TWO-TORSION SUBGROUPS OF CLASS GROUPS OF CUBIC FIELDS

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Abstract. We prove a generalization of a result of Bhargava regarding the average size $\text{Cl}(K)[2]$ as $K$ varies among cubic fields. For a fixed set of rational primes $S$, we obtain a formula for the average size of $\text{Cl}(K)/(S)[2]$ as $K$ varies among cubic fields with a fixed signature, where $\langle S \rangle$ is the subgroup of $\text{Cl}(K)$ generated by the classes of primes of $K$ above prime in $S$.

As a consequence, we are able to calculate the average sizes of $K_{2n}(\mathcal{O}_K)[2]$ for $n > 0$ and for the relaxed Selmer group $\text{Sel}_2^S(K)$ as $K$ varies in these same families.

1. Introduction

In addition to the Davenport-Heilbronn theorem, one of the few results proven concerning the distribution of class groups of number fields is a result of Bhargava in $[1]$ and extended by Bhargava and Varma in $[4]$ which states:

Theorem 1. When ordered by absolute discriminant,

(i) the average size of $\text{Cl}(K)[2]$ as $K$ ranges over totally real $S_3$-cubic fields is equal to $5/4$,
(ii) the average size of $\text{Cl}(K)[2]$ as $K$ ranges over complex $S_3$-cubic fields is equal to $3/2$, and
(iii) the average size of $\text{Cl}(K)^+[2]$ as $K$ ranges over totally real $S_3$-cubic fields is equal to $2$.

Theorem 1 may be thought of as an analogue of the classical Davenport-Heilbronn theorem regarding the average size of the 3-torsion subgroups of class groups of quadratic fields. In $[10]$, we generalized the Davenport-Heilbronn theorem to quotients of ideal class groups of quadratic fields by the subgroup generated by the classes of primes lying above a fixed set of rational primes $S$. The goal of this work is to do the same for Theorem 1.

Explicitly: Let $S$ be a finite set of rational primes. For each cubic field $K$, define $\text{Cl}(K)_S := \text{Cl}(K)/\langle S_K \rangle$, where $S_K$ is the set of primes of $\mathcal{O}_K$ lying above the primes in $S$ and $\langle S_K \rangle$ is the subgroup of $\text{Cl}(K)$ generated by the ideal classes of the primes in $S_K$. Define $\text{Cl}(K)^+_S$ similarly using the narrow class group $\text{Cl}(K)^+$ of $K$.

Theorem 2. When ordered by absolute discriminant,

(i) the average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over totally real $S_3$-cubic fields is equal to

\[ 1 + \frac{1}{2^{|S|}+2} \prod_{p \in S} \left( 1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right), \]

(ii) the average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over complex $S_3$-cubic fields is equal to

\[ 1 + \frac{1}{2^{|S|}+1} \prod_{p \in S} \left( 1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right), \]

and

(iii) the average size of $\text{Cl}(K)^+_S[2]$ as $K$ ranges over totally real $S_3$-cubic fields is equal to

\[ 1 + \frac{1}{2^{|S|}} \prod_{p \in S} \left( 1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right). \]
Corollary 3. If $S$ is non-empty, then

(i) a positive proportion of totally real $S_3$-cubic fields $K$ have $\text{Cl}(K)^+_S[2] = 0$,
(ii) a positive proportion of totally real $S_3$-cubic fields $K$ have $\text{Cl}(K)^+_S = \text{Cl}(K)$, and
(iii) a positive proportion of totally real $S_3$-cubic fields $K$ have $S_K$-units of all possible signatures.

In each case, the proportion of totally real $S_3$-cubic fields having the property claimed is at least

$$1 - \frac{1}{2^{\left|S\right|}} \prod_{p \in S} \left(1 + \frac{p^2 + 4}{4(p^2 + p + 1)}\right).$$

The local product appearing in Theorem 2 is a consequence of the fact that the decomposition type of any prime in $S$ will vary with $K$. By assuming a fixed decomposition type for each prime in $S$, we get a more natural answer that is consistent with Cohen and Lenstra’s model for quotients of class groups — see Remark 1.3.

Theorem 4. Let $S$ be a set of rational primes. For each $p \in S$, fix a rank 3 $\mathbb{Q}_p$-algebra $R_p$ and set

$$r = \sum_{p \in S} (r_p - 1),$$

where $r_p$ is the number of irreducible components of $R_p$. When ordered by absolute discriminant,

(i) the average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over totally real $S_3$-cubic fields with $K \otimes \mathbb{Q}_p \cong R_p$ for all $p \in S$ is equal to $1 + 2^{-r+2}$,
(ii) the average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over complex $S_3$-cubic fields with $K \otimes \mathbb{Q}_p \cong R_p$ for all $p \in S$ is equal to $1 + 2^{-(r+1)}$, and
(iii) the average size of $\text{Cl}(K)^+_S[2]$ as $K$ ranges over totally real $S_3$-cubic fields with $K \otimes \mathbb{Q}_p \cong R_p$ for all $p \in S$ is equal to $1 + 2^{-r}$.

We are able to use Theorem 4 to obtain results about $K_{2n}(\mathcal{O}_K)$ for $n > 0$. When $S = \{2\}$, a theorem of Rognes and Weibel relates $\dim_2 K_{2n}(\mathcal{O}_K)$ to $\text{Cl}(K)_S$ when $n \equiv 0, 1 \pmod{4}$ and to $\text{Cl}(K)^+_S$ when $n \equiv 2, 3 \pmod{4}$ — see Theorem 6.1. As a consequence, we obtain the following result about the average size of $K_{2n}(\mathcal{O}_K)[2]$ as $K$ varies over cubic fields.

