On the Efetov–Wegner terms by diagonalizing a Hermitian supermatrix

Mario Kieburg

Department of Physics and Astronomy, State University of New York, Stony Brook, NY 11794-3800, USA
E-mail: mario.kieburg@stonybrook.edu

Received 2 March 2011, in final form 19 May 2011
Published 17 June 2011
Online at stacks.iop.org/JPhysA/44/285210

Abstract
The diagonalization of Hermitian supermatrices is studied. Such a change of coordinates is inevitable if one wishes to find certain structures in random matrix theory. However, it still poses serious problems since until now the calculation of all Rothstein contributions, known as Efetov–Wegner terms in physics, was quite cumbersome. We derive the supermatrix Bessel function with all Efetov–Wegner terms for an arbitrary rotation invariant probability density function. As applications we consider representations of generating functions for Hermitian random matrices with and without an external field as integrals over eigenvalues of Hermitian supermatrices. All results are obtained with all Efetov–Wegner terms which were previously unknown in such an explicit and compact representation.

PACS numbers: 02.30.Cj, 02.30.Fn, 02.30.Px, 05.30.Ch, 05.30.—d, 05.45.Mt
Mathematics Subject Classification: 30G35, 58C50, 81Q50, 82D30

1. Introduction

Many eigenvalue correlations for random matrix ensembles such as matrix Green functions, the $k$-point correlation functions [1, 2], as well as the free energy [3] can be derived by generating functions. These functions are averages over ratios of characteristic polynomials for random matrices. They have diverse applications in physics as well as in many other fields. A few of them can be found in disordered physics [4], quantum chaos [5] and quantum chromodynamics [6].

A common approach to calculate generating functions is the supersymmetry method [1, 7–9]. Another approach is the orthogonal polynomial method [10]. In the supersymmetry method one maps integrals over ordinary matrices to integrals over supermatrices. Its advantage is the drastic reduction of the number of integration variables. Nevertheless there is a disadvantage. To date, it is not completely clear how to get the full structures
found with the orthogonal polynomial method for factorizing probability densities. For such probability densities the \( k \)-point correlation functions can be written as determinants and Pfaffians of certain kernels. This property was extended to the generating functions [11–14]. Unfortunately, the determinants and Pfaffians were not found for the full generating function after mapping the integrals over ordinary matrices to integrals over supermatrices. Only the determinantal expression of the \( k \)-point correlation function for rotation invariant Hermitian random matrix ensembles could be regained with the help of the supersymmetry method [1, 15]. Grönqvist et al studied this problem from another point of view in [11]. They started from the determinantal and Pfaffian expressions of the \( k \)-point correlation functions for Gaussian orthogonal, unitary and symplectic random matrix ensembles and showed that the kernels of the generating functions with two characteristic polynomials as integrals in superspace yield the known result.

Changing coordinates in superspaces causes serious problems since the Berezinian [16] playing the role of the Jacobian in superspace incorporates differential operators. These differential operators have no analog in ordinary space and are known in the mathematical literature as Rothstein’s vector fields [17]. For supermanifolds with boundaries, such differential operators yield boundary terms. In physics, these boundary terms are called Efetov–Wegner terms [4, 18–20]. They can be understood as corrections to the Berezinian without Rothstein’s vector fields. There were several attempts to explicitly calculate these vector fields or the corresponding Efetov–Wegner terms for diagonalizations of Hermitian supermatrices, but this was only successful for low-dimensional supermatrices, e.g. \((1 + 1) \times (1 + 1)\) Hermitian supermatrices [21–23]. For higher dimensions the calculation of all Efetov–Wegner terms becomes cumbersome. Also for other sets of supermatrices such changes in the coordinates were studied [24–26]. There are quite general formulas [17, 25, 27] for calculating the Efetov–Wegner terms but for concrete examples these formulas become cumbersome and one never came beyond the low-dimensional cases. As an example for such an approach we tried to derive all Efetov–Wegner terms by considering these terms as boundary terms resulting from partial integrations of differential operators which are equivalent to the integration over the Grassmann variables [20]. Though a quite compact form for a differential operator for the Hermitian supermatrices was found, we were unable to calculate all Efetov–Wegner terms. The order of such a differential operator is the number of pairs of Grassmann variables.

We derive the supermatrix Bessel function [28] for Hermitian supermatrices with all Efetov–Wegner terms. Thereby we apply a method called ‘supersymmetry without supersymmetry’ [13] on Hermitian matrix ensembles whose characteristic function factorizes in the eigenrepresentation of the random matrix. This approach uses determinantal structures of Berezinians without mapping into superspace. We combine this method with the supersymmetry method described in [1, 15]. The supermatrix Bessel function is then obtained by simply identifying the left-hand side with the right-hand side of the resulting matrix integrals. As a simple application we calculate the generating function of arbitrary, finite dimensional, rotation invariant Hermitian matrix ensembles with and without an external field. Hence, we generalize known results [1, 13, 15, 29] because they now contain all Efetov–Wegner terms which guarantee the correct normalizations.

Hermitian matrix ensembles in an external field serve as a simple model to describe quantum billiards with randomly distributed scatterers [30, 31]. In this model the random matrix plays the role of the scatterers and the kinetic part is the external field. Also in quantum chaos such ensembles were considered since they are ideal systems to study transitions between regularity and chaos [32, 33]. The hydrogen atom in a strong magnetic field is one popular example for such a transition [34]. In quantum chromodynamics a new random matrix
ensemble modeling the Dirac–Wilson operator was recently considered [35–37]. It consists of four blocks where those on the diagonal are Hermitian random matrices and those on the off-diagonal build the anti-Hermitian part. The coupling between these two parts is given by the lattice spacing. Out of convenience one often considers a Hermitized version of it where the original random matrix is multiplied by the $\gamma_5$ Dirac matrix. Then, the ensemble fits in the class we are considering.

Instead of using the supersymmetry method for calculating the correlations of Hermitian ensembles in an external field or intermediate ensembles one can also construct some evolution equations in combination with an initial value problem. Recently, Macedo-Junior and Macêdo [38] derived Fokker–Planck equations for a large class of these ensembles.

We organize the paper as follows. In section 2, we give an outline of our approach and introduce some basic quantities. Using the method ‘supersymmetry without supersymmetry’ we derive a determinantal structure for generating functions of rotation invariant Hermitian random matrices without an external field and with a factorizing characteristic function, in section 3. In section 4, we briefly present the results of the supersymmetry method. We also calculate the supermatrix representation of the generating function with one determinant in the numerator as in the denominator. For this generating function the corresponding supermatrix Bessel function with all Efetov–Wegner terms is known [39] since it only depends on a $(1+1) \times (1+1)$ Hermitian supermatrix. In section 5, we plug the result of section 4 into the result of section 3 and obtain the supermatrix Bessel function for arbitrary dimensional Hermitian supermatrices. Moreover, we discuss this result with respect to double Fourier transformations and, thus, Dirac distributions in superspace. In section 6, we first apply our result on arbitrary, rotation invariant Hermitian random matrices without an external field, and then in the presence of an external field. Details of the calculations are given in the appendices.

2. Outline

We want to generalize the well-known result for Gaussian superfunctions [20, 39]

$$
\int \exp \left( -\frac{1}{2} \text{Str} \sigma^2 + i \text{Str} \sigma \kappa \right) \, d[\sigma] \propto 1
$$

$$
- \frac{(\kappa_1 - \kappa_2)}{2\pi i} \int \frac{\exp \left( -\left[ s_1^2 + s_2^2 \right]/2 + i s_1 \kappa_1 + s_2 \kappa_2 \right)}{s_1 - i s_2} \, d[s].
$$

(1)

The matrix $\sigma$ is a Hermitian $(1+1) \times (1+1)$ supermatrix with the bosonic eigenvalue $s_1$, fermionic eigenvalue $i s_2$, and $\kappa = \text{diag}(\kappa_1, \kappa_2)$ is a diagonal $(1+1) \times (1+1)$ supermatrix. The first term on the right-hand side of equation (1) is the Efetov–Wegner term and the second one is the supergroup integral appearing after diagonalizing $\sigma = U \sigma U^\dagger$ with $U \in U(1/1)$. Indeed, this result can easily be generalized to higher dimensional supermatrices with Gaussian weights. However, considering Gaussian weights is not sufficient to generalize this result to arbitrary weights. Thus, we pursue another approach.

The main idea of our approach is the comparison of results for generating functions,

$$
Z_{k_1, k_2}^{(N)}(\kappa) = \int \text{Herm}^{(N)}(H) \frac{\prod_{j=1}^{k_1} \det(H - \kappa_j \mathbb{1}_N)}{\prod_{j=1}^{k_2} \det(H - \kappa_j \mathbb{1}_N)} \, d[H],
$$

(2)

obtained with and without supersymmetry. The matrix $\mathbb{1}_N$ is the $N$-dimensional unit matrix. The integration domain in equation (2) is the set Herm $(N)$ of $N \times N$ Hermitian matrices and the parameter $\kappa = \text{diag}(\kappa_{11}, \ldots, \kappa_{k_1,1}, \kappa_{12}, \ldots, \kappa_{k_2,2}) = \text{diag}(\kappa_1, \kappa_2)$ is chosen in such a way
that the integral exists. Usually they are chosen as real energies \( \epsilon \) with a small imaginary part \( i\epsilon \) and source variables \( J_j \), i.e. \( \epsilon J_j = x_j \mp J_j - i\epsilon \) [1, 15]. The differentiation with respect to \( J_j \) of \( \hat{Z}_{k_1/k_2}^{(N)} \) generates the matrix Green function which is intimately related to the \( k \)-point correlation functions [39]. The measure \( d[H] \) is defined as

\[
d[H] = \prod_{n=1}^{N} dH_{nn} \prod_{1 \leq m < n \leq N} d\text{Re} \ H_{mn} \, d\text{Im} \ H_{mn}
\]

and \( P^{(N)} \) is a probability density over \( N \times N \) Hermitian matrices.

In section 3, we show that \( \hat{Z}_{k_1/k_2}^{(N)} \) is a determinant:

\[
\hat{Z}_{k_1/k_2}^{(N)}(\kappa) \propto \det \left[ \begin{array}{c}
\hat{Z}_{1/1}^{(N)}(\kappa_{a1}, \kappa_{b2}) \\
\kappa_{a1} - \kappa_{b2}
\end{array} \right]
\]

if the Fourier transform

\[
\mathcal{F} P^{(N)}(\hat{H}) = \int_{\text{Herm}(N)} P^{(N)}(H) \exp[i \text{tr} \ H \hat{H}] \, d[H]
\]

factorizes in the eigenrepresentation of the Hermitian matrix \( \hat{H} \). These determinantal structures are similar to those found for generating functions with factorizing \( B^{(N)} \) in the eigenrepresentation of the Hermitian matrix \( H [12, 13] \).

Moreover, the integral of \( \hat{Z}_{k_1/k_2}^{(N)} \) can easily be mapped into superspace if the characteristic function \( \mathcal{F} P^{(N)} \) is rotation invariant, see [1, 15]. In this representation one does not integrate over ordinary matrices but over Wick-rotated Hermitian supermatrices [15, 20]. The integrand is almost rotation invariant apart from an exponential term which reflects a Fourier transformation in superspace. We aim at the full integrand with all Efetov–Wegner terms appearing by a diagonalization of a supermatrix. In particular, we want to find the distribution \( \hat{\varphi}_{k_1/k_2} \) which satisfies

\[
\int \exp[-i \text{Str} \ \kappa \rho] \, F(\rho) \, d[\rho] = \int \text{Ber}_{k_1/k_2}^{(2)}(r) \hat{\varphi}_{k_1/k_2}(-i r, \kappa) \, F(r) \, d[r]
\]

for an arbitrary rotation invariant superfunction \( F \). In equation (6) we diagonalize the \((k_1 + k_2) \times (k_1 + k_2)\) Hermitian supermatrix \( \rho \) to its eigenvalues \( r \). This diagonalization not only yields the Berezinian \( \text{Ber}_{k_1/k_2}^{(2)} \) but also the distribution \( \hat{\varphi}_{k_1/k_2} \). The distribution \( \hat{\varphi}_{k_1/k_2} \) is the integral over the supergroup \( U(k_1/k_2) \) and, additionally, all Efetov–Wegner terms. It is called the supermatrix Bessel function with Efetov–Wegner terms [20, 28].

We derive \( \hat{\varphi}_{k_1/k_2} \) in two steps. In the first step we combine the mapping into superspace with the determinant (4) for factorizing \( \mathcal{F} P^{(N)} \). This is a particular case of the identity (6) but it is sufficient to generalize its result to an arbitrary rotation invariant superfunction \( F \) in the second step.

