PERTURBATION THEORY FOR SECOND ORDER ELLIPTIC OPERATORS WITH BMO ANTISYMMETRIC PART

MARTIN DINDOŠ, ERIK SÄTTERQVIST, MARTIN ULMER

ABSTRACT. In the present paper we study perturbation theory for the $L^p$ Dirichlet problem on bounded chord arc domains for elliptic operators in divergence form with potentially unbounded antisymmetric part in BMO. Specifically, given elliptic operators $L_0 = \text{div}(A_0 \nabla)$ and $L_1 = \text{div}(A_1 \nabla)$ such that the $L^p$ Dirichlet problem for $L_0$ is solvable for some $p > 1$; we show that if $A_0 - A_1$ satisfies certain Carleson condition, then the $L^q$ Dirichlet problem for $L_1$ is solvable for some $q \geq p$. Moreover if the Carleson norm is small then we may take $q = p$. We use the approach first introduced in Fefferman-Kenig-Pipher ’91 on the unit ball, and build on Milakis-Pipher-Toro ’11 where the large norm case was shown for symmetric matrices on bounded chord arc domains. We then apply this to solve the $L^p$ Dirichlet problem on a bounded Lipschitz domain for an operator $L = \text{div}(A \nabla)$, where $A$ satisfies a Carleson condition similar to the one assumed in Kenig-Pipher ’01 and Dindoš-Petermichl-Pipher ’07 but with unbounded antisymmetric part.

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1. Introduction

The study of perturbations of elliptic operators in divergence form \( L := \text{div}(A \nabla \cdot) \) goes back to a result of Dahlberg [Dah86]. Specifically, given elliptic operators \( L_0 = \text{div}(A_0 \nabla) \) and \( L_1 = \text{div}(A_1 \nabla) \), where we know that the \( L^p \) Dirichlet problem for \( L_0 \) is solvable, he considered the discrepancy function \( \varepsilon(X) := |A_0(X) - A_1(X)| \), and showed that if the measure

\[
d\mu(Z) = \sup_{X \in B(Z, \delta(Z)/2)} \frac{\varepsilon(X)^2}{\delta(X)} dZ, \quad \text{with} \quad \delta(Z) := \text{dist}(Z, \partial\Omega),
\]

is a Carleson measure with vanishing Carleson norm, then the solvability of the \( L^p \) Dirichlet problem is transferred to \( L_1 := \text{div}(A_1 \nabla \cdot) \) with the same exponent \( p \).

Actually this was formulated in terms of properties of the corresponding elliptic measures \( \omega_0 \) and \( \omega_1 \) since we know that the \( L^p \) Dirichlet problem for \( L = \text{div}(A \nabla \cdot) \) is solvable iff the elliptic measure \( \omega \) associated with \( L \) belongs to the reverse H"older space \( B_p(d\sigma) \), where \( d\sigma \) is surface measure on \( \partial\Omega \). In this language Dahlberg has shows that if the Carleson norm of \( \mu \) is small, then \( \omega_0 \in B_p(d\sigma) \) implies \( \omega_1 \in B_p(d\sigma) \).

A natural question that arose was whether the condition on \( \mu \) could be relaxed to draw the weaker conclusion that \( \omega_0 \in A_\infty(d\sigma) \) implies \( \omega_1 \in A_\infty(d\sigma) \), where \( A_\infty(d\sigma) = \bigcup_{q>1} B_q(d\sigma) \); i.e. transferring solvability to \( L_1 \) but not necessarily with the same exponent. After some progress was made in [Fef89] it was finally shown in [FKP91].

To summarize two different types of results were established

(L) If the Carleson norm of \( \mu \) is bounded then \( \omega_0 \in A_\infty(d\sigma) \) implies \( \omega_1 \in A_\infty(d\sigma) \).

(S) If the Carleson norm of \( \mu \) is small then \( \omega_0 \in B_p(d\sigma) \) implies \( d\omega_1 \in B_p(d\sigma) \).

In [Dah86] and [FKP91] the results were only proved for symmetric matrices in the case \( \Omega = \mathbb{R}^n \subset \mathbb{R}^n \). Since then there has been some work to extend this result to more general domains. In [MPT11] the authors extend (L) to the case where \( \Omega \) is a bounded chord-arc domain (see Definition 2.4).

These results were recently generalized to 1-sided chord-arc domains in [CHM19] where the authors show both type (L) and (S) results. In the second part [CHMT20] they also prove a type (L) results for non-symmetric bounded matrices. Finally an (S) type result for bounded matrices was obtained in [AHMT21].

In this paper we relax the boundedness hypothesis on the coefficients and assume that an elliptic matrix \( A \) has (potentially unbounded) antisymmetric part. These operators where first studied in the elliptic case by Li and Pipher in [Li19], where they have shown that under the assumption that the antisymmetric part of the matrix \( A \) belongs to BMO space (and the symmetric part is bounded) then the usual elliptic theory holds for such operators and in particular we have the usual Harnack’s inequality, interior and boundary Hölder continuity, etc.

We note that in our approach we need to assume that \( \Omega \) is a chord-arc domain as we currently require the exterior cone condition (see Remark 2.19) to hold. There is an opportunity that new techniques as in [CHM20, AHMT21] will remove this assumption in the future.
Recall that a matrix $A$ is\emph{elliptic} that there exists $\lambda_0$ such that
\begin{equation}
\lambda_0|\xi|^2 \leq \xi^T A(X) \xi \leq \lambda_0^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } X \in \Omega.
\end{equation}
Note that even in the case where the matrix $A$ is not symmetric, ellipticity is only a condition on the symmetric part of the matrix $A^s$.

For the antisymmetric part $A^a$ we ask that $\|A^a\|_{\text{BMO}(\Omega)} \leq \Lambda_0$, i.e.
\begin{equation}
\sup_{Q \subset \Omega} \frac{\int_Q |A^a(Y) - (A^a)_Q| dY}{\text{vol}(Q)} \leq \Lambda_0.
\end{equation}

Our main results are as follows. We generalize (L) and (S) type results to operators as above in\ref{thm:1.5} and \ref{thm:1.6}. Instead of using the Carleson measure $\mu$ defined as in (1.1) which uses the $L^\infty$ norm, we introduce a more generalized version that allows $A$ to be unbounded, namely that
\[ d\mu'(Z) := \frac{\beta_r(Z)^2}{\delta(Z)} dZ \]
where
\begin{equation}
\beta_r(Z) := \left( \int_{B(Z, \delta(Z)/2)} |A_1 - A_0|^r \right)^{1/r},
\end{equation}
for some large fixed $1 \leq r < \infty$ which only depends on $n, \lambda_0$ and $\Lambda_0$. Recall that by the John-Nirenberg inequality a function in BMO belongs to all $L^r$ spaces $r < \infty$. Observe that even if we restrict ourselves to bounded matrices $A$ this new Carleson measure $\mu'$ has smaller Carleson norm than the original measure $\mu$. In particular, it follows that perturbation result\ref{thm:MPT11, Theorem 8.1} is a special case of \ref{thm:1.5}.

We are ready to state two perturbation results:

\begin{theorem}
Let $\Omega \subset \mathbb{R}^n$ be a bounded chord arc domain and $L_0 = \text{div}(A_0 \nabla \cdot)$ and $L_1 = \text{div}(A_1 \nabla \cdot)$ two elliptic operators that satisfy (1.2) and (1.3). Let $\omega_0$ and $\omega_1$ be the corresponding elliptic measures. Then there exists $1 \leq r = r(n, \lambda_0, \Lambda_0) < \infty$ such that if $d\mu'(Z) := \frac{\beta_r(Z)^2}{\delta(Z)} dZ$ is a Carleson measure then $\omega_0 \in A_\infty(\delta\sigma)$ implies $\omega_1 \in A_\infty(\delta\sigma)$. Thus if the $L^p$ Dirichlet problem for $L_0$ is solvable this implies solvability of the $L^q$ Dirichlet problem for $L_1$, for some $q > p$.
\end{theorem}

\begin{theorem}
Let $\Omega \subset \mathbb{R}^n$ be a bounded chord arc domain and $L_0 = \text{div}(A_0 \nabla \cdot)$ and $L_1 = \text{div}(A_1 \nabla \cdot)$ two elliptic operators that satisfy (1.2) and (1.3). Let $\omega_0$ and $\omega_1$ be the corresponding elliptic measures. Let $1 < p < \infty$ and assume that $\omega_0 \in B_p(\delta\sigma)$. Then there exists $1 \leq r = r(n, \lambda_0, \Lambda_0) < \infty$ and $\gamma = \gamma(n, p, [\omega_0]_{B_p}, \lambda_0, \Lambda_0) > 0$ such that if $\|\mu'\|_C \leq \gamma$ then $\omega_1 \in B_p(\delta\sigma)$. Thus if the $L^p$ Dirichlet problem for $L_0$ is solvable this implies solvability of the $L^p$ Dirichlet problem for $L_1$, for the same exponent $p$.
\end{theorem}

(The Carleson norm $\| \cdot \|_C$ is defined below in Definition 2.5.)

When we started to develop the above perturbation theory for unbounded operators we had in mind one particular application in the spirit of papers by Kenig and Pipher\ref{KP01} and Dindoš, Petermichl and Pipher\ref{DPP07} and extend such results to unbounded matrices.
To summarize [KP01, DPP07], it follows them that if \( \Omega \) is a Lipschitz domain and \( A : \Omega \to \mathbb{R}^{n \times n} \) is a bounded elliptic matrix such that
\[
d\hat{\mu}(X) := \delta(X)^{-1} \sup_{B(X, \delta(X)/2)} \text{osc}_{ij} |a_{ij}(Z)|^2
\]
is a Carleson measure then the \( L^p \) Dirichlet problem is solvable for some large \( p < \infty \). Additionally, if \( p \in (1, \infty) \) is given and both the Lipschitz character of our domain and the Carleson norm of \( \hat{\mu} \) is sufficiently small then we can conclude solvability of the \( L^p \) Dirichlet problem for this given value of \( p \). So again we have one large-Carleson and one small-Carleson type result.

To obtain this one needs perturbation results since in the paper [DPP07] mollification procedure is used to replace above Carleson condition with
\[
d\hat{\mu}(X) := \sup_{B(X, \delta(X)/2)} |\nabla A(Z)|^2 \delta(Z).
\]
This gives the authors better matrix to work with and get the conclusions. To deduce the same for the original matrix we apply our Theorems Theorem 1.5 and Theorem 1.6 and improve conclusions of [KP01, DPP07] to unbounded matrices.

We note that under a different assumption of so-called \( t \)-independence of the coefficients on \( A \) the solvability of the \( L^p \) Dirichlet problem for matrices with BMO antisymmetric part was shown in [HLMP22].

**Theorem 1.7.** Let \( \Omega \) be a bounded Lipschitz domain with Lipschitz character \( K > 0 \) (that is the Lipschitz constant of graphs describing \( \partial \Omega \) is bounded by \( K \)). Let \( L_0 = \text{div}(A \nabla \cdot \cdot) \) be an elliptic operator satisfying (1.2) and (1.3) let \( \alpha_r \) be
\[
(1.8) \quad \alpha_r(Z) := \left( \int_{B(Z, \delta(Z)/2)} |A - (A)_{B(Z, \delta(Z)/2)}|^r dY \right)^{1/r}.
\]
Then for every \( 1 < p < \infty \) there exists \( r = r(n, \lambda_0, \Lambda_0) > 1 \) and \( \varepsilon = \varepsilon(p) > 0 \) such that if
\[
\|\alpha_r(Z)^2 \delta(Z)^{-1} dZ\|_c < \varepsilon \quad \text{and} \quad K < \varepsilon,
\]
then \( \omega \in B_p(\sigma) \), i.e. the \( L^p \) Dirichlet problem is solvable for the operator \( L_0 \) in \( \Omega \).

Similarly, [KP01] can be improved as follows:

**Theorem 1.9.** Let \( \Omega \) be a bounded Lipschitz domain and \( L_0 = \text{div}(A \nabla \cdot \cdot) \) an elliptic operator satisfying (1.2) and (1.3). Consider \( \alpha_r \) defined as above in (1.8). Then there exists \( r = r(n, \lambda_0, \Lambda_0) > 1 \) such that if
\[
\|\alpha_r(Z)^2 \delta(Z)^{-1} dZ\|_c < \infty
\]
then the corresponding elliptic measure of \( L_0 \) belongs to \( \omega \in A_\infty(d\sigma) \), i.e. the \( L^p \) Dirichlet problem for \( L_0 \) is solvable for all \( p \in (p_0, \infty) \) where some \( p_0 > 1 \) is sufficiently large.

It follows the we now have a larger class of elliptic operators that solve the \( L^p \) Dirichlet problem on bounded Lipschitz domains than was previously known, since we are replacing the oscillation of \( A \) measured in \( L^\infty \) norm by an an \( L^p \) mean oscillation for some large \( p > 2 \).

It is worth noting that the study of boundary value problems for scalar elliptic operators has a long history. The reader might be interested to read more in the
survey paper [DP22] in this volume.

The paper is organized as follows: We start with Section 2 containing definitions and other preliminaries. In Section 3, we outline the proof of Theorem 1.5, which closely follows that of [MPT11] and [FKP91]; this contains a subsection with results needed to prove the key identity

\[ F(X) := u_1(X) - u_0(X) = \int_{\Omega} \varepsilon \nabla u_1(Y) \nabla Y G_0(X,Y) dY. \]

The meat of the proof of Theorem 1.5 consists of proving Lemma 3.3 and Lemma 5.3, which is done in Sections 4 and 5 respectively. With these results established, Theorem 1.6 follows (Section 6). Finally, in Section 7 we prove Theorem 1.9 and Theorem 1.7.

We note that the paper [FKP91] contains some gaps that were unfortunately carried over to [MPT11]; we have rectified those. For more details see remarks 4.1 and 4.10.

2. Preliminaries

Here and in the following sections we implicitly allow all constants to depend on \( n, \lambda_0 \) and \( \Lambda_0 \).

**Definition 2.1.** \( \Omega \subset \mathbb{R}^n \) satisfies the corkscrew condition with parameters \( M > 1, r_0 > 0 \) if, for each boundary ball \( \Delta := \Delta(Q,r) \) with \( Q \in \partial \Omega \) and \( 0 < r < r_0 \), there exists a point \( A(Q,r) \in \Omega \), called a corkscrew point relative to \( \Delta \), such that \( B(A(Q,r), M^{-1}r) \subset T(Q,r) \).

**Definition 2.2.** \( \Omega \) is said to satisfy the Harnack chain condition if there is a constant \( c_0 > 0 \) such that for each \( \rho > 0, \Lambda \geq 1, X_1, X_2 \in \Omega \) with \( \delta(X_j) \geq \rho \) and \( |X_1 - X_2| \leq \Lambda \rho \), there exists a chain of open balls \( B_1, \ldots, B_N \subset \Omega \) with \( N \leq \Lambda \) 1, \( X_1 \in B_1, X_2 \in B_N, B_i \cap B_{i+1} \neq \emptyset \) and \( c_0^{-1}r(B_i) \leq \text{dist}(B_i, \partial \Omega) \leq cr(B_i) \). The chain of balls is called a Harnack chain.

**Definition 2.3 (NTA).** \( \Omega \) is an Non-Tangentially Accessible domain if it satisfies the Harnack chain condition and \( \Omega, \mathbb{R}^n \setminus \bar{\Omega} \) both satisfy the corkscrew condition. If only \( \Omega \) satisfied the corkscrew condition then it is called a 1-sided NTA domain or uniform domain.

**Definition 2.4 (CAD).** Let \( \Omega \subset \mathbb{R}^n \). \( \Omega \) is called chord arc domain (CAD) if \( \Omega \) is a NTA set of locally finite perimeter and Ahlfors regular boundary, i.e. there exists \( C \geq 1 \) so that for \( r \in (0, \text{diam}(\Omega)) \) and \( Q \in \partial \Omega \)

\[ C^{-1}r^{n-1} \leq \sigma(B(Q,r)) \leq Cr^{n-1}. \]

Here \( B(Q,r) \) denotes the \( n \)-dimensional ball with radius \( r \) and center \( Q \) and \( \sigma \) denotes the surface measure. The best constant \( C \) in the condition above is called the Ahlfors regularity constant.

If we replace NTA domain with 1-sided NTA domain in the above definition then \( \Omega \) is called a 1-sided chord arc domain (1-sided CAD).

Throughout this paper \( \Omega \) will denote a bounded CAD.
Definition 2.5. For a measure \( \mu \) on \( \Omega \) if the quantity

\[
\| \mu \|_C := \sup_{\Delta \subset \partial \Omega} \frac{1}{\sigma(\Delta)} \int_{T(\Delta)} d\mu,
\]

is finite then \( \mu \) is said to be the Carleson measure and \( \| \mu \|_C \) its Carleson norm. Here the Carleson region \( T(\Delta) \) of a boundary ball \( \Delta = \Delta(Q,r) := B(Q,r) \cap \partial \Omega \) is defined as \( T(\Delta(Q,r)) = \overline{B(Q,r)} \cap \Omega \).

Proposition 2.6. Let \( b \) be a constant anti-symmetric matrix and let \( u \in W^{1,2}(E) \) and \( v \in W^{1,2}_0(E) \), with \( E \subset \Omega \) measurable. Then

\[
\hat{E} b \nabla u \cdot \nabla v = 0.
\]

Proof. Note that if \( b \) is a constant anti-symmetric matrix and \( E \subset \Omega \), then for \( u \in W^{1,2}(E) \) and \( \phi \in C^\infty_c(E) \) we have

\[
\hat{E} b \nabla u \cdot \nabla \phi = \int_E b_{ij} \partial_i u \partial_j \phi = \int_E u \partial_i (b_{ij} \partial_j \phi) = \int_E u b_{ij}^T \partial_j \phi = -\int_E u b_{ij} \partial_j \phi = -\int_E \partial_i (b_{ij} u) \partial_j \phi
\]

\[
= -\int_E b_{ij} \partial_i u \partial_j \phi = -\int_E b \nabla u \cdot \nabla \phi.
\]

Denoting by \( (A^a_i)_{E} \) the constant matrix of component-wise means of \( A \) on \( E \). It follows that for \( u, v \) as above

\[
\int_E A_i \nabla u \cdot \nabla v = \int_E (A_i - (A^a_i)_{E}) \nabla u \cdot \nabla v.
\]

2.1. Muckenhoupt and Reverse Hölder spaces. Let \( \mu \) be a doubling measure on \( \partial \Omega \) and let \( w : \partial \Omega \rightarrow [0,\infty) \). Furthermore, let \( 1 < p < \infty \) and let \( p' \) denote its Hölder conjugate i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \).

Definition 2.8 (Muckenhoupt spaces). We define the Muckenhoupt spaces \( A_1(\mu) \), \( A_p(\mu) \), \( A_\infty(\mu) \) by:

- \( w \in A_p(\mu) \) iff there exists \( C > 0 \) such that for all balls \( B \subset \mathbb{R}^n \)

\[
\left( \frac{1}{\mu(B)} \int_B w \, d\mu \right) \left( \frac{1}{\mu(B)} \int_B w^{1-p'} \, d\mu \right)^{p-1} \leq C < \infty.
\]

- \( w \in A_1(\mu) \) iff there exists \( C > 0 \) such that for \( \mu \)-a.e. \( x \in \mathbb{R}^n \) and balls \( B = B(x) \subset \mathbb{R}^n \) centered at \( x \)

\[
\frac{1}{\mu(B(x))} \int_{B(x)} w \, d\mu \leq Cw(x).
\]

- Finally, we set \( A_\infty(\mu) := \bigcup_{1 \leq p < \infty} A_p(\mu) \).

Definition 2.9 (Reverse Hölder spaces). We define the Reverse Hölder spaces \( B_p(\mu), B_\infty(\mu) \) by:
• \( w \in B_p(\mu) \) iff there exists \( C > 0 \) such that for all balls \( B \subset \mathbb{R}^n \)
\[
\left( \frac{1}{\mu(B)} \int_B w^p d\mu \right)^{1/p} \leq \frac{C}{\mu(B)} \int_B w d\mu.
\]
The best constant in the above estimate we shall denote by \([w]_{B_p}\).

• \( w \in B_\infty(\mu) \) iff there exists \( c > 0 \) such that for a.e. \( x \in \mathbb{R}^n \) and balls \( B = B(x) \subset \mathbb{R}^n \) centered at \( x \)
\[
cw(x) \leq \frac{1}{\mu(B)} \int_{B(x)} w d\mu.
\]

It is easy to see that the following hold:

• \( A_1(\mu) \subset A_p(\mu) \subset A_q(\mu) \subset A_\infty(\mu) \) for \( 1 \leq p < q < \infty \),
• \( B_\infty(\mu) \subset B_p(\mu) \) for \( 1 < p < q \leq \infty \), and
• \( A_\infty(\mu) = \bigcup_{p>1} B_p(\mu) \).

For more properties of these spaces we refer the reader to \([Gra09]\).

Suppose now that \( \nu \) is another doubling measure on \( \partial \Omega \). We say that \( \nu \in A_p(\mu) [B_q(\mu)] \) if \( \nu \ll \mu \) and the Radon-Nikodym \( w := \frac{d\nu}{d\mu} \in A_p(\mu) [B_q(\mu)] \).

2.2. The \( L^p \) Dirichlet boundary value problem.

**Definition 2.10.** Let \( u : \Omega \to \mathbb{R} \). The nontangential maximal function \( N[u] : \partial \Omega \to \mathbb{R} \) is defined as
\[
N_\alpha[u](Q) := \sup_{X \in \Gamma_\alpha(Q)} |u(X)|,
\]
where
\[
\Gamma_\alpha(Q) := \{ Y \in \Omega; |Y - Q| < \alpha \delta(Y) \},
\]
is the cone of aperture \( \alpha \) (for \( \alpha > 1 \)). Here \( \delta \) denotes the distance function to the boundary \( \partial \Omega \).