Theorem 5. For $n > 0$, the average size of $K_{2n}(\mathcal{O}_K)[2]$ as $K$ ranges over totally real (resp. complex) $S_3$-cubic fields is as follows:

| $n \equiv 0 \pmod{4}$ | $n \equiv 1 \pmod{4}$ | $n \equiv 2 \pmod{4}$, $n \equiv 3 \pmod{4}$ |
|---------------------|---------------------|---------------------|
| 59/28               | 118/7               | 20/7                |
| 33/14               | 33/7                | 33/14               |

Remark 1.1. Theorem 5 is an analogue of Theorem 1.2 in the author’s joint work with Jordan, Poonen, Skinner, and Zaytman which proves a similar result about the average size of $K_{2n}(\mathcal{O}_K)[3]$ as $K$ varies over quadratic fields.

Remark 1.2. We restrict our attention to even indexed $K$-groups, since for $n$ odd, $K_n(\mathcal{O}_K)[2]$ is entirely determined by the residue class of $n \pmod{8}$ and the number of real places of $K$ (see Theorem 0.7 in [11]).

We also obtain distribution results about $\text{Sel}^S_2(K)$, the relaxed 2-Selmer group of $K$ (defined in Section 5).
Theorem 6. When ordered by absolute discriminant,

(i) the average size of \( \text{Sel}^S(K) \) as \( K \) ranges over totally real \( S_3 \)-cubic fields is equal to

\[
2^{[S]+1} + 2^{[S]+3} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right)
\]

and

(ii) the average size of \( \text{Sel}^S_2(K) \) as \( K \) ranges over complex \( S_3 \)-cubic fields is equal to

\[
2^{[S]+1} + 2^{[S]+2} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right).
\]

Remark 1.3. In Theorem 4, \( r_p \) is the number of primes above \( p \) in \( K \). If the primes above \( p \) take classes uniformly at random in \( \text{Cl}(K) \) subject only to the relation arising from the factorization of \( p \mathcal{O}_K \) and all of the primes in \( S \) behave independently, then the subgroup \( \langle S_K \rangle \leq \text{Cl}(K) \) may be thought of as a group generated by \( r \) elements chosen uniformly at random from \( \text{Cl}(K) \). It is therefore natural to expect that for any finite abelian 2-group \( H \), the probability \( \text{Prob}(\text{Cl}(K)/\langle S \rangle [2^\infty] \simeq H) \) is equal to what Cohen and Lenstra dub the \( u \)-probability of \( H \) with \( u = r + r_\infty \), where \( r_\infty \) is equal to 2, 1, and 0 in cases (i), (ii), and (iii) of the theorem respectively. The average sizes appearing Theorem 4 are precisely the \( u \)-averages for these values of \( u \).

Remark 1.4. All of the results we present are stated in terms of \( S_3 \)-cubic fields. However, the results remain correct and can be proven with modified versions of the current proofs even if we remove the \( S_3 \) assumption.

We have nonetheless chosen to maintain the \( S_3 \)-assumption throughout the paper for the purposes of clarity. Instead, we present the following argument showing that the total contribution to the average size of \( \text{Cl}(K)[2] \) coming from cyclic cubic fields as \( K \) varies among all totally real cubic fields is zero.

It is well-known that the total number of cyclic cubic fields of discriminant less than \( X \) grows like \( O(X^{1/2}) \) [7]. By a result of Wong [12], the size of \( \text{Cl}(K)[2] \) in any cyclic cubic field \( K \) is bounded by \( O((|\text{Disc}(K)|^{3/8+\epsilon}) \) for any \( \epsilon > 0 \) (see also [3] for a better bound). As a result, the combined number of elements in \( \text{Cl}(K)[2] \) from all cyclic cubic fields with discriminant less than \( X \) is bounded by \( O(X^{7/8+\epsilon}) \). Since the number of totally real cubic fields of discriminant less than \( X \) grows like \( O(X) \), the total contribution to the average size of \( \text{Cl}(K)[2] \) coming from cyclic cubic fields must be zero.

1.1. Methods and Organization. The core idea behind Theorem 1 is how to use the geometry of numbers to count \( S_3 \)-quartic fields. The application to class groups as it originally appears in [11] arises from a bijection established by Heilbronn and slightly refined by Bhargava between the set of non-trivial two-torsion elements in the class group \( \text{Cl}(K) \) of an \( S_3 \)-cubic field and the set of nowhere overramified isomorphism classes of \( S_3 \)-quartic fields \( L \) with resolvent field \( K \) (see Definitions 2.1 and 2.2) that have a real place.

We establish a similar bijection in Section 2 between the set of non-trivial two-torsion elements in \( \text{Cl}(K)_S \) and the set of isomorphism classes of \( S_3 \)-cubic fields \( L \) with resolvent field \( K \) satisfying certain local conditions. Section 3 presents recent results of Bhargava used for counting the number of such fields having bounded discriminant [2] and Section 4 then describes how to apply these techniques to prove Theorem 2. Section 5 defines the relaxed 2-Selmer group of \( K \) and develops the machinery needed to prove Theorem 6. Finally, we prove Theorem 5 in Section 6.
1.2. Acknowledgement. I would like to thank Frank Thorne for suggesting the use of the Artin relation in the proof of Lemma 2.7 and for pointing out the result of Wong mentioned in Remark 1.4. I would also like to thank Alyson Deines for a number of useful suggestions.

1.3. Notation. We will use the following notation throughout this paper:

- $S$ will be a set of rational primes.
- $K$ will be a cubic field.
- $\mathcal{O}_K$ will be the ring of integers of $K$.
- $S_K$ will denote the set of primes of $\mathcal{O}_K$ lying above primes in $S$.
- $\mathcal{O}_K^{\times}$ will denote the $S_K$-units of $K$.
- $\text{Sel}_2^S(K)$ will be the 2-Selmer group of $K$ relaxed at the primes in $S_K$.
- $\text{Cl}(K)$ will be the ideal class group of $\mathcal{O}_K$.
- $\text{Cl}(K)_S$ will be the quotient $\text{Cl}(K)/\langle S_K \rangle$, where $\langle S_K \rangle$ is the subgroup of $\text{Cl}(K)$ generated by primes in $S_K$.

