On Hypergraph Lagrangians and Frankl-Füredi’s Conjecture

Hui Lei ∗ Linyuan Lu †

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Abstract

Frankl and Füredi conjectured in 1989 that the maximum Lagrangian, denoted by \( \lambda_r(m) \), among all \( r \)-uniform hypergraphs of fixed size \( m \) is achieved by the minimum hypergraph \( C_{r,m} \) under the colexicographic order. We say \( m \) in principal domain if there exists an integer \( t \) such that \( (t-1) \leq m \leq \binom{t}{r} - \binom{t-2}{r-2} \). If \( m \) is in the principal domain, then Frankl-Füredi’s conjecture has a very simple expression:

\[
\lambda_r(m) = \frac{1}{(t-1)^r} \binom{t-1}{r}.
\]

Many previous results are focusing on \( r = 3 \). For \( r \geq 4 \), Tyomkyn in 2017 proved that Frankl-Füredi’s conjecture holds whenever \( (t-1) \leq m \leq \binom{t}{r} - \binom{t-2}{r-2} - \delta_r t^{r-2} \) for a constant \( \delta_r > 0 \). In this paper, we improve Tyomkyn’s result by showing Frankl-Füredi’s conjecture holds whenever \( (t-1) \leq m \leq \binom{t}{r} - \binom{t-2}{r-2} - \delta'_r t^{r-2} - \frac{7}{3} \) for a constant \( \delta'_r > 0 \).

1 Introduction

The Lagrangians of hypergraphs are closely related to the Turán densities in the extremal hypergraph theory. Given an \( r \)-uniform hypergraph \( H \) on a vertex set \([n] := \{1, 2, \ldots, n\} \), the Lagrangian of \( H \), denoted by \( \lambda(H) \), is defined to be

\[
\lambda(H) = \max_{x \in \mathbb{R}_n^r : \|x\|_1 = 1} \sum_{\{i_1, i_2, \ldots, i_r\} \in E(H)} x_{i_1} x_{i_2} \cdots x_{i_r},
\]

\footnote{∗Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China, (hlei@mail.nankai.edu.cn). This author was supported in part by National Natural Science Foundation of China (No. 11771221).}

\footnote{†University of South Carolina, Columbia, SC 29208, (lu@math.sc.edu). This author was supported in part by NSF grant DMS 1600811.}
where the maximum is taken over on a simplex \( \{x \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0, \text{ and } \sum_{i=1}^n x_i = 1 \} \). A maximum point \( \bar{x}_0 \) is called an optimal legal weighting and the set of its nonzero coordinates in \( \bar{x}_0 \) is called a support of \( H \) (see section 2 for details.) One can show that \( r! \cdot \lambda(H) \) is the supremum of edge densities among all hypergraphs which are blow-ups of \( H \). It has important applications in the Turán theory.

The concept of Lagrangians of graphs was introduced by Motzkin and Straus \([5]\) in 1965, who proved that \( \lambda(H) = \frac{1}{2} \left( 1 - \frac{1}{\omega(H)} \right) \), where \( \omega(H) \) is the clique number of a graph \( H \). Their theorem implies Turán’s theorem.

Let \( \lambda_r(m) \) be the maximum of Lagrangians among all \( r \)-uniform hypergraphs with \( m \) edges. Then Motzkin-Straus’ result implies \( \lambda_2(m) = \frac{1}{2} \left( 1 - \frac{1}{t} \right) \) if \( \binom{t}{2} \leq m < \binom{t}{2} \) for some integer \( t \).

For any \( r \geq 2 \) and any \( m \geq 1 \), let \( C_{r,m} \) be the \( r \)-uniform hypergraph consisting of the first \( m \) sets in \( \binom{r}{3} \) in the colexicographic order (that is \( A < B \) if \( \max(A \Delta B) \in B. \) For example, for \( r = 3 \), the first 5 triple sets under the colexicographic order are given below:

\[
\{1,2,3\} < \{1,2,4\} < \{1,3,4\} < \{2,3,4\} < \{1,2,5\} < \cdots
\]

so \( C_{3,5} \) is the hypergraph on 5 vertices with the following edge set

\[
E(C_{3,5}) = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,5\}\}.
\]

One can easy to check that if \( m = \binom{t}{3} \) for some integer \( t \), then \( C_{r,m} \) is just the complete \( r \)-uniform hypergraph \( K_t^r \) (or \( [t]^{(r)} \) under Talbot’s notation \([7]\).)

In 1989, Frankl and Füredi made the following conjecture:

**Conjecture 1** \([1]\). For all \( r \geq 3 \) and any \( m \geq 1 \), we have \( \lambda_r(m) \leq \lambda(C_{r,m}) \).

Talbot \([7]\) pointed out that \( \lambda(C_{r,m}) \) remains a constant \( \lambda(C_{r,m}) = (t-1)/(t-1)^r \) for \( m \in \lfloor (t-1), (t^r) - (t-r-2) \rfloor \), and jumps for every \( m \in \lfloor (t^r), (t-r-2) \rfloor \). Tyomkyn called \( m = \binom{t}{3} \) the principal case. Here we refer the interval \( \lfloor (t^r), (t^r) - (t-r-2) \rfloor \) as the principal domain, and refer \( m = \binom{t}{3} \) the critical case. Most partial results on Frankl-Füredi’s conjecture occur in the principal domain.

