LOCALLY CONTINUOUSLY PERFECT GROUPS OF
HOMEOMORPHISMS

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Abstract. The notion of a locally continuously perfect group is intro-
duced and studied. This notion generalizes locally smoothly perfect groups
introduced by Haller and Teichmann. Next, we prove that the path con-
ected identity component of the group of all homeomorphisms of a man-
ifold is locally continuously perfect. The case of equivariant homeomor-
phism group and other examples are also considered.

1. Introduction

Recall that a group $G$ is perfect if it is equal to its own commutator subgroup
$[G, G]$. This means that for any $g \in G$ there exist $r \in \mathbb{N}$ and $h_i, \bar{h}_i \in G,$
$i = 1, \ldots, r$, such that

$$g = [h_1, \bar{h}_1] \cdots [h_r, \bar{h}_r].$$

When we consider the category of topological groups a fundamental question
arises whether $h_i, \bar{h}_i$ can be chosen to be continuously dependent on $g$. More
precisely, we introduce the following notion.

Definition 1.1. A topological group $G$ will be called locally continuously
perfect if there exist $r \in \mathbb{N}$, a neighborhood $U$ of $e$ in $G$ and continuous
mappings $S_i : U \to G$, $\bar{S}_i : U \to G$, $i = 1, \ldots, r$, such that

$$g = [S_1(g), \bar{S}_1(g)] \cdots [S_r(g), \bar{S}_r(g)]$$

for every $g \in U$. Moreover, we assume that $S_i(e) = e$ for all $i$. The smallest $r$
as above will be denoted by $r_G$.

Clearly any connected locally continuously perfect group is perfect.

An analogous notion of a locally smoothly perfect group in the category of
(possibly infinite dimensional) Lie groups have been studied in the paper by

Date: December 15, 2010.

1991 Mathematics Subject Classification. 57S05, 22A05, 22E65.

Key words and phrases. Perfect group, locally continuously perfect, locally smoothly per-
fected, uniformly perfect, manifold, homeomorphism, diffeomorphism, conjugation-invariant
norm, group of homeomorphisms, fragmentation, deformation.

Partially supported by the Polish Ministry of Science and Higher Education and the
AGH grant n. 11.420.04.
Haller and Teichmann [7], where, for the first time, the problem of smooth
dependence of \( h_i, \bar{h}_i \) on \( g \) in (1.1) was put forward. The main purpose of the
present paper is to show that the property of being locally continuously per-
fected is even more common for homeomorphism groups of manifolds than its
smooth counterpart in \([7]\) for diffeomorphism groups of manifolds. In both
cases very deep (but completely different from each other) facts are exploited:
our main result is based on deformations in the spaces of imbeddings of man-
ifolds (Edwards and Kirby [5]), while Haller and Teichmann used a simplicity
theorem of Herman (\([8]\)) on the diffeomorphism group of a torus and the small
denominator theory (or the KAM theory) in its background.

Throughout \( M \) is a topological metrizable manifold, possibly with bound-
ary, and \( \mathcal{H}(M) \) denotes the path connected identity component of the group
of all compactly supported homeomorphisms of \( M \) endowed with the graph
topology (\([10]\)) (or the majorant topology \([5]\)). By a ball in \( M \) we will mean
rel. compact, open ball imbedded with its closure in \( M \). Similarly we define
a half-ball if \( M \) has boundary. For \( M \) compact, let \( d_M \) is the smallest integer
such that \( M = \bigcup_{i=1}^{d_M} B_i \) where \( B_i \) is a ball or half-ball for each \( i \).

**Theorem 1.2.** If \( M \) is compact, then \( \mathcal{H}(M) \) is a locally continuously perfect
group (even more, it satisfies Def. 2.1 below) with \( r_{\mathcal{H}(M)} \leq d_M \). In particular,
\( \mathcal{H}(M) \) is perfect and simple (if \( M \) is connected).

The fact that \( \mathcal{H}(M) \) is perfect is an immediate consequence of \([13]\) and \([5]\),
Corollary 1.3. A special case was already proved by Fisher \([6]\). Note that
if we drop the compactness assumption, \( \mathcal{H}(M) \) is also perfect in view of an
argument based on Theorem 5.1 (also Theorem 5.1 in \([5]\)). The group \( \mathcal{H}(M) \)
is simple as well, see e.g. \([12]\).

**Theorem 1.3.** Let \( M \) be an open manifold such that \( M = \text{Int}(\bar{M}) \), where \( \bar{M} \)
is a compact manifold with boundary. Then \( \mathcal{H}(M) \) is a locally continuously perfect
group (and fulfils Def. 2.1). In particular, \( \mathcal{H}(M) \) is perfect and simple.
Furthermore, \( r_{\mathcal{H}(M)} \leq d_M + 2 \). Here \( d_M \) stands for the smallest integer such
that \( M = P \cup \bigcup_{i=1}^{d_M} B_i \) where \( B_i \) is an open ball for each \( i \) and \( P \) is a collar
neighborhood of the boundary.

It is doubtful whether \( \mathcal{H}(M) \) is locally continuously perfect without the
assumption of Theorem 1.3. Observe that Theorems 1.2 and 1.3 are also true
for isotopies (Corollary 3.3).

In the next three sections we present miscellaneous notions, examples, facts
and problems related to locally continuously perfect groups. The proofs of
Theorems 1.2 and 1.3, making use of subtle and difficult techniques of Ed-
wards and Kirby in \([5]\), are presented in section 5. The case of \( G \)-equivariant
homeomorphisms is investigated in section 6.
2. Relative notions and basic lemma

In order to describe the structure of homeomorphism groups of manifolds it will be useful to strengthen slightly Def.1.1.

