Fractal Weyl laws for quantum decay in generic dynamical systems

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Weyl’s law approximates the number of states in a quantum system by partitioning the energetically accessible phase-space volume into Planck cells. Here we show that typical resonances in generic open quantum systems follow a modified, fractal Weyl law, even though their classical dynamics is not globally chaotic but also contains domains of regular motion. Besides the obvious ramifications for quantum decay, this delivers detailed insight into quantum-to-classical correspondence, a phenomenon which is poorly understood for generic quantum-dynamical systems.

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Phase-space rules provide powerful universal relations for classical and quantum systems. A time-honored example is Sabine’s law, originally formulated in the context of room acoustics, which can be cast into the relation \( \tau_{\text{dwell}} = 4V/vA \) for the mean dwell time of a classical particle escaping from a container, expressed in terms of the volume \( V \) of the container, the area \( A \) of the opening, and the particle’s velocity \( v \) \[1\]. In quantum mechanics, Weyl’s law approximates the number \( \mathcal{N}(E) \) of states with energy \( E_n < E \) by the number of Planck cells \( h^d \) which fit into the accessible phase-space volume of the corresponding classical system (here \( h \) is Planck’s constant and \( d \) is the number of dimensions) \[2\]. For a quantum particle in a container (e.g., an electron in a quantum dot), \( \mathcal{N}(E) \propto E^{d/2} \) therefore follows a power law with a strictly determined exponent.

In reality, quantum systems are open, and the eigenstates acquire a finite lifetime—the resulting resonance states constitute a fundamental concept across many fields of physics. However, there are reasons to believe that the synthesis of both mentioned phase-space rules in open quantum systems can modify Weyl’s law. Evidence in this direction is provided by systems with globally chaotic classical dynamics, which exhibit a fractal Weyl law \( \mathcal{N}(E, \tau) \propto c \tau E^{d/2} \) for the number of resonances with \( E_n < E \) and lifetime \( \tau_n > \tau \) (where the cut-off value \( \tau \) only enters the shape function \( c_\tau \)) \[3, 4, 5, 6\]. Remarkably, the number \( d_H \) in the exponent is not an integer; instead, it is given by the dimension of the strange repeller, which for a chaotic system is a fractal \[7\]. While this observation already proofed suited to instigate a paradigmatic shift of the study of resonances in open quantum-chaotic systems, its direct practical consequences are necessarily limited: typical dynamical systems are not globally chaotic, which has profound consequences on their quantum dynamics \[8\].

Here, we show that fractal Weyl laws indeed apply much more generally to generic dynamical systems, for which regular and chaotic motion coexists in a mixed phase space \[8\]. The key which reveals the fractal Weyl law is to restrict the resonance counting to a window of typical lifetimes \( \tau < \tau_n < \tau^*, \) where \( \tau \ll \tau_{\text{dwell}} \ll \tau^* \).

We arrive at this conclusion by a combination of semi-classical arguments (based on a tailor-made phase-space representation of resonance wave functions which avoids problems with their mutual non-orthogonality) with numerical results for a paradigmatic quantum-dynamical model system, the open kicked rotator \[4, 9, 10, 11\].

We start our considerations with some general observations about quantum-to-classical correspondence. A basic ingredient in the derivation of the ordinary Weyl law in closed systems is the mutual orthogonality of energy eigenstates, which is guaranteed by the hermiticity of the Hamiltonian (equivalently, the unitarity of the time-evolution operator). Quantum-to-classical correspondence can then be exploited, e.g., by using a basis of semiclassically localized states \( |x \rangle \) which occupy Planck cells in phase space \( x = (q,p) \). Resonance wave functions, however, overlap with each other because open systems are necessarily represented by non-normal (neither hermitian nor unitary) operators. In open quantum maps, for example, resonances are associated to the spectrum of a truncated unitary matrix \( \mathcal{M} = QFQ \), composed of the unitary time-evolution operator \( F \) of the closed system and the projector \( Q = Q^2 \) onto the non-leaky part of Hilbert space (see Fig. \[1\] for an example of an open quantum map). Because of the truncation, the eigenvalues \( \mu_n = \exp(-iE_n - \gamma_n/2) \) of \( \mathcal{M} \) lie inside the unit disk of the complex plane, ensuring that the decay rates \( \gamma_n = 1/\tau_n \) are positive \[12\]. But since \( \mathcal{M} \) is not normal, the associated eigenstates—which now describe the resonance wave functions—are not orthogonal to each other. This circumstance, which is the root for the scarceness of analytical tools in non-normal problems, complicates the task of exploiting quantum-to-classical correspondence for the purpose of resonance counting.

