Finite-time High-probability Bounds for Polyak-Ruppert Averaged Iterates of Linear Stochastic Approximation

A. Durmus ∗, E. Moulines †, A. Naumov ‡, S. Samsonov §

March 30, 2023

Abstract

This paper provides a finite-time analysis of linear stochastic approximation (LSA) algorithms with fixed step size, a core method in statistics and machine learning. LSA is used to compute approximate solutions of a $d$-dimensional linear system $\bar{A}\theta = \bar{b}$ for which $(\bar{A}, \bar{b})$ can only be estimated by (asymptotically) unbiased observations $\{(A(Z_n), b(Z_n))\}_{n \in \mathbb{N}}$. We consider here the case where $\{Z_n\}_{n \in \mathbb{N}}$ is an i.i.d. sequence or a uniformly geometrically ergodic Markov chain. We derive $p$-th moment and high-probability deviation bounds for the iterates defined by LSA and its Polyak-Ruppert-averaged version. Our finite-time instance-dependent bounds for the averaged LSA iterates are sharp in the sense that the leading term we obtain coincides with the local asymptotic minimax limit. Moreover, the remainder terms of our bounds admit a tight dependence on the mixing time $t_{\text{mix}}$ of the underlying chain and the norm of the noise variables. We emphasize that our result requires the SA step size to scale only with logarithm of the problem dimension $d$.

1 Introduction

This paper is concerned with the linear stochastic approximation (LSA) algorithm for solving the linear system $\bar{A}\theta = \bar{b}$ with unique solution $\theta^\star$, based on a sequence of observations $\{(A(Z_n), b(Z_n))\}_{n \in \mathbb{N}}$. Here $A : Z \to \mathbb{R}^{d \times d}$, $b : Z \to \mathbb{R}^d$ are measurable functions, and $(Z_k)_{k \in \mathbb{N}}$ is

1. either an i.i.d. sequence taking values in a state space $(Z, Z)$ with distribution $\pi$ satisfying $\mathbb{E}[A(Z_1)] = \bar{A}$ and $\mathbb{E}[b(Z_1)] = \bar{b}$;

2. or a $Z$-valued ergodic Markov chain with unique invariant distribution $\pi$, such that $\lim_{n \to +\infty} \mathbb{E}[A(Z_n)] = \bar{A}$ and $\lim_{n \to +\infty} \mathbb{E}[b(Z_n)] = \bar{b}$.

For a fixed step size $\alpha > 0$, burn-in period $n_0 \in \mathbb{N}$, and initialization $\theta_0$, we consider the sequences of estimates $\{\theta_n\}_{n \in \mathbb{N}}$, $\{\bar{\theta}_n\}_{n \geq n_0 + 1}$ given by

$$
\theta_k = \theta_{k-1} - \alpha \{A(Z_k)\theta_{k-1} - b(Z_k)\}, \quad k \geq 1,
$$

$$
\bar{\theta}_n = (n-n_0)^{-1} \sum_{k=n_0}^{n-1} \theta_k, \quad n \geq n_0 + 1.
$$

∗Ecole Polytechnique, Paris, alain.durmus@polytechnique.edu.
†Ecole Polytechnique, Paris, eric.moulines@polytechnique.edu.
‡HSE University, Russia, anaumov@hse.ru.
§HSE University, Russia, svsamsonov@hse.ru.
With a slight abuse of notation we drop the dependence upon \( n_0 \) in \( \bar{\theta}_n \). The sequence \( \{\theta_k\}_{k \in \mathbb{N}} \) are the standard LSA iterates, while \( \{\bar{\theta}_n\}_{n \geq n_0+1} \) corresponds to the Polyak-Ruppert (PR) averaged iterates; see Ruppert (1988); Polyak and Juditsky (1992).

The LSA algorithm is central in statistics, machine learning, and linear systems identification, see e.g. the works Eweda and Macchi (1983); Widrow and Stearns (1985); Benveniste et al. (2012); Kushner and Yin (2003) and references therein. More recently, it has sparked a renewed interest in machine learning, especially for high-dimensional least squares and reinforcement learning (RL) problems; Bertsekas and Tsitsiklis (2003); Bottou et al. (2018); Sutton (1988); Bertsekas (2019); Watkins and Dayan (1992). The LSA and LSA-PR recursions (1) have been the subject of a wealth of work, and it is difficult to adequately acknowledge all contributions. Polyak and Juditsky (1992); Kushner and Yin (2003); Borkar (2008); Benveniste et al. (2012) provided asymptotic convergence guarantees (almost sure convergence, central limit theorem) under both i.i.d. and Markovian noise settings. In particular, it has been established that LSA-PR can accelerate LSA and satisfies a central limit theorem with an asymptotically minimax-optimal covariance matrix.

Although asymptotic convergence analysis is of theoretical interest, the current trend is to obtain nonasymptotic guarantees that take into account both the limited sample size and the dimension of the parameter space. For these reasons, non-asymptotic analysis of both i.i.d. and Markovian SA procedures has recently attracted much attention.

In the i.i.d. setting, Rakhlin et al. (2012); Nemirovski et al. (2009); Jain et al. (2018a, 2019) investigated the finite-time mean squared error. Moreover, Durmus et al. (2021b) provided tight high-probability bounds for the LSA sequence \( \{\theta_n\}_{n \in \mathbb{N}} \). For least squares regression problems with a symmetric matrix \( A(Z_n) \), Bach and Moulines (2013); Jain et al. (2018a) showed that for a constant step size, the mean squared error (MSE) of \( \theta_n - \theta^* \) converges as \( O(1/n) \). For general LSA, which includes instrumental variable methods for linear system identification and temporal differences in reinforcement learning (TD), Lakshminarayanan and Szepesvari (2018) showed a convergence rate of mean square error \( O(1/n) \). The LSA-PR procedure can also be viewed as a two-timescale SA algorithm, with Hoeffding-type non-asymptotic deviation bounds provided in Dalal et al. (2020). Mou et al. (2020) provides a nonasymptotic high probability bound for LSA-PR with independent observations. However, the proof of their main result Mou et al. (2020, Theorem 3) relies on tools from Markov chain theory that assume strong conditions for \( \{(A(Z_n), b(Z_n))_{n \in \mathbb{N}} \) and it is not clear how to adapt their method to the general case.

For the Markovian setting, the literature is scarcer. Assuming an upper bound on the mixing time of the Markov chain, a projected variant of the LSA was analyzed by Bhandari et al. (2021), yielding nonasymptotic rates of mean squared error (MSE) that are sharp in their dependence on sample size \( n \) but not on dimension \( d \). This result was later extended in Srikant and Ying (2019) with the analysis of LSA without the projection step. Srikant and Ying (2019) obtained the same convergence rate as Bhandari et al. (2021). In Chen et al. (2020), the authors obtained a sharp MSE bound for the last iteration of LSA assuming a \( V \)-uniformly ergodic Markov chain and decreasing step sizes \( \alpha_k = 1/k \). Recently, Mou et al. (2021) established \( p \)-moment bounds for the last iterates of LSA and showed that the mean-square error obtained with PR-LSA matches the local asymptotic minimax optimal limit.

**Contributions and organization of the paper.** Our main contribution is a unified framework for the finite-time analysis of LSA with i.i.d. and Markov noise dynamics, based on the stochastic expansion for LSA (1) introduced in Aguech et al. (2000). In this framework, we derive the finite time bounds for the \( p \)-th moment of \( \{\|\theta_n - \theta^*\|\}_{n \in \mathbb{N}} \) and \( \{\|\bar{\theta}_n - \theta^*\|\}_{n \in \mathbb{N}} \). The obtained bounds for PR-averaged LSA iterates are sharp in a sense that the leading term of these bounds coincides
with the one of the central limit theorem. Moreover, as a corollary, for a fixed tolerance parameter \( \delta \in (0,1) \) and the number of iterations \( n \), we provide high-probability bounds on the error of LSA-PR iterates. In the i.i.d. setup, our results extend and improve those obtained in Mou et al. (2020) by providing a better dependence with respect to on the problem dimension \( d \) and on the moment order \( p \) for the remainder terms. The improvement w.r.t. moment order dependence comes through a logarithmic dependence of the maximal allowed step size upon the dimension \( d \). In the case of Markovian noise, to the best of our knowledge, the results concerning the \( p \)-th moment of the LSA-PR error are novel. Moreover, in the Markovian setup the remainder terms of our bounds scale with the ratio \( n/t_{\text{mix}} \), which can not be improved in general even for the case of MSE bounds (see e.g. (Mou et al., 2021, Theorem 2)).

The paper is organized as follows. In Section 2 we introduce the decomposition of the error, which is key to our proof (see Aguech et al. (2000)), and formulate our main assumptions. In Section 3 we present our results for the independent case. In Section 4 we extend our results when \( \{Z_n\}_{n \in \mathbb{N}^*} \) is a uniformly geometrically ergodic Markov chain. The proofs are postponed to the appendix. For reader’s convenience the notations and key constants appearing in the text are summarized in Appendix A.

2 Stochastic expansions for LSA and LSA-PR

As an introduction, we present tools and some preliminary results relevant to our analysis of LSA and LSA-PR under both i.i.d. and Markovian noise dynamics. Using the definition (1) and some elementary algebra, we obtain

\[
\theta_n - \theta^* = (I - \alpha A(Z_n))(\theta_{n-1} - \theta^*) - \alpha \varepsilon(Z_n),
\]

where we have set

\[
\varepsilon(z) = \tilde{A}(z)\theta^* - \tilde{b}(z), \quad \tilde{A}(z) = A(z) - \bar{A}, \quad \tilde{b}(z) = b(z) - \bar{b}.
\]

We expand the recurrence above using the notation \( \Gamma^{(\alpha)}_{1:n} \) for the product of random matrices

\[
\Gamma^{(\alpha)}_{m:n} = \prod_{i=m}^{n} (I - \alpha A(Z_i)), \quad m, n \in \mathbb{N}^*, \quad m \leq n,
\]

with the convention, \( \Gamma^{(\alpha)}_{m:n} = I \) for \( m > n \). We arrive at the decomposition of the LSA error:

\[
\theta_n - \theta^* = \tilde{\theta}_n^{(\text{tr})} + \tilde{\theta}_n^{(\text{fl})},
\]

where we have defined

\[
\tilde{\theta}_n^{(\text{tr})} = \Gamma^{(\alpha)}_{1:n}\{\theta_0 - \theta^*\}, \quad \tilde{\theta}_n^{(\text{fl})} = -\alpha \sum_{j=1}^{n} \Gamma^{(\alpha)}_{j:n} \varepsilon(Z_j).
\]

Here \( \tilde{\theta}_n^{(\text{tr})} \) is a transient term (reflecting the forgetting of the initial condition) and \( \tilde{\theta}_n^{(\text{fl})} \) is a fluctuation term (reflecting misadjustment noise). In both i.i.d. and Markov noise dynamics, we proceed by treating the \( \tilde{\theta}_n^{(\text{tr})} \) and \( \tilde{\theta}_n^{(\text{fl})} \) terms separately.

Bounding the transient term. We first bound the \( p \)-th moments of \( \{\|\tilde{\theta}_n^{(\text{tr})}\|\}_{n \in \mathbb{N}^*} \) by proving that the sequence of random matrices \( \{A(Z_i)\}_{i \in \mathbb{N}^*} \) is exponentially stable (see Guo and Ljung
(1995); Ljung (2002)). Formally, this means that for some \( q \geq 1 \), there exist constants \( a_q, C_q > 0 \) and \( \alpha_{\infty,q} < \infty \), such that for any step size \( \alpha \leq \alpha_{\infty,q} \), \( m, n \in \mathbb{N}^* \), \( m \leq n \),

\[
\mathbb{E}[\| \Gamma_{m:n}^{(\alpha)} \|^q] \leq C_q \exp (-a_q \alpha (n - m)) .
\]  

(5)

Exponential stability is established in Proposition 2 for the i.i.d. setting and in Proposition 7 for the Markovian setting. Intuitively, exponential stability means that the \( q \)-th moment of the product of random matrices \( \Gamma_{m:n}^{(\alpha)} \) behaves similarly to the product of deterministic matrices \( (I - \alpha \bar{A})^{n-m} \) under the assumption that the matrix \( -\bar{A} \) is Hurwitz, i.e., for each eigenvalue \( \lambda \) of \( \bar{A} \) we have \( \text{Re}(\lambda) > 0 \). To handle the product \( (I - \alpha \bar{A})^{n-m} \), we use the Lyapunov contraction property: if the matrix \( -\bar{A} \) is Hurwitz, there exists a symmetric positive definite matrix \( Q \) such that \( I - \alpha \bar{A} \) is a strict contraction in \( \| \cdot \|_Q \) (see Appendix A for the relevant definitions). Precisely, the following result holds:

**Proposition 1** ((Durmus et al., 2021a, Proposition 1)). **Assume that \(-\bar{A}\) is Hurwitz. Then there exists a unique symmetric positive definite matrix \( Q \) satisfying the Lyapunov equation \( \bar{A}^\top Q + Q \bar{A} = I \). In addition, setting

\[
a = \| Q \|^{-1}/2, \quad \text{and} \quad \alpha_{\infty} = (1/2) \| \bar{A} \|^{-1} \| Q \|^{-1} \wedge \| Q \|,
\]

(6)

it holds for any \( \alpha \in [0, \alpha_{\infty}] \) that \( \| I - \alpha \bar{A} \|_Q^2 \leq 1 - a \alpha \), and \( a \alpha \leq 1/2 \).

**Bounding the fluctuation term.** For the fluctuation term \( \tilde{\theta}_n^{(0)} \) we use the perturbation expansion technique formalized in Aguech et al. (2000). We do not exploit the full power of this decomposition here and use only first- and second-order expansions. Higher order expansions are presented in Aguech et al. (2000). We emphasize that the considered error expansion remains the same in both the i.i.d. and Markovian noise dynamics, so that \( p \)-th moment bounds on the error norm can be obtained in both cases. In the following, we briefly sketch how the decomposition of \( \tilde{\theta}_n^{(0)} \) is constructed. By definition (4) of \( \tilde{\theta}_n^{(0)} \), it satisfies the recurrence

\[
\tilde{\theta}_n^{(0)} = (I - \alpha A(Z_n)) \tilde{\theta}_{n-1}^{(0)} - \alpha \varepsilon(Z_n) .
\]

(7)

Using the definition (2) of \( \bar{A}(\cdot) \) and an induction argument, it is easy to verify that the following decomposition holds for any \( n \in \mathbb{N} \):

\[
\tilde{\theta}_n^{(0)} = J_n^{(0)} + H_n^{(0)},
\]

(8)

where the latter terms are defined by the following pair of recursions

\[
J_n^{(0)} = (I - \bar{A}) J_{n-1}^{(0)} - \alpha \varepsilon(Z_n) ,
J_0^{(0)} = 0 ,
\]

(9)

\[
H_n^{(0)} = (I - \bar{A}(Z_n)) H_{n-1}^{(0)} - \alpha \bar{A}(Z_n) J_{n-1}^{(0)} ,
H_0^{(0)} = 0 .
\]

(10)

The recursion for \( J_n^{(0)} \) is obtained by replacing \( A(Z_n) \) by its mean \( \bar{A} \) in (7) (cf. (9) and (7)), and \( H_n^{(0)} \) is remainder term. As a result \( J_n^{(0)} \) becomes a linear statistic in \( \{ \varepsilon(Z_n) \}_{n \in \mathbb{N}} \), and is relatively easy to analyze with common concentration tools for i.i.d. variables or Markov chains, see e.g. (Vershynin, 2018, Section 2) and (Douc et al., 2018, Chapter 23). Interestingly, it can be shown that the term \( J_n^{(0)} \) is the leading term with respect to \( \alpha \) in the expansion (8). Moreover, the covariance matrix of \( J_n^{(0)} \) is closely related to the asymptotic covariance matrix appearing in
the central limit theorems for LSA with decreasing step size, see Durmus et al. (2021a) for further discussion on this topic. We can further expand the decomposition (8) since the recursion for the remainder term (10) resembles the one in (7). Hence, we can apply the same decomposition for $H_n^{(0)}$, and obtain

$$H_n^{(0)} = J_n^{(1)} + H_n^{(1)},$$

(11)

where we have set

$$J_n^{(1)} = (1 - \alpha \hat{A}) J_{n-1}^{(1)} - \alpha \hat{A}(Z_n) J_n^{(0)}, \quad J_0^{(1)} = 0,$$

$$H_n^{(1)} = (1 - \alpha A(Z_n)) H_{n-1}^{(1)} - \alpha \hat{A}(Z_n) J_n^{(1)}, \quad H_0^{(1)} = 0.$$  

(12)

Representation (12) can be elaborated further to decompose $H_n^{(1)}$, but this is not needed in this work. Combining (8) and (11), we obtain the decomposition which is the cornerstone of our analysis:

$$\theta_n - \theta^* = \tilde{\theta}^{(tr)}_n + J_n^{(0)} + J_n^{(1)} + H_n^{(1)},$$

(13)

where $J_n^{(0)}$, $J_n^{(1)}$, and $H_n^{(1)}$ are defined in (9) and (12), respectively. Following the arguments in Durmus et al. (2021a), this decomposition can be used to obtain sharp bounds on the $p$-th moment of the final LSA iterate $\theta_n$.

**Stochastic expansion for LSA-PR.** Our analysis of PR-LSA is based on another useful error representation. Using (1) and the definition of the noise term $\varepsilon(\cdot)$ in (3), we get that

$$\tilde{A}(\tilde{\theta}_n - \theta^*) = \{\alpha(n - n_0)\}^{-1} (\theta_{n_0} - \theta_n) - (n - n_0)^{-1} \sum_{t=n_0}^{n-1} e(\theta_t, Z_{t+1}),$$

(14)

$$e(\theta, z) = \tilde{A}(z) \theta - \tilde{b}(z) = \varepsilon(z) + \tilde{A}(z)(\theta - \theta^*).$$

(15)

We will establish in the sequel when bounding the error of the last LSA iterate, the term $\{\alpha(n - n_0)\}^{-1} (\theta_{n_0} - \theta_n)$ is small compared to the second term in (14) with a suitably chosen $n_0$. Moreover, in the i.i.d. case it can be shown that $\{e(\theta_t, Z_{t+1})\}_{t=0}^n$ are martingale increments, and the MSE bound of LSA-PR will follow directly from this observation (see Proposition 5). This property of $\{e(\theta_t, Z_{t+1})\}_{t=0}^n$ was also used in Mou et al. (2020).

To proceed with the $p$-th moment bounds of the LSA-PR method, we need to combine the extensions (14) and (13). That is, we write

$$\sum_{t=n_0}^{n-1} e(\theta_t, Z_{t+1}) = E_n^{tr} + E_n^{fl},$$

(16)

where we have set

$$E_n^{tr} = \sum_{t=n_0}^{n-1} \tilde{A}(Z_{t+1}) (\Gamma_{t+1}^{(0)}) \{\theta_0 - \theta^*\},$$

$$E_n^{fl} = \sum_{t=n_0}^{n-1} \varepsilon(Z_{t+1}) + \sum_{t=0}^{n-1} \sum_{t=n_0}^{n-1} \tilde{A}(Z_{t+1}) J_t^{(0)} + \sum_{t=n_0}^{n-1} \tilde{A}(Z_{t+1}) H_t^{(1)}.$$  

(17)

Based on the decompositions (14) and (16), our analysis of the PR recursion consists in bounding the terms $\{\alpha(n - n_0)\}^{-1} (\theta_{n_0} - \theta_n)$, $E_n^{tr}$ and $E_n^{fl}$ separately. For the first one, we use the bounds derived on the $p$-th moment of non-averaged LSA iterates $\theta_k - \theta^*$. For the second one, we use the exponential stability for the products of random matrices $\{\Gamma_{t+1}^{(0)}\}$ (see (5)). Finally, the fluctuation term $E_n^{fl}$ is dealt with the conditions we impose on the sequence $\{Z_n\}_{n\in\mathbb{N}^*}$.

We suppose from now on that the sample size $n$ is even, and fix the size of burn-in period $n_0 = n/2$. Thus we suppress the dependence upon $n_0$ in (1) and use simplified notation

$$\tilde{\theta}_n = (2/n) \sum_{k=n/2}^{n-1} \theta_k.$$  

5
Assumptions. Throughout this paper (both in case of i.i.d. and Markovian noise dynamics) we impose the following assumption regarding $z \mapsto \bar{A}(z)$ and $\bar{A}$:

**A 1.** $C_A = \sup_{z \in \mathbb{Z}} \|A(z)\| \vee \sup_{z \in \mathbb{Z}} \|\bar{A}(z)\| < \infty$ and the matrix $-\bar{A}$ is Hurwitz.

In particular, the condition that $-\bar{A}$ is Hurwitz implies that the linear system $\bar{A} \theta = \bar{b}$ has a unique solution $\theta^*$.

We further require the following assumptions on the noise term $\varepsilon(z)$ and the stationary distribution $\pi$ of the sequence $\{Z_n\}_{n \in \mathbb{N}^+}$:

**A 2.** $\int_Z A(z)d\pi(z) = \bar{A}$ and $\int_Z b(z)d\pi(z) = \bar{b}$. Moreover, $\|\varepsilon\|_{\infty} = \sup_{z \in \mathbb{Z}} \|\varepsilon(z)\| < +\infty$.

Our bounds in the i.i.d. noise case will also depend upon the covariance matrix of $\varepsilon(z)$, that is, $\Sigma_\varepsilon = \int_Z \varepsilon(z)\varepsilon(z)^\top d\pi(z)$.

Assumption A 2 can be generalized in certain directions. In particular, in Appendix C we provide the counterparts of the results of Section 3 under the assumption that the sequence $\{\varepsilon(Z_t)\}_{t \in \mathbb{N}^+}$ are i.i.d. sub-Gaussian random variables. The case of unbounded noise in the Markovian setting is much more technical and is left as a direction for future work.

3 Finite-time Moment and High-probability Bounds in the Independent Noise Setting

In addition to A 1 and A 2, we consider in this section the following assumption:

**IND 1.** $\{Z_k\}_{k \in \mathbb{N}}$ is a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution $\pi$.

The first result of this section provides $p$-th moment bounds for LSA-PR under IND 1. Using the notations of Proposition 1 and assuming A 1, we define for $q \geq 2$

\[
\kappa_Q = \lambda_{\max}(Q)/\lambda_{\min}(Q) , \quad b_Q = 2\sqrt{\kappa_Q} C_A , \quad \alpha_{q, \infty} = \alpha_{\infty} \wedge c_A / q , \quad c_A = a / \{2 b_Q^2\} .
\]

The quantity $\alpha_{q, \infty}$ defined above is a threshold on the step size that guarantees exponential stability for the $q$-th moment of the product of $\Gamma_{1,n}^{(\alpha)}$. Below we give our exponential stability result for the $p$-th moment of the product of $\Gamma_{1,n}^{(\alpha)}$.