**Definition 2.1.** A topological group $G$ is *locally continuously perfect* (in a stronger sense) if there are $r \in \mathbb{N}$, a neighborhood $U$ of $e \in G$, elements $h_1, \ldots, h_r \in G$ and continuous mappings $S_i : U \to G$ with $S_i(e) = e$, $i = 1, \ldots, r$, satisfying

\[
g = [S_1(g), h_1] \cdots [S_r(g), h_r]
\]

for all $g \in U$.

Let us "globalize" the notion of local continuous perfectness.

**Definition 2.2.** A group $G$ is called *uniformly perfect* (see, e.g., [23]), if $G$ is perfect and there is $r \in \mathbb{N}$ such that any $g \in G$ can be expressed by (1.1) for some $h_i, \tilde{h}_i \in G$, $i = 1, \ldots, r$. Next, a topological group $G$ is said to be *continuously perfect* if there exist $r \in \mathbb{N}$ and continuous mappings $S_i : G \to G$, $\tilde{S}_i : G \to G$, $i = 1, \ldots, r$, satisfying the equality (1.2) for all $g \in G$.

Of course, every continuously perfect group is uniformly perfect.

In early 1970’s Thurston proved that the identity component of the group of compactly supported diffeomorphism of class $C^\infty$ of a manifold $M$ is perfect and simple (see [21], [2]). The proof is based on Herman’s theorem [8]. Next, Thurston’s theorem was extended to groups of $C^r$-diffeomorphisms where $r \neq \dim(M) + 1$ ([14]), and to classical diffeomorphism groups ([2], [18]). Recently, the problem of uniform perfectness of diffeomorphism groups has been studied in [3], [23] and [19]. In contrast to the problem of perfectness and simplicity, the obtained results depend essentially on the topology of the underlying manifold. However, in most cases (with some exceptions presented below) difficult open problems arise whether the groups in question satisfy Def. 1.1, 2.1, or 2.2, or whether they are locally smoothly perfect.

The following type of fragmentations is important when studying groups of homeomorphisms.

**Definition 2.3.** Let $\mathcal{U}$ be an open covering of $M$. A subgroup $G \subset \mathcal{H}(M)$ is *locally continuously factorizable with respect to $\mathcal{U}$* if for any finite subcovering $(U_i)_{i=1}^d$ of $\mathcal{U}$, there exist a neighborhood $\mathcal{P}$ of $\text{id} \in G$ and continuous mappings $\sigma_i : \mathcal{P} \to G$, $i = 1, \ldots, d$, such that for all $f \in \mathcal{P}$ one has

\[
f = \sigma_1(f) \cdots \sigma_d(f), \quad \text{supp}(\sigma_i(f)) \subset U_i, \forall i.
\]

Given a subset $S \subset M$, by $\mathcal{H}_S(M)$ we denote the path connected identity component of the subgroup of all elements of $\mathcal{H}(M)$ with compact support contained in $S$.  

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Using the Alexander trick, we have that $\mathcal{H}(\mathbb{R}^n)$ coincides with the group of all compactly supported homeomorphisms of $\mathbb{R}^n$. In fact, if supp($g$) is compact, we define an isotopy $g_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in I$, from the identity to $g$, by

$$g_t(x) = \begin{cases} tg\left(\frac{x}{t}\right) & \text{for } t > 0 \\ x & \text{for } t = 0. \end{cases}$$

In particular, for every ball $B$ in $M$ the group $\mathcal{H}_B(M)$ consists of all homeomorphisms compactly supported in $B$.

The following fact, with a straightforward proof, plays a basic role in studies on homeomorphism groups.

**Lemma 2.4.** [13] (Basic lemma) Let $B \subset M$ be a ball and $U \subset M$ be an open subset such that $\text{cl}(B) \subset U$. Then there are $\varphi \in \mathcal{H}_U(M)$ and a continuous mapping $S : \mathcal{H}_B(M) \to \mathcal{H}_U(M)$ such that $h = [S(h), \varphi]$ for all $h \in \mathcal{H}_B(M)$.

**Proof.** First choose a larger ball $B'$ such that $\text{cl}(B) \subset B' \subset \text{cl}(B') \subset U$. Next, fix $p \in \partial B'$ and set $B_0 = B$. There exists a sequence of balls $(B_k)_{k=1}^{\infty}$ such that $\text{cl}(B_k) \subset B'$ for all $k$, the family $(B_k)_{k=0}^{\infty}$ is pairwise disjoint, locally finite in $B'$, and $B_k \to p$ when $k \to \infty$. Choose a homeomorphism $\varphi \in \mathcal{H}_U(M)$ such that $\varphi(B_{k-1}) = B_k$ for $k = 1, 2, \ldots$. Here we use the fact that $\mathcal{H}_U(M)$ acts transitively on the family of balls in $B'$ (c.f. [9]).

Now we define a continuous homomorphism $S : \mathcal{H}_B(M) \to \mathcal{H}_U(M)$ by the formula

$$S(h) = \varphi^k h \varphi^{-k} \quad \text{on } B_k, \quad k = 0, 1, \ldots$$

and $S(h) = \text{id}$ outside $\bigcup_{k=0}^{\infty} B_k$. It is clear that $h = [S(h), \varphi]$, as required. □

**Remark 2.5.** (1) Perhaps for the first time the above reasoning appeared in Mather’s paper [13]. Actually Mather proved also the acyclicity of $\mathcal{H}(\mathbb{R}^n)$. Obviously, [13] and Lemma 2.4 are no longer true for $C^1$ homeomorphisms. However, Tsuboi brilliantly improved this reasoning and adapted it for $C^r$-diffeomorphisms with small $r$, see [22].

