Measurement of stimulated Hawking emission in an analogue system

Silke Weinfurtner*, Edmund W. Tedford**, Matthew C. J. Penrice*, William G. Unruh*, and Gregory A. Lawrence**

*Department of Physics and Astronomy, University of British Columbia, Vancouver, Canada V6T 1Z1
**Department of Civil Engineering, University of British Columbia, 6250 Applied Science Lane, Vancouver, Canada V6T 1Z4.

There is a mathematical analogy between the propagation of fields in a general relativistic space-time and long (shallow water) surface waves on moving water. Hawking argued that black holes emit thermal radiation via a quantum spontaneous emission. Similar arguments predict the same effect near wave horizons in fluid flow. By placing a streamlined obstacle into an open channel flow we create a region of high velocity over the obstacle that can include wave horizons. Long waves propagating upstream towards this region are blocked and converted into short (deep water) waves. This is the analogue of the stimulated emission by a white hole (the time inverse of a black hole), and our measurements of the amplitudes of the converted waves demonstrate the thermal nature of the conversion process for this system. Given the close relationship between stimulated and spontaneous emission, our findings attest to the generality of the Hawking process.

I. INTRODUCTION

One of the most striking findings of general relativity is the prediction of black holes, accessible regions of no escape surrounded by an event horizon. In the early 70s, Hawking suggested that black holes evaporate via a quantum instability [1-3]. The study of classical and quantum fields around black holes shows that a pair of field excitations at temporal frequency $f$ are created, with positive and negative norm amplitudes $\alpha_f$, $\beta_f$ (Bogoliubov coefficients) related by,

$$\frac{|\beta_f|^2}{|\alpha_f|^2} = \exp\left(-\frac{4\pi^2 f}{g_H}\right)$$

where $g_H$ is the surface gravity of the black hole [1-4]. Positive norm modes are emitted, while negative ones are absorbed by the black hole, effectively reducing its mass. The surface gravity for a non-rotating black hole with a mass $M$ is given by $g_H = 1.01035/M[\text{kg/s}]$. Equation (1) is applicable for both stimulated and spontaneous emission, and at regimes where the quantum physics is dominant. A comparison of (1) with the Boltzmann-distribution allows one to associate a temperature $T$ with the black hole,

$$T = 1.2 \cdot 10^{-12} g_H[\text{K}] = 6.0 \cdot 10^{-8} \frac{M}{M_\odot}[\text{K}]$$

Here $M_\odot$ is a solar mass, and the smallest observed black holes are of this order. Thus black hole evaporation is clearly difficult to observe directly [5].

In 1981 Unruh showed [6] (see also [7,8]), that there is a mathematical analogy between the behaviour of classical and quantum fields in the vicinity of black hole horizons and sound waves in trans-sonic fluid flows. In 2002 it was argued that surface waves on an open channel flow with varying depth are an ideal toy model for black hole experiments [9]. The 1981 paper raised the possibility of doing experiments with these analogues. A difficulty with Hawking’s derivation is its apparent reliance on arbitrarily high frequencies (the trans-Planckian problem [10-13]). The dispersion relation of gravity waves creates a natural physical short wavelength cutoff, which obviates this difficulty. Thus the dependence of the Hawking effect on the high-frequency behaviour of the theory can be tested in such analogue experiments [14]. While numerical studies indicate that the effect is independent of short-wavelength physics, experimental verification of this would strengthen our faith in the process. The presence of this effect in our physical system, which exhibits turbulence, viscosity, flow separation, and non-linearities, would indicate the generic nature of the Hawking thermal process.
FIG. 1: — Conversion process. Dispersion relation for waves propagating against a flow typical of our experiments. A shallow water wave, $k_{in}$, sent upstream, is blocked by the flow and converted to a pair of deep water waves ($k_{out}^{+}$ and $k_{out}^{-}$) that are swept downstream.

II. FLUID SURFACE GRAVITY WAVES

The excitation spectrum of gravity waves on a slowly varying background flow is well understood and has a dispersion relation given by,

$$f^2 = \left(\frac{gk}{2\pi}\right) \tanh(2\pi kh)$$

with the frequency, $f = 1/\tau$, where $\tau$ is the wave period; the wavenumber, $k = 1/\lambda$, where $\lambda$ is the wavelength; $g$ is the gravitational acceleration, and $h$ the depth of the fluid. We neglect surface tension and viscosity. We classify waves according to the value of $2\pi kh$. For $2\pi kh < 1$ the dispersion relation can be approximated by $f = \sqrt{(gh)k}$. These shallow water waves have both group and phase speed approximately equal to $\sqrt{gh}$. For $2\pi kh > 1$, the dispersion relation is approximated by $f = \sqrt{gk/2\pi}$. The group speed of these deep water waves is approximately half the phase speed, and both vary as the square root of the wavelength. For a given water depth, both the group and phase speeds of deep water waves are less than the group and phase speeds of shallow water waves.

