Extremal Reversible Measures for the Exclusion Process

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Abstract

The invariant measures \( \mathcal{I} \) for the exclusion process have long been studied and a complete description is known in many cases. This paper gives characterizations of \( \mathcal{I} \) for exclusion processes on \( \mathbb{Z} \) with certain reversible transition kernels. Some examples for which \( \mathcal{I} \) is given include all finite range kernels that are asymptotically equal to \( p(x, x + 1) = p(x, x - 1) = 1/2 \). One tool used in the proofs gives a necessary and sufficient condition for reversible measures to be extremal in the set of invariant measures, which is an interesting result in its own right. One reason that this extremality is interesting is that it provides information concerning the domains of attraction for reversible measures.

Keywords: Exclusion process; Invariant measures; Extremal invariant measures; Domains of attraction

1 Introduction

Given a countable set \( S \) and a corresponding probability transition function \( p(x, y) \) satisfying \( \sup_y \sum_x p(x, y) < \infty \), IPS (Liggett(1985)) constructs and describes the exclusion process on \( \{0, 1\}^S \). Its generator is given by the closure of the operator \( \Omega \) on \( \mathcal{D}(\{0, 1\}^S) \), the set of all functions on \( \{0, 1\}^S \) that depend on finitely many coordinates. If \( f \in \mathcal{D}(\{0, 1\}^S) \) and \( \eta_{xy} \) is defined as

\[
\eta_{xy}(u) = \begin{cases} 
\eta(y) & \text{if } u = x \\
\eta(x) & \text{if } u = y \\
\eta(u) & \text{if } u \neq x, y
\end{cases}
\]

then

\[
\Omega f(\eta) = \sum_{\eta(x) = 1, \eta(y) = 0} p(x, y)[f(\eta_{xy}) - f(\eta)].
\]

The semigroup of this process will be denoted by \( S(t) \).

When \( p(x, y) = p(y, x) \) the process has been completely studied in that a full description of its invariant measures is known as well as their respective domains of attraction. The asymmetric

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exclusion process on the other hand has been much more elusive. General classes of invariant measures are known in the two cases where \( p(x, y) \) is doubly stochastic (i.e., \( \sum_{x \in S} p(x, y) = 1 \) for all \( y \in S \)) or when there exists a reversible measure \( \pi(x) > 0 \) on \( S \) (i.e., a measure satisfying \( \pi(x)p(x, y) = \pi(y)p(y, x) \)). However, a complete description of \( \mathcal{I} \) is known only in the three cases when either

(a) \( p(x, y) \) is reversible and positive recurrent for either the particles or holes (1’s or 0’s) (Liggett(1976))

(b) \( p(x, y) \) corresponds to certain random walks on \( \mathbb{Z} \) (Liggett(1976) and Bramson, Liggett, and Mountford(2002)) or

(c) \( p(x, y) \) corresponds to a birth and death chain on \( \mathbb{Z}^+ \) (Liggett(1976)).

Almost nothing is known about the domains of attraction concerning invariant measures in the asymmetric case, although we note here that there are some nice theorems concerning the case where \( p(x, y) \) is an asymmetric simple random walk on \( \mathbb{Z} \) (see Liggett(1999)).

Our purpose in this paper is to shed some more light on the problem of classifying \( \mathcal{I} \) and its respective domains of attraction for the asymmetric exclusion process when a reversible measure \( \pi(x) \) exists for \( p(x, y) \).

Before describing the invariant measures for case (a), we define some extremal reversible invariant measures \( \{\nu^{(n)}\} \) when a reversible measure \( \pi(x) \) satisfying

\[
\sum_{x} \pi(x)/[1 + \pi(x)]^2 < \infty
\]

exists. This family of extremal reversible measures was first discovered by Liggett. In particular, he breaks down \( \mathbb{B} \) into three cases and writes

1. If \( \sum_{x} \pi(x) < \infty \), let \( A_n = \{\eta : \sum_{x} \eta_x = n\} \) for nonnegative integers \( n \).

2. If \( \sum_{x} 1/\pi(x) < \infty \), let \( A_n = \{\eta : \sum_{x} [1 - \eta_x] = n\} \) for nonnegative integers \( n \).

3. If \( \sum_{x} \pi(x)/[1 + \pi(x)]^2 < \infty \), \( \sum_{x} \pi(x) = \infty \), and \( \sum_{x} 1/\pi(x) = \infty \), there exists a \( T \subset S \) for which \( \sum_{x \in T} \pi(x) < \infty \) and \( \sum_{x \notin T} 1/\pi(x) < \infty \). In this case, let

\[
A_n = \{\eta : \sum_{x \in T} \eta(x) - \sum_{x \notin T} [1 - \eta(x)] = n\}
\]

for integers \( n \).
1 INTRODUCTION

To define \( \{ \nu^{(n)} \} \), let \( \nu^c \) be the product measure with marginals \( \nu^c \{ \eta : \eta(x) = 1 \} = \frac{e^{c(x)}}{1 + e^{c(x)}} \). Liggett shows that the measures

\[
\nu^{(n)}(\cdot) = \nu^c(\cdot | A_n) \quad \text{for } n \in \mathbb{Z},
\]

\[
\nu^{(\infty)} = \text{the pointmass on } \eta(x) = 1,
\]

\[
\nu^{(-\infty)} = \text{the pointmass on } \eta(x) = 0
\]

are the unique stationary distributions for the positive recurrent Markov chains on \( A_n \). A simple consequence of Theorem B52 in Liggett(1999) is that the reversible measures \( \{ \nu^{(n)} \} \) are extremal in the set of invariant measures. For the first two cases in the trichotomy of (2) above, these are the only extremal invariant measures. These first two cases correspond exactly to (a) above. Note that changing \( T \) in the third case of the trichotomy above amounts to a relabeling of the sequence \( \{ \nu^{(n)}, n \in \mathbb{Z} \} \).

Whenever a reversible measure \( \pi(x) \) on \( S \) exists, the product measures \( \{ \nu^c \} \) are well-defined. Theorem VIII.2.1 in IPS tells us that these measures are invariant for the exclusion process. Applying Kakutani’s Dichotomy (e.g. page 244 of Durrett(1996)) we have that \( \sum_{x} \pi(x)/[1 + \pi(x)]^2 = \infty \) is a necessary and sufficient condition for the measures \( \{ \nu^c : 0 \leq c \leq \infty \} \) to be mutually singular. Since all the results in this paper concern the reversible measures \( \{ \nu^c \} \), we will assume throughout the rest of the paper that \( \pi(x) \) satisfying \( \pi(x)p(x, y) = \pi(y)p(y, x) \) exists.

In Section 3 we prove Theorem 2.1 which states that \( \sum_{x} \pi(x)/[1 + \pi(x)]^2 = \infty \) is exactly the situation in which the measures \( \nu^c \) are extremal invariant. Not only does this result have some nice applications, but knowing that an invariant measure is extremal in the set of invariant measures has always been an interesting question concerning particle systems. Examples of such results are Theorem III.1.17 in Liggett(1999) and Theorem 1.4 in Sethuraman(2001). The main reason extremality of invariant measures is interesting is its close connection with ergodicity. This is seen by the application Theorem III.1.17 in Liggett(1999) to prove Theorem III.4.8 in Liggett(1999) concerning the tagged particle process; it is again seen by the application of Theorem 1.4 in Sethuraman(2001) to certain central limit theorems given in Kipnis and Varadhan(1986). In particular, if the initial measure for a process is an extremal invariant measure then the process evolution is ergodic with respect to time shifts.

Sections 3 and 4 use Theorem 2.1 to extract information about the invariant measures of the process on \( \mathbb{Z} \). In particular, Section 3 modifies Liggett’s original proof of the result stated above equation (11) to obtain the following result:

**Theorem 1.1.** Let \( \mathbb{Z} \) be irreducible with respect to a transition kernel \( p(x, y) \) for which there exists a reversible measure \( \pi(x) \). Suppose \( q_i(z) \) is a transition kernel such that \( \sum_z zq_i(z) = 0 \) and \( \sum_z |z|q_i(z) < \infty \) for \( i = 1, 2 \), and suppose that

\[
\lim_{K \to \infty} \sum_{x \geq 0} \sum_{|z| \geq |x-K|} |p(x, x+z) - q_1(z)| = 0 \quad \text{and} \quad \lim_{K \to \infty} \sum_{x \leq 0} \sum_{|z| \geq |x+K|} |p(x, x+z) - q_2(z)| = 0. \quad (3)
\]

(a) If \( \sum_x \pi(x)/[1 + \pi(x)]^2 = \infty \) then \( \mathcal{J}_e = \{ \nu^c : 0 \leq c \leq \infty \} \).

(b) If \( \sum_x \pi(x)/[1 + \pi(x)]^2 < \infty \) then \( \mathcal{J}_e = \{ \nu^{(n)} \} \).

In essence the above theorem says that when the transition probabilities are asymptotically translation invariant and have an asymptotic mean of zero, the reversible measures are the only invariant
measures. Theorem 1.1 is merely an extension (in the case where \( \pi(x) \) exists) of the theorem proved by Liggett(1976) which is stated above equation (1).