2. Class Field Theory

Bhargava’s results about $\text{Cl}(K)[2]$ in [1] rely on a correspondence of Heilbronn detailed in [8]. We briefly describe this correspondence before establishing a similar correspondence for $\text{Cl}(K)_S[2]$ in Proposition 2.6.

Definition 2.1. Given an $S_4$-quartic field $L$, the cubic resolvent field $\text{Res}(L)$ is the unique (up to isomorphism) cubic subfield of the Galois closure $N$ of $L/\mathbb{Q}$.

While the resolvent field of $L$ is unique, non-isomorphic $L$ may share the same resolvent field.

Following Section 3.1 in [1], we make the following definition.

Definition 2.2. Let $p$ be a rational prime. A quartic field $L$ is called overramified at $p$ if $L \otimes \mathbb{Q}_p$ is either irreducible and ramified or the direct sum of two ramified fields. The field $L$ is called nowhere overramified if $L$ is not overramified at $p$ for any prime $p$.

Remark 2.3. Note that unlike Bhargava’s definition in [1], our definition of nowhere overramified does not include any restriction on the ramification of $L$ at infinity.

The fields $L$ and $\text{Res}(L)$ have the same discriminant precisely when $L$ is nowhere overramified.

Proposition 2.4. Let $K$ be an $S_3$-cubic field.

(A) The following are in bijective correspondence.

(i) The set of index two subgroups of $\text{Cl}(K)$.

(ii) The set of unramified quadratic extensions $F$ of $K$.

(iii) The set of isomorphism classes of nowhere overramified $S_4$-quartic fields $L$ having a real place with $\text{Res}(L) = K$.

(B) The following are in bijective correspondence.

(i) The set of index two subgroups of $\text{Cl}(K)^+$.

(ii) The set of quadratic extensions $F$ of $K$ that are unramified at all finite places.

(iii) The set of isomorphism classes of nowhere overramified $S_4$-quartic fields $L$ with $\text{Res}(L) = K$. 
Remark 2.5. In both Proposition 2.4 and Proposition 2.6 which follows, if \( \text{Cl}(K) \) be a complex cubic field, then for both (A) and (B), the correspondence (i) \( \leftrightarrow \) (ii) in (B) is detailed in [3] and the correspondence between (ii) and (iii) in (A) comes from restricting this correspondence to the sets in (A). While we do not include a proof of the equivalence between (ii) and (iii) here, we will however describe the maps yielding the correspondence.

We begin with the map (iii) \( \rightarrow \) (ii). Let \( N \) be the Galois closure of \( L/\mathbb{Q} \). By assumption, \( \text{Gal}(N/\mathbb{Q}) \simeq S_4 \). The group \( S_4 \) contains three distinct \( D_4 \) subgroups, all of which are conjugate. The resolvent field \( K = \text{Res}(L) \) may be taken to be the fixed field of any of these \( D_4 \) subgroups. The group \( D_4 \) contains two distinct Klein four subgroups, only one of which contains a transposition when \( D_4 \) is viewed as a subgroup of \( \text{Gal}(N/\mathbb{Q}) \simeq S_4 \). Letting \( V \) be that subgroup, the extension \( F/K \) is given by the fixed field of \( V \).

We next describe the map (ii) \( \rightarrow \) (iii). While the field \( F/\mathbb{Q} \) is not Galois, the lack of ramification (at finite primes) in \( F/K \) forces its Galois closure \( N \) to be an \( S_4 \)-extension of \( \mathbb{Q} \). The group \( S_4 \) contains four distinct \( S_3 \) subgroups, all of which are conjugate. The field \( L \) may be taken to be the fixed field of any of these \( S_3 \) subgroups. \( \square \)

Remark 2.5. In both Proposition 2.4 and Proposition 2.6 which follows, if \( K \) is taken to be a complex cubic field, then \( \text{Cl}(K) = \text{Cl}(K)^+ \) and parts (A) and (B) are equivalent.

Proposition 2.6. Let \( K \) be an \( S_3 \)-cubic field.

(A) The following are in bijective correspondence.

(i) The set of index two subgroups of \( \text{Cl}(K)_S \).

(ii) The set of unramified quadratic extensions \( F \) of \( K \) in which all primes in \( S_K \) split completely.

(iii) The set of isomorphism classes of nowhere overramified \( S_4 \)-quartic fields \( L \) such that \( \text{Res}(L) = K \), \( L \) has a real place, and \( L \otimes \mathbb{Q}_p \) has a component equal to \( \mathbb{Q}_p \) for all \( p \in S \).

(B) The following are in bijective correspondence.

(i) The set of index two subgroups of \( \text{Cl}(K)^+_S \).

(ii) The set of quadratic extensions \( F \) of \( K \) that are unramified at all finite places and in which all primes in \( S_K \) split completely.

(iii) The set of isomorphism classes of nowhere overramified \( S_4 \)-quartic fields \( L \) such that \( \text{Res}(L) = K \) and \( L \otimes \mathbb{Q}_p \) has a component equal to \( \mathbb{Q}_p \) for all \( p \in S \).

Proof. For both (A) and (B), the correspondence (i) \( \leftrightarrow \) (ii) is class field theory. The equivalences (ii) \( \leftrightarrow \) (iii) in (A) and (B) follow from the similar equivalences in Proposition 2.4 and Lemma 2.7 appearing below. \( \square \)

Lemma 2.7. Let \( K \) be an \( S_3 \)-cubic field field and \( F/K \) a quadratic extension unramified at all finite places. If \( L \) is the \( S_4 \)-quartic field corresponding to \( F/K \) under Heilbronn’s correspondence as in Proposition 2.4, then \( L \otimes \mathbb{Q}_p \) has a component equal to \( \mathbb{Q}_p \) if and only if all primes of \( K \) above \( p \) split in \( F/K \).

