CONTINUOUS QUIVERS OF TYPE $A$ (IV)
CONTINUOUS MUTATION AND GEOMETRIC MODELS OF E-CLUSTERS

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ABSTRACT. This is the final paper in the series Continuous Quivers of Type $A$. In this part, we generalize existing geometric models of type $A$ cluster structures to the new $E$-clusters introduced in part (III). We also introduce an isomorphism of cluster theories and a weak equivalence of cluster theories. Examples of both are given. We use the geometric models and isomorphisms of cluster theories to begin classifying continuous type $A$ cluster structures. We also introduce a continuous generalization of mutation. This encompasses mutation and (infinite) sequences of mutation. Then we link continuous mutation to our earlier geometric models. Finally, we introduce the space of mutations which generalizes the exchange graph of a cluster structure.

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INTRODUCTION

History. Cluster algebras were first introduced by Fomin and Zelevinsky in [11]. In particle physics they can be used to study scattering diagrams (see work of Golden, Goncharov, Spradlin, Vergu, and Volovich in [12]). The structure of cluster algebras was first categorized independently by two teams in 2006: Buan, Marsh, Reineke, Reiten, and Todorov in [8] and Caldero, Chapoton, and Schiffler in [9]. The first team’s construction provided a way to construct a cluster category from the category of finitely generated representations of a Dynkin quiver and the second team’s construction related the category to a geometric model. This geometric model on a polygon was extended to the infinity-gon by Holm and Jørgensen and the completed infinity-gon by Baur and Graz in [14] and [5], respectively. In [10], Fomin, Shapiro, and Thurston expanded on [9] and studies the relationship between triangulated surfaces and cluster algebras. We refer the reader to [1, Chapter 4.1] for the state of the art at the time of writing. A continuous construction, both categorically and geometrically, was introduced by Igusa and Todorov in [18]. Structures relating to clusters are still activity studied ([3, 25, 21, 22]). In particular, continuous structures were studied by Arkani-Hamed, He, Salvatori, and Thomas in [2] and by Kulkarni, Matherne, Mousavand and the author in [20].

In Part (I) of this series Igusa, Todorov, and the author introduced continuous quivers of type $A$, denoted $A_{R,S}$, which generalize quivers of type $A$ [15]. Results about decomposition of pointwise finite-dimensional representations of such a quiver and the category of finitely-generated representations (denoted $\text{rep}_k(A_{R,S})$) were proven. In Part (II) the author generalized the Auslander–Reiten quiver for finitely-generated representations of an $A_n$ quiver and its bounded derived category to
the Auslander–Reiten space for \( \operatorname{rep}_k(A_{\mathbb{R},S}) \) and its bounded derived category, denoted \( \mathcal{D}^b(A_{\mathbb{R},S}) \) [23]. Results were proven about constructions of extensions in \( \operatorname{rep}_k(A_{\mathbb{R},S}) \) and distinguished triangles in \( \mathcal{D}^b(A_{\mathbb{R},S}) \) in relation to the Auslander–Reiten space. In Part (III) Igusa, Todorov, and the author used Parts (I) and (II) to classify which continuous quivers of type \( A \) are derived equivalent, construct the new continuous cluster category (denoted \( \mathcal{C}(A_{\mathbb{R},S}) \)) with \( \mathbf{E} \)-clusters, and generalize the notion of cluster structures to cluster theories [16]. It was shown that each element in an \( \mathbf{E} \)-cluster has none or one choice of mutation and the result of mutation yielded another \( \mathbf{E} \)-cluster. It was also shown that some type \( A \) cluster theories (recovered from existing cluster structures) can be embedded in this new cluster theory.

**Contributions.** The final part of this series begins with a review of the relevant parts of the previous works. Then, we define an isomorphism of cluster theories and a weak equivalence of cluster theories (Definition 2.0.1). In Sections 2.1 and 2.2, we construct geometric models of \( \mathbf{E} \)-cluster theories from part (III) of this series [16]. We obtain an additive category \( \mathcal{C}(A_{\mathbb{R},S}) \) and a pairwise compatibility condition \( \mathbf{N}_{R,S} \) on its indecomposables that induces the cluster theory \( \mathcal{T}_{R,S}(\mathcal{C}(A_{\mathbb{R},S})) \). The purpose of the geometric models is to generalize triangulations of polygons and ideal triangulations of the hyperbolic plane, which encode several existing type \( A \) cluster structures [9, 18]. In particular, we want a connection to the cluster theory \( \mathcal{T}_E(\mathcal{C}(A_{\mathbb{R},S})) \) from [16]. We prove that the geometric models are “correct” in Theorem A and then use them to prove Theorem B.

**Theorem A** (Theorems 2.1.8 and 2.2.16). Let \( A_{\mathbb{R},S} \) be a continuous quiver of type \( A \). The pairwise compatibility condition \( \mathbf{N}_{R,S} \) induces the \( \mathcal{N}_{R,S} \)-cluster theory of \( \mathcal{C}(A_{\mathbb{R},S}) \) and there is an isomorphism of cluster theories \( (F, \eta) : \mathcal{T}_{N_{R,S}}(\mathcal{C}(A_{\mathbb{R},S})) \to \mathcal{T}_E(\mathcal{C}(A_{\mathbb{R},S})) \).

**Theorem B** (Corollary 2.3.7). Let \( A_{\mathbb{R},S} \) and \( A_{\mathbb{R},R} \) be continuous quivers of type \( A \) such that one of the following is true: (i) \( |S| = |R| \) and \( |S| < \infty \), (ii) \( S \) and \( R \) are both bounded on exactly one side, or (iii) both \( S \) and \( R \) are indexed by \( \mathbb{Z} \). Then \( \mathcal{T}_E(\mathcal{C}(A_{\mathbb{R},S})) \cong \mathcal{T}_E(\mathcal{C}(A_{\mathbb{R},R})) \).

In Section 2.4 we use the geometric models to show how one may visualize \( \mathbf{E} \)-mutations. Some of these pictures are different from the usual “swap diagonals on a quadrilateral” that appears for triangulations of polygons and ideal triangulations of the hyperbolic plane.

In Section 3 we define a continuous generalization of mutation (Definition 3.1.2) with two key motivations. The first is to unify various ways of describing a sequence of mutations (possibly infinite as in [5]). In Part (III), Igusa, Todorov, and the author show that the indecomposable objects that were projective in \( \operatorname{rep}_k(A_{\mathbb{R}}) \) form an \( \mathbf{E} \)-cluster but many of the elements are not \( \mathbf{E} \)-mutable [16, Examples 4.3.2 and 4.4.1]. The second motivation for continuous mutation is to work around this obstruction so that we may mutate the cluster of projectives into the cluster of injectives as one usually does for type \( A_n \). In Section 3.4 we show how mutations and continuous mutations can be interpreted with these geometric models.

We use continuous mutation to define mutation paths (Definition 3.3.2) and generalize the exchange graph of a cluster structure to the space of mutations for a cluster theory (Definition 3.5.2). For a cluster theory \( \mathcal{T}_P(\mathcal{C}) \), we denote its space of mutations by \( \mathbf{P}(\mathcal{C}) \).

**Theorem C** (Propositions 3.5.3, 3.5.5, and 3.5.6). Let \( \mathcal{T}_P(\mathcal{C}) \) be a cluster theory and \( \mathbf{P}(\mathcal{C}) \) its the space of mutations. Then \( \mathbf{P}(\mathcal{C}) \) is a non-Hausdorff topological space where each path begins and ends at a \( \mathbf{P} \)-cluster, up to homotopy.

In Definition 3.5.7 we define what it means for one cluster to be (strongly) reachable from another. We then show we have achieved the goal of working around the afore-mentioned obstruction.

**Theorem D** (Theorem 3.5.8). Consider the \( \mathbf{E} \)-cluster theory of \( \mathcal{C}(A_{\mathbb{R},S}) \) where \( A_{\mathbb{R},S} \) has the straight descending orientation. The cluster of injectives is strongly reachable from the cluster of projectives.
Future Work. The exchange graph of an \( A_n \) cluster structure is well-understood but the space of mutations for \( E \)-clusters poses difficult question due to continuous mutations (Section 3.5). However, preliminary calculations suggest the techniques to prove Theorem D may be generalized to all continuous quivers \( A_{R,S} \) of type \( A \) where \( |S| < \infty \).

It is not yet clear which \( E \)-cluster theories for continuous type \( A \) quivers are equivalent. Some theories are shown to be isomorphic (see Propositions 2.3.4 and 2.3.5) but the whole classification is still open (Section 2.3).

The next question to ask is, “What about continuous types other than \( A \)?” The next steps are continuous types \( \tilde{A} \) and \( D \). Each present their own complications to our constructions. If one performs a similar constructions for continuous type \( D \) then the resulting cluster theory should be similar to Igusa and Todorov’s construction in [17]. Preliminary work by the C. Paquette, E. Yıldırım, and the author show that continuous representations of type \( D \) decompose similarly to representations of a \( D_n \) quiver. Also, Hanson and the author have proven that representations of \( \tilde{A} \) decompose analogously to representations of \( \tilde{A}_n \) [13].

Changes to this Version. This version has been reorganized and party rewritten for readability. The original version of this paper also contained a section that related several different cluster theories of type \( A \), including the new \( E \)-cluster theory. Due to length and recent results, this section has been removed and then expanded into a separate work [24].

Conventions. Here we state conventions used throughout the paper. We have a fixed field \( k \) throughout. When we say a “Krull–Schmidt category” we mean a “skeletally small Krull–Schmidt additive category.”

Let \( a < b \in \mathbb{R} \cup \{ \pm \infty \} \). By the notation \( |a, b| \) we mean an interval subset of \( \mathbb{R} \) whose endpoints are \( a \) and \( b \). The ‘|’s indicate that the inclusion of \( a \) or \( b \) is not known or not relevant.

When we say “arc” in reference to a polygon with more than 3 sides, we mean what many call a “diagonal.” That is, a straight line from one vertex of the polygon to a nonadjacent vertex. This terminology better coincides with definitions in Section 2. We will still use “diagonals” for quadrilaterals when we talk about mutation.

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1. Prerequisites from This Series

In this section we revisit the most relevant definitions and theorems from parts (I) and (III) in this series, divided into two subsections.

1.1. Continuous Quivers of Type \( A \) and Their Representations. In this section we state relevant definitions and theorems from part (I) of this series. In particular, we provide a definition of a continuous quiver of type \( A \), its representations, and its indecomposables. The reader may use the picture in Figure 1 for intuition when reading the definition of a continuous quiver of type \( A \).

Definition 1.1.1. A \underline{quiver} of continuous type \( A \), denoted by \( A_{R,S} \), is a triple \((R, S, \preceq)\), where:

1. \( S \subset R \) is a discrete subset, possibly empty, with no accumulation points.
2. Order on \( S \cup \{ \pm \infty \} \) is induced by the order of \( R \), and \(-\infty < s < +\infty \) for \( \forall s \in S \).
3. Elements of \( S \cup \{ \pm \infty \} \) are indexed by a subset of \( \mathbb{Z} \cup \{ \pm \infty \} \) so that \( s_n \) denotes the element of \( S \cup \{ \pm \infty \} \) with index \( n \). The indexing must adhere to the following two conditions:
Definition 1.1.2. Let $A_{\mathbb{R}, S} = (\mathbb{R}, S \preceq)$ be a continuous quiver of type $A$. A representation $V$ of $A_{\mathbb{R}, S}$ is the following data:

- A vector space $V(x)$ for each $x \in \mathbb{R}$.
- For every pair $y \preceq x$ in $A_{\mathbb{R}}$ a linear map $V(x, y) : V(x) \to V(y)$ such that if $z \preceq y \preceq x$ then $V(x, z) = V(y, z) \circ V(x, y)$.

We say $V$ is pointwise finite-dimensional if $\dim V(x) < \infty$ for all $x \in \mathbb{R}$.

Definition 1.1.3. Let $A_{\mathbb{R}, S}$ be a continuous quiver of type $A$ and $I \subset \mathbb{R}$ be an interval. We denote by $M_I$ the representation of $A_{\mathbb{R}, S}$ defined as:

$$M_I(x) = \begin{cases} 1 & x \in I \\ 0 & \text{otherwise} \end{cases}$$

We call $M_I$ an interval indecomposable.

We require the two following results from [15] (the first recovers a result from [6]).

Theorem 1.1.4 (Theorems 2.3.2 and 2.4.13 in [15]). Let $A_{\mathbb{R}, S}$ be a continuous quiver of type $A$. For any interval $I \subset \mathbb{R}$, the representation $M_I$ of $A_{\mathbb{R}, S}$ is indecomposable. Any indecomposable pointwise finite-dimensional representation of $A_{\mathbb{R}, S}$ is isomorphic to $M_I$ for some interval $I$. Finally, any pointwise finite-dimensional representation $V$ of $A_{\mathbb{R}, S}$ is the direct sum of interval indecomposables.

Theorem 1.1.5 (Theorem 2.1.6 and Remark 2.4.16 in [15]). Let $P$ be a projective indecomposable in the category of pointwise finite-dimensional representations of a continuous quiver $A_{\mathbb{R}, S}$. Then there exists $a \in \mathbb{R} \cup \{\pm \infty\}$ such that $P$ is isomorphic to one of $P_a$, $P(a)$, or $P(a)$, given by:

$$P_a(x) = \begin{cases} 1 & x \leq a \\ 0 & \text{otherwise} \end{cases} \quad P(a)(x) = \begin{cases} 1 & x \leq a \\ 0 & \text{otherwise} \end{cases}$$

These allow us to define the category of finitely-generated representations:
**Definition 1.1.6.** Let $A_{\mathbb{R}, S}$ be a continuous quiver of type $A$. By $\text{rep}_k(A_{\mathbb{R}, S})$ we denote the full subcategory of pointwise finite-dimensional representations whose objects are finitely generated by the indecomposable projectives in Theorem 1.1.5.

By [15, Theorem 3.0.1], $\text{rep}_k(A_{\mathbb{R}, S})$ is Krull–Schmidt with global dimension 1.

1.2. Cluster Theories and Embeddings. In this section we state the cluster theories content we need from Part (III) [16]. However, we first need just one result from Part (II).

