Local density of solutions of time and space fractional equations

Alessandro Carbotti†, Serena Dipierro‡ and Enrico Valdinoci§

October 22, 2018

Abstract

We prove that any given function can be smoothly approximated by functions lying in the kernel of a linear operator involving at least one fractional component. The setting in which we work is very general, since it takes into account anomalous diffusion, with possible fractional components in both space and time. The operators studied comprise the case of the sum of classical and fractional Laplacians, possibly of different orders, in the space variables, and classical or fractional derivatives in the time variables.

This type of approximation results shows that space-fractional and time-fractional equations exhibit a variety of solutions which is much richer and more abundant than in the case of classical diffusion.

Contents

1 Introduction and main results 2
2 Sharp boundary behaviour for the time-fractional eigenfunctions 7
3 Sharp boundary behaviour for the time-fractional harmonic functions 9
4 Green representation formulas and solution of \((-\Delta)^{s}u = f\) in \(B_1\) with homogeneous Dirichlet datum 11
   4.1 Solving \((-\Delta)^{s}u = f\) in \(B_1\) for discontinuous \(f\) vanishing near \(\partial B_1\) . . . . . . 11
   4.2 Solving \((-\Delta)^{s}u = f\) in \(B_1\) for \(f\) Hölder continuous near \(\partial B_1\) . . . . . . . . . 15
5 Existence and regularity for the first eigenfunction of the higher order fractional Laplacian 16

---

*Supported by the Australian Research Council Discovery Project 170104880 NEW “Nonlocal Equations at Work”. The authors are members of INdAM/GNAMPA.

†Dipartimento di Matematica e Fisica, Università del Salento, Via Per Arnesano, 73100 Lecce, Italy. alessandro.carbotti@unisalento.it

‡Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley WA 6009, Australia. serena.dipierro@uwa.edu.au

§Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley WA 6009, Australia, and Istituto di Matematica Applicata e Tecnologie Informatiche, Consiglio Nazionale delle Ricerche, Via Ferrata 1, 27100 Pavia, Italy, and Dipartimento di Matematica, Università degli studi di Milano, Via Saldini 50, 20133 Milan, Italy. enrico@mat.uniroma3.it
6 Boundary asymptotics of the first eigenfunctions of $(-\Delta)^s$ 23
7 Boundary behaviour of $s$-harmonic functions 33
8 A result which implies Theorem 1.1 36
9 A pivotal span result towards the proof of Theorem 8.1 38
10 Every function is locally $\Lambda_{-\infty}$-harmonic up to a small error, and completion of the proof of Theorem 8.1 59
  10.1 Proof of Theorem 8.1 when $f$ is a monomial .......................... 59
  10.2 Proof of Theorem 8.1 when $f$ is a polynomial .......................... 62
  10.3 Proof of Theorem 8.1 for a general $f$ ................................. 63

1 Introduction and main results

In this paper we prove the local density of functions which annihilate a linear operator built by classical and fractional derivatives, both in space and time.

Nonlocal operators of fractional type present a variety of challenging problems in pure mathematics, also in connections with long-range phase transitions and nonlocal minimal surfaces, and are nowadays commonly exploited in a large number of models describing complex phenomena related to anomalous diffusion and boundary reactions in physics, biology and material sciences (see e.g. [BV16] for several examples, for instance in atom dislocations in crystals and water waves models). Furthermore, anomalous diffusion in the space variables can be seen as the natural counterpart of discontinuous Markov processes, thus providing important connections with problems in probability and statistics, and several applications to economy and finance (see e.g. [MVN68, Man12] for pioneer works relating anomalous diffusion and financial models).

On the other hand, the development of time-fractional derivatives began at the end of the seventeenth century, also in view of contributions by mathematicians such as Leibniz, Euler, Laplace, Liouville and many others, see e.g. [Fer18] and the references therein for several interesting scientific and historical discussions. From the point of view of the applications, time-fractional derivatives naturally provide a model to comprise memory effects in the description of the phenomena under consideration.

In this paper, the time-fractional derivative will be mostly described in terms of the so-called Caputo fractional derivative (see [Cap08]), which induces a natural “direction” in the time variable, distinguishing between “past” and “future”. In particular, the time direction encoded in this setting allows the analysis of “non anticipative systems”, namely phenomena in which the state at a given time depends on past events, but not on future ones. The Caputo derivative is also related to other types of time-fractional derivatives, such as the Marchaud fractional derivative, which has applications in modeling anomalous time diffusion, see e.g. [ACV16, AV18, Fer18]. See also [MR93, SKM93] for more details on fractional operators and several applications.

In this article, we will take advantage of the nonlocal structure of a very general linear operator containing fractional derivatives in some variables (say, either time, or space, or
both), in order to approximate, in the smooth sense and with arbitrary precision, any prescribed function. Remarkably, no structural assumption needs to be taken on the prescribed function: therefore this approximation property reveals a truly nonlocal behaviour, since it is in contrast with the rigidity of the functions that lie in the kernel of classical linear operators (for instance, harmonic functions cannot approximate a function with interior maxima or minima, functions with null first derivatives are necessarily constant, and so on).

The approximation results with solutions of nonlocal operators have been first introduced in [DSV17] for the case of the fractional Laplacian, and since then widely studied under different perspectives, including harmonic analysis, see [RS18, GSU16, Rül17, RS17a, RS17b]. The approximation result for the one dimensional case of a fractional derivative of Caputo type has been considered in [Buc17, CDV18], and operators involving classical time derivatives and additional classical derivatives in space have been studied in [DSV18].

The great flexibility of solutions of fractional problems established by this type of approximation results has also consequences that go beyond the purely mathematical curiosity. For example, these results can be applied to study the evolution of biological populations, showing how a nonlocal hunting or dispersive strategy can be more convenient than one based on classical diffusion, in order to avoid waste of resources and optimize the search for food in sparse environment, see [MV17, CDV17]. Interestingly, the theoretical descriptions provided in this setting can be compared with a series of concrete biological data and real world experiments, confirming anomalous diffusion behaviours in many biological species, see [VAB+96].

Another interesting application of time-fractional derivatives arises in neuroscience, for instance in view of the anomalous diffusion which has been experimentally measured in neurons, see e.g. [SWDSA06] and the references therein. In this case, the anomalous diffusion could be seen as the effect of the highly ramified structure of the biological cells taken into account, see [DV18].

In many applications, it is also natural to consider the case in which different types of diffusion take place in different variables: for instance, classical diffusion in space variables could be naturally combined to anomalous diffusion with respect to variables which take into account genetical information, see [RVD+13, Sef17].

Now, to state the main results of this paper, we introduce some notation. In what follows, we will denote the “local variables” with the symbol $x$, the “nonlocal variables” with $y$, the “time-fractional variables” with $t$. Namely, we consider the variables

\[
\begin{align*}
x &= (x_1, \ldots, x_n) \in \mathbb{R}^{p_1} \times \ldots \times \mathbb{R}^{p_n}, \\
y &= (y_1, \ldots, y_M) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_M} \\
\text{and} \quad t &= (t_1, \ldots, t_l) \in \mathbb{R}^l,
\end{align*}
\]

(1.1)

for some $p_1, \ldots, p_n$, $M$, $m_1, \ldots, m_M$, $l \in \mathbb{N}$, and we let

\[
(x, y, t) \in \mathbb{R}^N, \quad \text{where} \quad N := p_1 + \ldots + p_n + m_1 + \ldots + m_M + l.
\]

When necessary, we will use the notation $B_R^k$ to denote the $k$-dimensional ball of radius $R$, centered at the origin in $\mathbb{R}^k$; otherwise, when there are no ambiguities, we will use the usual notation $B_R$. 

3
Fixed $r = (r_1, \ldots, r_n) \in \mathbb{N}^{p_1} \times \ldots \times \mathbb{N}^{p_n}$, with $|r_i| \geq 1$ for each $i \in \{1, \ldots, n\}$, and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, we consider the local operator acting on the variables $x = (x_1, \ldots, x_n)$ given by

$$I := \sum_{i=1}^{n} \alpha_i \partial_{x_i}^r,$$

where the multi-index notation has been used.

Furthermore, given $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_M) \in \mathbb{R}^M$ and $s = (s_1, \ldots, s_M) \in (0, +\infty)^M$, we consider the operator

$$\mathcal{L} := \sum_{j=1}^{M} \mathcal{D}_j (-\Delta)_{y_j}^{s_j},$$

where each operator $(-\Delta)_{y_j}^{s_j}$ denotes the fractional Laplacian of order $2s_j$ acting on the set of space variables $y_j \in \mathbb{R}^{m_j}$. More precisely, for any $j \in \{1, \ldots, M\}$, given $h_j \in \mathbb{N}$ and $s_j \in (0, h_j)$, in the spirit of [AJS18a], we consider the operator

$$(-\Delta)_{y_j}^{s_j} u (x, y, t) := \int_{\mathbb{R}^{m_j}} \frac{(\delta_{h_j} u) (x, y, Y_j)}{|Y_j|^{m_j + 2s_j}} dY_j,$$

where

$$(\delta_{h_j} u) (x, y, t, Y_j) := \sum_{k=-h_j}^{h_j} (-1)^k \left( \frac{2h_j}{h_j - k} \right) u (x, y_1, \ldots, y_{j-1}, y_j + kY_j, y_{j+1}, \ldots, y_M, t).$$

In particular, when $h_j := 1$, this setting comprises the case of the fractional Laplacian $(-\Delta)_{y_j}^{s_j}$ of order $2s_j \in (0, 2)$, given by

$$(-\Delta)_{y_j}^{s_j} u (x, y, t) := c_{m_j, s_j} \int_{\mathbb{R}^{m_j}} \left( 2u(x, y, t) - u(x, y_1, \ldots, y_{j-1}, y_j + Y_j, y_{j+1}, \ldots, y_M, t) - u(x, y_1, \ldots, y_{j-1}, y_j - Y_j, y_{j+1}, \ldots, y_M, t) \right) \frac{dY_j}{|Y_j|^{m_j + 2s_j}},$$

where $s_j \in (0, 1)$ and $c_{m_j, s_j}$ denotes a multiplicative normalizing constant (see e.g. formula (3.1.10) in [BV16]).

It is interesting to recall that if $h_j = 2$ and $s_j = 1$ the setting in (1.4) provides a nonlocal representation for the classical Laplacian, see [AV18].

In our general framework, we take into account also nonlocal operators of time-fractional type. To this end, for any $\alpha > 0$, letting $k := \lceil \alpha \rceil + 1$ and $a \in \mathbb{R} \cup \{-\infty\}$, one can introduce
the left\textsuperscript{1} Caputo fractional derivative of order $\alpha$ and initial point $a$, defined, for $t > a$, as

$$\text{D}^\alpha_{t,a}u(t) := \frac{1}{\Gamma(k-\alpha)} \int_a^t \frac{\partial^k u(\tau)}{(t-\tau)^{\alpha-k+1}} d\tau,$$

where\textsuperscript{2} $\Gamma$ denotes the Euler’s Gamma function.

In this framework, fixed $\xi = (\xi_1, \ldots, \xi_l) \in \mathbb{R}^l$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in (0, +\infty)^l$ and $a = (a_1, \ldots, a_l) \in (\mathbb{R} \cup \{-\infty\})^l$, we set

$$\mathcal{D}_a := \sum_{h=1}^l \xi_h \text{D}^{\alpha_h}_{t_h,a_h}.$$  \hfill (1.7)

Then, in the notation introduced in (1.2), (1.3) and (1.7), we consider here the superposition of the local, the nonlocal, and the time-fractional operators, that is, we set

$$\Lambda_a := I + \mathcal{L} + \mathcal{D}_a.$$  \hfill (1.8)

With this, the statement of our main result goes as follows:

**Theorem 1.1.** Suppose that

then there exists $i \in \{1, \ldots, M\}$ such that $\xi_i \neq 0$ and $s_i \not\in \mathbb{N}$, or there exists $i \in \{1, \ldots, l\}$ such that $\xi_i \neq 0$ and $\alpha_i \not\in \mathbb{N}$.

(1.9)

Let $\ell \in \mathbb{N}$, $f : \mathbb{R}^N \to \mathbb{R}$, with $f \in C^\ell(B_1^N)$. Fixed $\epsilon > 0$, there exist

$$u = u_\epsilon \in C^\infty(B_1^N) \cap C(\mathbb{R}^N),$$

$$a = (a_1, \ldots, a_l) = (a_{1,\epsilon}, \ldots, a_{l,\epsilon}) \in (-\infty, 0)^l,$$

and $R = R_\epsilon > 1$

such that

$$\begin{cases}
\Lambda_a u = 0 & \text{in} \ B_1^N, \\
u = 0 & \text{in} \ \mathbb{R}^N \setminus B_R^N,
\end{cases}$$

and

$$\|u - f\|_{C^\ell(B_1^N)} < \epsilon.$$  \hfill (1.12)

\textsuperscript{1}In the literature, one often finds also the notion of right Caputo fractional derivative, defined for $t < a$ by

$$\left(\frac{-1}{k}\right) \frac{\Gamma(k-\alpha)}{(a-t)^{\alpha-k+1}} \int_0^{a-t} \partial^k u(\tau) d\tau.$$  \hfill (1.6)

Since the right time-fractional derivative boils down to the left one (by replacing $t$ with $2a - t$), in this paper we focus only on the case of left derivatives.

Also, though there are several time-fractional derivatives that are studied in the literature under different perspectives, we focus here on the Caputo derivative, since it possesses well-posedness properties with respect to classical initial value problems, differently than other time-fractional derivatives, such as the Riemann-Liouville derivative, in which the initial value setting involves data containing derivatives of fractional order.

\textsuperscript{2}For notational simplicity, we will often denote $\partial^k_t u = u^{(k)}$. 

\hfill 5
We observe that the initial points of the Caputo type operators in Theorem 1.1 also depend on $\epsilon$, as detailed in (1.10) (but the other parameters, such as the orders of the operators involved, are fixed arbitrarily).

We also stress that condition (1.9) requires that the operator $\Lambda_a$ contains at least one nonlocal operator among its building blocks in (1.2), (1.3) and (1.7). This condition cannot be avoided, since approximation results in the same spirit of Theorem 1.1 cannot hold for classical differential operators.

Theorem 1.1 comprises, as particular cases, the nonlocal approximation results established in the recent literature of this topic. Indeed, when

\[
\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_{M-1} = \epsilon_1 = \cdots = \epsilon_l = 0, \\
\beta_M = 1, \\
\text{and} \quad s \in (0, 1)
\]

we see that Theorem 1.1 recovers the main result in [DSV17], giving the local density of $s$-harmonic functions vanishing outside a compact set.

Similarly, when

\[
\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_{M} = \epsilon_1 = \cdots = \epsilon_{l-1} = 0, \\
\epsilon_l = 1, \\
\text{and} \quad D_a = D_{t,a}^\alpha, \quad \text{for some } \alpha > 0, \ a < 0
\]

we have that Theorem 1.1 reduces to the main results in [Buc17] for $\alpha \in (0, 1)$ and [CDV18] for $\alpha > 1$, in which such approximation result was established for Caputo-stationary functions, i.e., functions that annihilate the Caputo fractional derivative.

Also, when

\[
p_1 = \cdots = p_n = 1, \\
\epsilon_1 = \cdots = \epsilon_l = 0, \\
\text{and} \quad s_j \in (0, 1), \quad \text{for every } j \in \{1, \ldots, M\},
\]

we have that Theorem 1.1 recovers the cases taken into account in [DSV18], in which approximation results have been established for the superposition of a local operator with a superposition of fractional Laplacians of order $2s_j < 2$.

In this sense, not only Theorem 1.1 comprises the existing literature, but it goes beyond it, since it combines classical derivatives, fractional Laplacians and Caputo fractional derivatives altogether. In addition, it comprises the cases in which the space-fractional Laplacians taken into account are of order greater than 2.

As a matter of fact, this point is also a novelty introduced by Theorem 1.1 here with respect to the previous literature.

Theorem 1.1 was announced in [CDV18], and we have just received the very interesting preprint [Kry18] which also considered the case of different, not necessarily fractional, powers of the Laplacian, using a different and innovative methodology.

The rest of the paper is organized as follows. Sections 2 and 3 focus on time-fractional operators. More precisely, in Sections 2 and 3 we study the boundary behaviour of the
eigenfunctions of the Caputo derivative and of functions with vanishing Caputo derivative, respectively, detecting their singular boundary behaviour in terms of explicit representation formulas. These type of results are also interesting in themselves and can find further applications.

Sections 4–7 are devoted to some properties of the higher order fractional Laplacian. More precisely, Section 4 provides some representation formula of the solution of \((-\Delta)^s u = f\) in a ball, with \(u = 0\) outside this ball, for all \(s > 0\), and extends the Green formula methods introduced in [DG17] and [AJS18b].

Then, in Section 5 we study the boundary behaviour of the first Dirichlet eigenfunction of higher order fractional equations, and in Section 6 we give some precise asymptotics at the boundary for the first Dirichlet eigenfunction of \((-\Delta)^s\) for any \(s > 0\).

Section 7 is devoted to the analysis of the asymptotic behaviour of \(s\)-harmonic functions, with a “spherical bump function” as exterior Dirichlet datum.

Section 8 contains an auxiliary statement, namely Theorem 8.1, which will imply Theorem 1.1. This is technically convenient, since the operator \(\Lambda_a\) depends in principle on the initial point \(a\): this has the disadvantage that if \(\Lambda_a u_a = 0\) and \(\Lambda_b u_b = 0\) in some domain, the function \(u_a + u_b\) is not in principle a solution of any operator, unless \(a = b\). To overcome such a difficulty, in Theorem 8.1 we will reduce to the case in which \(a = -\infty\), exploiting a polynomial extension that we have introduced and used in [CDV18].

In Section 9 we make the main step towards the proof of Theorem 8.1. In this section, we prove that functions in the kernel of nonlocal operators such as the one in (1.8) span with their derivatives a maximal Euclidean space. This fact is special for the nonlocal case and its proof is based on the boundary analysis of the fractional operators in both time and space. Due to the general form of the operator in (1.8), we have to distinguish here several cases, taking advantage of either the time-fractional or the space-fractional components of the operators.

We conclude the paper with Section 10 in which we complete the proof of Theorem 8.1, using the previous approximation results and suitable rescaling arguments.

2 Sharp boundary behaviour for the time-fractional eigenfunctions

In this section we show that the eigenfunctions of the Caputo fractional derivative in (1.6) have an explicit representation via the Mittag-Leffler function. For this, fixed \(\alpha, \beta \in \mathbb{C}\) with \(\Re(\alpha) > 0\), for any \(z\) with \(\Re(z) > 0\), we recall that the Mittag-Leffler function is defined as

\[
E_{\alpha, \beta}(z) := \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}.
\]

The Mittag-Leffler function plays an important role in equations driven by the Caputo derivatives, replacing the exponential function for classical differential equations, as given by the following well-established result (see [GKMR14] and the references therein):

**Lemma 2.1.** Let \(\alpha \in (0, 1]\), \(\lambda \in \mathbb{R}\), and \(a \in \mathbb{R} \cup \{-\infty\}\). Then, the unique solution of the boundary value problem

\[
\begin{cases}
D_t^\alpha u(t) = \lambda u(t) & \text{for any } t \in (a, +\infty), \\
u(a) = 1
\end{cases}
\]
is given by \( E_{\alpha,1} (\lambda (t - a)^\alpha) \).

Lemma 2.1 can be actually generalized\(^3\) to any fractional order of differentiation \( \alpha \):

**Lemma 2.2.** Let \( \alpha \in (0, +\infty), \) with \( \alpha \in (k - 1, k] \) and \( k \in \mathbb{N}, \) \( a \in \mathbb{R} \cup \{-\infty\}, \) and \( \lambda \in \mathbb{R}. \) Then, the unique continuous solution of the boundary value problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
D^\alpha_{t,a} u(t) = \lambda u(t) & \text{for any } t \in (a, +\infty), \\
u(a) = 1, & \\
\partial_t^m u(a) = 0 & \text{for any } m \in \{1, \ldots, k - 1\}
\end{array} \right.
\end{align*}
\]  

(2.2)

is given by \( u(t) = E_{\alpha,1} (\lambda (t - a)^\alpha) \).

**Proof.** For the sake of simplicity we take \( a = 0. \) Also, the case in which \( \alpha \in \mathbb{N} \) can be checked with a direct computation, so we focus on the case \( \alpha \in (k - 1, k), \) with \( k \in \mathbb{N}. \)

We let \( u(t) := E_{\alpha,1} (\lambda t^\alpha). \) It is straightforward to see that \( u(t) = 1 + \mathcal{O}(t^k) \) and therefore
\[
u(0) = 1 \quad \text{and} \quad \partial_t^m u(0) = 0 \quad \text{for any } m \in \{1, \ldots, k - 1\}. \quad (2.3)
\]

We also claim that
\[
D^\alpha_{t,a} u(t) = \lambda u(t) \quad \text{for any } t \in (0, +\infty). \quad (2.4)
\]

To check this, we recall (1.6) and (2.1) (with \( \beta := 1 \)), and we have that
\[
D^\alpha_{t,a} u(t) = \frac{1}{\Gamma(k - \alpha)} \int_0^t \frac{u^{(k)}(\tau)}{(t - \tau)^{\alpha-k+1}} d\tau
\]

(2.6)

\[
= \frac{1}{\Gamma(k - \alpha)} \int_0^t \left( \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j (\alpha j - 1) \ldots (\alpha j - k + 1)}{\Gamma(\alpha j + 1)} \right) \frac{d\tau}{(t - \tau)^{\alpha-k+1}}
\]

\[
= \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j (\alpha j - 1) \ldots (\alpha j - k + 1)}{\Gamma(k - \alpha) \Gamma(\alpha j + 1)} \int_0^t \tau^{\alpha j - k} (t - \tau)^{k-\alpha-1} d\tau. \quad (2.5)
\]

Hence, using the change of variable \( \tau = t\sigma, \) we obtain that
\[
D^\alpha_{t,a} u(t) = \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j (\alpha j - 1) \ldots (\alpha j - k + 1)}{\Gamma(k - \alpha) \Gamma(\alpha j + 1)} \int_0^1 \sigma^{\alpha j - k} (1 - \sigma)^{k-\alpha-1} d\sigma. \quad (2.6)
\]

On the other hand, from the basic properties of the Beta function, it is known that if \( \Re(z), \) \( \Re(w) > 0, \) then
\[
\int_0^1 \sigma^{z-1} (1 - \sigma)^{w-1} d\sigma = \frac{\Gamma(z) \Gamma(w)}{\Gamma(z + w)}. \quad (2.6)
\]

In particular, taking \( z := \alpha j - k + 1 \in (\alpha - k + 1, +\infty) \subseteq (0, +\infty) \) and \( w := k - \alpha \in (0, +\infty), \) and substituting (2.6) into (2.5), we conclude that
\[
D^\alpha_{t,a} u(t) = \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j (\alpha j - 1) \ldots (\alpha j - k + 1)}{\Gamma(k - \alpha) \Gamma(\alpha j + 1)} \frac{\Gamma(\alpha j - k + 1) \Gamma(\alpha j - \alpha + 1)}{\Gamma(k - \alpha) \Gamma(\alpha j + 1)} \int_0^1 \sigma^{\alpha j - k} (1 - \sigma)^{k-\alpha-1} d\sigma.
\]

\[
= \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j (\alpha j - 1) \ldots (\alpha j - k + 1)}{\Gamma(\alpha j + 1)} \frac{\Gamma(\alpha j - k + 1) \Gamma(\alpha j - \alpha + 1)}{\Gamma(\alpha j + 1)} \int_0^1 \sigma^{\alpha j - k} (1 - \sigma)^{k-\alpha-1} d\sigma. \quad (2.7)
\]

\(^3\)It is easily seen that for \( k := 1 \) Lemma 2.2 boils down to Lemma 2.1.
Now we use the fact that \( z \Gamma (z) = \Gamma (z + 1) \) for any \( z \in \mathbb{C} \) with \( \Re (z) > -1 \), so, we have

\[
\alpha j (\alpha j - 1) \ldots (\alpha j - k + 1) \Gamma (\alpha j - k + 1) = \Gamma (\alpha j + 1).
\]

Plugging this information into (2.7), we thereby find that

\[
D^{\alpha}_{t,a}u(t) = \sum_{j=1}^{\infty} \frac{\lambda^j}{\Gamma (\alpha j - \alpha + 1)} t^{\alpha j - \alpha} = \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{\Gamma (\alpha j + 1)} t^{\alpha j} = \lambda u(t).
\]

This proves (2.4).

Then, in view of (2.3) and (2.4) we obtain that \( u \) is a solution of (2.2). Hence, to complete the proof of the desired result, we have to show that such a solution is unique. To this end, supposing that we have two solutions of (2.2), we consider their difference \( w \), and we observe that

\[
\begin{cases}
D^{\alpha}_{t,a}w(t) = \lambda w(t) & \text{for any } t \in (0, +\infty), \\
\partial^m_t w(0) = 0 & \text{for any } m \in \{0, \ldots, k-1\}.
\end{cases}
\]

By Theorem 4.1 in [SZ16], it follows that \( w \) vanishes identically, and this proves the desired uniqueness result.

The boundary behaviour of the Mittag-Leffler function for different values of the fractional parameter \( \alpha \) is depicted in Figure 1. In light of (2.1), we notice in particular that, near \( z = 0 \),

\[
E_{\alpha, \beta} (z) = \frac{1}{\Gamma (\beta)} + \frac{z}{\Gamma (\alpha + \beta)} + O(z^2)
\]

and therefore, near \( t = a \),

\[
E_{\alpha, 1} (\lambda (t-a)^\alpha) = 1 + \frac{\lambda (t-a)^\alpha}{\Gamma (\alpha + 1)} + O(\lambda^2 (t-a)^{2\alpha}).
\]

## 3 Sharp boundary behaviour for the time-fractional harmonic functions

In this section, we detect the optimal boundary behaviour of time-fractional harmonic functions and of their derivatives. The result that we need for our purposes is the following:

**Lemma 3.1.** Let \( \alpha \in (0, +\infty) \setminus \mathbb{N} \). There exists a function \( \psi : \mathbb{R} \to \mathbb{R} \) such that \( \psi \in C^\infty((1, +\infty)) \) and

\[
D^\alpha_0 \psi(t) = 0 \quad \text{for all } t \in (1, +\infty), \quad \text{(3.1)}
\]

and

\[
\lim_{\epsilon \to 0} \epsilon^{-\alpha} \partial^\ell \psi(1 + \epsilon t) = \kappa_{\alpha, \ell} t^{\alpha - \ell}, \quad \text{for all } \ell \in \mathbb{N}, \quad \text{(3.2)}
\]

for some \( \kappa_{\alpha, \ell} \in \mathbb{R} \setminus \{0\} \), where (3.2) is taken in the sense of distribution for \( t \in (0, +\infty) \).
Figure 1: Behaviour of the Mittag-Leffler function $E_{\alpha,1}(t^\alpha)$ near the origin for $\alpha = \frac{1}{100}$, $\alpha = \frac{1}{20}$, $\alpha = \frac{1}{3}$, $\alpha = \frac{2}{3}$, $\alpha = \frac{3}{2}$ and $\alpha = \frac{11}{2}$.

Proof. We use Lemma 2.5 in [CDV18], according to which (see in particular formula (2.16) in [CDV18]) the claim in (3.1) holds true. Furthermore (see formulas (2.19) and (2.20) in [CDV18]), we can write that, for all $t > 1$,

$$
\psi(t) = -\frac{1}{\Gamma(\alpha)\Gamma([\alpha] + 1 - \alpha)} \int_{[1, t] \times [0, 3/4]} \partial^{[\alpha]+1}\psi_0(\sigma)(\tau - \sigma)^{[\alpha]-\alpha}(t - \tau)^{\alpha-1} d\tau d\sigma,
$$

(3.3)

for a suitable $\psi_0 \in C^{[\alpha]+1}([0, 1])$.

