Identification and Estimation of A Rational Inattention Discrete Choice Model with Bayesian Persuasion

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Abstract

This paper studies the semi-parametric identification and estimation of a rational inattention model with Bayesian persuasion. The identification requires the observation of a cross-section of market-level outcomes. The empirical content of the model can be characterized by three moment conditions. A two-step estimation procedure is proposed to avoid computation complexity in the structural model. In the empirical application, I study the persuasion effect of Fox News in the 2000 presidential election. Welfare analysis shows that persuasion will not influence voters with high school education but will generate higher dispersion in the welfare of voters with a partial college education and decrease the dispersion in the welfare of voters with a bachelors degree.

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1 Introduction

In many applications of discrete choice models, econometricians usually assume the decision maker has the following random utility from choosing item $j$ among a choice set $J = 1,...,J$: $U_j = u_j + \epsilon_j$, where $u_j$ is the mean utility observed by the econometricians and $\epsilon_j$ is the utility shock known to the decision maker but not the econometrician. Decision makers in the model choose the item with the highest utility. When the unobserved shock follows the Type I extreme value distribution, we can solve the probability of choosing $j$ analytically. Aggregating the choice outcomes of the decision makers in the market we can get the market share of an item. This approach to studying market structure was initiated by [McFadden (1973)], and then adopted by [Berry, Levinsohn, and Pakes (1995)] (henceforth BLP) to study automobile markets, and became widely applied to other industries.

This model, however, is not easily adaptable to accommodate persuasion in a structural way. Take advertising as a form of persuasion. In the classical analysis of the effect of advertising, three approaches are adopted. The first is to model advertising as a feature of the item that enters mean utility $u_j = u_j(A)$, where the level of advertising $A$ affects the choice utility. The argument is that advertising is ‘persuasive’ and the individual will buy more of the advertised goods because their utility is distorted [Dorfman and Steiner (1954)]. This reduced form approach does not offer us much explanation of how advertisement influences decision making and market structure. The second approach is to model advertisement as the trigger of the consideration set change [Goeree (2008)]. The consideration set is the priori set of items that the decision maker chooses from. Advertisement thus serves as the trigger that puts a previously non-considered item into the consideration set. This approach views advertisement as the information revealing device that reveals the true $\epsilon_j$ to the decision maker which was previously $-\infty$ to the decision maker. If a good $j$ is already in the consideration set for all customers, the model of consideration set predicts that advertising has no effect on the market share. If a good is well known, the model of consideration set cannot explain why sellers advertise. The third approach is to view advertising as a signaling device to separate the high-quality product from the low-quality product [Nelson (1974) Bagwell and Ramey (1988)]. The degree of advertisement serves as
the signal that induces the separating equilibrium where only high-quality firms advertise. In particular, they assume the unobserved quality is common for all decision makers. However, this approach requires the decision maker in the model to have imperfect knowledge of $\epsilon_j$, which contradicts the assumption that $\epsilon_j$ is known by the decision maker.

Compared to the classical approaches to model persuasion, this paper develops an empirical model of persuasion using the Bayesian persuasion theory in [Kamenica and Gentzkow (2011)]. The Bayesian persuasion approach to model advertising differs from the previously mentioned informative view in two ways: 1. Decision makers in the model can have a different realization of the product quality; 2. The advertiser, who acts as the Bayesian persuader, does not always want to reveal their quality honestly. However, similar to the informative view, the Bayesian persuasion model assumes that the decision maker in the model only has a prior belief on the $\{\epsilon_j\}_{j=1}^J$ and the exact realization of $\{\epsilon_j\}_{j=1}^J$ is unknown. The decision maker’s prior distribution of $\{\epsilon_j\}_{j=1}^J$ comes from the reputation of the goods. The prior belief is likely to be common across decision makers. However, the standard Bayesian persuasion model assumes the decision makers only have access to the signal sent by the persuader to update their belief and no other sources of information are available. In the real world, decision makers will also search actively for information on the goods’ quality by themselves. For example, if a person wants to buy a car, he or she will have a test drive before making a decision. An extensive search of information can reduce the randomness of $\{\epsilon_j\}_{j=1}^J$ but at the time is costly. [Matejka and McKay (2015)] considers a model where the decision maker searches information of $\{\epsilon_j\}_{j=1}^J$ to maximize the expected utility after deducting the search cost. Their rational inattention discrete choice model can incorporate Bayesian persuasion by assuming persuaders send signals after the decision makers get their own information.

The analysis of the structural persuasion and information search model has largely been discussed under the assumption that the decision makers’ prior belief of $\{\epsilon_j\}_{j=1}^J$, denoted by $G$, is known to the economist. In empirical researches, the prior belief $G$ is unknown and should be estimated from data. A recent empirical study by [Xiang (2020)] assumes the decision makers’ prior distribution $G$ is normally distributed and analyzes the decision makers’ welfare change when a policy change induces the persuader to change the persuasion strategy. However, the empirical content of a parametric assumption on $G$ is unclear.
This paper follows Matejka and McKay (2015) to consider a rational inattention discrete model with Bayesian persuasion. I discuss the non-parametric identification of the prior distribution $G$ and parametric identification of persuader’s persuasion strategy when an econometrician observes the choice ratio at the market level across many independent markets. The independent markets are divided into two groups: the first group is not influenced by the persuader and the second group is influenced by the persuader. The prior distribution $G$ is identified from the choice ratio in the first group of markets. Given the identification of $G$, a parametric persuasion strategy is identified from the second group of markets. I characterize a set of moment conditions implied by the model, and the standard estimation method such as GMM can be applied easily.

For econometricians who already observe the market shares with and without the influence of persuasion, identifying the persuasion strategy is the first step to understanding the behavior of the persuader. If we assume the persuader use a persuasion strategy is to maximize some utility function, the identified persuasion strategy can help us understand the persuader’s objective function. Analysis in this paper leaves the persuader’s objective function as unknown and analyzes the behavior from the buyers’ side. A complete two-sided analysis will incorporate the persuader’s utility as a function of persuasion strategy and analyze the problem as a sequential game played between the persuader and the buyers.

For policymakers, given the knowledge of the prior belief $G$, they will be able to evaluate the effect of regulating the persuasion strategy. In the advertisement market, the policymakers for example can ban one seller from directly revealing information about his competitors’ products. Moreover, policymakers can also evaluate the effect of providing less costly information to the decision makers. In other words, policymakers can compete with existing persuaders in the markets to increase the decision makers’ welfare.

In the empirical application, I look at the 2000 presidential election in the United States. I treat the presidential candidates as voters’ choices and view voting statistical areas as separated markets. In 1996, Fox News was developed and then entered into approximately 30% of the towns in the United States by 2000. DellaVigna and Kaplan (2007) shows that Fox News motivated voters to vote for Republicans compared to voters in towns without Fox News. I take the data and analyze how Fox News persuaded voters in different towns.
The estimated results from the markets without Fox News show that the prior belief of the quality of the presidential candidates varies a lot with voters’ education level. Both voters with bachelor’s degrees and with only high school degrees prefer the Democratic party than the Republican Party. The estimated results also show that Fox News provided very little information to voters, but managed to manipulate the voting outcome by a significant margin. I also compare the welfare of voters with different education levels. Voters’ welfare is defined as the probability of choosing their first best choice, and their first best choice is the presidential candidate that will generate the highest utility to voters when the voters know the realization of $\{\epsilon_j\}_{j=1}^J$. The result shows that persuasion will not influence the welfare of voters with high school education but will generate higher dispersion in the welfare of voters with a partial college education and decrease the dispersion in the welfare of voters with a bachelors degree.

Another way to study the effect of persuasion is to model the presence of a persuader as a treatment status (Jun and Lee 2018). In their model, the presence of a Bayesian persuader is taken as treatment assignment and sharp bounds on the persuasion effect are given under various data generating processes. The treatment effect model does not specify the decision makers’ utility and thus analysis of the decision makers’ welfare before and after persuasion is not possible. The treatment effect model also makes it hard to consider policy counterfactual such as regulations on persuasion strategy or when the policymaker provides extra information in the market.

The rest of the paper is organized as follows. Section 2 introduces the rational inattention discrete choice model with persuasion. Section 3 discusses the data generating process and the identification strategy. Section 4 discusses the estimation strategy. Section 5 studies the 2000 presidential election and the effect of Fox News. Section 6 concludes.

2 The Model

I consider the standard random utility specification: a decision maker (DM) derives utility level $U_j$ from good $j$ from the choice set $\mathcal{J} = \{1, ..., J\}$:

$$U_j = u_j + \epsilon_j.$$
The $u_j$ is the mean utility of choosing good $j$ and $\epsilon_j$ is the individual specific random draw of utility shock. Throughout this section, I assume that the decision maker knows only $(u_1, ..., u_j)$ but not $(\epsilon_1, ..., \epsilon_j)$. The decision maker has a prior belief on the distribution $G$ on the utility shock: $(\epsilon_1, ..., \epsilon_J) \equiv \epsilon \sim G$. If there is no further information about the true utility shock $\epsilon$, the decision maker will choose the one with highest expected utility:

$$j \in a(G) \equiv \arg\max_{j \in J} E_G[u_j + \epsilon_j].$$ (2.1)

If $\arg\max_{j \in J} E_G[u_j + \epsilon_j]$ is not a singleton, we let $a(G)$ to be an arbitrary selection of maximizers. The maximized utility derived from the belief $G$ is given by

$$V(G) \equiv \max_{j \in J} E_G[u_{ij} + \epsilon_{ij}].$$ (2.2)

I will first introduce a rational inattention discrete choice model and then discuss how persuasion can be incorporated.

### 2.1 Rational Inattention Discrete Choice Model

The rational inattention discrete choice model in Matejka and McKay (2015) assumes that the decision maker can choose an information strategy to get a signal $s_{DM}$. The signal $s_{DM}$ updates the decision makers’ belief on the true utility shock $\epsilon$. The decision maker then choose the item with highest posterior mean according to (2.1). Following the notation in Matejka and McKay (2015), denote $u_j + \epsilon_j \equiv v_j$. Formally, the decision maker’s information strategy is a joint distribution of the true utility vector $v \in \mathbb{R}^J$ and the signal $s_{DM} \in \mathbb{R}^J$, denoted by $F(s_{DM}, v)$. The marginal distribution of the information strategy has to be consistent with the prior belief $G$. Once the decision maker is committed to the information strategy, the random shocks to utility are realized, and then the decision maker get a realized signal $s_{DM}$ from $F(s_{DM} | v)$. The decision maker updates his belief as $F(\epsilon | s_{DM})$, and chooses the item in $a(F(\epsilon | s_{DM}))$ according to (2.1).

Since the real utility shocks are not observed by the decision maker, the decision maker solves the following optimization problem to maximize his expected utility:

$$\max_{F \in \Delta(\mathbb{R}^{2J})} \int_{s_{DM}} \int_{v} V(F(\cdot | s_{DM}))F(ds_{DM} | v)G(dv) - c(F)$$ (2.3)
\[
\text{s.t. } \int_{s^{DM}} F(ds^{DM}, v) = G(v) \quad (2.4)
\]

where \( V(F(\cdot|s^{DM})) \) is determined by (2.2). The constraint (2.4) requires that the DM’s prior distribution \( G \) is consistent with the real state of the world. The cost of information \( c(F) \) is the mutual information between the shocks \( \epsilon \) and the signal \( s^{DM} \):

\[
c(F) = \lambda \{ H(G) - E_s[H(F(\cdot|s^{DM})]) \}, \quad (2.5)
\]

where the parameter \( \lambda \) is the unit cost of information, and \( E_s \) denote the expectation over the marginal distribution of \( F(s^{DM}, v) \). The entropy function \( H \) of a discrete distribution \( G \) is defined as:

\[
H(G) = - \sum_k P_k \log(P_k), \quad \text{where } P_k \text{ is the probability of the state } k.
\]

When \( G \) is continuously distributed, the differential entropy is defined as

\[
H(G) = - \int g(s) \log(g(s))ds.
\]

The use of entropy reduction as a measure of information cost is standard in the rational inattention literature. See De Oliveira et al. (2017) for the discussion of entropy cost. Moreover, the entropy number is related to the complexity of a random variable, and can be given a data compression interpretation. The mutual information in (2.5) can be interpreted as the number of binary questions asked by acquiring signal \( s \). Appendix A gives an example of data compression interpretation.

Let \( S_j^{DM} \equiv \{ s^{DM} \in \mathcal{R}^J : a(F(\cdot|s^{DM}) = j) \} \) be the set of signals that lead the DM to choose \( j \). Also denote

\[
P_j(v) \equiv \int_{S_j^{DM}} F(ds^{DM}|v) \quad (2.6)
\]

as the conditional choice probability of choosing item \( j \) when the realized utility vector is \( v \). Also define the unconditional choice probability of choosing \( j \) as

\[
P_j^0 = \int_v P_j(v)dG(v). \quad (2.7)
\]

This is the ex-ante probability of choosing \( j \) before the utility vector is realized.

A set of optimality condition to the problem (2.3)-(2.5) from Matejka and McKay (2015) is summarized in the following lemma.

\[1\] Note that the DM does not know the realization of \( v \). The conditional choice probability should be understood to be the choice probability when the actual utility vector is \( v \).
Lemma 2.1. If $\lambda > 0$ and $F$ is an optimal information strategy that solves (2.3)-(2.5), then the conditional and choice probability in (2.6) satisfies
\[ P_j(v) = \frac{P_j^0 e^{v_j/\lambda}}{\sum_{k \in J} P_k^0 e^{v_k/\lambda}} \text{ a.s.}, \quad (2.8) \]
\[ E_G[P_j(v)] = P_j^0. \quad (2.9) \]

The unconditional choice probability in (2.7) solves the following convex optimization problem:
\[
\max_{\{P_j^0\}_{j=1}^J} \int_v \lambda \log \left( \sum_{j=1}^J P_j^0 e^{v_j/\lambda} \right) G(dv) \\
\text{s.t. } \forall j : P_j^0 \geq 0, \\
\sum_{k=1}^J P_k^0 = 1. \quad (2.10)
\]

Conversely, if $\{P_j^0\}_{j=1}^J$ is the solution to (2.10), and $P_j(v)$ defined in (2.8) satisfies (2.9), then we can construct an information strategy $F$ such that:

- The signal $s^{DM}$ is supported on $J$ points: $\{s_1, \ldots, s_J\}$;
- The conditional distribution of $s^{DM}$ satisfies $Pr_F(s^{DM} = s_j) = P_j(v)$.

This information strategy $F$ solves the optimization problem (2.3)-(2.5).

