Evaluation of maximum safe-slope angle using the kinematical element method and its application

Rui Bao¹ and Ke Zhang*²,³

¹ Department of Safety and Environmental Protection, Kunming Prospecting Design Institute of China Nonferrous Metals Industry Co., Ltd., Kunming, Yunnan, China
² Faculty of Electric Power Engineering, Kunming University of Science and Technology, Kunming, Yunnan, China
³ Faculty of Civil and Architectural Engineering, Kunming University of Science and Technology, Kunming, Yunnan, China

*Corresponding author: zhangke@kust.edu.cn ORCID: http://orcid.org/0000-0002-7178-8304

Abstract: The determination of the maximum safe-slope angle with a planned slope height is a critical issue in slope design and construction. In this study, to overcome the drawbacks of the rule of thumb and trial-and-error methods, the optimization of slope angle is treated as a root-finding problem. The kinematical element method (KEM) within a rigorous theoretical framework is used to solve this problem using the modified false-position method. The critical noncircular-failure surface and the associated minimum factor of safety are located using the KEM. A rock slope with planar failure is given for demonstrating the validity of this method. The result shows that the maximum safe-slope angle obtained using the KEM is consistent with the analytical solution. The method proposed in this study has a satisfactory convergence speed. In addition, a bench-shape fill slope in an iron and steel base is used as a case study. The maximum safe-bench face angle of the fill slope, under self-weight condition, is 41.43°, and it decreased by 17% due to seismic loading. Finally, the effect of the slope height on the maximum safe-slope angle is analyzed, and strong correlations that display an exponential function are found. The critical failure surface associated with the maximum safe-slope angle becomes deeper as the slope height increases.

1. Introduction
The determination of reasonable slope angle is a major task in slope design [1]. The common requirement is to determine the maximum safe-slope angle with a planned slope height. Chen [2] conducted a field survey to illustrate the relationship between slope height, slope angle, and stable conditions. Moreover, the rule of thumb method is applied in routine-slope design, namely, common stability slope ratios for varying soil/rock conditions. However, if the slopes are not selected according to geology, there is little correlation among slope height, slope angle, and stability conditions [1].

For the plane failure in the rock slopes, an explicit analytical approach has been given. Therefore, the maximum safe-slope angle that will produce the required factor of safety can be calculated [3]. For the other types of slope failures, the factors of safety are expressed using the implicit functions, therefore, it is impossible to solve the maximum safe-slope angle directly. The trial-and-error method is used as the design methodology. Various slope angles are tested, and unsatisfactory ones are
eliminated until the desired result is achieved, thus, lacking effective and an efficient problem-solving mechanism.

The stability analysis method contributes to slope design, which can be classified into two groups: limit equilibrium and numerical analyses [4–6]. The limit equilibrium method (LEM) is suggested in many national and international design standards and codes. The LEM employs assumptions regarding inter-slice forces to render the problem determinate. In addition, the failure mass is assumed to be rigid and divided into a series of vertical slices.

The kinematical element method (KEM) is a limit analysis method within a more rigorous theoretical framework. The KEM has advantages over the LEM: (1) no assumption regarding the inter-element forces; (2) the kinematic analysis ensures the rationality of the directions of inter-element shear forces; (3) no assumption regarding the geometry of the failure mass and the inclination of inter-element interface; (4) the optimization algorithm ensures the success in locating the critical failure surface. Gussmann [7] solved the matrix formulation for the limit-load problem in soil mechanics using the KEM. Zhang and Cao [8–9] extended the KEM to investigate the seismic and rock slope stability. Qarmout et al. [10] introduced the KEM to tunnel-face stability analysis.

This study aims at extending the KEM to slope design. From a different perspective, the optimization design of slope angle was treated as a root-finding problem. A new methodology for optimizing the slope angle was proposed to provide a more rigorous and efficient problem-solving mechanism, overcoming the drawbacks of the rule of thumb and the trial-and-error methods.

2. Theoretical formulation

2.1. General description

In KEM, the failure mechanism consists of a finite number of rigid blocks [7–10], and each block is called a kinematic element. The safety factor is calculated by considering the sliding mass as rigid–perfectly plastic material. The stability analysis procedure includes discretization, kinematic analysis, static analysis, and optimization [7–10].

The concept of allowable factor of safety \( F_s \) is used to margin for error and uncertainty [11] and serves as the criterion for judging the slope stability. The slope satisfies the stability requirement if \( F_s \geq \left[ F_s \right] \). However, stabilization measures should be taken to achieve the required factor of safety if \( F_s < \left[ F_s \right] \).

