Distributed system identification with ADMM

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Abstract— This paper presents identification of both network connected systems as well as distributed systems governed by PDEs in the framework of distributed optimization via the Alternating Direction Method of Multipliers. This approach opens first the possibility to identify distributed models in a global manner using all available data sequences and second the possibility for a distributed implementation. The latter will make the application to large scale complex systems possible. In addition to outlining a new large scale identification method, illustrations are shown for identifying both network connected systems and discretized PDEs.

I. INTRODUCTION

Control of distributed systems has recently received a renewed interest. To just name a few examples we mention [2], [5], [9], [10]. The interest stems from the challenging applications that arose through the increase in dimensionality of the systems to be controlled. Such increase is stimulated by various developments, such as network communication enabling the operation of network connected systems and/or the increase in number of actuators and sensors for control. An example of a network connected systems is formation flying, [8], and an example of large scale sensor and actuator systems is the ongoing development of the new European Extreme Large telescope where both the primary mirror as well as the secondary mirror are devices with a number of sensors and actuators in the order of $10^8$ or more, [7].

A more recent development in the design of distributed controllers is the renewed interest in distributed optimization methods from the middle of the previous century, such as reported in [3].

Despite this vast interest and despite numerous developments in the area of distributed controller synthesis, appropriate modeling tools for deriving the necessary models from measured data sequences are still rather scarce. Most results are restricted to the identification of transfer functions. In the area of identification of two dimensional (2D) systems there is the work of [4] and more recently [1]. The last approach was developed to overcome the difficulty in applying transfer function estimation methods that relied on the impulse response of the system. The approach taken was to solve the distributed identification problem as a whole using the Alternating Direction Method of Multipliers. This approach allows us to both address problems where all input and output data are available, the latter occurs e.g in the identification of systems governed by PDEs. In Section III the the problem is put on a generic form, which is suitable for making use of the ADMM algorithm in Section IV. The distributed implementation is discussed briefly in Section V. Section VI illustrates the methodology for identifying ARX models connected in a feedback topology. The application for identifying discretized PDEs is discussed in Section VII. Numerical results are summarized in Section VIII. Finally, in Section IX conclusions are given together with directions for future research.

II. IDENTIFICATION PROBLEM

We are interested in distributed system identification of systems that are sparsely interconnected and where we do not measure all inputs and outputs of the system. To fix the ideas consider systems described by

$$\mathcal{J}_i(y_i, u_i, e_i, \theta_i) = 0, \quad i = 1, \ldots, M,$$

where \(\mathcal{J}_i\) is a possibly nonlinear mapping of the parameter vector \(\theta_i \in \mathbb{R}^p\), the input signal vector \(u_i = (u_i(1), \ldots, u_i(N))\), where \(u_i(k) \in \mathbb{R}^m\), the output signal vector \(y_i = (y_i(1), \ldots, y_i(N))\), where \(y_i(k) \in \mathbb{R}^m\), and the error vector \(e_i = (e_i(1), \ldots, e_i(N))\), where \(e_i(k) \in \mathbb{R}^p\).
We assume that we measure the goodness of a parameter $\theta$ for describing relationship between $u_i$ and $y_i$ with a function $f_i(y_i, u_i, \theta)$. For the purpose of the remaining part of this paper we will consider

$$f_i(y_i, u_i, \theta) = \| e_i \|_2^2$$

However, it should be easy to extend the result to other norms such as the nuclear norm.

We will assume that the systems are interconnected according to

$$u(k) = \Gamma y(k) + Bu_0(k) \quad (1)$$
$$y(0) = Cy(k) \quad (2)$$

where we assume that only $u_0(k) \in \mathbb{R}^{m_0}$ and $y_0(k) \in \mathbb{R}^{p_0}$ are measured. Here $u(k) = (u_1(k), \ldots, u_M(k))$ and $y(k) = (y_1(k), \ldots, y_M(k))$. We will also assume that $C$ has full row rank and that there exists a permutation matrix $P$ such that $CP = [I \ 0]$. We also assume that $[\Gamma \ B]$ has only 0–1 entries and that it has at least one non-zero entry in each row. The remaining signals are just given implicitly by the above equations. Notice that we do not assume that they are uniquely defined by these equations. However, we need to make the assumption that they are uniquely defined from the optimization problem