**Proposition 2** (Durmus et al. (2021a, Corollary 1)). Assume A 1 and IND 1. Then, for any $2 \leq p \leq q$, $\alpha \in (0, \alpha_{\infty}]$ and $n \in \mathbb{N}^*$, it holds

\[
\mathbb{E}^{1/p} \left[ \|\Gamma_{1,n}^{(\alpha)}\|^p \right] \leq \sqrt{\kappa_Q} d^{1/q} (1 - a\alpha + (q - 1)b_Q^2\alpha^2)^{n/2} .
\]

Moreover, for $\alpha \in (0, \alpha_{q, \infty}]$, it holds

\[
\mathbb{E}^{1/p} \left[ \|\Gamma_{1,n}^{(\alpha)}\|^p \right] \leq \sqrt{\kappa_Q} d^{1/q} (1 - a\alpha/2)^{n/2} .
\]

Proposition 2 implies that $\sup_{n \in \mathbb{N}} \mathbb{E}[(\|\Gamma_{1,n}^{(\alpha)}\|^p)_{1,n}] < +\infty$ for any $\alpha \in (0, \alpha_{q, \infty}]$ and $2 \leq p \leq q$. This condition connecting the choice of the step size $\alpha$ with $p$ and $q$ is unavoidable; see Durmus et al. (2021a, Example 1). Roughly speaking, the condition on the step size $\alpha \in (0, \alpha_{q, \infty}]$ in Proposition 2 requires scaling the step size $\alpha$ with $1/q$. (Srikant and Ying, 2019, Theorem 9) reports the same kind of dependence.
Remark 1. The flexibility achieved by using \( q \geq p \) allows us to control the dependence on dimension \( d \) independently of the choice of the \( p \)-th moment. In particular, we can choose \( q = c \log d \) with a suitable constant \( c > 0 \) and obtain \( \mathbb{E}^{1/p} [ [\Gamma^{(\alpha)}]^{1/p} ] \leq \sqrt{\kappa_{Q} e^{p}} \) for \( 2 \leq p \leq c \log d \). This comes at the expense of taking a maximum step size \( \alpha_{c \log d, \infty} \) which scales with \( 1/\log d \).

We can now state the main result of this section, which is a \( p \)-th moment bound for \( \mathbb{E}^{1/p} [ [\bar{A} (\bar{\theta}_{n} - \theta^{*})]^{p} ] \).

We write \( \lesssim_{d} \) for inequality up to a constant that depends on \( \kappa_{Q}, a, C_{A} \), and polylogarithmic factors in \( d \).

Theorem 1. Assume A 1, A 2, IND 1. Then, for any \( p \geq 2 \), even \( n \geq 2 \), step size

\[
\alpha(n, d, p) = (\alpha_{\infty} \wedge c_{A} / \{1 + \log d\}) (pn^{1/2})^{-1},
\]

and an initial parameter \( \theta_{0} \in \mathbb{R}^{d} \), it holds that

\[
\mathbb{E}^{1/p} [ [\bar{A} (\bar{\theta}_{n} - \theta^{*})]^{p} ] \lesssim_{d} \frac{\{\text{Tr} \sum_{\epsilon}^{1/2} p^{1/2}\}}{n^{1/2}} + \|\epsilon\|_{\infty} \left( \frac{p}{n^{3/4}} + \frac{p^{2}}{n} \right) + p\|\theta_{0} - \theta^{*}\| \exp \left\{ - \left( \alpha_{\infty} \wedge c_{A} \right) \sqrt{n} \right\}. \]

A generalization of Theorem 1 for an arbitrary step size \( \alpha \in (0, \alpha_{q, \infty}) \) is given below in Theorem 2, and its reformulation into high probability bounds is given in Corollary 1. The dependence of the step size \( \alpha(n, d, p) \) on the sample size \( n \) can be illustrated by optimizing the bounds provided by Theorem 2 (see (26)). For completeness, we provide versions of Theorem 2 and Corollary 1 with exact constants in Appendix B.6.

Compared to Mou et al. (2020, Theorem 3), Theorem 1 has a similar leading term, but with a different numerical factor. Instead of \( \text{Tr} \Sigma_{\epsilon} \), the factor in Mou et al. (2020, Theorem 3) is the covariance matrix associated with the \{\( \theta_{k} \}_{k \in \mathbb{N}} \) when viewed as a Markov chain: \( \Sigma_{\epsilon}^{(\alpha)} = n^{-1} \lim_{n \to +\infty} \mathbb{E}[\sum_{i=1}^{n} (\theta_{i} - \theta^{*})(\theta_{i} - \theta^{*})^{T}] \). Durmus et al. (2021a, Proposition 6) shows that \( \|\Sigma_{\epsilon}^{(\alpha)} - \Sigma_{\epsilon}\| = O(\alpha) \) with \( \alpha \to 0 \). It is worth noting that the conclusions of Mou et al. (2020, Theorem 3) regarding the choice of the optimal step size and the resulting high probability bounds differ slightly from ours since their optimization omits the dependence on the step size \( \alpha \) in the term \( \text{Tr}(\Sigma_{\epsilon}^{(\alpha)}) \).

Theorem 1 accounts for this additional factor in the optimization and leads to an optimal choice for \( \alpha \) of order \( n^{-1/2} \) and a residual term in \( n^{-3/4} \), while the optimal choice of \( \alpha \) from Mou et al. (2020, Theorem 3) has order \( n^{-1/3} \) and leads to a residual term in \( n^{-2/3} \). Moreover, Theorem 1 improves the scaling of the residual term with respect to \( p \), and, unlike Mou et al. (2020, Theorem 3), shows exponential forgetting of the initial condition. Finally, an inspection of the proof of Mou et al. (2020, Theorem 3) shows that it relies heavily on results on additive functionals of Markov chains developed in Joulin and Ollivier (2010), while the main result in our derivation is a Rosenthal inequality for martingales. Applying the results from Joulin and Ollivier (2010) requires log-Sobolev conditions for the noise distribution (\( \epsilon(Z_{n})_{n \in \mathbb{N}} \)), which are very restrictive. Mou et al. (2020, Theorem 3) does not apply to the general framework considered here.

Bounds for the non-averaged LSA iterates To bound \( \mathbb{E}^{1/p} [ [\bar{A} (\bar{\theta}_{n} - \theta^{*})]^{p} ] \) in Theorem 1, we first need to control the \( p \)-th moment of the last LSA iterate error \( \{\theta_{n} - \theta^{*} : n \in \mathbb{N}\} \). To this end, we use the decomposition (8) and rely on the following \( p \)-th moment bounds for the sequence \( \{J_{n}^{(0)} : n \in \mathbb{N}\} \).
Proposition 3. Assume A1, A2, and IND 1. Then, for any \( \alpha \in (0, \alpha_\infty) \), \( p \geq 2 \), and \( n \in \mathbb{N} \), it holds
\[
\mathbb{E}^{1/p} \left[ \| J_n^{(0)} \|^p \right] \leq D_1 \sqrt{\alpha ap} \| \varepsilon \|_\infty, \quad \text{where } D_1 = \sqrt{2\kappa_Q/a}.
\] (22)

The proof is deferred to Appendix B.1. The argument goes as follows. Expanding the recurrence (9), we represent
\[
J_n^{(0)} = -\alpha \sum_{k=1}^{n} (I - \alpha \hat{A})^{n-k} \varepsilon(Z_k).
\] (23)

Now the bound (22) follows from a Hoeffding-type bound for sums of independent random vectors (see (Pinelis, 1994, Theorem 3.1)) in combination with Proposition 1.

The constant \( D_1 \) is instance-dependent, since it depends on \( a, \kappa_Q \). Nevertheless, the product \( D_1 \| \varepsilon \|_\infty \) is scale-invariant, that is, if we multiply \( \hat{A} \) by a positive constant \( M \), both \( a \) and \( \| \varepsilon \|_\infty \) scales in the same way, leaving \( D_1 \| \varepsilon \|_\infty \) unchanged. This property of scale invariance holds for all constants in the bounds that appear in the following statements. We emphasize that \( J_n^{(0)} \) is the leading (with respect to the step size \( \alpha \)) term in the error decomposition (13). Indeed, (22) and the stability result (Proposition 2) are sufficient to obtain a rough bound \( \mathbb{E}^{1/p} [\| H_n^{(0)} \|^p] \leq C \sqrt{\alpha} \) for a constant \( C \geq 0 \). Combining these results gives the following \( p \)-th moment bound for the LSA error \( \| \theta_n - \theta^* \| \):

Proposition 4. Assume A1, A2, and IND 1. Then, for any \( p, q \in \mathbb{N} \), \( 2 \leq p \leq q \), \( \alpha \in (0, \alpha_{q,\infty}] \), \( n \in \mathbb{N} \), and \( \theta_0 \in \mathbb{R}^d \) it holds
\[
\mathbb{E}^{1/p} [\| \theta_n - \theta^* \|^p] \leq d^{1/q} \kappa_Q^{1/2} (1 - \alpha a/4)^n \| \theta_0 - \theta^* \| + d^{1/q} D_2 \sqrt{\alpha ap} \| \varepsilon \|_\infty,
\] (24)

where \( D_2 \) is given in (45).

The proof is given in Appendix B.2. It is based on the expansion (8), the stability result of Proposition 2, and bounds on \( J_n^{(0)} \) obtained in Proposition 3. We control the moments of \( H_n^{(0)} \) with Hölder’s inequality and a bound for \( J_n^{(0)} \). The bound thus obtained for \( H_n^{(0)} \) is not optimal: the dependence of \( H_n^{(0)} \) in \( \alpha \) is improved below. However, this preliminary bound is sufficient to obtain the \( p \)-th moment bound for \( \| \theta_n - \theta^* \| \), which is tight with respect to the dependence on the step size \( \alpha \). Note that at the expense of the logarithmic dependence of step size \( \alpha \) on the dimension, one can get rid of the dependence on dimension \( d \) in (24); see Remark 1.

MSE bound for LSA-PR We preface the proof of Theorem 1 by a separate bound on the mean square error of \( n \)-steps LSA-PR \( \hat{\theta}_n - \theta^* \). While this result could be a consequence of Theorem 1, we present here a separate and simpler derivation which leads to sharper bounds. Our strategy consists in using the decomposition (14) and the fact that \( \{ e(\theta_t, Z_{t+1}) \}_{t=0}^{n-1} \) is a martingale increment sequence. Thus, we get from (14) that
\[
(n/2) \mathbb{E} \left[ \| \hat{A} (\hat{\theta}_n - \theta^*) \|^2 \right] \leq 4n^{-1} \sum_{t=n/2}^{n-1} \mathbb{E} [\| e(\theta_t, Z_{t+1}) \|^2] + 4(\alpha^2 n)^{-1} \mathbb{E} [\| \theta_{n/2} - \theta_n \|^2].
\] (25)

Since by definition \( e(\theta_t, Z_{t+1}) = \varepsilon(Z_{t+1}) + \hat{A}(Z_{t+1})(\theta_t - \theta^*) \), the term \( T_1 \) contains the variance term \( \mathbb{E} [\| \varepsilon(Z_{t+1}) \|^2] = \text{Tr} \Sigma_e \), which is the leading term when the step size \( \alpha \) is small enough. This fact follows from Proposition 4, which implies that \( \mathbb{E} [\| \theta_t - \theta^* \|^2] \leq C \alpha \) up to the exponentially decreasing transient terms. Next result is obtained by deriving quantitative bounds for \( T_1 \) and \( T_2 \).
Proposition 5. Assume A1, A2, and IND1. Then, for any even \( n \geq 2 \), \( \alpha \in (0, \alpha_{\infty} \wedge c_A / \{2 + 2 \log d\}) \), \( \theta_0 \in \mathbb{R}^d \), it holds that

\[
(n/2) \mathbb{E} \left[ \| \tilde{A} (\tilde{\theta}_n - \theta^*) \|^2 \right] \leq 4 \text{Tr} \Sigma_{\varepsilon} + \Delta_{n, \alpha}^{(fl)} \| \varepsilon \|_{\infty} + e^{-\alpha n/4} \Delta_{n, \alpha}^{(tr)} \| \theta_0 - \theta^* \|^2,
\]

where \( \Delta_{n, \alpha}^{(fl)} \) and \( \Delta_{n, \alpha}^{(tr)} \) are given in (47).

The complete proof is postponed to Appendix B.3. In the previous statement, \( \Delta_{n, \alpha}^{(tr)} \) and \( \Delta_{n, \alpha}^{(fl)} \) correspond to the transient and fluctuation components of the LSA error. The initial condition’s exponential forgetting is represented by the transient term, and the fluctuations of the non-averaged LSA iterates \( \tilde{\theta}_n \) around \( \theta^* \) are captured by \( \Delta_{n, \alpha}^{(fl)} \). It is worth noting that

\[
\Delta_{n, \alpha}^{(tr)} \lesssim_d (\alpha n)^{-1} (1 + \alpha^{-1}), \quad \Delta_{n, \alpha}^{(fl)} \lesssim_d (\alpha n)^{-1} + \alpha .
\]

(26)

The above bounds can be simplified for a given choice of \( \alpha \) as a function of sample size \( n \). Optimizing the fluctuation error \( \Delta_{n, \alpha}^{(fl)} \) in (26) for a fixed sample size \( n \) suggests that \( \alpha \) should scale with \( n \) as \( n^{-1/2} \). Then, choosing

\[
\alpha(n, d) = (\alpha_{\infty} \wedge c_A / \{2 + 2 \log d\}) n^{-1/2},
\]

we obtain from Proposition 5 the MSE bound

\[
\mathbb{E} \left[ \| \tilde{A} (\tilde{\theta}_n - \theta^*) \|^2 \right] \lesssim_d \frac{\text{Tr} \Sigma_{\varepsilon}}{n} + \frac{\| \varepsilon \|_{\infty}^2}{n^{3/2}} + \| \theta_0 - \theta^* \|^2 \exp \left\{ -\frac{(\alpha_{\infty} \wedge c_A) \sqrt{n}}{8(1 + \log d)} \right\} .
\]

(27)

Note that the bound (27) has the same (optimal) leading term \( n^{-1} \text{Tr} \Sigma_{\varepsilon} \) as in Mou et al. (2021, Theorem 1), improving the dependence on sample size \( n \) in the remainder term. To compare with Mou et al. (2021, Theorem 1), we assume that \( \| \varepsilon \|_{\infty} \approx \sqrt{d} \). Then (27) yields a remainder term of order \( d/n^{3/2} \), while the second-order term in Mou et al. (2021, Theorem 1) scales as \( (d/n)^{4/3} \).

Outline of the proof of Theorem 1. To obtain Proposition 5, we only used the expansion (8). But this decomposition is not sufficient to show scale separation with respect to the step size \( \alpha \) between \( \{ J_n^{(0)} : n \in \mathbb{N} \} \) and \( \{ H_n^{(0)} : n \in \mathbb{N} \} \). More precisely, in the proof of Proposition 5, we only show that \( \sup_{n \in \mathbb{N}} \mathbb{E}^{1/p} \| H_n^{(0)} \|^p \leq C \alpha^{1/2} \) for \( \alpha \) small enough and a constant \( C \geq 0 \). To refine this bound, we use the expansion (11) to obtain that \( \sup_{n \in \mathbb{N}} \mathbb{E}^{1/p} \| H_n^{(0)} \|^p \leq C \alpha \), if \( \alpha \) is small enough, for a constant \( C \geq 0 \). We formalize this result in the following proposition.

Proposition 6. Assume A1, A2, and IND1. Then, for any \( \alpha \in (0, \alpha_{\infty}) \), \( p \geq 2 \), and \( n \in \mathbb{N} \), it holds

\[
\mathbb{E}^{1/p} \left[ \| J_n^{(1)} \|^p \right] \leq D_3 \alpha a p^{3/2} \| \varepsilon \|_{\infty} ,
\]

(28)

where \( D_3 \) is defined in (48). Moreover, for any \( 2 \leq p \leq q \) and \( \alpha \in (0, \alpha_{q,\infty}] \), \( n \in \mathbb{N} \),

\[
\mathbb{E}^{1/p} \left[ \| H_n^{(1)} \|^p \right] \leq D_4 \alpha a p^{3/2} d^{1/q} \| \varepsilon \|_{\infty} ,
\]

(29)

where \( D_4 \) is defined in (48).

The proof is provided in Appendix B.4. Using Proposition 6, we obtain \( p \)-th moment error bounds for LSA-PR.
Theorem 2. Assume A 1, A 2, IND 1. Then, for any p ≥ 2, even n ≥ 2, α ∈ (0, α p(1 + log d), ∞), \( \theta_0 \in \mathbb{R}^d \), it holds that
\[
(n/2)^{1/2}E^{1/p} \left[ \| A \left( \bar{\theta}_n - \theta^* \right) \|_p^p \right] \leq C_{Rm, 1} \{ \text{Tr} \Sigma_e \}^{1/2} p^{1/2} + \Delta_{n,p,\alpha}^{(fl)} \| e \|_\infty + e^{-\alpha n/8} \Delta_{n,p,\alpha}^{(tr)} \| \theta_0 - \theta^* \|,
\]
where \( C_{Rm,i}, i = 1, 2 \), are defined in Appendix A and \( \Delta_{n,p,\alpha}^{(tr)}, \Delta_{n,p,\alpha}^{(fl)} \) are given in (53).

The proof is postponed to Appendix B.5. Similarly to the proof of Theorem 2, we rely on the decomposition (16) but use Proposition 6 to bound p-th moments of the fluctuation term (17). Here again the leading term is \( \sum_{t=n/2}^n \varepsilon(Z_{t+1}) \). We bound the p-th moment of this sum with Rosenthal’s inequality for martingales from Pinelis (1994, Theorem 4.1) and using that \( \mathbb{E}[\| \varepsilon(Z) \|_2^2] = \text{Tr} \Sigma_e \). The dependence in \( p^{1/2} \) comes from the leading Gaussian term in this inequality. The other terms come from controlling the p-th moments in Rosenthal’s inequality and from majorizing the remainder terms. Simplified expressions for \( \Delta_{n,p,\alpha}^{(tr)} \) and \( \Delta_{n,p,\alpha}^{(fl)} \) are given by
\[
\Delta_{n,p,\alpha}^{(fl)} \lesssim \| \theta_0 - \theta^* \|_d \lesssim_d \sqrt{\{ \text{Tr} \Sigma_e \} \log(3e/\delta) + \Delta^{(HP)}(n, \theta_0, \delta)},
\]
where
\[
\Delta^{(HP)}(n, \theta_0, \delta) = n^{-1/4} \| e \|_\infty \log^{3/2}(3e/\delta)
\]
\[
+ (\log(3e/\delta) + \sqrt{n}) \| \theta_0 - \theta^* \| \exp \left\{ \frac{-(\alpha_\infty \land c_\Lambda) \sqrt{n}}{8(1 + \log d) \log(3e/\delta)} \right\}.
\]

For completeness, we give the statement of Corollary 1 with exact constants in Appendix B.6.

4 Finite-time Moment and High-probability Bounds in the Markovian Noise Setting

We now consider the Markov case. Let \((Z, dz)\) be a Polish space endowed with its Borel \( \sigma \)-field denoted by \( Z \) and let \((Z^N, Z^N)\) be the corresponding canonical space. Consider a Markov kernel \( Q \) on \( Z \times Z \) and denote by \( P_\xi \) and \( E_\xi \) the corresponding probability distribution and expectation with initial distribution \( \xi \). Without loss of generality, assume that \((Z_k)_{k \in \mathbb{N}}\) is the associated canonical process. By construction, for any \( A \in Z, P_\xi(Z_k \in A | Z_{k-1}) = Q(Z_{k-1}, A), P_\xi \)-a.s. In the case \( \xi = \delta_z, z \in Z, P_\xi \) and \( E_\xi \) are denoted by \( P_z \) and \( E_z \).

In this section we impose the following assumption on the mixing properties of \( Q \):

UGE 1. The Markov kernel \( Q \) admits \( \pi \) as an invariant distribution and is uniformly geometrically ergodic, that is, there exists \( t_{mix} \in \mathbb{N}^\ast \) such that for all \( k \in \mathbb{N}^\ast \),
\[
\Delta(Q^k) = \sup_{z, z' \in Z} (1/2)\| Q^k(z, \cdot) - Q^k(z', \cdot) \|_{TV} \leq (1/4)^{\left\lfloor k/t_{mix} \right\rfloor}.
\]
Here, $t_{\text{mix}}$ is the mixing time of $Q$. With (31) it is easy to see that
\[
\sum_{k=0}^{\infty} \Delta(Q^k) = \sum_{t=0}^{t_{\text{mix}}-1} \sum_{r=0}^{\infty} \Delta(Q^{t+r t_{\text{mix}}}) \leq (4/3)t_{\text{mix}} .
\]  
(32)

UGE 1 implies that $\pi$ is the unique invariant distribution of $Q$. UGE 1 is equivalent to the condition that $Q$ satisfies a uniform minorization condition (see Douc et al. (2018, Theorem 18.2.5)), i.e., there exists a probability measure $\nu$ such that for all $z \in Z$, $A \in Z$, $Q^{t_{\text{mix}}}(z, A) \geq (3/4)\nu(A)$. Under $A 1$, we define the quantity
\[
\alpha_{\infty}^{(M)} = \left[ \alpha_{\infty} \land \kappa_{Q}^{-1/2}C_{A}^{-1} \land a/(6\varepsilon_{Q} C_{A}) \right] \times \left[ 8\kappa_{Q}^{1/2}C_{A}/a \right]^{-1},
\]  
(33)
\[
C_{T} = 4(\kappa_{Q}^{1/2}C_{A} + a/6)^2 \times \left[ 8\kappa_{Q}^{1/2}C_{A}/a \right],
\]
where $\alpha_{\infty}$, $a, \kappa_{Q}$ are defined in (6) and (18), respectively. Now we use $\alpha_{\infty}^{(M)}$ and $C_{T}$ to define, for $q \geq 2$,
\[
\alpha_{q,\infty}^{(M)} = \alpha_{\infty}^{(M)} \land c_{A}^{(M)}/q , \quad c_{A}^{(M)} = a/\{12C_{T}\} .
\]  
(34)

We will see that $\alpha_{q,\infty}^{(M)} t_{\text{mix}}^{-1}$ is a natural counterpart of the stability threshold $\alpha_{q,\infty}$ from (19). Our goal now is to prove the counterpart of the stability result for the product of random matrices (cf. Proposition 2) under Markov conditions UGE 1. The main difference with the i.i.d. scenario is that the maximum step size which allows for matrix product stability scales with $t_{\text{mix}}^{-1}$. Similar scaling was reported in Srikant and Ying (2019) and Mou et al. (2021).