**Proposition 1.2.1 (Proposition 5.1.2 in [23]).** Let $A_{\mathbb{R}, S}$ be a continuous quiver of type $A$. Then $\mathcal{D}^b(A_{\mathbb{R}, S})$ is Krull–Schmidt. The indecomposable objects are shifts of indecomposables in $\text{rep}_k(A_{\mathbb{R}, S})$.

**Theorem 1.2.2 (Theorem A in [16]).** $A_{\mathbb{R}, S}$ and $A_{\mathbb{R}, R}$ be continuous quivers of type $A$. Then $\mathcal{D}^b(A_{\mathbb{R}, S})$ is triangulated equivalent to $\mathcal{D}^b(A_{\mathbb{R}, R})$ if and only if (i) $S$ and $R$ are both finite, (ii) $S$ and $R$ both bounded on exactly one side, or (iii) $S$ and $R$ are both indexed by $\mathbb{Z}$.

**Definition 1.2.3.** The category $\mathcal{C}(A_{\mathbb{R}, S})$ is the orbit category of the doubling of $\mathcal{D}^b(A_{\mathbb{R}, S})$ via almost-shift as in [16, Definition 3.1.2].

Importantly, the isomorphism classes of indecomposables in $\mathcal{C}(A_{\mathbb{R}, S})$ are the same as if we had took the orbit of $\mathcal{D}^b(A_{\mathbb{R}, S})$ by shift. That is, $V \cong V[1]$ for all indecomposables $V$ in $\mathcal{C}(A_{\mathbb{R}, S})$. The doubling process ensures $\mathcal{C}(A_{\mathbb{R}, S})$ is a triangulated category. Thus we have distinguished triangles in $\mathcal{C}(A_{\mathbb{R}, S})$ of the form $Q_V \to P_V \to V \to Q_V$ where $Q_V \to P_V \to V \to 0$ is the minimal projective resolution of $V$ in $\text{rep}_k(A_{\mathbb{R}, S})$. Furthermore, for indecomposables $V$ and $W$ in $\mathcal{C}(A_{\mathbb{R}, S})$, either $\text{Hom}_{\mathcal{C}(A_{\mathbb{R}, S})}(V, W) \cong k$ or $\text{Hom}_{\mathcal{C}(A_{\mathbb{R}, S})}(V, W) = 0$ [16, Proposition 3.1.2]. The authors of [16] then defined $g$-vectors following Jørgensen and Yakimov in [19].

**Definition 1.2.4.** Let $V$ be an indecomposable in $\mathcal{C}(A_{\mathbb{R}})$. The $\overline{g}$-vector of $V$ is the element $[P_V] - [Q_V]$ in $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}, S}))$ where $Q_V \to P_V \to V \to 0$ is the minimal projective resolution of $V$ in $\text{rep}_k(A_{\mathbb{R}, S})$.

**Definition 1.2.5.** For $[A] = \sum_i m_i[A_i]$ and $[B] = \sum_j n_j[B_j]$ in $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}, S}))$, define:

$$\langle [A], [B] \rangle := \sum_i \sum_j \langle m_i[A_i], n_j[B_j] \rangle.$$  

The Euler form is used to define $E$-compatibility and $E$-clusters.

**Definition 1.2.6.**

- Let $V$ and $W$ be two indecomposables in $\mathcal{C}(A_{\mathbb{R}, S})$ with $g$-vectors $[Q_V] - [P_V]$ and $[Q_W] - [P_W]$. We say $\{V, W\}$ is $E$-compatible if

$$\langle [P_V] - [Q_V], [P_W] - [Q_W] \rangle \geq 0 \quad \text{and} \quad \langle [P_W] - [Q_W], [P_V] - [Q_V] \rangle \geq 0.$$  

- A set $T$ is called $E$-compatible if for every $V, W \in T$ the set $\{V, W\}$ is $E$-compatible. If $T$ is maximally $E$-compatible then we call $T$ an $E$-cluster.

- Let $T$ be an $E$-cluster and $V \in T$ such that there exists $W \notin T$ where $\{V, W\}$ is not $E$-compatible but $(T \setminus \{V\}) \cup \{W\}$ is $E$-compatible. Then we say $V$ is $E$-mutable. The bijection (see Theorem 1.2.8) $T \to (T \setminus \{V\}) \cup \{W\}$ given by $V \mapsto W$ and $X \mapsto X$ if $X \neq V$ is called an $E$-mutation or $E$-mutation at $V$.

The following is used in Section 2.2.

**Proposition 1.2.7 (Proposition 4.2.4 in [16]).** Let $V$ and $W$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}, S})$ and let $\tilde{V}$ and $\tilde{W}$ be the respective indecomposables in $\text{rep}_k(A_{\mathbb{R}, S})$. Then, up to symmetry, $V$ and $W$ are not $E$-compatible if and only if there exists an extension $\tilde{V} \twoheadrightarrow E \twoheadrightarrow \tilde{W}$ such that $E \not\cong \tilde{V} \oplus \tilde{W}$. 

The words E-cluster and E-mutation are justified with the following theorem.

**Theorem 1.2.8** (Theorem 4.3.8 in [16]). Let $T$ be an E-cluster and $V \in T$ E-mutable with choice $W$. Then $(T \setminus \{V\}) \cup \{W\}$ is an E-cluster and any other choice $W'$ for $V$ is isomorphic to $W$.

The key difference between E-clusters and the usual cluster structures (such as those in [7]) is that not all $V$ in an E-cluster $T$ need be mutable. The authors only require there be none or one choice. This is generalized to the abstract notion of cluster theories.

**Definition 1.2.9** (Definition 4.1.1 in [16]). Let $C$ be a Krull–Schmidt additive category and $P$ a pairwise compatibility condition on its (isomorphism classes of) indecomposable objects. Suppose that for each (isomorphism class of) indecomposable $X$ in a maximally $P$-compatible set $T$ there exists none or one (isomorphism class of) indecomposable $Y$ such that $\{X, Y\}$ is not $P$-compatible but $(T \setminus \{X\}) \cup \{Y\}$ is $P$-compatible. Then

- We call the maximally $P$-compatible sets $P$-clusters.
- We call a function of the form $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ such that $\mu Z = Z$ when $Z \neq X$ and $\mu X = Y$ a $P$-mutation or $P$-mutation at $X$.
- If there exists a $P$-mutation $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ we say $X \in T$ is $P$-mutable.
- The subcategory $\mathcal{F}_P(C)$ of Set whose objects are $P$-clusters and whose morphisms are generated by $P$-mutations (and identity functions) is called the $P$-cluster theory of $C$.
- The functor $I_{P, C} : \mathcal{F}_P(C) \to \text{Set}$ is the inclusion of the subcategory.

**Remark 1.2.10.** We note three things immediately about Definition 1.2.9.

- The set $(T \setminus \{X\}) \cup \{Y\}$ must be maximally $P$-compatible, so this does not need to be checked in practice.
- Since $P$-clusters contain isomorphism classes of indecomposables as elements and $C$ is skeletally small, the category $\mathcal{F}_P(C)$ is small.
- Finally, the pairwise compatibility condition $P$ determines the cluster theory.

Thus we may say that $P$ induces the cluster theory.

**Definition 1.2.11** (Definition 4.1.4 in [16]). Let $C$ be a Krull–Schmidt category and $P$ a pairwise compatibility condition such that $P$ induces the $P$-cluster theory of $C$. If, for every $P$-cluster $T$ and $X \in T$, there is a $P$-mutation at $X$ then we call $\mathcal{F}_P(C)$ the tilting $P$-cluster theory.

Every cluster structure in the sense of [8, 7] yields an tilting cluster theory.

**Definition 1.2.12** (Definition 4.1.9 in [16]). Let $C$ and $D$ be two Krull–Schmidt categories with respective pairwise compatibility conditions $P$ and $Q$. Suppose these compatibility conditions induce the $P$-cluster theory and $Q$-cluster theory of $C$ and $D$, respectively.

Suppose there exists a functor $F : \mathcal{F}_P(C) \to \mathcal{F}_Q(D)$ such that $F$ is an injection on objects and an injection from $P$-mutations to $Q$-mutations. Suppose also there is a natural transformation $\eta : I_{P, C} \to I_{Q, D} \circ F$ whose morphisms $\eta_T : I_{P, C}(T) \to I_{Q, D} \circ F(T)$ are all injections. Then we call $(F, \eta) : \mathcal{F}_P(C) \to \mathcal{F}_Q(D)$ an embedding of cluster theories.

2. Geometric Models of E-clusters

In this section we construct geometric models of E-clusters. In Section 2.1 we address the straight descending orientation of $A_2$ and in Section 2.2 we address the rest of the orientations. See [24] for a more general version of this technique. We discuss the classification of cluster theories of continuous type $A$ in Section 2.3.

Recall that an isomorphism of categories $F : C \to D$ has an inverse functor $G : D \to C$ such that $GF = 1_C$ and $FG = 1_D$; the compositions are equal to the identity.
Definition 2.0.1. Let \( C \) and \( D \) be a Krull–Schmidt categories. Let \( P \) and \( Q \) be pairwise compatibility conditions in \( C \) and \( D \) such that they, respectively, induce the cluster theories \( \mathcal{T}_P(C) \) and \( \mathcal{T}_Q(D) \). A weak equivalence of cluster theories is an embedding of cluster theories \((F, \eta): \mathcal{T}_P(C) \to \mathcal{T}_Q(D)\) such that \( F \) is an isomorphism of categories. We instead say \((F, \eta)\) is an isomorphism of cluster theories if additionally each \( \eta_T \) is an isomorphism.

Remark 2.0.2. An isomorphism of categories is ordinarily a very stringent requirement. However, since every cluster theory is a groupoid the only real control we really have over comparing the “size” of each category is to insist they be identically the same via an isomorphism on objects. And, since clusters in a cluster theory are sets of isomorphism classes of objects in \( C \) and \( D \), respectively, we are already accounting for the type of equivalence with which we are familiar.

We use the following lemma in Sections 2.1 and 2.2.

Lemma 2.0.3. Let \( C \) and \( D \) be Krull–Schmidt categories. Let \( P \) be a pairwise compatibility condition in \( C \) such that \( P \) induces the cluster theory \( \mathcal{T}_P(C) \) and let \( Q \) be a pairwise compatibility condition in \( D \). Suppose

- there is a bijection \( \Phi: \text{Ind}(C) \to \text{Ind}(D) \) and
- for indecomposables \( A \) and \( B \) in \( C \), \( \{A, B\} \) is \( P \)-compatible if and only if \( \{\Phi(A), \Phi(B)\} \) is \( Q \)-compatible.

Then \( Q \) induces the cluster theory \( \mathcal{T}_Q(D) \) and \( \Phi \) induces an isomorphism of cluster theories \((F, \eta): \mathcal{T}_Q(D) \to \mathcal{T}_P(C)\).

Proof. Let \( T \) be a maximally \( Q \)-compatible set of \( D \)-indecomposables and let \( F(T) = \{\Phi^{-1}(A) \mid A \in T\} \). First we show \( F(T) \) is a \( P \)-cluster. Suppose \( \{X\} \subseteq F(T) \) is \( P \)-compatible. Then \( \{\Phi(X)\} \cup T \) is \( Q \)-compatible. However, \( T \) is maximally \( Q \)-compatible and so \( \Phi(X) \in T \) and \( X \in F(T) \).

Suppose there is \( A \in T \) and \( B \notin T \) such that \((T \setminus \{A\}) \cup \{B\}\) is \( Q \)-compatible. Then \( \{A, B\} \) is not \( Q \)-compatible since \( T \) is maximally \( Q \)-compatible. So \( \{\Phi^{-1}(A), \Phi^{-1}(B)\} \) is not \( P \)-compatible but \((F(T) \setminus \{\Phi^{-1}(A)\}) \cup \{\Phi^{-1}(B)\}\) is \( P \)-compatible. This is a \( P \)-mutation and so \((F(T) \setminus \{\Phi^{-1}(A)\}) \cup \{\Phi^{-1}(B)\}\) is a \( P \)-cluster. Then by a similar argument to beginning of this proof, \((T \setminus \{A\}) \cup \{B\}\) is maximally \( Q \)-compatible. Suppose there is \( C \notin T \) such that \((T \setminus \{A\}) \cup \{C\}\) is \( Q \)-compatible.

Again, \( \{A, C\} \) is not \( Q \)-compatible and \((T \setminus \{A\}) \cup \{C\}\) is maximally \( Q \)-compatible. However, this means \( \Phi^{-1}(B) = \Phi^{-1}(C) \) and so \( C = B \). Therefore, \( Q \) induces the cluster theory \( \mathcal{T}_Q(D) \).

We have already shown \( F \) is a functor. Suppose \( T \neq T' \). Then \( T \cap T' \subseteq T \) and \( T \cap T' \subseteq T' \) Using \( \Phi^{-1} \) we see \( F(T) \cap F(T') \subseteq F(T) \) and \( F(T) \cap F(T') \subseteq F(T') \) which means \( F(T) \neq F(T') \). Suppose \( L \) is a \( P \)-cluster. Then \( \{\Phi(X) \mid X \in L\} \) is a \( Q \)-cluster by a similar argument to that at the beginning of the proof. Therefore, \( F \) is an isomorphism of categories. Finally, for each \( Q \)-cluster \( T \), we define \( \eta_T: T \to F(T) \) by \( A \mapsto \Phi^{-1}(A) \). These are isomorphisms, as desired.

2.1. Straight orientation: \( A_\mathcal{R} \). In this section we construct a geometric model of the cluster theory \( \mathcal{T}_E(\mathcal{C}(A_\mathcal{R})) \) when \( A_\mathcal{R} \) has the straight descending orientation. With this orientation there is a single frozen indecomposable in every \( E \)-cluster (Definition 1.2.6): \( P_{+\infty} \). The geometric model of \( E \)-clusters of this orientation is a generalization of the models in [14, 5]. The generic arc in [5] is very similar to \( P_{+\infty} \).

It is straightforward to check that for \( M_{[a,b]} \) and \( M_{[c,d]} \) where \( a, b, c, d \) are all distinct the set \( \{M_{[a,b]}, M_{[c,d]}\} \) is not \( E \)-compatible if and only if \( a < c < b < d \) or \( c < a < d < b \). If \( a < c < b < d \) we can draw the crossing arcs from \( a \) to \( b \) and from \( c \) to \( d \), for the “macroscopic” perspective, in Figure 2, both of which are always \( E \)-compatible with \( P_{+\infty} = M_{(-\infty,+\infty)} \).

