MODE REGULARIZATION OF THE CONFIGURATION
SPACE PATH INTEGRAL FOR A PARTICLE IN CURVED
SPACE

F. BASTIANELLI
Dipartimento di Fisica, Università di Modena, via Campi 213-A, I-41100 Modena, and
INFN, Sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy
E-mail: bastiane@unimo.it

The proper definition and evaluation of the configuration space path integral for
the motion of a particle in curved space is a notoriously tricky problem. We
discuss a consistent definition which makes use of an expansion in Fourier sine
series of the particle paths. Salient features of the regularization are the Lee-Yang
ghost fields and a specific effective potential to be added to the classical action.
The Lee-Yang ghost fields are introduced to exponentiate the non-trivial path
integral measure and make the perturbative loop expansion finite order by order,
whereas the effective potential is necessary to maintain the general coordinate
invariance of the model. We also discuss a three loop computation which tests the
mode regularization scheme and reproduces consistently the perturbative De Witt
solution of the heat kernel.

1 Introduction

The Schrödinger equation for a particle moving in a curved space with metric
g_{\mu\nu}(x) has a wide range of applications going from non-relativistic diffusion
problems (described by its euclidean version) to the relativistic description of
particles in a curved space-time. However it cannot be solved exactly for an
arbitrary background metric and one has to use some kind of perturbation
theory. A very useful perturbative solution can be obtained by employing
an ansatz introduced by De Witt\(^1\), known as the heat kernel ansatz. This
ansatz makes use of a power series expansion in the time of propagation of the
particle. The coefficients of the power series are then determined iteratively
by requiring that the Schrödinger equation be satisfied perturbatively.

Equivalently, the solution of the Schrödinger equation should admit a path
integral representation, as proposed by Feynman fifty years ago. However, a
proper definition of the path integral in curved space is not straightforward,
and the history of this subject has been quite complicated and controversial\(^2\).
The essential complications arise mainly from: i) the non-trivial path integral
measure and ii) the proper discretization of the action necessary to regulate
the path integral. A quite extensive literature has been produced over the
years addressing especially the latter point.
In this talk we discuss a consistent method of defining the path integral by employing mode regularization, as it is standard in many calculations done in quantum field theory. The methods extends the one already employed by Feynman and Hibbs in discussing mode regularization of the path integral in flat space and has been introduced and successively refined in [4], where quantum mechanics was used mostly to compute one loop trace anomalies of certain quantum field theories. The key feature is to employ ghost fields to treat the non-trivial path integral measure as part of the action. These ghost fields have been named Lee-Yang ghosts and allow to take care of the non-trivial path integral measure at higher loops in a consistent manner. The path integral is then defined by expanding all fields, including the ghosts, in a sine expansion about the classical trajectories (or any trajectory which includes the correct boundary values) and integrating over the corresponding Fourier coefficients. The necessary regularization is obtained by integrating all coefficients up to a fixed mode $M$, which is eventually taken to infinity. A drawback of mode regularization is that it doesn’t respect general coordinate invariance of the target space: a particular non-covariant counterterm arising at two loops has to be used in order to restore that symmetry. General arguments based on power counting (quantum mechanics can be thought as a super-renormalizable field theory) plus the fact that the correct trace anomalies are obtained by the use of this path integral suggest that the mode regularization described above is consistent to any loop order without any additional input. We have tested this construction by a full three loop computation of the transition amplitude. The result is found to be right since it solves the correct Schrödinger equation at the required loop order. This gives a powerful check on the method of mode regularization for quantum mechanical path integrals on curved space and puts it on solid basis. In addition, the computation can be readily extended and compared to the time discretization method developed in refs. 7, which is also based on the use of the Lee-Yang ghosts. That method requires its own specific counterterm to restore general coordinate invariance and we checked that it gives the same correct answer.

In the next section we present the relevant items of mode regularization and give the three loop transition amplitude in Riemann normal coordinates.

2 Mode Regularization and the 3 Loop Transition Amplitude

The euclidean Schrödinger equation for a particle of unit mass moving in a $D$-dimensional space with metric $g_{\mu\nu}$ and coupled to a scalar potential $V$ is

$$\frac{-\partial}{\partial t}\Psi = \left[\frac{-1}{2}\nabla^2 + V(x)\right]\Psi. \quad (1)$$
It can be solved perturbatively by the heat kernel ansatz of De Wit:

\[ \Psi(x, y, t) = \frac{1}{(2\pi t)^{D/2}} e^{-\frac{\sigma(x, y)}{2t}} \sum_{n=0}^{\infty} a_n(x, y) t^n \]  

which depends parametrically on the point \( y^\mu \) specifying the boundary condition \( \Psi(x, y, 0) = g^{-1/2} \delta_D(x - y) \) and with \( \sigma(x, y) \) the Synge world function.