**Proposition 7.** Assume $A 1$ and UGE 1. Then, for any $2 \leq p \leq q$, $\alpha \in (0, \alpha_{\infty}^{(M)} t_{\text{mix}}^{-1})$, $n \in \mathbb{N}$, and probability distribution $\xi$ on $(Z, \mathcal{Z})$, it holds
\[
\mathbb{E}_{\xi}^{1/p} \left[ \|\Gamma_{1:n}^{(\alpha)}\|^{p} \right] \leq \sqrt{\kappa_{Q}} e^{2d^{1/q}} \{ -n\alpha a/6 + n(q - 1)\alpha^2 C_{T} \} ,
\]
where $\alpha_{\infty}^{(M)}$ is defined in (33). Moreover, for $\alpha \in (0, \alpha_{q,\infty}^{(M)} t_{\text{mix}}^{-1})$, it holds
\[
\mathbb{E}_{\xi}^{1/p} \left[ \|\Gamma_{1:n}^{(\alpha)}\|^{p} \right] \leq \sqrt{\kappa_{Q}} e^{d^{1/q}e^{-a\alpha n/12}} .
\]  
(35)

The proof is given in Appendix D.1. It is based on a simplification of the arguments in Durmus et al. (2021b) together with a new result about the matrix concentration for the product of random matrices, using a proof method introduced in Huang et al. (2021).

Similar to the i.i.d. case in Proposition 2, there is an unavoidable interaction between the choice of step size $\alpha$ and the maximum controlled moment $q$. Moreover, Remark 1 can be applied to obtain dimension-independent bounds for $\mathbb{E}_{\xi}^{1/p} \left[ \|\Gamma_{1:n}^{(\alpha)}\|^{p} \right]$.

With the above notations, we are ready to state and prove the Markov counterpart of Theorem 2. Under $A 2$ and UGE 1, we define the matrix $\Sigma_{\varepsilon}^{(M)}$ as
\[
\Sigma_{\varepsilon}^{(M)} = \mathbb{E}_{\pi}[\varepsilon(Z_{0})\varepsilon(Z_{0})^{T}] + 2 \sum_{\ell=0}^{\infty} \mathbb{E}_{\pi}[\varepsilon(Z_{0})\varepsilon(Z_{\ell})^{T}] .
\]  
(36)

For any initial probability measure $\xi$ on $(Z, \mathcal{Z})$, (Douc et al., 2018, Theorem 21.2.10) implies that $n a^{-1/2} \sum_{t=0}^{\infty} \varepsilon(Z_{t})$ converges in distribution to the zero-mean Gaussian distribution with covariance matrix $\Sigma_{\varepsilon}^{(M)}$. Hence, $\Sigma_{\varepsilon}^{(M)}$ is a counterpart of the covariance matrix $\Sigma_{\varepsilon}$, and we expect it to be the leading term in the bound for $\mathbb{E}_{\xi}^{1/p} \left[ \|A (\bar{\theta}_{n} - \theta^{*})\|^{p} \right]$. 

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Theorem 3. Assume A1, A2, and UGE 1. Then, for any \( p \geq 2 \), even \( n \geq 4 \sqrt{t_{\text{mix}}} \), step size
\[
\alpha^{(M)}(n, d, p, t_{\text{mix}}) = \left( \alpha^{(M)} \wedge c^{(M)}_{\theta, \kappa} \right) / \{ 1 + \log d \} \left( pn^{2/3} t_{\text{mix}}^{1/3} \right)^{-1},
\]
initial parameter \( \theta_0 \in \mathbb{R}^d \), and initial probability measure \( \xi \) on \((Z, \mathcal{Z})\), it holds that
\[
\mathbb{E}_{\xi}^{1/p} \left[ \| \tilde{A}(\tilde{\theta}_n - \theta^*) \|^p \right] \lesssim_d \left\{ \begin{array}{ll}
\frac{\{ \text{Tr} \Sigma^{(M)}_\xi \}^{1/2} \rho^{1/2}}{n^{1/2}} + \| \varepsilon \|_{\infty} \left( \frac{t_{\text{mix}}^{2/3} \rho \log n}{n^{2/3}} + \frac{t_{\text{mix}}^{2p/3}}{n} \right) \\
+ \eta \rho^{1/2} \| \theta_0 - \theta^* \| \exp \left\{ - \frac{(\alpha^{(M)} \wedge c^{(M)}_{\theta, \kappa}) n^{1/3}}{24 \rho t_{\text{mix}}^{1/3} (1 + \log d)} \right\}
\end{array} \right.
\]
(38)

As in i.i.d. case, we provide the generalization of Theorem 3 for the case of an arbitrary step size \( \alpha \in (0, \alpha^{(M)} t_{\text{mix}}^{-1}) \) in Theorem 4, together with the corresponding high-probability bounds (see Corollary 2). The expression for the step size (37) now differs from the i.i.d. case (21). Unsurprisingly, the stability result Proposition 7 requires to scale the step size with \( t_{\text{mix}}^{-1} \). At the same time, the optimization upon the sample size \( n \) in Theorem 4 suggests now that the optimal step size should scale as \( n^{-2/3} \). This is due to the fact that, unlike with the i.i.d. noise, the Polyak-Ruppert estimate is no longer an unbiased estimate of \( \theta^* \) (see Proposition 11 and the corresponding discussion).

Theorem 3 generalizes and improves the results of Mou et al. (2021, Theorem 1). First, Mou et al. (2021, Theorem 1) considers only the mean squared error, while in Theorem 3 we derive bounds for arbitrary \( p \)-th moments of the LSA-PR error. These bounds are further used to derive high probability bounds in Corollary 2 below. Second, the refined bound (38) for \( p = 2 \) yields the same leading term of order \{ \text{Tr} \Sigma^{(M)}_\xi \}^{1/2} n^{-1/2} \) and improves the dependence of the residual term on dimension. Indeed, for comparison with Mou et al. (2021, Theorem 1), we assume that \( \| \varepsilon \|_{\infty} \approx \sqrt{d} \). This leads to a residual term with a dependence of order \( d^{1/2} \) in Theorem 3 instead of \( d^{3/2} \) in Mou et al. (2021, Theorem 1). Moreover, the optimal step size \( \alpha \) in (37) scales with \( d \) as \( (1 + \log d)^{-1} \), unlike \( d^{-1/3} \) in Mou et al. (2021, Theorem 1).

Bounds on the non-averaged LSA iterates. Similar to the i.i.d. setting, we first obtain a preliminary bound on the \( p \)-th moment of the LSA error \( \| \tilde{\theta}_n - \theta^* \| \). In this preliminary result, we are especially interested in obtaining a sharp bound with respect to the step size \( \alpha \), for which we only rely on the first decomposition (8). We now give the bounds for \( \mathbb{E}_{\xi}^{1/p} [ \| J_n^{(0)} \| ] \) and \( \mathbb{E}_{\xi}^{1/p} [ \| \tilde{\theta}_n - \theta^* \| ] \), which match the corresponding bounds of Proposition 3 and Proposition 4 up to absolute constants and the factor \( \sqrt{t_{\text{mix}}} \).

Proposition 8. Assume A1, A2, and UGE 1. Then, for any \( \alpha \in (0, \alpha^{(M)}] \), \( p \geq 2 \), initial probability measure \( \xi \) on \((Z, \mathcal{Z})\), and \( n \in \mathbb{N} \), it holds
\[
\mathbb{E}_{\xi}^{1/p} [ \| J_n^{(0)} \| ] \leq D_1^{(M)} \sqrt{c_{\text{apt}}} t_{\text{mix}}^{1/2} \| \varepsilon \|_{\infty},
\]
(39)
where \( D_1^{(M)} \) is defined in (79).

The proof is provided in Appendix D.2. By definition (23), \( J_n^{(0)} \) is a linear statistics of the Markov chain \( (Z_k)_{k \in \mathbb{N}} \). Thus the desired result follows from a Mac-Diarmid type inequality under UGE 1 (see Paulin (2015, Corollary 2.10)). Note that the bound (39) is similar to the one established in Proposition 3 up to an additional \( \sqrt{t_{\text{mix}}} \) factor.
Proposition 9. Assume A 1, A 2, and UGE 1. Let $2 \leq p \leq q/2$ and $\alpha_{q,\infty}^{(M)}$ be defined in (34). Then, for any $\alpha \in (0, \alpha_{q,\infty}^{(M)} t_{\text{mix}}^{-1}]$, $\theta_0 \in \mathbb{R}^d$, initial probability measure $\xi$ on $(Z, \mathcal{Z})$, and $n \in \mathbb{N}$, it holds
\[
\mathbb{E}_{\xi}^{1/p} \left[ \|\theta_n - \theta^*\|^p \right] \leq \sqrt{\kappa_G e^d 1/q e^{-\alpha_{q,\infty}^{(M)} t_{\text{mix}}}} \|\theta_0 - \theta^*\| \left[ D_2^{(M)} d^{1/q} \sqrt{\alpha_{\text{mix}} t_{\text{mix}}} \|\varepsilon\|_{\infty} \right],
\]
where $D_2^{(M)}$ is defined in (82).

The proof is postponed to Appendix D.3. Proposition 9 improves the results obtained in Mou et al. (2021, Proposition 1). First, we derive a better scaling with respect to $p$ for the fluctuation term. Indeed, Mou et al. (2021, Proposition 1) implies that this term scales with $p^{3/2}$, while we obtain $p^{1/2}$. Moreover, the constraints on the step size $\alpha$ are relaxed. Proposition 9 holds for the step size $\alpha \lesssim 1/[p(1 + \log d) t_{\text{mix}}]$, while Mou et al. (2021, Proposition 1) requires that $\alpha \lesssim 1/[p^3 t_{\text{mix}}].$

Outline of the proof of Theorem 3 We state below a counterpart of Proposition 10. The objective, as in the i.i.d. case, is to obtain a better control of $\mathbb{E}_{\xi}^{1/p} \left[ \|H_n(0)\|^p \right]$, which we do here by using the decomposition (11).

Proposition 10. Assume A 1, A 2, and UGE 1. Then, for any $p \geq 2$, $\alpha \in (0, \alpha_{\infty}]$, and initial probability measure $\xi$ on $(Z, \mathcal{Z})$, it holds that
\[
\mathbb{E}_{\xi}^{1/p} \left[ \|J_n(1)\|^p \right] \leq \|\varepsilon\|_{\infty} (\alpha_{t_{\text{mix}}}^{(M)}) \left[ D_{J,1}^{(M)} \sqrt{\log (1/\alpha a)} p^2 + D_{J,2}^{(M)} (\alpha_{t_{\text{mix}}}^{(M)})^{1/2} p^{1/2} \right],
\]
where $D_{J,1}^{(M)}$ and $D_{J,2}^{(M)}$ are defined in (83) and (84) respectively. In addition, for any $p, q \geq 2$, satisfying $2 \leq p \leq q/2$, $\alpha \in (0, \alpha_{q,\infty}^{(M)} t_{\text{mix}}^{-1}]$, and initial probability measure $\xi$ on $(Z, \mathcal{Z})$, it holds that
\[
\mathbb{E}_{\xi}^{1/p} \left[ \|H_n(1)\|^p \right] \leq d^{1/q} \|\varepsilon\|_{\infty} (\alpha_{t_{\text{mix}}}^{(M)}) \left[ D_{H,1}^{(M)} \sqrt{\log (1/\alpha a)} p^2 + D_{H,2}^{(M)} (\alpha_{t_{\text{mix}}}^{(M)})^{1/2} p^{1/2} \right],
\]
where $D_{H,1}^{(M)}$ and $D_{H,2}^{(M)}$ are defined in (85).

The proof is postponed to Appendix D.4. Unlike the case of i.i.d.-noise, $J_n(1)$ is no longer a martingale, so we cannot directly apply Rosenthal-type inequalities to upper bound $\mathbb{E}_{\xi}^{1/p} \left[ \|J_n(1)\|^p \right]$. Instead, we rely on Berbee’s lemma (Rio (2017, Lemma 5.1)). The leading term in the bound of $\mathbb{E}_{\xi}^{1/p} \left[ \|J_n(1)\|^p \right]$ is $(\alpha_{t_{\text{mix}}}^{(M)}) \sqrt{\log (1/\alpha a)}$ in the Markov case instead of $a\alpha$ in the i.i.d. case. The factor $\sqrt{\log (1/\alpha a)}$ is a byproduct of Berbee’s inequality and is most likely an artifact of the proof.

With the above estimates, we are ready to state and prove the counterpart of Theorem 2 under UGE 1. Unlike the i.i.d. case, the Polyak-Ruppert estimate (1) is not an unbiased estimate for $\theta^*$. We quantify this resulting bias in our next result.

Proposition 11. Assume A 1, A 2, and UGE 1. Then, for any $\alpha \in (0, \alpha_{2(1+\log d),\infty}^{(M)} t_{\text{mix}}^{-1}]$, $\theta_0 \in \mathbb{R}^d$, initial probability measure $\xi$ on $(Z, \mathcal{Z})$, and even $n \geq 2$, it holds that
\[
\|\mathbb{E}_{\xi}[\tilde{\theta}_n] - \theta^*\| \leq D_4^{(M)} e^{-\alpha_{q,\infty}^{(M)} t_{\text{mix}}/24} \|\theta_0 - \theta^*\| + D_5^{(M)} \|\varepsilon\|_{\infty} (\alpha_{t_{\text{mix}}}^{(M)}) \sqrt{\log (1/\alpha a)} + D_6^{(M)} \|\varepsilon\|_{\infty} (\alpha_{t_{\text{mix}}}^{(M)})^{3/2},
\]
where $D_4^{(M)}, D_5^{(M)}, D_6^{(M)}$ are defined in (92).

The proof is given in Appendix D.5. Note that our bound on the bias scales as $O(\alpha_{t_{\text{mix}}}^{(M)})$ up to the factor $\sqrt{\log (1/\alpha a)}$. Similar bounds on the bias of LSA-PR procedure are provided in (Lau and...
and Meyn, 2022, Section 2). Moreover, one could conclude that \( \bar{\theta}_n \) is not an unbiased estimate of \( \theta^* \) using the decomposition (14). Indeed, we notice that in the fluctuation term \( E_n^{(fl)} \) defined in (17), the term \( \sum_{t=n/2}^{n-1} \tilde{A}(Z_{t+1})J_t^{(0)} \) has not mean zero. This comes in contrast to the i.i.d. case where \( \tilde{A}(Z_{t+1}) \) and \( J_t^{(0)} \) are independent and have zero mean. In fact, \( \| E_\xi [\tilde{A}(Z_{t+1})J_t^{(0)}] \| \) scales as \( O(\alpha t^{-1}_\text{mix}) \). The precise statement is given in Appendix D.6 in appendix.

Equipped with the above bounds, we can prove the \( p \)-th moment bound of the LSA-PR error under the Markovian noise dynamics.

**Theorem 4.** Assume **A 1**, **A 2**, and **UGE 1**. Then, for any \( p \geq 2 \), \( \alpha \in (0, \alpha_{\theta(\text{i.i.d.})}^{(M)}) \), even \( n \geq 4 \), \( \theta_0 \in \mathbb{R}^d \), and initial probability measure \( \xi \) on \( (Z, \mathcal{Z}) \), it holds

\[
\frac{(n/2)^{1/2}}{p} E_\xi^{1/p} \left[ \| \tilde{A}(\bar{\theta}_n - \theta^*) \| \right] \leq C \max \left\{ \frac{1}{4} \left( \frac{\text{tr} \{ \text{Sym} [\tilde{A}] \} }{1/2} \right)^{1/2}, \frac{\| \xi \|}{\| E_\xi [\tilde{A}Z_{t+1}J_t^{(0)}] \|} \right\} R_n^{(fl)} + R_n^{(tr)} \theta_0 - \theta^* \exp \left\{ -\alpha \frac{n}{24} \right\},
\]

where \( R_n^{(fl)} \) and \( R_n^{(tr)} \) are defined in (94).

The proof is postponed to Appendix D.7. The terms \( R_n^{(fl)} \) and \( R_n^{(tr)} \) correspond to the fluctuation and transient terms of the error decomposition, respectively. Simplified expressions of \( R_n^{(fl)} \) and \( R_n^{(tr)} \) are given by

\[
R_n^{(fl)} \lesssim d \left( p t^{-1/2}_\text{mix} (\alpha n)^{-1/2} + p^{3/4} t^{-1/4}_\text{mix} \right) + t^{-1/2}_\text{mix} \left( \log(1/\alpha) \right)^{1/2} p^{1/2} + (\alpha t^{-1}_\text{mix})^{1/2} (n^{-1/2} + \alpha n^{1/2}),
\]

\[
R_n^{(tr)} \lesssim d \alpha^{-1} n^{-1/2} + \alpha n^{1/2}.
\]

The bound for the transient term is similar to the i.i.d. case, only the numerical constants are affected. The expression for the fluctuation term is more complicated. Unlike the i.i.d. case, \( \{ e(\theta_t, Z_{t+1}) \} \) is no longer a martingale increment sequence (see the decomposition (14) and (15)). By (17) we have the decomposition

\[
\tilde{A}(\bar{\theta}_n - \theta^*) = 2(\alpha_n)^{-1} (\theta_n/2 - \theta_n) - 2n^{-1} E^{(0)}_{n/2,n} + 2n^{-1} \sum_{t=n/2}^{n-1} \varepsilon(Z_{t+1}) + 2n^{-1} \sum_{t=n/2}^{n-1} \tilde{A}(Z_{t+1})J_t^{(0)} + 2n^{-1} \sum_{t=n/2}^{n-1} \tilde{A}(Z_{t+1})H_t^{(0)}.
\]

The fluctuation term above consists of 3 summands. The first one is \( (n/2)^{-1} \sum_{t=n/2}^{n-1} \varepsilon(Z_{t+1}) \), which is an additive functional of the uniformly geometrically ergodic Markov chain \( \{ Z_t \} \), under **UGE 1**. Using a novel Rosenthal inequality from (Durmus et al., 2023, Theorem 1), we get that the leading term of the \( p \)-th moment of this quantity scales as \( \left( \frac{\text{tr} \{ \text{Sym} [\tilde{A}] \} }{1/2} \right)^{1/2} \). This is also the leading term in the bound (43). The remainder terms in this inequality are more involved than in the i.i.d. case, which explains the occurrence of the term of order \( p t^{3/4}_\text{mix} n^{-1/4} \). Regarding the second term in the fluctuation component, we already mentioned that in general \( E_\xi [\tilde{A}(Z_{t+1})J_t^{(0)}] \neq 0 \), in contrast to the i.i.d. case. Moreover, we provide the bound on the quantity \( E_\xi [\tilde{A}(Z_{t+1})J_t^{(0)}] \) in Proposition 16. This bound is of the same order (up to the factor \( \sqrt{\log 1/\alpha \alpha} \)) with respect to the step size \( \alpha \) as the bound for \( H_t^{(0)} \), that can be obtained through Proposition 10 (recall that \( H_n^{(0)} = J_n^{(1)} + H_n^{(1)} \)).
This fact can also be applied to control the last summand in the decomposition and explains why we did not consider the higher-order expansions for $H_{n,1}$.

The bound of Theorem 4 can be refined under the special choice of the step size $\alpha$. The fluctuation error term (44) suggests that $\alpha$ should scale with $n$ as $n^{-2/3}$, which explains the dependence of $\alpha^{(M)}(n, d, p, t_{\text{mix}})$ in (37) upon the sample size $n$. Similar to Corollary 1, the bound of Theorem 4 can be reformulated as a high probability bound using the Markov inequality.

**Corollary 2.** Assume $A_1$, $A_2$, $UGE_1$, and set $\delta \in (0, 1)$. Then, for any $\theta_0 \in \mathbb{R}^d$, sample size $n \in \mathbb{N}^*$, $n \geq 4 \vee t_{\text{mix}}$, and initial probability measure $\xi$ on $(Z, Z)$, choosing the step size $\alpha = \alpha^{(M)}(n, d, \log(3e/\delta), t_{\text{mix}})$ defined in (37), it holds with probability at least $1 - \delta$, that

$$n^{1/2}\|\hat{A}(\hat{\theta}_n - \theta^*)\| \lesssim_d \sqrt{\langle \text{Tr } \Sigma^{(M)} \rangle} \log(3e/\delta) + \epsilon_1^{(M)} R^{(HP)}(n, \theta_0, \delta, t_{\text{mix}}),$$

where

$$R^{(HP)}(n, \theta_0, \delta, t_{\text{mix}}) = \|\varepsilon\| \log(3e/\delta) \left( n^{-1/6} \log(n) t_{\text{mix}}^{2/3} + n^{-1/2} t_{\text{mix}} \log(3e/\delta) \right) + (n^{1/6} t_{\text{mix}}^{1/3} \log(3e/\delta) + n^{1/2}) \|\theta_0 - \theta^*\| \exp \left\{ \frac{\epsilon^{(M)}(n) \wedge \epsilon^{(M)}(n) n^{1/3}}{24t_{\text{mix}}^{1/3}(1 + \log d) \log(3e/\delta)} \right\}.$$

## A Notations and Constants

Denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{N}_- = \mathbb{Z} \setminus \mathbb{N}^*$. Let $d \in \mathbb{N}^*$ and $Q$ be a symmetric positive definite $d \times d$ matrix. For $x \in \mathbb{R}^d$, we denote $\|x\|_Q = \|x^\top Qx\|^{1/2}$. For brevity, we set $\|x\| = \|x\|_I$. We denote $\|A\|_Q = \max_{\|x\|_Q = 1} \|Ax\|_Q$, and the subscriptless norm $\|A\| = \|A\|_I$ is the standard spectral norm. For a function $g : Z \to \mathbb{R}^d$, we denote $\|g\|_\infty = \sup_{z \in Z} \|g(z)\|$.

