Clustering under Local Stability: Bridging the Gap between Worst-Case and Beyond Worst-Case Analysis *

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Abstract

Recently, there has been substantial interest in clustering research that takes a beyond worst-case approach to the analysis of algorithms. The typical idea is to design a clustering algorithm that outputs a near-optimal solution, provided the data satisfy a natural stability notion. For example, Bilu and Linial (2010) and Awasthi et al. (2012) presented algorithms that output near-optimal solutions, assuming the optimal solution is preserved under small perturbations to the input distances. A drawback to this approach is that the algorithms are often explicitly built according to the stability assumption and give no guarantees in the worst case; indeed, several recent algorithms output arbitrarily bad solutions even when just a small section of the data does not satisfy the given stability notion.

In this work, we address this concern in two ways. First, we provide algorithms that inherit the worst-case guarantees of clustering approximation algorithms, while simultaneously guaranteeing near-optimal solutions when the data is stable. Our algorithms are natural modifications to existing state-of-the-art approximation algorithms. Second, we initiate the study of local stability, which is a property of a single optimal cluster rather than an entire optimal solution. We show our algorithms output all optimal clusters which satisfy stability locally. Specifically, we achieve strong positive results in our local framework under recent stability notions including metric perturbation resilience (Angelidakis et al. 2017) and robust perturbation resilience (Balcan and Liang 2012) for the \( k \)-median, \( k \)-means, and symmetric/asymmetric \( k \)-center objectives.

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1 Introduction

Clustering is a fundamental problem in combinatorial optimization with numerous real-life applications in areas from bioinformatics to computer vision to text analysis and so on. The underlying goal is to group a given set of points to maximize similarity inside a group and minimize similarity among groups. A common approach to clustering is to set up an objective function and then approximately find the optimal solution according to the objective. Given a set of points $S$ and a distance metric $d$, common clustering objectives include finding $k$ centers to minimize the sum of the distance, or squared distance, from each point to its closest center ($k$-median and $k$-means, respectively), or to minimize the maximum distance from a point to its closest center ($k$-center). These popular objective functions are provably NP-hard to optimize \cite{23,28,32}, so research has focused on finding approximation algorithms. This has attracted significant attention in the theoretical computer science community \cite{5,15,16,17,18,23,33}.

Traditionally, the theory of clustering (and more generally, the theory of algorithms) has focused on the analysis of worst-case instances. While this approach has led to many elegant algorithms and lower bounds, it is often overly pessimistic of an algorithm’s performance on “typical” instances or real world instances. A rapidly developing line of work in the algorithms community, the so-called beyond worst-case analysis of algorithms (BWCA), considers designing algorithms for instances that satisfy some natural structural properties. BWCA has given rise to many positive results \cite{26,31,37}, especially for clustering problems \cite{6,7,10,30}. For example, the popular notion of $\alpha$-perturbation resilience, introduced by Bilu and Linial \cite{14}, informally states that the optimal solution does not change when the input distances are allowed to increase by up to a factor of $\alpha$. This definition seeks to capture a phenomenon in practice: the optimal solution is often significantly better than all other solutions, thereby the optimal solution does not change even when the input is slightly perturbed.

However, there are two potential downsides to this approach. The first downside is that many of these algorithms aggressively exploit the given structural assumptions, which can lead to contrived algorithms with no guarantees in the worst case. Therefore, a user can only use algorithms from this line of work if she is certain her data satisfies the assumption (even though none of these assumptions are computationally efficient to verify). The second downside is that while the algorithms return the optimal solution when the input is stable, there are no partial guarantees when most, but not all, of the data is stable. For example, the algorithms of Balcan and Liang \cite{12} and Angelidakis et al. \cite{3} return the optimal clustering when the instance is resilient to perturbations, however, both algorithms use a dynamic programming subroutine that is susceptible to errors which can propagate when a small fraction of the data does not satisfy perturbation resilience (see Appendix \ref{appendix} for more details). From these setbacks, two natural questions arise. (1) Can we find natural algorithms that achieve the worst-case approximation ratio, while outputting the optimal solution if the data is stable,\footnote{1 A trivial solution is to run an approximation algorithm and a BWCA algorithm in parallel, and output the better of the two solutions. We seek a more satisfactory and “natural” answer to this question.} and (2) Can we construct robust algorithms that still output good results even when only part of the data satisfies stability? The current work seeks to answer both of these questions.

1.1 Our results and techniques

In this work, we answer both questions affirmatively for a variety of clustering objectives under perturbation resilience. We present algorithms that simultaneously achieve state-of-the-art approximation ratios in the worst case, while outputting the optimal solution when the data is stable. All of our algorithms are natural modifications to existing approximation algorithms. To answer question (2), we define the notion of local perturbation resilience in Section \ref{sec:local}. This is the first definition that applies to an individual cluster, rather than the dataset as a whole. Informally, an optimal cluster satisfies $\alpha$-local perturbation resilience if it remains in the optimal solution under any $\alpha$ perturbation to the input. We show that every optimal cluster...
satisfying local stability will be returned by our algorithms. Therefore, given an instance with a mix of stable and non-stable data, our algorithms will return the optimal clusters over the stable data, and a worst-case approximation guarantee over the rest of the data. Specifically, we prove the following results.

**Approximation algorithms under local perturbation resilience** In Section 3 we introduce a condition that is sufficient for an $\alpha$-approximation algorithm to return the optimal $k$-median or $k$-means clusters satisfying $\alpha$-local perturbation resilience. Intuitively, our condition requires that the approximation guarantee must be true locally as well as globally. We show the popular local search algorithm satisfies the property for a sufficiently large search parameter. For $k$-center, we show that any $\alpha$-approximation algorithm for $k$-center will always return the clusters satisfying $\alpha$-local metric perturbation resilience, which is a slightly weaker notion of perturbation resilience.

**Asymmetric $k$-center** The asymmetric $k$-center problem admits an $O(\log^* n)$ approximation algorithm due to Vishnwanathan [38], which is tight [19]. In Section 4 we show a simple modification to this algorithm ensures that it returns all optimal clusters satisfying a condition slightly stronger than 2-local perturbation resilience (all neighboring clusters must satisfy 2-local perturbation resilience). The first phase of Vishwanathan’s approximation algorithm involves iteratively removing the neighborhood of special points called center-capturing vertices (CCVs). We show the centers of locally perturbation resilient clusters are CCVs and satisfy a separation condition, by constructing 2-perturbations in which neighboring clusters cannot be too close to the locally perturbation resilient centers without causing a contradiction. This allows us to modify the approximation algorithm by first removing the neighborhood around CCVs satisfying the separation condition. With a careful reasoning, we maintain the original approximation guarantee while only removing points from a single local perturbation resilient cluster at a time.

**Robust perturbation resilience** In Section 5 we consider $(\alpha, \epsilon)$-local perturbation resilience, which states that at most $\epsilon n$ points can swap into or out of the cluster under any $\alpha$-perturbation. For $k$-center, we show that any 2-approximation algorithm will return the optimal $(3, \epsilon)$-locally perturbation resilient clusters, assuming a mild lower bound on optimal cluster sizes. To prove this, we show that if points from two different locally perturbation resilient clusters are close to each other, then $k-1$ centers achieve the optimal value under a carefully constructed 3-perturbation. The rest of the analysis involves building up conditional claims dictating the possible centers for each locally perturbation resilient cluster under the 3-perturbation. We utilize the idea of a cluster-capturing center [11] along with machinery specific to handling local perturbation resilience to show that a locally perturbation resilient cluster must split into two clusters under the 3-perturbation, causing a contradiction. Finally, we show that the mild lower bound on the cluster sizes is necessary. Specifically, we show hardness of approximation for $k$-median, $k$-means, and $k$-center, even when it is guaranteed the clustering satisfies $(\alpha, \epsilon)$-perturbation resilience for any $\alpha \geq 1$ and $\epsilon > 0$. In fact, the result holds even for a stronger notion called $(\alpha, \epsilon)$-approximation stability. The hardness is based on a reduction from the general clustering instances, so the APX-hardness constants match the worst-case APX-hardness results of 2, 1.73, and 1.0013 for $k$-center [23], $k$-median [28], and $k$-means [32], respectively. This generalizes prior hardness results in BWCA [10, 11].

### 1.2 Related work

**Clustering** The first constant-factor approximation algorithm for $k$-median was given by Charikar et al. [16], and the current best approximation ratio is 2.675 by Byrka et al. [15]. Jain et al. proved $k$-median is NP-hard to approximate to a factor better than 1.73 [28]. For $k$-center, Gonzalez showed a tight 2-approximation algorithm [23]. For $k$-means, the best approximation ratio was recently lowered to 6.357 by
Ahmadian et al. [2], k-means was shown to be APX-hard by Awasthi et al. [8], and the constant was recently improved to 1.0013 [32].

**Perturbation resilience** Perturbation resilience was introduced by Bilu and Linial, who showed algorithms that outputted the optimal solution for max cut under $\Omega(\sqrt{n})$-perturbation resilience [13]. This result was improved by Markarychev et al. [34], who showed the standard SDP relaxation is integral for $\Omega(\sqrt{\log n \log \log n})$-perturbation resilient instances. They also show an optimal algorithm for minimum multiway cut under 4-perturbation resilience. The study of clustering under perturbation resilience was initiated by Awasthi et al. [7], who provided an optimal algorithm for center-based clustering objectives (which includes k-median, k-means, and k-center clustering, as well as other objectives) under 3-perturbation resilience. This result was improved by Balcan and Liang [12], who showed an algorithm for center-based clustering under $(1 + \sqrt{2})$-perturbation resilience. They also gave a near-optimal algorithm for k-median $(2 + \sqrt{3}, \epsilon)$-perturbation resilience, a robust version of perturbation resilience, when the optimal clusters are not too small. Balcan et al. [11] constructed algorithms for k-center and asymmetric k-center under 2-perturbation resilience and $(3, \epsilon)$-perturbation resilience, and they showed no polynomial-time algorithm can solve k-center under $(2 - \epsilon)$-approximation stability (a notion that is stronger than perturbation resilience) unless $NP = RP$. Recently, Angelidakis et al. [3], gave algorithms for center-based clustering under 2-perturbation resilience and minimum multiway cut with k terminals under $(2 - 2/k)$-perturbation resilience. They also define the more general notion of metric perturbation resilience. In Appendix [A] we discuss prior work in the context of local perturbation resilience. Perturbation resilience has also been applied to other problems, such as the traveling salesman problem, and finding Nash equilibria [10, 35].

**Approximation stability** Approximation stability is a related definition that is stronger than perturbation resilience. It was introduced by Balcan et al. [10], who showed algorithms that outputted nearly optimal solutions under $(\alpha, \epsilon)$-approximation stability for k-median and k-means when $\alpha > 1$. Balcan et al. [13] studied a relaxed notion of approximation stability in which a specified $\nu$ fraction of the data satisfies approximation stability. In this setting, there may not be a unique approximation stable solution. The authors provided an algorithm which outputted a small list of clusterings, such that all approximation stable clusterings are close to one clustering in the list. We remark the property itself is similar in spirit to local stability, although the solution/results are much different. Voevodski et al. [39] gave an algorithm for empirically clustering protein sequences using the min-sum objective under approximation stability, which compares favorably to popular clustering algorithms used in practice. Gupta et al. [25] showed algorithm for finding near-optimal solutions for k-median under approximation stability in the context of finding triangle-dense graphs.

**Other stability notions** Ostrovsky et al. show how to efficiently cluster instances in which the k-means clustering cost is much lower than the $(k-1)$-means cost [46]. Kumar and Kannan give an efficient clustering algorithm for instances in which the projection of any point onto the line between its cluster center to any other cluster center is a large additive factor closer to its own center than the other center [30]. This result was later improved along multiple axes by Awasthi and Sheffet [9]. There are many other works that show positive results for different natural notions of stability in various settings [4, 6, 25, 26, 30, 31, 37].

### 2 Preliminaries

A clustering instance consists of a set $S$ of $n$ points, as well as a distance function $d : S \times S \rightarrow \mathbb{R}_{\geq 0}$. For a point $u \in S$ and a set $A \subseteq S$, we define $d(u, A) = \min_{v \in A} d(u, v)$. The k-median, k-means, and k-center objectives are to find a set of points $X = \{x_1, \ldots, x_k\} \subseteq S$ called centers to minimize $\sum_{v \in S} d(v, X)$, $\sum_{v \in S} d(v, X)^2$, and $\max_{v \in S} d(v, X)$, respectively. We denote by $\text{Vor}_X(x)$ the Voronoi tile of $x$ induced
by $X$ on the set of points $S$, and we denote $\text{Vor}_X(X') = \bigcup_{x \in X'} \text{Vor}_X(x)$ for a subset $X' \subseteq X$. We refer to the Voronoi partition induced by $X$ as a clustering. Throughout the paper, we denote the clustering with minimum cost by $OPT = \{C_1, \ldots, C_k\}$, and we denote the optimal centers by $c_1, \ldots, c_k$, where $c_i$ is the center of $C_i$ for all $1 \leq i \leq k$.

All of the distance functions we study are metrics, except for Section 4, in which we study an asymmetric distance function. An asymmetric distance function satisfies all the properties of a metric space except for symmetry. In particular, an asymmetric distance function must satisfy the directed triangle inequality: for all $u, v, w \in S$, $d(u, w) \leq d(u, v) + d(v, w)$.

We formally define perturbation resilience, a notion introduced by Bilu and Linial [14]. $d'$ is called an $\alpha$-perturbation of the distance function $d$, if for all $u, v \in S$, $d(u, v) \leq d'(u, v) \leq \alpha d(u, v)$. (We only consider perturbations in which the distances increase because WLOG we can scale the distances to simulate decreasing distances.)

**Definition 1.** A clustering instance $(S, d)$ satisfies $\alpha$-perturbation resilience ($\alpha$-PR) if for any $\alpha$-perturbation $d'$ of $d$, the optimal clustering under $d'$ is unique and equal to $OPT$.

