DIFFERENTIAL EQUATIONS EXTENDED TO SUPERSPACE

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We present a simple SUSY $\mathcal{N}_S=2$ superspace extension of the differential equations in which the sought solutions are considered to be real superfields but maintaining the common derivative operators and the coefficients of the differential equations unaltered. In this way, we get selfconsistent systems of coupled differential equations for the components of the superfield. This procedure is applied to the Riccati equation, for which we obtain in addition the system of coupled equations corresponding to the components of the general superfield solution.

\textit{Introduction}

In the framework of quantum cosmology, Obregon et al \cite{1} developed a formalism to find supersymmetric actions based on local $\mathcal{N}_S=2$ SUSY transformations and the concept of superfield. In particular, they generalized the local time transformations, $\delta t = a(t)$ that leave invariant the action, to local grassmannian transformations involving the Grassmann time variables $\eta$ and $\bar{\eta}$

$$
\delta t = a(t) + \frac{i}{2} \eta \beta(t) + \frac{i}{2} \bar{\eta} \bar{\beta}(t)
$$

$$
\delta \eta = \frac{1}{2} \dot{\beta}(t) + \frac{1}{2} (\dot{a}(t) + ib(t)) \eta + \frac{i}{2} \dot{\beta} \eta \bar{\eta},
$$

$$
\delta \bar{\eta} = \frac{1}{2} \dot{\bar{\beta}}(t) + \frac{1}{2} (\dot{a}(t) - ib(t)) \bar{\eta} - \frac{i}{2} \dot{\bar{\beta}}(t) \eta \bar{\eta},
$$
where \( \eta \) is a grassmannian time coordinate and \( \bar{\eta} \) is its complex conjugate (we actually take \( \eta \propto \theta_1 + i\theta_2 \) and \( \bar{\eta} \propto \theta_1 - i\theta_2 \)). The parameter \( \beta(t) \) is the Grassmann complex parameter corresponding to the local SUSY \( \mathcal{N}_S=2 \) whereas \( b(t) \) is the parameter of the local U(1) rotation group of \( \eta \). Because of these generalized transformations the ordinary fields turn into superfields, i.e., \( f(t) \rightarrow \mathcal{F}(t, \eta, \bar{\eta}) \). We shall consider only real superfields, \( \mathcal{F}^\dagger = \mathcal{F} \), where the dagger operation is defined for grassmannian variables as \( (\eta \bar{\eta})^\dagger = \bar{\eta}^\dagger \eta^\dagger \).

Thus, for an arbitrary superfield

\[
\mathcal{N}(t, \eta, \bar{\eta}) = N(t) + i\eta \bar{\psi}(t) + i\bar{\eta} \psi(t) + \eta \bar{\eta} V(t)
\]

one has

\[
\mathcal{N}^\dagger(t, \eta, \bar{\eta}) = N^\dagger(t) - i\bar{\psi}^\dagger(t) \eta^\dagger - i\psi^\dagger(t) \bar{\eta}^\dagger + \bar{\eta}^\dagger \eta^\dagger V^\dagger(t).\]

The condition of reality implies

\[
N^\dagger = N, \quad \bar{\psi}^\dagger = \psi, \quad V^\dagger = V, \quad \eta^\dagger = \bar{\eta}.
\]

**Differential equations extended to superspace**

There are many works on superspace extensions of differential equations [2]. We consider here one of the simplest possible extensions by turning the dependent variable of any differential equation of arbitrary order \( n \) into a superfield but without changing the derivative operator as performed in the literature. Moreover, we maintain unchanged the coefficients of the superextended equation because in this way we get selfconsistent differential systems of coupled equations for the superfield components.

Thus, what we are doing is to take the general equation

\[
a_n(t) y^{(n)}(t) + \ldots + a_0(t) y(t) = F(t)
\]

and change it into

\[
a_n(t) \mathcal{Y}^{(n)}(t, \eta, \bar{\eta}) + \ldots + a_0(t) \mathcal{Y}(t, \eta, \bar{\eta}) = F(t),
\]
where the superscript \((n)\) means the \(n\)th order derivative \(\frac{d^n}{dt^n}\).

