FUSION PRODUCTS, COHOMOLOGY OF $GL_N$ FLAG MANIFOLDS AND KOSTKA POLYNOMIALS

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Abstract. This paper explains the relation between the fusion product of symmetric power $\mathfrak{sl}_n$ evaluation modules, as defined by Feigin and Loktev, and the graded coordinate ring $R_{\mu}$ which describes the cohomology ring of the flag variety $\mathcal{F}_{\mu'}$ of $GL_N$. The graded multiplicity spaces appearing in the decomposition of the fusion product into irreducible $\mathfrak{sl}_n$-modules are identified with the multiplicity spaces of the Specht modules in $R_{\mu}$. This proves that the Kostka polynomial gives the character of the fusion product in this case. In the case of the product of fundamental evaluation modules, we give the precise correspondence with the reduced wedge product, and thus the usual wedge space construction of irreducible level-1 $\mathfrak{sl}_n$-modules in the limit $N \rightarrow \infty$. The multiplicity spaces are $W(\mathfrak{sl}_n)$-algebra modules in this limit.

1. Introduction

The graded tensor product (fusion product) of finite-dimensional evaluation representations of the Lie algebra $g \otimes \mathbb{C}[t]$, where $g$ is a simple Lie algebra, was introduced by Feigin and Loktev in [FL]. The usual tensor product of irreducible finite-dimensional $g$-modules $\{\pi_{\mu(i)}(i), i = 1, ..., N\}$, with highest weights $\mu(i)$, decomposes into the direct sum of irreducible $g$-modules:

$$(\pi_{\mu(1)} \otimes \cdots \otimes \pi_{\mu(m)} \cong \bigoplus_{\lambda} K_{\lambda, \mu} \pi_{\lambda}$$

($\mu \equiv (\mu^{(1)}, ..., \mu^{(m)})$), where the integers $K_{\lambda, \mu}$, products of Clebsch Gordan coefficients, are the multiplicities of $\pi_\lambda$ in the tensor product.

The fusion product of $[FL]$ is a $g$-equivariant grading on this product, so that the multiplicities $K_{\lambda, \mu}$ become the generating functions for the dimensions of the graded components of the multiplicity space. It was conjectured $[FL]$ that these polynomials are related to the Kostka polynomials in the special cases where such polynomials are defined.

On the other hand, in the case where $\mu^{(i)} = \mu_i \omega_1$ are the highest weights of the symmetric power representation, and $\mu = (\mu_1, ..., \mu_m)$ with $\sum_i \mu_i = N$, it is known $[GP, CP]$ that Kostka polynomials give the graded dimensions of the cohomology ring of the partial flag variety $\mathcal{F}_{\mu'}$ of $GL_N$. The Weyl group, which is the symmetric group $S_N$, acts on this ring, preserving degree. As a graded space and as an $S_N$-module, the cohomology ring is isomorphic to the coordinate ring $R_{\mu}$, which is a certain finite-dimensional quotient of $\mathbb{C}[z_1, ..., z_N]$ (see section 3 for the precise definition) on which $S_N$ acts by permutation of variables. The decomposition of $R_{\mu}$ into irreducible...
The graded tensor product was introduced by Feigin and Loktev in [FL] for any simple Lie algebra \( \mathfrak{g} \). It is called the “fusion product” in that reference because it is motivated by the fusion product of affine algebra modules in conformal field theory, in the case where the level is generic.
Let $\hat{\mathfrak{g}} \simeq \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t] \otimes \mathbb{C}[z_1, \ldots, z_N]$ be the affine algebra associated with $\mathfrak{g}$, with central element $c$. On an irreducible $\hat{\mathfrak{g}}$-module, $c$ acts as a constant $k \in \mathbb{C}$ (the level). If $k$ is generic, the Weyl module $W_\mu = \text{Ind}_{\mathfrak{g}}^{\mathfrak{g} \otimes \mathbb{C}[t^{-1}]\mathbb{C}[t]} \pi_\mu$, induced from the finite dimensional irreducible $\mathfrak{g}$-module $\pi_\mu$ by the action of $\mathfrak{g} \otimes \mathbb{C}[t^{-1}]$, (where $\mathfrak{g} \otimes t\mathbb{C}[t]$ acts trivially on $\pi_\mu$ and $c$ acts as $k$) is an irreducible level-$k$ module. In this case, $\pi_\mu$ is called the top component of $W_\mu$.

Let $\{z_1, \ldots, z_N\}$ be distinct finite points in $\mathbb{C}P^1$. By a $\hat{\mathfrak{g}}$-module at the point $z_i$, we mean the module on which the element $x \otimes f(t)$ acts as $x \otimes f(t - z_i)$. There exists a level-$k$ action of $\hat{\mathfrak{g}}$ on the tensor product of $N$ level-$k$ modules $W_\mu(\mu)$ at the points $z_i$. The loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ acts by the usual coproduct with the action on $W_\mu(\mu)$ being at $z_i$, while cocycle on the product is the sum of the canonical cocycles. This action has the same level $k$ as the modules $W_\mu(\mu)$, and is referred to as the fusion action. The product of $\hat{\mathfrak{g}}$-modules with this action is called the fusion product. We denote it by $W_\mu(\mathbb{Z}) (\mathbb{Z} = (z_1, \ldots, z_N))$.

Let $\mathcal{A}$ denote space of regular functions on $\mathbb{C}P^1 \setminus \{z_1, \ldots, z_N\}$ vanishing at infinity, with possible poles at $z_i$. Then $\mathfrak{g} \otimes \mathcal{A}$ acts on $W_\mu(\mathbb{Z})$ (with trivial central charge) and the space of $\mathfrak{g} \otimes \mathcal{A}$-invariant functionals on $W_\mu(\mathbb{Z})$ is isomorphic to the dual of the top component, $\pi_\mu(\mu) \otimes \cdots \otimes \pi_\mu(N)$ in $W_\mu(\mathbb{Z})$, for generic $k$. Its dual space is the space of coinvariants.

On the other hand, the fusion action of $\mathfrak{g} \otimes \mathbb{C}[t]$ (in fact, of $\mathfrak{n}_- [t]$) on the tensor product of highest weight vectors $v_\mu^{(i)} \in \pi_\mu^{(i)}$ generates the top component $\pi_\mu^{(i)} \otimes \cdots \otimes \pi_\mu^{(N)}$ if $z_i$ are distinct. Moreover, there is a natural grading on $U(\mathfrak{g} \otimes \mathbb{C}[t])v_\mu^{(i)} \otimes \cdots \otimes v_\mu^{(N)}$, by degree in $t$ (see below for a precise description of the grading), which is $\mathfrak{g}$-equivariant. Therefore, this gives a graded version of the tensor product of the finite dimensional representations, which is described by the formula (1) with $K_{\mu, \mu}$ graded multiplicities. This is the idea behind the definition of the fusion product introduced in (1). Let us recall the precise definition.

Given $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, a partition with $\mu_1 \geq \cdots \geq \mu_n \geq 0$, $\mu_i \in \mathbb{Z}$, denote by $\pi_\mu$ the irreducible $\mathfrak{sl}_n$-module with $\mathfrak{sl}_n$-highest weight $(\mu_1 - \mu_2)\omega_1 + \cdots + (\mu_{n-1} - \mu_n)\omega_{n-1}$ where $\omega_i$ are the fundamental weights.

Let $z$ be a formal parameter. Denote by $\pi_\mu[z] \simeq \pi_\mu \otimes \mathbb{C}[z]$ the formal evaluation module of $\mathfrak{sl}_n[t] = \mathfrak{sl}_n \otimes \mathbb{C}[t]$, which acts in the usual way:

$$(x \otimes f(t)) \circ (w \otimes g(z)) = x \otimes f(t) g(z), \quad w \in \pi_\mu, \quad x \in \mathfrak{sl}_n.$$ 

All tensor products are over $\mathbb{C}$. The formal evaluation module is infinite-dimensional. The evaluation module at a complex number $a$ is the image of the evaluation map $\text{id} \otimes \phi_a : \pi_\mu \otimes \mathbb{C}[z] \to \pi_\mu$ corresponding to evaluation of the formal parameter $z$ at the complex number $a$. The evaluation module at a complex number is finite-dimensional. It is denoted in the same way, $\pi_\mu[a]$, and it will be made clear in what follows which type of evaluation module is under consideration.

