Weak versus strong wave turbulence in the MMT model

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Within the spirit of fluid turbulence, we consider the one-dimensional Majda-McLaughlin-Tabak (MMT) model that describes the interactions of nonlinear dispersive waves. We perform a detailed numerical study of the direct energy cascade in the defocusing regime. In particular, we consider a configuration with large-scale forcing and small scale dissipation, and we introduce three non-dimensional parameters: the ratio between nonlinearity and dispersion, $\epsilon$, and the analogues of the Reynolds number, $Re$, i.e. the ratio between the nonlinear and dissipative time-scales, both at large and small scales. Our numerical experiments show that (i) in the limit of small $\epsilon$ the spectral slope observed in the statistical steady regime corresponds to the one predicted by the Weak Wave Turbulence (WWT) theory. (ii) As the nonlinearity is increased, the WWT theory breaks down and deviations from its predictions are observed. (iii) It is shown that such departures from the WWT theoretical predictions are accompanied by the phenomenon of intermittency, typical of three dimensional fluid turbulence. We calculate the structure-function as well as the probability density function of the wave field at each scale and show that the degree of intermittency depends on $\epsilon$.

I. INTRODUCTION

Many physical phenomena are associated with the propagation of dispersive waves. While in some cases their dynamics is linear, many relevant situations manifest a non-negligible nonlinearity which produces complex patterns. When the number of degrees of freedom is large enough, such problems must be treated in a statistical manner. The Weak-Wave Turbulence (WWT) theory is a very general framework by which the statistical properties of a large number of incoherent and interacting waves can be studied. The theory, developed during the late sixties [1–4], is based on a systematic analytical approach that culminates in the so called wave kinetic equation that describes the evolution of the wave spectrum in time (homogeneity and weak nonlinearity are assumed). The wave kinetic equation is thus the analogue of the Boltzmann equation for classical particles and, in principle, should be able to give reliable predictions for the statistical distribution of energy as a function of wave numbers, as well as for various statistical observables.

The WWT theory has been applied to a variety of fields such as for example ocean waves [1 5 6], capillary waves [7 8] Alfven waves [9] or optical waves [10]. WWT constitutes hence an interdisciplinary tool suitable for investigating the statistical mechanics of a large number of interacting waves. A remarkable aspect of such a theory is that, in the presence of an external forcing and dissipation, exact solutions of the kinetic equation describing constant fluxes of its quadratic conserved quantities can be obtained analytically; this important result was achieved for the first time by V. Zakharov in 1966[1]. Despite the beauty of these theoretical results it is of paramount importance to verify if the assumptions behind the theory are realized in practice and thus if this approach is suitable to address physical issues of complex wave systems.

Almost 20 years ago a family of one-dimensional nonlinear dispersive wave equations, namely the MMT model [11], was introduced as a, in principle simple, model for assessing the validity of WWT theory; however, the results reported in [11] were somehow discouraging and it was reported that “the predictions of weak turbulence theory fail and yield a much flatter spectrum compared with the steeper spectrum observed in the numerical statistical steady state”. Deviations from the WWT predictions have been then observed numerically in [12] and have been associated to the presence of coherent structures such as quasi-solitons (see also [13]). The MMT model has become a “paradigm-like” for the verification of the WWT predictions. Indeed, this model offers the rare opportunity to analyse turbulent dynamics over a large range of scales, and it has been proven to display much of the features relevant for turbulent flows [27 28], even though it is a one-dimensional idealised model.

