Non-Archimedean Normal Operators

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Abstract

We describe some classes of linear operators on Banach spaces over non-Archimedean fields, which admit orthogonal spectral decompositions. Several examples are given.

MSC 2000. Primary: 47S10. Secondary: 11S80; 81Q99

1 INTRODUCTION

Non-Archimedean functional analysis is a well-developed branch of mathematics comparable to its classical counterpart dealing with spaces over $\mathbb{R}$ and $\mathbb{C}$; see, for example, the monographs [8, 22, 26, 31] and the survey papers [14, 24, 30]. This includes some basic information on non-Archimedean Banach spaces, and a rather complete theory of compact operators (Serre [28]). A new stimulus for the development of non-Archimedean operator theory was given by recent attempts to develop $p$-adic models of quantum mechanics with $p$-adic valued wave functions [16, 1]. In contrast to the classical situation, most of the interesting examples deal with bounded operators. In particular, there exist bounded $p$-adic representations of the canonical commutation relations of quantum mechanics [6, 18].

As in any kind of operator theory, a central problem is a construction and study of spectral decompositions. In the non-Archimedean case, there are several results in this direction [32, 7, 20]. In particular, analogs of spectral operators of scalar type were found. However, no class of operators resembling normal operators on Hilbert spaces (with orthogonal, in an appropriate non-Archimedean sense, spectral decompositions) is known. The main difficulties are the absence of nontrivial involutions on non-Archimedean fields coordinated with their algebraic structure, and the absence of inner products coordinated with the norms, on non-Archimedean Banach spaces. In [9], examples of symmetric matrices over the field $\mathbb{Q}_p$ of $p$-adic numbers are given, which cannot be diagonalized over any extension of $\mathbb{Q}_p$. For other examples of unusual behavior of $p$-adic matrices see [1]. Thus, already the non-Archimedean linear algebra is quite
different from the classical one ("exotic" exceptions appearing for some fields with infinite rank valuations will not be considered in this paper; see \[15\]).

In this paper we propose a new approach to the above problem. We consider separately the cases of finite-dimensional spaces (where lesser restrictions upon the underlying field are imposed) and infinite-dimensional spaces. For both situations, we obtain spectral theorems comparable to the classical ones; their conditions are especially transparent for finite matrices and compact operators. A number of examples are considered. As it could be expected, the structure substituting the non-existing ones, is the possibility of the reduction procedure – from a space over a non-Archimedean field to a space over its residue field.

Our method follows the well-known idea of deriving the spectral theorem from the representation theorem for an appropriate commutative Banach algebra. For our situation, the crucial result regarding a class of non-Archimedean Banach algebras was obtained by Berkovich \[7\].

The structure of this paper is as follows. In Section 2, we recall some notions and results from non-Archimedean analysis, especially from the theory of non-Archimedean Banach algebras. For the latter, we follow the approach by Berkovich \[7\]; for other methods and the history of this subject see \[11\]. For a detailed exposition of a variety of topics from non-Archimedean analysis, see \[25, 29\].

In Section 3, we describe the reduction of our problem to the study of the Banach algebra generated by a linear operator. The spectral theorem for the finite-dimensional case is proved in Section 4, while the infinite-dimensional case is considered in Section 5. Section 6 is devoted to examples.

2 PRELIMINARIES

2.1. Let \( \mathcal{A} \) be a ring with identity 1. A \textit{seminorm} on \( \mathcal{A} \) is a function \( \| \cdot \| : \mathcal{A} \to \mathbb{R}_+ \) possessing the following properties: \( \| 0 \| = 0, \| 1 \| = 1, \| f + g \| \leq \| f \| + \| g \|, \| fg \| \leq \| f \| \cdot \| g \| \), for any \( f, g \in \mathcal{A} \). A seminorm is a \textit{norm}, if the equality \( \| f \| = 0 \) holds only for \( f = 0 \). A seminorm is called \textit{multiplicative}, if \( \| fg \| = \| f \| \cdot \| g \| \) for any \( f, g \in \mathcal{A} \). A multiplicative norm is called a \textit{valuation}. Any norm defines a metric, thus a topology on \( \mathcal{A} \), in a standard way. A Banach ring is a normed ring complete with respect to its norm.

A seminorm \( \| \cdot \| \) is called \textit{non-Archimedean}, if \( \| f + g \| \leq \max(\| f \|, \| g \|) \), \( f, g \in \mathcal{A} \).

A \textit{valuation field} is a commutative Banach field whose norm is a valuation. In particular, a complete non-Archimedean field is a valuation field with a non-Archimedean valuation (in this terminology the completeness property is included in the notion of a valuation field). Below we consider only fields with nontrivial valuations, that is the valuations taking only the values 0 and 1 are excluded.

The simplest and most important example of a valuation field is the field \( \mathbb{Q}_p \) of \( p \)-adic numbers where \( p \) is a prime number. \( \mathbb{Q}_p \) is a completion of the field \( \mathbb{Q} \) of rational numbers with respect to the norm (the norm of a valuation field is called an absolute value) \( \| x \|_p = p^{-N} \) where the rational number \( x \neq 0 \) is presented as \( x = p^N \frac{\xi}{\eta} \), \( N, \xi, \eta \in \mathbb{Z} \), and \( p \) does not divide \( \xi, \eta \).

This absolute value can be extended to any wider field (see \[29\]), in particular, to an algebraic closure of \( \mathbb{Q}_p \), and then to the completion \( \mathbb{C}_p \) of the algebraic closure. It is important that \( \mathbb{C}_p \) is algebraically closed.
A Banach ring \( A \) (with the norm \( \| \cdot \| \) ) that is an algebra over a non-Archimedean field \( k \) (with the absolute value \( | \cdot | \) ) is called a non-Archimedean Banach algebra, if \( \| \lambda f \| = | \lambda | \cdot \| f \| \) for any \( \lambda \in k, f \in A \).

Below we have to deal with multiplicative seminorms on a Banach ring \( A \) where a norm \( \| \cdot \| \) defining the structure of a normed ring is already fixed. It will be convenient to denote such seminorms by \( | \cdot |_s \) where the meaning of the index will be clear later. In this situation, a multiplicative seminorm \( | \cdot |_s \) is called bounded if \( | f |_s \leq \| f \| \) for all \( f \in A \); this definition is in fact equivalent to a seemingly wider one, with the inequality \( | f |_s \leq C \| f \| \) (\( C > 0 \) does not depend on \( f \)); see [7].

Note that a bounded multiplicative seminorm \( | \cdot |_s \) on a non-Archimedean Banach algebra \( A \) has the property that \( | \lambda \cdot 1 |_s = | \lambda |_s \), so that it coincides with the absolute value on \( k \), if \( k \) is considered as a subfield of \( A \). Indeed, \( | \cdot |_s \) induces another absolute value on \( k \), and \( | \lambda |_s \leq | \lambda | \) for any \( \lambda \in k \). Since the valuation \( | \cdot | \) is nontrivial, we find that \( | \lambda |_s = | \lambda | \) for all \( \lambda \in k \) ([29, Exercise 9.C]; see also Proposition 11.2 in [10]).