(2) It is likely that the basic lemma is no longer true for $\mathcal{H}(M)$ instead of $\mathcal{H}_B(M)$ and $\mathcal{H}_U(M)$. In fact, consider $\mathcal{H}(\mathbb{R}^n)$ and a weaker Def.1.1. If one tried to repeat the proof of Lemma 2.4 then one would have continuous maps $S, \tilde{S} : \mathcal{H}(\mathbb{R}^n) \supset U \to \mathcal{H}(\mathbb{R}^n)$, where $\tilde{S}(h)$ would play a role of the shift homeomorphism $\varphi$ for $h \in \mathcal{H}(\mathbb{R}^n)$. But then $\tilde{S}(h)$ depends somehow on the support of $h$. On the other hand, there are arbitrarily close to id elements of $\mathcal{H}(\mathbb{R}^n)$ with arbitrarily large support. This would spoil the continuity of $\tilde{S}$.

Given a foliated manifold $(M, \mathcal{F})$, a mapping $h : M \to M$ is leaf preserving if $h(L) \subset L$ for all $L \in \mathcal{F}$. Let $\mathcal{H}(M, \mathcal{F})$ be the path connected identity component of the group of all leaf preserving homeomorphisms of $(M, \mathcal{F})$. 
Corollary 2.6. Let $\mathcal{F}_k = \{ \mathbb{R}^k \times \{ pt \} \}$, $k = 1, \ldots, n-1$, be the product foliation of $\mathbb{R}^n$. If $B = I \times \mathbb{R}^{n-k}$ and $U = J \times \mathbb{R}^{n-k}$, where $I, J \subset \mathbb{R}^k$ are open intervals such that $\text{cl}(I) \subset J$, then there exist $\varphi \in \mathcal{H}_U(\mathbb{R}^n, \mathcal{F}_k)$ and a continuous mapping $S : \mathcal{H}_B(\mathbb{R}^n, \mathcal{F}_k) \rightarrow \mathcal{H}_U(\mathbb{R}^n, \mathcal{F}_k)$ such that $h = [S(h), \varphi]$ for all $h \in \mathcal{H}_B(\mathbb{R}^n, \mathcal{F}_k)$.

Proof. Consider $\mathcal{H}_I(\mathbb{R}^k)$ and $\mathcal{H}_J(\mathbb{R}^k)$, and repeat the construction of $\varphi$ and $S(h)$ from the proof of Lemma 2.4. Then multiply everything by $\text{id}_{\mathbb{R}^{n-k}}$. □

Corollary 2.7. Assume that either

1. $M$ is a manifold with boundary and $B, U \subset M$ such that $B$ is a half-ball, $U$ is open with $\text{cl}(B) \subset U$; or
2. $M = N \times \mathbb{R}$, where $N$ is a manifold, and $B = N \times I$, $U = N \times J$ where $I, J \subset \mathbb{R}$ are open intervals with $\text{cl}(I) \subset J$.

Then the assertion of Lemma 2.4 holds.

The proof is analogous to the above.

3. Examples

First examples are provided by Theorems 1.2 and 1.3, and by Theorem 6.2 below.

1. For a smooth manifold $M$ let $\mathcal{D}^r(M)$ stand for the subgroup of all elements of $\text{Diff}^r(M)$ that can be joined to the identity by a compactly supported isotopy in $\text{Diff}^r(M)$, where $r = 1, \ldots, \infty$.

Let $G$ be a possibly infinite dimensional Lie group which is simultaneously a topological group. If $G$ is also locally smoothly perfect (see Def. 1 in Haller and Teichmann [7]) then obviously $G$ is locally continuously perfect (even satisfies Def. 2.1).

In particular, the following groups are locally smoothly perfect (and a fortiori locally continuously perfect).

1. Any finite dimensional perfect Lie group $G$; we then have $r_G \leq \dim G$ ([7]).
2. Any real semisimple Lie group $G$; then $r_G = 2$ ([7]).
3. $\mathcal{D}^\infty(\mathbb{T}^n)$ for the torus $\mathbb{T}^n$ with $r_{\mathcal{D}^\infty(\mathbb{T}^n)} \leq 3$ ([8]).
4. Let $M$ be a closed smooth manifold being the total space of $k$ locally trivial bundles with fiber $\mathbb{T}^{m(i)}$, $i = 1, \ldots, k$, such that the corresponding vertical distributions span $TM$. Then $\mathcal{D}^\infty(M)$ is locally smoothly perfect. In particular, the assumption is satisfied for odd dimensional spheres and for any compact Lie group $G$, and one has $r_{\mathcal{D}^\infty(S^3)} \leq 18$ and $r_{\mathcal{D}^\infty(G)} \leq 3(\dim G)^2$ (see [7]).

Remark 3.1. To obtain the theorem mentioned in (4) Haller and Teichmann used a fruitful method of decomposing diffeomorphisms into fiber preserving
ones. This method enabled an application of the deep Herman's theorem
stating that $D^\infty(\mathbb{T}^n)$ is not only perfect and simple but also locally smoothly
perfect. Notice that this method was already considered in the "generic" case
of $M = \mathbb{R}^n$ in [16] to study the (still open) problem of the perfectness of
$D^{n+1}(M^n)$ by means of the possible perfectness property of the group of leaf
preserving $C^r$-diffeomorphisms with $r$ large. For a hypothetical proof of such a
result one would apply Mather's proof of the simplicity of $D^r(M^n)$, $r \neq n + 1$,
from [14]. See also [14] III, [15], [11] for the problem of the perfectness of leaf
preserving diffeomorphism groups.

Notice as well that recently Tsuboi in [21] used a similar method in the
proofs of perfectness theorems for groups of real-analytic diffeomorphisms in
absence of the fragmentation property.

It seems likely that the groups $D^r(\mathbb{R}^n)$ with small $r$ (depending on $n$) would
be continuously perfect. By using his own method Tsuboi in [22] generalized Mather's method from [13], reproved the simplicity theorem for $C^r$-
diffeomorphisms with $1 \leq r \leq n$ (originally proved by Mather [14], II), and
showed the vanishing of lower order homologies of the groups $D^r(\mathbb{R}^n)$. However, a possible analysis of a very technical proof in [22] is beyond the scope of
the present paper. We may also ask whether $D^r(\mathbb{R}^n)$ is (locally) $C^r$-smoothly
perfect.

