Two sharp inequalities for operators in a Hilbert space

Abstract. In this paper we obtained generalisations of the L. V. Taikov’s and N. Ainulloev’s sharp inequalities, which estimate a norm of function’s first-order derivative (L. V. Taikov) and a norm of function’s second-order derivative (N. Ainulloev) via the modulus of continuity or the modulus of smoothness of the function itself and the modulus of continuity or the modulus of smoothness of the function’s second-order derivative. The generalisations are obtained on the power of unbounded self-adjoint operators which act in a Hilbert space. The moduli of continuity or smoothness are defined by a strongly continuous group of unitary operators.

Key words: Hilbert space, self-adjoint operator, modulus of smoothness, partition of unity, strongly continuous group of unitary operators.

1. Introduction

In 1976 L. V. Taikov obtained (see [7]) sharp inequalities, which estimate a norm of function’s derivative in terms of the modulus of continuity of function itself and the modulus of continuity of its second derivative for the cases of $2\pi$-periodic functions and functions, which are defined on real line, in space $L_p$, $1 \leq p < \infty$. Later, in 1991, N. Ainulloev under the same conditions obtained...
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(see [1]) sharp inequalities for a norm of function’s second derivative in terms of the second-order modulus of smoothness of the function itself and the second-order modulus of smoothness of the function’s second derivative. In this paper we obtain generalisations of mentioned results in the case when operators of (multiple) differentiation are substituted by powers of unbounded self-adjoint operators acting in a Hilbert space. The modulus of continuity and the modulus of smoothness of the elements of a Hilbert space are defined in terms of a strongly continuous group of unitary operators generated by this self-adjoint operator.

The paper has the following structure. In Section 2 we present necessary notations, some facts from spectral theory, definitions of the modulus of continuity and modulus of smoothness for elements of a Hilbert space, some properties of the Riemann integral for abstract functions. The main results of the paper are contained in Section 3.

2. Some necessary facts

Let $H$ be a Hilbert space over $\mathbb{C}$ with the inner product $(f, g)_H = (f, g)$, $f, g \in H,$ and norm $\|f\|_H = \|f\| := (f, f)^{\frac{1}{2}}.$ Let also $\mathcal{L}(H)$ be a space of continuous linear operators $G : H \to H.$ By a partition of unity (for details see, for example, Chapter VI §67 in [2], Chapter XIII §1 in [5]) we mean a one-parameter family of projecting operators $E_s \in \mathcal{L}(H), s \in \mathbb{R},$ satisfying the following properties:

(E1) $E_u E_v = E_{\min\{u,v\}}, \ u, v \in \mathbb{R};$

(E2) $\lim_{\tau \to s-0} E_\tau = E_s$ in the sense of strong convergence;

(E3) $\lim_{s \to +\infty} E_s = I, \ \lim_{t \to -\infty} E_t = 0$ in the sense of strong convergence;

(E4) $E_u \leq E_v$ if $u < v, \ u, v \in \mathbb{R}.$

Further, let $A$ be a self-adjoint operator with domain $D_A$ dense in $H.$ It is well known (Chapter VI §75 in [2], Chapter XIII §6 in [5]), that each self-adjoint operator $A$ has a uniquely defined partition of unity $E_s, s \in \mathbb{R},$ and for all $f \in D_A$ there exists a spectral decomposition

$$Af = \int_{-\infty}^{\infty} s \, dE_s f,$$

with domain

$$D_A = \{ f \in H : \|Af\|^2 < \infty \}$$

and the square of the norm

$$\|Af\|^2 = \int_{-\infty}^{\infty} s^2 \, d(E_s f, f).$$
Also (see [2], p. 252), in order for \( n \)-fold application of the self-adjoint operator \( A \) to be admitted to some \( f \in H \), it is necessary and sufficient that the inequality
\[
\int_{-\infty}^{\infty} s^{2n} d(E_s f, f) < \infty
\]
holds, and if inequality (3) holds, then next equalities are true
\[
A^k f = \int_{-\infty}^{\infty} s^k dE_s f, \quad \|A^k f\|^2 = \int_{-\infty}^{\infty} s^{2k} d(E_s f, f), \quad (k = 1, 2, ..., n). \tag{4}
\]

