Approximation of $k$-dimensional maps

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Abstract

In this paper we prove the equivalence of the questions of B.A. Pasynkov ([5]) and V.V. Uspenskij ([11]). We also get some partial results answering these questions in affirmative. As a corollary to these results we get an extension of the Hurewicz formula to the extensional dimension.

Keywords: $k$-dimensional map; Property $C$; Polyhedron; Extensional dimension; Bing compacta

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1 Introduction

All spaces are assumed to be separable metrizable. By a map we mean a continuous function, $I = [0; 1]$. If $K$ is a simplicial complex then by $|K|$ we mean the corresponding polyhedron. By a simplicial map we mean a map $f:|K| \longrightarrow |L|$ which sends simplices to simplices and is affine on them. We say that a map $f: X \longrightarrow Y$ has dimension at most $k$ ($\dim f \leq k$) if and only if the dimension of each of its fibers is at most $k$. We recall that a space $X$ is a $C$-space or has property $C$ if for any sequence $\{\alpha_n : n \in \mathbb{N}\}$ of open covers of $X$ there exists a sequence $\{\mu_n : n \in \mathbb{N}\}$ of disjoint families of open sets such that each $\mu_n$ refines $\alpha_n$ and the union of all systems $\mu_n$ is a cover of $X$. Each finite-dimensional paracompact space and each countable-dimensional metrizable space has property $C$. By a $C$-compactum we mean a compact $C$-space.

In [11] V.V. Uspenskij introduced the notion of a map admitting an approximation by $k$-dimensional simplicial maps. Following him we say that a map $f: X \longrightarrow Y$ admits approximation by $k$-dimensional simplicial maps if

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for every pair of open covers $\omega_X$ of the space $X$ and $\omega_Y$ of the space $Y$ there exists a commutative diagram of the following form

$$
\begin{array}{ccc}
X & \xrightarrow{\kappa_X} & |K| \\
|f| & \downarrow & \downarrow |p| \\
Y & \xrightarrow{\kappa_Y} & |L|
\end{array}
$$

where $\kappa_X$ is an $\omega_X$-map, $\kappa_Y$ is an $\omega_Y$-map and $p$ is a $k$-dimensional simplicial map between polyhedra $|K|$ and $|L|$.

In that paper V.V. Uspenskij proposed the following question and conjectured that in the general case the answer to it is ”no”.

($Q_1$) Does every $k$-dimensional map $f: X \rightarrow Y$ between compacta admit approximation by $k$-dimensional simplicial maps?

In [2] A.N. Dranishnikov and V.V. Uspenskij proved that light maps admit approximation by finite-to-one simplicial maps. In this paper we give some partial results answering the question of V.V. Uspenskij in affirmative.

**Theorem 1.1.** Let $f: X \rightarrow Y$ be a $k$-dimensional map between $C$-compacta. Then for any pair of open covers $\omega_X$ of the space $X$ and $\omega_Y$ of the space $Y$ there exists a commutative diagram of the following form

$$
\begin{array}{ccc}
X & \xrightarrow{\kappa_X} & |K| \\
|f| & \downarrow & \downarrow |p| \\
Y & \xrightarrow{\kappa_Y} & |L|
\end{array}
$$

where $\kappa_X$ is an $\omega_X$-map, $\kappa_Y$ is an $\omega_Y$-map and $p$ is a $k$-dimensional simplicial map between compact polyhedra $|K|$ and $|L|$. Furthermore, one can always assume that

$$
\dim |L| \leq \dim Y \quad \text{and} \quad \dim |K| \leq \dim Y + k.
$$

**Theorem 1.2.** $k$-dimensional maps between compacta admit approximation by $(k + 1)$-dimensional simplicial maps.

**Theorem 1.3.** $k$-dimensional maps of Bing compacta (i.e. compacta with each component hereditarily indecomposable) admit approximation by $k$-dimensional simplicial maps.
Remark. Theorem 1.1 was announced by V.V. Uspenskij at the 8th International Topology Symposium in Prague (1996), but the proof appeared only here.

It turned out that the question (Q1) is closely related to the next question proposed by B.A. Pasynkov in [5]. We recall that the diagonal product of two maps $f: X \to Y$ and $g: X \to Z$ is a map $f \triangle g: X \to Y \times Z$ defined by $f \triangle g(x) = (f(x), g(x))$.

