TWISTED SOLUTIONS TO A SIMPLIFIED ERICKSEN-LESLIE EQUATION

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ABSTRACT: In this article we construct global solutions to a simplified Ericksen–Leslie system on $\mathbb{R}^3$. The constructed solutions are twisted and periodic along the $x_3$-axis with period $d = 2\pi/\mu$. Here $\mu > 0$ is the twist rate. $d$ is the distance between two planes which are parallel to the $x_1x_2$-plane. Liquid crystal material is placed in the region enclosed by these two planes. Given a well-prepared initial data, our solutions exist classically for all $t \in [0, \infty)$. However these solutions become singular at all points on the $x_3$-axis and escape into third dimension exponentially while $t \to \infty$. An optimal blow up rate is also obtained.

I. INTRODUCTION

I.1. BACKGROUND AND MOTIVATION

Ericksen–Leslie equation is a hydrodynamical system describing nematic liquid crystal flow. For the sake of simplifying and meanwhile preserving the energy dissipative property of the original Ericksen–Leslie equation, a simplified version was proposed by Lin in [9]. With all the parameters in the system normalized to be 1, the simplified equation can be read as follows:

\[
\begin{align*}
\partial_t \phi + u \cdot \nabla \phi - \Delta \phi &= |\nabla \phi|^2 \phi & \text{in } \mathbb{R}^3 \times (0, \infty); \\
\partial_t u + u \cdot \nabla u - \Delta u &= -\nabla p - \nabla \cdot (\nabla \phi \odot \nabla \phi) & \text{in } \mathbb{R}^3 \times (0, \infty); \\
\text{div } u &= 0 & \text{in } \mathbb{R}^3 \times (0, \infty).
\end{align*}
\]

(1.1)

In (1.1), $\phi : \mathbb{R}^3 \times (0, \infty) \to S^2$ represents the macroscopic orientation of a nematic liquid crystal. $u : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3$ is the velocity field of fluid. $p : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}$ is the pressure. The stress tensor $\nabla \phi \odot \nabla \phi$ is defined with its $(i,j)$-th entry given by

\[
(\nabla \phi \odot \nabla \phi)_{ij} = \langle \partial_i \phi, \partial_j \phi \rangle.
\]

Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^3$. As one can see, the system (1.1) is a coupled system consisting of the non-homogeneous incompressible Navier-Stokes equation and the transported heat flow of harmonic map.

Many research works have been devoted to the study of (1.1). We refer readers to the survey article [11] by Lin-Wang and references therein. Most recently solutions of (1.1) with finite time singularity have also been constructed by authors in [8], where the spatial domain is a bounded open set in $\mathbb{R}^3$. In contrast to [8], in the current work, we are concerned with global solutions of (1.1) which become singular at $t = \infty$. Our motivation originates from the twisted ansatz for nematic liquid crystal in [14]. In fact our solutions are supposed to admit a special form which is given by

\[
u = \mathcal{V}(x_1, x_2, t) \quad \text{and} \quad \phi = e^{\mu x_3 R} \psi(x_1, x_2, t).
\]

(1.2)

Here $\mu > 0$ is called twist rate of nematic liquid crystal. $\mathcal{V}$ and $\psi$ are two 3-vectors. Particularly $\psi$ takes its value in $S^2$. $R$ in (1.2) denotes the generator of horizontal rotations, which can be represented by

\[
R = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

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For a given real number \( \alpha \), the exponential matrix of \( \alpha R \) equals to
\[
\begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Plugging the ansatz (1.2) into (1.1), we obtain the equation satisfied by \( \psi \) and \( \sigma \) as follows:
\[
\begin{align*}
\frac{D}{Dt} \psi - \Delta_2 \psi &= |\nabla^h \psi|^2 \psi - \mu \mathcal{Y}_3 R \psi + \mu^2 \left[ R^2 \psi + |R\psi|^2 \psi \right] & \text{in } \mathbb{R}^2 \times (0, \infty); \\
\frac{D}{Dt} \mathcal{Y}^h - \Delta_2 \mathcal{Y}^h &= -\nabla^h q - \nabla^h \cdot (\nabla^h \psi \odot \nabla^h \psi) & \text{in } \mathbb{R}^2 \times (0, \infty); \\
\frac{D}{Dt} \mathcal{Y}_3 - \Delta_2 \mathcal{Y}_3 &= -\mu \langle \Delta_2 \psi, R \psi \rangle & \text{in } \mathbb{R}^2 \times (0, \infty),
\end{align*}
\]
where \( \mathcal{Y}^h \) satisfies the incompressibility condition:
\[
\nabla^h \cdot \mathcal{Y}^h = 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty). \tag{1.4}
\]

In (1.3)-(1.4), \( \nabla^h \) and \( \Delta_2 \) are the gradient and Laplace operators on \( \mathbb{R}^2 \), respectively. \( \mathcal{Y}^h = (\mathcal{Y}_1, \mathcal{Y}_2)^t \) is the horizontal part of the vector field \( \mathcal{Y} \). \( \mathcal{Y}_3 \) is the third component of \( \mathcal{Y} \). The differential operator \( D/Dt \) is the material derivative \( \partial_t + \mathcal{Y}^h \cdot \nabla^h \).

When \( \mu = 0 \) and \( \mathcal{Y}_3 = 0 \), the system (1.3), (1.4) is then reduced to a 2D version of (1.1). Its vorticity formulation has been studied by authors in [3]. In fact a global weak solution is obtained in [3] under the assumption that the initial vorticity of fluid lies in \( L^1(\mathbb{R}^2) \) and the initial director field has finite Dirichlet energy. Furthermore in [4], when initial vorticity and director field are sufficiently close to
\[
\begin{pmatrix}
0, e^{m\theta R + \alpha_0 R} h \left( \frac{r}{\sigma_0} \right)
\end{pmatrix}
\]
in some norm space, then (1.1) in 2D admits a global classical solution which has the form
\[
\mathcal{Y}^h = f(r, t) \left( \frac{-x_2}{x_1} \right) \quad \text{and} \quad \psi = e^{m\theta R} \psi_\alpha(r, t).
\]
Here \( (r, \theta) \) is the polar coordinate on \( \mathbb{R}^2 \). \( m \) is an integer with \( |m| \geq 4 \). \( \sigma_0 > 0 \) and \( \alpha_0 \in \mathbb{R} \) are two constants. For any \( \rho > 0 \), \( h \) is a 3-vector defined by
\[
\begin{pmatrix}
h_1(\rho) \\
0 \\
h_3(\rho)
\end{pmatrix}, \quad \text{where } h_1(\rho) = \frac{2}{\rho^{[m]} + \rho^{-[m]}} \quad \text{and} \quad h_3(\rho) = \frac{\rho^{[m]} - \rho^{-[m]}}{\rho^{[m]} + \rho^{-[m]}}, \tag{1.6}
\]
The authors in [4] also show that when the absolute value of the initial circulation Reynolds number \( \omega \) is suitably small, then there exists a positive constant \( \sigma_\infty \) and an angular function \( \alpha(t) \) so that
\[
\psi \sim e^{m\theta R + \alpha(t) R} h \left( \frac{r}{\sigma_\infty} \right), \quad \text{as } t \to \infty.
\]
Moreover for some \( \alpha_d \in \mathbb{R} \), the following limit holds for the angular function \( \alpha(t) \):
\[
\lim_{t \to \infty} \left[ 4 \alpha(t) + m \omega \log t \right] = \alpha_d.
\]
The above limit implies that the director field keeps rotating around the \( x_3 \)-axis as \( t \) tends to \( \infty \), provided that \( |\omega| \) is small and nonzero. It can be easily checked that (1.5) gives a stationary solution to the vorticity formulation of (1.1) in 2D. The results in [4] then indicate that these stationary solutions are globally dynamically
unstable in the space $L^1(\mathbb{R}^2) \times H^1_0(\mathbb{R}^2;\mathbb{S}^2)$, where $e_3 \in \mathbb{S}^2$ is the north pole.

I.2. MAIN RESULTS

Inspired by the works [3]-[4], we construct in this article a global solution to (1.3)-(1.4) under the following $m$-equivariant ansatz:

$$\psi = \frac{W(r,t)}{r^2} \left( \begin{array}{c} \frac{-x_2}{x_1} \\ 0 \\ V(r,t) \end{array} \right) \quad \text{and} \quad \psi = e^{m\theta R} \varphi(r,t).$$

Here $W$, $V$ are two real-valued functions and $\varphi$ is an $\mathbb{S}^2$-valued vector field. They all depend on the variables $t$ and $r$ only. By this ansatz, the incompressibility condition (1.4) is automatically satisfied. Meanwhile the system (1.3) is reduced to

$$\begin{align*}
\partial_t \varphi + \left( \frac{mW}{r^2} + \mu V \right) R \varphi &= \Delta_2 \varphi + |\partial_r \varphi|^2 \varphi + \left( \frac{m^2}{r^2} + \mu^2 \right) \left[ R^2 \varphi + |R\varphi|^2 \varphi \right] \quad \text{in } \mathbb{R}^2 \times (0, \infty); \\
\partial_t W &= \partial_r W - \frac{1}{r} \partial_r W - m \left( \Delta_2 \varphi, R \varphi \right) \quad \text{in } \mathbb{R}^2 \times (0, \infty); \\
\partial_t V &= \Delta_2 V - m \left( \Delta_2 \varphi, R \varphi \right) \quad \text{in } \mathbb{R}^2 \times (0, \infty).
\end{align*}$$

(1.8)

If $(W, V, \varphi)$ is a global solution of (1.8), then it provides a global solution $(u, \phi)$ to (1.1) by the change of variables in (I.2) and (I.7).

Before stating the main theorem, we introduce some notations. The map $h$ in (1.6) generates a 2-parameter family of $m$-equivariant harmonic maps in 2D:

$$\{ e^{m\theta R} h^{\alpha, \sigma}(r) : \alpha \in \mathbb{R} \text{ and } \sigma > 0 \}.$$  

Here we simply call a map harmonic if it is a harmonic map from $\mathbb{R}^2$ to $\mathbb{S}^2$. For any $\alpha \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, $h^{\alpha, \sigma}$ in the set given above is defined by

$$h^{\alpha, \sigma}(r) := e^{\alpha R} h \left( \frac{r}{\sigma} \right), \quad \forall r > 0.$$  

It satisfies the boundary conditions $h^{\alpha, \sigma}(0) = -e_3$ and $h^{\alpha, \sigma}(\infty) = e_3$. Moreover $h^{\alpha, \sigma}$ attains the minimal Dirichlet energy $4\pi|m|$ in the class $\Sigma_m$ of all $m$-equivariant maps. Here $\Sigma_m$ is given as follows:

$$\Sigma_m := \left\{ \psi : \mathbb{R}^2 \to \mathbb{S}^2 \left| \psi = e^{m\theta R} \varphi(r), \| \nabla \psi \|_{L^2(\mathbb{R}^2)} < \infty, \varphi(0) = -e_3, \varphi(\infty) = e_3 \right. \right\}.$$  

Associated with an $m$-equivariant map $\psi = e^{m\theta R} \varphi(r) \in \Sigma_m$, we define

$$q = q[\varphi] := \partial_r \varphi - \frac{|m|}{r} \varphi \times R \varphi.$$  