For \( r = 3 \), Talbot \([7]\) proved that Conjecture \([1]\) holds whenever \( (t-1)/3 \leq m \leq (t^2 - 2t - 1)/3 - (2t - 3) \) for some \( t \in \mathbb{N} \). Tang, Peng, Zhang and Zhao \([8]\) extended the interval to \( (t-1)/3 \leq m \leq (t^2 - 2t - 1)/3 - 1/2(t - 1) \).
Recently, Tyomkyn [9] further extended the interval to \([\lfloor (t-1)/3 \rfloor, (t_3) - (t_{-2}) - \delta_3 t^{3/4}]\) for some constant \(\delta_3 > 0\). These results can be rephrased in terms of the gap (i.e., the number of missed values) in the principal domain: the gap drops from \(t - 1\), to \(\frac{1}{2}(t - 1)\), and further to \(O(t^{3/4})\). Recently, Lei, Lu, and Peng [3] further reduced the gap to \(O(t^{2/3})\).

For \(r \geq 4\), there are fewer results than at \(r = 3\). In 2017, Tyomkyn [9] proved the following theorem.

**Theorem 1 ([9]).** For \(r \geq 4\), there exists a constant \(\delta_r > 0\) such that for any \(m\) satisfying \((t_{r-1}) \leq m \leq (t_r) - (t_{r-2}) - \delta_r t^{r-2}\) we have

\[
\lambda_r(m) = \frac{(t_{r-1})}{(t - 1)^r}.
\]

Here is our main theorem.

**Theorem 2.** For \(r \geq 4\), there exists a constant \(\delta_r > 0\) such that for any \(m\) satisfying \((t_{r-1}) \leq m \leq (t_r) - (t_{r-2}) - \delta_r t^{r-2}\) we have

\[
\lambda_r(m) = \frac{(t_{r-1})}{(t - 1)^r}.
\]

Tyomkyn [9] proved that the gap can be reduced to \(O(t^{r-9/4})\) under an assumption that the hypergraphs have support on \(t\) vertices. We actually proved that the maximum hypergraphs have support on \(t\) vertices for sufficiently large \(t\) (see Lemma 4). Moreover, our gap \(O(t^{r-7/3})\) improves \(O(t^{r-9/4})\) on the exponent slightly.

Another related result is a smooth upper bound on \(\lambda_r(m)\). The following result, which was conjectured (and partially solved for \(r = 3, 4, 5\) and any \(m\); and for the case \(m \geq \frac{4(r-1)(r-2)}{r}\)) by Nikiforov [6], was completely proved by the second author.

**Theorem 3 ([4]).** For all \(r \geq 2\) and all \(m \geq 1\), if we write \(m = \binom{s}{r}\) for some real number \(s \geq r - 1\), then have

\[
\lambda_r(m) \leq ms^{-r}.
\]

The equality holds if and only if \(s\) is an integer and the hypergraph achieving \(\lambda_r(m)\) must be the complete \(r\)-uniform hypergraph \(K_r^s\) (possibly with some isolated vertices added.)

The paper will be organized as follows: the notation and previous lemmas will be given in Section 2. In Section 3, we prove a key lemma that the maximum hypergraphs have support \(t\) for \(t\) sufficiently large. Finally, the proof of Theorem 2 will be given in Section 4.
2 Notation and Preliminaries

Let $\mathbb{N}$ be the set of all positive integers and $[t]$ the set of first $t$ positive integers. For any integer $r \geq 2$ and a set $V$, we use $V^{(r)}$ (or $\binom{V}{r}$) to denote all $r$-subsets of $V$. An $r$-uniform hypergraph (or $r$-graph, for short) consists of a vertex set $V$ and an edge set $E \subseteq V^{(r)}$. Given an $r$-graph $G = (V, E)$ and a set $S \subseteq V$ with $|S| < r$, the $(r - |S|)$-uniform link hypergraph of $S$ is defined as $G_S = (V, E_S)$ with $E_S := \{ f \in V^{(r-|S|)} : f \cup S \in E \}$. We will denote the complement graph of $G_S$ by $G_S^c = (V, E_S^c)$ with $E_S^c := \{ f \in V^{(r-|S|)} : f \cup S \notin E \}$.

Define $G_{i,j}=(V,E_{i,j})$, where $E_{i,j} := \{ f \in E_i \setminus E_j : j \notin f \}$ and $G_{i,j}^c = (V,E_{i,j}^c)$, where $E_{i,j}^c := \{ f : f \cup \{i\} \in E^c \text{ but } f \cup \{j\} \in E \}$, i.e., $E_{i,j}^c = E_i^c \cap E_j$. Let $G - i$ be the $r$-graph obtained from $G$ by deleting vertex $i$ and those edges containing $i$. A hypergraph $G = (V, E)$ is said to cover a vertex pair $\{i,j\}$ if there exists an edge $e \in E$ with $\{i,j\} \subseteq e$. The $r$-graph $G$ is said to cover pairs if it covers every pair $\{i,j\} \subseteq V^{(2)}$.

