Area Regge Calculus and Discontinuous Metrics

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Abstract

Taking the triangle areas as independent variables in the theory of Regge calculus can lead to ambiguities in the edge lengths, which can be interpreted as discontinuities in the metric. We construct solutions to area Regge calculus using a triangulated lattice and find that on a spacelike hypersurface no such discontinuity can arise. On a null hypersurface however, we can have such a situation and the resulting metric can be interpreted as a so-called refractive wave.

1 Introduction

The Ponzano-Regge [1] model of quantum gravity in three dimensions begins with a labelling of the 1-simplices in a simplicial complex with irreducible representations of $SU(2)$. Using these labellings a state sum is constructed, which is the discrete analogue of the partition function of three dimensional quantum gravity. The sum over labellings of the 1-simplices in the state sum corresponds to an integration over edge lengths in the path integral. The action appearing in the exponential of the path integral is the Regge action [2], defined in terms of these edge lengths.

In four dimensions the labelling of the 1-simplices does not seem to be sufficient [3]. For example in the Barrett-Crane model [4] we label the 2-simplices with balanced irreducible representations of $SU(2) \times SU(2)$. The resulting state sum then takes the form of the path integral for the four dimensional Regge action, yet now the areas of the triangles appear to be taking the role of the independent variables.

This is then our motivation to study Regge calculus with triangle areas, rather than edge lengths, as the independent variables. This idea was originally suggested by Rovelli [5] in connection with loop quantum gravity.

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In Section 2 we will introduce the theory of area Regge calculus and discuss some of its potential consequences such as discontinuous metrics. In Section 3 we will construct full solutions of area Regge calculus to investigate the possibility of discontinuities along spacelike and null hypersurfaces. We then, in Section 4, need to introduce the theory of tensor distributions which allows us, in Section 5, to draw some interesting comparisons between area Regge calculus and the theory of gravity.

## 2 Area Regge Calculus

In area Regge calculus we begin with the Regge action on a 4-dimensional simplicial complex \([2]\),

\[
I = \sum_{\Delta} A_{\Delta} \epsilon_{\Delta}.
\]  

(2.1)

However, instead of taking the lengths of the 1-simplices to be the independent variables we consider the areas of the 2-simplices, i.e. the \(A_{\Delta}\), to be independent. Variation of the action with respect to the \(A_{\Delta}\) results in the field equations \(\epsilon_{\Delta} = 0\), which states that the deficit angle at each triangle is zero. If this were standard Regge calculus, a zero deficit condition would imply the space were locally flat. However, the situation here is less straightforward, as explained in \([6]\), which we shall now summarize.

Consider a single 4-simplex, which contains 10 edges and 10 triangles. The edge lengths uniquely specify the geometry of the simplex and hence determine the areas. However the reverse is not true; two 4-simplices can be constructed having the same triangle areas, yet different edge lengths. This problem can be overcome if we consider our 4-simplices to be in some sense close to a regular 4-simplex, analogous to choosing the principal value of a multi-valued function such as \(\sin^{-1} x\).

Now suppose we have two 4-simplices meeting at a tetrahedron. We now have a total of 16 triangles and 14 edges, hence a problem of over-determinism. Also note the shared tetrahedron has 4 triangles and 6 edges, hence an under-determined geometry. We can thus envisage a situation where calculating the edge lengths of the tetrahedron from the areas of one 4-simplex results in a different answer to if they were calculated from the other 4-simplex. We could interpret such a situation as a discontinuity of the metric.

We shall proceed by investigating under what circumstances such discontinuities can arise in solutions to area Regge calculus.
3 Hypersurface Discontinuities

We have seen in Section 2 how discontinuities might arise in area Regge calculus; however this was only for two adjoining simplices. We would like to be able to construct a full solution to area Regge calculus containing such discontinuities.

The simplest situation would be to restrict the discontinuity to some hypersurface $\Sigma$, separating $M$ into $M^\pm$. We triangulate $M$ such that we also triangulate $\Sigma$. Taking the areas of the triangles as the independent variables, we have the field equations $\epsilon_\Delta = 0$, i.e. zero deficit angles.

To restrict the discontinuity to $\Sigma$ we demand the areas are chosen to give well defined edge lengths in $M^\pm$ respectively. Combined with the zero deficit condition we see that both $M^\pm$ are flat.

We simplify the situation further by assuming $\Sigma$ is embedded in $M^\pm$ with no extrinsic curvature; this takes care of the zero deficit condition by virtue of the fact that the interior dihedral angles at those triangles in $\Sigma$ will be $\pi$ in both $M^\pm$. We work with a triangulated lattice, each lattice site being identical to its neighbours except across $\Sigma$.

The procedure we will take is as follows.

- Begin with a flat 3-lattice in $\partial M^-$.
- Transform the coordinates of the lattice points (thus keeping the lattice flat) in such a way that the areas of the triangles are unchanged. This gives the lattice on $\partial M^+$.
- Extend the 3-lattices on $\partial M^\pm$ to 4-lattices on $M^\pm$ and since the triangle areas agree we can identify the lattices on $\partial M^\pm$ giving the full 4-dimensional solution.

