Horizon Shells:
Classical Structure at the Horizon of a Black Hole

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We address the question of the uniqueness of the Schwarzschild black hole by considering the following question: How many meaningful solutions of the Einstein equations exist that agree with the Schwarzschild solution (with a fixed mass $m$) everywhere except maybe on a codimension one hypersurface? The perhaps surprising answer is that the solution is unique (and uniquely the Schwarzschild solution everywhere in spacetime) unless the hypersurface is the event horizon of the Schwarzschild black hole, in which case there are actually an infinite number of distinct solutions. We explain this result and comment on some of the possible implications for black hole physics.

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The Schwarzschild solution is arguably the single most important exact solution of the Einstein field equations. In particular, it is the prototypical black hole metric, exhibiting an event horizon which is a Killing horizon, and by the celebrated Israel theorem (1967) [1], the first black hole uniqueness (“no hair”) theorem, under certain regularity conditions it is the unique static black hole solution of the vacuum Einstein equations.

While these classical aspects of the Schwarzschild solution have thus been completely understood for a long time, the deep puzzles raised by the discovery of Hawking radiation and the thermodynamical properties of black holes continue to fuel heated debates regarding the nature of black hole microstates and the physics at or in the vicinity of a black hole horizon.

Many of these considerations suggest that at the quantum level the black hole horizon may be a special place not just globally but even locally in the space-time. In light of this let us take a step back and ask the seemingly innocuous purely classical

**Question:** How many meaningful solutions of the Einstein equations exist that agree with the Schwarzschild solution (with a fixed mass $m$) everywhere except perhaps on a codimension one hypersurface?

This question is a special case of a general problem analysed in [2], and, as we will explain, has the following (perhaps somewhat surprising) **Answer:** The solution is unique (and uniquely the Schwarzschild solution everywhere in spacetime) unless the hypersurface is the event horizon of the Schwarzschild black hole, in which case there are actually an infinite number of distinct solutions.

These solutions (almost all of which are new) are described in detail in [2]. In particular, this result shows that there can be a very rich structure at the horizon of a black hole already at the classical level, with potential implications for various aspects of black hole physics.

To set the stage, we need to explain what we mean by “meaningful solutions of the Einstein equations except on a codimension one hypersurface” in the above question. Thus let us imagine that we have two solutions of the Einstein equations that we want to glue (or solder) along a common boundary hypersurface $N$ to create a joint solution of the Einstein equations. Then the primordial (and obviously necessary) mathematical consistency condition, expressing the fact that at each point in the extended space-time there should be a unique metric, is the Israel junction condition that the two metrics should induce the same metric on $N$. 

[1] Israel, W. (1967). General relativity and gravitation: An introduction for physicists. Gulf Publishing Company.

[2] Bondi, H. (1965). The static black hole. General Relativity and Gravitation, 3, 1089-1095.
It turns out that this necessary mathematical condition is also sufficient to arrive at a physically meaningful solution of the Einstein equations, namely with non-singular particle propagation across the hypersurface, and with a delta-function localised contribution to the energy-momentum tensor,

$$G_{\alpha\beta} = (\text{bulk terms}) + S_{\alpha\beta}\delta_N$$

(and/or to the Weyl tensor). The generic solution of this type can be interpreted as describing a shell of matter on $N$ together with impulsive gravitational radiation in the case that $N$ is null. The general formalism of thin shells in general relativity (see e.g. [3] for a nice account, and [4, 5] for further details regarding the more subtle case of null shells) provides an algorithm to determine the shell energy momentum tensor $S_{\alpha\beta}$ and other intrinsic physical properties of the shell from the jumps in the normal derivatives of the metric across the shell.

So far this is completely general, and for generic hypersurfaces (including all timelike and spacelike hypersurfaces but also generic null hypersurfaces) such a soldering, when it exists, is unique, i.e. there is (up to isometries) a unique solution to the Israel junction condition. In particular, to return to the question raised at the beginning of this essay, in all these cases there is a unique solution to the Einstein equations that can be obtained by soldering two equal mass Schwarzschild black hole solutions across such a hypersurface, namely the Schwarzschild solution without a shell (equivalently, philosophical questions aside, with an empty shell).

However, it was already mentioned in [4] that for the particular case of Killing horizons of static black holes there is considerable freedom in the way that the two geometries are attached, and isolated examples of this phenomenon are already present in the literature (e.g. the Dray ’t Hooft shell on the horizon of a Schwarzschild black hole [6]). Intuitively (but not completely correctly) this freedom can be attributed to the fact that any null matter placed on the horizon will be infinitely redshifted relative to any point a finite distance from the horizon and will, in particular, have no impact on quantities like the ADM mass of the solution (unlike matter on a collapsing non-horizon shell, say).