The procedure described above incorporates determinants derived in [13, 40]. Without loss of generality let \( p \geq q \). Then, these determinants are

\[
\sqrt{\text{Ber}_{p/q}^{(2)}(\kappa)} = \prod_{1 \leq a \leq b \leq p} (\kappa_{a1} - \kappa_{b1}) \prod_{1 \leq a \leq b \leq q} (\kappa_{a2} - \kappa_{b2}) \prod_{a=1}^{p} \prod_{b=1}^{q} (\kappa_{a1} - \kappa_{b2})
\]

\[
\frac{1}{\sqrt{\text{Ber}_{p/q}^{(2)}(\kappa)}} = (-1)^{(p-1)/2} \det \left[ \begin{array}{c}
\kappa_{b1}^{p-1} & \kappa_{a1}^{q-p} \\
\kappa_{a2}^{p-1} & \kappa_{b2}^{q-p}
\end{array} \right]
\]

\[
\times \left| \begin{array}{c}
1 \leq a \leq p \\
1 \leq b \leq p
\end{array} \right|
\]
\[(9)\]

All three expressions find their applications in our discussion. For \( p = q \), we obtain the Cauchy determinant [39]

\[
\sqrt{\text{Ber}_{p/p}(\kappa)} = \det \begin{bmatrix} \frac{1}{2} \kappa a - \kappa b \end{bmatrix} \begin{bmatrix} 1 \leq a \leq q \end{bmatrix} \begin{bmatrix} 1 \leq b \leq p \end{bmatrix},
\]

whereas for \( q = 0 \) we have the Vandermonde determinant

\[
\sqrt{\text{Ber}_{p/0}(\kappa)} = \Delta_p(\kappa) = \det \begin{bmatrix} \frac{1}{2} \kappa a - \kappa b \end{bmatrix} \begin{bmatrix} 1 \leq a \leq p \end{bmatrix}.
\]

These determinants appear as square roots of Berezinians by diagonalizing Hermitian supermatrices. This also explains our notation for them. The upper index 2 results from the Dyson index \( \beta \) which is 2 throughout this work. Thus, we are consistent with our notation used in other articles [13–15, 41].

3. The determinantal structure and characteristic function

We consider the generating function (2). Let the probability density \( P(N) \) be rotation invariant. Then, the characteristic function (5) inherits this symmetry. We consider such characteristic functions which factorize in the eigenvalues \( \tilde{E} = \text{diag}(\tilde{E}_1, \ldots, \tilde{E}_N) \) of the matrix \( \tilde{H} \), i.e.

\[
\mathcal{F} P^{(N)}(\tilde{E}) = \prod_{j=1}^{N} f(\tilde{E}_j).
\]

Due to the normalization of \( P^{(N)} \), the function \( f : \mathbb{R} \rightarrow \mathbb{C} \) is unity at zero. We restrict ourselves to the case

\[
k_2 \leq k_1 \leq N.
\]

This case is sufficient for our purpose.

Let \( f \) be conveniently integrable and \( d = N + k_1 - k_2 \). In appendix A, we derive that the generating function in equation (2) has for a factorizing characteristic function (12) under condition (13) the determinantal expression

\[
Z_{k_1/k_2}^{(N)}(\kappa) = \frac{(-1)^{k_2(k_2 + 1)/2 + k_2}}{\sqrt{\text{Ber}_{k_2/k_2}(\kappa)}} \det \begin{bmatrix} Z_{1/0}^{(k_1)}(\kappa_{a_1}, \kappa_{b_1}) \end{bmatrix} \begin{bmatrix} 1 \leq a \leq k_1 \end{bmatrix} \begin{bmatrix} 1 \leq b \leq k_2 \end{bmatrix} \begin{bmatrix} 1 \leq a \leq k_1 \end{bmatrix} \begin{bmatrix} d+1 \leq b \leq N \end{bmatrix}
\]

for all \( \tilde{N} \in \{d, d+1, \ldots, N\} \). Note that equation (14) is indeed true for all values \( \tilde{N} \in \{d, d+1, \ldots, N\} \) because the determinant is skew symmetric. Hence, we may add any linear combination of the last \( k_1 - k_2 \) columns to the first \( k_2 \) columns.

With this result we take the first step towards obtaining all Efetov–Wegner terms for the diagonalization of a supermatrix. For non-normalized probability densities, we find equation (14) multiplied by \( f^\lambda(0) \) where \( \lambda = d(d + 1)/2 - (N - 1)N/2 - \tilde{N}k_2 \). We need the non-normalized version to analyze the integrals in section 5.
Equation (14) is also a very handy intermediate result. It yields a result of a simpler structure than the one for factorizing probability densities \[12, 13\]. Thus, all eigenvalue correlations for probability densities stemming from a characteristic function with the property (12) are determined by one and two point averages.

4. Integral representation in the superspace of the generating function and the Efetov–Wegner term of \(Z_{1/1}^{(N)}\)

In \([1, 9, 15, 41]\), it was shown that the generating function (2) can be mapped to an integral over supermatrices. Let \(\Sigma_{k_1/k_2}^{(0)}\) be the set of supermatrices with the form

\[
\rho = \begin{bmatrix} \rho_1 & \eta^\dagger \\ \eta & \rho_2 \end{bmatrix}.
\]

(15)

The boson–boson block \(\rho_1\) is an ordinary \(k_1 \times k_1\) positive definite Hermitian matrix and the fermion–fermion block \(\rho_2\) is an ordinary \(k_2 \times k_2\) Hermitian matrix. The off-diagonal block \(\eta\) comprises \(k_1 \times k_2\) independent Grassmann variables. We recall that \((\eta^\dagger)^\dagger = -\eta\) and the integration over one Grassmann variable is defined by

\[
\int \eta_{jm} \, d\eta_{jm} = \int \eta^\dagger_{jm} \, d\eta^\dagger_{jm} = \delta_{j1}, \quad j \in \{0, 1\}.
\]

(16)

We need the Wick-rotated set \(\Sigma_{k_1/k_2}^{(\psi)} = \tilde{\Pi}_\psi \Sigma_{k_1/k_2}^{(0)} \tilde{\Pi}_\psi\) to regularize the integrals below. The matrix \(\tilde{\Pi}_\psi = (\mathbb{1}_{k_1}, e^{i\psi/2} \mathbb{1}_{k_2})\) with \(\psi \in [0, \pi]\) is the generalized Wick rotation \([15, 20]\). We assume that a Wick rotation exists such that the characteristic function is a Schwartz function on the Wick-rotated real axis. Examples of such functions are given in \([15]\).

We define the supersymmetric extension \(\Phi\) of the characteristic function \(F_P\) with the help of a representation

\[
\mathcal{F} P_1(\text{tr} \, H^m, m \in \mathbb{N}) = \mathcal{F} P^{(N)}(H)
\]

as a function in matrix invariants:

\[
\Phi^{(k_1/k_2)}(\rho) = \mathcal{F} P_1(\text{Str} \, \rho^m, m \in \mathbb{N}).
\]

(18)

For a factorizing characteristic function (12) we want to consider only such extensions \(\Phi^{(k_1/k_2)}\) which factorize, too. In particular, we only use the extension

\[
\Phi^{(k_1/k_2)}(\rho) = \prod_{a=1}^{k_1} f(r_{a1}) \prod_{b=1}^{k_2} \frac{1}{f(e^{-i\psi} r_{b2})}.
\]

(19)

Indeed the extension of \(\mathcal{F} P^{(N)}\) to \(\Phi^{(k_1/k_2)}\) is not unique and, hence, the factorization property (19) does not hold for each extension, see \([15]\). However, for our purpose the choice (19) is sufficient at the moment. Later on we extend our result to arbitrary rotation invariant superfunctions such that it also applies to other rotation invariant extensions.

Let the signs of the imaginary parts for all \(\kappa_{j1}\) be negative. Assuming that \(\Phi\) is analytic in the fermion–fermion block \(\rho_2\), the generalized Hubbard–Stratonovich transformation \([1, 15, 41]\) shows that the integral equation (2) with condition (13) is

\[
Z_{k_1/k_2}^{(N)}(\kappa) = C_{k_1/k_2}^{(N)} \int_{\Sigma_{k_1/k_2}^{(\psi)}} \Phi^{(k_1/k_2)}(\hat{\rho}) \exp[-i \text{Str} \, \kappa \hat{\rho}] \det \rho_1 \prod_{j=1}^{k_2} \left( e^{-i\psi} \frac{\partial}{\partial r_{j2}} \right)^{d-1} \times e^{-i\psi} \delta(r_{j2}) \, d[\rho],
\]

(20)
where \( r_{j2} \) are the eigenvalues of \( \rho_2 \) and the matrix \( \hat{\rho} \) is given by
\[
\hat{\rho} = \begin{bmatrix}
\rho_1 & e^{i\psi/2} \eta^\dagger \\
e^{-i\psi/2} \eta & e^{i\psi} (\rho_2 + \eta \rho_1^{-1} \eta^\dagger)
\end{bmatrix}.
\] (21)

The constant is
\[
C_{k_1/k_2}^{(N)} = \frac{(-1)^{k_2(k_2+2N-1)/2}}{[(d-1)!]^k} \frac{\text{Vol}(U(N))}{\text{Vol}(U(d))} f^d(0).
\] (22)

Here, the volume of the unitary group \( U(N) \) is
\[
\text{Vol}(U(N)) = \prod_{j=1}^N \frac{2\pi^j}{(j-1)!}.
\] (23)

The term \( f(0) \) becomes unity if we consider normalized probability densities. The definition of the measure \( d[\rho] = d[\rho_1] d[\rho_2] d[\eta] \) is equal to the one in [41]:
\[
d[\rho_1] = \prod_{n=1}^{k_1} d\rho_{mn1} \prod_{1 \leq m < n \leq k_1} d\text{Re} \rho_{mn1} d\text{Im} \rho_{mn1},
\] (24)

\[
d[\rho_2] = e^{ik_2 \psi} \prod_{n=1}^{k_2} d\rho_{mn2} \prod_{1 \leq m < n \leq k_2} d\text{Re} \rho_{mn2} d\text{Im} \rho_{mn2},
\] (25)

\[
d[\eta] = e^{-ik_1 k_2 \psi} \prod_{n=1}^{k_1} \prod_{m=1}^{k_2} d\eta_{mn} d\eta^*_{mn}.
\] (26)

We use the conventional notation for the supertrace ‘Str’ and superdeterminant ‘Sdet’.

Let \( \rho \in \Sigma_{k_1/k_2}^{(N)} \) with the form (15). As long as the eigenvalues of the boson–boson block \( \rho_1 \) are pairwise different with those of the fermion–fermion block \( \rho_2 \), we may diagonalize the whole supermatrix \( \rho \) by an element \( U \in U(k_1/k_2) \). The corresponding diagonal eigenvalue matrix is \( \mathbf{r} = \text{diag} (r_{11}, \ldots, r_{k_1}, e^{i\psi} r_{k_1+1}, \ldots, e^{i\psi} r_{k_2}) = \text{diag} (r_1, e^{i\psi} r_2) \), i.e. \( \rho = U \mathbf{r} U^\dagger \). Due to Rothstein’s [17] vector field resulting from such a change of coordinates in the Berezin measure, we have
\[
d[\rho] \neq \text{Ber}_{k_1/k_2}^{(2)} (\mathbf{r}) d[\mathbf{r}] d\mu(U),
\] (27)

where \( d[\mathbf{r}] \) is the product of all eigenvalue differentials and \( d\mu(U) \) is the supersymmetric Haar measure of the unitary supergroup \( U(k_1/k_2) \). We have to consider some boundary terms since the Berezin integral is fundamentally connected with differential operators [17, 20, 42, 43].

We consider the generating function \( Z^{(N)}_{\mathbf{r}/1} \). Equation (20) is an integral over Dirac distributions. Hence, we cannot simply apply a Cauchy-like theorem [4, 18–20, 44, 45] but
for this integration domain we obtain an Efetov–Wegner term for \( Z^{(N)}_{1/1} \). Let the function \( 1/f \) be analytic at zero. Then, we have

\[
\frac{Z^{(N)}_{1/1}(\kappa)}{F^N(0)} = \left[ \frac{f(r_1)}{f(e^{i\psi}r_2)} \exp[-i \text{Str } \kappa r] \right]_{r=0}^r + \frac{(-1)^N}{(N-1)!} \int_{\mathbb{R} \times \mathbb{R}} \frac{\kappa_1 - \kappa_2}{r_1 - e^{i\psi}r_2} \frac{f(r_1)}{f(e^{i\psi}r_2)} \]

\[
\times \exp[-i \text{Str } \kappa r] r_1^N \left( e^{-i\psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) \, dr_1 \, dr_2 \] (28)

\[
= \frac{(-1)^N}{(N-1)!} \int_{\mathbb{R} \times \mathbb{R}} \left[ \frac{1}{r_1 - e^{i\psi}r_2} \left( \frac{\partial}{\partial r_1} + e^{-i\psi} \frac{\partial}{\partial r_2} \right) + \frac{\kappa_1 - \kappa_2}{r_1 - e^{i\psi}r_2} \right] \frac{f(r_1)}{f(e^{i\psi}r_2)} \]

\[
\times \exp[-i \text{Str } \kappa r] r_1^N \left( e^{-i\psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) \, dr_1 \, dr_2. \] (29)

In equation (29), we first integrate over \( r_2 \) and then over \( r_1 \). We derive this result in appendix B. The first summand on the right-hand side of equality (28) is 1 which is the Efetov–Wegner term. The second equality (29) is more convenient than equality (28) for the discussions in the ensuing section.

