 1$ be close enough to 1, to be fixed later. We define, for $j = 1, \ldots, J$, a functional $\mathcal{F}_j$ that is close to the Lyapunov functional $\mathcal{H}_j$:

$$\mathcal{F}_j(t) := F_j(t) + 2(b_j^2 - a_j^2) E_j(t) + \nu(a_j^2 + b_j^2)^2 M_j(t).$$

The following lemma states the almost-growth of $\mathcal{F}_j$:

Lemma 12  There exists $0 < \nu < 1$ close enough to 1 such that there exists $T_4 \geq T_2$ and $C > 0$ such that for any $t \geq T_4$,

$$\mathcal{F}_j(t) - \sum_{i=1}^{j} F[P_i] - 2(b_j^2 - a_j^2) \sum_{i=1}^{j} E[P_i] - \nu(a_j^2 + b_j^2)^2 \sum_{i=1}^{j} M[P_i] \leq Ce^{-2\sigma t}.$$
Remark 13 We emphasize the factor \( v < 1 \) in front of the \( M_j \) term, which make it a weakened version of the Lyapunov functional \( \mathcal{H}_j \). Therefore, as \( M_j \) enjoys strong monotonicity, the monotonicity of \( \mathcal{F}_j \) is a stronger result than merely that of \( \mathcal{H}_j \). This improvement is needed in order to deal with \( \int \tilde{P}_j \omega \), see Step 5.

Proof If \( b_j^2 - a_j^2 \geq 0 \), then Lemma 12 is an immediate consequence of Lemma 11. For the rest of the proof, we consider the case \( b_j^2 - a_j^2 < 0 \), which can only occur when \( j = 1 \), and which we assume for the rest of this proof. Let

\[
\nu = \nu_1 + \frac{2}{3} (1 - \nu_1) < 1,
\]

where \( \nu_1 \) is defined in (8).

In the proof of Lemma 11 (see [11, Lemma 4.11]), there hold the more precise bounds: given \( \omega > 0 \), there exist \( T_3' = T_3'(\omega) \geq T_2 \) such that for all \( t \geq T_3' \)

\[
\begin{align*}
\frac{d}{dt} F_1(t) &\geq -C e^{-2\nu \nu_1} + \frac{3}{2} \int u_{xx}^2 |\Phi_{1x}| + \frac{m_1}{2} \int u_{xx}^2 |\Phi_{1x}| - \omega \int (u_x^2 + u_x^2 + u^2) |\Phi_{1x}|, \\
-\frac{d}{dt} E_1(t) &\geq -C e^{-2\nu \nu_1} - \frac{3}{2} \int u_{xx}^2 |\Phi_{1x}| - \frac{m_1}{2} \int u_x^2 |\Phi_{1x}| - \omega \int (u_x^2 + u^2) |\Phi_{1x}|, \\
\frac{d}{dt} M_1(t) &\geq -C e^{-2\nu \nu_1} + \frac{3}{2} \int u_x^2 |\Phi_{1x}| + \frac{m_1}{2} \int u^2 |\Phi_{1x}| - \omega \int u^2 |\Phi_{1x}|.
\end{align*}
\]

Summing up the right linear combination, we infer:

\[
\begin{align*}
\frac{d}{dt} \mathcal{F}_1(t) &\geq -C e^{-2\nu \nu_1} + \frac{3}{2} \int u_{xx}^2 |\Phi_{1x}| \\
&\quad + \left( 3(b_1^2 - a_1^2) + \frac{m_1}{2} - \omega \right) \int u_{xx}^2 |\Phi_{1x}| \\
&\quad + \left( \frac{3}{2} \nu(a_1^2 + b_1^2)^2 + m_1(b_1^2 - a_1^2) - \omega \right) \int u_x^2 |\Phi_{1x}| \\
&\quad + \left( \frac{m_1}{2} \nu(a_1^2 + b_1^2)^2 - \omega \right) \int u^2 |\Phi_{1x}|.
\end{align*}
\]

