Self-bumping of deformation spaces of hyperbolic 3-manifolds

K. Bromberg* and J. Holt†

September 4, 2000

Abstract

Let \( N \) be a hyperbolic 3-manifold and \( B \) a component of the interior of \( AH(\pi_1(N)) \), the space of marked hyperbolic 3-manifolds homotopy equivalent to \( N \). We will give topological conditions on \( N \) sufficient to give \( \rho \in \overline{B} \) such that for every small neighborhood \( V \) of \( \rho \), \( V \cap B \) is disconnected. This implies that \( \overline{B} \) is not manifold with boundary.

1 Introduction

In this paper we study aspects of the topology of deformation spaces of Kleinian groups. The basic object of study is \( AH(\pi_1(N)) \), the space of isometry classes of marked, complete hyperbolic 3-manifolds homotopy equivalent to \( N \), where \( N \) is a compact, orientable, irreducible, atoroidal 3-manifold with boundary. The study of the global topology of \( AH(\pi_1(N)) \) was begun by Anderson, Canary and McCullough in [8] for the case in which \( N \) has incompressible boundary. They described necessary and sufficient criteria for two components of the interior of \( AH(\pi_1(N)) \) to "bump"; that is, to have intersecting closures. We address the question of when a component of the interior "self-bumps"; that is, if \( B \) denotes such a component, then when is there an element \( \rho \) in the closure of \( B \) such that for any sufficiently small neighborhood \( V \) of \( \rho \) in \( AH(\pi_1(N)) \) the set \( V \cap B \) is disconnected? In this paper we will establish the following result:

* Partially supported by a grant from the Rackham School of Graduate Studies, University of Michigan and by the Clay Mathematics Institute
† Partially supported by a National Science Foundation Postdoctoral Fellowship
Theorem 4.5. Let $N$ be a compact, orientable, atoroidal, irreducible 3-manifold with boundary. Suppose that $N$ contains an essential, boundary incompressible annulus whose core curve is not homotopic into a torus boundary component of $\partial N$. Let $B$ be a component of the interior of $AH(\pi_1(N))$. Then there is a representation $\rho \in B$ such that for any sufficiently small neighborhood $V$ of $\rho$ in $AH(\pi_1(N))$ the set $V \cap B$ is disconnected.

Note that this result applies even when $N$ has compressible boundary. In [11] McMullen, using projective structures and ideas of Anderson and Canary, proved Theorem 4.5 when $N$ is an oriented $I$-bundle over a surface. Our techniques avoid the use of projective structures, and furthermore, even in the $I$-bundle case we will find bumping representations that are not detected with McMullen’s methods. In a sequel, we will use the techniques developed here to study the topology of the space of projective structures with discrete holonomy.

We sketch the proof of Theorem 4.5 in the case where $N = S \times [0, 1]$ is an $I$-bundle over a closed surface of genus $\geq 2$. In this case the interior of $AH(\pi_1(N))$ consists of a single component of quasifuchsian structures on $M = int N$, which is usually denoted $QF(S)$.

To construct the representation where bumping occurs we start with a hyperbolic structure on $M$ with a curve removed. That is choose a simple closed curve $c$ on $S$ and let $\hat{M} = M - (c \times \{1/2\})$. Then give $\hat{M}$ a geometrically finite hyperbolic structure. Now, $\pi_1(\hat{M})$ has many conjugacy classes of subgroups isomorphic to $\pi_1(S)$, for example $S \times \{1/4\}$ and $S \times \{3/4\}$ each define such a subgroup. However, to find our bumping representation we choose a non-standard subgroup of $\pi_1(\hat{M})$ by wrapping $S$ around the removed curve (see Figure 1). Then the hyperbolic structure on $\hat{M}$ defines a representation of $\pi_1(\hat{M})$ and our choice of subgroup defines a representation $\rho_\infty$ of $\pi_1(S)$. The cover $M_\infty$ associated to this subgroup will be homeomorphic to $M$.

The next step is to construct an immersion $f : N \to \hat{M}_\infty$ in the homotopy class associated to $\rho_\infty$ and then use $f$ to pull back a hyperbolic structure $N_\infty$ on $N$. For each $\rho \in AH(\pi_1(N))$ there is a hyperbolic 3-manifold $M_\rho$ homeomorphic to $M$. Given a small neighborhood $V$ of $\rho_\infty$, for each $\rho \in V$ a general theorem allows us to construct a smoothly varying family of hyperbolic structures $N_\rho$ on the compact manifold $N$. Here $N_\rho$ has holonomy $\rho$ and $N_{\rho_\infty} = N_\infty$. Since $N_\rho$ and $M_\rho$ have the same holonomy there will be an isometric immersion $f_\rho$ of $N_\rho$ in $M_\rho$. If $\rho \in V \cap QF(S)$ then $c$ will have a geodesic representative $c_\rho$ in $M_\rho$ and there will be a canonical homeomorphism between $M_\rho - c_\rho$ and $\hat{M}$. Furthermore, geometric consid-
erations will show that the image of $f_\rho$ misses $c_\rho$ so we can view $f_\rho$ as a map to $\hat{M}$. In particular, we can compare the homotopy classes of the maps $f_\rho$ in $\hat{M}$.