Let \( O \subset k \) be the ring of integers, that is \( O = \{ \lambda \in k : | \lambda | \leq 1 \} \). The set \( P = \{ \lambda \in k : | \lambda | < 1 \} \) is a maximal ideal in \( O \). The quotient ring \( \hat{k} = O / P \) is in fact a field called the residue field of \( k \). In particular, if \( k \) is locally compact (such non-Archimedean fields are called local), then \( \hat{k} \) is a finite field. For example, if \( k = \mathbb{Q}_p \), then \( \hat{k} = \mathbb{F}_p \), the field with exactly \( p \) elements. If \( k = \mathbb{C}_p \), then \( \hat{k} \) is an algebraic closure of \( \mathbb{F}_p \). For \( k = \mathbb{Q}_p \), there is a standard notation \( O = \mathbb{Z}_p \); in this case \( P = p\mathbb{Z}_p \).

A vector space \( \mathcal{B} \) over a non-Archimedean valuation field \( k \) is called a Banach space, if it is endowed with a norm \( \| x \| \), \( x \in \mathcal{B} \), with values in \( \mathbb{R}_+ \), such that \( \| x \| = 0 \) if and only if \( x = 0 \), \( \| \lambda x \| = | \lambda | \cdot \| x \| \), \( \| x + y \| \leq \max(\| x \|, \| y \|) \) \((x, y \in \mathcal{B}, \lambda \in k)\), and \( \mathcal{B} \) is complete as a metric space where the metric is given by the norm. The conjugate space \( \mathcal{B}^* \) consists of all continuous linear functionals \( \mathcal{B} \to k \).

Below we will consider Banach spaces over \( k \) possessing orthonormal bases, that is such families \( \{ e_j \}_{j \in J} \) that each element \( x \in \mathcal{B} \) has a unique representation as a convergent series \( x = \sum c_j e_j, c_j \in k, | c_j | \to 0 \) (by the filter of complements to finite sets), and

\[ \| x \| = \sup_{j \in J} | c_j |. \]

Conditions for the existence of such bases (formulated for abstract spaces) are well known; see [25, 26]. To simplify matters, we will consider the finite-dimensional spaces \( k^n, n \in \mathbb{N} \), and the infinite-dimensional space \( c(J,k) \) of \( k \)-valued sequences \( \{ a_j \}, j \in J \), tending to zero by the filter of complements of finite subsets of the set \( J \); in both cases the supremum norm is used. In examples, we will deal with some function spaces with explicitly given bases.

2.2. Let \( A \) be a non-Archimedean commutative Banach algebra over a complete non-Archimedean field \( k \) with a nontrivial valuation. Its spectrum \( \mathcal{M}(A) \) is defined as the set of all bounded multiplicative seminorms on \( A \) (denoted \( | \cdot |_s \), \( s \in \mathcal{M}(A) \)). The set \( \mathcal{M}(A) \) is endowed with the weakest topology, with respect to which all the mappings \( \mathcal{M}(A) \to \mathbb{R}, | \cdot |_s \mapsto | T |_s \) \((T \in A)\) are continuous. The spectrum \( \mathcal{M}(A) \) is a nonempty Hausdorff compact topological space.

For \( | \cdot |_s \in \mathcal{M}(A) \), denote \( P_s = \{ T \in A : | T |_s = 0 \} \). The set \( P_s \) is a closed prime ideal of \( A \). The value \( | T |_s \) depends only on the residue class of \( T \) in \( A/P_s \). The resulting valuation on the integral domain \( A/P_s \) extends to a valuation on its fraction field \( \mathcal{F}_s \). Let the valuation field
\( \mathcal{H}(s) \) be the completion of \( \mathcal{F}_s \) with respect to the above valuation. Denote by \( T(s) \) the image of an element \( T \in \mathcal{A} \) in \( \mathcal{H}(s) \). The homomorphism
\[
\mathcal{A} \to \prod_{s \in \mathcal{M}(\mathcal{A})} \mathcal{H}(s), \quad T \mapsto (T(s))_{s \in \mathcal{M}(\mathcal{A})},
\]
is called the Gelfand transform.

Let \( K \) be an arbitrary non-Archimedean valuation field. A nonzero continuous homomorphism \( \chi : \mathcal{A} \to K \) is called a character of the ring \( \mathcal{A} \). Two characters \( \chi' : \mathcal{A} \to K' \) and \( \chi'' : \mathcal{A} \to K'' \) are called equivalent, if there exist such a character \( \chi : \mathcal{A} \to K \) and isometric monomorphisms \( K \to K' \) and \( K \to K'' \) that the diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\chi} & K' \\
\downarrow & & \downarrow \\
K & \xrightarrow{\chi'} & K''
\end{array}
\]
is commutative.

The spectrum \( \mathcal{M}(\mathcal{A}) \) may be interpreted as the set of equivalence classes of characters: a character \( \chi \) generates a seminorm \( T \mapsto |\chi(T)| \), while each seminorm \( s \in \mathcal{M}(\mathcal{A}) \) generates, via the Gelfand transform, a character \( T \mapsto T(s) \in \mathcal{H}(s) \). In [7], this description is given for general Banach rings, and ring characters are used. For commutative non-Archimedean Banach algebras (where \( P_s \) is an ideal of \( \mathcal{A} \) as an algebra, thus the Gelfand transform is an algebra homomorphism), it is sufficient to consider the algebra characters.

A non-Archimedean commutative Banach algebra \( \mathcal{A} \) over \( k \) is called uniform, if \( \|T^2\| = \|T\|^2 \) for any \( T \in \mathcal{A} \). The simplest example of a uniform algebra is the algebra \( C(M, k) \) of all continuous functions on a compact set with values in \( k \), endowed with the supremum norm.

The following result ([7, Corollary 9.2.7]) will be our main tool in the sequel.

**Theorem A** (Berkovich). Let \( \mathcal{A} \) be a uniform commutative Banach algebra over \( k \) with identity, such that all the characters of \( \mathcal{A} \) take values in \( k \). Then the space \( \mathcal{M}(\mathcal{A}) \) is totally disconnected, and the Gelfand transform gives an isomorphism \( \mathcal{A} \xrightarrow{\sim} C(\mathcal{M}(\mathcal{A}), k) \).

### 3 BANACH ALGEBRA OF A BOUNDED OPERATOR

#### 3.1

Let \( A \) be a bounded linear operator on a non-Archimedean Banach space \( \mathcal{B} \) over a non-Archimedean valuation field \( k \) with the absolute value \( |\cdot| \). Denote by \( \mathcal{L}_A \) the commutative closed subalgebra in the algebra \( \mathcal{L}(\mathcal{B}) \) of all bounded operators on \( \mathcal{B} \) generated by the operators \( A \) and \( I \) (the identical operator). The algebra \( \mathcal{L}_A \) is the closure, with respect to the operator
norm, of the algebra \( k[A] \) of polynomials of \( A \). The elements \( \lambda \in k \) are identified with the operators \( \lambda I \). Denote by \( \mathcal{M}_A \) the spectrum of the algebra \( \mathcal{L}_A \).

