The algebra of $q$-pseudodifferential symbols and the $q$-$W^{(n)}_{KP}$ algebra

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Abstract

In this paper we continue with the program to explore the topography of the space of $W$-type algebras. In the present case, the starting point is the work of Khesin, Lyubashenko and Roger on the algebra of $q$-deformed pseudodifferential symbols and their associated integrable hierarchies. The analysis goes on by studying the associated hamiltonian structures for which compact expressions are found. The fundamental Poisson brackets yield $q$-deformations of $W_{KP}$ and related $W$-type algebras which, in specific cases, coincide with the ones constructed by Frenkel and Reshetikhin. The construction underlies a continuous correspondence between the hamiltonian structures of the Toda lattice and the KP hierarchies.

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The literature concerning the so-called W algebras increases as the belief that the \( w \) stands for “wild”. In fact they are wild objects in that they still resist all efforts to achieve a clear and unified understanding of their physical meaning or at least of their geometrical origin. On the other side, the fascination about them stems from the way they underlie so many a priori disconnected physical and mathematical constructions: 2-dimensional conformal field theory [1], soliton systems [2], vertex-operator and Kac-Moody algebras [3], classical and quantum fluids [4][5], 2-D quantum gravity [6], generalized particle systems [7], and a long etcetera.

On the way to taming the W algebras different proposals have been pursued. On one hand, in the last years some effort has been posed in setting up a classification program. It has been realized that a natural arena to handle this program is the phase space of integrable soliton-systems, where very many of the known W algebras arise either as Poisson bracket algebras, or as symmetries of the evolution equations. On the other hand, searching for an interpretation of W algebras in physical terms, some simplifications have been produced, yielding somewhat simpler objects which still preserve many of the distinguishing features of W algebras. Among them, the presence of the Virasoro subalgebra plays a central role. Thus for example the “dispersionless” or “classical” limit in which the operator \( \partial \) is smoothly replaced by a commuting symbol \( \xi \) [8] has shed some light about the geometry of classical W-morphisms in relation to “area preserving diffeomorphisms” [9] and Hamiltonian mechanics [10].

Another interesting simplification should occur if we replaced the derivative \( \partial \) by the \( q \)-derivative \( \partial_q \). The \( q \)-derivative is in fact a difference operator, i.e., let \( F \) denote the ring of complex valued polynomials in \( z \) and \( z^{-1} (\mathbb{C}[z, z^{-1}]) \) and \( q \in \mathbb{C} \):

\[
\partial_q f(z) = \frac{f(qz) - f(z)}{z(q - 1)}
\]

\( \partial \) is recovered in the \( \lim_{q \to 1} \partial_q = \partial \). Using \( \partial_q \) instead of \( \partial \) provides a sort of short distance cut-off. For this reason it has been widely investigated in connection with the problem of regulating quantum field theories [12]. In the last years, a few works have been concerned with the issue of the \( q \)-deformed Virasoro an W algebras; in [13][14] we have listed the references we are aware of, where structures deserving such name have been constructed. The generic approach in them exploits heavily the use of the \( q \)-affine algebras, \( q \)-vertex operators, and a \( q \) deformed version the Miura transformation. The connection of these algebras with integrable systems remained unclear, until the recent work of E. Frenkel [15]. In this paper it is claimed that the \( q \)-deformed W algebras constructed in ref. [14] provide bi-hamiltonian structures for a particular set of differential–\( q \)-difference integrable systems, which naturally deserve the name of \( q \)-deformed KdV hierarchies.

\footnote{See also [11] for other interesting proposal.}
Our original motivation was to pursue the line of research developed in [16]. In this work the central object of study was the Lie algebra of so called $q$-pseudodifferential symbols $\partial_q$, its extensions and contractions, as well as the associated Lax systems. Actually, the $q$-deformed n-KdV integrable hierarchies defined there turn out to be the same as those in [15], albeit in a different basis. With respect to this work, ours is somewhat complementary in that we asked ourselves, first, what are the most general hierarchies that one could write in terms of Lax operators involving $q$-pseudodifferential symbols and, second, what are their hamiltonian structures. To perform the analysis, the unified framework described in [17] proved to be instrumental. As an output, a large class of $q$-deformations of classical $W$ algebras are found, including those of $W_{\text{KP}}, GD_n$, or the centrally extended $W_{1+\infty}$. In specific cases we find agreement with the results of ref. [15]. We also comment on some obstruction found when trying to define a $q$-deformation of $W_n$.

This paper is organized as follows: for completeness, sections 2 and 3 are devoted to the introductory material. In the former one, some basic notions about the algebra of $q$-pseudodifferential operators are included; the later gives an overview of the r-matrix approach to integrable systems. In both sections we have followed closely the clear expositions of refs. [16] and [17] respectively.

Section 4 is a straightforward application of the machinery of section 3. The analysis is performed in a twisted basis $T$, which we refer to as the “Toda lattice” basis. In particular, three tri-hamiltonian hierarchies of non-linear differential-difference equations are found. The Poisson brackets are explicitly computed and agree in special cases with those found in [15]. One of the advantages of the present formalism is the possibility of carrying out a transparent treatment of reductions. Some of them are investigated at the end of this section.

Section 5 is a re-elaboration of the previous findings in the basis $\partial_q$ introduced in section 1, and named $q$-KP basis after its direct relationship with the standard KP basis. The non-linear infinite dimensional algebra which we obtain and compute is connected with the $W^{(n)}_{\text{KP}}$ algebra [18] in the limit $q \rightarrow 1$; thereafter we name it, the $q$-$W^{(n)}_{\text{KP}}$ algebra. Reductions are treated at the end. Of utmost importance are the reductions of $q$-KP to $q$-KdV. We comment about the possibility to obtain several $q$-deformations of the Virasoro algebra within the present formalism.

Finally, in section 6 we bring the logarithm of the $q$-differential symbol $\log \partial_q$ into the game. We do this by formally continuing the order $n$ of the Lax operator to real values and taking afterwards a suitable limit $n \rightarrow 0$. The resulting algebra can be considered as a $q$-deformation of the centerful $W_{1+\infty}$ algebra.
It will be useful to define the “shift” \( \tau f(z) = f(qz) \), \( \tau^\beta f(z) = f(q^\beta z) \), \( \beta \in \mathbb{C} \). So, \( \partial_q \) is a \( q \)-derivative in the following sense

\[
\partial_q(fg) = \partial_q(f)g + \tau(f)\partial_q(g).
\]  

which can be proven by explicit computation. The actions of \( \tau \) and \( \partial_q \) are not commutative but rather \( q \)-commutative, i.e. \( \partial_q(\tau(f)) = q\tau(\partial_q(f)) \)

**Definition 2.2.** An algebra \( \Psi DO_q \) of \( q \)-pseudodifferential operators is a vector space of formal series

\[
\Psi DO_q = \{ A(x,\partial_q) = \sum_{-\infty}^{n} u_i(z)\partial^i_q \mid u_i \in F \}
\]

with respect to \( \partial_q \). The multiplication law in \( \psi DO_q \) is defined by the following rule: \( F \) is a subalgebra of \( \psi DO_q \) and there are commutation relations (\( u \in F \)):

\[
\begin{align*}
\partial_q u &= (\partial_q u) + \tau(u)\partial_q, \\
\partial_q^{-1} u &= \sum_{k \geq 0} (-1)^k q^{-k(k+1)/2} \left( \tau^{-k-1} \left( \partial^k_q u \right) \right) \partial_q^{-k-1},
\end{align*}
\]

Each term of the product of two Laurent series in \( \partial_q \) is found by applying these rules finite number of times. The formula (2.4) is built so that \( \partial_q^{-1} \partial_q u = \partial_q \partial_q^{-1} u = u \). For \( q = 1 \) these formulas recover the “classical” definition of multiplication law in the algebra of pseudodifferential operators \( \psi DO \).

