ON SHAKE SLICE KNOTS

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Abstract. Here we discuss $r$–shake slice knots, and their relation to corks, we then prove that $0$-shake slice knots are slice.

0. Introduction and the main theorem

A knot $K \subset S^3$ is slice if it is the boundary of a properly imbedded smooth disk $D^2 \subset B^4$. Let $K^r = B^4 \sim h_K(r)$ be the 4-manifold obtained by attaching a 2-handle to $B^4$ along the knot $K$ with framing $r$. Clearly $K$ is slice if and only if $K^0$ imbeds into $S^4$. To see this, write $S^4$ as union of two 4-balls $B^4_+ \sim B^4_+$ glued along boundaries, then use the slice disk in $B^4_+ \sim h_K(0) \subset S^4$ (compare [KM]).

Clearly $K^r$ is homotopy equivalent to $S^2$. We say that $K$ is $r$-shake slice if a generator of $H_2(K^r) = \mathbb{Z}$ is represented by a smoothly imbedded 2-sphere in $K^r$. An $r$-shaking of the knot $K$ is defined to be the link $L^r = \{K, K^+, \ldots, K^+, K^-, \ldots, K^-\}$, consisting of $K$ and an even number of oppositely oriented parallel copies of $K$ (pushed off by the framing $r$). Then by [A2] and [A1] we see that $K$ is $r$-shake slice if an $r$-shaking of $K$ bounds a disk $D_n$ with $2n$ holes for some $n \geq 0$ in $B^4$. Now define $K^r_n = B^4 \sim D_n \times B^2 / \sim$, where we glue $D_n \times B^2$ to $B^4$ along $\partial D_n \times B^2$ to the framed link $L^r \times D^2$ with 0-framing.

Figure 1. $K^0_n$

2020 Mathematics Subject Classification. 58D27, 58A05, 57R65.
It is known that, for each $r \neq 0$, there are $r$-shake slice knots which are not slice (Theorem 2.1 of [A1]). Also not every knot is $r$-shake slice (e.g. Theorem 1 of [A4], and [Y]). By the same argument of the first paragraph above, we see that $K$ is 0-shake slice if and only if $K_0^n$ imbeds into $S^4$ for some $n$. For example for $U^0_n \subset S^4$ for the unknot $U$. When $r \neq 0$ we also have: $K$ is $r$-shake slice $\Rightarrow K^r_n \subset S^4$.

A handlebody of $K_0^n$ can be drawn by generalizing the round handle attachment technique, applied to $L^0$ (e.g. 3.2 of [A1]), i.e. we attach 2-handle with 0-framing to the knot obtained by band summing the link $L$ with $2n + 1$ components $K \# (K_+ \# \ldots K_+) \# (K_- \# \ldots K_-)$ so that each band going through a 1-handle (circle with dot), as shown in Figure 2. After the obvious isotopies, from Figure 2 we obtain Figure 3.

![Figure 2. Handlebody of $K_0^n$](image1)

![Figure 3. Handlebody of $K_0^n$](image2)
Notice, Figure 3 shows $K_0^n$ is obtained by a single 2-handle attachment to $\tilde{\natural}_{2n}S^1 \times B^3$. Also note that 1-handles of Figure 3 are in the special position relative to the 2-handle (as described in 1.4 of [A1]), hence they can be slid over the 2-handle of $K^0$ as indicated by Figure 3 to obtain the handlebody of $K_0^n$ in Figure 4. Which shows that $K_0^n$ is also obtained by a single 2-handle attachment $(\gamma \# K)^0$ to the manifold which we called $L_0^n$ in Figure 5.

![Figure 4. $K_0^n$, (n=2)](image)

Figure 4. $K_0^n$, (n=2)

![Figure 5. $L_0^n$](image)

Figure 5. $L_0^n$

![Figure 6. $S^4 - L_0^n$](image)

Figure 6. $S^4 - L_0^n$

Figure 6 gives the complement of the “standard imbedding” $L_0^n \subset S^4$, which is given by the obvious carved ribbons of Figure 5. The choices of the carvings in Figure 5 determines the imbedding $L_0^n \subset S^4$, and $S^4 - L_0^n$ in Figure 6, but not vice versa. It is easy to see that, there is an involution $f : \partial L_0^n \to \partial L_0^n$ taking the loop $\gamma$ to $f(\gamma)$.