**Riccati equation**

Our preferred example is the Riccati equation

\[
\frac{dy}{dt} = a(t)y^2 + b(t)y + c(t).
\]

It is well known that the theory of Riccati equation is equivalent to the theory of the linear differential equations of the second order. This is due to the connection between the two through the direct transformation

\[
y(t) = -\frac{1}{a(t)} \frac{\dot{w}}{w}
\]
leading to

\[
\ddot{w}(t) - \left(\frac{\dot{a}}{a} + b(t)\right) \dot{w}(t) + a(t)c(t)w(t) = 0
\]
and the inverse one

\[
w(t) = \exp \left(- \int a(t)y(t)dt\right)
\]
by which one goes back to the Riccati equation. Another important result is the possibility to construct the general Riccati solution once one knows a particular solution through the famous Bernoulli "ansatz"

\[
y_g = y_p + \frac{1}{f},
\]
which introduced in the Riccati equation turns it into the following linear equation

\[
f' = -\left[b(t) + 2a(t)y_p(t)\right] f - a(t).
\]
The solution of the latter is

\[
f(t) = \frac{1}{I(t)} \left[ - \int^t a(x)I(x)dx + C \right],
\]
where \(I(t) = \exp \left(\int^t (b(x) + 2a(x)y_p(x))dx\right)\).
Extended Riccati equation

We now write the superspace extended Riccati equation as follows

\[ i \frac{1}{2} \{D_\eta, D_{\bar{\eta}}\} \mathcal{V}(t, \eta, \bar{\eta}) = a(t)\mathcal{V}^2(t, \eta, \bar{\eta}) + b(t)\mathcal{V}(t, \eta, \bar{\eta}) + c(t) \]  

(1)

where

\[ i \frac{1}{2} \{D_\eta, D_{\bar{\eta}}\} = \frac{d}{dt} \]

and

\[ D_\eta = \frac{\partial}{\partial \eta} + i\bar{\eta} \frac{\partial}{\partial t}, \quad D_{\bar{\eta}} = -\frac{\partial}{\partial \bar{\eta}} - i\eta \frac{\partial}{\partial t}, \]

with \( D_\eta \) and \( D_{\bar{\eta}} \) the supercovariant derivatives.

We write the superfield \( \mathcal{V} \) in the form

\[ \mathcal{V}(t, \eta, \bar{\eta}) = y(t) + i\eta \bar{\lambda}(t) + i\bar{\eta} \lambda(t) + \eta\bar{\eta}G(t) \]

(2)

Introducing (2) in (1) and identifying the corresponding components one gets the following system

\[ \dot{y} = ay^2 + by + c \]  

(3)

\[ \dot{\lambda} = (2ay + b)\lambda \]  

(4)

\[ \dot{\bar{\lambda}} = (2ay + b)\bar{\lambda} \]  

(5)

\[ \dot{G} = (ay + b)G + 2a\lambda \bar{\lambda} \]  

(6)

Since the two equations for the fermionic components are conjugate to one another it is sufficient to solve only one of them.

In parallel with the ordinary procedure we propose the generalized transformation

\[ \mathcal{V} = -\frac{1}{a} \hat{\mathcal{N}} = -\frac{1}{a} \mathcal{N}^{-1} \hat{\mathcal{N}} \]

(7)

where \( \mathcal{N}^{-1} \) is defined by \( \mathcal{N}^{-1}\mathcal{N} = \mathcal{N}\mathcal{N}^{-1} = 1 \). Considering \( \mathcal{N}(t, \eta, \bar{\eta}) = N(t) + i\eta \bar{\psi}(t) + i\bar{\eta} \psi(t) + \eta\bar{\eta}V(t) \) how can one obtain the superfield \( \mathcal{N}^{-1} \)? To answer this question, we write
\( \mathcal{N} = N + \Lambda \), where \( \Lambda = i \eta \bar{\psi}(t) + i \bar{\eta} \psi(t) + \eta \bar{\eta} V(t) \). Since \( \mathcal{N}^{-1} = \frac{1}{\mathcal{N}} = \frac{1}{N+\Lambda} = \frac{1}{N(1+\frac{\Lambda}{N})} \), expanding the latter expression we get

\[
\frac{1}{N(1+\frac{\Lambda}{N})} = \frac{1}{N} \sum_{k=0}^{\text{#supercharges}} (-1)^k \left( \frac{\Lambda}{N} \right)^k = \frac{1}{N} \left( 1 - \frac{\Lambda}{N} + \left( \frac{\Lambda}{N} \right)^2 \right). \tag{8}
\]

Substituting \( \Lambda \) in the latter formula one eventually gets

\[
\mathcal{N}^{-1} = \frac{1}{N} - i \eta \mathcal{N}^{-2} \bar{\psi} - i \bar{\eta} \mathcal{N}^{-2} \psi + \eta \bar{\eta}(2 \mathcal{N}^{-3} \bar{\psi} \psi - \mathcal{N}^{-2} V). \tag{9}
\]

Using (8) and (9) in (1) one can get after some tedious algebra the second order linear differential equations for all the components of the superfield \( \mathcal{N} \). For the first bosonic component we obtain naturally the standard equation

\[
\ddot{N} - \left( \frac{\dot{a}}{a} + b \right) \dot{N} + acN = 0. \tag{10}
\]