In addition, by Lemma 2.6 in [CDV18], we can write that

$$
\lim_{\epsilon \searrow 0} \epsilon^{-\alpha}\psi(1 + \epsilon) = \kappa,
$$

(3.4)

for some $\kappa \neq 0$. Now we set

$$(0, +\infty) \ni t \mapsto f_\epsilon(t) := \epsilon^{\ell - \alpha} \partial^\ell \psi(1 + \epsilon t).$$

We observe that, for any $\varphi \in C_0^\infty((0, +\infty))$,

$$
\int_0^{+\infty} f_\epsilon(t) \varphi(t) dt = \epsilon^{\ell - \alpha} \int_0^{+\infty} \partial^\ell \psi(1 + \epsilon t) \varphi(t) dt
$$

$$
= \epsilon^{-\alpha} \int_0^{+\infty} \frac{d^\ell}{dt^\ell}(\psi(1 + \epsilon t)) \varphi(t) dt
= (-1)^\ell \epsilon^{\ell - \alpha} \int_0^{+\infty} \psi(1 + \epsilon t) \partial^\ell \varphi(t) dt.
$$

(3.5)
Also, in view of (3.3),
\[
\epsilon^{-\alpha}\psi(1 + \epsilon t) \leq C\epsilon^{-\alpha}\int_{[1,1+\epsilon]} (1 + \epsilon t - \tau)^{\alpha-1} d\tau
\]

which is locally bounded in \(t\), where \(C > 0\) here above may vary from line to line.

As a consequence, we can pass to the limit in (3.5) and obtain that
\[
\lim_{\epsilon \downarrow 0} \int_0^{+\infty} f_\epsilon(t) \varphi(t) dt = (-1)^\ell \int_0^{+\infty} \epsilon^{-\alpha}\psi(1 + \epsilon t) \partial^\ell \varphi(t) dt.
\]

This and (3.4) give that
\[
\lim_{\epsilon \downarrow 0} \int_0^{+\infty} f_\epsilon(t) \varphi(t) dt = (-1)^\ell \int_0^{+\infty} t^\alpha \partial^\ell \varphi(t) dt = \kappa \alpha \ldots (\alpha - \ell + 1) \int_0^{+\infty} t^{\alpha-\ell} \varphi(t) dt,
\]

which establishes (3.2).

4 Green representation formulas and solution of \((-\Delta)^s u = f\) in \(B_1\) with homogeneous Dirichlet datum

Our goal is to provide some representation results on the solution of \((-\Delta)^s u = f\) in a ball, with \(u = 0\) outside this ball, for all \(s > 0\). Our approach is an extension of the Green formula methods introduced in [DG17] and [AJS18b]: differently from the previous literature, we are not assuming here that \(f\) is regular in the whole of the ball, but merely that it is Hölder continuous near the boundary and sufficiently integrable inside. Given the type of singularity of the Green function, these assumptions are sufficient to obtain meaningful representations, which in turn will be useful to deal with the eigenfunction problem in the subsequent Section 5.

4.1 Solving \((-\Delta)^s u = f\) in \(B_1\) for discontinuous \(f\) vanishing near \(\partial B_1\)

In this subsection, we want to extend the representation results of [DG17] and [AJS18b] to the case in which the right hand side is not Hölder continuous, but merely in a Lebesgue space, but it has the additional property of vanishing near the boundary of the domain. To this end, fixed \(s > 0\), we consider the polyharmonic Green function in \(B_1 \subset \mathbb{R}^n\), given, for every \(x \neq y \in \mathbb{R}^n\), by

\[
\mathcal{G}_s(x,y) := \frac{k(n, s)}{|x - y|^{n-2s}} \int_0^{r_0(x,y)} \frac{\eta^{s-1}}{(\eta + 1)^{n/2}} d\eta,
\]

where \(r_0(x,y) := \left(\frac{1 - |x|^2}{|x - y|^2} + \frac{1 - |y|^2}{|x - y|^2}\right)^{1/2}\),

\[
k(n, s) := \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}} 4\pi^2(s)},
\]

\[
(4.1)
\]
Given \( x \in B_1 \), we also set
\[
d(x) := 1 - |x|. \tag{4.2}
\]
In this setting, the main result of this subsection is the following:

**Proposition 4.1.** Let \( r \in (0, 1) \) and \( f \in L^2(B_1) \), with \( f = 0 \) in \( \mathbb{R}^n \setminus B_r \). Let
\[
u(x) := \begin{cases} 
\int_{B_1} G_s(x, y) f(y) \, dy & \text{if } x \in B_1, \\
0 & \text{if } x \in \mathbb{R}^n \setminus B_1.
\end{cases} \tag{4.3}
\]
Then:
\[
u \in L^1(B_1), \quad \|\nu\|_{L^1(B_1)} \leq C \|f\|_{L^1(B_1)}, \tag{4.4}
\]
for every \( R \in (r, 1) \), \( \sup_{x \in B_1 \setminus B_R} d^{-s}(x) |\nu(x)| \leq C_R \|f\|_{L^1(B_1)}, \tag{4.5}
\)
u satisfies \((-\Delta)^s u = f\) in \( B_1 \) in the sense of distributions, \tag{4.6}
and
\[
u \in W^{2s, 2}_{loc}(B_1). \tag{4.7}
\]
Here above, \( C > 0 \) is a constant depending on \( n, s \) and \( r \), \( C_R > 0 \) is a constant depending on \( n, s, r \) and \( R \), and \( C_\rho > 0 \) is a constant depending on \( n, s, r \) and \( \rho \).

When \( f \in C^\alpha(B_1) \) for some \( \alpha \in (0, 1) \), Proposition 4.1 boils down to the main results of [DG17] and [AJS18b].

**Proof of Proposition 4.1.** We recall the following useful estimate, see Lemma 3.3 in [AJS18b]:
for any \( \epsilon \in (0, \min\{n, s\}) \), and any \( \tilde{R}, \tilde{r} > 0 \),
\[
\frac{1}{\tilde{R}^{n-2s}} \int_0^{\tilde{r}/\tilde{R}^2} \frac{\eta^{s-1}}{(\eta + 1)^2} \, d\eta \leq \frac{2}{s} \frac{\tilde{r}^{s-(\epsilon/2)}}{\tilde{R}^{n-\epsilon}},
\]
and so, by (4.1) and (4.2), for every \( x, y \in B_1 \),
\[
G_s(x, y) \leq C \frac{d^{s-(\epsilon/2)}(x) d^{s-(\epsilon/2)}(y)}{|x - y|^{n-\epsilon}}
\]
for some \( C > 0 \). Hence, recalling (4.3),
\[
\int_{B_1} |\nu(x)| \, dx \leq \int_{B_1} \left( \int_{B_1} G_s(x, y) |f(y)| \, dy \right) \, dx \\
\leq C \int_{B_1} \left( \int_{B_1} \frac{|f(y)|}{|x - y|^{n-\epsilon}} \, dy \right) \, dx \\
= C \int_{B_1} \left( \int_{B_1} \frac{|f(y)|}{|x - y|^{n-\epsilon}} \, dx \right) \, dy \\
= C \int_{B_1} |f(y)| \, dy,
\]
up to renaming \( C > 0 \) line after line, and this proves (4.4).
Now, if \( x \in B_1 \setminus B_R \) and \( y \in B_r \), with \( 0 < r < R < 1 \), we have that
\[
|x - y| \geq |x| - |y| \geq R - r
\]
and accordingly
\[
r_0(x, y) \leq \frac{2d(x)}{(R - r)^2},
\]
which in turn implies that
\[
\mathcal{G}_s(x, y) \leq \frac{k(n, s)}{|x - y|^{n-2s}} \int_0^{2d(x)/(R - r)^2} \frac{\eta^{s-1}}{(\eta + 1)^2} \, d\eta, \leq C d^s(x),
\]
for some \( C > 0 \). As a consequence, since \( f \) vanishes outside \( B_r \), we see that, for any \( x \in B_1 \setminus B_R \),
\[
|u(x)| \leq \int_{B_r} \mathcal{G}_s(x, y) \, |f(y)| \, dy \leq C d^s(x) \int_{B_r} |f(y)| \, dy,
\]
which proves (4.5).

Now, we fix \( \tilde{r} \in (r, 1) \) and consider a mollification of \( f \), that we denote by \( f_j \in C^\infty_0(B_\tilde{r}) \), with \( f_j \to f \) in \( L^2(B_1) \) as \( j \to +\infty \). We also write \( \mathcal{G}_s \ast f \) as a short notation for the right hand side of (4.3). Then, by [DG17] and [AJS18b], we know that \( u_j := \mathcal{G}_s \ast f_j \) is a (locally smooth, hence distributional) solution of \( (-\Delta)^s u_j = f_j \). Furthermore, if we set \( \tilde{u}_j := u_j - u \) and \( \tilde{f}_j := f_j - f \) we have that
\[
\tilde{u}_j = \mathcal{G}_s \ast (f_j - f) = \mathcal{G}_s \ast \tilde{f}_j,
\]
and therefore, by (4.4),
\[
\|\tilde{u}_j\|_{L^1(B_1)} \leq C \|\tilde{f}_j\|_{L^1(B_1)},
\]
which is infinitesimal as \( j \to +\infty \). This says that \( u_j \to u \) in \( L^1(B_1) \) as \( j \to +\infty \), and consequently, for any \( \varphi \in C^\infty_0(B_1) \),
\[
\int_{B_1} u(x) (-\Delta)^s \varphi(x) \, dx = \lim_{j \to +\infty} \int_{B_1} u_j(x) (-\Delta)^s \varphi(x) \, dx
\]
\[
= \lim_{j \to +\infty} \int_{B_1} f_j(x) \varphi(x) \, dx = \int_{B_1} f(x) \varphi(x) \, dx,
\]
thus completing the proof of (4.6).

Now, to prove (4.7), we can suppose that \( s \in (0, +\infty) \setminus \mathbb{N} \), since the case of integer \( s \) is classical, see e.g. [GT01]. First of all, we claim that
\[
(4.7) \text{ holds true for every } s \in (0, 1).
\]
(4.8)

For this, we first claim that if \( g \in C^\infty(B_1) \) and \( v \) is a (locally smooth) solution of \( (-\Delta)^s v = g \) in \( B_1 \), with \( v = 0 \) outside \( B_1 \), then \( v \in W^{2s, 2}_{\text{loc}}(B_1) \), and, for any \( \rho \in (0, 1) \),
\[
\|v\|_{W^{2s, 2}(B_{\rho})} \leq C_\rho \|g\|_{L^2(B_1)}.
\]
(4.9)

This claim can be seen as a localization of Lemma 3.1 of [DK12], or a quantification of the last claim in Theorem 1.3 of [BWZ17]. To prove (4.9), we let \( R_- < R_+ \in (\rho, 1) \), and
From this and (4.11), we obtain (4.9), as desired. We let \( v^* := \eta \), and we recall formulas (3.2), (3.3) and (A.5) in [BWZ17], according to which

\[
(-\Delta)^s v^* - \eta (-\Delta)^s v = g^* \quad \text{in } \mathbb{R}^n,
\]

with

\[
\| g^* \|_{L^2(\mathbb{R}^n)} \leq C \| v \|_{W^{s,2}(\mathbb{R}^n)},
\]

for some \( C > 0 \).

Moreover, using a notation taken from [BWZ17] we denote by \( W_0^{s,2}(B_1) \) the space of functions in \( W^{s,2}(\mathbb{R}^n) \) vanishing outside \( B_1 \) and we consider the dual space \( W_0^{-s,2}(B_1) \). We remark that if \( h \in L^2(B_1) \) we can naturally identify \( h \) as an element of \( W_0^{-s,2}(B_1) \) by considering the action of \( h \) on any \( \varphi \in W_0^{s,2}(B_1) \) as defined by

\[
\int_{B_1} h(x) \varphi(x) \, dx.
\]

With respect to this, we have that

\[
\| h \|_{W_0^{-s,2}(B_1)} = \sup_{\| \varphi \|_{W_0^{s,2}(B_1)} = 1} \int_{B_1} h(x) \varphi(x) \, dx \leq \| h \|_{L^2(B_1)}. \tag{4.10}
\]

We notice also that

\[
\| v \|_{W^{s,2}(\mathbb{R}^n)} \leq C \| g \|_{W_0^{-s,2}(B_1)},
\]

in light of Proposition 2.1 of [BWZ17]. This and (4.10) give that

\[
\| v \|_{W^{s,2}(\mathbb{R}^n)} \leq C \| g \|_{L^2(B_1)}.
\]

Then, by Lemma 3.1 of [DK12] (see in particular formula (3.2) there, applied here with \( \lambda := 0 \), we obtain that

\[
\| v^* \|_{W^{2s,2}(\mathbb{R}^n)} \leq C \| \eta (-\Delta)^s v + g^* \|_{L^2(\mathbb{R}^n)}
\]

\[
\leq C \left( \| (-\Delta)^s v \|_{L^2(B_{R_1})} + \| g^* \|_{L^2(\mathbb{R}^n)} \right)
\]

\[
= C \left( \| g \|_{L^2(B_{R_1})} + \| g^* \|_{L^2(\mathbb{R}^n)} \right)
\]

\[
\leq C \| v \|_{W^{s,2}(\mathbb{R}^n)}.
\]

up to renaming \( C > 0 \) step by step. On the other hand, since \( v^* = v \) in \( B_\rho \),

\[
\| v \|_{W^{2s,2}(B_\rho)} = \| v^* \|_{W^{2s,2}(B_\rho)} \leq \| v^* \|_{W^{s,2}(\mathbb{R}^n)}.
\]

From this and (4.11), we obtain (4.9), as desired.

Now, we let \( f_j, \tilde{f}_j, u_j \) and \( \tilde{u}_j \) as above and make use of (4.9) to write

\[
\| u_j \|_{W^{2s,2}(B_\rho)} \leq C_\rho \| f_j \|_{L^2(B_1)}
\]

and

\[
\| \tilde{u}_j \|_{W^{2s,2}(B_\rho)} \leq C_\rho \| \tilde{f}_j \|_{L^2(B_1)}. \tag{4.12}
\]

As a consequence, taking the limit as \( j \to +\infty \) we obtain that

\[
\| u \|_{W^{2s,2}(B_\rho)} \leq C_\rho \| f \|_{L^2(B_1)}.
\]
that is (4.7) in this case, namely the claim in (4.8).

Now, to prove (4.7), we argue by induction on the integer part of $s$. When the integer part of $s$ is zero, the basis of the induction is warranted by (4.8). Then, to perform the inductive step, given $s \in (0, +\infty) \setminus \mathbb{N}$, we suppose that (4.7) holds true for $s - 1$, namely

$$G_{s-1} * f \in W_{loc}^{2s-2,2}(B_1). \quad (4.13)$$

Then, following [AJS18b], it is convenient to introduce the notation

$$[x, y] := \sqrt{|x|^2|y|^2 - 2x \cdot y + 1}$$

and consider the auxiliary kernel given, for every $x \neq y \in B_1$, by

$$P_{s-1}(x, y) := \frac{(1 - |x|^2)^{s-2}(1 - |y|^2)^{s-1}(1 - |x|^2|y|^2)}{|x, y|^n}. \quad (4.14)$$

We point out that if $x \in B_r$ with $r \in (0, 1)$, then

$$|x, y|^2 \geq |x|^2|y|^2 - 2|x||y| + 1 = (1 - |x||y|)^2 \geq (1 - r)^2 > 0. \quad (4.15)$$

Consequently, since $f$ is supported in $B_r$,

$$P_{s-1} * f \in C^\infty(\mathbb{R}^n). \quad (4.16)$$

Then, we recall that

$$-\Delta_s G_s(x, y) = G_{s-1}(x, y) - CP_{s-1}(x, y), \quad (4.17)$$

for some $C \in \mathbb{R}$, see Lemma 3.1 in [AJS18b].

As a consequence, in view of (4.13), (4.16), (4.17), we conclude that

$$-\Delta(G_s * f) = (-\Delta_s G_s) * f \in W_{loc}^{2s-2,2}(B_1).$$

This and the classical elliptic regularity theory (see e.g. [GT01]) give that $G_s * f \in W_{loc}^{2s,2}(B_1)$, which completes the inductive proof and establishes (4.7).

### 4.2 Solving $(-\Delta)^s u = f$ in $B_1$ for $f$ Hölder continuous near $\partial B_1$

The goal of this subsection is to extend the representation results of [DG17] and [AJS18b] to the case in which the right hand side is not Hölder continuous in the whole of the ball, but merely in a neighborhood of the boundary. This result is obtained here by superposing the results in [DG17] and [AJS18b] with Proposition 4.1 here, taking advantage of the linear structure of the problem.

**Proposition 4.2.** Let $f \in L^2(B_1)$. Let $\alpha, r \in (0, 1)$ and assume that

$$f \in C^\alpha(B_1 \setminus B_r). \quad (4.18)$$

In the notation of (4.1), let

$$u(x) := \begin{cases} \int_{B_1} G_s(x, y) f(y) \, dy & \text{if } x \in B_1, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_1. \end{cases} \quad (4.19)$$

This and the classical elliptic regularity theory (see e.g. [GT01]) give that $G_s * f \in W_{loc}^{2s,2}(B_1)$, which completes the inductive proof and establishes (4.7).
Then, in the notation of (4.2), we have that:

\[
\text{for every } R \in (r, 1), \quad \sup_{x \in B_1 \setminus B_R} d^{-s}(x) |u(x)| \leq C_R \left( \|f\|_{L^1(B_1)} + \|f\|_{L^\infty(B_1 \setminus B_n)} \right),
\]

(4.20)

\[ u \text{ satisfies } (-\Delta)^s u = f \text{ in } B_1 \text{ in the sense of distributions,} \quad (4.21) \]

and

\[ u \in W_{\text{loc}}^{2s,2}(B_1). \quad (4.22) \]

Here above, \( C > 0 \) is a constant depending on \( n, s \) and \( r \), \( C_R > 0 \) is a constant depending on \( n, s, r \) and \( R \) and \( C_\rho > 0 \) is a constant depending on \( n, s, r \) and \( \rho \).

Proof. We take \( r_1 \in (r, 1) \) and \( \eta \in C_0^\infty(B_{r_1}) \) with \( \eta = 1 \) in \( B_r \). Let also

\[ f_1 := f \eta \quad \text{and} \quad f_2 := f - f_1. \]

We observe that \( f_1 \in L^2(B_1) \), and that \( f_1 = 0 \) outside \( B_{r_1} \). Therefore, we are in the position of applying Proposition 4.1 and find a function \( u_1 \) (obtained by convolving \( G_s \) against \( f_1 \)) such that

\[
\text{for every } R \in (r_1, 1), \quad \sup_{x \in B_1 \setminus B_R} d^{-s}(x) |u_1(x)| \leq C_R \|f_1\|_{L^1(B_1)},
\]

(4.23)

\[ u_1 \text{ satisfies } (-\Delta)^s u_1 = f_1 \text{ in } B_1 \text{ in the sense of distributions,} \quad (4.24) \]

and

\[ u_1 \in W_{\text{loc}}^{2s,2}(B_1). \quad (4.25) \]

On the other hand, we have that \( f_2 = f(1 - \eta) \) vanishes outside \( B_1 \setminus B_r \) and it is Hölder continuous. Accordingly, we can apply Theorem 1.1 of [AJS18b] and find a function \( u_2 \) (obtained by convolving \( G_s \) against \( f_2 \)) such that

\[
\text{for every } R \in (r_1, 1), \quad \sup_{x \in B_1 \setminus B_R} d^{-s}(x) |u_2(x)| \leq C_R \|f_2\|_{L^\infty(B_1)},
\]

(4.26)

\[ u_2 \text{ satisfies } (-\Delta)^s u_2 = f_2 \text{ in } B_1 \text{ in the sense of distributions,} \quad (4.27) \]

and

\[ u_2 \in C_{\text{loc}}^{2s+\alpha}(B_1). \quad (4.28) \]

Then, \( f = f_1 + f_2 \), and thus, in view of (4.19), we have that \( u = u_1 + u_2 \). Also, \( u \) satisfies (4.20), thanks to (4.23) and (4.26), (4.21), thanks to (4.24) and (4.27), and (4.22), thanks to (4.25) and (4.28). \( \square \)

5 Existence and regularity for the first eigenfunction of the higher order fractional Laplacian

The goal of these pages is to study the boundary behaviour of the first Dirichlet eigenfunction of higher order fractional equations.

For this, writing \( s = m + \sigma \), with \( m \in \mathbb{N} \) and \( \sigma \in (0, 1) \), we define the energy space

\[ H_0^s(B_1) := \{ u \in H^s(\mathbb{R}^n) ; u = 0 \text{ in } \mathbb{R}^n \setminus B_1 \}, \]

(5.1)

endowed with the Hilbert norm

\[
\|u\|_{H_0^s(B_1)} := \left( \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2(B_1)}^2 + \mathcal{E}_s(u, u) \right)^{\frac{1}{2}},
\]

(5.2)
where
\[ E_s(u,v) = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}u(\xi) \overline{\mathcal{F}v}(\xi) \, d\xi, \] (5.3)

being \( \mathcal{F} \) the Fourier transform and using the notation \( \overline{z} \) to denote the complex conjugated of a complex number \( z \).

In this setting, we consider \( u \in H^s_0(B_1) \) to be such that
\[
\begin{cases}
(-\Delta)^s u = \lambda_1 u & \text{in } B_1, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \overline{B_1},
\end{cases}
\] (5.4)

for every \( s > 0 \), with \( \lambda_1 \) as small as possible.

The existence of solutions of (5.4) is ensured via variational techniques, as stated in the following result:

**Lemma 5.1.** The functional \( E_s(u,u) \) attains its minimum \( \lambda_1 \) on the functions in \( H^s_0(B_1) \) with unit norm in \( L^2(B_1) \).

The minimizer satisfies (5.4).

In addition, \( \lambda_1 > 0 \).

**Proof.** The proof is based on the direct method in the calculus of variations. We provide some details for completeness. Let \( s = m + \sigma \), with \( m \in \mathbb{N} \) and \( \sigma \in (0,1) \). Let us consider a minimizing sequence \( u_j \in H^s_0(B_1) \subseteq H^m(\mathbb{R}^n) \) such that \( \|u_j\|_{L^2(B_1)} = 1 \) and
\[ \lim_{j \to +\infty} E_s(u_j,u_j) = \inf_{u \in H^s_0(B_1), \|u\|_{L^2(B_1)} = 1} E_s(u,u) . \]

In particular, we have that \( u_j \) is bounded in \( H^s_0(B_1) \) uniformly in \( j \), so, up to a subsequence, it converges to some \( u_* \) weakly in \( H^s_0(B_1) \) and strongly in \( L^2(B_1) \) as \( j \to +\infty \).

The weak lower semicontinuity of the seminorm \( E_s(\cdot,\cdot) \) then implies that \( u_* \) is the desired minimizer.

Then, given \( \phi \in C_0^\infty(B_1) \), we have that
\[ E_s(u_* + \epsilon \phi, u_* + \epsilon \phi) \geq E_s(u_*, u_*) , \]
for every \( \epsilon \in \mathbb{R} \), and this gives that (5.4) is satisfied in the sense of distributions, and also in the classical sense by the elliptic regularity theory.

Finally, we have that \( E_s(u_*, u_*) > 0 \), since \( u_* \) (and thus \( \mathcal{F}u_* \)) does not vanish identically. Consequently,
\[ \lambda_1 = \frac{E_s(u_*, u_*)}{\|u_*\|^2_{L^2(B_1)}} = E_s(u_*, u_*) > 0 , \]

as desired.

Our goal is now to apply Proposition 4.2 to solutions of (5.4), taking \( f := \lambda u \). To this end, we have to check that condition (4.18) is satisfied, namely that solutions of (5.4) are Hölder continuous in \( B_1 \setminus B_r \), for any \( 0 < r < 1 \).

To this aim, we prove that polyharmonic operators of any order \( s > 0 \) always admit a first eigenfunction in the ball which does not change sign and which is radially symmetric. For this, we start discussing the sign property:
Lemma 5.2. There exists a nontrivial solution of (5.4) that does not change sign.

Proof. We exploit a method explained in details in Section 3.1 of [GGS10]. As a matter of fact, when $s \in \mathbb{N}$, the desired result is exactly Theorem 3.7 in [GGS10].

Let $u$ be as in Lemma 5.1. If either $u \geq 0$ or $u \leq 0$, then the desired result is proved. Hence, we argue by contradiction, assuming that $u$ attains strictly positive and strictly negative values. We define

$$
\mathcal{K} := \{w : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } \mathcal{E}_s(w, w) < +\infty, \text{ and } w \geq 0 \text{ in } B_1\}.
$$

Also, we set

$$
\mathcal{K}^* := \{w \in H^s_0(B_1) \text{ s.t. } \mathcal{E}_s(w, v) \leq 0 \text{ for all } v \in \mathcal{K}\}.
$$

We claim that

$$
\text{if } w \in \mathcal{K}^*, \text{ then } w \leq 0. \tag{5.5}
$$

To prove this, we recall the notation in (4.1), take $\phi \in C_0^\infty(B_1) \cap \mathcal{K}$, and let

$$
v_\phi(x) := \begin{cases} 
\int_{B_1} G_s(x, y) \phi(y) \, dy & \text{if } x \in B_1, \\
0 & \text{if } x \in \mathbb{R}^n \setminus B_1.
\end{cases}
$$

Then $v_\phi \in \mathcal{K}$ and it satisfies $(-\Delta)^s v_\phi = \phi$ in $B_1$, thanks to [DG17] or [AJS18b]. Consequently, we can write, for every $x \in B_1$,

$$
\phi(x) = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} v_\phi)(x).
$$

Hence, for every $w \in \mathcal{K}^*$,

$$
0 \geq \mathcal{E}_s(w, v_\phi) = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F} v_\phi(\xi) \mathcal{F} w(\xi) \, d\xi = \int_{\mathbb{R}^n} \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} v_\phi)(x) w(x) \, dx = \int_{B_1} \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} v_\phi)(x) w(x) \, dx = \int_{B_1} \phi(x) w(x) \, dx.
$$

Since $\phi$ is arbitrary and nonnegative, this gives that $w \leq 0$, and this establishes (5.5).

Furthermore, by Theorem 3.4 in [GGS10], we can write

$$
u = u_1 + u_2,
$$

with $u_1 \in \mathcal{K} \setminus \{0\}$, $u_2 \in \mathcal{K}^* \setminus \{0\}$, and $\mathcal{E}_s(u_1, u_2) = 0$.

We observe that

$$
\mathcal{E}_s(u_1 - u_2, u_1 - u_2) = \mathcal{E}_s(u_1, u_1) + \mathcal{E}_s(u_2, u_2) + 2\mathcal{E}_s(u_1, u_2) = \mathcal{E}_s(u_1, u_1) + \mathcal{E}_s(u_2, u_2).
$$

In the same way,

$$
\mathcal{E}_s(u, u) = \mathcal{E}_s(u_1 + u_2, u_1 + u_2) = \mathcal{E}_s(u_1, u_1) + \mathcal{E}_s(u_2, u_2),
$$

18
and therefore
\[ E_s(u_1 - u_2, u_1 - u_2) = E_s(u, u). \] (5.6)

On the other hand,
\[ \|u_1 - u_2\|^2_{L^2(B_1)} - \|u\|^2_{L^2(B_1)} = \|u_1 - u_2\|^2_{L^2(B_1)} - \|u_1 + u_2\|^2_{L^2(B_1)} = -4 \int_{B_1} u_1(x) u_2(x) \, dx. \]

As a consequence, since \( u_2 \leq 0 \) in view of (5.5), we conclude that
\[ \|u_1 - u_2\|^2_{L^2(B_1)} - \|u\|^2_{L^2(B_1)} \geq 0. \]

This and (5.6) say that the function \( u_1 - u_2 \) is also a minimizer for the variational problem in Lemma 5.1. Since now \( u_1 - u_2 \geq 0 \), the desired result follows.

Now, we define the spherical mean of a function \( v \) by
\[ v_\#(x) := \frac{1}{|S^{n-1}|} \int_{S^{n-1}} v(R_\omega x) \, d\mathcal{H}^{n-1}(\omega) \]
where \( R_\omega \) is the rotation corresponding to the solid angle \( \omega \in S^{n-1} \), \( \mathcal{H}^{n-1} \) is the standard Hausdorff measure, and \( |S^{n-1}| = \mathcal{H}^{n-1}(S^{n-1}) \). Notice that \( v_\#(x) = v_\#(R_\omega x) \) for any \( \omega \in S^{n-1} \), that is \( v_\# \) is rotationally invariant.