The heart of the proof is that we can find representations $\rho_0$ and $\rho_1$ in $V \cap QF(S)$, such that $f_0$ is homotopic in $\hat{M}$ to the original immersion $f$ while $f_1$ is homotopic to an embedding. If $\rho_0$ and $\rho_1$ are in the same component of $V \cap QF(S)$ then our smoothly varying family of hyperbolic structures $N_\rho$ will define a homotopy between $f_0$ and $f_1$ in $\hat{M}$. This contradiction proves the theorem.

To find the representation $\rho_0$ we take a small deformation of $\hat{M}$ that fills in the torus boundary to give a manifold homeomorphic to $M$. To find $\rho_1$ we take a small deformation of $M_\infty$ that resolves the rank one cusp. In this case the homeomorphism type is preserved. Since $M_0$ is geometrically very close to $\hat{M}$, $f_0$ will have the same homotopy class as $f$. In $M_\infty$, $f$ lifts to an embedding and therefore $f_1$ will be an embedding in $M_1$.

It is worthwhile to compare this result with the bumping of distinct components examined in \cite{1} and \cite{2}. As mentioned above in \cite{3}, necessary and sufficient conditions are given for components to bump. We will not state them here, but at the very least we need a manifold with more topology than an $I$-bundle so that the interior of $AH(\pi_1(N))$ will have more than one component. The construction of the bumping representation is then very similar to the one above.
We first remove a suitably chosen simple closed curve $c$ from $M = \text{int } N$ to obtain a new manifold $\hat{M}$. We then find a cover $M_\infty$ of $\hat{M}$ that is homotopy equivalent, but in this case not homeomorphic to, $M$. A hyperbolic structure on $\hat{M}$ defines a hyperbolic structure on $M_\infty$. As above we make a small deformation $M_0$ of $\hat{M}$ that will be homeomorphic to $M$ while a small deformation $M_1$ of $M_\infty$ will be homeomorphic to $M_\infty$. Although $M_0$ and $M_1$ are not homeomorphic, their holonomy representations $\rho_0$ and $\rho_1$ will both be near the holonomy representation $\rho_\infty$ of $M_\infty$. The next, and last, step is the real difference between the two arguments. As the components of the interior of $AH(\pi_1(N))$ are parameterized by the (marked) oriented homeomorphism types of $N$, $\rho_0$ and $\rho_1$ must be in distinct components that bump at $\rho_\infty$.

**Acknowledgments.**

The authors would like to thank Jeff Brock and Dick Canary for interesting and helpful discussions.

## 2 Preliminaries

A *Kleinian group* is a discrete, torsion free subgroup of the orientation preserving isometries of hyperbolic 3-space, $\mathbb{H}^3$. In the upper-half-space model of $\mathbb{H}^3$ the orientation-preserving isometries are identified with the group $\text{PSL}_2(\mathbb{C})$, so that a Kleinian group can be considered a discrete, torsion free subgroup of $\text{PSL}_2(\mathbb{C})$.

Let $\Gamma$ be a Kleinian group and set $M$ to be the quotient manifold $\mathbb{H}^3/\Gamma$. The convex core of $M$ is the smallest convex submanifold of $M$ whose inclusion in $M$ is a homotopy equivalence. If the convex core has finite volume, and $\Gamma$ is finitely generated then $\Gamma$ is called *geometrically finite*. In addition, a geometrically finite Kleinian group is *minimally parabolic* if every maximal parabolic subgroup is of rank 2. Let $R(\pi_1(N)) = \text{Hom}(\pi_1(N), \text{PSL}_2(\mathbb{C}))/\text{PSL}_2(\mathbb{C})$ be the space of conjugacy classes of representations of $\pi_1(N)$ in $\text{PSL}_2(\mathbb{C})$ where $N$ is a compact, orientable, atoroidal 3-manifold. The subset $AH(\pi_1(N)) \subset R(\pi_1(N))$ consists of the discrete, faithful representations of $\pi_1(N)$, modulo conjugacy. It is a result of Jørgensen [1] that $AH(\pi_1(N))$ is a closed subset of $R(\pi_1(N))$. By work of Marden [10] and Sullivan [12] the interior of $AH(\pi_1(N))$ is $MP(\pi_1(N))$, the minimally parabolic representations.

A representation $\rho \in AH(\pi_1(N))$ determines an oriented hyperbolic manifold $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$ along with a homotopy equivalence, $f_\rho : N \to M_\rho$. While in general $MP(\pi_1(N))$ will have many components, in this paper
our interest is the topology of the closure of a single component \( B \). Note that \( AH(\pi_1(N)) \) is determined only by the homotopy type of \( N \). We can therefore assume that \( N \) is chosen such that if \( \rho \) is in \( B \) then there is a homeomorphism from \( M_\rho \) to the interior of \( N \) that is a homotopy inverse for \( f_\rho \). We can also orient \( N \) such that this homeomorphism is orientation preserving. Then \( B \) will be the unique component of \( MP(\pi_1(N)) \) satisfying these two properties.