Heuristically, \( \omega \) can be neglected, so that the coefficients of the terms in \( \int u_{xx}^2 |\Phi_{1x}| \) and \( \int u_x^2 |\Phi_{1x}| \) are all positive. For the \( \int u_x^2 |\Phi_{1x}| \) term, we can play with \( \nu \) and \( m_1 \) to ensure its positivity (which is one purpose of the conditions (8) and (9)). However, the coefficient in front of \( \int u_{xx}^2 |\Phi_{1x}| \) could be truly negative: \( m_1 < \nu_2 \) could be only marginally positive, and \( b_1^2 - a_1^2 \) can take any real value (the only constraint is that it is larger than the speed \( \nu_1 \), on which we have no control, though). This term might be problematic, and will concentrate our efforts: roughly speaking, the idea is to bound it via interpolation with the other terms.

From the definition of \( m_1 \) given by (9), we have that

\[
\frac{3}{2} \nu(a_1^2 + b_1^2)^2 + m_1(b_1^2 - a_1^2) > \frac{3}{2} \nu'(a_1^2 + b_1^2)^2.
\]

where

\[
\nu' = \nu_1 + \frac{1 - \nu_1}{3}.
\]

We choose \( \omega \) small enough with respect to the previous choice (by choosing \( T_4 \) large enough) so that

\[
\frac{3}{2} \nu(a_1^2 + b_1^2)^2 + m_1(b_1^2 - a_1^2) - \omega \geq \frac{3}{2} \nu'(a_1^2 + b_1^2)^2.
\]
Knowing that \(a_1^2 + b_1^2 > 0\), we may choose \(\omega\) even smaller (with respect to \(m_1\) and \(\nu\)) so that

\[
\frac{m_1}{2} \nu (a_1^2 + b_1^2)^2 - \omega \geq 0
\]

and

\[
3(b_1^2 - a_1^2) + \frac{m_1}{2} - \omega \geq 3(b_1^2 - a_1^2).
\]

In the case when with the chosen values of \(m_1\), \(\nu\) and \(\omega\), \(3(b_1^2 - a_1^2) + \frac{m_1}{2} - \omega\) is positive, the desired conclusion is straightforward by integration. From now on, we place ourselves in the case when

\[
3(b_1^2 - a_1^2) + \frac{m_1}{2} - \omega < 0.
\]

Now, we want to bound above \(\int u_{xx}^2 |\Phi_{1x}|\). By integration by parts,

\[
\int u_{xx}^2 |\Phi_{1x}| = -\int u_x u_{xxx} |\Phi_{1x}| - \int u_x u_{xx} |\Phi_{1xx}|
\]

\[
\leq \sqrt{\int u_x^2 |\Phi_{1x}| \int u_{xxx}^2 |\Phi_{1x}|} + \frac{\sqrt{\sigma}}{2} \sqrt{\int u_x^2 |\Phi_{1x}| \int u_{xx}^2 |\Phi_{1x}|},
\]

because \(|\Phi_{1xx}| \leq \frac{\sqrt{\sigma}}{2} |\Phi_{1x}|\). We denote:

\[
X := \sqrt{\int u_{xx}^2 |\Phi_{1x}|}
\]

and

\[
A := \sqrt{\int u_x^2 |\Phi_{1x}| \int u_{xx}^2 |\Phi_{1x}|}.
\]

So, we have that

\[
X^2 \leq A + \epsilon X,
\]

where

\[
\epsilon := \frac{\sqrt{\sigma}}{2} \sqrt{\int u_x^2 |\Phi_{1x}|} \leq \frac{\sqrt{\sigma}}{2} \|u\|_{H^1},
\]

which can be as small as we want if we take \(\sigma\) small enough (for a given solution \(u\)). We deduce that

\[
X \leq \frac{\epsilon + \sqrt{\epsilon^2 + 4A}}{2} \leq \epsilon + \sqrt{A}.
\]

