Transformation Laws for Theta functions

Chongying Dong\textsuperscript{1} and Geoffrey Mason\textsuperscript{2}
Department of Mathematics, University of California, Santa Cruz, CA 95064

1 Introduction

We prove some results which extend the classical theory, due to Hecke-Schoeneberg [H], [S1], of the transformation laws of theta functions. Although our results are classical in nature, they were suggested by recent work involving modular-invariance in conformal field theory [DMN], and we shall say more about these connections in the last section of the present paper.

Let \( Q \) be a positive-definite, integral quadratic form of even rank \( f = 2r \) with theta-function

\[
\theta(Q, \tau) = \sum_{m \in \mathbb{Z}^f} e^{2\pi i Q(m) \tau}.
\]

(1.1)

One knows [S2] that \( \theta(Q, \tau) \) is a modular form of weight \( r \) and character \( \epsilon \) on the group \( \Gamma_0(N) \). Here \( N \) is the level of \( Q \) and \( \epsilon \) is the Dirichlet character given by the Jacobi symbol

\[
\epsilon(n) = \left( \frac{(-1)^r \det A}{n} \right)
\]

for \( n > 0 \), with \( A \) a Gram matrix of the bilinear form \( \langle , \rangle \) corresponding to \( Q \).

We fix a vector \( v \in \mathbb{C}^f \) and define

\[
\theta(Q, v, k, \tau) = \sum_{m \in \mathbb{Z}^f} \langle v, m \rangle^k e^{2\pi i Q(m) \tau}
\]

(1.2)

where \( k \) is a nonnegative integer. Obviously, \( \theta(Q, v, k, \tau) \) identically zero if \( k \) is odd, and coincides with \( \theta(Q, \tau) \) if \( k = 0 \).

We define

\[
\Theta(Q, v, \tau, X) = \sum_{n \geq 0} \frac{2^n \theta(Q, v, 2n, \tau)}{(2n)!} (2\pi i X)^n
\]

(1.3)

regarding this as a function on \( \mathfrak{h} \times \mathbb{C} \) (\( \mathfrak{h} \) is the complex upper half-plane) for fixed \( Q \) and \( v \).

Our main result may then be stated as follows:

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Theorem 1 \( \Theta(Q, v, \tau, X) \) satisfies the following transformation law for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in \( \Gamma_0(N) \):

\[
\Theta(Q, v, \gamma \tau, \frac{X}{(c\tau + d)^2}) = \epsilon(d)(c\tau + d)^r \exp \left( \frac{c(v, v)X}{c\tau + d} \right) \Theta(Q, v, \tau, X).
\] (1.4)

It is evident from (1.2) that scaling \( v \) (i.e., replacing \( v \) by \( \lambda v \) for a scalar \( \lambda \)) simply multiplies \( \theta(Q, v, k, \tau) \) by \( \lambda^k \). For this reason, there are essentially only two cases, namely

(a) \( v \) is a null vector i.e., \( \langle v, v \rangle = 0 \), or
(b) \( v \) is a unit vector i.e., \( \langle v, v \rangle = 1 \).

Suppose first that \( v \) is a null vector. Then Hecke \([H]\) proved that the function \( P_k(x) = \langle v, x \rangle^k \) is a spherical harmonic of degree \( k \) with respect to \( Q \), moreover every spherical harmonic of degree \( k \) is a linear combination of such functions. Thus in this case (1.2) is simply a theta function with spherical harmonic \( \theta(Q, P_k, \tau) \), and

\[
\Theta(Q, v, \tau, X) = \sum_{n \geq 0} \frac{2^n \theta(Q, P_{2n}, \tau)}{(2n)!} (2\pi i X)^n. \tag{1.5}
\]

The transformation law (1.4) then says exactly that \( \theta(Q, P_{2n}, \tau) \) is a modular form on \( \Gamma_0(N) \) of weight \( r + 2n \) and character \( \epsilon \). This is the theorem of Schoeneberg \([S1]\).