2. It is very likely that theorems analogous to Theorems 1.2 and 1.3 can be
obtained for the groups of Lipschitz homeomorphisms on Lipschitz manifolds.
See Theorem 2.2 and other results in Abe and Fukui [1].

3. Now we consider permanence properties of locally continuously perfect
groups. These properties provide further examples of such groups.

Let $H \subset G$ be a subgroup and $G$ be locally continuously perfect with
continuous mappings $S_i : U \to G$, $\tilde{S}_i : U \to G$, $i = 1, \ldots, r$, satisfying (1.2).
If $S_i(U \cap H) \subset H$ and $\tilde{S}_i(U \cap H) \subset H$ for all $i$ then $H$ is also continuously
perfect. Corollary 2.6 illustrates this situation.

If $G$ and $H$ are locally continuously perfect groups then so is its product
$G \times H$.

For a compact manifold $M$ and a topological group $G$, let $C(M, G)$ stand for
the group of continuous maps $M \to G$ with the pointwise multiplication and
the compact-open topology. Note that $C(M, G)$ can be viewed as an analogue
of the current group (c.f. [10]).

**Proposition 3.2.** If $G$ is locally continuously perfect then so is $C(M, G)$ and
$r_{C(M, G)} = r_G$.

**Proof.** Let $S_i : U \to G$, $\tilde{S}_i : U \to G$, $i = 1, \ldots, r$, be as in Def. 1.1. Set $U = \{f \in C(M, G) : f(M) \subset U\}$ and define continuous maps $S^r_i : U \to C(M, G)$,
S_C^i : U → \mathcal{C}(M, G), i = 1, \ldots, r, by the formulae \( S_C^i(f)(x) = S_i(f(x)) \), where \( f \in \mathcal{C}(M, G) \), \( x \in M \), and similarly for \( S_C^i \). It follows that

\[
\prod_i [S_C^i(f), S_C^i(f)](x) = \prod_i [S_i(f(x)), \tilde{S}_i(f(x))] = f(x)
\]

for all \( x \in M \). Thus \( f = \prod_i [S_C^i(f), S_C^i(f)] \), as required. Observe that for all \( x \in M \) and \( f \in \mathcal{C}(M, G) \), if \( f(x) = e \) then \( S_C^i(f)(x) = e \) for all \( i \). \( \square \)

For a topological group \( G \) we denote by \( \mathcal{P}G = \{ f : I → G : f(0) = e \} \) the path group.

**Corollary 3.3.** Theorems 1.2 and 1.3 hold for \( \mathcal{P}H(M) \). In other words, these theorems are true for isotopies.

Let \( G \) be a compact Lie group. Given a principal \( G \)-bundle \( p : M → B_M \) the gauge group \( \text{Gau}(M) \) is the group of all \( G \)-equivariant mappings of \( M \) over \( \text{id}_{B_M} \). That is, \( \text{Gau}(M) \) is the space of \( G \)-equivariant mappings \( \mathcal{C}(M, (G, \text{conj}))^G \). It follows that \( \text{Gau}(M) \) identifies with \( \mathcal{C}(B_M ↷ M[G, \text{conj}]) \), the space of sections of the associated bundle \( M[G, \text{conj}] \). Consequently, any \( f \in \text{Gau}(M) \) in a trivialization of \( p \) over \( B = B_i ⊂ B_M \) identifies with a mapping \( f^{(i)} : B_i → G \) such that \( f(x) = x.f^{(i)}(p(x)) \).

**Proposition 3.4.** Let \( G \) be a compact Lie group and let \( p : M → B_M \) be a principal \( G \)-bundle with \( B_M \) compact. Then \( \text{Gau}(M) \) is locally continuously perfect provided \( G \) is so, and \( r_{\text{Gau}(M)} = d_Mr_G \) where \( d_M \) is as in Theorem 1.2.

**Proof.** Let \( (B_i)_{i=1}^d \) be a covering of \( M \) by balls. Choose another covering by balls \( (B'_i)_{i=1}^d \) with \( \text{cl}(B'_i) ⊂ B_i \) for all \( i \). We identify \( \mathcal{L}(G) \), the Lie algebra of \( G \), with \( \mathbb{R}^q, q = \dim G \), by means of a basis \( (X_1, \ldots, X_q) \) of \( \mathcal{L}(G) \). Let \( \Phi : \mathbb{R}^q ⊃ V → U ⊂ G \) be a chart given by \( \Phi(t_1, \ldots, t_q) = (\exp t_1X_1)\ldots(\exp t_qX_q) \).

Suppose that \( h \in \text{Gau}(M) \) is so small that the image of \( h^{(1)} \) is in \( U \). We let \( \tilde{h}^{(1)} = \Phi^{-1} \circ h^{(1)} \). By using bump functions for \( (B'_1, B_1) \) we modify \( \tilde{h}^{(1)} \) and compose the resulting map with \( \Phi \). Consequently, we get \( g_1 \in \text{Gau}(M) \) such that \( \text{supp}(g_1) ⊂ p^{-1}(B_1) \) and \( g_1^{(1)} = h^{(1)} \) on \( B'_1 \). Moreover, \( g_1 \) depends continuously on \( h \). Now take \( f_1 = g_1^{-1}h \). Then \( f_1 \in \text{Gau}(M_1) \), where \( M_1 = M \setminus B'_1 \) is a compact manifold with boundary endowed with the coverings \( (B_i \cap M_1)_{i=1}^d \) and \( (B'_i \cap M_1)_{i=1}^d \). Note that \( f_1 = \text{id} \) on \( \partial B'_1 \). Taking possibly smaller \( h \) we may continue the procedure. Finally, we get a neighborhood \( U \) of \( e \in \text{Gau}(M) \) such that for all \( h \in U \) we get a uniquely determined decomposition \( h = h_1 \ldots h_d \) with \( \text{supp}(h_i) ⊂ p^{-1}(B_i) \) and \( h_i \) depending continuously on \( h \) for all \( i \).