Then, we have:

**Lemma 5.3.** Any positive power of the Laplacian commutes with the spherical mean, that is
\[((-\Delta)^s v)_\#(x) = (-\Delta)^s v_\#(x).\]

**Proof.** By density, we prove the claim for a function \( v \) in the Schwartz space of smooth and rapidly decreasing functions. In this setting, writing \( R_\omega^T \) to denote the transpose of the rotation \( R_\omega \), and changing variable \( \eta := R_\omega^T \xi \), we have that
\[ (-\Delta)^s v(R_\omega x) = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F} v(\xi) e^{2\pi i R_\omega x \cdot \xi} \, d\xi \]
\[ = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F} v(\xi) e^{2\pi i x \cdot R_\omega^T \xi} \, d\xi \]
\[ = \int_{\mathbb{R}^n} |\eta|^{2s} \mathcal{F} v(R_\omega \eta) e^{2\pi i x \cdot \eta} \, d\eta. \] (5.7)

On the other hand, using the substitution \( y := R_\omega^T x \),
\[ \mathcal{F} v(R_\omega \eta) = \int_{\mathbb{R}^n} v(x) e^{-2\pi i x \cdot R_\omega \eta} \, dx \]
\[ = \int_{\mathbb{R}^n} v(x) e^{-2\pi i R_\omega^T y \cdot \eta} \, dx \]
\[ = \int_{\mathbb{R}^n} v(R_\omega y) e^{-2\pi iy \cdot \eta} \, dy, \]

19
and therefore, recalling \((5.7)\),
\[
(\Delta^s v(\mathcal{R}_\omega x) = \int \int_{\mathbb{R}^n \times \mathbb{R}^n} |\eta|^{2s} v(\mathcal{R}_\omega y) e^{2\pi i (x-y) \cdot \eta} dy \, d\eta.
\]

As a consequence,
\[
\begin{align*}
\left( (\Delta^s v) \right)_1 (x) &= \frac{1}{|S_{n-1}|} \int_{S_{n-1}} (\Delta^s v(\mathcal{R}_\omega x) \, d\mathcal{H}^{n-1}(\omega) \\
&= \frac{1}{|S_{n-1}|} \int \int_{\mathbb{R}^n \times \mathbb{R}^n} |\eta|^{2s} v(\mathcal{R}_\omega y) e^{2\pi i (x-y) \cdot \eta} d\mathcal{H}^{n-1}(\omega) \, dy \, d\eta \\
&= \int_{\mathbb{R}^n} |\eta|^{2s} \mathcal{F}(v_\xi)(\eta) e^{2\pi i x \cdot \eta} \, d\eta \\
&= (\Delta^s v_\xi(x),
\end{align*}
\]
as desired.

It is also useful to observe that the spherical mean is compatible with the energy bounds. In particular we have the following observation:

**Lemma 5.4.** We have that
\[
\mathcal{E}_s (v_1, v_2) \leq \mathcal{E}_s (v, v).
\]
Moreover,
\[
\text{if } v \in H_0^1(B_1), \text{ then so does } v_\xi.
\]

**Proof.** We see that
\[
\mathcal{F}(v_\xi)(\xi) = \int_{\mathbb{R}^n} v_\xi(x) e^{-2\pi i x \cdot \xi} \, dx = \frac{1}{|S_{n-1}|} \int_{S_{n-1} \times \mathbb{R}^n} v(\mathcal{R}_\omega x) e^{-2\pi i x \cdot \xi} \, d\mathcal{H}^{n-1}(\omega) \, dx
\]
and therefore, taking the complex conjugated,
\[
\overline{\mathcal{F}(v_\xi)(\xi)} = \frac{1}{|S_{n-1}|} \int_{S_{n-1} \times \mathbb{R}^n} v(\mathcal{R}_\omega x) e^{2\pi i x \cdot \xi} \, d\mathcal{H}^{n-1}(\omega) \, dx.
\]
Hence, by \((5.3)\), and exploiting the changes of variables \(y := \mathcal{R}_\omega x \) and \(\tilde{y} := \mathcal{R}_\omega \tilde{x} \),
\[
\begin{align*}
\mathcal{E}_s (v_\xi, v_\xi) &= \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}(v_\xi)(\xi) \overline{\mathcal{F}(v_\xi)(\xi)} \, d\xi \\
&= \frac{1}{|S_{n-1}|^2} \int \int_{S_{n-1} \times S_{n-1} \times \mathbb{R}^n \times \mathbb{R}^n} |\xi|^{2s} v(\mathcal{R}_\omega x) v(\mathcal{R}_\omega \tilde{x}) e^{2\pi i (\tilde{x} - x) \cdot \xi} \, d\mathcal{H}^{n-1}(\omega) \, d\mathcal{H}^{n-1}(\tilde{\omega}) \, dx \, d\tilde{x} \, d\xi \\
&= \frac{1}{|S_{n-1}|^2} \int \int_{S_{n-1} \times S_{n-1} \times \mathbb{R}^n \times \mathbb{R}^n} |\xi|^{2s} v(y) \overline{v(\tilde{y})} e^{2\pi i y \cdot \mathcal{R}_\omega \xi} e^{-2\pi i y \cdot \mathcal{R}_\omega \xi} \, d\mathcal{H}^{n-1}(\omega) \, d\mathcal{H}^{n-1}(\tilde{\omega}) \, dy \, d\tilde{y} \, d\xi
\end{align*}
\]
\[
\frac{1}{|\mathbb{S}^{n-1}|^2} \iiint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n} |\xi|^{2s} \mathcal{F} v(R_{\omega} \xi) \mathcal{F} \bar{v}(\mathcal{R}_{\omega} \xi) \ d\mathcal{H}^{n-1}(\omega) \ d\mathcal{H}^{n-1}(\bar{\omega}) \ d\xi.
\]

Consequently, using the Cauchy-Schwarz Inequality, and the substitutions \( \eta := R_{\omega} \xi \) and \( \tilde{\eta} := \mathcal{R}_{\omega} \xi \),
\[
\mathcal{E}_s(v_1, v_2) \leq \frac{1}{|\mathbb{S}^{n-1}|^2} \left( \iiint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n} |\xi|^{2s} |\mathcal{F} v(R_{\omega} \xi)|^2 \ d\mathcal{H}^{n-1}(\omega) \ d\mathcal{H}^{n-1}(\bar{\omega}) \ d\xi \right)^{\frac{1}{2}}
\]
\[
\cdot \left( \iiint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n} |\xi|^{2s} |\mathcal{F} \mathcal{R}_{\omega} \xi|^2 \ d\mathcal{H}^{n-1}(\omega) \ d\mathcal{H}^{n-1}(\bar{\omega}) \ d\xi \right)^{\frac{1}{2}}
\]
\[
= \frac{1}{|\mathbb{S}^{n-1}|^2} \left( \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n} |\eta|^{2s} |\mathcal{F} v(\eta)|^2 \ d\mathcal{H}^{n-1}(\omega) \ d\mathcal{H}^{n-1}(\bar{\omega}) \ d\eta \right)^{\frac{1}{2}}
\]
\[
\cdot \left( \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{R}^n} |\tilde{\eta}|^{2s} |\mathcal{F} \tilde{v}(\tilde{\eta})|^2 \ d\mathcal{H}^{n-1}(\omega) \ d\mathcal{H}^{n-1}(\bar{\omega}) \ d\tilde{\eta} \right)^{\frac{1}{2}}
\]
\[
= \left( \int_{\mathbb{R}^n} |\eta|^{2s} |\mathcal{F} v(\eta)|^2 \ d\eta \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |\tilde{\eta}|^{2s} |\mathcal{F} \tilde{v}(\tilde{\eta})|^2 \ d\tilde{\eta} \right)^{\frac{1}{2}}
\]
\[
= \mathcal{E}_s(v, v).
\]

This proves (5.8).

Now, we prove (5.9). For this, we observe that
\[
\frac{\partial^\ell v_2}{\partial x_{j_1} \ldots \partial x_{j_\ell}}(x) = \frac{1}{|\mathbb{S}^{n-1}|} \sum_{k_1, \ldots, k_\ell=1}^n \int_{\mathbb{S}^{n-1}} \frac{\partial^\ell v}{\partial x_{k_1} \ldots \partial x_{k_\ell}}(R_{\omega} x) \ R_{\omega}^{k_j j_1} \ldots \ R_{\omega}^{k_j j_\ell} \ d\mathcal{H}^{n-1}(\omega),
\]
for every \( \ell \in \mathbb{N} \) and \( j_1, \ldots, j_\ell \in \{1, \ldots, n\} \), where \( R_{\omega}^{jk} \) denotes the \((j, k)\) component of the matrix \( R_{\omega} \). In particular,
\[
\left| \frac{\partial^\ell v_2}{\partial x_{j_1} \ldots \partial x_{j_\ell}}(x) \right| \leq C \sum_{k_1, \ldots, k_\ell=1}^n \int_{\mathbb{S}^{n-1}} \left| \frac{\partial^\ell v}{\partial x_{k_1} \ldots \partial x_{k_\ell}}(R_{\omega} x) \right| \ d\mathcal{H}^{n-1}(\omega),
\]
for some \( C > 0 \) only depending on \( n \) and \( \ell \), and hence
\[
\left\| \frac{\partial^\ell v_2}{\partial x_{j_1} \ldots \partial x_{j_\ell}}(x) \right\|_{L^2(B_1)}^2 \leq C \sum_{k_1, \ldots, k_\ell=1}^n \iiint_{\mathbb{S}^{n-1} \times B_1} \left| \frac{\partial^\ell v}{\partial x_{k_1} \ldots \partial x_{k_\ell}}(R_{\omega} x) \right|^2 \ d\mathcal{H}^{n-1}(\omega) \ dx
\]
\[
= C \sum_{k_1, \ldots, k_\ell=1}^n \iiint_{\mathbb{S}^{n-1} \times B_1} \left| \frac{\partial^\ell v}{\partial x_{k_1} \ldots \partial x_{k_\ell}}(y) \right|^2 \ d\mathcal{H}^{n-1}(\omega) \ dy
\]
\[
= C \sum_{k_1, \ldots, k_\ell=1}^n \left\| \frac{\partial^\ell v}{\partial x_{k_1} \ldots \partial x_{k_\ell}} \right\|_{L^2(B_1)}^2,
\]
up to renaming \( C \).

This, together with (5.2) and (5.8), gives (5.9), as desired. \( \square \)
With this preliminary work, we can now find a nontrivial, nonnegative and radial solution of \((5.4)\).

**Proposition 5.5.** There exists a solution of \((5.4)\) in \(H^s_0(B_1)\) which is radial, nonnegative and with unit norm in \(L^2(B_1)\).

**Proof.** Let \(u \geq 0\) be a nontrivial solution of \((5.4)\), whose existence is warranted by Lemma 5.2.

Then, we have that \(u_\sharp \geq 0\). Moreover,
\[
\int_{B_1} u_\sharp(x) \, dx = \frac{1}{|S^{n-1}|} \int_{S^{n-1} \times B_1} u(\mathcal{R}_\omega x) \, d\mathcal{H}^{n-1}(\omega) \, dx
\]
\[
= \frac{1}{|S^{n-1}|} \int_{S^{n-1} \times B_1} u(y) \, d\mathcal{H}^{n-1}(\omega) \, dy = \int_{B_1} u(y) \, dy > 0,
\]
and therefore \(u_\sharp\) does not vanish identically.

As a consequence, we can define
\[
u_* := \frac{u_\sharp}{\|u_\sharp\|_{L^2(B_1)}}.
\]
We know that \(u_* \in H^s_0(B_1)\), due to \((5.9)\). Moreover, in view of Lemma 5.3,
\[
(-\Delta)^s u_* = \left(-\Delta\right)^s \frac{u_\sharp}{\|u_\sharp\|_{L^2(B_1)}} = \frac{\left(-\Delta\right)^s u}{\|u\|_{L^2(B_1)}} = \frac{\lambda_1 u}{\|u\|_{L^2(B_1)}} = \lambda_1 u_*,
\]
which gives the desired result. \(\square\)

Now, we are in the position of proving the following result.

**Lemma 5.6.** Let \(s \geq 1\) and \(r \in (0, 1)\). If \(u \in H^s_0(B_1)\) and \(u\) is radial, then \(u \in C^\alpha(\mathbb{R}^n \setminus B_r)\) for any \(\alpha \in [0, \frac{1}{2}]\).

**Proof.** We write
\[
u(x) = u_0(|x|), \quad \text{for some } u_0 : [0, +\infty) \to \mathbb{R}
\]
and we observe that \(u \in H^s_0(B_1) \subset H^1(\mathbb{R}^n)\).

Accordingly, for any \(0 < r < 1\), we have
\[
\int_{\mathbb{R}^n \setminus B_r} |u(x)|^2 \, dx 
\geq \int_r^{+\infty} |u_0(\rho)|^2 \rho^{n-1} \, d\rho 
\geq \int_r^{+\infty} |u_0(\rho)|^2 \, d\rho \geq r^{n-1} \int_r^{+\infty} |u_0(\rho)|^2 \, d\rho
\]
\[
\int_{\mathbb{R}^n \setminus B_r} |\nabla u(x)|^2 \, dx 
\geq \int_r^{+\infty} |u_0'(\rho)|^2 \rho^{n-1} \, d\rho 
\geq \int_r^{+\infty} |u_0'(\rho)|^2 \, d\rho.
\]
Thanks to \((5.11)\) and \((5.12)\) we have that \(u_0 \in H^1((r, +\infty))\), with \(u_0 = 0\) in \([1, +\infty)\).

Then, from the Morrey Embedding Theorem, it follows that \(u_0 \in C^\alpha((r, +\infty))\) for any \(\alpha \in [0, \frac{1}{2}]\), which leads to the desired result. \(\square\)

**Corollary 5.7.** Let \(s \in (0, +\infty)\). There exists a radial, nonnegative and nontrivial solution of \((5.4)\) which belongs to \(H^s_0(B_1) \cap C^\alpha(\mathbb{R}^n \setminus B_{1/2})\), for some \(\alpha \in (0, 1)\).

**Proof.** If \(s \in (0, 1)\), the desired claim follows from Corollary 8 in \([DSV18]\).

If instead \(s \geq 1\), we obtain the desired result as a consequence of Proposition 5.5 and Lemma 5.6. \(\square\)
6 Boundary asymptotics of the first eigenfunctions of \((-\Delta)^s\)

In Lemma 4 of [DSV18], some precise asymptotics at the boundary for the first Dirichlet eigenfunction of \((-\Delta)^s\) have been established in the range \(s \in (0, 1)\).

Here, we obtain a related expansion in the range \(s > 0\) for the eigenfunction provided in Corollary 5.7. The result that we obtain is the following:

**Proposition 6.1.** There exists a nontrivial solution \(\phi_*\) of (5.4) which belongs to \(H^s_0(B_1) \cap C^{\alpha}(\mathbb{R}^n \setminus B_{1/2})\), for some \(\alpha \in (0, 1)\), and such that, for every \(e \in \partial B_1\) and \(\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n\),

\[
\lim_{\epsilon \to 0} \epsilon^{\beta - s} \partial^\beta \phi_*(e + \epsilon X) = (-1)^{|\beta|} k_* s (s - 1) \ldots (s - |\beta| + 1) e_1^{\beta_1} \ldots e_n^{\beta_n} (-e \cdot X)^{s - |\beta|},
\]

in the sense of distribution, with \(|\beta| := \beta_1 + \cdots + \beta_n\) and \(k_* > 0\).

The proof of Proposition 6.1 relies on Proposition 4.2 and some auxiliary computations on the Green function in (4.1). We start with the following result:

**Lemma 6.2.** Let \(0 < r < 1\), \(e \in \partial B_1\), \(s > 0\), \(f \in C^{\alpha}(\mathbb{R}^n \setminus B_r) \cap L^2(\mathbb{R}^n)\) for some \(\alpha \in (0, 1)\), and \(f = 0\) outside \(B_1\). Then the integral

\[
\int_{B_1} f(z) \frac{(1 - |z|^2)^s}{|z - e|^n} \, dz
\]

is finite.

**Proof.** We denote by \(I\) the integral in (6.1). We let

\[
I_1 := \int_{B_1 \setminus B_r} f(z) \frac{(1 - |z|^2)^s}{|z - e|^n} \, dz \quad \text{and} \quad I_2 := \int_{B_r} f(z) \frac{(1 - |z|^2)^s}{|z - e|^n} \, dz.
\]

Then, we have that

\[
I = I_1 + I_2. \tag{6.2}
\]

Now, if \(z \in B_1 \setminus B_r\), we have that

\[
f(z) \leq |f(z) - f(e)| \leq C |z - e|^\alpha,
\]

therefore

\[
I_1 \leq \int_{B_1 \setminus B_r} \frac{(1 - |z|^2)^s}{s |z - e|^{n-\alpha}} \, dz < \infty. \tag{6.3}
\]

If instead \(z \in B_r\),

\[
|z - e| \geq 1 - r > 0,
\]

and consequently

\[
I_2 \leq \frac{1}{s (1 - r)^n} \int_{B_r} f(z) \, dz < \infty. \tag{6.4}
\]

The desired result follows from (6.2), (6.3) and (6.4).

The next result gives a precise boundary behaviour of the Green function for any \(s > 0\) (the case in which \(s \in (0, 1)\) and \(f \in C^{\alpha}(\mathbb{R}^n)\) was considered in Lemma 6 of [DSV18], and in fact the proof presented here also simplifies the one in Lemma 6 of [DSV18] for the setting considered there).
Lemma 6.3. Let \( e, \omega \in \partial B_1, \epsilon_0 > 0 \) and \( r \in (0, 1) \). Assume that
\[
e + \epsilon \omega \in B_1,
\]
for any \( \epsilon \in (0, \epsilon_0] \). Let \( f \in C^\alpha(\mathbb{R}^n \setminus B_r) \cap L^2(\mathbb{R}^n) \) for some \( \alpha \in (0, 1) \), with \( f = 0 \) outside \( B_1 \).
Then
\[
\lim_{\epsilon \searrow 0} \epsilon^{-s} \int_{B_1} f(z) G_s(e + \epsilon \omega, z) \, dz = k(n, s) \int_{B_1} f(z) \frac{(-2e \cdot \omega)^s (1 - |z|^2)^s}{s|z - e|^n} \, dz,
\]
for a suitable normalizing constant \( k(n, s) > 0 \).

Proof. In light of (6.5), we have that
\[
1 > |e + \epsilon \omega|^2 = 1 + \epsilon^2 + 2 \epsilon e \cdot \omega,
\]
and therefore
\[
- \epsilon \cdot \omega > \frac{\epsilon}{2} > 0.
\]
Moreover, if \( r_0 \) is as given in (4.1), we have that, for all \( z \in B_1 \),
\[
 r_0(e + \epsilon \omega, z) = \frac{\epsilon(-\epsilon - 2e \cdot \omega)(1 - |z|^2)}{|z - e - \epsilon \omega|^2} \leq \frac{3\epsilon}{|z - e - \epsilon \omega|^2}.
\]
Also, a Taylor series representation allows us to write, for any \( t \in (-1, 1) \),
\[
\frac{t^{s-1}}{(t + 1)^{\frac{n}{2}}} = \sum_{k=0}^{\infty} \left( -\frac{n/2}{k} \right) t^{k+s-1}.
\]
We also notice that
\[
\left| \left( -\frac{n/2}{k} \right) \right| = \left| \frac{-\frac{n}{2} \left( \frac{n}{2} - 1 \right) \ldots \left( \frac{n}{2} - k + 1 \right)}{k!} \right| = \frac{n}{2} \left( \frac{n}{2} + 1 \right) \ldots \left( \frac{n}{2} + (k - 1) \right)
\leq \frac{n(n + 1) \ldots (n + (k - 1))}{k!} \leq \frac{(n + (k - 1))!}{k!} = (k + 1) \ldots (n + (k - 1))
\leq (n + k + 1)^{n+1}.
\]
This and the Root Test give that the series in (6.9) is uniformly convergent on compact sets in \((-1, 1)\).

As a consequence, if we set
\[
r_1(x, z) := \min \left\{ \frac{1}{2}, r_0(x, z) \right\},
\]
we can switch integration and summation signs and obtain that
\[
\int_0^{r_1(x, z)} \frac{t^{s-1}}{(t + 1)^{\frac{n}{2}}} \, dt = \sum_{k=0}^{\infty} c_k(r_1(x, z))^{k+s},
\]
where
\[ c_k := \frac{1}{k+s} \binom{-n/2}{k}. \]

Once again, the bound in (6.10), together with (6.11), give that the series in (6.12) is convergent.

Now, we omit for simplicity the normalizing constant \( k(n, s) \) in the definition of the Green function in (4.1), and we define
\[
\mathcal{G}(x, z) := |z - x|^{2s-n} \sum_{k=0}^{\infty} c_k(r_1(x, z))^{k+s}
\]
and
\[
g(x, z) := |z - x|^{2s-n} \int_{r_0(x, z)}^{\infty} \frac{t^{s-1}}{(t + 1)^{n}} \, dt.
\]

Using (4.1) and (6.12), and dropping dimensional constants for the sake of shortness, we can write
\[
\mathcal{G}_s(x, z) = \mathcal{G}(x, z) + g(x, z).
\]

Now, we show that
\[
g(x, z) \leq \begin{cases} 
C \chi(r, z) |z - x|^{2s-n} & \text{if } n > 2s, \\
C \chi(r, z) \log r_0(x, z) & \text{if } n = 2s, \\
C \chi(r, z) |z - x|^{2s-n}(r_0(x, z))^{s-\frac{n}{2}} & \text{if } n < 2s,
\end{cases}
\]
where \( \chi(r, z) = 1 \) if \( r_0(x, z) > \frac{1}{2} \) and \( \chi(r, z) = 0 \) if \( r_0(x, z) \leq \frac{1}{2} \). To check this, we notice that if \( r_0(x, z) \leq \frac{1}{2} \) we have that \( r_1(x, z) = r_0(x, z) \), due to (6.11), and therefore \( g(x, z) = 0 \).

On the other hand, if \( r_0(x, z) > \frac{1}{2} \), we deduce from (6.11) that \( r_1(x, z) = \frac{1}{2} \), and consequently
\[
g(x, z) \leq |z - x|^{2s-n} \int_{1/2}^{r_0(x, z)} t^{s-\frac{n}{2}} \, dt \leq \begin{cases} 
C |z - x|^{2s-n} & \text{if } n > 2s, \\
C \log r_0(x, z) & \text{if } n = 2s, \\
C |z - x|^{2s-n}(r_0(x, z))^{s-\frac{n}{2}} & \text{if } n < 2s,
\end{cases}
\]
for some constant \( C > 0 \). This completes the proof of (6.15).

Now, we exploit the bound in (6.15) when \( x = e + \epsilon \omega \). For this, we notice that if \( r_0(e + \epsilon \omega, z) > \frac{1}{2} \), recalling (6.8), we find that
\[
|z - e - \epsilon \omega|^2 \leq 6 \epsilon < 9 \epsilon,
\]
and therefore \( z \in B_{3\sqrt{\tau}}(e + \epsilon \omega) \).

Hence, using (6.15),
\[
\left| \int_{B_1} f(z) g(e + \epsilon \omega, z) \, dz \right| \leq \int_{B_{3\sqrt{\tau}}(e + \epsilon \omega)} |f(z)||g(e + \epsilon \omega, z)| \, dz 
\]
\[
\leq \begin{cases} 
C \int_{B_{3\sqrt{\tau}}(e + \epsilon \omega)} |f(z)||z - e - \epsilon \omega|^{2s-n} \, dz & \text{if } n > 2s, \\
C \int_{B_{3\sqrt{\tau}}(e + \epsilon \omega)} |f(z)| \log r_0(e + \epsilon \omega, z) \, dz & \text{if } n = 2s, \\
C \int_{B_{3\sqrt{\tau}}(e + \epsilon \omega)} |f(z)||z - e - \epsilon \omega|^{2s-n}(r_0(e + \epsilon \omega, z))^{s-\frac{n}{2}} \, dz & \text{if } n < 2s,
\end{cases}
\]
(6.17)
Now, if $z \in B_{3\sqrt{r}}(e + \epsilon \omega)$, then
$$|z - e| \leq |z - e - \epsilon \omega| + |\epsilon \omega| \leq 3\sqrt{r} + \epsilon < 4\sqrt{r}. \quad (6.18)$$
Furthermore, for a given $r \in (0, 1)$, we have that $B_{3\sqrt{r}}(e + \epsilon \omega) \subseteq \mathbb{R}^n \setminus B_r$, provided that $\epsilon$ is sufficiently small.

Hence, if $z \in B_{3\sqrt{r}}(e + \epsilon \omega)$, we can exploit the regularity of $f$ and deduce that
$$|f(z)| = |f(z) - f(e)| \leq C|z - e|^n.$$
This and $(6.18)$ lead to
$$|f(z)| \leq C\epsilon^{2s}, \quad (6.19)$$
for every $z \in B_{3\sqrt{r}}(e + \epsilon \omega)$.

Thanks to $(6.8)$, $(6.17)$ and $(6.19)$, we have that
$$\left| \int_{B_1} f(z) g(e + \epsilon \omega, z) dz \right| \leq \left\{ \begin{array}{ll}
C\epsilon^{2s} \int_{B_{3\sqrt{r}}(e + \epsilon \omega)} |z - e - \epsilon \omega|^{2s-n} dz & \text{if } n > 2s, \\
C\epsilon^{2s} \int_{B_{3\sqrt{r}}(e + \epsilon \omega)} \log \frac{3\epsilon}{|z - e - \epsilon \omega|^2} dz & \text{if } n = 2s, \\
C\epsilon^{2s-s/2} \int_{B_{3\sqrt{r}}(e + \epsilon \omega)} dz & \text{if } n < 2s
\end{array} \right.
\leq C\epsilon^{2s},$$
up to renaming $C$.

This and $(6.14)$ give that
$$\int_{B_1} f(z) G_s(e + \epsilon \omega, z) dz = \int_{B_1} f(z) G(e + \epsilon \omega, z) dz + o(\epsilon^s). \quad (6.20)$$
Now, we consider the series in $(6.13)$, and we split the contribution coming from the index $k = 0$ from the ones coming from the indices $k > 0$, namely we write
$$G(x, z) = G_0(x, z) + G_1(x, z),$$
with $G_0(x, z) := \frac{|z - x|^{2s-n}}{s} (r_1(x, z))^s$ \quad (6.21)
and $G_1(x, z) := |z - x|^{2s-n} \sum_{k=1}^{+\infty} c_k (r_1(x, z))^{k+s}$.

Firstly, we consider the contribution given by the term $G_1$. Thanks to $(6.11)$ and $(6.19)$, we have that
$$\left| \int_{B_1 \cap B_{3\sqrt{r}}(e + \epsilon \omega)} f(z) G_1(e + \epsilon \omega, z) dz \right| \leq \int_{B_{3\sqrt{r}}(e + \epsilon \omega)} |f(z)| G_1(e + \epsilon \omega, z) dz \leq C\epsilon^{2s} \int_{B_{3\sqrt{r}}(e + \epsilon \omega)} |z - e - \epsilon \omega|^{2s-n} \sum_{k=1}^{+\infty} |c_k| (r_1(e + \epsilon \omega, z))^{k+s} dz \leq C\epsilon^{2s} \int_{B_{3\sqrt{r}}(e + \epsilon \omega)} |z - e - \epsilon \omega|^{2s-n} \sum_{k=1}^{+\infty} |c_k| \left( \frac{1}{2} \right)^{k+s} dz \leq C\epsilon^{2s} \int_{B_{3\sqrt{r}}(e + \epsilon \omega)} |z - e - \epsilon \omega|^{2s-n} dz \leq C\epsilon^{2s+s},$$
26
up to renaming the constant $C$ step by step.

On the other hand, for every $z \in \mathbb{R}^n$,

$$|z| = |e + \epsilon \omega + z - e - \epsilon \omega| \geq |e + \epsilon \omega| - |z - e - \epsilon \omega| \geq 1 - \epsilon - |z - e - \epsilon \omega|.$$ 

Therefore, for every $z \in B_1 \setminus (B_r \cup B_{3\sqrt{\tau}}(e + \epsilon \omega))$, we can take $e_* := \frac{z}{|z|}$ and obtain that

$$|f(z)| = |f(z) - f(e_*)| \leq C|z - e_*|^\alpha = C(1 - |z|)^\alpha$$

$$\leq C(\epsilon + |z - e - \epsilon \omega|)^\alpha \leq C|z - e - \epsilon \omega|^\alpha,$$  \hspace{1cm} (6.23)

up to renaming $C > 0$.

