HOMOLOGICAL INVARIANTS IN CATEGORY $\mathcal{O}$ FOR THE GENERAL LINEAR SUPERALGEBRA

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Abstract. We study three related homological objects in the BGG category $\mathcal{O}$ for basic classical Lie superalgebras with specific focus on the general linear superalgebra. These are the projective dimension, associated variety and complexity of a module. We demonstrate connections between projective dimension and singularity of modules and blocks. Similarly we investigate the connection between complexity and atypicality. This creates concrete tools to describe singularity and atypicality as homological, and hence categorical, properties of a block. However, we also demonstrate how two integral blocks in category $\mathcal{O}$ with identical global categorical characteristics of singularity and atypicality will generally still be inequivalent. This principle implies that category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$ can contain infinitely many non-equivalent blocks, which we work out explicitly for $\mathfrak{gl}(3|1)$.

All of this is in sharp contrast with category $\mathcal{O}$ for Lie algebras, but also with the category of finite dimensional modules for superalgebras. Furthermore we characterise modules with finite projective dimension to be those with trivial associated variety. We also study the associated variety of Verma modules. To do this, we also classify the orbits in the cone of self-commuting odd elements under the action of an even Borel subgroup.

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1. Introduction

The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ associated to a triangular decomposition of a finite dimensional contragredient Lie (super)algebra is an important and intensively studied object in modern representation theory, see e.g. [Hu]. Category $\mathcal{O}$ for a basic classical Lie superalgebra $\mathfrak{g}$ is not yet as well understood as the corresponding category for semisimple Lie algebras and exhibits many novel features. However, for the particular case of the general linear superalgebra $\mathfrak{gl}(m|n)$, category $\mathcal{O}$ has a Kazhdan-Lusztig (KL) theory, introduced by Brundan in [Br1] and proved to be correct by Cheng, Lam and Wang in [CLW]. This determines the characters of simple modules algorithmically. Moreover, in this case, category $\mathcal{O}$ is Koszul as proved by Brundan, Losev and Webster in [BLW, Br2]. In the current paper we note that this theory is also a ‘KL theory’ in the abstract sense of [CPS2]. Also for the case of $\mathfrak{osp}(2m + 1|2n)$ a KL type theory has been introduced and established by Bao and Wang in [BW].

Our main focus in the current paper is the study of three homological invariants in category $\mathcal{O}$, specifically for $\mathfrak{gl}(m|n)$, and their applications to the open question concerning the classification non-equivalent blocks.
The associated variety of a module \( M \in \mathcal{O} \) is the set of self-commuting odd elements \( x \) of the Lie superalgebras for which \( M \) has non-trivial \( \mathbb{C}[x] \)-homology, see [DS, Se2]. Since the associated variety of a module in \( \mathcal{O} \) consists of orbits of the even Borel subgroup, we classify such orbits. Then we investigate the associated variety of Verma modules for \( \mathfrak{gl}(m|n) \) with distinguished Borel subalgebra, leading in particular to a complete description for the cases \( \mathfrak{gl}(1|n), \mathfrak{gl}(m|1) \) and \( \mathfrak{gl}(2|2) \).

The projective dimension of an object in an abelian category with enough projective objects is the length of a minimal projective resolution. Contrary to category \( \mathcal{O} \) for semisimple Lie algebras, category \( \mathcal{O} \) for \( \mathfrak{g} \) contains modules with infinite projective dimension. We obtain two characterisations for the abelian subcategory of modules having finite projective dimension. The first one (valid for \( \mathfrak{gl}(m|n) \)) is as the category of modules having trivial associated variety. The second one (valid for arbitrary \( \mathfrak{g} \)) is as an abelian category generated by the modules induced from the underlying Lie algebra. Then we determine the projective dimension of injective modules for \( \mathfrak{gl}(m|n) \) and use this to obtain the finitistic global homological dimension of the blocks, which builds on and extends some results of Mazorchuk in [CM2, Ma1, Ma2]. Concretely we show that this global categorical invariant of the blocks is determined by the singularity of the core of the central character. The results also provides a means to describe the level of dominance of a simple module in terms of the projective dimension of its injective envelope.

To deal with modules with infinite projective dimension we define the complexity of a module as the polynomial growth rate of a minimal projective resolution. The concrete motivation is to obtain a tool to homologically and categorically describe atypicality, similar to the description of singularity and dominance which followed from our study of projective dimension. We prove that our notion of complexity is well defined on category \( \mathcal{O} \) for any basic classical Lie superalgebra, meaning that complexity of all modules is finite. Then we study the complexity of Verma (for the distinguished Borel subalgebra) and simple modules for \( \mathfrak{gl}(m|n) \) and the relation with the degree of atypicality. Similar results for the category of finite dimensional weight modules of \( \mathfrak{gl}(m|n) \) have been obtained by Boe, Kujawa and Nakano in [BKN1, BKN2].

Integral blocks in category \( \mathcal{O} \) for Lie algebras are equivalent if and only if they have the same singularity, see [So]. Similarly, the blocks of the category of finite dimensional modules of basic classical Lie superalgebras depend (almost) only on the degree of atypicality, see [CS, Mar]. The classification of non-equivalent blocks in category \( \mathcal{O} \) for Lie superalgebras is an open question. Our main result here is the fact that a combination of the two aforementioned global categorical characteristics does not suffice to separate between non-equivalent blocks. Concretely we use our results on projective dimensions to materialise subtle local differences in blocks with similar global properties into categorical invariants. We do this explicitly for regular atypical blocks for \( \mathfrak{sl}(3|1) \), resulting in the fact that all such blocks are non-equivalent. In particular this implies that, contrary to category \( \mathcal{O} \) for Lie algebras and the category of finite dimensional weight modules for Lie superalgebras, category \( \mathcal{O} \) for Lie superalgebras can contain infinitely many non-equivalent blocks and equivalences between integral blocks will be very rare. This is summarised in the table [H] where \( \mathcal{F} \) represents the category of finite dimensional weight modules and \( \mathfrak{g}_0 \) a reductive Lie algebra.

We hope that our results can be applied to obtain a full classification of non-equivalent blocks in category \( \mathcal{O} \) for \( \mathfrak{gl}(m|n) \). Our results seem to suggest that equivalences would be rather exceptional. Furthermore in further work we aim to strengthen the equivalence between trivial associative variety and zero complexity for a module to a more general link between complexity and the associated variety and in particular to determine the complexity of simple modules.
The Lie superalgebra generated by the positive root vectors is denoted by $b$.

In this paper we work with basic classical Lie superalgebras, we refer to Chapters 1-4 of [MM] for concrete definitions. We denote a basic classical Lie superalgebra by $g$ and its even and odd part by $g_{\even} \oplus g_{\odd} = g$. A Borel subalgebra will be denoted by $b$, a Cartan subalgebra by $h$ and the set of positive roots by $\Delta^+$. The set of integral weights is denoted by $P_0 \subset \mathfrak{h}^*$. The Weyl group $W = W(g : h)$ is the same as the Weyl group $W(g_0 : h)$.

2. Preliminaries

In this paper we work with basic classical Lie superalgebras, we refer to Chapters 1-4 of [MM] for concrete definitions. We denote a basic classical Lie superalgebra by $g$ and its even and odd part by $g_{\even} \oplus g_{\odd} = g$. A Borel subalgebra will be denoted by $b$, a Cartan subalgebra by $h$ and the set of positive roots by $\Delta^+$. The set of integral weights is denoted by $P_0 \subset \mathfrak{h}^*$. The Weyl group $W = W(g : h)$ is the same as the Weyl group $W(g_0 : h)$.

2.1. Basic classical Lie superalgebras of type $A$. Mostly $g$ will be one of the following

$$ g = \begin{cases} g(m|n) & \text{if } m \neq n, \\ \mathfrak{sl}(m|n) & \text{if } m = n, \\ \mathfrak{psl}(n|n) & \text{if } m = n. \end{cases} $$

Then we use the standard $\mathbb{Z}$-grading $g = g_{-1} \oplus g_0 \oplus g_1$, with $g_0 = g_0$ and $g_1 = g_{-1} \oplus g_1$. We fix an element in the centre of $g_0$, $z \in \mathfrak{z}(g_0)$, that satisfies

$$ [z, X] = X, \quad \forall X \in g_1 \quad \text{and} \quad [z, Y] = -Y, \quad \forall Y \in g_{-1}. $$

The necessity of such an element is the reason we can not always include the other basic classical Lie superalgebras of type $A$, $\mathfrak{sl}(n|n)$ and $\mathfrak{psl}(n|n)$. In Remark 1.5 we provide an example of properties concerning the associated variety that fail for $\mathfrak{sl}(n|n)$.

Unless stated otherwise, the positive roots are chosen to be

$$ \Delta^+ = \begin{cases} \varepsilon_i - \varepsilon_j & \text{for } 1 \leq i < j \leq m, \\ \varepsilon_i - \delta_j & \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n, \\ \delta_i - \delta_j & \text{for } 1 \leq i < j \leq n. \end{cases} $$

The Lie superalgebra generated by the positive root vectors is denoted by $n$, so we have $b = h \oplus n$, $g_1 = n \cap g_1$ and set $n_0 := n \cap g_0$.

We define $\rho = \frac{1}{2} (\sum_{\alpha \in \Delta^+_0} \alpha) - \frac{1}{2} (\sum_{\gamma \in \Delta^+_1} \gamma)$, but it is more convenient to use

$$ \delta = -\sum_{i=1}^m i \varepsilon_i + \sum_{i=1}^n (m - i + 1) \delta_i $$

The paper is structured as follows. In Section 2 we recall some preliminary results. In Section 3 we study extensions between Verma and simple modules, in connection with Brundan’s Kazhdan-Lusztig theory. In Section 4 we obtain the characterisations for the modules with finite projective dimension. In Section 5 we study the self-commuting cone and associated varieties in relation with category $O$. In Section 6 we study projective dimensions and obtain the result on the non-equivalence of blocks. In Section 7 we introduce and study the notion of complexity. Finally, in the appendices we illustrate certain results for the example of $\mathfrak{gl}(2|1)$ and carry out some technicalities.
We fix a bijection between integral weights $P_0 \subset \mathfrak{h}^*$ and $\mathbb{Z}^{mn}$, by

$$P_0 \leftrightarrow \mathbb{Z}^{mn}, \quad \lambda \mapsto \mu^\lambda \quad \text{with} \quad \mu^\lambda_i = (\lambda + \delta, \varepsilon_i) \quad \text{and} \quad \mu^\lambda_{m+j} = (\lambda + \delta, \delta_j).$$

We will often silently use this identification $\lambda \mapsto \mu^\lambda$ between integral weights and $\mathbb{Z}^{mn}$. As in [BLW] we use the notation $\Lambda := \mathbb{Z}^{mn}$. The dot action

$$w \cdot \lambda = w(\lambda + \rho) - \rho = w(\lambda + \rho_0) - \rho_0 = w(\lambda + \delta) - \delta$$

of $w \in W$ on $\lambda \in \mathfrak{h}^*$ leads to the regular action of $W \cong S_m \times S_n$ on $\Lambda = \mathbb{Z}^{mn}$.

We also set

$$\Lambda^+ = \{\lambda \in \Lambda \mid w\lambda < \lambda, \forall w \in W\}.$$

This is the poset that describes the highest weight structure of the category of finite dimensional weight modules $\mathcal{F}$, see e.g. [Br1, Se1].

### 2.2. BGG category $\mathcal{O}$

Category $\mathcal{O}$ for a basic classical Lie superalgebra $\mathfrak{g}$ with Borel subalgebra $\mathfrak{b}$ is defined as the full subcategory of all $\mathfrak{g}$-modules, where the objects are finitely generated; $\mathfrak{b}$-semisimple and locally $U(\mathfrak{b})$-finite. Note that this definition does not depend on the actual choice of Borel subalgebra, only the even part $\mathfrak{b}_0 := \mathfrak{b} \cap \mathfrak{g}_0$. For each Borel subalgebra $\mathfrak{b}$ (with $\mathfrak{b} \cap \mathfrak{g}_0 = \mathfrak{b}_0$), this category has a (different) structure of a highest weight category. An alternative definition of $\mathcal{O}$ is as the full subcategory of all $\mathfrak{g}$-modules, where the objects $M$ satisfy $\text{Res}_{\mathfrak{b}_0}^{\mathfrak{g}_0} M \in \mathcal{O}_0$, for $\mathcal{O}_0$ the corresponding category for $\mathfrak{g}_0$.

Now we turn to this category for $\mathfrak{gl}(m|n)$. For an overview of the current knowledge we refer to the survey [Br2]. We denote the Serre subcategory of $\mathcal{O}$ generated by modules admitting the central character $\chi$ by $\mathcal{O}_\chi$. This subcategory does not need to be indecomposable. The indecomposable blocks of category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$ have been determined by Cheng, Mazorchuk and Wang in [CMW]. They also proved that every non-integral block is equivalent to an integral block of a direct sum of other general linear superalgebras, therefore we can restrict to integral blocks for several of our purposes. The Serre subcategory of $\mathcal{O}$ of modules with integral weight spaces is denoted by $\mathcal{O}_Z$. The blocks of category $\mathcal{O}$ can be described by linkage classes. The integral case is easily described; the linkage class $\xi$ generated by $\mu \in P_0$ is

$$\xi = [\mu] = \{\lambda \in P_0 \mid \chi_\mu = \chi_\lambda\}.$$

The indecomposable (see Theorem 3.12 in [CMW]) block $\mathcal{O}_\xi$ is then defined as the full Serre subcategory of $\mathcal{O}$ generated by the set of simple modules $\{L(\lambda) \mid \lambda \in \xi\}$. The degree of atypicality of the weights in the linkage class is denoted by $\sharp \xi$. Furthermore we denote the central character corresponding to the block by $\chi_\xi$.

The linkage classes in general correspond to $\mathfrak{h}^*$ modulo the equivalence relation generated by the super Bruhat order, see [Br1, BLW, CMW]. This Bruhat order $\leq$ on integral weights is the minimal partial order satisfying (we use $\leq$ for the usual dominance order)

- if $s \cdot \lambda \leq \lambda$ for a reflection $s \in W$ and $\lambda \in P_0$, we have $s \cdot \lambda \leq \lambda$;
- if $(\lambda + \rho, \gamma) = 0$ for $\lambda \in P_0$ and $\gamma \in \Delta^+_f$, we have $\lambda - \gamma \leq \lambda$.

The set $\Lambda \cong P_0$ equipped with this Bruhat order is the poset for $\mathcal{O}_Z$ as a highest weight category.

As in [GS] we define the core of $\chi_\xi$, which we denote by $\chi_\xi'$. This is the typical central character of $\mathfrak{gl}(m - \sharp \xi|n - \sharp \xi)$, corresponding to a weight in $\xi$ where $\sharp \xi$ labels on each side are removed in order to create a typical weight. We also fix $w_0^\xi \in W$ as the longest element in the subgroup of the Weyl group of $\mathfrak{gl}(m - k|n - k)$ which stabilises a dominant weight corresponding to $\chi_\xi'$. 

since the coefficients of $\delta$ are integers. The difference $\rho - \delta$ is orthogonal to all roots.
We will use the translation functors on \( \mathcal{O} \) introduced in \[Br1\] and studied further in \[BLW, Kn\]. Denote by \( U = \mathbb{C}^{m,n} \) the tautological module and let \( F \) (resp. \( E \)) be the exact endofunctor of \( \mathcal{O}_Z \) defined by tensoring with \( U \) (resp. \( U^* \)). Then \( \{ F_i \mid i \in \mathbb{Z} \} \) and \( \{ E_i \mid i \in \mathbb{Z} \} \) are the subfunctors of \( F \) and \( E \) corresponding to projection on certain blocks. According to Theorem 3.10 of \[BLW\], this defines an \( \mathfrak{sl}(\infty) \) tensor product categorification on \( \mathcal{O}_Z \).

Finally we introduce notation for some structural modules in category \( \mathcal{O} \). For each \( \lambda \in \mathfrak{h}^* \) we denote the Verma module (the \( \mathfrak{g} \)-module induced from the one dimensional \( \mathfrak{h} \)-module on which \( \mathfrak{h} \) acts through \( \lambda \)) by \( M(\lambda) \). Its simple top is denoted by \( L(\lambda) \). The indecomposable projective cover of \( L(\lambda) \) is denoted by \( P(\lambda) \) and the indecomposable injective hull by \( I(\lambda) \). The corresponding modules in \( \mathcal{O}^0 \) are denoted by \( L_0(\lambda), M_0(\lambda), P_0(\lambda), I_0(\lambda) \). We denote the Kac-type modules by \( K(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L_0(\mu) \) and the dual Kac modules by \( \overline{K}(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L_0(\mu) \) for any \( \mu \in \mathbb{Z}^{m,n} \). These modules are also co-induced, e.g.

\[
\begin{align*}
U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L_0(\mu) & \cong \text{Hom}_{U(\mathfrak{g}_0 + \mathfrak{g}_1)}(U(\mathfrak{g}), L_0(\mu - 2\rho_1)), \\
\end{align*}
\]

We denote the central character (morhpism \( Z(\mathfrak{g}) \to \mathbb{C} \)) corresponding to \( L(\lambda) \) by \( \chi_\lambda \).