## 5. Supermatrix Bessel function with all Efetov–Wegner terms

Let supermatrices in \( \tilde{\Sigma}^{(\phi)}_{k_1/k_2} \) be similar to those in \( \Sigma^{(\psi)}_{k_1/k_2} \) without the positive definiteness of the boson–boson block. We want to find the distribution \( \tilde{\varphi}_{k_1/k_2} \) which satisfies

\[
\int_{\mathbb{C}^{(\phi)}_{k_1/k_2}} \exp[-i \text{Str } \kappa \rho] F(\rho) \, d[\rho] = \int_{\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}} \text{Ber}^{(2)}_{k_1/k_2}(r) \tilde{\varphi}_{k_1/k_2}(-ir, \kappa) F(r) \, d[r], \] (30)

for an arbitrary sufficiently integrable, rotation invariant superfunction \( F \) analytic at zero. Recognizing that the integral expression (20) includes the supersymmetric Ingham–Siegel integral \([1, 15, 41]\), the generating function is apart from a shift in the fermion–fermion block and analyticity, a particular example of this type of integral.

First, we want to derive \( \tilde{\varphi}_{k_1/k_2} \) in equation (30) for factorizing superfunctions (19) and, then, extend this result to arbitrary \( F \). In appendix C, we show that the distribution \( \tilde{\varphi}_{k_1/k_2} \) defined by

\[
\int_{\mathbb{C}^{(\phi)}_{k_1/k_2}} F(\hat{\rho}) \exp[-i \text{Str } \kappa \hat{\rho}] \det^{d(\rho)} \prod_{j=1}^{k_1} \left( e^{-i\psi} \frac{\partial}{\partial r_{j2}} \right)^{d-1} \, e^{-i\psi} \delta(r_{j2}) \, d[\rho]

\[
= \int_{\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}} \text{Ber}^{(2)}_{k_1/k_2}(r) \tilde{\varphi}_{k_1/k_2}(-ir, \kappa) F(r) \det^{d(r)} \prod_{j=1}^{k_1} \left( e^{-i\psi} \frac{\partial}{\partial r_{j2}} \right)^{d-1} \, e^{-i\psi} \delta(r_{j2}) \, d[r] \] (31)
is given by

\[
\hat{\phi}_{k_1/k_2}(-ir, \kappa) = \frac{(-1)^{(k_1+k_2)(k_1+k_2-1)/2}(i\pi)^{(k_2-k_1)/2-(k_1+k_2)/2}}{2^{2k_1k_2}k_1!k_2!\sqrt{\text{Ber}_{k_1/k_2}^{(2)}(\kappa)\text{Ber}_{k_1/k_2}^{(2)}(r)}}
\times \sum_{\alpha_1 \in \Theta(k_1)} \sum_{\alpha_2 \in \Theta(k_2)} \exp\left[ -ir\sum_{j=1}^{k_1} k_1\text{Re}(\alpha_1(j)) + i\theta\sum_{j=1}^{k_2} k_2\text{Re}(\alpha_2(j)) \right]
\times \det\left[ \left\{ \text{e}^{-i\theta}\frac{\partial}{\partial \text{Re}(\alpha_1)} + \text{e}^{i\theta}\frac{\partial}{\partial \text{Re}(\alpha_2)} \right\} \chi_{\alpha_1,\alpha_2} \right]
\] (32)

for an arbitrary sufficiently integrable, rotation invariant superfunction \( F \) analytic at zero and factorizing in the eigenrepresentation of \( \rho \). The set of permutations over \( k \) elements is \( \Theta(k) \). The derivation of equation (32) incorporates equations (14) and (29).

In appendix D, we take the next step and generalize equation (32) to arbitrary rotation invariant superfunctions without the additional Dirac distributions in the integral, cf equation (20). The distribution \( \hat{\phi}_{k_1/k_2} \) defined by equation (30) is indeed the one defined in equation (32) for an arbitrary sufficiently integrable, rotation invariant superfunction \( F(\rho) \) analytic at zero. For such superfunctions and generalized Wick rotations \( \psi \in ]0, \pi[ \) this distribution alternatively has the form

\[
\hat{\phi}_{k_1/k_2}(-ir, \kappa) = \frac{(-1)^{(k_1+k_2)(k_1+k_2-1)/2}(i\pi)^{(k_2-k_1)/2-(k_1+k_2)/2}}{2^{2k_1k_2}k_1!k_2!\sqrt{\text{Ber}_{k_1/k_2}^{(2)}(\kappa)\text{Ber}_{k_1/k_2}^{(2)}(r)}}
\times \sum_{\alpha_1 \in \Theta(k_1)} \sum_{\alpha_2 \in \Theta(k_2)} \det\left[ \left\{ \text{e}^{-i\theta}\delta(\text{Re}(\alpha_1)) + \text{e}^{i\theta}\delta(\text{Re}(\alpha_2)) \right\} \chi_{\alpha_1,\alpha_2} \right]
\] (33)

with the distribution

\[
\chi(x) = \begin{cases} 
0, & x = 0, \\
1, & \text{else}.
\end{cases} 
\] (34)

Equation (33) is true because of the Cauchy-like theorem for \( (1 + 1) \times (1 + 1) \) Hermitian supermatrices, see [19, 20, 45]. One has to be careful on which half of the complex plane the general Wick rotation is lying. If \( \psi \in ]\pi, 2\pi[ \), then the minus sign changes to a plus in front of the Dirac distributions.

We emphasize that the distribution \( \hat{\phi}_{k_1/k_2}(r, \kappa) \) is not symmetric in exchanging its arguments \( r \) and \( \kappa \). Apart from the characteristic function \( \chi \), such a symmetry exists for the
supermatrix Bessel functions [20, 39, 46] which is \( \hat{\varphi}_{k_1/k_2}(r, \kappa) \) without the Dirac distributions, i.e.

\[
\varphi_{k_1/k_2}(-ir, \kappa) = \frac{(-1)^{(k_1+k_2+1)/2}}{2k_1k_2!} \sqrt{\text{Ber}_{k_1/k_2}^{(2)}(\kappa) \text{Ber}_{k_1/k_2}^{(2)}(r)} \times \sum_{a_1 \in \Theta(k_1)} \sum_{a_2 \in \Theta(k_2)} \det \left[ \exp(-t \kappa b_1 r_{ao_1(b_1)} + te^{i\psi} k_a 2 r_{ao_2(a_2)}) \right]^{1 \leq a \leq k_1} \chi(k_{b_1} - k_{a_2}) \bigg|_{1 \leq b \leq k_1}
\]

\[
= \frac{(-1)^{(k_1+k_2+1)/2}}{2k_1k_2!} \prod_{1 \leq a \leq k_1} \chi(k_{a_1} - k_{b_2}) \times \det \left[ \exp(-t \kappa a_1 r_{ao_1(b_1)}) \right]_{1 \leq a \leq k_1} \det \left[ \exp(te^{i\psi} k_a r_{ao_2(a_2)}) \right]_{1 \leq a \leq k_2}.
\]

The asymmetry \( \hat{\varphi}_{k_1/k_2}(r, \kappa) \neq \hat{\varphi}_{k_1/k_2}(\kappa, r) \) is mainly due to the diagonalization of \( \rho \) to \( r \), whereas the supermatrix \( \kappa \) is already diagonal. The characteristic function \( \chi \) in equations (33) and (35) is crucial because of the commutator

\[
\left[ \frac{\partial}{\partial r_{ao_1(b_1)}}, e^{-r \psi} \frac{\partial}{\partial r_{ao_2(a_2)}} \right] \exp(-t \kappa b_1 r_{ao_1(b_1)} + te^{i\psi} k_a 2 r_{ao_2(a_2)}) = -t (k_{b_1} - k_{a_2}) \exp(-t \kappa b_1 r_{ao_1(b_1)} + te^{i\psi} k_a 2 r_{ao_2(a_2)}) \chi(k_{b_1} - k_{a_2}).
\]

Indeed the set which is cut out by \( \chi \) is a set of measure zero and does not play any role when one integrates \( \varphi_{k_1/k_2}(-ir, \kappa) \) or \( \hat{\varphi}_{k_1/k_2}(-ir, \kappa) \) over \( \kappa \) with a function continuous around this set. However, it becomes important for the integral

\[
\int_{\mathbb{R}^{k_1 + k_2}} \varphi_{k_1/k_2}(is, r) \hat{\varphi}_{k_1/k_2}(-ir, \kappa) \text{Ber}_{k_1/k_2}^{(2)}(r) \text{Ber}_{k_1/k_2}^{(2)}(s) \, dr \, ds = \frac{\pi^{(k_1+k_2)^2}}{2^{k_1+k_2}k_1!k_2!} \prod_{1 \leq a \leq k_1} \chi(k_{a_1} - s_{b_2}) \bigg|_{1 \leq b \leq k_2} \det \left[ e^{i\psi} \delta(k_{a_1} - s_{b_2}) \right]_{1 \leq a \leq k_1} \bigg|_{1 \leq b \leq k_2}.
\]

where \( s = \text{diag}(s_{11}, \ldots, s_{k_1}, e^{-i\psi}s_{12}, \ldots, e^{-i\psi}s_{k_2}) \) and \( \kappa = \text{diag}(\kappa_{11}, \ldots, \kappa_{k_1}, e^{-i\psi}\kappa_{12}, \ldots, e^{-i\psi}\kappa_{k_2}) \) with \( s_{ab}, \kappa_{ab} \in \mathbb{R} \). This result is the correct one for the supermatrix Bessel function. The difference to other results [20, 39, 46] is the distribution \( \chi \) which guarantees that the Dirac distribution (37) in the eigenvalues of a Hermitian supermatrix \( \sigma \) vanishes if a bosonic eigenvalue of \( \kappa \) equals a fermionic one. On the first sight, this would also happen if we omit \( \chi \) due to the prefactor \( \sqrt{\text{Ber}_{k_1/k_2}^{(2)}(\kappa)} \) in equation (37) which vanishes on this set. Yet after applying the distribution (37) on functions over the set of diagonalized Hermitian
supermatrices comprising the Berezinian $\text{Ber}^{(2)}_{k_1/k_2}(s)$ in the measure, this term cancels out, i.e.

$$F(\kappa) = \int_{\mathbb{R}^{k_1+k_2}} F(s) \frac{\det[\delta(k_a - s_{b1})]_{1 \leq a, b \leq k_1} \det[\epsilon_{a1} \delta(k_a - s_{b2})]_{1 \leq a, b \leq k_2}}{k_1!k_2! \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(\kappa)}} \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(s)} \, ds

\neq F(\kappa) \prod_{1 \leq a \leq k_1, 1 \leq b \leq k_2} \chi(\kappa_a - e^{-i\theta} \kappa_b_2).$$

(38)

The reason why equation (37) has to vanish on this set becomes clear when we interpret equation (37) as an integral over the supergroup $U(k_1/k_2)$, i.e.

$$\int_{\mathbb{R}^{k_1+k_2}} \psi_{k_1/k_2}(s, r) \psi_{k_1/k_2}(-tr, \kappa) \text{Ber}^{(2)}_{k_1/k_2}(r) \, dr \sim \int_{U(k_1/k_2)} \delta(UsU^\dagger - \kappa) \, d\mu(U).$$

(39)

The measure $d\mu(U)$ is the Haar measure on $U(k_1/k_2)$ and the Dirac distribution is defined by two Fourier transformations

$$\delta(UsU^\dagger - \kappa) \sim \int \exp[i \text{Str} \rho(UsU^\dagger - \kappa)] \, d[\rho].$$

(40)

The Haar measure $d\mu$ of the supergroup $U(k_1/k_2)$ cannot be normalized as can be done for the ordinary unitary groups since the volume of $U(k_1/k_2)$ is zero for $k_1k_2 \neq 0$. This is also the reason why equation (37) has to vanish if one bosonic eigenvalue of $\kappa$ equals a fermionic one. Then, the integral (39) is rotation invariant under the subgroup $U(1/1)$ which has zero volume, too. This cannot be achieved without the distribution $\chi$ as was done in the well-known literature [20, 39, 46]. Interestingly, the replacement of $\psi_{k_1/k_2}$ by $\tilde{\psi}_{k_1/k_2}$ in equation (37) yields the full Dirac distribution

$$\int_{\mathbb{R}^{k_1+k_2}} \tilde{\psi}_{k_1/k_2}(s, r) \tilde{\psi}_{k_1/k_2}(-tr, \kappa) \text{Ber}^{(2)}_{k_1/k_2}(r) \, dr

= \frac{\pi^{(k_1-k_2)^2} \det[\delta(k_a - s_{b1})]_{1 \leq a, b \leq k_1} \det[\epsilon_{a1} \delta(k_a - s_{b2})]_{1 \leq a, b \leq k_2}}{2^{2k_1-k_1-k_2} k_1!k_2! \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(\kappa)}} \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(s)}.$$ 

(41)

which is similar to the one in equation (37). Equation (41) is derived in appendix E.