Let \( L = \text{div}(A \nabla \cdot) \), where \( A(X) \in \mathbb{R}^{n \times n} \) is a matrix, satisfying (1.2) and (1.3). We say that \( u \in W^{1,2}_{\text{loc}}(\Omega) \) is weak solution to the equation \( Lu = 0 \) in \( \Omega \) if
\[
\int_{\Omega} A \nabla u \nabla \varphi dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega),
\]
where \( C_0^\infty(\Omega) \) denotes the space of all smooth functions with compact support. We know (see e.g. \([Li19]\)) that if \( f \in C^0(\partial \Omega) \) then there exists a \( u \in W^{1,2}(\Omega) \cap C^0(\Omega) \) such that
\[
\begin{cases}
Lu = 0, & \text{in } \Omega, \\
u = f, & \text{on } \partial \Omega.
\end{cases}
\]

**Definition 2.12.** Let \( \alpha > 0 \). We say the \( L^p \) Dirichlet problem for the operator \( L \) is solvable, if for all boundary data \( f \in L^p(\partial \Omega) \cap C(\partial \Omega) \) the solution \( u \) as defined above satisfies the estimate
\[
\|N_\alpha(u)\|_{L^p(\partial \Omega)} \lesssim_\alpha \|f\|_{L^p(\partial \Omega)}.
\]
2.3. Elliptic measure. Recall that by Riesz theorem exists a measure $\omega^X$ such that for $u$ as above

$$u(X) = \int_{\Omega} f d\omega^X.$$ 

This is called the elliptic measure with pole at $X$. As noted in the introduction the $L^p$ Dirichlet problem is solvable iff $\omega \in B_{p}^\prime(d\sigma)$. For a proof see e.g. [Ken94] and the references therein.

2.4. Properties of solutions. In this sections we include some important results from Li’s thesis [Li19] that will be used later. These results hold for solutions on NTA domains. First we have reverse Holder’s and Caccioppoli’s inequalities for the gradient:

**Proposition 2.13** (Lemma 3.1.2). Let $u \in W^{1,2}_{\text{loc}}(\Omega)$ be a weak solution. Let $X \in \Omega$ and let $B_{R} = B_{R}(X)$ be such that $B_{R} \subset \Omega$ and let $0 < \sigma < 1$. Then there exists $p > 2$ such that

$$\left(\int_{B_{\sigma R}} |\nabla u|^p \right)^{1/p} \lesssim \left(\int_{B_{R}} |\nabla u|^2 \right)^{1/2}.$$ 

**Proposition 2.14.** For a $C = C(n, \lambda, \Lambda) < \infty$ we have for a solution $u$ and $B(X, 2R) \subset \Omega$

$$\int_{B(X,R)} |\nabla u(Z)|^2 dZ \leq \frac{C}{R^2} \int_{B(X,2R)} |u(Z)|^2 dZ.$$ 

**Proposition 2.15** (Lemma 3.1.4). Let $u \in W^{1,2}_{\text{loc}}(\Omega)$ be a weak solution of $Lu = 0$ and $B(X,2R) \subset \Omega$. Then

$$\sup_{B(X,R)} |u| \leq C(n, \lambda, \Lambda_0) \left(\int_{B(X,2R)} |u|^2 \right)^{1/2}.$$ 

Harnack’s inequality also does hold:

**Proposition 2.16** (Lemma 3.1.8). Let $u \in W^{1,2}_{\text{loc}}(\Omega)$ be a nonnegative weak solution and $B(X,2R) \subset \Omega$. Then

$$\sup_{B(X,R)} |u| \leq C(n, \lambda, \Lambda_0) \inf_{B(X,R)} |u|.$$ 

We also have the comparison principle.

**Proposition 2.17** (Proposition 4.3.6). Let $u, v \in W^{1,2}(T_{2r}(Q)) \cap C^0(T_{2r}(Q))$ be non-negative such that $Lu = Lv = 0$ in $T_{2r}(Q)$ and $u, v \equiv 0$ on $\Delta(Q,2r)$. Then

$$\frac{u(X)}{v(X)} \approx \frac{u(A_r(Q))}{v(A_r(Q))}, \quad X \in T_r(Q).$$

And the boundary Hölder estimate also holds.

**Proposition 2.18** (Lemma 3.2.5). Let $u \in W^{1,2}(\Omega)$ be a solution in $\Omega$ and $P \in \partial \Omega$. Suppose that $u$ vanishes on $\Delta(P, R)$. Then for $0 < r \leq R$ we have

$$\text{osc}_{T(P,r)} u \lesssim \Omega \left(\frac{r}{R}\right)^{\alpha} \sup_{T(P,R)} |u|.$$
Remark 2.19. The proof of this result uses the exterior corkscrew condition, i.e., that \( \mathbb{R}^n \setminus \bar{\Omega} \) satisfies Definition 2.1 and it is the reason why in the paper we assume that \( \Omega \) is a CAD rather than a 1-sided CAD domain.

An important corollary of the result above is the following lemma:

**Proposition 2.20.** Let \( u \geq 0 \) be a solution in \( \Omega \) that vanishes on \( \Delta(Q,2r) \). Then
\[
\sup_{T(\Delta(Q,r))} u \lesssim u(A(Q,r)).
\]
Here \( A(Q,r) \) is a corkscrew point inside \( \Omega \) w.r.t \( Q \) and \( r \) as defined by Definition 2.1.

This is Lemma 4.4 of [JK82], the only difference in our setting is that equation (4.5) in [JK82] follows from Proposition 2.18. After combining Proposition 2.20 with Proposition 2.18 we have:

**Proposition 2.21.** Let \( u \geq 0 \) be a solution that vanishes on \( \Delta(Q,R) \). Then there are \( C > 0, 1 > \alpha > 0 \) such that
\[
\sup_{T(\Delta(Q,r))} u \leq C \left( \frac{r}{R} \right)^\alpha u(A(Q,R)).
\]

2.5. Properties of the Green’s function. The paper [Li19] also gives us information on some properties of the Green’s function.

**Proposition 2.22** (Theorem 4.1.1). There exists a unique function (the Green’s function) \( G : \Omega \times \Omega \rightarrow \mathbb{R} \cup \{\infty\} \), such that
\[
G(\cdot, Z) \in W^{1,2}(\Omega \setminus B(Z,r)) \cap W^{1,1}_0(\Omega), \quad Z \in \Omega, \ r > 0,
\]
and
\[
(2.23) \quad \int_{\Omega} A(Y)\nabla YG(Y,Z)\nabla \phi(Y)dY = \phi(Z), \quad \phi \in W^{1,p}_0(\Omega) \cap C^0(\Omega), \quad p > n.
\]

**Proposition 2.24.**
\[
G(X,Y) \lesssim |X - Y|^{2-n}, \quad X,Y \in \Omega,
\]
and for any \( 0 < \theta < 1 \) we have
\[
G(X,Y) \gtrsim_{n,\lambda_0,\lambda_0,\theta} |X - Y|^{2-n}, \quad X,Y \in \Omega : |X - Y| < \theta \delta(Y).
\]

**Proposition 2.25.** Let \( L^* \) be the adjoint operator to \( L \) and let \( G^* \) be its Green’s function. Then
\[
G^*(X,Y) = G(Y,X), \quad X,Y \in \Omega.
\]

Finally we have the following relation between the Green’s function and the elliptic measure \( \omega^X \) which gives us that the elliptic measure must be doubling.

**Proposition 2.26** (Corollary 4.3.1).
\[
\omega^X(\Delta(Q,r)) \approx r^{n-2}G(X,A(Q,r)), \quad X \in \Omega \setminus B(Q,2r).
\]

**Proposition 2.27** (Corollary 4.3.2).
\[
\omega^X(\Delta(Q,2r)) \lesssim \omega^X(\Delta(Q,r)), \quad X \in \Omega \setminus B(Q,2r).
\]

As \( \delta(X) \) is a continuous function on \( \overline{\Omega} \), without loss of generality assume that \( 0 \in \Omega \) and that \( \delta(0) \geq \delta(X) \) for all \( X \in \Omega \). Let \( \omega^0 = \omega \).
Lemma 2.28. Then
\[ \omega(\Delta(X^*, \delta(X))) \approx \delta(X)^{n-2}G(0, X), \quad X \in \Omega \setminus B(0, \frac{1}{2}\delta(0)). \]

Proof: Let \( X \in \Omega \setminus B(0, \frac{1}{2}\delta(0)) \). To begin with note that if \( \delta(X) < \frac{1}{2}\delta(0) \), then \( 0 \notin B(X^*, 2\delta(X)) \) and hence the result immediately follows from Proposition 2.26. Assume therefore that \( \delta(X) \geq \frac{1}{2}\delta(0) \). Let \( Z \) be the point given by \( \partial B(X^*, \frac{1}{2}\delta(0)) \cap [X^*, X] \). Then \( \delta(Z) = |Z - X^*| = \frac{1}{2}\delta(0) \) and we may choose \( Z^* = X^* \). Thus \( 0 \notin T(Z^*, 2\delta(Z)) \) so Proposition 2.26 applies. We get that
\[ \omega(\Delta(Z^*, \delta(Z))) \approx \delta(Z)^{n-2}G(0, Z) \]

Next as our domain is CAD, there clearly exists a finite Harnack chain, from \( X \) to \( Y \) in \( B(X, \delta(X)) \setminus B(0, \delta(0)/4) \) whose length is independent of \( X \). Thus by Proposition 2.16 we deduce that
\[ G(0, X) \approx G(0, Z). \]

Finally we note that since \( \omega \) is doubling and \( 4\delta(Z) = \delta(X) \leq \delta(0) \) we have
\[ \omega(\Delta(X^*, \delta(X))) \approx \omega(\Delta(Z^*, \delta(Z))). \]

Thus combining (2.29), (2.30) and (2.31) yields the desired result. \( \square \)

Throughout this work \( G_i \) will denote the Green’s function of \( L_i \) for \( i = 0, 1 \). Furthermore, as above, we assume \( 0 \in \Omega \) and declare this to be the special point that is the “center of the domain” \( \Omega \) in the sense that \( \delta(0) = \max\{\delta(X); X \in \Omega\} \). We shorten notation and set \( G_0(Y) := G_0(0, Y) \).

2.6. Nontangential behaviour and the square function in chord arc domains. Recall that the nontangential maximal function is given by
\[ N_\alpha[u](Q) := \sup_{X \in \Gamma_\alpha(Q)} |u(X)|, \]
and the mean-valued nontangential maximal function is defined by
\[ \bar{N}_\alpha^n[u](Q) := \sup_{X \in \Gamma_\alpha(Q)} \left( \int_{B(X, \eta\delta(X)/2)} |u(Z)|^2 dZ \right)^{1/2}. \]

It is immediately clear that
\[ \bar{N}_\alpha^n[u](Q) \leq N_{\alpha+\eta/2}[u](Q). \]
We write \( \bar{N}_\alpha[u] = \bar{N}_\alpha^n[u] \) and drop the aperture \( \alpha \) when it is clear from the context.

Lemma 2.33. [Remark 7.2 in [MPT11]] Let \( \mu \) be a doubling measure on \( \partial \Omega \), where \( \Omega \) is a NTA domain. Let \( v : \Omega \to \mathbb{R} \) and let \( 0 < p < \infty, \alpha, \beta > 0, 2 > \eta > 0 \). Then
\[ \|\bar{N}_\alpha[v]\|_{L^p(\mu)} \approx \|\bar{N}_\beta[v]\|_{L^p(\mu)} \approx \|\bar{N}_\eta^n[v]\|_{L^p(\mu)}, \]
where the implied constant depends on the character of \( \Omega \), the doubling constant of \( \mu \) and \( \beta/\alpha \).

Now for \( L_i = \text{div}(A_i), \ i = 0, 1 \), where \( A_i \) satisfies (1.2) and (1.3), let \( u_i \) be the solution to the Dirichlet problem for \( L_i \) with boundary data \( f \in L^p(\partial \Omega) \cap C(\partial \Omega) \). We set \( F := u_1 - u_0 \) and note in clearly \( F = 0 \) on \( \partial \Omega \).
Lemma 2.34. Let $0 < \eta < 2$. For any solution $u$ of an elliptic PDE we have

\begin{equation}
N_\alpha[u](Q) \lesssim \tilde{N}_\alpha^0[u](Q)
\end{equation}

and furthermore

\begin{equation}
N_\alpha[F](Q) \lesssim \tilde{N}_\alpha^0[F](Q) + \tilde{N}_\alpha^0[u_0](Q).
\end{equation}

Proof. First, note that by Proposition 2.15 we have, for any solution $u$:

\[ |u(X)| \leq \sup_{B(X, \frac{1}{2} \delta(X)/4)} |u| \lesssim \left( \int_{B(X, \frac{1}{2} \delta(X)/2)} |u|^2 \right)^{1/2} \lesssim \left( \int_{B(X, \eta \delta(X)/2)} |u|^2 \right)^{1/2}, \]

and hence

\[ N_\alpha[u](Q) \lesssim \tilde{N}_\alpha^0[u](Q). \]

Using this and the triangle inequality we then have

\[ N_\alpha[F](Q) \leq N_\alpha[u_1](Q) + N_\alpha[u_0](Q) \lesssim \tilde{N}_\alpha^0[u_1](Q) + \tilde{N}_\alpha^0[u_0](Q) \]

\[ \lesssim \tilde{N}_\alpha^0[F](Q) + \tilde{N}_\alpha^0[u_0](Q). \]

□

We also have an “almost Caccioppoli inequality” for $F$.

Lemma 2.37.

\[ \int_{B(X, R)} |A|^2 dZ \lesssim \frac{1}{R^2} \int_{B(X, 2R)} (F^2 + u_0^2) dZ. \]

To see this one simply uses Caccioppoli for $u_0$ and $u_1$, and the triangle inequality

\[ \int_{B(X, R)} |A|^2 dZ \lesssim \int_{B(X, R)} (|\nabla u_1|^2 + |\nabla u_0|^2) dZ \lesssim \frac{1}{R^2} \int_{B(X, 2R)} (u_1^2 + u_0^2) dZ \]

\[ \lesssim \frac{1}{R^2} \int_{B(X, 2R)} (F^2 + u_0^2) dZ. \]

As is customary, we can define the square function of a function $u \in W^{1,2}_{loc}(\Omega)$ by

\[ S_\alpha[u](Q) := \left( \int_{\Gamma_\alpha(Q)} |\nabla u(X)|^2 \delta(X)^{2-n} dX \right)^{1/2}. \]

If we set $f := |\nabla u| \delta$, the square function can be considered to be a restriction of a more general operator

\[ A^{(\alpha)}[f](Q) := \left( \int_{\Gamma_\alpha(Q)} \frac{|f|^2}{\delta(X)^{2-n}} dX \right)^{1/2}. \]

More results on this operator can be found in [MPT11], in particular their Proposition 4.5:

Proposition 2.38. We have for $0 < p < \infty$ and two apertures $\alpha, \beta \geq 1$

\[ \|A^{(\alpha)}[f]\|_{L^p(\mu)} \approx \|A^{(\beta)}[f]\|_{L^p(\mu)}. \]

This holds for any doubling measure $\mu$ and in our case it means that the $L^p$ norms of square functions for cones of different aperture are comparable.
2.7. Dyadic decomposition of \( \partial \Omega \) and definition of decomposition \((\partial \Omega, 4R_0)\).
Recall that \( 0 \in \Omega \) and set \( R_0 = \min(\sqrt[\delta(0)]{1}, 1) \). As in \([MPT11]\) we consider a matrix \( A' \) with \( A' = A_1 \) on \( (\partial \Omega, R_0/2) := \{ Y \in \Omega : \delta(Y) < R_0/2 \} \) and \( A' = A_0 \) on \( \Omega \setminus (\partial \Omega, 2R_0) \). Then the following holds

**Lemma 2.39** (cf. Lemma 7.5 in \([MPT11]\)). If \( \omega' \) denotes the elliptic measure associated to \( \omega' = \text{div}(A'\nabla \cdot) \). Then \( \omega_1 \in B_{\rho}(\omega_0) \) iff \( \omega' \in B_{\rho}(\omega_0) \).

Thus without loss of generality we may assume that \( \beta_r(Y) = 0 \) for \( Y \in \Omega, \delta(Y) > 4R_0 \).

Recall the famous decomposition of M. Christ. By \([Chr90]\) there exists a family of “cubes” \( \{ Q^k_{\alpha} \subset \partial \Omega; k \in \mathbb{Z}, \alpha \in I_k \subset \mathbb{N} \} \) where each scale \( k \) decomposes \( \partial \Omega \) such that for every \( k \in \mathbb{Z} \):

\[
\sigma \left( \partial \Omega \setminus \bigcup_{\alpha \in I_k} Q^k_{\alpha} \right) = 0 \quad \text{and} \quad \omega_0 \left( \partial \Omega \setminus \bigcup_{\alpha \in I_k} Q^k_{\alpha} \right) = 0.
\]

Furthermore, the following properties hold:

1. If \( l \geq k \) then either \( Q^l_{\beta} \subset Q^k_{\alpha} \) or \( Q^l_{\beta} \cap Q^k_{\alpha} = \emptyset \).
2. For each \( (k, \alpha) \) and \( l < k \) there is a unique \( \beta \) so that \( Q^l_{\alpha} \subset Q^l_{\beta} \).
3. Each \( Q^k_{\alpha} \) contains a ball \( \Delta(Z^k_{\alpha}, 8^{-k-1}) \).
4. There exists a constant \( C_0 > 0 \) such that \( 8^{-k-1} \leq \text{diam}(Q^k_{\alpha}) \leq C_0 8^{-k} \).

The last listed property implies together with the Ahlfors regularity of the surface measure that \( \sigma(Q^k_{\alpha}) \approx 8^{-k(n-1)} \). Similarly, the doubling property **Proposition 2.27** of the elliptic measure guarantees us \( \omega_0(Q^k_{\alpha}) \approx \omega_0(B(Z^k_{\alpha}, 8^{-k-1})) \).

Now we can define a decomposition of \((\partial \Omega, 4R_0)\). For \( k \in \mathbb{Z}, \alpha \in I_k \), set

\[
I^k_{\alpha} := \{ Y \in \Omega; \lambda 8^{-k-1} < \delta(Y) < \lambda 8^{-k+1}, \exists P \in Q^{k}_{\alpha} : \lambda 8^{-k-1} < |P - Y| < \lambda 8^{-k+1} \},
\]

where \( \lambda \) is chosen so small that the \( \{ I^k_{\alpha} \}_{\alpha \in I_k} \) have finite overlaps and

\[
(\partial \Omega, 4R_0) \subset \bigcup_{\alpha, k \geq k_0} I^k_{\alpha}.
\]

The scale \( k_0 \) is chosen such that \( k_0 \) is the largest integer with \( 4R_0 < \lambda 8^{-k_0+1} \).

Additionally, for \( \varepsilon > 0 \) we set the scale \( k_\varepsilon \) as the smallest integer such that \( I^k_{\alpha} \subset (\partial \Omega, \varepsilon) \) for all \( k \geq k_\varepsilon \). The choices of \( k_0 \) and \( k_\varepsilon \) guarantee that

\[
(\partial \Omega, 4R_0) \setminus (\partial \Omega, \varepsilon) \subset \bigcup_{\alpha, k \geq k_0} I^k_{\alpha} \subset (\partial \Omega, 32R_0) \setminus (\partial \Omega, \varepsilon/8)
\]

Furthermore, we define the following enlarged decomposition

\[
\tilde{I}^k_{\alpha} := \{ Y \in \Omega; \lambda 8^{-k-2} < \delta(Y) < \lambda 8^{-k+2}, \exists P \in Q^{k}_{\alpha} : \lambda 8^{-k-2} < |P - Y| < \lambda 8^{-k+2} \},
\]

and it is clear that that \( I^k_{\alpha} \subset \tilde{I}^k_{\alpha} \) and that the \( \tilde{I}^k_{\alpha} \) have finite overlap. Observe that we can cover \( I^k_{\alpha} \) by balls \( \{ B(X_i, \lambda 8^{-k-3}) \}_{1 \leq i \leq N} \) with \( X_i \in I^k_{\alpha} \) such that \( |X_i - X_j| \geq \lambda 8^{-k-3}/2 \). Note here that \( N \) is independent of \( k \) and \( \alpha \).