Suppose that the algebra \( \mathcal{L}_A \) is uniform, and all its characters take values in \( k \). By Theorem A, the space \( \mathcal{M}_A \) is totally disconnected, and \( \mathcal{L}_A \) is isomorphic to \( C(\mathcal{M}_A, k) \). Locally constant functions \( \eta : \mathcal{M}_A \to k \), that is functions constant on a neighbourhood of each point, are finite linear combinations of characteristic functions \( \eta_\Lambda \) of disjoint open-closed subsets \( \Lambda \subset \mathcal{M}_A \). The set of all such functions is dense in \( C(\mathcal{M}_A, k) \) (see Corollary 9.2.6 in [7] or Theorem 26.2 in [29]). Thus, if \( f \in C(\mathcal{M}_A, k) \), then, for any \( \varepsilon > 0 \), there exists such a locally constant function

\[
f_\varepsilon(x) = \sum_{i=1}^{n_\varepsilon} c_i \eta_{\Lambda_i}(x), \quad \bigcup_{i=1}^{n_\varepsilon} \Lambda_i = \mathcal{M}_A,
\]

that \( \max_{x \in \mathcal{M}_A} |f(x) - f_\varepsilon(x)| < \varepsilon \). In particular, if \( x_i \in \Lambda_i \), then \( |f(x_i) - c_i| < \varepsilon \), whence

\[
\max_{x \in \mathcal{M}_A} \left| f(x) - \sum_{i=1}^{n_\varepsilon} f(x_i) \eta_{\Lambda_i}(x) \right| < \varepsilon. \tag{1}
\]

Under the isomorphism \( \mathcal{L}_A \cong C(\mathcal{M}_A, k) \), the characteristic functions \( \eta_\Lambda \) correspond to idempotent operators \( E(\Lambda) \in \mathcal{L}_A \), \( \|E(\Lambda)\| = 1 \) (if \( \Lambda \) is nonempty), and we may write the inequality (1) in the form

\[
\left\| f(A) - \sum_{i=1}^{n_\varepsilon} f(x_i) E(\Lambda_i) \right\| < \varepsilon
\]

where the operator \( f(A) \) corresponds to the function \( f \). Interpreting this approximation procedure as integration (compare with Appendix A5 in [29]) we may write

\[
f(A) = \int_{\mathcal{M}_A} f(\lambda)E(d\lambda), \quad f \in C(\mathcal{M}_A, k), \tag{2}
\]

where \( E(\cdot) \) is a finitely additive norm-bounded (by 1) projection-valued measure.

In particular, we have the decomposition of unity

\[
I = \int_{\mathcal{M}_A} E(d\lambda).
\]

If \( f \in \mathcal{B} \), then \( \|f\| \leq \sup_{\Lambda \subset \mathcal{M}_A} \|E(\Lambda)f\| \) (the supremum is taken over all nonempty open-closed subsets of \( \mathcal{M}_A \)). Since \( \|E(\Lambda)\| = 1 \) for any \( \Lambda \), we find that

\[
\|f\| = \sup_{\Lambda} \|E(\Lambda)f\|. \tag{3}
\]

If \( \mathcal{M}_A \) is finite or countable, this equality is a kind of the non-Archimedean orthogonality property of the expansion in eigenvectors.

By the construction, the spectral measure \( E(\Lambda) \) has the operator multiplicativity property: if \( \Lambda_1 \) and \( \Lambda_2 \) are open-closed sets, then \( E(\Lambda_1 \cap \Lambda_2) = E(\Lambda_1)E(\Lambda_2) \). Moreover, if \( \Lambda_1 \cap \Lambda_2 = \emptyset \),
the operators $E(\Lambda_1)$ and $E(\Lambda_2)$ are orthogonal in non-Archimedean sense, as elements of the Banach space $\mathcal{L}(\mathcal{B})$. Indeed, if $a_1, a_2 \in k$, then
\[ \|a_1 E(\Lambda_1) + a_2 E(\Lambda_2)\| = \|a_1 \eta_{\Lambda_1} + a_2 \eta_{\Lambda_2}\| = \sup_{\lambda \in \Lambda_1 \cup \Lambda_2} |a_1 \eta_{\Lambda_1}(\lambda) + a_2 \eta_{\Lambda_2}(\lambda)| = \max(|a_1|, |a_2|). \]

We will call an operator $A$ normal, if its Banach algebra $\mathcal{L}_A$ generates the functional calculus (2) with a multiplicative and orthogonal, in the and non-Archimedean sense, does not have a bounded inverse), while all possible values of a character $\chi$ are determined by a single value $\lambda = \chi(A)$, and $\lambda \in \sigma(A)$. It follows from the definition of the Gelfand transform that in this case the operator $\pi(A)$, where $\pi$ is an arbitrary polynomial, corresponds to the polynomial function $\{\pi(\lambda), \lambda \in \sigma(A)\}$. In particular, $A$ itself corresponds to $\pi(\lambda) = \lambda$, and we obtain the classical formula $A = \int_{\sigma(A)} \lambda E(d\lambda)$.

In the finite-dimensional case, the operator $E(\{\lambda\}), \lambda \in \sigma(A)$, is a projection onto the eigensubspace corresponding to an eigenvalue $\lambda$.

### The Finite-Dimensional Case

#### 4.1. Let $\mathcal{B} = k^n$, with the norm $\|(x_1, \ldots, x_n)\| = \max_{1 \leq i \leq n} |x_i|$. An operator $A$ is represented, with respect to its standard basis in $k^n$, by a matrix $(a_{ij})_{i,j=1}^n$. Its operator norm coincides with $\|A\| = \max_{i,j} |a_{ij}|$ (see [28]). Without restricting generality, we assume that $\|A\| = 1$.

Let $\hat{k}$ be the residue field of the field $k$. Together with the operator $A$, we consider its reduction, the operator $\mathfrak{A}$ on the $\hat{k}$-vector space $\hat{\mathcal{B}} = \hat{k}^n$ corresponding to the matrix $(\hat{a}_{ij})_{i,j=1}^n$, where $\hat{a}_{ij}$ is the image of $a_{ij}$ under the canonical mapping $O \to \hat{k}$. In invariant terms, we may define $\hat{\mathcal{B}} = \mathcal{B}_0/P\mathcal{B}_0$ ($\mathcal{B}_0$ is the closed unit ball in $\mathcal{B}$); $\mathfrak{A}$ is the operator induced by $A$ on $\hat{\mathcal{B}}$.

An operator $A$ will be called degenerate, if $\mathfrak{A} = \nu I$ where $\nu \in \hat{k}$, and $I$ denotes the identity operator on $\hat{\mathcal{B}}$ (in fact, we denote by $I$ all the identity operators). Otherwise $A$ will be called non-degenerate.

**Lemma 1.** If all $n$ eigenvalues of the operator $A$ belong to $k$, then all the characters of the Banach algebra $\mathcal{L}_A$ take their values in $k$, and $\mathcal{M}_A = \sigma(A)$.

**Proof.** Let $P_A$ be the characteristic polynomial of the operator $A$. By the Cayley-Hamilton theorem, $P_A(A) = 0$. If $\chi$ is a character of $\mathcal{L}_A$, then $P_A(\chi(A)) = 0$, that is $\chi(A)$ is a root of the characteristic equation, thus $\chi(A) \in \sigma(A) (\subset k)$. Then also $\chi(f(A)) \in k$, for any $f(A) \in \mathcal{L}_A$.\]
and each equivalence class of the characters is determined by the element $\chi(A)$. This means that $M_A = \sigma(A)$. ■

Now we can give a description of all non-degenerate normal operators.