We denote $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$. Let $A_1, \ldots, A_N$ be $d$-dimensional matrices. We denote $\prod_{i=1}^j A = A_1 \ldots A_j$ if $i \leq j$ and by convention $\prod_{i=1}^0 A = I$ if $i > j$. We say that a centered random variable (r.v.) $X$ is subgaussian with variance proxy factor $\sigma^2$ and denote $X \in SG(\sigma^2)$ if for all $\lambda \in \mathbb{R}$, $\log \mathbb{E}[e^{\lambda X}] \leq \lambda^2 \sigma^2/2$.

The readers can refer to the following table on the variables, constants and notations that are used across the paper for references.
| Variable | Description | Reference |
|----------|-------------|-----------|
| Q        | Solution of Lyapunov equation for $\bar{A}$ | Proposition 1 |
| $\kappa_Q$ | $\lambda^{-1}_{\min}(Q)\lambda_{\max}(Q)$ | Proposition 1 |
| $\alpha_{\kappa_Q}$ | Real part of minimum eigenvalue of $\bar{A}$ | Proposition 1 |
| $\Gamma^{(\alpha)}_{m,n}$ | Product of random matrices with step size $\alpha$ | (3) |
| $\varepsilon(Z_n)$ | Noise in LSA procedure | (2) |
| $\tilde{\theta}_n^{(T)}$, $\tilde{\theta}_n^{(R)}$ | Transient and fluctuation terms of LSA error | (4) |
| $\alpha_{\kappa_{\infty}}$ (resp. $\alpha_{\kappa_{\infty}}^{(M)}$) | Stability threshold for $\Gamma_{m,n}^{(\alpha)}$ to have bounded $p$-th moment under IND 1 (resp. UGE 1) | (19) |
| $J_n^{(0)}$, $H_n^{(0)}$ | Dominant term in $\tilde{\theta}_n^{(R)}$ | (9) |
| $J_n^{(1)}$, $H_n^{(1)}$, $\Sigma_\varepsilon$, $\Sigma_\varepsilon^{(M)}$ | Residual term $\tilde{\theta}_n^{(R)} - J_n^{(0)}$, Expansion of $H_n^{(0)}$, Noise covariance $\mathbb{E}[\varepsilon_1\varepsilon_1^\top]$, Asymptotic covariance matrix under Markovian noise | (11)-(12), A 2, (36) |
| $C_{Rn1} = 60e$ | Constant in martingale Rosenthal’s inequality | (Pinelis, 1994, Theorem 4.1) |
| $C_{Rn2} = 60$ | Constant in martingale Rosenthal’s inequality | (Pinelis, 1994, Theorem 4.1) |
| $C_{R1}^{(M)}$, $C_{R2}^{(M)}$ | Constants in Rosenthal’s inequality under UGE 1 | Theorem 6 |
| $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ | filtration $\mathcal{F}_t = \sigma(Z_s : 1 \leq s \leq t)$ with $\mathcal{F}_0 = \{\emptyset, Z\}$ | |
| $\mathbb{E}^{\mathcal{F}_t}$ | the conditional expectation with respect to $\mathcal{F}_t$ | |

B Independent case bounds

In the lemmas below we use the shorthand notations $\bar{A}_n, A_n, \varepsilon_n$ for $\bar{A}(Z_n), A(Z_n)$, and $\varepsilon(Z_n)$, respectively, where $\varepsilon(z) : \mathcal{Z} \rightarrow \mathbb{R}^d$ is defined in (2). For $t \in \mathbb{N}$, we define the filtration $\mathcal{F}_t = \sigma(Z_s : 1 \leq s \leq t)$, $\mathcal{F}_0 = \{\emptyset, Z\}$, and denote by $\mathbb{E}^{\mathcal{F}_t}$ the conditional expectation with respect to $\mathcal{F}_t$.

B.1 Proof of Proposition 3

Recall that the constant $D_1$ is defined as $D_1 = \sqrt{2\kappa_Q}/a$. With the decomposition (9), we expand $J_n^{(0)}$ as

$$J_n^{(0)} = \alpha \sum_{j=1}^n (I - \alpha \bar{A})^{n-j} \varepsilon_j =: \alpha \sum_{j=1}^n \eta_{n,j},$$

where $\eta_{n,j} = (I - \alpha \bar{A})^{n-j} \varepsilon_j$.

Proposition 1 implies that $\| (I - \alpha \bar{A})^{n-j} \| \leq \kappa_Q^{1/2}(1 - \alpha a)^{(n-j)/2}$. Hence, using Lemma 8 and A 2, we get for any $t \geq 0$ that

$$\mathbb{P}(\|J_n^{(0)}\| \geq t) \leq 2 \exp\left(-t^2/(2\sigma_{\alpha,n}^2)\right),$$

where

$$\sigma_{\alpha,n}^2 = \alpha^2 \kappa_Q \|\varepsilon\|_2^2 \sum_{j=1}^n (1 - \alpha a)^{n-j} \leq \alpha \kappa_Q \|\varepsilon\|_2^2 / a.$$ 

Combining this result with Lemma 7 yields (22).
B.2 Proof of Proposition 4

Define the constant $D_2$ as
\[ D_2 = (2\kappa_Q)^{1/2} a^{-1} (1 + 4\kappa_Q^{1/2} C_A a^{-1}) . \] (45)

Using the main expansion (8) and Minkowski’s inequality,
\[ \mathbb{E}^{1/p} \left[ \left\| \theta_n - \theta^* \right\|^p \right] \leq \mathbb{E}^{1/p} \left[ \left\| \Gamma^{(\alpha)}_{1:n}(\theta_0 - \theta^*) \right\|^p \right] + \mathbb{E}^{1/p} \left[ \left\| J_n^{(0)} \right\|^p \right] + \mathbb{E}^{1/p} \left[ \left\| H_n^{(0)} \right\|^p \right] . \] (46)

Applying Proposition 2, using that $\alpha a \leq 1/2$ by Proposition 1, and $(1-t)^{1/2} \leq 1-t/2$ for $t \in [0,1],$
\[ \mathbb{E}^{1/p} \left[ \left\| \Gamma^{(\alpha)}_{1:n}(\theta_0 - \theta^*) \right\|^p \right] \leq \kappa_Q^{1/2} d^{1/q} (1 - \alpha a/4)^n \| \theta_0 - \theta^* \| . \]

With Proposition 3, we get $\mathbb{E}^{1/p} \left[ \left\| J_n^{(0)} \right\|^p \right] \leq D_1 \sqrt{\alpha a p} \| \varepsilon \|_{\infty}$. It remains to bound $\mathbb{E}^{1/p} \left[ \left\| H_n^{(0)} \right\|^p \right]$.

Expanding the recurrence (9), we represent
\[ H_n^{(0)} = -\alpha \sum_{j=1}^{n} \Gamma^{(\alpha)}_{j+1:n} \tilde{A}(Z_j) J_{j-1}^{(0)} . \]

Using Minkowski’s inequality and since $\tilde{A}(Z_j)$ and $J_{j-1}^{(0)}$ are independent under $\text{IND}_1$, we obtain with A1, that
\[ \mathbb{E}^{1/p} \left[ \left\| H_n^{(0)} \right\|^p \right] \leq C_A \sum_{j=1}^{n} \mathbb{E}^{1/p} \left[ \left\| \Gamma^{(\alpha)}_{j+1:n} \right\|^p \right] \mathbb{E}^{1/p} \left[ \left\| J_{j-1}^{(0)} \right\|^p \right] . \]

Now (20) and Proposition 3 yield
\[ \mathbb{E}^{1/p} \left[ \left\| H_n^{(0)} \right\|^p \right] \leq c_1 d^{1/q} \sqrt{\alpha a p} \| \varepsilon \|_{\infty} , \text{ where } c_1 = 4D_1 C_A \kappa_Q^{1/2}/a . \]

Combining the bounds above in (46) completes the proof.

B.3 Proof of Proposition 5

Define
\[ \Delta_{n,\alpha}^{(tr)} = 32e\kappa_Q/(\alpha^2 n) + 128e\kappa_Q C_A^2/(7a n) , \quad \Delta_{n,\alpha}^{(fr)} = 64e D_2^2/(\alpha n) + 16\alpha C_A^2 D_2^2 . \] (47)

Let $q \geq 2$ be a number to be fixed later, and assume that $\alpha \in (0,\alpha_q,\infty)$. We need this additional degree of freedom to ensure that our bounds are dimension-free. Recall for $k \in \mathbb{N}$, $F_k = \sigma(Z_s : 1 \leq s \leq k), F_0 = \{\emptyset, Z\}$. Our proof is based on the decomposition (14). Under $\text{IND}_1$, $\mathbb{E}^{F_t} [e(\theta_t, Z_{t+1})] = 0$ $\mathbb{P}$-a.s., showing that $e(\theta_t, Z_{t+1})$ is an $F_t$-martingale increment. Thus, exploiting (14), we proceed with decomposition (25) and estimate the terms $T_1$ and $T_2$ separately. To control the remainder term $T_2$, we apply Proposition 4 with $p = 2$, and obtain
\[ T_2 \leq \frac{32d^{2/q} \kappa_Q \| \theta_0 - \theta^* \|^2 (1 - \alpha a/4)^n}{\alpha^2 n} + \frac{64d^{2/q} D_2^2 a \| \varepsilon \|_\infty^2}{\alpha n} . \]

Now we bound $T_1$. Recall that for $\theta \in \mathbb{R}^d$, $z \in \mathbb{Z}$, $e(\theta, z) = \varepsilon(z) + \tilde{A}(z)(\theta - \theta^*)$. Hence,
\[ \mathbb{E} \left[ \| e(\theta_t, Z_{t+1}) \|^2 \right] \leq 2 \operatorname{Tr} \Sigma_{\varepsilon} + 2 \mathbb{E} \left[ \| \tilde{A}(Z_{t+1}) \{ \theta_t - \theta^* \} \|^2 \right] , \]

where we used that $\mathbb{E} \left[ \| e_\varepsilon \|_\infty^2 \right] = \operatorname{Tr} \Sigma_{\varepsilon}$. Proposition 4 together with $\alpha \leq \alpha_{\infty}$ and $\alpha a \leq 1/2$ yields
\[ \sum_{t=n/2}^{n-1} \mathbb{E} \left[ \| e(\theta_t, Z_{t+1}) \|^2 \right] \leq n \operatorname{Tr} \Sigma_{\varepsilon} + 4\alpha a n d^{2/q} D_2^2 C_A^2 \| \varepsilon \|_\infty^2 + \frac{32d^{2/q} \kappa_Q C_A^2 \| \theta_0 - \theta^* \|^2 (1 - \alpha a/4)^n}{7a} . \]

It remains to combine the above bounds and to choose $q = 2(1 + \log d)$, and the elementary inequality $d^{2/(2+2\log d)} \leq e$. 

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B.4 Proof of Proposition 6

Define

\[ D_3 = 2\kappa_Q C_A/a^2, \quad \text{and} \quad D_4 = 4\kappa_Q^{1/2} C_A D_3/a. \]  

(48)

B.4.1 Moment bounds for \( J_n^{(1)} \)

We begin with the proof of (28). Expanding the recurrence (12) with \( \ell = 1 \) and using that \( J_{k-1}^{(0)} = -\alpha \sum \limits_{i=1}^{k-1}(1 - \alpha \bar{A})^{k-i-1} \varepsilon_i \) yields

\[ J_n^{(1)} = \alpha^2 \sum_{i=1}^{n-1} S_{i+1:n}^{(1)} \varepsilon_i, \quad \text{where} \quad S_{i+1:n}^{(1)} = \sum_{k=i+1}^{n}(1 - \alpha \bar{A})^{n-k} \tilde{A}_k(1 - \alpha \bar{A})^{k-1-i}. \]  

(49)

Recall for \( k \in \mathbb{N}, F_k = \sigma(Z_s : 1 \leq s \leq k), F_0 = \{\emptyset, \mathbb{Z}\}. \) It is easy to check that the sequence \( \{S_{i+1:n}^{(1)} \varepsilon_i\}_{i=1}^{n-1} \) is a martingale-difference with respect to the filtration \( (F_k)_{k \in \mathbb{N}}: \mathbb{E}[S_{i+1:n}^{(1)} \varepsilon_i \mid F_{i-1}] = 0. \) Applying the Burkholder inequality Osekowski (2012, Theorem 8.6) and the Minkowski inequality, we get

\[ \mathbb{E}[\|J_n^{(1)}\|p] \leq p^p \alpha^{2p} \mathbb{E}[(\sum_{i=1}^{n-1} \|S_{i+1:n}^{(1)} \varepsilon_i\|p)^{2/p}] \leq p^p \alpha^{2p} (\sum_{i=1}^{n-1} \mathbb{E}^{2/p}[\|S_{i+1:n}^{(1)} \varepsilon_i\|p]^p)^{2/p}. \]  

(50)

Let us denote \( v_i = \varepsilon_i/\|\varepsilon_i\| \). Then, using IND 1, we get

\[ \mathbb{E}[\|S_{i+1:n}^{(1)} \varepsilon_i\|^p] = \mathbb{E}[\|\varepsilon_i\|^p \mathbb{E}_{\mathcal{F}_i}[\|S_{i+1:n}^{(1)} \varepsilon_i\|^p]] \leq \mathbb{E}[\|\varepsilon_i\|^p] \sup_{u \in \mathbb{S}^{d-1}} \mathbb{E}[\|S_{i+1:n}^{(1)} u\|^p]. \]

A1 and Proposition 1 imply that \( \|(1 - \alpha \bar{A})^{n-k} \tilde{A}_k(1 - \alpha \bar{A})^{k-1-i}\| \leq \kappa_Q C_A (1 - \alpha a)^{(n-i-1)/2}. \) Hence, applying Lemma 8, we get for any \( t \geq 0 \) and \( u \in \mathbb{S}^{d-1} \) that

\[ \mathbb{P}(\|S_{i+1:n}^{(1)} u\| \geq t) \leq 2 \exp \left\{ -\frac{t^2}{2\kappa_Q^2 C_A^2 (n-i)(1 - \alpha a)^{n-i-1}} \right\}. \]

Applying Lemma 7, we get for any \( u \in \mathbb{S}^{d-1} \)

\[ \mathbb{E}^{2/p}[\|S_{i+1:n}^{(1)} u\|^p] \leq 2p C_A^2 \kappa_Q^2 (n-i)(1 - \alpha a)^{n-i-1}. \]  

(51)

Combining (50), (51), and A2, we get

\[ \mathbb{E}^{1/p}[\|J_n^{(1)}\|p] \leq 2\|\varepsilon\|_{\infty} p^{3/2} \alpha^2 C_A \kappa_Q (\sum_{i=1}^{n-1} (n-i)(1 - \alpha a)^{n-i-1})^{1/2} \leq D_3 a^p \varepsilon_{\infty}, \]

(52)

where \( D_3 \) is defined in (28). In the above we have used that \( \sum_{k=1}^{\infty} k\rho^{k-1} = (1 - \rho)^{-2} \) for \( \rho \in [0, 1) \) together with \( \alpha a \leq 1/2. \)

B.4.2 Moment bounds for \( H_n^{(1)} \)

The decomposition (12) implies that

\[ H_n^{(1)} = -\alpha \sum_{\ell=1}^{n} \Gamma^{(\alpha)}_{\ell+1:n} \tilde{A}_\ell J_{\ell-1}^{(1)}. \]
Hence, using Minkowski’s inequality together with IND 1,
\[
\mathbb{E}^{1/p}[\|H_n^{(1)}\|^p] \leq \alpha \sum_{t=1}^n \mathbb{E}^{1/p}[\|\ell_{t+1:n} \tilde{A}_t\|^p] \mathbb{E}^{1/p}[\|J_{t+1}^{(1)}\|^p].
\]
Applying Proposition 2 and (52), we get using the definition (29) of \(D_4\)
\[
\mathbb{E}^{1/p}[\|H_n^{(1)}\|^p] \leq \kappa_1^{1/2} C_A D_3 \alpha^2 a \frac{d^{1/q}}{p^{3/2}} \varepsilon \|\| \sum_{t=1}^n (1 - \alpha a/4)^n \leq D_4 d^{1/q} a \alpha p^{3/2} \varepsilon \|\|\).
\]

### B.5 Proof of Theorem 1 and Theorem 2

Define
\[
\Delta_{n,p,\alpha}^{(\emptyset)} = \frac{4e^{1/p} D_2(ap)^{1/2}}{(\alpha n)^{1/2}} + e^{1/p} C_A (D_3 + D_4) a \alpha p^{5/2} + \frac{2 C_{Rm,2} p}{n^{1/2}} + C_A D_1 (\alpha a)^{1/2} p^{3/2},
\]
\[
\Delta_{n,p,\alpha}^{(tr)} = e^{1/p} \kappa_1^{1/2} (4/(\alpha n^{1/2}) + 2^{-1/2} n^{1/2} C_A).
\]

We begin with the proof of Theorem 2. The result of Theorem 1 will directly follow from it using the step size \(\alpha\) fixed in (21).

**Proof.** Proof of Theorem 2. Let \(q \geq 2\) be a number to be fixed later, and assume that \(\alpha \in (0, \alpha_{q,\infty})\). The proof is based on exploiting the decomposition (14). Below we use shorthand notations \(\tilde{A}_t, A_t, \varepsilon_t\) for \(\tilde{A}(Z_t), A(Z_t), \) and \(\varepsilon(Z_t)\), respectively. Applying (14) and Minkowski’s inequality, we get

\[
(n/2) \mathbb{E}^{1/p} \left[\|\tilde{A}_t (\theta_n - \theta^*)\|^p\right] \leq T_1 + T_2, (54)
\]
\[
T_1 = \mathbb{E}^{1/p} \left[\|\sum_{t=n/2}^{n-1} \varepsilon_t Z_{t+1}\|^p\right], \quad T_2 = \alpha^{-1} \mathbb{E}^{1/p} [\|\theta_{n/2} - \theta_n\|^p].
\]

The term \(T_2\) is a remainder one, which is controlled with Proposition 4 and Minkowski’s inequality:
\[
T_2 \leq 2 \alpha^{-1} d^{1/q} \kappa_1^{1/2} (1 - \alpha a/4)^n/2 \|\theta_0 - \theta^*\| + 2 \alpha^{-1/2} d^{1/q} D_2(ap)^{1/2} \varepsilon \|\|.
\]

Now we proceed with the leading term \(T_1\). Using Minkowski’s inequality, (15) and (13),
\[
T_1 \leq \mathbb{E}^{1/p} \left[\|\sum_{t=n/2}^{n-1} \varepsilon_{t+1}\|^p\right] + \mathbb{E}^{1/p} [\|\sum_{t=n/2}^{n-1} \tilde{A}_{t+1} J_{t+1}^{(0)} \{\theta_0 - \theta^*\}\|^p] + \mathbb{E}^{1/p} [\|\sum_{t=n/2}^{n-1} \tilde{A}_{t+1} J_{t+1}^{(1)}\|^p] + \mathbb{E}^{1/p} [\|\sum_{t=n/2}^{n-1} \tilde{A}_{t+1} H_{t+1}^{(1)}\|^p].
\]

We first estimate the leading term \(\mathbb{E}^{1/p} [\|\sum_{t=n/2}^{n-1} \varepsilon_{t+1}\|^p]\). Applying Rosenthal’s inequality for martingales from Pinelis (1994, Theorem 4.1) and using that \(\mathbb{E}[\|\varepsilon(Z)\|^2] = \text{Tr} \Sigma_{\varepsilon}\), we get
\[
\mathbb{E}^{1/p} [\|\sum_{t=n/2}^{n-1} \varepsilon_{t+1}\|^p] \leq C_{Rm,1} p^{1/2} (n/2)^{1/2} \{\text{Tr} \Sigma_{\varepsilon}\}^{1/2} + C_{Rm,2} p \mathbb{E}^{1/p} [\max_t \|\varepsilon_{t+1}\|^p].
\]

With the assumption A 2, we get from the previous bound
\[
\mathbb{E}^{1/p} [\|\sum_{t=n/2}^{n-1} \varepsilon_{t+1}\|^p] \leq C_{Rm,1} p^{1/2} (n/2)^{1/2} \{\text{Tr} \Sigma_{\varepsilon}\}^{1/2} + C_{Rm,2} \|\varepsilon\|_{\infty} p.
\]
We now proceed with the other terms. The term \( \sum_{t=n/2}^{n-1} \tilde{A}_{t+1} \tilde{\theta}_t^{(t)} \) is upper-bounded using Minkowski’s inequality and the Proposition 2:

\[
\E^{1/p}\left[ \left\| \sum_{t=n/2}^{n-1} \tilde{A}_{t+1} \Gamma_t^{(\alpha)} \{ \theta_t - \theta^* \} \right\|^p \right] \leq C_A (n - n_0) \kappa_Q^{1/2} d^{1/q} (1 - \alpha a/4)^{n/2} \| \theta_0 - \theta^* \| .
\]

Note that the sequences \( \{ \tilde{A}_{t+1} \tilde{\theta}_t^{(0)} \}_{t=n/2}^{n-1} \), \( \{ \tilde{A}_{t+1} \tilde{\theta}_t^{(1)} \}_{t=n/2}^{n-1} \), and \( \{ \tilde{A}_{t+1} \tilde{\theta}_t^{(2)} \}_{t=n/2}^{n-1} \) are \((\mathcal{F}_t)_{t \in \mathbb{N}}\)-martingale increments. Hence, applying the Burkholder inequality Osekowski (2012, Theorem 8.6) and the Minkowski inequality,

\[
\E^{1/p}\left[ \left\| \sum_{t=n/2}^{n-1} \tilde{A}_{t+1} \tilde{\theta}_t^{(1)} \right\|^p \right] \leq p \left( \sum_{t=n/2}^{n-1} \E^{2/p}\left[ \left\| \tilde{A}_{t+1} \tilde{\theta}_t^{(1)} \right\|^p \right] \right)^{1/2} \leq C_A D_4 (n/2)^{1/2} p^{5/2} \alpha a^{1/2} \| \epsilon \|_{\infty},
\]

where the last inequality follows from Proposition 6. Similarly, using Proposition 3 and Proposition 6, we get

\[
\E^{1/p}\left[ \left\| \sum_{t=n/2}^{n-1} \tilde{A}_{t+1} \tilde{J}_t^{(0)} \right\|^p \right] \leq p \left( \sum_{t=n/2}^{n-1} \E^{2/p}\left[ \left\| \tilde{A}_{t+1} \tilde{J}_t^{(0)} \right\|^p \right] \right)^{1/2} \leq C_A D_1 (n/2)^{1/2} p^{3/2} (\alpha a)^{1/2} \| \epsilon \|_{\infty}.
\]

By the same reasoning, with Proposition 6, we get

\[
\E^{1/p}\left[ \left\| \sum_{t=n/2}^{n-1} \tilde{A}_{t+1} \tilde{J}_t^{(1)} \right\|^p \right] \leq C_A D_3 (n/2)^{1/2} p^{5/2} \alpha a \| \epsilon \|_{\infty}.
\]

It remains to choose \( q = p(1 + \log d) \) and combine the bounds above in (54).

