Time-dependent angularly averaged inverse transport

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Received 16 February 2009, in final form 7 May 2009
Published 18 June 2009
Online at stacks.iop.org/IP/25/075010

Abstract
This paper concerns the reconstruction of the absorption and scattering parameters in a time-dependent linear transport equation from knowledge of angularly averaged measurements performed at the boundary of a domain of interest. Such measurement settings find applications in medical and geophysical imaging. We show that the absorption coefficient and the spatial component of the scattering coefficient are uniquely determined by such measurements. We obtain stability results on the reconstruction of the absorption and scattering parameters with respect to the measured albedo operator. The stability results are obtained by a precise decomposition of the measurements into components with different singular behavior in the time domain.

1. Introduction

Inverse transport theory has many applications in e.g. medical and geophysical imaging. It consists of reconstructing optical parameters in a domain of interest from measurements of the transport solution at the boundary of that domain. The optical parameters are the total absorption (extinction) parameter \( \sigma(x) \) and the scattering parameter \( k(x, v', v) \), which measures the probability of a particle at position \( x \in X \subset \mathbb{R}^n \) to scatter from direction \( v' \in S^{n-1} \) to direction \( v \in S^{n-1} \), where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \).

The domain of interest is probed as follows. A known flux of particles enters the domain and the flux of outgoing particles is measured at the domain’s boundary. Several inverse theories may then be envisioned based on available data. In this paper, we assume availability of time-dependent measurements that are angularly averaged. Also the source term used to probe the domain is not resolved angularly in order to e.g. save time in the acquisition of data. More precisely, the incoming density of particles \( \phi(t, x, v) \) as a function of time \( t \), at position \( x \in \partial X \) at the boundary of the domain of interest and for incoming directions \( v \), is of the
form \( \phi_S(t, x, v) = \phi(t, x)S(x, v) \), where \( \phi(t, x) \) is arbitrary but \( S(x, v) \) is fixed. This paper is concerned with the reconstruction of the optical parameters from such measurements. We show that the attenuation coefficient is uniquely determined and that the spatial structure of the scattering coefficient can be reconstructed provided that scattering vanishes in the vicinity of the domain’s boundary (except in dimension \( n = 2 \) and when \( X \) is a disc, where our theory does not require \( k \) to vanish in the vicinity of \( \partial X \)). For instance, when \( k(x, v', v) = k_0(x)g(v', v) \) with \( g(v', v) \) known \textit{a priori}, then \( k_0(x) \) is uniquely determined by the measurements. Similar results were announced in [1] when measurements are available in the modulation frequency variable, which is the dual (Fourier) variable to the time variable.

Several other regimes have been considered in the literature. The uniqueness of the reconstruction of the optical parameters from knowledge of angularly resolved measurements both in the time-dependent and time-independent settings was proved in [7, 8]; see also [15] for a review. Stability in the time-independent case has been analyzed in dimension \( n = 2, 3 \) under smallness assumptions for the optical parameters in [12, 13] and in dimension \( n = 2 \) in [16]. Stability results in the presence of full, angularly resolved, measurements have been obtained in [3, 4, 17]. The intermediate case of angularly averaged measurements with angularly resolved sources was considered in [11]. The lack of stability of the reconstruction in the time independent setting with angularly averaged measurements and isotropic sources is treated in [6]. See also [2] for a recent review of results in inverse transport theory.

The remainder of the paper is structured as follows. In section 2, we recall known results on the transport equation, present our measurement setting and a decomposition of the resulting measurement operator (the averaged albedo operator) in proposition 2.1, and study the temporal behavior of the averaged albedo operator when the scattering coefficient vanishes in the vicinity of the boundary of the (convex) domain \( X \). Our main results on uniqueness and stability are presented in section 3. We show that the absorption coefficient and the spatial structure of the scattering coefficient (the phase function describing scattering from \( v \) to \( v' \) has to be known in advance) can be reconstructed stably from angularly averaged time-dependent data. The reconstruction of the scattering coefficient requires inversion of a weighted Radon transform in the general case. When \( X \) is a ball, i.e., when measurements are performed at the boundary of a sphere, then the scattering coefficient may be obtained by inverting a classical Radon transform. In section 4, we show that the results are significantly modified when \( k \) does not vanish at the boundary of the domain \( X \).

The mathematical derivation of the results is fairly technical and is based on a careful analysis of the temporal behavior of the decomposition of the albedo operator into components that are multi-linear in the scattering coefficient. We show that the ballistic and single scattering components can be separated from the rest of the data. These two components are then used to obtain the uniqueness and stability results. It turns out that the structure of single scattering is very different depending on whether \( k \) vanishes on \( \partial X \) or not. When \( k \) does not vanish on \( \partial X \), the main singularities of the single scattering component do not allow us to ‘see inside’ the domain as they only depend on values of \( k \) at the domain’s boundary in dimension \( n \geq 3 \). The singular structure of single scattering and the resulting stability estimates are presented in detail both when \( k \) vanishes on \( \partial X \) and when it does not. Since the case of non-vanishing \( k \) on \( \partial X \) is practically interesting mostly as a negative result (for then we are not able to reconstruct \( k \) inside the domain from such singularities in dimension \( n \geq 3 \)), we have presented the results without proofs in section 4 and refer the reader to [5] for the mathematical details. The proof of the results when \( k \) vanishes in the vicinity of \( \partial X \) are presented in detail in sections 5 and 6. In appendix A, we give the proof of elementary lemmas, which appear in section 5. In appendix B, we complete the proof of the technical but central proposition 2.1.
2. Forward problem and albedo operator

2.1. The linear Boltzmann transport equation

We now introduce notation and recall some known results on the linear transport equation. Let $X$ be a bounded convex open subset of $\mathbb{R}^n$, $n \geq 2$, with a $C^1$ boundary $\partial X$. Let $v(x)$ denote the outward normal unit vector to $\partial X$ at $x \in \partial X$. Let $\Gamma_\pm = \{(x, v) \in \partial X \times S^{n-1} | \pm v(x) \cdot v > 0\}$ be the sets of incoming and outgoing conditions. For $(x, v) \in \bar{X} \times S^{n-1}$, we define $\tau_\pm(x, v)$ and $\nu(x, v)$ by $\tau_\pm(x, v) := \inf \{s \in (0, +\infty) | x \pm sv \notin X\}$ and $\nu(x, v) := \tau_-(x, v) + \tau_+(x, v)$. For $x \in \partial X$, we define $\Sigma_{x,\pm} := \{v \in S^{n-1} | \pm v(x) \cdot v > 0\}$.

We consider two nonnegative (measurable) functions $\sigma : X \times S^{n-1} \to \mathbb{R}$ and $k : X \times S^{n-1} \times S^{n-1} \to \mathbb{R}$ and two convex open subsets $Y$ and $Z$ of $\mathbb{R}^n$ with $C^1$ boundary such that:

\[
Z \subseteq Y \subseteq X; \sigma \text{ is bounded and continuous on } Y \times S^{n-1} \text{ and supported on } \tilde{Y} \times S^{n-1};
\]

$k$ is bounded and continuous on $Z \times S^{n-1} \times S^{n-1}$ and supported on $\tilde{Z} \times S^{n-1} \times S^{n-1}$. \hspace{1cm} (2.1)

We consider also the real $\delta$ defined by $\delta := \inf_{(x, z) \in \partial X \times Z} |x - z|$ and we assume throughout this paper except in section 4 that $\delta > 0$. \hspace{1cm} (2.2)

In other words, except in section 4, $k$ vanishes in the $\delta$-vicinity of the boundary $\partial X$. In practice, this simply means that the array of detectors has to be located some distance away from the scattering region, which is not too restrictive an assumption.

Let $T > \eta > 0$. We consider the following linear Boltzmann transport equation:

\[
\frac{\partial u}{\partial t}(t, x, v) + v \cdot \nabla_x u(t, x, v) + \sigma(x, v)u(t, x, v) = \int_{S^{n-1}} k(x, v', v)u(t, x, v')dv', \hspace{1cm} (t, x, v) \in (0, T) \times X \times S^{n-1},
\]

\[
u_{|_{(0, T) \times \Gamma_+}}(t, x, v) = \phi(t, x, v),
\]

\[
u(0, x, v) = 0, \hspace{1cm} (x, v) \in X \times S^{n-1},
\]

where $\phi \in L^1((0, T), L^1(\Gamma_+, d\xi))$ and $\text{supp } \phi \subseteq [0, \eta]$. Signals are then emitted for a maximal duration $\eta$ and are recorded at the domain’s boundary for a duration $T$ that will be chosen sufficiently large so that information can travel through the domain $X$ and be measured. Here, $d\xi(x, v) = |v \cdot v(x)|dv \, d\mu(x)$, where $d\mu$ is the surface measure on $\partial X$ and $dv$ is the surface measure on $S^{n-1}$.

The theory for (2.3) is well developed; we refer the reader to [4, 7, 9]. For our purpose, it is sufficient to recall that the solution may be decomposed as

\[
u(t) = G_-(t)\phi + \int_0^t \sum_{m=1}^\infty H_{m-1}(t-s)A_2G_-(s)\phi \, ds,
\]

where we have defined the following operators:

\[
A_2f = \int_{S^{n-1}} k(x, v', v)f(x, v')dv', \hspace{1cm} H_m(t) = \int_0^t H_{m-1}(t-s)A_2U_1(s) \, ds, \hspace{1cm} m \geq 1
\]

\[
U_1(t) = H_0(t) = e^{\int_0^1 \sigma(x-tv, v)dv}f(x-tv, v)\chi_{[0, t_\tau_+(x,v)]}(t), \hspace{1cm} (x, v) \in X \times S^{n-1},
\]

\[
G_-(t) \phi(x, v) = e^{-\int_0^1 \sigma(x-tv, v)dv} \phi_0(x-t_\tau_+(x,v), x = \tau_+(x, v)v, v), \hspace{1cm} (t, x, v) \in (0, T) \times X \times S^{n-1},
\]
with \( \chi_{[0, r(x, v)]} \) defined on \( \mathbb{R} \) such that \( \chi_{[0, r(x, v)]}(t) = 1 \) when \( 0 \leq t < r_-(x, v) \) and \( \chi_{[0, r(x, v)]}(t) = 0 \) otherwise. Functions \( \phi \in L^1((0, \eta), L^1(\Gamma_-, d\xi)) \) are extended to \( t \in \mathbb{R} \) by 0 outside of the interval \((0, \eta)\). The first term \( G_-(t)\phi \) in the above series is the ballistic part of \( u(t) \) while the term corresponding to \( m \geq 1 \) is \( m \)-linear in the scattering kernel \( k \). The term corresponding to \( m = 1 \) is the single scattering term.

The albedo operator \( A \) given by the formula

\[
A\phi = u_{(0,T)\times\Gamma_+}, \quad \text{for} \quad \phi \in L^1((0, \eta), L^1(\Gamma_-, d\xi)) \quad \text{where} \quad u \quad \text{solves (2.3)},
\]

is also well-defined and bounded; see [4, 7] for a derivation of the albedo operator and for the reconstruction of the optical parameters when the full albedo operator is known. We assume here that only partial knowledge of the albedo operator is available from measurements.

2.2. The operator \( A_{S,W} \) and its distributional kernel

We now define more precisely the type of measurements we consider in this paper. The directional behavior of the source term is determined by a fixed function \( S(x, v) \), which is bounded and continuous on \( \Gamma_- \). We assume that the incoming conditions have the following structure:

\[
\phi_\delta(t', x', v') = S(x', v')\phi(t', x'), \quad t' \in (0, \eta), \quad (x', v') \in \Gamma_-.
\]

where \( \phi(t, x) \) is an arbitrary function in \( L^1((0, \eta) \times \partial X) \). We model the detectors by the kernel \( W(x, v) \), which we assume is a continuous and bounded function on \( \Gamma_+ \). The available measurements are therefore modeled by the availability of the averaged albedo operator \( A_{S,W} \) from \( L^1((0, \eta) \times \partial X, d\sigma(x)) \) to \( L^1((0, T) \times \partial X, d\mu(x)) \) and defined by

\[
A_{S,W}\phi(t, x) = \int_{\mathbb{R}^n_+} A(\phi_\delta)(t, x, v)W(x, v)(\nu(x) \cdot v) \, dv, \quad \text{for a.e.} \quad (t, x) \in (0, T) \times \partial X.
\]

The functions \( S \) and \( W \) are fixed throughout the paper. The case \( W = 1 \) corresponds to measurements of the current of exiting particles at the domain’s boundary.

The decomposition of the transport solution (2.4) translates into a similar decomposition of the albedo operator of the form

\[
A_{S,W}\phi(t, x) = \sum_{m=0}^{+\infty} A_{m,S,W}\phi(t, x),
\]

for \((t, x) \in (0, T) \times \partial X\), where we have defined

\[
A_{0,S,W}\phi(t, x) = \int_{\mathbb{R}^n_+} (\nu(x) \cdot v)W(x, v) (G_-(\cdot)\phi_\delta)_{(0,T)\times\Gamma_+} (t, x, v) \, dv,
\]

\[
A_{m,S,W}\phi(t, x) = \int_{\mathbb{R}^n_+} (\nu(x) \cdot v)W(x, v) \times \left( \int_{-\infty}^{t'} H_{m-1}(t-s)A_2G_-(s)\phi_\delta \, ds \right)_{(0,T)\times\Gamma_+} (t, x, v) \, dv,
\]

for a.e. \((t, x) \in (0, T) \times \partial X\) where \( \phi_\delta \) is defined by (2.6). The kernels of the operators \( A_{m,S,W} \) can be written explicitly. Consider the nonnegative (measurable) function \( E \) from \( \partial X \times \partial X \) to \( \mathbb{R} \) defined by

\[
E(x_1, x_2) = \exp \left(- \int_0^{\frac{|x_1-x_2|}{\sigma}} \sigma \left( \frac{1}{2} \frac{x_1-x_2}{|x_1-x_2|}, \frac{x_1-x_2}{|x_1-x_2|} \right) \, ds \right)
\]
for \((x_1, x_2) \in \partial X \times \partial X\). We use the same notation \(E(x_1, x_2)\) when \(x_1\) and \(x_2\) are either in \(X\) or on \(\partial X\). For \(m \geq 3\), we also define the nonnegative (measurable) function \(E(x_1, \ldots, x_m)\) by induction:

\[
E(x_1, \ldots, x_m) = E(x_1, \ldots, x_{m-1})E(x_{m-1}, x_m), \quad m \geq 3,
\]

which measures the total attenuation along the broken path \((x_1, \ldots, x_m) \in \partial X \times X^{m-2} \times \partial X\).

For \(m \in \mathbb{N}, m \geq 1\) and for any subset \(U\) of \(\mathbb{R}^m\) we denote by \(\chi_U\) the characteristic function from \(\mathbb{R}^m\) to \(\mathbb{R}\) defined by \(\chi_U(y) = 1\) when \(y \in U\) and \(\chi_U(y) = 0\) otherwise. We then obtain the following result on the structure of the kernels of the albedo operator.

