Arc-Search Infeasible Interior-Point Algorithm for Linear Programming

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Abstract

Mehrotra’s algorithm has been the most successful infeasible interior-point algorithm for linear programming since 1990. Most popular interior-point software packages for linear programming are based on Mehrotra’s algorithm. This paper proposes an alternative algorithm, arc-search infeasible interior-point algorithm. We will demonstrate, by testing Netlib problems and comparing the test results obtained by arc-search infeasible interior-point algorithm and Mehrotra’s algorithm, that the proposed arc-search infeasible interior-point algorithm is a more efficient algorithm than Mehrotra’s algorithm.

Keywords: Arc-search, infeasible interior-point algorithm, linear programming.

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1 Introduction

Interior-point method is now regarded as a mature technique of linear programming [1], following many important developments in 1980-1990, such as, a proposal of path-following method [2], the establishment of polynomial bounds for path-following algorithms [3] [4], the development of Mehrotra’s predictor-corrector (MPC) algorithm [5] and independent implementation and verification [6] [7], and the proof of the polynomiality of infeasible interior-point algorithm [8] [9]. Although many more algorithms have been proposed since then (see, for example, [10] [11] [12] [13] [14]), there is no significant improvement in the best polynomial bound for interior-point algorithms, and there is no report of a better algorithm than MPC for general linear programming problems.

In fact, the most popular interior-point method software packages implemented MPC, for example, PCx [16] and LIPSOL [17]. However, there were some interesting results obtained in recent years. For example, higher-order algorithms that used second or higher-order derivatives were demonstrated to improve the computational efficiency [5] [7]. Higher-order algorithms, however, had either a poorer polynomial bound than first-order algorithms [18] or did not even have a polynomial bound [5] [19]. This dilemma was partially solved in [20] which proved that higher-order algorithms can achieve the best polynomial bound. An arc-search interior-point algorithm for linear programming was devised in [20]. The algorithm utilized the first and second-order derivatives to construct an ellipse to approximate the central path. Intuitively, searching along this ellipse should generate a larger step size than searching along any straight line. Indeed, it was shown in [20] that the arc-search algorithm has the best polynomial bound and it may be very efficient in practical computation. This result was extended to prove a similar result for convex quadratic programming and the numerical test result was very promising [21].

The algorithms proposed in [20] [21] assume that the starting point is feasible and the central path does exist. Available Netlib test problems are limited because most Netlib problems may not even have an interior-point as noted in [22]. To better demonstrate the claims in the previous papers, we propose an infeasible arc-search interior-point algorithm in this paper, which allows us to test a lot more Netlib problems. The proposed algorithm keeps a nice feature developed in [20] [21], i.e., it searches optimizer along an arc (part of an ellipse). It also adopts some strategies used in MPC, such as using different step sizes for the vector of primal variables and the vector of slack variables. We will show that the proposed arc-search infeasible interior-point algorithm is very competitive in computation by testing all Netlib problems in standard form and comparing the results to those obtained by MPC. To make a fair comparison, both algorithms are implemented in MATLAB; for all test problems, the two Matlab codes use the same pre-processor, start from the same initial point, use the same parameters, and terminate with the same stopping criterion. Since the main cost in computation for both algorithms is to solve linear systems of equations which are exactly the same for both algorithms, and the arc-search infeasible interior-point algorithm uses less iterations in most tested problems than MPC, we believe that the proposed algorithm

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1 There are some noticeable progress focused on problems with special structures, for example [15].
is more attractive than the MPC algorithm.

The remaining of the paper is organized as follows. Section 2 briefly describes the problem. Section 3 presents the proposed algorithm and some simple but important properties. Section 4 discusses implementation details for both algorithms. Section 5 provides numerical results and compares the results obtained by both arc-search method and Mehrotra’s method. Conclusions are summarized in Section 6.

2 Problem Descriptions

Consider the Linear Programming in the standard form:

$$\min \quad c^T x, \quad \text{subject to } \quad Ax = b, \quad x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m, \ c \in \mathbb{R}^n$ are given, and $x \in \mathbb{R}^n$ is the vector to be optimized. Associated with the linear programming is the dual programming that is also presented in the standard form:

$$\max \quad b^T \lambda, \quad \text{subject to } \quad A^T \lambda + s = c, \ s \geq 0,$$

where dual variable vector $\lambda \in \mathbb{R}^m$, and dual slack vector $s \in \mathbb{R}^n$.

Throughout the paper, we will denote the residuals of the equality constraints (the deviation from the feasibility) by

$$r_b = Ax - b, \quad r_c = A^T \lambda + s - c,$$

the duality measure by

$$\mu = \frac{x^T s}{n},$$

the $i$th component of $x$ by $x_i$, the Euclidean norm of $x$ by $\|x\|$, the identity matrix of any dimension by $I$, the vector of all ones with appropriate dimension by $e$, the Hadamard (element-wise) product of two vectors $x$ and $s$ by $x \odot s$. To make the notation simple for block column vectors, we will denote, for example, a point in the primal-dual problem $[x^T, \lambda^T, s^T]^T$ by $(x, \lambda, s)$. We will denote the initial point of any algorithm by $(x^0, \lambda^0, s^0)$, the corresponding duality measure by $\mu_0$, the point after the $k$th iteration by $(x^k, \lambda^k, s^k)$, the corresponding duality measure by $\mu_k$, the optimizer by $(x^*, \lambda^*, s^*)$, the corresponding duality measure by $\mu_*$. For $x \in \mathbb{R}^n$, we will denote a related diagonal matrix by $X \in \mathbb{R}^{n \times n}$ whose diagonal elements are components of the vector $x$.

The central path $\mathcal{C}(t)$ of the primal-dual linear programming problem is parameterized by a scalar $t \geq 0$ as follows. For each interior point $(x, \lambda, s) \in \mathcal{C}(t)$ on the central path, there is a $t \geq 0$ such that

$$Ax = b$$

$$A^T \lambda + s = c$$

$$(x, s) \geq 0$$

$$x_i s_i = t, \quad i = 1, \ldots, n.$$
Therefore, the central path is an arc in $\mathbb{R}^{2n+m}$ parameterized as a function of $t$ and is denoted as
\[
\mathcal{C}(t) = \{(x(t), \lambda(t), s(t)) : t \geq 0\}.
\]
As $t \to 0$, the central path $(x(t), \lambda(t), s(t))$ represented by (6) approaches to a solution of LP represented by (1) because (6) reduces to the KKT condition as $t \to 0$.

Because of high cost of finding an initial feasible point and the central path described in (5), we consider a modified problem which allows infeasible initial point.

\[
\begin{align*}
Ax & - b = r_b & (7a) \\
A^T \lambda + s - c & = r_c & (7b) \\
(x, s) & \geq 0 & (7c) \\
x_i s_i & = t, \quad i = 1, \ldots, n. & (7d)
\end{align*}
\]

We search the optimizer along an infeasible central path neighborhood. The infeasible central path neighborhood $\mathcal{F}(\gamma)$ considered in this paper is defined as a collection of points that satisfy the following conditions,
\[
\mathcal{F}(\gamma(t)) = \{(x, \lambda, s) : \|(r_b(t), r_c(t))\| \leq \gamma(t)\|(r_b^0, r_c^0)\|, (x, s) > 0\},
\]
where $r_b(1) = r_b^0$, $r_c(1) = r_c^0$, $\gamma(t) \in [0, 1]$ is a monotonic function of $t$ such that $\gamma(1) = 1$ and $\gamma(t) \to 0$ as $t \to 0$. It is worthwhile to note that this central path neighborhood is the widest in any neighborhood considered in existing literatures.