However, on the “microscopic” scale things are different. Because we allow all types of intervals, we need two possible arc endpoints per \( x \in \mathcal{R} \), but only one endpoint at each \( -\infty \) and \( +\infty \).
Definition 2.1.1. Let $A_R$ have the straight descending orientation. In the set $\{-, +\}$ we consider $- < +$ and denote an arbitrary element by $\varepsilon$, $\varepsilon'$, etc. We give the set $\mathcal{E} := (\mathbb{R} \times \{-, +\}) \cup \{\pm \infty\}$ the total ordering where

- $-\infty < (x, \pm) < +\infty$ for all $x \in \mathbb{R}$ and
- $(x, \varepsilon) < (y, \varepsilon')$ if either $x < y$ or $x = y$ and $\varepsilon < \varepsilon'$.

For ease of notation we write $(-\infty, +)$ for $-\infty$ and $(+\infty, -)$ for $+\infty$. We also write $a$ to mean $(a, \varepsilon)$ for arbitrary $\varepsilon \in \{-, +\}$.

Definition 2.1.2. Let $\mathcal{A}$ be the set $\{([a, b]) \in \mathcal{E} \times \mathcal{E} \mid \overline{a} < \overline{b}\}$. We call $\mathcal{A}$ the set of arcs. Let $M_{[a, b]}$ be the indecomposable in $\mathcal{C}(A_R)$ that is the image of the indecomposable with the same name in $\text{rep}_k(A_R)$. We define $\Phi(M_{[a, b]})$ to be $([a, b])$ where

- $\overline{a} = (a, -)$ if $a \in [a, b]$ and $\overline{a} = (a, +)$ if $a \notin [a, b]$, and
- $\overline{b} = (b, -)$ if $b \notin [a, b]$ and $\overline{b} = (b, +)$ if $b \in [a, b]$.

This defines $\Phi : \text{Ind}(\mathcal{C}(A_R)) \rightarrow \mathcal{A}$. Note $\Phi(M_{[a, a]}) = ((a, -), (a, +))$.

We impose the following rule on our arcs.

Rule 2.1.3. Define a crossing function $c : \mathcal{A} \times \mathcal{A} \to \{0, 1\}$.

\[
(2.1.3) \quad c(\alpha, \beta) = \begin{cases} 
1 & \overline{\alpha} < \overline{\beta} \leq \overline{\alpha} < \overline{\beta} \text{ or } \overline{\alpha} < \overline{\beta} \leq \overline{\beta} < \overline{\alpha} \\
0 & \text{otherwise.}
\end{cases}
\]

If $\alpha \neq \beta$ and $c(\alpha, \beta) = 1$ we say $\alpha$ and $\beta$ cross. Otherwise, we say $\alpha$ and $\beta$ do not cross.

!! Notice the difference from the usual convention in the middle. If two arcs meet from opposing sides we still consider them to cross. This only happens on the “microscopic” scale. I.e., for $a < b < d$, $(\overline{a}, (b, -))$ and $((b, +), \overline{d})$ do not cross but any other combination of $+$ and $-$ for $\overline{b}$ cross (see Figure 3).

For the following proposition, recall the Definition 1.2.4.

Proposition 2.1.4. The map $\Phi$ in Definition 2.1.2 is a bijection.

Proof. Suppose $M_{[a, b]} \not\cong M_{[c, d]}$. Then $|a, b| \neq |c, d|$ and so one of the endpoints of the intervals must differ. I.e., even if $a = c$ and $b = d$ then $a \notin |a, b|$ or $a \notin |c, d|$ or $b \notin |a, b|$ or $b \notin |c, d|$. Then endpoints of the arcs associated to $M_{[a, b]}$ and $M_{[c, d]}$ are different. Let $\alpha = (\overline{a}, \overline{b})$ be an arc. Then
\( \alpha = \Phi(M_{[a,b]}) \) where \( a \in [a,b] \) if and only if \( \bar{a} = (a, -) \) and \( b \in [a,b] \) if and only if \( \bar{b} = (b, +) \).

Therefore \( \Phi \) is both injective and surjective and so bijective.

**Lemma 2.1.5.** Let \( M_{[a,b]} \neq M_{[c,d]} \) be indecomposables in \( \mathcal{C}(A_\mathbb{R}) \), \( \alpha = \Phi(M_{[a,b]}) \), and \( \beta = \Phi(M_{[c,d]}) \). Then \( \{M_{[a,b]}, M_{[c,d]}\} \) is \( \mathbf{E} \)-compatible if and only if \( \epsilon(\alpha, \beta) = 0 \).

**Proof.** Suppose \( \{M_{[a,b]}, M_{[c,d]}\} \) is not \( \mathbf{E} \)-compatible. As we have discussed, if \( a, b, c, d \) are all distinct then \( a < c < b < d \) or \( c < a < d < b \). In either case it follows that \( \alpha \) and \( \beta \) cross. Suppose \( a = c \). Since the \( \mathbf{g} \)-vectors of \( M_{[a,b]} \) and \( M_{[c,d]} \) are not \( \mathbf{E} \)-compatible, (Definition 2.1.6), we must have \( a \notin [a,b] \) and \( c \in [c,d] \) or vice versa.

Without loss of generality suppose \( a \notin [a,b] \) and \( c \in [c,d] \). Then either \( d < b \) or if \( d = b \) then \( d \notin [a,b] \) and \( b \in [a,b] \). In either case the arcs \( \alpha \) and \( \beta \) cross. We can perform a similar argument starting with \( b = d \) and see that \( \alpha \) and \( \beta \) cross.

Now suppose \( \alpha \) and \( \beta \) cross. Then \( \bar{a} < \bar{c} \leq \bar{b} < \bar{d} \) or \( \bar{c} < \bar{a} \leq \bar{d} < \bar{b} \). Without loss of generality assume the first. Then if \( a = c \), \( a \in [a,b] \) but \( c \notin [c,d] \). Similarly if \( b = d \) then \( b \notin [a,b] \) and \( d \in [c,d] \). In all cases we see that the \( \mathbf{g} \)-vectors of \( M_{[a,b]} \) and \( M_{[c,d]} \) are not \( \mathbf{E} \)-compatible and so the set \( \{M_{[a,b]}, M_{[c,d]}\} \) is not \( \mathbf{E} \)-compatible.

**Definition 2.1.6.** Let \( \mathcal{C}_{\mathbb{R}} \) be the additive category whose indecomposable objects are \( \mathcal{A} \). We define \( \text{Hom}_{\mathcal{C}_{\mathbb{R}}}(\alpha, \beta) \) and composition \( \alpha \overset{f}{\rightarrow} \beta \overset{g}{\rightarrow} \gamma \), for \( f \) and \( g \) nonzero, by

\[
\text{Hom}_{\mathcal{C}_{\mathbb{R}}}(\alpha, \beta) = \begin{cases} k & \text{if } \epsilon(\alpha, \beta) = 1 \\ 0 & \text{otherwise.} \end{cases}
\]

\[
g \circ f = \begin{cases} g \cdot f & \text{if } \alpha = \beta \text{ or } \beta = \gamma \\ 0 & \text{otherwise.} \end{cases}
\]

Extending bilinearly, we have a Krull–Schmidt category. For \( \alpha \neq \beta \), we define \( \{\alpha, \beta\} \) to be \( \mathbf{N}_{\mathbb{R}} \)-compatible if and only if \( \epsilon(\alpha, \beta) = 0 \).

**Corollary 2.1.7** (to Lemma 2.1.5). Let \( M_{[a,b]} \) and \( M_{[c,d]} \) be in \( \text{Ind}(\mathcal{C}(A_\mathbb{R})) \), \( \alpha = \Phi(M_{[a,b]}) \), and \( \beta = \Phi(M_{[c,d]}) \). Then \( \{\alpha, \beta\} \) is \( \mathbf{N}_{\mathbb{R}} \)-compatible if and only if \( \{M_{[a,b]}, M_{[c,d]}\} \) is \( \mathbf{E} \)-compatible.

**Proof.** The Corollary is immediately true if \( \alpha = \beta \) or \( M_{[a,b]} = M_{[c,d]} \). Suppose \( \alpha \neq \beta \). Then \( \{\alpha, \beta\} \) is \( \mathbf{N}_{\mathbb{R}} \)-compatible if and only if \( \epsilon(\alpha, \beta) = 0 \). Now apply Lemma 2.1.5.

**Theorem 2.1.8.** The pairwise compatibility condition \( \mathbf{N}_{\mathbb{R}} \) induces the \( \mathbf{N}_{\mathbb{R}} \)-cluster theory of \( \mathcal{C}_{\mathbb{R}} \) and \( \Phi \) induces the isomorphism of cluster theories \( (F, \eta) : \mathcal{F}_{\mathcal{N}_{\mathbb{R}}}(\mathcal{C}_{\mathbb{R}}) \rightarrow \mathcal{F}_{\mathbf{E}}(\mathcal{C}(A_\mathbb{R})) \).

**Proof.** We have shown there is a bijection \( \Phi : \text{Ind}(\mathcal{C}(A_\mathbb{R})) \rightarrow \text{Ind}(\mathcal{C}_{\mathbb{R}}) \) (Proposition 2.1.4) and that \( \{M_{[a,b]}, M_{[c,d]}\} \) is \( \mathbf{E} \)-compatible if and only if \( \{\Phi(M_{[a,b]}), \Phi(M_{[c,d]})) \) is \( \mathbf{N}_{\mathbb{R}} \)-compatible (Corollary 2.1.7). By Lemma 2.0.3, \( \mathcal{N}_{\mathbb{R}} \) induces the cluster theory \( \mathcal{F}_{\mathcal{N}_{\mathbb{R}}}(\mathcal{C}_{\mathbb{R}}) \) and we have the isomorphism of cluster theories given by \( F(T) := \{\Phi^{-1}(\alpha) \mid \alpha \in T\} \) and \( \eta_T(\alpha) := \Phi^{-1}(\alpha) \).

If we remove the arc \((-\infty, +), (+\infty, -)) \) from our geometric model, we still have a weak equivalence of cluster theories.

### 2.2. Other orientations.

We now construct a geometric model of \( \mathcal{F}_{\mathbf{E}}(\mathcal{C}(A_{\mathbb{R},S})) \) for orientations of \( A_{\mathbb{R},S} \) other than the straight orientation. We take inspiration from the model of representations in [4]. In the case of straight \( A_{\mathbb{R},S} \), we think of all the arcs as being directed: originating at the lower point and ending at the upper point. We update our pictures from Rule 2.1.3 to those in Figure 4. When \( A_{\mathbb{R},S} \) has the straight descending orientation, these are the only possibilities.

Now suppose \( A_{\mathbb{R},S} \) has some orientation other than straight descending or straight ascending. We construct a set of endpoints \( \mathcal{E} \) as the union of two sets: \( \mathcal{E}^\downarrow \) and \( \mathcal{E}^\uparrow \). Recall in the definition of a continuous quiver of type \( A \) (Definition 1.1.1) that sinks have even index, \( s_{2n} \), and sources have odd index, \( s_{2n+1} \). Recall also that if the sinks and sources of \( A_{\mathbb{R},S} \) are bounded below then \(-\infty\) is assigned the next available index below and similarly for \(+\infty\) when the sinks and sources are
bounded above. When the sinks and sources are not bounded below (above) we assign the index $-\infty$ to $-\infty$ ($+\infty$ to $+\infty$).

Recall $- < +$ in $\{-, +\}$ and we write $\varepsilon$ to mean an arbitrary element in $\{-, +\}$.

**Definition 2.2.1.** The sets $E^\downarrow$ and $E^\uparrow$ are defined as follows, where each $s_m$ in the notation is a sink or source in $A_{\mathbb{R},S}$ or one of $\pm\infty$ where appropriate:

$$E^\downarrow := \{(x \in \mathbb{R} \mid \exists \text{ a sink and source } s_{2m} < x < s_{2m+1} \} \cup \{(s_{2n-1}, s_{2n})\} \times \{-, +\}$$

$$E^\uparrow := \{(x \in \mathbb{R} \mid \exists \text{ a source and sink } s_{2m-1} < x < s_{2m} \} \cup \{(s_{2n}, s_{2n+1})\} \times \{-, +\}$$

We write $\bar{a}$ to mean an element $(a, \varepsilon)$ when $\varepsilon$ is unknown or understood from context. In this case $a$ may be in $\mathbb{R}$ or equal to some $(s_m, s_{m+1})$. We define a total order on $E := E^\downarrow \cup E^\uparrow$:

- We say $(x, \varepsilon) < (y, \varepsilon')$ if $x < y$ or $x = y$ and $\varepsilon < \varepsilon'$.
- We say $((s_m, s_{m+1}), \varepsilon) < ((s_n, s_{n+1}), \varepsilon')$ if $s_m < s_n$ or $s_m = s_n$ and $\varepsilon < \varepsilon'$.
- We say $(x, \varepsilon) < ((s_m, s_{m+1}), \varepsilon')$ if $x < s_m$ or if $s_m < x < s_{m+1}$ and $\varepsilon' = +$.
- We say $((s_m, s_{m+1}), \varepsilon) < (y, \varepsilon')$ if $s_{m+1} < y$ or if $s_m < y < s_{m+1}$ and $\varepsilon = -$.

The set $E^\downarrow$ has a maximal (respectively minimal) element if and only if $+\infty$ has an odd index (respectively $-\infty$ has an even index). Dually, $E^\uparrow$ has a maximal (respectively minimal) element if and only if $+\infty$ has an even index (respectively $-\infty$ has an odd index).

**Definition 2.2.2.** Let $A$ be the set $\{(\bar{a}, \bar{b}) \in E \times E \mid \bar{a} < \bar{b}\}$. We call $A$ the set of $\text{arcs}$. For an element $(\bar{a}, \bar{b}) \in A$, we call $\bar{a}$ and $\bar{b}$ the endpoints.

