Solution of linearized Ginzburg-Landau problem for mesoscopic superconductors by conformal mapping

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Abstract. We present a new method for the solution of linearized Ginzburg-Landau problem for mesoscopic superconducting nanostructures of arbitrary shapes in applied magnetic field. The method is based on the conformal mapping of the analytical solution for the disk and uses a specially designed superconducting gauge for the vector potential corresponding to the magnetic field. As a demonstration of the methods accuracy, we calculate the distribution of the order parameter in superconducting regular polygons and compare the obtained solutions with the available numerical results. We further consider an example of irregular polygon and show the evolution of the vortex patterns in function of the geometry of samples boundary. The obtained results will be compared with available experimental data on mesoscopic and nanoscopic superconductors.

The linearized Ginzburg-Landau (LGL) equation describes the distribution of the superconducting order parameter (Ψ) during the nucleation of the superconductivity, i.e. close to the normal-superconducting phase boundaries of the $T_c(H_{c2})$ diagram [1]. It can also be used to solve the non-linear Ginzburg-Landau problem[2, 3].

The LGL problem is defined by the equations

$$\frac{1}{2m}(-i\hbar \vec{\nabla} - 2e\vec{A})^2\Psi = \alpha \Psi$$

$$(-i\hbar \vec{\nabla} - 2e\vec{A})|_{n.b.}\Psi = 0,$$

where $\vec{A}$ is the vector potential corresponding to the applied magnetic field ($\vec{H} = \nabla \times \vec{A}$), $\Psi$ is the superconducting order parameter, $\alpha$ is the condensation energy density (in the linearized Ginzburg-Landau is an eigenvalue to be determined) and $n.b.$ means normal to the boundary. The most common method to solve Ginzburg-Landau (GL) problem in superconductivity (the linearized GL equation, non-linear GL equation, the GL equation coupled with the magnetic field equation, the time dependent GL, the two component GL ) is the finite difference method. However comparison with analytical and semi-analytical methods applied to the LGL equation has shown that the finite difference method needs a great number of points (with grids of at least 201 by 201 points [4]) to become sufficiently accurate to obtain vortex features like anti-vortex in the square [5]. Still, analytical and semi-analytical methods are only applicable, to the best of our knowledge, to a few geometries like the circle [6, 7], square [5], rectangle [8], equilateral triangle [9] and annular [10] geometries.

Most of the newly accessible bounded geometries by semi-analytical methods were analyzed by a superconducting gauge method [11]. Recently, a numeric method [12], not so efficient as the mentioned gauge method, was developed to solve the LGL problem for bounded geometries.

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using the finite element method and the superconducting gauge approach. The solutions for this problem were found for different geometries, including some highly symmetric, like the square, triangle, pentagon and five point stars.

Figure 1. Plots of the vector potential with the superconducting gauge fixed by the method described in the text.

We propose to solve the LGL problem in a generic polygon with the eigenfunctions of the Laplacian operator (with Neumann boundary condition), defined on the circle, as basis functions. We also propose to fix the gauge of the vector potential, that reduces the boundary condition of the Ginzburg-Landau problem to the Neumann boundary condition, by solving Laplace equation with a general Neumann boundary condition (that can be found by the Dini integral formula [13]).

First, we calculated the eigenfunctions of the Laplacian operator in the circle (with Neumann boundary condition). These well known solutions are

\[ \Psi_{l,n}(r,\theta) = J_l(u'_{l,n}) r e^{i l \theta} \]  \(\text{(3)}\)

where \(u'_{l,n}\) is the \(n\) zero of the \(l\) Bessel function derivative and \(J_l(u'_{l,n}) r\) is the \(l\) Bessel function rescaled to match the Neumann boundary conditions. The solutions for the circle (auxiliary space) were mapped onto the general polygon (original space) by the SC mapping. We have represented the functions with a mesh in the original space. Afterwards, this set of functions was orthogonalized via the Löwding orthogonalization process. Then, the new functions were used as a basis for the LGL problem. To apply the functions as a basis for the LGL equation, the superconducting gauge needs, first, to be fixed using the gauge condition[11], \(\vec{A}'|_{n.b.} = 0\), which means \(\vec{\nabla}|_{n.b.} S = -\vec{A}|_{n.b.}\), where \(\vec{A}' = \vec{A} + \vec{\nabla} S\).

