A note on hardness of promise hypergraph colouring

Marcin Wrochna  
University of Warsaw (Institute of Informatics), Poland  
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The goal of this short note is to show a slightly simpler proof of the following.

1 Theorem. (Dinur, Regev, Smith [DRS05]) PCSP($H_k, H_c$) is NP-hard for all $2 \leq k \leq c$.

That is, hardness of Promise Hypergraph Colouring (for $2 \leq k \leq c$ and 3-uniform hypergraphs). Here $H_c$ is the structure with domain $[c]$ and one 3-ary relation $\text{NAE}_c = [c]^3 \setminus \{(a, a, a) \mid a \in [c]\}$. The problem is: given a 3-uniform hypergraph, can we distinguish the case where it is $k$-colourable (admits homomorphism to $H_k$) from the case where it is not even $c$-colourable (does not admit a homomorphism to $H_c$; the promise is that the input falls into one of these two cases). The same proof applies to the search version: given a $k$-colourable 3-uniform hypergraph, find a $c$-colouring.

The proof in [DRS05] relies on a constructing a somewhat ad-hoc reduction and analysing it’s completeness and soundness. We recast this proof in the more recent algebraic framework for Promise CSPs [Bar+21], where it suffices to study polymorphisms associated with the problem (defined below) and then apply general theorems that give generic NP-hardness reductions. Thus instead of a more quantitative analysis, we use only constants everywhere (independent of the arity $L$ of the multivariate functions involved).  

1 We also replace the use of Schrijver graphs (defined below) and then apply general theorems that give generic NP-hardness reductions. Thus instead of a more quantitative analysis, we use only constants everywhere (independent of the arity $L$ of the multivariate functions involved).

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The two facts we need about colourings are the following.

2 Theorem. (Lovász [Lov78]) $\chi(KG(n, k)) = n - 2k + 2$.

3 Lemma. Let $\chi(G) > n$. Then $\chi(G \times K_{n+1}) > n$.

Proof. We show the contrapositive: suppose there is a homomorphism $G \times K_{n+1} \to K_n$. Equivalently $G \to K_{n+1}$. There is a homomorphism $K_{n+1} \to K_n$: simply map any (improper) $n$-colouring of $K_{n+1}$ to an arbitrary repeating colour. This shows $G \to K_n$.  

(Alternatively one could say both Kneser graphs and cliques have large chromatic number for topological reasons, so their tensor product does as well [SZ10]).

To begin the proof of Theorem 1, without loss of generality, let $k = 2$, $c \geq 2$. A polymorphism of arity $L$ is a function $f : [2]^L \to [c]$ such that for any three input row vectors satisfying $\text{NAE}_2$ column-wise, the outputs are in $\text{NAE}_c$. Let us consider an arbitrary such polymorphism $f$. We aim to show that $f$ in a way “distinguishes” a constant number of coordinates in $[L]$.  

For a set of colours $C \subseteq [c]$, we say a set $S \subseteq [L]$ is $C$-avoiding if fixing the inputs on $S$ of $f$ to 1 avoids (at least) $C$ in the output. That is, for every input $\bar{v} \in [2]^L$ with $\bar{v}|_S \equiv 1$ we have $f(\bar{v}) \notin C$. A set $S$ is $t$-avoiding if it is $C$-avoiding for some set $C$ of size $t$. One of the main ideas of [DRS05] is the following lemma.

4 Lemma. There is a $1$-avoiding set $S$ of constant size, $|S| \leq c$.

\[\text{On the other hand, while the proof in [DRS05] can be slightly strengthened to statements about independent sets or non-constant numbers of colours, the algebraic framework so far cannot do that directly.}\]
Proof. If \( L \leq c \), then \([L]\) is a \((c - 1)\)-avoiding set and we are done. Otherwise assume \( L > c \).

Let \( h \) be an integer satisfying \( L - c + 2 > 2h \geq L - c \). Consider inputs in \([2]^L\) of Hamming weight \( h \). The graph whose vertices are those inputs and whose edges are input pairs with disjoint supports is a Kneser graph \( KG(L, h) \). Since \( \chi(KG(L, h)) = L - 2h + 2 > c \), there are two inputs \( \bar{u}, \bar{v} \) with disjoint supports but the same colour \( f(\bar{u}) = f(\bar{v}) = b \). Let \( S := [L] \setminus (\text{supp}(\bar{u}) \cup \text{supp}(\bar{v})) \). Then any input \( \bar{w} \) with \( \bar{w}|_S \equiv 1 \) has \( f(\bar{w}) \neq b \). Thus \( S \) is 1-avoiding and \( |S| = L - 2h \leq c \). 

The problem with applying this directly is that there may be many disjoint 1-avoiding sets. However in that case, there are many disjoint sets that avoid the same colour \( b \), and we can use a similar argument to find two inputs with the same colour \( b \neq b \) and with large disjoint supports, which similarly implies a small set avoiding \( \{b, b'\} \). More generally the inductive step is as follows.

5 Lemma. Let \( 1 \leq t \leq c \) and \( L \) large enough \( (L \geq (c + 1)c + c) \). Suppose \( f \) has \( \binom{c}{t} \cdot c \) disjoint \( t \)-avoiding sets of size \( \leq c^t \). Then it has a \((t + 1)\)-avoiding set of size \( \leq c^{t+1} \).