**Proposition 2.1.** We have

\[
A_{m,S,W}(\phi)(t, x) = \int_{(0,0) \times \partial X} \gamma_m(t - t', x, x')\phi(t', x') \, dt' \, d\mu(x'), \tag{2.12}
\]

for \(m \geq 0\) and a.e. \((t, x) \in (0, T) \times \partial X\), where

\[
\gamma_0(\tau, x, x') := \frac{E(x, x')}{|x - x'|^{n-1}}W(x, v)S(x', v)(v(x) \cdot v)|v(x') \cdot v|_{v'}\delta(\tau - |x - x'|),
\]

\[
\gamma_1(\tau, x, x') := \chi_{(0,\infty)}(\tau - |x' - x|) \int_{S^{n-1}} (v(x) \cdot v)W(x, v)[E(x, x - sv, x')k(x - sv, v', v)
\]

\[
\times \chi_{(0,\infty)}(\tau - |x' - x|)S(x', v')|v(x') \cdot v'|_{v'}\delta(\tau - |x - x'|)
\]

\[
\times \frac{2^{n-2}(|\tau - (x - x') \cdot v|^{n-3}}{|x - x' - \tau v|^{2n-4}} \, dv,
\]

for \((\tau, x, x') \in \mathbb{R} \times \partial X \times \partial X\) and where \(\gamma_m\) for \(m \geq 2\) admits a similar, more complex, expression given in section 5 (see (5.21) and (5.22)).

Proposition 2.1 is proved in detail in appendix B. As in (2.4), \(\gamma_0\) is the kernel of the ballistic contribution to \(A_{S,W}\), \(\gamma_1\) that of the single scattering contribution and \(\gamma_m\) that of the contribution that is \(m\)-linear in \(k\).

### 2.3. Regularity of the albedo kernels

The reconstruction of the optical parameters is based on an analysis of the behavior in time of the kernels of the albedo operator. We define the scattering kernels

\[
\Gamma_0 = \sum_{m=0}^{+\infty} \gamma_m, \quad \Gamma_1 = \Gamma_0 - \gamma_1, \quad \Gamma_2 = \Gamma_1 - \gamma_2.
\]

Thus, \(\Gamma_k\) accounts for scattering of order at least \(k\) in the albedo operator. Our first result is the following.

**Theorem 2.2.** Under the assumption \(k \in L^\infty(X \times S^{n-1} \times S^{n-1})\) and under assumption (2.2), which implies that the scattering coefficient vanishes in the vicinity of \(\partial X\), we have

\[
(\tau - |x - x'|)^{-\frac{n}{4}}\gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X) \quad \text{when } n \geq 2,
\]

\[
\Gamma_2(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when } n = 2,
\]
for any \((v, x)\)
\[\left(1 + \ln \left(\frac{\tau + |x - x'|}{\tau - |x - x'|}\right)\right)^{-1} \Gamma_2(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when } n = 3,\]
\[(\tau - |x - x'|)^{\frac{1+n}{2}} \Gamma_2(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when } n \geq 4.\]  
(2.18) (2.19)

Theorem 2.2 is proved in section 5. The results in (2.17)–(2.19) quantify how 'smoother' multiple scattering is compared to the single scattering contribution considered in (2.16).

2.4. Single scattering contribution and weighted x-ray transform

We want to analyze the behavior of the function \(\gamma_1(\tau, x, x')\) given by the right-hand side of (2.14) for all \((\tau, x, x') \in \mathbb{R} \times \partial X \times \partial X\). Under hypothesis (2.2), i.e., when the scattering coefficient vanishes in the vicinity of where measurements are collected, theorem 2.3 below describes the behavior in time of the single scattering term as a function of \(E(x, x')\) for \((x, x') \in \partial X^2\) (which is uniquely determined by the ballistic term; see (2.13)) and the weighted x-ray transform \(P_{v_0}\) of \(x \mapsto k(x, v_0, v_0)\) (for a fixed \(v_0 \in \mathbb{S}^{n-1}\)), where \(P_{v_0}\) is defined by
\[P_{v_0}(v, x) = \int_{\tau \cdot (x, v)} \partial_0(v, tv + x) f(tv + x) \, dt,\]
for a.e. \((v, x) \in \mathbb{S}^{n-1} \times \partial X\) and \(f \in L^2(X, \sup_{v \in \mathbb{S}^{n-1}} \partial_0(v, x) \, dx)\), and where the weight \(\partial_0 : \mathbb{S}^{n-1} \times X \to \mathbb{R}\) is the function defined by
\[\partial_0(v, x) = (\tau \cdot (x, v)) \sqrt{n-1}, \quad (v, x) \in \mathbb{S}^{n-1} \times X.\]  
(2.20) (2.21)

**Theorem 2.3.** Let \((x, x'_0) \in \partial X^2\) be such that \(x'_0 + s(x - x'_0) \in Z\) for some \(s \in (0, 1)\). Define \(v_0 = \frac{x - x'_0}{|x - x'_0|}\) and \(t_0 = |x - x'_0|\) and let \(k_{v_0}(y) := k(y, v_0, v_0)\) for \(y \in X\). Then we have the following:

\[\gamma_1(\tau, x, x'_0) = \frac{1}{\sqrt{\tau - t_0}} \sqrt[4]{2W(x, v_0)S(x'_0, v_0)|v(x'_0) \cdot v_0|\sqrt{t_0^2} - (2\pi)^{\frac{n}{2}} \partial_0(v_0, x), \quad \text{as } \tau \to t_0^+, \quad \text{when } n = 2,\]
\[\gamma_1(\tau, x, x'_0) = (\tau - t_0)^{\frac{n-1}{2}} (2\pi)^{\frac{n}{2}} \partial_0(v_0, x) W(x, v_0)|v(x'_0) \cdot v_0|\sqrt{t_0^2} - (2\pi)^{\frac{n}{2}} \partial_0(v_0, x), \quad \text{as } \tau \to t_0^+, \quad \text{when } n \geq 3.\]  
(2.22) (2.23)

Theorem 2.3 is proved in section 6. Theorem 2.3 may remain valid under more general assumptions. For instance, when \(\sigma\) is bounded and continuous on \(X, k\) is continuous on \(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}\) and \(k(x, .., )\) decays sufficiently rapidly as \(x\) approaches the boundary \(\partial X\) for any \(x \in X\), then the same asymptotic expansion holds for \(\gamma_1\).

3. Uniqueness and stability results under condition (2.2)

We recall that \(\Gamma_0 = \sum_{m=0}^{\infty} \Gamma_m\) the distributional kernel of \(A_{\mathbb{S}^1}\) and that \(\Gamma_0 - \gamma_0 = \Gamma_1\) denotes the distributional kernel of the multiple scattering of \(A_{\mathbb{S}^1}\). Form the remainder of the paper,
we assume that the duration of measurement $T > \text{diam}(X) := \sup_{x,y \in X} |x - y|$ so that the singularities of the ballistic and single scattering contributions are indeed captured by the available measurements.

Let $(\sigma, k)$ be a pair of absorption and scattering coefficients that also satisfy (2.1) and (2.2) for the same $(Y, Z)$ related to $(\sigma, k)$. We denote by a superscript "$\tilde{}$" any object (such as the albedo operator $A$ or the distributional kernels $\Gamma_0$ and $\tilde{\gamma}_0$) associated with $(\sigma, k)$. Let $\| \cdot \|_{\partial X} := \| \cdot _{L^1((0,T) \times \partial X)}$.

3.1. Stability estimates in integral form

The following theorem presents stability results for the reconstruction of the attenuation coefficient and of the scattering coefficient when the latter vanishes in the vicinity of the boundary $\partial X$.

**Theorem 3.1.** Assume $m_{S,W} = \min \{ \inf_{\Gamma} S, \inf_{\Gamma} W \} > 0$. Let $(\sigma, k)$ and $(\tilde{\sigma}, k)$ satisfy conditions (2.1) and (2.2). Let $x_0' \in \partial X$. Then there exist constants $C_1 = C_1(m_{S,W}, X, Y)$ and $C_2 = C_2(m_{S,W}, X, Z)$ such that

\[
\int_{\partial X} |E - \tilde{E}|(x_0' + \tau(x_0', v_0, x_0')v_0, x_0')|v(x_0) \cdot v_0| \, dv_0 \leq C_1 \| A_{S,W} - \tilde{A}_{S,W} \|_{\partial X},
\]

\[
\leq C_2 \| \tau - |z - z'| \|_{L^\infty}^2 (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \|_{L^\infty},
\]

for $x \in \partial X$ such that $px_0' + (1 - p)x \in Z$ for some $p \in (0, 1)$ where $v_0' = \frac{v_0}{|x - x_0'|}$, $P_{\theta_0}$ is defined in (2.20), and $k_{\theta_0}(y) := k(y, v_0, v_0')$ for $y \in X$ ($\tilde{k}_{\theta_0}$ is defined similarly).

Theorem 3.1 is the main result of the paper. The first estimate (3.1) shows that integrals of the attenuation coefficients are stably determined by the measurements $A_{S,W}$. The attenuation coefficient may then stably be reconstructed by inverse Radon transform as we will see in (3.7) below. The second inequality shows that a weighted integral of the scattering coefficient is stably determined by an appropriate measure of the multiple scattering coefficient $\Gamma_1$. What may then be stably reconstructed in the scattering kernel will be made explicit in theorems 3.3 and 3.4. Theorem 3.1 is proved in section 6.

3.2. Uniqueness and stability when $X$ is a ball of $\mathbb{R}^n$

When $X$ is an open Euclidean ball of $\mathbb{R}^n$, which is important from the practical point of view in medical imaging as it is relatively straightforward to place sources and detectors on a sphere, we are able to invert the weighted x-ray transform $P_{\theta_0}f$ for functions of the form $f(x) \in L^2(X, \sup_{t \in S^{n-1}} \Phi(x,v) \, dx)$ using the classical inverse x-ray transform (inverse Radon transform in dimension $n = 2$). More general convex domains $X$, which require one to solve more complex weighted x-ray transforms, are considered in the following subsection.

Up to rescaling, we assume $X = B_n(0, 1)$, the ball in $\mathbb{R}^n$ centered at 0 of radius 1. Consider the x-ray transform $P$ defined by

\[
P f(v, x) = \int_{\Gamma, (x,v)} f(sv + x) \, ds \quad \text{for a.e. } (v,x) \in S^{n-1} \times \partial X,
\]

for $f \in L^2(X)$ (we extend $f$ by 0 outside $X$). We have the following result:
**Proposition 3.2.** When \( X = B_0(0, 1) \) we have

\[
P_{B_0}(v, x) = P(v) f(v, x), \quad \text{for a.e. } (v, x) \in S^{n-1} \times \partial X,
\]

for \( f \in L^2(X, \sup_{x \in S^{n-1}} \partial_{B_0}(v, x) \, dx) \) where \( \varrho(y) := (1 - |y|^2)^{-\frac{n-1}{2}}, \ y \in X \).

**Proof of proposition 3.2.** It is easy to see that

\[
\tau_+(t q + q v^\perp, v) = \sqrt{1 - q^2} \pm t,
\]

\[
\vartheta_0(v, x) = (1 - q^2 - t^2)^{-\frac{n-1}{2}} = (1 - |x|^2)^{-\frac{n-1}{2}},
\]

for \((t, q) \in \mathbb{R}^2, t^2 + q^2 \leq 1\) and for \((v, v^\perp) \in S^{n-1} \times S^{n-1}, v \cdot v^\perp = 0\), where \(x = tv + qv^\perp\) (we recall that \(\vartheta_0\) is defined by (2.21)). Then proposition 3.2 follows from the definition (2.20).

Assume that \((\sigma, k)\) satisfies (2.1) and (2.2). Assume also that \(k(x, v, v') = k_0(x)g(v, v')\) for a.e. \((x, v, v') \in X \times S^{n-1} \times S^{n-1}\) where \(g\) is a given continuous function on \(S^{n-1} \times S^{n-1}\), \(\inf_{v \in S^{n-1}} g(v, v) > 0\), and where \(k_0 \in L^\infty(X)\). Then from the decomposition of the angularly averaged albedo operator \(A_{S,W}\) (proposition 2.1) and from theorems 2.2, 2.3, and from proposition 3.2 and methods of reconstruction of a function from its x-ray transform, it follows that \((\sigma, k_0)\) can be reconstructed from the asymptotic expansion in time of \(A_{S,W}\) provided that \(\sigma = \sigma(x)\) and \(\min(\inf_{\Gamma_- S, \inf_{\Gamma_+ W}} > 0)\). In addition we have the following stability estimates.

**Theorem 3.3.** Assume \( X = B_0(0, 1) \) and \( m_{S,W} = \min(\inf_{\Gamma_- S}, \inf_{\Gamma_+ W}) > 0 \). Let \((\sigma, k)\) and \((\bar{\sigma}, \bar{k})\) satisfy conditions (2.1) and (2.2). Assume that \(\sigma, \bar{\sigma}\) do not depend on the velocity variable \((\sigma(x, v) = \sigma(x))\) and let \(M = \max(\|\sigma\|_{L^\infty(Y)}, \|\bar{\sigma}\|_{L^\infty(Y)})\). Assume \(k(x, v, v') = k_0(x)g(v, v')\) and \(\bar{k}(x, v, v') = \bar{k}_0(x)g(v, v'), g(v, v) > 0\), for \((x, v, v') \in X \times S^{n-1} \times S^{n-1}\) where \(g\) is an a priori known continuous function on \(S^{n-1} \times S^{n-1}\).

Then there exists \(C_3 = C_3(m_{S,W}, X, Y, M)\) and \(C_4 = C_4(m_{S,W}, X, Y, Z, M, g)\) such that

\[
\|\sigma - \bar{\sigma}\|_{L^{\frac{4}{3}}(Y)} \leq C_3\|\sigma - \bar{\sigma}\|_{L^\infty(Y)}\|A_{S,W} - \bar{A}_{S,W}\|_{H^{-\frac{1}{2}}(Y)};
\]

\[
\|k_0 - \bar{k}_0\|_{L^{\frac{4}{3}}(Z)} \leq C_4\|k_0 - \bar{k}_0\|_{L^\infty(Z)}\|A_{S,W} - \bar{A}_{S,W}\|_{H^{-\frac{1}{2}}(Y)} + \|\tau - |z - z'|\|_{L^\infty}^{\frac{4}{3}}(\|\Gamma_1 - \Gamma_1\|_{L^\infty} + \|\Gamma_1 - \Gamma_1\|_{L^\infty})(\tau, z, z')\|_{L^\infty}^{\frac{4}{3}}.
\]

Theorem 3.3 is proved as theorem 3.4 given below for a larger class of domains \(X\). Under the assumptions of theorem 3.3 and additional regularity assumptions on \((\sigma, k)\), we obtain stability estimates similar to those given in corollary 3.5 below for a larger class of domains \(X\). Note that \(\|\sigma - \bar{\sigma}\|_{L^\infty(Y)}\) and \(\|k_0 - \bar{k}_0\|_{L^\infty(Z)}\) are bounded a priori by positive constants. These and similar estimates below show how measurement errors translate into reconstruction errors. These are Hölder-type estimates in the sense that measurement errors of size \(\delta\) generate reconstruction errors of size \(\delta^\alpha\) for some \(\alpha > 0\). Such estimates should be compared to those obtained for other measurement settings in inverse transport theory; see e.g. [2].

### 3.3. Uniqueness and stability estimates for more general domains \(X\)

**Theorem 3.4.** Assume that the open subset \(X\) of \(\mathbb{R}^n\) is convex with a real analytic boundary and that \(\min(\inf_{\Gamma_- S}, \inf_{\Gamma_+ W}) > 0\). Let \((\sigma, k)\) and \((\bar{\sigma}, \bar{k})\) satisfy conditions (2.1) and
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(2.2). Assume also that \( \sigma, \hat{\sigma} \) do not depend on the velocity variable \( (\sigma(x, v) = \sigma(x)) \) and \( k(x, v, v') = k_0(x) g(x, v, v') \) and \( \hat{k}(x, v, v') = \hat{k}_0(x) g(x, v, v') \), \( g(x, v, v') > 0 \), for \( (x, v, v') \in X \times S^{n-1} \times S^{n-1} \) where \( g \) is an a priori known real analytic function on \( X \times S^{n-1} \times S^{n-1} \) and where \( \text{supp}k_0 \cup \text{supp} \hat{k}_0 \subseteq Z \), \( (k_0, \hat{k}_0) \in L^\infty(Z) \). Then estimates (3.7) and (3.8) still hold.