We can look at the cases of $\Sigma$ spacelike and $\Sigma$ null separately.

3.1 $\Sigma$ Spacelike

We work in coordinates $(t, x, y, z)$ with $\Sigma$ located at $t = 0$. Our flat 3-lattice in $\partial M^-$ is given by coordinates in $(x, y, z)$. The metric in this space is $ds^2 = dx^2 + dy^2 + dz^2$. We label the edges of the triangulated 3-lattice from 1 to 7 as shown in Figure 3.1 consistent with the binary notation of $[7, 8]$. Since the hypersurface is flat the coordinates of edges 3, 5, 6 and 7 are determined by those of edges 1, 2 and 4.

Let $x_i$ be the vector along the $i^{th}$ edge, then a transformation of the lattice is determined by a transformation of $x_1$, $x_2$ and $x_4$. This induces a linear transformation on $\mathbb{R}^3$, which we write as a matrix $T \in GL(3)$.
Now since each lattice site is identical we only have 6 independent triangle areas\(^1\), these are \(A_{13}, A_{15}, A_{17}, A_{26}, A_{27}\) and \(A_{37}\), where \(A_{ij}\) is the area of the triangle which has \(i\) and \(j\) as edges.

Under the transformation \(x_i \rightarrow x'_i = Tx_i\) we require that these six areas remain invariant. We can decompose \(T\) in order to analyse which transformations will satisfy this condition.

We proceed by first noting we can always split \(T\) as \(T = OP\), where \(O \in O(3)\) and \(P\) is some positive definite symmetric matrix. Since \(P\) is symmetric we can further decompose it as \(P = QDQ^T\), where \(Q \in O(3)\) and \(D\) is diagonal with positive entries (since \(P\) was positive definite). Thus we have decomposed \(T\) as

\[
T = OQDQ^T. \tag{3.1}
\]

Since both \(O\) and \(Q\) are orthogonal they will not transform the triangle areas. Thus we only require that \(D\) does not transform the triangle areas. So, let \(D = \text{diag}(a, b, c)\) with \(a, b, c > 0\), and consider a triangle based at the origin, with the other two vertices at \((X_A, Y_A, Z_A)\) and \((X_B, Y_B, Z_B)\). This triangle has an area \(A\) given by

\[
A^2 = \frac{1}{4} \left( \left| \begin{array}{cc} Y_A & Z_A \\ Y_B & Z_B \end{array} \right|^2 + \left| \begin{array}{cc} Z_A & X_A \\ Z_B & X_B \end{array} \right|^2 + \left| \begin{array}{cc} X_A & Y_A \\ X_B & Y_B \end{array} \right|^2 \right), \tag{3.2}
\]

\(^1\)Each lattice site has 12 triangles associated to it, however in the flat case each of these triangles has another which is always congruent to it, for example \(A_{13} = A_{23}, A_{17} = A_{67}\), etc.
and under a transformation by $D$ the area changes as

$$A'^2 = \frac{1}{4} \left( b^2 c^2 \begin{vmatrix} Y_A & Z_A & 2 \\ Y_B & Z_B & \end{vmatrix} + c^2 a^2 \begin{vmatrix} Z_A & X_A & 2 \\ Z_B & X_B & \end{vmatrix} + a^2 b^2 \begin{vmatrix} X_A & Y_A & 2 \\ X_B & Y_B & \end{vmatrix} \right).$$

(3.3)

Now, we require $A' = A$. Hence, if we let $\alpha = b^2 c^2 - 1$, $\beta = c^2 a^2 - 1$ and $\gamma = a^2 b^2 - 1$ we must have

$$\alpha \begin{vmatrix} Y_A & Z_A & 2 \\ Y_B & Z_B & \end{vmatrix} + \beta \begin{vmatrix} Z_A & X_A & 2 \\ Z_B & X_B & \end{vmatrix} + \gamma \begin{vmatrix} X_A & Y_A & 2 \\ X_B & Y_B & \end{vmatrix} = 0. \tag{3.4}$$

This condition has to be satisfied for all 6 triangles, giving 6 linear equations in $\alpha$, $\beta$ and $\gamma$. Letting $v^T = (\alpha, \beta, \gamma)$ we can write these in matrix form,

$$Gv = 0, \tag{3.5}$$

where $G$ is a $6 \times 3$ matrix, each row consisting of the squared determinants appearing in equation (3.4) for each of the six triangles.

We argue in appendix A.1 that the matrix $G$ will have rank 3. Hence the only solution to equation (3.5) will be $v = 0$. This in turn implies that $a = b = c = 1$ and thus only orthogonal transformations leave the 6 triangle areas independent.

### 3.2 $\Sigma$ Null

Now we wish to perform a similar analysis for the case when $\Sigma$ is null. We work in double null coordinates $(u, v, y, z)$ with $\Sigma$ located at $u = 0$. Our flat three lattice is given by coordinates in $(v, y, z)$. The metric in this space is simply $ds^2 = dy^2 + dz^2$, since $v$ is a null direction.