It is instructive to remind that the process of energy cascade observed in systems of interacting random waves is similar to the one in fluid turbulence. Indeed, fully-developed fluid turbulence is known to exhibit a direct energy cascade with a power-law energy spectrum very close to $k^{-5/3}$, $k$ being the spacial wave-number. Such a slope is consistent with the phenomenological Kolmogorov theory which assumes scale invariance. It turns out that such assumption is not supported by experimental data and numerical simulations; indeed, the three-dimensional fluid turbulence cascade is characterized by an anomalous scaling associated to the intermittent properties of the transfer and dissipation of the kinetic energy [29]. Intermittency has been widely studied in fluid turbulence; however, while exact results have been found for passive scalar dynamics in random flows [30], no analytical prediction (based on the Navier-Stokes equations) for the scaling of the structure functions is at the moment available.
The theory of Weak-Wave Turbulence can be considered as a mean-field approach, where the relevant observable is the wave action spectrum \( n(k, t) \), which becomes deterministic in the thermodynamic limit and thus ignores fluctuations. From this perspective, one would be tempted to rule out the presence of anomalous scaling and intermittency. Nonetheless, as anticipated, many numerical and experimental observations have pointed out the existence of such non trivial statistics. Notably in wave turbulence, the phenomenon of intermittency has been observed in different experiments of mechanically forced surface gravity waves by different groups, \([0, 20, 33, 35]\). Nonetheless the origin of such intermittency remains mysterious: in \([33]\) such phenomenon has been associated with numerous nonpropagating spikes or splashes and propagating breakers. In experiments reported in \([3]\) wave breaking or whitecaps did not occur but cusps (that can leave a signature on the spectrum) were observed on the fluid surface. In \([35]\) it has been observed that the degree of intermittency depends on the forcing parameter and is somehow independent from breaking and capillary bursts on steep gravity waves. More in general, in a number of papers \([14–17, 31, 32]\) a tentative explanation of the origin of intermittency in wave turbulence has been put forward; however, no quantitative predictions are so far available.

In this paper, we consider the MMT model in the defocusing regime with gravity-wave dispersion as a basic tool for studying the properties of wave turbulence. While simple, the model has been shown to be dynamically particularly rich, and to display high sensitivity to external forces and parameters \([27, 28]\). For this reason, here we stick to the same configuration examined in the first investigation \([11]\), to disentangle the various effects. In particular, we show that in the direct energy cascade regime the WWT theory offers an accurate prediction for wave spectral slope, provided the ratio between the nonlinear to the linear Hamiltonian is sufficiently small. When the ratio of nonlinear to linear energy is large enough, the prediction of the wave turbulence fails and, as shown in \([11]\), the spectrum becomes steeper than the WWT prediction. The calculation of the structure functions and of the probability density function (PDF) of increments of the wave field at different scales show that, while in the strongly nonlinear regime the dynamics is characterised by intermittency, in the weakly nonlinear case described by the WWT spectrum the wave field exhibits a quasi-self-similar scaling. The model cannot describe the phenomenon of wave breaking or the formations of cusps; therefore, it offers a unique tool to establish that the observed intermittency cannot be attributed solely to such singular or quasi-singular structures.

Our paper is organised as follows: we first describe the MMT model with forcing and dissipation and introduce the control parameters of the simulations. The numerical set-up is then presented and the results on the spectral slopes are shown. We then consider the structure function and the PDF of the wave field at different scales and discuss the intermittency properties of the defocusing MMT model. In the last Section, a discussion of the results and the final conclusions are reported.

II. THEORETICAL MODEL AND THE WWT PREDICTION

We consider the following model

\[
i \frac{\partial \psi}{\partial t} = \mathcal{L} \psi + |\psi|^2 \psi + \mathcal{F} + \mathcal{D},
\]

(1)

where \( \psi = \psi(x, t) \) and \( \mathcal{L} \) is a dispersive operator of the form \( \mathcal{L} \exp(ikx) = \omega(k) \exp(ikx) \); the dispersion relation is chosen as \( \omega = \sqrt{|k|} \). \( \mathcal{F}, \mathcal{D} \) are two terms that have been included in order to mimic forcing and dissipation; their specific form will be given in the next Section. The model in \([1]\) belongs to the MMT family of equations, which generalizes the nonlinear Schrödinger equation; our selection is based on the fact that, despite it has been shown in \([12]\) that for such member of the family the flux for the direct cascade predicted by the WWT has the correct sign and the cascade is local, numerical simulations have shown an unexpected distribution of energy in the spectral modes \([11]\).