For a slope with planned height, the slope angle is the most risky design parameter. Here, the factor of safety is a function of slope angle \((\alpha)\) and is designated \( F_s(\alpha) \). The design process is a tradeoff between stability and economics. For an optimum slope operation, the slope design should be close to the limit of its stability; the factor of safety, \( F_s(\alpha) \), for the numerical model is equal to the allowable factor of safety \( \left[ F_s \right] \):

\[
F_s(\alpha) = \left[ F_s \right]
\] (1)

Therefore, finding the slope angle associated with the allowable factor of safety \( \left[ F_s \right] \) causes the problem of finding a root of the following equation:

\[
f(\alpha) = F_s(\alpha) - \left[ F_s \right] = 0
\] (2)

The root \( \alpha = [\alpha] \) such that \( f([\alpha]) = 0 \) is called the maximum safe-slope angle.

There are two key issues in the slope design described above: the slope stability analysis and root-finding methods. The KEM was adopted to search the minimum factor of safety and critical failure surface. A variant of the false-position method was developed to find the root of the function, discussed in the following section.

2.2. Optimization design procedure

The common numerical root-finding methods include the bisection, Newton's, secant, and false-position methods. These methods and the KEM were integrated to optimize the slope angle. By
performing a series of slope design examples, we found that (1) the bisection method converges very slowly, however, it is guaranteed to converge; (2) the Newton's method requires computing the derivative of the function \( f(\alpha) \); however, this function is defined by an implicit equation in the KEM; (3) the idea of the secant method is similar to Newton’s method, where the tangent line is approximated using a secant line; however, the secant method has the disadvantage of the root not guaranteed to converge; (4) the false-position method is similar to the bisection method; however, the midpoint is replaced by a secant method-like approximation. In addition, the false-position method converges slowly to the root for some slope design examples, no better than the bisection method. To address this problem, the modified false-position method was presented to give a faster convergence. At every iteration, the function at the midpoint was first evaluated, and then the false-position method was applied to increase the speed of convergence.

The optimization design of slope angle by combining the modified false-position method and the KEM (Figure 1) is as follows:

![Flowchart of slope angle optimization.](image)

**Figure 1.** Flowchart of slope angle optimization.

**Step 1:** Initialize the interval of slope angle as \([\alpha_1, \alpha_2]\) and the termination tolerance as \(\varepsilon\).

**Step 2:** Calculate the midpoint \(\alpha_m = (\alpha_1 + \alpha_2)/2\). Set up the numerical model with slope angle, \(\alpha_m\), and calculate the factor of safety, \(F_s(\alpha_m)\), using the KEM.

**Step 3:** If \(|f(\alpha_m)| \leq \delta, [\alpha] = \alpha_m\), terminate the program.

**Step 4:** If \(f(\alpha_m) > 0\), a better approximation, \(\alpha_t\), following the secant line is given by:
\[ \alpha_i = \alpha_2 - \frac{f(\alpha_3)}{f(\alpha_2) - f(\alpha_m)} (\alpha_2 - \alpha_m) \]

Set up the numerical model with slope angle \( \alpha_s \) and calculate the factor of safety, \( F_s(\alpha) \), using the KEM. If \( f(\alpha_m) \leq \varepsilon \), \( \alpha = \alpha_s \) terminate the program. If \( f(\alpha) > 0 \), let \( \alpha_1 = \alpha \) go to step 2. Else, let \( \alpha_1 = \alpha_m, \alpha_2 = \alpha_s \), and go to step 2.

Step 5: If \( f(\alpha_m) < 0 \), calculate \( \alpha \):

\[ \alpha_i = \alpha_{i-1} - \frac{f(\alpha_3)}{f(\alpha_2) - f(\alpha_m)} (\alpha_2 - \alpha_m) \]

Set up the numerical model with slope angle \( \alpha_s \) and calculate the factor of safety, \( F_s(\alpha) \), using the KEM. If \( f(\alpha_m) \leq \varepsilon \), \( \alpha = \alpha_s \) terminate the program. If \( f(\alpha) > 0 \), let \( \alpha_2 = \alpha \) go to step 2. Else, let \( \alpha_1 = \alpha \), \( \alpha_2 = \alpha_m \), and go to step 2.

### 2.3. Verification

This example was used to examine the effectiveness and efficiency of the proposed method. A 175.26-m (\( H \)) high rock slope is to be excavated [3], as shown in Figure 2(a). The rock slope has a potential sliding surface that dips at an angle of 40° (\( \alpha \)). A horizontal seismic force is considered, where the seismic coefficient (\( k \)) is 0.10. The strength parameters of the sliding surface are as follows: cohesion, \( c = 137.8 \) kPa and friction angle, \( \phi = 32^\circ \). The unit weight of the rock (\( \gamma \)) is 24.8 kN·m\(^{-3}\). By setting the allowable factor of safety [\( F_s \)] to 1.0, the analytical equation for calculating the maximum safe-slope angle, [\( \alpha \)], is as follows [3]:

\[ [\alpha] = \arccot \left( \cot \beta - \frac{2W}{\gamma H} \right) \]

where \( W = \frac{cL}{\sin \beta + k \cos \beta - k \cos \beta \tan \phi + k \sin \beta \tan \phi} \) and \( L = H / \sin \beta \). Therefore, the analytical solution to the maximum safe-slope angle, [\( \alpha \)], is 49.95°, using equation (5).

