Abstract

The aim of these notes is to provide a reasonably short and “hands-on” introduction to the differential calculus on associative algebras over a field of characteristic zero. Following a suggestion of Ginzburg’s we call the resulting theory associative geometry. We argue that this formalism sheds a new light on some classic solution methods in the theory of finite-dimensional integrable dynamical systems.

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1 Introduction

The fundamental relationship between algebra and geometry is well known since the times of Fermat and Descartes. In the modern language of category theory this relationship is expressed in terms of equivalences of the form

\[
\text{Spa} \xrightarrow{\mathcal{O}} \text{Alg}^{\text{op}} \xleftarrow{\text{Spec}}
\]

where \text{Spa} is some category of spaces and \text{Alg} is some category of (commutative) algebras. The functor \mathcal{O} maps a space to the algebra of “regular” (in the appropriate sense) functions on it; the functor Spec maps an algebra to its spectrum, an object of the category \text{Spa} naturally associated with it. Two celebrated instances of this kind of construction are \textit{Gelfand duality}, relating compact Hausdorff spaces to commutative C*-algebras, and the basic duality of modern algebraic geometry, relating affine schemes to commutative rings (see [1, Chapter 1] for a detailed exposition of these and other examples of this kind).

In this framework it is natural to ask whether this picture can be broadened by taking as \text{Alg} some category of associative \textit{but not necessarily} commutative algebras. It became clear early on that a naive approach to this question is not viable: namely, one cannot simply extend in any non-trivial way the usual definition of spectrum coming from algebraic geometry to the category of all rings. This insight has been recently formalized as a set of actual no-go theorems [2, 3].

To cope with this problem many different strategies for doing geometry on particular classes of noncommutative algebras have been developed. In the approach pioneered by Alain Connes in the 1980s and popularized in the book [4], the idea is to use as a starting point the dictionary provided by the aforementioned Gelfand duality and interpret the theory of (not necessarily commutative) C*-algebras as a kind of “noncommutative topology”. When needed, this picture can be further refined by introducing analogues of smooth structures and Riemannian metrics. This gave rise to a very rich theory which is deeply rooted in functional analysis. We refer again the reader to [1] for a recent and very readable introduction to this field.

Another possible strategy to pursue is to generalize the usual algebro-geometric concepts to some (hopefully large) class of noncommutative rings. Here we encounter another important distinction, which corresponds to the classical split between projective and affine algebraic geometry. In the projective case, one is naturally led to study suitable classes of \textit{graded} noncommutative rings. This is the approach taken, for instance, by Artin and Zhang in their seminal paper [5]. The resulting theory, which is known as “noncommutative projective geometry”, is beautifully described in the surveys [6], [7] and [8].

The affine case can be further divided, following [9, Section 1], in two main strands. The first, that Ginzburg calls “noncommutative geometry in the small”, is best seen as a \textit{generalization} of conventional (affine) algebraic geometry. Here one is typically interested in some sort of noncommutative deformation of commutative algebras like superalgebras, rings of differential operators and universal enveloping algebras of Lie algebras. This kind of investigations is strictly related to the various mathematical approaches to the problem of \textit{quantization}.

On the other hand, the second approach (called “noncommutative geometry in the large” by Ginzburg) is a completely new theory that does \textit{not} reduce to the commutative one as a special case. In this approach one deals with \textit{generic} associative algebras, the basic examples being given by free ones (namely, algebras of noncommutative polynomials in a finite number of variables). In these notes we are going to explain in more detail this point of view; following a suggestion of Ginzburg’s, we shall refer to this approach by the name of \textit{associative geometry}.

To recover some degree of geometric intuition in this very general setting the following perspective, usually attributed to Kontsevich (see [10, Section 9]), is very helpful. Let \(K\) be a field
of characteristic zero. For each $d \in \mathbb{N}$ we have a representation functor

$$\text{Rep}_d : \text{AsAlg} \to \text{AffSch}$$

mapping each associative $K$-algebra $A$ to the affine scheme of $d$-dimensional representations of $A$ (that is, algebra morphisms $A \to \text{Mat}_{d,d}(K)$). According to Kontsevich, the “associative-geometric” objects on $A$ are precisely those objects which induce in a natural way a family of the corresponding (commutative) objects on each scheme in the sequence $(\text{Rep}_d(A))_{d \in \mathbb{N}}$. In other words, one can see associative-geometric objects on $A$ as “blueprints” for an infinite sequence of ordinary geometric objects defined on representation spaces of $A$, each scheme $\text{Rep}_d(A)$ giving an increasingly better approximation to the mysterious geometry determined by $A$.

The first aim of these notes is to provide a reasonably compact survey of the fundamental constructions and results at the basis of this circle of ideas. The second aim is to show how the resulting theory provides a new interpretation for some classic solution techniques in the field of finite-dimensional integrable dynamical systems.

In more detail, the paper is organized as follows. In section 2 we review the definition of the basic notions of differential calculus on a generic associative algebra over a field of characteristic zero. In particular in §2.3 we build the Karoubi-de Rham complex, whose elements play the role of “associative differential forms”. We analyze in detail a couple of examples, the associative affine spaces (§2.4) and the associative varieties determined by path algebras of quivers (§2.7). To deal with the latter it will be necessary to slightly refine the class of associative differential forms used by introducing a notion of differential calculus relative to a subalgebra (§2.6).

In section 3 we review the connection between the worlds of associative and commutative geometry. Following Kontsevich’s philosophy recalled above, one is led to consider the space of finite-dimensional representations of a (finitely generated) associative algebra. This can be interpreted as an affine scheme (or variety) on which a natural action of the general linear group is defined. We explain in some detail the basic process through which associative-geometric objects defined on the algebra $A$ induce GL-invariant geometric objects on representation spaces. The relative case, which is the appropriate one for quiver representation spaces, is treated in §3.3.

Finally in section 4 we survey the applications of this formalism to finite-dimensional integrable dynamical systems. We first review the development by Kontsevich and Ginzburg of the associative version of symplectic varieties (inducing ordinary symplectic structures on representation spaces) and the definition of the canonical associative symplectic structures on free algebras and quiver path algebras. Then we consider some simple examples of Hamiltonian systems on associative spaces, and show how their (trivial) solutions project down to interesting flows on some symplectic quotients of the corresponding representation spaces. This approach can be seen as a natural extension of the projection method introduced by Olshanetsky and Perelomov to solve the systems of Calogero-Moser type [11].

In order to keep our treatment within reasonable bounds we were forced to gloss over some more recent developments in associative geometry such as the introduction of bisymplectic structures [12] and double Poisson structures [13]. We hope to be able to cover these important topics (and their applications to integrable systems) in a sequel to these notes [14].

As it should be clear from the above summary, this paper is meant to be purely expository. Every construction we are going to review can be found in more advanced sources such as [9] and [15]. On the reader’s part we assume a basic acquaintance with the fundamental notions of algebraic (or differential) geometry, but little or no experience in dealing with noncommutative algebras. We also assume a reasonable amount of familiarity with basic category theory (especially universal constructions and adjoint functors), and (for the material in section 4) a working knowledge of ordinary symplectic geometry.
It should be stressed that I am not an expert in noncommutative geometry; the content of these notes merely reflects my understanding of a small part of this topic at the time of the deadline for submitting the manuscript. I hope this effort will be useful for other novices who intend to venture into this complex and fascinating field.

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2 Differential calculus on associative algebras

In this section we review the construction of the universal calculus of differential forms on associative algebras. Most of this material is taken from Ginzburg’s lectures [9] and may be found in many other standard references such as, for instance, [16, 17, 18].

2.1 Kähler differentials

In ordinary (commutative) geometry a fundamental role is played by differential forms. As the name implies, the concept originated in the theory of smooth spaces (differentiable manifolds), but was later exported in much more general settings by a purely algebraic construction, known as Kähler differentials. It turns out that this more abstract reformulation can easily be adapted to the associative context.

We start by recalling some definitions. Let $A$ be a (not necessarily commutative) ring. An $A$-bimodule is an abelian group $M$ equipped with two actions of $A$, one from the left and one from the right, which are compatible in the following sense:

\[ a \cdot (m \cdot b) = (a \cdot m) \cdot b \quad \text{for every } a, b \in A, m \in M. \tag{2.1} \]

An $A$-bimodule $M$ is called symmetric if the two actions coincide, that is $a \cdot m = m \cdot a$ for every $a \in A$ and $m \in M$. Clearly, if $A$ is commutative then every left (or right) $A$-module can be extended to a symmetric $A$-bimodule by simply defining the opposite action to be equal to the one given (but notice that, even in the commutative case, not every $A$-bimodule is of this kind). It follows that the categories of left $A$-modules, of right $A$-modules and of symmetric $A$-bimodules are all isomorphic when the ring $A$ is commutative, hence we can simply speak of “$A$-modules” and let the elements of $A$ act from whatever side we wish.

Now let $\mathbb{K}$ be a field of characteristic zero and $A$ a commutative $\mathbb{K}$-algebra, thought of as the algebra of “regular” functions $X \to \mathbb{K}$ for some space $X$. Given an $A$-module $M$, a $\mathbb{K}$-linear map $\delta : A \to M$ is called a derivation if

\[ \delta(ab) = \delta(a)b + a\delta(b) \quad \text{for every } a, b \in A. \tag{2.2} \]
Let us denote by Der\( (A, M) \) the set of derivations \( A \to M \), and consider the functor

\[
\text{Der}(A, \cdot): A\text{-Mod} \to \text{Set}
\]

which sends each \( A \)-module \( M \) to the set \( \text{Der}(A, M) \) and each \( A \)-linear map \( f: M \to N \) to its pushforward \( f^* \) (defined by \( f^*(\delta) = f \circ \delta \)). We claim that this functor has a universal element (see [19, p. 57]), namely there exists a pair \( (\Omega^1(A), d) \), where \( \Omega^1(A) \) is an \( A \)-module and \( d \in \text{Der}(A, \Omega^1(A)) \), such that for every other pair \( (M, \theta) \) of this kind there exists a unique \( A \)-linear map \( f_\theta: \Omega^1(A) \to M \) that makes the following diagram commute in \( A\text{-Mod} \):

\[
\begin{array}{ccc}
A & \xrightarrow{d} & \Omega^1(A) \\
\theta \downarrow & & \downarrow f_\theta \\
M & &
\end{array}
\]

To see this one can either build the required \( A \)-module “by hand”, taking as generators the formal symbols \( \{da\}_{a \in A} \) and imposing suitable relations between them (see e.g. [20, Chap. 16]), or proceed in a more explicit way by taking a suitable quotient of the kernel of the multiplication map of \( A \), seen as a \( K \)-linear map \( A \otimes A \to A \) (see e.g. [21, Chap. 9]).

The elements of \( \Omega^1(A) \) are called Kähler differentials and act as substitutes of differential forms in a purely algebraic context. When \( A \) is the coordinate algebra of a smooth algebraic variety, the Kähler differentials are precisely the usual differential forms with regular (that is, polynomial) coefficients; for singular varieties the two concepts no longer agree, and Kähler differentials behave in a better way\(^2\).

Suppose now that our \( K \)-algebra \( A \) is no longer commutative. Since the definition (2.2) of derivation makes perfect sense also when \( M \) is a non-symmetric \( A \)-bimodule, we can again define a functor

\[
\text{Der}(A, \cdot): A\text{-Bimod} \to \text{Set}
\]

in exactly the same manner as above, the only difference being that now the domain is the whole category of \( A \)-bimodules. We can then ask if the same universal problem previously used to define Kähler differentials has a solution in the new context, and the answer is affirmative.

**Theorem 1.** The functor (2.5) has a universal element: there exists a pair \( (\Omega_{nc}^1(A), d) \), where \( \Omega_{nc}^1(A) \) is an \( A \)-bimodule and \( d \in \text{Der}(A, \Omega_{nc}^1(A)) \), such that for every other pair \( (M, \theta) \) of this kind there exists a unique \( A \)-bimodule morphism \( f_\theta: \Omega_{nc}^1(A) \to M \) that makes the following diagram commute in \( A\text{-Bimod} \):

\[
\begin{array}{ccc}
A & \xrightarrow{d} & \Omega_{nc}^1(A) \\
\theta \downarrow & & \downarrow f_\theta \\
M & &
\end{array}
\]

The proof of this result is not difficult but would entail a long detour through topics like Hochschild cohomology which will have no further use in these notes. For this reason we omit it, referring the interested reader to [22, Section 10].

---

1. Here and in what follows we adopt the following convention: whenever we use a tensor product sign without a subscript, we mean a tensor product over \( K \).
2. We remark that more care is needed when interpreting Kähler differentials in non-algebraic contexts, see http://mathoverflow.net/q/60749.
Concretely, if we denote by \( \mu: A \otimes A \to A \) the multiplication map of \( A \) we can take \( \Omega_{\text{nc}}^1(A) \) to be the kernel of \( \mu \) (seen as a sub-\( A \)-bimodule of \( A \otimes A \), the bimodule structure being given by \( a.(a' \otimes a'') = aa' \otimes a'', (a' \otimes a'').b = a' \otimes a''b \) with the map \( d \) defined by

\[
da := 1 \otimes a - a \otimes 1 \quad \text{for every } a \in A.
\]

By construction, every \( \alpha \in \Omega_{\text{nc}}^1(A) \) may be written as a finite sum of the form

\[
\alpha = \sum_i a'_i \otimes a''_i
\]

for some \( a'_i, a''_i \in A \) such that \( \sum_i a'_i a''_i = 0 \). Using the map \( (2.7) \) we can write equivalently

\[
\alpha = \sum_i a'_i da''_i = \sum_i a'_i(1 \otimes a''_i - a''_i \otimes 1) = \sum_i a'_i \otimes a''_i - \sum_i a'_i a''_i \otimes 1 = \sum_i a'_i \otimes a''_i.
\]

Given a pair \((M, \theta)\) as in the statement of theorem \( \text{[1]} \) the map \( f_\theta \) is then defined by sending the element \( (2.8) \) to \( \sum_i a'_i \theta(a''_i) \in M \).

It is useful to reformulate the universal property expressed by theorem \( \text{[1]} \) as the existence of a natural isomorphism

\[
\text{Der}(A, \cdot) \simeq \text{A-Bimod}(\Omega_{\text{nc}}^1(A), \cdot).
\]

Again, this is exactly analogous to what happens in the commutative case, with the category \( \text{A-Bimod} \) replacing the category \( \text{A-Mod} \).

Particularly important is the case when \( M = \) the algebra \( A \) itself seen as an \( A \)-bimodule in the obvious way, that is by defining

\[
a \cdot x := ax \quad \text{and} \quad x \cdot b := bx \quad \text{for all } a, b, x \in A.
\]

When \( A \) is (commutative and) the algebra of regular functions on a smooth affine manifold \( X \), the derivations \( A \to A \) are in 1-1 correspondence with algebraic vector fields globally defined on \( X \). As we shall see in section \( \text{[3]} \) the same interpretation makes sense also in the associative context; we then take \( \text{Der}(A) := \text{Der}(A, A) \) to be the (linear) space of vector fields on the “associative variety” determined by \( A \). The natural isomorphism \( (2.9) \) then implies that for every \( \theta \in \text{Der}(A) \) there exists a unique \( A \)-bimodule map \( i_\theta: \Omega_{\text{nc}}^1(A) \to A \) such that \( \theta = i_\theta \circ d \), that is

\[
i_\theta(da) = \theta(a) \quad \text{for every } a \in A.
\]

Clearly this property specifies completely the action of \( i_\theta \) on every element of \( \Omega_{\text{nc}}^1(A) \).

Let us remark that the \( \mathbb{C} \)-linear space \( \text{Der}(A) \) has a natural structure of Lie algebra when equipped with the usual commutator bracket:

\[
[\theta_1, \theta_2] := \theta_1 \circ \theta_2 - \theta_2 \circ \theta_1.
\]

However, it cannot be equipped with the structure of a (left or right) \( A \)-module as soon as \( A \) fails to be commutative. The best one can do is to define an action of \( Z(A) \), the center of the algebra \( A \), on \( \text{Der}(A) \) as follows: given \( k \in Z(A) \) and \( \theta \in \text{Der}(A) \), we take as \( k \cdot \theta \) the map

\[
a \mapsto k\theta(a) \quad \text{for every } a \in A.
\]

This map is a derivation because

\[
(k \cdot \theta)(ab) = k(\theta(a)b + a\theta(b)) = k\theta(a)b + ka\theta(b)
\]

which coincides with

\[
(k \cdot \theta)(a)b + a(k \cdot \theta)(b) = k\theta(a)b + ak\theta(b)
\]

precisely because \( ka = ak \) for every \( a \in A \). This makes \( \text{Der}(A) \) a symmetric \( Z(A) \)-bimodule.
2.2 The complex $\Omega^*(A)$

From now on we are going to denote the bimodule of Kähler differentials for a generic associative algebra $A$ simply as $\Omega^1(A)$. In order to obtain a notion of $n$-form for every $n > 1$ we would like to build a cochain complex $\Omega^*(A)$ whose lower degree part reduces to the universal derivation $d: A \to \Omega^1(A)$ provided by theorem [1]. It turns out that the most convenient way to achieve this goal involves another kind of universal construction.

Recall that a $K$-algebra $A$ is said to be graded (over $\mathbb{N}$) if it comes equipped with a direct sum decomposition

\[ A = \bigoplus_{i \in \mathbb{N}} A_i \]  \hspace{1cm} (2.11)

such that $A_i A_j \subseteq A_{i+j}$. It follows that $A_0$ is a subalgebra of $A$ and each $A_i$ is an $A_0$-bimodule (not necessarily symmetric). A grading over $\mathbb{N}$ automatically induces also a $\mathbb{Z}/2$ grading, namely a decomposition of $A$ into an “even” and an “odd” part,

\[ A_+ := \bigoplus_{i \text{ even}} A_i \quad \text{and} \quad A_- := \bigoplus_{i \text{ odd}} A_i, \]  \hspace{1cm} (2.12)

such that $A_\pm A_\pm \subseteq A_\pm$, $A_\pm A_\mp \subseteq A_-$. A $K$-linear map $f: A \to A$ is said to be of degree $\ell$ if $f(A_i) \subseteq A_{i+\ell}$. A map $\delta: A \to A$ of degree $\ell$ which satisfies the graded Leibniz rule

\[ \delta(ab) = \delta(a)b + (-1)^{k\ell} a \delta(b) \]  \hspace{1cm} (2.13)

for every $a \in A_k$, $b \in A$ is called a graded derivation of degree $\ell$. Ordinary derivations of $A$ are exactly the graded derivations of degree zero. We shall speak of odd derivations for graded derivations of odd degree, and similarly for even derivations.