Given a sequence of $N$ evaluation representations $\{\pi_\mu^{(i)}[z_i], i = 1, \ldots, N\}$ corresponding to partitions $\mu^{(i)}$, let $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(N)})$ and define $V_\mu(\mathbb{Z})$ to be the tensor product:

$$V_\mu(\mathbb{Z}) = \pi_\mu^{(1)}[z_1] \otimes \cdots \otimes \pi_\mu^{(N)}[z_N]$$

$$\simeq (\pi_\mu^{(1)} \otimes \cdots \otimes \pi_\mu^{(N)}) \otimes \mathbb{C}[z_1, \ldots, z_N].$$
It contains a subspace
\[ F_\mu(Z) = U(n_\cdot \otimes \mathbb{C}[t]) \left( (v_{\mu(1)} \otimes \cdots \otimes v_{\mu(N)}) \otimes 1 \right), \]
where \( v_{\mu(i)} \) is the highest weight vector of \( \pi_{\mu(i)} \) and \( 1 \in \mathbb{C}[z_1, \ldots, z_N] \). The action is by the usual co-product on evaluation modules. When \( z_i \) are taken to be formal parameters, the space \( F_\mu(Z) \) is graded by homogeneous degree in the \( z_i \)'s. If, instead, \( z_i \) are specialized to be distinct complex numbers, we can define a filtration of the space as follows. Let
\[ F_\mu^{(m)}(Z) = U^{(m)}(n_\cdot \otimes \mathbb{C}[t]) v_1 \otimes v_2 \otimes \cdots \otimes v_N, \]
where \( U^{(m)} \) is spanned by homogeneous elements of degree \( m \) in \( t \). The subspaces
\[ F_\mu^{(\leq n)}(Z) = \oplus_{m \leq n} F_\mu^{(m)}(Z) \]
define a filtration of \( F_\mu(Z) \).

The fusion product can be defined in either setting. In the case where \( z_i \) are specialized to distinct complex numbers, the graded tensor product of \([FL]\) is the following:

**Definition 2.1.** The graded tensor product of evaluation modules at the distinct complex numbers \( \{z_1, \ldots, z_N\} \) is
\[ F^*_\mu = \text{gr } F_\mu(Z). \]

It is possible to define the graded tensor product in the setting where \( z_i \) are formal variables also. In this setting, the space \( F_\mu(Z) \) is an infinite-dimensional graded space. Define the subspace
\[ \tilde{F}_\mu^{(n)}(Z) = (\mathbb{C}[z_1, \ldots, z_N] \otimes C F_\mu^{(\leq n-1)}(Z)) \cap F_\mu^{(n)}(Z) \subset F_\mu^{(n)}(Z). \]
Then the graded tensor product can be defined as

**Definition 2.2.** The graded tensor product of formal evaluation modules is
\[ F^*_\mu = \oplus_{n \geq 0} F_\mu^{(n)}(Z) / \tilde{F}_\mu^{(n)}(Z). \]

The two definitions are clearly equivalent. In either case, the space \( F^*_\mu \) is a finite-dimensional \( \mathfrak{sl}_n \)-module, of dimension equal to the dimension of the tensor product of the finite dimensional \( \mathfrak{sl}_n \)-modules \( \pi_{\mu(i)} \). Since the action of \( \mathfrak{sl}_n \) does not change the degree, the graded components are \( \mathfrak{sl}_n \)-modules. The decomposition of the graded tensor product into irreducible \( \mathfrak{sl}_n \)-modules,
\[ F^*_\mu \simeq \bigoplus_{\lambda} \pi_\lambda \otimes M_{\lambda,\mu}, \]
defines the multiplicity spaces \( M_{\lambda,\mu} \). The multiplicity spaces are graded spaces, and the purpose of this paper is to give an explicit description of these spaces.

It was conjectured in \([FL]\) that the graded multiplicities,
\[ \text{ch}_q M_{\lambda,\mu} = \sum_i q^i \dim M_{\lambda,\mu}[i] \]
(for any graded space \( V \) we denote by \( V[i] \) its graded component) are related to the generalized Kostka polynomials of \([Sh, SW]\). In the paper \([SW]\), the generalized Kostka polynomial was defined for the case where \( \mu^{(i)} \) are partitions of the form
(j)m corresponding to rectangular Young diagrams, or highest weights mωj. One can expect to relate the q-dimension of Mλ,µ to the generalized Kostka polynomial in this case, although this is still a conjecture.

Assuming that this conjecture is correct, one can take this character to be the definition of a more general Kostka polynomial. This was the subject of [FJKLM], where the level-restricted version of the most general Kostka polynomial was defined as a quotient of the graded tensor product. Here, we discuss only the generic level case, which is obtained from [FJKLM] by taking the limit as the level k tends to infinity.

Remark 2.3. The level restriction is related to the fact that the decomposition of the fusion product of irreducible representations of \( \hat{g} \), in the case where k is an integer, is determined by the Verlinde rule and not by the usual Clebsch-Gordan rule. In that case the dimensions \( K_{\lambda,\mu} \) are smaller. We do not consider this case in this paper.

3. Polynomial \( S_N \)-representations and the Kostka polynomials

Here, we review some useful facts from [GP] about the decomposition of certain quotients of the space of polynomials in \( N \) variables into irreducible symmetric group modules. These quotients were obtained [GP, CP] as a description of the cohomology ring of the flag variety of \( GL_N \). The symmetric group is the Weyl group of \( GL_N \) so it acts on the cohomology ring. As a symmetric group module, the ring corresponding to the flag \( F_{\mu'} \) is isomorphic to \( \mathbb{C}S_N/S_\mu \) where \( S_\mu \) is the Young subgroup.

Let \( z_i \) be formal variables, and consider the action of \( S_N \) on the ring \( \mathbb{C}[z_1,...,z_N] \) by permutation of variables. Let \( \mu \) be a partition of \( N \), \( \mu = (\mu_1,...,\mu_m) \) with \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m > 0 \). Let \( \{a_1,...,a_m\} \) be distinct complex numbers, and let \( X = (x_1,x_2,...,x_N) \) denote the \( N \)-tuple of complex numbers with the first \( \mu_1 \) variables equal to \( a_1 \), the next \( \mu_2 \) equal to \( a_2 \), etc.

Denote by \( I(X) \) the ideal of functions in \( \mathbb{C}[z_1,...,z_N] \) which vanish under the evaluation map

\[
\phi_X : \mathbb{C}[z_1,...,z_N] \rightarrow \mathbb{C} \\
f(z_1,...,z_N) \mapsto f(x_1,...,x_N).
\]

This ring is invariant under the action of Young subgroup of \( S_N \) given by the partition \( \mu \), \( S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_m} \). Here, by \( S_{\mu_1} \) we mean the permutation group of the first \( \mu_1 \) variables \( (z_1,...,z_{\mu_1}) \) and so forth.

Let \( I_\mu \) be the intersection of the ideals in the \( S_N \)-orbit of \( I(X) \):

\[
I_\mu = \cap_{\sigma \in S_N} I(\sigma X),
\]

where \( S_N \) acts by permutation of the indices in \( (x_1,...,x_N) \). Functions in \( I_\mu \) vanish at all points in the \( S_N \)-orbit of \( (x_1,...,x_N) \).

The ideal \( I_\mu \) is \( S_N \)-invariant by definition, hence the quotient ring

\[
A_\mu = \mathbb{C}[z_1,...,z_N]/I_\mu
\]

is an \( S_N \)-module, isomorphic to the representation of \( S_N \) acting on left cosets of \( S_\mu \). The space \( A_\mu \) is filtered by the homogeneous degree in the variables \( z_j \). Define \( R_\mu \) to be the associated graded space:

\[
R_\mu = \text{gr} A_\mu.
\]
We give an explicit combinatorial description of $R_\mu$ below for completeness, but we will use it only in the special case when $\mu = (1)^N$, which is explained first.

In the case $\mu = (1)^N$, let $J_N$ be the ideal in the ring $\mathbb{C}[z_1, \ldots, z_N]$, generated by symmetric polynomials in $N$ variables, of degree greater than 0: For example, the generators of this ideal can be taken to be the elementary symmetric functions,
\[ E_1(z_1, \ldots, z_N), \ldots, E_N(z_1, \ldots, z_N), \]
where
\[ E_m(z_1, \ldots, z_N) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq N} z_{i_1} \cdots z_{i_m}. \]
(5)

\[ J_N = \langle E_1(z_1, \ldots, z_N), \ldots, E_N(z_1, \ldots, z_N) \rangle. \]

Since $J_N$ is invariant with respect to the action of $S_N$, $S_N$ acts on the quotient space

\[ R_N := R_{(1)^N} = \mathbb{C}[z_1, \ldots, z_N]/J_N. \]

This is a finite-dimensional vector space. In fact,

**Lemma 3.1.** [GP] As a symmetric group module, $R_N$ is isomorphic to the regular representation:

\[ R_N \simeq \bigoplus_{\lambda \vdash N} W_\lambda \otimes M_{\lambda,(1)^N} \]

where $W_\lambda$ is the irreducible $S_N$ Specht (left-) module, corresponding to the partition $\lambda$, and $M_{\lambda,(1)^N}$ is a multiplicity space, of dimension equal to the dimension of $W_\lambda$.