Proof. Let \( N \) be the Galois closure of \( L/\mathbb{Q} \). If \( L \otimes \mathbb{Q}_p \) has a component equal to \( \mathbb{Q}_p \), then \( \text{Gal}(N_F/\mathbb{Q}_p) \leq S_3 \) for all primes \( P \mid p \) of \( N \). Letting \( p \) be a prime of \( K \) above \( p \), we have \( \text{Gal}(N_P/K_p) \leq \text{Gal}(N/K) \). Since \( \text{Gal}(N_P/K_p) \) must simultaneously embed into a copy of \( S_3 \) and a copy of \( D_4 \) inside of \( S_4 \), we find that \( \text{Gal}(N_P/K_p) \) must be a subgroup of the Klein four group \( \text{Gal}(N/F) \). As a result, \( p \) splits in \( F/K \).
For the opposite direction, we rely on the Artin relation of zeta functions (see [8], for example)

\[ \zeta_L(s) = \frac{\zeta(s)\zeta_F(s)}{\zeta_K(s)}. \]

If all primes of \( K \) above \( p \) split in \( F/K \), then we have \( \zeta_F^{(p)}(s) = \zeta_K^{(p)}(s) \), where \( \zeta^{(p)}(s) \) denotes the part of the Euler product for \( \zeta(s) \) coming from primes above \( p \). As a result, if all primes of \( K \) above \( p \) split in \( F/K \), then (1) yields

\[ \zeta^{(p)}_L(s) = (1 - p^{-s})^{-1} \zeta^{(p)}_K(s). \]

Observe that if \( L \otimes \mathbb{Q}_p \) contains a ramified component, then \( K \otimes \mathbb{Q}_p \) must contain a ramified component of the same degree. As a result, (2) holds even if we restrict to the Euler factors coming from unramified primes. As a result, the Euler product for \( \zeta_L(s) \) contains a factor of \((1 - p^{-s})^{-1}\) coming from an unramified prime above \( p \), and the \( L \otimes \mathbb{Q}_p \) has a component equal to \( \mathbb{Q}_p \). \( \square \)

We therefore get the following corollary:

**Corollary 2.8.** If \( K \) is an \( S_3 \)-cubic field, then

\[
\begin{align*}
(i) & \quad |\text{Cl}(K)_S[2]| = 1 + \left| \left\{ \text{nowhere overramified } S_4\text{-quartic fields } L \text{ (up to iso.) such that } \text{Res}(L) = K, \text{ }\text{ }\text{ }\text{ }\text{ }\text{ }L \text{ has a real place, and } L \otimes \mathbb{Q}_p \text{ has a component equal to } \mathbb{Q}_p \text{ for all } p \in S \right\} \right| \\
(ii) & \quad |\text{Cl}(K)^+_S[2]| = 1 + \left| \left\{ \text{nowhere overramified } S_4\text{-quartic fields } L \text{ (up to iso.) such that } \text{Res}(L) = K \text{ and } L \otimes \mathbb{Q}_p \text{ has a component equal to } \mathbb{Q}_p \text{ for all } p \in S \right\} \right| .
\end{align*}
\]

**Proof.** Since \( \text{Cl}(K)_S \) is a finite abelian group, the number of index two subgroups of \( \text{Cl}(K)_S \) is the same as the number of non-trivial two-torsion elements of \( \text{Cl}(K)_S \). By Proposition 2.6, this is equal to the number of quartic fields \( L \) such that \( \text{Res}(L) = K \), \( L \) has a real place and \( L \otimes \mathbb{Q}_p \) has \( \mathbb{Q}_p \) component for all \( p \in S \). The result for \( \text{Cl}(K)^+_S \) follows similarly. \( \square \)

### 3. Counting Fields

In order to prove Theorems 2 and 4, we will need to be able to count the number of quartic fields of bounded discriminant satisfying a given set of local conditions.

For a set \( \Sigma_p \) of \( \mathbb{Q}_p \)-algebras, define \( \mu_p(\Sigma_p) \) as

\[
\mu_p(\Sigma_p) := \sum_{R \in \Sigma_p} \frac{p - 1}{p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{|\text{Aut}(R)|};
\]

where \( \text{Disc}_p(R) \) is the \( p \)-part of the discriminant of \( R \).

For each \( p \in S \), let \( \Sigma_p \) be a set of non-overramified rank 4 \( \mathbb{Q}_p \)-algebras and set \( \Sigma = (\Sigma_p)_{p \in S} \). For each \( i \in \{0, 2, 4\} \), define \( N^{(i)}(X, \Sigma) \) to be the number of nowhere overramified \( S_4 \)-quartic fields \( L \) (up to isomorphism) such that \( |\text{Disc}(L)| < X \), \( L \) has \( i \) real places, and \( L \otimes \mathbb{Q}_p \in \Sigma_p \) for all \( p \in S \).

We then have the following specialization of a theorem of Bhargava.
Theorem 3.1 (Theorem 1.3 in [2]). For each $i \in \{0, 2, 4\}$,
\begin{equation}
N^{(i)}(X, \Sigma) = \frac{1}{2n_i\zeta(3)} \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\mu_p} \cdot X + o(X)
\end{equation}
where $\mu_p = 1 - \frac{1}{p^i}$ and $n_i = \begin{cases} 8 & \text{if } i = 0 \\ 4 & \text{if } i = 2 \\ 24 & \text{if } i = 4 \end{cases}$.

