Envy-Free and Pareto-Optimal Allocations for Asymmetric Agents

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Abstract

We study the problem of allocating \( m \) indivisible items to \( n \) agents with additive utilities. It is desirable for the allocation to be both fair and efficient, which we formalize through the notions of envy-freeness and Pareto-optimality. While envy-free and Pareto-optimal allocations may not exist for arbitrary utility profiles, previous work has shown that such allocations exist with high probability assuming that all agents’ values for all items are independently drawn from a common distribution. In this paper, we consider a generalization of this model with asymmetric agents, where an agent’s utilities for the items are drawn independently from a distribution specific to the agent. We show that envy-free and Pareto-optimal allocations are likely to exist in this asymmetric model when \( m = \Omega(n \log n) \), matching the best bounds known for the symmetric setting. Empirically, an algorithm based on Maximum Nash Welfare obtains envy-free and Pareto-optimal allocations for small numbers of items.

1 Introduction

Imagine that the neighborhood children go trick-or-treating and return successfully, with a large heap of candy between them. They then try to divide the candy amongst themselves, but quickly reach the verge of a fight: Each has their own conception of which sweets are most desirable, and, whenever a child suggests a way of splitting the candy, another child feels unfairly disadvantaged. As a (mathematically inclined) adult in the room, you may wonder: Which distribution of candies should you suggest to keep the peace? And, is it even possible to find such a fair distribution?

In this paper, we study the classic problem of fairly dividing \( m \) items among \( n \) agents (Bouveret, Chevaleyre, and Maudet 2016), as exemplified by the scenario above. We assume that the items we seek to divide are goods (e.g., receiving an additional piece of candy never makes a child less happy), that items are indivisible (candy cannot be split or shared), and that the agents have additive valuations (roughly: a child’s value for a piece of candy does not depend on which other candies they receive).

We will understand an allocation to be fair if it satisfies two axioms: envy-freeness (EF) and Pareto-optimality (PO). First, fair allocations should be envy-free, which means that no agent should strictly prefer another agent’s bundle to their own. Indeed, if an allocation violates envy-freeness, the former agent has good reason to contest it as unfair. Second, fair allocations should be Pareto-optimal, i.e., there should be no way of reallocating items such that the reallocation makes some agent strictly better off and no agent worse off. Not only does this axiom rule out allocations whose wastefulness is unappealing; it is also arguably necessary to preserve envy-freeness: Indeed, if a chosen allocation is envy-free but not Pareto-optimal, rational agents can be expected to trade items after the fact, which might lead to a final allocation that is not envy-free after all. Unfortunately, even envy-freeness alone is not always attainable. For instance, if two agents like a single item, the agent who does not receive it will always envy the agent who does.

Motivated by the fact that worst-case allocation problems may not have fair allocations, a line of research in fair division studies asymptotic conditions for the existence of such allocations, under the assumption that the agents’ utilities are random rather than adversarially chosen (e.g. Dickerson et al. 2014; Manurangsi and Suksompong 2019). Specifically, these papers assume that all agents’ utilities for all items are drawn independently from a common distribution \( \mathcal{D} \), a model which we will call the symmetric model. Among the multiple algorithms shown to satisfy envy-freeness in this setting, only one is also Pareto-optimal: the (utilitarian) welfare-maximizing algorithm, which simply allocates each item to the agent who values it the most. This algorithm is Pareto-optimal, and it is also envy-free with high probability as the number of items \( m \) grows in \( \Omega(n \log n) \).

Since envy-free allocations may exist with only vanishing probability for \( m \in \Theta(n \log n / \log \log n) \) in the symmetric model (Manurangsi and Suksompong 2019), the above result characterizes almost tightly when envy-free and Pareto-optimal allocations exist in this model.

Zooming out, however, this positive result is unsatisfying in that, outside of this specific random model, the welfare-maximizing algorithm can hardly be called “fair”: For example, if an agent A tends to have higher utility for most items than agent B, the welfare-maximizing algorithm will allocate most items to agent A, which can cause large envy for agent B. In short, the welfare-maximizing algorithm leads to fair allocations only because the model assumes each agent

¹In fact, Dickerson et al. (2014) prove this result for a somewhat more general model than the one presented above, but their model assumes the key symmetry between agents that we discuss below.
Figure 1: The top panel shows probability density functions of five agents’ utility distributions. The bottom panel shows densities after scaling distributions by the given multipliers. When drawing an independent sample from each scaled distribution, each sample is the largest with probability 1/5.

Figure 2: Earlier and new results on when EF and EF+PO allocations are guaranteed to exist in both models.

Motivated by these limitations of prior work, this paper investigates the existence of fair allocations in an extension of the symmetric model, which we refer to as the asymmetric model. In this model, each agent $i$ is associated with their own distribution $D_i$, from which their utility for all items is independently drawn. Within this model, we aim to answer the question: When do envy-free and Pareto-optimal allocations exist for such asymmetric agents?

1.1 Our Techniques and Results

In Section 3, we study which results in the symmetric model generalize to the asymmetric model. In particular, we apply an analysis by Manurangsi and Suksompong (2021) to the asymmetric model in a black-box manner to prove envy-free allocations exist when $m \in \Omega(n \log n / \log \log n)$, which is tight with existing impossibility results on envy-freeness. However, this approach does not preserve Pareto-optimality.

Using a new approach, we prove in Section 4 that generalizing the random model from symmetric to asymmetric agents does not substantially decrease the frequency of envy-free and Pareto-optimal allocations. The key idea is to find a multiplier $\beta_i > 0$ for each agent such that, when drawing an independent sample $u_i$ from each utility distribution $D_i$, each agent $i$ has an equal probability of $\beta_i u_i$ being larger than the $\beta_j u_j$ of all other agents $j \neq i$. Figure 1 illustrates how five utility distributions can be rescaled in this way. While the existence of such multipliers follows from a fixed-point argument, we instead give a constructive proof that exploits local monotonicity properties of the mapping from multipliers to allocation probabilities. Besides the existence of multipliers, this approach also proves their uniqueness and allows us to approximate them algorithmically.

These multipliers define what we call the multiplier algorithm, which allocates each item to the agent $i$ whose utility weighted by $\beta_i$ is the largest. Put differently, the multiplier algorithm simulates the welfare-maximizing algorithm in an instance in which each agent $i$’s distribution is scaled by $\beta_i$. Just like the welfare-maximizing algorithm, the multiplier algorithm is Pareto-optimal, and the similarity between the two allows us to apply proof techniques developed for the welfare-maximizing algorithm and the symmetric setting. Specifically, after showing a constant-size gap between each agent’s expected utility for an item conditioned on them receiving the item and the agent’s expected utility conditioned on another agent receiving the item, the argument of Dickerson et al. (2014) shows that the multiplier algorithm is envy-free with high probability when $m \in \Omega(n \log n)$.

In Section 5, we empirically evaluate how many items are needed to guarantee envy-free and Pareto-optimal allocations for five agents with the distributions in Fig. 1. We find that the multiplier algorithm needs large numbers of items to ensure envy-freeness, that the round robin algorithm violates Pareto-optimality in almost all instances, and that a kind of Maximum Nash Welfare algorithm achieves both axioms already for few items.

1.2 Related work

The question of when fair allocations exist for random utilities was first raised by Dickerson et al. (2014), whose main result we have already discussed. Our paper also builds on work by Manurangsi and Suksompong (2019, 2021), who prove the lower bound on the existence of envy-free allocations mentioned in the introduction and that the classic round robin algorithm produces envy-free allocations in the symmetric model for slightly lower $m$ than the welfare-maximizing algorithm. A bit further afield, Suksompong (2016) and Amanatidis et al. (2017) study the existence of proportional and maximin-share allocations (two relaxations of envy-freeness) in the symmetric model, and Manurangsi and Suksompong (2017) study envy-freeness when items are allocated to groups rather than to individuals. None of these papers consider Pareto-optimality, perhaps because fair division yields few tools for simultaneously guaranteeing envy-freeness and Pareto-optimality.

The asymmetric model we investigate has been previously used, for example, by Kurokawa, Procaccia, and Wang (2016) to study the existence of maximin-share allocations. While part of their proof applies the results by Dickerson et al. to construct envy-free allocations in the asymmetric model, as do we, their allocation algorithm is not Pareto-optimal (see Section 3.3). Farhadi et al. (2019) also consider maximin-share allocations in the asymmetric model, for agents with weighted entitlements. Finally, Zeng and
Psomas (2020) study allocation problems in the asymmetric model, with the added complication that items arrive online. While they do consider and achieve Pareto-optimality, they only obtain approximate notions of envy-freeness.

2 Preliminaries

General Definitions. We consider a set $M$ of $m$ indivisible items being allocated to a group $N = \{1, \ldots, n\}$ of $n$ agents. Each agent $i \in N$ holds a utility $u_i(\alpha) \geq 0$ for each item $\alpha \in M$, indicating their degree of preference for the item. The collection of agent–item utilities make up a utility profile. An allocation $A = \{A_i\}_{i \in N}$ is a partition of the items into $n$ bundles: $M = A_1 \cup \cdots \cup A_n$, where agent $i$ gets the items in bundle $A_i$. Under our assumption that the agents’ utilities are additive, agent $i$’s utility for a subset of items $A \subseteq M$ is $u_i(A) = \sum_{\alpha \in A} u_i(\alpha)$.

An allocation $A = \{A_i\}_{i \in N}$ is said to be envy-free (EF) if $u_i(A_i) \geq u_i(A_j)$ for all $i, j \in N$, i.e., if each agent weakly prefers their own bundle to any other agent’s bundle. We say that an allocation $A = \{A_i\}_{i \in N}$ is Pareto dominated by another allocation $A' = \{A'_i\}_{i \in N}$ if $u_i(A_i) \leq u_i(A'_i)$ for all $i \in N$, with at least one inequality holding strictly. An allocation is Pareto-optimal (PO) if it is not Pareto dominated by any other allocation. An allocation is called fractionally Pareto-optimal (fPO) if it is not even Pareto dominated by any “fractional” allocation of items. For our purposes, it suffices to note that an allocation is fPO iff there exist multipliers $\{\beta_i > 0\}_{i \in N}$ such that each item $\alpha$ is allocated to an agent $i$ with maximal $\beta_i u_i(\alpha)$ (Negishi 1960).

Asymmetric Model. In our asymmetric model, each agent $i$ is associated with a utility distribution $D_i$, a nonatomic probability distribution over $[0, 1]$. The model assumes that the utilities $u_i(\alpha)$ for all $\alpha \in M$ are independently drawn from $D_i$. For simplicity, we just write $u_i$ as a random variable for $u_i(\alpha)$ if we are not talking about a specific item $\alpha$, where $u_i \sim D_i$. Let $f_i$ and $F_i$ denote the probability density function (PDF) and cumulative distribution function (CDF) of $D_i$. For our main result, we make the following assumptions on utility distributions: (a) Interval support: The support of each $D_i$ is an interval $[a_i, b_i]$ for $0 \leq a_i < b_i \leq 1$. (b) $(p, q)$-PDF-boundedness: For constants $0 < p < q$, the density of each $D_i$ is bounded between $p$ and $q$ within its support. These two assumptions are weaker than those by Manurangsi and Suksompong (2021), who also require the support of PDF to be $[0, 1]$.