By Theorem 24(i) and Corollary 13 of \[CM2\], for any \( \lambda \in \mathfrak{h}^* \) and \( N \in \mathcal{O} \) we have

\[
\text{Ext}^i_N(M(\lambda), N) \cong \text{Hom}_\mathbb{C}(\mathcal{O}_\lambda, \mathcal{H}(n, N)),
\]

with \( \mathcal{H}(n, -) \cong \mathcal{E}_n(\mathbb{C}, -) \), the Lie superalgebra cohomology of \( n \).

2.3. Brundan Kazhdan-Lusztig theory. We review a few items of \[Br1, Br2, BLW, CLW\], to which we refer for details, see also the survey \[Br2\].

Let \( V \) be the natural \( \mathfrak{sl}(\infty) \) module and \( W \) its dual. The Lie algebra \( \mathfrak{sl}(\infty) \) is generated by the Chevalley generators \( \{ e_i, f_i \mid i \in \mathbb{Z} \} \). Then \( \Lambda = \mathbb{Z}_+^{m,n} \) naturally parametrizes a monomial basis of \( V_\otimes m \otimes W_\otimes n \). We denote such a monomial basis by \( \{ v_\lambda \mid \lambda \in \Lambda \} \).

Identifying \( [M(\lambda)] \in K(\mathcal{O}_Z) \) with \( v_\lambda \) then leads to an isomorphism of vector spaces

\[
K(\mathcal{O}_Z^\mathcal{A}) \leftrightarrow V_\otimes m \otimes W_\otimes n,
\]

with \( K(\mathcal{O}_Z^\mathcal{A}) \) the Grothendieck group of the exact full subcategory \( \mathcal{O}_Z^\mathcal{A} \) of \( \mathcal{O}_Z \) with as objects the modules admitting a Verma flag. It follows immediately from the standard filtration of \( M(\lambda) \otimes U \) that we can define an \( \mathfrak{sl}(\infty) \)-action on \( K(\mathcal{O}_Z) \) by \( e_i [M] := [E_i M] \) and \( f_i [M] = [F_i M] \). The isomorphism (2.6) then becomes an \( \mathfrak{sl}(\infty) \)-module isomorphism.

For the quantised enveloping algebra \( U_q(\mathfrak{sl}(\infty)) \) we denote the corresponding module by \( V_\otimes m \otimes W_\otimes n \). It turns out that \( V_\otimes m \otimes W_\otimes n \) admits a Lusztig canonical basis. This basis is denoted by \( \{ \hat{b}_\mu, \mu \in \Lambda \} \) and the monomial basis by \( \{ \hat{v}_\lambda, \lambda \in \Lambda \} \). Then the polynomials \( d_{\lambda,\mu}(q) \) and the inverse matrix, the KL polynomials \( p_{\lambda,\nu}(q) \), are defined as

\[
\begin{align*}
\hat{b}_\mu &= \sum_{\lambda \in \Lambda} d_{\lambda,\mu}(q) \hat{v}_\lambda \quad \text{and} \quad \hat{v}_\lambda = \sum_{\mu \in \Lambda} p_{\lambda,\nu}(q) \hat{b}_\nu, \\
\end{align*}
\]

In \[Br1\], Brundan conjectured and in \[CLW\] Cheng, Lam and Wang proved that

\[
( P(\mu) : M(\lambda) ) = d_{\lambda,\mu}(1) = [M(\lambda) : L(\mu)].
\]

Furthermore Brundan, Losev and Webster proved in Theorem A of \[BLW\] that \( \mathcal{O} \) has a graded lift which is standard Koszul. This implies that the KL polynomials can be interpreted as a minimal projective resolution of the Verma module, or

\[
p_{\lambda,\nu}(q) = \sum_{k \geq 0} q^k \dim \text{Ext}^k_{\mathcal{O}}(M(\lambda), L(\nu)),
\]

see also Section 5.9 of \[BLW\]. For \( q = -1 \) equation (2.8) is a direct consequence of equation (2.7) and the Euler-Poincaré principle.
Take an interval $I \subset \mathbb{Z}$. The Lie algebra $\mathfrak{sl}(I) \cong \mathfrak{sl}(|I|+1)$ is generated by the Chevalley generators $\{e_i, f_i \mid i \in I\}$. We set $I_+ := I \cup (I+1)$. Then $\Lambda_I$ is the sub-poset of $\Lambda$, where all labels are in the interval $I_+$ and $\Lambda_I$ is in bijection with the monomial basis of the $\mathfrak{sl}(I)$-module $V_I \otimes^m \otimes W_I \otimes^n$. Since $W_I \cong \Lambda^{|I|}V_I$, this also corresponds to another poset of a highest weight category. These are the blocks of parabolic category $\mathcal{O}$ for $\mathfrak{gl}(m+|I|n)$ with Levi subalgebra $\mathfrak{gl}(1) \oplus \mathfrak{gl}(|I| \otimes^m)$, for the weights where the labels are in $I_+$, see Definition 3.13 in [LW]. We denote this category by $\mathcal{O}_I^\prime$. In Section 3.8 in [LW], two ideals $\Lambda_{\leq I}$ and $\Lambda_{< I}$ in the poset are constructed with the property $\Lambda_I = \Lambda_{\leq I} \setminus \Lambda_{< I}$. Then we have Serre (highest weight) subcategories $\mathcal{O}_{\leq I}$ and $\mathcal{O}_{< I}$ in $\mathcal{O}_{\leq I}$ and the quotient category $\mathcal{O}_I = \mathcal{O}_{\leq I}/\mathcal{O}_{< I}$. This is a highest weight category with poset $\Lambda_I$. By the general theory of such subquotients of highest weight categories, it follows that

$$\text{(2.9) } \text{Ext}^\ast_{\mathcal{O}_I}(M(\mu), L(\lambda)) \cong \text{Ext}^\ast_{\mathcal{O}_I^\prime}(M(\mu), L(\lambda))$$

if $\lambda, \mu \in \Lambda_I$, follows immediately from the general theory of (sub)quotients of highest weight categories, see Section 2.5 in [BLW].

It is proved in [BLW] that, through uniqueness of tensor product categorifications in [LW], the categories $\mathcal{O}_I$ and $\mathcal{O}_I^\prime$ are equivalent.

### 3. Extensions and Kazhdan-Lusztig theory

In this entire section we consider $\mathfrak{g} = \mathfrak{gl}(m|n)$. The results extend to $\mathfrak{sl}(m|n)$ if $m \neq n$.

#### 3.1. Length function and abstract Kazhdan-Lusztig theories

We define a length function $l : \Lambda \times \Lambda \to \mathbb{N}$ on a poset $\Lambda$ to be a function with domain $\{(\lambda, \mu) \mid \mu \preceq \lambda\}$ which satisfies $l(\lambda, \mu) = l(\lambda, \kappa) + l(\kappa, \mu)$ if $\mu \preceq \kappa \preceq \lambda$, with $l(\lambda, \mu) = 0$ if and only if $\lambda = \mu$. Note that, in principal, a length function should be a function $l' : \Lambda \to \mathbb{N}$ such that $l(\lambda, \mu) := l'(\lambda) - l'(\mu)$ satisfies the above properties. However, in our case, it is possible to construct such an $l'$ from our $l$ by the procedure in Section 3-g in [Br1]. And as in the expressions we use we will only need the difference in length between two comparable weights, we ignore this technicality.

Before going to $\mathfrak{gl}(m|n)$ we review this function for any (possibly singular) block in (possibly) parabolic category $\mathcal{O}$ for a semisimple Lie algebra. For a block of category $\mathcal{O}$ we set $l(\lambda, \mu) = l(\lambda) - l(\mu)$, with $l(\lambda), l(\mu)$ as defined in Theorem 3.8.1 in [CPS1]. For a parabolic category we just keep the same length function restricted to the poset of weights dominant for the Levi subalgebra. Also the Bruhat order is the restriction of the Bruhat order in the non-parabolic case.

Now we can define a length function on category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$. For the cases $\mathfrak{g} = \mathfrak{gl}(2|1)$ and $\mathfrak{g} = \mathfrak{gl}(1|2)$ this will be made explicit in Appendix $A$.

**Lemma 3.1.** For any two $\lambda, \mu \in \Lambda$ with $\mu \preceq \lambda$ and any interval $I$ such that $\lambda, \mu \in \Lambda_I$, set $l_I(\lambda, \mu) = l(\phi_I(\lambda), \phi_I(\mu))$.

This value $l_I(\lambda, \mu)$ does not depend on the particular interval $I$. This leads to a well-defined length function $1$ on $\Lambda$ defined by

$$1(\lambda, \mu) = l_I(\lambda, \mu) \quad \text{for any } I \text{ such that } \lambda, \mu \in \Lambda_I.$$

**Proof.** By the description in Section 2.2 in [BLW] and Section 3.5 in [LW] we have

$$\phi_{[a,b]}(\alpha_1, \ldots, \alpha_m|\beta_1, \ldots, \beta_n) = (\alpha_1, \ldots, \alpha_m, b+1, b, \ldots, b, a, b+1, b, \ldots, b, a, b, \ldots, a).$$
We have to prove that the length function \( l_{[a,b]} \) does not depend on \([a,b]\). In order to calculate the length function \( l_{[a,b]}(\lambda, \mu) \) we have to translate the weights \( \phi_{[a,b]}(\lambda) \) and \( \phi_{[a,b]}(\mu) \) away from the walls using the procedure in the following paragraph. Note that since \( \chi_\mu = \chi_\lambda \), \( \phi_{[a,b]}(\lambda) \) and \( \phi_{[a,b]}(\mu) \) are in the same orbit and hence contain the same labels with the same multiplicities.

Take a certain value \( x \) which appears \( p \) times (with \( p > 1 \)), then all labels strictly larger than \( x \) must be raised by \( p - 1 \); and a label equal to \( x \) must be raised by \( i - 1 \) if it is the \( i \)-th occurrence starting from the left. After a finite number of times of applying this procedure we obtain regular weights which we denote by \( \tilde{\varphi}_I(\lambda) \) and \( \tilde{\varphi}_I(\mu) \).

It then follows that if

\[
\tilde{\varphi}_{[a,b]}(\lambda) = (x_1, x_2, \cdots, x_{m+n(b-a+1)}) \quad \text{and} \quad z = x_{m+(n-1)(b-a+1)+1},
\]

then \( z \) is the highest label in \( \tilde{\varphi}_{[a,b]}(\lambda) \) and we have \( \tilde{\varphi}_{[a,b+1]}(\lambda) = (x_1, \cdots, x_m, z + 1, x_{m+1}, \cdots, x_{m+b-a+1}, z + 2, x_{m+b-a+2}, \cdots, x_{m+2(b-a+1)}, \cdots) \).

Since we have the same description of \( \tilde{\varphi}_{[a,b]}(\mu) \). The length function is the same when the labels \( z + 1, z + 2, \cdots \) are removed. \( \square \)

This definition has the following immediate consequence.

**Theorem 3.2.** For \( \lambda, \mu \in \Lambda \), we have

\[
\text{Ext}^i_\mathcal{O}(M(\mu), L(\lambda)) \neq 0 \quad \Rightarrow \quad \mu \preceq \lambda \quad \text{with} \quad i \leq 1(\lambda, \mu) \quad \text{and} \quad i \equiv 1(\lambda, \mu) \pmod{2}.
\]

**Proof.** Through equations (2.8), (2.9) and the fact that KL polynomials of \( \mathcal{O}_I \) correspond to those in \( \mathcal{O}'_I \), this can be reduced to the corresponding statement on extensions between standard and simple modules in singular parabolic blocks of category \( \mathcal{O} \) for Lie algebras.

By Lemma 3.1

This result is known in these categories. For blocks in non-parabolic category \( \mathcal{O} \) this is Theorem 3.8.1 in [CPS1]. In general, the Koszul dual (see [Ba]) statement concerns the radical filtration of standard modules, or equivalently the Koszul grading of the standard modules. The dual of the result in [CPS1] is then about regular parabolic blocks. The full (dual) result follows immediately from graded translation to the wall, see [S1]. \( \square \)

By applying the work of Cline, Parshall and Scott this leads to the following corollary.

**Corollary 3.3.** The highest weight category \( \mathcal{O} \) with length function \( 1 \) has a Kazhdan-Lusztig theory, according to Definition 2.1 in [CPS2]. Consequently, we have

\[
\sum_{k \geq 0} q^k \dim \text{Ext}^k_\mathcal{O}(L(\lambda), L(\mu)) = \sum_{\nu \in \Lambda} \nu_{\nu, \lambda}(q) p_{\nu, \mu}(q).
\]

**Proof.** This is a special case of Corollary 3.9 in [CPS2], using the duality on \( \mathcal{O} \). \( \square \)

Comparison with equation (2.8) yields

\[
(3.1) \quad \dim \text{Ext}^j_\mathcal{O}(L(\lambda), L(\mu)) = \sum_{i=0}^j \sum_{\nu \in \Lambda} \dim \text{Ext}^i_\mathcal{O}(M(\nu), L(\lambda)) \dim \text{Ext}^{j-i}_\mathcal{O}(M(\nu), L(\mu)).
\]

**Remark 3.4.** The analogue of equation (2.9) does not hold for

\[
\text{Ext}^*_\mathcal{O}(L(\mu), L(\lambda)) \leftrightarrow \text{Ext}^*_\mathcal{O}_I(L(\mu), L(\lambda)),
\]

which is confirmed by equation (3.1), as the summation over \( \nu \) goes out of \( \Lambda_I \). However, using the subsequent Lemma 3.3 it is possible to show that for each \( \lambda, \mu \in \Lambda_I \) and a fixed degree \( j \), there is an interval \( \bar{\lambda}_{\lambda, \mu, j} \), such that

\[
\dim \text{Ext}^j_\mathcal{O}(L(\mu), L(\lambda)) \cong \dim \text{Ext}^j_{\mathcal{O}_I \bar{\lambda}_{\lambda, \mu, j}}(L(\mu), L(\lambda)).
\]
Lemma 3.8. The result then follows from the analogue of equation (2.5) in the category.

Remark 3.5. (1) The length function 1 of Lemma 3.1 does not reduce to the length function for \( \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \), when restricted to one Weyl group orbit.
(2) The restriction of 1 to \( \Lambda^{++} \) does not correspond to the known length function, as defined by Brundan in Section 3-g. of [Br1]. It is impossible to find a length function on \( \Lambda \) with such a restriction.

Both properties are illustrated in Appendix A.

3.2. Further vanishing properties of extensions.

Lemma 3.6. Consider \( \lambda, \mu \in \mathfrak{h}^* \) and a simple reflection \( s \in W \).

(i) If \( s \cdot \lambda = \lambda \) and \( s \cdot \mu < \mu \), we have \( p_{\lambda,\mu} \equiv 0 \), so \( \text{Ext}_{\mathcal{O}}^j(M(\lambda), L(\mu)) = 0 \).

(ii) If \( s \cdot \lambda < \lambda \) and \( s \cdot \mu < \mu \), we have \( \text{Ext}_{\mathcal{O}}^j(M(s \cdot \lambda), L(\mu)) \cong \text{Ext}_{\mathcal{O}}^{j-1}(M(\lambda), L(\mu)) \).

Proof. We prove these results using the right exact twisting functors \( T_s \) and the left derived functors \( \mathcal{L}_i \), as studied in [CMW], [CMI]. For both (i) and (ii) we have

\[ \mathcal{L}_i T_s (M(\lambda)) \cong \delta_{i,0} M(s \cdot \lambda) \quad \text{and} \quad \mathcal{L}_i T_s (L(\mu)) \cong \delta_{i,1} L(\mu), \]

see Lemmata 5.4 and 5.7 and Theorem 5.12(i) in [CMI]. The combination of these results with Proposition 5.11 in [CMI], leads to

\[ \text{Ext}_{\mathcal{O}}^j(M(s \cdot \lambda), L(\mu)) \cong \text{Ext}_{\mathcal{O}}^{j-1}(M(\lambda), L(\mu)), \]

see e.g. the procedure in the proof of Proposition 3 in [Ma1]. This yields (i). In case \( s \cdot \lambda = \lambda \), we obtain by iteration that

\[ \text{Ext}_{\mathcal{O}}^j(M(\lambda), L(\mu)) = \text{Hom}_\mathcal{O}(M(\lambda), L(\mu)) = 0, \]

proving (i).