We also need to work with hyperbolic structures on the compact manifold \( N \) that may not extend to complete hyperbolic structures on an open manifold containing \( N \). We let \( \mathcal{H}(N) \) be the space of hyperbolic metrics on \( N \). Given two hyperbolic metrics on \( N \) the identity map will be a biLipschitz map between the two metrics. Given a structure, \( N' \in \mathcal{H}(N) \), a neighborhood \( N'(\epsilon) \) of \( N' \) consists of those structures in \( \mathcal{H}(N) \) for which the identity map from \( N' \) is a \((1+\epsilon)\)-biLipschitz map. The \( N'(\epsilon) \) are a basis of neighborhoods for \( N' \).

Theorem 1.7.1 in [6] describes the local structure of a neighborhood of \( N' \). We will need the following simple consequence of this theorem:

**Theorem 2.1** [6] The holonomy map \( \mathcal{H}(N) \rightarrow R(\pi_1(N)) \) is locally onto. Furthermore, for any neighborhood \( V \) of \( N' \), there exists a neighborhood \( U \subset V \), such that if \( N_0 \) and \( N_1 \) are hyperbolic structures in \( U \) with holonomy \( \rho_0 \) and \( \rho_1 \), respectively, and \( \rho_t, 0 \leq t \leq 1 \), is a path in the image of \( U \) then there is a path \( N_t \in U \), where each \( N_t \) has holonomy \( \rho_t \).

Now assume that \( \partial N \) contains at least one torus, \( T \). Choose a meridian and longitude for this torus such that elements of \( \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z} \) are determined by a pair of integers. Let \( (p, q) \) be a pair of relatively prime integers. Let \( N(p, q) \) denote the result of performing \( (p, q) \)-Dehn filling on \( N \) along this torus; that is, there exists an embedding \( d_{p,q} : N \rightarrow N(p, q) \) such that \( \overline{N(p, q)} - d_{p,q}(N) \) is a solid torus bounded by \( d_{p,q}(T) \) and the image of the \((p, q)\) curve on \( T \) is trivial in \( N(p, q) \). Let \( \gamma \) denote the core curve of of the solid torus. If \( N \) and \( N(p, q) \) have complete hyperbolic structures, \( M \) and \( M(p, q) \), on their interiors then \( M(p, q) \) is a hyperbolic Dehn filling of \( M \) if \( M(p, q) - d_{p,q}(M) \) contains the geodesic representative of \( \gamma \). Note that a hyperbolic structure \( M(p, q) \) may not be a hyperbolic Dehn filling of \( M \) if \( \gamma \) is not isotopic to its geodesic representative. Also note that the holonomy representation \( \rho \) for \( M(p, q) \) induces a non-faithful, holonomy representation, \( \rho_{p,q} \), for \( N \) via pre-composition with \( (d_{p,q})_* \).

If \( N \) has \( k \) torus boundary components, we can Dehn fill each of them. Let relatively prime integers, \( (p_i, q_i) \), be the Dehn filling coefficients for the \( i \)-
th torus and let \((p, q) = (p_1, q_1; \ldots; p_k, q_k)\). Then \(N(p, q)\) is the \((p, q)\)-Dehn filling of \(N\).

The following theorem has an extensive history. The interested reader should also see [3], [13], [4], and [7].

**Theorem 2.2** The Hyperbolic Dehn Surgery Theorem ([5])

Let \(M\) be a compact 3-manifold with \(k\) torus boundary components and assume \(M\) has a minimally parabolic hyperbolic structure with holonomy \(\rho\). We then have the following:

1. Except for a finite number of pairs for each \(i = 1, \ldots, k\), for each collection of relatively prime pairs \((p, q)\) there exist a geometrically finite hyperbolic \((p, q)\)-Dehn filling \(M(p, q)\) of \(M\).

2. \(\rho_{p, q} \to \rho\) as \(|p, q| \to \infty\) (\(|p, q| = |p_1| + |q_1| + \cdots + |p_k| + |q_k|\)).

3. If \(X\) is the complement of a neighborhood of the cusps and \(|p, q| > n\) then \(d_{p, q}|X\) is \(K_n\)-biLipschitz with \(K_n \to 1\) as \(n \to \infty\).

### 3 Wraps and twists

Let

\[ X = [-1, 1] \times [-1, 1] \times S^1 \]

and

\[ \hat{X} = X - ([-\frac{1}{3}, \frac{1}{3}] \times [-\frac{1}{3}, \frac{1}{3}] \times S^1). \]

We begin be defining maps of the annulus,

\[ A = [-1, 1] \times S^1 \]

into \(\hat{X} \subset X\). First we define \(w : A \to \hat{X}\) by

\[ w(x, \theta) = \left(-\frac{1}{2} \sin(\pi x), \frac{1}{2} \cos(\pi x), \theta \right). \]