Also, using (6.8), we see that, for any $k > 0$,

$$(r_0(e + \epsilon \omega, z))^{s + \frac{\alpha}{2}} \left(\frac{1}{2}\right)^k \leq \frac{C\epsilon^{s+\frac{\alpha}{2}}}{2^k|z - e - \epsilon \omega|^{2s+\frac{\alpha}{2}}}.$$  \hspace{1cm} (6.24)

This, (6.11) and (6.23) give that if $z \in B_1 \setminus (B_r \cup B_{3\sqrt{\tau}}(e + \epsilon \omega))$, then

$$|f(z)| \mathcal{G}_1(e + \epsilon \omega, z) \leq C|z - e - \epsilon \omega|^{\alpha+2s-n} \sum_{k=1}^{+\infty} |c_k| (r_1(e + \epsilon \omega, z))^{k+s}$$

$$= C|z - e - \epsilon \omega|^{\alpha+2s-n} \sum_{k=1}^{+\infty} |c_k| (r_1(e + \epsilon \omega, z))^{s+\frac{\alpha}{2}} (r_1(e + \epsilon \omega, z))^{k-s}$$

$$\leq C|z - e - \epsilon \omega|^{\alpha+2s-n} \sum_{k=1}^{+\infty} |c_k| (r_1(e + \epsilon \omega, z))^{s+\frac{\alpha}{2}} \left(\frac{1}{2}\right)^k$$

$$\leq C\epsilon^{s+\frac{\alpha}{2}} |z - e - \epsilon \omega|^{\frac{\alpha}{2}-n} \sum_{k=1}^{+\infty} |c_k| 2^k,$$

where the latter series is absolutely convergent thanks to (6.10).

This implies that, if we set $E := B_1 \setminus (B_r \cup B_{3\sqrt{\tau}}(e + \epsilon \omega))$, it holds that

$$\left| \int_E f(z) \mathcal{G}_1(e + \epsilon \omega, z) dz \right| \leq C\epsilon^{s+\frac{\alpha}{2}} \int_E |z - e - \epsilon \omega|^{\frac{\alpha}{2}-n} dz$$

$$\leq C\epsilon^{s+\frac{\alpha}{2}} \int_{B_1} |z - e - \epsilon \omega|^{\frac{\alpha}{2}-n} dz \leq C\epsilon^{s+\frac{\alpha}{2}} \int_{B_{3\sqrt{\tau}}} |z|^{\frac{\alpha}{2}-n} dz \leq C\epsilon^{s+\frac{\alpha}{2}}.$$  \hspace{1cm} (6.25)

Moreover, if $z \in B_r$, we have that

$$|e + \epsilon \omega - z| \geq 1 - \epsilon - r,$$

and therefore, recalling (6.24),

$$\sup_{z \in B_r} |\mathcal{G}_1(e + \epsilon \omega, z)| \leq |z - e - \epsilon \omega|^{2s-n} \sum_{k=1}^{+\infty} |c_k| (r_1(e + \epsilon \omega, z))^{s+\frac{\alpha}{2}} (r_1(e + \epsilon \omega, z))^{k-s}$$

$$\leq |z - e - \epsilon \omega|^{2s-n} \sum_{k=1}^{+\infty} |c_k| (r_1(e + \epsilon \omega, z))^{s+\frac{\alpha}{2}} \left(\frac{1}{2}\right)^k$$

$$\leq C |z - e - \epsilon \omega|^{-\frac{n}{2}} \sum_{k=1}^{+\infty} \frac{|c_k|}{2^k}$$

$$\leq C(1 - \epsilon - r)^{-\frac{n}{2}} \epsilon^{s+\frac{\alpha}{2}}.$$
up to renaming $C$.
As a consequence, we find that
\[
\left| \int_{B_r} f(z)G_1(e + \epsilon \omega, z)dz \right| \leq \sup_{z \in B_r} |G_1(e + \epsilon \omega, z)| \|f\|_{L^1(B_r)}
\]
\[
\leq \|f\|_{L^1(B_r)} (1 - \epsilon - r)^{-n - \frac{\alpha}{2}} e^{s + \frac{n}{2}}
\]
\[
\leq \|f\|_{L^1(B_r)} 2^{n + \frac{\alpha}{2}} (1 - r)^{-n - \frac{\alpha}{2}} e^{s + \frac{n}{2}}
\]
\[
= C e^{s + \frac{n}{2}},
\]
as long as $\epsilon$ is suitably small with respect to $r$, and $C$ is a positive constant which depends on $\|f\|_{L^1(B_r)}$, $r$, $n$ and $\alpha$.

Then, by (6.22), (6.25) and (6.26) we conclude that
\[
\int_{B_1} f(z)G_1(e + \epsilon \omega, z)dz = o(\epsilon^s). \tag{6.27}
\]

Inserting this information into (6.20), and recalling (6.21), we obtain
\[
\int_{B_1} f(z)G_s(e + \epsilon \omega, z)dz = \int_{B_1} f(z)G_0(e + \epsilon \omega, z)dz + o(\epsilon^s). \tag{6.28}
\]

Now, we define
\[D_1 := \{ z \in B_1 \text{ s.t. } r_0(e + \epsilon \omega, z) > 1/2 \}\]
and
\[D_2 := \{ z \in B_1 \text{ s.t. } r_0(e + \epsilon \omega, z) \leq 1/2 \}.
\]

If $z \in D_1$, then $z \in B_1 \setminus B_r$, thanks to (6.16), and hence we can use (6.17) and (6.19) and write
\[|f(z)G_0(e + \epsilon \omega, z)| \leq C e^{\frac{n}{2}} |z - e - \epsilon \omega|^{2s - n}.
\]

Then, recalling again (6.17),
\[
\left| \int_{D_1} f(z)G_1(e + \epsilon \omega, z)dz \right| \leq C e^{\frac{n}{2}} \int_{B_1 \setminus B_r} |z - e - \epsilon \omega|^{2s - n}dz = C e^{\frac{n}{2} + s}, \tag{6.29}
\]
up to renaming the constant $C > 0$. This information and (6.28) give that
\[
\int_{B_1} f(z)G_s(e + \epsilon \omega, z)dz = \int_{D_2} f(z)G_0(e + \epsilon \omega, z)dz + o(\epsilon^s).
\]

Now, by (6.8) and (6.11), if $z \in D_2$,
\[
G_0(e + \epsilon \omega, z) = \frac{|z - e - \epsilon \omega|^{2s - n}}{s}(r_0(e + \epsilon \omega))^s = \frac{e^{s(-\epsilon - 2e \cdot \omega)^s(1 - |z|^2)^s}}{s|z - e - \epsilon \omega|^n}.
\]

Hence, we have
\[
\lim_{\epsilon \to 0} e^{-s} \int_{B_1} f(z)G_s(e + \epsilon \omega, z)dz
\]
\[
= \lim_{\epsilon \to 0} e^{-s} \int_{D_2} f(z)G_0(e + \epsilon \omega, z)dz
\]
\[
= \lim_{\epsilon \to 0} \int_{\{2e(-\epsilon - 2e \cdot \omega)(1 - |z|^2) \leq |z - e - \epsilon \omega|^2\}} f(z) \frac{(-\epsilon - 2e \cdot \omega)^s(1 - |z|^2)^s}{s|z - e - \epsilon \omega|^n}dz. \tag{6.30}
\]
Now we set
\[
F_\epsilon(z) := \begin{cases} 
  f(z) \frac{(-\epsilon - 2\epsilon \cdot \omega)(1 - |z|^2)}{s|z - e - \epsilon \omega|^n} & \text{if } 2\epsilon(-\epsilon - 2\epsilon \cdot \omega)(1 - |z|^2) \leq |z - e - \epsilon \omega|^2, \\
  0 & \text{otherwise},
\end{cases}
\]  
(6.31)
and we prove that for any \(\eta > 0\) there exists \(\delta > 0\) independent of \(\epsilon\) such that, for any \(E \subset \mathbb{R}^n\) with \(|E| \leq \delta\), we have
\[
\int_{B_B \cap E} |F_\epsilon(z)| \, dz \leq \eta.
\]  
(6.32)
To this aim, given \(\eta\) and \(E\) as above, we define
\[
\rho := \min \left\{ \epsilon(-\epsilon - 2\epsilon \cdot \omega), \sqrt{2\epsilon(-\epsilon - 2\epsilon \cdot \omega)(1 - r)}, \left( \frac{2^{s+\alpha}s^2\epsilon^{s+\alpha}(-\epsilon - 2\epsilon \cdot \omega)^{\alpha}\eta}{3^{2s} \|f\|_{C^\alpha(B_1 \setminus B_r)} \|\partial B_1\|} \right)^{\frac{1}{2\alpha}} \right\}.
\]  
(6.33)
We stress that the above definition is well-posed, thanks to (6.7). In addition, using the integrability of \(f\), we take \(\delta > 0\) such that if \(A \subseteq B_1\) and \(|A| \leq \delta\) then
\[
\int_A |f(x)| \, dx \leq \frac{s\rho^n \eta}{2 \cdot 3^s}.
\]  
(6.34)
We set
\[
E_1 := E \cap B_\rho(e + \epsilon \omega) \quad \text{and} \quad E_2 := E \setminus B_\rho(e + \epsilon \omega).
\]  
(6.35)
From (6.31), we see that
\[
|F_\epsilon(z)| \leq \frac{|f(z)| \chi_\epsilon(z)}{2^s s \epsilon^s|z - e - \epsilon \omega|^{n-2s}},
\]
where
\[
\chi_\epsilon(z) := \begin{cases} 
  1 & \text{if } 2\epsilon(-\epsilon - 2\epsilon \cdot \omega)(1 - |z|^2) \leq |z - e - \epsilon \omega|^2, \\
  0 & \text{otherwise},
\end{cases}
\]
and therefore
\[
\int_{B_\rho \cap E_1} |F_\epsilon(z)| \, dz \leq \int_{B_\rho \cap E_1} \frac{|f(z)| \chi_\epsilon(z)}{2^s s \epsilon^s|z - e - \epsilon \omega|^{n-2s}} \, dz.
\]  
(6.36)
Now, for every \(z \in B_1 \cap E_1 \subseteq B_\rho(e + \epsilon \omega)\) for which \(\chi_\epsilon(z) \neq 0\), we have that
\[
2\epsilon(-\epsilon - 2\epsilon \cdot \omega)(1 - |z|^2) \leq |z - e - \epsilon \omega|^2 \leq \rho^2,
\]
and hence
\[
|z| \geq \sqrt{1 - \frac{\rho^2}{2\epsilon(-\epsilon - 2\epsilon \cdot \omega)}} \geq 1 - \frac{\rho^2}{2\epsilon(-\epsilon - 2\epsilon \cdot \omega)},
\]
which in turn gives that \(|z| \geq r\), recall (6.33).
From this and (6.36) we deduce that
\[
\int_{B_1 \cap E_1} |F_e(z)| \, dz \leq \int_{1-\frac{\rho^2}{2(\epsilon - 2e \cdot \omega)}}^{1} \|f\|_{C^0(B_1 \setminus B_{\rho})} (1 - |z|)^\alpha \, dz
\]
\[
\leq \frac{\|f\|_{C^0(B_1 \setminus B_{\rho})}}{2^{s+1} \epsilon^s} \int_{s=1}^{\infty} \left( \frac{2e(-\epsilon - 2e \cdot \omega)}{\epsilon^s} \right)^{\alpha} \, dz
\leq \frac{3^{2s} \|f\|_{C^0(B_1 \setminus B_{\rho})} |\partial B_1|}{2s+1} \epsilon^{s+1} (-\epsilon - 2e \cdot \omega)^\alpha \rho^2 \alpha
\leq \frac{\eta}{2},
\]
where (6.33) has been exploited in the last inequality.

We also point out that, by (6.31), (6.34) and (6.35),
\[
\int_{B_1 \cap E_2} |F_e(z)| \, dz \leq \int_{(B_1 \setminus B_{\rho(e+\omega)}) \cap E} |f(z)| \left( \frac{(-\epsilon - 2e \cdot \omega)^s(1 - |z|^2)^s}{|z - e - \epsilon \omega|^n} \right) \, dz
\leq \frac{3s}{s \rho^n} \int_{B_1 \cap E} |f(z)| \, dz
\leq \frac{\eta}{2},
\]
This, (6.35) and (6.37) give (6.32), as desired.

Notice also that the sequence \(F_e(z)\) converges pointwise to the function
\[
F(z) := f(z) \left( \frac{-2e \cdot \omega)^s(1 - |z|^2)^s}{s |z - e|^n} \right).
\]
Hence (6.30), (6.32) and the Vitali Convergence Theorem allow us to conclude that
\[
\lim_{\epsilon \searrow 0} \int_{B_1} f(z) G_\epsilon(e + \epsilon \omega, z) \, dz = \lim_{\epsilon \searrow 0} \int_{B_1} F_e(z) \, dz
= \int_{B_1} f(z) \left( \frac{-2e \cdot \omega)^s(1 - |z|^2)^s}{s |z - e|^n} \right) \, dz,
\]
which establishes the claim of Lemma 6.3 (notice that the finiteness of the latter quantity in (6.38) follows from (6.2)).

With this preliminary work, we can now establish the boundary behaviour of solutions which is needed in our setting. As a matter of fact, from Lemma 6.3 we immediately deduce that:

**Corollary 4.6.** Let \(e, \omega \in \partial B_1, \epsilon_0 > 0\) and \(r \in (0,1)\).

Assume that \(e + \epsilon \omega \in B_1\), for any \(\epsilon \in (0, \epsilon_0]\). Let \(f \in C^\alpha(\mathbb{R^n} \setminus B_r) \cap L^2(\mathbb{R^n})\) for some \(\alpha \in (0,1)\), with \(f = 0\) outside \(B_1\).

Let \(u\) be as in (4.19). Then,
\[
\lim_{\epsilon \searrow 0} \epsilon^{-s} u(e + \epsilon \omega) = k(n, s)(-2e \cdot \omega)^s \int_{B_1} f(z) \left( \frac{1 - |z|^2)^s}{s |z - e|^n} \right) \, dz,
\]
where \(k(n, s)\) denotes a positive normalizing constant.
Now we apply the previous results to detect the boundary growth of a suitable first eigenfunction. For our purposes, the statement that we need is the following:

**Corollary 6.5.** There exists a nontrivial solution \( \phi_* \) of (5.4) which belongs to \( H^s_0(B_1) \cap C^\alpha(\mathbb{R}^n \setminus B_{1/2}) \), for some \( \alpha \in (0,1) \), and such that, for every \( e \in \partial B_1 \),

\[
\lim_{\epsilon \searrow 0} \epsilon^{-s} \phi_*(e + \epsilon \omega) = k_* (-e \cdot \omega)^s, \tag{6.39}
\]

for a suitable constant \( k_* > 0 \).

Furthermore, for every \( R \in (r,1) \), there exists \( C_R > 0 \) such that

\[
\sup_{x \in B_1 \setminus B_R} d^{-s}(x) |\phi_*(x)| \leq C_R. \tag{6.40}
\]

**Proof.** Let \( \alpha \in (0,1) \) and \( \phi \in H^s_0(B_1) \cap C^\alpha(\mathbb{R}^n \setminus B_{1/2}) \) be the nonnegative and nontrivial solution of (5.4), as given in Corollary 5.7.

In the spirit of (4.19), we define

\[
\phi_*(x) := \begin{cases} 
\lambda_1 \int_{B_1} G_s(x,y) \phi(y) \, dy & \text{if } x \in B_1, \\
0 & \text{if } x \in \mathbb{R}^n \setminus B_1.
\end{cases}
\]

We stress that we can use Proposition 4.2 in this context, with \( f := \lambda_1 \phi \), since condition (4.18) is satisfied in this case.

Then, from (4.20) and (4.22), we know that \( \phi_* \in H^s_0(B_1) \) and, from (4.21),

\[
(-\Delta)^s \phi_* = \lambda_1 \phi \text{ in } B_1.
\]

In particular, we have that \( (-\Delta)^s(\phi - \phi_*) = 0 \) in \( B_1 \), and \( \phi - \phi_* \in H^s_0(B_1) \), which give that \( \phi - \phi_* \) vanishes identically. Hence, we can write that \( \phi = \phi_* \), and thus \( \phi_* \) is a solution of (5.4).

Now, we check (6.39). For this, we distinguish two cases. If \( e \cdot \omega \geq 0 \), we have that

\[
|e + \epsilon \omega|^2 = 1 + 2\epsilon e \cdot \omega + \epsilon^2 > 1,
\]

for all \( \epsilon > 0 \). Then, in this case \( e + \epsilon \omega \in \mathbb{R}^n \setminus B_1 \), and therefore \( \phi_*(e + \epsilon \omega) = 0 \). This gives that, in this case,

\[
\lim_{\epsilon \searrow 0} \epsilon^{-s} \phi_*(e + \epsilon \omega) = 0. \tag{6.41}
\]

If instead \( e \cdot \omega < 0 \), we see that

\[
|e + \epsilon \omega|^2 = 1 + 2\epsilon e \cdot \omega + \epsilon^2 < 1,
\]

for all \( \epsilon > 0 \) sufficiently small. Hence, we can exploit Corollary 6.4 and find that

\[
\lim_{\epsilon \searrow 0} \epsilon^{-s} \phi_*(e + \epsilon \omega) = \lambda_1 k(n,s)(-2e \cdot \omega)^s \int_{B_1} \phi(z) \frac{(1 - |z|^2)^s}{s|z - e|^n} \, dz, \tag{6.42}
\]

with \( k(n,s) > 0 \). Then, we define

\[
k_* := 2^n k(n,s) \int_{B_1} \phi(z) \frac{(1 - |z|^2)^s}{s|z - e|^n} \, dz. \]
We observe that $k_\ast$ is positive by construction, with $k(n, s) > 0$. Also, in light of Lemma 6.2, we know that $k_\ast$ is finite. Hence, from (6.41) and (6.42) we obtain (6.39), as desired.

It only remains to check (6.40). For this, we use (4.21), and we see that, for every $R \in (r, 1)$,

$$
\sup_{x \in B_1 \setminus B_R} d^{-s}(x) |\phi_\ast(x)| \leq C_R \lambda_1 \left( \|\phi\|_{L^1(B_1)} + \|\phi\|_{L^\infty(B_1 \setminus B_1)} \right),
$$

and this gives (6.40) up to renaming $C_R$. \hfill \Box

Now, we can complete the proof of Proposition 6.1, by arguing as follows.

**Proof of Proposition 6.1.** Let $\psi$ be a test function in $C^\infty_0(\mathbb{R}^n)$. Let also $R := \frac{r+1}{2} \in (r, 1)$ and

$$
g_\epsilon(X) := \epsilon^{-s}\phi_\ast(e + \epsilon X) \partial^3 \psi(X).
$$

We claim that

$$
\sup_{x \in \mathbb{R}^n} |g_\epsilon(X)| \leq C,
$$

for some $C > 0$ independent of $\epsilon$. To prove this, we distinguish three cases. If $e + \epsilon X \in \mathbb{R}^n \setminus B_1$, we have that $\phi_\ast(e + \epsilon X) = 0$ and thus $g_\epsilon(X) = 0$. If instead $e + \epsilon X \in B_R$, we observe that

$$
R > |e + \epsilon X| \geq 1 - \epsilon |X|,
$$

and therefore $|X| \geq \frac{1-R}{\epsilon}$. In particular, in this case $X$ falls outside the support of $\psi$, as long as $\epsilon > 0$ is sufficiently small, and consequently $\partial^3 \psi(X) = 0$ and $g_\epsilon(X) = 0$.

Hence, to complete the proof of (6.43), we are only left with the case in which $e + \epsilon X \in B_1 \setminus B_R$. In this situation, we make use of (6.40) and we find that

$$
|\phi_\ast(e + \epsilon X)| \leq C d^s(e + \epsilon X) = C (1 - |e + \epsilon X|)^s \\
\leq C (1 - |e + \epsilon X|)^s(1 + |e + \epsilon X|)^s = C (1 - |e + \epsilon X|^2)^s \\
= C \epsilon^s(-2e \cdot X - \epsilon |X|^2)^s \leq C \epsilon^s,
$$

for some $C > 0$ possibly varying from line to line, and this completes the proof of (6.43).

Now, from (6.43) and the Dominated Convergence Theorem, we obtain that

$$
\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \epsilon^{-s}\phi_\ast(e + \epsilon X) \partial^3 \psi(X)dX = \int_{\mathbb{R}^n} \lim_{\epsilon \searrow 0} \epsilon^{-s}\phi_\ast(e + \epsilon X) \partial^3 \psi(X)dX. \tag{6.44}
$$

On the other hand, by Corollary 6.5, used here with $\omega := \frac{X}{|X|}$, we know that

$$
\lim_{\epsilon \searrow 0} \epsilon^{-s}\phi_\ast(e + \epsilon X) = \lim_{\epsilon \searrow 0} \epsilon^{-s}\phi_\ast(e + \epsilon |X|\omega) = |X|^s \lim_{\epsilon \searrow 0} \epsilon^{-s}\phi_\ast(e + \epsilon \omega) \\
= k_\ast |X|^s (\epsilon - \epsilon \cdot \omega)_+^{\ast} = k_\ast (\epsilon - \epsilon \cdot X)_+^{\ast}.
$$

Substituting this into (6.44), we thus find that

$$
\lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \epsilon^{-s}\phi_\ast(e + \epsilon X) \partial^3 \psi(X)dX = k_\ast \int_{\mathbb{R}^n} (\epsilon - \epsilon \cdot X)_+^{\ast} \partial^3 \psi(X)dX.
$$

32
As a consequence, integrating by parts twice,
\[
\lim_{\epsilon \searrow 0} \epsilon^{s-|\beta|} \int_{\mathbb{R}^n} \partial^\beta \phi_\epsilon (e + \epsilon X) \psi(X) dX = \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \partial^\beta \left( \epsilon^{-s} \phi_\epsilon (e + \epsilon X) \right) \psi(X) dX
\]
\[
= (-1)^{|\beta|} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \epsilon^{-s} \phi_\epsilon (e + \epsilon X) \partial^\beta \psi(X) dX
\]
\[
= (-1)^{|\beta|} k_s \int_{\mathbb{R}^n} (-e \cdot X)^s \partial^\beta \psi(X) dX
\]
\[
= k_s \int_{\mathbb{R}^n} \partial^\beta (-e \cdot X)^s \psi(X) dX
\]
\[
= (-1)^{|\beta|} k_s (s-1) \ldots (s-|\beta|+1) \epsilon^{s-1} \int_{\mathbb{R}^n} (-e \cdot X)^{s-|\beta|} \psi(X) dX.
\]
Since the test function \(\psi\) is arbitrary, the claim in Proposition 6.1 is proved. \(\square\)

7 Boundary behaviour of \(s\)-harmonic functions

In this section we analyze the asymptotic behaviour of \(s\)-harmonic functions, with a “spherical bump function” as exterior Dirichlet datum.

The result needed for our purpose is the following:

Lemma 7.1. Let \(s > 0\). Let \(m \in \mathbb{N}_0\) and \(\sigma \in (0, 1)\) such that \(s = m + \sigma\).

Then, there exists \(\psi \in H^s(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)\) such that \((-\Delta)^s \psi = 0\) in \(B_1\), \((7.1)\)

and, for every \(x \in \partial B_{1-\epsilon}\),
\[
\psi(x) = k \epsilon^s + o(\epsilon^s), \quad (7.2)
\]
as \(\epsilon \searrow 0\), for some \(k > 0\).

Proof. Let \(\overline{\psi} \in C^\infty(\mathbb{R}, [0, 1])\) such that \(\overline{\psi} = 0\) in \(\mathbb{R} \setminus (2, 3)\) and \(\overline{\psi} > 0\) in \((2, 3)\). Let \(\psi_0(x) := (-1)^m \overline{\psi}(|x|)\). We recall the Poisson kernel
\[
\Gamma_s(x, y) := (-1)^m \frac{\gamma_{n, \sigma} (1 - |x|^2)^s}{{|x-y|^n} (|y|^2 - 1)^s},
\]
for \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}^n \setminus \overline{B}_1\), and a suitable normalization constant \(\gamma_{n, \sigma} > 0\) (see formulas (1.10) and (1.30) in [AJSb]). We define
\[
\psi(x) := \int_{\mathbb{R}^n \setminus B_1} \Gamma_s(x, y) \psi_0(y) dy + \psi_0(x).
\]

Notice that \(\psi_0 = 0\) in \(B_{3/2}\) and therefore we can exploit Theorem in [AJSb] and obtain that \((7.1)\) is satisfied (notice also that \(\psi = \psi_0\) outside \(B_1\), hence \(\psi\) is compactly supported).
Furthermore, to prove (7.2) we borrow some ideas from Lemma 2.2 in [DSV17] and we see that, for any \( x \in \partial B_1 \),

\[
\psi(x) = c(-1)^m \int_{\mathbb{R}^n \setminus B_1} \frac{\psi_0(y)(1 - |x|^2)}{|y|^2 - 1|y - x|^n} dy + \psi_0(x)
\]

\[
= c(-1)^m \int_{\mathbb{R}^n \setminus B_1} \frac{\psi_0(y)(1 - |x|^2)}{|y|^2 - 1|y - x|^n} dy
\]

\[
= c(1 - |x|^2)^s \int_2^3 \left[ \int_{\mathbb{S}^{n-1}} \frac{\rho^{n-1}\psi(\rho)}{(\rho^2 - 1)^s|x - \rho\omega|^n} d\omega \right] d\rho
\]

\[
= c(2\epsilon - \epsilon^2)^s \int_2^3 \left[ \int_{\mathbb{S}^{n-1}} \frac{\rho^{n-1}\psi(\rho)}{(\rho^2 - 1)^s(1 - \epsilon)e_1 - \rho\omega|^n} d\omega \right] d\rho
\]

\[
= 2^s c \epsilon^s \int_2^3 \left[ \int_{\mathbb{S}^{n-1}} \frac{\rho^{n-1}\psi(\rho)}{(\rho^2 - 1)^s|e_1 - \rho\omega|^n} d\omega \right] d\rho + o(\epsilon^s)
\]

\[
= ce^s + o(\epsilon^s),
\]

where \( c > 0 \) is a constant possibly varying from line to line, and this establishes (7.2).

**Remark 7.2.** As in Proposition 6.1, one can extend (7.2) to higher derivatives (in the distributional sense), obtaining, for any \( e \in \partial B_1 \) and \( \beta \in \mathbb{N}^n \)

\[
\lim_{\epsilon \to 0} \epsilon^{|\beta|-s} \partial^\beta \psi(e + \epsilon X) = k_\beta e_1^{\beta_1} \cdots e_n^{\beta_n} (-e \cdot X)^{s-|\beta|},
\]

for some \( k_\beta \neq 0 \).

Using Lemma 7.1, in the spirit of [DSV17], we can construct a sequence of \( \gamma \)-harmonic functions approaching \( (x \cdot e)^+ \) for a fixed unit vector \( e \), by using a blow-up argument. Namely, we prove the following:

**Corollary 7.3.** Let \( e \in \partial B_1 \). There exists a sequence \( v_{\epsilon,j} \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n) \) such that\((-\Delta)^s v_{\epsilon,j} = 0 \text{ in } B_1(e), v_{\epsilon,j} = 0 \text{ in } \mathbb{R}^n \setminus B_{4j}(e), \) and

\[ v_{\epsilon,j} \to \kappa(x \cdot e)^+ \text{ in } L^1(B_1(e)), \]

as \( j \to +\infty \), for some \( \kappa > 0 \).

**Proof.** Let \( \psi \) be as in Lemma 7.1 and define

\[
v_{\epsilon,j}(x) := j^s \psi \left( \frac{x}{j} - e \right).
\]

The \( s \)-harmonicity and the property of being compactly supported follow by the ones of \( \psi \). We now prove the convergence. To this aim, given \( x \in B_1(e) \), we write \( p_j := \frac{x}{j} - e \) and \( \epsilon_j := 1 - |p_j| \). Recall that since \( x \in B_1(e) \), then \( |x - e|^2 < 1 \), which implies that \( |x|^2 < 2x \cdot e \) and \( x \cdot e > 0 \) for any \( x \in B_1(e) \).

