ON THE STABILITY OF THE BRESSE SYSTEM WITH FRICTIONAL DAMPING

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Abstract. In this paper, we consider the Bresse system with frictional damping terms and prove some optimal decay results for the $L^2$-norm of the solution and its higher order derivatives. In fact, if we consider just one damping term acting on the second equation of the solution, we show that the solution does not decay at all. On the other hand, by considering one damping term alone acting on the third equation, we show that this damping term is strong enough to stabilize the whole system. In this case, we found a completely new stability number that depends on the parameters in the system. In addition, we prove the optimality of the results by using eigenvalues expansions. We have also improved the result obtained recently in [12] for the two damping terms case and get better decay estimates. Our obtained results have been proved under some assumptions on the wave speeds of the three equations in the Bresse system.

1. Introduction

In this paper, we consider the Cauchy problem of the Bresse system with frictional damping

\begin{align*}
\varphi_{tt} - (\varphi_x - \psi - lw)_x - k^2 l (w_x - l \varphi) &= 0, \\
\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi - lw) + \gamma_1 \psi_t &= 0, \\
w_{tt} - k^2 (w_x - l \varphi)_x - l (\varphi_x - \psi - lw) + \gamma_2 w_t &= 0,
\end{align*}

with the initial data

\begin{align*}
(\varphi, \varphi_t, \psi, \psi_t, w, w_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1),
\end{align*}

where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and $a, l, \gamma_1, \gamma_2$ and $k$ are positive constants. The functions $w(x, t), \varphi(x, t)$ and $\psi(x, t)$ are, respectively, the longitudinal displacements, the vertical displacement of the beam and the rotation angle of the linear filaments material.

The decay rate of the solution of the problem (1.1)-(1.2) has been first studied by Soufyane and Said-Houari in [12] and investigated the relationship between the frictional damping terms, the wave speeds of propagation and their influence on the decay rate of the solution. In addition, they showed that the $L^2$-norm of the solution decays with the following rate:

- For $a = 1$, we have

\begin{align*}
\| \partial_x^j U(t) \|_{L^2} \leq C (1 + t)^{-1/4 - j/2} \| U_0 \|_{L^1} + C (1 + t)^{-\ell/2} \| \partial_x^{j+\ell} U_0 \|_{L^2},
\end{align*}

- For $a \neq 1$, we have

\begin{align*}
\| \partial_x^j U(t) \|_{L^2} \leq C (1 + t)^{-1/4 - j/2} \| U_0 \|_{L^1} + C (1 + t)^{-\ell/4} \| \partial_x^{j+\ell} U_0 \|_{L^2},
\end{align*}

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where $C$ is a positive constant, $U(x, t) = (\varphi_x - \psi - lw, \varphi_t, \alpha \psi_x, \psi_t, k(w_x - l \varphi), w_x)^T(x, t)$ and $j$ and $\ell$ are positive integers. As we have seen both estimates contain some regularity losses. These regularity losses will make the nonlinear problem difficult to handle. See for instance [3] and [7] where similar difficulties hold for the Timoshenko system.

The main open questions stated in [12] were:

- Is it possible to remove the regularity loss in the above estimates, especially in (1.3)?
- Can we prove some decay estimates by considering just one damping term in the system. That is for $\gamma_1 = 0$ or $\gamma_2 = 0$?

The main goal of this paper is to give answers to the above questions. Indeed, we can summarize our results as follows:

First, for $\gamma_1 > 0$ and $\gamma_2 > 0$, we refined the decay estimates in (1.3) and (1.4) and instead, we obtained the following:

- For $a = 1$, we have
  \begin{equation}
  \|\partial^j U(t)\|_{L^2} \leq C (1 + t)^{-1/4-j/2} \|U_0\|_{L^1} + Ce^{-\alpha t} \|\partial^j U_0\|_{L^2},
  \end{equation}

- For $a \neq 1$, we have
  \begin{equation}
  \|\partial^j U(t)\|_{L^2} \leq C (1 + t)^{-1/4-j/2} \|U_0\|_{L^1} + C (1 + t)^{-\ell/2} \|\partial^j U_0\|_{L^2}.
  \end{equation}

In addition, we showed the optimality of the above estimates (1.5) and (1.6) by exploiting the eigenvalues expansion. Consequently, those estimates, under the same assumptions on the initial data are optimal and cannot be improved. The proof is essentially based on refinements of the Lyapunov functionals used in [12], which give the following decays for the Fourier image of the solution $\hat{U}(\xi, t)$:

\begin{equation}
|\hat{U}(\xi, t)|^2 \leq \begin{cases} 
Ce^{-\rho_1(\xi)t} |\hat{U}(\xi, 0)|^2, & \text{if } a = 1, \quad \rho_1(\xi) = \frac{\xi^2}{1 + \xi^4}, \\
Ce^{-\rho_2(\xi)t} |\hat{U}(\xi, 0)|^2, & \text{if } a \neq 1, \quad \rho_2(\xi) = \frac{\xi^2}{1 + \xi^2 + \xi^4}.
\end{cases}
\end{equation}

Since $\rho_1(\xi)$ is behaving like $|\xi|^2$ near zero, then the decay rate of the solution is the same as the one of the heat kernel. On the other hand, as $|\xi|$ goes to infinity, $\rho_1(\xi)$ does not go to zero, which keeps the dissipation still effective at infinity and prevents the regularity loss in (1.5). When $a \neq 1$, $\rho_2(\xi)$ has the same behavior as $\rho_1(\xi)$ near zero, but as $|\xi|$ tends to infinity, $\rho_2(\xi)$ goes to zero, which induces the regularity loss at infinity.

Second, for $\gamma_2 = 0$, we showed, despite the presence of the dissipation term $\gamma_1 \psi_t(x, t)$ in the second equation, that the solution $U(x, t)$ of the system does not decay at all. This is due to the weakness of the coupling term $(\varphi_x - \psi - lw)$ which is of order zero in the second equation. For instance, this can be viewed when $l = 0$, where the first two equations in the system (1.1) reduces to the Timoshenko system and the third one becomes a conservative decoupled wave equation

\begin{equation}
\begin{cases}
\varphi_{tt} - (\varphi_x - \psi)_x = 0, \\
\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \gamma_1 \psi_t = 0, \\
w_{tt} - k^2 w_{xx} = 0.
\end{cases}
\end{equation}

If we pick initial data $w_0$ and $w_1$ such that the corresponding solution $w(x, t) = 0$, for all $t > 0$, then the system (1.8) will decay like the Timoshenko one. In fact, we proved (see...
Theorem 3.1) the following estimate

\[ \| \partial_x^j U(t) \|_{L^2} \leq C \| \partial_x^j U_0 \|_{L^2} \cdot \]

Third, for \( \gamma_1 = 0 \), thanks to the strong coupling term \( k^2 (w_x - l \varphi)_x \), which is of order one, in the third equation, the effect of the dissipation term \( \gamma_2 w_t(x, t) \) will be propagated to the other equations of the system. More precisely, we proved that the solution decays with the following rate:

- For \( a = k = 1 \), we have

\[ \| \partial_x^j U(t) \|_{L^2} \leq C (1 + t)^{-1/4-j/2} \| U_0 \|_{L^1} + C (1 + t)^{-\ell/2} \| \partial_x^{j+\ell+1} U_0 \|_{L^2} \cdot \]

- For \( a \neq 1 \), we have

\[ \| \partial_x^j U(t) \|_{L^2} \leq C (1 + t)^{-1/4-j/2} \| U_0 \|_{L^1} + C (1 + t)^{-\ell/2} \| \partial_x^{j+\ell+3} U_0 \|_{L^2} \cdot \]

The above results were proved under the following additional assumptions on the coefficients of the system

\[ (k^2 - 1)l^2 \neq 1. \]

The above stability assumption is completely new in this framework and it is satisfied for instance if \( k = 1 \) or \( l = 0 \).

To prove the above estimates, we present a new method that is based solely on the eigenvalues expansion and it does not require the knowledge of eigenvectors and eigenspaces at all. (See Lemma 5.2). Indeed, we split the frequency space into three regions, low frequencies, middle frequencies and high frequencies, where in each region we derive the eigenvalues expansion and estimate the Fourier image of the solution in each region. In particular, this method avoids the use of the Jordan canonical form in computing the exponential of the matrix of the solution which is a heavy task to accomplish, especially, for large systems. The method that we introduce here seems new and can be easily extended to other problems.

Now, before closing this section, let us recall some related results. The initial boundary value problem associated to (1.1) has been considered by many peoples recently. Liu and Rao [5] investigated the Bresse system with two different dissipative mechanism, given by two temperatures coupled to the system. The authors proved that the exponential decay exists only when the velocities of the wave propagations are the same. If the wave speeds are different they showed that the energy of the system decays polynomially to zero with the rate \( t^{-1/2} \) or \( t^{-1/4} \), provided that the boundary conditions is of Dirichlet–Neumann–Neumann or Dirichlet–Dirichlet–Dirichlet type, respectively. This result was improved by Fatori and Muñoz Rivera [1], where they showed that, in general, the Bresse system is not exponentially stable but there exists polynomial stability with rates that depend on the wave speed propagations and the regularity of the initial data.

For the Cauchy problem, there are only few results. The first paper that dealt with the Cauchy problem for the Bresse system is [12], where the authors investigated the relationship between the frictional damping terms, the wave speeds of propagation and their influences on the decay rate of the solution. In addition, they showed (among other results) the estimates (1.3) and (1.4). The Bresse–Fourier system (Bresse system coupled with the Fourier law of heat conduction) has been investigated by Said-Houari and Soufyane in [11], the Bresse–Cattaneo by Said-Houari and Hamadouche [10] and the Bresse system in thermoelasticity of type III by Said-Houari and Hamadouche [9] where in these three systems, some decay
estimates have been proved under some appropriate assumptions on the coefficients of the systems.

This paper is organized as follows: in Section 2, we state the problem. Section 3 is devoted to the case $\gamma_1 > 0$ and $\gamma_2 = 0$, where we prove that the solution does not decay at all. In Section 4, we study the case $\gamma_1 > 0$ and $\gamma_2 > 0$, where we prove an optimal decay rate using the Lyapunov functional method. Finally in Section 5 we investigate the case where $\gamma_1 = 0$ and $\gamma_2 > 0$ and show, through a new method based on the use of the eigenvalues expansion combined with Fourier splitting method, the optimal decay rate of the solution.

