On the Statistical Properties of Localized Solutions of Klein-Fock-Gordon Equation

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Abstract

We consider the normalized axisymmetric solutions of Klein-Fock-Gordon equation with energy spectrum that lies below usual rest energy $mc^2$. It is shown that the gas of hypothetical particles, described by such solutions, would possess uncommon thermodynamic and kinetic characteristics, in particular, an anomalously high temperature of Bose-Einstein condensations as well as - in the case of charged particles - a conductivity, much exceeding that of conventional plasma.
1. Introduction

As it is well known, the motion of relativistic particle of mass $m$ can be described by Klein-Fock-Gordon (KFG) equation,

$$\Delta \Psi = c^{-2}\Psi_{tt} + \kappa^2 \Psi ; \kappa \equiv mc/h ,$$  \hspace{1cm} (1)

for its wave function, $\Psi$. As a root, just simplest, plane wave solutions of Eq.(1),

$$\Psi \propto \exp(ik \cdot r - i\omega t) ,$$  \hspace{1cm} (2)

are usually considered in empty - uniform and isotropic - space. They only depend on energy $h\omega$ and momentum $\hbar k$ of a particle and yield the standard relativistic dispersion relation,

$$\omega(k) = c\sqrt{\kappa^2 + |k|^2}$$  \hspace{1cm} (3)

However, Eq.(1) also admits more general stationary solutions, not uniform in a plane orthogonal to $k$. Basing on the correspondence principle, we can assume that $OZ$ axis, parallel to $k$ and coincident with classical trajectory of particle, is assigned in coordinate space. So - as a hypothesis - we can introduce into consideration the wave functions of a form

$$\Psi(x, y, z, t) = \Phi(\rho) \exp(ikz - i\omega t) ; \rho \equiv \sqrt{x^2 + y^2} ,$$  \hspace{1cm} (4)

localized nearby this axis. Axially symmetric solutions of KFG-equation and their role in causal propagator construction have been considered in [1]. In present communication we investigate some statistical and kinetic properties of the ensemble of hypothetic particles described by such solutions and demonstrate that these properties differ essentially from the statistics of conventional relativistic gas.

2. Dispersion relations

Radial function $\Phi(\rho)$ satisfies the equation

$$\Phi_{\rho\rho} + \rho^{-1}\Phi_\rho = (\kappa^2 + k^2 - \omega^2/c^2)\Phi \equiv q^2\Phi .$$  \hspace{1cm} (5)

In addition to usual limit conditions $\Phi(+\infty) \to 0 ; \Phi_\rho(+\infty) \to 0$ , it is natural to require $\Phi$ to be quadratically integrable in $z = const$ plane, i.e. to belong to Gilbert space (see [2] ). Such solutions, normalized to 1, exist only at positive values of parameter $q^2$ and are given by McDonald function (zeroth order modified Bessel function of the second kind):

$$\Phi(\rho) = (q/\pi)K_0(q\rho) .$$  \hspace{1cm} (6)
Aside from this, we have to require the stability of solution (i.e. reality of \( \omega \)) at every permitted values of \( k \) or, the same, to exclude "non-physical" states with superlight group speeds: \( v \equiv \partial \omega / \partial k \leq c \). As a result, we come to more general, than Eq.(3), dispersion relation:

\[
\omega(k, q) = c \sqrt{\kappa^2 + k^2 - q^2} ; 0 < q^2 \leq \kappa^2 .
\] (7)

With increasing of parameter \( q^2 \), localization becomes stronger and energy decreases. At maximal value \( q = \kappa \) the wave function is localized in a region with radius of order of Compton wavelength \( \hbar / mc \), and spectrum begins from zero energy, like that of massless particles: \( \omega(k, \kappa) = c |k| \).

The presence of anomalous branch of spectrum, lying under usual rest energy \( mc^2 \), is the result of singular eigen-functions admission. Bound solutions of Eq.(6) can exist only at \(-\infty < q^2 \leq 0 \); they are represented by non-normable (i.e. delocalized) Bessel functions \( J_0(|q| \rho) \), including the plane wave moving along \( OZ \) axis at \( q = 0 \). However, it is necessary to indicate that such admission does not contrary to well known principles of quantum mechanics: it is commonly accepted (see, e.g., \( ^3 \)) that the wave function – or rather \( |\Psi|^2 \) – may entirely have integrable singularities in domains of dimensionalities lower than its domain of definition. Seemingly, the most famous example of singular wave function application is relativistic theory of hydrogen atom \( ^3 \).