The tangent vector $q = q[\varphi] \in T_\varphi \mathbb{S}^2$ provides us with the information on the harmonicity of $\psi = e^{m\theta R} \varphi$. In fact if $q[\varphi] = 0$, then the vector field $\psi$ is a harmonic map in 2D. More properties on the tangent vector field $q[\varphi]$ can be found in [4]. We also study in this article the fluid with possibly nonzero circulation Reynolds number. Therefore we should introduce the Oseen part of the variable $W$ (see (I.7)). In fact we denote by $W^{os}$ the Oseen part of $W$, which is explicitly given by

$$W^{os}(r,t) = \omega \left( 1 - e^{-r^2/l(t)} \right) \quad \text{with } l(t) := 4t + r_0^2.$$  

(1.9)

Here $r_0 > 0$ is the initial core radius of the Oseen vortex. $\omega \in \mathbb{R}$ is the circulation Reynolds number. When $t = 0$, we simply denote by $W^{os}_{in}$ the function $W^{os}(.0)$. In the following statements $A \leq B$ means that there...
is a constant $c > 0$ so that $A \leq cB$. Here $c$ depends possibly on the structural parameters $m$, $\mu$, $\omega$, and $r_0$. Throughout the article for any $p \in [1, \infty]$, we use $L^p$ to denote the functional spaces $L^p(\mathbb{R}^d)$. Usually if integrand in an integration is a function of variable $r$, then we suppress the notation $r\,dr$ from the integration and simply employ the following agreement:

$$
\int_0^\infty f = \int_0^\infty f \, r\,dr.
$$

We also need an $X$-space, which is defined by

$$
X := \left\{ z : [0, \infty) \to \mathbb{C} \mid \|z\|_X < \infty \right\}.
$$

(1.10)

Here $\|\cdot\|_X$ is the following norm for functions in the space $X$:

$$
\|z\|^2_X := \int_0^\infty \left( |\partial_\rho z|^2 + \frac{|z|^2}{\rho^2} \right) \rho\,d\rho.
$$

(1.11)

Now we give our main results in this article.

**Theorem 1.1.** Suppose that $m$ is an integer satisfying $|m| \geq 3$. $\mu > 0$ is a given twist rate. Suppose that the initial velocity field of fluid satisfies $V_{in} \in L^2$ and

$$
W_{in} = W_{in}^{\text{os}} + W_{in}^* \quad \text{with} \quad \frac{W_{in}^*}{r} \in L^2.
$$

(1.12)

Given two constants $\Theta_{in} \in \mathbb{R}$, $\sigma_{in} > 0$ and two single-variable functions $z_{j, in}$ $(j = 1, 2)$ on $\mathbb{R}^+$ with

$$
\|z_{j, in}\|_{L^\infty} < 1/2 \quad \text{and} \quad \int_0^\infty z_{j, in} \, h_1 = 0,
$$

(1.13)

we assume the following representation for the $\mathbb{S}^2$-valued vector field $\varphi_{in}$:

$$
\varphi_{in}(r) = e^{\Theta_{in} R} \left\{ \begin{array}{l}
h(\rho) + \gamma_{in}(\rho) \, h(\rho) + z_{1, in}(\rho) \, e_2 + z_{2, in}(\rho) \, h(\rho) \times e_2 \\
\end{array} \right\},
$$

(1.14)

where $\rho = \frac{r}{\sigma_{in}}$. The function $\gamma_{in}(\cdot)$ is given by

$$
\gamma_{in} = \left( 1 - |z_{in}|^2 \right)^{1/2} - 1 \quad \text{with} \quad z_{in} = z_{1,in} + i z_{2,in}.
$$

For any given $\varepsilon \in (0, 1)$, there exists a positive constant $\delta_\varepsilon$ (depending on $m$, $\mu$, $\omega$, $r_0$ and $\varepsilon$) such that if

$$
\|z_{in}\|_X + \|V_{in}\|_{L^2} + \left\| \frac{W_{in}^*}{r} \right\|_{L^2} + \sigma_{in} + \left( \int_0^\infty |z_{in}(\rho)|^2 \rho\,d\rho \right)^{1/2} < \delta_\varepsilon,
$$

(1.15)

then the followings hold for the equation (1.8):

(i). Supplied with the initial data $(W_{in}, V_{in}, \varphi_{in})$, (1.8) admits a classical solution, denoted by $(W, V, \varphi)$, on the time interval $[0, \infty)$. For some $C^1$-regular time dependent parameter functions $(\sigma(t), \Theta(t))$, the vector field $\varphi$ can be expressed as

$$
\varphi(r,t) = e^{\Theta(t) R} \left\{ \begin{array}{l}
h(\rho) + \gamma(\rho,t) \, h(\rho) + z_1(\rho,t) \, e_2 + z_2(\rho,t) \, h(\rho) \times e_2 \\
\end{array} \right\},
$$

(1.16)

where $\rho = \frac{r}{\sigma(t)}$. For a fixed $t > 0$, $z_1(\rho,t)$ and $z_2(\rho,t)$ in the above expression are two functions in the space $X \cap L^2(\rho\,d\rho)$. $\gamma$ is a function given in (2,5). Let $z$ be the complexified function $z_1 + i z_2$. Then for all $t > 0$, $z(\cdot, t)$ satisfies the orthogonal condition:

$$
\int_0^\infty z(\rho, t) \, h_1(\rho) \rho\,d\rho = 0.
$$

(1.17)
Moreover the following estimate holds for the $X$-norm of $z$:

\[
\int_0^\infty \exp \left\{ \frac{2\mu^2}{m^2} s \right\} \| z(\cdot, s) \|_{X^2}^2 \, ds \leq 1;
\]  

(1.17)

(ii). The functions $W$ and $V$ can be decomposed into

\[ W = W_0^* + W_1^* + W_2^* \quad \text{and} \quad V = V_1 + V_2, \]

respectively. Moreover for all $t > 0$, $V_1$, $V_2$, $W_1^*$, $W_2^*$ satisfy the following time decay estimates:

\[
\| V_1 \|_{L^\infty}^2 + \left\| W_1^*/r \right\|_{L^\infty} \leq t^{-1}, \quad \| V_2 \|_{L^2} + \left\| W_2^*/r \right\|_{L^2} \leq (1 + t)^{-1};
\]  

(1.18)

(iii). The scaling function $\sigma(\cdot)$ decays exponentially. More precisely it holds

\[
(1 - \varepsilon) e^{-\frac{\mu^2}{m^2} t} \sigma_{in} \leq \sigma \leq (1 + \varepsilon) e^{-\frac{\mu^2}{m^2} t} \sigma_{in} \quad \text{for all } t \geq 0.
\]  

(1.19)

It is this estimate that gives us the blow-up of $\varphi$ at $t = \infty$.

I.3. SOME REMARKS ON THEOREM 1.1

We would like to point out three remarks on Theorem 1.1.

I. Motivation for the initial vector field $\varphi_{in}$ in (1.13).

If we decouple the fluid part from the system (1.8), then the first equation in (1.8) gives us the following heat flow of harmonic maps:

\[
\partial_t \varphi = \Delta_2 \varphi + |\partial_r \varphi|^2 \varphi + \left( \frac{m^2}{y^2} + \mu^2 \right) \left[ R^2 \varphi + |R\varphi|^2 \varphi \right], \quad \text{in } \mathbb{R}^2 \times (0, \infty).
\]  

(1.20)

In (1.13) we choose $\varphi_{in}$ to be a small perturbation of the harmonic map $h_{\Omega, \sigma_{in}}$ in some norm space. Obviously $h_{\Omega, \sigma_{in}}$ is not a stationary solution to the equation (1.20) if $\mu \neq 0$. However by applying the following change of variables:

\[
\varphi(r, t) = \Phi(y, s), \quad \text{where} \quad y = \frac{r}{\lambda(t)}, \quad s(t) = \int_0^t \lambda^{-2}(\tau) \, d\tau \quad \text{and} \quad \lambda(t) = \mu^{-1} e^{-\mu^2 t/m^2},
\]  

(1.21)

the equation (1.20) can be rewritten as

\[
\partial_s \Phi + \frac{y \partial_y \Phi}{2s + m^2} = \Delta_2 \Phi + |\partial_y \Phi|^2 \Phi + \left( \frac{m^2}{y^2} + \frac{m^2}{2s + m^2} \right) \left[ R^2 \Phi + |R\Phi|^2 \Phi \right].
\]  

(1.22)

Formally if we take $s \to \infty$, then a global solution $\Phi$ of the above equation should asymptotically approach to a solution of the equation

\[
\Delta_2 \Phi + |\partial_y \Phi|^2 \Phi + \frac{m^2}{y^2} \left( R^2 \Phi + |R\Phi|^2 \Phi \right) = 0.
\]  

(1.23)

This observation motivates us the choice of $\varphi_{in}$ in (1.13).

II. Results on the pure harmonic map heat flow.

With slight modifications to the proof of Theorem 1.1, we have the following results for (1.20), the heat flow of harmonic map:
Corollary 1.2. Let \( m \) be an integer satisfying \(|m| \geq 3\). \( \mu > 0 \) is a twist rate. Suppose that (1.12) - (1.13) hold for the initial vector field \( \varphi_m \). For any \( \varepsilon \in (0, 1) \), there exists a positive constant \( \delta_\varepsilon \) (depending on \( m \), \( \mu \) and \( \varepsilon \)) such that if
\[
\|z_m\|_\infty + \sigma_{in} + \left( \int_0^{\infty} |z_m(\rho)|^2 \rho d\rho \right)^{1/2} < \delta_\varepsilon,
\]
then the followings hold for the equation (1.20):

(i). Supplied with the initial data \( \varphi_{in} \), (1.20) admits a classical solution, denoted by \( \varphi \), on the interval \([0, \infty)\).

For some \( C^1\)-regular time dependent parameter functions \((\sigma(t), \Theta(t))\), the vector field \( \varphi \) can be expressed in terms of (1.15). Moreover the functions \( z_1 \) and \( z_2 \) in (1.15) satisfies (1.16) - (1.17):

(ii). The parameter function \( \sigma(\cdot) \) decays to 0 as \( t \to \infty \). The optimal decay rate is given in (1.19). Furthermore there exists a constant \( \Theta_\infty \in \mathbb{R} \) such that \( \Theta(t) \to \Theta_\infty \) as \( t \to \infty \).

For the simplified Ericksen-Leslie equation, we only have algebraic decay for \( V_1, V_2, W^*_1 \) and \( W^*_2 \) in (1.18). It is not enough to show the \( L^1 \)-integrability of \( \Theta' \) on \( \mathbb{R}^+ \). But for the pure heat flow of harmonic maps, we do have \( L^1 \)-integrability of \( \Theta' \) on \( \mathbb{R}^+ \), which gives us the convergence of \( \Theta(\cdot) \) in (ii) of Corollary 1.2.

III. Comparison with some known results.

Given \( m \) an integer satisfying \(|m| \geq 3\) and \( \mu > 0 \) a twist rate, our global solution \((W, V, \varphi)\) obtained by Theorem 1.1 gives a global solution \((u, \phi)\) to (1.1) through the change of variables in (1.2) and (1.7). \( u \) is homogeneous in terms of the variable \( x_3 \), while \( \phi \) is twisted and periodic along the \( x_3 \)-axis with period \( 2\pi/\mu \). Moreover the director field \( \phi \) blows up with an exponential rate at all points on the \( x_3 \)-axis, as \( t \) tends to \( \infty \).