Furthermore, we have for each \( Z \in B(X_i, 2\lambda 8^{-k-3}) \) that

\[
\lambda 8^{-k+2} \geq \delta(Z) > \lambda 8^{-k-1} - 2\lambda 8^{-k-3} = \frac{62}{8} \lambda 8^{-k-2},
\]

where
and hence

\[(2.42) \quad B(X_i, \lambda 8^{-k-3}) \subset B(Z, 3\lambda 8^{-k-3}) \subset B(Z, \frac{62}{8}\lambda 8^{-k-3}) \subset B(Z, \delta(Z)/8),\]

\[(2.43) \quad |B(X_i, \lambda 8^{-k-3})| \approx |I_\alpha^k| \approx B(Z, \delta(Z)/2),\]

and

\[(2.44) \quad I_\alpha^k \subset \bigcup_{i=1}^N B(X_i, \lambda 8^{-k-3}) \subset \bigcup_{i=1}^N B(X_i, 2\lambda 8^{-k-3}) \subset \tilde{I}_\alpha^k.\]

Note also that for \(Z \in (\partial\Omega, 4R_0)\) there exists \(Z^* \in \partial\Omega\) with 
\[|Z^* - Z| = \delta(Z)\]
and an \(I_\alpha^k\) such that
\[Z^* \in Q_k^\alpha\]
and
\[2\lambda 8^{-k-1} \leq \delta(Z) \leq \frac{1}{2}8^{k+1}.\]

Thus, for every \(X \in B(Z, \delta(Z)/4)\)
\[\lambda 8^{-k-1} \leq \delta(X) \leq \lambda 8^{-k+1}\]
and \(\lambda 8^{-k-1} \leq |Z^* - X| \leq \lambda 8^{k+1}.\)

Hence

\[(2.45) \quad B(Z, \delta(Z)/4) \subset I_\alpha^k.\]

Next, we note that

\[(2.46) \quad \tilde{I}_\alpha^k \subset T(\Delta(Z^*_\alpha, C_0 + 16\lambda)8^{-k}).\]

Lastly, we also observe that if \(P \in Q_\alpha^k\) and \(Z \in \tilde{I}_\alpha^k\) then, for \(M = 8^4,\)

\[(2.47) \quad Z \in \Gamma_M(P).\]

We are also going to need an intermediate decomposition in cubes \(\tilde{I}_\alpha^k\) defined by
\[\tilde{I}_\alpha^k := \left\{Y \in \Omega : \lambda \frac{14}{16}8^{-k-1} < \delta(Y) < \lambda \frac{18}{16}8^{-k+1}, \exists P \in Q_\alpha^k : \lambda \frac{14}{16}8^{-k-1} < |P - Y| < \lambda \frac{18}{16}8^{-k+1}\right\}.\]

The enlargement compared to \(I_\alpha^k\) here is chosen such that

\[(2.48) \quad I_\alpha^k \subset \bigcup_{i=1}^N B(X_i, \lambda 8^{-k-3}) \subset \bigcup_{i=1}^N B(X_i, 2\lambda 8^{-k-3}) \subset \tilde{I}_\alpha^k \subset \bigcup_{i=1}^N B(X_i, 4\lambda 8^{-k-3}) \subset \tilde{I}_\alpha^k.\]

All of this will be used to prove the following important proposition:

**Proposition 2.49.** With \(I_\alpha^k, \tilde{I}_\alpha^k\) as above we have:

\[
\left(\int_{I_\alpha^k} |\varepsilon(Y)|^r dY\right)^{1/r} \lesssim \frac{1}{\omega_0(Q_\alpha^k)} \left(\int_{\tilde{I}_\alpha^k} \delta(Z)^{-2}G_0(Z)\beta(Z)^2 dZ\right)^{1/2}.
\]
In particular, if \( \| \frac{\beta_r(Z)^2 G_0(Z)}{\delta(Z)^2} \|_c \leq \varepsilon_0^2 \), then
\[
\left( \int_{I_{k}^\alpha} |\varepsilon(Y)|^r dY \right)^{1/r} \lesssim \varepsilon_0.
\]

Proof: Using the covering of \( I_{k}^\alpha \) by balls defined above we have
\[
\left( \int_{I_{k}^\alpha} |\varepsilon(Y)|^r dY \right)^{1/r} \leq \sum_{i=1}^{N} \int_{B(X_{i}, \lambda_{8}^{-k-3})} \beta_r(Z)^2 dY
\]
\[
\leq \sum_{i=1}^{N} \left( \int_{B(X_{i}, \lambda_{8}^{-k-3})} \delta(Z)^{-n} \beta_r(Z)^2 dZ \right)^{1/2}
\]
\[
\leq \sum_{i=1}^{N} \left( \int_{B(X_{i}, \lambda_{8}^{-k-3})} \delta(Z)^{-2} G_0(Z) \beta_r(Z)^2 dZ \right)^{1/2}
\]
\[
\leq \left( \int_{I_{k}^\alpha} \delta(Z)^{-2} G_0(Z) \beta_r(Z)^2 dZ \right)^{1/2}.
\]

Furthermore, for the second part under the assumption that \( \| \frac{\beta_r(Z)^2 G_0(Z)}{\delta(Z)^2} \|_c \leq \varepsilon_0^2 \), we have that
\[
\left( \int_{I_{k}^\alpha} \delta(Z)^{-2} G_0(Z) \beta_r(Z)^2 dZ \right)^{1/2}
\]
\[
\leq \left( \frac{1}{\omega_0(Q_{k}^\alpha)} \int_{B(X_{i}, \lambda_{8}^{-k-3})} \delta(Z)^{-2} G_0(Z) \beta_r(Z)^2 dZ \right)^{1/2}
\]
\[
\leq \varepsilon_0.
\]

3. Proof of Theorem 1.5

We use a method inspired by [MPT11]. Recall that \( A_0 \) and \( A_1 \) satisfy (1.2) and (1.3), and that \( \omega_0 \) and \( \omega_1 \) denote their elliptic measures. Furthermore, \( F = u_1 - u_0 \) and \( \beta_r \) are as in (1.4). First, we need to prove
**Theorem 3.1.** Let $\Omega$ be a bounded CAD. There exists $\epsilon_0 = \epsilon_0(n, \lambda_0, \Lambda_0) > 0$ and $r > 0$, such that if

$$\sup_{\Delta \subset \partial \Omega} \left( \frac{1}{\omega_0(\Delta)} \int_{\Gamma(\Delta)} \beta_r^2(X) \frac{G_0(X)}{\delta(X)^2} dX \right)^{1/2} < \epsilon_0,$$

then $\omega_1 \in B_2(\omega_0)$.

This theorem corresponds to Theorem 2.9 in [MPT11] but with the discrepancy function $\alpha$ instead of $\beta_r$. Using this the authors of [MPT11] first prove

**Theorem 3.2** (Theorem 8.2). Let $\Omega$ be a bounded CAD and let

$$A(a)(Q) = \left( \int_{\Gamma(Q)} \frac{\alpha^2(X)}{\delta(X)^n} dX \right)^{1/2}.$$ 

If $\|A(a)\|_{L^\infty} \leq C < \infty$ and $\omega_0 \in A_\infty(\sigma)$ then $\omega_1 \in A_\infty(\sigma)$.

After that, they establish **Theorem 1.5** (which is Theorem 8.1 in their notation). If we replace $\alpha$ by $\beta_r$, we can conclude **Theorem 1.5** from **Theorem 3.2** and **Theorem 3.2** from **Theorem 3.1** in the same way as in [MPT11]. The only modification is the substitution of every discrepancy function $\alpha$ by $\beta_r$.

Hence it remains to prove **Theorem 3.1**. We first establish the following two lemmas.

**Lemma 3.3.** Let $\mu$ be a doubling measure on $\partial \Omega$. Under the assumptions of **Theorem 3.1** we have for every $0 < p < \infty$

$$\int_{\partial \Omega} \tilde{N}_\alpha[F]^p d\mu \lesssim \epsilon_0 \int_{\partial \Omega} M_{\omega_0}[S_{\tilde{M}}u_1]^p d\mu,$$

where the aperture $\tilde{M} = 2^8$ is at least twice as large as $\alpha$.

In particular this lemma holds with $\mu \in \{\omega_0, \sigma\}$.

**Lemma 3.4.** Under the assumptions of **Theorem 3.1** we have for any aperture $\alpha > 0$

$$\int_{\partial \Omega} S_{\tilde{M}}[F]^2 d\omega_0 \lesssim \int_{\partial \Omega} \left( \tilde{N}_\alpha[F]^2 + f^2 \right) d\omega_0.$$ 

These lemmas are versions of Lemma 2.9 and Lemma 2.10 in [FKP91] or Lemma 7.7 and Lemma 7.8 in [MPT11].

**Proof of Theorem 3.1:** Assume that **Lemma 3.3** and **Lemma 3.4** hold. Since

$$\|S_\alpha[u_0]\|_{L^2(d\omega_0)} \lesssim \|f\|_{L^2(d\omega_0)},$$

we have that

$$\int_{\partial \Omega} \tilde{N}_\alpha[F]^2 d\omega_0 \lesssim \int_{\partial \Omega} \epsilon_0^2 M_{\omega_0}[S_{\tilde{M}}u_1]^2 d\omega_0 \lesssim \epsilon_0^2 \int_{\partial \Omega} S_{\tilde{M}}[u_1]^2 d\omega_0$$

$$\leq \epsilon_0^2 \int_{\partial \Omega} S_{\tilde{M}}[F]^2 d\omega_0 + \epsilon_0^2 \int_{\partial \Omega} S_{\tilde{M}}[u_1]^2 d\omega_0$$

$$\lesssim \epsilon_0^2 \int_{\partial \Omega} f^2 d\omega_0 + \epsilon_0^2 \int_{\partial \Omega} S_{\tilde{M}}[F]^2 d\omega_0$$

$$\lesssim \epsilon_0^2 \int_{\partial \Omega} \tilde{N}_\alpha[F]^2 d\omega_0 + \epsilon_0^2 \int_{\partial \Omega} f^2 d\omega_0.$$
Thus, with $\varepsilon_0$ small enough we can hide the term $\tilde{N}_\alpha[F]^2$ and absorb it by the lefthand side. We get
\[ \| \tilde{N}_\alpha[F]\|_{L^2(\omega_0)} \lesssim \| f \|_{L^2(\omega_0)}. \]

By Lemma 2.34 we obtain
\[ \int_{\partial \Omega} N_\alpha[u_1]^2 d\omega_0 \lesssim \int_{\partial \Omega} \tilde{N}_\alpha[F]^2 d\omega_0 + \int_{\partial \Omega} \tilde{N}_\alpha[u_0]^2 d\omega_0 \lesssim \int_{\partial \Omega} f^2 d\omega_0. \]

Therefore $\omega_1 \in B_2(\omega_0)$ as desired. \hfill \Box

3.1. The difference function $F$. Our first proposition is a generalization of Lemma 3.12 in [CHM19] using the same strategy for its proof.

**Proposition 3.5.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded CAD and let $L_0, L_1$ be two elliptic operators. Let $u_0 \in W^{1,2}(\Omega)$ be a weak solution of $L_0 u_0 = 0$ in $\Omega$, and let $G_1$ be the Green’s function of $L_1$. Then
\[ \int_{\Omega} A_0(Y) \nabla_Y G_1(Y, X) \cdot \nabla u_0(Y) dY = 0 \quad \text{for a.e. } X \in \Omega. \]

**Proof.** We begin by fixing a point $X_0 \in \Omega$ and considering a cut-off function $\varphi \in C^0_c([-2,2])$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $[-1,1]$. For each $0 < \varepsilon < \delta(X_0)/16$ we set $\varphi_\varepsilon(X) = \varphi(|X - X_0|/\varepsilon)$ and $\psi_\varepsilon = 1 - \varphi_\varepsilon$. Furthermore let $G_1^{X_0} = G_1(Y, X_0)$. We see that
\[ \int_{\Omega} A_0 \nabla G_1^{X_0} \cdot \nabla u_0 dY = \int_{\Omega} A_0 \nabla (G_1^{X_0} \psi_\varepsilon) \cdot \nabla u_0 dY + \int_{\Omega} A_0 \nabla (G_1^{X_0} \psi_\varepsilon) \cdot \nabla u_0 dY. \]

Thanks to Proposition 2.22 we have that $G_1(\cdot, X_0)\psi_\varepsilon \in W^{1,2}_0(\Omega)$, which gives us (as $L_0 u_0 = 0$) that
\[ \int_{\Omega} A_0(Y) \nabla (G_1(Y, X_0)\psi_\varepsilon(Y)) \cdot \nabla u_0(Y) dY = 0. \]

Next we note that
\[ \int_{\Omega} A_0 \nabla (G_1^{X_0} \varphi_\varepsilon) \cdot \nabla u_0 dY = \int_{\Omega} A_0 \nabla G_1^{X_0} \cdot \nabla u_0 \varphi_\varepsilon dY + \int_{\Omega} A_0 \nabla \varphi_\varepsilon \cdot G_1^{X_0} \nabla u_0 dY =: I_\varepsilon(X_0) + II_\varepsilon(X_0). \]

Thus if we can show that $I_\varepsilon(X_0) + II_\varepsilon(X_0) \to 0$ as $\varepsilon \to 0$ for a.e. $X_0 \in \Omega$, then we have shown our claim. We start by considering the first term. Clearly,
\[ |I_\varepsilon(X_0)| \leq \int_{B(X_0,2\varepsilon)} |A_0||\nabla G_1^{X_0}||\nabla u_0| dY \]
\[ \leq \left( \int_{B(X_0,\delta(X_0)/8)} |A_0|^{r/\ell} dY \right)^{1/\ell} \left( \int_{B(X_0,2\varepsilon)} \left(||\nabla G_1^{X_0}|||\nabla u_0||^{\ell/\ell'} \right)^{\ell'/\ell'} dY \right)^{1/\ell'}, \]

for some $r > 2$ to be determined later. Notice that the first term is bounded since $A \in L^r_{loc}(\Omega)$. To deal with the second term we decompose the ball $B(X_0,2\varepsilon)$ into family of annuli $C_j(X_0, \varepsilon) = B(X_0, 2^{-j+1}\varepsilon) \setminus B(X_0, 2^{-j}\varepsilon), j \geq 0$. This gives us
\[ |I_\varepsilon(X_0)| \lesssim \left( \int_{B(X_0, 2r)} \left( |\nabla G^{X_0}_1(\nabla u_0)| \right)^{r'} \, dY \right)^{1/r'} \]

\[ \leq \left( \sum_{j=0}^{\infty} (2^{-j} \varepsilon)^n \frac{1}{C_j(X_0, \varepsilon)} \left( |\nabla G^{X_0}_1(\nabla u_0)| \right)^{r'} \, dY \right)^{1/r'} \]

\[ \lesssim \sum_{j=0}^{\infty} (2^{-j} \varepsilon)^{n/r'} \left( \int_{C_j(X_0, \varepsilon)} |\nabla G^{X_0}_1|^2 \, dY \right)^{1/2} \left( \int_{B(X_0, 2^{-j} \varepsilon)} |\nabla u_0|^2 \, dY \right)^{r-2 \over 2 r} \]

Using Proposition 2.13, Caccioppoli's inequality and Proposition 2.24 we get that for \( r \) sufficiently large we have:

\[ \left( \int_{B(X_0, 2^{-j} \varepsilon)} |\nabla u_0|^{2-n} \, dY \right)^{r-2 \over 2 r} \lesssim \left( \int_{B(X_0, 2^{-j+1} \varepsilon)} |\nabla u_0|^2 \, dY \right)^{1/2} \]

\[ \leq M||\nabla u_0||^2_{\Omega}(X_0)^{1/2} \]

and

\[ \left( \int_{C_j(X_0, \varepsilon)} |\nabla G^{X_0}_1|^2 \, dY \right)^{1/2} \leq \frac{1}{2^{-j} \varepsilon} \left( \int_{\bigcup_{j=j-1}^{j+1} C_i(X_0, \varepsilon)} |G^{X_0}_1|^2 \, dY \right)^{1/2} \]

\[ \lesssim \frac{1}{2^{-j} \varepsilon} \left( \int_{\bigcup_{j=j-1}^{j+1} C_i(X_0, \varepsilon)} (2^{-j} \varepsilon)^{2(2-n)} \, dY \right)^{1/2} \]

\[ = (2^{-j} \varepsilon)^{1-n} \]

Hence

\[ |I_\varepsilon(X_0)| \lesssim \sum_{j=0}^{\infty} (2^{-j} \varepsilon)^{1-(n-n/r')} M||\nabla u_0||^2_{\Omega}(X_0)^{1/2} \]

\[ = \sum_{j=0}^{\infty} (2^{-j} \varepsilon)^{1-n/r} M||\nabla u_0||^2_{\Omega}(X_0)^{1/2}. \]

Choosing \( r > 2n \) we get that

\[ |I_\varepsilon(X_0)| \lesssim \sum_{j=0}^{\infty} 2^{-j/2} \varepsilon^{1-n} M||\nabla u_0||^2_{\Omega}(X_0)^{1/2} \lesssim \sqrt{\varepsilon} M||\nabla u_0||^2_{\Omega}(X_0)^{1/2}. \]

For the second term we note that \( \|\nabla \varphi_\varepsilon\|_\infty \approx \varepsilon^{-1} \). Then Proposition 2.24 and Hölder's inequality give us:

\[ |II_\varepsilon(X_0)| \leq \varepsilon^{-1} \int_{B(X_0, 2\varepsilon)} |A_0||\nabla u_0||G^{X_0}_1| \, dY \]

\[ \lesssim \varepsilon^{-n+1} \int_{B(X_0, 2\varepsilon)} |A_0||\nabla u_0| \, dY \]
By \( \phi \)

Notice that, by taking a subsequence, we may assume that \( X \).

Proposition 3.6.

Combining both integrals we have

\[
\left( \int_{B(x_0,\delta(x_0)/8)} |A_0|^r \right)^{1/r} \left( \int_{B(x_0,2\varepsilon)} |\nabla u_0|^{r'} dY \right)^{1/r'} \leq \varepsilon \sqrt{\varepsilon} \left( \int_{B(x_0,2\varepsilon)} |\nabla u_0|^2 dY \right)^{1/2} \leq \sqrt{\varepsilon} M||\nabla u_0^2\chi_\Omega||_0(X_0)^{1/2}.
\]

for all \( \varepsilon > 0 \). Since \( \nabla u_0 \in L^2(\Omega) \) we have \( M||\nabla u_0^2\chi_\Omega|| \in L^{1,\infty}(\Omega) \) and thus \( M||\nabla u_0^2\chi_\Omega|| < \infty \) a.e. on \( \Omega \). Thus letting \( \varepsilon \to 0^+ \) finishes the proof.

Next we prove a result similar to Lemma 3.18 in [CHM19]. However, we note that the proof of this result is not actually given in [CHM19], the paper instead cites [HMT14] which was not available to us as it has not yet appeared anywhere.

Proposition 3.6. Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded CAD. Let \( L_0, L_1 \) be two elliptic operators, \( u_0, u_1 \in W^{1,2}(\Omega) \) be a pair of weak solutions of \( L_0 u_0 = 0 \), \( L_1 u_1 = 0 \) in \( \Omega \), with \( u_0 - u_1 \in W^{1,2}_0(\Omega) \) and \( G_0 \) be the Green’s function of \( L_0 \). Then for a.e. \( X \in \Omega \) we have

\[
F(X) := u_0(X) - u_1(X) = \int_{\Omega} A_0(Y) \nabla Y G_0(Y, X) \cdot \nabla (u_0 - u_1)(Y) dY.
\]

Proof. Fix \( X_0 \in \Omega \) and consider a cut-off function \( \vartheta \in C_c([-2,2]) \) such that \( 0 \leq \vartheta \leq 1 \) and \( \vartheta \equiv 1 \) on \([-1,1]\). For each \( 0 < \varepsilon < \delta(X_0)/16 \) we set \( \vartheta_\varepsilon(X) = \vartheta(|X - X_0|/\varepsilon) \)

and \( \psi_\varepsilon = 1 - \vartheta_\varepsilon \). Consider a sequence of functions \( \varphi_k \in C_0^\infty(\Omega) \) such that \( \varphi_k \to u_0 - u_1 \) in \( W^{1,2}(\Omega) \) with

\[
\|\varphi_{k+1} - (u_0 - u_1)\|_{W^{1,2}(\Omega)} \leq \frac{1}{2}\|\varphi_k - (u_0 - u_1)\|_{W^{1,2}(\Omega)}, \quad \forall k \geq 1.
\]

By Proposition 2.22

\[
\varphi_k(X_0) = \int_{\Omega} A_0(Y) \nabla Y G_0(Y, X_0) \cdot \nabla \varphi_k(Y) dY.
\]

By Proposition 2.22

Notice that, by taking a subsequence, we may assume that \( \varphi_k \to u_0 - u_1 \) a.e. It follows that

\[
\int_{\Omega} A_0(Y) \nabla Y G_0(Y, X_0) \nabla (\varphi_k - (u_0 - u_1))(Y) dY
\]

\[
= \int_{\Omega} A_0(Y) \nabla Y (G_0(Y, X_0) \psi_\varepsilon(Y)) \nabla (\varphi_k - (u_0 - u_1))(Y) dY
\]

\[
+ \int_{\Omega} A_0(Y) \nabla Y G_0(Y, X_0) \nabla (\varphi_k - (u_0 - u_1))(Y) \vartheta_\varepsilon(Y) dY
\]

\[
+ \int_{\Omega} A_0(Y) \nabla \vartheta_\varepsilon(Y) \nabla (\varphi_k - (u_0 - u_1))(Y) G_0(Y, X_0) dY
\]

\[
=: I^k_\varepsilon(X_0) + II^k_\varepsilon(X_0) + III^k_\varepsilon(X_0).
\]
We aim to show that
\begin{equation}
I^k_{\varepsilon}(X_0) + II^k_{\varepsilon}(X_0) + III^k_{\varepsilon}(X_0) \to 0, \quad \varepsilon \to 0,
\end{equation}
for a well chosen sequence \( k = k(\varepsilon) \) with property that \( k(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \).

Analogous to the proof of Proposition 3.5 we get that for a large \( k \)
\begin{equation}
|I^k_{\varepsilon}| \lesssim \|G_0(\cdot , X_0)\psi_\varepsilon\|_{W^{1,2}(\Omega)} \|\varphi_k - (u_0 - u_1)\|_{W^{1,2}(\Omega)} \leq \sqrt{\varepsilon},
\end{equation}
and
\begin{equation}
|II^k_{\varepsilon}|, |II^k_{\varepsilon}| \lesssim \sqrt{\varepsilon} M[|g_k|(X_0)]^{1/2},
\end{equation}
where \( g_k := |\nabla (\varphi_k - (u_0 - u_1))|^2 \chi_\Omega \).

