On the Links Between Neutral Directions, Buoyancy Forces, Energetics, Potential Vorticity, and Lateral Stirring in the Ocean: A First-Principles Approach

Rémi Tailleux, Gabriel Wolf

ABSTRACT: For over 80 years, lateral stirring in the ocean has been commonly assumed to occur preferentially along the N-neutral directions, a view at the origin of all forms of potential and quasi-neutral density variables used in oceanography. Yet, the validity of doing so has remained unclear as the theory of N-neutrality is solely rooted in heuristic parcel-based arguments lacking firm theoretical foundations. Furthermore, the only form of neutrality deducible from the rigorous analysis of the equations of motion so far, called P-neutrality, is only able to justify N-neutrality theoretically near an arbitrary fixed reference pressure but not globally. In this paper, we succeed in eliminating this difficulty over most of the ocean by reformulating the theory of P-neutrality in terms of the spatially varying reference pressure entering Lorenz theory of available potential energy and generalising it to the full Navier-Stokes equations. The improved form of P-neutrality, however, remains irreconcilable with N-neutrality in regions strongly departing from Lorenz reference state, such as the Southern Ocean, where the use of neutral rotated diffusion might hence be potentially invalid and the cause of spurious diapycnal mixing. The new theory establishes that energetics, buoyancy forces, available potential energy, potential vorticity, the Bernoulli function, nonlinear balance, and small-scale mixing all play important roles in constraining the preferential directions of lateral stirring in the ocean. Its potential importance for numerical ocean modelling and the theory of density variables used in water mass analyses are also discussed.

SIGNIFICANCE STATEMENT: The problem of how to define the preferential directions of lateral stirring in the ocean is given rigorous theoretical foundations for the first time in over 80 years. The new theory suggests that while the use of neutral rotated diffusion is reasonable in most of the ocean, it may be significantly in error in the Southern Ocean and Gulf Stream area, with potentially important consequences for the correct simulation of the water masses and the uptake of heat and carbon in numerical ocean models.

1. Introduction

That lateral stirring in the ocean is not isotropic but rather occurring along preferential directions defined by isopycnal surfaces of some kind has been a central tenet of physical oceanography for over 80 years. Indeed, such directions play a key role in a wide range of issues, such as the theory of water masses, the construction of layered models of the ocean circulation, the construction of rotated diffusion tensors for describing the turbulent mixing of heat, salt, and passive tracers, to name but a few of the most important ones. As a result, developing a theoretical understanding of what such directions are and how they arise from the governing equations of motion remains a central research issue in oceanography. To that end, energetics, buoyancy forces, available potential energy, the near materiality of potential temperature and salinity, potential density surfaces, neutral directions, potential vorticity, and P vectors, have all been invoked as being somehow involved in determining such preferential directions. In this paper, we are able to show for the first time how to assemble all these ingredients into a deductive and cohesive first-principles analysis of the Navier-Stokes equations governing oceanic motions, which had so far proved elusive.

Prior to developing a theory of lateral stirring for the oceans, one first needs to ascertain that the nature of lateral stirring occurs preferentially along isentropic surfaces, which for a simple fluid may be equivalently defined as the surfaces of constant potential temperature or constant potential density. To test the validity of such an idea, several authors have proposed to diagnose the turbulent tensor allowing one to express the turbulent fluxes of passive tracers to their mean gradients in numerical simulations resolving lateral stirring, and check that the eigendirections corresponding to lateral stirring align with isentropic directions, consistent with Redi’s (1982) suggestion that mixing in the oceans should be parameterised by means of rotated diffusion tensors. Abernathey et al. (2013) implemented such an idea in the idealised context of a re-entrant channel using a linear equation of state for density function of temperature only, simulating the turbulent evolution of 6 different passive tracers. Their main conclusion was that while the lateral eigendirections of the turbulent fluxes tensors could indeed be verified to approximately align with the isopycnal directions, this was true only in the ocean interior away from sources of diabatism and friction, as upon approaching the surface, these eigendirections were found to become increasingly horizontal and hence non-material.

Corresponding author: R. Tailleux, R.G.I.Tailleux@reading.ac.uk
In the oceans, the identification of what might be the seawater counterpart of the isentropic directions of a simple fluid is complicated by the thermobaric nonlinearity of the equation of state for seawater density (due to the pressure dependence of the thermal expansion coefficient), as well as by the infinite number of material invariants (defined by all functions $\gamma(S, \theta)$ of potential temperature $\theta$ and salinity $S$), e.g., Eden and Willebrand (1999); de Szoeke (2000); de Szoeke and Springer (2000). To make progress, one may first ask what is (are) the particular attribute(s) of the potential temperature of a simple fluid, if any, that might be responsible for its relevance to lateral stirring? If such an attribute (attributes) could be understood and isolated, the best seawater analogue might simply be the seawater variable possessing it (them). Physically, the attributes that seem to be the most relevant and hence the ones that have received the most attention in the literature are the materiality, neutrality, existence of a geostrophic streamfunction, and flatness at rest.

Physically, only the materiality and flatness-at-rest properties are known to carry over to seawater, the neutrality and existence of a geostrophic streamfunction being only enforceable approximately. As a result, it is not possible to simply look for the seawater variable possessing all the attributes of the potential temperature of a simple fluid, as such a variable does not exist. This means that it is essential to understand how each attribute relates to lateral stirring. Note here that for a particular attribute to be able to determine the properties of lateral stirring in some sense, it should a priori be satisfied by at most one seawater variable or class of seawater variables defining the same physical surfaces. It follows that the materiality property, which is naturally satisfied by an infinite number of seawater variables (all material functions $\gamma(S, \theta)$), can at best only be expected to modify the properties of lateral stirring, not to fully determine them. So far, it is the neutrality property that appears to have been regarded as the most important attribute in this regard, as perhaps first articulated by Sverdrup et al. (1942) (pp. 414–416, section entitled ‘Significance of $\sigma_r$ surfaces’). This work reveals that the neutrality property was originally defined as the property of the stratification and potential energy of a simple fluid to remain unaffected upon the interchange of two parcels belonging to the same surface of constant potential temperature. Physically, this amounts to define the preferential directions of lateral stirring as the directions along which the interchange of fluid parcels has a vanishing energy cost and therefore as the most energetically favourable directions in some sense. Sverdrup et al. (1942) recognised, however, that no seawater variable could be made to satisfy the neutrality property exactly, and therefore suggested that the best seawater analogue of the potential temperature of a simple fluid should be defined as the seawater variable best approaching it. Such a proposition prompted a quest for a density-like variable maximising neutrality in some sense that is still ongoing today, and appears to be at the origin of most density variables used in oceanography.

Early efforts focused on $\sigma_r$, before turning on $\sigma_g$, then on Lynn and Reid (1968) patched potential density (PPD), each successive variable improving in its neutrality properties. To obtain a continuous analogue of PPD, Jackett and McDougall (1997) completely reformulated the problem as that of constructing a globally defined variable $\gamma^n$ defined so that its gradient $\nabla \gamma^n$ be everywhere as parallel as feasible to the so-called neutral vector $\mathbf{N} = g(a \nabla \theta - \beta \nabla S)$, resulting in a function $\gamma^n = \gamma^n(S, \theta, x, y, p)$ being a hybrid function of both thermodynamic and geographical variables. Recently, Tailleux (2016a) showed how to construct a purely material approximation of $\gamma^n$, called thermodynamic neutrality density $\gamma^T$, by enforcing the flatness-at-rest property that had hitherto not been used. In doing so, Tailleux (2016a) established a link between the theory of lateral stirring and of density variables with Lorenz (1955) theory of available potential energy, of crucial importance to the new theory developed in this paper, which were further developed in Tailleux (2016b).