Now we define local perturbation resilience, a property of an optimal cluster rather than a dataset.

**Definition 2.** Given a clustering instance $(S, d)$ with optimal clustering $\mathcal{C} = \{C_1, \ldots, C_k\}$, an optimal cluster $C_i$ satisfies $\alpha$-local perturbation resilience ($\alpha$-LPR) if for any $\alpha$-perturbation $d'$ of $d$, the optimal clustering $\mathcal{C}'$ under $d'$ contains $C_i$.

We will sometimes refer to a center $c_i$ of an $\alpha$-LPR cluster $C_i$ as an $\alpha$-LPR center. Clearly, if a clustering instance is perturbation resilient, then every optimal cluster satisfies local perturbation resilience. Now we will show the converse is also true.

**Fact 3.** A clustering instance $(S, d)$ satisfies $\alpha$-PR if and only if each optimal cluster satisfies $\alpha$-LPR.

**Proof.** Given a clustering instance $(S, d)$, the forward direction follows by definition: assume $(S, d)$ satisfies $\alpha$-PR, and given an optimal cluster $C_i$, then for each $\alpha$-perturbation $d'$, the optimal clustering stays the same under $d'$, therefore $C_i$ is contained in the optimal clustering under $d'$. Now we prove the reverse direction. Given a clustering instance with optimal clustering $\mathcal{C}$, and given an $\alpha$-perturbation $d'$, let the optimal clustering under $d'$ be $\mathcal{C}'$. For each $C_i \in \mathcal{C}$, by assumption, $C_i$ satisfies $\alpha$-LPR, so $C_i \in \mathcal{C}'$. Therefore $\mathcal{C} = \mathcal{C}'$. \qed

In Section 4 we define a stronger version of Definition 2 specifically for $k$-center. Next, we define a more robust version of $\alpha$-PR and $\alpha$-LPR that allows a small change in the optimal clusters when the distances are perturbed. We say that two clusters $A$ and $B$ are $\epsilon$-close if they differ by only $en$ points, i.e., $|A \setminus B| + |B \setminus A| \leq en$. We say that two clusterings $\mathcal{C}$ and $\mathcal{C}'$ are $\epsilon$-close if $\min_\sigma \sum_{i=1}^k |C_i \setminus C'_\sigma(i)| \leq en$.

**Definition 4.** [12] A clustering instance $(S, d)$ satisfies $(\alpha, \epsilon)$-perturbation resilience ($(\alpha, \epsilon)$-PR) if for any $\alpha$-perturbation $d'$ of $d$, all optimal clusterings under $d'$ must be $\epsilon$-close to $OPT$.

**Definition 5.** Given a clustering instance $(S, d)$ with optimal clustering $\mathcal{C} = \{C_1, \ldots, C_k\}$, an optimal cluster $C_i$ satisfies $(\alpha, \epsilon)$-local perturbation resilience ($(\alpha, \epsilon)$-LPR) if for any $\alpha$-perturbation $d'$ of $d$, the optimal clustering $\mathcal{C}'$ under $d'$ contains a cluster $C'_i$ which is $\epsilon$-close to $C_i$.

We prove a statement similar to Fact 3 for $(\alpha, \epsilon)$-PR, but the $\epsilon$ error adds up among the clusters. See Appendix B for the proof.

**Lemma 6.** A clustering instance $(S, d)$ satisfies $(\alpha, \epsilon)$-PR if and only if each optimal cluster $C_i$ satisfies $(\alpha, \epsilon_i)$-LPR and $\sum_i \epsilon_i \leq 2en$. 

In all definitions thus far, we do not assume that the \( \alpha \)-perturbations satisfy the triangle inequality. Angelidakis et al. \cite{Angelidakis2015} recently studied the weaker definition in which the \( \alpha \)-perturbations must satisfy the triangle inequality, called metric perturbation resilience. All of our definitions can be generalized accordingly, and some of our results hold under this weaker assumption. To this end, we will sometimes take the metric completion \( d' \) of a non-metric distance function \( d'' \), by setting the distances in \( d' \) as the length of the shortest path on the graph whose edges are the lengths in \( d'' \).

## 3 Approximation algorithms under local perturbation resilience

In this section, we show that local search for \( k \)-median will always return the \((3 + \epsilon)\)-LPR clusters, and for \( k \)-means it will return the \((9 + \epsilon)\)-LPR clusters. We also show that any 2-approximation for \( k \)-center will return the 2-LPR clusters.

### k-median

We start by giving a condition on an approximate \( k \)-median solution, which is sufficient to show the solution contains all \( \alpha \)-LPR clusters.

**Lemma 7.** Given a \( k \)-median instance \((S, d)\) and a set of \( k \) centers \( X \), if for all sets \( Y \) of size \( k \),

\[
\sum_{v \in \text{Vor}_X(X \cap Y) \cup \text{Vor}_Y(Y \setminus X)} d(v, X) \leq \sum_{v \in \text{Vor}_Y(Y \setminus X)} \min(d(v, X), \alpha d(v, Y)) + \alpha \sum_{v \in \text{Vor}_X(X \setminus Y)} d(v, Y),
\]

then all \( \alpha \)-LPR clusters \( C_i \) are contained in the clustering defined by \( X \).

**Proof.** Given such a set of centers \( X \), we construct an \( \alpha \)-perturbation \( d' \) as follows. Increase all distances by a factor of \( \alpha \) except for the distances between each point \( v \) and its closest center in \( X \). Now our goal is to show that \( X \) is the optimal set of centers under \( d' \).

Given any other set \( Y \) of \( k \) centers, we consider four types of points: \( \text{Vor}_X(X \cap Y) \cap \text{Vor}_Y(X \cap Y) \), \( \text{Vor}_X(X \cap Y) \setminus \text{Vor}_Y(X \cap Y) \), \( \text{Vor}_Y(X \cap Y) \setminus \text{Vor}_X(X \cap Y) \), and \( \text{Vor}_X(X \setminus Y) \cap \text{Vor}_Y(Y \setminus X) \), which we denote by \( A_1 \), \( A_2 \), \( A_3 \), and \( A_4 \), respectively (see Figure \[1\]). The distance from a point \( v \in S \) to its center in \( Y \) might stay the same under \( d' \), or increase, depending on its type. For each point \( v \in A_1 \), \( d'(v, Y) = d(v, Y) = d(v, X) \) because these points have centers in \( X \cap Y \). For each point \( v \in A_3 \cup A_4 \), \( d'(v, Y) = \alpha d(v, Y) \) because their centers are in \( Y \setminus X \). The points in \( A_2 \) were originally closest to a center in \( Y \setminus X \), but might switch to their center in \( X \), since it is in \( X \cap Y \). Therefore, for each \( v \in A_2 \), \( d'(v, Y) = \min(d(v, X), \alpha d(v, Y)) \). Altogether,

\[
\sum_{v \in S} d'(v, Y) \geq \sum_{v \in A_1} d(v, X) + \sum_{v \in A_2} \min(d(v, X), \alpha d(v, Y)) + \alpha \sum_{v \in A_3 \cup A_4} d(v, Y).
\]

The cost of the clustering induced by \( X \) under \( d' \), using our assumption, is equal to

\[
\sum_{v \in S} d'(v, X) \leq \sum_{v \in A_1} d(v, X) + \sum_{v \in A_2} \min(d(v, X), \alpha d(v, Y)) + \alpha \sum_{v \in A_3 \cup A_4} d(v, Y).
\]

Therefore, \( X \) is the optimal set of centers under the perturbation \( d' \). Given an \( \alpha \)-LPR cluster \( C_i \), by definition, there exists \( x \in X \) such that \( \text{Vor}_X(x) = C_i \) under \( d' \), therefore by construction, \( \text{Vor}_X(x) = C_i \) under \( d \) as well. This proves the theorem. \( \square \)

Essentially, Lemma 7 claims that any \( \alpha \)-approximation algorithm will return the \( \alpha \)-LPR clusters as long as the approximation ratio is “uniform” across all clusters. For example, a 3-approximation algorithm that returns half of the clusters paying 1.1 times their cost in \( OPT \), and half of the clusters paying 5 times their...
(a) The bold lines represent center assignment in $X$, and the dotted lines represent center assignment in $Y$.

(b) $u$ is a CCV, so it is distance $2r^*$ to its entire cluster. Each directed arrow represents a distance of $r^*$.

Figure 1: The clustering setup for Lemma 7 and Theorem 8 (left), and an example of a center-capturing vertex (right).

cost in $OPT$, would fail the property. Luckily, the local search algorithm is well-suited for this property, due to the local optimum guarantee.

The local search algorithm starts with any set of $k$ centers, and iteratively replaces $t$ centers with $t$ different centers if it leads to a better clustering (see Algorithm 1). The number of iterations is $O\left(\frac{n}{\epsilon}\right)$.

The classic Local Search heuristic is widely used in practice, and many works have studied local search theoretically for $k$-median and $k$-means [5, 24, 29]. For more information on local search, see a general introduction by Aarts and Lenstra [1].

Algorithm 1 Local Search algorithm for $k$-median clustering

Input: $k$-median instance $(S, d)$, parameter $\epsilon$

1: Pick an arbitrary set of centers $C$ of size $k$.

2: While $\exists C'$ of size $k$ such that $|C \setminus C'| + |C' \setminus C| \leq \frac{2}{\epsilon}$ and $\text{cost}(C') \leq (1 - \frac{\epsilon}{2^k})\text{cost}(C)$

   • Replace $C$ with $C'$.

Output: Centers $C$

The next theorem utilizes Lemma 7 and a result by Cohen-Addad and Schwiegelshohn [21], who showed that local search returns the optimal clustering under a stronger version of $(3 + 2\epsilon)$-PR. For the formal proof of Theorem 8 see Appendix C.

Theorem 8. Given a $k$-median instance $(S, d)$, running local search with search size $\frac{1}{2}$ returns a clustering that contains every $(3 + 2\epsilon)$-LPR cluster, and it gives a $(3 + 2\epsilon)$-approximation overall.

The value of local perturbation resilience, $3 + 2\epsilon$, is tight due to a counterexample by Cohen-Addad et al. [20]. For $k$-means, we can use local search to return each $(9 + \epsilon)$-LPR cluster by using Lemma 7 (which works for squared distances as well) and adding the result from Kanungo et al. [29].

$k$-center Because the $k$-center objective takes the maximum rather than the average of all cluster costs, the equivalent of the condition in Lemma 7 is essentially satisfied by any $\alpha$-approximation algorithm. We will now prove an even stronger result. Any $\alpha$-approximation for $k$-center returns the $\alpha$-LPR clusters, even for metric perturbation resilience. First we state a lemma which allows us to reason about a specific class
of $\alpha$-perturbations which will be useful in this section as well as throughout the paper, for symmetric and asymmetric $k$-center. For the full proofs, see Appendix C.

**Lemma 9.** Given $\alpha \geq 1$ and an asymmetric $k$-center clustering instance $(S,d)$ with optimal radius $r^*$, let $d''$ denote an $\alpha$-perturbation such that for all $u,v$, either $d''(u,v) = \min(\alpha r^*, \alpha d(u,v))$ or $d''(u,v) = \alpha d(u,v)$. Let $d'$ denote the metric completion of $d''$. Then $d'$ is an $\alpha$-metric perturbation of $d$, and the optimal cost under $d'$ is $\alpha r^*$.

**Proof sketch.** First, $d'$ is a valid $\alpha$-metric perturbation of $d$ because for all $u,v$, $d(u,v) \leq d'(u,v) \leq d''(u,v) \leq \alpha d(u,v)$. To show the optimal cost under $d'$ is $\alpha r^*$, it suffices to prove that for all $u,v$, if $d(u,v) \geq r^*$, then $d'(u,v) \geq \alpha r^*$. This is true of $d''$ by construction, so we show it still holds after taking the metric completion of $d''$, which can shrink some distances. Given $u,v$ such that $d(u,v) \geq r^*$, there exists a path $u = u_0-u_1-\cdots-u_{s-1}-u_s = v$ such that $d''(u,v) = \sum_{i=0}^{s-1} d'(u_i, u_{i+1})$ and for all $0 \leq i \leq s-1$, $d'(u_i, u_{i+1}) = d''(u_i, u_{i+1})$. If one of the segments has length $\geq r^*$ in $d$, then it has length $\geq \alpha r^*$ in $d''$ and we are done. If not, all distances increase by exactly a factor of $\alpha$, so we sum up all distances to show $d'(u,v) \geq \alpha r^*$.

**Theorem 10.** Given an asymmetric $k$-center clustering instance $(S,d)$ and an $\alpha$-approximate clustering $C$, each $\alpha$-LPR cluster is contained in $C$, even under the weaker metric perturbation resilience condition.

**Proof sketch.** Similar to the proof of Lemma 7, we construct an $\alpha$-perturbation $d'$ and argue that $C$ becomes the optimal clustering under $d'$. Let $r^*$ denote the optimal $k$-center radius of $(S,d)$. First we define an $\alpha$-perturbation $d''$ by increasing the distance from each point $v \in S$ to its center $c$ in $C$ to $min\{\alpha r^*, \alpha d(v, C(v))\}$, and increase all other distances by a factor of $\alpha$. Then by Lemma 9, the metric completion of $d'$ of $d''$ has optimal cost $\alpha r^*$, and so $C$ is the optimal clustering. Now we finish off the proof in a manner identical to Lemma 7.

We remark that Theorem 10 generalizes the result from Balcan et al. [11]. Although Theorem 10 applies more generally to asymmetric $k$-center, it is most useful for symmetric $k$-center, for which there exist several $2$-approximation algorithms [22, 23, 27]. Asymmetric $k$-center is NP-hard to approximate to within a factor of $O(\log^* n)$ [19], so Theorem 10 only guarantees returning the $O(\log^* n)$-LPR clusters. In the next section, we show how to substantially improve this result.

