CONTACT STRUCTURES, CR YAMABE INvariant, AND CONNECTED SUM

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Abstract. We propose a global invariant $\sigma_c$ for contact manifolds which admit a strictly pseudoconvex CR structure, analogous to the Yamabe invariant $\sigma$. We prove that this invariant is non-decreasing under handle attaching and under connected sum. We then give a lower bound on $\sigma_c$ in a particular case.

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1. Introduction

The classical Yamabe problem is that of the existence, on a compact Riemannian manifold, of a metric conformal with a given metric and with constant scalar curvature [Sch84, LP87]. The Yamabe invariant $\sigma$ has been introduced by R. Schoen and O. Kobayashi in the wake of the resolution of the Yamabe problem [Kob87, Sch89]. It is built the following way: a sufficient condition for a metric $g$ to have constant scalar curvature is to minimize, among metrics of same volume in the same conformal class, the integral scalar curvature $S(g)$. This minimum is moreover always smaller than $S(g_{S^n})$, where $g_{S^n}$ is the standard metric on the sphere. The Yamabe invariant is then defined, for a compact differentiable manifold $M$, as the "max-min"$

\sigma(M) := \sup_{[g]} \inf_{\hat{g} \in [g]} \int_M \text{Scal}(\hat{g}) \, d\text{vol}_{\hat{g}},$

where Scal denotes the Riemannian scalar curvature, the supremum runs over all conformal classes of metrics $[g]$ on $M$ and the infimum runs over all metrics of volume 1 in $[g]$.

This global differential invariant is produced by looking at the given differentiable manifold $M$ through the prism of the conformal structures $[g]$ with which it can be equipped. Similarly, let us consider a compact contact manifold $(M, H)$ which admits a strictly pseudoconvex CR structure, which we will call an SPC manifold. To each CR structure corresponds a conformal class of positive contact forms $\theta$,
which leads to the\textit{ CR Yamabe problem}. This problem has been given a positive answer by D. Jerison and J. Lee, N. Gamara and R. Yacoub [JL87, JL89, Gam01, GY01]. One can thus define a \textit{contact Yamabe invariant} $\sigma_c$ as
\[
\sigma_c(M,H) := \sup_{J} \inf_{\hat{\theta} \in [\theta]} \int_{M} \text{Scal}_W(J, \hat{\theta}) \, \hat{\theta} \wedge (d\hat{\theta})^n,
\]
where $\text{Scal}_W$ denotes the \textit{Webster scalar curvature}, the supremum runs over the set $J$ of all complex structures $J$ such that $(M, H, J)$ is strictly pseudoconvex, and the infimum runs over all compatible pseudohermitian forms of volume 1. This invariant has been introduced by C.-T. Wu [Wu09], but up to our knowledge has not been studied since. This invariant is a genuine contact invariant in dimension 3, in the sense that all orientable 3-dimensional contact manifolds are SPC.

A construction by W. Wang, recently implemented by J.-H. Cheng and H.-L. Chiu, shows that the positivity of $Y_{CR}$ is preserved under handle attaching on a strictly pseudoconvex CR spherical manifold [Wan03, CC19]:

\textbf{Theorem 1.1 [Wan03, CC19].} Let $(M, H, J)$ be a compact spherical strictly pseudoconvex CR manifold with positive $Y_{CR}$. Let $(\tilde{M}, H, \tilde{J})$ be obtained from $(M, H, J)$ by CR handle attaching. Then $(\tilde{M}, H, \tilde{J})$ is spherical and $Y_{CR}(\tilde{M}, \tilde{H}, \tilde{J}) > 0$.

From a continuity argument detailed in Section 5.2, we generalize this result:

\textbf{Theorem 1.2.} Let $(M, H)$ be a compact SPC manifold. Let $(\tilde{M}, \tilde{H})$ be a manifold obtained from $(M, H)$ by SPC handle attaching, then
\[
\sigma_c(\tilde{M}, \tilde{H}) \geq \sigma_c(M, H).
\]
Moreover, we prove the contact analogue of a theorem due to Kobayashi [Kob87]:

\textbf{Theorem 1.3.} Let $(M_1, H_1)$ and $(M_2, H_2)$ be two compact SPC manifolds of dimension $2n + 1$. Let $(M_1, H_1) \# (M_2, H_2)$ be their SPC connected sum, then
\[
\sigma_c((M_1, H_1) \# (M_2, H_2)) \geq \begin{cases} 
- \left( |\sigma_c(M_1, H_1)|^{n+1} + |\sigma_c(M_2, H_2)|^{n+1} \right)^{\frac{1}{n+1}} & \text{if } \sigma_c(M_1, H_1) \leq 0 \\
\min (\sigma_c(M_1, H_1), \sigma_c(M_2, H_2)) & \text{otherwise}.
\end{cases}
\]

We also prove a weakened contact version of a theorem due to C. LeBrun and J. Petean, who, using the generalized Gauss-Bonnet theorem, have computed the Yamabe invariant for complex surfaces of general type [LeB96, Pet98]:

\textbf{Theorem 1.4.} Let $(M, H)$ be a circle bundle over a Riemann surface $\Sigma$ of positive genus admitting an Einstein pseudohermitian structure. Then
\[
\sigma_c(M, H) \geq -2\pi \sqrt{-\chi(\Sigma)}.
\]

Section 5 contains the proof of Theorems 1.2 and 1.3, and Section 6 contains the proof of Theorem 1.4.

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2. CR geometry

2.1. Generalities. Let \( n \in \mathbb{N}^* \) and \( M \) be a smooth differentiable manifold of real dimension \( 2n + 1 \). We assume that \( M \) is orientable. A CR structure is given on \( M \) by a complex subbundle \( T^{1,0}M \) of \( TM \otimes \mathbb{C} \) of complex dimension \( n \) verifying

\[
T^{1,0}M \cap T^{0,1}M = \{0\}
\]

where \( T^{0,1}M = \overline{T^{1,0}M} \), and which is stable under the Lie bracket.

Equivalently, let \( H \) be a Levi distribution, i.e. an orientable hyperplane distribution in \( TM \). Let \( J \) be a complex structure on \( H \), i.e. \( J \) is an endomorphism of \( H \) which satisfies \( J^2 = -\text{id}_H \) and is integrable: \( \forall X,Y \in \Gamma(H) \),

\[
[JX,Y] + [X,JY] \in \Gamma(H) \quad \text{and} \quad [JX,JY] - [X,Y] = J ([JX,Y] + [X,JY]),
\]

where \([ \cdot , \cdot ]\) denotes the Lie bracket. The existence of \( J \) also requires that \( H \) is orientable. A CR manifold is the triplet \( (M,H,J) \).

Let \( E := \{ \omega \in \Gamma(T^*M) \mid \ker \omega \supseteq H \} \simeq TM/H \). It is a real line subbundle of \( T^*M \), hence trivial since \( M \) is orientable. A pseudohermitian structure on \( M \) is a never-vanishing section \( \theta \) of \( E \) compatible with \( J \), i.e. such that

\[
d\theta(J\cdot,J\cdot) = d\theta(\cdot,\cdot) \quad \text{on} \quad TM \otimes \mathbb{C}.
\]

The associated Levi form \( \gamma \) is the Hermitian form on \( H \) given by \( \gamma := d\theta(\cdot,J\cdot) \).

**Definition 2.1.** A pseudohermitian structure \( \theta \) is said to be strictly pseudoconvex when its Levi form is definite positive and when the orientation of the associated volume form \( \theta \wedge d\theta^n \) coincides with the orientation of \( M \).