Despite its central role in oceanography, the neutrality property (referred to as N-neutrality in this paper) remains nevertheless poorly understood and unsatisfactorily defined, for being exclusively rooted in the ad-hoc and heuristic parcel-based arguments put forward by Sverdrup et al. (1942), McDougall (1987) or McDougall et al. (2014), which Tailleux (2015b) has shown to be often invalid or inaccurate in seawater. As a result, some of its most important theoretical aspects remain unclear, such as how to define precisely (i.e., in the context of the full Navier-Stokes equations) what one means by the directions along which lateral stirring minimally perturb the stratification and its potential energy or the buoyancy forces that McDougall (1987) claim to affect the interchange of parcels along directions other than the neutral directions.

To gain insights into the nature of the forces acting on fluid parcels evolving within the iso-surface of an arbitrary thermodynamic variable $\gamma$, de Szoeke (2000) proposed that one examines the horizontal momentum balance obtained by using $\gamma$ as a generalised vertical coordinate. For the hydrostatic primitive equations, this balance takes the form

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} + \nabla \gamma \cdot \mathbf{J} M = \Phi(\gamma) + \Phi_h$$

(his Eq. 2.15), where $\mathbf{u}$ is the horizontal component of the velocity field, $f$ is the Coriolis parameter, $\mathbf{k}$ is the vertical unit vector pointing upward, and $\Phi_h$ is the horizontal friction force. In (1), $M$ represents the so-called Montgomery potential, which may be defined as

$$M = M(S, \theta, p, z) = \int_{p_r}^p \nu(S, \theta, p') dp' + \Phi(z),$$

where $p_r$ is the reference pressure.
where $p_r$ is an arbitrary reference pressure, $\Phi(z) = g z$ is the geopotential height, $v(S, \theta, p)$ is the specific volume seen as a function of salinity $S$, potential temperature $S$ and pressure $p$, while $\mathbf{P}^{(\gamma)}$ is the iso-$\gamma$ component of a thermodynamic buoyancy force defined as the thermohaline gradient of $M$,

$$
\mathbf{P} = \frac{\partial M}{\partial \theta} \nabla \theta + \frac{\partial M}{\partial S} \nabla S = \frac{p - p_r}{\rho g} \overline{N},
$$

(3)

where $\overline{N}$ is a neutral-like vector defined by

$$
\overline{N} = g(\overline{\alpha} \nabla \theta - \overline{\beta} \nabla S),
$$

(4)

where $\overline{\alpha}$ and $\overline{\beta}$ are vertically-averaged values of the thermal expansion and haline contraction coefficients over the pressure range $[p_r, p]$. As shown further in this paper (see Appendix A), the expression of $\mathbf{P}$ underlying de Szoëke (2000)'s framework happens to be nearly identical to Ny- 

We explore the possibility of circumventing this difficulty by redefining $\mathbf{P}$ in terms of the physically-based spatially variable reference pressure $p_r = p_r(S, \theta)$ entering Lorenz (1955) theory of available potential theory, as recently illustrated in the context of spinicess by Tailleux (2021). To achieve full generality, we move away from the hydrostatic primitive equations that have been the basis for most theoretical discussions on the topic in favour of the full Navier-Stokes equations, hence retaining all compressible and non-hydrostatic effects as well as the full Earth rotation vector. The resulting newly defined $\mathbf{P}$ vector is denoted $\mathbf{P}_a$ in the following. By construction, lateral stirring taking place along the $\mathbf{P}_a$-neutral directions leave Tailleux (2018)'s definition of potential energy density $\Pi$ unaffected. In the absence of friction, it is also possible to use $\mathbf{P}_a$ to construct the following 3D velocity field vector

$$
v_{ia} = \frac{\mathbf{P}_a \times \nabla B}{\rho Q},
$$

(6)

which happens to define an exact solution of the steady-state Navier-Stokes equations approximately satisfying the continuity equation $\nabla(\rho v_{ia}) \approx 0$, where $B$ is a suitably defined Bernoulli function, while $Q = \omega_a \cdot \mathbf{P}_a / \rho = \omega_a \cdot \nabla B / \rho$ defines a generalised form of potential vorticity, $\omega_a$ being the absolute vorticity. Eq. (6) extends to the case of seawater what Gassmann (2014) calls the inactive wind in the context of dry atmospheric dynamics. Physically, $v_{ia}$ is a generalised and nonlinear form of geostrophic balance, referred to as the inactive wind because by construction, inactive wind trajectories neither affect the available potential energy nor kinetic energy of the fluid. As shown in this paper, redefining $\mathbf{P}$ in this way succeeds in making it approximately parallel to McDougall (1987) neutral vector $\mathbf{N}$ in most of the ocean with a few important exceptions in the polar and Gulf Stream regions, which leads us to speculate that the use of neutral rotated diffusion tensors might cause spurious diapycnal mixing in such regions.

The theoretical basis for the construction of the new $\mathbf{P}_a$ vector, along with that for the inactive wind solution (6), are detailed in Section 2. Section 3 discusses various alternative forms of $\mathbf{P}_a$ that may be more useful in practice. Section 4 systematically compares the differences between $\mathbf{P}_a$ and $\mathbf{N}$, and discusses the implications for the assessment of the neutral character of $\gamma^n$ and $\gamma^T$. Section 5 summarises the results and discusses their implications for the theory of lateral stirring.

2. Forces and neutral directions revisited

a. Classical view of forces

To justify the idea that lateral stirring preferentially takes place along ‘paths of least resistance’ in some sense, it is a priori necessary to develop a deep understanding of the forces driving fluid motion. As shown in this section, this is complicated by the fact that there appears to be no
unique way to write down the expression of such forces, which depends on the view adopted. In the classical view, for instance, it is customary to regard fluid motion as being driven by the Coriolis force, the pressure gradient, gravity, and friction, as is embodied in the standard form of the Navier-Stokes equations (NSE):

\[
\frac{D\mathbf{v}}{Dt} = -2\mathbf{\Omega} \times \mathbf{v} - \frac{1}{\rho} \nabla p - \nabla \Phi + \mathbf{F},
\]

(7)

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0,
\]

(8)

\[
\frac{D\eta}{Dt} = \dot{\eta}, \quad \frac{DS}{Dt} = \dot{S},
\]

(9)

\[
v = v(\eta, S, p) = \frac{\partial h}{\partial p},
\]

(10)

where \(\mathbf{v} = (u, v, w)\) is the 3D velocity field, \(p\) is pressure, \(\rho\) is density, \(\nu = 1/\rho\) is the specific volume, \(h = h(\eta, S, p)\) is the specific enthalpy, \(\eta\) is the specific entropy, \(\Phi(z) = g z\) is the geopotential with \(g\) the gravitational acceleration and \(z\) height increasing upward, \(\mathbf{\Omega}\) is Earth’s rotation vector, and \(\mathbf{F}\) denotes the frictional force.

