Nonlinear Evolution of \(q\)-Gaussian Laser Beams in Preformed Parabolic Plasma Channels

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Abstract. Theoretical investigation on nonlinear propagation characteristics of \(q\)-Gaussian laser beams in preformed parabolic plasma channels has been presented. The optical response of the channel has been modelled by ponderomotive nonlinearity. Following Virial theorem in W.K.B approximation, semi analytical solution of the wave equation for the field of laser beam has been obtained. Particularly the evolutions of beam width and axial phase of the laser beam have been investigated in detail. Self trapping of the laser beam arising as a consequence of the cancellation of diffraction broadening by nonlinear refraction of the laser beam has also been investigated.

1. Introduction
Laser [1] is one of the most revolutionary inventions of the 20th century science. When laser was invented it was called as solution in search of a problem. With the amelioration in technology the impact of laser on society has changed over time, and is still changing. Lasers now find widespread applications in an incredibly diverse range of duties, including printing papers, scanning bar codes, medical diagnostics and treatment, high speed optical communication, driving controlled nuclear fusion to quench humanity’s endless thirst for energy, particle acceleration for cancer treatment etc[2-5]. In most of these applications laser intensity is the key parameter that decides their ultimate breath. Currently, due to the light’s inherent wave property of diffraction and damage threshold of optical components of the laser system, the laser power has gotten into bottleneck at the order of few peta watts. Both the problems of diffraction and ionization induced damage can be obviated by producing tight focusing of the laser beam with the help of plasma channels[6,7].

In contrast to normal states of matter (solids, liquids and gases) plasmas are already broken down as they consist of free electrons and ions. Therefore, ideally plasmas have an infinite immunity against optical damages. Hence, plasma channels are ideal candidate for the purpose of enhancing the laser intensity via self-focusing. In plasma channels, the optical properties like index of refraction, dielectric function, absorption coefficient etc get modified in the presence of intense laser beam. The resulting change in index of refraction resembles to that of a converging lens and thus the beam automatically gets accumulated towards its axis. This phenomenon is known as self-focusing[8]. When self-focusing gets exactly balanced by diffraction the beam constitutes a spatial soliton i.e., it propagates in a self-trapped mode.

The modification in the transverse dimensions of the laser beam as a consequence of self-focusing originates another peculiar nonlinear optical phenomenon known as self-modulation of its spatial frequency also known as self-phase modulation (SPM). Self-phase modulation refers to change in the...
axial phase of a focused beam compared to a simple plane wave. Due to SPM the spacing between the
phase fronts of an optical beam increases due to reduction in local phase velocity.

The aim of present study is to present a theoretical investigation on nonlinear propagation
characteristics of q-Gaussian laser beams in preformed parabolic plasma channels.

2. Mathematical formulation

The model equation governing the evolution of an optical beam through a preformed plasma channel with
ponderomotive nonlinearity is

$$2ik_0 \frac{\partial A}{\partial z} = \nabla^2 A + \frac{1}{c^2} \left[ \omega^2 p_0 e^{-\beta AA^*} - \left( \frac{\omega^2 p_0}{r_{ch}} + \frac{r^2}{\omega^2 p_{ch}} \right) e^{-\beta AA^*} \right] A \quad (1)$$

with, $\omega^2 p_0 = \frac{4\pi e^2}{m} n_0$ and $\omega^2 p_{ch} = \frac{4\pi e^2}{m} \Delta n$. Here, $n_0$ is the axial electron density of the channel, $\Delta n$ is
the difference in the electron density at the edge of the channel and that at the axis and $r_{ch}$ if the radius
of the plasma channel.