### 3 Arc-Search Algorithm for Linear Programming

Starting from any point $(x^0, \lambda^0, s^0)$ in a central path neighborhood that satisfies $(x^0, s^0) > 0$, for $k \geq 0$, we consider a special arc parameterized by $t$ and defined by the current iterate as follows:

\[
\begin{align*}
Ax(t) - b & = tr_b^k & (9a) \\
A^T \lambda(t) + s(t) - c & = tr_c^k & (9b) \\
(x(t), s(t)) & > 0 & (9c) \\
x(t) \circ s(t) & = tx^k \circ s^k. & (9d)
\end{align*}
\]

Clearly, each iteration starts at $t = 1$; and $(x(1), \lambda(1), s(1)) = (x^k, \lambda^k, s^k)$. We want the iterate stays inside $\mathcal{F}(\gamma)$ as $t$ decreases. We denote the infeasible central path defined by (9) as
\[
\mathcal{H}(t) = \{(x(t), \lambda(t), s(t)) : t \geq \tau \geq 0\}.
\]
If this arc is inside $\mathcal{F}(\gamma)$ for $\tau = 0$, then as $t \to 0$, $(r_b(t), r_c(t)) := t(r_b^k, r_c^k) \to 0$; and equation (9d) implies that $\mu(t) \to 0$; hence, the arc will approach to an optimal solution of (1) because (9) reduces to KKT condition as $t \to 0$. To avoid computing the entire infeasible central path $\mathcal{H}(t)$, we will search along an approximation of $\mathcal{H}(t)$ and keep the
iterate stay in $F(\gamma)$. Therefore, we will use an ellipse $E(\alpha)$ in $2n + m$ dimensional space to approximate the infeasible central path $H(t)$, where $E(\alpha)$ is given by

$$E(\alpha) = \{(x(\alpha), \lambda(\alpha), s(\alpha)) : (x(\alpha), \lambda(\alpha), s(\alpha)) = \vec{a}\cos(\alpha) + \vec{b}\sin(\alpha) + \vec{c}\},$$

where the duality measure $\mu$ is evaluated at $(x(\alpha), \lambda(\alpha), s(\alpha))$.

Therefore, by taking the first derivative for (9) at $(x, \lambda, s)$, we will determine $\vec{a}, \vec{b}, \vec{c}$ and $\alpha_0$ such that the first and second derivatives of $E(\alpha)$ at $(x, \lambda, s)$ are the same as those of $H(t)$ at $(x(\alpha_0), \lambda(\alpha_0), s(\alpha_0))$. Therefore, by taking the first derivative for (9) at $(x(\alpha_0), \lambda(\alpha_0), s(\alpha_0)) = (x^k, \lambda^k, s^k) \in E$, we have

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} r_b^k \\ r_c^k \\ x^{k \circ s^k} \end{bmatrix},$$

(12)

These linear systems of equations are very similar to those used in [20] except that equality constraints in [5] are not assumed to be satisfied. By taking the second derivative, we have

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\lambda} \\ \ddot{s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2\dot{x} \circ \dot{s} \end{bmatrix},$$

(13)

Similar to [5], we modify (13) slightly to make sure that a substantial segment of the ellipse stays in $F(t)$, thereby making sure that the step size along the ellipse is significantly greater than zero,

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \end{bmatrix} \begin{bmatrix} \ddot{x}(\sigma_k) \\ \ddot{\lambda}(\sigma_k) \\ \ddot{s}(\sigma_k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma_k \mu_k e - 2\dot{x} \circ \dot{s} \end{bmatrix},$$

(14)

where the duality measure $\mu_k$ is evaluated at $(x^k, \lambda^k, s^k)$, and we set the centering parameter $\sigma_k$ satisfying $0 < \sigma_k < \sigma_{\text{max}} \leq 0.5$. We emphasize that the second derivatives are functions of $\sigma_k$ which is selected by using a heuristic of [5] to speed up the convergence of the proposed algorithm. Several relations follow immediately from (12) and (14).

**Lemma 3.1** Let $(\dot{x}, \dot{\lambda}, \dot{s})$ and $(\ddot{x}, \ddot{\lambda}, \ddot{s})$ be defined in (12) and (14). Then, the following relations hold.

$$s^T \dot{x} + x^T \dot{s} = n\mu, \quad s^T \ddot{x} + x^T \ddot{s} = \sigma \mu n - 2\dot{x} \circ \dot{s}, \quad \ddot{x}^T \ddot{s} = 0.$$

(15)

Equations (12) and (14) can be solved in either unreduced form, or augmented system form, or normal equation form as suggested in [1]. We solve the normal equations
for \((\dot{x}, \dot{\lambda}, \dot{s})\) and \((\ddot{x}, \ddot{\lambda}, \ddot{s})\) as follows:

\[
(AXS^{-1}A^T)\dot{\lambda} = AXS^{-1}r_c - b, \tag{16a}
\]
\[
\dot{s} = r_c - A^T\dot{\lambda}, \tag{16b}
\]
\[
\dot{x} = x - XS^{-1}\dot{s}, \tag{16c}
\]

and

\[
(AXS^{-1}A^T)\ddot{\lambda} = -AS^{-1}(\sigma\mu n - 2\dot{x} \circ \dot{s}), \tag{17a}
\]
\[
\ddot{s} = -A^T\ddot{\lambda}, \tag{17b}
\]
\[
\ddot{x} = S^{-1}(\sigma\mu e - X\ddot{s} - 2\dot{x} \circ \dot{s}). \tag{17c}
\]

Given the first and second derivatives defined by (12) and (14), an analytic expression of the ellipse that is used to approximate the infeasible central path is derived in [20].

**Theorem 3.1** Let \( (x(\alpha), \lambda(\alpha), s(\alpha)) \) be an arc defined by (11) passing through a point \((x, \lambda, s) \in E\), and its first and second derivatives at \((x, \lambda, s)\) be \((\dot{x}, \dot{\lambda}, \dot{s})\) and \((\ddot{x}, \ddot{\lambda}, \ddot{s})\) which are defined by (12) and (14). Then the ellipse approximation of \(H(t)\) is given by

\[
x(\alpha, \sigma) = x - \dot{x}\sin(\alpha) + \ddot{x}(\sigma)(1 - \cos(\alpha)). \tag{18}
\]
\[
\lambda(\alpha, \sigma) = \lambda - \dot{\lambda}\sin(\alpha) + \ddot{\lambda}(\sigma)(1 - \cos(\alpha)). \tag{19}
\]
\[
s(\alpha, \sigma) = s - \dot{s}\sin(\alpha) + \ddot{s}(\sigma)(1 - \cos(\alpha)). \tag{20}
\]

In the algorithm proposed below, we suggest taking step size \(\alpha^s_k = \alpha^\lambda_k\) which may not be equal to the step size of \(\alpha^x_k\).

**Algorithm 3.1**

*Data:* \(A, b, c,\) and step scaling factor \(\beta \in (0, 1)\).

*Initial point:* \(\lambda^0 = 0, x^0 > 0, s^0 > 0,\) and \(\mu_0 = \frac{x^0s^0}{n}\).