We want to finish this section with a remark about the relation of the result for the supermatrix Bessel function, see equation (33), and the differential operator derived by the author in an earlier work [20]. This differential operator is defined by

$$D_r^{(k_1+k_2)} F(r) = \int F(\rho) \, d[\eta]$$

(42)

for an arbitrary sufficiently integrable superfunction $F$ on the $(k_1+k_2) \times (k_1+k_2)$ supermatrices invariant under $U(k_1/k_2)$. It has the form

$$D_r^{(k_1+k_2)} = \frac{1}{(k_1k_2)! (4\pi)^{k_1+k_2}} \text{Str}_{k_1} \left( \epsilon_{a1} r_{a1} \right) \Delta_{k_1} \left( r_{a1} \right) \Delta_{k_2} \left( r_{a2} \right) \sum_{n=0}^{k_1+k_2} \binom{k_1k_2}{n} \left( \text{Str} \left( \frac{\partial^2}{\partial r^2} \right) \right)^{k_1k_2-n}

\times \prod_{1 \leq a \leq k_1, 1 \leq b \leq k_2} (r_{a1} - e^{\epsilon_{a1} r_{a2}}) \left( -\text{Str} \left( \frac{\partial^2}{\partial r^2} \right) \right)^n \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(r)},$$

(43)

where we define

$$\text{Str} \frac{\partial^2}{\partial r^2} = \sum_{a=1}^{k_1} \frac{\partial^2}{\partial r_{a1}^2} - e^{-2\epsilon_{a1}} \sum_{b=1}^{k_2} \frac{\partial^2}{\partial r_{b2}^2}.$$ 

(44)
Due to equation (42) the differential operator $D^{(k_1k_2)}_r$ is equivalent to the integration over all Grassmann variables of the supermatrix $\rho$.

The comparison of equations (42) and (43) with equations (30) and (33) for an arbitrary sufficiently integrable, rotation invariant superfunction $F$ and arbitrary diagonal supermatrix $\kappa$ yields

$$
D^{(k_1k_2)}_r = \frac{1}{(2\pi)^{k_1k_2}} \frac{1}{\Delta_{k_1}(r_1) \Delta_{k_2}(e^{ip}r_2)} \prod_{1 \leq a \leq k_1} \prod_{1 \leq b \leq k_2} \left( \frac{\partial}{\partial r_{a1}} + e^{-ip} \frac{\partial}{\partial r_{b2}} \right) \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(r)}.
$$

Thus, we have found a quite compact form for $D^{(k_1k_2)}_r$ which is easier to deal with than the one in equation (43).

6. Some applications for Hermitian matrix ensembles

In random matrix theory, generating functions such as

$$
Z_k^{(N)}(\kappa, \alpha H_0) = \int_{\text{Herm}(N)} p^{(N)}(H) \prod_{j=1}^{k} \frac{\det(H + \alpha H_0 - \kappa_{j1} I_N)}{\det(H + \alpha H_0 - \kappa_{j1} I_N)} d[H]
$$

are of paramount importance since they model Hermitian random matrices in external potentials [29, 47] or intermediate random matrix ensembles [30–33, 48, 49]. The matrix $H_0$ is an $N \times N$ Hermitian matrix and can be arbitrarily chosen or drawn from another random matrix ensemble. The external parameter $\alpha \in \mathbb{R}$ is the coupling constant between the two matrices $H$ and $H_0$ and yields the generating function (2) for $\alpha = 0$, i.e. $Z_k^{(N)}(\kappa, 0) = Z_{k/1}^{(N)}(\kappa)$.

The variables $\kappa_{j1}$ have to have an imaginary increment to guarantee the convergence of the integral, i.e. $\kappa_{j1} = \chi_{j1} - J_j - \text{i} \varepsilon$.

In subsection 6.1, we consider the generating function (46) with $\lambda = 0$. We will use the mapping of this integral to a representation in superspace which is shown in a previous work by the authors [13] and diagonalize the supermatrix. In a formalism similar to the case $\alpha = 0$ we will treat the more general case $\alpha \neq 0$ in subsection 6.2.

6.1. Hermitian matrix ensembles without an external field ($\alpha = 0$)

Omitting the index $N$ in $p^{(N)}$ we consider a normalized rotation invariant probability density $P$ with respect to $N \times N$ Hermitian random matrices.

Following the derivations made in [1, 15] we have to calculate the characteristic function $\mathcal{F}P$, see equation (5). Assuming that this can be done we recall the rotation invariance of $\mathcal{F}P$. This allows us to choose a representation of $\mathcal{F}P$ as a function in a finite number of matrix invariants, i.e. $\mathcal{F}P(H) = \mathcal{F}P(\text{tr } H, \ldots, \text{tr } H^N)$. A straightforward supersymmetric extension $\Phi$ of the characteristic function is the one in equation (17) and its Fourier transform is

$$
\mathcal{F}\Phi(\sigma) = 2^{2k(k-1)} \int_{\text{Sym}(\alpha)} \Phi(\rho) \exp(-\text{i} \text{Str } \rho \sigma) d[\rho].
$$

The Fourier transform is denoted by $Q$ in [1]. The supermatrices $\rho$ and $\sigma$ are Wick rotated with the phases $e^{ip}$ and $e^{-ip}$, respectively.

The supersymmetric integral for $Z_k^{(N)}(\kappa, \alpha H_0)$ is [1, 15]

$$
Z_k^{(N)}(\kappa, \alpha H_0) = \int_{\text{Sym}(\alpha)} \mathcal{F}\Phi(\sigma) \text{Sdet}^{-1}(\sigma \otimes I_N + \alpha I_{k+1} \otimes H_0 - \kappa) d[\sigma].
$$
Setting the coupling constant $\alpha$ to zero we also find

$$Z^{(N)}_k(\kappa) = 2^{2(k-1)} \int_{\Sigma^{(k)}_{\alpha E_0}} \mathcal{F}(\sigma) \exp[i \text{Str } \rho(\sigma - \kappa)] I_N(\rho) \, d[\sigma] \, d[\rho],$$

where the supersymmetric Ingham–Siegel integral is [1]

$$I^1_{\kappa} (\rho) = \int_{\Sigma^{(k)}_{\alpha E_0}} \text{Sdet}^{-N}(\sigma + i \varepsilon \mathbb{1}_{N(k+1)}) \exp(-i \text{Str } \rho \sigma + \varepsilon \text{Str } \rho) \, d[\sigma]$$

is the Heavyside distribution and $r_{j1}$ and $e^{i \psi} r_{j2}$ are the bosonic and fermionic eigenvalues of $\rho$, respectively. In appendix F, we perform the integration (50) with help of the results in section 5 and find

$$Z^{(N)}_k(\kappa) = \frac{\pi^{k-1}}{2^{2(k-1)}} \int_{\mathbb{R}^{2k}} \frac{d[x]}{\text{Ber}_{k/k}(\kappa)} \mathcal{F}(x) \det \left[ \frac{e^{i \psi} \delta(s_{b2}) \delta(s_{a2})}{\kappa - \kappa_{b2}} \right]^{N} + \frac{N}{2\pi} \left( e^{-i \psi} (\kappa - \kappa_{b2})^{N-1} \right) \chi(\kappa - \kappa_{b2}) 1_{1 \leq a,b \leq k}.$$

(51)

The first term in the determinant is the Efetov–Wegner term, whereas the second term can be understood as integrals over supergroups.

### 6.2. Hermitian matrix ensembles in the presence of an external field ($\alpha \neq 0$)

For Hermitian matrix ensembles in an external field it is convenient to consider the integral representation

$$Z^{(N)}_k(\kappa, \alpha H_0) = 2^{2(k-1)} \int_{\Sigma^{(k)}_{\alpha E_0}} \mathcal{F}(\sigma) \exp[-i \text{Str } \kappa \rho] \left[ \int_{\Sigma^{(k)}_{\alpha E_0}} \exp(-i \text{Str } \rho \sigma + \varepsilon \text{Str } \rho) \right]$$

$$\times \text{Sdet}^{-1}(\sigma \otimes I_N + \alpha \mathbb{1}_{N(k+1)} \otimes H_0 + i \varepsilon \mathbb{1}_{N(k+1)}) \, d[\sigma] \, d[\rho]$$

(52)

for the generating function, see equations (46) and (48). In appendix G we integrate this representation in two steps and obtain

$$Z^{(N)}_k(\kappa, \alpha H_0) = \frac{(-1)^{k(k-1)/2}}{\Delta_N(\alpha E_0) \text{Ber}_{k/k}(\kappa)}$$

$$\times \int_{\mathbb{R}^{2k}} \det \left[ B_1(r_{b1}, r_{a2}, \kappa_{b1}, \kappa_{a2}) \right]_{1 \leq a,b \leq k} \left[ B_2(r_{a2}, \kappa_{a2}) \right]_{1 \leq a \leq k}$$

$$\times \Phi(r) \, d[r].$$

(53)

where

$$B_1(r_{b1}, r_{a2}, \kappa_{b1}, \kappa_{a2}) = \frac{e^{-i \psi} \delta(r_{b1}) \delta(r_{a2})}{\kappa_{b1} - \kappa_{a2}},$$

$$+ \frac{I^{(N)}_{\kappa}(r_{b1}, e^{i \psi} r_{a2})}{2\pi (r_{b1} - e^{i \psi} r_{a2})} \exp(-i \kappa_{b1} r_{b1} + i e^{i \psi} \kappa_{a2} r_{a2}) \chi(\kappa_{b1} - \kappa_{a2})$$

(54)
\[ B_{b2}(r_{a2}, \kappa_{a2}) = \exp \left( ir \psi \kappa_{a2} r_{a2} \right) \left( -te^{-i \psi} \frac{\partial}{\partial r_{a2}} \right)^{b-1} e^{-i \psi \delta(r_{a2})} \prod_{j=1}^{k} \chi(\kappa_{j1} - \kappa_{a2}), \] (55)

\[ B_3(r_{b1}, \kappa_{b1}, \alpha E_a^{(0)}) = -t \exp \left( -i \kappa_{b1} r_{b1} \right) \Theta(r_{b1}) \sum_{n=N}^{\infty} \frac{(i \alpha E_a^{(0)} r_{b1})^{n} \prod_{j=1}^{k} \chi(\kappa_{b1} - e^{i \psi} \kappa_{j2})}{n!}. \] (56)

The case \( \alpha = 0 \) can be easily deduced from the result (53),

\[ Z_k^{(N)}(k) = \left( -1 \right)^{k(k-1)/2} \sqrt{\text{Ber}_{k/k}(k)} \int_{\mathbb{R}^{2k}} \det \left[ \begin{array}{c} e^{-i \psi \delta(r_{b1})} \delta(r_{a2}) \frac{1}{\kappa_{b1} - \kappa_{a2}} + t \frac{I_{1}^{(N)}(r_{b1}, e^{i \psi} r_{a2})}{2 \pi (r_{b1} - e^{i \psi} r_{a2})} \chi(\kappa_{b1} - \kappa_{a2}) \end{array} \right] \prod_{1 \leq n, b \leq k} \delta(r_{b1} - r_{b1}). \] (57)

Again we are able to distinguish the Efetov–Wegner terms from those terms corresponding to supergroup integrals. In the determinant of equation (57) and in the left-upper block of equation (53), see equation (54), the Dirac distributions are the contributions from the Efetov–Wegner terms. When we expand the determinants in the Dirac distributions we obtain the leading terms of \( Z_k^{(N)} \), \( Z_{k-1}^{(N)} \), ..., \( Z_0^{(N)} \) which are exactly those found in [1, 15]. Thus, we have found an expression which can be understood as a generator for all generating functions \( Z_k^{(N)} \).

The external matrix \( H_0 \) can also be drawn from another random matrix ensemble as it was done in [30–33, 48, 49]. However, we do not perform the calculation, here, since it is straightforward to those in an application of [13].

7. Remarks and conclusions

We derived the supermatrix Bessel function with all Efetov–Wegner terms for Hermitian supermatrices of arbitrary dimensions. We arrived at an expression from which one can easily deduce what the Efetov–Wegner terms are and which terms result from supergroup integrals. With this result we showed that the completeness and orthogonality relation for the supermatrix Bessel function without Efetov–Wegner terms slightly differs from the formerly assumed one [32]. It has to be zero on a set of measure zero and, thus, does not matter for smooth integrands but plays an important role if the integrand has singularities on this set.

We applied the supermatrix Bessel function with Efetov–Wegner terms to arbitrary, rotation invariant Hermitian random matrix ensembles with and without an external field. The already known leading terms [1, 13, 15, 29–33, 47–49] were obtained plus all Efetov–Wegner terms. The correction terms were previously unknown in this explicit form and yield new insights into the supersymmetric representation of the generating functions. In particular, the Efetov–Wegner terms become important for the matrix Green functions.

We also found an integral identity for the generating functions whose integrand can be easily expanded in the Efetov–Wegner terms. In such an expansion one obtains correlation functions related to \( k \)-point correlation functions which are of lower order than those corresponding to the originally considered generating function. Thus, it reflects the relation of Mehta’s definition [10] for the \( k \)-point correlation function and the one commonly used in the supersymmetry method [1] which was explained in [50].

We expect that similar results may also be derived for other supermatrices, e.g. diagonalization of complex supermatrices [51], as they would appear in non-Hermitian matrix ensembles [52] or as they emerge in the color–flavor transformation for circular ensembles [53]. Nevertheless we guess that the knowledge about the supergroup integrals as well as
about the ordinary group integrals is crucial. We are only able to obtain these compact results due to this knowledge.

**Acknowledgments**

I thank Thomas Guhr for fruitful discussions, Taro Nagao for helpful comments about applications as well as Martin Zirnbauer by pointing out the diploma thesis by Ralf Bundschuh. I acknowledge support from the Deutsche Forschungsgemeinschaft within Sonderforschungsbereich Transregio 12 ‘Symmetries and Universality in Mesoscopic Systems’.

