One-loop vertex integrals in heavy-particle effective theories

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Abstract

We give a complete analytical computation of three-point one-loop integrals with one heavy propagator, up to the third tensor rank, for arbitrary values of external momenta and masses.

1 Introduction

The study of the dynamics and spectroscopy of hadrons containing a heavy quark has been greatly simplified and systematized with the introduction of heavy quark effective theory (HQET) [1]. Heavy-particle theories along similar lines have also been successfully applied in other, related contexts. Thus, in those cases where a chiral approach to the strong interactions of heavy hadrons with light mesons is applicable, a combination of chiral and heavy-quark symmetries leads to heavy hadron chiral perturbation theory (HHChPT) [1]. A heavy-particle expansion has also been developed in the chiral-perturbative framework for nucleon-meson interactions, which constitutes the so-called heavy baryon chiral perturbation theory (HBChPT) [2].

In this paper we report on a complete analytic computation of three-point loop integrals with one heavy propagator, up to the third tensor rank, for arbitrary real values of masses and residual momenta. Together with the results for vertex integrals with two heavy propagators [3], this gives the set of all one-loop three-point integrals in heavy particle theories.

For the calculations we use the same method as in [3], namely, we compute the vertex integrals with heavy propagators as limits of ordinary three-point integrals. The latter are evaluated by closely following the method of [4], so that the calculation is standardized and the results expressed in terms of the usual logs and dilogs. Since we only have to compute to leading order in a certain limit, however, the calculations are drastically simplified as compared to ordinary vertex integrals.

In the next section, we define the integrals to be studied and give the calculational details and results for the scalar integral. We also discuss some particular cases which are of importance in applications and as cross-checks of our results. Then, in section 3 we consider the vector, second-, and third-rank tensor integrals, which are given in terms of scalar integrals.

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2 Vertex integrals

The three-point loop integrals we consider are of the form,

$$\mathcal{H}_{31\ldots3n}^{\alpha_1\ldots\alpha_n} = \frac{i\mu^{4-d}}{(2\pi)^d} \int d^d\ell \frac{\ell^{\alpha_1} \ldots \ell^{\alpha_n}}{(2\nu \cdot \ell - \delta M + i\varepsilon)((\ell - p)^2 - m^2 + i\varepsilon)((\ell - p')^2 - m'^2 + i\varepsilon)}. \quad (1)$$

Here $v^\mu$ is the velocity of the heavy lines, $p^\mu$ and $p'^\mu$ the residual momenta of the external heavy particles, and $\delta M$ the residual mass of the internal one, given by the mass splitting within the corresponding heavy-quark symmetry multiplet. $m$ and $m'$ are the masses of the light particles in the loop, which are pseudoscalar mesons in HHChPT and HBChPT, and (massless) gluons in HQET. The scalar integral is finite, whereas the vector and higher-rank tensors are divergent, with degree of divergence $n + d - 5$. We consider tensor integrals with $n = 1, 2, 3$ below. These are computed in terms of integrals with lower rank and fewer points.

By shifting the integration variable $\ell \to \ell + (p' + p)/2$, the scalar integral can be written as,

$$\mathcal{H}_3 = \frac{i}{(2\pi)^4} \int d^4\ell \frac{1}{(2\nu \cdot \ell - \Delta + i\varepsilon)((\ell + q)^2 - m^2 + i\varepsilon)((\ell - q)^2 - m'^2 + i\varepsilon)}, \quad (2)$$

with $\Delta = \delta M - v \cdot (p' + p)$, $q = (p' + p)/2$. Thus, $\mathcal{H}_3$ depends explicitly on $q^\mu$, through $q^2$ and $v \cdot q$, but depends on $(p' + p)^\mu$ only through $\Delta$. Since $\mathcal{H}_3$ is invariant under the transformation $q \to -q$, $m \leftrightarrow m'$ of its arguments, we can choose $v \cdot q > 0$ without loss of generality.

We will compute the scalar integral $\mathcal{H}_3$ as the large-mass limit of the auxiliary integral

$$\tilde{\mathcal{H}}_3 = \frac{i}{(2\pi)^4} \int d^4\ell \frac{1}{((\ell + Mv)^2 - (M + \Delta/2)^2 + i\varepsilon)((\ell + q)^2 - m^2 + i\varepsilon)((\ell - q)^2 - m'^2 + i\varepsilon)}. \quad (3)$$

Since both $\mathcal{H}_3$ and $\tilde{\mathcal{H}}_3$ are convergent, we clearly have, $\mathcal{H}_3 = \lim_{M \to \infty} M \tilde{\mathcal{H}}_3$. The calculation of $\tilde{\mathcal{H}}_3$ to leading order in $M$ will be carried out by closely following the standard method of $[4]$. We will only sketch the main steps of the calculation.