The commutation rule for \( \partial_q^n \) (with any integer \( n \)) and \( u(z) \) join these formulae in one

\[
\partial_q^n u = \sum_{k \geq 0} {n \choose k}_q \left( \tau^{n-k} \left( \partial^k_q u \right) \right) \partial_q^{n-k},
\]

where we use the following notation for \( q \)-numbers and \( q \)-binomials.

\[
(n)_q = \frac{q^n - 1}{q - 1}
\]

\[
{m \choose k}_q = \frac{(m)_q(m-1)_q \cdots (m-k+1)_q}{(1)_q(2)_q \cdots (k)_q}.
\]

The \( q \)-analog of the Leibnitz rule of multiplication of two \( q \)-pseudodifferential operators
\(A(x, \partial_q), B(x, \partial_q)\) can be written as the following operation on their symbols

\[
A(x, \partial_q) B(x, \partial_q) = \sum_{k \geq 0} \frac{1}{(k)!} \left( \frac{d^k}{d\partial_q^k} A \right) \ast \left( \partial_q^k B \right)
\]  

(2.6)

where for any complex value of \(\alpha\)

\[
\frac{d^k}{d\partial_q^k}(f \partial_q^k) = (\alpha)_q (\alpha - 1)_q \cdots (\alpha - k + 1)_q f \partial_q^{\alpha - k}
\]

and the \(\ast\) multiplication of symbols obeys the following commutation rule for the generators:

\[
f \ast \partial_q = f \partial_q, \quad \partial_q \ast f = \tau(f) \partial_q, \quad \partial_q^{-1} \ast f = \tau^{-1}(f) \partial_q^{-1}.
\]

(2.7)

This follows by a straightforward verification of the formula (2.6) for the product \(\partial_q^n u(z)\), which gives the same answer as (2.5).

Define the Lie algebra \(G_q\) as the set of all \(q\)-pseudodifferential symbols equipped with the commutator bracket \([A, B] = A B - B A\).

With this setup in mind, it is straightforward to construct a \(q\)-deformed analog of the KP hierarchy. The phase space for this dynamical system is the set \(\{ L_q = \partial_q + u_1(z) + u_2(z)\partial_q^{-1} + u_3(z)\partial_q^{-2} + \ldots \}\), and the equations of motion adopt the familiar Lax form

\[
\frac{dL_q}{dt_m} = \left[ L_q, \left( L_q^m \right)_+ \right] = \left[ \left( L_q^m \right)_-, L_q \right]
\]

(2.8)

Notice that unlike in the differential case, the potential \(u_1(z)\) has a nontrivial evolution. This is due to the fact that now the highest degree of the commutator of two \(q\)-pseudodifferential operators is the sum of their respective highest degrees, this being a consequence of the non commutativity of the multiplication of symbols as shown in (2.7).

§3 R-matrix approach to integrable systems

We recall here the rudiments of r-matrix and refer the interested reader to the literature [19]. In this section we shall follow closely the clear introduction given in [17]. A classical r-matrix on a Lie algebra \(g\) is a linear map \(\mathcal{R} : g \rightarrow g\) such that the modified bracket

\[
[a, b]_\mathcal{R} = [\mathcal{R}(a), b] + [a, \mathcal{R}(b)]
\]

is a Lie bracket, thus providing a second Lie algebra structure on \(g\). As was shown in [19] a sufficient condition for a linear map \(\mathcal{R}\) to be an r-matrix is given by the so-called modified Yang Baxter equation (m-YB(\(\alpha\)) for short).

\[
[\mathcal{R}(a), \mathcal{R}(b)] - \mathcal{R}([a, b]_\mathcal{R}) = -\alpha [a, b]
\]

(3.1)

where \(\alpha\) is any real number. Now let us assume, that in \(g\) there is an ad-invariant (under the natural Lie bracket \([,]\) in \(g\)) inner product \(\langle , \rangle : g \times g \rightarrow \mathbb{C}\) under which \(g\) can be
identified with its dual $g^\ast$. Immediately we know of a natural Poisson structure that lives on $C^\infty(g^\ast)$, namely the Lie-Poisson bracket arising from the modified Lie bracket $[\ ,\ ]_R$:

$$\{f_1, f_2\}_1(L) \equiv \langle L, [\mathcal{R}df_1, df_2] + [df_1, \mathcal{R}df_2] \rangle. \quad (3.2)$$

evaluated at a point $L \in g = g^\ast$. This Poisson bracket, termed linear after its dependence on $L$, is the first of a series of other “potential” Poisson brackets.

$$\{f_1, f_2\}_2 \equiv \langle L, [\mathcal{R}(L df_1 + df_1 L), df_2] + [df_1, \mathcal{R}(L df_2 + df_2 L)] \rangle \quad (3.3)$$

$$\{f_1, f_2\}_3 \equiv \langle L, [\mathcal{R}(L df_1 L), df_2] + [df_1, \mathcal{R}(L df_2 L)] \rangle \quad (3.4)$$

Using ad-invariance of the inner product, and the definition of the adjoint r-matrix as $\langle \mathcal{R}(a), b \rangle = \langle a, \mathcal{R}^\ast(b) \rangle$, we may encode the above “potential” Poisson brackets in terms of the associated Poisson map $J$, defined by

$$\{f_1, f_2\}_s(L) = \langle J_s^L(df_1), df_2 \rangle, \quad s = 1, 2, 3. \quad (3.5)$$

as follows

$$J_1^L(df) = [L, \mathcal{R}(df)] + \mathcal{R}^\ast([L, df])$$

$$J_2^L(df) = [L, \mathcal{R}(L df + df L)] + L \mathcal{R}^\ast([L, df]) + \mathcal{R}^\ast([L, df])L$$

$$J_3^L(df) = [L, \mathcal{R}(L df L)] + L \mathcal{R}^\ast([L, df])L \quad (3.6)$$

Now the crucial question: for what $\mathcal{R}$ will the above maps define hamiltonian maps? The findings of [20][17] specify that:

a) $J_1^L$ is hamiltonian for any r-matrix $\mathcal{R}$ on $g$.

b) $J_2^L$ is hamiltonian if $\mathcal{R}$ and its skew-symmetric combination $\frac{1}{2}(\mathcal{R} - \mathcal{R}^\ast)$ both satify the m-YB($\alpha$) equation (3.1).

c) $J_3^L$ is hamiltonian if $\mathcal{R}$ an r-matrix which satisfies m-YB($\alpha$) equation.

The three maps are related with one-another by simple deformations

$$J_1^{(2)} = J_2^{(2)} + 2\epsilon J_1^{(1)}$$

$$J_1^{(3)} = J_2^{(3)} + \epsilon J_2^{(2)} + \epsilon^2 J_1^{(1)}$$

where 1 is the generator of the center in $g$. This, by the way, shows the compatibility of the three “would be” Poisson structures.
The construction of integrable systems, that are hamiltonian with respect to the above brackets refers to the existence of a (possibly maximal) set of conserved functions in involution. Here, an important piece in the game is played by the set of Casimir (invariant) functions, \( C \in C^\infty(g^*) \) satisfying \( ad^*_L(C(L)) = 0 \) or, equivalently,

\[
ad_L(dC(L)) = [L, dC(L)] = 0
\]

If one has a chance to characterize the Casimir functions (in short, the centralizer of \( L \in g \)), then a short look at the form of \( J^{(s)} \) in (3.6) reveals that

(i) the associated hamiltonian flows adopt the Lax form

\[
\frac{dL}{dt} = J_L^{(1)}(dC) = [L, \mathcal{R}(dC)]
\]

\[
\frac{dL}{dt} = J_L^{(2)}(dC) = [L, \mathcal{R}(2L dC)]
\]

\[
\frac{dL}{dt} = J_L^{(3)}(dC) = [L, \mathcal{R}(L^2 dC)]
\]

(ii) the Casimir functions are in involution. For example when \( s = 1 \)

\[
\{C_1, C_2\}_1 = \langle [L, \mathcal{R}(dC_1)], dC_2 \rangle = -\langle [L, dC_2], \mathcal{R}(dC_1) \rangle = 0.
\]

A particular (partial) solution is given by the traces of powers of \( L \).

\[
C_p(L) \equiv \frac{1}{k} \text{Tr} \ (L^p), \quad dC_p(L) = L^{(k-1)}, \quad p = 1, 2, ...
\]

for this particular set of functions, the Lax equations are tri-hamiltonian

\[
\frac{dL}{dt_p} = [L, \mathcal{R}(L^p)] = J_L^{(1)}(dC_{p+1}) = J_L^{(2)}(dC_p) = J_L^{(3)}(dC_{p-1})
\]

In some cases (\( n \)-KdV), \( p \) may be a fraction of the order of \( L \).