An alternative proof of Lemma 3.6 follows from equation (2.8) and a Hochschild-Serre spectral sequence (see e.g. Section 16.6 in [Ma1]) reducing the statement to an \( \mathfrak{sl}(2) \) property. In the following, we will apply the Hochschild-Serre spectral sequence

\[ H^p(n_0, H^q(\mathfrak{g}_1, L(\mu))) \Rightarrow H^{p+q}(n, L(\mu)). \]

Lemma 3.7. The restriction of the polynomials \( p_{\lambda,\nu}(q) \) in equation (2.8) to \( \lambda, \nu \in \Lambda^{++} \) gives the KL polynomials of [Br1], [Se1] for the category \( \mathcal{F} \). Concretely, for \( \lambda, \nu \in \Lambda^{++} \)

\[ \dim \text{Ext}_{\mathcal{O}}^j(M(\lambda), L(\nu)) = \dim \text{Ext}_\mathcal{F}^j(K(\lambda), L(\nu)). \]

Proof. Equation (2.8) allows to use the spectral sequence (3.2). The fact that \( H^q(\mathfrak{g}_1, L(\nu)) \) is a finite dimensional \( \mathfrak{g}_0 \)-module and Kostant cohomology (see [Ko]) imply that the spectral sequence collapses, so we get

\[ \dim \text{Ext}_{\mathcal{O}}^j(M(\lambda), L(\nu)) = \dim \text{Hom}_{\mathfrak{g}_0}(C_\lambda, H^j(\mathfrak{g}_1, L(\nu))), \]

Since \( H^j(\mathfrak{g}_1, L(\nu)) \) is a semisimple \( \mathfrak{g}_0 \)-module, this equals \( \dim \text{Hom}_{\mathfrak{g}_0}(H^j(\mathfrak{g}_1, L(\nu))). \)

The result then follows from the analogue of equation (2.8) in the category \( \mathcal{F} \).

The combination of Lemmata 3.6 and 3.7 completely determines the KL polynomials \( p_{\lambda,\nu} \) in case \( \nu \in \Lambda^{++} \) in terms of the KL polynomials for \( \mathcal{F} \) introduced in [Se1] and determined explicitly in [Br1].

Lemma 3.8. For \( \mu, \lambda \in \Lambda \) in the same orbit of the Weyl group we have

\[ \dim \text{Ext}_{\mathcal{O}}^j(M(\mu), L(\lambda)) = \dim \text{Ext}_{\mathcal{O}_0}^j(M_0(\mu), L_0(\lambda)) \quad \text{and} \]

\[ \dim \text{Ext}_{\mathcal{O}}^1(L(\mu), L(\lambda)) = \dim \text{Ext}_{\mathcal{O}_0}(L_0(\mu), L_0(\lambda)). \]
Proof. The second equality follows from the first and equation \([3.1]\).

For the first equality we consider equation \([2.5]\) and spectral sequence \([3.2]\). We have

\[
\text{Hom}_b(C_{\lambda}, H^p(n_0, H^q(g_1, L(\mu)))) = 0 \quad \text{if} \quad q > 0,
\]

since \(H^q(g_1, L(\mu))\) is a subquotient of \(S^q(g_{-1} \otimes \Lambda g_{-1} \otimes L_0(\mu))\), which does not contain any weights in the orbit of \(\lambda\) and \(\mu\) if \(q > 0\) by the existence of \(z\) in equation \([2.1]\). Therefore the spectral sequence collapses, yielding

\[
\text{Hom}_b(C_{\lambda}, H^1(n, L(\lambda))) \cong \text{Hom}_b(C_{\lambda}, H^1(n_0, L(\lambda)^{\oplus 1})) \cong \text{Ext}^1_{\mathcal{O}_0}(M(\mu), L_0(\lambda)),
\]

concluding the proof. An alternative proof of the first equality follows from the algorithm to calculate the canonical basis in Section 3 of \([Br2]\). \(\square\)

Lemma 3.9. Consider \(\lambda, \mu \in \mathfrak{h}^*\), then we have

\[
\text{Ext}^1_{\mathcal{O}_0}(M(\lambda), L(\mu)) \neq 0 \implies \max\{j - l(w_0), 0\} \leq \mu(z) - \lambda(z) \leq j + \dim g_1,
\]

with \(z \in z(g_0) \subset \mathfrak{g}\) as in equation \([2.1]\).

Proof. We use the reformulation \([2.3]\) and spectral sequence \([3.2]\). This implies that the extension vanishes unless \(\lambda\) is in the Weyl group orbit of the highest weight of a simple \(g_0\)-subquotient of \(H^q(g_1, L(\mu))\) for \(j - l(w_0) \leq q \leq j\). This implies in particular that

\[
[S^q(g_{-1}) \otimes \Lambda(g_{-1}) \otimes L_0(\mu) : L_0(w \cdot \lambda)] \neq 0,
\]

for some \(w \in W\), which yields the condition on \(\mu(z) - \lambda(z)\). \(\square\)

3.3. Socle of the tensor space.

Theorem 3.10. The socle of the \(\mathfrak{sl}(\infty)\)-module \(V^{\otimes m} \otimes W^{\otimes n}\) contains \(\{b_\mu : \mu \in \Lambda\}\).

Furthermore, under the \(\mathfrak{sl}(\infty)\)-module morphism \(V^{\otimes m} \otimes W^{\otimes n} \cong K(O_\Delta^1)\), this socle corresponds to the subgroup of \(K(O_\Delta^1)\) generated by the projective modules, or by the tilting modules.

Proof. We denote the socle by \(V^{(m,n)}\). It is proved in Theorem 2.2 of \([PS]\) that \(V^{(m,n)}\) corresponds to the intersection of the kernel of all contractions of the form

\[
V^{\otimes m} \otimes W^{\otimes n} \twoheadrightarrow V^{\otimes m-1} \otimes W^{\otimes n-1}.
\]

So \(v_\lambda\) is in \(V^{(m,n)}\) if and only if \(\lambda\) is typical.

According to the finiteness discussion of the algorithm to compute Lusztig’s canonical basis in Section 3 of \([Br2]\) it follows that for any \(\nu \in \Lambda\),

\[
\hat{b}_\nu = \sum_{\lambda} f_\lambda(q) A_\lambda \hat{v}_\lambda,
\]

for a finite sum of typical (dominant) \(\lambda \in \Lambda\), certain \(A_\lambda \in U_q(\mathfrak{sl}(\infty))\) and \(f_\lambda(q) \in \mathbb{Z}[q, q^{-1}]\). Evaluating this in \(q = 1\) yields the first part.

To prove the second statement we consider the description of the socle in Theorem 2.1 of \([PS]\). This implies that the socle is the direct sum of highest weight modules (with respect to the system of positive roots introduced there), where the highest weight vectors are linear combinations of \(v_\lambda\) for typical \(\lambda \in \Lambda\). This implies that the socle is inside the submodule generated by the basis \(\{b_\mu : \mu \in \Lambda\}\). \(\square\)
4. Finiteness of homological dimension and the associated variety

In this section (except Lemma 4.3) \( \mathfrak{g} \) is of type \( A \) excluding \( \mathfrak{sl}(n|n) \) and \( \mathfrak{psl}(n|n) \).

In Section 4.4 of [Ma2], Mazorchuk proved that the finitistic global dimension of (parabolic) category \( \mathcal{O} \) is finite for classical Lie superalgebras. In this section we relate the finiteness of projective dimensions with the associated variety as defined in [DS, Se2]. For any \( M \in \mathfrak{g}\text{-mod} \) we define its associated variety as

\[
X_M = \{ x \in \mathfrak{g}_1 \mid [x, x] = 0 \mid xM \neq \ker_x M \}. \tag{4.1}
\]

Denote the full subcategory of \( \mathcal{O} \) of modules which are direct summands of modules of the form \( \text{Ind}_{\mathfrak{g}_0}^\mathfrak{g} N_0 \) with \( N \in \mathcal{O}^0 \) by \( \mathcal{A}_0 \). This category does not need to be abelian. By iteration we define \( \mathcal{A}_j \) as the full subcategory of \( \mathcal{O} \), containing the modules in \( \mathcal{A}_{j-1} \) as well as modules in \( \mathcal{O} \) which can be written as a kernel or cokernel of an injective or surjective morphism between two modules in \( \mathcal{A}_{j-1} \) or an extension of two modules in \( \mathcal{A}_{j-1} \). By taking the full subcategory of \( \mathcal{O} \) consisting of modules in some \( \mathcal{A}_j \) for \( j \in \mathbb{N} \), we obtain an abelian full Serre subcategory which we denote by \( \langle (\mathfrak{g}, 0) \rangle \mathcal{O} \). The equivalence between \((iii)\) and \((iv)\) in the following theorem actually implies firstly that \( \langle (\mathfrak{g}, 0) \rangle \mathcal{O} \cong \mathcal{A}_{2l(w_0)} \) and secondly that this category can also be obtained by a similar procedure using only cokernels.

**Theorem 4.1.** Take \( \mathfrak{g} \) a basic classical Lie superalgebra of type \( A \), excluding \( \mathfrak{sl}(n|n) \) and \( \mathfrak{psl}(n|n) \) and \( M \in \mathcal{O} \). The following conditions are equivalent:

\[
(i) \quad \text{pd}_{\mathcal{O}} M < \infty; \quad (ii) \quad X_M = \{0\}; \quad (iii) \quad M \in \langle (\mathfrak{g}, 0) \rangle \mathcal{O}; \quad (iv) \quad \text{pd}_{\mathcal{O}} M \leq 2l(w_0).
\]

Consequently we have \( \text{fin.dim} \mathcal{O} = 2l(w_0) \).

The statement on the finitistic global dimension will be improved upon, by determining it each for each indecomposable block individually in Theorem 4.3.

**Remark 4.2.** As the proof of Theorem 4.1 reveals, for arbitrary contragredient Lie superalgebras with reductive even part we still have the property

\[
(i) \iff (iii) \iff (iv) \implies (ii).
\]

However, already for \( \mathfrak{sl}(1|1) \) we have \((ii) \not\iff (i)\) by Remark 4.5.

The remainder of this section is devoted to proving this theorem.

**Lemma 4.3.** Let \( \mathfrak{g} \) be a basic classical Lie superalgebra, \( M \in \mathcal{O} \) and \( N_0 \in \mathcal{O}^0 \). We have

\[
(i) \quad \text{pd}_{\mathcal{O}} M \geq \text{pd}_{\mathcal{O}_{\mathfrak{g}_0}} \text{Res}_{\mathfrak{g}_0}^\mathfrak{g} M;
(ii) \quad \text{If } M \text{ is a direct summand of Ind}_{\mathfrak{g}_0}^\mathfrak{g} N_0, \text{ then } \text{pd}_{\mathcal{O}} M \leq \text{pd}_{\mathcal{O}_{\mathfrak{g}_0}} N_0;
(iii) \quad \text{pd}_{\mathcal{O}} \text{Ind}_{\mathfrak{g}_0}^\mathfrak{g} N_0 = \text{pd}_{\mathcal{O}_{\mathfrak{g}_0}} N_0.
\]

Consequently we have \( \text{fin.dim} \mathcal{O} = 2l(w_0) \).

**Proof.** The functors \( \text{Res}_{\mathfrak{g}_0}^\mathfrak{g} \) and \( \text{Ind}_{\mathfrak{g}_0}^\mathfrak{g} \) are exact and map projective modules to projective modules, this implies \((i)\) and \((ii)\).

Claim \((iii)\) follows from combining \((i)\) and \((ii)\) since the \( \mathfrak{g}_0\)-module \( C \) is a direct summand of \( \Lambda \mathfrak{g}_1 \), so \( N_0 \) is a direct summand \( \text{Res}_{\mathfrak{g}_0}^\mathfrak{g} \text{Ind}_{\mathfrak{g}_0}^\mathfrak{g} N_0 \cong \Lambda \mathfrak{g}_1 \otimes N_0 \).

Property \((iii)\) implies \( \text{fin.dim} \mathcal{O} \geq 2l(w_0) \) and the reversed inequality is Theorem 3 of [Ma2].

**Proposition 4.4.** Take \( \mathfrak{g} \) a basic classical Lie superalgebra of type \( A \), excluding \( \mathfrak{sl}(n|n) \) and \( \mathfrak{psl}(n|n) \). Assume that \( M \in \mathcal{O} \), is \( \mathfrak{g}_{-1}\)-free (resp. \( \mathfrak{g}_1\)-free), then \( M \) has a Kac flag (resp. dual Kac flag).
Proof. Take $M$ to be $\mathfrak{g}_{-1}$-free. The $\mathfrak{g}_0$-module $N := M/\mathfrak{g}_{-1}M$ decomposes according to the eigenvalues of $z$, in equation (2.1), as $N = \bigoplus_{\alpha \in \mathfrak{h}^*} N_\alpha$. Since $M$ is finitely generated there is only a finite amount of $\alpha$ for which $N_\alpha \neq 0$. We take $\alpha_0$ to be the highest of these. Then $N_{\alpha_0}$ is isomorphic to a $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-submodule of $\text{Res}^\mathfrak{g}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1} M$. Since $M$ is $\mathfrak{g}_{-1}$-free, we find

$$U(\mathfrak{g}) \otimes U(\mathfrak{g}_0 \oplus \mathfrak{g}_1) N_{\alpha_0} \hookrightarrow M.$$ 

Since $U(\mathfrak{g}) \otimes U(\mathfrak{g}_0 \oplus \mathfrak{g}_1) N_{\alpha_0}$ clearly has a filtration by Kac modules, the proof can be completed iteratively by considering the cokernel of the above morphism.

The proof for $\mathfrak{g}_1$-free $M$ is identical. \qed

Remark 4.5. For $\mathfrak{g} = \mathfrak{sl}(n|m)$, the element $z \in \mathfrak{z}(\mathfrak{g}_0)$ does not exist, which leads to counterexamples of Proposition 3.4 and Theorem 4.1. Consider $\mathfrak{sl}(1|1) = \langle x, y, e \rangle$ with

$$[x, x] = 0 = [y, y], \quad [x, y] = e, \quad [e, x] = 0 = [e, y].$$

The two dimensional module $V = \langle v_1, v_2 \rangle$ with action $x v_1 = v_2 = y v_1$ and with trivial action of $e$ satisfies $X_V = \{0\}$ but has no (dual) Kac flag and is not projective in $\mathcal{F}$.

Lemma 4.6. If $M \in \mathcal{O}$ admits a Kac flag and a dual Kac flag, then $\text{pd}_\mathcal{O} M < \infty$.

Proof. We will prove $\text{id}_\mathcal{O} M < \infty$, which is equivalent by Section 3 in [Ma2] (or a special case of Proposition 4.4). The proof could also be done immediately for projective dimension using an unconventional definition of the Kac modules. For any $\lambda, \mu \in \mathfrak{h}^*$, we prove

$$\text{Ext}_\mathcal{O}^j(K(\mu), \overline{K}(\lambda)) = 0 = \text{Ext}_\mathcal{O}^j(\overline{K}(\lambda), K(\mu)) \quad \text{for} \quad j > 2l(w_0).$$

Indeed, applying Frobenius reciprocity twice (using Theorem 24(i) in [CM2] and equation (2.4)) yields

$$\text{Ext}_\mathcal{O}^j(\overline{K}(\lambda), K(\mu)) = \text{Ext}_\mathcal{O}^j(L_0(\lambda), L_0(\mu - 2p_1)),$$

and a similar argument holds for the other equality.

In particular this implies that for any $\lambda, \mu \in \mathfrak{h}^*$ we have

$$\text{Ext}_\mathcal{O}^j(K(\mu), M) = 0 = \text{Ext}_\mathcal{O}^j(\overline{K}(\lambda), M) \quad \text{for} \quad j > 2l(w_0).$$

Since $M$ is finitely generated, the element $z \in \mathfrak{z}(\mathfrak{g}_0)$ in equation (2.1) has eigenvalues in an interval of finite length $p$. We prove that for any $\alpha \in \mathfrak{h}^*$,

$$\text{Ext}_\mathcal{O}^j(L(\alpha), M) = 0 \quad \text{if} \quad j > p + \dim \mathfrak{g}_1 + 4l(w_0).$$

Assume that the above extension does exist. The short exact sequence $N \hookrightarrow K(\alpha) \twoheadrightarrow L(\alpha)$ and the vanishing properties in (4.2) imply that there must be a $\beta \in \mathfrak{h}^*$ for which $L(\beta)$ is a subquotient of $N$ (and therefore satisfies $\beta(z) \leq \alpha(z) - 1$) such that

$$\text{Ext}_\mathcal{O}^{j-1}(L(\beta), M) \neq 0.$$ 

This procedure and the dual one using dual Kac modules can be repeated so that we come to the conclusion that there must exist $\kappa, \nu \in \mathfrak{h}^*$ with $\kappa(z) \geq \alpha(z) + j - 2l(w_0)$ and $\nu(z) \leq \alpha(z) - j + 2l(w_0)$ such that

$$\text{Ext}_\mathcal{O}^{2l(w_0)}(L(\kappa), M) \neq 0 \quad \text{and} \quad \text{Ext}_\mathcal{O}^{2l(w_0)}(L(\nu), M) \neq 0.$$

The combination of equation (3.1) and Lemma 3.9 yields that for any two $\mu, \mu'$ we have

$$\text{Ext}_\mathcal{O}^i(L(\mu), L(\mu')) = 0 \quad \text{unless} \quad |\mu(z) - \mu'(z)| \leq \dim \mathfrak{g}_1 + i.$$

Equation (4.3) therefore implies that both $\kappa(z)$ and $\nu(z)$ must lie in an interval of length $p + 2(\dim \mathfrak{g}_1 + 2l(w_0))$. However, the construction above implies that

$$\kappa(z) - \nu(z) \geq 2j - 4l(w_0), \quad \text{with} \quad j > p + \dim \mathfrak{g}_1 + 4l(w_0).$$
This means we have proved that
\[ \text{id}_\mathcal{O} M < p + \dim \mathfrak{g}_1 + 4l(w_0), \]
for some finite \( p \in \mathbb{N} \).

\[ \square \]

**Proof of Theorem 4.1.** The equivalence of (i) \( \iff \) (iv) follows from Lemma 4.3.

Assume that \( M \in \mathcal{O} \) is a direct summand of a module induced from one in \( \mathcal{O}^0 \). Lemma 4.3(ii) therefore yields \( \text{pd}_\mathcal{O} M < \infty \). Recall the categories \( \mathcal{A}_j \) from the beginning of this section. Lemma 6.9 in [Hu] or Section 2.3 in [CM2] show that if every module in \( \mathcal{A}_{j-1} \) has finite projective dimension in \( \mathcal{O} \), then so has every module in \( \mathcal{A}_j \). Hence we find (iii) \( \iff \) (i).