Note that the ideal $J_N$ is generated by homogeneous polynomials, so the quotient $R_N$ is a graded space. The symmetric group action preserves homogeneous degree in $z_i$, so the Specht modules are spanned by polynomials of fixed degree. Thus, the multiplicity spaces $M_{\lambda,(1)^N}$ are graded spaces and the $q$-character (or $q$-dimension)

\[ \text{ch}_q(M_{\lambda,(1)^N}) = \sum_i q^i \dim M_{\lambda,(1)^N}[i] \]

is the Kostka polynomial $\tilde{K}_{\lambda,(1)^N}(q)$, where

\[ \tilde{K}_{\lambda,\mu}(q) = q^{n(\mu)} K_{\lambda,\mu}(1/q). \]

Here, $n(\mu) = \sum_i (i-1) \mu_i$. The normalization factor ensures that this is a polynomial in $q$, as it should be: the lowest degree polynomial in $R_N$ is 1, of degree 0. Here, $K_{\lambda,\mu}(q)$ is the usual Kostka polynomial [Mac], defined, for example, as the transition matrix between Schur polynomials and Hall-Littlewood polynomials:

\[ S_\lambda(x) = \sum_{\mu \leq \lambda} K_{\lambda,\mu}(q) P_\mu(x, q). \]

For completeness, we recall here the description of the coordinate ring $R_\mu$ [GP] in the general case. The ring $R_\mu$ is the associated graded space of the quotient $A_\mu$. For a function $f(z_1, \ldots, z_N) \in I_\mu$, define the top term to be the term of highest homogeneous degree in $\{z_i\}$. Such terms generate an ideal $J_\mu$, and are obtained by taking the limit of $f(z_1, \ldots, z_N)$ as $x_i \to 0$ for all $i$. The ring $R_\mu$ is the quotient of $\mathbb{C}[z_1, \ldots, z_N]$ by the ideal $J_\mu$. It has an explicit combinatorial description as follows.

Let $d_k(\mu) = N - (\mu'_1 + \cdots + \mu'_{N-k})$, where $\mu'$ is the conjugate partition to $\mu$. Define the set

\[ C_\mu(z_1, \ldots, z_k) = \{ E_r(z_1, \ldots, z_k) : k - d_k < r \leq k \}. \]
If \( d_k(\mu) = 0 \) the set is empty. If \( k = N \), the set includes all elementary symmetric polynomials of positive degree in \( N \) variables. Let \( C_\mu \) be the set of polynomials in the \( S_N \)-orbit of \( C_\mu(z_1, \ldots, z_k) \) for all \( k \),

\[
C_\mu = \bigcup_{\sigma \in S_N} \bigcup_{k > 0} \sigma C_\mu(z_1, \ldots, z_k).
\]

For example, the set \( C_{(1,N)} \) is just the case \( k = N \) above, and the set \( C_{(N)} \) includes all polynomials other than 1.

**Theorem 3.2.** [GP] The ideal \( J_\mu \) is generated by \( C_\mu \). The graded space \( R_\mu \) is the quotient ring

\[
R_\mu = \mathbb{C}[z_1, \ldots, z_N]/J_\mu.
\]

The decomposition of \( R_\mu \) into irreducible \( S_N \)-modules,

\[
R_\mu \cong \bigoplus_{\lambda,\mu} \mathbb{C}[\mathbb{N}] W_\lambda \otimes M_{\lambda,\mu},
\]

gives the graded multiplicity spaces \( M_{\lambda,\mu} \).

**Theorem 3.3.** [GP]

\[
ch_q M_{\lambda,\mu} = \tilde{K}_{\lambda,\mu}(q).
\]

\( R_\mu \), as a representation of the symmetric group, is isomorphic to the representation of the symmetric group acting on left cosets of the Young subgroup \( S_\mu \), where \( S_\mu \) is the stabilizer of \( X \).

4. **Graded tensor product fundamental \( \mathfrak{sl}_n \)-modules**

The simplest example of the graded tensor product product of \( \mathfrak{sl}_n \)-representations is the \( N \)-fold product of the fundamental \( n \)-dimensional evaluation representation associated with \( \pi = \pi_{(1)} \) of highest weight \( \omega_1 \). We have \( \mu(i) = (1) \) for all \( i \in \{1, \ldots, N\} \), and

\[
V_N(\mathbb{Z}) := V_{(1^N)}(\mathbb{Z}) = \pi[z_1] \otimes \cdots \otimes \pi[z_N] \cong \pi^\otimes N \otimes \mathbb{C}[z_1, \ldots, z_N].
\]

We consider here the tensor product of formal evaluation modules.

This case is interesting for two reasons. First, it is isomorphic to the reduced finite-dimensional wedge product, which is recalled in the next section. Second, the inductive limit as \( N \to \infty \) gives a construction of the basic representations of \( \mathfrak{sl}_N \) [FL]. In terms of the cohomology ring of the flag variety, its character is related to the complete flag.

Denote by \( v_0, \ldots, v_{n-1} \) the standard basis of \( \pi \cong \mathbb{C}_n \), where \( v_0 \) is the highest weight vector. The subspace \( \mathcal{F}_N := U(\mathfrak{n}_- \otimes \mathbb{C}[\mathfrak{t}]) w_0 \), with \( w_0 = (v_0 \otimes \cdots \otimes v_0) \otimes 1 \in \pi^\otimes N \otimes \mathbb{C}[z_1, \ldots, z_N] \), can be characterized largely in terms of the action of the symmetric group \( S_N \). This is the approach taken in this section.

As a symmetric group module, the space \( V_N \) is a tensor product of two \( S_N \)-modules, say, \( V_N = \Pi \otimes G \), where \( \Pi = \pi^\otimes N \), on which \( S_N \) acts by permutation of factors in the tensor product, and \( G = \mathbb{C}[z_1, \ldots, z_N] \), on which \( S_N \) acts by permutation of variables as before. We consider the **diagonal left action** of symmetric group on this product. That is, for any \( \sigma \in S_N \),

\[
\sigma \circ ((v_{i_1} \otimes v_{i_2} \otimes \cdots) \otimes f(z_1, z_2, \ldots)) = (v_{\sigma(i_1)} \otimes v_{\sigma(i_2)} \otimes \cdots) \otimes f(\sigma(z_1), \sigma(z_2), \cdots)).
\]
By definition, $\mathfrak{sl}_n \otimes \mathbb{C}[t]$ acts on the tensor product by the coproduct, which commutes with the diagonal left action of $S_N$. In addition, the cyclic vector $w_0$ is $S_N$-invariant. Therefore, we have that
\[ \mathcal{F} \subset (\pi^{\otimes N} \otimes \mathbb{C}[z_1, \ldots, z_N])^{S_N}. \]
(This is true for any $N$-fold product of identical representations, not only the fundamental one.) Here, if $W$ is a vector space on which a group $G$ acts, by $W^G$ we mean the subspace on which $G$ acts trivially. Note that the $S_N$-action preserves homogeneous degree.

Next, consider the definition $\mathcal{F}_n$ of the graded tensor product. We can choose an $S_N$-invariant basis for the representatives of the quotient $\mathcal{F}_n^{(n)}/\mathcal{F}_n^{(n)}$. To see this, let $p^{(n)} \in \mathcal{F}_n^{(n)}$ and let $\tilde{p}^{(n)}$ be a representative of $p^{(n)}$ in the quotient. That is, there exists an expression
\[ p^{(n)} = \tilde{p}^{(n)} + \sum_i \sum_{m < n} f_i^{(n-m)}(z_1, \ldots, z_N) p_i^{(m)} \]
where $p_i^{(m)} \in \mathcal{F}_n^{(m)}$, and $f_i^{(n-m)}(z_1, \ldots, z_N)$ are homogeneous polynomials of degree $n - m$. Since both $p^{(n)}$ and each of the vectors $p_i^{(m)}$ are $S_N$-invariant, we have for any $\sigma \in S_N$:
\[ \tilde{p}^{(n)} - \sigma(\tilde{p}^{(n)}) = \sum_i \sum_{m < n} (f_i^{(n-m)}(z) - f_i^{(n-m)}(\sigma(z))) p_i^{(m)} \in \tilde{\mathcal{F}}_n^{(n)}. \]
Thus, the difference is zero modulo $\tilde{\mathcal{F}}_n^{(n)}$, so it vanishes in the quotient. Therefore, we can take as a basis for the coset representatives $S_N$-invariant vectors $\tilde{p}^{(n)}$.

This means that in choosing an $S_N$-invariant basis, the functions $f_i^{(n-m)}$ appearing in $\mathcal{F}_n^{(n)}$ are symmetric functions. That is,
\[ \tilde{\mathcal{F}}_n^{(n)} \simeq (\mathbb{C}[z_1, \ldots, z_N]^{S_N} \otimes \mathcal{F}_n^{(\leq n-1)}) \cap \mathcal{F}_n^{(n)}. \]
Since the graded space $\mathcal{F}_n^{(n)}$ has degree $n$, such polynomials must be of positive degree.