The classification of solutions to (3.1) has been achieved partially. A class of them fall into the following characterization: if \( g = g_+ \oplus g_- \) is a decomposition into Lie subalgebras, denoting by \( P_+ \ (P_-) \) the projection of \( g_+ \) (resp. \( g_- \)) along \( g_- \) (resp. \( g_+ \)), then \( \mathcal{R} = \frac{1}{2}(P_+ - P_-) \) satisfies the modified Yang Baxter equation (3.1) with \( \alpha = 1/4 \) since \( [a, b]_\mathcal{R} \) is easily calculated to be \( [a_+, b_+] - [a_-, b_-] \) in obvious notation.
We may give the particular form of (3.6) whenever adapted to the present situation.\textsuperscript{2}

\[
J^{(1)} (df) = [L, P_+ df] - P_+^* [L, df] = - [L, P_- df] + P_+^* [L, df]
\]

\[
J^{(2)} (df) = [L, P_+(L df + df L)] - L (P_+^* [L, df]) - (P_+^* [L, df]) L
= - [L, P_-(L df + df L)] + L (P_+^* [L, df]) + (P_+^* [L, df]) L
\]

\[
J^{(3)} (df) = [L, P_+(L df L)] - L(P_+^* [L, df]) L
= - [L, P_-(L df L)] + L(P_+^* [L, df]) L
\]

(3.7)

Moreover, if \(g_+\) and \(g_-\) are isotropic, then clearly \(R\) is skew-adjoint with respect to the inner product \(\langle R(a), b \rangle = \langle a, R^*(b) \rangle\), i.e. \(R^* = -R\). In this case \(P_\pm = P_\mp\) and the 3 structures in (3.6) reduce to the following form \((X \equiv df)\):

\[
J^{(1)}_L (z) = [L, X_+] - [L, X_-]
\]

\[
J^{(2)}_L (z) = L(XL)_+ - (LX)_+ L = -L(XL)_- + (LX)_- L
\]

\[
J^{(3)}_L (z) = [L, (LXL)_+] - L [L, X_+] L
\]

(3.8)

\[\text{§4 } \text{The "Toda lattice" basis}\]

Let us return to the \(q\)-deformation of the KP hierarchy that we showed in the introduction (2.8). Define, for \(q \neq 1\)

\[
T = z(q-1)\partial_q + 1
\]

\[
T^{-1} = \frac{1}{z(q-1)\partial_q + 1} = \sum_{i=1}^\infty - \frac{(-q)^i}{(q-1)^i} z^{-i} \partial_q^{-i}
\]

(4.1)

Any element of \(\Psi DO_q\) of the form (2.3) admits a similar expression in this “twisted” basis

\[
A = \sum_{-\infty}^{n} a_i(z) \partial_q^i = \sum_{-\infty}^{n} t_i(z) T^i
\]

(4.2)

Hence we will be describing the same algebra \(G_q\) in this basis. The relevant composition law is the following, which can be proven by elementary manipulations: For any \(f \in F\)

\[
T f = \tau(f) T
\]

(4.3)

in particular \(Tz = qzT\). We will use the notation \((T f)\) to mean that \(T\) acts only on \(f\), i.e.\((T f) \equiv T f T^{-1} = \tau(f)\).

\[\text{\textsuperscript{2} Hereafter we shall obviate the dependence of } J^{(s)}_L \text{ on } L, \text{ and write simply } J^{(s)}.\]
The algebraic approach to integrability relies heavily on the existence of an ad-invariant symmetric bilinear form. As a step in this direction, a linear functional $\int : f \in C$ is defined satisfying $\int \tau(f) = \int f$ for all $f \in F$. In agreement with this requirement, we further specify that $\int z^n = \delta_{n,0}$. A particular realization of this definition is given by the usual Riemann integration over $S^1$ of the Fourier basis functions $z^n = e^{in\theta}$, where the action of $\tau$ is seen as a shift of $(-i \log q)$ in $\theta$. Also $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$.

Now, let $A = \sum_i a_i T^i \in \Psi DO_q$. We define the residue $\text{res}_T : \Psi DO_q \to C$ by

$$\text{res}_T \left( \sum_i a_i T^i \right) = a_0$$

and the trace $\text{Tr} : \Psi DO_q \to C$ by

$$\text{Tr} A = \int \text{res}_T A$$

**Lemma 4.4.** The bilinear form $\langle , \rangle : \mathcal{G}_q \times \mathcal{G}_q \to C$, given by

$$\langle A, B \rangle = \text{Tr} AB = \int \text{res}_T AB$$

(4.5)

defines an ad-invariant bilinear symmetric inner product in $\mathcal{G}_q$

**Proof:** By direct computation and use of the defining “shift” invariance of $\int$ we find

$$\text{Tr} AB = \int \text{res}_T a_i T^i b_j T^j$$

$$= \int a_i \tau^i(b_{-i}) = \int b_{-i} \tau^{-i}(a_i)$$

$$= \int \text{res}_T b_j \tau^j(a_i) T^{j+i} = \int \text{res}_T b_j T^j a_i T^i$$

$$= \text{Tr} BA$$

We would like to stress that this bilinear product is the same (up to factors of $q$) as the one defined in [16], as we shall show in section 5. The previous lemma is fully equivalent to theorem 3.3 in that reference. With respect to this inner product, the adjoint of $\tau$ is $\tau^* = \tau^{-1}$, i.e. $(T^* f) = (T^{-1} f)$.

Let us investigate the possible splittings of the form $\mathcal{G}_q = \mathcal{G}_1 \oplus \mathcal{G}_2$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are Lie subalgebras. In view of the generic (graded) commutation relations

$$[t_i T^i, t_j T^j] = (t_i \tau^i(t_j) - t_j \tau^j(t_i)) T^{i+j}$$

we find only three possibilities as follows
1. \((\sigma = -1)\), \(G_q = G_{\geq 0} \oplus G_{\leq -1}\)

\[
G_{\geq 0} \equiv \left\{ \sum_{i \geq 0} t_i(z)T^i \right\} ; \quad G_{\leq -1} \equiv \left\{ \sum_{i \leq -1} t_i(z)T^i \right\}
\]

2. \((\sigma = +1)\), \(G_q = G_{\geq 1} \oplus G_{\leq 0}\):

\[
G_{\geq 1} \equiv \left\{ \sum_{j \geq 1} t_j(z)T^j \right\} ; \quad G_{\leq 0} \equiv \left\{ \sum_{j \leq 0} t_j(z)T^j \right\}
\]

3. \((\sigma = 0)\), \(G_q = G_{0^+} \oplus G_{0^-}\)

\[
G_{0^+} \equiv \left\{ \sum_{k \geq 0} t_k(z)T^k, t_0 \in zC[z] \right\} ;
\]
\[
G_{0^-} \equiv \left\{ \sum_{k \leq 0} t_k(z)T^k, t_0 \in z^{-1}C[z^{-1}] \right\}
\]

**Remark 4.6.** As mentioned in [16], the interest of the last case comes from the fact that, relative to the inner product defined in (4.5), it is the only one where \(G_{0^\pm}\) are isotropic. Hence \((G, G_{0^+}, G_{0^-})\) is a Manin triple and \((G, G_{0^+})\) a Lie double.

**The fundamental Poisson brackets**

In order to define the phase space where our dynamics will take place, let

\[
L = \sum_{i=m}^{n} t_i(z)T^i \quad (4.7)
\]

where \(n, m \in \mathbb{Z}\) and \(n > m\). We regard any of these difference operators as “points” on a manifold \(M^{(n,m)}_T\). The dynamics is governed by differential-\(q\)-difference equations derived from the usual Lax system

\[
\frac{dL}{dt_p} = [L, (L^p)_+] = [(L^p)_-, L] \quad (4.8)
\]

which is manifestly consistent for \(L\) of the form given in (4.7) Here \(\pm\) refers to any one of the \(\sigma = 0, \pm 1\) splittings defined above.
In the case of $\sigma = -1$ with $(n, m) = (1, -\infty)$ this system is none other than the simplest version of the Toda lattice hierarchy involving one set of time parameters [21][22]. Indeed, in these works the Toda lattice hierarchy is formulated in terms of a Lax operator of the form

$$L = e^\partial + \sum_{n=0}^{\infty} u_{n+1} e^{-n\partial}$$

which involves the difference operator $e^\partial$ acting as $e^{n\partial}u_i(x) = u_i(x+n)e^{n\partial}$. The isomorphism between both formulations is made patent after identifying $u_i(x)$ with $t_i(z = q^x \zeta)$ where $\zeta \in \mathbb{C}$ is any fixed complex number.