Now assume that \( \text{pd}_\mathcal{O} M < \infty \) holds. Since \( M \) has a finite resolution by projective modules and projective modules are direct summands of modules induced from projective modules in \( \mathcal{O}^0 \) we obtain \( M \in (\mathfrak{g} \mathfrak{g}_0) \mathcal{O} \). This proves (i) \( \Rightarrow \) (iii).

Since \( X_M \cap \mathfrak{g}_{\pm 1} = \{0\} \) implies that \( M \) is \( \mathfrak{g}_{\pm 1} \)-free, property (ii) \( \Rightarrow \) (i) follows from Proposition 4.4 and Lemma 4.6. Since every projective module is induced, it has trivial associated variety, see Lemma 2.2(1) in [DS]. If \( M \) has a finite resolution by projective modules, it has finite projective dimension as a \( \mathbb{C}[x] \)-module for any \( x \in \mathfrak{g}_1 \) with \( [x, x] = 0 \), so it is \( \mathbb{C}[x] \)-free. Thus \( X_M = \{0\} \) and we obtain (i) \( \Rightarrow \) (ii).

The last statement is a special case of Lemma 4.3. \[ \square \]

5. \( B_0 \)-orbits in the self-commuting cone and some results on associated variety in category \( \mathcal{O} \)

The associated variety (4.1) of an object \( M \in \mathcal{O} \) is intrinsically not a categorical invariant, contrary to projective dimension and complexity, although results as Theorem 4.1 indicate links with categorical invariants. So it is not possible to use the results in [CMW] to reduce to the integral case. Therefore throughout the entire section we consider weights in \( \mathfrak{h}^* \), not just in \( P_0 \).

In the first two subsections we will consider the associated variety of modules in category \( \mathcal{O} \) for \( \mathfrak{g} \) a basic classical Lie superalgebra in the list

\[ (5.1) \quad \mathfrak{sl}(m|n), m \neq n; \; \mathfrak{gl}(m|n); \; \mathfrak{osp}(m|2n); \; D(2, 1, \alpha); \; G(3); \; F(4), \]

with arbitrary Borel subalgebra. In the last three subsections we will focus on \( \mathfrak{g} = \mathfrak{gl}(m|n) \) with distinguished Borel subalgebra.

5.1. \( B_0 \)-orbits. Let \( X \) denote the set of all self-commuting odd elements in \( \mathfrak{g} \) and \( B_0 \) be the Borel subgroup of the algebraic group \( G_0 \) with Lie algebra \( \mathfrak{b}_0 \). If \( M \) is in category \( \mathcal{O} \), then the simply connected cover of \( B_0 \) acts on \( M \). Furthermore \( B_0 \) acts on \( X \) by adjoint action and thus the associated variety \( X_M \), as defined in equation (4.1), is a \( B_0 \)-invariant subvariety of \( X \). Therefore it is important to study \( B_0 \)-orbits in \( X \). It is proven in [DS] that \( X \) has finitely many \( G_0 \)-orbits. We will show in this subsection that the same is true for \( B_0 \)-orbits.

Let \( S = \{\alpha_1, \ldots, \alpha_k\} \) be a set of mutually orthogonal linearly independent isotropic roots and \( x_1, \ldots, x_k \) be some non-zero elements in the root subspaces \( \mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_k} \) respectively. Then \( x_S := x_1 + \cdots + x_k \in X \). For such an \( x_S \in X \), we say that its rank is \( k = |S| \). Let \( S \) denote the set of all subsets of mutually orthogonal linearly independent isotropic roots and \( X/B_0 \) denote the set of \( B_0 \)-orbits in \( X \). In [DS] it was proved that \( X/G_0 \cong S/W \), now we derive an analogous description of \( X/B_0 \).

Define the map

\[ \Phi : S \rightarrow X/B_0; \quad \Phi(S) := B_0 x_S, \quad \text{for all } S \in S. \]
We assume that \( \Phi(\emptyset) = 0 \). Note that \( \Phi \) does not depend on a choice of \( x_1, \ldots, x_k \) since any two such elements are conjugate under the action of a maximal torus in \( G_\emptyset \).

**Theorem 5.1.** Consider \( \mathfrak{g} \) in the list \( \{5.1\} \), the map \( \Phi \) is a bijection, so \( X/B_\emptyset \cong S \).

**Proof.** First, we will prove that \( \Phi \) is surjective, i.e. every \( B_\emptyset \)-orbit contains \( x_S \) for some \( S \in S \). Recall that Theorem 4.2 in [DS] implies that every \( G_\emptyset \)-orbit contains \( x_S \) for some \( S \in S \). Due to the Bruhat decomposition

\[
G_\emptyset = \bigsqcup_{w \in W} B_0 w B_\emptyset,
\]

it suffices to prove that for every \( S \in S \) and \( w \in W \), \( B_0 w B_\emptyset x_S \) is a union of \( B_0 x_{S'} \) for some \( S' \in S \). Moreover, using induction on the length of \( w \) (and \( B_0 s w B_\emptyset x_S \subseteq B_0 s B_\emptyset w B_\emptyset x_S \) for a simple reflection \( s \)), it is sufficient to prove the latter statement only in the case when \( w = r_\alpha \) is a simple reflection.

Let \( G_\alpha \) be the \( SL_2 \)-subgroup in \( G_\emptyset \) associated with the root \( \alpha \) and \( B_\alpha := G_\alpha \cap B_\emptyset \). Since

\[
B_0 r_\alpha B_\emptyset \subset B_0 G_\alpha,
\]

we have to show that \( G_\alpha x_S \) lies in a union of \( B_0 x_{S'} \) for some \( S' \in S \).

We need the following well-known facts about root system of basic superalgebras. By odd \( \alpha \)-chain we mean the maximal subset of odd roots of the form \( \beta + p_\alpha \) with \( p \in \mathbb{Z} \) for some \( \beta \in \Delta \).

1. The length of any odd \( \alpha \)-chain consisting of odd roots is at most 3;
2. If \( \beta \) is an isotropic root then either \( \beta + \alpha \) is not a root or \( \beta - \alpha \) is not a root;
3. If \( S \in S \), then at most 2 roots of \( S \) are not preserved by \( r_\alpha \).

We consider the three cases allowed by statement (3) individually. If all roots of \( S \) are preserved by \( r_\alpha \), then \( G_\alpha x_S = x_S \) and the statement is trivial.

Assume that there exists exactly one root \( \alpha_i \in S \), which is not preserved by \( r_\alpha \). Consider \( G_\alpha \)-submodule \( V_i \subset \mathfrak{g}_1 \) generated by \( x_i \). By (2) \( x_i \) and \( r_\alpha(x_i) \) are the lowest and the highest weight vectors in \( V_i \) and from representation theory of \( SL_2 \) we have

\[
G_\alpha x_i = B_\alpha x_i \cup B_\alpha r_\alpha(x_i).
\]

Since for any \( \alpha_j \in S \), \( j \neq i \) we have \( G_\alpha x_j = x_j \), we obtain

\[
G_\alpha x_S = B_\alpha x_S \cup B_\alpha x_{r_\alpha(S)} \subset B_0 x_S \cup B_0 x_{r_\alpha(S)}.
\]

Hence the statement is proved in this case.

Finally, assume that there are two distinct roots \( \alpha_i, \alpha_j \in S \) which are not preserved by \( r_\alpha \). Since this case is only possible for Lie superalgebras of defect greater than 1, we may assume that \( \mathfrak{g} \) is either general linear or orthosymplectic. In this case \( \alpha_i = \pm \varepsilon_a \pm \delta_b \) and \( \alpha_j = \pm \varepsilon_c \pm \delta_d \) for some \( a \neq c \) and \( b \neq d \). That implies that \( \alpha_i, \alpha_j \) and \( \alpha \) are roots of some root subalgebra \( g' \) isomorphic to \( \mathfrak{sl}(2|2) \). Furthermore, the corresponding subgroup \( G_\emptyset' \) preserves \( x_l \) for all \( l \neq i, j \). Therefore it suffices to check the analogous statement for \( G' \). It can be done by direct computation and we leave it to the reader.

Now we will prove that \( \Phi \) is injective, i.e. \( x_{S'} \in B_0 x_S \) implies \( S = S' \). Assume that \( x_{S'} = Ad_\alpha x_S \) for some \( \alpha \in B_0 \). Note \( |S| = |S'| \) by the property \( X/G_\emptyset \cong S/W \) of [DS]. Recall from the proof of Theorem 4.5 in [DS] that there exists an \( \mathfrak{sl}(1|1) \)-triple \( \{x_S, h, y\} \) such that \( h \in \mathfrak{h}, [h, x_S] = [h, y] = 0 \) and \( [x_S, y] = h \). Moreover \( y = y_1 + \cdots + y_k \) and \( h = h_1 + \cdots + h_k \) such that \( [x_i, y_j] = \delta_{ij} h_i \in \mathfrak{h} \). Let \( h' = Ad_\alpha h, y' = Ad_\alpha y \), then we have a similar \( \mathfrak{sl}(1|1) \)-triple \( \{x_{S'}, h', y'\} \). Because

\[
h' - h \in \bigoplus_{\alpha \in \Delta_0^+} \mathfrak{g}_\alpha
\]
and \( h' \in h \), we find \( \sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \alpha'_i \). However, by construction we have \( \alpha_i \leq \alpha'_i \) for each \( i \). This implies \( S = S' \). \( \square \)

For any \( g \)-module \( M \in \mathcal{O} \), we introduce the notation

\[
S(M) = \{ S \in \mathcal{S}[\Phi(S) \subset X_M] \}.
\]

In particular, Theorem [5.1] implies that for \( M, N \in \mathcal{O} \) we have \( S(M) = S(N) \) if and only if \( X_M = X_N \).

5.2. General properties of the associated variety for modules in category \( \mathcal{O} \). Recall form [DS] that if \( x \in X \) and \( M_x := \ker x/\text{im} x \) is a \( g_x \)-module, where \( g_x := \ker \text{ad} x/\text{im} \text{ad} x \). If \( g \) is a basic classical superalgebra, then \( g_x \) is also a basic classical superalgebra. For example, if \( g = gl(m|n) \) and the rank of \( x \) is \( k \), then \( g_x \) is isomorphic to \( gl(m-k|n-k) \). If \( x = x_S \) we identify \( g_x \) with the root subalgebra in \( g \), whose roots are orthogonal but not proportional to the roots from \( S \).

For a Lie superalgebra \( l \) we denote by \( Z(l) \) the center of the universal enveloping algebra \( U(l) \) and by \( Z(l) \) the set of central characters. In [DS] a map \( \phi : Z(g) \to Z(g_x) \) was introduced. Furthermore it was proved that all fibers of the dual map \( \bar{\phi} : \bar{Z}(g_x) \to \bar{Z}(g) \) are finite, more precisely any fiber consists of at most two points. If \( M \) admits a generalised central character \( \chi \) then \( M_x \) is a direct sum of \( g_x \)-modules admitting generalised central characters from \( \bar{\phi}^{-1}(\chi) \). The degree of atypicality of the central characters in \( \bar{\phi}^{-1}(\chi) \) is equal to the degree of atypicality of \( \chi \) minus the rank of \( x \), see e.g. equation (3) in [Se2].

In the case \( g = gl(m|n) \), the map \( \phi \) is injective and it maps a central character \( \chi' \) of \( g_x \) to the central character \( \chi \) of \( g \) with the same core.

Below we summarise the general properties of the functor \( g \)-mod to \( g_x \)-mod, which sends \( M \) to \( M_x \).

**Lemma 5.2.** Consider \( g \) in the list [5.1].

(1) For any \( g \)-modules \( M \) and \( N \) we have \( (M \otimes N)_x \cong M_x \otimes N_x \).

(2) Let \( M \) be a \( g \)-module from the category \( \mathcal{O} \) which admits a generalised central character with atypicality degree \( k \) and \( S \in \mathcal{S} \). If \( S \in S(M) \), then \( |S| \leq k \).

(3) Let \( M \) be from the category \( \mathcal{O} \), \( x = x_S \) for some \( S \in \mathcal{S} \) and \( b_x := b \cap g_x \), then \( b_x \) acts locally finitely on \( M_x \) and \( M_x \) is a weight module.

**Proof.** To prove (1) note that we have the obvious homomorphism \( M_x \otimes N_x \to (M \otimes N)_x \) of \( g_x \)-modules. To check that it is an isomorphism consider \( M \) and \( N \) as \( \mathbb{C}[x] \)-modules. Then

\[
M \cong M^{\mathcal{F}} \otimes M_x, \quad N \cong N^{\mathcal{F}} \otimes N_x,
\]

where \( M^{\mathcal{F}} \) and \( N^{\mathcal{F}} \) are free \( \mathbb{C}[x] \)-modules. Since \( M^{\mathcal{F}} \otimes N \) and \( M \otimes N^{\mathcal{F}} \) are free we obtain an isomorphism \( M_x \otimes N_x \cong (M \otimes N)_x \).

To show (2) use the map \( \phi \). If \( |S| > k \), then \( \bar{\phi}^{-1}(\chi) \) is empty and therefore \( M_x = 0 \).

Finally, (3) is trivial since \( M_x \) is a subquotient of \( M \), and \( b_x \) acts locally finitely and \( h \cap g_x \) diagonally on \( M \). \( \square \)

**Remark 5.3.** We believe that (3) can be strengthened. Namely, if \( M \) lies in the category \( \mathcal{O} \), then \( M_x \) belongs to the category \( \mathcal{O} \) for \( g_x \), i.e. \( M_x \) is finitely generated. But we do not have a proof of this at the moment.

5.3. On associated variety of Verma modules for \( gl(m|n) \). In this subsection we assume \( g = gl(m|n) \) and consider the distinguished Borel subalgebra \( b = b_0 \oplus g_1 \). We denote by \( M(\lambda) \) the Verma module with highest weight \( \lambda + \rho \) and set \( S(\lambda) := S(M(\lambda)) \). We start with the following technical lemma.
Lemma 5.4. Let \( s \) be \((1|2)\) dimensional superalgebra with odd generators \( \xi, \eta \) and even generator \( u \) satisfying \([\xi, \eta] = 0, [u, \xi] = \xi, [u, \eta] = -\eta\). Assume that \( M \) is an \( s \)-module semisimple over \( u \) and such that the spectrum of \( u \) in \( M \) is bounded from above, i.e. there exists \( \gamma_0 \) such that \( \text{Re}\gamma < \gamma_0 \) for any eigenvalue \( \gamma \) of \( u \). Then \( M_\eta = 0 \) implies \( M_{\eta+\xi} = 0 \).

Proof. Note that any \( u \)-eigen vector \( v \in M \) such that \( \xi v \neq 0 \) generates a projective \( s \)-submodule (in the category of \( s \)-modules semisimple over \( u \)). Hence we have a decomposition \( M = P \oplus L \) for some projective \( P \) and \( L \) such that \( \xi \eta L = 0 \). Obviously, \( P_{\xi+\eta} = 0 \) and we have to check only that \( L_{\xi+\eta} = 0 \). Since \( L_\eta = 0 \), \( L \) is free over \( \eta \), hence we have an \( u \)-invariant decomposition \( L = L' \oplus L'' \) such that \( \xi L' \subset L'' \), \( \eta L' = L'' \), \( \xi L'' = \eta L'' = 0 \) and \( \eta : L' \rightarrow L'' \) is an isomorphism. Let \( \zeta : L'' \rightarrow L' \) be the inverse of \( \eta \). Then

\[
\zeta(\xi + \eta) = 1d_{L'} + n', \quad (\xi + \eta)\zeta = 1d_{L''} + n'',
\]

where \([u, n'] = 2n'\) and \([u, n''] = 2n''\). The latter condition and the assumption on the spectrum of \( u \) imply that \( n' \) and \( n'' \) are locally nilpotent and hence \( 1d_{L'} + n' \) and \( 1d_{L''} + n'' \) are both invertible. Hence \( \text{Ker}(\xi + \eta) = L'' \) and \( \text{Im}(\xi + \eta) = L''' \). That implies \( M_{\eta+\xi} = 0 \).