By the representation of \( \varphi \) in (1.15), except at \( r = 0 \) where \( \varphi = -e_3 \), for all \( r > 0 \), \( \varphi(r, t) \) converges to \( e_3 \) as \( t \to \infty \). In other words our solution \( \phi \) escapes into third dimension for large time \( t \). In [4] the authors show that for \( \mu = 0 \), \( V \equiv 0 \) and \( |\omega| \) suitably small, where \( \omega \) is the circulation Reynolds number, the vector field \( \varphi \) should keep rotating around the \( x_3 \)-axis as \( t \to \infty \). However our results in Theorem 1.1 indicate that for \( \mu > 0 \), the associated vector field \( \phi \) should escape into third dimension exponentially. Compared with the blow-up of \( \varphi \) in Theorem 1.1, the oscillating effect from the swirling velocity field \( u \) can be ignored. Even when \( \mu > 0 \) is small, the system (1.8) should not be regarded as a perturbed system of the one with \( \mu = 0 \) (the case studied in [4]).

Interesting reads should also refer to [3] [7], where global solutions for harmonic map heat flow are constructed. The global solutions in [5] [7] are \( m \)-equivariant with \(|m| \geq 3\) and do not blow up. Now we compare our current work with [1] in which the solution obtained for the pure heat flow of harmonic maps also blows up at \( t = \infty \). But the blow-up result in [1] is due to a boundary condition on angular function of orientation variables. A suitably constructed barrier function is utilized in order to prevent the occurrence of bubbles at finite time. In our Theorem 1.1, the decay rate for the parameter function \( \sigma(\cdot) \) is given in (1.19). It is the non-zero twist rate \( \mu \) that makes our solution blow up at \( t = \infty \). The mechanism for our blow up in Theorem 1.1 is quite different from the work [1]. As for the other dynamical systems and some finite time blow up results, we refer readers to [3] [7] [8] [12] and references therein.

I.4. ORGANIZATION OF THE ARTICLE

The article is organized as follows: in Sect.II, we derive equations satisfied by the tangent vector \( q \) and the perturbation functions \( z_1, z_2 \) (see (1.15)). With these equations, in Sect.III, we discuss some fundamental energy estimates and estimates on the modulation parameters \((\sigma(t), \Theta(t))\). The proof of Theorem 1.1 and Corollary 1.2 are given in Sect.IV with a bootstrap argument.
II. EQUATIONS OF VARIABLES.

In this section we derive some equations that will be used in the study of (1.8). Given \((W, V, \varphi)\) a solution of (1.8), the vector field \(\varphi\) induces a covariant derivative

\[
D_\varphi \mathbf{e} = \mathbf{e}_r + \langle \mathbf{e}, \varphi \rangle \varphi, \quad \text{for all } \mathbf{e} \in T_\varphi S^2.
\]

Suppose that \(\mathbf{e}\) is the unique solution of the boundary value problem:

\[
D_\varphi \mathbf{e} = 0, \quad \mathbf{e} \big|_{r=\infty} = \mathbf{e}_2.
\]

Then \(\{\mathbf{e}, \varphi \times \mathbf{e}\}\) forms an orthonormal frame on \(T_\varphi S^2\). Therefore for some coefficient functions \(q_1\) and \(q_2\), it holds

\[
q[\varphi] = \hat{\varphi}_r - \frac{m}{r} \varphi \times \mathbf{R}_\varphi = q_1 \mathbf{e} + q_2 \varphi \times \mathbf{e}. \tag{2.1}
\]

Here and in what follows, we assume \(m \geq 3\) and denote by \(q\) the complexified function \(q_1 + i q_2\). Utilizing similar arguments as in [2, 4, 5], by (1.8) and (2.1), we have the following equations satisfied by the unknown variables \(q, V\) and \(W\):

\[
\begin{align*}
\hat{\varphi}_t q + iS q + iL^*_{m} \left[ \left( \frac{mW}{r^2} + \mu V \right) v \right] &= -L^*_{m} L_{m} q - \mu^2 L^*_{m} \left[ \varphi_3 v \right]; \\
L^*_{m} v &= \varphi_3 q; \\
\hat{\varphi}_r W &= \hat{\varphi}_r W - \frac{1}{r} \hat{\varphi}_r W + m \left( \hat{\varphi}_r + \frac{1}{r} \right) \langle q, iv \rangle; \\
\hat{\varphi}_r V &= \Delta_2 V + \mu \left( \hat{\varphi}_r + \frac{1}{r} \right) \langle q, iv \rangle. \tag{2.2}
\end{align*}
\]

Here \(L_m\) and its adjoint operator \(L^*_m\) on \(L^2\) are given by

\[
L_m = \hat{\varphi}_r + \frac{1}{r} - \frac{m \varphi_3}{r} \quad \text{and} \quad L^*_m = -\hat{\varphi}_r - \frac{m \varphi_3}{r}, \tag{2.3}
\]

respectively. \(v\) in (2.2) is a complex function \(v := v_1 + i v_2\) with \(v_1\) and \(v_2\) defined by

\[
v_1 = -\langle \mathbf{R}_\varphi, \varphi \times \mathbf{e} \rangle = \langle \mathbf{e}_3, \mathbf{e} \rangle \quad \text{and} \quad v_2 = \langle \mathbf{R}_\varphi, \mathbf{e} \rangle = \langle \mathbf{e}_3, \varphi \times \mathbf{e} \rangle, \tag{2.4}
\]

respectively. Moreover \(S\) in (2.2) can be read as follows:

\[
S = \int_{r}^{\infty} \left( L_m q + \mu^2 v \varphi_3 + i v \left( \frac{mW}{r^2} + \mu V \right) \right) d\tau. \tag{2.2}
\]

Without ambiguity, we also use \(\langle \cdot, \cdot \rangle\) to denote the standard inner product on the complex field \(\mathbb{C}\).

Letting \(\Theta\) and \(\sigma\) be two time dependent modulation parameters, we suppose that \(\varphi\) admits a decomposition as shown in (1.14). Here for \(z = z_1 + i z_2\) satisfying \(\|z\|_{L^\infty} \leq 1/2\), \(\gamma\) in (1.15) is given by

\[
\gamma = (1 - |z|^2)^{1/2} - 1. \tag{2.5}
\]

Moreover it holds

\[
|\gamma| \leq |z|^2 \quad \text{and} \quad |\hat{\varphi}_r \gamma| \leq |z| \cdot |\hat{\varphi}_r |. \tag{2.6}
\]

Now we plug (1.15) into (1.8) and obtain the following equation satisfied by \(z\):

\[
\hat{\varphi}_t z + \frac{1}{\sigma^2} N z = \text{Mod} + \text{HT}, \tag{2.7}
\]
where
\[
\text{Mod} := -(1 + \gamma) h_1 + i h_3 z \left( \Theta' + \mu V + \frac{mW}{r^2} \right) + \frac{\sigma'}{\sigma} \left\{ i(1 + \gamma) m h_1 + \rho \partial_\rho z \right\} + \mu^2 \left\{ i(1 + \gamma) h_1 h_3 + i h_1^2 z_2 - h_3^2 z \right\};
\]
\[
\text{HT} := \frac{i}{\sigma^2} \frac{2mh_1}{\rho} \partial_\rho \gamma
\]
\[+ \left( \frac{m^2}{\rho^2 \sigma^2} + \mu^2 \right) z \left\{ \frac{z_1^2}{\epsilon} + (\gamma h_1 - z_2 h_3)^2 + 2h_1(\gamma h_1 - z_2 h_3) \right\}
\[+ \frac{1}{\sigma^2} \left\{ \left( \partial_\rho \gamma - \frac{m h_1}{\rho} z_2 \right)^2 + \left( \partial_\rho \rho + \frac{m h_1}{\rho} \gamma \right)^2 + \frac{2m h_1}{\rho} \left( \partial_\rho \rho + \frac{m h_1}{\rho} \gamma \right) \right\} z.
\]

In (2.3), \( \Theta' \) and \( \sigma' \) represent the time derivatives of \( \Theta \) and \( \sigma \), respectively. The operator \( N \) is defined by
\[
-N := -L_h^* L_h = \partial_{\rho \rho} + \frac{1}{\rho} \partial_\rho + \frac{m^2}{\rho^2} \left( 2h_1^2 - 1 \right), \quad \text{where} \quad L_h = \partial_\rho + \frac{m}{\rho} h_3(\rho).
\]

Here \( L_h^* \) is the adjoint operator of \( L_h \) on \( L^2(\rho d\rho) \).

Before proceeding we give some preliminary results for later use. For the variable \( z \), we equip it with the norm in the space \( X \) (see (1.1)). By Sobolev embedding, we have
\[
z \text{ is continuous on } [0, \infty) \text{ with } z(0) = z(\infty) = 0 \text{ and satisfies } \|z\|_{L^\infty} \lesssim \|z\|_X.
\]

In light of (2.4) and (1.11), it follows
\[
|v|^2 = |R \varphi|^2 \lesssim h_1^2(\rho) + |z(\rho, t)|^2, \quad \text{provided that } \|z\|_{L^\infty} \leq 1/2.
\]

Now we consider the operator \( L_h \) in (2.4). Its kernel space is non-trivial and satisfies \( \text{Ker } L_h = \text{span } \{ h_1 \} \), where the function \( h_1 \) is defined in (1.6). Moreover \( L_h \) satisfies the following coercivity result:

**Lemma 2.3** (Lemma 2.4 in [6]). *Let \( m \) be an integer with \( |m| \geq 3 \). If \( z \in X \) satisfies
\[
\int_0^\infty z \ h_1 \ \rho \ d\rho = 0,
\]
then we have
\[
\|z\|^2_X \lesssim \int_0^\infty \|L_h \ z\|^2 \rho \ d\rho.
\]

Moreover we have the following equivalence of \( \|z\|_X \) and \( \|q\|_{L^2} \):

**Lemma 2.4.** *Suppose that \( z \in X \) and satisfies the orthogonality condition (2.12). \( q \) is the complexified function \( q_1 + iq_2 \), where \( q_1 \) and \( q_2 \) are defined in (2.1). There exists a positive constant \( \epsilon_0 = \epsilon_0(m) \) suitably small so that if \( \|z\|_X < \epsilon_0 \), then it holds
\[
\|q\|_{L^2} \lesssim \|z\|_X \lesssim \|q\|_{L^2}.
\]

The first inequality in Lemma 2.4 can be obtained by (1.15) and (2.1). The second inequality in Lemma 2.4 is due to Proposition 2.3 in [6]. To end this section, we state a coercivity result on the operator \( L_m \) (see (2.3)). We refer to Lemma 4.1 in [4] for the proof.

**Lemma 2.5.** *There exists a positive constant \( \epsilon_0 = \epsilon_0(m) \) suitably small so that if \( \|z\|_X < \epsilon_0 \), then it holds
\[
\int_0^\infty |\partial_\rho q|^2 + \left| \frac{q}{r^2} \right|^2 \lesssim \int_0^\infty \|L_m q\|^2.
\]
III. ESTIMATES OF ENERGY AND MODULATION PARAMETERS.

In this section we study various energy estimates related to a solution \((W, V, q)\) of (2.2).

III.1. FUNDAMENTAL ENERGY ESTIMATES.

Recalling the Oseen part \(W^\text{os}\) in (1.9), we have

\[
\partial_t W^\text{os} = \partial_{rr} W^\text{os} - r^{-1} \partial_r W^\text{os}.
\]

Therefore by the third equation in (2.2), \(W^* := W - W^\text{os}\) satisfies

\[
\partial_t W^* = \partial_{rr} W^* - \frac{1}{r} \partial_r W^* + m \left( \partial_r + \frac{1}{r} \right) \langle q, iv \rangle.
\]

(3.1)

In the following lemma, we give an energy identity related to \(q, v, V\) and \(W^*\).