3 Takeaways From the Symmetric Model

We begin by exploring whether results from the symmetric model generalize to our asymmetric model. As we show, existing arguments are successful in charting out the existence of EF allocations, but do not directly extend to PO.

3.1 Non-Existence of EF Allocations

Since the symmetric model is a special case of the asymmetric model—in which all $D_i$ are equal—the following impossibility result immediately applies:

**Proposition 1** (Manurangsi and Suksompong 2019\(^2\)). There exists $c > 0$ such that, if $m = ⌊(c \log n / \log \log n) + 1/2⌋ n$ and all utility distributions are uniform on $[0, 1]$, then, with high probability, no envy-free allocation exists.

Since EF allocations cannot be guaranteed for $m \in \Theta(n \log n/ \log \log n)$, the existence of EF and PO allocations requires $m$ to grow at least as fast.

3.2 Existence of EF Allocations

In the symmetric model, Manurangsi and Suksompong (2021) give an allocation algorithm that leads to EF allocations with high probability. As it turns out, the algorithm used for proving the existence, round robin, has a property that permits its extension to the asymmetric model: An agent’s allocation given a utility profile depends not on the cardinal information of the utility profile, but only on each agent’s ordinal preference order over items.

This dependence only on ordinal information allows us to prove that their result generalizes to the asymmetric model since, in a nutshell, an agent $i$’s envy of the other agents is indistinguishable between the asymmetric model and a symmetric model with common distribution $D_i$. This follows from the observations that (1) agent $i$’s envy only depends on their own utilities and the allocation, that (2) each allocation is Pareto optimal, and (3) the agents’ ordinal preferences are just independently and uniformly drawn permutations over the items. To our knowledge, we are the first to observe that the analysis by Manurangsi and Suksompong generalizes in this way, which improves on the previously best known upper bound of $m \in \Omega(n \log n)$ for the symmetric model due to Kurokawa, Procaccia, and Wang (2016). We defer the proof of the proposition to Appendix A.

**Proposition 2.** When distributions have interval support and are $(p, q)$-PDF-bounded, if $m \in \Omega(n \log n/ \log \log n)$, an envy-free allocation exists with high probability.

3.3 Towards EF+PO Allocations

While we find results on EF alone to be easily transferable to the asymmetric model, generalizing the existence result for EF and PO allocations by Dickerson et al. (2014) is more challenging since, in contrast to round robin, the welfare-maximizing algorithm requires cardinal information.

To see this difficulty, we consider how Kurokawa, Procaccia, and Wang (2016) apply the theorem of Dickerson et al. (2014) to prove the existence of EF allocations in the asymmetric model, and why this application violates PO. The core idea of their algorithm is to allocate each item to the agent for whom the item is in the highest percentile of their utility distribution, which we will call the maximum-percentile algorithm. It is easy to see that each agent has a probability $1/n$ of receiving each item, and it is not too hard to show that agents have higher expected utility for items they receive than for items allocated to other agents. This implies envy-freeness with high probability as $m \in \Omega(n \log n)$ following the proof by Dickerson et al.

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\(^2\)Here, we present a special case; the original result holds for different choices of distribution and leaves some flexibility in $m$. 

Unfortunately, this construction is unlikely to generate PO allocations: Consider a setting with two asymmetric agents, in which agent A’s utility is drawn, with 50% probability, uniformly between 0 and 1/4, and, with 50% probability, uniformly between 1/4 and 1; and in which agent B’s utility is drawn either uniformly between 0 and 3/4 or uniformly between 3/4 and 1, each with 50% probability. Conceptually, A’s utility distribution skews towards lower values, whereas B’s skews towards higher values. The black dots in Fig. 3 show random samples of these utilities, and the shaded region in the left plot marks the range of utilities in which the maximum-percentile algorithm allocates items to A. The left plot also highlights three specific items: two lie around the median utility for both agents and are given to A, and one of them lies around the top percentile for both agents and is given to B. The fact that ratio \( u_B(\alpha)/u_A(\alpha) \) is strictly greater for the “median items” (at roughly \((3/4)/(1/4) = 3\)) than for the “top item” (roughly 1) immediately implies that the allocation is not fPO: Agent B would profit from trading half of the top item against one of A’s median items (roughly, since \(3/4 > 1/2\)), and A would also profit from this trade (since \(1/2 > 1/4\)). In fact, a similar trade of whole items, which exchanges both median items against the top item, shows that the maximum-percentile allocation violates Pareto-optimality proper.

The most promising way to avoid violations of PO is to construct fPO allocations, since the characterization of fPO using multipliers in Section 4 provides useful structure that is not available for PO. As shown in the right panel in Fig. 3, this corresponds to choosing a line through the origin, allocating items below the line to A, and allocating items above the line to B. In fact, the plot shows the unique such line with the added property that a random-utility item is equally likely to be given to either agent. In the next section, we generalize this kind of allocation to arbitrary numbers of agents.

4 Existence of EF+PO Allocations

We now prove our main theorem:

**Theorem 3.** Suppose that all utility distributions have interval support and are \((p, q)\)-PDF-bounded for some \(p, q\). If \(m \in \Omega(n \log n)\) as \(n \to \infty\), an envy-free and (fractionally) Pareto-optimal allocation exists with high probability.

In Section 4.1, we prove that we can always find multipliers \(\{\beta_i\}_{i \in N}\) that equalize each agent’s probability of receiving a random-utility item from the multiplier algorithm (which allocates item \(\alpha\) to the agent with maximal \(\beta_i u_i(\alpha)\) and is trivially fPO). Next, in Section 4.2, we show that an agent’s expected utility for an item allocated to them is larger by a constant than their expected utility for an item allocated to another agent. In Section 4.3, we combine these properties to prove envy-freeness.

4.1 Existence of Equalizing Multipliers

We begin by proving the existence of multipliers that equalize the agents’ probabilities of receiving an item. These multipliers are unique, up to the fact that scaling all multipliers by a common factor does not change the agents’ probabilities. Assuming, without loss of generality, that \(\beta_n = 1\), we prove the following proposition.

**Proposition 4.** Let \(N = \{1, \ldots, n\}\) be a set of agents whose utility distributions have interval support. Normalizing \(\beta_n\) to 1, there exists a unique set of positive multipliers \(\{\beta_i\}_{i \in N}\) such that, for all \(i \in N\), \(P[\beta_i u_i = \max_{j \in N} \beta_j u_j] = 1/n\).

In Appendix D, we prove a variant of this proposition (without uniqueness and without assuming interval support) using Sperner’s lemma. Here, we instead present a constructive proof that we find to be more informative: it makes use of the dependency between multipliers and probabilities, its induction structure lends itself to implementation as an algorithm for computing the multipliers numerically, and it shows their uniqueness. The proposition follows from the following lemma (which strengthens the claim to allow proof by induction), by setting \(i := n - 1\) and \(\beta_n := 1\):

**Lemma 5.** Let \(N\) be as in Proposition 4, let \(i\) be an integer such that \(0 \leq i \leq n - 1\), and let there be an arbitrary set of fixed positive multipliers \(\beta_j\) for the agents \(j = i + 1, \ldots, n\). Then, there exists a unique set of positive multipliers \(\beta_1, \ldots, \beta_i\) such that, for each \(1 \leq j \leq i\), \(P[\beta_j u_j = \max_{k \in N} \beta_k u_k] = 1/n\).

**Proof.** By induction on \(i\). The case of \(i = 0\) is vacuously true, so we can from now on assume that \(i \geq 1\) and that the statement holds for \(i - 1\), which in particular means that, if we fix any \(\beta_i > 0\), the induction hypothesis will yield a uniquely defined set of positive multipliers \(\beta_1, \ldots, \beta_{i-1}\) such that each agent \(j < i\) has the largest weighted utility with probability \(1/n\). Crucially, this allows us to treat each \(\beta_j\) for \(j < i\) as a function \(\beta_j : \mathbb{R}_+ \to \mathbb{R}_+\) of \(\beta_i\), and, since all multipliers are now determined by \(\beta_i\) or are constant, the probability \(p_k\) of an agent \(k \in N\) receiving a random item is now also a function \(p_k : \mathbb{R}_+ \to [0, 1]\) of \(\beta_i\).

The core idea in proving the induction step is to use the bisection method to find a unique \(\beta_i\) for which \(p_k(\beta_i) = \)
1/n. The proof proceeds in three steps:
(1) Bracketing condition: Since, for small enough $\beta_i$, $p_i(\beta_i)$ is less than 1/n, and since, for large enough $\beta_i$, $p_i(\beta_i)$ is larger than 1/n, the bisecion method converges to a point $\beta_i^*$ such that, in every neighborhood of $\beta_i^*$, $p_i$ takes on values below and above 1/n.
(2) Continuity: Since $p_i$ is continuous, $p_i(\beta_i^*) = 1/n$.
(3) Monotonicity: Since $p_i$ is monotonic and, moreover, $p_i$ is strictly monotonic in a neighborhood of $\beta_i^*$, $\beta_i^*$ is the only $\beta_i$ satisfying $p_i(\beta_i) = 1/n$.

We show each step satisfied in the following paragraphs.

Bracketing Condition. A first encouraging observation is that
$$\lim_{\beta_i \to 0} p_i(\beta_i) = 1 - \frac{1}{n} \geq 2/n.$$ Note that, since all $p_j(\beta_i)$ for $j < i$ are constant at $1/n$, $\sum_{k=1}^{n} p_k(\beta_i) = 1 - \frac{1}{n}$. As $\beta_i \to \infty$, $\beta_i u_k$ exceeds the constant distributions $\beta_i u_k$ for all $k > i$ almost surely, which shows that $p_i(\beta_i) \to 0$ and that $p_i(\beta_i) \to 1 - \frac{1}{n}$.  

Continuity. Now we show the continuity of $p_i$ in $\beta_i$. To do this, we consider, in general, how the probability of an agent $k \in N$ having the largest value depends on $n$ multipliers, which we express as a function $\bar{p}_k: \mathbb{R}^n_{+} \to [0, 1]$.

$$\bar{p}_k(\beta_1, \ldots, \beta_n) = P \left[ \beta_k u_k = \max_{t \in N} \beta_t u_t \right] = \int_0^1 f_k(u) \prod_{t \in N \setminus \{k\}} F_t \left( \frac{\beta_k u_t}{\beta_t u_t} \right) \, du. \quad (1)$$

It is not to difficult to derive from the above integral that the $\bar{p}_k$ are continuous in the vector of multipliers $\beta_1, \ldots, \beta_n$ (Appendix B.2). However, more work is needed to show that $p_k$ varies continuously as a function of $\beta_i$, since we have not yet shown that the $\beta_j$ for $j < i$, whose values are plugged into $\bar{p}_k$, vary continuously as a function of $\beta_i$. As a step towards this continuity, we make a key monotonicity observation about Eq. (1): $p_k$ increases monotonically in the ratios $\{\beta_k/\beta_t\}_{t \in N}$, that is, whenever the $\beta_k/\beta_t$ weakly increase for all $t \in N$, since the CDFs $F_t$ are monotone increasing, $p_k$ also weakly increases. Moreover, the assumption of interval support allows us to show what we call the local strict monotonicity lemma (in the ratios $\{\beta_k/\beta_t\}_{t \in N}$): if we additionally know that some $\beta_k/\beta_t$ strictly increases, in which $p_k$ and $\bar{p}_k$ were positive before the change, then $p_k$ also strictly increases (see proof in Appendix B.1).