Lemma 5.5. If \( M \in \mathcal{O} \) is free over \( U(\mathfrak{g}_{\lambda}) \), then \( X_M \subset \mathfrak{g}_1 \) and therefore \( S \in \mathcal{S}(M) \) implies \( S \subset \Delta^*_1 \).

Proof. If \( x \in \mathfrak{g}_{\lambda} \), then \( M_x = 0 \) since \( M \) is free over \( \mathbb{C}[x] \). Let \( x \in X \). Then \( x \) can be written uniquely as \( x^+ + x^- \) with \( x^\pm \in \mathfrak{g}_{\pm 1} \). We claim that if \( x^- \neq 0 \), then \( M_x = 0 \). Indeed, we apply Lemma 5.3 with \( u = z, \xi = x^+ \) and \( \eta = x^- \) and use the fact that \( M_\eta = 0 \).

If \( \alpha \) is a root of \( \mathfrak{g} \), we denote by \( X_\alpha \) some non-zero element from the root space \( \mathfrak{g}_\alpha \) and set \( H_\alpha := [X_\alpha, X_{-\alpha}] \).

Lemma 5.6. Let \( \mathcal{S} \) be free on \( \mathfrak{h} \) and \( \mathcal{S}(M) \). Let \( S \) be a disjoint union of two subsets \( S_1 \) and \( S_{-1} \) and \( h \in \mathfrak{h}^* \) be an element of the Cartan subalgebra, non-negative on all even positive roots. Assume that \( \alpha(h) = i \) for all \( \alpha \in S_i \) where \( i = \pm 1 \). Then \( S_{-1} \in \mathcal{S}(M) \).

Proof. Follows again from Lemma 5.3. We write \( X = X^+ + X^- \), where \( X^\pm = \sum_{\alpha \in S_{\pm 1}} X_\alpha \), set \( \xi = X^+, \eta = X^-, u = h \).

Remark 5.7. More generally, it seems plausible that if \( S \in \mathcal{S}(M) \), then any subset \( S' \subset S \) is also in \( \mathcal{S}(M) \).

Lemma 5.8. Let \( S = \{\varepsilon_s - \delta_s \mid s \in [1, k]\} \) be a set of mutually orthogonal positive odd roots. Set \( a := \min\{\varepsilon_s \mid s \in [1, k]\} \) and \( b := \max\{\varepsilon_s \mid s \in [1, k]\} \) and let \( \mathfrak{g}' = \mathfrak{gl}(m-a+1|b) \) be the subalgebra of \( \mathfrak{g} \equiv \mathfrak{gl}(m|n) \) generated by \( X_{\pm (\varepsilon_i - \delta_j)} \) with \( i \geq a \) and \( j \leq b \). Let \( \lambda' \) be the restriction of \( \lambda \) to the Cartan subalgebra of \( \mathfrak{g}' \) and \( S' \) be the set of subsets of mutually orthogonal linearly independent odd roots in \( \mathfrak{g}' \) (clearly, \( S' \subset S \)). Then we have \( \mathcal{S}(\lambda') = \mathcal{S}(\lambda) \cap S' \).

Proof. Let \( S \in S' \) and \( x = x_S \). The Verma module \( M(\lambda) \) is isomorphic to \( M(\lambda') \otimes S(\mathfrak{gl}(\mathfrak{g}' + b)) \) as a \( \mathfrak{g}' \)-module and therefore as a \( \mathbb{C}[x] \)-module, with adjoint action on \( \mathfrak{g}/(\mathfrak{g}' + b) \). In particular \( M(\lambda') \) is a direct summand in \( M(\lambda) \). Therefore \( M(\lambda')_x \neq 0 \) implies \( M(\lambda)_x \neq 0 \). On the other hand, if \( M(\lambda')_x = 0 \), then \( M(\lambda)_x = 0 \) by Lemma 5.2 (1).

Corollary 5.9. Consider the set \( \{\varepsilon_s - \delta_s \} \) of all positive odd atypical roots for \( \lambda \in \mathfrak{h}^* \). For a set \( S \) of mutually orthogonal odd positive roots of the form \( \varepsilon_i - \delta_j \) with \( i > i_s \) and \( j < j_s \) for every \( s \), we have \( S \notin \mathcal{S}(\lambda) \).
Proof. This follows from Lemma 5.8 since \( \lambda' \) is a typical weight. \( \square \)

Lemma 5.10. Consider a set \( S = \{ \alpha_1, \cdots, \alpha_k \} \) of mutually orthogonal linearly independent isotropic roots. If \( S \in S(\lambda) \), then \( \langle \lambda, \alpha_i \rangle \in \mathbb{Z} \) for all \( i = 1, \ldots, k \).

Proof. Let \( h_i \in [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] \), such that \( \beta(h_i) = (\beta, \alpha_i) \) for any weight \( \beta \). For any \( t_1, \ldots, t_k \in \mathbb{C} \) consider the \( \mathfrak{sl}(1|1) \)-triple \( \{ x_S, y, h \} \), where \( h = t_1h_1 + \cdots + t_kh_k \). If \( S \in S(\lambda) \), then \( M \) cannot be a typical module for the \( \mathfrak{sl}(1|1) \)-triple, so there is a weight \( \mu \) such that \( \mu(h) = 0 \).

Since \( \lambda - \mu \) is an integral linear combination of roots, we have

\[
\lambda(h) = \sum_{i=1}^{k} t_i \langle \lambda, \alpha_i \rangle \in \sum_{i=1}^{k} \mathbb{Z} t_i.
\]

For generic choice of \( t_1, \ldots, t_k \) this implies \( \langle \lambda, \alpha_i \rangle \in \mathbb{Z} \) for all \( i = 1, \ldots, k \). \( \square \)

Lemma 5.11. Let \( \mathfrak{g} = \mathfrak{gl}(n|n) \) and \( \lambda \) be an integral regular dominant weight of degree of atypicality \( n \). Then \( X_M(\lambda) = \mathfrak{g}_1 \).

Proof. By Lemma 5.5 it suffices to prove \( \mathfrak{g}_1 \subset X_M(\lambda) \). For a module \( M \) in \( \mathcal{O} \), let \( M^+ \) denote the highest degree component with respect to the grading consistent with the standard \( \mathbb{Z} \)-grading of \( \mathfrak{g} \). As for any \( x \in \mathfrak{g}_1 \) we have \( \lambda(\lambda)^+ \subseteq \ker x \), it would suffice to prove \( \lambda(\lambda)^+ \not\subseteq xM(\lambda) \). We will prove this by using the property \( L(\lambda)^+ \not\subseteq xL(\lambda) \) for any \( x \in \mathfrak{g}_1 \), which is known to be true by Section 10 in [DS].

Consider the exact sequence \( M(\lambda) \xrightarrow{\pi} L(\lambda) \to 0 \) of graded \( \mathfrak{g} \)-modules. Now assume that \( M(\lambda)^+ \subseteq xM(\lambda) \), so any \( a \in M(\lambda)^+ \) can be written as \( a = xb \) for some \( b \in M(\lambda) \). Then clearly \( \pi(a) = x\pi(b) \) and, as the graded map \( \pi \) is surjective, we find the contradiction that \( L(\lambda)^+ \subseteq xL(\lambda) \). \( \square \)

Now we focus on the case \( |S| = 1 \).

Lemma 5.12. Let \( \alpha = \varepsilon_i - \delta_j \) be a positive odd root. If \( \langle \lambda + \rho, \alpha \rangle = 0 \), then \( \{ \alpha \} \in S(\lambda) \).

Proof. Consider the subset \( \Gamma \) of odd positive roots defined by

\[
\Gamma = \{ \varepsilon_p - \delta_q | p > i, q \leq j, \text{ or } p \geq i, q < j \}.
\]

We set \( \Sigma_{\Gamma} = \sum_{\gamma \in \Gamma} \gamma \), then we have \( \langle \Sigma_{\Gamma}, \alpha \rangle = -\langle \rho, \alpha \rangle \).

Let \( \gamma \in M(\lambda) \) be a highest weight vector and

\[
w := \prod_{\gamma \in \Gamma} X_{-\gamma}v.
\]

Then a quick check yields \( X_\alpha w = 0 \).

We claim that \( w \notin \text{im}X_\gamma \). Indeed, \( w \) is a weight vector of weight \( \mu = \lambda - \Sigma_{\Gamma} \). Assume \( w = X_\alpha w' \). Without loss of generality we may assume that \( w' \) is a weight vector of weight \( \mu - \alpha \). Note that the weight space \( M(\lambda)_{\mu - \alpha} \) is one-dimensional. Therefore we may assume that \( w' \) is proportional to \( X_{-\alpha}w \). On the other hand

\[
X_\alpha X_{-\alpha} w = H_\alpha w = 0
\]

since \( \langle \mu, \alpha \rangle = \langle \lambda - \Sigma_{\Gamma}, \alpha \rangle = \langle \lambda + \rho, \alpha \rangle = 0 \).

Therefore \( X_\alpha \in X_M(\lambda) \) and the lemma is proven. \( \square \)

Lemma 5.13. Let \( \gamma = \varepsilon_p - \delta_q \) be a positive odd root with \( \langle \lambda + \rho, \gamma \rangle = 0 \). Then for \( \alpha = \varepsilon_i - \delta_j \) with \( i \leq p, j \geq q, \langle \lambda + \rho, \varepsilon_i - \varepsilon_p \rangle \in \mathbb{Z}_{\geq 0} \) and \( \langle \lambda + \rho, \delta_q - \delta_j \rangle \in \mathbb{Z}_{\leq 0} \) we have \( \{ \alpha \} \in S(\lambda) \).
\textbf{Proof.} If \(i = p\) and \(j = q\), this is Lemma 5.12. Otherwise we set \(\beta = \varepsilon_i - \varepsilon_p\), \(\beta' = \delta_q - \delta_j\) and \(r = r_\beta r'_\beta\) (or \(r = r_\beta\) if \(\beta' = 0\) or \(r = r'_\beta\) if \(\beta = 0\)). By assumption and application of Verma’s theorem in Theorem 4.6 in [Hu], we have an embedding \(M(r \cdot \lambda) \subset M(\lambda)\).

Since \((r \cdot \lambda + \rho, \varepsilon_i - \delta_j) = 0\) we can repeat the construction of the proof of Lemma 5.12 of a vector \(w\) for the highest weight vector \(v\) of \(M(r \cdot \lambda)\). We again have \(X_{\alpha} w = 0\). Suppose that \(w = X_{\alpha} w\). We may assume that \(w\) has weight \(\nu = \mu - \alpha\), where \(\mu\) is the weight of \(w\). Although the corresponding weight space \(M(\lambda)\) is not one dimensional, any vector in this space is of form

\[X_{-\alpha} \prod_{\gamma \in \Gamma} X_{-\gamma} u\]

for some \(u \in U(\mathfrak{gl})v\) of weight \(r_\beta \cdot \lambda\). Therefore we have

\[X_{\alpha} X_{-\alpha} \prod_{\gamma \in \Gamma} X_{-\gamma} u \in H_\alpha \prod_{\gamma \in \Gamma} X_{-\gamma} u + \text{im} X_{-\alpha}.
\]

But \((\mu, \alpha) = 0\), hence \(H_\alpha \prod_{\gamma \in \Gamma} X_{-\gamma} u = 0\). Since \(w \notin \text{im} X_{-\alpha}\), we obtain that \(X_{\alpha} w\) is never \(w\). Thus, \(M(\lambda)_{X_{\alpha}} \neq 0\).

\textbf{Proposition 5.14.} Let \(M\) be a Verma or simple module. Then \(X_M = 0\) if and only if the highest weight of \(M\) is typical.

\textbf{Proof.} The case where \(M\) is a Verma module is an immediate consequence of Lemma 5.12 so we consider \(M \cong L(\lambda)\) simple for \(\lambda \in \mathfrak{h}^*\). Consider \((\lambda + \rho, \gamma) = 0\) for \(\gamma\) an odd root. If \(\gamma\) is simple the result follows immediately since \(X_{-\gamma} v = 0\) for a highest weight vector \(v\) (since \(U(\mathfrak{n}^+)X_{-\gamma} v = 0\) while \(v \notin \text{im} X_{-\gamma}\). If \(\gamma\) is not simple we can consider a sequence of odd reflections (see [Min]) to obtain a system of positive roots in which \(\gamma\) is simple. If one of these odd reflections is atypical for the simple module we can take the corresponding root as \(\gamma\), so we consider the situation where each odd reflection is typical. In the new system of roots (with corresponding half-sum \(\tilde{\rho}\)) the simple module will have highest weight \(\tilde{\lambda} = \lambda + \rho - \tilde{\rho}\), which thus satisfies \((\tilde{\lambda} + \rho, \tilde{\rho}) = 0\), so we end up in the setting where \(\gamma\) is simple.

\textbf{5.4. The cases \(\mathfrak{g} = \mathfrak{gl}(1|n)\) and \(\mathfrak{g} = \mathfrak{gl}(m|1)\).}

\textbf{Theorem 5.15.} Let \(\mathfrak{g} = \mathfrak{gl}(1|n)\) and \(\lambda\) be some atypical weight. Let \(p \leq n\) be such that \((\lambda + \rho, \varepsilon_i - \delta_p) = 0\) and \((\lambda + \rho, \varepsilon_i - \delta_j) \neq 0\) for all \(j < p\). Then \(\{\varepsilon_i - \delta_i\} \notin \mathcal{S}(\lambda)\) if and only if \(i \geq p\) and \((\lambda + \rho, \varepsilon_i - \delta_i) \notin \mathbb{Z}_{\leq 0}\).

\textbf{Proof.} If \(\alpha = \varepsilon_i - \delta_i\) for some \(i < p\), then \(\{\alpha\} \notin \mathcal{S}(\lambda)\), by Corollary 5.9. If \(\alpha = \varepsilon_i - \delta_i\) satisfies \(i \geq p\) and \((\lambda + \rho, \varepsilon_i - \delta_i) \in \mathbb{Z}_{< 0}\), then \(\{\alpha\} \in \mathcal{S}(\lambda)\), by Lemma 5.13.

Now let us assume that \(\alpha = \varepsilon_i - \delta_i\) for some \(i > p\), but \((\lambda + \rho, \alpha) \notin \mathbb{Z}_{\leq 0}\). Then we have that \((\lambda, \alpha) \notin \mathbb{Z} \text{ or } (\lambda, \alpha) > i - 1\). The first case is covered by Lemma 5.10. For the second case, according to Lemma 5.8 it suffices to consider the subalgebra \(\mathfrak{g}'\) and the Verma module \(M(\lambda')\) and to show that \(M(\lambda')_{X_{\alpha}} = 0\). Note that \((\beta, \alpha) \geq 0\) for any negative even root \(\beta\) of \(\mathfrak{g}'\) and \((\beta, \alpha) = -1\) for any negative odd root \(\beta \neq -\alpha\) of \(\mathfrak{g}'\). Therefore \((\mu, \alpha) \neq 0\) for any weight \(\mu\) of \(M(\lambda')\). Therefore \(M(\lambda')\) as a module over the \(\mathfrak{gl}(1|1)\)-subalgebra, generated by \(X_{\pm \alpha}\), is a direct sum of typical modules. Thus, \(M(\lambda')_{X_{\alpha}} = 0\).

\textbf{Remark 5.16.} The above theorem implies that in contrast with the finite-dimensional case for some \(M \in \mathcal{O}\) the associated variety \(X_M\) is not closed. For example, if \(\mathfrak{g} = \mathfrak{gl}(1|2)\) and \(\lambda = 3\delta_2\), then \(X_{\lambda}(\mathfrak{g}) = \mathfrak{g}_1 \setminus \mathbb{C}(\varepsilon_1 - \delta_2)\) is not closed.
The isomorphism $\mathfrak{gl}(1|n) \cong \mathfrak{gl}(n|1)$ links the highest weight structure of category $O$ with distinguished system of positive roots for $\mathfrak{gl}(1|n)$ to category $O$ with anti-distinguished system of positive roots for $\mathfrak{gl}(n|1)$. The following result is therefore not identical to Theorem 5.13 but can be proved similarly.

**Theorem 5.17.** Let $g = \mathfrak{gl}(m|1)$ and $\lambda$ be some atypical weight. Let $p \leq m$ be such that $(\lambda + \rho, \varepsilon_p - \delta_1) = 0$ and $(\lambda + \rho, \varepsilon_j - \delta_1) \neq 0$ for all $j > p$. Then $\{\varepsilon_i - \delta_1\} \in S(\lambda)$ if and only if $j \leq p$ and $(\lambda + \rho, \varepsilon_i - \delta_1) \in \mathbb{Z}_{\geq 0}$.

5.5. **The case $g = \mathfrak{gl}(2|2)$.** In this case we have four positive odd roots

$$\alpha = \varepsilon_2 - \delta_1, \beta = \varepsilon_1 - \delta_1, \gamma = \varepsilon_2 - \delta_2, \delta = \varepsilon_1 - \delta_2.$$

We represent weights by using the bijection $b^* = \mathbb{R}^{2|2}$, as a natural extension of equation (2.3).