This means that $\tilde{\mathcal{F}}_n^{(n)}$ is in contained in the ideal in $\mathcal{F}_n$ generated by symmetric polynomials of positive degree. Defining the ideal $\mathcal{I}$ by $\mathcal{I}^{(1)N} = V_n^{S_N}/\mathcal{J}$, we have shown that $\mathcal{J} \subset (\pi^{\otimes N} \otimes J_N)^{S_N}$, where $J_N$ is the ideal defined in (4).

We will show the equality: The graded component of degree $n$ of the ideal in $\mathcal{F}_n$ generated by symmetric functions of positive degree contains $\tilde{\mathcal{F}}_n^{(n)}$, $\mathcal{J} \cap \mathcal{F}_n^{(n)} \subset \tilde{\mathcal{F}}_n^{(n)}$. This follows from Lemma 4.1 which shows that any vector in $V_n(\mathbb{Z})^{S_N}$ can be expressed as a linear combination (with coefficients symmetric functions) of vectors in $\mathcal{F}_n$. That is, $V_n(\mathbb{Z})^{S_N} \simeq \mathcal{F}_n \otimes \mathbb{C}[z_1, \ldots, z_N]^{S_N}$. It follows that if $p^{(n)} \in \mathcal{F}_n^{(n)}$ and $p^{(n)} = \sum_{i=0}^{n-1} f_i^{(n-i)}$ with $f_i^{(n-i)}$ symmetric functions of degree $n - i > 0$, then $v_i \in \mathcal{F}_n^{(\leq i)} \otimes \mathbb{C}[z_1, \ldots, z_N]^{S_N}$ and hence $p^{(n)} \in \tilde{\mathcal{F}}_n^{(\leq n-1)}$.

**Lemma 4.1.** The space $V_n(\mathbb{Z})^{S_N}$ is generated by $\mathcal{F}_n$ as a $\mathbb{C}[z_1, \ldots, z_N]^{S_N}$-module.

**Proof.** First, note that in the case of the fundamental module, $\pi$ is generated by the action of the commuting elements
\[ \{b_i = E_{1,i+1}, i = 1, \ldots, n - 1\} \]
acting on $v_0$, with $b_j v_0 = v_j$. Moreover, $b_i b_j v_0 = 0$ for any $i, j$.  
We denote the vector \( v_{i_1} \otimes \cdots \otimes v_{i_N} \in \pi^\otimes N \) by \([i_1, \ldots, i_N]\). The space \( V_N(\pi)^{S_N} \) is clearly spanned by elements of the form

\[
(9) \quad v_N^{(m)} = \sum_{\sigma \in S_N} \sigma[n - 1, \ldots, n - 1, n - 2, \ldots, n - 2, \ldots, 0, \ldots, 0] \otimes z_{\sigma(1)}^{m_1} \cdots z_{\sigma(N)}^{m_N}.
\]

Here, \( m \) is an \( N \)-tuple of integers, and \( N \) is the \( n \)-tuple \((N_{n-1}, \ldots, N_0)\).

If \( m_j = 0 \) for all \( j > N - N_0 \), then \( v_N^{(m)} \in \mathcal{F}_N \). This is because \( \mathcal{F}_N \) is spanned by elements of the form

\[
(b_{i_1} \otimes t^{m_1}) \cdots (b_{i_k} \otimes t^{m_k}) w_0 = \sum_{\sigma \in S_N} \sigma[i_1, \ldots, i_k, 0, \ldots, 0] \otimes z_{\sigma(1)}^{m_1} \cdots z_{\sigma(k)}^{m_k}
\]

with \( n - 1 \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1 \). Thus in general, let \( \nu(m, N) \) denote the number of non-zero components \( m_j \) with \( j > N - N_0 \).

The lemma is proven by induction on \( \nu(m, N) \). If \( \nu(m, N) = 0 \) then \( v_N^{(m)} \in \mathcal{F}_N \). Suppose \( \nu(m, N) > 0 \), we will show that \( v_N^{(m)} \) can be expressed as a sum of the form

\[
(10) \quad \sum_{m'} f_{m'}(z_1, \ldots, z_N)v_{N}^{(m')}
\]

where \( f_{m'}(z_1, \ldots, z_N) \) are symmetric functions, and \( \nu(m', N) < \nu(m, N) \) for all \( m' \) occurring in the sum. Given this fact, it follows by induction that we can rewrite any element in \( V_N(\pi)^{S_N} \) in the form \( (10) \) with \( \nu(m', N) = 0 \), proving the lemma.

Define a partition of the set of integers \([1, \ldots, N]\) into subsets \( J_0, \ldots, J_{n-1} \), with \( J_{n-1} = \{1, \ldots, N_{n-1}\}, J_{n-2} = \{N_{n-1} + 1, \ldots, N_{n-1} + N_{n-2}\} \) and so forth. Denote the corresponding Young subgroup of \( S_N \) by \( S_J = S_{J_0} \times S_{J_1} \times \cdots \times S_{J_{n-1}} \). By \( m(J_i) \), denote the \( N_i \)-tuple composed of the elements \( m_j \) where \( j \in J_i \).

Let \( M(J; m(J)) \) be the monomial symmetric function in the variables \( \{z_i : i \in J\} \):

\[
M(J; m(J)) = \sum_{\sigma \in S_J} \prod_{i \in J} z_{\sigma(i)}^{m_i}.
\]

Up to a constant multiple, the expression \((9)\) for \( v_N^{(m)} \) is equal to

\[
(11) \quad \sum_{\sigma \in S_{N - J}} \sigma[n - 1, \ldots, 0] \otimes \prod_{i=0}^{n-1} M(\sigma(J_i); m(J_i)),
\]

because \( \sigma \in S_J \) acts trivially on the vector \([n - 1, \ldots, 0]\) and on \( \prod M(J_i, m(J_i)) \). The summation is over the left coset representatives.

One can always rewrite

\[
M(J_0; m(J_0)) = F_1 - F_2
\]

where

\[
F_1 = ((N_1 + \cdots + N_{n-1})!)^{-1} \sum_{\sigma \in S_N} \prod_{i \in J_0} z_{\sigma(i)}^{m_i}
\]

is a completely symmetric function. Since \( M(J_0; m(J_0)) \) is symmetric in each set of variables \( \{z_j : j \in J_i\} \) separately, so is \( F_2 \). But each monomial in \( F_2 \) has at least one
non-zero power of some $z_i$ with $i \in J_k$ with $k \in \{1, \ldots, N-1\}$. This is because all of the terms containing only $z_j$ ($j \in J_0$) are all contained in $F_1$. Therefore,

\begin{equation}
F_2 = \sum_{m'} c_{m'} \prod_{i=0}^{n-1} M(J_i, m'(J_i))
\end{equation}

with some constants $c_{m'}$, where $m'(J_0)$ contains fewer non-zero elements than $m(J_0)$.

Using this fact to rewrite (11), we have the difference of the two kinds of terms:

\[
\sum_{\sigma \in S_N/S_J} \sigma[n-1, \ldots, 0] \otimes \sigma \left( \prod_{i=1}^{n-1} M(J_i, m(J_i))(F_1 - F_2) \right)
\]

\[
= F_1 \sum_{\sigma \in S_N/S_J} \sigma[n-1, \ldots, 0] \otimes \sigma \prod_{i=1}^{n-1} M(J_i, m(J_i))
\]

\[- \sum_{\sigma \in S_N/S_J} \sigma[n-1, \ldots, 0] \otimes \sigma \prod_{i=1}^{n-1} M(J_i, m(J_i))F_2.
\]

Here, since $F_1$ is completely symmetric, it can be factored out of the summation, since $S_N$ acts on it trivially. The first term is proportional to

\begin{equation}
\sum_{\tau \in S_N, j \in J} z_{\tau(j)}^{m_j} \sum_{\sigma \in S_N/S_J} \sigma[n-1, \ldots, 0] \otimes \prod_{i=1}^{n-1} M(\sigma(J_i); m(J_i)),
\end{equation}

Let $m'_j = m_j$ if $j \in J_1 \cup \cdots \cup J_{n-1}$, and $m'_j = 0$ if $j \in J_0$. Then (13) is equal to

\[
f(z_1, \ldots, z_N)v_{m'}^N,
\]

where $f$ is a symmetric function. This term is in the $\mathbb{C}[z_1, \ldots, z_N]^{S_N}$-span of $\mathcal{F}_N(\mathbb{C})$.

The second term involves the function

\[
\prod_{i=1}^{n-1} M(J_i, m(J_i))F_2,
\]

which is again symmetric in each set of variables $\{m_j, j \in J_i\}$ for each $i$. Due to (12), the second term is therefore expressible as a sum of several terms of the same form (12) with $m''$ replacing $m'$, and where the number of non-zero elements in $m''(J_0)$ is the same as in $m'(J_0)$. It is strictly less than $\nu(m, N)$.