Thus, possibly shrinking \( U \), in view of Proposition 3.2 the claim follows. \( \square \)
4. Remarks on conjugation-invariant norms

The notion of the conjugation-invariant norm is a basic tool in studies on the structure of groups. Let $G$ be a group. A conjugation-invariant norm (or norm for short) on $G$ is a function $\nu : G \rightarrow [0, \infty)$ which satisfies the following conditions. For any $g, h \in G$

(1) $\nu(g) > 0$ if and only if $g \neq e$;
(2) $\nu(g^{-1}) = \nu(g)$;
(3) $\nu(gh) \leq \nu(g) + \nu(h)$;
(4) $\nu(hgh^{-1}) = \nu(g)$.

Recall that a group is called bounded if it is bounded with respect to any bi-invariant metric. It is easily seen that $G$ is bounded if and only if any conjugation-invariant norm on $G$ is bounded.

Let us introduce the following norm.

**Definition 4.1.** Let $G$ be a connected topological group and let $U$ be a neighborhood of $e \in G$.

(1) By $\tilde{U}$ we denote the "saturation" of $U$ w. r. t. $\text{conj}_g$ for $g \in G$ and the inversion $i$, that is $\tilde{U} = \bigcup_{g \in G} gUg^{-1} \cup gU^{-1}g^{-1}$. Then for $g \in G$, $g \neq e$, by $\mu^U(g)$ we denote the smallest $s \in \mathbb{N}$ such that $g = g_1 \ldots g_s$ with $g_i \in \tilde{U}$ for $i = 1, \ldots, s$. It is easily seen that $\mu^U$ is a conjugation-invariant norm.

(2) We say that $G$ is continuously decomposable with respect to $U$ if there are $s \in \mathbb{N}$ and continuous mappings $\varphi_i : G \rightarrow \tilde{U}$, $i = 1, \ldots, s$, such that $g = \varphi_1(g) \ldots \varphi_s(g)$ for all $g \in G$. In particular, $\mu^U(G) \leq s$.

Clearly if $U \subset V$ then $\mu^V \leq \mu^U$.

It is straightforward that if $G$ is locally continuously perfect and continuously decomposable w.r.t. $U$ as in Def. 1.1, then $G$ is continuously perfect. Likewise, if $G \subset H(M)$ satisfies Def. 2.3 and if $G$ is continuously decomposable w.r.t. $\mathcal{P}$ as in Def. 2.3, then $G$ is continuously factorizable. However, we do not have any example of a homeomorphism group being continuously decomposable. Notice that, in view of Lemma 2.4, for any two balls $B$ and $U$ in $M$ with $\text{cl}(B) \subset U$ the group $H_B(M)$ is continuously perfect "in the group $H_U(M)$", but not in itself.

Recall now two classical examples of conjugation-invariant norms.

For $g \in [G, G]$ by the commutator length of $g$, $\text{cl}_G(g)$, we mean the smallest $r$ as in (1.1). Observe that the commutator length $\text{cl}_G$ is a norm on $[G, G]$. In particular, if $G$ is a perfect group then $\text{cl}_G$ is a norm on $G$.

A subgroup $G \subset H(M)$ is factorizable if for every $g \in G$ there are $g_1, \ldots, g_d \in G$ with $\text{supp}(g_i) \subset B_i$, where each $B_i$ is a ball or a half-ball. Clearly any connected $G$ satisfying Def. 2.3 with respect to a family of balls is factorizable.
If \( G \) is factorizable then we may introduce the following \textit{fragmentation norm} \( \text{frag}_G \) on \( G \). For \( g \in G, g \neq \text{id} \), we define \( \text{frag}_G(g) \) to be the least integer \( d > 0 \) such that \( g = g_1 \ldots g_d \) with \( \text{supp}(g_i) \subseteq B_i \) for some ball or half-ball \( B_i \).

Let \( \nu \) be a conjugation-invariant norm on a topological group \( G \). Then \( \nu \) is called \textit{locally bounded with respect to} \( \nu \) if there are \( r \in \mathbb{N} \) and a symmetric neighborhood \( U \) of \( e \in G \) (i.e. \( U = U^{-1} \)) such that \( \nu(g) \leq r \) for all \( g \in U \).

The following obvious fact can be applied to \( \text{cl}_G \) or to \( \text{frag}_G \).

**Corollary 4.2.** Let \( G \) be a subgroup of \( \mathcal{H}(M) \). If \( G \) is locally bounded w. r. t. \( \nu \) and the norm \( \mu^\nu \) (Def. 4.1) is bounded where \( U \) is as above, then \( G \) is bounded w. r. t. \( \nu \).

5. Proofs of Theorems 1.2 and 1.3

The proofs depend on the deformation properties for the spaces of imbeddings obtained by Edwards and Kirby in [5]. See also Siebenmann [20]. First let us recall some notions and the main theorem of [5]. From now on \( M \) is a metrizable topological manifold and \( I = [0, 1] \). If \( U \) is a subset of \( M \), a \textit{proper imbedding} of \( U \) into \( M \) is an imbedding \( h : U \to M \) such that \( h^{-1}(\partial M) = U \cap \partial M \). An \textit{isotopy} of \( U \) into \( M \) is a family of imbeddings \( h_t : U \to M, t \in I \), such that the map \( h : U \times I \to M \) defined by \( h(x, t) = h_t(x) \) is continuous. An isotopy is \textit{proper} if each imbedding in it is proper. Now let \( C \) and \( U \) be subsets of \( M \) with \( C \subseteq U \). By \( I(U, C; M) \) we denote the space of proper imbeddings of \( U \) into \( M \) which equal the identity on \( C \), endowed with the compact-open topology.