*for* iteration \(k = 0, 1, 2, \ldots\)

1. **Step 1:** Calculate \((\dot{x}, \dot{\lambda}, \dot{s})\) using (16) and set
   \[
   \alpha^a_x := \arg \max\{\alpha \in [0, 1] | x - \alpha \dot{x} \geq 0\},
   \]
   \[
   \alpha^a_s := \arg \max\{\alpha \in [0, 1] | s - \alpha \dot{s} \geq 0\}. \tag{21a}
   \]

2. **Step 2:** Calculate \(\mu^a = \frac{(x + \alpha^a_x)^T(s + \alpha^a_s)}{n}\) and compute the centering parameter
   \[
   \sigma = \left(\frac{\mu^a}{\mu}\right)^3. \tag{22}
   \]

3. **Step 3:** Compute \((\ddot{x}, \ddot{\lambda}, \ddot{s})\) using (17).
Step 4: Set
\[
\alpha^x = \arg\max\{\alpha \in [0, \frac{\pi}{2}] | x^k - \dot{x} \sin(\alpha) + \ddot{x}(1 - \cos(\alpha)) \geq 0\}, \quad (23a)
\]
\[
\alpha^s = \arg\max\{\alpha \in [0, \frac{\pi}{2}] | s^k - \dot{s} \sin(\alpha) + \ddot{s}(1 - \cos(\alpha)) \geq 0\}. \quad (23b)
\]

Step 5: Scale the step size by \(
\alpha^x_k = \beta \alpha^x_k \) and \(
\alpha^s_k = \beta \alpha^s_k \) such that the update
\[
x^{k+1} = x^k - \alpha^x_k (\dot{x} - \ddot{x}) > 0, \quad (24a)
\]
\[
\lambda^{k+1} = \lambda^k - \dot{\lambda} \sin(\alpha^s_k) + \ddot{\lambda}(1 - \cos(\alpha^s_k)), \quad (24b)
\]
\[
s^{k+1} = s^k - \alpha^s_k (\dot{s} - \ddot{s}) > 0. \quad (24c)
\]

Step 6: Set \(k + 1 \rightarrow k\). Go back to Step 1.

end (for)

Remark 3.1 The main difference between the proposed algorithm and Mehrotra’s algorithm is in Steps 4 and 5 where the iterate moves along the ellipse instead of a straight line. More specifically, instead of using (23) and (24), Mehrotra’s method uses
\[
\alpha^x = \arg\max\{\alpha \in [0, 1] | x^k - \alpha (\dot{x} - \ddot{x}) \geq 0\}, \quad (25a)
\]
\[
\alpha^s = \arg\max\{\alpha \in [0, 1] | s^k - \alpha (\dot{s} - \ddot{s}) \geq 0\}. \quad (25b)
\]
and
\[
x^{k+1} = x^k - \alpha^x_k (\dot{x} - \ddot{x}) > 0, \quad (26a)
\]
\[
\lambda^{k+1} = \lambda^k - \alpha^s_k (\dot{\lambda} - \ddot{\lambda}), \quad (26b)
\]
\[
s^{k+1} = s^k - \alpha^s_k (\dot{s} - \ddot{s}) > 0. \quad (26c)
\]

Note that the end points of arc-search algorithm \((\alpha^x_k, \alpha^s_k) = (0, 0)\) or \((\alpha^x_k, \alpha^s_k) = (\frac{\pi}{2}, \frac{\pi}{2})\) in (23) and (24) are equal to the end points of Mehrotra’s formulae in (25) and (26); for any \((\alpha^x_k, \alpha^s_k)\) between \((0, \frac{\pi}{2})\), the ellipse is a better approximation of the infeasible central path. Therefore, the proposed algorithm should have a larger step size than Mehrotra’s method and be more efficient. This intuitive has been verified in our numerical test.

The following lemma shows that searching along the ellipse in iterations will reduce the residuals of the equality constraints to zero as \(k \rightarrow \infty\) provided that \(\alpha^x_k\) and \(\alpha^s_k\) are bounded below from zero.
Lemma 3.2 Let \( r_b^k = A x^k - b, r_c^k = A^T \lambda^k + s^k - c, \) and \( \nu_k = \prod_{j=0}^{k-1} (1 - \sin(\alpha_j^s)) \). Then, the following relations hold.

\[
\begin{align*}
\nu_k &= \prod_{j=0}^{k-1} (1 - \sin(\alpha_j^s)), \\
\nu_k &= \prod_{j=0}^{k-1} (1 - \sin(\alpha_j^s)) = r^0_b g_k, \\
(27a) \quad &r_b^k = r_b^{k-1} (1 - \sin(\alpha_{k-1}^x)) = \cdots = r_b^0 \prod_{j=0}^{k-1} (1 - \sin(\alpha_j^x)) = r^0_b g_k, \\
(27b) \quad &r_c^k = r_c^{k-1} (1 - \sin(\alpha_{k-1}^s)) = \cdots = r_c^0 \prod_{j=0}^{k-1} (1 - \sin(\alpha_j^s)) = r^0_c \nu_k.
\end{align*}
\]

Proof: From Theorem 3.1, searching along ellipse generates iterate as follows.

\[
\begin{align*}
x_{k+1} - x_k &= -\dot{x} \sin(\alpha_k^x) + \ddot{x}(1 - \cos(\alpha_k^x)), \\
\lambda_{k+1} - \lambda_k &= -\dot{\lambda} \sin(\alpha_k^s) + \ddot{\lambda}(1 - \cos(\alpha_k^s)), \\
s_{k+1} - s_k &= -\dot{s} \sin(\alpha_k^s) + \ddot{s}(1 - \cos(\alpha_k^s)).
\end{align*}
\]

In view of (12) and (14), we have

\[
\begin{align*}
r_{b}^{k+1} - r_{b}^{k} &= A(x_{k+1} - x_k) = A(-\dot{x} \sin(\alpha_k^x) + \ddot{x}(1 - \cos(\alpha_k^x)) \\
&= -A \dot{x} \sin(\alpha_k^x) = -r_{b}^{k} \sin(\alpha_k^x),
\end{align*}
\]

therefore, \( r_{b}^{k+1} = r_{b}^{k} (1 - \sin(\alpha_k^x)) \); this proves (27a). Similarly,

\[
\begin{align*}
r_{c}^{k+1} - r_{c}^{k} &= A^T(\lambda_{k+1} - \lambda_k) + (s_{k+1} - s_k) \\
&= A^T(-\dot{\lambda} \sin(\alpha_k^s) + \ddot{\lambda}(1 - \cos(\alpha_k^s)) - \dot{s} \sin(\alpha_k^s) + \ddot{s}(1 - \cos(\alpha_k^s)) \\
&= -(A^T \dot{\lambda} + \dot{s}) \sin(\alpha_k^s) + (A^T \ddot{\lambda} + \ddot{s})(1 - \cos(\alpha_k^s)) \\
&= -r_{c}^{k} \sin(\alpha_k^s),
\end{align*}
\]

therefore, \( r_{c}^{k+1} = r_{c}^{k} (1 - \sin(\alpha_k^s)) \); this proves (27b). □

To show that the duality measure decreases with iterations, we present the following lemma.

Lemma 3.3 Let \( \alpha_x \) be the step length for \( x(\sigma, \alpha) \) and \( \alpha_s \) be the step length for \( s(\sigma, \alpha) \) and \( \lambda(\sigma, \alpha) \) defined in Theorem 3.1. Assume that \( \alpha_x = \alpha_s := \alpha \), then, the updated duality measure can be expressed as

\[
\mu(\alpha) = \mu[1 - \sin(\alpha) + \sigma(1 - \cos(\alpha))] + (\dot{x}^T r_c - \dot{\lambda}^T r_b) \sin(\alpha)(1 - \cos(\alpha)) \\
+ (x^T r_c - \ddot{\lambda}^T r_b)(1 - \cos(\alpha))^2.
\]
Proof: First, from (12) and (14), we have
\[ \dot{x}^T A^T \dot{\lambda} - \dot{x}^T \dot{s} = \dot{x}^T r_c, \]
this gives
\[ \dot{x}^T \dot{s} = \dot{\lambda}^T r_b - \dot{x}^T r_c. \]

Similarly,
\[ \ddot{x}^T \dot{s} = -\ddot{x}^T A^T \dot{\lambda} = -\ddot{\lambda}^T r_b, \]
\[ \ddot{x}^T \dot{s} = \ddot{x}^T \dot{s} + \ddot{x}^T A^T \dot{\lambda} = \ddot{x}^T r_c. \]