Suppose next that \( v \) is a unit vector. Then the transformation law (1.4) says that \( \Theta(Q, v, \tau, X) \) is a (holomorphic) Jacobi-like form of weight \( r \), level \( N \), character \( \epsilon \) in the sense of Zagier \([Z]\).

When \( v \) is a unit vector, \( \theta(Q, v, k, \tau) \) will not be modular, but one can suitably combine two Jacobi-like forms to produce a sequence of modular forms (loc. cit.). For example, there is a holomorphic Jacobi-like form of weight 0 of particular interest, namely

\[
\tilde{E}_2(\tau, X) = \sum_{n \geq 0} (-1)^n \frac{E_2(\tau)^n}{n!} (2\pi i X)^n \tag{1.6}
\]

where

\[
E_2(\tau) = -\frac{1}{12} + 2 \sum_{n \geq 1} \sigma_1(n) q^n
\]

is the usual “unmodular” Eisenstein series of “weight” 2 normalized as indicated. Using this together with Theorem 1 yields

Theorem 2 Let notation be as before, and suppose that \( v \) is a unit vector. Set

\[
\gamma(t, k) = 2^{-t} \binom{k}{t} \binom{k-t}{t} t!
\] (1.7)
\[ \Psi(Q, v, 2k, \tau) = \sum_{t=0}^{k} \gamma(t, 2k) E_2(\tau)^t \theta(Q, v, 2k - 2t, \tau). \]  

(1.8)

Then \( \Psi(Q, v, 2k, \tau) \) is a holomorphic modular form on \( \Gamma_0(N) \) of weight \( 2k + r \) and character \( \epsilon \).

One knows that if \( k > 0 \) and \( v \) is a null vector then in fact \( \theta(Q, P_k, \tau) \) is a cusp form.

In the same spirit we have the following supplement to Theorem 2:

**Theorem 3**  
Let notation and assumptions be as in Theorem 2, and \( k \geq 2 \). Then

\[ \Psi(Q, v, 2k, \tau) - \gamma(k, 2k)(-\frac{1}{12})^k \theta(Q, \tau) E_{2k}(\tau) \]  

(1.9)

is a cusp form on \( \Gamma_0(N) \) of weight \( 2k + r \) and character \( \epsilon \). Here,

\[ E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \sigma_{k-1}(n)q^n \]

is the usual Eisenstein series.

It is forms of the general shape (1.9) that appear as partition function in certain conformal field theories [DMN], and whose existence led us to the results of the present paper.

The paper is organized as follows: in Section 2 we discuss Jacobi-like forms and the proof of Theorem 2. Section 3 is devoted to the proof of Theorems 1 and 3, which follows in general outline the original proof of Schoeneberg. In Section 4 we discuss connections with conformal field theory and state some further results which will be proved in [DMN].

### 2 Jacobi-like forms

We are interested in holomorphic functions \( \phi(\tau, X) \) on \( \mathfrak{h} \times \mathbb{C} \) of the form

\[ \phi(\tau, X) = \sum_{n \geq 0} \phi^{(n)}(\tau)(2\pi i X)^n \]  

(2.1)

and which satisfy

\[ \phi(\gamma \tau, \frac{X}{(c\tau + d)^2}) = \chi(d)(c\tau + d)^k \exp \left( \frac{cmX}{c\tau + d} \right) \phi(\tau, X) \]  

(2.2)

for some \( m \in \mathbb{C} \), integer \( k \) and Dirichlet character \( \chi(\text{mod } N) \), and for all \( \gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \). By scaling \( X \), the essential cases correspond to \( m = 0 \) and \( m = 1 \). As long as
\( \phi \) is holomorphic at the cusps, the case \( m = 0 \) means precisely that each \( \phi^{(n)}(\tau) \) is a holomorphic modular form on \( \Gamma_0(N) \) of weight \( k + 2n \) and character \( \chi \). The case \( m = 1 \) means that \( \phi \) is a holomorphic Jacobi-like form on \( \Gamma_0(N) \) of weight \( k \) and character \( \chi \) (cf. [Z]).

