Proof of the tree module property for exceptional representations of tame quivers

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Abstract

This document serves as an arXiv entry point for the appendix to the paper [13] (the ancillary file e6_proof.pdf – “Proof of the tree module property for exceptional representations of the quiver $\tilde{E}_6$”) and the appendix to the paper [12] (the ancillary file d6_proof.pdf – “Proof of the tree module property for exceptional representations of the quiver $\tilde{D}_6$”). The ancillary files contain the computer generated part of the proofs of the main results in [13] respectively [12], giving a complete and general list of tree representations corresponding to exceptional modules over the path algebra of the canonically oriented Euclidean quiver $\tilde{E}_6$, respectively $\tilde{D}_6$. The proofs (involving induction and symbolic computation with block matrices) were partially generated by a purposefully developed computer software, outputting in a detailed step-by-step fashion as if written “by hand”.

We also give here a short theoretical introduction and an overview of the computational method used to prove the formulas given in the papers [13] and [12].

1 Basic notions of representation theory of algebras

Let $Q = (Q_0, Q_1, s, t)$ be a quiver, that is, a directed graph, where $Q_0$ is the set of vertices, $Q_1$ is the set of arrows and $s, t : Q_1 \to Q_0$ are functions which attach to an arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$.

We often write shortly $Q = (Q_0, Q_1)$. Let $k$ be a field and consider the path algebra $kQ$. The category mod-$kQ$ of finite dimensional right modules over $kQ$ can be identified with the category rep-$kQ$ of the finite dimensional $k$-representations of the quiver $Q$ (therefore we will use the terms “module” and “representation” interchangeably).

Recall that a $k$-representation $M = (M_i, M_\alpha)$ of $Q$ is defined as a set of finite dimensional $k$-spaces $\{M_i \mid i \in Q_0\}$ corresponding to the vertices together with $k$-linear maps $\{M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)} \mid \alpha \in Q_1\}$ corresponding to the arrows. Given two representations $M = (M_i, M_\alpha)$ and $N = (N_i, N_\alpha)$ of the quiver $Q$, a morphism of representations $f : M \to N$ consists of a family of $k$-linear maps (corresponding to the vertices) $f_i : M_i \to N_i$, such that $N_\alpha f(s(\alpha)) = f(t(\alpha))M_\alpha$ for all $\alpha \in Q_1$.

The dimension vector of a representation $M = (M_i, M_\alpha)$ is

$$\dim M = (d_i)_{i \in Q_0} \in \mathbb{Z}Q_0 \text{ where } d_i = \dim_k M_i,$$

which is treated as an $n$-dimensional row vector where $n = |Q_0|$. In this case the length of $M$ is $\ell(M) = \sum_{i \in Q_0} d_i$.

There are five types of so-called Euclidean (or tame) quivers: $\tilde{A}_m$, $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

The Euler form of an arbitrary acyclic quiver $Q$ is the bilinear form defined on $\mathbb{Z}Q_0$ as

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$
Its quadratic form $qq$ (called \\textit{Tits form}) is independent from the orientation of $Q$ and in the tame case it is positive semi-definite with radical $\mathbb{Z}\delta$, where $\delta$ is a minimal positive imaginary root of the corresponding Kac–Moody root system. A vector $x \in \mathbb{Z}Q_0$ is called real root if $qq(x) = 1$, imaginary root if $qq(x) = 0$ and it is positive if $x_i \in \mathbb{N}$ for all $i \in Q_0$. For two (dimension) vectors $d, d' \in \mathbb{Z}Q_0$ we say that $d \leq d'$ if $d_i \leq d'_i$ for all $i \in Q_0$.

Let $P(i)$ and $I(i)$ be the indecomposable projective respectively injective module corresponding to the vertex $i$. The Cartan matrix $C_Q$ is a matrix with the $j$-th column being equal with $\dim \Phi_{Q(j)}$. The Coxeter matrix is defined as $\Phi_Q = -C_Q^{-1}C_Q^{-1}$. Then $\Phi_Q \delta = \delta$ and the Euler form satisfies $(a, b) = a \left( C_Q^{-1} \right)^t b' = -\langle b, \Phi_Q a \rangle$, where $a, b \in \mathbb{Z}Q_0$.

