The Geometry of N=1 and N=2 Real
Supersymmetric Non-Linear $\sigma$-Models
in the Atiyah-Ward Space-Time

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Abstract

We analyse the structure of N=1 and N=2 supersymmetric non-linear $\sigma$-models built up with a pair of real superfields defined in the superspace of Atiyah-Ward space-time. The geometry arising has new features such as the existence of a locally product structure (N=1 case) and a set of automorphisms of the tangent space that is isomorphic to the split-quaternionic algebra (N=2 case).

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1 Introduction

Bosonic non-linear (N=0) $\sigma$-models come in evidence in Physics as the structure of scalar fields that appear in a theory with a spontaneously broken symmetry. In all these models, the scalars define a mapping from the underlying space-time into a Riemannian manifold parametrised by coordinates that are the scalar fields themselves. Non-linear supersymmetric $\sigma$-models are then natural generalisations of the bosonic ones in which the scalar fields are now N=1 superfields, i.e., they provide a representation of N=1 supersymmetry. The first formulation of supersymmetric $\sigma$-models in superspace was given by Zumino \[1\]. He succeeded to write a supersymmetric invariant action for a N=1, D=(3+1) model. In that reference, it was shown that the scalar superfields span a Kähler manifold. Later on, Hull et all \[2\] also succeeded to write a N=2 supersymmetric $\sigma$-model in superspace. They showed how to derive a N=2 supersymmetric action over a Hyperkähler manifold making use of its quaternionic structures. The same constructions can also be performed for a space-time with signature D=(2+2), the so-called Atiyah-Ward space-time \[4\], \[5\]. The common aspect in all these constructions is the use of a pair $(\phi, \phi^*)$ of complex scalar superfields and their conjugates. The fact that the double covering of the isometry group of Atiyah-Ward space-time is $SL(2, \mathbb{R})$ bring us new features to the formulation of supersymmetric models \[6\], \[7\]. Here the supersymmetric chiral and antichiral sectors, for example, are no more related by complex conjugation. Hence, we can work consistently with a pair of real scalar superfields of different chiralities. We can also define Majorana-Weil spinors. We named such models as real supersymmetric $\sigma$-models. As a result, we obtain a geometry that is different from that one which appears in the formulation with a pair of complex superfields and their conjugates: in the N=1 case we obtain a manifold that presents some characteristics between a Kähler and a locally-product manifold (the latter have already been found by Gates et all in their formulation of twisted supersymmetric $\sigma$-models \[8\]), while in the N=2 case we obtain a manifold that admits a set of automorphisms of the tangent bundle that is parametrized by the split-quaternionic algebra. This paper is organised as follows: in Sections 2 and 3, we shall show how physical requirements determine the characteristics of the geometry of the N=1 and N=2 supersymmetric $\sigma$-models, respectively. In Section 4, we review some definitions concerning the split-quaternionic algebra, we establish definitions similar to

\[1\] In ref.\[3\] the same ideas were developed working with the component approach and in 2 dimensions.
the quaternion case and finally we present a mathematical formulation of our results.

## 2 The Real N=1 Supersymmetric σ-Model

Let us consider a set of 2n real superfields \( X^I \equiv (X^i, X^i) = (\Phi^i, \Xi^i) \), \( i = i + n; \ i = 1, \ldots, n \), respectively chiral and antichiral whose expansion in components read off as (we are using the same notations of [1]),

\[
\Phi^i = A^i + i\theta^i \bar{\psi}^i + i\theta^2 F^i + i\bar{\theta} \bar{\theta} A^i + \frac{1}{2} \theta^2 \bar{\theta} \bar{\theta} \psi^i - \frac{1}{4} \theta^2 \bar{\theta}^2 \bar{\partial}_\mu \partial^\mu A^i, \quad (1)
\]

\[
\Xi^i = B^i + i\theta \bar{\chi}^i + i\bar{\theta} \bar{G}^i + i\theta \bar{\theta} \bar{B}^i + \frac{1}{2} \bar{\theta} \bar{\theta} \bar{\partial} \bar{\chi}^i - \frac{1}{4} \bar{\theta} \bar{\theta} \bar{\partial} \bar{\partial} \bar{B}^i, \quad (2)
\]