Starting from the usual Feynman parametrization for $\tilde{\mathcal{H}}_3$, after integrating over $d^4\ell$ and over the Feynman parameter multiplying the last propagator in $[3]$, we obtain,

$$\tilde{\mathcal{H}}_3 = \frac{1}{(4\pi)^2} \int_0^1 dz \int_0^1 dx \frac{1}{D} + O\left(\frac{1}{M^2}\right), \quad (4)$$

$$D = M^2 z^2 + Mz(-2(1 - 2x)v \cdot q + \Delta) + xm^2 + (1 - x)m'^2 - 4x(1 - x)q^2 - i\varepsilon.$$

In this expression we have retained only the leading terms in $M$ in the coefficients of the polynomial in the denominator. Clearly, only the region where $z = O(M^{-1})$ will contribute to $\tilde{\mathcal{H}}_3$ to $O(M^{-1})$, so we have also set the upper limit of the integral over $x$ to 1 instead of 1 - $z$.

We notice that if we make the change of variable $\lambda = Mz$ in $[4]$ and let $M \to \infty$, we are led to the HQET parametrization for $\mathcal{H}_3$, confirming the validity of the approximations leading to $[3]$.

Following $[4]$, we make the change of variable $z \to z - \alpha x/M$, with $\alpha$ one of the roots to $(\alpha v - q)^2 = 0$,

$$\alpha_{\pm} = 2(v \cdot q \pm |q|). \quad (5)$$

We henceforth assume that $q^2 < (v \cdot q)^2$, so that $|q| \equiv \sqrt{(v \cdot q)^2 - q^2} > 0$ is real. For the change of variable we choose $\alpha = \alpha_+ > 0$, leading to,

$$\tilde{\mathcal{H}}_3 = \frac{1}{(4\pi)^2} \int_0^1 dx \int_{\alpha x/M}^{1+\alpha x/M} dz \frac{1}{M^2 z^2 - (4M|q|z - z_0)x + Mz(-2v \cdot q + \Delta) + m'^2 - i\varepsilon}. \quad (6)$$
with \( z_0 = -(m'^2 - m^2) - \alpha_+(\Delta - 2|q|) \). Exchanging the order of integration,

\[
\tilde{\mathcal{H}}_3 = \frac{1}{(4\pi)^2} \left\{ \int_0^{\alpha/M} dz \int_0^{Mz/\alpha} dx + \int_0^{1} dz \int_0^{1} dx + \int_0^{1+\alpha/M} dz \int_{\alpha/(z-1)}^{1} dx \right\} \times \frac{1}{M^2z^2 - (4M|q|z - z_0)x + Mz(-2v \cdot q + \Delta) + m'^2 - i\varepsilon}.
\] (7)

In the last integral in this equation \( z = \mathcal{O}(1) \) over the entire domain of integration, so that the integral itself is \( \mathcal{O}(M^{-2}) \) and can be neglected. We will drop it henceforth. Integrating over \( dx \) we then have,

\[
\tilde{\mathcal{H}}_3 = \frac{1}{(4\pi)^2} \int_0^{\alpha/M} dz \frac{-1}{4M|q|z - z_0} \log \left( \frac{Y}{Z} \right) + \frac{1}{(4\pi)^2} \int_0^{1} dz \frac{-1}{4M|q|z - z_0} \log \left( \frac{X}{Z} \right)
\] (8)

\[
X = M^2z^2 + Mz(-2v \cdot q + \Delta) + m'^2 - i\varepsilon - (4M|q|z - z_0)
\]

\[
Y = M^2z^2 + Mz(-2v \cdot q + \Delta) + m'^2 - i\varepsilon - \frac{M}{\alpha}(4M|q|z - z_0)
\]

\[
Z = M^2z^2 + Mz(-2v \cdot q + \Delta) + m'^2 - i\varepsilon
\]

We notice that there is no singularity at the zero of the denominator of the integrand in (8), \( z = z_0/(4M|q|) \), since \( X = Y = Z \) there and the logs vanish. The roots of these polynomials in \( z \) are given by,

\[
x_{1,2} = v \cdot p' + 2|q| - \frac{\delta M}{2} \pm \sqrt{\left( v \cdot p' - \frac{\delta M}{2} \right)^2 - m^2 + i\varepsilon}
\]

\[
y_{1,2} = \frac{1}{2\alpha_-} \left( 4q^2 + m'^2 - m^2 \pm \sqrt{\left( 4q^2 - (m' + m)^2 \right) \left( 4q^2 - (m' - m)^2 \right) + i\varepsilon} \right)
\] (9)

\[
z_{1,2} = v \cdot p' - \frac{\delta M}{2} \pm \sqrt{\left( v \cdot p' - \frac{\delta M}{2} \right)^2 - m'^2 + i\varepsilon},
\]

where in the expression for \( y_2 \) we denoted \( \sigma \equiv \text{sgn}(q^2) \). The fact that \( X, Y, \) and \( Z \) are all equal at \( z = z_0/(4M|q|) \) leads to the important identity,

\[
\left( \frac{z_0}{4|q|} - x_1 \right) \left( \frac{z_0}{4|q|} - x_2 \right) = \frac{\alpha_-}{\alpha_+} \left( \frac{z_0}{4|q|} - y_1 \right) \left( \frac{z_0}{4|q|} - y_2 \right) = \left( \frac{z_0}{4|q|} - z_1 \right) \left( \frac{z_0}{4|q|} - z_2 \right)
\] (10)