Thus,

\[
\int u_{xx}^2 |\Phi_{1x}| \leq \left( \frac{\sigma}{4} \sqrt{\int u_x^2 |\Phi_{1x}|} + \sqrt{\sigma} \left( \int u_x^2 |\Phi_{1x}| \int u_{xx}^2 |\Phi_{1x}| \right)^{\frac{1}{2}} \right)
\]

\[
+ \frac{\sqrt{\int u_{xxx}^2 |\Phi_{1x}|}}{\sqrt{\int u_x^2 |\Phi_{1x}|}}.
\]
So,

\[
(3(b_1^2 - a_1^2) + \frac{m_1}{2} - \omega) \int u_{x,x}^2 |\Phi_{1x}| \geq 3(b_1^2 - a_1^2) \sqrt{\int u_x^2 |\Phi_{1x}|} \int u_{x,x}^2 |\Phi_{1x}|
\]

\[
+ 3(b_1^2 - a_1^2) \sqrt{\alpha} \left( \int u_x^2 |\Phi_{1x}| \right) \left( \int u_{x,x}^2 |\Phi_{1x}| \right)^{\frac{1}{2}}
\]

\[
+ 3(b_1^2 - a_1^2) \frac{\sigma}{4} \int u_x^2 |\Phi_{1x}|.
\]

On the other hand, we have, for a choice of \( \nu_2, \nu_3 > 0 \) such that \( \nu_1 + \nu_2 + \nu_3 = \nu' \) that

\[
\frac{3}{2} \int u_{x,x,x}^2 |\Phi_{1x}| + \left( \frac{3}{2} \nu'(a_1^2 + b_1^2)^2 \right) \int u_x^2 |\Phi_{1x}|
\]

\[
\geq \frac{3}{2} \nu_1 \int u_{x,x,x}^2 |\Phi_{1x}| + \left( \frac{3}{2} \nu_1 (a_1^2 + b_1^2)^2 \right) \int u_x^2 |\Phi_{1x}|
\]

\[
+ \frac{3}{2} (1 - \nu_1) \int u_{x,x,x}^2 |\Phi_{1x}| + \left( \frac{3}{2} \nu_2 (a_1^2 + b_1^2)^2 \right) \int u_x^2 |\Phi_{1x}|
\]

\[
+ \left( \frac{3}{2} \nu_3 (a_1^2 + b_1^2)^2 \right) \int u_x^2 |\Phi_{1x}|
\]

\[
\geq 2 \sqrt{\int u_x^2 |\Phi_{1x}|} \int u_{x,x,x}^2 |\Phi_{1x}|
\]

\[
+ 4 \left( \frac{3}{2} (1 - \nu_1) \int u_{x,x,x}^2 |\Phi_{1x}| \right) \left( \left( \frac{1}{2} \nu_2 (a_1^2 + b_1^2)^2 \right) \int u_x^2 |\Phi_{1x}| \right)^{\frac{1}{2}}
\]

\[
+ \left( \frac{3}{2} \nu_3 (a_1^2 + b_1^2)^2 \right) \int u_x^2 |\Phi_{1x}|
\]

\[
\geq 3 \nu_1 (a_1^2 + b_1^2) \sqrt{\int u_x^2 |\Phi_{1x}|} \int u_{x,x,x}^2 |\Phi_{1x}|
\]

\[
+ 2 \cdot 3^{\frac{1}{4}} (1 - \nu_1) \left( \frac{3}{2} \nu_2 (a_1^2 + b_1^2)^2 \right) \left( \int u_x^2 |\Phi_{1x}| \right)^{\frac{3}{4}} \left( \int u_{x,x,x}^2 |\Phi_{1x}| \right)^{\frac{1}{4}}
\]

\[
+ \frac{3}{2} \nu_3 (a_1^2 + b_1^2)^2 \int u_x^2 |\Phi_{1x}|.
\]