In that case, \( \theta \) is a contact form, and \( (M,H,J) \) is a contact manifold. A contact form on \( (M,H,J) \) which is a strictly pseudoconvex pseudohermitian structure will be called positive. A CR manifold admitting an positive contact form is called SPCR, and a contact manifold admitting an SPCR structure is called SPC. We will always assume that \( H \) is a contact distribution.

**Definition 2.2.** Given an SPC manifold \( (M,H) \), we define

\[
\mathcal{J} = \{ \text{Compatible complex structures } J \text{ on } H \mid (M,H,J) \text{ is an SPCR manifold} \}.
\]

In dimension \( 2n+1 = 3 \), \( T^{1,0}M \) is of complex rank 1, the integrability of \( J \) is thus automatic. The set \( \mathcal{J} \) is defined by purely algebraic conditions, and it is moreover contractible. Indeed, considering \( J_0, J_1 \) in \( \mathcal{J} \), let, for \( i \) in \( \{0,1\} \), \( \gamma_i := d\theta(\cdot,J_i) \). For \( t \) in \( [0,1] \), the metric \( \gamma_t := (1-t)\gamma_0 + t\gamma_1 \) gives a complex structure \( J_t \) compatible with \( \theta \), with \( J_0 = J_0 \) and \( J_1 = J_1 \). The set \( \mathcal{J} \) is therefore always non-empty if \( M \) is orientable.

The Reeb field of a contact form \( \theta \) is the unique vector field \( R \in TM \) verifying \( \theta(R) = 1 \) and \( \iota_RD\theta = 0 \). We get a pseudohermitian decomposition of the tangent space

\[
TM = \mathbb{R}R \oplus H,
\]

and a pseudohermitian projection \( \pi_0 : TM \rightarrow H \). Note that this projection depends on \( \theta \). An admissible coframe is a set of \( (1,0) \)-forms \( (\theta_1, \ldots, \theta_n) \) whose restriction to \( T^{1,0}M \) forms a basis for \( (T^{1,0}M)^* \) and such that, for all \( \alpha \) in \( \{1, \ldots, n\} \), \( \theta^\alpha(R) = 0 \). Then \( d\theta = ih_{\alpha\beta}\theta^\alpha \wedge \theta^\beta \), where \( \theta^\beta := \overline{\theta^\beta} \) and \( (h_{\alpha\beta}) \) is a positive definite Hermitian
We then have, for the dual frame $\gamma(T_\alpha, T_\beta) = h_{\alpha \beta}$.

**Proposition 2.3** [Tan75, Web78]. Let $(M, H, J, \theta)$ be a strictly pseudoconvex pseudohermitian manifold. There is a unique linear connection $\nabla^0$ on $M$, called the Tanaka-Webster connection, which parallelizes the Levi distribution $H$, the Reeb field $R$, the complex structure $J$, and the Webster metric $g_{J, \theta}$, and whose torsion $T^\theta$ verifies

$$\forall X, Y \in H, \quad T^\theta(X, Y) = d\theta(X, Y)R \quad \text{and} \quad T^\theta(R, JX) + JT^\theta(R, X) = 0.$$ 

In other words, if $\theta^1, \ldots, \theta^n$ is an admissible coframe with $d\theta = ih_{\alpha \beta} \theta^\alpha \wedge \theta^\beta$, then the connection forms $\omega^\alpha_\beta$ and the torsion forms $\tau_\alpha = A_{\alpha \beta} \theta^\beta$ of the Tanaka-Webster connection are defined by

$$d\theta^\beta = \theta^\alpha \wedge \omega^\beta_\alpha + \theta \wedge \tau_\beta, \quad \omega^\alpha_\beta + \omega^\beta_\alpha = dh_{\alpha \beta}, \quad A_{\alpha \beta} = A_{\beta \alpha},$$

where indices are raised and lowered with $h_{\alpha \beta}$, i.e. $\omega_{\alpha \beta} = h_{\sigma \rho} \omega^\sigma_\alpha \omega^\rho_\beta$ [Web78, Lee88].

We then have, for the dual frame $(T_1, \ldots, T_n)$ to $(\theta^1, \ldots, \theta^n)$, $\nabla^0 T_\alpha = \omega^\alpha_\beta \otimes T_\beta$.

Due to the first condition in Theorem 2.3, the torsion of the Tanaka-Webster connection is nonvanishing; however, we define:

**Definition 2.4.** The pseudohermitian torsion $\tau$ of the Tanaka-Webster connection is the operator $\tau = i\mathcal{R}^\theta$. If $\tau$ vanishes, $(M, H, J, \theta)$ is called normal.

Note that the definition of Tanaka-Webster connection implies that the pseudohermitian torsion is always trace-free as an endomorphism of the real vector bundle $H$.

Let $\mathcal{R}^\theta$ be the curvature tensor field corresponding to the Tanaka-Webster connection. It can be decomposed into vertical, mixed, and horizontal terms. The vertical and mixed terms only depend on $\tau$ and its first derivatives. The horizontal part gives the Webster curvature tensor. Let $\text{Ric}_W(J, \theta)$ be its Ricci tensor, and $\text{Scal}_W(J, \theta)$ be its scalar curvature, called the Webster scalar curvature. In other words, the curvature forms $\Pi_\alpha^\beta = d\omega^\alpha_\beta - \omega^\beta_\sigma \wedge \omega^\alpha_\sigma$ verify

$$\Pi_\alpha^\beta = \mathcal{R}^\theta_{\alpha \beta \rho \sigma} \theta^\rho \wedge \theta^\sigma \mod \theta.$$

We then have

$$\text{Ric}_W(J, \theta)(T_\alpha, T_\beta) = \mathcal{R}^\theta_{\alpha \beta, \rho \sigma} \text{ and } \text{Scal}_W(J, \theta) = h^{\alpha \beta} \text{Ric}_W(J, \theta)(T_\alpha, T_\beta).$$

**Definition 2.5** [CY13, Wan15]. A strictly pseudoconvex pseudohermitian manifold $(M, H, J, \theta)$ is said to be pseudo-Einstein if

$$\text{Ric}_W = \frac{1}{n} \text{Scal}_W \gamma \text{ if } n \geq 2, \quad \text{Scal}_{W,1} = i\mathcal{T}_{1,1} \text{ if } n = 1.$$ 

A normal, pseudo-Einstein contact form is called an Einstein contact form.

**Example 2.6.** The sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ can be endowed with the contact form

$$\theta_0 = i \left( z_j d\bar{z}^j - z_j d\bar{z}^j \right) |_{S^{2n+1}}.$$

The induced CR structure $(S^{2n+1}, H_0, J_0)$ is called the standard CR structure of $S^{2n+1}$. The pseudohermitian manifold $(S^{2n+1}, H_0, J_0, \theta_0)$ is Einstein, with constant positive Webster scalar curvature $\text{Scal}_W(J_0, \theta_0) = \frac{n(n + 1)}{2}$. A CR manifold is called spherical if it is locally CR equivalent to $(S^{2n+1}, H_0, J_0)$. 

matrix. If $(T_1, \ldots, T_n)$ is the dual frame to $(\theta^1, \ldots, \theta^n)$ on $T^{1,0} M$, then $\gamma(T_\alpha, T_\beta) = h_{\alpha \beta}$. 