By way of examples, let \( \tilde{E}_2(\tau, X) \) be as in (1.4). The particular normalization of \( E_2(\tau) \) that we are using satisfies the well-known transformation law

\[
E_2(\gamma \tau) = (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i}.
\]

(2.3)

It follows that \( \tilde{E}_2(\tau, X) = \exp(-2\pi i E_2(\tau)X) \) satisfies

\[
\tilde{E}_2(\gamma \tau, \frac{X}{(c\tau + d)^2}) = \exp \left( -2\pi i \left( (c\tau + d)^2 E_2(\tau) - \frac{c(c\tau + d)}{2\pi i} \right) \frac{X}{(c\tau + d)^2} \right)
\]

\[
= \exp \left( \frac{cX}{c\tau + d} \right) \tilde{E}_2(\tau, X),
\]

so that \( \tilde{E}_2(\tau, X) \) is a holomorphic Jacobi-like form of level 1 and weight 0.

Now let \( \Theta(Q, v, \tau, X) \) be as in (1.3) with \( v \) a unit vector \( v \). Then \( \tilde{E}_2(\tau, -X) \Theta(Q, v, \tau, X) \) satisfies (2.2) with \( m = 0 \) (we are assuming the truth of Theorem 1 at this point). We have

\[
\tilde{E}_2(\tau, -X) \Theta(Q, v, \tau, X) = \sum_{k \geq 0} f^{(k)}(\tau)(2\pi i X)^k
\]

where

\[
f^{(k)}(\tau) = \frac{2^k}{(2k)!} \sum_{t=0}^{k} \gamma(t, 2k) E_2(\tau)^t \theta(Q, v, 2k - 2t, \tau)
\]

(2.4)

and \( \gamma(t, 2k) \) is as in (1.7). It follows, granting holomorphy at the cusps for now, that \( f^{(k)}(\tau) \) is a holomorphic modular form on \( \Gamma_0(N) \) of weight \( r + 2k \) and character \( \epsilon \). So Theorem 2 follows from Theorem 1.

There are other Jacobi-like forms that one could use in place of \( \tilde{E}_2(\tau, X) \) in order to construct modular forms involving the \( \theta(Q, v, n, \tau) \). For example, we could take the Cohen-Kuznetsov Jacobi-like form

\[
\sum_{n \geq 0} \frac{f^{(n)}(\tau)}{n!(n + k - 1)!} (2\pi i X)^n
\]

for \( f \) a modular form of weight \( k \), as described in [Z]. We will not pursue this possibility here: it is the forms \( \Psi(Q, v, 2k, \tau) \) of Theorem 2 that we need in [DMN].
3 Proofs of Theorem 1 and 3

In this section we present the proofs of Theorem 1 and 3. They follow in general outline the proof of Schoeneberg [S2]. We therefore adopt notation similar to (loc. cit.) and omit some details. In particular, we have \( \langle x, y \rangle = x'Ay \) for \( x, y \in \mathbb{C}^f \), where \( x' \) denotes the transpose of the column vector \( x \).

Let \( A \) and \( Q \) be as in Section 1. For \( x = (x_1, ..., x_f) \) and a scalar \( \lambda \) we set

\[
\theta_\lambda(A, x) = \sum_{m \in \mathbb{Z}^f} \exp(2\lambda Q(m + x)). \tag{3.1}
\]

For \( l = (l_1, ..., l_f) \in \mathbb{C}^f \) we let \( \mathcal{L} \) be the linear differential operator

\[
\mathcal{L} = \sum_{i=1}^f l_i \frac{\partial}{\partial x_i}. \tag{3.2}
\]

**Lemma 3.1** Let \( k \geq 0 \) be an integer. Then

\[
\mathcal{L}^k(\theta_\lambda(A, x)) = \sum_{i=0}^{[k/2]} \sum_{m \in \mathbb{Z}^f} \gamma(i, k)(2\lambda)^{k-i}(2Q(l))^i(l'A(m + x))^{k-2i} \exp(2\lambda Q(m + x)) \tag{3.3}
\]

where \( \gamma(i, k) \) is defined by (1.7).