**Lemma 5.18.** Let $\lambda$ be a weight with degree atypicality 1. Then up to the shift by the weight $(t, t|t, t)$ we have the following options.

1. $\mu^\lambda = (0, a|b, 0)$ with $a \neq b$ and $ab \neq 0$. Then $S(\lambda) = \{\{\delta\}, \emptyset\}$.
2. $\mu^\lambda = (0, 0|a, b)$ with $a \neq b$ and $a \neq 0$. If $b \notin \mathbb{Z}_{\geq 0}$, then $S(\lambda) = \{\{\beta\}, \emptyset\}$. If $a \in \mathbb{Z}_{\geq 0}$, then $S(\lambda) = \{\{\gamma\}, \emptyset\}$.
3. $\mu^\lambda = (a, 0|b, 0)$ with $a \neq b$ and $b \geq 0$. If $a \notin \mathbb{Z}_{\geq 0}$, then $S(\lambda) = \{\{\gamma\}, \emptyset\}$. If $a \in \mathbb{Z}_{\geq 0}$, then $S(\lambda) = \{\{\beta\}, \emptyset\}$. If $a \notin \mathbb{Z}_{\geq 0}$, then $S(\lambda) = \{\{\alpha\}, \emptyset\}$. If $a \in \mathbb{Z}_{\geq 0}$, then $S(\lambda) = \{\{\beta\}, \{\alpha\}, \emptyset\}$.

**Proof.** The proof of this lemma is a straightforward application of Lemma 5.13 and reduction to the case of $\mathfrak{gl}(1|2)$. We leave it as an exercise to the reader. □

**Lemma 5.19.** Let $\lambda$ be a weight with degree atypicality 2 and $A(\lambda)$ denote the set of all odd positive roots atypical to $\lambda$.

1. If $\lambda$ is regular dominant integral, then $S(\lambda)$ is the set of all subsets of mutually orthogonal roots in $\Delta^\perp$.
2. If $\lambda$ is regular integral and neither dominant nor antidominant, then $S(\lambda) = \{\{\delta\}, \{\beta, \gamma\}, \{\gamma\}, \{\beta\}, \emptyset\}$.
3. If $\lambda = -\rho$, then $S(\lambda)$ is the set of all subsets of mutually orthogonal roots in $\Delta^\perp$.
4. If $\lambda$ is regular, non-integral or antidominant integral, then $S(\lambda)$ is the set of all subsets of $A(\lambda)$.

**Proof.** We first observe that (1) is a particular case of Lemma 5.11.

Next, we prove (2). We assume that $\mu^\lambda = (a, 0|a, 0)$ with positive integral $a$, the case $(0, a|0, a)$ being similar. Then $S(\lambda)$ contains $\{\beta\}$ and $\{\gamma\}$ by Lemma 5.12 and $\{\delta\}$ by Lemma 5.13. On the other hand, $\{\alpha\} \notin S(\lambda)$ by Lemma 5.8. Also we can apply Lemma 5.10 to $S = \{\alpha, \delta\}$ with $h = \varepsilon_1 - \varepsilon_2$ and conclude that $\{\alpha, \delta\} \notin S(\lambda)$. It remains to show that $\{\beta, \gamma\} \in S(\lambda)$. For this we take $w = X_{-\alpha}v$, where $v$ denotes the highest weight vector, and let $x = X_\beta + X_\gamma$. Then $xw = 0$ and we claim that $w \notin im x$ by the same argument in the proof of Lemma 5.13 since we have

$$X_\beta X_{-\beta}X_{-\alpha}v = (\lambda - \alpha, \beta)X_{-\alpha}v = 0, \quad X_\gamma X_{-\gamma}X_{-\alpha}v = (\lambda - \alpha, \gamma)X_{-\alpha}v = 0.$$

Let us prove (3) now. As $\mu^\lambda = (0, 0|0, 0)$, Lemma 5.12 implies that $S(\lambda)$ contains all singletons. Furthermore, $\{\beta, \gamma\} \in S(\lambda)$ by the same argument as above. To prove that $\{\alpha, \delta\} \in S(\lambda)$ set $x = X_\alpha + X_\delta$. Let $M$ denote the projection of $M\{0, -1|0, 0\} \otimes U$ on the
most atypical block. Lemma 5.2 (1),(2) implies that $M_\epsilon = 0$. On the other hand, $M$ has a filtration by three Verma modules, $M(0,0,0,0)$, $M(0,-1,-1,0)$ and $M(0,-1|0,-1)$. From the previous cases we have $M(0,-1|1,0) \neq 0$ and $M(0,-1|0,-1) = 0$. Thus, we must have $M(0,0,0,0)_x \neq 0$.

Finally, let us deal with (4). Here we have several subcases to consider. If $A(\lambda) = \{\beta, \gamma\}$, then we may assume $\lambda = (a,0,a,0)$ with non-integral $a$. Any subset of $A(\lambda)$ is in $S(\lambda)$ by the same argument as in the previous case. Furthermore, $\{\alpha\}, \{\delta\} \notin S(\lambda)$ by Lemma 5.9 and $\{\alpha, \delta\} \notin S(\lambda)$ by Lemma 5.6.

If $A(\lambda) = \{\alpha, \delta\}$, then we may assume $\mu^\lambda = (-a,0,0,-a)$ with $a \in \mathbb{Z}_{>0}$. Then $\{\beta\}, \{\gamma\} \notin S(\lambda)$ by Lemma 5.9. Moreover, Lemma 5.8 implies that $\{\beta, \gamma\} \notin S(\lambda)$. On the other hand, $\{\alpha\}, \{\delta\} \in S(\lambda)$ by Lemma 5.12. Finally, to prove that $\{\alpha, \delta\} \in S(\lambda)$ we use the same trick with translation functor as in (3). More precisely, we set again $x = X_\alpha + X_\delta$ and consider the projection $M(-a-1,0,0,-a) \otimes U$ on the most atypical block. Now $M$ is filtered by two Verma modules: $M(-a-1,0,0,-a-1) and M(-a,0,0,-a)$. Using the result for $a = 0$ we obtain by induction in $a$ that $M(-a,0,0,-a)_x \neq 0$. □

6. Projective dimensions and blocks of category $\mathcal{O}$

In this entire section we consider $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{g} = \mathfrak{sl}(m|n)$. We denote by $\mathfrak{b} : W \to \mathbb{N}$ Lusztig's $a$-function, see [Lu].

6.1. Projective dimensions of structural modules.

Theorem 6.1. We have the following connection between projective dimensions of structural modules in the categories $\mathcal{O}$ and $\mathcal{O}^0$, for $\lambda \in \mathfrak{h}^*$:

(i) $pd_\mathcal{O} L(\lambda) = pd_\mathcal{O} \tilde{L}(\lambda)$ if $\lambda$ is typical, otherwise $pd_\mathcal{O} L(\lambda) = \infty$;

(ii) $pd_\mathcal{O} M(\lambda) = pd_\mathcal{O} \tilde{M}(\lambda)$ if $\lambda$ is typical, otherwise $pd_\mathcal{O} M(\lambda) = \infty$;

(iii) $pd_\mathcal{O} I(\lambda) = pd_\mathcal{O} \tilde{I}(\lambda)$.

This implies that for $\lambda \in P_0$, $pd_\mathcal{O} I(\lambda) = a(w_0 x_\lambda)$ with $x_\lambda$ the longest Weyl group element such that $x_\lambda < \lambda$. $\lambda$ is dominant.

Proof. Properties (i) and (ii) for typical $\lambda$ follow from the fact that Brundan’s KL polynomials for typical weights correspond to those for $\mathfrak{g}_0$. Properties (i) and (ii) for atypical $\lambda$ follow from the combination of Proposition 5.12 and Theorem 4.4 (i) $\leftrightarrow$ (ii).

According to Lemma 4.3, to prove (iii) it suffices to prove that $I_0(\lambda)$ is a direct summand of $\tilde{I}(\lambda)$, with $\lambda := \lambda + 2p_1$ (as $pd_\mathcal{O} \tilde{L}(\lambda)$ and $pd_\mathcal{O} \tilde{I}(\lambda)$) and that $I(\lambda)$ is a direct summand of $\tilde{I}(\lambda)$. Both the induced and restricted module are injective and $Ind_{\mathfrak{g}_0}^\mathfrak{b} \cong Coind_{\mathfrak{g}_0}^\mathfrak{b}$, so the properties

$\text{Hom}_{\mathfrak{g}_0}(L_0(\lambda), \text{Res}_{\mathfrak{g}_0}^\mathfrak{b} \tilde{I}(\lambda)) \cong \text{Hom}_{\mathfrak{g}}(\text{Ind}_{\mathfrak{g}_0}^\mathfrak{b} L_0(\lambda), I(\lambda)) \neq 0$ and

$\text{Hom}_{\mathfrak{g}}(L(\lambda), \text{Ind}_{\mathfrak{g}_0}^\mathfrak{b} \tilde{I}(\lambda)) \cong \text{Hom}_{\mathfrak{g}_0}(\text{Res}_{\mathfrak{g}_0}^\mathfrak{b} L(\lambda), I_0(\lambda)) \neq 0$ conclude the proof of (iii).

For the last statement use the result on projective dimensions for type $A$ Lie algebras in Theorem 16 of [Ma1] which, by translation to the wall, extends easily to singular integral blocks. □

The projective dimensions of simple and Verma modules in category $\mathcal{O}^0$ have been calculated for integral regular weights by Mazorchuk in [Ma1]. For singular blocks some preliminary results were obtained in [CM2].

This leads to the following immediate consequences.

Corollary 6.2. Consider $\lambda \in \mathfrak{h}^*$, then

(i) $dim L(\lambda) < \infty \Leftrightarrow pd_\mathcal{O} I(\lambda) = 2l(w_0)$;
Therefore preserve these two types of simple modules.

Property (ii) was first obtained through other methods in Theorem 2.22 of [BLW].

6.2. Finitistic global dimension of blocks.

Theorem 6.3. The finitistic global homological dimension of the block $\mathcal{O}_\xi$ for an integral linkage class $\xi$ is given by

$$\text{fin.dim}\mathcal{O}_\xi = 2a(w_0 w_0^\xi).$$

Proof. The proof of Theorem 3 in [Ma2] implies that fin.dim$\mathcal{O}_\xi$ for a classical Lie super-algebra is equal to the highest projective dimension of an injective module. The result therefore follows immediately from Theorem 6.1.

6.3. Blocks in category $\mathcal{O}$. In this subsection we demonstrate a principle which is responsible for the fact that different integral blocks in category $\mathcal{O}$ will almost never be equivalent, even when they have the same degree of atypicality and singularity of core.

The origin of this new phenomenon is that each atypical integral block contains simple objects which are more regular or more singular. The category behaves differently ‘around’ these objects. The ‘distance’ between these objects in the category is determined by how far the core is distanced from the walls of the Weyl chamber. We make this explicit for blocks for $\mathfrak{sl}(3|1)$ with regular core, by using our results on projective dimensions.

In order to avoid the obvious equivalences of blocks coming from the centre $\mathfrak{z}(\mathfrak{g})$, see Lemma 3.5 in [CMW], we consider $\mathfrak{g} = \mathfrak{sl}(3|1)$ rather than $\mathfrak{gl}(3|1)$. We use the fact that an equivalence of abelian categories is always given in terms of exact functors.

Theorem 6.4. Consider $\mathfrak{g} = \mathfrak{sl}(3|1)$. No two atypical integral blocks $\mathcal{O}_\xi$ with regular core $\chi_\xi$ are equivalent.

Proof. We use the notation of $\mathfrak{gl}(3|1)$-weights, silently making the relevant identification.

The integral linkage classes with regular core are given by $\xi^p = [(p, 1, 0)|i]$, parametrised by $p \in \mathbb{N}$ with $p > 1$. We label the set $\Lambda^{++} \cap \xi^p$ as $\{\lambda^p_i \mid i \in \mathbb{Z}\}$ with

- $\lambda^p_0 = (p, 1, i|i)$ for $i \leq 0$;
- $\lambda^p_{i+1} = (p, i + 1, 1|i + 1)$ for $0 < i < p - 1$;
- $\lambda^p_{i+2} = (i + 2, p, 1|i + 2)$ for $i \geq p - 1$.

From Corollary 6.2 we know that finite dimensional and anti-dominant simple modules are categorically defined. Any equivalence of categories between $\mathcal{O}_{\xi^p}$ and $\mathcal{O}_{\xi'^p}$ must therefore preserve these two types of simple modules.

Thus we can construct a categorical invariant as follows: the Ext$^1$-quiver of the subcategory of finite dimensional modules in $\mathcal{O}_\xi$, where in each node $\lambda \in \Lambda^{++}$ we write the number of anti-dominant simple subquotients in $P(\lambda)$. This number is denoted by $b\lambda$.

Based on the subsequent Lemma 6.7 we know that the number of anti-dominant simple subquotients in the $P(\lambda)$ is equal to twice the number of Verma modules in its standard filtration. A direct application of the bumping procedure, see Example 3.3 in [Br2], to compute $d_{\nu,\lambda}$ in equation (2.7) yields this number. It turns out that the length of the standard filtration of $P(\lambda)$ is one plus the number of times the atypical root of $\lambda$ must be added before another regular weight is obtained.

The Ext$^1$-quiver of the category of finite dimensional weight modules of $\mathfrak{sl}(3|1)$ is well-known to be of Dynkin type $A_{\infty}$, which follows e.g. from Appendix A and [GS]. The combination of these results yields the following graphs: for $p \geq 3$ we have

\[
\begin{align*}
\cdots \bullet \lambda^p_{-2} &= 4 & \bullet \lambda^p_{-1} &= 4 & \bullet \lambda^p_0 &= 6 & \bullet \lambda^p_{1} &= 4 & \bullet \lambda^p_{2} &= 4 & \cdots \\
\cdots \bullet \lambda^p_{-4} &= 4 & \bullet \lambda^p_{-3} &= 4 & \bullet \lambda^p_{-2} &= 6 & \bullet \lambda^p_{-1} &= 4 & \bullet \lambda^p_0 &= 4 & \cdots,
\end{align*}
\]

where $\lambda^p_i$ denote the weights $\lambda^p_i$ of $\mathfrak{sl}(3|1)$-weights.

\[\text{(ii) } \lambda \text{ is anti-dominant } \Leftrightarrow \text{pd}_\mathcal{O} I(\lambda) = 0.\]
meaning \( p - 3 \) nodes between the two exceptional nodes and if \( p = 2 \) we have
\[
\cdots \cdot \lambda^2_{2} = 4 \longrightarrow \cdot \lambda^2_{1} = 4 \longrightarrow \cdot \lambda^2_{0} = 8 \longrightarrow \cdot \lambda^2_{1} = 4 \longrightarrow \cdot \lambda^2_{2} = 4 \longrightarrow \cdots
\]
Since each diagram is different from the others the result follows.

\[\square\]

**Corollary 6.5.** Category \( \mathcal{O}_\mathbb{Z} \) for a basic classical Lie superalgebra can contain infinitely many nonequivalent blocks.

**Remark 6.6.** An alternative homological invariant would be to take \([P(\lambda) : L(\lambda)]\) rather than the number of antidermoninant simple modules. By BGG reciprocity and the fact that in the \( \mathfrak{sl}(3|1) \) case the standard filtration of \( P(\lambda) \) is multiplicity free, \([P(\lambda) : L(\lambda)]\) corresponds to the number of Verma modules in the standard filtration of \( P(\lambda) \).

**Lemma 6.7.** Any atypical integral Verma module of \( \mathfrak{sl}(3|1) \) contains exactly two simple subquotients which have an anti-dominant highest weight.

**Proof.** We use the property \( M(\mu) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 + \mathfrak{g}_1)} M_0(\mu) \) for \( \mu, \lambda \in \Lambda \). Since \( \mathfrak{g}_{-1} \) is a finite dimensional \( \mathfrak{g}_0 \)-module, the only possibility for \( K(\lambda) \) to have an anti-dominant simple subquotient is if \( \lambda \) is anti-dominant. Since every Verma module \( M_0(\mu) \) has exactly one anti-dominant simple subquotient, it suffices to prove the lemma for anti-dominant Verma modules \( M(\mu) = K(\mu) \).

Lemma 6.10(i) in [CMN] implies that the \( \mathfrak{g}_{-1} \)-depth of anti-dominant atypical simple modules for \( \mathfrak{sl}(3|1) \) is 2. This implies that any anti-dominant simple subquotient \( L(\nu) \) of \( K(\mu) \) with \( \nu \neq \mu \) must satisfy \( \nu \in W \cdot (\mu - \gamma) \) with \( \gamma \in \Delta^+_\mathfrak{f} \) atypical for \( \mu \). This leaves only one possibility besides \( \mu \), which we denote by \( \nu \). We claim that \([K(\mu) : L(\nu)] \leq 1\). This follows from looking at weight spaces corresponding to the weight \( \mu - 2p_1 \). In \( K(\mu) \) this has dimension one, in \( L(\mu) \) dimension zero and in \( L(\nu) \) dimension one. The fact that there appears at least two anti-dominant simple modules follows from the fact that both the socle and top of \( K(\mu) \) must be anti-dominant.

\[\square\]

**Remark 6.8.** Classically the equivalences between regular integral blocks are often given by translation functors, see Section 7.8 in [Hu]. It is interesting to note how this fails in the super case. The functor \( F_p \) maps the block corresponding to \((p, 1, 0|0)\) to the one corresponding to \((p + 1, 1, 0|0)\). According to Theorem 2.4 in [Kn], this maps every simple module to a simple module with only three exceptions: the singular modules \( L(p, p, 1|p) \), \( L(p, 1, p|p) \) and \( L(1, p, p|p) \).