**Example 2.2.3.** Let $A_{\mathbb{R},S}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$. Then $-\infty = s_{-3}$ and $+\infty = s_3$. The set $E^\downarrow$ has a maximum element and $E^\uparrow$ has a minimum element. In Figure 5 we draw $E^\downarrow$ and $E^\uparrow$ using piece-wise linear curves in the plane and draw arcs on the “macroscopic” scale as lines between two points in $E$. For example, let $\alpha = (\bar{a}, \bar{b})$ where $s_{-2} < a < s_{-1}$ and $s_1 < b < s_2$. Since $\bar{a} < \bar{b}$, we draw $\alpha$ oriented from $\bar{a}$ to $\bar{b}$.

Recall that if the sinks and sources of $A_{\mathbb{R},S}$ are unbounded below (respectively above) then no indecomposable in $\text{rep}_k(A_{\mathbb{R},S})$ may have $-\infty$ as a lower endpoint (respectively $+\infty$ as an upper
endpoint) of its support. Thus if we have $M_{a,b}$ and $a = -\infty$ (respectively $b = +\infty$) then we know the sinks and sources of $A_{R,S}$ are bounded below (respectively above).

**Definition 2.2.4.** We now define $\Phi : \text{Ind}(C(A_{R,S})) \rightarrow A$. Let $M_{[a,b]}$ be an indecomposable in $C(A_{R,S})$. We define $\overline{x}$ and $\overline{y}$ in $E$ beginning with $\overline{x}$.

- If $a \in R$ is neither a sink nor a source then $\overline{x} = (a, \varepsilon)$ where $\varepsilon = -$ if and only if $a \in [a, b]$.
- If $a = -\infty = s_m$ then $\overline{x} = (\{s_m, s_{m+1}\}, -)$.
- If $-\infty < a = s_m$ and $a \in [a, b]$ then $\overline{x} = \{(s_m, s_{m+1}), -\}$.
- If $-\infty < a = s_m$ and $a \notin [a, b]$ then $\overline{x} = (\{s_{m-1}, s_m\}, +)$.

Now, $\overline{y}$.

- If $b \in R$ is neither a sink nor a source then $\overline{y} = (b, \varepsilon)$ where $\varepsilon = -$ if and only if $b \in [a, b]$.
- If $b = +\infty = s_n$ then $\overline{y} = (\{s_{n-1}, s_n\}, +)$.
- If $+\infty > b = s_n$ and $b \in [a, b]$ then $\overline{y} = (\{s_{n-1}, s_n\}, +)$.
- If $+\infty > b = s_n$ and $b \notin [a, b]$ then $\overline{y} = (\{s_n, s_{n+1}\}, -)$.

**Proposition 2.2.5.** The function $\Phi$ in Definition 2.2.4 is a bijection.

**Proof.** Let $M_{[a,b]} \neq M_{[c,d]}$ be indecomposables in $C(A_{R,S})$. Let $(\overline{x}, \overline{y}) = \Phi(M_{[a,b]})$ and $(\overline{x}, \overline{w}) = \Phi(M_{[c,d]})$. Using the definition it is straightforward to check that if $a \neq c$ or $b \neq d$ then $(\overline{x}, \overline{y}) \neq (\overline{x}, \overline{w})$. Now suppose $a = c$ and $b = d$. Since $M_{[a,b]} \neq M_{[c,d]}$ the endpoints of $[a, b]$ and $[c, d]$ must differ by at least one point. By symmetry and possibly reversing the roles of $M_{[a,b]}$ and $M_{[c,d]}$, assume $a \in [a, b]$ and $c \notin [a, b]$. Then $\overline{x} \neq \overline{y}$ and so $(\overline{x}, \overline{y}) \neq (\overline{x}, \overline{w})$. Thus, $\Phi$ is injective.

Let $\alpha = (\overline{x}, \overline{y})$ be an arc in $A$. We now construct an interval $[a, b]$ such that $\Phi(M_{[a,b]}) = \alpha$. If $x \in R$ and $x$ is neither a sink nor a source then we let $a = x$ and $a \in [a, b]$ if and only if $\varepsilon = -$. If $y \in R$ and $y$ is neither a sink nor a source then we let $b = y$ and $b \in [a, b]$ if and only if $\varepsilon' = +$.

Suppose $x = (s_m, s_{m+1})$. If $\varepsilon = +$, then either $y \in R$ is greater than $s_{m+1}$ or $b = (s_n, s_{n+1})$ where $n > m$. In this case we let $a = s_{m+1}$ and $a \notin [a, b]$. If $\varepsilon = -$, then either either $y \in R$ is greater than $s_m$ or $y = (s_n, s_{n+1})$ where $n \geq m$; if $n = m$ then $\overline{y} = (s_{m-1}, s_m)$. In this case if $s_m = -\infty$ then we let $a = -\infty$ and note $a \notin [a, b]$. If $s_m > -\infty$ then we let $a = s_m$ and $a \in [a, b]$. We perform the dual constructions for $\overline{y}$ and $b$ as well.

Now we have $a \leq b$ and the requirements for $[a, b]$ to contain either $a$ or $b$. We need to check $a = b$ to ensure that in this case $a, b \in [a, b]$ by our construction. If $a = b \in R$ is neither a sink nor a source then $\alpha = \{(a, -), (a, +)\}$ and so $[a, b] = [a, a]$. If $a = b \in R$ is a sink or a source let $s_n = a = b$. Then we have that $[a, b] = \{s_n\}$. Thus $\Phi$ is surjective and so bijective.

Our rules for crossing are more complicated than before. The cases are: straightforward (Rule 2.2.6), the “macroscopic” (Rule 2.2.8), and the “microscopic” (Rule 2.2.10).

**Rule 2.2.6.** Let $\alpha$ and $\beta$ be arcs with endpoints in $E$.

- If both $\alpha$ and $\beta$ have endpoints in $E^\downarrow$ then we follow Rule 2.1.3.
- If both $\alpha$ and $\beta$ have endpoints in $E^\downarrow$ then we follow Rule 2.1.3.
- If $\alpha$ has endpoints in $E^\downarrow$ and $\beta$ has endpoints in $E^\uparrow$ then we say $\alpha$ and $\beta$ do not cross.

**Example 2.2.7** (Example of Rule 2.2.6). Let $A_{R,S}$ have sinks $s_{-2} = -2, s_0 = 0, s_2 = 2$ and sources $s_{-1} = -1, s_1 = 1$ with $-\infty = s_{-3}$ and $+\infty = s_3$ as in Example 2.2.3. Let $\alpha^\downarrow$ and $\beta^\downarrow$ be crossing arcs with endpoints in $E^\downarrow$. Let $\alpha^\uparrow$ and $\beta^\uparrow$ be crossing arcs with endpoints in $E^\uparrow$. In Figure 6 we see why the “upper” and “lower” arcs to not cross.

**Rule 2.2.8.** Let $\alpha$ and $\beta$ be arcs in $A$. Suppose $\alpha = (\overline{x}, \overline{b})$ has one endpoint in $E^\downarrow$ and the other in $E^\uparrow$. Let $\beta = (\overline{c}, \overline{d})$. We assume $\overline{x}, \overline{b}, \overline{c},$ and $\overline{d}$ are all distinct.

1. Suppose $\overline{x} \in E^\downarrow$. If $\overline{x} < \overline{c} < \overline{d}$, where $\{\overline{x}\} = \{\overline{x}, \overline{b}\} \cap E^\downarrow$, we say $\alpha$ and $\beta$ cross.
2. Suppose $\overline{x} \in E^\uparrow$. If $\overline{x} < \overline{c} < \overline{d}$, where $\{\overline{x}\} = \{\overline{x}, \overline{b}\} \cap E^\uparrow$, we say $\alpha$ and $\beta$ cross.
Figure 6. A visual depiction of why “upper” and “lower” arcs (in Example 2.2.6) do not cross. We see $\alpha \downarrow$ and $\beta \downarrow$ cross each other but cross neither $\alpha \uparrow$ nor $\beta \uparrow$.

Figure 7. Consider $\alpha$ from Example 2.2.9. We have $\delta$ and $\eta$, examples of arcs with endpoints in both $E \downarrow$ and $E \uparrow$, that do not cross $\alpha$. We see $\delta$ goes “top to bottom” and $\eta$ goes "bottom to top." We have $\beta$ and $\zeta$, also with mixed endpoints, that cross $\alpha$. We see $\beta$ goes “top to bottom" and $\zeta$ goes “bottom to top.” Finally, we have $\gamma$, with both endpoints in $E \uparrow$, that crosses $\alpha$ but not $\beta$. From Rule 2.2.8.

(3) Suppose either (i) $\overline{a}, \overline{c} \in E \downarrow$ and $b, \overline{d} \in E \uparrow$ or (ii) $\overline{a}, \overline{c} \in E \uparrow$ and $b, \overline{d} \in E \downarrow$. If $\overline{a} < \overline{c} < \overline{d} < b$ or $\overline{c} < \overline{a} < \overline{d} < b$ we say $\alpha$ and $\beta$ cross.

(4) Suppose either (i) $\overline{a}, \overline{d} \in E \downarrow$ and $\overline{b}, \overline{c} \in E \uparrow$ or (ii) $\overline{a}, \overline{d} \in E \uparrow$ and $\overline{b}, \overline{c} \in E \downarrow$. If $\overline{a} < \overline{d}$ and $\overline{c} < \overline{b}$ we say $\alpha$ and $\beta$ cross.

Notice that $\overline{a} < \overline{d}$ and $\overline{c} < \overline{b}$ is not enough for Rule 2.2.8(3). In this case, if $\overline{c} < \overline{a} < \overline{d} < b$, for example, then $\alpha$ and $\beta$ do not cross. However, if $\alpha$ and $\beta$ cross, we must have $\overline{a} < \overline{d}$ and $\overline{c} < \overline{b}$. See Example 2.2.9 and Figure 7.

Example 2.2.9 (Example of Rule 2.2.8). Let $A_{\mathbb{R},s}$ have sinks $s_{-2} = -2$, $s_0 = 0$, $s_2 = 2$ and sources $s_{-1} = -1$, $s_1 = 1$ with $-\infty = s_{-3}$ and $+\infty = s_3$ as in Example 2.2.3. For $\alpha$ in $A$ such that one endpoint is in $E \downarrow$ and the other in $E \uparrow$, see Figure 7 for several examples of Rule 2.2.8.

The only case not covered by Rules 2.2.6 and 2.2.8 is when two arcs share an endpoint.

Rule 2.2.10. Let $\alpha = (\overline{a}, \overline{b})$ and $\beta = (\overline{c}, \overline{d})$ be in $A$. We have four cases: $\overline{a} = \overline{c}$, $\overline{a} = \overline{d}$, $\overline{b} = \overline{c}$, and $\overline{b} = \overline{d}$. (If two equalities hold at once we have $\alpha = \beta$.)

- If $\overline{a} = \overline{d}$ or $\overline{b} = \overline{c}$ then we say $\alpha$ and $\beta$ cross.
- If $\overline{a} = \overline{c}$ or $\overline{b} = \overline{d}$ then we say $\alpha$ and $\beta$ do not cross.

See Figure 8 for a visual depiction of this rule.
Definition 2.2.11. Define the crossing function $c : A \times A \to \{0, 1\}$ by

$$c(\alpha, \beta) = \begin{cases} 
1 & (\alpha = \beta) \text{ or } (\alpha \text{ and } \beta \text{ cross according to Rules 2.1.3, 2.2.8, and 2.2.10}) \\
0 & (\alpha \neq \beta) \text{ and } (\alpha \text{ and } \beta \text{ do not cross according to Rules 2.1.3, 2.2.8, and 2.2.10}).
\end{cases}$$

For $\alpha \neq \beta$, if $c(\alpha, \beta) = 1$ we say $\alpha$ and $\beta$ cross. Otherwise, we say $\alpha$ and $\beta$ do not cross.

We are now ready to prove the following lemma.

Lemma 2.2.12. Let $M_{[a,b]} \neq M_{[c,d]}$ be indecomposables in $\mathcal{C}(A_{\mathbb{R}}, S)$. Then $\{M_{[a,b]}, M_{[c,d]}\}$ is $E$-compatible if and only if $c(\Phi(M_{[a,b]}), \Phi(M_{[c,d]})) = 0$.

Proof. Setup. Let $\alpha = (\overline{a}, \overline{b}) = \Phi(M_{[a,b]})$ and $\beta = (\overline{c}, \overline{d}) = \Phi(M_{[c,d]})$ and note that by Lemma 2.2.5, $\alpha \neq \beta$. We note that Rules 2.2.6, 2.2.8, and 2.2.10 cover all possible combinations of endpoints for $\alpha$ and $\beta$. We show that if $c(\alpha, \beta) = 1$ then $\{M_{[a,b]}, M_{[c,d]}\}$ is not $E$-compatible and if $c(\alpha, \beta) = 0$ then $\{M_{[a,b]}, M_{[c,d]}\}$ is $E$-compatible. We follow the order in which the rules were stated.

Rule 2.2.6. If the endpoints of $\alpha$ and $\beta$ are all contained in $\mathcal{E}^\downarrow$ or all contained in $\mathcal{E}^\uparrow$, then if and only if statement follows from arguments similar to those in the proof of Lemma 2.1.5. Without loss of generality, suppose $\alpha$ has endpoints in $\mathcal{E}^\downarrow$ and $\beta$ has endpoints in $\mathcal{E}^\uparrow$. Then $c(\alpha, \beta) = 0$. If $a = s_m$ then $a$ is a source and if $b = s_n$ then $b$ is a sink. Dual statements for $c$ and $d$ are true as well. Using Definition 1.2.6 and Proposition 1.2.7 we see that $\{M_{[a,b]}, M_{[c,d]}\}$ is $E$-compatible.