Proof. See Theorem 1 and Lemma 2 in Matejka and McKay (2015). \qed

Lemma 2.1 shows that solving the optimization problem (2.3)-(2.5) is equivalent to solve the optimization problem (2.10). We do not observe the DM’s optimal information strategy. Instead, we observe their choice outcome. When we aggregate the choice outcome to the market level, it becomes the conditional and unconditional choice probability.

We should note that the conditional choice probability (2.8) takes a Logit-like choice probability form. However, the rational inattention discrete choice model does not imply the usual I.I.A constraints on the choice probability. Matejka and McKay (2015) discusses the two equivalent conditions on the conditional choice probability (2.8).
2.2 A Sequential Persuasion Game

Consider a persuader that tries to influence the choice probability by choosing a persuasion strategy and sending a realized signal. The persuader is also called he information designer (ID) in the Bayesian persuasion literature.

**Definition 1.** A persuasion strategy is a joint distribution \( \tilde{F}(s^{ID}, v) \) of the signal \( s^{ID} \in \mathcal{R}^J \) sent by the ID and the utility vector such that

\[
\int \tilde{F}(s^{ID}, v) ds^{ID} = G(v).
\]

I consider a sequential persuasion game between the decision makers and the information designer in the following order:

1. The information designer chooses an persuasion strategy and then sends the realized signal \( s^{ID} \) to the decision maker;

2. The decision maker updates his belief to the intermediate distribution:

\[
\tilde{G}_{s^{ID}} \equiv \tilde{G}(v|s^{ID} = s^{ID}) = \frac{G(v) \times \tilde{F}(s^{ID}|v)}{\int_v \tilde{F}(v, s) dv}; \tag{2.11}
\]

3. The decision maker solves optimization \((2.3)-(2.5)\) with the intermediate belief \( \tilde{G}_{s^{ID}} \);

4. The decision maker gets a realized signal \( s^{DM} \) from his optimal information strategy \( F \). He then makes the choice based on the updated belief \( F(v|s^{DM}) \).

For the persuasion strategy to work, it is assumed that the DM who receives the signal knows the joint distribution \( \tilde{F}(s^{ID}, v) \).

**Assumption 2.1.** The persuasion strategy \( \tilde{F}(s^{ID}, v) \) is common knowledge.

The assumption \([2.1]\) on \( \tilde{F} \) is satisfied when there is an underlying equilibrium determining how the information designer chooses the persuasion strategy. For example, information designer can have an objective function \( M : \Delta(\mathcal{R}^{2J}) \to \mathcal{R} \) so that \( \tilde{F} = \arg \max_{F \in C} M(F) \) where \( C \subset \Delta(\mathcal{R}^{2J}) \) is some constrained persuasion strategy set. When the objective function \( M \) and the constrained set \( C \) is known by the DM, the decision maker can solve the information designer’s optimization problem to get \( \tilde{F} \). This paper does not tackle the information
designer’s objective function. The objective function for the information designer is not easy to formulate. In the marketing context, the trade-off is between higher marketing cost of persuasion and higher sales. In the context of political persuasion, the goal of persuasion is not to maximize voting share but to increase voting share until it exceeds 50%. Also, the media that conducts persuasion may also care about other aspects of persuasion since their persuasion strategy can influence their audience ratings.

The setting of the persuasion game is different from the setting in [Bloedel and Segal (2018)]. In their setting, the decision maker chooses an information strategy to understand the signal send by the sender. In other words, the decision maker in their model pays attention cost to understand the signal from the sender and cannot acquire a signal about the true utility by himself. In my formulation, there is no cost to understand the signal $s_{ID}$ from the sender and there is a cost incurred by acquiring information about the true utility vector.

**Remark 2.1.** The persuasion strategy $\tilde{F}$ and the information strategy $F$ lie in the same space. The effect of persuasion is limited because decision makers can acquire their own information. While the ID can distort the prior distribution of the utility vector $\nu$ through $\tilde{F}$, the decision maker’s information strategy $F(s_{DM}, \nu)$ can still provide information to the decision maker.

### 3 Identification

In this section, I discuss a data generating process that allows us to non-parametrically identify the prior belief $G$ and parametrically identify the persuasion strategy $\tilde{F}$. To allow for the heterogeneity of decision makers’ preferences, I assume the utility of an individual $i$ in market $m$ in the demographic group $k$ takes the following additively separable form:

$$U_{ikjm} = u_1(x_j^m, \beta) + u_2(x_j^m, \nu_{ik}, \alpha) + \epsilon_{j,m}, \quad (3.1)$$

where $x_j^m$ is the characteristics of product $j$ in market $m$; $k$ is the index for people of demographic group $k$ with demographic characteristics $\nu_{ik}$; $m$ is the index for market. The utility function $u_1, u_2$ is of known parametric form, and $\alpha, \beta$ are two vectors to be estimated,
but the distribution of $G$ is left as non-parametric. The utility (3.1) assumes that the decision
makers’ demographic and product characteristics only influence their mean utility but not
the utility shocks. Here I assume that all DMs in the same market $m$ realize the same
$\epsilon_m = (\epsilon_{1m}, ..., \epsilon_{jm})$ since the random shock vector $\epsilon_m$ in equation (3.1) does not depend on
the individual index $i$. In particular, if individual $i_1$ and $i_2$ are in the same market, and they
have the same demographic characteristics, they should have the same realized utility vector.
This specification is reasonable when the shock is market-specific. For example, when we
want to study the voting decision, the market realization of $\epsilon_m$ can be the real payoff of
candidate $j$’s policy on town $m$’s local industry. In the automobile industry, this market
level states of the world may come from the local road condition, climate, or geographic
topology.

Notation

Throughout this section, I use $\tilde{}$ to denote probability quantities related to markets with the
presence of a persuader. Also, I drop the super-scrip on $s^{ID}$, and use $s$ to denote signals sent
by the information designer whenever there is no confusion. I use $m$ to denote the index for
markets, $j$ to denote the index of products, $k$ to denote the index of different demographic
groups.

3.1 Data Generating Process

In many data sets, we do not observe individual choices. Instead, we observe the market
share, which is the aggregated individual choices. Across different markets, I assume that
the prior distribution on $\epsilon_{j,m}$ is the same $G$.

Assumption 3.1. (Data) (i) We observe a binary variable $\chi^m$ such that $\chi^m = 1$ if and
only if the persuader is present in market $m$; (ii). The demographic heterogeneity $v_k$ is
discrete and supported on $K$ points. For each market $m$, the distribution of demographic
heterogeneity $D^m = (d^m_1, ..., d^m_K)$ is observed, where $d^m_k$ is the proportion of DMs in group $k$
in market $m$; (iii). We observe the market characteristics $X^m$ in each market $m$ and the
market share vector $ms^m = (ms^m_1, ..., ms^m_J)$, where $ms^m_j$ is the market share of product $j$. 

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The following is the assumption on the markets where there is no persuader.

**Assumption 3.2. (DGP without Persuasion)** For markets with $\chi^m = 0$, the data generating process satisfies:

1. *Common prior*: $\epsilon^m \sim G$;

2. *Independent random utility shock*: $\epsilon^m \perp X^m$

3. *Independent demographic distribution*: $D^m \perp (\epsilon^m, X^m)$

4. *The choice set $J$ and information cost $\lambda$ are the same across different markets.*

Assumption 3.2 imposes that the mean of $\epsilon^m$ is independent of the product characteristics and is normalized to be zero. If there is any unobserved characteristics that is correlated with $X^m_j$, the unobserved effects are captured by the observed $X^m_j$.

For markets with a persuader, I assume that the persuader is the same across these markets and the persuader use the same persuasion strategy. Moreover, I assume that the persuasion strategy is a joint distribution of $\epsilon^m$ and $s^{ID}$. This specification is different from Definition 1 and the persuader uses the same persuasion strategy even if the product characteristics $X^m_j$ may vary across markets.

**Assumption 3.3. (DGP with Persuasion)** For markets with $\chi^m = 1$, the data generating process satisfies:

1. $(\epsilon, X^m_j, D^m)$ and $(\lambda, J)$ satisfy the conditions in Assumption 3.2;

2. There is a uniform persuader across markets with $\chi^m = 1$ and the persuasion strategy $\tilde{F}^k(s^{ID}, \epsilon)$ can depend on the demographic groups $k$ but not the market;

3. The persuasion signal $s^{ID}_{km} \sim \tilde{F}^k(s^{ID}|\epsilon_m)$ and the signal $s^{ID}_{km}$ is independent of each other across demographic groups and markets;

4. *Signal Independence*: $(s^{ID}_k, \epsilon^m) \perp D^m$. 

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Normalization

Since the permutation of the item index does not matter, I call the last item $J$ the outside option. Note that in the discrete choice model, only the relative difference of utility matters for the DM. Therefore, we can normalize the utility of outside option $J$ to be zero $U_{k,J} = 0$. Also, when $u_1, u_2$ is homogeneous of degree one with respect to $\alpha, \beta$, the vector $(\alpha, \beta, \lambda, G)$ is not identified. Indeed, we can consider a model with $(c\alpha, c\beta, c\lambda, cG)$, where $cG$ is the distribution of $c\epsilon_m$. The model $(c\alpha, c\beta, c\lambda, cG)$ will generate the same choice probability (2.6) and (2.7). Since linear specification of utility is frequently used in the applied literature, I assume $u_1, u_2$ is homogeneous of degree one with respect to $\alpha, \beta$.

Assumption 3.4. (Normalization) The utility functions $u_1, u_2$ are homogeneous of degree 1 with respect to $(\alpha, \beta)$, and $\lambda = 1$.

3.2 The Identified Set

The parameters of interests include the mean utility parameters $(\alpha, \beta)$, the prior belief $G$, and the persuasion strategy $\tilde{F}$. For markets without persuasion, we are also interested in $P_{j,k}^{0,k}(X)$, which is demographic group $k$’s unconditional choice probability of choosing $j$ when the product characteristics are $X$. If we want to evaluate the overall effect of persuasion across different markets, we want to compare the post-persuasion market share with $P_{j,k}^{0,k}(X)$.

I first define the identified set of $(\alpha, \beta, G, P_{j,k}^{0,k}(X))$ from the rational inattention discrete choice model.

Definition 2. Let $F_{\chi=0}$ denote the conditional distribution of $(D^m, X^m, m_s^m)$ conditioned on $\chi^m = 0$. The identified set of $(\alpha, \beta, G, P_{j,k}^{0,k}(X))$ under the rational inattention discrete choice model, denoted by $\Gamma_I$, is the collection of $(\alpha, \beta, \{P_{j,k}^{0,k}(X)\}_{j,k}, G)$ that satisfies the following constraints:

1. Given $(\alpha, \beta, G)$, $\{P_{j,k}^{0,k}(X)\}_{j,k}$ solves the individuals optimization problem (2.10) with

$$v^m_j = u_1(x^m_j, \beta) + u_2(x^m_j, \nu_{ik}, \alpha) + \epsilon^m_j,$$

(3.2)
2. The unconditional mean of the conditional choice probability is the unconditional choice probability:

\[ E_G \left[ \frac{\mathcal{P}_{j}^{0,k}(X)e^{v_j^m}/\lambda}{\sum_{k\in\mathcal{J}} \mathcal{P}_{j}^{0,k}(X)e^{v_k^m}/\lambda} \right] = \mathcal{P}_{j}^{0,k}(X); \]  

(3.3)

3. Consider the mapping:

\[ \mathcal{P}_{j}^{m}(\alpha, \beta, \epsilon, D_{j}^{m}, X_{j}^{m}, \{\mathcal{P}_{j}^{0,k}(X)\}_{j,k}) = \sum_{k} d_{k}^{m} \frac{\mathcal{P}_{j}^{0,k}(X)e^{v_j^m}/\lambda}{\sum_{k\in\mathcal{J}} \mathcal{P}_{j}^{0,k}(X)e^{v_k^m}/\lambda} \]  

(3.4)

where \( v_j^m \) is defined in (3.2). Then \( (D_{j}^{m}, X_{j}^{m}, \mathcal{P}_{j}^{m}(\alpha, \beta, \epsilon, D_{j}^{m}, X_{j}^{m}, \{\mathcal{P}_{j}^{0,k}(X)\}_{j,k}) \) has the same distribution as \( \mathcal{F}_{\chi=0} \).

The first two conditions in Definition 2 corresponds to the optimization condition (2.10) and the condition (2.9) in Lemma 2.1. Equation (3.4) calculates the market share of product \( j \) as the weighted average of different demographic groups’ choice probability. The third condition in Definition 2 requires the model predicted market share is consistent with the observed data distribution.

I then define the identified set of the persuasion strategy \( \bar{F} \).

**Definition 3.** Let \( \mathcal{F}_{\chi=1} \) denote the conditional distribution of \( (D_{j}^{m}, X_{j}^{m}, ms_{j}^{m}) \) conditioned on \( \chi_{j}^{m} = 1 \). Given the value of \( (\alpha, \beta, G) \), and a persuasion strategy \( \bar{F} \), we consider the map:

\[ \bar{\mathcal{P}}_{j,s}^{k}(\epsilon; X^{m}) = \frac{\bar{\mathcal{P}}_{j,s}^{0,k}(X^{m})e^{v_j^m}}{\sum_{l=1}^{j} \bar{\mathcal{P}}_{l,s}^{0,k}(X^{m})e^{v_l^m}} \]  

(3.5)

where \( \bar{\mathcal{P}}_{j,s}^{0,k}(X^{m}) \) solves the individual optimization problem (2.10) when his belief is \( \bar{G}(\epsilon) = \bar{F}(\epsilon|s^{ID} = s) \). The identified set of the persuasion strategy is the set of \( \bar{F}(s^{ID}, \epsilon) \) such that

\[ (D_{j}^{m}, X_{j}^{m}, \sum_{k} d_{k}^{m} \bar{\mathcal{P}}_{j,s}^{k}(\epsilon; X^{m})) \]

has the same distribution as \( \mathcal{F}_{\chi=1} \).

The identified set in Definition 3 is conditioned on the vector \( (\alpha, \beta, G) \). This is because in markets with the persuader, there are two types of the unobserved heterogeneity: the utility shock \( \epsilon^m \) and the realization of the signal \( s^{ID} \). In contrast, in markets without the persuader,
only the utility shock $\epsilon^m$ exists. Therefore, knowing the prior distribution $G$ reduces the randomness and makes the problem of identifying the persuasion strategy tractable.

The identified set of $(\alpha, \beta, G)$ in Definition 2 is defined through the rational inattention discrete choice model only, and it ignores the empirical content of the subsequent persuasion stage. There are two reasons to define the identified set in this way. First, if we only have data on markets without any persuader, i.e. $\chi^m = 0$ for all markets, the identified set defined in Definition 2 can still be used. Second, the unobserved persuasion signal $s^{ID}$ in the persuasion stage makes it hard to characterize the empirical content of the whole persuasion game. I will stick with these two definitions and characterize the corresponding moment conditions.