It is also known (see Chapter VI §73 in [2], Chapter XIII §7 in [5]), that the self-adjoint operator \( A \) corresponding to the partition of unity \( E_s \) generates a one-parameter unitary group \( U_t \in \mathcal{L}(H), t \in \mathbb{R} \), i.e., an operator-valued function defined on \( \mathbb{R} \) which values are unitary operators in \( H \) with the representation
\[
U_t f = \int_{-\infty}^{\infty} e^{its} dE_s f = e^{itA} f, \tag{5}
\]
and the following properties hold:

(U1) \( U_t U_s = U_{t+s}, \ t, s \in \mathbb{R} \);

(U2) \( U_0 = I, \ U_{-1} = U_{-t} \);

(U3) \( U_t \) is strongly continuous: \( U_s f \to U_t f \), for \( s \to t, \ t, s \in \mathbb{R}, \ f \in H \);

(U4) \( U_t \) is strongly continuously differentiable: \( U_t' f = \lim_{h \to 0} \frac{1}{h} (U_{t+h} - U_t) f \)
\[
= iU_t A f, \quad f \in D_A.
\]

The considered strongly continuous unitary group, being one of the generalisations of the shift operator, allows one to construct for the elements of \( H \) such characteristics as the modulus of continuity and the modulus of smoothness. An overview of the results obtained in this direction and further references can be found in [6]. Let us define in \( H \) the operators of the \( k \)-th symmetric difference with step \( t, t \in \mathbb{R}, \ k \in \mathbb{N} \):
\[
\Delta_t^k = (U_{\frac{t}{2}} - U_{-\frac{t}{2}})^k.
\]
In particular
\[
\Delta_t^1 = (U_{\frac{t}{2}} - U_{-\frac{t}{2}}), \quad \Delta_t^2 = (U_t - 2I + U_{-t}),
\]
and for each \( f \in H \)
\[
\Delta_t^1 f = \int_{-\infty}^{\infty} \left\{ e^{its} - e^{-its} \right\} dE_s f,
\]
\[
\|\Delta_t^1 f\|^2 = \int_{-\infty}^{\infty} \left| e^{its} - e^{-its} \right|^2 d(E_s f, f);
\]
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\[ \Delta^2_t f = \int_{-\infty}^{\infty} \{e^{its} - 2 + e^{-its}\}dE_sf, \]
\[ \|\Delta^2_t f\|^2 = \int_{-\infty}^{\infty} |e^{its} - 2 + e^{-its}|^2 d(E_s f, f). \]

By the modulus of smoothness of order \( k, k \in \mathbb{N}, \) of an element \( f \in H \) we mean the quantity
\[ \omega_k(f, \delta) := \sup_{|t| \leq \delta} \| \Delta^k_t f \|, \]
for \( k = 1 \) we have in (6) the modulus of continuity of an element \( f \in H. \)

In conclusion of this section, we note some properties of the Riemann integral of an abstract function \( x(t) \), continuous on the interval \([a, b]\) with values in \( H \) (for more details see, for example, Chapter VI §25 in [8]), which we will need along with such natural properties as the fairness of Newton-Leibniz formula and integration by parts:

(R1) \( \left( \int_a^b F(t)dt \right) x = \int_a^b F(t)x dt, \quad F(t) \in \mathcal{L}(H), \quad x \in H; \)

(R2) \( \left( \int_a^b F(t)dt \right) G = \int_a^b F(t)G dt, \quad F(t) \in \mathcal{L}(H), \quad G \in \mathcal{L}(H); \)

(R3) \( u(t) = \int_a^t x(s)ds \) is continuously differentiable on \([a, b]\), and \( u'(t) = x(t), \quad t \in [a, b], \quad x(t) \in H; \)

(R4) \( \| \int_a^b x(t)dt \| \leq \int_a^b \| x(t) \| dt, \quad x(t) \in H. \)

3. Two sharp inequalities

In this section we prove two following theorems, which generalise results of [7] and [1]. Some methods, which are using next, are based on spectral decomposition and were considered, for instance, in [3], [4].

**Theorem 1.** For each \( f \in D_{A^2} \) and real \( \lambda > 0 \) the inequality
\[ \| Af \| \leq \frac{1}{2} \int_0^{\pi} \| \Delta_{2t}^1 A^2 f \| (1 - \sin \lambda t) dt + \frac{\lambda^2}{2} \int_0^{\pi} \| \Delta_{2t}^1 f \| \sin \lambda t dt, \quad (7) \]
holds and, hence, the second one
\[ \| Af \| \leq \frac{1}{2} \int_0^{\pi} \omega_1 (A^2 f, 2t) (1 - \sin \lambda t) dt + \frac{\lambda^2}{2} \int_0^{\pi} \omega_1 (f, 2t) \sin \lambda t dt \quad (8) \]
also holds.