The following result is easily proved using the graded Leibniz rule and induction.

**Lemma 2.** Let $A$ be a graded algebra. If two derivations $A \to A$ of a fixed degree coincide on a set of generators for $A$ then they are equal.

A morphism of graded algebras from $A$ to $B$ is a morphism of $K$-algebras $f: A \to B$ which has degree zero ($f(A_i) \subseteq B_i$ for every $i \in \mathbb{N}$).

A differential graded algebra, or dg-algebra for short, is a pair $(D,d)$ consisting of a graded algebra $D$ and a derivation $d: D \to D$ of degree 1 such that $d \circ d = 0$. The map $d$ is called the differential of the dg-algebra; we shall denote by $d_n: D_n \to D_{n+1}$ the restriction of $d$ to the homogeneous component of degree $n$ in $D$.

A morphism of dg-algebras from $(D,d)$ to $(E,d')$ is a morphism of graded algebras $f: D \to E$ that intertwines the two differentials, which means that the diagram

\[
\begin{array}{ccc}
D_n & \xrightarrow{d_n} & D_{n+1} \\
\downarrow f|_{D_n} & & \downarrow f|_{D_{n+1}} \\
E_n & \xrightarrow{d'_n} & E_{n+1}
\end{array}
\]  \hspace{1cm} (2.14)

commutes for every $n \in \mathbb{N}$. The category obtained by taking as object the dg-algebras over $K$ and as arrows the dg-algebra morphisms will be denoted by $\mathbb{K}$-$\text{dga}$.

As we already noted, the degree zero part of a graded $K$-algebra is itself a $K$-algebra. It follows that there exists a restriction functor

\[ (\cdot)_0: \mathbb{K}$-$\text{dga} \to \mathbb{K}$-$\text{Alg} \]  \hspace{1cm} (2.15)
which sends a generic dg-algebra \((D, d)\) to its degree zero part \(D_0\) and a morphism of dg-algebras \(f: (D, d) \to (E, d')\) to the restriction \(f|_{D_0}\) (which can be seen as a map with codomain \(E_0\) because \(f\) preserves the grading). It turns out that this functor possesses a left adjoint, which enables us to canonically construct a dg-algebra extending any given \(K\)-algebra of “degree zero” elements.

\[ A \xrightarrow{1} D(A)_0 \xleftarrow{\psi} \Gamma_0 \]

**Theorem 3.** For every \(K\)-algebra \(A\) there exists a universal morphism from \(A\) to \((\cdot)_0\), that is a pair \((D(A), i)\) consisting of a dg-algebra \(D(A)\) and a \(K\)-algebra morphism \(i: A \to D(A)_0\) such that for every other pair \((\Gamma, \psi)\) of this kind there exists a unique morphism of dg-algebras \(u_\psi: D(A) \to \Gamma\) that makes the following diagram commute in \(K\text{-Alg}\):

This result was first proved in a seminal paper by Cuntz and Quillen [22]. The dg-algebra \(D(A)\) is called the universal differential envelope of the \(K\)-algebra \(A\) and admits a very explicit description that we illustrate next.

Let us set \(\bar{A} := A/K\), as a quotient of vector spaces over \(K\); for every \(a \in A\) we shall denote by \(\bar{a}\) its image along the canonical projection \(A \to \bar{A}\). Now define

\[ D(A)_n := A \otimes \bar{A} \otimes \cdots \otimes \bar{A} \tag{2.17} \]

and let \(D(A) := \bigoplus_{n \in \mathbb{N}} D(A)_n\). Then define \(d_n: D(A)_n \to D(A)_{n+1}\) as follows:

\[ d_n(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) := 1 \otimes a_0 \otimes \cdots \otimes \bar{a}_n. \tag{2.18} \]

Finally we must give a product on \(D(A)\) compatible with the grading. We do this by defining a map \(D(A)_n \times D(A)_{m-1-n} \to D(A)_{m-1}\) via the following prescription:

\[ (a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) (a_{n+1} \otimes \bar{a}_{n+2} \otimes \cdots \otimes \bar{a}_m) := (-1)^n a_0 a_1 \otimes \bar{a}_2 \otimes \cdots \otimes \bar{a}_m + \sum_{i=1}^{n} (-1)^{n-i} a_0 \otimes \bar{a}_i \otimes \cdots \otimes \bar{a}_{i+1} \otimes \cdots \otimes \bar{a}_m, \tag{2.19} \]

where the \(a_1\) that figures in the first term on the right-hand side is any representative for \(\bar{a}_1\) (it is easy to check that the result does not depend on the particular representative chosen).

Notice in particular the non-trivial action of an element of \(D(A)_0 = A\) on \(D(A)_n\) from the right (the action from the left is the obvious one):

\[ (a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) a_{n+1} = (-1)^n a_0 a_1 \otimes \bar{a}_2 \otimes \cdots \otimes \bar{a}_{n+1} + \]

\[ + \sum_{i=1}^{n} (-1)^{n-i} a_0 \otimes \bar{a}_i \otimes \cdots \otimes \bar{a}_{i+1} \otimes \cdots \otimes \bar{a}_m. \]

For instance when \(n = 1\) we have

\[ a_0 \otimes \bar{a}_1 = a a_0 \otimes \bar{a}_1 \text{ but } (a_0 \otimes \bar{a}_1)b = -a_0 a_1 \otimes \bar{b} + a_0 \otimes \bar{a}_1 b. \]

In [22] the authors prove (1) that the pair \((D(A), d)\) is a dg-algebra, and (2) that the pair \((D(A), i)\), where \(i\) is the identity map \(A \to D(A)_0 = A\), solves the universal problem (2.10).

\[ \text{Recall our convention about tensor products: } \otimes \text{ sign means by default } \otimes_K. \]
Let us clarify the relationship between $\mathcal{D}(A)$ and the Kähler differentials of $A$. Consider the degree zero component of the map (2.18); as $\mathcal{D}(A)_0 = A$, it is a derivation from $A$ to the $A$-bimodule $\mathcal{D}(A)_1$. It can be shown (see [11, Theorem 10.7.1]) that the pair $(\mathcal{D}(A)_1, d_0)$ is also a universal element for the functor $\text{Der}(A, \cdot)$. The universal property of the pair $(\Omega^1(A), d)$ then implies that there exists a unique isomorphism of $A$-bimodules

$$\psi: \Omega^1(A) \to \mathcal{D}(A)_1$$

such that $\psi \circ d = d_0$; that is, $\psi$ sends $da = 1 \otimes a - a \otimes 1 \in \Omega^1(A)$ to $1 \otimes \overline{a} \in A \otimes \bar{A} = \mathcal{D}(A)_1$. Its inverse is given by

$$a_0 \otimes \overline{a_1} \mapsto a_0 \otimes a_1 - a_0a_1 \otimes 1$$

(it is easy to verify that the right-hand side does not depend on the particular representative chosen for $\overline{a_1}$). We conclude that the $A$-bimodule of degree 1 elements in $\mathcal{D}(A)$ gives another realization of the Kähler differentials for $A$.

To understand the nature of the complex $\mathcal{D}(A)$ in higher degrees let us briefly review the notion of tensor algebra of a bimodule. Given a $\mathbb{K}$-algebra $A$ and an $A$-bimodule $M$, the tensor algebra of $M$ is defined to be the $\mathbb{K}$-vector space

$$T_A(M) := \bigoplus_{i \in \mathbb{N}} T_A^i(M)$$  \hspace{1cm} (2.20)

where $T_A^0(M) := A$, $T_A^1(M) := M$ and, for every $i > 1$,

$$T_A^i(M) := M \otimes_A \cdots \otimes_A M.$$  \hspace{1cm} \text{i times}

Clearly $T_A(M)$ is a $\mathbb{K}$-algebra graded over $\mathbb{N}$, by taking $T_A^i(M)$ as the homogeneous component of degree $i$. It will be useful to interpret the tensor algebra construction as an adjoint functor; to explain this, however, a little aside is necessary.

In commutative algebra, by an algebra over a commutative ring $R$ it is usually meant a pair $(A, \eta)$ where $A$ is a (not necessarily commutative) ring and $\eta: R \to A$ is a morphism of rings whose image is contained in the center of $A$ (see e.g. [23, p. 121]). This gives to $A$ the structure of a left $R$-module (or right $R$-module, or symmetric $R$-bimodule) by defining $r.a := \eta(r)a$ and/or $a.r := a\eta(r)$; the two expressions always coincide precisely because $\eta(r)$ belongs to the center of $A$.

When $R$ is no longer assumed to be commutative it is natural (and in fact necessary, if we want non-symmetric bimodules) to drop the constraint on the image of $\eta$; hence for us an algebra over the ring $R$ will be a pair $(A, \eta)$ consisting of a ring $A$ and a morphism of rings $\eta: R \to A$. Again, this means that $A$ is automatically equipped with the structure of a (not necessarily symmetric) $R$-bimodule defined by

$$r.a.r' = \eta(r)a\eta(r') \quad \text{for every } a \in A, r, r' \in R.$$  \hspace{1cm} (2.21)

By an unfortunate mismatch in terminology, the category of algebras over $R$ (in this “non-central” sense) is exactly what category theorists call the category of objects under $R$ in $\text{Rng}$ (see e.g. [19, p. 45]); we shall denote it by $R \downarrow \text{Rng}$.\footnote{Another popular notation is $R/\text{Rng}$. Notice that we keep using the notation $R\text{-Alg}$ for the category of “usual” algebras over a commutative ring (or field) $R$.}

With this confusing point understood, let us return to the tensor algebra of an $A$-bimodule $M$. Since the map

$$\eta: A \to T_A(M)$$  \hspace{1cm} (2.22)


defined by sending each \( a \in A \) to the corresponding degree zero tensor is an (injective) morphism of \( \mathbb{K}\)-algebras, the tensor algebra \( T_A(M) \) is naturally an algebra over \( A \), that is an object of the category \( A \downarrow \mathbb{K}\text{-Alg} \). It follows that we can define a functor

\[
T_A : A\text{-Bimod} \to A \downarrow \mathbb{K}\text{-Alg}
\]

by sending each \( A\)-bimodule \( M \) to its tensor algebra and each morphism of \( A\)-bimodules \( f : M \to N \) to the map \( T_A(f) : T_A(M) \to T_A(N) \) defined on decomposable tensors of degree \( i \) as

\[
m_1 \otimes_A \cdots \otimes_A m_i \mapsto f(m_1) \otimes_A \cdots \otimes_A f(m_i)
\]

(2.24)

(as is well known, tensors of this form generate the whole of \( T_A^k(M) \)). In the other direction we have the “partially forgetful” functor \( U : A \downarrow \mathbb{K}\text{-Alg} \to A\text{-Bimod} \) that, given an algebra \( B \) over \( A \), forgets the product in \( B \) but keeps the \( A\)-bimodule structure. It is easy to check that, for every \( A\)-bimodule \( M \) and every \( A\)-algebra \( B \), there is a natural isomorphism

\[
A \downarrow \mathbb{K}\text{-Alg}(T_A(M), B) \simeq A\text{-Bimod}(M, U(B))
\]

(2.25)

and this means exactly that the functor \( T_A \) is left adjoint to \( U \). Hence the tensor algebra construction (2.20) can be seen as the universal way to “enhance” the structure of an \( A\)-bimodule to that of a full algebra over \( A \). Again, this is exactly analogous to what happens in the corresponding commutative situation, where the tensor algebra of a module gives a functor \( R\text{-Mod} \to R\text{-Alg} \) which is left adjoint to the forgetful functor \( R\text{-Alg} \to R\text{-Mod} \).

**Lemma 4.** Let \( M \) be an \( A\)-bimodule and \( B \) be an algebra over \( A \). Every morphism of \( A\)-bimodules \( f : M \to B \) can be extended in a unique manner to a (even or odd) derivation \( \delta_f : T_A(M) \to B \) such that:

1. the restriction of \( \delta_f \) to \( T_A^0(M) \simeq A \) is a specified map \( A \to B \), and
2. the restriction of \( \delta_f \) to \( T_A^1(M) \simeq M \) coincides with \( f \).

The proof boils down to a straightforward induction, where the inductive step uses the (possibly graded) Leibniz rule to reduce by one the degree of the tensor on which \( \delta_f \) operates.

Consider now the tensor algebra determined by the bimodule of Kähler differentials of \( A \),

\[
\Omega^*(A) := T_A(\Omega^1(A)).
\]

(2.26)

A tensor of degree \( k \) in \( \Omega^*(A) \) may be written as a linear combination of terms of the form

\[
a_1.db_1 \otimes_A a_2.db_2 \otimes_A \cdots \otimes_A a_k.db_k.a_{k+1}
\]

where each \( a_i, 2 \leq i \leq k \) may be freely moved across the tensor product sign. Clearly this looks quite different from a typical element of \( \mathcal{D}(A)_k \). This notwithstanding, we have the following:

**Theorem 5.** The graded algebras \( \Omega^*(A) \) and \( \mathcal{D}(A) \) are isomorphic.

**Proof.** Let \( \psi : \Omega^1(A) \to \mathcal{D}(A)_1 \) be the \( A\)-bimodule isomorphism defined above. By composing \( \psi \) with the embedding \( \mathcal{D}(A)_1 \hookrightarrow \mathcal{D}(A) \) we obtain an \( A\)-bimodule map \( \Omega^1(A) \to \mathcal{D}(A) \). Using the natural isomorphism (2.23), this corresponds to a morphism \( \Psi : T_A(\Omega^1(A)) \to \mathcal{D}(A) \) of algebras over \( A \) which is defined on decomposable tensors by

\[
\Psi(a_1 \otimes_A \cdots \otimes_A a_n) = \psi(a_1) \cdots \psi(a_n)
\]
To conclude it is sufficient to show that $\Psi$ is invertible. Its inverse can be defined as follows: given a decomposable element $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ in $D(A)_n$, we write it as the product

$$(a_0 \otimes a_1)(1 \otimes a_2) \cdots (1 \otimes a_n)$$

and send it to

$$\psi^{-1}(a_0 \otimes a_1) \otimes_A \psi^{-1}(1 \otimes a_2) \otimes_A \cdots \otimes_A \psi^{-1}(1 \otimes a_n)$$

It is clear that the map so defined is an inverse for $\Psi$.

We can use the isomorphism $\Psi$ to transfer the differential (2.18) naturally defined on $D(A)$ on the tensor algebra $\Omega^\bullet(A)$, as the map $d := \Psi^{-1} \circ d \circ \Psi$. For instance, a generic element $\alpha = \sum_i a_i \cdot db_i$ in $\Omega^1(A)$ corresponds via the map $\psi$ to $\sum_i a_i \otimes b_i$ in $D(A)_1$. Its differential in $D(A)_2$ is

$$\sum_i 1 \otimes a_i \otimes b_i = \sum_i (1 \otimes a_i)(1 \otimes b_i)$$

and $\Psi^{-1}$ maps this element back to

$$d\alpha = \sum_i d a_i \otimes_A d b_i \in \Omega^1(A) \otimes_A \Omega^1(A) \quad (2.27)$$

Notice that we could certainly use lemma 4 to directly define a map $d: \Omega^\bullet(A) \to \Omega^\bullet(A)$ which restricts to the universal derivation (2.7) on tensors of degree zero and acts as in equation (2.27) on tensors of degree 1. However, it is then a non-trivial endeavor to show that $d \circ d = 0$. On the contrary, the proof of this fact is immediate in the universal differential envelope. Thus, thanks to theorem 5 we can have the best of both worlds.

### 2.3 The Karoubi-de Rham complex

At this point it would seem natural to interpret the pair $(\Omega^\bullet(A), d)$ as the complex of differential forms on the “associative variety” determined by the algebra $A$. There are two problems with this idea. The first one is that the cohomology of this complex turns out to be (rather trivial and) entirely independent from $A$, as the following result shows.

**Theorem 6.** The cohomology of the complex $(\Omega^\bullet(A), d)$ is given by

$$H^k(\Omega^\bullet(A)) = \begin{cases} \mathbb{K} & \text{if } k = 0 \\ 0 & \text{otherwise}. \end{cases} \quad (2.28)$$

**Proof.** It is convenient to work in $(D(A), d)$. From the expression (2.18) it is clear that $H^0(\Omega^\bullet(A)) = \ker d_0 = \mathbb{K}1$. On the other hand, for every $n > 1$ the map $d_n$ admits the following factorization:

$$\mathbb{K}1 \otimes \overline{A}^{\otimes(n+1)} \xrightarrow{d_n} A \otimes \overline{A}^{\otimes(n+1)} \quad (2.29)$$

But $\ker d_{n+1}$ is exactly $\mathbb{K}1 \otimes \overline{A}^{\otimes(n+1)}$, hence the triviality of $H^k(\Omega^\bullet(A))$ for $k > 0$. \qed
The second drawback is that, even when the algebra $A$ is commutative, $\Omega^\bullet(A)$ does not coincide with the usual dg-algebra of differential forms on the corresponding affine variety. Indeed, the algebra $\Omega^\bullet(A)$ is not graded-commutative: for instance $da \, db \neq -db \, da$ in general. To recover this property we need to take a quotient of $\Omega^\bullet(A)$ in which such relations are imposed by hand.

In order to do this let us recall that the graded commutator on a graded algebra $D$ is the map $[\cdot, \cdot]: D \times D \to D$ defined on homogeneous elements $a \in D_i$, $b \in D_j$ by

$$[[a, b]] := ab - (-1)^{ij}ba$$

(2.30)

and then extended by linearity on the whole of $D$. Notice that $[[a, b]]$ coincides with the ordinary commutator $[a, b]$ as soon as at least one of $a$ and $b$ has even degree, whereas for two elements of odd degree we have $[[a, b]] = ab + ba$ instead. In the sequel the following compatibility property between derivations and graded commutators, which is immediate to check, will be rather useful.

**Lemma 7.** Let $D$ be a graded algebra and $\delta: D \to D$ be a graded derivation. Then $\delta$ maps a graded commutator in a linear combination of graded commutators.