The above monotonicity properties allow us to bound local changes in the $\beta_j(\beta_i)$ for $j < i$, which we will subsequently use to prove continuity. Consider increasing agent $i$’s multiplier from $\beta_i$ to $\beta_i' \geq \beta_i$, resulting in the set of multipliers changing from $\{\beta_k\}_{k \in N}$ to $\{\beta_k'\}_{k \in N}$. We will refer to the corresponding probabilities for each $k \in N$ as $p_k = p_k(\beta_i)$ and $p_k' = p_k(\beta_i')$, respectively. We will show, for all $j < i$, that
$$1 \leq \beta_j'/\beta_j \leq \beta_i'/\beta_i. \quad (2)$$

For the sake of contradiction, suppose that the left inequality did not hold for some $j$, and let $j$ denote an agent $j < i$ with minimal $\beta_j'/\beta_j$, which is less than 1 by assumption. For this agent $j$, $\beta_j'/\beta_j \leq \beta_j/\beta_k$ for all $k \in N$, due to the definition of $j$ and to the fact that $\beta_j'/\beta_j \geq 1 > \beta_j/\beta_j$ for all $k \geq i$. To use our local strict monotonicity property, we furthermore note that $p_j' = 1/n > 0$, that $p_j' > 0$ for some $k \geq i$ (since $m_k = p_k = 1 - \frac{1}{n}$), and that, as noted just above, $\beta_j/\beta_k > \beta_j'/\beta_k$ for this $k$. By the local strict monotonicity lemma in Appendix B.1, this would imply that $1/n = p_j' < p_j = 1/n$, a contradiction. The right-hand inequality in Eq. (2) follows from a symmetric argument, by considering the agent $j < i$ with maximum $\beta_j'/\beta_j$.

Finally, we are ready to prove the continuity of $\beta_j(\beta_i)$ for all $j < i$. Fix any $\beta_i > 0$ and $\epsilon > 0$, and set $\delta = \frac{\beta_i}{\min(\beta_i, \epsilon)}$. We show that, for any $\beta_i' \in [\beta_i - \delta, \beta_i + \delta]$, $\beta_j(\beta_i') \in [\beta_j(\beta_i) - \epsilon, \beta_j(\beta_i) + \epsilon]$. We refer to the~local strict monotonicity lemma.

Case 1: If $\beta_i' \in [\beta_i, \beta_i + \delta]$ from Eq. (2) we have
$$1 \leq \frac{\beta_j(\beta_i')}{\beta_j(\beta_i)} \leq \frac{\beta_j'}{\beta_j} \Rightarrow \beta_j(\beta_i') \in [\beta_j(\beta_i) - \epsilon, \beta_j(\beta_i) + \epsilon].$$

Case 2: If $\beta_i' \in [\beta_i - \delta, \beta_i]$, reversing the role of $\{\beta_k\}_{k \in N}$ and $\{\beta_k'\}_{k \in N}$ in Eq. (2) gives
$$\frac{\beta_j'}{\beta_j} \leq \frac{\beta_j(\beta_i')}{\beta_j(\beta_i)} \leq 1 \Rightarrow \beta_j(\beta_i') \in [\beta_j(\beta_i) - \epsilon, \beta_j(\beta_i)],$$

which establishes the continuity of $\beta_j(\cdot)$. From the continuity of $\beta_j(\cdot)$, it directly follows that $p_i(\beta_i)$ is continuous in $\beta_i$, since it is the concatenation of continuous functions $\bar{p}_i(\beta_1(\beta_i), \beta_2(\beta_i), \ldots, \beta_{i-1}(\beta_i), \beta_i, \ldots, \beta_n)$.

Monotonicity. It only remains to prove uniqueness, which follows from the monotonicity properties of $p_i(\cdot)$. Whenever $\beta_i$ increases, by Eq. (2), all ratios $\beta_j/\beta_i$ for $j \leq i$ weakly increase, and the ratios $\beta_j/\beta_k$ for $k > i$ strictly decrease. Then, since $p_i$ is monotone increasing in $\{\beta_j/\beta_t\}_{t \in N}$, $p_i$ increases, indicating that $p_i$ is monotone increasing with $\beta_i$.

Furthermore, for any $\beta_i$ such that $0 < p_i(\beta_i) < 1 - \frac{1}{n}$ and, in particular, where $p_i(\beta_i) = 1/n$ since $\sum_{k=i}^{n} p_k = 1 - \frac{1}{n}$, there exists some agent $k > i$ with $p_k > 0$. Meanwhile, when $\beta_i$ increases, $\beta_j/\beta_k$ for such $k$ strictly increases, and by the local strict monotonicity lemma, $p_k$ strictly increases. Thus we conclude that $p_i$ is strictly monotone in a neighborhood around any $\beta_i^*$ such that $p_i(\beta_i^*) = 1/n$, which implies that there is only one $\beta_i^*$.

Conclusion. Combining all previous results, the multipliers found by the bisecion method are the unique set of multipliers guaranteeing $p_i = p_{i-1} = \cdots = p_1 = 1/n$, which concludes the induction step and proves the claim.

$^3$That is, $\beta_k = \beta_k'$ for all $k > i$ are the fixed multipliers, and $\beta_j = \beta_j(\beta_i)$ and $\beta_j' = \beta_j(\beta_i')$ for $j < i$ are the values functionally determined by $\beta_i$ and $\beta_i'$, respectively.
4.2 Gap between Expected Utilities

By scaling the utilities using the multipliers, we essentially bring a key property used in Dickerson et al.’s proof to the asymmetric model: as does the welfare-maximizing algorithm in the symmetric setting, the multiplier algorithm gives a random item to each agent with equal probability. On its own, however, this property is not enough to provide envy-freeness. After all, allocating each item to a uniformly chosen agent would satisfy the same symmetry, but agent i’s utility for any bundle $A_i$ — including their own — would follow the same distribution for this algorithm, which makes envy quite likely, no matter how large the numbers of items. We must therefore show that the multiplier algorithm allocates in a better way than blind randomness: Hopefully, the fact that an agent receives items because they had relatively large weighted utility means that the items in their bundle are skewed towards items that the agent likes more. Formally, we hope to prove a constant positive gap between $\mathbb{E} [u_i | \beta_i u_i = \max_{k \in \mathbb{N}} \beta_k u_k]$ and $\mathbb{E} [u_i | \beta_j u_j = \max_{k \in \mathbb{N}} \beta_k u_k]$ for all $i \neq j \in \mathbb{N}$.

This gap is easy to prove in the symmetric model, since the former expectation is the expectation of the $n$th order statistic among $n$ samples while the latter expectation is at most the expectation for each sample. By contrast, in the asymmetric model, all distributions are different and, in the multiplier algorithm, who receives an item depends on the multipliers, which to our knowledge do not have a closed form. Despite these challenges, we can prove such a gap, using the assumptions of interval support and $(p, q)$-PDF-boundedness. Interval support will suffice to show a positive gap for any collection of $n$ utility distributions, which implies that, for large enough $m$, the multiplier algorithm is envy-free. Adding the assumption of $(p, q)$-PDF-boundedness, we derive a uniform gap (depends only on $p, q$) for all distributions having this property, which allows us to prove envy-freeness even as $n \to \infty$.

Before we go into the bounds, it is instructive to see why the interval support property is required for the multiplier approach. Consider a case with two agents: Agent A’s utility is uniformly distributed on $[1/4, 3/4]$, whereas agent B’s distribution is uniform on $[0, 1/4] \cup [3/4, 1]$, whose support is the union of two disjoint intervals. The multipliers for these two agents may both equal 1 since the welfare-maximizing algorithm already allocates items with equal probability. But, then, $\mathbb{E} [u_A | u_A \geq u_B] = \mathbb{E} [u_A]$, since the condition that $u_A \geq u_B$ only tells us that $u_B$ is taken from the left interval in its support $([0, 1/4])$ but $u_A$ is still distributed uniformly in $[1/4, 3/4]$. Hence, without assuming interval support, the gap we aim to bound may be zero.

Once we assume interval support, however, we can prove a positive gap for any fixed collection of agents and their multipliers (Appendix B.4).

Proposition 6. Fix a set of agents whose utility distributions have interval support, and let $\{\beta_i\}_{i \in \mathbb{N}}$ denote the multipliers from Proposition 4. Then, for all $i \neq j$,

\[ \mathbb{E} [u_i | \beta_i u_i = \max_{k \in \mathbb{N}} \beta_k u_k] > \mathbb{E} [u_i | \beta_j u_j = \max_{k \in \mathbb{N}} \beta_k u_k]. \]

The above bound shows a gap for any particular set of agents, but, to bound the probability of EF when the number and distributions of agents change, we need a uniform constant lower bound for all $n$ and all utility distributions involved. Obtaining such a bound requires some additional restriction on which distributions are allowed, and $(p, q)$-PDF-boundedness is a natural choice for this: On the one hand, excessively low probability densities can make the gap grow arbitrarily small, although still $> 0$. To see this, consider a variant of the counter-example above, in which we increase the density of agent B’s distribution in $[1/4, 3/4]$ to an small, positive constant $\epsilon$. It is easy to see that, in this example, allowing arbitrarily low (positive) densities can cause the gap to become arbitrarily small. On the other hand, the gap could vanish as a result of excessively high rather than low densities. Indeed, in a scenario where agent A’s utility is uniform on $[0, 1]$ and agent B’s distribution is uniform on $[1/2 - \epsilon, 1/2 + \epsilon]$, as $\epsilon \to 0^+$ and agent B’s density grows unboundedly, the positive gap for this agent goes to zero. Assuming that all densities (in the support) lie between some constants $p > 0$ and $q$ avoids these problematic cases.

A more subtle way in which the assumption of $(p, q)$-PDF-boundedness helps is that it allows us to bound the multipliers, since, if some ratio $\beta_i / \beta_j$ of multipliers were too large, $(p, q)$-boundedness would imply that $i$’s probability of getting a random item is larger than $j$’s, contradicting the choice of multipliers (proof in Appendix B.5).

Lemma 7. For all $i, j \in \mathbb{N}$, we have that $\frac{1}{q} \leq \frac{\beta_i}{\beta_j} \leq 2q$.