**Theorem 1.** Let an operator $A$ be non-degenerate, all $n$ its eigenvalues belong to $k$, and its reduction $\mathfrak{A}$ be diagonalizable, that is $\mathfrak{A}$ possess an eigenbasis in $\mathfrak{B}$. Then $A$ is a normal operator.

**Proof.** By Theorem B and Lemma 1, it suffices to prove that the algebra $\mathcal{L}_A$ is uniform.

First of all, for the operator $A$ with $\|A\| = 1$, the condition $\|A^2\| = \|A\|^2 (= 1)$ is equivalent to the fact that $\mathfrak{A}^2 \neq 0$. By our conditions, there exists such an invertible operator $U$ on $\hat{\mathfrak{B}}$ that

$$\mathfrak{A} = U^{-1} \text{diag}(\xi_1, \ldots, \xi_n)U,$$

$0 \neq (\xi_1, \ldots, \xi_n) \in \hat{k}^n$.

Then $\mathfrak{A}^2 = U^{-1} \text{diag}(\xi_1^2, \ldots, \xi_n^2)U \neq 0$.

In a similar way, consider an operator $f(A)$, $f \in k[t]$, $f(t) = \sum_{j=0}^N a_j t^j$, $a_j \in k$ (it is sufficient to prove the uniformity identity for such operators). Let $K$ be the splitting field of the polynomial $f$, that is

$$f(t) = a_N \prod_{j=1}^N (t - t_j), \quad t_j \in K,$$

whence

$$f(A) = a_N \prod_{j=1}^N (A - t_j I).$$

The operator $A$ is assumed to be extended onto the space $K^n$ where it corresponds to the same matrix, and its reduction has the same eigenbasis.

Now we have only to prove that, for each $j$,

$$\|(A - t_j I)^2\| = \|A - t_j I\|^2.$$

If $|t_j| < 1$ (we use the extension of the absolute value from $k$ to $K$), then $\|(A - t_j I)^2\| = \|A^2 - 2t_j A + t_j^2 I\| = \|A^2\| = 1$, and $\|A - t_j I\| = \|A\| = 1$. If $|t_j| > 1$, then $\|2t_j A\| \leq |t_j| < |t_j|^2$

whence $\|(A - t_j I)^2\| = |t_j|^2 = \|A - t_j I\|^2$.

Let us consider the case where $|t_j| = 1$. Let $\hat{t}_j$ be the image of $t_j$ in the residue field of $K$. Then the reduction of the matrix $A - t_j I$ has the form

$$U^{-1} \text{diag}(\xi_1 - \hat{t}_j, \xi_2 - \hat{t}_j, \ldots, \xi_n - \hat{t}_j)U,$$

so that the reduction of $(A - t_j I)^2$ equals

$$U^{-1} \text{diag}((\xi_1 - \hat{t}_j)^2, (\xi_2 - \hat{t}_j)^2, \ldots, (\xi_n - \hat{t}_j)^2)U.$$

Both the reductions are different from zero, due to the non-degeneracy of the operator $A$. Therefore $1 = \|A - t_j I\|^2 = \|(A - t_j I)^2\|$. ■
Corollary. If all \( n \) eigenvalues of the operator \( A \) belong to \( k \), and its reduction \( \mathfrak{A} \) has \( n \) different eigenvalues from \( k \), then the operator \( A \) is normal.

The proof follows from the fact \([10]\) that an operator on \( \hat{k}^n \) with all different eigenvalues is diagonalizable.

4.2. It is clear that the non-degeneracy assumption cannot be dropped. For example, the operator \( A = I + B \), where \( B \) is a non-diagonalizable operator and \( \|B\| < 1 \), is not normal and has the reduction \( I \).

In the case of a local field, we can describe the structure of degenerate operators.

Proposition 1. Let \( k \) be a non-Archimedean local field, \( \|A\| = 1 \), and \( \mathfrak{A} = \gamma I \), \( \gamma \in \hat{k} \). Then there exists such \( g \in k \), \( |g| = 1 \), that either \( A = gI \), or \( A = gI + A_0 \), where \( \|A_0\| < 1 \), and the operator \( \lambda_0 A_0 \) with such \( \lambda_0 \in k \) that \( \|\lambda_0 A_0\| = 1 \), is non-degenerate.

Proof. Let \( g_1 \in k \), \( |g_1| = 1 \), be an arbitrary inverse image of \( \gamma \) under the canonical mapping \( O \setminus P \to O/P \cong \hat{k} \). If \( A = g_1 I \), then the proof is finished. Otherwise \( A = g_1 I + A_1 \) where \( \|A_1\| < 1 \). Choose \( \lambda_1 \in k \) in such a way that \( \|\lambda_1 A_1\| = 1 \); then \( |\lambda_1| = q^{m_1} \), \( m_1 \in \mathbb{N} \) (\( q \) is the cardinality of the residue field \( \hat{k} \)). If \( \lambda_1 A_1 \) is non-degenerate, the proof is finished. Otherwise we find that \( \lambda_1 A_1 = g_2 I + A_2 \), \( \|A_2\| < 1 \), \( |g_2| = 1 \), that is \( A_1 = \lambda_1^{-1} g_2 I + \lambda_1^{-1} A_2 \),

\[
A = (g_1 + \lambda_1^{-1} g_2) I + \lambda_1^{-1} A_2.
\]

If the continuation of this procedure does not produce, at a certain stage, a non-degenerate operator, then, for each \( n \), we obtain the representation

\[
A = (g_1 + \lambda_1^{-1} g_2 + \lambda_1^{-1} \lambda_2^{-1} g_3 + \cdots + \lambda_1^{-1} \lambda_2^{-1} \cdots \lambda_{n-1}^{-1} g_n) I + \lambda_1^{-1} \lambda_2^{-1} \cdots \lambda_{n-1}^{-1} A_n
\]

where \( |\lambda_1^{-1} \cdots \lambda_{n-1}^{-1}| = q^{-m_1} \cdots q^{-m_{n-1}} \leq q^{-n+1} \to 0 \), as \( n \to \infty \), \( |g_n| = 1 \) for all \( n \), \( \|A_n\| < 1 \).

This means that \( A = g_0 I \) where

\[
g_0 = g_1 + \lambda_1^{-1} g_2 + \lambda_1^{-1} \lambda_2^{-1} g_3 + \cdots + \lambda_1^{-1} \lambda_2^{-1} \cdots \lambda_{n-1}^{-1} g_n + \cdots ,
\]

and the series converges in \( k \). ■

5 INFINITE-DIMENSIONAL OPERATORS

5.1. In this section we assume that \( k \) is a complete non-Archimedean algebraically closed field with a nontrivial valuation.

Let \( A \) be a bounded linear operator on the Banach space \( \mathcal{B} = c(J,k) \) (see Sect. 2.1). We assume that \( A \) is an analytic operator with a compact spectrum \([22]\), that is the spectrum \( \sigma(A) \subseteq k \) is a compact set, and the resolvent \( R_z(A) = (A - zI)^{-1} \) has the property that, for any \( h \in \mathcal{B}^* \), \( g \in \mathcal{B} \), the function \( z \mapsto \langle h, R_z(A) g \rangle \) belongs to the space \( H_0(k \setminus \sigma(A)) \) of Krasner analytic functions.