Now let $A$ be any associative algebra. The **Karoubi-de Rham complex** of $A$ [25] is the graded vector space over $K$ given by the quotient

$$\text{DR}^\bullet(A) := \frac{\Omega^\bullet(A)}{[[\Omega^\bullet(A), \Omega^\bullet(A)]]}$$

(2.31)

where $[[\Omega^\bullet(A), \Omega^\bullet(A)]]$ denotes the linear subspace in $\Omega^\bullet(A)$ generated by all the elements of the form $[[a, b]]$ for $a, b \in \Omega^\bullet(A)$. This is indeed a graded subspace, so that the quotient (2.31) makes sense in the category of graded vector spaces over $K$ (it does not make sense in the category graded $K$-algebras, since $[[\Omega^\bullet(A), \Omega^\bullet(A)]]$ is not an ideal). We shall take the elements of $\text{DR}^\bullet(A)$ as the associative-geometric counterpart of differential forms.

In general it is not easy to explicitly describe these objects. In degree zero, however, we have obviously

$$\text{DR}^0(A) = \frac{A}{[A, A]}$$

(2.32)

where $[A, A]$ is the linear subspace spanned by commutators in $A$. Classically a 0-form is just a function, so it is natural to regard $\text{DR}^0(A)$ as the linear space of “regular functions” on the associative variety determined by $A$.

The homogeneous component $\text{DR}^1(A)$ of the quotient (2.31) is also easy to describe: as the only degree 1 relations in $[[\Omega^\bullet(A), \Omega^\bullet(A)]]$ are of the form $a\beta - \beta a$ for some $a \in A$ and $\beta \in \Omega^1(A)$, we have that

$$\text{DR}^1(A) = \frac{\Omega^1(A)}{[A, \Omega^1(A)]}.$$  

(2.33)

As soon as $n \geq 2$ things get more complicated: for example $\text{DR}^2(A)$ is defined by relations coming from both the subspaces

$$[A, \Omega^2(A)] = \text{span}_K \{ a\omega - \omega a \}_{a \in A, \omega \in \Omega^2(A)}$$

and

$$[[\Omega^1(A), \Omega^1(A)] = \text{span}_K \{ a\beta + \beta a \}_{a, \beta \in \Omega^1(A)}.$$  

Some applications of these “non-standard” differential forms on commutative algebras can be found in [24].

Cognoscenti will recognize $\text{DR}^0(A)$ and $\text{DR}^1(A)$ as the degree zero Hochschild homology of the $A$-bimodules $A$ and $\Omega^1(A)$, respectively.
Clearly, without further information on the algebra \( A \) there is little hope for an explicit description of these quotients.

It follows from lemma \([7]\) that the differential of the complex \( \Omega^\bullet(A) \) maps a graded commutator to a linear combination of graded commutators, and so descends to a map

\[
d : \text{DR}^\bullet(A) \to \text{DR}^\bullet(A). \tag{2.34}
\]

Moreover, this map still obeys the fundamental relation \( d \circ d = 0 \). This means that the pair \((\text{DR}^\bullet(A), d)\) qualifies as a differential graded vector space, that is a graded vector space equipped with a map increasing the degree by one and whose square vanishes. On the other hand, it does \textit{not} qualify as a dg-algebra, since elements of \( \text{DR}^\bullet(A) \) cannot be meaningfully multiplied. (For the same reason, the map \((2.34)\) is not itself a derivation.)

However, the apparent lack of an “exterior product” operation between associative differential forms is not as serious a problem as it may seem. The reason is that many constructions involving such products can be performed at the level of the dg-algebra \( \Omega^\bullet(A) \) and then pushed down to its quotient \( \text{DR}^\bullet(A) \). Let us show how this works in practice by setting up a “differential calculus” for associative differential forms, following \([9, \text{Section 11}]\).

As anticipated in \((2.1)\), the role of \( \text{vector fields} \) will be played by derivations \( \theta : A \to A \). For every vector field \( \theta \in \text{Der}(A) \) we have the \( A \)-bimodule map \( i_\theta : \Omega^1(A) \to \Omega^\bullet(A) \) defined by the equality \((2.10)\). By composing with the embedding \( A \hookrightarrow \Omega^\bullet(A) \) we can see \( i_\theta \) as a morphism of \( A \)-bimodules from \( \Omega^1(A) \) to \( \Omega^\bullet(A) \). Then we can use lemma \([4]\) to extend this map to a derivation of degree \(-1\) on \( \Omega^\bullet(A) \) (that is, an odd derivation mapping each \( \Omega^n(A) \) to \( \Omega^{n-1}(A) \)) which vanishes on tensors of degree zero. The resulting map

\[
i_\theta : \Omega^\bullet(A) \to \Omega^\bullet(A) \tag{2.35}
\]

will be called the \textit{interior product} on \( \Omega^\bullet(A) \). For instance its action on a generic degree two elements is

\[
i_\theta(a_1db_1a_2db_2a_3) = a_1\theta(b_1)a_2db_2a_3 - a_1db_1a_2\theta(b_2)a_3 \tag{2.36}
\]

and one can readily verify that \( i_\theta \circ i_\theta = 0 \). More generally, we have

\[
i_\theta(da_1 \ldots da_n) = \sum_{i=1}^n (-1)^{i-1}da_1 \ldots \theta(a_i) \ldots da_n. \tag{2.37}
\]

It follows from lemma \([7]\) that each map \( i_\theta \) descends to a map on the Karoubi-de Rham complex that we still denote in the same way,

\[
i_\theta : \text{DR}^\bullet(A) \to \text{DR}^\bullet(A). \tag{2.35}
\]

We notice that the following equality holds for every pair of derivations \( \theta, \eta \in \text{Der}(A) \):

\[
i_\theta \circ i_\eta = -i_\eta \circ i_\theta. \tag{2.38}
\]

In particular the natural “pairing” map \( \Omega^1(A) \times \text{Der}(A) \to A \) defined by \( (\alpha, \theta) \mapsto i_\theta(\alpha) \) descends to a \( \mathbb{K} \)-linear map

\[
\langle \cdot, \cdot \rangle : \text{DR}^1(A) \times \text{Der}(A) \to \text{DR}^0(A) \tag{2.39}
\]

representing the operation of contraction between a 1-form and a vector field, resulting in a regular function.

Now that we have both an exterior differential and an interior product on \( \Omega^\bullet(A) \) it is straightforward to define a companion “Lie derivative” operator using Cartan’s formula:

\[
\mathcal{L}_\theta := d \circ i_\theta + i_\theta \circ d \tag{2.40}
\]
By definition, $L_\theta$ is a degree 0 (hence even) derivation. Explicitly, it acts as follows:

$$L_\theta(a_0 da_1 \ldots da_n) = \theta(a_0) da_1 \ldots da_n + \sum_{i=1}^{n} a_0 da_1 \ldots d\theta(a_i) \ldots da_n$$

(2.41)

Moreover, using lemma 2 one can verify by a direct calculation on 1-forms (which generate $\Omega^\bullet(A)$ as an algebra) that the following familiar identities hold:

$$L_\theta \circ i_\eta - i_\eta \circ L_\theta = i_{[\theta,\eta]}$$

(2.42)

$$L_\theta \circ L_\eta - L_\eta \circ L_\theta = L_{[\theta,\eta]}$$

(2.43)

By lemma 7 each map $L_\theta$ descends to the complex $\text{DR}^\bullet(A)$, where all the previous identities continue to hold. In particular, the identity (2.42) applied to a 1-form $\alpha \in \text{DR}^1(A)$ can be interpreted as an analogue of the classical fact that “Lie derivatives distribute inside contractions”:

$$L_\theta(i_\eta(\alpha)) = i_\eta(L_\theta(\alpha)) + i_{[\theta,\eta]}(\alpha)$$

where the commutator $[\theta,\eta]$ is interpreted as the action of $L_\theta$ on the derivation $\eta$.

Contrary to what happens for $\Omega^\bullet(A)$, computing the cohomology of the complex $\text{DR}^\bullet(A)$ for a given associative algebra $A$ is usually a nontrivial problem. The next result is sometimes useful in this connection. It states that, when the algebra $A$ itself is graded in positive degrees only, each cohomology group of $\text{DR}^\bullet(A)$ depends only on the subalgebra of degree zero elements in $A$.

**Theorem 8.** Suppose the algebra $A$ is graded over $\mathbb{N}$. For every $k \geq 0$ there is an isomorphism

$$H^k(\text{DR}^\bullet(A_0)) \simeq H^k(\text{DR}^\bullet(A)).$$

(2.44)

The proof closely mimics the usual argument leading to the Poincaré lemma for ordinary de Rham cohomology; the reader may find the details in [9] (Theorem 11.4.7).

### 2.4 Associative affine space

As the first (and simplest) example of an associative variety we consider the **associative affine $n$-dimensional space**, that is the associative space which corresponds to the free associative algebra on $n$ generators:

$$A = \mathbb{K}(x_1, \ldots, x_n).$$

(2.45)

To deal with this case it is useful to adopt the following “coordinate-free” approach. Let $V$ be an $n$-dimensional vector space with basis $(e_1, \ldots, e_n)$ and let $(x_1, \ldots, x_n)$ denote a basis of the dual space $V^*$. Then the tensor algebra of $V^*$ (over $\mathbb{K}$)

$$T(V^*) = \mathbb{K} \oplus V^* \oplus (V^* \otimes V^*) \oplus \ldots$$

is isomorphic to $A$, as the reader can easily check; the tensor product is simply the concatenation of words in the letters $x_1, \ldots, x_n$. This point of view is useful because the resulting formalism is automatically invariant under every affine automorphism of the algebra $A$.

Now let $M$ be an $A$-bimodule. Every $\mathbb{K}$-linear map $V^* \to M$ can be extended to a derivation $A \to M$ using the Leibniz rule, and every element of $\text{Der}(A, M)$ arises in this way (because the generators of $A$ belong to $V^*$). On the other hand, the duality theorem for finite-dimensional

---

\[\text{It goes without saying that free associative algebras have many more automorphisms other than affine ones; their description is a classic (and difficult) problem in associative algebra.}\]
vector spaces implies that the space of linear maps $V^* \to M$ is canonically isomorphic to $M \otimes V$. We conclude that
\[
\text{Der}(A, M) \simeq M \otimes V
\]
and $\Omega^1(A)$ is just the free $A$-bimodule generated by $V^*$,
\[
\Omega^1(A) \simeq A \otimes V^* \otimes A.
\]

The pairing map $\Omega^1(A) \times \text{Der}(A) \to A$ defined by $(\alpha, \theta) \mapsto i_\theta(\alpha)$ is then given by
\[
\langle a \otimes \varphi \otimes b, c \otimes v \rangle = \langle \varphi, v \rangle_V acb
\]
where $(\cdot, \cdot)_V$ denotes the pairing between $V$ and $V^*$.

Notice that $\Omega^1(A)$ is indeed isomorphic to $A \otimes \bar{A}$, as per general results, since $V^* \otimes A = V^* \otimes (K \oplus V^* \oplus V^* \otimes 2 \oplus \ldots) \simeq \bigoplus_{i>0} V^* \otimes i \simeq T(V^*)/K = \bar{A}$.

It follows that, for every $p \geq 1$,
\[
\Omega^p(A) \simeq A \otimes \bar{A}^p \simeq A \otimes (V^* \otimes A)^{\otimes p}.
\]

Now let us study the Karoubi-de Rham complex of $A$ starting from its component of degree zero,
\[
\text{DR}^0(A) = \frac{A}{[A, A]}.
\]

It is not difficult to prove that two words in $A$ differ by a commutator if and only if their letters are related by a cyclic permutation. It follows that $\text{DR}^0(A)$ can be identified with the $K$-linear space generated by the necklace words in the letters $x_1, \ldots, x_n$, that is ordinary words considered modulo cyclic permutations of their letters. These are well known combinatorial objects (see e.g. [26, Chapter 15]).

In degree 1, quotienting the free bimodule (2.47) by the linear subspace $[A, \Omega^1(A)]$ implies that we can always move an element of $A$ acting from the right to the left, as
\[
a \otimes \varphi \otimes b = ba \otimes \varphi \otimes 1 + [a \otimes \varphi \otimes 1, b]
\]
and the second term is killed by the projection onto $\text{DR}^1(A)$. It follows that
\[
\text{DR}^1(A) \simeq A \otimes V^*
\]
as a $K$-linear space. In particular for every $\varphi \in V^*$ there is the 1-form $d\varphi = 1 \otimes \varphi \in \text{DR}^1(A)$. The pairing (2.48) becomes
\[
\langle a \otimes \varphi, c \otimes v \rangle = \langle \varphi, v \rangle_V ac \mod [A, A]
\]
Unfortunately, even in this very special case there is no easy description of a generic associative $p$-form for $p \geq 2$. On the other hand, the cohomology of the complex $\text{DR}^*(A)$ is readily computed, as first shown by Kontsevich in [10].

**Theorem 9.** Let $A$ be a free algebra. Then
\[
H^k(\text{DR}^*(A)) = \begin{cases} K & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases}
\]
This is an immediate consequence of theorem 8. In fact, \( A = T(V^*) \) is exactly a positively graded algebra whose degree zero part is \( K \), hence there is an isomorphism

\[
H^k(\text{DR}^\bullet(A)) \cong H^k(\text{DR}^\bullet(K))
\]

and the right-hand side is trivial for \( k > 0 \) and equal to \( K \) for \( k = 0 \).

We also have a notion of “partial derivative” of a regular function along a direction in the dual vector space \( V \). Indeed, for every \( v \in V \) we can define a map \( \partial_v : \text{DR}^0(A) \to A \) by

\[
\partial_v f := \langle df, 1 \otimes v \rangle.
\] (2.50)

In particular when \( v = e_j \) is an element of the basis \( \{e_1, \ldots, e_n\} \) the resulting map is called the necklace derivative with respect to the generator \( x_j \). It is natural to denote this map by \( \frac{\partial}{\partial x_j} \),

\[
\frac{\partial}{\partial x_j} x_i = (1 \otimes x_i, 1 \otimes e_j) = \delta_{ij}
\]

where \( \delta_{ij} \) is the Kronecker symbol (\( \delta_{ij} = 1 \) when \( i = j \), 0 otherwise).

More generally, given a necklace word \( f \in \text{DR}^0(A) \) represented by the (ordinary) word \( x_{i_1} \ldots x_{i_\ell} \) for a suitable set of indices \( i_1, \ldots, i_\ell \in \{1, \ldots, n\} \), we have that

\[
df = \sum_{k=1}^\ell x_{i_1} \ldots x_{i_{k-1}} dx_{i_k} x_{i_{k+1}} \ldots x_{i_\ell} = \sum_{k=1}^\ell x_{i_{k+1}} \ldots x_{i_\ell} x_{i_1} \ldots x_{i_{k-1}} dx_{i_k}
\]

where the second equality holds in \( \text{DR}^1(A) \). It follows that

\[
\frac{\partial f}{\partial x_j} = \left( \sum_{k=1}^\ell x_{i_{k+1}} \ldots x_{i_\ell} x_{i_1} \ldots x_{i_{k-1}} \otimes x_{i_k}, 1 \otimes e_j \right) = \sum_{k=1}^\ell \delta_{i_kj} x_{i_{k+1}} \ldots x_{i_\ell} x_{i_1} \ldots x_{i_{k-1}}
\] (2.51)

It is easy to check that this result does not depend on the particular representative chosen for \( f \).

Finally let us derive an analogue of the usual formula for the differential of a function in terms of partial derivatives. Given \( f \in \text{DR}^0(A) \) we have \( df = 1 \otimes \bar{f} \) and since \( \bar{f} \in \bar{A} \simeq V^* \otimes A \) we can write

\[
\bar{f} = \sum_{i=1}^n x_i \otimes a_i \quad \text{for some } a_1, \ldots, a_n \in A.
\]

Then \( df = \sum_{i=1}^n 1 \otimes x_i \otimes a_i \in \Omega^1(A) \), which projects down to

\[
df = \sum_{i=1}^n a_i \otimes x_i
\] (2.52)

in \( \text{DR}^1(A) \). Substituting into (2.50) with \( v = e_i \) we see that the coefficients \( a_i \) are exactly the necklace derivatives \( \frac{\partial f}{\partial x_i} \), so that

\[
df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \otimes x_i \quad \text{for every } f \in \text{DR}^0(A).
\] (2.53)
2.5 The Quillen complex

Let us consider the map $\Omega^1(A) \to [A, A]$ given by

$$a_0 da_1 \mapsto [a_0, a_1].$$

This is well-defined because if $a'_1 = a_1 + \lambda$ for some $\lambda \in K$ then $[a_0, a'_1] = [a_0, a_1]$ (as $K$ is contained in the center of $A$). Moreover, given $a \in A$ and $\beta = b_0 db_1 \in \Omega^1(A)$ we have that the commutator

$$a\beta - \beta a = ab_0 db_1 - (b_0 db_1)a = ab_0 db_1 + b_0 b_1 da - b_0 db(da)$$

is sent to

$$[ab_0, b_1] + [b_0 b_1, a] - [b_0, b_1 a] = ab_0 b_1 - b_1 ab_0 + b_0 b_1 a - ab_0 b_1 - b_0 b_1 a + b_1 ab_0 = 0.$$

We conclude that there is a well-defined map

$$b: DR^1(A) \to [A, A]$$

(2.54)

given by $udv \mapsto [u, v]$. It is easy to check that

$$b \circ d = d \circ b = 0.$$ 

(2.55)

If we define

$$\overline{A} := \frac{A}{K + [A, A]} \cong \tilde{A}[A, A]$$

we can consider the sequence

$$0 \to \overline{A} \to DR^1(A) \to [A, A] \to 0.$$

(2.56)

By virtue of (2.55) this sequence is also a complex; it is called the Quillen complex.

**Lemma 10.** Suppose the algebra $A$ is free. Then the complex (2.56) is exact.

This result is proved for instance in [9, Lemma 11.5.3] and has the following important consequence. Let $\alpha \in DR^1(A)$ be a 1-form on an associative affine space, and write $\alpha = \sum_{i=1}^k a_i dx_i$. Clearly

$$b(\alpha) = \sum_{i=1}^k [a_i, x_i]$$

and lemma [10] implies that $\alpha$ is exact if and only if the right-hand side vanishes. On the other hand, $\alpha$ is exact if and only if there exists a (non-constant) necklace word $f \in \overline{A}$ such that $\alpha = df$, in which case, as we saw above, $a_i$ is just the necklace derivative $\frac{\partial f}{\partial x_i}$. Putting all together, we obtain:

**Theorem 11.** Let $A$ be a free algebra and $\{g_1, \ldots, g_k\}$ be a subset of a set of generators for $A$. There exist words $u_1, \ldots, u_k \in A$ such that

$$\sum_{i=1}^k [u_i, g_i] = 0$$

if and only if there exists $f \in \overline{A}$ such that $u_i = \frac{\partial f}{\partial x_i}$ for every $i \in \{1, \ldots, k\}$. 