The following Proposition 1 gives another handle description of $L_0^n$ (Figure 7), which consists of bunch of trivially carved discs from $B^4$ (hence they are imbedded into $S^4$ uniquely), along with a 2-handle.
Proposition 1. The handlebody of Figure 7 describes $L^0_n$.

Proof. Slide the dotted circles $a_1, a_2, ..., a_n$ over the 2-handle (as described in Section 1.4 of [A1]), then cancel the large 1- and 2-handle pair to get $L^0_n$ (Figure 5). □

![Figure 7. $L^0_n$](image1)

Theorem 2. 0-Shake slice knots are slice.

Proof. If a knot $K \subset S^3$ is 0-shake slice, then $K^0_n \subset S^4$ for some $n$. Recall that $K^0_n$ is obtained by attaching a 2-handle to $L^0_n$ along $\gamma \# K$ with 0-faming, i.e. $K^0_n = L^0_n \circ 2$-handle along $\gamma \# K \subset S^4$, which implies $\gamma \# K$ bounds a smooth disk $D$ in the complement $C := S^4 - L^0_n$.

![Figure 8. Cork twisting $S^4$. Carving a disks from $L^0_n$ is equivalent to attaching 2-handle to its complement $C$](image2)

From Figure 7 we see $f(\gamma)$ bounds a smooth disk in $L^0_n$ (the dual of the 2-handle), which implies $f(\gamma \# K)$ bounds a singular disk $\tilde{D}$ inside $L^0_n$ with $cone(K)$ singularity. If $\gamma \# K$ coincided with $f(\gamma \# K)$ we would be finished, because $D \sim_0 \tilde{D}$ would be a singular $S^2$ inside $S^4$ with $cone(K)$ singularity, which would imply $K^0_n \subset S^4$. Since this is not the case, we will complete the proof by performing a “cork twisting operation” to $S^4$ along $\partial L^0_n$, which throws $\gamma \# K$ to the $f(\gamma \# K)$ in $\partial L^0_n$, and then showing this doesn’t change the smooth type of $S^4$. 
To see twisting $S^4$ along $\partial L_n^0$ via $f$ gives $S^4$ back, we recall $f(\gamma)$ bounds a smooth disk $D \subset L_n^0$, and let $N(D)$ be its tubular neighborhood. Notice $L_n^0 - N(D) \approx \sharp_{2n+1} B^3 \times S^1$, and upside down $\sharp_{2n+1} B^3 \times S^1$ are 3-handles which are attached uniquely. Hence up to 3-handles, the cork twisting $S^4$ along $L_n^0$ via $f$, is the operation described in Figure 8
\[ C \sim h_{f(\gamma)}^2 \Rightarrow C \sim h_\gamma^2 \]
To sum up, the first three steps of Figure 9 describe this operation:
\[ L_n^0 \mapsto L_n^0 - N(D) \mapsto C \sim h_{f(\gamma)}^2 \mapsto C \sim h_\gamma^2 \]

In Figure 9 the circle with big yellow-dot denotes the complement of the 2-handle of $L_n^0$. In the upside down picture, this corresponds to the complement of a slice 2-disk $D$ (the core of the downside up 2-handle).