As a consequence

\[
|p_j|^2 = \left| \frac{x}{j} - e \right|^2 = \frac{|x|^2}{j^2} + 1 - 2 \frac{x}{j} \cdot e = 1 - 2 \frac{x}{j} (x \cdot e)_+ + o \left( \frac{1}{j} \right) (x \cdot e)_+^2,
\]

34
and so
\[ \epsilon_j = \frac{(1 + o(1))}{j} (x \cdot e)_+. \]

Therefore, using (7.2),
\[
v_{e,j}(x) = j^s \psi(p_j)
= j^s \kappa(\epsilon_s^j + o(\epsilon_s^j))
= j^s \left( \frac{\kappa}{j^s} (x \cdot e)^s_+ + o\left(\frac{1}{j^s}\right) \right)
= \kappa(x \cdot e)^s_+ + o(1).
\]

Integrating over \( B_1(e) \), we obtain the desired \( L^1 \)-convergence.

Now, we show that, as in the case \( s \in (0,1) \) proved in Theorem 3.1 of [DSV17], we can find an \( s \)-harmonic function with an arbitrarily large number of derivatives prescribed at some point.

**Proposition 7.4.** For any \( \beta \in \mathbb{N}^n \), there exist \( p \in \mathbb{R}^n \), \( R > r > 0 \), and \( v \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n) \) such that

\[
\begin{cases}
(-\Delta)^s v = 0 & \text{in } B_r(p), \\
v = 0 & \text{in } \mathbb{R}^n \setminus B_R(p),
\end{cases}
\]
and
\[
D^\alpha v(p) = 0 \quad \text{for any } \alpha \in \mathbb{N}^n \quad \text{with } |\alpha| \leq |\beta| - 1,
\]
\[
D^\alpha v(p) = 0 \quad \text{for any } \alpha \in \mathbb{N}^n \quad \text{with } |\alpha| = |\beta| \quad \text{and } \alpha \neq \beta
\]
and
\[
D^\beta v(p) = 1.
\]

**Proof.** Let \( Z \) be the set of all pairs \((v, x) \in (H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)) \times B_r(p)\) that satisfy (7.3) for some \( R > r > 0 \) and \( p \in \mathbb{R}^n \).

To each pair \((v, x) \in Z\) we associate the vector \( (D^\alpha v(x))_{|\alpha| \leq |\beta|} \in \mathbb{R}^{K'} \), for some \( K' = K_{|\beta|} \) and consider \( V \) to be the vector space spanned by this construction, namely we set
\[
V := \left\{(D^\alpha v(x))_{|\alpha| \leq |\beta|}, \quad \text{with } (v, x) \in Z\right\}.
\]
We claim that
\[
V = \mathbb{R}^{K'}.
\]

To check this, we suppose by contradiction that \( V \) lies in a proper subspace of \( \mathbb{R}^{K'} \). Then, \( V \) must lie in a hyperplane, hence there exists
\[
c = (c_\alpha)_{|\alpha| \leq |\beta|} \in \mathbb{R}^{K'} \setminus \{0\}
\]
which is orthogonal to any vector \( (D^\alpha v(x))_{|\alpha| \leq |\beta|} \) with \((v, x) \in Z\), that is
\[
\sum_{|\alpha| \leq |\beta|} c_\alpha D^\alpha v(x) = 0.
\]
We notice that the pair \((v_{e,j}, x)\), with \(v_j\) as in Corollary 7.3, \(e \in \partial B_1\) and \(x \in B_1(e)\), belongs to \(\mathcal{Z}\). Consequently, fixed \(\xi \in \mathbb{R}^n \setminus \mathring{B}_{1/2}\) and set \(e := \frac{\xi}{|\xi|}\), we have that (7.6) holds true when \(v := v_{e,j}\) and \(x \in B_1(e)\), namely

\[
\sum_{|\alpha| \leq |\beta|} c_{\alpha} D^\alpha v(x) = 0.
\]

Let now \(\varphi \in C_0^\infty(B_1(e))\). Integrating by parts, by Corollary 7.3 and the Dominated Convergence Theorem, we have that

\[
0 = \lim_{j \to +\infty} \int_{\mathbb{R}^n} \sum_{|\alpha| \leq |\beta|} c_{\alpha} D^\alpha v_{e,j}(x) \varphi(x) \, dx = \lim_{j \to +\infty} \int_{\mathbb{R}^n} \sum_{|\alpha| \leq |\beta|} (-1)^{|\alpha|} c_{\alpha} v_{e,j}(x) D^\alpha \varphi(x) \, dx
\]

\[
= \kappa \int_{\mathbb{R}^n} \sum_{|\alpha| \leq |\beta|} (-1)^{|\alpha|} c_{\alpha}(x \cdot e)^s_+ D^\alpha \varphi(x) \, dx = \kappa \int_{\mathbb{R}^n} \sum_{|\alpha| \leq |\beta|} c_{\alpha} D^\alpha (x \cdot e)^s_+ \varphi(x) \, dx.
\]

This gives that, for every \(x \in B_1(e)\),

\[
\sum_{|\alpha| \leq |\beta|} c_{\alpha} D^\alpha (x \cdot e)^s_+ = 0.
\]

Moreover, for every \(x \in B_1(e)\),

\[
D^\alpha (x \cdot e)^s_+ = s(s-1) \cdots (s-|\alpha|+1)(x \cdot e)^{s-|\alpha|}_+ e_{a_1}^{\alpha_1} \cdots e_{a_n}^{\alpha_n}.
\]

In particular, for \(x = \frac{e}{|\xi|} \in B_1(e)\),

\[
D^\alpha (x \cdot e)^s_+|_{x = \frac{e}{|\xi|}} = s(s-1) \cdots (s-|\alpha|+1)|\xi|^{-s} e_{a_1}^{\alpha_1} \cdots e_{a_n}^{\alpha_n}.
\]

And, using the usual multi-index notation, we write

\[
\sum_{|\alpha| \leq |\beta|} c_{\alpha} s(s-1) \cdots (s-|\alpha|+1)\xi^\alpha = 0, \tag{7.7}
\]

for any \(\xi \in \mathbb{R}^n \setminus B_{1/2}\). The identity (7.7) describes a polynomial in \(\xi\) which vanishes for any \(\xi\) in an open subset of \(\mathbb{R}^n\). As a result, the Identity Principle for polynomials leads to

\[
c_{\alpha} s(s-1) \cdots (s-|\alpha|+1) = 0,
\]

for all \(|\alpha| \leq |\beta|\).

Consequently, since \(s \in \mathbb{R} \setminus \mathbb{N}\), the product \(s(s-1) \cdots (s-|\alpha|+1)\) never vanishes, and so the coefficients \(c_{\alpha}\) are forced to be null for any \(|\alpha| \leq |\beta|\). This is in contradiction with (7.5), and therefore the proof of (7.4) is complete.

From this, the desired claim in Proposition 7.4 plainly follows.

\[\square\]

8 A result which implies Theorem 1.1

We will use the notation

\[
\Lambda_{-\infty} := \Lambda_{(-\infty, \ldots, -\infty)}, \tag{8.1}
\]

that is we exploit (1.8) with \(a_1 := \cdots := a_l := -\infty\). This section presents the following statement:
Theorem 8.1. Suppose that

either there exists \( i \in \{1, \ldots, M\} \) such that \( \beta_i \neq 0 \) and \( s_i \not\in \mathbb{N} \),
or there exists \( i \in \{1, \ldots, l\} \) such that \( c_i \neq 0 \) and \( \alpha_i \not\in \mathbb{N} \).

Let \( \ell \in \mathbb{N} \), \( f : \mathbb{R}^N \to \mathbb{R} \), with \( f \in C^\ell(\overline{B_N^1}) \). Fixed \( \epsilon > 0 \), there exist

\[ u = u_\epsilon \in C^\infty (B^1_N) \cap C (\mathbb{R}^N), \]
\[ a = (a_1, \ldots, a_l) = (a_1, \epsilon, \ldots, a_l, \epsilon) \in (-\infty, 0)^l, \]
and \( R = R_\epsilon > 1 \)
such that:

- for every \( h \in \{1, \ldots, l\} \) and \((x, y, t_1, \ldots, t_{h-1}, t_{h+1}, \ldots, t_l)\)

\[ \Lambda_{-\infty} u = 0 \quad \text{in} \quad B^1_{N-l} \times (-1, +\infty)^l, \]
\[ u(x, y, t) = 0 \quad \text{if} \quad |(x, y)| \geq R, \] (8.3)

\[ \partial_{t_h}^{a_h} u(x, y, t) = 0 \quad \text{if} \quad t_h \in (-\infty, a_h), \quad \text{for all} \quad h \in \{1, \ldots, l\}, \] (8.4)

and

\[ \|u - f\|_{C^\ell(B^1_N)} < \epsilon. \] (8.5)

The proof of Theorem 8.1 will basically occupy the rest of this paper, and this will lead us to the completion of the proof of Theorem 1.1. Indeed, we have that:

Lemma 8.2. If the statement of Theorem 8.1 holds true, then the statement in Theorem 1.1 holds true.

Proof. Assume that the claims in Theorem 8.1 are satisfied. Then, by (8.2) and (8.4), we are in the position of exploiting Lemma A.1 in [CDV18] and conclude that, in \( B^1_{N-l} \times (-1, +\infty)^l \),

\[ D_{t_h, a_h}^{a_h} u = D_{t_h, -\infty}^{a_h} u, \]

for every \( h \in \{1, \ldots, l\} \). This and (8.3) give that

\[ \Lambda_{a} u = \Lambda_{-\infty} u = 0 \quad \text{in} \quad B^1_{N-l} \times (-1, +\infty)^l. \] (8.6)

We also define

\[ a := \min_{h \in \{1, \ldots, l\}} a_h \]

and take \( \tau \in C_0^\infty([-a - 2, 3]) \) with \( \tau = 1 \) in \([-a - 1, 1]\). Let

\[ U(x, y, t) := u(x, y, t) \tau(t_1) \ldots \tau(t_l). \] (8.7)

Our goal is to prove that \( U \) satisfies the theses of Theorem 1.1. To this end, we observe that \( u = U \) in \( B^1_N \), therefore (1.12) for \( U \) plainly follows from (8.5).
In addition, from (1.6), we see that \( D_{\alpha_h}^\alpha \) at a point \( t_h \in (-1,1) \) only depends on the values of the function between \( a_h \) and 1. Since the cutoffs in (8.7) do not alter these values, we see that \( D_{\alpha_h}^\alpha U = D_{\alpha_h}^\alpha u \) in \( B_1^N \), and accordingly \( \Lambda_a U = \Lambda_a u \) in \( B_1^N \). This and (8.6) say that
\[
\Lambda_a U = 0 \quad \text{in} \quad B_1^N. \tag{8.8}
\]
Also, since \( u \) in Theorem 8.1 is compactly supported in the variable \((x,y)\), we see from (8.7) that \( U \) is compactly supported in the variables \((x,y,t)\). This and (8.8) give that (1.11) is satisfied by \( U \) (up to renaming \( R \)). \qed

9  A pivotal span result towards the proof of Theorem 8.1

In what follows, we let \( \Lambda_{-\infty} \) be as in (8.1), we recall the setting in (1.1), and we use the following multi-indices notations:
\[
i = (i, I, J) = (i_1, \ldots, i_n, I_1, \ldots, I_M, J_1, \ldots, J_l) \in \mathbb{N}^N
\]
and \( \partial^i w = \partial^{i_1}_{x_1} \ldots \partial^{i_n}_{x_n} \partial^{J_1}_{y_1} \ldots \partial^{J_M}_{y_M} \partial^{J_1}_{t_1} \ldots \partial^{J_l}_{t_l} w. \) \tag{9.1}

Inspired by Lemma 5 of [DSV18], we consider the span of the derivatives of functions in \( \ker \Lambda_{-\infty} \), with derivatives up to a fixed order \( K \in \mathbb{N} \). We want to prove that the derivatives of such functions span a maximal vectorial space.

For this, we denote by \( \partial^K w(0) \) the vector with entries given, in some prescribed order, by \( \partial^i w(0) \) with \( |i| \leq K \).

We notice that \( \partial^K w(0) \in \mathbb{R}^{K'} \) for some \( K' \in \mathbb{N} \), \tag{9.2}
with \( K' \) depending on \( K \).

Now, we adopt the notation in formula (1.4) of [CDV18], and we denote by \( \mathcal{A} \) the set of all functions \( w = w(x,y,t) \) such that for all \( h \in \{1, \ldots, l\} \) and all \( (x,y,t_1, \ldots, t_{h-1}, t_{h+1}, \ldots, t_l) \in \mathbb{R}^{N-1} \), the map \( \mathbb{R} \ni t_h \mapsto w(x,y,t) \) belongs to \( C^\infty((a_h, +\infty)) \cap C^{\alpha_h}(a_h), \) and (8.4) holds true for some \( a_h \in (-2,0) \).

We also set
\[
\mathcal{H} := \left\{ w \in C(\mathbb{R}^N) \cap C_0(\mathbb{R}^{N-1}) \cap C^\infty(N) \cap \mathcal{A}, \text{ for some neighborhood } N \text{ of the origin, such that } \Lambda_{-\infty} w = 0 \text{ in } N \right\}
\]
and, for any \( w \in \mathcal{H} \), let \( \mathcal{V}_K \) be the vector space spanned by the vector \( \partial^K w(0) \).

By (9.2), we know that \( \mathcal{V}_K \subseteq \mathbb{R}^{K'} \). In fact, we show that equality holds in this inclusion, as stated in the following\( ^4 \) result:

Lemma 9.1. It holds that \( \mathcal{V}_K = \mathbb{R}^{K'} \).

The proof of Lemma 9.1 is by contradiction. Namely, if \( \mathcal{V}_K \) does not exhaust the whole of \( \mathbb{R}^{K'} \) there exists \( \theta \in \partial B_1^{K'} \) \tag{9.3}
\(^4\)Notice that results analogous to Lemma 9.1 cannot hold for solutions of local operators: for instance, pure second derivatives of harmonic functions have to satisfy a linear equation, so they are forced to lie in a proper subspace. In this sense, results such as Lemma 9.1 here reveal a truly nonlocal phenomenon.
such that
\[ V_K \subseteq \{ \zeta \in \mathbb{R}^{K'} \text{ s.t. } \theta \cdot \zeta = 0 \}. \tag{9.4} \]

In coordinates, recalling (9.1), we write \( \theta \) as \( \theta_i = \theta_{i,1,3} \), with \( i \in \mathbb{N}^{p_1+\cdots+p_m} \), \( I \in \mathbb{N}^{m_1+\cdots+m_M} \) and \( \mathcal{J} \in \mathbb{N}^l \). We consider

a multi-index \( \vec{I} \in \mathbb{N}^{m_1+\cdots+m_M} \) such that it maximizes \( |I| \)
among all the multi-indexes \((i, I, \mathcal{J})\) for which \( |i| + |I| + |\mathcal{J}| \leq K \)
and \( \theta_{i,1,3} \neq 0 \) for some \((i, \mathcal{J})\).

Some comments on the setting in (9.5). We stress that, by (9.3), the set \( \mathcal{S} \) of indexes \( I \) for which there exist indexes \((i, \mathcal{J})\) such that \( |i| + |I| + |\mathcal{J}| \leq K \) and \( \theta_{i,1,3} \neq 0 \) is not empty. Therefore, since \( \mathcal{S} \) is a finite set, we can take

\[ S := \sup_{I \in \mathcal{S}} |I| = \max_{I \in \mathcal{S}} |I| \in \mathbb{N} \cap [0, K]. \]

Hence, we consider a multi-index \( \vec{I} \) for which \( |\vec{I}| = S \) to obtain the setting in (9.5). By construction, we have that

- \(|i| + |\vec{I}| + |\mathcal{J}| \leq K\),
- if \(|I| > |\vec{I}|\), then \( \theta_{i,1,3} = 0\),
- and there exist multi-indexes \( i \) and \( \mathcal{J} \) such that \( \theta_{i,1,3} \neq 0 \).

As a variation of the setting in (9.5), we can also consider

a multi-index \( \vec{\mathcal{J}} \in \mathbb{N}^l \) such that it maximizes \(|\mathcal{J}|\)
among all the multi-indexes \((i, I, \mathcal{J})\) for which \(|i| + |I| + |\mathcal{J}| \leq K \)
and \( \theta_{i,1,3} \neq 0 \) for some \((i, I)\).

In the setting of (9.5) and (9.6), we claim that there exists an open set of \( \mathbb{R}^{p_1+\cdots+p_m} \times \mathbb{R}^{m_1+\cdots+m_M} \times \mathbb{R}^l \) such that for every \((\vec{x}, \vec{y}, \vec{z})\) in such open set we have that

\[ \begin{align*}
\text{either} & \quad 0 = \sum_{|i|+|I|+|\mathcal{J}| \leq K} C_{i,I,J} \theta_{i,1,3} \vec{y}^{i/3} \vec{z}^{I/3}, \quad \text{with} \quad C_{i,I,J} \neq 0, \\
\text{or} & \quad 0 = \sum_{|i|+|I|+|\mathcal{J}| \leq K} C_{i,I,J} \theta_{i,1,3} \vec{y}^{i/3} \vec{z}^{I/3}, \quad \text{with} \quad C_{i,I,J} \neq 0. \tag{9.7}
\end{align*} \]

In our framework, the claim in (9.7) will be pivotal towards the completion of the proof of Lemma 9.1. Indeed, let us suppose for the moment that (9.7) is established and let us complete the proof of Lemma 9.1 by arguing as follows.

Formula (9.7) says that \( \theta \cdot \partial^K w(0) \) is a polynomial which vanishes for any triple \((\vec{x}, \vec{y}, \vec{z})\)
in an open subset of \( \mathbb{R}^{p_1+\cdots+p_m} \times \mathbb{R}^{m_1+\cdots+m_M} \times \mathbb{R}^l \). Hence, using the identity principle of polynomials, we have that each \( C_{i,I,J} \theta_{i,1,3} \) is equal to zero whenever \(|i| + |I| + |\mathcal{J}| \leq K\) and either \(|I| = |\vec{I}|\) (if the first identity in (9.7) holds true) or \(|\mathcal{J}| = |\vec{\mathcal{J}}|\) (if the second identity in (9.7) holds true). Then, since \( C_{i,I,J} \neq 0 \), we conclude that each \( \theta_{i,1,3} \) is zero as long as either \(|I| = |\vec{I}|\) (in the first case) or \(|\mathcal{J}| = |\vec{\mathcal{J}}|\) (in the second case), but this contradicts either
the definition of $\overline{I}$ in (9.5) (in the first case) or the definition of $\tilde{I}$ in (9.6) (in the second case). This would therefore complete the proof of Lemma 9.1.

In view of the discussion above, it remains to prove (9.7). To this end, we distinguish the following four cases:

1. there exist $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, M\}$ such that $a_i \neq 0$ and $\hat{b}_j \neq 0$,
2. there exist $i \in \{1, \ldots, n\}$ and $h \in \{1, \ldots, l\}$ such that $a_i \neq 0$ and $c_h \neq 0$,
3. we have that $a_1 = \cdots = a_n = 0$, and there exists $j \in \{1, \ldots, M\}$ such that $\hat{b}_j \neq 0$,
4. we have that $a_1 = \cdots = a_n = 0$, and there exists $h \in \{1, \ldots, l\}$ such that $c_h \neq 0$.

Notice that cases 1 and 3 deal with the case in which space fractional diffusion is present (and in case 1 one also has classical derivatives, while in case 3 the classical derivatives are absent).

Similarly, cases 2 and 4 deal with the case in which time fractional diffusion is present (and in case 2 one also has classical derivatives, while in case 4 the classical derivatives are absent).

Of course, the case in which both space and time fractional diffusion occur is already comprised by the previous cases (namely, it is comprised in both cases 1 and 2 if classical derivatives are also present, and in both cases 3 and 4 if classical derivatives are absent).

**Proof of (9.7), case 1.** For any $j \in \{1, \ldots, M\}$ we denote by $\tilde{\phi}_{\ast,j}$ the first eigenfunction for \((-\Delta)^{s_j}_{B_j}\) vanishing outside $B_{1}^{m_j}$ given in Corollary 5.7. We normalize it such that \(\|\tilde{\phi}_{\ast,j}\|_{L^2(B_{1}^{m_j})} = 1\), and we write $\lambda_{\ast,j} \in (0, +\infty)$ to indicate the corresponding first eigenvalue (which now depends on $s_j$), namely we write

\[
\begin{cases}
(-\Delta)^{s_j}_{B_j}\tilde{\phi}_{\ast,j} = \lambda_{\ast,j}\tilde{\phi}_{\ast,j} & \text{in } B_{1}^{m_j}, \\
\tilde{\phi}_{\ast,j} = 0 & \text{in } \mathbb{R}^{m_j} \setminus B_{1}^{m_j}.
\end{cases}
\]  
(9.8)

Up to reordering the variables and/or taking the operators to the other side of the equation, given the assumptions of case 1, we can suppose that

\[a_1 \neq 0\]  
(9.9)

and

\[\hat{b}_M > 0\]  
(9.10)

In view of (9.9), we can define

\[R := \left( \frac{1}{|\alpha_1|} \left( \sum_{j=1}^{M-1} |\hat{b}_j|\lambda_{\ast,j} + \sum_{h=1}^{l} |c_h| \right) \right)^{1/|\alpha_1|}.
\]  
(9.11)

Now, we fix two sets of free parameters

\[\underline{\alpha}_1 \in (R + 1, R + 2)^{p_1}, \ldots, \underline{\alpha}_n \in (R + 1, R + 2)^{p_n},\]  
(9.12)

and

\[\underline{\ell}_{\ast,1} \in \left( \frac{1}{2}, 1 \right), \ldots, \underline{\ell}_{\ast,l} \in \left( \frac{1}{2}, 1 \right).
\]  
(9.13)
We also set
\[ \lambda_j := \lambda_{*,j} \text{ for } j \in \{1, \ldots, M - 1\}, \quad (9.14) \]
where \( \lambda_{*,j} \) is defined as in (9.8), and
\[ \lambda_M := \frac{1}{\mathcal{G}_M} \left( \sum_{j=1}^{n} |c_j| \lambda^{r_j} - \sum_{j=1}^{M-1} \delta_j \lambda_j - \sum_{h=1}^{l} c_{h,j} \lambda_{*,h} \right). \quad (9.15) \]

Notice that this definition is well-posed, thanks to (9.10). In addition, from (9.12), we can write
\[ \lambda^{r_j} = \lambda^{r_{j1}} \ldots \lambda^{r_{jp_j}} \geq 0. \quad (9.16) \]
From this, (9.11) and (9.13), we deduce that
\[
\sum_{j=1}^{n} |c_j| \lambda^{r_j} \geq |\lambda| \lambda^{r_1} \geq |\lambda| (R + 1)^{|r_1|} > |\lambda| R^{|r_1|}
\]
and consequently, by (9.15),
\[ \lambda_M > 0. \quad (9.17) \]

We also set
\[ \omega_j := \begin{cases} 1 & \text{if } j = 1, \ldots, M - 1, \\ \lambda_{*,M}^{1/2s_M} & \text{if } j = M. \end{cases} \quad (9.18) \]
Notice that this definition is well-posed, thanks to (9.17). In addition, by (9.8), we have that, for any \( j \in \{1, \ldots, M\} \), the functions
\[ \phi_j (y_j) := \tilde{\phi}_{*,j} \left( \frac{y_j}{\omega_j} \right) \quad (9.19) \]
are eigenfunctions of \((-\Delta)^{s_j})\) in \( B_{\omega_j} \) with external homogenous Dirichlet boundary condition, and eigenvalues \( \lambda_j \): namely, we can rewrite (9.8) as
\[
\begin{cases}
(-\Delta)^{s_j} \phi_j = \lambda_j \phi_j & \text{in } B_{\omega_j}, \\
\phi_j = 0 & \text{in } \mathbb{R}^m \setminus \overline{B}_{\omega_j}. \end{cases} \quad (9.20) \]
Now, we define
\[ \psi_{*,h}(t_h) := E_{\alpha_h,1}(t_h^{\alpha_h}), \quad (9.21) \]
where \( E_{\alpha,1} \) denotes the Mittag-Leffler function with parameters \( \alpha := \alpha_h \) and \( \beta := 1 \) as defined in (2.1).
Moreover, we consider \( a_h \in (-2, 0) \), for every \( h = 1, \ldots, l \), to be chosen appropriately in what follows (the precise choice will be performed in (9.40)), and, recalling (9.13), we let
\[ \dot{t}_{*,h} := t_{*,h}^{1/\alpha_h}, \quad (9.22) \]
and we define
\[ \psi_h(t_h) := \psi_{*,h}(\frac{t}{h}(t_h - a_h)) = E_{\alpha_h,1}(\frac{t}{h}(t_h - a_h)^{\alpha_h}). \]  
(9.23)

We point out that, thanks to Lemma 2.2, the function in (9.23), solves
\[
\begin{cases}
D_{t_h, a_h}^{\alpha_h} \psi_h(t_h) = \frac{t}{h}(t_h - a_h) \psi_h(t_h) & \text{in } (a_h, +\infty), \\
\psi_h(a_h) = 1, \\
\partial_{t_h}^{\alpha_h} \psi_h(a_h) = 0 & \text{for every } m \in \{1, \ldots, [\alpha_h]\}. 
\end{cases}
\]  
(9.24)

Moreover, for any \( h \in \{1, \ldots, l\} \), we define
\[ \psi_h^*(t_h) := \begin{cases} \psi_h(t_h) & \text{if } t_h \in [a_h, +\infty) \\
1 & \text{if } t_h \in (-\infty, a_h). \end{cases} \]  
(9.25)

Thanks to (9.24) and Lemma A.3 in [CDV18] applied here with \( b := a_h, a := -\infty, u := \psi_h, u_* := \psi_h^* \), we have that \( \psi_h^* \in C_{-\infty}^{k, \alpha_h} \), and
\[ D_{t_h, -\infty}^{\alpha_h} \psi_h^*(t_h) = D_{t_h, a_h}^{\alpha_h} \psi_h(t_h) = \frac{t}{h}(t_h - a_h) \psi_h(t_h) = \frac{t}{h} \psi_h^*(t_h) \]  
in every interval \( I \Subset (a_h, +\infty) \).
(9.26)

We observe that the setting in (9.25) is compatible with the ones in (8.2) and (8.4).

From (2.1) and (9.23), we see that
\[ \psi_h(t_h) = \sum_{j=0}^{+\infty} \frac{\frac{(a_h - \alpha_h j)}{2} (\alpha_h + 1 \cdots (\alpha_h - j_h + 1)(t_h - a_h)^{\alpha_h j - 3_h}}{\Gamma(\alpha_h + 1)}. \]

Consequently, for every \( \mathcal{I}_h \in \mathbb{N} \), we have that
\[ \partial_{t_h}^{\mathcal{I}_h} \psi_h(t_h) = \sum_{j=0}^{+\infty} \frac{\frac{(a_h - \alpha_h j)}{2} (\alpha_h + 1 \cdots (\alpha_h - j_h + 1)(t_h - a_h)^{\alpha_h j - 3_h}}{\Gamma(\alpha_h + 1)}. \]  
(9.27)

Now, we define, for any \( i \in \{1, \ldots, n\} \),
\[ \bar{\mathcal{I}}_i := \begin{cases} \frac{\alpha_i}{|\alpha_i|} & \text{if } \alpha_i \neq 0, \\
1 & \text{if } \alpha_i = 0. \end{cases} \]

We notice that
\[ \bar{\mathcal{I}}_i \neq 0 \text{ for all } i \in \{1, \ldots, n\}, \]  
(9.28)

and
\[ \alpha_i \bar{\mathcal{I}}_i = |\alpha_i|. \]  
(9.29)

Now, for each \( i \in \{1, \ldots, n\} \), we consider the multi-index \( r_i = (r_{i1}, \ldots, r_{ip_i}) \in \mathbb{N}^{p_i} \). This multi-index acts on \( \mathbb{R}^{p_i} \), whose variables are denoted by \( x_i = (x_{i1}, \ldots, x_{ip_i}) \in \mathbb{R}^{p_i} \). We let \( \mathbf{v}_{i1} \) be the solution of the Cauchy problem
\[
\begin{cases}
\partial_{x_{i1}}^{\beta_1} \mathbf{v}_{i1} = -\bar{\mathcal{I}}_i \mathbf{v}_{i1} \\
\partial_{x_{i1}}^{\beta_1} \mathbf{v}_{i1}(0) = 1 & \text{for every } \beta_1 \leq r_{i1} - 1. 
\end{cases}
\]  
(9.30)

We notice that the solution of the Cauchy problem in (9.30) exists at least in a neighborhood of the origin of the form \([-\rho_{i1}, \rho_{i1}]\) for a suitable \( \rho_{i1} > 0 \).
Moreover, if \( p_i \geq 2 \), for any \( \ell \in \{2, \ldots, p_i\} \), we consider the solution of the following Cauchy problem:

\[
\begin{align*}
\partial_{x_i}^{\rho} \psi_{x_i} &= \psi_{x_i} \\
\partial_{x_i}^{\beta} \psi_{x_i}(0) &= 1 \quad \text{for every } \beta \leq r_{i\ell} - 1.
\end{align*}
\] (9.31)

As above, these solutions are well-defined at least in a neighborhood of the origin of the form \([-\rho_{i\ell}, \rho_{i\ell}]\), for a suitable \( \rho_{i\ell} > 0 \).