On the other hand, following refs. 3, 4, 5, 6, we can write the transition amplitude for the particle to propagate from the initial point \( x_i^\mu \) at time \( t_i \) to the final point \( x_f^\mu \) at time \( t_f \) as follows

\[ \langle x_f^\mu, t_f | x_i^\mu, t_i \rangle = \int Dx D\alpha Dc D\bar{c} \exp \left[ -\frac{1}{\beta} S \right] \]

\[ S = \int_{-1}^{0} d\tau \left[ \frac{1}{2} g_{\mu\nu}(x) (\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + \beta^2 \left( V(x) + V_{MR}(x) \right) \right] \]

\[ V_{MR} = \frac{1}{8} R - \frac{1}{24} g^{\mu\nu} g^{\alpha\beta} g_{\gamma\delta} \Gamma_{\mu\alpha\gamma} \Gamma_{\nu\beta\delta} \]

\[ Dx D\alpha Dc D\bar{c} = (2\pi^D)^{-3/2} \prod_{m=1}^{\infty} \prod_{\mu=1}^{D} m \, dx_m^\mu da_m^\mu db_m^\mu dc_m^\mu. \]

For commodity we have shifted and rescaled the time parameter in the action, \( t = t_f + \beta \tau \) with \( \beta = t_f - t_i \), so that \( -1 \leq \tau \leq 0 \). The total time of propagation \( \beta \) plays the role of the Planck constant \( \hbar \) (which we have set to one) and counts the number of loops. In the loop expansion generated by \( \beta \) the potentials \( V \) and \( V_{MR} \) start contributing only at two loops. The full action \( S \) includes terms proportional to the Lee-Yang ghosts, namely the commuting ghosts \( a^\mu \) and the anticommuting ghosts \( b^\mu \) and \( c^\mu \). Their effect is to reproduce a formally covariant measure. The regulated measure is obtained by expanding all fields in a sine series around a background value

\[ \phi^\mu(\tau) = \phi_{bg}^\mu(\tau) + \phi_{qu}^\mu(\tau); \quad \phi_{qu}^\mu(\tau) = \sum_{m=1}^{\infty} \phi_{m}^\mu \sin(\pi m \tau) \]

where \( \phi = (x^\mu, a^\mu, b^\mu, c^\mu) \) and \( \phi_{bg} = (x_{bg}^\mu, 0, 0, 0) \), and defining it as product of the standard measures for the coefficients \( \phi_{m}^\mu \) as in (3). The regularization is obtained by integrating up to a fixed mode number \( M \) that eventually is taken to infinity. This way the path integral (3) solves eq. (3). In fact, one can start computing it by using the standard perturbative expansion and using...
Riemann normal coordinates centred at $x^f_i$. Ambiguities in Feynman diagrams are resolved by mode regularization. Thus one gets the transition amplitude at 3 loops which is shown to solves eq. (8) at the required loop order. The result can be cast in a manifestly symmetric form by using symmetrized quantities $A \equiv \frac{1}{2}[A(x_i) + A(x_f)]$ and with $\xi^\mu \equiv x^\mu_i - x^\mu_f$ considered of order $\beta$. 

$$\langle x^\mu, t_f|x^\mu_i, t_i \rangle = \frac{1}{(2\pi\beta)^{2}} \exp \left[ -\frac{1}{2\beta} \xi^\mu \xi^\nu g_{\mu\nu} - \frac{1}{12} \xi^\alpha \xi^\beta R_{\alpha\beta} - \beta \left( \frac{1}{12} R + V \right) \right]$$

$$+ \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta \left( \frac{1}{360} R_{\alpha\mu\beta\gamma} R_{\delta\nu\delta} + \frac{1}{120} \nabla_{\gamma} \nabla_{\delta} R_{\alpha\beta} \right)$$

$$+ \beta \xi^\alpha \xi^\beta \left( \frac{1}{360} R_{\alpha\mu\lambda\beta} R_{\rho\nu\rho\delta} - \frac{1}{720} R_{\alpha\mu\beta\lambda} R_{\rho\nu} - \frac{1}{720} R_{\alpha\mu} R_{\beta\nu} \right)$$

$$- \frac{1}{240} \nabla^\mu \nabla^\nu R_{\alpha\mu\beta} + \frac{1}{160} \nabla_{\alpha} \nabla_{\beta} R_{\mu\nu} + \frac{1}{12} \nabla_{\alpha} \nabla_{\beta} V$$

$$+ \beta^2 \left( \frac{1}{720} R_{\alpha\mu\beta}^2 - \frac{1}{720} R_{\alpha\beta}^2 - \frac{1}{120} \nabla^2 R - \frac{1}{12} \nabla^2 V \right) + O(\beta^3).$$

By comparing this result with the expression in (2) one can extract the leading terms of the coefficients $a_0, a_1, a_2$ for non-coinciding points.

Acknowledgements

I wish to thank O. Corradini, K. Schalm and P. van Nieuwenhuizen for their invaluable collaboration at various stages of this research.

References

1. B. De Witt in *Relativity, Groups and Topology*, ed. B. De Witt and C. De Witt (Gordon and Breach, N.Y., 1964); *Relativity, Groups and Topology II*, ed. B. De Witt and R. Stora (North Holland, Amsterdam, 1984).
2. see ch. 24 of: L. Schulman, *Techniques and Applications of Path Integration*. (J. Wiley and Sons, New York 1981).
3. F. Bastianelli, *Nucl. Phys.* B 376, 113 (1992).
4. F. Bastianelli and P. van Nieuwenhuizen, *Nucl. Phys.* B 389, 53 (1993).
5. F. Bastianelli, K. Schalm and P. van Nieuwenhuizen, *Phys. Rev.* D 58, 044002 (1998).
6. F. Bastianelli and O. Corradini, [hep-th/9810119](http://arxiv.org/abs/hep-th/9810119).
7. J. de Boer, B. Peeters, K. Skanderis and P. van Nieuwenhuizen, *Nucl. Phys.* B 446, 211 (1995); *Nucl. Phys.* B 459, 631 (1996).