### B.6 Version of Theorem 1 and Corollary 1 with exact constants

**Corollary 3.** Assume A, A, IND I and let \( n \geq 2, p \geq 2 \) and consider the step size \( \alpha = \alpha(n, d, p) \) specified in (21). Then it holds that

\[
(n/2)^{1/2} \E^{1/p}\left[ \left\| \tilde{A} \left( \tilde{\theta}_n - \theta^* \right) \right\|^p \right] \leq C_{\text{Rm},1} \{ \text{Tr} \Sigma_\epsilon \}^{1/2} p^{1/2} \left( \frac{c_3 (1 + \log d)^{1/2} p^{1/4} + c_4 p}{n^{1/2}} \right)^{1/2} \leq C_A (n/2)^{1/2} p^{1/2} \alpha\| \epsilon \|_{\infty},
\]

where \( c_3, c_4 \) and \( c_5 \) are given by

\[
\begin{align*}
    c_3 &= \frac{4 \alpha^{1/2} D_2}{(\alpha_\infty \land c_A)^{1/2}} + 2 C_{\text{Rm},2} + (\alpha_\infty \land c_A)^{1/2} a^{1/2} C_A D_1, \\
    c_4 &= C_A (D_3 + D_4) a (\alpha_\infty \land c_A), \\
    c_5 &= \frac{4}{\alpha_\infty \land c_A} + C_A, \\
    c_6 &= \frac{4}{\alpha_\infty \land c_A} + C_A. 
\end{align*}
\]

Moreover, let us fix \( \delta \in (0, 1) \). Then for any \( \theta_0 \in \mathbb{R}^d, n \in \mathbb{N}, \) with \( \alpha = \alpha(n, d, \log(3e/\delta)) \) defined in (21), it holds with probability at least \( 1 - \delta \), that

\[
n^{1/2} \left\| \tilde{A} \left( \tilde{\theta}_n - \theta^* \right) \right\| \leq 3 e \sqrt{2} \sqrt{\{ \text{Tr} \Sigma_\epsilon \} \log(3e/\delta) + c_2 \Delta^{(\text{HP})}(n, \theta_0, \delta)} \times
\]

where

\[
c_2 = 3e \sqrt{2} \left( (c_3 + c_4)(1 + \log d)^{1/2} \lor c_5 (1 + \log d) \right) .
\]

**Proof.** Proof. The inequality (56) follows from Theorem 2 after substituting the step size \( \alpha(n, d, p) \) given in (21). Now Corollary 1 with \( c_2 \) defined in (57) follows from the Markov inequality applied with \( p = \log(3e/\delta) > 2 \).
C Independent case bounds under sub-Gaussian noise assumption

Assumption A 2 can be relaxed to a sub-Gaussian-type conditions on the noise variable \( \varepsilon(Z) \). Consider the following assumption:

**A 3.** For any \( u \in \mathbb{S}^{d-1} \), and \( \lambda \in \mathbb{R} \), \( \log \{ \mathbb{E}[\exp(\lambda u^T \varepsilon(Z))] \} \leq \lambda^2 \sigma _\varepsilon^2 /2 \), where \( Z \) is a random variable with distribution \( \pi \).

Note that A 2 implies A 3, and A 3 can be written more concisely as \( u^T \varepsilon(Z) \in \text{SG}(\sigma _\varepsilon^2) \) for any \( u \in \mathbb{S}^{d-1} \). For instance, this condition holds when \( \varepsilon(Z_{t+1}) \) is an outer product of sub-Gaussian random variables in the canonical coordinates; see Mou et al. (2021, Assumption 2). Note that, for any \( u \in \mathbb{S}^{d-1} \), and \( t \geq 0 \),

\[
\mathbb{P}(|u^T \varepsilon(Z)| \geq t) \leq 2 \exp(-(t^2/(2\sigma _\varepsilon^2))) .
\] (58)

Below we state the counterpart of Proposition 3 and Proposition 4.

**Proposition 12.** Assume A 1, IND 1 and A 3. Then, for any \( \alpha \in (0,\alpha_\infty] \), \( p \geq 2 \), \( u \in \mathbb{S}^d \) and \( n \in \mathbb{N} \),

\[
\mathbb{E}^{1/p}[|u^T J_n^{(0)}|^p] \leq D_1 \sqrt{\alpha a p \sigma _\varepsilon^2} ,
\]

(59)

where \( D_1 \) is given in (22). Moreover, for any \( p, q \in \mathbb{N} \), \( 2 \leq p \leq q \), \( \alpha \in (0,\alpha_\infty] \), \( n \in \mathbb{N} \), \( u \in \mathbb{S}^d \) and \( \theta_0 \in \mathbb{R}^d \),

\[
\mathbb{E}^{1/p}[|u^T (\theta - \theta^*)|^p] \leq d^{1/q} \alpha^{1/2} (1 - \alpha a /4)^n \| \theta_0 - \theta^* \| + D d^{1/q} \sqrt{\alpha a p \sigma _\varepsilon^2} ,
\]

(60)

where the constant \( D \) is given by

\[
D = (2\kappa Q)^{1/2} a^{-1} (1 + 4\kappa Q^{1/2} C a^{-1})
\]

**Proof.** Proof. We first show the bound (59). Expanding (9), we get for any \( u \in \mathbb{S}^{d-1} \), that

\[
J_n^{(0)} = \alpha n \sum_{j=1}^{n} \eta_{n,j}, \quad \text{where } \eta_{n,j} = u^T (I - \alpha \bar{A})^{n-j} \varepsilon_j .
\] (61)

Note that \( \{\eta_{n,j}\}_{j=1}^{n} \) are sub-Gaussian random variables. With Proposition 1, for any \( \lambda \in \mathbb{R} \),

\[
\log \mathbb{E}[\exp(\lambda \eta_{n,j})] \leq (1/2) \lambda^2 \| u^T (I - \alpha \bar{A})^{n-j} \| \sigma _\varepsilon^2 \leq (1/2) \lambda^2 \kappa Q (1 - \alpha a)^{n-j} \sigma _\varepsilon^2 .
\]

Hence, \( \eta_{n,j} \in \text{SG}(\sigma _{n,j}^2) \), where \( \sigma _{n,j}^2 = \kappa Q (1 - \alpha a)^{n-j} \sigma _\varepsilon^2 \). IND 1 and (61) imply that \( u^T J_n^{(0)} \) is also sub-Gaussian random variable, that is,

\[
u^T J_n^{(0)} \in \text{SG}(\sigma _{\alpha,n}^2) , \quad \sigma _{\alpha,n}^2 = \alpha^2 \sum_{j=1}^{n} \sigma _{n,j}^2 \leq a^{-1} \kappa Q \sigma _\varepsilon^2 \alpha .
\]

Using (58) and applying Lemma 7, we obtain for \( p \geq 2 \) that

\[
\mathbb{E}^{1/p}[|u^T J_n^{(0)}|^p] \leq D_1 \sigma _\varepsilon \sqrt{\alpha p} , \quad \text{where } D_1 = 2\kappa Q^{1/2} a^{-1} .
\]

Now the proof of the bound (60) follows the same line as the proof of Proposition 4 and is omitted. 

\[\square\]

\[\text{\textsuperscript{1}The condition can be further relaxed to cover heavier-tail setting in which } \varepsilon(Z_{t+1}) \text{ has only a finite number of moments or is sub-exponential (instead of sub-gaussian).}\]
Proposition 13. Assume A 1, IND 1 and A 3. Then, for any even \( n \geq 2 \), \( \alpha \in (0, \alpha_\infty \land \{c_A / \{2 + 2 \log d\}\}) \), \( \theta_0 \in \mathbb{R}^d \), \( u \in \mathbb{S}^{d-1} \), it holds
\[
(n/2) \mathbb{E} \left[ |u^\top \mathbf{A} (\tilde{\theta}_n - \theta^*)|^2 \right] \leq 4u^\top \Sigma_\epsilon u + \Delta_{n,\alpha}^{(n)} \sigma_\epsilon + e^{-\alpha_n/4} \Delta_{n,\alpha}^{(n)} \|\theta_0 - \theta^*\|^2 ,
\]
where \( \Delta_{n,\alpha}^{(n)} \), \( \Delta_{n,\alpha}^{(n)} \) are given in (47).

Proof. The proof follows the same line as Proposition 5.

Proposition 14. Assume A 1, A 3, and IND 1. Then, for any \( \alpha \in (0, \alpha_\infty], \ p \geq 2 \), \( u \in \mathbb{S}^{d-1} \) and \( n \in \mathbb{N} \), it holds
\[
\mathbb{E}^{1/p} [u^\top J_n^{(1)} |p] \leq D_3 \alpha p \sigma_\epsilon , \quad \text{where} \ \ D_3 = 4 \kappa_Q C_A / a^2 .
\]
Moreover, for any \( 2 \leq p \leq q \) and \( \alpha \in (0, \alpha_\infty], \ n \in \mathbb{N} \),
\[
\mathbb{E}^{1/p} [u^\top H_n^{(1)} |p] \leq D_4 \alpha p d / q \sigma_\epsilon , \quad \text{where} \ \ D_4 = 4 \kappa_Q C_A / d / a^2 .
\]

Proof. We begin with the bound (62). We use (49). The sequence \( \{S_{i+1:n}\}^{n-1}_{i=1} \) is a martingale-difference with respect to the filtration \( (\mathcal{F}_k)_{k \in \mathbb{N}} \). Applying Burkholder’s inequality Os-ekowski (2012, Theorem 8.6) and Minkowski’s inequality, we get
\[
\mathbb{E}[|u^\top J_n^{(1)} |p] \leq p^\alpha \alpha^{2p} \| \sum_{i=1}^{n-1} (u^\top S_{i+1:n}^{(1)} \epsilon_i)^2 \|^{p/2} \\
\leq p^\alpha \alpha^{2p} \| \sum_{i=1}^{n-1} \mathbb{E}^{2/p} [|u^\top S_{i+1:n}^{(1)} \epsilon_i |p] \|^{p/2} .
\]
Set \( v_{i+1:n} = [S_{i+1:n}^{(1)}]^T u \). Then, using IND 1, we get
\[
\mathbb{E}[|v_{i+1:n} \epsilon_i |p] \leq \mathbb{E}[|v_{i+1:n} |p] \sup_{v \in \mathbb{S}^{d-1}} \mathbb{E}[|v^\top \epsilon_i |p] .
\]
Using the same arguments as in Proposition 6, we get
\[
\mathbb{P}(\|v_{i+1:n} \| \geq t) \leq 2 \exp \left\{ - \frac{t^2}{2 \kappa_Q C_A (n - i) (1 - \alpha a)^{n-i-1}} \right\} .
\]
Hence, applying Lemma 7, we get for any \( u \in \mathbb{S}^{d-1} \)
\[
\mathbb{E}^{2/p} [\|v_{i+1:n} \| |p] \leq 4p C_A^2 \kappa_Q (n - i) (1 - \alpha a)^{n-i-1} .
\]
Combining (64), (65), and A 2, we get
\[
\mathbb{E}^{1/p} [u^\top J_n^{(1)} |p] \leq 4 \sigma_\epsilon \alpha^{2p} C_A \kappa_Q (\sum_{i=1}^{n-1} (n - i) (1 - \alpha a)^{n-i-1})^{1/2} \\
\leq D_3 \alpha p \sigma_\epsilon .
\]

We now consider (63). Recall that \( H_n^{(1)} = -\alpha \sum_{\ell=1}^{n} \Gamma_{\ell+1:n}^{(\alpha)} \tilde{A}_{\ell} L_{\ell-1}^{(1)} \). Hence, using Minkowski’s inequality together with IND 1,
\[
\mathbb{E}^{1/p} [|H_n^{(1)} |p] \leq \alpha \sum_{\ell=1}^{n} \mathbb{E}^{1/p} \| \Gamma_{\ell+1:n}^{(\alpha)} \tilde{A}_{\ell} L_{\ell-1}^{(1)} \| |p| \sup_{u \in \mathbb{S}^{d-1}} \mathbb{E}^{1/p} [\| u^\top L_{\ell-1}^{(1)} |p] .
\]
Applying Proposition 2 and (66), we get using the definition (29) of \( D_4 \)
\[
\mathbb{E}^{1/p} [|H_n^{(1)} |p] \leq \kappa_Q^{1/2} d^{1/q} C_A D_3 \alpha^2 \sigma_\epsilon / p^2 \sum_{\ell=1}^{n} (1 - \alpha a/4)^n \leq D_4 d^{1/q} \sigma_\epsilon \alpha a p^2 .
\]
Using the bounds of Proposition 13, we obtain the $p$-th moment error bound for LSA-PR procedure similarly to Theorem 2. Proceeding as in (53), we introduce the fluctuation and transient components of the LSA-PR error

$$
\Delta^{(f)}_{n,p,\alpha} = \frac{4e^{1/p} D_2 p^{1/2}}{(\alpha n)^{1/2}} + e^{1/p} C_A (D_3 + D_4) \alpha p^3 + \frac{3\sqrt{2} C_{Rm,2} \sqrt{\log\{\eta n\}} p^{3/2}}{n^{1/2}} + C_A D_1 \alpha^{1/2} p^{3/2},
$$

$$
\Delta^{(tr)}_{n,p,\alpha} = e^{1/p} C_Q \left( 2\sqrt{2/(\alpha n^{1/2})} + 2^{-1/2} n^{1/2} C_A \right).
$$

**Theorem 5.** Assume $A_1$, IND 1, and $A_2$. Then, for any even $n \geq 2$, $p \geq 2$, $\alpha \in (0,\alpha_\infty \wedge c_A/(2(1 + \log d)))$, $\theta_0 \in \mathbb{R}^d$, $u \in S_{d-1}$, it holds

$$
(n/2)^{1/2} E^{1/p} \left[ \left| u^\top A (\hat{\theta}_n - \theta^*) \right|^p \right] \leq C_{Rm,1} \left\{ u^\top \Sigma \epsilon u \right\}^{1/2} p^{1/2} + \sigma_\epsilon \Delta^{(f)}_{n,p,\alpha} + \Delta^{(tr)}_{n,p,\alpha} (1 - \alpha a/4)^{n/2} \|\theta_0 - \theta^*\|,
$$

where $C_{Rm,i}$, $i = 1, 2$ are defined in Appendix A.

**Proof.** The proof follows the lines of Theorem 2 and is omitted. The only difference with the mentioned proof is related with the term $\Delta^{(f)}_{n,p,\alpha}$ in (67).

**D Markov case bounds**

**D.1 Proof of Proposition 7**

We first provide a result on the product of dependent random matrices. The proof is based on Huang et al. (2021). Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{N}}, \mathbb{P})$ be a filtered probability space. For the matrix $B \in \mathbb{R}^{d \times d}$ we denote by $(\sigma_\ell(B))_{\ell=1}^d$ its singular values. For $q \geq 1$, the Shatten $q$-norm is denoted by $\|B\|_q = \left\{ \sum_{\ell=1}^d \sigma_\ell^q(B) \right\}^{1/q}$. For $p, q \geq 1$ and a random matrix $X$ we write $\|X\|_{q,p} = \left\{ \mathbb{E}[\|X\|_q^p] \right\}^{1/p}$.

**Proposition 15.** Let $\{Y_\ell\}_{t \in \mathbb{N}}$ be a sequence of random matrices adapted to the filtration $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$ and $P$ be a positive definite matrix. Assume that for each $\ell \in \mathbb{N}^+$ there exist $\mathfrak{m}_\ell \in (0, 1)$ and $\sigma_\ell > 0$ such that

$$
\|\mathbb{E}^{t-1}[Y_\ell]\|_P \leq 1 - \mathfrak{m}_\ell \text{ and } \|Y_\ell - \mathbb{E}^{t-1}[Y_\ell]\|_P \leq \sigma_\ell \quad \mathbb{P}-a.s. .
$$

Define $Z_n = \prod_{t=0}^n Y_\ell = Y_n Z_{n-1}$, for $n \geq 1$. Then, for any $2 \leq p \leq q$ and $n \geq 1$,

$$
\|Z_n\|_{q,p} \leq \kappa_P \prod_{\ell=1}^n (1 - \mathfrak{m}_\ell + (q - 1) \sigma_\ell^2) \|P^{1/2}Z_0P^{1/2}\|_{q,p}^{1/2},
$$

where $\kappa_P = \lambda_{\max}(P)/\lambda_{\min}(P)$ and $\lambda_{\max}(P), \lambda_{\min}(P)$ correspond to the largest and smallest eigenvalues of $P$.  

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Proof. Let $n \in \mathbb{N}^*$ and $2 \leq p \leq q$. We begin with the decomposition

$$Z_n = Y_n Z_{n-1} = (Y_n - E^{\delta_{n-1}}[Y_n])Z_{n-1} + E^{\delta_{n-1}}[Y_n]Z_{n-1}.$$ 

Let us define $f_P : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ as $f_P(B) = P^{1/2}BP^{-1/2}$. Therefore, for any $n \in \mathbb{N}$, it holds $f_P(Z_n) = A_n + B_n$, where

$$A_n = f_P((Y_n - E^{\delta_{n-1}}[Y_n])Z_{n-1}) \quad \text{and} \quad B_n = f_P(E^{\delta_{n-1}}[Y_n])f_P(Z_{n-1}).$$

Since $E[B_n] = E[B_n] = 0$, Huang et al. (2021, Proposition 4.3) implies that

$$\|f_P(Z_n)\|_{q,p}^2 \leq \|B_n\|_{q,p}^2 + (q-1)\|A_n\|_{q,p}^2. \quad (68)$$

It remains to bound the two terms on the right-hand side. To this end, we use Hiai and Petz (2014, Theorem 6.20) which implies that for any $B_1, B_2 \in \mathbb{R}^{d \times d}$,

$$\|B_1B_2\|_{q,p} \leq \|B_1\|\|B_2\|_{q,p}. \quad (69)$$

Combining (69) with $\|B\|_p = \|f_P(B)\|_p$, and $\|Y_n - E^{\delta_{n-1}}[Y_n]\|_p \leq \sigma_n$, we get

$$\|A_n\|_{q,p} = \left(\mathbb{E}\left[\|f_P(Y_n - E^{\delta_{n-1}}[Y_n])f_P(Z_{n-1})\|_q^p\right]\right)^{1/p} \leq \left(\mathbb{E}\left[\|Y_n - E^{\delta_{n-1}}[Y_n]\|_p^p\|f_P(Z_{n-1})\|_q^p\right]\right)^{1/p} \leq \sigma_n\|f_P(Z_{n-1})\|_{q,p}. \quad (70)$$

Similarly, applying $\|E^{\delta_{n-1}}[Y_n]\|_p^2 \leq 1 - m_n$

$$\|B_n\|_{q,p}^2 = \left(\mathbb{E}\left[\|f_P(E^{\delta_{n-1}}[Y_n])f_P(Z_{n-1})\|_q^p\right]\right)^{2/p} \leq \left(\mathbb{E}\left[\|E^{\delta_{n-1}}[Y_n]\|_p^p\|f_P(Z_{n-1})\|_q^p\right]\right)^{2/p} \leq (1 - m_n)\|f_P(Z_{n-1})\|_{q,p}^2. \quad (71)$$

Combining (70) and (71) in (68) yields

$$\|f_P(Z_n)\|_{q,p} \leq (1 - m_n + q-1)\sigma_n^2\|f_P(Z_{n-1})\|_{q,p}^2 \leq \prod_{i=1}^n (1 - m_i + (q-1)\sigma_i^2)\|f_P(Z_0)\|_{q,p}^2.$$ 

The proof is completed using (69) which implies that

$$\|Z_n\|_{q,p} = \|P^{-1/2}f_P(Z_n)P^{1/2}\|_{q,p} \leq \sqrt{\kappa_P}\|f_P(Z_n)\|_{q,p}. \quad \square$$

In the lemmas below we aim to prove the bound (78). Recall that $Y_1 = \prod_{i=1}^h (1 - \alpha A(Z_i))$.

**Lemma 1.** Assume $A$ and $UGE 1$. Then for any $\alpha \in (0, \alpha^M_{\infty} t_{mix}^{-1}]$ with $\alpha^M_{\infty}$ defined in (33), and any probability $\xi$ on $(Z, \mathcal{Z})$,

$$\|E_\xi[Y_1]\|_Q^2 \leq 1 - a \alpha h / 6, \quad \text{where} \quad h = 1 \vee \left[8\kappa_Q^1/2 C_A \frac{t_{mix}}{a}\right]. \quad (72)$$

**Proof.** Proof. We decompose the matrix product $Y_1$ as follows:

$$Y_1 = I - \alpha h \tilde{A} - S_1 + R_1, \quad (73)$$

where $S_1 = \alpha \sum_{k=1}^h \{A(Z_k) - \tilde{A}\}$ is linear statistics in $\{A(Z_k)\}_{k=1}^h$, and the remainder $R_1$ collects the higher-order terms in the products.
Finally, Proposition \(\text{Applying the definition of } R\) 
\[
R_1 = \sum_{r=2}^{h} (-1)^r \alpha^r \sum_{(i_1, \ldots, i_r) \in I_r} \prod_{u=1}^{r} A(Z_{i_u}) .
\]
with \(I_r = \{(i_1, \ldots, i_r) \in \{1, \ldots, h\}^r : i_1 < \cdots < i_r\}\). Using \(\|M\|_Q = \|Q^{1/2}M^{-1/2}\|\), it is straightforward to check that \(\mathbb{P}\text{-a.s. it holds}
\[
\|R_1\|_Q \leq \sum_{r=2}^{h} (\alpha \kappa_Q^{1/2} C_A)^r \binom{h}{r} \leq (\kappa_Q^{1/2} C_A \alpha h)^2 (1 + \kappa_Q^{1/2} C_A \alpha)^h = T_2 .
\]
On the other hand, using \(UGE 1\), we have for any \(k \in \mathbb{N}^*\), that
\[
\|E_\xi[A(Z_k) - \bar{A}]\| = \sup_{u, v \in \mathbb{E}^{d-1}} [E_\xi[u^T A(Z_k) v - u^T \bar{A} v] \leq C_A \Delta(Q^k) .
\]
Hence, with the triangle inequality and (32),
\[
\|E_\xi[S_1]\| \leq \alpha \kappa_Q^{1/2} \sum_{k=1}^{h} \|E_\xi[A(Z_k) - \bar{A}]\| \leq \alpha \kappa_Q^{1/2} C_A \sum_{k=1}^{h} \Delta(Q^k)
\leq (4/3) \alpha t_{\text{mix}} \kappa_Q^{1/2} C_A = T_1 .
\]
This result combined with (74) in (73) implies that
\[
\|E_\xi[Y_1]\|_Q \leq \|I - \alpha h \bar{A}\|_Q + T_1 + T_2 .
\]
First, by definition (72) of \(h\), we have
\[
T_1 \leq \alpha a h / 6 .
\]
With the definition of \(\alpha^{(M)}_\infty\) in (33), \(\alpha \leq \alpha^{(M)}_\infty \leq (\kappa_Q^{1/2} C_A h)^{-1} \wedge [a/(6e \kappa_Q C_A^2 h)]\), and
\[
T_2 \leq (\kappa_Q^{1/2} C_A \alpha h)^2 e \leq \alpha a h / 6 .
\]
Finally, Proposition 1 implies that, for \(\alpha h \leq \alpha_\infty\),
\[
\|I - \alpha h \bar{A}\|_Q \leq 1 - \alpha a h / 2 .
\]
Combining (75), (76), and (77) yield \(\|E_\xi[Y_1]\|_Q \leq 1 - \alpha a h / 6\), and the statement follows. \(\square\)

**Lemma 2.** Assume A1 and UGE 1, and let \(\alpha \in (0, \alpha^{(M)}_\infty t_{\text{mix}}^{-1}]\). Then, for any probability \(\xi\) on \((Z, \mathcal{Z})\), we have
\[
\|Y_1 - E_\xi[Y_1]\|_Q \leq C_\sigma \alpha h , \text{ where } C_\sigma = 2(\kappa_Q^{1/2} C_A + a/6) ,
\]
and \(h\) is given in (72).