The latter space is defined as follows. For a given \( r > 0 \), choose a covering of \( \sigma(A) \) by a minimal possible number of non-intersecting open balls \( D_i(r) \) of radius \( r \) with centers \( a_i \in \sigma(A) \).
Let \( D(r) \) be the union of these balls. The space \( H_0(k \setminus \sigma(A)) \) consists of all such functions \( \varphi : k \setminus \sigma(A) \to k, \varphi(\infty) = 0 \), that, for each \( r > 0 \), \( \varphi \) can be uniformly on \( k \setminus D(r) \) approximated by rational functions with possible poles in \( D(r) \).

A spectral theory of analytic operators with compact spectra was developed by Vishik [32], and we will use some of his results. In fact, we will deal with a more narrow class of scalar type operators which satisfy an additional condition

\[
\|R_z(A)\| \leq \frac{C}{\text{dist}(z, \sigma(A))}, \quad C > 0. \tag{4}
\]

In this case, there exists such a projection-valued finitely additive bounded measure \( \mu_A \) on the Boolean algebra of open-closed subsets of \( \sigma(A) \), such that

\[
\langle \mu_A(u), u^j \rangle = \int_{\sigma(A)} u^j \mu_A(du) = A^j, \quad j = 0, 1, 2, \ldots. \tag{5}
\]

More generally, the expression

\[
\langle \mu_A, f \rangle = \int_{\sigma(A)} f(u)\mu_A(du)
\]

defines a continuous mapping from \( C(\sigma(A), k) \) to \( \text{End} \mathcal{B} \). The measure \( \mu_A \) is uniformly bounded in the operator norm; however in general one cannot assert the crucial property \( \|\mu_A(\Lambda)\| = 1, \Lambda \neq \emptyset \).

The above approximation property of the resolvent can be made explicit: \( R_z(A) \) can be approximated, uniformly on \( k \setminus D(r) \), by the rational operator-functions

\[
R_N(z) = \sum_{i \in I} \sum_{j=1}^N A_{ij}(a_i - z)^{-j}, \quad N \to \infty,
\]

where \( I \) is a finite set (depending on \( r \)),

\[
A_{ij} = \langle \mu_A(u), \eta(r, i, j - 1, u) \rangle,
\]

\[
\eta(r, i, j, u) = \begin{cases} (u - a_i)^j, & \text{if } u \in D_i(r), \\ 0, & \text{if } u \notin D_i(r). \end{cases}
\]

By Kaplansky’s theorem (see Theorem 43.3 in [29]), each of the continuous functions \( \eta \) can be uniformly approximated by polynomials. Then it follows from (5) that, for any fixed \( z \notin \sigma(A) \), the operator \( R_z(A) \) can be approximated, in the operator norm, by polynomials of the operator \( A \). In other words, \( R_z(A) \) belongs to the Banach algebra \( \mathcal{L}_A \) of the operator \( A \). Similarly, approximating the characteristic function of an open compact set uniformly by polynomials and using (5) we find that the values of \( \mu_A \) belong to \( \mathcal{L}_A \).

5.2. Below we assume that \( A \) is an analytic operator with a compact spectrum, \( \|A\| = 1 \), and the set \( J \) of indices is infinite. Let us consider the matrix representation of the operator.
With respect to the standard orthonormal basis in the sequence space \( \mathcal{B} = c(J, k) \), the operator \( A \) corresponds to an infinite matrix \((a_{ij})_{i,j\in J}\). The operator norm \( \|A\| = \sup_{\|x\|\leq 1} \|Ax\| \) equals \( \sup_{i,j} |a_{ij}| \) (the double sequence \( \{a_{ij}\} \) is bounded, and \( |a_{ij}| \to 0 \) for any fixed \( j \), and \( i \to \infty \), by the filter of complements to finite subsets of \( J \); see [28]).

Just as for finite matrices, we can define the reduction \( \mathfrak{A} \) of the operator \( A \). This is an operator on the space \( \hat{k}_0^\infty \) of all such sequences with elements from \( \hat{k} \) that there is only a finite number of nonzero elements in each sequence. The operator \( \mathfrak{A} \) is determined by the infinite matrix \((\alpha_{ij})\) where \( \alpha_{ij} \) is the image of \( a_{ij} \) under the canonical mapping \( O \to \hat{k} \). We say that \( \mathfrak{A} \) is diagonalizable, if \( \mathfrak{A} \) possesses an eigenbasis (in algebraic sense) in \( \hat{k}_0^\infty \).

The operator \( A \) is called non-degenerate, if \( \mathfrak{A} \) is non-scalar: \( \mathfrak{A} \neq \gamma I \) for any \( \gamma \in \hat{k} \).

**Lemma 2.** If the operator \( A \) is non-degenerate, and its reduction is diagonalizable, then \( A \) is a scalar type operator.

**Proof.** Let \( z \in \hat{k} \). We consider three possible cases.

1) \(|z| > \|A\| (= 1)\). If \( \zeta \in \sigma(A) \), then \( |\zeta| \leq \|A\| \) (see Section 4.1 in [22]). Then

\[
\|A - zI\| = |z| = |(z - \zeta) + \zeta| = |z - \zeta| \geq \inf_{\zeta \in \sigma(A)} |z - \zeta| = \dist(z, \sigma(A)).
\]

2) \(|z| < \|A\|\). For \( \zeta \in \sigma(A) \), we have \( |z - \zeta| \leq \|A\|, \|A - zI\| = \|A\| \geq |z - \zeta| \geq \dist(z, \sigma(A)) \).

3) \(|z| = \|A\| = 1\). Then \( \|A - zI\| = \|B - I\| \) where \( B = z^{-1}A, \|B\| = 1 \). Suppose that \( B \) is given by an infinite matrix \((b_{ij})\). The reduced operator \( \tilde{B} \) equals \( U^{-1} \diag(\beta_1, \ldots, \beta_n, \ldots)U \) where \( U \) is the operator on \( \hat{k}_0^\infty \) transforming the bases, and there are at least two different elements among \( \beta_1, \ldots, \beta_n, \ldots \). We have

\[
\tilde{B} - I = U^{-1} \diag(\beta_1 - 1, \ldots, \beta_n - 1, \ldots)U \neq 0,
\]

so that \( \|B - I\| = 1 \), and for any \( \zeta \in \sigma(A) \),

\[
\|A - zI\| = |z| \geq |z - \zeta| \geq \dist(z, \sigma(A)),
\]

and we come to (4) in this case too.

**5.3.** Let us prove the result extending Theorem 1, for the case of a complete algebraically closed field \( k \) with a nontrivial valuation, to the infinite-dimensional situation.

**Theorem 2.** If \( A \) is an analytic operator with compact spectrum, \( A \) is non-degenerate, and its reduction is diagonalizable, then \( A \) is normal and \( M_A = \sigma(A) \).

**Proof.** As we have seen, it follows from Lemma 2 that, for each \( z \notin \sigma(A) \), the resolvent \( R_z(A) \) belongs to the algebra \( \mathcal{L}_A \). Let \( \chi : \mathcal{L}_A \to \hat{k} \) be a character of \( \mathcal{L}_A \) with values in a possibly wider field \( K \supset k \). Denote \( \beta = \chi(A) \).

