ANALYSIS OF THE GRADIENT FOR THE STOCHASTIC FRACTIONAL HEAT EQUATION WITH SPATIALLY-COLORED NOISE IN $\mathbb{R}^d$

RAN WANG

Abstract: Consider the stochastic partial differential equation

$$\frac{\partial}{\partial t} u_t(x) = -(-\Delta)^{\frac{\alpha}{2}} u_t(x) + b(u_t(x)) + \sigma(u_t(x)) \dot{F}(t, x), \quad t \geq 0, x \in \mathbb{R}^d,$$

where $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with the power $\alpha/2 \in (1/2, 1]$, and the driving noise $\dot{F}$ is a centered Gaussian field which is white in time and with a spatial homogeneous covariance given by the Riesz kernel. We study the detailed behavior of the approximation spatial gradient $u_t(x) - u_t(x - \varepsilon e)$ at any fixed time $t > 0$, as $\varepsilon \downarrow 0$, where $e$ is the unit vector in $\mathbb{R}^d$. As applications, we deduce the law of iterated logarithm and the behavior of the $q$-variations of the solution in space.

Keyword: Stochastic heat equation; Fractional Brownian motion; Fractional Laplacian; Gradient estimates.

MSC: 60H15; 60G17.

1. Introduction

Consider the following stochastic fractional heat equation

$$\frac{\partial}{\partial t} u_t(x) = -(-\Delta)^{\frac{\alpha}{2}} u_t(x) + b(u_t(x)) + \sigma(u_t(x)) \dot{F}(t, x), \quad t \geq 0, x \in \mathbb{R}^d,$$

(1.1)

where $b$ and $\sigma$ are assumed to be Lipschitz continuous functions, $\alpha \in (1, 2]$ is a fixed “spatial scaling” parameter, $-(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian with the power $\alpha/2$, and the driving noise $\dot{F}$ is a centered Gaussian field which is white in time and with a spatial homogeneous covariance given by the Riesz kernel $f(x) = c_{1,1} \|x\|^{-(d-\gamma)}$ of the form:

$$\mathbb{E} \left[ \dot{F}(t, x) \dot{F}(s, y) \right] = c_{1,1} \delta_0(t - s) \|x - y\|^{-(d-\gamma)}, \quad 0 < \gamma < d,$$

(1.2)

where $\delta_0$ denotes the Dirac delta function and

$$c_{1,1} = \frac{2^{d-\gamma} \pi^{\frac{d}{2}} \Gamma ((d - \gamma)/2)}{\Gamma (\gamma/2)}.$$  

(1.3)

Throughout this paper, we assume that

Condition 1.1.  
(a) $\alpha \in (1, 2]$, $d \geq 1$ and $\gamma \in ((d - \alpha)_+, d)$. 

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(b) Let \( u_0 = \{ u_0(x) \}_{x \in \mathbb{R}^d} \) be a random field. Assume that there exist real numbers \( k_0 > \max\{ 2, 2/(2 + d - \alpha - \gamma) \} \), \( \eta_0 \in (\frac{\alpha - d + \gamma}{2}, 1] \) and \( c_{1,2} > 0 \) such that
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \left[ |u_0(x)|^{k_0} \right] < \infty \tag{1.4}
\]
and
\[
\mathbb{E} \left[ |u_0(x) - u_0(x + h)|^{k_0} \right] \leq c_{1,2} \| h \|_{\eta_0 k_0}, \tag{1.5}
\]
uniformly for all \( x, h \in \mathbb{R}^d \).

(c) We assume also that \( b \) and \( \sigma : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous, that is,
\[
|b(r) - b(v)| \leq L_b |r - v|, \quad |\sigma(r) - \sigma(v)| \leq L_\sigma |r - v| \quad \text{for } r, v \in \mathbb{R}. \tag{1.6}
\]

Under Condition 1.1, by the theory of Dalang [6], there exists a unique continuous solution of (1.1), satisfying that for any \( T > 0, k \in [2, k_0] \),
\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \mathbb{E} \left[ |u_t(x)|^k \right] < +\infty. \tag{1.7}
\]
See, for instance, [2] and references therein for details.

When \( d = 1 \) and \( F \) is the space-time white noise, Foondun et al. [11] studied the detailed behavior of the approximated spatial gradient \( u_t(x) - u_t(x - \varepsilon) \) at any fixed time \( t > 0 \), as \( \varepsilon \downarrow 0 \). In [11], the authors proved that
\[
u_t(x) - u_t(x - \varepsilon) \approx c_{1,3} \sigma(u_t(x)) \left[ B^H(x) - B^H(x - \varepsilon) \right],
\]
where \( c_{1,3} = (2\Gamma(\alpha)|\cos(\alpha\pi/2)|)^{-1/2} \), and \( B^H \) denotes a fractional Brownian motion (fBm, for short) with Hurst index \( H = (\alpha - 1)/2 \in (0, 1/2] \).

In this paper, we extend the result in [11] to the multidimensional situation with the time-white and space-colored noise. In this case, the solution will be related to the isotropic multiparameter fractional Brownian motion (also known as the Lévy fBm) \( \{ B^H(x) \}_{x \in \mathbb{R}^d} \), which is defined as a centered Gaussian process, starting from zero, with covariance function
\[
\mathbb{E} \left[ B^H(x) B^H(y) \right] = \frac{1}{2} \left( \| x \|^{2H} + \| y \|^{2H} - \| x - y \|^{2H} \right) \quad (x, y \in \mathbb{R}^d). \tag{1.8}
\]
Here, \( H \in (0, 1) \) is Hurst index.

Our main result is the following.

**Theorem 1.2.** Under Condition 1.1, for every fixed \( t > 0 \), there exists an isotropic multiparameter fBm \( \{ B^{(\alpha - d + \gamma)/2}(x) \}_{x \in \mathbb{R}^d} \), with Hurst index \( (\alpha - d + \gamma)/2 \) such that for all \( \lambda > 0 \),
\[
\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^d} \sup_{|e| = 1} \mathbb{P} \left( \left| \frac{u_t(x) - u_t(x - \varepsilon e)}{B^{(\alpha - d + \gamma)/2}(x) - B^{(\alpha - d + \gamma)/2}(x - \varepsilon e)} - c_{\alpha, \gamma, d} \sigma(u_t(x)) \right| > \lambda \right) = 0, \tag{1.9}
\]
where \( c_{\alpha, \gamma, d} \) is the following numerical constant:
\[
c_{\alpha, \gamma, d} = (2\pi)^{-d/2} \left( \int_{\mathbb{R}^d} |w|^{-(\alpha + \gamma)} (1 - \cos(w \cdot e)) dw \right)^{1/2}, \tag{1.10}
\]
with \( e \) being a unit vector in \( \mathbb{R}^d \).
By using the arguments in the proofs of Corollaries 1.2, 1.3 and 1.5 in [11], we can obtain the following results.

**Corollary 1.3.** Assume that Condition 1.1 holds. Choose and fix \( t > 0, \mathbf{x} \in \mathbb{R}^d \) and the unit vector \( \mathbf{e} \in \mathbb{R}^d \). Then with probability one,

\[
\lim_{\varepsilon \to 0} \sup_{i \in I^d} \frac{u_t(\mathbf{x}) - u_t(\mathbf{x} - \varepsilon \mathbf{e})}{\sqrt{2\varepsilon^{\alpha-d+\gamma} \log \log(1/\varepsilon)}} = c_{\alpha,\gamma,d} |\sigma(u_t(\mathbf{x}))|.
\] (1.11)

**Corollary 1.4.** Assume that Condition 1.1 holds. Choose and fix \( t > 0, \mathbf{x} \in \mathbb{R}^d \) and the unit vector \( \mathbf{e} \in \mathbb{R}^d \). Then for all \( a \in \mathbb{R} \),

\[
\lim_{\varepsilon \to 0} \sup_{i \in I^d} \mathbb{P} \left( \frac{u_t(\mathbf{x}) - u_t(\mathbf{x} - \varepsilon \mathbf{e})}{\varepsilon^{(\alpha-d+\gamma)/2}} \leq a \right) = \mathbb{P} \left( \left| \sigma(u_t(\mathbf{x})) \right| \times \tilde{\mathcal{N}} \leq \frac{a}{c_{\alpha,\gamma,d}} \right),
\] (1.12)

where \( \tilde{\mathcal{N}} \) denotes a standard Gaussian random variable, independent of \( u_t(\mathbf{x}) \).