$$\min_{\theta} \sum_{i=1}^M f_i(y_i, u_i, \theta), \quad \text{s.t.} \quad [1-2] \quad \text{and} \quad \theta = E\theta_0,$$

where $y = (y_1, \ldots, y_M)$, $u = (u_1, \ldots, u_M)$, $\theta = (\theta_1, \ldots, \theta_M)$, $\theta_0 \in \mathbb{R}^r$ and $E \in \mathbb{R}^{r \times r}$, with $q = \sum_{i=1}^M q_i$. The solution of this problem will jointly minimize the goodness of the fit of the parameters $\theta$. We also restrict the parameters of the different sub-models to be related to one another by imposing the constraint $\theta = E\theta_0$, where $E$ has full column rank. This is typically the case for models that come from spatial discretization of partial differential equations. We may of course generalize the above problem by taking some other linear combinations of the functions $f_i$.

### III. Optimization Problem

We will now cast the above problem as an optimization problem on the form

$$\begin{align*}
\min_{\theta} & f(z, \theta) \\
\text{s.t.} & \quad z = Ax + b \\
& \quad \theta = E\theta_0
\end{align*} \quad (3)$$

where $A \in \mathbb{R}^{(m+p)N \times nN}$ has full column rank. To this end we immediately define $z_i = (y_i, u_i)$ and let $f(z, \theta) = \sum_{i=1}^M f_i(y_i, u_i, \theta)$, where $z = (z_1, \ldots, z_M) \in \mathbb{R}^{(m+p)N}$ with $m = \sum_{i=1}^M m_i$ and $p = \sum_{i=1}^M p_i$. Let $\bar{y}(k)$ be defined via

$$y(k) = P\bar{y}(k) = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \bar{y}_1(k) \\ \bar{y}_2(k) \end{bmatrix}$$

where $CP_1 = I$. Because of this $\bar{y}_2(k)$ has dimension $n = \sum_{i=1}^M p_i - p_0$. We define

$$\Gamma = [\hat{\Gamma}_1 \ \hat{\Gamma}_2] = [\Gamma P_1 \ \Gamma P_2]$$

Then it holds that

$$u(k) = \hat{\Gamma}_1 y_0(k) + \hat{\Gamma}_2 \bar{y}_2(k) + Bu_0(k)$$

We let $x(k) = \bar{y}_2(k)$. Then we may write

$$y(k) = P_1 y_0(k) + P_2 x(k)$$

We introduce

$$z(k) = \begin{bmatrix} y(k) \\ u(k) \end{bmatrix}$$

We also let $z_0(k) = (y_0(k), u_0(k))$. From this it follows that

$$z(k) = \begin{bmatrix} P_2 \\ \hat{\Gamma}_2 \end{bmatrix} x(k) + \begin{bmatrix} P_1 & 0 \end{bmatrix} z_0(k)$$

We now introduce a permutation matrix $Q$ such that

$$Qz(k) = \begin{bmatrix} y_1(k) \\ u_1(k) \\ \vdots \\ y_M(k) \\ u_0(k) \end{bmatrix}$$

We also let $y_0 = (y_0(1), \ldots, y_0(N))$, $u_0 = (u_0(1), \ldots, u_0(N))$, $z_0 = (y_0, u_0)$, $x_i = (x_i(1), \ldots, x_i(N))$, $x = (x_1, \ldots, x_n)$, $A = (Q\hat{A}) \otimes I_N$, and $B = (Q\hat{B}) \otimes I_N$. Then it holds that

$$z = Ax + Bz_0$$

Hence $b = Bz_0$ in (3). From this we realize that $A$ is a sparse matrix containing only 0–1 entries, and that it is a very sparse matrix if $\Gamma$ is sparse. Moreover, it follows that $A$ has full column rank, since $P_2$ has full column rank.