Then, we define

\[ \bar{\rho}_i := \min\{\rho_{i1}, \ldots, \rho_{ipi}\} = \min_{\ell \in \{1, \ldots, p_i\}} \rho_{i\ell}. \]

In this way, for every \( x_i = (x_{i1}, \ldots, x_{ipi}) \in B^p_{\bar{\rho}_i} \), we set

\[ \psi_i(x_i) := \psi_{i1}(x_{i1}) \ldots \psi_{ipi}(x_{ipi}). \] (9.32)

By (9.30) and (9.31), we have that

\[
\begin{align*}
\partial_{x_i}^{\rho} \psi_i &= -\partial_i \psi_i \\
\partial_{x_i}^{\beta} \psi_i(0) &= 1 \quad \text{for every } \beta = (\beta_1, \ldots, \beta_p) \in \mathbb{N}^p \\
&\quad \text{such that } \beta_\ell \leq r_{i\ell} - 1 \text{ for each } \ell \in \{1, \ldots, p_i\}.
\end{align*}
\] (9.33)

Now, we define

\[ \rho := \min\{\bar{\rho}_1, \ldots, \bar{\rho}_n\} = \min_{i \in \{1, \ldots, n\}} \bar{\rho}_i. \]

We take

\[ \tau \in C^\infty_0 \left( B^p_{\rho/(R+2)} \right), \]

with \( \tau = 1 \) in \( B^p_{\rho/(2(R+2))} \), and, for every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n} \), we set

\[ \tau_i(x_1, \ldots, x_n) := \tau(\tau_1 \otimes x_1, \ldots, \tau_n \otimes x_n). \] (9.34)

We recall that the free parameters \( \tau_1, \ldots, \tau_n \) have been introduced in (9.12), and we have used here the notation

\[ \tau_i \otimes x_i = (\tau_{i1}, \ldots, \tau_{ipi}) \otimes (x_{i1}, \ldots, x_{ipi}) := (\tau_{i1} x_{i1}, \ldots, \tau_{ipi} x_{ipi}) \in \mathbb{R}^{p_i}, \]

for every \( i \in \{1, \ldots, n\} \).

We also set, for any \( i \in \{1, \ldots, n\} \),

\[ v_i(x_i) := \bar{\tau}_i(\tau_i \otimes x_i). \] (9.35)

We point out that if \( x_i \in B^p_{\bar{\rho}_i/(R+2)} \) we have that

\[ |\tau_i \otimes x_i|^2 = \sum_{\ell=1}^{p_i} (\tau_{i\ell} x_{i\ell})^2 \leq (R+2)^2 \sum_{\ell=1}^{p_i} x_{i\ell}^2 < \bar{\rho}_i^2, \]

thanks to (9.12), and therefore the setting in (9.35) is well-defined for every \( x_i \in B^p_{\bar{\rho}_i/(R+2)} \).

Recalling (9.33) and (9.35), we see that, for any \( i \in \{1, \ldots, n\} \),

\[ \partial_{x_i}^{\rho} v_i(x_i) = \tau_i \partial_{x_i}^{\rho} \tau_i(\tau_i \otimes x_i) = -\partial_i \tau_i \tau_i(\tau_i \otimes x_i) = -\partial_i \tau_i \tau_i v_i(x_i). \] (9.36)
We take \( e_1, \ldots, e_M \), with
\[
e_j \in \partial B_{\omega_j}^m,
\]
and we introduce an additional set of free parameters \( Y_1, \ldots, Y_M \) with
\[
Y_j \in \mathbb{R}^m \quad \text{and} \quad e_j : Y_j < 0.
\]
We let \( \epsilon > 0 \), to be taken small possibly depending on the free parameters \( e_j, Y_j \) and \( \frac{1}{n} \), and we define
\[
w(x, y, t) := \tau_1(x) v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_M(y_M + eM + eY_M) \times \psi_1^*(t_1) \cdots \psi_n^*(t_n),
\]
where the setting in (9.19), (9.25), (9.34) and (9.35) has been exploited.

We also notice that \( w \in C((\mathbb{R}^N) \cap C_0(\mathbb{R}^{N-l}) \cap \mathcal{A} \). Moreover, if
\[
a = (a_1, \ldots, a_l) := \left( -\frac{\epsilon}{l_1}, \ldots, -\frac{\epsilon}{l_l} \right) \in \mathbb{R}^l
\]
and \((x, y)\) is sufficiently close to the origin and \( t \in (a_1, +\infty) \times \cdots \times (a_l, +\infty) \), we have that
\[
\Lambda_{-\infty} w(x, y, t)
= \left( \sum_{i=1}^n \partial_i \partial_{x_i}^r + \sum_{j=1}^M \hat{\partial}_j (-(\Delta)^{\rho_j} + \sum_{h=1}^l \hat{c}_h D_{\hat{h}, -\infty}^{\rho_h}) \right) w(x, y, t)
= \sum_{i=1}^n \partial_i v_1(x_1) \cdots v_{i-1}(x_{i-1}) \partial_{x_i} \psi_1^*(t_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_M(y_M + eM + eY_M) \psi_1^*(t_1) \cdots \psi_n^*(t_n)
+ \sum_{j=1}^M \hat{\partial}_j v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_j-1 (y_{j-1} + e_{j-1} + eY_{j-1}) \times (-\Delta)^{\rho_j} \phi_j(y_j + e_j + eY_j) \phi_{j+1}(y_{j+1} + e_{j+1} + eY_{j+1}) \cdots \phi_M(y_M + eM + eY_M) \times \psi_1^*(t_1) \cdots \psi_j^*(t_j)
+ \sum_{h=1}^l \hat{c}_h v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_M(y_M + eM + eY_M) \psi_1^*(t_1) \cdots \psi_{h-1}^*(t_{h-1}) \times D_{\hat{h}, -\infty}^{\rho_h} \psi_h^*(t_h) \psi_{h+1}(t_{h+1}) \cdots \psi_n^*(t_n)
= -\sum_{i=1}^n \partial_i \partial_{x_i}^r \psi_1^*(t_1) \cdots \psi_n^*(t_n) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_M(y_M + eM + eY_M) \psi_1^*(t_1) \cdots \psi_n^*(t_n)
+ \sum_{j=1}^M \hat{\partial}_j \lambda_j v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_M(y_M + eM + eY_M) \psi_1^*(t_1) \cdots \psi_n^*(t_n)
+ \sum_{h=1}^l \hat{c}_{h, \hat{h}} v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_M(y_M + eM + eY_M) \psi_1^*(t_1) \cdots \psi_n^*(t_n)
= \left( -\sum_{i=1}^n \partial_i \partial_{x_i}^r + \sum_{j=1}^M \hat{\partial}_j \lambda_j + \sum_{h=1}^l \hat{c}_{h, \hat{h}} \right) \left( -\sum_{i=1}^n \partial_i \partial_{x_i}^r + \sum_{j=1}^M \hat{\partial}_j \lambda_j + \sum_{h=1}^l \hat{c}_{h, \hat{h}} \right) w(x, y, t),
\]
thanks to (9.20), (9.26) and (9.36).

Consequently, making use of (9.14), (9.15) and (9.29), if \((x, y)\) lies near the origin and \(t \in (a_1, +\infty) \times \cdots \times (a_t, +\infty)\), we have that

\[
\Lambda_{-\infty} w(x, y, t) = \left( -\sum_{i=1}^{N} |c_i| \xi_i^\nu + \sum_{j=1}^{M-1} \hat{\gamma}_j \lambda_j + \hat{\gamma}_M \lambda_M + \sum_{h=1}^{l} c_h \xi_h \right) w(x, y, t)
\]

\[
= \left( -\sum_{i=1}^{N} |c_i| \xi_i^\nu + \sum_{j=1}^{M-1} \hat{\gamma}_j \lambda_j + \hat{\gamma}_M \lambda_M + \sum_{h=1}^{l} c_h \xi_h \right) w(x, y, t) = 0.
\]

This says that \(w \in \mathcal{H}\). Thus, in light of (9.4) we have that

\[
0 = \theta \cdot \partial^K w(0) = \sum_{|\nu| \leq K} \theta_i \partial^\nu w(0) = \sum_{|\nu| \leq K, \nu \neq 0} \theta_i \partial^\nu w(0).
\]

(9.41)

Now, we recall (9.32) and we claim that, for any \(i \in \{1, \ldots, n\}\), any \(\ell \in \{1, \ldots, p_j\}\) and any \(i_j \in \mathbb{N}\), we have that

\[
\partial^i \pi_{x_j}(0) \neq 0.
\]

(9.42)

We prove it by induction over \(i_j\). Indeed, if \(i_j \in \{0, \ldots, r_j-1\}\), then the initial condition in (9.30) (if \(\ell = 1\)) or (9.31) (if \(\ell \geq 2\)) gives that \(\partial^i \pi_{x_j}(0) = 1\), and so (9.42) is true in this case.

To perform the inductive step, let us now suppose that the claim in (9.42) still holds for all \(i_j \in \{0, \ldots, i_0\}\) for some \(i_0\) such that \(i_0 \geq r_j - 1\). Then, using the equation in (9.30) (if \(\ell = 1\)) or in (9.31) (if \(\ell \geq 2\)), we have that

\[
\partial^{i_0 + 1} \pi_{x_j} = \partial^{i_0 + 1 - r_j} \partial^r \pi_{x_j} = -a_j \partial^{i_0 + 1 - r_j} \pi_j,
\]

(9.43)

with

\[
\tilde{a}_j := \begin{cases} a_j & \text{if } \ell = 1, \\ -1 & \text{if } \ell \geq 2. \end{cases}
\]

Notice that \(\tilde{a}_j \neq 0\), in view of (9.28), and \(\partial^{i_0 + 1 - r_j} \pi_j(0) \neq 0\), by the inductive assumption. These considerations and (9.43) give that \(\partial^{i_0 + 1} \pi_j(0) \neq 0\), and this proves (9.42).

Now, using (9.32) and (9.42) we have that, for any \(j \in \{1, \ldots, n\}\) and any \(i_j \in \mathbb{N}^p\),

\[
\partial^{i_j} \pi_j(0) \neq 0.
\]

This, (9.12) and the computation in (9.36) give that, for any \(j \in \{1, \ldots, n\}\) and any \(i_j \in \mathbb{N}^p\),

\[
\partial^{i_j} v_j(0) = \partial^{i_j} \pi_j(0) \neq 0.
\]

(9.44)

We also notice that, in light of (9.25), (9.39) and (9.41),

\[
0 = \sum_{|\nu| + |\nu| \leq K} \theta_i \partial^{i_j} v_j(0) \cdots \partial^{i_n} v_n(0) \partial^{i_j} \psi_1(0) \cdots \partial^{i_n} \psi_n(0).
\]

(9.45)
Now, by (9.19) and Proposition 6.1 (applied to \( s := s_j, \beta := I_j, e := \frac{\epsilon_j y_j}{\omega_j} \in \partial B_1^{m_j} \), due to (9.37), and \( X := \sum \frac{\text{\(s_j \))}}{\omega_j} \), we see that, for any \( j = 1, \ldots, M \),

\[
\omega_j^{\left|I_j\right|} \lim_{\epsilon \searrow 0} \epsilon^{\left|I_j\right| - s_j} \partial_{\tilde{y}_j}^I \phi_j \left( e_j + \epsilon Y_j \right) = \lim_{\epsilon \searrow 0} \epsilon^{\left|I_j\right| - s_j} \partial_{\tilde{y}_j}^I \phi_j \left( \frac{e_j + \epsilon Y_j}{\omega_j} \right) = \frac{\epsilon_j Y_j}{\omega_j} \),
\]

(9.46)

with \( \kappa_j \neq 0 \), in the sense of distributions (in the coordinates \( Y_j \)).

Moreover, using (9.27) and (9.40), it follows that

\[
\partial_{\nu}^h \psi_h(0) = \sum_{j=0}^{+\infty} \frac{j^h \alpha_j (\alpha_j j - 1) \ldots (\alpha_j j - \mathcal{I}_h + 1) (0 - \alpha_j)^{\alpha_j j - 3_h}}{\Gamma (\alpha_j j + 1)}
\]

\[
= \sum_{j=0}^{+\infty} \frac{j^h \alpha_j (\alpha_j j - 1) \ldots (\alpha_j j - \mathcal{I}_h + 1) e^{\alpha_j j - 3_h}}{\Gamma (\alpha_j j + 1) \tilde{I}_h^{\alpha_j j - 3_h}}
\]

\[
= \sum_{j=1}^{+\infty} \frac{j^h \alpha_j (\alpha_j j - 1) \ldots (\alpha_j j - \mathcal{I}_h + 1) e^{\alpha_j j - 3_h}}{\Gamma (\alpha_j j + 1) \tilde{I}_h^{\alpha_j j - 3_h}}.
\]

Accordingly, recalling (9.22), we find that

\[
\lim_{\epsilon \searrow 0} \epsilon^{-h - \alpha_h} \partial_{\nu}^h \psi_h(0) = \lim_{\epsilon \searrow 0} \sum_{j=1}^{+\infty} \frac{j^h \alpha_j (\alpha_j j - 1) \ldots (\alpha_j j - \mathcal{I}_h + 1) e^{\alpha_j j - 1}}{\Gamma (\alpha_j j + 1) \tilde{I}_h^{\alpha_j j - 3_h}}
\]

\[
= \frac{\tilde{I}_h^{\alpha_h - 1} \ldots (\alpha_h - \mathcal{I}_h + 1) e^{\alpha_h}}{\Gamma (\alpha_h + 1)}
\]

(9.47)

Also, recalling (9.5), we can write (9.45) as

\[
0 = \sum_{\left|\mathcal{I}\right| \leq \sum \frac{\mathcal{T}_j}{|\mathcal{I}_j|} \leq K} \theta_{i, \mathcal{I}, \mathcal{G}} \partial_{x_1}^{\mathcal{G}} v_1(0) \ldots \partial_{x_n}^{\mathcal{G}} v_n(0) \partial_{y_1}^{\mathcal{G}} \phi_1 (e_1 + \epsilon Y_1) \ldots \partial_{y_M}^{\mathcal{G}} \phi_M (e_M + \epsilon Y_M)
\]

\[
\times \partial_{\nu}^h \psi_1(0) \ldots \partial_{\nu}^h \psi_M(0).
\]

(9.48)

Moreover, we define

\[
\Xi := |\mathcal{I}| - \sum_{j=1}^{M} s_j + |\mathcal{I}| - \sum_{h=1}^{\mathcal{I}} \alpha_h,
\]

Then, we multiply (9.48) by \( \epsilon^\Xi \in (0, +\infty) \), and we send \( \epsilon \) to zero. In this way, we obtain from (9.46), (9.47) and (9.48) that

\[
0 = \lim_{\epsilon \searrow 0} \epsilon^\Xi \sum_{\left|\mathcal{I}\right| \leq \sum \frac{\mathcal{T}_j}{|\mathcal{I}_j|} \leq K} \theta_{i, \mathcal{I}, \mathcal{G}} \partial_{x_1}^{\mathcal{G}} v_1(0) \ldots \partial_{x_n}^{\mathcal{G}} v_n(0) \partial_{y_1}^{\mathcal{G}} \phi_1 (e_1 + \epsilon Y_1) \ldots \partial_{y_M}^{\mathcal{G}} \phi_M (e_M + \epsilon Y_M)
\]

\[
\times \partial_{\nu}^h \psi_1(0) \ldots \partial_{\nu}^h \psi_M(0)
\]

\[
= \lim_{\epsilon \searrow 0} \sum_{\left|\mathcal{I}\right| \leq \sum \frac{\mathcal{T}_j}{|\mathcal{I}_j|} \leq K} \epsilon^{\Xi - |\mathcal{I}|} \theta_{i, \mathcal{I}, \mathcal{G}} \partial_{x_1}^{\mathcal{G}} v_1(0) \ldots \partial_{x_n}^{\mathcal{G}} v_n(0)
\]

46
\[ x^{[H] - \alpha_1} \partial^{[t]}_y \phi_1 (e_1 + e Y_1) \ldots e^{[M] - \alpha_1} \partial^{[M]}_y \phi_M (e_M + e Y_M) \\
\times e^{\alpha_1 - \alpha_1} \partial^{[t]}_t \psi_1 (0) \ldots e^{\alpha_1 - \alpha_1} \partial^{[t]}_t \psi_n (0) \]
\[ = \sum_{|i| + |j| + |\beta| \leq K} \tilde{C}_{i, j, \beta} \theta_{i, j, \beta} \partial^{[i]}_x v_1 (0) \ldots \partial^{[i]}_x v_n (0) \]
\times e^{I_1} \ldots e^{I_M} (-e_1 \cdot Y_1)^{s_1 - |I_1|} \ldots (-e_M \cdot Y_M)^{s_M - |I_M|} \tilde{\ell}_1 \ldots \tilde{\ell}_l ,
\]
for a suitable $\tilde{C}_{i, j, \beta} \neq 0$ (strictly speaking, the above identity holds in the sense of distribution with respect to the coordinates $Y$ and $\tilde{\ell}$, but since the left hand side vanishes, we can consider it also a pointwise identity).

Hence, recalling (9.44),
\[ 0 = \sum_{|i| + |j| + |\beta| \leq K} C_{i, j, \beta} \theta_{i, j, \beta} \tilde{\ell} = \sum_{|i| + |j| + |\beta| \leq K} C_{i, j, \beta} \theta_{i, j, \beta} \tilde{\ell} x^{\tilde{\ell} i_1} \ldots x^{\tilde{\ell} i_l} \]
\[ \times e^{I_1} \ldots e^{I_M} (-e_1 \cdot Y_1)^{s_1 - |I_1|} \ldots (-e_M \cdot Y_M)^{s_M - |I_M|} \tilde{\ell}_1 \ldots \tilde{\ell}_l , \]
(9.49)
for a suitable $C_{i, j, \beta} \neq 0$.

We observe that the equality in (9.49) is valid for any choice of the free parameters $(\tilde{x}, Y, \tilde{\ell})$ in an open subset of $\mathbb{R}^{p_1 + \ldots + p_n} \times \mathbb{R}^{m_1 + \ldots + m_M} \times \mathbb{R}^l$, as prescribed in (9.12), (9.13) and (9.38).

Now, we take new free parameters, $\tilde{y}_1, \ldots, \tilde{y}_M$ with $\tilde{y}_j \in \mathbb{R}^{m_j} \setminus \{0\}$, and we define
\[ e_j := \frac{\omega_j \tilde{y}_j}{|\tilde{y}_j|} \quad \text{and} \quad Y_j := -\frac{\tilde{y}_j}{|\tilde{y}_j|^2} . \]
(9.50)
We stress that the setting in (9.50) is compatible with that in (9.38), since
\[ e_j \cdot Y_j = \frac{\omega_j \tilde{y}_j}{|\tilde{y}_j|} \cdot \frac{\tilde{y}_j}{|\tilde{y}_j|^2} = -\frac{\omega_j}{|\tilde{y}_j|} < 0, \]
thanks to (9.18). We also notice that, for all $j \in \{1, \ldots, M\}$,
\[ e_j (\tilde{y}_j)^{s_j - |I_j|} = \frac{\omega_j \tilde{y}_j^{s_j - |I_j|}}{|\tilde{y}_j|^{s_j - |I_j|}} = \tilde{y}_j^{s_j - |I_j|} , \]
and hence
\[ e^l (\tilde{y}_1)^{s_1 - |I_1|} \ldots (\tilde{y}_M)^{s_M - |I_M|} = \tilde{y}_j^t . \]
Plugging this into formula (9.49), we obtain the first identity in (9.7), as desired. Hence, the proof of (9.7) in case 1 is complete.

**Proof of (9.7), case 2.** Thanks to the assumptions given in case 2, we can suppose that formula (9.9) still holds, and also that
\[ c_1 > 0 . \]
(9.51)
In addition, for any \( j \in \{1, \ldots, M\} \), we consider \( \lambda_j \) and \( \phi_j \) as in (9.20).

Then, we define
\[
R := \left( \frac{1}{|c_1|} \left( \sum_{h=1}^{l-1} |c_h| + \sum_{j=1}^{M} |\hat{\phi}_j| \lambda_j \right) \right)^{1/|r_1|}.
\] (9.52)

We notice that, in light of (9.9), the setting in (9.52) is well-defined.

Now, we fix two sets of free parameters \( \bar{\lambda}_1, \ldots, \bar{\lambda}_n \) as in (9.12) and \( \bar{\xi}_{*,1}, \ldots, \bar{\xi}_{*,l} \) as in (9.13), here taken with \( R \) as in (9.52). Moreover, we define
\[
\lambda := \frac{1}{c_{t_{*,l}}} \left( \sum_{j=1}^{n} |a_j| \bar{\xi}_j \sum_{j=1}^{M} \hat{\phi}_j \lambda_j - \sum_{h=1}^{l-1} c_h \bar{\xi}_{*,h} \right).
\] (9.53)

We notice that (9.53) is well-defined, thanks to (9.13) and (9.51). Furthermore, recalling (9.12), (9.16) and (9.52), we find that
\[
\sum_{i=1}^{n} |a_i| \bar{\xi}_i \geq |a_1| \bar{\xi}_1 > |a_1|(R + 1)|r_1| > |a_1|R|r_1|
\]
\[
= \sum_{h=1}^{l-1} |c_h| + \sum_{j=1}^{M} |\hat{\phi}_j| \lambda_j \geq \sum_{h=1}^{l-1} c_h \bar{\xi}_{*,h} + \sum_{j=1}^{M} \hat{\phi}_j \lambda_j.
\]

Consequently, by (9.53),
\[
\lambda > 0.
\] (9.54)

Hence, we can define
\[
\overline{\lambda} := \lambda^{1/\alpha_l}.
\] (9.55)

Moreover, we consider \( a_h \in (-2, 0) \), for every \( h \in \{1, \ldots, l\} \), to be chosen appropriately in what follows (the exact choice will be performed in (9.62)), and, using the notation in (9.21) and (9.22), we define
\[
\psi_h(t_h) := \psi_{*,h}(\bar{\xi}_h(t_h - a_h)) = E_{\alpha_{*,1}}(\bar{\xi}_{*,h}(t_h - a_h)^{\alpha_h}) \quad \text{if } h \in \{1, \ldots, l - 1\}
\] (9.56)
and
\[
\psi(t_l) := \psi_{*,l}(\overline{\lambda}_l(t_l - a_l)) = E_{\alpha_{*,1}}(\lambda_{*,l}(t_l - a_l)^{\alpha_l}).
\] (9.57)

We recall that, thanks to Lemma 2.2, the function in (9.56) solves (9.24) and satisfies (9.27) for any \( h \in \{1, \ldots, l - 1\} \), while the function in (9.57) solves
\[
\begin{aligned}
D_{t_{*,l}^m a_{l}^m}^\alpha \psi_l(t_l) &= \lambda_{*,l}^m \psi_l(t_l) \quad \text{in } (a_l, +\infty), \\
\psi_l(a_l) &= 1, \\
\partial_{t_{*,l}^m}^m \psi_l(a_l) &= 0 \quad \text{for every } m \in \{1, \ldots, \alpha_l\}.
\end{aligned}
\] (9.58)

As in (9.25), we extend the functions \( \psi_h \) constantly in \((-\infty, a_h)\), calling \( \psi_h^* \) this extended function. In this way, Lemma A.3 in [CDV18] translates (9.58) into
\[
D_{t_{*,l}^m a_{l}^m}^\alpha \psi_h^*(t_h) = \bar{\xi}_{*,h}^m \psi_h(t_h) = \bar{\xi}_{*,h}^m \psi_h^*(t_h) \quad \text{in every interval } I \subset (a_h, +\infty).
\] (9.59)

Now, we let \( \epsilon > 0 \), to be taken small possibly depending on the free parameters, and we exploit the functions defined in (9.34) and (9.35), provided that one replaces the positive constant \( R \) defined in (9.11) with the one in (9.52), when necessary.
With this idea in mind, for any \( j \in \{1, \ldots, M\} \), we let
\[
e_j \in \partial B_1^{m_j},
\]
and we define
\[
w(x, y, t) := \tau_1(x) v_1(x) \cdots v_n(x) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M) \times \psi_1^*(t_1) \cdots \psi_t^*(t_t),
\]
where the setting in (9.20), (9.34), (9.35), (9.38), (9.56) and (9.57) has been exploited.

We also notice that \( w \in C(\mathbb{R}^N) \cap C_0(\mathbb{R}^{N-l}) \cap \mathcal{A} \). Moreover, if
\[
a = (a_1, \ldots, a_l) := \left(-\frac{\epsilon}{l_1}, \ldots, -\frac{\epsilon}{l_l}\right) \in \mathbb{R}^l
\]
and \((x, y)\) is sufficiently close to the origin and \( t \in (a_1, +\infty) \times \cdots \times (a_l, +\infty)\), we have that
\[
\Lambda_{-\infty} w(x, y, t)
= \left( \sum_{i=1}^n \mathcal{A}_i \partial_x^{r_i} + \sum_{j=1}^M \mathcal{B}_j (\Delta)^{s_j} + \sum_{h=1}^l \mathcal{C}_h D_{l_h, -\infty}^{\alpha_h} \right) w(x, y, t)
= \sum_{i=1}^n \mathcal{A}_i v_1(x_1) \cdots v_{i-1}(x_{i-1}) \partial_{x_i}^{r_i} v_i(x_i) v_{i+1}(x_{i+1}) \cdots v_n(x_n)
\times \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M) \psi_1^*(t_1) \cdots \psi_{t-1}^*(t_{t-1}) \psi_t^*(t_t)
+ \sum_{j=1}^M \mathcal{B}_j v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_{j-1}(y_{j-1} + e_{j-1} + \epsilon Y_{j-1})
\times (\Delta)^{s_j} \phi_j(y_j + e_j + \epsilon Y_j) \phi_{j+1}(y_{j+1} + e_{j+1} + \epsilon Y_{j+1}) \cdots \phi_M(y_M + e_M + \epsilon Y_M)
\times \psi_1^*(t_1) \cdots \psi_{t-1}^*(t_{t-1}) \psi_t^*(t_t)
+ \sum_{h=1}^l \mathcal{C}_h v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M)
\times D_{l_h, -\infty}^{\alpha_h} \psi_h^*(t_h) \psi_{h+1}^*(t_{h+1}) \cdots \psi_{t-1}^*(t_{t-1}) \psi_t^*(t_t)
= \sum_{i=1}^n \mathcal{A}_i \sum_{j=1}^M \mathcal{B}_j \sum_{h=1}^l \mathcal{C}_h v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M)
\times \psi_1^*(t_1) \cdots \psi_{t-1}^*(t_{t-1}) \psi_t^*(t_t)
+ \sum_{j=1}^M \mathcal{B}_j \sum_{i=1}^n \mathcal{A}_i v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M)
\times \psi_1^*(t_1) \cdots \psi_{t-1}^*(t_{t-1}) \psi_t^*(t_t)
+ \sum_{h=1}^l \mathcal{C}_h \sum_{i=1}^n \mathcal{A}_i v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M)
\times \psi_1^*(t_1) \cdots \psi_{t-1}^*(t_{t-1}) \psi_t^*(t_t)
+ \mathcal{I}_l \mathcal{A}_i \sum_{j=1}^M \mathcal{B}_j \sum_{h=1}^l \mathcal{C}_h v_1(x_1) \cdots v_n(x_n) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M)
\times \psi_1^*(t_1) \cdots \psi_{t-1}^*(t_{t-1}) \psi_t^*(t_t)
\]
\(^5\)Comparing (9.60) with (9.37), we observe that (9.37) reduces to (9.60) with the choice \( \omega_j := 1 \).
\[
\begin{align*}
\psi^*_{t_1} \cdots \psi^*_{t_{l-1}} (t_{l-1}) \psi^*_{t_l} (t_l)
\begin{pmatrix}
- \sum_{i=1}^n \alpha_i \overline{\tau}_i \tau_i
\end{pmatrix} + \sum_{j=1}^M \lambda_j \nabla_j + \sum_{h=1}^{l-1} \gamma_h \lambda_h + \epsilon \Lambda_{t*,l} (0) \end{pmatrix} w(x, y, t),
\end{align*}
\]

thanks to (9.20), (9.24), (9.36) and (9.59).