Theorem 3.4 is proved in section 6. Now we give stability estimates under additional regularity assumptions on the optical parameters and on \( X \). Assume that \( X \) is convex with a C^1 boundary. Let \( g \) be an a priori known real analytic function on \( X \times S^{n-1} \times S^{n-1} \), \( g(x, v, v') > 0 \) for \( (x, v, v') \in X \times S^{n-1} \times S^{n-1} \). Let \( r_1 > 0, r_2 > 0 \). Consider the class

\[
N := \left\{ (\sigma, k) \in H^{\frac{\kappa}{2}}(Y) \times L^\infty(Z) | \sigma \in L^\infty(Y), k \in L^\infty(Z), \|\sigma\|_{H^{\frac{\kappa}{2}}(Y)} \leq M_1, \right. \\
\left. k = k_0 g, \text{supp} k_0 \subseteq Z, \|k_0\|_{H^{\frac{\kappa}{2}}(Z)} \leq M_2 \right\}. 
\]

(3.9)

Note that there exists a function \( D_1 : \mathbb{N} \times (0, +\infty) \to (0, +\infty) \) such that

\[
\|\sigma\|_{L^\infty(Y)} \leq D_1(n, r_1) \|\sigma\|_{H^{\frac{\kappa}{2}}(Y)} \leq D_1(n, r_1) M_1, \\
\|k_0\|_{L^\infty(Z)} \leq D_1(n, r_2) \|k_0\|_{H^{\frac{\kappa}{2}}(Z)} \leq D_1(n, r_2) M_2, \\
\|k\|_{L^\infty(Z)} \leq g \|L^\infty(Z)\| \|k_0\|_{L^\infty(Z)} \leq D_1(n, r_2) M_2 \|g\|_{L^\infty(Z)},
\]

(3.10)

for \( (\sigma, k) \in N \). We also use the classical interpolation formula

\[
\|f\|_{H^{s}(\Omega)} \leq \|f\|_{H^{\frac{s}{2}}(\Omega)}^{\frac{s}{2}} \|f\|_{H^{\frac{s}{2}}(\Omega)}^{\frac{s}{2}},
\]

(3.11)

for \( s_1 < s < s_2 \) and for \( (\Omega, s_1, s_2) \in \left\{ (Y, -\frac{1}{2}, \frac{n}{2} + r_1), (Z, -\frac{1}{2}, \frac{n}{2} + r_2) \right\} \). Using theorem 3.4 and (3.10), and applying (3.11) on \( f = \sigma - \hat{\sigma} \) and \( f = k_0 - \hat{k}_0 \) we obtain the following result.

**Corollary 3.5.** Let \( (\sigma, \hat{\sigma}, \tilde{k}) \in N \). Then, for \( -\frac{1}{2} < s \leq \frac{n}{2} + r_1 \) and for \( 0 < r < r_1 \), there exists \( C_5 = C_5(m_{S,W}, X, Y, M_1, r_1, s) \) such that

\[
\|\sigma - \hat{\sigma}\|_{H^s(\Omega)} \leq C_5 \|\sigma - \hat{\sigma}\|_{L^\infty(\Omega)}^{\frac{s}{2}} \|AS_w - \tilde{A}_S,w\|_{n,T}^{\frac{s}{2}}, \tag{3.12}
\]

\[
\|\sigma - \sigma\|_{H^\frac{\kappa}{2}(Y)} \leq C_6 \|AS_w - \tilde{A}_S,w\|_{n,T}^{\kappa}, \tag{3.13}
\]

where \( (\kappa, \kappa') = (\frac{n+2(r_2-s)}{n+1+2r_2}, \frac{2(r_2-s)}{n+1+2r_2}) \) and \( C_6 = C_5^{\frac{\kappa}{2}} D_1(n, r)^{\frac{\kappa'}{2}} \) (\( D_1(n, r) \) is defined by (3.10)).

In addition, there exists \( C_7 = C_7(m_{S,W}, X, Y, Z, g, M_1, r_1, M_2, r_2, s) \) such that

\[
\|k_0 - \tilde{k}_0\|_{H^s(Z)} \leq C_7 \|k_0 - \tilde{k}_0\|_{L^\infty(Z)}^{\frac{s}{2}} \|AS_w - \tilde{A}_S,w\|_{n,T} + \|\sigma - \hat{\sigma}\|_{L^\infty(\Omega)}^{\frac{s}{2}} \|\Gamma_1 - \tilde{\Gamma}_1\|_{L^\infty(\Omega)}^{\frac{s}{2}}, \tag{3.14}
\]

\[
\|k_0 - \tilde{k}_0\|_{L^\infty(Z)} \leq C_8 \|\sigma - \hat{\sigma}\|_{L^\infty(\Omega)}^{\frac{s}{2}} + \|\sigma - \hat{\sigma}\|_{L^\infty(\Omega)}^{\frac{s}{2}} \|\Gamma_1 - \tilde{\Gamma}_1\|_{L^\infty(\Omega)}^{\frac{s}{2}}, \tag{3.15}
\]

for \( -\frac{1}{2} < s \leq \frac{n}{2} + r_2 \) and \( 0 < r < r_2 \), where \( (\kappa, \kappa') = (\frac{n+2(r_2-s)}{n+1+2r_2}, \frac{2(r_2-s)}{n+1+2r_2}) \) and

\[
C_8 = C_7^{\frac{\kappa}{2}} D_1(n, r)^{\frac{\kappa'}{2}} \) (\( D_1(n, r) \) is defined by (3.10)).
Remark 3.6.

(i) Theorem 3.4 and corollary 3.5 remain valid when: \( X \) is only assumed to be convex with \( C^2 \) boundary; the weight \( \theta_\sigma \) defined by (2.21) (resp. the function \( g \) which appears in the assumptions of theorem 3.4 and corollary 3.5) is sufficiently close (in the \( C^2 \) norm) to an analytic weight \( \theta_{0,a} \) on the vicinity of \( \overline{Z} \times S^{n-1} \) (resp. an analytic function \( g_a \) on the vicinity of \( \overline{Z} \times S^{n-1} \times S^{n-1} \)); see proof of theorem 3.4 and [10, theorem 2.3].

(ii) When \( n = 3 \) then under hypothesis (2.2), we have
\[
\left\| (\tau - |z - z'|^{1/2} (\Gamma_1_0 + \Gamma_1^1)(\tau, z, z') \right\|_{L^n} = \left\| \sum_{m=1}^\infty (A_{m,S,W} - \tilde{A}_{m,S,W}) \right\|_{L^1((0, \eta) \times \partial X), L^\infty((0, T) \times \partial X)}.
\]
where the distribution kernel of the bounded operator \( \sum_{m=1}^\infty (A_{m,S,W} - \tilde{A}_{m,S,W}) \) from \( L^1((0, \eta) \times \partial X) \) to \( L^1((0, T) \times \partial X) \) is given by \( \Gamma_1_0 + \Gamma_1^1 \). Therefore when \( n = 3 \) and under condition (2.2), the right-hand side of the stability estimates (3.8) and (3.14) can be expressed with operator norms only (instead of using a norm on the distributional kernel of the multiple scattering).

4. Non-vanishing scattering on \( \partial X \)

Throughout this section, we consider the class of optical parameters \((\sigma, k)\) such that \((\sigma, k)\) satisfies (2.1) with
\[
X = Y = \overline{Z} (\text{thus } \delta = 0); \quad \text{the function } k \text{ is continuous on } \overline{X} \times S^{n-1} \times S^{n-1} \quad \text{.} \quad (4.1)
\]
Under assumptions (2.1) and (4.1), the albedo operator \( A_{S,W} \) is defined as in section 2 and the decomposition given in proposition 2.1 still holds. The behavior in time of the measurement operator \( A_{S,W} \) is however significantly modified when \( k \) does not vanish on \( \partial X \). Because they appear to be less interesting practically, the results of this section are given without proofs. We refer the reader to [5] for the details.

4.1. Behavior in time of the averaged albedo operator

The analog of theorem 2.2 is given by theorems 4.1 and 4.2.

**Theorem 4.1.** Under the conditions \( k \in L^\infty(X \times S^{n-1} \times S^{n-1}) \), the following holds:
\[
\sqrt{\tau^2 - |x - x'|^2} \gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when} \quad n = 2, \\
\frac{\tau |x - x'|}{\ln \left( \frac{\tau + |x - x'|}{\tau - |x - x'|} \right)} \gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when} \quad n = 3, \\
\tau |x - x'|^{n-2} \gamma_1(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X), \quad \text{when} \quad n \geq 4. \quad (4.4)
\]

The results in theorem 4.2 quantify how ‘smoother’ multiple scattering is compared to single scattering contribution considered in (4.2)–(4.4).

**Theorem 4.2.** Assume that \( k \in L^\infty(X \times S^{n-1} \times S^{n-1}) \). Then we have
\[
\text{statement (2.17) still holds,} \quad \text{when} \quad n = 2; \quad (4.5)
\]
Theorem 4.4. Let \( x, x_0 \in \partial \mathbb{R}^3 \) be such that \( x + s(x - x_0) \in \mathbb{R}^3 \) for some \( s \in (0, 1) \). Set \( v_0 = \frac{x - x_0}{|x - x_0|} \) and \( t_0 = |x - x_0| \). Then under condition (4.1), we have the following results.

(2.22) still holds when \( n = 2 \),

\[
\gamma_1(\tau, x, x_0) = \ln \left( \frac{1}{\tau - t_0} \right) \frac{\pi}{t_0} W(x, v_0) \mathbb{S}(x_0, v_0)(v(x) \cdot v_0)|v(x_0) - v_0|E(x, x_0) \times (k(x, v_0, v_0) + k(x_0, v_0, v_0)) + o\left( \ln \left( \frac{1}{\tau - t_0} \right) \right),
\]

as \( \tau \to t_0^+ \), when \( n = 3 \),

\[
\gamma_1(\tau, x, x_0) = t_0^{1-n} E(x, v_0) \left[ \mathbb{S}(x_0, v_0)|v(x_0) - v_0| \int_{\mathbb{R}^3} \frac{W(x, v)(v(x) - v)}{1 - v \cdot v_0} \, dv \right]
+ W(x, v_0)(v(x) - v_0) \int_{\mathbb{R}^3} \frac{k(x_0, v', v_0)S(x_0, v')|v(x_0) - v'|}{1 - v' \cdot v_0} \, dv' + o(1),
\]

as \( \tau \to t_0^+ \), when \( n \geq 4 \).

Note that the asymptotic expansion in time of \( \gamma_1 \) depends on the values of \( k \) on \( \partial \mathbb{R}^3 \) in dimension \( n \geq 3 \), and no longer on \( k \) inside \( \mathbb{R}^3 \). Such singularities are thus useless in the reconstruction of the scattering coefficient inside \( \mathbb{R}^3 \). The proof of these results can be found in [5].

4.2. Stability results

The singularities exhibited in the preceding results still allow us to perform stable reconstructions when \( k \) does not vanish on \( \partial \mathbb{R}^3 \). The analog of theorem 3.1 is as follows.

Theorem 4.4.3. Let \( (\sigma, k) \) and \( (\tilde{\sigma}, \tilde{k}) \) satisfy (2.1) and (4.1). Let \( x_0 \in \partial \mathbb{R}^3 \). Then we have

\[
\int_{\partial \mathbb{R}^3} \left[ \frac{|E - \tilde{E}|(x, x_0)}{|x - x_0|} W(x, v_0) \mathbb{S}(x', v_0)(v(x') \cdot v_0)|v(x_0) - v_0| \right] \, d\mu(x)
\leq \| A_{\mathbb{R}^3} - \tilde{A}_{\mathbb{R}^3} \|_{L^\infty},
\]

(4.10)

Let \( x \in \partial \mathbb{R}^3 \) be such that \( px_0 + (1 - p)x \in \mathbb{R}^3 \) for some \( p \in (0, 1) \). Set \( v_0 = \frac{x - x_0}{|x - x_0|} \) and \( t_0 = |x - x_0| \). When \( n = 2 \), we have

\[
W(x, v_0)S(x_0, v_0)(v(x) - v_0)|v(x_0) - v_0|E(x, x_0)P_{0, k_0}k_0(v_0, x) - \tilde{E}(x, x_0)P_{0, \tilde{k}_0}\tilde{k}_0(v_0, x)
\leq 1/2 \| \sqrt{\tau^2 - |z - \tilde{z}|^2} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, \tilde{z}) \|_{L^\infty},
\]

(4.11)
where \( \| \cdot \|_{L^\infty} := \| \cdot \|_{L^\infty((0,T) \times \partial \Omega \times \partial \Omega)} \). When \( n = 3 \), then
\[
|E(x, x_0')(k(x, v_0, v_0) + k(x_0', v_0, v_0)) - \tilde{E}(x, x_0')(\tilde{k}(x, v_0, v_0) + \tilde{k}(x_0', v_0, v_0))|
\leq \frac{1}{\pi} \| \frac{\tau |z - \tilde{z}|}{\ln (\frac{1 + |v - \tilde{v}|}{1 - |v - \tilde{v}|})} \|_{L^\infty} \times \| W(x, v_0)(v(x) \cdot v_0)|_{L^\infty} \|_{L^\infty} \times \| S(x_0', v_0)(v(x_0') \cdot v_0)|_{L^\infty} \|_{L^\infty}.
\]
(4.12)

When \( n \geq 4 \), then
\[
S(x_0', v_0)(v(x_0') \cdot v_0) \int_{S^{n-1}_+} \frac{W(x, v)((v(x) \cdot v_0)}{1 - v \cdot v_0} (E(x, x_0')k(x, v_0, v) - \tilde{E}(x, x_0')(\tilde{k}(x, v_0, v_0)) dv
+ \int_{S^{n-1}_-} \frac{S(x_0', v_0)(v(x_0') \cdot v_0)}{1 - v' \cdot v_0} \times (E(x, x_0')k(x_0', v', v_0) - \tilde{E}(x, x_0')(\tilde{k}(x_0', v', v_0)) dv
\leq \| \tau |z - \tilde{z}|^{n-2}(\Gamma_1 - \tilde{\Gamma}_1)\|_{L^\infty}.
\]
(4.13)

Note that if we assume that \( \inf_{\Gamma_+} S > 0 \) and \( \inf_{\Gamma_-} W > 0 \), then (4.10) and (3.1) are equivalent by performing the change of variables \( \tilde{v} = \frac{v_0 - \tilde{v}}{1 - v_0 \cdot \tilde{v}} \) in (4.10). Note also that (4.11) is similar to but different from (3.2) in dimension \( n = 2 \). Results in (4.12) and (4.13) show that the spatial structure of \( k \) may be stably reconstructed at the domain’s boundary in \( n \geq 3 \). They do not imply stable reconstruction of \( k \) inside the domain.