Again we transform the lattice by changing the three linearly independent vectors defining it; this induces a $GL(3)$ transformation on the $(v, y, z)$ space, which is given by the matrix $\tilde{T}$. We proceed to decompose this transformation, as we did in the spacelike case, but in a different fashion. Note now that areas of triangles do not depend on the $v$ coordinates of their vertices, hence we begin by writing $\tilde{T}$ as

$$\begin{pmatrix} a & b^T \\ c & E \end{pmatrix} = \begin{pmatrix} a - b^T E^{-1} c & b^T E^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}, \tag{3.6}$$

where $E \in GL(2)$ and $b, c$ are 2-component vectors. We can decompose $E$ as we did for $T$ in the spacelike case, giving $E = \tilde{O} \tilde{P} = \tilde{O} \tilde{Q} \tilde{D} \tilde{Q}^T$, with
\( O, \tilde{Q} \in O(2) \) and \( \tilde{D} \) diagonal with positive entries. Hence \( \tilde{T} \) now takes the form

\[
\tilde{T} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{O}\tilde{Q} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}^T \end{pmatrix},
\]

(3.7)

which can be rearranged to the form

\[
\tilde{T} = \begin{pmatrix} * & * \\ 0 & \tilde{O}\tilde{Q} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & \tilde{D} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}^T \end{pmatrix},
\]

(3.8)

where \( d = (\tilde{O}\tilde{Q})^T c \), and elements denoted by an asterisk (*) will not be relevant to our calculations.

We consider a triangle based at the origin with its other vertices at \((V_A, Y_A, Z_A)\) and \((V_B, Y_B, Z_B)\). The area of the triangle is given by

\[
A = \frac{1}{2} \begin{vmatrix} Y_A & Z_A \\ Y_B & Z_B \end{vmatrix},
\]

(3.9)

which does not depend on the \( v \) coordinates of the vertices, as stated earlier. We see that the first and third matrices in the decomposition of equation (3.8) will not alter the areas of the triangle, so all we require is that the second matrix does not either.

Let \( d^T = (c, d) \) and \( \tilde{D} = \text{diag}(a, b) \) with \( a, b > 0 \), then what we require is that the triangle areas remain invariant under \( y' = ay + cv \) and \( z' = bz + dv \). For our triangle’s coordinates this gives the condition

\[
(ab - 1) \begin{vmatrix} Y_A & Z_A \\ Y_B & Z_B \end{vmatrix} - bc \begin{vmatrix} Z_A & V_A \\ Z_B & V_B \end{vmatrix} - ad \begin{vmatrix} V_A & Y_A \\ V_B & Y_B \end{vmatrix} = 0.
\]

(3.10)

Again we demand that the 6 triangle areas, \( A_{13}, A_{15}, A_{17}, A_{26}, A_{27} \) and \( A_{37} \), remain invariant under the transformation. Thus we get 6 equations in our 3 unknowns \( \tilde{v}^T = (ab - 1, -bc, -ad) \), which we can write as

\[
\tilde{G}\tilde{v} = 0,
\]

(3.11)

where \( \tilde{G} \) is the \( 6 \times 3 \) matrix whose six rows contain the determinants appearing in equation (3.10) for each of the six triangles. We show in Appendix A.2 that the matrix \( \tilde{G} \) has rank 3, and hence the only solution to equation (3.11) is \( \tilde{v} = 0 \). Thus we have \( ab = 1 \) and \( c = d = 0 \), which leads to the most general form of \( \tilde{T} \) leaving all six triangle areas invariant as

\[
\tilde{T} = \begin{pmatrix} * & * \\ 0 & \tilde{O}\tilde{Q} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{D} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{Q}^T \end{pmatrix},
\]

(3.12)
where again * denotes elements which will not be relevant to us, and the matrix $\tilde{D}$ takes the form

$$\tilde{D} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

(3.13)

with $a > 0$.

### 3.3 Metric discontinuities

We can interpret these solutions to area Regge calculus as defining a metric on the space $M$ with a possible discontinuity at $\Sigma$. We began with a lattice in $\Sigma$, defined on a metric which we shall call $g$. The edge lengths of the lattice were defined in terms of this metric by $l^2 = x^T g x$. We then transformed the lattice with a linear transformation, which sent $x \rightarrow T x$. Thus the lengths of the edges were transformed as $l^2 \rightarrow l'^2 = (T x)^T g (T x)$. Equivalently, we can think of this as defining a new metric on $\Sigma$ given by

$$g' = T^T g T.$$

(3.14)

Edge lengths are then calculated using the old coordinates $l'^2 = x^T g' x$.

For the case of $\Sigma$ spacelike, our initial metric was $g = 1_3$, and we found that our transformation $T$ had to be orthogonal. Thus, the new metric is given by $g' = T^T 1_3 T = 1_3$. So we see that in this case there is no difference in the metrics and the edge lengths will then be well defined across $\Sigma$.