Suppose \( X \) is a space with subsets \( A \) and \( B \). A \textit{deformation of} \( A \) \textit{into} \( B \) is a continuous mapping \( \varphi : A \times I \to X \) such that \( \varphi|_{A \times 0} = \text{id}_A \) and \( \varphi(A \times 1) \subseteq B \). If \( \mathcal{P} \) is a subset of \( I(U; M) \) and \( \varphi : \mathcal{P} \times I \to I(U; M) \) is a deformation of \( \mathcal{P} \), we may equivalently view \( \varphi \) as a map \( \varphi : \mathcal{P} \times I \times U \to M \) such that for each \( h \in \mathcal{P} \) and \( t \in I \), the map \( \varphi(h, t) : U \to M \) is a proper imbedding.

If \( W \subseteq U \), a deformation \( \varphi : \mathcal{P} \times I \to I(U; M) \) is \textit{modulo} \( W \) if \( \varphi(h, t)|_W = h|_W \) for all \( h \in \mathcal{P} \) and \( t \in I \).

Suppose \( \varphi : \mathcal{P} \times I \to I(U; M) \) and \( \psi : \mathcal{Q} \times I \to I(U; M) \) are deformations of subsets of \( I(U; M) \) and suppose that \( \varphi(\mathcal{P} \times 1) \subseteq \mathcal{Q} \). Then the \textit{composition} of \( \psi \) with \( \varphi \), denoted by \( \psi \star \varphi \), is the deformation \( \psi \star \varphi : \mathcal{P} \times I \to I(U; M) \) defined by

\[\psi \star \varphi(h, t) = \begin{cases} 
\varphi(h, 2t) & \text{for } t \in [0, 1/2]\n\psi(\varphi(h, 1), 2t - 1) & \text{for } t \in [1/2, 1].\n\end{cases}\]

The main result in [5] is the following

**Theorem 5.1.** Let \( M \) be a topological manifold and let \( U \) be a neighborhood in \( M \) of a compact subset \( C \). For any neighborhood \( \mathcal{Q} \) of the inclusion \( i : U \subset M \) in \( I(U; M) \) there are a neighborhood \( \mathcal{P} \) of \( i \in I(U; M) \) and a deformation
Proposition 5.2. Let $M$ be compact and let $(U_i)_{i=1}^d$ be an open cover of $M$. Then there exist $P$, a neighborhood of the identity in $P(M)$, and continuous mappings $\sigma_i : P \to P(M)$, $i = 1, \ldots, d$, such that $h = \sigma_1(h) \cdots \sigma_d(h)$ and supp$(\sigma_i(h)) \subset U_i$ for all $i$ and all $h \in P$; that is $P(M)$ satisfies Def. 2.3.

Proof. (See also [5].) First we have to shrink the cover $(U_i)_{i=1}^d$ $d$ times, that is we choose an open $U_{i,j}$ for every $i = 1, \ldots, d$ and $j = 0, \ldots, d$ with $U_{i,0} = U_i$ such that $\bigcup_{j=1}^d U_{i,j} = M$ for all $j$ and such that $\text{cl}(U_{i,j+1}) \subset U_{i,j}$ for all $i, j$. We make use of Theorem 5.1 $d$ times with $q = 1$. Namely, for $i = 1, \ldots, d$ we have a neighborhood $P_i$ of the identity in $I(M, \bigcup_{\alpha=1}^{\alpha-1} U_{\alpha,i-1}; M)$ and a deformation $\varphi_i : P_i \times I \to P(M)$ which is modulo $M \setminus U_{i,0}$ and which takes its values in $I(M, \bigcup_{\alpha=1}^{\alpha-1} \text{cl}(U_{\alpha,i}); M)$ and such that $\varphi_i(id, t) = id$ for all $t$. Here we apply Theorem 5.1 with $C = \text{cl}(U_{i,i})$, $U = U_{i,0}$, $D_1 = \bigcup_{\alpha=1}^{\alpha-1} \text{cl}(U_{\alpha,i})$ and $V_1 = \bigcup_{\alpha=1}^{\alpha-1} U_{\alpha,i-1}$. Taking a neighborhood $P$ of $id$ small enough, we have that $\varphi_d \cdots \varphi_1$ restricted to $P \times I$ is well defined. For every $h \in P$ we set $h_0 = h$ and $\varphi_i = \varphi_1 \cdots \varphi_1(h, 1)$, $i = 1, \ldots, d$. It follows that $h_d = id$ and $h = \prod_{i=1}^d h_i h_{i-1}^{-1}$. It suffices to define $\sigma_i : P \to P(M)$ by $\sigma_i(h) = h_i h_{i-1}^{-1}$ for all $i$.

Proof of Theorem 1.2. Choose any finite cover $(U_i)_{i=1}^d$ of $M$ by balls and half-balls. Next, fix another cover of $M$ by balls and half-balls $(B_i)_{i=1}^d$ with $\text{cl}(B_i) \subset U_i$ for all $i$. Then apply Proposition 5.2 to $(B_i)_{i=1}^d$, Lemma 2.4 and Corollary 2.7(1) to each couple $(B_i, U_i)$.

Recall the notion of the graph topology. Let $X$ and $Y$ be Hausdorff spaces and let $P(X,Y)$ be the space of all continuous mappings $X \to Y$. For $f \in P(X,Y)$ by graph$_f : X \to X \times Y$ we denote the graph mapping. The graph topology on $P(X,Y)$ is given by the basis of all sets of the form $\{f \in P(X,Y) : \text{graph}_f(X) \subset U\}$, where $U$ runs over all open sets in $X \times Y$. The graph topology is Hausdorff since it is finer than the compact-open topology. If $X$ is paracompact and $(Y, d)$ is a metric space then for $f \in P(X,Y)$ one has a basis of neighborhoods of the form $\{g \in P(X,Y) : d(f(x), g(x)) < \varepsilon(x), \forall x \in X\}$, where $\varepsilon$ runs over all positive continuous functions on $X$.