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2.6 Relative differential forms

In order to show other interesting examples of associative varieties it is necessary to generalize slightly the differential calculus set up in §2.3 by developing the notion of differential forms relative to a subalgebra, which was also introduced by Cuntz and Quillen in [22].

Let us assume that the associative algebra $A$ contains a commutative subalgebra $B$ that we interpret as an enlarged subspace of “scalars”. Then it is natural to require for a derivation defined on $A$ to vanish not only on $K$ but on the whole of $B$; that is, given an $A$-bimodule $M$ we should consider the set

$$\text{Der}_B(A, M) := \{ \theta \in \text{Der}(A, M) \mid \theta(b) = 0 \text{ for all } b \in B \}.$$  

This defines a subfunctor of the functor (2.3),

$$\text{Der}_B(A, \cdot) : A\text{-Bimod} \to \text{Set}.$$  

It turns out that this subfunctor also has a universal element: there exists a pair $(\Omega_1(A/B), d)$ consisting of an $A$-bimodule $\Omega_1(A/B)$, whose elements will be called the Kähler differentials of $A$ relative to $B$, and a derivation $d \in \text{Der}_B(A, \Omega_1(A/B))$ vanishing on $B$, such that for every other pair $(M, \theta)$ of this kind there exists a unique $A$-bimodule morphism $f_\theta : \Omega_1(A/B) \to M$ such that $f_\theta \circ d = \theta$.

Let us consider the tensor algebra over $A$ of this bimodule,

$$\Omega^\bullet(A/B) := T_A(\Omega_1(A/B)).$$  

To equip this (graded) algebra with a differential we need again to identify it with a suitably “relativized” version of the universal differential envelope introduced in §2.2. Namely, we consider the category $B\text{-dga}$ having as objects the differential graded algebras over the commutative algebra $B$ and as arrows the dg-algebra morphisms $f : (D, d) \to (E, d')$ such that $f|_B = \text{id}_B$. We have a functor

$$\cdot_0 : B\text{-dga} \to B\text{-Alg}$$  

sending a dg-algebra $(D, d)$ over $B$ to its degree zero part, which is an algebra over $B$; the functor (2.15) corresponds to the case $B = K$. It turns out that also in this more general setting the functor $(\cdot)_0$ has a left adjoint: for every $B$-algebra $A$ there exists a pair $(D(A/B), i)$ consisting of a dg-algebra $D(A/B)$ over $B$ and a $B$-algebra morphism $i : A \to D(A/B)_0$ such that for every other pair $(\Gamma, \psi)$ of this kind there exists a unique morphism of dg-algebras $u_\psi : D(A/B) \to \Gamma$ that makes the diagram

$$A \xymatrix{ & D(A/B)_0 \ar[dl]_i \ar[dr]^{u_\psi} \ar[d]_{\psi} & \\
 & \Gamma_0 & }$$  

commute in $B\text{-Alg}$. The algebra $D(A/B)$ can be defined in a way that closely parallels the construction of $D(A)$,

$$D(A/B) := \bigoplus_{n \in \mathbb{N}} \underbrace{D(A/B)_n}_{n \text{ times}} \quad \text{with} \quad D(A/B)_n := A \otimes_B \underbrace{\overline{A} \otimes_B \cdots \otimes_B \overline{A}}_{n \text{ times}},$$  

the only difference being that now the tensor products are over $B$ and $\overline{A}$ is defined to be

$$\overline{A} := A/B.$$  

(2.58)
In particular, $\Omega^1(A/B)$ is isomorphic to $A \otimes B A/B$. The differential is still given by the formula (2.18), and the product by the formula (2.19). Naturally enough, the pair $(\mathcal{D}(A/B), i)$ is called the universal differential envelope of $A$ relative to $B$.

The crucial result is that theorem 5 generalizes to the new setting:

**Theorem 12.** The graded algebras $\Omega^\bullet(A/B)$ and $\mathcal{D}(A/B)$ are isomorphic.

The proof of this fact can be found in [9] (Theorem 10.7.1). This means that we have a differential

$$d: \Omega^\bullet(A/B) \to \Omega^{\bullet+1}(A/B)$$

extending to every degree the universal derivation $d: A \to \Omega^1(A/B)$. Theorem 6 generalizes as follows:

$$H^k(\Omega^\bullet(A/B)) = \begin{cases} B & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(2.59)

Let us define the Karoubi-de Rham complex of $A$ relative to $B$ as the graded vector space

$$\text{DR}^\bullet(A/B) := \frac{\Omega^\bullet(A/B)}{[\Omega^\bullet(A/B), \Omega^\bullet(A/B)]}.$$  

(2.60)

The “absolute” Karoubi-de Rham complex of $A$ then coincides with $\text{DR}^\bullet(A/k)$. Also note that $\text{DR}^0(A/B) = A/[A, A]$ does not actually depend on $B$, so that the choice of scalars in $A$ does not affect the space of regular functions on the associative variety determined by $A$.

The differential calculus introduced in §2.3 readily extends to the relative case: for every derivation $\theta \in \text{Der}_B(A)$ relative to $B$ we have a degree $-1$ “interior product”

$$i_\theta: \text{DR}^\bullet(A/B) \to \text{DR}^\bullet(A/B)$$

and a degree 0 “Lie derivative”

$$\mathcal{L}_\theta: \text{DR}^\bullet(A/B) \to \text{DR}^\bullet(A/B)$$

whose concrete expressions are still given by (2.37) and (2.41), respectively.

The constructions in §2.5 generalize as follows. The recipe $uv \mapsto [u, v]$ still defines a map $\Omega^1(A/B) \to [A, A]$ that descends to a map $b: \text{DR}^1(A/B) \to [A, A]$. Moreover, if we define $\overline{\text{DR}}^0(A/B)$ to be the linear space

$$\overline{\text{DR}}^0(A/B) := \frac{A}{B + [A, A]}$$

then we can again write a “relative” Quillen complex as follows:

$$0 \longrightarrow \overline{\text{DR}}^0(A/B) \xrightarrow{d} \text{DR}^1(A/B) \xrightarrow{b} [A, A] \longrightarrow 0.$$  

(2.61)

However, this sequence may no longer be exact even when $A$ is free.

### 2.7 Quiver path algebras

Another important class of examples of associative varieties arises by considering path algebras of quivers; let us briefly review their construction.
A quiver is a directed graph with no constraints on the kind and the number of its edges; in particular it may have loops and/or multiple edges between the same pair of vertices, as for instance in

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\Rightarrow \\
\end{array}
\]

It is convenient to identify a quiver \( Q \) with the set of its edges; we shall denote its set of vertices by \( I_Q \), or simply by \( I \) if the particular quiver we are referring to is clear from the context. Given an edge \( \xi \) of a quiver \( Q \) we shall denote by \( s(\xi) \) its starting vertex, or source, and by \( t(\xi) \) its ending vertex, or target.

A path in a quiver \( Q \) is a finite sequence of continuous edges in \( Q \), or equivalently a word of the form \( \xi_1 \ldots \xi_\ell \) for some \( \ell \in \mathbb{N} \) where \( \xi_1, \ldots, \xi_\ell \in Q \) and \( s(\xi_i) = t(\xi_{i+1}) \) for every \( 1 \leq i \leq \ell - 1 \). (In keeping with standard practice, the edges that make up a path are written down going from the right to the left.) The maps \( s \) and \( t \) may be extended from edges to paths in the obvious way: \( s(\xi_1 \ldots \xi_\ell) = s(\xi_\ell) \) and \( t(\xi_1 \ldots \xi_\ell) = t(\xi_1) \).

The path algebra of a quiver \( Q \) over the field \( K \), denoted \( KQ \), is the \( K \)-linear space generated by all the paths in \( Q \) equipped with the product defined as follows: given two paths \( p_1 \) and \( p_2 \), their product \( p_1 \cdot p_2 \) is the concatenation of the two words \( p_1 \) and \( p_2 \) if \( s(p_1) = t(p_2) \) (that is, \( p_2 \) ends at the same vertex where \( p_1 \) starts) and zero otherwise. It is clear that \( KQ \) is an associative algebra over \( K \) which is not commutative in general.

We shall be concerned only with quivers having a finite vertex set, say \( I = \{1, \ldots, m\} \). For every \( i \in I \) we shall denote by \( e_i \) the trivial (length zero) path at the vertex \( i \). Obviously, each \( e_i \) is an idempotent element of \( KQ \). Moreover, the set \((e_1, \ldots, e_m)\) is a complete set of mutually orthogonal idempotents for \( KQ \), in the sense that

\[ e_i e_j = 0 \quad \text{when} \quad i \neq j \quad \text{and} \quad \sum_{i \in I} e_i = 1 \]

where 1 is the unit of the path algebra. It follows that, as a vector space, the path algebra decomposes as a direct sum of the form

\[ KQ = \bigoplus_{i,j \in I} (KQ)_{ji} \]

(2.62)

where \((KQ)_{ji} := e_j KQ e_i\) is the linear subspace of \( KQ \) spanned by all the paths \( i \to j \) in \( Q \).

The decomposition can be seen equivalently as follows. Denote by \( B \) the subalgebra of \( KQ \) generated by the idempotents \((e_i)_{i \in I}\). This algebra is isomorphic to \( K^m \), seen as a (commutative) \( K \)-algebra with the product defined componentwise. The embedding \( B \hookrightarrow KQ \) then makes \( KQ \) an algebra over \( B \) (in the sense explained in [22]), and in fact it is easy to check that

\[ KQ \simeq T_B(E_Q) \]

(2.63)

where \( E_Q \) is the \( B \)-bimodule spanned (as a \( K \)-linear space) by the arrows in \( Q \), with left and right actions defined by

\[ e_i \xi = \begin{cases} 
\xi & \text{if} \ t(\xi) = i \\
0 & \text{otherwise}
\end{cases} \quad \xi e_i = \begin{cases} 
\xi & \text{if} \ s(\xi) = i \\
0 & \text{otherwise}
\end{cases} \quad \text{for every} \ \xi \in Q, \ i \in I. \]

This \( B \)-bimodule structure is just a compact way to package all the incidence relations described by the quiver \( Q \).
The associative geometry of quiver path algebras has been studied in [27, 28]; let us review
the main results of these papers. Fix a quiver $Q$ and let $A := \mathbb{K}Q$. In order to obtain a good
theory one has to work relatively to the subalgebra $B \subseteq A$ introduced above, using the relative
differential calculus reviewed in \textsection 2.6. Intuitively, this means that derivations and differential
forms defined on $A$ must keep the vertices of the quiver fixed.

Let us start, then, by considering the complex $\Omega^\bullet(A/B)$, seen as the universal differential
envelope $\mathcal{D}(A/B)$. We would like to find a (linear) basis for the homogeneous component of
degree $n$,

$$\mathcal{D}(A/B)_n = A \otimes_B A/B \otimes_B \cdots \otimes_B A/B.$$ 

An element of this space may be written as

$$p_0 \otimes_B dp_1 \otimes_B \cdots \otimes_B dp_n$$

where $p_0, \ldots, p_n \in A$ and $p_1, \ldots, p_n$ are paths of nonzero length (so that their projection in $A/B$
is nonzero). Suppose that the path $p_{k+1}$ ends at vertex $i$ (that is, $e_ip_{k+1} = p_{k+1}$) and the path
$p_k$ starts at vertex $j$ ($p_ke_j = p_k$); then

$$dp_k \otimes_B dp_{k+1} = dp_k.e_j \otimes_B e_i.dp_{k+1} = dp_k.e_j e_i \otimes_B dp_{k+1}$$

which is zero unless $i = j$. Clearly these are the only possible relations between elements of $\mathcal{D}(A/B)_n$, so as a basis for this space we can take the set of decomposable tensors of the form

\textsection 2.6 where $p_1, \ldots, p_n$ are paths of length $\geq 1$ and $s(p_k) = t(p_{k+1})$ for every $0 \leq k \leq n$.

Consider now the relative Karoubi-de Rham complex $\mathcal{D}^\bullet(A/B)$, starting as usual from the
component of degree zero. Let us call a path $\xi_1, \ldots \xi_n \in A$ an (oriented) cycle if

$$t(\xi_1) = s(\xi_n)$$

that is, the path ends at the same vertex where it begins. A necklace word in the path algebra $A$ is a cycle considered up to cyclic permutations of its component arrows. As in the free algebra
case, it is not difficult to prove the following result (see [28, Lemma 3.4]):

\textbf{Lemma 13.} A basis for the linear space $\mathcal{D}^0(A)$ is given by the set of necklace words in $A$.

This gives a description for regular functions on the associative variety determined by $\mathbb{K}Q$
quite analogous to the one obtained in \textsection 2.4 for the associative affine space.

Now let us consider 1-forms. As we saw above, a basis for $\Omega^1(A/B)$ is provided by expressions
of the form $p_0dp_1$ with $s(p_0) = t(p_1)$. If $s(p_1) \neq t(p_0)$ then $p_0dp_1 = [p_0, dp_1]$ which is killed by the
projection onto $\mathcal{D}^1(A/B)$, so the paths $p_0$ and $p_1$ must form a cycle in $Q$. Then an induction
argument over the length of $p_1$ (see [28, Lemma 3.5]) shows that it suffices to consider the case
where $p_1$ has length 1, which means that it is an edge of the quiver. Summing up, we have the
linear isomorphism

$$\mathcal{D}^1(A/B) \simeq \bigoplus_{\xi: i \mapsto j} A_{ij}d\xi.$$ \hspace{1cm} (2.65)

An interesting consequence of these results is that the necklace derivative operators introduced
in \textsection 2.6 can be generalized to the path algebra of any quiver. Indeed, using the isomorphism
\textsection 2.6 the differential of any regular function $f \in \mathcal{D}^0(A)$ can be written in a unique way as

$$df = \sum_{\xi \in Q} p_\xi d\xi$$ \hspace{1cm} (2.66)
where each \( p_\xi \) is a path which goes in opposite direction with respect to \( \xi \). It follows that we can define a map
\[
\frac{\partial}{\partial \xi} : \text{DR}^0(A) \to A_{ij} \mapsto A
\]
by sending each \( f \in \text{DR}^0(A) \) to the path \( p_\xi \). We can then rewrite formula (2.66) as
\[
df = \sum_{\xi \in Q} \frac{\partial f}{\partial \xi} d\xi.
\]
This expression neatly generalizes formula (2.53), to which it reduces when \( Q \) is the quiver with a single vertex and \( n \) loops.

In practice, the action of a necklace derivative \( \frac{\partial}{\partial \xi} \) on a necklace word \( f \) is computed in the same manner as in the free algebra case: for each occurrence of the arrow \( \xi \) in \( f \) we write down the path obtained by removing that arrow from the necklace (starting from the arrow immediately after it) and then take the sum of all the resulting paths. Explicitly, \( \eta_1 \ldots \eta_\ell \in A \) is a representative for \( f \) with \( \eta_1, \ldots, \eta_\ell \in Q \) then
\[
\frac{\partial f}{\partial \xi} = \sum_{k=1}^\ell \delta_{\eta_k \xi} \eta_{k+1} \ldots \eta_{\ell} \eta_1 \ldots \eta_{k-1}.
\]
The cohomology of the Karoubi-de Rham complex of \( A \) can also be explicitly calculated, as first shown in [27, 28].

**Theorem 14.** Let \( A \) be the path algebra of a quiver \( Q \). Then
\[
H^k(\text{DR}^\bullet(A/B)) = \begin{cases} B & \text{for } k = 0 \\ 0 & \text{for } k \geq 1. \end{cases}
\] (2.67)

This result shows that if the “right” choice for the subalgebra of scalar functions is made then the associative variety determined by a quiver has the same cohomology as a contractible space, exactly as it happens for associative affine spaces (theorem 9).

3 Representation spaces

In this section we review the connection between geometric objects defined on an associative algebra \( A \) and the corresponding objects defined on the representation spaces of \( A \), thereby making contact between the associative and the commutative worlds. Our main references for this part are [9], [15] and [29].

3.1 Representation spaces and their quotients

From now on we suppose that the associative algebra \( A \) is finitely generated, that is there exists a natural number \( n \in \mathbb{N} \) such that \( A \) may be presented as a quotient
\[
A \simeq \mathbb{k}\langle x_1, \ldots, x_n \rangle/I
\]
of the free algebra on \( n \) generators by a two-sided ideal \( I \). This implies that the dg-algebra \( \Omega^\bullet(A) \) is also finitely generated, and that each homogeneous component \( \Omega^k(A) \) is finitely generated as an \( A \)-bimodule.
For every \( d \in \mathbb{N} \) a \textbf{\( d \)-dimensional representation} of \( A \) is a morphism of \( \mathbb{K} \)-algebras

\[
\rho: A \to \text{Mat}_{d,d}(\mathbb{K}),
\]

where \( \text{Mat}_{d,d}(\mathbb{K}) \) denotes the algebra of \( d \times d \) matrices with entries in \( \mathbb{K} \). It is natural to interpret such matrices as linear endomorphisms of \( \mathbb{K}^d \) expressed in the canonical basis; then each representation \( \rho \) defines a left \( A \)-module structure on \( \mathbb{K}^d \) by putting

\[
a.v := \rho(a)v \quad \text{for every } a \in A, \ v \in \mathbb{K}^d.
\]

Conversely, suppose \( V \) is a \( d \)-dimensional vector space over \( \mathbb{K} \) equipped with the structure of a left \( A \)-module. Then the choice of a basis \( E \) for \( V \) determines, by the same rule, a map \( \rho: A \to \text{Mat}_{d,d}(\mathbb{K}) \). Moreover, for every pair \( a, b \in A \) one has that

\[
\rho(ab)v = ab.v = a.(b.v) = a.\rho(b)v = \rho(a)\rho(b)v \quad \text{for every } v \in \mathbb{K}^d.
\]

It follows that \( \rho(ab) = \rho(a)\rho(b) \) for every \( a, b \in A \), that is the map \( \rho \) is a morphism of \( \mathbb{K} \)-algebras, hence a \( d \)-dimensional representation of \( A \) in the original sense.