![Figure 2](image-url)
bisection and false-position methods make slow progress toward the root. Therefore, the result indicates that the modified false-position method is of better convergence speed than the other two methods.

3. Engineering application

A new fill slope is to be designed and constructed in a new iron and steel base, formed from clay fill having a density, cohesion, and friction angle of 19.8 kN·m⁻³, 30 kPa, and 16.5°, respectively. The slope will consist of two benches \( (H_1 = 10 \text{ m} \text{ and } H_2 = 7.5 \text{ m}) \) with a 3.75-m \( (B) \) wide berm, extending from the slope crest (elevation: 1532 m) to the slope base (elevation: 1514.5 m), as (Figure 3). The two bench face angles are set to the same value. A borehole measurement indicates that the groundwater level is at an elevation of 1501.1 m. The seismic coefficient \( (k) \) is 0.05. An allowable factor of safety \( [F_s] \) of 1.30 is considered reasonable for the slope design, as suggested by the technique code for building slope engineering\(^{[12]}\).

Two cases were considered: a self-weight condition \( (k = 0) \); a self-weight and seismic loading condition \( (k = 0.05) \). The optimization has been made by starting from an interval of bench face angle, \([30^\circ, 50^\circ]\), and the termination tolerance is given as \( \varepsilon = 0.005 \). There is a convergence after the first iteration, for \( k = 0 \), while the second iterations are needed to meet the termination tolerance requirement, for \( k = 0.05 \). Based on the proposed method, the maximum safe-bench face angles of the fill slope for two cases \( (k = 0 \text{ and } 0.05) \) are 41.43° and 34.38°, respectively. The maximum safe-bench face angle is decreased by 17% due to seismic loading. The factors of safety and locations of critical failure surfaces associated with the maximum safe-bench face angles are shown in Figure 4. The differences between the factors of safety associated with the maximum safe-bench face angles and the allowable factor of safety are small.

4. Role of slope height

The slope height contributes to the determination of the maximum safe-slope angle. With the increase in the slope height, there needs to be a corresponding decrease in the slope angle\(^{[1]}\). The geotechnical parameters of the fill slope example were used to conduct a sensitivity analysis of the slope height, and the slope height was varied from 10 to 30 m with 5-m increments.
Figure 4. Factors of safety and locations of the critical failure surfaces associated with the maximum safe-bench face angles.

Figure 5 is a plot of the slope height versus the location of the critical failure surface associated with the maximum safe-slope angle. Figure 6 is a plot of the slope height versus the maximum safe-slope angle. It is indicated that the critical failure surface associated with the maximum safe-slope angle deepens as the slope height increases. Given the slope height, the maximum safe-slope angle decreases due to seismic loading. In this study, the curve-fitting results in Figure 6 show that the relation of the slope height and the maximum safe-slope angle for two cases \((k = 0\) and \(0.05\)) appears to be an exponential problem with a correlation coefficient, \(R^2\), of 1.00. The relationship between the slope height and maximum safe-slope angle is given by equation (6), providing a guideline for construction fill slope.

\[
[a] = \begin{cases} 
142.10e^{-0.14H} + 22.42 & \text{for } k = 0 \\
155.56e^{-0.15H} + 18.58 & \text{for } k = 0.05 
\end{cases}
\]  

(6)

Figure 5. Slope height versus the location of critical failure surface associated with the maximum safe-slope angle.

Figure 6. Slope height versus maximum safe-slope angle.
5. Conclusions
(1) A new slope design methodology, integrating the modified false-position method and the KEM, was presented with a rigorous theoretical framework. For the slope with a planned height, the maximum safe-slope angle can be calculated accurately and fast.
(2) A rock slope example with plane failure was used to demonstrate the validity of the new methodology. The maximum safe-slope angle obtained using this method is consistent with its analytical solution. The convergence speed of this method is faster than those of the bisection and false-position methods.
(3) For a fill slope in an industrial site, the maximum safe-bench face angle is decreased by 17% due to seismic loading. The parametric analysis indicates that the relationship between the slope height and maximum safe-slope angle can be expressed using an exponential function, providing a guideline for slope construction.

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