Using these relations with (11) and Lemmas 3.1, we have
\[ \mu(\alpha) = \frac{(x - \dot{x} \sin(\alpha_x) + \dot{\lambda}(1 - \cos(\alpha_x)))^T \left( (s - \dot{s} \sin(\alpha) + \dot{s}(1 - \cos(\alpha)) \right)}{n} \]
\[ \approx \frac{x^T \dot{s} \sin(\alpha_x) + s^T \dot{x} \sin(\alpha_x)}{n} + \frac{x^T \dot{s}(1 - \cos(\alpha_x)) + s^T \dot{x}(1 - \cos(\alpha_x))}{n} \]
\[ \approx \frac{x^T \dot{s} \sin(\alpha) + s^T \dot{x} \sin(\alpha)}{n} \]
\[ = \mu[1 - \sin(\alpha) + \sigma(1 - \cos(\alpha))] + \frac{x^T \dot{s} \sin^2(\alpha) - 2x^T \dot{s}(1 - \cos(\alpha))}{n} \]
\[ \approx \mu[1 - \sin(\alpha) + \sigma(1 - \cos(\alpha))] + \frac{(\dot{x}^T r_c - \dot{\lambda}^T r_b) \sin(\alpha)(1 - \cos(\alpha))}{n} \]
\[ + (\dot{x}^T r_c - \ddot{\lambda}^T r_b)(1 - \cos(\alpha))^2. \] (32)

This finishes the proof.

The following simple result clearly holds.

Lemma 3.4 For \( \alpha \in \left[0, \frac{\pi}{2}\right] \),
\[ \sin^2(\alpha) = 1 - \cos^2(\alpha) \geq 1 - \cos(\alpha) \geq \frac{1}{2} \sin^2(\alpha). \]

Remark 3.2 In view of Lemma 3.3, if \( \sin(\alpha) \) is bounded below from zeros, then \( r_b \rightarrow 0 \) and \( r_c \rightarrow 0 \) as \( k \rightarrow \infty \). Therefore, in view of Lemmas 3.3 and 3.4, we have, as \( k \rightarrow \infty \),
\[ \mu(\alpha) \approx \mu[1 - \sin(\alpha) + \sigma(1 - \cos(\alpha))] \leq \mu[1 - \sin(\alpha) + \sigma \sin^2(\alpha)] < \mu \]
provided that \( \dot{\lambda}, \ddot{\lambda}, \dot{\lambda} \), and \( \ddot{x} \) are bounded. This means that equation \( \mu(\alpha) < \mu \) for any \( \alpha \in (0, \frac{\pi}{2}) \) as \( k \rightarrow \infty \). As a matter of fact, in all numerical test, we have observed the decrease of the duality measure in every iteration even for \( \alpha_x \neq \alpha_s \).

Positivity of \( x(\sigma, \alpha_x) \) and \( s(\sigma, \alpha_s) \) is guaranteed if \( (x, s) > 0 \) holds and \( \alpha_x \) and \( \alpha_s \) are small enough. Assuming that \( \dot{x}, \dot{s}, \ddot{x}, \) and \( \ddot{s} \) are bounded, the claim can easily be seen from the following relations
\[ x(\sigma, \alpha_x) = x - \dot{x} \sin(\alpha_x) + \dot{\lambda}(1 - \cos(\alpha_x)) > 0, \] (33)
\[ s(\sigma, \alpha_s) = s - \dot{s} \sin(\alpha_s) + \dot{\lambda}(1 - \cos(\alpha_s)) > 0. \] (34)
4 Implementation details

In this section, we discuss factors that are normally not discussed in the main body of algorithms but affect noticeably, if not significantly, the effectiveness and efficiency of the infeasible interior-point algorithms. These factors have been discussed in widespread literatures, and they are likely implemented differently from code to code. We will address all of these implementation topics and provide detailed information of our implementation. As we will compare arc-search method and Mehrotra’s method, to make a meaningful and fair comparison, we will implement everything discussed in this section the same way for both methods, so that the only differences of the two algorithms in our implementations are in Steps 4 and 5, where the arc-search method uses formulae (23) and (24) and Mehrotra’s method uses (25) and (26).

4.1 Initial point selection

Initial point selection has been known an important factor in the computational efficiency for most infeasible interior-point algorithms. However, many commercial software packages do not provide sufficient details, for example, [16, 17]. We will use the methods proposed in [5, 7]. We compare the duality measures obtained by these two methods and select the initial point with smaller duality measure as we guess this selection will reduce the number of iterations.

4.2 Pre-process

Pre-process or pre-solver is a major factor that can significantly affect the numerical stability and computational efficiency. Many literatures have been focused on this topic, for example, [1 24 25 26]. As we will test all linear programming problems in standard form in Netlib, we focus on the strategies only for the standard linear programming problems in the form of (1) and solved in normal equations. We will use $A_i, \cdot$ for the ith row of $A$, $A, \cdot, j$ for the jth column of $A$, and $A, i, j$ for the element at $(i, j)$ position of $A$. While reducing the problem, we will express the objective function into two parts, $c^T x = f_{obj} + \sum_k c_k x_k$. The first part $f_{obj}$ at the beginning is zero and is updated all the time as we reduce the problem (remove some $c_k$ from $c$); the terms in the summation in the second part are continuously reduced and $c_k$ are updated as necessary when we reduce the problem.

The first 6 pre-process methods presented below were reported in various literatures, such as [1 24 25 16 26]; the rest of them, to the best of our knowledge, are not reported anywhere.

1. Empty row

If $A_{i,} = 0$ and $b_i = 0$, this row can be removed. If $A_{i,} = 0$ but $b_i \neq 0$, the problem is infeasible.

---

2 Some strategies are specifically designed for solving augmented system form, for example, [27]
2. Duplicate rows
If there is a constant $k$ such that $A_{i,i} = k A_{j,j}$ and $b_i = k b_j$, a duplicate row can be removed. If $A_{i,i} = k A_{j,j}$ but $b_i \neq k b_j$, the problem is infeasible.

3. Empty column
If $A_{.,i} = 0$ and $c_i \geq 0$, $x_i = 0$ is the right choice for the minimization, the $i$th column $A_{.,i}$ and $c_i$ can be removed. If $A_{.,i} = 0$ but $c_i < 0$, the problem is unbounded as $x_i \to \infty$.

4. Duplicate columns
If $A_{.,i} = A_{.,j}$, then $A x = b$ can be expressed as $A_{.,i} (x_i + x_j) + \sum_{k \neq i,j} A_{.,k} x_k = b$. Moreover, if $c_i = c_j$, $c^T x$ can be expressed as $c_i (x_i + x_j) + \sum_{k \neq i,j} c_k x_k$. Since $x_i \geq 0$ and $x_j \geq 0$, we have $(x_i + x_j) \geq 0$. Hence, a duplicate column can be removed.

5. Row singleton
If $A_{i,i}$ has exact one nonzero element, i.e., $A_{i,k} \neq 0$ for some $k$, and for $\forall j \neq k$, $A_{i,j} = 0$; then $x_k = b_i / A_{i,k}$ and $c^T x = c_k b_i / A_{i,k} + \sum_{j \neq k} c_j x_j$. For $\ell \neq i$, $A_{\ell,i} x = b_\ell$ can be rewritten as $\sum_{j \neq k} A_{\ell,j} x_j = b_\ell - A_{\ell,k} b_i / A_{i,k}$. This suggests the following update: (i) if $x_k < 0$, the problem is infeasible, otherwise, continue, (ii) $f_{opt} + c_k b_i / A_{i,k} \to f_{opt}$, (iii) remove $c_k$ from $c$, and (iv) $b_\ell - A_{\ell,k} b_i / A_{i,k} \to b_\ell$. With these changes, we can remove the $i$th row and the $k$th column.