At equilibrium, Eq. (7) shows that the pressure and density fields are solutions of

\[
\nabla p = -\rho \nabla \Phi = -\rho g \mathbf{k},
\]

(11)

which yields the familiar result that \(p\) and \(\rho\) must be in hydrostatic balance and functions of \(z\) only, \(\mathbf{k}\) being the unit normal vector pointing upward. To reduce the dominance of the vertical variations that do not affect horizontal dynamics, \(p\) and \(\rho\) are commonly redefined as perturbations \(p' = p - p_0(z)\), \(\rho' = \rho - \rho_0(z)\), to reference hydrostatic pressure and density fields \(p_0(z)\) and \(\rho_0(z)\), in which case (7) becomes:

\[
\frac{D\mathbf{v}}{Dt} = -2\mathbf{\Omega} \times \mathbf{v} - \frac{1}{\rho} \nabla p' + b \mathbf{k} + \mathbf{F},
\]

(12)

As a result, fluid parcels now appear to be driven by the anomalous pressure gradient and vertical buoyancy force \(b \mathbf{k}\) rather than the pressure gradient and gravity, where the buoyancy \(b\) is defined by

\[
b = - \frac{g(\rho - \rho_0(z))}{\rho}.
\]

(13)

We call (12) the Pressure Gradient/Vertical Buoyancy Force (PGVBF) form of specific momentum balance. If \(\rho_0(z)\) are \(\rho_0(z)\) are specified in terms of the actual stratification — for instance via horizontal averaging or via computing the state of minimum potential energy entering Lorenz [1955] theory of available potential energy — (12) may then be regarded as a data-assimilated version of (7) that enhances its predictive powers to some extent, as it is now often possible to anticipate that upward (downward) motion will follow from \(b > 0\) \((b < 0)\).

The meso-scale eddies doing the bulk of the lateral stirring in the ocean are large enough to be affected by Earth rotation, meaning that they are expected to be in approximate geostrophic balance away from the equator, viz.,

\[
\mathbf{u} \approx \frac{1}{\rho f} \mathbf{k} \times \nabla h p'.
\]

(14)

Geostrophy (14) only constrains the horizontal component \(\mathbf{u}\) of the total velocity field \(\mathbf{v}\), however, so that even though it provides a mathematically explicit expression linking part of the forces to part of the velocity field, it is not sufficient in itself to say much about the preferential directions followed by fluid parcel trajectories. In the Boussinesq approximation, the obtention of the missing vertical component \(w\) generally proceeds from vertically integrating the continuity equation \(\nabla \cdot \mathbf{v} + w_z = 0\) from the knowledge of the horizontal velocity field. This is not necessarily accurate, however, as the horizontal divergence \(\nabla \cdot \mathbf{v}\) is dominated by the much smaller and harder to predict ageostrophic terms neglected in (14); moreover, errors may also increase with the length of the integration path. It is easy to see, therefore, that predicting the full 3D velocity field \(\mathbf{v}\) from the numerical integration of the Boussinesq approximation to (12) to ensure that the resulting \(\mathbf{v}\) follows the ‘right’ directions is a priori technically challenging due to the non-local character of the vertical velocity determination. The considerations developed in the subsequent sections suggest that part of the difficulty is attributable to the forces entering the PGVBF specific momentum balance being sub-optimal for the purposes of constructing explicit expressions of the full velocity field.

b. Alternative view of forces based on Crocco’s equation

The main alternative description of the forces driving fluid motion and also the main focus of this paper is obtained by making use of the total differential of specific enthalpy

\[
dh = T d\eta + \mu dS + \frac{1}{\rho} dp
\]

(15)

to get rid of the pressure gradient term \(\rho^{-1} \nabla p\) in favour of terms containing the gradients of entropy, salinity, and enthalpy, where \(\mu\) is the relative chemical potential, thus leading to

\[
\frac{D\mathbf{v}}{Dt} = -2\mathbf{\Omega} \times \mathbf{v} - \nabla (h + \Phi) + T \nabla \eta + \mu \nabla S + \mathbf{F}.
\]

(16)

Eq. (16) is often referred to as Crocco’s equation in the fluid mechanics literature, after Crocco [1937]. Such a manipulation is of course also well known in the atmospheric and oceanic literatures, as it is the starting point for recasting the equations of motion in isentropic coordinates,
The quantity $h + \Phi$ is called the static energy in the atmospheric literature, which for a dry atmosphere is generally written in the form $c_p T + g z$, with $c_p$ the specific heat capacity of air at constant pressure.

An important difference between (16) and (7) relates to the way each approach characterises equilibrium resting states. Indeed, for such states, (16) reduces to

$$\nabla (h + \Phi) = T \nabla \eta + \mu \nabla S, \quad (17)$$

and predicts that the static energy $h + \Phi$ must be a material function of $\eta$ and $S$ only. In other words, whereas (7) tends to emphasise the geometric height $z$ to describe equilibrium resting states, (16) emphasises thermodynamic coordinates $\eta$ and $S$. In a non-resting state, the static energy becomes formally a function of $(\eta, S, p, z)$. Yet, it is well known that for approximately hydrostatic motions, adiabatic and isohaline changes in specific enthalpy $dh = \rho^{-1} dp$ tend to be cancelled out by changes in gravitational energy $d\Phi$ so that $\rho^{-1} dp + d\Phi \approx 0$. In practice, $h + \Phi$ therefore tends to behave approximately as a purely material function of $S$ and $\eta$ even for a non-resting state, which explains why it is in general the quantity used in atmospheric studies for studying the poleward energy transport, e.g., Liang et al. (2018).

In the same way that $p$ can be redefined as a perturbation relative to a reference pressure $p_0(z)$ to eliminate the part of the pressure that does not contribute to horizontal dynamics, one may redefine the static energy relative to a reference pressure $p_0(z)$ to its actual pressure and is referred to as Available Compressible Energy (ACE) here, while $\Pi_2$ represents the work needed to bring a fluid parcel from its reference position $z_r$ in Lorenz reference state to its actual position $z$, and is called the APE density. Tailleux (2018) shows that at leading order in perturbation pressure $p-p_0(z)$ and displacement $z-z_r$, $\Pi_1$ and $\Pi_2$ approximate to

$$\Pi_1 \approx \frac{(p - p_0)^2}{2 \rho b_c^2}, \quad \Pi_2 \approx \frac{N_r^2(z_r)(z-z_r)^2}{2} \quad (22)$$

where $c_r$ is the speed of sound, $N_r^2 = -(g/\rho_r)(\rho'_r(z_r) + \rho_r g c_w^2(S,S_p)\rho_r) (e.g.,$ see Eq. (2.19) of Tailleux (2018)), with the suffixes ‘$b$’ and ‘$r$’ indicating that a thermodynamic quantity is estimated at the reference pressures $p_0(z)$ and $p_0(z_r)$ respectively.

As a result, it is possible to write $M$ in terms of $\Pi$ as follows:

$$M = \Pi + \frac{p - p_0(z)}{\rho}. \quad (23)$$

In the literature, $M$ is generally referred to as the Montgomery potential, our definition (18) being the same as that used by de Szoeke (2000), except for the use of Lorenz reference depth instead of the fixed reference depth $z_r = 0$.