**Appendix A. Derivation of equation (14)**

We plug the characteristic function (5) in equation (2) and diagonalize \( \tilde{H} \). This yields

\[
Z^{(N)}_{k_1/k_2}(\kappa) = \frac{V_N}{2^N \pi^N} \int_{Herm(N)} \int_{\mathbb{R}^N} \exp \left[ -i \text{tr} \tilde{H} \tilde{E} \right] \frac{\prod_{j=1}^{k_1} \det(H - \kappa j_2 I_N)}{\prod_{j=1}^{k_2} \det(H - \kappa j_1 I_N)} \times \prod_{j=1}^{N} f(\tilde{E}_j) \Delta_{N}^2(\tilde{E}) \ d[\tilde{E}] \ d[H].
\]  

(A.1)

where the constant is

\[
V_N = \frac{1}{N!} \prod_{j=1}^{N} \pi \frac{1}{(j - 1)!}.
\]  

(A.2)

The diagonalization of \( H \) yields the matrix Bessel function \([54, 55]\) according to the unitary group \( U(N) \):

\[
\varphi_N(E, \tilde{E}) = \int_{U(N)} \exp[-i \text{tr} EU \tilde{E} U^\dagger] \ d\mu(U) = \prod_{j=1}^{N} t^{j-1}(j-1)! \det[\exp(-iE_a \tilde{E}_b)]_{1 \leq a,b \leq N} \frac{\Delta_N(\tilde{E}) \Delta_N(E)}{\Delta_N(E) \Delta_N(\til{E})}. 
\]

(A.3)

The measure \( d\mu \) is the normalized Haar measure. Thus, we find

\[
Z^{(N)}_{k_1/k_2}(\kappa) = \frac{t^{N(N-1)/2}}{(2\pi)^N N!} \prod_{j=0}^{N} j! \int_{\mathbb{R}^{2N}} \det[\exp(-iE_a \til{E}_b)]_{1 \leq a,b \leq N} \prod_{a=1}^{N} f(\tilde{E}_a) \prod_{b=1}^{k_1} (E_a - \kappa b_1) \Delta_N(\til{E}) \Delta_N(E) \ d[\til{E}] \ d[E].
\]

(A.4)

Please note that we integrate first over the variables \( \tilde{E} \) and then over \( E \). Here, one can easily check that the normalization is \( Z^{(N)}_{k_1/k_2}(0) = f^N(0) \). Since determinants are skew-symmetric, we first expand the Vandermonde determinant \( \Delta_N(\til{E}) \) and then the determinant of the exponential functions. We have

\[
Z^{(N)}_{k_1/k_2}(\kappa) = \frac{(-1)^{N(N-1)/2}}{(2\pi)^N} \prod_{j=1}^{N} (j-1)! \int_{\mathbb{R}^{2N}} \prod_{a=1}^{N} f(\til{E}_a) \exp(-iE_a \til{E}_a) \til{E}_a^{k_1-1} \prod_{b=1}^{k_2} (E_a - \kappa b_2) \prod_{b=1}^{k_1} (E_a - \kappa b_1) \Delta_N(E) \ d[\til{E}] \ d[E].
\]

(A.5)
Following the ideas in [13], we extend the integrand by a square root Berezinian and find with the help of equation (9) the determinant

\[
Z_{N/k_1/k_2}(k) = \frac{(-1)^{k_1(k_2-1)/2+k_2+1}k_1 N(N-1)/2}{(2\pi)^N \prod_{j=1}^N (j-1)!} \frac{1}{\sqrt{\text{Ber}^{(2)}_{k_1/k_2}(k)}}
\times \det \left\{ \left[ \begin{array}{c} \int_{\mathbb{R}^2} f(E_2) \frac{E_1^b - 1}{E_2} \exp[-t E_2 E_1] \delta(E_1) \mathrm{d}[E] \\ \int_{\mathbb{R}^2} f(E_2) E_1^{a-1} \delta(E) \delta(E_1) \mathrm{d}[E] \end{array} \right]_{1 \leq a < k_1 \atop 1 \leq b \leq k_2} \right\}.
\]

We define the sign of the imaginary parts of \(\kappa_1\) by

\[
L_j = \frac{\text{Im} \kappa_1}{|\text{Im} \kappa_1|}.
\]

Integrating over \(E_1\), equation (A.6) reads

\[
Z_{N/k_1/k_2}(k) = \frac{(-1)^{k_1(k_2-1)/2+k_2+1}k_1 N(N-1)/2}{(2\pi)^N \prod_{j=1}^N (j-1)!} \frac{1}{\sqrt{\text{Ber}^{(2)}_{k_1/k_2}(k)}}
\times \det \left\{ \left[ \begin{array}{c} \int_{\mathbb{R}^2} f(E_2) \frac{E_1^b - 1}{E_2} \exp[-t E_2 E_1] \Theta(L_a E) \delta(E_1) \mathrm{d}[E] \\ \int_{\mathbb{R}^2} f(E_2) E_1^{a-1} \frac{\partial}{\partial E} \delta(E) \delta(E_1) \mathrm{d}[E] \end{array} \right]_{1 \leq a < k_1 \atop 1 \leq b \leq k_2} \right\}.
\]

In the lower-right block we use the following property of the integral:

\[
\int_{\mathbb{R}} f(E) E^{b-1} \left( \frac{\partial}{\partial E} \right)^{a-1} \delta(E) \mathrm{d}E = 0 \quad \text{for } b > a.
\]

Since \(d = N + k_2 - k_1 \leq N\), cf equation (13), the last \(N-d\) columns in the lower-right block in the determinant (A.8) are zero. The matrix

\[
\mathbf{M} = \left[ \int_{\mathbb{R}} f(E) E^{b-1} \left( \frac{\partial}{\partial E} \right)^{a-1} \delta(E) \mathrm{d}E \right]_{1 \leq a, b \leq d}
\]
is a lower triangular matrix with diagonal elements

\[ M_{jj} = \int_{\mathbb{R}} f(E) E^{j-1} \left( i \frac{\partial}{\partial E} \right)^{j-1} \delta(E) \, dE = (-i)^{j-1} (j-1)! \]  

(A.11)

Thus, the determinant of this matrix is

\[ \det M = (-i)^{N(N-1)/2} \prod_{j=1}^{N} (j-1)! \]  

(A.12)

We pull the matrix \( M \) out the determinant (A.8) and find

\[ Z(N)_{k_1/k_2}(\kappa) = (\frac{(-1)^{k_2(k_2-1)/2 + k_2+1/k_1}}{\sqrt{\text{Ber}_{k_1/k_2}(\kappa)}}) \times \det \left[ K^{(d)}(\kappa_{a_1}, \kappa_{b_2}) \right]_{1 \leq a_1 \leq k_1}^{1 \leq b_2 \leq k_2} \int_{\mathbb{R}} f(E) E^{b-1} \exp[-i\kappa_{a_1} E] \Theta(L_a E) \, dE \]  

(A.13)

where

\[ K^{(d)}(\kappa_{a_1}, \kappa_{b_2}) = \frac{1}{\kappa_{a_1} - \kappa_{b_2}} - t L_a \sum_{m,n=1}^{d} \int_{\mathbb{R}} f(E) E^{m-1} \exp[-i\kappa_{a_1} E] \Theta(L_a E) \, dE M_{mm}^{-1} \kappa_{b_2}^{n-1} \]  

(A.14)

Again we use the fact that the determinant is skew-symmetric which also allows us to write

\[ Z^{(N)}_{k_1/k_2}(\kappa) = (\frac{(-1)^{k_2(k_2-1)/2 + k_2+1/k_1}}{\sqrt{\text{Ber}_{k_1/k_2}(\kappa)}}) \times \det \left[ K^{(N)}(\kappa_{a_1}, \kappa_{b_2}) \right]_{1 \leq a_1 \leq k_1}^{1 \leq b_2 \leq k_2} \int_{\mathbb{R}} f(E) E^{b-1} \exp[-i\kappa_{a_1} E] \Theta(L_a E) \, dE \]  

(A.15)

for an arbitrary \( \tilde{N} \in \{d, d+1, \ldots, N\} \). For the cases \((k_1/k_2) = (1/1)\) and \((k_1/k_2) = (1/0)\), we identify

\[ Z^{(N)}_{1/1}(\kappa_{a_1}, \kappa_{b_2}) = (\kappa_{a_1} - \kappa_{b_2}) K^{(N)}(\kappa_{a_1}, \kappa_{b_2}), \]  

(A.16)

\[ Z^{(N)}_{1/0}(\kappa_{a_1}) = -t L_a \int_{\mathbb{R}} f(E) E^{N-1} \exp[-i\kappa_{a_1} E] \Theta(L_a E) \, dE. \]  

(A.17)

This yields the result (14).

Appendix B. Derivation of equation (29)

This derivation is similar to the one for the supermatrix Bessel function with the Efetov–Wegner term in section V.A of [20]. We consider the integral

\[ Z^{(N)}_{k_1/k_2}(\kappa) = \frac{(-1)^{N} 2\pi}{(N-1)!} \int_{\mathbb{R}} \Phi^{(1)(1)}(\hat{\rho}) \exp[-i\text{Str} \hat{\rho}] r^N \left( e^{-i\psi \frac{\partial}{\partial r^2}} \right)^{N-1} e^{-i\psi \delta(r_2)} \, d[\rho]. \]  

(B.1)
As in [20], we exchange the integration over the Grassmann variables by a differential operator which yields

$$\frac{Z_{1/1}^{(N)}(\kappa)}{f^{N}(0)} = \frac{(-1)^N}{(N-1)!} \int_{\mathbb{R} \times \mathbb{R}} r_1^N \left( e^{-i \psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) \left[ \frac{\kappa_1 - \kappa_2}{r_1 - e^{i \psi} r_2} \right] \left[ \frac{f(r_1)}{f(e^{i \psi} r_2)} \exp[-\imath \text{Str} \kappa r] \right] dr_1 dr_2. \tag{B.2}$$

The term

$$Z_1 = \frac{\imath (-1)^N}{(N-1)!} \int_{\mathbb{R} \times \mathbb{R}} \frac{\kappa_1 - \kappa_2}{r_1 - e^{i \psi} r_2} \frac{f(r_1)}{f(e^{i \psi} r_2)} \exp[-\imath \text{Str} \kappa r] r_1^N \left( e^{-i \psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) \, dr_1 dr_2 \tag{B.3}$$

contains the supermatrix Bessel function with respect to U (1/1) [20, 39, 46]. The second term

$$Z_2 = \frac{(-1)^N}{(N-1)!} \int_{\mathbb{R} \times \mathbb{R}} r_1^N \left( e^{-i \psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) \left[ \frac{1}{r_1 - e^{i \psi} r_2} \left( \frac{\partial}{\partial r_1} + e^{-i \psi} \frac{\partial}{\partial r_2} \right) - \frac{e^{-i \psi}}{r_1} \frac{\partial}{\partial r_2} \right] \times \left[ \frac{f(r_1)}{f(e^{i \psi} r_2)} \exp[-\imath \text{Str} \kappa r] \right] dr_1 dr_2 \tag{B.4}$$

has to yield the Efetov–Wegner term. By partial integration, we evaluate the Dirac distribution and omit the generalized Wick rotation. Thus, equation (B.2) becomes

$$Z_2 = -\frac{1}{(N-1)!} \int_{\mathbb{R}} \left[ \left\{ \sum_{j=0}^{N-1} \frac{(N-1)!}{j!} r_1^{j+1} \left( \frac{\partial}{\partial r_1} \right)^j \right\} - r_1^{N-1} \frac{\partial}{\partial r_2} N \right] \times \left[ \frac{f(r_1)}{f(r_2)} \exp[-\imath \text{Str} \kappa r] \right] \bigg|_{r_2=0} \, dr_1. \tag{B.5}$$

For all terms up to $j = 0$ we perform a partial integration in $r_1$ and find a telescope sum. Hence, we have

$$Z_2 = -\int_{\mathbb{R}} \left[ \frac{f(r_1)}{f(r_2)} \exp[-\imath \text{Str} \kappa r] \right] \bigg|_{r_2=0} \, dr_1 = 1. \tag{B.6}$$

This is indeed the Efetov–Wegner term.

The second equality (29) follows from

$$\int_{\mathbb{R} \times \mathbb{R}} \frac{1}{r_1 - e^{i \psi} r_2} \left( \frac{\partial}{\partial r_1} + e^{i \psi} \frac{\partial}{\partial r_2} \right) \left( \frac{f(r_1)}{f(e^{i \psi} r_2)} \exp[-\imath \text{Str} \kappa r] r_1^N \left( e^{-i \psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) \right) dr_1 dr_2 \nonumber$$

$$= \int_{\mathbb{R} \times \mathbb{R}} r_1^N \left( e^{-i \psi} \frac{\partial}{\partial r_2} \right)^{N-1} \delta(r_2) \left[ \frac{1}{r_1 - e^{i \psi} r_2} \left( \frac{\partial}{\partial r_1} + e^{i \psi} \frac{\partial}{\partial r_2} \right) \right] \times \left[ \frac{f(r_1)}{f(e^{i \psi} r_2)} \exp[-\imath \text{Str} \kappa r] \right] \, dr_1 dr_2 + \int_{\mathbb{R} \times \mathbb{R}} \frac{f(r_1)}{f(e^{i \psi} r_2)} \exp[-\imath \text{Str} \kappa r] \times \left[ \frac{N r_1^{N-1}}{r_1 - e^{i \psi} r_2} \left( e^{-i \psi} \frac{\partial}{\partial r_2} \right)^{N-1} \right] \delta(r_2) \, dr_1 dr_2$$

18
The next step is to pull all factors of $f$ through the determinant. Identifying the remaining terms of the integrands (B.4) and the distribution $\left(e^{-\psi} \frac{\partial}{\partial r_2}\right)^{-1} \delta(r_2)$ out the determinant. Identifying the remaining terms of the integrands for all $f$ we obtain equation (B.5).