We next define a sequence of maps \(w_n : A \to \hat{X}\) for each \(n > 0\). For each \(t\) and \(t'\) with \(-1 \leq t < t' \leq 1\) we let \(h_{t, t'} : ([t, t'] \times S^1) \to A\) be a homeomorphism that satisfies the conditions, \(h_{t, t'}(t, \theta) = (-1, \theta)\) and \(h_{t, t'}(t', \theta) = (1, \theta)\). To define \(w_n\) we choose real numbers, \(t_0, \ldots, t_n\) with \(-\frac{1}{3} = t_0 < t_1 < \cdots < t_n = \frac{1}{3}\), and let

\[
\begin{align*}
    w_n(x, \theta) &= \begin{cases} 
    \left(\frac{3}{2}x + \frac{1}{2}, -\frac{1}{2}, \theta\right) & \text{if } -1 \leq x < -\frac{1}{3} \\
    w \circ h_{t_i, t_{i+1}} & \text{if } t_i \leq x < t_{i+1} \\
    \left(\frac{3}{2}x - \frac{1}{2}, -\frac{1}{2}, \theta\right) & \text{if } \frac{1}{3} \leq x \leq 1.
    \end{cases}
\end{align*}
\]

6
Figure 2: The image of $A$ under the map $w_1$ in a cross section of $\hat{X}$.

The map $w_n$ wraps the annulus $n$ times around the missing core of $\hat{X}$. For $n = 0$, we define $w_0$ by $w_0(x, \theta) = (x, -1/2, \theta)$.

Our next family of maps, $t_{n,m} : \hat{X} \rightarrow \hat{X}$, are homeomorphisms which Dehn twist $\hat{X}$. They are defined by the following formula:

$$t_{n,m} = \begin{cases} 
(x, y, \theta) & \text{if } -1 \leq x < -\frac{1}{3} \text{ or } \frac{1}{3} < x \leq 1 \\
(x, y, \theta + 3n\pi(x + \frac{1}{3})) & \text{if } -\frac{1}{3} \leq x \leq \frac{1}{3} \text{ and } y > \frac{1}{3} \\
(x, y, \theta + 3m\pi(x + \frac{1}{3})) & \text{if } -\frac{1}{3} \leq x \leq \frac{1}{3} \text{ and } y < -\frac{1}{3}.
\end{cases}$$

**Lemma 3.1** The maps $w_n$ and $t_{k(n+1),kn} \circ w_n$ are homotopic rel $\partial A$ for any positive integer $n$ and any integer $k$.

**Proof**

Let $\hat{X}_{\frac{1}{3}} = ([-\frac{1}{3}, \frac{1}{3}] \times [-1,1] \times S^1) \cap \hat{X}$ denote the middle-third of $\hat{X}$; it has two components, the upper half and the lower half. The image of $A$ under the map $w_n$ intersects $\hat{X}_{\frac{1}{3}}$, so that $w_n^{-1}(w_n(A) \cap \hat{X}_{\frac{1}{3}})$ consists of $2n + 1$ essential sub-annuli of $A$; $n$ of the annuli map into the upper half of the middle third, while $n + 1$ of the annuli map into the lower half. On the each of the $n + 1$ annuli mapping into the lower half, $t_{k(n+1),kn}$ is a $kn$-Dehn twist, while on the $n$ upper annuli $t_{k(n+1),kn}$ is a $-k(n + 1)$-Dehn twist. Therefore the total affect of $t_{k(n+1),kn}$ is a $kn(n + 1) - k(n + 1)n = 0$-Dehn twist and $w_n$ is homotopic to $w_n \circ t_{k(n+1),kn}$ rel $\partial A$ (see Figure 3).

[proof of Lemma 3.1]
Figure 3: By identifying the top and bottom of the squares on the left we obtain (two copies of) the annulus $A$. The preimage of the $w_1(A) \cap \hat{X}_{\frac{1}{2}}$ is the three dashed annuli. The effect of $t_{2,1}$ on $A$, is two dehn twists on the center annuli and a single dehn twist in the opposite direction on the two outside annuli. As we see from the picture in the lower left, the net effect on $A$ is a map that is homotopic to the identity.

We now relate the maps $t_{n,m}$ to the Dehn filling of $\hat{X}$. As our coordinates for Dehn filling we choose the meridian to be the unique homotopy class that is trivial in $X$ and the longitude to be the curve $\{\frac{1}{3}\} \times \{\frac{1}{3}\} \times S^1$. Recall the Dehn filling maps $d_{1,k}: \hat{X} \to \hat{X}(1,k)$.

Lemma 3.2 For each $t_{n,m}$ there exists a homeomorphism $h_{n,m}: \hat{X}(1,n-m) \to \hat{X}(1,0)$ such that $d_{1,0} \circ t_{n,m} = h_{n,m} \circ d_{1,n-m}$.

Proof

The map $t_{n,m}$ takes the $(1,n-m)$-curve to the $(1,0)$-curve so $d_{1,0} \circ t_{n,m}$ takes the $(1,n-m)$ to a trivial curve in $\hat{X}(1,0)$. On the image of $\hat{X}$ in $\hat{X}$, we define $h_{n,m}$ to satisfy the equation, $d_{1,0} \circ t_{n,m} = h_{n,m} \circ d_{1,n-m}$. Since the
image of the $(1, n - m)$ curve is trivial in $\tilde{X}(1, n - m)$, $h_{n,m}$ extends to a homeomorphism.