Following the one-parameter case, we define the \( q \)-variation of the \( d \)-dimensional random field \( \{\xi(x)\}_{x \in \mathbb{R}^d} \) as the limit in probability, as \( n \to \infty \), of the sequence

\[
V_{[A_1,A_2]}^{n,q}(\xi) := \sum_{i=0}^{n-1} |\xi(x_{i+1}) - \xi(x_i)|^q,
\] (1.13)

where \( x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \) with \( x_i^{(j)} = A_1 + \frac{i}{n}(A_2 - A_1) \) for \( i = 0, 1, \ldots, n \) and \( j = 1, \ldots, d \). See, e.g., [12].

For any \( a \in \mathbb{R} \), denote \( \mathbf{a} := (a, \ldots, a) \in \mathbb{R}^d \).

**Corollary 1.5.** Assume that Condition 1.1 holds. If \( \varphi : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, then for all non random reals \( A_2 > A_1 \) and \( t > 0 \),

\[
\lim_{n \to \infty} \sum_{A_1, A_2} \varphi(u_t(x_j^{(n)})) \left| u_t(x_{j+1}^{(n)}) - u_t(x_j^{(n)}) \right|^{2/(\alpha-d+\gamma)}
= c_{1,d} \sqrt{\varphi(u_t(\mathbf{a}))} \sigma(u_t(\mathbf{a}))^{2/(\alpha-d+\gamma)} da,
\] (1.14)

almost surely and in \( L^2(\Omega, \mathbb{P}) \), where \( c_{1,d} = c_{\alpha,d,\gamma}^{2/(\alpha-d+\gamma)} \mathbb{E} \left| \tilde{\mathcal{N}} \right|^{2/(\alpha-d+\gamma)} \) with \( \tilde{\mathcal{N}} \) being a standard Gaussian random variable.

Particularly, taking \( \varphi \equiv 1 \), we get the \( 2/(\alpha - d + \gamma) \)-variation of \( \{u_t(x)\}_{x \in \mathbb{R}^d} \):

\[
\lim_{n \to \infty} V_{[A_1,A_2]}^{2n,2/(\alpha-d+\gamma)}(u_t) = \sqrt{d}c_{1,d} \int_{A_1}^{A_2} \sigma(u_t(\mathbf{a}))^{2/(\alpha-d+\gamma)} da,
\] (1.15)

almost surely and in \( L^2(\Omega, \mathbb{P}) \).

The rest of this paper is organized as follows. In Section 2, we first introduce the stochastic integral and give some facts about the linear stochastic heat equation taking from [12], then we prove the Hölder continuity of the solution. In Section 3, we give some estimates of the localization of the solution. Section 4 is devoted to prove the main result of this paper.
2. Preliminaries

2.1. Stochastic integral. We first define precisely the driving noise that appears in (1.1), which is borrowed from [8]. Let \( D_{\mathbb{R}^d} \) be the space of \( C^\infty \)-test functions with compact support. Then \( F = \{ F(\phi), \phi \in D(\mathbb{R}^{d+1}) \} \) is an \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \)-valued mean zero Gaussian process with covariance

\[
\mathbb{E} [F(\phi)F(\psi)] = c_{1,1} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \phi(r,y)\|y-z\|^{-(d-\gamma)}\psi(r,z). \tag{2.1}
\]

Using elementary properties of the Fourier transform (see [6]), this covariance can also be written as

\[
\mathbb{E} [F(\phi)F(\psi)] = \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} d\xi \|\xi\|^{-\gamma} \mathcal{F}\phi(r,\cdot)(\xi)\overline{\mathcal{F}\psi(r,\cdot)}(\xi), \tag{2.2}
\]

where \( \mathcal{F}f(\cdot)(\xi) \) denotes the Fourier transform of \( f \), that is,

\[
\mathcal{F}f(\cdot)(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} f(x) dx.
\]

Following Walsh [16] and Dalang [6], a rigorous formulation of (1.1) through the notion of mild solution as follows. Let \( M = \{ M_t(A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d) \} \) be the worthy martingale measure obtained as an extension of the process \( \tilde{F} \) as in Dalang and Frangos [7] (also see Dalang and Quer-Sardanyons [9]). Then a mild solution of (1.1) is jointly measurable \( \mathbb{R} \)-valued process \( u = \{ u(t,x) \}_{t \geq 0,x \in \mathbb{R}^d} \), adapted to the natural filtration generated by \( M \), such that

\[
\begin{align*}
u_t(x) = & \int_{\mathbb{R}^d} G^\alpha_t(x,y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} G^\alpha_{t-s}(x,y)b(u_s(y))dsdy \
& + \int_0^t \int_{\mathbb{R}^d} G^\alpha_{t-s}(x,y)\sigma(u_s(y))M(ds,dy), \tag{2.3}
\end{align*}
\]

where \( G^\alpha_t(x,y) := G^\alpha_t(x-y) \) is the Green kernel associated to the operator \(-(-\Delta)^{\frac{\alpha}{2}}\) on \( \mathbb{R}^d \), which is defined via its Fourier transform

\[
(\mathcal{F}G^\alpha_t)(\cdot)(\xi) = e^{-t\|\xi\|^\alpha}, \quad \xi \in \mathbb{R}^d, \tag{2.4}
\]

for \( \alpha \in (1,2] \), and the stochastic integral is interpreted in the sense of Walsh [16]. We note that the covariance measure \( M \) is

\[
Q([0,t] \times A \times B) := \langle M(A), M(B) \rangle_t = t \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy 1_A(x)\|x-y\|^{-(d-\gamma)}1_B(y)
\]
and its dominating measure $K \equiv Q$. In particular, by (2.1), (2.2) and (2.4), we have
\[
\mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{R}^d} G^\alpha_{t-s}(x, y) M(ds, dy) \right)^2 \right] = c_{1,1} \int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \ G^\alpha_{t-s}(x, y) \|y - z\|^{-(d-\gamma)} G^\alpha_{t-s}(x, z) \\
= \int_0^t ds \int_{\mathbb{R}^d} d\xi \|\xi\|^{-\gamma} |G^\alpha_{t-s}(-\cdot)(\xi)|^2 \\
= \int_0^t ds \int_{\mathbb{R}^d} d\xi \|\xi\|^{-\gamma} e^{-2(t-s)\|\xi\|^\alpha} \\
= c_{2,1} \int_0^t s^{-(d-\gamma)/\alpha} ds,
\]
where $c_{2,1} = \int_{\mathbb{R}^d} \|\xi\|^{-\gamma} e^{-2\|\xi\|^\alpha} d\xi < \infty$. The integral $ds$ in the last term of (2.5) is finite if and only if $d - \gamma < \alpha$.

For any random variable $\xi \in L^p(\Omega)$ with $p \geq 1$, let $\|\xi\|_{L^p(\Omega)} := (\mathbb{E}|\xi|^p)^{1/p}$. For any $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$, let
\[
\langle \phi, \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^d} d\xi \|\xi\|^{-\gamma} \mathcal{F}\phi(\cdot)(\xi) \overline{\mathcal{F}\psi(\cdot)(\xi)} \\
= c_{1,1} \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \ \phi(y) \|y - z\|^{-(d-\gamma)} \psi(z).
\]
Denote by $\mathcal{H}$ the Hilbert space obtained by the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to the inner product $\langle \phi, \psi \rangle_{\mathcal{H}}$ defined by (2.6).