Factorizing the polynomials in the arguments of the logarithms in (8), we get

\[
\tilde{\mathcal{H}}_3 = \frac{1}{(4\pi)^2} \int_0^{\alpha/M} dz \frac{-1}{4M|q|z - z_0} \left\{ \log \left[ \frac{\alpha_-}{\alpha_+} \left( z - \frac{y_1}{M} \right) \left( z - \frac{y_2}{M} \right) \right] - \log \left[ \left( z - \frac{z_1}{M} \right) \left( z - \frac{z_2}{M} \right) \right] \right\}
\]

\[
+ \frac{1}{(4\pi)^2} \int_0^{1} dz \frac{-1}{4M|q|z - z_0} \left\{ \log \left[ \left( z - \frac{x_1}{M} \right) \left( z - \frac{x_2}{M} \right) \right] - \log \left[ \left( z - \frac{z_1}{M} \right) \left( z - \frac{z_2}{M} \right) \right] \right\}. \quad \text{(11)}
\]

In order to be able to distribute the integrals inside the braces without introducing spurious singularities at the zero of the denominator, we add and subtract the value of each logarithm at \( z = z_0/(4M|q|) \), with the help of identity (10). We can then use the fact that \( x_{1,2} \) have imaginary parts of opposite sign, and analogously \( y_{1,2} \) and \( z_{1,2} \), to split the logarithms. (In the case of the logarithm containing \( y_{1,2} \), proper account must be taken of the fact that there is a factor \( \alpha_-/\alpha_+ \) in
In HQET the heavy quark-gluon vertex at one loop [5] is expressed in terms of integrals of the form (1) with \( m_H \) chosen to vanish. The scalar integral \( \delta M/3 \) has a cut at \( x = 1 \) containing \( y \). The integral of the logarithm containing \( y \) is evaluated in terms of dilogarithms, with the result,

\[
\mathcal{H}_3 = \lim_{M \to \infty} M \delta \mathcal{H}_3 = \frac{1}{(4\pi)^2} \frac{1}{4|q|} \sum_{j=1,2} \left( F_1(y_j) + F_2(x_j) - F_3(z_j) \right)
\]

\[
F_1(x) = \text{Li}_2 \left( \frac{z_0 - 4q|x|}{z_0 - 4q|x|} \right) - \text{Li}_2 \left( \frac{z_0}{z_0 - 4q|x|} \right)
\]

\[
F_2(x) = -\text{Li}_2 \left( \frac{z_0 - 4q|x|}{z_0 - 4q|x|} \right) - \frac{1}{2} \log^2 \left( \frac{z_0 - 4q|x|}{\mu^2} \right)
\]

\[
F_3(x) \equiv F_1(x) + F_2(x) = -\text{Li}_2 \left( \frac{z_0}{z_0 - 4q|x|} \right) - \frac{1}{2} \log^2 \left( \frac{z_0 - 4q|x|}{\mu^2} \right)
\]

In this equation the functions \( F_2 \) and \( F_3 \) are defined in terms of an arbitrary mass parameter \( \mu > 0 \), analogous to the dimensional regularization mass unit. \( \delta \mathcal{H}_3 \) does not depend on \( \mu \) since, by virtue of the identity (10), all \( \mu \) dependence cancels in the sum in (12). In principle, \( \mu \) can be eliminated from \( \delta \mathcal{H}_3 \), but we prefer not to do so. In appendix A we discuss the form of \( F_2 \) in more detail, and give several equivalent forms for it, some of which will be used below. (12) is then our final result for \( \delta \mathcal{H}_3 \).

The cuts of \( \delta \mathcal{H}_3 \) as a function of external momenta can be easily found from (11). As \( M \to \infty \), the integral of the logarithm containing \( y_{1,2} \) has a cut at \( (m' + m)^2 < 4q^2 < +\infty \), that of the log containing \( x_{1,2} \) has a cut at \( \delta M/2 + m < v \cdot p < +\infty \), and the integral of the log containing \( z_{1,2} \) has a cut at \( \delta M/2 + m' < v \cdot p' < +\infty \).

2.1 The massless case

In HQET the heavy quark-gluon vertex at one loop [3] is expressed in terms of integrals of the form (1) with \( m' = 0 = m \), and \( \delta M = 0 \) if the heavy quark residual mass term in the Lagrangian is chosen to vanish. The scalar integral \( \delta \mathcal{H}_3 \) has been given in [3] in that case, for \( q^2 < 0 \), \( v \cdot p' < 0 \), \( v \cdot p < 0 \). It is then of interest to evaluate our result (12) in the massless case, both because of its applicability to HQET and to cross-check our calculation with that of [3].