If operator \( A \) is such, that for any real \( u, v, \) \( 0 < u < v \leq \lambda \) the condition
\[ (E_v - E_u) D_{A^2} \neq \{ \theta \} \quad (9) \]
is satisfied (see, for example, [3], [4]), then for any \( \lambda > 0 \) both inequalities are sharp in the sense, that the inequalities stop to be true for a certain \( f \in D_{A^2}, \)
if the right-hand side of (7), (8) is multiplied by a factor \( (1 - \tau) \) for arbitrary small \( \tau > 0. \)
Proof. Following [7], we define the operator $\tilde{F} \in \mathcal{L}(H)$ as

$$(\tilde{F})h := \left( \frac{\lambda}{2} \int_{0}^{\pi} (U_t + U_{-t}) \cos \lambda t dt \right) h,$$

$\lambda > 0$, and for $\|Af\|$ we can write the inequality

$$\|Af\| \leq \|Af - (\tilde{F})Af\| + \|(\tilde{F})Af\|. \quad (10)$$

Let us estimate each component of the right-hand side of (10). For $\|Af - (\tilde{F})Af\|$, taking into account (R1)–(R4), (U2), (U4) and applying integration by parts, we have:

$$\|Af - (\tilde{F})Af\| = \left\|Af - \left( \frac{\lambda}{2} \int_{0}^{\pi} (U_t + U_{-t}) \cos \lambda t dt \right) Af \right\|$$

$$= \left\| \frac{\lambda}{2} \int_{0}^{\pi} (2Af - U_t Af - U_{-t} Af) \cos \lambda t dt \right\|$$

$$= \left\| \frac{\lambda}{2i} \int_{0}^{\pi} \left( U_t f - U_{-t} f - 2i Af \right) \cos \lambda t dt \right\|$$

$$= \left\| \frac{\lambda}{2i} \left( iU_t Af - (-iU_{-t} Af) - 2i Af \right) \right\|$$

$$\leq \frac{1}{2} \int_{0}^{\pi} \| (U_t - U_{-t}) A^2 f \| (1 - \sin \lambda t) dt$$

$$\leq \frac{1}{2} \int_{0}^{\pi} \| \Delta_2 A^2 f \| (1 - \sin \lambda t) dt. \quad (11)$$

For $\|(\tilde{F})Af\|$, taking into account (R1)–(R3), (U4) and the Newton-Leibniz
formula, we have
\[
\| (\tilde{F}) Af \| = \left\| \frac{\lambda}{2} \int_0^{\pi} \left( U_t + U_{-t} \right) \cos \lambda t dt \right\| A f \\
= \left\| \frac{\lambda}{2} \int_0^{\pi} \left( U_t Af + U_{-t} Af \right) \cos \lambda t dt \right\| \\
= \left\| \frac{\lambda}{2i} \int_0^{\pi} \left( U_t' f - U_{-t} f \right) \cos \lambda t dt \right\| \\
\leq \frac{\lambda^2}{2} \int_0^{\pi} \| (U_t - U_{-t}) f \| \sin \lambda t dt = \frac{\lambda^2}{2} \int_0^{\pi} \| \Delta_{2t} f \| \sin \lambda t dt. \quad (12)
\]

Summing up the obtained estimates, we have inequality (7), and, hence, according to (6) estimate (8) immediately follows.