**Proof:** One sees that there is an equality of the form

\[
\mathcal{L}^k(\theta_\lambda(A, x)) = \sum_{i=0}^{[k/2]} \sum_{m \in \mathbb{Z}^f} \gamma_\lambda(i, k)(2\lambda)^{k-i}(2Q(l))^i(l'A(m + x))^{k-2i} \exp(2\lambda Q(m + x)) \tag{3.4}
\]

for some scalars \( \gamma_\lambda(i, k), 0 \leq i \leq \lfloor k/2 \rfloor, k \geq 0 \). Setting \( \gamma_\lambda(i, k) = 0 \) for values of \( i \) and \( k \) not in these ranges, \( \gamma_\lambda(i, k) \) satisfies a recursion relation, namely

\[
\gamma_\lambda(i, k + 1) = (k + 2 - 2i)\gamma_\lambda(i - 1, k) - 2\lambda\gamma_\lambda(i, k), \gamma_\lambda(0, 0) = 1. \tag{3.5}
\]

We can solve the recursion, and find that \( \gamma_\lambda(i, k) = (2\lambda)^{k-i}\gamma(i, k) \). The lemma follows.

\[\Box\]

Now one knows (e.g. page 205 of [S2]) that the following transformation law holds: for \( \tau \) in the upper half-plane and for a suitable determination of the square root,

\[
\sum_{m \in \mathbb{Z}^f} \exp(2\pi i \tau Q(m + x)) = \theta_{\pi i \tau}(A, x)
\]

\[
= \frac{1}{(\sqrt{-i\tau})^f (\det A)^{1/2}} \sum_{m \in \mathbb{Z}^f} \exp\left(-\frac{\pi i}{\tau} m'A^{-1}m + 2\pi im'x\right). \tag{3.6}
\]
We apply the operator $L^k$ to both sides of (3.6), using Lemma 3.1, to obtain

$$\sum_{j=0}^{[k/2]} \sum_{m \in \mathbb{Z}} (2\pi i)^{k-j} \gamma(j, k)(2Q(l))^j (l' A(m + x))^{k-2j} \exp(2\pi i \tau Q(m + x))$$

$$= \frac{1}{(\sqrt{-i\tau})^{l'(\det A)^{1/2}}} \sum_{m \in \mathbb{Z}} (2\pi i m')^k \exp\left(-\frac{\pi i}{\tau} m'A^{-1}m + 2\pi i m'x\right). \quad (3.7)$$

Recall that $N$ is the level of $A$. Following [S2], we replace $x$ by $h/N$, $\tau$ by $-1/\tau$ and $m$ by $Am_1/N$ on the r.h.s. of (3.7). Remembering that $f = 2r$ and setting $D = \det A$, we get

$$\frac{(2\pi i)^{k-r} N^{-k}}{i^r \sqrt{D}} \sum_{m_1 \in \mathbb{Z}^f} (l' Am_1)^k \exp(2\pi i \tau Q(m_1)/N^2 + 2\pi i m_1 Ah/N^2)$$

$$= \sum_{j=0}^{[k/2]} \sum_{m \in \mathbb{Z}} N^{2j} \left(-\frac{2\pi i}{\tau}\right)^{k-j} \gamma(j, k)(2Q(l))^j (l' Am)^{k-2j} \exp\left(-\frac{2\pi i}{\tau} \frac{Q(m)}{N^2}\right). \quad (3.8)$$

Note that if $l$ is a null vector, only the term with $j = 0$ survives on the r.h.s. of (3.8), which then reduces to equation (12) of [S2], page 209.

We discuss some transformation formulas. For $h \in \mathbb{Z}^f$, $Ah \equiv 0 \pmod{N}$ and $0 \leq j \leq k$, we set

$$\theta(A, h, l, k, \tau) = \frac{1}{N^k} \sum_{m \in \mathbb{Z}} (l' Am)^k \exp(2\pi i \tau Q(m)/N^2). \quad (3.9)$$

It is also convenient to introduce

$$\Theta(A, h, l, k, j, \tau) = \frac{(-i)^{r+2k} \sqrt{D}}{\sqrt{D}} \exp(2\pi i g' Ah/N^2) \theta(A, g, l, k - 2j, \tau). \quad (3.10)$$

**Remark 3.2** (i) It is clear that $\Theta(A, h, l, k, j, \tau)$ depends only on $k - 2j$, rather than both $k$ and $j$. However it is convenient to keep the notation as it is.