**Lemma 3.1.** Suppose that \(q, v, V\) and \(W^*\) satisfy (2.2) and (3.1). Then it holds

\[
\frac{1}{2} \frac{d}{dt} E + \int_0^\infty \left| L_m q + \mu^2 v \varphi_3 \right|^2 + \left( \partial_r V \right)^2 + \left( \partial_r W^* \right)^2 + \frac{1}{r^2} \left( \frac{W^*}{r^2} \right) \langle q, iv \rangle = 0.
\]

(3.2)

where

\[
E := E^* + \mu^2 \int_0^\infty \langle v \rangle^2 \quad \text{with} \quad E^* := \int_0^\infty \langle q \rangle^2 + V^2 + \frac{\langle W^* \rangle^2}{r^2}.
\]

**Proof.** Multiplying \(W^*/r\) on both sides of (3.1) and integrating over \((0, \infty)\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty \left( \frac{W^*}{r^2} \right)^2 + \int_0^\infty \left( \partial_r W^* \right)^2 + \frac{1}{r^2} \int_0^\infty \left( \frac{W^*}{r^2} \right) \langle q, iv \rangle = 0.
\]

(3.3)

Multiplying \(rV\) on both sides of the last equation in (2.2) and integrating over \((0, \infty)\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty V^2 + \int_0^\infty \left( \partial_r V \right)^2 + \mu \int_0^\infty \partial_r V \langle q, iv \rangle = 0.
\]

(3.4)

By taking inner product with \(rq\) on both sides of the first equation in (2.2) and integrating over \((0, \infty)\), it follows

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty \langle q \rangle^2 + \int_0^\infty \left| L_m q \right|^2 + \int_0^\infty \left( \partial_r V \right)^2 + \mu \int_0^\infty \langle q, iv \rangle = -\mu^2 \int_0^\infty \langle L_m q, \varphi_3 v \rangle.
\]

(3.5)

Using the definition of \(L_m^*\) in (2.3) and the second equation in (2.2), one can show that

\[
iL_m \left[ \frac{mW}{r^2} + \mu V \right] = -i v \left( \frac{mW}{r^2} + \mu V \right) + \left( \frac{mW}{r^2} + \mu V \right) iL_m v
\]

(3.6)

Applying this equality to (3.5) yields

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty \langle q \rangle^2 + \int_0^\infty \left| L_m q \right|^2 - \int_0^\infty \left( \frac{mW}{r^2} + \mu V \right) \langle q, iv \rangle = -\mu^2 \int_0^\infty \langle L_m q, \varphi_3 v \rangle.
\]

(3.7)

This combined with (3.3)-(3.4) implies the following identity

\[
\frac{1}{2} \frac{d}{dt} E^* + \int_0^\infty \left| L_m q \right|^2 + \left( \partial_r V \right)^2 + \left( \partial_r W^* \right)^2 + \mu^2 \int_0^\infty \langle L_m q, \varphi_3 v \rangle = \int_0^\infty \left( \frac{mW^\text{os}}{r^2} \right) \langle q, iv \rangle.
\]

(3.8)
Lemma 3.2. Suppose that

Applying this estimate to the right-hand side of (3.2), we obtain

Solving this ODE inequality for $E$ gives us

$\nu$ of

Here in the first equality above we have used the fact that

Taking inner product with $r \varphi$ in (2.5), we can rewrite the equality (3.9) as follows:

As a corollary it follows

By using (3.2), the energy $E$ can be uniformly bounded for all $t > 0$. That is

Lemma 3.2. Suppose that $q, v, V$ and $W^*$ satisfy (2.2) and (3.1). Then we have, for all $t > 0$, that

Proof. Using integration by parts and the second equation of (2.2), we have

Multiplying both sides above by $\mu^2$ and summing with (3.8), we deduce (3.2).

By using (3.2), the energy $E$ can be uniformly bounded for all $t > 0$. That is

the estimate in (3.11) then implies

Applying this estimate to the right-hand side of (3.2), we obtain

Solving this ODE inequality for $E$ gives us

which can be applied to the right-hand side of (3.11) and yields the energy inequality (3.10).

As a corollary it follows
Corollary 3.3. With the same assumptions as in Lemma 3.2, for any positive constant $\epsilon_*$, there exists a small positive constant $\delta_*$ such that if (1.14) holds, then for all $t > 0$, we have

\[
\begin{aligned}
E(t) + \int_0^t \int_{\mathbb{R}^2} |L_m q + \mu^2 v \varphi_3|^2 + (\varphi_r V)^2 + \frac{(|\varphi_r W^*|^2)}{r^2} < \epsilon_*^2.
\end{aligned}
\tag{3.14}
\]

Here $\delta_*$ depends on $m$, $\mu$, $\omega$, $r_0$ and $\epsilon_*$. 

**Proof.** In light of the definition of $E$ in Lemma 3.1 and the energy estimate in Lemma 3.2, we only need to bound the $L^2$-norms of $v_{in}$ and $q_{in}$. Here $v_{in}$ and $q_{in}$ are initial functions of $v$ and $q$ at $t = 0$. By (2.11), it holds

\[
\begin{aligned}
\int_0^\infty |v_{in}|^2 \lesssim \sigma^2_{in} \int_0^\infty (h^2 + |z_{in}|^2) \rho d\rho \lesssim \sigma^2_{in} + \sigma^2_{in} \int_0^\infty |z_{in}|^2 \rho d\rho.
\end{aligned}
\tag{3.15}
\]

Applying Lemma 2.4 then yields

\[
\begin{aligned}
\|q_{in}\|_{L^2} \lesssim \|z_{in}\|_{L^2}.
\end{aligned}
\]

Therefore by (1.14), the $L^2$-norms of $v_{in}$ and $q_{in}$ can be small, provided that $\delta_*$ is suitably small. The proof then follows. \qed

### III.2. DECOMPOSITION OF THE VARIABLES $V$ AND $W^*$.

In the remainder of this article, we decompose $V$ into $V_1 + V_2$, where $V_1$ and $V_2$ satisfy respectively the following initial value problems:

\[
\begin{cases}
\partial_t V_1 = \Delta_2 V_1 & \text{in } \mathbb{R}^2 \times (0, \infty); \\
V_1 = V_{in} & \text{at } t = 0;
\end{cases}
\tag{3.16}
\]

and

\[
\begin{cases}
\partial_t V_2 = \Delta_2 V_2 + \mu (\partial_r + r^{-1}) \langle q, iv \rangle & \text{in } \mathbb{R}^2 \times (0, \infty); \\
V_2 = 0 & \text{at } t = 0.
\end{cases}
\tag{3.17}
\]

A standard application of the heat kernel on $\mathbb{R}^2$ implies the following lemma.

**Lemma 3.4.** For all $t > 0$, it holds

\[
\begin{aligned}
\|V_1(\cdot, t)\|_{L^2} \lesssim t^{-1/2} \|V_{in}\|_{L^2}.
\end{aligned}
\]

Now we consider some estimates of $V_2$. In what follows we use $\mathcal{F}_n$ to denote the standard Fourier transformation on $\mathbb{R}^n$. Firstly we estimate the Fourier transform of $V_2$ on $\mathbb{R}^2$.

**Lemma 3.5.** For all $t > 0$ and $\xi \in \mathbb{R}^2$, it holds

\[
\begin{aligned}
|\mathcal{F}_2[V_2](\xi)| \lesssim |\xi| \int_0^t \|q\|_{L^2} \|v\|_{L^2} ds.
\end{aligned}
\]

**Proof.** Taking Fourier transform $\mathcal{F}_2$ on both sides of the equation (3.17), we obtain for all $\xi \in \mathbb{R}^2$ that

\[
\begin{aligned}
(\partial_t + |\xi|^2) \mathcal{F}_2[V_2] = \mu \mathcal{F}_2 \left[(\partial_r + r^{-1}) \langle q, iv \rangle \right].
\end{aligned}
\]

Letting $\nabla^{(2)}$ be the gradient operator on $\mathbb{R}^2$, then we have

\[
\begin{aligned}
(\partial_r + r^{-1}) \langle q, iv \rangle = \nabla^{(2)} \cdot (x r^{-1} \langle q, iv \rangle) \quad \text{for } x \in \mathbb{R}^2.
\end{aligned}
\]
The above two equalities together with Hölder’s inequality imply

$$|\mathcal{F}_2 [V_2] (\xi)| = \left| \mu \int_0^t e^{-i(t-s)\xi^2} \sum_{j=1}^2 \xi_j \mathcal{F}_2 [x_j r^{-1} \langle q, iv \rangle] \, ds \right| \lesssim |\xi| \int_0^t \|x r^{-1} \langle q, iv \rangle\|_{L^1(\mathbb{R}^2)} \, ds \lesssim |\xi| \int_0^t \|q\| r \, dr \, ds \lesssim |\xi| \int_0^t \|q\|_{L^2} \|v\|_{L^2} \, ds.$$ 

This finishes the proof. \( \square \)

Now we give an energy estimate for \( V_2 \) based on Schonbek’s Fourier splitting method (see [13]).

**Lemma 3.6.** For all \( t > 0 \) and \( R_* > 0 \), it holds

$$\frac{d}{dt} \int_0^\infty V_2^2 + R_*^2 \int_0^\infty V_2^2 \lesssim \|z\|_X^2 + R_*^6 \left( \int_0^t \|q\|_{L^2} \|v\|_{L^2} \, ds \right)^2.$$ 

**Proof.** By Plancherel’s theorem, it turns out

$$\int_{\mathbb{R}^2} |\nabla (V_2)|^2 \, dx = \int_{\mathbb{R}^2} |\xi|^2 \left| \mathcal{F}_2 [V_2] \right|^2 \, d\xi \geq R_*^2 \int_{\mathbb{R}^2 \setminus D_{R_*}} \left| \mathcal{F}_2 [V_2] \right|^2 \, d\xi = R_*^2 \int_{\mathbb{R}^2} V_2^2 \, dx - R_*^2 \int_{D_{R_*}} \left| \mathcal{F}_2 [V_2] \right|^2 \, d\xi.$$ 

Here \( D_{R_*} \) is the disk in \( \mathbb{R}^2 \) with radius \( R_* \) and center 0. This estimate then yields

$$\int_0^\infty (\partial_\tau V_2)^2 \geq R_*^2 \int_0^\infty V_2^2 - \frac{R_*^2}{2\pi} \int_{D_{R_*}} \left| \mathcal{F}_2 [V_2] \right|^2 \, d\xi \quad (3.18)$$

since

$$\int_{\mathbb{R}^2} |\nabla (V_2)|^2 \, dx = 2\pi \int_0^\infty (\partial_\tau V_2)^2.$$ 

Multiplying \( V_2 \) on both sides of the equation \( 3.17 \) and integrating over \( \mathbb{R}^2 \), we deduce

$$\frac{1}{2} \frac{d}{dt} \int_0^\infty V_2^2 + \int_0^\infty (\partial_\tau V_2)^2 = -\mu \int_0^\infty \langle q, iv \rangle \partial_\tau V_2,$$

which implies in light of Young’s inequality

$$\frac{d}{dt} \int_0^\infty V_2^2 + \int_0^\infty (\partial_\tau V_2)^2 \leq \mu^2 \int_0^\infty |\langle q, iv \rangle|^2.$$ 

Applying \( 3.18 \) to the above estimate yields

$$\frac{d}{dt} \int_0^\infty V_2^2 + R_*^2 \int_0^\infty V_2^2 \leq \mu^2 \int_0^\infty |\langle q, iv \rangle|^2 + R_*^2 \int_{D_{R_*}} \left| \mathcal{F}_2 [V_2] \right|^2 \, d\xi.$$ 