### IV. Alternating Direction Methods of Multipliers

We now define the augmented Lagrangian for the optimization problem in (3):

$$L_\rho(x, \theta_0, z, \lambda, \mu) = f(z, \theta) + \lambda^T(z - Ax - b) + \mu^T(\theta - E\theta_0) + \frac{\rho}{2}\|z - Ax - b\|^2 + \frac{\rho}{2}\|\theta - E\theta_0\|^2$$

where $\lambda \in \mathbb{R}^{(m+p)N}$ and $\mu \in \mathbb{R}^r$. We will from now on assume that $f$ is bi-convex in $z$ and $\theta$. Hence there might be several local optima to the optimization problem. The Alternating Method of Multipliers (ADMM) can often successfully be applied to these type of problems. However, there is no guarantee for convergence even to local optima. The method performs alternating optimization steps where we need to solve $\min_{x, \theta_0, z} L_\rho$ for fixed $\theta$ and $\min_{\theta} L_\rho$ for fixed $x, \theta_0, z$. Both these problems are convex, and moreover we will see that they can be solved by solving linear system of equations. There are also trivial steps in which $(\lambda, \mu)$ and possibly also $\rho$ are updated.
We will now justify the bi-convexity assumption by making the assumption that $\mathcal{L}(y, u, e, \theta)$ is linear in the signals such that we may express $e_i$ as
\[
e_i = T_i(\theta)z_i
\]
for some matrix $T_i$ which depends linearly on $\theta_i$. Then
\[
f_i(z_i, \theta_i) = \|T_i(\theta_i)z_i\|^2_2
\]
From now on we will suppress the $\theta_i$ dependence in $T_i$.

We first consider the case of optimizing with respect to $(x, \theta, z)$, which separates into two independent optimization problems, one for $(x, z)$ and one for $\theta_i$. For $\theta_i$ the augmented Lagrangian is strictly convex, and hence the unique minimum is given by the solution of
\[
\frac{\partial L_p}{\partial \theta_i} = E^T \mu + \rho E^T (\theta - \hat{E} \theta_i) = 0
\]
or equivalently of
\[
\rho E^T \hat{E} \theta_i = E^T (\mu + \rho \theta)
\]
Before we continue with the other variables we realize that if $\mu$ is initialized as zero, then the fact that $E^T \mu + \rho E^T (\theta - \hat{E} \theta_i) = 0$ together with the updated rule for $\mu$ in Table I implies that $E^T \theta = 0$, and hence (4) may be simplified to
\[
E^T \hat{E} \theta_i = E^T \theta
\]
Then for $(x, z)$ we get with similar arguments the equations:
\[
\begin{align*}
\frac{\partial L_p}{\partial x_i} &= 2E^T x + \rho \lambda + \rho \mu = 0 \\
\frac{\partial L_p}{\partial z_i} &= 2E^T z + \rho \lambda + \rho \mu = 0
\end{align*}
\]
where $T = \text{blkdiag}(T_i)$.

We now turn our interest to solving $\min(\theta) L_p$ for fixed $(x, \theta, z)$. We notice that the gradient of the Lagrangian with respect to $\theta$ is given by
\[
\frac{\partial L_p}{\partial \theta} = \frac{\partial f_i}{\partial \theta} + \rho \theta + \mu - \rho E \theta_i
\]
which should be zero for the optimal $\theta$. Since $T$ is linear in $\theta$ the above equation is a linear system of equations. Notice that $\frac{\partial E^T}{\partial \theta}$ is block diagonal, and hence the above equations distribute nicely over $i$. We will later on for a specific model derive more explicit equations for updating $\theta$.

We summarize the ADMM algorithm in Table I. The residuals and tolerances in the stopping criterion in step 5 are defined as follows [3]:
\[
\begin{align*}
r_p &= \langle z - Ax - b, \theta - \hat{E} \theta_i \rangle \\
r_d &= \rho \langle A^T (z_{\text{prev}} - z), E^T (\theta_{\text{prev}} - \theta) \rangle \\
e_p &= \sqrt{(m + \rho) N + q \rho} \\
&+ \epsilon_{\text{rel}} \max \{ \| (Ax, E \theta_i) \|_2, \| \theta_i \|_2, \| \theta \|_2 \} \\
e_d &= \sqrt{n N + r \rho} + \epsilon_{\text{rel}} \| (A^T \lambda, E^T \mu) \|_2
\end{align*}
\]

Typical values for the relative and absolute tolerances are $\epsilon_{\text{rel}} = 10^{-3}$ and $\epsilon_{\text{abs}} = 10^{-6}$. The vectors $z_{\text{prev}}$ and $\theta_{\text{prev}}$ in (9) are the values of $z$ and $\theta$ in the previous iteration.