(Q2) Let $f: X \to Y$ be a $k$-dimensional map between compacta. Does there exist a map $g: X \to I^k$ such that $\dim(f \triangle g) \leq 0$?

In this paper we prove the following theorem which states that the questions (Q1) and (Q2) are equivalent.

**Theorem 1.4.** Let $f: X \to Y$ be a map between compacta. Then $f$ admits approximation by $k$-dimensional maps if and only if there exists a map $g: X \to I^k$ such that $\dim(f \triangle g) \leq 0$.

There are a lot of papers devoted to Pasynkov’s question ([5], [6], [7], [9], [8], [3], [4], [10]). In [5] Pasynkov announced the following theorem to which the proof appeared much later in [6] and [7].

**Theorem 1.5.** Let $f: X \to Y$ be a $k$-dimensional map between finite dimensional compacta. Then there exists a map $g: X \to I^k$ such that $\dim(f \triangle g) \leq 0$.

In [9] Torunczyk proved the following theorem which is closely related to the theorem proved by Pasynkov.

**Theorem 1.6.** Let $f: X \to Y$ be a $k$-dimensional map between finite dimensional compacta. Then there exists a $\sigma$-compact $A \subseteq X$ such that $\dim A \leq k - 1$ and $\dim f|_{X \setminus A} \leq 0$.

One can prove that for any map $f: X \to Y$ between compacta the statement of theorem 1.5 holds (for $f$) if and only if the statement of theorem 1.6 holds (for $f$) (the proof can be found in [3]).

We improve the argument used by Torunzyk in [9] to prove the next theorem and the implication "$$\Leftarrow$$" of theorem 1.4.
Theorem 1.7. Let $f: X \rightarrow Y$ be a $k$-dimensional map between $C$-compacta. Then

(i) there exists a $\sigma$-compact subset $A \subset X$ such that $\dim A \leq k - 1$ and $\dim f|_{X\setminus A} \leq 0$.

(ii) there exists a map $g: X \rightarrow I^k$ such that $\dim (f \triangle g) \leq 0$.

Remark. After the paper had been already submitted for publication, the author learned that using different techniques from those of this paper V. Valov and H. Murat Tuncali [10] obtained a more general result in the class of all metrizable spaces.

In the next corollary by $e - \dim(X)$ we mean the extensional dimension of a compact space $X$ introduced by A.N. Dranishnikov in [1].

Corollary. Let $f: X \rightarrow Y$ be a $k$-dimensional map between $C$-compacta. Then $e - \dim(X) \leq e - \dim(Y \times I^k)$.

Proof. From [2] it follows that the extensional dimension cannot be lowered by 0-dimensional maps so the corollary is an immediate consequence of theorem 1.7.

One can understand the statement of the previous corollary as a generalization of the classical Hurewicz formula.

2 Proofs

Further in this section we assume that every space $X$ is given with a fixed metric $\rho_X$ which generates the same topology on it. By $\rho_X(A, B)$ we mean the distance between subsets $A$ and $B$ in the space $X$, namely, $\rho_X(A, B) = \inf\{\rho_X(a, b) | a \in A, b \in B\}$. The closure of a subset $A$ will be denoted by $[A]$.

Lemma 2.1. Let $f: X \rightarrow Y$ be a map between compacta. Suppose that for any closed disjoint subsets $B$ and $C$ of $X$ there exists a closed subset $T$ of $X$ such that $\dim T \leq k - 1$ and for any $y \in Y$ the set $T$ separates $f^{-1}(y)$ between $B$ and $C$. Then there exists a $\sigma$-compact subset $A \subset X$ with $\dim A \leq k - 1$ such that $\dim f|_{X\setminus A} \leq 0$. 
Proof. Take a countable open base $\mathcal{B} = \{U_\gamma \mid \gamma \in \Gamma\}$ on $X$ such that the union of any finite number of sets from $\mathcal{B}$ is again a member of $\mathcal{B}$. Define the set $\Lambda \subset \Gamma \times \Gamma$ by the requirement: $(\gamma, \mu) \in \Lambda$ if and only if $[U_\gamma] \cap [U_\mu] = \emptyset$. Note that $\Lambda$ is countable. By assumption for every pair $(\gamma, \mu) \in \Lambda$ there exists a set $T_{(\gamma, \mu)}$ of dimension at most $k-1$ separating every fiber $f^{-1}(y)$ between $[U_\gamma]$ and $[U_\mu]$. Now define $A = \bigcup\{T_{(\gamma, \mu)} \mid (\gamma, \mu) \in \Lambda\}$. By definition $A$ is $\sigma$-compact and, obviously, $\dim A \leq k - 1$. It is also easy to see that $\dim f|_{X \setminus A} \leq 0$. Indeed, by the additivity property of the base $\mathcal{B}$ for every pair of disjoint closed subsets $G$ and $H$ of a given fiber $f^{-1}(y)$ there exists a pair $(\gamma, \mu) \in \Lambda$ such that $G \subset U_\gamma$ and $H \subset U_\mu$. Then $T_{(\gamma, \mu)} \subset A$ separates $f^{-1}(y)$ between $G$ and $H$. So, $\dim(f^{-1}(y) \setminus A) \leq 0$. \hfill $\square$