Remark that only for $m = 0$ and the splitting $\sigma = -1$, or $n = 0$ and $\sigma = +1$ equations (4.8) are empty since in this case the commutator vanishes identically. The non-trivial flows may come then from fractional powers of $L$ $^3$. For example, let $n = N$ and $m = 0$, then

$$\frac{dL}{dt_p} = [L, \left(L^{p/N}\right)_+] = \left(L^{p/N}\right)_-, L$$

are non-trivial differential difference equations as long as $p$ is not a multiple of $N$. An analogous way to characterize these flows is to consider an operator of the form $L \in \mathcal{M}_T^{1,-\infty}$ of the form $L = t_0 T + t_1 + t_2 T^{-1} + \ldots$, constrained to satisfy $L_N^N = 0$.

On $\mathcal{M}_T^{(n,m)}$ the (linear) functionals of interest have the form $f_X(L) = \text{Tr} LX$ with

$$X = \sum_{j=m}^{n} T^{-j} x_j(z).$$

Clearly $f_X$ adopts the form of an euclidean scalar product $f_X = \int \sum_{i=m}^{n} t_i x_i$. Defining the gradient $d : F \to \mathcal{G}_q$ by

$$\langle df, \delta L \rangle \equiv \frac{d}{d\epsilon} f_X(L + \epsilon \delta L) \bigg|_{\epsilon=0}$$

it turns out that $df_X(L) = X$.

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$^3$ This issue is more delicate than in the usual context of KdV, and requires a careful definition of the ring of functions [15][16]. We will not dwell here with this aspect, certainly important from the point of view of integrability.
We are interested in the fundamental Poisson brackets among the fields \( t_i(z) \). Since the Poisson maps \( J^{(s)}(df_X) \) are linear in \( df_X = X \) we may expand

\[
J^{(s)}(X) = \sum_{i=m}^{n} \sum_{j=m}^{n} (J^{(s)}_{ij}) x_i(z) T^j
\]  

(4.9)

where \( J^{(s)}_{ij} \) is some function of q-difference operators \( T \). Plugging this back in (3.5) we obtain

\[
\{ f_X, f_Y \} = \int (J_{ij} x_j y_i) = \int x_i(J^{*}_{ji} y_j) = - \int x_i(J_{ij} y_j)
\]

where the last equation follows from the antisymmetry \( \{ f_X, f_Y \} = - \{ f_Y, f_X \} \) which implies that \( J^{*}_{ji} = -J_{ij} \). Finally, comparing this expression with

\[
\{ f_X, f_Y \}_r = \int x_i(z) \int \{ u_i(z) , u_j(w) \}_s y_j(w)
\]

shows that

\[
\{ u_i(z) , u_j(w) \}_s = -(J^{(s)}_{ij}(z) \delta(z/w))
\]

(4.10)

where \( \delta(z/w) = \sum_{j\in \mathbb{Z}} (z/w)^j \), and the operators \( J_{ij} \) act at \( z \).

It is time to analyse in detail the potential Poisson structures on \( \mathcal{M}_{T}^{(n,m)} \). We will do this by taking into consideration, case by case, the three possible splittings of \( \mathcal{G}_{q} \): \( \sigma = 0, \pm 1 \). Notice that for all cases, the linear and cubic brackets in (3.2) and (3.4) define Poisson brackets, since \( R = \frac{1}{2}(P_{\geq 0} - P_{\leq 1}) \), with \( P_{\pm} \) in each case the relevant projection operators, yields automatically an r-matrix obeying the m-YB(\( \frac{1}{4} \)) equation (3.1). Therefore, further analysis is only required for the quadratic bracket.

1. (\( \sigma = -1 \)): \( \mathcal{G} = \mathcal{G}_{\geq 0} \oplus \mathcal{G}_{\leq -1} \)

This splitting is, among the three, the most analogous to the one of the standard KP hierarchy. Notice however the important difference: now the subalgebra \( \mathcal{G}_{\geq 0} \) is \emph{not isotropic}, and in consequence the r-matrix is not anti-selfadjoint. Thus, whether the “antisymmetric” combination \( \frac{1}{2}(R - R^*) \) satisfies the m-YB(\( \frac{1}{4} \)) equation as well, must be checked independently. In more concrete terms, let

\[
R = \frac{1}{2}(P_{\geq 0} - P_{\leq -1}).
\]

In view of the definition of the inner product (4.5) \( R^* = \frac{1}{2}(P_{\leq 0} - P_{\geq 1}) \), and therefore

\[
\frac{1}{2}(R - R^*) = \frac{1}{2}(P_{\geq 1} - P_{\leq -1}).
\]

(4.11)

It follows from an easy calculation that this linear map satisfies (3.1) with \( \alpha = 1/4 \) as well. Hence all three brackets in (3.2) and (3.4) are Poisson brackets. Using the general
For the particular case of $\mathcal{M}_T^{(n,0)}$ this expression also appears in [15].
For $J^{(2)}_{ij}$ an analogous computation yields

$$J^{(2)}_{ij} = 2 \min(n,i) \sum_{k=\max(m,i+j-n)}^{\min(n,i)} \left( t_k T^{k-j} t_{i+j-k} - t_{i+j-k} T^{i-k} t_k \right) + t_i \left( 1 + T^i \right) \left( 1 - T^{-j} \right) t_j$$

or, again

$$\{t_i(z), t_j(w)\}_{(2)} = 2 \min(n,i) \sum_{k=\max(m,i+j-n)}^{\min(n,i)} \left( t_{i+j-k}(z) t_k(w) \delta \left( \frac{q^j - k}{w} \right) - t_k(z) t_{i+j-k}(w) \delta \left( \frac{z}{q^j - k w} \right) \right) - t_i(z) t_j(w) \sum_{l \in \mathbb{Z}} \left( \frac{z}{w} \right)^l \left( 1 + q^i \right) \left( 1 - q^{-j} \right)$$

2. $(\sigma = +1)$: $\mathcal{G}_q = \mathcal{G}_{\geq 1} \oplus \mathcal{G}_{\leq 0}$

This situation is symmetric with respect to the one above. Notice that at the level of the algebra, this splitting transforms into the previous one upon the substitution $T \to T^{-1}$. Therefore, the formulas obtained from (4.12)-(4.14) can be adapted to the present case by a simple replacement $q \to q^{-1}$ and $m \leftrightarrow n$.

3. $(\sigma = 0)$: $\mathcal{G}_q = \mathcal{G}_{0+} \oplus \mathcal{G}_{0-}$

This is the standard case of a Lie bialgebra. The three maps in (3.8) automatically define Poisson brackets. The fundamental ones are a slight modification of the ones above, and involve an additional operator $p_{\pm}$, which projects any element $f \in F$ into its Taylor and Laurent parts respectively, i.e. $p_{+} z^m = z^m$ iff $m \geq 1$ and zero otherwise, and viceversa. $p_{+}$ and $p_{-}$ are mutually adjoint with respect to the inner product defined with $f$ and commute with $T$.

As before, the linear structure is a direct sum of two subalgebras, spanned by the fields $t_0^+ \equiv p_{+} t_0$ and $\{t_i, i = 1, \ldots, n\}$ on ones side, and $t_0^- \equiv p_{-} t_0$ and $\{t_i, i = -1, -2, \ldots, m\}$ on the other. Therefore as long as $i, j \geq 1$ but $n \geq i + j$

$$J^{(1)}_{ij} = -(t_{i+j} T^i - T^{-j} t_{i+j}) \quad (4.17)$$

The same expression with opposite sign holds if $-1 \geq i, j$ with $i + j \geq m$. Finally

$$J^{(1)}_{0j} = -\Theta(j - 1) p_{+} (T^{-j} - 1) t_j + \Theta(-j - 1) p_{-} (T^{-j} - 1) t_i \quad (4.18)$$

and $J^{(1)}_{0i} = -J^{(1)*}_{0i}$. In all other cases $J_{ij} = 0$. In formula (4.18) $\Theta$ stands for the usual step function $\Theta(x) = 1$ iff $x \geq 0$ and 0 otherwise. The quadratic brackets are computed along the same lines:

$$J^{(2)}_{ij} = 2 \min(n,i) \sum_{k=\max(m,i+j-n)}^{\min(n,i)} \left( t_k T^{k-j} t_{i+j-k} - t_{i+j-k} T^{i-k} t_k \right) + 2 t_i (1 - T^{-j}) p_{-} t_j$$
Some reductions

Let us focus on the $\sigma = -1$ splitting (the case $\sigma = +1$ follows a symmetric pattern). Remark that as, far as the Lax equations are concerned, the field $t_n$ is not dynamical. Let $\tilde{M}_T^{(n,m)}$ represent the submanifold of $M_T^{(n,m)}$ defined by the constraint $t_n = 1$ (or any constant). From $J^{(1)}$ in (4.12) we observe that, as long as $n \geq 1$, the highest positive order of $J^{(1)}$ is $n - 1$ and therefore the hamiltonian map is automatically tangent to the constraint submanifold. When $m = 0$ this is also the case for a similar constraint on the lowest field $t_0 = constant$; indeed (4.12) shows that in this case the contribution of $J^{(1)}(X)$ to order zero is $[L_0, X_0] = 0$. In few words, both constraints are first class, and the Poisson brackets are defined by simple restriction of (4.16).