## 4 Asymmetric $k$-center

In this section, we show that a slight modification to the $O(\log^* n)$ approximation algorithm of Vishwanathan [38] leads to an algorithm that maintains its performance in the worst case, while returning each cluster $C_i$ with the following property: $C_i$ is $2$-LPR, and all nearby clusters are $2$-LPR as well. This result also holds for metric perturbation resilience. We start by formally giving the stronger version of Definition 2.

**Definition 11.** An optimal cluster $C_i$ satisfies $\alpha$-strong local perturbation resilience ($\alpha$-SLPR) if for each $j$ such that there exists $u \in C_i$, $v \in C_j$ and $d(u,v) \leq r^*$, then $C_j$ is $\alpha$-LPR.

**Theorem 12.** Given an asymmetric $k$-center clustering instance $(S,d)$ of size $n$, Algorithm 3 returns each $2$-SLPR cluster exactly. For each $2$-LPR cluster $C_i$, Algorithm 3 outputs a cluster that is a superset of $C_i$ and does not contain any other $2$-LPR cluster. These statements hold for metric perturbation resilience as well. Finally, the overall clustering returned by Algorithm 3 is an $O(\log^* n)$-approximation.
Approximation algorithm for asymmetric $k$-center We start with a recap of the $O(\log^* n)$-approximation algorithm of Vishwanathan [38]. This was the first nontrivial algorithm for asymmetric $k$-center, and the approximation ratio was later proven to be tight [19]. To explain the algorithm, it is convenient to think of asymmetric $k$-center as a set covering problem. Given an asymmetric $k$-center instance $(S, d)$, define the directed graph $D_{(S, d)} = (S, A)$, where $A = \{(u, v) \mid d(u, v) \leq r^*\}$. For a point $v \in S$, we define $\Gamma^+(v)$ and $\Gamma^-(v)$ as the set of vertices with an arc to and from $v$, respectively. The asymmetric $k$-center problem is equivalent to finding a subset $C \subseteq S$ of size $k$ such that $\cup_{c \in C} \Gamma(c) = S$. We also define $\Gamma^-(v)$ and $\Gamma^+(v)$ as the set of vertices which have a path of length $\leq x$ to and from $v$ in $D_{(S, d)}$, respectively, and we define $\Gamma^+_x(A) = \bigcup_{v \in A} \Gamma^+_x(v)$ for a set $A \subseteq S$, and similarly for $\Gamma^-_x(A)$. It is standard to assume the value of $r^*$ is known; since it is one of $O(n^2)$ distances, the algorithm can search for the correct value in polynomial time.

Vishwanathan’s algorithm crucially utilizes the following concept.

**Definition 13.** Given an asymmetric $k$-center clustering instance $(S, d)$, a point $v \in S$ is a center-capturing vertex (CCV) if $\Gamma^-(v) \subseteq \Gamma^+(v)$. In other words, for all $u \in S$, $d(u, v) \leq r^*$ implies $d(v, u) \leq r^*$.

As the name suggests, each CCV $v \in C_i$, “captures” its center, i.e. $c_i \in \Gamma^+(v)$ (see Figure 1b). Therefore, $v$’s entire cluster is contained inside $\Gamma^+_x(v)$, which is a nice property that the approximation algorithm exploits. At a high level, the approximation algorithm has two phases. In the first phase, the algorithm iteratively picks a CCV $v$ arbitrarily and removes all points in $\Gamma^+_x(v)$. This continues until there are no more CCVs. For every CCV picked, the algorithm is guaranteed to remove an entire optimal cluster. In the second phase, the algorithm runs $\log^* n$ rounds of a greedy set-cover subroutine on the remaining points. See Algorithm 2. To prove the second phase terminates in $O(\log^* n)$ rounds, the analysis crucially assumes there are no CCVs among the remaining points. We refer the reader to [38] for these details.

| Algorithm 2 $O(\log^* n)$ APPROXIMATION ALGORITHM FOR ASYMMETRIC $k$-CENTER |
|----------------------------------------------------------------|
| **Input:** Asymmetric $k$-center instance $(S, d)$, optimal radius $r^*$ (or try all possible candidates) |
| Set $C = \emptyset$. |
| **Phase I:** Pull out arbitrary CCVs |
| While there exists an unmarked CCV |
| - Pick an unmarked CCV $c$, add $c$ to $C$, and mark all vertices in $\Gamma^+_2(c)$ |
| **Phase II:** Recursive set cover |
| Set $A_0 = S \setminus \Gamma^+_5(C)$, $i = 0$. While $|A_i| > k$: |
| - Set $A_{i+1}' = \emptyset$. While there exists an unmarked point in $A_i$: |
| - Pick $v \in S$ which maximizes $\Gamma^+_5(v) \cap A_i$ |
| - Mark all points $\Gamma^+_5(v) \cap A_i$ and add $v$ to $A_{i+1}'$. |
| - Set $A_{i+1} = A_{i+1}' \cup A$ and $i = i + 1$ |
| **Output:** Centers $C \cup A_{i+1}$ |

Robust algorithm for asymmetric $k$-center We show a small modification to Vishwanathan’s approximation algorithm leads to simultaneous guarantees in the worst case and under local perturbation resilience. We show that each 2-LPR center is itself a CCV, and displays other structure which allows us to distinguish it from non-center CCVs. This suggests a simple modification to Algorithm 2 instead of picking CCVs arbitrarily, we first pick CCVs which display the added structure, and then when none are left, we go back to picking regular CCVs. However, we need to ensure that a CCV chosen by the algorithm marks points from at most one 2-LPR cluster, or else we will not be able to output a separate cluster for each 2-LPR
cluster. Thus, the difficulty in our argument is carefully specifying which CCVs the algorithm picks, and which nearby points get marked by the CCVs, so that we do not mark other LPR clusters and simultaneously maintain the guarantee of the original approximation algorithm, namely that in every round, we mark an entire optimal cluster. To accomplish this tradeoff, we start by defining two properties. The first property will determine which CCVs are picked by the algorithm. The second property is used in the proof of correctness, but is not used explicitly by the algorithm. We give the full details of the proofs in Appendix D.

Definition 14. (1) A point $c$ satisfies CCV-proximity if it is a CCV, and each point in $\Gamma^-(c)$ is closer to $c$ than any CCV outside of $\Gamma^+(c)$. That is, for all points $v \in \Gamma^-(c)$ and CCVs $c' \notin \Gamma^+(c)$, $d(c, v) < d(c', v)$. 
(2) An optimal center $c_i$ satisfies center-separation if any point within distance $r^*$ of $c_i$ belongs to its cluster $C_i$. That is, for all $v \notin C_i$, $c_i \notin \Gamma^+(v)$.

Lemma 15. Given an asymmetric $k$-center clustering instance $(S, d)$ and a 2-LPR cluster $C_i$, $c_i$ satisfies CCV-proximity and center-separation. Furthermore, given a CCV $c \in C_i$, a CCV $c' \notin C_i$, and a point $v \in C_i$, we have $d(c, v) < d(c', v)$.

Proof sketch. Given an instance $(S, d)$ and a 2-LPR cluster $C_i$, we show that $c_i$ has the desired properties.

Center Separation: Assume there exists a point $v \in C_j$ for $j \neq i$ such that $d(v, c_i) \leq r^*$. The idea is to construct a 2-perturbation in which $v$ becomes the center for $C_i$, since the distance from $v$ to each point in $C_i$ is $\leq 2r^*$ by the triangle inequality. Define $d''$ by increasing all distances by a factor of 2, except for the distances between $v$ and each point $u \in C_i$, which we increase to $\min(2r^*, 2d(v, u))$. By Lemma 9, the metric completion $d'$ of $d''$ is a 2-metric perturbation with optimal cost $2r^*$, so we can replace $c_i$ with $v$ in the set of optimal centers under $d'$. However, now $c_i$ switches to a different cluster, contradicting 2-LPR.

Final property: Given CCVs $c \in C_i$, $c' \in C_j$, and a point $v \in C_i$, assume $d(c', v) \leq d(c, v)$. Again, we will use a perturbation to construct a contradiction. Since $c$ and $c'$ are CCVs and thus distance $2r^*$ to their clusters, we can construct a 2-metric perturbation with optimal cost $2r^*$ in which $c$ and $c'$ become centers for their respective clusters. Then $v$ switches clusters, so we have a contradiction.

CCV-proximity: By center-separation and the definition of $r^*$, we have that $\Gamma^-(c_i) \subseteq C_i \subseteq \Gamma^+(c_i)$, so $c_i$ is a CCV. Now given a point $v \in \Gamma^-(c_i)$ and a CCV $c \notin \Gamma^+(c_i)$, from center-separation and definition of $r^*$, $v \in C_i$ and $c \in C_j$ for $j \neq i$. Then from the property in the previous paragraph, $d(c_i, v) < d(c, v)$. □

Now we can modify the algorithm so that it first chooses CCVs satisfying CCV-proximity. The other crucial change is instead of each chosen CCV $c$ marking all points in $\Gamma_2^-(c)$, it instead marks all points $v$ such that $v \in \Gamma_2^+(c')$ for some $c' \in \Gamma^-(c)$. See Algorithm 3. Note this new way of marking preserves the guarantee that each CCV $c \in C_i$ marks its own cluster, because $c_i \in \Gamma^-(c)$. It also allows us to prove that each CCV $c$ satisfying CCV-proximity can never mark an LPR center $c_i$ from a different cluster. Intuitively, if $c$ marks $c_i$, then there exists a point $v \in \Gamma^-(c_i) \cap \Gamma^-(c)$, but there can never exist a point $v$ distance $\leq r^*$ to two points satisfying CCV-proximity, since both would need to be closer to $v$ by definition. Finally, the last property in Lemma 15 allows us to prove that when the algorithm computes the Voronoi tiles after Phase 1, all points will be correctly assigned. Now we are ready to prove Theorem 12.

Proof sketch of Theorem 12. First we explain why Algorithm 3 retains the approximation guarantee of Algorithm 2. Given any CCV $c \in C_i$ chosen in Phase I, $c$ marks its entire cluster by definition, and we start Phase II with no remaining CCVs. This condition is sufficient for Phase II to return an $O(\log^* n)$ approximation (Theorem 3.1 from [38]).

Next we claim that for each 2-LPR cluster $C_i$, there exists a cluster outputted by Algorithm 3 that is a superset of $C_i$ and does not contain any other 2-LPR cluster. To prove this claim, we first show there exists a point from $C_i$ satisfying CCV-proximity that cannot be marked by any point from a different cluster in...
Algorithm 3 Robust algorithm for asymmetric $k$-center

**Input:** Asymmetric $k$-center instance $(S,d)$, distance $r^*$ (or try all possible candidates)

Set $C = \emptyset$. Redefine $d$ using the shortest path length in $D_{(S,d)}$, breaking ties by distance to first common vertex in the shortest path.

**Phase I: Pull out special CCVs**

- While there exists an unmarked CCV:
  - Pick an unmarked point $c$ which satisfies CCV-proximity. If no such $c$ exists, then pick an arbitrary unmarked CCV instead. Add $c$ to $C$.
  - For all points $c' \in \Gamma^-(c)$, mark all points in $\Gamma^+(c')$.
- For each $c \in C$, let $V_c$ denote $c$’s Voronoi tile of the marked points induced by $C$.

**Phase II: Recursive set cover**

Run Phase II as in Algorithm 2, outputting $A_{i+1}$. Compute the Voronoi tile for each center in $C \cup A_{i+1}$, but a point in $V_c$ must remain in $c$’s Voronoi tile.

Phase I. From Lemma 15, $c_i$ satisfies CCV-proximity and center-separation. If a point $c \notin C_i$ marks $c_i$, then $\exists v \in \Gamma^-(c) \cap \Gamma^-(c_i)$. By center-separation, $c \notin \Gamma^-(c_i)$. Then from the definition of CCV-proximity, both $c$ and $c_i$ must be closer to $v$ than the other, causing a contradiction. At this point, we know a point $c \in C_i$ will always be chosen by the algorithm in Phase I. To finish the claim, we show that each point $v$ from $C_i$ is closer to $c$ than to any other point $c' \notin C_i$ chosen in Phase I. Since $c$ and $c'$ are both CCVs, this follows directly from Lemma 15. However, it is possible that a center $c' \in A_{i+1}$ is closer to $v$ than $c$ is to $v$, causing $c'$ to “steal” $v$: this is unavoidable. Therefore, we forbid the algorithm from decreasing the size of the Voronoi tiles of $C$ after Phase I.

Finally, we claim that Algorithm 3 returns each 2-SLPR cluster exactly. Given a 2-SLPR cluster $C_i$, by our previous argument, the algorithm chooses a CCV $c \in C_i$ such that $C_i \subseteq V_c$. It is left to show that $V_c \subseteq C_i$. The intuition is that since $C_i$ is 2-SLPR, its neighboring clusters were also marked in Phase I, and these clusters “shield” $V_c$ from picking up superfluous points in Phase II. Specifically, there are two cases. If there exists $v \in V_c \subseteq C_i$ that was marked in Phase I, then we can prove that $v$ comes from a 2-LPR cluster, so $v \in V_c$ contradicts our previous argument. If there exists $v \in V_c \subseteq C_i$ from Phase II, then the shortest path in $D_{(S,d)}$ from $c$ to $v$ is length at least 5 (see Figure 2a). The first point $u' \in C_j$, $j \neq i$ on the shortest path must come from a 2-LPR cluster, and we prove that $v$ is closer to $C_j$’s cluster using CCV-proximity.

\[\square\]

5 Robust local perturbation resilience

In this section, we show that any 2-approximation algorithm for $k$-center returns the optimal $(3, \epsilon)$-SLPR clusters, provided the clusters are size $> 2\epsilon n$. Our main structural result is the following theorem.

**Theorem 16.** Given a $k$-center clustering instance $(S,d)$ with optimal radius $r^*$ such that all optimal clusters are size $> 2\epsilon n$ and there are at least three $(3, \epsilon)$-LPR clusters, then for each pair of $(3, \epsilon)$-LPR clusters $C_i$ and $C_j$, for all $u \in C_i$ and $v \in C_j$, we have $d(u,v) > r^*$.