where \( A^i \) and \( B^i \) are real scalar fields, \( \psi^i \) and \( \bar{\chi}^i \) are Majorana-Weyl spinors and \( F^i \) and \( G^i \) are real scalar auxiliary fields. A scalar superfield is chiral (\( \Phi \)) or antichiral (\( \Xi \)) if it satisfies respectively

\[
\bar{D}_a \Phi^i = 0, \quad \bar{D}_a \Phi^{*i} = 0, \quad \text{and} \quad D_\alpha \Xi^i = 0, \quad D_\alpha \Xi^{*i} = 0, \quad (3)
\]

with

\[
D_\alpha = \partial_\alpha - i\partial_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha}, \quad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - i\partial_{\alpha\dot{\alpha}} \theta^\alpha, \quad (4)
\]

\[
\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu, \quad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \quad [D_\alpha, \partial_\mu] = [\bar{D}_{\dot{\alpha}}, \partial_\mu] = 0.
\]

Following Zumino's work [1] we write the action for the non-linear σ-model as,

\[
S = 2 \int d^4x \ d^2\theta \ d^2\bar{\theta} \ K(\Phi^i, \Xi^i), \quad (5)
\]

\( K \) being a real scalar function of the 2n chiral/antichiral superfields. After eliminating the auxiliary fields \( F^i, G^i \), by using their equations of motion, we have the action expressed in component form as

\[
S = \int d^4x \ d^2\theta \ d^2\bar{\theta} \{2g_{ij} \partial_\mu A^i \partial^\mu B^j - \frac{i}{2} g_{ij} \bar{\psi}^i \sigma^\mu D_\mu \bar{\chi}^j - \frac{i}{2} g_{ij} \bar{\chi}^j \bar{\sigma}^\mu D_\mu \psi^i + \frac{1}{8} R_{imjn} \bar{\chi}^m \bar{\chi}^n \bar{\psi}^i \psi^j \}, \quad (6)
\]

with

\[
D_\mu \psi^i = \partial_\mu \psi^i + g^{ik} \partial_\mu g_{kj} \psi^j \partial_\mu A^l
\]

\[
D_\mu \bar{\chi}^i = \bar{\partial}_\mu \bar{\chi}^i + g^{ik} \bar{\partial}_\mu g_{kj} \bar{\chi}^j \partial_\mu B^l
\]

\[
R_{imjn} = \partial_\mu \partial_\nu g_{nj} - g^{kl} \partial_\mu g_{kj} \partial_\nu g_{nl}. \quad (7)
\]
The superfields $\Phi^i$ and $\Xi^i$ span a riemannian manifold $\mathcal{M}$ whose metric comes from the kinetic term of the scalar fields in (3), that is

$$g_{IJ} = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}, \quad g_{ij} = \frac{\partial^2 K}{\partial A^i \partial B^j}.$$  \hfill (8)

Adopting a complex coordinates system $(Z^i, \tilde{Z}^i) \equiv (A^i + iB^i, A^i - iB^i)$ for the manifold spanned by the scalar superfields we get a metric that is non-hermitian. In order to characterize the geometry which lies under our construction we have also to analyse how the action behaves under a transformation of coordinates. Let us consider “holomorphic” transformations, i.e.,

$$(\Phi^i, \Xi^i) \rightarrow (\Phi'^i, \Xi'^i) \equiv (\Phi'^i(\Phi) = e^{\lambda \kappa} \Phi^i, \Xi'^i(\Xi) = e^{\lambda \tau} \Xi^i),$$  \hfill (9)

with $\mathcal{K}_a = (\kappa_a(\Phi), \tau_a(\Xi)) \equiv (\kappa_a \partial_i, \tau_a \partial_{\tilde{i}})$ holomorphic killing vectors. This is equivalent to the existence of a locally-product structure on the manifold $\mathcal{M}$, i.e. the existence of a mapping on the tangent space of $\mathcal{M}$ satisfying,

$$I : T\mathcal{M} \rightarrow T\mathcal{M}$$

$$I^2 = 1.$$  \hfill (10)