Instead of using a fixed penalty parameter $\rho$, one can vary $\rho$ to improve the speed of convergence. An example of such a scheme is to adapt $\rho$ at the end of each ADMM iteration as follows [3]
\[
\rho := \begin{cases} 
\tau \rho & \| r_p \|_2 > \mu \| r_d \|_2 \\
\rho / \tau & \| r_d \|_2 > \mu \| r_p \|_2 \\
\rho & \text{otherwise}.
\end{cases}
\]
This scheme depends on parameters $\mu > 1$, $\tau > 1$ (for example, $\mu = 10$ and $\tau = 2$).

V. DISTRIBUTED IMPLEMENTATION

We have so far seen that the equations for updating $\theta$ in (7) can be carried out distributively over $i = 1, \ldots, M$ by solving
\[
\frac{\partial L_p}{\partial \theta_i} = \frac{\partial f_i}{\partial \theta_i} + \rho \theta_i + \mu_i - \rho \hat{E} \theta_i
\]
\[
= 2 \frac{\partial E^T}{\partial \theta_i} T_i(\theta) z_i + \rho \theta_i + \mu_i - \rho \hat{E} \theta_i = 0
\]
because $\frac{\partial E^T}{\partial \theta_i}$ and $T(\theta)$ are block diagonal. In the right hand side we are however interested in explaining the term $\hat{E} \theta_i$, further. It is not uncommon that $E$ is an incidence matrix of zeros and ones describing what component of $\theta_i$ is related to each component in $\theta$. We write
\[
E = \begin{bmatrix} E_1 \\ \vdots \\ E_M \end{bmatrix}
\]
where the partitioning is done conformable with the partitioning of $\theta$. In a graph setting we consider each component of $\theta_i$ to be represented by its index in the vertex set $V_0 = \{1, \ldots, q_i\} \subset \mathbb{Z}$ and each component of $\theta_i$ to be represented by its index in the vertex set $V_i = \{1, \ldots, q_i\} \subset \mathbb{Z}$. The $i$th graph has a directed edge $e \in V_0 \times V_i$ if and only $(E_i)_e = 1$. We denote the set of all edges of the graph by $\mathcal{E}_0$. It then follows that we may write
\[
2 \frac{\partial E^T}{\partial \theta_i} T_i(\theta) z_i + \rho \theta_i + \mu_i - \rho \hat{\theta}_i = 0
\]
where $\hat{\theta}_i = \theta_i^\circ$ if $(j, k) \in \mathcal{E}_0$ and zero otherwise. Hence for each $i$ information is needed only from the components of $\theta_i$ that are defining $\theta_i$.

We will now discuss how also (5) and (6) distribute over $i$. First we consider (5). The out degree $d_{0,i}(j)$ of a vertex

TABLE I
ADMM ALGORITHM

1) Set $x = 0$, $\theta_i = 0$, $z = b$, $\lambda = 0$, $\mu = 0$, $\rho = 1$ and $\theta_0$ to a good guess.
2) Update $(x, \theta, z) := \text{argmin}_{x, \theta, z} \mathcal{L}(x, \theta, z, \theta, \lambda)$.
3) Update $\theta := \text{argmin}_\theta \mathcal{L}(x, \theta, z, \theta, \lambda)$.
4) Update $(\lambda, \mu) := (\lambda + \rho (z - Ax - b), \mu + \rho (\theta - \hat{E} \theta_i))$.
5) Terminate if $\| r_p \|_2 \leq \epsilon_p$ and $\| r_d \|_2 \leq \epsilon_d$ (see (5) and (12)). Otherwise, go to step 2.
\( j \in \mathcal{G}_0 \) is the number of edges that emerges from it in graph \( \mathcal{G}_0 \). It follows that

\[
E^T E = \text{diag}(d_0(j))
\]

where \( d_0(j) = \sum_{i=1}^{M} d_0(i,j) \). We now realize that we can updated each component in \( \theta_0 \) using the formula

\[
\theta_{0,j} = \frac{1}{d_0(j)} \sum_{(j,k) \in \mathcal{G}_0} \theta_{j,k}, \quad j \in \mathcal{G}_0
\]

We see that we only sum over those components of \( \theta \) which are defined by \( \theta_{0,j} \), and that the computations can be performed locally for each component of \( \theta_0 \).