For the case of $\Sigma$ null, our initial metric was of the form

$$g = \begin{pmatrix} 0 & 0 \\ 0 & 1_2 \end{pmatrix},$$

(3.15)

and we found that $T$ must take the form given in equation (3.12). Using this we can calculate the form of the new metric, which turns out to be

$$g' = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{Q} \tilde{D}^2 \tilde{Q}^T \end{pmatrix},$$

(3.16)

where $\tilde{D}$ takes the form given in equation (3.13). More explicitly, letting $\tilde{Q}$ be a rotation through angle $\theta$, we have

$$g' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 \cos^2 \theta + a^{-2} \sin^2 \theta & (a^2 - a^{-2}) \cos \theta \sin \theta \\ 0 & (a^2 - a^{-2}) \cos \theta \sin \theta & a^2 \sin^2 \theta + a^{-2} \cos^2 \theta \end{pmatrix}.$$  

(3.17)

This solution is precisely the metric of a refractive wave spacetime [9]. Note that if we take $a$ to be close to unity, i.e. $a = 1 + \varepsilon/2$, then to first
order the difference in metrics on the hypersurface will take the form

\[ g' - g = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & -\cos 2\theta \end{pmatrix}, \tag{3.18} \]

and we can clearly see that the angle \( \theta \) can be interpreted as the polarization of this wave, and the value \( a \) as its strength.

4 Tensor Distributions

In order to handle discontinuous quantities properly we need to use distributions. The standard idea of a distributions can be extended to tensor valued distributions on arbitrary manifolds. There have been many different approaches to this idea \cite{10, 11, 12}; we follow here Geroch and Traschen \cite{12}.

Just as we can define distributions on \( \mathbb{R} \) as linear maps from a space of test functions to \( \mathbb{R} \), so we define tensor distributions on a manifold \( M \) as linear maps from a space of test fields to \( \mathbb{R} \). The test fields in this case are tensor densities of weight +1 with compact support, and we write the linear map as

\[ \hat{T}^{a\ldots}_{b\ldots} : \phi_{a\ldots}^{b\ldots} \rightarrow \langle \hat{T}^{a\ldots}_{b\ldots}, \phi_{a\ldots}^{b\ldots} \rangle, \tag{4.1} \]

which we need to satisfy certain continuity conditions, which are discussed more in \cite{12}.

Every smooth tensor field \( S^{a\ldots}_{b\ldots} \) on \( M \) defines a distribution by

\[ \langle \hat{S}^{a\ldots}_{b\ldots}, \phi_{a\ldots}^{b\ldots} \rangle = \int S^{a\ldots}_{b\ldots} \phi_{a\ldots}^{b\ldots} dV. \tag{4.2} \]

The fact that we use densities, instead of just tensors as the test fields means we do not need to introduce a volume element for this integral, as is done in \cite{10, 11}.

We can define the derivative of distributions, which is consistent with that for smooth tensors, by

\[ \langle \partial_c \hat{T}^{a\ldots}_{b\ldots}, \phi_{a\ldots}^{c\ldots} \rangle = -\langle \hat{T}^{a\ldots}_{b\ldots}, \partial_c \phi_{a\ldots}^{c\ldots} \rangle. \tag{4.3} \]

If we introduce a smooth connection on \( M \) then we have the corresponding result

\[ \langle \nabla_c \hat{T}^{a\ldots}_{b\ldots}, \phi_{a\ldots}^{c\ldots} \rangle = -\langle \hat{T}^{a\ldots}_{b\ldots}, \nabla_c \phi_{a\ldots}^{c\ldots} \rangle. \tag{4.4} \]
We can use the derivative of distributions to define the \textit{weak derivative} of locally integrable tensors\(^2\). A locally integrable tensor field \(T_{b...}^a\) defines a distribution \(\hat{T}_{b...}^a\), the same way as for a smooth tensor field in equation (4.2). The weak derivative of \(T_{b...}^a\) is - if it exists - a locally integrable tensor field \(R_{b...}^{a...}\) such that \(\hat{R}_{b...}^{a...} = \nabla_c \hat{T}_{b...}^a\).

The curvature tensor is nonlinear in the metric, and since multiplication of distributions does not usually make sense, we need to investigate which distributional metrics lead to a well-defined curvature tensor.

Choosing an arbitrary smooth connection \(\nabla_c\), with curvature tensor \(\rho_{abcd}\), we can write the curvature tensor for a smooth metric \(g_{ab}\) as

\[
R_{abcd} = \rho_{abcd} + 2Q_{[a}^e \rho_{b]c}^d + 2\nabla_c Q_{ab},
\]

(4.5)

where

\[
Q_{bc} = \frac{1}{2} g^{ae} (2\nabla_b g_{c} - \nabla_c g_{b} - \delta n_c \Delta_d g_{ab}),
\]

(4.6)

Equation (4.5) continues to make sense if we relax the smoothness conditions on \(g_{ab}\). Geroch and Traschen show that a distributional curvature tensor can still be defined for a so-called \textit{regular metric}. A metric is said to be regular if (i) \(g_{ab}\) and \(g^{ab}\) exist everywhere and are locally bounded, and (ii) the weak first derivative of \(g_{ab}\) exists and is locally square integrable.