The holomorphicity of the killing vectors is a consequence of imposing the vanishing of the Lie derivative of $I$ along $\mathcal{K}$: $\mathcal{L}_\mathcal{K} I = 0$. In the canonical coordinate system defined by $X^i$ we have the locally-product structure written as

$$I^I_{\ J} = \begin{pmatrix} I^i_{\ j} & 0 \\ 0 & I^i_{\ j} \end{pmatrix} = \begin{pmatrix} \delta^i_{\ j} & 0 \\ 0 & -\delta^i_{\ j} \end{pmatrix}.$$  \hfill (11)

The metric (8) is antidiagonal because of the relation, \footnote{Equivalently the $\theta, \tilde{\theta}$ components, $A^i, B^i$, span the same manifold and we can use indistinctly $(\Phi^i, \Xi^i)$ or $(A^i, B^i)$ to denote local coordinates on $\mathcal{M}$.}

$$I^M_N I^I_{\ J} g_{IJ} = -g_{MN}.$$  \hfill (12)

This allows us to define a sympletic 2-form in the same way as for a Kählerian manifold, $w = (w_{IJ}) \equiv I g$. Since $w$ is closed, the metric is derived by a scalar function $K(\Phi, \Xi)$ according to (8). Therefore, we have the following assertion:

\footnote{The fact of an antidiagonal metric forbids the manifold of being a locally-product space. In others term: in a locally-product space we have $I^M_N I^I_{\ J} g_{IJ} = g_{MN}$ instead of (12).}

3
The $N=1$ manifold, i.e., the target manifold associated to the $N=1$ real supersymmetric $\sigma$-model is a $2n$-dimensional Riemannian manifold that admits a locally-product structure $I$, a metric $g$ and a 2-form $w$ such that:

(i) $g(IU,IV) = -g(U,V)$ (the metric is “anti-hermitian”)
(ii) $w(U,V) = g(IU,V)$ is closed.

It should be observed that all the geometric content of the manifold $M$ is encoded in the following assumptions:

(i) we have an action given by (3)
(ii) the coordinates transform holomorphically.

The manifold $M$ obviously shares common properties of a Khäler and a locally-product manifold. Like a Khäler manifold, $M$ also admits a metric that is hybrid [9] and there is also a symplectic 2-form $w$ that fix the form of the metric as derivatives of a scalar potential $K$. The similarity with a locally-product space comes from the existence in both of them of a locally-product structure.

The Levi-Civitta connexion on $M$ assumes the same form as in the Khälerian case, $\Gamma^H_{ij} \equiv (\Gamma^h_{ij}, \Gamma^\hat{h}_{i\hat{j}})$ where, $\Gamma^h_{ij} = g^{hr} \partial_i g_{jr}$, $\Gamma^\hat{h}_{i\hat{j}} = g^{\hat{h}r} \partial_i g_{\hat{j}r}$. The canonical locally-product structure has zero covariant derivative $\nabla I = 0$ and it is also integrable [9].

3 The Real $N=2$ Supersymmetric $\sigma$-Model

The $N=2$ supersymmetric transformation of the model can be realized explicitly in a superspace $N=1$ if we write it as follows [2], [3], [10], [11]:

$$\delta \Phi^i = \bar{D}^2(\epsilon \Omega^i), \quad \delta \Xi^i = D^2(\zeta \mathcal{V}^i),$$

(13)

where $\Omega^i = \Omega^i(\Phi, \Xi)$ and $\mathcal{V}^i = \mathcal{V}^i(\Phi, \Xi)$ are considered as generic functions of superspace for a moment but later on they will be related to the split quaternionic structures. The parameters $\epsilon$ and $\zeta$ are real superfields. The requirement of being a supersymmetry transformation implies that

$$\delta_1 \delta_2 - \delta_2 \delta_1 \approx \partial$$

(14)

and this gives us the relations

$$\Omega^i_{,jk} \Omega^j_{,r} - \Omega^i_{,jr} \Omega^j_{,k} = 0, \quad \mathcal{V}^i_{,jr} \mathcal{V}^j_{,k} - \mathcal{V}^i_{,jk} \mathcal{V}^j_{,r} = 0$$

(15)

$$\Omega^i_{,ij} \mathcal{V}^j_{,k} = \delta^i_k, \quad \mathcal{V}^i_{,ij} \Omega^j_{,k} = \delta^i_k$$