A normal, pseudo-Einstein contact form is called an Einstein contact form.
2.2. Circle bundles over a Riemann surface. We recall here a construction detailed by D. Burns and C. Epstein [BE88], that will be useful in Section 6. Let us consider a compact Riemann surface $\Sigma$ with a Hermitian metric $\gamma$. Let $T^{1,0}\Sigma$ be the holomorphic tangent bundle to $\Sigma$, and let $M$ be the unit circle bundle in $T^{1,0}\Sigma$. $M$ is then a $U(1)$-bundle over $\Sigma$, whose dual coframe gives a canonical one-form $\Theta^1$ on $M$. Moreover, since $\dim \Sigma = 2$, $\gamma$ is automatically a Kähler metric, hence there is a unique torsion-free connection form $\Theta^1$ such that $d\Theta^1 = \Theta^1 \wedge \Theta^1$. We then have

$$d\Theta^1 = K \Theta^1 \wedge \Theta^1,$$

where $K$ is the Gauss curvature of $\Sigma$. If $K$ never vanishes, an associated normal strictly pseudoconvex pseudohermitian structure $(J, \theta)$ is given on $M$ by $\theta = i \operatorname{sign}(K) \Theta^1$ and $\theta^1 = \sqrt{|K|} \Theta^1$, so that $d\theta = i \theta^1 \wedge \theta^1$. Moreover, we have

$$d\theta^1 = \Theta^1 \wedge \left( \frac{1}{2} \frac{|K|}{|K|} \theta^1 - \frac{1}{2} \frac{|K|}{|K|} \theta - i \operatorname{sign}(K) \theta \right).$$

If $K$ is constant, then $\omega^1 = -i \operatorname{sign}(K) \theta$, hence $\operatorname{Scal}_W(J, \theta) = \operatorname{sign}(K)$.

Note that, by the following result, all non-spherical SPCR compact $3$-manifolds which admit a normal contact form are such bundles or finite quotients of them, i.e. Seifert bundles.

**Proposition 2.7** [Bel01]. Let $(M, H, J)$ be a compact normal SPCR $3$-manifold. Then $(M, H, J)$ is either a finite quotient of the standard sphere or of a circle bundle over a Riemann surface of positive genus.

3. The contact Yamabe invariant

3.1. The CR Yamabe problem. Let $(M, H, J, \theta)$ be a compact strictly pseudoconvex pseudohermitian manifold of dimension $2n + 1$. We already mentioned that the set of positive contact forms on $(M, H, J)$ is a conformal class:

$$[\theta] = \{ u^{\frac{2}{n}} \theta \mid u \in C^\infty(M, \mathbb{R}^*_+) \}.$$

Here, the choice of the exponent $\frac{2}{n}$ is made to simplify further conformal change formulas.

**Definition 3.1.** Let $\pi_b : TM \rightarrow H$ be the pseudohermitian projection. The *horizontal gradient* is the operator $\nabla_b := \pi_b \nabla^\theta$. The *sublaplacian* is $\Delta_b := \text{div}(\nabla_b \cdot)$.

The similarity between conformal and CR geometry can be seen through the variation of the Webster scalar curvature under conformal changes of $\theta$: given a conformal factor $u$ in $C^\infty(M, \mathbb{R}^*_+)$, we have

$$\text{Scal}_W(J, u^{\frac{2}{n}} \theta) = u^{-\frac{n+2}{n}} \left( 2 \frac{n+1}{n} \Delta_b u + \text{Scal}_W(J, \theta) u \right).$$

Therefore, $u^{\frac{2}{n}} \theta$ has constant Webster curvature $\text{Scal}_W \equiv \lambda$ if and only if

$$2 \frac{n+1}{n} \Delta_b u + \text{Scal}_W(J, \theta) u = \lambda u^{\frac{n+2}{n}},$$

which we will call the *CR Yamabe equation*. This equation may be compared with the Riemannian Yamabe equation for a manifold of dimension $2n + 2$. 
By analogy with the conformal case, the CR Yamabe problem is the following question: is there a constant Webster scalar curvature positive contact form in the conformal class $\theta$?

As in the conformal case, a sufficient condition is that there exists a contact form which realizes the infimum of the CR invariant

$$Y_{CR}(M, H, J) := \inf_{\hat{\theta} \in [\theta]} S_W(M, J, \hat{\theta}),$$

where

$$[\theta]_1 := \{\hat{\theta} \in [\theta], \text{Vol}(M, \hat{\theta}) := \int_M \hat{\theta} \wedge d\hat{\theta}^n = 1\},$$

and

$$S_W(M, J, \hat{\theta}) := \int_M \text{Scal}(J, \hat{\theta}) \hat{\theta} \wedge d\hat{\theta}^n$$
denotes the integral Webster scalar curvature. The functional $Y_{CR}$ is maximal for the standard sphere:

**Theorem 3.2 [JL87]**. $Y_{CR}(M, H, J) \leq Y_{CR}(\mathbb{S}^{2n+1}, H_0, J_0) = 2\pi n(n + 1)$.

A positive contact form minimizing $Y_{CR}$ is called a Yamabe contact form. The CR Yamabe problem has been given a positive answer by the following results of D. Jerison and J. Lee, and N. Gamara and R. Yacoub:

**Theorem 3.3 [JL87]**. If $Y_{CR}(M, H, J) < Y_{CR}(\mathbb{S}^{2n+1}, H_0, J_0)$, then there is a Yamabe contact form.

**Theorem 3.4 [JL89]**. If $n \geq 2$ and $(M, H, J)$ is not spherical, then $Y_{CR}(M, H, J) < Y_{CR}(\mathbb{S}^{2n+1}, H_0, J_0)$.

**Theorem 3.5 [Gam01, GY01]**. If $n = 1$ or $(M, H, J)$ is spherical, then the CR Yamabe problem has a solution.

The proof of this last theorem uses a technique of critical points at infinity initiated by A. Bahri. Note that the positive contact forms found this way are not necessarily Yamabe contact forms. However, J.-H. Cheng, A. Malchiodi and P. Yang have shown that Yamabe contact forms always exist on SPCR 3-manifolds with non-negative CR Paneitz operator [CMY17].

On Einstein strictly pseudoconvex pseudohermitian manifolds, the following result by X. Wang ensures that all constant Webster scalar curvature contact forms are Einstein:

**Theorem 3.6 [Wan15]**. Let $(M, H, J)$ be a compact SPCR manifold which admits an Einstein contact form $\theta$. If $\hat{\theta} = u^\pm \theta \in [\theta]$ has constant Webster scalar curvature, then $\hat{\theta}$ is Einstein. Moreover, if $(M, H, J)$ is non-spherical, then $u$ is constant.

### 3.2. The contact Yamabe invariant

The resolution of the CR Yamabe problem, cf. Section 3.1, leads naturally to the consideration of the following quantity:

**Definition 3.7 [Wu09]**. Let $(M, H)$ be a compact SPC manifold. Let $\mathcal{J}$ be the set of complex structures $J$ on $(M, H)$ such that $(M, H, J)$ is SPCR. The contact Yamabe invariant $\sigma_c(M, H)$ is defined by

$$\sigma_c(M, H) := \sup_{\mathcal{J}} Y_{CR}(M, H, J).$$
As mentioned in the introduction, $\sigma_c$ is an actual contact invariant in dimension 3, in the sense that, since $\mathcal{J}$ is always non-empty, all contact 3-manifolds are SPC. Note that few contact invariants are currently available: they are necessarily global by Darboux’s theorem, and most of them come from homological considerations. In higher dimension, for some contact structures, due to the obstructions on the integrability of complex structures and on their compatibility with a given contact form, the set $\mathcal{J}$ might be empty.