Now consider a discontinuous metric. We split the manifold as \(M = M^+ \cup M^-\) with \(M^+ \cap M^- = \Sigma\), where \(\Sigma\) is a codimension 1 smooth hypersurface. The metric is then given by smooth tensor fields \(g^\pm_{ab}\) defined on \(M^\pm\) respectively. This metric clearly satisfies condition (i); to calculate the weak first derivative we note that this metric defines the distribution

\[
g_{ab} = \Theta^+ g_{ab}^+ + \Theta^- g_{ab}^-,
\]

(4.7)

where \(\Theta^\pm\) are the step functions on \(M^\pm\). The first derivative of this is

\[
\nabla_c g_{ab} = \Theta^+ \nabla_c g_{ab}^+ + \Theta^- \nabla_c g_{ab}^- + \delta n_c [g_{ab}],
\]

(4.8)

where \(\delta\) is the delta function on the boundary \(\partial M^-\), \(n_c\) is the normal to this boundary, and \([g_{ab}] = g_{ab}^+|_{\partial M^-} - g_{ab}^-|_{\partial M^-}\). Since there is no locally integrable tensor field which gives rise to the delta function we see that if \([g_{ab}] \neq 0\) then the weak derivative of \(g_{ab}\) does not exist, hence condition (ii) is not met, and the metric is not regular.

It is worth pointing out that distributional valued metrics appear quite frequently in the literature, in particular with regard to impulsive gravitational waves. Occasionally it is more convenient to write the metric for

\(^2\)A tensor field \(T_{b...}^a\) is locally integrable if \(T_{b...}^a\phi_{a...}\) is Lebesgue measurable and its integral converges for all test fields \(\phi_{a...}\). \(T_{b...}^a\) is locally square integrable if \(T_{b...}^a T_{a...}^d\) is locally integrable.
such solutions in a coordinate system which is not continuous. When this is done the metric may for instance contain delta functions, making it unbounded and hence not regular. However this is not a problem because the coordinate system was considered discontinuous, and a transformation of the coordinates back to a continuous coordinate system yields a regular metric. For a nice discussion of this see [13]. Here, we are assuming the coordinate system to be continuous, and as such the discontinuities cannot be transformed away with such discontinuous coordinate transformations.

Another problem with a discontinuous metric is the ambiguity of parallel transport. This is due to the fact there is no well defined metric connection. The standard formula for the metric connection will fail due to ill-defined multiplication of distributions, so the best we can do is a so-called regularly discontinuous connection [11]. Here the connection components are continuous, and metric compatible, in M yet not across the boundary, which results in the following rule for differentiation of the metric,

$$\nabla_c g_{ab} = \delta n_c [g_{ab}].$$

(4.9)

Despite the lack of a natural rule for parallel transport, we can still define metric geodesics. Since we are just extremizing the path length this problem is well defined. The result is familiar from optics, where we can describe the path of light through a medium with a metric representing the time of travel along the path. Where the refractive index of a medium changes abruptly we have a discontinuity of this metric, and we know that the path of light is refracted at this point. Exactly the same occurs here, with the geodesic refracted at the boundary by the rule

$$g^+(X^+, T) = g^-(X^-, T), \quad \forall T \in T_p \Sigma,$$

(4.10)

$$g^+(X^+ , X^+) = g^-(X^-, X^-),$$

(4.11)

where $X^\pm$ is the tangent to the geodesic on either side of the boundary.

It is interesting then that spaces which are solutions to area Regge calculus may not have a well defined notion of parallel transport. However it seems we would be able to define geodesics in the space.

5 Comparisons with General Relativity

We have shown in Section 4 that a discontinuity in the metric does not make sense in general relativity, since the curvature tensors cannot be defined. However, it is interesting to analyse the linearized Einstein equations, since these allow distributional solutions of all kinds. This will provide us with an interesting comparison with our area Regge calculus solutions.
We shall work with the perturbation about flat space $g_{ab} = \eta_{ab} + \varepsilon h_{ab}$ and write the linearized equations as

$$G^{abcd}_{ef} h^{cd}_{ab} = 0,$$

(5.1)

where $G^{abcd}_{ef} = \eta^{ac} \delta_e^d \delta_f^b + \eta^{ac} \delta_f^d \delta_e^b - \eta^{bd} \delta_e^d \delta_f^a - \eta_{ef} \eta^{ac} \eta^{bd}$ is a constant tensor and $\overline{h}_{ab} = h_{ab} - \frac{1}{2} \eta_{ab} h$ is the trace reversed perturbation.