In absence of forcing and dissipation the equation \([1]\) preserves the number of particles \( N \), and the Hamiltonian \( H \), which can be written as:

\[
H = H_{\text{lin}} + H_{\text{nl}} = \int \left| \frac{\partial \psi^{1/4}}{\partial x^{1/4}} \right|^2 dx + \frac{1}{2} \int |\psi|^4 dx
\]

(2)

The Hamiltonian is written as the sum of two terms that account for a linear contribution, \( H_{\text{lin}} \), (first term in r.h.s.) and a nonlinear one, \( H_{\text{nl}} \). The crucial assumption of WWT is that \( H_{\text{nl}}/H_{\text{lin}} \ll 1 \). This ratio thus constitutes the small parameter which allows the perturbative approach at the basis of the development of the WWT theory. It is important to underline that in front of the nonlinear term the coefficient is taken as positive: this implies that the model is defocusing. The case with opposite sign (focusing) is modulationally unstable and its dynamics is dominated by bright solitary waves and coherent structures \([27, 28]\). Here, we devote our attention only to the defocusing case, as it has been found to be more pathological with respect to WTT predictions.

By imposing a forcing confined at large scales (small \( k \)) and a dissipation only at small scales (large \( k \)), a direct cascade in \( k\)-space, characterised by a constant flux of linear energy, \( H_{\text{lin}} \) is predicted by the WWT once a stationary state is reached. The spectrum is expected to exhibit the following power law:

\[
n_k \sim k^{-1} \text{ WWT direct cascade },
\]

(3)

where \( n_k = \langle |\psi_k|^2 \rangle \) and the brackets \( \langle ... \rangle \) imply ensemble average. The verification of such predictions has failed
so far [11][13]. In particular a spectrum of the following type

\[ n_k \sim k^{-5/4} \quad \text{MMT direct cascade}, \]  

(4)

has been revealed in numerical simulations as a new final statistical steady state, consistently with a different closure proposed on heuristic grounds. Hereafter, this spectrum will be referred to as the MMT spectrum. In the following Section we will test the validity of the above predictions.

### III. RESULTS

#### A. Forcing and Dissipation

In order to observe the direct cascade, we have performed numerical simulations with the following deterministic instability-type forcing written in Fourier space:

\[ F_k = f \psi_k, \quad k \in [k_{\text{min}}, k_{\text{max}}], \]  

(5)

where \( f \) is a constant. Dissipation is imposed at very large and small scales using the following terms

\[ D_k = D_k^- + D_k^+ = (\nu^- |k|^{-m} + \nu^+ |k|^n) \psi_k, \]  

(6)

with \( m, n > 0 \).

#### B. Control parameters of the simulations

From equation (1) it is possible to establish four different time scales associated with dispersion, nonlinearity, high and low wave number dissipation:

\[ \tau_{\text{disp}} = \frac{1}{k_0^{1/2}}, \quad \tau_{\text{nl}} = \frac{1}{|\psi_0|^2}, \quad \tau_{\text{disp}}^+ = \frac{1}{\nu^+ k_0^n}, \quad \tau_{\text{dis}}^- = \frac{k_0^m}{\nu^-}. \]  

(7)

\( k_0 \) and \( |\psi_0| \) are a characteristic wave number and amplitude, respectively. We can then define the following “dimensionless” parameters as the ratio between the dispersive/dissipative and nonlinear time scales:

\[ \epsilon_0 = \frac{|\psi_0|^2}{k_0^{1/2}}, \quad Re_0^+ = \frac{|\psi_0|^2}{\nu^+ k_0^n}, \quad Re_0^- = \frac{k_0^m |\psi_0|^2}{\nu^-}. \]  