Let

$$\partial_0 X = [-1, 1] \times \{-1, 1\} \times S^1 \subset X$$

and

$$\partial_1 X = \{1\} \times [-1, 1] \times S^1 \subset X.$$ 

Also assume that $N$ is a compact manifold with boundary and that $N$ contains an essential, boundary incompressible annulus. Then there is a pairwise embedding of $(X, \partial_0 X)$ in $(N, \partial N)$ such that $\partial_1 X$ is an essential, boundary incompressible annulus. Identify $A$ with the lower half of $\partial_0 X$; that is, the annulus $[-1, 1] \times \{-1\} \times S^1$. Let $c = \{0\} \times \{0\} \times S^1$ be the core curve of $X$ and let $\tilde{M} = M - c$ where $M$ is the interior of $N$.

For each integer $n \geq 0$ we define an immersion $s_n : N \rightarrow M \subset N$ as follows. The map $s_n$ is homotopic to the identity map and a homeomorphism onto its image outside of $X$. We also require that $s_n(N) \cap c = \emptyset$ and that $s_n$ restricted to $A$ is homotopic to $w_n$ rel $\partial A$. This completely defines $s_n$ up to homotopy in $M$. We call any map that satisfies these properties a **shuffle immersion**.

![Diagram](image.png)

Figure 4: The map $s_1$ immerses $N$ in $M$ and is not homotopic to an embedding in $\tilde{M}$.

**Lemma 3.3** A shuffle immersion $s_n$ satisfies the following properties:
1. If \( n \neq m \) then \( s_n \) and \( s_m \) are not homotopic in \( \hat{M} \).

2. For each integer \( k \), there is an orientation preserving homeomorphism \( h_k : \hat{M}(1, k) \rightarrow M \) such that \( s_n \) and \( h_k \circ d_{1,k} \circ s_n \) are homotopic in \( \hat{M} \). Here, \( M = \hat{M}(1, 0) \).

3. The cover of \( \hat{M} \) associated to \( (s_n)_*(\pi_1(N)) \) is homeomorphic to \( M \) and \( s_n \) lifts to an embedding \( \hat{s}_n : N \rightarrow M \) which is homotopic to \( s_0 \) in \( \hat{M} \).

Proof

1. If \( n \neq m \), the maps \( (s_n)_*(\pi_1(N)) \) and \( (s_m)_*(\pi_1(N)) \) are non-conjugate subgroups of \( \pi_1(\hat{M}) \) and therefore the maps \( s_n \) and \( s_m \) are not homotopic.

2. On \( \hat{X}(1, k) \subset \hat{M}(1, k) \) we let \( h_k = h_k(n+1),kn \). Using Lemma 3.2, we see that \( h_k \) extends to a homeomorphism from \( \hat{M}(1, k) \) to \( M \). By Lemma 3.1, \( s_n \) and \( h_k \circ d_{1,k} \circ s_n \) are homotopic in \( \hat{M} \).

3. This is an easy exercise in 3-manifold topology which we leave to the reader.

proof of Lemma 3.3

4 Self-bumping

We now use the topology we developed in \( \S 3 \). With the same assumptions as in \( \S 3 \) we fix a shuffle immersion \( f = s_d \) with \( d > 0 \). Note that such a shuffle immersion exists if and only if \( N \) contains an essential, boundary incompressible annulus. However, for \( M \) and \( \hat{M} \) to support complete hyperbolic structures we need to make further topological restrictions. Namely, \( N \) must be irreducible and atoroidal and the simple closed curve \( c \) must be primitive and not homotopic to a torus boundary component of \( \partial N \). Then \( M \) and \( \hat{M} \) satisfies the conditions of Thurston’s hyperbolization theorem (see Lemma 2.5.10 in \( \S 3 \)) and we fix a minimally parabolic hyperbolic structure \( \hat{M}_\infty \) on \( \hat{M} \) with holonomy representation \( \hat{\rho}_\infty \). We also let \( N_\infty \) be the hyperbolic metric on \( N \) obtained as the pull-back by \( f \) of the metric \( \hat{M}_\infty \) on \( \hat{M} \).

We now set up a notational system that will hold for the remainder of the paper. For an index \( \alpha \), \( N_\alpha \) is a hyperbolic structure on \( N \) and \( \rho_\alpha \) will be the associated holonomy representation. Let \( M \) be the interior of
As we noted in the introduction, if \( \rho_\alpha \in AH(\pi_1(N)) \) then \( M_\alpha \) is a complete hyperbolic structure, marked by \( N \). As \( N_\alpha \) has the same holonomy as \( M_\alpha \) there will be an an isometric immersion, \( f_\alpha : N_\alpha \to M_\alpha \), with \( f_\alpha \) a homotopy equivalence. In other words, \( f_\alpha \) is the marking map. Let \( c_\alpha \) denote the geodesic representative of \( c \) in \( M_\alpha \).