3.3 Moment Equality Model

Recall that given $(\alpha, \beta)$ and the unconditional choice probability $\{P_{0,k}(X)\}_{j,k}$, the model predicted market share is given by (3.4). Following BLP, I denote $\delta_m^j = u_1(x^m_j, \beta) + \epsilon_m^j$ and let $\delta^m = (\delta_1^m, ..., \delta_J^m)$. Then the predicted market share in (3.4) can be written as:

$$P_m^j(\alpha, \beta, \epsilon, D_m, X^m, \{P_{0,k}(X)\}_{j,k}) = \sum_k P_{0,k}(X^m)e^{\delta_m^j + u_2(x^m_j, \nu_k, \alpha)} \delta_m^k$$

$$\equiv ms^*(X^m, \delta^m, \alpha, D_m, \{P_{0,k}(X)\}_{j,k}),$$

where the utility of the outside option is normalized to zero, so $\delta_J^m = u_2(x^m_J, \nu_k, \alpha) = 0$. Consider a mapping $T : \mathbb{R}^{J-1} \rightarrow \mathbb{R}^{J-1}$ such that

$$[T[X^m, ms^m, \alpha, D_m, \{P_{0,k}(X)\}_{j,k}](\delta^m)]_j = \delta_j + \log(ms^m_j) - \log(ms^*_j(X^m, \delta, \alpha, D_m, \{P_{0,k}(X)\}_{j,k})),$$

where $[T]_j$ is the $j$-th entry in the output vector. The input $ms^m$ will be the observed market share. When $T(\delta^m) = \delta^m$, the observed market share $ms^m$ equals the model predicted market share. This map is a contraction mapping whenever the outside option has a nonzero unconditional choice probability. As a result, there exists a unique market level $\delta^m$ that matches the observed market share with the model predicted market share.

Lemma 3.1. Suppose in a market we have $ms^m_j > 0$, then the mapping defined by (3.6) is a contraction mapping. Let $\delta^{*m}$ denote the fixed point of the contraction mapping (3.6). As a
result, the unobserved heterogeneity $\delta^*_m$ is a function of observables $x^m$, $D_k$, $ms^m$ and the parameters $\alpha$ and $P^{0,k}_j(X)$.

Now I state the first identification result of the prior distribution $G$.  

**Proposition 1.** For each $(\alpha, \beta, \{P^{0,k}_j(X)\}_{j,k})$ in the identified set $\Gamma_I$ defined in Definition 2, there exists a unique $G^*$ such that $(\alpha, \beta, G^*, \{P^{0,k}_j(X)\}_{j,k}) \in \Gamma_I$. In particular, for any measurable set $B$, define the set

$$MS(B; x^m, D_k, ms^m, \alpha, P^{0,k}_j(X), \beta) \equiv \{ms^m : \delta^m(x^m, D^m, ms^m, \alpha, P^{0,k}_j(X)) - [u_1(x^m_j, \beta)]_{j=1}^J \in B\},$$

where $\delta^m$ is defined in Lemma 3.4 and $[u_1(x^m_j, \beta)]_{j=1}^J = [u_1(x^m_1, \beta), ..., u_1(x^m_J, \beta)]'$. The $G^*$ satisfies

$$Pr_{G^*}(e^m \in B) = Pr_{F_{X=0}}(ms^m \in MS(B; x^m, D_k, ms^m, \alpha, P^{0,k}_j(X), \beta)).$$  \hspace{1cm} (3.7)

**Proof.** I prove this statement by contradiction. Suppose there exists a $G' \neq G^*$ such that $(\alpha, \beta, G', \{P^{0,k}_j(X)\}_{j,k})$ is also in the identified set. Suppose there exists a positively measured set $B'$ such that

$$Pr_{G^*}(e^m \in B') \neq Pr_{G'}(e^m \in B').$$

I claim the distribution of $ms^m$ implied by $G'$, denoted by $F'_{X=0}$ is different from $F_{X=0}$. By equation (3.4),

$$Pr_{F'_{X=0}}(ms^m \in MS(B'; x^m, D_k, ms^m, P^{0,k}_j(X), \beta)) = Pr_{G'}(e^m \in B') = Pr_{G^*}(e^m \in B') = Pr_{F_{X=0}}(ms^m \in MS(B'; x^m, D_k, ms^m, \alpha, P^{0,k}_j(X), \beta)).$$

Therefore, $G'$ cannot generate the same data distribution $F_{X=0}$, so $G'$ is not in the identified set by Definition 2. \qed

**Proposition 1** states that once we know $(\alpha, \beta, \{P^{0,k}_j(X)\}_{j,k})$, the distribution $G$ is point identified. This is similar to the identification strategy in the first price auction models (Guerre et al., 2000). The $\delta^m - [u_1(x^m_j, \beta)]_{j=1}^J$ is the pseudo value of $e^m$, similar to the pseudo value that is constructed from bids in the auction model.
Proposition 2. Suppose assumptions 3.2, 3.4. Suppose the unconditional choice probability \(\{P_{0,j}^k(X)\}_{j,k}\) are uniformly bounded away from zero and one. Each \((\alpha, \beta, \{P_{0,j}^k(X)\}_{j,k}, G)\) in the identified set \(\Gamma_I\) defined in Definition 2 satisfies:

1. Constraint on unconditional choice probability:

\[
E \left[ ms^m - \left( \begin{array}{ccc}
P_{1,1}^0 (X^m) & \cdots & P_{1,K}^0 (X^m) \\
\vdots & \ddots & \vdots \\
P_{J,1}^0 (X^m) & \cdots & P_{J,K}^0 (X^m)
\end{array} \right) \left( \begin{array}{c}
d_1^m \\
\vdots \\
d_K^m
\end{array} \right) \bigg| (D^m), X^m \right] = 0; \tag{3.8}
\]

2. Instrument constraint:

\[
E[\delta_j^* (ms^m, X^m, D^m, \alpha, \{P_{j}^{0,k}(X)\}_{j,k}) - u_1 (X_{j}^m, \beta)|X^m, D^m] = 0, \quad \forall j = 1, \ldots, J - 1; \tag{3.9}
\]

3. Optimality constraint on \(\{P_{j}^{0,k}(X)\}_{j,k}\) \(\forall j = 1, 2, \ldots, J - 1 \quad k = 1, \ldots, K:\)

\[
E \left[ \frac{e^{\delta_j^m + u_2 (x^m_j, \nu_k, \alpha)}}{\sum_{l \in J} P_{l}^{0,k} (X) e^{\delta_l^m + u_2 (x^m_l, \nu_k, \alpha)}} - 1 \bigg| X^m \right] = 0; \tag{3.10}
\]

4. \(G\) satisfies equation (3.7).

The first moment equality (3.8) is equivalent to condition (3.3) in Definition 2 since \(ms^m\) is the conditional choice probability while \(P_{j}^{0,k}\) is the unconditional choice probability. The second moment inequality (3.9) is the consequence of conditions 2 and 3 in Assumption 3.2. The third moment inequality is the first order condition of (2.10).

Remark 3.1. The identification results are different from the results in BLP in several ways. First, we need the number of markets to be large to identify the unconditional choice probability for different demographic groups from (3.8). From the identified unconditional choice probability, we can proceed to identify coefficients on the product and demographic heterogeneous characteristics \(\alpha\) and \(\beta\). Second, in BLP we assume there is a vector of unobserved product heterogeneity \(\xi = (\xi_1, \ldots, \xi_J)\) that can be recovered by matching market shares and model prediction. In the rational inattention discrete choice model, we recover a vector of market-specific utility shock \(\epsilon\). Third, the prior distribution of \(\epsilon\) is the structural
object that we are interested in, but the distribution of $\xi$ in BLP is not of fundamental interest.

**Remark 3.2.** If the price of item $j$, denoted by $q_j$, enters in the product heterogeneity $X_j$, then the price is likely to be correlated with the unobserved market realized utility shock. For example, when sellers know the realization of $\epsilon$, they may set a price accordingly. In this case, the assumption $E[\epsilon^m|X^m] = 0$ fails. In this case, we may want to find an instrument for $q_j$. The choice of instruments for the price is discussed in BLP.

Definition 3 of the identified set of persuasion strategy is conditioned on the value of $(\beta, \alpha, G)$. If $(\beta, \alpha, G)$ is point identified from Proposition 2, we can assume that $(\beta, \alpha, G)$ is known by the econometrician and plug the identified $(\beta, \alpha, G)$ into Definition 3. If $(\beta, \alpha, G)$ is not point identified, we can do analysis by considering that each point in the identified set $\Gamma_I$ as the true value separately.

For a point $(\alpha, \beta, G)$ in the identified set $\Gamma_I$, equation (3.5) defines the conditional choice probability of demographic group $k$ choosing item $j$ when they receive a persuasion signal $s$ from the ID. The $\tilde{P}^0_{j,s}$ is the unconditional choice probability solved from (2.3)-(2.5) when the intermediate belief is $F(\epsilon|s)$. The choice probability $\tilde{P}^0_{j,s}$ is conditioned on the signal $s^{ID}$, but unconditional on the utility shock.

The observed market share $\tilde{ms}^m$ is a linear combination of different demographic groups’ conditional choice probability:

$$\tilde{m}s^m_j = (\tilde{P}^1_{j,s}(\epsilon, X^m), ... \tilde{P}^K_{j,s}(\epsilon, X^m))(d^m_1, ..., d^m_K)'.$$  \hfill (3.11)

Conditioned on $(d^m_1, ..., d^m_K)$, we can take expectation on both sides of (3.11) to get:

$$E[\tilde{m}s_j - (\tilde{P}^1_{j,s}(\epsilon, X^m), ... \tilde{P}^K_{j,s}(\epsilon, X^m))(d^m_1, ..., d^m_K)'|D^m, X^m] = 0, \ \forall j = 1, ... J.$$  \hfill (3.12)

Since we do not observe the realization of the persuasion signal and the realization of the utility shock in each market, we can integrate it out. Let

$$h^k_j(X^m; \tilde{F}^k) := \int_{(s, \epsilon)} \tilde{P}^k_{j,s}(\epsilon, X^m)d\tilde{F}(s, \epsilon)$$

$$= \int_{(s, \epsilon)} \tilde{P}^k_{j,s}(\epsilon, X^m)d\tilde{F}(\epsilon|s)d\tilde{F}^k(s)$$

$$= \int_s \tilde{P}^0_{j,s}(X^m)d\tilde{F}^k(s)$$  \hfill (3.13)
be the unconditional choice probability for demographic group \( k \) under persuasion strategy \( \tilde{F}(s, \epsilon; \theta) \). The third equality holds because \( G(\epsilon | s; \theta) = \tilde{F}(\epsilon | s; \theta) \) by Bayes’ rule.

**Proposition 3.** Under assumption 3.2 - 3.3, for each \((\alpha, \beta, G)\), the true persuasion strategy parameter \( \theta_0 \) must satisfy the moment condition

\[
E[\tilde{m}_j - \sum_{k=1}^{K} h_j^k(X^m; \tilde{F}) d_k^m | D^m, X^m] = 0 \quad \forall j = 1 \ldots J - 1.
\] (3.14)

**Proof.** By assumption 3.2, the independence of demographic distribution \( D^m \) and \((\epsilon^m, X^m)\):

\[
E[\tilde{P}^0_{j,s}(\epsilon^m, X^m)|D^m, X^m] = \tilde{P}^0_{j,s}(X^m).
\]

Then by (3.12), we have

\[
E[\tilde{m}_j - (\tilde{P}^0_{j,1}(X^m), \ldots, \tilde{P}^0_{j,K}(X^m))(d_1^m, \ldots, d_K^m)' | D^m, X^m] = 0.
\] (3.15)

Since the signal \( s^{ID} \perp D^m, X^m \) by assumption 3.3, we have \( E[\tilde{P}^0_{j,s}(X^m)|D^m, X^m] = h_j^k(X^m, \tilde{F}) \). The result follows.

The effective number of conditional moment equality is \( J - 1 \) since I have the constraint that \( \sum \tilde{m}_j = 1 \). We should be careful with the persuasion strategy in Bayesian persuasion. The value of a signal in persuasion strategy itself has no meaning beyond the context of a communication game. For example, if \( \tilde{F}_1 \) is the distribution of \((\epsilon, s^{ID})\) and is the persuasion strategy used by the persuader, then let \( \tilde{F}_2 \) be the distribution of \((\epsilon, s^{ID} + \Delta)\), where \( \Delta \) is an arbitrary vector that lies in the same space as \( s^{ID} \). \( \tilde{F}_2 \) as a persuasion strategy is not different from \( \tilde{F}_1 \) since the value of the signal does not matter.

In practice, we can consider the case where the persuasion strategy is indexed by a finite-dimensional parameter \( \theta \): \( \tilde{F}^k(s^{ID}, \epsilon; \theta) \), and the support of \( s^{ID} \) is finite. The persuasion strategy can depend on the demographic group \( k \). There are several justifications for the use of a parametric persuasion strategy. First, when there are only two choices, the optimal persuasion strategy is to use a cut-off rule, see Kamenica and Gentzkow (2016). In this case, the parameter \( \theta \) is the cutoff points, and signals only take two values. Second, in many empirical contexts, it is costly to design complex persuasion strategies. For example, an online advertisement can only send a simple signal within a few seconds. If the cost of
signal increase with the number of parameters and support points of signals, it is natural to restrict the persuasion strategy to parametric form. Third, a parametric persuasion strategy with discrete signal support facilitates a clear interpretation of the meaning of the signals. In Kamenica and Gentzkow (2011), signals are interpreted as action recommendations.

Discussion of Moment Condition (3.14)

One issue with the moment condition (3.14) is that it does not guarantee the identification of persuasion parameters $\theta$. For example, consider the case where there is only one demographic group $K = 1$ and no product characteristics heterogeneity across markets $X^m = X \ \forall m$. In this case, moment condition (3.14) implies $h_j(\tilde{F}) = E[\tilde{m} s_j]$. If $\tilde{F}$ is indexed by a parameter $\theta$ and $h_j(\tilde{F}(\theta))$ is not monotone in $\theta$, then $\theta$ is not necessarily point identified.