(16)
The norm of \( \hat{\lambda} \) is given by Eq. (18).

\[
\hat{D}^2 \Omega^i = 0, \quad D^2 \psi = 0
\]

(17)

\[
\hat{D}^2 \epsilon = 0, \quad D^2 \epsilon = 0, \quad \partial^\mu \epsilon = 0
\]

(18)

\[
D^2 \zeta = 0, \quad \hat{D}_\alpha \zeta = 0, \quad \partial^\mu \zeta = 0.
\]

(19)

Eqs. (18, 19) determine the parameters \( \epsilon \) and \( \lambda \) as respectively spacetime constant antichiral/chiral real superfields.

The invariance of the action (17) under the N=2 supersymmetry transformations (13) implies:

\[
K_{\hat{ij}} \Omega^i \hat{\alpha} + K_{\hat{ij}} \Omega^i \hat{\beta} = 0, \quad K_{\hat{ij}} \psi^i \hat{\alpha} + K_{\hat{ij}} \psi^i \hat{\beta} = 0
\]

(20)

\[
K_{\hat{ij}} \Omega^i + K_{\hat{ij}} \psi^i = 0, \quad K_{\hat{ij}} \psi^i \hat{\nu} + K_{\hat{ij}} \psi^i \hat{\nu} = 0
\]

(21)

\[
K_{\hat{ij}} \Omega^i + K_{\hat{ij}} \psi^i = 0, \quad K_{\hat{ij}} \psi^i \hat{\nu} + K_{\hat{ij}} \psi^i \hat{\nu} = 0.
\]

(22)

These set of relations have a geometrical interpretation that will be made clear after the discussion on the next section.

4 Split-Quaternionic Analysis

4.1 Basic Properties of Split Quaternions

We will present here some results concerning the split-quaternionic algebra. They follow essentially the same development of [12][13]. Let \( H' \) be the algebra over \( R \) generated by \([\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3]\) with \( \hat{e}_0 \) being the identity and the others elements satisfying the relations

\[
\hat{e}_1 \hat{e}_1 = \hat{e}_0 \quad \hat{e}_1 \hat{e}_2 = \hat{e}_3 \quad \hat{e}_1 \hat{e}_3 = \hat{e}_2
\]

\[
\hat{e}_2 \hat{e}_1 = -\hat{e}_3 \quad \hat{e}_2 \hat{e}_2 = \hat{e}_0 \quad \hat{e}_2 \hat{e}_3 = -\hat{e}_1
\]

\[
\hat{e}_3 \hat{e}_1 = -\hat{e}_2 \quad \hat{e}_3 \hat{e}_2 = \hat{e}_1 \quad \hat{e}_3 \hat{e}_3 = -\hat{e}_0.
\]

A generic element of \( H' \) is then written as \( \hat{q} = (q^0, q^1, q^2, q^3) \equiv q^0 \hat{e}_0 + q^1 \hat{e}_1 + q^2 \hat{e}_2 + q^3 \hat{e}_3 \), \( q^0, q^1, q^2, q^3 \in R \). Addition and product of elements in \( H' \) are given naturally as \( \hat{q} + \hat{p} = (q^0 + p^0, ..., q^3 + p^3) \) while the product is defined by (23). The multiplication by a real \( \lambda \) is given by \( \lambda \hat{q} = \hat{q} \lambda = (\lambda q^0, ..., \lambda q^3) \). Complex conjugation on \( H' \) is defined in the following way: \( \hat{q} = (q^0, q^1, q^2, q^3) \rightarrow \hat{q}^* = (q^0, -q^1, -q^2, -q^3) \) and it satisfies, \( (\lambda \hat{q} + \zeta \hat{p})^* = \lambda \hat{q}^* + \zeta \hat{p}^* \), \( (\hat{q} \hat{p})^* = \hat{q}^* \hat{p}^* \), \( (\hat{q}^*)* = \hat{q} \). In particular, \( \hat{q} \hat{q}^* = \hat{q}^* \hat{q} = (q^0^2 - q^1^2 - q^2^2 + q^3^2) \hat{e}_0 \).