Considering $h_{ab}$ as a tensor distribution (from now on we will drop hats), we can find solutions to equation (5.1) by its action on an arbitrary test tensor density $\phi_{ef}$.

$$\langle G^{abcd}_{ef} \overline{h}_{ab,cd} , \phi_{ef} \rangle = 0$$

(5.2)

$$\Rightarrow G^{abcd}_{ef} (\overline{h}_{ab}, \phi_{ef,cd}) = 0.$$  

(5.3)

Substituting in the form of our desired discontinuous solution

$$h_{ab} = \Theta^+ p_{ab} + \Theta^- q_{ab},$$

(5.4)

we obtain the integral

$$G^{abcd}_{ef} \int_{M^+} \overline{p}_{ab} \phi_{ef,cd} dV + (q\text{-term}) = 0.$$  

(5.5)

Integrating by parts twice gives

$$G^{abcd}_{ef} \left[ \int_{M^+} \overline{p}_{ab,cd} \phi_{ef} dV + \int_{\partial M^+} (\overline{p}_{ab} n_c \phi_{ef,d} - \overline{p}_{ab,c} n_d \phi_{ef}) dS \right] + (q\text{-term}) = 0,$$

(5.6)

where $n^a$ is the normal to the hypersurface $\Sigma$. However we should be careful, since the metric is discontinuous, the normal vector will also be discontinuous across the hypersurface. In fact we can define $n^a_\pm = n^a + \varepsilon m^a_\pm$ and then solve for $m^a_\pm$. Fortunately, as the discontinuity in the normal direction is $O(\varepsilon)$ it will not come into our calculations, but will be important later.

Now $\phi_{ef}$ was arbitrary, hence choosing $\text{supp}(\phi_{ef}) \subset M^+$ only the first integral contributes, hence we require the linearized Einsteins equations to be satisfied in $M^+$, i.e.

$$G^{abcd}_{ef} \overline{p}_{ab,cd} = 0.$$  

(5.7)

Similarly $q_{ab}$ satisfies the linearized Einstein equations in $M^-$. 

The more interesting result comes from the surface integral. To deal with this we must split the derivative into parallel and perpendicular parts,
\[ \partial_c = \partial^\|_c + n_c \partial^\perp. \]

Integrating the parallel derivative components by parts we obtain

\[
G^{abcd}_{ef} \int_{\partial M^+} \left[ (\overline{p}_{ab} n_c n_d) \partial^\perp \phi^{ef} - (\overline{p}_{ab,c} n_d + \partial^\|_d (\overline{p}_{ab} n_c)) \phi^{ef} \right] dS + (q\text{-term}) = 0
\]

(5.8)

Note that we can choose \( \phi^{ef} |_\Sigma \) and \( \partial^\perp \phi^{ef} |_\Sigma \) independently hence we obtain two junction conditions satisfied on the surface \( \Sigma \), given by

\[
G^{abcd}_{ef} \overline{\omega}_{ab} n_c n_d = 0 \quad (5.9)
\]

\[
G^{abcd}_{ef} (\overline{\omega}_{ab,n_c} d + \overline{\omega}_{ab,c} n_d - \partial^\perp (\overline{\omega}_{ab} n_c) n_d) = 0, \quad (5.10)
\]

where we have defined \( \overline{\omega}_{ab} = \overline{p}_{ab} |_\Sigma - \overline{q}_{ab} |_\Sigma \) and derivatives as \( \overline{\omega}_{ab,c} = \overline{p}_{ab,c} |_\Sigma - \overline{q}_{ab,c} |_\Sigma \). To interpret these conditions we look at two important cases.

### 5.1 \( \Sigma \) Spacelike

Junction conditions at spacelike surfaces in general relativity are well known and were given by Israel \[14\]: we show here that the above method reproduces these results.

We work in coordinates \((x^0, x^1, x^2, x^3)\) with \( \eta_{ab} = \text{diag}(1, -1, -1, -1) \) and take \( \Sigma \) to be given by \( x^0 = 0 \), hence \( n_c = (1, 0, 0, 0) \). Equation (5.9) can be split in \((0,0), (0,i), \) and \((i,j)\) components, where \( i,j,... = 1,2,3 \). The \((0,0)\) and \((0,i)\) components are automatically satisfied, and the \((i,j)\) components give the condition

\[
\omega_{ij} = 0, \quad (5.11)
\]

hence the induced metrics on \( \Sigma \) must agree. Note that there is no restriction on the \((0,0)\) and \((0,i)\) components of the discontinuity. This is related to the discontinuity in the normal to the hypersurface, mentioned earlier. A direct calculation reveals that the normal takes the form \( n^{\pm} = (1 - \frac{1}{2} \varepsilon h_{00}^{\pm}, \varepsilon h_{0i}^{\pm}) \) on each side of the hypersurface. Thus the discontinuity in the normal direction is equivalent to the discontinuity in the \((0,0)\) and \((0,i)\) components of the metric. Note also that this discontinuity could be transformed away with a continuous (yet not \( C^1 \)) coordinate transformation, though such a transformation will change the differentiable structure of the manifold.

Equation (5.10) can also be split, with the \((i,j)\) components giving the only condition, which we can write as

\[
\omega_{ij,0} - \omega_{0,i,j} - \omega_{0,j,i} - \eta_{ij} (\omega_{k,0}^{k,0} - 2 \omega_{0,k}^{k}) = 0. \quad (5.12)
\]
which is simply the linearized form of the Israel matching condition in the vacuum,

\[ [K_{ij} - \eta_{ij}K] = 0. \] (5.13)

Hence the extrinsic curvatures, \( K_{ij} \), of \( \Sigma \) must agree from both sides.