Later in this section, we will show that both added conditions (the lower bound on the size of the clusters, and that there are at least three $(3, \epsilon)$-LPR clusters) are necessary. Since the distance from each point to its closest center is $\leq r^*$, a corollary of Theorem 16 is that any 2-approximate solution must contain the optimal $(3, \epsilon)$-SLPR clusters, as long as the 2-approximation satisfies two sensible conditions: (1) for every edge
Proof of Theorem 12. The arrows represent distances of \( \leq r^* \).

(c) is a CCC for \( C_j \), and \( c_j \) is a CCC2 for \( C_j \). The black disks have radius \( r^* \).

Figure 2: The proof of Theorem 12 (left), and an example of a cluster-capturing center (right).

d(\( u, v \)) \leq 2r^* in the 2-approximation, \( \exists w \) s.t. \( d(\( u, w \)) \) and \( d(\( w, v \)) \) are \( \leq r^* \), and (2) there cannot be multiple clusters outputted in the 2-approximation that can be combined into one cluster with the same radius. Both of these properties are easily satisfied using quick pre- or post-processing steps. We may also combine this result with Theorem 10 to obtain a more powerful result for \( k \)-center.

Theorem 17. Given a \( k \)-center clustering instance \((S, d)\) such that all optimal clusters are size > \( 2en \) and there are at least three \((3, \epsilon)\)-LPR clusters, then any 2-approximate solution satisfying conditions (1) and (2) must contain all optimal 2-LPR clusters and \((3, \epsilon)\)-SLPR clusters.

Proof. Given such a clustering instance, then Theorem 16 ensures that there is no edge of length \( r^* \) between points from two different \((3, \epsilon)\)-LPR clusters. Given a \((3, \epsilon)\)-SLPR cluster \( C_i \), it follows that there is no point \( v \notin C_i \) such that \( d(v, C_i) \leq r^* \). Therefore, given a 2-approximate solution \( C \) satisfying condition (1), any \( u \in C_i \) and \( v \notin C_i \) cannot be in the same cluster. Furthermore, by condition (2), \( C_i \) must not be split into two clusters. Therefore, \( C_i \in \mathcal{C} \). The second part of the statement follows directly from Theorem 10.

Proof idea for Theorem 16 The proof consists of two parts. The first part is to show that if two points from different LPR clusters are close together, then all points in the clustering instance must be near each other, in some sense (Lemma 22). The second part of the proof consists of showing that the points from three LPR clusters must be reasonably far from one another; therefore, we achieve the final result by contradiction.

Here is the intuition for part 1. Assume that there are points \( u \in C_i \) and \( v \in C_j \) from different LPR clusters, but \( d(\( u, v \)) \leq r^* \). Then by the triangle inequality, the distance from \( u \) to \( C_i \cup C_j \) is less than \( 3r^* \). We show that under a suitable 3-perturbation, we can replace \( c_i \) and \( c_j \) with \( u \) in the set of optimal centers. So, there is a 3-perturbation in which the optimal solution uses just \( k - 1 \) centers. However, as pointed out in [11], we are still a long way off from showing a contradiction. Since the definition of local perturbation resilience reasons about sets of \( k \) centers, we must add a “dummy center”. But adding any point as a dummy center might not immediately result in a contradiction, if the voronoi partition “accidentally” outputs the LPR clusters. To handle this problem, we use the notion of a cluster-capturing center [11].

\footnote{For condition (1), before running the algorithm, remove all edges of distance > \( r^* \), and then take the metric completion of the resulting graph. For condition (2), given the radius \( \hat{r} \) of the outputted solution, for each \( v \in S \), check if the ball of radius \( \hat{r} \) around \( v \) captures multiple clusters. If so, combine them.}
intuitively, a center which is within \( r^* \) of most of the points of a different optimal cluster (see Figure 2b). This allows us to construct perturbations and control which points become centers for which clusters. We show all of the points in the instance are close together, in some sense.

The second part of the argument diverges from all previous work in perturbation resilience, since finding a contradiction under local perturbation resilience poses a novel challenge. From the previous part of the proof, we are able to find two noncenters \( p \) and \( q \), which are collectively close to all other points in the dataset. Then we construct a 3-perturbation such that any size \( k \) subset of \( \{c_i\}_{i=1}^k \cup \{p, q\} \) is an optimal set of centers. Our goal is to show that at least one of these subsets must break up a LPR cluster, causing a contradiction. There are many cases to consider, so we build up conditional structural claims dictating the set of centers. Our goal is to show that at least one of these subsets must break up a LPR cluster, causing a contradiction under \( \alpha, \epsilon \)-perturbation resilience by Balcan et al. [11]. We start with the following fact.

**Fact 18.** Given a \( k \)-center clustering instance \((S, d)\) such that all optimal clusters have size \( > 2\epsilon n \), let \( d' \) denote an \( \alpha \)-perturbation with optimal centers \( C' = \{c'_1, \ldots, c'_k\} \). Let \( C' \) denote the set of \((\alpha, \epsilon)\)-LPR clusters. Then there exists a one-to-one function \( f : C' \rightarrow C' \) such that for all \( C_i \in C' \), \( f(C_i) \) is the center for more than half of the points in \( C_i \) under \( d' \).\(^5\)

In words, for any set of optimal centers under an \( \alpha \)-perturbation, each LPR cluster can be paired to a unique center. This follows simply because all optimal clusters are size \( > 2\epsilon n \), yet under a perturbation, \( < \epsilon n \) points can switch out of each LPR cluster. Next, we give the following definition, which will be a key point in the first part of the proof.

**Definition 19.** [11] A center \( c_i \) is a first-order cluster-capturing center (CCC) for \( C_j \) if for all \( x \neq j \), for all but \( \epsilon n \) points \( v \in C_j \), \( d(c_i, v) < d(c_x, v) \) and \( d(c_i, v) \leq r^* \). \( c_i \) is a second-order cluster-capturing center (CC2) for \( C_j \) if there exists an \( \ell \) such that for all \( x \neq j, \ell \), for all but \( \epsilon n \) points \( v \in C_j \), \( d(c_i, v) < d(c_x, v) \) and \( d(c_i, v) \leq r^* \). Then we say that \( c_i \) is a CCC2 for \( c_j \) discounting \( c_\ell \). See Figure 2b.

Intuitively, a center \( c_x \) is a CCC for \( C_y \) if \( c_x \) is a valid center for \( C_y \) when \( c_y \) is removed from the set of optimal centers. This is particularly useful when \( C_y \) is \((\alpha, \epsilon)\)-LPR, since we can combine it with Fact 18 to show that \( c_x \) is the unique center for the majority of points in \( C_y \). Another key idea in our analysis is the following concept.

**Definition 20.** A set \( C \subseteq S \) \((\beta, \gamma)\)-hits \( S \) if for all \( s \in S \), there exist \( \beta \) points in \( C \) at distance \( \leq \gamma r^* \) to \( s \).

We present the following lemma to demonstrate the usefulness of Definition 20 although this lemma will not be used until the second half of the proof of Theorem 16.

**Lemma 21.** Given a \( k \)-center clustering instance \((S, d)\), given \( z \geq 0 \), and given a set \( C \subseteq S \) of size \( k + z \) which \((z + 1, \alpha)\)-hits \( S \), there exists an \( \alpha \)-perturbation \( d' \) such that all size \( k \) subsets of \( C \) are optimal sets of centers under \( d' \).

\(^5\)A non local version of this fact appeared in [11].
Proof. Consider the following perturbation \( d'' \).

\[
d''(s, t) = \begin{cases} 
\min(\alpha r^*, \alpha d(s, t)) & \text{if } s \in C \text{ and } d(s, t) \leq \alpha r^* \\
\alpha d(s, t) & \text{otherwise.}
\end{cases}
\]

This is an \( \alpha \)-perturbation by construction. Define \( d' \) as the metric completion of \( d'' \). Then by Lemma 9, \( d' \) is an \( \alpha \)-metric perturbation with optimal cost \( \alpha r^* \). Given any size \( k \) subset \( C' \subseteq C \), then for all \( v \in S \), there is still at least one \( c \in C' \) such that \( d(c, v) \leq \alpha r^* \), therefore by construction, \( d'(c, v) \leq \alpha r^* \). It follows that \( C' \) is a set of optimal centers under \( d' \).

Now we are ready to prove the first half of Theorem 16, stated in the following lemma. For the full details, see Appendix E.

Lemma 22. Given a \( k \)-center clustering instance \((S, d)\) such that all optimal clusters are size \( > 2en \) and there exist two points at distance \( r^* \) from different \((3, \epsilon)\)-LPR clusters, then there exists a partition \( S_x \cup S_y \) of the non-centers \( S \setminus \{c_\ell\}_{\ell=1}^k \) such that for all pairs \( p \in S_x, q \in S_y, \{c_\ell\}_{\ell=x}^{x'} \cup \{p, q\} \) \((3, 3)\)-hits \( S \).

Proof sketch. This proof is split into two main cases. The first case is the following: there exists a CCC2 for a \((3, \epsilon)\)-LPR cluster, discounting a \((3, \epsilon)\)-LPR cluster. In fact, in this case, we do not need the assumption that two points from different LPR clusters are close. If there exists a CCC to a \((3, \epsilon)\)-LPR cluster, denote the CCC by \( c_x \) and the cluster by \( C_y \). Otherwise, let \( c_x \) denote a CCC2 to a \((3, \epsilon)\)-LPR cluster \( C_y \), discounting a \((3, \epsilon)\)-LPR center \( c_x \). Then \( c_x \) is at distance \( \leq r^* \) to all but \( en \) points in \( C_y \). Therefore, \( d(c_x, c_y) \leq 2r^* \) and so \( c_x \) is at distance \( \leq 3r^* \) to all points in \( C_y \). Now we can create a \( 3 \)-perturbation \( d' \) by increasing all distances by a factor of 3 except for the distances between \( c_x \) and each point \( v \in C_y \), which we increase to \( \min(3r^*, 3d(c_x, v)) \). Then by Lemma 9, \( d' \) is a \( 3 \)-perturbation with optimal cost \( 3r^* \). Therefore, given any non-center \( v \in S \), the set of centers \( \{c_\ell\}_{\ell=x}^{x'} \cup \{c_y\} \cup \{v\} \) achieves the optimal score, and from Fact 18 one of the centers in \( \{c_\ell\}_{\ell=x}^{x'} \cup \{c_y\} \cup \{v\} \) must be the center for the majority of points in \( C_y \) under \( d' \). If this center is \( c_\ell, \ell \neq x, y \), then by definition, \( c_\ell \) is a CCC for the \((3, \epsilon)\)-LPR cluster, \( C_y \), which creates a contradiction because \( \ell \neq x \). Therefore, either \( v \) or \( c_x \) must be the center for the majority of points in \( C_y \) under \( d' \).

If \( c_x \) is the center for the majority of points in \( C_y \), then because \( C_y \) is \((3, \epsilon)\)-LPR, the corresponding cluster must contain fewer than \( en \) points from \( C_x \). Furthermore, since for all \( \ell \neq x \) and \( u \in C_x, d(u, c_x) < d(u, c_x) \), it follows that \( v \) must be the center for the majority of points in \( C_x \). Therefore, every non-center \( v \in S \) is at distance \( \leq r^* \) to all of the majority of points in either \( C_x \) or \( C_y \).

Now partition all the non-centers into two sets \( S_x \) and \( S_y \), such that \( S_x = \{ p \mid \text{for the majority of points } q \in C_x, d(p, q) \leq r^* \} \) and \( S_y = \{ p \mid p \notin S_x \text{ and for the majority of points } q \in C_y, d(p, q) \leq r^* \} \). Given \( p, q \in S_x \), then \( d(p, q) \leq 2r^* \) since both points are close to more than half the points in \( C_x \). Similarly, any two points \( p, q \in S_y \) are \( \leq 2r^* \) apart.

Now we claim that \( \{c_\ell\}_{\ell=x}^{x'} \cup \{p, q\} \) \((3, 3)\)-hits \( S \) for any pair \( p \in S_x, q \in S_y \). This is because a point \( v \in C_x \) is \( 3r^* \) from \( S_x \) to \( c_x \) and \( c_x \) and a point \( v \in C_x \) is \( 3r^* \) to \( c_x \). The latter follows because \( d(c_x, c_y) \leq 2r^* \). Similar statements are true for \( S_y \) and \( C_y \).

Now we turn to the other case. Assume there does not exist a CCC2 to a LPR cluster, discounting a LPR center. In this case, we need to use the assumption that there exist \((3, \epsilon)\)-LPR clusters \( C_x \) and \( C_y \), and \( p \in C_x, q \in C_y \) such that \( d(p, q) \leq r^* \). Then by the triangle inequality, \( p \) is distance \( \leq 3r^* \) to all points in \( C_x \) and \( C_y \). Again we construct a \( 3 \)-perturbation \( d' \) by increasing all distances by a factor of 3 except for the distances between \( p \) and \( v \in C_x \cup C_y \), which we increase to \( \min(3r^*, 3d(s, t)) \). By Lemma 9, \( d' \) has optimal cost \( 3r^* \). Then given any non-center \( s \in S \), the set of centers \( \{c_\ell\}_{\ell=x}^{x'} \cup \{c_x, c_y\} \cup \{p, s\} \) achieves the optimal score.

From Fact 18, one of the centers in \( \{c_\ell\}_{\ell=x}^{x'} \cup \{c_x, c_y\} \cup \{p, s\} \) must be the center for the majority of points in \( C_x \) under \( d' \). If this center is \( c_\ell \) for \( \ell \neq x, y \), then \( c_\ell \) is a CCC2 for \( C_x \) discounting \( c_y \), which...
contradicts our assumption. Similar logic applies to the center for the majority of points in \( C_y \). Therefore, \( p \) and \( s \) must be the centers for \( C_x \) and \( C_y \). Since \( s \) was an arbitrary non-center, all non-centers are distance \( \leq r^* \) to all but \( \epsilon n \) points in either \( C_x \) or \( C_y \).