Analysing this issue more systematically, in [2] we found that there is a (continuously) infinite number of distinct ways to solder two geometries together along a null hypersurface whenever the induced metric on the null hypersurface is invariant under translations along its null generators. This condition is of course satisfied by Killing horizons of stationary black holes, but also by Rindler horizons, and more generally by other quasi-local notions of horizons such as non-expanding horizons and isolated horizons (see e.g. [7] for a review).

A concrete illustration of this is provided by the Schwarzschild metric in (ingoing)
Eddington-Finkelstein coordinates,

\[ ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2, \quad f(r) = 1 - \frac{2m}{r}. \quad (2) \]

The metric induced on a hypersurface of the form \( r = r(v) \) (say), will depend non-trivially on \( v \) and/or \( dv \), unless one makes the special horizon choice \( r(v) = 2m \). In this case, and this case only, the induced (degenerate) metric

\[ ds^2|_{r=2m} = (2m)^2d\Omega^2 \quad (3) \]

is manifestly invariant under arbitrary transformations

\[ v \rightarrow F(v, y^A) \quad (4) \]

of the null coordinate \( v \) (\( y^A \) are the angular coordinates), extended to or induced by a suitable coordinate transformation on one side of the shell. This provides one with an infinite number of distinct solutions to the Israel junction condition, and any such soldering transformation can be regarded as generating a shell from the empty shell (pure Schwarzschild solution). This shell will be non-trivial unless the soldering transformation is a pure isometry, i.e. a constant shift of \( v \).

The resulting shells will in general carry a conserved (null) matter energy momentum tensor (composed of energy density \( \mu \), energy currents or momentum densities \( j^A \) and pressure \( p \)), as well as impulsive gravitational waves travelling along the shell, and following the null shell algorithm these quantities can be expressed explicitly in terms of the first and second derivatives of \( F \). For example the null energy density of the shell \( \mu \) is given by

\[ \mu = \frac{1}{8\pi m F_v} \left( e^{-F/4m} \Delta^{(2)} e^{F/4m} + F_v - 1 \right) \quad (5) \]

(with \( \Delta^{(2)} \) the Laplacian on the 2-sphere).

To summarise, for any function (soldering transformation) \( F(v, y^A) \) we obtain a solution to the Einstein equation describing a shell living on the horizon of a Schwarzschild black hole. They generalise the known solutions of this kind in the following way:

- These solutions can be considered as significant generalisations of the Dray ’t Hooft null shell [6], to which they reduce for a special soldering transformation which is a constant shift of the Kruskal-Szekeres coordinate \( V \),

\[ V = e^{v/4m} \rightarrow V + b \iff F(v, y^A) = 4m \log \left( e^{v/4m} + b \right) \quad (6) \]

(this leads to a shell with constant energy density \( \mu \) and zero pressure, currents and gravitational wave components).
- As all of the solutions are non-singular on the shell (i.e. do not have any point particle singularities), they also provide one with a wide array of smoothed out versions and generalisations of Dray ’t Hooft impulsive gravitational waves [8].

- Finally these solutions reduce to those discussed by Barrabes and Israel in [4] in the special case $F = F(v)$.

Whatever the ultimate significance or role of these solutions may be, we believe that it is at the very least good to be aware of their existence, and we conclude this essay with some remarks on the possible implications of this result for black hole physics.

1. The most conservative interpretation is perhaps that these shells are bookkeeping devices that (in the spirit of the method of images or perhaps the membrane paradigm) encode faithfully the effects of coordinate transformations performed on one side of the horizon. Performing such one-sided coordinate transformations may be relevant to a complete description of the physics of black holes for observers in the exterior region as required by the principle of black hole complementarity.

2. A class of soldering transformations that may be of particular interest, especially in light of the observation in [9] that black holes must carry supertranslation hair, and the subsequent Hawking - Perry - Strominger proposal [10, 11] (cf. also the analysis by Compère and Long [12]) are the horizon analogues of BMS supertranslations at $I^+$, (infinitesimally) of the form

$$v \rightarrow v + T(x^A) ,$$

which are singled out by commuting with time ($v$) translations. For these one finds that the corresponding shells have zero pressure, but non-trivial $\mu$ and $j^A$ as well as non-trivial impulsive gravitational shock waves travelling along the horizon.

3. As circumstantial evidence for the suggestion that these shells may be more than an abstract bookkeeping device, we note that the Goldstone (-Bogoliubov) mode of spontaneously broken supertranslation invariance identified in [11] (cf. also [13]) is precisely the null energy density $\mu$ of the corresponding shell, suggesting a concrete physical origin and interpretation of the null matter and radiation propagating on the horizon shell.

Clearly there are also many other issues, both classical and quantum, that remain to be explored.
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