To show how to rewrite (10) in terms of $M$, let us take the gradient of (18), recalling that $p'_r(z_r) = -\rho_0(g)g$ by construction and accounting for (19), which yields

$$\nabla M = \frac{1}{\rho} \nabla p + \nabla \Phi + (T - T_r) \nabla \eta + (\mu - \mu_r) \nabla S \quad (24)$$

where $T_r = T(\eta, S, p_0(z_r))$ and $\mu_r = \mu(\eta, S, p_0(z_r))$ are reference values for the in-situ temperature and relative chemical potential respectively. By inserting (24) into (7) allows one to rewrite (10) in the form:

$$\frac{Dv}{Dt} = -2\Omega \times v - \nabla M + P_a + \mathbf{F}, \quad (25)$$

as first introduced by Tailleux (2013). As is now well understood, Lorenz (1955) global APE theory can actually be shown to derive from a local principle, which in the case of a multi-component compressible stratified fluid is the local available energy $\Pi = \Pi_1 + \Pi_2$ recently derived by Tailleux (2018), where $\Pi_1$ and $\Pi_2$ are the local potential energy sub-reservoirs

$$\Pi_1 = h(\eta, S, p) - h(\eta, S, p_0(z)) + \frac{p_0(z) - p}{\rho}, \quad (20)$$

$$\Pi_2 = h(\eta, S, p_0(z)) - h(\eta, S, p_0(z_r)) + \Phi(z) - \Phi(z_r). \quad (21)$$

In the same way that $p$ can be redefined as a perturbation relative to a reference pressure $p_0(z)$ to eliminate the part of the pressure that does not contribute to horizontal dynamics, one may redefine the static energy relative to a reference static energy function of $(\eta, S, p, z)$ only to similarly eliminate the passive part of the static energy. In most of the literature so far, such as in the aforementioned de Szoeke (2000) and Nycander (2011), this has generally been done by defining the reference static energy as the static energy reference to an arbitrary fixed reference pressure often chosen as the surface mean atmospheric pressure. Based on the similarity of the problematic with the Lorenz (1955) theory of available potential energy (APE), however, this paper proposes to explore the consequences of defining the perturbation static energy as the difference between the static energy of the actual state minus that of Lorenz reference state (that is, a state of minimum potential energy obtainable from the actual state by means of a re-arrangement of mass conserving salt and entropy, e.g., Saenz et al. (2015)), viz.,

$$M = h(\eta, S, p) + \Phi(z) - h(\eta, S, p_0(z)) - \Phi(z_r) \quad (18)$$

where $p_0(z)$ and $p_0(z_r)$ are the hydrostatic pressure and density reference profiles defining Lorenz reference state, while $z_r$ is the reference position of a fluid parcel in Lorenz reference state, defined as a solution of so-called level of neutral buoyancy (LNB) equation

$$\rho(\eta, S, p_0(z_r)) = \rho_0(z_r), \quad (19)$$

e.g., de Szoeke (2000).
where $\mathbf{P}_a$ is the thermodynamic buoyancy force vector defined by

$$
\mathbf{P}_a = \frac{\partial \Pi}{\partial \eta} \nabla \eta + \frac{\partial \Pi}{\partial S} \nabla S = (T - T_r) \nabla \eta + (\mu - \mu_r) \nabla S.
$$

We call (16) and (25) the Energy Gradient/Thermodynamic Buoyancy Force (EGTBF) form of the momentum balance. Because the reference pressure $p_0$ and depth $z_r$ represent empirical information about the fluid considered, (25) can also be regarded as a ‘data assimilated’ version of (16). It is important to note, however, that the amount of empirical information entering (25) is a priori considerably larger than that entering (12). Indeed, the empirically determined reference pressure and density fields $p_0(z)$ and $\rho_0(z)$ entering (12) do not in themselves tell us much about the fluid. In contrast, the construction of the reference static energy requires the knowledge of the reference position $z_r = z_r(S, \theta)$ of all fluid parcels, which in practice entails performing a full census of the water masses, as shown by Saenz et al. (2015), as well as solving the LNB equation (19) for each fluid parcel. As a result, $z_r$ contains information about the relative placement of each fluid parcel in the water column informed by mass conservation calculations not contained in $p_0(z)$ or $\rho_0(z)$ (at least, without performing additional calculations). The ability of the EGTBF form of momentum balance to ingest much more empirical information about the fluid than the PGVBTF form appears to be responsible for its superior predictive capabilities illustrated next.

c. Properties of the EGTBF momentum balance

Next, we seek to establish to what extent the preferential directions for lateral stirring can be linked to the directions perpendicular to the thermodynamic buoyancy force $\mathbf{P}_a$. To that end, we first need to discuss the energetics of (25). This is facilitated by rewriting the latter in vector invariant form as follows

$$
\frac{\partial \mathbf{v}}{\partial t} = -\omega_a \times \mathbf{v} - \nabla \mathbf{B}_\Pi + \mathbf{P}_a + \mathbf{F},
$$

obtained by using the well known identity $(\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \times \mathbf{v} + \nabla (\mathbf{v} \cdot \mathbf{v})/2$, where $\omega_a = 2\Omega \times \mathbf{v} + \nabla \times \mathbf{v}$ is the absolute vorticity, while

$$
\mathbf{B}_\Pi = \frac{\mathbf{v}^2}{2} + M = \frac{\mathbf{v}^2}{2} + \Pi + \frac{p - p_0}{\rho}
$$

represents an APE-based form of the well known Bernoulli function. Note here that in the literature, the Bernoulli function is more traditionally defined as

$$
\mathbf{B}_h = \frac{\mathbf{v}^2}{2} + h + \Phi.
$$

Multiplying (27) by $\mathbf{v}$ yields the following balance equation for kinetic energy

$$
\frac{\partial \mathbf{v}^2}{\partial t}/2 = -\mathbf{v} \cdot \nabla \mathbf{B}_\Pi + \mathbf{v} \cdot \mathbf{P}_a + \mathbf{v} \cdot \mathbf{F}.
$$

To shed light on the work $\mathbf{v} \cdot \mathbf{P}_a$ against the thermodynamic buoyancy force $\mathbf{P}_a$, we take the lagrangian derivative of (20) and (21), which after some manipulation can be shown to yield

$$
\frac{\partial \Pi_1}{\partial t} + \rho' \frac{\partial \mu}{\partial t} + \mathbf{v} \cdot \mathbf{P}_{a1} = (T - T_b) \eta + (\mu - \mu_b) \dot{S},
$$

$$
\frac{\partial \Pi_2}{\partial t} + \mathbf{v} \cdot \mathbf{P}_{a2} = (T_b - T_r) \eta + (\mu_b - \mu_r) \dot{S}
$$

where $\mathbf{P}_{a1}$ and $\mathbf{P}_{a2}$ given by

$$
\mathbf{P}_{a1} = \frac{\partial \Pi_1}{\partial \eta} \nabla \eta + \frac{\partial \Pi_1}{\partial S} \nabla S = (T - T_b) \nabla \eta + (\mu - \mu_b) \nabla S,
$$

$$
\mathbf{P}_{a2} = \frac{\partial \Pi_2}{\partial \eta} \nabla \eta + \frac{\partial \Pi_2}{\partial S} \nabla S = (T_b - T_r) \nabla \eta + (\mu_b - \mu_r) \nabla S,
$$

so that $\mathbf{P}_{a1} + \mathbf{P}_{a2} = \mathbf{P}_a$. As a result, summing (31) and (32) yields

$$
\frac{\partial \Pi}{\partial t} + \rho' \frac{\partial \mu}{\partial t} + \mathbf{v} \cdot \mathbf{P}_a = (T - T_r) \eta + (\mu - \mu_r) \dot{S} = \dot{\Pi}.
$$

Eq. (35) is important because it shows that a non-zero work $\mathbf{v} \cdot \mathbf{P}_a$ can either support mixing or a change in local available potential energy $\Pi$. Nyckander (2011) previously derived a similar result by showing that the work $\mathbf{v} \cdot \mathbf{P}$ against his $\mathbf{P}$ vector could either support mixing or a change in dynamic enthalpy $h^\ddagger$.