As in the conformal case, the contact Yamabe invariant characterizes manifolds which admit a structure with positive curvature:

**Proposition 3.8** [Wan03]. Let $(M, H)$ be a compact SPC manifold. Then $\sigma_c(M, H) > 0$ if and only if there exists a strictly pseudoconvex pseudohermitian structure $(J, \theta)$ on $(M, H)$ with positive Webster scalar curvature.

Finally, let us recall the following lemma, that will be essential in Section 5.1.

**Lemma 3.9.** Let $(M, H, J)$ be an SPCR manifold. The infimum in Definition (1) of $Y_{CR}(M, H, J)$ may be taken over the space $L_c(M, \mathbb{R}^+)$ of non-negative Lipschitz functions with compact support on $M$.

**Proof.** Indeed, $Y_{CR}(M, H, J) = \inf_{u \in C^\infty(M, \mathbb{R}^+)} Q_\theta(u)$ where $\theta$ is any positive contact form on $(M, H, J)$ and

$$Q_\theta(u) := \frac{\int_M \left( 2^{\frac{n+1}{n}} |\nabla u|^2 + \text{Scal}_W(J, \theta) u^2 \right) \theta \wedge d\theta^n}{\left( \int_M u^{2^{\frac{n+1}{n}}} \theta \wedge d\theta^n \right)^{\frac{n}{n+1}}} ,$$

hence $Q_\theta$ is continuous in the Sobolev space $W^{1,2}(M)$. Since $C^\infty(M)$ is dense in $W^{1,2}(M)$, since $Q_\theta(|u|) = Q_\theta(u)$ for all $u \in C^\infty(M)$, and since a nonnegative Lipschitz function can be arbitrarily approximated in $W^{1,2}$ norm by a positive smooth function, we have $Y_{CR}(M, H, J) = \inf_{u \in L_c(M, \mathbb{R}^+)} Q_\theta(u)$.

4. CR handle attaching on a spherical manifold

We recall here a handle attaching process on spherical SPCR manifolds, compatible with the CR structure, which is due to W. Wang [Wan03]. If the handle is attached between two distinct connected components, this provides a connected sum of the components.

Let either $\left( M = M_1 \sqcup M_2, H, J, \widehat{\theta} \right)$ be a disjoint union of two connected spherical strictly pseudoconvex pseudohermitian manifolds, and $p_1 \in M_1$, $p_2 \in M_2$; or $\left( M, H, J, \widehat{\theta} \right)$ be a connected spherical strictly pseudoconvex pseudohermitian manifold, and $p_1, p_2 \in M$.

Let $M_0 = M \setminus \{p_1, p_2\}$. For $i$ in $\{1, 2\}$, let $U_i \subset M$ be a neighbourhood of $p_i$ and

$$\varphi_i : U_i \to B(0, 2) := \{ \xi \in \mathbb{H}^{2n+1}, \|\xi\|_H < 2 \}$$

local coordinates such that $\varphi_i(p_i) = 0$. Let us denote, for $0 < r < 1$,

$$U_i(r) = \{ x \in U_i, \|\varphi_i(x)\|_H < r \},$$

$$U_i(r, 1) = \{ x \in U_i, \; r < \|\varphi_i(x)\|_H < 1 \}.$$
Since $M$ is spherical around $p_1$ and $p_2$, there exists $\lambda \in C^\infty(M_0, \mathbb{R}_+)$ such that, denoting $\theta = \lambda \hat{\theta}$ on $M_0$,

$$\forall i \in \{1, 2\}, \forall \xi \in B(0, 2) \setminus \{0\}, \quad \varphi_{\lambda \theta}(\xi) = \|\xi\|_{\mathbb{H}^2}^2 \theta(\xi),$$

that is, $(M_0, H, J, \theta)$ has cylindrical ends. Indeed, we can define a mapping

$$\Phi : B(0, 1) \rightarrow \mathbb{R}_+ \times \Sigma^{2n}$$

$$\xi \mapsto \left(\log \frac{1}{\xi}, \frac{\xi}{\|\xi\|_{\mathbb{H}^2}}\right),$$

where $\Sigma^{2n} := \{\xi \in \mathbb{H}^{2n+1}, \|\xi\|_{\mathbb{H}} = 1\}$, and $\hat{\theta} := \Phi_*(\|\cdot\|_{\mathbb{H}^2}^2 \theta_{\mathbb{H}})$. Then

$$(B(0, 1), H_{\mathbb{H}}, J_{\mathbb{H}}, \|\cdot\|_{\mathbb{H}^2}^2 \theta_{\mathbb{H}}) \simeq (\mathbb{R}_+ \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}),$$

where the equivalence is pseudohermitian. Denoting $\tilde{M} := M \setminus U_1(1) \cup U_2(1),

$$(M_0, H, J, \theta) \simeq (\mathbb{R}_+ \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}) \cup (\tilde{M}, H, J, \theta) \cup (\mathbb{R}_+ \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}). \quad (2)$$

Now, let us denote, for $r \in (0, 1)$ and $A \in U(n)$, by $\psi_{r,A} : U_1(r, 1) \rightarrow U_2(r, 1)$ the mapping

$$\psi_{r,A} = \varphi_2^{-1} \circ \delta_r \circ R \circ U_A \circ \varphi_1,$$

where

$$\delta_r : (z, t) \mapsto (rz, r^2t),$$

$$U_A : (z, t) \mapsto (Az, t),$$

and

$$R : (z, t) \mapsto \left(\frac{-z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2}\right)$$

denote respectively dilations, unitary transformations and inversion in $\mathbb{H}^{2n+1}$. Let $(M_{r,A}, H_{r,A}, J_{r,A}, \theta_{r,A})$ be the pseudohermitian manifold formed from $M$ by removing $\bar{U}_1(r)$ and $\bar{U}_2(r)$, and by identifying $U_1(r, 1)$ with $U_2(r, 1)$ along $\psi_{r,A}$. Let

$$\pi_{r,A} : M \setminus \bar{U}_1(r) \cup \bar{U}_2(r) \rightarrow M_{r,A}$$

be the corresponding projection. Since

$$\delta_r^* \varphi_{\lambda \theta} = \frac{1}{r^2\|\cdot\|_{\mathbb{H}^2}^2} r^2 \theta_{\mathbb{H}} = \varphi_{\lambda \theta}$$

and

$$R^* \varphi_{\lambda \theta} = \frac{1}{\|R(\cdot)\|_{\mathbb{H}^2}} R^* \theta_{\mathbb{H}} = \frac{\|\cdot\|_{\mathbb{H}^2}^2}{\|\cdot\|_{\mathbb{H}^2}^2} \theta_{\mathbb{H}} = \varphi_{\lambda \theta},$$

the gluing preserves $\theta$ on $U_i(r, 1)$. Hence,

$$\pi_{r,A}^* \theta_{r,A} = \theta \quad \text{on } M \setminus \bar{U}_1(r) \cup \bar{U}_2(r).$$

We have in fact

$$(M_{r,A}, H_{r,A}, J_{r,A}, \theta_{r,A}) \simeq (\tilde{M}, H, J, \theta) \cup ([0, l] \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}) \quad (3)$$

where $l = \log \left(\frac{1}{r}\right) \in (0, +\infty)$. 