2. **Statement of the problem**

In this section, we state the problem and introduce some materials that will be needed later. Let us first rewrite system (1.1)-(1.2) as a first order system of the form

\[
\begin{cases}
U_t + AU_x + LU = 0, \\
U (x,0) = U_0,
\end{cases}
\]

where $A$ is a real symmetric matrix and $L$ is a non-negative (non-symmetric) definite matrix. To this end, we introduce the following variables:

\[
v = \varphi_x - \psi - lw, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad \phi = k (w_x - l\varphi), \quad \eta = w_t.
\]

Consequently, system (1.1) can be rewritten into the following first order system

\[
\begin{cases}
v_t - u_x + y + l\eta = 0, \\
u_t - v_x -lk\phi = 0, \\
z_t - ay_x = 0, \\
y_t - az_x - v + \gamma_1 y = 0, \\
\phi_t - k\eta_x + lk\phi = 0, \\
\eta_t - k\phi_x - lv + \gamma_2 \eta = 0
\end{cases}
\]

and the initial conditions (1.2) takes the form

\[
(v, u, z, y, \phi, \eta) (x,0) = (v_0, u_0, z_0, y_0, \phi_0, \eta_0),
\]

where

\[
v_0 = \varphi_{0x} - \psi_0 - lw_0, \quad u_0 = \varphi_1, \quad z_0 = a\psi_{0x}, \quad y_0 = \psi_1, \quad \phi_0 = kw_{0x} - lk\varphi_0, \quad \eta = w_1.
\]

System (2.3)-(2.4) is equivalent to system (2.1) with

\[
U = \begin{pmatrix} v \\ u \\ z \\ y \\ \phi \\ \eta \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k & 0 \\ 0 & 0 & 0 & 0 & -k & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & l \\ 0 & 0 & 0 & 0 & -lk & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma_1 & 0 & 0 \\ 0 & lk & 0 & 0 & 0 & 0 \\ -l & 0 & 0 & 0 & \gamma_2 & 0 \end{pmatrix}.
\]

and $U_0 = (v_0, u_0, z_0, y_0, \phi_0, \eta_0)^T$. 

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By taking the Fourier transform of (2.1) we obtain the following Cauchy problem of a first order system

(2.5) \[
\begin{cases}
\hat{U}_t + i\xi A \hat{U} + L\hat{U} = 0, \\
\hat{U}(\xi, 0) = \hat{U}_0.
\end{cases}
\]

The solution of (2.5) is given by

\[\hat{U}(\xi, t) = e^{\Phi(i\xi)t}\hat{U}_0(\xi),\]

where

(2.6) \[\Phi(\zeta) = -(L + \zeta A), \quad \zeta = i\xi \in \mathbb{C}.\]

3. Non decaying solutions for \(\gamma_2 = 0\)

In this section, we assume that \(\gamma_2 = 0\) and show that the damping term \(\gamma_1 \psi_t\) is not strong enough to obtain a decay rate of the solution. Our main result in this section reads as follows:

**Theorem 3.1** (No decay rates). Let \(s\) be a nonnegative integer. Let \(U(x, t)\) be the solution of (2.1). Assume that \(U_0 \in H^s(\mathbb{R})\), then the following estimate holds:

(3.1) \[\|\partial_x^j U(t)\|_{L^2} \leq C \|\partial_x^j U_0\|_{L^2}, \quad j = 0, 1, \ldots, s.\]

where \(C\) is a positive constant.

**Proof.** To prove (3.1), we use the dissipation of the energy

\[\hat{E}(\xi, t) = \frac{1}{2} |\hat{U}(\xi, t)|^2\]

of (2.5) which satisfies (see Lemma 4.2)

(3.2) \[\frac{d\hat{E}(\xi, t)}{dt} = -\gamma_1 |\hat{y}|^2, \quad \forall t \geq 0.\]

Hence, the solution of (2.5) satisfies

(3.3) \[|\hat{U}(\xi, t)|^2 \leq |\hat{U}(\xi, 0)|^2, \quad \forall t \geq 0.\]

Therefore, (3.1) follows from the Plancherel theorem.

To justify the nondecay rate in (3.1), we show that there exists at least one eigenvalue \(\lambda(\zeta)\) of \(\Phi\) such that \(\text{Re}(\lambda(\zeta)) = 0\). We compute the characteristic polynomial of \(\Phi(\zeta)\) to get

\[
f(\lambda, \zeta) = \det(\lambda I - \Phi(\zeta))
= \lambda^6 + \gamma_1 \lambda^5 + \left\{ (k^2 + 1)(l^2 - \zeta^2) + 1 - a^2 \zeta^2 \right\} \lambda^4
+ \gamma_1 (k^2 + 1)(l^2 - \zeta^2) \lambda^3
+ (l^2 - \zeta^2) \left\{ k^2(l^2 - \zeta^2) + \left( k^2 - a^2(k^2 + 1) \zeta^2 \right) \right\} \lambda^2
+ \gamma_1 k^2(l^2 - \zeta^2)^2 \lambda - a^2 k^2 \zeta^2(l^2 - \zeta^2)^2.
\]

(3.4) The above polynomial can be rewritten as

(3.5) \[\left( \lambda^2 + k^2(l^2 - \zeta^2) \right) \left\{ \lambda^4 + \gamma_1 \lambda^3 + (l^2 + 1 - \zeta^2(a^2 + 1)) \lambda^2 + \gamma_1 (l^2 - \zeta^2) \lambda - a^2 \zeta^2(l^2 - \zeta^2) \right\}.\]
Since $\zeta = i\xi$, it is clear from (3.5) that for all $\xi \in \mathbb{R}$, the matrix $\Phi(\zeta)$ has two pure imaginary roots. Consequently, according to the stability theory of linear ODE systems (see [13, p.71]) the solution of (2.5) doesn’t go to zero.

4. Optimal decay rates for $\gamma_i > 0$, $i = 1, 2$

In this section, we consider the system

$$
\begin{align*}
\varphi_{tt} - (\varphi_x - \psi - lw)_x - k^2 l (w_x - l \varphi) &= 0, \\
\psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi - lw) + \gamma_1 \psi_t &= 0, \\
w_{tt} - k^2 (w_x - l \varphi)_x - l (\varphi_x - \psi - lw) + \gamma_2 w_t &= 0,
\end{align*}
$$

(4.1)

with the initial data

$$
(\varphi, \varphi_t, \psi, \psi_t, w, w_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1),
$$

(4.2)

where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ and $a, l, \gamma_1, \gamma_2$ and $k$ are positive constants. The main result in this section reads as:

**Theorem 4.1** (Optimal decay rates). Let $s$ be a nonnegative integer. Let $U(x, t)$ be the solution of (2.1). Assume that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, then the following decay estimates hold:

- For $a = 1$, we have
  $$
  \left\| \partial_x^j U(t) \right\|_{L^2} \leq C (1 + t)^{-1/4 - j/2} \left\| U_0 \right\|_{L^1} + C e^{-ct} \left\| \partial_x^j U_0 \right\|_{L^2}, \quad j = 0, 1, \ldots, s.
  $$

- For $a \neq 1$, we have
  $$
  \left\| \partial_x^j U(t) \right\|_{L^2} \leq C (1 + t)^{-1/4 - j/2} \left\| U_0 \right\|_{L^1} + C (1 + t)^{-\ell/2} \left\| \partial_x^{j+\ell} U_0 \right\|_{L^2}, \quad j = 0, 1, \ldots, s-\ell.
  $$

where $C$ and $c$ are two positive constants.

The proof of Theorem 4.1 is based on some pointwise estimates in the Fourier space (Lemma 4.3) and given in Subsection 4.2. The optimality of the decay rates is given in Subsection 4.3.

4.1. Pointwise estimates in the Fourier space. In this section, we use the Lyapunov functional method to show Theorem 4.1. We will prove later that the Lyapunov functional method agrees with the eigenvalues expansion (See Proposition 4.4) which shows the optimality of the estimates given in Theorem 4.1.

System (2.5) can be rewritten into the following form

$$
\begin{align*}
\dot{v}_t - i\xi \dot{u} + \dot{y} + l \dot{\eta} &= 0, \\
\dot{u}_t - i\xi \dot{v} - lk \dot{\phi} &= 0, \\
\dot{z}_t - ai\xi \dot{\gamma} &= 0, \\
\dot{\gamma}_t - ai\xi \dot{z} - \dot{\nu} + \gamma_1 \dot{\eta} &= 0, \\
\dot{\phi}_t - i\xi k \dot{\eta} + lk_0 \dot{u} &= 0, \\
\dot{\eta}_t - i\xi k \dot{\phi} - l \dot{\nu} + \gamma_2 \dot{\eta} &= 0.
\end{align*}
$$

(4.5)

Let us now define the following energy functional

$$
\hat{E}(\xi, t) = \frac{1}{2}(|\dot{v}|^2 + |\dot{u}|^2 + |\dot{z}|^2 + |\dot{\gamma}|^2 + |\dot{\phi}|^2 + |\dot{\eta}|^2)(\xi, t).
$$

(4.6)
The next lemma states that the energy $\hat{E}(\xi, t)$ of the entire system (4.5) (or equivalently system (2.5)) is a non-increasing function. More precisely we have the following result.

**Lemma 4.2.** Let $(\hat{v}, \hat{u}, \hat{z}, \hat{\gamma}, \hat{\phi}, \hat{\eta})$ be the solution of (2.5), then the energy $\hat{E}(\xi, t)$ is a non-increasing function and satisfies, for all $t \geq 0$,

\begin{equation}
\frac{d\hat{E}(\xi, t)}{dt} = -\gamma_1 |\hat{y}|^2 - \gamma_2 |\hat{\eta}|^2.
\end{equation}

**Proof.** Multiplying the first equation in (4.5) by $\hat{\bar{v}}$, the second equation by $\hat{\bar{u}}$, the third equation by $\hat{\bar{z}}$, the fourth equation by $\hat{\bar{\gamma}}$, the fifth equation by $\hat{\bar{\phi}}$, the sixth equation by $\hat{\bar{\eta}}$, adding these equalities and taking the real part, then (4.7) holds. □

The following lemma is crucial for the proof of Theorem 4.1. With this lemma in hand, we can show the decay estimates of the solution.

**Lemma 4.3.** Let $\hat{U}(\xi, t)$ be the solution of (2.5). Then for any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following pointwise estimates:

\begin{equation}
|\hat{U}(\xi, t)|^2 \leq \begin{cases} 
C e^{-c\rho_1(\xi)t} |\hat{U}(\xi, 0)|^2, & \text{if } a = 1 \\
C e^{-c\rho_2(\xi)t} |\hat{U}(\xi, 0)|^2, & \text{if } a \neq 1,
\end{cases}
\end{equation}

where

\begin{equation}
\rho_1(\xi) = \frac{\xi^2}{1 + \xi^2}
\end{equation}

and

\begin{equation}
\rho_2(\xi) = \frac{\xi^2}{1 + \xi^2 + \xi^4}.
\end{equation}

Here $C$ and $c$ are two positive constants.