(ii) Note that if $h = 0$ then $\theta(A, h, l, k, \tau)$ is just the function $\theta(A, l, k, \tau) = \theta(Q, l, k, \tau)$.

**Theorem 3.3** We have

$$\theta(A, h, l, k, -1/\tau) = \sum_{j=0}^{[k/2]} \left(\frac{Q(l)\tau}{\pi i}\right)^j \gamma(j, k)\Theta(A, h, l, k, j, \tau). \quad (3.11)$$
Proof: As $Ah \equiv 0 \mod N$, we can split-off an exponential factor from the l.h.s. of (3.8). Then using (3.9), (3.8) can be written in the form

$$
\frac{(2\pi i)^k}{i^r \sqrt{D}} \sum_{g \mod N} \exp(2\pi ig'Ah/N^2)\theta(A, g, l, k, \tau)
$$

$$
= \sum_{j=0}^{[k/2]} \left( \frac{-2\pi i}{\tau} \right)^{k-j} \gamma(j, k)(2Q(l))^j \theta(A, h, l, k - 2j, -1/\tau). \quad (3.12)
$$

We shall prove Theorem 3.3 by induction on $k$. If $k = 0$ it reduces to a standard transformation law (equation 17II of [S2], page 210). In the general case, (3.12) yields

$$
\theta(A, h, l, k, -1/\tau) = \frac{(-i)^{r+k} \tau}{\sqrt{D}} \sum_{g \mod N} \exp(2\pi ig'Ah/N^2)\theta(A, g, l, k, \tau)
$$

$$
- \sum_{j=1}^{[k/2]} \gamma(j, k) \left( \frac{-Q(l)\tau}{\pi i} \right)^j \theta(A, h, l, k - 2j, -1/\tau)
$$

$$
= \Theta(A, h, l, k, 0, \tau)
$$

$$
- \sum_{j=1}^{[k/2]} \gamma(j, k) \left( \frac{-Q(l)\tau}{\pi i} \right)^j \sum_{t=0}^{[(k-2j)/2]} \gamma(t, k - 2j) \left( \frac{Q(l)\tau}{\pi i} \right)^t \Theta(A, h, l, k - 2j, t, \tau).
$$

As $\Theta(A, h, l, k - 2j, t, \tau) = \Theta(A, h, l, k, t + j, \tau)$, we see that $\theta(A, h, l, k, -1/\tau)$ is equal to

$$
\Theta(A, h, l, k, 0, \tau) + \sum_{u=1}^{[k/2]} \left( \frac{Q(l)\tau}{\pi i} \right)^u \beta(u, k) \Theta(A, h, l, k, u, \tau) \quad (3.13)
$$

where

$$
\beta(u, k) = - \sum_{j=1}^{[k/2]} (-1)^j \gamma(j, k) \gamma(u - j, k - 2j). \quad (3.14)
$$

From (1.7) and (3.14) we see that

$$
\beta(u, k) = \gamma(u, k) - \sum_{j=0}^{[k/2]} (-1)^j \gamma(j, k) \gamma(u - j, k - 2j) = \gamma(u, k) - \sum_{j=0}^{[k/2]} (-1)^j \gamma(u, k) \left( \frac{u}{j} \right)
$$

i.e., $\beta(u, k) = \gamma(u, k)$. Now (3.13) implies the desired equality (3.11). □

We now proceed to our main transformation formula, which is the following:

**Theorem 3.4** If \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) lies in $\Gamma_0(N)$, and if $d > 0$, then

$$
(c\tau + d)^{-(r+k)}\theta(A, h, l, k, \frac{a\tau + b}{c\tau + d})
$$

$$
= \exp(2\pi iQ(h)ab/N^2)\epsilon(d) \sum_{j=0}^{[k/2]} \left( \frac{Q(l)c}{\pi i(c\tau + d)} \right)^j \gamma(j, k)\theta(A, bh, l, k - 2j, \tau). \quad (3.15)
$$
In particular, taking $h = 0$, if $d > 0$ then

$$(cτ + d)^{-(r+k)}θ(A, l, k, aτ + b/cτ + d) = ϵ(d) \sum_{j=0}^{[k/2]} \left( \frac{Q(l)c}{πi(cτ + d)} \right)^j γ(j, k)θ(A, l, k - 2j, τ). \quad (3.16)$$