(8)

where \( Re_0^+ \) is the analogue of the Reynolds number in fluid mechanics. In terms of these parameters, Eq. (1) takes the following form:

\[ \frac{i}{\epsilon_0} \frac{\partial \psi}{\partial t} = \frac{1}{\epsilon_0} \mathcal{L} \psi + |\psi|^2 \psi + F + \frac{1}{Re_0^+} D^+ + \frac{1}{Re_0^-} D^-, \]  

(9)

where all the variables and the operators \( \mathcal{L}, F \) and \( D \) are now “dimensionless”. Given a computational domain, \( \epsilon_0, Re_0^+ \) and the forcing amplitude represent our control parameters of the simulations. In order to extract the different physical effects, we shall keep constant the forcing and the Reynolds numbers and change only \( \epsilon_0 \). The latter parameter indeed controls the ratio between nonlinear to linear waves, and therefore represents the key parameter in the WWT theory. We will also monitor the degree of nonlinearity of the asymptotic steady state of our simulation by considering the ratio of nonlinear to linear Hamiltonian for the equation (9):

\[ \epsilon = \epsilon_0 \frac{\int |\psi|^4 dx}{2 \int |\partial \psi|^{4/3}/\partial x^{1/3}|^2 dx}. \]  

(10)

Note that the relation between \( \epsilon_0 \) and \( \epsilon \) calculated when a stationary state has reached is not obvious; in the next Section, we will find out that the relation between \( \epsilon \) and \( \epsilon_0 \) is quasi-linear.

#### C. Numerical set-up

Equation (9) has been solved through a Strang splitting pseudo-spectral method. In the unforced and undamped case, the method guarantees a conservation of the number of particles and Hamiltonian with a high degree of accuracy. Simulations have been performed in a periodic box of size \( L = 2\pi \) with a number of modes set at 213. The time-marching is carried out with \( \Delta t = 0.01 \). The coefficients \( \epsilon_0 \) and \( Re_0^+ \) and \( f \) are selected and the simulation is run until the total number of particles, the linear and the nonlinear Hamiltonian reach a stationary state. The simulations are performed in the following way: at time \( t = 0 \) the wave-field is initialised with \( \psi(x,0) = f \). Among many of the simulations performed, here, for the sake of clarity, we report two of them, i.e. RUN1 and RUN2, characterized by different nonlinearity; the parameters of the simulations are reported in Table I. In Fig. 1 the evolution of the number of particles for two numerical simulations is plotted. The figure shows that in both cases a stationary state is reached. It is also possible to appreciate that the fluctuations of the
number of particles are not small, especially for the more nonlinear case.

After that transient, the spectra, and all the other statistical observables are computed averaging over time until a satisfactory convergence is reached. Over the same amount of time, the averaged value of $\epsilon$ is calculated in order to characterize the numerical experiment.

### D. Spectra

In Fig. 2, spectra obtained averaging for times larger than 10000 are shown. Both spectra show a power law over an inertial range of one decade. The figure shows that RUN1, characterized by a lower value of $\epsilon_0$, displays a spectral slope that is in agreement with the WWT prediction, while the spectral slope for RUN2 appears steeper (see the inset in Fig. 2). Notably, RUN2 evidences a statistical steady state in agreement with the MMT spectrum.

In order to appreciate the dependence of the spectral slope, $\gamma$ is the absolute value, on $\epsilon$ we have performed a number of numerical simulations by keeping $f$ and $Re_0^+$ constant and changing $\epsilon_0$ from 0.1 to 12.5; a fit is then performed in the range of $k \in [20, 100]$. The results are shown in Fig. 3 where the spectral slope $\gamma$ is plotted as a function of $\epsilon$. Interestingly, for $\epsilon \lesssim 0.2$, the slope is in agreement with the prediction of the WWT $\gamma = 1$. Increasing $\epsilon$, the slope of the spectrum changes continuously, attaining the MMT prediction $\gamma = 5/4$ for high nonlinearity ($\epsilon \gtrsim 0.6$). In the inset we show the relation between the control parameter $\epsilon_0$ and the effective degree of nonlinearity of the simulation $\epsilon$; for the parameters chosen, the plot indicates a linear dependence between the two numbers. In order to look for other sig-