In [9] Schtzhold and Unruh argued that the equation of motion of shallow water waves can be cast into a wave equation on a curved spacetime background if the speed of the background flow varies. Assuming a steady, incompressible flow the velocity

$$v(x) = \frac{q}{h(x)}$$

Here the two-dimensional flow rate $q$ is fixed. The dispersion relation in the presence of a non-zero background velocity becomes,

$$(f + vk)^2 = \left(\frac{gk}{2\pi}\right) \tanh(2\pi kh)$$

In Fig. 1, the dispersion relation is plotted for a flow typical of our experiments. Only the branch corresponding to waves propagating against the flow is plotted. For low frequencies, there are three possible waves, which we denote according to wavenumber. The first, $k_{in}^{+}$, is a shallow water wave with both positive phase and group velocities, and corresponds to the wave that we generate in our experiments. The second, $k_{out}^{+}$, has positive phase velocity, but negative group velocity. Both waves, $k_{in}^{+}$ and $k_{out}^{+}$, are on the positive norm branch of the dispersion relation. The
third, \( k_{\text{out}} \), has both negative phase and group velocities, and it lies on the negative norm branch. In our experiment, generated shallow water waves move into a region where they are blocked by a counter-current, and converted into the other two waves. The goal of our experiment was the measurement of the relative amplitudes of the outgoing positive and negative norm modes to test the validity of (1). (Further conversion from deep-water waves to capillary waves [15,16] are also possible but are not studied here.)

The conversion from shallow water to deep water waves occurs where a counter-current become sufficiently strong to block the upstream propagation of shallow water waves [16-19]. It is this that creates the analogy with the white hole horizon in general relativity. That is, there is a region that the shallow water waves cannot access, just as light cannot enter a white hole horizon. Note that while our experiment is on white hole horizon analogues, it is because they are equivalent to the time inverse of black hole analogues that we can apply our results to the black hole situation.

III. EXPERIMENTAL PROCEDURE

Our experiments were performed in a 6.2 m long, 0.15 m wide and 0.48 m deep flume (Fig. 2), and were partly motivated by experiments in similar flumes [15-20]. We created a spatially varying background flow by placing a 1.55 m long and 0.106 m high obstacle in the flume. This obstacle was modelled after an airplane wing with a flat top and a maximum downstream slope of 5.2 degrees designed to prevent flow separation. We used particle image velocimetry [21] to determine \( q \), and to verify the absence of flow separation. Shallow water waves of approximately 2 mm amplitude were generated 2 m downstream of the obstacle, by a vertically oscillating mesh, which partially blocked the flow as it moved in and out of the water. The intake reservoir had flow straighteners and conditioners to dissipate surface waves produced by the ingoing flow. The flume was transparent to allow photography through the walls, and the experimental area was covered to exclude exterior light.

We measured and analysed the variations in water surface height using essentially the same techniques as in [22]. The water surface was illuminated using laser-induced fluorescence, and photographed with a high-resolution (1080p) monochrome camera. The camera was set up such that the pixel size was 1.3 mm, the imaged area was 2 m wide and 0.3 m high, and the sampling rate was 20 Hz. The green (532 nm) 0.5 W laser light passed through a Powell lens to create a thin (\( \approx 2 \) mm) light sheet (Fig. 3a). Rhodamine-WT dye was dissolved in the water, which fluoresced to create a sharp (\( < 0.2 \) mm) surface maximum in the light intensity. We interpolated the intensity of light between neighbouring pixels to determine the height of the water surface to subpixel accuracy.

To determine the ambient wave noise in our facility, and to check the effectiveness of our procedures, we conducted an experiment without the obstacle in place and with no wave generation. The space and time Fourier transform of the noise match the dispersion relation for this flow (\( q = 0.039 m^2/s \) and \( h = 0.24 m \)) extremely well (Fig. 3b). In general, the amplitude of the Fourier components has a noise level of less than 0.2 mm away from the dispersion curves. The apparently elevated noise energy crossing the \( k \) axis at \( f = 3.1 \) Hz is due to the second transverse mode branch of the dispersion relation (the first transverse mode has a node at the location of the light sheet).

