ON THE VINOGRA DOV MEAN VALUE

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Abstract. A discussion of recent work of C. Demeter, L. Guth and the author of the proof of the Vinogradov Main Conjecture using the decoupling theory for curves.

1. Introduction and statements

For \( k, s \in \mathbb{N} \) and \( x \in \mathbb{R}^k \), denote \( (e(t) = e^{2\pi it}) \)

\[
f_k(x, N) = \sum_{1 \leq n \leq N} e(nx_1 + n^2x_2 + \cdots + n^kx_k)
\]

(1.1)

and

\[
J_{s,k}(N) = \int_{[0,1]^k} |f_k(x, N)|^{2s} \, dx_1 \cdots dx_k.
\]

(1.2)

By orthogonality, \( J_{s,k}(N) \) counts the number of integral solutions of the system

\[
n_1^j + \cdots + n_s^j = n_{s+1}^j + \cdots + n_{2s}^j \quad (1 \leq j \leq k)
\]

(1.3)

where \( 1 \leq n_i \leq N \) \( (1 \leq i \leq 2s) \).

The evaluation of \( J_{s,k}(N) \) is a central problem of importance to several classical issues in analytic number theory, including the Waring problem, bounds on Weyl sums and zero-free regions for the Riemann zeta function. The introduction of the mean value (1.2) and its significance to number theory go back to the seminal work of I.M. Vinogradov (cf. [Vi2]). This approach is referred to as ‘Vinogradov’s Method’.

Following T. Wooley, we call ‘Main Conjecture’ the statement

\[
J_{s,k}(N) \ll N^{\varepsilon}(N^s + N^{2s-\frac{1}{2}k(k+1)}) \quad \text{for all } \varepsilon > 0
\]

(1.4)
and $2s = k(k + 1)$ the critical exponent. We note that indeed both $N^s$ and $N^{2s - \frac{1}{2} k(k+1)}$ are obvious lower bounds (up to multiplicative constants).

Vinogradov’s original argument [Vi1] for estimating $J_{s,k}(N)$ was refined by means of Linnik’s $p$-adic approach [Li] and the work of Karatsuba [Ka] and Stechkin [St], leading to the following bound for $s \geq k$

$$J_{s,k}(N) \leq D(s,k)N^{2s - \frac{1}{2} k(k+1) + \eta_{s,k}} \quad (1.5)$$

with

$$\eta_{s,k} = \frac{1}{2} k^2 \left( 1 - \frac{1}{k} \right)^{[s/k]} \quad \text{and} \quad D(s,k) = \min(k^{csk}, k^{ck^3}). \quad (1.6)$$

The latter leads to an asymptotic formula

$$J_{s,k}(N) \sim C(s,k)N^{2s - \frac{1}{2} k(k+1)} \quad (1.7)$$

provided

$$s \geq k^2(2 \log k + \log \log k + 5) \quad (1.8)$$

(see [A-C-K]).

Major progress towards the Main Conjecture was achieved by T. Wooley based on his efficient congruencing method.

**Theorem 1.** (see Theorem 4.1 in [W6], based on [W1], [W2], [F-W], [W3], [W4], [W5]).

The Main Conjecture for $J_{s,k}(N)$ holds when

(i) $k = 1, 2, 3$.

(ii) $1 \leq s \leq D(k)$, where $D(4) = 8$, $D(5) = 10$, . . . and

$$D(k) = \frac{1}{2} k(k + 1) - \frac{1}{3} k + O(k^{3/2}) \quad (1.9)$$

(iii) $k \geq 3$ and $s \geq k(k - 1)$

The reader is referred to the survey paper [W6] for a detailed discussion. It should be noted that prior to [W5], the Main Conjecture was only known for $k \leq 2$. 
Based on a more general harmonic analysis principle - the so-called ‘decoupling theorem’ for curves - the full Main Conjecture was finally established by C. Demeter, L. Guth and the author in the fall of 2015 (see [BDG]).

**Theorem 2.** ([BDG].) The Main Conjecture for $J_{s,k}(N)$ holds.

For $s > \frac{k}{2}k(k + 1)$, the prefactor $N^\varepsilon$ in (1.4) may be dropped and one has the asymptotic formula (1.7). In what follows, we will review some consequences to the Waring problem and Weyl syms, improving on earlier results. Next, we will formulate the underlying harmonic analysis result with a brief discussion (the reader will find complete proofs in [BDG]) and conclude with some further comments.