Let us proceed to the proof, that if assumption (9) is satisfied, then inequality (7) is sharp. Assume to the contrary, that there exists \( \tau \in (0, 1) \) such that for any \( f \in D_{A^2} \), \( \lambda > 0 \),
\[
\| Af \| \leq (1 - \tau) \left\{ \frac{1}{2} \int_0^{\pi} \| \Delta_{2t} A^2 f \| \sin \lambda t dt + \frac{\lambda^2}{2} \int_0^{\pi} \| \Delta_{2t} f \| \sin \lambda t dt \right\}. 
\]
(13)

We choose \( \epsilon > 0 \) from the condition
\[
\epsilon < \tau \lambda
\]
and set \( u = \lambda - \epsilon, v = \lambda \). Using (9), we consider an element \( g_{\lambda, \epsilon} \in D_{A^2} \) as follows:
\[
g_{\lambda, \epsilon} = \int_{\lambda-\epsilon}^{\lambda} dE_s g_{\lambda, \epsilon}.
\]

For \( \| A g_{\lambda, \epsilon} \|^2 \) the following estimate from below is satisfied:
\[
\| A g_{\lambda, \epsilon} \|^2 = \int_{\lambda-\epsilon}^{\lambda} s \cdot \frac{d}{ds} (E_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) \geq (\lambda - \epsilon)^2 \| g_{\lambda, \epsilon} \|^2,
\]
hence,
\[
\| A g_{\lambda, \epsilon} \| \geq (\lambda - \epsilon) \| g_{\lambda, \epsilon} \|. \quad (15)
\]

Next, for \( g_{\lambda, \epsilon} \) we estimate from above each component of the right-hand side of (7). For all \( 0 \leq t \leq \frac{\pi}{2}, \lambda > 0 \), we have
\[
\| \Delta_{2t} A^2 g_{\lambda, \epsilon} \|^2 = \int_{\lambda-\epsilon}^{\lambda} s^2 |e^{its} - e^{-its}|^2 d (E_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) \\
= \int_{\lambda-\epsilon}^{\lambda} 4s^4 \sin^2 t s d (E_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) \leq 4 \lambda^4 \sin^2 \lambda t \| g_{\lambda, \epsilon} \|^2 
\]
(16)
Thus, the estimate of the first component of right-hand side of (7) is:

\[
\frac{1}{2} \int_0^{\frac{\pi}{2}} \| \Delta_2 t A^2 g_{\lambda, \epsilon} \| (1 - \sin \lambda t) dt \leq \lambda^2 \| g_{\lambda, \epsilon} \| \int_0^{\frac{\pi}{2}} \sin \lambda t (1 - \sin \lambda t) dt.
\]  

(17)

For the second component we have:

\[
\| \Delta_2 t g_{\lambda, \epsilon} \|^2 = \int_{\lambda - \epsilon}^{\lambda} | e^{its} - e^{-its} |^2 d(E_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) = \int_{\lambda - \epsilon}^{\lambda} 4 \sin^2 t s d(E_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) \leq 4 \sin^2 \lambda t \| g_{\lambda, \epsilon} \|^2 \tag{18}
\]

or

\[
\| \Delta_2 t g_{\lambda, \epsilon} \| \leq 2 \sin \lambda t \| g_{\lambda, \epsilon} \|,
\]

and, hence, the estimate for the second component is written as:

\[
\frac{\lambda^2}{2} \int_0^{\frac{\pi}{2}} \| \Delta_2 t A^2 g_{\lambda, \epsilon} \| \sin \lambda t dt \leq \lambda^2 \| g_{\lambda, \epsilon} \| \int_0^{\frac{\pi}{2}} \sin^2 \lambda t dt \tag{19}
\]

Taking into account (13), (15), (17) and (19) we get:

\[
(\lambda - \epsilon) \| g_{\lambda, \epsilon} \| \leq (1 - \tau) \lambda^2 \| g_{\lambda, \epsilon} \| \int_0^{\frac{\pi}{2}} \sin \lambda t dt
\]

or

\[
(\lambda - \epsilon) \leq (1 - \tau) \lambda.
\]

Therefore, we obtained the contradiction with (14), and inequality (7) is sharp.

To prove sharpness of inequality (8) it is sufficiently to notice that follow

\[
\omega_1 (g_{\lambda, \epsilon}, 2t) = \sup_{|s| \leq 2t} \| \Delta_1 g_{\lambda, \epsilon} \| = \| \Delta_1 g_{\lambda, \epsilon} \|
\]

\[
\omega_1 (A^2 g_{\lambda, \epsilon}, 2t) = \sup_{|s| \leq 2t} \| \Delta_1 A^2 g_{\lambda, \epsilon} \| = \| \Delta_1 A^2 g_{\lambda, \epsilon} \|
\]

(we used monotony of functions \( \| \Delta_2 t g_{\lambda, \epsilon} \| \) and \( \| \Delta_2 t A^2 g_{\lambda, \epsilon} \| \) by argument \( t \) (see corresponding representation in (18) and (16))).