We begin by noting that

$$θ(A, h, l, k, τ + 1) = \exp(2πiQ(h)/N^2)θ(A, h, l, k, τ), \quad (3.17)$$

and also if $c > 0$ then

$$θ(A, h, l, k, τ) = \sum_{g \mod cN \equiv h(N)} θ(cA, g, l, k, cτ). \quad (3.18)$$

$(3.17)$ is immediate from $(3.9)$ (remembering that $Ah = 0 \pmod{N}$); $(3.18)$ follows as in equation 18 of [S2], page 211.

Using $(3.17)$, $(3.18)$ and Theorem 3.3 we calculate for $γ = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ in $SL(2, \mathbb{Z})$ with $c > 0$ that

$$θ(A, h, l, k, γτ) = θ(A, h, l, k, c^{-1}(a - (cτ + d)^{-1}))$$

$$= \sum_{g \mod cN \equiv h(N)} θ(cA, g, l, k, a - (cτ + d)^{-1})$$

$$= \sum_{g \mod cN \equiv h(N)} \exp(2πiacQ(g)/c^2N^2)θ(cA, g, l, k, -(cτ + d)^{-1})$$

$$= \sum_{g \mod cN \equiv h(N)} \sum_{j=0}^{[k/2]} \exp(2πiaQ(g)/cN^2) \left( \frac{Q(l)c(cτ + d)}{πi} \right)^j γ(j, k)θ(cA, g, l, k, j, cτ + d)$$

$$= \sum_{g \mod cN \equiv h(N)} \sum_{j=0}^{[k/2]} \sum_{q \mod cN \equiv 0(cN)} \exp(2πiaQ(g)/cN^2) \left( \frac{Q(l)c(cτ + d)}{πi} \right)^j γ(j, k) \frac{(-i)^{r+2(k-2j)}}{\sqrt{cD}} \cdot (cτ + d)^{r+k-2j} \exp(2πig'Aq/cN^2)θ(cA, q, l, k - 2j, cτ + d)$$

$$= \frac{(cτ + d)^{r+k}(-i)^{r+2k}}{c^p√D} \sum_{g \mod cN \equiv h(N)} \sum_{q \mod cN \equiv 0(N)} \sum_{j=0}^{[k/2]} \exp(2πiaQ(g) + dQ(q) + g'Aq)/cN^2) \cdot \gamma(j, k) \left( \frac{cQ(l)}{(cτ + d)πi} \right)^j θ(cA, q, l, k - 2j, cτ) \quad (3.19)$$

where we have used the fact that $cA$ has level $cN$. 

8
Following [S2], page 213 we write

\[ \phi_{h,q} = \sum_{g \equiv h(N)} \sum_{g \equiv cN} \exp(2\pi i(aQ(g) + dQ(q) + g'Aq)/cN^2) \]  \hspace{1cm} (3.20)

and note that \( \phi_{h,q} \) depends on \( q \) only modulo \( N \). Then (3.19) can be put into the form

\[
\frac{(c\tau + d)^{r+k}(-i)^{r+2k}}{c^r \sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{q \equiv cN \atop A_q \equiv 0(N)} \phi_{h,q} \gamma(j, k) \left( \frac{cQ(l)}{(c\tau + d)\pi i} \right)^j \theta(cA, q, l, k - 2j, c\tau)
\]

\[
= \frac{(c\tau + d)^{r+k}(-i)^{r+2k}}{c^r \sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{q \equiv cN \atop A_q \equiv 0(N)} \phi_{h,q} \gamma(j, k) \left( \frac{cQ(l)}{(c\tau + d)\pi i} \right)^j \cdot \sum_{q \equiv cN \atop A_q \equiv 0(N), q \equiv q_1(N)} \theta(cA, q, l, k - 2j, c\tau).
\]

