Research Article

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A combinatorial proof of the Gaussian product inequality in the MTP$_2$ case

Abstract: A combinatorial proof of the Gaussian product inequality (GPI) is given under the assumption that each component of a centered Gaussian random vector $X = (X_1, \ldots, X_d)$ of arbitrary length can be written as a linear combination, with coefficients of identical sign, of the components of a standard Gaussian random vector. This condition on $X$ is shown to be equivalent (in a non-trivial way) to the assumption that the density of the random vector $(|X_1|, \ldots, |X_d|)$ is multivariate totally positive of order 2, abbreviated MTP$_2$, for which the GPI is already known to hold. In addition to giving a new characterization of the MTP$_2$ class for nonsingular centered Gaussian random vectors, the paper highlights a new link between the GPI and the monotonicity of a certain ratio of gamma functions.

Keywords: Complete monotonicity, gamma function, Gaussian product inequality, Gaussian random vector, moment inequality, multinomial, multivariate normal, polygamma function.

MSC: Primary 60E15; Secondary 05A20, 33B15, 62E15, 62H10, 62H12

1 Introduction

The Gaussian product inequality (GPI) is a long-standing conjecture which states that for any centered Gaussian random vector $X = (X_1, \ldots, X_d)$ of dimension $d \in \{1, 2, \ldots\}$ and every integer $m \in \mathbb{N}$, one has

$$E \left( \prod_{i=1}^{d} X_i^{2m} \right) \geq \prod_{i=1}^{d} E(X_i^{2m}). \quad (1)$$

This inequality is known to imply the real polarization problem conjecture in functional analysis [12] and it is related to the so-called U-conjecture to the effect that if $P$ and $Q$ are two non-constant polynomials on $\mathbb{R}^d$ such that the random variables $P(X)$ and $Q(X)$ are independent, then there exist an orthogonal transformation $L$ on $\mathbb{R}^d$ and an integer $k \in \{1, \ldots, d-1\}$ such that $P \circ L$ is a function of $(X_1, \ldots, X_k)$ and $Q \circ L$ is a function of $(X_{k+1}, \ldots, X_d)$; see, e.g., [6, 12] and references therein.

Inequality (1) is well known to be true when $m = 1$; see, e.g., Frenkel [5]. Karlin and Rinott [7] also showed that it holds when the random vector $|X| = (|X_1|, \ldots, |X_d|)$ has a multivariate totally positive density of order 2, denoted MTP$_2$. As stated in Remark 4.1 of their paper, the latter condition is verified, among others, in dimension $d = 2$ for all nonsingular Gaussian random pairs.

Interest in the problem has recently gained traction when Lan et al. [9] established the inequality in dimension $d = 3$. Hope that the result might be true in general is also fueled by the fact, established by Wei [21], that for any reals $\alpha_1, \ldots, \alpha_d \in (-1, 0)$, one has

$$E \left( \prod_{i=1}^{d} |X_i|^{2\alpha_i} \right) \geq \prod_{i=1}^{d} E(|X_i|^{2\alpha_i}). \quad (2)$$

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Li and Wei [11] have further conjectured that the latter inequality holds for all reals $\alpha_1, \ldots, \alpha_d \in [0, \infty)$ and any centered Gaussian random vector $X$.

The purpose of this paper is to report a combinatorial proof of inequality (2) in the special case where the reals $\alpha_1, \ldots, \alpha_d$ are nonnegative integers and when each of the components $X_1, \ldots, X_d$ of the centered Gaussian random vector $X$ can be written as a linear combination, with coefficients of identical sign, of the components of a standard Gaussian random vector. A precise statement of this assumption is given as Condition (IV) in Section 2 and the proof of the main result, Proposition 2, appears in Section 3. It is then shown in Section 4, see Proposition 3, that this condition is actually equivalent to the assumption that the random vector $|X|$ is MTP$_2$.

As inequality (2) for all reals $\alpha_1, \ldots, \alpha_d \in [0, \infty)$ is already known to be true for MTP$_2$ random vectors, this paper’s contribution does not reside in the result itself but in the way in which it is proved using (i) an original combinatorial argument closely related to the complete monotonicity of multinomial probabilities previously shown by Ouimet [13] and Qi et al. [14]; and (ii) a new characterization of the MTP$_2$ class for nonsingular centered Gaussian random vectors, given in Proposition 3.