Notice that there is a certain amount of arbitrariness in this correspondence between representations of \( A \) and left \( A \)-modules, which is given by the choice of the basis \( E \). Choosing a different basis \( E' \) amounts to the choice of a different isomorphism \( V \cong \mathbb{K}^d \), leading to a different map \( \rho': A \to \text{Mat}_{d,d}(\mathbb{K}) \). The two maps \( \rho \) and \( \rho' \) are then related by the equality

\[
\rho'(a) = g\rho(a)g^{-1} \quad \text{for every } a \in A,
\]

where \( g \in \text{GL}_{d}(\mathbb{K}) \) is the invertible \( d \times d \) matrix which realizes the change of basis from \( E \) to \( E' \). We say in this case that the representations \( \rho \) and \( \rho' \) are \textbf{equivalent}. One of the basic goals in the representation theory of associative algebras is to classify the equivalence classes of finite-dimensional representations of \( A \), or (equivalently) the \( A \)-modules of finite dimension\(^8\).

In order to attack this problem from a geometric perspective let us consider the space of all \( d \)-dimensional representation of the algebra \( A \),

\[
\text{Rep}_d^A : = \mathbb{K} \text{-Alg}(A, \text{Mat}_{d,d}(\mathbb{K})).
\]

We shall show, following [15 Chapter 2], how this space can be seen in a natural way as an \textbf{affine scheme}. It is convenient to start from the special case in which the algebra \( A \) is free, say on the \( n \) generators \( x_1, \ldots, x_n \). Then any \( d \)-dimensional representation of \( A \) can be specified simply by picking a \( n \)-tuple of \( d \times d \) matrices \( (X_1, \ldots, X_n) \) and declaring that \( X_i \) is the image of the generator \( x_i \) for every \( i = 1 \ldots n \). It follows that

\[
\text{Rep}_d^A = \underbrace{\text{Mat}_{d,d}(\mathbb{K}) \oplus \cdots \oplus \text{Mat}_{d,d}(\mathbb{K})}_{n \text{ times}} = \text{Mat}_{d,d}(\mathbb{K})^\oplus n
\]

This is clearly an affine algebraic variety (hence, in particular, a scheme); the corresponding coordinate algebra, that we are going to denote \( \mathcal{A}_{n,d} \), is isomorphic to the polynomial ring over \( \mathbb{K} \) generated by the \( nd^2 \) indeterminates \( (x_{i,j,k})_{i=1 \ldots n,j,k=1 \ldots d} \) representing each entry in a generic \( n \)-tuple of \( d \times d \) matrices:

\[
X_1 = \begin{pmatrix}
  x_{1,11} & \cdots & x_{1,1d} \\
  \vdots & \ddots & \vdots \\
  x_{1,d1} & \cdots & x_{1,dd}
\end{pmatrix}
\quad \ldots \quad
X_n = \begin{pmatrix}
  x_{n,11} & \cdots & x_{n,1d} \\
  \vdots & \ddots & \vdots \\
  x_{n,d1} & \cdots & x_{n,dd}
\end{pmatrix}
\]

\(^8\text{It is important to note that there exist associative algebras having no finite-dimensional representations. For such algebras one must necessarily consider representations of a more general kind, for example as linear operators on an infinite-dimensional Hilbert space.}\)
Now let us return to the general case of a finitely generated algebra $A$, presented as in $\text{(3.3)}$. Then a $d$-dimensional representation of $A$ may again be specified by a $n$-tuple of $d \times d$ matrices $(X_1, \ldots, X_n)$ such that the map $K\langle x_1, \ldots, x_n \rangle \to \text{Mat}_{d,d}(K)$ defined by $x_i \mapsto X_i$ descends to the quotient $\text{(3.4)}$. But this happens if and only if the matrices $(X_1, \ldots, X_n)$ satisfy every relation determined by the ideal $I$, that is the equation

$$p(X_1, \ldots, X_n) = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}$$

holds for every $p \in I$ (seen as a noncommutative polynomial in $n$ indeterminates).

On the other hand, for each $p \in I$ we can interpret the left-hand side of equation $\text{(3.7)}$ as the evaluation of the noncommutative polynomial $p$ on the generic $n$-tuple of matrices $\text{(3.6)}$. The $d^2$ entries of this matrix are then polynomials in the indeterminates $x_{i,j}$ that generate the algebra $A_{n,d}$. Let us denote by $J_A$ the ideal of $A_{n,d}$ generated by all these polynomials as $p$ varies in $I$. The above argument then shows that the set of $d$-dimensional representations of $A$ coincides with the zero locus of the ideal $J_A$ in $\text{Mat}_{d,d}(K)^{\otimes n}$; it is thus an affine scheme (of finite type over $K$), as claimed.

Notice that, since the ideal $J_A$ defined above is not necessarily radical, the scheme $\text{Rep}_d^A$ is not a variety in general. However, this will always be the case for the particular examples we shall be interested in (namely, free algebras and quiver path algebras). For this reason, in what follows we shall usually avoid the more sophisticated scheme-theoretic point of view and regard $\text{Rep}_d^A$ simply as an affine algebraic variety.

To translate the notion of equivalent representations in geometric terms we note that the correspondence $\text{(3.3)}$ is naturally interpreted as the definition of a (left) action of the group $\text{GL}_d(K)$ on $\text{Rep}_d^A$: given $g \in \text{GL}_d(K)$ and $\rho \in \text{Rep}_d^A$, the representation $g.\rho$ is defined by

$$(g.\rho)(a) := g\rho(a)g^{-1} \quad \text{for every } a \in A.$$  

$\text{(3.8)}$

In fact the center of $\text{GL}_d(K)$ acts trivially, so that strictly speaking the group acting is rather its quotient $G_d := \text{PGL}_d(K)$. Clearly, two representations are equivalent if and only if they are related by the action of an element $g \in G_d$. We conclude that the equivalence classes of $d$-dimensional representations of $A$ are in one to one correspondence with the orbits of the group action $\text{(3.3)}$ on $\text{Rep}_d^A$. The fundamental goal becomes then to describe those orbits.

The modern approach to the study of group actions on affine algebraic varieties goes under the name of geometric invariant theory. It is obviously impossible for us to do justice to this huge topic here. We direct the reader to the standard reference $\text{[30]}$ for a comprehensive treatment; see also $\text{[31]}$ for a more concise introduction. We shall content ourselves with briefly summarizing some results which shed some light on the above-mentioned problem.

First of all, we remind the reader about the standard notion of quotient in the algebro-geometric context. Let $G$ be an algebraic group acting on an affine variety $X$. A categorical quotient for this action (in the category of affine algebraic varieties) is an affine variety $X//G$ together with a morphism $\pi : X \to X//G$ such that:

1. $\pi$ is $G$-invariant: $\pi(g.x) = \pi(x)$ for every $g \in G$ and $x \in X$;

2. $\pi$ is universal among such morphisms: for every $G$-invariant morphism $f : X \to Z$ there exists a unique morphism $k : X//G \to Z$ such that $f = k \circ \pi$.

As for any universal construction, if a categorical quotient exists then it is unique up to a unique isomorphism.
One of the main results of geometric invariant theory is that when $G$ is a reductive algebraic group acting on an affine variety $X$, the categorical quotient $X//G$ always exists. This follows from a basic result known as the Nagata-Hilbert theorem:

**Theorem 15.** Let $G$ be a reductive algebraic group acting on the affine algebraic variety $X$. Then the subalgebra $\mathbb{K}[X]^G \subseteq \mathbb{K}[X]$ consisting of $G$-invariant regular functions on $X$ is finitely generated.

The construction of the quotient then proceeds in the following way. We choose a set $(f_1,\ldots,f_m)$ of generators for $\mathbb{K}[X]^G$ and consider the morphism $X \to \mathbb{K}^m$ defined by $x \mapsto (f_1(x),\ldots,f_m(x))$.

The image $Y \subseteq \mathbb{K}^m$ of this morphism is closed and independent of the chosen generating set. The induced surjective morphism $\pi: X \to Y$ is clearly $G$-invariant, and it can be shown that it is also universal (in the sense of point 2 above). Thus the variety $Y$ is isomorphic to the categorical quotient $X//G$ for the given action.

It is important to note that the fibers of the quotient map $\pi$ will not consist, in general, of single orbits. However it can be proved that each fiber of $\pi$ contains a unique closed orbit, so that the variety $X//G$ may be seen as a moduli space for closed $G$-orbits.

Let us return now to the specific setting of representation spaces. As the group $G_d$ is reductive and the variety $\text{Rep}_A^d$ is affine, the previous results can be applied in a straightforward way. Moreover, closed orbits in $\text{Rep}_A^d$ are characterized by the following fundamental result, due to M. Artin [32].

**Theorem 16.** The orbit of a representation $\rho$ is closed in $\text{Rep}_A^d$ if and only if the corresponding $A$-module is semisimple.

Putting all together, it follows that for each $d \in \mathbb{N}$ there exists an affine algebraic variety (or scheme)

$$\mathcal{R}_A^d := \text{Rep}_A^d // G_d,$$

equipped with a surjective morphism $\pi: \text{Rep}_A^d \to \mathcal{R}_A^d$, that parametrizes equivalence classes of semisimple $d$-dimensional representations of $A$.

Let us remark that, depending on the situation at hand, the categorical quotient (3.9) may not be the best choice as a quotient space for $\text{Rep}_A^d$; for instance it can be too small, or too big, or too singular. It is possible, and often useful, to define a more general class of quotients by taking a $G_d$-invariant open subset in $\text{Rep}_A^d$ defined by suitable (semi)stability conditions and looking for a variety which parametrizes those $G_d$-orbits which are closed in this subset (see again [30] for the general theory of these "GIT quotients"). In the next section we shall see a particular case of this approach, which takes advantage of some additional structure on $\text{Rep}_A^d$ (namely a symplectic form) in order to construct smaller and more tractable quotient spaces.

### 3.2 The correspondence between the associative and the commutative worlds

We shall now explain, following [9, Section 12], how each associative-geometric object defined on the algebra $A$ induces a corresponding $G_d$-invariant object on the space of $d$-dimensional representations of $A$, and consequently a geometric object on its quotient spaces.

---

9 A linear algebraic group is called reductive if it does not contain any closed normal unipotent subgroup. Many commonly used groups are reductive, including all semisimple groups and general linear groups.

10 For scheme-theoretically inclined readers it is perhaps easier to think about the categorical quotient as the spectrum of the ring of invariants $\mathbb{K}[X]^G$; the morphism $\pi$ is then obtained from the algebra embedding $\mathbb{K}[X]^G \hookrightarrow \mathbb{K}[X]$ by duality.
Let us start from regular functions. By definition, the space of representations of $A$ comes equipped with an evaluation map

$$\text{Rep}_d^A \times A \to \text{Mat}_{d,d}(\mathbb{K})$$

given by $(\rho, a) \mapsto \rho(a)$. Keeping the second argument of this map fixed we see that every $a \in A$ determines a matrix-valued function on $\text{Rep}_d^A$, that is a map $\hat{a}: \text{Rep}_d^A \to \text{Mat}_{d,d}(\mathbb{K})$ given by $\rho \mapsto \rho(a)$ for every $\rho \in \text{Rep}_d^A$. Taking the trace of the resulting matrix we obtain a genuine function $\hat{a}: \text{Rep}_d^A \to \mathbb{K}$. Explicitly,

$$\hat{a}(\rho) = \text{tr} \hat{a}(\rho) = \text{tr} \rho(a).$$

Since $a$ can be expressed as a polynomial in some generating set $\{x_1, \ldots, x_n\}$ for $A$ we see that $\hat{a}(\rho)$ can be expressed as a polynomial in the entries of the matrices $\rho(x_1), \ldots, \rho(x_n)$. But these entries generate the coordinate algebra of $\text{Rep}_d^A$, hence $\hat{a}$ is a regular function on $\text{Rep}_d^A$. It follows that the correspondence $a \mapsto \hat{a}$ defines a map

$$\phi: A \to \mathbb{K}[\text{Rep}_d^A]. \quad (3.10)$$

Observe that this map is $\mathbb{K}$-linear, since for every $a, b \in A$ one has

$$\hat{a} + \hat{b}(\rho) = \text{tr} \rho(a + b) = \text{tr}(\rho(a) + \rho(b)) = \text{tr} \rho(a) + \text{tr} \rho(b) = \hat{a}(\rho) + \hat{b}(\rho),$$

whence $\phi(a + b) = \phi(a) + \phi(b)$, and similarly for every $\lambda \in \mathbb{K}$ and $a \in A$ one has

$$\lambda \hat{a}(\rho) = \text{tr} \rho(\lambda a) = \text{tr}(\lambda \rho(a)) = \lambda \text{tr} \rho(a) = \lambda \hat{a}(\rho),$$

whence $\phi(\lambda a) = \lambda \phi(a)$.

Moreover, the map $\phi$ vanishes on the linear subspace $[A, A] \subseteq A$. To check this it is sufficient to show that $\hat{c} = 0$ for every $c \in A$ that can be written as a commutator, say $c = [a, b]$. But then

$$\hat{c}(\rho) = \text{tr} \rho(c) = \text{tr}(\rho(a) \rho(b) - \rho(b) \rho(a)) = 0$$

by the cyclicity of the trace. It follows that the map $\phi$ descends from $A$ to DR$^0(A)$.

Finally, we claim that the image of $\phi$ is contained in the subalgebra of $\mathbb{K}[\text{Rep}_d^A]$ consisting of $G_d$-invariant functions. To see this let us start by noting that DR$^0(A)$ is generated, as a linear space, by the necklace words in $A$. Then it suffices to show that for every necklace word $a = a_1 \ldots a_{\ell}$ the regular function $\hat{a}$ is constant along $G_d$-orbits. Let us take $\rho \in \text{Rep}_d^A$, $g \in G_d$ and define $\rho' := g \rho$; then

$$\hat{a}(\rho') = \text{tr} \rho'(a_1) \ldots \rho'(a_{\ell}) = \text{tr} \rho(a_1)g^{-1}g\rho(a_2)g^{-1} \ldots g\rho(a_{\ell})g^{-1} = \text{tr} \rho(a_1) \ldots \rho(a_{\ell}) = \hat{a}(\rho)$$

as claimed. We conclude that there is a well defined linear map

$$\phi: \text{DR}^0(A) \to \mathbb{K}[\text{Rep}_d^A]^{G_d} \quad (3.11)$$

which sends a necklace word $a_1 \ldots a_{\ell}$ to the $G_d$-invariant function

$$\rho \mapsto \text{tr} \rho(a_1) \ldots \rho(a_{\ell})$$

on $\text{Rep}_d^A$. In this sense each regular function on the associative variety determined by $A$ induces a corresponding regular function on each quotient space $\mathcal{R}_d^A$. 

26
It should be stressed that the map (3.11) is far from being surjective in general. However, it follows from the first fundamental theorem of matrix invariants (see e.g. [15, Theorem 1.6]) that the image of \( \phi \) generates \( K[\text{Rep}_d^A G_d] \) as an algebra.

We can now clear up the mystery regarding the lack of a product between regular functions on associative varieties. The point is, of course, that the map (3.10) is not a morphism of algebras: given \( a, b \in A \) we have that
\[
\hat{a} \hat{b}(\rho) = \text{tr} \rho(ab) = \text{tr} \rho(a) \rho(b),
\]
which is different from the regular function on \( \text{Rep}_d^A \) given by
\[
\hat{a}(\rho) \cdot \hat{b}(\rho) = \text{tr} \rho(a) \cdot \text{tr} \rho(b)
\]
as the trace of a product is not the product of the traces. Obviously there is a well-defined product between invariant functions on each representation space \( \text{Rep}_d^A \), but this product does not come from the multiplication map on the algebra \( A \) (nor it is easily expressible in terms of the latter).

To further elaborate on this point let us consider in detail the case of associative affine space, \( A = K\langle x_1, \ldots, x_n \rangle \). As explained in the previous subsection, \( \text{Rep}_d^A \) can be identified with the linear space \( \text{Mat}_{d,d}(K)^{\oplus n} \). The map (3.11) then sends a generic necklace word \( f = x_{i_1} \ldots x_{i_\ell} \) (with \( i_1, \ldots, i_\ell \in \{1, \ldots, n\} \)) to the \( G_d \)-invariant function
\[
\hat{f}(X_1, \ldots, X_n) = \text{tr} X_{i_1} \ldots X_{i_\ell}.
\]

Not every invariant function on \( \text{Mat}_{d,d}(K)^{\oplus n} \) is of this form; for instance there is no hope of getting the function \( \text{tr} X_1 \cdot \text{tr} X_2 \) from an element of \( \text{DR}_d^0(A) \). Moreover, even the functions in the image of \( \phi \) are subject to a certain set of relations depending on \( d \) (see [15, Chapter 1]). For instance when \( d = 2 \) one has the relation
\[
\text{tr} X_1 X_2 X_3 + \text{tr} X_2 X_1 X_3 - \text{tr} X_1 \text{tr} X_2 X_3 - \text{tr} X_2 \text{tr} X_1 X_3 + \text{tr} X_1 \text{tr} X_2 \text{tr} X_3 - \text{tr} X_1 X_2 X_3 = 0.
\]
These relations are also invisible at the level of the linear space \( \text{DR}_d^0(A) \).