If we want to satisfy gauge condition while keeping the Coulomb gauge, the function \(S\) must be a solution of the problem:

\[ \Delta S = 0 \]  \(\text{(4)}\)

\[ \vec{\nabla}|_{n.b.} S = -\vec{A}|_{n.b.}, \]  \(\text{(5)}\)

where \(\vec{A}\) is a general vector potential. In general, this problem is only solvable when

\[ \int_{\text{boundary}} \vec{A}|_{n.b.} = 0, \]  \(\text{(6)}\)

where the integration is made over the boundary. This condition is fulfilled if the original vector potential is in the Coulomb gauge, \(\vec{\nabla}.\vec{A} = 0\). To solve the new problem, we have mapped the equation back to the auxiliary space (circle). The solution to this problem is given by the Dini integral formula [13]

\[ S(u(r, \theta), v(r, \theta)) = \frac{1}{4\pi} \int_{\phi=0}^{2\pi} |w'(z)| |\vec{A}(u, v)|_{n.b.} \]

\[ \log (r^2 - 2r \cos (\theta - \phi) + 1) d\phi, \]  \(\text{(7)}\)
for the unitary circle, where $\phi$ is a integration angle. This method makes it possible to fix the superconducting gauge for any magnetic field distribution, such that the vector potential is tangential to the boundary.

Fig. 1 and Fig. 2 show the vector potential and the distribution of the order parameter for the triangle, square, pentagon and hexagon geometry using this method and Fig. 3 below presents the same quantities but for a deformed hexagon.

**Figure 2.** (Color online) On the top, plots of the distribution of the superconducting order parameter $|\Psi|$, in the natural logarithmic scale, for the triangle, square, pentagon, hexagon, and circle geometries for different vorticities $L = 0, 1, 2, 3, 4, 5$ and $6$, and magnetic field in magnetic flux quanta is $\Phi/\Phi_0 = 1, 3, 4, 5.5, 7, 8, 9$, respectively (counting from left to right). The color scale goes from red (dark grey in the black and white version) where superconductivity is strong, through yellow (light grey), to blue (dark grey). This color scheme is present in all figures that show the density of the superconducting condensate. The vortices appear as blue (dark grey) spots in the middle of the sample. On the bottom, zooms (of anti-vortex patterns) from the areas enclosed by white dashed boxes, corresponding from left to right, to the pairs (geometry, number of vortices, vorticity number) = (triangle, 3, 2) ; (triangle, 6, 5) ; (square, 4, 3) ; (pentagon, 5, 4) ; (hexagon, 6, 5)
Figure 3. Fig. 3a: Vector potential with the superconducting gauge fixed (the vector potential lines are parallel to the boundary), as described in the text. Fig. 3b (Color online): Distribution of the superconducting order parameter, in the natural logarithmic scale, in the deformed hexagon for vorticities $L = 0, 1, 2, 3, 4, 5$ and $6$ and magnetic field in magnetic flux quanta $\Phi/\Phi_0 = 1, 3, 4, 5.5, 7, 8, 9$, respectively (counting from left to right, from top to bottom). The vortices appear as blue (or in black and white version as dark gray) spots in the middle of the sample.

1. Conclusions
The proposed method extends naturally the previously developed superconducting gauge method. The use of basis functions that already satisfy the boundary conditions is a key element present in the original method and the extended one. These basis functions are different for each of these methods, but only few number of basis functions are needed to get a very good approximation of the solution in both methods. These methods were compared for the case of the triangle and the square geometries. The solutions of the extended method match the ones of the original method for these geometries. With this new method, new geometries are now accessible to semi-analytical treatment. The application of the latter in the present work allowed to observe certain common features shared by the solutions of LGL equation calculated for regular polygons. The solutions of LGL equation for different polygons presented here have demonstrated that the proposed method is efficient (uses only a few basis functions) and fairly accurate. Moreover it can be equally applied to irregular polygons with comparable efficiency.

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