The norm of \( \hat{q} \) is the real number \( |\hat{q}| \equiv q^0^2 - q^1^2 - q^2^2 + q^3^2 \), that can be zero even if \( \hat{q} \neq 0 \), so that \( H' \) is not a division algebra.
Consider now $H^n = H' \times \ldots \times H'$ as the set of elements $Q = (\hat{q}^{1}, \ldots, \hat{q}^{n}), \hat{q}^i \in H', i = 1, \ldots, n$. We endow $H^n$ with a structure of right $H'$-module by defining the operations $\mathcal{P} + Q = (\hat{p}^{1} + \hat{q}^{1}, \ldots, \hat{p}^{n} + \hat{q}^{n}), \mathcal{P} \hat{q} = (\hat{q}^{\hat{p}^{1}} \hat{p}^{1}, \ldots, \hat{q}^{\hat{p}^{n}} \hat{p}^{n}), \forall \mathcal{P}, Q \in H^n, \forall \hat{q} \in H'$. From these definitions we have the following properties: $(\mathcal{P} + Q)\hat{q} = \mathcal{P}\hat{q} + Q\hat{q}$, $Q(\hat{p} + \hat{q}) = Q\hat{p} + Q\hat{q}$. $H^n$ is generated by the elements $\{E^1, \ldots, E^n\}$, $E_i = (\delta_{ij} \hat{e}_0), j = 1, \ldots, n$. Then we have $Q = \sum E_i \hat{q}^i$.

The symplectic product on $H'$ is a bilinear map $<, >: H' \times H' \rightarrow H'$, defined by $(\mathcal{P}, Q) \rightarrow <\mathcal{P}, Q > = \sum \hat{p}^{ij} \hat{q}^{*ij}$, and satisfies, $<\mathcal{P}\lambda, Q > = <\mathcal{P}, Q\lambda^* >$.

An endomorphism in $H^n$ is a mapping $\sigma : H^n \rightarrow H^n$, such that $\sigma(\mathcal{P}_1 + \mathcal{P}_2) = \sigma(\mathcal{P}_1) + \sigma(\mathcal{P}_2), \sigma(\mathcal{P}\hat{q}) = \sigma(\mathcal{P})\hat{q}$ and, in particular, a linear endomorphism is completely determined when it is given its action on the basis $\{E_i\}$, $\sigma(E_i) \equiv E_j \sigma_{ij}$, therefore $\sigma\mathcal{P} = \sum E_i \sigma_{ij} \hat{p}^j$. The association $\sigma \leftrightarrow \sigma_{ij}$ is a bijection and it allows us to represent the action of a linear endomorphism in $H^n$ by means of the matrices $\sigma \equiv (\sigma_{ij}) \in \mathcal{M}_{n \times n}(H')$ with coefficients in $H'$.

Every linear endomorphism of $H^n$ preserving the symplectic form $<, >$ is said to be a symplectic transformation. The set of such transformations defines the symplectic group. It is convenient to deal with the symplectic product as a bilinear in $C^{2n}$, since this will permit us to characterize the symplectic group in terms of a subgroup of $GL(2n, C)$, the so-called linear split-symplectic group.

### 4.2 The Linear Split-Symplectic Group

In order to define the linear split-symplectic group we remember that $\tilde{H} = [\hat{e}_0, \hat{e}_3]$ is a subalgebra of $H$ that has inverse. $\tilde{H}$ is also isomorphic to $C$ by the map: $\hat{q} = q^0 \hat{e}_0 + q^3 \hat{e}_3 \leftrightarrow z = q^0 + iq^3$. Then, we can define the action of $C$ on $H'$ as $(\hat{q}, z) \rightarrow \hat{q} z := (q^0 x - q^3 y) \hat{e}_0 + (q^1 x + q^2 y) \hat{e}_1 + (q^3 x + q^0 y) \hat{e}_2 + (q^2 y - q^1 x) \hat{e}_3$, $(z = x + iy)$. We can also write $\hat{q}_k \equiv \hat{e}_0 z_k^* - \hat{e}_1 z_{k+n},$ with $z_k = q^0_k \hat{e}_0 - q^3_k \hat{e}_3, z_{k+n} = -q^1_k \hat{e}_0 - q^2_k \hat{e}_3$, that associates $Q = (\hat{q}_k) \in H^n$ with $\hat{Q} = Z = (z_k) \equiv (z_k, z_{k+n}) \in C^{2n}$. Let us consider then $Q = (\hat{q}_k), \mathcal{P} = (\hat{p}_k) \in H^n$, to which corresponds $\hat{Q} = Z = (z_k) \equiv (z_k, z_{k+n}), \hat{P} = (w_k) \equiv (w_k, w_{k+n}) \in C^{2n}$. Given the symplectic transformation $\sigma = (\sigma_{ij})$ we have associated a transformation $\tilde{\sigma}$ of $GL(2n, C)$ in $C^{2n}, \hat{Q} \rightarrow \tilde{\sigma} \hat{Q}$. Since $\sigma$ is symplectic we have that $<\sigma Q, \sigma P >= <Q, P >.$
Let us define now 2-forms on $H^n$ by writing