**Lemma 4.1** Let \( V \) be a small neighborhood of \( \rho_\infty \). Then for each \( N_\alpha \) near \( N_\infty \) with \( \rho_\alpha \in V \cap MP(\pi_1(N)) \), \( f_\alpha(N_\alpha) \cap c_\alpha = \emptyset \).

**Proof**

By compactness there exists a \( K \) such that for any \( p \in N \) we can find a non-trivial simple closed curve \( \gamma_p \) through \( p \), and not homotopic to \( c \), with length \( < K \) in \( N_\infty \). We choose \( V \) small enough such that all structures in the neighborhood are 2-biLipschitz from \( N_\infty \). The Margulis lemma implies that there exists an \( \epsilon \) such that, for any complete hyperbolic 3-manifold, if a homotopically non-trivial simple closed curve intersects a homotopically distinct geodesic of length \( < \epsilon \) it has length \( > 3K \). Furthermore, since the length of curves is continuous on \( R(\pi_1(N)) \), we can further shrink \( V \) so that the curve, \( c_\alpha \), has length \( < \epsilon \) and therefore \( f_\alpha(\gamma_p) \), which has length \( < 2K \), does not intersect \( c_\alpha \); implying that \( p \notin c_\alpha \).

**proof of Lemma [4.1]**

Recall that \( B \) is a component of \( MP(\pi_1(N)) \) so that the marking map \( f_\alpha \) has a homotopy inverse which is an orientation preserving homeomorphism between \( M_\alpha \) and \( M \) if and only if \( \rho_\alpha \in B \).

**Lemma 4.2** For the shuffle immersion \( f \), there exists a sequence of hyperbolic structures \( N_k \) with holonomy representations \( \rho_k \), such that:

1. \( N_k \to N_\infty \).
2. \( \rho_k \to \rho_\infty \).
3. There exist homeomorphisms \( h_k : M_k \to M \) such that \( h_k \) is a homotopy inverse for \( f_k|_M \), \( h_k(c_k) = c \) and \( f \) and \( h_k \circ f_k \) are homotopic in \( \hat{M} \). In particular, \( \rho_k \in B \).

**Proof**

1. For large \( n \), let \( M_n = \hat{M}_\infty(1,n) \) be the manifolds obtained by performing hyperbolic Dehn surgery on \( \hat{M}_\infty \) as in Theorem [2.2]. Since
Proof of Lemma 4.2

The following lemma will be used to detect when two representations are not contained in the same component of $V \cap B$.

**Lemma 4.3** Let $U$ be a neighborhood of $N_\infty$ that satisfies the conclusion of Theorem 2.1 and Lemma 4.1, and let $V$ be the image of $U$ under the holonomy map. Let $N_0$ and $N_1$ be hyperbolic structures in $U$ with holonomy $\rho_0$ and $\rho_1$, both in $V \cap MP(\pi_1(N))$. Also assume that $h_i : M_i \to M$, $i = 0, 1$, are homeomorphisms that are homotopy inverses of $f_i|_M$ and $h_i(c_i) = c$. If $\rho_t$, $0 \leq t \leq 1$, is a path in $V \cap MP(\pi_1(M))$ then $h_0 \circ f_0$ and $h_1 \circ f_1$ are homotopic in $M$.

**Proof**

By Theorem 2.1 we have a path of structures $N_t$ in $V$ with holonomy $\rho_t$. The $\rho_t$ are in the same component of $MP(\pi_1(M))$ as $\rho_0$ and $\rho_1$ are in, so there are homeomorphisms, $h_t : M_t \to M$, that are homotopy inverses of $f_t|_M$. We can assume that the push-forward of the hyperbolic metrics on $M_t$ to $M$ is a continuously changing family of metrics on $M$. Furthermore, as all the $c_t$ are short geodesics, they will be simple. Hence $h_t(c_t)$ is an isotopy of $c$ in $M$. We can therefore modify the $h_t$ such that $h_t(c_t) = c$. Then $h_t \circ f_t$ will vary continuously in $t$. By Lemma 4.1, $f_t(N_t) \cap c_t = \emptyset$ so $h_t \circ f_t$ is a homotopy between $h_0 \circ f_0$ and $h_1 \circ f_1$ in $M$. 

Proof of Lemma 4.3
We next apply Lemma 4.3 to show that distinct shuffle immersion force $V \cap B$ to be disconnected.

**Lemma 4.4** Let $f, f' : N \to \hat{M} \subset M$, be distinct shuffle immersions. Assume that there exists minimally parabolic structures $\hat{M}_\infty$ and $\hat{M}'_\infty$ on $\hat{M}$ such that the pulled-back hyperbolic structures $N_\infty$ and $N'_\infty$ are isometric and hence define the same holonomy representation, $\rho_\infty$. Then for every small neighborhood $V$ of $\rho_\infty$, $V \cap B$ is disconnected.

**Proof**

Let $M_n, N_n, f_n, h_n,$ and $\rho_n$ and $M'_n, N'_n, f'_n, h'_n,$ and $\rho'_n$ be the hyperbolic structures, isometric immersions and holonomy representations given by Lemma 4.2 for $f$ and $f'$, respectively. Choose an open neighborhood $V$ of $\rho_\infty$ given by Lemma 4.1.