4.3. Improved results when \( X \) is a ball

First consider the case \( n = 2 \) and \( X = B_2(0, 1) \). Let \( (\sigma, k) \) satisfy (2.1) and (4.1). Assume that \( k(x, v, v') = k_0(x)g(v, v') \) for a.e. \( (x, v, v') \in X \times S^{n-1} \times S^{n-1} \) where \( g \) is a given continuous function on \( S^{n-1} \times S^{n-1} \), \( \inf_{v \in S^{n-1}} g(v, v) > 0 \), and where \( k_0 \in L^\infty(X) \). Then from the decomposition of the angularly averaged albedo operator \( A_{S,W} \) (proposition 2.1) and from theorems 4.1, 4.2, 4.3, and from proposition 3.2 and methods of reconstruction of a function from its x-ray transform, it follows that \( (\sigma, k_0) \) can be reconstructed from the asymptotic expansion in time of \( A_{S,W} \) provided that \( \sigma = \sigma(x) \) and \( \inf_{(x,v) \in \Gamma}\ S(x', v') > 0 \) and \( \inf_{(x,v) \in \Gamma}\ W(x, v) > 0 \). In addition we have the following stability estimates.

**Theorem 4.5.** Assume \( X = B_2(0, 1) \), and \( \min(\inf_{\Gamma_+} S, \inf_{\Gamma_-} W) > 0 \). Let \( (\sigma, k) \) and \( (\tilde{\sigma}, \tilde{k}) \) satisfy conditions (2.1) and (4.1). Assume that \( \sigma, \tilde{\sigma} \) do not depend on the velocity variable \( (\sigma(x, v) = \sigma(x)) \) and let \( M = \max(\|\sigma\|_{L^\infty(Y)}, \|\tilde{\sigma}\|_{L^\infty(Y)}) \). Assume \( k(x, v, v') = k_0(x)g(v, v') \) and \( \tilde{k}(x, v, v') = \tilde{k}_0(x)g(v, v'), g(v, v) > 0 \), for \( (x, v, v') \in X \times S^{n-1} \times S^{n-1} \) where \( g \) is an a priori known continuous function on \( S^{n-1} \times S^{n-1} \). Then we have

(i) (3.7) still holds;
(ii) when \( n = 2 \), there exists \( C_8 = C_8(S, W, X, Y, Z, M, g) \) such that
\[
\| \rho(k_0 - \tilde{k}_0) \|_{H^{-\frac{1}{2}}(X)} \leq C_8 \| k_0 - \tilde{k}_0 \|_{L^\infty}(\| k_0 \|_{L^\infty}(A_{S,W} - \tilde{A}_{S,W})_{0,T}) + \| \sqrt{\tau^2 - |z - \tilde{z}|^2(\Gamma_1 - \tilde{\Gamma}_1)} \|_{L^\infty} \frac{1}{2}.
\]
(4.14)

Theorem 4.5 can be proved as theorem 3.4. Under the assumptions of theorem 4.5 and additional regularity assumptions on \( (\sigma, k) \) one obtains stability estimates similar to those given in corollary 3.5.
5. Proof of theorem 2.2

For $0 < b < a$, we recall that

$$\int_0^{2\pi} \frac{1}{a - b \sin(\Omega)} \, d\Omega = \frac{2\pi}{\sqrt{a^2 - b^2}}. \tag{5.1}$$

5.1. Proof of (2.16)

First, we give an estimate on the single scattering term. From (2.14), it follows that

$$|\chi_1(\tau, x, x')| \leq 2^{n-2} \|W\|_\infty \|S\|_\infty \|k\|_\infty G_1(\tau, x, x')$$

for a.e. $(\tau, x, x') \in \mathbb{R} \times \partial X \times \partial X$, where

$$G_1(\tau, x, x') = \frac{\chi(0, x)}{\tau - |x - x'|} \int_{\mathbb{S}^{n-1}} \chi_{\text{supp} k}(x - sv) \left| \frac{(\tau - |x - x'|) \cdot v}{|\tau - |x - x'| - (x - x')^2|^{n/2}} \right| \, dv.$$  \tag{5.3}

Let $(\tau, x, x') \in (0, T) \times \partial X \times \partial X$ be such that $x \neq x'$ and $\tau > |x - x'|$. Assume without loss of generality $x' - x = |x - x'| (1, 0, \ldots, 0)$. Let $v \in \mathbb{S}^{n-1}$ and $\varepsilon := \frac{\tau - |x - x'|}{2(\tau - |x - x'| - v^2)}$. Straightforward computations give $s + |x - x' - sv| = \tau$. Using (2.2) we obtain that

if $\tau < \delta$ or $s > \tau - \delta$, then $x - sv \notin \text{supp} k$. \tag{5.4}

Let $n = 2$. From (5.3) and (5.1), it follows that

$$G_1(\tau, x, x') \leq \int_0^{2\pi} \frac{1}{\tau - |x - x'| \sin(\Omega)} \, d\Omega = \frac{2\pi}{\sqrt{\tau^2 - |x - x'|^2}} \leq \frac{2\pi}{\sqrt{\delta \tau - |x - x'|}}, \tag{5.5}$$

for $\tau \geq \delta$. Using (5.3) and (5.4), we obtain

if $\tau < \delta$ then $G_1(\tau, x, x') = 0$. \tag{5.6}

Combining (5.2) with (5.5) and (5.6), we obtain (2.16) for $n = 2$.

Let $n \geq 3$ and $\tau \geq \delta$ (the case $\tau < \delta$ is already considered in (5.6)). Performing the change of variables $r = \frac{\tau - |x - x'|^2}{2(\tau - |x - x'| \sin(\Omega))}$, with $v = \Phi(\Omega, \omega) := (\sin(\Omega), \cos(\Omega) \omega), \Omega \in (-\frac{\pi}{2}, \frac{\pi}{2}), \omega \in \mathbb{S}^{n-2}$ on the right-hand side of (5.3), we obtain

$$G_1(\tau, x, x') = 2^{n-1} \int_0^{\frac{1}{\sqrt{2}}} \int_{\mathbb{S}^{n-1}} \chi_{\text{supp} k}(x - sv) \left| \frac{\sqrt{r(|x - x'| - r)^{n-3}}}{(\frac{|x - x'|}{\sqrt{2}r} + r)^{n/2}(\frac{|x - x'|}{\sqrt{2}r} - r)^{n/2}} \right| \, dv \, dr.$$ \tag{5.7}

Now assume $\tau > \frac{\delta}{2} + |x - x'|$. Then

$$|x - x'|^{2-n} \int_0^{\frac{1}{\sqrt{2}}} \int_{\mathbb{S}^{n-1}} \frac{\sqrt{r(|x - x'| - r)^{n-3}}}{(\frac{|x - x'|}{\sqrt{2}r} + r)^{n/2}(\frac{|x - x'|}{\sqrt{2}r} - r)^{n/2}} \, dv \, dr \leq \left( \frac{\delta}{4} \right)^{4-2n} |x - x'|^{2-n} \int_0^{1} \sqrt{r(1 - r)^n} - 3 \, dr \leq \left( \frac{\delta}{4} \right)^{4-2n}.$$
Therefore using (5.7) we obtain

\[(\tau - |x - x'|)^{-\frac{\nu}{2}} G_1(\tau, x, x') \leq 2^{\nu - 2} \text{Vol}_{n-2}(\mathbb{S}^{n-2})(T + \text{diam}(X))^{\frac{\nu}{2}} (\frac{\delta}{2})^{4-2n}. \tag{5.8}\]

Finally assume \(\delta \leq \frac{\tilde{\delta}}{2} + |x - x'|\) and \(|x - x'| < \tau \leq T\). From (5.7) and (5.4), it follows that

\[(\tau - |x - x'|)^{-\frac{\nu}{2}} G_1(\tau, x, x') \leq \frac{\text{Vol}_{n-2}(\mathbb{S}^{n-2})(T + \text{diam}(X))^{\frac{\nu}{2}}}{2^{\nu - 2} |x - x'|^{n-2}} \times \int_{r_{\tau}(\tau, x, x')} \sqrt{r(\tau - |x - x'| - r)^{n-3}} \, dr, \tag{5.9}\]

\[r_{\tau}(\tau, x, x') := \frac{|x - x'| + \delta - \tau}{2}, \quad r_{\delta}(\tau, x, x') := \frac{\tau - \delta + |x - x'|}{2}. \tag{5.10}\]

Let

\[\int_{r_{\tau}(\tau, x, x')} \frac{\sqrt{r(\tau - |x - x'| - r)^{n-3}}}{(\frac{\tau - |x - x'|}{2} + r)^{n-2}} \, dr \]

\[= 2 \int_{r_{\tau}(\tau, x, x')} \frac{\sqrt{r(\tau - |x - x'| - r)^{n-3}}}{\tau - |x - x'| + r} \, dr \]

\[\leq 2 (\frac{\tau}{2})^{2-n} |x - x'|^{n-3} \int_{r_{\tau}(\tau, x, x')} \frac{1}{(\frac{\tau - |x - x'|}{2} + r)^{n-2}} \, dr = 2^{\nu - 1} (\frac{\delta}{2})^{3-n} |x - x'|^{n-3} \]

\[\times \int_{r_{\tau}(\tau, x, x')} \frac{1}{(\frac{\tau - |x - x'|}{2} + r)^{n-2}} \, dr. \tag{5.11}\]

Using (5.10) we obtain

\[\int_{r_{\tau}(\tau, x, x')} \frac{1}{(\frac{\tau - |x - x'|}{2} + r)^{n-2}} \, dr = C(n, \tau) := \begin{cases} \ln \left(\frac{\tau}{\delta}\right), & \text{if } n = 3, \\ \frac{1}{n-3} \left(\left(\frac{\delta}{2}\right)^{3-n} - \left(\frac{\tau}{2}\right)^{3-n}\right) & \text{otherwise.} \end{cases} \tag{5.12}\]

From (5.9), (5.11), (5.12) and the estimates \(\delta \leq \tau < \frac{\tilde{\delta}}{2} + |x - x'|\), it follows that

\[(\tau - |x - x'|)^{-\frac{\nu}{2}} G_1(\tau, x, x') \leq 2^{\nu} \text{Vol}_{n-2}(\mathbb{S}^{n-2})\delta^{-(n-1)}(T + \text{diam}(X))^{\frac{\nu}{2}} C(n, T), \tag{5.13}\]

where the constant \(C(n, T)\) is defined in (5.12). Combining (5.2) with (5.6), (5.8) and (5.13), we obtain (2.16) for \(n \geq 3\).

### 5.2. Preliminary results for the proof of (2.17)–(2.19)

To prove theorem 2.2, (2.17)–(2.19), we need the explicit expressions for \(\gamma_m, m \geq 2\), given below and the following lemmas 5.1 and 5.2, whose proof is given in appendix A.

Let \(m \geq 1\) and \(z, z' \in \mathbb{R}^n\) such that \(z \neq z'\). Let \(\mu \geq 0\). We denote by \(E_{m, n}(\mu, z, z')\) the subset of \((\mathbb{R}^n)^m\) defined by

\[E_{m, n}(\mu, z, z') = \{(y_1, \ldots, y_m) \in (\mathbb{R}^n)^m | |y_1| + \cdots + |y_m| + |z - z' - y_1 - \cdots - y_m| < \mu\}. \tag{5.14}\]

When \(\mu \leq |z - z'|\), then \(E_{m, n}(\mu, z, z') = \emptyset\).
**Lemma 5.1.** Let \( n \geq 2 \). Let \( (\mu, z, z') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \) be such that \( \mu > |z - z'| > 0 \). Then
\[
\text{Vol}_n(E_{1,\mu}(\mu, z, z')) \leq \frac{\text{Vol}_{n-2}(S^{n-2})\pi (\mu + |z - z'|) \left( \frac{\sqrt{\mu^2 - |z - z'|^2}}{2} \right)^{n-1}}{4}.
\] (5.15)

**Lemma 5.2.** Let \( n \geq 2 \) and \( \delta > 0 \). Let \( N \) denote the nonnegative measurable function from \((0, T) \times \partial X \times \mathbb{R}^n\) to \([0, +\infty[\) defined by
\[
N(\mu, z, z') = \chi(0, +\infty)(\mu - |z - z'|) \int_{|\tau - \nu| = 1} (\mu - (z - z') \cdot \nu)^{n-3} d\nu,
\] (5.16)
for \((\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n\). When \( n = 2 \), then
\[
C(N, 2) := \sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n} \int_{y \in E_{1,\mu}(\mu, z, z')} \frac{N(\mu - |y|, z, z' + y)}{|y|} dy < \infty.
\] (5.17)
When \( n = 3 \), then
\[
C_1(N, 3) := \sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n} (\mu - |z - z'|)^{-1} \left( 1 + \ln \left( \frac{\mu + |z - z'|}{\mu - |z - z'|} \right) \right)^{-1}
\times \int_{y \in E_{1,\mu}(\mu, z, z')} N(\mu - |y|, z, z' + y) dy < \infty,
\] (5.18)
\[
C_2(N, 3) := \sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n} (\mu - |z - z'|)^{-1} |\mu - z' - y| \left( 1 + \ln \left( \frac{\mu + |z - z'|}{\mu - |z - z'|} \right) \right)^{-2}
\times \int_{y \in E_{1,\mu}(\mu, z, z')} \frac{N(\mu - |y|, z, z' + y)}{|y|^2} dy < \infty.
\] (5.19)
When \( n \geq 4 \), then
\[
C(N, n) := \sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n} |\mu - z - z'|^{n-2} N(\mu, z, z') < \infty.
\] (5.20)

The explicit expression for \( \gamma_m, m \geq 2 \), is given by
\[
\gamma_2(\tau, x, x') := \int_{y \in E_{1,\mu}(\tau, x, x')} \left( (\psi(x) \cdot v) W(x, v) |E(x, x - (\tau - |y| - s_1) v, x' + y, x') \right.
\times \chi(0, +\infty)(\tau - |y| - s_1) k(x - (\tau - s_1 - |y|) v, v_1, v)
\times k(x' + v, v_1) S(x', v_1) [v(x') \cdot v_1] \left. \right|_{v_1 = \frac{|(\tau - y') - (\tau - s_2 - y')}{\tau - s_1 - s_2 - y'}} |\tau - y'| - |x - x' - y| - v_1 \left. \right|_{v' = v_1} \right)
\times \frac{2^{n-2} (\tau - |y| - (x - x' - y) \cdot v)^{n-3}}{|x - x' - y| - (\tau - |y|) v|^{2n-4} |y|^{n-1}} dy \, dv,
\] (5.21)
and for $\tau \in \mathbb{R}$ and a.e. $(x',x) \in \partial X \times \partial X$ and for $m \geq 3$:

$$\gamma_m(\tau, x, x') := \int_{E_{m-1, x}(\tau, x, x')} (v(x) \cdot v) W(x, v) \int_{S_{x'}^n} (v_2^{n-1} \cdots v_m^{n-1} |x - x' - y_2 - \cdots - y_m| v_2^{n-4} \cdots v_m^{n-4}) \\times \sum_{i=2}^m \sum_{j=2}^m \phi_i(x) \phi_j(x') (\tau - s_1 - \cdots - \tau_{m-1} - s_m) \\times \text{Vol}(S^{m-1}) v_2^{m-4} \cdots v_m^{m-4} \quad \text{for } i=2, \ldots, m-1$$

(5.22)

$5.3.$ Proof of (2.17)–(2.19)

Let $\tau > 0$ and let $x, x' \in \partial X$ such that $|x - x'| < \tau$ and $x \neq x'$. We set $t_0 = |x - x'|$. Using spherical coordinates $(y_j = s_i, \theta_i, \phi_i, s_i, \theta_i, \phi_i) \in (0, +\infty) \times S^{m-1}, i = 2, \ldots, m$ and (5.14), we obtain

$$\int_{E_{m-1, x}(\tau, x, x')} \frac{\text{Vol}(S^{m-1})}{|y_2^{m-1} \cdots y_m^{m-1}|} \text{Vol}(S^{m-1}) \frac{\tau^{m-1}}{(m-1)!} \frac{1}{s_1^{m-1} s_2^{m-1} \cdots s_m^{m-1}} ds_2 \cdots ds_m$$

(5.23)

for all $m \in \mathbb{N}, m \geq 2$. Note that using (2.2) and (5.14), we obtain

$$x' + \sum_{i=2}^m y_i \in \text{supp } k \quad \Rightarrow \quad \delta \leq |x - x'| - \sum_{i=2}^m y_i < \tau - \sum_{i=2}^m |y_i|,$$

(5.24)

$$x' + \sum_{i=2}^m y_i \in \text{supp } k \quad \Rightarrow \quad \delta \leq |y_m|,$$

(5.25)

for $(y_2, \ldots, y_m) \in E_{m-1, x}(\tau, x, x')$ (recall that $(x', x) \in \partial X^2$).