From now on we assume that $G$ is an $r$-graph on the vertex set $[n]$. Given a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $x_f = x_{i_1}x_{i_2}\cdots x_{i_r}$ if $f = \{i_1, i_2, \ldots, i_r\}$. The weight polynomial of $G$ is given by

$$w(G, \vec{x}) = \sum_{e \in E(G)} x_e.$$ 

We call $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ a legal weighting for $G$ if $x_i \geq 0$ for any $i \in [n]$ and $\sum_{i=1}^n x_i = 1$. The set of all legal weightings forms a standard simplex in $\mathbb{R}^n$. We call a legal weighting $\vec{x}_0$ optimal if $w(G, \vec{x})$ reaches the maximum at $\vec{x} = \vec{x}_0$ in this simplex. The maximum value of $w(G, \vec{x})$, denoted by $\lambda(G)$, is called the Lagrangian of $G$.

**Lemma 1** ([1] [9]). Suppose that $G \subseteq [n]^{(r)}$ and $\vec{x} = (x_1, \ldots, x_n)$ is a legal weighting. For all $1 \leq i < j \leq n$, we have

(i) Suppose that $G$ does not cover the pair $\{i, j\}$. Then $\lambda(G) \leq \max\{\lambda(G - i), \lambda(G - j)\}$. In particular, $\lambda(G) \leq \lambda([n-1]^{(r)})$.

(ii) Suppose that $m,t \in \mathbb{N}$ satisfy $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2}$. Then

$$\lambda(C_{r,m}) = \lambda([t-1]^{(r)}) = \frac{1}{(t-1)^r} \binom{t-1}{r}.$$ 

(iii) $w(G_i, \vec{x}) \leq (1-x_i)^{r-1} \lambda(G_i)$ for any $i \in [n]$. 


Let \( E \subset \mathbb{N}^r \), \( e \in E \) and \( i, j \in \mathbb{N} \) with \( i < j \). We define

\[
L_{ij}(e) = \begin{cases} 
(e \setminus \{j\}) \cup \{i\}, & \text{if } i \notin e \text{ and } j \in e; \\
e, & \text{otherwise},
\end{cases}
\]

and

\[
C_{ij}(E) = \{L_{ij}(e): e \in E\} \cup \{e: e, L_{ij}(e) \in E\}.
\]

We say that \( E \) is left-compressed if \( C_{ij}(E) = E \) for every \( 1 \leq i < j \).

From now on, we always assume \( \binom{t-1}{r} \leq m < \binom{t}{r} \) for some integer \( t \). Let \( G \) be a graph with \( e(G) = m \) which satisfies \( \lambda(G) = \lambda_r(m) \) and let \( \overrightarrow{x} \) be an (optimal) legal weighting attaining the Lagrangian of \( G \). Without loss of generality, we can assume \( x_i \geq x_j \) for all \( i < j \) and \( \overrightarrow{x} \) has the minimum possible number of non-zero entries, and let \( T \) be this number.

Suppose that \( G \) achieves a strictly larger Lagrangian than \( C_{r,m} \). Then we have

\[
\lambda(G) > \frac{1}{(t-1)^r} \binom{t-1}{r},
\]

which in turn implies \( T \geq t \), otherwise \( \lambda(G) \leq \lambda([t-1]^r) \).

**Lemma 2** ([1,2,7]). Let \( G, T \), and \( \overrightarrow{x} \) be as defined above. Then

(i) \( G \) can be assumed to be left-compressed and to cover pairs.

(ii) For all \( 1 \leq i \leq T \) we have

\[
w(G_i, \overrightarrow{x}) = r\lambda(G_i).
\]

(iii) For all \( 1 \leq i < j \leq T \) we have

\[
(x_i - x_j)w(G_{i,j}, \overrightarrow{x}) = w(G_{i\setminus j}, \overrightarrow{x}).
\]

**Lemma 3** ([9]). Let \( G, T \), and \( \overrightarrow{x} \) be as defined above. Then for sufficiently large \( t \),

(i) \( T = t + C \) for some constant \( C = C(r) \).

(ii) \( x_1 < \frac{1}{t-\alpha} \) for some constant \( \alpha = \alpha(r) > 0 \).

(iii) If \( T = t \), then \( x_1 < \frac{1}{t-r+1} \).

Here is our key lemma.

**Lemma 4.** Let \( G, T \), and \( \overrightarrow{x} \) be as defined above. There is a constant \( t_0 : = t_0(r) \) such that if \( t \geq t_0 \) then \( T = t \).
3 Proof of Lemma 4

We need several lemmas before we prove Lemma 4.