According to the proof of Lemma 5.4 in [10] or Proposition 2.1 in [15], we have the following Burkholder-Davis-Gundy inequality for the stochastic convolution driven by the time-white and space-colored noise.

**Proposition 2.1.** ([10]) Let $\{\sigma(t, x)\}_{(t, x) \in [0, T] \times \mathbb{R}^d}$ be a predictable random field such that the following stochastic integral is well-defined. Then for any $t \in [0, T], \ x \in \mathbb{R}^d$ and $p \geq 2$,
\[
\left\| \int_0^t \int_{\mathbb{R}^d} G^\alpha_{t-s}(x, y) \sigma(s, y) M(ds, dy) \right\|_{L^p(\Omega)}^2 \leq 4p \left\| G^\alpha_{t-\cdot}(x, \cdot) \right\|_{L^p(\Omega)}^2 \|\sigma(\cdot, \cdot)\|_{L^p(\Omega)}^2 ds. \quad (2.7)
\]

### 2.2. The fractional Green kernel.

Let us recall some useful properties of the kernel $G^\alpha_t(x, y) = G^\alpha_t(x - y)$ defined through (2.4). For details, we refer to [2, 5]. It is well known that $G^\alpha_t(\cdot)$ is the probability transition density function of a rotationally invariant $d$-dimensional stable process $\{L^\alpha_t\}_{t \geq 0}$. By the scaling property of $L^\alpha_t = t^{1/\alpha}L^\alpha_1$, it is easy to see that
\[
G^\alpha_t(x) = t^{-d/\alpha} G^\alpha_1 \left( t^{-1/\alpha} x \right) \quad (t > 0, \ x \in \mathbb{R}^d). \quad (2.8)
\]

When $\alpha = 2$, $L^\alpha_t$ is the standard Brownian motion, and
\[
G^\alpha_t(x, y) = \frac{1}{(2\pi t)^{d/2}} \exp \left\{ -\frac{\|x - y\|^2}{2t} \right\}.
\]
When $\alpha \in (1, 2)$, by [4, Theorem 2.1], there exist some finite constants $0 < K'_\alpha < K_\alpha < \infty$ such that for all $t > 0, x, y \in \mathbb{R}^d$,
\begin{equation}
K'_\alpha \left( \frac{t}{(t^{1/\alpha} + \|x - y\|^{d+\alpha})^{d+\alpha}} \right)^{d+\alpha} \leq G'_t(x, y) \leq K_\alpha \left( \frac{t}{(t^{1/\alpha} + \|x - y\|^{d+\alpha})^{d+\alpha}} \right)^{d+\alpha}.
\end{equation}

By [5, Lemma 2.2], one has that for every $T \geq 1$, there exists a constant $c_{2.2} > 0$ such that for all $0 < t \leq T$ and $x, y, z \in \mathbb{R}^d$,
\begin{equation}
|G'_t(x, y) - G'_t(x, z)| \leq c_{2.2} \left( \frac{\|y - z\|}{t^{1/\alpha}} \wedge 1 \right) \times (G'_t(x, y) + G'_t(x, z)).
\end{equation}

2.3. **Linearization of the stochastic heat equation.** Let us consider the following linearization of the stochastic heat equation:
\begin{equation}
\frac{\partial}{\partial t} Z_t(x) = -(-\Delta)^{\frac{\alpha}{2}} Z_t(x) + F(t, x),
\end{equation}
subject to $Z_0(x) = 0$ for all $x \in \mathbb{R}^d$. The solution of (2.11) can be written as
\begin{equation}
Z_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x, y) M(ds, dy), \quad t > 0, x \in \mathbb{R}^d.
\end{equation}

It is well-known that the mild solution (2.12) is well-defined if and only if $d < \alpha + \gamma$. Moreover, in this case, for every $T > 0, p \geq 2$,
\begin{equation}
\sup_{t \in [0, T], x \in \mathbb{R}^d} \mathbb{E}[\|Z_t(x)\|^p] < +\infty.
\end{equation}

See, e.g., Proposition 4.1 in [12] or Remark 5.4 in [3].

The following bounds are well-known; see, e.g., Propositions 3.2 and 3.3 in [13] or references therein.

**Lemma 2.2.** ([13])

(a) Fix $T > 0$ and $k \geq 2$. There exists a constant $c_{2.3} > 0$ such that
\begin{equation}
\|Z_t(x) - Z_t(x - h)\|_{L^k(\Omega)} \leq c_{2.3} |h|^{\frac{\alpha - d + \gamma}{2}},
\end{equation}
uniquely for all $t \in [0, T]$ and $x, h \in \mathbb{R}^d$.

(b) Fix $T > 0$ and $k \geq 2$. There exists a constant $c_{2.4} > 0$ such that
\begin{equation}
\|Z_t(x) - Z_s(x)\|_{L^k(\Omega)} \leq c_{2.4} |t - s|^{\frac{\alpha - d + \gamma}{2\alpha}},
\end{equation}
uniquely for all $t, s \in [0, T]$ and $x \in \mathbb{R}^d$.

**Lemma 2.3.** ([12, Proposition 4.3]) Fix $t > 0$. Then the process $\{Z_t(x)\}_{x \in \mathbb{R}^d}$ defined by (2.12) has the same finite-dimensional distribution as
\begin{equation}
c_{\alpha, \gamma, d} B^{(\alpha - d + \gamma)/2}(x) + S_t(x) \quad (x \in \mathbb{R}^d),
\end{equation}
where $B^{(\alpha - d + \gamma)/2}$ is an isotropic multiparameter fBm with Hurst index $(\alpha - d + \gamma)/2$, $c_{\alpha, \gamma, d}$ is the constant defined by (1.10), and $\{S_t(x)\}_{x \in \mathbb{R}^d}$ is a Gaussian process with $C^\infty$-paths which is independent of $B^{(\alpha - d + \gamma)/2}$. Moreover, for any $0 < T_0 < T_1 < \infty$, there exists a constant $c_{2.5} := c_{2.5}(T_0, T_1) > 0$ such that
\begin{equation}
\|S_t(x) - S_t(y)\|_{L^2(\Omega)} \leq c_{2.5} \|x - y\| \quad (x, y \in \mathbb{R}^d),
\end{equation}
where
uniformly for all \( t \in [T_0, T_1] \).

According to [12, Remark 3.1] and [14, Proposition 1], we have the following result.

**Lemma 2.4.** ([12, 14]) Fix \( x \in \mathbb{R}^d \). Then the process \( \{Z_t(x)\}_{t \geq 0} \) defined by (2.12) has the same finite-dimensional distribution as

\[
c_{2,6} B^{(\alpha-d+\gamma)/(2\alpha)}(t) + S(t) \quad (t \in \mathbb{R}_+),
\]

where \( B^{(\alpha-d+\gamma)/(2\alpha)} \) is an fBm with Hurst index \((\alpha-d+\gamma)/(2\alpha)\),

\[
c_{2,6} = \left[ (2\pi)^{-d/2} \frac{2-(d-\gamma)/\alpha}{(1-(d-\gamma)/\alpha)^{-1}} \int_{\mathbb{R}^d} d\xi \|\xi\|^{-\gamma} e^{-\|\xi\|^2} \right]^{1/2},
\]

and \( \{S(t)\}_{t \in \mathbb{R}_+} \) is a Gaussian process with \( C^\infty \)-paths, which is independent of \( B^{(\alpha-d+\gamma)/(2\alpha)} \). Moreover, for any \( 0 < T_0 < T_1 < \infty \), there exists a constant \( c_{2,7} := c_{2,7}(T_0, T_1) > 0 \) such that

\[
\|S(t) - S(s)\|_{L^2(\Omega)} \leq c_{2,7}|t-s|,
\]

uniformly for all \( s, t \in [T_0, T_1] \).

By the properties of the (isotropic multiparameter) fBm and Lemmas 2.3 and 2.4, it is easily to obtain the following results, which will be very important in our later needs.