As shown in the next section, $\mathbf{P}_{a1}$ may be approximated as

$$
\mathbf{P}_{a1} \approx \frac{p - p_0(z)}{\rho g} \mathbf{N},
$$

which implies that the directions that it defines can be identified with those defined by McDougall (1987) standard neutral vector $\mathbf{N}$. From (33), it follows that the part of the potential energy left unaffected by lateral stirring along the standard neutral directions is the Available Compressible Energy (ACE) reservoir $\Pi_1$, thus clarifying the issue for the first time since its first introduction by Sverdrup et al. (1942). This result is important, because in the case where $\mathbf{P}_{a1}$ and $\mathbf{P}_{a2}$ differ, lateral stirring along the standard neutral directions must necessarily require external work in order to support the unavoidable changes in the ACE density reservoir $\Pi_2$ predicted by (34).
By combining (35) with (30), the following evolution equation for the Bernoulli function \( B_{II} \) is obtained
\[
\frac{\partial B_{II}}{\partial t} + \mathbf{v} \cdot \nabla B_{II} = \dot{B}_{II} + \frac{1}{\rho} \frac{\partial p'}{\partial t} \quad (37)
\]
where the term
\[
\dot{B}_{II} = (T - T_r) \eta + (\mu - \mu_r) \mathcal{S} + \mathbf{F} \cdot \mathbf{v} = \dot{\Pi} + \mathbf{F} \cdot \mathbf{v} \quad (38)
\]
represents the diabatic nonconservation of the Bernoulli function.

To understand the benefits of formulating the EGTBF form of momentum balance in terms of \( B_{II} \) rather than in terms of \( B_h \), consider the case of a steady-state solution of (27) or (16) in the absence of diabatic sources of heat and salt as is well known, it is apparent that \( B_{II} \) is significantly more affected by diabatic effects than \( B_h \). In that case, the respective evolution equations for \( B_{II} \) and \( B_h \) become
\[
\mathbf{v} \cdot \nabla B_{II} = T \eta + \mu \mathcal{S}, \quad (39)
\]
\[
\mathbf{v} \cdot \nabla B_h = (T - T_r) \eta + (\mu - \mu_r) \mathcal{S}. \quad (40)
\]
Eqs. (39) and (40) show that although both forms of Bernoulli functions are conserved following parcel motion in the absence of diabatic sources of heat and salt as is well known, it is apparent that \( B_{II} \) is significantly more affected by diabatic effects than \( B_h \). To quantify this, let us neglect the effect of salinity for simplicity so as to be able to obtain the following simple expression for the ratio
\[
\frac{\mathbf{v} \cdot \nabla B_{II}}{\mathbf{v} \cdot \nabla B_h} = \frac{T - T_r}{T} \approx \frac{\alpha T}{\rho c_p} (\rho - \rho_r) \approx -\frac{\alpha g}{c_p} (z - z_r). \quad (41)
\]
Interestingly, this ratio has the form of a thermodynamic efficiency-like factor. Using \( \alpha = 10^{-4} \text{K}^{-1} \), \( c_p = 4000 \text{J} \text{K}^{-1} \text{kg}^{-1} \), \( g = 10 \text{m.s}^{-2} \), and \( |z-z_r| = 2000 \text{m} \), yields \( |T - T_r|/T = O(5 \times 10^{-4}) \), meaning that the ratio (41) is very small even for large departures from Lorenz reference state. This is a very important result as it suggests that \( B_{II} \), unlike \( B_h \), can be regarded as approximately conserved along steady fluid parcel trajectories in the absence of friction even if small scale mixing of heat and salt is retained, the same conclusion also likely applying to \( \mathbf{P}_a \) versus \( \mathbf{P}_h \), so that
\[
\mathbf{v} \cdot \nabla B_{II} \approx 0, \quad \mathbf{v} \cdot \mathbf{P}_a \approx 0. \quad (42)
\]
To the extent that (42) is satisfied, (27) appears to have the special property that all forces save for friction appearing in the EGTBF form of momentum balance are perpendicular or nearly perpendicular to the full 3D velocity field \( \mathbf{v} \). In the literature, there has been much discussion of how energetics and the near materiality of \( \theta \) and \( S \) constrain fluid motion, and how to exploit such constraints to improve ocean models. The EGTBF form of momentum balance appears to provide a natural answer to this question, with energy conservation being associated with the force \( \nabla B_{II} \), and the near material conservation of \( \theta \) and \( S \) being associated with the force \( \mathbf{P}_a \). It follows that the averaging and numerical implementation of the EGTBF form of momentum balance should naturally account for both constraints, which is not necessarily the case with the PGVBF form of momentum balance, a topic left for future investigations.

d. Balanced solutions of EGTBF momentum balances

To further demonstrate the superiority of the EGTBF form of specific momentum balance over the PGVBF form, we return to the problem of constructing explicit balanced solutions of the velocity field. To that end, we consider the full nonlinear balance equation, which in the absence of friction takes the form
\[
\omega_a \times \mathbf{v} = \mathbf{P}_a - \nabla B_{II} = \mathbf{P}_h - \nabla B_h, \quad (43)
\]
with one additional aim being to contrast the results obtained with \( (B_{II}, \mathbf{P}_a) \) versus \( (B_h, \mathbf{P}_h) \). Mathematically, the general solution of (43) may be written as the sum of a particular solution \( \mathbf{v}_{ia} \) plus an undetermined part belonging to the null-space spanned by linear combinations of \( \omega_a \).
\[
\mathbf{v} = \mathbf{v}_{ia} + \lambda \omega_a, \quad (44)
\]
where \( \lambda \) is an arbitrary scalar field to be determined. Because (43) is under-determined, the existence of a solution requires that the relevant solvability condition is met, viz.,
\[
\omega_a \cdot \mathbf{P}_a = \omega_a \cdot \nabla B_{II}, \quad \omega_a \cdot \mathbf{P}_h = \omega_a \cdot \nabla B_h. \quad (45)
\]
(obtained by multiplying (43) by \( \omega_a \)). In (44), the expressions for the explicit parts \( \mathbf{v}_{ia} \) are given by
\[
\mathbf{v}_{ia}^{\Pi} = \frac{\mathbf{P}_a \times \nabla B_{II}}{\rho \mathbf{Q}_a}, \quad \mathbf{v}_{ia}^h = \frac{\mathbf{P}_h \times \nabla B_h}{\rho \mathbf{Q}_h}, \quad (46)
\]
where
\[
\mathbf{Q}_a = \frac{\omega_a \cdot \nabla B_{II}}{\rho}, \quad \mathbf{Q}_h = \frac{\omega_a \cdot \nabla B_h}{\rho} \quad (47)
\]
are generalised forms of potential vorticity constructed around the the two Bernoulli functions \( B_{II} \) and \( B_h \). In general, \( \mathbf{v}_{ia}^h \) and \( \mathbf{v}_{ia}^{\Pi} \) define different vector fields, their difference necessarily belonging to the null-space. In other words, there must exist some scalar field \( \delta \lambda \) such that
\[
\mathbf{v}_{ia}^h - \mathbf{v}_{ia}^{\Pi} = \delta \lambda \omega_a. \quad (48)
\]
Since for large scale flows \( \omega_a \) tends to be dominated by the vertical component of the planetary vorticity \( f_k \), (48) imply that \( \mathbf{v}_{ia}^h \) and \( \mathbf{v}_{ia}^{\Pi} \) approximately differ by the magnitude of their vertical velocity component.
Physically, the undetermined part $\lambda \omega_a$ is in general needed for $v$ to satisfy the continuity equation and hence for $v$ to represent an admissible solution of $\nabla \cdot (\rho v_{ia})$. To establish to what extent such undetermined part is needed here, let us evaluate the divergence $\nabla \cdot (\rho v_{ia})$ of the explicit inactive wind solution $v_{ia}$. To that end, we multiply $\nabla \cdot (\rho v_{ia})$ by $\rho$ and take the divergence, which yields