Rule 2.2.8. Suppose $\alpha$ has both endpoints in $\mathcal{E}^\downarrow$ and $\beta$ has one endpoint each in $\mathcal{E}^\downarrow$ and $\mathcal{E}^\uparrow$. For now we assume all four endpoints of $\alpha$ and $\beta$ are distinct. Suppose $\overline{a} < \overline{b}, \overline{c} \in \mathcal{E}^\downarrow$, and $\overline{d} \in \mathcal{E}^\uparrow$. If $\overline{a} < \overline{b} < \overline{d}$ then $c(\alpha, \beta) = 1$ and one verifies there exists a distinguished triangle

$$M_{[a,b]} \to M_{[a,d]} \oplus M_{[c,b]} \to M_{[c,d]} \to$$

in $\mathcal{C}(A_{\mathbb{R}}, S)$. By Proposition 1.2.7, $\{M_{[a,b]}, M_{[c,d]}\}$ is not $E$-compatible. If $\overline{a} < \overline{b} < \overline{d}$ then $c(\alpha, \beta) = 1$ and one verifies there exists a distinguished triangle

$$M_{[a,b]} \to M_{[a,d]} \oplus M_{[c,b]} \to M_{[c,d]} \to$$

in $\mathcal{C}(A_{\mathbb{R}}, S)$ and by the same proposition $\{M_{[a,b]}, M_{[c,d]}\}$ is not $E$-compatible. If $\overline{a} < \overline{b}$ or $\overline{b} < \overline{d}$ we know $c(\alpha, \beta) = 0$ and it is straightforward to check that the $g$-vectors of $M_{[a,b]}$ and $M_{[c,d]}$ are $E$-compatible. Thus, $\{M_{[a,b]}, M_{[c,d]}\}$ is $E$-compatible.

Now we check when $\alpha$ and $\beta$ each have one endpoint in $\mathcal{E}^\downarrow$ and the other in $\mathcal{E}^\uparrow$. Suppose $c(\alpha, \beta) = 1$. For Rule 2.2.8(3), and without loss of generality, let $\overline{a}, \overline{c} \in \mathcal{E}^\downarrow$ and $\overline{b}, \overline{d} \in \mathcal{E}^\uparrow$. Up to symmetry, we have $\overline{a} < \overline{b} < \overline{d}$ and so $a < c < b < d$ in $\mathbb{R}$. One then verifies there exists a distinguished triangle

$$M_{[a,b]} \to M_{[a,d]} \oplus M_{[c,b]} \to M_{[c,d]} \to$$

in $\mathcal{C}(A_{\mathbb{R}}, S)$. Again using Proposition 1.2.7 we see $\{M_{[a,b]}, M_{[c,d]}\}$ is not $E$-compatible.

For Rule 2.2.8(4), and without loss of generality, let $\overline{a}, \overline{d} \in \mathcal{E}^\downarrow$ and $\overline{b}, \overline{c} \in \mathcal{E}^\uparrow$. Then $\overline{a} < \overline{b}$ and $\overline{b} < \overline{c}$ and one verifies there exists a distinguished triangle

$$M_{[c,d]} \to M_{[c,b]} \oplus M_{[a,d]} \to M_{[a,b]} \to$$
in \(C(A_{\mathbb{R},S})\). Again, \(\{M_{[a,b]}, M_{[c,d]}\}\) is not \(E\)-compatible.

Now suppose \(c(\alpha, \beta) = 0\). If \(x > y\) or \(x < y\), one verifies the \(g\)-vectors of \(M_{[a,b]}\) and \(M_{[c,d]}\) are \(E\)-compatible. If \(x < y\) and \(x < y\), then, up to symmetry \(x, z \in \mathcal{E}^1\) and \(y, w \in \mathcal{E}^1\). This means that \(x < y\) and \(y < z\) or that \(z < x\) and \(w < y\). Again one can check the \(g\)-vectors to see that \(M_{[a,b]}\) and \(M_{[c,d]}\) are \(E\)-compatible.

Rule 2.2.10. Now we assume \(\alpha\) and \(\beta\) share an endpoint.

If \(x = z\), then a straightforward calculation shows the \(g\)-vectors of \(M_{[a,b]}\) and \(M_{[c,d]}\) are \(E\)-compatible. Symmetrically, if \(y = w\), then \(\{M_{[a,b]}, M_{[c,d]}\}\) is \(E\)-compatible.

Next suppose \(x = w = (e, \varepsilon)\), for \(e \in \mathbb{R}\). Then \(M_{[a,b]} = M_{[e,b]}\) and \(M_{[c,d]} = M_{[c,e]}\). In particular, \(e \in [e, b]\) if and only if \(e \notin [e, e]\). Then one verifies the following is a distinguished triangle in \(C(A_{\mathbb{R}})\):

\[
M_{[c,d]} \rightarrow M_{[c,b]} \rightarrow M_{[a,b]} \rightarrow.
\]

By Proposition 1.2.7 again we see \(\{M_{[a,b]}, M_{[c,d]}\}\) is not \(E\)-compatible.

Finally, suppose \(x = w = ((s_n, s_{n+1}), \varepsilon)\) and note that \(\gamma \neq x\). If \(\varepsilon = -\) then \([a, b] = [s_n, b]\) and \([c, d] = [c, s_n]\). If \(\varepsilon = +\) then \([a, b] = [s_{n+1}, b]\) and \([c, d] = [c, s_{n+1}]\). In either case, one verifies we have the following distinguished triangle in \(C(A_{\mathbb{R},S})\):

\[
M_{[c,d]} \rightarrow M_{[c,b]} \rightarrow M_{[a,b]} \rightarrow.
\]

By Proposition 1.2.7 again we see \(\{M_{[a,b]}, M_{[c,d]}\}\) is not \(E\)-compatible.

Conclusion. For each of Rules 2.2.6, 2.2.8, and 2.2.10 we have shown (i) if \(c(\alpha, \beta) = 1\) then \(\{M_{[a,b]}, M_{[c,d]}\}\) is not \(E\)-compatible and (ii) if \(c(\alpha, \beta) = 0\) then \(\{M_{[a,b]}, M_{[c,d]}\}\) is \(E\)-compatible. \(\square\)

Definition 2.2.13. Let \(C_{\mathbb{R},S}\) be an additive category whose indecomposable objects are \(A\). Define Hom spaces and composition of morphisms similarly to 2.1.2. This also yields a Krull–Schmidt category. We say \(\{\alpha, \beta\}\) is \(N_{\mathbb{R},S}\)-compatible if and only if \(\alpha = \beta = \varepsilon \alpha, \beta = 0\).

Remark 2.2.14. Notice \(N_{\mathbb{R},S}\)-compatible is equivalent to Hom-orthogonal, not Ext-orthogonal.

Corollary 2.2.15 (to Lemma 2.2.12). Let \(M_{[a,b]}\) and \(M_{[c,d]}\) be indecomposables in \(C(A_{\mathbb{R}})\). Then \(\{\Phi(M_{[a,b]}), \Phi(M_{[c,d]}), \Phi(M_{[a,b]}), \Phi(M_{[c,d]}), \Phi(M_{[a,b]}, \Phi(M_{[c,d]}))\}\) is \(N_{\mathbb{R},S}\)-compatible if and only if \(\{M_{[a,b]}, M_{[c,d]}\}\) is \(E\)-compatible.

Theorem 2.2.16. Let \(A_{\mathbb{R},S}\) be a continuous quiver of type \(A\). The pairwise compatibility condition \(N_{\mathbb{R},S}\) induces the \(N_{\mathbb{R},S}\)-cluster theory of \(C_{\mathbb{R},S}\) and \(\Phi\) induces an isomorphism of cluster theories \((F, \eta) : \mathcal{F}_{\mathbb{R},S}(C_{\mathbb{R},S}) \rightarrow \mathcal{F}_E(C(A_{\mathbb{R},S}))\).

Proof. By Proposition 2.2.5 and Definition 2.2.13 we have a bijection \(\Phi : \text{Ind}(C(A_{\mathbb{R},S})) \rightarrow \text{Ind}(C_{\mathbb{R},S})\). The set \(\{M_{[a,b]}, M_{[c,d]}\}\) is \(E\)-compatible if and only if \(\{\Phi(M_{[a,b]}), \Phi(M_{[c,d]}), \Phi(M_{[a,b]}), \Phi(M_{[c,d]}), \Phi(M_{[a,b]}, \Phi(M_{[c,d]}))\}\) is \(N_{\mathbb{R},S}\)-compatible, by Corollary 2.2.15. Thus by Lemma 2.0.3 \(N_{\mathbb{R},S}\) induces the cluster theory \(\mathcal{F}_{\mathbb{R},S}(C_{\mathbb{R},S})\) and we have the isomorphism of cluster theories given by \(F(T) := \{\Phi^{-1}(\alpha) \mid \alpha \in T\}\) and \(\eta T(\alpha) := \Phi^{-1}(\alpha)\). \(\square\)

2.3. On the Classification of Cluster Theories of Continuous Type \(A\). In this section we identity some cluster theories of continuous type \(A\) which are isomorphic. We show there are at least four isomorphism classes of such cluster theories. The following notation is useful.

Notation 2.3.1. Let \(\mathcal{F}_P(C)\) and \(\mathcal{F}_Q(D)\) be two cluster theories. If there is an isomorphism of cluster theories \((F, \eta) : \mathcal{F}_P(C) \rightarrow \mathcal{F}_Q(D)\) then we say \(\mathcal{F}_P(C)\) is isomorphic to \(\mathcal{F}_Q(D)\) and write \(\mathcal{F}_P(C) \cong \mathcal{F}_Q(D)\).

Notation 2.3.2. Let \(A_{\mathbb{R},S}\) be a continuous quiver of type \(A\).

- By \((A_{\mathbb{R},S})^{-1}\) we denote the continuous quiver \(A_{\mathbb{R},R}\) where, if \(- \infty \neq s_0\), each source \(r_n\) in \(A_{\mathbb{R},R}\) is equal to a sink \(s_{n-1}\) and similarly for sinks in \(R\). If \(- \infty = s_0\) in \(A_{\mathbb{R},S}\), then each source \(r_n\) in \(A_{\mathbb{R},R}\) is instead equal to a source \(r_{n+1}\) and similarly for sinks in \(R\).
• By \(- (A_{R,S})\) we denote the continuous quiver \(A_{R,R}\) where each sink \(r_{2n}\) in \(A_{R,R}\) is equal to the sink \(- s_{-2n}\) in \(A_{R,S}\) and similarly for sources.

**Remark 2.3.3.** Notice that \(- (A_{R,S}) = A_{R,S}\). Furthermore, if \(- \infty = s_0\) or \(- \infty = s_{-1}\) then \(((A_{R,S})^{-1})^{-1}) = A_{R,S}\). If \(- \infty \neq s_0\) and \(- \infty \neq s_{-1}\), then we still have \(\text{rep}_k(A_{R,S})\) is equivalent to \(\text{rep}_k((A_{R,S})^{-1})^{-1}\) as \(k\)-linear abelian categories.

Finally, we see \((A_{R,S})^{-1} = (((A_{R,S})^{-1})^{-1})^{-1}\) and so \(\text{rep}_k((-(A_{R,S}))^{-1})\) is equivalent to \(\text{rep}_k((-(A_{R,S}))^{-1})\) as abelian categories.

**Proposition 2.3.4.** Let \(A_{R,S}\) be a continuous quiver of type \(A\) and \(A_{R,R} = (A_{R,S})^{-1}\). Then \(\mathcal{T}_{N_{R,S}}(C_{R,S}) \cong \mathcal{T}_{N_{R,R}}(C_{R,R})\).

**Proof.** Denote the sets of endpoints and arcs for \(A_{R,S}\) by \(\mathcal{E}_S\) and \(\mathcal{A}_S\). Denote the sets of endpoints and arcs for \(A_{R,R}\) by \(\mathcal{E}_R\) and \(\mathcal{A}_R\). Let the respective crossing functions be \(\epsilon_S\) and \(\epsilon_R\). There is an order preserving bijections \(g : \mathcal{E}_S^\downarrow \cong \mathcal{E}_R^\uparrow\) and \(h : \mathcal{E}_S^\uparrow \cong \mathcal{E}_R^\downarrow\). Let \(f : \mathcal{E}_S \cong \mathcal{E}_R\) be the bijection that is \(g\) on \(\mathcal{E}_S^\uparrow\) and \(h\) on \(\mathcal{E}_S^\downarrow\).

Notice Rules 2.1.3, 2.2.8, and 2.2.10 are symmetric with respect to \(\mathcal{E}_R^\uparrow\) and \(\mathcal{E}_R^\downarrow\), except at sinks and sources. Let \(\alpha = ([\pi, \eta])\) and \(\beta = ([\tau, \varpi])\) be in \(\mathcal{A}_S\). Let \(\gamma = (f([\pi, \eta]))\) and \(\delta = (f([\tau, \varpi]))\). Then \(\epsilon_S(\alpha, \beta) = 1\) if and only if \(\epsilon_R(\gamma, \delta) = 1\). Now apply Lemma 2.0.3.

**Proposition 2.3.5.** Let \(A_{R,S}\) be a continuous quiver of type \(A\) and \(A_{R,R} = -(A_{R,S})\). Then \(\mathcal{T}_{N_{R,S}}(C_{R,S}) \cong \mathcal{T}_{N_{R,R}}(C_{R,R})\).

**Proof.** Let \(\mathcal{E}_S, \mathcal{A}_S, \epsilon_S, \mathcal{E}_R, \mathcal{A}_R,\) and \(\epsilon_R\) be as in the proof of Proposition 2.3.4. Then we have order reversing bijections \(g : \mathcal{E}_S^\uparrow \cong \mathcal{E}_R^\downarrow\) and \(h : \mathcal{E}_S^\downarrow \rightarrow \mathcal{E}_R^\uparrow\). Let \(f : \mathcal{E}_S \cong \mathcal{E}_R\) be the bijection that is \(g\) on \(\mathcal{E}_S^\uparrow\) and \(h\) on \(\mathcal{E}_S^\downarrow\). Now proceed by a similar argument to Proposition 2.3.4.

**Theorem 2.3.6.** Let \(A_{R,S}\) be a continuous quiver of type \(A\). Then there is a diagram of isomorphisms of cluster theories:

\[
\begin{array}{cccc}
\mathcal{T}_E(C(A_{R,S})) & \cong & \mathcal{T}_E(C((A_{R,S})^{-1})) & \cong \\
\cong & & \cong & \\
\mathcal{T}_E(C(-(A_{R,S}))) & \cong & \mathcal{T}_E(C(((A_{R,S})^{-1})^{-1})). & \\
\end{array}
\]

**Proof.** Apply Propositions 2.3.4 and 2.3.5 and Remark 2.3.3.

**Corollary 2.3.7.** Let \(A_{R,S}\) and \(A_{R,S}\) be continuous quivers of type \(A\) such that one of the following is true: (i) \(|S| = |R|\) and \(|S| < \infty\), (ii) \(S\) and \(R\) are both bounded on exactly one side, or (iii) both \(S\) and \(R\) are indexed by \(\mathbb{Z}\). Then \(\mathcal{T}_E(C(A_{R,S})) \cong \mathcal{T}_E(C(A_{R,R})))\).