This combined with Lemma 3.5 shows that

$$\frac{d}{dt} \int_0^\infty V_2^2 + R_*^2 \int_0^\infty V_2^2 \leq \int_0^\infty |\langle q, iv \rangle|^2 + R_*^6 \left( \int_0^t \|q\|_{L^2} \|v\|_{L^2} \, ds \right)^2 \lesssim \int_0^\infty \left( |\partial_\rho z|^2 + \|z\|_X^2 \right) \, d\rho + R_*^6 \left( \int_0^t \|q\|_{L^2} \|v\|_{L^2} \, ds \right)^2,$$

In the second expression above we have used the fact that

$$|\langle q, iv \rangle| \lesssim \sigma^{-1} \left( |\partial_\rho z| + \|z\|_X \right).$$

The proof then follows. \( \square \)
In light of (3.1), it turns out that \( W^*/r^2 \) satisfies
\[
\partial_t \left( \frac{W^*}{r^2} \right) = \Delta_4 \left( \frac{W^*}{r^2} \right) + mr^{-2} \left( \partial_r + r^{-1} \right) \langle q, iv \rangle \quad \text{on } \mathbb{R}^4 \times (0, \infty),
\]
where \( \Delta_4 \) is the Laplace operator on \( \mathbb{R}^4 \). Then we decompose \( W^* \) into \( W^*_1 + W^*_2 \), where \( W^*_1/r^2 \) and \( W^*_2/r^2 \) satisfy respectively the following initial value problems:
\[
\begin{align*}
\partial_t \left( \frac{W^*_1}{r^2} \right) &= \Delta_4 \left( \frac{W^*_1}{r^2} \right) \quad \text{in } \mathbb{R}^4 \times (0, \infty); \\
W^*_1/r^2 &= W^*_{in}/r^2 \quad \text{at } t = 0;
\end{align*}
\]
and
\[
\begin{align*}
\partial_t \left( \frac{W^*_2}{r^2} \right) &= \Delta_4 \left( \frac{W^*_2}{r^2} \right) + mr^{-2} \left( \partial_r + r^{-1} \right) \langle q, iv \rangle \quad \text{in } \mathbb{R}^4 \times (0, \infty); \\
W^*_2/r^2 &= 0 \quad \text{at } t = 0.
\end{align*}
\]
By a standard application of the heat kernel on \( \mathbb{R}^4 \), the following estimate holds for \( W^*_1/r^2 \):

**Lemma 3.7.** For all \( t > 0 \), it holds
\[
\left\| W^*_1/r^2(\cdot, t) \right\|_{L^2} \lesssim t^{-1} \left\| W^*_{in}/r \right\|_{L^2}.
\]

Moreover we also have the following lemma concerning the Fourier transform of \( W^*_2/r^2 \) on \( \mathbb{R}^4 \):

**Lemma 3.8.** For all \( t > 0 \) and \( \eta \in \mathbb{R}^4 \), it holds
\[
\left| \mathcal{F}_4 [ W^*_2/r^2 ] (\eta) \right| \lesssim |\eta| \int_0^t \| q \|_{L^2} \| v \|_{L^2} \, ds.
\]

**Proof.** Taking Fourier transform \( \mathcal{F}_4 \) on both sides of (3.21), we have, for all \( \eta \in \mathbb{R}^4 \), that
\[
\partial_t \mathcal{F}_4 [ W^*_2/r^2 ] + |\eta|^2 \mathcal{F}_4 [ W^*_2/r^2 ] = m \mathcal{F}_4 [ r^{-2} \left( \partial_r + r^{-1} \right) \langle q, iv \rangle ].
\]
For all \( x \in \mathbb{R}^4 \) and \( r = |x| \), it satisfies
\[
r^{-2} \left( \partial_r + r^{-1} \right) \langle q, iv \rangle = \nabla^{(4)} \cdot \left( x r^{-3} \langle q, iv \rangle \right),
\]
where \( \nabla^{(4)} \) denotes the gradient operator on \( \mathbb{R}^4 \). Then the above two equalities yield
\[
\left| \mathcal{F}_4 [ W^*_2/r^2 ] (\eta) \right| = \left| m \int_0^t e^{-\langle t-s \rangle} |\eta|^2 \sum_{j=1}^4 \eta_j \mathcal{F}_4 [ x_j r^{-3} \langle q, iv \rangle ] \, ds \right| \lesssim \left| \eta \right| \int_0^t \| x r^{-3} \langle q, iv \rangle \|_{L^1(\mathbb{R}^4)} \, ds \lesssim \left| \eta \right| \int_0^t \| q \|_{L^2} \| v \|_{L^2} \, dr \, ds.
\]
This finishes the proof by Hölder’s inequality. \( \square \)

Based on Schonbek’s Fourier splitting method, we have the following energy estimate for \( W^*_2/r^2 \):

**Lemma 3.9.** For all \( t > 0 \) and \( R_\ast > 0 \), it holds
\[
\frac{d}{dt} \int_0^\infty \frac{(W^*_2)^2}{r^2} \, ds + R_\ast \int_0^\infty \frac{(W^*_2)^2}{r^2} \, ds \lesssim \int_0^\infty \frac{|q|^2}{r^2} \, ds + R_\ast \left( \int_0^t \| q \|_{L^2} \| v \|_{L^2} \, ds \right)^2.
\]
Lemma 3.10. Suppose that Corollary 3.3 holds with a small positive constant \( \epsilon^* \). Then the above two estimates, together with an use of Lemma 3.8, yield

\[
\text{we have }
\]

Here we have also used the fact that \( L_{R^*} \subset R^4 \) is the ball of radius \( R^* \) and center 0. Denoting by \( \omega_4 \) the surface area of the unit sphere in \( R^4 \), we have

\[
\int_{R^4} \left| \nabla \left( \frac{W^*}{r^2} \right) \right|^2 dx = \omega_4 \int_0^{\infty} \left| \partial_r \left( \frac{W^*}{r^2} \right) \right|^2 r^3 dr = \omega_4 \int_0^{\infty} \left( \partial_r W^* \right)^2 \frac{r}{r^2}.
\] (3.22)

Then the above two estimates, together with an use of Lemma 3.8 yield

\[
- \int_0^{\infty} \partial_r \left( \frac{W^*}{r^2} \right)^2 r^3 dr + R^2 \int_0^{\infty} \left( \frac{W^*}{r^2} \right)^2 \lesssim R^2 \int_{B_0^*} \left| \nabla \left[ \frac{W^*}{r^2} \right] \right|^2 d\eta \lesssim R^8 \left( \int_0^{\infty} \| q \|_{L^2}^2 \| v \|_{L^2}^2 ds \right)^2.
\] (3.23)

Multiplying \( W^* / r^2 \) on both sides of the equation (3.21) and integrating over \( R^4 \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^{\infty} \left( \frac{W^*}{r^2} \right)^2 + \int_0^{\infty} \left| \partial_r \left( \frac{W^*}{r^2} \right) \right|^2 r^3 dr = -m \int_0^{\infty} \partial_r \left( \frac{W^*}{r^2} \right) \langle q, iv \rangle.
\] (3.24)

By Young’s inequality and the fact that \( |v| \leq 1 \), it follows

\[
\frac{d}{dt} \int_0^{\infty} \left( \frac{W^*}{r^2} \right)^2 + \int_0^{\infty} \left| \partial_r \left( \frac{W^*}{r^2} \right) \right|^2 r^3 dr \leq m^2 \int_0^{\infty} \left| \frac{1}{r} \langle q, iv \rangle \right|^2 \leq m^2 \int_0^{\infty} \frac{|q|^2}{r^2}.
\] (3.25)

The proof then follows by adding (3.23) to the estimate (3.25).

III.3. ESTIMATE FOR MODULATION PARAMETERS.

Now we study the modulation parameters \((\sigma, \Theta)\).

Lemma 3.10. Suppose that Corollary 3.3 holds with a small positive constant \( \epsilon^* \). Moreover we assume that \( z \) is orthogonal to \( h_1 \) in the sense of (1.16). Then the following estimates hold for the modulation parameters \((\sigma, \Theta)\):

\[
\| z \| X^1 \Theta \| \leq \| z \| X^1 t^{-1/2} + \| z \| X^1 + \epsilon^* \sigma^{-2} \| z \| X^1 + \epsilon^* \left( \sigma' \sigma + \frac{\mu^2}{m^2} \right) + \epsilon^* \left( \sigma' \sigma + \frac{\mu^2}{m^2} \right) + \epsilon^* \int_0^{\infty} V_2^2 + \frac{(\partial_r W^*)^2}{r^2},
\] (3.26)

and

\[
\frac{\sigma' \sigma + \mu^2}{m^2} \leq \| z \| X^1 + \| z \| X^1 t^{-1/2} + \sigma^{-2} \| z \| X^1 + \epsilon^* \int_0^{\infty} V_2^2 + \frac{(\partial_r W^*)^2}{r^2}.
\] (3.27)

Proof. Taking \( L^2(\rho d\rho)\)-inner product with \( h_1 \) on both sides of (2.7) and using (1.16), we have

\[
\int_0^{\infty} \text{Mod} \cdot h_1 \rho d\rho = -\int_0^{\infty} \text{HT} \cdot h_1 \rho d\rho.
\] (3.28)

Here we have also used the fact that \( L_h[h_1] = 0 \). The real part of the left-hand side above is given by

\[
\text{Re} \int_0^{\infty} \text{Mod} \cdot h_1 \rho d\rho = \frac{\sigma'}{\sigma} \int_0^{\infty} h_1 \partial_\rho z_1 \rho^2 d\rho - \frac{\mu^2}{r^2} \int_0^{\infty} h_1 h_1^2 z_1 \rho d\rho
\]

\[
- \int_0^{\infty} \left( (1 + \gamma) h_1^2 - h_1 h_3 z_2 \right) \left( \Theta' + \mu V + \frac{mW}{r^2} \right) \rho d\rho.
\] (3.29)
The imaginary part of the left-hand side of (3.28) can be read as follows:

\[ \text{Im} \int_0^\infty \text{Mod} \cdot h_1 \rho d\rho = - \int_0^\infty z_1 h_1 h_3 \left[ \Theta' + \mu V + \frac{mW}{r^2} \right] \rho d\rho \\
+ \frac{\sigma'}{\sigma} \int_0^\infty \left[ (1 + \gamma) m h_1^2 + h_1 \rho \partial_\rho z_2 \right] \rho d\rho \\
+ \mu^2 \int_0^\infty \left[ (1 + \gamma) h_1^3 h_3 + h_1^3 z_2 - h_1 h_3^2 z_2 \right] \rho d\rho. \] (3.30)

Firstly we estimate the right-hand side of (3.29). Direct calculations show that

\[ \frac{\sigma'}{\sigma} \int_0^\infty h_1 \partial_\rho z_1 \rho^2 d\rho - \mu^2 \int_0^\infty h_1 h_3^2 z_1 \rho d\rho \\
= \left[ \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right] \int_0^\infty h_1 \partial_\rho z_1 \rho^2 d\rho - \mu^2 \int_0^\infty \left\{ h_3^2 z_1 + \frac{1}{m^2} \rho \partial_\rho z_1 \right\} h_1 \rho d\rho. \]