We now consider (6). We notice that we can first solve

\[
(2T_i^T T_i + \rho I) z_i = \rho (Ax)_i - \mu_i
\]

with respect to \( x \), where \( r = \lambda - \rho h \). Then we can solve

\[
(2T_i^T T_i + \rho I) z_i = \rho (Ax)_i - \mu_i
\]

with respect to \( z_i \) for \( i = 1, \ldots, M \). The latter equation clearly distributes over \( i \) for the left hand side, and for the right hand side we are interested in what information about \( x \) that is needed for each block \( i \), i.e. what \( (Ax)_i \) is. We remember that \( A = (Q\Lambda) \otimes I_N \), that \( \Lambda \) is a zero one matrix, and that \( Q \) is a permutation matrix. Hence \( A \) is also a zero one matrix. We let \( \tilde{A} = Q\Lambda \), and we partition it as

\[
\tilde{A} = \begin{bmatrix} \tilde{A}_1 \\ \vdots \\ \tilde{A}_M \end{bmatrix}
\]

where the partitioning is done conformable with \( z \). Then \( (Ax)_i = (\tilde{A}_i \otimes I_N) x \), and hence we may rewrite (13) as

\[
(2T_i^T T_i + \rho I) z_i = \rho (Ax)_i - \mu_i, \quad i = 1, \ldots, M
\]

Hence we are able to update each \( z_i \) locally with information only from those components of \( x \) which are used to explain \( z_i \).

We now turn our interest to-wards (13) and define \( X_i = I - \rho (2T_i^T T_i + \rho I)^{-1} \) and \( X = \text{blkdiag} X_i \). We then realize that \( A^T X A = \sum_{i=1}^{M} (\tilde{A}_i^T X \tilde{A}_i) \otimes I_N \), and hence

\[
x = -\frac{1}{\rho} \left\{ \left( \sum_{i=1}^{M} \tilde{A}_i^T X \tilde{A}_i \right)^{-1} \left( \sum_{i=1}^{M} \tilde{A}_i^T X_i \right) \otimes I_N \right\} r
\]

We see that we need global information in order to carry out the update of \( x \). However, we also realize that the matrix that needs to be inverted only has dimension \( n \), which is typically low.

**VI. FEEDBACK CONNECTION OF ARX-MODELS**

In this section we will give a description of a simple feedback connection of three ARX models:

\[
y_i(k) + a_{1,i}y_i(k-1) + a_{2,i}y_i(k-2) + b_{1,i}u(k-1) + b_{2,i}u(k-2) = e_i(k)
\]

where \( k = 1, \ldots, N \) and \( i = 1, 2, 3 \). We let \( \theta_i = (a_i, b_i) \in \mathbb{R}^4 \), and we define \( \theta_0 \) such that we may take \( E = I \), i.e. the parameters of the models are not constrained in any way. The interconnection matrices are given by

\[
\Gamma = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

Moreover we measure all outputs, i.e. \( C = I \). We may write

\[
e_i = \Phi_i \theta_i + y_i
\]

where \( \Phi_i = \begin{bmatrix} S_{y_i} & S_{y_i} & S_{u} & S_{u} \end{bmatrix} \), where \( S \) is a shift matrix. Hence (7) may be equivalently written as

\[
(\Phi_i^T \Phi_i + \rho I) \theta_i = \rho E \theta_0 - \mu_i - 2\Phi_i y_i
\]

where \( \Phi = \text{blkdiag} \Phi_i \). The distributed version is

\[
(\Phi_i^T \Phi_i + \rho I) \theta_i = \rho E \theta_0 - \mu_i - 2\Phi_i y_i, \quad i = 1, \ldots, M
\]

We remark that for this example the dimension \( n \) of the \( x \)-variable is zero.

**VII. DISCRETIZED PARTIAL DIFFERENTIAL EQUATION**

We will also consider a model that comes from a spatial discretization of a partial differential equation, which is defined as

\[
y_i(k) + (u_i(k) + (y_i(k-1) - y_i(k-2))(21) = (b_i)_{1} u_i(k) + e_i(k), \quad i = 1, \ldots, M
\]

where \( u_1(k), u_M(k) \in \mathbb{R}^2 \), \( u_3(k), u_{M-1}(k) \in \mathbb{R}^4 \), and \( u_i(k) \in \mathbb{R}^2 \) for \( i = 3, \ldots, M-2 \), and where \( y_1(k), e_i(k) \in \mathbb{R} \). The dimensions of \( a_i \) and \( b_i \) are compatible with the signal dimensions. The inputs are partially feedbacks from the neighboring systems according to