**Proof.** The proof is based on a delicate Fourier energy method. We do the proof in two main steps.

**Step 1. Exhibiting dissipation of the other terms:**

As we have seen from the estimate (4.7), only two components of the solution are damped through the energy dissipation inequality. So, our main goal in this first step is to find some appropriate functionals, that give some dissipative terms to the other components of the vector solution. Indeed, we have from [12], the identity

\begin{equation}
\begin{aligned}
\frac{d}{dt} \mathcal{F}(\xi, t) + a^2 i^2 \xi^2 (|\hat{\bar{z}}|^2 - |\hat{\bar{y}}|^2) - \xi^2 |\hat{\bar{y}}|^2 + \xi^2 |\hat{\bar{\gamma}}|^2 \\
= -a l^2 \gamma_1 Re(i\xi \hat{\bar{z}}\hat{\gamma}) + \gamma_2 a l Re(i\xi \hat{\bar{\gamma}}\hat{\bar{y}}) + (1 - a^2) l \xi^2 Re(\hat{\bar{y}}) \\
\quad + (a^2 - 1) Re(i\xi^3 \hat{\bar{u}}),
\end{aligned}
\end{equation}

where

\begin{equation}
\mathcal{F}(\xi, t) := l a \left\{ l Re(i\xi \hat{\bar{z}}\hat{\gamma}) + Re(i\xi \hat{\bar{y}}\hat{\bar{z}}) \right\} - \xi^2 \left\{ Re(\hat{\bar{\gamma}}) + Re(a \hat{\bar{u}}) \right\}.
\end{equation}
Also, from [12], we have
\[
\frac{d}{dt} \mathcal{K}(\xi, t) + k_0 \xi^2 (|\hat{\phi}|^2 - |\hat{\eta}|^2) = \Re(i \xi [l \hat{\eta}] - \gamma_2 \Re(i \xi \hat{\eta} \hat{\phi}) + \Re(l \alpha \xi^2 \hat{\phi}) + l \Re(\gamma_1 i \xi \hat{\phi} \hat{y})
\]
(4.13)
- \Re(\gamma_1 i \xi \hat{\phi} \hat{y})
(4.19)
where
(4.14)
\[\mathcal{K}(\xi, t) = \Re(-i \xi \hat{\phi} \hat{\eta}) + l \Re(-i \xi \hat{\phi} \hat{\eta}).\]

A simple application of Young's inequality gives
\[
\frac{d}{dt} \mathcal{K}(\xi, t) + (k_0 - \epsilon_1) \xi^2 |\hat{\phi}|^2
\]
(4.15)
\[\leq C(\epsilon_1, \epsilon_1')(1 + \xi^2)(|\hat{\eta}|^2 + |\hat{y}|^2) + C(\epsilon_1) \xi^2 |\hat{\zeta}|^2 + \epsilon_1' \xi^2 |\hat{u}|^2,
\]
where \(\epsilon_1, \epsilon_1'\) are arbitrary small positive constants.

Concerning (4.11), we have the following estimates:
- For \(a = 1\), we have as above, for any \(\epsilon_2, \epsilon_2'\) positive:
\[
\frac{d}{dt} \mathcal{P}(\xi, t) + (a^2 l^2 - \epsilon_2) \xi^2 |\hat{\zeta}|^2 + \xi^2 |\hat{\phi}|^2
\]
(4.16)
\[\leq C(\epsilon_2)(1 + \xi^2)(|\hat{\eta}|^2 + |\hat{y}|^2).
\]
- For \(a \neq 1\), we obtain, instead of (4.16),
\[
\frac{d}{dt} \mathcal{P}(\xi, t) + (a^2 l^2 - \epsilon_2) \xi^2 |\hat{\zeta}|^2 + \xi^2 |\hat{\phi}|^2
\]
(4.17)
\[\leq C(\epsilon_2, \epsilon_2')(1 + \xi^2 + \xi^4)(|\hat{\eta}|^2 + |\hat{y}|^2) + \epsilon_2' \xi^2 |\hat{u}|^2,
\]
where we used the estimate:
\[|(a^2 - 1) \Re(i \xi^3 \hat{\phi} \hat{\eta})| \leq \epsilon_2' \xi^2 |\hat{u}|^2 + C(\epsilon_2') \xi^4 |\hat{y}|^2.
\]

Next, multiplying the first equation in (4.5) by \(i \xi \hat{u}\), the second equation by \(-i \xi \hat{v}\), adding the results and taking the real part, we get
\[
\frac{d}{dt} \Re(i \xi \hat{v} \hat{u}) + \xi^2 (|\hat{u}|^2 - |\hat{v}|^2)
\]
(4.18)
\[= -\Re(i \xi \hat{v} \hat{u}) - \Re(i \xi \hat{v} \hat{u}) - \Re(i \xi k_0 \hat{\phi} \hat{v})\]

Multiplying the first equation in (4.5) by \(-\hat{\eta}\) and the sixth equation by \(-\hat{v}\), then taking the real part after adding the two results, we obtain
\[
-\frac{d}{dt} \Re(i \hat{v} \hat{\eta}) + l |\hat{v}|^2 - l |\hat{\eta}|^2
\]
(4.19)
\[= -\Re(i \xi \hat{v} \hat{u}) + \Re(i \hat{v} \hat{u}) - \Re(i \xi k_0 \hat{\phi} \hat{v}) + \gamma_2 \Re(i \hat{v} \hat{\eta}).
\]

Summing up (4.18) + l(4.19), we get
\[
\frac{d}{dt} \mathcal{P}(\xi, t) + \xi^2 (|\hat{u}|^2 - |\hat{v}|^2) + l^2 |\hat{v}|^2 - l^2 |\hat{\eta}|^2
\]
\[= -\Re(i \xi \hat{v} \hat{u}) - 2 \Re(i \xi k_0 \hat{\phi} \hat{v}) + l \Re(i \hat{v} \hat{\eta}) + l \gamma_2 \Re(i \hat{v} \hat{\eta}).
\]
Furthermore, we pick \( \epsilon \).

Applying Young’s inequality, we obtain for any \( \epsilon_3, \epsilon_4 > 0 \),

\[
\frac{d}{dt} \mathcal{P}(\xi, t) + \xi^2 (1 - \epsilon_3) |\dot{\hat{u}}|^2 + (l^2 - \epsilon_4) |\dot{\hat{v}}|^2 \leq \xi^2 |\dot{\hat{v}}|^2 + C(\epsilon_3, \epsilon_4) (|\dot{\hat{y}}|^2 + |\dot{\hat{\eta}}|^2) + C(\epsilon_4) \xi^2 |\dot{\hat{\theta}}|^2.
\]

(4.21)

Step 2. Building the appropriate Lyapunov functional:

In this step, we make the appropriate combination of the above obtained functionals to build a Lyapunov functional \( \mathcal{L}(\xi, t) \). To construct this functional, we need to take into account two main things. First, this functional should satisfy the estimate (4.24) and second, it should verify another estimate of the form

\[
c_1 \sigma(\xi) \dot{E}(\xi, t) \leq \mathcal{L}(\xi, t) \leq c_2 \sigma(\xi) \dot{E}(\xi, t),
\]

where \( c_1 \) and \( c_2 \) are two positive constants and \( \sigma(\xi) \) is a function depending on \( \xi \) only.

Hence, we define for \( a = 1 \), the Lyapunov functional \( \mathcal{L}_1(\xi, t) \) as follows:

\[
\mathcal{L}_1(\xi, t) := d_0 (1 + \xi^2) \dot{E}(\xi, t) + d_1 \mathcal{P}(\xi, t) + d_2 \mathcal{K}(\xi, t) + \mathcal{P}(\xi, t),
\]

where \( d_0, d_1, d_2 \) and \( d_3 \) are positive constants that will be fixed later.

The derivative of (4.22) with respect to \( t \) and the use of (4.7), (4.15), (4.16) and (4.21) lead to

\[
\frac{d}{dt} \mathcal{L}_1(\xi, t) + \left\{ d_1 (a^2 l^2 - \epsilon_2) - d_2 C(\epsilon_1) \right\} \xi^2 |\dot{\hat{z}}|^2 \\
+ (d_1 - 1) \xi^2 |\dot{\hat{v}}|^2 + \left\{ d_2 (k_0 - \epsilon_1) - C(\epsilon_4) \right\} \xi^2 |\dot{\hat{\theta}}|^2 \\
+ \left\{ (1 - \epsilon_3) - d_2 \epsilon'_1 \right\} \xi^2 |\dot{\hat{u}}|^2 + (l^2 - \epsilon_4) |\dot{\hat{v}}|^2 \\
\leq \left\{ C(\epsilon_1, \epsilon'_1, \epsilon_2, \epsilon_3, \epsilon_4, d_1, d_2) - d_0 \min(\gamma_1, \gamma_2) \right\} (1 + \xi^2) (|\dot{\hat{y}}|^2 + |\dot{\hat{\eta}}|^2).
\]

(4.23)

We choose the constants in the above formula as follows: fix \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) small enough such that

\[
\epsilon_1 < k_0, \quad \epsilon_2 < a^2 l^2, \quad \epsilon_3 < 1, \quad \epsilon_4 < l^2.
\]

After that, we fix \( d_2 \) large enough such that

\[
d_2 > \frac{C(\epsilon_4)}{k_0 - \epsilon_1}.
\]

Then, we select \( d_1 \) large enough such that

\[
d_1 > \max \left( 1, \frac{d_2 C(\epsilon_1)}{a^2 l^2 - \epsilon_2} \right).
\]

Furthermore, we pick \( \epsilon'_1 \) small enough such that

\[
\epsilon'_1 < \frac{1 - \epsilon_3}{d_2}.
\]

Finally, once all the above constants are fixed, we take \( d_0 \) large enough such that

\[
d_0 > \frac{C(\epsilon_1, \epsilon'_1, \epsilon_2, \epsilon_3, \epsilon_4, d_1, d_2)}{\min(\gamma_1, \gamma_2)}.
\]
Hence, we find a positive constant \( c_0 > 0 \), such that
\[
\frac{d}{dt} \mathcal{L}_1(\xi, t) + c_0 \xi^2 \hat{E}(\xi, t) \leq 0, \quad \forall t \geq 0.
\]
(4.24)

Since
\[
\hat{E}(\xi, t) = \frac{1}{2} |\hat{U}(\xi, t)|^2
\]
then, for \( d_0 \) large enough, there exist two positive constants \( c_1 \) and \( c_2 \) such that for all \( t \geq 0 \),
\[
c_1(1 + \xi^2) \hat{E}(\xi, t) \leq \mathcal{L}_1(\xi, t) \leq c_2(1 + \xi^2) \hat{E}(\xi, t).
\]
(4.25)

On the other hand, there exists a constant \( c_3 > 0 \), such that
\[
\frac{d}{dt} \mathcal{L}_1(\xi, t) + c_3 \xi^2 \frac{\xi^2}{1 + \xi^2} \hat{E}(\xi, t) \leq 0, \quad \forall t \geq 0.
\]
(4.26)

Integrating (4.27) and using once again (4.25) and (4.26), then (4.8) holds for \( a = 1 \).

Next, for \( a \neq 1 \), we define another Lyapunov Functional
\[
\mathcal{L}_2(\xi, t) = d_0(1 + \xi^2 + \xi^4) \hat{E}(\xi, t) + d_1 \mathcal{F}(\xi, t) + d_2 \mathcal{K}(\xi, t) + \mathcal{P}(\xi, t).
\]
(4.27)

Now, arguing as above and choosing the constants exactly as before, except for the new constant \( \epsilon_2' \) which should be small enough, we get
\[
\frac{d}{dt} \mathcal{L}_2(\xi, t) + c_4 \frac{\xi^2}{(1 + \xi^2 + \xi^4)} \hat{E}(\xi, t) \leq 0, \quad \forall t \geq 0,
\]
(4.28)

for some \( c_4 > 0 \). Which leads to the second estimate in (4.8). We omit the details.

4.2. Decay estimates: Proof of Theorem 4.1. In this subsection, we show the decay estimate of the \( L^2 \)-norm of the solution of (2.1). These decay estimates are optimal, since they agree with the asymptotic expansion of the eigenvalues given in Subsection 4.3. In addition, Theorem 4.1 improves the result of Theorem 6.1 in [12].