**Appendix C. Derivation of equation (32)**

Let the characteristic function and, hence, the superfunction $\Phi^{(k_1/k_2)}$ be factorizable, cf equation (19). To show identity (32) we plug equations (20) and (29) into the result (14) for $N = d$. We find

$$\int_{\mathbb{R}^d_+ \times \mathbb{R}^d} \text{Ber}^{(2)}_{k_1/k_2}(r) \Phi^{(k_1/k_2)}(r, \kappa) \prod_{j=1}^{k_1} f(r_{j1}) \prod_{j=1}^{k_2} f(e^\psi r_{j2}) \det r_1 \prod_{j=1}^{k_1} \left(e^{-\psi} \frac{\partial}{\partial r_{j1}}\right)^{d-1} \delta(r_{j2}) d[r]$$

$$= \left(-1\right)^{(k_1+k_2)(k_1+k_2)-1/2} \left(2\pi\right)^{(k_2-k_1)^2/2-(k_1+k_2)/2} 2^{k_1} \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(\kappa)} \det \left\{ A(\kappa_{ab}, \kappa_{a2}) \right\}_{1 \leq a \leq k_1, 1 \leq b \leq k_2, 1 \leq a \leq N, 1 \leq b \leq k_1} \int_{\mathbb{R}^d} f(r_1) r_1^{d-1} e^{-i\kappa_{ab} r_1} d[r_1]$$

with

$$A(\kappa_{ab}, \kappa_{a2}) = \int_{\mathbb{R}_+ \times \mathbb{R}} \frac{\exp(-i\kappa_{ab} r_1 + re^\psi \kappa_{a2} r_2)}{(\kappa_{ab} - \kappa_{a2})(r_1 - e^\psi r_2)} \left(\frac{\partial}{\partial r_1} + e^{-\psi} \frac{\partial}{\partial r_2}\right)^d \delta(r_2) d[r].$$

The next step is to pull all factors of $f$, the monomials $r_1^d$ and the distribution $\left(e^{-\psi} \frac{\partial}{\partial r_2}\right)^{-1} \delta(r_2)$ out the determinant. Identifying the remaining terms of the integrands for all $f$ we obtain equation (32).

**Appendix D. Derivation of the supermatrix Bessel function with Efetov–Wegner terms**

We use the result of appendix C as an ansatz in equation (30). To prove that this ansatz is indeed the result we are looking for, we construct a boundary value problem in a weak sense. We consider the left-hand side of equation (30) with the supermatrix

$$\sigma = \begin{bmatrix} [\sigma_{ab}]_{1 \leq a, b \leq k_1} & [\chi^{*}_a]_{1 \leq a \leq k_1} \\ [\chi_a]_{1 \leq a \leq k_2} & [\sigma_{ab2}]_{1 \leq a, b \leq k_2} \end{bmatrix}$$

(D.1)
with non-zero entries everywhere instead of a diagonal supermatrix $\kappa$. Then, the action of the differential operator

$$\text{Str} \frac{\partial^2}{\partial \sigma^2} = \sum_{a=1}^{k_1} \frac{\partial^2}{\partial \sigma_{aa}^2} + 2 \sum_{1 \leq a < b \leq k_1} \frac{\partial^2}{\partial \sigma_{ab} \partial \sigma_{ba}^*} = \sum_{a=1}^{k_1} \frac{\partial^2}{\partial \sigma_{aa}^2} - 2 \sum_{1 \leq a < b \leq k_2} \frac{\partial^2}{\partial \sigma_{ab} \partial \sigma_{ba}^*}$$

(D.2)

on the left-hand side of equation (30) yields

$$\text{Str} \frac{\partial^2}{\partial \sigma^2} \int_{\Sigma_{k_1}^{(0)}(\kappa)} F(\rho) \exp[-i \text{Str} \sigma \rho] \, d[\rho] = - \int_{\Sigma_{k_1}^{(0)}(\kappa)} F(\rho) \text{Str} \, \rho^2 \exp[-i \text{Str} \sigma \rho] \, d[\rho].$$

Since the integrand is rotation invariant the integral only depends on the eigenvalues of the supermatrix $\sigma$. This leads to a differential equation in the diagonal supermatrix $\kappa$. With the differential operator $\text{Str} \frac{\partial^2}{\partial \kappa^2}$ defined similar to equation (44) we have

$$\frac{1}{\sqrt{\text{Ber}^{(2)}_{k_1/k_2}(\kappa)}} \text{Str} \frac{\partial^2}{\partial \kappa^2} \sqrt{\text{Ber}^{(2)}_{k_1/k_2}(\kappa)} \int_{\Sigma_{k_1}^{(0)}(\kappa)} F(\rho) \exp[-i \text{Str} \kappa \rho] \, d[\rho]$$

$$= - \int_{\Sigma_{k_1}^{(0)}(\kappa)} F(\rho) \text{Str} \, \rho^2 \exp[-i \text{Str} \kappa \rho] \, d[\rho],$$

(D.4)

cf [32].

The boundaries of $\Sigma_{k_1}^{(0)}(\kappa)$ are given by $\Sigma_{k_1-1/k_2-1}^{(0)}$ canonically embedded in $\Sigma_{k_1/k_2}^{(0)}$ if one bosonic eigenvalue of a supermatrix in $\Sigma_{k_1/k_2}^{(0)}$ equals a fermionic one, i.e. there are two numbers $a \in \{1, \ldots, k_1\}$ and $b \in \{1, \ldots, k_2\}$ with $\kappa_{a1} = \kappa_{b2}$. For these cases we may use the Cauchy-like integral theorems for Hermitian supermatrices [19, 20, 45]. Without loss of generality, we consider the case $\kappa_{a1} = \kappa_{b2}$ and have

$$\int_{\Sigma_{k_1}^{(0)}(\kappa)} F(\rho) \exp[-i \text{Str} \kappa \rho] \, d[\rho]$$

$$= (-1)^{k_1 - k_2 - 1} \int_{\Sigma_{k_1-1/k_2-1}^{(0)}} F(\rho) \exp[-i \text{Str} \kappa \mid_{\kappa_{a1} = \kappa_{b2} = 0} \rho] \, d[\rho].$$

(D.5)

Here, we use the same symbol for the restriction of $F$ on $\Sigma_{k_1-1/k_2-1}^{(0)}$.

The boundary condition (D.5) for the distribution (33) can be readily checked. For the differential equation (D.4) we expand the determinant (33) in $l \leq k_2$ rows and columns in the upper block. Apart from a constant prefactor each term is given by

$$g(\kappa, r) = \sqrt{\text{Ber}^{(2)}_{k_1/l}(\kappa_\theta)} \sqrt{\text{Ber}^{(2)}_{k_1-l/k_2-l}(r_{\omega})} \prod_{j=1}^{l} e^{-i \psi \delta(r_{\omega_{(1)j}}) \delta(r_{\omega_{(2)j}})}$$

$$\times \prod_{a=1}^{k_1} \exp(-i \kappa_{a1} r_{\omega_{(1)1}}) \prod_{a=1}^{k_2} \exp(i \psi \kappa_{a2} r_{\omega_{(2)2}}),$$

(D.6)

where $\kappa_{a1} = \text{diag} (\kappa_{a1(1)}, \ldots, \kappa_{a1(l_1)}, \kappa_{a2(1)}, \ldots, \kappa_{a2(l_2)})$ and $r_{\omega_{(1)}} = \text{diag} (r_{\omega_{(1)1}}, \ldots, r_{\omega_{(1)l_1}}, r_{\omega_{(2)1}}, \ldots, r_{\omega_{(2)l_2}})$ with the permutations $\omega_{(1)}, \omega_{(2)} \in \mathfrak{S}(k_1)$.
and \(a_2, \beta_2 \in \mathbb{R}(k_2)\). The action of the distribution \(g(\kappa, r)\) on \(\text{Str} r^2\) is

\[
\text{Str} r^2 = \frac{1}{\text{Ber}_{k_1/k_2}(\kappa)} \frac{\partial^2}{\partial \kappa^2} \text{Str} \frac{\text{Ber}_{k_1/k_2}(\kappa) \text{Ber}_{k_1/k_2-1}(r_\omega)}{\text{Ber}_{k_1/k_2}(r)} \left( \prod_{j=1}^l e^{-\omega} \delta(r_{\omega(j)1}) \delta(r_{\omega(j)2}) \right) \times \prod_{a=l+1}^{k_2} \exp(-t \kappa_{\alpha(a)1} r_{\alpha(a)1}) \prod_{a=l+1}^{k_2} \exp(t e^{i\phi} \kappa_{\alpha(b)2} r_{\alpha(b)2}) \text{Str} r_{\omega}^2
\]

We consider the integral

\[
I = \int_{\mathbb{R}^{1+k_2}} \left\{ \frac{-2 \pi e^{-i\phi} \delta(r_{b1}) \delta(s_{a2})}{\kappa_{b1} - e^{i\phi}\kappa_{a2}} \frac{\exp(-t \kappa_{b1} r_{b1} + t \kappa_{a2} r_{a2})}{r_{b1} - e^{i\phi} r_{a2}} \chi(\kappa_{b1} - e^{-i\phi}\kappa_{a2}) \right\} \text{det} \left\{ \begin{array}{c} \{ r_{a}^{-1} \exp(-t \kappa_{b1} r_{b1}) \} & 1 \leq a \leq k_2 \\ {1 \leq b \leq k_3} \end{array} \right\} \times \text{det} \left\{ \begin{array}{c} \{ s_{b}^{-1} \exp(t r_{b1} s_{b1}) \} & 1 \leq a \leq k_2 \\ {1 \leq b \leq k_3} \end{array} \right\}
\]

because all other terms are zero due to the Dirac distributions. The differential operator in equation (D.4) acts on \(g(\kappa, r)\) as

\[
\frac{\partial^2}{\partial \kappa^2} \text{Str} \frac{\text{Ber}_{k_1/k_2}(\kappa) \text{Ber}_{k_1/k_2-1}(r_\omega)}{\text{Ber}_{k_1/k_2}(r)} \left( \prod_{j=1}^l e^{-\omega} \delta(r_{\omega(j)1}) \delta(r_{\omega(j)2}) \right) \times \prod_{a=l+1}^{k_2} \exp(-t \kappa_{\alpha(a)1} r_{\alpha(a)1}) \prod_{a=l+1}^{k_2} \exp(t e^{i\phi} \kappa_{\alpha(b)2} r_{\alpha(b)2}) \text{Str} r_{\omega}^2
\]

We split the differential operator \(\text{Str} \frac{\partial^2}{\partial \kappa^2}\) into a part acting on \(\kappa_{\beta}\) and a part for the remaining variables in \(\kappa\). The term for the latter variables acts on the exponential functions in equation (D.8) and contributes the term \(-\text{Str} r_{\omega}^2\). For the term according to \(\kappa_{\beta}\) we use the identity

\[
\text{Str} \frac{\partial^2}{\partial \kappa_{\beta}^2} \text{Str} \frac{\text{Ber}_{k_1/k_2}(\kappa) \text{Ber}_{k_1/k_2-1}(r_\omega)}{\text{Ber}_{k_1/k_2}(r)} \left( \prod_{j=1}^l e^{-\omega} \delta(r_{\omega(j)1}) \delta(r_{\omega(j)2}) \right) \times \prod_{a=l+1}^{k_2} \exp(-t \kappa_{\alpha(a)1} r_{\alpha(a)1}) \prod_{a=l+1}^{k_2} \exp(t e^{i\phi} \kappa_{\alpha(b)2} r_{\alpha(b)2}) = 0.
\]

Thus, the differential equation is also fulfilled by \(\varphi_{b_1/k_2}\).

Appendix E. Double Fourier transform

We consider the integral

\[
I = \int_{\mathbb{R}^{1+k_2}} \left\{ \begin{array}{c} \{ -2 \pi e^{-i\phi} \delta(r_{b1}) \delta(s_{a2}) \} & \frac{\exp(-t \kappa_{b1} r_{b1} + t \kappa_{a2} r_{a2})}{r_{b1} - e^{i\phi} r_{a2}} \chi(\kappa_{b1} - e^{-i\phi}\kappa_{a2}) \right\} \begin{array}{c} \{ r_{a}^{-1} \exp(-t \kappa_{b1} r_{b1}) \} & 1 \leq a \leq k_2 \\ {1 \leq b \leq k_3} \end{array} \times \begin{array}{c} \{ s_{b}^{-1} \exp(t r_{b1} s_{b1}) \} & 1 \leq a \leq k_2 \\ {1 \leq b \leq k_3} \end{array}
\]

\[21\]
We omit the two sums over the permutation groups, see equation (33). They do not contribute any additional new information of the calculation and the missing terms can be regained by permuting the indices of the eigenvalues in $s$ or $\tilde{\omega}_s$.