Due to its nonlinear nature eq. (1) does not possess any exact analytical solution. Hence, in order to
have physical insight into the propagation dynamics of the laser beam we have used a semi analytical
method known as Hamiltonian formulation. According to Virial theory[9] evolution of a laser beam
through a nonlinear medium is a variational problem characterized by the Hamiltonian

$$H = \int \mathcal{H} rdz$$

with

$$\mathcal{H} = \frac{1}{2k_0} [\nabla^2 A]^2 - \frac{1}{c^2} \int [\omega^2 p_0 e^{-\beta AA^*} - \left( \frac{\omega^2 p_0}{r_{ch}} + \frac{r^2}{\omega^2 p_{ch}} \right) e^{-\beta AA^*}] d(AA^*)$$

The basic idea of Hamiltonian formulation is then the selection of a trial function containing the physical
parameters of interest. This trial function characterizes the actual solution of the problem as close as
possible. The Hamiltonian formulation then recasts the original problem of solving a PDE into that of
solving a set of ODEs governing the evolution of these parameters. In the present analysis we assume
$A(r; z)$ takes the form of the function given by[10]

$$A(r, z) = E_{00} f \left( 1 + \frac{r^2}{qr_0 f^2} \right)^{\frac{q}{2}} e^{i\theta} \quad (2)$$

where, the parameter $f(z)$ can be referred to as dimensionless beam width parameter. The
phenomenological parameter $q$ is related to the deviation of irradiance over the beam cross section from
ideal Gaussian profile and hence, it is termed as deviation parameter. The function $\theta(z)$ is known as
axial phase of the laser beam.

By using Hamilton’s equations of motion by treating ($f, \frac{df}{dz}, \theta, \frac{d\theta}{dz}$) as generalized coordinates we get
following set of coupled equations for beam width and axial phase of the laser beam

$$\frac{d^2 f}{d\xi^2} = \left( 1 - \frac{1}{q} \right) \left( 1 + \frac{1}{q} \right) \frac{1}{f^3} - \left( 1 - \frac{1}{q} \right) \left( 1 - \frac{2}{q} \right) \left( f \frac{df}{d\xi} \right)^2 \quad (3)$$

$$\frac{d\theta}{dz} = \frac{1}{f^2} \left( 1 - \frac{1}{q} \right) - \frac{1}{2f^2} \left( 1 - \frac{1}{q} \right) \left( 1 - \frac{2}{q} \right) - \frac{1}{2} \left( 1 - \frac{1}{q} \right) I_4 \quad (4)$$

where

$$I_4 = \frac{\beta E_{00}^2}{c^2} \left( \frac{\omega_{p0}^2 r_0^2}{c^2} - I_1 + f^2 \frac{\omega_{pch}^2 r_0^2}{c^2} \frac{r_2^*}{r_{ch}^2} I_2 \right) + \frac{\omega_{p0}^2 r_0^2}{c^2} \frac{r_2^*}{r_{ch}^2} f^2 I_3$$
\[ I_1 = \int_0^\infty t \left( 1 + \frac{t}{q} \right)^{-2q-1} e^{-\frac{\beta E_0^2}{r^2} (1 + \frac{1}{q})^{-q}} \, dt \]
\[ I_2 = \int_0^\infty t^2 \left( 1 + \frac{t}{q} \right)^{-2q-1} e^{-\frac{\beta E_0^2}{r^2} (1 + \frac{1}{q})^{-q}} \, dt \]
\[ I_3 = \int_0^\infty t \left( 1 + \frac{t}{q} \right)^{-q} e^{-\frac{\beta E_0^2}{r^2} (1 + \frac{1}{q})^{-q}} \, dt \]
\[ I_4 = \int_0^\infty \left( 1 + \frac{1}{q} \right) \left\{ \frac{\omega_0^2 r_0^2}{c^2} e^{-\frac{\beta E_0^2}{r^2}} - \left[ \frac{\omega_0^2 r_0^2}{c^2} + \frac{\omega_{pch} r_0^2}{r_{ch}^2} \right] t e^{-\frac{\beta E_0^2}{r^2} (1 + \frac{1}{q})^{-q}} \right\} \, dt \]
\[ t = \frac{r^2}{r_0^2 f^2} \]
\[ \xi = \frac{k_0^2}{z} \]

3. Self-Trapped Mode

If while entering into the plasma channel, the laser beam possesses a plane wavefront i.e., if \( f = 1 \) and \( \frac{df}{d\xi} = 0 \) at \( \xi = 0 \), then the condition \( \frac{df}{d\xi} = 0 \) will maintain will maintain their values throughout the journey of the beam through the channel. Such mode of propagation, when there is no change in the beam width of the laser beam, is called self trapped mode or spatial optical soliton. Hence, for \( \frac{df}{d\xi} = \frac{d^2 f}{d\xi^2} = 0 \), eq.(3) gives the relation between dimensionless beam width \( r_e \) and the critical beam intensity \( \beta E_{00}^2 \) as

\[ r_e^2 = \frac{1}{I'_1} \left[ \frac{1}{1 + \frac{1}{q}} - \frac{\omega_0^2 r_0^2}{c^2} \right] \frac{1}{\beta E_{00}^2} - \frac{\omega_{pch} r_0^2}{r_{ch}^2} I'_2 \]