$$\nabla \cdot (\rho v_{ia}) = -\frac{(P \times \nabla B)}{Q^2} \cdot \nabla Q + \frac{(\nabla \times P) \cdot \nabla B}{Q},$$

(49)

which holds regardless of which $(B, P)$ representation is used. Eq. (49) shows that the degree to which $v_{ia}$ conserves mass depends on the degree of conservation of $Q$ following the inactive wind trajectories. According to the standard theory of potential vorticity, it is possible to show that in the present case, $Q$ satisfies

$$\frac{D(\omega_a \cdot \nabla B)}{Dt} = \omega_a \cdot \nabla \frac{DB}{Dt} + \nabla B \cdot (\nabla \times (P + F)) \rho$$

(50)

where $\omega_a = \omega_a / \rho$, e.g., see Eq. (4.95) of Vallis (2007). To proceed, let us assume that the undetermined part $\lambda \omega_a$ is not needed, and see whether this leads to a contradiction. If that were possible, one could write $v_{ia} \cdot \nabla Q = DQ / Dt$, in which case (50) would imply

$$\nabla \cdot (\rho v_{ia}) = -\frac{\omega_a \cdot \nabla \hat{B}}{\omega_a \cdot \nabla B}.$$  

(51)

Eq. (51) is a key result of this paper, because it shows that in the absence of friction and diabatic sources/sinks of entropy and salt, $v_{ia}$ naturally satisfies the continuity equation regardless of how the $P$ vectors and Bernoulli functions $B$ are defined. This is important, because it suggests that the EGTBF form of momentum balance represents a more useful re-arrangement of the forces acting on fluid parcels than the PGVBF form of momentum balance, at least as far as achieving theoretical insights on oceanic motions is concerned. If diabatic effects are retained, $v_{ia}$ no longer satisfies the continuity equation exactly, but the degree to which it does is controlled by the ratio $\omega_a \cdot \nabla \hat{B} / (\omega_a \cdot \nabla B)$, which scaling arguments indicate is smaller by the factor $(T - T_r) / T$ when based in terms of $P_a$ instead of $P_b$. This suggests that in the presence of small-scale mixing, $v_{ia}$ is therefore a better approximation to the full 3D velocity field $v$ than $v_{ia}$ and hence that $P_a$ is a better indicator of the preferential directions of lateral stirring than any other form of $P$ vector. To be fully rigorous, however, it would be better to establish the results for the time-averaged version of the above equations, i.e., for

$$\bar{\omega}_a \times \nabla = \bar{P}_a - \omega_a \times \nabla + \nabla B_\perp + \bar{F},$$

(52)

but this is significantly more complicated and therefore left as an open question for future investigations.

3. Alternative expressions for $P_a$

a. $P_a$ for different temperature variables

Because specific entropy is rarely used in practice, the following provides alternative expressions for $P_a$ in terms of potential temperature $\theta$, Conservative Temperature $\Theta$, and in-situ temperature $T$, which should prove more useful in practice. To obtain these, we make use of the following passage relationships

$$Td\eta + \mu dS = \frac{Te_{\rho \theta}}{\theta} d\theta + \left(\mu - T \frac{\partial \mu}{\partial \theta}\right) dS$$

$$= \frac{Te_{\rho \theta}}{\theta} d\Theta + \left(\mu - T \frac{\partial \mu}{\partial \Theta}\right) dS,$$

(53)

which have been discussed previously in Tailleux (2010, 2015a) for instance, where $\Gamma = aT / (\rho c_p \theta)$ is the adiabatic lapse rate. As a result, one may write

$$P_a = (T - T_r) \nabla \eta + (\mu - \mu_r) \nabla S$$

$$= \left(T - T_r\right) \frac{Te_{\rho \theta}}{\theta} \nabla \theta + \left(\mu - \mu_r - (T - T_r) \frac{\partial \mu}{\partial \theta}\right) \nabla S$$

$$= \left(T - T_r\right) \frac{Te_{\rho \theta}}{\theta} \nabla \Theta + \left(\mu - \mu_r - (T - T_r) \frac{\partial \mu}{\partial \Theta}\right) \nabla S$$

(54)

where $c_{\rho \theta} = c_p (S, \theta, 0)$ is the constant reference specific heat capacity underlying TEOS-10. It may be worth mentioning that although all expressions (54) are independent of the 4 arbitrary constants entering the Gibbs function for seawater (see IOC (2010) for details), this is the case of the individual components of $P_a$ only for the $(S, \theta)$ and $(S, T)$ representations.

b. Comparison with $N$

To understand how the directions defined by $P_a$ differ from those defined by the standard neutral vector $N = g(\alpha \nabla \beta - \nabla S)$, the simplest is to regard specific enthalpy $h = h(S, \theta, p)$ as a function of salinity, potential temperature and pressure and to write its total differential in the form

$$dh = \frac{\partial h}{\partial \theta} d\theta + \frac{\partial h}{\partial S} dS + \frac{\partial h}{\partial p} dp.$$  

(55)
From the associated Maxwell relationships (i.e., the equality of the cross-derivatives), namely

\[
\frac{\partial^2 h}{\partial \theta \partial p} = \frac{\partial v}{\partial \theta} = \frac{\alpha}{\rho}, \quad \frac{\partial^2 h}{\partial S \partial p} = \frac{\partial \alpha}{\partial S} = \frac{\beta}{\rho},
\]

\(P_a\) may be rewritten as follows

\[
P_a = \int_{p_r}^{p} \frac{\hat{\alpha}}{\rho} (S, \theta, p') \, dp' \, \nabla \theta - \int_{p_r}^{p} \frac{\hat{\beta}}{\rho} (S, \theta, p') \, dp' \, \nabla S
\]

\[
= \frac{(p - p_r)}{p} \left( \overline{\alpha} \nabla \theta - \overline{\beta} \nabla S \right) = \frac{p - p_r}{\rho} \overline{N},
\]

where \(\overline{\alpha}\) and \(\overline{\beta}\) are the following vertically-averaged thermal expansion and haline contraction coefficients:

\[
\overline{\alpha} = \frac{\overline{\beta}}{p - p_0(z_r)} \int_{p_0}^{p} \frac{\hat{\alpha}}{\rho} (S, \theta, p') \, dp',
\]

\[
\overline{\beta} = \frac{\overline{\beta}}{p - p_0(z_r)} \int_{p_0(z_r)}^{p} \frac{\hat{\beta}}{\rho} (S, \theta, p') \, dp'.
\]

As is well known, the thermobaric nonlinearity of the equation of state for seawater causes \(\alpha\) to depend more strongly on pressure than \(\beta\). As a result, the differences between P-neutral and N-neutral directions should increase with \(|p - p_r|\) and therefore be the largest in the Southern Ocean and other regions with strong horizontal density gradients, as quantified in more details in next section.