All background material required to understand the contribution and put it in perspective is provided in Section 2. The statements and proofs of the paper’s results are then presented in Sections 3 and 4. The paper concludes with a brief discussion in Section 5. For completeness, a technical lemma due to Ouimet [13], which is used in the proof of Proposition 2, is included in the Appendix.

## 2 Background

First recall the definition of multivariate total positivity of order 2 (MTP$_2$) on a set $S \subseteq \mathbb{R}^d$.

**Definition 1.** A density $f : \mathbb{R}^d \rightarrow [0, \infty)$ supported on $S$ is said to be multivariate totally positive of order 2, denoted MTP$_2$, if and only if, for all vectors $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in S$, one has

$$f(x \lor y)f(x \land y) \geq f(x)f(y),$$

where $x \lor y = (\max(x_1, y_1), \ldots, \max(x_d, y_d))$ and $x \land y = (\min(x_1, y_1), \ldots, \min(x_d, y_d))$.

Densities in this class have many interesting properties, including the following result, which corresponds to Eq. (1.7) of Karlin and Rinott [7].

**Proposition 1.** Let $Y$ be an MTP$_2$ random vector on $S$, and let $\varphi_1, \ldots, \varphi_r$ be a collection of nonnegative and (component-wise) non-decreasing functions on $S$. Then

$$\mathbb{E} \left\{ \prod_{i=1}^r \varphi_i(Y) \right\} \geq \prod_{i=1}^r \mathbb{E} \{ \varphi_i(Y) \}.$$

In particular, let $X = (X_1, \ldots, X_d)$ be a $d$-variate Gaussian random vector with zero mean and nonsingular covariance matrix $\text{var}(X)$. Suppose that the following condition holds.

(1) The random vector $|X| = (|X_1|, \ldots, |X_d|)$ belongs to the MTP$_2$ class on $[0, \infty)^d$.

Under Condition (1), the validity of the GPI conjecture (2) for all reals $\alpha_1, \ldots, \alpha_d \in [0, \infty)$ follows from Proposition 1 with $r = d$ and maps $\varphi_1, \ldots, \varphi_d$ defined, for every vector $y = (y_1, \ldots, y_d) \in [0, \infty)^d$ and integer $i \in \{1, \ldots, d\}$, by

$$\varphi_i(y) = y_i^{2\alpha_i}.$$

When $X = (X_1, \ldots, X_d)$ is a centered Gaussian random vector with covariance matrix $\text{var}(X)$, Theorem 3.1 of Karlin and Rinott [7] finds an equivalence between Condition (1) and the requirement
that the off-diagonal elements of the inverse of \( \text{var}(X) \) are all nonpositive up to a change of sign for some of the components of \( X \). The latter condition can be stated more precisely as follows using the notion of signature matrix, which refers to a diagonal matrix whose diagonal elements are \( \pm 1 \).

(II) There exists a \( d \times d \) signature matrix \( D \) such that the covariance matrix \( \text{var}(DX)^{-1} \) only has nonpositive off-diagonal elements.

Two other conditions of interest on the structure of the random vector \( X \) are as follows.

(III) There exists a \( d \times d \) signature matrix \( D \) such that the covariance matrix \( \text{var}(DX) \) has only nonnegative elements.

(IV) There exists a \( d \times d \) signature matrix \( D \) and a \( d \times d \) matrix \( C \) with entries in \( [0, \infty) \) such that the random vector \( DX \) has the same distribution as the random vector \( CZ \), where \( Z \sim \mathcal{N}_d(0_d, I_d) \) is a \( d \times 1 \) Gaussian random vector with zero mean vector \( 0_d \) and identity covariance matrix \( I_d \).

Recently, Russell and Sun \[19\] used Condition (III) to show that, for all integers \( d \in \mathbb{N} \), \( n_1, \ldots, n_d \in \mathbb{N}_0 = \{0, 1, \ldots \} \) and \( k \in \{1, \ldots, d - 1 \} \), and up to a change of sign for some of the components of \( X \),

\[
E \left( \prod_{i=1}^d X_i^{2n_i} \right) \geq E \left( \prod_{i=1}^k X_i^{2n_i} \right) E \left( \prod_{i=k+1}^d X_i^{2n_i} \right). \tag{3}
\]

This result was further extended by Edelmann et al. \[4\] to the case where the random vector \((X_1^2, \ldots, X_d^2)\) has a multivariate gamma distribution in the sense of Krishnamoorthy and Parthasarathy \[8\]. Moreover, Condition (III) was considered by Bølviken \[3\] in the context of the Gaussian correlation inequality (GCI) conjecture, and Proposition 2.2 in that author’s paper shows that Condition (III) implies Condition (II).