We circumvent this problem by applying standard phase space methods to an alternative spectral decomposition, the Schur decomposition \[13\] \( \mathcal{M} = UTU^\dagger \), which delivers the same eigenvalues (occupying the diagonal of the upper triangular matrix \( T \)) but associates to them an orthogonal basis set \( U \). For definiteness assume that all eigenvalues are ordered by their modulus, \( |\mu_1| \leq |\mu_2| \leq |\mu_3| \leq \ldots \leq |\mu_M| \). The first \( r \) rows \( u_r \) of
$U$ then form a complete basis for the $r$ fastest decaying resonance states. Quantum-to-classical correspondence can now be exploited by considering the Husimi representation \( \mathcal{H}_r(x) = \sum_{m=1}^{r} |\langle x | u_m \rangle|^2 \) of this subspace. This provides insight into the regions in classical phase space which support the quickly decaying quantum resonances. An analogous construction can be based on the opposite ordering, $|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \ldots \geq |\mu_M|$, which focuses on the slowest decaying resonance wave functions. Because of non-orthogonality, the resulting ‘fast’ and ‘slow’ Husimi-Schur representations carry independent information on the resonance wave functions.

Figure 1 demonstrates the viability of this method for a paradigm of quantum chaos, the kicked rotator [9] with unitary time-evolution operator

$$F_{nm} = (iM)^{-1/2} e^{\frac{i\pi}{M} (m-n)^2 - \frac{i\pi}{2M} \left( \cos \frac{2\pi m}{M} + \cos \frac{2\pi n}{M} \right)},$$

which classically reduces to the (symmetrized) standard map $p' = p + k \cos(2\pi q + \pi p)$ mod 1, $q' = q + p'/2 + p'/2 \mod 1$. The matrix dimension $M = h^{-1}$ determines the inverse effective Planck constant [13], while the kicking strength $k$ determines the nonlinearity. For $k = 0$ the classical system is integrable, while for $k \gtrsim 7$ the dynamics is globally chaotic (resonances in the chaotic variant have been studied in Refs. [4, 10]).

In Figure 1 the kicking strength is set to $k = 2$, for which the phase space of the closed system is mixed, as shown by trajectory segments (black dots) in panel (a). The color-coded areas superimposed on the phase space indicate the classical initial conditions for escape after 1, 2, 3, or 4 iterations when an opening is placed at $0 < q < 0.2$ ($\tau_{\text{dwell}} = 5$). Panel (b) shows the ordinary Husimi representation of quickly decaying resonance eigenfunctions with $0 < |\mu_n| < 0.1$ ($M = 1280$). Panels (c) and (d) show the ‘fast’ Husimi-Schur representation ($0 < |\mu_n| < 0.1$, $M = 160$ and $M = 1280$), while panels (e) and (f) display the ‘slow’ Husimi-Schur representation (0.98 $< |\mu_n| < 1$; $M = 160$ and $M = 1280$) [13].

The ordinary Husimi representation demonstrates that the quickly decaying resonance eigenfunctions are all localized in the region of escape after a single iteration. However, since these eigenfunctions strongly overlap, Planck-cell partitioning their phase-space support significantly underestimates their number (predicting $\approx 25$ instead of $r = 44$ short-lived states for $M = 160$, and $\approx 200$ instead of $r = 568$ short-lived states for $M = 1280$). In contrast, the fast Husimi-Schur representation clearly maps out the classical escape zones and uncovers that the domain of quantum-to-classical correspondence increases with increasing $M$. The slow Husimi-Schur representation maps out stable phase-space regions that are classically decoupled from the opening by impenetrable dynamical barriers; these regions do not significantly change as $M$ increases.

By construction, Planck-cell partitioning of the support of the Husimi-Schur representations accurately estimates the underlying number of resonances. The fast Husimi representation therefore uncovers a proliferation of anomalously short-lived states driven by the emerging quantum-to-classical correspondence — among the total count of $M$ resonances, the fraction of short-lived states increases as $M$ increases. On the other hand, the slow Husimi representation shows that the fraction of anomalously long-living resonances supported by classically uncoupled regions remains fixed, which corresponds to the
The ordinary Weyl law for this effectively closed-off part of the system. Crucially, we are led to conclude that the remaining fraction of resonances with typical lifetime (chosen here to satisfy \(0.1 < |\mu_n| < 0.98\)) decreases as \(M\) increases, and therefore cannot follow an ordinary Weyl law.

A detailed understanding of the resonance distribution can be obtained by counting the resonances in fixed lifetime windows and comparing these counts for different values of \(M\). We adopt probabilistic terminology and proceed in two steps. In the first step we determine the fraction \(P(\mu) = \text{prob}(|\mu_n| > \mu)\) of resonances with lifetime exceeding a lower threshold \(\tau = -1/(2 \ln \mu)\). This defines a monotonously decreasing function interpolating between \(P(0) = 1\) and \(P(1) = 0\). In the second step, we extract the fraction of resonances within an interval of typical lifetimes \((0.1 < |\mu_n| < 0.98)\), which follows from \(P_{\text{typ}} = \text{prob}(|\mu_n| \in [0.1, 0.98]) = P(0.1) - P(0.98)\) (our results and conclusions do not depend on the chosen window as long as it stays in the range of typical life times).