3. Energy density of states

Now consider the ideal gas of \( N \) particles with dispersion law (7) in space volume \( V \), and calculate the phase volume \( N(\omega) \), i.e. quantity of states with energies not exceeding \( \hbar \omega \). In space of wave numbers with cylindrical coordinates \((k, q, \varphi)\) the corresponding domain is restricted by the surface of revolution

\[
q(k, \omega) = \sqrt{\kappa^2 - \omega^2/c^2 + k^2} \quad (8)
\]

and, according to inequality (7), by cylinder \( q = \kappa \) (see Fig. 1). So one can find

\[
N(\omega) = \frac{V}{8\pi^3} \int_0^{2\pi} d\varphi \int_{-\omega/c}^{\omega/c} dk \int_{q(dk)}^{\kappa} qdq = \frac{V\omega^3}{6\pi^2c^3} ,
\] (9)

and, by definition, the density of anomalous (localized) states with \( \hbar \omega < mc^2 \) is

\[
g_L(\omega) \equiv \partial N(\omega) / \partial \omega = V\omega^2 / 2\pi^2c^3 \quad (10)
\]

(see Fig. 2). Indicate that it does not depend on the mass of particle.

Eqs. (7) and (10) are the key results: the presence of activationless spectrum and possibility of high speeds at low energies predetermine an anomalous statistical properties of the gas of particles described by localized solutions of KFG equation.
4. Bose-Einstein condensation

We can find the degeneracy temperature of anomalous gas considering the effect of Bose-Einstein condensation. As is known, it means that the maximal → i.e. corresponding to zero value of chemical potential µ − number of excited bosons becomes less than their given total number N at temperatures lower than some critical one, \( T_C \). So the rest particles are "condensing" in ground state (with minimal energy). Critical − or degeneracy − temperature separates (by order of value) the ranges of application of classical and quantum, in given case Bose statistics.

The equation for \( T_C \) (in energy units) can be written in form

\[
N = \int_0^\infty \frac{d\omega g(\omega)}{\exp(\hbar\omega/T) - 1}
\]

(see, e.g.,[4]). Since under "earthly" laboratory conditions we alwais have \( T << mc^2 \), the expression (10) for \( g(\omega) \) can be very exactly substituted to Eq.(11) at any \( \omega \). As a result we find

\[
N \approx \frac{V}{2\pi^2} \left( \frac{T}{\hbar c} \right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{V}{\pi^2} \left( \frac{T}{\hbar c} \right)^3 \zeta(3),
\]

where \( \zeta(x) \) is the Riemann function, \( \zeta(3) \approx 1.202 \). Critical temperature for localized states is

\[
T_C \approx h\epsilon(N\pi^2/V\zeta(3))^{1/3} \approx 2\hbar\epsilon(N/V)^{1/3}
\]

and does not depend on mass; the share of condensed particles is

\[
N_0(T)/N \approx 1 - (T/T_C)^3.
\]

Remind that for usual Bose gas the corresponding quantities are given by expressions

\[
(T_C)_{cl} \approx (N/V)^{2/3}\hbar^2/m;
\]

\[
(N_0(T)/N)_{cl} = 1 - (T/(T_C)_{cl})^{3/2}.
\]

At rational densities of particles, when \( N/V \ll \kappa^3 \approx 10^{31}cm^{-3} \), the degeneracy temperature of anomalous gas (13) exceeds much the usual one: \( T_C/(T_C)_{cl} \sim \kappa(V/N)^{1/3} \) (and the same is the ratio \( mc^2/T_C \)). E.g., even at very small densities, \( N/V \sim 10^9cm^{-3} \) one can obtain the "room" degeneracy temperature \( T_C \sim 300^\circ K \).
5. Thermodynamic functions

Using the density states found (10) and standard formulas of statistical mechanics \[^1, ^5\] one can easily calculate all thermodynamic functions of anomalous gas consisting of localized particles. For instance, the average intrinsic energy related to a particle is

\[
E(T) \approx \frac{\hbar}{N} \int_0^\infty \frac{d\omega g_L(\omega)}{\exp((\hbar \omega - \mu)/T) - 1},
\]