Now let us turn our attention to the vector fields. In order to establish a correspondence between associative vector fields on \( A \) (that is, derivations \( A \to A \)) and ordinary (algebraic) vector fields on \( \text{Rep}_d^A \) we need a description for the tangent space to a point \( \rho \in \text{Rep}_d^A \). In fact it is not difficult to prove (see [9, §12.4]) that there is an isomorphism
\[
T_\rho \text{Rep}_d^A \simeq \text{Der}(A, \text{Mat}_{d,d}(K))
\]
where the \( A \)-bimodule structure on \( \text{Mat}_{d,d}(K) \) is given by the left and right actions defined, respectively, by
\[
a.M := \rho(a)M \quad \text{and} \quad M.b := M \rho(b)
\]
for every \( a, b \in A \) and \( M \in \text{Mat}_{d,d}(K) \). More explicitly, a tangent vector to \( \text{Rep}_d^A \) at \( \rho \) is specified by a \( K \)-linear map \( \varphi : A \to \text{Mat}_{d,d}(K) \) such that
\[
\varphi(ab) = \varphi(a)\rho(b) + \rho(a)\varphi(b) \quad \text{for every} \ a, b \in A.
\]
Now let \( \theta \) be a derivation \( A \to A \); we wish to define a corresponding global vector field \( \hat{\theta} \) on \( \text{Rep}_d^A \). This means that for every \( \rho \in \text{Rep}_d^A \) we need to specify a derivation \( \hat{\theta}_\rho \) from \( A \) to \( \text{Mat}_{d,d}(K) \) (with the \( A \)-bimodule structure induced by \( \rho \)). We set
\[
\hat{\theta}_\rho(a) := \rho(\theta(a)) = \hat{\theta}(a)(\rho).
\]
The derivation property is easy to check:

\[ \tilde{\theta}_\rho(ab) = \rho(\theta(ab)) = \rho(\theta(a)b + a\theta(b)) = \rho(\theta(a))\rho(b) + \rho(a)\rho(\theta(b)) = \tilde{\theta}_\rho(a)\rho(b) + \rho(a)\tilde{\theta}_\rho(b). \]

We also have the following important result, whose (non-trivial) proof can be found in [10, Proposition 12.4.4].

**Theorem 17.** For any finitely generated associative algebra \( A \) the map \( \text{Der}(A) \to \Gamma(\mathcal{T} \text{Rep}_A^1) \) defined by \( \theta \mapsto \tilde{\theta} \) is a morphism of Lie algebras.

Here \( \mathcal{T} \text{Rep}_A^1 \) denotes the tangent sheaf to \( \text{Rep}_A^1 \) and \( \Gamma(\mathcal{T} \text{Rep}_A^1) \) its space of global sections.

It can be worthwhile to see explicitly how this correspondence works in the case of the \( n \)-th polynomial ring \( \mathbb{K} \langle x_1, \ldots, x_n \rangle \). A derivation on a free algebra is completely specified by sending each generator \( x_i \) to any chosen element \( f_i \in A \). Let us write the derivation defined in this way as

\[ \theta(x_1, \ldots, x_n) = (f_1, \ldots, f_n). \]

The corresponding vector field on \( \text{Rep}_A^1 = \text{Mat}_{d,d}(\mathbb{K})^\oplus_n \), then, is simply

\[ \hat{\theta}(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) \]

where we have used the fact that the tangent space to a linear space is canonically isomorphic to the linear space itself.

Finally let us consider the correspondence between associative \( p \)-forms on \( A \) (that is, elements of the Karoubi-De Rham complex \( \text{DR}^p(A) \)) and ordinary differential forms on \( \text{Rep}_A^1 \). We shall denote by \( \Omega^p \text{Rep}_A^1 \) the sheaf of (ordinary) differential \( p \)-forms on the affine variety \( \text{Rep}_A^1 \). Notice that \( \Omega^* \text{Rep}_A^1 \) is a dg-algebra when equipped with the ordinary exterior differential and exterior product.

We start from Kähler differentials. Given \( \alpha \in \Omega^1(A) \), say \( \alpha = a_0 da_1 \), we consider the *matrix-valued differential form* on \( \text{Rep}_A^1 \) whose value at \( \rho \in \text{Rep}_A^1 \) is given by the matrix product

\[ \tilde{\alpha}_\rho := \tilde{a}_0(\rho) \cdot \tilde{d}a_1(\rho) \]

where \( \tilde{d}a_1 \) is the differential of the matrix-valued function \( \tilde{a}_1: \text{Rep}_A^1 \to \text{Mat}_{d,d}(\mathbb{K}) \). (In other words, the \((i,j)\) entry of the matrix \( \tilde{d}a_1(\rho) \) is the differential of \( \tilde{a}_1(\rho)_{ij} \), seen as a function of \( \rho \).) Clearly \( \tilde{\alpha} \) is an element of \( \Gamma(\Omega^1 \text{Rep}_A^1) \otimes \text{Mat}_{d,d}(\mathbb{K}) \), that is a matrix-valued global section of the sheaf of 1-forms on \( \text{Rep}_A^1 \). Exactly as we did above for regular functions, we can turn \( \tilde{\alpha} \) into a scalar-valued differential form by taking traces. We denote by \( \hat{\alpha} \) the corresponding global section of \( \Omega^1 \text{Rep}_A^1 \):

\[ \hat{\alpha}_\rho = \text{tr} \tilde{a}_0(\rho) \tilde{d}a_1(\rho). \]

It is again immediate to check that the correspondence \( \alpha \mapsto \hat{\alpha} \) defines a linear map \( \Omega^1(A) \to \Gamma(\Omega^1 \text{Rep}_A^1) \) which vanishes on the linear subspace \( [A, \Omega^1(A)] \subseteq \Omega^1(A) \) and is constant along \( G_d \)-orbits. This correspondence then induces a map

\[ \text{DR}^1(A) \to \Gamma(\Omega^1 \text{Rep}_A^1)^{G_d} \]

which realizes the correspondence between associative 1-forms and \( G_d \)-invariant differential forms on \( \text{Rep}_A^1 \).

A similar procedure works for differential forms of degree \( p > 1 \): given a \( p \)-form \( \omega \in \Omega^p(A) \), say \( \omega = a_0 da_1 \ldots da_p \), there is a corresponding matrix-valued differential \( p \)-form on \( \text{Rep}_A^1 \) (that is, an element of \( \Gamma(\Omega^p \text{Rep}_A^1) \otimes \text{Mat}_{d,d}(\mathbb{K}) \)) whose value at \( \rho \) is

\[ \tilde{\omega}_\rho := \tilde{a}_0(\rho) \cdot \tilde{d}a_1(\rho) \wedge \cdots \wedge \tilde{d}a_p(\rho) \]
where the exterior product \( \wedge \) is extended from 1-forms to \( d \times d \) matrices of 1-forms in the obvious way.\(^{11}\)

Taking the trace of the resulting matrix we get the scalar-valued \( p \)-form

\[
\tilde{\omega}_\rho = \text{tr} \left( \tilde{a}_0(\rho) \cdot d\tilde{a}_1(\rho) \wedge \cdots \wedge d\tilde{a}_p(\rho) \right).
\]

The map \( \omega \mapsto \tilde{\omega} \) vanishes on the subspace of \( \Omega^p(A) \) spanned by graded commutators and is constant along the orbits of the group \( G_d \). Thus we obtain a map

\[
\text{DR}^*(A) \to \Gamma(\Omega^* \text{Rep}_A^A)_{G_d}
\]

sending associative differential forms to \( G_d \)-invariant differential forms on \( \text{Rep}_A^d \).

For example when \( A = \mathbb{K}(x_1, x_2) \) the associative 1-form \( \alpha = x_2dx_1 \) corresponds to the differential form

\[
\hat{\alpha}_{(x_1, x_2)} = \text{tr} X_2dX_1
\]
on \( \text{Mat}_{d,d}(\mathbb{K}) \oplus \text{Mat}_{d,d}(\mathbb{K}) \), where \( dX_1 \) is the matrix of differentials \( (dx_{1,ij})_{i,j=1 \ldots d} \). When \( d = 2 \) the corresponding coordinate expression for \( \hat{\alpha} \) is

\[
\text{tr} \begin{pmatrix} x_{1,11} & x_{1,12} \\ x_{2,21} & x_{2,22} \end{pmatrix} \begin{pmatrix} dx_{1,11} & dx_{1,12} \\ dx_{1,21} & dx_{1,22} \end{pmatrix} = x_{1,11}dx_{1,11} + x_{1,12}dx_{1,21} + x_{2,21}dx_{1,12} + x_{2,22}dx_{1,22}.
\]

Similarly, given the 2-form \( \omega = x_1dx_2dx_1 \) (or rather its equivalence class in \( \text{DR}^2(A) \)) the corresponding 2-form on \( \text{Mat}_{d,d}(\mathbb{K}) \oplus \text{Mat}_{d,d}(\mathbb{K}) \) reads

\[
\tilde{\omega}_{(x_1, x_2)} = \text{tr}(X_1dX_2 \wedge X_1dX_1).
\]

When \( d = 2 \) the corresponding coordinate expression in the basis consisting of the 2-forms \( dx_{i,jk} \wedge dx_{\ell,pq} \) involves 16 terms, and the count goes up very quickly as \( d \) increases. Already from these simple examples the convenience in dealing with associative forms compared to ordinary ones is rather evident.

### 3.3 Quiver representation spaces

We now consider in particular the case when \( A \) is the path algebra of a quiver. In this connection let us note that quivers are a fundamental tool in the representation theory of associative algebras; we refer the interested reader to the textbook [33] for more information about this topic.

Let \( A = \mathbb{K}Q \) be the path algebra of a quiver \( Q \) with vertex set \( I = \{1, \ldots, m\} \). Recall the important role played by the (finite-dimensional, semisimple, commutative) subalgebra \( B \subseteq A \) spanned by the complete set of idempotents \( (e_i)_{i \in I} \) corresponding to trivial paths in \( Q \). As we saw in §2.7 the path algebra \( A \) can be seen as a tensor algebra over \( B \); it is then natural to consider only those representations of \( A \) which keep track of this structure.

Observe now that \( B \)-algebra structures on \( \text{Mat}_{d,d}(\mathbb{K}) \) are in one to one correspondence with direct sum decompositions of the linear space \( V := \mathbb{K}^d \),

\[
V = \bigoplus_{i \in I} V_i,
\]
such that \( \sum_i \dim V_i = d \). Explicitly, one defines a morphism \( B \to \text{Mat}_{d,d}(\mathbb{K}) \) by sending the idempotent \( e_i \) to the matrix representing the map \( j_i \circ \pi_i : V \to V_i \), where \( \pi_i : V \to V_i \) is the

\(^{11}\)Namely, \((A \wedge B)_{ij} = \sum_{k=1}^d A_{ik} \wedge B_{kj} \). Notice that the resulting product is no longer skew-symmetric.
canonical projection and \( j_i : V_i \to V \) is the canonical immersion of the \( i \)-th factor. As the only invariants of the decomposition (3.13) are the dimensions of the subspaces \( V_i \), we conclude that each \( B \)-algebra structure on \( \text{Mat}_{d,d}(K) \) is completely specified by a vector

\[
d = (d_1, \ldots, d_m) \in \mathbb{N}^m
\]

such that \( d_1 + \cdots + d_m = d \). This \( m \)-tuple of natural numbers is called the **dimension vector** of the representation.

Now we would like to characterize the space of representations of \( A \) with a fixed dimension vector \( d \). To this end recall the natural isomorphism (2.25) given by the universal property of the tensor algebra. In the present situation, it can be used to obtain a bijection

\[
B \downarrow \mathbb{K}\text{-Alg}(T_B(E_Q), \text{Mat}_{d,d}(K)) \cong B\text{-Bimod}(E_Q, U(\text{Mat}_{d,d}(K)))
\]

between the set of \( B \)-algebra morphisms from \( T_B(E_Q) \simeq A \) to \( \text{Mat}_{d,d}(K) \) (that is, representations of the path algebra which respect the \( B \)-algebra structure) and the set of \( B \)-bimodule morphisms \( E_Q \to \text{Mat}_{d,d}(K) \). Such a morphism is completely determined by sending each arrow \( \xi : i \to j \) in \( Q \) to a linear map \( \rho(\xi) : V_i \to V_j \) between the subspaces corresponding to the source and target vertices of \( \xi \). It follows that the space of representations of \( A \) with dimension vector \( d = (d_1, \ldots, d_m) \) coincides with the linear space

\[
\text{Rep}(Q,d) := \bigoplus_{i,j \in I} \bigoplus_{\xi : i \to j} \text{Mat}_{d_j, d_i}(K).
\]

(3.14)

The notion of equivalence between representations must also be slightly adjusted, in order to preserve the chosen \( B \)-algebra structure on \( \text{Mat}_{d,d}(K) \). Namely, we consider the subgroup of \( \text{GL}(V) = \text{GL}_{d_i}(K) \) consisting of the endomorphisms of \( V \) which preserve the direct sum decomposition (3.13). This means acting on each subspace \( V_i \) with a copy of the general linear group \( \text{GL}(V_i) = \text{GL}_{d_i}(K) \), hence the subgroup in question is

\[
\prod_{i \in I} \text{GL}_{d_i}(K).
\]

(3.15)

Explicitly, the action of an \( m \)-tuple \( g = (g_1, \ldots, g_m) \) on a point \( \rho \in \text{Rep}(Q,d) \) is

\[
\rho(\xi) \mapsto g_j \rho(\xi) g_i^{-1} \quad \text{for every } \xi : i \to j \in Q.
\]

It is immediate to note that the subgroup \( H \) consisting of \( m \)-tuples of the form \( (\lambda I_{d_1}, \ldots, \lambda I_{d_m}) \) for some \( \lambda \in \mathbb{K}^* \) acts trivially on \( \text{Rep}(Q,d) \), so that we can just as well consider the group

\[
G_d := \left( \prod_{i \in I} \text{GL}_{d_i}(K) \right) / H.
\]

(3.16)

We are now in the same situation already considered in §3.1, namely, we have the action of the linear reductive group (3.16) on the affine algebraic variety (3.14). We can thus consider the corresponding categorical quotient,

\[
\text{Rep}(Q,d) / G_d,
\]

(3.17)

whose points correspond to closed \( G_d \)-orbits in \( \text{Rep}(Q,d) \), that is equivalence classes of semisimple representations of \( Q \) with dimension vector \( d \). These spaces have been extensively studied in the literature, starting from the seminal paper [34].
As remarked at the end of §3.1, one can also consider more general GIT quotients of \( \text{Rep}(Q, d) \) obtained by imposing suitable (semi)stability conditions. The reader can hardly do better than consult [29] for a detailed review of the moduli problem for quiver representations.

The correspondence between associative and commutative objects described in §3.2 generalizes to the above setting as soon as we consider derivations and differential forms on \( A \) relative to the subalgebra \( B \). For instance, the condition that a derivation \( \theta: A \to A \) vanishes on \( B \) is exactly what is needed in order to insure that the induced vector field \( \hat{\theta} \) on \( \text{Rep}(Q, d) \) preserves the chosen \( B \)-bimodule structure on \( \text{Mat}_{d,d}(K) \). As regards differential forms, the recipe [31, 24] defines a map

\[
\text{DR}^\bullet(A/B) \to \Gamma(\Omega^\bullet \text{Rep}(Q, d))^{G_d}
\]

relating relative differential forms with ordinary differential forms on \( \text{Rep}(Q, d) \) which are invariant with respect to the \( B \)-bimodule preserving group \( G_d \subseteq G_d \).

4 Associative symplectic geometry and applications

In this section we review the idea, introduced by Kontsevich [10] and developed by Ginzburg [27], of considering the associative analogue of symplectic structures, which play a fundamental role in the Hamiltonian approach to dynamical systems. Using the differential calculus for associative algebras reviewed in section 2, every proof from standard symplectic geometry can be translated verbatim to the new context (at least insofar it only uses the dg-algebraic properties of the de Rham complex).

In the second part of the section we briefly survey some applications of the resulting formalism to the theory of finite-dimensional integrable systems. In particular we shall recover the solution of some models of Calogero-Moser type by the classical projection method of Olshanetsky and Perelomov.

4.1 Symplectic structures on associative varieties

We shall follow the very clear exposition in [9, Section 14]. Let \( A \) be an associative algebra over the field \( K \) of characteristic zero. Given a 2-form \( \omega \in \text{DR}^2(A) \) we can define a \( K \)-linear map

\[
\omega^\flat: \text{Der}(A) \to \text{DR}^1(A)
\]

by \( \theta \mapsto i_\theta(\omega) \). The 2-form \( \omega \) is said to be nondegenerate if this map is a bijection, in which case we denote its inverse by \( \omega^\sharp: \text{DR}^1(A) \to \text{Der}(A) \). By definition, \( \omega^\sharp \) maps a 1-form \( \alpha \) to the unique derivation such that \( i_{\omega^\sharp(\alpha)}(\omega) = \alpha \).

An associative symplectic variety is a pair \( (A, \omega) \) consisting of an associative algebra \( A \) and an associative 2-form \( \omega \) which is closed (\( d\omega = 0 \in \text{DR}^3(A) \)) and nondegenerate in the above sense.

Let \( (A, \omega) \) be an associative symplectic variety. A derivation \( \theta \in \text{Der}(A) \) is called symplectic if \( \mathcal{L}_\theta(\omega) = 0 \). We shall denote by \( \text{Der}^\omega(A) \) the linear subspace of \( \text{Der}(A) \) consisting of symplectic derivations. This is a Lie subalgebra of \( \text{Der}(A) \) since, given two symplectic derivations \( \theta \) and \( \eta \), we have by (2.43) that

\[
\mathcal{L}_{[\theta,\eta]}(\omega) = [\mathcal{L}_\theta(\omega), \mathcal{L}_\eta(\omega)] = 0.
\]

Lemma 18. A derivation \( \theta \) is symplectic if and only if \( i_\theta(\omega) \) is closed in \( \text{DR}^1(A) \).

\[12\] It must be mentioned that the idea of relating finite-dimensional integrable systems to integrable equations on associative algebras goes back at least to the pioneering works [35, 36].
The standard proof via Cartan’s formula \(^2\) goes through in the obvious way. It follows that the image of the isomorphism \(^1\) restricted to \(\text{Der}^2(A)\) coincides with the linear subspace of closed 1-forms in \(\text{DR}^1(A)\).

To each regular function \(f \in \text{DR}^0(A)\) we can associate the (obviously closed) 1-form \(df \in \text{DR}^1(A)\), hence the corresponding derivation\(^3\)

\[
\theta_f := -\omega^2(df)
\]

is symplectic, and has every right to be called the **Hamiltonian derivation** determined by \(f\). Thus we have defined a \(\mathbb{K}\)-linear map

\[
\theta : \text{DR}^0(A) \to \text{Der}^2(A)
\]

which sends every regular function on the associative symplectic variety \((A, \omega)\) to the corresponding Hamiltonian derivation.