We first look for an upper bound on $|y_m(\tau, x, x')|, m \geq 2$. Using the explicit expression for the multiple scattering kernels $\gamma_m(\tau, x, x')$ (see (5.21) and (5.22)) and the fact that $\sigma$ is a nonnegative function, we obtain that

$$|y_2(\tau, x, x')| \leq 2^{m-2} \|W_\infty\| S_{\infty}\|S\|_\infty \langle G_2(\tau, x, x') \rangle,$$

(5.26)

where $N, E_{1, x}(\tau, x, x')$ and $G_2$ are defined by (5.16), (5.14) and

$$G_2(\mu, z, z') := \int_{E_{1, x}(\mu, z, z')} \frac{N(\mu - |y|, z, z' + y)}{|y|^{n-1}} dy,$$

(5.27)

for $(\mu, z, z') \in (0, +\infty) \times \partial X \times \mathbb{R}^n$. We also obtain

$$|y_m(\tau, x, x')| \leq 2^{m-2} \|W_\infty\| S_{\infty}\|S\|_\infty \langle G_m(\tau, x, x') \rangle,$$

(5.28)

where

$$G_m(\tau, x, x') := \int_{E_{m-1, x}(\tau, x, x')} \frac{G_2(\tau - \sum_{j=3}^m |y_j|, x, x' + \sum_{j=3}^m y_j)}{|y_3|^{n-1} \cdots |y_m|^{n-1}} dy_3 \cdots dy_m,$$

(5.29)

and $G_2$ (resp. $E_{m-2, x}(\tau, x, x')$) is defined by (5.27) (resp. (5.14)).
We now prove (2.17). Assume \( n = 2 \). From (5.26) and (5.17) we obtain
\[
|y(\tau, x, x')| \leq 2^{n-2} \|W\|_\infty \|S\|_\infty \|k\|_\infty^2 C(N, 2).
\]
(5.30)
Then using (5.29), (5.17) and (5.23), we obtain
\[
G_m(\tau, x, x') \leq C(N, 2) \int_{\mathcal{E}_{n-2}(\tau, x, x')} \frac{dy_3 \cdots dy_m}{|y_3| \cdots |y_m|} = C(N, 2)(2\pi)^{m-2} \frac{\tau^{m-2}}{(m-2)!}, \quad \text{for } m \geq 3.
\]
(5.31)
Finally combining (5.31) and (5.28), we obtain
\[
|y_m(\tau, x, x')| \leq 2^{n-2} C(N, 2)(2\pi)^{m-2} \|W\|_\infty \|S\|_\infty \|k\|_\infty^m \frac{\tau^{m-2}}{(m-2)!}, \quad \text{for } m \geq 3.
\]
(5.32)
Statement (2.17) follows from (5.30) and (5.32).

We prove (2.18). Assume \( n = 3 \). Using (5.24)–(5.25) (with \( m = 2 \)) and (5.27), we obtain
\[
G_2(\tau, x, x') \leq \delta^{-2} \int_{\tau - |y| \geq \delta, \tau - |x - y| \geq \delta, |y| \geq \delta} N(\tau - |y|, x, x' + y) \, dy.
\]
(5.33)
Therefore using (5.18) we obtain
\[
\sup_{(s, z, z') \in (0, T) \times \partial X \times \partial X} (s - \tau - z - z') (1 + \ln \frac{s + |z - z'|}{s - |z - z'|})^{-1} G_2(s, z, z') \leq \delta^{-2} C_1(N, 3) < \infty.
\]
(5.34)
Now assume \( n = 3 \) and \( m \geq 3 \). Using (5.29) and (5.19) and the estimate \( \sup_{\tau \in (0, 1)} (1 - \ln(\tau))^2 < \infty \) we obtain
\[
G_m(\tau, x, x') \leq D \int_{(y_3, \ldots, y_m) \in \mathcal{E}_{n-2}(\tau, x, x')} \frac{1}{|y_3|^{n-1} \cdots |y_m|^{n-1}} \times \frac{|x - x' - y_3 - \cdots - y_m + \tau - \sum_{i=3}^m |y_i| |x - x' - y_3 - \cdots - y_m| \cdots dy_m \cdots dy_3}{(\tau - \sum_{i=3}^m |y_i|)|x - x' - y_3 - \cdots - y_m|}.
\]
(5.35)
where \( D := \sup_{\tau \in (0, 1)} (1 - \ln(\tau))^2 C_2(N, 3) \). If \( m = 3 \), then using (5.24)–(5.25) with \( \langle y_2, \ldots, y_m \rangle \) replaced by \( \langle y_3, \ldots, y_m \rangle \), we obtain
\[
G_3(\tau, x, x') \leq 2\tau \delta^{-4} D \text{Vol}(\mathcal{E}_{1,3}(\tau, x, x'))
\]
(5.36)
(we also used the estimate \( |x - x' - y_3| + \tau - |y_3| \leq 2\tau \) for \( y_3 \in \mathcal{E}_{1,3}(\tau, x, x') \)). If \( m \geq 4 \), then using (5.24) and (5.25) with \( \langle y_2, \ldots, y_m \rangle \) replaced by \( \langle y_3, \ldots, y_m \rangle \), we obtain
\[
G_m(\tau, x, x') \leq 2\tau \delta^{-4} D \text{Vol}(\mathcal{E}_{1,3}(\tau, x, x')) \int_{(y_3, \ldots, y_{m-1}) \in \mathcal{E}_{m-3,3}(\tau, x, x')} \frac{dy_3 \cdots dy_{m-1}}{|y_3|^{n-1} \cdots |y_{m-1}|^{n-1}} \times \frac{\tau^{m-3}}{(m-3)!} \text{Vol}(\mathcal{E}_{1,3}(\tau, x, x'))
\]
(5.37)
(we also used the estimate \( |x - x' - y_3 - \cdots - y_m| + \tau - |y_3| - \cdots - |y_m| \leq 2\tau \) for \( (y_3, \ldots, y_m) \in \mathcal{E}_{m-2,3}(\tau, x, x') \) and we used (5.23)). Statement (2.18) follows from (5.26), (5.28), (5.34) and (5.36) and (5.37) and (5.15).
We prove (2.19). Let $n \geq 4$ and $m \geq 2$. Using (5.27), (5.29) and (5.30), we obtain

\[ G_m(\tau, x, x') \leq C(N, n) \int_{(y_2, \ldots, y_m) \in \Omega_{n-1}(\tau, x, x')} \frac{1}{|y_2|^n \cdots |y_m|^n} \]

\[ \times \prod_{i=m+2}^n \frac{dy_2 \cdots dy_m}{x - x' - \sum_{i=2}^m y_i} (\tau - \sum_{i=2}^m |y_i|). \] \hspace{1cm} (5.38)

From (5.38) and (5.24) and (5.25), it follows that

\[ G_m(\tau, x, x') \leq 2\delta^{-2n+2}C(N, n) \text{Vol}(E_{1,0}(\tau, x, x')) \] \hspace{1cm} (5.39)

for $m = 2$, and

\[ G_m(\tau, x, x') \leq 2\delta^{-2n+2}C(N, n) \int_{(y_2, \ldots, y_m) \in \Omega_{n-1}(\tau, x, x')} \frac{dy_2 \cdots dy_m}{|y_2|^n \cdots |y_m|^n} \]

\[ \times \frac{\tau^{m-2}}{(m-2)!} \] \hspace{1cm} (5.40)

for $m \geq 3$ (we also use (5.23) to prove (5.40)). From (5.39), (5.40) and (5.15) it follows that

\[ G_m(\tau, x, x') \leq 2\delta^{-2n+2}C(N, n) \text{Vol}_{n-2}(S^{n-2}) \pi (\mu + |z - \bar{z}|) \]

\[ \times \left( \frac{\sqrt{\mu^2 - |z - \bar{z}|^2}}{2} \right)^n \frac{\tau^{m-2}}{(m-2)!} \] \hspace{1cm} (5.41)

for $m \geq 2$. Statement (2.19) follows from (5.26), (5.28) and (5.41).

6. Proof of theorems 2.3, 3.1 and 3.4

Proof of theorem 2.3. For the sake of simplicity and without loss of generality we assume $v_0 = (1, 0, \ldots, 0)$. Assume that conditions (2.1) and (2.2) are satisfied. For $n \geq 2$ consider the following open subset of $(0, +\infty) \times S^{n-1} \times S^{n-1}$,

\[ D := \{(s, v, v') \in (0, +\infty) \times S^{n-1}_{\chi_0} \times S^{n-1}_{\bar{v}_0} : s \in (0, \tau, (x, v))\}. \] \hspace{1cm} (6.1)

Then we introduce the bounded function $\Psi_n$ on $D$ defined by

\[ \Psi_n(s, v, v') = 2^{n-2}W(x, v)(v(x) \cdot v)E(x, x - s v, x_0')k(x - s v, v', v)S(x_0', v') |v(x_0') \cdot v'|, \]

for $(s, v, v') \in D$. Note that from convexity of $X$ it follows that $\tau_{\bar{z}}$ is continuous on $\Gamma_{\bar{z}}$ and $E(x, x - s v, x_0') = e^{\int_{\bar{z}} \sigma(s - x, v) dp - \int_{\bar{z}} \sigma(s, x, v') dp}$ for $v \in S_{x, v}$ and $0 < s < \tau_{\bar{z}}(x, v)$. Under (2.1) and (2.2) we obtain that

\[ \Psi_n(s, v, v') = 0 \text{ for } (s, v, v') \in (0, +\infty) \times S^{n-1}_{\chi_0} \times S^{n-1}_{\bar{v}_0} \text{ such that } x - s v \notin \bar{Z}, \]

and the function $\Psi_n$ is continuous at any point $(s, v, v') \in D$ such that $x - s v \in Z$. \hspace{1cm} (6.3)

We first prove (2.22) for $n = 2$. Let $\tau > \tau_0$. From (6.2), (6.14), it follows that

\[ \gamma_1(\tau, x, x_0') = \int_{-\tau}^{\tau - \alpha_0} \frac{1}{\tau - t_0 \cos(\Omega)} [X_{1,0}(\tau, (x, v))(s) \Psi_2(s, v, v')] d\Omega \]

\[ = \gamma_{1,1}(\tau, x, x_0') + \gamma_{1,2}(\tau, x, x_0'). \] \hspace{1cm} (6.4)
where $\mathbb{S}^{n-1} = \{ \cos \Omega, \sin \Omega \} = \{ \cos \Omega, \sin \Omega \} \in (0, \pi)$ and

$$\gamma_{1,1}(\tau, x, x_0') = \int_0^\tau \frac{X(\Omega, \sin \Omega)}{\tau - \lambda \cos \Omega} \left[ X(0, r \cos \Omega)(s) \Psi(\nu(x), \tau, \nu'(s)) \right]_{\nu(s) = \gamma} \frac{d\Omega}{\sqrt{\tau - \lambda \cos \Omega}},$$

$$\gamma_{1,2}(\tau, x, x_0') = \int_{-\tau}^0 \frac{X(\Omega, \sin \Omega)}{\tau - \lambda \cos \Omega} \left[ X(0, r \cos \Omega)(s) \Psi(\nu(x), \tau, \nu'(s)) \right]_{\nu(s) = \gamma} \frac{d\Omega}{\sqrt{\tau - \lambda \cos \Omega}}.$$  \hspace{1cm} (6.5)

We now prove that

$$\sqrt{\tau - t_0} \gamma_{1,i}(\tau, x, x_0') \to \frac{W(x, v_0) S(x_0', v_0) |v(x_0') \cdot v_0| E(x_0', x)}{\sqrt{2\pi t_0}} \times \int_0^t \frac{1}{s(t_0 - s)} k(x - s v_0, v_0, v_0) \frac{1}{\sqrt{k(t_0 - s)}} \frac{d\omega}{ds},$$

as $\tau \to t_0^+$, \hspace{1cm} (6.7)

for $i = 1, 2$. Then adding (6.7) for $i = 1$ and $i = 2$, we obtain (2.22). We only prove (6.7) for $i = 1$ since the proof for $i = 2$ is similar. Let $\tau > t_0$. Using the change of variables $s = \frac{\tau - t_0^2}{2(\tau - t_0 \cos \Omega)} = \frac{\tau - t_0^2}{2}$, we obtain

$$\gamma_{1,1}(\tau, x, x_0') = \frac{1}{\sqrt{\tau^2 - t_0^2}} \int_0^{\tau_0} \frac{X(\Omega, \sin \Omega)}{\tau - \lambda \cos \Omega} \left[ X(0, r \cos \Omega)(s) \right]_\tau \frac{1}{\sqrt{s(t_0 - s)}} \frac{d\omega}{ds},$$

$$\gamma_{1,2}(\tau, x, x_0') = \frac{1}{\sqrt{\tau^2 - t_0^2}} \int_{-\tau}^0 \frac{X(\Omega, \sin \Omega)}{\tau - \lambda \cos \Omega} \left[ X(0, r \cos \Omega)(s) \right]_\tau \frac{1}{\sqrt{s(t_0 - s)}} \frac{d\omega}{ds},$$

where

$$v(s, \tau) = (\cos \Omega(s, \tau), \sin \Omega(s, \tau)), \quad \Omega(s, \tau) = \arccos \left( \frac{\tau - s}{\sqrt{\tau^2 - t_0^2}} \right),$$

$$v'(s, \tau) = \frac{\nu_0(1, 0) - \left( s + \frac{\tau}{2} \right)}{\frac{\tau}{2} - s} v(s, \tau).$$

Let $s \in (0, t_0)$. From (6.9), it follows that

$$v(s, \tau) \to (1, 0) \quad \text{as} \quad \tau \to t_0^+, \quad v'(s, \tau) \to (1, 0) \quad \text{as} \quad \tau \to t_0^+.$$  \hspace{1cm} (6.10)

Note that using the definition of $v_0$ and using the assumption $x_0' + \varepsilon(x - x_0') \in X$ for some $\varepsilon \in (0, 1)$ we obtain $t_0 = \tau(x, v_0)$, see also that the function $s \mapsto \frac{1}{\sqrt{s(t_0 - s)}} \frac{d\omega}{ds}, s \in (0, t_0)$, is integrable in $(0, t_0)$. Therefore, using (6.3), the boundedness of $\Psi$ on $D$ and the Lebesgue dominated convergence theorem, we obtain (6.7). This proves (2.22).