Let us write the representation

\[
A = \int_{\sigma(A)} \lambda \mu_A(d\lambda).
\]

(6)
Since $\mu_A$ is norm-uniformly bounded, the integral converges in the operator norm, that is $A$ is approximated by linear combinations of values of $\mu_A$ with coefficients from $\sigma(A) \subset k$. Let us apply the character $\chi$ to both sides of (6). Note that $[\mu_A(\Lambda)]^2 = \mu_A(\Lambda)$ for any open-closed subset $\Lambda \subset \sigma(A)$, so that $\chi(\mu_A(\Lambda))$ equals 0 or 1. It follows that $\beta \in k$.

Next, let us apply $\chi$ to both sides of the equality $(A - zI)R_z(A) = I$, $z \notin \sigma(A)$. We get that $(\beta - z)\chi(R_z(A)) = 1$, so that $\beta \neq z$. Thus, we have proved that an arbitrary character $k$ takes its values in $k$ and, more specifically, in $\sigma(A)$. Therefore $M_A \subset \sigma(A)$.

Just as in the proof of Theorem 1, we show that the algebra $L_A$ is uniform. By Theorem A, $L_A$ is isomorphic to $C(M_A, k)$. We find that the operator $A$ is normal and corresponds, under this isomorphism, to the multiplication operator $\varphi(m) \mapsto m\varphi(m)$, $\varphi \in C(M_A, k)$. Obviously, its spectrum is a subset of $M_A$. Thus, we have proved that $\sigma(A)$ and $M_A$ coincide as sets.

The topology on $\sigma(A)$ induced by its identification with $M_A$ is the weakest topology, for which all the functions from $C(M_A, k)$ are continuous. Since $M_A$ is a compact Hausdorff space, and polynomials $\pi : M_A (= \sigma(A)) \to k$ separate its points, the topology on $M_A$ coincides with the one determined by these polynomials ([27], Proposition 7.1.8; this proposition is formulated for real- or complex-valued functions but remains valid for our case). On the other hand, defining on $\sigma(A)$ a topology by the same polynomials and taking into account that $\sigma(A)$ is compact in $k$, we find similarly that the above topology of $\sigma(A)$ coincides with the topology of $\sigma(A)$ as a subset of $k$.  

5.5. Let us consider the case of a compact, or a completely continuous operator $A$, that is a norm limit of a sequence of finite rank operators. There exists also an alternative definition involving a generalization of the notion of a compact set called a compactoid; see [30, 31].

There is also a description of compact operators in terms of their matrices $(a_{ij})_{i,j \in J}$. Let $r_j(A) = \sup_{i \in J} |a_{ij}|$. Then $A$ is compact, if and only if $r_j(A) \to 0$, as $j \to \infty$ by the filter of complements of finite sets.

Suppose, as before, that $\|A\| = 1$. For a degenerate operator $A$, $r_j(A) = 1$ for all indices $j$. Therefore a compact operator on an infinite-dimensional space is always non-degenerate. Moreover, it is an analytic operator with a compact spectrum [32]. Thus, we have the following result.

**Corollary.** If a compact operator is such that its reduction is diagonalizable, then it is normal.

In general, the spectrum of a compact operator is at most countable, with 0 as the only possible accumulation point. Every nonzero element of the spectrum is an eigenvalue of finite multiplicity. For a compact normal operator, we have, just as in the classical case, that, for any $f \in \mathcal{B}$, the vector $Af$ can be expanded into a convergent series with respect to an orthogonal system of eigenvectors of the operator $A$. In particular, a normal compact operator always has a nonzero eigenvalue – if the spectral measure is concentrated at the origin, then $Af = 0$ for any $f \in \mathcal{B}$.

6. **EXAMPLES**

6.1. The first nontrivial example of a $p$-adic normal operator is quite explicit and does not
require any general theorems. This is a counterpart of the number operator coming from a
\( p \)-adic representation of the canonical commutation relations of quantum mechanics. In the
model given in [18], \( B = C(\mathbb{Z}_p, \mathbb{C}_p) \),
\[
(a^+ f)(x) = xf(x - 1), \quad (a^- f)(x) = f(x + 1) - f(x), \quad x \in \mathbb{Z}_p.
\]
The operators (7) are bounded and satisfy the relation \([a^-, a^+] = I\). Let \( A = a^+ a^-\), so that
\[
(Af)(x) = x\{f(x) - f(x - 1)\}.
\]
Then \( AP_n = nP_n, n \geq 0, \) where
\[
P_n(x) = \frac{x(x - 1) \cdots (x - n + 1)}{n!}, \quad n \geq 1; \quad P_0(x) \equiv 1,
\]
is the Mahler basis, an orthogonal basis in \( C(\mathbb{Z}_p, \mathbb{C}_p) \cong c(\mathbb{Z}_+, \mathbb{C}_p) \). Thus \( A \) is normal. An
equivalent, though more complicated, construction was given a little earlier in [6].

In fact, a general version of the above construction involves a Banach space \( B \) with an
orthonormal basis \( \{e_n\}_{n=0}^\infty \), and the linear operators \( \alpha^+, \alpha^- \) on \( B \) acting on the basis vectors as follows:
\[
\alpha^- e_n = e_{n-1}, \quad n \geq 1; \quad \alpha_- e_0 = 0;
\]
\[
\alpha^+ e_n = (n + 1)e_{n+1}, \quad n \geq 0.
\]
Then \([\alpha^-, \alpha^+] = I\) and \((\alpha^+ \alpha^-) e_n = ne_n, n \geq 0\). In the language of [6], \( B \) is a Banach
space (with an appropriate norm) generated by the normalized Hermite polynomials, and the
operators \( \alpha^\pm \) are given in terms of the differentiation and multiplication by the independent
variable.

It is not clear whether the position and momentum operators defined in [3, 4, 5] (or perhaps
their modifications) are normal. This subject deserves further study which goes beyond the
scope of this paper.

Returning to the representation (7), note that, as it was mentioned in [18], the relation
\([a^-, a^+] = I\) admits nonequivalent bounded representations. In particular, let us take, instead
of \( a^- \), the operator
\[
(a' f)(x) = f(x + 1), \quad x \in \mathbb{Z}_p.
\]
Then \([a', a^+] = I, A' = a^+ a'\) is the multiplication operator on \( C(\mathbb{Z}_p, \mathbb{C}_p) \), that is
\[
(A' f)(x) = xf(x), \quad x \in \mathbb{Z}_p.
\]
It is clear that \( A' \) has no eigenvalues. Nevertheless we have the following result.

**Proposition 2.** The operator \( A' \) is normal.

**Proof.** It is obvious that \( \sigma(A') \subset \mathbb{Z}_p \), so that \( A' \) has a compact spectrum. Let us prove that
the operator \( A' \) is analytic. We have to check that, for any \( h \in C(\mathbb{Z}_p, \mathbb{C}_p) \), and any \( \mathbb{C}_p \)-valued
bounded measure \( \mu \) on \( \mathbb{Z}_p \), the function
\[
\varphi(z) = \langle R_z(A') h, \mu \rangle = \int_{\mathbb{Z}_p} \frac{1}{x - z} h(x) \mu(dx)
\]

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admits, on the set $V_z = \{ z \in \mathbb{C}_p : |z|_p > 1 + r \}$ with an arbitrary fixed $r > 0$, the uniform approximation by rational functions with possible poles in the complement of $V_z$.