The principal used in the proof of Theorem 6.4 that the singular objects appear at different positions in the two blocks, is also responsible for the problematic behaviour of the translation functor. This principle namely causes translation onto the walls for some modules and translation out of the wall for other.

### 7. Complexity in category \( \mathcal{O} \)

In this section we introduce the notion of complexity in category \( \mathcal{O} \) for basic classical Lie superalgebras, as the rate of polynomial growth of a minimal projective resolution of a module. We prove that this is well-defined, i.e. it is finite for every module. Then we study the relation between degree of atypicality and complexity of Verma and simple modules for \( \mathfrak{gl}(m|n) \). Similar results for the category \( \mathcal{F} \) have been obtained by Boe, Kujawa and Nakano in [BKN1, BKN2].

#### 7.1. Definition and basic properties

The usual notion of complexity, as introduced by Alperin, measures the rate of growth of the dimension in a minimal projective resolution. Since the projective objects in category \( \mathcal{O} \) are infinite dimensional we need to consider the number of indecomposable projective objects. This variation has also been studied for the
category of finite dimensional modules of $\mathfrak{gl}(m|n)$ in Section 9 in [BKN2] and is (contrary to the original approach) a categorical invariant.

**Definition 7.1.** For $M \in \mathcal{O}$ we define $c_\mathcal{O}(M)$, the complexity of $M$ in category $\mathcal{O}$, as

$$c_\mathcal{O}(M) = r \left( \sum_{\mu \in b^*} \dim \text{Ext}^*(M, L(\mu)) \right).$$

The rate of growth $r(e^*)$ of a sequence of numbers $e^*$ is defined as the smallest non-negative integer $k$ such that there is a constant $C > 0$ such that $e^j \leq Cj^{k-1}$ for all $j > 0$. In case the $e^j$ are not finite or no such integer exists, we set $r(e^*) = \infty$.

By definition, the complexity of a module is zero if and only if it has finite projective dimension. Immediate from the definition we have the following properties.

**Lemma 7.2.** Consider a short exact sequence $A_1 \twoheadrightarrow A_2 \twoheadrightarrow A_3$ in category $\mathcal{O}$, then

$$c_\mathcal{O}(A_i) \leq \max\{c_\mathcal{O}(A_j), c_\mathcal{O}(A_k)\}$$

for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$.

As main results of this subsection we prove that this notion of complexity is well-defined for category $\mathcal{O}$ for basic classical Lie superalgebras, that translation functors cannot increase complexity and that the duality functor preserves complexity.

**Proposition 7.3.** For any $M \in \mathcal{O}$, $c_\mathcal{O}(M)$ is finite dimensional, more precisely $c_\mathcal{O}(M) \leq \dim \mathfrak{g}_1$.

**Proof.** We prove this by induction on the (finite) projective dimension of $\text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M$ in category $\mathcal{O}^0$. Assume that the property holds for any $K \in \mathcal{O}$ with $\text{pd}_{\mathcal{O}} \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} K < p$. Denote the projective cover of an $M \in \mathcal{O}$, with $\text{pd}_{\mathcal{O}} \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M = p$, by $P$ and the kernel of the morphism $P \rightarrow M$ by $N$. Since $\text{pd}_{\mathcal{O}} \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} N < p$ and $c_\mathcal{O}(P) = 0$, the induction step and Lemma 7.2 imply that

$$c_\mathcal{O}(M) \leq c_\mathcal{O}(N) \leq \dim \mathfrak{g}_1.$$

It remains to be proved that $c_\mathcal{O}(M) \leq \dim \mathfrak{g}_1$ in case $\text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M$ is projective in $\mathcal{O}^0$. We consider the Chevalley-Eilenberg resolution of $\mathcal{C}$ for $(\mathfrak{g}, \mathfrak{g}_0)$-relative homological algebra. It was proved explicitly in Proposition 2.4.1 of [BKN1] that this is a $(\mathfrak{g}, \mathfrak{g}_0)$-projective resolution of $\mathcal{C}$. Tensoring this resolution with $M$ yields a resolution

$$\cdots \rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} (S^j(\mathfrak{g}_1) \otimes \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M) \rightarrow \cdots \rightarrow U(\mathfrak{g}) \otimes_{\mathfrak{g}_0} (\mathfrak{g}_1 \otimes \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M) \rightarrow M \rightarrow 0.$$

Since $\text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M$ is projective in $\mathcal{O}^0$, this is a projective resolution in $\mathcal{O}$, which implies

$$\sum_{\mu \in b^*} \dim \text{Ext}^1_\mathcal{O}(M, L(\mu)) \leq \sum_{\mu \in b^*} \dim \text{Hom}_{\mathfrak{g}_0} (S^j(\mathfrak{g}_1) \otimes \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M, \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} L(\mu)).$$

Lemma [3.1] applied to $\mathfrak{g}_0$ and Lemma [3.2] allow to conclude

$$\sum_{\mu \in b^*} \dim \text{Ext}^j_\mathcal{O}(M, L(\mu)) \leq C \tilde{C} \dim S^j(\mathfrak{g}_1) q,$$

with $q$ the number of indecomposable projective modules of $\mathcal{O}^0$ in $\text{Res}^\mathfrak{g}_{\mathfrak{g}_0} M$. The result thus follows from the fact that the polynomial grow rate of $\dim S^j(\mathfrak{g}_1)$ is $\dim \mathfrak{g}_1$. \hfill $\square$

**Proposition 7.4.** Consider any finite dimensional module $V$ and translation functor

$$T^V_{\chi, \chi'}: \mathcal{O}_\chi \rightarrow \mathcal{O}_{\chi'}: \quad M \in \mathcal{O}_\chi \mapsto T^V_{\chi, \chi'}(M) = (M \otimes V)_{\chi'} \in \mathcal{O}_{\chi'}.$$

Then we have $c_\mathcal{O}(T^V_{\chi, \chi'}(M)) \leq c_\mathcal{O}(M)$.
Theorem 7.7. There are constants \( c_\mathcal{O}(M) \). Its rate of polynomial growth is \( c_\mathcal{O}(M) \). This projective resolution is mapped by the exact functor \( T^e_V \) to a (not necessarily minimal) projective resolution of \( T^e_V(\lambda') \). The polynomial rate of that resolution is smaller or equal to \( c_\mathcal{O}(M) \), by Lemma [B.1]. □

Corollary 7.5. Consider a translation functor \( T = T^e_V \) with adjoint \( \tilde{T} = T^e_V \) and \( M \in \mathcal{O}_\chi \). If for \( M' := TM \) we have \( \tilde{T}M' \cong M \), then \( c_\mathcal{O}(M) = c_\mathcal{O}(M') \).

Proposition 7.6. Take \( M \in \mathcal{O} \). The polynomial growth rate of a minimal injective (co)resolution of \( M \) is the same as that of a minimal projective resolution. This means

\[
c_\mathcal{O}(M) = r \left( \sum_{\mu \in \mathfrak{h}^*} \dim \operatorname{Ext}^\bullet(L(\mu), M) \right)
\]

and \( c_\mathcal{O}(M) = c_\mathcal{O}(M'^\vee) \), with \( \vee \) the duality functor, see Section 3.2 in [Hu].

Proof. First we show there is a constant \( N \) such that any minimal projective resolution of an indecomposable injective module in \( \mathcal{O} \) contains at most \( N \) indecomposable projective modules. By taking the maximum of finitely many non-isomorphic injective modules in finitely many non-equivalent blocks of \( \mathcal{O}^0 \), we obtain such a constant \( N_0 \) for \( \mathcal{O}^0 \). Any indecomposable injective module is a summand of the module induced from an indecomposable injective module in \( \mathcal{O}^0 \). Equation (B.1) and Lemma B.1 then imply we can take \( N = C \dim \Lambda g \| N_0 \).

Using the iterated cone construction (see e.g. [MO]) we can replace every injective module in \( \mathcal{O} \) by the dual resolution of \( M \) by its projective resolution. After deleting all trivial direct summands we obtain a projective resolution of \( M \). The previous paragraph then implies that the polynomial growth rate cannot have increased.

By symmetry we thus obtain that the polynomial growth rate of a minimal injective resolution of \( M \) is the same as that of a minimal projective resolution. □

7.2. Complexity of Verma modules for \( \mathfrak{g}(m|n) \). In this subsection we identify \( \lambda \in \mathcal{P}_0 \) with \( \mu^\lambda \in \mathbb{Z}^{m|n} \) as in equation (2.3) and by the labels of \( \lambda \) we mean the entries of the vector \( \mu^\lambda \).

The complexity of Verma modules is in principle determined by the KL polynomials. Using equation (2.8) we introduce the notation

\[
(7.1) \quad p^j_\lambda = \sum_{\nu \in \Lambda} \dim \operatorname{Ext}^j_\mathcal{O}(M(\lambda), L(\nu)) = \frac{1}{j!} \sum_{\nu \in \Lambda} \left( \frac{\partial^j}{\partial q^{j}} P_{\lambda,\nu}(q) \right)_{q=0}.
\]

Theorem 7.7. There are constants \( C_{k,m,n} \), such that for any \( \lambda \in \Lambda \) with \( z[\lambda] = k \) we have

\[
\sum_{\nu \in \Lambda} \dim \operatorname{Ext}^j_\mathcal{O}(M(\lambda), L(\nu)) \leq C_{k,m,n} j^{k-1}.
\]

The complexity of a (dual) Verma module satisfies \( c_\mathcal{O}(M(\lambda)) = z[\lambda] = c_\mathcal{O}(M(\lambda)^\vee) \) if \( \lambda \) is regular and \( c_\mathcal{O}(M(\lambda)) = c_\mathcal{O}(M(\lambda)^\vee) \leq z[\lambda] \) if \( \lambda \) is singular.

Theorem 7.8. Any module \( M \in \mathcal{O}_\xi \) that is either \( \mathfrak{g}_1 \)-free or \( \mathfrak{g}_{-1} \)-free satisfies \( c_\mathcal{O}(M) \leq \xi \).

The remainder of this subsection is devoted to the proof of these theorems, but first we observe that the corresponding property for category \( \mathcal{F} \) as derived in [BKN2] can be made even more precise.
Lemma 7.9. For any $\lambda \in \Lambda^+$ with $\sharp[\lambda] = k$, we have
\[ \sum_{\nu \in \Lambda^+} \dim \operatorname{Ext}^j_F(K(\lambda), L(\nu)) = \binom{k + j - 1}{k - 1} = \frac{1}{(k - 1)!} (j^{k-1} + \frac{1}{2} k(k - 1) j^{k-2} + \cdots). \]

Proof. Theorem 4.51 and Corollary 3.39(ii) in [Br1] imply that
\[ \sum_{\nu \in \Lambda^+} \dim \operatorname{Ext}^j_F(K(\lambda), L(\nu)) = \sharp\{\theta \in \mathbb{N}^k \mid |\theta| = j\}, \]
which proves the statement. \qed

For the subsequent proofs, we divide the atypical weights into four mutually exclusive types. For $\lambda \in \Lambda$, we set $a_\lambda$ equal to the highest label which appears on both sides.

(a) There is no label in $\lambda$ higher than $a_\lambda$.
(b) There is a label in $\lambda$ higher than $a_\lambda$, but no label equal to $a_\lambda + 1$.
(c) There is a label equal to $a_\lambda + 1$, but only one occurrence of $a_\lambda$ on each side.
(d) There is a label equal to $a_\lambda + 1$, as well as multiple occurrences of $a_\lambda$ on some side.

We also set $w(\lambda)$ equal to the number of labels in $a_\lambda$ strictly higher than $a_\lambda$. So $w(\lambda) = 0$ iff $\lambda$ satisfies (a).

Furthermore, we denote by $P[k]$ for $0 \leq k \leq \min(m, n)$ the property that there is a constant $C_{k,m,n}$ such that for every weight $\mu \in \Lambda$ with $\sharp[\mu] = k$, the inequality $P^j_\mu \leq C_{k,m,n} k^{-j}$ holds for all $j > 0$. We will also use the constant $C$ from Lemma [B.1].

Lemma 7.10. Assume that property $P[k - 1]$ holds and consider $\lambda \in \Lambda$ with $\sharp[\lambda] = k$.

1. If $\lambda$ satisfies (a), we have
\[ p^j_\lambda \leq (m + n)^2 C_{k-1,m,n} \sum_{l=0}^j (j-l)^{k-2} \leq (m + n)^2 C_{k-1,m,n} j^{k-1}. \]

2. If $\lambda$ satisfies (b), denote the lowest label in $\lambda$ strictly higher than $a_\lambda$ by $b_\lambda$, set $d := b_\lambda - a_\lambda - 1 > 0$. Then we have
\[ p^j_\lambda \leq (m + n)^2 C_{k-1,m,n} j^{k-1} + \sum_{i=1}^y p^{j-d}_{\lambda(i)}. \]

For some $y < m+n$ with $\lambda(i) \in \Lambda$ satisfying (c) and $w(\lambda(i)) = w(\lambda)$ and $\sharp[\lambda(i)] = k$.

Proof. Set $a := a_\lambda$. We refer to the side with strictly most occurrences of $a$ as the big side and the other as the small side. If there is an equal number of $a$ on each side, we choose the big and small side randomly. Denote the number of $a$'s appearing on the big side by $y$.

We create $\lambda'$ by raising one of the occurrences of $a$ on the small side to $a + 1$. As $\lambda$ satisfies (a) or (b), by construction $\lambda'$ has degree of atypicality $k - 1$. If the small side is the left-hand side we set $T = E_a$, otherwise $T = E_a$. Then $TM(\lambda')$ has a standard filtration of length $y + 1$, where $\lambda$ is the lowest weight appearing. The $y$ other weights which we denote by $\{\lambda(i) \mid i = 1, \cdots, y\}$, are obtained from $\lambda$ by raising one occurrence of $a$ on the big side and always the fixed occurrence on the small side. Thus we can define a $\widetilde{M} \in O$ by the short exact sequence
\[ 0 \to \widetilde{M} \to TM(\lambda') \to M(\lambda) \to 0. \]

The long exact sequence obtained by applying $\operatorname{Hom}_O(\cdot, L)$ and Lemma [B.1] imply that
\[ p^j_\lambda \leq \sum_{i=1}^y p^{j-1}_{\lambda(i)} + (m + n) C_{k-1,m,n} j^{k-2}. \]
If $\lambda$ satisfies (a), so do the $\lambda^{(i)}$, so we can continue this procedure with the added simplification that the highest label in $\lambda^{(i)}$ appears only once on each side, the analogue of $y$ is equal to 1 in the following steps. This proves (1).

If $\Lambda$ satisfies (b), we can repeat this procedure $d$ times, where again only the first time we will need a constant $y$. This yields

$$p^{(i)}_{\lambda} \leq (m + n)C_{1, m, n} \sum_{l=0}^{d-1} (j - i)^{k-2} + \sum_{i=1}^{y} p^{(i-d)}_{\lambda(i)}$$

with $\lambda^{(i)} \in \Lambda$ obtained from $\lambda$ by adding $d$ to our arbitrarily chosen occurrence of $\alpha_{\lambda}$ on the small side and adding $d$ to the $i$th occurrence of $\alpha_{\lambda}$ on the big side. By construction, these satisfy (c). This completes the proof of (2).

**Lemma 7.11.** Assume that $\lambda \in \Lambda$ is of atypicality degree $k$ and satisfies (c), then we have

$$p^{(i)}_{\lambda} \leq (m + n)C \left( p^{(i)}_{\lambda'} + p^{(i-1)}_{\lambda''} \right),$$

with $\lambda', \lambda'' \in \Lambda$ satisfying $\sharp[\lambda'] = \sharp[\lambda''] = k$, $w(\lambda') < w(\lambda)$ and $w(\lambda'') < w(\lambda)$.

**Proof.** We define $\lambda'$ as obtained from $\lambda$ by raising the occurrence of $\alpha_{\lambda}$ on the side where no $\alpha_{\lambda} + 1$ appears by 1 and $\lambda''$ as obtained from $\lambda$ by raising both occurrences of $\alpha_{\lambda}$ by 1. By definition there is a short exact sequence

$$0 \to M(\lambda'') \to TM(\lambda') \to M(\lambda) \to 0$$

with $T = E_a$ if $\alpha_{\lambda} + 1$ appears on the right and $T = F_a$ otherwise. The long exact sequence and Lemma [B.1] then imply

$$p^{(i)}_{\lambda} \leq (m + n)C p^{(i)}_{\lambda'} + p^{(i-1)}_{\lambda''}.$$

The properties of $\lambda', \lambda''$ follow from construction. □

**Lemma 7.12.** Assume that $\lambda \in \Lambda$ with $\sharp[\lambda] = k$ satisfies (d). Then we have

$$p^{(i)}_{\lambda} \leq (m + n)^{m+n} C_{\lambda, \lambda'}^p p^{(i)}_{\lambda'+},$$

for some $\lambda' \in \Lambda$ satisfying (b), $\sharp[\lambda'] = k$ and $w(\lambda') = w(\lambda)$.