**Theorem 2.** For each \( f \in D_{A^2} \) and real \( \lambda > 0 \) the inequality

\[
\| A^2 f \| \leq \frac{\lambda}{\pi - 2} \left\{ \int_0^{\frac{\pi}{2}} \| \Delta_2^2 f \| (1 - \sin \lambda t) dt + \lambda^2 \int_0^{\frac{\pi}{2}} \| \Delta_2 f \| \sin \lambda t dt \right\},
\]  

(20)
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holds and, hence, the second one

\[ \|A^2 f\| \leq \frac{\lambda}{\pi - 2} \left\{ \int_0^{\pi/2} \omega_2 (A^2 f, t) (1 - \sin \lambda t) \, dt + \lambda^2 \int_0^{\pi/2} \omega_2 (f, t) \sin \lambda t \, dt \right\} \]

(21)

also holds.

Inequalities (20) and (21) become sharp under the same condition (9) as in Theorem 1.

Proof. The ideas of the previous proof are applied here. Let us consider the operator

\[ (\tilde{F}) h := \left( \frac{\lambda}{\pi - 2} \int_0^{\pi/2} (U_t + U_{-t}) (1 - \sin \lambda t) \, dt \right) h, \]

\[ \lambda > 0, \] and estimate each component of the right-hand side of the inequality

\[ \|A^2 f\| \leq \|A^2 f - (\tilde{F}) A^2 f\| + \|(\tilde{F}) A^2 f\|. \]

(22)

We have

\[ \|A^2 f - (\tilde{F}) A^2 f\| = \left\| A^2 f - \left( \frac{\lambda}{\pi - 2} \int_0^{\pi/2} (U_t + U_{-t}) (1 - \sin \lambda t) \, dt \right) A^2 f \right\| \]

\[ = \left\| \frac{\lambda}{\pi - 2} \int_0^{\pi/2} \{ (2I - U_t - U_{-t}) A^2 f \} (1 - \sin \lambda t) \, dt \right\| \]

\[ \leq \frac{\lambda}{\pi - 2} \int_0^{\pi/2} \| \Delta^2 f \| (1 - \sin \lambda t) \, dt. \]

(23)

To estimate the second component we applied integration by parts twice:

\[ \|(\tilde{F}) A^2 f\| = \left\| \left( \frac{\lambda}{\pi - 2} \int_0^{\pi/2} (U_t + U_{-t}) (1 - \sin \lambda t) \, dt \right) A^2 f \right\| \]

\[ = \left\| \frac{\lambda}{\pi - 2} \int_0^{\pi/2} (U_t A^2 f + U_{-t} A^2 f) (1 - \sin \lambda t) \, dt \right\| \]

\[ = \left\| \frac{\lambda}{\pi - 2} \int_0^{\pi/2} (-U''_t f - U''_{-t} f) (1 - \sin \lambda t) \, dt \right\| \]

\[ = \left\| \frac{\lambda^2}{\pi - 2} \int_0^{\pi/2} (U'_t f + U'_{-t} f - 2U'_{0} f) \cos \lambda t \, dt \right\| \]

\[ = \left\| \frac{\lambda^3}{\pi - 2} \int_0^{\pi/2} (U_t f + U_{-t} f - 2U_{0} f) \sin \lambda t \, dt \right\| \]

\[ \leq \frac{\lambda^3}{\pi - 2} \int_0^{\pi/2} \| (U_t - 2I + U_{-t}) f \| \sin \lambda t \, dt \]

\[ = \frac{\lambda^3}{\pi - 2} \int_0^{\pi/2} \| \Delta^2 f \| \sin \lambda t \, dt. \]

(24)
Summing up the obtained results, we have inequality (20), and, hence, taking
into account (6), estimate (21) follows.