Similar to Case 1, we now partition all the non-centers into two sets \( S_x \) and \( S_y \), such that \( S_x = \{ u \mid \text{for the majority of points } v \in C_x, \ d(u, v) \leq r^* \} \) and \( S_y = \{ u \mid u \notin S_x \text{ and for the majority of points } v \in C_y, \ d(u, v) \leq r^* \} \). As before, each pair of points in \( S_x \) are distance \( \leq 2r^* \) apart, and similarly for \( S_y \).

Again we must show that \( \{ c_\ell \}_{\ell=1}^k \cup \{ p, q \} \) \((3, 3)\)-hits \( S \) for each pair \( p \in S_x, q \in S_y \). It is no longer true that \( d(c_x, c_y) \leq 2r^* \), however, we can prove that for both \( S_x \) and \( S_y \), there exist points from two distinct clusters each. From the previous paragraph, given a non-center \( s \in C_i \) for \( i \neq x, y \), we know that \( p \) and \( s \) are centers for \( C_x \) and \( C_y \). With an identical argument, given \( t \in C_j \) for \( j \neq x, y, i \), we can show that \( q \) and \( t \) are centers for \( C_x \) and \( C_y \). It follows that \( S_x \) and \( S_y \) both contain points from at least two distinct clusters. Now given a non-center \( s \in C_i \), \( \text{WLOG } s \in S_x \), then there exists \( j \neq i \) and \( t \in C_j \cap S_x \). Then \( c_i, c_j, \) and \( p \) are \( 3r^* \) to \( s \) and \( c_i, c_j, \) and \( p \) are \( 3r^* \) to \( c_i \). In the case where \( i = x \), then \( c_i, c_j, \) and \( p \) are \( 3r^* \) to \( c_i \). This concludes the proof.

Now we move to the second half of the proof of Theorem \[16\]. Recall that our goal is to show a contradiction assuming two points from different LPR clusters are close. From Lemma \[21\] and Lemma \[22\] we know there is a set of \( k + 2 \) points, and any size \( k \) subset is optimal under a suitable perturbation. And by Lemma \[18\], each size \( k \) subset must have a mapping from LPR clusters to centers. Now we state a fact which states these mappings are derived from a ranking of all possible center points by the LPR clusters. In other words, each LPR cluster \( C_x \) can rank all the points in \( S \), so that for any set of optimal centers for an \( \alpha \)-perturbation, the top-ranked center is the one whose cluster is \( \epsilon \)-close to \( C_x \). We defer the proof to Appendix \[E\].

**Fact 23.** Given a \( k \)-center clustering instance \((S, d)\) such that all optimal clusters have size \( > 2\epsilon n \), and an \( \alpha \)-perturbation \( d' \) of \( d \), let \( C' \) denote the set of \((\alpha, \epsilon)\)-LPR clusters. For each \( C_x \in C' \), there exists a bijection \( R_{x,d'} : S \to [n] \) such that for all sets of \( k \) centers \( C \) that achieve the optimal cost under \( d' \), then \( c = \arg \min_{c' \in C} R_{x,d'}(c') \) if and only if \( \text{Vor}_C(c) \) is \( \epsilon \)-close to \( C_x \).

For an LPR cluster \( C_x \) and a subset \( S' \subseteq S \) of size \( n' \), we also define \( R_{x,d',S'} : S' \to [n'] \) as the ranking specific to \( S' \). Now, using Fact \[23\] with the previous Lemmas, we can try to give a contradiction by showing that there is no set of rankings for the LPR clusters that is consistent with all the optimal sets of centers guaranteed by Lemmas \[21\] and \[22\]. The following lemma gives relationships among the possible rankings. These will be our main tools for contradicting LPR and thus finishing the proof of Theorem \[16\]. Again, the proof is provided in Appendix \[E\].

**Lemma 24.** Given a \( k \)-center clustering instance \((S, d)\) such that all optimal clusters are size \( > 2\epsilon n \), and given non-centers \( p, q \in S \) such that \( C = \{ c_\ell \}_{\ell=1}^k \cup \{ p, q \} \) \((3, 3)\)-hits \( S \), let the set \( C' \) denote the set of \((3, \epsilon)\)-LPR clusters. Define the \( 3 \)-perturbation \( d' \) as in Lemma \[21\]. The following are true.

1. Given \( C_x \in C' \) and \( c_i \) such that \( i \neq x, R_{x,d'}(c_x) < R_{x,d'}(c_i) \).
2. There do not exist \( s \in C \) and \( C_x, C_y \in C' \) such that \( x \neq y, R_{x,d',C}(s) + R_{y,d',C}(s) \leq 4 \).
3. Given \( C_t \) and \( C_x, C_y \in C' \) such that \( x \neq y \neq i, \text{if} R_{x,d',C}(c_i) \leq 3, \text{then} R_{y,d',C}(p) \geq 3 \) and \( R_{y,d',C}(q) \geq 3 \).

We are almost ready to bring everything together to give a contradiction. Recall that Lemma \[22\] allows us to choose a pair \((p, q)\) such that \( \{ c_\ell \}_{\ell=1}^k \cup \{ p, q \} \) \((3, 3)\)-hits \( S \). For an arbitrary choice of \( p \) and \( q \), we may not end up with a contradiction. It turns out, we will need to make sure one of the points comes from an LPR cluster, and is very high in the ranking list of its own cluster. This motivates the following fact, which is the final piece to the puzzle.
**Fact 25.** Given a k-center clustering instance \((S, d)\) such that all optimal clusters are size \(> 2\epsilon n\), given an \((\alpha, \epsilon)\)-LPR cluster \(C_x\), and given \(i \neq x\), then there are fewer than \(\epsilon n\) points \(s \in C_x\) such that \(d(c_i, s) \leq \min(r, \alpha d(c_x, s))\).

**Proof.** Assume the fact is false. Then let \(B \subseteq C_x\) denote a set of size \(\epsilon n\) such that for all \(s \in B\), \(d(c_i, s) \leq \min(r, \alpha d(c_x, s))\). Construct the following perturbation \(d'\). For all \(s \in B\), let \(d'(c_x, s) = \alpha d(c_x, s)\). For all other pairs \(s, t\), set \(d'(s, t) = d(s, t)\). This is clearly an \(\alpha\)-perturbation by construction. Then the original set of optimal centers still achieves cost \(r\) under \(d'\) because for all \(s \in B\), \(d'(c_i, s) \leq r\). Clearly, the optimal cost under \(d'\) cannot be \(< r\). It follows that the original set of optimal centers \(C\) is still optimal under \(d'\). However, all points in \(B\) are no longer in \(\text{Vor}_{C_x}(c_x)\) under \(d'\), contradicting the fact that \(C_x\) is \((\alpha, \epsilon)\)-LPR.

Now we are ready to prove Theorem 16.

**Proof of Theorem 16.** Assume towards contradiction that there are two points at distance \(\leq r^*\) from different \((3, \epsilon)\)-LPR clusters. Then by Lemma 22 there exists a partition \(S_1, S_2\) of non-centers of \(S\) such that for all pairs \(p \in S_1, q \in S_2\), \(\{c_i\}_{i=1}^k \cup \{p, q\}\) is \(3, 3\)-hit \(S\). Given three \((3, \epsilon)\)-LPR clusters \(C_x, C_y, C_z\), let \(c_{i^x}, c_{i^y}, c_{i^z}\) denote the optimal centers ranked highest by \(C_x, C_y, C_z\) respectively. Define \(p = \text{argmin}_{s \in C_x} d(c_i, s)\), and WLOG let \(p \in S_1\). Then pick an arbitrary point \(q\) from \(S_2\), and define \(C = \{c_i\}_{i=1}^k \cup \{p, q\}\). Define \(d'\) as in Lemma 21. We claim that \(R_{x,d',C}(p) < R_{x,d',C}(c_{i^x})\): from Fact 25, there are fewer than \(\epsilon n\) points \(s \in C_x\) such that \(d(c_i, s) \leq \min(r^* 3d(c_i, s))\). Among each remaining point \(s \in C_x\), we will show \(d'(p, s) \leq d'(c_{i^x}, s)\). Recall that \(d(p, s) \leq d(p, c_x) + d(c_x, s) \leq 2r^*\), so \(d'(p, s) = \min(3r^*, 3d(p, s))\). There are two cases to consider.

Case 1: \(d(c_{i^x}, s) > r^*\). Then by construction, \(d'(c_{i^x}, s) \geq 3r^*\), and so \(d'(p, s) \leq d'(c_{i^x}, s)\).

Case 2: \(d(c_{i^x}, s) < d(c_x, s)\). Then \(d'(p, s) \leq 3d(p, s) \leq 3d(p, c_x) + d(c_x, s) \leq 6d(c_x, s) \leq 2d(c_{i^x}, s)\), and this proves our claim.

Because \(R_{x,d',C}(p) < R_{x,d',C}(c_{i^x})\), it follows that either \(R_{x,d',C}(p) \leq 2\) or \(R_{x,d',C}(q) \leq 2\), since the top two can only be \(c_x, p, q\). The rest of the argument is broken up into cases.

Case 1: \(R_{x,d',C}(c_{i^x}) \leq 3\). From Lemma 24 then \(R_{y,d',C}(p) \geq 3\) and \(R_{y,d',C}(q) \geq 3\). It follows by process of elimination that \(R_{y,d',C}(c_y) = 1\) and \(R_{y,d',C}(c_{iy}) = 2\). Again by Lemma 24, \(R_{x,d',C}(p) \geq 3\) and \(R_{x,d',C}(q) \geq 3\), contradicting a case.

Case 2: \(R_{x,d',C}(c_{i^x}) > 3\) and \(R_{y,d',C}(c_{i^y}) \leq 3\). Then \(R_{x,d',C}(p) \leq 3\) and \(R_{x,d',C}(q) \leq 3\). From Lemma 24 \(R_{x,d',C}(p) \geq 3\) and \(R_{y,d',C}(q) \geq 3\), therefore we have a contradiction. Note, the case where \(R_{x,d',C}(c_{i^x}) > 3\) and \(R_{x,d',C}(c_{i^y}) \leq 3\) is identical to this case.

Case 3: The final case is when \(R_{x,d',C}(c_{i^x}) > 3\), \(R_{y,d',C}(c_{i^y}) > 3\), and \(R_{x,d',C}(c_{i^z}) > 3\). For each \(i \in \{x, y, z\}\), the top three for \(C_i\) in \(C\) is a permutation of \(\{c_1, p, q\}\). Then each \(i \in \{x, y, z\}\) must rank \(p\) or \(q\) in the top two, so by the Pigeonhole Principle, either \(p\) or \(q\) is ranked top two by two different LPR clusters, contradicting Lemma 24. This completes the proof.

We note that Case 3 in Theorem 16 is the reason why we need to assume there are at least three \((3, \epsilon)\)-LPR clusters. If there are only two, \(C_x\) and \(C_y\), it is possible that there exist \(u \in C_x\), \(v \in C_y\) such that \(d(u, v) \leq r^*\). In this case, for \(p, q, d', C\) as defined in the proof of Theorem 16 if \(C_x\) ranks \(c_x, p, q\) as its top three and \(C_y\) ranks \(c_y, q, p\) as its top three, then there is no contradiction.

**APX-Hardness under approximation stability** Now we show the lower bound on the cluster sizes in Theorem 17 is necessary, by showing hardness of approximation even when it is guaranteed the clustering satisfies \((\alpha, \epsilon)\)-perturbation resilience for \(\alpha \geq 1\) and \(\epsilon > 0\). In fact, this hardness holds even under the strictly stronger notion of approximation stability [10]. We say that two clusterings \(C\) and \(C'\) are \(\epsilon\)-close if only an \(\epsilon\)-fraction of the input points are clustered differently, i.e., \(\min_{c} \sum_{i=1}^{k} |C_i \setminus C'_i| \leq \epsilon n\).
Definition 26. A clustering instance $(S, d)$ satisfies $(\alpha, \epsilon)$-approximation stability ($(\alpha, \epsilon)$-AS) if any clustering $C'$ (not necessarily a Voronoi partition) such that $\text{cost}(C') \leq \alpha \text{cost}(OPT)$ is $\epsilon$-close to $OPT$.

The hardness is based on a reduction from the general clustering instances, so the APX-hardness constants match the non-stable APX-hardness results.

Theorem 27. Given $\alpha \geq 1$, $\epsilon > 0$, it is NP-hard to approximate $k$-center to 2, $k$-median to 1.73, or $k$-means to 1.0013, even when it is guaranteed the instance satisfies $(\alpha, \epsilon)$-approximation stability.

This theorem generalizes hardness results from [10] and [11]. Also, because of Fact [3], a corollary is that unless $P = NP$, there is no efficient algorithm which outputs each $(\alpha, \epsilon)$-LPR cluster for $k$-center (showing the condition on the cluster sizes in Theorem 17 is necessary).

Proof. Given $\alpha \geq 1$, $\epsilon > 0$, assume there exists a $\beta$-approximation algorithm $A$ for $k$-median under $(\alpha, \epsilon)$-approximation stability. We will show a reduction to $k$-median without approximation stability. Given a $k$-median clustering instance $(S, d)$ of size $n$, we will create a new instance $(S', d')$ for $k' = k + n/\epsilon$ with size $n' = n/\epsilon$ as follows. First, set $S' = S$ and $d' = d$, and then add $n/\epsilon$ new points to $S'$, such that their distance to every other point is $2\epsilon n \max_{u \in S} d(u, v)$. Let $OPT$ denote the optimal solution of $(S, d)$. Then the optimal solution to $(S', d')$ is to use $OPT$ for the vertices in $S$, and make each of the $n/\epsilon$ added points a center. Note that the cost of $OPT$ and the optimal clustering for $(S', d')$ are identical, since the added points are distance 0 to their center. Given a clustering $C$ on $(S, d)$, let $C'$ denote the clustering of $(S', d')$ that clusters $S$ as in $C$, and then adds $n/\epsilon$ extra centers on each of the added points. Then the cost of $C$ and $C'$ are the same, so it follows that $C$ is a $\beta$-approximation to $(S, d)$ if and only if $C'$ is a $\beta$-approximation to $(S', d')$. Next, we claim that $(S', d')$ satisfies $(\alpha, \epsilon)$-approximation stability. Given a clustering $C'$ which is an $\alpha$-approximation to $(S', d')$, then there must be a center located at all $n/\epsilon$ of the added points, otherwise the cost of $C'$ would be $> \alpha OPT$. Therefore, $C'$ agrees with the optimal solution on all points except for $S$, therefore, $C'$ must be $\epsilon$-close to the optimal solution. Now that we have established a reduction, the theorem follows from hardness of 1.73-approximation for $k$-median [28]. The proofs for $k$-center and $k$-means are identical, using hardness from [23] and [32], respectively.