Suppose that $G$ does not cover the pair $\{i, j\}$. Let $G_{ij}$ be an $r$-uniform hypergraph obtained from $G$ by gluing the vertices $i$ and $j$ as follows:

1. Let $v$ be a new fat vertex (by gluing $i$ and $j$.) Then $G_{ij}$ has the vertex set $(V(G) \setminus \{i, j\}) \cup \{v\}$.

2. The edge set of $G_{ij}$ consists of all edges in $G$ not containing $i$ or $j$, plus the edges of form $\{f \cup \{v\}: f \in E_i \cup E_j\}$.

The following lemma is similar to Lemma 1 (i), but has the advantage of being deterministic.

**Lemma 5.** Suppose that $G$ does not cover the pair $\{i, j\}$. Then $\lambda(G) \leq \lambda(G_{ij})$.

**Proof.** Let $\vec{x}$ be an optimal legal weighting of $G$. Define a legal weighting $\vec{y}$ of $G_{ij}$ by $y_v = x_i + x_j$ and $y_k = x_k$ if $k \neq v$. Then we have

$$w(G_{ij}, \vec{y}) - w(G, \vec{x}) = \sum_{f \in E_i \cup E_j} y_f y_v - \sum_{f \in E_i} x_f x_i - \sum_{f \in E_j} x_f x_j$$

$$= \sum_{f \in E_i \cup E_j} x_f (x_i + x_j) - \sum_{f \in E_i} x_f x_i - \sum_{f \in E_j} x_f x_j$$

$$= \sum_{f \in E_i \setminus E_j} x_f x_j + \sum_{f \in E_j \setminus E_i} x_f x_i$$

$$\geq 0.$$

Then we have

$$\lambda(G) = w(G, \vec{x}) \leq w(G_{ij}, \vec{y}) \leq \lambda(G_{ij}).$$

Lemma 6. For any $k \in [T - 1]$, we have

$$x_{T-k} > \frac{k - C_0}{k + 1} x_1,$$

where $C_0 = C + \alpha - 1$. 

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Proof. Observe that
\[ 1 = x_1 + \cdots + x_{T-k-1} + x_{T-k} + \cdots + x_T \]
\[ < (T-k-1)x_1 + (k+1)x_{T-k} \]
\[ < \frac{T-k-1}{t-\alpha} + (k+1)x_{T-k}. \] (by Lemma 3 (ii))

Solving \( x_{T-k} \) and applying Lemma 3 (i) and (ii), we have
\[ x_{T-k} > \frac{t-\alpha - T + k + 1}{k+1} \cdot \frac{1}{t-\alpha} = \frac{k-C_0}{k+1} \cdot \frac{1}{t-\alpha} > \frac{k-C_0}{k+1} x_1. \]

Lemma 7. There exists a constant \( \beta \) such that for any subset \( S \subseteq [T]^{(r-2)} \), we have
\[ \sum_{f \in S} (x_1^{r-2} - x_f) < \beta x_1. \] (2)

Proof. Let \( \beta \) be a constant \( > \frac{C+\alpha}{(r-3)!!} \). We will prove it by contradiction. Suppose that there is \( S \subseteq [T]^{(r-2)} \) such that
\[ \sum_{f \in S} (x_1^{r-2} - x_f) \geq \beta x_1. \]

We have
\[ \sum_{f \in S} x_f \leq |S|x_1^{r-2} - \beta x_1. \] (3)

Thus,
\[ 1 = (x_1 + x_2 + \cdots + x_T)^{r-2} \]
\[ \leq (r-2)! \sum_{f \in S} x_f + (T^{r-2} - (r-2)!!|S|)x_1^{r-2} \]
\[ \leq (r-2)!(|S|x_1^{r-2} - \beta x_1) + (T^{r-2} - (r-2)!!|S|)x_1^{r-2} \] (by (3))
\[ = (Tx_1)^{r-2} - (r-2)!! \cdot \beta x_1. \]

On the other hand, note \( \frac{1}{t-\alpha} > x_1 \geq \frac{1}{t+C} \). We have
\[ (Tx_1)^{r-2} - (r-2)!! \beta x_1 < \left( 1 + \frac{C+\alpha}{t-\alpha} \right)^{r-2} - (r-2)!! \cdot \beta \frac{1}{t+C} \]
\[ = 1 + (r-2)\frac{C+\alpha}{t-\alpha} + O(t^{-2}) - (r-2)!! \cdot \beta \frac{1}{t-\alpha} + O(t^{-2}) \]
\[ = 1 - \frac{(r-2)!!}{t-\alpha} \left( \beta - \frac{C+\alpha}{(r-3)!!} \right) + O(t^{-2}) \]
\[ < 1. \]
Contradiction.

Let \( s = \max\{i : \{T - i - (r - 2), \ldots, T - i - 1, T - i, T\} \in E^c\} \) and \( S = \{T - s, T - s + 1, \ldots, T - 1, T\} \). We have the following lemma.

Lemma 8. For \( s \) and \( S \) defined above, we have

1. Any non-edge \( f \in E^c \) must intersect \( S \) in at least two elements.
2. We have
   \[
   x_{T - 1} \geq \gamma x_1, 
   \]
   where \( \gamma := \prod_{k=0}^{r-2} \left( 1 - \frac{C_0 + 1}{s + k + 1} \right) \).