Moreover, because our algebra is hereditary, for two modules $M, N \in \text{mod-}kQ$ we get

\[
\langle \dim M, \dim N \rangle = \dim_k \text{Hom}_{kQ}(M, N) - \dim_k \text{Ext}_{kQ}^1(M, N).
\]

(1.1)

The \textit{Auslander–Reiten translates} are defined as

\[
\tau = D \text{Ext}_{kQ}^1(-, kQ) \quad \text{and} \quad \tau^{-1} = \text{Ext}_{kQ}^1(D(kQ), -)
\]

where $D = \text{Hom}_k(-, k)$.

An indecomposable module $M$ is \textit{preprojective} if there exists a positive integer $s$ such that $\tau^s(M) = 0$, while it is called \textit{preinjective} if $\tau^{-s}(M) = 0$. The indecomposable $M$ is \textit{regular} if it is neither preinjective nor preprojective.

From now on let $Q$ be a tame quiver. For $Q$, the structure of the category mod-$kQ$ and its \textit{Auslander–Reiten quiver} is well-known. Up to isomorphism, the indecomposable preprojective modules are $\tau^sP(i)$, while the indecomposable preinjectives are $\tau^{-s}I(i)$, where $s \in \mathbb{N}$ and $i \in Q_0$. In the sequel we use the somewhat more convenient notation $P(s, i)$ to denote the preprojective indecomposable module $\tau^sP(i)$ and $I(s, i)$ to denote the preinjective indecomposable module $\tau^{-s}I(i)$. The following is true concerning the dimension vectors of preprojective, respectively preinjective indecomposables:

\[
\dim P(s, i) = \Phi_Q^{-s} \cdot \dim P(i) \quad \text{and} \quad \dim I(s, i) = \Phi_Q^s \cdot \dim I(i).
\]

(1.2)

The category of regular modules is an abelian, exact subcategory which decomposes into a direct sum of serial categories with Auslander–Reiten quiver of the form $\mathbb{Z}k\infty/r$, called \textit{tubes} of rank $r$. A tube of rank 1 is called \textit{homogeneous}, otherwise it is called \textit{non-homogeneous}.

A very important fact is that mod-$kQ$ is a \textit{Krull–Schmidt category}, meaning that every module can be written as a direct sum of indecomposables in a unique way (up to order and isomorphism).

It is well-known that the dimension vector $x$ of an indecomposable is either a positive real root (i.e. $qq(x) = 1$) or a positive imaginary root (i.e. $qq(x) = 0$). It is also known that for every positive real root $x$ there is a unique (up to isomorphism) indecomposable $M$ with $\dim M = x$ (in fact these indecomposables are all the preprojectives, all the preinjectives and the non-homogeneous regular indecomposables with dimension different from a multiple of $\delta$).

An indecomposable module $M$ is called \textit{exceptional} if it has no self-extensions (i.e. if $\dim_k \text{Ext}_{kQ}^1(M, M) = 0$). This means that its dimension is a positive real root (called \textit{exceptional root}) and $\dim_k \text{End}_{kQ}(M) = 1$. We know that the exceptional indecomposable modules are all the preprojectives, all the preinjectives and the regular non-homogeneous indecomposables with dimension vector falling below $\delta$ (see [5]).

For more details concerning the notions presented in this section we refer to [2, 1, 23, 26].

2 \hspace{1em} Tree representations

An indecomposable module $M = (M_i, M_\alpha)$ is called a \textit{tree module} if there is a basis $B$ such that the matrices of the linear maps $M_\alpha$, written in basis $B$ consist only of elements 0 and 1, moreover, the total number of non-zero elements is $\ell(M) - 1$, where $\ell(M) = \sum_{i \in Q_0} d_i$ with $\dim M = (d_i)_{i \in Q_0}$. Equivalently, $M$ is a tree module if there exists a basis $B$ such that the associated coefficient quiver is a tree (for details see [19]).