6 Conclusion

In this work, we initiate the study of clustering under local stability. We define local perturbation resilience, a property of a single optimal cluster rather than the instance as a whole. We give algorithms that simultaneously achieve guarantees in the worst case, as well as guarantees when the data is stable.

Specifically, we show that local search outputs the optimal $(3 + \epsilon)$-LPR clusters for $k$-median and the $(9 + \epsilon)$-LPR clusters for $k$-means. For $k$-center, we show that any 2-approximation outputs the optimal 2-LPR clusters, as well as the optimal $(3, \epsilon)$-LPR clusters when assuming the optimal clusters are not too small. We provide a natural modification to the asymmetric $k$-center approximation algorithm of Vishwanathan [38] to prove it outputs all 2-SLPR clusters. Finally, we show APX-hardness of clustering under $(\alpha, \epsilon)$-approximation stability for any $\alpha \geq 1$, $\epsilon > 0$. It would be interesting to find other approximation algorithms satisfying the condition in Lemma [7] and in general to further study local stability for other objectives and conditions.
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A Prior algorithms in the context of local stability

In this section, we discuss previous algorithms in the context of our new local stability framework. All of these results are useful under the standard definition of perturbation resilience for which they were designed. However, we will see that the prior algorithms (except for the one designed specifically for k-center) use the global structure of the data, which causes the algorithms to behave poorly when a fraction of the dataset is not perturbation resilient.

We start with the recent algorithm of Angelidakis et al. [3] to optimally cluster 2-perturbation resilient instances for any center-based objective (which includes k-median, k-means, and k-center). The algorithm is intuitively simple to describe. The first step is to create a minimum spanning tree \(T\) on the dataset. The second step is to perform dynamic programming on \(T\) to find the k-clustering with the lowest cost. The key fact is that 2-perturbation resilience implies that each cluster \(C_i\) is a connected subtree in \(T\). This fact is partially preserved for each 2-LPR cluster \(C_i\). For example, it can be made to hold if nearby clusters are also 2-LPR. However, the dynamic programming step heavily relies on the entire dataset satisfying stability. For instance, consider a dataset in which there are \(k\) clusters in a line, and all clusters are 2-LPR except for the cluster in the center. It is possible for the non-LPR cluster causes an offset in the dynamic program, which forces the minimum cost pruning to split each 2-LPR cluster in two, and merge each half with half of its
neighboring cluster. Therefore, none of the LPR clusters are returned by the algorithm. We conclude that
the dynamic programming step is not robust with respect to perturbation resilience.

Next, we consider the algorithm of Balcan and Liang [12] to optimally cluster \((1 + \sqrt{2})\)-perturbation resilient instances for center-based objectives. This was the leading algorithm for perturbation resilient instances until the recent result by Angelidakis et al. This algorithm also utilizes a dynamic programming step, although on a different type of tree. The algorithm starts with a linkage procedure, which the authors call closure linkage. It starts with \(n\) singleton sets, and merges the two sets with the minimum closure distance, which is the minimum radius that covers the sets and satisfies a margin condition that exploits the perturbation resilient structure. The merge procedure is iterated until all sets merge into a single set containing all \(n\) points. Thus, the algorithm creates a tree \(\mathcal{T}\) where the leaves are the \(n\) singleton sets, each internal node is a set of points, and the root is the set of all \(n\) datapoints. If the dataset satisfies \((1 + \sqrt{2})\)-perturbation resilience, then each optimal cluster appears as a node in the tree. Then dynamic programming on \(\mathcal{T}\) to find the minimum \(k\)-pruning outputs the optimal clusters. Just as in the previous algorithm, the initial guarantee is partially satisfied for LPR clusters. For example, if a cluster \(C_i\) and all nearby clusters satisfy \((1 + \sqrt{2})\)-LPR, then \(C_i\) will appear as a node in \(\mathcal{T}\). However, the dynamic programming step is again not robust to just a few non-LPR clusters. For example, consider a dataset where \(k/2\) clusters are LPR, \(k/2\) clusters are non-LPR. It is possible that the minimum closure distance for each non-LPR cluster contains all \(k/2\) non-LPR clusters. Then the non-LPR clusters are grouped together in a single merge step, forcing the dynamic program to split every LPR cluster in half so that \(k\) clusters are outputted. In this example, the LPR clusters can be very far away from the non-LPR clusters, but the algorithm still fails.

Finally, we consider the algorithm of Balcan et al. [11] which returns the optimal asymmetric \(k\)-center solution under 2-perturbation resilience. This algorithm starts by finding a subset of the datapoints which “behave symmetrically.” We note that their symmetric set is equivalent to the set of all CCVs. Next, the algorithm uses a margin condition to separate out the optimal clusters in the symmetric set, and greedily attaches the non-symmetric points at the end. The first part of the algorithm will correctly output all LPR clusters within the symmetric set. However, when the non-symmetric points are reattached, many points from non-LPR clusters can attach to LPR clusters, forcing the final outputted clusters to have a huge radius. There is no easy fix to this algorithm, since the LPR clusters might “steal” the centers of non-LPR clusters, so the optimal radius of the remaining points may increase significantly.

In conclusion, existing algorithms for \(k\)-median and \(k\)-means heavily rely on the global structure provided by perturbation resilience. If just a single cluster does not satisfy local perturbation resilience, then the global structure is broken and the algorithm may not output any of the optimal clusters. Therefore, algorithms which exploit the local structure of perturbation resilience, such as in Section 3 are needed to have guarantees that are robust with respect to the level of perturbation resilience of the dataset. For \(k\)-center, the nature of the objective ensures that all algorithms have local guarantees, in some sense. However, it is still nontrivial to exploit the structure of the clusters satisfying local perturbation resilience, while ensuring the guarantees for the rest of the dataset are not arbitrarily bad (which is what we accomplish in Section 4).

**B Proofs from Section 2**

In this section, we give a proof of Lemma 6.

**Lemma 6 (restated).** A clustering instance \((S, d)\) satisfies \((\alpha, \epsilon)\)-PR if and only if each optimal cluster \(C_i\) satisfies \((\alpha, \epsilon_i)\)-LPR and \(\sum \epsilon_i \leq 2\epsilon n\).

**Proof.** Given a clustering instance \((S, d)\) satisfying \((\alpha, \epsilon)\)-PR, given an \(\alpha\)-perturbation \(d'\) and optimal clustering \(C'_1, \ldots, C'_k\) under \(d'\), then there exists \(\sigma\) such that \(\sum_{i=1}^{k} |C_i \setminus C'_{\sigma(i)}| \leq \epsilon n\). WLOG, we let \(\sigma\) equal the identity permutation. Now we claim that \(\sum_{i=1}^{k} |C_i \setminus C'_i| = \sum_{i=1}^{k} |C'_i \setminus C_i|\). Intuitively, this is true because
$C_1, \ldots, C_k$ and $C'_1, \ldots, C'_k$ are both partitions of the same point set $k$. Formally,

$$\sum_{i=1}^{k} |C_i \setminus C'_i| = \sum_{i=1}^{k} |(C_i \cup C'_i) \setminus C'_i|$$

because $C'_i \subseteq C'_i$

$$= \sum_{i=1}^{k} (|C_i \cup C'_i| - |C'_i|)$$

because $C'_i \subseteq (C_i \cup C'_i)$

$$= \sum_{i=1}^{k} |C_i \cup C'_i| - \sum_{i=1}^{k} |C'_i|$$

$$= \sum_{i=1}^{k} |C_i \cup C'_i| - \sum_{i=1}^{k} |C_i|$$

because $\sum_{i=1}^{k} |C'_i| = \sum_{i=1}^{k} |C_i|$

$$= \sum_{i=1}^{k} (|C_i \cup C'_i| - |C_i|)$$

$$= \sum_{i=1}^{k} |(C_i \cup C'_i) \setminus C_i|$$

because $C_i \subseteq (C_i \cup C'_i)$

$$= \sum_{i=1}^{k} |C'_i \setminus C_i|$$

because $C_i \subseteq C_i$

Now for each $i$, set $\epsilon_i = |C_i \setminus C'_i| + |C'_i \setminus C_i|$. Clearly, this ensures $C_i$ satisfies $(\alpha, \epsilon_i)$-LPR. Finally, we have

$$\sum_i \epsilon_i = \sum_i (|C_i \setminus C'_i| + |C'_i \setminus C_i|)$$

$$= 2 \sum_i |C_i \setminus C'_i|$$

$$\leq 2en.$$

Now we prove the reverse direction. Given a clustering instance $(S, d)$ such that each cluster $C_i$ satisfies $(\alpha, \epsilon_i)$-LPR and $\sum_i \epsilon_i \leq 2en$, given an $\alpha$-perturbation $d'$ and optimal clustering $C'_1, \ldots, C'_k$ under $d'$, then for each $i$, $|C_i \setminus C'_i| + |C'_i \setminus C_i| \leq \epsilon_i$. Therefore,

$$\sum_i |C_i \setminus C'_i| = \frac{1}{2} \sum_i (|C_i \setminus C'_i| + |C'_i \setminus C_i|)$$

$$\leq \epsilon n.$$

This concludes the proof.

C Proofs from Section 3

In this section, we give the details from Section 3.

Theorem 8 (restated). Given a $k$-median instance $(S, d)$, running local search with search size $\frac{1}{2}$ returns a clustering that contains every $(3 + 2\epsilon)$-LPR cluster, and it gives a $(3 + 2\epsilon)$-approximation overall.
Proof. Given a $k$-median instance $(S, d)$, let $X$ denote a set of centers obtained by running local search with search size $\frac{1}{4}$. Let $Y$ denote a different set of $k$ centers. As in the proof of Lemma 7, let $A_1$, $A_2$, $A_3$, and $A_4$ denote $\text{Vor}_X(X \cap Y) \cap \text{Vor}_Y(X \cap Y)$, $\text{Vor}_X(X \cap Y) \setminus \text{Vor}_Y(X \cap Y)$, $\text{Vor}_Y(X \cap Y) \setminus \text{Vor}_X(X \cap Y)$, and $\text{Vor}_X(X \cap Y) \cap \text{Vor}_Y(Y \setminus X)$, respectively (see Figure 4a). From Cohen-Addad and Schwiegelshohn [21], from Lemmas B.3 and B.4, we have the following.

$$\sum_{v \in A_2 \cup A_4} d(v, X) \leq \sum_{v \in A_2 \cup A_4} d(v, Y) + 2(1 + \epsilon) \sum_{v \in A_3 \cup A_4} d(v, Y)$$

Given a point $v \in A_2$, then its center in $X$ is from $X \cap Y$, and its center from $Y$ is in $Y \setminus X$. We deduce that $d(v, Y) \leq d(v, X)$, otherwise $v$’s center from $Y$ would be the same as in $X$. Similarly, for a point $v \in A_3$, we can conclude that $d(v, X) \leq d(v, Y)$, by definition of $A_3$.

Therefore,

$$\sum_{v \in A_2 \cup A_3 \cup A_4} d(v, X) \leq \sum_{v \in A_3} d(v, X) + \sum_{v \in A_2 \cup A_4} d(v, X)$$

$$\leq \sum_{v \in A_3} d(v, Y) + \left( \sum_{v \in A_2 \cup A_4} d(v, Y) + 2(1 + \epsilon) \sum_{v \in A_3 \cup A_4} d(v, Y) \right)$$

$$\leq \sum_{v \in A_2} d(v, Y) + (3 + 2\epsilon) \sum_{v \in A_3 \cup A_4} d(v, Y)$$

$$\leq \sum_{v \in A_2} \min(d(v, X), (3 + 2\epsilon)d(v, Y)) + (3 + 2\epsilon) \sum_{v \in A_3 \cup A_4} d(v, Y)$$

Now the proof follows from Lemma 7. 

Lemma 9 (restated). Given $\alpha \geq 1$ and an asymmetric $k$-center clustering instance $(S, d)$ with optimal radius $r^*$, let $d'$ denote an $\alpha$-perturbation such that for all $u, v$, either $d''(u, v) = \min(\alpha r^*, \alpha d(u, v))$ or $d''(u, v) = \alpha d(u, v)$. Let $d'$ denote the metric completion of $d''$. Then $d'$ is an $\alpha$-metric perturbation of $d$, and the optimal cost under $d'$ is $\alpha r^*$.

Proof. By construction, $d'(u, v) \leq d''(u, v) \leq \alpha d(u, v)$. Since $d$ satisfies the triangle inequality, we have that $d(u, v) \leq d'(u, v)$, so $d'$ is a valid $\alpha$-metric perturbation of $d$.