In Appendix B.7, we derive the desired constant gap:

Proposition 8. There exists a constant $C_{p,q} > 0$ that only depends on $p$ and $q$, such that, for any collection of agents whose utility distributions are $(p, q)$-PDF-bounded and have interval support, for all $i \neq j \in \mathbb{N}$,

\[ \mathbb{E} [u_i | \beta_i u_i = \max_{k \in \mathbb{N}} \beta_k u_k] - \mathbb{E} [u_i | \beta_j u_j = \max_{k \in \mathbb{N}} \beta_k u_k] \geq C_{p,q} > 0. \]

4.3 The Multiplier Algorithm Satisfies EF

Combining these two results, we prove our main result, Theorem 3, in Appendix B.8, i.e., that the multiplier algorithm is EF with high probability. Up to one technical difference, this proof follows the one by Dickerson et al. (2014): Propositions 4 and 8 imply an $\Omega(1/n)$ gap between the expected utility contribution of each item to $u_i(A_i)$ and $u_i(A_j)$, and thus, by concentration, $u_i(A_j) \geq u_i(A_j)$. Specifically, the multiplier allocation is envy-free with probability at least $1 - 2 \exp \left(2 \log n - \text{const}(p,q) m/n \right)$, which goes to 1 for appropriate $m \in \Omega(\log n)$. When all distributions have interval support, but no PDF-boundedness is guaranteed, from Proposition 6, the allocation is EF when $m \to \infty$. 
5 Empirical Results

After characterizing the existence of envy-free and Pareto-optimal allocations from an asymptotic angle, we now empirically investigate allocation problems for a concrete set of agents. Aiming for a diverse and challenging set of utility distributions, we choose the five example distributions shown in Fig. 1. Though the densities of these distributions are not bounded, as discussed in Section 4.2, the fact that all of them have the unit interval as their support ensures envy-freeness with high probability as $m \to \infty$.

We compute multipliers for these five distributions by implementing the constructive proof of Lemma 5. While this algorithm scales exponentially in the number of agents (due to the $n-1$ nested bisections), it takes less than one hour to compute multipliers to a high level of precision: Using these multipliers, each agent’s probability of receiving a random item lies within $2 \cdot 10^{-6}$ of the target probability $1/5$.

As shown by the solid line in Fig. 4, however, the multiplier algorithm requires huge numbers of items to be reliably envy-free: \footnote{Dickerson et al. assume that $\mathbb{E}[u_j | \beta_j u_i = \max_{k \in N} \beta_k u_k] \geq \mu^* > \mu \geq \mathbb{E}[u_j | \beta_j u_j = \max_{k \in N} \beta_k u_k]$ for all $i \neq j$ and two constants $\mu, \mu^*$. Proposition 8 essentially allows for $\mu, \mu^*$ to depend on $i$, but this does not substantially change the proof.} When allocating $m = 1,000$ items to the five agents, the allocation is still only envy-free in 57% of instances, and it requires $m = 10,000$ items for this probability to reach 99%. This slow convergence is in particular due to the agent labeled as agent E in Fig. 1, whose utility distribution is $\text{Beta}(5, 1)$. Whereas, for all other agents $j$, the difference $\mathbb{E}[u_j | \beta_j u_j = \max_{k \in N} \beta_k u_k] - \mathbb{E}[u_j]$ lies between 0.22 and 0.42, this difference is only 0.06 for agent E, which explains why many items are required for this agent to prefer their own bundle in most random instances. Even between 0.22 and 0.42, this difference is only 0.06 for agent E, as shown by the solid line in Fig. 4, this convergence is fast—much faster than what one would expect given the $\log \log n$ advantage over the multiplier argument we saw in theoretical analysis. Unfortunately, though, the round robin algorithm’s probability of producing a Pareto-optimal allocation is essentially zero except for very small $m$ (dash-dotted line), which matches our theoretical predictions (Appendix C).

A third algorithm finally shows that envy-free and Pareto-optimal allocations exist even for small numbers of items: by rounding\footnote{Specifically, we allocate each item to the agent who receives the largest share of the item in the fractional allocation.} the fractional Maximum Nash Welfare (MNW) allocation, we obtain an allocation that is guaranteed to be Pareto-optimal (even fractionally Pareto-optimal). We also find it to be envy-free nearly as often as round robin (dashed line in Fig. 4). We known of no theoretical guarantees for MNW satisfying envy-freeness in a randomized model; given its excellent properties in worst-case instances (Cara- giannis et al. 2019), this seems a promising future direction.

6 Discussion

In this paper, we showed that EF and PO allocations are likely to exist for random utilities even if different agents’ utilities follow different distributions. Given that the known asymptotic bounds for the existence of EF+PO allocations are equal in the asymmetric and in the symmetric model, we see no evidence that the asymmetry of agent utilities would make EF+PO allocations substantially rarer to exist, up to, possibly, a $\log \log n$ gap that remains open in both models.

The most interesting idea coming out of this paper is the technique of finding equalizing multipliers, which might be of use in more general settings. Notably, the alternative existence proof in Appendix D mainly uses the continuity of the function mapping multipliers to probabilities, and in particular does not use the independence between the agents’ utilities. Thus, the multiplier technique might apply to random models where the agents’ utilities exhibit some correlation, as long as that the gap in expected utilities can still be bounded. In the limit of infinitely many items, we can think of the multiplier technique as a way to find an allocation of divisible goods that is Pareto-optimal and balanced, i.e., where every agent receives an equal amount of items. In future work, we hope to explore if this construction extends to arbitrary sets of divisible items.
Acknowledgements

We would like to thank Bailey Flanigan and Ariel Procaccia for valuable comments and suggestions on the paper. Also we thank Dravyansh Sharma, Jamie Tucker-Foltz, Ruixiao Yang and Zizhao Zhang for helpful technical discussions.

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Appendix

A Proof of Proposition 2

Proposition 2. When distributions have interval support and are \((p, q)\)-PDF-bounded, if \(m \in \Omega(n \log n / \log \log n)\), an envy-free allocation exists with high probability.

Proof. For any pair of agents \(i, j \in N\), we will prove the probability of the event that \(i\) envies \(j\) in our asymmetric model is in \(O(1/m^3)\). Consider a symmetric model with distribution \(D_i\) for all agents. Let \(P^S[T]\) be the probability that some event \(T\) occurs in this symmetric model, and \(P^A[T]\) be the probability that some event \(T\) occurs in the asymmetric model.

For every utility profile, define its ordinal profile as \(\{O_k\}_{k \in N}\) where \(O_k\) is a permutation of items \(M\), in descending order according to \(u_k(\alpha)\) for \(\alpha \in M\). Let \(\mathcal{O}\) be the set of all possible ordinal profiles, which contains \((m!)^n\) elements. Since in both models, all agents are independent and the utility for all items are drawn independently, all ordinal profiles have the same probability to appear, that is,

\[
\forall \tilde{O} \in \mathcal{O}, P^S[\{O_k\}_{k \in N} = \tilde{O}] = P^A[\{O_k\}_{k \in N} = \tilde{O}] = \frac{1}{(m!)^n}.
\]

Since the allocation allocated by round robin algorithm is uniquely determined by the ordinal profile, let \(\{A_k(\tilde{O})\}_{k \in N}\) denote the resulting allocation given ordinal profile \(\tilde{O}\). We can express \(P^S[i \text{ envies } j]\) as

\[
P^S[i \text{ envies } j] = \sum_{\tilde{O} \in \mathcal{O}} P^S[\{O_k\}_{k \in N} = \tilde{O}] \cdot P^S[i \text{ envies } j \mid \{O_k\}_{k \in N} = \tilde{O}]
\]

\[
= \frac{1}{(m!)^n} \sum_{\tilde{O} \in \mathcal{O}} P^S[i \text{ envies } j \mid \{O_k\}_{k \in N} = \tilde{O}].
\]

Similarly, we can express \(P^A[i \text{ envies } j]\) as

\[
P^A[i \text{ envies } j] = \frac{1}{(m!)^n} \sum_{\tilde{O} \in \mathcal{O}} P^A[i \text{ envies } j \mid \{O_k\}_{k \in N} = \tilde{O}].
\]

For all \(\tilde{O} \in \mathcal{O}\),

\[
P^S[i \text{ envies } j \mid \{O_k\}_{k \in N} = \tilde{O}] = P^S\left[\{u_i(\alpha) \sim D_i\}_{\alpha \in M} : \sum_{\alpha \in A_i(\tilde{O})} u_i(\alpha) < \sum_{\alpha \in A_j(\tilde{O})} u_i(\alpha) \mid O_i = \tilde{O}_i\right]
\]

\[
= P^A[i \text{ envies } j \mid \{O_k\}_{k \in N} = \tilde{O}].
\]

Hence we have \(P^A[i \text{ envies } j] = P^S[i \text{ envies } j]\). The following lemma is implied by the proof for Thm 3.1 in (Manurangsi and Suksompong 2021).

Lemma 9 (Manurangsi and Suksompong 2021). In the symmetric model, if \(m \in \Omega(n \log n / \log \log n)\) and the common distribution \(D\) is \((p, q)\)-PDF-bounded on \([0, 1]\), then for any pair of agents \(i, i'\), the probability that \(i\) envies \(i'\) in the round robin allocation is at most \(O(1/m^3)\).

PDF-bounded on \([0, 1]\) Here we follow the assumptions that (Manurangsi and Suksompong 2021) made on the distributions: PDF-bounded on \([0, 1]\). Since \(D_i\) is \((p, q)\)-PDF-bounded, the lemma indicates that \(P^S[i \text{ envies } j] = O(1/m^3)\). Thus, by our earlier arguments, the probability that agent \(i\) envies agent \(j\) in our asymmetric model is also in \(O(1/m^3)\). Applying a union bound over all pairs \(i, j\), we know that the allocation is envy-free in the asymmetric model with probability at least \(1 - O(1/m)\) when \(m \in \Omega(n \log n / \log \log n)\).

Interval support and PDF-bounded Moreover, we make slight modification (on constant level) to the proof in (Manurangsi and Suksompong 2021) to generalize Lemma 9 and get bounded envy probability when only assuming interval support and \((p, q)\)-PDF-bounded.

We first review the main idea of the proof for Thm 3.1 in their paper. For two agents \(i, i'\), let \(X_i^{i,i'}\) denote \(i\)'s value for the item that \(i'\) gets in the \(t\)th round, and \(X_i^t\) denote \(i\)'s value for her own item in the \(t\)th round. While the maximum possible envy can be (when \(i'\) chooses before \(i\) in each round and gets 1 more item than \(i\))

\[
u_i(M_{i'}) - u_i(M_i) = X_i^{i,i'} - \sum_{t} (X_i^t - X_{i + 1}^{i,i'}) \leq 1 - \sum_{t} X_i^t \cdot \left(1 - Y_{i + 1}^{i,i'}\right), \tag{3}
\]

This result is proven in (Manurangsi and Suksompong 2021).
where the last inequality follows from $Y_{t+1}^{i,t} < 1$ for all $t \geq 1$, the gap in the first $T$ rounds is sufficient for the envy to be negative with high probability.\footnote{Note that we allow negative envy, whereas some works define envy to be $\max \{0, u_i(A_j) - u_i(A_i)\}$. When envy is $-e$, we can also say the negative envy is $e$.} Manurangsi and Suksompong choose such $T$ that with high probability $(1 - O(1/m^3))$, events (E1) $\sum_{t=1}^{T} Y_{t}^{i,t} \geq T - 2$ and (E2) $X_{t}^{i} < 1/2$ for all $t \leq T$, do not happen. Then it can be seen that when neither E1 nor E2 happen, the envy in Eq. (3) is non-positive.