**Proof.** Using (73), we obtain
\[
\|Y_1 - E_\xi[Y_1]\|_Q \leq \alpha \sum_{k=1}^{h} \|A(Z_k) - E_\xi[A(Z_k)]\|_Q + \|R_1 - E_\xi[R_1]\|_Q .
\]
Applying the definition of \(R_1\) in (74), the definition of \(h, \alpha^{(M)}_\infty\), and \(T_2\) in (76), we get from the above inequalities
\[
\|Y_1 - E_\xi[Y_1]\|_Q \leq 2\alpha \kappa_Q^{1/2} C_A h + \alpha a h / 3 ,
\]
and the statement follows. \(\square\)
We have now all we need to show Proposition 7.

Proof. Proof of Proposition 7 Denote by $h \in \mathbb{N}$ a block length, the value of which is determined later. Define the sequence $j_0 = 0, j_{\ell+1} = \min(j_\ell + h, n)$. By construction $j_{\ell+1} - j_\ell \leq h$. Let $N = \lceil n/h \rceil$. Now we introduce the decomposition

$$
\Gamma_{1:n}^{(\alpha)} = \prod_{\ell=1}^N Y_\ell, \quad \text{where} \quad Y_\ell = \prod_{i=j_{\ell-1}}^j (I - \alpha A(Z_i)), \quad \ell \in \{1, \ldots, N\}.
$$

Using a crude bound $\|Y_N\| \leq (1 + \alpha C_A)^h$, we get

$$
\mathbb{E}_{\xi}[\|\Gamma_{1:n}^{(\alpha)}\|^{1/p}] \leq (1 + \alpha C_A)^h \mathbb{E}_{\xi}[\|\prod_{\ell=1}^{N-1} Y_\ell\|^{1/p}].
$$

Now we aim to bound $\mathbb{E}_{\xi}^{1/p}[\|\prod_{\ell=1}^{N-1} Y_\ell\|^{1/p}]$ with the technique introduced in Proposition 15. To do so, we define, for $\ell \in \{1, \ldots, N-1\}$, the filtration $\mathcal{H}_\ell = \sigma(Z_k : k \leq j_\ell)$ and establish almost sure bounds on $\mathbb{E}_{\xi}^{1/p}[Y_\ell|\mathcal{H}_{\ell-1}]$ and $\|Y_\ell - \mathbb{E}_{\xi}^{1/p}[Y_\ell]||_Q$ for $\ell \in \{1, \ldots, N-1\}$. More precisely, by the Markov property, it is sufficient to show that there exist $\epsilon \in (0, 1]$ and $\sigma > 0$ such that for any probabilities $\xi, \xi'$ on $(Z, \mathcal{Z})$,

$$
\|\mathbb{E}_{\xi'}[Y_1]\|_Q^2 \leq 1 - \epsilon \quad \text{and} \quad \|Y_1 - \mathbb{E}_{\xi'}[Y_1]\|_Q \leq \sigma, \quad \text{P}_{\xi'}\text{-a.s.}.
$$

Such bounds require the blocking procedure, since (78) not necessarily holds with $h = 1$. Set $h = \lceil 8\kappa_Q^{1/2} C_A / a \rceil_{\text{mix}}$.

Applying Lemma 1 and Lemma 2, we show that (78) hold with $m = a\alpha h/6$ and $\sigma = C_\sigma a\alpha h$, with $C_\sigma = 2(\kappa_Q^{1/2} C_A + a/6)$. Then, applying Proposition 15,

$$
\mathbb{E}_{\xi}^{1/p}[\|\Gamma_{1:n}^{(\alpha)}\|^{1/p}] \leq \mathbb{E}_{\xi}^{1/q}[\|\Gamma_{1:n}^{(\alpha)}\|^{q}] \leq \sqrt{\kappa_Q d^{1/q} e^{a\alpha C_A} h (1 - a\alpha h/6 + (q - 1) C_\sigma^2 h^2)^{N-1}}
$$

$$
\leq \sqrt{\kappa_Q d^{1/q} e^{a\alpha C_A} h e^{-a\alpha h(N-1)/6 + (q - 1) \alpha^2 C_\sigma^2 h^2 (N-1)}}
$$

$$
\leq \sqrt{\kappa_Q d^{1/q} e^{a\alpha C_A + a/6} e^{-a\alpha h/6 + (q - 1) \alpha^2 C_\sigma^2 h}}
$$

$$
\leq \sqrt{\kappa_Q d^{1/q} e^{-a\alpha h/6 + (q - 1) \alpha^2 C_\sigma^2 h}}.
$$

Here we used that by definition of $h$ and since $\alpha \in (0, \alpha^{(M)}_{t_{\text{mix}}^{-1}}]$, $a\alpha C_A \leq 1$, and $a\alpha h/6 \leq 1$ by (72).

D.2 Proof of Proposition 8

Define

$$
D_{1}^{(M)} = 2^{7/2} \kappa_Q^{1/2} a^{-1} \{ e^{-1/4} + \sqrt{2\pi} e C_A a^{-1} \}.
$$

We first apply the Abel transform to $J_n^{(0)}$. Using the representation (9), we obtain that

$$
J_n^{(0)} = \alpha \sum_{j=1}^n (I - \alpha \bar{A})^{n-j} \varepsilon(Z_j)
$$

$$
= \alpha (I - \alpha \bar{A})^{n-1} \sum_{k=1}^{n-1} \varepsilon(Z_k) - \alpha^2 \sum_{j=1}^{n-1} (I - \alpha \bar{A})^{n-j-1} \bar{A} \sum_{k=j+1}^{n} \varepsilon(Z_k).
$$

(80)
Note that \( \pi(\varepsilon) = 0 \), and for any \( z \in \mathbb{Z} \), A 2 implies \( \|\varepsilon(z)\| \leq \|\varepsilon\|_{\infty} \). Hence, applying Lemma 9, we get for any \( j \in \mathbb{N} \) and \( t > 0 \), that
\[
\mathbb{P}_{\varepsilon}(\|\sum_{k=j+1}^{n} \varepsilon(Z_k)\| \geq t) \leq 2 \exp\{-t^2/(2\beta_{n-j}^2)\},
\]
where for \( \ell \in \mathbb{N}_* 
\]
\[
\beta_{\ell} = 8\sqrt{\ell \text{mix}}\|\varepsilon\|_{\infty}.
\]
Lemma 7 and (81) imply that, for any \( p \geq 2 \),
\[
\mathbb{E}_{\varepsilon}^{1/p}[\|\sum_{k=j+1}^{n} \varepsilon(Z_k)\|^p] \leq 2^{7/2} \sqrt{(n-j)p\text{mix}}\|\varepsilon\|_{\infty}.
\]
Then, applying Minkowski’s inequality to (80), we get
\[
\mathbb{E}_{\varepsilon}^{1/p}[\|J_n^{(0)}\|^p] \leq 2^{7/2} \alpha \|(I - \alpha \mathbf{A})^{n-1}\| \|\sqrt{n\text{pt}}\|\|\varepsilon\|_{\infty}
\]
\[
+ 2^{7/2} \alpha^2 \sum_{j=1}^{n-1} \|(I - \alpha \mathbf{A})^{n-j-1}\mathbf{A}\| \sqrt{(n-j)p\text{mix}}\|\varepsilon\|_{\infty}.
\]
Using A 1 and Proposition 1, for \( j \in \{1, \ldots, n\} \), \( \|(I - \alpha \mathbf{A})^{n-j}\| \leq \sqrt{kQ}(1 - \alpha a)^{(n-j)/2} \). Note also that, since \( aa \leq 1/2 \),
\[
\sum_{j=1}^{n-1} (1 - aa)^{(n-j-1)/2} \sqrt{n-j} \leq e^{aa} \sum_{k=1}^{n-1} \exp\{-aa(k+1)/2\} \sqrt{k}
\]
\[
\leq \frac{2^{3/2}e^{aa}}{(aa)^{3/2}} \int_{0}^{+\infty} \exp\{-y\} \sqrt{y} dy \leq \frac{2^{1/2}kQ^{1/2}e^{1/2}}{(aa)^{3/2}}.
\]
It remains to combine the previous bounds with an elementary inequality, using \( aa \leq 1/2 \), for any \( x > 0 \),
\[
(1 - aa)^{(x-1)/2} \sqrt{x} \leq e^{aa/2} \exp\{-aa/2\} \sqrt{x} \leq e^{1/4} \sup_{u \geq 0} \{ue^{-u}\}^{1/2} \leq \frac{1}{(aa)^{1/2}e^{1/4}}.
\]
Combining the bounds above yield (39) with the constant \( \mathcal{D}_1^{(M)} \) defined in (79).

### D.3 Proof of Proposition 9

Define
\[
\mathcal{D}_2^{(M)} = \mathcal{D}_1^{(M)}(1 + 24\sqrt{2}e^2 \sqrt{kQ} C_\mathbf{A} a^{-1}),
\]
where \( \mathcal{D}_1^{(M)} \) is given in (79). Proceeding as in (46), we get
\[
\mathbb{E}_{\varepsilon}^{1/p}[\|\theta_n - \theta^*\|^p] \leq \mathbb{E}_{\varepsilon}^{1/p}[\|\Gamma_{1:n}^{(\alpha)}(\theta_0 - \theta^*)\|^p] + \mathbb{E}_{\varepsilon}^{1/p}[\|J_n^{(0)}\|^p] + \mathbb{E}_{\varepsilon}^{1/p}[\|H_n^{(0)}\|^p].
\]
The first two terms are bounded using (35) and Proposition 8, respectively. Regarding the last one, the recurrence (9), \( H_n^{(0)} = -\alpha \sum_{j=1}^{n} \Gamma_{j+1:n}^{(\alpha)}(\tilde{Z}_j)J_{j-1}^{(0)} \), and Minkowski’s inequality yields
\[
\mathbb{E}_{\varepsilon}^{1/p}[\|H_n^{(0)}\|^p] \leq \alpha \sum_{j=1}^{n} \mathbb{E}_{\varepsilon}^{1/2p}[\|\Gamma_{j+1:n}^{(\alpha)}\|^2]^{1/2p}\{\mathbb{E}_{\varepsilon}[\|\tilde{Z}_jJ_{j-1}^{(0)}\|^p]\}^{1/2p}.
\]
Using Proposition 8 and \( e^{-x} \leq 1 - x/2 \), valid for \( x \in [0, 1] \), we get
\[
\mathbb{E}_{\varepsilon}^{1/p}[\|H_n^{(0)}\|^p] \leq \alpha d^{1/q}\|\varepsilon\|_{\infty} e^2 \sqrt{kQ} C_\mathbf{A} \mathcal{D}_1^{(M)} \sqrt{2kQ \text{mix}} \sum_{j=1}^{n}(1 - aa/24)^{n}.
\]
This completes the proof.
D.4 Proof of Proposition 10

Define

\[ D_{j,l}^{(M)} = 64\kappa_Q C_A a^{-2} \left( (\sqrt{2} + \kappa_Q^{1/2})/\sqrt{2} \log 2 + 2\pi^{1/2}\kappa_Q^{1/2} + \kappa_Q^{1/2}/\sqrt{\log 2} \right) \]  
\[ D_{j,2}^{(M)} = (128/3)\kappa_Q^{3/2} C_A a^{-2} \]  
\[ D_{H,1}^{(M)} = 96a^{-1} C_A e^{2\kappa_Q^{1/2}D_{j,l}^{(M)}} \quad \text{and} \quad D_{H,2}^{(M)} = 48a^{-1} C_A e^{2\kappa_Q^{1/2}D_{j,2}^{(M)}}. \]

We preface the proof of this proposition by giving a statement of the Berbee lemma, which plays an essential role. Consider the extended measurable space \( \bar{Z}_N = Z^N \times [0, 1] \), equipped with the \( \sigma \)-field \( \bar{Z}_N = Z^N \otimes \mathcal{B}(\{0, 1\}) \). For each probability measure \( \xi \) on \( (Z, \mathcal{F}) \), we consider the probability measure \( \bar{\mathbb{P}}_\xi = \mathbb{P}_\xi \otimes \text{Unif}([0, 1]) \) and denote by \( \bar{E}_\xi \) the corresponding expected value. Finally, we denote by \( \bar{Z}_k \) the canonical process \( Z_k : ((z_i)_{i \in N}, u) \in \bar{Z}_N \rightarrow z_k \) and \( U : ((z_i)_{i \in N}, u) \in \bar{Z}_N \rightarrow u \). Under \( \bar{\mathbb{P}}_\xi \), \( \{\bar{Z}_k\}_{k \in N} \) is by construction a Markov chain with initial distribution \( \xi \) and Markov kernel \( \bar{Q} \) independent of \( U \). The distribution of \( U \) under \( \bar{\mathbb{P}}_\xi \) is uniform over \( [0, 1] \).

**Lemma 3.** Assume \( \text{UGE 1} \), let \( m \in N^* \) and \( \xi \) be a probability measure on \( (Z, \mathcal{F}) \). Then, there exists a random process \( \bar{Z}_k^{*} \) defined on \( \bar{Z}_N, \bar{Z}_N, \bar{\mathbb{P}}_\xi \) such that for any \( k \in N \),

(a) \( \bar{Z}_k^{*} \) is independent of \( \bar{F}_{k+m} = \sigma\{\bar{Z}_\ell : \ell \geq k + m\}; \)

(b) \( \bar{\mathbb{P}}_\xi(\bar{Z}_k^{*} \neq \bar{Z}_k) \leq \Delta(\bar{Q}^m) \);

(c) the random variables \( \bar{Z}_k^{*} \) and \( \bar{Z}_k \) have the same distribution under \( \bar{\mathbb{P}}_\xi \).

**Proof.** Berbee’s lemma Rio (2017, Lemma 5.1) ensures that for any \( k \), there exists \( \bar{Z}_k^{*} \) satisfying (a), (c) and \( \bar{\mathbb{P}}_\xi(\bar{Z}_k^{*} \neq \bar{Z}_k) = \beta_\xi(\sigma(\bar{Z}_k), \bar{F}_{k+m}) \). Here for two sub-\( \sigma \)-fields \( \mathcal{G}, \mathcal{H} \) of \( \bar{Z}_N \),

\[
\beta_\xi(\mathcal{G}, \mathcal{H}) = \frac{1}{2} \sup_{j \in J} \sum_{i \in I} \left| \bar{\mathbb{P}}_\xi(A_i \cap B_j) - \bar{\mathbb{P}}_\xi(A_i)\bar{\mathbb{P}}_\xi(B_j) \right|, 
\]

and the supremum is taken over all pairs of partitions \( \{A_i\}_{i \in I} \in \mathcal{G}^I \) and \( \{B_j\}_{j \in J} \in \mathcal{H}^J \) of \( \bar{Z}_N \) with \( I \) and \( J \) finite. Applying Douc et al. (2018, Theorem 3.3) with \( \text{UGE 1} \) completes the proof. \( \square \)

**Proof.** Proof of Proposition 10. Recall that \( J_n^{(1)} = \alpha^2 \sum_{\ell=1}^{n-1} S_{\ell+1:n} \varepsilon(Z_\ell) \), where

\[ S_{\ell+1:n} = \sum_{k=\ell+1}^{n} (I - \alpha \hat{A})^{n-k} \hat{A}(Z_k)(I - \alpha \hat{A})^{k-1-\ell}. \]

(86)

We first set a constant block size \( m \in N^*, m \geq t_{\text{mix}} \) (to be determined later). In order to proceed with \( S_{\ell+1:n} \varepsilon(Z_\ell) \), we split \( S_{\ell+1:n} \) into a part measurable with respect to \( \mathcal{F}_m^\alpha = \sigma(Z_k : k \geq m + \ell) \) and a remainder term. Indeed, using its definition (86),

\[ S_{\ell+1:n} = (I - \alpha \hat{A})^{n-m-\ell} S_{\ell+1:n} + S_{\ell+1:n} (I - \alpha \hat{A})^m. \]

Let \( N = \lfloor (n-1)/m \rfloor \). With these notations, we can decompose \( J_n^{(1)} \) as a sum of three terms: \( J_n^{(1)} = T_1 + T_2 + T_3 \), with

\[ T_1 = \alpha^2 \sum_{\ell=1}^{m(N-1)} (I - \alpha \hat{A})^{n-m-\ell} S_{\ell+1:n} \varepsilon(Z_\ell) \]

\[ T_2 = \alpha^2 \sum_{\ell=1}^{m(N-1)} S_{\ell+1:n} (I - \alpha \hat{A})^m \varepsilon(Z_\ell), \]

\[ T_3 = \alpha^2 \sum_{\ell=m(N-1)+1}^{n-1} S_{\ell+1:n} \varepsilon(Z_\ell). \]
We bound the terms $T_1$, $T_2$ and $T_3$ separately. Using Minkowski’s inequality together with Proposition 1, Lemma 4, and the definition (86), we get

$$
\mathbb{E}^{1/p}_\xi[\|T_1\|_p] \leq \alpha^2 \sum_{\ell=1}^{m(N-1)} \kappa_Q^{1/2} (1 - \alpha \eta)^{(n-m-\ell)/2} \mathbb{E}^{1/p}_\xi[\|S_{\ell+1: \ell+m} \varepsilon(Z_{\ell})\|_p]
$$

$$
\leq 16 \alpha^2 \frac{3}{2} C_A D_S^{(M)} \|\varepsilon\|_\infty \sqrt{mt_{\text{mix}} \{\text{Tr }\Sigma_\varepsilon\}} \sum_{\ell=1}^{m(N-1)} (1 - \alpha \eta)^{(n-\ell-1)/2}
$$

$$
\leq 32 \kappa_Q^{3/2} C_A \alpha^{-1} \|\varepsilon\|_\infty \sqrt{mt_{\text{mix}}},
$$

where for the last inequality, we additionally used that $\sqrt{1-x} \leq 1 - x/2$ for $x \in [0, 1]$. Similarly, with Minkowski’s inequality and Lemma 4, we bound $T_3$:

$$
\mathbb{E}^{1/p}_\xi[\|T_3\|_p] \leq \alpha^2 \sum_{\ell=m(N-1)+1}^{n-1} \mathbb{E}^{1/p}_\xi[\|S_{\ell+1:n} \varepsilon(Z_{\ell})\|_p]
$$

$$
\leq 16 \sqrt{2} \alpha^2 \kappa_Q C_A \sqrt{mt_{\text{mix}} p} \|\varepsilon\|_\infty \sum_{\ell=m(N-1)+1}^{n-1} (1 - \alpha \eta)^{(n-\ell-1)/2}
$$

$$
\leq 32 \sqrt{2} \kappa_Q C_A \alpha^{-1} \sqrt{mt_{\text{mix}}} \|\varepsilon\|_\infty.
$$

In the bound above we used that $n - 1 - m(N - 1) \leq 2m$. Combining the above,

$$
\mathbb{E}^{1/p}_\xi[\|T_1\|_p] + \mathbb{E}^{1/p}_\xi[\|T_3\|_p] \leq c_1^{(M)} \alpha a \sqrt{mt_{\text{mix}}} \|\varepsilon\|_\infty,
$$

where $c_1^{(M)} = 32 \kappa_Q C_A \alpha^{-2}(\sqrt{2} + \kappa_Q^{1/2})$. It remains to bound $\mathbb{E}^{1/p}_\xi[\|T_2\|_p]$. We switch to the extended space $(\hat{Z}_N, \hat{Z}_T, \hat{E}_\varepsilon)$ and, using Lemma 3, we get that $\mathbb{E}^{1/p}_\xi[\|T_2\|_p] = \tilde{\mathbb{E}}^{1/p}_\xi[\|\tilde{T}_2\|_p]$ with

$$
\tilde{T}_2 = \alpha^2 \sum_{\ell=1}^{m(N-1)} \check{S}_{\ell+m+1:n} (1 - \alpha \bar{A})^m \varepsilon(\check{Z}_{\ell}).
$$

Here $\check{S}_{\ell+m+1:n}$ is a counterpart of $S_{\ell+m+1:n}$ defined on the extended space, that is,

$$
\check{S}_{\ell+m+1:n} = \sum_{k=\ell+m+1}^n (1 - \alpha \bar{A})^{n-k} \bar{A}(\check{Z}_k) (1 - \alpha \bar{A})^{k-1-\ell}.
$$