This is why, we deduce that

\[
\frac{d}{dt} \mathcal{F}_1(t) \geq -Ce^{-2\nu t}
\]

\[
+ (3(b_1^2 - a_1^2) + 3 \nu_1 (a_1^2 + b_1^2)) \sqrt{\int u_x^2 |\Phi_{1x}|} \int u_{x,x,x}^2 |\Phi_{1x}|
\]

\[
+ \left( 3(b_1^2 - a_1^2) \sqrt{\alpha} + 2 \cdot 3^{\frac{1}{4}} (1 - \nu_1) \left( \frac{3}{2} \nu_2 (a_1^2 + b_1^2)^2 \right) \left( \int u_x^2 |\Phi_{1x}| \right)^{\frac{3}{4}} \left( \int u_{x,x,x}^2 |\Phi_{1x}| \right)^{\frac{1}{4}}
\]

\[
+ \left( \frac{3}{2} \nu_3 (a_1^2 + b_1^2)^2 \right) \int u_x^2 |\Phi_{1x}|.
\]
To finish, we remark that the coefficients in front of the integrals in (20) are all non negative: indeed

\[ 3(b_1^2 - a_1^2) + 3v_1(a_1^2 + b_1^2) \geq 0, \]

by definition of \( v_1 \) given in (8):

\[ 3(b_1^2 - a_1^2)\sqrt{\sigma} + 2 \cdot 3^{\frac{1}{2}}(1 - v_1)\frac{3}{2}v_2^2(a_1^2 + b_1^2)^{\frac{3}{2}} \geq 0 \]

and

\[ 3(b_1^2 - a_1^2)\frac{\sigma}{4} + \frac{3}{2}v_3(a_1^2 + b_1^2)^2 \geq 0. \]

by choosing \( \sigma > 0 \) small enough.

Thus,

\[ \frac{d}{dt}F_1(t) \geq -Ce^{-2\sigma t}. \]

We obtain the desired conclusion by integration. \( \square \)

**Step 4: Bound from above for \( H_j(t) \)**

**Lemma 14** For any \( t \geq T_4 \), we have that

\[ H_j(t) \leq Ce^{-2\sigma t} + o\left( \int \left( w^2 + w_x^2 \right) \Phi_j \right) \]

**Proof** From Lemma 11, we know that for any \( t \geq T_1 \),

\[ M_j(t) - \sum_{i=1}^j M[P_i] \leq Ce^{-2\sigma t}. \]

By summing this fact with the fact from the Lemma 12, we obtain that for any \( t \) large enough:

\[ \mathcal{H}_j(t) - \sum_{i=1}^j F[P_i] - 2\left(b_j^2 - a_j^2\right)\sum_{i=1}^j E[P_i] - \left(a_j^2 + b_j^2\right)^2 \sum_{i=1}^j M[P_i] \leq Ce^{-2\sigma t}. \]

\[ \text{(21)} \]

From (21) and Lemma 10, we obtain the desired conclusion for any \( t \geq T_4 \). \( \square \)

Let us recall that \( H_j \) enjoys a crucial coercivity property:

**Proposition 15** (Coercivity of \( H_j \)) There exists \( \mu > 0 \), and \( T_5 \geq T_4 \) such that, for \( t \geq T_5 \),

\[ H_j(t) \geq \mu \int \left( w_{xx}^2 + w_x^2 + w^2 \right) \Phi_j(t) - \frac{1}{\mu} \left( \int \tilde{P}_j w \sqrt{\Phi_j} \right)^2. \]

**Proof** One can argue as in the proof of Proposition 4.10, Step 7 in [11] (see also Proposition 2.13). (We choose to keep the \( L^2 \) scalar product of \( \tilde{P}_j \) with \( w \sqrt{\Phi_j} \) because coercivity is derived from the original coercivity (related to the linearization around soliton or breather of the relevant conservation law see [1, 11]) via localization argument on \( w \sqrt{\Phi_j} \). Obviously, we could have stated coercivity up to the scalar product of \( \int \tilde{P}_j w \)). \( \square \)

Using the above two results, and the induction hypothesis we prove as in [11, Step 6] that for \( t \geq T_5 \),

\[ \int \left( w^2 + w_x^2 + w_{xx}^2 \right) \Phi_j \leq Ce^{-2\sigma t} + C \left( \int \tilde{P}_j w \sqrt{\Phi_j} \right)^2. \]

\[ \text{(22)} \]
Remark 16 Here the choice of modulating only by translation gives a writing simplification with respect to [11], where the scalar product only occurred when \( P_j \) was a breather: we do not need to make this distinction now.

