Acoustic waves in polydispersed bubbly liquids

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Abstract. The propagation of acoustic waves in polydispersed mixtures of liquid with two sorts of gas bubbles each of which has its own bubble size distribution function is studied. The system of the differential equations of the perturbed motion of a mixture is presented, the dispersion relation is obtained. Equilibrium speed of sound, low-frequency and high-frequency asymptotes of the attenuation coefficient are found. Comparison of the developed theory with known experimental data is presented.

1. Introduction

Various problems of acoustics in mixtures of liquids with bubbles of gas or vapor are considered in monographs [1], [2]. Fundamental physical processes involved in bubble dynamics are described in books [3], [4]. Some aspects of propagation of acoustic waves in bubbly liquids are presented in the book [5]. Commander and Prosperetti [6] reviewed the formulation of a rigorous model for the propagation of pressure waves in bubbly liquids. The theoretical results for the phase speed, attenuation and transmission coefficient through a layer of bubbly liquid were compared with available experimental data. The propagation of acoustic waves in a mixture of a liquid with monodispersed bubbles [7] and in a mixture of a liquid with two sorts monodispersed bubbles [8] is investigated. In this work the dynamics of weak perturbations in mixtures of liquid with two groups of polydispersed gas bubbles of different sizes and various compositions is considered. The developed theory is compared with known experimental data.

2. The theoretical model

2.1. Basic equations

The set of linearized differential equations determining the propagation of sound waves in a mixture of liquid with two groups of polydispersed gas bubbles has the form:

\[
\frac{\partial p_1'}{\partial t} + \rho_{10} \frac{\partial v'}{\partial x} = 0, \quad \frac{\partial p_{2a}'}{\partial t} + \rho_{20} \frac{\partial v'}{\partial x} = 0, \quad \frac{\partial p_{2b}'}{\partial t} + \rho_{20} \frac{\partial v'}{\partial x} = 0
\]

\[
\frac{\partial n_a'}{\partial t} + n_{a0} \frac{\partial v'}{\partial x} = 0, \quad \frac{\partial n_b'}{\partial t} + n_{b0} \frac{\partial v'}{\partial x} = 0, \quad \rho_{10} \frac{\partial v'}{\partial t} + \frac{\partial p_1'}{\partial x} = 0
\]

\[
\left\langle \frac{\partial p_{2a}'}{\partial t} \right\rangle = -3\gamma_{2a} p_0 \left\langle \frac{1}{a} \frac{\partial a'}{\partial t} \right\rangle + \left\langle q_a p_{2a} \right\rangle
\]
\[
\left\langle \frac{\partial p'_{2b}}{\partial t} \right\rangle_b = -3\gamma_{2b}p_0 \left\langle \frac{1}{b} \frac{\partial b'}{\partial t} \right\rangle_b + \left\langle q_0 p'_{2b} \right\rangle_b
\]

(1)

\[
\frac{\partial \alpha'}{\partial t} = w'_{Aa} + w'_{Ra}, \quad \frac{\partial b'}{\partial t} = w'_{Ab} + w'_{Rb}
\]

\[
a \frac{\partial w'_{Ra}}{\partial t} + 4\nu_1 \frac{w'_{Ra}}{\alpha} = \frac{p'_{2a} - p'_1}{\rho_1^{a}}, \quad b \frac{\partial w'_{Rb}}{\partial t} + 4\nu_1 \frac{w'_{Rb}}{b} = \frac{p'_{2b} - p'_1}{\rho_1^{b}}
\]

\[
w'_{Aa} = \frac{p'_{2a} - p'_1}{\rho_1^{a}C_1(\alpha_2^a)^{\beta}}, \quad w'_{Ab} = \frac{p'_{2b} - p'_1}{\rho_1^{b}C_1(\alpha_2^b)^{\beta}}, \quad p'_1 = C_1^2 \rho_1^{b}
\]

\[
\langle h \rangle_l = \frac{1}{\rho_2^{l}} \int_{l_{\text{min}}}^{l_{\text{max}}} N_0(l)g_0(l)hd\ell = \int_{l_{\text{min}}}^{l_{\text{max}}} N_0(l)g_0(l)hd\ell / \int_{l_{\text{min}}}^{l_{\text{max}}} N_0(l)g_0(l)dl
\]

\[
g_0(l) = \frac{4}{3\pi^3} \int_0^{\rho_2^{l}} q_1 = 3(\gamma_{2l} - 1)(i\omega)\left(\frac{y_1 \coth y_1 - 1}{y_1^2}\right), \quad y_1 = \sqrt{-\frac{i\omega l^2}{\kappa_{2l}}}, \quad l = a, b
\]

Hereafter, the subscripts 1 and 2 refer to the parameters of the liquid and gas phases, respectively. The primes designate the perturbations of parameters, and 0 means the initial unperturbed state. The variables with the subscript \(a\) refer to gas bubbles of radius \(a\), those with the subscript \(b\) refers to gas bubbles with a different radius \(b\), \(x\) is the coordinate, \(t\) is the time, \(\rho^o\) and \(\rho\) are the true and average density of the mixture, \(v\) is the velocity, \(p\) is the pressure, \(n\) is the number of bubbles per unit volume, \(\gamma\) is the adiabatic exponent, \(w\) is the bubble radial-motion velocity, \(\alpha\) is the volume content, \(\nu_1\) is the kinematic viscosity of the liquid, \(\omega\) is the frequency of perturbations, \(\kappa\) is the thermal diffusivity, \(C_1\) is the speed of sound in the carrying phase, \(N_0(a)\) and \(N_0(b)\) are the gas-bubble-size distribution functions. In Ref.[9], the value of \(\beta = 1/3\); as is shown below, in some cases it is better to use \(\beta = 1/6\).