**Proof.** We define $\lambda'$ as obtained from $\lambda$ by adding 1 to every label strictly bigger than $\alpha_{\lambda}$. By composing the appropriate $E_i$ and $F_i$ ($w(\lambda)$ in total) into a translation functor $T$ we have $M(\lambda)^{\otimes d} = TM(\lambda')$ for some number $d$. Lemma [B.1] then implies

$$p^{(i)}_{\lambda} \leq (m + n)^w(\lambda) C_{\lambda, \lambda'} w(\lambda) p^{(i)}_{\lambda'},$$

which proves the lemma. □

**Lemma 7.13.** Assume that for a fixed $gl(m|n)$ and some $k$ with $1 \leq k \leq \min\{m, n\}$, property $P[k - 1]$ holds, then $P[k]$ holds.

**Proof.** We will prove by induction on $w \in [0, m + n - 2k]$ that there is a constant $C_{k, m, n}$ such that if $\lambda$ is of atypicality degree $k$ and $w(\lambda) \leq w$, then $p^{(i)}_{\lambda} \leq C_{k, m, n} w(\lambda)^{k-1}$ holds for all $j > 0$. This proves the lemma for $C_{k, m, n} = C_{k, m, n}^{(m+n-2k)}$.

If $w(\lambda) = 0$, then $\lambda$ satisfies (a), so Lemma (7.10) implies this result with $C_{k, m, n}^{(0)} = (m + n)^2 CC_{k-1, m, n}$. Now assume that the property holds for all $w$ up to $\tilde{w}$ and consider $\lambda \in \Lambda$ with $\sharp[\lambda] = k$ and $w(\lambda) = \tilde{w} + 1$.

(i) If $\lambda$ satisfies (c), then the induction hypothesis and Lemma (7.10) imply

$$p^{(i)}_{\lambda} \leq D_1 j^{k-1} \quad \text{with} \quad D_1 := 2(m + n)CC_{k, m, n}^{(\tilde{w})}.$$
(ii) If $\lambda$ satisfies (b), then (i) and Lemma 7.10(2) imply
\[ p_\lambda^j \leq (m + n)D_1(j - d)^{k-1} + (m + n)^2CC_{k-1,m,n}j^{k-1} \leq D_2j^{k-1} \]
for $D_2 := (m + n)D_1 + (m + n)^2CC_{k-1,m,n}$.
(iii) If $\lambda$ satisfies (d), then (ii) and Lemma 7.12 imply
\[ p_\lambda^j \leq D_3j^{k-1} \quad \text{with} \quad D_3 := (m + n)^{m+n}C^{m+n}D_2. \]
Thus we can take $C_{k,m,n}^{(\lambda+1)} = D_3$ which concludes the proof. 

Proof of Theorem 7.7 First we note that there is a constant $C_{0,m,n}$, such that property $P[0]$ holds. This follows from the equivalence with blocks in $O^0_\mathbb{Z}$, since there are finitely many non-equivalent blocks each containing finitely many Verma modules. Lemma 7.13 thus iteratively proves the first statement in the theorem and thus also $c_O(M(\lambda)) \leq \sharp[\lambda]$.

Now we establish the equality for regular weights. First we take $\kappa \in \Lambda^{++}$ and use Lemma 7.7 to obtain
\[ \sum_{\nu \in \Lambda^{++}} \dim \text{Ext}^j_O(M(\kappa), L(\nu)) = \sum_{\nu \in \Lambda^{++}} \text{Ext}^j_F(K(\kappa), L(\nu)). \]
This has polynomial growth rate $\sharp[\kappa]$ by Lemma 7.9. Now for any $\kappa \in \Lambda^{++}$ and $w \in W$ we consider the subsequence of $\{7.1\}$
\[ \sum_{\nu \in \Lambda^{++}} \dim \text{Ext}^j_O(M(w\kappa), L(\nu)) = \sum_{\nu \in \Lambda^{++}} \dim \text{Ext}^{j-\ell(w)}_O(M(\kappa), L(\nu)), \]
where the equality follows from Lemma 3.6(ii). This proves that $p_{w\kappa}^j$ has polynomial growth rate at least $\sharp[\kappa]$.

The statement on dual Verma modules then follows from Proposition 7.6.

Proof of Theorem 7.8 First we prove that $c_O(K(\nu)) \leq \sharp[\lambda]$ for any $\lambda \in \Lambda$. For an antidominant $\mu$, we have $M(\mu) = K(\mu)$, so the result follows from Theorem 7.7. Then we use (finite) induction by considering the Bruhat order $\prec_0$ for $g_0$ on $P_0$. Assume that $c_O(K(\nu)) \leq k$ for all $\nu \prec_0 \lambda$ with $\sharp[\lambda] = k$. Then the module $N$ defined by the exact sequence
\[ 0 \to N \to M(\lambda) \to K(\lambda) \to 0, \]
has a filtration by $K(\nu)$ with $\nu \prec_0 \lambda$. By the induction hypothesis and Lemma 7.2 we have $c_O(N) \leq \sharp[\lambda]$. Lemma 7.2 and Theorem 7.7 then imply $c_O(K(\lambda)) \leq \sharp[\lambda]$.

Now take an arbitrary module which is $g_{-1}$-free. By Proposition 7.4 this module has a filtration by Kac modules, the result thus follows from Lemma 7.2. All results are also valid for the anti-distinguished system of positive roots, which proves the claim for $g_1$. 

7.3. Complexity of simple modules for $\mathfrak{gl}(m|n)$. Also the complexity of simple modules is in principle determined by the KL polynomials, see Corollary 3.3.

We prove the following relation between the complexity of a simple module and its $\mathfrak{n}$-cohomology.

**Proposition 7.14.** For any $\lambda \in \Lambda$, we have
\[ \max\{c_O(M(\lambda)), r(\dim H^*(\mathfrak{n}, L(\lambda)))\} \leq c_O(L(\lambda)) \leq \sharp[\lambda] + r(\dim H^*(\mathfrak{n}, L(\lambda))). \]
**Proof.** Set $\sharp[\lambda] = k$, by equation (3.1) and Theorem 7.7 we have
\[ \sum_{\mu \in \Lambda} \text{Ext}^j_O(L(\lambda), L(\mu)) \leq \sum_{i=0}^{j} C_{k,m,n}(j-i)^{k-1} \sum_{\kappa} \dim \text{Ext}^j_O(M(\kappa), L(\lambda)), \]
Recall equation (2.4). Consider the smallest $p$ for which
\[
\dim H^i(n, L(\lambda)) = \sum_{\kappa \in \Lambda} \dim \text{Ext}_\mathcal{O}^i(M(\kappa), L(\lambda)) \leq C \lambda^p - 1 \quad \forall i \in \mathbb{N},
\]
for some constant $C$. Then
\[
\sum_{\mu \in \Lambda} \text{Ext}_\mathcal{O}^j(L(\lambda), L(\mu)) \leq \sum_{i=0}^j C_{k,m,n}(j-i)^k \lambda^{p-1} \leq CC_{k,m,n}^{j+p-1}
\]
implies the second inequality. The first inequality follows from the subsequences corresponding to the extreme terms ($i = 0$ and $i = j$) in the summation (3.1).

For finite dimensional simple modules we can improve the estimates.

**Proposition 7.15.** If $\kappa \in \Lambda^{++}$, we have
\[
2\sharp[\kappa] \leq c_\mathcal{O}(L(\kappa)) \leq \sharp[\kappa] + r \left( \sum_{\nu \in \Lambda^{++}} \dim \text{Ext}_{\mathcal{F}}^i(K(\nu), L(\kappa)) \right).
\]

**Proof.** Equation (3.1) gives the following subsequence of $\sum_{\mu \in \Lambda} \dim \text{Ext}_\mathcal{O}^i(L(\kappa), L(\mu))$:
\[
(7.3) \quad \sum_{\lambda, \nu \in \Lambda^{++}} \sum_{i=0}^j \dim \text{Ext}_\mathcal{O}^i(M(\lambda), L(\kappa)) \dim \text{Ext}_{\mathcal{F}}^{j-i}(M(\lambda), L(\nu)).
\]
By Lemma 3.7 and KL theory in $\mathcal{F}$, see Theorem 4.51 and Corollary 4.52 in [Br1], we then find that the subsequence (7.3) is equal to $\sum_{\mu \in \Lambda^{++}} \text{Ext}_\mathcal{F}^i(L(\kappa), L(\nu))$. This has polynomial rate of growth $2\sharp[\kappa]$ by Theorem 9.1.1 in [BKN2], proving the first inequality.

By Lemma 3.6(i) and (ii) and Lemma 8.7 we have
\[
\dim H^i(n, L(\kappa)) = \sum_{i=0}^{l(\bar{\omega})} (\sharp W(i)) \sum_{\lambda \in \Lambda^{++}} \dim \text{Ext}_{\mathcal{F}}^{j-i}(K(\lambda), L(\kappa)),
\]
with $\sharp W(i)$ the number of elements in $W$ of length $i$. This proves the second inequality by Proposition 7.14.

We end this subsection with a conjecture.

**Conjecture 7.16.** For any $\lambda \in \Lambda$ we have $c_\mathcal{O}(L(\lambda)) = 2\sharp[\lambda]$ and $c_\mathcal{O}(M(\lambda)) = \sharp[\lambda]$.

If this conjecture is true we obtain for an integral block $\mathcal{O}_\xi$
- a categorical interpretation of the singularity, by the finitistic global dimension $2\alpha(w_\lambda w_\lambda^\circ)$, see Theorem 6.3
- a categorical interpretation of the atypicality, by the global complexity $2\sharp \xi$,

7.4. **Link between complexity and associated variety.** We note two explicit connections between complexity in category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$ and the associated variety, which follow from Theorem 7.1, Lemma 7.2(1) and Theorem 7.8.

**Proposition 7.17.**
1. For any $M \in \mathcal{O}$ we have $c_\mathcal{O}(M) = 0 \iff S(M) = \{0\}$.
2. If $M \in \mathcal{O}$ is $\mathfrak{g}_{-1}$-free and admits a generalised central character of atypicality degree $k$ we have both $c_\mathcal{O}(M) \leq k$ and $|S| \leq k$ for any $S \in S(M)$.

This results seem to suggest that there must be some deeper connection between complexity and the associated variety. Similar connections appear in [BKN2].
Appendix A. Example: $\mathfrak{gl}(2|1)$

Lusztig’s canonical basis for the principal block in category $\mathcal{O}$ for $\mathfrak{gl}(2|1)$ has been explicitly calculated in Section 9.5 in [CW], see also Example 3.4 in [Br2].

Applying the procedure in Subsection 3.1 reveals that the Bruhat order is transitively generated by relations $\mu \preceq \lambda$ where $l(\lambda, \mu) = 1$. In other words, all coverings in the Bruhat order correspond to length 1. These relations are represented by the arrows on the left in the graph underneath.

We also consider the length function for $\mathfrak{gl}(1|2)$, this case is naturally isomorphic to $\mathfrak{gl}(2|1)$ with antidistinguished Borel subalgebra. Our definition of the length here does not lead to coverings with length 1, the length of the coverings is denoted on the arrow on the right in the graph.

![Graph showing Bruhat order and length function for $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$](image)

The length function for arbitrary related weights can be read from these graphs. The case $\mathfrak{gl}(2|1)$ provides an example of Remark 3.5(2), since $l((10|1), (0|−1|−1)) = 3$, while $l(10|1) − l(0|−1|−1) = 1$ for Brundan’s length $l$ on $\Lambda^+$ in Section 3-g of [Br1]. The case $\mathfrak{gl}(1|2)$ clearly provides an example of Remark 3.5(1).
The interesting KL polynomials for $\mathfrak{gl}(2|1)$, meaning around the singular point \((00|0)\), are given by

$$\hat{v}_{(10|1)} = \sum_{k=1}^{\infty} (-q)^{k-1} \hat{b}_{(k0|k)}, \quad \hat{v}_{(01|1)} = \sum_{k=1}^{\infty} (-q)^{k-1} \hat{b}_{(0k|k)} + \sum_{k=1}^{\infty} (-q)^k \hat{b}_{(k0|k)},$$

$$\hat{v}_{(00|0)} = \sum_{k=0}^{\infty} (-q)^k \hat{b}_{(0k|k)}, \quad \hat{v}_{(0-1|-1)} = \hat{b}_{(0-1|-1)} + \sum_{k=0}^{\infty} (-q)^{k+1} \hat{b}_{(0k|k)} + \sum_{k=1}^{\infty} (-q)^k \hat{b}_{(k0|k)},$$

$$\hat{v}_{(-10|-1)} = \hat{b}_{(-10|-1)} - q \hat{b}_{(0-1|-1)} + \sum_{k=1}^{\infty} (-q)^{k+1} \hat{b}_{(k0|k)}.$$  

In particular, these confirm Theorem 3.2 and Lemmata 3.9 and 3.6. These equations also imply that the Ext$^1$-quiver for the principal block in category $\mathcal{O}$ for $\mathfrak{gl}(2|1)$ is obtained by replacing all arrows in the left diagram by $\leftrightarrow$ and adding one $\leftrightarrow$ between \((10|1)\) and \((0-1|-1)\). In particular this implies the well-known property that the Ext$^1$-quiver of the principal block of $\mathcal{F}$ is of Dynkin type $A_\infty$.

**Appendix B. Existence of upper bounds**

We fix an arbitrary basic classical Lie superalgebra $\mathfrak{g}$.

**Lemma B.1.** There exists a constant $C$, such that the number of indecomposable projective modules in $P(\lambda) \otimes V$ is bounded by $C \dim V$ for any $\lambda \in \mathfrak{h}^*$ and $\mathfrak{g}$-module $V$.

**Proof.** We denote this number by $N_{\lambda,V} := \sum_{\mu \in \mathfrak{h}^*} \dim \text{Hom}_P(P(\lambda) \otimes V, L(\mu))$. This amounts to calculating for how many $\mu$ the product $L(\mu) \otimes V^*$ has $L(\lambda)$ as a subquotient. This is smaller than the number of times $\text{Res}^g_{\mathfrak{g}_0} M(\mu) \otimes V^*$ has $L_0(\lambda)$ as a subquotient, so

$$N_{\lambda,V} \leq \sum_{\mu \in \mathfrak{h}^*} \dim (\mathfrak{g}_-1 \otimes V^*)_\mu [M_0(\mu) : L_0(\lambda)]$$

$$= \sum_{\mu, \kappa \in \mathfrak{h}^*} \dim (\mathfrak{g}_-1 \otimes V^*)^{\mu+\kappa} [M_0(\mu + \kappa) : L_0(\lambda)].$$

The terms in the right-hand side are zero unless $\lambda$ is in the Weyl group orbit of $\mu + \kappa$, so

$$N_{\lambda,V} \leq \sum_{w \in \mathfrak{w}} \sum_{\mu \in \mathfrak{h}^*} \dim (\mathfrak{g}_-1 \otimes V^*)_{w\lambda - \mu} [M_0(w \circ \lambda) : L_0(\lambda)]$$

$$\leq d \mathfrak{g} W \dim (\mathfrak{g}_-1) \dim V$$

with $\mathfrak{g} W$ the order of the Weyl group and $d$ the maximal length of a $\mathfrak{g}_0^*$-Verma module. \(\square\)

**Lemma B.2.** There is a constant $\tilde{C}$, such that for an arbitrary $\nu \in \mathfrak{h}^*$ we have

$$\sum_{\mu \in \mathfrak{h}^*} \dim \text{Hom}_P(P(\nu) : L_0(\nu)] \leq \tilde{C},$$

with $L_0(\nu)$ the simple $\mathfrak{g}_0$-module with highest weight $\nu$.

**Proof.** This sum is equal to

$$\sum_{\mu \in \mathfrak{h}^*} \dim \text{Hom}_P(P(\nu) : L_0(\nu)] \leq \tilde{C},$$

with $P_0(\nu)$ the projective cover of $L_0(\nu)$ in $\mathcal{O}^0$. This is the number of indecomposable projective modules in $\mathcal{O}$ in the decomposition of $\text{Ind}_{\mathfrak{g}_0} P_0(\nu)$, which is smaller than the number of indecomposable projective modules in $\mathcal{O}^0$ in the decomposition of

$$\text{Res}_{\mathfrak{g}_0} \text{Ind}_{\mathfrak{g}_0} P_0(\nu) \cong \mathfrak{g}_1 \otimes P_0(\nu).$$
The result therefore follows from Lemma B.1 applied to $\mathfrak{g}_0$, with $\tilde{C} = C \dim \Lambda g_1$. □

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