Defining \( w' \equiv v \cdot p', w = v \cdot p, Q = 2|q| = |p' - p| > 0 \), \( \sigma_{w'} = \text{sgn}(w') \) and \( \sigma_w = \text{sgn}(w) \), and setting \( m' = m = \delta M = 0 \), from (12) we get,

\[
\delta \mathcal{H}_3 = \frac{1}{(4\pi)^2} \frac{1}{4|q|} \left\{ \text{Li}_2 \left( \frac{w' + w - Q}{w' + w + Q + i\varepsilon w} \right) - \frac{1}{2} \log^2 \left( \frac{1}{\mu^2}(w' + w + Q)(w' - w - Q) - i\varepsilon \sigma_w \right) \right. \\
- \text{Li}_2 \left( \frac{w' + w + Q}{w' + w - Q - i\varepsilon w} \right) - \frac{1}{2} \log^2 \left( \frac{1}{\mu^2}(w' - w + Q)(w' + w - Q) + i\varepsilon \sigma_w \right) \\
- \text{Li}_2 \left( \frac{(w' - w + Q)(w' + w - Q)}{(w' - w - Q)(w' + w + Q) - i\varepsilon \sigma_w} \right) + \frac{1}{2} \log^2 \left( \frac{1}{\mu^2}(w' + w - Q)(w' - w + Q) - i\varepsilon \sigma_{w'} \right) \\
+ \text{Li}_2 \left( \frac{(w' + w + Q)(w' - w + Q)}{(w' + w - Q)(w' - w - Q) + i\varepsilon \sigma_{w'}} \right) \right\}.
\]

This expression is in full numerical agreement with eq. (24) of [3] in the region where \( q^2, w \) and \( w' \) are all negative. Also, [3] has cuts at \( w' > 0, w > 0 \) and \( q^2 > 0 \), as it should.

4
2.2 The limit $q \to 0$

In the limit $q \to 0$ the integral $\mathcal{H}_3$ can be easily computed as the difference of two-point integrals. Using the notation of \cite{3} (see appendix \cite{3}), we have,

$$\mathcal{H}_3|_{q=0} = \frac{1}{m^2 - m^2} \left( \mathcal{I}_2(\Delta, m^2) - \mathcal{I}_2(\Delta, m^2) \right)$$

$$= \frac{1}{(4\pi)^2} \frac{1}{m^2 - m^2} \left\{ \frac{\Delta}{2} \log \frac{m^2}{m^2} + m' F(\xi') - mF(\xi) \right\}$$

$$\mathcal{F}(x) \equiv \sqrt{x^2 - 1 + i\varepsilon} \left[ \log \left( x - \sqrt{x^2 - 1 + i\varepsilon} \right) - \log \left( x + \sqrt{x^2 - 1 + i\varepsilon} \right) \right],$$

where $\xi' = \Delta/(2m')$, $\xi = \Delta/(2m)$.

In order to obtain the value at $q = 0$ of $\mathcal{H}_3$ directly from \cite{12}, we have to expand the functions in the sum to first order in $q^\mu$, due to the factor $1/|q|$ in front of the sum.

We consider first the terms $\sum_j (F_1(y_j) - F_1(z_j))$ in \cite{12}. In the limit $q \to 0$ the two dilogarithms in the definition of $F_1$ (see \cite{12}) cancel each other up to terms of $\mathcal{O}(|q|^2)$, so these terms give a vanishing contribution to $\mathcal{H}_3$ at $q = 0$. It should be noted, however, that the expansion about $q = 0$ of $y_{1,2}$ is singular because of the factor $1/\alpha_{1,2}$ in their definition \cite{12}. Thus, we expand first about $|q| = 0$, $q^0 > 0$ and take the limit $q^0 \to 0$ afterwards. There are no such complications in the case of $F_1(z_{1,2})$.

In order to expand the remaining terms $\sum_j (F_2(x_j) - F_2(z_j))$ in \cite{12} about $q = 0$, we use the form \cite{13} for $F_2(x)$. To lowest order, we get,

$$\mathcal{H}_3 = \frac{1}{4|q|} \sum_{j=1,2} (F_2(x_j) - F_2(z_j)) + O(q)$$

$$= \sum_{j=1,2} \left\{ \frac{\bar{x}_j}{z_0} + \frac{x_j}{\bar{z}_0} \log \left( -\frac{x_j}{\bar{z}_0} \right) + \frac{\bar{x}_j}{x_0} - \frac{x_j}{\bar{z}_0} \log \left( \frac{\bar{x}_j}{x_0} \right) \right\}.$$ (15)

Here, $\bar{x}_j, x_j, z_0$ are the zeroth-order terms in the expansions of $x_j, z_j, z_0$ about $q = 0$, respectively. Using the notation of \cite{12}, we have,

$$\bar{x}_{1,2} = m \left( -\xi \pm \sqrt{\xi^2 - 1 + i\varepsilon} \right), \quad x_{1,2} = m' \left( -\xi' \pm \sqrt{\xi'^2 - 1 + i\varepsilon} \right), \quad z_0 = -m'^2 + m^2.$$ (16)

Substituting \cite{16} into \cite{15} we recover \cite{12}.