There exists integers $n$ and $m$ such that $\rho_n, \rho'_m \in V$. The intersection $V \cap B$ is an open subset of the manifold $B$ so the connected components of $V \cap B$ are path connected. If $\rho_n$ and $\rho'_m$ are in the same component of $V \cap B$ then Lemma 4.3 implies that $h_n \circ f_n$ and $h'_m \circ f'_m$ are homotopic in $\hat{M}$. On the other hand, by Lemma 4.2, $h_n \circ f_n$ and $h'_m \circ f'_m$ are homotopic in $\hat{M}$ to $f$ and $f'$, respectively. Since, $f$ and $f'$ aren’t homotopic in $\hat{M}$ we have a contradiction.

We now prove our main theorem.

**Theorem 4.5** Let $N$ be a compact, orientable, atoroidal, irreducible 3-manifold with boundary. Suppose that $N$ contains an essential, boundary incompressible annulus whose core curve is not homotopic into a torus boundary component of $\partial N$. Let $B$ be a component of the interior of $AH(\pi_1(N))$. Then there is a representation $\rho$ in $\overline{B}$ such that for any sufficiently small neighborhood $V$ of $\rho$ in $AH(\pi_1(N))$ the set $V \cap B$ is disconnected.

**Proof**

We recall our standing assumption that if $\rho \in B$ then the marking map $f_\rho : N \to M_\rho$ has a homotopy inverse that is a homeomorphism onto the interior of $N$. If we want to show self-bumping at a different component $B'$ we find a new manifold $N'$ homotopy equivalent to $N$ such that $N'$ and $B'$ have the above property. With the exceptions of $N$ being irreducible and atoroidal, all of the topological assumptions we have made depend only on the homotopy type of $N$. Since a hyperbolic manifold is automatically irreducible and atoroidal, $N'$ will also be atoroidal, irreducible and contain
an essential, boundary incompressible annulus. In particular if one component of $MP(\pi_1(N))$ self-bumps then every component of $MP(\pi_1(N))$ will self-bump.

By Lemma 3.3, there is a non-trivial shuffle immersion $f : N \rightarrow \hat{M} \subset M$ and $f$ lifts to an embedding $f'$ in the cover $M'$ associated to $f_\ast(\pi_1(N))$, with $M'$ homeomorphic to $\hat{M}$. Let $\hat{M}_\infty$ be a minimally parabolic structure on $\hat{M}$ which defines a hyperbolic structure $M'_\infty$ on $M' = M$. We use $f$ to pull back a hyperbolic structure $N_\infty$ and then $f'_\infty : N_\infty \rightarrow M'_\infty$ is an isometric immersion and $f'_\infty : N_\infty \rightarrow M'_\infty$ is an isometric embedding. The holonomy, $\rho_\infty(c)$, of $c$ will be parabolic so by an application of the second Klein-Maskit combination we can find another parabolic $\gamma$ such that the free product of $\rho_\infty(\pi_1(N))$ and $\gamma$ is a uniformization $\hat{M}'_\infty$ of $\hat{M}$ such that $M'_\infty$ covers $M'_\infty$ and $f'_\infty$ descends to an embedding. Therefore $f$ and $f'$ satisfy the conditions of Lemma 4.4 which implies the theorem.

**proof of Theorem 4.5**

**Corollary 4.6** \(\overline{B}\) is not a manifold.

**Proof**

If \(\overline{B}\) is a manifold then Theorem 4.5 implies that $\rho_\infty$ is in the interior of \(\overline{B}\), since it cannot be in the boundary. However, in [12], Sullivan proves that the interior of \(\overline{B}\) is $B$. Since $\rho_\infty$ is not in $B$, \(\overline{B}\) is not a manifold.

**proof of Corollary 4.6**

In Theorem 4.3 we characterized when the components of $MP(\pi_1(N))$ self-bump. To do so we constructed a representation where this self-bumping occurs. In our next theorem we describe a sufficient condition for a representation to be a point of self-bumping. To describe it we will assume some knowledge of Kleinian groups.

We now allow $N$ to contain more than one copy of $X$. In particular, assume that there are $m$ disjoint, pairwise embeddings of $(X, \partial X)$ in $(N, \partial N)$, labeled, $X_1, \ldots, X_m$. As before we assume that each $\partial_1 X_i$ is an essential, boundary incompressible annulus and that each core curve, $c_i$ is primitive and not homotopic to a boundary torus. We further assume that the $c_i$ are homotopically distinct. For each $i$, $1 \leq i \leq m$, choose an integer, $n_i \geq 0$. There is then a shuffle immersion, $s_{n_1, \ldots, n_m}$, that wraps $N$ around $c_i$, $n_i$ times.