The lemma follows by induction. \hfill \Box

**Remark 4.2.** The lemma simply states that $V_N(\mathbb{C})^{S_N} = U(\mathfrak{sl}_n \otimes \mathbb{C}[t])w_0$.

The lemma means that any element of degree $i$ in $V_N(\mathbb{C})^{S_N}$ which is zero modulo symmetric functions of positive degree is also zero modulo $\mathcal{F}^{(i)}$. Therefore, $\mathcal{F}_{(1)^N}^{*} \simeq V_N(\mathbb{C})^{S_N}/(\pi \otimes J_N)^{S_N}$ as $\mathfrak{sl}_n$-modules and as graded vector spaces. As a consequence,

**Theorem 4.3.**

\[
\mathcal{F}^*_{(1)^N} \simeq (\pi \otimes R_N)^{S_N}.
\]
Recall that as $S_N$-modules
\[ R_N \simeq \bigoplus_{\lambda \vdash N} W_\lambda \otimes M_{\lambda,(1)^N}, \]
where $M_{\lambda,\mu}$ is a graded multiplicity space and $W_\lambda$ is the irreducible Specht module. Also, by the Schur-Weyl duality, the action of $S_N$ on $\pi^\otimes N$ centralizes the action of $\mathfrak{sl}_n$, and the tensor product decomposes as
\[ \pi^\otimes N \simeq \bigoplus_{\mu \vdash N, \ell(\mu) \leq n} \pi_\mu \otimes W_\mu, \]
where $\pi_\mu$ is the irreducible representation of $\mathfrak{sl}_n$, and $W_\mu$ is an irreducible $S_N$-module.

Putting all these facts together,
\[ \left( \bigoplus_{\mu \vdash N, \ell(\mu) \leq n} \pi_\mu \otimes W_\mu \right) \otimes \left( \bigoplus_{\lambda \vdash N} W_\lambda \otimes M_{\lambda,(1)^N} \right)^{S_N} \simeq \bigoplus_{\mu,\lambda} \pi_\mu \otimes M_{\lambda,(1)^N} \otimes (W_\mu \otimes W_\lambda)^{S_N}. \]
The coefficient of the trivial representation in the tensor product $W_\mu \otimes W_\lambda$ is equal to 1 if and only if $\lambda = \mu$. Thus, we have

**Theorem 4.4.** As graded vector spaces and $\mathfrak{sl}_n$-modules, the graded tensor product of the basic representations is equal to
\[ \mathcal{F}^{(1)^N} \simeq \bigoplus_{\mu \vdash N, \ell(\mu) \leq n} \pi_\mu \otimes M_{\mu,(1)^N}, \]
where $M_{\mu,(1)^N}$ is the graded multiplicity space of the irreducible $S_N$-representation $W_\mu$ in the quotient ring $R_N$.

Finally in terms of characters,

**Corollary 4.5.**
\[ \text{ch}_q \mathcal{F}^{(1)^N} = \sum_{\mu \vdash N, \ell(\mu) \leq n} \text{ch}(\pi_\mu) \tilde{K}_{\mu,(1)^N}(q). \]

### 5. The Reduced Wedge Space

The $N$-fold graded tensor product of the fundamental representations is closely related to the wedge product of evaluation representations with formal evaluation parameters. Recall the wedge product construction of the basic representations of the affine Lie algebra $\hat{\mathfrak{sl}}_n$: They are obtained as the quotient of the semi-infinite wedge product, stabilized at infinity, by the image of a Heisenberg algebra, which can be described as the central extension of the loop algebra of the center of $\mathfrak{gl}_n$, acting on the wedge product.

This suggests a finite-dimensional analogue of the $N$-fold wedge product, which is the quotient of the finite-dimensional wedge product by the image of the same Heisenberg algebra, acting with central charge zero. We call this the reduced wedge product. In the appropriate limit $N \to \infty$, this reduced wedge product becomes the irreducible level-1 $\hat{\mathfrak{sl}}_n$-module.\(^1\)

We show here that the reduced wedge product and the graded tensor product of Section 4 are isomorphic, as $\mathfrak{sl}_n$-modules and as graded vector spaces.

\(^1\)The construction explained here appeared previously in a preprint by the author and E. Stern (unpublished).
Recall that there are \( n \) irreducible, integrable representations of \( \widehat{\mathfrak{sl}}_n \) with level \( k = 1 \), \( L(\Lambda_i) \), with highest weight vector of weight \( \Lambda_i \), the fundamental affine weight, with \( i = 0, ..., n - 1 \). Let us recall the wedge space construction of \( L(\Lambda_i) \).

Let \( \pi \) be the fundamental \( \mathfrak{sl}_n \)-module as before, considered now as a \( \mathfrak{gl}_n \)-module, with basis \( v_0, ..., v_{n-1} \) with \( v_0 \) the highest weight vector. Let \( z \) be a formal variable, and \( \pi(z) \simeq \pi \otimes \mathbb{C}[z, z^{-1}] \) the evaluation module of \( \widehat{\mathfrak{gl}}_n \). The space \( \pi(z) \land \pi(z) \) can be realized in \( \pi(z) \otimes \pi(z) \) as the span of vectors of the form

\[
v_i z^m \land v_j z^n = v_i z^m \otimes v_j z^n - v_j z^n \otimes v_i z^m
\]

where \( z_i \) refers to the parameter \( z \) appearing in the \( i \)-th factor in the tensor product. Therefore if we again consider the diagonal action of the symmetric group on the space \( \pi^{\otimes N} \otimes \mathbb{C}[z_1^{\pm 1}, ..., z_N^{\pm 1}] \), the \( N \)-fold wedge product of fundamental evaluation representations can be realized as the subspace on which the symmetric group \( S_N \) acts as the alternating representation. The algebra \( \widehat{\mathfrak{gl}}_n \) acts on the \( N \)-fold wedge product by the usual co-product, with central charge 0.

Denote the basis of \( \pi(z) \) by \( v_i^k = v_i \land z^k \). The semi-infinite wedge space \( F^{(j+1,k)} \)

\[
(\pi(z) \land \pi(z)) \cong \pi(z) \otimes \pi(z)
\]

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vector of the semi-infinite wedge product $F^{(i,k)}$ after the $N$th factor. For example, in the case $k = -m$, this vector is
\[
(15) \quad \overline{w}_i^{-m} = v_{i-1}^{-m} \wedge v_{i-1}^{-m} \wedge \cdots \wedge v_0^{-m} \wedge v_{n-1}^{-m} \wedge \cdots \wedge v_0^{-m+1} \wedge \cdots \wedge v_0^0 \wedge \cdots \wedge v_0^0, 
\]
and $U(\mathfrak{gl}_n \otimes \mathbb{C}[t^{-1}]) \overline{w}_i^{-m} = \wedge^{mn+i} \pi[z^{-1}]$. The isomorphism with $F_{m}^{(i,-m)}$ is the wedge product with the highest weight vector of $F^{(m,1)}$, $w_n^1$. Note that the choice of $k$ is immaterial as the difference is only in overall normalization of the character (or equivalently, multiplication by an overall monomial in $z_i^{\pm 1}$) and we choose a $k$-independent convention for the normalization of the character as follows.

To be consistent with the grading used in Section 4, we define the $q$-character of $F_m^{(i,k)}$ by taking the degree of $t$ to be 1. This means
\[
\text{ch}_q F_m^{(i,k)} = \sum_{j \leq 0} q^j \dim(F_m^{(i,k)}[j]) 
\]
where we normalize the highest weight vector for any $k$, $i$ to have degree 0, and $t$ (or $z_i$) has degree 1. This is a polynomial in $q^{-1}$, which we compute below.

The irreducible module $L(A_i)$ is the quotient of the semi-infinite wedge product with respect to the image of $\mathcal{H}_\alpha$. The vector $w_{n-1}^{k+1+m}$ is not in the image of $\mathcal{H}_\alpha$, so the quotient of $F_m^{(i,k)}$ by the image of the Heisenberg algebra is obtained by taking the quotient of the finite wedge product by the image of $\mathcal{H}_\alpha$, then taking the wedge product with $w_{n-1}^{k+1+m}$. That is,
\[
F_m^{(i,k)} / U(\mathcal{H}_\alpha) F_m^{(i,k)} \simeq \bigwedge^N \pi[z^{-1}] / U(\mathcal{H}_\alpha)(\bigwedge^N \pi[z^{-1}]) 
\]
(where the isomorphism is multiplication by a certain monomial in $z_i$, and wedging with $w_{n-1}^{k+1+m}$), and
\[
\left( \lim_{m \to \infty} F_m^{(i,k)} / U(\mathcal{H}_\alpha) F_m^{(i,k)} \right) / U(\mathcal{H}_\alpha) F_m^{(i,k)} = \lim_{m \to \infty} \left( F_m^{(i,k)} / U(\mathcal{H}_\alpha) F_m^{(i,k)} \right). 
\]
Therefore, one can reverse the order of taking the limit $m \to \infty$ and taking the quotient by the image of $\mathcal{H}_\alpha$.