For constant $c > 1$, consider changing E1 to E1': (E1') $\sum_{t=1}^{T} Y_{t}^{i,t} \geq T - 2c$, then E1' and E2 give us negative envy of at least $c - 1$. We can still bound the probability that E1' or E2 occur in $O(1/m^3)$, by multiplying the value of $T$ set in Manurangsi and Suksompong’s proof by a factor of $c$, while keeping othervaluations as they did. Then by Lemma 2.4 in their paper, the upper bound of the probability that E1' occurs is the upper bound for E1 to the power of $c$, which is still in $O(1/m^3)$. Meanwhile, the upper bound of the probability that E2 occurs is still in $O(1/m^3)$, for $m$ that is sufficiently large. Hence we show that for some constant $c > 1$, when the distribution is PDF-bounded on $[0, 1]$, the probability that $i$ envies $i'$ more than $1 - c$ in the round robin allocation is at most $O(1/m^3)$.

Now we use such result to further prove the envy probability is bounded by $O(1/m^3)$ when the distribution $D$ is PDF-bounded and has interval support instead of $[0, 1]$ support. The method is to use affine transformation to transform $D$’s support interval $[a, b]$ into $[0, 1]$, mapping the original utility $u$ to $u' = (u - a)/(b - a)$ and the original distribution $D$ to $D'$. The PDF $f_D$ now becomes $f_{D'}(u) = (b - a) \cdot f_D((b - a)u + a)$. Since $D$ is $(p, q)$-PDF-bounded, the length of its support, $b - a$, must be at least $1/q$. Thus for any $u \in [0, 1]$, $p/q \leq f_{D'}(u) \leq q$, indicating that $D'$ is PDF-bounded on $[0, 1]$. Since the affine transform does not change the ordinal profile, the round robin allocation under the transformed utility profile, where all original utilities are transformed by the affine transformation: $u \mapsto (u - a)/(b - a)$, is the same as the one under the original utility profile. For the same round robin allocation, suppose the envy that $i$ holds for $i'$ in the transformed utility profile is $e$, then the envy in the original utility profile becomes $e'$:

$$
e' = \begin{cases} (b - a) e & \text{if } i \text{ and } i' \text{ get same number of items} \\ (b - a) e + a & \text{if } i' \text{ gets one more item than } i \\ (b - a) e - a & \text{if } i \text{ gets one more item than } i' \end{cases}$$

Then for $i$ to envy $i'$ in the original utility profile, i.e., $e' > 0$, it must be true that $e > -a/(b - a)$. Since the utilities in the transformed utility profile can be considered as drawn randomly from $D'$, which is PDF-bounded on $[0, 1]$, by our previous result, the probability that $e > -a/(b - a) = 1 - c$ is in $O(1/m^3)$. Hence the probability that $i$ envies $i'$ in the round robin allocation for distribution $D$ is still in $O(1/m^3)$, generalizing the result Lemma 9 to only assuming interval support and PDF-bounded for the distributions. Finally, similarly, we get $\mathbb{P}[\exists j \text{ envies } j] = O(1/m^3)$ and that the allocation is envy-free in the asymmetric model with high probability, when distributions have interval support and are $(p, q)$-PDF-bounded.

\[\square\]

B Proofs used in Existence Result of Envy-free and Pareto-optimal Allocations

B.1 Proof of Local Strict Monotonicity

We formalize the notion of local strict monotonicity as the following lemma.

**Lemma 10.** Assuming the distributions for all agents have interval support. For any agent $j \in N$, let $p_j > 0$, $p'_j$ be the probability that agent $j$ receives each item under $\{\beta'_i\}_{i \in N}$ and $\{\beta'_i\}_{i \in N}$. If $\forall i \in N$, $\beta'_j/\beta'_k \geq \beta_j/\beta_k$ and there exists some $k \in N$ such that $p_k > 0$ and $\beta'_j/\beta'_k > \beta_j/\beta_k$, then $p'_j > p_j$.

**Proof.** First we can express $p'_j, p_j$ as follows:

$$p'_j = \mathbb{P}[\exists i \in N: \beta'_j u'_i \leq \beta'_i u'_i] = \int_0^1 f_j(u) \prod_{i \in N \setminus \{j\}} F_i \left(\frac{\beta'_j}{\beta'_i} u \right) du, \quad$$

$$p_j = \int_0^1 f_j(u) \prod_{i \in N \setminus \{j\}} F_i \left(\frac{\beta_j}{\beta_i} u \right) du.$$

Suppose $D_j$’s support interval is $[\underline{u}_j, \overline{u}_j]$. Now take

$$u = \max_{u} \{u : \exists i \in N, F_i \left(\frac{\beta_j}{\beta_i} u \right) = 0\}.$$
From \( F_j \left( \frac{\partial_j}{\partial_i} u_{ij} \right) = 0 \), then by definition of \( u \), it is true that \( u_j \leq u \). We also have \( u < \beta_j \), since otherwise we can find agent \( i \in N \), such that \( F_i \left( \frac{\partial_i}{\partial_i} u \right) = 0 \) for all possible value of \( u_j = u \in [u_j, \beta_j] \), making \( p_j = 0 \). Then we take

\[
\beta = \min_{u} \{ u : F_k \left( \frac{\beta_k}{\beta_i} u \right) = 1 \}.
\]

We argue that \( \beta > u \). Consider otherwise, then we can find \( u_0 \in [\beta, u] \) and \( i \in N \), where

\[
F_i \left( \frac{\beta_i}{\beta_j} u_0 \right) = 0, \quad F_k \left( \frac{\beta_k}{\beta_i} u_0 \right) = 1
\]

\[
\Rightarrow u_i \geq \frac{\beta_j}{\beta_i} u_0, \quad u_k \leq \frac{\beta_j}{\beta_k} u_0 \Rightarrow \beta_i u_i \geq \beta_k u_k,
\]

which will make \( p_k = 0 \) since all distributions are non-atomic, contradicting our assumption that \( p_k > 0 \). Combining the earlier arguments, we know that \( \max \{ u_j, u \} < \min \{ \beta_j, \beta \} \). Then for any \( u \in (\max \{ u_j, u \}, \min \{ \beta_j, \beta \}) = I \),

\[
f_j(u) > 0, \quad F_i \left( \frac{\beta_i}{\beta_j} u \right) > 0, \quad \forall i \in N,
\]

\[
0 < F_k \left( \frac{\beta_k}{\beta_i} u \right) < 1 \Rightarrow F_k \left( \frac{\beta'_{k}'}{\beta_j} u \right) > F_k \left( \frac{\beta_j}{\beta_k} u \right).
\]

The last inequality follows from the fact that the derivative \( F_k'(u) > 0 \) for \( u \) that satisfies \( 0 < F_k(u) < 1 \) (guaranteed by Interval support property, \( 0 < F_k(u) < 1 \) just means \( u \) is in the support interval), and that \( \beta'_{j}/\beta'_{k} > \beta_{j}/\beta_{k} \). Then

\[
\int_{u \in I} f_j(u) \prod_{i \in N \setminus \{j\}} F_i \left( \frac{\beta'_{k}}{\beta_i} u \right) du > \int_{u \in I} f_j(u) \prod_{i \in N \setminus \{j\}} F_i \left( \frac{\beta_j}{\beta_i} u \right) du.
\]

For \( u \) in the rest of the range, we have that

\[
\frac{\beta'_{k}}{\beta_i} \geq \frac{\beta_j}{\beta_i} \Rightarrow F_i \left( \frac{\beta'_{k}}{\beta_i} u \right) \geq F_i \left( \frac{\beta_j}{\beta_i} u \right), \quad \forall i \in N \setminus \{j\}.
\]

Then the integral in this range for \( p'_j \) is greater or equal to that for \( p_j \). Hence we have showed the strict ordering \( p'_j > p_j \). \( \square \)

**B.2 Continuity of Probability Function \( p_i(\cdot) \)**

**Lemma 11.** For all \( i \in N \), the probability function \( p_i(\cdot) \) defined by

\[
p_i(\beta_1, \ldots, \beta_n) = \mathbb{P} \left[ \beta_i u_i = \max_{j \in N} \beta_j u_j \right]
\]

is continuous on \( (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+ \).

**Proof.** We have

\[
p_i(\beta_1, \ldots, \beta_n) = \int_0^1 f_i(u) \prod_{j \neq i} F_j \left( \frac{\beta_i}{\beta_j} u \right) du,
\]

then

\[
|p_i(\beta'_1, \ldots, \beta'_n) - p_i(\beta_1, \ldots, \beta_n)| \leq \int_0^1 f_i(u) \cdot \left| \prod_{j \neq i} F_j \left( \frac{\beta'_{k}}{\beta_i} u \right) - \prod_{j \neq i} F_j \left( \frac{\beta_j}{\beta_i} u \right) \right| du
\]

\[
\leq \sum_{j \neq i} \int_0^1 f_i(u) \cdot \left| F_j \left( \frac{\beta'_{k}}{\beta_i} u \right) - F_j \left( \frac{\beta_j}{\beta_i} u \right) \right| du.
\]

For nonatomic distribution \( \mathcal{D}_j \), its cumulative distribution function \( F_j(\cdot) \) is continuous, also the function \( g_u(x, y) = \frac{xu}{y} \) is continuous when \( x, y \neq 0 \). Thus

\[
\lim_{(\beta'_1, \beta'_n) \to (\beta_1, \beta_n)} F_j \left( \frac{\beta'_{k}}{\beta_j} u \right) = F_j \left( \frac{\beta_j}{\beta_j} u \right),
\]
and we have
\[
\lim_{(\beta_1', \ldots, \beta_n') \to (\beta_1, \ldots, \beta_n)} |p_i(\beta_1', \ldots, \beta_n') - p_i(\beta_1, \ldots, \beta_n)| \\
\leq \sum_{j \neq i} \int_0^1 f_i(u) \lim_{(\beta_1', \beta_j') \to (\beta_1, \beta_j)} \left| F_j \left( \frac{\beta_j'}{\beta_j} u \right) - F_j \left( \frac{\beta_j}{\beta_j} u \right) \right| du = 0.
\]

Therefore function \( p_i(\cdot) \) is continuous on \( \mathbb{R}_+^n \).

\[\square\]

**B.3 Inequalities for Expectations**

**Lemma 12.** For any pair of agents \( i, j \in N \), the following inequalities between expectations hold (assuming that the conditions in the conditional expectations can be met):

\[
\mathbb{E} \left[ u_i \bigg| \beta_j u_j = \max_{k \in N} \beta_k u_k \right] \leq \mathbb{E} [u_i] \tag{4}
\]

and

\[
\mathbb{E} \left[ u_i \bigg| \beta_i u_i = \max_{k \in N} \beta_k u_k \right] \geq \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j \right] \tag{5}
\]

**Proof.** Let \( v_j = \mathbbm{1} [\beta_j u_j = \max_{k \in N} \beta_k u_k] \), which is a random variable taking value from \( \{0, 1\} \). Then we have

\[
\mathbb{E} \left[ u_i \big| \beta_j u_j = \max_{k \in N} \beta_k u_k \right] = \mathbb{E} [u_i \big| v_j = 1] = \int_0^1 f_{u_i|v_j=1}(u) \cdot \mathbb{E} [u_i \big| v_j = 1, u_j = u] \, du.
\]

Given \( v_j = 1 \), for all \( u \in [0, 1] \), it holds that

\[
\mathbb{E} [u_i \big| v_j = 1, u_j = u] = \mathbb{E} \left[ u_i \big| \beta_i u_i \leq \beta_j u_j \right] \leq \mathbb{E} [u_i].
\]

Thus

\[
\int_0^1 f_{u_i|v_j=1}(u) \cdot \mathbb{E} [u_i \big| v_j = 1, u_j = u] \, du \leq \int_0^1 f_{u_i|v_j=1}(u) \cdot \mathbb{E} [u_i] \, du = \mathbb{E} [u_i].
\]

Therefore Eq. (4) holds.