3 Tensor integrals

In this section we consider tensor three-point integrals, up to the third rank. These are given in terms of integrals with smaller ranks and fewer points, by using the well-known method of \cite{3}. The general tensor integral has the form \cite{3}. We will restrict ourselves to integrals of the standard form

$$H_{3}^{\alpha_1 \cdots \alpha_n}(v, q; \Delta, m, m') = \frac{i\mu^{d-4}}{(2\pi)^{d}} \int d^d \ell \int d^d \ell' \frac{\ell^{\alpha_1} \cdots \ell^{\alpha_n}}{(2v \cdot \ell - \Delta + i\varepsilon)((\ell + q)^2 - m^2 + i\varepsilon)((\ell - q)^2 - m'^2 + i\varepsilon)}.$$ (17)

By shifting $\ell \to \ell + (p' + p)/2$ in \cite{3} we can express $H_{3}^{\alpha_1 \cdots \alpha_n}$ in terms of $H_{3}^{\alpha_1 \cdots \alpha_n}$ as,

$$H_{3}^{\alpha_1 \cdots \alpha_n} = H_{3}^{\alpha_1 \cdots \alpha_n}(v, q; \Delta, m, m') + \sum_{j=1}^{n} r^{(\alpha_1} \cdots r^{\alpha_j)} H_{3}^{\alpha_j+1 \cdots \alpha_n}(v, q; \Delta, m, m'),$$ (18)
where \( r^\mu \equiv 1/2(p^\prime + p)^\mu \), and \( A^{\alpha_1 \alpha_2 \cdots \alpha_s} = A^{\alpha_1 \alpha_2 \cdots \alpha_s} + A^{\alpha_2 \cdots \alpha_s \alpha_1} + \cdots + A^{\alpha_s \alpha_1 \cdots \alpha_{s-1}} \). Clearly, the scalar integral \( \mathcal{H}_3 = \mathcal{H}_3 \). The standard form \([17]\) has the advantage that it depends explicitly on only two four-vectors, \( v^\mu \) and \( q^\mu \), instead of three as in \([1]\).

### 3.1 Vector integral

For the vector integral we write \( H_3^\alpha(v; q; \Delta, m_1, m_2) = V_1 v^\alpha + V_2 q^\alpha / |q| \), with \( |q| \equiv \sqrt{(v \cdot q)^2 - q^2} \), and \( V_{1,2} = V_{1,2}(v \cdot q, q^2, \Delta, m_1, m_2) \). If \( |q| = 0 \), then \( q^\alpha \propto v^\alpha \) and we can set \( V_2 = 0 \). If \( |q| \neq 0 \), then,

\[
|q|^2 V_1 = -q^2 v_0 H_3^\alpha + v \cdot q q_0 H_3^\alpha, \quad |q| V_2 = v \cdot q v_0 H_3^\alpha - q_0 H_3^\alpha,
\]

with,

\[
v_0 H_3^\alpha(v; q; \Delta, m_1, m_2) = \frac{1}{2} B_0 (4q^2, m_1, m_2) + \frac{\Delta}{2} H_2 (v; q; \Delta, m_1, m_2)
\]

\[
q_0 H_3^\alpha(v; q; \Delta, m_1, m_2) = \frac{1}{4} T_2 (\Delta - 2v \cdot q, m_2) - \frac{1}{4} T_2 (\Delta + 2v \cdot q, m_1)
\]

\[
+ \frac{m_1^2 - m_2^2}{4} H_3 (v; q; \Delta, m_1, m_2).
\]

The scalar two-point integrals \( T_2 \) and \( B_0 \) are given in the appendix.

### 3.2 Second-rank tensor integral

The decomposition of \( H_3^{\alpha \beta} \) in terms of form factors can be written as,

\[
H_3^{\alpha \beta}(v, q; \Delta, m_1, m_2) = T_1 g^{\alpha \beta} + T_2 v^\alpha v^\beta + T_3 q^\alpha q^\beta + T_4 \frac{v^{\alpha \beta}}{|q|^2}.
\]

The components of \( H_3^{\alpha \beta} \) are obtained by direct computation,

\[
F_1 \equiv g_{\alpha \beta} H_3^{\alpha \beta} = \frac{1}{2} T_2 (\Delta - 2v \cdot q, m_2) + \frac{1}{2} T_2 (\Delta + 2v \cdot q, m_1) + \frac{m_1^2 + m_2^2 - 2q^2}{2} H_3
\]

\[
F_2 \equiv v_\alpha v_\beta H_3^{\alpha \beta} = -v \cdot q B_1 (4q^2, m_1, m_2) + \frac{1}{2} \left( \frac{\Delta}{2} - v \cdot q \right) B_0 (4q^2, m_1, m_2) + \frac{\Delta^2}{4} H_3
\]

\[
F_3 \equiv q_\alpha q_\beta H_3^{\alpha \beta} = q_\alpha q_\beta \left( 4q^2 + m_1^2 + m_2^2 \right) - q_\alpha q_\beta \left( 2 B_1 (4q^2, m_1, m_2) - B_0 (4q^2, m_1, m_2) + \frac{\Delta^2}{4} H_3 \right)
\]