**Figure 9.** Cork twisting $S^4$ along $L_n^0$

Intersection patterns of this 2-disk $D$ with the 2-handles of the third picture are dictated by the figure itself. Notice that this disk doesn’t go over the 2-handle $h_\gamma^2(=A)$ because $h_\gamma^2$ is attached later on. In the forth picture, 2-handle denoted by B (and any piece of D which might be going over it) slides over the 2-handle $h_\gamma^2$ to the other side of $h_\gamma^2$. Hence this carved disk $D$ in the last picture doesn’t go over the 2-handle denoted by A; it lies entirely in $S^2 \times B^2$ (after adding 3-handles), and it has standard boundary $\partial B^2$. So by [G] it is isotopic to the meridional disk $p \times B^2$. So it cancels the middle 2-handle and results $B^4$ with a 2-handle attached to it, with boundary $S^2 \times S^1$. So by “Property R” it is $S^2 \times B^2$, which later gets cancelled by a 3 handle to give $S^4$. $\square$
Remark 1. Start with the handlebody of $L_0^n$ in Figure 7. Its 1-handles $\#_{2n+1}B^3 \times S^1$ can be imbeded into $S^4$ unique way, then by varying the imbedding of its 2-handle in the complement we can otain many different imbeddings $L_0^n \hookrightarrow S^4$. By decomposing $S^4 = B^4 \cup B^4_\bot$, we can place 1-handles as carved out $B^4_\bot$, and vary the imbedding of its 2-handle in complement. For example, if we take the standard imbedding $L_0^n$ (Figure 5) we get its complement $S^4 - L_0^n$ (Figure 6), and $f$ maps $h_2^2(f(\gamma)) \mapsto h_2^2$. From Figure 10 we see that $C \sim h_2^2(\gamma) \approx C \sim h_2^2 \approx S^4$.

![Figure 10. Cork twisting $C \sim h_2^2 \mapsto C \sim h_2^2(f(\gamma)) \approx S^4$.](image)

1. Relation to corks

The manifold $L_0^n$ has appeared in solution of many 4-manifold problems. For example, it appeared in solution of Zeeman Conjecture [A6], it appeared in construction of diffeomorphic disk pairs $(B^4, D_1)$, $(B^4, D_2)$ that are not isotopic to each other rel boundary [A7]. It also appear in $h$-cobordisms between homotopy 4-spheres as “protocorks” [A8]. Here we will show that it can be used to generate new corks, from which we can construct new exotic 4-manifolds (see [A3], [A4], [A5]).

Theorem 3. The diffeomorphism $f : \partial L_0^n \rightarrow \partial L_0^n$ can not extend to a self diffeomorphism of $L_0^n \rightarrow L_0^n$.

Proof. We prove this by associating to $(L_0^n, f)$ a cork: If $f$ extended to a diffeomorphism $F : L_0^n \rightarrow L_0^n$, then we could extend $F$ to a self diffeomorphism of the manifold $W_n$, obtained from $L_0^n$ by attaching $-1$ framed 2-handles to $a_1, \ldots, a_n, b_1, \ldots, b_n$, since $f$ fixes these dotted circles. For simplicity, we continue to call $F|_{\partial W_n} = f$. Figure 11 is the handle picture of $W_n$, and Figure 12 is the Legendrian handle picture, from
where we can check that it is a Stein manifold and calculate the TB invariant of \( \gamma \), so by adjunction inequality it can not bound a smooth disk. Since \( f(\gamma) \) bounds a smooth disk, the argument of Theorem 9.3 of [A1] shows \( (W_n, f) \) is a cork and \( f \) is a cork automorphism, so \( F \) can’t exist. \( W_n \) is a variant of the “positron cork” of [AM]. \[ \square \]

\[ \text{Figure 11. } W_n \]

\[ \text{Figure 12. } W_n \]

\[ \text{Figure 13. Exotic homotopy } S^1 \times B^3 \text{ rel boundary} \]
Remark 2. \((L^0_n, f)\) is a basic universal object, which appears in construction of many exotic smooth 4-manifolds. Notice that the rel boundary exotic homotopy \(S^1 \times B^3\) of [A6] (Figure 13) can be obtained from \(L^0_1\) by attaching \(-1\) framed 2-handle to \(b_1\) in Figure 7, which was carved out from the first cork introduced in [A4]. Notice the similarity between the construction of \(K^0_n\) and construction of infinite order corks by altering concordances [A7].

Remark 3. However remote, there is some similarity between constructing handlebody of \(K^0_n\) from \(K\) and constructing “spine manifolds” from bounding manifolds, where shaking corresponds to taking disjoint union with spheres (Fact 3.2 of [AK]).

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