### E. Intermittency

Following the studies on the velocity field in Navier-Stokes turbulence [37], we consider as the relevant tool for analysing the intermittent behavior of the wave field the structure functions, defined as:

$$S_p(r) = \langle |\delta \psi(r)|^p \rangle = \int |\delta \psi(r)|^p P(\delta \psi)d\delta \psi$$

with $\delta \psi = \Re(\psi(x + r) - \psi(x))$ where $\Re$ denotes the real part. $P(\delta \psi)$ is the PDF of the random field $\delta \psi$. Assuming for example that this probability is Gaussian at all scales with variance $S_2(r) = \sigma^2(r)$, then we have

$$S_p(r) = \sigma(r)^p \int \left| \frac{\delta \psi(r)}{\sigma} \right|^p P \left( \frac{\delta \psi(r)}{\sigma} \right) d \frac{\delta \psi(r)}{\sigma},$$

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| $f$ | $k_{\text{min}}$ | $k_{\text{max}}$ | $\epsilon_0$ | $Re_0^+$ | $Re_0^-$ |
|-----|----------------|----------------|---------|---------|--------|
| RUN1 | 0.02 | 4 | 7 | 0.5 | $5 \times 10^{22}$ | $2 \times 10^{-3}$ |
| RUN2 | 0.02 | 4 | 7 | 12 | $5 \times 10^{22}$ | $2 \times 10^{-3}$ |
where the integral is a constant that depends only on $p$. This property holds for any random process that is self-similar, i.e., whose statistical properties are scale invariant. The scaling properties of the structure functions can be related to those of the spectrum by observing that the two-point correlator $C(r) = \langle \psi(x)\psi(x+r) \rangle$ is the Fourier transform of the spectrum. For the Wiener-Kintchine theorem [37], if the wave spectrum is characterized by a power law $k^{-\gamma}$ with $1 < \gamma < 3$, then also $S_2(r)$ scales as $r^{2\zeta_2}$, with $\zeta_2 = -1$. It should be noted that, if WWT is verified, i.e., $\gamma = 1$, then the relation between the exponent of the second-order structure function and the spectral slope does not hold.

In general, since the spectral slopes change for different values of $\epsilon$, we may expect that also $S_2(r)$ would scale differently with $\epsilon$. Therefore, assuming self-similarity, we can only make the prediction that the relative scaling is universal. Indeed, since $S_p \sim r^{\zeta_p(\epsilon)} \sim S_2^{p/2} \sim r^{p\zeta_2(\epsilon)}$, we expect that $\zeta_p(\epsilon)/\zeta_2(\epsilon) = p/2$, i.e., $\epsilon$-independent and linear in $p$. If the PDF is not self-similar, higher moments do not follow any simple scaling relation with respect to the second one and the integral in (12) depends on $r$; therefore, we have that $\zeta_p(\epsilon)/\zeta_2(\epsilon)$ is a nonlinear (possibly $\epsilon$ dependent) function of $p$. In this case, we encounter an anomalous scaling, typical of turbulent flows [29]. From a technical point of view, the calculation of the scaling exponents requires some attention. It is rather difficult to get an accurate estimate of $\zeta_p$ from the scaling of $S_p(r)$ versus $r$. The most appropriate procedure is to plot higher order structure functions versus a reference one and get directly the relative scaling exponent, hoping in this way to get rid of any spurious dependence due to limited statistics or finite-size effects. This technique, known as the Extended Self-Similarity (ESS) and developed in the nineties [38], has been widely used in fluid turbulence to analyze low-to-moderate Re statistics. In particular, for three-dimensional turbulence, it is meaningful to plot structure functions as a function of $S_3(r)$. The reason is that Kolmogorov 4/5 law assures, for the Navier-Stokes equations, that $S_3 \sim r$. Since we are not aware of any analogous prediction for the present model, we compute the ESS exponents with respect to $S_2(r)$.