Let $n \geq 3$ and prove (2.23). From (6.2) and (2.14), it follows that

$$\gamma_{1,1}(\tau, x, x_0') = \int_{\mathbb{S}^{n-1}} \frac{(\tau - t_0 v_0 \cdot v)^{n-3}}{|t_0 v_0 - \tau v|^{2n-4}} X(0, +\infty)(v(x) \cdot v) \Psi_0(s, v, v') \frac{d\nu}{\nu_0 - s},$$

for $\tau > |x - x_0'|$. \hspace{1cm} (6.11)

Let $\Phi(\Omega, \omega) = (\sin \Omega, \cos \Omega \omega_1, \ldots, \cos \Omega \omega_{n-1})$ for $\Omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\omega = (\omega_1, \ldots, \omega_{n-1}) \in \mathbb{S}^{n-2}$. Using spherical coordinates we obtain

$$\gamma_{1,1}(\tau, x, x_0') = \int_{\mathbb{S}^{n-1}} \frac{\cos \Omega_{n-2}}{\cos \Omega} \frac{(\tau - t_0 \sin \Omega)^{n-3}}{|t_0^2 + \tau^2 - 2t_0 \tau \sin \Omega|^{n-2}} \times \frac{X(0, +\infty)(v(x) \cdot \Phi(\Omega, \omega)) \Psi_0(s, \Phi(\Omega, \omega), v')}{\frac{d\omega}{\nu_0 - s}} d\omega d\Omega,$$  \hspace{1cm} (6.12)
for $\tau > t_0$. Performing the change of variables $r = \frac{\tau^2 - |x|^2}{2(\tau - t_0 \sin \Theta)} - \frac{\tau - t_0}{2}$ on the first integral on the right-hand side of (6.12), we obtain

$$
\gamma_1(\tau, x, x_0^0) = 2^{-n} \left( \frac{2}{n} \right)^{n-2} \frac{1}{t_0^n} \int_0^{t_0} \frac{\sqrt{r(t_0 - r)^{n-3}}}{(t_0 - r)^n} \left( \frac{\tau - t_0}{2} - r \right)^{n-2} \times \int_{S(t, x)} \left[ \chi(\Phi(\Omega, \omega)) \Psi_{\beta} \left( r + \frac{\tau - t_0}{2}, \Phi(\Omega, \omega), \nu' \right) \right]_{\Omega = \arcsin \left( \frac{\tau - t_0}{2} \right)} \, d\omega \, dr.
$$

(6.13)

Therefore using (6.13), (6.3) and (6.2) and using Lebesgue dominated convergence theorem, we obtain (2.23). This concludes the proof of theorem 2.3. □

**Proof of theorem 3.1.** We now prove (3.1). Let $x_0' \in \partial X$. For $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$ and $\varepsilon_3 \in (0, +\infty)$, let $(f_{\varepsilon_3}, g_{\varepsilon_3}) \in C^4(\partial X) \times C^4(\mathbb{R})$ satisfy

$$
g_{\varepsilon_3} \geq 0, \quad f_{\varepsilon_3} \geq 0, \quad \text{supp } g_{\varepsilon_3} \subseteq (0, \min(\varepsilon_2, \eta)), \quad \text{supp } f_{\varepsilon_3} \subseteq \{x' \in \partial X \mid \|x' - x_0'\| < \varepsilon_1\},
$$

(6.14)

$$
\int_0^{\eta} g_{\varepsilon_3}(t') \, dt' = 1, \quad \int_{\partial X} f_{\varepsilon_3}(x') \, d\mu(x') = 1.
$$

(6.15)

Therefore, $\phi_{\varepsilon_3} := g_{\varepsilon_3} f_{\varepsilon_3}$ is an approximation of the delta function at $(0, x_0') \in \mathbb{R} \times \partial X$ for $\varepsilon := (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$. Let $\psi_{\varepsilon_3} \in L^\infty((0, T) \times \partial X)$ be defined by

$$
\psi_{\varepsilon_3}(t, x) = \chi(-\varepsilon_3, \varepsilon_3)(t - |x - x_0'|)(2\chi(0, +\infty)((E - \bar{E})(x, x_0') - 1), (t, x) \in (0, T) \times \partial X,
$$

(6.16)

for $\varepsilon_3 > 0$. From (2.12) and (2.15) it follows that

$$
\int_{(0,T)\times\partial X} \psi_{\varepsilon_3}(t, x)(A_{\varepsilon, \omega} - \bar{A}_{\varepsilon, \omega}) \phi_{\varepsilon_3}(t, x) \, d\mu(x) = I_0(\psi_{\varepsilon_3}, \phi_{\varepsilon_3}) + \int_{(0,T)\times\partial X \times (0,\eta) \times \partial X} \psi_{\varepsilon_3}(t, x) \phi_{\varepsilon_3}(t', x')(\Gamma_1 - \bar{\Gamma_1})(t - t', x, x') \, d\mu(x) \, dt' \, d\mu(x'),
$$

(6.17)

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$ and $\varepsilon_3 \in (0, +\infty)$, where

$$
I_0(\psi_{\varepsilon_3}, \phi_{\varepsilon_3}) = \int_{(0,T)\times\partial X \times (0,\eta) \times \partial X} \psi_{\varepsilon_3}(t, x) \phi_{\varepsilon_3}(t - |x - x'|, x') \frac{E(x, x') - \bar{E}(x, x')}{|x - x'|^{n-1}} \times \left| W(x, v)S(x', v)(v(x) \cdot v)(v'(x') \cdot v') \right|_{\gamma = |v'|} \, d\mu(x) \, d\mu(x').
$$

(6.18)

From (2.16)–(2.19) it follows that

$$
(t - |x - x'|)^{\frac{n}{2}} (\Gamma_1 - \bar{\Gamma_1})(t, x, x') \in L^\infty((0, T) \times \partial X \times \partial X).
$$

(6.19)

Combining (6.18) and the equality $\|\phi_{\varepsilon_3}\|_{L^1((0,\eta),\partial X)} = 1$ and the estimate $\|\psi_{\varepsilon_3}\|_{L^\infty((0,T)\times\partial X)} \leq 1$ and (6.20) we obtain

$$
I_0(\psi_{\varepsilon_3}, \phi_{\varepsilon_3}) \leq \|A_{\varepsilon, \omega} - \bar{A}_{\varepsilon, \omega}\|_{\eta,T} + C \Delta_1(\psi_{\varepsilon_3}, \phi_{\varepsilon_3}).
$$

(6.20)
for \( \epsilon = (\epsilon_1, \epsilon_2) \in (0, +\infty)^2 \) and \( \epsilon_3 \in (0, +\infty) \), where \( C = \| (\tau - |x - x'|) \|_{L^\infty(0,T)} \) and

\[
\Delta_1(\psi_{\epsilon_1}, \phi_\epsilon) = \int_{(0,T) \times \partial X_\epsilon \times \partial X_\epsilon} \psi_{\epsilon_1}(t,x)\phi_\epsilon(t',x') \\
\times (t - t' - |x - x'|)^{-\frac{3}{2}} d\mu(x) d\nu(x').
\]  

Note that the function \( \Phi_{1,\epsilon_3} : [0, \eta] \times \partial X \rightarrow \mathbb{R} \) defined by

\[
\Phi_{1,\epsilon_3}(t',x') := \int_{(0,T) \times \partial X_\epsilon} \psi_{\epsilon_1}(t,x)\phi_\epsilon(t' - t' - |x - x'|)^{-\frac{3}{2}} d\mu(x),
\]

is continuous on \([0, \eta] \times \partial X\) for \( \epsilon_3 \in (0, +\infty) \). Therefore, from (6.14)–(6.16) and the equality

\[
\Delta_1(\psi_{\epsilon_1}, \phi_\epsilon) = \int_{(0,T) \times \partial X} \psi_{\epsilon_1}(t',x')\Phi_{1,\epsilon_3}(t',x') d\mu(x),
\]

it follows that

\[
\lim_{\epsilon_3 \rightarrow 0^+} \lim_{\epsilon_1 \rightarrow 0} \Delta_1(\psi_{\epsilon_1}, \phi_\epsilon) = \lim_{\epsilon_3 \rightarrow 0^+} \Phi_{1,\epsilon_3}(0, x_0) = 0
\]  

(we also used (6.23), (6.17) and the Lebesgue dominated convergence theorem to prove that

\[
\lim_{\epsilon_3 \rightarrow 0^+} \Phi_{1,\epsilon_3}(0, x_0) = 0.
\]

Note that under condition (2.2) the function \( \Phi_{0,\epsilon_2,\epsilon_3} : \partial X \rightarrow \mathbb{R} \) defined by

\[
\Phi_{0,\epsilon_2,\epsilon_3}(x') = \int_{(0,T) \times \partial X_\epsilon} \psi_{\epsilon_1}(t,x)g_{\epsilon_2}(t - |x - x'|) \\
\times \left[ (x,v)S(x',v)(x' \cdot v) \right]|_{v = \frac{\nu(x')}{|\nu(x')|}} d\mu(x),
\]

is continuous on \( \partial X \) for \((\epsilon_2, \epsilon_3) \in (0, +\infty)^2 \). Therefore, from the equality \( I_0(\psi_{\epsilon_1}, \phi_\epsilon) = \int_{\partial X} \Phi_{0,\epsilon_2,\epsilon_3}(x') f_{\epsilon_3}(x') d\mu(x') \) (see (6.19)) it follows that

\[
\lim_{\epsilon_3 \rightarrow 0^+} I_0(\psi_{\epsilon_1}, \phi_\epsilon) = \Phi_{0,\epsilon_2,\epsilon_3}(x_0), \quad \text{for} \quad (\epsilon_2, \epsilon_3) \in (0, +\infty)^2.
\]

Thus, using the Lebesgue dominated convergence theorem and (6.25) we obtain

\[
\lim_{\epsilon_1 \rightarrow 0^+} \lim_{\epsilon_2 \rightarrow 0^+} \lim_{\epsilon_3 \rightarrow 0^+} \|E(x, x') - \tilde{E}(x, x_0')\|_{L^\infty(0,T)} \\
\times \left[ (x,v)S(x',v)(x' \cdot v) \right]|_{v = \frac{\nu(x')}{|\nu(x')|}} d\mu(x).
\]  

Combining (6.27), (6.24) and (6.21) we obtain the formula (4.10). Using (4.10) and the estimates \( \inf_{\Gamma_1} S > 0 \) and \( \inf_{\Gamma_1} W > 0 \) and the change of variables \( x = x_0' + \tau(x_0',v_0) \quad y(x',v) \rightarrow \frac{y(x',v) - x_0'}{v_0} \) we obtain (3.1) where the constant \( C_1 \), which appears on the right-hand side of (3.1), is given by \( C_1 = (\inf_{\Gamma_1} S \inf_{\Gamma_1} W)^{-1} \).

We now prove (3.2). Let \( x \in \partial X \) be such that \( px'_0 + (1 - p)x \in Z \) for some \( p \in (0, 1) \). We set \( t_0 = |x - x'_0| \) and \( v_0 = \frac{\tau - x'_0}{|\tau - x'_0|} \). From (2.16), (2.17), (2.18) and (2.19) it follows that

\[
\|E(x, x_0') - \hat{E}(x, x_0')\|_{L^\infty(0,T)} \\
\times \left[ (x,v)S(x',v)(x' \cdot v) \right]|_{v = \frac{\nu(x')}{|\nu(x')|}} d\mu(x).
\]

for \( \tau > |x - x'_0| \). From (2.17)–(2.19) it turns out that \( \lim_{x \rightarrow x'_0} (\tau - |x - x'_0|)^{-\frac{3}{2}} |\Gamma_1 - \hat{\Gamma}_2| = 0 \). Therefore applying (2.22) and (2.23) on the left-hand side of (6.28) we
we now prove (3.2). Let
\[ X := \{ x \in \mathbb{R}^d : \| x \|_\infty \leq 1 \} \]
be a bounded convex subset of \( \mathbb{R}^d \) with \( C_1 \) boundary and \( Z \subset X \). Therefore (3.2) follows from (6.29) where the constant \( C_2 \) which appears on the right-hand side of (3.2) is given by \( C_2 = \frac{2^{2d} \text{diam}(X)}{e^2 C_1 \text{diam}(X)} \). Theorem 3.1 is proved.

**Proof of theorem 3.4.** We first prove (3.7). We extend \( \sigma \) and \( \bar{\sigma} \) by \( 0 \) outside \( Y \). For a bounded and continuous function \( f \) on \( Y \) consider the X-ray transform \( P f : S^{d-1} \times \mathbb{R}^d \rightarrow \mathbb{R} \) defined by (3.3) (we extend \( f \) by \( 0 \) outside \( Y \)). We recall the following estimate:
\[ \| f \|_{H^{-1}(Y)} \leq \left( \int_{\mathbb{R}^d} \int_{\Pi_v} |P f(v, x)|^2 \, dx \, dv \right)^{1/2}, \]
where \( \Pi_v := \{ x \in \mathbb{R}^d : x \cdot v = 0 \} \) for \( v \in \mathbb{R}^{d-1} \). Note that using the estimate \( \| \sigma \|_{\infty} \leq M \), we obtain
\[ \int_{\Pi_v} \sigma(x_0 + sv, v) \, dv \leq M \tau_\varepsilon(x_0', v) \leq M \text{diam}(X), \quad \text{for } (x_0', v) \in \Gamma_- \]
Replacing \( \sigma \) by \( \bar{\sigma} \) on the left-hand side of (6.31) we obtain an estimate similar to (6.31) for \( \bar{\sigma} \). Therefore using the estimate \( |e^{t_1} - e^{t_2}| \geq e^{-M \text{diam}(X)} |t_1 - t_2| \) for \( (t_1, t_2) \in [0, +\infty)^2 \), \( \max(t_1, t_2) \leq M \text{diam}(X) \), we obtain
\[ |e^{-\int_0^{t_1} \sigma(x_0 + sv, v) \, dv} - e^{-\int_0^{t_1} \bar{\sigma}(x_0 + sv, v) \, dv}| \geq e^{-M \text{diam}(X)} |P(\sigma - \bar{\sigma})(v, x_0')|, \]
for \( (x_0', v) \in \Gamma_- \). Integrating the left-hand side of (3.1) over \( \partial X \) and using (6.32), we obtain
\[ \int_{\Gamma_-} |P(\sigma - \bar{\sigma})(v, x_0')| \, d\xi(v, x_0') \leq e^{M \text{diam}(X)} \text{Vol}(\partial X) C_1 \| A_{S,W} - \bar{A}_{S,W} \|_{0,T}, \]
where \( C_1 \) is the constant that appears on the right-hand side of (3.1). Note that using that \( X \) is a convex open subset of \( \mathbb{R}^d \) with \( C_1 \) boundary we obtain \( \int_{\Gamma_-} |P(\sigma - \bar{\sigma})(v, x_0')| \, d\xi(v, x_0') = \int_{\mathbb{R}^d} \int_{\Pi_v} |P(\sigma - \bar{\sigma})(v, x)| \, dx \, dv \). Therefore using (6.33) and the estimate \( |P(\sigma - \bar{\sigma})(v, x)|^2 \leq \| \sigma - \bar{\sigma} \|_{L_\infty}^2 \text{diam}(X) |P(\sigma - \bar{\sigma})(v, x)| \) for \( (v, x) \in T^{d-1} \) (see (6.31) and the estimates \( \sigma \geq 0, \bar{\sigma} \geq 0 \) we obtain
\[ \left( \int_{\mathbb{R}^d} \int_{\Pi_v} |P(\sigma - \bar{\sigma})(v, x)|^2 \, dx \, dv \right)^{1/2} \leq C_3 \| \sigma - \bar{\sigma} \|_{L_\infty} \| A_{S,W} - \bar{A}_{S,W} \|_{0,T}, \]
where \( C_3 = (\text{diam}(X) e^{M \text{diam}(X)} \text{Vol}(\partial X) C_1)^{1/2} \). Combining (6.34) and (6.30), we obtain (3.7).