Condition (3) may seem somewhat daunting, but note that if \( \lim_{x \to \infty} p(x, x + z) = q_1(z) \), \( \lim_{z \to -\infty} p(x, x + z) = q_2(z) \), and \( p(x, y) \) has finite range (i.e. \( p(x, y) \equiv 0 \) if \( |x - y| > n \) for some \( n \)), then (3) and \( \sum_z |q_1(z) < \infty \) are both automatically satisfied. Also, the below condition which is somewhat easier to grasp than (3) implies (3):

\[
\sum_{x \geq 0} \sum_z |p(x, x + z) - q_1(z)| < \infty \quad \text{and} \quad \sum_{x \leq 0} \sum_z |p(x, x + z) - q_2(z)| < \infty.
\]

A typical situation for which the theorem holds is when the transition rates are nearest-neighbor and are given by \( p(x, x + 1) = p(x, x - 1) = 1/2 \) except for finitely many \( x \).

Note that the premises of the theorem together with the assumption that a reversible \( \pi(x) \) exists imply that \( q_1(z) \) must be symmetric. To see this suppose \( q_1(z) \) is not symmetric. Also, assume that \( q_1(z_1) > q_1(-z_1) > 0 \) for some \( z_1 \in \mathbb{N} \). We can do this without loss of generality since \( q_1(z) > 0 \) implies \( q_1(-z) > 0 \) by the reversibility of \( \pi(x) \). The mean zero assumption tells us there exists \( z_2 \in \mathbb{N} \) such that \( q_1(z_2) < q_1(-z_2) \). If \( z_3 \) is a multiple of both \( z_1 \) and \( z_2 \) then since \( p(x, x + z) \rightarrow q_1(z) \) we can find \( x_1 \) so that for \( x > x_1 \), \( \pi(x) < \pi(x + z_3) \). But we can also find \( x_2 \) so that for \( x > x_2 \), \( \pi(x) > \pi(x + z_3) \), a contradiction. So \( q_1(z) \) must be symmetric. The proof that \( q_2(z) \) is symmetric follows similarly.

The proof of the above theorem follows Liggett’s original outline and does not actually require Theorem 2.1. However, the usefulness of Theorem 2.1 is seen in the simplification of one part of Liggett’s original proof.

In Section 4 we prove a theorem concerning the nearest-neighbor exclusion process on \( \mathbb{Z} \). For the statement of the theorem we will need the following definitions.

Let \( L^- \) be the set of limit points of \( \{ \pi(x), x < 0 \} \) and \( L^+ \) be the set of limit points of \( \{ \pi(x), x > 0 \} \).

**Theorem 1.2.** Suppose that \( \inf_{|x-y|=1} p(x, y) > 0 \) for a nearest-neighbor exclusion process on \( \mathbb{Z} \). Then nonreversible invariant measures can exist only when either (a) \( L^- = \{ 0 \} \) and \( L^+ = \{ \infty \} \) or (b) \( L^- = \{ \infty \} \) and \( L^+ = \{ 0 \} \).

The above theorem in no way guarantees the existence of nonreversible invariant measures as seen by the following example. Let

\[
\begin{align*}
p(-1, -2) &= p(-1, 0) = p(0, -1) = p(0, 1) = 1/2, \\
p(x, x + 1) &= 1 - p(x, x - 1) = \frac{|x| + 1}{2|x|} \quad \text{otherwise.}
\end{align*}
\]

This transition gives us situation (a) in the theorem above. The reversible invariant measures \( \{ \nu^x \} \) certainly exist, but it is easy to see that condition (b) of Theorem 1.1 is satisfied by \( \{ \pi \} \), therefore there are no nonreversible invariant measures.

A curious aside is as follows. If in this example we start this process off with initial measure \( \nu_0 \) and take the limit of some converging sequence of measures

\[
\frac{1}{T_n} \int_0^{T_n} \nu_0 S(t) \, dt
\]

(5)
then Theorem I.1.8 in IPS says that this limit is an invariant measure for the process. In view of the previous discussion, this limit must converge to some mixture of the extremal invariant measures \( \{ \nu^{(n)}, 0 \leq n \leq \infty \} \). It would be interesting indeed to find out which mixture \( \text{5} \) converges to. Note here that we started off with an initial state that concentrates on an uncountable number of states, but the limiting distribution concentrates on a countable number of states (which may very well be just the point masses of all 0’s and all 1’s).

If \( p(x, y) \) is an asymmetric random walk kernel with nonzero mean then we have one of the situations described in the theorem above, and one might correctly guess that there exists some nonreversible invariant measure. In fact, a well-known result of Liggett(1976) proves that the measures \( \{ \nu^\rho : 0 \leq \rho \leq 1 \} \) are invariant measures. Since any limit of \( \text{5} \) is invariant, we intuitively might have expected this. More precisely, if there were no nonreversible measures then this limit would presumably be a mixture of the reversible measures \( \nu^c \). But it is intuitive that there is no mixture of \( \nu^c \)'s to which this limit could converge, leading us to believe that the limit converges to some other measure.

We note here that the set of measures in \( \text{6} \) is the same as the set of measures in \( \text{1} \) but are of an entirely different nature. In the setting of \( \text{1} \) the measures \( \{ \nu^\rho : 0 \leq \rho \leq 1 \} \) are reversible and constitute the entire set of extremal invariant measures. On the other hand, under the current setting, the measures \( \{ \nu^\rho : 0 \leq \rho \leq 1 \} \) are not reversible and

\[
\mathcal{I}_e = \{ \nu^\rho : 0 \leq \rho \leq 1 \} \cup \{ \nu^c : 0 \leq c \leq \infty \}.
\]

The discussion in the previous paragraphs might make us wonder for which transition kernels a nonreversible invariant measure exists. To gain more insight into the situation we introduce a concept known as the flux of an invariant measure \( \mu \). We will continue to assume that the transition probabilities are nearest-neighbor, but we will no longer assume they are translation invariant. Define

\[
\text{flux}(\mu) = p(x, x + 1)\mu\{ \eta : \eta(x) = 1, \eta(x + 1) = 0 \} - p(x + 1, x)\mu\{ \eta : \eta(x) = 0, \eta(x + 1) = 1 \}.
\]

Let \( 1_x(\eta) = \eta(x) \) be the indicator function of \( \{ \eta(x) = 1 \} \). By computing the positive and negative terms of the left-hand side of \( \int \Omega_1 d\mu = 0 \) it can be seen that \( \text{flux}(\mu) \) is independent of \( x \).

When an invariant measure \( \mu \) is reversible it can easily be seen from \( \text{7} \) that \( \text{flux}(\mu) = 0 \). So if an invariant measure exists whose flux is nonzero it must be nonreversible. For the process with \( p(x, x + 1) > 1/2 \) and \( p(x, x - 1) = 1 - p(x, x + 1) \), the invariant measures \( \{ \nu^\rho : 0 \leq \rho \leq 1 \} \) all have a positive flux with the flux being maximized when \( \rho = 1/2 \) (a full discussion of this can be found in either Janowski and Lebowitz(1994) or Part III of Liggett(1999)). This positive flux is the reason why \( \text{6} \) is fundamentally different from \( \text{1} \). It would be quite nice if one could prove that some nonreversible invariant measure exists whenever \( p(x, x + 1) > 1/2 + \epsilon \) for all \( x \). The \( \epsilon \) here serves the role of providing some positive flux in the limit.

Finally, Section \( \text{5} \) will apply Theorem \( \text{2.1} \) to give information concerning the domains of attraction (in the Cesaro sense) of reversible measures in the case where \( \sum_x \pi(x)/[1 + \pi(x)]^2 = \infty \). The results of Section \( \text{5} \) only give sufficient conditions for Cesaro convergence to an invariant measure, but are nonetheless interesting since so little is known concerning domains of attraction for the asymmetric
2 Extremal reversible measures

In this section we state and prove Theorem 2.1. The common technique used in the proof of this theorem and in the proofs of most of the other results in this paper is the coupling technique. We now define the basic coupling of $\eta_t$ and $\xi_t$ which lets the two exclusion processes move together as much as possible. The generator for this coupling is the closure of the operator $\tilde{\Omega}$ defined on $D(\{0, 1\}^S)$:

$$
\tilde{\Omega}f(\eta, \xi) = \sum_{\eta(x)=\xi(x)=1, \eta(y)=\xi(y)=0} p(x, y)[f(\eta_{xy}, \xi_{xy}) - f(\eta, \xi)] \\
+ \sum_{\eta(x)=\xi(x)=1, \eta(y)=\xi(y)=0 \text{ and } (\eta(y)=1 \text{ or } \xi(x)=0)} p(x, y)[f(\eta_{xy}, \xi) - f(\eta, \xi)] \\
+ \sum_{\xi(x)=\xi(y)=0 \text{ and } (\eta(y)=1 \text{ or } \eta(x)=0)} p(x, y)[f(\eta, \xi_{xy}) - f(\eta, \xi)].
$$

**Theorem 2.1.** Suppose $S$ is irreducible with respect to $p(x, y)$. Then the measures $\nu^c$ are extremal invariant if and only if $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$.