$$\omega_i(\mathcal{P}, \mathcal{Q}) \equiv \frac{1}{2}(\langle \mathcal{P} \hat{e}_i, \mathcal{Q} \rangle + \langle \mathcal{Q}, \mathcal{P} \hat{e}_i \rangle) =$$

$$= \frac{1}{2} \{ \bar{e}_0 \sum_{k,i=1}^{2n} (z_{k}^* (1)^{i+1} (\mathcal{I}_k^1 \mathcal{I}_k^2) w_i - w_k^* (1)^{i+1} (\mathcal{I}_k^1 \mathcal{I}_k^2) z_i)$$

$$+ \bar{e}_1 \sum_{k,i=1}^{2n} (z_{k}^* (1)^{i+1} (\mathcal{I}_k^1 \mathcal{I}_k^2) w_i - w_k^* (1)^{i+1} (\mathcal{I}_k^1 \mathcal{I}_k^2) z_i) \}.$$  

They have the properties,

$$\omega_1(\mathcal{P}, \mathcal{Q}) = -\omega_1(\mathcal{P} \hat{e}_1, \mathcal{Q} \hat{e}_1) = \omega_1(\mathcal{P} \hat{e}_2, \mathcal{Q} \hat{e}_2) = -\omega_1(\mathcal{P} \hat{e}_3, \mathcal{Q} \hat{e}_3)$$  

$$\omega_2(\mathcal{P}, \mathcal{Q}) = \omega_2(\mathcal{P} \hat{e}_1, \mathcal{Q} \hat{e}_1) = -\omega_2(\mathcal{P} \hat{e}_2, \mathcal{Q} \hat{e}_2) = -\omega_2(\mathcal{P} \hat{e}_3, \mathcal{Q} \hat{e}_3)$$  

$$\omega_3(\mathcal{P}, \mathcal{Q}) = \omega_3(\mathcal{P} \hat{e}_1, \mathcal{Q} \hat{e}_1) = \omega_3(\mathcal{P} \hat{e}_2, \mathcal{Q} \hat{e}_2) = \omega_3(\mathcal{P} \hat{e}_3, \mathcal{Q} \hat{e}_3).$$

The group $sp(1)$ corresponds to the set of unit-split-quaternions, i.e., $\{ \hat{\lambda} \in H; |\hat{\lambda}| = 1 \}$ and it acts on $\omega_i$ as $\hat{\lambda} \omega_i \equiv \omega_i(\hat{\lambda} \mathcal{P}, \hat{\lambda} \mathcal{Q}).$ It follows then,

$$\omega_1(\hat{\mathcal{P}} \hat{\lambda}, \mathcal{Q} \hat{\lambda}) = (\lambda^0 \mathcal{P} - \lambda^1 \mathcal{Q} + \lambda^2 \mathcal{Q} + \lambda^3 \mathcal{Q}) - 2(\lambda^0 \mathcal{P} - \lambda^1 \mathcal{Q} + \lambda^2 \mathcal{Q} + \lambda^3 \mathcal{Q})$$

$$- 2(\lambda^0 \mathcal{Q} + \lambda^1 \mathcal{Q} + \lambda^2 \mathcal{Q} + \lambda^3 \mathcal{Q}) \omega_2(\mathcal{P}, \mathcal{Q}) \omega_3(\mathcal{P}, \mathcal{Q})$$