Concerning applications to the zeta-function, our work as it stands does not lead to further progress. The reason for this is that we did not explore the effect of large $k$ (possibly depending on $N$) and in the present form is likely very poor. A similar comment applies to Wooley’s approach.

References in this paper are far from exhaustive and only serve the purpose of this exposé.

2. THE ASYMPTOTIC FORMULA IN WARING’S PROBLEM

Denote $R_{s,k}(n)$ the number of representations of the positive integer $n$ as sum of $s$ $k$th powers. For $s$ sufficiently large, one has the asymptotic formula

$$R_{s,k}(n) = \frac{\Gamma(1 + \frac{1}{k})^s}{\Gamma(\frac{s}{k})}S_{s,k}(n)n^{\frac{s}{k} - 1} + o(n^{\frac{s}{k} - 1})$$

(2.1) where

$$S_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{a=1}^{q} \sum_{(a,q)=1}^{q} \left( \frac{1}{q} \sum_{r=1}^{q} e_q(ar^k) \right) e_q(-na)$$

(2.2) is the singular series.

Denote $\tilde{G}(k)$ the smallest integer $s$ for which (2.1) holds. Based on heuristic applications of the circle method, one expects $\tilde{G}(k) = k + 1$ for $k \geq 3$, but known results are far weaker. The Vinogradov main value theorem plays a crucial role in the minor arcs analysis (see in particular [W7]). Moreover, in
the implications of (1.4) to $\tilde{G}(k)$ are worked out, at that time conjectural. Recording Theorem 4.1 in [W7], one obtains therefore the bound

**Theorem 3.** For $k \geq 3$,

$$\tilde{G}(k) \leq k^2 + 1 - \max_{1 \leq j \leq k-1} \left[ \frac{kj - 2j}{k + 1 - j} \right]. \quad (2.3)$$

denoting $[t]$ the smallest integer no smaller then $t$.

In particular $\tilde{G}(4) = 15$, $\tilde{G}(k) \leq k^2 - 2(k \geq 5)$, $\tilde{G}(k) \leq k^2 - 3(k \geq 8), \ldots$

$$\tilde{G}(k) \leq k^2 + 1 - \left\lfloor \frac{\log k}{\log 2} \right\rfloor \quad (k \geq 3). \quad (2.4)$$

This is an improvement of all previously known bounds on $\tilde{G}(k)$, except for Vaughan’s $\tilde{G}(3) \leq 8$ ([Vau1]).

As we will see later, the bound (2.4) may be further improved for large $k$, due to the fact that our results also enable a certain improvement in Hua’s lemma.

For the record, we note that Wooley obtained $\tilde{G}(5) \leq 28$, $\tilde{G}(6) \leq 43$, $\tilde{G}(7) \leq 61, \ldots$

$$\tilde{G}(k) \leq (1.5407 \ldots + o(1))k^2 \text{ for large } k. \quad (2.5)$$

3. Weyl Sums

Recalling (1.1), Weyl’s theorem states (see [Vau2] for instance).

**Theorem 4.** (H. Weyl). With the notation (1.1), assume $(a,q) = 1$ and $|x_k - \frac{a}{q}| \leq \frac{1}{q^2}$. Then

$$|f_k(x,N)| \ll N^{1+\varepsilon}(q^{-1} + N^{-1} + qN^{-k})^{2^{1-k}}. \quad (3.1)$$

It is well-known that for large $k$, Vinogradov’s method leads to substantially better results. As a consequence of Theorem 2, one gets (cf. [Vau2]).
Theorem 5. Again with the notation (1.1), let \( k \geq 3, 2 \leq j \leq k \) and assume

\[ |x_j - \frac{a}{q}| \leq \frac{1}{q^2}, (a, q) = 1. \]

Then

\[ |f_k(x, N)| \ll N^{1+\varepsilon} (q^{-1} + N^{-1} + qN^{-j})^{\sigma(k)} \quad \text{with} \quad \sigma(k) = \frac{1}{k(k-1)}. \quad (3.2) \]

Theorem 5 improves Weyl’s bound and later refinements due to Heath-Brown [H-B] and Robert-Sargos [R-S] for \( k \geq 7 \).