**Proof.** The discussion on page 383 of IPS shows that if $\sum_x \pi(x)/[1 + \pi(x)]^2 < \infty$ then the measures $\nu^c$ are not extremal invariant giving us one direction of the theorem. We will prove the other direction.

Assume throughout that $0 < c < \infty$. Since the measures $\nu^c$ are invariant and since all bounded continuous functions can be approximated uniformly by functions that depend on finitely many coordinates then by Theorem B52 in Liggett(1999), we need only show that for any two functions $f$ and $g$ which depend on finitely many coordinates

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T E^{\nu^c} f(\eta_t)g(\eta_t)dt = \int f d\nu^c \int g d\nu^c.
$$
We claim that to show the above equation holds, it is enough to show that for any finite $A \subset S$ and for $\mu^c_{i,A}(\cdot) = \nu^c(\cdot|\{\eta(x) = 1\forall x \in A\})$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu^c_{i,A} S(t) dt = \nu^c.$$  \hspace{1cm} (8)

To see this define the measures $\mu^c_{\zeta,A}(\cdot) = \nu^c(\cdot|\{\eta(x) = \zeta(x)\forall x \in A\})$ where $\zeta$ is a configuration on $\{0,1\}^A$. We can write the measure $\nu^c$ as a linear combination

$$\nu^c = \sum_{\zeta \in \{0,1\}^A} a_\zeta \mu^c_{\zeta,A}$$

where we use the convention that $\zeta = i$ is the configuration in $\{0,1\}^A$ such that $\zeta(x) = i$ for all $x \in A$. For

$$f_A = \begin{cases} 1 & \text{when } \eta(x) = 1 \text{ for all } x \text{ in the finite set } A \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E^{\nu^c} f_A(\eta_0) g(\eta_t) dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T a_1 \int S(t) g(\eta_t) d\mu^c_{i,A} dt = \int f_A d\nu^c \int g d\nu^c$$

which proves the claim.

Define $\mu^c_{0,A}$ similarly to the way we defined $\mu^c_{i,A}$. If we assume a fixed $A$ then we can drop the subscript $A$ so as to write $\mu^c_i = \mu^c_{i,A}$. The rest of the proof will now argue that (8) holds.

Choose $\delta > 0$ and couple the processes $\eta_t$ and $\xi_t$ using the basic coupling starting with measures $\mu^c_0$ and $\mu^c_1$ so that $\eta_0$ and $\xi_0$ disagree only for $x \in A$. In particular, since the basic coupling is the coupling which allows $\eta_t$ and $\xi_t$ to move together as much as possible, then $\eta_t$ and $\xi_t$ can differ at most at $n$ sites where $|A| = n$.

If there exists $\bar{T}$ such that for all $T > \bar{T}$

$$\frac{1}{T} \int_0^T [\mu^c_1 S(t)\{\xi(0) = 1\} - \mu^c_0 S(t)\{\eta(0) = 1\}] dt \leq \delta$$

then we must have that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu^c_1 S(t)\{\xi(0) = 1\} dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu^c_0 S(t)\{\eta(0) = 1\} dt.$$  \hspace{1cm} (8)

Keeping in mind the way that $\eta_t$ and $\xi_t$ are coupled, irreducibility then tells us that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu^c_1 S(t) dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu^c_0 S(t) dt.$$  \hspace{1cm} (9)

But the measure $\nu^c$ lies stochastically between the left-hand side and the right-hand side of the equation above, so in fact we must have that (9) holds.

We can therefore assume to the contrary that there exists a $\delta > 0$ and a sequence $\{T_n\}$ such that

$$\frac{1}{T_n} \int_0^{T_n} [\mu^c_1 S(t)\{\xi(0) = 1\} - \mu^c_0 S(t)\{\eta(0) = 1\}] dt > \delta$$
for all $n$.

Pick $\epsilon > 0$ so that
\[ \nu^{\epsilon+\epsilon}\{\xi(0) = 1\} - \nu^{\epsilon-\epsilon}\{\eta(0) = 1\} < \delta/3. \]

Using the basic coupling once more, couple the processes $\eta_t$ and $\xi_t$ starting in the measures $\mu^c_1$ and $\nu^{\epsilon+\epsilon}$ so that $\nu_1\{\{\eta_t, \xi_t\} : \eta(x) \leq \xi(x) \text{ for all } x \in \mathbb{Z}\setminus A\} = 1$ where $\nu_1$ is the coupling measure. If $\hat{\nu}^{\epsilon} = \nu^{\epsilon}(|\{\eta : \eta(x) = 0 \text{ for some } x \in A\})$ then
\[ \nu^{\epsilon} = \gamma \mu^c_1 + (1 - \gamma)\hat{\nu}^{\epsilon} \]
for $\gamma = \nu^{\epsilon}\{\eta : \eta(x) = 1 \forall x \in A\}$. Couple the measures $\hat{\nu}^{\epsilon}$ and $\nu^{\epsilon+\epsilon}$ in a way similar to $\nu_1$ so that we get another coupling measure $\nu_2$.

Choose a subsequence $\{T_{nk}\}$ so that we can define some limiting invariant measure
\[ \omega_1 = \lim_{k \to \infty} \frac{1}{T_{nk}} \int_0^{T_{nk}} \lambda_1 S(t) dt. \]
Let $\nu_1^c$ be the $\eta$-marginal measure of $\omega_1$ so that in particular
\[ \nu_1^c = \lim_{k \to \infty} \frac{1}{T_{nk}} \int_0^{T_{nk}} \mu_1^c S(t) dt. \]

To complete the proof of the theorem we will need the following lemma:

**Lemma 2.2.** $\nu^{\epsilon+\epsilon} \geq \nu_1^c$.

**Proof of lemma.** Let $f_x(\eta, \xi) = [1 - \eta(x)] \xi(x)$, $D_m = \{(\eta, \xi) : \eta(x) > \xi(x) \text{ at exactly } m \text{ sites}\}$, and $D = \bigcup_{m \geq 1} D_m$. If $\nu^{\epsilon+\epsilon} \nless \nu_1^c$ then it must be that $\omega_1(D) > 0$. We claim that this implies
\[ \int_D \sum_x f_x d\omega_1 = 0. \]

To prove the claim, assume to the contrary that $\int_D \sum_x f_x d\omega_1 > 0$ so that there exist sites for which $\eta(x) < \xi(x)$. Let $M$ be the largest $m$ for which $\omega_1(D_m) > 0$. Then by the irreducibility condition and by the fact that there exist sites for which $\eta(x) < \xi(x)$ we have $\omega_1(D_M) < \omega_1(D_M)$ for $t > 0$. But this is a contradiction to the invariance of $\omega_1$ proving the claim.

Now if the two processes $\eta_t$ and $\xi_t$ have the measures $\nu^{\epsilon}$ and $\nu^{\epsilon+\epsilon}$ respectively then let $\omega$ be the coupling measure for $\{(\eta_t, \xi_t)\}$ which concentrates on $\nu^{\epsilon} \leq \nu^{\epsilon+\epsilon}$. For this coupling, the $\omega$ probability that $f_x(\eta, \xi) = 1$ for a given $x$ is equal to the left-hand side below:
\[ \frac{(c + \epsilon)\pi(x)}{1 + (c + \epsilon)\pi(x)} - \frac{c\pi(x)}{1 + c\pi(x)} > \frac{\epsilon\pi(x)}{[1 + (c + \epsilon)\pi(x)]^2}. \]
Since $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$, by the Borel-Cantelli Lemma the $\omega$ probability that $\sum_x f_x = \infty$ is equal to 1. The measure $\omega_1$ is absolutely continuous with respect to $\omega$ since
\[ \omega = \gamma \lambda_1 + (1 - \gamma)\lambda_2 = \gamma \omega_1 + (1 - \gamma) \lim_{k \to \infty} \frac{1}{T_{nk}} \int_0^{T_{nk}} \lambda_2 S(t) dt \]
where $\lambda_2$ is as defined above. Therefore $\int_E \sum_x f_x \omega_1 = \infty$ for any set $E$ with positive $\omega_1$ measure which contradicts $\int_D \sum_x f_x \omega_1 = 0$ so it must be that $\omega_1(D) = 0$ proving the lemma.

We now turn back to the proof of the theorem. Since by the lemma we have $\nu^{e+\varepsilon} \geq \nu_1^e$, then there exists a $K$ such that for all $k > K$

$$\frac{1}{T_{nk}} \int_{0}^{T_{nk}} \mu_1^e S(t) \{\eta(0) = 1\} dt - \nu^{e+\varepsilon} \{\xi(0) = 1\} < \delta/3.$$ 

If $\nu_0^e$ is some limiting measure of

$$\frac{1}{T_{nk}} \int_{0}^{T_{nk}} \mu_0^e S(t) dt$$

then an argument similar to that used in Lemma 2.2 shows that $\nu^{e-\varepsilon} \leq \nu_0^e$. There then exists an $L$ such that for $l > L$

$$\nu^{e-\varepsilon} \{\eta(0) = 1\} - \frac{1}{T_{nk}} \int_{0}^{T_{nk}} \mu_0^e S(t) \{\xi(0) = 1\} dt < \delta/3.$$ 

Altogether we have for $l > L$,

$$\frac{1}{T_{nk}} \int_{0}^{T_{nk}} [\mu_1^e S(t) \{\xi(0) = 1\} - \mu_0^e S(t) \{\eta(0) = 1\}] dt < \delta$$

which contradicts inequality (3) so it must be that (3) holds completing the proof of the theorem.