Now given $u, v$ such that $d(u, v) \geq r^*$, we will prove that $d'(u, v) \geq \alpha r^*$. By construction, $d''(u, v) \geq \alpha r^*$. Then since $d'$ is the metric completion of $d''$, there exists a path $u = u_0 - u_1 - \cdots - u_{s-1} - u_s = v$ such that $d'(u, v) = \sum_{i=0}^{s-1} d'(u_i, u_{i+1})$ and for all $0 \leq i \leq s - 1$, $d'(u_i, u_{i+1}) = d''(u_i, u_{i+1})$. If there exists an $i$ such that $d''(u_i, u_{i+1}) \geq \alpha r^*$, then $d'(u, v) \geq \alpha r^*$ and we are done. Now assume for all $0 \leq i \leq s - 1$, $d''(u_i, u_{i+1}) < \alpha r^*$. Then by construction, $d'(u_i, u_{i+1}) = d''(u_i, u_{i+1}) = \alpha d(u_i, u_{i+1})$, and so $d'(u, v) = \sum_{i=0}^{s-1} d'(u_i, u_{i+1}) = \alpha \sum_{i=0}^{s-1} d(u_i, u_{i+1}) \geq \alpha d(u, v) \geq \alpha r^*$.

We have proven that for all $u, v$, if $d(u, v) \geq r^*$, then $d'(u, v) \geq \alpha r^*$. Assume there exists a set of centers $C' = \{c'_1, \ldots, c'_k\}$ whose $k$-center cost under $d'$ is $< \alpha r^*$. Then for all $i$ and $s \in \text{Vor}_{C', d'}(c'_i)$, $d'(c'_i, s) < r^*$, implying $d(c'_i, s) < \alpha r^*$ by construction. It follows that the $k$-center cost of $C'$ under $d$ is $r^*$, which is a contradiction. Therefore, the optimal cost under $d'$ must be $\alpha r^*$. 

Theorem 10 (restated). Given an asymmetric $k$-center clustering instance $(S, d)$ and an $\alpha$-approximate clustering $\tilde{C}$, each $\alpha$-LPR cluster is contained in $C$, even under the weaker metric perturbation resilience condition.
Proof. Given an $\alpha$-approximate solution $C$ to a clustering instance $(S, d)$, and given an $\alpha$-LPR cluster $C_i$, we will create an $\alpha$-perturbation as follows. For all $v \in S$, set $d''(v, C(v)) = \min\{\alpha r^*, \alpha d(v, C(v))\}$. For all other points $u \in S$, set $d''(v, u) = \alpha d(v, u)$. Then by Lemma 9, the metric completion $d'$ of $d''$ is an $\alpha$-perturbation of $d$ with optimal cost $\alpha r^*$. By construction, the cost of $C$ is $\leq \alpha r^*$ under $d'$, therefore, $C$ is an optimal clustering. Denote the set of centers of $C$ by $C$. By definition of $\alpha$-LPR, there exists $v_i \in C$ such that $\text{Vor}_C(v_i) = C_i$ in $d'$. Now, given $v \in C_i$, $\text{argmin}_{u \in C} d'(u,v) = v_i$, so by construction, $\text{argmin}_{u \in C} d(u,v) = v_i$. Therefore, $\text{Vor}_C(v_i) = C_i$, so $C_i \in C$.

\[\square\]

D Proofs from Section 4

In this section, we give the details of the proofs from Section 4.

Lemma 15 (restated). Given an asymmetric $k$-center clustering instance $(S, d)$ and a 2-LPR cluster $C_i$, $c_i$ satisfies CCV-proximity and center-separation. Furthermore, given a CCV $c \in C_i$, a CCV $c' \not\in C_i$, and a point $v \in C_i$, we have $d(c,v) < d(c',v)$.

Proof. Given an instance $(S, d)$ and a 2-metric perturbation resilient cluster $C_i$, we show that $c_i$ has the desired properties.

Center Separation: Assume there exists a point $v \in C_j$ for $j \neq i$ such that $d(v, c_i) \leq r^*$. The idea is to construct a 2-perturbation in which $v$ becomes the center for $C_i$.

$$d''(s,t) = \begin{cases} \min(2r^*, 2d(s,t)) & \text{if } s = v, t \in C_i \\ 2d(s,t) & \text{otherwise.} \end{cases}$$

$d''$ is a valid 2-perturbation of $d$ because for each point $u \in C_i$, $d(v,u) \leq d(v,c_i) + d(c_i,u) \leq 2r^*$. Define $d'$ as the metric completion of $d''$. Then by Lemma 9, $d'$ is a 2-metric perturbation with optimal cost $2r^*$. The set of centers $\{c_i\}_{i=1}^k \setminus \{c_i\} \cup \{v\}$ achieves the optimal cost, since $v$ is distance $2r^*$ from $C_i$, and all other clusters have the same center as in $\text{OPT}$ (achieving radius $2r^*$). But in this new optimal clustering, $c_i$’s center is a point in $\{c_i\}_{i=1}^k \setminus \{c_i\} \cup \{v\}$, none of which are from $C_i$. We conclude that $C_i$ is no longer an optimal cluster, contradicting 2-LPR.

Final property Next we prove the final part of the lemma. Given a CCV $c$ from a 2-LPR cluster $C_i$, a CCV $c' \in C_j$ such that $j \neq i$, and a point $v \in C_i$, and assume $d(c',v) \leq d(c,v)$. We will construct a perturbation in which $c$ and $c'$ become centers of their respective clusters, and then $v$ switches clusters. Define the following perturbation $d''$.

$$d''(s,t) = \begin{cases} \min(2r^*, 2d(s,t)) & \text{if } s = c, t \in C_i \text{ or } s = c', t \in C_j \cup \{v\} \\ 2d(s,t) & \text{otherwise.} \end{cases}$$

$d''$ is a valid 2-perturbation of $d$ because for each point $u \in C_i$, $d(c,v) \leq d(c,c_i) + d(c_i,u) \leq 2r^*$, for each point $u \in C_j$, $d(c',v) \leq d(c',c_j) + d(c_j,u) \leq 2r^*$, and $d(c',v) \leq d(c,v) \leq d(c,c_i) + d(c_i,v) \leq 2r^*$. Define $d'$ as the metric completion of $d''$. Then by Lemma 9, $d'$ is a 2-metric perturbation with optimal cost $2r^*$. The set of centers $\{c_i\}_{i=1}^k \setminus \{c_i, c_j\} \cup \{c, c'\}$ achieves the optimal cost, since $c$ and $c'$ are distance $2r^*$ from $C_i$ and $C_j$, and all other clusters have the same center as in $\text{OPT}$ (achieving radius $2r^*$). Then since $d'(c',v) < d(c,v)$, $v$ will not be in the same optimal cluster as $c_i$, causing a contradiction.

CCV-proximity: By center-separation, we have that $\Gamma^-(c_i) \subseteq C_i$, and by definition of $r^*$, we have that $C_i \subseteq \Gamma^+(c_i)$. Therefore, $\Gamma^-(c_i) \subseteq \Gamma^+(c_i)$, so $c_i$ is a CCV. Now given a point $v \in \Gamma^-(c_i)$ and a CCV $c \not\in \Gamma^+(c_i)$, from center-separation and definition of $r^*$, $v \in C_i$ and $c \in C_j$ for $j \neq i$. Then from the property in the previous paragraph, $d(c,v) < d(c,v)$. \[\square\]
**Theorem 12 (restated).** Given an asymmetric \( k \)-center clustering instance \((S, d)\) of size \( n \), Algorithm 3 returns each 2-SLPR cluster exactly. For each 2-LPR cluster \( C_i \), Algorithm 3 outputs a cluster that is a superset of \( C_i \) and does not contain any other 2-LPR cluster. These statements hold for metric perturbation resilience as well. Finally, the overall clustering returned by Algorithm 3 is an \( O(\log^* n) \)-approximation.

**Proof of Theorem 12.** First we explain why Algorithm 3 retains the approximation guarantee of Algorithm 2. Given any CCV \( c \in C_i \) chosen in Phase I, since \( c \) is a CCV, then \( c_i \in \Gamma^+(c) \), and by definition of \( r^* \), \( C_i \subseteq \Gamma^+(c_i) \). Therefore, each chosen CCV always marks its cluster, and we start Phase II with no remaining CCVs. This condition is sufficient for Phase II to return an \( O(\log^* n) \) approximation (Theorem 3.1 from [23]).

Next we claim that for each 2-LPR cluster \( C_i \), there exists a cluster outputted by Algorithm 3 that is a superset of \( C_i \) and does not contain any other 2-LPR cluster. To prove this claim, we first show there exists a point from \( C_i \) satisfying CCV-proximity that cannot be marked by any point from a different cluster in Phase I. From Lemma 15, \( c \) satisfies CCV-proximity and center-separation. If a point \( c \notin C_i \) marks \( c_i \), then \( \exists v \in \Gamma^-(c) \cap \Gamma^-(c_i) \). By center-separation and by definition of CCV, \( c \notin \Gamma^-(c_i) \) and \( c \notin \Gamma^+(c) \). Then from the definition of CCV-proximity for \( c \) and \( c_i \) and \( c \), we have \( d(c, v) < d(c_i, v) \) and \( d(c_i, v) < d(c, v) \), so we have reached a contradiction. At this point, we know a point \( c \in C_i \) will always be chosen by the algorithm in Phase I. To finish the claim, we show that each point \( v \) from \( C_i \) is closer to \( c \) than to any other point \( c' \notin C_i \) chosen in Phase I. Since \( c \) and \( c' \) are both CCVs, this follows directly from the last property in Lemma 15. However, it is possible that a center \( c' \in A_{i+1} \) is closer to \( v \) than \( c \) is to \( v \), causing \( c' \) to “steal” \( v \); this is unavoidable. Therefore, we forbid the algorithm from decreasing the size of the Voronoi tiles of \( C \) after Phase I.

Finally, we claim that Algorithm 3 returns each 2-SLPR cluster exactly. Given a 2-SLPR cluster \( C_i \), by our previous argument, there exists a CCV \( c \in C_i \) from Phase I satisfying CCV-proximity such that \( C_i \subseteq V_c \). First we assume towards contradiction that there exists a point \( v \in \Gamma^-(c) \setminus C_i \). Let \( v \in C_j \). Since \( c \) is a CCV, then \( v \in \Gamma^+(c) \), so \( C_j \) must be 2-LPR by definition of 2-SLPR. By Lemma 15, \( c_j \) is a CCV and \( d(c_j, v) < d(c, v) \). But this violates CCV-proximity of \( c \), so we have reached a contradiction. Therefore, \( \Gamma^-(c) \subseteq C_i \). Now assume there exists \( v \in V_c \setminus C_i \) at the end of the algorithm.

Case 1: \( v \) was marked by \( c \) in Phase I. Let \( v \in C_j \). Then there exists a point \( u \in \Gamma^-(c) \) such that \( v \in \Gamma^+(u) \). Then \( u \in C_i \) and \( v \in \Gamma^+(u) \), so \( C_j \) must be 2-LPR. Since \( v \) is from a different 2-LPR cluster, it cannot be contained in \( V_c \), so we have a contradiction.

Case 2: \( v \) was not marked by \( c \) in Phase I. Denote the shortest path in \( D_{(S,d)} \) from \( c \) to \( v \) by \( c = v_0 v_1 \cdots v_{L-1} v_L = v \). Let \( v_l \in C_j \) denote the first vertex on the shortest path that is not in \( C_i \) (such a vertex must exist because \( v \notin C_i \)). See Figure 2a. Then \( v_{l-1} \in C_i \) and \( d(v_{l-1}, v_L) \leq r^* \), so \( C_j \) is 2-LPR. Let \( c' \) denote the CCV chosen in Phase I such that \( C_j \subseteq V_{c'} \). If \( v_L \) is not on the shortest path \( c' \leadsto v \), then that shortest path must be shorter than \( c \leadsto v \), and we are done. If \( v_{l-1} \) is on the shortest path \( c' \leadsto v \), then since \( v_L \in C_{j}, d(c', v_L) \leq r^* \), so the shortest path must start with \( c' \leadsto v_{l-1} \). If \( d(c, v_{l-1}) > r^* \), then we are done because \( c' \) is then closer to \( v \). Therefore, the distance from both \( c \) and \( c' \) to \( v_L \) is in \((r^*, 2r^*)\). We can set up a 2-perturbation as in the proof of the final property of Lemma 15 so that \( c \) and \( c' \) become the centers of their respective clusters, and \( v_L \) switches clusters.

\[
d''(s, t) = \begin{cases} \min(2r^*, 2d(s, t)) & \text{if } s = c, t \in C_i \text{ or } s = c', t \in C_j \cup \{v_L\} \\ 2d(s, t) & \text{otherwise.} \end{cases}
\]

\( d'' \) is a valid 2-perturbation of \( d \) because for each point \( u \in C_i \), \( d(c, u) \leq d(c, c_i) + d(c_i, u) \leq 2r^* \), for each point \( u \in C_j \), \( d(c', v_L) \leq d(c', c_j) + d(c_j, v_L) \leq 2r^* \), and \( d(c', v_L) \leq d(c, v_L) \leq d(c, c_i) + d(c_i, v_L) \leq 2r^* \). Define \( d' \) as the metric completion of \( d'' \). Then by Lemma 9, \( d' \) is a 2-metric perturbation with optimal cost \( 2r^* \). The set of centers \( \{c_i\}_{i=1}^k \setminus \{c_i, c_j\} \cup \{c, c'\} \) achieves the optimal cost, since \( c \) and \( c' \) are distance \( 2r^* \).
from $C_i$ and $C_j$, and all other clusters have the same center as in $OPT$ (achieving radius $2r^*$). Then since $d(c, v_j)$ and $d(c', v'_t)$ are both in $(r^*, 2r^*)$, by construction $d'(c, v_t) = d'(c', v'_t)$. This contradicts 2-LPR, since $v_t$ can switch clusters to $C_i$.

We conclude that $v_t$ is the first common vertex on the shortest paths $c-v$ and $c'-v$. Since $c$ and $c'$ are both CCVs, $d(c', v'_t) < d(c, v_t)$. Therefore, $v$ cannot be in $V_c$, so we have reached a contradiction. This completes the proof.

\qed

\section{Proofs from Section 5}

In this section, we give the details of the proofs from Section 16.