Proof. By the choice of \( s \), we know \( \{T - s - (r - 2), \ldots, T - s - 1, T - s, T\} \in E^c \) but \( \{T - s - (r - 1), \ldots, T - s - 2, T - s - 1, T\} \in E \). We now prove item 1 by contradiction. Suppose not, say there is \( f = \{i_1, i_2, \ldots, i_{r - 2}\} \in E^c \) such that \( i_1 < i_2 < \cdots < i_{r - 1} < T - s \). Since \( G \) is left-compressed, \( \{T - s - (r - 1), \ldots, T - s - 2, T - s - 1, T\} \in E^c \). Contradiction!

Since \( G \) covers the pair \( \{T - 1, T\} \), there is an \((r - 2)\)-tuple \( \{i_1, i_2, \ldots, i_{r - 2}\} \in E_{T - 1, T} \). We have

\[
x_{i_1} \cdots x_{i_{r - 2}} x_{T - 1} x_T - x_{T - s - (r - 2)} x_{T - s - (r - 3)} \cdots x_{T - s} x_T \geq 0. \tag{5}
\]

Otherwise by replacing the edge \( \{i_1, i_2, \ldots, i_{r - 2}, T - 1, T\} \) with the non-edge \( \{T - s - (r - 2), T - s - (r - 3), \ldots, T - s, T\} \), we get another \( r \)-graph with the same number of edges whose Lagrangian is strictly greater than the Lagrangian of \( G \). Contradiction!

Combining Inequalities (5) and (1), we get

\[
x_{T - 1} \geq \frac{x_{T - s - (r - 2)} x_{T - s - (r - 3)} \cdots x_{T - s}}{x_1^{r - 2}} > \gamma x_1. \tag{6}
\]

We have the following estimation of \( \gamma \):

\[
\gamma = \prod_{k=0}^{r-2} \left( 1 - \frac{C_0 + 1}{s + k + 1} \right) = 1 - \frac{(C_0 + 1)(r - 1)}{s} + O \left( \frac{1}{s^2} \right). \tag{6}
\]

Lemma 9. If \( T > t \), then \( |E^c_{T \setminus (T - 1)}| - |E_{T - 1, T}| \geq \frac{T - r}{r - 1}(T - 2) \).
Proof. Since $G$ is left-compressed, then

$$|E_T| \leq \frac{rm}{T} \leq \frac{r}{T} \left( \frac{t}{r} \right) \left( \frac{t-1}{r-1} \right) \leq \frac{t}{T} \left( \frac{T-2}{r-1} \right).$$

In the last step, we apply the assumption $t \leq T-1$. Thus, we have

$$|E^c_{T,T} | - |E_{T-1,T}| = \left( \frac{T-2}{r-1} \right) - (|E^c_{T,T}| + |E_{T-1,T}|)$$

$$= \left( \frac{T-2}{r-1} \right) - |E_T|$$

$$\geq \left( \frac{T-2}{r-1} \right) - \frac{t}{T} \left( \frac{T-2}{r-1} \right)$$

$$= \frac{T-t}{T} \left( \frac{T-2}{r-1} \right)$$

$$= \frac{(T-t)(T-r)}{(r-1)T} \left( \frac{T-2}{r-2} \right)$$

$$\geq \frac{T-r}{(r-1)T} \left( \frac{T-2}{r-2} \right).$$

Proof of Lemma 4: Let $G, \overline{x}, m, t$, and $T$ be as defined before. Let $\eta := \left\lceil 2 \sqrt{\frac{C_0+1}{(r-1)!} t r^{-5/2}} \right\rceil$.

We will prove Lemma 4 by contradiction. Assume $T > t$. By Lemma 9, we have

$$|E^c_{T,T} | - |E_{T-1,T}| \geq \frac{T-r}{(r-1)T} \left( \frac{T-2}{r-2} \right) > \eta. \tag{7}$$

This is possible for $t$ sufficiently large since $\frac{T-r}{(r-1)} \left( \frac{T-2}{r-2} \right) = \Theta(t^{-2})$ and $\eta = \Theta(t^{-2.5})$.

Let $b = |E_{T-1,T}|$. Pick any $F \subseteq \{ f \cup \{ T \} : f \in E^c_{T,T} \}$ of size $b + \eta$.

This is possible because of Inequality (7).