Then by Hölder’s inequality, we can deduce from the last equality that

\[ \left| \frac{\sigma'}{\sigma} \int_0^\infty h_1 \partial_\rho z_1 \rho^2 d\rho - \mu^2 \int_0^\infty h_1 h_3^2 z_1 \rho d\rho \right| \lesssim \| z_1 \|_X + \| z_1 \|_X \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right|. \] (3.31)

Here we have used the fact that \( \rho h_1 \in L^2(\rho d\rho) \) since \( m \geq 3 \). To estimate the last integral on the right-hand side of (3.29), we split it into the sum I.1 + I.2 + I.3 + I.4, where

\[ \text{I.1} = -\Theta' \int_0^\infty \left[ (1 + \gamma) h_1^2 - h_1 h_3 z_2 \right] \rho d\rho, \quad \text{I.2} = -\int_0^\infty \frac{mW^{\text{os}}}{r^2} \left[ (1 + \gamma) h_1^2 - h_1 h_3 z_2 \right] \rho d\rho, \]

\[ \text{I.3} = -\int_0^\infty \frac{mW^*}{r^2} \left[ (1 + \gamma) h_1^2 - h_1 h_3 z_2 \right] \rho d\rho, \quad \text{I.4} = -\int_0^\infty \mu V \left[ (1 + \gamma) h_1^2 - h_1 h_3 z_2 \right] \rho d\rho. \]

By Lemma 2.3 and Corollary 3.3, it follows that

\[ \| z \|_{L^\infty} \lesssim \| z \|_X \lesssim \| q \|_{L^2} < \epsilon^* \]. \] (3.32)

Hence we obtain

\[ | \Theta' | \lesssim | \text{I.1} | \], \] (3.33)

provided that a positive constant \( \epsilon^* \) is suitably small. From (3.12), it turns out

\[ | \text{I.2} | \lesssim t^{-1} \int_0^\infty h_1 \rho d\rho \lesssim t^{-1}. \] (3.34)

In light of the following estimate

\[ \left\| \frac{W^*}{r} \right\|_{L^\infty}^2 \lesssim \left\| \frac{W^*}{r} \right\|_X^2 = \int_0^\infty \left\| \partial_r \left( \frac{W^*}{r} \right) \right\|^2 + \left\| \frac{W^*}{r^2} \right\|^2 = \int_0^\infty \left( \frac{\partial_r W^*}{r^2} \right)^2, \] (3.35)

it can be shown that

\[ | \text{I.3} | \lesssim | \text{I.2} | \left\| \frac{W^*}{r} \right\|_{L^\infty} \int_0^\infty h_1 d\rho \lesssim \sigma^{-1} \left( \int_0^\infty \left( \frac{\partial_r W^*}{r^2} \right)^2 \right)^{1/2}. \] (3.36)

As for the integral I.4, it is bounded by

\[ | \text{I.4} | \lesssim \| V_1 \|_{L^\infty} \int_0^\infty h_1 \rho d\rho + \int_0^\infty | V_2 | h_1 \rho d\rho \lesssim \| V_1 \|_{L^\infty} + \sigma^{-1} \left( \int_0^\infty V_2^2 \right)^{1/2}. \] (3.37)
Applying the estimates (3.31) and (3.33)-(3.37) to (3.28)-(3.29), we get
\[ |\Theta'| \leq l^{-1} + \|V_1\|_{L^\infty} + \|z_1\|_X + \|z_1\|_X \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \]
\[ + \sigma^{-1} \left( \int_0^\infty V_2^2 \right)^{1/2} + \sigma^{-1} \left( \int_0^\infty \frac{\left( \partial_r \hat{W}^\ast \right)^2}{r^2} \right)^{1/2} + \left| \int_0^\infty HT \cdot h_1 \rho \, d\rho \right|. \] (3.38)

By the definition of HT in (2.19) and (2.20), it satisfies
\[ |HT| \leq |z|^2 + \sigma^{-2} \left( \left| \overline{\partial}_r \rho z \right|^2 + \left| \frac{z}{\rho^2} \right|^2 \right), \] (3.39)

which combined with (2.10) implies
\[ \left| \int_0^\infty HT \cdot h_1 \rho \, d\rho \right| \leq \int_0^\infty |z|^2 h_1 \rho \, d\rho + \sigma^{-2} \int_0^\infty \left( \left| \overline{\partial}_r \rho z \right|^2 + \left| \frac{z}{\rho^2} \right|^2 \right) h_1 \rho \, d\rho \]
\[ \leq \|z\|^2_X + \left( z \right)^2_X. \] (3.40)

Applying this estimate to (3.38) and using Lemma 3.1, we get
\[ |\Theta'| \leq l^{-1} + t^{-1/2} + \|z\|_X + \|z\|_X \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \]
\[ + \sigma^{-1} \left( \int_0^\infty V_2^2 \right)^{1/2} + \sigma^{-1} \left( \int_0^\infty \frac{\left( \partial_r \hat{W}^\ast \right)^2}{r^2} \right)^{1/2} + \sigma^{-2} \left| \int_0^\infty \rho \overline{\partial}_r \rho \, d\rho \right|. \]

By multiplying \|z\|_X on both sides above and utilizing Hölder’s inequality, the estimate (3.26) follows. Here we also used (3.32).

Next we establish (3.27) by estimating each term on the right-hand side of (3.30). In light of (2.10), it can be shown that
\[ \left| \Theta' \right| \int_0^\infty h_1 \rho \, d\rho \right| \leq \|z_1\|_{L^\infty} \left| \Theta' \right| \int_0^\infty h_1 \rho \, d\rho \leq \|z_1\|_X \left| \Theta' \right|. \] (3.41)

Utilizing (2.10), (3.31) and (3.37), by similar arguments as for (3.31), (3.33) and (3.37), we have
\[ \left| \int_0^\infty mW_{r2} \h_1 \rho \, d\rho \right| \leq \|z_1\|_{L^\infty} \left| \int_0^\infty mW_{r2} \h_1 \rho \, d\rho \right| \leq l^{-1} \|z_1\|_X, \] (3.42)

and
\[ \left| \int_0^\infty mW_{r2} \h_1 \rho \, d\rho \right| \leq \|z_1\|_X \|V_1\|_{L^\infty} + \|z_1\|_X \|V_1\|_{L^\infty} \left( \int_0^\infty V_2^2 \right)^{1/2}. \] (3.43)

Thus by (3.41)-(3.44), the first integral on the right-hand side of (3.30) can be estimated as follows:
\[ \left| \int_0^\infty \h_1 \rho \, d\rho \right| \leq \|z_1\|_X \left| \Theta' \right| + l^{-1} \|z_1\|_X + \|z_1\|_X \|V_1\|_{L^\infty} \]
\[ + \sigma^{-1} \|z_1\|_X \left( \int_0^\infty \frac{\left( \partial_r \hat{W}^\ast \right)^2}{r^2} \right)^{1/2} + \|z_1\|_X \sigma^{-1} \left( \int_0^\infty V_2^2 \right)^{1/2}. \] (3.45)

By using the following equality
\[ \int_0^\infty \h_1 \rho \, d\rho = m^{-1} \int_0^\infty \h_1 \rho \, d\rho, \]

we have

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the last two integrals on the right-hand side of (3.30) can be rewritten as

\[
\left[\frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2}\right] \int_0^\infty \left\{ (1 + \gamma)m h_1^2 + h_1 \rho \partial_\rho z_2 \right\} \rho \, d\rho \\
+ \mu^2 \int_0^\infty \left\{ \gamma h_1^2 h_3 + h_1^2 z_2 - h_1 h_3^2 z_2 \right\} \rho \, d\rho - \frac{\mu^2}{m^2} \int_0^\infty \left\{ \gamma m h_1^2 + h_1 \rho \partial_\rho z_2 \right\} \rho \, d\rho.
\]

(3.46)

In light of the smallness of \( \|z\|_X \) in (3.32), it holds

\[
\left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \lesssim \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \int_0^\infty \left\{ (1 + \gamma)m h_1^2 + h_1 \rho \partial_\rho z_2 \right\} \rho \, d\rho.
\]

(3.47)

By Hölder’s inequality and the fact that \( \rho h_1 \in L^2(\rho \, d\rho) \), the second line in (3.46) can be bounded from above by \( \|z\|_X \) up to a constant depending on \( m \) and \( \mu \). This estimate combined with (3.28), (3.30) and (3.45)-(3.47) then yields

\[
\left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \lesssim \|z\|_X \Theta' + \|z\|_X \left[ V_1 \right]_{L^\infty} + \|z\|_X \\
+ \|z\|_X \sigma^{-1} \left( \int_0^\infty V_2^2 \right)^{1/2} + \sigma^{-1} \|z\|_X \left( \int_0^\infty \left( \Theta \, W^* \right)^2 \right)^{1/2} + \left| \int_0^\infty HT \cdot h_1 \rho \, d\rho \right|.
\]

Applying (3.26) and (3.40) to the above estimate, by Lemma 3.1 and Hölder’s inequality, we deduce (3.27).

Here we also have used the smallness of \( \|z\|_X \) in (3.32).

\[ \square \]

III.4. \( L^2 \)-ESTIMATE OF THE VARIABLE \( z \).

In the next lemma we derive an energy estimate for the variable \( z \).

**Lemma 3.11.** With the same assumption as in Lemma 3.10, we have

\[
\frac{d}{dt} \left[ \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho \right] + \mu^2 \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho + c_\ast \|z\|_X^2 \\
\lesssim \sigma^2 \|z\|_X + \sigma^2 \|z\|_X t^{-1/2} + \sigma^2 \varepsilon^{-1} \int_0^\infty V_2^2 + \left( \partial_\rho W^* \right)^2.
\]

Here \( c_\ast \) is a positive constant.

**Proof.** Taking \( L^2(\rho \, d\rho) \)-inner product with \( z \) on both sides of (2.7), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty |z|^2 \rho \, d\rho + \sigma^{-2} \int_0^\infty |L_h z|^2 \rho \, d\rho = \int_0^\infty \langle \text{Mod}, z \rangle \rho \, d\rho + \int_0^\infty \langle HT, z \rangle \rho \, d\rho.
\]

(3.48)

Through integration by parts, the first integral on the right-hand side above can be rewritten as

\[
\int_0^\infty \langle \text{Mod}, z \rangle \rho \, d\rho = - \left[ \frac{\sigma'}{\sigma} + \mu^2 \right] \int_0^\infty |z|^2 \rho \, d\rho - \Theta' \int_0^\infty \gamma z_1 h_1 \rho \, d\rho \\
- \int_0^\infty \frac{m W^*}{r^2} (1 + \gamma) z_1 h_1 \rho \, d\rho - \int_0^\infty \frac{m W^*}{r^2} (1 + \gamma) z_1 h_1 \rho \, d\rho - \int_0^\infty \mu V (1 + \gamma) z_1 h_1 \rho \, d\rho \\
+ m \left[ \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right] \int_0^\infty \gamma z_2 h_1 \rho \, d\rho + \mu^2 \int_0^\infty \langle z, i(1 + \gamma) h_1 h_3 + i h_1^2 z_2 + h_1^2 z - i m^{-1} \gamma h_1 \rangle \rho \, d\rho.
\]
In light of (3.12), the second integral on the right-hand side of (3.49) can be estimated by
\[ \frac{1}{2} \frac{d}{dt} \left[ \int_{0}^{\infty} |z|^2 \rho d\rho \right] + \mu^2 \sigma^2 \int_{0}^{\infty} |z|^2 \rho d\rho + \int_{0}^{\infty} |L_n z|^2 \rho d\rho = \]
\[ - \sigma^2 \Theta' \int_{0}^{\infty} \gamma_{1} h_1 \rho d\rho - \sigma^2 \int_{0}^{\infty} \frac{m W^{\text{os}}}{r^2} (1 + \gamma) z_{1} h_1 \rho d\rho - \sigma^2 \int_{0}^{\infty} \frac{m W^{*}}{r^2} (1 + \gamma) z_{1} h_1 \rho d\rho \]
\[ - \sigma^2 \int_{0}^{\infty} \mu V (1 + \gamma) z_{1} h_1 \rho d\rho + m \sigma^2 \left[ \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right] \int_{0}^{\infty} \gamma z_{2} h_1 \rho d\rho \]
\[ + \mu^2 \sigma^2 \int_{0}^{\infty} \langle z, i(1 + \gamma) h_1 h_3 + i h_1^2 z_2 + h_1^2 z - im^{-1} \gamma h_1 \rangle \rho d\rho + \sigma^2 \int_{0}^{\infty} \langle HT, z \rangle \rho d\rho. \]
(3.49)