To detect the stimulated Hawking process, we sent shallow water waves toward the effective white hole horizon, which sits on the lee side of the obstacle. We conducted a series of experiments, with \( q = 0.045 m^2/s \) and \( h = 0.194 m \), and examined 9 different ingoing frequencies between 0.02 and 0.67 Hz, with corresponding still water wavelengths between 69 and 2.1 meters. This surface was imaged at 20 frames per second, for about 200 s. In all cases we analysed a period of time which was an exact multiple of the period of the ingoing wave, allowing us to carry out sharp temporal
FIG. 3: — Surface wave detection. a, Diagram of light-sheet projection for surface wave detection: (1) water with dye, (2) Powell lens, (3) light sheet, and (4) fluorescing water surface. b, Fourier transform of water surface in flat bottom flume without waves; \( q = 0.039 \mathrm{~m}^2/\mathrm{s} \) and \( h = 0.24 \mathrm{~m} \). Fluctuations lie on upstream (red line) and downstream (black line) branches of the dispersion relation. Just visible at \( f = \pm 3.1 \mathrm{~Hz}, k = 0 \) are the second transverse mode branches of the dispersion relation. Off the dispersion curves, the background noise amplitudes are less than 0.1 mm.

frequency filtering of the signals (i.e., eliminating spectral leakage).

\[
\xi = \int_0^x \frac{dx}{v(x)} \quad (6)
\]

The analysis of the surface wave data was facilitated by introducing the convective derivative operator \( \partial_t + v(x)\partial_x \). We redefine the spatial coordinate using, where \( x \) is the distance downstream from the right hand edge of the flat portion of the obstacle. The \( \xi \) coordinate has dimensions of time, and its associated wave number \( \kappa \) has units of Hz. The convective derivative becomes \( \partial_t + \partial_\xi \), or, in Fourier transform space, \( f + \kappa \). This is the term that enters the conserved norm. From equations (35), (36) and (87) of reference [9] we find that the conserved norm has the form

\[
\int \frac{|A(f, \kappa)|^2}{f + \kappa} d\kappa \quad (7)
\]

where \( A(f, \kappa) \) is the \( t, \xi \) Fourier transform of the vertical displacement of the wave. In using this coordinate system the outgoing waves have an almost uniform wavelength even over the obstacle slope.

IV. RESULTS

We will illustrate the pair-wave creation process by presenting the results for \( f_m = 0.185 \mathrm{~Hz} \). In this case, we analyzed images from exactly 18 cycles, measuring the free surface along approximately 2 m of the flow including the obstacle. After converting to \( \xi \)-coordinates (6), we calculated the two-dimensional Fourier transformation as displayed in Fig. 4a. Note that the amplitudes of the Fourier transform at frequencies above and below 0.185 Hz are very small, indicating that the noise level is small.

As expected, there are three peaks, one corresponding to the ingoing shallow water wavelength around \( \xi = 0 \), and the other two corresponding to converted deep water waves peaked near \( \kappa_{\text{out}}^+ = 9.7 \mathrm{~Hz} \) and \( \kappa_{\text{out}}^- = -10.5 \mathrm{~Hz} \). The former is a positive norm and the latter a negative norm outgoing wave, see equation (7).

In Fig. 4b we plot the wave characteristics (amplitude as function of \( t \) and \( \xi \)) filtered to give only the temporal 0.185 Hz band. Figures 4c and 4d are the characteristic plots where we further filter to include only \( \kappa < -1 \mathrm{~Hz} \) and...
$\kappa > 1 \text{ Hz}$ respectively. These are the negative and positive norm outgoing components without the central peak of the ingoing wave (because of their very long wavelengths and the rapid change in wavelength as they ascend the slope, the incoming waves have a very broad Fourier transform). Recall that since we are only interested in counter-propagating waves, we defined positive phase and group speeds as pointing to the left. As expected from the dispersion relationship, see Fig. (1), the negative norm waves have negative phase velocity, while the positive norm waves have positive phase velocity. The complex structure in the characteristics of Fig. 4b arises because of the interference between the three components, the original ingoing wave, and the positive and negative norm outgoing waves. In Fig. 4b, we see that the ingoing wave is blocked around $\xi = 0$, with only a small component penetrating into the region over the top of the obstacle $\xi < 0$.