6. Free variable
If $A_{.,i} = -A_{.,j}$ and $c_i = -c_j$, then we can rewrite $A x = b$ as $A_{.,i} (x_i - x_j) + \sum_{k \neq i,j} A_{.,k} x_k$, and $c^T x = c_i (x_i - x_j) + \sum_{k \neq i,j} c_k x_k$. The new variable $x_i - x_j$ is a free variable which can be solved if $A_{\alpha,i} \neq 0$ for some row $\alpha$ (otherwise, it is an empty column which has been discussed). This gives

$$x_i - x_j = \frac{1}{A_{\alpha,i}} \left( b_\alpha - \sum_{k \neq i,j} A_{\alpha,k} x_k \right).$$

For any $A_{\beta,i} \neq 0$, $\beta \neq \alpha$, $A_{\beta,i} x = b_\beta$ can be expressed as

$$A_{\beta,i} (x_i - x_j) + \sum_{k \neq i,j} A_{\beta,k} x_k = b_\beta,$$

or

$$\frac{A_{\beta,i}}{A_{\alpha,i}} \left( b_\alpha - \sum_{k \neq i,j} A_{\alpha,k} x_k \right) + \sum_{k \neq i,j} A_{\beta,k} x_k = b_\beta,$$

or

$$\sum_{k \neq i,j} \left( A_{\beta,k} - \frac{A_{\beta,i} A_{\alpha,k}}{A_{\alpha,i}} \right) x_k = b_\beta - \frac{A_{\beta,i} b_\alpha}{A_{\alpha,i}}.$$

Also, $c^T x$ can be rewritten as

$$\frac{c_i}{A_{\alpha,i}} \left( b_\alpha - \sum_{k \neq i,j} A_{\alpha,k} x_k \right) + \sum_{k \neq i,j} c_k x_k,$$
\[
\frac{c_i b_i}{A_{\alpha,i}} + \sum_{k \neq i,j} \left( c_k - \frac{c_i A_{\alpha,k}}{A_{\alpha,i}} \right) x_k.
\]

This suggests the following update: (i) \( f_{\text{obj}} + \frac{c_i b_i}{A_{\alpha,i}} \rightarrow f_{\text{obj}} \), (ii) \( c_k - \frac{c_i A_{\alpha,k}}{A_{\alpha,i}} \rightarrow c_k \), (iii) \( A_{\beta,k} - \frac{A_{\beta,i} A_{\alpha,k}}{A_{\alpha,i}} \rightarrow A_{\beta,k} \), (iv) \( b_{\beta} - \frac{A_{\beta,i} d_{\alpha}}{A_{\alpha,i}} \rightarrow b_{\beta} \), (v) delete \( A_{\alpha,i} \), \( b_{\alpha} \), \( m - 1 \rightarrow m \), delete \( A_{\alpha,i} \), \( A_{\beta,j} \), \( c_i \), \( c_j \), and \( n - 2 \rightarrow n \).

7. Fixed variable defined by a single row
If \( b_i < 0 \) and \( A_{i,i} \geq 0 \) with at least one \( j \) such that \( A_{i,j} > 0 \), then, the problem is infeasible. Similarly, if \( b_i > 0 \) and \( A_{i,i} \leq 0 \) with at least one \( j \) such that \( A_{i,j} < 0 \), then, the problem is infeasible. If \( b_i = 0 \), but either \( \max(A_{i,:}) \leq 0 \) or \( \min(A_{i,:}) \geq 0 \), then for any \( j \) such that \( A_{i,j} \neq 0 \), \( x_j = 0 \) has to hold. Therefore, we can remove all such rows in \( A \) and \( b \), and such columns in \( A \) and \( c \).

8. Fixed variable defined by multiple rows
If \( b_i = b_j \), but either \( \max(A_{i,:} - A_{j,:}) \leq 0 \) or \( \min(A_{i,:} - A_{j,:}) \geq 0 \), then for any \( k \) such that \( A_{i,k} - A_{j,k} \neq 0 \), \( x_k = 0 \) has to hold. This suggests the following update: (i) remove \( k \)th columns of \( A \) and \( c \) if \( A_{i,k} - A_{j,k} \neq 0 \), and (ii) remove either \( i \)th or \( j \)th row depending on which has more nonzeros. The same idea can be used for the case when \( b_i + b_j = 0 \).

9. Positive variable defined by signs of \( A_{i,:} \) and \( b_i \)
Since
\[
x_i = \frac{1}{A_{\alpha,i}} \left( b_{\alpha} - \sum_{k \neq i} A_{\alpha,k} x_k \right),
\]
if the sign of \( A_{\alpha,i} \) is the same as \( b_{\alpha} \) and opposite to all \( A_{\alpha,k} \) for \( k \neq i \), then \( x_i \geq 0 \) is guaranteed. We can solve \( x_i \), and substitute back into \( Ax = b \) and \( c^T x \). This suggests taking the following actions: (i) if \( A_{\beta,i} \neq 0 \), \( b_{\beta} - \frac{A_{\beta,i} b_{\alpha}}{A_{\alpha,i}} \rightarrow b_{\beta} \), (ii) moreover, if \( A_{\alpha,k} \neq 0 \), then \( A_{\beta,k} - \frac{A_{\beta,i} A_{\alpha,k}}{A_{\alpha,i}} \rightarrow A_{\beta,k} \), (iii) \( f_{\text{obj}} + \frac{c_i b_i}{A_{\alpha,i}} \rightarrow f_{\text{obj}} \), (iv) \( c_k - \frac{c_i A_{\alpha,k}}{A_{\alpha,i}} \rightarrow c_k \), and (v) remove the \( \alpha \)th row and \( i \)th column.

10. A singleton variable defined by two rows
If \( A_{i,:} - A_{j,:} \) is a singleton and \( A_{i,k} - A_{j,k} \neq 0 \) for one and only one \( k \), then \( x_k = \frac{b_i - b_j}{A_{i,k} - A_{j,k}} \). This suggests the following update: (i) if \( x_k \geq 0 \) does not hold, the problem is infeasible, (ii) if \( x_k \geq 0 \) does hold, for \( \forall \ell \neq i,j \) and \( A_{\ell,k} \neq 0 \), \( b_{\ell} - A_{\ell,k} x_k \frac{b_i - b_j}{A_{i,k} - A_{j,k}} \rightarrow b_{\ell} \), (iii) remove either the \( i \)th or the \( j \)th row, and remove the \( k \)th column from \( A \), (iv) remove \( c_k \) from \( c \), and (v) update \( f_{\text{obj}} + c_k \frac{b_i - b_j}{A_{i,k}} \rightarrow f_{\text{obj}} \).

We have tested all these ten pre-solvers, and they all work in terms of reducing the problem sizes and making the problems easier to solve in most cases. But pre-solvers 2,4,6,8,10 are observed to be significantly more time consuming than pre-solvers 1,3,5,7,9. Moreover, our experience shows that pre-solvers 1,3,5,7,9 are more
efficient in reducing the problem sizes than pre-solvers 2, 4, 6, 8, and 10. Therefore, in our implementation, we use only pre-solvers 1, 3, 5, 7, and 9 for all of our test problems.

Remark 4.1 Our extensive experience (by testing Netlib problems with various combinations of the pre-solves and comparing results composed of the first five columns of Table 1 in the next section and the corresponding columns of Table 1 in [16]) shows that the set of our pre-process methods uses less time and reduces the problem size more efficiently than the set of pre-process methods discussed and implemented in [16].