Thus sup \( |I^k_{\varepsilon}| + |II^k_{\varepsilon}| + |III^k_{\varepsilon}| \lesssim \sqrt{\varepsilon} \max \left\{ 1, \sup_k g_k \right\} \)
\begin{equation}
\tag{3.9}
\end{equation}
Now, since \( g_k \in L^1(\mathbb{R}^n) \) we have that \( M[|g_k|] \in L^1(\mathbb{R}^n) \) with the bound
\begin{equation}
\|M[|g_k|]\|_{L^1(\mathbb{R}^n)} \lesssim \|g_k\|_{L^1(\Omega)} \leq \|\varphi_k - (u_0 - u_1)\|_{W^{1,2}(\Omega)}^2 \lesssim 2^{-k},
\end{equation}
Thus sup \( k \) \( M[|g_k|]^{1/2} \leq \infty \) a.e. allowing us to let \( \varepsilon \to 0 \) in (3.9) and yielding (3.8) for a.e. \( X_0 \) as desired.
\[ \square \]

4. Proof of Lemma 3.3

Let \( Q \in \partial \Omega \) and \( X \in \Gamma_0(Q) \). Let \( G_0 \) be the Green’s function corresponding to \( L_0 \) and \( G_0^* \) its adjoint. As before let \( F = u_1 - u_0 \) and note that by Proposition 3.6, Proposition 3.5 and Proposition 2.25 we have
\begin{equation}
F(X) = u_1(X) - u_0(X) = \int_{\Omega} A^T_0 \nabla Y G_0(Y, X) \nabla (u_1 - u_0)(Y) dY
\end{equation}
\begin{equation}
= \int_{\Omega} A_0 \nabla u_1(Y) \nabla Y G_0(X, Y) dY - \int_{\Omega} A_0 \nabla u_0(Y) \nabla Y G_0(X, Y) dY
\end{equation}
\begin{equation}
= \int_{\Omega} A_0 \nabla u_1(Y) \nabla Y G_0(X, Y) dY - \int_{\Omega} A_1 \nabla u_1(Y) \nabla Y G_0(X, Y) dY
\end{equation}
\begin{equation}
= \int_{\Omega} \varepsilon \nabla u_1(Y) \nabla Y G_0(X, Y) dY.
\end{equation}

Remark 4.1. Note that in [MPT11] and [FKP91] the authors claim that the above holds simply by the Green’s function property and integration by parts. However, the Green’s function property (2.23) only holds for \( \varphi \in W^{1,p}_0(\Omega) \) with \( p > n \geq 2 \) and so cannot be applied directly unless we have shown statements such as Propositions 3.5 and 3.6. This is true even if \( A_0, A_1 \) are bounded and symmetric.

We split \( F \) into two terms (first of which is the near part close to \( X \)).
\begin{equation}
F = F_1 + F_2, \quad F_1(Z) = \int_{B(X)} \nabla Y G_0(Z, Y) \cdot \varepsilon(Y) \nabla u_1(Y) dY,
\end{equation}
and then split \( F_1 \) further and write it as
\begin{equation}
F_1 = \tilde{F}_1 + \tilde{F}_1, \quad \tilde{F}_1(Z) = \int_{B(X)} \nabla Y \tilde{G}_0(Z, Y) \cdot \varepsilon(Y) \nabla u_1(Y) dY.
\end{equation}
Here $B(X) := B(X, \delta(X)/4)$ and $\tilde{G}_0$ denotes the “local Green function” for $L_0$ on $2B(X)$. We also set $K(Z,Y) := G_0(Z,Y) - \tilde{G}_0(Z,Y)$. Since $\mu$ is a doubling measure we have by Lemma 2.33 that

$$\|\tilde{N}_\lambda[F]\|_L^P(d\mu) \leq \|\tilde{N}_\lambda[F_1]\|_L^P(d\mu) + \|\tilde{N}_\lambda[F_2]\|_L^P(d\mu)$$

$$\lesssim \|\tilde{N}_\lambda^{1/2}[\tilde{F}_1]\|_L^P(d\mu) + \|\tilde{N}_\lambda^{1/2}[\tilde{F}_2]\|_L^P(d\mu) + \|\tilde{N}_\lambda^{-1/4}[F_2]\|_L^P(d\mu).$$

Hence to conclude that Lemma 3.3 holds it is enough to show that the pointwise bound

$$\left(\int_{B(X)} |\tilde{F}_1|^2\right)^{1/2} + \left(\int_{B(X)} |\tilde{F}_2|^2\right)^{1/2} + \left(\int_{B(X,\delta(X))/8} |F_2|^2\right)^{1/2} \lesssim \varepsilon_0 S_{\tilde{M}}[u_1](Q)$$

is true for almost every $Q \in \partial\Omega$ and $X \in \Gamma_\delta(Q)$. We shall consider each of the terms above separately.

### 4.1. The “local term” $F_1$

Consider $\phi \in C_c^\infty(\mathbb{B}^n)$ non-negative with $\int_{\mathbb{B}^n} \phi = 1$ and set $\phi_m = (\frac{2m}{m(\delta(X))})^n \phi(2mX/\delta(X))$. Define

$$\tilde{A}_m := A_1 \ast \phi_m, \quad \varepsilon^m := \varepsilon \ast \phi_m, \quad \tilde{L}_m := \text{div}(\tilde{A}_m \nabla),$$

and let $r > 1$ be large enough that Proposition 2.13 holds with $p = \frac{2r}{r-2}$. Let $\tilde{u}_m$ be the weak solution to the Dirichlet problem

$$\tilde{L}_m v = 0, \text{ in } 2B(X), \quad u_1 - v \in W_0^{1,2}(2B(X)).$$

We know that $A_0, A_1 \in L^r(2B(X))$, wherefore

$$\tilde{A}_m \rightarrow A_1, \quad \varepsilon^m \rightarrow \varepsilon, \quad \text{in } L^r(2B(X)).$$

Moreover $A_1^* \in L^\infty(2B(X))$, hence by the ellipticity of our original $A_1$ for large $m$ there exists $\lambda_{0,m} > 0$ such that

$$\lambda_{0,m} |\xi|^2 \leq \xi^T \tilde{A}_m \xi \leq \lambda_{0,m}^{-1} |\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } X \in 2B(X),$$

with $\lambda_{0,m} \rightarrow \lambda_0$ as $m \rightarrow \infty$ and so the ellipticity constant does not depend on $m$. Next we will show that

$$\|\nabla(\tilde{u}_m - u_1)\|_{L^2(2B(X))} \lesssim \|\tilde{A}_m - A_1\|_{L^r(2B(X))}^2,$$
and in particular it follows that $\nabla \tilde{u}_m \to \nabla u_1$ in $L^2(2B(X))$. Clearly for $m$ large enough we have that

$$
\int_{2B(X)} |\nabla (\tilde{u}_m - u_1)|^2 \lesssim 2\lambda_0^{-1} \int_{2B(X)} \tilde{A}^m \nabla (\tilde{u}_m - u_1) \cdot \nabla (\tilde{u}_m - u_1)
$$

$$
= -2\lambda_0^{-1} \int_{2B(X)} (\tilde{A}^m) \nabla u_1 \cdot \nabla (\tilde{u}_m - u_1)
$$

$$
= 2\lambda_0^{-1} \int_{2B(X)} (A_1 - \tilde{A}^m) \nabla u_1 \cdot \nabla (\tilde{u}_m - u_1)
$$

$$
\leq 2\lambda_0^{-1} \|\tilde{A}^m - A_1\|_{L^r(2B(X))} \|\nabla u_1\|_{L^\frac{2r}{r-2}(2B(X))} \left( \int_{2B(X)} |\nabla \tilde{u}_m - \nabla u_1|^2 \right)^{1/2}
$$

$$
\leq 2\lambda_0^{-2} \|\tilde{A}^m - A_1\|_{L^r(2B(X))}^2 \|\nabla u_1\|_{L^\frac{2r}{r-2}(2B(X))}^2 + \frac{1}{2} \left( \int_{2B(X)} |\nabla \tilde{u}_m - \nabla u_1|^2 \right)
$$

$$
\lesssim \|\tilde{A}^m - A_1\|_{L^r(2B(X))}^2 \|\nabla u_1\|_{L^\frac{2r}{r-2}(2B(X))}^2.
$$

By Proposition 2.13 we have that

$$
\|\nabla u_1\|_{L^\frac{2r}{r-2}(2B(X))}^2 \lesssim \|\nabla u_1\|_{L^2(3B(X))}^2 \leq \|\nabla u_1\|_{L^2(\Omega)}^2 < \infty,
$$

and hence (4.2) follows.

The reason we consider approximations of $\tilde{F}_1$, namely

$$
\tilde{F}_m(Z) := \int_{B(X)} \nabla Y \tilde{G}_0(Z, Y) \cdot \tilde{\varepsilon}^m(Y) \nabla \tilde{u}_m(Y) dY
$$

and

$$
\tilde{F}_m^r(Z) := \int_{B(X)} \nabla Y \tilde{G}_0^r(Z, Y) \cdot \tilde{\varepsilon}^m(Y) \nabla \tilde{u}_m(Y) dY
$$

(4.3)

$$
= -\int_{B(X)} \text{div}(\tilde{\varepsilon}^m \nabla \tilde{u}_m) \tilde{G}_0(Z, Y) dY + \int_{\partial B(X)} \tilde{\varepsilon}^m \nabla \tilde{u}_m \cdot \nu \tilde{G}_0(Z, Y) dY
$$

is that for the term $\tilde{F}_1$ we have few situations where derivative hit terms that do not have required regularity. This is not true for mollified coefficients as those are smooth. Here, in the formula above $\tilde{G}_0^r(Z, Y) \in W^{1,2}_0(2B(X))$ is the unique function that satisfies

$$
\int_{2B(X)} A_0(Y) \nabla \tilde{G}_0^r(Z, Y) \nabla \phi(Y) dY = \int_{B(Z, \rho)} \phi(Y) dY \ \forall \phi \in W^{1,2}_0(2B(X)),
$$

which exists by the Lax-Milgram theorem. From (4.3), it is clear that $\tilde{F}_m^r$ is continuous on $2B(X) \setminus \frac{1}{2} B(X)$ with $\tilde{F}_m^r = 0$ on $\partial 2B(X)$. Next note that by [GW82]

$$
\|\tilde{G}_0^r\|_{L^\frac{2r}{r-2}(2B(X))} \lesssim 1,
$$
where the implied constant is independent of \( \rho \). Hence \( \| \tilde{G}_0^\rho \|_{L^1(2B(X))} \lesssim 1 \) and therefore

\[
\| \hat{F}_m^\rho \| \lesssim \| \text{div}(\hat{\varepsilon}^m \nabla \hat{u}_m) \|_{L^\infty} + \| \hat{\varepsilon}^m \nabla \hat{u}_m \|_{L^\infty} \lesssim m 1. \tag{4.4}
\]

This allows us to claim that \( \hat{F}_m^\rho \in L^\infty(2B(X)) \). Finally by Minkowski’s integral inequality

\[
\left\| \int_{B(X)} |\text{div}(\hat{\varepsilon}^m \nabla \hat{u}_m)| \| \nabla \tilde{G}_0^\rho(Z,Y) |dY \right\|_{L^2(2B(X))} \\
\leq \int_{B(X)} |\text{div}(\hat{\varepsilon}^m \nabla \hat{u}_m)| \| \nabla \tilde{G}_0^\rho(\cdot,Y) \|_{L^2(2B(X))} |dY \lesssim \rho \int_{B(X)} |\text{div}(\hat{\varepsilon}^m \nabla \hat{u}_m)| |dY| \lesssim m |B(X)| \lesssim X 1,
\]

and similarly

\[
\left\| \int_{\partial B(X)} |\hat{\varepsilon}^m \nabla \hat{u}_m| \| \nabla \tilde{G}_0^\rho(Z,Y) |dY \right\|_{L^2(2B(X))} \lesssim_{\rho,X} 1.
\]

Thus \( \| \nabla \hat{F}_m^\rho \|_{L^2(2B(X))} \lesssim_{\rho,X} 1 \) and we can conclude that \( \hat{F}_m^\rho \in W^{1,2}_0(2B(X)) \). Next,

\[
\int_{2B(X)} |\nabla \hat{F}_m^\rho|^2 \lesssim \int_{2B(X)} A_0 \nabla \hat{F}_m^\rho \cdot \nabla \hat{F}_m^\rho \\
= - \int_{B(X)} \text{div}(\hat{\varepsilon}^m \nabla \hat{u}_m) \left( \int_{2B(X)} A_0 \nabla \tilde{G}_0^\rho(Z,Y) \cdot \nabla \hat{F}_m^\rho |dZ| \right) |dY| \\
+ \int_{\partial B(X)} \nu \cdot \hat{\varepsilon}^m \nabla \hat{u}_m \left( \int_{2B(X)} A_0 \nabla \tilde{G}_0^\rho(Z,Y) \cdot \nabla \hat{F}_m^\rho |dZ| \right) |dY|
\]

\[
= - \int_{B(X)} \text{div}(\hat{\varepsilon}^m \nabla \hat{u}_m) \left( \int_{B(Y,\rho)} \hat{F}_m^\rho \right) |dY| \\
+ \int_{\partial B(X)} \nu \cdot \hat{\varepsilon}^m \nabla \hat{u}_m \left( \int_{B(Y,\rho)} \hat{F}_m^\rho \right) |dY|
\]

\[
= \int_{B(X)} \hat{\varepsilon}^m \nabla \hat{u}_m \cdot \nabla \left( \int_{B(Y,\rho)} \hat{F}_m^\rho \right) |dY| \\
= \int_{B(X)} \hat{\varepsilon}^m \nabla \hat{u}_m \cdot \left( \int_{B(Y,\rho)} \nabla \hat{F}_m^\rho \right) |dY|
\]

Therefore

\[
\int_{2B(X)} |\nabla \hat{F}_m^\rho|^2 \leq C \int_{B(X)} \hat{\varepsilon}^m \nabla \hat{u}_m \cdot \left( \int_{B(Y,\rho)} \nabla \hat{F}_m^\rho \right) \\
\leq \frac{1}{2} \int_{B(X)} \left( \int_{B(Y,\rho)} \nabla \hat{F}_m^\rho \right)^2 + C \int_{B(X)} |\hat{\varepsilon}^m|^2 |\nabla \hat{u}_m|^2.
\]
Since the term
\[ \int_{B(X)} \left( \int_{B(Y,\rho)} \nabla \hat{F}_m^\rho \right)^2 \leq \int_{B(X)} \int_{B(Y,\rho)} |\nabla \hat{F}_m^\rho|^2 \leq \int_{2B(X)} |\nabla \hat{F}_m^\rho|^2, \]
it can be absorbed by the left side of the expression above and thus giving us that
\[ \int_{2B(X)} |\nabla \hat{F}_m^\rho|^2 \lesssim \left( \int_{2B(X)} |\nabla \hat{u}_m|^2 \right)^{2/r} \]
We also note that
\[ \left( \int_{2B(X)} |\nabla \hat{u}_m|^2 \right)^{2/r} \rightarrow \left( \int_{B(X)} |\nabla \hat{u}_m|^2 \right)^{2/r}, \quad m \rightarrow \infty \]
We know that if \( \delta(X) \geq 4R_0 \), then \( \varepsilon = 0 \) on \( B(X) \) nd there is nothing to show. Hence may assume that \( \delta(X) \leq 4R_0 \) and use (2.45) and Proposition 2.49, to obtain
\[ \left( \int_{2B(X)} |\nabla \hat{u}_m|^2 \right)^{2/r} \leq \left( \int_{2B(X)} |\nabla \hat{u}_m|^2 \right)^{2/r, \quad m \rightarrow \infty}. \]
For the remaining term we have by Proposition 2.13
\[ \left( \int_{2B(X)} |\nabla \hat{u}_m|^2 \right)^{2/r} \rightarrow \int_{2B(X)} |\nabla \hat{u}_m|^2 \rightarrow \int_{2B(X)} |\nabla u_1|^2, \quad m \rightarrow \infty, \]
and by the Poincare inequality for \( \hat{F}_m^\rho \) we have
\[ \int_{B(X)} |\hat{F}_m^\rho|^2 \lesssim \int_{2B(X)} |\hat{F}_m^\rho|^2 \lesssim \delta(X)^{2-n} \int_{2B(X)} |\nabla \hat{F}_m^\rho|^2. \]
Collecting all terms together then yields:
\[ \lim_{m \rightarrow \infty} \left( \delta(X)^{2-n} \int_{2B(X)} |\nabla \hat{F}_m^\rho|^2 \right) \lesssim \varepsilon_0^2 \int_{2B(X)} |\nabla u_1|^2 \delta(Z)^{2-n} dZ \]
\[ \varepsilon \geq \varepsilon_0 S_M[u_1](Q_0)^2. \]
To get statement on the original \( \hat{F}_1 \) we let \( \rho \rightarrow 0 \) and then \( m \rightarrow \infty \). We claim that
\[ \int_{B(X)} |\hat{F}_m^\rho|^2 \rightarrow \int_{B(X)} |\hat{F}_m|^2, \quad \rho \rightarrow 0, \]
and
\[ \int_{B(X)} |\hat{F}_m|^2 \rightarrow \int_{B(X)} |\hat{F}_1|^2, \quad m \rightarrow \infty. \]
Having this (4.5) and (4.6) combined give us
\[ \left( \int_{B(X)} |\hat{F}_1|^2 \right)^{1/2} \lesssim \varepsilon_0 S_M[u_1](Q), \]
as desired.

We start by proving (4.7). As this is a claim at every \( X \) we allow the implicit constants below to depend on \( X \). We know from [Li19] that for a fixed \( Z, \tilde{G}_0(Z, \cdot) \rightarrow \)
\( \tilde{G}_0(Z, \cdot) \) weakly in \( W_{0}^{1,1+\eta}(2B(X)) \) for small \( \eta > 0 \). Hence we also have weak convergence in \( W^{1,1+\eta}(B(X)) \). For any \( v \in W^{1,1+\eta}(B(X)) \)

\[
\left| \int_{B(X)} \nabla v \cdot \epsilon^m \nabla \hat{u}_m dY \right| \leq \| \epsilon^m \nabla \hat{u}_m \|_{L^\infty(B(X))} \| \nabla v \|_{L^1(B(X))} \lesssim_{m,X} \| v \|_{W^{1,1+\eta}(B(X))},
\]

and in particular, this means that

\[
\int_{B(X)} \nabla \tilde{G}_0^\rho(Z, \cdot) \cdot \epsilon^m \nabla \hat{u}_m dY \to \int_{B(X)} \nabla \tilde{G}_0^\rho(Z, \cdot) \cdot \epsilon^m \nabla \hat{u}_m dY, \quad \rho \to 0,
\]

e.q., \( \hat{F}_m^\rho(Z) \to \hat{F}_m(Z) \). Thus using (4.4) and the dominated convergence theorem we conclude that

\[
(4.9) \quad \int_{B(X)} |\hat{F}_m^\rho|^2 \to \int_{B(X)} |\hat{F}_m|^2, \quad \rho \to 0.
\]

To establish (4.8) we consider the pointwise difference of the two functions at \( Z \in B(X) \). We have that

\[
(\hat{F}_m - \hat{F}_1)(Z) = \int_{B(X)} \nabla_Y \tilde{G}_0(Z, Y) \cdot (\epsilon^m - \epsilon)(Y) \nabla \hat{u}_m(Y) dY
\]

\[
+ \int_{B(X)} \nabla_Y \tilde{G}_0(Z, Y) \cdot \epsilon(Y) \nabla(\hat{u}_m - u_1)(Y) dY
\]

\[
=: I^m(Z) + II^m(Z).
\]