Our key results are presented in Fig. 5. Figure 5a shows the amplitude of the spatial Fourier transform at three selected ingoing frequencies. As the frequency increases, the ratio of the negative norm peak to positive norm peak decreases. Furthermore, the location of the positive norm peak moves slightly toward zero as the frequency increases, while the negative norm peak moves away from zero. This is to be expected from the location of the allowed spatial wavenumber from the dispersion plot, see Fig. 1. The red curve in Fig. 5a shows the Fourier transform in the adjacent temporal frequency bands for the sample case of 0.185 Hz. This is a representation of the noise, and is a factor of at least 10 lower than the signal in the 0.185 frequency band.

To test whether or not the negative norm wave creation was due to non-linearities we repeated the runs at all frequencies with 50% larger amplitudes. The converted wave amplitudes did, in fact, scale linearly.

The crucial question is: Does the ratio of the negative to positive norm outgoing waves scale as predicted by the thermal hypothesis of equation (1)? This is shown to be the case in Fig. 5b, where the norm ratios are plotted as a function of ingoing frequency. To calculate the norm of the outgoing waves we integrate $\int |A(f, \kappa)|^2/(f + \kappa) d\kappa$ over the peaks to evaluate $|\alpha_f|^2$ or $|\beta_f|^2$. In Fig. 5b the points represent the natural log of the ratios of these areas for each of the input frequencies we tested. The thermal hypothesis is strongly supported, with linear regression giving an inverse slope of 0.11 Hz and an offset close to zero. The slope corresponds to a temperature of $T = 5 \cdot 10^{-12}$ K.
We see from Fig. 4c,d that the region of “wave blocking” where the ingoing wave is converted to a pair of outgoing waves, is not a phase velocity horizon (where the phase velocity in the laboratory frame goes to zero). This is true even for the very lowest frequencies. The usual derivation of the temperature from the surface gravity relies on this conversion occurring at a phase velocity horizon. This makes the calculation of the surface gravity, and thus the predicted temperature uncertain. In our case estimates of the surface gravity give a predicted temperature of the same order as the measured temperature. What is important is that the conversion process does exhibit the thermal form predicted for the Hawking process.

V. SUMMARY

We have conducted a series of experiments to verify the stimulated Hawking process at a white hole horizon in a fluid analogue gravity system. These experiments demonstrate that the pair-wave creation is described by a Boltzmann-distribution, indicating that the thermal emission process is a generic phenomenon. It survives fluid-dynamical properties, such as turbulence and viscosity that, while present in our system, are not included when deriving the analogy. The ratio is thermal despite the different dispersion relation from that used by Hawking in his black hole derivation. This increases our trust in the ultraviolet independence of the effect, and our belief that the effect depends only on the low frequency, long wavelength aspects of the physics. When the thermal emission was originally discovered by Hawking, it was believed to be a feature peculiar to black holes. Our experiments, and prior numerical work [6,12], demonstrate that this phenomenon seems to be ubiquitous, and not something that relies on quantum gravity or Planck-scale physics.

While our experiments measure only the stimulated emission from this white hole analogue, it has been known since Einstein’s work [23] that there is a very close relation between spontaneous and stimulated emission from a quantum system. Furthermore the time reversal invariance of the theory leads to the equivalence of black and white hole horizons. It would still be exciting to measure the spontaneous emission from a black hole. While finding small black holes to test the prediction directly is beyond experimental reach, such measurements might be achievable in other analogue models, like Bose Einstein condensates, or optical fibre systems [24-28].
Acknowledgments

We thank Mauricio Richartz for his help during the initial stages of this project. WGU and GAL thank the Natural Sciences and Engineering Research Council for grants which supported this work. WGU thanks the Canadian Institute for Advanced Research, and GAL thanks the Canada Research Chairs program, for their support of this research. SW was supported by a Marie Curie Fellowship EMERGENT-2007-SW. We thank Piyush Jain and Ralf Schützhold for comments on the final manuscript and Trevor Guerard for help with fig 2. We thank the Department of Civil Engineering for the use of the flume and experimental space, which displaced undergraduate teaching. Their willingness to make do with other equipment and space made our experiments possible.