This result is then completely analogous to that in area Regge calculus where we found that we could not have a discontinuity in the metric, and also the extrinsic curvature had to match due to the zero deficit angle condition.

5.2 \( \Sigma \) Null

Junction conditions for null hypersurfaces have been studied in [15, 16]; however all assume the induced metrics on the null hypersurface agree. Following on from [9], and since we are only considering the linearized case, we do not make such an assumption, and see what the field equations demand.

We work in coordinates \((u, v, y, z)\) with metric \(ds^2 = 2dudv - dy^2 - dz^2\). Let \( \Sigma \) be given by \( u = 0 \) hence \( n_c = (1, 0, 0, 0) \), but note \( \partial^\perp = \frac{\partial}{\partial u} = l^c\partial_c \), where \( l_c = (0, 1, 0, 0) \). Despite this the above integrations by parts can still be done, giving the same equations (5.9, 5.10).

Let indices \( \alpha, \beta, ... \) represent the spatial \((y, z)\) directions. Solving equation (5.9), the only non-trivial solutions are the \((u,u)\), \((u,\alpha)\) and \((\alpha,\beta)\) components, which can be combined into the following one equation

\[ \omega_{ab}n^b = 0. \] (5.14)

This is exactly the condition obtained by Barrett [9] which states that the areas of spacelike 2-surfaces is continuous across \( \Sigma \). This is therefore the linearized metric of a refractive wave spacetime. Solving for \( \omega_{ab} \), we can see explicitly the form of the metric discontinuity.

\[
\omega_{ab} = \begin{pmatrix}
\omega_{uu} & \omega_{uv} & \omega_{uy} & \omega_{uz} \\
\omega_{uv} & 0 & 0 & 0 \\
\omega_{uy} & 0 & \omega_{yy} & \omega_{yz} \\
\omega_{uz} & 0 & \omega_{yz} & -\omega_{yy}
\end{pmatrix}
\] (5.15)

Note again that the \( \omega_{uu} \) terms can be transformed away with a continuous transformation, yet the \( \omega_{a\beta} \) terms cannot.

This result is also completely analogous to that obtained in area Regge calculus, where we found a discontinuity which, in the linearized situation of Equation (3.18), is precisely of this form.

Solving equation (5.10) and using equation (5.14) to simplify the results we obtain four conditions, which we write in the following suggestive forms
\[ \eta^{\alpha \beta}(\omega_{\alpha \beta, u} + \omega_{\beta u, \alpha} - \omega_{\alpha \beta, u}) = 0, \quad (5.16) \]
\[ \omega^\beta_{\alpha, \beta} + (\omega_{\alpha \beta, u} - \omega_{\beta u, \alpha} - \omega_{\alpha \beta, u}) = 0, \quad (5.17) \]
\[ \omega_{u \alpha, \beta} + \omega_{u \beta, \alpha} - \omega_{\alpha \beta, u} = 0, \quad (5.18) \]
\[ \omega_{v \alpha, \beta} + \omega_{v \beta, \alpha} - \omega_{\alpha \beta, v} = 0. \quad (5.19) \]

Now, for a null hypersurface we can define various fundamental forms, see [15] for more details. These are the internal second fundamental form \( \chi_{\alpha \beta} \), the external second fundamental form \( \psi_{\alpha \beta} \), the normal fundamental form \( \eta_{\alpha} \), and a quantity \( \omega \). In this case these are given by

\[ \chi_{\alpha \beta} = \nabla_{\alpha} n_{\beta}, \quad (5.20) \]
\[ \psi_{\alpha \beta} = \nabla_{\alpha} l_{\beta}, \quad (5.21) \]
\[ \eta_{\alpha} = \nabla_{\alpha} l_{\nu}, \quad (5.22) \]
\[ \omega = \nabla_{\nu} l_{\nu}, \quad (5.23) \]

which we can calculate for this linearized situation, and hence rewrite equations \((5.16-5.19)\) in terms of these quantities

\[ [\eta^{\alpha \beta}, \psi_{\alpha \beta}] = 0, \quad (5.24) \]
\[ [\Gamma_{\alpha \beta} + 2\eta_{\alpha}] = 0, \quad (5.25) \]
\[ [\omega] = 0, \quad (5.26) \]
\[ [\chi_{\alpha \beta}] = 0. \quad (5.27) \]

The quantity \( \Gamma_{\alpha \beta} \) in the second of these equations does not appear to be interpretable in terms of our fundamental forms.

In the case where the induced metric is assumed to be continuous these conditions reduce to those obtained by Clarke and Dray [15], namely that \([\eta^{\alpha \beta} \psi_{\alpha \beta}] = [\eta_{\alpha}] = [\omega] = 0\). However, here we have the more general case allowing for the discontinuous metrics present in the linearized theory.