We further decompose $\tilde{T}_2 = \tilde{T}_{2,1} + \tilde{T}_{2,2}$, where

$$
\tilde{T}_{2,1} = \alpha^2 \sum_{k=0}^{N-2} \sum_{i=1}^m \check{S}_{(k+1)m+i+1:n} (1 - \alpha \bar{A})^m \varepsilon(\check{Z}_{km+i}),
$$

$$
\tilde{T}_{2,2} = \alpha^2 \sum_{k=0}^{N-2} \sum_{i=1}^m \check{S}_{(k+1)m+i+1:n} (1 - \alpha \bar{A})^m \{\varepsilon(\check{Z}_{km+i}) - \varepsilon(\check{Z}_{km+i})\}.
$$

We begin with bounding $\tilde{T}_{2,2}$. Set $V_\ell = \varepsilon(\check{Z}_\ell) - \varepsilon(\check{Z}_\ell^*)$ and $\check{F}_\ell = \sigma(\check{Z}_k, \check{Z}_k^* : k \leq \ell)$. Using Lemma 3 we get with the convention $0/0 = 0$,

$$
\tilde{\mathbb{E}}^{1/p}_\xi[\|\check{S}_{(k+1)m+i+1:n} (1 - \alpha \bar{A})^m V_{km+i}\|_p]
$$

$$
= \tilde{\mathbb{E}}^{1/p}_\xi[\|\check{S}_{(k+1)m+i+1:n} (1 - \alpha \bar{A})^m V_{km+i}\{\check{Z}_{km+i} \neq \check{Z}_{km+i}^*\}\|_p]
$$

$$
\leq \tilde{\mathbb{E}}^{1/p}_\xi[\|V_{km+i}\|_p] \tilde{\mathbb{E}}^{1/p}_\xi[\|\check{S}_{(k+1)m+i+1:n} (1 - \alpha \bar{A})^m V_{km+i}/\|V_{km+i}\|\|_p]
$$

$$
\leq \tilde{\mathbb{E}}^{1/p}_\xi[\|V_{km+i}\|_p] \sup_{u \in S^{d-1}, \ell' \in P(Z)} \tilde{\mathbb{E}}^{1/p}_\xi[\|\check{S}_{(k+1)m+i+1:n} (1 - \alpha \bar{A})^m u\|_p],
$$

29
where $P(Z)$ is the set of probability measure on $(Z, \mathcal{Z})$. Applying Lemma 5 and Proposition 1, for any $u \in S^{d-1}$ and probability measure $\xi'$,

\[
\mathbb{E}_\xi^{1/p}[\|S_{(k+1)m+i+1:n}(I - \alpha \tilde{A})^mu\|^p] 
\leq 168^{3/2} C_A [(n - (k + 1)m - i) t_{\text{mix}} (1 - \alpha a)^{n - km - i - 1}]^{1/2}.
\]

Moreover, under A2 and UGE 1, $\|V_{km+i}\| \leq 2\|\varepsilon\|\{\bar{Z}_{km+i} \neq \bar{Z}_{km+i}\}$, and $\mathbb{P}_\xi(\bar{Z}_{km+i} \neq \bar{Z}_{km+i}) \leq \Delta(Q^m) \leq (1/4)^{m/\text{mix}}$ by Lemma 3 and UGE 1. Combining the bounds above,

\[
\mathbb{E}_\xi^{1/p}[\|\tilde{S}_{(k+1)m+i+1:n}(I - \alpha \tilde{A})^mV_{km+i}\|^p] \leq 328^{3/2} C_A \|\varepsilon\|\{1/4\}^{1/p}[\|\tilde{A}^m\|^{m/\text{mix}} \times (n - (k + 1)m - i) (1 - \alpha a)^{(n - km - i - 1)} t_{\text{mix}}]^{1/2}.
\]

Substituting (89) into the definition (88) of $\tilde{T}_{2,2}$, and using

\[
\sum_{\ell=1}^{m(N-1)} \sqrt{n - \ell}(1 - \alpha a)(n - \ell + 1)^{1/2} \leq \int_0^{\infty} \ell^{1/2} e^{-\alpha \ell t/2} dt = 2^{3/2}(\alpha a)^{-3/2} \Gamma(3/2),
\]

we get

\[
\mathbb{E}_\xi^{1/p}[\|\tilde{T}_{2,2}\|^p] \leq c_2^{(M)} (\alpha a)^{1/2} (1/4)^{1/p}[\|\tilde{A}^m\|^{m/\text{mix}} \sqrt{t_{\text{mix}}}]^{1/2} \|\varepsilon\|_\infty,
\]

where $c_2^{(M)} = 64\pi^{1/2} K^3 C_A a^{-2}$. To obtain (90) we have additionally used that $m \geq 1$ and $\alpha a \leq 1/2$.

Now we bound $\tilde{T}_{2,1}$. Define the function $g(z) : Z \mapsto \mathbb{R}$, $g(z) = (I - \alpha \tilde{A})^m \varepsilon(z)$. A 2 and Proposition 1 imply $\|g\|_\infty \leq K^2 (1 - \alpha a)^m/\|\varepsilon\|_\infty$ and $\pi(g) = 0$. Then we apply Lemma 3 and Lemma 6, and obtain

\[
\mathbb{E}_\xi^{1/p}[\|\tilde{S}_{(k+1)m+i+1:n}\|^p] \leq c_2^{(M)} (\alpha a)^{1/2} (1/4)^{1/p}[\|\tilde{A}^m\|^{m/\text{mix}} \sqrt{t_{\text{mix}}}]^{1/2} \|\varepsilon\|_\infty,
\]

Assumption UGE 1 with $\pi(g) = 0$ implies $\|\xi Q^{km+i} g\| \leq \Delta(Q^{km+i}) \|g\|_\infty$. Combining it with Lemma 5,

\[
\sum_{i=1}^{m} \sum_{k=0}^{N-2} \|\xi Q^{km+i} g\| \sup_{u \in S^{d-1}} \mathbb{E}_\xi^{1/p}[\|S_{(k+1)m+i+1:n} u\|^p]
\]

\[
\leq 16K^3 C_A (1 - \alpha a)^{(m - 1)/2} \sup_{x \geq 1} \{x(1 - \alpha a)^x \}^{1/2} \sqrt{t_{\text{mix}}}/\|\varepsilon\|_\infty \sum_{\ell=0}^{\infty} \Delta(Q^{\ell})
\]

\[
\leq \frac{64}{3\varepsilon^{1/2}} (\alpha a)^{-1/2} K^3 C_A (1 - \alpha a)^{(m - 1)/2} t_{\text{mix}}^{3/2} \|\varepsilon\|_\infty,
\]

where we have used for the last inequality (32), $\alpha a \leq 1/2$, and $\sup_{x \geq 1} \{x(1 - \alpha a)^x \}^{1/2} \leq e^{-1/2}(\alpha a)^{-1/2}$. Jensen’s inequality together with Lemma 5 yields

\[
\sum_{i=1}^{m} \{\sum_{k=0}^{N-2} \sup_{u \in S^{d-1}} \mathbb{E}_\xi^{2/p}[\|S_{(k+1)m+i+1:n} u\|^p]\}^{1/2}
\]

\[
\leq \sqrt{m} \left\{ \sum_{\ell=1}^{m(N-1)} \sup_{u \in S^{d-1}} \mathbb{E}_\xi^{2/p}[\|S_{\ell+m+n} u\|^p]\right\}^{1/2}
\]

\[
\leq 16K^3 C_A (m t_{\text{mix}})^{1/2} \left\{ \sum_{\ell=1}^{m(N-1)} (n - \ell - m)(1 - \alpha a)^{n - \ell - m - 1}\right\}^{1/2}
\]

\[
\leq 16\sqrt{2} K^3 C_A (m t_{\text{mix}})^{1/2} (\alpha a)^{-1}.
\]
Combining the bounds above with \( \|g\|_\infty \leq \kappa_Q^{1/2} (1 - \alpha a)^{m/2} \|\varepsilon\|_\infty \), we get
\[
\mathbb{E}_\xi^{1/p} \|\tilde{T}_{2,1}\|_p^p \leq [32 \sqrt{2} \kappa_Q^{3/2} C_A a^{-2}] \sqrt{mt_{\text{mix}} \alpha p} \|\varepsilon\|_\infty^{3/2} + [(64 / 3e^{-1/2}) \kappa_Q^{3/2} C_A a^{-2}] (\alpha a p)^{3/2} \|\varepsilon\|_\infty^{1/2}.
\] (91)

Now the proof is completed combining (87), (90), and (91), setting
\[
m = t_{\text{mix}} \left\lceil \frac{p \log (1/\alpha a)}{2 \log(2)} \right\rceil,
\]
and using \( p^{1/2} \leq p \) and \( t_{\text{mix}} \leq t_{\text{mix}}. \) Indeed, with this choice of \( m, \) \( (1/4) (1/p) \lceil m / t_{\text{mix}} \rceil \leq \sqrt{\alpha a}, m \geq t_{\text{mix}}. \) In addition, note that \( m \leq 2t_{\text{mix}} p \log(1/\alpha a)/(2 \log(2)) \) using \( \alpha a \leq 1/2 \) and \( p \geq 2. \)

We now prove (41). The decomposition (12) implies \( H_n^{(1)} = -\alpha \sum_{\ell=1}^n \Gamma_{\ell+1:n}^\alpha \tilde{A} (Z_{\ell}) J_{\ell-1}^{(1)} \). Hence, with Minkowski’s and Holder’s inequalities,
\[
\mathbb{E}_\xi^{1/p} \|H_n^{(1)}\|_p^p \leq \alpha \sum_{\ell=1}^n \mathbb{E}_\xi^{1/2p} \|\Gamma_{\ell+1:n}^\alpha \tilde{A} (Z_{\ell})\|_2^{2p} \mathbb{E}_\xi^{1/2p} \|J_{\ell-1}^{(1)}\|_2^{2p}.
\]

Applying Proposition 7 and (40), we get
\[
\mathbb{E}_\xi^{1/p} \|H_n^{(1)}\|_p^p \leq 4e^2 \kappa_Q^{1/2} C_A \Delta_{j,1}^{(M)} t_{\text{mix}} d^{1/q} p^2 \alpha^2 a \sqrt{\log (1/\alpha a)} \|\varepsilon\|_\infty \sum_{\ell=1}^n e^{-\alpha a n/12} \\
+ 2e^2 \kappa_Q^{1/2} d^{1/q} \Delta_{j,2}^{(M)} p \alpha a \|\varepsilon\|_\infty \sum_{\ell=1}^n e^{-\alpha a n/12}.
\]

Now the proof follows from elementary bound \( e^{-x} \leq 1 - x/2, x \in [0,1]. \)

\[\square\]

### D.5 Proof of Proposition 11

Define the constants
\[
\begin{align*}
D_4^{(M)} &= 48 \kappa_Q^{1/2} e^3, \\
D_5^{(M)} &= 4e (D_{j,1}^{(M)} + D_{H,1}^{(M)}) + D_7^{(M)} / \sqrt{\log 2}, \\
D_6^{(M)} &= \sqrt{2e} (D_{j,2}^{(M)} + D_{H,2}^{(M)}). 
\end{align*}
\] (92)

Note first that
\[
\|\mathbb{E}_\xi [\tilde{\theta}_n] - \theta^*\| \leq (2/n) \sum_{t=n/2}^{n-1} \|\mathbb{E}_\xi [\tilde{\theta}_t] - \theta^*\|.
\]

Proceeding as in (46), we get for each \( t \in \{n/2, \ldots, n\} \) that
\[
\|\mathbb{E}_\xi [\tilde{\theta}_t] - \theta^*\| = \sup_{u \in \mathbb{R}^{d-1}} \left\{ \mathbb{E}_\xi [u^\top \Gamma_{1:t}^{(\alpha)} (\theta_0 - \theta^*)] + \mathbb{E}_\xi [u^\top J_{t}^{(0)}] + \mathbb{E}_\xi [u^\top H_{t}^{(0)}] \right\}.
\]

Now we bound each term above separately. For the first one note that
\[
\|\mathbb{E}_\xi [u^\top \Gamma_{1:t}^{(\alpha)} (\theta_0 - \theta^*)]\| \leq \mathbb{E}_\xi^{1/2} \left[ \|\Gamma_{1:t}^{(\alpha)}\|^2 \right] \|\theta_0 - \theta^*\| \leq \sqrt{\kappa_Q e^3 e^{-\alpha a t/12}} \|\theta_0 - \theta^*\|.
\]
Using the representation (9), Proposition 1, and UGE 1, we get that
\[ |E_\xi[u^\top J_t^{(0)}]| = \alpha |E_\xi[u^\top \sum_{j=1}^t (I - \alpha \tilde{A})^{t-j} \varepsilon(Z_j)]| \leq \alpha \kappa_Q^{1/2} C_A \|\varepsilon\|_\infty \sum_{t=0}^\infty \Delta(Q^t) \leq D_7^{(M)} a t_{\text{mix}} \|\varepsilon\|_\infty . \]

Moreover, applying the Cauchy-Schwartz inequality and Proposition 10, we get
\[ |E_\xi[u^\top H_t^{(0)}]| \leq E_\xi^{1/2}[\|H_t^{(0)}\|^2] \leq E_\xi^{1/2}[\|J_t^{(1)}\|^2] + E_\xi^{1/2}[\|H_t^{(1)}\|^2] \]
\[ \leq 4e(D_{J_1}^{(M)} + D_{H_1}^{(M)}) \|\varepsilon\|_\infty (a t_{\text{mix}}) \sqrt{\log(1/\alpha a)} + \sqrt{2}e(D_{J_2}^{(M)} + D_{H_2}^{(M)}) \|\varepsilon\|_\infty (a t_{\text{mix}})^{3/2} . \]

Combining the bounds above yields (42).

**D.6 Proof of Proposition 16**

**Proposition 16.** Assume A 1, A 2, and UGE 1. Then, for any \( \alpha \in (0, \alpha_\infty) \), \( t \in \mathbb{N}^* \) and initial probability measure \( \xi \) on \( (Z, \mathcal{Z}) \), it holds that
\[ \|E_\xi[\tilde{A}(Z_{t+1})J_t^{(0)}]\| \leq D_t^{(M)} a t_{\text{mix}} \|\varepsilon\|_\infty , \tag{93} \]
where the constant \( D_t^{(M)} \) is given by
\[ D_t^{(M)} = (4/3)\kappa_Q^{1/2} C_A a^{-1} . \]

**Proof.** Using (9), we get
\[ \|E_\xi[\tilde{A}(Z_{t+1})J_t^{(0)}]\| = \sup_{u \in \mathbb{S}^{d-1}} E_\xi[\alpha u^\top \tilde{A}(Z_{t+1}) \sum_{j=1}^t (I - \alpha \tilde{A})^{t-j} \varepsilon(Z_j)] . \]

Define for \( z \in Z \) and \( j \in \{1, \ldots, t\} \), the function \( g_{j,t}(z) : Z \mapsto \mathbb{R}^d \) as
\[ g_{j,t}(z) = \int_Z \tilde{A}(z')(1 - \alpha \tilde{A})^{t-j} \varepsilon(z) Q^{t-j+1}(z, dz') \]

Using that \( \pi(\tilde{A}) = 0 \) together with Proposition 1 and UGE 1, for any \( u \in \mathbb{S}^{d-1}, \)
\[ |u^\top g_{j,t}(z)| \leq \kappa_Q^{1/2} (1 - \alpha a)^{(t-j)/2} C_A \|\varepsilon\|_\infty \Delta(Q^{t-j+1}) . \]

Using the Markov property of \( (Z_k)_{k \in \mathbb{N}} \) and the definition of \( t_{\text{mix}} \) (see UGE 1), we get from the previous bound that
\[ |E_\xi[\alpha u^\top \tilde{A}(Z_{t+1}) \sum_{j=1}^t (1 - \alpha \tilde{A})^{t-j} \varepsilon(Z_j)]| \leq \alpha \kappa_Q^{1/2} C_A \|\varepsilon\|_\infty \sum_{t=0}^\infty \Delta(Q^t) \]
\[ \leq D_t^{(M)} a t_{\text{mix}} \|\varepsilon\|_\infty , \]
and (93) follows.
D.7 Proof of Theorem 3 and Theorem 4

Define the quantities

\[
R^{(8)}_{n,p,\alpha,t_{\text{mix}}} = \frac{8D_2^{(M)} e^{1/p} \sqrt{npt_{\text{mix}}}}{\sqrt{\alpha n}} + \frac{2^{1/2} C_{\text{Ros,1}}^{(M)} t_{\text{mix}}^{3/4} p \log_2(2p) n^{1/4}}{n^{1/2}} + \frac{2C_{\text{Ros,2}}^{(M)} t_{\text{mix}} p \log_2(2p)}{n^{1/2}} \\
+ 8e^{1/p} (D_{j,1}^{(M)} + D_{H,1}^{(M)}) \alpha t_{\text{mix}} \sqrt{\log(1/\alpha)p^2} (\alpha^{-1} n^{-1/2} + n^{1/2} C_A) \\
+ 8e^{1/p} (D_{j,2}^{(M)} + D_{H,2}^{(M)}) (\alpha t_{\text{mix}})^{3/2} p^{1/2} (\alpha^{-1} n^{-1/2} + n^{1/2} C_A)
\]

(94)

\[
R^{(tr)}_{n,p,\alpha,t_{\text{mix}}} = e^{2 + 1/p} \kappa Q \left( \frac{4}{\alpha n^{1/2}} + 2^{-1/2} n^{1/2} C_A \right).
\]

We begin with the proof of Theorem 4. The result of Theorem 3 will directly follow from it using the step size \( \alpha \) fixed in (37).

**Proof.** Proof of Theorem 4. Let \( p \geq 2 \) and \( q \geq p \) be a number to be fixed later and assume in addition that \( \alpha \in (0, e_{q,\infty} t_{\text{mix}}^{-1}) \). Below we use shorthand notations \( \tilde{A}_t, A_t, \varepsilon_t \) for \( A(Z_t), A(Z_t), \) and \( \varepsilon(Z_t) \), respectively. Proceeding as in (54) and (55), we decompose the \( p \)-th moment of LSA-PR error as

\[
(n/2)E^{1/p}_\xi \left[ \parallel A(\theta_n - \theta^*) \parallel^p \right] \leq E^{1/p}_\xi \left[ \parallel \sum_{t=n/2}^{n-1} \varepsilon_{t+1} \parallel^p \right] + T_1^{(M)} + T_2^{(M)} + T_3^{(M)}
\]

(95)

where we have set \( T_1^{(M)} = \alpha^{-1} E^{1/p}_\xi \left[ \parallel \theta_{n/2} - \theta_n \parallel^p \right], T_2^{(M)} = E^{1/p}_\xi \left[ \parallel \sum_{t=n/2}^{n-1} \tilde{A}_{t+1}^{(\alpha)} \{ \theta_0 - \theta^* \} \parallel^p \right] \) and

\[
T_3^{(M)} = E^{1/p}_\xi \left[ \parallel \sum_{t=n/2}^{n-1} \tilde{A}_{t+1}^{(0)} \{ \tilde{\theta}_t \} \parallel^p \right] + E^{1/p}_\xi \left[ \parallel \sum_{t=n/2}^{n-1} \tilde{A}_{t+1}^{(0)} \{ H_t \} \parallel^p \right].
\]

Now we bound each term in the decomposition (95). We begin with the first term, which is linear statistics of uniformly geometrically ergodic Markov chain. Applying Theorem 6, we obtain

\[
E^{1/p}_\xi \left[ \parallel \sum_{t=n/2}^{n-1} \varepsilon_{t+1} \parallel^p \right] \leq C_{\text{Ren,1}} p^{1/2} n^{1/2} \left\{ \text{Tr} \Sigma^{(M)}_\varepsilon \right\}^{1/2} + C_{\text{Ros,1}}^{(M)} \| \varepsilon \|_{\text{\tiny{\infty}}} (n/2)^{1/4} t_{\text{mix}}^{3/4} p \log_2(2p) + C_{\text{Ros,2}}^{(M)} \| \varepsilon \|_{\text{\tiny{\infty}} t_{\text{mix}} p \log_2(2p)}.
\]

Applying Proposition 9 and Minkowski’s inequality, we get

\[
T_1^{(M)} \leq 2 \alpha^{-1} \sqrt{\kappa Q} e^{2d_{1/q}} d^{1/q} e^{-aan/24} \parallel \theta_0 - \theta^* \parallel + 2D_2^{(M)} d^{1/q} \alpha^{-1/2} \sqrt{ap t_{\text{mix}}} \| \varepsilon \|_{\text{\tiny{\infty}}}.
\]

Applying Proposition 7, Minkowski’s inequality, and using A 1, we get

\[
T_2^{(M)} \leq (n/2) \sqrt{\kappa Q} e^{2d_{1/q}} C_A e^{-aan/24} \parallel \theta_0 - \theta^* \parallel.
\]

It remains to proceed with \( T_3^{(M)} \). Using the representation (9),

\[
\sum_{t=n/2}^{n-1} H_t^{(0)} = \sum_{t=n/2}^{n-1} \left( \{ 1 - \alpha A(Z_{t+1}) \} H_t^{(0)} - \alpha \sum_{t=n/2}^{n-1} \tilde{A}_{t+1}^{(0)} J_t^{(0)} \right)
\]

which yields

\[
\sum_{t=n/2}^{n-1} \tilde{A}_{t+1}^{(0)} J_t^{(0)} = \alpha^{-1} (H_{n/2}^{(0)} - H_n^{(0)}) - \sum_{t=n/2}^{n-1} A(Z_{t+1}) H_t^{(0)}.
\]

Applying again Minkowski’s inequality, we get

\[
E^{1/p}_\xi \left[ \parallel \sum_{t=n/2}^{n-1} \tilde{A}_{t+1}^{(0)} J_t^{(0)} \parallel^p \right] \leq \{ 2 \alpha^{-1} + (n/2) C_A \} \sup_{t \in N^*} E^{1/p}_\xi \left[ \parallel H_t^{(0)} \parallel^p \right].
\]

Now it remains to combine the bounds above in (95), use Proposition 10, and set \( q = p(1+\log d) \). \( \square \)
E  Technical bounds: Markov case

Recall that $$S_{\ell+1:t+m}$$ is defined, for $$\ell, m \in \mathbb{N}^*$$, as

$$S_{\ell+1:t+m} = \sum_{k=1}^{t+m} B_k(Z_k),$$

with $$B_k(z) = (I - \alpha A)^{\ell + m - k} \mathbf{A}_k(z)(I - \alpha A)^{k - 1 - \ell}.$$ \hfill (96)

Lemma 4. Assume $$A_1, A_2$$, and $$UGE_1$$. Then, for any $$p \geq 2$$, any initial probability $$\xi$$ on $$(Z, \mathcal{Z})$$, $$\ell, m \in \mathbb{N}^*$$, it holds that

$$\mathbb{E}^{1/p}_\xi(\|S_{\ell+1:t+m}e(Z_t)\|^p) \leq D_S^{(M)} m^{1/2}(1 - \alpha a)^{(m-1)/2} \sqrt{t_{\text{mix}}(P)} \|e\|_\infty,$$

where $$D_S^{(M)} = 16\kappa Q C_A$$.