5. A CR Kobayashi Inequality

5.1. Non-decreasing of $\sigma_c$ under SPC handle attaching. This part follows a method developed in the conformal setting by O. Kobayashi [Kob87]. It has been adapted in the CR setting by W. Wang, and more recently by J.-H. Cheng and H.-L. Chiu [Wan03, CCI19]. A similar technique has been implemented in the quaternionic context [SW16]. Theorem 1.2 is a direct consequence of the following result:

**Theorem 5.1.** Let $(M, H, J)$ be a compact SPCR manifold. Let $(\tilde{M}, \tilde{H}, \tilde{J})$ be a manifold obtained from $(M, H, J)$ by CR handle attaching, then

$$Y_{CR}(\tilde{M}, \tilde{H}, \tilde{J}) \geq Y_{CR}(M, H, J).$$

**Proof.** Let $p_1, p_2 \in M$. We use the following lemma, that will be proved in Section 5.2.

**Lemma 5.2.** We may assume that $(M, H, J)$ is spherical around $p_1$ and $p_2$.

Under this assumption, we can apply the construction of Section 4 to $M$. Let $(M_{r,A}, H_{r,A}, J_{r,A})$ be obtained from $(M, H, J)$ by CR handle attaching.

By definition of $Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A})$, there exists a function $f_i \in C^\infty(M_{r,A}, \mathbb{R}_+)$ such that

$$S_W(M_{r,A}, J_{r,A}, f_i^{2/n} \theta_{r,A}) = \int_{M_{r,A}} \left( 2 \left( \frac{n+1}{n} \right) |\nabla^{r,A} f_i|^2 + \text{Scal}_W(J_{r,A}, \theta_{r,A}) f_i^2 \right) \theta_{r,A} \wedge d\theta_{r,A}$$

$$< Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{1}{1+l},$$

and

$$\int_{M_{r,A}} f_i^{2 \left( \frac{n+1}{n} \right)} \theta_{r,A} \wedge d\theta_{r,A} = 1.$$  

**Lemma 5.3.** There exists $l_* \in [0, l]$ such that

$$\int_{\{l_*\} \times \Sigma^{2n}} \left( |d_b f_i|^2 + f_i^2 \right) dS_{\Sigma^{2n}} \leq C \frac{1}{l_*},$$

where $C$ is a constant independent of $l$.

**Proof.** Let $C_1 = \frac{1}{\min_{\tilde{M}} \langle \text{Vol}(\tilde{M}, \theta) \rangle^{\frac{n}{n+1}}}$. Hölder’s inequality yields

$$\int_{\tilde{M}} f_i^2 \theta \wedge d\theta^n \leq \text{Vol}(\tilde{M}, \theta)^{\frac{n+1}{n}},$$

and then, using decomposition (3),

$$\int_{[0,l] \times \Sigma^{2n}} \left( 2 \left( \frac{n+1}{n} \right) |d_b f_i|^2 + \text{Scal}_W(J, \theta) f_i^2 \right) \theta \wedge d\theta^n \leq Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{1}{1+l} + C_1.$$

Consequently there exists $l_* \in [0, l]$ such that

$$\int_{\{l_*\} \times \Sigma^{2n}} \left( 2 \left( \frac{n+1}{n} \right) |d_b f_i|^2 + \text{Scal}_W(J, \theta) f_i^2 \right) \theta \wedge d\theta^n \leq \frac{1}{l} \left( Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{1}{1+l} + C_1 \right).$$

The lemma is obtained with $C = \frac{Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + 1 + C_1}{\min \left( 2 \left( \frac{n+1}{n} \right), \min_{\{l_*\} \times \Sigma^{2n}} \text{Scal}_W(J, \theta) \right)}.$ \hfill $\square$
We therefore decompose
\[(M_0, H, J, \theta) \cong (I^* \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta})\]
\[\cup (M_{r,A} \setminus \{I^*\} \times \Sigma^{2n}, H_{r,A}, J_{r,A}, \theta_{r,A})\]
\[\cup ([I^* - I^* + 1] \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta})\]
and extend \(f_t\) to \(M_0\) as follows: \(F_t = f_t\) on \(M_{r,A} \setminus \{I^*\} \times \Sigma^{2n}\) and
\[
F_t : (s, \xi) \mapsto \begin{cases} 
(l^* + s) f_t(l^*, \xi) & \forall (s, \xi) \in [I^*, I^* + 1] \times \Sigma^{2n}, \\
0 & \forall (s, \xi) \in [I^* + 1, \infty) \times \Sigma^{2n}, \\
0 & \forall (s, \xi) \in [I^* - I^* + 1, \infty) \times \Sigma^{2n}, \\
(l - I^* - s) f_t(l^*, \xi) & \forall (s, \xi) \in [I^* - I^* + 1, I^* + 1] \times \Sigma^{2n}.
\end{cases}
\]

We thus obtain from (4) and Lemma 5.3
\[
S_W(M_0, J, F_t^{2/n} \theta) = S_W(M_{r,A}, J_{r,A}, f_t^{2/n} \theta_{r,A})
+ \int_{[I^*, I^* + 1] \times \Sigma^{2n}} \left(2 \left(\frac{n + 1}{n}\right) |d_b f_t|^2 + \text{Scal}_W(J, \tilde{\theta}) f_t^2 \right) \tilde{\theta} \wedge d\tilde{\theta}^n
+ \int_{[I^* - I^* + 1, \infty) \times \Sigma^{2n}} \left(2 \left(\frac{n + 1}{n}\right) |d_b f_t|^2 + \text{Scal}_W(J, \tilde{\theta}) f_t^2 \right) \tilde{\theta} \wedge d\tilde{\theta}^n
= S_W(M_{r,A}, J_{r,A}, f_t^{2/n} \theta_{r,A})
+ 2 \int_{\Sigma^{2n}} \left(\frac{2n + 1}{3} \right) |d_b f_t(l^*, \cdot)|^2 + \left(2 \left(\frac{n + 1}{n}\right) + \frac{1}{3} \text{Scal}_W(J, \tilde{\theta}) \right) f_t^2 \right) dS_{\Sigma^{2n}}
\leq S_W(M_{r,A}, J_{r,A}, f_t^{2/n} \theta_{r,A})
+ 2 \int_{(I^*) \times \Sigma^{2n}} \left(\frac{2n + 1}{3} \right) |d_b f_t|^2 + \left(2 \left(\frac{n + 1}{n}\right) + \frac{1}{3} \text{Scal}_W(J, \tilde{\theta}) \right) f_t^2 \right) dS_{\Sigma^{2n}}
\leq Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{B}{l},
\]
where \(B\) is a constant independent of \(l\), and
\[
\int_{M_0} F_t^{2(n + 1)/n} \tilde{\theta} \wedge d\tilde{\theta}^n > 1.
\]

Since the infimum in the Yamabe functional may be taken over all nonnegative Lipschitz functions with compact support as conformal factors by Lemma 3.9, we get that
\[
Y_{CR}(M_0, H, J) \leq Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{B}{l},
\]
which, for \(l\) sufficiently large, yields the desired inequality. \(\square\)

5.2. Local sphericity. In this section, we prove the following technical lemma, which is essential for the proof of Theorem 5.1.

Lemma 5.4. Let \((M, H, J)\) be a compact SPCR manifold. Given \(p_1\) and \(p_2\) in \(M\), there is a 1-parameter family of complex structures \((J_\theta)\) in \(\mathcal{J}\) \(C^0\)-converging to...
to $J$ such that $(M, H, J_t)$ is spherical around $p_1$ and $p_2$, and $Y_{CR}(M, H, J_t) \xrightarrow{t \to 0} Y_{CR}(M, H, J)$.