\[
F_4 \equiv v_\alpha q_\beta H_3^{\alpha \beta} = -q^2 B_1 (4q^2, m_1, m_2) - \frac{q^2}{2} B_0 (4q^2, m_1, m_2) + \frac{\Delta}{2} q_\alpha H_3
\]

where, on both sides of these equations the arguments \((v, q; \Delta, m_1, m_2)\) of \( H_3^{\alpha \beta} \) have been omitted for brevity. To one-loop level, the form factors in \([21]\) are given in terms of the \( F_s \) as,

\[
2|q|^2 T_1 = \left( 1 + \frac{\epsilon}{2} \right) \left( 1 \right) \left( |q|^2 F_1 + q^2 F_2 + F_3 - 2v \cdot q F_4 \right)
\]

\[
2|q|^2 T_2 = \left( 1 + \frac{\epsilon}{2} \right) q^2 F_1 + \left( 3 + \frac{\epsilon}{2} \right) \frac{q^4}{|q|^2} F_2 + \frac{1}{|q|^2} \left( 2(v \cdot q)^2 + \left( 1 + \frac{\epsilon}{2} \right) q^2 \right) F_3 - \left( 6 + \epsilon \right) \frac{q^2 v \cdot q}{|q|^2} F_4
\]
\[ 2|q|^2 T_3 = \left(1 + \frac{\epsilon}{2}\right) |q|^2 F_1 + \left(2(v \cdot q)^2 + \left(1 + \frac{\epsilon}{2}\right) q^2\right) F_2 + \left(3 + \frac{\epsilon}{2}\right) F_3 - (6 + \epsilon)v \cdot q F_4 \]  
\[ 2|q|^2 T_4 = -\left(1 + \frac{\epsilon}{2}\right) v \cdot q |q| F_1 - \left(3 + \frac{\epsilon}{2}\right) \frac{g^2 v \cdot q}{|q|} F_2 - \left(3 + \frac{\epsilon}{2}\right) \frac{v \cdot q}{|q|} F_3 + \frac{1}{|q|} \left(2q^2 + (4 + \epsilon)(v \cdot q)^2\right) F_4 . \]

Equations (22) and (23) give an explicit expression for \( H_3^{\alpha \beta} \).

### 3.3 Third-rank tensor integral

The decomposition in terms of form factors can be written as,

\[
H_3^{\lambda \mu \nu}(v, q; \Delta, m_1, m_2) = S_1 g^{\lambda \mu \nu} + S_2 g^{\lambda \mu \nu} + S_3 v^{\lambda \mu \nu} + S_4 v^{\lambda \mu \nu} + S_5 v^{\lambda \mu \nu} + S_6 q^{\lambda \mu \nu} + S_7 q^{\lambda \mu \nu} + \cdots
\]

These form-factors are given implicitly by the following linear relations,

\[
(d + 2)S_1 + (d + 2)v \cdot q S_2 + 3v \cdot q S_3 + (q^2 + 2(v \cdot q)^2) S_4 + S_5 + q^2 v \cdot q S_6 = g_{\lambda \mu \nu} v \cdot H_3^{\lambda \mu \nu}
\]

\[
(d + 2)v \cdot q S_1 + (d + 2)q^2 S_2 + (q^2 + 2(v \cdot q)^2) S_3 + 3q^2 v \cdot q S_4 + q \cdot q S_5 + q^4 S_6 = g_{\lambda \mu \nu} q \cdot H_3^{\lambda \mu \nu}
\]

\[
3v \cdot q S_1 + (q^2 + 2(v \cdot q)^2) S_2 + (q^2 + 2(v \cdot q)^2) S_3 + (2q^2 + (v \cdot q)^2) v \cdot q S_4
\]

\[
+ v \cdot q S_5 + q^2 v \cdot q S_6 = v_{\lambda \mu \nu} q \cdot H_3^{\lambda \mu \nu}
\]

\[
(q^2 + 2(v \cdot q)^2) S_1 + 3q^2 (v \cdot q) S_2 + v \cdot q ((v \cdot q)^2 + 2q^2) S_3 + q^2 (q^2 + 2(v \cdot q)^2) S_4
\]

\[
+ (v \cdot q)^2 S_5 + q^2 v \cdot q S_6 = v_{\lambda \mu \nu} q \cdot H_3^{\lambda \mu \nu}
\]

\[
3S_1 + 3v \cdot q S_2 + 3v \cdot q S_3 + 3(v \cdot q)^2 S_4 + S_5 + (v \cdot q)^3 S_6 = v_{\lambda \mu \nu} v \cdot H_3^{\lambda \mu \nu}
\]

\[
3q^2 v \cdot q S_1 + 3q^4 S_2 + 3(v \cdot q)^2 q^2 S_3 + 3(v \cdot q)q^4 S_4 + (v \cdot q)^3 S_5 + q^6 S_6 = q_{\lambda \mu \nu} q \cdot H_3^{\lambda \mu \nu},
\]

where the right-hand side can be obtained by direct computation.
4 Final remarks

In phenomenological applications, the exact functional dependence of Feynman integrals on masses and residual momenta is usually not needed. Often, the first few terms in a series expansion in some of the parameters provides the required accuracy. We believe, however, that the exact analytic computation presented here does not require more calculational effort than approximate schemes. It has the added advantage of being valid over the entire physical region for internal and external masses. Applications of the results presented here to loop graphs in heavy baryon chiral perturbation theory which have a calculable dependence on the ratio of the $\Delta$-nucleon mass difference to the pion mass will be given elsewhere.