For $J^{(2)}$, things are more involved. Notice in fact from the expression (4.13), that the highest order of $J^{(2)}(X)$ is $n$, i.e. the same as that of $L$. Therefore, in order to define Poisson brackets on $\tilde{M}_T^{(n,m)}$ we would follow the standard prescription for second class constraints due to Dirac. However, instead of plugging here the formula of the Dirac brackets we will pause briefly to describe how they appear in our context. Given the projection map, such as $M_T^{(n,m)} \rightarrow \tilde{M}_T^{(n,m)}$ that sets $t_n \rightarrow 1$, at each point $L$, the induced projection of vectors on the tangent subspace is unique. This is not so for 1-forms. To see this notice that if we want to compute Poisson brackets of functions $f, g$ on $\tilde{M}_T^{(n,m)}$ via (4.13) we first need to extend them to $M_T^{(n,m)}$. This extension, being non-unique, renders the component $x_n = \frac{\delta f}{\delta t_n}$ in the gradient

$$df(L) = \sum_{k=m}^{n} T^{-k} x_k$$

undefined. Therefore some additional structure is required in order to specify the cotangent subspace. Since we have $J^{(2)}$ at hand, a map from 1-forms to vectors, we may fix this ambiguity by demanding that the associated hamiltonian vector fields be tangent to $\tilde{M}_T^{(n,m)}$. In other words, we fix $x_n$ by the requirement that $J_2(df(L))$ should have no term of order $n$. This form of computing the algebra is fully equivalent to the Dirac bracket prescription as we show next [23]. The demand that $J(z)$ should stay tangent to the constraint manifold implies for $df(L)$ that

$$\sum_{j=m}^{n} J_{nj} x_j = 0$$

and this may be solved for $x_n = -\sum_{j=m}^{n-1} J_{nn}^{-1} J_{nj} x_j$. Plugging this back into (4.9) we have

$$J(z) = \sum_{i,j=m}^{n-1} (\tilde{J}_{ij} x_j) T^i$$

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where
\[ \tilde{J}_{ij} = J_{ij} - J_{in} J_{nj}^{-1} J_{nj}, \quad i, j = 1, \ldots, n - 1. \] (4.19)

are the corresponding Dirac brackets on the constraint surface. For the explicitly reduced brackets we find a non-local expression as follows
\[ \tilde{J}^{(2)}_{ij} = 2 \sum_{k = \max(m, i + j - n)}^{\min(n, i)} \left( t_k T^{k-j} t_{i+j-k} - t_{i+j-k} T^{i-k} t_k \right) + 2 t_i \frac{(1 - T^{-n})(1 - T^{-j})}{(1 - T^{-n})} t_j \] (4.20)

Indeed, the interest in the reduction \( t_n = 1 \) stemmed from the fact that the Lax flows (4.8) stabilize this constraint. Likewise, if \( m = 0 \) the Lax equation for \( t_0 \) is trivial, hence we may want to set it also to a constant. However, in contrast to the previous case, the contribution of \( J^{(2)}_0(X) \) to order zero is \( 2 L_0(X) L_0 - 2(LX)_0 L_0 + 2L_0 [L, X]_0 = 0 \); in other words, for all \( j \), \( J^{(2)}_{0j} \) vanishes and therefore this constraint is first-class and does not lead to any modification of the algebra. If we put \( m = 0 \), (4.20) is equivalent to formula (3.6) in [15].

§5 The q-KP basis

We recall that our main purpose is to construct a q-deformation of the algebra \( \mathcal{W}_{KP} \). For this reason it will be interesting to reformulate the findings of the previous section in terms of the basis \( \partial_q \), i.e.
\[ \partial_q = \frac{1}{z(q-1)(T-1)} \] (5.1)

Written in this basis, the limit \( q \to 1 \) should yield directly \( \mathcal{W}_{KP}^{(n)} \) in [18]. We recall here the relevant formulae for the change of basis.
\[ T = z(q - 1) \partial_q + 1 \]
\[ T^{-1} = \frac{1}{z(q - 1) \partial_q + 1} = -\sum_{i=1}^{\infty} \frac{(-q)^i}{(q - 1)^i z^{-i}} \partial_q^{-i} \] (5.2)

These imply in particular, that the phase space \( \mathcal{M}_{T,(n,m)} \) will be coordinatized now by q-pseudodifferential operators \( L \), of the form
\[ L = \sum_{j=-\infty}^{n} u_j(z) \partial_q^j, \] (5.3)

\((m + n)\) being still the number of degrees of freedom. Yet the manifold of all q-pseudodifferential operators of the form (5.3), which we will denote by \( \mathcal{M}_{\partial_q}^{(n)} \), is much bigger than \( \mathcal{M}_{T,(n,m)}^{(n)} \). Rather we have that the set of all these spaces \( \mathcal{M}_{T,m}^{(n,m)}, \ m = 1, 2, 3... \) is dense in \( \mathcal{M}_{\partial_q}^{(n)} \).
Notice that (5.2) involves a specific choice of the expansion point, namely around \( \partial_q = \infty \). Other choices may lead to different \( W \)-algebras. In the \( q \)-KP basis we may grade \( \mathcal{G}_q \) by the scaling dimension: if \( z \) has degree \(-1\), \( \partial_q \) will have \(+1\) and we may make \( L \) homogeneous of a certain degree, \( n \), by further assignment of degree \( j \) to \( u_j \). This gives us a chance to look for a \( q \)-deformation of the Virasoro algebra in the subalgebra spanned by the counterpart of the energy momentum tensor (the field \( u_2 \) in the context of the classical \( W_{\text{KP}} \) algebra), which will be a particular \( x \)-dependent combination of various fields in the Toda basis (where the grading was a different one).

In order to study the hamiltonian structures we have to re-define the residue and trace functionals in the new basis. The point is the following; let

\[
L(T)_{\geq 0} = \sum_{i=0}^{n} t_i T^i = \sum_{i=0}^{n} u_i \partial_q^i = L(\partial_q)_{\geq 0}
\]

where in each case the projection is performed with respect to the relevant basis. Making use of (5.1) we may write \( t_0 \) in terms of \( u_i \):

\[
t_0(u_i) = (-1)^m u_m \frac{z^m (q-1)^m}{z^m (q-1)^m} + (-1)^{m-1} u_{m-1} \frac{z^{m-1} (q-1)^{m-1}}{z^{m-1} (q-1)^{m-1}} + \ldots - \frac{u_1}{z(q-1)} + u_0 \quad (5.4)
\]

If \( L(T) \equiv t_i T^i \), \( t_0 = \text{res}_T L(T) \). How can we manage extract \( t_0(u_i) \) out of \( L(\partial_q) \) as given by the right hand side of (5.4)? Notice that we may take advantage of the fact that the projections (in the respective basis) \( (\ )_{\geq 0} \) and \( (\ )_{\leq -1} \) commute with the change of basis \( T \leftrightarrow \partial_q \), hence

\[
(L(T))_0 = (L(T) T^{-1})_{-1} = (L(T)_{\geq 0} T^{-1})_{-1} = \frac{z(q-1)}{q} (L(\partial_q)_{\geq 0} T^{-1}(\partial_q))_{-1}
\]

In the last expression \( T^{-1}(\partial_q) \) stands for the second relation in (5.2). Thus

\[
t_0(u_i) = \frac{z(q-1)}{q} \text{res}_{\partial_q} (L(\partial_q) T^{-1}).
\]

where we have introduced the symbol \( \text{res}_{\partial_q} a_i \partial_q^i \equiv a_{-1} \). Concerning the ad-invariant symmetric bilinear product, we find the same expression that was considered in ref [16] (modulo a constant factor)

\[
\langle A, B \rangle = \langle B, A \rangle = \int \text{res}_T AB = \frac{q-1}{q} \int z \text{res}_{\partial_q} AB T^{-1}
\]