In [19] Ringel proves that exceptional modules are tree modules. The proof is based on a result by Schofield (see [21]), stating that if $M$ is an exceptional module that is not simple, then there are exceptional modules $X, Y$.
with the properties $\text{Hom}_{kQ}(X,Y) = \text{Hom}_{kQ}(Y,X) = \text{Ext}_{kQ}^1(Y,X) = 0$ and an exact sequence of the following form: $0 \longrightarrow vY \longrightarrow M \longrightarrow uX \longrightarrow 0$, where $u$ and $v$ are positive integers and the notation $uY$ means $Y \oplus \cdots \oplus Y$ ($u$ times). There are precisely $s(M) - 1$ such sequences where $s(M)$ is the number of nonzero components in $\dim M$. We call these short exact sequences Schofield sequences and the pair $(X,Y)$ a Schofield pair (associated to $M$). Note that the original proof of Schofield assumes an algebraically closed field, but Ringel gives a proof in [20] which works in arbitrary field $k$. Proposition 6 from [25] states that if $X$, $Y$, $M$ are exceptional indecomposables such that $u \dim X + v \dim Y = \dim M$, then we have a Schofield sequence $0 \longrightarrow vY \longrightarrow M \longrightarrow uX \longrightarrow 0$, if and only if $(\dim X, \dim Y) = 0$. This means that Schofield sequences and pairs depend only on the dimensions of indecomposables, thus their existence condition is field independent. Also note that although the short exact sequences used in our proofs are Schofield sequences (as above, with $v = u = 1$), we do not use here the results from [21] or [20] to construct them, but every short exact sequence used throughout the proofs is written (and verified) using Lemma 5 (working also over an arbitrary field $k$).

Although tree representations for some particular quivers are known, the proof in [19] does not give an explicit method for constructing them in general.

In [8] Gabriel gave a full list of indecomposable representations for the Dynkin quivers using $0-1$-matrices. All the given representations (excepting 4 of them) were tree representations. Tree representations in these four cases were given by Crawley-Boevey [4].

Regarding the Euclidean case, Mróz gave a full list of the indecomposable tree representations for the quiver of type $\tilde{D}_4$ with four subspace orientation in [15]. His results were generalized by Lőrinczi and Szántó, giving a full list of tree representations for the quiver of type $\tilde{D}_6$ with a particular non-canonical orientation (see [14]). We note that these representations were proved for path algebras over algebraically closed fields only, moreover in the paper indecomposability was checked only for some random representations from the list (so the checking was not complete). Analogous problems are considered for canonical algebras in [6], for nilpotent operators in [7] and for poset representations in [9].

Concerning the $\tilde{D}_m$, and $\tilde{E}_8$ cases, indecomposable representations for preinjectives and preprojectives were given by Kussin, Kędzierski and Meltzer in [10] and [11], respectively (however, those representations are not tree representations). Their aim was not to give explicit tree representations in particular, but to describe a general method for obtaining indecomposable (not necessarily tree) representations in tame cases.

Our first aim was to study tree representations and to develop a computational method which produces rigorously proved explicit tree formulas (in a “ready to consume” form) and which is also “scalable” (can be performed in a timely manner for all possible families of exceptional modules). Our second aim was to use the method in producing a complete and explicit list of tree representations for all families of exceptionals, which can be easily introduced and used in any computer algebra system, without bothering about the way they were obtained. Given the nature of the problem (the number of cases to be considered and the amount of block-matrix arithmetic to be performed) the best we could come up with was the idea presented in Subsection 1.3 from [13] (which, to our knowledge, is new and completely different from the method(s) used by Mróz, Kussin, Kędzierski and Meltzer) and to develop a special proof assistant software performing the matrix-crunching and producing a rather lengthy, nevertheless completely general, formal and correct proof of every formula listed in Part II of the auxiliary documents.