Proof. This will follow from Theorem 1.3 in [2]. For each prime $p$, define $\hat{\Sigma}_p$ to be the set of all rank 4 non-overramified $\mathbb{Q}_p$-algebras. An easy computation shows that $\mu_p(\Sigma_p) = 1 - \frac{1}{p^i}$. We then set $\hat{\Sigma} = (\hat{\Sigma}_p)_p$.

For each prime $p \in S$, let $\Sigma_p$ be as above and for $p \not\in S$, set $\Sigma_p = \hat{\Sigma}_p$. Applying Theorem 1.3 in [2], we then get
\begin{equation}
N^{(i)}(X, \Sigma) = \frac{1}{n_i} \prod_p \mu_p(\hat{\Sigma}_p) \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\mu_p(\hat{\Sigma}_p)} = N^{(i)}(X, \hat{\Sigma}) \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\mu_p(\hat{\Sigma}_p)}
\end{equation}

However, $N^{(i)}(X, \hat{\Sigma})$ is simply the number of nowhere totally ramified $S_4$-quartic fields $L$ (up to isomorphism) such that $|\text{Disc}(L)| < X$ and $L$ has $i$ real places. By Lemma 27 in [1], this is known to be $\frac{1}{2n_i\zeta(3)} \cdot X + o(X)$, so the result follows.

We may similarly count the number of cubic fields of bounded discriminant satisfying a set of local conditions. For a set of local conditions $\Sigma = (\Sigma_p)_{p \in S}$ where each $\Sigma_p$ is a set of rank 3 $\mathbb{Q}_p$-algebras and each $i \in \{1, 3\}$, define $M^{(i)}(X, \Sigma)$ to be the number of $S_3$-cubic fields $K$ (up to isomorphism) such that $|\text{Disc}(K)| < X$, $K$ has $i$ real places, and $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$.

Theorem 3.2 (Theorem 1.3 in [2]). For each $i \in \{1, 3\}$,
\begin{equation}
M^{(i)}(X, \Sigma) = \frac{1}{2m_i\zeta(3)} \prod_{p \in S} \frac{\mu_p(\Sigma_p)}{\mu_p} \cdot X + o(X)
\end{equation}
where $m_1 = 2$, $m_3 = 6$, and $\mu_p$ is as in Theorem 3.1.

The proof of Theorem 3.2 is extremely similar to that of Theorem 3.1 so we omit it.

4. DEALING WITH LOCAL CONDITIONS

In general, if $K$ is an $S_3$-cubic field and $L$ an $S_4$-quartic field such that the resolvent $\text{Res}(L) = K$, then $K \otimes \mathbb{Q}_p$ does not determine $L \otimes \mathbb{Q}_p$. However, if we further assume that $L \otimes \mathbb{Q}_p$ has a $\mathbb{Q}_p$ component, then this is no longer the case.

Lemma 4.1. Let $K$ be an $S_3$-cubic field and $L$ an $S_4$-quartic field such that the resolvent $\text{Res}(L) = K$. If $p$ is a prime such that $L \otimes \mathbb{Q}_p$ has a component equal to $\mathbb{Q}_p$, then $L \otimes \mathbb{Q}_p \simeq (K \otimes \mathbb{Q}_p) \oplus \mathbb{Q}_p$.

Proof. Let $N$ be the Galois closure of $L/\mathbb{Q}$. Since $L \otimes \mathbb{Q}_p$ has a component equal to $\mathbb{Q}_p$, we have $\text{Gal}(N_p/\mathbb{Q}_p) \leq S_3$ for any prime $P$ of $N$ above $P$.

Let $\tilde{K}$ be the Galois closure of $K/\mathbb{Q}$ contained in $N/\mathbb{Q}$ and let $p$ be any prime above $p$ in $\tilde{K}$. The Galois group of $N/\tilde{K}$ is the unique order four normal subgroup $V$ of $S_4$. Therefore, if $P$ is any prime of $N$ above $p$, we have $\text{Gal}(N_P/\tilde{K}_p) \leq V$. However, the subgroups $V$ and
Let $\mathcal{S}_3$ of $\mathcal{S}_4$ intersect trivially. As a result, we have $\text{Gal}(N_p/\tilde{K}_p) = 0$ and $p$ splits completely in $N/\tilde{K}$.

We therefore see that $N \otimes \mathbb{Q}_p = (\tilde{K} \otimes \mathbb{Q}_p)^4$. Taking $\text{Gal}(N/L)$ invariants, we get that $L \otimes \mathbb{Q}_p = (K \otimes \mathbb{Q}_p) \oplus \mathbb{Q}_p$. \qed

Lemma 4.1 motivates the following definition. Given a rank $3$ $\mathbb{Q}_p$-algebra $R_p$, we define a rank $4$ $\mathbb{Q}_p$-algebra $\tilde{R}_p$ as $\tilde{R}_p = R_p \oplus \mathbb{Q}_p$. We then get the following corollary.

Proposition 4.2. Let $\Sigma = (\Sigma_p)_{p \in S}$ where each $\Sigma_p$ is a set of rank $3$ $\mathbb{Q}_p$-algebras. Set

$$\tilde{\mu}(\Sigma) = \prod_{p \in S} \mu_p(\Sigma_p)/\mu_p(\Sigma),$$

where $\Sigma_p = \{R_p : R_p \in \Sigma_p\}$. Then

(i) the average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over totally real $\mathcal{S}_3$-cubic fields with $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$ is equal to $1 + \frac{1}{2}\tilde{\mu}(\Sigma)$

(ii) the average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over complex $\mathcal{S}_3$-cubic fields with $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$ is equal to, and $1 + \frac{1}{2}\tilde{\mu}(\Sigma)$

(iii) the average size of $\text{Cl}(K)_S^+[2]$ as $K$ ranges over totally real $\mathcal{S}_3$-cubic fields with $K \otimes \mathbb{Q}_p \in \Sigma_p$ for all $p \in S$ is equal to $1 + \tilde{\mu}(\Sigma)$.