Sharpness of inequality (20) is proved in the same way as in the previous
case. Let condition (9) be satisfied. Assume, that there exists \( \tau \in (0, 1) \) such
that for any \( f \in D_{A^2}, \lambda > 0, \)
\[
\|Af\| \leq \frac{(1 - \tau)\lambda}{\pi - 2} \left\{ \int_0^{\frac{\pi}{2}} \|\Delta_t^2 A^2 f\| (1 - \sin \lambda t) dt + \lambda^2 \int_0^{\frac{\pi}{2}} \|\Delta_t^2 f\| \sin \lambda t dt \right\}. \tag{25}
\]
We choose \( \epsilon > 0 \) from the condition
\[
\epsilon < (1 - \sqrt{1 - \tau})\lambda \tag{26}
\]
and set \( u = \lambda - \epsilon, \ v = \lambda \). As before, we consider an element \( g_{\lambda, \epsilon} \in D_{A^2} \) as
follows:
\[
g_{\lambda, \epsilon} = \int_{\lambda - \epsilon}^{\lambda} dE_s g_{\lambda, \epsilon}.
\]
For all \( 0 \leq t \leq \frac{\pi}{2\lambda}, \lambda > 0, \) we can write the estimate from below for \( \|A^2 g_{\lambda, \epsilon}\|:
\[
\|A^2 g_{\lambda, \epsilon}\|^2 = \int_{\lambda - \epsilon}^{\lambda} s^4 (dE_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) \geq (\lambda - \epsilon)^4 \|g_{\lambda, \epsilon}\|^2,
\]
hence,
\[
\|A^2 g_{\lambda, \epsilon}\| \geq (\lambda - \epsilon)^2 \|g_{\lambda, \epsilon}\|. \tag{27}
\]
Next, we obtain the estimate from above for right-hand side of (20). Indeed,
\[
\|\Delta_t^2 A^2 g_{\lambda, \epsilon}\|^2 = \int_{\lambda - \epsilon}^{\lambda} s^4 |e^{its} - 2 + e^{-its}|^2 (dE_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon})
\]
\[
= \int_{\lambda - \epsilon}^{\lambda} 4s^4 (1 - \cos ts)^2 (dE_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) \leq 4\lambda^4 (1 - \cos \lambda t)^2 \|g_{\lambda, \epsilon}\|^2 \tag{28}
\]
or
\[
\|\Delta_t^2 A^2 g_{\lambda, \epsilon}\| \leq 2\lambda^2 (1 - \cos \lambda t) \|g_{\lambda, \epsilon}\|.
\]
Therefore,
\[
\frac{\lambda}{\pi - 2} \int_0^{\frac{\pi}{2}} \|\Delta_t^2 A^2 g_{\lambda, \epsilon}\| (1 - \sin \lambda t) dt \leq \frac{2\lambda^3}{\pi - 2} \|g_{\lambda, \epsilon}\| \int_0^{\frac{\pi}{2}} (1 - \cos \lambda t) (1 - \sin \lambda t) dt. \tag{29}
\]
For the second one, we also have
\[
\|\Delta_t^2 g_{\lambda, \epsilon}\|^2 = \int_{\lambda - \epsilon}^{\lambda} |e^{i\lambda s} - 2 + e^{-i\lambda s}|^2 (dE_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon})
\]
\[
= \int_{\lambda - \epsilon}^{\lambda} 4 (1 - \cos ts)^2 (dE_s g_{\lambda, \epsilon}, g_{\lambda, \epsilon}) \leq 4 (1 - \cos \lambda t)^2 \|g_{\lambda, \epsilon}\|^2 \tag{30}
\]
or
\[ \| \Delta^2_t g_{\lambda, \epsilon} \| \leq 2 (1 - \cos \lambda t) \| g_{\lambda, \epsilon} \|. \]

Then,
\[ \frac{\lambda^3}{\pi - 2} \int_0^{\frac{\pi}{2}} \| \Delta^2_t g_{\lambda, \epsilon} \| \sin \lambda t dt \leq \frac{2\lambda^3}{\pi - 2} \| g_{\lambda, \epsilon} \| \int_0^{\frac{\pi}{2}} (1 - \cos \lambda t) \sin \lambda t dt. \]  (31)

Taking into account (25), (27), (29) and (31) we get:
\[ (\lambda - \epsilon)^2 \| g_{\lambda, \epsilon} \| \leq (1 - \tau) \frac{\lambda^3}{\pi - 2} \| g_{\lambda, \epsilon} \| \int_0^{\frac{\pi}{2}} (1 - \cos \lambda t) dt \]
or
\[ (\lambda - \epsilon)^2 \leq (1 - \tau) \lambda^2, \]
and we obtained the contradiction with (26). Therefore, inequality (20) is sharp. Sharpness of inequality (21) can be proved by analogy with previous proof.

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