Let $\mathcal{F} = \mathbb{N}^0 \cup \bigcup \{\mathbb{N}^k : k \geq 1\}$ be the union of all finite sequences of positive integers plus empty sequence $\mathbb{N}^0 = \{\ast\}$. For every $i \in \mathcal{F}$ let us denote by $|i|$ the length of the sequence $i$ and by $(i, p)$ the sequence obtained by adding to $i$ a positive integer $p$.

**Lemma 2.2.** Let $f : X \to Y$ be a map between compacta. Let $B$ and $C$ be closed disjoint subsets of $X$. Suppose that for every $i \in \mathcal{F}$ there are sets $U(i), V(i)$ and $F(i)$ such that:

(a) $F(i)$ is closed in $Y$, the sets $U(i)$ and $V(i)$ are open subsets of $X$ and $[U(i)] \cap [V(i)] = \emptyset$;

(b) $U(\ast) \supset B, V(\ast) \supset C$ and $F(\ast) = Y$;

(c) $U(i, p) \supset U(i) \cap f^{-1}(F(i, p))$ and $V(i, p) \supset V(i) \cap f^{-1}(F(i, p))$ for every $p \in \mathbb{N}$;

(d) $F(i) \subseteq \bigcup\{F(i, p) : p \in \mathbb{N}\}$ and $\operatorname{diam} F(i) < \frac{1}{|i|}$;

(e) the set $E(i) = f^{-1}(F(i)) \setminus (U(i) \cup V(i))$ admits an open cover of order $k$ and diameter \(\frac{1}{1+|i|}\).

(f) in notations of (e) the family $\{E(i, p) : p \in \mathbb{N}\}$ is discrete in $X$.

Then there exists a closed subset $T$ of $X$ such that $\dim T \leq k - 1$ and for any $y \in Y$ the set $T$ separates $f^{-1}(y)$ between $B$ and $C$.

**Proof.** We define the set $T$ in the following way:

$$T_n = \bigcup \{E(i) : |i| = n\} \quad \text{and} \quad T = \cap \{T_n : n \geq 0\}.$$ 

From property (e) it follows that $\dim T \leq k - 1$. Let us show that for every $y \in Y$ the set $T$ separates $f^{-1}(y)$ between $B$ and $C$. For every $y \in Y$ there exists a sequence $\{i_n : n \in \mathbb{N}\}$ such that

$$\{y\} = F(i_1) \cap F(i_1, i_2) \cap \ldots$$

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Then \( f^{-1}(y) \setminus T \subseteq U(y) \cup V(y) \) is a desired partition. Here we denote by \( U(y) \) and \( V(y) \) the following sets

\[
U(y) = f^{-1}(y) \cap \bigcup \{ U(i_1, \ldots, i_p) : p \in \mathbb{N} \},
\]

\[
V(y) = f^{-1}(y) \cap \bigcup \{ V(i_1, \ldots, i_p) : p \in \mathbb{N} \}.
\]

\[\square\]

**Lemma 2.3.** Let \( f: X \rightarrow Y \) be a \( k \)-dimensional map between \( C \)-compacta and \( \epsilon \) be any positive number. Let \( U \) and \( V \) be open subsets of \( X \) with \( [U] \cap [V] = \emptyset \), and \( F \) be a closed subset of \( Y \) such that \( f(U) \cap f(V) \supseteq F \). Then there exist families of sets \( \{U_p\}, \{V_p\}, \) and \( \{F_p\} \) for \( p \in \mathbb{N} \) such that:

1. \( F_p \) is closed in \( Y \), the sets \( U_p \) and \( V_p \) are open subsets of \( X \) and \( [U_p] \cap [V_p] = \emptyset \);
2. \( U_p \supseteq U \cap f^{-1}(F_p), V_p \supseteq V \cap f^{-1}(F_p) \);
3. \( F \subseteq \bigcup \{ F_p : p \in \mathbb{N} \} \) and \( \text{diam} F_p < \epsilon \);
4. the set \( E_p = f^{-1}(F_p) \setminus (U_p \cup V_p) \) admits an open cover of order \( k \) and diameter \( \epsilon \);
5. in notations of (4) the family \( \{ E_p : p \in \mathbb{N} \} \) is discrete in \( X \).

**Proof.** Let \( \{ W_l : l \in \mathbb{N} \} \) be a countable sequence of open disjoint sets such that each of them separates \( X \) between \([U]\) and \([V]\). For every \( y \in F \) let \( P_l(y) \subset W_l \) be a closed \((k-1)\)-dimensional set separating \( f^{-1}(y) \) between \([U]\) and \([V]\). Let \( Q_l(y) \subset W_l \) be an open neighborhood of \( P_l(y) \) admitting a finite open cover of size \( \epsilon \) and order \( k \). As the map \( f \) is closed there exists a neighborhood \( G_l(y) \) of \( y \) in \( F \) such that \( f^{-1}([G_l(y)]) \cap Q_l(y) \) separates \( f^{-1}([G_l(y)]) \) between \([U]\) and \([V]\) and \( \text{diam} G_l(y) < \epsilon \). For every \( l \in \mathbb{N} \) the family \( \alpha_l = \{ G_l(y) : y \in F \} \) is an open cover of \( F \). As \( F \) is a \( C \)-compactum there exists a finite sequence of finite disjoint open families of sets \( \{ \mu_l : l \leq N \} \) such that each family \( \mu_l \) refines the cover \( \alpha_l \) and \( \mu = \cup \{ \mu_l : l \leq N \} \) is an open cover of \( F \). Further, for every \( G \in \mu \) there are open subsets \( U(G) \) and \( V(G) \) of \( X \) with disjoint closures such that

\[
U(G) \supset f^{-1}(G) \cap [U], \quad V(G) \supset f^{-1}(G) \cap [V]
\]

and if \( G \) is a member of \( \mu_l \) then

\[
f^{-1}(G) \setminus (U(G) \cup V(G)) \subset Q_l(y)
\]
for some \( y \in F \). Let \( \{F(G) : G \in \mu\} \) be a closed shrinking of the cover \( \mu \) and let

\[
E(G) = f^{-1}(F(G)) \setminus (U(G) \cup V(G))
\]

for \( G \in \mu \). Then for \( G \in \mu_i \) and \( H \in \mu_m \) we have \( E(G) \cap E(H) \subset W_i \cap W_m = \emptyset \). So the family \( \{E(G) : G \in \mu\} \) is discrete in \( X \). Let us enumerate the members of \( \mu \): \( G_1, G_2, \ldots \). To get the desired sets we set

\[
F_p = F(G_p), \quad U_p = U(G_p), \quad V_p = V(G_p).
\]

\[ \square \]

**Lemma 2.4.** Let \( f : X \to Y \) be a \( k \)-dimensional map between \( C \)-compacta. Then for any closed disjoint subsets \( B \) and \( C \) of \( X \) and for any \( i \in F \) there exist sets \( U(i) \), \( V(i) \) and \( F(i) \) satisfying (a)–(f) of Lemma 2.2.

**Proof.** We will construct the sets \( U(i) \), \( V(i) \) and \( F(i) \) by induction on \( |i| \).

First set \( F(*) = Y \) and \( U(*) = U, V(*) = V \) for some open subsets \( U \) and \( V \) of \( X \) with \([U] \cap [V] = \emptyset \). Assume the sets \( U(i), V(i) \) and \( F(i) \) are already constructed and satisfy the conditions (a)-(f) of Lemma 2.2. Now to get the sets \( U(i,p) \), \( V(i,p) \) and \( F(i,p) \) for all \( p \in \mathbb{N} \) apply Lemma 2.3 to the sets \( U = U(i) \), \( V = V(i) \), \( F = F(i) \) and to \( \epsilon = \frac{1}{1+i} \).

\[ \square \]

**Lemma 2.5.** Let \( f : X \to Y \) be a map between compacta admitting approximations by \( k \)-dimensional maps. Then for any closed disjoint subsets \( B \) and \( C \) of \( X \) there exist sets \( U(i) \), \( V(i) \) and \( F(i) \) satisfying (a)–(f) of Lemma 2.2.