Let $G'$ be an $r$-graph obtained from $G$ by deleting all edges in $\{ f \cup \{ T-1, T \} : f \in E_{T-1,T} \}$ and adding all $r$-tuples in $F$ as edges. The main proof is to show the following inequality:

$$w(G, \overline{x}) \leq w(G', \overline{x}). \tag{8}$$

Now we will prove Inequality (8). We divide it into two cases.
Case 1: $\beta s < \eta x_1^{r-3}$. By Lemma 8 item 1, every non-edge intersects $S$ in at least two elements. This implies $F$ can be partitioned into $s - 1$ parts:

$$F = \bigcup_{i=2}^{s} \{ f \cup \{ T - i, T \} : f \in F_i \},$$

where $F_i := \{ f \in [T - i - 1]^{(r-2)} : f \cup \{ T - i, T \} \in F \}$. We have

$$w(G', \bar{x}) = w(G, \bar{x}) - \sum_{e \in E_{T-1,T}} x_e x_{T-1} x_T + \sum_{f \in F} x_f$$

$$\geq w(G, \bar{x}) + \eta x_1^{r-2} x_{T-1} x_T - \sum_{f \in F} (x_1^{r-2} x_{T-1} x_T - x_f)$$

$$\geq w(G, \bar{x}) + \eta x_1^{r-2} x_{T-1} x_T - \sum_{i=2}^{s} \sum_{f \in F_i} (x_1^{r-2} - x_{f'}) x_{T-1} x_T$$

$$\geq w(G, \bar{x}) + \eta x_1^{r-2} x_{T-1} x_T - x_{T-1} x_T \sum_{i=2}^{s} \beta x_1 \quad \text{(by (2))}$$

$$> w(G, \bar{x}) + (\eta x_1^{r-3} - s \beta) x_1 x_{T-1} x_T$$

$$> w(G, \bar{x}).$$

Case 2: $\beta s \geq \eta x_1^{r-3}$. Since $x_1 \geq \frac{1}{r}$, we have

$$s \geq \frac{\eta x_1^{r-3}}{\beta} \geq \frac{1}{\beta T^{r-3}}. \quad (9)$$

We first prove an inequality:

$$\gamma^{r-2} \eta > (1 - \gamma^{r-2}) b. \quad (10)$$

$$\gamma^{r-2} \eta - (1 - \gamma^{r-2}) b$$

$$\geq \gamma^{r-2} \eta - (1 - \gamma^{r-2}) \left( \frac{T - 2}{r - 2} \right)$$

$$> \gamma^{r-2} \left( \eta - (\gamma^{r-2} - 1) \frac{T^{r-2}}{(r - 2)!} \right)$$

$$> \gamma^{r-2} \left( \eta - \left( \frac{(C_0 + 1)(r - 1)(r - 2)}{s} + O \left( \frac{1}{s^2} \right) \right) \frac{T^{r-2}}{(r - 2)!} \right) \quad \text{(by (6))}$$

$$= \frac{\gamma^{r-2}}{s} \left( s \eta - \left( \frac{(C_0 + 1)(r - 1)(r - 2)}{s} + O \left( \frac{1}{s} \right) \right) \frac{T^{r-2}}{(r - 2)!} \right) \quad \text{(using (9))}$$

$$\geq \frac{\gamma^{r-2}}{s} \left( \frac{1}{\beta T^{r-3}} \eta^2 - \left( \frac{(C_0 + 1)(r - 1)(r - 2)}{s} + O \left( \frac{1}{s} \right) \right) \frac{T^{r-2}}{(r - 2)!} \right)$$

$$> 0.$$
In the last step, we apply the definition of $\eta$ and get
\[
\eta^2 \geq 4 \frac{(C_0 + 1)(r - 1)(r - 2)}{(r - 2)!} \beta^{2r-5} > 2 \frac{(C_0 + 1)(r - 1)(r - 2)}{(r - 2)!} \beta^{2r-5}.
\]
Now we are ready to estimate $w(G', \vec{x})$:

\[
w(G', \vec{x}) = w(G, \vec{x}) - \sum_{e \in E_{T-1,T}} x_e x_{T-1} x_T + \sum_{f \in F} x_f \\
> w(G, \vec{x}) - bx_1^r x_{T-1} x_T + (b + \eta)x_{T-1}^{-1} x_T \\
= w(G, \vec{x}) + x_{T-1} x_T ((b + \eta)x_{T-1}^{-1} - bx_1^r) \\
\geq w(G, \vec{x}) + x_{T-1} x_T ((b + \eta)\gamma^{-2} x_1^{-2} - bx_1^{-r-2}) \quad \text{(by (4))} \\
= w(G, \vec{x}) + x_{T-1}^{-2} x_{T-1} x_T (\gamma^{-2} - (1 - \gamma^{-2})b) \quad \text{(by (10))} \\
> w(G, \vec{x}).
\]

Therefore, Inequality (8) holds in any circumstances.

Note that $G'$ does not cover the pair $\{T-1, T\}$. Applying Lemma 5, we have

\[
\lambda(G) = w(G, \vec{x}) < w(G', \vec{x}) \leq \lambda(G') \leq \lambda(G/(T-1)_T).
\]

By the construction of $G'$, the added edges are from $F \subseteq \{f \cup \{T\} : f \in E_{T \setminus (T-1)}\}$. These edges have the form of $f \cup \{T\}$, where $f \cup \{T-1\}$ is also an edge in $G$. After gluing $T-1$ and $T$ together, both edges $f \cup \{T\}$ and $f \cup \{T-1\}$ are glued into one edge $f \cup \{v\}$. We have

\[
|E(G'/(T-1)_T)| \leq |E(G')| - |F| \leq |E(G)|.
\]

Contradiction! This completes the proof of Lemma 4.

\[\square\]

4 Proof of Theorem 2

Assume $t \geq t_0$. Let $G = (V, E)$ be an $r$-graph with $m$ edges satisfying \(\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r} \) and $\lambda(G) = \lambda_r(m) > \lambda([t-1]^r))$. Let $\vec{x} = (x_1, \ldots, x_n)$ be an optimal legal weighting for $G$ that uses exactly $T$ nonzero weights (i.e., $x_1 \geq \cdots \geq x_T > x_{T+1} = \cdots = x_n = 0$). By Lemma 4, we may assume $T = t$. In addition, we may assume $G$ is left-compressed by Lemma 2(i).