To show (4.3), we have from (4.9) that
\[
\rho_1(\xi) \geq \begin{cases} 
  c \xi^2, & \text{if } \xi \leq 1, \\
  c, & \text{if } \xi \geq 1.
\end{cases}
\]
(4.30)

Applying the Plancherel theorem and using the first estimate in (4.8), we obtain
\[
\| \partial_x^j U(t) \|^2_{L^2} \leq \int_{\mathbb{R}} |\xi|^{2j} |\hat{U}(\xi, t)|^2 d\xi \leq C \int_{\mathbb{R}} |\xi|^{2j} e^{-c_0(\xi^2)t} |\hat{U}(\xi, 0)|^2 d\xi
\]
\[
= C \int_{|\xi| \leq 1} |\xi|^{2j} e^{-c_0(\xi^2)t} |\hat{U}(\xi, 0)|^2 d\xi + C \int_{|\xi| \geq 1} |\xi|^{2j} e^{-c_0(\xi^2)t} |\hat{U}(\xi, 0)|^2 d\xi
\]
(4.31)

Exploiting (4.30), we infer that
\[
I_1 \leq C \|\hat{U}_0\|_{L^\infty} \int_{|\xi| \leq 1} |\xi|^{2j} e^{-c_0(\xi^2)t} d\xi \leq C (1 + t)^{-\frac{1}{2}(1+2j)} \|U_0\|_{L^1}^2,
\]
(4.32)
where we have used the inequality

\[(4.33) \int_0^1 |\xi|^\sigma e^{-c\xi^2t}d\xi \leq C (1 + t)^{-(\sigma+1)/2}.\]

In the high frequency region (\(|\xi| \geq 1\)), we have

\[I_2 \leq e^{-ct} \int_{|\xi| \geq 1} |\xi|^{2j}|\hat{U}(\xi, 0)|^2d\xi \leq e^{-ct} \|\partial_x U_0\|_{L^2}^2\]

which leads to the estimates in (4.3).

Second, assume that \(a \neq 1\). As above,

\[\|\partial^j U(t)\|_{L^2}^2 \leq C \int_{|\xi| \leq 1} |\xi|^{2j} e^{-c\rho_2(\xi)t}|\hat{U}(\xi, 0)|^2d\xi + C \int_{|\xi| \geq 1} |\xi|^{2j} e^{-c\rho_2(\xi)t}|\hat{U}(\xi, 0)|^2d\xi\]

\[(4.34) = L_1 + L_2.\]

Now using the second estimate in (4.8) and the fact that \(\rho_2(\xi) \geq \frac{1}{4} \xi^2\) for \(|\xi| \leq 1\), we have by the same method as in the proof of the estimate of \(I_1\),

\[(4.35) L_1 \leq C (1 + t)^{-1/2-j} \|U_0\|_{L^1}^2.\]

To estimate the term \(L_2\), we use the inequality \(\rho_2(\xi) \geq c\xi^{-2}\) for \(|\xi| \geq 1\) to obtain

\[(4.36) L_2 \leq C \sup_{|\xi| \geq 1} \left(|\xi|^{-2\ell} e^{-c\xi^{-2t}}\right) \int_{|\xi| \geq 1} |\xi|^{2(j+\ell)}|\hat{U}(\xi, 0)|^2d\xi \leq C (1 + t)^{-\ell} \|\partial^{j+\ell}_x U_0\|_{L^2}^2.\]

Inserting the estimates (4.35) and (4.36) into (4.34), then (4.4) is obtained. This finishes the proof of Theorem 4.1.

4.3. Optimality of the decay rates. To prove the optimality of the decay rate in Theorem 4.1, we use the following proposition based on eigenvalues expansion.

**Proposition 4.4.** Let \(\lambda_j(\xi), 1 \leq j \leq 6\) be the eigenvalues of \(\Phi(\xi)\). Then as \(|\xi| \rightarrow 0\)

\[(4.37) \text{Re}(\lambda_j)(i\xi) = \begin{cases} -\frac{a^2l^2}{\gamma_1 l^2 + \gamma_2} |\xi|^2 + O(|\xi|^3), & \text{for } j = 1, \\ -\text{Re}(\beta_j)|\xi|^2 + O(|\xi|^3), & \text{for } j = 2, 3, \\ \text{Re}(r_j) + O(|\xi|), & \text{for } j = 4, 5, 6. \end{cases}\]

For \(|\xi| \rightarrow +\infty\)

\[\bullet \text{ For } a = 1, \text{ we get}

\[(4.38) \text{Re}(\lambda_j)(i\xi) = \begin{cases} \text{Re}(\delta_j) + O(|\xi|^{-1}), & \text{for } j = 1, 2, \\ -\gamma_1 2 + O(|\xi|^{-1}), & \text{for } j = 3, 4, \\ -\gamma_2 2 + O(|\xi|^{-1}), & \text{for } j = 5, 6. \end{cases}\]
• For $a \neq 1$, we have

$$\Re(\lambda_j)(i\xi) = \begin{cases} -\kappa_j |\xi|^{-2} + O(|\xi|^{-3}), & \text{for } j = 1, 2 \\ -\gamma_1 \frac{1}{2} + O(|\xi|^{-1}), & \text{for } j = 3, 4, \\ -\gamma_2 \frac{1}{2} + O(|\xi|^{-1}), & \text{for } j = 5, 6. \end{cases}$$

(4.39)

Proof. For $\gamma_i > 0$, $i = 1, 2$, the system (4.1)-(4.2) is equivalent to (2.1) with

$$U = \begin{pmatrix} v \\ u \\ z \\ \phi \\ \eta \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & -k & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & l \\ 0 & 0 & 0 & 0 & -lk & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma_1 & 0 & 0 \\ 0 & lk & 0 & 0 & 0 & 0 \\ -l & 0 & 0 & 0 & 0 & \gamma_2 \end{pmatrix}$$

and $U_0 = (v_0, u_0, z_0, y_0, \phi_0, \eta_0)^T$. Observe that, since $L$ is not symmetric, the general theory for hyperbolic systems does not apply.

Recall that

(4.40) \[ \Phi(\zeta) = -(L + \zeta A), \quad \zeta = i\xi \in \mathbb{C}. \]

Let us denote by $\lambda_j(\zeta)$, $1 \leq j \leq 6$ the eigenvalues of $\Phi(\zeta)$, then we compute the characteristic equation as

$$\det(\lambda I - \Phi(\zeta)) = \lambda^6 + (\gamma_1 + \gamma_2)\lambda^5 + \left\{(k^2 + 1)(l^2 - \zeta^2) + \gamma_1\gamma_2 + 1 - a^2\zeta^2\right\}\lambda^4$$

$$+ \left\{\gamma_1(k^2 + 1)(l^2 - \zeta^2) + \gamma_2((k^2l^2 + 1) - (1 + a^2)\zeta^2)\right\}\lambda^3$$

$$+ \left[\gamma_1\gamma_2(k^2l^2 - \zeta^2) + (l^2 - \zeta^2)\left\{k^2(l^2 - \zeta^2) + (k^2 - a^2(k^2 + 1)\zeta^2)\right\}\right]\lambda^2$$

$$+ \left\{\gamma_1k^2(l^2 - \zeta^2)^2 + k^2l^2\gamma_2 - a^2k^2l^2\gamma_2\zeta^2 + a^2\gamma_2\zeta^4\right\}\lambda - a^2k^2\zeta^2(l^2 - \zeta^2)^2.$$

(4.41)

It is legitimate to do an asymptotic expansion of the eigenvalues. Indeed, since the dependence on $\zeta$ of $\Phi$ is analytic then by [4, p.63] the eigenvalues depends also analytically on $\zeta$.

• Behavior of $\lambda_j(\zeta)$ when $|\zeta| \to 0$.

First, when $|\zeta| \to 0$, then $\lambda_j(\zeta)$ has the following asymptotic expansion:

$$\lambda_j(\zeta) = \lambda_j^{(0)} + \lambda_j^{(1)}\zeta + \lambda_j^{(2)}\zeta^2 + \ldots, \quad 1 \leq j \leq 6.$$ 

(4.42)

Notice that $\lambda_j^{(0)}$ are the eigenvalues of the matrix $-L$ and satisfy, with $y = \lambda_j^{(0)}$, the equation

$$y \left(y^2 + k^2l^2\right) \left(y^3 + (\gamma_1 + \gamma_2)y^2 + (l^2 + 1 + \gamma_1\gamma_2)y + \gamma_1l^2 + \gamma_2\right) = 0.$$ 

Consequently, we have from the above equation that

$$\left\{ \begin{array}{ll}
\lambda_j^{(0)} = 0, & \text{for } j = 1, \\
\lambda_j^{(0)} = \pm ikl, & \text{for } j = 2, 3, \\
\lambda_j^{(0)} = r_j, & \text{for } j = 4, 5, 6, 
\end{array} \right.$$
where \( r_j \) are the solutions of the algebraic equation
\[
(4.43) \quad g(X) = X^3 + (\gamma_1 + \gamma_2)X^2 + (l^2 + 1 + \gamma_1 + \gamma_2)X + \gamma_1 l^2 + \gamma_2 = 0.
\]
It is well known that an algebraic equation of an odd degree with real coefficients has at least one real root \( r_1 \). Now, in order to know the location of \( r_1 \), we consider the equation (4.43) with \( X \in \mathbb{R} \). Then, it is clear that
\[
g(-\gamma_1)g(0) = -\gamma_1 l^2 + \gamma_2 \left( \gamma_1^2 + \gamma_2^2 + 1 + (\gamma_1 + \gamma_2)^2 \right) < 0.
\]
Therefore, equation (4.43) has at least one real root \( X = r_1 \) in the interval \((-\gamma_1, 0)\).

In this case, we may rewrite equation (4.43) in the form
\[
(4.44) \quad g(X) = (X - r_1) \left( X^2 + (\gamma_1 + \gamma_2 + r_1)X + l^2 + 1 + \gamma_1 + \gamma_2 + (\gamma_1 + \gamma_2)r_1 + r_1^2 \right).
\]
Now, let us denote by \( r_2 \) and \( r_3 \), the other two roots. Then, we have
\[
r_1 + r_2 + r_3 = -(\gamma_1 + \gamma_2),
\]
and
\[
r_1 r_2 r_3 = -(\gamma_1 l^2 + \gamma_2).
\]
Since \( r_1 \) is a real root, then the coefficients of (4.44) are real and therefore
\[
Re(r_2) = Re(r_3).
\]
This implies that
\[
Re(r_2) = Re(r_3) = -\frac{1}{2}(r_1 + \gamma_1 + \gamma_2) < 0.
\]
If \( r_2 \) and \( r_3 \) are real, then they satisfy
\[
r_2 + r_3 = -(\gamma_1 + \gamma_2 + r_1) < 0, \quad \text{and} \quad r_2 r_3 = -\frac{(\gamma_1 l^2 + \gamma_2)}{r_1} > 0,
\]
which implies \( r_2, r_3 < 0 \).