On the vector $\overline{w}_i^{-m}$ (and on any vector in the finite wedge product), $U(\mathcal{H}_\alpha)$ acts as multiplication by symmetric polynomials in $N$ variables $z_i^{-1}$ of positive degree. Thus, the quotient of the finite wedge product with cyclic vector (15) can be described as follows. Let $\pi[z^{-1}] = \pi \otimes \mathbb{C}[z^{-1}]$. The $N$-fold wedge product of such modules is an infinite-dimensional space (recall $z_i$ are formal variables)
\[
(16) \quad \wedge^N \pi[z^{-1}] \simeq (\pi \otimes \mathbb{C}[z_1^{-1}, \ldots, z_N^{-1}])^A 
\]
where the superscript $A$ means the subspace on which the diagonal action of $S_N$ is as on the alternating representation: $\sigma \in S_N$ acts by $(-1)^{\text{sgn}(\sigma)}$.

The reduced wedge space is a quotient of this space by the image of the Heisenberg algebra $\mathcal{H}_\alpha$. But the generator $B_a = \text{id} \otimes t^{-a}$ ($a > 0$) acts on the wedge product by multiplication by $z_1^{-a} + \cdots + z_N^{-a}$. Such polynomials form a basis for the space of symmetric functions in $N$ variables of positive degree, so they generate the ideal $J_N(Z^{-1})$ (i.e. with $z_i$ replaced by $z_i^{-1}$) in $\mathbb{C}[z_1^{-1}, \ldots, z_N^{-1}]$. The image of the Heisenberg algebra in (16) is the ideal generated by multiplication by elements in $J_N(Z^{-1})$. As in the previous section, we have
\[
\wedge^N \pi[z^{-1}] / \text{Im} \mathcal{H}_\alpha \simeq (\pi \otimes \mathbb{C}[Z^{-1}])^A, 
\]
where $R_N Z^{-1}$ is the ring $\mathfrak{k}$ with $z_i$ replaced by $z_i^{-1}$. Again, we use the theorem of \[\text{GP}\] and the Schur-Weyl duality to re-write this as

$$\wedge^N \pi [z^{-1}] / \text{Im } \mathcal{H} \cong \bigoplus_{\mu : \ell(\mu) \leq n} \pi_\mu \otimes M_{\lambda,(1)^N} (Z^{-1}) \otimes (W_\mu \otimes W_\lambda)^A.$$ 

The summand is non-zero if and only if $\mu = \lambda'$ (the conjugate partition of $\mu$). Thus we have

**Lemma 5.1.**

$$\wedge^N \pi [z^{-1}] / \text{Im } \mathcal{H} \cong \bigoplus_{\ell(\mu) \leq n} \pi_\mu \otimes M_{\mu',(1)^N} (Z^{-1}).$$

Therefore,

$$\text{ch}_q \wedge^N \pi [z^{-1}] / \text{Im } \mathcal{H} = \sum_{\mu : \ell(\mu) \leq n} \text{ch}(\pi_\mu) q^{-N(N-1)/2} K_{\mu',(1)^N}(q)$$

$$= \sum_{\mu : \ell(\mu) \leq n} \text{ch}(\pi_\mu) K_{\mu,(1)^N}(q^{-1}),$$

where we used a well-known identity for the Kostka polynomial.

**Lemma 5.2.**

$$\text{ch}_q F^{(i,k)} in \mathcal{H} = q^{n(\mu_0')}, \sum_{\ell(\mu) \leq n} \text{ch}(\pi_\mu) K_{\mu,(1)^N}(q^{-1})$$

with $\mu_0' = ((n)^m, i)$, $N = mn+i$ and $k \in \mathbb{Z}$.

**Proof.** Apart from the argument above, the only additional information is the normalization, which is determined by the convention that the highest weight vector of $F^{(i,k)}$ should have degree 0. This highest weight vector is in the $\mathfrak{sl}_n$-module with highest weight $w_i$, if $i < n$, or the trivial one if $i = n$. This corresponds to $\pi_{\mu_0}$ with $\mu_0' = ((n)^m, i)$ in the decomposition (17). On the other hand the Kostka polynomial $K_{\mu,(1)^N}(q^{-1})$ behaves as a power series in $q^{-1}$ of the form $q^{-n(\mu')}(1 + O(q^{-1}))$. The lemma follows.

Thus, up to an overall normalization factor, we see that the characters of the reduced wedge space and the graded tensor product (Corollary 4.5) are the same.

The difference in the normalization factor is due to the fact that the cyclic vector, which is normalized in each case to have degree 0, is different. The wedge space is generated by the action of $\mathfrak{sl}_n \otimes \mathbb{C}[t^{-1}]$ on the highest weight vector with respect to the $\mathfrak{sl}_n$ action, whereas the graded tensor product is generated by the action of $\mathfrak{sl}_n \otimes \mathbb{C}[t]$ on a cyclic vector which, in the limit $N \to \infty$, is deep inside the representation – it is an extremal vector at infinity, in the language of \[\text{FS}\].

6. **THE LIMIT $N \to \infty$**

It was explained in \[\text{FL} \ | \text{FF}\] that the irreducible level-1 $\widehat{\mathfrak{sl}_n}$-modules can be obtained as the inductive limit of the graded tensor product. The same is true for the reduced wedge product, by construction. Let us consider the behavior of the characters. In this section, we change conventions from $q$ to $q^{-1}$ in the character (so that the character
is a polynomial or power series in $q$ not $q^{-1}$) since this is the standard convention in the literature.

In the limit $N (or m) \to \infty$, for any finite-dimensional $\pi_\mu$, there is a well-defined limit

$$
\lim_{m \to \infty} q^{-n(\mu_0)} K_{\mu,(1)^m}(q).
$$

This limit is a character of the algebra which centralizes the action of $\mathfrak{sl}_n$ on the irreducible level-1 module – the $W$-algebra of $\mathfrak{sl}_n$ [FKRW]. That is, as an $\mathfrak{sl}_n \otimes W(\mathfrak{sl}_n)$-module,

$$
L(\Lambda_i) \simeq \bigoplus_{\mu} \pi_\mu \otimes M_\mu,
$$

where $\mu$ is a partition of length less than $n$, such that $|\mu| \equiv i \mod n$, and $M_\mu$ is an irreducible $W(\mathfrak{sl}_n)$-module whose character is described below.

The $W$-algebra of $\mathfrak{gl}_n$ is the central extension of the algebra of differential operators on $\mathbb{C}[t, t^{-1}]$ (including multiplication by polynomials in $t$). These act as symmetric differential operators on the $N$-fold tensor product of evaluation modules $\pi(z)$, acting only on $\mathbb{C}[z_{\pm 1}, \ldots, z_{\pm N}]$ and not on the vector space. Since such differential operators commute with the action of the symmetric group, they also act on the wedge product. Therefore they act on the multiplicity space $M_{\mu',(1)^n}$, where the Heisenberg subalgebra in $W$ acts trivially by definition. The statement that the Kostka polynomials tend to the specialized character of an irreducible $W$-algebra module in the limit $m \to \infty$ means that in the limit, the $W$ algebra acts irreducibly on the multiplicity space. This action in the limit has central charge $n$.

The character formulas of [FKRW] are obtained by using the hook formula for the Kostka polynomial (Another approach to computing them is the functional realization, see [FJKLM2] for the case of $\mathfrak{sl}_2$):

$$
(18) \quad K_{\mu,(1)^n} = q^{n(\mu')} \frac{(q)_n}{\prod_{x \in \mu} (1 - q^{h(x)})},
$$

where $h(x)$ is the hook length of the box $x$ in the Young diagram of shape $\mu$: if $x$ has coordinates $i, j$,

$$
h(x) = \mu_i - j + \mu'_j - i + 1.
$$

Here we use the standard notation $(q)_m = \prod_{i=1}^m (1 - q^i)$. It is not difficult to see that

$$
\prod_{x \in \mu} (1 - q^{h(x)}) = \prod_{i=1}^n (q)_{\mu_i + i} \prod_{1 \leq i < j \leq n} (1 - q^{\mu_i - \mu_j + j - i})^{-1}
$$

if the length of $\mu$ is $n$.