We proceed as in the proof of Proposition 3.6 and consider a cut-off function \( \vartheta \in C_c([-2, 2]) \) such that \( 0 \leq \vartheta \leq 1 \) and \( \vartheta \equiv 1 \) on \([-1, 1]\). For each \( 0 < s < \delta(X)/16 \) we set \( \vartheta_s(Y) := \vartheta(|Y - Z|/s) \) and \( \psi_s := 1 - \vartheta_s \). This allows us to write

\[
I^m(Z) = \int_{B(X)} \nabla_Y \left( \tilde{G}_0(Z, Y) \psi_s(Y) \right) \cdot (\epsilon^m - \epsilon)(Y) \nabla \hat{u}_m(Y) dY
\]

\[
+ \int_{B(X)} \nabla_Y \tilde{G}_0(Z, Y) \vartheta_s(Y) \cdot (\epsilon^m - \epsilon)(Y) \nabla \hat{u}_m(Y) dY
\]

\[
+ \int_{B(X)} \tilde{G}_0(Z, Y) \nabla \vartheta_s(Y) \cdot (\epsilon^m - \epsilon)(Y) \nabla \hat{u}_m(Y) dY
\]

\[
=: I^m_s(Z) + I^m_s(Z) + I^m_s(Z),
\]

and

\[
II^m(Z) = \int_{B(X)} \nabla_Y \left( \tilde{G}_0(Z, Y) \psi_s(Y) \right) \cdot \epsilon(Y) \nabla(\hat{u}_m - u_1)(Y) dY
\]

\[
+ \int_{B(X)} \nabla_Y \tilde{G}_0(Z, Y) \vartheta_s(Y) \cdot \epsilon(Y) \nabla(\hat{u}_m - u_1)(Y) dY
\]

\[
+ \int_{B(X)} \tilde{G}_0(Z, Y) \nabla \vartheta_s(Y) \cdot \epsilon(Y) \nabla(\hat{u}_m - u_1)(Y) dY
\]

\[
=: II^m_s(Z) + II^m_s(Z) + II^m_s(Z),
\]
For $\tilde{I}_s^m$ and $\tilde{I}_s^m$ we can use Hölder’s inequality to get
\[
\tilde{I}_s^m(Z) \lesssim \|\nabla(\tilde{G}_0(Z, \cdot)\psi_s)\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))} \|\tilde{\epsilon}_m - \tilde{\epsilon}\|_{L^r(B(X))} \|\nabla \tilde{u}_m\|_{L^2(B(X))}
\lesssim \|\nabla(\tilde{G}_0(Z, \cdot)\psi_s)\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))} \|\tilde{\epsilon}_m - \tilde{\epsilon}\|_{L^r(B(X))},
\]
and
\[
\tilde{I}_s^m(Z) \lesssim \|\nabla(\tilde{G}_0(Z, \cdot)\psi_s)\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))} \|\tilde{\epsilon}\|_{L^r(B(X))} \|\nabla(\tilde{u}_m - u_1)\|_{L^2(B(X))}
\lesssim \|\nabla(\tilde{G}_0(Z, \cdot)\psi_s)\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))} \|\nabla(\tilde{u}_m - u_1)\|_{L^2(B(X))}.
\]
By the chain rule we see that
\[
\|\nabla(\tilde{G}_0(Z, \cdot)\psi_s)\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))} \lesssim \|\nabla(\tilde{G}_0(Z, \cdot))\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))}
+ \|\nabla \psi_s\|_{L^\infty} \|\tilde{G}_0(Z, \cdot)\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))},
\]
Since $\|\nabla \psi_s\|_{L^\infty} \lesssim \frac{1}{s}$ by Proposition 2.24 and $\tilde{G}_0(Z, \cdot)\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))} \lesssim 1$ and therefore by Proposition 2.13 and Proposition 2.22
\[
\|\nabla(\tilde{G}_0(Z, \cdot))\|_{L^{\frac{2s}{s+r}}(B(X) \setminus B(Z,s))} \lesssim \|\nabla \tilde{G}_0(Z, \cdot)\|_{L^2(B(X) \setminus B(Z,s))}
\lesssim \|\tilde{G}_0(Z, \cdot)\|_{L^2(B(X) \setminus B(Z,s))} \lesssim 1.
\]
Since $\|\tilde{\epsilon}_m - \tilde{\epsilon}\|_{L^r(B(X))}, \|\nabla(\tilde{u}_m - u_1)\|_{L^2(B(X))} \to 0$ for a fixed $s > 0$ we may therefore choose $m = m(s)$ so that
\[
|\tilde{I}_s^m(s)| + |\tilde{I}_s^m(s)| \lesssim \sqrt{s}.
\]
For the remaining terms, estimates similar to the ones in the proofs of Proposition 3.6 and Proposition 3.5 give us:
\[
|\tilde{I}_s^m(Z)|, |\tilde{I}_s^m(Z)| \lesssim \sqrt{s}M[\|\nabla \tilde{u}_m\|_2^2 \chi_{\frac{B}{2}}(B(X))](Z)^{1/2},
\]
and
\[
|\tilde{I}_s^m(Z)|, |\tilde{I}_s^m(Z)| \lesssim \sqrt{s}M[\|\nabla(\tilde{u}_m - u_1)\|_2^2 \chi_{\frac{B}{2}}(B(X))](Z)^{1/2}
\lesssim \sqrt{s}M[\|\nabla \tilde{u}_m\|_2^2 \chi_{\frac{B}{2}}(B(X))](Z)^{1/2} + \sqrt{s}M[\|\nabla u_1\|_2^2 \chi_{\frac{B}{2}}(B(X))](Z)^{1/2}.
\]
Putting everything together, we get for some $m = m(s)$
\[
\int_{B(X)} |\tilde{F}_m - \tilde{F}_1(Z)| \lesssim \int_{B(X)} (\tilde{I}_s^m + \tilde{I}_s^m + \tilde{I}_s^m + \tilde{I}_s^m + \tilde{I}_s^m)
\lesssim \sqrt{s} + \int_{B(X)} \sqrt{s}M[\|\nabla \tilde{u}_m\|_2^2 \chi_{\frac{B}{2}}(B(X))](Z)^{1/2} + \int_{B(X)} \sqrt{s}M[\|\nabla u_1\|_2^2 \chi_{\frac{B}{2}}(B(X))](Z)^{1/2}.
\]
Since $\nabla u_1, \nabla \tilde{u}_m \in L^{\frac{2s}{s+r}}(\frac{B}{2}(B(X)))$, by Hölder and the fact that the maximal function $\|M\|_{L^p \to L^p} < \infty$ for $p > 1$ is bounded we may conclude that
\[ \int_{B(X)} \sqrt{s} M \| \nabla u_1 \|^2 \chi_{B(X)}^{1/2} \leq \sqrt{s} \left( \int_{B(X)} M \| \nabla u_1 \|^2 \chi_{B(X)}^{-r/2} \right)^{-1/r} \]

\[ \leq \sqrt{s} \left( \int_{B(X)} |\nabla u_1|^2 \right)^{1/2} \leq \sqrt{s} \left( \int_{B(X)} |\nabla u_1|^2 \right)^{1/2} \]

and similarly

\[ \int_{B(X)} \sqrt{s} M \| \nabla \hat{u}_m \|^2 \chi_{B(X)}^{1/2} \leq \sqrt{s}. \]

Hence

\[ \int_{B(X)} |\hat{F}_m - \hat{F}_1| \leq \sqrt{s}. \]

Because the implicit constant in this inequality depends on \( m \) or \( s \) we conclude that (4.8) must hold. Thus

\[ \left( \int_{B(X)} |\hat{F}_1|^2 \right)^{1/2} \leq \varepsilon_0 S_M[u_1](Q), \]

as desired.

**Remark 4.10.** This involved approximation argument is not shown in either [MPT11] nor [FKP91]. Instead they argue that \( \hat{F}_1 \) satisfies the equation

\[ \begin{cases} L_0 \hat{F}_1 = \text{div}[\varepsilon \nabla u_1 \chi_{B(X)}], & \text{in } 2B(X), \\ \hat{F}_1 = 0, & \text{on } 2B(X). \end{cases} \]  

(4.11)

There are two problems with this. The first one is that it is not clear if the weak derivative \( \nabla \hat{F}_1 \) even exists in \( L^1_{\text{loc}}(2B(X)) \) due to the low regularity of the Green’s function. The second issue is that even if we could claim that \( \hat{F}_1 \in W^{1,2}_{0}(2B(X)) \), the Green’s function property (2.23) only holds for \( \varphi \in W^{1,p}_0(2B(X)) \) with \( p > n \geq 2 \) and so it would not apply for this case.

Next we consider bounds for \( \hat{F}_1(Z) \). For a large \( r > 1 \) and \( Z \in B(X) \) we have that

\[ \hat{F}_1(Z) = \int_{B(X)} \nabla_Y K(Z, Y) \cdot \varepsilon(Y) \nabla u_1(Y) dY \]

\[ \leq \delta(X)^n \left( \int_{B(X)} |\varepsilon|^r dY \right)^{1/r} \left( \int_{B(X)} |\nabla u_1(Y)|^{2r} dY \right)^{\frac{r-2}{r}} \cdot \left( \int_{B(X)} |\nabla_Y K(Z, Y)|^2 dY \right)^{1/2}. \]

The first two terms are handled as we did above for \( \hat{F}_1 \). Note that \( L_0 K(Z, \cdot) = 0 \) in \( 2B(X) \) and \( K(Z, \cdot) \geq 0 \), so we may use Caccioppoli (Proposition 2.14) and Harnack
(Proposition 2.16) to deduce that

\[
\int_{B(\mathbf{X})} |\nabla_Y K(Z,Y)|^2 dY \lesssim \delta(X)^{-1} \left( \int_{\frac{1}{2}B(\mathbf{X})} |K(Z,Y)|^2 dY \right)^{1/2}
\]

\[
\lesssim \delta(X)^{-1} \sup_{Y \in \frac{1}{2}B(\mathbf{X})} |K(Z,Y)|
\]

\[
\lesssim \delta(X)^{-1} \inf_{Y \in \frac{1}{2}B(\mathbf{X})} |K(Z,Y)|
\]

\[
\lesssim \delta(X)^{-n-1} \int_{\frac{1}{2}B(\mathbf{X})} |K(Z,Y)|dY.
\]

Bounds in Proposition 2.24 apply on \( K \) as it is the sum of two Green’s functions. Hence

\[
\int_{\frac{1}{2}B(\mathbf{X})} |K(Z,Y)| \lesssim \int_{\frac{1}{2}B(\mathbf{X})} |Z - Y|^{2-n} dY = \int_0^{\frac{1}{2}B(\mathbf{X})} tdt \approx \delta(X)^2/2.
\]

Combining this with the previous estimate we obtain

\[
|\tilde{F}_1(Z)| \lesssim \delta(X)^n \varepsilon_0 \left( \int_{B(\mathbf{X})} |\nabla u_1(Y)|^2 dY \right)^{1/2} \delta(X)^{-n+1}
\]

\[
\lesssim \varepsilon_0 \left( \int_{B(\mathbf{X})} |\nabla u_1(Y)|^2 dY \right)^{1/2} \delta(X)^{2-n} \lesssim \varepsilon_0 S_{M}(u_1)(Q_0).
\]

4.2. The “away” term \( F_2 \). The aim is to consider a fixed point \( Z \in B(\mathbf{X}, \delta(X)/8) \). We integrate over \( Y \in \Omega \setminus B(\mathbf{X}) \) and by triangle inequality we therefore must have \( |Z - Y| \geq \delta(X)/8 \) for all such points \( Y \). We would like to obtain a pointwise bound

\[
F_2(Z) \lesssim C \varepsilon_0 M_{\omega_0} S_{M}(u_1)(Q).
\]

Let \( X^* \in \partial \Omega \) be a point such that \( |X^* - X| = \delta(X) \). We consider the following decompositions of the boundary and the domain:

\[
\Delta_j := \Delta(X^*, 2j^{-1}\delta(X)), \quad \Omega_j := \Omega \cap B(X^*, \delta(X)2j^{-1}), \quad R_j := \Omega \setminus (\Omega_{j-1} \cup B(\mathbf{X})),
\]

\[
A^j := A(X^*, 2j^{-1}\delta(X)).
\]

for \( j = -1, 0, 1, \ldots, N \) where \( N \) is chosen so that \( 2^{14}R_0 \leq 2^{-1j-1}\delta(X) < 2^{15}R_0 \). Let

\[
\begin{align*}
F_2^0(Z) := & \int_{\Omega_0} \varepsilon(Y) \nabla_Y G_0(Z,Y) \cdot \nabla u_1(Y) dY, \\
F_2^j(Z) := & \int_{R_j} \varepsilon(Y) \nabla_Y G_0(Z,Y) \cdot \nabla u_1(Y) dY.
\end{align*}
\]
This decomposes $F_2$ into the following terms:

$$
|F_2(Z)| = \left| \int_{\Omega \setminus B(X)} \varepsilon(Y) \nabla_Y G_0(Z, Y) \cdot \nabla u_1(Y) dY \right|
\leq \left| \int_{\partial R_0} \varepsilon(Y) \nabla_Y G_0(Z, Y) \cdot \nabla u_1(Y) dY \right|
+ \sum_{j=1}^N \left| \int_{R_j} \varepsilon(Y) \nabla_Y G_0(Z, Y) \cdot \nabla u_1(Y) dY \right|
+ \int_{(\partial \Omega, 4R_0) \setminus (B(X) \cup B(X^*, 2^{15}R_0))} |\varepsilon(Y)| |\nabla_Y G_0(Z, Y)||\nabla u_1(Y)| dY
= |F_2^0(Z)| + \sum_{j=1}^N |F_2^j(Z)| + J.
$$

Starting with estimating $F_2^0(Z)$ we have that

$$
|F_2^0(Z)| \leq \int_{\Omega \cap (\partial \Omega, 4R_0)} |\varepsilon(Y)| |\nabla_Y G_0(Z, Y)||\nabla u_1(Y)| dY = \lim_{\varepsilon \to 0} \int_{(\Omega \cap (\partial \Omega, R_0)) \setminus (\partial \Omega, \varepsilon)} |\varepsilon(Y)| |\nabla_Y G_0(Z, Y)||\nabla u_1(Y)| dY.
$$

Since we can cover $(\partial \Omega, 4R_0) \setminus (\partial \Omega, \varepsilon)$ by the decomposition introduced in Subsection 2.7, by (2.40), we can write

$$
\int_{(\Omega \cap (\partial \Omega, R_0)) \setminus (\partial \Omega, \varepsilon)} |\varepsilon(Y)| |\nabla_Y G_0(Z, Y)||\nabla u_1(Y)| dY
\leq \sum_{Q^k_{\alpha} \subset \partial \Omega} \int_{I^k_{\alpha} \setminus \{0\}} |\varepsilon(Y)| |\nabla_Y G_0(Z, Y)||\nabla u_1(Y)| dY
\leq \sum_{Q^k_{\alpha} \subset 3 \Delta_0} \int_{I^k_{\alpha}} |\varepsilon(Y)| |\nabla_Y G_0(Z, Y)||\nabla u_1(Y)| dY
\leq \sum_{Q^k_{\alpha} \subset 3 \Delta_0} \text{diam}(Q^k_{\alpha})^n \left( \int_{I^k_{\alpha}} |\varepsilon(Y)|^r dY \right)^{1/r} \left( \int_{I^k_{\alpha}} |\nabla_Y G_0(Z, Y)|^2 dY \right)^{1/2}
\cdot \left( \int_{I^k_{\alpha}} |\nabla u_1(Y)|^{\frac{2r}{r-2}} dY \right)^{\frac{r-2}{2r}}.
$$

(4.12)
Using the ball covering that we have introduced in subsection 2.7 and its properties (2.43) and (2.42) together with Proposition 2.13 we obtain

\[
\left( \int_{I_{k}^{i}} |\nabla u_{1}|^{2} \right)^{\frac{r-2}{r}} \leq \sum_{i=1}^{N} \left( \int_{B(X_{i},\lambda^{8-k-3})} |\nabla u_{1}|^{2} \right)^{\frac{r-2}{r}} \\
\leq \sum_{i=1}^{N} \left( \int_{B(X_{i},2\lambda^{8-k-3})} |\nabla u_{1}|^{2} \right)^{\frac{r-2}{r}} \\
\leq \left( \int_{I_{k}^{i}} |\nabla u_{1}|^{2} \right)^{\frac{r-2}{r}}.
\]

(4.13)

We estimate the Green’s function using Caccioppoli’s inequality

\[
\left( \int_{I_{k}^{i}} |\nabla Y_{G_{0}(Z,Y)}|^{2} dY \right)^{\frac{1}{2}} \leq \text{diam}(Q_{n}^{\alpha})^{-1} \left( \int_{I_{k}^{i}} |G_{0}(Z,Y)|^{2} dY \right)^{\frac{1}{2}} \\
\leq \text{diam}(Q_{n}^{\alpha})^{-1} \left( \int_{I_{k}^{i}} \frac{|G_{0}(Z,Y)|^{2}}{|G_{0}(Y)|^{2}} |G_{0}(Y)|^{2} dY \right)^{\frac{1}{2}} \\
\leq \text{diam}(Q_{n}^{\alpha})^{-1} \left( \inf_{Y \in I_{k}^{i}} \frac{|G_{0}(Z,Y)|^{2}}{|G_{0}(Y)|^{2}} \right) \left( \int_{I_{k}^{i}} |G_{0}(Y)|^{2} dY \right)^{\frac{1}{2}}.
\]

For the last term we further use the comparison principle and the doubling property of the elliptic measure:

\[
\text{diam}(Q_{n}^{\alpha})^{-1} \left( \int_{I_{k}^{i}} |G_{0}(Y)|^{2} dY \right)^{\frac{1}{2}} \approx \text{diam}(Q_{n}^{\alpha})^{-1} \left( \int_{I_{k}^{i}} \frac{\omega_{0}(Q_{n}^{\alpha})^{2}}{\text{diam}(Q_{n}^{\alpha})^{2n-3}} dY \right)^{\frac{1}{2}} \\
\leq \omega_{0}(Q_{n}^{\alpha}) \text{diam}(Q_{n}^{\alpha})^{-n+1}.
\]

Since we can cover 5\Delta_{0} with \( N \) balls \( B(Q_{i},\delta(X)/4) \) such that \( |Q_{i}-Q_{j}| < \delta(X)/4, Q_{i} \in 5\Delta_{0} \) we see that \( \Omega_{0} \cap (\partial\Omega,\delta(X)/8) \subset \bigcup_{i} B(Q_{i},\delta(X)/4) \). Note that \( N \) here is independent of \( X \) and \( \delta(X) \). Let \( \tilde{A}_{i} = A(Q_{i},\delta(X)/4) \). By the comparison principle for \( Y \in B(Q_{i},\delta(X)/4) \) we have that

\[
\frac{|G_{0}(Z,Y)|}{|G_{0}(Y)|} \approx \frac{|G_{0}(Z,\tilde{A}_{i})|}{|G_{0}(\tilde{A}_{i})|}.
\]

By the Harnack’s inequality for all \( Y \in \Omega_{0} \setminus (\partial\Omega,\delta(X)/16) \)

\[
\frac{|G_{0}(Z,Y)|}{|G_{0}(Y)|} \approx \frac{|G_{0}(Z,\tilde{A}_{i})|}{|G_{0}(\tilde{A}_{i})|}.
\]

Since \( \tilde{A}_{i} \in \Omega_{0} \setminus (\partial\Omega,\delta(X)/16) \) also have the same estimates for all \( Y \in \Omega_{0} \cap (\partial\Omega,\delta(X)/16) \), that is

\[
\frac{|G_{0}(Z,Y)|}{|G_{0}(Y)|} \approx \frac{|G_{0}(Z,\tilde{A}_{i})|}{|G_{0}(\tilde{A}_{i})|} \approx \frac{|G_{0}(Z,A_{0})|}{|G_{0}(A_{0})|}.
\]

Hence we may use the comparison principle for the Green’s function to obtain

\[
\frac{|G_{0}(Z,A_{0})|}{|G_{0}(A_{0})|} \approx \frac{\omega_{0}^{2}(\Delta^{*}(\delta(X)/2))}{\omega_{0}(\Delta^{*}(\delta(X)/2))} \leq \frac{1}{\omega_{0}(\Delta_{0})}.
\]

(4.14)
After we put all pieces together we finally have for the gradient of $G_0$:

\[
\left( \int_{I^k_h} |\nabla_Y G_0(Z,Y)|^2 dY \right)^{1/2} \lesssim \omega_0(Q^k_\alpha) \text{diam}(Q^k_\alpha)^{-n+1} \frac{\omega_0(\Delta_0)}{\omega_0(\Delta_0)}.
\]

Next, we consider the term of (4.12) containing $\varepsilon$ function. By Proposition 2.49 we have

\[
\omega_0(Q^k_\alpha) \left( \int_{I^k_h} |\varepsilon'|^r dY \right)^{1/r} \leq \omega_0(Q^k_\alpha)^{1/2} \left( \int_{I^k_h} \frac{G_0(Z)\beta_\varepsilon(Z)^2}{\delta(Z)^2} dZ \right)^{1/2},
\]

and hence (4.12) can be further estimated by

\[
\sum_{Q^k_\alpha \in 3\Delta_0} \text{diam}(Q^k_\alpha)^n \left( \int_{I^k_h} |\varepsilon(Z)|^r dY \right)^{1/r} \left( \int_{I^k_h} |\nabla_Y G_0(Z,Y)|^2 dY \right)^{1/2} \cdot \left( \int_{I^k_h} |\nabla u_1(Z)\|^2 dY \right)^{r-2/r}.
\]

\[
(4.16)
\]

\[
\lesssim \frac{1}{\omega_0(\Delta_0)} \sum_{Q^k_\alpha \in 3\Delta_0} \omega_0(Q^k_\alpha)^{1/2} \left( \int_{I^k_h} \frac{G_0(Z)\beta_\varepsilon(Z)^2}{\delta(Z)^2} dZ \right)^{1/2} \left( \int_{I^k_h} |\nabla u_1(Z)\|^2 \delta^{2-n} \right)^{1/2}.
\]

For the purposes of the stopping time argument below we define

\[
Tu_1(Z) = |\nabla u_1(Z)|^2 \delta(Z)^{2-n}
\]

and the super-level sets

\[
O_j = \{ P \in 3\Delta_0; T \in (\partial (\partial T_u_1(P)) \cap (\partial 1 \cdot 4 R_0)) \} \geq 2^j \}
\]

We say a dyadic boundary cube $Q^k_\alpha$, $k R_0 \leq k \leq k_\varepsilon$ belongs to $J_j$, if

\[
\omega_0(O_j \cap Q^k_\alpha) \geq \frac{1}{2} \omega_0(Q^k_\alpha) \quad \text{and} \quad \omega_0(O_{j+1} \cap Q^k_\alpha) < \frac{1}{2} \omega_0(Q^k_\alpha),
\]

and belongs to $J_\infty$, if

\[
\omega_0(Q^k_\alpha \cap \{ T \in u_1 = 0 \}) \geq \frac{1}{2} \omega_0(Q^k_\alpha).
\]

Furthermore, let $M_{\omega_0}$ be the uncentered Hardy-Littlewood maximal function and let

\[
\tilde{O}_j = \{ M_{\omega_0}(\varepsilon O_j) > 1/2 \}
\]

and observe that for $Z \in Q^k_\alpha \in J_j$

\[
M_{\omega_0}(\varepsilon O_j)(Z) \geq \frac{\omega_0(Q^k_\alpha \cap O_j)}{\omega_0(Q^k_\alpha)} \geq \frac{1}{2}
\]

and hence $Q^k_\alpha \subset \tilde{O}_j$. Thus, we also have

\[
\omega_0(Q^k_\alpha \cap \tilde{O}_j \cap O_{j+1}) = \omega_0(Q^k_\alpha \cap O_{j+1}) \geq \frac{1}{2} \omega_0(Q^k_\alpha).
\]
The weak $L^1$ boundedness of the maximal function therefore implies that

$$\omega_0(\tilde{O}_j \setminus O_{j+1}) \leq \omega_0(\tilde{O}_j) = \omega_0(M\omega_0(\chi_{O_j}) > 1/2)) \lesssim \|\chi_{O_j}\|_{L^1(\omega_0)} = \omega_0(O_j).$$