(16)

where chemical potential \(\mu(T)\) is determined by equation of the form (11) but with \(\hbar \omega - \mu\) in exponent instead of \(\hbar \omega\). At temperatures lower than \(T_C\) we may assume \(\mu(T) \approx 0\), so similarly to Eq.(12) we obtain

\[
E(T) \approx \frac{3\zeta(4)}{\zeta(3)} T \approx 2.701 \times T;
\]

\[
C_V \approx 2.701 \times B,
\]

(17)

where \(C_V\) is a specific heat; \(B \approx 1.380 \times 10^{-16} \text{erg/}^o\text{K}\), the Boltzmann constant. For the conventional gas (with usual density of states) at \(T < (T_C)_{cl} \ll mc^2\) one could obtain

\[
E(T) \approx \frac{3\zeta(5/2)}{2\zeta(3/2)} T \approx 0.770 \times T
\]

\[
C_V \approx 0.770 \times B
\]

(18)

(see \[^4, ^5\]). In case \(T >> T_C\) we can neglect 1 in denominator of expressions for \(E\) and \(N\), so corresponding results for anomalous gas are

\[
\mu(T) \approx T \ln \left(\frac{\pi^2 N}{V} \left(\frac{\hbar c T}{2}\right)^3\right) \approx -3T \ln \left(\frac{T}{T_C}\right)
\]

\[
E \approx 3T; \quad C_V \approx 3B;
\]

(19)

Indicate for comparison, that thermodynamic quantities of usual gas, consisting of only delocalized particles, attain the values (19) just in ultra-relativistic limit, when \(T >> mc^2\). At real temperature, if it exceeds \(T_C\), the chemical potential (with substitution \((T_C)_{cl}\) instead of \(T_C\)), intrinsic energy and specific heat of usual gas are sharply half of expressions (19) (see \[^4\]).
6. Conductivity

Rigorously, only electrically neutral scalar bosons can be described by KFG equation. However, in a case of charged particle every component of its spinor wave function must satisfy the same equation. Therefore (with understandable provisos) we have a right to consider properties of a gas of charged particles with the dispersion law (7).

Let us assume, as it has been done in elementary theory of metal, that each of them carries the charge $e$, that – on average – is compensated by distributed opposite charge of massive medium, almost transparent for carriers. Using the well known results of physic kinetics, we can calculate the conductivity of such system. In the limit of high frequencies $\Omega$, much more than reciprocal relaxation time of carriers, one can obtain

$$\sigma(\Omega) = \frac{2e^2}{\hbar \Omega} \int_0^\infty d\omega \frac{\partial F}{\partial \omega} \langle v^2 \rangle,$$

where

$$F(\omega, \mu) = \left( 1 + \exp \left( \frac{\hbar \omega - \mu}{T} \right) \right)^{-1}$$

is Fermi distribution; $v = \partial \omega/\partial k = c^2 k/\omega$, the group velocity of particle; its square is averaged over the isoenergetic surface $S(\omega)$ has been described in Section 3:

$$\langle v^2 \rangle \equiv (2\pi)^{-3} \int dS \frac{v^2}{\sqrt{v^2 + (\partial \omega/\partial q)^2}} = (2\pi)^{-3} \int d\omega' \frac{q}{\partial \omega/\partial q} \frac{\omega'^2}{3\pi^2 c}$$

(the specific form of dispersion law (7) has been used in these transformations). As a result

$$\sigma(\Omega) = \frac{2ie^2T^2}{3\pi^2\hbar^3c\Omega} \int_0^\infty \frac{xdx}{\exp(x - \mu/T) + 1}. \quad (22)$$

The chemical potential of anomalous Fermi gas is defined, as above, from a constancy of total number of particles:

$$N = 2 \int_0^\infty d\omega F(\omega, \mu)g_L(\omega). \quad (23)$$

In degenerated limit this gives us

$$\mu \cong \hbar c (3\pi^2N/V)^{1/3}, \quad T << T_C, \quad (24)$$
and for non-degenerate gas \(( T >> T_C \) ) the expression (19) remains, naturally, in force (with doubled volume \( V \) if the spin is 1/2). Finally we obtain for the conductivity:

\[
\sigma(\Omega) \cong \frac{iN_e^2}{m\Omega V} \cdot \kappa \left( \frac{V}{3\pi^2N} \right)^{1/3}, \quad T << T_C;
\]

\[
\sigma(\Omega) \cong \frac{iN_e^2}{m\Omega V} \cdot \frac{mc^2}{3T}, \quad T >> T_C.
\]  \hspace{1cm} (25)

The first factor in these formulas is the classical value of \( \sigma(\Omega) \). Thus, in a wide region of temperature and carriers concentration values,

\[
T, \ T_C << mc^2,
\]  \hspace{1cm} (26)

the high-frequency conductivity of the gas considered, as well as the square of plasma frequency, proportional to it, would much exceed the corresponding characteristics of conventional plasma.