**Corollary 2.5.** (a) Fix \( T_1 > T_0 > 0 \). Then,

\[
\mathbb{E} \left[ |Z_t(x) - Z_t(x - \varepsilon e)|^2 \right] = c_{\alpha,\gamma,d}^2 \varepsilon^{\alpha-d+\gamma} + O(\varepsilon^2) \quad (\varepsilon \downarrow 0),
\]

uniformly for all \( t \in [T_0, T_1] \), \( x \in \mathbb{R}^d \) and the unit vector \( e \) in \( \mathbb{R} \), where \( c_{\alpha,\gamma,d} \) is defined by (1.10).

(b) Fix \( T_1 > T_0 > 0 \). Then

\[
\mathbb{E} \left[ |Z_t(x) - Z_s(x)|^2 \right] = c_{2,6}^2 |t-s|^\frac{(\alpha-d+\gamma)}{\alpha} + O(|t-s|^2) \quad (|t-s| \downarrow 0),
\]

uniformly for all \( t, s \in [T_0, T_1] \) and \( x \in \mathbb{R}^d \), \( c_{2,6} \) is defined by (2.15).

### 2.4. Hölder continuity of the solution.

**Proposition 2.6.** Under Condition 1.1, for any \( k \in [2, k_0] \) and \( T > 0 \), there exists a constant \( c_{2,8} > 0 \) such that

\[
\|u_{t+\varepsilon}(x) - u_t(x + h)\|_{L^k(\Omega)} \leq c_{2,8} \left( \varepsilon^{\frac{\alpha-d+\gamma}{2\alpha}} + \|h\|^{\frac{\alpha-d+\gamma}{2}} \right),
\]

uniformly for all \( t \in [0, T] \), \( \varepsilon \in (0, 1) \), \( x, h \in \mathbb{R}^d \) with \( \|h\| \leq 1 \).

Throughout this paper, we decompose the solution \( u \) in the following three terms:

\[
u_t(x) = \xi_t(x) + X_t(x) + Y_t(x),
\]

where

\[
\xi_t(x) := \int_{\mathbb{R}^d} G^\alpha_t(x,y) u_0(y) dy;
\]

\[
X_t(x) := \int_0^t \int_{\mathbb{R}^d} G^\alpha_{t-s}(x,y) \sigma(u_s(y)) M(ds, dy);
\]

\[
Y_t(x) := \int_0^t \int_{\mathbb{R}^d} G^\alpha_{t-s}(x,y) \sigma(u_s(y)) M(ds, dy);
\]
Y_t(x) := \int_0^t \int_{\mathbb{R}^d} G^\alpha_{t-s}(x, y) b(u_s(y)) ds dy. \quad (2.22)

Proof of Proposition 2.6. For the first term \{\xi_t(x)\}_{t \in [0, T], x \in \mathbb{R}^d}, by \cite[Proposition 2.6]{13}, there exists a constant \(c_{2.9} > 0\) such that
\[
\|\xi_{t+h}(x+h) - \xi_t(x)\|_{L^\kappa(\Omega)} \leq c_{2.9} \left( \varepsilon^2 + \|h\| \right)^{\eta_0}, \quad (2.23)
\]
uniformly for all \(t \in [0, T], \varepsilon \in (0, 1)\) and \(x, h \in \mathbb{R}^d\).

The second term \(X = \{X_t(x)\}_{t \in [0, T], x \in \mathbb{R}^d}\) solves the following equation:
\[
\frac{\partial}{\partial t} X_t(x) = -(-\Delta)^{-\alpha/2} X_t(x) + \sigma(u_t(x)) \dot{F}(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (2.24)
\]
with vanishing initial condition \(X_0(x) = 0\) for every \(x \in \mathbb{R}^d\).

By the Burkholder-Gavis-Gundy inequality (2.7) and (2.5), we have
\[
\frac{\partial}{\partial t} X_t(x) = -(-\Delta)^{-\alpha/2} X_t(x) + \sigma(u_t(x)) \dot{F}(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (2.25)
\]
uniformly for all \(t \in [0, T]\) and \(x, h \in \mathbb{R}^d\).

Since \(b\) is uniformly global Lipschitz continuous, applying Minkowski’s inequality, there exists a constant \(c_{2.10} > 0\) such that
\[
\|Y_t(x+h) - Y_t(x)\|^2_{L^\kappa(\Omega)} \leq c_{2.10} \int_0^t \sup_{w \in \mathbb{R}^d} \|u(s, w+h) - u(s, w)\|_{L^\kappa(\Omega)}^2 \|G^\alpha_{t-s}(x, \cdot)\|_{L^\kappa(\Omega)}^2 ds, \quad (2.26)
\]
uniformly for all \(t \in [0, T], x, h \in \mathbb{R}^d\).

Set
\[
\Lambda_h(t) := \sup_{w \in \mathbb{R}^d} \|u(t, w+h) - u(t, w)\|_{L^\kappa(\Omega)}^2, \quad 0 \leq t \leq T.
\]

By (2.3), (2.25) and (2.26), there exists a constant \(c_{2.11} > 0\) such that
\[
\Lambda_h(t) \leq c_{2.11} \left( h^{2\eta_0} + \int_0^t \left( 1 + s^{-(d-\gamma)/\alpha} \right) \Lambda_h(s) ds \right),
\]
uniformly in \(t \in [0, T]\) and \(h \in \mathbb{R}^d\). By the fractional Grönwall’s lemma, we have that for any \(t \in [0, T]\) and \(\|h\| \leq 1\),
\[
\Lambda_h(t) \leq c_{2.12}(T) \|h\|^{2\eta_0}, \quad (2.27)
\]
Lemma 2.7. For every $x, y \in \mathbb{R}^d$ and for every $k \in [2, k_0]$, there exists a constant $c_{2,14} > 0$ such that

$$
\sup_{t \in [0,T]} \mathbb{E} \left[ \left| Y_t(x) - Y_t(y) \right|^k \right] \leq c_{2,14} \|x - y\|^{k-1}.
$$

Proof. For any $t \in [0,T]$ and $x, y \in \mathbb{R}^d$, by Hölder’s inequality and (2.10), we have that for any $k \in [2, k_0]$, 

$$
\mathbb{E} \left[ \left| Y_t(x) - Y_t(y) \right|^k \right] 
\leq \left( \int_0^t ds \int_{\mathbb{R}^d} dz \left| G_{t-s}^{\alpha}(x, z) - G_{t-s}^{\alpha}(y, z) \right| \right)^{k-1} 
\times \left\{ \left. \int_0^t ds \int_{\mathbb{R}^d} dz \left| G_{t-s}^{\alpha}(x, z) - G_{t-s}^{\alpha}(y, z) \right| \cdot \mathbb{E} \left[ \left| b(u_s(z)) \right|^k \right] \right\} 
\leq c_{2,15} \|x - y\|^{k-1} \cdot \left( 1 + \sup_{(s,z) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left| u_s(z) \right|^k \right] \right),
$$

where $c_{2,15} > 0$. This, together with (1.7), completes the proof. \hfill \Box

3. Localization

For any $\varepsilon > 0$ and unit vector $e$ in $\mathbb{R}^d$, define

$$
(\nabla_\varepsilon f)(x) := f(x) - f(x - \varepsilon e) \quad (x \in \mathbb{R}^d),
$$

as a substitute for the approximate spatial gradient of the function $f : \mathbb{R}^d \to \mathbb{R}$.