Wooley had proven (3.2) with \( \sigma(k) = \frac{1}{2(k-1)(k-2)} \), see [W6].

4. The Decoupling Theorem for curves

It turns out that in fact (1.4) is a consequence of a more general harmonic analysis principle that we discuss next.

Let \( \Gamma = \{(t, t^2, \ldots, t^k) : 0 \leq t \leq 1\} \) be the moment curve (or, more generally a non-degenerate curve in \( \mathbb{R}^k \)). Given \( g : [0, 1] \to \mathbb{C} \) and an interval \( J \subset [0, 1] \), define the extension operator

\[ E_J g(x) = \int_J g(t)e(tx_1 + t^2x_2 + \cdots + t^kx_k)dt \quad x = (x_1, \ldots, x_k) \in \mathbb{R}^k. \quad (4.1) \]

Given a ball \( B = B(c_B, R) \) in \( \mathbb{R}^k \), denote \( \omega_B \) the weight function

\[ \omega_B(x) = \left(1 + \frac{|x - c_B|}{R}\right)^{-100k}. \quad (4.2) \]

Theorem 6. ([BDG]). Let \( k \geq 2 \) and \( 0 < \delta \leq 1 \). For each ball \( B \subset \mathbb{R}^k \) of radius at least \( \delta^{-k} \), one has the inequality

\[ \|E_{[0,1]}g\|_{L^{k(k+1)}(\omega_B)} \ll \delta^{-\varepsilon} \left( \sum_{J \subset [0,1], |J| = \delta} \|E_Jg\|_{L^{k(k+1)}(\omega_B)}^2 \right)^{\frac{1}{2}} \quad (4.3) \]

where \( J \) runs over a partition of \([0,1]\) in \( \delta \)-intervals.

Remarks.

(i) Decoupling inequalities of the type were previously established in [BD1] for smooth hypersurfaces in \( \mathbb{R}^k \) with non-vanishing curvature. In particular,
the case $k = 2$ of Theorem 6 already appears in [BD1]. We also refer the reader to [BD1] for the analysis background of the decoupling problem.

(ii) The exponent $k(k+1)$ in (4.3) is best possible. Let us point out that there is a similar decoupling inequality for $2 \leq p < k(k+1)$, though for $k \geq 3$ this is not just a consequence of interpolation.

(iii) Our decoupling inequalities for curves appear in [B1], [BD2], [B2]. In particular, the reader is referred to [B2] for an application to exponential sums and the Lindelöf hypothesis for the Riemann-zeta function.

(iv) The weight function $\omega_B$ (rather than $1_B$) is a (necessary) technical issue but will often be ignored in our later discussion for simplicity.

It is easy to deduce Theorem 2 from Theorem 6. One first observes that the decoupling theorem implies the following discretized version.

**Theorem 7.** For each $1 \leq n \leq N$, let $\frac{n-1}{N} < t_n < \frac{n}{N}$ and let $R > N^k$. For each $p \geq 1$, one has

$$\left\{ \frac{1}{|B_R|} \int \left| \sum_{n=1}^{N} a_n e(t_n x_1 + t_n^2 x_2 + \cdots + t_n^k x_k) \right|^p \omega_B(x) dx_1 \cdots dx_k \right\}^{1/p} \ll N^{\varepsilon} \left( 1 + N^{1 \frac{1-k(k+1)}{p}} \right) \left( \sum |a_n|^2 \right)^{1/2}.$$  

(4.4)

(the case $p < k(k+1)$ is obtained by interpolation with $p = 2$ and $p > k(k+1)$ with the obvious $p = \infty$ bound).

Taking $a_n = 1$ and $p = 2s$, it follows from (4.4) that the system of inequalities

$$|t_{n_1}^j + \cdots + t_{n_s}^j - t_{n_{s+1}}^j - \cdots - t_{n_{2s}}^j| < N^{-k} \quad (1 \leq j \leq k)$$  

(4.5)

has at most $N^{\varepsilon} \left( N^s + N^{2s-k(k+1)} \frac{k(k+1)}{2} \right)$ solutions in $1 \leq n_1, \ldots, n_{2s} \leq N$. Specifying $t_n = \frac{n}{N}$, Theorem 2 follows immediately.
5. Elements of the Proof of Theorem 6

Most techniques involved in proving decoupling theorems had previously been developed in the study of the restriction and Kakeya problems in harmonic analysis. These include wave packet decomposition, parabolic rescaling and the use of multi-linear analysis. In what follows, we make a few mostly superficial comments on how they appear in the context of curves.