Now we estimate each term on the right-hand side above. Using (3.50) and (2.10), we have
\[ \int_{0}^{\infty} \gamma_{1} h_1 \rho d\rho \leq \|z\|^3_{X}. \]
(3.50)

It then follows
\[ \sigma^2 \Theta' \int_{0}^{\infty} \gamma_{1} h_1 \rho d\rho \leq \sigma^2 \|z\|^3_{X} |\Theta'|. \]
(3.51)

In light of (3.12), the second integral on the right-hand side of (3.49) can be estimated by
\[ \sigma^2 \int_{0}^{\infty} \frac{m W^{\text{os}}}{r^2} (1 + \gamma) z_{1} h_1 \rho d\rho \leq \sigma^2 \|z\|_{X}. \]
(3.52)

Using similar estimates as in (3.36)–(3.37), we obtain
\[ \sigma^2 \int_{0}^{\infty} \frac{m W^{*}}{r^2} (1 + \gamma) z_{1} h_1 \rho d\rho \leq \sigma \|z\|_{X} \left( \int_{0}^{\infty} \frac{(\partial_{z} W^{*})^2}{r^2} \right)^{1/2} \]
\[ \leq \epsilon_{*} \|z\|^3_{X} + \sigma^2 \epsilon_{*}^{-1} \int_{0}^{\infty} \frac{(\partial_{z} W^{*})^2}{r^2}, \]
(3.53)

and
\[ \sigma^2 \int_{0}^{\infty} \mu V (1 + \gamma) z_{1} h_1 \rho d\rho \leq \sigma^2 \|z\|_X \|V_1\|_{L^\infty} + \sigma \|z\|_X \left( \int_{0}^{\infty} V_2^2 \right)^{1/2} \]
\[ \leq \epsilon_{*} \|z\|^3_{X} + \sigma^2 \|z\|_X \|V_1\|_{L^\infty} + \sigma^2 \epsilon_{*}^{-1} \int_{0}^{\infty} V_2^2. \]
(3.54)

We are left to study the last three integrals on the right-hand side of (3.49). Firstly by a similar argument as for the estimate (3.50), we have
\[ \int_{0}^{\infty} \gamma z_{2} h_1 \rho d\rho \leq \sigma ^2 \|z\|^3_{X} |\sigma' + \frac{\mu^2}{m^2}|. \]
(3.55)

Moreover by (2.10), (3.39) and (3.32), it holds
\[ \mu^2 \sigma^2 \int_{0}^{\infty} \langle z, i(1 + \gamma) h_1 h_3 + i h_1^2 z_2 + h_1^2 z - im^{-1} \gamma h_1 \rangle \rho d\rho \leq \sigma^2 \|z\|_{X}, \]
(3.56)

and
\[ \sigma^2 \int_{0}^{\infty} \langle HT, z \rangle \rho d\rho \leq \sigma^2 \int_{0}^{\infty} |z|^3 \rho d\rho + \|z\|_{L^\infty} \int_{0}^{\infty} \left( |\partial_{\rho} z|^2 + \frac{|z|^2}{\rho^2} \right) \rho d\rho \]
\[ \leq \epsilon_{*} \sigma^2 \int_{0}^{\infty} |z|^2 \rho d\rho + \epsilon_{*} \|z\|^3_{X}. \]
(3.57)
By applying (3.51)-(3.57) to the right-hand side of (3.49) and employing the coercivity of $L$, operator given in Lemma 2.3, it turns out
\[
\frac{1}{2} \frac{d}{dt} \left[ \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho \right] + \frac{1}{2} \mu^2 \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho + c_1 \|z\|_X^2 \\
\leq \sigma^2 \|z\|_X^2 |\Theta'| + \sigma^2 \|z\|_X^2 \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| + \sigma^2 \|z\|_X \\
+ \sigma^2 \|z\|_X \left\{ t^{-1} + \|V_1\|_{L^\infty} \right\} + \sigma^2 \varepsilon_1^{-1} \int_0^\infty V_2^2 + \frac{(\partial_\rho W^*)^2}{r^2},
\]
provided that a positive constant $\varepsilon$ is suitably small. Here $c_1$ is a positive constant. By Lemma 3.4 Lemma 3.10 and the above estimate, the proof then follows. Here we also have used the smallness of $\|z\|_X$ in (3.32). \[\square\]

IV. PROOF OF THE MAIN THEOREM.

This section is devoted to the proof of our main results.

**Proof of Theorem 1.1** The local existence of (1.8) with initial data satisfying (1.12)-(1.14) can be obtained by methods in [3]-[4]. We omit it here. Moreover for the local solution obtained, denoted by $\phi$, the assumption (A.1) holds for some $T$ is an arbitrary constant.

For some $C^1$-regular parameter functions $\sigma(\cdot)$ and $\Theta(\cdot)$, (1.15) holds for $\varphi$ with the perturbation function $\rho$ satisfying (1.16). In addition we can also assume

(A.1). $$(1 - \varepsilon/2) e^{-\frac{\mu^2}{m^2} t} \sigma_{in} \leq \sigma \leq (1 + \varepsilon/2) e^{-\frac{\mu^2}{m^2} t} \sigma_{in}, \quad \forall t \in [0, T];$$

(A.2). $$\int_0^\infty V_2^2 \leq \varepsilon^{3/2} \left( \frac{1}{1 + t} \right)^2, \quad \forall t \in [0, T].$$

Here $\varepsilon \in (0, 1)$ is an arbitrary constant. $\epsilon$ is given in Corollary 3.3. In fact by the continuity of the parameter function $\sigma(\cdot)$, the assumption (A.1) holds for some $T > 0$. Utilizing Corollary 3.3 and the estimate given below:

$$\|V_1\|_{L^2} \leq \|V_{in}\|_{L^2} < \delta < \epsilon_*,$$

we get (A.2) for some $T > 0$. One should notice that the last estimate holds since $V_1$ satisfies the heat equation (3.10). Now we separate the following arguments into four steps.

**Step 1.** Multiplying $\exp \left\{ \frac{\mu^2}{m^2} t \right\}$ on both sides of the estimate in Lemma 3.11 and utilizing (A.1)-(A.2), we obtain

\[
\frac{d}{dt} \left[ \exp \left\{ \frac{\mu^2}{m^2} t \right\} \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho \right] + \epsilon \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho + c_* \exp \left\{ \frac{\mu^2}{m^2} t \right\} \|z\|_X^2 \leq \epsilon \sigma^2 \int_0^\infty (\partial_\rho W^*)^2 \left( \frac{1}{r^2} \right)
\]

Here we also have used the smallness of $\|z\|_X$ in (3.32). Fixing $t \in [0, T]$ and integrating the above estimate from 0 to $t$, by Corollary 3.3, we have

$$\exp \left\{ \frac{\mu^2}{m^2} t \right\} \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho + c_* \int_0^t \exp \left\{ \frac{\mu^2}{m^2} s \right\} \|z\|_X^2 ds \leq \epsilon \sigma^2 \int_0^\infty (\partial_\rho W^*)^2 \left( \frac{1}{r^2} \right)$$

(4.1)
In the next we multiply \( \exp \left\{ \frac{2 \mu^2}{m^2} t \right\} \) on both sides of the estimate in Lemma 3.11 and employ (A.1)-(A.2). It then follows

\[
\frac{d}{dt} \left[ \exp \left\{ \frac{2 \mu^2}{m^2} t \right\} \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho \right] + c_e \exp \left\{ \frac{2 \mu^2}{m^2} t \right\} \|z\|_X^2
\]

\[
\leq \sigma_{in}^2 \|z\|_X + \sigma_{in}^2 \|z\|_X t^{-1/2} + \sigma_{in}^2 \epsilon_* \frac{1}{2} \left( 1 + t^{-2} \right) + \sigma_{in}^2 \epsilon_*^{-1} \int_0^\infty \frac{\left( \partial_t W^* \right)^2}{t^3}.
\]

Here we have used the assumption \( m \geq 3 \) so that \( m^2 \geq 2 \). Integrating the above estimate from 0 to \( t \) and using Corollary 3.33 we obtain

\[
\exp \left\{ \frac{2 \mu^2}{m^2} t \right\} \sigma^2 \int_0^t |z|^2 \rho \, d\rho + c_e \int_0^t \exp \left\{ \frac{2 \mu^2}{m^2} s \right\} \|z\|_X^2 \, ds
\]

\[
\leq \sigma_{in}^2 \epsilon_* \sigma_{in}^2 \int_0^t |z_{in}(\rho)|^2 \rho \, d\rho + \sigma_{in}^2 \int_0^t \|z\|_X \, ds + \sigma_{in}^2 \int_0^t \|z\|_X s^{-1/2} \, ds.
\]

By Hölder’s inequality and (4.1), it holds

\[
\int_0^t \|z\|_X \, ds = \left( \int_0^t \exp \left\{ \frac{-\mu^2}{2m^2} s \right\} \exp \left\{ \frac{\mu^2}{2m^2} s \right\} \|z\|_X \, ds \right)^{1/2} \left( \int_0^t \exp \left\{ \frac{-\mu^2}{m^2} s \right\} \, ds \right)^{1/2}
\]

\[
\leq \left( \sigma_{in}^2 \epsilon_* \int_0^t |z_{in}(\rho)|^2 \rho \, d\rho \right)^{1/2}.
\]

Moreover in light of (3.32), the above estimate implies that

\[
\int_0^t \|z\|_X s^{-1/2} \, ds \leq \begin{cases} 
\epsilon_* \text{,} & \text{if } t \leq 1; \\
\epsilon_* + \int_1^t \|z\|_X \, ds \leq \epsilon_* + \left( \sigma_{in}^2 \epsilon_* \sigma_{in}^2 \int_0^t |z_{in}(\rho)|^2 \rho \, d\rho \right)^{1/2}, & \text{if } t > 1.
\end{cases}
\]

Applying (4.3) to (4.2) then yields

\[
\exp \left\{ \frac{2 \mu^2}{m^2} t \right\} \sigma^2 \int_0^\infty |z|^2 \rho \, d\rho + c_e \int_0^t \exp \left\{ \frac{2 \mu^2}{m^2} s \right\} \|z\|_X^2 \, ds
\]

\[
\leq \sigma_{in}^2 \epsilon_* \left[ \epsilon_* + \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho + \left( \sigma_{in}^2 \epsilon_* \sigma_{in}^2 \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho \right)^{1/2} \right].
\]

It then follows from the assumption (A.1) that

\[
\int_0^t \sigma^{-2} \|z\|_X^2 \, ds \leq \epsilon_* \left[ \epsilon_* + \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho + \left( \sigma_{in}^2 \epsilon_* \sigma_{in}^2 \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho \right)^{1/2} \right].
\]

**Step 2.** Integrating the estimate (3.27) from 0 to \( t \), by (3.3) - (1.10), Corollary 3.33 and the assumption (A.2), we obtain

\[
\int_0^t \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \, ds \leq \epsilon_* \left[ \epsilon_* + \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho + \left( \sigma_{in}^2 \epsilon_* \sigma_{in}^2 \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho \right)^{1/2} \right].
\]

Since it holds

\[
\left| \ln \left( \frac{\sigma}{\sigma_{in} \epsilon_{im}^2} \right) \right| = \left| \ln \sigma + \frac{\mu^2}{m^2} t - \ln \sigma_{in} \right| \leq \int_0^t \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \, ds,
\]
we then have, from the above two estimates, that
\[
(1 - \varepsilon/4) e^{-\frac{m^2}{6m^2} t} \sigma \leq \sigma \leq (1 + \varepsilon/4) e^{-\frac{m^2}{2m^2} t} \sigma, \quad \forall t \in [0, T].
\] (4.8)

Here we need \( \epsilon_a \) and \( \delta_a \) in (1.14) to be sufficiently small. The smallness depends on \( m, \mu, \omega, r_0 \) and \( \varepsilon \). The assumption (A.1) is then improved.