Step 5: Bound from above for \( \left| \int \tilde{P}_j w \sqrt{\Phi_j} \right| \)

We now show that the seemingly problematic scalar product \( \int \tilde{P}_j w \) is actually quadratic (which the second part of the induction hypothesis \( P_j \)).

Lemma 17 For any \( t \geq T_5 \),

\[
\left| \int \tilde{P}_j w \sqrt{\Phi_j} \right| \leq C e^{-2\nu t} + C \int (w^2 + w_x^2) \Phi_j. \tag{23}
\]

Proof The proof follows the lines of that of Step 7 of Proposition 4.10 in [11, Section 4.2], and we only sketch it.

First, we observe that it is enough to prove the same bound on

\[
\left| \sum_{i=1}^{j} \int \tilde{P}_i w \right|
\]

because the induction hypothesis \( P_{j-1} \) (10) takes care of the terms \( i \leq j - 1 \), and the localization \( \sqrt{\Phi_j} \) causes an error of size \( O(e^{-2\nu t}) \). Then, the idea is to go back to Lemma 11, and work on \( M_j \) and on \( E_j, F_j \) separately: both make the scalar products \( \int \tilde{P}_i w \) appear, but with opposite signs.

On the one hand, due to (13) and (16), we may deduce that from Lemmas 11 and 10 that

\[
\sum_{i=1}^{j} \int \tilde{P}_i w \leq C e^{-2\nu t}. \tag{24}
\]

On the other hand, due to (14)–(15), and using the elliptic equation satisfied by \( \tilde{P}_j \) and the induction assumption, we may infer that for \( t \geq T_5 \),

\[
-(1 - \nu)(a_j^2 + b_j^2) \sum_{i=1}^{j} \int \tilde{P}_i w
\]

\[
= O(e^{-2\nu t}) + O \left( \int (w^2 + w_x^2) \Phi_j \right)
\]

\[
+ F_j(t) - \sum_{i=1}^{j} \left( F[P_i] + 2(b_j^2 - a_j^2)E[P_i] + \nu(a_j^2 + b_j^2)^2 M[P_i] \right).
\]

Observe the factor \( 1 - \nu > 0 \) that we placed: this make the functional \( F_j \) appear, and not merely the Lyapunov \( H_j \) (for which the linear terms cancel). Therefore, we are now in position of using the monotonicity (19), and we obtain that

\[
-(1 - \nu)(a_j^2 + b_j^2) \sum_{i=1}^{j} \int \tilde{P}_i w \leq C e^{-2\nu t} + C \int (w^2 + w_x^2) \Phi_j. \tag{25}
\]
Estimates (24) and (25) yield together that
\[
\left| \sum_{i=1}^{j} \int \tilde{P}_i w \right| \leq Ce^{-2\sigma t} + C \int (w^2 + w_x^2) \Phi_j,
\]
which gives the desired conclusion. \(\square\)