There are several restrictions that help to tighten the identified set of $\tilde{F}$. The first is to impose the persuasion strategy is the same for certain demographic groups, i.e. $\tilde{F}^k(s|\epsilon) = \tilde{F}^{k'}(s|\epsilon)$ for some $k \neq k'$. Then demographic variation will tighten the bounds on the persuasion strategy. This is because different demographic groups’ choice probability can have different sensitivity to the same persuasion strategy. The second is to impose the parameter $\theta$ to be of lower dimension smaller than $J$. The variation of the choice probability across different products can tighten the bounds of the parameter that indexes the persuasion strategy. Third, the variation of product characteristics across markets can also tighten the bounds on $\tilde{F}$. This is because if in a market $m$ the $j$-th product characteristics $x^m_j$ generates large utility to decision makers, persuasion strategy is unlikely to change the market share a lot.

Point Identification Assumption

It is worthwhile to discuss the assumptions under which parameters $(\alpha, \beta, P^{0,k}_j, G)$ and $\theta$ are point identified. Note that the moment conditions constructed in (3.8)- (3.10) are similar to the moment conditions appeared in BLP, except that I have extra parameters $P^{0,k}_j(X)$ to identify. Note that the moment condition for $P^{0,k}_j(X)$ is similar to the moment condition for linear regression, so if $E[F_k F'_k | X]$ is invertible $X - a.s.$, then $P^{0,k}_j(X)$ is identified. The global sufficient primitive conditions for identification of moment conditions (3.9)- (3.10) are
not easy to interpret, because the fixed point $\delta^*$ in Lemma 3.1 is highly non-linear in its arguments. In a similar situation in BLP, they assume the moment conditions are sufficient to identify the utility parameter.

**Assumption 3.5. (Identification Assumption)**

1. $E[F_kF'_k|X]$ is invertible, $X - a.s.$
2. At the true parameter $\{P_{j,k}^{0}(X)\}_{j,k}$, there is a unique $(\alpha, \beta)$ such that moment conditions (3.9) and (3.10) hold.

The second requirement in Assumption 3.5 is not as restrictive as it seems. In particular, if there is only one demographic group, the fixed point in Lemma 3.1 is given by

$$\delta_j^* = \log \frac{ms_j^m}{ms_j^m} - \log \frac{P_{j}^{0}(X^m)}{P_{j}^{0}(X^m)}, \tag{3.16}$$

and moment condition (3.9) becomes

$$E \left[ \log \frac{ms_j^m}{ms_j^m} - \log \frac{P_{j}^{0}(X^m)}{P_{j}^{0}(X^m)} - u_1(X^m_j, \beta) \bigg| X^m \right] = 0.$$

If $\{P_{j,k}^{0}(X)\}_{j,k}$ is identified from moment condition 3.8 and $u_1$ is a linear function, then $\beta$ is point identified.

Now suppose the persuasion is parametric and indexed by $\theta$. The assumptions to guarantee that $\theta$ is identified up to $(\alpha, \beta, G)$ is easier to write down. The discussion of (3.14) shows that $h_{j,k}^{0}(x^m) \equiv h_{j}^{k}(x^m, \theta_0)$ is identified if $E[D^mD^m|X]$ is invertible, $X - a.s.$, where $\theta_0$ is the true value of $\theta$. Then the identified set of the persuasion strategy is then the set of the $\theta^*$ such that $h_{j}^{k}(x; \theta^*) = h_{j,k}^{0}(x)$ for all $j, k$ and $x \in supp(X)$.

**Assumption 3.6.** The matrix $E[D^mD^m|X]$ is invertible for $X - a.s.$

### 4 Estimation

Under Assumption 3.5, $(\alpha, \beta, G, \{P_{j,k}^{0}(X)\}_{j,k})$ is point identified from moment conditions (3.8), (3.9) and (3.10). When the product characteristics $X$ are continuously distributed,

---

\[^{2}\text{We say } \theta \text{ is identified up to } (\alpha, \beta, G) \text{ if the data generating process allow us to point identify } \theta \text{ for each given parameter } (\alpha, \beta, G).\]
$P_{j,k}(X)$ in moment condition (3.8) needs to be estimated non-parametrically. However, in some empirical settings, the product characteristics are discrete and standard estimators of moment equality such as GMM estimator can be implemented directly. In this section, I discuss the estimation of $(\alpha, \beta, G, \{P_{j,k}(X)\})_{j,k}$ when the characteristics $X$ are discrete.

**Assumption 4.1.** The product characteristics $X^m$ are discretely distributed and supported on $L$ points: $\{x(1), \ldots, x(L)\}$, and the probability $\inf_{l=1, \ldots, L} Pr(X^m = x(l)) > 1/C$ for some constant $C > 0$.

Under Assumption 4.1, the analysis of moment conditions (3.8), (3.9) and (3.10) can be done conditioned on the value of $X^m$ separately. Since the demographic characteristics $v_k$ are also discrete, the most general utility function of (3.1) under discrete $v_k$ and $X$ can be re-written as

$$u_{ijkm} = \alpha_{j,l}(k) \text{ if } (x^m_j)_{j=1}^J = x(l),$$

where $\alpha_{j,l}(k)$ is the mean utility of product $j$ for a demographic group $k$ individual in a market with characteristics $x(l)$. Any parametric assumption on the utility $u_1$ and $u_2$ in (3.1) can be imposed as constraints on the value of $\alpha_{j,l}(k)$.

Even if $v_k$ is distributed on $K$ discrete points, the random vector $D^m$ is continuously distributed. Moment conditions (3.8), (3.9) are still conditioned on $D^m$ and we need to transform them into unconditional moment conditions. Moment condition (3.8) is linear in the elements of $D^m$ and the optimal instrument will be $d^m_j, \ldots, d^m_k$, and we can define $P_{j,k}^0(x(l))$ as $P_{j,k}(l)$. For moment condition (3.9), we can use $D^m$ and its second order power terms $\{(d^m_j)^t : j = 1, \ldots, J, t = 1, 2\}$ as instruments to form unconditional moment conditions.

Let $\alpha$ denote the vector of $\{\alpha_{j,l}(k)\}_{j,k,l}$ and $P$ denote the vector of $\{P_{j,k}^0(l)\}_{j,k,l}$. Let $\gamma(ms^m, D^m, X^m, \alpha, P)$ denote the moment unconditional conditions. The standard GMM estimator of $(\alpha, P)$ is given by

$$(\hat{\alpha}, \hat{P}) = \arg \min \left[ \frac{1}{M} \sum_{m=1}^M \gamma(ms^m, D^m, X^m, \alpha, P) \right]^{-1} \hat{W} \left[ \frac{1}{M} \sum_{m=1}^M \gamma(ms^m, D^m, X^m, \alpha, P) \right],$$

(4.1)

3The utility parameter $\beta$ cannot be separated from $\alpha_{j,l}(k)$, so I normalize $u_1 \equiv 0$ for all $j$. 

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where $M$ is the number of markets without the persuader, and $\hat{W}$ is any positive semi-definite weighting matrix. Standard asymptotic normality results\footnote{For example, see Theorem 3.4 of Newey and McFadden (1994).} on the GMM estimator can be applied if the moment condition satisfies some regularity conditions.

**Assumption 4.2.** Suppose the following conditions hold: (i). The true parameter value $(\alpha_0, P_0)$ lies in the interior of the parameter space; (ii). $\gamma(m_{s}^{m}, D^{m}, X^{m}, \cdot, \cdot)$ is continuously differentiable on the interior of the parameter space for $(m_{s}^{m}, D^{m}, X^{m})$; (iii). $\gamma(m_{s}^{m}, D^{m}, X^{m}, \alpha^{0}, P^{0})$ has finite second moment; (iv) $E[|\nabla_{(\alpha, P)}\gamma(m_{s}^{m}, D^{m}, X^{m}, \alpha^{0}, P^{0})|$ has rank $\text{dim}((\alpha, P))$; (v). There exists a integrable function $b$ such that

$$\left|\nabla_{(\alpha, P)}\gamma(m_{s}^{m}, D^{m}, X^{m}, \alpha, P)\right| < b(m_{s}^{m}, D^{m}, X^{m}).$$

Conditions (i), (iii) and (iv) are assumptions on the true value of the parameter of interest $(\alpha_0, P_0)$, which are not verifiable without observing the data distribution. Conditions (ii) and (v) are assumptions on the derivatives of the moment conditions. It is difficult to verify (ii) and (v) because $\delta^{m*}$ as a function of $\alpha$ and $P$ is defined through the contraction mapping (3.6). General primitive conditions on the rational inattention model to guarantee that $\delta^{m*}$ is continuously differentiable in $\alpha, P$ are hard to find. However, when there is no demographic heterogeneity, the $\delta^{m*}$ in Lemma 3.1 has a closed form solution (3.16). In this case, the moment conditions (3.9) and (3.10) can be rewritten as

$$E\left[\left(\log \frac{m_{s}^{m}}{m_{s}^{m}_{j}} - \log \frac{P_{j}^{0}(x(l))}{P_{j}^{0}(x(l))} - \alpha_j(l)\right) 1(X^{m} = x(l))\right] = 0,$$

$$E\left[\left(\sum_{l \in J} P_{l}^{0,k}(X) \frac{m_{s}^{m}}{m_{s}^{m}_{j}} / \frac{P_{j}^{0}(x(l))}{P_{j}^{0}(x(l))} - 1\right) 1(X^{m} = x(l))\right] = 0.$$

If there exists a constant $C > 0$ such that $P_{j}^{0}(x(l)) > 1/C$ holds for all $j, l$, then conditions (ii) and (iv) holds.

**Lemma 4.1.** Suppose assumption 4.2 holds. Denote $B_0 = E[\nabla_{(\alpha, P)}\gamma(m_{s}^{m}, D^{m}, X^{m}, \alpha, P)]$. Then

$$\sqrt{M}[(\hat{\alpha}, \hat{P}) - (\alpha_0, P_0)] \to_d N(0, \Sigma),$$

\footnote{For example, see Theorem 3.4 of Newey and McFadden (1994).}
where \( \Sigma = (B'WB_0)^{-1}B'WAB_0(B'WB_0)^{-1} \), and

\[
\Lambda = E[\gamma(ms^m, D^m, X^m, \alpha, P)\gamma(ms^m, D^m, X^m, \alpha, P)^\prime].
\]

Recall that the moment condition for persuasion strategy in (3.14) is derived for each identified value of \((\alpha, \beta, G)\). Now I give an estimator of the persuasion strategy when the estimated \((\hat{\alpha}, \hat{P})\) in Lemma 4.1 are directly plugged into (3.14). This is a two-step estimation procedure and will not be efficient. I will discuss the complexity of the joint estimation of moment conditions 3.8-3.10 and (3.14) after the plug-in estimator of the persuasion strategy is introduced.

Given the estimated \((\hat{\alpha}, \hat{P})\), we can construct a sample of estimated realized utility

\[
\hat{v}^m_{j,k}(x(l)) = \sum_{l=1}^{L} \left[ \delta_j(ms^m, X^m, D^m, \hat{\alpha}, \hat{P}^0) + \hat{\alpha}^k_j(X^m) \right] 1(X^m = x(l))
\]  

(4.2)
corresponding to (3.2) and a sample of utility shock

\[
\hat{\epsilon}^m_j = \delta_j(ms^m, X^m, D^m, \hat{\alpha}, \hat{P}^0).
\]  

(4.3)

Fixing the demographic group \(k\) and the characteristics \(x(l)\), the distribution of \(\hat{v}^m_{j,k}(x(l))\) conditioned on \(k\) and \(x(l)\) is an estimated distribution of realized utility.

To form moment condition (3.14), we first need the unconditional choice probability \(\tilde{P}^0_{j,s}(X^m)\) in (3.13) for each demographic group \(k\) and for each product characteristics. To get an estimator of \(\tilde{P}^0_{j,s}(X^m)\), denoted by \(\tilde{\tilde{P}}^0_{j,s}(X^m)\), we need to solve optimization problem (2.10) with an estimated prior belief. I look at the empirical counterparts of optimization problem (2.10) under persuasion strategy \(\tilde{F}(s^{ID}, \epsilon; \theta)\) conditioned on markets with \(X^m = x(l)\):

\[
\frac{1}{\sum_{m'=1}^{M(x(l))} \tilde{F}^k(s^{ID} = s|\epsilon^{m'}; \theta)} \max_{(\tilde{\tilde{P}}^0_{j,s})_{j=1}^{J}} \sum_{m=1}^{M(x(l))} \log(\sum_{j=1}^{J} \tilde{\tilde{P}}^{0,k}_{j,s}(x(l)) e^{\hat{v}^m_{j,k}}) \times \tilde{F}^k(s^{ID} = s|\epsilon^{m}; \theta)
\]  

\[
\text{s.t. } \forall j: \tilde{\tilde{P}}^{0,k}_{j,s}(x(l)) \geq 0,
\]  

\[
\sum_{j=1}^{J} \tilde{\tilde{P}}^{0,k}_{j,s}(x(l)) = 1,
\]  

(4.4)

where \(M(x(l))\) is the number of markets such that \(X^m = x(l)\). I implicitly imposed that the marginal distribution of \(\tilde{F}^k(\epsilon^m)\) is the empirical distribution of \(\hat{\epsilon}^m\), and by Bayes’ rule
\[ \sum_{m=1}^{M} N(s \mid \epsilon_m; \theta) \] is the posterior belief when the DM receive a signal \( s \). Let \( \tilde{P}_{j,s}^{0,k}(x(l)) \) be the solution to (4.4), and denoted the vector \( (\tilde{P}_{j,s}^{0,k}(x(l)))_{j,k,s,l} \) as \( \tilde{P}_s \).

After solving \( \hat{P}_{j,s}^{0,k}(x(l)) \), we can now write the empirical version of moment condition (3.14). Let \( N \) be the number of markets with persuasion. Denote \( \forall \ l = 0, \ldots, L \) and \( \forall \ j = 1, \ldots, J - 1 \):

\[
g_{l,j,k}(\theta, \tilde{m}s^m, D^m, X^m, \tilde{P}_s) = [\tilde{m}s^m_j - \sum_{d=1}^{K} h_d^k(\theta, \tilde{P}_s, x(l))d^m_k]d^m_k 1(X^m = x(l)), \tag{4.5}
\]

\[
h^k_j(\theta, \tilde{P}_s, x(l)) = \sum_s \left[ \tilde{P}_{j,s}^{0,k}(x(l)) \sum_{m=1}^{N(x(l))} \tilde{F}(s \mid \epsilon_m; \theta) \right] \tag{4.6}
\]

where \( \tilde{m}s^m \) is a vector of share observation in market \( m \), and \( N(x(l)) \) is the number of markets with persuasion such that \( X^m = x(l) \). Then we can estimate \( \theta \) by the usual GMM estimator:

\[
\hat{\theta} = \arg \min (\frac{1}{N} \sum_{m=1}^{N} g^m(\theta))^tW_2(\frac{1}{N} \sum_{m=1}^{N} g^m(\theta)), \tag{4.7}
\]

where \( g^m(\theta) \) is the vector of moment functions \( (g_{l,j})_{l,j} \) in (4.5).