4.3 Matrix scaling

For ill-conditioned matrix \( A \) where the ratio \( \frac{\max |A_{ij}|}{\min \{|A_{kl}|, A_{kl} \neq 0\}} \) is big, scaling is believed to be a good practice, for example, see [16]. PCx adopted a scaling strategy proposed in [28]. Let \( \Phi = \text{diag}(\phi_1, \ldots, \phi_m) \) and \( \Psi = \text{diag}(\psi_1, \ldots, \psi_n) \) be the diagonal scaling matrices of \( A \). The scaling for matrix \( A \) in [28, 16] is equivalent to minimize

\[
\sum_{A_{ij} \neq 0} \log^2 \left| \frac{A_{ij}}{\phi_i \psi_j} \right|.
\]

Different methods are proposed to solve this problem [16, 28]. Our extensive experience with these methods and some variations (by testing all standard problems in Netlib and comparing the results) makes us to believe that although scaling can improve efficiency and numerical stability of infeasible interior-point algorithms for many problems, but over all, it does not help a lot. There are no clear criteria on what problems may benefit from scaling and what problems may be adversely affected by scaling. Therefore, we decide to not use scaling in all our test problems.

4.4 Removing row dependency from \( A \)

Theoretically, convergence analyses in most existing literatures assume that the matrix \( A \) is full rank. Practically, row dependency causes some computational difficulties. However, many real world problems including some problems in Netlib have dependent rows. Though using standard Gaussian elimination method can reduce \( A \) into a full rank matrix, the sparse structure of \( A \) will be destroyed. In [29], Andersen reported an efficient method that removes row dependency of \( A \). The paper also claimed that not only the numerical stability is improved by the method, but the cost of the effort can also be justified. One of the main ideas is to identify most independent rows of \( A \) in a cheap and easy way and separate these independent rows from those that may be dependent. A variation of Andersen’s method can be summarized as follows.

First, it assumes that all empty rows have been removed by pre-solver. Second, matrix \( A \) often contains many column singletons (the column has only one nonzero), for example, slack variables are column singletons. Clearly, a row containing a column singleton cannot be dependent. If these rows are separated (temporarily removed) from rest rows of \( A \), new column singletons may appear and more rows may be separated.
This process may separate most rows from rest rows of \( A \) in practice. Permutation operations can be used to move the singletons to the diagonal elements of \( A \). The dependent rows are among the rows left in the process. Then, Gaussian elimination method can be applied with pivot selection using Markowitz criterion \([30, 31]\). Some implementation details include (a) break ties by choosing element with the largest magnitude, and (b) use threshold pivoting.

Our extensive experience makes us to believe that although Andersen’s method may be worthwhile for some problems and significantly improve the numerical stability, but it may be expensive for many other problems. We choose to not use this function unless we feel it is necessary when it is used as part of handling degenerate solutions discussed later. For the safe of fair comparison between two algorithmss, we will make it clear in our test report what algorihtm and/or problems use this function and what algorihtm and/or problems do not use this function.

### 4.5 Linear algebra for sparse Cholesky matrix

Similar to Mehrotra’s algorithm, the majority of the computational cost of our proposed algorithm is to solve sparse Cholesky system \((16)\) and \((17)\), which can be expressed as an abstract problem as follows.

\[
AD^2A^T u = v, \tag{35}
\]

where \( D = X^{\frac{1}{2}}S^{-\frac{1}{2}} \) is identical in \((16)\) and \((17)\), but \( u \) and \( v \) are different vectors. Many popular LP solvers \([16, 17]\) call a software package \([32]\) which uses some linear algebra specifically developed for the sparse Cholesky decomposition \([33]\). However, MATLAB does not yet have this function to call. This is the major difference of our implementation comparing to other popular LP solvers, which is most likely the main reason that our test results are slightly different from test results reported in other literatures.

### 4.6 Handling degenerate solutions

An important result in linear programming \([34]\) is that there always exist strictly complementary optimal solutions which meet the conditions \( x^* \circ s^* = 0 \) and \( x^* + s^* > 0 \). Therefore, the columns of \( A \) can be partitioned as \( B \subseteq \{1, 2, \ldots, n\} \), the set of indices of the positive coordinates of \( x^* \), and \( S \subseteq \{1, 2, \ldots, n\} \), the set of indices of the positive coordinates of \( s^* \), such that \( B \cup N = \{1, 2, \ldots, n\} \) and \( B \cap N = \{1, 2, \ldots, n\} \). Thus, we can partition \( A = (A_B, A_N) \), and define the primal and dual optimal faces by

\[
P_* = \{ x : A_B x_B = b, x \geq 0, x_N = 0 \},
\]

and

\[
D_* = \{ (\lambda, s) : A_N^T \lambda + S_N = c_N, s_B = 0, s \geq 0 \}.
\]

However, not all optimal solutions in linear programming are strictly complementary. A simple example is provided in \([11]\ Page 28]. Although many interior-point algorithms
are proved to converge to strictly complementary solutions, this claim may not be true for Mehrotra’s method and arc-search method proposed in this paper.

Recall that the problem pair (1) and (2) is called to have a primal degenerate solution if a primal optimal solution \( x^* \) has less than \( m \) positive coordinates, and have a dual degenerate solution if a dual optimal solution \( s^* \) has less than \( n - m \) positive coordinates. The pair \((x^*, s^*)\) is called degenerate if it is primal or dual degenerate. This means that as \( x^k \to x^* \), equation (35) can be written as
\[
(A_B X_B S_B^{-1} A_B^T)u = v,
\]
If the problem converges to a primal degenerate solution, then the rank of \((A_B X_B S_B^{-1} A_B^T)\) is less than \( m \) as \( x^k \to x^* \). In this case, there is a difficulty to solve (36). Difficulty caused by degenerate solutions in interior-point methods for linear programming has been realized for a long time [35]. We have observed this troublesome incidence in quite a few Netlib test problems. Similar observations was also reported in [36]. Though we don’t see any special attention or report on this troublesome issue from some widely cited papers and LP solvers, such as [5, 6, 7, 17, 16], we noticed from [1, page 219] that some LP solvers [17, 16] twisted the sparse Cholesky decomposition code [32] to overcome the difficulty.

In our implementation, we use a different method to avoid the difficulty because we do not have access to the code of [32]. After each iteration, minimum \( x^k \) is examined. If \( \min\{x^k\} \leq \epsilon_x \), then, for all components of \( x \) satisfying \( x_i \leq \epsilon_x \), we delete \( A_i, x_i, s_i, c_i \), and the \( i \)th component of \( rc \); use the method proposed in Subsection 4.4 to check if the updated \( A \) is full rank and make the updated \( A \) full rank if it is necessary.

The default \( \epsilon_x \) is \( 10^{-6} \). For problems that needs a different \( \epsilon_x \), we will make it clear in the report of the test results.

### 4.7 Step scaling parameter

A fixed step scaling parameter is used in PCx [16]. A more sophisticated step scaling parameter is used in LIPSOL according to [1, Pages 204-205]. In our implementation, we use an adaptive step scaling parameter which is given below
\[
\beta = 1 - e^{-(k+2)},
\]
where \( k \) is the number of iterations. This parameter will approach to one as \( k \to \infty \).

### 4.8 Terminate criteria

The main stopping criterion used in our implementations of arc-search method and Mehrotra’s method is similar to that of LIPSOL [17]

\[
\frac{\|r^k_b\|}{\max\{1, \|b\|\}} + \frac{\|r^k_c\|}{\max\{1, \|c\|\}} + \frac{\mu_k}{\max\{1, \|c^T x^k\|, \|b^T \lambda^k\|\}} < 10^{-8}.
\]

In case that the algorithms fail to find a good search direction, the programs also stop if step sizes \( \alpha^x_k < 10^{-8} \) and \( \alpha^s_k < 10^{-8} \).
Finally, if due to the numerical problem, \( r_b^k \) or \( r_c^k \) does not decrease but \( 10r_{b}^{k-1} < r_b^k \) or \( 10r_{c}^{k-1} < r_c^k \), the programs stop.