Our approach, which is based on well-known methods and results for vertex integrals in renormalizable theories [4, 6], streamlines the computation so that it can be easily reproduced and verified, and adapted to other, more complicated one-loop diagrams. Another important feature of this approach is that contact with the unitary cuts of the diagram, as given by Cutkosky rules, is explicitly kept at every step of the calculation [4].

Acknowledgements

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A Calculation of $F_2$

In this appendix we discuss the calculation of the second integral in (3). We consider first the basic integral, whose computation is standard,

$$\int_{\alpha/M}^{1} \frac{dz}{4M|q|} \left\{ \log \left( \frac{z - x}{M} \right) - \log \left( \frac{z_0 - 4|q|/M}{x} \right) \right\} = \frac{\pi^2}{6} + \text{Li}_2 \left( \frac{4|q|/M}{z_0 - 4|q|x} \right)$$

$$+ \log \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right) \log \left( \frac{\alpha - x}{\mu} M \right) - \log \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right) \log \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right) + \frac{1}{2} \log^2 \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right)$$

$$- \log \left( \frac{4M|q|/z_0 - 4|q|x}{z_0 - 4|q|x} \right) \log \left( \frac{4M|q|}{z_0 - 4|q|x} \right) \right) + \mathcal{O} \left( \frac{1}{M^2} \right) \equiv \frac{1}{4M|q|} F_2(x) + \mathcal{O} \left( \frac{1}{M^2} \right), \quad (A.1)$$

where $F_2(x)$ is defined by this equation. In (A.1) we have made use of the relation \[ \text{Li}_2(x) = -\text{Li}_2(1/x) - \pi^2/6 - 1/2 \log^2(-x) \] in order to obtain the leading $M$ dependence of $\text{Li}_2(4M|q|/(z_0 - 4|q|x))$.

Denoting by $G$ the second integral in (3), we have,

$$G = \frac{1}{4M|q|} \sum_{j=1,2} (F_2(x_j) - F_2(z_j)) + \mathcal{O} \left( \frac{1}{M^2} \right). \quad (A.2)$$

Clearly, those terms in $F_2(x)$ which do not depend on $x$ will cancel in (A.2), so they can be omitted. Also, due to the identity (10), $G$ does not depend on the value of $M$, which can be replaced by an arbitrary mass $\mu > 0$ that remains constant as $M \rightarrow \infty$. Thus, in (A.2) we can write,

$$F_2(x) = \text{Li}_2 \left( \frac{4|q|/M}{z_0 - 4|q|x} \right) + \log \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right) \log \left( \frac{\alpha - x}{\mu} M \right) - \log \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right) \log \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right) + \frac{1}{2} \log^2 \left( \frac{z_0 - 4|q|/M}{z_0 - 4|q|x} \right)$$

$$+ \sum_{j=1,2} (F_2(x_j) - F_2(z_j)) + \mathcal{O} \left( \frac{1}{M^2} \right). \quad (A.3)$$

The last two logarithms in this equation can be rewritten according to the identity $\log^2(z) - 2 \log(-z) \log(z) = -\pi^2 - 2 \log^2(z)$, which is valid for the principal determination of the log. Dropping the constant term, we see that in (A.2) we can write,

$$F_2(x) = \text{Li}_2 \left( \frac{4|q|/\mu}{z_0 - 4|q|x} \right) + \log \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right) \log \left( \frac{\alpha - x}{\mu} \right) - \log \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right) \log \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right) + \frac{1}{2} \log^2 \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right). \quad (A.4)$$

This equation can still be rewritten by completing the last two terms to the square of a sum of logs. Once again discarding $x$-independent terms that do not contribute to (A.2), we get,

$$F_2(x) = \text{Li}_2 \left( \frac{4|q|/\mu}{z_0 - 4|q|x} \right) + \log \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right) \log \left( \frac{\alpha - x}{\mu} \right) + \frac{1}{2} \log^2 \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right). \quad (A.5)$$

From (A.4), by means of the identity (4)

$$\text{Li}_2 \left( \frac{4|q|/\mu}{z_0 - 4|q|x} \right) = \text{Li}_2 \left( 1 - \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right)$$

$$= \frac{\pi^2}{6} - \text{Li}_2 \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right) - \log \left( \frac{z_0 - 4|q|/\mu}{z_0 - 4|q|x} \right) \log \left( \frac{4|q|/\mu}{z_0 - 4|q|x} \right),$$

we obtain $F_2(x)$ as given by (3), up to constant terms. All of these expressions for $F_2(x)$ lead to the same results when substituted into (3.3) or (4).
B  Loop integrals