We now prove (3.8). Let \( f \in L^2(X) \), \( \text{supp } f \subseteq Z \). We consider the weighted x-ray transform of \( f \), \( P_\theta f \), defined by
\[ P_\theta f(x, v) = \int_0^{t(x,v)} f(pv + x) \theta(pv + x, v) \, dp, \quad \text{for a.e. } (x, v) \in \Gamma_- \]
where $\vartheta : X \times S^{n-1} \to (0, +\infty)$ is the analytic function given by

$$
\vartheta(x,v) = (\tau_-(x,v)\tau_+(x,v))^{-\frac{1}{\omega}} g(x,v,v), \quad \text{for} \quad (x,v) \in X \times S^{n-1}.
$$

(6.36)

From [10, theorem 2.2] and from [14, theorem 4] we obtain

$$
\int_0^{\tau_-(x_0, v_0)} \frac{\tilde{\vartheta}(x_0' + p n_0, v_0', v_0')}{p \tilde{\vartheta}^{1-\frac{1}{\omega}}(\tau_+(x_0', v_0') - p)} dp \leq \|\tilde{\vartheta}\|_{L^\infty(Z)} \int_0^{\tau_-(x_0, v_0)} \frac{1}{p \tilde{\vartheta}^{1-\frac{1}{\omega}}(\tau_+(x_0', v_0') - p)} dp.
$$

(6.37)

We use the estimate

$$
|P_\theta(k_0 - \tilde{k}_0)(x_0', v_0')| \leq e^{P \theta(t_0, x_0')} |P_\theta(\tilde{k}_0(x_0', v_0')) - e^{-P \theta(t_0, x_0')}| + e^{P \theta(t_0, x_0')} |e^{-P \theta(t_0, x_0')} - e^{-P \theta(t_0, x_0')}| P_\theta(k_0(x_0, v_0') - e^{-P \theta(t_0, x_0')} P_\theta(\tilde{k}_0(x_0', v_0')).
$$

(6.39)

Integrating both sides of inequality (6.39) over $v_0' \in S^{n-1}_{x_0'}$ and using the estimate $e^{P \theta(t_0, x_0')} \leq e^{M \text{diam}(X)}$, and using (6.38), (3.1) and (3.2), we obtain

$$
\int_{S^{n-1}_{x_0'}} |P_\theta(k_0 - \tilde{k}_0)(x_0', v_0')| |v(x_0') \cdot v| dv \leq \delta^{-\frac{1}{\omega}(n-1)} \text{diam}(X)
$$

$$
\times e^{M \text{diam}(X)} C_1 \|\tilde{\vartheta}\|_{L^\infty} \|A_{x,w} - \tilde{A}_{x,w}\|_{\partial X} + \frac{\text{Vol}(S^{n-1}) e^{M \text{diam}(X)} C_2}{2}
\times \|\tilde{\vartheta}\|_{L^\infty((0,T) \times \partial X \times \partial X)}
$$

(6.40)

where $C_1$ and $C_2$ are the constants that appear on the right-hand sides of (3.1) and (3.2).

From the estimate $|P_\theta(k_0 - \tilde{k}_0)(x_0', v_0')| \leq \|k - \tilde{k}\|_{L^\infty(Z)} \delta^{-\frac{1}{\omega}(n-1)} \text{diam}(X)$ for a.e. $(x_0', v_0') \in \Gamma_-$ (see (6.38)), it follows that

$$
\|P_\theta(k_0 - \tilde{k}_0)\|_{L^2(\Gamma_-, d\xi)}^2 \leq \|k - \tilde{k}\|_{L^\infty(Z)} \delta^{-\frac{1}{\omega}(n-1)} \text{diam}(X)
$$

$$
\times \int_{S^{n-1}_{x_0'}} \int_{\Gamma_-} |P_\theta(k_0 - \tilde{k}_0)(x_0', v_0')| |v(x_0') \cdot v| dv d\mu(x_0').
$$

(6.41)

Combining (6.40), (6.41) and (6.37), we obtain (3.8).

\[\square\]

Acknowledgments

This paper was funded in part by grant NSF DMS-0554097. The authors would like to thank Ian Langmore and François Monard for stimulating discussions on the inverse transport problem.

Appendix A. Proof of some lemmas

We recall the following change of variables for the proof of lemmas 5.1, 5.2.
Lemma 5.1 is proved.

Proof of lemma 5.2. We first prove (5.17). Let $n = 2$. Note that

$$N(\mu, z, z') = \int_0^{2\pi} \frac{1}{\mu - |z - z'| \sin(\Omega)} \, d\Omega,$$

for $(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n, \mu > |z - z'|$. Therefore, using (5.1) we obtain

$$N(\mu, z, z') = \frac{2\pi}{\sqrt{\mu^2 - |z - z'|^2}},$$

for $(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n, \mu > |z - z'|$. Now let $(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n$ be fixed with $\mu > |z - z'|$. Set $t_0 = |z - z'|$. Using (A.1) ("$t = \mu'$, $t_0\nu = z - z'$"), we
obtain
\[
\int_{\mathcal{E}_1(\mu, z, z')} \frac{2\pi}{|y|\sqrt{(\mu - |y|)^2 - |z' - y|^2}} \, dy
= \int_{\mathcal{E}_1(\mu, t_0(1,0), 0)} \frac{2\pi}{|y|\sqrt{(\mu - |y|)^2 - t_0(1,0) - y^2}} \, dy
= 4\pi \int_0^\mu \int_0^{2\pi} G_{2,2}(\mu, s, \phi) \, d\phi \, ds,
\]
where
\[G_{2,2}(\mu, s, \phi) := \frac{(s - t_0 \cos(\phi))}{\sqrt{s^2 - t_0^2 \sqrt{\mu - s} \sqrt{\mu - t_0 \cos(\phi)}}}.
\]

for \(\phi \in (0, 2\pi)\) and \(s \in (t_0, \mu)\). From (A.7) and the estimates \(\mu - t_0 \cos(\phi) \geq s - t_0 \cos(\phi), s + t_0 \geq s - t_0 \cos(\phi)\), it follows that
\[G_{2,2}(\mu, s, \phi) \leq \frac{1}{\sqrt{s - t_0 \sqrt{\mu - s}}},
\]
for \(\phi \in (0, 2\pi)\) and \(s \in (t_0, \mu)\). Performing the change of variables \(s = t_0 + \varepsilon(\mu - t_0)\) we have
\[\int_0^\mu \frac{1}{\sqrt{s - t_0 \sqrt{\mu - s}}} \, ds = \int_0^1 \frac{1}{\sqrt{\varepsilon(1 - \varepsilon)}} \, d\varepsilon < +\infty,
\]
for \(s \in (t_0, \mu)\). Combining (A.5), (A.6), (A.8) and (A.9), we obtain
\[\sup_{(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^2} \frac{N(\mu - |y|, z, z' + y)}{|y|} \, dy \leq 8\pi^2 \int_0^1 \frac{1}{\sqrt{\varepsilon(1 - \varepsilon)}} \, d\varepsilon < \infty.
\]
(A.10)

Statement (5.17) follows from (A.10).

We prove (5.18). Let \(n = 3\). Note that
\[N(\mu, z, z') = 2\pi \int_{\mathbb{R}^3} d\Omega(\mu + |z - z'| \sin(\Omega)) \, d\Omega,
\]
for \((\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3, \mu > |z - z'|\). Therefore
\[N(\mu, z, z') = 2\pi \frac{1}{\mu |z - z'|} \ln \left(\frac{\mu + |z - z'|}{\mu + |z - z'|} \right).\]
(A.11)

for \((\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3, \mu > |z - z'|\). Now let \((\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^3\) be fixed with \(\mu > |z - z'|\). Set \(t_0 = |z - z'|\). Using (A.11) and (A.1), we obtain
\[
\int_{y \in E_1(\mu, z, z')} N(\mu - |y|, z, z' + y) \, dy
\leq 2\pi \delta^{-2} \int_{y \in E_1(\mu, z, z')} \ln \left(\frac{\mu - |y| + |z - z' - y|}{\mu - |y| - |z - z' - y|} \right) \, dy
= \frac{\pi^2}{2\delta^2} \int_0^\mu \int_0^{2\pi} G_{2,3,a}(\mu, s, \phi) \, d\phi \, ds,
\]
where
\[G_{2,3,a}(\mu, s, \phi) := (s^2 - t_0^2 \cos^2(\phi) \sin(\phi)) \ln \left(\frac{\mu - t_0 \cos(\phi)}{\mu - s} \right).
\]
(A.13)
for $\varphi \in (0, \pi)$ and $s \in (t_0, \mu)$. Using (A.13) and the estimates $\ln\left(\frac{\mu - t_0 \cos(\varphi)}{\mu - s}\right) \leq \ln\left(\frac{\mu - t_0 \cos(\varphi)}{\mu - t_0 \cos(\varphi)}\right) \leq \mu^2$, we obtain

$$\int_{t_0}^{\mu} \ln\left(\frac{\mu + t_0}{\mu - s}\right) ds = (\mu - t_0) \ln\left(\frac{\mu + t_0}{\mu - t_0}\right) + \mu - t_0. \quad (A.15)$$

Combining (A.12), (A.14) and (A.15) we obtain

$$\int_{y \in E_1, |z - z'| = \delta} \frac{N(\mu - |y|, z, z' + y) dy}{|y|^2} = \int_{y \in E_1, |z - z'| = \delta} \frac{2\pi}{|y|^2} \ln\left(\frac{\mu + t_0}{\mu - t_0}\right) \left(\ln\left(\frac{\mu + t_0}{\mu - t_0}\right) + 1\right) dy$$

which proves (5.18).

We next prove (5.19). Let $n = 3$. Let $(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n$ be fixed, $\mu > |z - z'|$. Set $t_0 = |z - z'|$. Using (A.11) and (A.1), we obtain

$$\int_{y \in E_1, |z - z'| = \delta} \frac{N(\mu - |y|, z, z' + y) dy}{|y|^2} = \int_{y \in E_1, |z - z'| = \delta} \frac{2\pi}{|y|^2} \ln\left(\frac{\mu + t_0}{\mu - t_0}\right) \left(\ln\left(\frac{\mu + t_0}{\mu - t_0}\right) + 1\right) dy$$

where

$$G_{2,3,b}(\mu, s, \varphi) := \frac{\sin(\varphi) \ln\left(\frac{\mu - t_0 \cos(\varphi)}{\mu - s}\right)}{(s + t_0 \cos(\varphi))(2\mu - s - t_0 \cos(\varphi))} = \ln\left(\frac{\mu - t_0 \cos(\varphi)}{\mu - s}\right) \left(\frac{\sin(\varphi)}{2\mu(s + t_0 \cos(\varphi))} + \frac{\sin(\varphi)}{2\mu(2\mu - s - t_0 \cos(\varphi))}\right) \quad (A.18)$$

for $\varphi \in (0, \pi)$ and $s \in (t_0, \mu)$. From (A.18) and the estimates $2\mu - s - t_0 \cos(\varphi) \geq \mu - t_0 \cos(\varphi), 0 \leq \ln\left(\frac{\mu - t_0 \cos(\varphi)}{\mu - s}\right) \leq \ln\left(\frac{\mu + t_0 \cos(\varphi)}{\mu - s}\right)$, it follows that

$$G_{2,3,b}(\mu, s, \varphi) \leq \ln\left(\frac{\mu + t_0}{\mu - s}\right) \left(\frac{\sin(\varphi)}{2\mu(s - t_0 \cos(\varphi))} + \frac{\sin(\varphi)}{2\mu(\mu - t_0 \cos(\varphi))}\right).$$

for $\varphi \in (0, \pi)$ and $s \in (t_0, \mu)$. Therefore

$$\int_{t_0}^{\mu} G_{2,3,b}(\mu, s, \varphi) ds \leq \ln\left(\frac{\mu + t_0}{\mu - s}\right) \left(\ln\left(\frac{s + t_0}{s - t_0}\right) + \ln\left(\frac{\mu + t_0}{\mu - t_0}\right)\right)$$

$$\leq \ln\left(\frac{\mu + t_0}{\mu - s}\right) \left(\ln\left(\frac{\mu + t_0}{s - t_0}\right) + \ln\left(\frac{\mu + t_0}{\mu - t_0}\right)\right). \quad (A.19)$$
Using the estimate $\ln \frac{\mu + t_0}{\mu - t_0} \leq \ln \left( \frac{2(\mu + t_0)}{\mu - t_0} \right)$ for $s \in (t_0, \frac{t_0 + t_0}{2})$, we obtain
\[
\int_{t_0}^{\mu} \ln \left( \frac{\mu + t_0}{s - t_0} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right) \, ds = 2 \int_{t_0}^{\mu} \ln \left( \frac{\mu + t_0}{s - t_0} \right) \ln \left( \frac{\mu + t_0}{\mu - s} \right) \, ds \\
\leq 2 \ln \left( \frac{2(\mu + t_0)}{\mu - t_0} \right) \int_{t_0}^{\mu} \ln \left( \frac{\mu + t_0}{s - t_0} \right) \, ds \\
\leq 2(\mu - t_0) \ln \left( \frac{\mu + t_0}{\mu - t_0} \right) + 2 \ln 2 \left( \ln \left( \frac{\mu + t_0}{\mu - t_0} \right) + 1 \right).
\] (A.20)

Combining (A.17)–(A.20) and (A.15), we obtain
\[
\int_{y \in E_1(\mu, z, z')} N(\mu - |y|, z, z' + y) \, dy \\
\leq 4\pi^2 \frac{\mu - t_0}{\mu t_0} \left( 3 \ln \left( \frac{\mu + t_0}{\mu - t_0} \right) + 2 \ln 2 \right) \left( \ln \left( \frac{\mu + t_0}{\mu - t_0} \right) + 1 \right). \quad (A.21)
\]

Statement (5.19) follows from (A.21).

We now prove (5.20). Let $n \geq 4$ and let $(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n$ be such that $\mu > |z - z'|$ (we recall that $N(\mu, z, z') = 0$ if $\mu \leq |z - z'|$). Using spherical coordinates, we obtain
\[
N(\mu, z, z') = \text{Vol}_{n-2}(S^{n-2}) \int_{|z - z'|}^{\infty} \frac{(\mu - |z - z'| \sin(\Omega))^{n-3}}{|z - z'|^{n-2}} \, d\Omega.
\] (A.22)

Performing the change of variables $r = \frac{\mu^2 - |z - z'|^2}{2(\mu - |z - z'| \sin(\Omega))}, \quad \frac{\mu - |z - z'|}{2}$, we obtain
\[
N(\mu, z, z') = \frac{\text{Vol}_{n-2}(S^{n-2})}{|z - z'|^{n-2}} \int_0^{\frac{|z - z'|^2}{\mu - |z - z'| \sin(\Omega)}} \frac{(\mu^2 - r)^{n-3}}{(\mu^2 - r)^{\frac{n-2}{2}} (\mu^2 - r) \frac{n-2}{2}} \, dr \\
= \frac{2 \text{Vol}_{n-2}(S^{n-2})}{|z - z'|^{n-2}} \left( \frac{\mu^2 - |z - z'|^2}{2} \right)^{\frac{n-1}{2}} \int_0^{\frac{|z - z'|^2}{\mu - |z - z'| \sin(\Omega)}} \frac{1}{(\mu^2 - r)^n \frac{n-2}{2}} \frac{1}{(\mu^2 - r)^{\frac{n-2}{2}}} \, dr \\
\leq \frac{2 \text{Vol}_{n-2}(S^{n-2})}{|z - z'|^{n-2}} \left( \frac{\mu^2 - |z - z'|^2}{2} \right)^{\frac{n-1}{2}} \int_0^{\frac{|z - z'|^2}{\mu - |z - z'| \sin(\Omega)}} \frac{1}{(\mu^2 - r)^n} \, dr \\
\leq \frac{2^{n-1}}{n-3} \text{Vol}_{n-2}(S^{n-2}) (|z - z'|^{2-n} \frac{\mu + |z - z'|}{2\mu})^{n-1} \mu^{-1} \\
\leq \frac{2^{n-1}}{n-3} \text{Vol}_{n-2}(S^{n-2}) |z - z'|^{2-n} \mu^{-1}. \quad (A.23)
\]

Appendix B. Proof of proposition 2.1

We start with the derivations of (2.13) and (2.14). From (2.9) and the definition of $G_-$, we obtain $A_{0, w, \phi}(t, x) = \int_{\Omega_{\tau -} \cap \Omega_{\omega}} (v(x) \cdot v) W(x, v) E(x, x - \tau(x, y)) S(x - \tau(x, y)) v \phi(t -
\[ \tau_\ast (x, v), x = \tau_\ast (x, v) v \, dv, (t, x) \in (0, T) \times \partial X \text{ and for } \phi \in L^1((0, \eta) \times \partial X). \] Therefore, performing the change of variables \('x' = x - \tau_\ast (x, v) v (dv = \frac{|v(x')|}{|x'-x|} \, d\mu(x') \) and \( \tau_\ast (x, v) = |x'-x| \), we obtain (2.13).