Proof. By combining Corollary 2.8 with Lemma 4.1 we have

$$\sum_{\text{K cubic, } K \otimes \mathbb{Q}_p \simeq R_p \text{ for all } p \in S} |\text{Cl}(K)_S[2]| = M^{(3)}(X, \Sigma) + N^{(4)}(X, \tilde{\Sigma}).$$

Part (i) then follows from Theorems 3.1 and 3.2. Part (ii) follows from an essentially identical calculation.

For part (iii), again by Corollary 2.8 combined with Lemma 4.1 we have

$$\sum_{\text{K cubic, } K \otimes \mathbb{Q}_p \simeq R_p \text{ for all } p \in S} |\text{Cl}(K)_S^+[2]| = M^{(3)}(X, \Sigma) + N^{(4)}(X, \tilde{\Sigma}) + N^{(0)}(X, \tilde{\Sigma}).$$

The result then follows from Theorems 3.1 and 3.2 \qed

The quotients $\mu_p(\Sigma_p)/\mu_p(\Sigma_p)$ have a nice formula when either $\Sigma_p = \{R_p\}$ or $\Sigma_p$ contains all rank $3$ $\mathbb{Q}_p$-algebras (up to isomorphism).

Lemma 4.3. If $\Sigma_p = \{R_p\}$, then $\mu_p(\Sigma_p)/\mu_p(\Sigma_p) = 2^{-(r_p-1)}$, where $R_p$ is the number of irreducible components of $R_p$.

Proof. Since $\tilde{R}_p = R_p \oplus \mathbb{Q}_p$, we have $\text{Disc}_p(\tilde{R}_p) = \text{Disc}_p(R_p)$, so

$$\mu_p(\{\tilde{R}_p\})/\mu_p(\{R_p\}) = |\text{Aut}(R_p)|/|\text{Aut}(\tilde{R}_p)| = n!/((n+1)! = 1/(n+1),$$

where $n$ is the number of components of $R_p$ equal to $\mathbb{Q}_p$. Examination shows that this is equal to $2^{-(r_p-1)}$. \qed
Lemma 4.4. If $\Sigma_p$ contains all rank 3 $\mathbb{Q}_p$-algebras, then $\mu_p(\overline{\Sigma}_p)/\mu_p(\Sigma_p) = \frac{1}{2} \left( 1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right)$.

Proof. By Lemma 4.3 we have $\mu_p(\{\overline{R}_p\})/\mu_p(\{R_p\}) = 2^{-(r-1)}$ for each $R_p$ with $r$ irreducible components. Letting $\Sigma_{p,r}$ denote the set of all rank 3 $\mathbb{Q}_p$-algebras with $r$ irreducible components, we get

$$\mu_p(\overline{\Sigma}_p) = \sum_{R_p \in \Sigma_p} \mu_p(\overline{\Sigma}_{p,r}) = \sum_{r=1}^{3} 2^{-(r-1)} \mu_p(\Sigma_{p,r}).$$

Calculating $\mu_p(\Sigma_{p,r})$ for each $r \in \{1, 2, 3\}$, we find

$$\mu_p(\Sigma_{p,1}) = \frac{p^3 - p^2 + 3p - 3}{3p^3}, \mu_p(\Sigma_{p,2}) = \frac{p^2 + p - 2}{2p^2}, \text{ and } \mu_p(\Sigma_{p,3}) = \frac{p - 1}{6p}.$$

As a result, we get $\mu_p(\overline{\Sigma}_p) = \frac{p^3 - p^2 + 4p - 8}{8p^3}$, so

$$\mu_p(\overline{\Sigma}_p)/\mu_p(\Sigma_p) = \frac{p^3 - p^2 + 4p - 8}{8p^3} - \frac{p^3}{p^3 - 1} = \frac{5p^2 + 4p + 8}{8(p^2 + p + 1)} = \frac{1}{2} \left( 1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right).$$

\qed

We are now able to prove Theorems 2 and 4.

Proof of Theorem 2. For each $p \in S$, let $\Sigma_p$ be the set of all rank 3 $\mathbb{Q}_p$ algebras. The result then follows from combining Proposition 4.2 with Lemma 4.4.

Corollary 3 will now follow from Theorem 2.

Proof of Corollary 3. Let $\lambda$ be the proportion of totally real $\mathcal{S}_3$-cubic fields $K$ having $\text{Cl}(K)_S^+ \neq \{2\}$. We then have $\text{Avg}(\text{Cl}(K)_S^+ \{2\}) \geq \lambda + 2 \cdot (1 - \lambda) = 2 - \lambda$. By Theorem 2 we have

$$\text{Avg}(\text{Cl}(K)_S^+ \{2\}) = 1 + \frac{1}{2|S|} \prod_{p \in S} \left( 1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right),$$

so it follows that

$$\lambda \geq 1 - \frac{1}{2|S|} \prod_{p \in S} \left( 1 + \frac{p^2 + 4}{4(p^2 + p + 1)} \right).$$

If $S$ is non-empty, then the right-hand side of (5) is at least $\frac{5}{14}$, proving (i).

To see (ii), observe that the kernel of the surjection $\text{Cl}(K)_S^+ \to \text{Cl}(K)_S$ is a subgroup of $\text{Cl}(K)_S^+$ having order at most two. As a result, if $\text{Cl}(K)_S^+$ has odd order, then $\text{Cl}(K)_S^+ = \text{Cl}(K)_S$. The result then follows from (i).

Finally, we note that $K$ has $S_K$-units of all signatures if and only if $\text{Cl}(K)_S^+ = \text{Cl}(K)_S$. Part (iii) of the corollary therefore follows from (ii).