**Proof.** The sets \( U(i) \), \( V(i) \) and \( F(i) \) will be constructed by induction on \( |i| \).

First set \( F(*) = Y \) and \( U(*) = U, V(*) = V \) for some open subsets \( U \) and \( V \) of \( X \) with \([U] \cap [V] = \emptyset \). Assume the sets \( U(i), V(i) \) and \( F(i) \) are already constructed and satisfy the conditions (a)-(f) of Lemma 2.2.

Take \( \epsilon = \min\left\{\frac{\rho(U(i),V(i))}{4},\frac{1}{1+i}\right\} \). By assumption, there exists a commutative diagram of the following form:

\[
\begin{array}{ccc}
X & \xrightarrow{\kappa_X} & |\mathcal{K}| \\
\downarrow f & & \downarrow p \\
Y & \xrightarrow{\kappa_Y} & |\mathcal{L}|,
\end{array}
\]

where \( \kappa_X \) and \( \kappa_Y \) are maps with \( \epsilon \)-small fibers. Let \( G = \kappa_X([U(i)]) \), \( H = \kappa_X([V(i)]) \) and \( F = \kappa_Y(F(i)) \). Note that \( G \cap H = \emptyset \). Let \( U \) and \( V \) be
open subsets of $|\mathcal{K}|$ with $U \supseteq G$, $V \supseteq H$ and $[U] \cap [V] = \emptyset$. Let $\lambda_1$ be a Lebesgue number of some open covering on $|\mathcal{K}|$ whose preimage under the map $\kappa_X$ is an $\epsilon$-small covering on $X$. Let $\lambda_2$ be a number defined similarly for $|\mathcal{L}|$ and $\kappa_Y$. Let $\lambda = \min\{\lambda_1, \lambda_2\}$. Apply Lemma 2.3 to the sets $U$, $V$, $F$ and to $\lambda$ to produce the sets $U_p$, $V_p$ and $F_p$ for all $p \in \mathbb{N}$ satisfying conditions (1)-(5) of Lemma 2.3. Now set $U(i,p) = \kappa_X^{-1}(U_p)$, $V(i,p) = \kappa_X^{-1}(V_p)$ and $F(i,p) = \kappa_Y^{-1}(F_p)$. Since taking a preimage preserves intersections, unions and subtractions, the sets $U(i,p)$, $V(i,p)$ and $F(i,p)$ satisfy the conditions (a)-(f).

Proof of theorem 1.7. The statements (i) and (ii) are equivalent ([3]), so, it is sufficient to prove only (i). But (i) immediately follows from Lemmas 2.1, 2.2 and 2.4.

Let $\gamma$ be an open cover on $X$. Then by $N_\gamma$ we mean the nerve of the cover $\gamma$. Let $\{a_\alpha : \alpha \in \mathcal{A}\}$ be some partition of unity on $X$ subordinated to the locally finite cover $\gamma$. Then the canonical map defined by the partition of unity $\{a_\alpha : \alpha \in \mathcal{A}\}$ is a map $\kappa: X \to |N_\gamma|$ defined by the following formula

$$\kappa(x) = \sum_{\alpha \in \mathcal{A}} a_\alpha(x) \cdot \alpha.$$ 

If $\tau$ is some triangulation of the polyhedron $P$, then by $St(a, \tau)$ we mean the star of the vertex $a \in \tau$ with respect to triangulation $\tau$, i.e. the union of all open simplices having $a$ as a vertex.

Proof of theorem 1.4. Let $f: X \to Y$ be a map between compacta admitting approximation by $k$-dimensional simplicial maps. By [3] to show that there exists a map $g: X \to I_k$ with $\dim(f \triangle g) \leq 0$ it is sufficient to find a $\sigma$-compact subset $A$ in $X$ of dimension at most $k - 1$ such that $\dim f|_{X \setminus A} \leq 0$. The existence of such subset $A$ follows immediately from Lemmas 2.1, 2.2 and 2.3.