(5.5)

In this basis the natural integral functional is \( \int_{-1} \equiv \int z \) which, in spite of not being scale invariant \( \int_{-1} \tau(f) = q^{-1} \int_{-1} f \) satisfies the desirable property that \( \int_{-1}(\partial_q f) = 0 \). With
respect to the above inner product, the adjoints of $\tau$ and $\partial_q$ are easy to compute, yielding
\[
\tau^* = \frac{1}{q} \tau^{-1} ; \quad \partial_q^* = -\partial_q^* \sigma^{-1} .
\] (5.6)

For later use we shall introduce the following compact notation:
\[
\Omega(A) \equiv \frac{z(q-1)}{q} \text{res}_{\partial_q}(A)
\]

Next, we must characterize the three possible splittings of $\mathcal{G}_q$ ($\sigma = 0, \pm 1$) in the $\partial_q$ basis:

The untwisted basis is naturally adapted to the case $\sigma = -1$
\[
\mathcal{G}_{\geq 0}(T) = \mathcal{G}_{\geq 0}(\partial_q) ; \quad \mathcal{G}_{\leq -1}(T) = \mathcal{G}_{\leq -1}(\partial_q).
\]

$\sigma = +1$ looks a little bit more contrived
\[
\mathcal{G}_{\geq 1}(T) = \tilde{\mathcal{G}}_{\geq 0}(\partial_q) \equiv \{ L = \sum_{j=0}^{m} u_j \partial_q^j | \text{res}_{\partial_q}(LT^{-1}) = 0 \} \\
\mathcal{G}_{\leq 0}(T) = \mathcal{G}_{\leq 0}(\partial_q) = \{ L = \sum_{j=-\infty}^{0} u_j \partial_q^j \} \\
\]

Lastly, the characterization of $\sigma = 0$ in the untwisted basis makes this splitting very unnatural
\[
\mathcal{G}_{0+}(\partial_q) \equiv \{ L = \sum_{j=0}^{m} u_j \partial_q^j | z \text{res}_{\partial_q}(LT^{-1}) \in zC[z] \} \\
\mathcal{G}_{0-}(\partial_q) \equiv \{ L = \sum_{j=-\infty}^{0} u_j \partial_q^j | u_0 \in z^{-1}C[z^{-1}] \} \\
\]

We want to consider again the Poisson maps (3.7). Now in order to compute the analog of (4.12)-(4.14) we have to say what the relevant projection operators are. From the form of the scalar product (5.5) it is clear that
\[
P_{\geq 0}L = L_{\geq 0} ; \quad P_{\geq 0}^*L = (LT^{-1})_{\leq -1}T \\
P_{\leq -1}L = L_{\leq -1} ; \quad P_{\leq -1}^*L = (LT^{-1})_{\geq 0}T \\
\] (5.7)

We may simplify these expressions, reminding that the projections $(\ )_{\geq 0}$ and $(\ )_{\leq -1}$
commute with the change of basis $T \leftrightarrow \partial_q$. So for $L = t_i T^i = u_j \partial^j_q$

$$P_{\geq 0}^* L = (L(\partial_q)T^{-1}(\partial_q))_{\leq -1} T = (L(T)T^{-1})_{\leq -1} T = (L(T))_{\leq 0}$$

$$= (L(T))_{\leq -1} + t_0$$

$$= (L(\partial_q))_{\leq -1} + \Omega(L)$$

where we made use of (5.2). Similarly

$$P_{\leq -1}^* L = (L(\partial_q)T^{-1})_{\geq 0} T = (L(T)T^{-1})_{\geq 0} T = L(T)_{\geq 1} T^{-1} T$$

$$= L(T)_{\geq 0} - t_0$$

$$= L(\partial_q)_{\geq 0} - \Omega(L)$$

With these results the antisymmetric part of the r-matrix is

$$\frac{1}{2}(\mathcal{R} - \mathcal{R}^*) = \frac{1}{2}(P_{\geq 0} - P_{\leq -1} - \Omega) = \mathcal{R} - \frac{1}{2} \Omega \tag{5.8}$$

It is not evident that this expression also satisfies the m-YB($\frac{1}{4}$) equation. However an explicit computation shows that the only non-vanishing contribution to (3.1) has the form $\Omega([a_{\geq 0}, b_{\geq 0}])$, which vanishes. An easy way to convince oneself of this fact is that when written in the $T$ basis this is $\text{res}_T [a(T)_{\geq 0}, b(T)_{\geq 0}] = 0$.

With all this information, it is now an easy exercise to find the explicit expressions for (3.7) as adapted to the present case. Let again $X \equiv df$:

$$J^{(1)}(X) = [L, X_{\geq 0}]_{\leq -1} - [L, X_{\leq -1}]_{\geq 0} + \Omega([L, X]) \tag{5.9}$$

$$J^{(2)}(X) = 2L(XL)_{\geq 0} - 2(LX)_{\geq 0} L + L \Omega([L, X]) + \Omega([L, X]) L$$

$$= -2L(XL)_{\leq -1} + 2(LX)_{\leq -1} L + L \Omega([L, X]) + \Omega([L, X]) L \tag{5.10}$$

$$J^{(3)}(X) = [L, (LXL)_{\geq 0}] - L [L, X]_{\geq 0} L + L \Omega([L, X]) L \tag{5.11}$$

Notice that as compared with the analogous expressions for the $W_{\text{KP}}$ algebra [18], the ones above present additional terms which vanish in the limit $q \to 1$. However these terms are not active whenever $f$ is a Casimir function, and hence, in particular for the Lax-hamiltonian flows.

In order to compute the algebra of fundamental Poisson brackets we have to describe the manifold and the class of functionals for which $J^{(i)}$, $i = 1, 2, 3$ describe tangent maps. We will work on $\mathcal{M}^{(n)}_{\partial_q}$ whose points are parameterized as

$$L^{(n)} = \sum_{i=0}^{\infty} u_i \partial^{n-i} \quad (n \in \mathbb{Z}) \tag{5.12}$$

Accordingly, in order to define linear functionals of the form $f_X = \int_{-1}^{1} u_i x_i$ as $f_X = \int_{-1}^{1} \text{res}_{\partial_q} LXT^{-1}$ our gradient 1-forms will be $q$-pseudodifferential operators of the
form

$$X \equiv df_X = \sum_{j=0}^{\infty} \partial_q^{-n-1} T x_j.$$

After a straightforward computation, we list the full set of fundamental brackets for $J^{(1)}$ as follows: first we have that for all $j$: $J^{(1)}_{ij} = J^{(1)}_{0ij} = J^{(1)}_{j0} = 0$ if $i, j \geq n + 1$:

$$J^{(1)}_{ij} = \sum_{k=0}^{i+j-n-1} \binom{i-n-1}{k} \frac{q^{k+1}}{q} x^{n-i-1}(q-1) u_{i+j-n-k} (-\partial_q)^k x + \frac{1}{q} u_{i+j-n-k-1} (-\partial_q)^k T^{n-i}$$

$$- \sum_{k=0}^{i+j-n} \binom{j-n}{k} \frac{q-1}{q} T^{j-n-k} \partial_q^k u_{i+j-n-k} x$$

$$- \sum_{k=0}^{i+j-n-1} \binom{j-n-1}{k} \frac{1}{q} T^{j-n-k-1} \partial_q^k u_{i+j-n-k-1}. \quad (5.13)$$

If however $1 \leq i, j \leq n - 1$ the same expression (5.13) is valid with the opposite sign. Finally when $\overline{j = n}$:

$$J^{(1)}_{in} = \sum_{k=0}^{i-n-1} \binom{i-n-1}{k} \frac{q^{k+1}}{q} x^{n-i-1}(q-1) u_{i-k} (-\partial_q)^k x T^{n-i} - \frac{1}{q}(q-1) x u_i \Theta(i - (n + 1))$$

$$+ \left\{ - \sum_{k=0}^{i-1} \binom{i-n-1}{k} \frac{q^{k+1}}{q} x^{n-i-1}(q-1) u_{i-k} (-\partial_q)^k x + \frac{1}{q} u_{i-k-1} (-\partial_q)^k \right\} \Theta(n - i).$$

$$+ \frac{q-1}{q} x u_i + \sum_{k=0}^{i-1} \binom{-1}{k} \frac{1}{q} T^{-k-1} \partial_q^k u_{i-k-1} \delta_{i,n} \quad (5.14)$$