Proof of Theorem 1.3. In view of the assumption $M$ is the interior of a compact, connected manifold $M$ with non-empty (not necessarily connected)
boundary $\partial$. Then $\partial$ admits a collar neighborhood, that is an open subset $P$ of $M$, where $P = \partial \times (0, 1)$. Here $\partial \times [0, 1]$ is imbedded in $M$, and $\partial \times \{1\}$ is identified with $\partial$.

Take a finite family of balls $(B_i)_{i=1}^d$ in $M$ and a collar neighborhood $P$ of $\partial$ such that $M = \bigcup_i B_i \cup P$. We wish to check that $\mathcal{H}(M)$ fulfils Def. 2.3 for $\{B_1, \ldots, B_d, P\}$. We define balls $B_{i,j}$ for every $i = 1, \ldots, d$ and $j = 0, \ldots, d$ with $U_{i,0} = U_i$ such that $\bigcup_{i=1}^d U_{i,j} \cup P = M$ for all $j$ and such that $\text{cl}(U_{i,j+1}) \subset U_{i,j}$ for all $i,j$. Now proceeding like in the proof of Prop.5.2 there exist a neighborhood $\mathcal{P}$ of $\text{id} \in \mathcal{H}(M)$ and continuous mappings $\sigma_i : \mathcal{P} \to \mathcal{H}(M)$, where $i = 0, \ldots, d$, such that for all $h \in \mathcal{P}$ we have

$$h = \sigma_0(h)\sigma_1(h) \ldots \sigma_d(h), \quad \text{supp}(\sigma_0(h)) \subset P, \quad \text{supp}(\sigma_i(h)) \subset B_i$$

for $i = 1, \ldots, d$. Fix a sequence of reals from $(0, 1)$

$$\bar{a}_1 < \bar{a}_1 < a_1 < b_1 < \bar{b}_1 < \bar{b}_1 < \bar{b}_1 < \cdots < \bar{b}_k < \bar{b}_k < \cdots$$

tending to 1. For $j = 1, 2, \ldots$, let $C_j = \partial \times [a_j, b_j], \quad V_j = \partial \times (\bar{a}_j, \bar{b}_j)$ and $U_j = \partial \times (a_j, b_j)$. In view of Theorem 5.1, for every $j$ there exist a neighborhood $\mathcal{P}_j$ of the inclusion $i_j : C_j \subset U_j$ in $I(U_j; M)$ and a deformation $\varphi_j : \mathcal{P}_j \times I \to I(U_j, C_j; M)$ which is modulo of $M \setminus V_j$ and such that $\varphi(i_j, t) = i_j$ for all $t \in I$.

Shrinking $\mathcal{P}$ if necessary, for any $h \in \mathcal{P}$ we may have that $\sigma_0(h)|_{U_j} \in \mathcal{P}_j$ for all $j$. Put $U = \bigcup U_j, \quad V = \bigcup V_j$ and let $D = \bigcup \partial \times I_j$, where $I_j$ is an arbitrary open interval with $\text{cl}(I_j) \subset (a_j, b_j)$ for all $j$. Therefore there are a neighborhood $\mathcal{P}$ of $\text{id} \in \mathcal{H}(M)$ in the graph topology and a continuous mapping $\sigma_0^1 : \mathcal{P} \to \mathcal{H}(M)$ given by

$$\sigma_0^1(h)|_{U_j} = \varphi_j(\sigma_0(h)|_{U_j}), \quad j = 1, 2, \ldots,$$

and $\sigma_0^1(h)|_{M \setminus U} = \sigma_0(h)|_{M \setminus U}$. It follows that $\sigma_0^1(h) = \sigma_0(h)$ on $M \setminus V$ and that $\text{supp}(\sigma_0^1(h)) \subset M \setminus \text{cl}(D)$. Set $\sigma_0^2 = (\sigma_0^1)^{-1} \sigma_0$. Then $\sigma_0^2 : \mathcal{P} \to \mathcal{H}(M)$ is continuous, and $\sigma_0 = \sigma_0^1 \sigma_0^2$ with $\text{supp}(\sigma_0^2) \subset U$. Thus we get a decomposition $h = \sigma_0^1(h)\sigma_0^2(h)\sigma_1(h) \ldots \sigma_d(h)$ for all $h \in \mathcal{P}$. By applying Lemma 2.4 to $\sigma_i$ and Corollary 2.7(2) to $\sigma_0^2$, the claim follows.

6. THE CASE OF $G$-EQUIVARIANT HOMEOMORPHISMS

Let $G$ be a compact Lie group acting on $M$. Let $\mathcal{H}_G(M)$ be the group of all equivariant homeomorphisms of $M$ which are isotopic to the identity through compactly supported equivariant isotopies. Suppose now that $G$ acts freely on $M$. Then $M$ can be regarded as the total space of a principal $G$-bundle $p : M \to B_M = M/G$ (c.f. [3]).

Let $\mathcal{C}_c(\mathbb{R}^m)$ (resp. $\mathcal{C}_c^o(\mathbb{R}^m)$) denote the space of continuous maps $u : \mathbb{R}^m \to \mathbb{R}$ with compact support (resp. contained in $B$). Consider the semi-direct product group $\mathcal{H}(\mathbb{R}^m) \times \tau \mathcal{C}_c(\mathbb{R}^m)$, where $\tau_h(u) = u \circ h^{-1}$ for $h \in \mathcal{H}(\mathbb{R}^m)$ and
Let $u \in C_c(\mathbb{R}^m)$. Then we have
\[(h_1, u_1) \cdot (h_2, u_2) = (h_1 \circ h_2, u_1 \circ h_2^{-1} + u_2)\]
for all $h_1, h_2 \in H(\mathbb{R}^m)$ and $u_1, u_2 \in C_c(\mathbb{R}^m)$. For $(h, u) \in H(\mathbb{R}^m) \times \tau C_c(\mathbb{R}^m)$ we have $(h, u) = (h, 0) \cdot (\text{id}, u) = (\text{id}, u_1) \cdot (h, 0)$, where $u_1 = u \circ h$. We may treat $h, u$ as elements of $H(\mathbb{R}^m) \times \tau C_c(\mathbb{R}^m)$.