### 6 Conclusions and Outlook

We have shown how to construct non-trivial solutions to area Regge calculus. In particular we have shown the existence of solutions with a discontinuity in the metric along a null hypersurface. These solutions are precisely the refractive wave spacetimes introduced by Barrett [9].

The relation between our results for area Regge calculus and those of the linearized Einstein equations is also apparent. At spacelike hypersurfaces both theories predict a matching of the induced metrics, and of the extrinsic...
curvature. At null hypersurfaces, both theories predict the discontinuous metric of a refractive wave spacetime. We would also like to be able to compare the curvature matching conditions in this case.

This comparison with linearized gravity fits in nicely with the ideas of Rocek and Williams [3]. They calculate the weak field expansion of area Regge calculus and show the dynamics of the theory to be equivalent to that of edge length variable Regge calculus. Since at the perturbative level, in the long wavelength limit, Regge calculus is equivalent to Einstein’s theory [7, 8], so also will area Regge calculus.

Despite this perturbative/dynamical relation, area Regge calculus is clearly not so simply associated with the full theory of general relativity. The refractive wave solution of area Regge calculus is not a solution of the full non-linear theory of relativity, since for such a metric the curvature tensor will not be well defined.

The next step in the study of area Regge calculus would be to look for more complex and interesting solutions. It appears to be straightforward to combine refractive wave solutions so long as they are moving parallel or antiparallel, the result being a spacetime with multiple discontinuities. In double null coordinates \((u, v, y, z)\) the discontinuities are along surfaces of constant \(u\), and surfaces of constant \(v\). These discontinuities will thus intersect, and we can check that the field equations are still satisfied where this occurs. We can also consider geodesics which start parallel, but travel either side of such an intersection to a new region of space. Here the geodesics will no longer be necessarily parallel, which reveals the non trivial nature of these solutions.

More general solutions to area Regge calculus can presumably be found. Whether we can relate the properties of these solutions, such as geodesic behaviour, to properties of solutions to general relativity remains to be seen.

The ultimate goal will be to relate area Regge calculus back to the quantum theories of gravity from which it was born, in order to understand to a greater extent what these theories are telling us about the nature of space, time and gravitation.

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A Appendix

A.1 Rank of $G$

We show here the matrix $G$ of Equation (3.5) has rank 3. First we can define a vector $s_{AB}$ by

$$s_{AB}^T = \begin{pmatrix} Y_A Z_A Z_B & Z_B X_B X_B Y_B \end{pmatrix}^2.$$

(A.1)

Then $G$ takes the form

$$G^T = (s_{13}, s_{15}, s_{17}, s_{26}, s_{27}, s_{37}).$$

(A.2)

Now we know the coordinates of vertices 3, 5, 6 and 7 are given in terms of those of vertices 1, 2 and 4. So, using the properties of determinants we have various relations to simplify the matrix. We can then use column operations on the matrix $G^T$ to put it in the form

$$G^T = (s_{24}, s_{41}, s_{12}, *, *, *).$$

(A.3)

Without loss of generality we choose the coordinates of our lattice basis vector as $x_1^T = (a, b, c)$, $x_2^T = (0, d, e)$ and $x_4^T = (0, 0, f)$ where $a, d, f \neq 0$. The matrix $G$ will then take the form

$$G = \begin{pmatrix} d^2 f^2 & 0 & 0 \\
* & a^2 f^2 & 0 \\
* & * & a^2 d^2 \\
* & * & * \\
* & * & * \\
* & * & * \\
\end{pmatrix},$$

(A.4)

thus, since $a, d, f \neq 0$, this matrix will have rank 3.

A.2 Rank of $\tilde{G}$

We show here that the matrix $\tilde{G}$ of Equation (3.11) has rank 3. First define the vector $r_{AB}$ by

$$r_{AB}^T = \begin{pmatrix} Y_A Z_A Y_B Z_B & Z_A V_A V_B & V_A Y_A Y_B \end{pmatrix}.$$

(A.5)

Then we can write $\tilde{G}$ in the form

$$\tilde{G}^T = (r_{13}, r_{15}, r_{17}, r_{26}, r_{27}, r_{37}).$$

(A.6)

Now we know that the coordinates of the vertices 3, 5, 6 and 7 can be given in terms of those of vertices 1, 2 and 4. So, using the properties of
determinants we have for example \( r_{13} = r_{11} + r_{12} = r_{12} \), and likewise for the other entries of \( \tilde{G} \). Thus we can use column operations to put \( \tilde{G}^T \) in the form

\[
\tilde{G}^T = (r_{24}, r_{41}, r_{12}, 0, 0, 0).
\]

The first three rows of \( G \) are then just the cofactor matrix of

\[
S = \begin{pmatrix} V_1 & Y_1 & Z_1 \\ V_2 & Y_2 & Z_2 \\ V_4 & Y_4 & Z_4 \end{pmatrix},
\]

and since the three vectors along the edges 1, 2 and 4 are linearly independent we have \( \det S \neq 0 \). Hence, the cofactor matrix of \( S \) will also be invertible, which implies that \( \tilde{G} \) will have rank 3.

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