**Proof.** If there is an order- and indexing-preserving bijection \(S \cong R\) then \(\text{rep}_k(A_{R,S})\) and \(\text{rep}_k(A_{R,R})\) are equivalent as abelian categories and \(\mathcal{T}_E(C(A_{R,S})) \cong \mathcal{T}_E(C(A_{R,R})))\). Now apply Theorem 2.3.6.

The classification of cluster theories in Corollary 2.3.7 is nearly the classification of derived categories in Theorem 1.2.2.

We have two remaining isomorphisms of cluster theories we would like:

1. Any isomorphism between \(\mathcal{T}_{N_{R,S}}(C_{R,S})\) and \(\mathcal{T}_{N_{R,R}}(C_{R,R})\) where \(A_{R,S}\) has an even number \(\geq 2\) of sinks and sources in \(\mathbb{R}\) and \(A_{R,R}\) has an odd number of sinks and sources in \(\mathbb{R}\).
2. An isomorphism between \(\mathcal{T}_{N_{R}}(C_{R})\) and \(\mathcal{T}_{N_{S}}(C_{S})\) where \(A_{R}\) has no sinks or sources in \(\mathbb{R}\) and \(A_{S}\) has an even number \(\geq 2\) of sinks and sources in \(\mathbb{R}\).

We immediately share the unfortunate news:
Proposition 2.3.8. Let $A_{\mathbb{R}}$ be a continuous quiver of type $A$ with straight descending or straight ascending orientation. Let $A_{\mathbb{R},S}$ be a continuous quiver of type $A$ with at least one sink or source in $\mathbb{R}$. Then there is no isomorphism of cluster theories $\mathcal{F}_{N_{\mathbb{R}}}(C_{\mathbb{R}}) \to \mathcal{F}_{N_{\mathbb{R},S}}(C_{\mathbb{R},S})$.

Proof. The arc $\alpha$ corresponding to the indecomposable $M_{(-\infty, +\infty)}$ in $\mathcal{C}(A_{\mathbb{R}})$ is in every $N_{\mathbb{R}}$-cluster of $\mathcal{F}_{N_{\mathbb{R}}}(C_{\mathbb{R}})$. The arcs corresponding to the projectives from $\text{rep}_{k}(A_{\mathbb{R},S})$ form an $N_{\mathbb{R},S}$-cluster; this is similarly true for the arcs corresponding to the injectives from $\text{rep}_{k}(A_{\mathbb{R},S})$. However, there are not projective-injective objects in $\text{rep}_{k}(A_{\mathbb{R},S})$ and so these two clusters share no elements. Therefore, there cannot be such an isomorphism of cluster theories. \qed

This leaves us with at least four isomorphism classes of cluster theories of continuous type $A$: (i) no sinks or sources in $\mathbb{R}$, (ii) finitely-many sinks and sources in $\mathbb{R}$, (iii) half-bounded sinks and sources in $\mathbb{R}$, and (iv) unbounded sinks and sources in $\mathbb{R}$. However, it is not clear whether (ii) is just one class, separate classes for even and odd numbers, or a separate class for all numbers.

Open Questions:

- Does there exist a weak equivalence of cluster theories
  $$\mathcal{F}_{N_{\mathbb{R}}}(C_{\mathbb{R}}) \to \mathcal{F}_{N_{\mathbb{R},S}}(C_{\mathbb{R},S}) \quad \text{or} \quad \mathcal{F}_{N_{\mathbb{R},S}}(C_{\mathbb{R},S}) \to \mathcal{F}_{N_{\mathbb{R}}}(C_{\mathbb{R}}),$$
  where $A_{\mathbb{R}}$ has no sinks or sources in $\mathbb{R}$ and $A_{\mathbb{R},S}$ has an even number $\geq 2$ of sinks and sources in $\mathbb{R}$?

- Does there exist an isomorphism of cluster theories or weak equivalence of cluster theories
  $$\mathcal{F}_{N_{\mathbb{R},S}}(C_{\mathbb{R},S}) \to \mathcal{F}_{N_{\mathbb{R},R}}(C_{\mathbb{R},R}) \quad \text{or} \quad \mathcal{F}_{N_{\mathbb{R},R}}(C_{\mathbb{R},R}) \to \mathcal{F}_{N_{\mathbb{R},S}}(C_{\mathbb{R},S}),$$
  where $A_{\mathbb{R},S}$ has an odd number $n$ of sinks and sources in $\mathbb{R}$ and $A_{\mathbb{R},R}$ has $n + 1$ sinks and sources in $\mathbb{R}$?

2.4. Connection to $E$-Mutations. Let $A_{\mathbb{R},S}$ be a continuous quiver of type $A$. In this section we use geometric models to draw a $N_{\mathbb{R},S}$-mutation corresponding to an $E$-mutation. Because of our rules on crossing, mutation is not as clearly described as swapping diagonals of a quadrilateral. However, we can make similar descriptions. Let us begin with the “microscopic” scale. Let $A_{\mathbb{R},S}$ be a continuous quiver of type $A$ with at least one sink or source in $\mathbb{R}$. Let $a < b \in \mathbb{R}$ such that neither $a$ nor $b$ is a sink or source and $(a, \varepsilon), (b, \varepsilon) \in E^\downarrow$, for any $\varepsilon \in \{-, +\}$.

Let $T$ be an $N_{\mathbb{R},S}$-cluster such that $((a, -), (b, +)), ((a, +), (b, +)), ((a, +), (b, -)) \in T$. These correspond to the indecomposables $M_{(a,b)}$, $M_{(a,b)}$, and $M_{(a,b)}$, respectively, in $\mathcal{C}(A_{\mathbb{R},S})$. We can mutate at $\{(a, +), (b, +)\}$ to obtain $(T \setminus \{(a, +), (b, +)\}) \cup \{(a, -), (b, -)\}$. The picture one should have in mind is Figure 9.
We now move to the “macroscopic” scale. In $C(A_{\mathbb{R},S})$, we know that if $\{M_{[a,b]}M_{[c,d]}\}$ is not $E$-compatible then, up to reversing the roles of the indecomposables, we have the following distinguished triangle in $C(A_{\mathbb{R},S})$:

$$M_{[a,b]} \rightarrow M_{[a,d]} \oplus M_{[c,b]} \rightarrow M_{[c,d]} \rightarrow$$

where one of $M_{[a,d]}$ or $M_{[c,b]}$ may be 0. Now suppose we are $E$-mutating in some cluster at $M_{[a,b]}$ and obtain $M_{[c,d]}$, the left picture in Figure 10. If the middle object in the distinguished triangle is not an indecomposable, then, in the geometric model, we have two of the four sides of the quadrilateral we see in triangulations of polygons.

However, we do not know if we have the indecomposables corresponding to the two dotted arcs that complete the quadrilateral. The dotted arcs may be incompatible with $M_{[a,b]}$ and/or $M_{[c,d]}$. For example, if $b \in [a,b]$ then there is no arc with $(b,-)$ as a lower endpoint that is compatible with either of the arcs corresponding to $M_{[a,b]}$ and $M_{[c,b]}$. In the case where one of $M_{[c,b]}$ or $M_{[a,d]}$ is 0, we instead have the right picture in Figure 10. If some of the endpoints are in $E_\downarrow$ and others in $E_\uparrow$ then we instead draw pictures such as those in Figure 11.

**Figure 10.** Depictions of mutation on the “macroscopic” scale where we replace red arcs with blue arcs. Notice the arc orientations pointing from the lower element to the upper element. We only need the orientations in the right picture.

**Figure 11.** Depiction of macroscopic mutation where the arcs have endpoints mixed above and below. We mutate the red arc to the blue arc. The orientations of the arcs in the left picture do not matter but we need these in the right picture.

3. Continuous Mutation and Mutation Paths

This section is dedicated to the definition of a continuous mutation and the basic properties of continuous mutations. These generalize the familiar notion of mutation in a cluster structure. We define this new type of mutation for all cluster theories (Definition 1.2.9) though we use type $A$ cluster theories for our examples. Notably, for a cluster theory $T_P(C)$, any $P$-mutation can be thought of as a continuous $P$-mutation (see Example 3.2.1). In Section 3.4, we show how to interpret
continuous mutations via geometric models (Sections 2.1 and 2.2) using continuous type A as an example. The final subsection of this section is dedicated to the space of mutations (Definition 3.1.4), which generalizes the exchange graph of a cluster structure. We pose questions related specifically to the C-cluster theory of an arbitrary continuous quiver of type A at the end.

For Section 3 we fix C a Krull–Schmidt category and P a pairwise compatibility condition on the indecomposables in C such that P induces the P-cluster theory of C (Definition 1.2.9).

3.1. Continuous Mutation. In this section we define continuous mutation. In order to better interpret continuous mutations, we need to define a trivial mutation.

Definition 3.1.1. A trivial P-mutation is an identity function id_T : T → T, for any P-cluster T.

Definition 3.1.2. Let T and T′ be P-clusters and μ : T → T′ a bijection. We call μ a continuous P-mutation if it satisfies the following four properties.

- There is a set S ⊂ T such that μX = X if and only if X /∈ S.
- Let S′ = μ(S). For all μX ∈ S′, (i) μX /∈ T and (ii) {X, μX} is not P-compatible.
- There exist injections f_μ : S → [0, 1] and g_μ : S′ → [0, 1] such that (g_μ ∘ μ)|S = (f_μ)|S.
- For any subinterval J ⊂ [0, 1], where 0 ∈ J and 1 /∈ J, the following is a P-cluster:

\[(T \setminus f_μ^{-1}(J)) \cup g_μ^{-1}(J) = (T′ \setminus g_μ^{-1}(J)) \cup f_μ(J),\]

where J = [0, 1] \ J.

We need to justify the word ‘mutation.’ We do this with Propositions 3.1.3 and 3.1.4. The first states that every continuous mutation can be reversed. The second states that we may consider a continuous mutation a collection of mutations, one each at time t, for all t ∈ [0, 1].

The proof of the following proposition is a straightforward application of the definition.

Proposition 3.1.3. Let μ : T → T′ be a continuous P-mutation. Then μ^{-1} : T′ → T is also a continuous P-mutation.

Proposition 3.1.4. Let μ : T → T′ be a continuous P-mutation. For every t ∈ [0, 1], the following bijection (T \setminus f_μ^{-1}([0, t))) \cup g_μ^{-1}([0, t)) → (T \setminus f_μ^{-1}([0, t))) \cup g_μ^{-1}([0, t)) is a P-mutation:

\[X \mapsto \begin{cases} X & \text{if } X /\not\in f^{-1}(t) \\ g^{-1}(t) & \text{if } X = f^{-1}(t). \end{cases}\]

Proof. In the case (T \setminus f_μ^{-1}([0, t))) \cup g_μ^{-1}([0, t)) = (T \setminus f_μ^{-1}([0, t))) \cup g_μ^{-1}([0, t)) we have a trivial P-mutation. Suppose (T \setminus f_μ^{-1}([0, t))) \cup g_μ^{-1}([0, t)) \neq (T \setminus f_μ^{-1}([0, t))) \cup g_μ^{-1}([0, t)). Since f_μ and g_μ are injections, f_μ^{-1}([0, t)) differs from f_μ^{-1}([0, t)) by at most one element and by assumption they differ by at least one element; thus differing by exactly one element. This is similarly true for g_μ^{-1}([0, t)) and g_μ^{-1}([0, t)). By definition, μ(f_μ^{-1}(t)) = g_μ^{-1}(t) and {f_μ^{-1}(t), g_μ^{-1}(t)} are not P-compatible. Therefore, we have a P-mutation.

We conclude with this final definition that is useful in asking questions about the classification of E-clusters in C(A_{R,s}) (Definition 1.2.6) at the end of Section 3.5.2.

Definition 3.1.5. Let Z = \{1, ..., n\} or Z = Z_{>0}. For each i ∈ Z let μ_i be a continuous P-mutation such that the target of μ_i is the source of μ_{i+1} when i, i + 1 ∈ Z. We call \{μ_i\}_{i∈Z} a sequence of continuous P-mutations. If each μ_i mutates only one element of T_i we may also say that \{μ_i\} is a sequence of P-mutations.
3.2. Examples. In this section we highlight two existing examples of continuous mutations that do not feel so continuous followed by a new example. The first (Example 3.2.1) shows that a mutation, in the traditional sense, can be thought of as a continuous mutation. The second (Example 3.2.3) describes an infinite sequence of mutations. While these both exist in the literature, the contribution is that continuous mutation unifies the way to describe these types of mutations. We conclude with Proposition 3.2.5, which, as far as the author knows, does not exist anywhere in the literature.

Example 3.2.1. Let $\mathcal{C}$ be a Krull–Schmidt category with pairwise compatibility condition $\mathbf{P}$ on indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. Let $\mu : T \to (T \setminus \{X\}) \cup \{Y\}$ be a $\mathbf{P}$-mutation. Furthermore, let $S = \{X\}$, $S' = \{Y\}$, and $T' = (T \setminus \{X\}) \cup \{Y\}$. Finally, let $f : \{X\} \to [0,1]$ and $g : \{Y\} \to [0,1]$ each send $X$ and $Y$ to $\frac{1}{2}$, respectively. This meets the requirements for the definition of a continuous mutation.

The second example is based on the completed infinity-gon from [5].

Definition 3.2.2. Let $\mathcal{E} = \mathbb{Z} \cup \{-\infty, +\infty\}$ with the usual total ordering. Let

$$
\mathcal{A} = \{(i,j) \in \mathcal{E} \times \mathcal{E} \mid \exists k \in \mathcal{E} \text{ s.t. } i < k < j\} \setminus \{(-\infty, +\infty)\}.
$$

Define the crossing function $c : \mathcal{A} \times \mathcal{A} \to \{0,1\}$ by

$$
c((i,j),(i',j')) = \begin{cases} 
1 & \text{if } ((i,j) = (i',j')) \text{ or } (i < i' < j < j') \text{ or } (i' < i < j < j') \\
0 & \text{otherwise.}
\end{cases}
$$

We define $\mathcal{C}(\mathbb{N}_{\infty})$ to be the additive category whose indecomposable objects are $\mathcal{A}$. Define Hom spaces and composition as in Definitions 2.1.2 and 2.2.13. We again obtain a Krull–Schmidt category. For $\alpha \neq \beta$, we say $\{\alpha, \beta\}$ is $\mathbb{N}_{\infty}$-compatible if and only if $c(\alpha, \beta) = 0$.