### 4. P-neutrality versus N-neutrality

As established above, the magnitudes of \(|\mathbf{v} \cdot \nabla B_\Pi|\) and \(|\mathbf{v} \cdot \mathbf{P}_a|\) are both energetically constrained to be small in the absence of significant sources of mechanical energy sources of stirring, which is generally assumed to be the case in the ocean interior away from direct atmospheric influences. As a result, \(\nabla B_\Pi\) and \(\mathbf{P}_a\) both define previously undocumented neutral directions that a priori differ from the standard N-neutral directions considered by oceanographers so far. As a result, both are expected to constrain the preferential directions of lateral stirring in the ocean, but at this stage, which one should be regarded as imposing the most stringent constraints is difficult to ascertain rigorously without a more detailed analysis of local energetics and of the processes determining local temporal variations in \(B_\Pi\) and \(\Pi\), left for a subsequent study.

Because the preferential directions of lateral stirring have been so far primarily linked to the directions minimising the work against buoyancy forces, we limit ourselves in this paper to assessing the differences between the newly introduced P-neutrality and N-neutrality. Nonetheless, the fact that \(\nabla B_\Pi\) also define neutral directions cannot be ignored and warrants further investigation. If the neutral directions associated with \(\nabla B_\Pi\) could be established as more relevant than \(\mathbf{P}_a\) or \(\mathbf{N}\) for defining the preferential directions of lateral stirring, it would have the immense advantage of also making it possible to regard the Bernoulli iso-surfaces \(B_\Pi = \text{constant}\) as the natural definition of lateral stirring surfaces, thus eliminating the need for constructing density-like surfaces maximising either P-neutrality or N-neutrality. Being able to confirm the merits of such a possibility would require a paradigm shift in the study of lateral stirring, however. Importantly, it would provide a theoretical justification for the construction of lateral stirring surfaces that are not necessarily purely material functions of \(S\) and \(\theta\), which has been a source of controversy and confusion in the field.

As shown in the previous section, the differences between the directions defined by \(\mathbf{N}\) and \(\mathbf{N}\) are controlled by the magnitude of the difference \(|p - p_r|\), which previous estimates of Lorenz reference state have showed to be small over a large fraction of the ocean, e.g., Tailleux (2013); Saenz et al. (2013). Because in those regions, P-neutrality is expected to be indistinguishable from N-neutrality, what matters to challenge the theory of N-neutrality are the regions where the two forms of neutrality remain irreconcilable. Physically, this is important to establish: 1) where the use of neutral rotated diffusion might be significantly in error and therefore a previously unrecognised cause of spurious diapycnal mixing; 2) whether the superior neutrality properties of \(\gamma^n\) over \(\gamma^p\) in terms of N-neutrality still holds according to P-neutrality, in order to determine which density variable is potentially the better one for the study of ocean circulation studies.

To proceed, we need a way to evaluate P-neutrality. In the following, this is done by estimating \(\mathbf{N}\) in terms of the following formula:

\[
\overline{\mathbf{N}} \approx \frac{1}{z - z_r} \left[ \nabla_{lr} h(\Theta, S_R, p_0(z)) - \nabla_{lr} h(\Theta, S_R, p_0(z_r)) \right],
\]

where \(\nabla_{lr}\) is the ‘locally-referenced’ gradient, that is, the gradient calculated by ignoring the pressure dependence. The enthalpy and reference composition salinity were estimated using the TEOS-10 Matlab routines gsw_enthalpy_CT_exact and gsw_SR_from_SP, with the Conservative Temperature \(\Theta\) evaluated from temperature and reference composition salinity using gsw_CT_from_t.

Given two density variables with diapycnal vectors \(\mathbf{A}\) and \(\mathbf{B}\), we define a notional effective diffusivity

\[
K_f(\mathbf{A}, \mathbf{B}) = K_f \sin^2(\mathbf{A}, \mathbf{B}),
\]

e.g., Hochet et al. (2019), with \(K_f = 1000 \text{m}^2 \cdot \text{s}^{-1}\), where \(<\mathbf{A}, \mathbf{B}\>\) is the angle between the two vectors. The following calculations are based on climatology for temperature and salinity of Gouretski and Koltermann (2004), using a similar methodology as in Tailleux (2021).
First, Fig. 1 shows the statistics of $K_f$ ($\mathbf{N}, \mathbf{N}$) for the whole ocean, which are further contrasted for the main oceanic basins (left panel) versus for the polar oceans (right panel). As expected, the latter are where the main differences between P-neutrality and N-neutrality are found, consistent with the result that this is also where the actual state of the ocean differs most from Lorenz reference state and hence where $|p - p_r|$ values are the largest.

Next, Fig. 2 compares the neutrality of $\gamma^N$ (top panels) and of $\gamma^T_{\text{analytic}}$ (bottom panels) when evaluated in terms of N-neutrality (Left panels) versus P-neutrality (right panels). This shows that while the neutrality of $\gamma^N$ appears to outperform that of $\gamma^T_{\text{analytic}}$ when assessed in terms of N-neutrality, the opposite holds when the assessment is done in terms of P-neutrality. A detailed understanding of the relative merits of P-neutrality versus that of N-neutrality is therefore essential to decide on the best choice of density variable for the tracing of water masses for instance.

Finally, Fig. 3 provides an illustration of the results depicted in Fig. 1 by confirming that the polar regions (as well as the Gulf Stream region to some extent) are those where P-neutrality differs most from N-neutrality.

5. Summary and conclusions

In this paper, we have developed a new theoretical framework that suggests that to understand the nature of the preferential directions of lateral stirring in the ocean, it is necessary to rewrite the specific momentum balance of the full Navier-Stokes equations in their EGTBF form, viz.,

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{\omega}_a \times \mathbf{v} - \nabla \mathbf{B} + \mathbf{P}_a + \mathbf{F},$$

(62)

where $\mathbf{B}$ is a new form of Bernoulli function (denoted $B_{\Pi}$ in the core of the paper, and $\mathbf{P}_a$ a new APE-based thermo-dynamic buoyancy force similar in nature to the dynamic enthalpy-based $\mathbf{P}$ vector of Nycander (2011). The main key result here is that $\mathbf{v}$ is energetically constrained to remain close to being orthogonal to both $\nabla \mathbf{B}$ and $\mathbf{P}_a$, as the result of Bernoulli’s theorem as well as to avoid the development of restoring buoyancy forces that would locally increase the available potential energy of the fluid. In this regard, the argument is reminiscent to that originally proposed by McDougall (1987) in support of N-neutrality, namely that lateral stirring along directions other than those perpendicular to the standard $\mathbf{N}$ vector would necessarily give rise to restoring buoyancy forces. The main difference is that whereas McDougall (1987) derived N-neutrality from heuristic parcel-based arguments (which themselves can be traced back to those originally put forward by Sverdrup et al. (1942)), we obtained $\nabla \mathbf{B}$ and $\mathbf{P}_a$ by means of a fully rigorous and deductive analysis of the full Navier-Stokes equations. The result that $\mathbf{v}$ should always remain close to being perpendicular to both $\nabla \mathbf{B}$ and $\mathbf{P}_a$ is further vindicated by the fact that the following solution of the nonlinear balance equation (obtained by discarding $\partial_t \mathbf{v}$ and $\mathbf{F}$ in (62))

$$\mathbf{v}_{ia} = \frac{\mathbf{P}_a \times \nabla \mathbf{B}}{\omega_{ia} \cdot \nabla \mathbf{B}}.$$