For Eq. (5), let \( v_{i,j} = \mathbbm{1} [\beta_i u_i = \max_{k \in N \setminus \{i,j\}} \beta_k u_k] \) be a random variable taking value in \([0, 1]\), and \( v_i = \mathbbm{1} [\beta_i u_i = \max_{k \in N} \beta_k u_k] \). Then by substituting the condition we have

\[
\mathbb{E} \left[ u_i \big| \beta_i u_i = \max_{k \in N} \beta_k u_k \right] = \mathbb{E} [u_i \big| v_i = 1] = \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j, \beta_i u_i \geq v_{i,j} \right]
\]

\[
= \int_0^1 f_{v_{i,j}|v_i=1}(u) \cdot \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j, \beta_i u_i \geq v_{i,j}, v_{i,j} = u \right] \, du.
\]

Since \( v_{i,j}, u_i, u_j \) are independent, for all \( u \in [0, 1] \) it holds that

\[
\mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j, \beta_i u_i \geq v_{i,j}, v_{i,j} = u \right] = \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j, \beta_i u_i \geq u \right] \geq \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j \right].
\]

Hence

\[
\int_0^1 f_{v_{i,j}|v_i=1}(u) \cdot \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j, \beta_i u_i \geq v_{i,j}, v_{i,j} = u \right] \, du
\]

\[
\geq \int_0^1 f_{v_{i,j}|v_i=1}(u) \cdot \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j \right] \, du = \mathbb{E} \left[ u_i \big| \beta_i u_i \geq \beta_j u_j \right].
\]

Therefore Eq. (5) holds. \[\square\]
For all $i \neq j$, \[
\mathbb{E} \left[ u_i \left| \beta_i u_i = \max_{k \in N} \beta_k u_k \right. \right] > \mathbb{E} \left[ u_i \left| \beta_j u_j = \max_{k \in N} \beta_k u_k \right. \right].
\]

**Proof.** Let $X := \beta_i u_i$ and $Y := \beta_j u_j$ denote the scaled random variables, then it still holds that: (a) $X$ and $Y$ are independent, (b) $X$ and $Y$ have interval support, and (c) both $\mathbb{P}[X > Y]$ and $\mathbb{P}[Y > X]$ are at least $1/n > 0$. From Appendix B.3, we know that
\[
\mathbb{E} \left[ u_i \left| \beta_i u_i = \max_{k \in N} \beta_k u_k \right. \right] \geq \mathbb{E} \left[ u_i \left| \beta_j u_j \geq \beta_j u_j \right. \right] = \frac{1}{\beta_i} \mathbb{E} \left[ X \left| X > Y \right. \right],
\]
\[
\mathbb{E} \left[ u_i \left| \beta_j u_j = \max_{k \in N} \beta_k u_k \right. \right] \leq \mathbb{E} \left[ u_i \right] = \frac{1}{\beta_i} \mathbb{E} \left[ X \right].
\]

Then it suffices to show that $\mathbb{E} \left[ X \left| X > Y \right. \right] > \mathbb{E} \left[ X \right]$. Let $I$ denote the intersection of the support intervals of $X$ and $Y$, excluding both endpoints. This intersection is a nonempty interval and both $\mathbb{P}[X \in I]$ and $\mathbb{P}[Y \in I]$ must have positive probability, since, else, one variable’s support would entirely lie below or above the other variable’s support, which would contradict the above observation that $0 < \mathbb{P}[X > Y] < 1$. Since for all $y \in I$,
\[
\mathbb{E} \left[ X \right] = \frac{\mathbb{P}[X > y]}{>0} \cdot \mathbb{E} \left[ X \left| X > y \right. \right] + \frac{\mathbb{P}[X \leq y]}{>y} \cdot \mathbb{E} \left[ X \left| X \leq y \right. \right].
\]

Hence
\[
\mathbb{E} \left[ X \left| X > Y, Y \in I \right. \right] = \int_0^{\beta_j} f_{Y|X,Y \in I}(y) \cdot \mathbb{E} \left[ X \left| X > Y, Y \in I, Y = y \right. \right] dy
\]
\[
= \int_0^{\beta_j} f_{Y|X,Y \in I}(y) \cdot \mathbb{E} \left[ X \left| X > y \right. \right] dy
\]
\[
> \int_0^{\beta_j} f_{Y|X,Y \in I}(y) \cdot \mathbb{E} \left[ X \right] dy = \mathbb{E} \left[ X \right].
\]

For all $y \notin I$, $\mathbb{E} \left[ X \left| X > y \right. \right] \geq \mathbb{E} \left[ X \right]$, from which we have
\[
\mathbb{E} \left[ X \left| X > Y, Y \notin I \right. \right] = \int_0^{\beta_j} f_{Y|X,Y \notin I}(y) \cdot \mathbb{E} \left[ X \left| X > Y, Y \notin I, Y = y \right. \right] dy
\]
\[
= \int_0^{\beta_j} f_{Y|X,Y \notin I}(y) \cdot \mathbb{E} \left[ X \left| X > y \right. \right] dy
\]
\[
\geq \int_0^{\beta_j} f_{Y|X,Y \notin I}(y) \cdot \mathbb{E} \left[ X \right] dy = \mathbb{E} \left[ X \right].
\]

Then, we can bound
\[
\mathbb{E} \left[ X \left| X > Y \right. \right] = \frac{\mathbb{P}[Y \in I \left| X > Y \right.]}{>0} \cdot \mathbb{E} \left[ X \left| X > Y, Y \in I \right. \right] + \frac{\mathbb{P}[Y \notin I \left| X > Y \right.]}{\geq \mathbb{E}[X]} \cdot \mathbb{E} \left[ X \left| X > Y, Y \notin I \right. \right]
\]
\[
> \mathbb{E} \left[ X \right],
\]

which shows a positive gap.

\[\Box\]

**B.5 Proof of Lemma 7**

**Lemma 7.** For all $i, j \in N$, we have that $\frac{1}{2q} \leq \frac{\beta_i}{\beta_j} \leq 2q$. 


Proof. By symmetry, it suffices to show one side of the inequality: \( \beta_i/\beta_j \leq 2q \). We prove this by contradiction: suppose there is a pair of \( i, j \in N \) such that \( \beta_i/\beta_j > 2q \). Then we have the following inequalities for the probability \( p_i, p_j \) of \( i \) and \( j \) getting each item:

\[
p_i = P\left[ \beta_i u_i = \max_{k \in N} \beta_k u_k \right] \geq P\left[ \beta_i u_i = \max_{k \in N} \beta_k u_k \cap \beta_i u_i \geq \beta_j \right]
\]

\[
= P[\beta_i u_i \geq \beta_j] \cdot P\left[ \beta_i u_i = \max_{k \in N} \beta_k u_k \left| \beta_i u_i \geq \beta_j \right. \right] \geq P[\beta_i u_i \geq \beta_j] \cdot P\left[ \beta_j \geq \max_{k \in N, k \neq i, j} \beta_k u_k \right],
\]

and

\[
p_j = P\left[ \beta_j u_j = \max_{k \in N} \beta_k u_k \right] = P[\beta_j u_j \geq \beta_i u_i] \cdot P\left[ \beta_j u_j = \max_{k \in N} \beta_k u_k \left| \beta_j u_j \geq \beta_i u_i \right. \right]
\]

\[
\leq P[\beta_j \geq \beta_i u_i] \cdot P\left[ \beta_j \geq \max_{k \in N, k \neq i, j} \beta_k u_k \right],
\]

where the last inequality follows from \( u_j \leq 1 \). Since

\[
P[\beta_i u_i \leq \beta_j] = P[u_i \leq \beta_j/\beta_i] \leq P[u_i < 1/(2q)] < 1/2,
\]

we have

\[
P[\beta_i u_i \geq \beta_j] > P[\beta_j \geq \beta_i u_i] \Rightarrow p_i > p_j,
\]

which contradicts to the fact that \( p_i = p_j \) under equalizing multipliers. \( \square \)

B.6 Lower Bound: Length of the Intersection of Support Intervals

Lemma 13. For any \( i, j \in N \) and \( h(u) = f_j \left( \frac{\beta_j}{\beta_i} u \right) \), the interval \( I^* = \text{supp}(h) \cap \text{supp}(f_i) \) has length at least \( L_{p,q} \), which only depends on \( p \) and \( q \).

Proof. We consider the random variables \( u_i \sim D_i, u_j \sim D_j \) after scaled: \( \bar{u}_i = \beta_i u_i, \bar{u}_j = \beta_j u_j \), which have PDF: \( \bar{f}_i(u) = \frac{1}{\beta_i} f_i \left( \frac{u}{\beta_i} \right) \) and \( \bar{f}_j(u) = \frac{1}{\beta_j} f_j \left( \frac{u}{\beta_j} \right) \). Without loss of generality we will assume that the \( \min_{k \in N} \beta_k = 1 \), then by Lemma 7 we know that \( \max_{k \in N} \beta_k \leq 2q \).

We will first prove that \( \text{supp}(\bar{f}_i) \cap \text{supp}(\bar{f}_j) \) has lower bounded length. Note that \( \bar{f}_i \) is \((p/\beta_i, q/\beta_i)\)-PDF-bounded and \( \bar{f}_j \) is \((p/\beta_j, q/\beta_j)\)-PDF-bounded.

If one of the two support intervals contain the other, then the length of their intersection will be at least \( \min\{\beta_i/q, \beta_j/q\} \geq 1/q \).

Otherwise, if \( \text{supp}(\bar{f}_i) \) lies on the right of \( \text{supp}(\bar{f}_j) \), suppose their intersection is \([a, b]\) (there will not be a vacant intersection since then \( \beta_j u_j \) would always be smaller than \( \beta_i u_i \)), then it always holds that \( \beta_i u_i \geq a, \beta_j u_j \leq b \). We claim that the length of \( [a, b] \) is at least \( 1/(2q) \). Otherwise, consider

\[
p_j = P\left[ \beta_j u_j = \max_{k \in N} \beta_k u_k \right]
\]

\[
= P[\beta_i u_i \leq b] \cdot P\left[ \beta_j u_j = \max_{k \in N} \beta_k u_k \left| \beta_i u_i \leq b \right. \right] \leq P[\beta_i u_i \leq b] \cdot P\left[ \max_{k \in N, k \neq i} \beta_k u_k \leq b \right],
\]

while

\[
p_i = P\left[ \beta_i u_i = \max_{k \in N} \beta_k u_k \right]
\]

\[
\geq P[\beta_i u_i \geq b] \cdot P\left[ \beta_j u_j \geq \max_{k \in N} \beta_k u_k \left| \beta_i u_i \geq b \right. \right] \geq P[\beta_i u_i \geq b] \cdot P\left[ \max_{k \in N, k \neq i} \beta_k u_k \leq b \right].
\]
If the length of \([a, b]\) is less than \(1/(2q)\), we have
\[
\Pr[\beta_i u_i \leq b] = \Pr[\beta_i u_i \in [a, b]] < \frac{q}{\beta_i} \cdot \frac{1}{2q} \leq \frac{1}{2}
\]
which indicates that \(\Pr[\beta_i u_i \leq b] < \Pr[\beta_i u_i \geq b]\). Then we have the contradiction where \(p_j < p_i\) while it should be true that \(p_j = p_i\). The argument is symmetric for the case where \(\text{supp}(f_i)\) lies on the left of \(\text{supp}(f_j)\), which also gives the same lower bound \(1/(2q)\) on the length.