Another application of (3.18) now yields

**Lemma 3.5** Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \) with \( c > 0 \). Then

\[
(c\tau + d)^{-(r+k)} \theta(A, h, l, k; \gamma\tau)
\]

\[
= \frac{(-i)^{r+2k}}{c^r \sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{q \equiv cN \atop A_q \equiv 0(N)} \phi_{h,q} \gamma(j, k) \left( \frac{cQ(l)}{(c\tau + d)\pi i} \right)^j \theta(A, q, l, k - 2j, \tau). \]  \hspace{1cm} (3.21)

We continue the calculation, but now assuming also that \( d \equiv 0 \pmod{N} \). As in (loc. cit.) we now find that (3.21) can be written as follows:

\[
(c\tau + d)^{-(r+k)} \theta(A, h, l, k; \gamma\tau) = \frac{(-i)^{r+2k}}{c^r \sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{q \equiv cN \atop A_q \equiv 0(N)} \exp(-2\pi ih' A q_1 b / N^2) \cdot \gamma(j, k) \left( \frac{cQ(l)}{(c\tau + d)\pi i} \right)^j \theta(A, q_1, l, k - 2j, \tau). \]  \hspace{1cm} (3.22)

Replacing \( \tau \) by \(-1/\tau\) in (3.22) and using Theorem 3.3 again leads to

\[
\left( \frac{d\tau - c}{\tau} \right)^{-(r+k)} \theta(A, h, l, k; b\tau - a \frac{d\tau - c}{d\tau - c})
\]

\[
= \frac{(-i)^{r+2k}}{c^r \sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{q \equiv cN \atop A_q \equiv 0(N)} \exp(-2\pi ih' A q_1 b / N^2) \gamma(j, k). \]
where $\delta$ is the Kronecker delta and $g, bh$ are considered modulo $N$. Thus (3.23) reduces to

\[
(d\tau - c)^{-(r+k)}\theta(A, h, l, k; \frac{b\tau - a}{d\tau - c})
\]

\[
= \frac{(-1)^r \phi_{h,0}}{c^r \sqrt{D}} \sum_{j=0}^{[k/2]} \sum_{u=0}^{[(k-2j)/2]} \frac{\gamma(j + u, k)}{Aq_1 \equiv 0(N)} \cdot \exp(2\pi i(g - bh)Aq_1/N^2) \cdot \frac{(\frac{Q(l)}{\pi i\tau})^j}{\sqrt{D}}
\]

\[
\cdot (j + u) \left( \frac{c}{d\tau - c} \right) \frac{(Q(l)\tau)^j}{(d\tau - c)^j} \theta(A, g, l, k - 2(j + u), \tau).
\]
\[
\phi_{h,o} = \exp(2\pi i Q(h)ab/N^2)e(d).
\]

Now (3.24) implies all assertions of Theorem 3.4.

Finally, it is known (page 215 et seq of [S2]) that

\[
\theta(A, bh, l, k - 2t, \tau). \quad (3.24)
\]

which is the required (1.4) in the case \( d > 0 \). The general case follows easily.

Finally, we consider behavior at the cusps. This will lead to the proof of Theorem 3 as well as the proof of Theorem 2 initiated in Section 2.

To check the expansion of \( \Psi(Q, l, k, \tau) \) at the finite cups, we use Lemma 3.5. Thus for \( c > 0 \),

\[
\Psi(Q, l, k, \tau) = (c\tau + d)^{-\langle r+k \rangle} \Psi(Q, l, k, \frac{a\tau + b}{c\tau + d})
\]

\[
= \sum_{t=0}^{[k/2]} \gamma(t, k) (c\tau + d)^{-\langle r+k \rangle - 2t} E_2(a\tau + b) (c\tau + d)^{-\langle r+k-2t \rangle} \theta(A, l, k - 2t, \frac{a\tau + b}{c\tau + d})
\]