Let us recall the solution $Z$ to Eq. (2.12). Consider the approximate gradient operator

$$
(\nabla_\varepsilon Z_t)(x) := Z_t(x) - Z_t(x - \varepsilon e) = \int_{(0,t) \times \mathbb{R}^d} (\nabla_\varepsilon G_{t-s}^{\alpha}) (x, y) M(ds, dy).
$$

Throughout, let us choose and fix parameters

$$
\beta > 1 \quad \text{and} \quad \delta := 1 + \beta^{1+\frac{2}{\alpha-d+\gamma}}.
$$

Then, we define a family of space-time boxes as follows: For every $t \geq 0, x \in \mathbb{R}^d$ and $\varepsilon > 0$,

$$
B_\beta(x, t; \varepsilon) = [t - \beta \varepsilon^\alpha, t] \times B(x; \varepsilon \delta),
$$
where $B(x; \varepsilon \delta) = \{y \in \mathbb{R}^d : \|x - y\| \leq \varepsilon \delta\}$.

The following is a generalization of Proposition 4.1 in [11].

**Proposition 3.1.** Choose and fix $T > 0$. There exists a positive constant $c_{3,1}$ such that

$$
\mathbb{E}\left[\left(\nabla_{\varepsilon e} Z_t(x) - \int_{B_{\beta}(x, t; \varepsilon)} (\nabla_{\varepsilon e} G_{t-s}^{\alpha}) (x, y) M(ds, dy)\right)^2\right] \leq c_{3,1} \varepsilon^{-d+\gamma} \beta^{-\frac{d+2-\alpha-\gamma}{\alpha}}, \quad (3.5)
$$

simultaneously for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\varepsilon \in (0, 1)$, $\beta > 1$ and the unit vector $e$ in $\mathbb{R}^d$.

**Proof.** By the stationarity of $\{Z_t(x)\}_{x \in \mathbb{R}^d}$ and the independence of $\sigma\{M(s, x)\}_{s \in [0, t], x \in \mathbb{R}^d}$ and $\sigma\{M(s, x) - M(t, x)\}_{s \geq t, x \in \mathbb{R}^d}$, we have

$$
Q := \mathbb{E}\left[\left(\nabla_{\varepsilon e} Z_t(x) - \int_{B_{\beta}(x, t; \varepsilon)} (\nabla_{\varepsilon e} G_{t-s}^{\alpha}) (x, y) M(ds, dy)\right)^2\right] = \mathbb{E}\left[\left(\int_{[0, t) \times \mathbb{R}^d \setminus B_{\beta}(0, t; \varepsilon)} (\nabla_{\varepsilon e} G_{t-s}^{\alpha}) (y) M(ds, dy)\right)^2\right] = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} (\nabla_{\varepsilon e} G_{t-s}^{\alpha}) (y) M(ds, dy)\right]^2 + \mathbb{E}\left[\int_{t-\beta \varepsilon}^t \int_{\|y\| > \delta} (\nabla_{\varepsilon e} G_{t-s}^{\alpha}) (y) M(ds, dy)\right]^2 = Q_1 + Q_2.
$$

Because $\mathcal{F} (\nabla_{\varepsilon e} G_{t-s}^{\alpha}) (\cdot)(\xi) = e^{-(t-s)\|\xi\|^\alpha} (1 - e^{-2\pi \varepsilon \xi \cdot e})$, (2.5) implies that

$$
Q_1 = \int_{\beta \varepsilon}^t ds \int_{\mathbb{R}^d} d\xi \ e^{-2s\|\xi\|^\alpha} [1 - \cos (2\pi \varepsilon \xi \cdot e)]\|\xi\|^{-\gamma} \leq 2\pi^2 \varepsilon^2 \int_{\beta \varepsilon}^\infty ds \int_{\mathbb{R}^d} d\xi \ e^{-2s\|\xi\|^\alpha}\|\xi\|^{2-\gamma} = \pi^2 \varepsilon^2 \int_{\mathbb{R}^d} e^{-2\beta \varepsilon \alpha\|\xi\|^\alpha}\|\xi\|^{2-\alpha-\gamma} d\xi = \pi^2 \int_{\mathbb{R}^d} e^{-\|\xi\|^\alpha}\|\xi\|^{2-\alpha-\gamma} d\xi \cdot (2\beta)^{-\frac{1+2-\alpha-\gamma}{\alpha}} \varepsilon^{-d+\gamma}, \quad (3.7)
$$

where $1 - \cos(a) \leq a^2/2$, $a \in \mathbb{R}$, has been used in the second step.
Let us estimate $Q_2$ with a little more effort. First, by the Hölder inequality, we have that for $p := \max\left\{ \frac{2\alpha}{\alpha - d + \gamma}, \frac{d}{\gamma} \right\}$,

$$Q_2 = \int_0^{\beta \varepsilon \alpha} ds \int_{|y| > \varepsilon \delta} dy \int_{|z| > \varepsilon \delta} dz \left( \nabla \varepsilon G_s^\alpha (y) \right) \left( y - z \right)^{-d + \gamma} \left( \nabla \varepsilon G_s^\alpha (z) \right)$$

$$\leq \left\{ \int_0^{\beta \varepsilon \alpha} ds \int_{|y| > \varepsilon \delta} dy \int_{|z| > \varepsilon \delta} dz \left( \left( \nabla \varepsilon G_s^\alpha (y) \right) \left( \nabla \varepsilon G_s^\alpha (z) \right) \right)^{1/p} \right\}^{p - 1}$$

$$= Q_{2.1} \cdot Q_{2.2}.$$

Next, we use the argument in the proof of (4.15) in [11] to estimate $Q_{2.1}$. Notice that

$$\int_{|y| > \varepsilon \delta} \left( \nabla \varepsilon G_s^\alpha (y) \right) dy \leq \int_{|y| > \varepsilon \delta} G_s^\alpha (y) dy + \int_{|y| > \varepsilon \delta} G_s^\alpha (y - \varepsilon \varepsilon) dy. \tag{3.9}$$

Let $L^{(t)} = \left\{ L^{(t)} \right\}_{t > 0}$ be the rotationally invariant $\alpha$-stable distribution. If $\eta \in [0, 1]$, then

$$\int_{|y| > \varepsilon \delta} G_s^\alpha (y - \eta \varepsilon \varepsilon) dy \leq \mathbb{P} \left\{ \left| L^{(t)} \right| \geq \varepsilon (\delta - \eta) \right\}. \tag{3.10}$$

Since $\delta = 1 + \beta^{1 + \frac{2}{\alpha - d + \gamma}}$ [see (3.3)], by the triangle inequality and the scaling of $L^{(t)}$, we have

$$\sup_{0 < \eta < 1} \int_{|y| > \varepsilon \delta} G_s^\alpha (y - \eta \varepsilon \varepsilon) dy \leq \mathbb{P} \left\{ \left| L^{(t)} \right| \geq \varepsilon \beta^{1 + \frac{2}{\alpha - d + \gamma}} \right\}$$

$$= \mathbb{P} \left\{ \left| L^{(t)} \right| \geq \varepsilon \beta^{1 + \frac{2}{\alpha - d + \gamma}} s^{-1/\alpha} \right\} \tag{3.11}$$

$$\leq \varepsilon^{-\alpha} \beta^{-\alpha (1 + \frac{2}{\alpha - d + \gamma})} s,$$

where we have used a well-known bound on the tail of the rotationally invariant $\alpha$-stable distribution

$$\mathbb{P} \left( \left| L^{(t)} \right| > \lambda \right) \leq \text{const} \cdot \lambda^{-\alpha} \quad \text{for all} \quad \lambda > 0.$$

See, e.g., [1].