### 3 The asymptotically mean zero process on \(\mathbb{Z}\)

In this section we prove Theorem 1.1. To do so we will need to define $\tilde{I}$ as the set of invariant measures for the basic coupling and $\tilde{I}_e$ as its extreme points.

Recall that $f_x(\eta, \xi) = [1 - \eta(x)]\xi(x)$. In order to simplify the notation we further define the functions

$$h_{yx}(\eta, \xi) = [1 - \eta(y)][1 - \xi(y)]f_x(\eta, \xi), \quad g_{yx}(\eta, \xi) = \eta(y)\xi(y)f_x(\eta, \xi),$$

and $f_{yx}(\eta, \xi) = \eta(y)[1 - \xi(y)]f_x(\eta, \xi)$.

In particular, for $T$ a finite subset of $\mathcal{S}$ we have

$$\tilde{\Omega} \left( \sum_{x \in T} f_x(\eta, \xi) \right) = - \sum_{x \in T, y \in \mathcal{S}} (p(x, y) + p(y, x)) f_{yx}(\eta, \xi)$$

$$+ \sum_{x \in T, y \notin T} [p(x, y)g_{xy} - p(y, x)g_{yx}] + \sum_{x \in T, y \notin T} [p(y, x)h_{xy} - p(x, y)h_{yx}].$$

(10)
Proof of Theorem 1.1. Let \( \nu \in \mathcal{I} \). Then \( \int \Omega(\sum_{x\in T} f_x) d\nu = 0 \) for each finite \( T \subset \mathbb{Z} \) so that for \( T_{m,n} = \{ x \in \mathbb{Z} : m \leq x \leq n \} \) we get

\[
\sum_{x \in T_{m,n}, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yx} d\nu = \sum_{x \in T_{m,n}, y \notin T_{m,n}} p(x, y) \int (g_{xy} - h_{yx}) d\nu + \sum_{x \in T_{m,n}, y \notin T_{m,n}} p(y, x) \int (h_{xy} - g_{yx}) d\nu. \tag{11}
\]

Notice that the left-hand side of this equation is increasing in \( n \) and \( -m \), so that when we take the limit as \( n \to \infty \) or as \( -m \to \infty \), a limit exists.

Choosing \( \epsilon > 0 \) we can find \( N \) so that for \( n > N \):

\[
\sum_{x > n + N, z < n - x} p(x, x + z) \leq \sum_{x > n + N, z < n - x} |p(x, x + z) - q_1(z)| + \sum_{|z| > N} |z| q_1(z) < \frac{\epsilon}{3}
\]

\[
\sum_{0 < z < n, z < n - x + N} p(x, x + z) \leq \sum_{0 < z < n, z < n - x + N} |p(x, x + z) - q_1(z)| + \sum_{|z| > N} |z| q_1(z) < \frac{\epsilon}{3}
\]

\[
\sum_{x \leq 0, z > n + N - x} p(x, x + z) \leq \sum_{x \leq 0, z > n + N - x} |p(x, x + z) - q_2(z)| + \sum_{|z| > N} |z| q_2(z) < \frac{\epsilon}{3}
\]

and

\[
\sum_{x < -n, z < -n - x} p(x, x + z) \leq \sum_{x < -n, z < -n - x} |p(x, x + z) - q_2(z)| + \sum_{|z| > N} |z| q_2(z) < \frac{\epsilon}{3}
\]

\[
\sum_{-n < x < 0, z < -n - N} p(x, x + z) \leq \sum_{-n < x < 0, z < -n - N} |p(x, x + z) - q_2(z)| + \sum_{|z| > N} |z| q_2(z) < \frac{\epsilon}{3}
\]

\[
\sum_{x \geq 0, z < -n - N - x} p(x, x + z) \leq \sum_{x \geq 0, z < -n - N - x} |p(x, x + z) - q_1(z)| + \sum_{|z| > N} |z| q_1(z) < \frac{\epsilon}{3}
\]

Since the construction of the exclusion process assumes that \( \sup_y \sum_x p(x, y) \) is finite (See IPS Chapter VIII) and since \( \int (g_{xy} - h_{yx}) d\nu \leq 1 \), the right-hand side sums in (11) above are absolutely convergent for any fixed \( n \) and \( m \).

Now by the inequalities above and by (9) we can pass to the limit in (11) so as to write

\[
\lim_{m \to -\infty} \lim_{n \to \infty} \sum_{x \in T_{m,n}, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yx} d\nu = \lim_{n \to \infty} \sum_{x \in T_{[0,n]}, y > n} \left[ q_1(y - x) \int (g_{xy} - h_{yx}) d\nu + q_1(x - y) \int (h_{xy} - g_{yx}) d\nu \right]
\]

\[
+ \lim_{m \to -\infty} \sum_{x \in T_{[m,0]}, y < m} \left[ q_2(y - x) \int (g_{xy} - h_{yx}) d\nu + q_2(x - y) \int (h_{xy} - g_{yx}) d\nu \right].
\]
The right-hand side above is equal to
\[
\lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} \sum_{x \in T_{[0,n], y>m}} \left[ q_1(y - x) \int (g_{xy} - h_{yx}) d\nu + q_1(x - y) \int (h_{xy} - g_{yx}) d\nu \right] + \lim_{k \to \infty} \frac{1}{k} \sum_{m=-1}^{-k} \sum_{x \in T_{[m,0], y<m}} \left[ q_2(y - x) \int (g_{xy} - h_{yx}) d\nu + q_2(x - y) \int (h_{xy} - g_{yx}) d\nu \right].
\]

We will devote the next few paragraphs to showing that these limits are in fact equal to zero.

Define the measures \( \nu^+ \) and \( \nu^- \) by choosing a subsequence \( n_j \) so that the following limits exist:
\[
\nu^+ = \lim_{j \to \infty} \frac{1}{n_j} \sum_{1 \leq x \leq n_j} \nu_x
\]
\[
\nu^- = \lim_{j \to \infty} \frac{1}{n_j} \sum_{-1 \geq x \geq n_j} \nu_x
\]
where \( \nu_x \) is the \( x \) translate of \( \nu \). In the partial sums of (12) above, for \( j \) large enough each term
\[
q_i(y - x) \int (g_{xy} - h_{yx}) d\nu
\]
appears \( |y - x| \) times when \( q_i(y - x) > 0 \), so we can write (12) as
\[
\sum_{z \in \mathbb{Z}^+} [zq_1(z) \int (g_{oz} - h_{zo}) d\nu^+ - zq_1(-z) \int (g_{zo} - h_{oz}) d\nu^+] + \sum_{z \in \mathbb{Z}^-} [-zq_2(z) \int (g_{oz} - h_{zo}) d\nu^- + zq_2(-z) \int (g_{zo} - h_{oz}) d\nu^-]
\]

Now consider two coupled processes with transition rates equal to \( q_1(z) \) and \( q_2(z) \) respectively. The measures \( \nu^+ \) and \( \nu^- \) are translation invariant and are also invariant measures for the coupled process with respect to \( q_1(z) \) and \( q_2(z) \) respectively. In particular if \( \tilde{\Omega}_i \) is the generator for the coupled process of \( q_i(z) \), \( \tilde{\Omega} \) is the generator for the coupled process of \( p(x,y) \), and
\[
f_{(A,B)} = \begin{cases} 
1 & \text{when } \eta(x) = \xi(y) = 1 \text{ for all } x \text{ in the finite set } A, y \text{ in the finite set } B \\
0 & \text{otherwise}
\end{cases}
\]
then
\[
\int \tilde{\Omega}_1 f_{(A,B)} d\nu^+ = \lim_{k \to \infty} \frac{1}{n_k} \sum_{1 \leq x \leq n_k} \int \tilde{\Omega}_1 f_{(A+x,B+x)} d\nu = \lim_{k \to \infty} \frac{1}{n_k} \sum_{1 \leq x \leq n_k} \int \tilde{\Omega} f_{(A+x,B+x)} d\nu \]
where \( A + x \) is the \( x \) translate of \( A \).
By Lemma VIII.3.2 in IPS we have $\int f_{xy}d\nu^+ = 0$ for all $x,y$. We can therefore write $\nu^+$ as $\nu^+ = \nu_1 + (1-\lambda)\nu_2$ where $\nu_1$ concentrates on $\{\eta, \xi : \eta < \xi\}$ and $\nu_2$ on $\{\eta, \xi : \eta \geq \xi\}$. Then

$$\int (g_{oz} - h_{zo})d\nu^+ = \lambda \int (g_{oz} - h_{zo})d\nu_1$$

$$= \lambda \nu_1 \{\eta, \xi : \eta(0) = 1, \eta(z) = 0\} - \nu_1 \{\eta, \xi : \eta(0) = \xi(0) = 1, \eta(z) = \xi(z) = 0\}$$

Because $\nu^+$ is translation invariant and invariant for the process with rates $q_1(z)$, $\nu_1$ is also since $\nu_1$ and $\nu_2$ are mutually singular and $\nu^+ = \nu_1 + (1-\lambda)\nu_2$. By Theorem VIII.3.9 in IPS, the marginals of $\nu_1$ are exchangeable, thus the right-hand side above is equal to a constant $c^+$ as is the expression $\int (g_{zo} - h_{zo})d\nu^+$. Similarly we have that $\int (g_{zo} - h_{zo})d\nu^+ - \int (g_{zo} - h_{zo})d\nu^-$ are equal to a constant $c^-$. Now by the mean zero assumption, we have that expression (13) is equal to 0, but since (12) and (14) are equal, we have in fact that

$$\sum_{y \in T} (p(x, y) + p(y, x)) \int f_{xy}d\nu = 0$$

(14)

for each finite $T \subset \mathbb{Z}$.