Therefore we conclude that \(I = [a, b] = \text{supp}(\tilde{f}_i) \cap \text{supp}(\tilde{f}_j)\) has length at least \(1/(2q)\). Then \(f_j\) is supported on \([a/\beta_j, b/\beta_j]\) and \(f_i\) is supported on \([a/\beta_i, b/\beta_i]\). Thus
\[
\forall u \in \left[\frac{a}{\beta_i}, \frac{b}{\beta_j}\right], \quad h(u) = f_j(u/\beta_j) > 0 \\
\Rightarrow [a/\beta_i, b/\beta_i] \subseteq \text{supp}(h),
\]
and
\[
\forall u \in \left[\frac{a}{\beta_i}, \frac{b}{\beta_i}\right], \quad f_i(u) > 0 \Rightarrow [a/\beta_i, b/\beta_i] \subseteq \text{supp}(f_i).
\]
Therefore the interval \([a/\beta_i, b/\beta_j]\) \(\subseteq \text{supp}(h) \cap \text{supp}(f_i)\), with a length of at least \(1/q \cdot 1/(2q) = 1/(2q^2)\). This proves our lemma that the intersection \(I^*\) has length at least \(L_{p,q} = 1/(2q^2)\).

\[\square\]

### B.7 Proof of Proposition 8

**Proposition 8.** There exists a constant \(C_{p,q} > 0\) that only depends on \(p\) and \(q\), such that, for any collection of agents whose utility distributions are \((p, q)\)-PDF-bounded and have interval support, for all \(i \neq j \in N\),
\[
\mathbb{E}\left[u_i \left| \beta_i u_i = \max_{k \in N} \beta_k u_k\right\rangle\right] - \mathbb{E}\left[u_i \left| \beta_j u_j = \max_{k \in N} \beta_k u_k\right\rangle\right] \\
\geq C_{p,q} > 0.
\]

**Proof.** In Appendix B.3 we show that
\[
\mathbb{E}\left[u_i \left| \beta_j u_j = \max_{k \in N} \beta_k u_k\right\rangle\right] \leq \mathbb{E}[u_i]
\]
and
\[
\mathbb{E}\left[u_i \left| \beta_i u_i = \max_{k \in N} \beta_k u_k\right\rangle\right] \geq \mathbb{E}\left[u_i \left| \beta_i u_i \geq \beta_j u_j\right\rangle\right].
\]
Then it suffices to show that there is a constant gap between \(\mathbb{E}[u_i \mid \beta_i u_i > \beta_j u_j]\) and \(\mathbb{E}[u_i]\). Let
\[
\mathcal{P} = \Pr[\beta_i u_i \geq \beta_j u_j] = \int_0^1 f_i(u) F_j\left(\frac{\beta_i}{\beta_j} u\right) du,
\]
and
\[
\Delta\mathbb{E} = \mathbb{E}\left[u_i \left| \beta_i u_i \geq \beta_j u_j\right\rangle\right] - \mathbb{E}[u_i],
\]
then we have
\[
\Delta\mathbb{E} = \frac{1}{\mathcal{P}} \int_0^1 u f_i(u) \left(F_j\left(\frac{\beta_i}{\beta_j} u\right) - \mathcal{P}\right) du.
\]
Let \(g(u) = F_j\left(\frac{\beta_i}{\beta_j} u\right) - \mathcal{P}\), and \(h(u) = f_j\left(\frac{\beta_i}{\beta_j} u\right)\). It is clear that \(g(u)\) is monotonically increasing, moreover, we can lower bound the derivative of \(g(u)\) on the support of \(h(u)\), which is an interval and we denote this range as \(\text{supp}(h)\):
\[
g'(u) = \frac{\beta_i}{\beta_j} f_j\left(\frac{\beta_i}{\beta_j} u\right) \geq \frac{1}{2q} \cdot \frac{p}{2q} = \frac{p}{4q^2}, u \in \text{supp}(h).
\]
The inequality is derived from Lemma 7. Let \(\text{supp}(f_i)\) denote the support interval of \(f_i(u)\). In Appendix B.6 we show a lower bound, \(L_{p,q}\) which only depends on \(p\) and \(q\), on the length of the interval \(I^* = \text{supp}(h) \cap \text{supp}(f_i)\). We consider such interval
\[ I^* = [l^*, r^*] \subseteq [0, 1] \] with midpoint \( m^* = (l^* + r^*)/2 \), and we know that \( r^* - l^* \geq L_{p,q} \). From the previous analysis, we know that for any \( u \in I^* \), it always holds that \( f_i(u) \geq p \) and \( g'(u) \geq D_{p,q} = p/(4q^2) \).

Since
\[ \int_0^1 f_i(u) g(u) \, du = 0, \tag{6} \]
combined with \( g(u) \)'s continuity and monotonicity, there exists a point \( u^* \in [0, 1] \) where \( g(u^*) = 0 \). The interval \( I^* \) must have at least half of its length that lies on the left or right side of \( u^* \), without loss of generality we assume that \( u^* \leq m^* \) and interval \([m^*, r^*]\) lies on the right of \( u^* \). Let
\[ c_1 = \int_{u^*}^{m^*+r^*} f_i(u) g(u) \, du, \quad c_2 = \int_{m^*+r^*}^1 f_i(u) g(u) \, du. \]
When \( u \in [u^*, m^*+r^*/2] \), we have \( f_i(u) \geq 0, g(u) \geq g(u^*) = 0 \), hence \( c_1 \geq 0 \). While \( u \in [m^*-r^*/2, m^*] \), we have that \( f_i(u) \geq p \) and \( g(u) \geq g(u^*) + \frac{L_{p,q}}{2} \cdot D_{p,q} \)
\[ \geq g(u^*) + \frac{L_{p,q}}{4} \cdot D_{p,q} = \frac{L_{p,q} D_{p,q}}{4}. \]
Then we can lower bound \( c_2 \) by a positive constant \( G_{p,q} \):
\[ c_2 \geq \int_{m^*-r^*/2}^{m^*} f_i(u) g(u) \, du \geq \frac{r^* - m^*}{2} \cdot p \cdot \frac{L_{p,q} D_{p,q}}{4} \geq \frac{p L_{p,q} D_{p,q}}{16} = G_{p,q}. \]
Eq. (6) indicates that
\[ \int_0^{u^*} f_i(u) g(u) \, du = -(c_1 + c_2). \]
Then we have
\[ \int_0^{u^*} f_i(u) g(u) \, du \geq -u^* (c_1 + c_2), \]
\[ \int_{u^*}^{m^*+r^*/2} f_i(u) g(u) \, du \geq u^* c_1, \]
\[ \int_{m^*+r^*/2}^1 f_i(u) g(u) \, du \geq \frac{m^* + r^*}{2} c_2, \]
and we can lower bound \( \Delta E \) by
\[ \Delta E \geq -u^* (c_1 + c_2) + u^* c_1 + \frac{m^* + r^*}{2} c_2 \]
\[ = \left( \frac{m^* + r^*}{2} - u^* \right) c_2 \geq \frac{L_{p,q} G_{p,q}}{4}. \]
Therefore we have finished the proof with
\[ C_{p,q} = \frac{L_{p,q} G_{p,q}}{4} = \frac{p^2}{2048q^8}. \]

### B.8 Envy-free: Combining Previous Results

For any two agents \( i, j \in N \), and each item \( \alpha \in M \), let \( X_\alpha \) denote its contribution to \( u_i(A_i) \) and \( Y_\alpha \) denote its contribution to \( u_i(A_j) \). In particular, \( X_\alpha = 0 \) if \( \alpha \notin A_i \) and \( X_\alpha = u_i(\alpha) \) if \( \alpha \in A_i \); while \( Y_\alpha = 0 \) if \( \alpha \notin A_j \) and \( Y_\alpha = u_i(\alpha) \) if \( \alpha \in A_j \). Then
following from Proposition 4 and Proposition 8, we have

\[
\mathbb{E}[X_\alpha] = \mathbb{P}[\alpha \in A_i] \cdot \mathbb{E}[u_i(\alpha) \mid \alpha \in A_i]
\]

\[
= \mathbb{P}\left[\beta_i u_i = \max_{k \in N} \beta_k u_k\right] \cdot \mathbb{E}\left[u_i \mid \beta_i u_i = \max_{k \in N} \beta_k u_k\right]
\]

\[
= \frac{1}{n} \cdot \mathbb{E}\left[u_i \mid \beta_i u_i = \max_{k \in N} \beta_k u_k\right].
\]

\[
\mathbb{E}[Y_\alpha] = \mathbb{P}[\alpha \in A_j] \cdot \mathbb{E}[u_i(\alpha) \mid \alpha \in A_j]
\]

\[
= \mathbb{P}\left[\beta_j u_j = \max_{k \in N} \beta_k u_k\right] \cdot \mathbb{E}\left[u_i \mid \beta_j u_j = \max_{k \in N} \beta_k u_k\right]
\]

\[
= \frac{1}{n} \cdot \mathbb{E}\left[u_i \mid \beta_j u_j = \max_{k \in N} \beta_k u_k\right].
\]

\[
\Rightarrow \mathbb{E}[X_\alpha] - \mathbb{E}[Y_\alpha] \geq \frac{C_{p,q}}{n}.
\]

Note that \(X_\alpha, Y_\alpha\) are independently and identically distributed for all \(\alpha \in M\), and

\[
u_i(A_i) = \sum_{\alpha \in M} X_\alpha, \quad \nu_i(A_j) = \sum_{\alpha \in M} Y_\alpha.
\]

Thus we can bound \(\nu_i(A_i)\) by Chernoff’s bound: for any \(\alpha\),

\[
\mathbb{P}\left[\nu_i(A_i) < \left(1 - \frac{C_{p,q}}{2n\mathbb{E}[X_\alpha]}\right) m \mathbb{E}[X_\alpha]\right]
\]

\[
\leq \exp\left(-\frac{mC_{p,q}^2}{8n^2\mathbb{E}[X_\alpha]}\right) \leq \exp\left(-\frac{mC_{p,q}^2}{8n}\right).
\]

where the last inequality follows from \(\mathbb{E}[X_\alpha] \leq 1/n\). Similarly we can bound \(\nu_i(A_j)\):

\[
\mathbb{P}\left[\nu_i(A_j) > \left(1 + \frac{C_{p,q}}{2n\mathbb{E}[Y_\alpha]}\right) m \mathbb{E}[Y_\alpha]\right]
\]

\[
\leq \exp\left(-\frac{mC_{p,q}^2}{12n^2\mathbb{E}[Y_\alpha]}\right) \leq \exp\left(-\frac{mC_{p,q}^2}{12n}\right).
\]

Then we can use union bound to bound the probability \(\mathcal{P}_{ij}\) that neither of the above two events happen. With probability

\[
\mathcal{P}_{ij} \geq 1 - \exp\left(-\frac{mC_{p,q}^2}{8n}\right) - \exp\left(-\frac{mC_{p,q}^2}{12n}\right)
\]

\[
\geq 1 - \frac{2}{mn^2}\exp\left(2\log n - \frac{mC_{p,q}^2}{8n}\right),
\]

we have \(\nu_i(A_i) \geq \nu_i(A_j)\). Again we use union bound on the probability that for any \(i, j \in N\), \(\nu_i(A_i) \geq \nu_i(A_j)\) (it just means that the allocation is envy-free):

\[
\mathcal{P} \geq 1 - 2\exp\left(2\log n - \frac{mC_{p,q}^2}{8n}\right),
\]

which suggests that the allocation is envy-free with probability at least \(1 - 2\exp\left(2\log n - \text{const}(p, q) m/n\right)\).