5 Numerical Tests

In this section, we first examine a simple problem and show graphically what feasible central path and infeasible central path look like, why ellipsoidal approximation may be a better approximation to infeasible central path than a straight line, and how arc-search is carried out for this simple problem. Using a plot, we can easily see that searching along the ellipse is more attractive than searching along a straight line. We then provide the numerical test results of larger scale Netlib test problems to validate our observation from this simple problem.

5.1 A simple illustrative example

Let us consider

Example 5.1

\[
\begin{align*}
\min \ x_1, \quad & s.t. \ x_1 + x_2 = 5, \ x_1 \geq 0, \ x_2 \geq 0.
\end{align*}
\]

The feasible central path \((x, s)\) defined in (5) satisfies the following conditions:

\[
\begin{align*}
\begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda + \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
x_1s_1 &= \mu, \quad x_2s_2 = \mu.
\end{align*}
\]

The optimizer is given by \( x_1 = 0, \ x_2 = 5, \ \lambda = 0, \ s_1 = 1, \ and \ s_2 = 0. \ The \ feasible \ central \ path \ of \ this \ problem \ is \ given \ analytically \ as\)

\[
\begin{align}
\lambda &= \frac{5 - 2\mu - \sqrt{(5 - 2\mu)^2 + 20\mu}}{10}, \quad (38a) \\
s_1 &= 1 - \lambda, \quad s_2 = -\lambda, \quad x_1s_1 = \mu, \quad x_2s_2 = \mu. \quad (38b)
\end{align}
\]

The feasible and infeasible central paths are arcs in 5-dimensional space \((\lambda, x_1, s_1, x_2, s_2)\). If we project the central paths into 2-dimensional subspace spanned by \((x_1, x_2)\), they are arcs in 2-dimensional subspace. Figure 1 shows the first two iterations of Algorithm 3.1 in the 2-dimensional subspace spanned by \((x_1, x_2)\). In Figure 1, the initial point \((x_1^0, x_2^0)\) is marked by ‘x’ in red; the optimal solution is marked by ‘*’ in red; \((\dot{x}, \dot{s}, \dot{\lambda})\) is calculated by using (12); \((\ddot{x}, \ddot{s}, \ddot{\lambda})\) is calculated by using (14); the projected feasible central path \(C(t)\) near the optimal solution is calculated by using (38) and is plotted as a continuous line in black; the infeasible central path \(H(t)\) starting from current iterate is calculated by using (9) and plotted as the dotted lines in blue; and the projected
ellipsoidal approximations $\mathcal{E}(\alpha)$ are the dotted lines in green (they may look like continuous line some times because many dots are used). In the first iteration, the iterate 'x' moves along the ellipse (defined by in Theorem 3.1) to reach the next iterate marked as 'o' in red because the calculation of infeasible central path (the blue line) is very expensive and ellipse is cheap to calculate and a better approximation to the infeasible central path than a straight line. The rest iterations are simply the repetition of the process until it reaches the optimal solution $(s^*, x^*)$. Only two iterations are plotted in Figure 1.

It is worthwhile to note that in this simple problem, the infeasible central path has a sharp turn in the first iteration which may happen a number of times for general problem as discussed in [37]. The arc-search method is expected to perform better than Mehrotra’s method in iterations that are close to the sharp turns. In this simple problem, after the first iteration, the feasible central path $\mathcal{C}(t)$, the infeasible central path $\mathcal{H}(t)$, and the ellipse $\mathcal{E}(\alpha)$ are all very close to each other and close to a straight line.

5.2 Netlib test examples

The algorithm developed in this paper is implemented in a Matlab function. Mehrotra’s algorithm is also implemented in a Matlab function. They are almost identical. Both algorithms use exactly the same initial point, the same stopping criteria, the same
pre-process, and the same parameters. The only difference of the two implementations are that arc-search method searches optimizer along an ellipse and Mehrotra’s method searches optimizer along a straight line. Numerical tests for both algorithms have been performed for all Netlib LP problems that are presented in standard form. The iteration numbers used to solve these problems are listed in Table 1. Only one Netlib problem Osa60 (m = 10281 and n = 232966) presented in standard form is not included in the test because the PC computer used for the testing does not have enough memory to handle this problem.