In Fig. 5, the ratio between $\zeta_p/\zeta_2$ is shown for the two simulations discussed in the previous Section. We observe that both $RUN2$ exhibit a degree of intermittency larger than $RUN1$. The fact that the discrepancy with respect to self similar scaling increases with nonlinearity is not a peculiarity of this one-dimensional model under investigation; indeed, our results are consistent with results obtained in very different systems [6, 24]. The nonlinear behaviour of $\zeta_p/\zeta_2$ seen in Fig. 5 implies that for any structure function of order $p > 1$, the ratio $S_p(r)/S_2(r)^p/2 \sim r^{p\zeta_2(\epsilon)}$ increases as $r \to 0$, or at least down to the dissipation range. In particular, this is true for the flatness $F = S_4(r)/S_2^2(r)$. For a Gaussian field $F = 3$, and in general for any self-similar field $F$ does not depend on $r$. On the other hand, an increasing flatness at small separation indicates that large fluctuations are relatively more frequent at those scales, a phenomenon called intermittency [29] in the fluid turbulence literature. It should be noted that $\zeta_p$ is constrained by general exact results to be a concave and non-decreasing function of $p$ for any field with bounded values [29]. The inset of Fig. 5 shows that the $F(r)$ is constant for the weakly-nonlinear

**FIG. 4.** $|\psi(k, \omega)|^2$ (a) for very small nonlinearity $\epsilon \approx 0$, $RUN1$; (b) for large nonlinearity $\epsilon \simeq 1$, $RUN2$.

**FIG. 5.** The ESS scaling exponents $\zeta_p/\zeta_2$ as a function of $p$ for $RUN1$ and $RUN2$: while a linear dependence $\zeta_p/\zeta_2 = p/2$ would indicate self-similarity (consistent with the WWT theory), our simulations show an increasingly anomalous behaviour as the nonlinearity parameter $\epsilon$ increases. Inset: the flatness $S_4(r)/S_2(r)^2$ (see text). For the weakly non-linear $RUN1$ the flatness is always close to the Gaussian value 3; higher non-linearities produce more intense fluctuations at smaller scales, the very definition of intermittency.
values of the nonlinearity $\epsilon$. The statistics from RUN1 (low non-linearity) is shown in panel (a). In this case the PDFs cannot be distinguished from a Gaussian distribution at all separations but at large ones, where they display sub-Gaussian tails. In contrast, panel (b) shows that RUN2 (high non-linearity) displays strongly non-Gaussian statistics at small scales, with wider and wider tails with decreasing separations. Gaussian statistics are still found at intermediate separations. Again, distributions with sub-Gaussian tails are observed at large scales. The scale-dependence of the tails of the distributions is indeed the signature of intermittency. Furthermore this is clearly triggered by large enough non-linearity, in agreement with our previous remarks on Fig. 5.

**IV. CONCLUSIONS**

Our numerical results on the defocusing MMT model with $\sqrt{|k|}$ dispersion indicate that if the nonlinear interactions are sufficiently small, the statistical steady dynamics is well described by the WWT theory. Furthermore, the probability density function of the wave-field is quasi-Gaussian at all scales.

On the other hand, when nonlinearity starts to become large enough, we have shown that the dynamics is even richer than previously depicted, confirming that the MMT model is a particularly complex model. In particular, we have shown that there is a continuous transition from weak to strong turbulence as the nonlinearity is increased. In this case, WWT ceases to represent correctly the system and anomalous scaling is observed, with spectra which differ from the WWT predictions. Interestingly, the MMT spectrum is just one of the various possible statistical steady states.

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