Since $T = t$, by Lemma 5(iii), we have

\[
x_1 < \frac{1}{t - r + 1}, \quad \text{(11)}
\]
and we may improve Lemma 6 as follows.

**Lemma 10.** For any \( k \in [t-1] \), we have

\[
x_{t-k} > \frac{k-r+2}{k+1}x_1.
\] (12)

Let \( T = t, C = 0, \alpha = r - 1, C_0 = r - 2, \) and \( \beta = \frac{r}{(r-3)!} \) in the proof of Lemma 7. Then we can get the following Lemma improving Lemma 7.

**Lemma 11.** For any subset \( S \subseteq [t]^{(r-2)} \), we have

\[
\sum_{f \in S} (x_{r-2}^r - x_f) < \frac{r}{(r-3)!}x_1.
\] (13)

With \( T = t \), we have \( s = \max \{i: \{t-i-(r-2), \ldots, t-i-1, t-i, t\} \in E^c\} \) and \( S = \{t-s, t-s+1, \ldots, t-1, t\} \). Lemma 8 can be improved to:

**Lemma 12.** For \( s \) and \( S \) defined above, we have

1. Any non-edge \( f \in E^c \) must intersect \( S \) in at least two elements.
2. We have

\[
x_{t-1} > \gamma x_1,
\] (14)

where \( \gamma := \prod_{k=0}^{r-2} \left( 1 - \frac{r-1}{s+k+1} \right) \).

**Lemma 13.** Let \( \gamma \) be defined in Lemma 12. Consider two functions:

\[
g_1(s) := \binom{s+1}{2} \frac{r \gamma^{-(r-2)}}{r-3)!} t^{r-3},
\] (15)

\[
g_2(s) := \binom{t-2}{r-2} (\gamma^{-(r-2)} - 1).
\] (16)

Then

\[
\min \{g_1(s), g_2(s)\} = O(t^{r-7/3}).
\] (17)

**Proof.** By Equation (6) (with \( C_0 = r - 2 \)), we have

\[
\gamma = 1 - \frac{(r-1)^2}{s} + O\left( \frac{1}{s^2} \right).
\]

When \( s \leq t^{1/3} \), we have

\[
g_1(s) = \binom{s+1}{2} \frac{r \gamma^{-(r-2)}}{r-3)!} t^{r-3} = O(t^{r-7/3}).
\]
When $s \geq \frac{t^1}{3}$, we have
\[ g_2(s) = \left(\frac{t-2}{r-2}\right)(\gamma-(r-2)-1) \]
\[ = \left(\frac{t-2}{r-2}\right)\left(\frac{(r-1)^2(r-2)}{s} + O\left(\frac{1}{s^2}\right)\right) \]
\[ = O(t^r-\frac{7}{3}). \]

Thus, Equality (17) holds.

**Proof of Theorem 2:** We assume $t_0 \geq t_0$ so that $T=t$ holds. Let $\eta := \left\lceil \min\{f(s), g(s)\} \right\rceil = O(t^r-\frac{7}{3})$.

**Claim 1.** For $t \geq \eta$, if $(t-1) \leq m \leq (t) - (t-2) - \eta$, then $\lambda_r(m) = \frac{(t-1)}{(t-1)^r}$. We will prove this claim by contradiction. Assume there is a graph $G$ with $m$ edges, where $(t-1) \leq m \leq (t) - (t-2) - \eta$, and $\lambda(G) = \lambda_r(m) > \frac{(t-1)}{(t-1)^r}$. Let $B$ be any family of $|E_{t-1,t}|$ many non-edges of $G$ which does not contain both $t$ and $t-1$. This is possible since $G$ has at least $(t-2) + \eta$ non-edges. Let $G'$ be an $r$-graph obtained from $G$ by deleting all edges in $E_{t-1,t}$ and adding all $r$-tuples in $B$ as edges. Then $G'$ has exactly $m$ edges.

By Lemma 12 item 1, any non-edge in $B$ must intersect $S$ in at least two elements. For any $\{i,j\} \subseteq S(2)$ with $i < j$, define
\[ B_{ij} := \{\{i_1, \ldots, i_{r-2}\} : \{i_1, \ldots, i_{r-2}, i, j\} \in B \text{ and } i_1 < \cdots < i_{r-2} < i < j\}. \]

Then we have
\[ B = \bigcup_{\{i,j\}\subseteq S(2)} \{f \cup \{i,j\} : f \in B_{ij}\}. \]

We allow some $B_{ij}$ to be emptysets. (For example, $B_{t-1,t} = \emptyset$.)