Let $\hat{M} = M - C$. If $\hat{\rho}$ is a minimally parabolic, geometrically finite uniformization of $\hat{M}$ then the space of all minimally parabolic hyperbolic
structures on \( \hat{M} \), with the same marking, is \( QD(\hat{\rho}) \), the quasiconformal deformation space of \( \hat{\rho} \). The image of \( (s_{n_1, \ldots, n_m})_*(\pi_1(N)) \) in \( \pi_1(M) \) defines a Kleinian subgroup \( \Gamma = \hat{\rho}(\pi_1(\hat{M})) \) that uniformizes \( M \), and a representation \( \rho = \hat{\rho} \circ (s_{n_1, \ldots, n_m})_* \), with image \( \Gamma \). If \( \hat{\rho}' \) is another representation in \( QD(\hat{\rho}) \) then \( \hat{\rho}' \circ (s_{n_1, \ldots, n_m})_* \) is in \( QD(\rho) \), the quasiconformal deformation space of \( \rho \). Therefore \( (s_{n_1, \ldots, n_m})_* \) defines a map between \( QD(\hat{\rho}) \) and \( QD(\rho) \).

Our previous work shows the following:

**Theorem 4.7** All representations in \( QD(\rho) \) in the image of \( QD(\hat{\rho}) \) under \( (s_{n_1, \ldots, n_m})_* \) are points of self-bumping for \( B \) if \( n_i \neq 0 \) for some \( i \).

Note that \( \rho \) will not be minimally parabolic, for the \( c_i \) will all be parabolic in \( \Gamma = \rho(\pi_1(N)) \). Let \( c'_i = \{0\} \times \{1\} \times S^1 \subset \partial_0 X_i \). The quotient of the domain of discontinuity for \( \Gamma \) will be a conformal structure on \( \partial N - \bigcup c'_i \). As the pinched curves in \( \partial N \) are determined by the embeddings of the \( X_i \), if \( s_{n'_1, \ldots, n'_m} \) is another shuffle immersion then the image of \( (s_{n'_1, \ldots, n'_m})_* \) will be the same quasiconformal deformation space, \( QD(\rho) \). (While these maps have the same range, \( (s_{n_1, \ldots, n_m})_*(\hat{\Gamma}) \neq (s_{n'_1, \ldots, n'_m})_*(\hat{\Gamma}) \). On the other hand, each \( X_i \) has an involution which swaps the two components of \( \partial_0 X_i \). By performing this involution on some (possibly all) of the \( X_i \) we get a new family of shuffle immersions. The bumping representations associated to these shuffle immersions will then lie in a different quasi-conformal deformation space.

We also remark that even in the case where \( N \) is an \( I \)-bundle, Theorem 4.7 is stronger than McMullen’s result in [11]. In McMullen’s theorem, all the \( c'_i \) must lie in the same component of \( \partial N \). Here we have no such restriction.

We close with the following conjecture.

**Conjecture 4.8** A representation \( \rho \) is a point of self-bumping for \( B \) if and only if there is a non-empty collection of curves \( \mathcal{C} \) (as above) in \( M \), a shuffle immersion \( s \) with respect to \( \mathcal{C} \), and a uniformization \( \hat{\rho} \) of \( \hat{M} = M - \mathcal{C} \) so that \( \rho = \hat{\rho} \circ s_* \).

**References**

[1] J.W. Anderson and R. D. Canary. Algebraic limits of Kleinian groups which rearrange the pages of a book. *Invent. Math.*, 126:205–214, 1996.

[2] J.W. Anderson, R.D. Canary, and D. McCullough. On the topology of deformation spaces of Kleinian groups. to
be published in Annals of Math., preprint available from http://www.math.lsa.umich.edu/~canary/.

[3] R. Benedetti and C. Petronio. Lectures on hyperbolic geometry. Universitext. Springer-Verlag, 1992.

[4] F. Bonahon and J.P. Otal. Variétés hyperboliques à géodésiques arbitrairement courtes. Bull. L.M.S., 20:255–261, 1988.

[5] K. Bromberg. Hyperbolic Dehn surgery on geometrically infinite 3-manifolds. preprint, 2000.

[6] R. D. Canary, D. B. A Epstein, and P. Green. Notes on notes of Thurston. In D. B. A Epstein, editor, Analytical and geometric aspects of hyperbolic space, volume 111 of London Math. Soc. Lecture Note series, pages 3–92. Cambridge University Press, 1987.

[7] T.D. Comar. Hyperbolic Dehn surgery and convergence of Kleinian groups. PhD thesis, University of Michigan, 1996.

[8] J. Holt. The global topology of deformation spaces of Kleinian groups. PhD thesis, University of Michigan, 2000.

[9] T. Jørgensen. On discrete groups of Möbius transformations. Amer. J. Math., 98:739–749, 1976.

[10] A. Marden. The geometry of finitely generated Kleinian groups. Annals of Math., 99:383–462, 1974.

[11] C. McMullen. Complex earthquakes and Teichmüller theory. J. Amer. Math. Soc., 11(2):283–320, 1998.

[12] D.P. Sullivan. Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity of Kleinian groups. Acta Math., 155:243–260, 1985.

[13] W. Thurston. The geometry and topology of 3-manifolds. Princeton lecture notes, 1977.