5.1. Wave Packet Decomposition. Let $J \subset [0,1]$ be a small interval and $\tau = \{ \gamma(t) = (t,t^2,\ldots,t^k); t \in J \} \subset \Gamma$ the corresponding arc. Then, roughly speaking, $|E_J g|$ may be viewed as ‘essentially constant’ on translates of the geometric polar $\tau$ of the convex hull of $\tau$. Thus if $|J| = \delta$, these are $\frac{1}{\delta} \times \frac{1}{\delta^2} \times \cdots \times \frac{1}{\delta^k}$-boxes oriented according to the Frenet basis of $\Gamma$.

5.2. Parabolic Rescaling. Take $k = 2$ and $J = [t_0, t_0 + \sigma] \subset [0,1]$. Write for $t = t_0 + \sigma t' \in J$

$$x_1 t + x_2 t^2 = x_1 t_0 + x_2 t_0^2 + \sigma(x_1 + 2x_2 t_0) t' + \sigma^2 x_2 (t')^2 \quad (5.1)$$

and make a change of variables $x'_1 = \sigma(x_1 + 2t_0 x_2), x'_2 = \sigma^2 x_2$.

The map $(x_1, x_2) \mapsto (x'_1, x'_2)$ maps $B_R$ to an $\sigma R \times \sigma^2 R$ size ellipse which we cover with $\sigma^2 R$-balls.

In general, $(x_1, \ldots, x_k) \mapsto (x'_1, \ldots, x'_k)$ maps $B_R$ to an ellipsoid covered by $\sigma^k R$-balls. Next, denote $K_p(\delta)$ the best constant for which a decoupling inequality

$$\|E_{[0,1]} g\|_{L^p(B)} \leq K_p(\delta) \left( \sum_{|J| = \delta} \|E_J g\|_{L^p(B)}^2 \right)^{\frac{1}{2}} \quad (5.2)$$

with $B$ a $\delta^{-k}$-ball holds. It follows then from the previous discussion that if $J \subset [0,1], |J| = \sigma > \delta$, then (5.2) will hold with $K_p(\delta)$ replaced by $K_p(\frac{\delta}{\sigma})$ if supp $g \subset J$. 
5.3. Multilinear Analysis. The reduction of (4.3), which in some sense is a linear statement, to multi-linear expressions is crucial as it allows us to exploit transversality. This technique, which is basically simple, goes back to the joint work [BG] of L. Guth and the author. All available results on decoupling make use of this procedure.

Continuing our high-level discussion, the left side of (4.3) will be replaced by certain multi-linear quantities which we describe next. Define

\[ D_q(\Delta, \delta_1) = \prod_{i=1}^M \left[ \sum_{J_i, |J_i|=\delta_1} \| E_J g \|_{L^q_{w,\Delta}}^2 \right]^{\frac{1}{2M}} \]  

(5.3)

where

- \( M = M_k \) is an appropriate integer (\( M_2 = 2 \))
- \( J_1, \ldots, J_M \subset [0,1] \) are fixed \( O(1) \)-separated intervals
- \( \Delta = R \)-ball, \( R > \delta_1^{-1} \) and \( L^q_{w,\Delta} \) is the normalized \( L^q \)-norm on \( \Delta \).

Let \( B \) be a (fixed) large ball and define further for \( 2 \leq q \leq p \)

\[ \tilde{D}_q(M, \delta_1) = \left[ \text{Average}_{\Delta = R \text{-ball} \subset B} D_q(\Delta, \delta_1)^p \right]^{\frac{1}{p}} \]  

(5.4)

Hence \( \tilde{D}_p(R, \delta_1) \leq D_p(B, \delta_1) \). The strategy is to bound \( \tilde{D}_q(R, \delta_1) \) by gradually decreasing \( \delta_1 \) and increasing \( R \). Note that from the previous discussion, one has for \( \delta < \delta_1, |B| > \delta^{-k} \)

\[ D_p(B, \delta_1) \leq K_p \left( \frac{\delta}{\delta_1} \right) D_p(B, \delta). \]  

(5.5)

Clearly, from basic orthogonality, if \( \delta_1 > \frac{1}{R} \), then

\[ D_2(\Delta, \delta_1) \lesssim D_2(\Delta, \frac{1}{R}) \]  

(5.6)

(a rigorous justification requires in fact replacing \( \Delta \) by a weight function \( w_\Delta \) of the type (4.2)).