By (3.9)-(3.11), we have

$$Q_{2.1} \leq \left( 3^{-1} \varepsilon \beta^{-\left( 2\alpha \left( 1 + \frac{2}{\alpha - d + \gamma} \right) - 3 \right)} \right)^{1/p}. \tag{3.12}$$

By using elementary inequality of

$$2 |\langle f, g \rangle_H| \leq \|f\|^2_H + \|g\|^2_H, \quad \tag{3.13}$$
we have

\[
Q_{22} = \left\{ \int_0^{\beta \varepsilon^\alpha} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left| (\nabla_{\varepsilon \varepsilon} G_s^\alpha)(y) \right| \left( y - z \right)^{-\frac{\mu}{p-1} (d-\gamma)} \left| (\nabla_{\varepsilon \varepsilon} G_s^\alpha)(z) \right| \right\}^{\frac{p-1}{p}}
\]

\[
\leq \left\{ 4 \int_0^{\beta \varepsilon^\alpha} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \, G_s^\alpha(y) \left| y - z \right|^{-d + \frac{\mu}{p-1} (p\gamma - d)} G_s^\alpha(z) \right\}^{\frac{p-1}{p}}
\]

\[
= \left\{ 4c_{3,2} \int_0^{\beta \varepsilon^\alpha} ds \int_{\mathbb{R}^d} d\xi \left\| \xi \right\|^{-\frac{\mu}{p-1} (p\gamma - d)} \right\| \mathcal{F} G_s^\alpha(\cdot)(\xi) \right\|^{\frac{p-1}{p}}
\]

\[
= \left\{ 4c_{3,2} \int_0^{\beta \varepsilon^\alpha} ds \int_{\mathbb{R}^d} d\xi \left\| \xi \right\|^{-\frac{\mu}{p-1} (p\gamma - d)} e^{-2\|\xi\|^a} \right\}^{\frac{p-1}{p}}
\]

\[
= \left\{ 4c_{3,2} c_{3,3} \int_0^{\beta \varepsilon^\alpha} s^{\frac{\mu}{p-1} (d-\gamma)} ds \right\}^{\frac{p-1}{p}}
\]

\[
= \left[ 1 - \frac{p(d-\gamma)}{(p-1)\alpha} \right]^{\frac{p-1}{p}} \left( 4c_{3,2} c_{3,3} \right)^{\frac{p-1}{p}} (\beta \varepsilon^\alpha)^{\frac{p-1}{p}(d-\gamma)/\alpha},
\]

where the constant \( c_{3,2} \in (0, \infty) \) is of the form \( c_{1,1} \) in (1.3) replacing \( \gamma \) by \( (p\gamma - d)/(p-1) \), and \( c_{3,3} = \int_{\mathbb{R}^d} \left\| \xi \right\|^a^{-\frac{\mu}{p-1} (p\gamma - d)} e^{-2\|\xi\|^a} d\xi < \infty. \)

We put (3.12) and (3.14) together in order to see that

\[
Q_2 \leq \text{const} \cdot \varepsilon^{\alpha - d + \gamma} \beta^{-\left(2\alpha(1+\frac{2}{\alpha-d+\gamma})^{-3}/p+(p-1)/p-(d-\gamma)/\alpha \right)}, \quad (3.15)
\]

Finally, we can combine our bounds for \( Q_1 \) and \( Q_2 \) in order deduce the inequality,

\[
Q \leq \text{const} \cdot \varepsilon^{\alpha - d + \gamma} \left( \beta^{-\frac{d+2-\alpha-\gamma}{\alpha}} + \beta^{-\left(2\alpha(1+\frac{2}{\alpha-d+\gamma})^{-3}/p+(p-1)/p-(d-\gamma)/\alpha \right)} \right). \quad (3.16)
\]

Elementary analysis of the exponent of \( \beta \) show that

\[
- \frac{d + 2 - \alpha - \gamma}{\alpha} \geq - \left( 2\alpha \left( 1 + \frac{2}{\alpha-d+\gamma} \right) - 3 \right) / p + p - 1 - \frac{d-\gamma}{\alpha},
\]

because \( p \geq \frac{2\alpha}{\alpha-d+\gamma} \). Since \( \beta > 1 \), we get the desired result (3.5).

The proof is complete. \( \square \)

Since Corollary 2.5 says that \( \mathbb{E} \left( \| (\nabla_{\varepsilon \varepsilon} Z_t)(x) \|^{2} \right) \approx \text{const} \cdot \varepsilon^{\alpha - d + \gamma} \) when \( \varepsilon \ll 1 \), Proposition 3.1 shows that

\[
\nabla_{\varepsilon \varepsilon} Z_t(x) \approx \int_{B_{\beta}(x,t;\varepsilon)} \left( \nabla_{\varepsilon \varepsilon} G_{t-s}^\alpha \right)(x,y) M(ds,dy) \text{ when } \beta \gg 1 \text{ and } \varepsilon \ll 1. \quad (3.17)
\]

That is, the most of contribution to the stochastic integral in (3.2) comes from the region \( B_{\beta}(x,t;\varepsilon) \). This is the so-called strong localization property in [11].

The next result describes the strong localization property the solution to the nonlinear heat equation (1.1).
Lemma 3.2. Choose and fix $T > 0$ and $k \in [2, k_0]$. Then, there exists a constant $c_{3,4} > 0$ such that
\[
\left\| \left( \nabla_{\varepsilon} X_t \right) (x) - \int_{B_{\beta}(x,t_{\varepsilon})} \left( \nabla_{\varepsilon} G_{t-s}^\beta (x,y) \sigma(u_s(y)) \right) M(ds,dy) \right\|_{L^k(\Omega)} \leq c_{3,4} \varepsilon^{-\frac{d+\gamma}{2}} \beta^{-\frac{2d-\alpha-\gamma}{\alpha}},
\]
simultaneously for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\varepsilon \in (0,1)$, $\beta > 1$ and the unit vector $e$ in $\mathbb{R}^d$.

Proof. As in the proof of Proposition 3.1, we have
\[
Q := \left\| \left( \nabla_{\varepsilon} X_t \right) (x) - \int_{B_{\beta}(x,t_{\varepsilon})} \left( \nabla_{\varepsilon} G_{t-s}^\beta (x,y) \sigma(u_s(y)) \right) M(ds,dy) \right\|_{L^k(\Omega)}^2
= \left\| \int_{[0,t] \times \mathbb{R}^d} \left( \nabla_{\varepsilon} G_{t-s}^\beta (y) \sigma(u_s(y)) \right) M(ds,dy) \right\|_{L^k(\Omega)}^2
\leq c_{2,2} k \varepsilon^2 \int_0^{t-\beta_{\varepsilon}} (t-s)^{-\frac{2}{\alpha}} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left( G_{t-s}^\alpha (y) + G_{t-s}^\alpha (y - \varepsilon e) \right) \cdot \|y - z\|^{-(d-\gamma)} \cdot G_{t-s}^\alpha (z)
\leq c_{1,1} c_{2,2} k \varepsilon^2 \int_0^{t-\beta_{\varepsilon}} (t-s)^{-\frac{2}{\alpha}} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz G_{t-s}^\alpha (y) \cdot \|y - z\|^{-(d-\gamma)} \cdot G_{t-s}^\alpha (z)
\leq c_{1,1} c_{2,2} k \varepsilon^2 \int_0^{t-\beta_{\varepsilon}} (t-s)^{-\frac{2}{\alpha}} ds \int_{\mathbb{R}^d} d\xi \|\xi\|^{-\gamma} e^{-2(t-s)|\xi|^\alpha}
\leq c_{3,5} \varepsilon^{-d+\gamma} \beta^{-\frac{2d-\alpha-\gamma}{\alpha}},
\]
where $c_{3,5} > 0$.