In the following section, it will be shown how Condition (IV) can be used to give a combinatorial proof of a weak form of inequality (3). It will then be seen in Section 4 that Condition (II) implies Condition (IV), thereby proving the equivalence between Conditions (I)–(IV) and hence providing a new characterization of the MTP\(_2\) condition for nonsingular centered Gaussian random vectors. For clarity, the relations between Conditions (I)–(IV) are summarized in Fig. 1.
3 A combinatorial proof of the GPI conjecture

The following result, which is this paper’s main result, shows that the extended GPI conjecture of Li and Wei \[2\] given in \[2\] holds true under Condition (IV) when the reals $\alpha_1, \ldots, \alpha_d$ are nonnegative integers. This result also follows from inequality \[2\], due to Russell and Sun \[13\], but the argument below is completely different from the latter authors’ derivation based on Condition (III).

**Proposition 2.** Let $X = (X_1, \ldots, X_d)$ be a $d$-variate centered Gaussian random vector. Assume that there exist a $d \times d$ signature matrix $D$ and a $d \times d$ matrix $C$ with entries in $[0, \infty)$ such that the random vector $DX$ has the same distribution as the random vector $CZ$, where $Z \sim N_d(0_d, I_d)$ is an $d$-dimensional standard Gaussian random vector. Then, for all integers $n_1, \ldots, n_d \in \mathbb{N}_0 = \{0, 1, \ldots\}$,

$$
E \left( \prod_{i=1}^d X_i^{2n_i} \right) \geq \prod_{i=1}^d E \left( X_i^{2n_i} \right).
$$

**Proof.** In terms of $Z$, the claimed inequality is equivalent to

$$
E \left\{ \prod_{i=1}^d \left( \sum_{j=1}^d c_{ij} Z_j \right)^{2n_i} \right\} \geq \prod_{i=1}^d \left\{ \sum_{j=1}^d c_{ij} Z_j \right\}^{2n_i}.
$$

(4)

For each integer $j \in \{1, \ldots, d\}$, set $K_j = k_{1j} + \cdots + k_{dj}$ and $L_j = \ell_{1j} + \cdots + \ell_{dj}$, where $k_{ij}$ and $\ell_{ij}$ are nonnegative integer-valued indices to be used in expressions (5) and (6) below.

By the multinomial formula, the linearity of expectations, and the mutual independence of the components of the random vector $Z$, the left-hand side of inequality (4) can be expanded as follows:

$$
E \left\{ \prod_{i=1}^d \sum_{k_{i1}, \ldots, k_{id} \in \mathbb{N}_0^d: k_{i1} + \cdots + k_{id} = 2n_i} \binom{2n_i}{k_{i1}, \ldots, k_{id}} \prod_{j=1}^d c_{ij}^{k_{ij}} Z_j^{k_{ij}} \right\}
$$

$$
= \sum_{k_{i1}, \ldots, k_{id} \in \mathbb{N}_0^d: k_{i1} + \cdots + k_{id} = 2n_i} \prod_{i=1}^d E(K_j) \prod_{j=1}^d c_{ij}^{k_{ij}}.
$$

(5)

Given that the coefficients $c_{ij}$ are all nonnegative by assumption, and exploiting the fact that, for every integer $j \in \{1, \ldots, d\}$ and $m \in \mathbb{N}_0$,

$$
E(Z_j^{2m}) = \frac{(2m)!}{2^m m!},
$$

one can bound the left-hand side of inequality (4) from below by

$$
\sum_{\ell_{11} + \cdots + \ell_{id} = 2n_1} \cdots \sum_{\ell_{d1} + \cdots + \ell_{id} = 2n_d} \left\{ \prod_{j=1}^d E \left( Z_j^{2L_j} \right) \right\} \prod_{j=1}^d c_{ij}^{2\ell_{ij}}
$$

$$
= \sum_{\ell_{11} + \cdots + \ell_{id} = n_1} \cdots \sum_{\ell_{d1} + \cdots + \ell_{id} = n_d} \left\{ \prod_{j=1}^d \binom{2L_j}{2\ell_{ij}} \right\} \prod_{j=1}^d c_{ij}^{2\ell_{ij}}.
$$