Now suppose there exists a map $g: X \to I_k$ such that $\dim(f \triangle g) \leq 0$. For every $(y, t) \in Y \times I_k$ there exists a finite disjoint family of open sets $\nu_{(y,t)} = \{V_\gamma : \gamma \in \Gamma_{(y,t)}\}$ such that $(f \triangle g)^{-1}(y, t) \subset \bigcup \nu_{(y,t)}$ and $\nu_{(y,t)} \succ \omega_X$. Let $O_{(y,t)}$ be an open neighborhood of $(y, t)$ in $Y \times I_k$ such that $(f \triangle g)^{-1}O_{(y,t)} \subset \bigcup \nu_{(y,t)}$. Let $\varsigma = \{U_\alpha : \alpha \in \mathcal{A}\}$ and $\iota = \{I_\delta : \delta \in \mathcal{D}\}$ be finite open covers of the spaces $Y$ and $I_k$ such that:

$$(a_1) \ \varsigma \succ \omega_Y;$$
The partition of unity \( \{ u_\alpha : \alpha \in A \} \) on \( Y \) subordinated to the cover \( \zeta \) gives rise to the canonical map \( \mu : Y \to |N_\zeta| \). Then the map \( \mu \times \text{id} : Y \times I^k \to |N_\zeta| \times I^k \) is an \((\zeta \times \iota)\)-map. By \( \pi : |N_\zeta| \times I^k \to |N_\zeta| \) we denote the projection. Let \( \tau \) and \( \theta \) be such triangulations on polyhedra \( |N_\zeta| \times I^k \) and \( |N_\zeta| \) respectively such that the following conditions are satisfied:

\[ (1') \pi : |N_\zeta| \times I^k \to |N_\zeta| \text{ is a simplicial map relative to the triangulations } \tau \text{ and } \theta; \]

\[ (2') \{ (\mu \times \text{id})^{-1}(\text{St}(a, \tau)) : a \in \tau \} \supset \zeta \times \iota; \]

\[ (3') \{ \mu^{-1}(\text{St}(b, \theta)) : b \in \theta \} \supset \zeta. \]

Let us define \( \xi = \{ \text{St}(a, \tau) : a \in \tau \} \) and \( \zeta = \{ \text{St}(b, \theta) : b \in \theta \} \). Define the partition of unity \( \{ w_\alpha : \alpha \in \tau \} \) on \( |N_\xi| \times I^k \) subordinated to the cover \( \xi \) by letting \( w_\alpha(z) \) be the barycentric coordinate of \( z \in |N_\xi| \times I^k \) with respect to the vertex \( a \in \tau \). Analogously, define the partition of unity \( \{ v_b : b \in \theta \} \) on \( |N_\zeta| \) subordinated to the cover \( \zeta \). Note that the projection \( \pi : |N_\xi| \times I^k \to |N_\xi| \) sends the stars of the vertices of the triangulation \( \tau \) to the stars of the vertices of the triangulation \( \theta \). That is why there is a simplicial map \( \varpi : N_\xi \to N_\zeta \) between the nerves of the covers \( \xi \) and \( \zeta \). Moreover, the following diagram commutes.

\[
\begin{array}{ccc}
|N_\xi| \times I^k & \xrightarrow{\psi} & |N_\zeta| \\
\pi \downarrow & & \downarrow \varpi \\
|N_\xi| & \xrightarrow{\phi} & |N_\zeta|.
\end{array}
\]

Here \( \psi \) and \( \phi \) are canonical maps defined by the partitions of unity \( \{ w_\alpha : \alpha \in \tau \} \) and \( \{ v_b : b \in \theta \} \) respectively. Let us remark that \( \dim \varpi \leq k \). By \( W_\alpha \) we will denote the set \((\mu \times \text{id})^{-1}(\text{St}(a, \tau))\), by \( \lambda \) the cover \( \{ W_\alpha : a \in \tau \} \) and \( \eta \) is \( \mu^{-1}(\zeta) \). Further, we set \( w_\alpha^* = w_\alpha \circ (\mu \times \text{id}) \) for each \( a \in \tau \) and \( v_b^* = v_b \circ \mu \) for each \( b \in \theta \). The partitions of unity \( \{ w_\alpha^* : a \in \tau \} \) on \( Y \times I^k \) and \( \{ v_b^* : b \in \theta \} \) on \( Y \) are subordinated to the covers \( \lambda \) and \( \eta \) respectively. We set \( \mathcal{K}' = N_\lambda \) and \( \mathcal{L} = N_\eta \). The simplicial complexes \( \mathcal{K}' \) and \( \mathcal{L} \) are isomorphic to \( N_\xi \) and \( N_\zeta \) that is why the simplicial map \( q : |\mathcal{K}'| \to |\mathcal{L}| \) is defined and the following diagram commutes.