We investigate the solutions of this set of equations having the form of progressive waves for perturbations \(\phi' (\phi' = \rho_1, \rho_2, \rho_2, \rho_1, \psi, T_1...):\)

\[
\phi' = A\phi \exp[i(K_s x - \omega t)], \quad K_s = K + iK_{ss}
\]

(2)

\[
C_p = \omega / K, i^2 = -1
\]

where \(K_s\) is the complex wave number, \(K_{ss}\) is the attenuation coefficient, and \(C_p\) is the phase velocity.

2.2. Dispersion relation
From the condition of existence of the nontrivial solution of type (2) for set (1) of linear equations, the following dispersion relation is obtained:

\[
\left(\frac{K_s}{\omega}\right)^2 = \frac{1}{C_f^2} + \frac{3\alpha_1^{20}\alpha_1^{10}\rho_1^{\alpha}(Q_0)_a}{3\gamma_{2a}p_0 - (Q_0S_a)_a} + \frac{3\alpha_1^{b}20\alpha_1^{10}\rho_1^{\alpha}(Q_0)_b}{3\gamma_{2b}p_0 - (Q_0S_b)_b}
\]

(3)

\[
C_f = \frac{C_1}{\alpha_1^{10}}, \quad Q_l = 1 + \frac{q_l}{i\omega}, \quad S_l = \frac{i\omega^2 h_l \rho_1^{\alpha}}{1 + h_l t_l}
\]

\[
h_l = \frac{4\nu_1}{l^2} - i\omega, \quad t_l = \frac{1}{C_1(\alpha_1^{20})^2}, \quad l = a, b
\]
The low-frequency and high-frequency asymptotes of the attenuation coefficient $K_{ss}$ valid for the frequencies $\omega \ll \kappa_{2l}/l_{a,3}^2$ and $\omega \gg \kappa_{2l}/l_{a,3}^2$, respectively, are deduced from dispersion relation (3), where

$$l_{i,j} = \left[\frac{P_i}{P_j}\right]^{1/(i-j)}, i \neq j, l = a, b$$

Here $l_{i,j}$ are the average bubble radii [10], [11]. When obtaining the asymptotes, we neglected the kinematic viscosity $\nu_1$ of the liquid.

The low-frequency asymptote is written as follows:

$$K_{ss}^{(0)}(\omega) = \frac{C_0}{2} \left( \frac{\alpha_{10} a_{20}^{\gamma_2} - 1}{15 \rho_0^{\gamma_2} a_{20}^{\gamma_2}} + \frac{\alpha_{10} b_{20}^{\gamma_2} - 1}{15 \rho_0^{\gamma_2} b_{20}^{\gamma_2}} \right) \omega^2$$

$$C_0 = \left( \frac{1}{C_f^2} + \frac{\rho_2 a_{10}^{\gamma_2} a_{20}^{\gamma_2}}{p_0} + \frac{\rho_2 b_{10}^{\gamma_2} b_{20}^{\gamma_2}}{p_0} \right)^{-1/2}$$

where $C_0$ is the equilibrium speed of sound. It can be seen that these are the average radii $a_{5,3}$ and $b_{5,3}$ previously introduced during the analysis in Ref.[10, 11] of the acoustics of gas suspensions that are characteristic in this asymptote and describe the interphase heat exchange. In the propagation of sound waves, the attenuation coefficient is directly proportional to the square of the frequency of perturbations.

The high-frequency asymptote has the following form:

$$K_{ss}^{(\infty)}(\omega) = C_f \frac{(H_1 + H_2)}{2} - C_f^2 (H_1 + H_2) \frac{H_3^2}{4\omega^2}$$

$$H_1 = \frac{3 \alpha_{20}^{\alpha_1} a_{10}^{\alpha_1}/a_{4,3}^{\alpha_1}}, H_2 = \frac{3 \alpha_{20}^{\beta_1} b_{10}^{\alpha_1}/b_{4,3}^{\alpha_1}}, h_a = (C_1(\alpha_{20}^{\beta_1}))^{-1}$$

$$h_b = (C_1(\alpha_{20}^{b_1}))^{-1}, H_3 = \frac{(H_1 + H_2)^2 C_f^2}{4} - (H_1 + H_5)$$

$$H_4 = 3 \alpha_{20}^{\alpha_1} a_{10}^{\alpha_1}(\rho_0^{\alpha_1} - 3 p_0 b_{20}^{\alpha_1}/\gamma_{20}), (\rho_0^{\alpha_1} a_{20}^{\gamma_2})$$

$$H_5 = 3 \alpha_{20}^{b_1} a_{10}^{\beta_1}(\rho_0^{\beta_1} - 3 p_0 b_{20}^{b_1}/\gamma_{20}), (\rho_0^{\beta_1} b_{20}^{b_1})$$