(6)
The right-hand side of (4) can be expanded in a similar way. Upon using the fact that $E(Y^{2m}) = (2m)!σ^{2m}/(2^m m!)$ for every integer $m \in \mathbb{N}_0$ when $Y \sim \mathcal{N}(0, σ^2)$, one finds

$$\prod_{i=1}^{d} \left( \sum_{j=1}^{d} c_{ij} Z_j \right)^{2n_i} = \prod_{i=1}^{d} \frac{(2n_i)!}{2^{n_i} n_i!} \left( \sum_{j=1}^{d} c_{ij}^2 \right)^{n_i} = \prod_{i=1}^{d} \frac{(2n_i)!}{2^{n_i} n_i!} \sum_{\ell_i \in \mathbb{N}_0^d: \ell_i = n_i} \left( \prod_{j=1}^{d} c_{ij} \right)^{n_i} \prod_{j=1}^{d} 2^{l_{ij}}.$$  

$$= \prod_{\ell_i \in \mathbb{N}_0^d: \ell_i = n_i} \cdots \prod_{\ell_i \in \mathbb{N}_0^d: \ell_i = n_i} \prod_{j=1}^{d} \frac{(2n_i)!}{2^{n_i} n_i!} \left( \prod_{j=1}^{d} c_{ij} \right)^{n_i} \prod_{j=1}^{d} 2^{l_{ij}}. \tag{6}$$

Next, compare the coefficients of the corresponding powers $c_{ij}^{2l_{ij}}$ in expressions (5) and (6). In order to prove inequality (4), it suffices to show that, for all integer-valued vectors $\ell_1, \ldots, \ell_d \in \mathbb{N}_0^d$ satisfying $\ell_1 + \cdots + \ell_id = n_i$ for every integer $i \in \{1, \ldots, d\}$,

$$\prod_{j=1}^{d} \frac{(2L_{ij})!}{2^{L_{ij}} L_{ij}!} \geq \prod_{j=1}^{d} \frac{L_{ij}!}{2^{L_{ij}} L_{ij}!}.$$  

Taking into account the fact $2^{L_{1j}+\cdots+L_{dj}} = 2^{n_1+\cdots+n_d}$, and after cancelling some factorials, one finds that the above inequality reduces to

$$\prod_{j=1}^{d} \frac{(2L_{ij})!}{2^{L_{ij}} L_{ij}!} \geq \prod_{j=1}^{d} \frac{L_{ij}!}{2^{L_{ij}} L_{ij}!}. \tag{7}$$

Therefore, the proof is complete if one can establish inequality (7). To this end, one can assume without loss of generality that the integers $L_1, \ldots, L_d$ are all non-zero; otherwise, inequality (7) reduces to a lower-dimensional case. For any given integers $L_1, \ldots, L_d \in \mathbb{N}$ and every integer $j \in \{1, \ldots, d\}$, define the function

$$a \mapsto g_j(a) = \frac{\Gamma(aL_j + 1)}{\prod_{i=1}^{d} \Gamma(aL_{ij} + 1)},$$

on the interval $(-1/L_j, \infty)$, where $\Gamma$ denotes Euler’s gamma function.

To prove inequality (7), it thus suffices to show that, for every integer $j \in \{1, \ldots, d\}$, the map $a \mapsto \ln\{g_j(a)\}$ is non-decreasing on the interval $[0, \infty)$. Direct computations yield, for every real $a \in [0, \infty)$,

$$\frac{d}{da} \ln\{g_j(a)\} = L_j \psi(aL_j + 1) - \sum_{i=1}^{d} \ell_{ij} \psi(aL_{ij} + 1),$$

$$\frac{d^2}{da^2} \ln\{g_j(a)\} = L_j^2 \psi(aL_j + 1) - \sum_{i=1}^{d} \ell_{ij}^2 \psi'(aL_{ij} + 1),$$

where $\psi = (\ln \Gamma)'$ denotes the digamma function. Now call on the integral representation (11) p. 260