\[
u_1(k) = \begin{bmatrix} u_{0,1}(k) \\ y_2(k) \\ y_3(k) \end{bmatrix}, \quad \cdots, \quad u_M(k) = \begin{bmatrix} y_{M-3}(k) \\ y_{M-2}(k) \\ y_M(k) \end{bmatrix}
\]

\[
u_{M-1}(k) = \begin{bmatrix} y_{M-3}(k) \\ y_{M-2}(k) \\ y_{m}(k) \end{bmatrix}, \quad \cdots, \quad u_2(k) = \begin{bmatrix} u_{0,2}(k) \\ y_3(k) \\ y_4(k) \end{bmatrix}
\]

\[
u_i(k) = \begin{bmatrix} u_{0,i}(k) \\ y_{i-1}(k) \\ y_{i-1}(k) \\ y_{i-1}(k) \end{bmatrix}, \quad i = 3, \ldots, M-2
\]

\[
u_M(k) = \begin{bmatrix} y_{M-3}(k) \\ y_{M-2}(k) \\ y_{M}(k) \end{bmatrix}
\]
where \( u_0, \theta_0 \) are measured inputs. This defines \( \Gamma \) and \( B \). Moreover we measure every second output \( y_i(k) \), i.e.

\[
C = \begin{bmatrix}
\begin{array}{c}
\epsilon_1^T \\
\epsilon_2^T \\
\vdots \\
\epsilon_{M-2}^T \\
\epsilon_M^T
\end{array}
\end{bmatrix}
\]

where \( \epsilon_i \) is the \( i \)th unit vector with abuse of notation. We will also assume that \( M \geq 5 \) and that \( M \) is an odd integer. We let \( \theta_0 = (a_0, b_0) \in \mathbb{R}^3 \), \( \theta_i = (a_i, b_i) \in \mathbb{R}^{2+M_i} \). We then define the constraints \( a_i = a_0 \) and

\[
\begin{align*}
b_1 &= b_0 \\
bb(k) &= \begin{bmatrix}
\epsilon_1^T \\
\epsilon_2^T \\
\vdots \\
\epsilon_{i-1}^T \\
\epsilon_i^T \\
\epsilon_{i+1}^T \\
\epsilon_{i+2}^T \\
\vdots \\
\epsilon_M^T
\end{bmatrix}b_0, \quad i = 3, \ldots, M - 2 \\
b_{M-1}(k) &= \begin{bmatrix}
\epsilon_1^T \\
\epsilon_2^T \\
\vdots \\
\epsilon_{M-2}^T \\
\epsilon_M^T
\end{bmatrix}b_0 \\
b_{M}(k) &= \begin{bmatrix}
\epsilon_1^T \\
\epsilon_2^T \\
\vdots \\
\epsilon_{M-1}^T
\end{bmatrix}b_0
\end{align*}
\]

where \( \epsilon_i \) is the \( i \)th unit vector in \( \mathbb{R}^3 \). This defines \( E \), and the overall model. We now define

\[
\Phi_i = \begin{bmatrix}
S_i \\
S_i^2y_i \\
-U^T
\end{bmatrix}
\]

where \( S \) is a shift matrix of compatible dimension and where \( U \) is such that \( S^m_{ui} = \text{vec}(U) \) with \( \text{vec} \) being the vectorization operator. Here \( S \) has different dimension depending on where it appears. Then

\[
e_i = \Phi_i \theta_i + y_i
\]

and hence (20) may be equivalently written as

\[
(\Phi^T \Phi + \rho I) \theta = \rho E \theta_0 - \mu - 2\Phi y
\]

where \( \Phi = \text{blkdiag} \Phi_i \). The distributed version is

\[
(\Phi^T \Phi + \rho I) \theta_i = \rho \theta_i - \mu_i - 2\Phi_i y_i, \quad i = 1, \ldots, M
\]

VIII. NUMERICAL EXPERIMENTS

All implementations have been carried out in MATLAB R2013b. The computations have been run on an Intel Core i5 CPU M 250 4 GHz with 4 GB of RAM.