Consequently, making use of (9.29) and (9.53), when \((x, y)\) is near the origin and \(t \in (a_1, +\infty) \times \cdots \times (a_l, +\infty)\), we have that

\[
\Lambda_{-\infty} w(x, y, t) = \left( - \sum_{i=1}^n \alpha_i \overline{\tau}_i \tau_i + \sum_{j=1}^M \lambda_j \nabla_j + \sum_{h=1}^{l-1} \gamma_h \lambda_h + \epsilon \Lambda_{t*,l} (0) \right) w(x, y, t) = 0.
\]

This says that \(w \in \mathcal{H}\). Thus, in light of (9.4) we have that

\[
0 = \theta \cdot \partial^K w(0) = \sum_{|i| \leq K} \theta_i \partial^i w(0) = \sum_{|i| + |j| + |\beta| \leq K} \theta_{i,j,\beta} \partial^i \partial^j \partial^\beta w(0).
\]

Hence, in view of (9.44) and (9.61),

\[
0 = \sum_{|i| + |j| + |\beta| \leq K} \theta_{i,j,\beta} \partial^i \partial^j \partial^\beta w(0) = \sum_{|i| + |j| + |\beta| \leq K} \theta_{i,j,\beta} \partial^i \partial^j \partial^\beta w(0).
\]

Moreover, using (2.1), (9.57) and (9.62), it follows that

\[
\partial^j \psi^* (0) = \sum_{j=0}^{+\infty} \lambda^{j}_{l,j} (\alpha l j - 1) \cdots (\alpha l j - 3) (0 - a_1)^{a l j - 3} \Gamma (\alpha l j + 1)
\]

\[
\begin{align*}
= \sum_{j=0}^{+\infty} \lambda^{j}_{l,j} (\alpha l j - 1) \cdots (\alpha l j - 3) e^{a l j - 3} \Gamma (\alpha l j + 1) \frac{1}{\lambda^{j}_{l,j} (\alpha l j - 1) \cdots (\alpha l j - 3) \Gamma (\alpha l j + 1)}
\end{align*}
\]

Accordingly, by (9.22), we find that

\[
\lim_{\epsilon \to 0} \epsilon^{\alpha l j - \alpha} \partial^j \psi^* (0) = \lim_{\epsilon \to 0} \sum_{j=1}^{+\infty} \lambda^{j}_{l,j} (\alpha l j - 1) \cdots (\alpha l j - 3) (0 - a_1)^{a l j - 3} \Gamma (\alpha l j + 1) \frac{1}{\lambda^{j}_{l,j} (\alpha l j - 1) \cdots (\alpha l j - 3) \Gamma (\alpha l j + 1)}
\]

\[
\begin{align*}
= \lambda^{j}_{l,j} (\alpha l j - 1) \cdots (\alpha l j - 3) (0 - a_1)^{a l j - 3} \Gamma (\alpha l j + 1) \frac{1}{\lambda^{j}_{l,j} (\alpha l j - 1) \cdots (\alpha l j - 3) \Gamma (\alpha l j + 1)}
\end{align*}
\]

Hence, recalling (9.6), we can write (9.63) as

\[
0 = \sum_{|i| + |j| + |\beta| \leq K} \theta_{i,j,\beta} \partial^i \partial^j \partial^\beta w(0) = \sum_{|i| + |j| + |\beta| \leq K} \theta_{i,j,\beta} \partial^i \partial^j \partial^\beta w(0)
\]

\[
\times \partial^j \psi^* (e_1 + e Y_1) \cdots \partial^j \psi^* (e M + e Y_M) \partial^j \psi^* (0) \cdots \partial^j \psi^* (0).
\]
Moreover, we define
\[
\Xi := |\mathcal{F}| - \sum_{h=1}^{l} \alpha_{h} + |I| - \sum_{j=1}^{M} s_{j}.
\]
Then, we multiply (9.65) by \( \epsilon^{\Xi} \in (0, +\infty) \), and we send \( \epsilon \) to zero. In this way, we obtain from (9.47), used here for \( h \in \{1, \ldots, l-1\} \), (9.64) and (9.65) that
\[
0 = \lim_{\epsilon \to 0} \epsilon^{\Xi} \sum_{|i|+|I|+|\bar{I}| \leq \bar{K}} \frac{\theta_{i,I,\bar{I}}}{|\mathcal{P}|} \frac{\chi_{a}}{\epsilon} \prod_{\overline{m}} \frac{\delta_{i_{m}}^{1} \psi_{1}(0) \ldots \delta_{i_{n}}^{\bar{m}} \psi_{n}(0) \prod_{i,I}}{\epsilon^{\Xi} Y_{I}} \frac{\partial_{i_{m}}^{1} \psi_{1}(0) \ldots \partial_{i_{n}}^{\bar{m}} \psi_{n}(0)}{\epsilon^{\Xi} Y_{I}} 
\]
for a suitable \( \tilde{C}_{i,I,\bar{I}} \). We stress that \( \tilde{C}_{i,I,\bar{I}} \neq 0 \), thanks also to (9.46), applied here with \( \omega_{j} := 1 \), \( \tilde{\phi}_{*,j} := \phi_{j} \) and \( e_{j} \) as in (9.60) for any \( j \in \{1, \ldots, M\} \).

Hence, recalling (9.54),
\[
0 = \sum_{|i|+|I|+|\bar{I}| \leq \bar{K}} C_{i,I,\bar{I}} \frac{\theta_{i_{1},\ldots,i_{l},I_{1},\ldots,I_{M},\bar{I}_{1},\ldots,\bar{I}_{\bar{m}}}}{|\mathcal{P}|} \frac{\chi_{a}}{\epsilon} \prod_{\overline{m}} \frac{\delta_{i_{m}}^{1} \psi_{1}(0) \ldots \delta_{i_{n}}^{\bar{m}} \psi_{n}(0) \prod_{i,I}}{\epsilon^{\Xi} Y_{I}} \frac{\partial_{i_{m}}^{1} \psi_{1}(0) \ldots \partial_{i_{n}}^{\bar{m}} \psi_{n}(0)}{\epsilon^{\Xi} Y_{I}} \prod_{i,I}
\]
for a suitable \( C_{i,I,\bar{I}} \neq 0 \).

We observe that the equality in (9.66) is valid for any choice of the free parameters \( (\overline{\gamma}, Y,\hat{\xi}) \) in an open subset of \( \mathbb{R}^{p_{1}+\cdots+p_{n}} \times \mathbb{R}^{m_{1}+\cdots+m_{M}} \times \mathbb{R}^{l} \), as prescribed in (9.12), (9.13) and (9.38).

Now, we take new free parameters \( \overline{\gamma}_{j} \) with \( \overline{\gamma}_{j} \in \mathbb{R}^{m_{j}} \setminus \{0\} \) for any \( j = 1, \ldots, M \), and perform in (9.66) the same change of variables done in (9.50), obtaining that
\[
0 = \sum_{|i|+|I|+|\bar{I}| \leq \bar{K}} C_{i,I,\bar{I}} \frac{\theta_{i_{1},\ldots,i_{l},I_{1},\ldots,I_{M},\bar{I}_{1},\ldots,\bar{I}_{\bar{m}}}}{|\mathcal{P}|} \frac{\chi_{a}}{\epsilon} \prod_{\overline{m}} \frac{\delta_{i_{m}}^{1} \psi_{1}(0) \ldots \delta_{i_{n}}^{\bar{m}} \psi_{n}(0) \prod_{i,I}}{\epsilon^{\Xi} Y_{I}} \frac{\partial_{i_{m}}^{1} \psi_{1}(0) \ldots \partial_{i_{n}}^{\bar{m}} \psi_{n}(0)}{\epsilon^{\Xi} Y_{I}} \prod_{i,I}
\]
for some \( C_{i,I,\bar{I}} \neq 0 \).

Hence, the second identity in (9.7) is obtained as desired, and the proof of Lemma 9.1 in case 2 is completed. \( \square \)
Proof of (9.7), case 3. We divide the proof of case 3 into two subcases, namely either

\begin{equation}
\text{there exists } h \in \{1, \ldots, l\} \text{ such that } c_h \neq 0, \tag{9.67}
\end{equation}

or

\begin{equation}
c_h = 0 \text{ for every } h \in \{1, \ldots, l\}. \tag{9.68}
\end{equation}

We start by dealing with the case in (9.67). Up to relabeling and reordering the coefficients \(c_h\), we can assume that

\begin{equation}
c_1 \neq 0. \tag{9.69}
\end{equation}

Also, thanks to the assumptions given in case 3, we can suppose that

\begin{equation}
\bar{\theta}_M < 0, \tag{9.70}
\end{equation}

and, for any \(j \in \{1, \ldots, M\}\), we consider \(\lambda_{*,j}\) and \(\tilde{\phi}_{*,j}\) as in (9.8). Then, we take \(\omega_j := 1\) and \(\phi_j\) as in (9.19), so that (9.20) is satisfied. In particular, here we have that

\begin{equation}
\lambda_j = \lambda_{*,j} \quad \text{and} \quad \phi_j = \tilde{\phi}_{*,j}. \tag{9.71}
\end{equation}

We define

\begin{equation}
R := \frac{1}{|c_1|} \sum_{j=1}^{M-1} |\bar{\theta}_j| \lambda_{*,j}. \tag{9.72}
\end{equation}

We notice that, in light of (9.69), the setting in (9.72) is well-defined.

Now, we fix a set of free parameters

\begin{equation}
\bar{l}_{*,1} \in (R + 1, R + 2), \ldots, \bar{l}_{*,l} \in (R + 1, R + 2). \tag{9.73}
\end{equation}

Moreover, we define

\begin{equation}
\lambda_M := \frac{1}{\bar{\theta}_M} \left( - \sum_{j=1}^{M-1} \bar{\theta}_j \lambda_{*,j} - \sum_{h=1}^{l} |c_h| \bar{l}_{*,h} \right). \tag{9.74}
\end{equation}

We notice that (9.74) is well-defined thanks to (9.70). From (9.72) we deduce that

\begin{align*}
\sum_{h=1}^{l} |c_h| \bar{l}_{*,h} + \sum_{j=1}^{M-1} \bar{\theta}_j \lambda_{*,j} &\geq |c_1| \bar{l}_{*,1} - \sum_{j=1}^{M-1} |\bar{\theta}_j| \lambda_{*,j} \\
&> |c_1| R - \sum_{j=1}^{M-1} |\bar{\theta}_j| \lambda_{*,j} = 0.
\end{align*}

Consequently, by (9.70) and (9.74),

\begin{equation}
\lambda_M > 0. \tag{9.75}
\end{equation}

Now, we define, for any \(h \in \{1, \ldots, l\}\),

\begin{equation}
\overline{c}_h := \begin{cases} 
\frac{|c_h|}{|c_h|} & \text{if } c_h \neq 0, \\
1 & \text{if } c_h = 0.
\end{cases} \tag{9.76}
\end{equation}

We notice that

\begin{equation}
\overline{c}_h \neq 0 \text{ for all } h \in \{1, \ldots, l\}, \tag{9.76}
\end{equation}

52
Moreover, we consider $a_h \in (-2,0)$, for every $h = 1, \ldots, l$, to be chosen appropriately in what follows (see (9.85) for a precise choice).

Now, for every $h \in \{1, \ldots, l\}$, we define

$$\psi_h(t_h) := E_{\alpha_h,1}(\zeta_{h+1,h}^+(t_h - a_h)^\alpha_h),$$

(9.78)

where $E_{\alpha_h,1}$ denotes the Mittag-Leffler function with parameters $\alpha := \alpha_h$ and $\beta := 1$ as defined in (2.1). By Lemma 2.2, we know that

$$D_{t_h,a_h}^{\alpha_h} \psi_h(t_h) = \zeta_{h+1,h}^+ \psi_h(t_h) \quad \text{in} \quad (a_h, +\infty),$$

(9.79)

and we consider again the extension $\psi_h^*$ given in (9.25). By Lemma A.3 in [CDV18], we know that (9.79) translates into

$$D_{t_h,-\infty}^{\alpha_h} \psi_h^*(t_h) = \zeta_{h+1,h}^+ \psi_h^*(t_h) \quad \text{in every interval} \quad I \in (a_h, +\infty).$$

(9.80)

Now, we consider auxiliary parameters $\xi_h, e_j$ and $Y_j$ as in (9.22), (9.37) and (9.38). Moreover, we introduce an additional set of free parameters

$$\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}. $$

(9.81)

We let $\epsilon > 0$, to be taken small possibly depending on the free parameters. We take $\tau \in C^\infty(\mathbb{R}^{p_1+\cdots+p_n}, [0 + \infty))$ such that

$$\tau(x) := \begin{cases} \exp(\xi \cdot x) & \text{if} \quad x \in B_1^{p_1+\cdots+p_n}, \\ 0 & \text{if} \quad x \in \mathbb{R}^{p_1+\cdots+p_n} \setminus B_2^{p_1+\cdots+p_n}, \end{cases}$$

(9.82)

where

$$\xi \cdot x := \sum_{j=1}^n \xi_i \cdot x_i$$

denotes the standard scalar product.

We notice that, for any $i \in \mathbb{N}^{p_1} \times \cdots \times \mathbb{N}^{p_n}$,

$$\partial_{x}^i \tau(0) = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \tau(0) = \text{e}^{i_1} \cdots \text{e}^{i_{p_1}} \cdots \text{e}^{i_{p_1}} \cdots \text{e}^{i_{p_n}} = \text{e}^{i}. $$

(9.83)

We define

$$w(x,y,t) := \tau(x) \phi_1(y_1 + e_1 + eY_1) \cdots \phi_M(y_M + e_1 + eY_M) \psi_1^*(t_1) \cdots \psi_l^*(t_l),$$

(9.84)

where the setting in (9.20) has also been exploited.

We also notice that $w \in C(\mathbb{R}^N) \cap C_0(\mathbb{R}^{N-l}) \cap A$. Moreover, if

$$a = (a_1, \ldots, a_l) := \left(-\frac{\epsilon}{\xi_1}, \ldots, -\frac{\epsilon}{\xi_l}\right) \in \mathbb{R}^l $$

(9.85)
and \((x, y)\) is sufficiently close to the origin and \(t \in (a_1, +\infty) \times \cdots \times (a_l, +\infty)\), we have that

\[
\Lambda_{-\infty} w(x, y, t) = \left( \sum_{j=1}^M \partial_j (-\Delta)^{s_j} + \sum_{h=1}^l c_h D_{t_h, -\infty}^{\alpha_h} \right) w(x, y, t)
\]

\[
= \sum_{j=1}^M \partial_j \tau(x) \phi_1 (y_1 + e_1 + \epsilon Y_1) \ldots \phi_{j-1} (y_{j-1} + e_{j-1} + \epsilon Y_{j-1}) (-\Delta)^{s_j} \phi_j (y_j + e_j + \epsilon Y_j)
\]

\[
\times \phi_{j+1} (y_{j+1} + e_{j+1} + \epsilon Y_{j+1}) \ldots \phi_M (y_M + e_M + \epsilon Y_M) \psi^*_1 (t_1) \ldots \psi^*_l (t_l)
\]

\[
+ \sum_{h=1}^l c_h \tau(x) \phi_1 (y_1 + e_1 + \epsilon Y_1) \ldots \phi_M (y_M + e_M + \epsilon Y_M) \psi^*_1 (t_1) \ldots \psi^*_h (t_{h-1})
\]

\[
\times D_{t_h, -\infty}^{\alpha_h} \psi^*_h (t_h) \psi^*_h (t_{h+1}) \ldots \psi^*_l (t_l)
\]

\[
= \sum_{j=1}^M \partial_j \lambda_j \tau(x) \phi_1 (y_1 + e_1 + \epsilon Y_1) \ldots \phi_M (y_M + e_M + \epsilon Y_M) \psi^*_1 (t_1) \ldots \psi^*_l (t_l)
\]

\[
+ \sum_{h=1}^l c_h \lambda_h \tau(x) \phi_1 (y_1 + e_1 + \epsilon Y_1) \ldots \phi_M (y_M + e_M + \epsilon Y_M) \psi^*_1 (t_1) \ldots \psi^*_h (t_{h-1})
\]

\[
= \left( \sum_{j=1}^M \partial_j \lambda_j + \sum_{h=1}^l c_h \lambda_h \right) w(x, y, t),
\]

thanks to (9.20) and (9.80).

Consequently, making use of (9.71), (9.74) and (9.77), if \((x, y)\) is near the origin and \(t \in (a_1, +\infty) \times \cdots \times (a_l, +\infty)\), we have that

\[
\Lambda_{-\infty} w(x, y, t) = \left( \sum_{j=1}^M \partial_j \lambda_{s,j} + \partial_M \lambda_M + \sum_{h=1}^l c_h \lambda_{s,h} \right) w(x, y, t) = 0.
\]

This says that \(w \in \mathcal{H}\). Thus, in light of (9.4) we have that

\[
0 = \theta \cdot \partial^K w (0) = \sum_{|i| \leq K} \theta_i \partial^i w (0) = \sum_{|i| + |l| + |\gamma| \leq K} \theta_{i,l,\gamma} \partial_x^i \partial_y^l \partial_t^\gamma w (0).
\]

From this and (9.84), we obtain that

\[
0 = \sum_{|i| + |l| + |\gamma| \leq K} \theta_{i,l,\gamma} \partial_x^i \tau (0) \partial_y^l \phi_1 (e_1 + \epsilon Y_1) \ldots \partial_y^l \phi_M (e_M + \epsilon Y_M) \partial_t^\gamma \psi_1 (0) \ldots \partial_t^\gamma \psi_1 (0). \tag*{(9.86)}
\]

Moreover, using (9.78) and (9.85), it follows that, for every \(\mathcal{J}_h \in \mathbb{N}\)

\[
\partial_{t_h}^{\lambda_h} \psi_h (0) = \sum_{j=0}^{+\infty} \xi_h^{ij} \alpha_h (\alpha_h j - 1) \ldots (\alpha_h j - \mathcal{J}_h + 1) (0 - \alpha_h)^{\alpha_h j - \mathcal{J}_h} \frac{\Gamma (\alpha_h j + 1)}{\Gamma (\alpha_h j + \mathcal{J}_h) \xi_h^{\alpha_h j - \mathcal{J}_h}}.
\]

54
Accordingly, recalling (9.22), we find that

$$
\lim_{\epsilon \to 0} e^{\alpha h} \partial_{yh}(\epsilon \partial_y \phi_1(\epsilon e_Y) \ldots \partial_{yM} \phi_M(\epsilon e_Y)) \partial_{y1} \psi_1(0) \ldots \partial_{y1} \psi_1(0) = \lim_{\epsilon \to 0} e^{\alpha h} \partial_{yh}(\epsilon \partial_y \phi_1(\epsilon e_Y) \ldots \partial_{yM} \phi_M(\epsilon e_Y)) \partial_{y1} \psi_1(0) \ldots \partial_{y1} \psi_1(0).
$$

(9.88)

Moreover, we define

$$
\Xi := |\bar{T}| - \sum_{j=1}^{M} s_j + |\mathcal{I}| - \sum_{h=1}^{I} \alpha_h.
$$

Then, we multiply (9.88) by $\epsilon^\Xi \in (0, +\infty)$, and we send $\epsilon$ to zero. In this way, we obtain from (9.46), (9.83), (9.87) and (9.88) that

$$
0 = \sum_{|i| + |l| + |\mathcal{I}| \leq K} C_{i,l,\mathcal{I}} \theta_{i,l,\mathcal{I}} \partial_{x} \tau(0) \partial_{y} \phi_1(\epsilon e_Y) \ldots \partial_{yM} \phi_M(\epsilon e_Y) \partial_{y1} \psi_1(0) \ldots \partial_{y1} \psi_1(0) \partial_{y1} \psi_1(0)
$$

$$
\times e^{\alpha h} \partial_{y1} \psi_1(0) \ldots e^{\alpha h} \partial_{y1} \psi_1(0)
$$

$$
= \sum_{|i| + |l| + |\mathcal{I}| \leq K} C_{i,l,\mathcal{I}} \theta_{i,l,\mathcal{I}} \partial_{x} \tau(0) \partial_{y} \phi_1(\epsilon e_Y) \ldots \partial_{yM} \phi_M(\epsilon e_Y) \partial_{y1} \psi_1(0) \ldots \partial_{y1} \psi_1(0) \partial_{y1} \psi_1(0)
$$

$$
\times e^{\alpha h} \partial_{y1} \psi_1(0) \ldots e^{\alpha h} \partial_{y1} \psi_1(0)
$$

$$
= \sum_{|i| + |l| + |\mathcal{I}| \leq K} C_{i,l,\mathcal{I}} \theta_{i,l,\mathcal{I}} \partial_{x} \tau(0) \partial_{y} \phi_1(\epsilon e_Y) \ldots \partial_{yM} \phi_M(\epsilon e_Y) \partial_{y1} \psi_1(0) \ldots \partial_{y1} \psi_1(0) \partial_{y1} \psi_1(0)
$$

$$
\times e^{\alpha h} \partial_{y1} \psi_1(0) \ldots e^{\alpha h} \partial_{y1} \psi_1(0)
$$

for a suitable $C_{i,l,\mathcal{I}} \neq 0$.

We observe that the latter equality is valid for any choice of the free parameters $(\Xi, Y, \mathcal{I})$ in an open subset of $\mathbb{R}^{n+ \ldots + n} \times \mathbb{R}^{m_1 + \ldots + m_M} \times \mathbb{R}^l$, as prescribed in (9.38), (9.73) and (9.81).

Now, we take new free parameters $\mathcal{I} j$ with $\mathcal{I} j \in \mathbb{R}^{m_M} \setminus \{0\}$ for any $j = 1, \ldots, M$, and perform in the latter identity the same change of variables done in (9.50), obtaining that

$$
0 = \sum_{|i| + |l| + |\mathcal{I}| \leq K} C_{i,l,\mathcal{I}} \theta_{i,l,\mathcal{I}} \partial_{y} \phi_1(\epsilon e_Y) \ldots \partial_{yM} \phi_M(\epsilon e_Y) \partial_{y1} \psi_1(0) \ldots \partial_{y1} \psi_1(0) \partial_{y1} \psi_1(0)
$$

for some $C_{i,l,\mathcal{I}} \neq 0$. This completes the proof of (9.7) in case (9.67) is satisfied.
Hence, we now focus on the case in which (9.68) holds true. For any \( j \in \{1, \ldots, M\} \), we consider the function \( \psi \in H^{s_j}(\mathbb{R}^{m_j}) \cap C^0(\mathbb{R}^{m_j}) \) constructed in Lemma 7.1 and we call such function \( \phi_j \), to make it explicit its dependence on \( j \) in this case. We recall that

\[
(-\Delta)^{s_j} \phi_j(y_j) = 0 \quad \text{in} \quad B_1^{m_j}.
\]

(9.89)

Also, for every \( j \in \{1, \ldots, M\} \), we let \( \epsilon_j \) and \( Y_j \) be as in (9.37) and (9.38). Thanks to Lemma 7.1 and Remark 7.2, for any \( I_j \in \mathbb{N}^{m_j} \), we know that

\[
\lim_{\epsilon \downarrow 0} e^{i|\epsilon_j| - s_j} \partial_{y_j}^i \phi_j(e_j + \epsilon Y_j) = \kappa_{s_j} e^{i|\epsilon_j|},
\]

(9.90)

for some \( \kappa_{s_j} \neq 0 \).

Moreover, for any \( h = 1, \ldots, l \), we define \( \tau_h(t_h) \) as

\[
\tau_h(t_h) := \begin{cases} 
eq e^{t_h} & \text{if} \quad t_h \in [-1, +\infty), \\ e^{-t_h} \sum_{i=0}^{t_h-1} \frac{1}{i!} t_h^i (t_h + 1)^i & \text{if} \quad t_h \in (-\infty, -1), \\ \end{cases}
\]

(9.91)

where \( \ell = (\ell_1, \ldots, \ell_l) \in (1, 2)^l \) are free parameters.

We notice that, for any \( j \in \{1, \ldots, M\} \) and \( J_h \in \mathbb{N}^l \),

\[
\partial_{h_j}^{\ell_j} \tau_h(0) = \ell_j^\ell.
\]

(9.92)

Now, we define

\[
w(x, y, t) := \tau(x) \phi_1(y_1 + e_1 + \epsilon Y_1) \cdots \phi_M(y_M + e_M + \epsilon Y_M) \tau_1(t_1) \cdots \tau_l(t_l),
\]

(9.93)

where the setting of (9.19), (9.82) and (9.91) has been exploited. We have that \( w \in \mathcal{A} \). Moreover, we point out that, since \( \tau, \phi_1, \ldots, \phi_M \) are compactly supported, we have that \( w \in C(\mathbb{R}^{N}) \cap C_0(\mathbb{R}^{N-1}) \), and, using Proposition 7.4, for any \( j \in \{1, \ldots, M\} \), it holds that \( \phi_j \in C^\infty(\mathcal{N}_j) \) for some neighborhood \( \mathcal{N}_j \) of the origin in \( \mathbb{R}^{m_j} \). Hence \( w \in C^\infty(\mathcal{N}) \).

Furthermore, using (9.89), when \( y \) is in a neighborhood of the origin we have that

\[
\Lambda_{-\infty} w(x, y, t) = \tau(x) \left( \tilde{\phi}_1(\cdot - \Delta)^{s_1}_{y_1} \phi_1(y_1 + e_1 + \epsilon Y_1) \right) \cdots \left( \tilde{\phi}_M(\cdot - \Delta)^{s_M}_{y_M} \phi_M(y_M + e_M + \epsilon Y_M) \right) \tau_1(t_1) \cdots \tau_l(t_l) = 0,
\]

(9.94)

which gives that \( w \in \mathcal{H} \).