\textbf{Lemma 22 (restated).} Given a $k$-center clustering instance $(S, d)$ such that all optimal clusters are size greater than $2en$ and there exist two points at distance $r^*$ from different $(3, \epsilon)$-LPR clusters, then there exists a partition $S_x \cup S_y$ of the non-centers $S \setminus \{c\}_{i=1}^k$ such that for all pairs $p \in S_x$, $q \in S_y$, $\{c\}_{i=1}^k \cup \{p, q\}$ is a $(3, \delta)$-hits $S$.

\textbf{Proof.} This proof is split into two main cases. The first case is the following: there exists a CCC2 for a $(3, \epsilon)$-LPR cluster, discounting a $(3, \epsilon)$-LPR cluster. In fact, in this case, we do not need the assumption that two points from different LPR clusters are close. If there exists a CCC to a $(3, \epsilon)$-LPR cluster, denote the CCC by $c_x$ and the cluster by $C_y$. Otherwise, let $c_x$ denote a CCC2 of a $(3, \epsilon)$-LPR cluster $C_y$, discounting a $(3, \epsilon)$-LPR center $c_z$. Then $c_x$ is at distance at most $r^*$ to all but $en$ points in $C_y$. Therefore, $d(c_x, c_y) \leq 2r^*$ and so $c_x$ is at distance at most $3r^*$ to all points in $C_y$. Consider the following perturbation $d''$.

$$d''(s, t) = \begin{cases} \min(3r^*, 3d(s, t)) & \text{if } s = c_x, t \in C_y \\ 3d(s, t) & \text{otherwise.} \end{cases}$$

This is a 3-perturbation because for all $v \in C_y$, $d(c_x, v) \leq 3r^*$. Define $d'$ as the metric completion of $d''$. Then by Lemma 9 $d'$ is a 3-metric perturbation with optimal cost $3r^*$. Given any non-center $v \in S$, the set of centers $\{c\}_{i=1}^k \cup \{c_y\} \cup \{v\}$ achieves the optimal score, since $c_x$ is at distance $3r^*$ from $C_y$, and all other clusters have the same center as in $OPT$ (achieving radius $3r^*$). Therefore, from Fact 18, one of the centers in $\{c\}_{i=1}^k \cup \{c_y\} \cup \{v\}$ must be the center for the majority of points in $C_y$ under $d'$. If this center is $c_\ell$, $\ell \neq x, y$, then for the majority of points $u \in C_y$, $d(c_\ell, u) \leq r^*$ and $d(c_\ell, u) < d(c_x, u)$ for all $z \neq \ell, y$. Then by definition, $c_\ell$ is a CCC for the $(3, \epsilon)$-LPR cluster, $C_y$. But then by construction, $\ell$ must equal $x$, so we have a contradiction. Note that if some $c_\ell$ has for the majority of $u \in C_y$, $d(c_\ell, u) \leq d(c_x, u)$ (non-strict inequality) for all $z \neq \ell, y$, then there is another equally good partition in which $c_\ell$ is not the center for the majority of points in $C_y$, so we still obtain a contradiction. Therefore, either $v$ or $c_x$ must be the center for the majority of points in $C_y$ under $d'$.

If $c_x$ is the center for the majority of points in $C_y$, then because $C_y$ is $(3, \epsilon)$-LPR, the corresponding cluster must contain fewer than $en$ points from $C_x$. Furthermore, since for all $\ell \neq x$ and $u \in C_x$, $d(u, c_x) < d(u, c_\ell)$, it follows that $v$ must be the center for the majority of points in $C_x$. Therefore, every non-center $v \in S$ is at distance at most $r^*$ to the majority of points in either $C_x$ or $C_y$.

Now partition all the non-centers into two sets $S_x$ and $S_y$, such that $S_x = \{p \mid \text{for the majority of points } q \in C_x, d(p, q) \leq r^*\}$ and $S_y = \{p \mid p \notin S_x \text{ and for the majority of points } q \in C_y, d(p, q) \leq r^*\}$. Given $p, q \in S_x$, there exists an $s \in C_x$ such that $d(p, q) \leq d(p, s) + d(s, q) \leq 2r^*$ (since both points are close to more than half of points in $C_x$). Similarly, any two points $p, q \in S_y$ are at distance $2r^*$ apart.

For now, assume that $S_x$ and $S_y$ are both nonempty. Given a pair $p \in S_x$, $q \in S_y$, we claim that $\{c_\ell\}_{\ell=1}^k \cup \{p, q\}$ is a $(3, \delta)$-hits $S$. Given a point $s \in C_i$ such that $i \neq x, y$, WLOG $s \in S_x$. Then $c_i, p,$ and $c_x$ are all distance $3r^*$ to $s$. Furthermore, $c_i, c_x$ and $p$ are all distance $3r^*$ to $c_i$. Given a point $s \in C_x$, then
Assume the lemma is false. Then there exists an $α,ϵ$-LPR cluster, discounting a $(3,ϵ)$-LPR cluster. Given any non-center $s ∈ S$, the set of centers $\{c_{ℓ}\}_{ℓ=1}^k \cup \{p,q\}$ achieves the optimal score, since $p$ is distance $3r^*$ from $C_x ∪ C_y$, and all other clusters have the same center as in $OPT$ (achieving radius $3r^*$).

From Fact 23, one of the centers in $\{c_{ℓ}\}_{ℓ=1}^k \cup \{p,q\}$ must be the center for the majority of points in $C_x$ under $d'$. If this center is $c_{ℓ}$ for $ℓ ≠ x$, then for the majority of points $t ∈ C_x$, $d(c_{ℓ}, t) ≤ r^*$ and $d(c_{ℓ}, t) < d(c_z, t)$ for all $z ≠ ℓ, x, y$. So by definition, $c_{ℓ}$ is a CCC2 for $C_x$ discounting $c_y$, which contradicts our assumption. Similar logic applies to the center for the majority of points in $C_y$. Therefore, $p$ and $s$ must be the centers for $C_x$ and $C_y$. Since $s$ was an arbitrary non-center, all non-centers are distance $≤ r^*$ to all but $en$ points in either $C_x$ or $C_y$.

Similar to Case 1, we now partition all the non-centers into two sets $S_x$ and $S_y$, such that $S_x = \{u |$ for the majority of points $v ∈ C_x, d(u, v) ≤ r^*\}$ and $S_y = \{u | u ∉ S_x$ and for the majority of points $v ∈ C_y, d(u, v) ≤ r^*\}$. As before, each pair of points in $S_x$ are distance $≤ 2r^*$ apart, and similarly for $S_y$. It is no longer true that $d(c_x, c_y) ≤ 2r^*$, however, we can prove that for both $S_x$ and $S_y$, there exist points from two distinct clusters each. From the previous paragraph, given a non-center $s ∈ C_i$ for $i ≠ x, y$, we know that $p$ and $s$ are centers for $C_x$ and $C_y$. With an identical argument, given $t ∈ C_j$ for $j ≠ x, y, i$, we can show that $q$ and $t$ are centers for $C_x$ and $C_y$. It follows that $S_x$ and $S_y$ both contain points from at least two distinct clusters.

Now we finish the proof by showing that for each pair $p ∈ S_x, q ∈ S_y$, $\{c_{ℓ}\}_{ℓ=1}^k \cup \{p,q\}$ $(3,3)$-hits $S$. Given a non-center $s ∈ C_i$, WLOG $s ∈ S_x$, then there exists $j ≠ i$ and $t ∈ C_j ∩ S_x$. Then $c_i, c_j, p$ are $3r^*$ to $s$ and $c_i, c_x, p$ are $3r^*$ to $c_i$. In the case where $i = x$, then $c_i, c_j, p$ are $3r^*$ to $c_i$. This concludes the proof.

Fact 23 (restated). Given a $k$-center clustering instance $(S, d)$ such that all optimal clusters have size $> 2en$, and an $α$-perturbation $d'$ of $d$, let $C'$ denote the set of $(α,ϵ)$-LPR clusters. For each $C_x ∈ C'$, there exists a bijection $R_{x,d'} : S → [n]$ such that for all sets of $k$ centers $C$ that achieve the optimal cost under $d'$, then $c = argmin_{c' ∈ C} R_{x,d'}(c')$ if and only if $Vor_C(c)$ is $ϵ$-close to $C_x$.

Proof. Assume the lemma is false. Then there exists an $(α,ϵ)$-LPR cluster $C_i$, two distinct points $u, v ∈ S$, and two sets of $k$ centers $C$ and $C'$ both containing $u$ and $v$, and both sets achieve the optimal score under an $α$-perturbation $d'$, but $u$ is the center for $C_i$ in $C$ while $v$ is the center for $C_i$ in $C'$. Then $Vor_C(u)$ is $ϵ$-close to $C_i$; similarly, $Vor_C'(v)$ is $ϵ$-close to $C_i$. This implies $u$ is closer to all but $en$ points in $C_i$ than $v$, and $v$ is closer to all but $en$ points in $C_i$ than $u$. Since $|C_i| > 2en$, this causes a contradiction.
Lemma 24 (restate). Given a $k$-center clustering instance $(S, d)$ such that all optimal clusters are size $> 2\epsilon n$, and given non-centers $p, q \in S$ such that $C = \{c_i\}_{i=1}^k \cup \{p, q\}$ $(3, \epsilon)$-hits $S$, let the set $C'$ denote the set of $(3, \epsilon)$-LPR clusters. Define the 3-perturbation $d'$ as in Lemma 27. The following are true.

1. Given $C_x \in C'$ and $C_i$ such that $i \neq x$, $R_{x, d'}(c_x) < R_{x, d'}(c_i)$.
2. There do not exist $s \in C$ and $C_x, C_y \in C'$ such that $x \neq y$, and $R_{x, d', C}(s) + R_{y, d', C}(s) \leq 4$.
3. Given $C_i$ and $C_x, C_y \in C'$ such that $x \neq y \neq i$, if $R_{x, d', C}(c_i) \leq 3$, then $R_{y, d', C}(p) \geq 3$ and $R_{y, d', C}(q) \geq 3$.

Proof. 1. By definition of the optimal clusters, for each $s \in C_x$, $d(c_s, s) < d(c_i, s)$, and therefore by construction, $d'(c_s, s) < d'(c_i, s)$. It follows that $R_{x, d'}(c_x) < R_{x, d'}(c_i)$.

2. Assume there exists $s \in C$ and $C_x, C_y \in C'$ such that $R_{x, d', C}(s) + R_{y, d', C}(s) \leq 4$.

Case 1: $R_{x, d', C}(s) = 1$ and $R_{y, d', C}(s) \leq 3$. Define $u$ and $r$ such that $R_{y, d', C}(u) = 1$ and $R_{y, d', C}(r) = 2$. (If $u$ or $r$ is equal to $s$, then redefine it to an arbitrary center in $C \setminus \{s, u, r\}$.) Consider the set of centers $C' = C \setminus \{u, r\}$ which is optimal under $d'$ by Lemma 21. By Fact 23, $s$ is the center for the majority of points in both $C_x$ and $C_y$, causing a contradiction.

Case 2: $R_{x, d', C}(s) = 2$ and $R_{y, d', C}(s) = 2$. Define $u$ and $v$ such that $R_{x, d', C}(u) = 1$ and $R_{y, d', C}(v) = 1$. (Again, if $u$ or $v$ is equal to $s$, then redefine it to an arbitrary center in $C \setminus \{s, u, v\}$.) Consider the set of centers $C' = C \setminus \{u, v\}$ which is optimal under $d'$ by Lemma 21. However, by Fact 23, $s$ is the center for the majority of points in both $C_x$ and $C_y$, causing a contradiction.

3. Assume $R_{x, d', C}(c_i) \leq 3$.

Case 1: $R_{x, d', C}(c_i) = 2$. Then by Lemma 24 part 1, $R_{x, d', C}(c_x) = 1$. Consider the set of centers $C' = C \setminus \{c_x, r\}$, which is optimal under $d'$. By Fact 23, $\text{Vor}_{C'}(c_i)$ must be $\epsilon$-close to $C_x$. In particular, $\text{Vor}_{C'}(c_i)$ cannot contain more than $\epsilon n$ points from $C_i$. But by definition, for all $j \neq i$ and $s \in C_i, d(c_i, s) < d(c_j, s)$. It follows that $\text{Vor}_{C'}(c_i)$ must contain all but $\epsilon n$ points from $C_i$. Therefore, for all but $\epsilon n$ points $s \in C_i$, for all $j, d'(q, s) < d'(c_j, s)$. If $R_{y, d', C}(q) \leq 2$, then $C_y$ ranks $c_y$ or $p$ number one. Then for the set of centers $C' = C \setminus \{c_y, r\}$, $\text{Vor}_{C'}(q)$ contains more than $\epsilon n$ points from $C_y$ and $C_i$, contradicting the fact that $C_y$ is $(3, \epsilon)$-LPR. Therefore, $R_{y, d', C}(q) \geq 3$. The argument to show $R_{y, d', C}(p) \geq 3$ is identical.

Case 2: $R_{x, d', C}(c_i) = 3$. If there exists $j \neq i, x$ such that $R_{x, d', C}(c_i) = 2$, then WLOG we are back in case 1. By Lemma 24 part 1, $R_{x, d', C}(c_x) \leq 2$. Then either $p$ or $q$ are ranked top two, WLOG $R_{x, d', C}(p) \leq 2$. Consider the set $C' = C \setminus \{c_x, p\}$. Then as in the previous case, $\text{Vor}_{C'}(c_i)$ must be $\epsilon$-close to $C_x$, implying for all but $\epsilon n$ points $s \in C_i$, for all $j, d'(q, s) < d'(c_j, s)$. If $R_{y, d', C}(q) \leq 2$, again, $C_y$ ranks $c_y$ or $p$ as number one. Let $C' = C \setminus \{c_y, r\}$, and then $\text{Vor}_{C'}(q)$ contains more than $\epsilon n$ points from $C_y$ and $C_i$, causing a contradiction. Furthermore, if $R_{y, d', C}(p) \leq 2$, then we arrive at a contradiction by Lemma 24 part 2. 

□