By irreducibility, if $\int f_{xy}d\nu > 0$ for some $x,y$ then $\int f_{xy}d\nu > 0$ for all $x,y$. Choose $x_0$ and $y_0$ such that $p(x_0, y_0) + p(y_0, x_0) > 0$. By (14) and the nonnegativity of $\int f_{xy}d\nu$, we must have that $\int f_{x_0y_0}d\nu = 0$ and thus $\int f_{xy}d\nu = 0$ for all $x,y$. Therefore $\nu \in I$ implies that

$$\nu\{\{(\eta, \xi) : \eta < \xi\} \text{ or } \eta \geq \xi\} = 1.$$

If $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$ we can use Theorem 2.1 to pick $\mu \in I$ and $\nu^c \in I_c$. On the other hand if $\sum_x \pi(x)/[1 + \pi(x)]^2 < \infty$ we can use the analysis in the introduction to pick $\mu \in I$ and $\nu^{(n)} \in I_c$. Since $\nu\{\{(\eta, \xi) : \eta < \xi\} \text{ or } \eta \geq \xi\} = 1$, Proposition VIII.2.13 in IPS tells us there exists a coupling with invariant measure $\nu$ where $\nu$ has marginals $\mu \leq \nu^c$ or $\mu \geq \nu^c$ in the first case and marginals $\mu \leq \nu^{(n)}$ or $\mu \geq \nu^{(n)}$ in the second case.

Take first the case where $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$. Supposing that $\mu \neq \nu^0 \neq \nu^\infty$, we have that there exists a $c_0$ for which $\nu^{c_1} \leq \mu$ for all $c_1 < c_0$ and $\mu \leq \nu^{c_2}$ for all $c_2 > c_0$. By the continuity of the one parameter family of measures $\{\nu^c\}$ it must be that $\mu = \nu^{c_0}$.

If $\sum_x \pi(x)/[1 + \pi(x)]^2 < \infty$ then we have three cases (i), (ii), and (iii) as given in the introduction. Theorem VIII.2.17 in IPS proves the first two cases so we will consider only (iii). If $\mu \neq \nu^{(-\infty)} \neq \nu^{(\infty)}$ then there exists an $n \in \mathbb{Z}$ such that either $\mu = \nu^{(n)}$ or $\nu^{(n)} < \mu < \nu^{(n+1)}$. If the latter is true then $\mu$ concentrates on $A = \{\eta : \sum_{x \in T} \eta(x) < \infty, \sum_{x \in T} [1 - \eta(x)] < \infty\}$ for some $T \subset S$ which means that it must be a mixture of stationary distributions for the Markov chains on $A_n$ as described in the introduction. But $\mu \in I$ so it must in fact be equal to some $\nu^{(n)}$ completing the proof.

We include in this section two more results which have proofs similar to that of Theorem 1.1. We first need the following definition: given transition probabilities $p(x, y)$ define the boundary of a set
Proof. Choose \( \mu \in \mathcal{I}_c \). If \( \lim_{n \to \infty} \sum_{x \in \partial T_n} \pi(x) = 0 \) then couple \( \xi_t \) with \( \eta_t \) so that they have the measures \( \mu \) and \( \nu^c \) respectively. If \( \lim_{n \to \infty} \sum_{x \in \partial T_n} 1/\pi(x) = 0 \) then couple them vice versa. We will prove the case in which \( \lim_{n \to \infty} \sum_{x \in \partial T_n} \pi(x) = 0 \). The other case follows similarly.

By (10),

\[
\sum_{x \in T_n, y \in S} [p(x, y) + q(x, x)] \int f_{yx} d\nu = \sum_{x \in T_n, y \notin T_n} p(x, y) \int (g_{xy} - h_{yx}) d\nu + \sum_{x \in T_n, y \notin T_n} p(y, x) \int (h_{xy} - g_{yx}) d\nu.
\]

Just as in the above proof, the left-hand side of this equation is increasing in \( n \) so that a limit exists as \( n \to \infty \). The right-hand side above goes to 0 as \( n \to \infty \) since

\[
\sum_{x \in T_n, y \notin T_n} p(x, y) \int (g_{xy} - h_{yx}) d\nu + \sum_{x \in T_n, y \notin T_n} p(y, x) \int (h_{xy} - g_{yx}) d\nu
\]

\[
\leq \sum_{x \in T_n, y \notin T_n} \int f_{yx} d\nu + \sum_{x \in T_n, y \notin T_n} \int f_{xy} d\nu
\]

\[
\leq C \sum_{y \in \partial T_n} \int f_{yx} d\nu + \sum_{y \in \partial T_n} \int f_{yx} d\nu \leq C \sum_{y \in \partial T_n} \pi(y) + \sum_{y \in \partial T_n} \pi(y).
\]

Here \( C = \sup_y \sum_x p(x, y) \) which is finite by the assumptions in the introduction.

Irreducibility now gives us \( \int f_{xy} d\nu = 0 \) for all \( x, y \). The rest of the proof just follows that of Theorem 1.1. \( \square \)

Note that if we change the hypothesis \( \sum_x \pi(x)/|1 + \pi(x)|^2 = \infty \) to \( \sum_x \pi(x)/|1 + \pi(x)|^2 < \infty \) then Theorem VIII.2.17 in IPS says that \( \mathcal{I}_c = \{ \nu^{(n)} : 0 \leq n \leq \infty \} \).

**Corollary 3.2.** If in Theorem 1.1 we replaced condition 3 with the condition that \( p(x, y) \) has finite range, \( \lim_{x \to \infty} p(x, x + z) = q_1(z) \), and \( \lim_{x \to -\infty} \pi(x) \) equals 0 or \( \infty \) (or alternatively \( \lim_{x \to -\infty} p(x, x + z) = q_2(z) \), and \( \lim_{x \to -\infty} \pi(x) \) equals 0 or \( \infty \) then the result still holds.

**Proof.** Replace expression (12) in the proof of Theorem 1.1 with

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{n=1}^{k} \sum_{x \in [n, n+1], y > n} \left[ q_1(y - x) \int (g_{xy} - h_{yx}) d\nu + q_1(x - y) \int (h_{xy} - g_{yx}) d\nu \right]
\]

\[
+ \lim_{m \to -\infty} \sum_{x \in T_m: 0, y < m} \left[ p(x, y) \int (g_{xy} - h_{yx}) d\nu + \pi(x) \int (h_{xy} - g_{yx}) d\nu \right].
\]
The proofs of Theorem 1.1 and Proposition 3.1 imply that this expression is 0. The rest is proven above.

Before moving on to the next section let us discuss what the above results tell us in the case where \( p(x, y) \) has finite range on \( \mathbb{Z} \). Proposition 3.1 together with Theorem VIII.2.17 in IPS says that if \( \lim_{|x| \to \infty} \pi(x) \) equals 0 or \( \infty \) then the reversible measures are the only invariant measures. If the limits \( \lim_{x \to \infty} \pi(x) \) and \( \lim_{x \to -\infty} \pi(x) \) exist and one of them is nonzero and finite, then the combination of Theorem 1.1 and Corollary 3.2 imply that the only invariant measures are the reversible ones. All together we have the following: if \( \pi(x) \) exists and has limits in both directions for the finite range exclusion process on \( \mathbb{Z} \), then unless the limit is 0 in one direction and \( \infty \) in the other direction, the only invariant measures are the reversible ones. Of course, as seen in an example in the introduction, it is also possible to have \( \lim_{x \to \infty} p(x, x + z) = q_1(z) \) and \( \lim_{x \to -\infty} p(x, x + z) = q_2(z) \) as given in Theorem 1.1 and at the same time have the limit of \( \pi(x) \) to be 0 in one direction, \( \infty \) in the other. In those cases Theorem 1.1 rules out nonreversible invariant measures. A similar comment can be made for Corollary 3.2. We remind the reader, however, that if the transition probabilities are translation invariant with a drift so that the limit of \( \pi(x) \) is 0 in one direction and \( \infty \) in the other direction, then Liggett(1976) tells us that \( \{ \nu_\rho : 0 \leq \rho \leq 1 \} \) is a class of nonreversible invariant measures.