The importance of knowing explicit formulas for tree representations stems from a number of advantageous properties. In case of tree representations, the matrices involved are the “sparsest possible” (i.e. containing the minimal number of non-zero elements), thus reducing the storage and running time complexity in computer implementations. As mentioned before, the exceptional modules are determined by their dimension vectors up to isomorphism, so having a formula for each of them gives a “nice” representative of each isomorphism class. In fact, we could say that tree representations are the “canonical” forms of these modules, analogously to the canonical form of matrix pencils or canonical forms of matrices (for example the Jordan normal form). An example of nice consequences of knowing such sparse forms is the paper [16] by Mróz, where such matrix forms of modules were applied to obtain formulas for the multiplicities of the preprojective and preinjective indecomposables appearing in the decomposition of an arbitrary $\tilde{D}_4$ module.

It is very important to realize that the tree representations given remain valid independently on the underlying field
of the representation. That is, the $1 - 0$ matrices listed in the ancillary files withstand a replacement of the base field $k$ in mod-$k \Delta(Q)$ such that if $M \in \text{mod-}k \Delta(Q)$ is an exceptional tree representation, then $M' \in \text{mod-}k' \Delta(Q)$ is also an exceptional tree representation where $\dim M = \dim M'$, and every matrix $M_\alpha$ from the first representation is formally the same as the corresponding matrix $M'_\alpha$ from the second one.

3 Proving the field independent tree module property

In this section we describe the method used to prove the tree module property for every representation given in the lists in Part II of the ancillary documents, both from the theoretical and practical perspective. The method presented here is general (in the sense that it could be applied to any tame quiver), so as stated before, $Q$ denotes an arbitrary tame quiver and $k$ an arbitrary field. We just state the results here, the proofs are to be found in [13].

We will use the ‘field independent’ qualifier in relation to representations and short exact sequences in the following precise manner:

Definition 1. Let $M \in \text{mod-}kQ$ an (exceptional) indecomposable module. We say that:

1. The module $M$ is field independent (exceptional) indecomposable if in the corresponding representation $M = (M_\alpha, M_\alpha)$ all the elements in the matrices $M_\alpha$ are either 0 or 1 and for any field $k'$ if we consider a module $M' \in \text{mod-}k'Q$ such that $\dim M = \dim M'$ and every matrix $M'_\alpha$ from the corresponding representation $M' = (M'_\alpha, M'_\alpha)$ is formally the same as $M_\alpha$ (for all arrows $\alpha$), then $M'$ is also (exceptional) indecomposable in mod-$k'Q$.

2. The module $M$ has the field independent tree property if it is a tree module in mod-$kQ$ and it is also a field independent (exceptional) indecomposable module (i.e. if we consider the corresponding representation with formally the same matrices over any other field $k'$, we still get an exceptional indecomposable tree module in mod-$k'Q$).

3. A short exact sequence of the form $0 \longrightarrow Y \overset{f}{\longrightarrow} Z \overset{g}{\longrightarrow} X \longrightarrow 0$ is field independent (with $X, Y, Z \in \text{mod-}kQ$) if all the elements in the matrices of the representations $X, Y$ and $Z$ are either 0 or 1, all the elements in the matrices $f_i$ and $g_i$ of the embedding $f = (f_i)_{i \in \Omega_k}$ respectively the projection $g = (g_i)_{i \in \Omega_k}$ are either 0 or 1 or −1 and in any field $k'$ the sequence $0 \longrightarrow Y' \overset{f'}{\longrightarrow} Z' \overset{g'}{\longrightarrow} X' \longrightarrow 0$ is also exact, where $X', Y', Z' \in \text{mod-}k'Q$, $f' : Y' \rightarrow Z'$, $g' : Z' \rightarrow X'$ correspond in order to $X, Y, Z$, $f : Y \rightarrow Z$, $g : Z \rightarrow X$ with the respective dimension vectors unchanged and with all matrices (both from the representations and from the morphisms) being formally the same when considering them over $k'$ instead of $k$.

The following proposition and lemmas constitute the theoretical elements of the technique used to prove the formulas in a field independent way:

Lemma 2. For a module $M \in \text{mod-}kQ$ we have $M$ is exceptional indecomposable if and only if $\dim_k \text{End}_{kQ}(M) = 1$ and $\dim M \neq \delta$.