The expansion in the first determinant yields

\[
I = \sum_{l=0}^{k_1} \sum_{a_1 \in \Theta(k_1)} \sum_{a_2 \in \Theta(k_2)} \frac{\text{sign} \omega_1 \omega_2}{(ll')^2(k_1 - l)(k_2 - l)!} \int_{\mathbb{R}^{k_1 + 2}} \det \left[ \begin{array}{c} -2\pi e^{-i\phi} \delta(r_{a_1(b)1}) \delta(r_{a_2(a)2}) \\ \kappa_{a_1(b)1} - e^{-i\phi} \kappa_{a_2(a)2} \end{array} \right]_{1 \leq a,b \leq l} \times \det \left[ \begin{array}{c} \exp \left( -i\kappa_{a_1(b)1} r_{a_1(b)1} + i\kappa_{a_2(a)2} r_{a_2(a)2} \right) \\ r_{a_1(b)1} - e^{i\phi} r_{a_2(a)2} \end{array} \right]_{1 \leq a \leq k_1, 1 \leq b \leq k_2} \times \det \left[ \begin{array}{c} \exp \left( i r_{a_1(b)1} - r_{a_2(a)2} \right) s_{a_1(b)1} - e^{-i\phi} s_{a_2(a)2} \\ r_{a_1(b)1} - e^{i\phi} r_{a_2(a)2} \end{array} \right]_{1 \leq a \leq k_1, 1 \leq b \leq k_2} \times \frac{d[r]}{\sqrt{\text{Ber}^{(2)}_{k_1/k_2}(r) \left( \text{Ber}^{(2)}_{k_1/k_2}(\tilde{\omega}_s) \right) \text{Ber}^{(2)}_{k_1/k_2}(s)}}. \tag{E.1}
\]

where the function ‘sign’ yields 1 for an even permutation and −1 for an odd one. The permutations in the indices of the $r$ are absorbed in the integration. We remark that the remaining integral goes over $k_1 + k_2 - 2l$ variables because we have already used the Dirac distributions.
With the help of the formula
\[\int_{\mathbb{R}^2} \frac{2\pi e^{i\Psi} \delta(s_{\omega_1(1)}) \delta(s_{\omega_2(2)})}{r_{b1} - e^{i\Psi} r_{a2}} \exp \left(-ik_{\omega_1(1)} r_{b1} + i k_{\omega_2(2)} r_{a2} \right) \, dr \]
\[= \int_{\mathbb{R}^2} \frac{2\pi i e^{i\Psi} \delta(s_{\omega_1(1)}) \delta(s_{\omega_2(2)})}{(k_{\omega_1(1)} - e^{-i\Psi} k_{\omega_2(2)})(r_{b1} - e^{i\Psi} r_{a2})} \left( \frac{\partial}{\partial r_{b1}} + e^{-i\Psi} \frac{\partial}{\partial r_{a2}} \right) \exp \left(-ik_{\omega_1(1)} r_{b1} + i k_{\omega_2(2)} r_{a2} \right) \, dr]}
\[= \frac{(2\pi i)^2 e^{i\Psi} \delta(s_{\omega_1(1)}) \delta(s_{\omega_2(2)})}{k_{\omega_1(1)} - e^{-i\Psi} k_{\omega_2(2)}} \]  
(E.3)
we integrate and sum expression (E.2) up. This yields
\[\int_{\mathbb{R}^1 \otimes s^2} \tilde{\psi}_{k_1/k_2}(s, r) \tilde{\psi}_{k_1/k_2}(-ir, \tilde{\kappa}) \text{Ber}_{k_{1/k_2}}^{(2)}(r) \, dr \]
\[= \frac{(-1)^{k_1(k_1-1)/2}(k_2-k_1)^2}{2^{2k_1-k_2-k_1} k_1! k_2! \sqrt{\text{Ber}_{k_{1/k_2}}^{(2)}(\tilde{\kappa}) \text{Ber}_{k_{1/k_2}}^{(2)}(\kappa)}} \]
\[\times \sum_{a_1 \in \Theta(k_1)} \sum_{a_2 \in \Theta(k_2)} \det \left[ B_{ab} \right] \]  
(E.4)
with
\[B_{ab} = \frac{e^{i\Psi} \delta(s_{\omega_1(1)}) \delta(s_{\omega_2(2)})}{k_{b1} - e^{-i\Psi} k_{a2}} \left[ 1 - \chi(k_{b1} - e^{-i\Psi} k_{a2}) \right] + e^{i\Psi} \frac{\delta(k_{b1} - s_{\omega_1(1)}) \delta(k_{a2} - s_{\omega_2(2)})}{s_{\omega_1(1)} - e^{-i\Psi} s_{\omega_2(2)}} \chi(k_{b1} - e^{-i\Psi} k_{a2}). \]  
(E.5)
This is the result (41). The index \(a\) goes from 1 to \(k_2\) in the upper block and from 1 to \(k_1 - k_2\) in the lower block, whereas \(b\) takes the values from 1 to \(k_1\) in both blocks.

Appendix F. Calculations for subsection 6.1

We diagonalize the supermatrices \(\sigma\) and \(\rho\) in equation (49) and have for the generating function
\[Z_k^{(N)}(\kappa) = \frac{1}{(2\pi i)^{2k}} \int_{\mathbb{R}^{2k}} \Phi(s) I_k^{(N)}(r) \, dr \]
\[= \det \left[ -\frac{2\pi e^{-i\Psi} \delta(s_{b1}) \delta(s_{a2})}{r_{b1} - e^{i\Psi} r_{a2}} + \exp \left(-ik_{b1} r_{b1} + i e^{i\Psi} k_{a2} r_{a2} \right) \chi(k_{b1} - k_{a2}) \right]_{1 \leq a, b \leq k} \]
\[\times \det \left[ \frac{2\pi e^{i\Psi} \delta(s_{b1}) \delta(s_{a2})}{r_{b1} - e^{i\Psi} r_{a2}} - \exp \left(i r_{b1} s_{b1} - i r_{a2} s_{a2} \right) \chi(r_{b1} - e^{i\Psi} r_{a2}) \right]_{1 \leq a, b \leq k}. \]  
(F.1)
This expression is not well defined because the supersymmetric Ingham–Siegel is at zero not well defined. We recall that the supersymmetric Ingham–Siegel integral factorizes in each eigenvalue of the supermatrix \(r\), cf equation (50). To understand equation (F.1) we have to know what \(I_k^{(N)}(0)\) is. Since the supersymmetric Ingham–Siegel integral is a distribution, we consider an arbitrary rotation invariant, sufficiently integrable superfunction \(f\) on the set of
\[(1 + 1) \times (1 + 1)\) Hermitian supermatrices. Then, we have

\[
\int_{\Sigma_{N}(\rho)} f(\rho) I_{1}^{(N)}(\rho) \, d[\rho] = \int_{\Sigma_{N}(\rho)} \left( \int_{\Sigma_{N}(\rho)} f(\rho) \exp(-t \text{Str } \rho \sigma + \varepsilon \text{Str } \rho) \, d[\rho] \right) \times \text{Sdet}^{-N}(\sigma + t \mathbb{I}_{+1}) \, d[\sigma] \\
= t \int_{\Sigma_{N}(\rho)} f(\rho) \exp(\varepsilon \text{Str } \rho) \, d[\rho] \\
= f(0) \\
= -t f(0) I_{1}^{(N)}(0)
\]

with help of the Cauchy-like integral theorem for \((1 + 1) \times (1 + 1)\) Hermitian supermatrices, see [19, 20]. Please note that the constant resulting from the Cauchy-like integral theorem converts to the complex conjugate when the generalized Wick rotation is complex conjugated. The last equality in equation (F.2) is the Cauchy-like integral theorem formally applied to the left-hand side of equation (F.2). Hence, we conclude that \(I_{1}^{(N)}(0) = i\) in a distributional sense. Using this result we find

\[
Z_{k}^{(N)}(\kappa) = \frac{(-1)^{k(k+1)/2}}{2^{k(k+1)} \pi^{2k}} \sum_{l=0}^{k} \sum_{\alpha_{1}, \alpha_{2} \in \Theta(k)} \frac{\text{sign } \alpha_{1} \alpha_{2}}{[l![(k-l)!]^2} \int_{\mathbb{R}^{2a}} \mathcal{F}(s) \\
\times \text{det} \left[ \exp\left(-t \kappa_{\alpha_{1}(b)1} \Gamma_{\alpha_{1}(b)1} + t e^{i \psi} \kappa_{\alpha_{2}(a)2} \Gamma_{\alpha_{2}(a)2}\right) \right] \\
\times I_{1}^{(N)}(r_{\alpha_{1}(b)1}, e^{i \psi} r_{\alpha_{2}(a)2}) \chi(\kappa_{\alpha_{1}(b)1} - \kappa_{\alpha_{2}(a)2})_{l+1 \leq a, b \leq k} \\
\times \text{det} \left[ \frac{2 \pi \varepsilon^{i \psi} \delta(\omega_{b1}) \delta(\omega_{a2})}{s_{b1} - e^{i \psi} s_{a2}} \chi(r_{b1} - e^{i \psi} r_{a2}) \right]_{1 \leq a, b \leq l} \\
\times \int_{\mathbb{R}^{2l-2i}} \mathcal{F}(s) \\
\times \prod_{a, b = l+1}^{k} \exp\left(-t \kappa_{\alpha_{1}(b)1} \Gamma_{\alpha_{1}(b)1} + t e^{i \psi} \kappa_{\alpha_{2}(a)2} \Gamma_{\alpha_{2}(a)2}\right) I_{1}^{(N)}(r_{b1}, e^{i \psi} r_{a2}) \\
\times \text{det} \left[ \frac{2 \pi \varepsilon^{i \psi} \delta(\omega_{b1}) \delta(\omega_{a2})}{s_{b1} - e^{i \psi} s_{a2}} \right]_{l+1 \leq a, b \leq k} \\
\times \int_{\mathbb{R}^{2l-2i}} \mathcal{F}(s) \\
\times \prod_{a, b = l+1}^{k} \exp\left(-t \kappa_{\alpha_{1}(b)1} \Gamma_{\alpha_{1}(b)1} + t e^{i \psi} \kappa_{\alpha_{2}(a)2} \Gamma_{\alpha_{2}(a)2}\right) I_{1}^{(N)}(r_{b1}, e^{i \psi} r_{a2}) \\
\times \text{det} \left[ \frac{2 \pi \varepsilon^{i \psi} \delta(\omega_{b1}) \delta(\omega_{a2})}{s_{b1} - e^{i \psi} s_{a2}} \right]_{l+1 \leq a, b \leq k} \\
\times \int_{\mathbb{R}^{2l-2i}} \mathcal{F}(s) \\
= \frac{1}{2^{k(k-1)} k} \sum_{l=0}^{k} \sum_{\alpha_{1}, \alpha_{2} \in \Theta(k)} \frac{\text{sign } \alpha_{1} \alpha_{2}}{[l![(k-l)!]^2} \prod_{a, b = l+1}^{k} \chi(\kappa_{\alpha_{1}(b)1} - \kappa_{\alpha_{2}(a)2})
\]
Appendix G. Calculations for subsection 6.2

In the first step, we derive the Fourier transform of the superdeterminant in equation (52). Let $U$ with

\[ \text{J. Phys. A: Math. Theor. 44 (2011) 285210} \]

We perform the sum and use the identity

\[ \int \frac{d[r] d[s]}{\sqrt{\text{Ber}_{k/l}(r)}} \mathcal{F}(s) \det \left[ \begin{array}{c} -e^{i\Psi} \delta(s_{ab1}) \delta(s_{ab2}) \\ \kappa_{ab1} - \kappa_{ab2} \end{array} \right]_1 \leq a, b \leq l \]

\[ \times \left[ \begin{array}{c} -e^{i\Psi} \delta(s_{ab1}) \delta(s_{ab2}) \\ 1 - \left( \frac{\kappa_{ab2}}{\kappa_{ab1}} \right)^N \end{array} \right] \]

\[ = \frac{N (e^{-i\Psi} s_{ab1} - \kappa_{ab1})^{N-1}}{2\pi i (e^{-i\Psi} s_{ab1} - \kappa_{ab1})^{N+1}} (s_{ab1} - e^{-i\Psi} \kappa_{ab2}) \int_1 \leq a, b \leq k . \]

(F.3)

We perform the sum and use the identity

\[ 1 - \left[ 1 - \left( \frac{\kappa_{ab2}}{\kappa_{ab1}} \right)^N \right] \chi(\kappa_{ab1} - \kappa_{ab2}) = \left( \frac{\kappa_{ab2}}{\kappa_{ab1}} \right)^N . \]

(F.4)

Then we have the result (51).