The main lemma in [17] (Lemma 2.1), which has an elementary but rather sophisticated proof, can be reformulated for our purpose as follows.

**Lemma 6.1.** Let $B$ be a ball in $\mathbb{R}^m$. There are homeomorphisms $\varphi^-, \varphi^+, \psi^-, \psi^+$ from $H(\mathbb{R}^m)$, depending on $B$, and continuous mappings
\[
v_1^-, v_1^+, v_2^-, v_2^+: C_B(\mathbb{R}^m) \to C_c(\mathbb{R}^m)
\]
such that
\[
u = [\varphi^-, v_1^-(u)]^{-1} [\varphi^+, v_1^+(u)]^{-1} [\psi^-, v_2^-(u)][\psi^+, v_2^+(u)]
\]
for all $u \in C_B(\mathbb{R}^m)$ in the semi-direct product group $H(\mathbb{R}^m) \times \tau C_c(\mathbb{R}^m)$.

In view of Lemma 6.1 we have

**Theorem 6.2.** If $B_M$ is compact then the group $H_G(M)$ is locally continuously perfect. Moreover, $r_{H_G(M)} \leq (4 \dim G + 1)d_{BM}$.

**Proof.** Let $(B_i)^d_{i=1}$ be a covering by balls of $B_M$. Let $P : H_G(M) \to H(B_M)$ be the homomorphism given by $P(h)(p(x)) = p(h(x))$, where $x \in M$. Let $h \in U$, where $U$ is a neighborhood of id $\in H_G(M)$. Then for small enough $P(h)$ can be decomposed as $P(h) = g_1 \cdots g_d$ such that $g_i \in H_B(B_M)$, $i = 1, \ldots, d$. Then each $g_i$ can be lifted to $h_i \in H_G(M)$, i.e. $P(h_i) = g_i$. Thus, due to Theorem 1.2 it suffices to consider $f = hh_1^{-1} \cdots h_d^{-1} \in \ker P = \text{Gau}(M)$.

Proceeding as in the proof of Prop. 3.4, we can write $f = f_1 \cdots f_d$ where $f_i \in \text{Gau}(M)$ and $\text{supp}(f_i) \subset p^{-1}(B_i)$. Shrinking $U$ we assume that $f_i \in H(\mathbb{R}^m) \times \tau C_c(\mathbb{R}^m, \mathbb{R}^q)$ for all $i$, where $q = \dim G$. We can extend the semi-direct product structure from $H(\mathbb{R}^m) \times \tau C_c(\mathbb{R}^m)$ to $H(\mathbb{R}^m) \times \tau C_c(\mathbb{R}^m, \mathbb{R}^q)$, where $C_c(\mathbb{R}^m, \mathbb{R}^q)$ is the space of compactly supported $\mathbb{R}^q$-valued functions, by the formulae $(h, (v_1, \ldots, v_q)) = (\text{id}, (v_1, \ldots, v_q) \circ h) \cdot (h, 0)$ and $(\text{id}, (v_1, \ldots, v_q)) = (\text{id}, v_1) \cdots (\text{id}, v_q)$. In view of Lemma 6.1, each $(\text{id}, v_i)$ is written as a product of four commutators from $H(\mathbb{R}^m) \times \tau C_c(\mathbb{R}^m, \mathbb{R}^q)$ with factors depending continuously on $f$. This completes the proof. \hfill \Box

**Corollary 6.3.** Let $M$ be a topological $G$-manifold with one orbit type. Then $H_G(M)$ is a locally continuously perfect group.

**Proof.** Indeed, if $H$ is the isotropy group of a point of $M$ then $M^H = \{x \in M : H \text{ fixes } x\}$ is a free $N^G(H)/H$-manifold, where $N^G(H)$ is the normalizer of $H$ in $G$. Since $H_G(M)$ is isomorphic and homeomorphic to $H_{N(H)/H}(M^H)$, the corollary follows from Theorem 6.2.
To explain the relation $H_G(M) \cong H_{N/H}(M^H)$, recall basic facts on the $G$-spaces with one orbit type (see, Bredon [3], section II, 5). Let $G$ a compact Lie group and let $X$ be a $T_{34} G$-space with one orbit type $G/H$ (that is, all isotropy subgroups are conjugated to $H$). Set $N = N^G(H)$ and $X^H = \{ x \in X : h.x = x, \forall h \in H \}$. Then we have the homeomorphism $G \times X \ni [g, x] \mapsto g(x) \in X$. That is, the total space of the bundle over $G/N$ with the standard fiber $X^H$ associated to the principal $N$-bundle $G \to G/N$ is $G$-equivalent to $X$. In particular, the inclusion $X^H \subset X$ induces a homeomorphism $X^H/N \cong X/G$.

Denote $K = N/H$. Given an arbitrary $G$-space $Y$, there is a bijection $\kappa_{X,Y}$ between $G$-equivariant mappings $X \to Y$ and $K$-equivariant mappings $X^H \to Y^H$ such that $\kappa_{X,Y}(f) = f|_{X^H}$.

Notice that $K$ acts freely on $X^H$ and the homeomorphism $X^H/N \cong X/G$ induces the homeomorphism $X^H/K \cong X/G$. In particular, we get the principal $K$-bundle $\pi_X : X^H \to X/G$, where $\pi_X$ is the restriction to $X^H$ of the projection $\pi : X \to X/G$. □

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