In what follows, I derive the consistency of \( \hat{\theta} \) when the persuasion strategy has a smooth parametric form \( \tilde{F}(s^{ID}; \theta) \) and the signal \( s^{ID} \) is discrete.

**Assumption 4.3.** The persuasion strategy satisfies that \( \exists C > 0 \) for all \( s \) value:

1. \( \tilde{F}(s \mid \epsilon; \theta) \) is differentiable with respect to \( \epsilon \), and the gradient is uniformly bounded in \( \theta \):

\[
\sup_{\theta \in \Theta, s} \left| \frac{\partial \tilde{F}(s \mid \epsilon; \theta)}{\partial \epsilon_j} \right| < C;
\]

2. The \( \delta^{sm}(ms^m, D^m, X^m; \alpha, (P_{j}^{0,k}(X^m))_{j,k}) \) defined in Lemma 3.1 satisfies

\[
|\frac{\partial \delta^{sm}}{\partial \kappa}| < C \ \forall \kappa \in \{\alpha_j^k(l), (P_{j}^{0,k}(X^m)) : j, k, l\}
\]

for all values of \( ms^m, D^m, X^m \).

3. The partial derivatives with respect to the elements of \( \theta \) satisfy

\[
\sup_{\epsilon, s, i} \left| \frac{\partial \tilde{F}(s \mid \epsilon; \theta)}{\partial \theta_i} \right| < C.
\]
The following condition imposes that the instruments $Z(D^m)$ point identify the parameter $\theta$. The point identification conditions are discussed in section 3.

**Assumption 4.4.** Let $g(\theta) = \left( g_{l,j}(\theta, ms^m, D^m, X^m, \hat{P}_s) \right)_{l=1,\ldots,L}^{j=1,\ldots,J-1}$, and define $L(\theta) = g(\theta)^W_2 g(\theta)$. The following identification condition hold for all $\zeta > 0$

$$\sup_{d(\theta,\theta_0) > \zeta} L(\theta) - L(\theta_0) > 0.$$ 

**Proposition 4.** Under assumptions 4.2 - 4.4 and technical assumption C.1, $\hat{\theta}$ is a consistent estimator of $\theta_0$.

**Remark 4.1.** The asymptotic distribution of $\hat{\theta}$ is not derived in this paper. There are two difficulties in deriving the asymptotic distribution of $\theta$. The unconditional choice probability vector under persuasion $\hat{P}_s$ is estimated using the sample of markets without persuasion. The sampling error of $\hat{P}_s$ comes from two aspects: (i) $\hat{P}_s$ is estimated from the empirical version (4.4) of the optimization problem (2.10); (ii) the utility shocks in (4.4) are constructed from the estimator $\hat{\alpha}$. Another difficulty comes from the fact that $P_{j,k}^{0,s}(x(l))$ can be local to the boundary to the parameter space under the true persuasion strategy, i.e. $P_{j,k}^{0,s}(x(l)) \approx \frac{1}{\sqrt{n}}$ for some $(j,k,l)$. In this case, the sampling distribution of $P_{j,k}^{0,s}(x(l))$ is hard to derive and the influence of the sampling error on $\hat{\theta}$ is hard to derive.

**Joint Estimation and Two Step Estimation**

In this section, I briefly discuss how to estimate the persuasion strategy parameter $\theta$ and preference parameters $(\alpha, \{P_{j,k}^0(x(l))\}, G)$ jointly using moment conditions (3.8)-(3.10) and (3.14). The objective function of joint GMM estimation is just the simple stack of $\gamma^m$ in (4.1) and $\mathbf{g}^m$ in (4.7). For each $(\alpha, \{P_{j,k}^0(x(l))\}, \theta)$ in the parameter space, we need to find the $\delta^m$ for each market, and construct the pseudo sample of $\{e^m\}_{m=1}^M$. Given the pseudo sample of $\{e^m\}_{m=1}^M$, we then solve the optimization problem (4.4) to get $h^k_j$ in (4.6). Given $e^m$ and $h^k_j$, we can evaluate the value of the joint GMM objective function at this $(\alpha, \{P_{j,k}^0(x(l))\}, \theta)$.

The joint GMM estimation procedure introduces two extra computational burden compared with the two step estimation procedure. First, the fixed point $\delta^m$ needs to calculated at each $(\alpha, \{P_{j,k}^0(x(l))\}_{j,k,l})$ parameter evaluation in the joint estimation. In contrast, in
the two-step estimation, we find the fixed point for each \((\alpha, \{P_{j,k}^0(x(l))\}_{j,k,l})\). If the dimension of \(\theta\) is large, the extra parameter \(\theta\) can introduce significant computational burden to the joint estimation. Second, the optimization problem (4.4) needs to be solved at each \((\alpha, \{P_{j,k}^0(x(l))\}_{j,k,l}, \theta)\) in the joint estimation. In contrast, we plug the estimator \((\hat{\alpha}, \{\hat{P}_{j,k}^0(x(l))\}_{j,k,l})\) into (3.14), and the optimization problem (4.4) only needs to be solved for each \(\theta\). Plugging in the estimator \((\hat{\alpha}, \{\hat{P}_{j,k}^0(x(l))\}_{j,k,l})\) reduces the dimension of the parameter space for the second step GMM estimation.

Joint estimation of \((\alpha, \{P_{j,k}^0(x(l))\}_{j,k,l}, \theta)\) also makes the inference of \((\alpha, \{P_{j,k}^0(x(l))\}_{j,k,l})\) difficult. The discussion under Proposition 4 reveals the difficulty of deriving the asymptotic distribution of \(\hat{\theta}\). The difficulty comes from the unknown limit distribution of \(\hat{P}_{j,k}^{0,s}(x(l))\) when \(P_{j,k}^{0,s}(x(l))\) is local to zero. The same issue will happen to \((\hat{\alpha}, \{\hat{P}_{j,k}^0(x(l))\}_{j,k,l})\) if we estimate all moment conditions jointly.

5 Application: Fox News and the 2000 Presidential Election

In this section, I apply the rational inattention, discrete choice model, with persuasion to the effect of Fox News on the 2000 presidential election (DellaVigna and Kaplan, 2007).

Fox News started the distribution of its channel in 1996 and its twenty-four-hour cable program penetrated about 20% of the towns in the United States by Nov, 2000. Fox News channel is perceived to provide political views that are right to the mainstream news channel such as ABC and CNN. In the empirical application, I treat the entry of Fox News into the local cable markets as the presence of the persuader. The DMs’ prior distribution \(G\) is understood as the prior belief on the presidential candidates under mainstream news channels. The goal is to estimate the preference parameters of each demographic group and the persuasion strategy used by Fox News in these markets. The estimated persuasion strategy can reveal the degree of bias in Fox News program.
5.1 Data

The election outcome data are taken from DellaVigna and Kaplan (2007) and the demographic data are a mixture of the original demographic data in DellaVigna and Kaplan (2007) and the 2000 U.S. census data. Each observation consists of a vector of presidential election vote results and a vector of demographic statistics that correspond to a town and an indicator for the presence of Fox News. The presidential election vote result includes the total votes cast, the number of votes for the Democratic Party, and the number of votes for the Republican Party. The demographic statistics include the number of people that are above 18 years old, the gender ratio, the ethnic group decomposition (African American, Hispanic, Asian, etc), and the decomposition by education level. The education level statistics are for eligible voters (18+ years old), but ethnic group statistics incorporate both the adults and children.

The original demographic data in Vigna and Kaplan is flawed. In about 15% of the towns, the number of votes cast is more than the number of residents above 18 years old. The issue happens when the town name corresponds to multiple administrative levels. For example, there are some names used for two different townships and cities but in different counties, their match tends to get wrong. I re-match the voting data with the 2000 U.S. census data to deal with this issue but the problem is not solved completely. There are still about 5% of the towns that have the inconsistency of votes and adults. As mentioned in DellaVigna and Kaplan (2007), this may be due to flaws in the process of collecting the election data.

I follow the data selection procedure in DellaVigna and Kaplan (2007) to discard towns: 1. without CNN news channel; 2. the number of precincts in 2000 differs from that in 1996 by 20%; 3. the total number of votes in 2000 differs from that in 1996 by 100%; 4. with multiple cabal systems; 5. the number of people with high school and above is more than the number of adults; 6. the number of votes is greater than the number of adults.

Throughout the application, I assume the choice set includes $J = 3$ options: $\{\text{Rep}, \text{Dem}, \text{Out}\}$.

The education level variable in their data set is not correct for some towns. For example, the proportion of residents with no more than high school education and the proportion of residents with more than high school education sum to greater than 1.
All votes not cast for the two major parties candidates or adults not registered to vote are grouped into the Out option.

5.2 Market Assumptions and Justification

First I separate markets into two groups: with persuasion and without persuasion. If Fox News is available in the town, I assume the town is under the influence of the persuader. This assumes that the presence of Fox News influences the whole town. Since I only use observation of towns with one unique cable company, if the cable company includes Fox News, everyone in the town should have access to the channel. While some residents may not watch the channel, the contents of the news program can be spread through workplaces and places of entertainment. This also assumes that towns without Fox News cannot be influenced by persuasion. This assumption suits the historical context in 2000 where the fixed broadband subscription in the United States accounts for around 2.5% of the population, so streaming of Fox News is not accessible to major voters in the towns without Fox News.

The key assumption on market without persuasion is the assumption 3.2. The i.i.d assumption on $\epsilon_m$ assumes that there is no spatial correlation conditioned on the observed characteristics in the town. This variation in $\epsilon_m$ may come from the geographic location differences of towns and the composition of industries in towns. For example, a policy of cleaner fuel may generate different perceptions in the coal mining towns and forest zone. The independence assumption $\epsilon_m \perp D^m$ assumes the composition of demographics does not influence the prior belief.

For markets with persuasion, assumption 3.3 requires that Fox News use the same persuasion strategy for all towns, regardless of the demographic composition. This assumption is justified because Fox News is a national program, so the perception of persuasion strategy should be similar for all towns.\footnote{Note that this is not a restriction on the entry decision. In fact, Fox News can endogenously choose the town they wanted to provide channels but this is out of the scope of this paper. The model aims to estimate the persuasion strategy used by Fox News but does not model its utility to justify the persuasion and the entry. As long as the persuasion strategy is the same for all towns, the identification argument goes through whether the entry was chosen optimally or exogenous.} Last, the assumption that the persuader draw persuasion
signal $s^{ID}_D \sim_{i.i.d} \tilde{F}^k$ says that the signals should be independent for all towns. This assumption is hard to justify since Fox News is a national program. However, Fox News reports on different aspects of the candidates (e.g. foreign policy, economic policy), and each town may only focus on one aspect of a candidate, which may result in an $i.i.d$ persuasion signal across towns.

### 5.3 The Specification

I assume there are no product characteristics across towns. The utility is $u^n_{kj} = \alpha_{j,k} + \epsilon^n_j$, where the parameters $\alpha_{j,k}$ are the mean utility of candidate $j$ that differ across demographic group $k$. The utility for the outside option is normalized to be zero. I partition the decision makers in each town based on their education level at the time of the election: \{ High School and Lower, College Partial, College Complete \}\footnote{A finer partition of the demographics is desired, but the U.S. census data do not provide the joint distribution of education with other demographic characteristics.} The segment of education level can reflect the differences in income levels and the political spectrum. The estimators and the 95\% confidence intervals are reported in table 1.

| Choice $j$ | Rep | Dem |
|-----------|-----|-----|
| High School | -0.1318 [-0.1540, -0.1050] | -0.0983 [-0.0983, 0.0707] |
| College Partial | 0.1369 [0.0816, 0.1848] | 0.0693 [0.0693, 0.1725] |
| College Complete | 0.0306 [0.0079, 0.0538] | 0.0529 [0.0529, 0.0857] |

Table 1: Estimated Mean Preference Parameters

The estimation result shows several interesting observations. First, the group with partial college degree has a slightly lower preference for the Democratic Party than the Republican Party. The partial college group includes eligible voters who earn degrees from community college or technical colleges. So we see that both highly educated group and the least
educated group prefer the Democratic Party\textsuperscript{8}, but the middle class seems to be indifferent between these two parties. Second, the College Partial group has a higher willingness to vote. However, this does not imply the College Partial group vote more to the Democratic Party than those who complete college education. Table \ref{table:2} reports the estimated unconditional choice probability for each demographic group.

\begin{table}[h]
\centering
\caption{Unconditional Choice Probability: With and Without Fox News}
\begin{tabular}{lcccc}
\hline
 & High School & & & \\
 & No Fox & With Fox & No Fox & With Fox & No Fox & With Fox \\
Rep & 0.1998 & 0.1610 & 0.5082 & 0.5488 & 0.3031 & 0.3415 \\
Dem & 0.1891 & 0.2086 & 0.2925 & 0.2498 & 0.3974 & 0.3634 \\
\hline
\end{tabular}
\end{table}

The result in table \ref{table:2} cannot be generated by a random utility model with Logit shock. By random utility model with Logit shock, we would predict that the College Partial group vote more for the Democratic Party than College Complete group would do, because $\alpha_{Dem,College\,Partial} > \alpha_{Dem,College\,Complete}$. The estimated density of the prior distribution $G$ is given in Figure ??.

\textsuperscript{8}Note that the confidence interval of $\alpha_{Dem,k}$ does not intersect with $\alpha_{Rep,k}$ for $k \in \{High\,school,\,College\,Complete\}$.
For the identification of persuasion, I use two different parametric persuasion strategies that differ across the three demographic groups. This can be true if the news programs are designed to target different demographic groups. I restrict the persuader to have only two signals to send. A two-signal persuasion strategy is easy to interpret. A '+' signal means '1 is good' if it only conveys information on $\epsilon_1$, and it means '1 is better than 2' if it compares the $\epsilon_1$ with $\epsilon_2$. A '-' signal means the contrary.\footnote{Two-signal persuasion strategy is also justified by Gitmez and Molavi (2018), where the politician in their model has full control of the news media and voters are heterogeneous in their belief.}

The persuasion strategy of the high school education group is given by

$$P_{r_FHS}(S^{TD} = -|\epsilon) = \begin{cases} 
1 & \text{if } \epsilon_{rep} < \epsilon_{dem} \\
\frac{1}{\theta_{hs}}(\epsilon_{rep} - \epsilon_{dem})^2 & \text{if } \epsilon_{rep} \geq \epsilon_{dem},
\end{cases}$$

and the persuasion strategy of the college partial and college complete group is given by

$$P_{r_{FCollege}}(S^{TD} = -|\epsilon) = \begin{cases} 
0 & \text{if } \epsilon_{rep} > \epsilon_{dem} \\
1 - \frac{1}{\theta_{hs}}(\epsilon_{rep} - \epsilon_{dem})^2 & \text{if } \epsilon_{rep} \leq \epsilon_{dem}.
\end{cases}$$
I use the same parametric family for demographic group with education higher than high school but treat the least educated group separately. This is because Table 2 shows that only the least educated group has decreased unconditional choice probability for the Republican Party and increased unconditional choice probability for the Democratic Party after Fox News entered into their towns.