| Problem | before prep | after prep | Arc-search | Mehrotra |
|---------|-------------|------------|------------|----------|
|         | m | n | m | n | iter | objective | infeas | iter | objective | infeas |
| Adlittle | 56 | 138 | 54 | 136 | 15 | 2.2549e+05 | 1.0e-07 | 15 | 2.2549e+05 | 3.4e-08 |
| Afrio | 27 | 51 | 8 | 32 | 9 | -464.7531 | 1.0e-11 | 9 | -464.7531 | 8.0e-12 |
| Agg | 488 | 615 | 391 | 479 | 18 | -3.5992e+07 | 5.0e-06 | 22 | -3.5992e+07 | 5.2e-05 |
| Agg2 | 516 | 758 | 514 | 755 | 18 | -2.0239e+07 | 4.6e-07 | 20 | -2.0239e+07 | 5.2e-07 |
| Agg3 | 516 | 758 | 514 | 755 | 17 | 1.0312e+07 | 3.1e-08 | 18 | 1.0312e+07 | 8.8e-09 |
| Bandm | 305 | 472 | 192 | 347 | 19 | -158.6280 | 3.2e-11 | 22 | -158.6280 | 8.3e-10 |
| Beaconfd | 173 | 295 | 57 | 147 | 10 | 3.3592e+04 | 1.4e-12 | 11 | 3.3592e+04 | 1.4e-10 |
| Blend | 74 | 114 | 49 | 89 | 12 | -30.8121 | 1.0e-09 | 14 | -30.8121 | 4.9e-11 |
| Bnl1 | 643 | 1586 | 429 | 1314 | 32 | 1.9776e+03 | 2.7e-09 | 35 | 1.9776e+03 | 3.4e-09 |
| Bnl2 | 2324 | 4486 | 1007 | 3066 | 32 | 1.8112e+03 | 5.4e-10 | 37 | 1.8112e+03 | 9.3e-07 |
| Brandy | 220 | 303 | 113 | 218 | 20 | 1.5185e+03 | 3.0e-06 | 19 | 1.5185e+03 | 6.2e-08 |
| Degen2* | 444 | 757 | 440 | 753 | 16 | -1.4352e+03 | 1.9e-08 | 17 | -1.4352e+03 | 2.0e-10 |
| Degen3* | 1503 | 2604 | 1490 | 2591 | 22 | -9.8729e+02 | 7.0e-05 | 22 | -9.8729e+02 | 1.2e-09 |
| Qap800 | 525 | 1208 | 487 | 991 | 26 | 5.5608e+05 | 4.3e-05 | 31 | 5.5608e+05 | 7.7e-04 |
| Israel | 174 | 316 | 174 | 316 | 23 | -8.9664e+05 | 7.4e-08 | 29 | -8.9665e+05 | 1.8e-08 |
| Lotfi | 153 | 366 | 113 | 326 | 14 | -25.2647 | 3.5e-10 | 18 | -25.2647 | 2.7e-07 |
| Maros27 | 3136 | 9408 | 2152 | 7440 | 18 | 1.4972e+06 | 1.6e-08 | 21 | 1.4972e+06 | 6.4e-09 |
| Osa17* | 1118 | 25067 | 1081 | 25030 | 37 | 5.3574e+05 | 4.2e-07 | 35 | 5.3574e+05 | 1.5e-07 |
| Osa14 | 2337 | 54797 | 2300 | 54760 | 35 | 1.1065e+06 | 2.0e-09 | 37 | 1.1065e+06 | 3.0e-08 |
| Osa30 | 4350 | 104374 | 4313 | 104337 | 32 | 2.1421e+06 | 1.0e-08 | 36 | 2.1421e+06 | 1.3e-08 |
| Qap12 | 3192 | 8856 | 3048 | 8712 | 22 | 5.2289e+02 | 1.9e-08 | 24 | 5.2289e+02 | 6.2e-09 |
| Qap15* | 6330 | 22275 | 6105 | 22050 | 27 | 1.0411e+03 | 3.9e-07 | 44 | 1.0410e+03 | 1.5e-05 |
| Qap8* | 912 | 1632 | 848 | 1568 | 12 | 2.0350e+02 | 1.2e-12 | 13 | 2.0350e+02 | 7.1e-09 |
| Sc15 | 105 | 163 | 44 | 102 | 10 | -52.2021 | 3.8e-12 | 11 | -52.2021 | 9.8e-11 |
| Sc205 | 205 | 317 | 89 | 201 | 13 | -52.2021 | 3.7e-10 | 12 | -52.2021 | 8.8e-11 |
| Sc50a | 50 | 78 | 19 | 47 | 10 | -64.5751 | 3.4e-12 | 9 | -64.5751 | 8.3e-08 |
| Sc50b | 50 | 78 | 14 | 42 | 8 | -70.0000 | 1.0e-10 | 8 | -70.0000 | 9.1e-07 |
| Scagr25 | 471 | 671 | 343 | 543 | 19 | -1.4753e+07 | 5.0e-07 | 18 | -1.4753e+07 | 4.6e-09 |
| Scagr7 | 129 | 185 | 91 | 147 | 15 | -2.3314e+06 | 2.7e-09 | 17 | -2.3314e+06 | 1.1e-07 |
| Scfxm1+ | 330 | 600 | 238 | 500 | 20 | 1.8417e+04 | 3.1e-07 | 21 | 1.8417e+04 | 1.6e-08 |
| Scfxm2 | 660 | 1200 | 479 | 1003 | 23 | 3.6600e+04 | 2.3e-06 | 26 | 3.6600e+04 | 2.0e-06 |
| Scfxm3+ | 990 | 1800 | 720 | 1506 | 24 | 5.4901e+04 | 1.9e-06 | 23 | 5.4901e+04 | 9.8e-08 |
| Sctap | 490 | 1275 | 115 | 893 | 23 | 9.0430e+02 | 1.2e-11 | 30 | 9.0430e+02 | 1.8e-10 |
| Sctap1 | 77 | 760 | 77 | 760 | 12 | 8.6666 | 1.0e-10 | 13 | 8.6666 | 8.7e-14 |
| Sctap2 | 1090 | 2500 | 1033 | 2443 | 20 | 1.7248e+03 | 2.1e-10 | 21 | 1.7248e+03 | 4.4e-07 |
| Sctap3 | 1480 | 3340 | 1408 | 3268 | 20 | 1.4240e+03 | 5.7e-08 | 22 | 1.4240e+03 | 5.9e-07 |
Several problems have degenerate solutions which make them difficult to solve or need significantly more iterations. We choose to use the option described in Section 4.6 to solve these problems. For problems marked with '+', this option is called only for Mehrotra’s method. For problems marked with '*', both algorithms need to call this option for better results. For problems with '**', in addition to call this option, the default value of $10^{-6}$ has to be changed to $10^{-4}$ for Mehrotra’s method. We need to keep in mind that although using the option described in Section 4.6 reduces the iteration count significantly, these iterations are significantly more expensive. Therefore, simply comparing iteration counts for problem(s) marked with '+' will lead to a conclusion in favor of Mehrotra’s method (which is what we will do in the following discussions).

Since the major cost in each iteration for both algorithms are solving linear systems of equations, which are identical in these two algorithms, we conclude that iteration numbers is a good measure of efficiency. In view of Table 1, it is clear that Algorithm 3.1 uses less iterations than Mehrotra’s algorithm to find the optimal solutions for majority tested problems. Among 51 tested problems, Mehrotra’s method uses fewer iterations (7 iterations in total) than arc-search method for only 6 problems (brandy, osa_07, sc205, sc50a, scagr25, scfxm3), while arc-search method uses fewer iterations (126 iterations in total) than Mehrotra’s method for 40 problems. For the rest 5 problems, both methods use the same number of iterations. Arc-search method is numerically more stable than Mehrotra’s method because for problems scfxml, scfxml3, ship08s, ship12s, arc-search method does not need to use the option described in Section 4.6 but Mehrotra’s method need to use the option to solve the problems. For problem ship08l, Mehrotra’s method need to adjust parameter in the option to find the optimizer but arc-search method does not need to adjust the parameter.

For this problem, Mehrotra’s method needs to use the option described in Section 4.6 but arc-search method does not need to. As a result, Mehrotra’s method uses noticeably more CPU time than arc-search method.

| Share1b | 117 | 253 | 102 | 238 | 22 | -7.6589e+04 | 6.5e-08 | 25 | -7.6589e+04 | 1.5e-06 |
| Share2b | 96  | 162 | 87  | 153 | 13 | -4.1573e+02 | 4.9e-11 | 15 | -4.1573e+02 | 7.9e-10 |
| Ship04l | 402 | 2166 | 292 | 1905 | 17 | 1.7933e+06 | 5.2e-11 | 18 | 1.7933e+06 | 2.9e-11 |
| Ship04s | 402 | 1506 | 216 | 1281 | 17 | 1.7933e+06 | 4.9e-11 | 15 | 1.7933e+06 | 7.9e-10 |
| Ship08l** | 778 | 4363 | 470 | 3121 | 17 | 1.9090e+06 | 3.7e-08 | 19 | 1.9090e+06 | 4.5e-12 |
| Ship08s+ | 778 | 2467 | 274 | 1600 | 17 | 1.9201e+06 | 2.9e-09 | 20 | 1.9201e+06 | 4.5e-12 |
| Ship12l* | 1151 | 5533 | 610 | 4171 | 19 | 1.4702e+06 | 4.7e-13 | 20 | 1.4702e+06 | 1.0e-08 |
| Ship12s+ | 1151 | 2869 | 340 | 1943 | 17 | 1.4892e+06 | 1.0e-10 | 19 | 1.4892e+06 | 2.1e-13 |
| Stocfor1* | 117 | 165 | 34 | 82 | 14 | -4.1132e+04 | 2.8e-10 | 15 | -4.1132e+04 | 1.1e-10 |
| Stocfor2 | 2157 | 3045 | 766 | 1654 | 22 | -3.9024e+04 | 2.1e-09 | 22 | -3.9024e+04 | 1.6e-09 |
| Stocfor3 | 16675 | 23541 | 5974 | 12840 | 34 | -3.9976e+04 | 4.7e-08 | 38 | -3.9976e+04 | 6.4e-08 |
| Truss | 1000 | 8806 | 1000 | 8806 | 22 | 4.5882e+05 | 1.7e-07 | 36 | 4.5882e+05 | 9.5e-06 |

Table 1: Numerical results for test problems in Netlib
6 Conclusions

This paper proposes an arc-search interior-point path-following algorithm that searches optimizers along the ellipses that approximate infeasible central paths. The proposed algorithm is different from Mehrotra’s method only in search paths. Both arc-search method and Mehrotra’s method are implemented in Matlab so that the two methods use exactly same initial point, the same pre-process, the same parameters, and the same stopping criteria. By doing this, we can compare the two algorithms in a fair and controlled way. Numerical test is conducted for Netlib problems for both methods. The results show that the proposed arc-search method is more efficient and reliable than the well-known Mehrotra’s method.

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