Now, using equation (4.41) and (4.42), by equating coefficients of like powers of \( \zeta \), we obtain
\[
\begin{align*}
\lambda_j^{(1)} &= 0, & \text{for } & j = 1, 2, 3 \\
\lambda_j^{(2)} &= \frac{a^2 l^2}{\gamma_1 l^2 + \gamma_2}, & \text{for } & j = 1, \\
\lambda_j^{(2)} &= \beta_j, & \text{for } & j = 2, 3,
\end{align*}
\]
where \( \beta_j \) is the solution of the equation
\[
A + iB + \beta_j(C + iD) = 0,
\]
where
\[
\begin{align*}
A &= k^2 l^2 \left( k^2 (k^2 + l^2) - k^4 l^2 + \gamma_1 \gamma_2 \right) \\
B &= k^3 l^3 \left( \gamma_1 (k^2 - 1) + \gamma_2 \right) \\
C &= 2 k^2 l^2 \left( \gamma_1 l^2 (k^2 - 1) + \gamma_2 (k^2 l^2 - 1) \right) \\
D &= 2 k^3 l^3 \left( l^2 (k^2 - 1) - 1 - \gamma_1 \gamma_2 \right).
\end{align*}
\]
We need to show that $Re(\beta_j) > 0$. In order to prove this, it is enough to verify that $AC + BD < 0$.

Hence,

$$K = AC + BD = \left( k^2(1 + l^2) - k^4l^2 + \gamma_1\gamma_2 \right) \left( \gamma_1 l^2(k^2 - 1) + \gamma_2(k^2l^2 - 1) \right) + k^2l^2 \left( \gamma_1(k^2 - 1) + \gamma_2 \right) \left( l^2(k^2 - 1) - 1 - \gamma_1\gamma_2 \right).$$

Factorizing by $k^4l^4\gamma_2$, we deduce

$$K = -2k^4l^4\gamma_2 \left( \gamma_1\gamma_2 + (k^2 - 1)^2 l^2 \gamma_1^2 + k^2 \left( (k^2 - 1) l^2 - 1 \right)^2 \right) < 0,$$

which concludes the proof of (5.13)

- **Behavior of $\lambda_j(\zeta)$ when $|\zeta| \to \infty$.**

For $|\zeta| \to \infty$ and following [2], we consider the characteristic equation in the form

$$\Psi(\zeta) = \zeta^6 \det (\mu I + (A + \zeta^{-1}L))$$

$$= \mu^6 + (\gamma_1 + \gamma_2)\zeta^{-1}\mu^5 + \left\{ (k^2 + 1)(l^2\zeta^2 - 1) + (\gamma_1\gamma_2 + 1)\zeta^2 - a^2 \right\} \mu^4$$

$$+ \left\{ \gamma_1(k^2 + 1)(l^2\zeta^2 - 1) + \gamma_2 \left( (k^2l^2 + 1)\zeta^2 - (1 + a^2) \right) \right\} \zeta^{-1}\mu^3$$

$$+ \left[ \gamma_1\gamma_2(k^2l^2\zeta^2 - 1)\zeta^2 - (l^2\zeta^2 - 1) \right] \left\{ \mu^2 \left( (k^2l^2\zeta^2 - 1) + \left( (k^2\zeta^2 - a^2(k^2 + 1) \right) \right) \right\} \zeta^{-1}\mu - a^2k^2(l^2\zeta^2 - 1)^2 = 0,$$

where $\mu(\zeta^{-1})$ is the eigenvalues of (4.45). Moreover, we have the relation

$$\lambda_j(\zeta) = \zeta \mu_j(\zeta^{-1}).$$

Now, for $|\zeta|^{-1} \to 0$, we have the asymptotic expansion of $\mu_j(\zeta^{-1})$ in the form (for simplicity, we put $\nu = \zeta^{-1}$)

$$\mu_j(\nu) = \mu_j^{(0)} + \nu \mu_j^{(1)} + \nu^2 \mu_j^{(2)} + \ldots, \quad 1 \leq j \leq 6. \quad (4.46)$$

Plugging (4.46) into (4.45) and equating coefficients of like powers of $\nu$, we get

$$\begin{cases} 
\mu_j^{(0)} = \pm 1, & \text{for } j = 1, 2 \\
\mu_j^{(0)} = \pm a, & \mu_j^{(1)} = -\frac{\gamma_1}{2} & \text{for } j = 3, 4, \\
\mu_j^{(0)} = \pm k, & \mu_j^{(1)} = -\frac{\gamma_2}{2} & \text{for } j = 5, 6.
\end{cases}$$

- For $a = 1$,

$$\mu_j^{(1)} = \delta_j = \frac{1}{4} \left( -\gamma_1 \pm \sqrt{\gamma_1^2 - 4} \right), \quad \text{for } j = 1, 2$$

It is clear that $Re(\delta_j) < 0$. 

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• For $a \neq 1$, we have

\[
\begin{align*}
\mu_j^{(1)} &= 0, \\
\mu_j^{(2)} &= \pm \frac{l^2(1-a^2) + 1}{2(a^2 - 1)}, \quad \text{for } j = 1, 2 \\
\mu_j^{(3)} &= \frac{(a^2 - 1)^2 l^2 \gamma_2 + \gamma_1}{2(a^2 - 1)^2} = \kappa_j > 0,
\end{align*}
\]

which concludes the proof of Proposition 4.4.

5. The case $\gamma_1 = 0$ and $\gamma_2 > 0$

In this section, we investigate the case where $\gamma_1 = 0$ and $\gamma_2 > 0$. In this case the only acting damping term $\gamma_2 w_t$ of the whole system is in the third equation. We prove that the effect of this damping term will be propagated to the other components of the solution which will lead eventually, to the convergence of the solution to zero. This requires to assume more regularity on the initial data than the case where $\gamma_i > 0$, $i = 1, 2$. Moreover, our result has been proved under a new extra assumption on the coefficients of the system and it reads as follows.

**Theorem 5.1.** Assume that $\gamma_1 = 0$ and

\[(k^2 - 1) l^2 - 1 \neq 0.\]

Let $s$ be a nonnegative integer and $U(x,t)$ be the solution of (2.1). Assume that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, then the following decay estimates hold for $t$ large enough:

- For $a = k = 1$, we have

\[
\| \partial^j_x U(t) \|_{L^2} \leq C (1 + t)^{-1/4-j/2} \| U_0 \|_{L_1} + C (1 + t)^{-\ell/2} \| \partial_x^{j+\ell+1} U_0 \|_{L^2}, \quad 0 \leq j \leq s - \ell - 1.
\]

- For $a \neq 1$, we have

\[
\| \partial^j_x U(t) \|_{L^2} \leq C (1 + t)^{-1/4 - j/2} \| U_0 \|_{L_1} + C (1 + t)^{-\ell/2} \| \partial_x^{j+\ell+3} U_0 \|_{L^2}, \quad 0 \leq j \leq s - \ell - 3.
\]

The proof of Theorem 5.1 will be given in Subsection 5.5. However, it seems difficult to build appropriate Lyapunov functionals in this case, as we did in Section 4.1. Instead, we rely on asymptotic expansion of the eigenvalues of the matrix $\Phi(\zeta)$ and on the behavior of the Fourier image of the solution in low frequencies $\Upsilon_L = \{|\xi| < \nu \ll 1\}$, middle frequencies $\Upsilon_M = \{ \nu \leq |\xi| \leq N \}$ and high frequencies $\Upsilon_H = \{|\xi| > N \gg 1\}$ regions. To estimate the Fourier image of the solution, we compute the exponential of the matrix $\Phi(\zeta)t$ thanks to the following lemma which avoids the use of eigenvectors and eigenspaces.

**Lemma 5.2** ([6]). Assume that $A$ is an $n \times n$ matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (real or complex) written in some arbitrary but specified order and they are not necessary distinct. Then

\[
e^{tA} = \sum_{j=0}^{n-1} r_{j+1}(t) P_j
\]
where

\[ P_0 = I, \quad P_j = \prod_{k=1}^{j}(A - \lambda_k I), \quad j = 1, \ldots, n \]

and \( r_1(t), \ldots, r_n(t) \) are (real or complex) functions defined by the following equations

\[
\begin{cases}
  r'_1(t) = \lambda_1 r_1(t), & r_1(0) = 1, \\
  r'_2(t) = \lambda_2 r_2(t) + r_1(t), & r_2(0) = 0, \\
  \vdots \\
  r'_n(t) = \lambda_n r_n(t) + r_{n-1}(t), & r_n(0) = 0.
\end{cases}
\]

(5.4)

Notice that Lemma 5.2 enables us to compute the exponential of the matrix \( \Phi(\zeta) t \) without the use of the canonical Jordan form which requires the knowledge of the eigenvectors and the corresponding eigenspaces.

5.1. Asymptotic expansion of the eigenvalues. In this subsection, we perform an asymptotic expansion of the eigenvalues of \( \Phi(\zeta) \) in the low and high frequencies.

**Proposition 5.3.** Let \( \gamma_1 = 0, \gamma_2 > 0 \) and \( \lambda_j(\zeta), 1 \leq j \leq 6 \) be the eigenvalues of \( \Phi(\zeta) \). Assume that

\[
(k^2 - 1)t^2 - 1 \neq 0.
\]

Then, as \( |\xi| \to 0 \), we have

\[
Re(\lambda_j)(i\xi) = \begin{cases} \frac{-a^2 t^2}{2\gamma_2}|\xi|^2 + O(|\xi|^3), & \text{for } j = 1, \\
-\beta|\xi|^2 + O(|\xi|^3), & \text{for } j = 2, 3, \\
Re(\sigma_j) + O(|\xi|), & \text{for } j = 4, 5, 6. \end{cases}
\]

(5.6)

On the other hand, as \( |\xi| \to +\infty \), we obtain

- **For** \( a = k = 1 \),

\[
Re(\lambda_j)(i\xi) = \begin{cases} \frac{-t^2\gamma_2}{2}|\xi|^{-2} + O(|\xi|^{-3}), & \text{for } j = 1, 2, \\
-\frac{\gamma_2}{6} + O(|\xi|^{-1}), & \text{for } j = 3, 4, \\
-\frac{\gamma_2}{2} + O(|\xi|^{-1}), & \text{for } j = 5, 6. \end{cases}
\]

(5.7)

- **For** \( a \neq 1 \), we have

\[
Re(\lambda_j)(i\xi) = \begin{cases} \frac{-t^2\gamma_2}{2}|\xi|^{-2} + O(|\xi|^{-3}), & \text{for } j = 1, 2, \\
-\frac{t^2\gamma_2}{2(a-1)^2(a+1)^2}|\xi|^{-4} + O(|\xi|^{-5}), & \text{for } j = 3, 4, \\
-\frac{\gamma_2}{2} + O(|\xi|^{-1}), & \text{for } j = 5, 6. \end{cases}
\]

(5.8)
Proof. \footnote{Behavior of $\lambda_j(\zeta)$ when $|\zeta| \to 0$} For $\gamma_1 = 0$, the characteristic polynomial of (4.41) takes the form

$$\det (\lambda I - \Phi(\zeta)) = \lambda^6 + \gamma_2 \lambda^5 + \left\{ (k^2 + 1)(l^2 - \zeta^2) + 1 - a^2 \zeta^2 \right\} \lambda^4$$

$$+ \gamma_2 \left\{ (k^2l^2 + 1) - (1 + a^2)\zeta^2 \right\} \lambda^3$$

$$+ \left[ (l^2 - \zeta^2) \left\{ k^2(l^2 - \zeta^2) + \left( k^2 - a^2(k^2 + 1)\zeta^2 \right) \right\} \lambda^2$$

$$+ \gamma_2 \left\{ k^2l^2 - a^2k^2l^2\zeta^2 + a^2\zeta^4 \right\} \lambda - a^2k^2\zeta^2(l^2 - \zeta^2)^2. \tag{5.9}$$

For $|\zeta| \to 0$, $\lambda_j(\zeta)$ has the asymptotic expansion as in (4.42). Consequently, we have from the above equation that

$$\begin{cases}
\lambda_j^{(0)} = 0, & \lambda_j^{(1)} = 0, & \lambda_j^{(2)} = \frac{a^2l^2}{\gamma_2}, & \text{for} & j = 1, \\
\lambda_j^{(0)} = \pm ik, & \lambda_j^{(1)} = 0, & \lambda_j^{(2)} = \hat{\beta} \pm i\hat{\delta} & \text{for} & j = 2, 3, \\
\lambda_j^{(0)} = \sigma_j, & \text{for} & j = 4, 5, 6,
\end{cases}$$

with

$$\hat{\beta} = \frac{\gamma_2 k^2 \left((k^2 - 1)l^2 - 1\right)^2}{2 \left( \gamma_2^2 (k^2l^2 - 1)^2 + k^2l^2 \left((k^2 - 1)l^2 - 1\right)^2 \right)},$$

and

$$\hat{\delta} = -\frac{kl \left( \gamma_2^2 (k^2l^2 - 1) + (k - k \left(k^2 - 1\right)l^2)^2 \right)}{2 \left( \gamma_2^2 (k^2l^2 - 1)^2 + k^2l^2 \left((k^2 - 1)l^2 - 1\right)^2 \right)}.$$

It is clear that under the assumption (5.5), $\hat{\beta} > 0$.