The ‘-’ signal in the persuasion strategy for the high school group can either mean when the Republican party is indeed worse than the Democratic party, or it can mean with a small probability that the Republican party is better.

The persuasion strategy for the eligible voters with at least a partial college education has a better interpretation. The positive signal $S^{ID} = +$ can be read as ‘the Republican is better than the Democratic’. A positive signal is always sent when the Republican is indeed better, i.e. $\epsilon_{\text{rep}} > \epsilon_{\text{dem}}$, and a fake positive signal can also be sent when $\epsilon_{\text{rep}} < \epsilon_{\text{dem}}$, but the probability decays as the difference becomes larger in absolute value.

The estimated persuasion strategy parameters are reported in Table 3 and I plot the probability of the "+" signal for the two persuasion strategies in Figure 2. We should note that the persuasion strategy parameter $\theta$ is very close to 1 and the entropy of the marginal distribution of the signal is close to zero. The close-to-zero entropy indicates that the signal sent by Fox News does not carry much information. However, the relative scale of entropy is still significantly large compared with the utility parameter $\alpha_{jk}$ for all three groups.

### Table 3: Persuasion Strategy

|                | High School | College Partial and Complete |
|----------------|-------------|------------------------------|
| Estimator $\hat{\theta}$ | 0.9620      | 0.9452                       |
| Entropy        | 0.0553      | 0.0378                       |

*Note: The entropy numbers are calculated based on the marginal distribution of signal.*

The overall fit of the persuasion model can be seen from the difference between the data unconditional choice probability and the unconditional choice probability predicted by the persuasion strategy. Table 4 shows that the model predicts the unconditional choice probability quite well except for the high school group’s unconditional choice probability of
choosing the Republican party.

Table 4: Unconditional Choice Probability in Towns with Fox News: Model vs Data

|         | High School | College Partial | College Complete |
|---------|-------------|-----------------|------------------|
|         | Model       | Data            | Model            | Data            |
| Rep     | 0.1853      | 0.1610          | 0.5427           | 0.5488          |
| Dem     | 0.2090      | 0.2086          | 0.3708           | 0.3634          |

5.4 Welfare Analysis

Costly information acquisition can lead the decision maker to choose the second-best choice with some probability. If information is free (i.e., $\lambda = 0$, or decision maker can perfectly observe $(\epsilon_{\text{rep}}, \epsilon_{\text{dem}})$), the decision maker should be able to choose the one that maximizes
his utility. This is defined as the first-best outcome. Persuasion signal has two influences on decision makers: persuasion signal provides extra information that reduce the entropy of belief, but it also intentionally leads some decision makers to make wrong decisions. In this section, I analyze the welfare by asking what is the percentage of voters that cast votes consistent with their first best choice before and after Fox News enters into their town. Formally, the first best choice $j_{k, m}^{fb}$ in a town $m$ is defined as

$$j_{k, m}^{fb} = \arg \max_{j \in J} \alpha_{j, k} + \epsilon_j^m$$

and $\mathcal{P}_{j=fb}^k(\alpha + \epsilon^m)$ is the proportion of voters that make the correct choice in the rational inattention model without persuasion in town $m$, and $\sum_s \tilde{F}(s|\epsilon)\mathcal{P}_{j=fb,s}^k(\alpha + \epsilon^m)$ is the proportion of voters that make the correct choice with Fox News Persuasion. Since we have the estimated prior distribution $G(\epsilon_{rep}, \epsilon_{dem})$, we get the distribution of $\mathcal{P}_{j=fb}^k(\alpha + \epsilon^m)$ and $\sum_s \tilde{F}(s|\epsilon)\mathcal{P}_{j=fb,s}^k(\alpha + \epsilon^m)$. The estimated distribution (across towns) can be seen in figure 3. The patterns are quite different for the three groups. For voters with high school education, persuasion does not really help them to make better decisions overall. For voters with a partial college education, persuasion generates higher dispersion in the distribution of voters that vote for their first best choice. It should be noted that even if the persuasion strategy is the same for voters with a partial and full college education, the persuasion strategy tightens the distribution of the first best choice for voters who complete a college education.
Figure 3: Distribution of percentage of voters that achieve their first best choice

6 Conclusion

In this paper, I study the identification of the rational inattention discrete choice model with Bayesian Persuasion. I derive the conditional moment conditions that identify the mean utility of each product and prior distribution. I also show the identification of a parametric
persuasion strategy when the persuader plays a sequential game with decision makers in the model. In the empirical application, I studied the effect of Fox News in persuading voters to vote for the Republican Party. I also analyze the welfare change for voters before and after the influence of Fox News.

For future research, we should derive a method to unify the supply-side model with the identified persuasion strategy. If the supply side, which is Fox News in the context, is rational when it chooses the persuasion strategy, the optimal strategy should reveal constraints on its utility parameters. Such parameters are crucial when we conduct a counterfactual analysis on the supply side. For instance, in the IO context, the preference for persuasion strategy would allow us to model the non-price competition.

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A Appendix 1: Data Compression Interpretation of Entropy Cost

The entropy of a discrete random variable is closely related to the expected number of binary questions needed to be asked to determine the realization. Consider the following example:

- $X$ is supported on 4 points: $X_1 = (H, H)$, $X_2 = (H, L)$, $X_3 = (L, H)$ and $X_4 = (L, L)$.
- The probability of each realization is $P_1 = P_4 = 1/3$ and $P_2 = P_3 = 1/6$.
- Consider two ways of asking questions:

1. Q1: The state is: (A) First component is H; (B) the First component is L. Q2: (A) Second component is H; (B) the Second component is L.
2. Q1: The state is: (A) Both high; (B) Both low; (C) Neither. Q2: The state is: (A) (H,L); (B) (L,L).

Using the first approach, we need to ask two binary questions for sure to pin down the realization. Using the second approach we are expected to ask one 3-adic question for sure, and with 1/3 probability we need another binary question. If we consider that a 3-adic question is equivalent to $\log_2 3$ binary questions\(^{10}\), the expected number of binary questions we need to ask is $\log_2 3 + 1/3 = -2/3 \times \log_2(1/3) + 1/3 \times \log_2(1/6)$ which is the entropy number. In many examples, the entropy number cannot be coded with an integer number of binary questions, but nonetheless the entropy number is a good approximation for the complexity of the random variable.

Now, consider the entropy cost function we defined in (2.5). The entropy $H(G)$ is interpreted as the number of binary questions of the prior distribution. Now, given the signal $s$ the DM acquire from the world, the number of binary questions is reduced to $H(F(\cdot|s))$.

---

\(^{10}\)One way to understand this conversion is the following. Suppose we have $N$ binary questions that can cover all possible states of the world, the cardinality of states of the world is approximately $2^N$. In the same case, suppose we need $M$ 3-adic question to cover all states of the world. By setting $2^N \approx 3^M$, we find the $N = M \times \log_3 3$. The more rigorous conversion argument can be established using large scale data compression theory. See Cover (2006), Chapter 5.
Since ex ante, the DM do not know the realization of $s$, the expected number of questions remained is $E_s[H(F\cdot|s)]$. Therefore, the difference of entropy $H(G) - E_s[H(F\cdot|s)]$ is interpreted as the expected number of binary questions that is answered by the signal $s$, and the unit cost of information $\lambda = 0$ is interpreted as the market price for asking a binary question.

The interpretation still works when we consider $s$ being discrete but the state $v$ is continuous. Let’s consider an example where $X \sim U[-1, 1]$ and $Y = 1$ when $X \geq 0$ and $Y = 0$ when $X < 0$. Given $X$ is negative with probability 0.5, $Y$ answers one binary question whether $X$ is negative or not. Direct calculation shows that $H(X) = 1$ and $H(X|Y) = 0$, so the mutual information $I(X; Y) = H(X) - H(X|Y) = 1$.

When the pair $(s, v)$ is continuously distributed, the data compression argument needs to be modified slightly. The approach is to take a quantization of the random variable. The quantization of $v$ is to slice the support of $v$ with cubes of side length $\Delta$. As the quantization length $\Delta \to 0$, the entropy of the discrete random vector, denoted as $V(\delta)$, will converge to the differential entropy of $v$ in the following sense:

$$H(V(\delta)) + \log J \to H(G(v)) \quad \text{as} \quad \Delta \to 0$$

where $J$ is the dimension of $v$. We can perform the same quantization for the signal variable $s$. When we calculate the entropy difference $H(G) - E_s[H(F\cdot|s)]$, which is called the mutual information, the effect of quantization will disappear. See Cover and Thomas (2006), Chapter 8 for discussion of quantization. Then we can use the interpretation for the data compression on the quantized version of $(v, s)$.

\section*{B Appendix 2: Proofs of Section 3}

\subsection*{B.1 The Contraction Mapping Lemma 3.1}

\textit{Proof.} The proof is a minor adaptation of [Berry et al., 1995]. To show the operator $T$ is a contraction mapping, it suffice to show that the conditions of theorem 1 in BLP holds. Let $T_j : R^{J-1} \to R$ denote the $j-th$ component of the mapping $T : R^{J-1} \to R^{J-1}$ defined in
I use the following notation for the proof:

\[
P_j^k(\delta, X, P^{0,k}, \nu_k, \alpha) = \frac{P_j^{0,k}(X)e^{\delta_j + u_2(X^m, \nu_k, \alpha)}}{\sum_{l \in J} P_l^{0,k}(X)e^{\delta_l + u_2(X^m, \nu_k, \alpha)}}.
\]  

(B.1)

First note that:

\[
\frac{\partial T_j}{\partial \delta_j} = 1 - \frac{1}{m s_j^*} \sum_k P_j^k(\delta, X, P^{0,k}, \nu_k, \alpha) \times (1 - P_j^k(\delta, X, P^{0,k}, \nu_k, \alpha))d_k
\]

\[
\geq 1 - \frac{1}{m s_j^*} \sum_k P_j^k(\delta, X, P^{0,k}, \nu_k, \alpha)d_k \geq 0
\]

\[
\frac{\partial T_j}{\partial \delta_l} = \frac{1}{m s_j^*} \sum_k P_j^k(\delta, X, P^{0,k}, \nu_k, \alpha)P_l^k(\delta_l, X, P^{0,k}, \nu_k, \alpha)d_k \geq 0
\]

and for any \(j = 1, ..., J - 1\):

\[
\sum_{l < j} \frac{\partial T_j}{\partial \delta_l} = 1 - \frac{1}{m s_j^*} \sum_k \left[ P_j^k(\delta, X, P^{0,k}, \nu_k, \alpha) \times (1 - \sum_{l = 1}^{J-1} P_l^k(\delta_l, X, P^{0,k}, \nu_k, \alpha))d_k \right]
\]

\[
= 1 - \frac{1}{m s_j^*} \sum_k \left[ P_j^k(\delta, X, P^{0,k}, \nu_k, \alpha) \times P_j^k(\delta_j, X, P^{0,k}, \nu_k, \alpha)d_k \right],
\]

where \(m s_j^* \equiv m s_j^*(X^m, \delta^m, \alpha, D^m, \{P_j^{0,k}(X)\}_{j,k})\). For the outside option \(J, \delta_J = 0\) holds because we assume the choice utility of the outside option is zero by normalization. By assumption, the unconditional choice probability of the outside option is non-zero for all \(X\) and all \(k\), i.e. \(P_l^{0,k}(X) > 0\). The Logit form \(B.1\) implies the conditional choice probability \(P_j^k(\delta_j, X, P^{0,k}, \nu_k, \alpha) > 0\) must hold, which further implies:

\[
\sum_{l = 1}^{J-1} \frac{\partial T_j}{\partial \delta_l} < 1.
\]

This verifies the condition 1 of the contraction mapping theorem in Appendix 1 of BLP.

To verify condition 2 of the contraction mapping theorem in BLP, I rewrite equation (3.6) by plug in the expression of \(m s_j^*\) into the mapping \(T\):

\[
[T(\delta)]_j = \log(m s_j^m) - \log \left( \sum_k P_j^{0,k} e^{u_2(X^m_j, \nu_k, \alpha)} \frac{P_j^{0,k} e^{u_2(X^m_j, \nu_k, \alpha)}}{P_j^{0,k} e^{u_2(X^m_j, \nu_k, \alpha)}}d_k \right)
\]

The function \(T\) is bounded from below by \(\log(m s_j^m) - \log \left( \sum_k P_j^{0,k} e^{u_2(X^m_j, \nu_k, \alpha)} \frac{P_j^{0,k} e^{u_2(X^m_j, \nu_k, \alpha)}}{P_j^{0,k} e^{u_2(X^m_j, \nu_k, \alpha)}}d_k \right)\) when \(\delta_j \to -\infty\).
For condition 3 of the contraction mapping theorem in BLP, for any \( j \), I set \( \bar{\delta}_j \):

\[
\bar{\delta}_j = \arg\min_{\delta_j} \left[ ms_j^m - \sum_k \frac{P_{j}^{0,k}(X)}{P_j^{0,k}(X) + P_j^{0,k}(X)e^{\delta_j + u_2(x_j^m, \nu_k, \alpha)}}d_k \right]^2
\]

which is the solution of \( \delta_j \) to match the market share of the outside option when \( \delta_k = -\infty \) for all \( k \neq j \).

**Remark B.1.** The extra condition that the outside option is chosen with positive unconditional choice probability is not required in the proof of Berry et al. (1995), because when the shock is supported on unbounded space, the outside option will always have a positive choice probability. The last step is also slightly different from Berry (1994) where the unconditional choice probability \( P_j^{0,k} \) appears in the denominator.

**B.2 Proof of Proposition 2**

*Proof.* Since all three moment conditions are conditioned on \( X^m \), and by assumption 3.2, the product characteristics \( X^m \) is independent of the random utility shocks \( \epsilon^m \) and demographic distribution vector \( D^m \), I prove the proposition conditioned on the value of \( X^m \) and drop \( X^m \) moment condition expressions whenever there is no confusion.