Returning to the commutative world, it follows from the discussion in section \([^3\)]\) that for every \(d \in \mathbb{N}\) the pair consisting of the affine variety \(\text{Rep}_{\mathfrak{a}}^d\) and the induced 2-form \(\tilde{\omega} \in \Omega^2 \text{Rep}_{\mathfrak{a}}^d\) qualifies as an (ordinary) symplectic variety. Moreover, for every \(f \in \text{DR}^0(A)\) the derivation \([^4\)]\) induces precisely the Hamiltonian vector field on \(\text{Rep}_{\mathfrak{a}}^d\) determined by the function \(\tilde{f} \in \mathbb{K}[\text{Rep}_{\mathfrak{a}}^d]\). All these geometric objects are automatically invariant with respect to the action \([^5\)]\) of the group \(G_d\), hence they descend to every quotient of \(\text{Rep}_{\mathfrak{a}}^d\) with respect to that action. As we shall see later in this section, by working directly at the associative-geometric level it is possible to treat in a unified way any family of dynamical systems whose phase space can be obtained by a quotient process of this kind.

Now let us look for the associative version of the **Poisson bracket** naturally associated to a symplectic form. Using the above definitions and the results established in section \([^2\)]\) it is a straightforward task to verify that the following chain of equalities holds for every \(f, g \in \text{DR}^0(A)\):

\[
\mathcal{L}_{\theta_f}(g) = i_{\theta_f(\,dg)} = i_{\theta_f(\,i_{\theta_g}(\omega))} = i_{\theta_f(i_{\theta_g}(\omega))} = -i_{\theta_g(df)} = -\mathcal{L}_{\theta_g}(f),
\]

where the various Lie derivative and contraction operators involved are seen as maps on \(\text{DR}^* A\), as discussed at the end of \([^2\)]\). Let us define the **Poisson bracket** of \(f\) and \(g\), denoted \(\{f, g\}\), to be the regular function on \(A\) resulting from any of the expressions in equation \([^4\)]\). Equivalently, this defines a \(\mathbb{K}\)-bilinear map

\[
\{ \cdot, \cdot \} : \text{DR}^0(A) \times \text{DR}^0(A) \to \text{DR}^0(A).
\]

It follows immediately from \([^4\)]\) that this bracket operation on \(\text{DR}^0(A)\) is skew-symmetric. The easiest way to prove that it also satisfies the Jacobi identity is to first make the connection with the commutator bracket on the corresponding symplectic derivations.

Let us start by noting that, quite generally, given \(\gamma, \eta \in \text{Der}(A)\) and using equation \([^2\)]\) we have

\[
i_{[\gamma, \eta]} = \mathcal{L}_\gamma \circ i_\eta - i_\eta \circ \mathcal{L}_\gamma = (d \circ i_\gamma + i_\gamma \circ d ) \circ i_\eta - i_\eta \circ (d \circ i_\gamma + i_\gamma \circ d).
\]

Then by taking \(\gamma = \theta_f\), \(\eta = \theta_g\) and applying \(i_{\theta_f, \theta_g}\) to \(\omega\) we get

\[
i_{\theta_f, \theta_g}(\omega) = (d \circ i_{\theta_f}(\omega)) + i_{\theta_f}(d\circ i_{\theta_g}(\omega)) - i_{\theta_g}(d \circ i_{\theta_f}(\omega)) + i_{\theta_g}(d\circ i_{\theta_f}(\omega))
\]

\[
= -d\theta_g(\omega) - i_{\theta_f}(d\circ i_{\theta_g}(\omega)) + i_{\theta_g}(d\circ i_{\theta_f}(\omega))
\]

\[
= -d\{f, g\}.
\]

\(^{13}\)Beware: many authors define \(\theta_f\) with the opposite sign.
But \( \theta_{\{f,g\}} \) is, by definition, the only derivation such that \( i_{\theta_{\{f,g\}}} (\omega) = -d\{f,g\} \), hence
\[
\theta_{\{f,g\}} = [\theta_f, \theta_g] \tag{4.9}
\]
and since the bracket \([\cdot, \cdot]\) satisfies the Jacobi identity, the same holds true for \(\{\cdot, \cdot\}\). Summing up, we have proved the following:

**Theorem 19.** The pair \((\text{DR}^0(A), \{\cdot, \cdot\})\) is a Lie algebra, and the map \(f \mapsto \theta_f\) is a Lie algebra morphism from it to \((\text{Der}^\omega(A), [\cdot, \cdot])\).

We conclude that the space of regular functions on an associative symplectic variety \((A, \omega)\) is naturally equipped with a Lie algebra structure.

Classically, the Poisson bracket has also the essential feature of being a *derivation* in both arguments with respect to the associative (and commutative) product on the coordinate ring of a symplectic variety; in other words, it determines a *Poisson algebra* structure on that ring. In the present setting it makes no sense to impose such a condition on the bracket \((4.6)\), as there is no associative product on \(\text{DR}^0(A)\). However, the induced bracket
\[
\{\hat{f}, \hat{g}\} := \hat{\{f, g\}}
\]
defined on the image of the map \((3.11)\) (which generates the algebra of invariants) by
\[
\{\hat{f}, \hat{g}\} := \{\hat{f}, \hat{g}\}
\]
is a genuine Poisson bracket\(^{14}\) on \(K[\text{Rep}^A_d G_d] \times K[\text{Rep}^A_d G_d] \to K[\text{Rep}^A_d G_d]
\]
defined on the image of the map \((3.11)\) (which generates the algebra of invariants) by
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\]
is a genuine Poisson bracket\(^{14}\) on \(K[\text{Rep}^A_d G_d] \times K[\text{Rep}^A_d G_d] \to K[\text{Rep}^A_d G_d]
\]
and in fact coincides with the Poisson bracket obtained by inverting the \(G_d\)-invariant symplectic form \(\hat{\omega}\).

To conclude this quick review of associative symplectic geometry let us display the analogue of the familiar four-terms exact sequence of Lie algebras associated to a symplectic variety.

**Lemma 20.** The map \((4.4)\) fits into the following exact sequence of Lie algebras:
\[
0 \longrightarrow H^0(\text{DR}^*(A)) \longrightarrow \text{DR}^0(A) \xrightarrow{\theta} \text{Der}^\omega(A) \longrightarrow H^1(\text{DR}^*(A)) \longrightarrow 0. \tag{4.10}
\]

Here the linear spaces \(H^0(\text{DR}^*(A))\) and \(H^1(\text{DR}^*(A))\) are seen as Lie algebras by equipping them with the zero bracket.

### 4.2 Some examples of associative symplectic varieties

We now review a few examples of symplectic structures on the associative varieties introduced in section 2. These examples are exactly the symplectic structures studied by Ginzburg and Bocklandt-Le Bruyn in \[27, 28\].

Let us start by looking for symplectic structures on associative affine spaces. Let \(A\) be a free algebra, seen again as the tensor algebra \(T(V^*)\) of a \(n\)-dimensional vector space \(V^*\) with dual space \(V\). By definition, an associative symplectic structure on \(A\) is given by a 2-form \(\omega \in \text{DR}^2(A)\) which is closed and nondegenerate. The nature of closed 2-forms on \(A\) is clarified by the following result.

**Theorem 21.** When the algebra \(A\) is free the subspace of closed forms in \(\text{DR}^2(A)\) is canonically isomorphic to \([A, A]\)\(^{14}\).

\(^{14}\)The map \((4.6)\) is thus an example of an “\(H_0\)-Poisson structure” as introduced by Crawley-Boevey in \[37\], and also comes from a *double Poisson bracket* on \(A\) in the sense of Van den Bergh \[13\]. Lack of space forces us to postpone a discussion of these important notions to the second part of these notes \[14\].

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A structure on the associative affine space $A$ associative symplectic structure on $\omega$ follows that the map $\alpha$ only Actually it is not hard to show that these are the Liouville 1-form $\alpha$ where $\omega$ where $\omega$ is the vector space isomorphism $V$. Then every derivation $\theta \in A \otimes V$ may be expressed as
\[
\theta = \sum_{j=1}^{k} (a_j \otimes e_j + b_j \otimes f_j) \quad \text{for some } a_1, \ldots, a_k, b_1, \ldots, b_k \in A.
\]
An easy computation shows that
\[
\omega^\theta(\omega) = i_\theta(\omega) = \sum_{i=1}^{k} (\theta(y_i)dx_i - \theta(x_i)dy_i) = \sum_{i=1}^{k} (b_i dx_i - a_i dy_i) = (id_A \otimes \omega_V^\theta)(\theta),
\]
where $\omega_V^\theta$ is the vector space isomorphism $V \to V^*$ induced by the symplectic form $\omega_V$. It follows that the map $\omega^\theta$ is also an isomorphism, and the associative 2-form $\omega^\theta$ defines an associative symplectic structure on $A$. We shall call this 2-form the canonical symplectic structure on the associative affine space $A$ (with respect to the chosen set of generators $x_1, \ldots, x_k, y_1, \ldots, y_k$). Notice that all these 2-forms are related by (affine) automorphisms of $A$. Actually it is not hard to show that these are the only possible associative symplectic forms on $A$; in particular, odd-dimensional associative affine spaces have no symplectic forms.

The subspace of symplectic derivations for the canonical symplectic structure on $A$ is easy to characterize. For the sake of simplicity we shall consider only the associative plane, $A = \mathbb{K}(x, y)$; the adaptation to the higher-dimensional cases is immediate. The symplectic form \eqref{eq:canonical_symplectic_form} reads $\omega = dx dy$. The derivation defined by $\theta(x, y) = (f_1, f_2)$ is symplectic if and only if
\[
L_\theta(\omega) = df_1 dy + dx df_2 = 0.
\]
Using the isomorphism given by theorem \ref{thm:adjoint_representation} this is equivalent to the condition
\[
[f_1, y] - [f_2, x] = 0. \tag{4.12}
\]
By theorem 11 the above equation is solved by all the pairs of the form \((\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x})\). Hence the generic symplectic derivation of \((A,\omega)\) is given by

\[
\theta(x, y) = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)
\]

(4.13)

for some \(f \in DR^0(A)\). This correspondence is bijective and a Lie algebra isomorphism, as follows from exactness of the sequence (4.10) (where \(H^0(DR^\bullet(A)) = K\) and \(H^1(DR^\bullet(A)) = 0\) by virtue of theorem 9).

The reason for calling the symplectic structure (4.11) “canonical” becomes clear when we look at the induced symplectic structure on the space \(\text{Rep}_d = \text{Mat}_{d,d}(K) \oplus \mathbb{N}\). Given \(\rho \in \text{Rep}_d\), let us define \(X_i := \rho(x_i)\) and \(Y_i := \rho(y_i)\). Then

\[
\hat{\omega}_\rho = \text{tr}(dY_1 \wedge dX_1 + \cdots + dY_k \wedge dX_k).
\]

This can be interpreted as the canonical symplectic form on the cotangent bundle

\[T^* \text{Mat}_{d,d}(K)^{\oplus k}\]

if we identify a point \((X_1,\ldots,X_k,Y_1,\ldots,Y_k) \in \text{Rep}_d\) with the point \((X_1,\ldots,X_k,\zeta_1,\ldots,\zeta_k) \in T^* \text{Mat}_{d,d}(K)^{\oplus k}\), where for each \(i = 1\ldots k\) the linear functional \(\zeta_i\) is defined by

\[\zeta_i(M) = \text{tr}MY_i\quad\text{for every } M \in \text{Mat}_{d,d}(K).
\]

(4.14)

A second source of examples comes from a class of associative symplectic structures on quiver path algebras. In order to describe them it is useful to define the following “doubling” operation. Given a quiver \(Q\) the \textbf{opposite} of \(Q\), denoted by \(Q^{\text{op}}\), is the quiver with the same vertices as \(Q\) and, for each arrow \(\xi : i \to j\) in \(Q\), an arrow \(\xi^* : j \to i\) going in the opposite direction. The \textbf{double} of \(Q\), denoted \(\overline{Q}\), is the quiver having the same set of vertices as \(Q\) and the arrows of \(Q\) and \(Q^{\text{op}}\).

Now let \(Q\) be any quiver and denote by \(A := \mathbb{K}\overline{Q}\) the path algebra of its double. We continue to denote by \(B\) the subalgebra of \(A\) spanned by the trivial paths. We consider the associative 2-form on \(A\) given by (the equivalence class in \(DR^2(A/B)\) of)

\[
\omega_Q := \sum_{\xi \in Q} d\xi^* d\xi \in \Omega^2(A/B).
\]

(4.15)

Notice that the sum runs over all the arrows in the original quiver \(Q\). This 2-form is closed, being the differential of \(\alpha_Q := \sum_{\xi \in Q} \star_d d\xi\). Furthermore, an argument similar to the one used above for the 2-form (4.11) (using the expression of the path algebra as a tensor algebra of the \(B\)-bimodule \(E_Q\)) shows that the map

\[
\omega^\sharp_Q(\theta) = \sum_{\xi \in Q} (\theta(\xi^*) d\xi - \theta(\xi) d\xi^*)
\]

is invertible, so that \(\omega_Q\) is also nondegenerate.

It follows that every quiver \(Q\) gives origin to an associative symplectic variety \((\mathbb{K}\overline{Q},\omega_Q)\). By analogy with the previous case, we shall call the 2-form (4.15) the \textbf{canonical symplectic form associated to the quiver} \(Q\). One reason is that, when \(Q\) is the quiver with one vertex and \(k\) loops \(x_1,\ldots,x_k\), the 2-form \(\omega_Q\) coincides with the 2-form (4.11). But the main reason is that the
induced symplectic structures on representation spaces may again be interpreted as canonical symplectic forms on the cotangent bundle

\[ T^* \text{Rep}(Q, d) \]

where a point \((\rho(\xi), \rho(\xi^*))_{\xi \in Q} \in \text{Rep}(Q, d)\) is identified with the point in \(T^* \text{Rep}(Q, d)\) determined on the base by the matrices \((\rho(\xi))_{\xi \in Q}\) and on the fiber by the linear functionals corresponding to the matrices \((\rho(\xi^*))_{\xi \in Q}\) via the isomorphism (4.14).

From the relative version of the sequence (4.10),

\[ 0 \rightarrow H^0(\text{DR}^*(A/B)) \rightarrow \text{DR}^0(A) \xrightarrow{\theta} \text{Der}_B^\omega(A) \rightarrow H^1(\text{DR}^*(A/B)) \rightarrow 0 \]

(where \(\text{Der}_B^\omega(A)\) denotes the Lie subalgebra of \(\text{Der}_B(A)\) consisting of symplectic derivations) we get, using the description for the cohomology of the complex \(\text{DR}^*(A)\) provided by theorem 14, the following short exact sequence of Lie algebras:

\[ 0 \rightarrow B \rightarrow \text{DR}^0(A) \xrightarrow{\theta} \text{Der}_B^\omega(A) \rightarrow 0 \quad (4.16) \]

It follows that the Lie algebra \(\text{Der}_B^\omega(A)\) of symplectic derivations can be identified with the quotient space \(\overline{\text{DR}^0(A)}\). The generic symplectic derivation of \(A\) can be written as

\[ \theta(\xi, \xi^*) = \left( \frac{\partial f}{\partial \xi^*_i}, -\frac{\partial f}{\partial \xi_i} \right) \quad (4.17) \]

where \(f \in \overline{\text{DR}^0(A)}\) and the index \(i\) runs over the arrows in the quiver \(Q\).

### 4.3 Free motion on the associative plane and the rational Calogero-Moser system

From now on we specialize to the case \(K = \mathbb{C}\), the field of complex numbers. We are going to describe some examples of dynamics on the (complex) associative plane \(A = \mathbb{C}\langle x, y \rangle\) equipped with the canonical symplectic form \(\omega = dy dx\), and the corresponding flows on representation spaces.

Let us start from the simplest possible system, namely the Hamiltonian describing the free motion on \((A, \omega)\):

\[ H = \frac{1}{2}y^2. \quad (4.18) \]

Clearly \(dH = y \, dy\), so that the symplectic derivation determined by \(H\) is

\[ \theta_H(x, y) = (y, 0). \]

This derivation induces an Hamiltonian vector field on each manifold \(\text{Rep}_d^A \simeq T^* \text{Mat}_{d,d}(\mathbb{C})\) equipped with the canonical symplectic form \(\hat{\omega}(X,Y) = \text{tr}(dY \wedge dX)\). The resulting flow is rather trivial, and is given by

\[ \Phi_t(X, Y) = (X + ty, Y). \quad (4.19) \]

Things become much more interesting on certain quotient spaces of \(\text{Rep}_d^A\) with respect to the natural action of \(G_d = \text{PGL}_{d}(\mathbb{C})\). Since we want to reduce the symplectic form \(\hat{\omega}\) along with the manifold, it is natural to consider a symplectic reduction (or Marsden-Weinstein quotient) of the symplectic vector space \((\text{Rep}_d^A, \hat{\omega})\) (see for instance [30, 33, 39] and many other sources).
basic ingredient of this process is the so-called moment (or momentum) map of the $G_d$-action which in our case is the map

$$\mu: \text{Rep}_A^d \to \mathfrak{sl}_d(\mathbb{C})$$ defined by $\mu(X,Y) = [X,Y]$.

The reduction then proceeds as described by the following:

**Theorem 22.** Let $\mathcal{O}$ be an adjoint orbit in the Lie algebra $\mathfrak{sl}_d(\mathbb{C})$ and consider the categorical quotient

$$M_\mathcal{O} := \mu^{-1}(\mathcal{O})/\!/G_d$$

with projection map $\pi: \mu^{-1}(\mathcal{O}) \to M_\mathcal{O}$. Suppose that the action of the group $G_d$ on $\mu^{-1}(\mathcal{O}) \subseteq \text{Rep}_A^d$ is free. Then the variety $M_\mathcal{O}$ is smooth and there exists a unique symplectic form $\omega_\mathcal{O}$ on $M_\mathcal{O}$ such that

$$\pi^* (\omega_\mathcal{O}) = i^*(\hat{\omega})$$

where $i$ is the canonical immersion $\mu^{-1}(\mathcal{O}) \hookrightarrow \text{Rep}_A^d$.

We are going to use this result to recover the phase space and the dynamics of the rational Calogero-Moser system from the free motion on the associative plane. In fact this is precisely the example that motivated the initial development of associative symplectic geometry by Ginzburg in [27]. As this particular reduction is explained in a number of excellent sources [42, 43, 39], we shall be quite brief.