Remark 3. We know that exactly these are the exceptional modules in the tame case: the preprojective indecomposables, the preinjective indecomposables and the regular non-homogeneous indecomposable modules with dimension vector falling below $\delta$.

Proposition 4. Let $X, Y, X', Y' \in \text{mod-}kQ$ be indecomposable modules. If $M \in \text{mod-}kQ$ such that

(a) there is an exceptional $Z \in \text{mod-}kQ$ such that $(X, Y)$ and $(X', Y')$ are Schofield pairs associated to $Z$,

(b) there exist two short exact sequences

\[
0 \longrightarrow Y \longrightarrow M \longrightarrow X \longrightarrow 0
\]

and

\[
0 \longrightarrow Y' \longrightarrow M \longrightarrow X' \longrightarrow 0,
\]
(c) $X \nsubseteq X'$ or $Y \nsubseteq Y'$,
(d) $\dim_k \text{Ext}^1_{kQ}(X,Y) = \dim_k \text{Ext}^1_{kQ}(X',Y') = 1$

then $M$ is exceptional indecomposable.

Lemma 5. Let $X, Y, Z \in \text{mod-}kQ$ and $f = (f_i)_{i \in Q_0}, g = (g_i)_{i \in Q_0}$ families of $k$-linear maps $f_i : Y_i \rightarrow Z_i, g_i : Z_i \rightarrow X_i$. Then there is a short exact sequence

$$0 \rightarrow Y \xrightarrow{f} Z \xrightarrow{g} X \rightarrow 0$$

if and only if the following conditions hold (we identify the maps $f_i$ and $g_i$ with their matrices in the canonical basis):

(a) the matrices $f_i$ (respectively $g_i$) have maximal column (respectively row) ranks,
(b) $f_{t(a)}Y_\alpha = Z_\alpha f_{s(a)}$ and $g_{t(a)}Z_\alpha = X_\alpha g_{s(a)}$, for all $\alpha \in Q_1$,
(c) $g_if_i = 0$, for all $i \in Q_0$,
(d) $\dim Z = \dim X + \dim Y$.

The embedding $f : Y \rightarrow Z$ can be given via a family of maximal (column) rank matrices $f_i$ ($i \in Q_0$) satisfying $f_{t(a)}Y_\alpha = Z_\alpha f_{s(a)}$ for all $\alpha \in Q_1$, while the projection $g : Z \rightarrow X$ can be given via a family of maximal (row) rank matrices $g_i$ ($i \in Q_0$) satisfying $g_{t(a)}Z_\alpha = X_\alpha g_{s(a)}$ for all $\alpha \in Q_1$.

Lemma 6. If $X, Y \in \text{mod-}kQ$ are indecomposable modules such that $X$ is regular and $Y$ is preprojective, or $X$ is preinjective and $Y$ is regular or both of them are preprojectives (or preinjectives) and there is a path in the Auslander–Reiten quiver from the vertex corresponding to $Y$ to the vertex corresponding to $X$, then $\dim_k \text{Ext}^1_{kQ}(X,Y) = - (\dim X, \dim Y)$.

We are now ready to describe the process of proving the formulas from Part II of the ancillary document.

The process of proving the field independent tree property

Suppose we have formulas defining families of matrices $(M_\alpha^{(n)})_{\alpha \in Q_1}$ depending on some $n \in \mathbb{N}$. The elements of the matrices $M_\alpha^{(n)}$ are either 0 or 1, so they can be considered over an arbitrary field $k$. We want to prove that the representation of the quiver $Q$ given as $M = M(n) = (M_1^{(n)}, M_2^{(n)})$ has the field independent tree property (where the dimension of each $k$-space $M_j^{(n)}$ is in accordance with the column and row sizes of the matrices $M_\alpha^{(n)}$, thus the formulas also determine $\dim(M)$). Suppose that $\dim M$ is such that it coincides with the dimension vector of an exceptional indecomposable (see Lemma 2 and Remark 3). Suppose also that the number of elements equal to 1 in the matrices $M_\alpha^{(n)}$ is exactly $\ell(M) - 1$. So, in order to prove the field independent tree module property, we need only to show that $M$ is field independent indecomposable. We may use one of the following lines of reasoning:

1. Prove that $\dim_k \text{End}_{kQ}(M) = 1$ in any field $k$ and use Lemma 2. This may be done by writing the matrix $A$ of the homogeneous system of linear equations defining $\text{End}_{kQ}(M)$ and showing that the corank of $A$ is one (i.e. the solution space is one dimensional). In order to compute the rank of $A$, it must be echelonized (brought to row echelon form) using elementary operations on rows and/or columns in a “field independent way”. This means that every single elementary operation used in the process of echelonizing $A$ must be such that the elements in the resulting matrix are either 0, 1 or $-1$ and the result is exactly the same if performed in any field $k$. For example if in the case of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ we perform the elementary row operation $r_2 \leftarrow r_2 - r_1$, then we get $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, if performed in $\mathbb{R}$, or $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, if performed in $\mathbb{Z}_2$. Hence it has different ranks if considered over different fields. A crucial element of this proof is to ensure something like this never happens, but the result of every single elementary operation performed is formally the same matrix, independently of the field it is considered in.
(2) **Perform an induction on** \( n \) **using method** (1) **above** (typically for \( n = 0 \), but the structure of the block matrices depending on \( n \) might require to make additional proofs for small values of \( n \)). Then suppose the formula gives field independent exceptional indecomposables \( M^{(n)} = (M^{(n)}_i, M^{(n)}_a) \) for all \( n' < n \). Find two pairs of modules \((X, Y)\) and \((X', Y')\) conforming to all requirements of Proposition 4, such that any of these four representations is obtained either using formula \( M^{(n')} \) for some \( n' < n \) (or some permuted version of it) or some other formulas proved already to give field independent exceptional indecomposables. If the quiver \( Q \) presents some symmetries, then a permuted version of the formula \( M^{(n')} = (\tilde{M}^{(n')}_i, \tilde{M}^{(n')}_a) \) may also be used in the induction step, where \((\tilde{M}^{(n')}_i)_{i \in Q_0} = (M^{(n')}_{\sigma(i)})_{i \in Q_0} \) and \((\tilde{M}^{(n')}_{i \rightarrow j})_{(i \rightarrow j) \in Q_1} = (M^{(n')}_{\sigma(i) \rightarrow \sigma(j)})_{(i \rightarrow j) \in Q_1} \) for some permutation \( \sigma \). One has to construct here the two field independent short exact sequences of the form \( 0 \to Y \to M^{(n)} \to X \to 0 \) and \( 0 \to Y' \to M^{(n)} \to X' \to 0 \) in order to show their existence. Once the matrices of the morphisms are constructed, Lemma 5 can be used to prove that indeed these form short exact sequences in any field \( k \). We emphasize that conditions (a), (b) and (c) from Lemma 5 must be verified in a “field independent way”: the rank of the matrices must be checked using field independent echelonization as explained before, and the result of the matrix arithmetic operations used in (b) and (c) must be formally the same, independently of the underlying field.

(3) **Perform a direct proof**, making use of Proposition 4. Use two pairs of modules \((X, Y)\) and \((X', Y')\) conforming to all requirements of Proposition 4, such that any of these four representations are obtained by some formulas showed already to give field independent exceptional indecomposables, and prove the existence of the two field independent short exact sequences \( 0 \to Y \to M^{(n)} \to X \to 0 \) and \( 0 \to Y' \to M^{(n)} \to X' \to 0 \) by constructing them using Lemma 5 in the “field independent way”.

**Remark 7.** Note that in methods (2) and (3) the condition \( \dim \operatorname{Ext}^1_Q(X, Y) = \dim \operatorname{Ext}^1_Q(X', Y') = 1 \) required by (d) from Proposition 4 may be checked by simply computing \(-\dim X, \dim Y\) and \(-\dim X', \dim Y'\), if both pairs are such that Lemma 6 may be applied in their case.