On the other hand $\sigma_j$, 4 $\leq$ $j$ $\leq$ 6 are the solutions $Z$ of the cubic equation

$$(5.10) Z^3 + \gamma_2 Z^2 + (l^2 + 1)Z + \gamma_2 = 0.$$ We can easily show as before, that $\text{Re}(\sigma_j) < 0$. Now, we want to see the multiplicity of the roots of (5.10). As we have seen before, equation (5.10) has at least one negative real root. In order to know the nature of the other two roots of the equation (in general)

$$ax^3 + bx^2 + cx + d = 0$$
we use the Cardano method and investigate the sign of

$$D = Q^3 + R^2,$$

with

$$Q = \frac{3ac - b^2}{9a^2} \quad \text{and} \quad R = \frac{9abc - 2b^3 - 27a^2d}{54a^3}.$$ Now, for equation (5.10), we have

$$Q = \frac{3(l^2 + 1) - \gamma_2^2}{9}, \quad R = \frac{9\gamma_2(l^2 + 1) - 2\gamma_2^3 - 27\gamma_2}{54}.$$ We define

$$D = Q^3 + R^2.$$ Thus, we have the following cases

- If $D > 0$, then (5.10) has one real and two complex conjugate roots.
• If $D < 0$, there are three distinct real roots.
• If $D = 0$, there is one real root and another real root of double multiplicity.

To do so, we compute:

$$D = \frac{1}{108} \left( 4\gamma_2^4 - \left( l^4 + 20l^2 - 8 \right) \gamma_2^2 + 4 \left( l^2 + 1 \right)^3 \right).$$

In order to determine the sign of $D$, we consider the quadratic polynomial

$$\Lambda = 4\omega^2 - \left( l^4 + 20l^2 - 8 \right) \omega + 4 \left( l^2 + 1 \right)^3, \quad \omega = \gamma_2^2.$$

Now, it is clear that if $l^2 < 8$, then the discriminant of $\Lambda$ is:

$$\Delta = \left( l^4 + 20l^2 - 8 \right)^2 - 64 \left( l^2 + 1 \right)^3 = l^2 \left( l^2 - 8 \right) < 0.$$

Therefore, $\Lambda > 0$, hence $D > 0$. Consequently, (5.10) has one real root and two complex conjugate roots. In this case

$$\lambda_4 = \sigma_4, \quad \lambda_5 = \sigma_5 + i\sigma_5, \quad \lambda_6 = \sigma_5 - i\sigma_5,$$

with $\sigma_i < 0$, $i = 4, 5$.

If $l^2 > 8$, then we consider the equation

$$4\gamma_2^4 + 4 \left( l^2 + 1 \right)^3 - \gamma_2^2 \left( l^4 + 20l^2 - 8 \right) = 0,$$

written as

$$\gamma_2^2 = \frac{1}{8} \left( -8 + 20l^2 + l^4 \pm l\sqrt{(-8 + l^2)^3} \right) > 0.$$

We put

$$\hat{\gamma}_1 = \frac{1}{8} \left( -8 + 20l^2 + l^4 + l\sqrt{(-8 + l^2)^3} \right), \quad \hat{\gamma}_2 = \frac{1}{8} \left( -8 + 20l^2 + l^4 - l\sqrt{(-8 + l^2)^3} \right).$$

Then, for $\gamma_2^2 \in (0, \hat{\gamma}_1) \cup (\hat{\gamma}_2, \infty)$, then $D > 0$ and then we have the same situation as in (5.11).

If $\gamma_2^2 \in (\hat{\gamma}_1, \hat{\gamma}_2)$, then (5.10) has three real roots $\sigma_4 \neq \sigma_5 \neq \sigma_6$.

If $l^2 = 8$, then $D = 0$, and in this case (5.10) has three real roots $\sigma_4 \neq \sigma_5 = \sigma_6$.

Consequently, for $|\xi| \to 0$, we have

- for $l^2 < 8$ or ($l^2 > 8$ and $\gamma_2^2 \in (0, \hat{\gamma}_1) \cup (\hat{\gamma}_2, \infty)$)

$$\lambda_j(i\xi) = \begin{cases} 
-\frac{a^2l^2}{\gamma_2} \xi^2 + O(|\xi|^3), & \text{for } j = 1, \\
-\beta \xi^2 \pm i(\ell k - \hat{\delta} \xi^2) + O(|\xi|^3), & \text{for } j = 2, 3, \\
\sigma_4 + O(|\xi|), & \text{for } j = 4, \\
\sigma_5 + i\sigma_5 + O(|\xi|), & \text{for } j = 5, 6.
\end{cases}
$$

- For $l^2 > 8$ and $\gamma_2^2 \in (\hat{\gamma}_1, \hat{\gamma}_2)$. In this case, we have the following expansion for $|\xi| \to 0$,

$$\lambda_j(i\xi) = \begin{cases} 
-\sigma_0 \xi^2 + O(|\xi|^3), & \text{for } j = 1, \\
-\beta \xi^2 \pm i(\ell k - \hat{\delta} \xi^2) + O(|\xi|^3), & \text{for } j = 2, 3, \\
\sigma_i + O(|\xi|), & \text{for } j = 4, 5, 6.
\end{cases}
$$
• For \( l^2 = 8 \), we have
\[
\lambda_5(i\xi) = \lambda_6(i\xi) = \sigma_5
\]
and \( \lambda_j(i\xi), j = 1, 2, 3, 4 \) are the same as before.

• **Behavior of \( \lambda_j(\zeta) \) when \( |\zeta| \to \infty \).**

Set \( \lambda_j(\zeta) = \zeta \mu_j(\zeta^{-1}) \). Hence, for \( |\zeta| \to \infty \) then (4.45) takes the form
\[
\Psi(\zeta^{-1}) = \zeta^6 \det (\mu I + (A + \zeta^{-1}L))
\]
\[
= \mu^6 + \gamma_2 \zeta^{-1} \mu^5 + \left\{ (k^2 + 1)(l^2 \zeta^{-2} - 1) + \zeta^{-2} - a^2 \right\} \mu^4
\]
\[
+ \gamma_2 \left\{ (k^2 l^2 + 1) \zeta^{-2} - (1 + a^2) \right\} \zeta^{-1} \mu^3
\]
\[
+ \left[ (l^2 \zeta^{-2} - 1) \left\{ k^2 (l^2 \zeta^{-2} - 1) + (k^2 \zeta^{-2} - a^2 (k^2 + 1)) \right\} \right] \mu^2
\]
\[
+ \gamma_2 \left\{ k^2 l^2 \zeta^{-4} - a^2 k^2 l^2 \zeta^{-2} + a^2 \right\} \zeta^{-1} \mu - a^2 k^2 (l^2 \zeta^{-2} - 1)^2 = 0.
\]

We assume that \( \mu_j(\zeta^{-1}) \), have the asymptotic expansion (4.46), then we get by a direct computation, using (5.15):

For \( a = k = 1 \)
\[
\begin{cases}
\mu_j^{(0)} = \pm 1, & \mu_j^{(1)} = 0, & \mu_j^{(2)} = \frac{l^2}{2}, & \mu_j^{(3)} = \frac{l^2 \gamma_2}{4}, & \text{for } j = 1, 2 \\
\mu_j^{(0)} = \pm a, & \mu_j^{(1)} = -\frac{\gamma_2}{6} & \text{for } j = 3, 4, \\
\mu_j^{(0)} = \pm k & \mu_j^{(1)} = -\frac{\gamma_2}{2} & \text{for } j = 5, 6.
\end{cases}
\]

For \( a \neq 1 \), we have
\[
\begin{cases}
\mu_j^{(2)} = \pm \frac{l^2 (1 - a^2) + 1}{2(a^2 - 1)}, & \mu_j^{(3)} = \frac{l^2 \gamma_2}{2}, & \text{for } j = 1, 2 \\
\mu_j^{(1)} = 0, & \mu_j^{(2)} = \pm \frac{a}{2 (1 - a^2)}, & \mu_j^{(3)} = 0, & \text{for } j = 3, 4, \\
\mu_j^{(4)} = \pm \frac{a (a^2 (4 l^2 - 1) - 4 l^2 - 3)}{8(a^2 - 1)^3}, & \mu_j^{(5)} = -\frac{l^2 \gamma_2}{2 (a - 1)^2 (a + 1)^2} & \text{for } j = 3, 4, \\
\mu_j^{(0)} = \pm k & \mu_j^{(1)} = -\frac{\gamma_2}{2} & \text{for } j = 5, 6.
\end{cases}
\]

Consequently, we deduce (5.7) and (5.8). \( \square \)

5.2. **The estimates in the low frequency region \( \Upsilon_L \).** For \( \xi \in \Upsilon_L \), we have the following estimates.