Proof. Proof. Now, with $$\mathcal{F}_t = \sigma\{Z_j, j \leq \ell\}$$, it holds that

$$\mathbb{E}^{1/p}_\xi(\|S_{\ell+1:t+m}e(Z_t)\|^p) = \mathbb{E}^{1/p}_\xi(\|e(Z_t)\|^p \mathcal{F}_t [S_{\ell+1:t+m}e(Z_t)/\|e(Z_t)\|])$$

$$\leq \mathbb{E}^{1/p}_\xi(\|e(Z_t)\|^p \sup_{u \in \mathbb{S}^{d-1}, \ell' \in \mathcal{P}(Z)} \mathbb{E}_{\ell'}[\|S_{\ell+1:t+m}u\|^p]),$$

where $$\mathcal{P}(Z)$$ denotes the set of probability measure on $$(Z, \mathcal{Z})$$. Combining the above bounds with Lemma 5 and A 2 yields the statement. \hfill \Box

Lemma 5. Assume $$A_1, A_2$$, and $$UGE_1$$. For any $$\ell, m \in \mathbb{N}^*$$, $$t \geq 0$$, $$u \in \mathbb{S}^{d-1}$$, and initial probability $$\xi$$ on $$(Z, \mathcal{Z})$$, it holds that

$$\mathbb{P}_\xi(\|S_{\ell+1:t+m}u\| \geq t) \leq 2 \exp\left\{-\frac{t^2}{2\gamma_m^2}\right\},$$

where $$\gamma_m = 8\kappa Q C_A [mt_{\text{mix}}(1 - \alpha a)^{m-1}]^{1/2}$$.

Moreover,

$$\sup_{u \in \mathbb{S}^{d-1}} \mathbb{E}^{1/p}_\xi(\|S_{\ell+1:t+m}u\|^p) \leq 16\kappa Q C_A [mt_{\text{mix}}(1 - \alpha a)^{m-1}]^{1/2}.$$ \hfill \Box

Proof. Proof. Define $$g_k(z) : Z \mapsto \mathbb{R}^d$$ as $$g_k(z) = B_k(z)u$$ where $$B_k$$ is given in (96). Note that under A 1 and applying Proposition 1, $$\pi(g_k) = 0$$ and $$\sup_{z \in Z} \|g_k(z)\| \leq \kappa Q C_A (1 - \alpha a)^{(m-1)/2}$$ for any $$k \in \{\ell + 1, \ldots, \ell + m\}$$. The proof then follows from Lemma 9 and Lemma 7. \hfill \Box

Lemma 6. Let $$(\Omega, \mathcal{G}, \mathbb{P})$$ be a probability space, $$\{W_k, W_k^\ast\}_{k \in \mathbb{N}}$$ be a sequence of $$\mathbb{Z}^2$$-valued random variables, and $$\{\mathbf{A}_k\}_{k \in \{2, \ldots, N+1\}}$$ be a sequence of $$d \times d$$ random matrices. Denote $$\mathcal{G}_k = \sigma(W_\ell, \ell \geq k)$$ for $$k \in \mathbb{N}^*$$. Assume that for $$k \in \mathbb{N}^*$$, that $$\mathbf{A}_k$$ is $$\mathcal{G}_k$$-measurable and $$\sigma(W_k^\ast)$$ and $$\mathcal{G}_k$$ are independent. Then, for any family of measurable functions $$\{g_k\}_{k=1}^N$$ from $$Z$$ to $$\mathbb{R}^d$$, with $$\max_{k \in \{1, \ldots, N\}} \|g_k\|_{\infty} \leq 1$$, and $$p \geq 2$$,

$$\mathbb{E}^{1/p}(\|\sum_{k=1}^N \mathbf{A}_{k+1}g_k(W_k^\ast)\|^p) \leq 2p\left\{\sum_{k=1}^N \sup_{u \in \mathbb{S}^{d-1}} \mathbb{E}^{2/p}(\|\mathbf{A}_{k+1}u\|^p)\right\}^{1/2} + \mathbb{E}^{1/p}(\|\sum_{k=1}^N \mathbf{A}_{k+1} \mathbb{E}^{\mathcal{G}_{k+1}}[g_k(W_k^\ast)]\|^p).$$

Proof. Proof. Applying Minkowski’s inequality,

$$\mathbb{E}^{1/p}(\|\sum_{k=1}^N \mathbf{A}_{k+1}g_k(W_k^\ast)\|^p) \leq \mathbb{E}^{1/p}(\|\sum_{k=1}^N \mathbf{A}_{k+1} \mathbb{E}^{\mathcal{G}_{k+1}}[g_k(W_k^\ast)]\|^p)$$

$$+ \mathbb{E}^{1/p}(\|\sum_{k=1}^N \mathbf{A}_{k+1}\{g_k(W_k^\ast) - \mathbb{E}^{\mathcal{G}_{k+1}}[g_k(W_k^\ast)]\}\|^p).$$

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The sequence \( \{ \tilde{A}_k(g_k(W_k^*)) - \mathbb{E}^{\mathcal{G}_{k+1}}[g_k(W_k^*)] \}_k \) is a reversed martingale difference sequence with respect to \( \{ \mathcal{G}_k \}_{k \geq 1} \). Hence, applying the Burkholder inequality (see Osekowski (2012, Theorem 8.6)), we obtain
\[
\mathbb{E}^{1/p}[\| \sum_{k=1}^N \tilde{A}_k(g_k(W_k^*)) - \mathbb{E}^{\mathcal{G}_{k+1}}[g_k(W_k^*)] \|^p] \leq p (\sum_{k=1}^N \mathbb{E}^{2/p}[\| \tilde{A}_k(g_k(W_k^*)) - \mathbb{E}^{\mathcal{G}_{k+1}}[g_k(W_k^*)] \|^p]^{1/2}).
\]

\[\Box\]

F. Technical lemmas

In this section we first provide a sharp Rosenthal inequality for the Markov chain \( \{ Z_n \}_{n \in \mathbb{N}} \) under \textbf{UGE 1}. This result is due to (Durmus et al., 2023, Theorem 1). Under \textbf{UGE 1}, it is known (see e.g., (Douc et al., 2018, Theorem 21.2.10)) that, for bounded functions \( f: \mathbb{Z} \to \mathbb{R}^d \), linear statistics \( n^{-1/2} \sum_{i=0}^{n-1} (f(Z_i) - \pi(f)) \) converges in distribution to the zero-mean Gaussian distribution with variance given by
\[
\sigma_n^2(f) = \lim_{n \to \infty} n^{-1} \mathbb{E}[\| \sum_{i=0}^{n-1} (f(Z_i) - \pi(f)) \|^2].
\]

**Theorem 6.** Assume \textbf{UGE 1}. Then, for any measurable function \( f: \mathbb{Z} \to \mathbb{R}^d \), \( \|f\|_\infty \leq 1 \), \( p \geq 2 \), and \( n \geq 1 \), it holds
\[
\mathbb{E}^{1/p}[\| \sum_{i=0}^{n-1} f(Z_i) - \pi(f) \|^p] \leq C_{Rm,1} \sqrt{2} p^{1/2} n^{1/2} \sigma_n(f) + C_{Ros,1} n^{1/4} \sigma^2 \log(2p) + C_{Ros,2} t \log(2p),
\]
where
\[
C_{Ros,1} = \frac{16 \sqrt{19}}{3 \sqrt{3}} C_{Rm,1}^{5/2}, \quad C_{Ros,2} = 64 (C_{Rm,1}^{1/2} + C_{Rm,2}),
\]
where the constants \( C_{Rm,1}, C_{Rm,2} \) are given in Appendix A and \( \sigma_n^2(f) \) is defined in (97).

Now we provide a standard moment bounds for sub-Gaussian random variable, which is proven for completeness.

**Lemma 7.** Let \( X \) be an \( \mathbb{R}^d \)-valued random variable satisfying \( \mathbb{P}(\| X \| \geq t) \leq 2 \exp(-t^2/(2\sigma^2)) \) for any \( t \geq 0 \) and some \( \sigma^2 > 0 \). Then, for any \( p \geq 2 \), it holds that \( \mathbb{E}[\| X \|^p] \leq 2p^{p/2} \sigma^p \).

**Proof.** Proof. Using Fubini’s theorem and the change of variable formula,
\[
\mathbb{E}[\| X \|^p] = \int_0^\infty p t^{p-1} \mathbb{P}(\| X \| \geq t) \, dt = p 2^{p/2} \sigma^p \Gamma(p/2),
\]
where \( \Gamma \) is the Gamma function. It remains to apply the bound \( \Gamma(p/2) \leq (p/2)^{p/2-1} \), which holds for \( p \geq 2 \) due to Anderson and Qiu (1997, Theorem 1.5).

Now we present the general version of Hoeffding inequality for martingale-difference sequences, taking values in Banach spaces. This result is due to Pinelis (1994, Theorem 3.5). Below we specify this inequality to the special case of sum of zero-mean independent random vectors.

**Lemma 8.** Let \( X_1, \ldots, X_n \in \mathbb{R}^d \) be independent random vectors satisfying \( \| X_i \| \leq \beta_i \) \( \mathbb{P} \)-a.s. and \( \mathbb{E}[X_i] = 0 \), \( i \in \{1, \ldots, n\} \). Then, for any \( t \geq 0 \), it holds
\[
\mathbb{P} \left( \| \sum_{i=1}^n X_i \| \geq t \right) \leq 2 \exp \left\{ - \frac{t^2}{2 \sum_{j=1}^n \beta_j^2} \right\}.
\]
The result above can be generalized for bounded $\mathbb{R}^d$-valued functions of the Markov chains with kernel satisfying UGE 1.

**Lemma 9.** Assume UGE 1. Let $\{g_i\}_{i=1}^n$ be a family of measurable functions from $Z$ to $\mathbb{R}^d$ such that $\|g\|_\infty = \max_{i \in \{1, \ldots, n\}} \|g_i\|_\infty < \infty$ and $\pi(g_i) = 0$ for any $i \in \{1, \ldots, n\}$. Then, for any initial probability $\xi$ on $(Z, Z)$, $n \in \mathbb{N}$, $t \geq 0$, it holds

$$
P_\xi \left( \left\| \sum_{i=1}^n g_i(Z_i) \right\| \geq t \right) \leq 2 \exp \left\{ - \frac{t^2}{2u_n^2} \right\}, \text{ where } u_n = 8\|g\|_\infty \sqrt{n} \sqrt{t_{mix}}. \quad (98)$$

**Proof.** The function $\varphi(z_1, \ldots, z_n) := \|\sum_{i=1}^n g_i(z_i)\|$ on $Z^n$ satisfies the bounded differences property. Moreover, $(1/2) \sup_{z, z' \in Z} \|Q^{mix}(z, \cdot) - Q^{mix}(z', \cdot)\|_{TV} \leq 1/4$ by definition of $t_{mix}$. Thus, applying Paulin (2015, Corollary 2.10), we get for $t \geq E_\xi[\| \sum_{i=1}^n g_i(Z_i) \|]$

$$
P_\xi \left( \left\| \sum_{i=1}^n g_i(Z_i) \right\| \geq t \right) \leq \exp \left\{ - \frac{2(t - E_\xi[\| \sum_{i=1}^n g_i(Z_i) \|])^2}{9n\|g\|_\infty^2 t_{mix}} \right\}. \quad (98)$$

It remains to upper bound $\mathbb{E}_\xi[\| \sum_{i=1}^n g_i(Z_i) \|]$. Note that

$$
\mathbb{E}_\xi[\| \sum_{i=1}^n g_i(Z_i) \|^2] = \sum_{i=1}^n \mathbb{E}_\xi[\|g_i(Z_i)\|^2] + 2 \sum_{k=1}^{n-1} \sum_{\ell=1}^{n-k} \mathbb{E}_\xi[g_k(Z_k)^T g_{k+\ell}(Z_{k+\ell})].
$$

and, using UGE 1 and $\pi(g_{k+\ell}) = 0$, we obtain

$$
\mathbb{E}_\xi[g_k(Z_k)^T g_{k+\ell}(Z_{k+\ell})] = \left| \int_Z g_k(z)^T \left( Q^\ell_{g_{k+\ell}}(z) - \pi(g_{k+\ell}) \right) Q^k(dz) \right| \leq \|g\|_\infty^2 \Delta(Q^\ell) .
$$

Together with (32), this implies

$$
\sum_{k=1}^{n-1} \sum_{\ell=1}^{n-k} \mathbb{E}_\xi[g_k(Z_k)^T g_{k+\ell}(Z_{k+\ell})] \leq \sum_{k=1}^{n-1} \|g\|_\infty^2 \Delta(Q^\ell) \leq (4/3)\|g\|_\infty^2 t_{mix} n .
$$

Combining the bounds above, we upper bound $\mathbb{E}_\xi[\| \sum_{i=1}^n g_i(Z_i) \|]$ as

$$
\mathbb{E}_\xi[\| \sum_{i=1}^n g_i(Z_i) \|] \leq \left( \mathbb{E}_\xi[\| \sum_{i=1}^n g_i(Z_i) \|^2] \right)^{1/2} \leq 2\sqrt{n}\|g\|_\infty \sqrt{t_{mix}} =: v_n
$$

Plugging this result in (98), we obtain that

$$
P_\xi \left( \left\| \sum_{i=1}^n g_i(Z_i) \right\| \geq t \right) \leq \begin{cases} 1, & t < v_n, \\ \exp \left\{ - \frac{2(t - v_n)^2}{3v_n^2} \right\}, & t \geq v_n \end{cases}. \quad (99)$$

Now it is easy to see that right-hand side of (99) is upper bounded by $2 \exp\{-t^2/(8v_n^2)\}$ for any $t \geq 0$, and the statement follows.
References

R. Aguech, E. Moulines, and P. Priouret. On a perturbation approach for the analysis of stochastic tracking algorithms. *SIAM Journal on Control and Optimization*, 39(3):872–899, 2000.

G. D. Anderson and S.-L. Qiu. A monotonicity property of the gamma function. *Proc. Amer. Math. Soc.*, 125(11):3355–3362, 1997. ISSN 0002-9939. doi: 10.1090/S0002-9939-97-04152-X. URL https://doi.org/10.1090/S0002-9939-97-04152-X.

F. Bach and E. Moulines. Non-strongly-convex smooth stochastic approximation with convergence rate $o(1/n)$. In C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 26. Curran Associates, Inc., 2013. URL https://proceedings.neurips.cc/paper/2013/file/7fe1f8abaad094e0b5cb1b01d712f708-Paper.pdf.

A. Benveniste, M. Méoultry, and P. Priouret. *Adaptive algorithms and stochastic approximations*, volume 22. Springer Science & Business Media, 2012.

D. Bertsekas. *Reinforcement learning and optimal control*. Athena Scientific, 2019.

D. P. Bertsekas and J. N. Tsitsiklis. Parallel and distributed computation: numerical methods. 2003.

J. Bhandari, D. Russo, and R. Singal. A finite time analysis of temporal difference learning with linear function approximation. *Operations Research*, 69(3):950–973, 2021. doi: 10.1287/opre.2020.2024. URL https://doi.org/10.1287/opre.2020.2024.

V. S. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.

L. Bottou, F. E. Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. *Siam Review*, 60(2):223–311, 2018.

S. Chen, A. Devraj, A. Busic, and S. Meyn. Explicit mean-square error bounds for monte-carlo and linear stochastic approximation. In *International Conference on Artificial Intelligence and Statistics*, pages 4173–4183. PMLR, 2020.

G. Dalal, B. Szorenyi, and G. Thoppe. A tale of two-timescale reinforcement learning with the tightest finite-time bound. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 3701–3708, 2020.

R. Douc, E. Moulines, P. Priouret, and P. Soulier. *Markov chains*. Springer Series in Operations Research and Financial Engineering. Springer, 2018. ISBN 978-3-319-97703-4.

A. Durmus, E. Moulines, A. Naumov, S. Samsonov, K. Scaman, and H.-T. Wai. Tight high probability bounds for linear stochastic approximation with fixed stepsize. In M. Ranzato, A. Beygelzimer, K. Nguyen, P. S. Liang, J. W. Vaughan, and Y. Dauphin, editors, *Advances in Neural Information Processing Systems*, volume 34, pages 30063–30074. Curran Associates, Inc., 2021a. URL https://proceedings.neurips.cc/paper/2021/file/fc95fa5740ba01a870cfa52f671fe1e4-Paper.pdf.

A. Durmus, E. Moulines, A. Naumov, S. Samsonov, and H.-T. Wai. On the stability of random matrix product with markovian noise: Application to linear stochastic approximation and td
learning. In M. Belkin and S. Kpotufe, editors, Proceedings of Thirty Fourth Conference on Learning Theory, volume 134 of Proceedings of Machine Learning Research, pages 1711–1752. PMLR, 15–19 Aug 2021b. URL https://proceedings.mlr.press/v134/durmus21a.html.

A. Durmus, E. Moulines, A. Naumov, S. Samsonov, and M. Sheshukova. Rosenthal-type inequalities for linear statistics of Markov chains. arXiv preprint arXiv:2303.05838, 2023.

E. Eweda and O. Macchi. Quadratic mean and almost-sure convergence of unbounded stochastic approximation algorithms with correlated observations. Ann. Inst. H. Poincaré Sect. B (N.S.), 19(3):235–255, 1983. ISSN 0020-2347.

L. Guo and L. Ljung. Exponential stability of general tracking algorithms. IEEE Transactions on Automatic Control, 40(8):1376–1387, 1995.

F. Hiai and D. Petz. Introduction to Matrix Analysis and Applications. Universitext. Springer International Publishing, 2014. ISBN 9783319041506.

D. Huang, J. Niles-Weed, J. A. Tropp, and R. Ward. Matrix concentration for products. Foundations of Computational Mathematics, pages 1–33, 2021.

P. Jain, S. Kakade, R. Kidambi, P. Netrapalli, and A. Sidford. Parallelizing stochastic gradient descent for least squares regression: mini-batching, averaging, and model misspecification. Journal of Machine Learning Research, 18, 2018a.

P. Jain, S. M. Kakade, R. Kidambi, P. Netrapalli, and A. Sidford. Accelerating stochastic gradient descent for least squares regression. In Conference On Learning Theory, pages 545–604. PMLR, 2018b.

P. Jain, D. Nagaraj, and P. Netrapalli. Making the last iterate of sgd information theoretically optimal. In A. Beygelzimer and D. Hsu, editors, Proceedings of the Thirty-Second Conference on Learning Theory, volume 99 of Proceedings of Machine Learning Research, pages 1752–1755, Phoenix, USA, 25–28 Jun 2019. PMLR.

A. Joulin and Y. Ollivier. Curvature, concentration and error estimates for markov chain monte carlo. The Annals of Probability, 38(6):2418–2442, 2010.

H. Kushner and G. G. Yin. Stochastic approximation and recursive algorithms and applications, volume 35. Springer Science & Business Media, 2003.

C. Lakshminarayanan and C. Szepesvari. Linear stochastic approximation: How far does constant step-size and iterate averaging go? In A. Beygelzimer and F. Perez-Cruz, editors, Proceedings of the Twenty-First International Conference on Artificial Intelligence and Statistics, volume 84 of Proceedings of Machine Learning Research, pages 1347–1355. PMLR, 2018.

C. K. Lauand and S. Meyn. Bias in stochastic approximation cannot be eliminated with averaging. In 2022 58th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pages 1–4, 2022. doi: 10.1109/Allerton49937.2022.9929369.

L. Ljung. Recursive identification algorithms. Circuits, Systems and Signal Processing, 21(1):57–68, 2002.

W. Mou, C. J. Li, M. J. Wainwright, P. L. Bartlett, and M. I. Jordan. On linear stochastic approximation: Fine-grained polyak-ruppert and non-asymptotic concentration. In Conference on Learning Theory, pages 2947–2997. PMLR, 2020.
W. Mou, A. Pananjady, M. J. Wainwright, and P. L. Bartlett. Optimal and instance-dependent guarantees for markovian linear stochastic approximation. *arXiv preprint arXiv:2112.12770*, 2021.

A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on optimization*, 19(4):1574–1609, 2009.

A. Osekowski. *Sharp Martingale and Semimartingale Inequalities*. Monografie Matematyczne 72. Birkhäuser Basel, 1 edition, 2012. ISBN 3034803699, 9783034803694.

D. Paulin. Concentration inequalities for Markov chains by Marton couplings and spectral methods. *Electronic Journal of Probability*, 20(none):1 – 32, 2015. doi: 10.1214/EJP.v20-4039. URL https://doi.org/10.1214/EJP.v20-4039.

I. Pinelis. Optimum Bounds for the Distributions of Martingales in Banach Spaces. *The Annals of Probability*, 22(4):1679 – 1706, 1994. doi: 10.1214/aop/1176988477. URL https://doi.org/10.1214/aop/1176988477.

B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM journal on control and optimization*, 30(4):838–855, 1992.

A. Rakhlin, O. Shamir, and K. Sridharan. Making gradient descent optimal for strongly convex stochastic optimization. In *Proceedings of the 29th International Conference on International Conference on Machine Learning*, pages 1571–1578, 2012.

E. Rio. *Asymptotic Theory of Weakly Dependent Random Processes*. Springer, 2017.

D. Ruppert. Efficient estimations from a slowly convergent robbins-monro process. Technical report, Cornell University Operations Research and Industrial Engineering, 1988.

R. Srikant and L. Ying. Finite-Time Error Bounds For Linear Stochastic Approximation and TD Learning. In *Conference on Learning Theory*, 2019.

R. S. Sutton. Learning to predict by the methods of temporal differences. *Machine Learning*, 3(1): 9–44, Aug 1988. ISSN 1573-0565. doi: 10.1007/BF00115009.

R. Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2018.

C. J. Watkins and P. Dayan. Q-learning. *Machine learning*, 8(3-4):279–292, 1992.

B. Widrow and S. D. Stearns. Adaptive signal processing prentice-hall. *Englewood Cliffs, NJ*, 1985.