In addition, using (9.5), (9.83) and (9.92), we have that

\[
0 = \theta \cdot \partial^K w(0) = \sum_{|\ell| \leq K} \theta_{i, I, J} \partial_{y_j}^{\ell_j} \partial_{\theta}^{\ell_j} w(0) = \sum_{|\ell| \leq K} \theta_{i, I, J} \partial_{y_j}^{\ell_j} \partial_{\theta}^{\ell_j} w(0)
\]

\[
= \sum_{|\ell| \leq K} \theta_{i, I, J} \varpi^{i} \partial_{y_j}^{\ell_j} \phi_1(e_1 + \epsilon Y_1) \cdots \partial_{y_M}^{\ell_M} \phi_M(e_M + \epsilon Y_M) \ell^\ell.
\]

(9.95)

Hence, we set

\[
\Xi := |J| - \sum_{j=1}^M s_j.
\]
we multiply the latter identity by $\epsilon^2$ and we exploit (9.90). In this way, we find that

$$
0 = \lim_{\epsilon \to 0} \sum_{|i| \leq K, |j| \leq |I|} e^{[|I| - |j|]} \theta_{i,I,j} \sum_{s} e^{[|I|] - s} \partial_{j}^{|I|,1} \phi_{1}(e_{1} + \epsilon Y_{1}) \ldots e^{[|I|] - s} \partial_{j}^{|I|,1} \phi_{M}(e_{M} + \epsilon Y_{M}) \frac{1}{2},
$$

which is (9.91). Then, we define

$$
\psi_{h} = \sum_{|i| \leq K, |j| = |I|} e^{[|I| - |j|]} \theta_{i,I,j} \sum_{s} e^{[|I|] - s} \partial_{j}^{|I|,1} \phi_{1}(e_{1} + \epsilon Y_{1}) \ldots e^{[|I|] - s} \partial_{j}^{|I|,1} \phi_{M}(e_{M} + \epsilon Y_{M}) \frac{1}{2},
$$

and consequently

$$
0 = \sum_{|i| \leq K, |j| = |I|} \theta_{i,I,j} \sum_{s} e^{[|I| - |j|]} \partial_{j}^{|I|,1} \phi_{1}(e_{1} + \epsilon Y_{1}) \ldots e^{[|I|] - s} \partial_{j}^{|I|,1} \phi_{M}(e_{M} + \epsilon Y_{M}) \frac{1}{2}.
$$

(9.94) Now we take free parameters $y_{j} \in \mathbb{R}^{m_{1} + \ldots + m_{M}} \setminus \{0\}$ and we perform the same change of variables in (9.50). In this way, we deduce from (9.94) that

$$
0 = \sum_{|i| + |j| + |\alpha| \leq K, |\beta| = |I|} C_{i,I,\alpha} \theta_{i,I,\alpha} \psi_{h}^{(y_{j})} \frac{1}{2},
$$

for some $C_{i,I,\alpha} \neq 0$, and the first claim in (9.7) is proved in this case as well. \(\square\)

**Proof of (9.7), case 4.** Notice that if there exists $j \in \{1, \ldots, M\}$ such that $\partial_{j} \neq 0$, we are in the setting of case 3. Therefore, we assume that $\partial_{j} = 0$ for every $j \in \{1, \ldots, M\}$.

We let $\psi$ be the function constructed in Lemma 3.1. For each $h \in \{1, \ldots, l\}$, we let $\overline{\psi}_{h}(t_{h}) := \psi(t_{h})$, to make the dependence on $h$ clear and explicit. Then, by formulas (3.1) and (3.2), we know that

$$
D_{h,0}^{\alpha} \overline{\psi}_{h}(t_{h}) = 0 \quad \text{in} \quad (1, +\infty)
$$

and, for every $\ell \in \mathbb{N}$,

$$
\lim_{\epsilon \to 0} \epsilon^{\ell} \alpha \partial_{t_{h}}^{\ell} \overline{\psi}_{h}(1 + \epsilon t_{h}) = \kappa_{h,\ell} t_{h}^{\alpha_{h} - \ell},
$$

in the sense of distribution, for some $\kappa_{h,\ell} \neq 0$.

Now, we introduce a set of auxiliary parameters $\xi = (\xi_{1}, \ldots, \xi_{l}) \in (1, 2)^{l}$, and fix $\epsilon$ sufficiently small possibly depending on the parameters. Then, we define

$$
a = (a_{1}, \ldots, a_{l}) := \left(-\frac{\epsilon}{\xi_{1}} - 1, \ldots, -\frac{\epsilon}{\xi_{l}} - 1\right) \in (-2, 0)^{l},
$$

and

$$
\psi_{h}(t_{h}) := \overline{\psi}_{h}(t_{h} - a_{h}).
$$

(9.98) With a simple computation we have that the function in (9.98) satisfies

$$
D_{h,a_{h}}^{\alpha} \psi_{h}(t_{h}) = D_{h,0}^{\alpha} \overline{\psi}_{h}(t_{h} - a_{h}) = 0 \quad \text{in} \quad (1 + a_{h}, +\infty) = \left(-\frac{\epsilon}{\xi_{h}}, +\infty\right),
$$

(9.99)
where the multi-index notation has been used.

\[ \epsilon^{\ell-a_h} \partial^\ell_{t_h} \psi_h(0) = \epsilon^{\ell-a_h} \partial^\ell_{t_h} \overline{\psi}_h(-a_h) = \epsilon^{\ell-a_h} \partial^\ell_{t_h} \overline{\psi}_h \left(1 + \frac{\epsilon}{\ell} \right) \rightarrow \kappa_{h,\ell} \epsilon^{\ell-a_h}, \]  

in the sense of distributions, as \( \epsilon \searrow 0 \).

Moreover, since for any \( h = 1, \ldots, l \), \( \psi_h \in C^\infty_{a_h} \), we can consider the extension

\[ \psi_h^*(t_h) := \begin{cases} 
\psi_h(t_h) & \text{if } t_h \in [a_h, +\infty), \\
\sum_{i=0}^{k_h-1} \frac{\psi_h^{(i)}(a_h)}{i!}(t_h - a_h)^i & \text{if } t_h \in (-\infty, a_h),
\end{cases} \]  

and, using Lemma A.3 in [CDV18] with \( u := \psi_h \), \( a := -\infty \), \( b := a_h \) and \( u_\ast := \psi_h^* \), we have that

\[ \psi_h^* \in C^\infty_{-\infty} \quad \text{and} \quad D^\alpha_{h, -\infty} \psi_h^* = D^\alpha_{h, a_h} \psi_h = 0 \quad \text{in every interval } I \subset \left(-\frac{\ell}{h}, +\infty\right). \]  

Now, we fix a set of free parameters \( \underline{y} = \left( y_1, \ldots, y_M \right) \in \mathbb{R}^{m_1 + \ldots + m_M} \), and consider \( \tau \in C^\infty(\mathbb{R}^{m_1 + \ldots + m_M}) \), such that

\[ \tau(y) := \begin{cases} 
\exp \left( \underline{y} \cdot y \right) & \text{if } y \in B_1^{m_1 + \ldots + m_M}, \\
0 & \text{if } y \in \mathbb{R}^{m_1 + \ldots + m_M} \setminus B_2^{m_1 + \ldots + m_M},
\end{cases} \]  

where

\[ \underline{y} \cdot y = \sum_{j=1}^M \underline{y}_j \cdot y_j, \]

denotes the standard scalar product.

We notice that, for any multi-index \( I \in \mathbb{N}^{m_1 + \ldots + m_M} \),

\[ \partial^I_y \tau(0) = \underline{y}^I, \]  

where the multi-index notation has been used.

Now, we define

\[ w(x, y, t) := \tau(x) \tau(y) \psi_1^*(t_1) \ldots \psi_l^*(t_l), \]  

where the setting in (9.82), (9.101) and (9.103) has been exploited.

Using (9.102), we have that, for any \( (x, y) \) in a neighborhood of the origin and \( t \in \left(-\frac{\ell}{h}, +\infty\right)^l \),

\[ \Lambda_{-\infty} w(x, y, t) = \tau(x) \tau(y) \left( \zeta_1 D^\alpha_{t_1, -\infty} \psi_1^*(t_1) \right) \ldots \psi_l^*(t_l) + \ldots + \tau(x) \tau(y) \psi_1^*(t_1) \ldots \left( \zeta_1 D^\alpha_{t_1, -\infty} \psi_l^*(t_l) \right) = 0. \]

We have that \( w \in \mathcal{A} \), and, since \( \tau \) and \( \underline{\tau} \) are compactly supported, we also have that \( w \in C(\mathbb{R}^N) \cap C_0(\mathbb{R}^{N-l}) \). Also, from Lemma 3.1, for any \( h \in \{1, \ldots, l\} \), we know that \( \overline{\psi}_h \in C^\infty((1, +\infty)) \), hence \( \psi_h \in C^\infty \left( \left[-\frac{\ell}{h}, +\infty\right) \right) \). Thus, \( w \in C^\infty(\mathcal{N}) \), and consequently \( w \in \mathcal{H} \).

58
Recalling (9.6), (9.83), and (9.104), we have that
\[
0 = \theta \cdot \partial^K w(0) = \sum_{|i| \leq K} \theta_{i,I,j} \partial_i^x \partial_j^y \partial_l^z w(0) = \sum_{|i| \leq K, |j| \leq |I|} \theta_{i,I,j} \partial_i^x \partial_j^y \partial_l^z w(0)
\]
\[
= \sum_{|i| \leq K, |j| \leq |I|} \theta_{i,I,j} \xi_i \psi_i(0) \ldots \partial_l^z \psi_l(0).
\]  
(9.106)

Hence, we set
\[
\Xi := |I| - \sum_{h=1}^l \alpha_h,
\]
we multiply the identity in (9.106) by \(\epsilon^\Xi\) and we exploit (9.100). In this way, we find that
\[
0 = \lim_{\epsilon \to 0} \sum_{|i| \leq K, |j| \leq |I|} \epsilon^{|I|-|\alpha|} \theta_{i,I,j} \xi_i \psi_i(0) \ldots \epsilon^{3-I-\alpha_i} \partial_l^z \psi_l(0)
\]
\[
= \sum_{|i| \leq K, |j| \leq |I|} \theta_{i,I,j} \kappa_{1,3_1} \ldots \kappa_{l,3_l} \xi_i \psi_i(0) \ldots \xi_l \psi_l(0)
\]
\[
= \xi_1 \ldots \xi_l \sum_{|i| \leq K, |j| \leq |I|} \theta_{i,I,j} \kappa_{1,3_1} \ldots \kappa_{l,3_l} \xi_i \psi_i(0) \ldots \xi_l \psi_l(0)
\]
and consequently
\[
0 = \sum_{|i| \leq K, |j| \leq |I|} \theta_{i,I,j} \kappa_{1,3_1} \ldots \kappa_{l,3_l} \xi_i \psi_i(0) \ldots \xi_l \psi_l(0)
\]
and the second claim in (9.7) is proved in this case as well. \(\Box\)

10 Every function is locally \(\Lambda_{-\infty}\)-harmonic up to a small error, and completion of the proof of Theorem 8.1

In this section we complete the proof of Theorem 8.1 (which in turn implies Theorem 1.1 via Lemma 8.2). By standard approximation arguments we can reduce to the case in which \(f\) is a polynomial, and hence, by the linearity of the operator \(\Lambda_{-\infty}\), to the case in which is a monomial. The details of the proof are therefore the following:

10.1 Proof of Theorem 8.1 when \(f\) is a monomial

We prove Theorem 8.1 under the initial assumption that \(f\) is a monomial, that is
\[
f(x, y, t) = x_1 \ldots x_n y_1 \ldots y_M t_1 \ldots t_l = \frac{x^i y^j t^z}{i! j! l!} = \frac{(x, y, t)^e}{e!},
\]  
(10.1)
where \(e! := i_1! \ldots i_n! j_1! \ldots j_M! l_1! \ldots l_l!\) and \(I_\beta := I_{\beta,1}! \ldots I_{\beta,m_\beta}!\) and \(I_\chi := i_\chi_1! \ldots i_{\chi,p_\chi}!\) for all \(\beta = 1, \ldots, M\). and \(\chi = 1, \ldots, n\). To this end, we argue as follows. We consider \(\eta \in (0, 1)\), to
be taken sufficiently small with respect to the parameter $\epsilon > 0$ which has been fixed in the statement of Theorem 8.1, and we define

$$T_\eta(x, y, t) := \left(\eta^{r_1} x_1, \ldots, \eta^{r_n} x_n, \eta^{s_1} y_1, \ldots, \eta^{s_M} y_M, \eta^{\alpha_1} t_1, \ldots, \eta^{\alpha_l} t_l\right).$$

We also define

$$\gamma := \sum_{j=1}^{n} \left| i_j \right| r_j + \sum_{j=1}^{M} \left| I_j \right| 2s_j + \sum_{j=1}^{l} \frac{J_j}{\alpha_j}, \quad (10.2)$$

and

$$\delta := \min \left\{ \frac{1}{r_1}, \ldots, \frac{1}{r_n}, \frac{1}{2s_1}, \ldots, \frac{1}{2s_M}, \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_l} \right\}. \quad (10.3)$$

We also take $K_0 \in \mathbb{N}$ such that

$$K_0 \geq \frac{\gamma + 1}{\delta} \quad (10.4)$$

and we let

$$K := K_0 + |i| + |I| + |J| + \ell = K_0 + |i| + \ell, \quad (10.5)$$

where $\ell$ is the fixed integer given in the statement of Theorem 8.1.

By Lemma 9.1, there exist a neighborhood $\mathcal{N}$ of the origin and a function $w \in C\left(\mathbb{R}^N \cap C_0(\mathbb{R}^{N-l}) \cap C^\infty(\mathcal{N}) \cap \mathcal{A}\right)$ such that

$$\Lambda_{-\infty} w = 0 \text{ in } \mathcal{N}, \quad (10.6)$$

and such that

all the derivatives of $w$ in 0 up to order $K$ vanish,

with the exception of $\partial^\nu w(0)$ which equals 1,  \quad (10.7)

being $\nu$ as in (10.1). Recalling the definition of $\mathcal{A}$ on page 38, we also know that

$$\partial^{k_h}_{t_h} w = 0 \text{ in } (-\infty, a_h), \quad (10.8)$$

for suitable $a_h \in (-2, 0)$, for all $h \in \{1, \ldots, l\}$.

In this way, setting

$$g := w - f, \quad (10.9)$$

we deduce from (10.7) that

$$\partial^\sigma g(0) = 0 \quad \text{for any } \sigma \in \mathbb{N}^N \text{ with } |\sigma| \leq K.$$  \quad (10.10)

Accordingly, in $\mathcal{N}$ we can write

$$g(x, y, t) = \sum_{|\tau| \geq K+1} x^{\tau_1} y^{\tau_2} t^{\tau_3} h_\tau(x, y, t),$$

for some $h_\tau$ smooth in $\mathcal{N}$, where the multi-index notation $\tau = (\tau_1, \tau_2, \tau_3)$ has been used.

Now, we define

$$u(x, y, t) := \frac{1}{\eta^\tau} w(T_\eta(x, y, t)). \quad (10.11)$$

60
In light of (10.8), we notice that $\partial^h u = 0$ in $(-\infty, a_h/\eta_\alpha)$, for all $h \in \{1, \ldots, l\}$, and therefore $u \in C_c(\mathbb{R}^N) \cap C_{00}(\mathbb{R}^{N-1}) \cap C^\infty(T_\eta(N)) \cap \mathcal{A}$. We also claim that
\begin{equation}
T_\eta([-1, 1]^{N-1} \times (a_1, +\infty) \times \ldots \times (a_l, +\infty)) \subseteq \mathcal{N}.
\end{equation}

To check this, let $(x, y, t) \in [-1, 1]^{N-1} \times (a_1 + \infty) \times \ldots \times (a_l + \infty)$ and $(X, Y, T) := T_\eta(x, y, t)$. Then, we have that $|X_1| = \eta^{1/2}|x| \leq \eta^{1/2}$, $|Y_1| = \eta^{1/2}|y| \leq \eta^{1/2}$, $T_1 = \eta^{1/2}t_1 > a_1\eta^{1/2} > -1$, provided $\eta$ is small enough. Repeating this argument, we obtain that, for small $\eta$,
\begin{equation}
(X, Y, T) \text{ is as close to the origin as we wish.}
\end{equation}

From (10.13) and the fact that $\mathcal{N}$ is an open set, we infer that $(X, Y, T) \in \mathcal{N}$, and this proves (10.12).

Thanks to (10.6) and (10.12), we have that, in $B_1^{N-1} \times (-1, +\infty)^l$,
\begin{align*}
\eta^{\gamma-1} \Lambda_{\infty} u (x, y, t) &= \sum_{j=1}^n \partial_j \partial_j^{\gamma} w (T_\eta(x, y, t)) + \sum_{j=1}^M \partial_j (-\Delta)^{\gamma} w (T_\eta(x, y, t)) + \sum_{j=1}^l \partial_j D_{t_\eta,-\infty}^{\gamma} w (T_\eta(x, y, t)) \\
&= \Lambda_{\infty} w (T_\eta(x, y, t)) \\
&= 0.
\end{align*}

These observations establish that $u$ solves the equation in $B_1^{N-1} \times (-1 + \infty)^l$ and $u$ vanishes when $|(x, y)| \geq R$, for some $R > 1$, and thus the claims in (8.3) and (8.4) are proved.

Now we prove that $u$ approximates $f$, as claimed in (8.5). For this, using the monomial structure of $f$ in (10.1) and the definition of $\gamma$ in (10.2), we have, in a multi-index notation,
\begin{equation}
\frac{1}{\eta^{\gamma}} f (T_\eta(x, y, t)) = \frac{1}{\eta^{\gamma} t!} (\eta^{\gamma} x)^{(l \frac{1}{2})} (\eta^{\gamma} y)^{(l \frac{1}{2})} \frac{1}{t!} x^i y^j t^k = f(x, y, t).
\end{equation}

Consequently, by (10.9), (10.10), (10.11) and (10.14),
\begin{align*}
 u(x, y, t) - f(x, y, t) &= \frac{1}{\eta^{\gamma}} g \left( \eta^{\gamma} x_1, \ldots, \eta^{\gamma} x_n, \eta^{\gamma} y_1, \ldots, \eta^{\gamma} y_M, \eta^{\gamma} t_1, \ldots, \eta^{\gamma} t_1 \right) \\
&= \sum_{|r| \geq K+1} \eta^{n\gamma} x^{\tau_1} y^{\tau_2} t^{\tau_3} h_x \left( \eta^{\gamma} x, \eta^{\gamma} y, \eta^{\gamma} t \right),
\end{align*}

where a multi-index notation has been used, e.g. we have written
\begin{equation}
\frac{\tau_i}{r} := \left( \frac{\tau_{1,1}}{r_{1}}, \ldots, \frac{\tau_{1,n}}{r_{n}} \right) \in \mathbb{R}^n.
\end{equation}

Therefore, for any multi-index $\beta = (\beta_1, \beta_2, \beta_3)$ with $|\beta| \leq \ell$,
\begin{align*}
\partial^\beta (u(x, y, t) - f(x, y, t)) &= \partial_x^{\beta_1} \partial_y^{\beta_2} \partial_t^{\beta_3} (u(x, y, t) - f(x, y, t)) \\
&= \sum_{|\beta_1| + |\beta_2| + |\beta_3| = |\beta|} c_{r, \beta} \eta^{\kappa_\beta} x^{\tau_1 - \beta_1} y^{\tau_2 - \beta_2} t^{\tau_3 - \beta_3} \partial_x^{\beta_1} \partial_y^{\beta_2} \partial_t^{\beta_3} h_x \left( \eta^{\gamma} x, \eta^{\gamma} y, \eta^{\gamma} t \right),
\end{align*}

(10.15)
where
\[
\kappa_{\tau, \beta} := \left| \frac{\tau_1}{r} + \frac{\tau_2}{2s} + \frac{\tau_3}{\alpha} \right| - \gamma + \left| \frac{\beta'_1}{r} + \frac{\beta'_2}{2s} + \frac{\beta'_3}{\alpha} \right|,
\]
for suitable coefficients \( c_{\tau, \beta} \). Thus, to complete the proof of (8.5), we need to show that this quantity is small if so is \( \eta \). To this aim, we use (10.3), (10.4) and (10.5) to see that
\[
\kappa_{\tau, \beta} \geq \left| \frac{\tau_1}{r} \right| + \left| \frac{\tau_2}{2s} \right| + \left| \frac{\tau_3}{\alpha} \right| - \gamma \geq \delta (|\tau_1| + |\tau_2| + |\tau_3|) - \gamma \geq K\delta - \gamma \geq K_0\delta - \gamma \geq 1.
\]
Consequently, we deduce from (10.15) that
\[
\| u - f \|_{C^r(B_1^N)} \leq C\eta
\]
for some \( C > 0 \). By choosing \( \eta \) sufficiently small with respect to \( \epsilon \), this implies the claim in (8.5). This completes the proof of Theorem 8.1 when \( f \) is a monomial.

10.2 Proof of Theorem 8.1 when \( f \) is a polynomial

Now, we consider the case in which \( f \) is a polynomial. In this case, we can write \( f \) as
\[
f(x, y, t) = \sum_{j=1}^{J} c_j f_j(x, y, t),
\]
where each \( f_j \) is a monomial, \( J \in \mathbb{N} \) and \( c_j \in \mathbb{R} \) for all \( j = 1, \ldots, J \).

Let
\[
c := \max_{j \in \{1, \ldots, J\}} c_j.
\]
Then, by the work done in Subsection 10.1, we know that the claim in Theorem 8.1 holds true for each \( f_j \), and so we can find \( u_j \in (-2, 0)^l \), \( u_j \in C^\infty(\bar{B}_1^N) \cap C(\mathbb{R}^N) \cap A \) and \( R_j > 1 \) such that \( \Lambda_{-\infty} u_j = 0 \) in \( B_1^{N-l} \times (-1, +\infty)^l \), \( \| u_j - f_j \|_{C^r(\bar{B}_1^N)} \leq \epsilon \) and \( u_j = 0 \) if \( |(x, y)| \geq R_j \).

Hence, we set
\[
u(x, y, t) := \sum_{j=1}^{J} c_j u_j(x, y, t),
\]
and we see that
\[
\| u - f \|_{C^r(\bar{B}_1^N)} \leq \sum_{j=1}^{J} |c_j| \| u_j - f_j \|_{C^r(\bar{B}_1^N)} \leq cJ\epsilon.
\]
(10.16)
Also, \( \Lambda_{-\infty} u = 0 \) thanks to the linearity of \( \Lambda_{-\infty} \) in \( B_1^{N-l} \times (-1, +\infty)^l \). Finally, \( u \) is supported in \( B_R^{N-l} \) in the variables \( (x, y) \), being
\[
R := \max_{j \in \{1, \ldots, J\}} R_j.
\]
This proves Theorem 8.1 when \( f \) is a polynomial (up to replacing \( \epsilon \) with \( cJ\epsilon \)).
10.3 Proof of Theorem 8.1 for a general $f$

Now we deal with the case of a general $f$. To this end, we exploit Lemma 2 in [DSV17] and we see that there exists a polynomial $\tilde{f}$ such that

$$\|f - \tilde{f}\|_{C^r(B^N_1)} \leq \epsilon. \quad (10.17)$$

Then, applying the result already proven in Subsection 10.2 to the polynomial $\tilde{f}$, we can find $a \in (-\infty, 0)^l$, $u \in C^\infty(B^N_1) \cap C(\mathbb{R}^N) \cap A$ and $R > 1$ such that

$$\Lambda_{-\infty} u = 0 \quad \text{in} \quad B^N_{1-l} \times (-1, +\infty)^l, $$

$$u = 0 \quad \text{if} \quad |(x, y)| \geq R,$$

$$\partial^{k_h}_{t_h} u = 0 \quad \text{if} \quad t_h \in (-\infty, a_h), \quad \text{for all} \quad h \in \{1, \ldots, l\},$$

and $$\|u - \tilde{f}\|_{C^r(B^N_1)} \leq \epsilon.$$

Then, recalling (10.17), we see that

$$\|u - f\|_{C^r(B^N_1)} \leq \|u - \tilde{f}\|_{C^r(B^N_1)} + \|f - \tilde{f}\|_{C^r(B^N_1)} \leq 2\epsilon.$$

Hence, the proof of Theorem 8.1 is complete. \hfill \Box

References

[AJS18a] Nicola Abatangelo, Sven Jarohs, and Alberto Saldaña, *Positive powers of the Laplacian: From hypersingular integrals to boundary value problems*, Commun. Pure Appl. Anal. **17** (2018), no. 3, 899–922, DOI 10.3934/cpaa.2018045. MR3809107

[AJS18b] _____, *Green function and Martin kernel for higher-order fractional Laplacians in balls*, Nonlinear Anal. **175** (2018), 173–190, DOI 10.1016/j.na.2018.05.019. MR3830727

[AJSa] _____, *On the loss of maximum principles for higher-order fractional Laplacians*, to appear on Proc. Amer. Math. Soc., DOI 10.1090/proc/14165.

[AJSb] _____, *Integral representation of solutions to higher-order fractional Dirichlet problems on balls*, to appear on Commun. Contemp. Math., DOI 10.1142/S0219199718500025.

[AV18] Nicola Abatangelo and Enrico Valdinoci, *Getting acquainted with the fractional Laplacian*, Springer INdAM Series (2018).

[ACV16] Mark Allen, Luis Caffarelli, and Alexis Vasseur, *A parabolic problem with a fractional time derivative*, Arch. Ration. Mech. Anal. **221** (2016), no. 2, 603–630, DOI 10.1007/s00205-016-0969-z. MR3488533

[BWZ17] Umberto Biccari, Mahamadi Warma, and Enrique Zuazua, *Local elliptic regularity for the Dirichlet fractional Laplacian*, Adv. Nonlinear Stud. **17** (2017), no. 2, 387–409, DOI 10.1515/ans-2017-0014. MR3641649

[Buc16] Claudia Bucur, *Some observations on the Green function for the ball in the fractional Laplace framework*, Commun. Pure Appl. Anal. **15** (2016), no. 2, 657–699, DOI 10.3934/cpaa.2016.15.657. MR3461641

[Buc17] _____, *Local density of Caputo-stationary functions in the space of smooth functions*, ESAIM Control Optim. Calc. Var. **23** (2017), no. 4, 1361–1380, DOI 10.1051/cocv/2016056. MR3716924

[BV16] Claudia Bucur and Enrico Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016. MR3469920
[RVD+13] Benjamin M. Regner, Dejan Vučinić, Cristina Domnisoru, Thomas M. Bartol, Martin W. Hetzer, Daniel M. Tartakovsky, and Terrence J. Sejnowski, Anomalous diffusion of single particles in cytoplasm, Biophys. J. 104 (2013), no. 8, 1652–1660, DOI 10.1016/j.bpj.2013.01.049.

[ROS14] Xavier Ros-Oton and Joaquim Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. (9) 101 (2014), no. 3, 275–302, DOI 10.1016/j.matpur.2013.06.003 (English, with English and French summaries). MR3168912

[ROS17] ______, Boundary regularity estimates for nonlocal elliptic equations in $C^1$ and $C^{1,\alpha}$ domains, Ann. Mat. Pura Appl. (4) 196 (2017), no. 5, 1637–1668, DOI 10.1007/s10231-016-0632-1. MR3694738

[Rül17] Angkana Rüland, Quantitative invertibility and approximation for the truncated Hilbert and Riesz Transforms, ArXiv e-prints (2017), available at 1708.04285.

[RS17a] Angkana Rüland and Mikko Salo, The fractional Calderón problem: low regularity and stability, ArXiv e-prints (2017), available at 1708.06294.

[RS17b] ______, Quantitative approximation properties for the fractional heat equation, ArXiv e-prints (2017), available at 1708.06300.

[RS18] ______, Exponential instability in the fractional Calderón problem, Inverse Problems 34 (2018), no. 4, 045003, 21, DOI 10.1088/1361-6420/aaae5a. MR3774704

[SKM93] Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev, Fractional integrals and derivatives, Gordon and Breach Science Publishers, Yverdon, 1993. Theory and applications; Edited and with a foreword by S. M. Nikol’ski˘ı; Translated from the 1987 Russian original; Revised by the authors. MR1347689

[SWDSA06] F. Santamaria, S. Wils, E. De Schutter, and G. J. Augustine, Anomalous diffusion in Purkinje cell dendrites caused by spines, Neuron. 52 (2006), no. 4, 635–648, DOI 10.1016/j.neuron.2006.10.025.

[SZ16] Chung-Sik Sin and Liancun Zheng, Existence and uniqueness of global solutions of Caputo-type fractional differential equations, Fract. Calc. Appl. Anal. 19 (2016), no. 3, 765–774, DOI 10.1515/fca-2016-0040. MR3563609

[Seff17] William Seffens, Models of RNA interaction from experimental datasets: framework of resilience, Applications of RNA-Seq and Omics Strategies, 2017, DOI 10.5772/intechopen.69452.

[Val09] Enrico Valdinoci, From the long jump random walk to the fractional Laplacian, Bol. Soc. Esp. Mat. Apl. SeMA 49 (2009), 33–44. MR2584076

[VAB+96] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, and H. E. Stanley, Lévy flight search patterns of wandering albatrosses, Nature 381 (1996), 413–415, DOI 10.1038/381413a0.