Denote

$$\Delta = \frac{h(x)}{x - z} - \frac{h(y)}{y - z}, \quad x, y \in \mathbb{Z}_p, \ z \in V_z.$$ 

Then, for any $\varepsilon > 0$, there exists such $\delta > 0$, independent of $z \in V_z$, that $|\Delta| < \varepsilon$, as soon as $|x - y|_p < \delta$.

Indeed,

$$\Delta = \frac{y h(x) - x h(y)}{(x - z)(y - z)} - \frac{z}{(x - z)(y - z)} [h(x) - h(y)],$$

so that

$$|\Delta|_p \leq \frac{1}{|(x - z)(y - z)|_p} \max \{|y h(x) - x h(y)|_p, |z|_p |h(x) - h(y)|_p\}.$$ 

We have $|x - z|_p = |y - z|_p = |z|_p$,

$$|y h(x) - x h(y)|_p \leq \max \{|y - x|_p |h(x)|_p, |x|_p |h(x) - h(y)|_p\},$$

whence

$$|\Delta|_p \leq (1 + r)^{-1} \max \{|x - y|_p \|h\|, |h(x) - h(y)|_p\} < \varepsilon,$$

if $|x - y|_p$ is small enough.

The integral in (8) is a limit, as $n \to \infty$, of the Riemann sums

$$\sum \frac{h(x_0 + x_1p + \cdots + x_np^n)}{x_0 + x_1p + \cdots + x_np^n - z} \mu(B(x_0 + x_1p + \cdots + x_np^n, p^{-n-1}))$$ 

where the sum is taken over all $x_0, \ldots, x_n \in \{0, 1, \ldots, p - 1\}$, $B(\cdot, p^{-n-1})$ is a closed ball in $\mathbb{Z}_p$ with a center at the appropriate point and radius $p^{-n-1}$. The difference of two Riemann sums of this kind, the above one and the one with $l > n$ substituted for $n$, which corresponds to the decomposition of balls into non-intersecting subballs, is a sum of the expressions like the above $\Delta$ multiplied by the uniformly bounded measures of the balls (this is similar to the standard justification of the $p$-adic integration procedure; see Chapter 2 in [L7]). Each of these Riemann sums is a rational function of $z$ with poles in $\mathbb{Z}_p$, and the above estimate of $\Delta$ proves the uniform approximation of the function $\varphi$ by such rational functions. Thus, the operator $A'$ is analytic.

Let us use the identity

$$t P_n(t) = n P_n(t) + (n + 1) P_{n+1}(t), \quad n = 0, 1, 2, \ldots$$

([29], Example 52.B). We find that, with respect to the basis $\{P_n\}$, the matrix of $A'$ has the form

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 3 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & n & n + 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}.
$$

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In the reduction procedure, the block
\[
\begin{pmatrix}
  p & p+1 & 0 & \ldots & 0 & 0 \\
  0 & p+1 & p+2 & \ldots & 0 & 0 \\
  & & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & \ldots & 2p-2 & 2p-1 \\
  0 & 0 & 0 & \ldots & 0 & 2p-1 \\
\end{pmatrix}
\]
(as well as similar subsequent blocks; here we do not show zeroes to the right) is transformed into
\[
\begin{pmatrix}
  0 & 1 & 0 & \ldots & 0 & 0 \\
  0 & 1 & 2 & \ldots & 0 & 0 \\
  & & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & \ldots & p-2 & p-1 \\
  0 & 0 & 0 & \ldots & 0 & p-1 \\
\end{pmatrix}
\]
(9)

Therefore the reduction $A'$ of the operator $A'$ is an infinite direct sum of the finite blocks (9), each of which is diagonalizable having all different eigenvalues. Thus, $A'$ is normal. ■

6.2. Let us give an example of a non-degenerate operator with a non-diagonalizable reduction. Let, as before, $\mathcal{B} = C(\mathbb{Z}_p, \mathbb{C}_p)$. Consider the operator $(A_1 f)(t) = f(t-1)$, $t \in \mathbb{Z}_p$. Since
\[
P_n(t-1) = \sum_{j=0}^{n} P_{n-j}(-1)P_j(t)
\]
([29], Proposition 47.2), the matrix of $A_1$ has the form
\[
\begin{pmatrix}
  1 & 0 & 0 & \ldots & 0 & 0 & \ldots \\
  P_1(-1) & 1 & 0 & \ldots & 0 & 0 & \ldots \\
  P_2(-1) & P_1(-1) & 1 & \ldots & 0 & 0 & \ldots \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  P_n(-1) & P_{n-1}(-1) & P_{n-2}(-1) & \ldots & P_1(-1) & 1 & \ldots \\
\end{pmatrix}
\]

It is known ([29], Exercise 47.C) that $|P_n(-1)|_p = 1$, $n = 0, 1, 2, \ldots$. Thus the reduction $\mathfrak{A}_1$ has the form
\[
\begin{pmatrix}
  1 & 0 & 0 & \ldots \\
  \theta_{21} & 1 & 0 & \ldots \\
  \theta_{31} & \theta_{32} & 1 & \ldots \\
\end{pmatrix}
\]
where $\theta_{ij} \neq 0$ for any $i > j$. If $u = (u_0, u_1, u_2, \ldots)^T$ is an eigenvector, then the eigenvalue must equal 1, and it is easy to check that $u = 0$ (for a finite matrix of this form, an eigenvector would have a nonzero last component, but here the last component is absent). The more so, $\mathfrak{A}_1$ is non-diagonalizable.

6.3. The last counter-example prompts us to change the Banach space, in order to accommodate such natural operators. More generally, we will consider the affine dynamical system
\( T(x) = \alpha x + \beta, \; x \in \mathbb{Z}_p, \) where \( \alpha, \beta \in \mathbb{Z}_p, \; |\alpha|_p = 1. \) Properties of this transformation have been studied by a number of authors, see [12] and references therein; note also the paper [23] where an operator generated by \( T \) on a space of complex-valued functions on \( \mathbb{Z}_p \) is considered.

Let \( \mu_{p^n} \subset \mathbb{C}_p \) be the set of all roots of 1 of order \( p^n, \) \( \Gamma = \bigcup_{n=0}^{\infty} \mu_{p^n}. \) Following [13], we consider the space \( B_2 = L^1(\Gamma, \mathbb{C}_p) \) of functions

\[
 f(x) = \sum_{\gamma \in \Gamma} c_\gamma \gamma^x, \; x \in \mathbb{Z}_p,
\]

where \( c_\gamma \in \mathbb{C}_p, \; |c_\gamma|_p \to 0 \) by the filter of complements to finite sets. Note that if \( \zeta \in \mu_{p^n}, \) then

\[
 |\zeta - 1|_p = p^{-\frac{1}{(p-1)p^{n-1}}} < 1
\]

([25], Sect. 3.4.2), so that \( \gamma^x \) is well-defined. See [25] [29] regarding the definition and properties of the mapping \( \gamma \mapsto \gamma^x. \)

The norm in \( B_2 \) is given by \( \|f\| = \sup_{\gamma \in \Gamma} |c_\gamma|. \) The functions \( \gamma \mapsto \gamma^x, \; \gamma \in \Gamma, \) form an orthonormal basis of \( B_2. \)

Let \( A_2 \) be the operator on \( B_2 \) of the form

\[
 (A_2 f)(x) = f(T(x)) = \sum_{\gamma \in \Gamma} c_\gamma \gamma^x \cdot \gamma^{\alpha x}, \; x \in \mathbb{Z}_p.
\]

Considering the structure of \( A_2, \) we begin with the following lemma.