$$\psi'(z) = \int_{0}^{\infty} \frac{te^{-(z+1)t}}{e^t - 1} dt,$$

valid for every real $z \in (0, \infty)$ to write

$$\frac{d^2}{da^2} \ln\{g_j(a)\} = \int_{0}^{\infty} \frac{(L_j t)e^{-a(L_j t)}}{e^t - 1} L_j dt = \sum_{i=1}^{d} \int_{0}^{\infty} \frac{(L_{ij} t)e^{-a(L_{ij} t)}}{e^t - 1} \ell_{ij} dt$$

$$= \int_{0}^{\infty} \frac{se^{-as}}{e^{s/L_j} - 1} - \sum_{i=1}^{d} \frac{1}{(e^{s/L_j}L_j/\ell_{ij} - 1)} ds. \tag{8}$$
Given that \((\ell_{i1} + \cdots + \ell_{id})/L_j = 1\) by construction, the quantity within braces in Eq. (8) is always nonnegative by Lemma 1.4 of Ouimet [13]; this can be checked upon setting \(y = e^{s/L_j}\) and \(u_i = \ell_{ij}/L_j\) for every integer \(i \in \{1, \ldots, d\}\) in that paper’s notation. Alternatively, see p. 516 of Qi et al. [14]. Therefore,

\[
\forall a \in [0, \infty) \quad \frac{d^2}{da^2} \ln \{g_j(a)\} \geq 0.
\]  

In fact, the map \(a \mapsto \frac{d^2}{da^2} \ln \{g_j(a)\}\) is even completely monotonic. Moreover, given that

\[
\frac{d}{da} \ln \{g_j(a)\} \bigg|_{a=0} = L_j \psi(1) - \sum_{i=1}^{d} \ell_{ij} \psi(1) = 0 \times \psi(1) = 0,
\]

one can deduce from inequality (9) that

\[
\forall a \in [0, \infty) \quad \frac{d}{da} \ln \{g_j(a)\} \geq 0.
\]

Hence the map \(a \mapsto \ln \{g_j(a)\}\) is non-decreasing on \([0, \infty)\). This concludes the argument. \(\Box\)

4 A new characterization of the MTP\(_2\) condition

This paper’s second main result proves that Condition (II) implies Condition (IV). In view of Fig. 1 one may then conclude that Conditions (I)–(IV) are equivalent.

**Proposition 3.** Let \(\Sigma\) be a symmetric positive definite matrix with Cholesky decomposition \(\Sigma = CC^\top\). If the off-diagonal entries of \(\Sigma^{-1}\) are all nonpositive, then the elements of \(C\) are all nonnegative.

**Proof.** The proof is by induction on the dimension \(d\) of \(\Sigma\). The claim trivially holds when \(d = 1\). Assume that it is verified for some integer \(n \in \mathbb{N}\), and fix \(d = n + 1\). Given the assumptions on \(\Sigma\), one can write

\[
\Sigma^{-1} = \begin{pmatrix} a & v^\top \\ v & B \end{pmatrix}
\]

in terms of a real \(a \in (0, \infty)\), an \(n \times 1\) vector \(v\) with nonpositive components, and an \(n \times n\) matrix \(B\) with nonpositive off-diagonal entries.

Given that \(\Sigma\) is symmetric positive definite by assumption, so is \(\Sigma^{-1}\), and hence so are \(B\) and \(B^{-1}\). Moreover, the off-diagonal entries of \(B = (B^{-1})^{-1}\) are nonpositive and hence by induction, the factor \(L\) in the Cholesky decomposition \(B^{-1} = LL^\top\) has nonnegative entries. Letting \(w = a - v^\top LL^\top v\) denote the Schur complement, which is strictly positive, one has

\[
\Sigma^{-1} = \begin{pmatrix} a & v^\top \\ v & (LL^\top)^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{w} & \sqrt{w} L \\ 0_n & (L^\top)^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{w} & 0_n^\top \\ 0_n & L^{-1} \end{pmatrix},
\]

where \(0_n\) is an \(n \times 1\) vector of zeros. Accordingly,

\[
\Sigma = \begin{pmatrix} \sqrt{w} & 0_n^\top \\ L^\top v & L^{-1} \end{pmatrix}^{-1} \begin{pmatrix} \sqrt{w} & \sqrt{w} L \\ 0_n & (L^\top)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1/\sqrt{w} & 0_n^\top \\ -LL^\top v/\sqrt{w} & L \end{pmatrix} \begin{pmatrix} 1/\sqrt{w} & 0_n^\top \\ -LL^\top v/\sqrt{w} & L \end{pmatrix}^\top = CC^\top.
\]