(63)

which is exactly perpendicular to both $\nabla \mathbf{B}$ and $\mathbf{P}_a$, satisﬁes the continuity equation $\nabla \cdot (\rho \mathbf{v}_{ia}) = 0$ exactly in the absence of diabatic effects, and approximately so otherwise. In other words, it is an admissible explicit solution of the full Navier-Stokes equations in the absence of diabatic effects. Interestingly, the degree to which $\mathbf{v}_{ia}$ satisfies the continuity equation is related to the conservation properties of the Bernoulli-based PV $Q_{\Pi} = \omega_{ia} \cdot \nabla \mathbf{B} / \rho$, which does not appear to have been considered as a possible way to construct potential vorticity before.

By construction, the neutral directions defined by $\mathbf{P}_a$ approximately coincide with the standard N-neutral directions everywhere where the actual pressure of a fluid parcel is close to its Lorenz reference pressure $p_r$. As it happens, this is the case in most of the ocean except in the polar and Gulf-Stream regions. This is a remarkable result, because the present framework is the first one in over 80 years to succeed in vindicating the use of the N-neutral directions from first principles over most of the ocean. However, the fact that regions exist where P-neutrality remains irreconcilable with N-neutrality clearly establishes that N-neutrality is not the correct principle for defining the preferential directions of lateral stirring, and hence that the use of N-neutral rotated diffusion tensors in numerical ocean models is likely to be in error in regions such as the Southern Ocean, and potentially a cause of spurious diapycnal mixing there. Note that contrary to what is generally assumed, (62) allows one to define at least two set of neutral directions rather than just one. Even though we limited ourselves in this paper to the discussion of $\mathbf{P}_a$, owing to it being the quantity the closest to $\mathbf{N}$, one should not exclude the possibility at this stage that $\nabla \mathbf{B}$ might actually be a better candidate for defining the preferential directions of lateral stirring. This would have the advantage of naturally defining the lateral stirring surfaces as the iso-surfaces of the Bernoulli function $\mathbf{B}$, thus providing a natural resolution to a longstanding issue in the field. We plan on examining this possibility in details in a subsequent study.

It is important to note that the validity and usefulness of our theoretical framework and conclusions is conditional to that of APE theory, since Tailleux (2018) potential energy density $\Pi$ is central to the construction of both $\nabla \mathbf{B}$ and $\mathbf{P}_a$. In this paper, this is taken for granted, as our work and that of others over the past 15 years or so, e.g., Tailleux (2009, 2010); Tailleux and Roulet (2010); Hughes et al. (2009), seem to leave little doubt that the concept of APE is vital for correctly understanding ocean energetics and how surfaces buoyancy fluxes power the ocean circulation. The present results, together with the apparent connection between Jackett and McDougall (1997) neutral density $\gamma^N$ and
Lorenz reference density established by Tailleux (2016a), or Tailleux (2021)'s successful use of APE theory for constructing an improved theory of spiciness, all suggest that the usefulness of APE goes beyond simply clarifying ocean energetics, but that it is also fundamental for elucidating the nature of the forces driving oceanic motions, for constructing explicit approximations of the full 3D velocity field, and more generally for clarifying many aspects of the ocean circulation that have remained unclear so far, as we hope to further demonstrate in the future.

As a final note, let us conclude by pointing out that on a conceptual level, the present theory can be regarded as unifying the previous frameworks of de Szoeke (2000) and Nycander (2011) (see Appendix A also) whose connection had remained hitherto unnoticed, while also extending and generalising them in important new ways. These two studies pioneered the development of P-neutrality in important and complementary ways, and were both determinant in the development and formulation of the present ideas.
Fig. 1. p.d.f. of the effective diapycnal diffusivity-like metric measuring the angle between $P_{\Delta z}$ and $N$ for the Pacific, Atlantic, and Indian oceans (left panel) versus for the polar oceans (right panel), the statistics for the whole ocean being also shown in the background in both panels.

Fig. 2. Effective diapycnal mixing like metric quantifying (a) the N-neutrality of $\gamma^N$; (b) the P-neutrality of $\gamma^N$; (c) the N-neutrality of $\gamma^T_{analytic}$; d) the P-neutrality of $\gamma^T_{analytic}$. 
Fig. 3. Vertically-averaged value of the effective diapycnal diffusivity like metric illustrating the geographical distribution of the differences between P-neutrality and P-neutrality.
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Data availability statement. The hydrographic data used in this study are publicly available. Software used to produce the figures are available upon request to the authors.

APPENDIX A

Links with Nycander (2011) and de Szoeke (2000)

The definition of Montgomery potential \( \Phi \) discussed in this paper, if used in conjunction with a fixed reference depth \( z_r \), instead of Lorenz reference depth, is formally equivalent to that previously used by de Szoeke (2000), in which case the \( \mathbf{P} \) vector that it leads to can be shown to coincide with Nycander (2011)'s definition. To clarify these connections, let us first rewrite (18) in the form

\[
M = h(\eta, S, p) - h(\eta, S, P_0(z)) + h_r^+(\eta, S, z),
\]

where

\[
h_r^+(\eta, S, z) = -\int_{z_r}^z b(\eta, S, P_0(z')) dz'.
\]

can be identified as a compressible equivalent of the dynamic enthalpy first introduced by Young (2010) and Nycander (2010) in the particular case \( z_r = 0 \) (with Nycander calling the variable ‘effective potential energy’), with \( b \) the buoyancy defined by

\[
b = b(\eta, S, z) = -g\left(1 - \frac{\rho_0(z)}{\rho(\eta, S, P_0(z))}\right).
\]

The use of a fixed \( z_r \) does not formally alter the forms of the specific momentum balance (16) and \( \mathbf{P} \) vector (26), which still may be written as

\[
\frac{Du}{Dt} + f k \times u + \nabla M = \mathbf{P}^\gamma + \mathbf{F}_h
\]

where \( \mathbf{P} \) represents the isopycnal projection of \( \mathbf{P} \). In other words, the isopycnal projection of Nycander’s \( \mathbf{P} \) vector can be identified with the isopycnal buoyancy force discussed by de Szoeke (2000), although this connection was overlooked by Nycander (2011).

APPENDIX B

Baroclinic production term

In this paper, we established the following identities:

\[
\frac{1}{\rho} \nabla \rho + \nabla \Phi = \nabla (h + \phi) - \mathbf{P}_h = \nabla (h + \Phi - \mathbf{B}_r) - \mathbf{P}_a.
\]

Taking the curl yields the following equivalent expressions for the baroclinic production term

\[
\frac{1}{\rho^2} \nabla \rho \times \nabla p = \frac{1}{\rho^2} \mathbf{N} \times \nabla p = \nabla \times \mathbf{P}_h = \nabla \times \mathbf{P}_a,
\]

where we defined \( \mathbf{N} = \rho \nabla S + \rho \phi \nabla \theta \).
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