In other words, to prove Theorem 5.1, we may assume that $(M, H, J)$ is spherical around $p_1$ and $p_2$. This lemma is a direct consequence of the two following results:

**Lemma 5.5.** Let $(M, H, J, \theta)$ be a compact strictly pseudoconvex pseudohermitian manifold. Let $(J_t)$ be a 1-parameter family of complex structures in $C^0$-converging to $J$ such that $\text{Scal}_W(J_t, \theta) \xrightarrow{t \to 0} \text{Scal}_W(J, \theta)$. Then $Y_{CR}(M, H, J_t) \xrightarrow{t \to 0} Y_{CR}(M, H, J)$.

**Lemma 5.6** [CCH19]. Let $(M, H, J, \theta)$ be a compact strictly pseudoconvex pseudohermitian manifold. Let $p_1$ and $p_2$ be two points in $M$. Let $(\varepsilon_t)$ be a 1-parameter family of positive numbers decreasing to 0. There is a 1-parameter family of complex structures $(J_t)$ in $C^0$-converging to $J$ such that for all $t$, $J_t$ coincides with $J$ outside an $\varepsilon_t$-neighbourhood $U'_t$ of $\{p_1, p_2\}$, $J_t$ is spherical inside $U_t \subset U'_t$, and $|\text{Scal}_W(J_t, \theta) - \text{Scal}_W(J, \theta)| \leq \varepsilon_t$.

**Proof of Lemma 5.5.** We adapt from the conformal case a proof due to L. Bérard Bergery [Ber83]. Let us denote

$$F = \left\{ u \in C^\infty(M, \mathbb{R}^*_+) \mid \int_M u^2 \left( \frac{n+1}{n} \right)^{\frac{n}{n+1}} \theta \wedge d\theta^n = 1 \right\}.$$  

By definition, $Y_{CR}(M, H, J_t) = \inf_{u \in F} I_t(u)$ where

$$I_t(u) = 2 \left( \frac{n+1}{n} \right) \int_M |d_b u_t|^2 \theta \wedge d\theta^n + \int_M u_t^2 \text{Scal}_W(J_t, \theta) \theta \wedge d\theta^n.$$

Let $\varepsilon > 0$. For each $t$ there exists $u_t$ in $F$ such that

$$Y_{CR}(M, H, J_t) \leq I_t(u_t) \leq Y_{CR}(M, H, J_t) + \varepsilon.$$

Let $\eta > 0$ and $K > 0$ be such that, for all $t$ in $[-\eta, \eta]$,

$$|\text{Scal}_W(J_t, \theta) - \text{Scal}_W(J, \theta)| \leq \varepsilon,$$

and

$$\frac{1}{1 + \varepsilon} \leq \frac{1}{|\omega|^2_0} \leq 1 + \varepsilon \quad \forall \omega \in \Omega^1(M).$$

In particular, $I_t(u_t) \leq I_t(1) + \varepsilon \leq K + \varepsilon$. Now, by Hölder’s inequality,

$$\int_M u_t^2 \theta \wedge d\theta^n \leq 1,$$

hence we have

$$I_0(u_t) = 2 \left( \frac{n+1}{n} \right) \int_M |d_b u_t|^2 \theta \wedge d\theta^n + \int_M u_t^2 \text{Scal}_W(J_0, \theta) \theta \wedge d\theta^n$$

$$\leq 2 \left( \frac{n+1}{n} \right) (1 + \varepsilon) \int_M |d_b u_t|^2 \theta \wedge d\theta^n + \int_M u_t^2 \text{Scal}_W(J_t, \theta) \theta \wedge d\theta^n$$

$$+ \int_M u_t^2 |\text{Scal}_W(J, \theta) - \text{Scal}_W(J_t, \theta)| \theta \wedge d\theta^n$$

$$\leq 2 \left( \frac{n+1}{n} \right) (1 + \varepsilon) \int_M |d_b u_t|^2 \theta \wedge d\theta^n + \int_M u_t^2 \text{Scal}_W(J_t, \theta) \theta \wedge d\theta^n + \varepsilon$$

$$\leq (1 + \varepsilon) I_t(u_t) + \varepsilon(K + 1),$$
and similarly
\[ I_t(u_0) \leq (1 + \varepsilon) I_0(u_0) + \varepsilon(K + 1). \]

Then, for all \( t \) in \([-\eta, \eta]\),
\[
\frac{1}{1 + \varepsilon} Y_{\text{CR}}(M, H, J) - \frac{\varepsilon}{1 + \varepsilon} (K + 1) - \varepsilon \leq Y_{\text{CR}}(M, H, J_t) \leq (1 + \varepsilon)(Y_{\text{CR}}(M, H, J) + \varepsilon) + \varepsilon(K + 1).
\]

\( \square \)

\textbf{Remark 5.7.} Since \( Y_{\text{CR}}(M, H, J) \) only depends on derivatives up to order 2 of \( J \), the supremum in \( \sigma_c(M, H) \) may be taken over all \( C^2 \) complex structures on \((M, H)\). Therefore, in the following proof, gluing complex structures only needs to be considered up to \( C^2 \)-regularity.

\textbf{Proof of Lemma 5.6 [CCH10].} We follow a construction due to O. Biquard and Y. Rollin [BR09]. We assume that for all \( t \), \( B(p_1, \varepsilon_t) \cap B(p_2, \varepsilon_t) = \emptyset \), where the distances are taken with respect to the Webster metric. For a given \( t \), let \( U'_t \) be an \( \varepsilon_t \)-neighbourhood of \( \{p_1, p_2\} \), and let \( x = \min(d(\cdot, p_1), d(\cdot, p_2)) \) on \( M \). There is a smooth cut-off function \( w_t : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \chi_t := w_t \circ x = 0 \) on some \( U_t \subset U'_t \), \( \chi_t = 1 \) outside \( U'_t \), and for all \( x \in \mathbb{R}^+ \), \( |xw_t'(x)| \leq \varepsilon_t \) and \( |x^2w_t''(x)| \leq \varepsilon_t \) (cf. [Kob87, Sublemma 3.4.]). Indeed, we may take \( w_t \) as a smoothing of \( \tilde{w}_t \) defined by

\[
\tilde{w}_t(x) = \begin{cases} 
0 & \forall x \leq \varepsilon_t e^{-\frac{2}{\varepsilon_t}} \\
1 - \frac{\varepsilon_t}{2} \log \left( \frac{\varepsilon_t}{x} \right) & \forall x \in [\varepsilon_t e^{-\frac{2}{\varepsilon_t}}, \varepsilon_t] \\
1 & \forall x \geq \varepsilon_t.
\end{cases}
\]