More generally, if \( q \leq d(d+1), d < k \) and \( R > \delta_1^{-d} \), one has

\[ D_q(\Delta, \delta_1) \ll R^\varepsilon D_q(\Delta, R^{-\frac{1}{2}}) \]  

(5.7)
by appealing to the decoupling theorem in dimension $d$ (exploiting only the variables $x_1, \ldots, x_k$), assuming the latter already obtained.

We also note the following interpolation property, which is immediate from Hölder’s inequality. Let $q_1 \leq q \leq q_2$ and $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$. Then

$$D_q(\Delta, \delta_1) \leq D_{q_1}(\Delta, \delta_1)^{1-\theta} D_{q_2}(\Delta, \delta_1)^\theta$$

and similarly for $\tilde{D}_q$.

Next, the ball inflation, i.e. the increment of $R$, uses essentially transversality which comes with the multi-linear structure of (5.3). The main inequality writes

$$\tilde{D}_{d,k}^{-d}(\delta_1^{-d}, \delta_1) \ll \tilde{D}_{d,k}^{-d-1}(\delta_1^{-d-1}, \delta_1) \text{ for } 1 \leq d < k$$

and follows from wave packet decomposition as explained in (4.1) and multi-linear Kakeya type estimates originating from the work [BCT]. Note that in (5.9) and keeping in mind (5.4), we are essentially trading an $L^p$-norm for an $L^{d,p}$-norm. This is possible by exploiting certain transversality properties. The key result is the Brascamp-Lieb inequality that underlies the multi-linear Kakeya theory and we formulate next.

**Theorem 8.** (Brascamp-Lieb, see [BDG] for related references).

Let $d \leq k$ and for $1 \leq i \leq M$, let $V_i$ be a $d$-dimensional subspace of $\mathbb{R}^k$. Denote $\pi_i : \mathbb{R}^k \to V_i$ the orthogonal projection. We assume the following transversality condition

$$\frac{d}{k} \dim V \leq \frac{1}{M} \sum_{i=1}^{M} \dim(\pi_i V)$$

for all linear subspaces $V$ of $\mathbb{R}^k$.

Then the quantity

$$\sup_{g_i \in L^1(V_i)} \frac{\|\prod_{i=1}^{M} |g_i \circ \pi_i|^{1/M} \|_{L^{k/4}(\mathbb{R}^k)}}{\|\prod_{i=1}^{M} \|g_i\|_{L^1(V_i)}^{1/M}}$$

is finite.
In the present application, the spaces $V_i$ are obtained as $V_i = [\gamma'(t_i), \ldots, \gamma^{(d)}(t_i)]$ with $t_i \in J_i$ ($1 \leq i \leq M$) introduced above and condition (5.10) for appropriate $M$ results from the assumption that the curve $\Gamma$ is non-degenerate.

Let $p < k(k + 1)$ be sufficiently close to $k(k + 1)$.

Let $\delta_0 = \delta^u$ with $u > 0$ fixed and arbitrarily small. Starting from $\tilde{D}_2(\delta_0^{-1}, \delta_0)$, it follows from (5.9) that

$$\tilde{D}_2(\delta_0^{-1}, \delta_0) \leq \tilde{D}_k(\delta_0^{-1}, \delta_0) \ll \tilde{D}_k(\delta_0^{-2}, \delta_0). \quad (5.12)$$