For the \( \psi \) component we get

\[
\ddot{\psi} + \left( -a^{-1} \dot{a} - N^{-2} \ddot{N} + N^{-1} \dot{N} - b \right) \dot{\psi} + \left( a^{-1} \dot{a} N^{-1} \ddot{N} - N^{-1} \dot{N} + b \right) = 0, \tag{11}
\]

and identically for the \( \bar{\psi} \) component in view of the reality condition. On the other hand, the equation for the \( V \) component is more complicated and we do not include it here. We mention that it is possible to write the inverse generalized transformation to recover from these equations the extended Riccati one. One should write

\[
\mathcal{N} = e^{-\int^t a \gamma dt} \tag{12}
\]

and using \( \gamma = y + \Gamma(t, \eta, \bar{\eta}) \) we have

\[
\mathcal{N} = e^{-\int^t a \gamma dt} \left[ 1 - \int^t a \Gamma dt + \frac{1}{2} \int^t \int^t a(u) a(v) \Gamma(u, \eta, \bar{\eta}) \Gamma(v, \eta, \bar{\eta}) dudv \right]. \tag{13}
\]

**The general superfield solution of the extended Riccati equation**

For the general superfield we proceed again by analogy with the standard calculus. We write
where $\mathcal{B}$ is a superfield satisfying the extended Riccati equation

$$i \frac{1}{2} \{D_\eta, D_{\bar{\eta}}\} \mathcal{B}(t, \eta, \bar{\eta}) = a(t) \mathcal{B}^2(t, \eta, \bar{\eta}) + b(t) \mathcal{B}(t, \eta, \bar{\eta}) + c(t)$$

(15)

Which is the superequation $\mathcal{D}$ satisfies? Substituting (14) in the extended Riccati equation we get

$$i \frac{1}{2} \{D_\eta, D_{\bar{\eta}}\} (\mathcal{B} + \mathcal{D}^{-1}) = a(t)(\mathcal{B}^2 + 2\mathcal{B}\mathcal{D}^{-1} + \mathcal{D}^{-2}) + b(t)(\mathcal{B} + \mathcal{D}^{-1}) + c(t) ,$$

(16)

where we have used the fact that two superfields commute. Since $\mathcal{B}$ satisfies the Riccati superequation we can simplify (16) to the form

$$- \mathcal{D}^{-2} \mathcal{\dot{D}} = 2a\mathcal{B} \mathcal{D}^{-1} + b \mathcal{D}^{-1} + a \mathcal{D}^{-2} .$$

(17)

Multiplying by $-\mathcal{D}^2$ we obtain

$$\mathcal{\dot{D}} = -(2a\mathcal{B} + b) \mathcal{D} - a ,$$

(18)

which is completely equivalent to the ordinary case. We used $\frac{d}{dt} \mathcal{D}^{-1} = -\mathcal{D}^{-2} \mathcal{\dot{D}}$.

A case of special interest is $b = 0$ that corresponds to the Riccati-Schroedinger connection in supersymmetric quantum mechanics. For this case we get the system

$$\dot{B} = aB^2 + c$$

(19)

$$\dot{\phi} = 2aB\phi$$

(20)

$$\dot{\bar{\phi}} = 2aB\bar{\phi}$$

(21)

$$\dot{A} = 2aBA + 2a\phi\bar{\phi}$$

(22)

$$\dot{D} = -2aBD - a$$

(23)

$$\dot{\psi} = -2a(B\psi + D\phi)$$

(24)

$$\ddot{\psi} = (B\dot{\psi} + D\dot{\phi})$$

(25)

$$\dot{U} = -2a(BU + \bar{\phi}\psi + \bar{\psi}\phi + AD)$$

(26)
where
\[ B(t, \eta, \bar{\eta}) = B + i\eta \bar{\varphi} + i\bar{\eta} \varphi + \eta \bar{\eta} A, \]
\[ D(t, \eta, \bar{\eta}) = D + i\eta \bar{\psi} + i\bar{\eta} \psi + \eta \bar{\eta} U. \]

**Conclusion**

Although the procedure of superspace extension presented here is quite simple, it appears to be quite interesting for physical applications related to supersymmetric quantum mechanics and other fields of physics. Recently, we have used this scheme for the Riccati equation of constant coefficients that appears in FRW barotropic cosmologies for which we have solved explicitly the system of coupled differential equations generated by our procedure [3]. In addition, we have interpreted the obtained solutions as the components of a Hubble superfield parameter.
REFERENCES

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