Proof of Theorem 4. For each $p \in S$, let $\Sigma_p = \{R_p\}$. By Lemma 4.3, for each $p \in S$, we have $\mu_p(\overline{\Sigma}_p)/\mu_p(\Sigma_p) = 2^{-(r_p-1)}$. Letting $r = \sum_{p \in S}(r_p - 1)$, we have $\mu(\Sigma) = 2^{-r}$ and the result follows from Proposition 4.2.
5. Selmer Groups

**Definition 5.1.** Let $S$ be a set of rational primes. The 2-Selmer group of $K$ relaxed at $S$, denoted $\text{Sel}^2_S(K)$ is defined as

$$\text{Sel}^2_S(K) := \{ \alpha \in K^\times/(K^\times)^2 : \text{val}_p(\alpha) \equiv 0 \pmod{2} \text{ for all } p \not\in S_K \}.$$ 

where $S_K$ is the set of primes of $K$ lying about $S$.

A standard result (see Section 8.3.2 of [5], for example) shows that $\text{Sel}^2_S(K)$ sits in the short exact sequence

$$0 \to \mathcal{O}_{K,S}^\times/(\mathcal{O}_{K,S}^\times)^2 \to \text{Sel}^2_S(K) \to \text{Cl}(K)_S[2] \to 0,$$

where $\mathcal{O}_{K,S}^\times$ is the $S_K$ units of $\mathcal{O}_K$.

Define a function $\nu_S$ on $S_3$-cubic fields by $\nu_S(K) = \sum_{p \in S} r_p(K)$, where $r_p(K)$ is the number of irreducible components of $K \otimes \mathbb{Q}_p$. Dirichlet’s unit theorem then tells us that $|\text{Sel}^2_S(K)| = 2^{\nu_S(K) + 3} |\text{Cl}(\mathcal{O}_K)_S[2]|$ if $K$ is totally real and $|\text{Sel}^2_S(K)| = 2^{\nu_S(K) + 2} |\text{Cl}(\mathcal{O}_K)_S[2]|$ if $K$ is complex.

5.1. **An averaging result for** $\nu_S(K)$. To compute the average size of $\text{Sel}^2_S(K)$, we would there like to calculate the average value of $2^{\nu_S(K)} |\text{Cl}(K)_S[2]|$. We begin with the following lemma.

**Lemma 5.2.** When ordered by absolute discriminant, the average value of $2^{\nu_S(K)}$ as $K$ ranges over totally real (resp. complex) $S_3$-cubic fields is $2^{[S]} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right)$.

**Proof.** We will only prove the totally real case, since the complex case is identical. We proceed by induction on the set $S$. For the base case, consider the case where $S$ contains a single prime $p$.

For each $r \in \{1, 2, 3\}$, define $\rho_p(r)$ to be the proportion of totally real $S_3$-cubics $K$ such that $r_p = r$. By Theorem 3.2, we have $\rho_p(r) = \frac{\mu_p(\Sigma_{p,r})}{\mu_p}$ where $\Sigma_{p,r}$ is as in the proof of Lemma 4.4 and $\mu_p = 1 - \frac{1}{p^r}$. We then get

$$\text{Avg}(2^r) = \sum_{r=1}^{3} 2^r \cdot \rho_p(r) = \frac{p^3}{p^3 - 1} \left( 2 \cdot \frac{p^3 - p^2 + 3p - 3}{3p^3} + 4 \cdot \frac{p^2 + p - 2}{2p^2} + 8 \cdot \frac{p - 1}{6p} \right) = 4 - \frac{2}{p^2 + p + 1} = 2 \cdot \left( 2 - \frac{1}{p^2 + p + 1} \right).$$

For the inductive step, let $S' = S \cup \{p'\}$ for some prime $p' \not\in S$. We then have

$$\text{Avg}(2^{\nu_{S'}(K)}) = \sum_{s} \left( 2^s \cdot \text{Prob}(\nu_S(K) = s) \cdot \sum_{n=1}^{3} 2^n \cdot \text{Prob}(r_{p'} = r|\nu_S(K) = s) \right).$$
By Theorem 3.2, the splitting type of $K \otimes \mathbb{Q}_p$ is independent from the splitting type of $K \otimes \mathbb{Q}_p$ for all $p \in S$, and therefore independent of $\nu_S(K)$. As a result, we get

\[
\text{(9)} \quad \text{Avg}(2^{\nu_S(K)}) = \left( \sum_s 2^s \cdot \text{Prob}(\nu_S(K) = s) \right) \left( \sum_{n=1}^3 2^n \cdot \rho_p(r) \right)
\]

\[
= \text{Avg}(2^{\nu_S(K)}) \left( \sum_{n=1}^3 2^n \cdot \rho_p(r) \right) = \text{Avg}(2^{\nu_S(K)}) \cdot 2 \cdot \left( 2 - \frac{1}{p^2 + p' + 1} \right),
\]

where the final equality follows from (7). By the inductive hypothesis, we have

\[
\text{(10)} \quad \text{Avg}(2^{\nu_S(K)}) = 2^{|S|} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right)
\]

Combining (9) with (10) yields the result. \qed

5.2. Averaging for fixed $\nu_S(K)$. In the setting of Theorem 4, we are given a fixed rank 3 $\mathbb{Q}_p$-algebra $R_p$ for each $p \in S$ and we are able to compute the average size of of $\text{Cl}(K)_S[2]$ as we range over $S_3$-cubic fields $K$ with a fixed signature such that $K \otimes \mathbb{Q}_p \simeq R_p$ for every $p \in S$.