**Proposition 5.4.** There exists two positive constants \( \hat{c}_1 \) and \( \hat{c}_2 \) such that the solution \( \hat{U}(\xi, t) \) of (2.5) satisfies for \( t \) large enough in \( \Upsilon_L \) the following estimate:
\[
|\hat{U}(\xi, t)| \leq \hat{c}_1 e^{-\hat{c}_2 |\xi|^2 t} |\hat{U}(\xi, 0)|, \quad \forall t \geq 0.
\]

**Proof.** We discuss the following cases

**Case 1:** \( l^2 < 8 \) or \( l^2 > 8 \) and \( \gamma_2 \in (0, \gamma_1) \cup (\gamma_2, \infty) \) where \( \gamma_1 \) and \( \gamma_2 \) are defined in (5.12).
In this case, we have (by neglecting the small terms) that in $\Upsilon_L$, the eigenvalues are:

$$\lambda_j(i\xi) = \begin{cases} 
-\sigma_0 \xi^2 + O(|\xi|^3), & \text{for } j = 1, \\
-\beta \xi^2 \pm i(\text{Re} \delta \xi^2) + O(|\xi|^3), & \text{for } j = 2, 3, \\
\sigma_4 + O(|\xi|), & \text{for } j = 4, \\
\sigma_5 \pm i\delta_5 + O(|\xi|), & \text{for } j = 5, 6.
\end{cases}$$

(5.17)

where $\sigma_0 = \frac{a^2}{2} l^2 \gamma^2$. Using the above eigenvalues, we may find the functions $r_j(t), 1 \leq j \leq 6$ as solutions of (5.4). Indeed, we have

$$r_j(t) = \begin{cases} 
eq 1 \frac{e^{\lambda_1 t}}{\lambda_1}, \\
\int_0^t e^{\lambda_j (t-s)} r_{j-1}(s) ds & \text{for } 2 \leq j \leq 6.
\end{cases}$$

Consequently, since all the eigenvalues are of multiplicity one we deduce for $\xi \neq 0$

$$r_j(t) = \sum_{i=1}^{j} \left( \prod_{k=1, (k \neq i)}^{j} e^{\lambda_i t} \frac{1}{\lambda_i - \lambda_k} \right), \quad 1 \leq j \leq 6.$$  

(5.18)

For instance, we have

$$r_1(t) = e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \quad r_3(t) = \frac{e^{\lambda_1 t}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{e^{\lambda_2 t}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{e^{\lambda_3 t}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}.$$  

Hence, using (5.3), we deduce that

$$\hat{U}(\xi, t) = e^{\Phi(i\xi)t} \hat{U}_0(\xi) = \sum_{j=0}^{5} r_{j+1}(t) P_j \hat{U}_0(\xi).$$

(5.19)

By doing some tedious computations, we may show that for $|\xi| \to 0$

$$\left| \prod_{k=1, (k \neq i)}^{j} \frac{1}{\lambda_i - \lambda_k} \right| \leq \Lambda,$$

(5.20)

where $\Lambda$ is a positive constant and $1 \leq j \leq 6$. For example, we have

$$\left| \frac{1}{\lambda_2 - \lambda_1} \right| \leq \kappa_1,$$

$$\left| \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \right| \leq \kappa_2,$$

$$\left| \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \right| \leq \kappa_3,$$

where $\kappa_i, 1 \leq i \leq 3$ are positive constants. From (5.18) and (5.20), we deduce

$$|r_j(t)| \leq c_1 e^{-c|\xi|^2 t}, \quad j \in \{1, 2, 3, 4, 5, 6\}.$$  

(5.21)
On the other hand, since $|\xi|$ is close to zero we have
\begin{equation}
|P_j| \leq C, \quad j \in \{1, 2, 3, 4, 5, 6\}.
\end{equation}
Consequently, we obtain from (5.19) that for $|\xi| \to 0$
\begin{equation}
|e^{\Phi(i\xi)t}| \leq Ce^{-|\xi|^2t},
\end{equation}
which leads to (5.16).

**Case 2:** $l^2 > 8$ and $\gamma_2^2 \in (\hat{\gamma}_1, \hat{\gamma}_2)$.
In this case, the eigenvalues of $\Phi(\xi)$ has the following expansion for $|\xi| \to 0$,
\begin{equation}
\lambda_j(i\xi) = \begin{cases} 
-\sigma_0 \xi^2 + O(|\xi|^3), & \text{for } j = 1, \\
-\hat{\beta} \xi^2 \pm i(lk - \hat{\delta} \xi^2) + O(|\xi|^3), & \text{for } j = 2, 3, \\
\sigma_i + O(|\xi|), \quad Re(\sigma_i) < 0, & \text{for } j = 4, 5, 6.
\end{cases}
\end{equation}
One can easily show, as previously, that
\begin{equation}
|e^{\Phi(i\xi)t}| \leq Ce^{-|\xi|^2t}.
\end{equation}

**Case 3:** $l^2 = 8$.
In this case, the main difference with the other cases is the presence of an eigenvalue with multiplicity 2, namely:
\[\lambda_5(i\xi) = \lambda_6(i\xi) = \sigma_5, \quad \text{for } \xi \in \Upsilon_L.\]
Consequently, $r_6(t)$ in the case $l^2 = 8$ becomes
\begin{equation}
r_6(t) = \sum_{i=1}^{4} \left( \prod_{k=1, (k \neq i)}^{4} \frac{e^{\lambda_k t}}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_5)^2} \right) - e^{\lambda_5 t} \sum_{i=1}^{4} \left( \prod_{k=1, (k \neq i)}^{4} \frac{1}{(\lambda_i - \lambda_k)(\lambda_i - \lambda_5)^2} \right) + te^{\lambda_5 t} \left( \prod_{k=1}^{4} \frac{1}{\lambda_i - \lambda_5} \right).
\end{equation}
We can prove as before that
\[|r_6(t)| \leq Ce^{-c|\xi|^2t} + Cte^{\sigma_5 t}.\]
Hence, for $t$ large enough there exist $C > 0$ and $c_0 > 0$ such that
\[te^{\sigma_5 t} \leq Ce^{-c_0 t}.
\]
Consequently, we have from (5.19) that for $|\xi| \to 0$ and $t$ large enough
\[|e^{\Phi(i\xi)t}| \leq Ce^{-\tilde{c}|\xi|^2t},
\]
for some $\tilde{c} > 0$. This leads to (5.16) and ends the proof of Proposition 5.4. \(\square\)
5.3. The estimates in the high frequency region $\Upsilon_H$. For the high frequency region, we have the following estimates.

**Proposition 5.5.** Assume that $(k^2 - 1)l^2 - 1 \neq 0$. Then, there exists two positive constants $\hat{c}_3$ and $\hat{c}_4$ such that the solution $\hat{U}(\xi, t)$ of (2.5) satisfies in $\Upsilon_H$ the estimates:

- If $a = k = 1$, then
  \begin{equation}
  |\hat{U}(\xi, t)| \leq \hat{c}_3|\xi|^2e^{-\hat{c}_4|\xi|^{-2}t}|\hat{U}(\xi, 0)|, \quad \forall t \geq 0.
  \end{equation}

- If $a \neq 1$, then we have
  \begin{equation}
  |\hat{U}(\xi, t)| \leq \hat{c}_3|\xi|^6e^{-\hat{c}_4|\xi|^{-2}t}|\hat{U}(\xi, 0)|, \quad \forall t \geq 0.
  \end{equation}

**Proof.** Now, for $|\xi| \to \infty$, we have the following expansion of the eigenvalues:

For $a = k = 1$, we get

\begin{equation}
\lambda_j(i\xi) = \begin{cases} 
\pm i\xi \pm \frac{i^2}{2}\xi^{-1} - \frac{l^2 \gamma_2}{2}\xi^{-2} + O(|\xi|^{-3}), & \text{for } j = 1, 2, \\
\pm i\xi - \frac{\gamma_2}{6} + O(|\xi|^{-1}), & \text{for } j = 3, 4, \\
\pm i\xi - \frac{\gamma_2}{2} + O(|\xi|^{-1}), & \text{for } j = 5, 6.
\end{cases}
\end{equation}

Using the same method as before, we have for $|\xi| \to \infty$

\begin{equation}
\hat{U}(\xi, t) = e^{\Phi(i\xi)t}\hat{U}_0(\xi) = \sum_{j=0}^{5} r_{j+1}(t)P_j\hat{U}_0(\xi),
\end{equation}

where $r_j(t)$ are given by (5.18) and $P_j$ are defined as before.

First of all, it is straightforward to see that for $|\xi| \to \infty$ and for all $1 \leq j \leq 6$, we have

\begin{equation}
|e^{\lambda_j(i\xi)t}| \leq Ce^{-\hat{c}_4|\xi|^{-2}t}.
\end{equation}

On the other hand, we have

\begin{equation}
\begin{cases}
\left|\frac{1}{\lambda_2 - \lambda_1}\right| \leq C|\xi|^{-1}, \\
\sum_{i=1}^{3} \left( \Pi_{k=1, (k \neq i)} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|^{-1} \\
\sum_{i=1}^{4} \left( \Pi_{k=1, (k \neq i)} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|^{-2} \\
\sum_{i=1}^{5} \left( \Pi_{k=1, (k \neq i)} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|^{-2} \\
\sum_{i=1}^{6} \left( \Pi_{k=1, (k \neq i)} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|^{-3}.
\end{cases}
\end{equation}

For the matrices $P_j$, $0 \leq j \leq 5$, we have as before

\[ |P_0| = |I| \leq C, \]

and since

\[ |\Phi(i\xi)| = |-(L + i\xi A)| \leq C(1 + |\xi|), \]

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then, we have
\[ |P_1| = |\Phi(i\xi) - \lambda_1 I| = |-(L + i\xi A) - \lambda_1 I| \leq C|\xi|, \]
for \(|\xi| \to \infty\). Also we have that
\[ |P_2| = |P_1(\Phi(i\xi) - \lambda_2 I)| \leq C|P_1|.(|\Phi(i\xi)| + |\lambda_2|) \leq C|\xi|^2. \]
Hence, we can deduce easily in the same fashion that for \(|\xi| \to \infty\)
\[ |P_j| \leq C|\xi|^j \quad \text{for all} \quad j \in \{0, 1, 2, 3, 4, 5\}. \]
Consequently, since all the eigenvalues are of multiplicity one when \(|\xi|\) is large and by using (5.28), (5.29), (5.30) and the above estimates, we obtain
\[ |\hat{U}(\xi, t)| \leq \sum_{j=0}^{5} |r_{j+1}(t)||P_j||U_0(\xi)| \leq C|\xi|^2 e^{-c|\xi|^{-2}}|U_0(\xi)|, \]
for \(|\xi| \to \infty\).

For \(a \neq 1\), we obtain
\[ \lambda_j(i\xi) = \begin{cases} \pm i\frac{l^2(1 - a^2) + 1}{2(a^2 - 1)}\xi^{-1} - \frac{l^2\gamma_2}{2}\xi^{-2} + O(|\xi|^{-3}), & \text{for} \quad j = 1, 2. \\ \pm i\frac{a}{2(1 - a^2)}\xi^{-1} \pm i\frac{a}{8(a^2 - 1)^3}\xi^{-3} - \frac{l^2\gamma_2}{2(a - 1)^2(a + 1)^2}\xi^{-4} + O(|\xi|^{-5}), & \text{for} \quad j = 3, 4, \\ \pm ik\xi - \frac{\gamma_2}{2} + O(|\xi|^{-1}), & \text{for} \quad j = 5, 6. \end{cases} \]
In this case, we have
\[ \begin{cases} \left| \frac{1}{\lambda_2 - \lambda_1} \right| \leq C|\xi|, \\ \sum_{i=1}^{3} \left( \prod_{k=1, (k \neq i)}^{3} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|^2, \\ \sum_{i=1}^{4} \left( \prod_{k=1, (k \neq i)}^{4} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|^3, \\ \sum_{i=1}^{5} \left( \prod_{k=1, (k \neq i)}^{5} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|^2, \\ \sum_{i=1}^{6} \left( \prod_{k=1, (k \neq i)}^{6} \frac{1}{(\lambda_i - \lambda_k)} \right) \leq C|\xi|. \end{cases} \]
Also, in this case, we have for the matrices \(P_j\), \(0 \leq j \leq 5\), as before
\[ |P_j| \leq |\xi|^j, \quad 1 \leq j \leq 5. \]
Consequently, since
\[ |e^{\lambda_j(i\xi)}| \leq C e^{-c\xi^{-4}t}, \quad 1 \leq j \leq 6, \]
then, we have by the same method as above
\[ |\hat{U}(\xi, t)| \leq \sum_{j=0}^{5} |r_{j+1}(t)||P_j||U_0(\xi)| \]
\[ \leq C |e^{0} e^{-|\xi|^{-4}t}||U_0(\xi)|, \]
for $|\xi| \to \infty$. This proves (5.26), and concludes the proof of Proposition 5.5. \(\square\)

5.4. The estimates in the middle frequency region \(\Upsilon_M\).

For the middle frequency region, we need to show first that for \(\xi \in \Upsilon_M\), the matrix \(\Phi(i\xi)\) has no pure imaginary eigenvalue. We have the following lemma.