The rest of the brackets can be computed making use of the identity $J_{ij} = -J_{ji}^*$, and (5.6)

Concerning reductions, as long as $n \geq 1$, the highest order of $J^{(1)}(X)$ is $n - 1$ and thereafter its action is tangent to the submanifold defined by $u_0 = \text{constant}$. Again this reduction is therefore first-class. Two other consistent reductions of $L$ are of the form $L = L_{\geq 0}$ with $J^{(1)}(X) = -[L_{\geq 0}, X_{\leq -1}]_{\geq 0} + \Omega([L_{\geq 0}, X_{\leq -1}])$ or $L = L_{\leq 0}$, in which case $J^{(1)}(X) = [L_{\leq 0}, X_{\geq -1}]_{\leq -1} + \Omega([L_{\leq 0}, X_{\geq -1}])$. The relevant explicit form of the Poisson brackets can be obtained in each case from (5.9) after suitably setting to zero the corresponding fields $u_i$ and its duals $x_i$. 

$$- 20 -$$
Written in this basis, the formula (5.13) exhibits a nested sequence of subalgebras \( N = 1, 2, \ldots \), spanned by \( \{u_{n+N+k}, \ k = 0, 1, 2, \ldots\} \). In the continuum limit \( q \to 1 \) these contract to the nested set truncations of the centerless \( W_{1+\infty} \) algebra known as \( W_{-N+\infty} \) [18].

For \( J^{(2)} \) we have in turn

\[
J^{(2)}_{ij} = \sum_{k=0}^{i-1} \sum_{l=0}^{k} \left[ \begin{array}{c} l-k-1 \\ l \end{array} \right] q^{(l-1)(k+1)} u_{j+k-l} \partial_q^{k} T^{-k} u_{i-k-1} \\
- \sum_{k=0}^{i} \sum_{l=0}^{k+j-i-k-1} \sum_{m=0}^{j-n-1} \left[ \begin{array}{c} j-n-1 \\ l \end{array} \right] q^{(l-1)(k+1)} \left[ \begin{array}{c} n-m \\ i-k-m-1 \end{array} \right] q^{(l-1)(l-j-n+1)+(i+l-k-m-2)(i-k-n-1)} u_{m} \partial_q^{i+l-k-m-1} T^{j+k-i-l+1} u_{j+k-l} \\
+ \sum_{k=0}^{i} \sum_{l=0}^{k+j-i-k} \left[ \begin{array}{c} l-k-1 \\ l \end{array} \right] q^{(l-1)(k+1)} (q-1) x u_{j+k-l} \partial_q^{k} T^{-k} u_{i-k} \\
- \sum_{k=0}^{i} \sum_{l=0}^{k} \left[ \begin{array}{c} j-n-1 \\ l \end{array} \right] q^{(l-1)(l-j-n+1)+(i+l-k-m-1)(i-k-n)} q^{(i-1)} u_{m} \partial_q^{i+l-k-m} T^{j+k-i-l} u_{j+k-l} \\
- (1-q^{-1}) x u_{i} u_{j} + \sum_{k=0}^{i} \sum_{l=0}^{j} \left[ \begin{array}{c} j-n-1 \\ l \end{array} \right] q^{(l-1)(l-j-n+1)+(i+l-k-1)(i-n)} (q-1) x u_{k} \partial_q^{i+l-k} T^{j-i-l} u_{j-i} \\
+ \sum_{k=0}^{i-1} \sum_{l=0}^{j} \left[ \begin{array}{c} j-n-1 \\ l \end{array} \right] q^{(l-1)(l-j-n+1)+(i+l-k-1)(i-n-1)} (q^{n-i+1}-1) u_{k} \partial_q^{i+l-k} T^{j-i-l+1} u_{j-l} \\
+ \sum_{k=0}^{j} \left[ \begin{array}{c} n-k \\ i-k \end{array} \right] q^{(i-k)(i-n)} (1-q^{-1}) u_{k} \partial_q^{i-k} T^{n-i} x u_{j} \\
- \sum_{k=0}^{j} \left[ \begin{array}{c} j-n-1 \\ k \end{array} \right] q^{(k-1)(k-j+n+1)} q^{(i-1)} x u_{i} \partial_q^{k} T^{j-k-n} u_{j-k} \\
(5.15)
\]

This expression reduces in the limit \( q \to 1 \) to the one of the \( W_{KP} \) algebra. Contrarily to \( J^{(1)} \), \( J^{(2)}(X) \) does not stabilize the field \( u_{0} \); i.e., from (5.10) we see that the highest order of \( J^{(2)}(X) \) is the same as that of \( L \). Therefore the constraint \( u_{0} = 1 \) is second class. The same discussion that was developed in the \( T \) basis holds here \textit{mutatis mutandis}. We will refrain from giving the explicit form of the reduced Poisson brackets, whose computation follows again the standard Dirac’s recipe.
Reductions. Where is $q$-$W_n$?

Let us consider here the very important reductions of $q$-KP to $q$-KdV. The expressions in (5.9) and (5.10) are perfectly consistent when applied to purely $q$-differential operators $L = u_0 \partial_q^n + u_1 \partial_q^{n-1} + \ldots + u_n$. The related algebras are simply obtained by restricting the subindices of the fields appearing in (5.13) (5.14) and (5.15) to take values in the range $i, j \in [0, n]$, and neglecting all other fields. In strict sense, these algebras should be considered as deformations of $GD_n$, the second Gelfand-Dickey bracket over the phase-space of Lax operators of the form $L = \partial_q^n + u_1 \partial_q^{n-1} + \ldots + u_n$. Hence we will name them $q$-$GD_n$ algebras.

An important point arises here: as compared with $GD_n$, $q$-$GD_n$ contain an additional generator $u_0$. In the limit $q \to 1$ this field decouples because $\lim_{q \to 1} J_{0j} = 0$, $\forall j$ and we may set $u_0 = 1$. One could argue that in order to construct a true $q$-deformation of $GD_n$ which involves exactly $n$ generators we should first reduce $u_0 = 0$ via Dirac brackets. However the projection involved in the reduction is not a continuous step and nothing guarantees that the resulting algebra will still recover the desired limit when $q \to 1$.

Let us give an example of this phenomenon by considering the simplest Lax operator $L = u_0 \partial_q + u_1$. The Poisson brackets for $u_0$ and $u_1$ generate $q$-$GD_1$, whose brackets are given by

\[
J_{00}^{(2)} = \frac{q-1}{2q} u_0 (T - T^{-1}) z u_0 \\
J_{01}^{(2)} = \frac{1}{2q} u_0 (T - T^{-1}) u_0 \\
J_{10}^{(2)} = \frac{1}{2q} u_0 (qT - q^{-1}T^{-1}) u_0 \\
J_{11}^{(2)} = \frac{1}{2zq(q-1)} u_0 (T - T^{-1}) u_0
\]

which in the limit $q \to 1$ reproduce the free boson algebra $GD_1$ after $u_0$ is set to 1, i.e. $J_{11}^{(2)} \to \partial$, and $J_{0i}^{(2)} \to 0$. However if we insisted in reducing $u_0 = 1$ before taking the limit, the Dirac formula gives us a vanishing answer for $\tilde{J}_{11}^{(2)}$:

\[
\tilde{J}_{11}^{(2)} = J_{11}^{(2)} - J_{10}^{(2)} (J_{00}^{(2)})^{-1} J_{01}^{(2)} \\
= u_0 \frac{1}{2q(q-1)} \left( \frac{1}{z}(T - T^{-1}) - (qT - q^{-1}T^{-1}) \frac{1}{z} \right) u_0 \\
= 0
\]

One cannot cure this result by multiplying the starting brackets by global factors of $(q-1)$, because the Dirac bracket is homogeneous under such rescalings. This vanishing result is also independent of any $q$-dependent redefinition of the field $u_0$. 

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We expect that a similar discussion applies to the classical $W_n$ algebras although we do not have a general proof. These algebras arise as hamiltonian reductions of $GD_n$ where the generator $u_1$ is set to 0. The first generator, $u_2$, closes a linear subalgebra which is non other than the ubiquitous Virasoro algebra. It is in this sense that $W_n$ algebras are sometimes defined as (non-linear) extensions of the Virasoro algebra. A continuous $q$-deformation of $W_n$ in $n−1$ fields $u_2,...,u_n$ would present the same problems that we have exposed above in the case of $GD_n$. The naive procedure, of starting from $q$-$GD_n$ and reducing $u_0 = 1$ and $u_1 = 0$ may spoil the continuous correspondence with $W_n$ in the limit $q \to 1$. We feel this is an important point that deserves further attention.