As seen in Theorem 4, the average size does not depend on the collection of $R_p$, only on $r = \sum_{p \in S} (r_p - 1)$. As a result, we get the following:

**Proposition 5.3.** Let $s$ be an integer with $|S| \leq s \leq 3|S|$.

(i) The average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over totally real $S_3$-cubic fields with $\nu_S(K) = s$ is equal to $1 + 2^{-(s-|S|)+2}$,

(ii) the average size of $\text{Cl}(K)_S[2]$ as $K$ ranges over complex $S_3$-cubic fields with $\nu_S(K) = s$ is equal to $1 + 2^{-(s-|S|)+1}$, and

(iii) the average size of $\text{Cl}(K)_S^+[2]$ as $K$ ranges over totally real $S_3$-cubic fields with $\nu_S(K) = s$ is equal to $1 + 2^{-(s+|S|)}$.

**Proof.** We have $\sum_{p \in S} (r_p - 1) = \nu_S(K) - |S|$. The result then follows directly from Theorem 4. \qed

We then get the following corollary:

**Corollary 5.4.** Let $s$ be an integer with $|S| \leq s \leq 3|S|$. When ordered by absolute discriminant,

(i) the average value of $2^s |\text{Cl}(K)_S[2]|$ as $K$ ranges over totally real cubic fields with $\nu_S(K) = s$ is equal to $2^s + 2^{3|S|-2}$,

(ii) the average value of $2^s |\text{Cl}(K)_S[2]|$ as $K$ ranges over complex cubic fields with $\nu_S(K) = s$ is equal to $2^s + 2^{3|S|-1}$, and

(iii) the average value of $2^s |\text{Cl}(K)_S^+[2]|$ as $K$ ranges over totally real cubic fields with $\nu_S(K) = s$ is equal to $2^s + 2^{3|S|}$.

**Proof.** This follows immediately from Proposition 5.3. \qed
5.3. Proof of Theorem 6. We are finally able to calculate the average of $2^{\nu_S(K)}|\text{Cl}(K)_S[2]|$.

**Proposition 5.5.** When ordered by absolute discriminant,

(i) the average of $2^{\nu_S(K)}|\text{Cl}(K)_S[2]|$ as $K$ ranges over totally real $S_3$-cubic fields is equal to

$$2^{[S]-2} + 2^{|S|} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right),$$

(ii) the average of $2^{\nu_S(K)}|\text{Cl}(K)_S[2]|$ as $K$ ranges over complex $S_3$-cubic fields is

$$2^{[S]-1} + 2^{|S|} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right),$$

and

(iii) the average of $2^{\nu_S(K)}|\text{Cl}(K)_S[2]|$ as $K$ ranges over totally real $S_3$-cubic fields is equal to

$$2^{|S|} + 2^{|S|} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right).$$

**Proof.** We only prove part (i), since the proofs of parts (ii) and (iii) are nearly identical. For each integer $s$ with $|S| \leq s \leq 3|S|$, define $\rho(s)$ to be the proportion of $S_3$-cubics $K$ such that $\nu_S(K) = s$. By Corollary 5.4, we then have

$$\text{Avg}(2^{\nu_S(K)}|\text{Cl}(K)_S[2]|) = \sum_{s=|S|}^{3|S|} (2^s + 2^{[S]-2}) \cdot \rho(s) = \sum_{s=|S|}^{3|S|} 2^{[S]-2} \cdot \rho(s) + \sum_{s=|S|}^{3|S|} 2^s \cdot \rho(s)$$

$$= 2^{[S]-2} + \sum_{s=|S|}^{3|S|} 2^s \cdot \rho(s) = 2^{[S]-2} + 2^{|S|} \prod_{p \in S} \left( 2 - \frac{1}{p^2 + p + 1} \right),$$

where the final equality follows from Lemma 5.2. □

We are now able to prove Theorem 6.

**Proof of Theorem 6.** As noted in the beginning of this section, we have $|\text{Sel}_2^S(K)| = 2^{\nu_S(K)+3}|\text{Cl}(O_K)_S[2]|$ if $K$ is totally real and $|\text{Sel}_2^S(K)| = 2^{\nu_S(K)+2}|\text{Cl}(O_K)_S[2]|$ if $K$ is complex. The result then follows from Proposition 5.5. □

6. K-groups

We are able to use Theorem 6 to study the average size of $K_{2n}(O_K)[2]$ by appealing to the following result of Rognes and Weibel [11].

**Theorem 6.1 (Theorem 0.7 in [11]).** Let $K$ be a number field and set $S = \{2\}$. For even $n > 0$, the 2-rank of $K_{2n}(O_K)[2]$ is given by

$$\dim_{\mathbb{F}_2} K_{2n}(O_K)[2] = \begin{cases} \dim_{\mathbb{F}_2} \text{Cl}(K)_S[2] + r_p - 1 & n \equiv 0 \pmod{4} \\ \dim_{\mathbb{F}_2} \text{Cl}(K)_S[2] + r_1 + r_p - 1 & n \equiv 1 \pmod{4} \\ \dim_{\mathbb{F}_2} \text{Cl}(K)_S[2] + r_p - 1 & n \equiv 2 \pmod{4}, n \equiv 3 \pmod{4} \end{cases}$$

where $r_p$ is the number of places above 2 in $K$.

To prove Theorem 6 we therefore want to calculate the average value of $2^{\nu_S} \cdot |\text{Cl}(K)_S[2]|$. 

Lemma 6.2. Let $S = \{2\}$. When ordered by absolute discriminant,

(i) the average value of $2^{r_p} |\text{Cl}(K)_S[2]|$ as $K$ ranges over totally real $S_3$-cubic fields is $\frac{59}{11}$,

(ii) the average value of $2^{r_p} |\text{Cl}(K)_S[2]|$ as $K$ ranges over complex $S_3$-cubic fields is $\frac{33}{7}$, and

(iii) the average value of $2^{r_p} |\text{Cl}(K)_S^+[2]|$ as $K$ ranges over totally real $S_3$-cubic fields is $\frac{40}{7}$.

Proof. This follows immediately from Proposition 5.5.

We are now able to prove Theorem 5.1

Proof of Theorem 5.1 Combine Theorem 6.1 with Lemma 6.2.

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