$$= (\lambda^0 \mathcal{P} - \lambda^1 \mathcal{Q} + \lambda^2 \mathcal{Q} + \lambda^3 \mathcal{Q}) + (\lambda^0 \mathcal{P} - \lambda^1 \mathcal{Q} + \lambda^2 \mathcal{Q} + \lambda^3 \mathcal{Q}) \omega_2(\mathcal{P}, \mathcal{Q}) +$$

$$+ (\lambda^0 \mathcal{P} - \lambda^1 \mathcal{Q} + \lambda^2 \mathcal{Q} + \lambda^3 \mathcal{Q}) \omega_3(\mathcal{P}, \mathcal{Q}).$$

4.3 The Fundamental 4-Form

Let us define now 2-forms on $4.3$ The Fundamental 4-Form

$$\omega$$ and it acts on

$$C$$ represent the split quaternionic algebra in $C^{2n}.$

In particular one has, $\mathcal{P} = \mathcal{P} \hat{\lambda} \rightarrow \mathcal{P} = (\lambda^0 \mathcal{I} + \lambda^1 (-1)^{i+1} \mathcal{I}_i) \hat{\mathcal{P}}$ where

$$(\mathcal{I}_i) = \left[ \begin{array}{ccc} 0 & -1 & \lambda \\ -1 & 0 & -i \\ \lambda & i & 0 \end{array} \right]$$

represent the split quaternionic algebra in $C^{2n}.$
The action of $\sigma \in sp(n)$ on $\omega_i$ is defined by $\omega_i \rightarrow \sigma \omega_i : \sigma \omega_i(P, Q) \equiv \omega_i(\sigma P, \sigma Q)$ and since the symplectic product is invariant by the action of $sp(n)$ we have the forms $\omega_i$ also invariant by $sp(n)$. We can also define an action of $sp(n)sp(1)$ on $\omega_i$ as $(\sigma, \hat{\lambda})\omega_i(P, Q) = \omega_i(\sigma P \hat{\lambda}, \sigma Q \hat{\lambda})$, where on the right hand side we are supposed to do first the multiplication by $sp(1)$ and latter by $sp(n)$.

Finally, we define in $H^n$ a 4-form $\Lambda \equiv \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 - \omega_3 \wedge \omega_3$. The action of $sp(n)sp(1)$ is defined by the corresponding action of $sp(n)sp(1)$ on each $\omega_i$, $(\sigma, \hat{\lambda})\Lambda \equiv (\sigma, \hat{\lambda})\omega_1 \wedge (\sigma, \hat{\lambda})\omega_1 + (\sigma, \hat{\lambda})\omega_2 \wedge (\sigma, \hat{\lambda})\omega_2 - (\sigma, \hat{\lambda})\omega_3 \wedge (\sigma, \hat{\lambda})\omega_3$, and from eqs. (31,32,33) we have that $\Lambda$ is invariant by $sp(n)sp(1)$.

4.4 The N=2 Manifold

We follow here the same definitions as was given in [14], but we adapt it to the split-quaternionic case. Let $M$ be a smooth 4n-dimensional manifold ($n \geq 1$) and $TM$ its tangent bundle. Consider $G$ a 3-dimensional subbundle of $Hom(TM, TM)$ that has fiber $G_x$ generated by the automorphisms $\{I_1, I_2, I_3\}$ satisfying the split-quaternionic algebra. The bundle $G$ is called an almost split-quaternion structure in $M$ and $(M, G)$ is an almost split-quaternion manifold.

$M$ admits a metric $g$ such that $g(sV, V') + g(V, sV') = 0$ for all cross-section $s \in \Gamma(G)$ and any vector fields $V, V' \in TM$. This means $I_1$ and $I_2$ are almost “anti-hermitian” relative to $g$ while $I_3$ is almost hermitian. We call $(M, g)$ an almost split-quaternion metric structure and $(M, g, G)$ an almost split-quaternion metric manifold. The almost split-quaternion structure $G$ is said to be integrable if, given any neighborhood $U$ in $M$, there exists a system of local coordinates $X = (x^k, \hat{x}^k)$ in which the split-quaternionic structures are written as $I$