Baur and Graz proved in [5] that $\mathbb{N}_{\infty}$ induces the $\mathbb{N}_{\infty}$-cluster theory of $\mathcal{C}(\mathbb{N}_{\infty})$.

Baur and Graz define a $T$-admissible sequence of arcs $\{\alpha_i\}$ is one where $\alpha_1$ is $\mathbb{N}_{\infty}$-mutable in $T_1 = T$ and each $T_i$ for $i > 1$ is obtained by mutating $\alpha_{i-1}$ which must be mutable in $T_{i-1}$. Note this sequence may be infinite so long as there is a first arc in the sequence. Baur and Graz note that mutating along a $T$-admissible sequence does not always result in a $\mathbb{N}_{\infty}$-cluster. I.e., the colimit of such a sequence of mutations may not be a $\mathbb{N}_{\infty}$-cluster.

Example 3.2.3. Let $T$ be an $\mathbb{N}_{\infty}$-cluster in $\mathcal{C}(\mathbb{N}_{\infty})$ and $\{\alpha_i\}$ a $T$-admissible sequence of arcs. Since each $\mathbb{N}_{\infty}$-mutation $\mu_i : T_i \to T_{i+1}$ is also a continuous $\mathbb{N}_{\infty}$-mutation any admissible sequence of arcs yields a sequence of continuous $\mathbb{N}_{\infty}$-mutations.

Now suppose $\{\alpha_i\} \subseteq T$ and the result of mutating along $\{\alpha_i\}$ yields an $\mathbb{N}_{\infty}$-cluster $T'$. Then we let $S = \{\alpha_i\}$ and let $f : S \to [0,1]$ be given by $\alpha_i \mapsto 1 - \frac{1}{i+1}$. Let $S' = \{\mu_i(\alpha_i)\}$ and let $g : S' \to [0,1]$ be given by $\mu_i(\alpha_i) \mapsto 1 - \frac{1}{i+1}$. We now have a continuous $\mathbb{N}_{\infty}$-mutation.

In general, a $T$-admissible sequence of arcs can be “grouped” into intervals of arcs which each belong to the first cluster of the group. This yields a sequence of $\mathbb{N}_{\infty}$-mutations in a somewhat minimal way. Of course, this does not work if $\{\alpha_i\} \subseteq T$ and mutation along $\{\alpha_i\}$ does not result in an $\mathbb{N}_{\infty}$-cluster.

Remark 3.2.4. Let $\mathcal{C}$ be a Krull–Schmidt category with pairwise compatibility condition $\mathbf{P}$ on indecomposables such that $\mathbf{P}$ induces the $\mathbf{P}$-cluster theory of $\mathcal{C}$. As seen in Example 3.2.3 it might be possible to construct a sequence of (continuous) $\mathbf{P}$-mutations that does not yield a $\mathbf{P}$-cluster. The authors of [5] provide a way to complete their compatible sets for their cluster theory.

Proposition 3.2.5. Let $A_{\mathbb{R}}$ have the straight descending orientation, $\text{Proj}$ be the $\mathbf{E}$-cluster containing all the projectives from $\text{rep}_{k}(A_{\mathbb{R}})$, and $\text{Inj}$ be the $\mathbf{E}$-cluster containing the injectives from $\text{rep}_{k}(A_{\mathbb{R}})$. There is a sequence of continuous mutations $\{\mu_1, \mu_2\}$ from $\text{Proj}$ to $\text{Inj}$.
Proof. Recall that every indecomposable in \( C(A_\mathbb{R}) \) comes from an indecomposable \( M_f \) in \( \text{rep}_\mathbb{R}(A_\mathbb{R}) \) (Definition 1.1.3, Theorem 1.1.5, Proposition 1.2.1, and [16, Proposition 3.1.4]). Recall also that \([a,b]\) means the inclusion of \( a \) or \( b \) is either indeterminate or clear from context (see Conventions in the introduction) and Theorem 1.1.4). Note that \( \mathcal{P}\text{proj} \cap \mathcal{I}n\text{j} = \{M_{(-\infty,+\infty)}\}\).

We construct two continuous \( E \)-mutations to mutate \( \mathcal{P}\text{proj} \) to \( \mathcal{I}n\text{j} \). First, let \( S_1 = \mathcal{P}\text{proj} \) and define \( f_1 : \mathcal{P}\text{proj} \to \{0,1\} \) in two parts. For \( M_{(-\infty,x)} \) and \( M_{(-\infty,x]} \) in \( \mathcal{P}\text{proj} \), we let
\[
f_1(M_{(-\infty,x)}) = \frac{1}{2} \left( \frac{\tan^{-1} x}{2\pi} + \frac{1}{4} \right) \quad f_1(M_{(-\infty,x]}) = 1 - \left( \frac{\tan^{-1} x}{2\pi} + \frac{3}{4} \right).
\]
The “middle” \( E \)-cluster is
\[
T_2 := \{M_{(-\infty,+\infty)}\} \cup \{M_{[x,+\infty)}, M_{[x,x]} \mid x \in \mathbb{R}\}.
\]
We then define \( g_1 : T_2 \to [0,1] \) to match with \( f_1 \):
\[
g_1(M_{[x,x]}) = \frac{1}{2} \left( \frac{\tan^{-1} x}{2\pi} + \frac{1}{4} \right) \quad g_1(M_{[x,+\infty)}) = 1 - \left( \frac{\tan^{-1} x}{2\pi} + \frac{3}{4} \right).
\]
Both \( f_1 \) and \( g_1 \) are injections and we may define \( \mu_1(M) = g^{-1}(f(M)) \) and obtain the continuous \( E \)-mutation \( \mu_1 : \mathcal{P}\text{proj} \to T_2 \).

Now let \( S_2 = \{M_{[x,x]} \mid x \in \mathbb{R}\} \subset T_2 \) and \( S'_2 = \{M_{(x,+\infty)} \mid x \in \mathbb{R}\} \subset \mathcal{I}n\text{j} \). We define \( f_2 : T_2 \to [0,1] \) and \( g_2 : \mathcal{I}n\text{j} \to [0,1] \) by
\[
f_2(M_{[x,x]}) = \tan^{-1} x = \frac{1}{2} = g_2(M_{[x,+\infty)}).
\]
We define \( \mu_2(M) \) to be \( M \) if \( M \notin S_2 \) and \( g^{-1}(f(M)) \) if \( M \in S_2 \). This gives the continuous \( E \)-mutation \( \mu_2 : T_2 \to \mathcal{I}n\text{j} \). Thus we have a sequence of continuous \( E \)-mutations \( \{\mu_1, \mu_2\} \) to mutate the projectives into the injectives.

\[\square\]

3.3. Mutation Paths. In this section we define mutation paths, which should be thought of as a generalization of a sequence of mutations. At first we formally define a long sequence of continuous mutations (Definition 3.3.1) and then move on to mutation paths in general (Definition 3.3.2). Note also that a continuous mutation is an example of a mutation path (Example 3.3.5) just as a mutation is an example of a continuous mutation.

A mutation path should be thought of as a generalization of a path of mutations in the exchange graph of a cluster structure. This is formalized in Section 3.5. As before, our definitions are for any cluster theory but our interest is in \( E \)-cluster theories of \( A_\mathbb{R} \) quivers.

**Definition 3.3.1.** Let \( \overline{\pi} = \{i\mu \mid iT_0 \to iT_1\}_{i \in \mathbb{Z}} \) be a collection of continuous mutations such that \( iT_1 = i+1T_0 \). This yields a diagram in \( \text{Sets} \):
\[
\cdots \to i-1T_1 \xrightarrow{i-1\mu} iT_0 \xrightarrow{i\mu} iT_1 = i+1T_0 \xrightarrow{i+1\mu} i+2T_0 \to \cdots
\]
If this diagram has a limit and colimit we call \( \overline{\pi} \) a long sequence of continuous mutations and we call the limit and colimit the source and target of \( \overline{\pi} \), respectively.

**Definition 3.3.2.** Define a category \( \mathcal{I} \) whose objects are pairs \((x,i) \in [0,1] \times \{0,1\} \). Consider \([0,1] \) and \([0,1] \) with their respective usual total ordering. Morphisms in \( \mathcal{I} \) are defined by
\[
\text{Hom}_\mathcal{I}((s,i),(t,j)) := \begin{cases} \{\ast\} & \text{if } s < t \text{ or } (s = t \text{ and } i \leq j) \\ \emptyset & \text{otherwise.} \end{cases}
\]
Let \( \overline{\pi} : \mathcal{I} \to \text{Sets} \) be a functor such that \( \overline{\pi}^* : \overline{\pi}(s,0) \to \overline{\pi}(s,1) \) is a (possibly trivial) \( P \)-mutation in \( \mathcal{F}_P(C) \). Then we call \( \overline{\pi} \) a \( P \)-mutation path.
Remark 3.3.3. The reader may notice that the target of the functor is not \( \mathcal{R}_p(C) \), but just \( \text{Sets} \). This is because we have not defined \( \mathcal{R}_p(C) \) (in Definition 1.2.9) to be closed under any kind of transfinite composition. However, transfinite composition is indeed sometimes defined in \( \text{Sets} \). For example, if every set in a diagram has the same cardinality and every morphism is a bijection, the transfinite composition is well-defined (and in this case is also a bijection). We only ensure the smallest morphisms \((s, 0) \rightarrow (s, 1)\) are in \( \mathcal{R}_p(C) \).

Proposition 3.3.4. Let \( \overline{p} : I \rightarrow \text{Sets} \) be a \( P \)-mutation path.

Let \( \overline{p}^{-1} : I \rightarrow \text{Sets} \) be a functor given by
\[
\overline{p}^{-1}(s, i) := \overline{p}(1 - s, 1 - i)
\]
\[
\overline{p}^{-1}((s_i) \rightarrow (t, j)) := \overline{p}((1 - t, 1 - j) \rightarrow (1 - s, 1 - i)).
\]

Then \( \overline{p}^{-1} \) is also a \( P \)-mutation path.

Proof. Since \( \mathcal{R}_p(C) \) is a groupoid inside \( \text{Sets} \) the definition of \( \overline{p}^{-1} \) amounts to reversing the order of the objects and taking the inverse morphism between each pair of objects in the image. \( \square \)

Example 3.3.5. Let \( \mu : T \rightarrow T' \) be a continuous \( P \)-mutation. Let \( \bar{\mu} : I \rightarrow \text{Sets} \) be defined in the following way. On objects,
\[
\overline{\mu}(s, 0) = (T \setminus f^{-1}[0, s]) \cup g^{-1}[0, s)
\]
\[
\overline{\mu}(s, 1) = (T \setminus f^{-1}[0, s]) \cup g^{-1}[0, s].
\]

By Proposition 3.1.4, for each \( s \in [0, 1] \), \( \mu \) defines a \( P \)-mutation \( \overline{\mu}(s, 0) \rightarrow \overline{\mu}(s, 1) \). Define \( \overline{\mu} : \overline{\mu}(s, 0) \rightarrow \overline{\mu}(s, 1) \) to be precisely this \( P \)-mutation. Thus each continuous \( P \)-mutation is a \( P \)-mutation path.

Below we construct some variables \( i_s, a_s, b_s, \) and \( t_s \) for each \( s \in [0, 1] \). We use these to show how a long sequence of continuous mutations can be considered as a mutation path.

Construction 3.3.6. Let \( \overline{\mu} \) be a long sequence of continuous mutations and fix \( 0 < \varepsilon < 1 \). For each \( s \in (0, 1) \), there exists \( i \in \mathbb{Z} \) such that
\[
\left( \frac{\tan^{-1} i}{\pi} + \frac{1}{2} \right) \leq s < \left( \frac{\tan^{-1}(i + 1)}{\pi} + \frac{1}{2} \right).
\]

Note that since the right inequality is strict, there is a unique such \( i \) for each \( s \in (0, 1) \). Denote it by \( i_s \). Let
\[
a_s := \left( \frac{\tan^{-1} i_s}{\pi} + \frac{1}{2} \right)
\]
\[
b_s := \left( \frac{\tan^{-1}(i + 1)}{\pi} + \frac{1}{2} \right).
\]

Note that if \( i_s = i_{s'} \) for \( s \) and \( s' \) then \( a_s = a_{s'} \) and \( b_s = b_{s'} \). We now define \( t_s \):
\[
t_s := \begin{cases} 
0 & s \in [a_s, (1 - \varepsilon)a_s + \varepsilon b_s] \\
(s - (1 - \varepsilon)a_s - \varepsilon b_s)/(1 - 2\varepsilon)(b_s - a_s) & s \in [(1 - \varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1 - \varepsilon)b_s] \\
1 & s \in (\varepsilon a_s + (1 - \varepsilon)b_s, b_s).
\end{cases}
\]

We provide a picture to make the variables easier to understand for \( s \in \left[\frac{1}{2}, \frac{3}{4}\right] \):

Proposition 3.3.7. Let \( \overline{\mu} \) be a (long) sequence of continuous mutations. Then \( \overline{\mu} \) is also a mutation path.
Figure 12. Schematic of $t_s$ for $s \in \left[ \frac{1}{4}, \frac{3}{4} \right]$.

**Proof.** We may consider $\overline{\mu}$ as a functor $\mathcal{I} \to \text{Sets}$ in the following way. We now make our assignment on objects:

$$(s, 0) \mapsto \begin{cases} iT_0 = i_{-1}T_1 & s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s] \\ (i_s T \setminus i^{-1} f^{-1}[0, t_s]) \cup i g^{-1}[0, t_s] & s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s] \\ iT_1 = i_{+1}T_0 & s \in (\varepsilon a_s + (1-\varepsilon)b_s, b_s) \end{cases}$$

$$(s, 1) \mapsto \begin{cases} iT_0 = i_{-1}T_1 & s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s] \\ (i_s T \setminus i^{-1} f^{-1}[0, t_s]) \cup i g^{-1}[0, t_s] & s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s] \\ iT_1 = i_{+1}T_0 & s \in (\varepsilon a_s + (1-\varepsilon)b_s, b_s). \end{cases}$$

When $s \in [(1-\varepsilon)a_s + \varepsilon b_s, \varepsilon a_s + (1-\varepsilon)b_s]$ we see by Proposition 3.1.4 that the morphism $*: (s, 0) \to (s, 1)$ is sent to a (possibly trivial) P-mutation. When $s \in [a_s, (1-\varepsilon)a_s + \varepsilon b_s) \cup (\varepsilon a_s + (1-\varepsilon)b_s, b_s)$ the morphism $*: (s_0) \to (s_1)$ is sent to the trivial P-mutation on $\overline{\mu}(s, 0)$. This defines a mutation path.