**Constraint on \( P_j^{0,k} \)**

For each market \( m \), we observe only the market share vector

\[
ms^m = (ms_1^m, ... ms_J^m)'
\]

and the demographic distribution

\[
D^m = (d_1^m, ... d_K^m)
\]

where \( d_k^m \) is the share of people in demographic group \( k \) in market \( m \). Then in market \( m \), the observation \( ms^m \) satisfies:

\[
ms_j^m = \sum_{k=1}^{K} \mathcal{P}_j^k(\epsilon^m)d_k^m \quad \forall j = 1, ..., J.
\]

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If we take expectation with respect to the $G$ distribution and the demographic distribution on both sides of the above equation, we have

$$E_G[m_s^m_j - (P^1_j(\epsilon^m), \ldots P^K_j(\epsilon^m))(d_1^m, \ldots d_K^m)|D^m] = 0$$

By assumption (3.2) $(d_1^m, \ldots d_K^m) \perp (\epsilon^m, X^m)$, we have

$$E_G[P^k_j(\epsilon^m)d_k^m|(d_1^m, \ldots d_K^m)] = d_k^m E_G[P^k_j(\epsilon^m)] = d_k^m E_G[P^{0,k}_j].$$

Use the linearity of expectation we can rewrite the above equation as:

$$E[m_s^m_j - (P^{0,1}_j, \ldots P^{0,K}_j)(d_1^m, \ldots d_K^m)|D^m] = 0.$$ 

This is the moment condition (3.8).

**Independent $\epsilon$ constraint**

Lemma (3.6) establishes $\delta^m$ as a function of $(\alpha, \beta, P^{0,k}_j)$. So we can write the $\epsilon$ as the difference of $\delta$ and $u_1$. The moment condition (3.9) then comes directly from the assumption that $\epsilon^m \perp D^m$ in assumption (3.2).

**Optimality constraint**

Lastly, I derive the condition that is implied by the fact that $P^{0,k}_j$ solves the optimization problem (2.10). Since $P^{0,k}_j$ uniformly bounded away from zero and one, so the first order condition of (2.10) is

$$\int_\epsilon \left( \sum_{l=1}^J p^{0,k}_l e^{\delta_l^m + u_2(x_l^m, \nu_k, \alpha)}dG(\epsilon) = 1. \right.$$

Note that the optimization (2.10) is a convex optimization so the first order condition is sufficient to characterize the solution. So the above first order condition can be transformed into the condition:

$$E \left[ \frac{e^{\delta_j^m + u_2(x_j^m, \nu_k, \alpha)}}{\sum_{l \in J} p^{0,k}_l e^{\delta_l^m + u_2(x_l^m, \nu_k, \alpha)} - 1} \right] = 0,$$

which is the moment condition (3.10).
C Proofs of Proposition 4

Some Notations

Fix a $\theta$ and a persuasion strategy $\tilde{F}(s; \theta)$. Recall that I use $\hat{P}_0^{0,k}(x(l); \theta)$ to denote the estimated unconditional choice probability under persuasion signal $s$ solved from (4.4) and use $\hat{P}_s(\theta)$ to denote the vector of all $j, k, l, s$. I use $\tilde{P}_0^{0,k}(x(l); \theta)$ to denote the true unconditional choice probability under persuasion solved from (2.10), and use $\tilde{P}_s(\theta)$ to denote the vector of all $j, k, l, s$. I use $P^0$ to denote the true unconditioned choice probabilities without persuasion that corresponds to the moment condition (3.8), and use $\hat{P}_0^0$ to denote its estimator. I use $G$ to denote the empirical distribution of $\hat{\epsilon}$ and use $G$ to denote the true distribution of $\epsilon$. I use $B_r(\cdot)$ to denote a neighborhood of radius $r$ near $(\cdot)$.

C.1 Some Lemmas

Assumption C.1. Fixing the index $k, l, s$, let

$$M_n(\{P_j\}_{j=1}^J, \theta) = \frac{1}{M(x(l))} \sum_{m=1}^M \sum_{j=1}^J P_j e^{\alpha^k_m(x(l)) + \epsilon_m} \tilde{F}_k(s|\epsilon^m; \theta) \mathbb{1}(X_m = x(l)),$$

where $M(x(l)) = \sum_{m=1}^M \mathbb{1}(X_m = x(l))$. Suppose assumptions in Proposition 4 hold, then

$$\inf_{\theta \in \Theta} \left[ M_n(\{\tilde{P}_j^{0,k}(x(l); \theta)\}_{j=1}^J, \theta) - \sup_{(P_j)_{j=1}^J \in \Delta^J} M_n(\{P_j\}_{j=1}^J, \theta) \right] = -o_p(1),$$

where $\Delta^J = J$ dimensional probability simplex.

Remark C.1. The $M_n$ differs from the objective function of (4.4) because the $\alpha^k_m(x(l))$ is the true value of $\alpha$, while we use $\tilde{\alpha}$ in (4.4). This Lemma shows that $\tilde{P}_s(\theta)$ is also the $o_p(1)$-maximizer of $M_n$. 

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Proof. Define

\[ \hat{M}_n(\{P_j\}_{j=1}^J, \theta) = \frac{1}{M(\mathbf{x}(l))} \sum_{m=1}^{M(\mathbf{x}(l))} \sum_{j=1}^J P_j \hat{e}^{k}_{j,0}(\mathbf{x}(l)) + \epsilon^m \hat{F}_k(s|\epsilon^m, \theta) \mathbb{I}(X^m = \mathbf{x}(l)), \]

which is the objective function in (4.4), and \( \{\hat{P}_{j,s}(\mathbf{x}(l))\}_{j=1}^J \) is the maximizer of the above objective function in the simplex \( \Delta^{J-1} \). Let \( \{P^*_j(\theta)\}_{j=1}^J \) be the maximizer of \( M_n(\{P_j\}_{j=1}^J, \theta) \), then we have

\[ \hat{M}_n(\{\hat{P}^{0,k}_{j,s}(\mathbf{x}(l))\}_{j=1}^J, \theta) \geq \hat{M}_n(\{P^*_j(\theta)\}_{j=1}^J, \theta) \]

\[ = \frac{1}{M(\mathbf{x}(l))} \sum_{m=1}^{M(\mathbf{x}(l))} \sum_{j=1}^J P^*_j(\theta) e^{k}_{j,0}(\mathbf{x}(l)) + \epsilon^m \hat{F}_k(s|\epsilon^m; \theta) \mathbb{I}(X^m = \mathbf{x}(l)) \]

\[ = \frac{1}{M(\mathbf{x}(l))} \sum_{m=1}^{M(\mathbf{x}(l))} \sum_{j=1}^J P^*_j(\theta) f^m_j(1) \mathbb{I}(X^m = \mathbf{x}(l)) \]

(C.1)

where the function \( f^m_j \) is defined in the following:

\[ f^m_j(t) = e^{\alpha^{k}_{j,0}(\mathbf{x}(l)) + t(\hat{\alpha}^{k}_{j,0}(\mathbf{x}(l)) - \alpha^{k}_{j,0}(\mathbf{x}(l))) + \epsilon^m + t(\epsilon^m - \epsilon^m)} \hat{F}_k(s|\epsilon^m + t(\epsilon^m - \epsilon^m); \theta). \]

By mean value theorem, we can find a \( t^m_j \in [0, 1] \) such that \( f^m_j(1) = f^m_j(0) + (f^m_j)'(t^m_j) \). The derivatives with respect to \( t \) is

\[ (f^m_j)'(t) = f^m_j(t) \left[ \hat{\alpha}^{k}_{j,0}(\mathbf{x}(l)) - \alpha^{k}_{j,0}(\mathbf{x}(l)) + \epsilon^m - \epsilon^m \right] \]

\[ + e^{\alpha^{k}_{j,0}(\mathbf{x}(l)) + t(\hat{\alpha}^{k}_{j,0}(\mathbf{x}(l)) - \alpha^{k}_{j,0}(\mathbf{x}(l))) + \epsilon^m + t(\epsilon^m - \epsilon^m)} \sum_{i=1}^J \frac{\partial \hat{F}_k}{\partial \epsilon_i}(\epsilon^m - \epsilon^m). \]

(C.2)

Now I bound the term \( \hat{\epsilon}^m_j - \epsilon^m_j \):

\[ |(\hat{\epsilon}^m_j - \epsilon^m_j)| = |\delta^*_j(\mathbf{m}s^m, D^m, X^m, \alpha, \hat{P}^0) - \delta^*_j(\mathbf{m}s^m, D^m, X^m, \alpha, P^0)| \]

\[ = \left| \sum_{j,k} \frac{\partial \delta^*_j}{\partial \alpha^k_j(l)}(\hat{\alpha}^k_j(l) - \alpha^k_j(l)) + \sum_{j,k} \frac{\partial \delta^*_j}{\partial \hat{P}^0_{j,k}(l)}(\hat{P}^0_{j,k}(l) - P^0_{j,k}(l)) \right| \]

\[ \leq J \times K \times C \max_{j,k} \left\{ \max\left\{ \alpha^k_j(l) - \hat{\alpha}^k_j(l), P^0_{j,k}(l) - \hat{P}^0_{j,k}(l) \right\} \right\} \]

(C.3)

where the inequality holds by Assumption 4.3. Moreover, by Assumption 4.3 \( \frac{\partial \hat{F}_k}{\partial \epsilon_i} < C \)

also holds. Now, denote the term \( \max_{j,k} \left\{ \max\left\{ \alpha^k_j(l) - \hat{\alpha}^k_j(l), P^0_{j,k}(l) - \hat{P}^0_{j,k}(l) \right\} \right\} \) by \( o^*_\alpha, P \),

combining (C.2) and (C.3), we have

\[ |(f^m_j)'(t)| \leq 2J^2KC^2 e^{\alpha^{k}_{j,0}(\mathbf{x}(l)) + t(\hat{\alpha}^{k}_{j,0}(\mathbf{x}(l)) - \alpha^{k}_{j,0}(\mathbf{x}(l))) + \epsilon^m + t(\epsilon^m - \epsilon^m)} \times o^*_\alpha, P. \]
Now we substitute the mean value expansion of \( f_j^m(1) \) back to (C.1) to get
\[
\frac{1}{M(x(l))} \sum_{m=1}^{M(x(l))} \sum_{j=1}^{J} P^*_j(\theta) f_j^m(1) \mathbb{1}(X^m = x(l)) = \frac{1}{M(x(l))} \sum_{m=1}^{M(x(l))} \sum_{j=1}^{J} P^*_j(\theta) f_j^m(0) \mathbb{1}(X^m = x(l)) \]
\[
\geq \frac{1}{M(x(l))} \sum_{m=1}^{M(x(l))} \sum_{j=1}^{J} P^*_j(\theta) f_j^m(0) \mathbb{1}(X^m = x(l)) \]
\[
- 2J^2KC^2 |\alpha^*_{\alpha,p}| \frac{1}{M(x(l))} \sum_{m=1}^{M(x(l))} \sum_{j=1}^{J} P^*_j(\theta) e^{\alpha_{j,0}(x(l)) + t(\alpha_{j,0}(x(l))) + e_j^m + t(\epsilon_j^m - \epsilon_j^m)} \mathbb{1}(X^m = x(l)) \]

By Lemma 4.1, \( |\alpha^*_{\alpha,p}| = o_p(1) \) and
\[
\sum_{m=1}^{M(x(l))} \sum_{j=1}^{J} P^*_j(\theta) e^{\alpha_{j,0}(x(l)) + t(\alpha_{j,0}(x(l))) + e_j^m + t(\epsilon_j^m - \epsilon_j^m)} \mathbb{1}(X^m = x(l)) \]
\[
\to_p E \left[ \sum_{j=1}^{J} P^*_j(\theta) e^{\alpha_{j,0}(x(l)) + e_j^m} \mathbb{1}(X^m = x(l)) \right] \leq E \left[ \sum_{j=1}^{J} e^{\alpha_{j,0}(x(l)) + e_j^m} \mathbb{1}(X^m = x(l)) \right],
\]
where the last inequality holds because \( P^*_j(\theta) \leq 1 \). The observation is that
\[
E \left[ \sum_{j=1}^{J} e^{\alpha_{j,0}(x(l)) + e_j^m} \mathbb{1}(X^m = x(l)) \right]
\]
is independent of the parameter \( \theta \).
\[
\frac{1}{M(x(l))} \sum_{m=1}^{M(x(l))} \sum_{j=1}^{J} P^*_j(\theta) f_j^m(1) \mathbb{1}(X^m = x(l)) \geq \frac{1}{M(x(l))} \sum_{m=1}^{M(x(l))} \sum_{j=1}^{J} P^*_j(\theta) f_j^m(0) \mathbb{1}(X^m = x(l)) - o_p(1) \]
\[
= \sup_{(P_j^j)_{j=1}^J} M_n(\{P_j^j\}_{j=1}^J, \theta) - o_p(1),
\]
where the last equality holds by definition of \( M_n(\{P_j^j\}_{j=1}^J, \theta) \) and \( \{P_j^j(\theta)\}_{j=1}^J \) is the maximizer of \( M_n(\{P_j^j\}_{j=1}^J, \theta) \). In particular, the \( o_p(1) \) term \( 2J^2KC^2 |\alpha^*_{\alpha,p}| E \left[ \sum_{j=1}^{J} e^{\alpha_{j,0}(x(l)) + e_j^m} \mathbb{1}(X^m = x(l)) \right] \) is independent of \( \theta \), so the result in the Lemma follows. □
Lemma C.2. \( \sup_{\theta \in \Theta, (P_j)_{j=1}^J} |M_n(\{P_j\}_{j=1}^J, \theta) - M(\{P_j\}_{j=1}^J, \theta)| = o_p(1) \)

Proof. Let \( ((P_j)_{j=1}^J, \theta) \) and \( ((\bar{P}_j)_{j=1}^J, \bar{\theta}) \) be two values in the set \( \Theta \times \Delta^J \).

\[
\begin{align*}
&\sum_{j=1}^J P_j e^{a_j^k(x(l)) + c_j^m} \bar{F}^k(s | e^m, \bar{\theta}) \mathbb{1}(X^m = x(l)) - \sum_{j=1}^J \bar{P}_j e^{a_j^k(x(l)) + c_j^m} \bar{F}^k(s | e^m, \bar{\theta}) \mathbb{1}(X^m = x(l)) \\
\leq& (1) \left( \sum_{i=1}^{dim(\theta)} (\bar{\theta}_i - \theta_i) \frac{\partial \bar{F}^k}{\partial \theta_i} + \sup_j |P_j - \bar{P}_j| \right) \sum_{j=1}^J e^{a_j^k(x(l)) + c_j^m} \mathbb{1}(X^m = x(l)) \\
\leq& (2) C \times dim(\theta) \|(\bar{P}_j)_{j=1}^J, \bar{\theta}) - (P_j)_{j=1}^J, \theta)\|_\infty \sum_{j=1}^J e^{a_j^k(x(l)) + c_j^m} \mathbb{1}(X^m = x(l)) \\
\leq& C \times C_1 \times dim(\theta) \|(\bar{P}_j)_{j=1}^J, \bar{\theta}) - (P_j)_{j=1}^J, \theta)\|_\infty \sum_{j=1}^J e^{a_j^k(x(l)) + c_j^m} \mathbb{1}(X^m = x(l))
\end{align*}
\]

where \( \| \cdot \|_\infty \) is the sup norm on a vector, and \( C_1 \) is a constant such that \( \| \cdot \|_\infty \leq \| \cdot \|_2 \).