**Step 6: Conclusion**

From (22) and (23), we deduce that for all \(t \geq T_5\),
\[
\int (w^2 + w_x^2 + w_{x}^2) \Phi_j \leq Ce^{-2\sigma t}.
\]
Plugging this in (23), an immediate localization argument shows that
\[
\left| \int \tilde{P}_j w \right| \leq Ce^{-2\sigma t}.
\]
Using also \(\mathcal{P}_{j-1}\), this proves \(\mathcal{P}_j (10)\), and the induction is complete. Thus, for \(j = J\), we get
\[
\forall t \geq T_5, \quad \|u(t) - P(t)\|_{H^2} \leq Ce^{-\sigma t}.
\]
Now, recall that from \(\|u(t) - P(t)\|_{H^2} \to 0\) and (11), there hold
\[
|y_{1,k}(t)| + |y_{2,k}(t)| + |y_{0,l}(t)| \to 0 \text{ as } t \to +\infty.
\]
Then, by integration from (12) on \([t, +\infty)\) that for any \(t \geq T_5\),
\[
|y_{1,k}(t)| + |y_{2,k}(t)| + |y_{0,l}(t)| \leq Ce^{-\sigma t}.
\]
So, by the mean-value theorem,
\[
\|u(t) - P(t)\|_{H^2} \leq \|w(t)\|_{H^2} + \|\tilde{P}(t) - P(t)\|_{H^2}
\]
\[
\leq \|w(t)\|_{H^2} + \sum_{k=1}^{K} (|y_{1,k}(t)| + |y_{2,k}(t)|) + \sum_{l=1}^{L} |y_{0,l}(t)|
\]
\[
\leq Ce^{-\sigma t}.
\]
This concludes the proof of Proposition 7.

**2.2 Proof of Theorem 6**

Let us recall the uniqueness result of [11] in the class of super-polynomial convergence to the profile.

**Proposition 18** Given breathers (1) and solitons (2), whose velocities satisfy (3), let \(P\) be the sum of the considered solitons and breathers given in (4).

There exists \(N > 0\) large enough such that the multi-breather from [11] \(p \in \mathcal{C}(T^*, +\infty), H^2\) associated to \(P\) is the unique solution \(u \in \mathcal{C}([T_0, +\infty), H^2(\mathbb{R}))\) of (mKdV) such that
\[
\|u(t) - P(t)\|_{H^2} = O \left( \frac{1}{t^N} \right) \quad \text{as } t \to +\infty
\]

Theorem 6 is then an immediate consequence of Propositions 18 and 7.
References

1. Alejo, M.A., Muñoz, C.: Nonlinear stability of MKdV breathers. Commun. Math. Phys. 324, 233–262 (2013)
2. Côte, R., Friederich, X.: On smoothness and uniqueness of multi-solitons of the non-linear Schrödinger equations. Commun. Partial Differ. Equ. 46, 2325–2385 (2021)
3. Côte, R., Muñoz, C.: Multi-solitons for nonlinear Klein-Gordon equations. Forum Math. Sigma. 2, e15 (2014)
4. Côte, R., Martel, Y.: Multi-travelling waves for the nonlinear Klein-Gordon equation. Trans. Amer. Math. Soc. 370, 7461–7487 (2018)
5. El Dika, K., Martel, Y.: Stability of N solitary waves for the generalized BBM equations. Dyn. Partial Differ. Equ. 1, 401–437 (2004)
6. Kenig, C.E., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Commun. Pure Appl. Math. 46, 527–620 (1993)
7. Martel, Y., Merle, F.: Multi solitary waves for nonlinear Schrödinger equations. Ann. Inst. H. Poincaré C Anal. Non Linéaire 23, 849–864 (2006)
8. Martel, Y.: Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. Amer. J. Math. 127, 1103–1140 (2005)
9. Ming, M., Rousset, F., Tzvetkov, N.: Multi-solitons and related solutions for the water-waves system. SIAM J. Math. Anal. 47, 897–954 (2015)
10. Perelman, T., Fridman, A., El’yashevich, M.: A modified Korteweg-de Vries equation in electrohydrodynamics. Sov. Phys. JETP 39, 643–646 (1974)
11. Semenov, A.: On the uniqueness of multi-breathers of the modified Korteweg–de Vries equation. Rev. Mat. Iberoam. (2022). https://doi.org/10.4171/RMI/1363 arXiv:2109.03584(2021)
12. Valet, F.: Asymptotic K-soliton-like solutions of the Zakharov-Kuznetsov type equations. Trans. Amer. Math. Soc. 374, 3177–3213 (2021)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.