The $\varepsilon$ “padding” in Construction 3.3.6 is necessary to prove Proposition 3.3.7. If we did not have the “padding” we would attempt to assign two P-mutations, or their composition, to morphisms such as $*: (\frac{1}{2}, 0) \to (\frac{1}{2}, 1)$.

**Remark 3.3.8.** Let $\overline{\mu}$ be a long sequence of continuous P-mutations. We see in Proposition 3.3.7 that for a fixed $\varepsilon$ the the inverse path $\overline{\mu}^{-1}$ agrees with the inverse sequence $\{-i\mu\}_{i \in \mathbb{Z}}$. Thus when working with a long sequence of continuous mutations we need not be specific about which inverse we take as long as an $\varepsilon$ has been chosen.

**Definition 3.3.9.** Let $\overline{\mu}_1, \overline{\mu}_2 : \mathcal{I} \to \text{Sets}$ be two P-mutation paths and suppose $\overline{\mu}_1(1, 0) = \overline{\mu}_2(0, 0)$ and $\overline{\mu}_1(1, 1) = \overline{\mu}_2(0, 1)$.

We define the composition of P-mutation paths, denoted $\overline{\mu}_1 \cdot \overline{\mu}_2$ in the following way:

$$\overline{\mu}_1 \cdot \overline{\mu}_2(s, i) := \begin{cases} \overline{\mu}_1(2s, i) & 0 \leq s \leq \frac{1}{2} \\ \overline{\mu}_2(2s - 1, i) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

$$\overline{\mu}_1 \cdot \overline{\mu}_2((s, 0) \to (s, 1)) := \begin{cases} \overline{\mu}_1((2s, 0) \to (2s, 1)) & 0 \leq s \leq \frac{1}{2} \\ \overline{\mu}_2((2s - 1, 0) \to (2s - 1, 1)) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

**Proposition 3.3.10.** Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be P-mutation paths such that

$$\overline{\mu}_1(1, 0) = \overline{\mu}_2(0, 0) \quad \text{and} \quad \overline{\mu}_1(1, 1) = \overline{\mu}_2(0, 1).$$

Then $\overline{\mu}_1 \cdot \overline{\mu}_2$ is a P-mutation path.
Proof. By assumption the definitions agree at $\frac{1}{2}$. For $0 \leq s < \frac{1}{2}$ and $\frac{1}{2} < t \leq 1$, the morphism $ar{m}_1 \cdot \bar{m}_2^* : \bar{m}_1 \cdot \bar{m}_2(s, i) \to \bar{m}_1 \cdot \bar{m}_2(t, j)$ is the composition

$$\bar{m}_1 \cdot \bar{m}_2(s, i) \to \bar{m}_1 \cdot \bar{m}_2 \left(\frac{1}{2}, 0\right) \to \bar{m}_1 \cdot \bar{m}_2 \left(\frac{1}{2}, 1\right) \to \bar{m}_1 \cdot \bar{m}_2(t, j).$$

\[\square\]

Remark 3.3.11. The composition of two long sequences of continuous mutations as in Definition 3.3.9 is not a long sequence of continuous mutations as in Proposition 3.3.7.

3.4. Connection to $N_{\mathbb{R}^2}^S$-mutations. The more interesting pictures of $N_{\mathbb{R}^2}^S$ mutations (Section 2.2) are those of continuous mutations. In this section we use our geometric models to show how one may picture a continuous $E$-mutation by drawing the corresponding continuous $N_{\mathbb{R}^2}^S$-mutation.

In particular, those continuous mutations that cannot be described as any type of sequence of mutations, which is discrete. Consider $A_\mathbb{R}$ with straight descending orientation. Let $T$ be $\{P_x, M_{[x, x]} \mid x \in \mathbb{R}\} \cup \{P_{+\infty}\}$ and $\phi : \mathbb{R} \to (0, 1)$ be some order reversing bijection. Let $f : \{P_x \mid x \in \mathbb{R}\} \to [0, 1]$ be given by $P_x \mapsto \phi(x)$. Let $g : \{I_x \mid x \in \mathbb{R}\} \to [0, 1]$ be given by $I_x \mapsto \phi(x)$ and let $T' = \{I_x, M_{[x, x]} \mid x \in \mathbb{R}\} \cup \{P_{+\infty}\}$. Then we have a continuous mutation $T \to T'$.

We would like to show what this looks like in terms of arcs. Of course, we cannot depict each of the mutations at time $t$ for all $t \in (0, 1)$, as we do not have uncountably-many pages. However, we can think of the process as an animation and take a few select frames so that we have the general idea. In Figure 13, we only show 6 frames. One could make a proper animation at a sufficiently high frame rate to get the full effect.

![Figure 13. Six frames depicting a continuous $N_{\mathbb{R}^2}^S$-mutation.](image)

Figure 13. Six frames depicting a continuous $N_{\mathbb{R}^2}^S$-mutation. (All arcs have orientation left to right.) In the frames between $t = 0$ and $t = 1$ we mutate the red arc to the blue arc. The first and sixth frames are $T$ and $T'$, respectively. The other four frames are at time $\frac{i}{5}$ for $i \in \{1, 2, 3, 4\}$. We include $\approx 40$ arcs of the uncountably many in the same way one includes level curves in a topographical map.

3.5. Space of Mutations. In this section we define the space of mutations (Definition 3.5.2) which generalizes the exchange graph of a cluster structure. The intent is to view mutation paths (Definition 3.3.2) as paths in a topological space just as a sequence of mutations of a cluster structure forms a path in the exchange graph. This majority of this section is for cluster theories in general. However, its purpose is to study $E$-clusters in the future and so we return our attention to $E$-clusters at the end of the section.
Since the class of indecomposable objects in \( \mathcal{C} \) form a set and \( \mathbb{C} \) is Krull–Schmidt, we see \( \mathbb{C} \) is small. In particular, the class of morphisms in \( \mathcal{C} \) is a set. Denote the set of mutations by \( (\mathcal{R}_P(\mathcal{C}))_1 \).

**Definition 3.5.1.** Let \( \overline{\pi} \) be a \( P \)-mutation path and denote by \( p_{\overline{\pi}} \) the induced function from \([0,1]\) to \((\mathcal{R}_P(\mathcal{C}))_1\).

**Definition 3.5.2.** We define the set \( P(\mathcal{C}) \subset (\mathcal{R}_P(\mathcal{C}))_1 \) to be the set containing all (trivial) \( P \)-mutations.

We give the set of \( P \)-mutations a topology in the following way. Consider \([0,1]\) with the usual topology. A set \( U \subseteq P(\mathcal{C}) \) is called open if, for all \( p_{\overline{\pi}} : [0,1] \to P(\mathcal{C}) \) induced by a \( P \)-mutation \( \overline{\pi} \), \( p_{\overline{\pi}}^{-1}(U) \) is open in \([0,1]\). We call \( P(\mathcal{C}) \) the space of \( P \)-mutations.

**Proposition 3.5.3.** Then the open sets in Definition 3.5.2 form a topology on \( P(\mathcal{C}) \).

**Proof.** Trivially, both \( \emptyset \) and \( P(\mathcal{C}) \) are open. Suppose \( p_{\overline{\pi}} : [0,1] \to P(\mathcal{C}) \) is induced by a \( P \)-mutation path \( \overline{\pi} \). Let \( \{U_1, \ldots, U_n\} \) be open in \( P(\mathcal{C}) \). Since \( \bigcap_{i=1}^{n} p_{\overline{\pi}}^{-1}(U_i) = p_{\overline{\pi}}^{-1} \left( \bigcap_{i=1}^{n} U_i \right) \), we see that \( \bigcap_{i=1}^{n} U_i \) is open in \( P(\mathcal{C}) \). Now consider a collection \( \{U_\alpha\} \) of open sets in \( P(\mathcal{C}) \). Since \( \bigcup_{\alpha} p_{\overline{\pi}}^{-1}(U_\alpha) = p_{\overline{\pi}}^{-1} \left( \bigcup_{\alpha} U_\alpha \right) \), we see that \( \bigcup_{\alpha} U_\alpha \) is open in \( P(\mathcal{C}) \). This concludes the proof. \( \square \)

**Remark 3.5.4.** We consider a \( P \)-cluster \( T \) to be the trivial mutation \( T \to T \) in \( P(\mathcal{C}) \). We wish to consider paths that start and end at clusters rather than at mutations (see Proposition 3.5.6).

**Proposition 3.5.5.** The space of \( P \)-mutations is not Hausdorff.

**Proof.** Let \( \mu : T \to (T \setminus \{X\}) \cup \{Y\} \) be a \( P \)-mutation. Let \( \overline{\mu} \) be the \( P \)-mutation path that induces the path \( p_{\overline{\mu}} \) given by

\[
p_{\overline{\mu}}(t) = \begin{cases} 
T & t < 0 \\
\mu & t = 1
\end{cases}
\]

Let \( U \) be an open set that contains \( \mu \). If \( T \notin U \) then \( p_{\overline{\mu}}^{-1}(U) \) is not open. This would be a contradiction and so \( T \in U \). Thus, for any \( P \)-mutation \( \mu : T \to T' \) and open set \( U \) containing \( \mu \), \( T, T' \in U \) as well. Therefore, \( P(\mathcal{C}) \) is not Hausdorff. \( \square \)

**Proposition 3.5.6.** Let \( p : [0,1] \to P(\mathcal{C}) \) be a path in \( P(\mathcal{C}) \). Then there is a path \( q : [0,1] \to P(\mathcal{C}) \) whose endpoints are clusters (see Remark 3.5.4) such that \( p \) and \( q \) are homotopic.

**Proof.** Let \( p : [0,1] \to P(\mathcal{C}) \) be a path in \( P(\mathcal{C}) \), let \( T_0 \) be the source of \( p(0) \), and let \( T_1 \) the target of \( p(1) \).

For any \( 0 < \varepsilon << \frac{1}{2} \), let \( q_\varepsilon : [0,1] \to P(\mathcal{C}) \) be the path given by:

\[
q_\varepsilon(t) = \begin{cases} 
T_0 & \text{if } t < \varepsilon \\
T_1 & \text{if } (1-\varepsilon) < t \\
p \left( (t - \frac{1}{2})(1 - 2\varepsilon) + \frac{1}{2} \right) & \text{if } \varepsilon \leq t \leq (1 - \varepsilon)
\end{cases}
\]

We see that \( q_\varepsilon \) is homotopic to the composition of three paths. The first is constant at \( T_0 \) except the last point is \( p(0) \). The second is \( p \). The third is constant at \( T_1 \) except the first point is \( p(1) \). In particular, the first and third path are induced by \( P \)-mutation paths. Thus, \( q_\varepsilon \) is indeed a path. We just say \( q_0 = p \).

Fix a \( 0 < \varepsilon << \frac{1}{2} \). Let \( H : [0,1] \times [0,1] \to P(\mathcal{C}) \) be given by:

\[
H(t, s) := q_{\varepsilon s}(t).
\]

Let \( U \) be open in \( P(\mathcal{C}) \). If the inverse image of \( U \) does not contain \( p(0) \) or \( p(1) \) then \( H^{-1}(U) \) is open in \([0,1] \times [0,1] \).
Now suppose $U$ contains $p(0)$. By the proof of Proposition 3.5.5 we see that $T_0 \in U$ as well. Similarly, if $p(1) \in U$ then $T_1 \in U$. Therefore, if $U$ is open in $P(\mathcal{C})$ then $H^{-1}(U)$ is open in $[0,1] \times [0,1]$, completing the proof. \qed

**Definition 3.5.7.** Let $T_1$ and $T_2$ be $P$-clusters of $\mathcal{C}$.

1. We say $T_2$ is reachable from $T_1$ if there is a path $p : [0,1] \to P(\mathcal{C})$ such that $p(0) = T_1$ and $p(1) = T_2$.

2. We say $T_2$ is strongly reachable from $T_1$ if there is a $P$-mutation path $\mathbf{p}$ that (i) comes from a long sequence of continuous $P$-mutations and (ii) induces a path $p_{\mathbf{p}} : [0,1] \to P(\mathcal{C})$ such that $p_{\mathbf{p}}(0) = T_1$ and $p_{\mathbf{p}}(1) = T_2$.

**Theorem 3.5.8.** Let $A_\mathbb{R}$ be the continuous quiver of type $A$ with straight descending orientation. The cluster of injectives, $I_{nj}$ is strongly reachable from the cluster of projectives, $Proj$.

**Proof.** In Proposition 3.2.5 we see there is a sequence of $E$-mutations $\{\mu_1, \mu_2\}$ to mutate $Proj$ to $I_{nj}$. Choose some $0 < z << \frac{1}{2}$ and note that a sequence of $E$-mutations is also a long sequence of $E$-mutations. Then, as in Proposition 3.3.7, we have a $E$-mutation path $\mathbf{p}$ with source $Proj$ and target $I_{nj}$. \qed

Open Questions. Let $A_{R,S}$ be a continuous quiver of type $A$ and $\mathcal{C}(A_{R,S})$ the $E$-cluster theory of $\mathcal{C}$ (Definition 1.2.6).

- Is the space $E(\mathcal{C}(A_{R,S}))$ path connected?
- If $E(\mathcal{C}(A_{R,S}))$ is not path-connected, what do its path components look like? What does the path component containing the cluster of projectives look like?
- If $E(\mathcal{C}(A_{R,S}))$ is not path-connected, are there clusters $T$ and $T'$ that are reachable but not strongly reachable from one another.
- If $E(\mathcal{C}(A_{R,S}))$ is path connected, which clusters are strongly reachable from the cluster of projectives?

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