7. Conclusion

Thus, some unusual results follow the hypothesis of "quasi-classical" KFG-particles described by localized wave functions, with energies below \( mc^2 \). In particular, Bose-Einstein condensation of a gas of such particles would be possible at very high temperatures; in a case of plasma containing charged localized particles, it would possess the anomalously high conductivity and plasmons activation energy.

However, it seems that there is no evident physical principle to cut-off the filling of localized states. Maybe, expected reduction of symmetry of wave functions (in comparison with the plane waves) could be more probable in a presence of external axisymmetric potential, i.e. if particles move inside microscopic cavities like carbon nanotubes \([6]\). The corresponding boundary problem – in cylindrical scalar potential – is solved in Appendix.

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8. Appendix

Consider the stationary axisymmetric solutions of Eq.(1) in the form of

\[ \Psi = \Phi(\rho) \exp(ikz - i\omega t) \]  

in cylindrical bore \( \rho \equiv \sqrt{x^2 + y^2} \leq R \) restricted by (for simplicity) infinitely high potential barrier. Therefore, we assume the given energy of particle (\( \hbar \omega \geq 0 \)) and projection of its momentum onto bore axis. Radial function \( \Phi(\rho) \) must satisfy the equation

\[ \Phi_{\rho\rho} + \rho^{-1}\Phi_{\rho} = Q^2\Phi \]  

where

\[ Q^2 = \kappa^2 + k_z^2 - \omega^2/c^2 , \]  

and the boundary condition \( \Phi(R) = 0 \). Aside from it, by analogy with Section 2, we require the normalization integral

\[ M(R) \equiv 2\pi \int_0^\infty \rho |\Phi(\rho)|^2 d\rho \]  

to be finite at any radius of the domain of free movement of particles, including \( R \to \infty \). It is not difficult to certaine that the linear combination of modified Bessel functions of first and second kind,

\[ \Phi(\rho) \propto K_0(Q\rho) - \frac{K_0(QR)}{I_0(QR)} I_0(Q\rho) , \]  

8
satisfies both these conditions, just at positive values of the parameter $Q^2$.

Putting, as above, the additional physical condition, $v_z \equiv \partial \omega / \partial k \leq c$, i.e. considering the sub-light particles only, we obtain the continuous spectrum

$$\omega(k_z, Q) = c\sqrt{\kappa^2 + k_z^2 - Q^2} : 0 < Q^2 \leq \kappa^2,$$

(32)

similar to Eq.(7). Its form leads to expression (10) for energy density of localized states and so to the rest conclusions obtained above (taking into account that in this case the expressions for conductivity in Section 6 are related to an electric field directed along the canal).

Indicate once more that anomalous (inverse) spectrum (32) with the "effective mass" depending on particle localization parameter, $m^* = \sqrt{m^2 - (\hbar Q/c)^2}$, has been conditioned by quite reasonable quantum-mechanical requirement $M(R \to \infty) < \infty$ (see (30)). If, quite the contrary, we put a more habitual condition $|\Phi| < \infty$ (on the analogy of classical mechanics problems), we obtain at once the usual discrete spectrum

$$\omega(k_z, n) = c\sqrt{\kappa^2 + k_z^2 + (\xi_n/R)^2}$$

(33)

parametrized by zeros $\xi_n$ of Bessel function and lying higher than $mc^2$.

The same consideration can easily be carried out also for a spherical potential well. For localized states described by singular solutions of KFG-equation within the spherical cavity, one can find the energy density

$$g_L(\omega) = \frac{V}{2\pi^2 c^3} \omega \sqrt{\kappa^2 c^2 - \omega^2}$$

(34)

and, then, the temperature of Bose-Einstein condensation

$$T_C \approx \frac{\pi \hbar}{\sqrt{2 \zeta(2)}} \frac{\hbar c N}{m V},$$

$$\zeta(2) \approx 1.645.$$  

(35)

If, as assumed, $\kappa \sqrt{V/N} >> 1$, it much exceeds the classical value (15).