By using the Burkholder-Davis-Gundy inequality (2.7), (2.10) and (3.13), we have
\[
Q_1 \leq 2k \int_0^{t-\beta_{\varepsilon}} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left( \nabla_{\varepsilon} G_{t-s}^\beta (y) \right) \sigma(u_s(y)) \left( \nabla_{\varepsilon} G_{t-s}^\beta (z) \right) \sigma(u_s(z)) \leq c_{2,2} k \varepsilon^2 \int_0^{t-\beta_{\varepsilon}} (t-s)^{-\frac{2}{\alpha}} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left( G_{t-s}^\alpha (y) + G_{t-s}^\alpha (y - \varepsilon e) \right) \cdot \|y - z\|^{-(d-\gamma)} \cdot G_{t-s}^\alpha (z)
\leq c_{1,1} c_{2,2} k \varepsilon^2 \int_0^{t-\beta_{\varepsilon}} (t-s)^{-\frac{2}{\alpha}} ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz G_{t-s}^\alpha (y) \cdot \|y - z\|^{-(d-\gamma)} \cdot G_{t-s}^\alpha (z)
\leq c_{1,1} c_{2,2} k \varepsilon^2 \int_0^{t-\beta_{\varepsilon}} (t-s)^{-\frac{2}{\alpha}} ds \int_{\mathbb{R}^d} d\xi \|\xi\|^{-\gamma} e^{-2(t-s)|\xi|^\alpha}
\leq c_{3,6} \varepsilon^{-d+\gamma} \beta^{-\frac{2d-\alpha-\gamma}{\alpha}},
\]
where $c_{3,6} > 0$.
where \( c_{3,6} = 4c_{1,1}^{-1}c_{2,2}k \int_{\mathbb{R}^d} \| \xi \|^{-\gamma} e^{-2|\xi|^a} \, d\xi < \infty \).

Putting (3.20) and (3.21) together and using the Minkowski inequality, we obtain (3.18). The proof is complete. \( \square \)

**Lemma 3.3.** Choose and fix \( T > 0, k \in [2, k_0] \) and \( \beta := \varepsilon^{-l} \) for some \( l > 0 \). Then, there exists a constant \( c_{3,7} > 0 \) such that

\[
\left\| \int_{B_\beta(x; t; \varepsilon)} (\nabla \varepsilon G_{t-s}^\alpha) (x, y) \left[ \sigma(u_s(y)) - \sigma(u_s(\bar{x})) \right] M(ds, dy) \right\|_{L^k(\Omega)} \leq c_{3,7} \varepsilon^{\alpha-d+\gamma} \beta^{1+\alpha-d+\gamma},
\]

simultaneously for all \( x \in \mathbb{R}^d, t \in [0, T], \varepsilon \in (0, 1), l > 0, \bar{x} \in B(x; \varepsilon \delta) \) and the unit vector \( e \) in \( \mathbb{R}^d \).

**Proof.** Let \( t \in [0, T] \) and \( \bar{x} \in B(x; \varepsilon \delta) \). We first consider a related quantity \( Q_1 \), where

\[
Q_1 := \left\| \int_{B_\beta(x; t; \varepsilon)} (\nabla \varepsilon G_{t-s}^\alpha) (x, y) \left[ \sigma(u_s(y)) - \sigma(u_t-\beta \varepsilon^\alpha)(\bar{x}) \right] M(ds, dy) \right\|_{L^k(\Omega)}^2.
\]

Since \( u_t-\beta \varepsilon^\alpha(\bar{x}) \) is measurable with respect to the white noise \([0, t-\beta \varepsilon^\alpha] \times \mathbb{R}^d\), it is independent of the white noise of \( B_\beta(x; t; \varepsilon) \). Therefore, by the Burkholder-Davis-Gundy inequality (2.7), we have

\[
Q_1 \leq 2k \int_{t-\beta \varepsilon^\alpha}^t ds \int_{B(x; \varepsilon \delta)} dy \int_{B(x; \varepsilon \delta)} dz \left\| \nabla \varepsilon G_{t-s}^\alpha (x, y) \right\|_{L^k(\Omega)} \left\| \sigma(u_s(y)) - \sigma(u_t-\beta \varepsilon^\alpha(\bar{x})) \right\|_{L^k(\Omega)}^2 \cdot \|y-z\|^{-(d-\gamma)} \cdot \left\| (\nabla \varepsilon G_{t-s}^\alpha)(x, z) \right\|_{L^k(\Omega)} \cdot \left\| \sigma(u_s(z)) - \sigma(u_t-\beta \varepsilon^\alpha(\bar{x})) \right\|_{L^k(\Omega)}.
\]

By Hölder’s inequality, we have that for any \( p > \max\{d, \frac{\alpha^2}{\alpha-d+\gamma}\} \),

\[
Q_1 \leq 2k \int_{t-\beta \varepsilon^\alpha}^t ds \left[ \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left\| (\nabla \varepsilon G_{t-s}^\alpha)(x, y) \cdot (\nabla \varepsilon G_{t-s}^\alpha)(x, z) \right\|^{\frac{1}{p}} \cdot \sup_{(s, y) \in B_\beta(x; t; \varepsilon)} \left\| \sigma(u_s(y)) - \sigma(u_t-\beta \varepsilon^\alpha(\bar{x})) \right\|_{L^k(\Omega)}^2 \cdot \left[ \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left| (\nabla \varepsilon G_{t-s}^\alpha)(x, y) \cdot (\nabla \varepsilon G_{t-s}^\alpha)(x, z) \right|^{\frac{p}{p-1}} \cdot \|y-z\|^{(d-\gamma)/p} \cdot \left| (\nabla \varepsilon G_{t-s}^\alpha)(x, z) \right| \right]^\frac{p}{p-1}.
\]

By (2.10) and (3.13), we have

\[
\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left| (\nabla \varepsilon G_{t-s}^\alpha)(x, y) \right| \cdot \left| (\nabla \varepsilon G_{t-s}^\alpha)(x, z) \right| \\
\leq c_{2,2} \frac{\varepsilon}{(t-s)^\alpha} \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left( G_{t-s}^\alpha(x-y) + G_{t-s}^\alpha(x-y) \right) \left( G_{t-s}^\alpha(x-z) + G_{t-s}^\alpha(x-z) \right) \\
\leq 4c_{2,2} \frac{\varepsilon}{(t-s)^\alpha}.
\]

By Proposition 2.6, we have that for \( \eta := (\alpha - d + \gamma)/(2\alpha) \),

\[
\sup_{(s, y) \in B_\beta(x; t; \varepsilon)} \left\| \sigma(u_s(y)) - \sigma(u_t-\beta \varepsilon^\alpha(\bar{x})) \right\|_{L^k(\Omega)}^2 \leq c_{2,8} \varepsilon^{2\alpha \eta} \left( (2\delta)^{\alpha \eta} + \beta \eta \right)^2.
\]
By (3.13), we obtain that

\[
\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \left| \nabla \varphi G^\alpha_{t-s}(x,y) \cdot |y-z|^{-\frac{p}{p-1}(d-\gamma)} \cdot |\nabla \varphi G^\alpha_{t-s}(x,z)| \right|
\]

\[
\leq 4 \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz \ G^\alpha_{t-s}(x-y) \|y-z\|^{-\frac{p}{p-1}(d-\gamma)} G^\alpha_{t-s}(x-z)
\]

\[
= c_{3,8} \int_{\mathbb{R}^d} d\xi \left\| \xi \right\|^{-\frac{1}{p-1}(p\gamma-d)} e^{-2(t-s)|\xi|^\alpha}
\]

\[
= c_{3,8} (t-s)^{-\frac{1}{p-1}(p\gamma-d)} e^{-2|\xi|^\alpha} d\xi,
\]

where the constant \(c_{3,8}\) is in \((0, \infty)\), the integral \(\int_{\mathbb{R}^d} \left\| \xi \right\|^{-\frac{1}{p-1}(p\gamma-d)} e^{-2|\xi|^\alpha} d\xi\) is finite because of \(\frac{p\gamma-d}{p-1} < d\). Thus,

\[
Q_1 \leq c_{3,9} \varepsilon^{\frac{2\alpha}{p}} \left[ (2\delta)^{\alpha \eta} + \beta^\eta \right]^2 \int_{t-\beta^\alpha}^t (t-s)^{-\frac{\alpha}{p} \cdot \frac{d-\gamma}{\alpha}} ds
\]

\[
= c_{3,9} \left(1 - \frac{\alpha}{p} \cdot \frac{d-\gamma}{\alpha}\right)^{-1} \varepsilon^{2(\alpha-d+\gamma)} \left[ (2\delta)^{\alpha \eta} + \beta^\eta \right]^2 \cdot \beta^{1-\frac{d-\gamma}{\alpha}} \cdot \varepsilon^{-\frac{\alpha^2}{p} + \frac{1}{p}}. \tag{3.24}
\]