Appendix G. Calculations for subsection 6.2

In the first step, we derive the Fourier transform of the superdeterminant in equation (52). Let the entries of the diagonal $(k + k) \times (k + k)$ supermatrix $r$ and the entries of the diagonal $N \times N$ matrix $E_0$ be the eigenvalues of the supermatrix $\rho$ and the Hermitian matrix $H_0$, i.e. $\rho = U_r U_r^\dagger$ with $U \in U(k/k)$ and $H_0 = V E_0 V^\dagger$ with $V \in U(N)$. Then, the Fourier transform is

\[ J = \int_{\mathbb{R}^k} \frac{d[r]d[s]}{\sqrt{\text{Ber}_{k/l}(r)}} \mathcal{F}(s) \det^{-1} (\sigma \otimes \mathbb{1}_N + \alpha \mathbb{1}_{k+k} \otimes H_0 + i\epsilon \mathbb{1}_{N(k+k)}) d[\sigma] \]

\[ = \frac{i}{2^{k/2}} \int_{\mathbb{R}^2} d[r]d[s] \mathcal{F}(s) \det^{-1} (\sigma \otimes \mathbb{1}_N + \alpha \mathbb{1}_{k+k} \otimes H_0 + i\epsilon \mathbb{1}_{N(k+k)}) d[\sigma] \]

With help of identity (8) we find

\[ J = \frac{i}{2^{k/2}} \exp(i\epsilon \text{Str} r) \sum_{a=0}^{k/2} \sum_{i=0}^{k} \frac{\sigma \otimes \mathbb{1}_N}{\Delta_N(\alpha E_0) \sqrt{\text{Ber}_{k/l}(r)}} \chi(r_{ab1} - e^{-i\Psi} r_{ab2}) \]

\[ \times \int_{\mathbb{R}^2} \mathcal{F}(s) \det^{-1} (\sigma \otimes \mathbb{1}_N + \alpha \mathbb{1}_{k+k} \otimes E_0 + i\epsilon \mathbb{1}_{N(k+k)}) d[\sigma] \]

\[ = \frac{i}{2^{k/2}} \exp(i\epsilon \text{Str} r) \frac{1}{\Delta_N(\alpha E_0) \sqrt{\text{Ber}_{k/l}(r)}} \sum_{a=0}^{k/2} \sum_{i=0}^{k} \frac{\sigma \otimes \mathbb{1}_N}{\Delta_N(\alpha E_0) \sqrt{\text{Ber}_{k/l}(r)}} \chi(r_{ab1} - e^{-i\Psi} r_{ab2}) \]

\[ \times \int_{\mathbb{R}^2} d[r]d[s] \mathcal{F}(s) \det^{-1} (\sigma \otimes \mathbb{1}_N + \alpha \mathbb{1}_{k+k} \otimes E_0 + i\epsilon \mathbb{1}_{N(k+k)}) d[\sigma] \]

\[ = \frac{i}{2^{k/2}} \exp(i\epsilon \text{Str} r) \frac{1}{\Delta_N(\alpha E_0) \sqrt{\text{Ber}_{k/l}(r)}} \sum_{a=0}^{k/2} \sum_{i=0}^{k} \frac{\sigma \otimes \mathbb{1}_N}{\Delta_N(\alpha E_0) \sqrt{\text{Ber}_{k/l}(r)}} \chi(r_{ab1} - e^{-i\Psi} r_{ab2}) \]

\[ \times \int_{\mathbb{R}^2} d[r]d[s] \mathcal{F}(s) \det^{-1} (\sigma \otimes \mathbb{1}_N + \alpha \mathbb{1}_{k+k} \otimes E_0 + i\epsilon \mathbb{1}_{N(k+k)}) d[\sigma] \]

25
In the upper-left block both indices $a$ and $b$ run from $l + 1$ to $k$, whereas in the lower-right block the range is from $1$ to $N$. In the upper-right block $a$ goes from $l + 1$ to $k$ and $b$ goes from $1$ to $N$, whereas it is vice versa in the lower-left block. We sum all terms in equation (G.2) up and pull the integrations into the determinant. Then, we have

\[
J = \frac{1}{2^k \pi^k \Delta_N(\alpha E_0) \sqrt{\Ber_{k/k}(r)}} \det \left[ \begin{array}{cc} A_1(r_{b1}, e^{i\psi} r_{a2}) & A_{b2}(e^{i\psi} r_{a2}) \\ A_{b3}(r_{b1}, \alpha E_a^{(0)}) & \{(-\alpha E_a^{(0)})^{b-1}\} \end{array} \right]_{1\leq a, b \leq N},
\]

where

\[
A_1(r_{b1}, e^{i\psi} r_{a2}) = \exp \left( i r_{b1} - e^{i\psi} r_{a2} \right) \left( \frac{2\pi}{r_{b1} - e^{i\psi} r_{a2}} \right) + \int_{\mathbb{R}^2} \exp \left( -i s r_{b1} + i s r_{a2} \right) \left( \frac{e^{i\psi} s_2 + \imath \epsilon}{s_1 + \imath \epsilon} \right)^N \chi(r_{b1} - e^{i\psi} r_{a2}) d[s]
\]

\[
= -2\pi i t \frac{r_{b1} - e^{i\psi} r_{a2}}{N} (N - 1)! \chi(r_{b1} - e^{i\psi} r_{a2})
\]

\[
A_{b2}(e^{i\psi} r_{a2}) = \exp (-e^{i\psi} r_{a2}) \int_{\mathbb{R}} e^{(i s_2 r_{a2})} (e^{i\psi} s_2 + \imath \epsilon)^{b-1} e^{-i\psi} d[s] \prod_{j=1}^k \chi(r_{j1} - e^{i\psi} r_{a2})
\]

\[
= 2\pi \left( \frac{e^{-i\psi}}{\partial r_{a2}} \right)^{b-1} e^{-i\psi} \delta(r_{a2}) \prod_{j=1}^k \chi(r_{j1} - e^{i\psi} r_{a2}),
\]

\[
A_{b3}(r_{b1}, \alpha E_a^{(0)}) = \exp (\imath r_{b1}) \int_{\mathbb{R}} e^{i s_1 + i \epsilon + \alpha E_a^{(0)}} (\frac{-\alpha E_a^{(0)}}{s_1 + \imath \epsilon})^N d[s] \prod_{j=1}^k \chi(r_{b1} - e^{i\psi} r_{j2})
\]

\[
= 2\pi i \Theta(r_{b1}) \sum_{n=0}^{\infty} \frac{((\alpha E_a^{(0)}) r_{b1})^n}{n!} \prod_{j=1}^k \chi(r_{b1} - e^{i\psi} r_{j2}).
\]

Surprisingly, this part of our result agrees with the one in [13] (apart from a forgotten $2\pi i$ in the upper-left block of equation (6.9) in [13] and the characteristic functions $\chi(r_{b1} - e^{i\psi} r_{a2})$) although we omitted all Efetov–Wegner terms in this work.

The second step is to diagonalize the supermatrix $\rho$ in equation (52):
(G.7) The range of the indices in the second determinant is the same as in equation (G.3). Expanding the first determinant we have

\[
Z_k^{(N)}(\kappa, \alpha H_0) = (-1)^{(k+1)/2} \int_{\mathbb{R}^{2^k}} \det \left[ \frac{-2\pi e^{-i\psi} \delta(r_{b1})\delta(r_{a2})}{\kappa_{b1} - \kappa_{a2}} \right] \\
\times \exp \left( -t_\kappa \kappa_2 \Gamma_{a2} + t e^{\psi} \kappa_2 \Gamma_{a2} \right) \chi(\kappa_{b1} - \kappa_{a2}) \\
\times \det \left[ \frac{-2\pi e^{-i\psi} \delta(r_{b1})\delta(r_{a2})}{\kappa_{b1} - \kappa_{a2}} \right] \\
\times \frac{\Theta(r_{b1}) \sum_{n \leq N} \frac{\left(\alpha E_n \right)^n}{n!} \prod_{j=1}^k \chi(r_{b1} - e^{\psi} r_{j2})}{\Delta_N(\alpha E_0) \sqrt{\text{Ber}^{(2)}_{k/k}(\kappa)} \sqrt{\text{Ber}^{(2)}_{k/k}(r)}}.
\]

(G.8) When integrating and summing up we use the normalization (F.2) of the supersymmetric Ingham–Siegel integral and arrive at the result (53).

References

[1] Guhr T 2006 J. Phys. A: Math. Gen. 39 13191
[2] Zimbauser M R 2006 The Supersymmetry Method of Random Matrix Theory (Encyclopedia of Mathematical Physics vol 5) ed J P Françoise, G L Naber and S T Tsun (Oxford: Elsevier) p 151
[3] Verbaarschot J M and Zimbauser M R 1985 J. Phys. A: Math. Gen. 18 1093
[4] Efetov K B 1995 Supersymmetry in Disorder and Chaos 1st edn (Cambridge: Cambridge University Press)
[5] Haake F 2001 Quantum Signatures of Chaos 2nd edn (Berlin: Springer)
[6] Verbaarschot J M 2004 AIP Conf. Proc. 744 277 (arXiv:hep-th/0410211)
[7] Lehmann N, Saher D, Sokolov V V and Sommers H J 1995 Nucl. Phys. A 582 223
[8] Efetov K B, Schwiete G and Takahashi K 2004 Phys. Rev. Lett. 92 026807
[9] Littlmann P, Sommers H-J and Zimbauser M R 2008 Commun. Math. Phys. 283 343
[10] Mehta M L 2004 Random Matrices 3rd edn (New York: Academic)
[11] Grünewald J, Guhr T and Kohler H 2004 J. Phys. A: Math. Gen. 37 2331
[12] Borodin A and Strahov E 2006 Commun. Pure Appl. Math. 59 161
[13] Kieburg M and Guhr T 2010 J. Phys. A: Math. Theor. 43 075201
[14] Kieburg M and Guhr T 2010 J. Phys. A: Math. Theor. 43 135204
[15] Kieburg M, Grönqvist J and Guhr T 2009 J. Phys. A: Math. Theor. 42 275205
[16] Brezin E 1987 Introduction to Superanalysis 1st edn (Dordrecht: Reidel)
[17] Rothstein M J 1987 Trans. Am. Math. Soc. 299 387
[18] Efetov K B 1983 Adv. Phys. 32 53
[19] Wegner F 1983 unpublished notes
[20] Kieburg M, Kohler H and Guhr T 2009 J. Math. Phys. 50 013528
[21] Guhr T 1993 J. Math. Phys. 34 2523
[22] Guhr T 1993 J. Math. Phys. 34 2541
[23] Guhr T 1993 Nucl. Phys. A 560 223
[24] Zirnbauer M R 1991 Commun. Math. Phys. 141 503
[25] Bundschuh R A 1993 Ensemblemittelung in ungeordneten mesoskopischen Leitern: Superanalytische Koordinatensysteme und ihre Randterme Diploma Thesis University of Cologne, Germany
[26] Zirnbauer M R 1996 J. Math. Phys. 37 4986
[27] Palzer W 2010 Integration on supermanifolds Diploma Thesis University of Paderborn, Germany
[28] Guhr T and Kohler H 2002 J. Math. Phys. 43 2741
[29] Guhr T 2006 J. Phys. A: Math. Gen. 39 12527
[30] Stockmann H-J 2002 J. Phys. A: Math. Gen. 35 5165
[31] Guhr T and Stockmann H-J 2004 J. Phys. A: Math. Gen. 37 2175
[32] Guhr T 1996 Ann. Phys., NY 250 145
[33] Guhr T 1996 Phys. Rev. Lett. 76 2258
[34] Friedrich H and Wintgen D 1989 Phys. Rep. 183 37
[35] Damgaard P H, Splittorff K and Verbaarschot J J M 2010 Phys. Rev. Lett. 105 162002
[36] Akemann G, Damgaard P H, Splittorff K and Verbaarschot J J M 2010 Talk presented at the 28th Int. Symp. on Lattice Field Theory, Lattice2010 (Villasimius, Italy) PoS(Lattice 2010)092 (arXiv:1011.5118)
[37] Akemann G, Damgaard P H, Splittorff K and Verbaarschot J J M 2011 Phys. Rev. D 83 085014
[38] Macedo-Junior A F and Macêdo A M S 2006 Nucl. Phys. B 752 439
[39] Guhr T 1991 J. Math. Phys. 32 336
[40] Basor E L and Forrester P J 1994 Math. Nachr. 170 5
[41] Kieburg M, Sommers H-J and Guhr T 2009 J. Phys. A: Math. Theor. 42 275206
[42] De Bie H and Sommen F 2007 J. Phys. A: Math. Theor. 40 7193
[43] De Bie H, Eelbode D and Sommen F 2009 J. Phys. A: Math. Theor. 42 245204
[44] Constantinescu F 1988 J. Stat. Phys. 50 1167
[45] Constantinescu F and de Groote H F 1989 J. Math. Phys. 30 981
[46] Guhr T 1996 Commun. Math. Phys. 176 555
[47] Brezin E and Hikami S 1998 Phys. Rev. E 58 7176
[48] Pandey A and Mehta M L 1983 Commun. Math. Phys. 87 449
[49] Johansson K 2007 Probab. Theory Relat. Fields 138 75
[50] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 Phys. Rep. 299 189
[51] Guhr T and Wettig T 1996 J. Math. Phys. 37 6395
[52] Feinberg J and Zee A 1997 Nucl. Phys. B 504 579
[53] Zirnbauer M R 1996 J. Phys. A: Math. Gen. 29 7113
[54] Harish-Chandra 1958 Am. J. Math. 80 241
[55] Zirnbauer C and Zuber J B 1980 J. Math. Phys. 21 411