Let us denote by $\mathcal{O}_\nu$ the adjoint orbit in $\mathfrak{sl}_d(\mathbb{C})$ of the $d \times d$ matrix $\nu = i\tau$

$$\nu = i\tau \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}$$

for some $\tau \in \mathbb{C}^*$ (the orbits corresponding to different choices of $\tau$ are isomorphic; notice that these are precisely the adjoint orbits of minimal nonzero dimension in $\mathfrak{sl}_d(\mathbb{C})$). It can be proved (see e.g. [39, Theorem 1.22]) that the action of $G_d$ on the inverse image $\mu^{-1}(\mathcal{O}_\nu)$ is free. We are thus in a position to apply theorem [22] obtaining a smooth symplectic variety of dimension $2d$ that we denote by

$$C_d := \mu^{-1}(\mathcal{O}_\nu)/\!/PGL_d(\mathbb{C}).$$

As explained for instance in [39, Section 2.7] the variety $C_d$ is naturally interpreted as the completed phase space of the (complexified) Calogero-Moser system. In order to recover the usual interpretation in terms of particles moving on a (complex) line, let us restrict to the open dense subset $U \subset C_d$ consisting of equivalence classes of pairs where the matrix $X$ is diagonalizable (in which case it automatically has distinct eigenvalues). Then a point in $U$ can be represented by a pair $(X,Y)$ in which the first matrix is diagonal, say $X = \text{diag}(q_1, \ldots, q_d)$, where all the $q_i$’s are distinct. An easy computation then shows that the matrix $Y$ must have the form

$$Y = \begin{pmatrix} p_1 & \frac{i\tau}{q_1 - q_2} & \cdots & \frac{i\tau}{q_1 - q_d} \\ \frac{i\tau}{q_2 - q_1} & p_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{i\tau}{q_d - q_1} & \cdots & \cdots & p_d \end{pmatrix}$$

\[\text{Here and in what follows we shall tacitly identify the Lie algebra } \mathfrak{sl}_d(\mathbb{C}) \text{ with its dual using the nonsingular bilinear form } (X,Y) \mapsto \text{tr } XY.\]
for some complex numbers $p_1, \ldots, p_d$. The correspondence $(X, Y) \mapsto (q_1, \ldots, q_d, p_1, \ldots, p_d)$ sets up a bijection between $U$ and the cotangent bundle to the space
\[ \mathbb{C}^{(d)} := (\mathbb{C}^d \setminus \Delta)/S_d \]
of $d$-tuples of unordered distinct complex numbers (here $\Delta$ denotes the “big diagonal” in $\mathbb{C}^d$, namely the union of all the hyperplanes $x_i = x_j$ for $i \neq j \in \{1 \ldots n\}$). In these coordinates, the reduced symplectic form on $\mathcal{C}_d$ (restricted to $U$) reads
\[ \omega_{O_{\nu}} = \sum_{i=1}^{d} dp_i \wedge dq_i. \]
We thus have a symplectic isomorphism $U \simeq T^*\mathbb{C}^{(d)}$, and the Hamiltonian induced by the necklace word (4.18) becomes, in the new coordinates,
\[ \hat{H}(q_i, p_i) = \frac{1}{2} \sum_{i=1}^{d} p_i^2 + \frac{\tau^2}{2} \sum_{i \neq j=1}^{d} \frac{1}{(q_i - q_j)^2} \]
which is the Hamiltonian of the rational Calogero-Moser system with coupling constant $\tau$.

Notice that by construction the variables $(q_1, \ldots, q_d)$ can be identified with the eigenvalues of the matrix $X$ at each instant of time. Since $X$ evolves according to the very simple law (4.19), the positions of the $d$ particles at time $t$ are completely determined by finding the eigenvalues of the matrix
\[ X(t) = X(0) + tY(0) \]
that is, by finding the $d$ roots of an algebraic equation.

This method of solving the rational Calogero-Moser system is well known: it goes back to the seminal papers by Olshanetsky and Perelomov (see [11] and references therein), who called it the projection method. Their basic idea is to consider a geodesic motion in some “big” Riemannian symmetric space (for which the solution curves can be explicitly written down using the exponential map) and then project these curves on some suitable quotient space in order to reproduce the dynamics of a nonlinear system with a smaller number of degrees of freedom.

The formalism of associative symplectic geometry sheds a new light on this classic procedure, seamlessly incorporating it in a much more general mechanism for producing an infinite family of dynamical systems (one for each $d \in \mathbb{N}$) starting from a single associative variety equipped with a symplectic form and a Hamiltonian function.

To show the fruitfulness of this new point of view let us consider a slight variation of the symplectic quotient (4.22) obtained by replacing the orbit $O_{\nu}$ defined above with an adjoint orbit $O$ of higher dimension in the Lie algebra $\mathfrak{sl}_d(\mathbb{C})$. In this case there are some additional complications due to the fact that the action of $G_d$ on the inverse image $\mu^{-1}(O)$ is no longer free in general, and the ordinary theory of Marsden-Weinstein reduction does not apply. However one can resort to the more general theory of singular symplectic reductions (see e.g. [38]), in which case the quotient (4.20) exists as a stratified symplectic space. This approach is taken, for instance, in [44]. It turns out that the reduced dynamics is then confined on a smooth symplectic stratum inside the singular quotient space. Moreover, there exists a dense open subset $U$ and a system of coordinates $(q_i, p_i)_{i=1 \ldots d}$ and $(\lambda_{ij})_{i \neq j=1 \ldots d}$ on it such that the function induced by the Hamiltonian (4.18) reads as follows:
\[ \hat{H}(q_i, p_i, \lambda_{ij}) = \frac{1}{2} \sum_{i=1}^{d} p_i^2 + \frac{1}{2} \sum_{i \neq j=1}^{d} \frac{\lambda_{ij} \lambda_{ji}}{(q_i - q_j)^2} \]
This describes another class of solvable many-body models known as *rational Calogero-Moser systems with spin*.

### 4.4 Other systems obtained from motions on the associative plane

Let us show some other examples of integrable dynamical systems which may be obtained by symplectic reduction from a motion defined on the associative plane. Consider first the “harmonic oscillator” Hamiltonian

$$H = \frac{1}{2} (y^2 + \omega^2 x^2).$$

where $\omega \in \mathbb{C}$ is a constant. By formula (4.13), the symplectic derivation determined by $H$ is

$$\theta_H(x, y) = (y, -\omega^2 x).$$

On the symplectic vector space $(\text{Rep}_d^A, \hat{\omega})$ the induced Hamiltonian function reads

$$\hat{H}(X, Y) = \frac{1}{2} \text{tr}(Y^2 + \omega^2 X^2)$$

and Hamilton’s equations are $\dot{X} = Y$, $\dot{Y} = -\omega^2 X$. These equations can also be integrated easily; the corresponding flow is given by

$$\Phi_t(X, Y) = (X \cos(\omega t) + Y \omega^{-1} \sin(\omega t), Y \cos(\omega t) - X \omega \sin(\omega t)).$$

(4.24)

Now let us descend to the symplectic quotient (4.22). Restricting once again to the dense open subset $U \simeq T^*\mathbb{C}^d$ with canonical coordinates $(q_1, \ldots, q_d, p_1, \ldots, p_d)$ we see that the function (4.23) becomes

$$\hat{H}(q_i, p_i) = \frac{1}{2} \sum_{i=1}^d p_i^2 + \frac{\tau^2}{2} \sum_{i \neq j=1}^d \frac{1}{(q_i - q_j)^2} + \frac{\omega^2}{2} \sum_{i=1}^d q_i^2.$$

This is the Hamiltonian of the rational Calogero-Moser system with the addition of an external harmonic potential. This model is also completely integrable in the Liouville sense [11, 43]. The position of the particles at time $t$ are simply the eigenvalues of the first matrix in the pair (4.24),

$$X(t) = X(0) \cos(\omega t) + Y(0) \omega^{-1} \sin(\omega t).$$

By performing a symplectic reduction along a higher-dimensional adjoint orbit we can similarly obtain a version of the rational Calogero-Moser systems with spin variables and an external harmonic potential.

More generally, we could consider a generic Hamiltonian in “standard” form

$$H = \frac{1}{2} y^2 + p(x)$$

(4.25)

where $p$ is a polynomial that will play the role of an external potential for the Calogero-Moser particles after the reduction step. The symplectic derivation determined by (4.25) is

$$\theta_H(x, y) = (y, -p'(x)).$$

Notice that for this particular class of examples the noncommutativity of the variables $x$ and $y$ is totally irrelevant. The induced flow on representation spaces is then given by the solutions to the matrix differential equation

$$\ddot{X} + p'(X) = 0.$$
Of course, the difficulty in this case is to explicitly solve this equation (which in general amounts to a system of $d^2$ coupled nonlinear ODEs; for the harmonic potential these equations become linear and decoupled).

The motions determined by non-standard Hamiltonians on $(A,\omega)$ are also of considerable interest. For instance let us take, following [39, Section 2.8],

$$H = \frac{1}{2}xyxy.$$  \hfill (4.26)

In this case $dH = yxydx + xyxdy$ and the associated symplectic derivation is

$$\theta_H(x,y) = (xyx, -yxy).$$

Descending to the symplectic quotient $C_d$ and restricting to the usual open subset $U \simeq T^*C(d)$

we see that the necklace word (4.26) induces the function

$$\hat{H}(q_i, p_i) = \frac{1}{2} \sum_{i=1}^{d} q_i^2 p_i^2 + \frac{\tau^2}{2} \sum_{i\neq j=1}^{d} \frac{q_i q_j}{(q_i - q_j)^2}.$$  \hfill (4.27)

By further restricting to the open subset

$$U' := \{(q_i, p_i) \in U \mid q_i > 0 \text{ for every } i = 1 \ldots d\}$$

and performing the change of variables

$$\theta_i := \log q_i \quad \text{and} \quad \tilde{p}_i := q_i p_i,$$

the function (4.27) becomes

$$\hat{H}(\theta, \tilde{p}) = \frac{1}{2} \sum_{i=1}^{d} \tilde{p}_i^2 + 2\tau^2 \sum_{i \neq j=1}^{d} \left( \sinh \frac{\theta_i - \theta_j}{2} \right)^{-2}$$

which is the Hamiltonian of the hyperbolic Calogero-Moser system. With a similar change of variables the system with trigonometric potential can also be obtained.

Finally let us note that this mechanism for producing families of solvable dynamical systems is by no means limited to Hamiltonian evolution equations. In fact every derivation $\theta \in \text{Der}(A)$, not necessarily symplectic, will give rise to a $\text{GL}_d$-invariant vector field on each representation space $\text{Rep}_d^A$. If we are able to explicitly solve the corresponding matrix ODEs, thus obtaining an explicit expression for the integral curves of this vector field, we can again project these solution curves on suitable lower-dimensional quotients of $\text{Rep}_d^A$ (not necessarily obtained by symplectic reduction) in order to get a solvable system with a smaller number of degrees of freedom.

A similar process has been used quite effectively in a series of papers by Calogero and his coworkers (see [45] and references therein). Following the exposition in [45], the idea is to start from a matrix differential equation of second order

$$\ddot{X} = F(X, \dot{X})$$  \hfill (4.28)

whose solutions can be written explicitly. The function $F$ is assumed to be $\text{PGL}_d$-equivariant, that is

$$gF(U, \dot{U})g^{-1} = F(gUg^{-1}, g\dot{U}g^{-1}) \quad \text{for every } g \in \text{PGL}_d(\mathbb{C}).$$

Each solution of (4.28) defines a curve $X = X(t)$; we only consider those solutions for which the matrix $X$ is diagonalizable with distinct eigenvalues at all times. Then we can look once
again at the corresponding motion of the eigenvalues \((q_1, \ldots, q_d)\) of the matrix \(X\). In general, the evolution equations for these eigenvalues will not be self-contained. However, in some particular cases the supplementary unknowns can be consistently expressed, by means of an appropriate \textit{ansatz}, as functions of the \(q_i\)’s and their derivatives. The resulting system of \(d\) scalar second order ODEs can be interpreted as the equations of motion for a dynamical system consisting of \(n\) point particles on the complex line subject to nonlinear interactions of various sorts.

Let us sketch the natural interpretation of this construction from the point of view of associative geometry. At each instant of time, the pair of matrices \((X(t), \dot{X}(t))\) defines a point in \(\text{Rep}_d^A\). The evolution equation (4.28) can then be interpreted as the definition of a (not necessarily symplectic) derivation on the associative affine plane. The resulting \(G_d\)-invariants flows on \(\text{Rep}_d^A\) clearly descend to the categorical quotient \(\text{Rep}_d^A//G_d\); however, they can be further projected on a \(2d\)-dimensional submanifold \(M\) inside \(\text{Rep}_d^A//G_d\) by means of the particular \textit{ansatz} which is used to get rid of the additional unknowns. The resulting flow on \(M\) then defines the evolution of the reduced dynamical system. We plan to provide some detailed examples of this reduction process in a future work.

### 4.5 Integrable systems related to quiver varieties

To conclude let us present a family of dynamical systems whose phase space may be obtained as a quotient of the representation spaces of a quiver with more than one vertex, thus leaving the realm of associative affine spaces. As these systems were introduced by Gibbons and Hermsen in [46] we shall call them \textit{Gibbons-Hermsen systems}.

For every natural number \(r \geq 1\) let \(Q_r\) denote the quiver\(^{16}\) with two vertices 1 and 2, a loop \(a\) at 1, an arrow \(x: 2 \to 1\) and (if \(r > 1\)) \(r - 1\) arrows \(y_2, \ldots, y_r: 1 \to 2\) (notice that there is no \(y_1\)). Let \(\overline{Q}_r\) denote the double of this quiver; it has an additional loop \(a^*\) at 1, an arrow \(x^*: 1 \to 2\) and \(r - 1\) arrows \(y_2^*, \ldots, y_r^*: 1 \to 2\).

The canonical symplectic form determined by the quiver \(Q_r\) is

\[
\omega_r := da^*da + dx^*dx + \sum_{i=2}^{r} dy_i^*dy_i
\]

We are going to consider free motion on \((\mathbb{C} \overline{Q}_r, \omega_r)\), described by the Hamiltonian \(H = \frac{1}{2}a^{*2}\).

Let us consider representations of \(\overline{Q}_r\) with dimension vector \((n, 1)\). A point in this space is given by \(2 + 2r\) matrices. We shall denote by:

- \(X\) and \(Y\) the \(n \times n\) matrices representing \(a\) and \(a^*\);
- \(v_1\) the \(n \times 1\) matrix representing \(x\) and \(w_1\) the \(1 \times n\) matrix representing \(x^*\);
- \(w_2, \ldots, w_r\) the \(1 \times n\) matrices representing \(y_2, \ldots, y_r\) and \(v_2, \ldots, v_r\) the \(n \times 1\) matrices representing \(y_2^*, \ldots, y_r^*\).

The natural action of the group \(G_{(n,1)} \cong \text{GL}_n(\mathbb{C})\) on this data is given by

\[
g.(X, Y, v_\alpha, w_\alpha) = (gXg^{-1}, gYg^{-1}, gv_\alpha, w_\alpha g^{-1})
\]

and the flow induced by the Hamiltonian \(\frac{1}{2}a^{*2}\) is

\[
\Phi_t(X, Y, v_\alpha, w_\alpha) = (X + tY, Y, v_\alpha, w_\alpha)
\]

\(^{16}\)Readers of [47] should note that the quivers described here are not the “zigzag” quivers; rather, it is the family of quivers briefly considered in Appendix B of that paper.
The moment map relative to the action \(4.29\) is the map \(\text{Rep}(Q_r, (n, 1)) \to \mathfrak{gl}_n(\mathbb{C})\) defined by

\[
\mu(X, Y, v, w, \alpha) = [X, Y] + v_1 w_1 - \sum_{i=2}^r v_i w_i
\]

In order to recover the phase space of the Gibbons-Hermsen system we must consider the (trivial) adjoint orbit in \(\mathfrak{gl}_n(\mathbb{C})\) given by the single point \(\tau I\). We trust the reader to verify that the \(2nr\)-dimensional variety given by symplectic quotient

\[
C_{n,r} := \mu^{-1}(\tau I) / GL_n(\mathbb{C})
\]

coincides with the symplectic quotient construction considered in [46].

Let us denote by \(U\) the open dense subset of \(C_{n,r}\) consisting of equivalence classes of \((2 + 2r)\)-tuples for which the matrix \(X\) is diagonalizable with distinct eigenvalues. The points of this subset may be parametrized by a set of \(2n\) complex numbers \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) and \(n\) additional points \((f_i, e_i) \in A_r\), where \(A_r\) is the algebraic variety defined by

\[
A_r := \{ (\xi, \eta) \in \text{Mat}_{1,r}(\mathbb{C}) \times \text{Mat}_{r,1}(\mathbb{C}) \mid (\xi, \eta) = 1 \} / \mathbb{C}^*
\]

with \(\lambda \in \mathbb{C}^*\) acting as \((\xi, \eta) \mapsto (\lambda \xi, \lambda^{-1} \eta)\). The reduced symplectic form restricted to \(U\) reads

\[
\omega = \sum_{i=1}^n (dy_i \wedge dx_i + \tau de_i \wedge df_i)
\]

and the Hamiltonian \(\hat{H} = \frac{1}{2} \text{tr} Y^2\) projects down to

\[
H = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{\tau^2}{2} \sum_{i \neq j=1}^n \frac{(f_i, e_j)(f_j, e_i)}{(q_i - q_j)^2}
\]

The resulting dynamics is connected to the one described by the Calogero-Moser systems with spin, although in this case the number of internal degrees of freedom is higher.

Let us emphasize that the system considered above is just a single example involving a particular family of quivers, a particular choice of the dimension vector for the corresponding representation spaces, and a particular choice of Hamiltonian function. Clearly, many variations on this theme are possible. Actually, one could argue that every quiver possess a large family of “natural” dynamical systems defined on the corresponding representation spaces; these systems may frequently be explicitly solvable and/or integrable in the Liouville sense.

This possibility was considered by Nekrasov in his survey [48] on many-body integrable systems obtained by symplectic reduction. In [48, Section 5.3] Nekrasov explicitly poses the problem of determining which dynamical systems of this form are integrable, both in the smooth and in the holomorphic setting (problems 5.20 and 5.22). To the best of our knowledge, these problems are still wide open.

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\[17\] Explicitly, the matrices \(X, P, F\) and \(E\) used by Gibbons and Hermsen correspond, respectively, to \(X, Y\), the \(n \times r\) matrix \((-v_1, v_2, \ldots, v_r)\) and the \(r \times n\) matrix \((w_1, w_2, \ldots, w_r)^T\).
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