4 The nearest-neighbor process on \( \mathbb{Z} \)

We now restrict our attention to the nearest-neighbor case. More specifically, assume throughout this section that we are dealing with the irreducible nearest-neighbor exclusion process on \( \mathbb{Z} \) \( (p(x, y) = 0 \) if and only if \( |x - y| > 1 \) ). In this case, a reversible \( \pi(x) \) always exists so we need not make this assumption. Similar to the discussion at the end of the last section, we will show that if \( \inf_{|x - y| = 1} p(x, y) > 0 \) then the only possible nonreversible measures are in the case where the limit of \( \pi(x) \) is 0 in one direction and \( \infty \) in the other direction.

In order to prove the next two propositions we need the following lemma which appears in a slightly different form as Corollary 5.2 in Liggett(1976):

**Lemma 4.1 (Liggett).** If \( \inf_{|x - y| = 1} p(x, y) > 0 \) and \( \nu \in \tilde{\mathcal{I}}_\nu \), then exactly one of the following holds:

1. \( \nu\{ (\eta, \xi) : \eta = \xi \} = 1 \),
2. \( \nu\{ (\eta, \xi) : \eta \leq \xi, \eta \neq \xi \} = 1 \),
3. \( \nu\{ (\eta, \xi) : \eta \geq \xi, \eta \neq \xi \} = 1 \),
4. \( \nu(B) = 1 \),
5. \( \nu\{ (\eta, \xi) : (\xi, \eta) \in B \} = 1 \),

where \( B = \{ (\eta, \xi) : \exists x \in \mathbb{Z} \text{ such that } \eta(y) \leq \xi(y) \text{ for all } y < x, \eta(y) < \xi(y) \text{ for some } y < x, \eta(z) \geq \xi(z) \text{ for all } z \geq x, \eta(z) > \xi(z) \text{ for some } z \geq x \} \).
Proposition 4.2. If \( \inf_{|x-y|=1} \rho(x, y) > 0 \) and \( \pi(x) \) has some finite, nonzero limit point as \( x \) goes to \( \infty \) and some finite, nonzero limit point as \( x \) goes to \( -\infty \), then \( \mathcal{I}_c = \{ \nu^c : 0 \leq c \leq \infty \} \).

**Proof.** The assumptions imply that \( \sum_x \pi(x)/(1 + \pi(x))^2 = \infty \) so Theorem 2.1 tells us \( \mathcal{I}_c \subset \{ \nu^c : 0 \leq c \leq \infty \} \). We will show the reverse containment.

Choose a sequence \( \{n_k\} \) extending in both directions so that finite, nonzero limits of \( \pi(n_k) \) exist. For a measure \( \mu \) on \( \{0, 1\}^\infty \) the set of limit points \( L_+ \) of \( \{\mu(\xi(n_k) = 1), k > 0\} \) satisfies one of the following properties:

(i) \( L_+ = \{1\} \) or \( L_+ = \{0\} \).

(ii) \( L_+ = \{1, 0\} \).

(iii) \( L_+ \) contains some limit point between 0 and 1.

The same is true for the set of limit points \( L_- \) of \( \{\mu(\xi(n_k) = 1), k < 0\} \).

Now suppose we couple \( \nu^c \) with another extremal invariant measure \( \mu_c \), the two measures corresponding to the processes \( \eta \) and \( \xi \), respectively. Since Theorem 2.1 tells us \( \nu^c \) is extremal, Section VIII.2 in IPS implies there exists a coupling measure such that \( \nu \in \mathcal{I}_c \).

If \( \mu_c \) satisfies condition (i) for both \( L_+ \) and \( L_- \), then there are two possibilities: either \( L_+ = L_- \) or \( L_+ \neq L_- \). Suppose first that \( L_+ = L_- = \{1\} \) for \( \mu_c \). In this case we have that \( \mu_c(\xi(z) = 1) < 1 \) for some \( z \) then we can choose \( c < \infty \) large enough so that \( \nu^c(\eta(z) = 1) > \mu_c(\xi(z) = 1) \). But this contradicts the assumption that \( \nu^c(\eta(n_k) = 1) = c\pi(n_k)/[1 + c\pi(n_k)] \) has limits less than 1 for \( k \) going to \( \infty \) and \( -\infty \). To see this suppose the coupling measure satisfies \( \nu(B) = 1 \) as defined in Lemma 4.1. Given

\[
0 < \epsilon < 1 - \lim_{k \to \infty} c\pi(n_k)/[1 + c\pi(n_k)]
\]

we can choose \( K \) large enough so that

\[
1 - \epsilon < \nu((\eta, \xi) : \exists x < K \text{ such that } \eta(y) \leq \xi(y) \forall y < x, \eta(y) < \xi(y) \text{ for some } y < x,
\eta(z) \geq \xi(z) \forall z \geq x, \eta(z) > \xi(z) \text{ for some } z \geq x).
\]

This, however, contradicts \( L_+ = 1 \). Similarly we cannot have that \( \nu((\eta, \xi) : (\xi, \eta) \in B) = 1 \). So Lemma 4.1 tells us that \( \eta \leq \xi \) which contradicts \( \nu^c(\eta(z) = 1) > \mu_c(\xi(z) = 1) \). It must be that \( \mu_c = \nu^\infty \). A similar argument shows that if \( L_+ = L_- = \{0\} \) for \( \mu_c \) then \( \mu_c = \nu^0 \).

Consider the second case where \( L_+ \neq L_- \); without loss of generality we will assume that \( L_- = \{0\} \).

We claim that given \( \epsilon > 0 \), we can find \( n \) such that \( \mu_c(\xi(n) = 0) < \epsilon \) and \( \mu_c(\xi(n+1) = 0) < \epsilon \). To see this suppose that for some \( \epsilon > 0 \) there exists no \( n \) for which this is true. Then since \( L_+ = \{1\} \), there are infinitely many \( x > 0 \) for which \( \mu_c(\xi(x) = 0) < \epsilon/4 \) and infinitely many \( y > 0 \) for which \( \mu_c(\xi(y) = 0) \geq \epsilon \). Choosing \( \nu^c \) so that \( \lim_{k \to \infty} c\pi(n_k)/[1 + c\pi(n_k)] = 1 - \epsilon/2 \) gives us a contradiction to Lemma 4.1 and thus proves the claim.

Given the same \( \epsilon > 0 \) we can choose \( m < n \) so that \( \mu_c(\xi(m-1) = 1) < \epsilon \). Since we have that
for all \( \mu \) either

By Lemma 4.1 either \( \mu \) satisfies condition (ii) for either \( c < c_0 \leq \infty \) so that

\[
\sum_{m \leq x \leq n, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yx} d\nu
\]

\[
= \sum_{x=m \text{ or } n, y=m-1 \text{ or } n+1} [p(x, y) \int (g_{xy} - h_{yx}) d\nu + p(y, x) \int (h_{xy} - g_{yx}) d\nu]
\]

which is increasing in \( n \) and \( -m \).

Due to our choice of \( m \) and \( n \) above, \( \int h_{n,n+1} d\nu < \epsilon \) and \( \mu \{ \xi(m-1) = 1 \} < \epsilon \); moreover \( P(A) - P(A \cap B \cap C) \leq P(B^c) + P(C^c) \) implies that \( \nu^c \{ \eta(n+1) = 1, \eta(n) = 0 \} - \int g_{n+1,n} d\nu < 2\epsilon \)

so that

\[
\sum_{m \leq x \leq n, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yx} d\nu
\]

\[
< p(n, n+1) \int g_{n,n+1} d\nu - p(n+1, n) \int g_{n+1,n} d\nu + 3\epsilon
\]

\[
< p(n, n+1) \nu^c \{ \eta(n) = 1, \eta(n+1) = 0 \}
\]

\[
- p(n+1, n) \nu^c \{ \eta(n) = 0, \eta(n+1) = 1 \} + 5\epsilon.
\]

By the reversibility of \( \nu^c \)

\[
p(n, n+1) \nu^c \{ \eta(n) = 1, \eta(n+1) = 0 \} = p(n+1, n) \nu^c \{ \eta(n) = 0, \eta(n+1) = 1 \}
\]

so equation (16) is in fact equal to 0. Since we have assumed here that \( L_- = \{0\} \) and \( L_+ = \{1\} \) for \( \mu_e \), then choosing \( 0 < c < \infty \) gives us a contradiction.

Suppose \( \mu_e \) satisfies condition (ii) for either \( L_+ \) or \( L_- \) so that either \( L_+ = \{0,1\} \) or \( L_- = \{0,1\} \). Choose \( \nu^c \) with \( 0 < c < \infty \). Again we contradict Lemma 4.1.