In this appendix we give a list of loop integrals used in the foregoing. More complete calculations can be found, e.g., in [3, 4, 6, 7, 8] and references therein. Divergent integrals are separated in a dimensional-regularization pole term and a finite remainder. \( \mu = \mu \sqrt{4\pi e^{-\gamma E}} \).

\[
A_0(m) = \frac{i\mu^\epsilon}{(2\pi)^d} \int d^d \ell \frac{1}{(\ell^2 - m^2 + i\epsilon)} = -\frac{m^2}{8\pi^2 \epsilon} + \frac{m^2}{16\pi^2} a_0(m^2)
\]

\[
a_0(m^2) = \log \left( \frac{m^2}{\mu^2} \right) - 1
\]

\[
B_0(p^2, m_1, m_2) = \frac{i\mu^\epsilon}{(2\pi)^d} \int d^d \ell \frac{1}{(\ell^2 - m_1^2 + i\epsilon)(\ell^2 - m_2^2 + i\epsilon)}
\]

\[
B_0(p^2, m_1, m_2) = -\frac{1}{8\pi^2 \epsilon} + \frac{1}{16\pi^2} b_0(p^2, m_1, m_2)
\]

\[
b_0(p^2, m_1^2, m_2^2) = \int_0^1 dx \log \left( \left(1 - x\right) \frac{m_1^2}{\mu^2} + x \frac{m_2^2}{\mu^2} - x(1-x) \frac{p^2}{\mu^2} - i\epsilon \right)
\]

\[
B_1^\mu(p, m_1, m_2) = \frac{i\mu^\epsilon}{(2\pi)^d} \int d^d \ell \frac{\ell^\mu}{(\ell^2 - m_1^2 + i\epsilon)((\ell + p)^2 - m_2^2 + i\epsilon)} = p^\mu B_1(p^2, m_1, m_2)
\]

\[
p^2 B_1(p^2, m_1, m_2) = \frac{1}{2} \left( A_0(m_1) - A_0(m_2) - (p^2 + m_1^2 - m_2^2) B_0(p^2, m_1, m_2) \right)
\]

\[
B_2^{\mu\nu}(p, m_1, m_2) = \frac{i\mu^\epsilon}{(2\pi)^d} \int d^d \ell \frac{\ell^\mu \ell^\nu}{(\ell^2 - m_1^2 + i\epsilon)((\ell + p)^2 - m_2^2 + i\epsilon)} = G \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + L \frac{p^\mu p^\nu}{p^2}
\]

\[
G = \frac{1}{2} A_0(m_2) + m_1^2 B_0(p^2, m_1, m_2) + \frac{1}{2} (m_1^2 - m_2^2 + p^2) B_1(p^2, m_1, m_2)
\]

\[
L = \frac{1}{2} A_0(m_2) - \frac{1}{2} (m_1^2 - m_2^2 + p^2) B_1(p^2, m_1, m_2)
\]

\[
I_2(\Delta, m) = \frac{i\mu^\epsilon}{(2\pi)^d} \int d^d \ell \frac{1}{(2v \cdot \ell - \Delta + i\epsilon)(\ell^2 - m^2 + i\epsilon)}
\]

\[
I_2(\Delta, m) = \frac{\Delta}{8\pi^2 \epsilon} \left( \frac{2}{\epsilon} + \log \left( \frac{\mu^2}{m^2} \right) + 2 \right) + \frac{m}{16\pi^2} \mathcal{F} \left( \frac{\Delta}{2m} \right) \quad \text{(see (4))}
\]

\[
I_2^\mu(\Delta, m) = \frac{i\mu^\epsilon}{(2\pi)^d} \int d^d \ell \frac{\ell^\mu}{(2v \cdot \ell - \Delta + i\epsilon)(\ell^2 - m^2 + i\epsilon)} = F(\Delta, m) v^\mu
\]

\[
F(\Delta, m) = \frac{1}{2} A_0(m) + \frac{\Delta}{2} I_2(\Delta, m)
\]

\[
I_1^\mu(\Delta, m) = \frac{i\mu^\epsilon}{(2\pi)^d} \int d^d \ell \frac{\ell^\mu \ell^\nu}{(2v \cdot \ell - \Delta + i\epsilon)(\ell^2 - m^2 + i\epsilon)} = I_0(\Delta, m) g^{\mu\nu} + I_1(\Delta, m) v^\mu v^\nu
\]

\[
(d - 1) I_0(\Delta, m) = -\frac{\Delta}{4} A_0(m) + \left( m^2 - \frac{\Delta^2}{4} \right) I_2(\Delta, m)
\]

\[
(d - 1) I_1(\Delta, m) = \frac{d\Delta}{4} A_0(m) - \left( m^2 - \frac{d\Delta^2}{4} \right) I_2(\Delta, m)
\]