The proof process described is extremely cumbersome, time-consuming and error-prone if performed by a human, therefore we have implemented a proof assistant software to help us in carrying it out. The proof assistant can perform any of the steps (1), (2) or (3) based on some input given in a \( \LaTeX \) file. The input data consists of the formulas \((M^{(n)}_a)_{a \in Q_1}\) defining the representations and the choice for the short exact sequences required in (2) and (3), together with the families of matrices defining the morphisms. All this data must be given in a \( \LaTeX \) document with a well-defined structure, in order for the proof assistant to be able to parse it and extract the relevant information. The matrices are given either as “usual matrices” (of fixed size, with elements equal to either 1, -1 or 0), or symbolic block-matrices of variable size, depending on the parameter \( n \in \mathbb{N} \). Every block-matrix is built using the following three types of blocks: zero block of size \( n_1 \times n_2 \), the identity block \( I_n \) and a block denoted by \( E_n \) having ones on the secondary diagonal and zeros everywhere else (note that \( E_n^2 = I_n \) in every field). We have used the document processor \( \LaTeX \) to edit the input document and export it to \( \LaTeX \) (in this way ensuring a syntactically correct \( \LaTeX \) file).

These are the steps performed by the software:

- It reads and stores the data \( M^{(n)} = (M^{(n)}_i, M^{(n)}_a) \) defining the representations \( M^{(n)} \).
- Computes the total number of elements equal to 1 in the matrices \( M^{(n)}_a \) and compares it against \( \ell(M^{(n)}) \) to ensure their number is exactly \( \ell(M^{(n)}) - 1 \).
- If instructed to perform along method (1), it computes the matrix \( A \) of the homogeneous system of linear equations defining \( \operatorname{Ext}^1_Q(M^{(n)}) \) and shows that it can be brought to echelon form by performing exactly the same elementary operations resulting in exactly the same matrix (formally) if considered in any field. In this way it ensures that the corank of \( A \) is one independently of the field. Note that it can perform in this mode only with formulas where \( n \) has any given concrete value.
If instructed (and given sufficient data) it performs all checks required by methods (2) or (3) based on Proposition 4. First it checks in the list provided in [25] to see that both pairs \((X, Y)\) and \((X', Y')\) are Schofield pairs associated to \(Z \in \text{mod-}kQ\) exceptional indecomposable such that \(\dim Z = \dim M^{(n)}\), then verifies conditions (c) and (d) from Proposition 4. It is ensured that the requirements of Lemma 6 are met and condition (d) is validated as mentioned in Remark 7. Finally, it ensures the existence of two short exact sequences of the form

\[
0 \rightarrow Y \xrightarrow{f} M^{(n)} \xrightarrow{g} X \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Y' \xrightarrow{f'} M^{(n)} \xrightarrow{g'} X' \rightarrow 0
\]

by reading the matrices of the morphisms \(f, f', g\) and \(g'\) and showing that every elementary operation and block-matrix arithmetic may be performed in a field independent way in order to fulfill every requirement of Lemma 5.

Every single operation performed by the proof assistant software is written to this output \LaTeX{} document. Everything (including the elementary operations and the details of computing the block matrix sums and products) is output a detailed step-by-step fashion as if written "by hand". In this way one does not have to believe in the correctness of the implementation, because the complete proof is "on paper" and every single step may be crosschecked and verified by a human mathematician.

4 About this document

The purpose of this document is to give an overview of the computational method used to prove the formulas given in the papers [13] and [12] and also to serve as an entry point on arXiv to the quite lengthy proofs given as separate files. The documents containing the complete proofs have considerable sizes, so they are given as attached ancillary documents:

- the file named e6_proof.pdf has the title “Proof of the tree module property for exceptional representations of the quiver \(\tilde{E}_6\)” and is the appendix to the paper [13];
- the file named d6_proof.pdf has the title “Proof of the tree module property for exceptional representations of the quiver \(\tilde{D}_6\)” and is the appendix to the paper [12].

The ancillary files contain the output generated by the proof assistant software. Being relatively self-contained materials, the introductory text from the current document is also present in the appendices.

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