Now, we consider the difference between $w(G', \vec{x})$ and $w(G, \vec{x})$. On the
one hand,

\[ w(G', \vec{x}) = w(G, \vec{x}) - \sum_{e \in E_{t-1,t}} x_e x_{t-1} x_t + \sum_{f \in B} x_f \]

\[ \geq w(G, \vec{x}) - \sum_{f \in B} (x_f^{r-2} x_{t-1} x_t - x_f) \]

\[ \geq w(G, \vec{x}) - \sum_{\{i,j\} \in S^{(2)}} \sum_{f' \in B_{ij}} (x_f^{r-2} - x_{f'}) x_{t-1} x_t \]

\[ > w(G, \vec{x}) - \left( \frac{s + 1}{2} \right) r \frac{r}{(r - 3)!} x_{t-1} x_t \]

(by \(13\))

\[ > w(G, \vec{x}) - \left( \frac{s + 1}{2} \right) r \frac{r - 3}{(r - 3)!} x_{t-1} x_t. \]

(since \(x_1 \geq \frac{1}{t}\))

\[ > w(G, \vec{x}) - \left( \frac{s + 1}{2} \right) \gamma^{(r-2)} \frac{r - 3}{(r - 3)!} x_{t-1} x_t. \]

(by \(14\))

\[ \geq w(G, \vec{x}) - g_1(s) x_{t-1} x_t. \]

On the other hand,

\[ w(G', \vec{x}) = w(G, \vec{x}) - \sum_{e \in E_{t-1,t}} x_e x_{t-1} x_t + \sum_{f \in B} x_f \]

\[ > w(G, \vec{x}) - \sum_{f \in B} (x_f^{r-2} x_{t-1} x_t - x_f^{r-2} x_{t-1} x_t) \]

\[ > w(G, \vec{x}) - \left( \frac{t - 2}{r - 2} \right) (x_f^{r-2} - x_f^{r-2}) x_{t-1} x_t \]

\[ = w(G, \vec{x}) - \left( \frac{t - 2}{r - 2} \right) (\gamma^{(r-2)} - 1) x_{t-1} x_t. \]

(by \(14\))

\[ \geq w(G, \vec{x}) - g_2(s) x_{t-1} x_t. \]

Thus, we have

\[ w(G', \vec{x}) > w(G, \vec{x}) - \min\{g_1(s), g_2(s)\} x_{t-1} x_t \geq w(G, \vec{x}) - \eta x_{t-1} x_t. \quad (18) \]

Note that \(G'\) does not cover \(\{t - 1, t\}\) and \(G'\) still has \(\eta\) non-edges which does not contain both \(t - 1\) and \(t\). Let \(G''\) be an \(r\)-graph obtained from \(G'\) by adding these \(\eta\) \(r\)-tuples as edges. We have

\[ w(G'', \vec{x}) \geq w(G', \vec{x}) + \eta x_{t-r+1} x_{t-r+2} \cdots x_t \]

\[ > w(G, \vec{x}) + \eta x_{t-1} x_t + \eta x_{t-r+1} x_{t-r+2} \cdots x_t \]

\[ > w(G, \vec{x}). \]
Note that $G''$ still does not cover the pair $\{t-1, t\}$. We have
\[
\lambda(G) = w(G, \bar{x}) < w(G'', \bar{x}) \leq \lambda(G'') \leq \lambda([t-1]^r),
\]
a contradiction. Claim I is proved.

Finally we can choose a constant $\delta_r$ large enough so that the following two conditions hold:

- $\delta_r t^{r-7/3} > \eta$ for all $t \geq t_0$,
- and $\delta_r t^{r-7/3} > \left(\frac{t-2}{r-1}\right)$ for $1 \leq t \leq t_0$.

When $t \geq t_0$, we have
\[
\lambda_r(m) \leq \lambda\left(C_{r,\left(\begin{array}{c} t-1 \\ r \end{array}\right)} - \delta_r t^{r-7/3}\right) \leq \lambda\left(C_{r,\left(\begin{array}{c} t-2 \\ r-2 \end{array}\right)} - \eta\right) = \frac{(t-1)}{(t-1)^r}.
\]
When $1 \leq t \leq t_0$, we have
\[
\lambda_r(m) \leq \lambda\left(C_{r,\left(\begin{array}{c} t-1 \\ r \end{array}\right)}\right) = \frac{(t-1)}{(t-1)^r}.
\]
Since $m \geq \left(\begin{array}{c} t-1 \\ r \end{array}\right)$, we have
\[
\lambda_r(m) \geq \lambda_r\left(\left(\begin{array}{c} t-1 \\ r \end{array}\right)\right) = \frac{(t-1)}{(t-1)^r}.
\]
Thus,
\[
\lambda_r(m) = \frac{(t-1)}{(t-1)^r}.
\]
This completes the proof of Theorem 2.

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