Combining all the above arguments we have that either \( \mu_e = \nu^0, \mu_e = \nu^\infty \) for some \( c \) satisfying (iii) in some direction. Assuming the latter we can, without loss of generality, choose \( 0 < c_0 < \infty \) so that

\[
\lim_{k \to \infty} c_0 \pi(n_k) / [1 + c_0 \pi(n_k)] = \lim_{l \to \infty} \mu_e \{ \xi(n_k) = 1 \}.
\]

For all \( c > c_0 \),

\[
\lim_{k \to \infty} c \pi(n_k) / [1 + c \pi(n_k)] > \lim_{l \to \infty} \mu_e \{ \xi(n_k) = 1 \}.
\]

By Lemma 4.1 either \( \mu_e \leq \nu^c \) or \( \nu(B) = 1 \) where \( B \) is defined in the lemma. Similarly, for all \( c < c_0 \), either \( \mu_e \geq \nu^c \) or \( \nu \{ (\xi, \eta) : (\xi, \eta) \in B \} = 1 \). Combining these two arguments gives \( \nu^{c_1} \leq \mu_e \leq \nu^{c_2} \) for all \( c_1 < c_0 < c_2 \). By the continuity of the one parameter family of measures \( \nu^c \), \( \mu_e = \nu^0 \). □

**Proposition 4.3.** If \( \inf_{|x-y|=1} p(x, y) > 0 \), then \( \lim_{x \to -\infty} \pi(x) = \infty \), and \( \pi(x) \) has a finite, nonzero limit point as \( x \) goes to \( -\infty \), then \( \mathcal{I}_e = \{ \nu^c : 0 \leq c \leq \infty \} \).

**Proof.** Again, by Theorem 2.1 we need only show that \( \mathcal{I}_e \subset \{ \nu^c : c \in [0, \infty] \} \).

We argue first that without loss of generality we can assume the limit points of \( \{ \pi(x), x < 0 \} \) are bounded above. Assume to the contrary that \( \infty \) is a limit point. Then for any \( R > 0 \) we can
Using the argument at the end of the proof of Proposition 4.2, we have that for all 
contradiction to \( L \)
The same is true for the set \( c > c \)
exists. Since we can assume that the limit points of \( \pi(x), x < 0 \) are bounded above.
Couple \( \nu^e \) with another extremal invariant measure \( \mu_e \), the two measures corresponding to the 
processes \( \eta_t \) and \( \xi_t \) respectively. As argued above there exists a coupling measure such that \( \nu \in \mathcal{I}_e \).
Let \( L^- \) be the the set of limit points of \( \{ \mu_e \{ \xi(x) = 1 \}, x < 0 \} \). Note that \( L^- \) is slightly different 
from \( L_- \) described in Proposition 4.2 in that \( L_- \) is the set of limit points for a subset of \( \{ \mu_e \{ \xi(x) = 1 \}, x < 0 \} \). \( L^- \) satisfies one of the following properties:

(i) \( L^- \) contains some point between 0 and 1.
(ii) \( L^- = \{ 1, 0 \} \).
(iii) \( L^- = \{ 1 \} \).
(iv) \( L^- = \{ 0 \} \).

The same is true for the set \( L^+ \) of limit points of \( \{ \mu_e \{ \xi(x) = 1 \}, x > 0 \} \).

Suppose \( L^- \) satisfies (i). Choose a sequence \( x_n \to -\infty \) so that \( 0 < \lim_{n \to \infty} \mu_e \{ \xi(x_n) = 1 \} < 1 \) exists. Since we can assume that the limit points of \( \{ \pi(x), x < 0 \} \) are all finite, there exists a 
subsequence \( \{ x_{n_k} \} \) such that \( \lim_{k \to \infty} \pi(x_{n_k}) < \infty \) exists.

Consider the two cases where \( \lim_{k \to \infty} \pi(x_{n_k}) = 0 \) and \( \lim_{k \to \infty} \pi(x_{n_k}) > 0 \). Assume the 
latter case first. Choose \( 0 < c_0 < \infty \) so that

\[
\lim_{k \to \infty} c_0 \pi(x_{n_k})/[1 + c_0 \pi(x_{n_k})] = \lim_{n \to \infty} \mu_e \{ \xi(x_n) = 1 \}.
\]

For all \( c > c_0 \),

\[
\lim_{k \to \infty} c \pi(x_{n_k})/[1 + c \pi(x_{n_k})] > \lim_{n \to \infty} \mu_e \{ \xi(x_n) = 1 \}.
\]

Using the argument at the end of the proof of Proposition 4.2 we have that for all \( c_1 < c_0 < c_2 \), 
\( \nu^{c_1} \leq \mu_e \leq \nu^{c_2} \). Consequently, it must be that \( \mu_e = \nu^{c_0} \).

Now assume that \( \lim_{k \to \infty} \pi(x_{n_k}) = 0 \) so that for all \( 0 < c < \infty \) the coupling satisfies either 
\( \nu^c \leq \mu_e \) or \( \nu \{ B \} = 1 \) where \( B \) is given in Lemma 4.4. If \( \nu^c \leq \mu_e \) for all \( 0 < c < \infty \) then \( \mu_e = \nu^\infty \), a 
contradiction to \( L^- \) satisfying (i). So it must be that \( \nu \{ B \} = 1 \).

We claim that for any \( r < 1 \) there exists \( m < 0 \) such that \( \mu_e \{ \xi(m) = 1 \} > r \) and \( \mu_e \{ \xi(m-1) = 1 \} > r \). By the hypothesis of the theorem we can choose a sequence \( \{ x_l \} \) going to \( -\infty \) so that 
\( 0 < \lim_{l \to \infty} \pi(x_l) < \infty \) exists. If \( \inf_{|x-y|=1} p(x, y) > p \) then choose \( c \) so that

\[
\lim_{l \to \infty} \frac{c \pi(x_l)}{1 + c \pi(x_l)} > r + \frac{1 - r}{2}.
\]

Since \( \pi(x_l - 1) > r \pi(x_l) \), it follows that the set of limit points of \( \{ \frac{c \pi(x_l)}{1 + c \pi(x_l-1)} \}, l > 0 \) is bounded 
below by \( r + \frac{1 - r}{2} \). Now since \( \nu \{ B \} = 1 \) there exists a \( K \) such that \( l > K \) implies \( \mu_e \{ \xi(x_l) = 1 \} > r \) 
and \( \mu_e \{ \xi(x_l - 1) = 1 \} > r \) which proves the claim.
Since we have that $\nu \in \hat{\nu}$, then $\int \hat{\nu}(\sum_{x \in T} f_x) d\nu = 0$ for each finite $T \subset \mathbb{Z}$. By (10),

$$\sum_{m \leq x \leq n, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yz} d\nu = \sum_{x = m, y = m - 1 \text{ or } n + 1} [p(x, y) \int (g_{xy} - h_{yz}) d\nu + p(y, x) \int (h_{xy} - g_{yz}) d\nu]$$

which is increasing in $n$ and $-m$.

Using the claim above along with the fact that $\lim_{x \to \infty} \pi(x) = \infty$, we can argue just as we argued in the case where $L_+ \neq L_+$ of (i) in Proposition 4.2 to get

$$\sum_{m \leq x \leq n, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yz} d\nu < p(m, m - 1) \int g_{m, m - 1} d\nu - p(m - 1, m) \int g_{m - 1, m} d\nu + 3\epsilon$$

By the reversibility of $\nu^c$, the left-hand side is just $5\epsilon$, but this contradicts $\nu(B) = 1$ for small $\epsilon$.

Suppose $L^-$ satisfies condition (ii). Choosing $\nu^c$ with $0 < c < \infty$ gives us a contradiction to Lemma 4.1.

If $L^-$ satisfies condition (iii) then we will handle the two cases (a) $L^+ = \{1\}$ and (b) $L^+ \neq \{1\}$. Consider case (a) first. If we switch the coupling so that $\mu_c$ corresponds to $\eta$ and $\nu^c$ corresponds to $\xi$, then we have that the left-hand side of the following inequality goes to 0:

$$\sum_{|x| = n, |y| = n + 1} (p(x, y) + p(y, x)) \int f_{yz} d\nu$$

By (10) and by irreducibility we get $\int f_{xyz} d\nu = 0$ for all $x, y$. Therefore the measure $\mu_c$ must lie stochastically above all $\nu^c$ for all finite $c$ and must therefore be equal to $\nu^\infty$.

If (b) holds then we refer the reader to the argument given above in the case where $L^-$ satisfies (i) and $\lim_{k \to \infty} \pi(x_{n_k}) = 0$.

Finally suppose that (iv) holds so that $L^- = \{0\}$. If $L^+$ satisfies (i) or (ii) then by Lemma 4.1 $\mu_c \leq \nu^c$ for all $c > 0$ so that $\mu_c = \nu^0$, a contradiction. If $L^+$ satisfies (iv) then similarly $\mu_c = \nu^0$. Let $L^+$ satisfy (iii) so that $L^+ = \{1\}$. For a given $z$ choose $c$ small enough so that $\nu^c(\eta(z) = 1) < \mu_c(\xi(z) = 1)$. We thus have that $\nu(\eta, \xi) : (\xi, \eta) \in B = 1$ as given in Lemma 4.1. But by (10) and (17), for a given $\epsilon > 0$ we can find $-m$ and $n$ large enough so that

$$\sum_{m \leq x \leq n, y \in \mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yz} d\nu < \epsilon$$

which of course contradicts $\nu((\eta, \xi) : (\xi, \eta) \in B) = 1$. \qed
Proof of Theorem 1.2. Note first that since \( \inf_{|x-y|=1} p(x,y) > 0 \) then it cannot be that \( \mathcal{L}^- \) or \( \mathcal{L}^+ \) is equal to \( \{0, \infty\} \). In light of this fact, if either \( \mathcal{L}^- \) or \( \mathcal{L}^+ \) contains a finite, nonzero point then Proposition 4.1 and analogs of Proposition 4.3 imply there are no nonreversible measures. If \( \mathcal{L}^+ = \mathcal{L}^- = \{0\} \) or \( \mathcal{L}^+ = \mathcal{L}^- = \{\infty\} \) then Proposition 3.1 implies there are no nonreversible measures.