§ 6 Analitic continuation

Notice that the expression for the algebra $q$-$W^{(n)}_{KP}$ as given in (5.15) admits analytic continuation to complex values of $n = \alpha \in \mathbb{C}$. This happens in contrast with the first structure, given in (5.13)(5.14), where $n$ appears explicitly in the limits of sumatories. The best way to understand this is by implementing the analytic continuation right from the beginning. Actually the whole formalism is susceptible of such a continuation along the lines advocated in ref. [18] and [24]. Hence $q$-$W^{(\alpha)}_{KP}$ describes a two-parameter family of nonlinear $W_{\infty}$ type algebras.

There is an important technical question concerning the triviality of such deformation parameters, i.e., whether the algebras for different pairs $(q, \alpha)$ and $(q', \alpha')$ are isomorphic or not. At least in the continuum case $q = 1$ we know positively that $\alpha$ represents a non-trivial deformation parameter [25].

The second issue we intend to address in this concern is the possibility of connecting the linear and quadratic structures by a suitable contraction of the parameter $\alpha$. In [18] the limit $\alpha \to 0$ was shown to yield an extension of the linear algebra $W_{1+\infty}$ by means of the Khesin-Kravchenko cocycle [26]. This fact was also understood in [27] and in [24] from a Poisson-Lie group theoretical point of view.

In more concrete terms, let us introduce a second parameter $\beta$ and define $L^{(\alpha, \beta)} \in \mathcal{M}^{(\alpha)}_{\partial_q}$ such that

\[ L^{(\alpha, \beta)} = \beta \partial^\alpha_q + \sum_{j=0}^{\infty} u_j(z) \partial^{\alpha-j}_q \equiv \beta \partial^\alpha_q + L^{(\alpha)}. \]  

(6.1)

Correspondingly, the 1-forms $X$ will look as

\[ X = \sum_{j=0}^{\infty} \partial^j_q \alpha - 1 \ T \ x_j. \]

We will be interested in the following “scaling” limit in which $\alpha$ tends to 0 and $\beta$ to $\infty$ in such a way that $\alpha \beta = c$ a finite constant. It will be convenient to normalize $J^{(2)}$ in the
following form

\[ J_{L(\alpha, \beta)}^{(2)}(z) = \frac{1}{\beta} \left\{ L(\alpha, \beta)(XL(\alpha, \beta))_{\geq 0} - (L(\alpha, \beta)X)_{\geq 0}L(\alpha, \beta) \right\} + \frac{1}{2}L(\alpha, \beta) \Omega\left(L(\alpha, \beta), X\right) + \frac{1}{2} \Omega\left(L(\alpha, \beta), X\right) L(\alpha, \beta) \]

Plugging (6.1) in this expression we may first gather all the terms quadratic in \( \beta \partial^\alpha q \):

\[ \beta \left( \partial^\alpha q (X \partial q)_{\geq 0} - (\partial^\alpha q X)_{\geq 0} \partial^\alpha q + \frac{1}{2} \partial^\alpha q (\partial^\alpha q, X) \right) + \frac{1}{2} \Omega\left(\partial^\alpha q, X\right) \partial^\alpha q \]

Expanding \( \partial^\alpha q = 1 + \alpha \log \partial q + O(\alpha^2) \) the surviving terms in the desired limit yield

\[ c \left[ \log \partial q, X_{\geq 0} \right] - c \left[ \log \partial q, X \right]_{\geq 0} + c \Omega\left(\left[ \log \partial q, X\right]\right) = c \left[ \log \partial q, X_{\geq 0}\right]_{-1} + c \Omega\left(\left[ \log \partial q, X\right]\right)\]

In the linear terms the \( \beta \) dependence cancels out and we obtain

\[ \left[L(0), X_{\geq 0}\right] - \left[L(0), X\right]_{\geq 0} + \Omega\left(\left[L(\alpha), X\right]\right) = \left[L(0), X_{\geq 0}\right]_{-1} + \Omega\left(\left[L(0), X\right]\right) \]

In summary, the limiting hamiltonian structure yields

\[ J_{1+\infty,q}(X) = \left[c \log \partial q + L(0), X_{\geq 0}\right]_{-1} + \Omega\left(\left[c \log \partial q + L(0), X\right]\right) \quad (6.2) \]

Consistency of \( J(X) \) as a tangent map demands that \( L \) be of the form

\[ L = \log \partial q + u_0 + \sum_{i=1}^{\infty} u_i \partial^{-i} \]

The expression \( \log \partial q \) has to be understood as an outer automorphism of the Lie algebra of \( q \)-pseudodifferential symbols. Its action can be defined and computed as a limit:

\[ \left[ \log \partial q, f \partial^p q \right] = \lim_{\alpha \to 0} \frac{1}{\alpha} (\partial^\alpha q f \partial^p q \partial^{-\alpha} q - f \partial^p q) \]

\[ = \log q f^\tau \partial^p q - \sum_{k \geq 1} \frac{\log q}{(q - 1)} \left[ -1 \right]_q q^k \tau^{-k} (\partial^k q f) \partial^{p-k} q \]

The notation in (6.2) intends to make explicit that this algebra is a \( q \)-deformation of the centrally extended \( W_{1+\infty} \) algebra. We will not write down explicitly the Poisson brackets here. They agree with the ones given in (5.13) and (5.14) with \( n = 0 \) except for the central terms, which are the only ones that acquire corrections proportional to \( \log q \).
§7  CONCLUSIONS AND OUTLOOK

The picture of an atlas of $W$ algebras is slowly emerging. In this landscape, $W_\infty$ algebras provide natural landmarks and, among them, the algebra $W_{\text{KP}}$ is a cornerstone. In ref.\,[28] this algebra was shown to be related with a large amount of the known classical $W$-type algebras by continuous deformation or truncation. The main result of the present paper is that a lot of points in that atlas admit yet another deformation, parameterized by $q$. Of particular importance are $q$-$W_{\text{KP}}^{(\alpha)}$, $q$-$GD_n$, and $q$-$W_{1+\infty}$.

It has been amusing to observe how many structures that worked fine for the algebra of pseudodifferential operators, are robust enough to resist their implementation in the algebra of $q$-deformed pseudodifferential operators, as well. It certainly points out that perhaps other well known results could be exported. To be more precise, we think about issues like the dressing transformation, the embedding of the Lie algebra of differential operators into $W_n$\,[29] or the Kupershmidt-Wilson-Yu theorem\,[30]. In fact, concerning this last important theorem, a straightforward implementation of the proof given in\,[31] for a quadratic structure of the form (3.3) works fine in the case of an isotropic splitting. This requirement is only fulfilled in the present work for the splitting $\sigma = 0$ and hence there is a $q$-deformed version of the Kupershmidt-Wilson-Yu theorem in this case. For $\sigma = \pm 1$ we have not been able to establish a similar result. In this respect we should mention that a proposal for a $q$-deformed Miura transformation has appeared in\,[14]. Its connection to some peculiar way to factorize the Lax operator has been addressed in\,[15].

We should emphasize the existence of three consistent splittings ($\sigma = 0, \pm 1$), for the algebra $\Psi\text{DO}_q$. They all yield integrable hamiltonian systems and thereafter $W$ type algebras. In references\,[32][33] the m-KdV hierarchy was investigated in the scalar Lax formalism. It was recognized that this system is related to a nonstandard splitting of the algebra of ordinary $\Psi\text{DO}$. Indeed $L = L_{\geq k} + L_{< k}$ yields consistent subalgebras for $k = 0, 1$ and 2. It would be interesting to find out whether a possible $q$-deformation of these non-standard splittings could be related to the cases $\sigma = 0$ and $\sigma = +1$ in this paper.

The connection of the KP hierarchy with the Toda lattice hierarchy is a subject of recent interest which has received the attention of different groups\,[34][22][35]. We believe that our approach is substantially different to these and closer, at least in spirit, to the lattice deformation of\,[36]. We expect that the powerful techniques that have been used in this paper can be implemented also in the context of the Calogero-Sutherland model, especially in the formulation that makes use of the exchange operators\,[37].
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