Recall that \(w\) is strictly positive and all the entries of \(L\) and \(-v\) are nonnegative. Hence, all the elements of \(C\) are nonnegative, and the argument is complete. \(\Box\)

5 Discussion

Proposition 2 holds irrespective of whether the centered Gaussian random vector \(X\) is absolutely continuous or not. When \(X\) is nonsingular, the equivalence between Conditions (I)–(IV) implies that it is a
special case of Proposition 1, stated as Eq. (1.7) of Karlin and Rinott [7]. Proposition 2 is also implied by inequality 3 and its extension to the multivariate gamma setting, stated as Lemma 2.2 of Russell and Sun [19] and Theorem 2.1 of Edelmann et al. [4], respectively. However, the proof of the present result is entirely different from theirs.

Beyond its intrinsic interest, the approach to the proof of the GPI presented herein, together with its connection with the complete monotonicity of multinomial probabilities previously shown by Ouimet [13] and Qi et al. [14], hints to a deep relationship between the MTP2 class for the multivariate gamma distribution of Krishnamoorthy and Parthasarathy [8], the range of admissible parameter values for their infinite divisibility, and the complete monotonicity of their Laplace transform; see the work of Royen on the GCI conjecture [15–18] and Theorems 1.2 and 1.3 of Scott and Sokal [20]. These topics, and the proof or refutation of the GPI in its full generality, provide interesting avenues for future research.

Appendix: Technical lemma

The following result, used in the proof of Proposition 2, extends Lemma 1 of Alzer [2] from the case \( d = 1 \) to an arbitrary integer \( d \in \mathbb{N} \). It was already reported by Ouimet [13], see his Lemma 4.1, but its short statement and proof are included here to make the article more self-contained.

**Lemma A.1.** For every integer \( d \in \mathbb{N} \), and real numbers \( y \in (1, \infty) \) and \( u_1, \ldots, u_{d+1} \in (0, 1) \) such that \( u_1 + \cdots + u_{d+1} = 1 \), one has

\[
\frac{1}{y-1} > \sum_{i=1}^{d+1} \frac{1}{y^{1/u_i} - 1}. \tag{A.1}
\]

**Proof.** The proof is by induction on the integer \( d \). The case \( d = 1 \) is the statement of Lemma 1 of Alzer [2]. Fix an integer \( d \geq 2 \) and assume that inequality (A.1) holds for every smaller integer. Fix any reals \( y \in (1, \infty) \) and \( u_1, \ldots, u_d \in (0, 1) \) such that \( \|u\|_1 = u_1 + \cdots + u_d < 1 \). Write \( u_{d+1} = 1 - \|u\|_1 > 0 \). Calling on Alzer’s result, one has

\[
\frac{1}{y-1} > \frac{1}{y^{1/\|u\|_1} - 1} + \frac{1}{y^{1/(1-\|u\|_1)} - 1}.
\]

Therefore, the conclusion follows if one can show that

\[
\frac{1}{y^{1/\|u\|_1} - 1} > \sum_{i=1}^{d} \frac{1}{y^{1/u_i} - 1}.
\]

Upon setting \( z = y^{1/\|u\|_1} \) and \( v_i = u_i/\|u\|_1 \), one finds that the above inequality is equivalent to

\[
\frac{1}{z - 1} > \sum_{i=1}^{d} \frac{1}{z^{1/v_i} - 1},
\]

which is true by the induction assumption. Therefore, the argument is complete. \( \Box \)

**Acknowledgments.** C. Genest’s research is funded in part by the Canada Research Chairs Program (Grant no. 950–231937) and the Natural Sciences and Engineering Research Council of Canada (RGPIN–2016–04720). F. Ouimet received postdoctoral fellowships from the Natural Sciences and Engineering Research Council of Canada and the Fond québécois de la recherche – Nature et technologies (B3X supplement and B3XR). Thanks to Donald Richards for comments on an earlier version of this note.
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