We use this to further estimate (4.16). Applying the above decomposition and the Cauchy-Schwarz inequality we get

$$\sum_{Q_k^\alpha \subset 3\Delta_0 \atop k_0 \leq k \leq k_e} \omega_0(Q_k^\alpha)^{1/2} \left( \int_{I_k^\alpha} \frac{G_0(Z)\beta_r(Z)^2}{\delta(Z)^2} dZ \right)^{1/2} \left( \int_{I_k^\alpha} |\nabla u_1|^2 \delta^{2-n} dY \right)^{1/2} \lesssim \frac{1}{\omega_0(\Delta_0)} \sum_j \left( \sum_{Q_k^\alpha \subset J_j \atop k_0 \leq k \leq k_e} \int_{I_k^\alpha} \frac{G_0(Z)\beta_r(Z)^2}{\delta(Z)^2} dZ \right)^{1/2} \cdot \left( \sum_{Q_k^\alpha \subset J_j \atop k_0 \leq k \leq k_e} \omega_0(Q_k^\alpha) \int_{I_k^\alpha} |\nabla u_1|^2 \delta^{2-n} dY \right)^{1/2}.$$

Since for every two cubes $Q_k^\alpha, Q_k^\beta$, $l \leq k$ either contain each other, i.e., $Q_k^\beta \subset Q_k^\alpha$ or are disjoint $Q_k^\beta \cap Q_k^\alpha = \emptyset$, there is a disjoint collection of cubes in $J_j$ such that their union covers all the other cubes. We call them the top cubes. We observe that for any such top cube $Q_k^\alpha$ and its subcube $Q_k^\beta \subset Q_k^\alpha$ we have $I_k^\beta \subset T(\Delta(z_n^\alpha, (C_0 + 16\lambda)s^{-k}))$ and the overlap of these Carleson regions of different top cubes $Q_k^\alpha$ is finite. We also know that the overlap of the $I_k^\alpha$ is finite. Hence

$$\sum_{Q_k^\alpha \subset J_j} \int_{I_k^\alpha} \frac{G_0(Z)\beta_r(Z)^2}{\delta(Z)^2} dZ = \sum_{Q_k^\alpha \subset J_j} \sum_{Q_k^\beta \subset Q_k^\alpha} \int_{I_k^\beta} \frac{G_0(Z)\beta_r(Z)^2}{\delta(Z)^2} dZ \lesssim \sum_{Q_k^\alpha \subset J_j} \int_{T(\Delta(z_n^\alpha, (C_0 + 16\lambda)s^{-k}))} \frac{G_0(Z)\beta_r(Z)^2}{\delta(Z)^2} dZ \lesssim \sum_{Q_k^\alpha \subset J_j} \varepsilon_0^2 \omega_0(Q_k^\alpha) \leq 2\varepsilon_0^2 \omega_0(O_j).$$

Here we have used Proposition 2.49 in the penultimate step and the property of cubes in $J_j$ in the last step. Denote $S_M'(Q) = \Gamma_M(Q) \setminus B_{2\varepsilon(Q)} \cap (\partial\Omega, 4R_0)$. Then

$$\sum_{Q_k^\alpha \subset J_j} \omega_0(Q_k^\alpha) \int_{I_k^\alpha} Tu_1(Z) dZ = \sum_{Q_k^\alpha \subset J_j} \sum_{Q_k^\beta \subset Q_k^\alpha} \omega_0(Q_k^\beta) \int_{I_k^\beta} Tu_1(Z) dZ$$
Taking $Y \in \Omega$, we observe that $\omega_0((\tilde{O}_j \setminus O_{j+1}) \cap Q^0_\beta) \int_{\tilde{I}^0_\beta} T u_1(Z) dZ \lesssim \int_{(\tilde{O}_j \setminus O_{j+1}) \cap Q^0_\beta} T u_1(Z) dZ d\omega_0(P)$.

\[
\lesssim \sum_{Q^\kappa_\beta \in I_j} \sum_{Q^\mu_\beta \subset Q^\kappa_\beta \text{ disj. top cubes}} \int_{(\tilde{O}_j \setminus O_{j+1}) \cap Q^\mu_\beta} T u_1(Z) dZ d\omega_0(P)
\]

We put everything together to finally get the following estimate for (4.12):

\[
\int_{\Omega_0 \setminus (\partial \Omega, \varepsilon)} |\varepsilon||G_0||\nabla u_1| dY \lesssim \frac{1}{\omega_0(\Delta_0)} \sum_j \varepsilon_0 \omega_0(O_j)^{1/2} \left( \int_{(O_j \setminus O_{j+1})} T \varepsilon u_1^2 d\omega_0 \right)^{1/2}
\]

\[
\leq \varepsilon_0 \frac{1}{\omega_0(\Delta_0)} \sum_j 2^{j+1} \omega_0(O_j)^{1/2} \omega_0((\tilde{O}_j \setminus O_{j+1}) \cap Q^0_\beta)^{1/2}
\]

\[
\leq \varepsilon_0 \frac{1}{\omega_0(\Delta_0)} \sum_j 2^{j+1} \omega_0(O_j)
\]

\[
\leq \varepsilon_0 \frac{1}{\omega_0(\Delta_0)} \int_{3 \Delta_0} T \varepsilon u_1 d\omega_0
\]

\[
= \varepsilon_0 \frac{1}{\omega_0(\Delta_0)} \int_{3 \Delta_0} \left( \int_{(G_0(\Delta_0 \setminus B_4(\Delta_0 \cap (\partial \Omega, \varepsilon))) \cap (\partial \Omega, \varepsilon))} |\nabla u_1(Z)|^2 \delta(Z)^{-\alpha} dY \right)^{1/2} d\omega_0(P).
\]

Taking $\varepsilon \to 0$ finally yields

\[
F^0_2(Z) = \int_{\Omega_0} |\varepsilon||\nabla G_0||\nabla u_1| dY \lesssim \varepsilon_0 \frac{1}{\omega_0(\Delta_0)} \int_{3 \Delta_0} S_M u_1 \omega_0 \lesssim M_{\omega_0} [S_M(u)](Q_0).
\]

Our next objective are the terms $F^j_2$ for $j \geq 1$. We split the region $R_j$ and write it as a union of two subregions:

$R_j = (R_j \cap (\partial \Omega, 2^{j-6} \delta(X))) \cup (R_j \setminus (\partial \Omega, 2^{j-6} \delta(X))) := V_j \cup W_j$.

For $Y \in R_j$, we observe that $|A_{j-1} - Y| \geq \delta(X)/4$ since $A_{j-1} \in \tilde{\Omega}_1$. Now, by the Harnack’s inequality $G_0(Z,Y) \approx G_0(A_{j-1},Y)$ where the implicit constants in the Harnack’s inequality are independent of the points $Z$ and $X$. Using the boundary
Hölder continuity of the nonnegative solution $G_0(\cdot, Y)$ in $\Omega_{j-2}$ in Proposition 2.21 we get that

$$G_0(Z, Y) \approx G_0(A_{j-2}, Y) \leq \sup_{Z \in T(\Omega_{j-1})} G_0(\tilde{Z}, Y) \lesssim \left( \frac{2^{-2j}\delta(X)}{2^{-3}\delta(X)} \right)^{\beta} G_0(A_{j-2}, Y) \lesssim 2^{-j\alpha} G_0(A_{j-2}, Y). \quad (4.17)$$

Assume now that $Y \in V_j$. We proceed similarly to the above case for $\Omega_0$. We have a bound analogous to (4.12), namely that

$$\int_{V_j \setminus (\partial \Omega_j \cup \mathcal{C})} |\varepsilon(Y)||\nabla Y G_0(Z, Y)||\nabla u_1(Y)|dY \leq \sum_{Q_k^\alpha \subset \frac{3}{2} \Delta_j \setminus \frac{3}{2} \Delta_{j-2}} \text{diam}(Q_k^\alpha)^n \left( \int_{I_k^\alpha} |\varepsilon(Y)|^r dY \right)^{1/r} \left( \int_{I_k^\alpha} |\nabla Y G_0(Z, Y)|^2 dY \right)^{1/2} \cdot \left( \int_{I_k^\alpha} |\nabla u_1(Y)|^{2r} dY \right)^{-1/2} \left( \int_{I_k^\alpha} |G_0(Z, Y)|^2 dY \right)^{1/2} \lesssim \text{diam}(Q_k^\alpha)^{-1} \left( \int_{I_k^\alpha} |G_0(Z, Y)|^2 dY \right)^{1/2}.$$

Instead of (4.15) we have a similar argument. By Caccioppoli’s inequality we have

$$\left( \int_{I_k^\alpha} |\nabla Y G_0(Z, Y)|^2 dY \right)^{1/2} \lesssim \text{diam}(Q_k^\alpha)^{-1} \left( \int_{I_k^\alpha} |G_0(Z, Y)|^2 dY \right)^{1/2} \lesssim \text{diam}(Q_k^\alpha)^{-1} \left( \int_{I_k^\alpha} |G_0(Y)|^2 |G_0(Y)|^2 dY \right)^{1/2}.$$

We can cover $\frac{3}{2} \Delta_j \setminus \frac{3}{2} \Delta_{j-2}$ with at most $N$ balls $B(Q_i, 2^{j-5}\delta(X))$ such that $|Q_i - Q_j| < 2^{j-5}\delta(X)$, $Q_i \subset \frac{3}{2} \Delta_j \setminus \frac{3}{2} \Delta_{j-2}$ and $I_k^\alpha \subset \bigcup_i B(Q_i, 2^{j-5}\delta(X))$ when $Q_k^\alpha \subset \frac{3}{2} \Delta_j \setminus \frac{3}{2} \Delta_{j-2}$. Note that $N$ is again independent of $X$ and $\delta(X)$. Let $\tilde{A}_i = A(Q_i, 2^{j-5}\delta(X))$ and since $\text{dist}(I_k^\alpha, A_{j-2}) \geq 2^{j-3} - 2^{j-3} - 2^{j-5} \geq 2^{j-5}\delta(X)$ the comparison principle applies and gives us for $Y \in B(Q_i, 2^{j-5}\delta(X))$:

$$\frac{|G_0(A_{j-2}, Y)|}{|G_0(Y)|} \approx \frac{|G_0(A_{j-2}, \tilde{A}_i)|}{|G_0(\tilde{A}_i)|}.$$

Since $\delta(\tilde{A}_i) = 2^{j-5}\delta(X)$ we can use Harnack to obtain

$$\frac{|G_0(A_{j-2}, \tilde{A}_i)|}{|G_0(A_i)|} \approx \frac{|G_0(A_{j-2}, A_j)|}{|G_0(A_j)|}.$$

Hence by Proposition 2.26 we have that

$$\frac{|G_0(A_{j-2}, A_j)|}{|G_0(A_j)|} \lesssim \frac{\omega_0^{A_j}(\Delta_{j-2})}{\omega_0(\Delta_j)} \lesssim \frac{1}{\omega_0(\Delta_j)}. \quad (4.18)$$

Combined with the above estimate with (4.17) we therefore have

$$\frac{G_0(Z, Y)}{G_0(Y)} \approx 2^{-j\beta} \frac{G_0(A_{j-2}, Y)}{G_0(Y)} \lesssim 2^{-j\beta} \frac{1}{\omega_0(\Delta_j)},$$
and hence
\[
\left( \int_{I_0^j} |\nabla_Y G_0(Z,Y)|^2 dY \right)^{1/2} \lesssim \frac{1}{\omega_0(\Delta_j)} 2^{-j\beta} \text{diam}(Q^0_\alpha)^{-1} \left( \int_{I_0^j} |G_0(Y)|^2 dY \right)^{1/2}.
\]

We again apply the stopping time argument but this time with the sets \( O_j = \{ P \in \frac{3}{2}\Delta_j \setminus \frac{1}{2}\Delta_{j-2}; T_z u_1(P) > 2^j \} \). This leads to the estimate
\[
\int_{V_j \setminus (\partial \Omega, \varepsilon)} |\varepsilon(Y)||\nabla_Y G_0(Z,Y)||\nabla u_1(Y)| dY \lesssim 2^{-jn} \varepsilon_0 \frac{1}{\omega_0(\Delta_j)} \int_{\frac{3}{2}\Delta_j \setminus \frac{1}{2}\Delta_{j-2}} \left( \int_{D_{M,\epsilon, R_0}} |\nabla u_1(Z)|^2 \delta(Z)^{-n} d\omega \right)^{1/2} d\omega_0(P),
\]
where \( D_{M,\epsilon, R_0} = (\Gamma_M(P) \setminus B_\varepsilon(P)) \cap (\partial \Omega, 4R_0) \). Again, taking \( \varepsilon \to 0 \) yields:
\[
\int_{V_j} |\varepsilon||\nabla_Y G_0||\nabla u_1| dY \lesssim \varepsilon_0 \frac{1}{\omega_0(\Delta_j)} \int_{\frac{3}{2}\Delta_j \setminus \frac{1}{2}\Delta_{j-2}} S_M u_1(P) d\omega_0(P) \leq M_{\omega_0} [S_M(u)](Q_0).
\]

This takes care of the subregion \( V_j \). When \( Y \in W_j \) we cover \( W_j \) by at most \( N \) balls \( B_{jl} := B(X_l^j, 2^{j-3} \delta(X)) \) with \( X_l^j \in W_j \). Again, \( N \) is independent of \( j \). Using the comparison principle for the Green's function, the doubling the elliptic measure and the fact that \( G_0(A_{j-2}) \approx G_0(Y) \) using Harnack we have
\[
G_0(A_{j-2}, Y) = \frac{G_0(A_{j-2}, Y)}{G_0(A_{j-2})} G_0(Y) = \frac{G_0(Y, A_{j-2})}{G_0(A_{j-2})} G_0(Y) \lesssim \frac{1}{\omega_0(\Delta_j)} G_0(Y).
\]
Since (4.17) still applies we therefore have \( G_0(Z,Y) \lesssim 2^{-j\alpha} \frac{1}{\omega_0(\Delta_j)} G_0(Y) \). By Proposition 2.20 we have \( G_0(Y) \approx G_0(A_{j+1}) \) and therefore
\[
\left( \int_{B_{jl}} |\nabla_Y G_0(Z,Y)|^2 dY \right)^{1/2} \lesssim \frac{2^{-j\beta} 2^{-j+7} \delta(X)}{\omega_0(\Delta_j)} \left( \int_{B_{jl}} |G_0(Y)|^2 dY \right)^{1/2} \leq \frac{2^{-j\beta}}{\omega_0(\Delta_j)} 2^{-j} \delta(X) G_0(A_{j+1}).
\]

For every \( Y \in B_{jl} \) we have estimates \( \delta(Y) \geq (2^{j-6} - 2^{j-8}) \delta(X) \geq 2^{j-7} \delta(X) \) and \( |Y - X^*| \leq (2^j + 2^{j-8}) \delta(X) \leq 2^{j+1} \delta(X) \) therefore have for \( M = 2^8 \)
\[
B_{jl} \subset \tilde{B}_{jl} \subset \Gamma_M(X^*) \subset \Gamma_M(Q_0),
\]
where \( \tilde{B}_{jl} = B(X_l^j, 2^{j-8} \delta(X)) \) is an enlarged ball. Since the finite ball covers \( (B_{jl}) \) and \( \tilde{B}_{jl} \) can be chosen to have finite overlap. This implies and estimate similar to
(4.12), namely that
\[
\int_{W_j} |\varepsilon||\nabla Y G_0||\nabla u_1|dY \leq \sum_l \int_{B_{jl}} |\varepsilon||\nabla Y G_0||\nabla u_1|dY \\
\leq \sum_l (2^{j-7}\delta(X))^n \left( \int_{B_{jl}} |\varepsilon(Y)|^r dY \right)^{1/r} \left( \int_{B_{jl}} |\nabla Y G_0(Z,Y)|^2dY \right)^{1/2} \\
\cdot \left( \int_{B_{jl}} |\nabla u_1(Y)|^{2/r} dY \right)^{2/r} \\
\lesssim \sum_l (2^j\delta(X))^n \left( \int_{B_{jl}} |\varepsilon(Y)|^r dY \right)^{1/r} \left( \int_{B_{jl}} |\nabla Y G_0(Z,Y)|^2dY \right)^{1/2} \\
\cdot \left( \int_{B_{jl}} |\nabla u_1(Y)|^{2/r} dY \right)^{2/r} .
\]

By Proposition 2.13 we get a statement analogous to (4.13):
\[
\left( \int_{B_{jl}} |\nabla u_1(Y)|^{2/r} dY \right)^{2/r} \lesssim \left( \int_{B_{jl}} |\nabla u_1(Y)|^2dY \right)^{1/2} .
\]

Next we consider term containing the function \(\varepsilon\). We see that \(|B_{jl}| \approx \delta(Y)^n \approx (2^j\delta(X))^n\) and \(B_{jl} \subset B(Y,\delta(Y)/2)\) for \(Y \in B_{jl}\). Hence by the Harnack \(G_0(Y) \approx G_0(A_{j+1})\) and therefore
\[
\left( \int_{B_{jl}} |\varepsilon(Y)|^r dY \right)^{1/r} \lesssim \int_{B_{jl}} \left( \int_{B_{jl}} |\varepsilon(Y)|^r dY \right)^{1/r} dZ \\
\lesssim \int_{B_{jl}} \left( \int_{B(Z,\delta(Z)/2)} |\varepsilon(Y)|^r dY \right)^{1/r} dZ \\
\lesssim \left( \frac{1}{(2^j\delta(X))^{n/2}G_0(A_{j+1})} \int_{B_{jl}} \beta_\varepsilon(Y)^2 \frac{G_0(Y)dY}{\delta(Y)^2} \right)^{1/2} .
\]
and therefore
\[
\sum_l (2^j\delta(X))^n \left( \int_{B_{jl}} |\varepsilon(Y)|^r dY \right)^{1/r} \left( \int_{B_{jl}} |\nabla Y G_0(Z,Y)|^2dY \right)^{1/2} \\
\cdot \left( \int_{B_{jl}} |\nabla u_1(Y)|^{2/r} dY \right)^{2/r} \\
\lesssim \sum_l \left( \frac{2^{-j\beta}}{\omega_0(\Delta_j)} (2^j\delta(X))^{-n/2}G_0(A_{j+1})^{1/2} \left( \int_{B_{jl}} \beta_\varepsilon(Y)^2 \frac{G_0(Y)dY}{\delta(Y)^2} \right)^{1/2} \right)^{1/r} \\
\cdot \left( \int_{B_{jl}} |\nabla u_1(Y)|^2dY \right)^{1/2} .
\]
Next, by the comparison principle, Proposition 2.26, Cauchy-Schwarz the assumptions of Theorem 3.1 we have

\[
\sum_{l} \frac{2^{-j\beta}}{\omega_0(\Delta_j)} \left( (2^j \delta(X))^{n/2} G_0(A_{j+1}) \right)^{1/2} \left( \int_{B_{j,l}} \frac{\beta_r(Y)}{\delta(Y)^2} G_0(Y) dY \right)^{1/2} 
\cdot \left( \int_{B_{j,l}} |\nabla u_1(Y)|^2 dY \right)^{1/2}
\lesssim \sum_{l} \frac{2^{-j\beta}}{\omega_0(\Delta_j)} \left( (2^j \delta(X))^{n-2} G_0(A_{j+1}) \right)^{1/2} \left( \int_{B_{j,l}} \frac{\beta_r(Y)}{\delta(Y)^2} G_0(Y) dY \right)^{1/2} 
\cdot \left( \int_{B_{j,l}} |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2}
\lesssim 2^{-j\beta} \left( \sum_{l} \frac{1}{\omega_0(\Delta_{j+1})} \int_{B_{j,l}} \frac{\beta_r(Y)}{\delta(Y)^2} G_0(Y) dY \right)^{1/2} 
\cdot \left( \sum_{l} \int_{B_{j,l}} |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2}
\lesssim 2^{-j\beta} \left( \frac{1}{\omega_0(\Delta_{j+1})} \int_{\Omega_{j+1}} \frac{\beta_r(Y)}{\delta(Y)^2} G_0(Y) dY \right)^{1/2} 
\cdot \left( \int_{\Omega_{j+1} \setminus (\partial \Omega, 2^{j-7} \delta(X))} |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2}
\lesssim 2^{-j\beta} \varepsilon_0 S_M^k(u_1)(Q_0).
\]

Combining together the estimates for the subregions $V_j$ and $W_j$ and summing over $j$’s we get

\[
\sum_{j=1}^{N} |F_j^2(Z)| \lesssim \sum_{j=1}^{N} \int_{R_j} |\varepsilon(Y)||\nabla Y G(Z,Y)||\nabla u_1(Y)| dY
\lesssim \sum_{j=1}^{N} 2^{-j\beta} \varepsilon_0 M_{\omega_0} [S_M^k(u_1)](Q_0) \lesssim \varepsilon_0 M_{\omega_0} [S_M^k(u_1)](Q_0).
\]