Proof. By assumption, \( f \) has \( \geq c + 1 \) disjoint sets \( S_1, \ldots, S_{c+1} \subseteq [L] \) that avoid the same \( C \subseteq [c] \) of size \( |C| = t \). Let \( R := [L] \setminus (S_1 \cup \cdots \cup S_{c+1}) \). Let \( h \) be an integer satisfying \( |R| - c + 2 > 2h \geq |R| - c \). Consider inputs in \([2]^L\) whose support consists of exactly one of \( S_1, \ldots, S_{c+1} \), plus exactly \( h \) elements of \( R \). The graph whose vertices are those inputs and whose edges are input pairs with disjoint supports is isomorphic to \( K_{c+1} \times KG(R, h) \). Since \( \chi(K_{c+1} \times KG(R, h)) > c \), there are two inputs \( \bar{u}, \bar{v} \) with disjoint support but the same colour \( f(\bar{u}) = f(\bar{v}) = b' \). Since fixing any \( S_i \) to 1 avoids \( C \), we have \( b' \notin C \). Let \( S := [L] \setminus (\text{supp}(\bar{u}) \cup \text{supp}(\bar{v})) \). Then any input \( \bar{w} \) with \( \bar{w}|_S \equiv 1 \) has \( f(\bar{w}) \neq b' \). Moreover, \( \bar{w}|_S \equiv 1 \) implies \( w|_S \equiv 1 \) for \((c + 1) - 2 \geq 1\) different \( i \), hence \( f(\bar{w}) \notin C \).

Thus \( S \) is \( C \cup \{b'\}\)-avoiding. Finally \( |S| \leq ((c + 1) - 2) \cdot c^t + |R| - 2h \leq (c - 1) \cdot c^t + c \leq c^{t+1} \).

Note that there cannot be any \( c \)-avoiding set, since \( f \) cannot avoid all colours. So there is a maximum \( t = t(f) \) with \( 1 \leq t \leq c \) such that \( f \) has some \( t \)-avoiding set of size \( \leq c^t \). Let \( \text{sel}(f) \subseteq [L] \) be the sum of a maximal family of disjoint \( t(f) \)-avoiding sets of size \( \leq c^t \). By the above lemma \( |\text{sel}(f)| \leq \binom{c}{t} \cdot c^t \leq (2c)^c \) (technically if \( L < (c + 1)c + c \) take \( \text{sel}(f) = [L] \)) instead.

We show that our selection \( \text{sel}(f) \subseteq [L] \) of constant size \( \leq (2c)^c \), independent of \( L \) is somewhat consistent for different \( f \), in the following very weak sense. Let \( g \) be a minor of \( f \) for some \( \pi: [L] \to [L'] \); that is, \( g(x_1, \ldots, x_L) = f(x_{\pi(1)}, \ldots, x_{\pi(L)}) \). We denote this as \( f \xrightarrow{\pi} g \). Suppose for a moment that \( t(f) = t(g) = t \). Clearly for every \( t \)-avoiding set \( S \) of \( f \), \( \pi(S) \) is a \( t \)-avoiding set of \( g \) of at most the same size. Hence \( \pi(\text{sel}(f)) \) intersects \( \text{sel}(g) \), by maximality of the family selected for \( g \).

Consider now a chain of minors \( f_0 \xrightarrow{\pi_{0,1}} f_1 \xrightarrow{\pi_{1,2}} f_2 \cdots \xrightarrow{\pi_{c-1,c}} f_c \). For \( i, j \) we let \( \pi_{i,j} \) be the composition \( \pi_{j-1,j} \circ \cdots \circ \pi_{i,i+1} \) and observe that \( f_i \xrightarrow{\pi_{i,j}} f_j \). Since there are only \( c - 1 \) possibilities for what \( t(f) \) can be, we conclude that in any such sequence of minors, there exist \( i, j \) such that \( t(f_i) = t(f_j) \), and hence \( \pi_{i,j}(\text{sel}(f_i)) \) intersects \( \text{sel}(f_j) \).

Thus we have found a selection \( \text{sel}(f) \subseteq [L] \) of constant size for each polymorphism \( f \), which is “consistent” on chains of minors of length \( c \). This is a sufficient condition for NP-hardness by the following, “layered, baby” corollary of the PCP theorem.

6 Theorem. ([BWZ21]) Suppose there are constants \( k, \ell \) and there is an assignment \( \text{sel}(f) \subseteq [L] \) to every polymorphism \( f \) of a Promise CSP, such that for every \( \ell \)-chain of minors there are \( i, j \) such that \( \pi_{i,j}(\text{sel}(f_i)) \cap \text{se}(f_j) \neq \emptyset \). Then the Promise CSP is NP-hard.

(The version in Theorem 5.22 of [Bar+21] would also be sufficient.) This concludes the proof of Theorem 1.
Discussion

Barto and Kozik [BK22] recently found a purely combinatorial, self-contained proof of Theorem 6, without using the original PCP theorem itself. The fact that it implies Theorem 1 (among now many other examples) highlights that the “baby” version, despite its humble name, is quite powerful, making its new proof all the more intriguing.

We remark that the idea of using “layered” versions of PCP (here they appear as “chains of minors”) comes from [DRS05]. While it is a relatively simple extension of the non-layered versions, it is now a crucial ingredient of several NP-hardness results, such as in [BWZ21]. We refer the reader to [BK22] for a more detailed discussion of all these versions of the PCP theorem and its corollaries.

On a final note, we stress that at the core of our proof is still Lovász’ topological proof of Theorem 2 (relying on the Borsuk-Ulam theorem), as used in Lemma 4. These ideas from [DRS05] were extended in other results on Promise CSPs, see e.g. [ABP20]. An interesting open problem is whether they can be connected with the other use of topology in proving hardness of Promise CSPs, presented in [Kro+20].

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