In lemma 17, $|\mu|$ depends on $m$. However for any module $\pi_\mu$ which is finite-dimensional in the limit $m \to \infty$, the parameter $\overline{m} = (\mu_0)_n - \mu_n < \infty$ is finite. In this case, $\pi_\mu \simeq \pi_{\overline{m}}$ as an $\mathfrak{sl}_n$-module, where $\overline{m}$ is finite in the limit $m \to \infty$, with parts

$$
\overline{\mu}_j = \mu_j - (m - \overline{m}).
$$

Then

$$
q^{-n(\mu_0)} K_{\mu,(1)^n}(q) = q^{n(\overline{m}) - \overline{m}(i - (\overline{m} - 1)n/2)} (q)_{mn+i} \prod_{1 \leq i < j \leq n} (1 - q^{\overline{m} - \overline{m} + j - i}) \prod_{i=1}^n (q)_{\overline{m} + i}.
$$
In the limit \( m \to \infty \) this becomes:

\[
\lim_{m \to \infty} q^{-n(\mu^\ell)} K_{\mu, (1)^N} (q) = \frac{q^{n(\mu^\ell) - m(i-(m-1)n/2)}}{(q)_\infty} \prod_{1 \leq i < j \leq n} (1 - q^{\mu_i - \mu_j + j-i}).
\]

In [FKRW], the irreducible \( W(\mathfrak{sl}_n) \)-modules \( M_\mu \) are normalized so that their specialized character is

\[
\chi_\mu(q) = q^{n(\mu^\ell) + |\mu|} \prod_{1 \leq i < j \leq n} (1 - q^{\mu_i - \mu_j + j-i})
\]

(where we factored out the character of the Heisenberg subalgebra \( 1/(q)_\infty \)). With this convention, we have

\[
\operatorname{ch}_q L(\Lambda_i) = \sum_{\mu: \mu_n = 0, |\mu| \equiv i \mod n} \operatorname{ch}(\pi_\mu) q^{-n(\mu^\ell)} \chi_\mu(q),
\]

where \( \pi_0 = ((n)^m, i) \) depends on \( \mu \).

7. Tensor Product of Symmetric-Power \( \mathfrak{sl}_n \)-Modules

The construction of section 4 turns out to be sufficient to deduce the structure of the graded tensor product of any set of symmetric power evaluation modules, parameterized by highest weights \( \mu, \omega \) (\( \mu \in \mathbb{Z}_{>0} \)). In this section we denote \( \mu := \mu = (\mu_1, \ldots, \mu_m) \), and we consider the fusion product of evaluation modules at a complex number.

Recall that any finite-dimensional irreducible \( \mathfrak{sl}_n \)-module \( \pi_\lambda \) is isomorphic to the image of a Young symmetrizer \( c_{t(\lambda)} \in \mathbb{C} S_N \) acting on \( \pi^{\otimes |\lambda|} \), where \( \pi \) is the fundamental module. In general, given a Young diagram \( \lambda \), and choosing any standard tableau \( t(\lambda) \) of shape \( \lambda \) on the letters \( 1, \ldots, |\lambda| = \sum \lambda_i \), let \( R_{t(\lambda)} \in S_N \) be the stabilizer of the rows of the tableau, and \( C_{t(\lambda)} \) the stabilizer of the columns. The Young symmetrizer is

\[
c_{t(\lambda)} = \sum_{\sigma \in C_{t(\lambda)}} \operatorname{sign}(\sigma) \sum_{\sigma' \in R_{t(\lambda)}} \sigma'.
\]

Similarly, the evaluation module \( \pi_\lambda[a] \) is isomorphic to the image of the composite map \( c_{t(\lambda)} \otimes \phi_a \) acting on \( \otimes_{i=1}^{|\lambda|} \pi[z_i] \), where \( \phi_a \) is the evaluation of the polynomial at the point \( z_i = a \) for all \( i \) (since the isomorphism is independent of the particular tableau \( t(\lambda) \), we will drop the tableau notation). We use this to construct the general tensor product of symmetric power evaluation modules, and explain how the graded tensor product can be obtained as a quotient of this.

Let \( V_N(Z) = \pi_1[z_1] \otimes \cdots \otimes \pi_N[z_N] \) as before, with \( z_i \) formal. Choose a partition of \( N, \mu = (\mu_1, ..., \mu_m) \), with \( \mu_i \) positive integers (not necessarily ordered) and \( \mu_1 + \cdots + \mu_m = N \). Let \( A = (a_1, ..., a_m) \), with \( a_i \) distinct complex numbers. Let \( \mathfrak{X} = (x_1, ..., x_N) \) be an \( N \)-tuple of complex numbers with the first \( \mu_1 \) variables being equal to \( a_1 \), the next \( \mu_2 \) equal to \( a_2 \), etc.

The symmetric group \( S_N \) acts on \( \mathfrak{X} \) by permuting indices, resulting in some orbit \( O_\mathfrak{X} \) in \( \mathbb{C}^N \). The stabilizer of the \( N \)-tuple \( \mathfrak{X} \) is the Young subgroup \( S_\mu \in S_N \), \( S_\mu = S_{\{1, \ldots, \mu_1\}} \times \cdots \times S_{\{\mu_1 + \cdots + \mu_{m-1}, \ldots, \mu_m\}} \), where \( S_{\{1, \ldots, n\}} \) is the group of permutations of
the elements \((1, \ldots, n)\). Corresponding to this subgroup we have the partial Young symmetrizer in \(\mathbb{C}[S_N]\),
\[
y_\mu = \sum_{\sigma \in S_\mu} \sigma.
\]
(The tableau under consideration is the standard one, with \((1, \ldots, \mu_1)\) in the first row, and so forth.) Since \(y_\mu\) is a partial Young symmetrizer, the image of \(y_\mu\) acting on \(\pi^\otimes N\) is, in general, not irreducible. In fact,
\[
y_\mu(\pi^\otimes N) \simeq \pi_{\mu_1} \otimes \pi_{\mu_2} \otimes \cdots \otimes \pi_{\mu_m} \simeq \bigoplus_{\lambda \vdash n} K_{\lambda, \mu} \pi_\lambda
\]
where \(K_{\lambda, \mu}\) is the Kostka number \(K_{\lambda, \mu}(1)\), and \(\mu\) is the partition with parts \(\mu_i\) obtained by ordering the integers \(\mu_i\).

Define the evaluation map, \(\phi_X\) acting on \(\mathbb{C}[z_1, \ldots, z_N]\) to be the evaluation at the point \(z_i = x_i\) for all \(i\):
\[
\phi_X(f(z_1, \ldots, z_N)) = f(x_1, \ldots, x_N).
\]
Therefore the map \(\nu_{\mu, A} : V_N(Z) \rightarrow V_\mu(A) = \pi_{\mu_1} [a_1] \otimes \cdots \otimes \pi_{\mu_m} [a_m]\) is the composition of maps
\[
(19) \quad V_N(Z) \xrightarrow{y_\mu \otimes \text{id}} (\pi^\otimes N)^{S_\mu} \otimes \mathbb{C}[z_1, \ldots, z_N] \xrightarrow{\text{id} \otimes \phi_X} V_\mu(A).
\]
(The order of the maps is not important as they act on different spaces.) Here, by \((\pi^\otimes N)^{S_\mu}\), we mean the image of \(y_\mu\), which is invariant under the action of \(S_\mu\). This is a surjective map, in particular, the second map, \(\text{id} \otimes \phi_X\) is surjective. Therefore,
\[
(20) \quad V_\mu(A) \simeq (\pi^\otimes N)^{S_\mu} \otimes \mathbb{C}[z_1, \ldots, z_N] / \ker \phi_X.
\]
Let \(I_X\) be the ideal in \(\mathbb{C}[z_1, \ldots, z_N]\) of functions which vanish at the evaluation point \(z_i = x_i\). The kernel of \(\phi_X\) is \((\pi^\otimes N)^{S_\mu} \otimes I_X\). Notice that \(I_X\) is invariant under the action of the Young subgroup \(S_\mu\).

As for the grading, one can define the grading on the space \(V_\mu(A)\) to be the grading inherited from the preimage in \(V_N(Z)\). Therefore the isomorphism (20) is an isomorphism of filtered spaces. The associated graded space of the RHS is isomorphic to the LHS as a graded vector space.

Next, consider the space \(\mathcal{F}_\mu(Z)\) generated by the action of \(U(\mathfrak{sl}_n[t])\) on the tensor product of highest weight vectors \(v_{\mu_i}\) of \(\pi_{\mu_i}\). In the special case under consideration here, with \(\pi_{\mu_i}\) corresponding to the symmetric product representation, the cyclic vector \(w_\mu = v_{\mu_1} \otimes \cdots \otimes v_{\mu_m}\) of \(\mathcal{F}_\mu\), is the image under \(\nu_{\mu, A}\) of \(w_0 = v_0 \otimes \cdots \otimes v_0 \in \mathcal{F}_N\). In addition, each of the modules \(\pi_{\mu_i}\) is generated by the action of the same commuting generators \(b_i\) on the highest weight vector. It follows that
\[
\mathcal{F}_\mu(A) = \nu_{\mu, A} (\mathcal{F}_N(Z)).
\]
Note that the evaluation map preserves grading by definition.