\[
= \sum_{t=0}^{[k/2]} \gamma(t, k) \left( E_2(\tau) - \frac{c}{2\pi i (c\tau + d)} \right)^t \left( -i \right)^{r+2(k-2t)} \left( \frac{c}{c^\prime \sqrt{D}} \right) \sum_{j=0}^{(k-2t)/2} \sum_{\substack{q \equiv 0 \mod N \quad Aq \equiv 0(N) \quad \phi_0, q}} \gamma(j, k - 2t) \left( \frac{cQ(l)}{\pi i (c\tau + d)} \right)^j \theta(A, q, l, k - 2t - 2j, \tau)
\]
\[
\frac{(-i)^{r+2k}}{c^r \sqrt{D}} \sum_{q \mod N} \sum_{t=0}^{[k/2]} \sum_{j=0}^{[(k-2t)/2]} \phi_{0,q} \gamma(j + t, k) \binom{j + t}{j} \left( \frac{c}{2\pi i(c\tau + d)} \right)^j .
\]
\[
(2Q(l))^j \left( E_2(\tau) - \frac{c}{2\pi i(c\tau + d)} \right)^{t} \theta(A, q, l, k - 2t - 2j, \tau)
\]
\[
= \frac{(-i)^{r+2k}}{c^r \sqrt{D}} \sum_{q \mod N} \phi_{0,q} \sum_{s=0}^{[k/2]} \gamma(s, k) E_2(\tau)^s \theta(A, q, l, k - 2s, \tau). \tag{3.25}
\]

This shows that \( \Psi(Q, l, k, \tau) \) is holomorphic at the cusps, and thus complete the proof of Theorem 2.

As for Theorem 3, the value of \( \Psi(Q, v, 2k, \tau) \) at \( i\infty \) is the same as that of the function
\[
\gamma(k, 2k) E_2(\tau)^k \theta(A, v, 0, \tau) = \gamma(k, 2k) E_2(\tau)^k \theta(Q, \tau)
\]
which has \( q \)-expansion \( \gamma(k, 2k)(-1/12)^k (1 + \cdots) \). Thus
\[
\Psi(Q, v, 2k, \tau) - \gamma(k, 2k)(-1/12)^k \theta(Q, \tau) E_{2k}(\tau)
\]
certainly vanishes at \( i\infty \). From (3.25), the value at a general cusp is
\[
\frac{(-i)^{r}}{c^r \sqrt{D}} \sum_{q \mod N} \phi_{0,q} \gamma(k, 2k)(-1/12)^k .
\]
But \( \frac{(-i)^{r}}{c^r \sqrt{D}} \sum_{q \mod N} \phi_{0,q} \) is the value of \( \theta(Q, \tau) \) at the same cusp (cf. equation (24) of [S2], page 213), whence it is clear that \( \Psi(Q, v, 2k, \tau) - \gamma(k, 2k)(-1/12)^k \theta(Q, \tau) E_{2k}(\tau) \)
vanishes at every cusp if \( k \geq 2 \). This completes the proof of Theorem 3.

4 Concluding comments

We have already mentioned that the previous results were motivated by conformal field theory, more precisely by the problem of calculating 1-point correlation functions for vertex operator algebras [DMN]. For earlier results in this direction, see [DLM] and [DM].

This perspective also enables us to prove the following result (see [DMN]): let the notation be as before, and suppose that the lattice \( \mathbb{Z}^l \) contains a root \( \alpha \) i.e., \( Q(\alpha) = 1 \). Then the cusp form of Theorem 3 (with \( v = \frac{\alpha}{\sqrt{2}} \)) is identically zero. That is, we have
\[
\theta(Q, \frac{\alpha}{\sqrt{2}}, 4, \tau) + 6 E_2(\tau) \theta(Q, \frac{\alpha}{\sqrt{2}}, 2, \tau) + 3 E_2(\tau)^2 \theta(Q, \tau) = \frac{1}{48} E_4(\tau) \theta(Q, \tau). \tag{4.1}
\]
It is interesting that in [Z], Zagier raises the question of whether there is a relation between Jacobi-like forms and vertex operator algebras. The present work together with [DMN] certainly suggests that this question continues to be one which is worth exploring.

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