### 3. Comparison between theory and experiment

The data shown in Figs. 1 and 2 enable us to compare the theory with the experimental data [12] for the dependences of the sonic speed and the attenuation coefficient on the perturbation frequency $f = \omega/(2\pi)$. Experimental data 1 were obtained for the values of $a_0 = 1.9 \times 10^{-3}$ m, $\alpha_{20}^{a_0} = 5.8 \times 10^{-4}$; theoretical curve 2 was obtained for $(a \in [1.9 \times 10^{-3}, 2 \times 10^{-3}])$, Fig.1) and $(a \in [1.9 \times 10^{-3}, 3 \times 10^{-3}])$, Fig.2). $\alpha_{20}^{a_0} = 5.8 \times 10^{-4}$, $\alpha_{20}^{b_0} = 0$, and disregarding the acoustic depression $(w_{\alpha_0}^a = 0)$; and curve 3 was found at the same values as curve 2, but with taking into account the acoustic depression. The bubble-size distribution function is $N_0(a) = a^{-3}$. It can be seen that the acoustic depression substantially affects the behavior of the curves, which, in turn, agrees well with the experimental data.

In real experiments, it is reasonably difficult to obtain a monodisperse bubble medium, when all bubbles have the same radius. It is very likely to have a group of very small bubbles with a low volume content, which renders, as is shown below, a substantial effect on the dynamics of propagation of waves.

In Fig.3a, we show experimental data 1 obtained for the following values: $a_0 = 2.6 \times 10^{-3}$ m and $\alpha_{20}^{a_0} = 0.01$; curve 2 corresponds to $a_0 = 2.6 \times 10^{-3}$ m and $\alpha_{20}^{a_0} = 0.01$; curve 3 corresponds
Figure 1. Comparison of the theory with the experimental data for the dependences of the sonic speed on the frequency of perturbations with taking into account the acoustic depression (solid line) and disregarding the acoustic depression (dashed line).

Figure 2. Comparison of the theory with the experimental data for the dependences of the attenuation coefficient on the frequency of perturbations with taking into account the acoustic depression (solid line) and disregarding the acoustic depression (dashed line).

to \( \alpha^2_{20} = 0.01 \), \( a \in [2.6 \times 10^{-3}, 5.6 \times 10^{-3}] \), \( \alpha^2_{40} = 2 \times 10^{-4} \) and \( b \in [6.5 \times 10^{-4}, 8.5 \times 10^{-4}] \); and curve 4 corresponds to the same values as curve 3, but with the modified acoustic depression, which is presented below.

Experimental data 1 in Fig.3b are obtained for the following values: \( a_0 = 2.1 \times 10^{-3} \) m, \( \alpha^2_{20} = 0.0053 \); curve 2 for \( a_0 = 2.1 \times 10^{-3} \) m, \( \alpha^2_{20} = 0.0053 \); curve 3 for \( \alpha^2_{20} = 0.0053 \), \( a \in [2.1 \times 10^{-3}, 4.1 \times 10^{-3}] \), \( \alpha^2_{40} = 10^{-4} \), and \( b \in [6.5 \times 10^{-4}, 8.5 \times 10^{-4}] \) m; curve 4 for the same values as curve 3, but with the modified acoustic depression.

Figure 3. Comparison of the theory with the experimental data for the dependences of the attenuation coefficient on the frequency of perturbations.

The bubble-size distribution functions are

\[
N_0(a) = a^2 \exp(-K_a a), K_a = 3/a_{5,3}
\]
\[ N_0(b) = b^2 \exp(-K_b b), K_b = 3/b_{5,3} \]

It is shown that the two-faction composition of the polydispersed bubble liquid results in the occurence of two peaks for the attenuation coefficient, which is observed in the experiment (Figs.3, curves 3 and 4). The experiment data show that the modification of the formula for the acoustic depressions \( w'_Aa \) and \( w'_Ab \) from set (1) by replacing the exponents \( \beta = 1/3 \) for \( \beta = 1/6 \) of the volume bubble contents \( \alpha_{20}^A \) and \( \alpha_{20}^B \), respectively, results in an improvement of theoretical and experimental values of the attenuation coefficient (Fig.3, curves 4). The terms \( w'_Aa \) and \( w'_Ab \) are also determined from the solution of the problem on the acoustic depression of the spherical bubble in the carrying liquid [9].

4. Conclusions
Theory of the propagation of acoustic waves in polydispersed bubbly liquids is developed. It is shown, that the account of small bubbles with small volume content leads to a better agreement of theory and experiment.

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