We still have one more term $J$ to tackle. First, we observe that

\[
(\partial \Omega, 4R_0) \setminus (B(X) \cup B(X^*, 2^{15} R_0)) \subset \bigcup_{Q_k \subset \partial \Omega, \Delta_{2^{14} R_0}} I_{\alpha_k}^k.
\]
To see this consider any $Y \in (\partial \Omega, 4R_0) \setminus (B(X) \cup B(X^*, 2^{15}R_0))$. Since the collection $\{I^k_{\alpha} \}_{\alpha,k}$ covers $(\partial \Omega, 4R_0)$ we have that $Y \in I^k_{\alpha}$. We know that that there exists $P^k_{\alpha} \in Q^k_{\alpha}$ such that $8/\lambda \leq |P^k_{\alpha} - Y| \leq 8\lambda$. For such $P \in I^k_{\alpha}$ we have

$$|P - X^*| \geq |Y - X^*| - |P - P^k_{\alpha}| - |P^k_{\alpha} - Y| \geq 2^{15}R_0 - C_08^{-k} - \lambda 8^{-k+1}$$

$$\geq 2^{15}R_0 - 8R_0 - 8R_0 \geq 2^{14}R_0,$$

and hence $Q^k_{\alpha} \subset \partial \Omega \setminus \Delta_{2^{14}R_0}$. We again start with an estimate analogous to (4.12):

$$\int_{(\partial \Omega, 4R_0) \setminus (B(X) \cup B(X^*, 2^{15}R_0))} |\varepsilon(Y)||\nabla G_0(Z,Y)||\nabla u_1(Y)|dY$$

$$\leq \sum_{Q^k_{\alpha} \subset \partial \Omega \setminus \Delta_{2^{14}R_0} \atop k_0 \leq k} \int_{I^k_{\alpha}} |\varepsilon(Y)||\nabla G_0(Z,Y)||\nabla u_1(Y)|dY$$

$$\leq \sum_{Q^k_{\alpha} \subset \partial \Omega \setminus \Delta_{2^{14}R_0} \atop k_0 \leq k} \text{diam}(Q^k_{\alpha})^n \left( \int_{I^k_{\alpha}} |\varepsilon(Y)|^r dY \right)^{1/r} \left( \int_{I^k_{\alpha}} |\nabla G_0(Z,Y)|^2 dY \right)^{1/2}$$

$$\cdot \left( \int_{I^k_{\alpha}} |\nabla u_1(Y)|^{2/n} dY \right)^{n/2}.$$

Since $Y \in I^k_{\alpha}$ is far away from $Z$ and $0$ we can use Harnack’s inequality to conclude that $G_0(Z,Y) \approx G_0(Y)$, where the implicit constants are independent of $Y$. Again, using the Cacciopoli’s inequality, Proposition 2.26 and the fact that $\omega_0$ is doubling we have as a replacement of (4.15):

$$\left( \int_{I^k_{\alpha}} |\nabla G_0(Z,Y)|^2 dY \right)^{1/2} \lesssim \left( \int_{I^k_{\alpha}} |\nabla G_0(Y)|^2 dY \right)^{1/2} \lesssim \text{diam}(Q^k_{\alpha})^{-1} \left( \int_{I^k_{\alpha}} |G_0(Y)|^2 dY \right)^{1/2} \lesssim \frac{\omega_0(3Q^k_{\alpha})}{\text{diam}(Q^k_{\alpha})^{n-1}} \lesssim \frac{\omega_0(Q^k_{\alpha})}{\text{diam}(Q^k_{\alpha})^{n-1}}.$$

Next step is again the stopping time argument analogous to the case $F_2^0$ with sets $O_j = \{ P \in \partial \Omega \setminus \Delta_{2^{14}R_0}: T_{e_u}(P) > 2^j \}$. This gives us the final estimate of this section:

$$J = \int_{(\partial \Omega, 4R_0) \setminus (B(X) \cup B(X^*, 2^{15}R_0))} |\varepsilon(Y)||\nabla G_0(Z,Y)||\nabla u_1(Y)|dY$$

$$\lesssim \varepsilon_0 \int_{\partial \Omega \setminus \Delta_{2^{14}R_0}} S_M(u_1) dP \lesssim \varepsilon_0 \frac{1}{\omega_0(\partial \Omega)} \int_{\partial \Omega} S_M(u_1) dP$$

$$\lesssim \varepsilon_0 M \omega_0 |S_M(u_1)|(Q_0).$$

5. Proof of Lemma 3.4

To prove Lemma 3.4 we need establish a “good-λ” inequality. To shorten our notation let

$$h[F, u_1] := N_M[F] S_M[F] + N_M[F] S_M[u_1] + \tilde{N}_M[u_0] S_M[u_1], \quad M := 8\bar{M}.$$


Lemma 5.1. There exists $0 < \gamma < 1$ such that for all $\lambda > 0$
\[ \omega_0 \left( \left\{ S_M[F] > 2\lambda, \ h[F, u_1] \leq (\lambda \gamma)^2 \right\} \right) \lesssim \gamma^2 \omega_0 \left( \left\{ S_M[F] > \lambda \right\} \right). \]

Also, because $\omega_0 \in A_\infty(d\sigma)$ a similar "good-\(\lambda\)" inequality holds for $\sigma$ as well:

Corollary 5.2. There exists $0 < \eta < 1, C > 0$ and $0 < \gamma < 1$ such that for all $\lambda > 0$
\[ \sigma \left( \left\{ S_M[F] > 2\lambda, \ h[F, u_1] \leq (\lambda \gamma)^2 \right\} \right) \lesssim \gamma^\eta \sigma \left( \left\{ S_M[F] > \lambda \right\} \right). \]

Proof. Consider $q$ for which $\omega_0 \in A_q(d\sigma)$. Then
\[ \frac{\sigma(E)}{\sigma(\Delta)} \lesssim \left( \frac{\omega_0(E)}{\omega_0(\Delta)} \right)^{1/q}, \quad E \subset \Delta \subset \partial \Omega, \quad \Delta \text{ cube.} \]

Next take a Whitney decomposition of $\left\{ S_M[F] > \lambda \right\} = \bigcup_j \Delta_j$, where $\Delta_j \subset \partial \Omega$ are cubes, and set
\[ E_j := \Delta_j \cap \left\{ S_M[F] > 2\lambda, \ h[F, u_1] \leq (\lambda \gamma)^2 \right\}. \]

Then
\[ \sigma(E_j) = \sigma(\Delta_j) \frac{\sigma(E_j)}{\sigma(\Delta_j)} \lesssim \sigma(\Delta_j) \left( \frac{\omega_0(E_j)}{\omega_0(\Delta_j)} \right)^{1/q} \lesssim \sigma(\Delta_j) \left( \frac{\gamma \omega_0(\Delta_j)}{\omega_0(\Delta_j)} \right)^{1/q} \]
\[ \lesssim \gamma^{1/q} \sigma(\Delta_j). \]

This proves our corollary. \( \square \)

Lemma 3.4 is a consequence of the following lemma.

Lemma 5.3.

\[ \int_{\partial \Omega} S_M[F]^q d\omega_0 \lesssim \int_{\partial \Omega} f^q d\omega_0 + \int_{\partial \Omega} N_\alpha[F]^q d\omega_0. \] (5.4)

Moreover if $\omega_0 \in B_p(d\sigma)$ we have

\[ \int_{\partial \Omega} S_M[F]^q d\sigma \lesssim \int_{\partial \Omega} f^q d\sigma + \int_{\partial \Omega} N_\alpha[F]^q d\sigma, \] (5.5)

Proof. We take $\mu \in \{ \sigma, \omega_0 \}$ since the proof works analogously for both measures. The "good-\(\lambda\)-inequality" of Lemma 5.1 or Corollary 5.2 implies that
\[ \int_{\partial \Omega} S_M[F]^q d\mu = q2^{q-1} \int_0^\infty \lambda^{q-1} \mu(\{ S_M[F] > 2\lambda \}) d\lambda \]
\[ \lesssim \int_0^\infty \lambda^{q-1} \mu(\{ S_M[F] > 2\lambda, \ h[F, u_1] \leq (\lambda \gamma)^2 \}) d\lambda. \]
\[ + \int_0^\infty \lambda^{\alpha - 1} \mu(\{N_M[F]S_{\hat{M}}[F] > (\lambda \gamma)^2\}) d\lambda \]
\[ + \int_0^\infty \lambda^{\alpha - 1} \mu(\{N_M[F]S_{\hat{M}}[u_1] > (\lambda \gamma)^2\}) d\lambda \]
\[ + \int_0^\infty \lambda^{\alpha - 1} \mu(\{N_M[u_0]S_{\hat{M}}[u_1] > (\lambda \gamma)^2\}) d\lambda \]
\[ \leq \gamma^\alpha \int_{\partial \Omega} S_{\hat{M}}[F]^q d\mu + C_\gamma \left( \int_{\partial \Omega} (S_{\hat{M}}[F]N_M[F])^{q/2} d\mu + \int_{\partial \Omega} (N_M[u_0]S_{\hat{M}}[u_1])^{q/2} d\mu \right). \]

Because \( \|S_{\hat{M}}[F]\|_{L^q(\mu)} \approx \|S_M[F]\|_{L^q(\mu)} \) thank to Proposition 2.38, we choose \( \gamma \) sufficiently small so that the first term of the last line can be absorbed by the lefthand side. Next,
\[ \int_{\partial \Omega} (S_{\hat{M}}[F]N_M[F])^{q/2} d\mu \leq \rho \int_{\partial \Omega} S_{\hat{M}}[F]^q d\mu + C_\rho \int_{\partial \Omega} N_M[F]^q d\mu. \]

Hence again for a sufficiently small \( \rho \) the square function term can be hidden on the lefthand side. We treat the other terms similarly, using the fact that \( S_{\hat{M}}[u_1] \lesssim S_{\hat{M}}[F] + S_{\hat{M}}[u_0] \), and then apply (2.36) to obtain
\[ \int_{\partial \Omega} S_{\hat{M}}[F]^q d\mu \lesssim \int_{\partial \Omega} \tilde{N}_{\hat{M}}[u_0]^q d\mu + \int_{\partial \Omega} N_{\hat{M}}[F]^q d\mu + \int_{\partial \Omega} S_{\hat{M}}[u_0]^q d\mu. \]

Finally, note that \( \omega_0 \in B_{\rho}(\mu) \) which implies
\[ \|S_{\hat{M}}[u_0]\|_{L^q(\mu)} \approx \|\tilde{N}_{\hat{M}}[u_0]\|_{L^q(\mu)} \lesssim \|f\|_{L^q(\mu)}, \]

therefore with the help of Lemma 2.33 we have
\[ \int_{\partial \Omega} S_{\hat{M}}[F]^q d\mu \lesssim \int_{\partial \Omega} f^q d\mu + \int_{\partial \Omega} N_{\hat{M}}[F]^q d\mu \lesssim \int_{\partial \Omega} f^q d\mu + \int_{\partial \Omega} N_\alpha[F]^q d\mu. \]

This means that (5.5) holds. Since \( \omega_0 \in B_2(\omega_0) \) we also get (5.4). \( \square \)

It remains to establish Lemma 5.1.

5.1. **Proof of Lemma 5.1.** Consider a decomposition of \( \{S_{\hat{M}}[F] > \lambda\} \) into a union of Whitney balls \( \Delta_j \). We set
\[ E_j := \Delta_j \cap \left\{ N_{\hat{M}}[F]S_{\hat{M}}[F], N_{\hat{M}}[F]S_{\hat{M}}[u_1], \tilde{N}_{\hat{M}}[u_0]S_{\hat{M}}[u_1] \leq (\lambda \gamma)^2 \right\} \]
and in what follows we drop the subscript \( j \). By Lemma 1 of [DJK84] we know that for every \( \tau > 0 \) there exists a \( \gamma > 0 \) such that for the truncated square function
\[ S_{\tau\gamma}[F]^2(Q) := \int_{\Gamma_{\tau\gamma}(Q)} \left| \nabla F(X) \right|^2 \delta(X)^{2-n} dX > \frac{\lambda^2}{4} \]
holds for all points \( Q \in E \), where \( \Gamma_{\tau\gamma}(Q) := \Gamma_{\tau\gamma}(Q) \cap B(Q, \tau r) \). Let \( \tilde{\Omega} = \bigcup_{Q \in E} \Gamma_{\tau\gamma}(Q) \) be a sawtooth region. We would like to define a partition of unity on it. Recall from subsection 2.7 the family of balls \( B(X_\lambda, 8^{-k-3})_{1 \leq i \leq N} \) covering \( I_\alpha^k \) and denote their center points by \( X_{\alpha,i}^k \). We claim the existence of a family \( (\eta_{\alpha,i}^k)_k \) with the following properties
(1) \( \eta_{\alpha,i}^k \in C_0^\infty (\tilde{I}_\alpha^k) \)
(2) $0 \leq \eta^k, l, \leq 1$

(3) $\eta^0, l \equiv 1$ on $B(X^k, l, 2\lambda 8^{l, 3})$ and $\eta^0, l \equiv 0$ outside of $B(X^k, l, 2\lambda 8^{l, 3})$

(4) $\|\nabla \eta^k, l\|_{L^\infty} \approx \frac{1}{\text{diam}(Q^k, \lambda)}$

(5)

$$\sum_{\alpha, k, l} \eta^k, l = 1 \quad \text{on } \Gamma^* M(Q)$$

and

$$\sum_{\alpha, k, l} \eta^k, l = 0 \quad \text{on } \Omega \setminus \Gamma^* M(Q).$$

Let $D_k(Q) := \{ I^k \mid I^k \cap \Gamma^* M(Q) \neq \emptyset \}$, $D(Q) = \bigcup_k D_k(Q)$, where we let $k_0$ be the scale from which on $D_k(Q) = \emptyset$ for all $k \geq k_0$. We can observe that for the choice $M := 8M$ and for all $I^k \in D(Q)$ we have $I^k \subset \Gamma^* M(K)$. Therefore

$$\omega_0(E) \lesssim \frac{1}{\lambda^2} \int_E S_{\tau_r}[F]^2 d\omega_0 = \frac{1}{\lambda^2} \int_E \int_{\Omega} \|\nabla F\|^2 \delta^{2-n} \chi_{\Gamma^* M(Q)} dX d\omega(Q)$$

$$= \frac{1}{\lambda^2} \int_{\Omega} \|\nabla F\|^2 \delta^{2-n} \left( \int_E \chi_{\Gamma^* M(Q)} d\omega(Q) \right) dX$$

$$\lesssim \frac{1}{\lambda^2} \int_{\Omega} \|\nabla F\|^2 \delta^{2-n} \omega_0(\Delta(X^*, \delta(X))) dX$$

$$\lesssim \frac{1}{\lambda^2} \int_{\Omega} \|\nabla F\|^2 G_0 dX.$$

In the last line we have used the comparison principle, Proposition 2.26 and writing $G_0(X) = G(0, X)$. Continuing our estimate we have

$$\frac{1}{\lambda^2} \int_{\Omega} \|\nabla F\|^2 G_0 dX \approx \frac{1}{\lambda^2} \int_E \int_{\Gamma^* M(Q)} \delta^{1-n} \|\nabla F\|^2 G_0 dX d\sigma(Q)$$

$$\leq \frac{1}{\lambda^2} \int_E \int_{\Gamma^* M(Q)} \delta^{1-n} \|\nabla F\|^2 \left( G_0 \sum_{k, l, \alpha} \eta^k, l \right) dX d\sigma(Q)$$

$$\lesssim \frac{1}{\lambda^2} \int_E \sum_{k, l, \alpha} \int_{I^k} \delta(X)^{1-n} \|\nabla F\|^2 \left( G_0 \eta^k, l \right) dX d\sigma(Q)$$

$$\lesssim \frac{1}{\lambda^2} \int_E \sum_{k, l, \alpha} \text{diam}(Q^k)^{1-n} \int_{I^k} \|\nabla F\|^2 \left( G_0 \eta^k, l \right) dX d\sigma(Q)$$

$$\lesssim \frac{1}{\lambda^2} \int_E \sum_{k, l, \alpha} \text{diam}(Q^k)^{1-n} \int_{I^k} A_0 \nabla F \cdot \nabla F \left( G_0 \eta^k, l \right) dX d\sigma(Q).$$
By the penultimate line we have used the fact that \( \delta(X) \approx \text{diam}(Q^k_\alpha) \). Consider some fixed \( k, l, \alpha \). Since \( G_0 \eta^{k,l}_\alpha, FG_0 \eta^{k,l}_\alpha \in W^{1,2}(I^k_\alpha) \) we have

\[
\int_{I^k_\alpha} A_0 \nabla \cdot \nabla F(G_0 \eta^{k,l}_\alpha) = \int_{I^k_\alpha} \text{div}(A_0 \nabla (F^2)) G_0 \eta^{k,l}_\alpha - F \text{div}(A_0 \nabla u_0) G_0 \eta^{k,l}_\alpha \\
+ F \text{div}(A_1 \nabla u_1) G_0 \eta^{k,l}_\alpha + \text{div}(\varepsilon \nabla u_1) F G_0 \eta^{k,l}_\alpha
= \int_{I^k_\alpha} \text{div}(A_0 \nabla (F^2)) G_0 \eta^{k,l}_\alpha + \int_{I^k_\alpha} \text{div}(\varepsilon \nabla u_1) F G_0 \eta^{k,l}_\alpha,
\]

and therefore

\[
\text{diam}(Q^k_\alpha)^{1-n} \int_{I^k_\alpha} A_0 \nabla \cdot \nabla F(G_0 \eta^{m,l}_\alpha) = \text{diam}(Q^k_\alpha)^{1-n} \int_{I^k_\alpha} \text{div}(A_0 \nabla (F^2)) G_0 \eta^{k,l}_\alpha + \text{diam}(Q^k_\alpha)^{1-n} \int_{I^k_\alpha} \text{div}(\varepsilon \nabla u_1) F G_0 \eta^{k,l}_\alpha
=: I^{k,l,\alpha} + II^{k,l,\alpha}.
\]

The term \( I^{k,l,\alpha} \) we handle using integration by parts

\[
|I^{k,l,\alpha}| = \left| \text{diam}(Q^k_\alpha)^{1-n} \int_{I^k_\alpha} \text{div}(A_0 \nabla (F^2)) G_0 \eta^{k,l}_\alpha \right|
= \left| \text{diam}(Q^k_\alpha)^{1-n} \int_{I^k_\alpha} A_0 \nabla (F^2) \nabla (G_0 \eta^{k,l}_\alpha) \right|
\leq \text{diam}(Q^k_\alpha)^{1-n} N_M[F](Q) \left| \int_{I^k_\alpha} A_0 \nabla F \nabla (G_0 \eta^{k,l}_\alpha) \right|.
\]

By (2.7) we may assume that \( (A_0^0)_{I^k_\alpha} = 0 \) and therefore

\[
|I^{k,l,\alpha}| = \text{diam}(Q^k_\alpha) N_M[F](Q) \left| \int_{I^k_\alpha} A_0 \nabla F \nabla (G_0 \eta^{k,l}_\alpha) \right|
\leq \text{diam}(Q^k_\alpha) N_M[F](Q) \left( \int_{I^k_\alpha} |A_0|^r \right)^{1/r} \left( \int_{I^k_\alpha} |\nabla F|^\frac{r}{2} \right)^{\frac{r}{2}}
\cdot \left( \int_{I^k_\alpha} (\nabla (G_0 \eta^{k,l}_\alpha))^2 \right)^{1/2}
\lesssim \text{diam}(Q^k_\alpha) N_M[F](Q) \left( \int_{I^k_\alpha} |\nabla F|^\frac{r}{2} \right)^{\frac{r}{2}} \left( \int_{I^k_\alpha} (\nabla (G_0 \eta^{k,l}_\alpha))^2 \right)^{1/2}.
\]
We use the fact that \( \|\nabla \eta_{k,l}^{\alpha}\|_{L^\infty} \approx \frac{1}{\text{diam}(Q_{\beta}^m)} \), and apply Proposition 2.14 and Proposition 2.26:

\[
\left( \int_{I_{\alpha}^m} |\nabla (G_0 \eta_{k,l}^{\alpha})|^2 \right)^{1/2} \lesssim \left( \int_{I_{\alpha}^m} |\nabla G_0|^2 + \frac{|G_0|^2}{\delta^2} \right)^{1/2} \\
\lesssim \frac{1}{\text{diam}(Q_{\beta}^m)} \left( \int_{I_{\alpha}^m} |\nabla G_0|^2 \right)^{1/2} \\
\lesssim \frac{1}{\text{diam}(Q_{\beta}^m)} \left( \int_{I_{\alpha}^m} \left( \frac{\omega_0(\Delta(X, \delta(X)))}{\delta(X)^{n-2}} \right)^2 \right)^{1/2} \\
\lesssim \omega_0(\Delta(Q, \text{diam}(Q_{\beta}^m))) \\
\lesssim \text{diam}(Q_{\beta}^m)^{n-1} \\
(5.6) \quad \lesssim \int_{\Delta(Q, \text{diam}(Q_{\beta}^m))} k d\sigma \leq M[k](Q).
\]

The definition of \( F \) and Proposition 2.13 then give us

\[
\left( \int_{I_{\alpha}^m} |\nabla F|^{\frac{r-2}{r}} \right)^{\frac{r}{r-2}} \leq \left( \int_{I_{\alpha}^m} |\nabla u_0|^{\frac{r-2}{r}} \right)^{\frac{r}{r-2}} + \left( \int_{I_{\alpha}^m} |\nabla u_1|^{\frac{r-2}{r}} \right)^{\frac{r}{r-2}} \\
\lesssim \left( \int_{I_{\alpha}^m} |\nabla u_0|^2 \right)^{1/2} + \left( \int_{I_{\alpha}^m} |\nabla u_1|^2 \right)^{1/2} \\
\lesssim \left( \int_{I_{\alpha}^m} |\nabla F|^2 \right)^{1/2} + \left( \int_{I_{\alpha}^m} |\nabla u_1|^2 \right)^{1/2}.
\]

Finally, after putting all terms together we have

\[
|I^{k,l,\alpha}| \lesssim N_{\alpha}^1[F](Q) M[k](Q) \left( \left( \int_{I_{\alpha}^m} |\nabla F|^2 \delta^{2-n} \right)^{1/2} + \left( \int_{I_{\alpha}^m} |\nabla u_1|^2 \delta^{2-n} \right)^{1/2} \right).
\]

Next, consider \( II^{k,l,\alpha} \). Again, by integration by parts

\[
II^{k,l,\alpha} = \text{diam}(Q_{\alpha}^k)^{1-n} \int_{I_{\alpha}^m} \text{div}(\varepsilon \nabla u_1) F G_0 \eta_{k,l}^{\alpha} \\
= \text{diam}(Q_{\alpha}^k) \int_{I_{\alpha}^m} \varepsilon \nabla u_1 \nabla G_0 \eta_{k,l}^{\alpha} + \text{diam}(Q_{\alpha}^k) \int_{I_{\alpha}^m} \varepsilon \nabla u_1 F \nabla (G_0 \eta_{k,l}^{\alpha}) \\
= II_1^{m,l,\beta} + II_2^{m,l,\beta}.
\]