Next, we estimate the cost of estimating \(u_t(\bar{x})\) by \(u_{t-\beta^\alpha}(\bar{x})\). By the Hölder inequality, (2.18) and Lemma 2.2(a), we have

\[
Q_2 := \left\| \left[ \sigma (u_t(\bar{x})) - \sigma (u_{t-\beta^\alpha}(\bar{x})) \right] \int_{B(t_\delta, t \varepsilon)} (\nabla \varphi G^\alpha_{t-s}) (x,y) M(ds, dy) \right\|_{2k}^2
\]

\[
\leq L^2_\sigma \left\| u_t(\bar{x}) - u_{t-\beta^\alpha}(\bar{x}) \right\|_{2k}^2 \cdot \left\| \int_0^t \int_{\mathbb{R}^d} (\nabla \varphi G^\alpha_{t-s}) (x,y) M(ds, dy) \right\|_{2k}^2 \tag{3.25}
\]

\[
\leq c_{2,7} L^2_\sigma (\beta \varepsilon^\alpha)^{2\eta} \cdot \varepsilon^{-d+\gamma}.
\]

Since \(\delta = 1 + \beta^{1-\frac{d-\gamma}{\alpha-d+\gamma}} \leq 2\beta^{1-\frac{d-\gamma}{\alpha-d+\gamma}}\), we can conclude from (3.24) and (3.25) that

\[
\sqrt{Q_1} + \sqrt{Q_2} \leq c_{3,10} \varepsilon^{\alpha-d+\gamma} \left[ \beta^{\frac{(\alpha-1)(\alpha-d+\gamma)}{2\alpha}} \cdot \varepsilon^{-\frac{\alpha^2}{p} + \frac{1}{p}} + \beta^{-1} \right], \tag{3.26}
\]

where \(c_{3,10} \in (0, \infty)\). Since \(\beta = \varepsilon^{-l}\) for some \(l > 0\), taking \(p\) large enough such that \(\beta^{\frac{(\alpha-1)(\alpha-d+\gamma)}{2\alpha}} \cdot \varepsilon^{-\frac{\alpha^2}{p} + \frac{1}{p}} < 1\), we obtain that

\[
\sqrt{Q_1} + \sqrt{Q_2} \leq 2c_{3,10} \varepsilon^{\alpha-d+\gamma} \beta^{1+\alpha-d+\gamma}. \tag{3.27}
\]

This and Minkowski’s inequality together imply this lemma. \(\square\)

4. Proof of Theorem 1.2

Proof of Theorem 1.2. For simplicity of writing, we assume that \(\alpha \in (1, 2)\), while the proof in the case of \(\alpha = 2\) is relatively simple, which is omitted here.
According to the proof of Theorem 1.1 in [11], it is sufficient to get the following estimate: For any fixed $T_1 > T_0 > 0$,
\[
\sup_{x \in \mathbb{R}} \sup_{t \in [T_0, T_1]} \sup_{|e|=1} \left\| (\nabla_{\varepsilon e} u_t) (x) - c_{\alpha,d,\gamma}(u_t(\tilde{x})) (\nabla_{\varepsilon e} B^{(\alpha-d+\gamma)/2}) (x) \right\|_k \leq A \varepsilon^{\frac{\alpha-d+\gamma}{2} + \eta},
\] (4.1)
where $\tilde{x} \in B(x; \varepsilon \delta)$, $k \in [2, k_0]$, $A > 0$ and $\eta > 0$.

Recall the decomposition of $u$ in (2.19). By using the estimates in (2.23) and (2.29), we know that to prove (4.1), it suffices to prove that
\[
\sup_{x \in \mathbb{R}} \sup_{t \in [T_0, T_1]} \sup_{|e|=1} \left\| (\nabla_{\varepsilon e} X_t) (x) - c_{\alpha,d,\gamma}(u_t(\tilde{x})) (\nabla_{\varepsilon e} B^{(\alpha-d+\gamma)/2}) (x) \right\|_k \leq I_1 + I_2,
\] (4.2)
where $\tilde{x} \in B(x; \varepsilon \delta)$, $k \in [2, k_0]$, $A > 0$ and $\eta > 0$.

Let us split the preceding expectation in two parts as follows:
\[
\left\| (\nabla_{\varepsilon e} X_t) (x) - c_{\alpha,d,\gamma}(u_t(\tilde{x})) (\nabla_{\varepsilon e} B^{(\alpha-d+\gamma)/2}) (x) \right\|_k \leq I_1 + I_2,
\]
where
\[
I_1 := \left\| (\nabla_{\varepsilon e} X_t) (x) - \sigma(u_t(\tilde{x})) (\nabla_{\varepsilon e} Z_t) (x) \right\|_k; \quad I_2 := \left\| \sigma(u_t(\tilde{x})) (\nabla_{\varepsilon e} Z_t) (x) - c_{\alpha,d,\gamma}(u_t(\tilde{x})) (\nabla_{\varepsilon e} B^{(\alpha-d+\gamma)/2}) (x) \right\|_k.
\] (4.3)
(4.4)

By Lemma 2.2(a) and Proposition 2.6, there exists a constant $c_{4,1} > 0$ such that
\[
\left\| (\nabla_{\varepsilon e} Z_t) (x)[\sigma(u_t(\tilde{x})) - \sigma(u_t(x))] \right\|_k \leq \left\| (\nabla_{\varepsilon e} Z_t) (x) \right\|_{2k} \cdot L_\sigma \cdot \|u_t(\tilde{x}) - u_t(x)\|_{2k} \leq c_{4,1} \varepsilon^{\alpha-d+\gamma}\beta^{1 + \frac{\alpha-d+\gamma}{2}}.
\] (4.5)

By Corollary 3.2 and Lemma 3.3, we have
\[
\left\| (\nabla_{\varepsilon e} X_t) (x) - \sigma(u_t(x)) (\nabla_{\varepsilon e} Z_t) (x) \right\|_k \leq c_{4,2} \varepsilon^{\frac{\alpha-d+\gamma}{2} \beta d + \frac{d+2-\sigma-a-\gamma}{2a}} + c_{4,3} \varepsilon^{\alpha-d+\gamma}\beta^{1 + \frac{\alpha-d+\gamma}{2}}.
\] (4.6)

where $c_{4,2}$ and $c_{4,3}$ denote the finite constants that do not depend on the values of $x \in \mathbb{R}^d$, $t \in [T_0, T_1]$, $\varepsilon \in (0, 1)$ and the unit vector $e$ in $\mathbb{R}^d$.

Define $\beta := \varepsilon^{-l} > 1$. Since the left-hand side of (4.6) does not depend on $\beta$, we can optimize the right-hand side over $l > 0$, to find that the best bound in (4.6) is obtained when
\[
l := \frac{\alpha(\alpha - d + \gamma)}{(\alpha - 2 + \alpha - l) + \alpha(\alpha - d + \gamma)}.
\] (4.7)

This particular choice yields
\[
I_1 \leq c_{4,4} \varepsilon^{\frac{\alpha-d+\gamma}{2} \beta d + \frac{d+2-\sigma-a-\gamma}{2a}},
\] (4.8)
for some constant $c_{4,4} \in (0, \infty)$.

By Lemma 2.3 and using the same argument in the proof of Theorem 1.1 in [11], we obtain that there exists a constant $c_{4,5} \in (0, \infty)$ such that
\[
I_2 \leq c_{4,5} \varepsilon, \quad \text{uniformly for all } \varepsilon > 0.
\] (4.9)
The estimates (4.8) and (4.9) together imply the desired result (4.1).

The proof is complete. □

By using the arguments in [11] and using the estimates in the proof of Theorem 1.2, we can obtain Corollaries 1.3-1.5, whose proofs are omitted here.

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Ran Wang, School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China.

Email address: rwang@whu.edu.cn