**Lemma 7.1.** **Representatives of the graded tensor product** \(\mathcal{F}_{\mu}^*\) **are a subset of the image of representatives of** \(\mathcal{F}_{(1)^N}^*\) **under the evaluation map:**
\[
\mathcal{F}_{\mu}^* \subset \nu_{\mu, A} (\mathcal{F}_{(1)^N}^*)
\]
where by \(\mathcal{F}_{\mu}^*\), we mean the set of representatives of \(\mathcal{F}_{\mu}^*\) in \(F_\mu(A)\) etc.
Proof. This follows from the definition \((2.2)\), because \(\nu_{\mu,\mathcal{A}}(\tilde{F}^{(n)}_N) \subset \tilde{F}^{(n)}_\mu\). To see this, suppose \(p^{(n)} \in \tilde{F}^{(n)}_N\). Then it can be expressed in the form \((8)\) with \(\tilde{p}^{(n)} = 0\). Therefore,
\[
\nu_{\mu,\mathcal{A}}(p^{(n)}) = \sum_{m < n} \tilde{f}_i^{(n-m)}(x_1, \ldots, x_N) \nu_{\nu,\mathcal{A}}(p^{(m)}_i),
\]
Recall that \(\nu_{\nu,\mathcal{A}}(\tilde{F}^{(n)}_N) = \tilde{F}^{(n)}_\mu\) since the grading is preserved. Therefore \(\nu_{\nu,\mathcal{A}}(p^{(m)}_i) \in \tilde{F}^{(m)}_\mu(\mathcal{A})^{(m)}\) on the right hand side. This means that
\[
\nu_{\nu,\mathcal{A}}(p^{(n)}) \in \mathbb{C}[a_1, \ldots, a_m] \otimes \tilde{F}^{(n)}_\mu(\mathcal{A})^{(n)} \cap \tilde{F}^{(n)}_\mu(\mathcal{A})^{(n)} = \tilde{F}^{(n)}_\mu(\mathcal{A})^{(n)}.
\]
\(\square\)

The result of Section 4 showed that \(\mathcal{F}^*_{(1)N}\) is the quotient of \(V^SN_N\) by the ideal generated by symmetric functions of positive degree. We will take consecutive quotients of \(V^SN_N(\mathbb{Z})\): First by the kernel of the evaluation map and then by \(\langle J_N \rangle\). Therefore, consider the evaluation map \(\phi_X\) acting on
\[
(21) \quad V^SN_N(\mathbb{Z}) = (\pi^\otimes N \otimes \mathbb{C}[z_1, \ldots, z_N])^{S_N}.
\]

Lemma 7.2.

(22) \[
\phi_X(\pi^\otimes N \otimes \mathbb{C}[z_1, \ldots, z_N])^{S_N} = \nu_{\mu,\mathcal{A}}(\pi^\otimes N \otimes \mathbb{C}[z_1, \ldots, z_N])^{S_N}.
\]

Proof. Consider any \(w \in \pi^\otimes N\) and \(f(z_1, \ldots, z_N) \in \mathbb{C}[z_1, \ldots, z_N]\). Th set of all such \(w, f\) gives a spanning set of vectors for \(V^SN_N(\mathbb{Z})^{S_N}\) after complete symmetrization:
\[
v = \sum_{\sigma \in S_N} \sigma(w) \otimes \sigma f(z_1, \ldots, z_N) \in V^SN_N(\mathbb{Z})^{S_N}.
\]
Acting with \(\phi_X\), we have
\[
\phi_X(v) = \sum_{\sigma \in S_N} \sigma(w) \otimes \phi_X \sigma f(z_1, \ldots, z_N)
\]
\[
= \sum_{\sigma \in S_N} \sigma(w) \otimes \phi_{\sigma^{-1}(x)} f(z_1, \ldots, z_N)
\]
\[
= \sum_{\sigma \in S_N} \sum_{\tau \in S_N} \tau \sigma(w) \otimes \phi_{\sigma^{-1}(x)} f(z_1, \ldots, z_N)
\]
\[
= y_\mu \sum_{\sigma \in S_N \setminus S_N} \sigma(w) \otimes \phi_X \sigma f(z_1, \ldots, z_N),
\]
(here, the sum is over coset representatives). In the third line, we used the fact that \((\tau \sigma)^{-1} = \sigma^{-1} \tau\) for \(\tau \in S_N\). Since \(y_\mu^2 = y_\mu\), the lemma follows. \(\square\)

Let \(I_\mu\) be the symmetrization of \(I_X\):
\[
I_\mu = \bigcap_{\sigma \in S_N} I_{\sigma(x)}.
\]
Polynomials in \(I_\mu\) are those which vanish everywhere on the \(S_N\)-orbit of the point \(X\). As is clear from the proof of the previous lemma, the kernel of \(\phi_X\) acting on \(V^SN_N\) is the symmetrization, \((\pi^\otimes N \otimes I_\mu)^{S_N}\). We have
Lemma 7.3.\[ (\pi^N \otimes \mathbb{C}[z_1, \ldots, z_N])^{S_N} / \ker \phi_X = (\pi^N \otimes \mathbb{C}[z_1, \ldots, z_N] / I_\mu)^{S_N} = (\pi^N \otimes A_\mu)^{S_N} \]

where \(A_\mu\) is the quotient of the ring of polynomials \(\mathbb{C}[z_1, \ldots, z_N]\) by the ideal of functions which vanish on the \(S_N\)-orbit of \(X\).

The generating polynomials of the ideal \(I_\mu\) are not of homogeneous in degree in \(z_i\), and therefore the quotient (23) is a filtered space, with the filtration inherited from \(V_N(\mathbb{Z})^{S_N}\). In this picture, \(a_i\) are treated as distinct complex numbers. The graded components of (23) have the same dimensions as the graded components of the image of the evaluation map, since it preserves grading.

The quotient (23) is isomorphic to the image of \(V_N(\mathbb{Z})^{S_N} \simeq F_N(\mathbb{Z}) \otimes \mathbb{C}[z_1, \ldots, z_N]^{S_N}\), not \(F_N(\mathbb{Z})\) itself, so it is not equal to \(F_\mu(A)\). However, its associated graded space is the graded tensor product \(F^*\):\[ F^* \simeq \text{gr} (\pi^N \otimes A_\mu)^{S_N} \]

\[ \text{Proof.} \] Taking the associated graded space is equivalent to taking the limit \(a_i \to 0\) for all \(i\). In this limit, (see Prop. 3.1 of [GP]) \(I_\mu\) is identical to \(J_\mu\), which contains the ideal \(J_N\) generated by symmetric functions of positive degree. Thus, taking the quotient by \(J_N\) of (23) and then taking the associated graded space is equivalent to simply taking the associated graded space. \(\square\)

As \(S_N\)-modules, \(A_\mu \simeq R_\mu\), and the decomposition into irreducible \(S_N\)-modules has the same form as \(R_\mu\):

\[ A_\mu \simeq \oplus_\lambda W_\lambda \otimes \tilde{M}_{\lambda, \mu}. \]

The multiplicity spaces \(\tilde{M}_{\lambda, \mu}\) are filtered vector spaces, by degree in \(z_i\).

\[ (\pi^N \otimes A_\mu)^{S_N} \simeq (\pi^N \otimes (\oplus_\lambda W_\lambda \otimes \tilde{M}_{\lambda, \mu}))^{S_N}. \]

As in Section 4, we conclude that (24) is equal to

\[ ((\bigoplus_{\ell(\lambda) \leq n} \pi_\lambda \otimes W_\lambda) \otimes (\bigoplus_{\lambda'} W_{\lambda'} \otimes \tilde{M}_{\lambda', \mu}))^{S_N} \simeq \bigoplus_{\ell(\lambda) \leq n, \lambda \geq \mu} \pi_\lambda \otimes \tilde{M}_{\lambda, \mu}, \]

where the last step follows from the same reasoning as Theorem 4.4. From Lemma 7.3, equation (25) and the fact that \(R_\mu = \text{gr} A_\mu\).

Corollary 7.5.

\[ F^* \simeq \bigoplus_{\lambda \vdash \mu, \ell(\lambda) \leq n} \pi_\lambda \otimes M_{\lambda, \mu}. \]

The restriction \(\lambda \geq \mu\) turns out to be unnecessary, as the space \(M_{\lambda, \mu}\) is zero otherwise.

The main result of this section therefore is the conclusion that:
Corollary 7.6. If $\mu = (\mu_1, \ldots, \mu_m)$ is a partition of $N$ into positive integers, then

\begin{equation}
ch_q J^* \mu = \sum_{\lambda \vdash N \ell(\lambda) \leq n} ch_\pi \tilde{K}_{\lambda, \mu}(q).
\end{equation}

where $\tilde{K}_{\lambda, \mu}(q)$ is defined in [7].

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