C Negative Result for Round Robin Algorithm

Suppose agent 1 has uniform distribution on \([0, 6, 1]\), and agent \(n\) has uniform distribution on \([0, 1]\). Assume in the round robin algorithm, there are a total of \(t\) rounds. Agent 1 gets the first item, and agent \(n\) gets the last item. With probability \(p_1 = 1/3\), event A happens, where agent \(n\) values the item that agent 1 gets in the first round at least \(2/3\).

Consider the last two items that agent \(n\) gets, let \(X_{t-1}, X_t\) denote agent \(n\)’s utility on them. From Lemma 3.2 in (Manurangis and Sukosmpong 2021) we know their distribution is \(X_{t-1} \sim D_{\leq X_{t-1}}^{max(n+1)}\), \(X_t \sim D_{\leq X_{t-1}}^{max(1)}\), where \(D_{\leq X_{t-1}}^{max(k)}\) denote the distribution of the maximum of \(k\) samples drawn from \(D\) truncated at \(T\). \(D_{\leq X_{t-1}}^{max(n+1)}\) is stochastically dominated by \(D_{\leq X_{t-1}}^{max(n+1)}\), which
is the \((n + 1)\)th order statistic of \((n + 1)\) samples. Hence, with probability at least \(p_2 = (1/3)^{n+1}\), event \(B\) happens, where \(X_{t-1} \leq 1/3\), and since \(X_t \leq X_{t-1}\) we also have \(X_t \leq 1/3\).

The probability that both events happen is at least \(p_1 \cdot p_2 = (1/3)^{n+2}\), since event \(A\) and event \(B\) are independent. When both event \(A\) and event \(B\) happen, consider agent 1 trading the first item she gets for the last two items agent \(n\) gets, then agent 1’s utility strictly increase, since \(0.6 + 0.6 > 1\); at the same time, agent \(n\)’s utility does not decrease, since \(X_{t-1} + X_t \leq 2/3\). Then the original allocation is Pareto dominated by the allocation after this trade. Therefore, when \(n = \Theta(1)\), this means that with constant probability, the allocation with round robin algorithm is not Pareto-optimal, no matter how large \(m\) is. As the dash-dotted line in Fig. 4 suggests, for larger \(m\), such trade for Pareto improvement is more prevalent.

### D Alternative Proof for Existence of Multipliers with Sperner’s Lemma

For readers that might be interested, we provide a proof for the existence of multipliers based on Sperner’s Lemma. This proof does not take advantage of the local structure (monotonicity property) between the multipliers and probabilities, and as a result, it cannot argue about the uniqueness of the multipliers and cannot give an algorithm for locating the multipliers. On the bright side, this existence proof does not use any assumption on the distributions, i.e., interval support and independent—both necessary for the proof in Section 4.1. This suggests that the set of equalizing multipliers exist for arbitrary nonatomic distributions in \([0, 1]\), even when there is dependency between them.

**Lemma 14.** There exists a set of multipliers \(\{\beta_i\}_{i \in N}\) that for all \(i \in N\),

\[
P(\beta_i u_i = \max_{j \in N} \beta_j u_j) = 1/n.
\]

**Proof.** Without loss of generality we assume all multipliers add up to 1, then \((\beta_1, \beta_2, \ldots, \beta_n)\) falls in a \((n - 1)\)-dimensional simplex \(S\). Now we define the coloring function \(f : S \to N\), which maps each set of multipliers to an agent with the highest probability of having the largest scaled utility under the set of multipliers:

\[
f(\beta_1, \ldots, \beta_n) = \arg \max_{i \in N} \beta_i u_i = \max_{j \in N} \beta_j u_j.
\]

It is clear that the \(n\) vertices of the simplex are colored with \(n\) different “colors”, since agent \(i\) will have probability 1 of having the largest scaled utility when \(\beta_i = 1\) and \(\beta_j = 0, \forall j \neq i\).

Now we divide the simplex \(S\) into \(2^{n-1}\) smaller simplices by cutting on the midpoint on each edge. Let \(S_1\) denote the set of smaller simplices. Then we further divide each simplex in \(S_1\) into \(2^{n-1}\) even smaller simplices, resulting in a total of \(2^{2(n-1)}\) simplices and we denote them by \(S_2\). We repeat the procedure and divide the original simplex into \(S_1, S_2, S_3, \ldots\) containing smaller and smaller simplices.

By Sperner’s Lemma, in each \(S_i\) \((i \geq 1)\), there are always an odd number of simplices that are colored with \(n\) colors, indicating the existence of a simplex \(s_i \in S_i\) whose vertices are colored with all \(n\) colors. Now consider the sequence of such simplices: \(s_1, s_2, s_3, \ldots\), and let the \(i^{th}\) vertex be the vertex that is mapped to \(i\) by \(f\). We can represent each simplex with a \(n^2\)-dimensional vector:

\[
(\beta^1, \beta^2, \ldots, \beta^n) = (\beta^1_1, \ldots, \beta^1_n; \ldots; \beta^n_1, \ldots, \beta^n_n) \in \mathbb{R}^{n^2}.
\]

where \(\beta^i\) is the vector of multipliers in the \(i^{th}\) vertex and \(\beta^i_j\) is the \(j^{th}\) multiplier in the \(i^{th}\) vertex. Let \(s^j_i\) denote the subvector \(\beta^i\) in \(s^j\).

Since all of them are bounded between \((0, 0, \ldots, 0)\) and \((1, 1, \ldots, 1)\), by Bolzano-Weierstrass Theorem, there must be a convergent subsequence \(s_{a_1}, s_{a_2}, s_{a_3}, \ldots\) that converges to a simplex \((\beta^1, \beta^2, \ldots, \beta^n)\) in the space:

\[
\lim_{t \to \infty} s_{a_1} = (\beta^1, \beta^2, \ldots, \beta^n).
\]

Since the simplices in \(\{s_i\}_{i \geq 2}\) get smaller and smaller with a ratio of \(1/2\), we have

\[
\lim_{t \to \infty} s^1_{a_1} = \lim_{t \to \infty} s^2_{a_1} = \cdots = \lim_{t \to \infty} s^n_{a_1}.
\]

Then it can be deduced that \(\beta^1 = \cdots = \beta^n = \beta^*\).

We argue that \(\beta^* \in \mathbb{R}_+^n\). Since otherwise, if \(\beta^*_i = 0\) for some \(i \in N\), then \(\beta^*_i\), the \(i^{th}\) multiplier in the \(i^{th}\) vertex grows arbitrarily small as the sequence \(\{s_{a_1}\}\) goes. This would lead to the probability that agent \(i\) having the largest scaled utility goes to 0, contradicting the fact that agent \(i\) should have the highest probability of having largest scaled utility, with probability at least \(1/n\).

Now we claim that all agents have equal probability of having the largest scaled utility under \(\beta^*\). Define

\[
p_i(\beta_1, \ldots, \beta_n) = P(\beta_i u_i = \max_{j \in N} \beta_j u_j).
\]
As shown in Appendix B.2, $p_i(\cdot)$ is a continuous function in $\mathbb{R}^n_+$. Then for all $i \in N$,

$$\frac{1}{n} \leq \lim_{t \to \infty} p_i(s_{\alpha^t}) = p_i(\lim_{t \to \infty} s_{\alpha^t}) = p_i(\beta^*) .$$

Since $\sum_{i=1}^n p_i(\beta^*) = 1$, we must have

$$p_1(\beta^*) = \cdots = p_n(\beta^*) = \frac{1}{n} .$$

This shows that $\beta^*$ is the set of multipliers we are looking for, hence the existence.

\[\square\]

### E Details on Empirical Results

#### E.1 Setup

The five utility distributions we investigate are (labeled as in Fig. 1):

$$D_A = \text{Beta}(1/2, 1/2), \quad D_B = \text{Beta}(1, 3), \quad D_C = \text{Beta}(2, 5), \quad D_D = \text{Beta}(2, 2), \quad D_E = \text{Beta}(5, 1).$$

We chose them based on the illustration displayed on top of the Wikipedia page on Beta distributions\textsuperscript{10} at the time of writing. Code for all our experiments can be found at https://github.com/pgoelz/asymmetric. We implemented the multiplier algorithm and the main experiments in Python (3.7.10). We rely on Scipy for evaluating integrals (\texttt{integrate.quad}) and for the PDF and CDF of the beta distributions. We optimize fractional Maximum Nash Welfare using cvxpy (1.1.14), which in turn calls the MOSEK solver (9.2.9). To check Pareto-optimality, we use the Gurobi solver (9.0.3). Finally, we use Numpy in version 1.17.3.

All experiments were run on a MacBook Pro with a 3.1 GHz Dual-Core i5 processor and 16 GB RAM, running macOS 10.15.7. To verify the allocation probabilities, and for Fig. 3, we use Mathematica 12.0.0 on the same machine (Mathematica code is included in the above Git repository).

#### E.2 Divisibility by $n$

In the experiment displayed in Fig. 4 in the body of the paper, we evaluate the following sequence of items: $m \in \{5, 10, 20, 100, 200, 500, 1000, 2000, 5000, 10000\}$. This progression is a natural way to explore the space of $m$ on a logarithmic axis, but comes with one caveat: All $m$ are divisible by $n$, a special case in which envy-free allocations are known to appear at lower $m$ than in the general case (Manurangsi and Suksompong 2019). To verify that our empirical findings are robust to $m$ that are not multiples of $n = 5$, we repeat the experiment for the first values of $m$, but shifting each $m$ by 3 as follows: $m \in \{8, 13, 23, 53, 103, 203, 503, 1003\}$. We see that this shift in $m$ causes the round robin algorithm’s to converge towards envy-freeness at a slightly slower rate, which now makes the Maximum Nash Welfare algorithm converge faster than round robin, and also makes the round robin algorithm be even less likely to be Pareto-optimal. Nonetheless, the large trends identified in the body of the paper all persist, and we see no notable difference due to the offset when $m \geq 50$.

\textsuperscript{10}See https://en.wikipedia.org/wiki/Beta_distribution, accessed on September 8, 2021. The figure is https://commons.wikimedia.org/wiki/File:Beta_distribution_pdf.svg.