**Step 3.** Now we improve (A.2). By Lemma 3.6 and Corollary 3.3 it holds
\[
\frac{d}{dt} \int_0^\infty V_2^2 + R_2^2 \int_0^\infty V_2^2 \lesssim \|z\|_X^2 + R_2^6 \epsilon_a^2 \left( \int_0^t \|v\|_{L^2} \, ds \right)^2, \quad \forall t \in [0, T] \text{ and } R_2 > 0.
\] (4.9)

In light of (3.15), (A.1) and (4.5), the \( L^2 \)-norm of \( \mathcal{L} \) satisfies
\[
\int_0^\infty |v|^2 \lesssim \sigma^2 + \sigma^2 \int_0^\infty |\rho| \, d\rho \lesssim \sigma_\infty^2 \exp \left\{ \frac{2\mu^2 t}{m^2} \right\}, \quad \forall t \in [0, T].
\] (4.10)

This estimate then implies
\[
\int_0^t \|v\|_{L^2} \, ds \lesssim \sigma_\infty \exp \left\{ \frac{\mu^2 t}{m^2} \right\} \lesssim \sigma_\infty.
\] (4.11)

Plugging this estimate into the right-hand side of (4.9), we get
\[
\frac{d}{dt} \int_0^\infty V_2^2 + \frac{1}{t} \int_0^\infty V_2^2 \lesssim \|z\|_X^2 + R_2^6 \epsilon_\infty^2 \sigma_\infty^2.
\]

Multiplying \( (1 + t)^3 \) on both sides above and integrating from 0 to \( t \), we have, with an use of (4.1), that
\[
(1 + t)^3 \int_0^\infty V_2^2 \lesssim \epsilon_\infty^2 \sigma_\infty^2 \int_0^t \|z\|_X^2 (1 + s)^3 \, ds
\]
\[
\lesssim \epsilon_\infty^2 \sigma_\infty^2 t + \epsilon_\infty^{1/2} \sigma_\infty^2 + \sigma_\infty^2 \int_0^\infty |z_\infty(\rho)| \, d\rho.
\]

One should notice that as defined in (3.17), the initial value of \( V_2 \) equals to 0. This fact is also applied in the last estimate. Now we choose \( \epsilon_a \) and \( \delta_a \) in (1.14) to be sufficiently small. The smallness depends on \( m, \mu, \omega, r_0 \) and \( \varepsilon \). The above estimate immediately implies
\[
\int_0^\infty V_2^2 \lesssim \frac{\epsilon_\infty^2}{(1 + t)^2}, \quad \forall t \in [0, T].
\] (4.12)

This improves the assumption (A.2). By the above arguments, we can extend the local solution \( (W, V, \varphi) \) globally in time. Furthermore by taking \( t \to \infty \), the estimate (1.14) follows from (1.10).

**Step 4.** In this last step we give a time decay estimate for the \( L^2 \)-norm of \( W_2 \). Let
\[
E_2 := \int_0^\infty \|q\|^2 + V_2^2 + \frac{(W_2)^2}{r^2}.
\]

Adding the equalities (3.17), (3.19) and (3.24), by (3.22), we have
\[
\frac{1}{2} \frac{d}{dt} E_2 + \int_0^\infty \left| L_m q \right|^2 + (\partial_r V_2)^2 + \frac{(\partial_r W_2)^2}{r^2} = -\mu^2 \int_0^\infty \langle L_m q, \varphi_3 v \rangle
\]
\[
- \int_0^\infty \left[ \frac{m W_{\text{ros}}}{r^2} + \frac{m V_1}{r^2} + \mu V_1 \right] \langle L_m q, iv \rangle.
\] (4.13)
Here we have used integration by parts to obtain the last integral above (refer to (3.6) and (3.11)). In light of the bound \(3.12\) and Lemmas \(3.3, 7\) for all \(t > 0\), it holds

\[
\left\| \frac{m W^{\infty}}{r^2} + \frac{m W_1^*}{r^2} + \mu V_1 \right\|_{L^\infty} \leq t^{-1/2} \| V_{in} \|_{L^2} + t^{-1} \left[ 1 + \left\| \frac{W_1^*}{r} \right\|_{L^2} \right].
\]

Applying this estimate and Young’s inequality to the right-hand side of (4.13), we obtain

\[
\frac{d}{dt} E_2^* + \int_0^\infty | L_m q |^2 + (\partial_r W_2^*)^2 + \beta \frac{(\partial_r W_2^*)^2}{r^2} \leq (1 + t^{-2}) \int_0^\infty | v |^2.
\]  

(4.14)

By \(3.32\), the coercivity estimate in Lemma \(2.5\) holds. Thus Lemmas \(3.9, 2.5\) assert that

\[
d \int_0^\infty (\frac{W_2^*}{r^2})^2 + R_2^* \int_0^\infty (\frac{W_2^*}{r^2})^2 \leq \int_0^\infty | L_m q |^2 + R_2^* \left( \int_0^t \| q \|_{L^2} \| v \|_{L^2} \right)^2, \quad \text{for all } t > 0 \text{ and } R_2^* > 0.
\]

Multiplying a small positive constant \(c_1\) on both sides above and summing the resulting estimate with the estimate (4.14), we get

\[
(1 + c_1) \frac{d}{dt} \int_0^\infty (\frac{W_2^*}{r^2})^2 + c_1 R_2^* \int_0^\infty (\frac{W_2^*}{r^2})^2 + \frac{d}{dt} \int_0^\infty | q |^2 + V_2^2
\]

\[
\leq (1 + t^{-2}) \int_0^\infty | v |^2 + R_2^* \left( \int_0^t \| q \|_{L^2} \| v \|_{L^2} \right)^2.
\]

By (4.10)–(4.11) and Corollary \(3.3\) this estimate implies

\[
d \int_0^\infty (\frac{W_2^*}{r^2})^2 + c_2 R_2^* \int_0^\infty (\frac{W_2^*}{r^2})^2 + c_3 \frac{d}{dt} \int_0^\infty | q |^2 + V_2^2 \leq \sigma_{m}^2 (t + t^3) \exp \left\{ \frac{2 \mu^2}{m^2} t \right\} + R_2^* \sigma_{m}^2 \sigma_{m}^2,
\]

where \(c_2\) and \(c_3\) are two positive constants. Now we take \(c_2 R_2^* = 3 t^{-1}\) and multiply \(t^{3}\) on both sides above. It then turns out

\[
d \int_0^\infty (\frac{W_2^*}{r^2})^2 + c_3 t^{3} \frac{d}{dt} \int_0^\infty | q |^2 + V_2^2 \leq \sigma_{m}^2 (t + t^3) \exp \left\{ \frac{2 \mu^2}{m^2} t \right\} + t^{-1} c_2^2 \sigma_{m}^2.
\]

By integrating the above estimate from \(1\) to \(t\), it follows, for all \(t > 1\), that

\[
t^3 \int_0^\infty (\frac{W_2^*}{r^2})^2 + c_3 \int_1^t s^3 \left( \frac{d}{ds} \int_0^s | q |^2 + V_2^2 \right) ds \lesssim E_2^* (1) + 1 + t.
\]  

(4.15)

For the second integral on the left-hand side above, we have, through integration by parts, that

\[
\int_1^t s^3 \left( \frac{d}{ds} \int_0^s | q |^2 + V_2^2 \right) ds \geq t^3 \int_0^\infty | q |^2 + V_2^2 - E_2^* (1) - 3 \int_1^t s^2 \left( \int_0^s | q |^2 + V_2^2 \right) ds
\]

\[
\geq - E_2^* (1) - \int_1^t s^2 \| q \|_{L^2}^2 ds - \int_1^t s^2 \| V_2 \|_{L^2}^2 ds, \quad \forall \ t > 1.
\]

Since (A.2) holds for all \(t > 0\), this fact together with (4.17), Lemma \(2.3\) and the above estimate, gives us

\[
\int_1^t s^3 \left( \frac{d}{ds} \int_0^s | q |^2 + V_2^2 \right) ds \geq - \left[ E_2^* (1) + 1 + t \right].
\]

Plugging this estimate into (4.15) then yields

\[
t^3 \int_0^\infty (\frac{W_2^*}{r^2})^2 \lesssim E_2^* (1) + 1 + t \quad \text{for all } t > 1.
\]  

(4.16)
Since $V_1$ and $W_1^*$ solve (3.16) and (3.20), respectively, it satisfies
\[
\int_0^\infty V_1^2 \leq \int_0^\infty V_{in}^2 \text{ and } \int_0^\infty \left( \frac{W_1^*}{r^2} \right)^2 \leq \int_0^\infty \left( \frac{W_{in}}{r^2} \right)^2.
\]
These estimates combined with Lemma 3.2 show that
\[
E_2^*(t) \lesssim E^*(t) + \int_0^\infty \left( \frac{W_1^*}{r^2} \right)^2 \lesssim E(0) \text{ for all } t > 0.
\]

Applying this estimate to (4.16) yields the last estimate in (1.18). The proof is finished.

In the end we prove Corollary 1.2.

Proof of Corollary 1.2 By the proof of Theorem 1.1 it remains to obtain the convergence of the rotation parameter $\Theta$ as $t \to \infty$. Using the same derivations as for (3.38) and employing (3.40), we have
\[
|\Theta'| \lesssim \|\tau\|_X + \|z\|_X \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| + |\sigma^{-2} z\|_X^2.
\]
Here one just needs to take $V \equiv 0$, $W^* \equiv 0$ and $\omega = 0$ in the proof of (3.38). Integrating the above estimate from 0 to $\infty$, by the estimates in (4.2), (4.10) and (4.17), we can show that
\[
\int_0^\infty |\Theta'| \, ds \lesssim \epsilon_2^{1/2} + \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho + \left( \epsilon_8^{1/2} + \sigma_8 \right) \int_0^\infty |z_{in}(\rho)|^2 \rho \, d\rho \right)^{1/2}.
\]
This implies that $\Theta(t)$ converges to some constant $\Theta_\infty$ as $t \to \infty$. The proof is finished.

Conflict of interest: WE DECLARE THAT THERE IS NO CONFLICT OF INTERESTS.

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