Next, use (5.8) with $q = \frac{p}{k}$, $q_1 = 2$, $q_2 = \frac{2p}{k}$ and (5.9) with $d = 2$ to get

$$\tilde{D}_k(\delta_0^{-2}, \delta_0) \ll \tilde{D}_2(\delta_0^{-2}, \delta_0)^{1-\theta_1} \tilde{D}_k(\delta_0^{-3}, \delta_0)^{\theta_1} \quad (5.13)$$

for some $0 < \theta_1 < 1$. The second factor in (5.13) is further processed interpolating between $q_1 = 6$ and $q_2 = \frac{3p}{k}$. Applying (5.7) with $d = 2, q = 6$ and (5.9) leads to

$$\tilde{D}_k(\delta_0^{-3}, \delta_0) \ll \tilde{D}_6(\delta_0^{-3}, \delta_0)^{\frac{3}{2}-\theta_2} \tilde{D}_k(\delta_0^{-4}, \delta_0)^{\theta_2} \quad (4.14)$$

for some $0 < \theta_2 < 1$. Next,

$$\tilde{D}_6(\delta_0^{-3}, \delta_0^{\frac{3}{2}}) \leq \tilde{D}_2(\delta_0^{-3}, \delta_0^{3})^{1-\psi_2} \tilde{D}_k(\delta_0^{-\frac{3}{2}}, \delta_0^{\frac{3}{2}})^{\psi_2} \quad (5.15)$$

for some $0 < \psi_2 < 1$. The above are the first few steps of an interpolation scheme that together with inequality (5.5) and a bootstrap argument eventually permits us to estimate $K_p(\delta) \ll \delta^{-\varepsilon}$ for $p < k(k + 1)$.

The sole purpose of the above discussion is to give the reader some sense of how the proof of Theorem 6 works, again referring to [BDG] for the full account and further references.

6. Some Further Comments

6.1. An Improvement of Hua’s Inequality. We point out another arithmetical consequence of Theorem 6 related to Hua’s lemma. Recall the statement (of [Vau2]).
Theorem 9. (Hua). For \( k \geq 1 \), denote
\[
S(x) = \sum_{1 \leq n \leq N} e(n^k x). \quad (6.1)
\]
Then for \( 1 \leq \ell \leq k \) we have
\[
\int_0^1 |S(x)|^{2\ell} dx \ll N^{2\ell-\ell+\varepsilon} \quad \text{for all } \varepsilon > 0. \quad (6.2)
\]
We sketch the proof of the following

Theorem 10. Let \( S(x) \) be defined by (6.1) and \( s \leq k \) a positive integer. Then
\[
\int_0^1 |S(x)|^{s(s+1)} dx \ll N^{s^2+\varepsilon} \quad \text{for all } \varepsilon > 0. \quad (6.3)
\]
Clearly Theorem 10 improves upon Theorem 9 for \( \ell \geq 5 \).

Proof of Theorem 10.

We apply the decoupling theorem to the non-degenerate curve in \( \mathbb{R}^s \)
\[
\Gamma = \{(t^k, t^{s-1}, \ldots, t), 1 \leq t \leq 2\}. \quad (6.4)
\]
The discretized version analogous to Theorem 7 implies
\[
N^{-s} \int_{[-N,N]^s} \left| \sum_{n=N}^{2N} e\left(\frac{n}{N} x + \left(\frac{n}{N}\right)^{s-1} x_{s-1} + \cdots + \frac{n}{N} x_1\right)\right|^{s(s+1)} dx_1 \cdots dx_{s-1} dx \ll N^{1/2s(s+1)+\varepsilon}. \quad (6.5)
\]
Rescaling and use of periodicity gives
\[
\int_{[-1,1]} \int_{[0,1]^{s-1}} \left| \sum_{n=N}^{2N} e\left(\frac{n^k}{N^{k-s}} x + n^{s-1} x_{s-1} + \cdots + nx_1\right)\right|^{s(s+1)} dx_1 \cdots dx_{s-1} dx \ll N^{s(s+1)+\varepsilon}. \quad (6.6)
\]
Denote \( K_r = K_r(t) \) the kernel on \( T = \mathbb{R}/\mathbb{Z} \) which Fourier transform \( \hat{K}_r \) is trapezoidal, satisfying \( \hat{K}_r(n) = 1 \) for \( |n| \leq r \) and \( \text{supp} \hat{K}_r \subset [-2r, 2r] \). Hence \( \|K_r\|_1 \leq 3 \). Multiply the integrand in (6.6) by
\[
K_{2N}(x_1)K_{2N^2}(x_2) \cdots K_{2N^{s-1}}(x_{s-1}) \quad (6.7)
\]
and perform the integration in \(x_1, \ldots, x_{s-1}\). Since \((6.7) \leq C_s N^{\frac{1}{2}s(s-1)}\), it follows from (6.6) that

\[
\int \left| \sum_{n=N}^{2N} e\left(\frac{n^k}{N^{k-s}x}\right)e^{is(s+1)x}\right| \, dx \ll N^{s^2+\varepsilon}. \tag{6.8}
\]