First, we consider \( II_1^{k,l,\alpha} \). For \( X \in I_{\alpha}^k \) by Proposition 2.26 and Proposition 2.27 we obtain

\[
G_0(X) \approx \frac{\omega_0(\Delta(X, \delta(X)))}{\delta(X)^{n-2}} \approx \frac{\omega_0(\Delta(Q, \text{diam}(Q_{\alpha}^k)))}{\text{diam}(Q_{\alpha}^k)^{n-2}} \\
= \text{diam}(Q_{\alpha}^k) \int_{\Delta(Q, \text{diam}(Q_{\alpha}^k))} k(Y) dY \leq \text{diam}(Q_{\alpha}^k) M[k](Q).
\]
Hence we have

\[ |II_{1,k,l,\alpha}| \leq M[k](Q) \operatorname{diam}(Q^k_\alpha)^2 \left( \int_{I^k_\alpha} |\nabla u_1|^2 \right)^{1/2} \left( \int_{I^k_\alpha} |\varepsilon|^r \right)^{1/r} \left( \int_{I^k_\alpha} |\nabla F|^{5/2} \right)^{\frac{5}{3}}. \]

For the last term by Lemma 2.37 and Proposition 2.13 we have

\[ \left( \int_{I^k_\alpha} |\nabla F|^{5/2} \right)^{\frac{5}{3}} \lesssim \left( \int_{sI^k_\alpha} |\nabla F|^2 \right)^{1/2} + \left( \int_{sI^k_\alpha} |\nabla u_0|^2 \right)^{1/2}. \]

\[ \lesssim \frac{1}{\operatorname{diam}(Q^k_\alpha)} \left( \left( \int_{I^k_\alpha} |F|^2 \right)^{1/2} + \left( \int_{I^k_\alpha} |u_0|^2 \right)^{1/2} \right) \]

\[ \leq \frac{1}{\operatorname{diam}(Q^k_\alpha)} \left( N_M[F](Q) + \tilde{N}_M(u_0)(Q) \right). \]

In the previous calculation we have chosen \( s > 1 \) (independent of \( k \) and \( \alpha \)) to be a small constant such that the enlargement \( s\tilde{I}^k_\alpha \) of \( \tilde{I}^k_\alpha \) is a sufficiently small enlargement, so that it is between the sets \( \tilde{I}^k_\alpha \) and \( \hat{I}^k_\alpha \); that is

\[ \tilde{I}^k_\alpha \subset s\tilde{I}^k_\alpha \subset s^2\tilde{I}^k_\alpha \subset \hat{I}^k_\alpha. \]

Since \( \left( \int_{\tilde{I}^k_\alpha} |\varepsilon|^r \right)^{1/r} \) is bounded by Proposition 2.49 and Proposition 7.1 of [MPT11] we then get

\[ |II_{1,k,l,\alpha}| \lesssim (\tilde{N}_M[F] + \tilde{N}_M(u_0))(Q)M[k](Q) \left( \int_{\tilde{I}^k_\alpha} |\nabla u_1|^2 \delta^{2-n} \right)^{1/2}. \]

Next, we consider \( II_{2,k,l,\alpha} \). We have

\[ |II_{2,k,l,\beta}| \leq N_M[F](Q) \operatorname{diam}(Q^k_\alpha) \left( \int_{\tilde{I}^k_\alpha} |\nabla u_1|^2 \delta^{2-n} \right)^{1/2} \left( \int_{\tilde{I}^k_\alpha} |\varepsilon|^r \right)^{1/r} \left( \int_{\tilde{I}^k_\alpha} |\nabla(G_0\eta^{k,\alpha})|^2 \right)^{1/2}. \]

The term \( \left( \int_{\tilde{I}^k_\alpha} |\varepsilon|^r \right)^{1/r} \) is again bounded, and using Proposition 2.13 and (5.6) we then have

\[ |II_{2,k,l,\beta}| \lesssim N_M[F](Q)M[k](Q) \left( \int_{\tilde{I}^k_\alpha} |\nabla u_1|^2 \delta^{2-n} \right)^{1/2}. \]

It remains to add up together all terms.

\[ \omega_0(E) \lesssim \frac{1}{\lambda^2} \int_E \sum_{k,l,\alpha} I^{k,l,\alpha} + II^{k,l,\alpha} d\sigma(Q). \]
\begin{align*}
\lesssim \frac{1}{\lambda^2} \int_E \left[ \sum_{k,l,\alpha \in D(Q)} N_M[F](Q) M[k](Q) \left( \left( \int_{I^k_\alpha} |\nabla F|^2 \right)^{1/2} + \left( \int_{I^k_\alpha} |\nabla u_1|^2 \right)^{1/2} \right) 
+ (N_M[F] + \tilde{N}_M[u_0])(Q) M[k](Q) \left( \int_{I^k_\alpha} |\nabla u_1|^2 \right)^{1/2} 
+ N_M[F](Q) M[k](Q) \left( \int_{I^k_\alpha} |\nabla u_1|^2 \right)^{1/2} \right] d\sigma(Q)
\lesssim \frac{1}{\lambda^2} \int_E M[k] \cdot \left[ N_M[F] S_M(F) + N_M[F] S_M(u_1) + \tilde{N}_M[u_0] S_M(u_1) \right] d\sigma(Q)
\lesssim \gamma^2 \int_E M[k](Q) d\sigma(Q).
\end{align*}

In the last line we used the properties of the set $E$. For the last step we use the fact that $k \in B_p(d\sigma)$, the Hölder’s inequality and boundedness of the maximal function.

\begin{equation}
\gamma^2 \int_E M[k](Q) d\sigma(Q) \leq \gamma^2 |\Delta| \int_{\Delta} M[k](Q) d\sigma(Q) 
\leq \gamma^2 |\Delta| \left( \int_{\Delta} M[k](Q)^p d\sigma(Q) \right)^{1/p} 
\lesssim \gamma^2 |\Delta| \left( \int_{\Delta} k^p d\sigma(Q) \right)^{1/p} 
\leq \gamma^2 |\Delta| \int_{\Delta} k d\sigma(Q) = \gamma^2 \omega_0(\Delta).
\end{equation}

Hence we have $\omega_0(E) \lesssim \gamma^2 \omega_0(\Delta)$, or more precisely $\omega_0(E_j) \lesssim \gamma^2 \omega_0(\Delta_j)$ as this is true for all $j$. Summing up in $j$ gives us the desired good-$\lambda$ inequality.

### 6. Proof of Theorem 1.6

Most of the work to prove Theorem 1.6 is already done. Recall that we want to show that $\omega_1 \in B_p(d\sigma)$ which is equivalent to

$$
\|\hat{N}_\alpha(u_1)\|_{L^q(\partial \Omega, d\sigma)} \lesssim \|f\|_{L^q(\partial \Omega, d\sigma)}, \quad \text{for} \quad \frac{1}{q} + \frac{1}{p} = 1.
$$

We assume that $\omega_0 \in B_p(d\sigma)$ which is equivalent to $\sigma \in A_q(d\omega)$. Using this, Lemma 3.3 and Lemma 5.3 imply:

\begin{align*}
\int_{\partial \Omega} \hat{N}_\alpha[F]^q d\sigma & \lesssim \int_{\partial \Omega} \varepsilon_0^q M \omega_0[S_M u_1]^q d\sigma \\
& \lesssim \varepsilon_0^q \int_{\partial \Omega} S_M[u_1]^q d\sigma
\end{align*}
\[ \lessapprox \varepsilon_0^q \int_{\partial \Omega} S_M[F]^q d\sigma + \varepsilon_0^q \int_{\partial \Omega} S_M[u_0]^q d\sigma \]
\[ \lessapprox \varepsilon_0^q \int_{\partial \Omega} S_M[F]^q d\sigma + \int_{\partial \Omega} f^q d\sigma \]
\[ \lessapprox \varepsilon_0^q \int_{\partial \Omega} \bar{N}_\alpha[F]^q d\sigma + \int_{\partial \Omega} \bar{N}_\alpha[u_0]^q d\sigma + \int_{\partial \Omega} f^q d\sigma. \]
\[ \lessapprox \varepsilon_0^q \int_{\partial \Omega} \bar{N}_\alpha[F]^q d\sigma + \int_{\partial \Omega} f^q d\sigma. \]

By Lemma 2.33, and with \( \varepsilon_0 \) sufficiently small, we can hide the first term of the righthand side by moving it to the lefthand side. Hence
\[ \| \bar{N}_\alpha[F] \|_{L^q(d\sigma)} \lesssim \| f \|_{L^q(d\sigma)}. \]
Moreover we also have \( \| \bar{N}_\alpha[u_0] \|_{L^q(d\sigma)} \lesssim \| f \|_{L^q(d\sigma)} \) because \( \omega_0 \in B_p(d\sigma) \). Thus
\[ \int_{\partial \Omega} \bar{N}_\alpha[u_1]^q d\sigma \lesssim \int_{\partial \Omega} \bar{N}_\alpha[F]^q d\sigma + \int_{\partial \Omega} \bar{N}_\alpha[u_0]^q d\sigma \lesssim \int_{\partial \Omega} \bar{N}_\alpha[F]^q d\sigma + \int_{\partial \Omega} f^q d\sigma \]
\[ \lesssim \int_{\partial \Omega} f^q d\sigma. \]
From this \( \omega_1 \in B_p(d\sigma) \).

7. Operators with coefficients satisfying Carleson condition

In this section \( \Omega \) will be a bounded Lipschitz domain with Lipschitz constant \( K \). We consider the operator \( L = \text{div}(A \nabla \cdot) \), where \( A \) is \( \lambda_0 \)-elliptic with \( \| A^\eta \|_{\text{BMO}(\Omega)} \leq \Lambda_0 \) and recall that
\[ \alpha_r(Z) := \left( \int_{B(Z, \delta(Z)/2)} |A - (A)_{B(Z, \delta(Z)/2)}|^r \right)^{1/r}. \]
The aim of this section is to prove Theorem 1.9 and Theorem 1.7. But first we prove a similar slightly weaker result:

**Theorem 7.1.** Let \( \Omega \) be a bounded Lipschitz domain with Lipschitz constant \( K \) and suppose that \( \bar{A} \) is elliptic with BMO antisymmetric part. Moreover suppose that the weak derivative of coefficients exists and consider
\[ \hat{\alpha}_\eta(Z) := \delta(Z) \sup_{X \in B(Z, \eta \delta(Z))} |\nabla \bar{A}(X)|, \quad 0 < \eta < 1/2. \]

(i) If
\[ \| \hat{\alpha}_\eta(Z)^2 \delta(Z)^{-1} dZ \|_c < \infty \]
then the \( L^p \) Dirichlet problem is solvable for some \( 1 < p < \infty \),

(ii) For each \( 1 < p < \infty \), there exists an \( \varepsilon = \varepsilon(p) > 0 \), such that if
\[ \| \hat{\alpha}_\eta(Z)^2 \delta(Z)^{-1} dZ \|_c < \varepsilon \quad \text{and} \quad K < \varepsilon, \]
then the \( L^p \) Dirichlet problem is solvable.

Statement (ii) follows from [DPP07, Theorem 2.2]; though it is stated there for bounded matrices it holds in our case as well. To prove (i) we apply the following theorem (see [DPP17, Theorem 1.3] or [KKPT14, Theorem 4.1]):
Theorem 7.2. If
\[ \| |\nabla u|^2 \delta dX\|_C \lesssim \| f\|_{L^\infty(\partial \Omega)}, \quad \forall f \in C(\partial \Omega), \] (7.3)
then \( \omega \in A_\infty(\sigma) \).

It remains to show (7.3). Let \( \Delta \subset \partial \Omega \) be a boundary ball with \( \text{diam} \Delta \leq \gamma \), where \( \gamma \) is taken small enough that \( T(\Delta) \) lies above a Lipschitz graph. Note that by [DP18, Cor 5.2] we have
\[ \hat{T}(\Delta) |\nabla u|^2 \delta dX \lesssim \hat{\alpha}^2 \Delta (|f|^2 + N_\alpha(u)^2) d\sigma. \]
(Again although \( \hat{A} \) is assumed to be bounded in [DP18] this assumption is not necessary this Corollary to hold as it only uses ellipticity and boundedness of the symmetric part of the matrix). Thus \( f \in C(\partial \Omega) \) and the maximum principle imply
\[ \int_{T(\Delta)} |\nabla u|^2 \delta dX \lesssim \| f\|_{L^\infty(\partial \Omega)} \lesssim \| f\|_{L^\infty(\Delta)}. \]
Dividing both sides by \( \sigma(\Delta) \) and taking supremum over \( \Delta \) yields (7.3). \( \square \)

7.1. Proofs of Theorem 1.9 and Theorem 1.7. In order to prove Theorem 1.9 and Theorem 1.7 we have the following strategy. First we construct a matrix \( \hat{A} \) from \( A \). The objective is to improve regularity of coefficients in order to use Theorem 7.1 for \( \hat{A} \). We then deduce solvability for the original matrix \( A \) by applying our perturbation results.

To begin with, let \( B(X) = B(X, \delta(X)/2) \) and set
\[ \hat{B}(X) := B(X, \frac{\delta(X)}{2}), \]
so that
\[ \bigcup_{X \in \hat{B}(Z)} \hat{B}(X) \subset B(Z). \]

In order to apply Theorem 7.1 our matrix needs to be differentiable. Thus we define \( \hat{A} \) from \( A \) using a mollification procedure. Consider \( \phi \in C_\infty(\mathbb{R}^n) \) to be nonnegative with \( \int_{\mathbb{R}^n} \phi = 1 \), and \( \phi_t(X) = t^{-n} \phi(X/t) \).

Let \( \delta(X) \) be a smooth version of the distance function and
\[ \hat{A}(X) := (\phi_\delta(X) * A)(X). \]

Clearly \( \hat{A}(X) \) is differentiable with
\[ \nabla \hat{A}(X) = \int_{\Omega} (A(Y) - b) \nabla X \phi_\delta(X)(X - Y) dY, \quad \text{for any} \ b \in \mathbb{R}^{n \times n}. \]

If we can show that
\[ \| \hat{\alpha}(Z)^2 \delta(Z)^{-1} dZ \|_c \lesssim \| \alpha_r(Z)^2 \delta(Z)^{-1} dZ \|_c, \]
holds for \( \hat{\alpha} := \hat{\alpha}^2 \) clearly Theorem 7.1 implies that:

Lemma 7.7. Let \( \Omega \) be a bounded Lipschitz domain with Lipschitz constant \( K > 0 \) Let \( \alpha_r \) be defined like in (1.8) and let \( \omega \) be the elliptic measure of the operator \( L = \text{div}(\hat{A} \nabla \cdot) \). Then there exists \( 1 < r = r(n, \lambda_0, \Lambda_0) < \infty \) such that
(i) If
\[ \|\alpha_r(Z)^2\delta(Z)^{-1}dZ\|_C < \infty \]
then \( \omega \in A_\infty(\sigma) \), i.e. the \( L^p \) Dirichlet problem for \( \hat{A} \) is solvable for some \( 1 < p < \infty \).

(ii) For every \( 1 < p < \infty \) there exists an \( \varepsilon = \varepsilon(p) > 0 \), such that if
\[ \|\alpha_r(Z)^2\delta(Z)^{-1}dZ\|_C < \varepsilon \quad \text{and} \quad K < \varepsilon, \]
then \( \omega \in B_p(\sigma) \), i.e., the \( L^p \) Dirichlet problem for \( \hat{A} \) is solvable.

To prove (7.6) it suffices to show that
\[ \hat{\alpha}(Z) \lesssim \alpha_r(Z). \]
Take \( b = (A)_B(Z) \) in (7.5). Then
\[ |\nabla \hat{A}(X)| \leq \int_{B(X)} |A(Y) - (A)_B(Z)| |\nabla \phi(X) (X - Y)| dY. \]
Let us estimate the gradient term inside the integral.
\[ \delta(X)^{n+1} |\nabla \phi(X)| \lesssim \left| \nabla \delta(X) \phi \left( \frac{X - Y}{\delta(X)} \right) \right| + \left| \nabla \phi \left( \frac{X - Y}{\delta(X)} \right) \right| \left| \nabla \delta(X) \right| \lesssim 1, \]
since \( |X - Y| \leq \delta(X) \) and \( |\nabla \delta| \leq 1 \). It follows that
\[ |\nabla \phi(X)| \lesssim \delta(X)^{-1}. \]
This implies that for any \( X \in \hat{B}(Z) \) we have that
\[ |\nabla \hat{A}(X)| \lesssim \frac{1}{\delta(X)^{n+1}} \int_{B(X)} |A(Y) - (A)_B(Z)| dY \]
\[ \leq \frac{1}{\delta(X)^{n+1}} \int_{B(Z)} |A(Y) - (A)_B(Z)| dY \]
\[ \approx \frac{1}{\delta(Z)} \left( \int_{B(Z)} |A(Y) - (A)_B(Z)|^r dY \right)^{1/r} \]
\[ \leq \frac{1}{\delta(Z)} \left( \int_{B(Z)} |A(Y) - (A)_B(Z)|^r dY \right)^{1/r} = \frac{1}{\delta(Z)} \alpha_r(Z). \]
From this
\[ \hat{\alpha}(Z) = \delta(Z) \sup_{X \in \hat{B}(Z)} |\nabla \hat{A}| \lesssim \alpha_r(Z), \]
as desired.

It remains to apply our two perturbation results for \( A_0 = \hat{A} \) and \( A_1 = A \). Clearly \( \hat{A} \lambda_0 \)-elliptic. Moreover we can see that \( \|\hat{A}\|_{BMO(\Omega)} \lesssim \Lambda_0 \).

To see this we distinguish two cases. First, consider a ball \( B \subset \Omega \) such that \( B \not\subset \hat{B}(X) \) is true for all \( X \in \Omega \). Then we can find a cover with balls \( (\hat{B}(X_i))_i \) such that the balls \( \hat{B}(X_i) \) have finite overlap, and \( |\bigcup_i \hat{B}(X_i)| \lesssim |B| \). The constants in the last inequality are independent of \( B \). By Lemma 2.1 of [Jon80] we know that \( |(A)_{\hat{B}(X)} - (A)_{B(Z)}| \lesssim \Lambda_0 \) for all \( X \in \hat{B}(Z) \). Hence
\[
\int_B |\hat{A} - (A)_B|dX \leq \int_B |\hat{A} - A|dX + \int_B |A - (A)_B|dX \\
\leq \int_B \left| \int_{B(X)} (A(Y) - A(X)) \frac{\phi \left( \frac{X - Y}{\delta(X)^{\nu}} \right)}{\delta(X)^{\nu}} dY \right| dX + \Lambda_0 \\
\leq \int_B \frac{1}{|B|} \sum_i \int_{B(X_i)} |A(X) - (A)_{B(X_i)}| dY + |(A)_{B(X_i)} - (A)_{B(X_i)}|dX + 2\Lambda_0 \\
\leq \int_B |A(X) - (A)_{B(X)}|dX + 2\Lambda_0 \\
\lesssim \sum_i |B(X)| \int_{B(X_i)} |A(X) - (A)_{B(X_i)}| + \Lambda_0 dX + 2\Lambda_0 \\
\lesssim \lambda_0 \left( 2 + \frac{1}{|B|} \sum_i |B(X_i)| \right) \lesssim \Lambda_0.
\]

The second case is if \( B \subset \hat{B}(X_1) \) for some \( X_1 \in \Omega \). Then we have
\[
\int_B |\hat{A} - (A)_{B(X_1)}|dX \leq \int_B \left| \int_{B(X)} (A(Y) - (A)_{B(X)} \frac{\phi \left( \frac{X - Y}{\delta(X)^{\nu}} \right)}{\delta(X)^{\nu}} dY \right| dX \\
\leq \int_B \int_{B(X)} |A(Y) - (A)_{B(X_1)}|dY dX \\
\leq \int_B \int_{B(X_1)} |A(Y) - (A)_{B(X_1)}|dY dX \lesssim \Lambda_0.
\]

Thus we can conclude that
\[
\inf_{M \in \mathbb{R}^{n \times n} \text{ constant}} \int_B |\hat{A}(X) - M|dX \lesssim \Lambda_0,
\]
which implies \( \|\hat{A}\|_{BMO(\Omega)} \lesssim \Lambda_0 \). It follows that we indeed may apply our perturbation results.

Let
\[
\beta_r(Z) := \left( \int_{B(Z)} |\hat{A}(Y) - A(Y)|^r dY \right)^{1/r}.
\]

Our next objective is to show that
\[
(7.9) \quad \|\beta_r(Z)\|_C \lesssim \|\alpha_r(Z)\|_C.
\]

Assume for the moment this is indeed true. Then Lemma 7.7 and Theorem 1.5 imply Theorem 1.9. Similarly, Lemma 7.7 and Theorem 1.6 imply Theorem 1.7. Thus if we establish (7.9) we are done.
We start by observing that the following estimate holds:

\[
\left( \int_{B(Z)} |\hat{A} - A|^r \right)^{1/r} \leq \left( \int_{B(Z)} |\hat{A} - (A)_{B(Z)}|^r \right)^{1/r} + \left( \int_{B(Z)} |A - (A)_{B(Z)}|^r \right)^{1/r} = \left( \int_{B(Z)} |\hat{A} - (A)_{B(Z)}|^r \right)^{1/r} + \alpha_r(Z).
\]

The last term already has the required form. For the first term we see that

\[
\left( \int_{B(Z)} |\hat{A} - (A)_{B(Z)}|^r \right)^{1/r} \leq \left( \int_{B(Z)} \left| A(Y) - (A)_{B(Z)} \right| dY \right)^{1/r}
\]

\[
\leq \|\phi\|_{L^\infty} \left( \int_{B(Z)} \left| \delta(X) - \delta(X)(X-Y) \right| dY \right)^{1/r}
\]

\[
\leq \left( \int_{B(Z)} |A(Y) - (A)_{B(Z)}|^r dY \right)^{1/r} = \alpha_r(Z).
\]

\[ \square \]

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