**Lemma 5.6.** The matrix \(\Phi(i\xi) = -L - i\xi A\) has no pure imaginary eigenvalue in \(\Upsilon_M\).

**Proof.** We consider the characteristic equation (5.9). We argue by contradiction. Assume that there exists an eigenvalue \(\lambda_0(\xi) = i\alpha\) a solution of (5.9) with \(\alpha \in \mathbb{R}\). We plug \(\lambda_0\) into (5.9) and split the real and imaginary parts. Hence,

the real part:
\[ \alpha^6 - (\xi^2 (a^2 + k^2 + 1) + (k^2 + 1) \xi^2 + 1) \alpha^4 + (l^2 + \xi^2) \{ k^2 ((a^2 + 1) \xi^2 + l^2 + 1) + a^2 \xi^2 \} \alpha^2 - a^2 k^2 \xi^2 (l^2 + \xi^2)^2 = 0. \]
(5.33)

The imaginary part:
\[ \alpha [\alpha^4 - ((a^2 + 1) \xi^2 + k^2 l^2 + 1) \alpha^2 + (k^2 l^2 (a^2 \xi^2 + 1) + a^2 \xi^4)] = 0. \]
(5.34)

On one hand, if \(\alpha = 0\) in equation (5.34), then, we have from (5.33)
\[ a^2 k^2 \xi^2 (l^2 + \xi^2)^2 = 0, \]
which is a contradiction since \(\xi \in \Upsilon_M\). On the other hand, we get from (5.34) that
\[ S := \alpha^4 - ((a^2 + 1) \xi^2 + k^2 l^2 + 1) \alpha^2 + (k^2 l^2 (a^2 \xi^2 + 1) + a^2 \xi^4) = 0. \]

On the other hand, equation (5.33) can be rewritten as
\[ \alpha^2 [S - (\xi^2 k^2 l^2 + l^2)\alpha^2 + (l^2 + \xi^2) (k^2 \xi^2 + k^2 l^2) + \xi^2 (k^2 + l^2 a^2)] - a^2 k^2 \xi^2 (l^2 + \xi^2)^2 = 0 \]
Thus, we obtain
\[ (\xi^2 k^2 + l^2)\alpha^4 - \{(l^2 + \xi^2) (k^2 \xi^2 + k^2 l^2) + \xi^2 (k^2 + l^2 a^2)\} \alpha^2 + a^2 k^2 \xi^2 (l^2 + \xi^2)^2 = 0. \]
Solving the above equation, we obtain
\[ \alpha_{1,2}^2 = \frac{a^2 l^2 \xi^2 + k^2 (l^2 + \xi^2)^2 + \xi^2}{2(k^2 \xi^2 + l^2)} \pm \sqrt{\frac{a^2 k^2 \xi^2 (l^2 + \xi^2)^2}{k^2 \xi^2 + l^2} + \frac{a^2 l^2 \xi^2 + k^2 (l^2 + \xi^2)^2 + \xi^2}{4(k^2 \xi^2 + l^2)^2}} \]
This leads to \(\alpha_2^2 < 0\), which is a contradiction. \(\square\)

**Lemma 5.7.** There is a constant \(C > 0\), such that
\[ \text{Re}(\lambda_j(i\xi)) < -C < 0, \]
for all \(\xi \in \Upsilon_M\), where \(\lambda_j(i\xi), 1 \leq j \leq 6\) are the eigenvalues of the matrix \(\Phi(i\xi)\).
Proof. Since by [4, p.63] the eigenvalues are analytic in $\zeta = i\xi$, it follows by continuity and Lemma 5.6 that if the real part of the eigenvalues is negative on the boundary and it cannot be zero inside the domain (by Lemma 5.6) then the real part of the eigenvalues must be negative inside the domain also. Which concludes the proof of Lemma 5.7. \qed

Proposition 5.8. Assume that $\Phi(i\xi)$ has an eigenvalue of multiplicity $m$. Then, there exist two positive constant $\hat{c}_5$ and $C$ such that the solution $\hat{U}(\xi,t)$ of (2.5) satisfies in $\Upsilon_M$ the estimates:

\begin{equation}
|\hat{U}(\xi,t)| \leq \hat{c}_5(1 + |\xi|^{2m}t^m)e^{-Ct}|\hat{U}(\xi,0)|, \quad \forall t \geq 0.
\end{equation}

Proof. First, if the eigenvalues of $\Phi(i\xi)$ are simple then, it is not hard to see that for $\xi \in \Upsilon_M$,

\begin{equation}
\left|\sum_{i=1}^{j} \left( \prod_{k=1, k \neq i}^{j} \frac{1}{(\lambda_i - \lambda_k)} \right) \right| \leq C, \quad 1 \leq j \leq 6.
\end{equation}

Also,

\begin{equation}
|P_j| \leq C.
\end{equation}

Thus, using (5.35) together with (5.37) and (5.38), we deduce that (5.39) holds true.

Second, let us discuss, for example, the case of one double roots. Assume that there exists $\xi_0 \in \Upsilon_M$ such that $\lambda_5(i\xi_0) = \lambda_6(i\xi_0)$ (that is $m = 1$ in (5.39)). Hence, as we have seen in the proof of Proposition 5.4 (case 3), and by (5.35)

\begin{equation}
|r_j(t)| \leq C_1e^{-Ct}, \quad 1 \leq j \leq 5
\end{equation}

and

\begin{equation}
|r_6(t)| \leq C_1(1 + t)e^{-Ct}.
\end{equation}

On the other hand, we can show that

\begin{equation}
|P_j(i\xi_0)| \leq C, \quad 0 \leq j \leq 5.
\end{equation}

In particular

\begin{equation}
|P_5(i\xi_0)| \leq C \leq C|\xi_0|^2.
\end{equation}

Therefore, by collecting the above estimates, we get

\begin{equation}
|\hat{U}(\xi_0,t)| \leq (C_3 + |\xi_0|^2t)e^{-Ct}|\hat{U}(\xi_0,0)|, \quad \forall t \geq 0.
\end{equation}

Now, let us consider, for $\xi$ varying in small neighborhood of $\xi_0$, the problem

\begin{equation}
\hat{U}_t - \Phi(i\xi_0)\hat{U} = (\Phi(i\xi) - \Phi(i\xi_0))\hat{U}.
\end{equation}

Let $M(t,s,\xi_0)$ be the fundamental matrix of the operator $\partial_t - \Phi(i\xi_0)$. Then, it does satisfy the system

\begin{align*}
\partial_t M(t,s,\xi_0) - \Phi(i\xi_0)M(t,s,\xi_0) &= 0, \\
M(s,s,\xi_0) &= I.
\end{align*}

Due to (5.39), then the estimate

\begin{equation}
|M(t,s,\xi_0)| \leq (C_3 + |\xi_0|^2t)e^{-C(t-s)}
\end{equation}

holds for all $0 \leq s \leq t$. If $\xi$ is near $\xi_0$, then it holds that

\begin{equation}
|\Phi(i\xi) - \Phi(i\xi_0)| \leq \varepsilon.
\end{equation}
for some $\varepsilon > 0$. As in [8, Proposition 3.3], by Duhamel’s principle and Gronwall’s inequality, we can obtain from (5.40) the estimate

$$|\hat{U}(\xi, t)| \leq (C_3 + |\xi|^2 t) e^{-(C_3 \varepsilon)t}.$$ 

Hence, for $\varepsilon$ small enough, we deduce (5.39). \hfill \Box

5.5. **Proof of Theorem 5.1.** In this section, we show the decay estimate of the solution. To achieve this, we use the pointwise estimates obtained above.

Applying the Plancherel theorem and using the first estimate in (4.8), we obtain

$$\|\partial_x^2 U(t)\|_2^2 = \int_{\mathbb{R}} |\xi|^{2j} |\hat{U}(\xi, t)|^2 d\xi$$

Using Proposition 5.4, we have in the low frequency region

$$\int_{\mathcal{T}_L} |\xi|^{2j} |\hat{U}(\xi, t)|^2 d\xi \leq C \int_{\mathcal{T}_L} |\xi|^{2j} e^{-c|\xi|^2t} |\hat{U}(\xi, 0)|^2 d\xi$$

$$\leq C \|\hat{U}_0\|_{L^\infty}^2 \int_{|\xi| \leq 1} |\xi|^{2j} e^{-c|\xi|^2t} d\xi$$

(5.42)

In the high frequency region we distinguish two cases:

First, we assume that $a = k = 1$, then, we get, by using (5.25)

$$\int_{\mathcal{T}_H} |\xi|^{2j} |\hat{U}(\xi, t)|^2 d\xi \leq C \int_{\mathcal{T}_H} |\xi|^{2(j+2)} e^{-c|\xi|^2t} |\hat{U}(\xi, 0)|^2 d\xi$$

$$\leq C \sup_{|\xi| \geq 1} \left( |\xi|^{-2t} e^{-c|\xi|^2t} \right) \int_{|\xi| \geq 1} |\xi|^{2(j+\ell+2)} |\hat{U}(\xi, 0)|^2 d\xi$$

(5.43)

Second, for $a \neq 1$, we have, buy using (5.26),

$$\int_{\mathcal{T}_H} |\xi|^{2j} |\hat{U}(\xi, t)|^2 d\xi \leq C \int_{\mathcal{T}_H} |\xi|^{2(j+6)} e^{-c|\xi|^2t} |\hat{U}(\xi, 0)|^2 d\xi$$

$$\leq C \sup_{|\xi| \geq 1} \left( |\xi|^{-2t} e^{-c|\xi|^2t} \right) \int_{|\xi| \geq 1} |\xi|^{2(j+\ell+6)} |\hat{U}(\xi, 0)|^2 d\xi$$

(5.44)

For the middle frequency region, we have by using Proposition 5.8,

$$\int_{\mathcal{T}_M} |\xi|^{2j} |\hat{U}(\xi, t)|^2 d\xi \leq C \int_{\mathcal{T}_M} |\xi|^{2j} (1 + |\xi|^{2m_\ell}) e^{-Ct|\hat{U}(\xi, 0)|^2 d\xi}$$

(5.45)

Collecting (5.42), (5.43), (5.44) and (5.45), then the estimates in Theorem 5.1 are fulfilled.
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