[1] Hawking, S.W. Black Hole Explosions. Nature 248, 30 (1974).
[2] Hawking, S.W. The Analogy between Black-Hole Mechanics and Thermodynamics. Annals of the New York Academy of Sciences 4268, (1973).
[3] Hawking, S.W. Particle Creation by Black Holes. Commun. Math. Phys. 43, 199 (1975).
[4] Unruh, W.G. Notes on black hole evaporation. Phys. Rev. D 14, 870 (1976).
[5] Carr, B. J. & Giddings, S. B. Quantum Black Holes. Scientific American 292, 48-55 (May 2005).
[6] Unruh, W.G. Experimental black hole evaporation. Phys. Rev. Lett. 46, 1351-1353 (1981).
[7] Barcel, C. & Liberati, S. & Visser M. Analogue gravity. Matt Living Rev. 8, 12 (2005).
[8] Jacobson, T. A. & Parentani, R. An echo of black holes. Scientific American 17, 12-19 (2007).
[9] Schützhold, R. & Unruh, W.G. Gravity wave analogs of black holes. Phys. Rev. D 66, 044019 (2002).
[10] Unruh, W.G. Dumb Holes and the Effects of High Frequencies on Black Hole Evaporation. Phys. Rev. D 6, 2827-2838 (1995).
[11] Jacobson, T. Outgoing Black Hole Modes. Phys. Rev. D 53, 7082-7088 (1996).
[12] Corley, S. & Jacobson, T. High Frequency Dispersion. Phys. Rev. D 54, 1568-1586 (1996).
[13] Corley, S. & Jacobson, T. Lorentz violation and Hawking radiation. Phys. Rev. D 59, 124011 (1999).
[14] Visser, M. Essential and inessential features of Hawking radiation. Int. J. Mod. Phys. 12, 649-661 (2003).
[15] Badulin, S. I., Pokazeyev, K. V. & Rozenberg, A. D. A Laboratory Study of the Transformation of Regular Gravity-Capillary Waves on Inhomogeneous Flows. Izvestiya, Atmospheric and Oceanic Physics 19, 10 (1983).
[16] Rousseaux, G., Massa, P., Mathis, C., Coullet, P., Philbin, T. G. & Leonhardt, U. Horizon effects with surface waves on moving water. arXiv:1004.5546v1 [gr-qc].
[17] Suastika, I. K. 2004 Wave blocking PhD Thesis Technische Universiteit Delft, The Netherlands Online at http://repository.tudelft.nl/file/275166/201607
[18] Unruh, W.G. Dumb holes: analogues for black holes. Phil. Trans. R. Soc. A 366, 2905-2913 (2008).
[19] Rousseaux, G., Mathis C., Massa, Philippe, P., Thomas G., & Leonhardt, U. Observation of negative-frequency waves in a water tank: a classical analogue to the Hawking effect? New Journal of Physics 10 053015 (2008)
[20] Lawrence, G. A. Steady flow over an obstacle. Journal of Hydraulic Engineering-ASCE 113(8), 981-991 (1987).
[21] Adrian, R.J. (1991). "Particle-imaging techniques for experimental fluid mechanics". Annual Review of Fluid Mechanics 23: 261–304. doi:10.1146/annurev.fl.23.010191.001401
[22] Tedford, E. W. & Pieters, R. & Lawrence, G. A. Symmetric Holmboe instabilities in a laboratory exchange flow. J. Fluid Mech. 636, 137-153 (2009).
[23] Einstein, A. Zur Quantentheorie der Strahlung. Phys. Zeitschr. XVIII (1917).
[24] Dimopoulos, S. & Landsberg, G. Black Holes at the LHC. Phys. Rev. Lett. 87 161602 (2001).
[25] Jain, P., Bradley, A. S. & Gardiner, C. W. The Quantum de Laval Nozzle: stability and quantum dynamics of sonic horizons in a toroidally trapped Bose gas containing a superflow. Phys. Rev. A 76, 023617 (2007).
[26] Belgioioso, F., Cacciatori, S. L., Ortenzi, G., Sala, V. G. & Faccio, D. Quantum radiation from superluminal refractive index perturbations. Phys. Rev. Lett. 104 140403 (2010).
[27] Carusotto, I., Fagnocchi, S., Recati, A., Balbinot, R. & Fabbri, A. Numerical observation of Hawking radiation from acoustic black holes in atomic Bose-Einstein condensates. New J. Phys. 10 103001 (2008).
[28] Horstmann, B., Reznik, B., Fagnocchi, S. & Cirac, I. J. Hawking Radiation from an Acoustic Black Hole on an Ion Ring. Phys. Rev. Lett. 104 (2010).