If \( \dim M = 3 \), then all almost complex structures are formally integrable. Let us take \( i \) in \( \{1, 2\} \). Let \( \psi_i : U_i \to U_\mathbb{H} \) be a contactomorphism identifying a neighbourhood \( U_i \) of \( p_i \) in \( M \) with a neighbourhood \( U_\mathbb{H} \) of 0 in \( \mathbb{H}^3 \) such that \( \psi_i(p_i) = 0 \) and, denoting \( \tilde{J}_i := (\psi_i)_* J \) and \( \tilde{\theta}_i := (\psi_i)_* \theta \), such that \( j_{\mathbb{H}}^1(\tilde{J}_i) = j_{\mathbb{H}}^1(\tilde{\theta}_i) \) and \( \text{Scal}_W(\tilde{J}_i, \tilde{\theta}_i) = \text{Scal}_W(\tilde{J}_i, \tilde{\theta}_i) \) at 0. We assume that \( U_1 \cap U_2 = \emptyset \). For \( t \) large enough that \( U'_t \subset U_1 \cap U_2 \), let \( \chi_{t,i} := (\psi_i^{-1})^* \chi_t|_{U_i} \). For \((z_1, y)\) in \( \mathbb{H}^3 \), let \( \tilde{J}_{t,i}(z_1, y) := \tilde{J}_i(\chi_{t,i}z_1, \chi_{t,i}^2y) \). Then \( \tilde{J}_{t,i} \) coincides with \( J_{\mathbb{H}} \) inside \( \psi_i(U_i \cap U_t) \), and with \( \tilde{J}_i \) outside \( \psi_i(U'_t \cap U_i) \). Therefore, the complex structure \( J_t \) defined on \( M \) by

\[
\forall i \in \{1, 2\}, \quad J_t := \psi_i^* \tilde{J}_{t,i} \quad \text{on} \quad U_i, \quad J_t := J \quad \text{elsewhere},
\]

has the desired properties. In particular, we have \(|J_t - J| = O(x^2), |\nabla(J_t - J)| = O(x), \text{ and } |\nabla^2(J_t - J)| = O(1)|. We then use Formula (4.7) in [CT00]: for \( E \in \mathcal{E}(J) := \{ E \in \text{End}(H) \mid E J + J E = 0 \} \), we have

\[
\partial_J \text{Scal}_W(J, \theta)(E) = \frac{i}{2}(E_{11,12} - E_{12,11}) - \frac{1}{2}(\tau_{11}E_{11} + \tau_{12}E_{12}).
\]

In our case, for some constant \( C \), and for \( t \) sufficiently small, we then have

\[
|\text{Scal}_W(J_t, \theta) - \text{Scal}_W(J, \theta)| \leq C \left( |w'_t||J_t - J| + |w_t||\nabla(J_t - J)| + |w_t||\nabla^2(J_t - J)| \right) \leq C \varepsilon_t.
\]
If \( \dim M \geq 5 \), then, since \( M \) is compact, \( (M, H, J) \) is embeddable. Let us consider an ACH manifold \((X, g)\) with CR infinity \((M, H, J)\). Let \( J_X \) be a complex structure on \( X \) and let \( z = (z_1, \ldots, z_{n+1}) \) be complex coordinates near \( \{p_1, p_2\} \). Then, by the normal form theorem of Chern and Moser, there is a boundary defining function \( r \) on \( X \) such that
\[
 r(z) = r_0(z) + O_{1 \leq j \leq n} (|z_j|^4),
\]
where \( r_0(z) = \Re(z_{n+1}) - \frac{1}{4} \sum_{1 \leq j \leq n} |z_j|^2 \) is a boundary defining function for the Heisenberg group \([CM74]\). We glue the defining functions as follows:
\[
 r_t = (1 - \chi_t) r_0 + \chi_t r.
\]
The corresponding contact form is given by \( \theta_t = i (\overline{\theta} - \widehat{\theta}) r_t \). The induced complex structure \( J_t \) on \( M \) is then given by the relation \( d\theta_t(J_t, \cdot) = d\theta_t(\cdot, \cdot) \). By construction, \( J_t \) is spherical inside \( U_t \) and coincides with \( J \) outside \( U_t \), and \( (J_t) \) \( C^0 \)-converges to \( J \). Moreover, since \( |r_t - r| = O(x^4) \) and \( |\nabla(r_t - r)| = O(x^3) \), we have, for some constant \( C \),
\[
 |\text{Scal}_V(J_t, \theta - \text{Scal}_V(J, \theta)| \leq C \left( |w'_t| \left( |\nabla^2(r_t - r)| + |\nabla^3(r_t - r)| \right) + |w''_t||\nabla^2(r_t - r)| \right) \leq C \epsilon_t. 
\]

**Example 5.8.** If \((M, H) = (S^{2n+1}, H_0)\), using Theorem 3.2 we thus have the equality
\[
 \sigma_c(S^1 \times S^{2n}, H_0) = \sigma_c(S^{2n+1}, H_0).
\]

**5.3. Disjoint union.** In the case of a connected sum, Theorem 1.2 can be written the following way:

**Theorem 5.9.** Let \((M_1, H_1)\) and \((M_2, H_2)\) be two compact SPC manifolds of dimension \(2n+1\). Let \((M_1, H_1) \# (M_2, H_2)\) be their SPC connected sum, then
\[
 \sigma_c((M_1, H_1) \# (M_2, H_2)) \geq \sigma_c((M_1, H_1) \cup (M_2, H_2)).
\]

Alongside with the hereunder computation of the right-hand side, this gives Theorem 1.3.

**Proposition 5.10.** Let \((M_1, H_1)\) and \((M_2, H_2)\) be two compact SPC manifolds of dimension \(2n+1\). Then
\[
 \sigma_c((M_1, H_1) \cup (M_2, H_2)) = \begin{cases} 
 - \left( |\sigma_c(M_1, H_1)|^{n+1} + |\sigma_c(M_2, H_2)|^{n+1} \right)^{\frac{1}{n+1}} & \text{if } \sigma_c(M_1, H_1) \leq 0 \text{ and } \sigma_c(M_2, H_2) \leq 0, \\
 \min(\sigma_c(M_1, H_1), \sigma_c(M_2, H_2)) & \text{otherwise.} 
\end{cases}
\]

**Proof.** Let us consider a unit volume strictly convex pseudohermitian structure \((J, \theta)\) on \((M_1, H_1) \cup (M_2, H_2)\). Let us denote, for \( i \) in \( \{1, 2\} \), \( J_i := J |_{M_i} \), \( \theta_i := \theta |_{M_i} \), \( Y_i = Y_{CR}(M_i, H_i, J_i) \), and \( \lambda_i \) in \( \mathbb{R}_+^* \) which verifies
\[
 \text{Vol}(M_i, \lambda_i, \theta_i) = \lambda_i^{n+1} \text{Vol}(M_i, \theta_i) = 1.
\]
We recall that $S_W$ denotes the integral Webster scalar curvature. Since, for $i$ in \{1, 2\}, $\text{Scal}_W(J, \theta_i) = \lambda_i^{-1} \text{Scal}_W(J, \theta_i)$, we have

$$S_W(M_1 \sqcup M_2, J, \theta) = S_W(M_1, J, \theta_1) + S_W(M_2, J, \theta_2)$$

$$= \lambda_1^n S_W(M_1, J_1, \lambda_1 \theta_1) + \lambda_2^n S_W(M_2, J_2, \lambda_2 \theta_2)$$

$$= \text{Vol}(M_1, \theta_1) \frac{n}{n+1} S_W(M_1, J_1, \lambda_1 \theta_1) + \text{Vol}(M_2, \theta_2) \frac{n}{n+1} S_W(M_2, J_2, \lambda_2 \theta_2)$$

$$\geq \text{Vol}(M_1, \theta_1) \frac{n}{n+1} Y_1 + \text{Vol}(M_2, \theta_2) \frac{n}{n+1} Y_2,$$

with equality when $\lambda_1 \theta_1$ and $\lambda_2 \theta_2$ are Yamabe contact forms on $(M_1, H_1, J_1)$ and $(M_2, H_2, J_2)$ respectively. Optimizing the right-hand side under the constraint $\text{Vol}(M_1, \theta_1) + \text{Vol}(M_2, \theta_2) = 1$ yields