\[
\begin{array}{ccc}
Y \times I^k & \xrightarrow{\varphi} & |\mathcal{K}'| \\
pr \downarrow & & \downarrow q \\
Y & \xrightarrow{\kappa_Y} & |\mathcal{L}|.
\end{array}
\]
Here \( \varphi \) and \( \kappa_Y \) are canonical maps defined by the partitions of unity \( \{ w_a^* : a \in \tau \} \) and \( \{ v_b^* : b \in \theta \} \) respectively. Obviously, \( \dim q \leq k \).

Recall that \( \varphi \) is an \( \{ O(y,t) \} \)-map. For every \( a \in \tau \) pick a point \((y,t)_a\) such that \( W_a \subset O(y,t)_a \). Let \( w_a^{**} = w_a^* \circ (f \triangle g) \) and \( B_a = \Gamma(y,t)_a \). Then

\[
\text{supp } (w_a^{**}) \subset (f \triangle g)^{-1}(W_a) \subset (f \triangle g)^{-1}(O(y,t)_a).
\]

Consequently, \( \bigcup \nu(y,t)_a \supset \text{supp}(w_a^{**}) \). As the family \( \nu(y,t)_a \) is disjoint, there exists a family of non-negative functions \( \{ b_{\beta} : \beta \in B_a \} \) such that \( w_a^{**} = \sum_{\beta \in B_a} b_{\beta} \) and \( \text{supp}(b_{\beta}) \subset V_{\beta} \). Let \( B = \bigcup B_a : a \in \tau \) and \( V = \{ V_{\beta} \cap (f \triangle g)^{-1}(W_a) : a \in \tau, \beta \in B_a \} \). The family \( \{ b_{\beta} : \beta \in B \} \) is a partition of unity on \( X \) subordinated to the cover \( V \). Let \( K \) be the nerve of the cover \( V \) and \( \kappa_X : X \rightarrow |K| \) the canonical map defined by the partition of unity \( \{ b_{\beta} : \beta \in B \} \). We define a simplicial map \( p' : |K| \rightarrow |K'| \) by requiring that the vertex \( \beta \in B_a \) goes to \( a \). Clearly, the map \( p' \) is finite-to-one. Indeed, no two vertices in \( B_a \) are connected by an edge. That is why the restriction of \( p' \) to any simplex is a homeomorphism. Finally we define the desired \( k \)-dimensional simplicial map \( p : |K| \rightarrow |L| \) as the composition \( p = q \circ p' \). Moreover by (b_1) we have \( \dim |L| \leq \dim Y \) and \( \dim |K| \leq \dim Y + k \) since \( p \) is \( k \)-dimensional.

\[
\text{Proof of theorem 1.1.} \quad \text{The theorem is an immediate consequence of theorems 1.4 and 1.7 and the remark at the end of the proof of theorem 1.4.} \]

The following results of M. Levin and Y. Sternfeld are needed to prove theorems 1.2 and 1.3.

**Theorem 2.1 ([3]).** Let \( f : X \rightarrow Y \) be a \( k \)-dimensional map between compacta. Then there exists a map \( g : X \rightarrow \mathbb{I}^{k+1} \) such that \( \dim(f \triangle g) \leq 0 \).

**Theorem 2.2 ([8]).** Let \( f : X \rightarrow Y \) be a \( k \)-dimensional map of Bing compacta. Then there exists a map \( g : X \rightarrow \mathbb{I}^k \) such that \( \dim(f \triangle g) \leq 0 \).

\[
\text{Proof of theorem 1.2.} \quad \text{The theorem is an immediate consequence of theorems 2.1 and 1.4.} \]

\[
\text{Proof of theorem 1.3.} \quad \text{The theorem is an immediate consequence of theorems 2.2 and 1.4.} \]

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