A. Feedback Connection of ARX-Models

All ARX models have been defined as \( a_i = (-1.5, 0.7) \) and \( b_i = (-0.1, 0.1) \) for \( i = 1, 2, 3 \). The input \( u_0 \) has been taken as a sequence of independent \( \pm 1 \)-variables. The error vector \( e \) has been generated from a zero mean normal density function with standard deviation \( \sigma = 1 \). Then the closed loop signals have been computed from the equations

\[
\begin{align*}
(b_{\text{diag}}(T_{x,i}) + \text{blkdiag}(T_{a,i})(\Gamma \otimes I_N))y &= (27) \\
(- \text{blkdiag}(T_{a,i})(B \otimes I_N))u_0 + e &= (28) \\
u &= (\Gamma \otimes I_N)y + (B \otimes I_N)u_0 (29) \\
y_0 &= (C \otimes I_N)y (30)
\end{align*}
\]

The value of \( N \) has been 300. We have used the default settings for the ADMM algorithm as detailed above. The initial guess for \( \theta_0 \) was the zero vector.

We repeated the optimization 100 times. The mean value of the estimated parameters were

\[
\begin{align*}
m_{\theta_0} &= [-1.4988 \quad 0.7013 \quad -0.964 \quad 0.965]^T \\
m_{\theta_2} &= [-1.4934 \quad 0.6923 \quad -0.1068 \quad 0.1071]^T \\
m_{\theta_3} &= [-1.4897 \quad 0.6896 \quad -0.1105 \quad 0.1084]^T
\end{align*}
\]

with standard deviations

\[
\begin{align*}
\sigma_{\theta_1} &= [0.0371 \quad 0.0385 \quad 0.0321 \quad 0.0315]^T \\
\sigma_{\theta_2} &= [0.0457 \quad 0.0473 \quad 0.0342 \quad 0.0349]^T \\
\sigma_{\theta_3} &= [0.0435 \quad 0.0408 \quad 0.0476 \quad 0.0473]^T
\end{align*}
\]

We see that the model parameters are estimated accurately.

B. Discretized Partial Differential Equation

The dynamical system considered has been \( a_0 = (0.7, 0.9) \) and \( b_0 = (0.5, -0.5, 0.5) \). The input \( u_0 \) has been taken as a sequence of independent \( \pm 1 \)-variables. The error vector \( e \) has been generated from a zero mean normal density function with standard deviation \( \sigma = 1 \). Then the closed loop signals have been computed in the same way as for the previous example. The value of \( N \) has been 100 and the value of \( M \) has been 15. We have used the default settings for the ADMM algorithm as detailed above except for \( \epsilon_{\text{rel}} = 10^{-1} \) and \( \epsilon_{\text{abs}} = 10^{-4} \), which provided good enough solutions. The initial guess for \( \theta_0 \) was the true value of its components perturbed with a value drawn from a zero mean normal density with standard deviation \( 0.1 \).

We repeated the optimization 10 times and we report in Table III computer time, and the number of iterations in the ADMM algorithm for the different runs. The mean value of the estimated parameters were

\[
\begin{align*}
m_{\theta_0} &= [0.7017 \quad 0.8950 \quad 0.4958 \quad -0.4966 \quad 0.4957]^T \\
\sigma_{\theta_0} &= [0.0075 \quad 0.0110 \quad 0.0212 \quad 0.0089 \quad 0.0086]^T
\end{align*}
\]

It is seen that the proposed algorithm computes good estimates of the true parameters in reasonable time. It should be stressed that we have not made use of parallel or distributed implementations. Hence the computational times should be possible to decrease significantly. It should also be noted that the our results rely on a good initial guess of \( \theta_0 \).
TABLE II

| Run nr | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|--------|------|------|------|------|------|------|------|------|------|------|
| Iterations | 177  | 107  | 77   | 135  | 34   | 84   | 306  | 95   | 177  | 105  |
| Time (s)    | 515.9| 306.2| 219.4| 406.6| 106.8| 246.3| 3164.8| 262.5| 496.9| 305.9|

IX. SUMMARY

To summarize it looks like it should be possible to solve identification problems of interconnected systems where we do not measure all input or output signals in a distributed way. An open question is how much need to be measured to have a unique solution. Also can this framework be used to solve identification problems for state space descriptions when one impose structure on the system matrices? Our framework addresses as a special case distributed estimation of signals by assuming that $\theta$ is known. We admit that in case no good guess of the true parameters are available to initialize the ADMM algorithm, it may fail to find the global optimal solution. It may instead be trapped in a local minimum. Future research will investigate the possibility to use continuation methods to remedy this flaw.

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