$$S_W(M_1 \sqcup M_2, J, \theta) \geq \begin{cases} - \left( |Y_1|^{n+1} + |Y_2|^{n+1} \right)^{\frac{1}{n+1}} & \text{if } Y_1 \leq 0 \text{ and } Y_2 \leq 0, \\ \text{min}(Y_1, Y_2) & \text{otherwise}, \end{cases}$$

with equality when $\lambda_1 \theta_1$ and $\lambda_2 \theta_2$ are Yamabe contact forms and, in the first case, when $\frac{1}{\text{Vol}(M_1, \theta_1)} = 1 + \left( \frac{Y_2}{Y_1} \right)^{n+1}$, and, in the second case, at the limit $\text{Vol}(M_i, \theta_i) \to 0$, where $i \in \{1, 2\}$ verifies $Y_i = \max(Y_1, Y_2)$. Consequently,

$$Y_{CR}((M_1, H_1, J_1) \sqcup (M_2, H_2, J_2)) = \begin{cases} - \left( |Y_1|^{n+1} + |Y_2|^{n+1} \right)^{\frac{1}{n+1}} & \text{if } Y_1 \leq 0 \text{ and } Y_2 \leq 0, \\ \text{min}(Y_1, Y_2) & \text{otherwise}, \end{cases}$$

hence the result. \hfill \Box

6. A CR Gauss-Bonnet-LeBrun Formula

We prove in this part Theorem 1.3. Let us first recall some facts on the Burns-Epstein invariant of a CR manifold. Let $(M, H, J)$ be a compact SPCR 3-manifold. The Burns-Epstein invariant $\mu(M, H, J)$ is defined as the evaluation of a well-chosen de Rham cohomology class on the fundamental class $[M]$ in $H_3(M, \mathbb{R})$ [BESS]. In particular, we have the following estimates.

**Proposition 6.1** [Mar15]. The Burns-Epstein invariant of a compact SPCR 3-manifold $(M, H, J)$ admitting a pseudo-Einstein contact form $\theta$ is given by

$$\mu(M, H, J) = \frac{1}{4 \pi^2} \int_M \left( |\tau(J, \theta)|^2 - \frac{1}{4} \text{Scal}_W(J, \theta)^2 \right) \theta \wedge d\theta.$$

**Proposition 6.2** [BESS]. The Burns-Epstein invariant value of a circle bundle $(M, H, J)$ over a Riemann surface $\Sigma$ is

$$\mu(M, H, J) = -\frac{1}{4} |\chi(\Sigma)| + \frac{1}{12 \pi} \int_{\Sigma} (\Delta \log |\text{Scal}_W(J, \theta_J)|)^2 d\text{vol}_\Sigma,$$

where $\theta_J$ is the unique normal contact form on $(M, H, J)$ and $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$.

We now prove the CR analogue of a result due to C. LeBrun [LeB99].
Proposition 6.3. Let \((M, H, J)\) be a compact SPCR manifold of dimension \(2n + 1\) admitting a Yamabe contact form. Then
\[
|Y_{CR}(M, H, J)|^{n+1} = \inf_{\theta \in [\theta]} \int_M |\text{Scal}_W(J, \hat{\theta})|^{n+1} \hat{\theta} \wedge d\hat{\theta}^n,
\]
and the infimum is realized by Yamabe contact forms.

Proof. By Hölder’s inequality, for all \(\hat{\theta} \in [\theta]\),
\[
\left( \int_M |\text{Scal}_W(J, \hat{\theta})|^{n+1} \hat{\theta} \wedge d\hat{\theta}^n \right)^{\frac{1}{n+1}} \geq \frac{\int_M \text{Scal}_W(J, \hat{\theta}) \hat{\theta} \wedge d\hat{\theta}^n}{\left( \int_M \hat{\theta} \wedge d\hat{\theta}^n \right)^{\frac{n}{n+1}}},
\]
with equality if and only if \(\text{Scal}_W(J, \hat{\theta})\) is a non-negative constant. If \(Y_{CR}(M, H, J) \geq 0\), the claim follows from the fact that there exists a Yamabe contact form.

If \(Y_{CR}(M, H, J) < 0\), let \(\hat{\theta} \in [\theta]\) be a Yamabe contact form. Let us consider \(\hat{\theta} \in [\theta]\) and \(u \in C^\infty(M, \mathbb{R}^n)\) such that \(\hat{\theta} = u^2 \hat{\theta}\). Then
\[
\text{Scal}_W(J, \hat{\theta}) u^{\frac{n+2}{n}} = 2 \left( \frac{n+1}{n} \right) \Delta_b u + \text{Scal}_W(J, \hat{\theta}) \cdot u,
\]
so that
\[
\int_M \text{Scal}_W(J, \hat{\theta}) u^{\frac{n+2}{n}} \wedge d\hat{\theta}^n = \int_M \left( 2 \left( \frac{n+1}{n} \right) \frac{\Delta_b u}{u} + \text{Scal}_W(J, \hat{\theta}) \right) \hat{\theta} \wedge d\hat{\theta}^n
\]
\[
= \int_M \left( -2 \left( \frac{n+1}{n} \right) \left| d_b u \right|^2 + \text{Scal}_W(J, \hat{\theta}) \right) \hat{\theta} \wedge d\hat{\theta}^n \leq S_W(M, J, \hat{\theta}),
\]
and by Hölder’s inequality,
\[
\left( \int_M |\text{Scal}_W(J, \hat{\theta})|^{n+1} \hat{\theta} \wedge d\hat{\theta}^n \right)^{\frac{1}{n+1}} \geq \frac{\int_M \text{Scal}_W(J, \hat{\theta}) u^2 \hat{\theta} \wedge d\hat{\theta}^n}{\left( \int_M \hat{\theta} \wedge d\hat{\theta}^n \right)^{\frac{n}{n+1}}}
\]
\[
\geq - \frac{S_W(M, J, \hat{\theta})}{\left( \int_M \hat{\theta} \wedge d\hat{\theta}^n \right)^{\frac{n}{n+1}}}
\]
\[
= |Y_{CR}(M, H, J)|,
\]
with equality if and only if \(u\) is a constant, which proves the desired equality. \(\square\)

This proposition yields the following estimate on \(Y_{CR}\), which implies Theorem 1.4.

Corollary 6.4. Let \((M, H, J)\) be a circle bundle over a Riemann surface \(\Sigma\) of positive genus admitting an Einstein contact form. Then
\[
Y_{CR}(M, H, J) = -2\pi\sqrt{-\chi(\Sigma)}.
\]

Proof. Let \(\theta\) be an Einstein contact form on \((M, H, J)\). By Propositions 6.1 and 6.2,
\[
\int_M \text{Scal}_W(J, \theta)^2 \theta \wedge d\theta = -16\pi^2 \mu(M, H, J) = 4\pi^2 |\chi(\Sigma)|.
\]
Then by Proposition 6.3
\[
Y_{CR}(M, H, J)^2 \leq 4\pi^2 |\chi(\Sigma)|. \tag{6}
\]
If $\Sigma$ is a torus, this implies that $Y_{CR}(M, H, J) = 0$. Otherwise, $(M, H, J)$ admits a contact form of negative Webster scalar curvature, hence $Y_{CR}(M, H, J) \leq 0$ by Proposition 3.8. In all cases, $Y_{CR}(M, H, J) \leq Y_{CR}(S^3, H_0, J_0)$. By Theorems 3.3 and 3.6, $\theta$ is thus a Yamabe contact form, hence the inequality (6) is an equality. □
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