(10)

The \( \beta E_{00}^2 \) vs. re curve is known as critical curve that for a given values of \( q \) and \( \Delta n \), divides the \((\beta E_{00}^2, re)\) plane into three regions shown in fig.(3). Each region of \((\beta E_{00}^2, re)\) plane characterizes a different regime of propagation of the laser beam.

The laser beam for which the point \((\beta E_{00}^2, re)\) lies on the critical curve defined by eq.(10), will \( \frac{df}{d\xi} \) vanish at \( \xi = 0 \). This simply means that during the journey of the laser beam through the channel there will be no change in the curvature of the wavefront i.e., \( \frac{df}{d\xi} \) will remain constant and value of this constant will be equal to initial value, that we have taken to be zero. Hence, \( \frac{df}{d\xi} = \frac{d^2 f}{d\xi^2} = 0 \) at \( \xi = 0 \) indicates that \( \frac{df}{d\xi} = 0 \) for \( \xi > 0 \) also. Physically, this means that there will be no change in the beam width of the laser beam during its propagation. In this mode the laser beam is said to constitute a spatial soliton. Thus, the region of space lying on the critical curve corresponds to self channelling of the laser beam.

If the point \((\beta E_{00}^2, re)\) lies above the critical curve, then the initial value of \( \frac{d^2 f}{d\xi^2} \) will be positive and hence \( f \) will increase monotonically with distance. This mode of propagation is known as self broadening of the beam.

If the point \((\beta E_{00}^2, re)\) lies below the critical curve then initial value of \( \frac{d^2 f}{d\xi^2} \) will be negative and thus \( f \) will decrease with distance. This mode is known as self focusing of the laser beam. Thus, the region below the critical curve corresponds to self focusing.
4. Results and Discussion
In solving eqs. (3) and (4) it has been assumed that while entering into the channel the laser beam is collimated and is having plane wave front. Mathematically these conditions define the boundary conditions $f=1, \frac{df}{d\xi}=\theta = 0$ at $\xi=0$. In the present study eqs.(3) and (4) have been solved for a typical set of parameters: $\omega_0 = 1.78 \times 10^{14} \text{rad sec}^{-1}$, $r_0 = 15 \mu m$, $n_0 = 1.5 \times 10^{18} \text{cm}^{-3}$, $\Delta n = 10 \text{cm}^{-3}$, $E_{00} = 6 \times 10^9 \text{V m}^{-1}$ and $q=(3,4,\infty)$.

The graphical curves in fig.1 show the effect of deviation parameter $q$ on evolution of the beam radius along the length of plasma channel. It is observed that as the laser beam propagates down the plasma channel its radius changes harmonically with distance and the amplitude of these oscillations decreases with increase in the value of deviation parameter $q$.

The graphs in fig.2 depict the behavior of axial phase of the laser beam through the plasma channel. It can be seen that the axial phase decreases monotonically with distance by showing periodical steps at the positions of minimum beam radius i.e., as the beam propagates from one focus to another, its axial phase remains almost constant however, at the position of minimum radius its axial phase reduces abruptly. It is further observed that increase in the value of deviation parameter $q$ reduces the steepness of the steps indicating that the axial phase of the laser beams with higher value of $q$ reduces comparatively at a slower rate.

Fig.3 shows the effect of deviation parameter $q$ on self-trapping of the laser beam. It is observed that smaller is the initial radius of the laser beam higher is the critical intensity required for self-trapping. Also, with increase in deviation parameter $q$ the critical curves shift upwards indicating the difficulty in self-trapping the beams with higher value of $q$.

![Fig1: Effect of $q$ on beam width.](image1)

![Fig2: Effect of $q$ on axial phase.](image2)

![Figure 3. Effect of $q$ on self-trapping.](image3)
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