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (34)$$

An almost split-quaternion manifold which have integrable $G$ is said to be a split-quaternion manifold. On $M$ we also define the 2-forms $w_i(V, V') = g(I_i V, V')$ and the 4-form $\Lambda = w_1 \wedge w_1 + w_2 \wedge w_2 - w_3 \wedge w_3$.

\[5\] In [13] is discussed the conditions for integrability of split-quaternionic structures. They found that a necessary and sufficient condition of integrability is that at least two of the Niejenhuis tensors $N(I_k, I_k)$ and the curvature $R$ of the affine connection vanish.
The analogue definition of a quaternion Kählerian manifold also exists in the split quaternionic case. It is obtained naturally if we impose the condition $\nabla s \in \Gamma(G)$, $\forall s \in \Gamma(G)$ and $\forall V \in TM$ with $\nabla$ the Riemannian connection on $M$. This is equivalent to the equations:

$$
\begin{align*}
\nabla_V I_1 &= r_3(V)I_2 + r_2(V)I_3 \\
\nabla_V I_2 &= -r_3(V)I_1 - r_1(V)I_3 \\
\nabla_V I_3 &= r_2(V)I_1 - r_1(V)I_2,
\end{align*}
$$

(35)

with $r_i$ being 1-forms. This set of equations are also equivalent to the condition $\nabla_V \Lambda = 0$.

We define now the N=2 manifold, i.e., the target manifold of the N=2 real $\sigma$-model. It constitutes the extension of hyperKähler manifold to the split-quaternionic case. Let $(\mathcal{M}, g, \mathcal{G})$ be a split-quaternion metric manifold. It will define an N=2 manifold iff, $\forall x \in \mathcal{M}$, $\mathcal{G}_x$ satisfies $\nabla_V I_i = 0$, $i = 1, 2, 3$.

Now, it is straightforward to show that this is indeed the manifold satisfying the N=2 supersymmetry constraints (15-22). Indeed, from the superfields $\Omega$, $V$ introduced in (13) we identify

$$I_2 = \begin{pmatrix} 0 & \Omega^i_j \\ \psi^j_i & 0 \end{pmatrix},$$

(36)

$I_1$ is given by (11) and $I_3 \equiv I_1 I_2$. Eq.(13) is associated to the integrability condition and corresponds to the requirement of $N(I_2, I_2) = 0$. Eq.(16) comes from the fact that $\mathcal{G}$ is a split-quaternion structure and so that $I_2^2 = 1$. Eqs.(20,21) corresponds to $\nabla_V I_2 = 0$ (with the Riemannian connection restricted to the Levi-Civitta) and finally eq.(22) is a consequence of $g(sV, V') + g(V, sV') = 0$, i.e. of $I_2$ be anti-hermitian relative to $g$.

5 Concluding Remarks

The geometric content of the real models obtained here presents new features such as a locally-product structure instead of a complex structure in the N=1-model and split-quaternionic structures replacing the quaternionic ones in the N=2-extension. The emergence of this geometric structure is determined only by the physical requirements that there is an action which is supersymmetric invariant and that the transformations of the scalar superfields being restricted to be holomorphic. In these real models, the possible couplings with vector superfields no longer correspond to simply gauging the isometry group since now the full set of locally-product structure does not leave the metric in-
variant. It may be even possible that further restrictions can arise in order to achieve a
gauge-invariant model. Also, the characterisation of those manifolds in terms of holon-
omy groups (see [16]) can also be developed and compared with the definitions we have
gotten from a purely tensorial analysis, the starting point to this being the construction
of the fundamental 4-form Λ. Finally, the analysis of a similar process of generating new
hyperKähler manifolds using the quotient process of [17] would deserve some investigation.

Acknowledgements

M.C. thanks Profs. T.Kori, K.Ozawa, and T.Suzuki and Y.Homma from the mathematics
department of Waseda University for many discussions. MC is also grateful to the japanese
monbusho for his schoolarship and to Mr. Hassu and all the staff of the Centro Cultural
e Informativo of the japanese Consulate at Rio de Janeiro. Finally, he expresses his
gratitude to Miss Ying Chen and Aruvaru Buraun for many helpful conversations.

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