Numerical results for the system with opening at \(0 < q < 0.2\) are shown in Fig. 2. Panel (a) shows \(P(\mu)\) for various values of \(M\). As a function of \(\mu\), \(P\) decreases very sharply at the two extreme ends of the graph. At \(\mu \approx 0\) we witness the influence of extremely short-lived resonances, while at \(\mu \approx 1\) we observe the states which have very long lifetimes. Applicability of the ordinary Weyl law would entail that modulo small fluctuations, \(P(\mu)\) is independent of \(M\), since the uncertainty-limited resolution of phase space increases uniformly when the Planck cell shrinks. The plot, however, shows that the body of the function \(P\) drops as \(M\) increases. This is due to the proliferation of the short-lived resonances \((\mu \approx 0)\), whose relative fraction among all resonances increases with increasing \(M\), in agreement with the expanding domain of support of the fast Husimi-Schur representation. In the region of long-living states \((\mu \approx 1)\), on the other hand, \(P\) does not depend significantly on \(M\), in agreement with the observed \(M\)-independent support of long-living resonances in the slow Husimi-Schur representation.

Complementing these trends for short and long lifetimes, the body of \(P\) becomes flatter as \(M\) increases. As shown in Fig. 2(b), the fraction \(P_{\text{typ}}\) of resonance with typical lifetime therefore decreases with increasing \(M\). The linear fit in this double-logarithmic plot demonstrates that this trend closely follows a power law \(P_{\text{typ}} \propto M^{-0.50}\). The typical resonances therefore obey a fractal Weyl law, \(N_{\text{typ}} = MP_{\text{typ}} \propto M^{0.50}\).

Figure 3 shows that the fractal Weyl law remains intact when the opening is shifted to \(0.2 < q < 0.4\), so that it couples to a larger part of the regular regions in phase space [see Fig. 3(a)]. In this case \(P_{\text{typ}} \propto M^{-0.19}\), and therefore \(N_{\text{typ}} \propto M^{0.81}\). Compared to the situation in Fig. 2, the fraction of long-living states is now reduced, in keeping with the shrunken size of the classically uncoupled phase-space region. The fast Husimi-Schur representation [Fig. 3(b)] shows that the newly coupled parts support additional short-lived resonances. Consequently,
the states with typical lifetime are still associated to the chaotic regions.

For globally chaotic systems, the fractal Weyl law can be understood by associating the proliferation of short-lived resonance to quasi-deterministic decay following classical escape routes [4]. This quasi-deterministic decay requires classical-to-quantum correspondence, which is lost exponentially on a time scale given by the Ehrenfest time $\tau_E = \lambda^{-1} \ln M$. The probability to reside within the system decays exponentially, too, and is governed by $\tau_{dwell}$. The power law for the fractal Weyl law therefore arises from the combination of two exponential laws, based on the relation $\exp(-\tau_E/\tau_{dwell}) = M^{-1/\lambda\tau_{dwell}} \propto M^{\delta-1/\lambda\tau_{dwell}}$ for the part of phase space where classical-to-quantum correspondence does not apply; this region supports $N \propto M^{d-1/\lambda\tau_{dwell}}$ resonances (delivering an accurate estimate for $d\mu$ if the opening is sufficiently small [7]). For generic dynamical systems, our results confirm the association of short-lived resonances to quasi-deterministic escape routes, while the resonances of typical lifetime are now associated to chaotic regions dominated by sticking motion, where classical power-law decay $\propto t^{-\alpha}$ rule in place of the exponential decay [17]. This can be reconciled with the fractal Weyl law when one assumes that the loss of quantum-to-classical correspondence in these regions is similarly modified into a power law $\propto t^{\alpha}$, so that the Ehrenfest time takes the algebraic form $\tau_E \propto M^{1/\beta}$ [18]. The fractal Weyl law then arises from the combination of two power laws, based on the relation $\tau_E^{\alpha/\beta} \propto M^{-\alpha/\beta}$ for the part of phase space where classical-to-quantum correspondence does not apply and dynamics is dominated by transient sticking motion.

In summary, we have shown that typical resonances in open quantum systems obey a modified, fractal Weyl law, a phenomenon previously associated only to the non-generic case of systems with a globally chaotic classical limit. We unravelled this law by introducing the concept of a Husimi-Schur representation, a phase-space representation of resonance wave functions which captures maximal information on quantum-to-classical correspondence (circumventing the problem that resonance eigenfunctions are not orthogonal to each other). The formation of anomalously short-lived states that drives the departure from the ordinary Weyl law originates in quasi-deterministic decay along classical escape routes, whose phase-space support expands as one approaches the classical limit. The fractal Weyl law emerges from the interplay of the transient sticking of chaotic trajectories to the stable components and the algebraic violation of quantum-to-classical correspondence, two mechanisms which are intimately related to a mixed phase space.

The kicked rotator used here for illustration can be realized with atoms that are driven by pulsed optical waves [19]. Experimentally, direct evidence of the fractal Weyl law is more likely to come from autonomous (non-driven) systems such as microwave resonators, in which advanced techniques allow for the accurate determination of complex resonance frequencies [20]; a mixed phase space is obtained for any generic smooth resonator shape. While the details of a mixed phase space constitute a unique fingerprint of a given dynamical systems, the general features (a hierarchy of stability islands embedded into chaotic domains where long-time transients arise from sticking motion) are remarkably robust. Based on this general phenomenology we expect that the fractal Weyl law for typical resonances is a generic feature of open quantum-dynamical systems.

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