Note that inequality (6.8) is essentially optimal and implies the weaker statement

\[
\int_0^1 \left| \sum_{n=N}^{2N} e(n^kx)e^{is(s+1)x}\right| \, dx \ll N^{s^2+\varepsilon}. \tag{6.9}
\]

This proves (6.3).

Returning to the discussion in §2 and [W7], we point out that in the treatment of the minor arcs in the circle method, besides Vinogradov’s inequality also Hua’s lemma (Theorem 9) is involved in deriving Theorem 3 (see §3 in [W7]). Hence Theorem 10 is expected to produce further improvements in bounding \(\tilde{G}(k)\), which we discuss next (referring to [W7] for details).

Following [W7], define the set \(\mathcal{M} = \mathcal{M}_k\) of minor arcs as the set of real numbers \(x \in [0,1]\) with the property that, whenever \(a \in \mathbb{Z}, q \in \mathbb{Z}_+, (a,q) = 1\) satisfy \(|qx - a| \leq (2k)^{-1}N^{1-k}\), then \(q > (2k)^{-1}N\).

Injecting (1.4) with \(s = \frac{1}{2}k(k+1)\) in Theorem 2.1 of [W7] implies

\[
\int_{\mathcal{M}} |S(x)|^{k(k+1)} \, dx \ll N^{k^2-1-\varepsilon}. \tag{6.10}
\]

Inequality (6.10) is then interpolated with (6.2) or alternatively (6.3) in order to establish an inequality of the form

\[
\int_{\mathcal{M}} |S(x)|^{s_0} \, dx < N^{s_0-k-\tau} \tag{6.11}
\]

for some \(\tau > 0\) and as small as possible exponent \(s_0 \in \mathbb{Z}_+\) that will provide a bound on \(\tilde{G}(k)\).

Taking \(s < k\) a parameter, let \(s_0 \in \mathbb{Z}_+, s(s+1) \leq s_0 \leq k(k+1)\). Interpolation between (6.10) and (6.3) gives

\[
\int_{\mathcal{M}} |S(x)|^{s_0} \, dx \ll N^{s_0-\eta+\varepsilon} \tag{6.12}
\]
with
\[ \eta = (1 - a)(k + 1) + as \quad \text{and} \quad a = \frac{k(k + 1) - s_0}{k(k + 1) - s(s + 1)}. \quad (6.13) \]

Hence, in order to obtain (6.11), we are lead to the condition \( \eta > k \), which translates in
\[ s_0 > k^2 - \frac{k - s - 1}{k + 1 - s}. \quad (6.14) \]

Consequently, we proved

**Theorem 11.**
\[ \tilde{G}(k) \leq k^2 + 1 - \max_{s \leq k} \left[ s \frac{k - s - 1}{k - s + 1} \right]. \quad (6.15) \]

The reader will verify that (6.15) improves over Theorem 3 for \( k > 12 \) and moreover implies that for large \( k \)
\[ \tilde{G}(k) < k^2 - k + O(\sqrt{k}) \quad (6.16) \]
rather than (2.4).

### 6.2. Generalizations of Vinogradov's Inequality.

Mean value estimates for multi-dimensional Weyl sums using efficient congruencing were obtained in [PPW]. One could reasonably expect that a complete understanding of decoupling phenomena for surfaces in \( \mathbb{R}^k \) will also lead to progress and perhaps optimal results in this more general setting. Presently, we only reached a satisfactory understanding of decoupling for co-dimension one surfaces and for curves. A decoupling theorem for 2-dimensional surfaces in \( \mathbb{R}^k \) was established in [BD3] implying in particular results on 2-dimensional cubic Weyl sums but that are likely not optimal. The recent developments around curves obtained in [BDG] almost surely will further contribute in this direction.

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