5 A result concerning domains of attraction

Theorem 5.1. Let \( \sum_x \pi(x)/[1 + \pi(x)]^2 = \infty \) and let \( \omega \) be a probability measure on \([0, \infty]\). Also, assume that \( \nu_c \) is a family of invariant measures indexed by \( c \geq 0 \) each of which is in \( \mathcal{L}_c \). Suppose \( \{\mu_c\} \) is a family of probability measures on \( \{0, 1\}^S \) such that for each \( 0 \leq c \leq \infty, \mu_c \) is absolutely continuous with respect to \( \nu_c \). If

\[
\mu = \int_0^\infty \mu_c \omega(dc) \text{ and } \nu = \int_0^\infty \nu_c \omega(dc)
\]

then \( \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu S(t) dt \) exists and is equal to \( \nu \).

Proof. For a fixed \( c \) we first prove that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_c S(t) dt = \nu_c.
\]

By the compactness of \( \mathcal{P} \) we can choose a sequence of times such that

\[
\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \mu_c S(t) dt
\]

converges in distribution to some measure \( \lambda \). Pick a continuous (and therefore bounded) function \( f \) on \( \{0, 1\}^S \) with \( \|f\| \leq 1 \) and let \( g \) be the Radon-Nikodym derivative of \( \mu_c \) with respect to \( \nu_c \). Given \( \epsilon > 0 \) we have that for \( n \) large enough

\[
\frac{1}{t_n} \int_0^{t_n} (S(t) f) g \nu_c dt - \int f d\lambda < \epsilon/3.
\]

We can choose a simple function

\[
\hat{g} = \sum_{k=1}^N c_k 1_{E_k}
\]

approximating \( g \) such that \( \cup_k E_k = \{0, 1\}^S, \hat{g} \geq 0, \int \hat{g} d\nu_c = 1, \) and \( \int |g - \hat{g}| d\nu_c < \epsilon/3. \) Since \( \|S(t)f\| \leq \|f\| \leq 1 \) this gives us

\[
\frac{1}{t_n} \int_0^{t_n} (S(t) f) g \nu_c dt - \frac{1}{t_n} \int_0^{t_n} (S(t) f) \hat{g} d\nu_c dt | \leq \int |g - \hat{g}| d\nu_c < \epsilon/3.
\]
Without loss of generality we can henceforth assume that \( \nu_c(E_k) > 0 \) for each \( k \). Define the measure \( \mu_k \) concentrating on \( E_k \) by letting
\[
\mu_k(A) = \frac{\nu_c(A)}{\nu_c(E_k)}
\]
for all \( A \subset E_k \) and \( \mu_k = 0 \) otherwise. If we think of \( \hat{g} \) as the Radon-Nikodym derivative of some measure \( \lambda \) with respect to \( \nu_c \) then we can write
\[
\sum_{k=1}^{N} \nu_c(E_k)\mu_k = \nu_c \quad \text{and} \quad \sum_{k=1}^{N} c_k\nu_c(E_k)\mu_k = \lambda.
\]
We can now find a subsequence \( \{t_{n_l}\} \) such that the following limits exist for each \( k \):
\[
\lim_{l \to \infty} \frac{1}{t_{n_l}} \int_{0}^{t_{n_l}} \mu_k S(t) dt = \nu_k.
\]
Moreover, Proposition I.1.8 in IPS tells us \( \nu_k \in \mathcal{I} \). Since \( \nu_c \) is extremal invariant and since \( \sum_{k \geq 1} \nu_c(E_k)\nu_k = \nu_c \), it must be that \( \nu_k = \nu_c \) for each \( k \). This then yields
\[
\sum_{k=1}^{N} c_k\nu_c(E_k)\nu_k = \lim_{l \to \infty} \frac{1}{t_{n_l}} \int_{0}^{t_{n_l}} \lambda_c S(t) dt = \nu_c
\]
which gives us
\[
|\frac{1}{t_{n_l}} \int_{0}^{t_{n_l}} (S(t)f)\hat{g} \, d\nu_c dt - \int f \, d\nu_c| < \epsilon/3
\]
for \( l \) large enough.

Combining the three inequalities we have
\[
|\int f \, d\lambda - \int f \, d\nu_c| < \epsilon.
\]
But \( \epsilon > 0 \) is arbitrary so it must be that \( \int f \, d\lambda = \int f \, d\nu_c \) for each continuous \( f \) with \( \|f\| \leq 1 \) which implies that \( (20) \) is equal to \( \nu_c \). Now let \( M_n \) be the closure of the set of measures
\[
\left\{ \frac{1}{T} \int_{0}^{T} \mu_c S(t) dt : T \geq n \right\}.
\]
Using the compactness of \( \mathcal{P} \) along with the fact that \( \{t_n\} \) is an arbitrary sequence of times causing convergence in \( (20) \), we have that \( \bigcap_{n \in \mathbb{N}} M_n = \nu_c \) proving \( (19) \).

To finish the proof note that since \( \|S(t)f\| \leq \|f\| \), we can use the Dominated Convergence Theorem together with Fubini’s Theorem to show that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \int_{0}^{\infty} S(t) f \, d\mu_c \omega(\omega) dt = \int f \, d\nu.
\]
Corollary 5.2. Suppose $\sum_x \pi(x)/[1 + \pi(x)]^2 = \infty$. If $\sum_x |\alpha(x) - \beta(x)| < \infty$ then
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \nu_\alpha S(t) dt = \nu^c. \tag{21}
\]

Proof. Let $\beta(x) = \frac{e\pi(x)}{1 + e\pi(x)}$, $m_x = \min\{\alpha(x), \beta(x)\}$, and $M_x = \max\{\alpha(x), \beta(x)\}$. We then have
\[
1 - |\alpha(x) - \beta(x)| = 1 - M_x + m_x \\
\leq [(1 - M_x)(1 - m_x)]^{1/2} + (m_x M_x)^{1/2} \\
= [(1 - \alpha(x))(1 - \beta(x))]^{1/2} + (\alpha(x)\beta(x))^{1/2}.
\]

Since $\sum_x |\alpha(x) - \beta(x)| < \infty$ then
\[
\prod_x [(\alpha(x)\beta(x))^{1/2} + [(1 - \alpha(x))(1 - \beta(x))]^{1/2}] \geq \prod_x (1 - |\alpha(x) - \beta(x)|) > 0.
\]

An application of Kakutani’s Dichotomy tells us that $\nu_\alpha$ is absolutely continuous with respect to $\nu^c$ which completes the proof. \qed

We remark here that if $\alpha(x)$ and $\beta(x)$ are both bounded away from 0 and 1 then Kakutani’s Dichotomy tells us that $\sum_x [\alpha(x) - \beta(x)]^2 < \infty$ is a necessary and sufficient condition for $\nu_\alpha$ to be absolutely continuous with respect to $\nu^c$ (e.g. page 245 of Durrett(1996)).

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References

[1] Andjel, E. D. (1986) Convergence to a nonextremal equilibrium measure in the exclusion process. *Probab. Th. Rel. Fields*, 73, 127-134.

[2] Andjel, E. D., Bramson, M. D., Liggett, T. M. (1988) Shocks in the asymmetric exclusion process. *Probab. Th. Rel. Fields*, 78, 231-247.

[3] Bramson, M. D., Liggett, T. M., Mountford, T. S. (2002) Characterization of stationary measures for one dimensional exclusion processes. *Ann. Probab.*, 30.

[4] Durrett, R. (1996) *Probability: Theory and Examples*, second edition. Duxbury Press, Belmont, CA.

[5] Janowski, S. A., Lebowitz, J. L. (1994) Exact results for the asymmetric simple exclusion with a blockage. *J. Statist. Phys.*, 77, 35-51.
REFERENCES

[6] Kipnis, C., Varadhan, S. R. S. (1986) Central limit theorem for additive functionals of reversible Markov processes. *Comm. Math. Phys.*, 104, 1-19.

[7] Liggett, T. M. (1976) Coupling the simple exclusion process. *Ann. Probab.*, 4, 339-356.

[8] Liggett, T. M. (1985) *Interacting Particle Systems*. Springer-Verlag, New York.

[9] Liggett, T. M. (1999) *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes*. Springer-Verlag, Berlin Heidelberg.

[10] Sethuraman, S. (2001) On extremal measures for conservative particle systems. *Ann. Inst. H. Poincare*, 37, 139-154.