Inequality (1) follows by mean value theorem and inequality (2) follows by Assumption 4.3. Then by Theorem 2.7.11 in [Wellner and van der Vaart (2013)], we have the uniform convergence.

 Lemma C.3. If Condition [C.4] holds, then \( \sup_{\theta \in \Theta} |\hat{P}_s(\theta) - \bar{P}_s(\theta)| = o_p(1) \)

Proof. Lemma C.1 and C.2 implies that

\[
\sup_{\theta \in \Theta} |M_n(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) - M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta)| = o_p(1).
\]

So we have

\[
\sup_{\theta} M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) - M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) \\
\leq \sup_{\theta} M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) - M_n(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) + o_p(1) = o_p(1),
\]

where the last equality hold by Lemma C.2. By assumption C.1, the event

\[
d \left( (P_j)_{j=1}^J, (\hat{P}_{j,s}^0(x(l); \theta))_{j=1}^J \right) > \kappa
\]

is contained in the event \( \sup_{\theta} M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) - M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) > \kappa \), therefore

\[
Pr \left( d \left( (P_j)_{j=1}^J, (\hat{P}_{j,s}^0(x(l); \theta))_{j=1}^J \right) > \kappa \right) < Pr(\sup_{\theta} M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) - M(\{\hat{P}_{j,s}^0(x(l))\}_{j=1}^J, \theta) > \kappa) \to 0
\]

The result follows by taking the union over finite index \( k = 1, \ldots, K \) and \( l = 1, \ldots, L \).
Lemma C.4. Let $\mathbb{P}^k(s|\theta) \equiv \frac{1}{M} \sum_{m=1}^{M} \tilde{F}^k(s|\tilde{\epsilon}_m^m; \theta)$ and let $\tilde{F}^k(s|\theta) \equiv \int \tilde{F}^k(s|\epsilon; \theta)dG(\epsilon)$. The following hold under assumption 4.3: $\sup_{\theta \in \Theta} |\tilde{F}^k(s|\theta) - \tilde{F}(s|\theta)| = o_p(1)$.

Proof. We look at the following expansion

$$
\left| \tilde{F}^k(s|\theta) - \tilde{F}(s|\theta) \right| = \frac{1}{M} \sum_{m=1}^{N} \tilde{F}(s|\tilde{\epsilon}_m^m; \theta) - \tilde{F}(s|\epsilon^m; \theta) + E_\epsilon(\tilde{F}(s|\epsilon^m; \theta)) \\
= \left| \frac{1}{M} \sum_{m=1}^{M} \sum_{j=1}^{J} \partial \tilde{F}(\epsilon_j^m - \epsilon_j^m) \right| + \left| \frac{1}{M} \sum_{m=1}^{M} \left[ \tilde{F}(s|\epsilon_m; \theta) - E_\epsilon(\tilde{F}(s|\epsilon_m; \theta)) \right] \right| \\
\leq \frac{C}{M} \sum_{m=1}^{M} \sum_{j=1}^{J} (\epsilon_j^m - \epsilon_j^m) + \left| \frac{1}{M} \sum_{m=1}^{M} \left[ \tilde{F}(s|\epsilon_m; \theta) - E_\epsilon(\tilde{F}(s|\epsilon_m; \theta)) \right] \right|,
$$

(C.4)

where the last inequality holds by assumption 4.3. Now we use the expansion of $\epsilon_j^m$ in (C.3) to get

$$
\left| \frac{C}{M} \sum_{m=1}^{M} \sum_{j=1}^{J} (\epsilon_j^m - \epsilon_j^m) \right| \leq J K C^2 \max_{j,k} \left\{ \max \{ \alpha_j^k(l) - \alpha_j^k(l), \hat{P}_{j,s}^0(l) - P_{j,s}^0(l) \} \right\} = o_p(1)
$$

Note that $F(s|\epsilon_m; \theta)$ is a Donsker class indexed by $\theta$ by assumption 4.3, which implies

$$
\sup_{\theta \in \Theta} \left| \frac{1}{M} \sum_{m=1}^{M} \left[ \tilde{F}(s|\epsilon_m; \theta) - E_\epsilon(\tilde{F}(s|\epsilon_m; \theta)) \right] \right| = o_p(1).
$$

Combined the two terms in (C.4) we can get $\sup_{\theta} |\tilde{F}^k(s|\theta) - \tilde{F}(s|\theta)| = o_p(1)$.

Lemma C.5. $\sup_{\theta \in \Theta} |\tilde{F}^k(s|\theta) - \tilde{F}^k(s|\theta)| = o_p(1)$.

Proof. This follows directly from Lemma C.3 and C.4.

Lemma C.6. Consider

$$
g_{i,j,k}^s(\theta, \tilde{\mathbb{M}}^m, D^m, X^m, \hat{P}_s) = [\tilde{m} \hat{s}^m_j - \sum_{d=1}^{K} \tilde{h}_{i,j,k}^s(\theta, \hat{P}_s, x(l))d^m_{i,d}(X^m = x(l)], \quad (C.5)
$$

$$
h_{i,j,k}^s(\theta, \hat{P}_s, x(l)) = \sum \left[ \hat{P}_{j,s}^0(x(l), \theta) \tilde{F}^k(s|\theta) \right]. \quad (C.6)
$$

The equation (C.5) and (C.6) differ from (4.5) and (4.6) because (C.5) and (C.6) use the true unconditional choice probability instead of the estimator. Define

$$
L_n(\theta) = \left( \frac{1}{N} \sum_{m=1}^{N} g^*(\theta) \right)^{'} W \left( \frac{1}{N} \sum_{m=1}^{N} g^*(\theta) \right),
$$

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where $g^*(\theta)$ collects $g_{l,j}^*$ for all $l,j,k$ indices. Then $\hat{\theta}$ is an $o_p(1)$ minimizer of $L_n(\theta)$, i.e. $L_n(\hat{\theta}) \leq \min_{\theta} L_n(\theta) + o_p(1)$.

**Proof.** Note that $\hat{\theta} = \arg \min \hat{L}_n(\theta)$, where $\hat{L}_n(\theta)$ is the objective function of (4.7).

I first denote $\Delta_{l,j,k}(\theta) = g_{l,j,k}^*(\theta, \tilde{m}s^m_n, D^m, X^m, \tilde{P}_s) - g_{l,j,k}(\theta, \tilde{m}s^m_n, D^m, X^m, \tilde{P}_s)$, where $g_{l,j}$ is defined in (4.5). Using the expression of $g_{l,j,k}^*$ and $g_{l,j,k}$, we have

$$\sup_{\theta \in \Theta} \left| \Delta_{l,j,k}(\theta) \right| \leq \sup_{\theta \in \Theta} \left| \sum_{d=1}^K \sum_s \left( \hat{P}_{j,s}^{0,k}(x(l); \theta) \tilde{F}_k(s|\theta) - \hat{P}_{j,s}^{0,k}(x(l); \theta) \tilde{F}_k(s|\theta) \right) 1(X^m = x(l)) \right| = o_p(1). \quad (C.7)$$

The difference $L_n(\theta) - \hat{L}_n(\theta) = \Delta(\theta)'W \Delta(\theta)$, where $\Delta(\theta) = (\Delta_{l,j,k}(\theta))_{l,j}$. Then by (C.7),

$$\sup_{\theta} \left| L_n(\theta) - \hat{L}_n(\theta) \right| \leq \|\Delta(\theta)\|_2^2 \max \text{eig}(W) = o_p(1).$$

Now I look at $L_n(\hat{\theta})$. Suppose we can find $\theta^*$ such that $L_n(\theta^*) \leq \inf_{\theta \in \Theta} L_n(\theta) + o_p(1)$

$$L_n(\hat{\theta}) = \hat{L}_n(\hat{\theta}) + L_n(\hat{\theta}) - \hat{L}_n(\hat{\theta})$$

$$\leq (1) \hat{L}_n(\theta^*) + L_n(\hat{\theta}) - \hat{L}_n(\hat{\theta})$$

$$= L_n(\theta^*) + \underbrace{L_n(\hat{\theta}) - \hat{L}_n(\hat{\theta}) - [L_n(\theta^*) - \hat{L}_n(\theta^*)]}_{o_p(1)}$$

$$= (2) L_n(\theta^*) + o_p(1) \leq \inf_{\theta \in \Theta} L_n(\theta) + o_p(1)$$

where inequality (1) holds by the definition of $\hat{\theta}$, and (2) equality holds because we have shown $\sup_{\theta} \left| L_n(\theta) - \hat{L}_n(\theta) \right| \leq \|\Delta(\theta)\|_2^2 \max \text{eig}(W) = o_p(1)$.

**Lemma C.7.** Let $L(\theta) = E[g^*(\theta)]'W E[g^*(\theta)]$. Then $\sup_{\theta \in \Theta} |L_n(\theta) - L(\theta)| = o_p(1)$

**Proof.** Define the difference

$$\Delta_{l,j,k}^*(\theta) = \frac{1}{N} \sum_{m=1}^N \left( \tilde{m}s^m_j \mathbb{1}(X^m = x(l)) d^m_k - E[\tilde{m}s^m_j \mathbb{1}(X^m = x(l)) d^m_k] \right) +$$

$$+ \sum_{k'=1}^K \left( \frac{1}{N} \sum_{m=1}^N d^m_k \mathbb{1}(X^m = x(l)) d^m_k - E[d^m_k \mathbb{1}(X^m = x(l)) d^m_k] \right) \mathbb{P}_{1,s}^{0,k}(\theta) \tilde{F}_k(s|\theta)$$

Observe that $\mathbb{P}_{1,s}^{0,k}(\theta) \tilde{F}_k(s|\theta) \in [0,1]$ because it is the product of two probability quantities. Moreover, the $d^m_k \in [0,1]$. Therefore, we can bound $\Delta^*(\theta)$, which is the vector of $\Delta_{l,j,k}^*$ for all $l,j,k$ indices by

$$\|\Delta^*(\theta)\|_2 \leq JKL \left| \frac{1}{N} \sum_{m=1}^N \tilde{m}s^m_j - E[\tilde{m}s^m_j] \right| + JK^2 L \max_{k,k'} \left| \frac{1}{N} \sum_{m=1}^N d^m_k d^m_{k'} - E[d^m_k d^m_{k'}] \right|. \quad (C.8)$$
The right hand side of (C.8) does not depend on \( \theta \). By apply weak law of large numbers to the sample means of \( \tilde{m}_m s_j^m \) and \( d_k^m d_k' \), we have \( \sup_{\theta \in \Theta} ||\Delta^*(\theta)||_2 = o_p(1) \). Then notice that 
\[
L(\theta) - L_n(\theta) = \Delta^*(\theta)' W \Delta^*(\theta),
\]
so we have 
\[
\sup_{\theta \in \Theta} |L(\theta) - L_n(\theta)| \leq \max \text{eig}(W)||\Delta^*(\theta)||_2^2 = o_p(1).
\]

C.2 Main Proof of Proposition 4

Proof. The consistency of \( \hat{\theta} \) follows by the identification assumption

\[
\sup_{d(\theta, \theta_0) > \zeta} L(\theta) - L(\theta_0) > 0
\]

where \( \hat{\theta} \) is an \( o_p(1) \) minimizer of \( L_n \) by Lemma [C.6]. Moreover we have the uniform convergence of \( \sup_{\theta \in \Theta} |L_n(\theta) - L(\theta)| = o_p(1) \) by Lemma [C.7]. Conditions of Theorem 5.7 in Van der Vaart (2000) are satisfied, so \( \hat{\theta} \to_p \theta \).

D Discussion of Computation

The estimators in the main text are constructed in two steps. While the joint estimation of \( (\alpha, \beta, P, \theta) \) is possible, the computational burden is heavy. Markets with persuasion also provide identification power to the first stage parameter \( (\alpha, \beta, P) \), but this requires me to use contraction mapping each time I search over a higher dimensional parameter space when including \( \theta \). Also, the estimation of \( \theta \) requires solving the empirical optimization problem (4.4) for given first stage parameters. For the two-step estimation, I just plug in the first stage estimator and solve the (4.4) for different values of \( \theta \), while for joint estimation the optimization problem needs to be repeated for each guessed value of \( (\alpha, \beta, P) \).

The computational burden also comes from the contraction mapping because I need to iterate over \( M \) markets. So here I use the following trick to convert the \( M \) contraction mappings to one single contraction mapping.

**Proposition 5.** Let \( T^m(\delta) : R^d \to R^d \) be a contraction mapping for each \( m = 1 \ldots M \). Then \( T \equiv (T^1, \ldots, T^M) : R^{dM} \to R^{dM} \) is a contraction mapping acting on \( (\delta^1, \ldots, \delta^M) \).
Proof. Let $C_m < 1$ be the contraction constant that $|T_m(\delta_1) - T_m(\delta_2)| < C_m|\delta_1 - \delta_2|$. Then $C(M) = \max C_m < 1$ is the contraction constant for $T$.

The computational burden for this combined $T$ can be potentially high because: 1. even though iteration on matrix is faster than iteration over $M$ markets, the iteration on matrix is still slow when $M$ is large; 2. the uniform contraction constant $C(M)$ can be close to one and number of iteration to achieve certain tolerance level may be large. The following algorithm is helpful to reduce the running time:

- Set up a tolerance level $tol$ and a threshold integer $K_{thr}$. Run the iteration on $T$, and count the number of $\delta^m$ such that $|T(\delta^m) - \delta^m| < tol$, denote this number as $K_{con}$.

- When $K_{con} > K_{thr}$, collect the remaining markets index and construct the new contraction $T' = \{T^m\}_{\text{remain}}$. Iterate until convergence.

- Multiple threshold to decide remaining markets can be set up to further boost the speed.

This algorithm exploits the fact that the contraction mapping is simply the stacking of individuals. The intuition is that if $C^m \in \{C^{small}, C^{large}\}$, and when the number of markets falls into $C^{large}$ group is relatively small, the algorithm avoids the slow iteration on $C^{large}$ and also avoid excessive iteration on markets with $C^{small}$. This algorithm boost the computation speed of contraction mapping by a factor of 10 in the empirical application.