From the definition of \( A_2 \) and \( G_\ast \) we note that \( A_2 G_\ast (s) \phi_3 (z, w) := \int_{\mathbb{S}^{n-1}} k(z, v', w) E(z, -\tau_\ast (z, v') v') S(-\tau_\ast (z, v') v') \phi(s - \tau_\ast (z, v'), z - \tau_\ast (z, v') v') \, dv', \) for a.e. \((z, w) \in X \times \mathbb{S}^{n-1} \) and for \( \phi \in L^1((0, \eta) \times \partial X). \) Performing the change of variables \('x' = z - \tau_\ast (z, v') v' \), we obtain the equality \( (A_2 G_\ast (s) \phi_3) (z, w) = \int_{\mathbb{S}^{n-1}} k(z, v', w) S(x', \tau'(x')) |v'(x') \cdot v'| \left| T(dT) \right| \mu(x') \phi(s - |z - x'|, x') \, d\mu(x'), \) for a.e. \((z, w) \in X \times \mathbb{S}^{n-1} \) and \( \phi \in L^1((0, \eta) \times \partial X). \) Using also the definition of \( A_{1,5, w} \) (see (2.10) for \( m = 1 \)) we obtain the following equality for any \( \phi \in L^1((0, \eta) \times \partial X) \) and for a.e. \((t, x) \in (0, T) \times \partial X\)

\[ A_{1,5, w} (\phi) (t, x) = \int_{\mathbb{S}^{n-1}} \int_{0}^{t} \int_{\partial X} [k(x - (t-s)v, v', v) S(x', v')] \cdot v' (v(x') \cdot v) E(x, x - (t-s)v, v') \left| x - (t-s)v - x' \right|^{m-1} \chi(x, v, \tau(x, v))(t-s) \times \phi(s - |x - (t-s)v - x'|, x') W(x, v) \, d\mu(x') \, dx \, dv. \] (B.1)

Then, performing the changes of variables \('s' = t - x \) and \('t' = t - s - |x - sv - x'| \) \((s = \frac{t'-t}{s(t'-s-|x'-x'|)^2}, \frac{dv'}{s(t'-s-|x'-x'|)^2}) \), we obtain (2.14).

In order to prove (5.21) and (5.22), we introduce and prove proposition B.1 below, which gives the distributional kernel of the operators \( H_m \) defined in section 2.1. Let \( \tilde{E} \) denote the nonnegative measurable function from \( \mathbb{R}^n \times \mathbb{R}^n \) to \( \mathbb{R} \) defined by

\[ \tilde{E}(x_1, x_2) = e^{-|x_1|^{\alpha+1}|x_2|^{\alpha+1}} |(x_1-s_1)|^{\frac{\alpha}{1+\alpha}} |(x_2-s_2)|^{\frac{\alpha}{1+\alpha}} \, ds_1 ds_2, \text{ for a.e. } (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n. \] (B.2)

where \( \Theta(x_1, x_2) = 1 \) if \([px_1 + (1-p)x_2] p \in [0, 1] \subseteq X \) and \( \Theta(x_1, x_2) = 0 \) otherwise. For \( m \geq 3 \), we define recursively the nonnegative measurable real function \( \tilde{E}(x_1, \ldots, x_m) \) by the formula

\[ \tilde{E}(x_1, \ldots, x_m) = \tilde{E}(x_1, \ldots, x_{m-1}) \tilde{E}(x_{m-1}, x_m), \] (B.3)

for \((x_1, \ldots, x_m) \in (\mathbb{R}^n)^m\).

**Proposition B.1. We have**

\[ H_m (t) \phi(x, v) = \int_{X \times \mathbb{S}^{n-1}} \beta_m (t, x, v, x', v') \phi(x', v') \, dx' \, dv', \] (B.4)

for \( t \in (0, T) \) and a.e. \((x, v) \in X \times \mathbb{S}^{n-1}\) and for \( m \geq 2 \), where

\[ \beta_2 (t, x, v, x', v') = \int_0^t \chi(0,t-s_2) |(x' - (x - s_2 v'))|^{2n-2} |(x - s_2 v' - x' - (t - s_2)v)|^{2n-4} \times \left[ E(x, x - (t-s_1), s_2 v' - x', s_2 v') \right] k(x, (s_2 v' - x', v)) ds_2, \] (B.5)

\[ \beta_m (t, x, v, x', v') = \int_{\mathbb{S}^{n-1}} \sum_{j=0}^{m-1} \chi(0,t-s_{j+1}) \chi(0,t-s_{j+2}) \left[ (x' + s_m v') + \sum_{j=0}^{m-1} s_j v_j - x \right] \times 2^{n-2} \left( t - \sum_{j=2}^{m} s_j - (x' - \sum_{j=2}^{m-1} s_j v_j - s_m v') v^{m-3} \right) \left| x - x' - \sum_{j=2}^{m} s_j v_j - s_m v' - (t - \sum_{j=2}^{m} s_j)v \right|^{2n-4} \, ds_{j+1}. \]
\[
\begin{aligned}
&\times \left[ \hat{E} \left( x, x - \left( t - \sum_{j=1}^{m} s_j \right) u, x' + s_m u' + \sum_{j=1}^{m-1} s_j u_j, x' + s_m u' \right)
+ \sum_{j=m+1}^{\infty} s_j u_j, \ldots, x' + s_m u' \right) k \left( x - \left( t - \sum_{j=1}^{m} s_j \right) u, v, v_1, v \right)
\times k \left( x' + s_m u' + \sum_{j=2}^{\infty} s_j u_j, v_2, v_1 \right) \cdots k \left( x' + s_m u' + \sum_{j=m+1}^{\infty} s_j u_j, v_{i+1}, v_i \right) \cdots \\
&\times k \left( x' + s_m u', v', v_{m-1} \right) \right]
\end{aligned}
\]
\[
\int_{\gamma^{-1}} \phi \left( x, x - \left( t - s_1 - s_2 \right) u - s_1 v_1 \right) d_s \cdot d_s m \cdot d u_2 \cdots d v_{m-1}, \quad m \geq 3.
\] (B.6)

**Proof of proposition B.1.** Note that

\[
H_2(t) \phi(x, v) = \left( \int_0^t \int_0^{\gamma^{-1}} U_1(t - s - s_2) A_2 U_1(s) A_2 U_1(s) \phi(x, v) d s d s_1 \right) (x, v)
= \left( \int_0^t \int_0^{\gamma^{-1}} U_1(t - s - s_2) A_2 U_1(s) A_2 d s \right) U_1(s_2) \phi(x, v) \right) (x, v)
= \int_0^t \int_0^{\gamma^{-1}} \hat{E}(x, x - (t - s - s_2) u) \int_{\gamma^{-1}} k(x - (t - s - s_2) u, v, v_1, v)
\times \hat{E}(x, x - (t - s_1 - s_2) u, x - (t - s_1 - s_2) v - s_1 v_1)
\times \int_{\gamma^{-1}} k(x - (t - s_1 - s_2) v - s_1 v_1, v_2, v_1)
\times \hat{E}(x, x - (t - s_1 - s_2) v - s_1 v_1, x - (t - s_1 - s_2) v - s_1 v_1 - s_2 v_2)
\times \phi(x - (t - s_1 - s_2) v - s_1 v_1 - s_2 v_2, v_2) d v_2 d v_1 d s_1 d s_2,
\]
for \( t \in (0, T) \) and \((x, v) \in X \times \mathbb{S}^{n-1}\), where the functions \( \hat{E} \) are defined by (B.2)–(B.3).

Using the change of variables \( y(s_1, t) = (t - s_1 - s_2) v + s_1 v_1 \) we obtain

\[
H_2(t) \phi(x, v) = \int_0^t \int_{\mathbb{S}^{n-1}} \left[ \hat{E}(x, x - (t - s_1 - s_2) u, x - y, x - y - s_2 v_2) \right.
\times k(x - (t - s_1 - s_2) u, v, v_1, v) k(x - y, v, v_1, v) \left. \right] d_y \left( y - (t - s_2) v \right)^{2n-4} \phi(x - y - s_2 v_2, v_2) d y d v_2 d s_2.
\]

Hence we obtain (B.4) for \( m = 2 \). Note that

\[
(H_3(t) \phi)(x, v) = \int_0^t H_2(t - s_3) A_2 U_1(s_3) \phi(x, v) d s_3
= \int_0^t \int_{X \times \mathbb{S}^{n-1}} \beta_2(t - s_3, x, v, x_2, v_2) (A_2 U_1(s_3)) \phi(x_2, v_2) d x_2 d v_2 d s_3
= \int_{X \times \mathbb{S}^{n-1}} \int_0^t \beta_2(t - s_3, x, v, x_2, v_2) \int_{\mathbb{S}^{n-1}} k(x_2, v', v_2) \hat{E}(x_2, x_2 - s_3 v') \times \phi(x_2 - s_3 v', v') d v' d s_2 d v_2.
\]
Hence performing the change of variables \( 'x' = x_2 - s_1 v' \) (\( dx' = dx_2 \)) and using definitions (B.5) and (B.6) (for \( m = 3 \)) we obtain
\[
(H_3(t)\phi)(x, v) = \int_{\mathbb{R}^{n-1}} \beta_3(t, s, x, v, 'x', 'v')\phi(x_2, v_2) dx' dv'.
\] (B.7)

The proof of (B.6) follows by induction from (B.7) and the recurrence formula \( H_m(t) = \int_0^t H_{m-1}(t-s) A_2 U_1(s) ds \) for \( t \geq 0 \).

**Proof of (5.21) and (5.22).** We recall that
\[
(A_2 G_{-}(s)\phi_3)(z, w) = \int_{\partial X} [k(z, v', w) S(x', v') v(x') \cdot v']_{v'=\frac{z-x'}{|z-x'|}}
\times \frac{E(z, x')}{|z-x'|^{n-1}} \phi(s - |z-x'|, x') d\mu(x'),
\] (B.8)
for a.e. \((z, w) \in \mathbb{R} \times \mathbb{R}^{n-1}\) and \( \phi \in L^1((0, \eta) \times \partial X) \).

Let \( m = 2 \). Then from (2.10) and the definition of the operator \( H_1 \), it follows that
\[
A_{2,2}(t)\phi(t, x) = \int_{\mathbb{R}^{n-1}} (v(x) \cdot v) W(x, v) \int_0^{\frac{t}{s}} \int_{\mathbb{R}^{n-1}} [k(x - (t-s)v) v, v_1, v) \times k(x - (t-s-v_{1-v}) v_1, v_1) S(x', v') v(x') \cdot v']_{v'=\frac{z-x'}{|z-x'|}}
\times E(z, x') \phi(s - |z-x'|, x') dx' dv' \times |x - (t-s-v_{1-v}) v - s v_1 - x'|^{n-1} \chi_{X^2}(x - (t-s-v_{1-v}) v - s v_1 - x') d\mu(x') dv_1 ds dv.
\] Performing the change of variables \( y(s_1, v_1) = (t-s-v_{1-v}) v + s v_1 \), we obtain
\[
A_{2,2}(t)\phi(t, x) = \int_{\mathbb{R}^{n-1} \times \partial X \times \mathbb{R}^{n-1}} (v(x) \cdot v) W(x, v) \int_0^{\frac{t}{s}} \int_{\mathbb{R}^{n-1}} [k(x - (t-s-v_{1-v}) v, v_1, v) \times k(x - (t-s-v_{1-v}) v_1, v_1) S(x', v') v(x') \cdot v']_{v'=\frac{z-x'}{|z-x'|}}
\times E(z, x') \phi(s - |z-x'|, x') ds dv dy d\mu(x') dv.
\] (B.9)
Performing the change of variables \( 'y' = x_2 - y \) and \( t' = s - |y| \) we obtain (5.21).

Let \( m = 3 \). Then from (2.10), (B.4) (for \( 'm' = 2 \)) and (B.8) it follows that
\[
A_{3,3}(t)\phi(t, x) = \int_{\mathbb{R}^{n-1}} (v(x) \cdot v) W(x, v) \int_0^{\frac{t}{s}} \int_{\mathbb{R}^{n-1}} [k(x_2, v_2) S(x', v') v(x') \cdot v']_{v'=\frac{z-x'}{|z-x'|}}
\times E(x_2, x') d\mu(x') dx_2 dv_2 ds dv,
\] (B.10)
for \( t \in (0, T) \) and \( x \in \partial X \). From (B.10) and (B.5) we obtain
\[
A_{3,3}(t)\phi(t, x) = \int_{\mathbb{R}^{n-1}} (v(x) \cdot v) W(x, v) \int_{\mathbb{R}^{n-1} \times \partial X} \int_0^{\frac{t}{s}} \int_{\mathbb{R}^{n-1}} [k(x_2, v_2) S(x', v') v(x') \cdot v']_{v'=\frac{z-x'}{|z-x'|}}
\times E(x_2, x') d\mu(x') dx_2 dv_2 ds dv,
\] (B.10)
Performing the change of variables $y_2 = s_2 v_2$ and $y_3 = x - x'$ we obtain (5.22) for $m = 3$.

Let $m \geq 3$. From (2.10), (B.4) and (B.8) it follows that

$$
A_{m+1, S, W}(\phi)(t, x) = \int_{\mathbb{R}^{n-1}} \int_0^t (v(x) \cdot \nu) \mathcal{W}(x, v) \int \int_{(-\infty, t - |x_m - x'|) \times \mathbb{R}^{n-1}} \beta_m(t - t' - |x_m - x'|, x, v, x_m, v_m) [k(x_m, x', v, v_m) S(x', v') |v(x') - v'|]_{\nu = -|x_m - x'|}^{\nu = |x_m - x'|} d\nu d\nu_m d\nu_m \, dv.
$$

where

$$
A_{m+1}(t, x, x') := \int_{\mathbb{R}^{n-1}} \int_0^t (v(x) \cdot \nu) \mathcal{W}(x, v) \int \int_{X \times \mathbb{R}^{n-1}} \chi(0, +\infty)(\tau - |x_m - x'|) d\tau \, dv.
$$

Performing the change of variables $y_i = s_i v_i$, $i = 2, \ldots, m$, and $y_{m+1} = x_{m+1} - x'$, we obtain (5.22) for $m \geq 4$.\hfill $\square$
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