**Lemma 3.** For each \( n, \) the mapping \( S_\alpha \gamma = \gamma^\alpha \) is a one-to-one mapping of the set \( \mu_{p^n} \) onto itself.

**Proof.** We have \( (\gamma^\alpha)^{p^n} = \gamma^{\alpha p^n} = 1, \) so that \( S_\alpha : \mu_{p^n} \to \mu_{p^n}. \) Suppose that \( \gamma_1^\alpha = \gamma_2^\alpha, \gamma_1, \gamma_2 \in \mu_{p^n}, \) so that \( \gamma_1^3 = 1, \gamma_3 = \gamma_1 \gamma_2^{-1} \in \mu_{p^n}. \) Writing the canonical representation

\[
 \alpha = \alpha_0 + \alpha_1 p + \cdots + \alpha_n p^n + \alpha_{n+1} p^{n+1} + \cdots, \; \alpha_j \in \{0, 1, \ldots, p - 1\} \; (j \geq 0), \; \alpha_0 \neq 0,
\]

we find that

\[
 1 = \gamma_3^\alpha = \gamma_3^{\alpha_0 + \alpha_1 p + \cdots + \alpha_n p^n},
\]

so that \( \gamma_3 \) is a root of unity of order \( \alpha_0 + \alpha_1 p + \cdots + \alpha_n p^n. \) The last number is not divisible by \( p. \) Since the sets of nontrivial roots of unity of order \( p^n \) and of an order not divisible by \( p \) do not intersect ([29], Lemma 33.1), we have \( \gamma_3 = 1, \) so that \( \gamma_1 = \gamma_2. \)

By Lemma 3, the mapping \( \gamma \mapsto \gamma^\alpha \) defines, for each \( n, \) a permutation \( \Lambda_n \) of the set \( \mu_{p^n} \) agreed with the filtration

\[
 \{1\} = \mu_1 \subset \mu_p \subset \mu_{p^2} \subset \ldots.
\]

We will assume that, for each \( n \geq 1, \) the finite set \( \mu_{p^n} \) is numbered in such a way that elements of \( \mu_{p^{n-1}} \) go first. The permutations \( \Lambda_n \) determine a permutation \( \Lambda \) of the whole set \( \Gamma \) coinciding with \( \Lambda_n \) on \( \mu_{p^n}. \) For each \( n, \) the permutation \( \Lambda \) preserves also the set \( \mu_{p^{n+1}} \setminus \mu_{p^n}. \)
After a change of variables, we see that

\[(A_2f)(x) = \sum_{\nu \in \Gamma} c_{\Lambda^{-1}(\nu)} B(\nu) \nu^x, \quad x \in \mathbb{Z}_p,\]

where \(B(\nu) = [\Lambda^{-1}(\nu)]^3\), and \(B(\nu) \in \mu_{p^n}\), if \(\nu \in \mu_{p^n}\).

For each \(n\), the permutation \(\Lambda^{-1}_n\) defines a linear transformation \(L_n\) on \(p^n\)-tuples \((c_1, \ldots, c_{p^n})\) described by a zero-one permutation matrix \[21\]. Thus, the operator \(A_2\) is represented by an infinite block matrix with nonzero diagonal blocks \(L_n\) multiplied from the right by the diagonal matrix \(\text{diag}(B(\nu), \nu \in \Gamma)\).

In order to prove the normality of \(A_2\), it is sufficient to prove it for each finite block. In the reduction process, we obtain the same permutation matrix; the image of \(B(\nu)\) equals 1. Some diagonal block may be degenerate only if it is equal to \(I\).

It is known \([21\), Sect. 4.10.8\] that a finite permutation matrix can be brought to a block-diagonal form determined by a cycle structure of the permutation; the transforming matrix is a permutation matrix itself (so that the construction makes sense over fields like an algebraic closure of \(\mathbb{F}_p\)). Each diagonal block has roots of unity of an appropriate order as its eigenvalues. Thus, if \(\alpha \neq 1\), the diagonal subblocks are either non-degenerate or diagonal, and the reduction of each of them is a diagonalizable operator. If \(\alpha = 1\), then \(A_2\) is diagonal. We have proved the following result.

**Proposition 3.** The operator \(A_2\) is normal.

6.4. Many new examples of normal operators can be constructed via a dilation procedure described below.

Let an operator \(A\) on \(k^n\) be given by a matrix \((a_{ij})\). Extending the space \(k^n\) to \(k^{n+1}\), we will denote by \(P\) the projection operator: if \(y = (\xi, y_1, \ldots, y_n) \in k^{n+1}\), then \(Py = (y_1, \ldots, y_n)\).

An operator \(B\) on \(k^{n+1}\) is called a 1-dilation of \(A\), if \(Ax = PB\bar{x}\) for all \(x \in k^n\), where \(x = (x_1, \ldots, x_n), \bar{x} = (0, x_1, \ldots, x_n)\).

Suppose that \(A\) is a Jordan cell

\[
A = \begin{pmatrix}
a & 1 & 0 & \ldots & 0 & 0 \\
0 & a & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & a & 1 \\
0 & 0 & 0 & \ldots & 0 & a \\
\end{pmatrix}, \quad a \in k, |a| \leq 1.
\]

Define an operator \(B\) on \(k^{n+1}\), setting

\[
B = \begin{pmatrix}
a & 1 & 0 & \ldots & 0 & 0 \\
0 & a & 1 & \ldots & 0 & 0 \\
0 & 0 & a & 1 & \ldots & 0 \\
0 & 0 & 0 & \ldots & a & 1 \\
-1 & 0 & 0 & \ldots & 0 & a \\
\end{pmatrix}
\]
(we added the first row and the first column). Then
\[
B(0, x_1, x_2, \ldots, x_n)^T = (x_1, (Ax)_1, \ldots, (Ax)_n),
\]
so that \( B\tilde{x} = Ax \). Let us check the normality of \( B \).

The reduction \( \tilde{B} \) of the matrix \( B \) has the same form, with \( \tilde{a} \in \hat{k} \) substituted for \( a \). In order to find the eigenvalues of \( \tilde{B} \), we note expanding along the first column, that \( \det(\tilde{B} - \lambda I) = (\tilde{a} - \lambda)^{n+1} - (-1)^n \). Thus the eigenvalues equal \( \tilde{a} + \gamma_j \), \( j = 1, \ldots, n + 1 \), where \( \gamma_j \) are the roots of \( (-1)^n \) of degree \( n + 1 \).

Therefore, if the field \( k \) is complete and algebraically closed, then, by Theorem 1, \( B \) is a normal 1-dilation of \( A \). We obtain further examples, considering finite or infinite direct sums of such operators.

**ACKNOWLEDGEMENT**

This work was supported in part by the Ukrainian Foundation for Fundamental Research, Grant 29.1/003.

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